Analysis & PDE

Volume 16 No. 4 2023

msp

Analysis & PDE
msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2023 is US $405/year for the electronic version, and $630/year (+$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS AND PDE
Vol. 16 (2023), No. 4, pp. 891–926
DOI: 10.2140/apde.2023.16.891

STRONG SEMICLASSICAL LIMITS FROM HARTREE AND HARTREE–FOCK TO VLASOV–POISSON EQUATIONS

LAURENT LAFLECHE AND CHIARA SAFFIRIO

We consider the semiclassical limit from the Hartree to the Vlasov equation with general singular interaction potential including the Coulomb and gravitational interactions, and we prove explicit bounds in the strong topologies of Schatten norms. Moreover, in the case of fermions, we provide estimates on the size of the exchange term in the Hartree–Fock equation and also obtain a rate of convergence for the semiclassical limit from the Hartree–Fock to the Vlasov equation in Schatten norms. Our results hold for general initial data in some Sobolev space and any fixed time interval.

1. Introduction

The Vlasov equation is a kinetic equation describing the time evolution of the probability density of particles in interaction, such as particles in a plasma or in a galaxy. The problem of deriving this equation from the dynamics of N quantum interacting particles in a joint mean-field and semiclassical approximation is a classical question in mathematical physics, and the first rigorous results were obtained in the 1980s (see [Narnhofer and Sewell 1981; Spohn 1981]).

We study here the semiclassical limit from the Hartree and Hartree-Fock equations towards the Vlasov equation, i.e., the limit corresponding to a regime in which the Planck constant h becomes negligible. For any fixed time interval, we obtain quantitative Schatten norm estimates between the solutions of the quantum equations (Hartree and Hartree-Fock) and the Weyl quantization of the solution of the Vlasov equation. In particular, these imply the convergence of the Wigner transform of the quantum equations towards the solution of the Vlasov equation.

1A. *Context and state of the art.*

Vlasov equation. The Vlasov equation is a nonlinear transport equation for the probability density $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$;

$$\partial_t f + \xi \cdot \nabla_x f + E \cdot \nabla_\xi f = 0, \tag{1}$$

where $t \in \mathbb{R}_+$ denotes the time variable, $x \in \mathbb{R}^d$ denotes the space variable and $\xi \in \mathbb{R}^d$ denotes the momentum variable. In the above equation, $E := -\nabla K * \rho_f$ is the self-induced mean-field force field

Saffirio acknowledges the support of the Swiss National Science Foundation through the Eccellenza project PCEFP2_181153 and of the NCCR SwissMAP.
MSC2020: primary 35Q41, 35Q55, 82C10; secondary 35Q83, 82C05.
Keywords: Hartree equation, Hartree–Fock equation, Vlasov equation, Coulomb interaction, gravitational interaction, semiclassical limit.

created by the pair interaction potential $K : \mathbb{R}^d \to \mathbb{R}$ through the formula

$$-(\nabla K * \rho_f)(t,x) = -\int_{\mathbb{R}^d} \nabla K(x-y)\rho_f(t,y)\,\mathrm{d}y,$$

where ρ_f is the spatial density associated to f, namely

$$\rho_f(t,x) = \int_{\mathbb{R}^d} f(t,x,\xi)\,\mathrm{d}\xi.$$

When K is the Green's function for the Laplace operator, (1) is called the Vlasov-Poisson system because K can be obtained as a solution to the Poisson equation $-\Delta K = \rho_f$, thus linking the Vlasov equation to the Poisson equation. In this case, in dimension 3, K corresponds to the Coulomb potential

$$K(x) = \frac{1}{4\pi|x|},$$

but our method allows us to consider more general attractive and repulsive potentials. To simplify the presentation, we will look at homogeneous potentials of the form $K(x) = \pm|x|^{-a}$ or at $K(x) = \pm\ln(|x|)$, and we will then indicate how to generalize our results to a class of Sobolev spaces (see page 899).

The well-posedness of the Vlasov equation (1) is due to Dobrushin [1979] for smooth interaction potentials $K \in C_c^2(\mathbb{R}^d)$. Concerning singular interactions, the cases of Coulomb and gravitational potentials have been tackled first in [Iordanskiĭ 1961] and [Ukai and Okabe 1978] for $d = 1$ and $d = 2$, respectively. In $d = 3$, the well-posedness for small data has been proven in [Bardos and Degond 1985] and later extended to general initial data by Pfaffelmoser [1992] and by Lions and Perthame [1991]. In recent years, improvements on the conditions of propagation of momenta and on the uniqueness condition have been addressed in [Desvillettes et al. 2015; Holding and Miot 2018; Loeper 2006; Miot 2016; Pallard 2012; 2014]. The setting of this paper will be close to the setting of the paper by Lions and Perthame [1991]; that is the one that best suits the comparison with the quantum dynamics because of its Eulerian viewpoint.

The Vlasov equation (1) is supposed to emerge as a joint mean-field (weakly interacting particles at high density) and semiclassical limit from the dynamics of N interacting quantum particles. This was first proven in [Narnhofer and Sewell 1981] and [Spohn 1981] for analytic and C^2 interaction potentials, respectively, using the BBGKY approach in the fermionic setting. The case of bosons interacting through a smooth pair potential has been studied in [Graffi et al. 2003] in the mean-field limit combined with a semiclassical limit through the analysis of the dynamics of factored WKB states.

Hartree and Hartree–Fock equations. It is well known that the many-body dynamics can be approximated in the mean-field limit by the Hartree equation

$$i\hbar\partial_t \boldsymbol{\rho} = [H, \boldsymbol{\rho}], \tag{2}$$

an evolution equation for the density operator $\boldsymbol{\rho} = \boldsymbol{\rho}(t)$, a nonnegative bounded operator on the space $L^2(\mathbb{R}^d)$ with $\mathrm{Tr}(\boldsymbol{\rho}) = 1$. In (2), $\hbar = \frac{h}{2\pi}$ is the reduced Planck constant, and H is the Hamiltonian

$$H = -\tfrac{1}{2}\hbar^2\Delta + K * \rho, \tag{3}$$

where Δ is the Laplace operator, K is the pair interaction potential, $\rho(x) = \boldsymbol{\rho}(x, x)$ is the diagonal of the integral kernel of the trace class operator $\boldsymbol{\rho}$ and $K * \rho$ is identified with the operator of multiplication by the function $x \mapsto K * \rho(x)$.

In the case of fermions, a more precise mean-field approximation for the many-body quantum dynamics is given by the Hartree-Fock equation

$$i\hbar\partial_t \boldsymbol{\rho} = [H_{\mathrm{HF}}, \boldsymbol{\rho}], \tag{4}$$

with $H_{\mathrm{HF}} = -\hbar^2\Delta + K * \rho - \mathsf{X}$, where X is the so-called exchange term defined as the operator with integral kernel

$$\mathsf{X}(x, y) = K(x - y)\boldsymbol{\rho}(x, y). \tag{5}$$

We recall that the interest in the mean-field regime is due to the fact that many systems of interest in quantum mechanics are usually made of large numbers of particles, which typically range between 10^2 and 10^{23}, while the above equations only describe the behavior of one typical particle in a system of infinitely many particles. The mathematical literature on this subject is rather extensive. See for example [Bardos et al. 2000; 2002; Chen et al. 2011; 2018; Erdős and Yau 2001; Fröhlich et al. 2009; Golse and Paul 2017; 2019; Golse et al. 2016; 2018; Grillakis et al. 2010; Kuz 2015; Mitrouskas et al. 2019; Pickl 2011; Rodnianski and Schlein 2009] for the case of bosons, and [Bach et al. 2016; Benedikter et al. 2014; 2016a; Elgart et al. 2004; Fröhlich and Knowles 2011; Petrat 2017; Petrat and Pickl 2016; Porta et al. 2017; Saffirio 2018] for the case of fermions.

Semiclassical limit. The Hartree and Hartree–Fock equations are quantum models. It is therefore natural to investigate their semiclassical limit as $\hbar \to 0$. First results in this direction provide the convergence from the Hartree dynamics towards the Vlasov equation in the abstract sense, without rate of convergence and in weak topologies, but including the case of singular interaction potentials, such as the Coulomb interaction (see [Figalli et al. 2012; Gasser et al. 1998; Lions and Paul 1993; Markowich and Mauser 1993]). Explicit bounds on the convergence rate in stronger topologies were established in [Amour et al. 2013a; 2013b; Athanassoulis et al. 2011; Benedikter et al. 2016b; Golse and Paul 2017; Pezzotti and Pulvirenti 2009]. They all deal with smooth interaction potentials. More recently, the case of singular interactions, including the Coulomb potential, has been considered in [Lafleche 2019; 2021], where the convergence from the Hartree to the Vlasov equation is achieved in weak topology using the quantum Wasserstein–Monge–Kantorovich distance, providing explicit bounds on the convergence rate. In strong topology (trace norm and Hilbert–Schmidt norm), explicit bounds on the convergence from the Hartree dynamics to the Vlasov equation with inverse power law of the form $K(x) = |x|^{-a}$ with $a \in \left(0, \frac{1}{2}\right)$ have been proven in [Saffirio 2020b], and a proof that includes the Coulomb potential has been provided in [Saffirio 2020a] but under restrictive assumptions on the initial data.

Key novelties. The aim of this paper is to establish a strong convergence result from both the Hartree and the Hartree–Fock equations towards the Vlasov dynamics for a large class of regular initial states. Our results apply to a wide class of initial data which are smooth as $h \to 0$, thus giving a thorough answer to the question of strong convergence of the Hartree equation to the Vlasov system for singular interactions, at least in the case of mixed states converging to smooth solutions of the Vlasov equation.

With respect to the results present in the literature, there are several novelties: Apart from the large class of initial data for whose evolution we can establish strong convergence with explicit rate towards the Vlasov equation, our techniques allow us to consider inverse power law potentials that are more singular than the Coulomb potential, and our methods easily extend to very general nonradially symmetric potentials. Moreover, the topology we consider is not only the one induced by the trace or Hilbert–Schmidt norm (as it is for instance in [Saffirio 2020b]), but the ones induced by semiclassical Schatten norms $\mathcal{L}^p$, for all $p \in [1, \infty)$. These are obtained by a refinement on the estimate for the $\mathcal{L}^p$-norms of the commutator $[K(\cdot - z), \boldsymbol{\rho}]$ and a careful analysis of the propagation in time of initial conditions leading to bound the quantity

$$\left\| \operatorname{diag} \left| \left[\frac{x}{i\hbar}, \boldsymbol{\rho} \right] \right| \right\|_{L^p(\mathbb{R}^d)}$$

uniformly in $\hbar$, for $p > 3$. This requires using kinetic interpolation inequalities as in [Lafleche 2019] and an extension of the Calderón–Vaillancourt theorem for Weyl quantization.

Finally, we extend our results to the Hartree–Fock equation (4), thus proving the strong convergence of the Hartree–Fock dynamics to the Vlasov equation. As a corollary, we get explicit estimates on the difference between the Hartree and Hartree–Fock dynamics in Schatten norms, thus giving a rigorous proof of the fact that the exchange term in the Hartree–Fock dynamics is also subleading with respect to the direct term when the interaction potential is singular (this was proved in [Benedikter et al. 2014] in the case of smooth potentials).

Open problems. Our work gives good answers to the problem of the semiclassical limit from the Hartree and Hartree–Fock equations to the Vlasov equation with general singular potentials in the context of mixed states. However, a certain number of questions related to the derivation of the Vlasov equation from quantum dynamics remain open.

(i) To our knowledge, the mean-field limit from a system of N quantum particles interacting through a singular potential in the case of mixed states is open in both the bosonic and the fermionic settings.

(ii) In the bosonic setting, where N and $\hbar$ are independent parameters, the joint mean-field and semiclassical limit is an open problem when the interaction is singular. Namely, no uniform convergence in the semiclassical parameter $\hbar$ has been proven so far.

(iii) We believe our results give optimal bounds on the convergence rate in trace norm $\mathcal{L}^1$. The question whether the bounds we obtain for the semiclassical Hilbert–Schmidt norm $\mathcal{L}^2$ are optimal, and thus for the L^2 convergence of the associated Wigner functions, is open. The exact same question can be asked about the bounds in Theorem 1.6 for the convergence of the Hartree–Fock equation to the Vlasov equation. In both cases, we believe the bounds we get are not optimal and there is room for improvement.

Structure of the paper. The paper is structured as follows:

• We state our main result in Section 1B and include additional comments and generalizations in Section 1C.

- In Section 2 we explain our strategy. We introduce a semiclassical notion of regularity (Section 2A) and then explain our method to get the semiclassical limit by making a comparison with the classical Vlasov dynamics, finding a new stability estimate for the Vlasov system (Section 2B).
- Section 3 contains the main results concerning the regularity of the Weyl transform of a solution to the Vlasov equation, which will be crucial to prove the theorems stated in Section 1B.
- Section 4 is devoted to proving Theorems 1.1 and 1.4, dealing with the semiclassical limit from the Hartree equation under the assumption that the regularity proved in Section 3 holds.
- In Section 5 we present the proof of Theorem 1.6 about the semiclassical limit from the Hartree–Fock equation, based on additional estimates on the exchange term.
- Two appendices on the propagation of regularity for the Vlasov equation and on basic operator identities complement the paper.

1B. ***Main results.***

Operators and function spaces. We denote by $L^p = L^p(\mathbb{R}^d)$ the classical Lebesgue spaces and by $L^{p,q} = L^{p,q}(\mathbb{R}^d)$ the classical Lorentz spaces for $(p,q) \in [1,\infty]^2$; see for example [Bergh and Löfström 1976]. In particular, $L^{p,p} = L^p$. We define the space of positive and trace class operators by

$$\mathcal{L}^1_+ := \{\boldsymbol{\rho} \in \mathcal{L}(L^2),\ \boldsymbol{\rho} = \boldsymbol{\rho}^* \geq 0,\ \mathrm{Tr}(\boldsymbol{\rho}) < \infty\},$$

where $\mathcal{L}(L^2)$ denotes the space of linear operators on L^2, and the quantum Lebesgue norms (or semiclassical Schatten norms) $\mathcal{L}^p$ by

$$\|\boldsymbol{\rho}\|_{\mathcal{L}^p} := h^{-d/p'} \|\boldsymbol{\rho}\|_p = h^{-d/p'} (\mathrm{Tr}(|\boldsymbol{\rho}|^p))^{1/p},$$

where $\|\boldsymbol{\rho}\|_p$ denotes the usual Schatten norm (i.e., without dependency in h) and $p' = p/(p-1)$ denotes the conjugate exponent.

In this work, we consider the semiclassical limit to solutions of the Vlasov equation with regular data in the sense that the initial condition will be bounded in some weighted Sobolev space. Therefore, we write the following for smooth polynomial weight functions,

$$\langle y \rangle := \sqrt{1 + |y|^2},$$

and for $\sigma \in \mathbb{N}$, we define the spaces $W^{\sigma,p}_k(\mathbb{R}^{2d})$ as the spaces equipped with the norm

$$\|f\|_{W^{\sigma,p}_k(\mathbb{R}^{2d})} := \|\langle z \rangle^k f(z)\|_{L^p(\mathbb{R}^{2d})} + \|\langle z \rangle^k \nabla^\sigma_z f(z)\|_{L^p(\mathbb{R}^{2d})},$$

where $z = (x, \xi)$ with $\langle z \rangle^2 = 1 + |x|^2 + |\xi|^2$. We also use the standard notation when $\sigma = 0$ or $p = 2$:

$$L^p_k(\mathbb{R}^{2d}) := W^{0,p}_k(\mathbb{R}^{2d}), \qquad H^\sigma_k(\mathbb{R}^{2d}) := W^{\sigma,2}_k(\mathbb{R}^{2d}).$$

When $\mathbb{R}^{2d}$ is replaced by $\mathbb{R}^d$, as for Lebesgue spaces, we will use shortcut notation and write H^n instead of $H^n(\mathbb{R}^d)$, for example, and C^∞_c for the space of smooth compactly supported functions on $\mathbb{R}^d$.

Wigner and Weyl transforms. We can associate to each density operator $\boldsymbol{\rho}$ a function of the phase space called the Wigner transform, which is defined (for $h=1$) by

$$w(\boldsymbol{\rho})(x,\xi) := \int_{\mathbb{R}^d} e^{-2i\pi y\cdot\xi}\boldsymbol{\rho}\big(x+\tfrac{1}{2}y, x-\tfrac{1}{2}y\big)\,\mathrm{d}y = \mathcal{F}(\tilde{\boldsymbol{\rho}}_x)(\xi),$$

where $\tilde{\boldsymbol{\rho}}_x(y) = \boldsymbol{\rho}\big(x+\frac{1}{2}y, x-\frac{1}{2}y\big)$ and we used the following convention for the Fourier transform:

$$\mathcal{F}(u)(\xi) := \int_{\mathbb{R}^d} e^{-2i\pi x\cdot\xi}u(x)\,\mathrm{d}x.$$

This function of the phase space is not a probability distribution, however, since it is generally not nonnegative. We refer to [Lions and Paul 1993] for more properties of the Wigner transform. Given $\boldsymbol{\rho}$, we will write its semiclassical Wigner transform as

$$w_\hbar(\boldsymbol{\rho})(x,\xi) := \frac{1}{h^d}w(\boldsymbol{\rho})\bigg(x,\frac{\xi}{h}\bigg).$$

Conversely, to each function of the phase space, we can associate an operator through the Weyl transformation, which is the inverse of the Wigner transform. It is defined as the operator $\boldsymbol{\rho}_\hbar^W(g)$ such that for any $\varphi\in C_c^\infty$,

$$\boldsymbol{\rho}_\hbar^W(g)\varphi := \iint_{\mathbb{R}^{2d}} g\big(\tfrac{1}{2}(x+y),\xi\big)e^{-i(y-x)\cdot\xi/\hbar}\varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi.$$

Theorems. We state our theorems, starting with our main result.

Theorem 1.1. *Let $d\in\{2,3\}$, $a\in\big(\max\{\frac{1}{2}d-2,-1\}, d-2\big]$ and K be given by either*

$$K(x) = \frac{\pm 1}{|x|^a} \quad \text{or} \quad K(x) = \pm\ln(|x|). \tag{6}$$

In the second case we set $a:=0$. Let $f\geq 0$ be a solution of the Vlasov equation (1) and $\boldsymbol{\rho}\geq 0$ be a solution of the Hartree equation (2) with respective initial conditions

$$f^{\mathrm{in}} \in W_m^{\sigma+1,\infty}(\mathbb{R}^{2d})\cap H_\sigma^{\sigma+1}(\mathbb{R}^{2d}), \tag{7}$$

$$\boldsymbol{\rho}^{\mathrm{in}}\in\mathcal{L}^1, \tag{8}$$

where $(m,\sigma)\in(4\mathbb{N})\times(2\mathbb{N})$ satisfies $m>d$ and $\sigma>m+d/(\mathfrak{b}-1)$ with $\mathfrak{b}=d/(a+1)$. If $a\leq 0$, we also require $\mathrm{Tr}((|x|^2-\hbar^2\Delta)\boldsymbol{\rho}^{\mathrm{in}})$ to be bounded. Then there exist $\lambda_f(t)\in C^0(\mathbb{R}_+,\mathbb{R}_+)$ and $C_f(t)\in C^0(\mathbb{R}_+,\mathbb{R}_+)$ depending only on d, a and the initial condition of the solution of the Vlasov equation such that

$$\mathrm{Tr}(|\boldsymbol{\rho}-\boldsymbol{\rho}_f|) \leq (\mathrm{Tr}(|\boldsymbol{\rho}^{\mathrm{in}}-\boldsymbol{\rho}_f^{\mathrm{in}}|)+C_f(t)\hbar)e^{\lambda_f(t)}, \tag{9}$$

where $\boldsymbol{\rho}_f=\boldsymbol{\rho}_\hbar^W(f)$ and $\boldsymbol{\rho}_f^{\mathrm{in}}=\boldsymbol{\rho}_{f^{\mathrm{in}}}$. Upper bounds for the functions λ_f and C_f are given by

$$\lambda_f(t)\leq C_{d,a}\int_0^t \|\nabla_\xi f\|_{W^{n_0,\infty}(\mathbb{R}^{2d})\cap H_\sigma^\sigma(\mathbb{R}^{2d})}\,\mathrm{d}s,$$

$$C_f(t)\leq C_{d,a}\int_0^t \|\rho_f(s)\|_{L^1\cap H^\nu}\|\nabla_\xi^2 f(s)\|_{H_m^m(\mathbb{R}^{2d})}e^{-\lambda_f(s)}\,\mathrm{d}s,$$

where $\nu=\big(\frac{1}{2}m+a+2-d\big)_+$ and $n_0=2\lfloor\frac{1}{2}d\rfloor+2$, and remain bounded at any time $t\geq 0$.

Remark 1.2. Condition (6) includes in particular the Coulomb or Newton potential in dimensions $d=3$ and $d=2$. In these cases, the conditions of regularity (7) of the initial data of the Vlasov equation become $f^{\text{in}} \in W_4^{13,\infty}(\mathbb{R}^{2d}) \cap H_{12}^{13}(\mathbb{R}^{2d})$ when $d=3$ and $a=1$, and $f^{\text{in}} \in W_4^{9,\infty}(\mathbb{R}^{2d}) \cap H_8^{9}(\mathbb{R}^{2d})$ when $d=2$ and $a=0$. These conditions are of course not optimal: for example, we ask that $m/2$ and σ be even numbers to simplify some computations.

Remark 1.3. To see more explicitly that (9) gives a good semiclassical approximation estimate, one can take $\boldsymbol{\rho}^{\text{in}}$ and $\boldsymbol{\rho}_f^{\text{in}}$ such that $\operatorname{Tr}(|\boldsymbol{\rho}^{\text{in}} - \boldsymbol{\rho}_f^{\text{in}}|) \leq C\hbar$ and fix some $T>0$, which yields the existence of a constant $C_T>0$ such that for any $t \in [0, T]$,

$$\operatorname{Tr}(|\boldsymbol{\rho} - \boldsymbol{\rho}_f|) \lesssim C_T \hbar. \tag{10}$$

The theorem also implies the convergence of the spatial density of particles $\rho \to \rho_f$ in L^1. Indeed, by duality we have

$$\|\rho - \rho_f\|_{L^1} = \sup_{\substack{O \in L^\infty(\mathbb{R}^d) \\ \|O\|_{L^\infty} \leq 1}} \left| \int O(x)(\rho(x) - \rho_f(x))\,\mathrm{d}x \right| \leq \operatorname{Tr}(|\boldsymbol{\rho} - \boldsymbol{\rho}_f|), \tag{11}$$

since every bounded function $x \mapsto O(x)$ also defines a multiplication operator with operator norm $\|O\|_{L^\infty}$.

From the bound in Theorem 1.1 we obtain estimates in other semiclassical Lebesgue spaces.

Theorem 1.4. *Take the same assumptions and notations as in Theorem 1.1, define $\mathfrak{b} = d/(a+1)$ and assume moreover that*

$$f^{\text{in}} \in W_\sigma^{\sigma+1,\infty}(\mathbb{R}^{2d}) \cap H_\sigma^{\sigma+1}(\mathbb{R}^{2d})$$

and that $\sigma > n_0 + d/\mathfrak{b}$. Then for any $p \in [1, \mathfrak{b})$,

$$\|\boldsymbol{\rho} - \boldsymbol{\rho}_f\|_{\mathcal{L}^p} \leq \|\boldsymbol{\rho}^{\text{in}} - \boldsymbol{\rho}_f^{\text{in}}\|_{\mathcal{L}^p} + (\operatorname{Tr}(|\boldsymbol{\rho}^{\text{in}} - \boldsymbol{\rho}_f^{\text{in}}|) + c(t)\hbar)e^{\lambda(t)}, \tag{12}$$

where c and λ are continuous functions on $\mathbb{R}_+$ depending on d, a, p and f^{in}. For any $q \in [\mathfrak{b}, \infty)$, assuming also that $\boldsymbol{\rho}^{\text{in}} \in \mathcal{L}^\infty$, this leads to the estimate

$$\|\boldsymbol{\rho} - \boldsymbol{\rho}_f\|_{\mathcal{L}^q} \leq c_2(t)(\|\boldsymbol{\rho}^{\text{in}} - \boldsymbol{\rho}_f^{\text{in}}\|_{\mathcal{L}^p}^{p/q} + \operatorname{Tr}(|\boldsymbol{\rho}^{\text{in}} - \boldsymbol{\rho}_f^{\text{in}}|)^{p/q} + \hbar^{p/q})e^{(p/q)\lambda(t)}, \tag{13}$$

where $\boldsymbol{\rho}_f = \boldsymbol{\rho}_\hbar^W(f)$, $\boldsymbol{\rho}_f^{\text{in}} = \boldsymbol{\rho}_{f^{\text{in}}}$ and $c_2 \in C^0(\mathbb{R}_+, \mathbb{R}_+)$ can be computed explicitly and depend on the initial conditions.

Remark 1.5. In particular, if we assume $\boldsymbol{\rho}^{\text{in}} = \boldsymbol{\rho}_f^{\text{in}}$, or more generally

$$\operatorname{Tr}(|\boldsymbol{\rho}^{\text{in}} - \boldsymbol{\rho}_f^{\text{in}}|) \leq C\hbar \quad \text{and} \quad \|\boldsymbol{\rho}^{\text{in}} - \boldsymbol{\rho}_f^{\text{in}}\|_{\mathcal{L}^2} \leq C\hbar,$$

then we have a rate of the form $\hbar^{\mathfrak{b}/2-\varepsilon}$ with $\varepsilon > 0$ arbitrarily small. For the Coulomb potential in dimension $d=3$, the estimate reads

$$\|f_{\boldsymbol{\rho}} - f\|_{L^2(\mathbb{R}^{2d})} = \|\boldsymbol{\rho} - \boldsymbol{\rho}_f\|_{\mathcal{L}^2} \leq C_T \hbar^{3/4-\varepsilon},$$

for any $t \in [0, T]$ for some fixed $T>0$, where $f_{\boldsymbol{\rho}} = w_\hbar(\boldsymbol{\rho})$ is the Wigner transform of $\boldsymbol{\rho}$. Notice that Theorem 1.1 does not imply convergence of the operators but is only a quantitative estimate, where both $\boldsymbol{\rho}$ and $\boldsymbol{\rho}_\hbar^W(f)$ depend on $\hbar$. As operators, they both for instance converge to 0 in operator norm since by hypothesis $\|\boldsymbol{\rho}\|_\infty \sim C h^d$. On the contrary, the above equation is both a quantitative estimate and a convergence result since f is a fixed element which does not depend on $\hbar$. Thus it implies the convergence of $f_{\boldsymbol{\rho}}$ to f in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^2(\mathbb{R}^{2d}))$.

With the same assumptions in the case $d = 2$, the Coulomb kernel is of the form $K(x) = C \ln(|x|)$ and $\mathfrak{b} = 2$, implying that (12) holds for any $p \in [1, 2)$ and that we almost get the conjectured optimal rate of convergence for $p = 2$,

$$\|f_{\boldsymbol{\rho}} - f\|_{L^2(\mathbb{R}^{2d})} \le C_T \hbar^{1-\varepsilon}.$$

Our third result concerns the Hartree–Fock equation. In this case, we combine $p = 1$ and $p > 1$ in one theorem.

Theorem 1.6. *Let $\boldsymbol{\rho}$ be a solution of the Hartree–Fock equation (4) and f be a solution of the Vlasov equation (1) which satisfy the same initial conditions as in Theorem 1.1, and if $p > 1$, the same initial conditions as in Theorem 1.4. If $a > 0$, we also assume that the solution has finite kinetic energy, i.e.,*

$$-\operatorname{Tr}(\hbar^2 \Delta \boldsymbol{\rho}^{\mathrm{in}})$$

is bounded uniformly with respect to $\hbar$. Then, for any $p \in [1, \mathfrak{b})$, there exist functions $c \in C^0(\mathbb{R}_+, \mathbb{R}_+)$ and $\lambda \in C^0(\mathbb{R}_+, \mathbb{R}_+)$ depending on d, a, p and f^{in} such that

$$\|\boldsymbol{\rho} - \boldsymbol{\rho}_f\|_{\mathcal{L}^p} \le \|\boldsymbol{\rho}^{\mathrm{in}} - \boldsymbol{\rho}_f^{\mathrm{in}}\|_{\mathcal{L}^p} + (\operatorname{Tr}(|\boldsymbol{\rho}^{\mathrm{in}} - \boldsymbol{\rho}_f^{\mathrm{in}}|) + c(t)\hbar^{\min\{1,\tilde{s}-1\}})e^{\lambda(t)},$$

where $\boldsymbol{\rho}_f = \boldsymbol{\rho}_\hbar^W(f)$, $\boldsymbol{\rho}_f^{\mathrm{in}} = \boldsymbol{\rho}_{f^{\mathrm{in}}}$ and $\tilde{s} = d - a_+ - d\big(\frac{1}{2} - \frac{1}{p}\big)_+$. For $q \in [\mathfrak{b}, \infty)$, assuming again that $\boldsymbol{\rho}^{\mathrm{in}} \in \mathcal{L}^\infty$, we still have the estimate

$$\|\boldsymbol{\rho} - \boldsymbol{\rho}_f\|_{\mathcal{L}^q} \le c_2(t)(\|\boldsymbol{\rho}^{\mathrm{in}} - \boldsymbol{\rho}_f^{\mathrm{in}}\|_{\mathcal{L}^p}^{p/q} + \operatorname{Tr}(|\boldsymbol{\rho}^{\mathrm{in}} - \boldsymbol{\rho}_f^{\mathrm{in}}|)^{p/q} + \hbar^{(p/q)\min\{1,\tilde{s}-1\}})e^{(p/q)\lambda(t)},$$

where $c_2(t)$ can be computed explicitly and depends on the initial conditions.

1C. *Discussion.*

Higher singularities. For $a > d - 2$, we have no propagation of regularity and therefore our results hold true only in a conditional form. Namely, if the solution to the Vlasov equation is sufficiently regular, then the bounds of Theorem 1.1 and Theorem 1.4 are still satisfied. More precisely, if $d = 3$, such conditional results hold for any $a \in (1, 2)$. As for Theorem 1.6, a conditional result is still true. However, due to the need to control the exchange term X, we can only address a smaller class of potentials. In particular, in dimension $d = 3$ we have $a \in \big(1, \frac{3}{2}\big)$. Our results in dimensions 2 and 3 can be summarized as follows:

settings:	$d = 2$		$d = 3$		
	$a \in (-1, 0]$	$a \in (0, 1)$	$a \in \big(-\frac{1}{2}, 1\big]$	$a \in \big(1, \frac{3}{2}\big)$	$a \in \big[\frac{3}{2}, 2\big)$
Hartree	global	conditional	global	conditional	conditional
Hartree–Fock	global	conditional	global	conditional	no results

General class of potentials. All our results generalize to more general nonradial pair interactions. For $s \in (0, d)$, define the weak Sobolev space $\dot{H}^{s,1}_w$ as the completion of C^∞_c with respect to the norm

$$\|u\|_{\dot{H}^{s,1}_w} := \|\Delta^{s/2} u\|_{\mathrm{TV}},$$

where $\|\cdot\|_{\mathrm{TV}}$ denotes the total variation norm over the space $\mathcal{M}$ of bounded measures. By the formula of the inverse of the powers of the Laplacian, we deduce that it is the space of functions that can be written as

$$u(x) = \int_{\mathbb{R}^d} \frac{1}{|x-w|^{d-s}} \mu(\mathrm{d}w), \tag{14}$$

for some measure $\mu \in \mathcal{M}$.

Notice that this space contains the interaction kernel

$$K(x) = \frac{1}{|x|^a} \quad \text{with } a = d - s,$$

when $a > 0$, which follows by taking $\mu = \delta_0$. In particular, for the Coulomb potential in dimension $d = 3$,

$$\frac{1}{|x|} \in \dot{H}^{2,1}_w.$$

However, this space contains also more general potentials. It contains for example the Sobolev space $\dot{H}^{s,1} = \dot{F}^s_{2,1}$ which is defined by the norm $\|\Delta^{s/2} u\|_{L^1}$. When $n \in \mathbb{N}$, then $\dot{H}^{n,1} = \dot{W}^{n,1}$ is a classical homogeneous Sobolev space.

The proof for more general potentials follows mainly from the fact that the equation and most of our estimates depend linearly on K. As an example, Proposition 5.1 is proved with this class of potentials. Hence, all our results also hold with the assumption $K \in \dot{H}^{d-a}_w$ instead of $K(x) = |x|^{-a}$ when $a > 0$, except Theorem 1.6, since we need an assumption on K^2 to prove inequalities (39a) and (39b). For this theorem, the assumption $K(x) = |x|^{-a}$ can therefore be replaced by $K \in \dot{H}^{d-a}_w$ and $K^2 \in \dot{H}^{d-2a}_w$ when $a \geq 0$.

From Hartree to Hartree–Fock. Notice that Theorem 1.4 and Theorem 1.6 give a semiclassical estimate between the solutions of the Hartree equation (2) and the solutions of the Hartree–Fock equation (4). Indeed, let $\boldsymbol{\rho}_{\mathrm{H}}$ and $\boldsymbol{\rho}_{\mathrm{HF}}$ be solutions to the Hartree equation and the Hartree–Fock equation, respectively, and let $\boldsymbol{\rho}_f$ be a solution to the Weyl transformed Vlasov equation. Then, for $p \in [1, \infty)$, we have

$$\|\boldsymbol{\rho}_H - \boldsymbol{\rho}_{\mathrm{HF}}\|_{\mathcal{L}^p} \leq \|\boldsymbol{\rho}_H - \boldsymbol{\rho}_f\|_{\mathcal{L}^p} + \|\boldsymbol{\rho}_{\mathrm{HF}} - \boldsymbol{\rho}_f\|_{\mathcal{L}^p},$$

where the first term on the right-hand side is bounded by Theorem 1.4 and the second term on the right-hand side can be estimated by Theorem 1.6.

Well-posedness. One of the strengths of the method is that our strong regularity assumptions that must be independent of $\hbar$ only concern the solutions of the Vlasov equation. Our assumptions on the solution of the Hartree equation imply the global well-posedness of solutions, as proved in [Castella 1997], where the trace norm corresponds to the $L^2(\lambda)$-norm; see also [Ginibre and Velo 1980; 1985; Lions and Paul 1993]. Even if these assumptions are weak, observe however that the operator $\boldsymbol{\rho}^{\mathrm{in}}$ has to be at a finite trace norm distance from the operator $\boldsymbol{\rho}^{\mathrm{in}}$ which by construction is bounded in higher Sobolev spaces (as can be deduced from Proposition 3.3). The additional moment bound when $a \leq 0$ ensures that the energy is finite, which allows us to propagate the space moments; see e.g., [Lafleche 2019, Remark 3.1]. This is sufficient to give a meaning to the pair interaction potential which is growing at infinity in this case.

2. Strategy

The strategy of this paper consists in getting the semiclassical analogue of the estimates of classical mechanics, and in particular the case of kinetic models. The quantum analogue of the classical momentum variable ξ is the operator

$$\boldsymbol{p} := -i\hbar\nabla,$$

which is an unbounded operator on L^2. From this we get in particular that $|\boldsymbol{p}|^2 := \boldsymbol{p}^*\boldsymbol{p} = -\hbar^2\Delta$ and we can express the Hamiltonian (3) as $H = \frac{1}{2}|\boldsymbol{p}|^2 + V(x)$.

2A. *Quantum gradients of the phase space.* Since our method uses regular initial conditions, we define the following operators which are the quantum equivalent of the gradient with respect to the variables x and ξ of the phase space:

$$\begin{aligned}\boldsymbol{\nabla}_x\boldsymbol{\rho} &:= [\nabla, \boldsymbol{\rho}] = \left[\frac{\boldsymbol{p}}{i\hbar}, \boldsymbol{\rho}\right],\\ \boldsymbol{\nabla}_\xi\boldsymbol{\rho} &:= \left[\frac{x}{i\hbar}, \boldsymbol{\rho}\right].\end{aligned}$$

These formulas can be seen from the point of view of the correspondence principle as the quantum equivalent of the Poisson bracket definition of the classical gradients. Another point of view is to observe that they are Weyl quantizations, since we have

$$\begin{aligned}\boldsymbol{\nabla}_x\boldsymbol{\rho} &= \boldsymbol{\rho}_\hbar^W(\nabla_x w_\hbar(\boldsymbol{\rho})),\\ \boldsymbol{\nabla}_\xi\boldsymbol{\rho} &= \boldsymbol{\rho}_\hbar^W(\nabla_\xi w_\hbar(\boldsymbol{\rho})).\end{aligned}$$

One should not confuse $\nabla \in \mathcal{L}(L^2)$ with $\boldsymbol{\nabla}_x \in \mathcal{L}(\mathcal{L}(L^2))$. In Section 3, we prove that if a function on the phase space is sufficiently smooth in the classical sense, then its Weyl quantization also has some smoothness in the semiclassical sense.

2B. *The classical case: L^1 weak-strong stability.* In the classical case, the method we use to prove the semiclassical limit, which is the content of Sections 4 and 5 can be seen as an equivalent of the following L^1 weak-strong stability estimate for the Vlasov equation, which says that we need to have control of the gradient of only one of the solutions to get a bound on the integral of their difference.

We use the shortcut notation $L^p_x L^{q,r}_\xi = L^p(\mathbb{R}^d, L^{q,r}(\mathbb{R}^d))$ for functions on the phase space of the form $f = f(x,\xi)$. The next proposition can be seen as the classical equivalent of Theorem 1.1.

Proposition 2.1. *Let $\mathfrak{b} \in (1,\infty]$ and $\nabla K \in L^{\mathfrak{b},\infty}$, and assume f_1 and f_2 are two solutions of the Vlasov equation* (1) *in $L^\infty([0,T], L^1(\mathbb{R}^{2d}))$ for some $T > 0$. Then, under the condition*

$$\nabla_\xi f_2 \in L^1([0,T], L^{\mathfrak{b}',1}_x L^1_\xi), \tag{15}$$

one has the stability estimate

$$\|f_1 - f_2\|_{L^1(\mathbb{R}^{2d})} \leq \|f_1^{\mathrm{in}} - f_2^{\mathrm{in}}\|_{L^1(\mathbb{R}^{2d})} \exp\biggl(C \int_0^T \|\nabla_\xi f_2\|_{L_x^{\mathfrak{b}',1} L_\xi^1} \,\mathrm{d}t\biggr),$$

where $C = \|\nabla K\|_{L^{\mathfrak{b},\infty}}$.

Remark 2.2. In the case of the Coulomb interaction and $\mathfrak{b} = \frac{3}{2}$, the condition on f_2 becomes

$$\int_{\mathbb{R}^d} |\nabla_\xi f_2| \,\mathrm{d}\xi \in L^1([0,T], L_x^{3,1}),$$

which by real interpolation follows if

$$\|\nabla_\xi f_2\|_{L_\xi^1} \in L^1([0,T], L_x^{3+\varepsilon} \cap L_x^{3-\varepsilon}),$$

for some $\varepsilon \in (0, 2]$. In particular, the case $\varepsilon = 2$ yields $(3-\varepsilon, 3+\varepsilon) = (1, 5)$, which corresponds to the classical equivalent of the hypotheses required on the solutions in [Saffirio 2020a]. A quantum version of this hypothesis can also be found in [Porta et al. 2017].

Remark 2.3. This result allows ∇K to be more singular than the case of the Coulomb potential. However, it is a conditional result, since one still has to show that condition (15) holds. If the potential is the Coulomb potential or a less singular potential, then one can prove that this condition holds if the data is initially in some weighted Sobolev space by Proposition A.1 in Appendix A. If the potential is more singular than the Coulomb potential, then it is not clear that there are cases such that condition (15) is satisfied.

Proof of Proposition 2.1. Let $f := f_1 - f_2$ and define $\rho_k = \int_{\mathbb{R}^d} f_k \,\mathrm{d}\xi$ and $E_k = -\nabla V_k = -\nabla K * \rho_k$ for $k \in \{1, 2\}$. Then

$$\partial_t f + \xi \cdot \nabla_x f + E_1 \cdot \nabla_\xi f = (E_2 - E_1) \cdot \nabla_\xi f_2,$$

so that by defining $\rho := \rho_1 - \rho_2$ we obtain

$$\begin{aligned} \partial_t \iint_{\mathbb{R}^{2d}} |f| \,\mathrm{d}x \,\mathrm{d}\xi &= \iint_{\mathbb{R}^{2d}} (\nabla K * \rho \cdot \nabla_\xi f_2) \operatorname{sgn}(f) \,\mathrm{d}x \,\mathrm{d}\xi \\ &= -\int_{\mathbb{R}^d} \rho \nabla K \,\dot{*} \biggl(\int_{\mathbb{R}^d} \operatorname{sgn}(f) \nabla_\xi f_2 \,\mathrm{d}\xi\biggr) \leq \|f\|_{L^1} \biggl\| \nabla K * \int_{\mathbb{R}^d} |\nabla_\xi f_2| \,\mathrm{d}\xi \biggr\|_{L^\infty}, \end{aligned}$$

where the notation $\dot{*}$ indicates that we perform the dot product of vectors inside the convolution. We conclude by noticing that by Hölder's inequality for Lorentz spaces (see for example [Hunt 1966, (2.7)]), for any $g \in L^{\mathfrak{b}',1}$,

$$\|\nabla K * g\|_{L^\infty} \leq \sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\nabla K(z - \cdot) g| \leq \|\nabla K\|_{L^{\mathfrak{b},\infty}} \|g\|_{L^{\mathfrak{b}',1}}, \tag{16}$$

so the result follows by taking $g = \|\nabla_\xi f_2\|_{L_\xi^1}$ and then using Grönwall's lemma. □

The next proposition is the classical equivalent of the first part of Theorem 1.4.

Proposition 2.4. *Let $\mathfrak{b} > 1$ and $\nabla K \in L^{\mathfrak{b},\infty}$, and assume f_1 and f_2 are two solutions of the Vlasov equation (1) in $L^\infty([0,T], L^1(\mathbb{R}^{2d}))$ for some $T > 0$. Then, if $\nabla_\xi f_2 \in L^1([0,T], L^{q,1}_x L^p_\xi)$,*

$$\|f_1 - f_2\|_{L^p(\mathbb{R}^{2d})} \leq \|f_1^{\mathrm{in}} - f_2^{\mathrm{in}}\|_{L^p(\mathbb{R}^{2d})} + C\|f_1^{\mathrm{in}} - f_2^{\mathrm{in}}\|_{L^1(\mathbb{R}^{2d})} \int_0^t \|\nabla_\xi f_2\|_{L^{q,1}_x L^p_\xi} \,\mathrm{d}t \exp\left(C \int_0^T \|\nabla_\xi f_2\|_{L^{\mathfrak{b}',1}_x L^1_\xi} \,\mathrm{d}t\right),$$

where $C = \|\nabla K\|_{L^{\mathfrak{b},\infty}}$ and

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{\mathfrak{b}}. \tag{17}$$

Remark 2.5. Formula (17) implies $p \leq \mathfrak{b}$. In the case of the Coulomb interaction in dimension $d = 3$ we have $\mathfrak{b} = \frac{3}{2}$, thus the estimate works at most with $p = \frac{3}{2}$.

Proof. We define the two-parameter semigroup $S_{t,s}$ such that $S_{s,s} = 1$ and

$$\partial_t S_{t,s} g = \Lambda_t S_{t,s} g,$$

where

$$\Lambda_t S_{t,s} g := -\xi \cdot \nabla_x S_{t,s} g - E_1(t) \cdot \nabla_\xi S_{t,s} g,$$

with $E_1(t) = E_1(t,x) = -\nabla K * \rho_1$ and $\rho_1(t,x) = \int f_1(t,x,\xi)\,\mathrm{d}\xi$. Now observe that the flow property of $S_{t,s}$ implies that $\partial_s S_{t,s} = -S_{t,s}\Lambda_s$. Thus, using the notation

$$\tilde{\Lambda}_t := -\xi \cdot \nabla_x - E_2(t) \cdot \nabla_\xi$$

and taking $f_1(s) = f_1(s,x,\xi)$ and $f_2(s) = f_2(s,x,\xi)$ to be two solutions of the Vlasov equation, we get

$$\begin{aligned}\partial_s S_{t,s}(f_1 - f_2)(s) &= -S_{t,s}\Lambda_s(f_1 - f_2)(s) + S_{t,s}\Lambda_s f_1(s) - S_{t,s}\tilde{\Lambda}_s f_2(s) \\ &= S_{t,s}(\Lambda_s - \tilde{\Lambda}_s) f_2(s) = S_{t,s}((E_2(s) - E_1(s)) \cdot \nabla_\xi f_2(s)),\end{aligned}$$

and by integrating with respect to s and writing $f := f_1 - f_2$ and $E := E_1 - E_2$ we obtain the Duhamel formula

$$f(t) = S_{t,0} f^{\mathrm{in}} + \int_0^t S_{t,s}(E(s) \cdot \nabla_\xi f_2(s))\,\mathrm{d}s.$$

Since the semigroup $S_{t,s}$ preserves all Lebesgue norms of the phase space, taking the L^p-norm yields

$$\|f(t)\|_{L^p_{x,\xi}} \leq \|f^{\mathrm{in}}\|_{L^p_{x,\xi}} + \int_0^t \|E(s) \cdot \nabla_\xi f_2(s)\|_{L^p_{x,\xi}}\,\mathrm{d}s.$$

To bound the expression inside the time integral we write

$$\begin{aligned}\|E(s) \cdot \nabla_\xi f_2(s)\|_{L^p_{x,\xi}} = \|(\rho * \nabla K) \cdot \nabla_\xi f_2(s)\|_{L^p_{x,\xi}} &\leq \int_{\mathbb{R}^d} |\rho(z)| \, \|\nabla K(\cdot - z) \cdot \nabla_\xi f_2(s)\|_{L^p_{x,\xi}}\,\mathrm{d}z \\ &\leq \int_{\mathbb{R}^d} |\rho(z)| \, \Big\| |\nabla K(\cdot - z)| \|\nabla_\xi f_2(s)\|_{L^p_\xi} \Big\|_{L^p_x}\,\mathrm{d}z \\ &\leq \|\rho\|_{L^1} \|\nabla K\|_{L^{\mathfrak{b},\infty}} \|\nabla_\xi f_2(s)\|_{L^{q,1}_x L^p_\xi},\end{aligned}$$

where we again used Hölder's inequality for Lorentz spaces. $\square$

3. Regularity of the Weyl transform

In this section, we prove that if the solution f of the Vlasov equation is sufficiently well-behaved, then we can obtain uniform in $\hbar$ bounds for the quantum equivalent of the norm $\|\nabla_\xi f\|_{L^p_x L^1_\xi}$ expressed in terms of the Weyl transform of f.

Proposition 3.1. *Let $n, n_1 \in \mathbb{N}$ be even numbers such that $n > \frac{1}{2}d$, and define $\sigma := 2n + n_1$ and $n_0 = 2\lfloor \frac{1}{2}d \rfloor + 2$. Then, for any $f \in W^{n_0+1,\infty}(\mathbb{R}^{2d}) \cap H^{\sigma+1}_\sigma(\mathbb{R}^{2d})$, there exists a constant $C_{d,n_1} > 0$ depending only on d and n_1 such that*

$$\|\mathrm{diag}(|\nabla_\xi \boldsymbol{\rho}^W_\hbar(f)|)\|_{L^p} \leq C_{d,n_1} \|\nabla_\xi f\|_{W^{n_0,\infty}(\mathbb{R}^{2d}) \cap H^\sigma_\sigma(\mathbb{R}^{2d})}$$

for any $p \in [1, 1 + n_1/d]$.

The strategy is to use a special case of the quantum kinetic interpolation inequality proved in Theorem 6 of [Lafleche 2019]. For the operator $|\nabla_\xi \boldsymbol{\rho}|$, this special case reads

$$\|\mathrm{diag}(|\nabla_\xi \boldsymbol{\rho}|)\|_{L^p} \leq C(\mathrm{Tr}(|\nabla_\xi \boldsymbol{\rho}|\,|\boldsymbol{p}|^{n_1}))^\theta \|\nabla_\xi \boldsymbol{\rho}\|^{1-\theta}_{\mathcal{L}^\infty}, \tag{18}$$

where $p = 1 + n_1/d$ and $\theta = 1/p$. The corresponding kinetic inequality is

$$\|\nabla_\xi f\|_{L^p_x(L^1_\xi)} \leq C\left(\iint_{\mathbb{R}^{2d}} |\nabla_\xi f|\,|\xi|^{n_1}\,\mathrm{d}x\,\mathrm{d}\xi\right)^\theta \|\nabla_\xi f\|^{1-\theta}_{L^\infty_{x,\xi}}.$$

To do this, we will need to compare the multiplication of the Weyl transform of a phase space function g by $|\boldsymbol{p}|^n$ and $|x|^n$, with the Weyl transform of the multiplication of g by $|\xi|^n$ and $|x|^n$. This makes error terms appear involving derivatives of g. For example, in the case $n = 2$,

$$\boldsymbol{\rho}^W_\hbar(g)|\boldsymbol{p}|^2 = \boldsymbol{\rho}^W_\hbar\big(|\xi|^2 g + \tfrac{1}{2} i\hbar\xi\cdot\nabla_x g - \tfrac{1}{4}\hbar^2\Delta_x g\big) \quad \text{and} \quad \boldsymbol{\rho}^W_\hbar(g)|x|^2 = \boldsymbol{\rho}^W_\hbar\big(|x|^2 g + i\hbar\xi\cdot\nabla_\xi g + \tfrac{1}{4}\hbar^2\Delta_\xi g\big).$$

More generally, one can obtain similar identities when $n \in \mathbb{N}$. In order to write them, we introduce the standard multi-index notation

$$\alpha := (\alpha_{\mathrm{i}})_{\mathrm{i}\in[\![1,d]\!]} \in \mathbb{N}^d, \qquad |\alpha| := \sum_{\mathrm{i}=1}^d \alpha_{\mathrm{i}}, \qquad \alpha! := \alpha_1!\,\alpha_2!\cdots\alpha_d!,$$

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2}\cdots x_d^{\alpha_d}, \qquad \partial^\alpha_x := \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\cdots\partial^{\alpha_d}_{x_d}, \qquad \alpha \leq \beta \iff \forall \mathrm{i} \in [\![1,d]\!], \alpha_{\mathrm{i}} \leq \beta_{\mathrm{i}}.$$

We then obtain the following set of identities.

Lemma 3.2. *For any $n \in 2\mathbb{N}$ and any tempered distribution g of the phase space,*

$$\boldsymbol{\rho}^W_\hbar(g)|\boldsymbol{p}|^n = \sum_{|\alpha+\beta|=n} a_{\alpha,\beta}\big(\tfrac{1}{2}i\hbar\big)^{|\beta|}\boldsymbol{\rho}^W_\hbar(\xi^\alpha\partial^\beta_x g), \tag{19a}$$

$$\boldsymbol{\rho}^W_\hbar(g)|x|^n = \sum_{|\alpha+\beta|=n} b_{\alpha,\beta}(-i\hbar)^{|\beta|}\boldsymbol{\rho}^W_\hbar(x^\alpha\partial^\beta_\xi g), \tag{19b}$$

$$\boldsymbol{\rho}^W_\hbar(g)|\boldsymbol{p}|^{n_1}|x|^n = \sum_{\substack{|\alpha+\beta|=n_1\\|\alpha'+\beta'|=n}} a_{\alpha,\beta}b_{\alpha',\beta'}(-i\hbar)^{|\beta'|}\big(\tfrac{1}{2}i\hbar\big)^{|\beta|}\boldsymbol{\rho}^W_\hbar(x^{\alpha'}\partial^{\beta'}_\xi(\xi^\alpha\partial^\beta_x g)), \tag{19c}$$

where the coefficients $a_{\alpha,\beta}$ and $b_{\alpha,\beta}$ are nonnegative and do not depend on $\hbar$.

Proof. By definition of the Weyl transform, we deduce that for any $\varphi \in C_c^\infty$,

$$\begin{aligned}\boldsymbol{\rho}_\hbar^W(g)|\boldsymbol{p}|^n\varphi &= (i\hbar)^n \iint_{\mathbb{R}^{2d}} g\big(\tfrac{1}{2}(x+y),\xi\big)e^{-i(y-x)\cdot\xi/\hbar}\Delta^{n/2}\varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi \\ &= (i\hbar)^n \iint_{\mathbb{R}^{2d}} \Delta_y^{n/2}\big(g\big(\tfrac{1}{2}(x+y),\xi\big)e^{-i(y-x)\cdot\xi/\hbar}\big)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi.\end{aligned}$$

With the multi-index notation, we can expand the powers of the Laplacian of a product of functions in the following way:

$$\Delta^{n/2}(fg) = \sum_{|\alpha+\beta|=n} a_{\alpha,\beta}\partial^\alpha f\partial^\beta g,$$

where the $a_{\alpha,\beta}^n$ are nonnegative constants depending on n and on the multi-index α such that

$$\sum_{|\alpha+\beta|=n} a_{\alpha,\beta} = (4d)^n.$$

Thus we deduce that the integral kernel κ of the operator $\boldsymbol{\rho}_\hbar^W(g)|\boldsymbol{p}|^n$ is given by

$$\kappa(x,y) = \sum_{|\alpha+\beta|=n} a_{\alpha,\beta}(i\hbar)^{n-|\alpha|}\int_{\mathbb{R}^d} 2^{-|\beta|}\partial_x^\beta g\big(\tfrac{1}{2}(x+y),\xi\big)\xi^\alpha e^{-i(y-x)\cdot\xi/\hbar}\,\mathrm{d}\xi,$$

which yields

$$\boldsymbol{\rho}_\hbar^W(g)|\boldsymbol{p}|^n = \sum_{|\alpha+\beta|=n} a_{\alpha,\beta}\big(\tfrac{1}{2}i\hbar\big)^{|\beta|}\boldsymbol{\rho}_\hbar^W(\xi^\alpha\partial_x^\beta g).$$

This proves (19a). To prove the second identity, we write $u := \frac{1}{2}(x+y)$ and $v := y-x$, so that the integral kernel κ_2 of the operator $\boldsymbol{\rho}_\hbar^W(g)|x|^2$ is given by

$$\begin{aligned}\kappa_2(x,y) &= \iint_{\mathbb{R}^{2d}} g\big(\tfrac{1}{2}(x+y),\xi\big)e^{-i(y-x)\cdot\xi/\hbar}|y|^n\,\mathrm{d}\xi \\ &= \iint_{\mathbb{R}^{2d}} g(u,\xi)e^{-iv\cdot\xi/\hbar}\big(\big|u+\tfrac{1}{2}v\big|^2\big)^{n/2}\,\mathrm{d}\xi \\ &= \iint_{\mathbb{R}^{2d}} g(u,\xi)e^{-iv\cdot\xi/\hbar}\Bigg(\sum_{\mathrm{i}=1}^d\big(u_\mathrm{i}^2+\tfrac{1}{4}v_\mathrm{i}^2+u_\mathrm{i}v_\mathrm{i}\big)\Bigg)^{n/2}\,\mathrm{d}\xi.\end{aligned}$$

By the multinomial theorem, this can be written as

$$\begin{aligned}\kappa_2(x,y) &= \sum_{|\alpha+\beta|=n} b_{\alpha,\beta}\iint_{\mathbb{R}^{2d}} u^\alpha g(u,\xi)v^\beta e^{-iv\cdot\xi/\hbar}\,\mathrm{d}\xi \\ &= \sum_{|\alpha+\beta|=n} b_{\alpha,\beta}\iint_{\mathbb{R}^{2d}} u^\alpha g(u,\xi)(i\hbar)^{|\beta|}\partial_\xi^\beta e^{-iv\cdot\xi/\hbar}\,\mathrm{d}\xi \\ &= \sum_{|\alpha+\beta|=n} b_{\alpha,\beta}(-i\hbar)^{|\beta|}\iint_{\mathbb{R}^{2d}} u^\alpha\partial_\xi^\beta g(u,\xi)e^{-iv\cdot\xi/\hbar}\,\mathrm{d}\xi,\end{aligned}$$

where we used integration by parts $|\beta|$ times to get the last line, and the $b_{\alpha,\beta}$ are nonnegative constants that satisfy

$$\sum_{|\alpha+\beta|=n} b_{\alpha,\beta} = \left(\tfrac{9}{4}d\right)^{n/2}.$$

In term of operators, this leads to

$$\boldsymbol{\rho}_\hbar^W(g)|x|^n = \sum_{|\alpha+\beta|=n} b_{\alpha,\beta}(-i\hbar)^{|\beta|}\boldsymbol{\rho}_\hbar^W(x^\alpha \partial_\xi^\beta g),$$

which is (19b). To get the last identity, we combine the first two to get

$$\begin{aligned}\boldsymbol{\rho}_\hbar^W(g)|\boldsymbol{p}|^{n_1}|x|^n &= \sum_{|\alpha+\beta|=n_1} a_{\alpha,\beta}\left(\tfrac{1}{2}i\hbar\right)^{|\beta|}\boldsymbol{\rho}_\hbar^W(\xi^\alpha\partial_x^\beta g)|x|^n \\ &= \sum_{\substack{|\alpha+\beta|=n_1\\ |\alpha'+\beta'|=n}} a_{\alpha,\beta}b_{\alpha',\beta'}(-i\hbar)^{|\beta'|}\left(\tfrac{1}{2}i\hbar\right)^{|\beta|}\boldsymbol{\rho}_\hbar^W(x^{\alpha'}\partial_\xi^{\beta'}(\xi^\alpha\partial_x^\beta g)).\end{aligned}$$ ☐

From this lemma, we deduce the following $\mathcal{L}^2$ inequalities.

Proposition 3.3. *Let $n \in 2\mathbb{N}$, and let g be a function of the phase space. Then there exists a constant $C > 0$ depending only on d and n such that*

$$\|\boldsymbol{\rho}_\hbar^W(g)|\boldsymbol{p}|^n\|_{\mathcal{L}^2} \le (4d)^n\left(\|g|\xi|^n\|_{L^2(\mathbb{R}^{2d})} + \left(\tfrac{1}{2}\hbar\right)^n\|\nabla_x^n g\|_{L^2(\mathbb{R}^{2d})}\right), \tag{20a}$$

$$\|\boldsymbol{\rho}_\hbar^W(g)|x|^n\|_{\mathcal{L}^2} \le \left(\tfrac{9}{4}d\right)^n(\|g|x|^n\|_{L^2(\mathbb{R}^{2d})} + \hbar^n\|\nabla_\xi^n g\|_{L^2(\mathbb{R}^{2d})}), \tag{20b}$$

$$\begin{aligned}\|\boldsymbol{\rho}_\hbar^W(g)|\boldsymbol{p}|^{n_1}|x|^n\|_{\mathcal{L}^2} \le C\big(&\|(1+|x|^n|\xi|^{n_1})g\|_{L^2(\mathbb{R}^{2d})} + \hbar^{n_1}\||x|^n\nabla_x^{n_1}g\|_{L^2(\mathbb{R}^{2d})} \\ &+ \hbar^n\||\xi|^{n_1}\nabla_\xi^n g\|_{L^2(\mathbb{R}^{2d})} + \hbar^{n+n_1}\|\nabla_x^{n_1}\nabla_\xi^n g\|_{L^2(\mathbb{R}^{2d})}\big).\end{aligned} \tag{20c}$$

Proof. By (19a) and the fact that $\|\boldsymbol{\rho}_\hbar^W(u)\|_{\mathcal{L}^2} = \|u\|_{L^2(\mathbb{R}^{2d})}$ for any $u \in L^2(\mathbb{R}^{2d})$, we obtain

$$\|\boldsymbol{\rho}_\hbar^W(g)|\boldsymbol{p}|^n\|_{\mathcal{L}^2} \le \sum_{|\alpha+\beta|=n} a_{\alpha,\beta}\left(\tfrac{1}{2}\hbar\right)^{|\beta|}\|\xi^\alpha\partial_x^\beta g\|_{L^2(\mathbb{R}^{2d})}. \tag{21}$$

Then, for any multi-index α and β such that $|\alpha+\beta| = n$, define $\hat{g}(y,\xi)$ to be the Fourier transform of $g(x,\xi)$ with respect to the variable x, and use the fact that the Fourier transform is unitary in L_x^2 to get

$$\begin{aligned}\left(\tfrac{1}{2}\hbar\right)^{|\beta|}\|\xi^\alpha\partial_x^\beta g\|_{L^2(\mathbb{R}^{2d})} = \left(\tfrac{1}{2}h\right)^{|\beta|}\|\xi^\alpha y^\beta\hat{g}\|_{L^2(\mathbb{R}^{2d})} &\le \frac{|\alpha|}{n}\||\xi|^n\hat{g}\|_{L^2(\mathbb{R}^{2d})} + \frac{|\beta|}{n}\left(\tfrac{1}{2}h\right)^{|\beta|}\||y|^n g\|_{L^2(\mathbb{R}^{2d})} \\ &\le \||\xi|^n g\|_{L^2(\mathbb{R}^{2d})} + \left(\tfrac{1}{2}\hbar\right)^n\|\nabla_x^n\hat{g}\|_{L^2(\mathbb{R}^{2d})}.\end{aligned}$$

Moreover, as remarked in the proof of Lemma 3.2,

$$\sum_{|\alpha+\beta|=n} a_{\alpha,\beta} = (4d)^n,$$

from which we obtain (20a). Formulas (20b) and (20c) can be proved in the same way. ☐

Moreover, we can bound weighted $\mathcal{L}^1$-norms using $\mathcal{L}^2$-norms with bigger weights. This is the content of the following proposition where we recall the notation $\langle y\rangle = \sqrt{1+|y|^2}$ for the weights.

Proposition 3.4. *Let n, $n_1 \in \mathbb{N}$ be even numbers such that $n > \frac{1}{2}d$, and define $k := n + n_1$. Assume $\boldsymbol{\rho} := \boldsymbol{\rho}_\hbar^W(g)$ is the Weyl transform of a function $g \in H^{n+k}_{n+k}(\mathbb{R}^{2d})$. Then the following inequality holds*:

$$\operatorname{Tr}(|\boldsymbol{\rho}|\,|\boldsymbol{p}|^{n_1}) \leq C(\|\langle\xi\rangle^k\langle x\rangle^n g\|_{L^2(\mathbb{R}^{2d})} + \hbar^k\|\langle x\rangle^n\nabla_x^k g\|_{L^2(\mathbb{R}^{2d})} + \hbar^n\|\langle\xi\rangle^k\nabla_\xi^n g\|_{L^2(\mathbb{R}^{2d})} + \hbar^{k+n}\|\nabla_x^k\nabla_\xi^n g\|_{L^2(\mathbb{R}^{2d})}).$$

Proof. First notice that since the sum of eigenvalues is always smaller than the sum of singular values (see for example [Simon 2005, (3.1)]), we have that

$$\operatorname{Tr}(|\boldsymbol{\rho}|\,|\boldsymbol{p}|^{n_1}) \leq \operatorname{Tr}(|\,|\boldsymbol{\rho}|\,|\boldsymbol{p}|^{n_1}|),$$

and from the definition of $|AB|$ if A and B are two operators, we see that $|AB| = (B^*A^*AB)^{1/2} = |\,|A|B|$, so that $\operatorname{Tr}(|\,|\boldsymbol{\rho}|\,|\boldsymbol{p}|^{n_1}|) = \operatorname{Tr}(|\boldsymbol{\rho}|\boldsymbol{p}|^{n_1}|)$. Defining $\boldsymbol{m}_n := (1 + |\boldsymbol{p}|^n)(1 + |x|^n)$, we deduce from the Cauchy–Schwarz inequality that

$$\operatorname{Tr}(|\boldsymbol{\rho}|\,|\boldsymbol{p}|^{n_1}) \leq \operatorname{Tr}(|\boldsymbol{\rho}|\boldsymbol{p}|^{n_1}|) \leq \|\boldsymbol{\rho}|\boldsymbol{p}|^{n_1}\boldsymbol{m}_n\|_2\|\boldsymbol{m}_n^{-1}\|_2. \tag{22}$$

To control the second factor on the right-hand side, we observe that it is of the form $\boldsymbol{m}_n^{-1} = w(x)w(-i\hbar\nabla)$ with $w(y) = (1 + |y|^n)^{-1}$, so its Hilbert–Schmidt norm can be computed (see e.g., [Simon 2005, (4.7)]) as

$$\|\boldsymbol{m}_n^{-1}\|_2 = (2\pi)^{-d/2}\|w\|_{L^2}\|w(\hbar\,\cdot\,)\|_{L^2} = C_{d,n}h^{-d/2},$$

where $C_{d,n} = \|w\|_{L^2}^2$ is finite since $n > \frac{1}{2}d$. Therefore, by the definition of the $\mathcal{L}^2$-norm, (22) leads to

$$\operatorname{Tr}(|\boldsymbol{\rho}|\,|\boldsymbol{p}|^{n_1}) \leq C_{d,n}\|\boldsymbol{\rho}|\boldsymbol{p}|^{n_1}\boldsymbol{m}_n\|_{\mathcal{L}^2} \leq C_{d,n}\|\boldsymbol{\rho}(|\boldsymbol{p}|^{n_1} + |\boldsymbol{p}|^{n_1}|x|^n + |\boldsymbol{p}|^{n+n_1} + |\boldsymbol{p}|^{n+n_1}|x|^n)\|_{\mathcal{L}^2}.$$

To get the result, we take $\boldsymbol{\rho} = \boldsymbol{\rho}_\hbar^W(g)$ and use Proposition 3.3 to bound the right-hand side of the above inequality by weighted classical L^2-norms of g. □

We can now prove the main proposition of this section following the strategy explained at the beginning of this section.

Proof of Proposition 3.1. We use an improvement of the Calderón–Vaillancourt theorem for Weyl operators proved by Boulkhemair [1999] which states that if $g \in W^{n_0,\infty}(\mathbb{R}^{2d})$ with $n_0 = 2\lfloor\frac{1}{2}d\rfloor + 2$, then $\boldsymbol{\rho}_1^W(g)$ is a bounded operator on L^2 and its operator norm is bounded by

$$\|\boldsymbol{\rho}_1^W(g)\|_{\mathscr{B}(L^2)} \leq C\|g\|_{W^{n_0,\infty}(\mathbb{R}^{2d})}. \tag{23}$$

Since $\boldsymbol{\rho}_\hbar^W(g) = h^d\boldsymbol{\rho}_1^W(g(\,\cdot\,, h\,\cdot\,))$, and, for $h \leq 1$,

$$\|g(\,\cdot\,, h\,\cdot\,)\|_{W^{n_0,\infty}(\mathbb{R}^{2d})} \leq \|g\|_{W^{n_0,\infty}(\mathbb{R}^{2d})},$$

by taking $g = \nabla_\xi f$ we deduce from (23) and the definition of the $\mathcal{L}^\infty$-norm that

$$\|\nabla_\xi\boldsymbol{\rho}_\hbar^W(f)\|_{\mathcal{L}^\infty} \leq C\|\nabla_\xi f\|_{W^{n_0,\infty}(\mathbb{R}^{2d})},$$

uniformly in $\hbar$. Moreover, taking $g = \nabla_\xi f$ in Proposition 3.4 yields

$$\operatorname{Tr}(|\nabla_\xi\boldsymbol{\rho}_\hbar^W(f)|\,|\boldsymbol{p}|^{n_1}) \leq C\|\nabla_\xi f\|_{H^\sigma_\sigma(\mathbb{R}^{2d})}.$$

The result then follows by combining these two inequalities to bound the right-hand side of the interpolation inequality (18). □

4. Proof of Theorems 1.1 and 1.4

We start this section by proving a stability estimate similar to the inequality used in the classical case, and then use the results of Section 3 and the propagation of regularity for the Vlasov equation to get the proof of Theorem 1.1 and then the proof of Theorem 1.4. The conditional result is stated here.

Proposition 4.1. *Let $K = 1/|x|^a$ with $a \in \big(\big(\frac{1}{2}d-2\big)_+, d-1\big)$, and assume $\boldsymbol{\rho}$ is a solution of the Hartree equation (2) with initial condition $\boldsymbol{\rho}^{\mathrm{in}} \in \mathcal{L}^1_+$ and $f \geq 0$ is a solution of the Vlasov equation satisfying*

$$f \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, W^{n_0+1,\infty}(\mathbb{R}^{2d}) \cap H^{\sigma+1}_\sigma(\mathbb{R}^{2d})), \tag{24a}$$

$$\rho_f \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^1 \cap H^\nu), \tag{24b}$$

where $n_0 = 2\big\lfloor \frac{1}{2}d \big\rfloor + 2$ and $n,\ n_1 \in 2\mathbb{N}$ are such that $n > \frac{1}{2}d$ and $n_1 \geq d/(\mathfrak{b}-1)$, and we use the notation $\sigma = 2n + n_1,\ \nu = (n+a+2-d)_+$ and $\mathfrak{b} = d/(a+1)$. Then

$$\mathrm{Tr}(|\boldsymbol{\rho} - \boldsymbol{\rho}_f|) \leq (\mathrm{Tr}(|\boldsymbol{\rho}^{\mathrm{in}} - \boldsymbol{\rho}^{\mathrm{in}}_f|) + C_f(t)\hbar)e^{\lambda_f(t)},$$

where

$$\lambda_f(t) = C_{d,n_1,a} \int_0^t \|\nabla_\xi f\|_{W^{n_0,\infty}(\mathbb{R}^{2d}) \cap H^\sigma_\sigma(\mathbb{R}^{2d})}\, \mathrm{d}s,$$

$$C_f(t) = C_{d,n_1,a} \int_0^t \|\rho_f(s)\|_{L^1 \cap H^\nu} \|\nabla^2_\xi f(s)\|_{H^{2n}_{2n}(\mathbb{R}^{2d})} e^{-\lambda_f(s)}\, \mathrm{d}s.$$

Remark 4.2. It is actually sufficient to assume that $f \geq 0$ when $t = 0$ since this implies that it holds at any time $t \geq 0$.

In analogy with the classical case (see the proof of Proposition 2.4), we introduce the two-parameter semigroup $\mathcal{U}_{t,s}$ such that $\mathcal{U}_{s,s} = 1$ and defined for $t > s$ by

$$i\hbar\partial_t \mathcal{U}_{t,s} = H(t)\mathcal{U}_{t,s},$$

where H is the Hartree Hamiltonian (3). We consider the quantity

$$i\hbar\partial_t(\mathcal{U}^*_{t,s}(\boldsymbol{\rho}(t) - \boldsymbol{\rho}_f(t))\mathcal{U}_{t,s}) = \mathcal{U}^*_{t,s}[K * (\rho(t) - \rho_f(t)), \boldsymbol{\rho}_f(t)]\mathcal{U}_{t,s} + \mathcal{U}^*_{t,s} B_t\, \mathcal{U}_{t,s},$$

where B_t is an operator defined through its integral kernel by

$$B_t(x,y) = \big((K*\rho_f)(x) - (K*\rho_f)(y) - (\nabla K * \rho_f)\big(\tfrac{1}{2}(x+y)\big)\cdot(x-y)\big)\boldsymbol{\rho}_f(x,y). \tag{25}$$

Using Duhamel's formula and taking the Schatten p-norm (recall that $\mathcal{U}_{t,s}$ is a unitary operator), we get

$$\begin{aligned}&\|\boldsymbol{\rho}(t) - \boldsymbol{\rho}_f(t)\|_p \\ &\quad\leq \|\boldsymbol{\rho}^{\mathrm{in}} - \boldsymbol{\rho}^{\mathrm{in}}_f\|_p + \frac{1}{\hbar}\int_0^t \|B_t\|_p\, \mathrm{d}s + \frac{1}{\hbar}\int_0^t\!\!\int |\rho(s,z) - \rho_f(s,z)|\, \|[K(\cdot - z), \boldsymbol{\rho}_f(s)]\|_p\, \mathrm{d}z\, \mathrm{d}s.\end{aligned} \tag{26}$$

We now take $p = 1$, i.e., the trace norm, and we have to bound each term on the right-hand side of (26) in order to obtain a Grönwall type inequality which will prove Proposition 4.1. Note that we will then again use (26) with $p > 1$ together with Theorem 1.1 to prove Theorem 1.4.

4A. ***The commutator inequality.*** Generalizing [Porta et al. 2017, Lemma 3.1], we obtain the quantum equivalent of (16), which is the following inequality for the trace norm of the commutator of K and a trace class operator $\boldsymbol{\rho}$.

Theorem 4.3. *Let $a \in (-1, d-1)$ and $K(x) = 1/|x|^a$ or $K(x) = \ln(|x|)$ when $a = 0$. Then*

$$\nabla K \in L^{\mathfrak{b},\infty} \quad \text{with } \mathfrak{b} = \mathfrak{b}_a := \frac{d}{a+1}.$$

Let $\mathfrak{b}'$ be the conjugated Hölder exponent of $\mathfrak{b}$. Then for any $\varepsilon \in (0, \mathfrak{b}'-1]$, there exists a constant $C > 0$ such that

$$\mathrm{Tr}(|[K(\cdot - z), \boldsymbol{\rho}]|) \leq Ch \|\mathrm{diag}(|\nabla_\xi \boldsymbol{\rho}|)\|_{L^{\mathfrak{b}'-\varepsilon}}^{1/2+\tilde{\varepsilon}} \|\mathrm{diag}(|\nabla_\xi \boldsymbol{\rho}|)\|_{L^{\mathfrak{b}'+\varepsilon}}^{1/2-\tilde{\varepsilon}},$$

for any $\tilde{\varepsilon} \in (0, \varepsilon/(2\mathfrak{b}'))$ and with the additional assumption $\varepsilon < \mathfrak{b}_3' - \mathfrak{b}'$ if $d \geq 4$.

Remark 4.4. In the case of the Coulomb interaction and $d = 3$, we have $K(x) = 1/|x|$, $\mathfrak{b} = \mathfrak{b}_1 = \frac{3}{2}$ and $\nabla K \in L^{3/2,\infty}$. Thus for any $\varepsilon \in (0, 2]$, there exists a constant $C > 0$ such that

$$\mathrm{Tr}(|[K(\cdot - z), \boldsymbol{\rho}]|) \leq Ch \|\mathrm{diag}(|\nabla_\xi \boldsymbol{\rho}|)\|_{L^{3-\varepsilon}}^{1/2+\tilde{\varepsilon}} \|\mathrm{diag}(|\nabla_\xi \boldsymbol{\rho}|)\|_{L^{3+\varepsilon}}^{1/2-\tilde{\varepsilon}},$$

for any $\tilde{\varepsilon} \in \left(0, \frac{1}{6}\varepsilon\right)$.

Theorem 4.3 is a corollary of the slightly more precise proposition that follows.

Proposition 4.5. *For any $\delta \in ((1/\mathfrak{b}_1' - 1/\mathfrak{b}')_+, 1 - 1/\mathfrak{b}')$ and $q \in (\mathfrak{b}'/(1-\delta\mathfrak{b}'), \infty]$, there exists a constant $C > 0$ such that*

$$\mathrm{Tr}(|[K(\cdot - z), \boldsymbol{\rho}]|) \leq Ch \|\mathrm{diag}(|\nabla_\xi \boldsymbol{\rho}|)\|_{L^p}^{\theta} \|\mathrm{diag}(|\nabla_\xi \boldsymbol{\rho}|)\|_{L^q}^{1-\theta}, \tag{27}$$

where $1/p = 1/\mathfrak{b}' + \delta$ and $\theta = \delta/(1/p - 1/q)$ and with the additional assumption that $q < \mathfrak{b}_3'$ if $d \geq 4$.

Proof of Theorem 4.3. We will decompose the potential as a combination of Gaussian functions (see e.g., [Lieb and Loss 2001, 5.9 (3)]). By using the definition of the gamma function and a simple change of variable, when $a > 0$ one obtains, for any $r > 0$,

$$\frac{1}{\omega_a r^{a/2}} = \frac{1}{2}\int_0^\infty t^{a/2-1} e^{-\pi r t}\,\mathrm{d}t, \tag{28}$$

where $\omega_a = 2\pi^{a/2}/\left(\Gamma\left(\frac{1}{2}a\right)\right)$. Taking $r = |x|^2$ leads directly to the decomposition

$$\frac{1}{\omega_a |x|^a} = \frac{1}{2}\int_0^\infty t^{a/2-1} e^{-\pi |x|^2 t}\,\mathrm{d}t.$$

Now when $a \in (-2, 0)$, take (28) with $a+2$ instead of a, integrate it with respect to r, exchange the integrals and then again replace r by $|x|^2$. This yields a similar decomposition of the form

$$\frac{1}{\omega_a |x|^a} = \frac{1}{2}\int_0^\infty t^{a/2-1} (e^{-\pi |x|^2 t} - 1)\,\mathrm{d}t.$$

In order to treat the case of the logarithm, do the same steps with $a = 0$ to obtain

$$-\ln(|x|) = \frac{1}{2}\int_0^\infty t^{a/2-1} (e^{-\pi |x|^2 t} - e^{-\pi t})\,\mathrm{d}t.$$

In all these cases, defining $\omega_0 := 1$, we get the identity

$$\frac{1}{\omega_a}(K(x)-K(y)) = \frac{1}{2}\int_0^\infty t^{a/2-1}(e^{-\pi|x|^2 t} - e^{-\pi|y|^2 t})\,\mathrm{d}t.$$

Following the idea of [Porta et al. 2017] but using this new decomposition, we write

$$\begin{aligned}\frac{1}{\omega_a}(K(x)-K(y)) &= \frac{1}{2}\int_0^\infty t^{a/2-1}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}(e^{-\pi\theta|x|^2 t}e^{-\pi(1-\theta)|y|^2 t})\,\mathrm{d}\theta\,\mathrm{d}t \\ &= -\pi\int_0^\infty t^{a/2}\int_0^1 (x-y)\cdot(x+y)e^{-\pi\theta|x|^2 t}e^{-\pi(1-\theta)|y|^2 t}\,\mathrm{d}\theta\,\mathrm{d}t,\end{aligned}$$

from which we get

$$\frac{K(x-z)-K(y-z)}{-\pi\omega_a} = \int_0^1\int_0^\infty t^{a/2}(x-y)\cdot(\phi_\theta(x)\varphi_{1-\theta}(y)+\varphi_\theta(x)\phi_{1-\theta}(y))\,\mathrm{d}t\,\mathrm{d}\theta,$$

where we defined $\varphi_k(x) := e^{-k\pi|x-z|^2 t}$ and $\phi_k(x) := (x-z)\varphi_k(x)$. Thus, using the fact that the integral kernel of $\nabla_\xi\boldsymbol{\rho}$ is $((x-y)/(i\hbar))\boldsymbol{\rho}(x,y)$ and exchanging θ by $1-\theta$ in the second term of the integral, we obtain

$$\frac{1}{i\pi\hbar\omega_a}[K(\cdot-z),\boldsymbol{\rho}] = \int_0^1\int_0^\infty t^{a/2}(\phi_\theta\cdot\nabla_\xi\boldsymbol{\rho}\varphi_{1-\theta}+\varphi_{1-\theta}\nabla_\xi\boldsymbol{\rho}\cdot\phi_\theta)\,\mathrm{d}t\,\mathrm{d}\theta.$$

Noticing that $(\phi_\theta\cdot\nabla_\xi\boldsymbol{\rho}\varphi_{1-\theta})^* = \varphi_{1-\theta}\nabla_\xi\boldsymbol{\rho}\cdot\phi_\theta$, we can now estimate the trace norm by

$$\frac{1}{h|\omega_a|}\|[K(\cdot-z),\boldsymbol{\rho}]\|_1 \le \int_0^1\int_0^\infty t^{a/2}\|\phi_\theta\cdot\nabla_\xi\boldsymbol{\rho}\varphi_{1-\theta}\|_1\,\mathrm{d}t\,\mathrm{d}\theta. \tag{29}$$

Then, by decomposing the self-adjoint operator $\nabla_\xi\boldsymbol{\rho}$ on an orthogonal basis $(\psi_j)_{j\in J}$, we can write $\nabla_\xi\boldsymbol{\rho} = \sum_{j\in J}\lambda_j|\psi_j\rangle\langle\psi_j|$ and get

$$\begin{aligned}\|\phi_\theta\cdot\nabla_\xi\boldsymbol{\rho}\varphi_{1-\theta}\|_1 &\le \sum_{j\in J}|\lambda_j|\,\||\phi_\theta\psi_j\rangle\langle\psi_j\varphi_{1-\theta}|\|_1 \\ &\le \sum_{j\in J}|\lambda_j|\,\|\phi_\theta\psi_j\|_{L^2}\|\psi_j\varphi_{1-\theta}\|_{L^2},\end{aligned}$$

where we used the fact that $\||u\rangle\langle v|\|_1 = \|u\|_{L^2}\|v\|_{L^2}$. Thus, by the Cauchy–Schwarz inequality for series,

$$\begin{aligned}\|\phi_\theta\cdot\nabla_\xi\boldsymbol{\rho}\varphi_{1-\theta}\|_1 &\le \Big(\sum_{j\in J}|\lambda_j|\,\|\phi_\theta\psi_j\|_{L^2}^2\Big)^{1/2}\Big(\sum_{j\in J}|\lambda_j|\,\|\psi_j\varphi_{1-\theta}\|_{L^2}^2\Big)^{1/2} \\ &\le \Big(\int_{\mathbb{R}^d}|\phi_\theta|^2\rho_1\Big)^{1/2}\Big(\int_{\mathbb{R}^d}|\varphi_{1-\theta}|^2\rho_1\Big)^{1/2},\end{aligned}$$

with the notation $\rho_1 = \operatorname{diag}(|\nabla_\xi\boldsymbol{\rho}|) = \sum_{j\in J}|\lambda_j||\psi_j|^2$. By the integral Hölder's inequality, this yields

$$\|\phi_\theta\cdot\nabla_\xi\boldsymbol{\rho}\varphi_{1-\theta}\|_1 \le \|\phi_\theta\|_{L^{2p'}}\|\varphi_{1-\theta}\|_{L^{2q'}}\|\rho_1\|_{L^p}^{1/2}\|\rho_1\|_{L^q}^{1/2}, \tag{30}$$

where $(p,q)\in[1,\infty]^2$ can depend on the parameter t, which will help us to obtain the convergence of the integral in (29). We can now compute explicitly the integrals of the functions ϕ and φ:

$$\|\phi_\theta\|_{L^{2p'}}^{2p'} = \int_{\mathbb{R}^d} |x-z|^{2p'} e^{-2\pi\theta|x-z|^2p't}\,\mathrm{d}x = \frac{\omega_d}{\omega_{d+2p'}}\frac{1}{(2\theta p't)^{(d+2p')/2}},$$

$$\|\varphi_{1-\theta}\|_{L^{2q'}}^{2q'} = \int_{\mathbb{R}^d} e^{-2\pi(1-\theta)|x-z|^2q't}\,\mathrm{d}x = \frac{1}{(2(1-\theta)q't)^{d/2}}.$$

Combining these two formulas with (29) and (30) leads to

$$\|[K(\cdot-z),\boldsymbol{\rho}]\|_1 \leq h\int_0^\infty \frac{C_{d,a,p'}\|\rho_1\|_{L^p}^{1/2}\|\rho_1\|_{L^q}^{1/2}}{t^{(1/2)(d/(2p')+d/(2q')+1-a)}}\int_0^1 \frac{\mathrm{d}\theta}{\theta^{(d+2p')/(4p')}(1-\theta)^{d/(4q')}}\,\mathrm{d}t,$$

with

$$C_{d,a,p'} = |\omega_a|\left(\frac{\omega_d}{\omega_{d+2p'}}\right)^{1/(2p')}(2p')^{-(d+2p')/(4p')}(2q')^{-d/(4q')}.$$

We observe that the integral over θ is converging as soon as

$$\frac{1}{p'} < \frac{2}{d} = \frac{1}{\mathfrak{b}_1} \qquad\text{and}\qquad \frac{1}{q'} < \frac{4}{d} = \frac{1}{\mathfrak{b}_3}. \tag{31}$$

In order to get a finite integral of the variable t, we cut the integral into two parts. The first for $t\in(0,R)$ and the second for $t\in(R,\infty)$, for a given $R>0$. Then we choose p and q such that

$$\frac{1}{2}\left(\frac{d}{2p'}+\frac{d}{2q'}+1-a\right)<1 \quad\text{for } t\in(0,R) \qquad\text{and}\qquad \frac{1}{2}\left(\frac{d}{2p'}+\frac{d}{2q'}+1-a\right)>1 \quad\text{for } t\geq R,$$

or equivalently, since $\mathfrak{b}=d/(a+1)$,

$$\frac{1}{2}\left(\frac{1}{p'}+\frac{1}{q'}\right)<\frac{1}{\mathfrak{b}} \quad\text{for } t\in(0,R) \qquad\text{and}\qquad \frac{1}{2}\left(\frac{1}{p'}+\frac{1}{q'}\right)>\frac{1}{\mathfrak{b}} \quad\text{for } t\geq R.$$

However, this has to be compatible with the constraint (31). Therefore, when $t\in(0,R)$, we can in particular take $q=p_0$ with $p_0<\min(\mathfrak{b}',\mathfrak{b}_1')$. When $t\geq R$, then we can also take for example $p=p_0>\frac{1}{2}\mathfrak{b}'$ and then any q such that

$$\frac{2}{\mathfrak{b}}-\frac{1}{p_0'}<\frac{1}{q'}<\frac{4}{d} \qquad\text{and}\qquad \frac{1}{q'}\leq 1. \tag{32}$$

Notice that the condition $1/q'<4/d$ is only used when $d\geq 4$ and can be rewritten as $q<\mathfrak{b}_3'$. Such a pair (p_0,q) exists as long as $a\leq\frac{1}{2}d$ and $a<2$. By defining $\delta:=1/p_0-1/\mathfrak{b}'$, then these conditions are equivalent to

$$\left(\frac{1}{\mathfrak{b}_1'}-\frac{1}{\mathfrak{b}'}\right)_+<\delta<1-\frac{1}{\mathfrak{b}'}, \qquad \frac{1}{p_0}=\frac{1}{\mathfrak{b}'}+\delta, \qquad \frac{1}{q}<\frac{1}{\mathfrak{b}'}-\delta.$$

With these p and q, we deduce that there exists a constant C depending on $d,\ a,\ p_0$ and q such that

$$\|[K(\cdot-z),\boldsymbol{\rho}]\|_1 \leq Ch(R^{(d/2)(1/\mathfrak{b}-1/p_0')}\|\rho_1\|_{L^{p_0}} + R^{(d/2)(1/\mathfrak{b}-1/(2p_0')-1/(2q'))}\|\rho_1\|_{L^{p_0}}^{1/2}\|\rho_1\|_{L^q}^{1/2}).$$

Optimizing with respect to R yields

$$\operatorname{Tr}(|[K(\cdot - z), \boldsymbol{\rho}]|) \leq C h \|\rho_1\|_{L^{p_0}}^{\theta_0} \|\rho_1\|_{L^q}^{1-\theta_0}, \tag{33}$$

where $\theta_0 = (1/p_0 - 1/\mathfrak{b}')/(1/p_0 - 1/q)$. In order to obtain an equation of the form (27), we can define $\varepsilon := q - \mathfrak{b}'$, which is positive by (32) and the fact that $p_0 < \mathfrak{b}'$. The condition $q < \mathfrak{b}_3'$ when $d \geq 4$ then reads as $\varepsilon < \mathfrak{b}_3' - \mathfrak{b}'$. We can also define $p := \mathfrak{b}' - \varepsilon \geq 1$. Then by a direct computation and again using (32) we obtain

$$p_0 - p = p_0 + q - 2\mathfrak{b}' > 0,$$

so that $p < p_0 < \mathfrak{b}' < q$ and by interpolation of Lebesgue spaces,

$$\|\rho_1\|_{L^{p_0}} \leq \|\rho_1\|_{L^p}^{\theta_1} \|\rho_1\|_{L^q}^{1-\theta_1},$$

where $\theta_1 = (1/p_0 - 1/q)/(1/p - 1/q)$. Noticing that $\theta_0\theta_1 = (1/p_0 - 1/\mathfrak{b}')/(1/p - 1/q)$ and that we can take $1/p_0$ as close as we want to $1/p$, there exists ε_1 such that we can choose p_0 such that

$$\theta_0\theta_1 + \varepsilon_1 = \frac{1/p - 1/\mathfrak{b}'}{1/p - 1/q} = \frac{1}{2} + \frac{\varepsilon}{2\mathfrak{b}'}.$$

Therefore, the last inequality combined with (33) leads to (27). □

The following proposition is an extension of Theorem 4.3 to $\mathcal{L}^p$ spaces, for $p < \mathfrak{b}$. Notice however that the right-hand side here is expressed in terms of weighted quantum Lebesgue norms, which makes the inequality weaker than the one in Theorem 4.3.

Proposition 4.6. *Let $d \geq 2$, $a \in \left(-1, \min\left(2, \frac{1}{2}d\right)\right)$ and $1 \leq p < \mathfrak{b} := d/(a+1)$. Then for any $\varepsilon \in (0, q-1)$ and $n > a+1$, there exists a constant $C > 0$ such that*

$$\|[K(\cdot - z), \boldsymbol{\rho}]\|_{\mathcal{L}^p} \leq C h \|\nabla_\xi \boldsymbol{\rho} \boldsymbol{m}_n\|_{\mathcal{L}^{q+\varepsilon}}^{1/2+\tilde{\varepsilon}} \|\nabla_\xi \boldsymbol{\rho} \boldsymbol{m}_n\|_{\mathcal{L}^{q-\varepsilon}}^{1/2-\tilde{\varepsilon}},$$

where $\tilde{\varepsilon} = \varepsilon/q$, $\boldsymbol{m}_n = 1 + |\boldsymbol{p}|^n$ and with

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{\mathfrak{b}}.$$

Proof. First we do the same decomposition as for the $\mathcal{L}^1$ case but then take a $\mathcal{L}^p$-norm in (29). This yields

$$\frac{1}{h|\omega_a|} \|[K(\cdot - z), \boldsymbol{\rho}]\|_p \leq \int_0^\infty \int_0^1 t^{a/2} \|\phi_\theta \cdot \nabla_\xi \boldsymbol{\rho} \varphi_{1-\theta}\|_p \, \mathrm{d}\theta \, \mathrm{d}t. \tag{34}$$

In order to bound this integral, we will cut it into two parts corresponding to $t \in (0, R)$ and $t \geq R$, and we take $1/q > 1/p - 1/\mathfrak{b}$ when t is small and $1/q < 1/p - 1/\mathfrak{b}$ in the second case. Using the hypotheses, we can find $(\alpha, \beta) \in [2, \infty)^2$ and $(n_\alpha, n_\beta) \in (d/\alpha, \infty) \times (d/\beta, \infty)$ such that $\alpha > d$, $\beta > \frac{1}{2}d$, $n_\alpha + n_\beta = n$ and $1/\alpha + 1/\beta = 1/p - 1/q$. Then we define $\boldsymbol{m}_k := 1 + |\boldsymbol{p}|^k$ and multiply and divide by $\boldsymbol{m}_{n_\alpha}$ and $\boldsymbol{m}_{n_\beta}$. This yields

$$\begin{aligned}\|\phi_\theta \cdot \nabla_\xi \boldsymbol{\rho} \varphi_{1-\theta}\|_p &= \|(\phi_\theta \boldsymbol{m}_{n_\alpha}^{-1}) \cdot \boldsymbol{m}_{n_\alpha} \nabla_\xi \boldsymbol{\rho} \boldsymbol{m}_{n_\beta} \boldsymbol{m}_{n_\beta}^{-1} \varphi_{1-\theta}\|_p \\ &\leq \|\phi_\theta \boldsymbol{m}_{n_\alpha}^{-1}\|_\alpha \|\boldsymbol{m}_{n_\beta}^{-1} \varphi_{1-\theta}\|_\beta \|\boldsymbol{m}_{n_\alpha} \nabla_\xi \boldsymbol{\rho} \boldsymbol{m}_{n_\beta}\|_q,\end{aligned}$$

where we twice used Hölder's inequality for operators to obtain the second line from the right side of the first. We notice that $\phi_\theta \boldsymbol{m}_{n_\alpha}^{-1}$ is of the form $g(-i\nabla)f(x)$, so that since $\alpha \geq 2$, by the Kato–Seiler–Simon inequality (see e.g., [Simon 2005, Theorem 4.1]),

$$\|\phi_\theta \boldsymbol{m}_{n_\alpha}^{-1}\|_\alpha \leq (2\pi)^{-d/\alpha} \|\phi_\theta\|_{L^\alpha} \|m_{n_\alpha}^{-1}(\hbar\,\cdot\,)\|_{L^\alpha},$$

with $m_{n_\alpha}^{-1}(\hbar x) = (1+|\hbar x|^{n_\alpha})^{-1}$. By the change of variable $y = \hbar x$ in the last integral, and using the fact that $C_{d,n_\alpha,\alpha} := \|m_{n_\alpha}^{-1}\|_{L^\alpha} < \infty$, this yields

$$\|\phi_\theta \boldsymbol{m}_{n_\alpha}^{-1}\|_\alpha \leq C_{d,n_\alpha,\alpha} h^{-d/\alpha} \|\phi_\theta\|_{L^\alpha}.$$

Then a direct computation of the integral of ϕ_θ yields

$$\|\phi_\theta \boldsymbol{m}_{n_\alpha}^{-1}\|_\alpha \leq C_{d,n_\alpha,\alpha} h^{-d/\alpha} \left(\frac{\omega_d}{\omega_{d+\alpha}}\right)^{1/\alpha} \frac{1}{(\alpha\theta t)^{(d+\alpha)/(2\alpha)}}.$$

By the same proof but replacing ϕ_θ by $\varphi_{1-\theta}$, if $\beta \geq 2$, we have

$$\|\boldsymbol{m}_{n_\beta}^{-1} \varphi_{1-\theta}\|_\beta \leq C_{d,n_\beta,\beta} h^{-d/\beta} \frac{1}{(\beta(1-\theta)t)^{d/(2\beta)}}.$$

Therefore, (34) leads to

$$\begin{aligned}\|[K(\cdot\, - z), \boldsymbol{\rho}]\|_p &\leq \int_0^\infty \frac{C_{\boldsymbol{\rho}} h^{1-d(1/\alpha+1/\beta)}}{t^{(1/2)(d/\alpha+d/\beta+1-a)}} \left(\int_0^1 \frac{\mathrm{d}\theta}{\theta^{(d+\alpha)/(2\alpha)}(1-\theta)^{d/(2\beta)}}\right) \mathrm{d}t \\ &\leq \int_0^\infty \frac{C_{\boldsymbol{\rho}} h^{1+d/p'-d/q'}}{t^{(d/2)(1/p-1/q-1/\mathfrak{b})+1}} \left(\int_0^1 \frac{\mathrm{d}\theta}{\theta^{(d+\alpha)/(2\alpha)}(1-\theta)^{d/(2\beta)}}\right) \mathrm{d}t,\end{aligned}$$

where

$$C_{\boldsymbol{\rho}} = \left(\frac{\omega_d}{\omega_{d+\alpha}}\right)^{1/\alpha} \frac{C_{d,n_\alpha,\alpha} C_{d,n_\beta,\beta}}{\beta^{d/(2\beta)} \alpha^{(d+\alpha)/(2\alpha)}} \|\boldsymbol{m}_{n_\alpha} \nabla_\xi \boldsymbol{\rho} \boldsymbol{m}_{n_\beta}\|_q.$$

The integrals in θ and t converge since

$$\alpha > d \quad \text{and} \quad \beta > \frac{d}{2}, \qquad \frac{1}{p} - \frac{1}{q} < \frac{1}{\mathfrak{b}} \quad \text{if } t \in [0, R], \qquad \frac{1}{p} - \frac{1}{q} > \frac{1}{\mathfrak{b}} \quad \text{if } t \in (R, \infty).$$

Then observe that as proved in Appendix B (see (56)),

$$\|\boldsymbol{m}_{n_\alpha} \nabla_\xi \boldsymbol{\rho} \boldsymbol{m}_{n_\beta}\|_q \leq \|\nabla_\xi \boldsymbol{\rho} \boldsymbol{m}_{n_\beta} \boldsymbol{m}_{n_\alpha}\|_q = \|\nabla_\xi \boldsymbol{\rho} \boldsymbol{m}_n\|_q,$$

and we conclude the proof by taking the optimal R as in the proof of Theorem 4.3. □

4B. *Bound for the error term.* In this section, we will prove that the operator B_t defined by (25) is small when h goes to 0.

Proposition 4.7. *Under the hypotheses of Theorem 1.1, if $p \in [1, 2]$ and $n \in 2\mathbb{N}$ with $n > \frac{1}{2}d$, then*

$$\|B_t\|_{\mathcal{L}^p} \leq C\hbar^2 \|\rho_f\|_{L^1 \cap H^\nu} \|\nabla_\xi^2 f\|_{H^{2n}_{2n}(\mathbb{R}^{2d})},$$

where $\nu = (n+a+2-d)_+$ and C is independent from $\hbar$.

Proof. Recalling the notation $E = -\nabla K * \rho$ as in [Saffirio 2020a; 2020b], we write a decomposition of B_t as

$$\frac{1}{i\hbar}B_t(x,y) = \int_0^1 E((1-\theta)x+\theta y) - E\big(\tfrac{1}{2}(x+y)\big)\,\mathrm{d}\theta \cdot \nabla_\xi \boldsymbol{\rho}_f(x,y)$$
$$= i\hbar \int_0^1\int_0^1 \big(\theta-\tfrac{1}{2}\big)\nabla E\big(((1-\theta)x+\theta y)\theta' + \tfrac{1}{2}(x+y)(1-\theta')\big)\,\mathrm{d}\theta\,\mathrm{d}\theta' : \nabla_\xi^2 \boldsymbol{\rho}_f(x,y)$$
$$= i\hbar \int_0^1\int_0^1 \big(\theta-\tfrac{1}{2}\big)\nabla E(a_{\theta,\theta'}x + b_{\theta,\theta'}y)\,\mathrm{d}\theta\,\mathrm{d}\theta' : \nabla_\xi^2 \boldsymbol{\rho}_f(x,y)$$

where $a_{\theta,\theta'} = \frac{1}{2}(\theta'+1) - \theta\theta'$, $b_{\theta,\theta'} = \frac{1}{2}(1-\theta') + \theta\theta'$ and " : " denotes the double contraction of tensors. In terms of the Fourier transform of ∇E, this yields

$$\frac{1}{i\hbar}B_t(x,y) = i\hbar \int_0^1\int_0^1\int_{\mathbb{R}^d} \big(\theta-\tfrac{1}{2}\big)e^{2i\pi z\cdot(a_{\theta,\theta'}x+b_{\theta,\theta'}y)}\widehat{\nabla E}(z)\,\mathrm{d}\theta\,\mathrm{d}\theta'\,\mathrm{d}z : \nabla_\xi^2 \boldsymbol{\rho}_f(x,y).$$

Defining e_ω as the operator of multiplication by the function $x \mapsto e^{2i\pi\omega z\cdot x}$, we obtain

$$\frac{1}{i\hbar}B_t = i\hbar \int_0^1\int_0^1\int_{\mathbb{R}^d} \big(\theta-\tfrac{1}{2}\big)\widehat{\nabla E}(z) : e_{a_{\theta,\theta'}}(\nabla_\xi^2 \boldsymbol{\rho}_f)e_{b_{\theta,\theta'}}\,\mathrm{d}\theta\,\mathrm{d}\theta'\,\mathrm{d}z,$$

and since e_ω is a bounded (unitary) operator, taking the quantum Lebesgue norms yields

$$\frac{1}{\hbar}\|B_t\|_{\mathcal{L}^p} \leq \hbar \int_0^1\int_0^1\int_{\mathbb{R}^d} \big|\theta-\tfrac{1}{2}\big||\widehat{\nabla E}(z)|\|e_{a_{\theta,\theta'}}(\nabla_\xi^2 \boldsymbol{\rho}_f)e_{b_{\theta,\theta'}}\|_{\mathcal{L}^p}\,\mathrm{d}\theta\,\mathrm{d}\theta'\,\mathrm{d}z \leq \tfrac{1}{2}\hbar\|\widehat{\nabla E}\|_{L^1}\|\nabla_\xi^2 \boldsymbol{\rho}_f\|_{\mathcal{L}^p}.$$

- Now to bound $\|\widehat{\nabla E}\|_{L^1}$, we can use the fact that for any $n > \frac{1}{2}d$, the Fourier transform maps H^n continuously into L^1 to get

$$\|\widehat{\nabla E}\|_{L^1} \leq C_{d,n}\|\nabla^2 K * \rho_f\|_{H^n}.$$

If $a = d-2$, then by the continuity of $\nabla^2 K * \cdot$ in H^n, we get $\|\widehat{\nabla E}\|_{L^1} \leq C\|\rho_f\|_{H^n}$. Otherwise, if $a \in \big(\frac{1}{2}d-2, d\big)\setminus\{2\}$, we get

$$\|\widehat{\nabla E}\|_{L^1} \leq C_{d,n,a}\|(1+|x|^n)|x|^{a+2-d}\widehat{\rho_f}\|_{L^2} \leq C_{d,n,a}\|\rho_f\|_{\dot{H}^{a+2-d}\cap\dot{H}^{n+a+2-d}} \leq C_{d,n,a}\|\rho_f\|_{L^1\cap H^{(n+a+2-d)_+}},$$

where we used the fact that if $\alpha \in \big(-\frac{1}{2}d, 0\big)$, then by Sobolev's inequalities $L^{p^*} \supset \dot{H}^\alpha$ with $1/p^* = \frac{1}{2} - \alpha/d$, and then $L^2 \cap L^1 \subset L^{p^*}$ since $p^* \in [1,2]$.

- Finally, to bound $\|\nabla_\xi^2 \boldsymbol{\rho}_f\|_{\mathcal{L}^p}$, we interpolate it between the $\mathcal{L}^1$ and the $\mathcal{L}^2$ norms to get

$$\|\nabla_\xi^2 \boldsymbol{\rho}_f\|_{\mathcal{L}^p} \leq \|\nabla_\xi^2 \boldsymbol{\rho}_f\|_{\mathcal{L}^2}^\theta \|\nabla_\xi^2 \boldsymbol{\rho}_f\|_{\mathcal{L}^1}^{1-\theta} = \|\nabla_\xi^2 f\|_{L^2(\mathbb{R}^{2d})}^\theta \|\nabla_\xi^2 \boldsymbol{\rho}_f\|_{\mathcal{L}^1}^{1-\theta}, \tag{35}$$

where $\theta = 2/p'$. Then using the fact that $\nabla_\xi^2 \boldsymbol{\rho}_f = \boldsymbol{\rho}_\hbar^W(\nabla_\xi^2 f)$, we can use Proposition 3.4 with $g = \nabla_\xi^2 f$, $n_1 = 0$ and $n > \frac{1}{2}d$ to get

$$\|\nabla_\xi^2 \boldsymbol{\rho}_f\|_{\mathcal{L}^1} \leq C\|\nabla_\xi^2 f\|_{H_{2n}^{2n}(\mathbb{R}^{2d})},$$

which using (35) implies that $\|\nabla_\xi^2 \boldsymbol{\rho}_f\|_{\mathcal{L}^p} \leq C\|\nabla_\xi^2 f\|_{H_{2n}^{2n}(\mathbb{R}^{2d})}$. □

4C. *Proof of Proposition 4.1.* We can now use the bounds on the commutator and the error terms proved in previous sections to prove the stability estimate of Proposition 4.1.

For $p=1$, inequality (26) yields

$$\begin{aligned}\operatorname{Tr}(|\boldsymbol{\rho}(t)-\boldsymbol{\rho}_f(t)|)&\\ \le \operatorname{Tr}(|\boldsymbol{\rho}^{\mathrm{in}}-\boldsymbol{\rho}_f^{\mathrm{in}}|)&+\frac{1}{\hbar}\int_0^t \operatorname{Tr}(|B_s|)\,\mathrm{d}s+\frac{1}{\hbar}\int_0^t\!\!\int |\rho(s,z)-\rho_f(s,z)|\operatorname{Tr}(|[K(\cdot-z),\boldsymbol{\rho}_f(s)]|)\,\mathrm{d}z\,\mathrm{d}s.\end{aligned}$$

Proposition 4.7 gives a bound on the second term on the right-hand side, whereas Theorem 4.3 allows us to bound the last term on the right-hand side uniformly in z. Moreover, because of (11), we have

$$\|\rho-\rho_f\|_{L^1}\le \operatorname{Tr}(|\boldsymbol{\rho}-\boldsymbol{\rho}_f|).$$

Altogether, we obtain for some small $\varepsilon>0$ to be chosen later,

$$\begin{aligned}\operatorname{Tr}(|\boldsymbol{\rho}-\boldsymbol{\rho}_f|)\le \operatorname{Tr}(|\boldsymbol{\rho}^{\mathrm{in}}-\boldsymbol{\rho}_f^{\mathrm{in}}|)+C\hbar\int_0^t \|\rho_f(s)\|_{L^1\cap H^{(n+a+2-d)_+}}\|\nabla_\xi^2 f(s)\|_{H^{2n}_{2n}(\mathbb{R}^{2d})}\,\mathrm{d}s&\\ +C\int_0^t \operatorname{Tr}(|\boldsymbol{\rho}(s)-\boldsymbol{\rho}_f(s)|)\|\operatorname{diag}(|\nabla_\xi\boldsymbol{\rho}_f(s)|)\|_{L^{\mathfrak{b}'+\varepsilon}\cap L^{\mathfrak{b}'-\varepsilon}}\,\mathrm{d}s&,\end{aligned}$$

where $\mathfrak{b}'=d/(d-(a+1))$. We then use Proposition 3.1 to bound the L^p-norm of the diagonal for $p=\mathfrak{b}'+\varepsilon$ and $p=\mathfrak{b}'-\varepsilon$ by

$$\|\operatorname{diag}(|\nabla_\xi\boldsymbol{\rho}_f|)\|_{L^p}\le \mathcal{C}_{d,n_1}\|\nabla_\xi f\|_{W^{n_0,\infty}(\mathbb{R}^{2d})\cap H^{2n+n_1}_{2n+n_1}(\mathbb{R}^{2d})},$$

where since $n_1>d/(\mathfrak{b}-1)=d(\mathfrak{b}'-1)$ we can choose ε such that $\mathfrak{b}'+\varepsilon\le 1+n_1/d$. We conclude by using Grönwall's lemma. □

4D. *Proof of Theorem 1.1.* To prove this theorem, it remains to prove that the assumptions (24a) and (24b) are satisfied with our choice of initial conditions, which will imply the result by Proposition 4.1. But these bounds are only about the solution of the classical Vlasov equation for which the long time existence of regular solutions is known. More precisely, we prove the regularity needed in our case in Proposition A.1 in Appendix A. With our assumptions on the initial data, we have $f^{\mathrm{in}}\in W^{\sigma+1,\infty}_m(\mathbb{R}^{2d})$ with $m>d$. Moreover, since $f^{\mathrm{in}}\in H^{\sigma+1}_\sigma(\mathbb{R}^{2d})$ with $\sigma>m+d/(\mathfrak{b}-1)$, by Hölder's inequality we deduce in particular that $f^{\mathrm{in}}\in L^2_\sigma(\mathbb{R}^{2d})$ which by Hölder's inequality yields

$$\iint_{\mathbb{R}^{2d}} f^{\mathrm{in}}|\xi|^{n_1}\,\mathrm{d}x\,\mathrm{d}\xi<\infty$$

for some $n_1>d/(\mathfrak{b}-1)$. Therefore, Proposition A.1 indeed leads to

$$f\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+,W^{\sigma+1,\infty}_m(\mathbb{R}^{2d})\cap H^{\sigma+1}_\sigma(\mathbb{R}^{2d})),$$

where we notice that $\sigma>n_0:=2\lfloor\frac12 d\rfloor+2$. Finally, the H^ν bound for ρ also follows from Hölder's inequality since $\sigma>\frac12 d$, so that

$$\|\nabla^{\lceil\nu\rceil}\rho\|_{L^2}\le\left\|\int_{\mathbb{R}^d}|\nabla_x^{\lceil\nu\rceil}f|\,\mathrm{d}\xi\right\|_{L^2}\le C_{d,\sigma}\|\langle\xi\rangle^\sigma\nabla_x^{\lceil\nu\rceil}f\|_{L^2(\mathbb{R}^{2d})}\le C\|f\|_{H^{\sigma+1}_\sigma(\mathbb{R}^{2d})},$$

where the last inequality follows from the fact that $\lceil\nu\rceil\le\sigma+1$. □

4E. *Proof of Theorem 1.4.* We now prove Theorem 1.4 using the results of Propositions 4.6 and 4.7.

Recall inequality (26). The bound (11) yields

$$\|\boldsymbol{\rho}-\boldsymbol{\rho}_f\|_{\mathcal{L}^p} \le \|\boldsymbol{\rho}^{\mathrm{in}}-\boldsymbol{\rho}_f^{\mathrm{in}}\|_{\mathcal{L}^p} + \frac{1}{\hbar}\int_0^t \|B_t\|_{\mathcal{L}^p}\,\mathrm{d}s + \frac{1}{\hbar}\int_0^t \operatorname{Tr}(|\boldsymbol{\rho}-\boldsymbol{\rho}_f|)\sup_z \|[K(\cdot - z),\boldsymbol{\rho}_f]\|_{\mathcal{L}^p}\,\mathrm{d}s.$$

The second term on the right-hand side can be estimated thanks to Proposition 4.7 and can then be bounded as in the case $p=1$. The last term on the right-hand side is bounded by Proposition 4.6 by terms of the form $\|\nabla_\xi \boldsymbol{\rho}_f \boldsymbol{m}_n\|_{\mathcal{L}^q}$ with $\boldsymbol{m}_n = 1+|\boldsymbol{p}|^n$, $n > a+1 = d/\mathfrak{b}$ and $1/q$ close to $1/p - 1/\mathfrak{b}$. When $a < \frac{1}{2}(d-2)$, then $q \le 2$ and we can bound them by interpolating between $\mathcal{L}^1$ and $\mathcal{L}^2$ weighted norms, yielding

$$\|\nabla_\xi \boldsymbol{\rho}_f \boldsymbol{m}_n\|_{\mathcal{L}^q} \le \|\nabla_\xi \boldsymbol{\rho}_f \boldsymbol{m}_n\|_{\mathcal{L}^2}^{2/q'} \|\nabla_\xi \boldsymbol{\rho}_f \boldsymbol{m}_n\|_{\mathcal{L}^1}^{1-2/q'},$$

and we can then bound these terms by Propositions 3.3 and 3.4. When $q > 2$, this strategy is no longer possible; however, by the property of the Weyl transform and Calderón–Vaillancourt–Boulkhemair inequality (23) we know that

$$\|\boldsymbol{\rho}_\hbar^W(g)\|_{\mathcal{L}^2} = \|g\|_{L^2(\mathbb{R}^{2d})} \qquad \text{and} \qquad \|\boldsymbol{\rho}_\hbar^W(g)\|_{\mathcal{L}^\infty} \le C_d \|g\|_{W^{n_0,\infty}(\mathbb{R}^{2d})},$$

where $n_0 = 2\lfloor \frac{1}{2}d \rfloor + 2$. Therefore, this time, we interpolate between the $\mathcal{L}^2$- and $\mathcal{L}^\infty$-norm to get

$$\|\boldsymbol{\rho}_\hbar^W(g)\|_{\mathcal{L}^q} \le C_d^\theta \|\boldsymbol{\rho}_\hbar^W(g)\|_{\mathcal{L}^\infty}^\theta \|\boldsymbol{\rho}_\hbar^W(g)\|_{\mathcal{L}^2}^{1-\theta} \le C_d^\theta \|g\|_{W^{n_0,\infty}(\mathbb{R}^{2d})}^\theta \|g\|_{L^2(\mathbb{R}^{2d})}^{1-\theta}, \tag{36}$$

where $\theta = 1 - 2/q$ is close to $2/p' - 1/\mathfrak{b}' \pm \varepsilon$. Using Lemma 3.2, we see that $\nabla_\xi \boldsymbol{\rho}_f \boldsymbol{m}_n$ can be written as a linear combination of terms of the form $\boldsymbol{\rho}_\hbar^W(\xi^\alpha \partial_x^\beta \nabla_\xi f) =: \boldsymbol{\rho}_\hbar^W(g_{\alpha,\beta})$, where α and β are multi-indices satisfying $|\alpha+\beta| \le n$. Therefore, taking $g = g_{\alpha,\beta}$ in (36) for each $g_{\alpha,\beta}$, we obtain a control in terms of weighted Sobolev norms of the solution f of the classical solution of the Vlasov equation (1) of the form $\|f\|_{W_\sigma^{\sigma+1,\infty}(\mathbb{R}^{2d})\cap H_\sigma^{\sigma+1}(\mathbb{R}^{2d})}$ with $\sigma > n_0 + d/\mathfrak{b}$, which can be controlled as in the proof of Theorem 1.1. We can therefore conclude by Grönwall's lemma that (12) holds.

Now we prove (13). Consider (12) and the bound

$$\|\boldsymbol{\rho}-\boldsymbol{\rho}_f\|_{\mathcal{L}^\infty} \le \|\boldsymbol{\rho}\|_{\mathcal{L}^\infty} + \|\boldsymbol{\rho}_f\|_{\mathcal{L}^\infty}. \tag{37}$$

As for the first term on the right-hand side, we know that all $\mathcal{L}^p$-norms are propagated by the Hartree equation, therefore $\|\boldsymbol{\rho}\|_{\mathcal{L}^\infty} = \|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^\infty}$ and hence it is bounded by assumption. In the second term on the right-hand side we again use the Calderón–Vaillancourt–Boulkhemair inequality (23). Hence, if $f \in W^{n_0,\infty}(\mathbb{R}^{2d})$ and $\boldsymbol{\rho}^{\mathrm{in}} \in \mathcal{L}^\infty$, the $\mathcal{L}^\infty$-norm of the difference $\boldsymbol{\rho}-\boldsymbol{\rho}_f$ is bounded uniformly in $\hbar$. To conclude, we bound the $\mathcal{L}^q$-norm using the $\mathcal{L}^\infty$-norm and the $\mathcal{L}^p$-norm with $p = \mathfrak{b} - \varepsilon$, for $\varepsilon > 0$ small enough, and get

$$\|\boldsymbol{\rho}-\boldsymbol{\rho}_f\|_{\mathcal{L}^q} \le \|\boldsymbol{\rho}-\boldsymbol{\rho}_f\|_{\mathcal{L}^p}^{p/q} \|\boldsymbol{\rho}-\boldsymbol{\rho}_f\|_{\mathcal{L}^\infty}^{1-p/q},$$

since $q \in (p,\infty)$. Then (12) yields

$$\|\boldsymbol{\rho}-\boldsymbol{\rho}_f\|_{\mathcal{L}^q} \le C(t)(\|\boldsymbol{\rho}^{\mathrm{in}}-\boldsymbol{\rho}_f^{\mathrm{in}}\|_{\mathcal{L}^p}^{p/q} + \operatorname{Tr}(|\boldsymbol{\rho}^{\mathrm{in}}-\boldsymbol{\rho}_f^{\mathrm{in}}|)^{p/q} + \hbar^{p/q})e^{\lambda(t)},$$

where C is a constant which depends on the dimension of the space d, on $\|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^\infty}$ and on f^{in}. □

5. Proof of Theorem 1.6

We recall that $\mathsf{X} = \mathsf{X}_{\boldsymbol{\rho}}$ is the operator of the time-dependent integral kernel $\mathsf{X}_{\boldsymbol{\rho}}(x, y) = K(x-y)\boldsymbol{\rho}(x, y)$, where $\boldsymbol{\rho}$ is the integral kernel of the operator $\boldsymbol{\rho}$. Under the conditions of Theorem 1.6, the associated energy is bounded and we have the following inequalities.

Proposition 5.1. *Let $a \in [0, d)$, $s := d - a$ and $\boldsymbol{\rho}$ be a positive trace class operator. Then if $K \in \dot{H}^{s,1}_w$, we have that*

$$\operatorname{Tr}(\mathsf{X}\boldsymbol{\rho}) \leq C h^s \|K\|_{\dot{H}^{s,1}_w} \||\boldsymbol{p}|^{a/2}\boldsymbol{\rho}\|^2_{\mathcal{L}^2}. \tag{38}$$

Moreover, if $a \in \left[0, \frac{1}{2}d\right)$ and $K^2 \in \dot{H}^{2s-d,1}_w$, then for any $p \in [1, 2]$ and $q = (2p)/(2-p) \in [2, \infty]$ there exists a constant such that for any operator $\boldsymbol{\rho}_2$,

$$\|\mathsf{X}_{\boldsymbol{\rho}}\boldsymbol{\rho}_2\|_{\mathcal{L}^p} \leq C h^s \|K^2\|^{1/2}_{\dot{H}^{2s-d,1}_w} \||\boldsymbol{p}|^{a/2}\boldsymbol{\rho}\|_{\mathcal{L}^2} \|\boldsymbol{\rho}_2\|_{\mathcal{L}^q}. \tag{39a}$$

When $p \in [2, \infty]$ we still have

$$\|\mathsf{X}_{\boldsymbol{\rho}}\boldsymbol{\rho}_2\|_{\mathcal{L}^p} \leq C h^{s+d(1/p-1/2)} \|K^2\|^{1/2}_{\dot{H}^{2s-d,1}_w} \||\boldsymbol{p}|^{a/2}\boldsymbol{\rho}\|_{\mathcal{L}^2} \|\boldsymbol{\rho}_2\|_{\mathcal{L}^\infty}, \tag{39b}$$

where in both (39a) and (39b) the constants C depend only on s and d.

Remark 5.2. We can control the weighted $\mathcal{L}^2$-norms by the inequality

$$\||\boldsymbol{p}|^{a/2}\boldsymbol{\rho}\|^2_{\mathcal{L}^2} \leq \|\boldsymbol{\rho}\|_{\mathcal{L}^\infty} \operatorname{Tr}(|\boldsymbol{p}|^a\boldsymbol{\rho}). \tag{40}$$

Notice that we cannot deduce it immediately by Hölder's inequality for the Schatten norms because it would give us $\operatorname{Tr}(||\boldsymbol{p}|^a\boldsymbol{\rho}|)$ instead of $\operatorname{Tr}(|\boldsymbol{p}|^a\boldsymbol{\rho})$ on the right-hand side. However, by definition of the absolute value for operators and by cyclicity of the trace, we get

$$\||\boldsymbol{p}|^{a/2}\boldsymbol{\rho}\|^2_2 = \operatorname{Tr}(\boldsymbol{\rho}|\boldsymbol{p}|^a\boldsymbol{\rho}) = \operatorname{Tr}(\boldsymbol{\rho}\boldsymbol{\rho}^{1/2}|\boldsymbol{p}|^a\boldsymbol{\rho}^{1/2}) = \operatorname{Tr}(\boldsymbol{\rho}||\boldsymbol{p}|^{a/2}\boldsymbol{\rho}^{1/2}|^2).$$

Now, Hölder's inequality gives

$$\operatorname{Tr}(\boldsymbol{\rho}||\boldsymbol{p}|^{a/2}\boldsymbol{\rho}^{1/2}|^2) \leq \|\boldsymbol{\rho}\|_\infty \operatorname{Tr}(||\boldsymbol{p}|^{a/2}\boldsymbol{\rho}^{1/2}|^2) = \|\boldsymbol{\rho}\|_\infty \operatorname{Tr}(|\boldsymbol{p}|^a\boldsymbol{\rho}),$$

which leads to (40) by the definition of $\mathcal{L}^2$ and $\mathcal{L}^\infty$.

Proof of Proposition 5.1. We first prove (38) and then use it to show (39a) and (39b).

- *Proof of inequality (38).* Use (14) to get

$$\iint_{\mathbb{R}^{2d}} K(x-y)|\boldsymbol{\rho}(x, y)|^2 \,\mathrm{d}x\,\mathrm{d}y = c_{d,a} \int_{\mathbb{R}^d} \left(\iint_{\mathbb{R}^{2d}} \frac{|\boldsymbol{\rho}(x, y)|^2}{|x-y-w|^a} \,\mathrm{d}x\,\mathrm{d}y \right) Q(\mathrm{d}w)$$

for some measure Q such that $\|Q\|_{\mathrm{TV}} = \|K\|_{\dot{H}^{s,1}_w}$. This leads to

$$\mathcal{E}_{\mathsf{X}} \leq c_{d,a} \sup_{w \in \mathbb{R}^d} \left(\iint_{\mathbb{R}^{2d}} \frac{|\boldsymbol{\rho}(x, y)|^2}{|x-y-w|^a} \,\mathrm{d}x\,\mathrm{d}y \right) \|Q\|_{\mathrm{TV}}.$$

Now we concentrate on bounding the double integral. First we observe that by the change of variable $z = x - y - w$ we have

$$\mathcal{E}_a := \iint_{\mathbb{R}^{2d}} \frac{|\boldsymbol{\rho}(x,y)|^2}{|x-y-w|^a} \,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathbb{R}^{2d}} \frac{1}{|z|^a} |\boldsymbol{\rho}(z+y+w,y)|^2 \,\mathrm{d}z\,\mathrm{d}y.$$

Then, by the Hardy–Rellich inequality (see e.g., [Yafaev 1999]), since $a \in [0,d)$, for any $\varphi \in H^{a/2}$, we have that

$$\int_{\mathbb{R}^d} \frac{|\varphi(z)|^2}{|z|^a} \,\mathrm{d}z \le \mathcal{C}_{d,a} \int_{\mathbb{R}^d} |\Delta^{a/4}\varphi(z)|^2 \,\mathrm{d}z.$$

Therefore, taking $\varphi(z) = \boldsymbol{\rho}(z+y+w,y)$ in the above inequality and integrating with respect to y yields

$$\mathcal{E}_a \le \mathcal{C}_{d,a} \iint_{\mathbb{R}^{2d}} |\Delta_z^{a/4}\boldsymbol{\rho}(z+y+w,y)|^2 \,\mathrm{d}z\,\mathrm{d}y = \mathcal{C}_{d,a} \iint_{\mathbb{R}^{2d}} |\Delta_x^{a/4}\boldsymbol{\rho}(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y.$$

Recalling that $\Delta_x^{a/4}\boldsymbol{\rho}(x,y)$ is nothing but the integral kernel of the operator $\hbar^{-a/2}|\boldsymbol{p}|^{a/2}\boldsymbol{\rho}$ and using the definition of the $\mathcal{L}^2$-norm, we obtain

$$\mathcal{E}_a \le C_{d,a} h^{d-a} \| |\boldsymbol{p}|^{a/2}\boldsymbol{\rho}\|_{\mathcal{L}^2}^2,$$

where $C_{d,a} = (2\pi)^a \mathcal{C}_{d,a}$.

- *Proof of inequality* (39a). Since $\mathsf{X}_{\boldsymbol{\rho}}$ is a positive operator, $\mathsf{X}_{\boldsymbol{\rho}}^2 = |\mathsf{X}_{\boldsymbol{\rho}}|^2$. Moreover, denoting by $\tilde{\mathsf{X}}_{\boldsymbol{\rho}}$ the exchange operator associated to the kernel K^2, the following interesting property holds:

$$\mathrm{Tr}(\mathsf{X}_{\boldsymbol{\rho}}^2) = \iint_{\mathbb{R}^{2d}} K(x-y)^2 |\boldsymbol{\rho}(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y = \mathrm{Tr}(\tilde{\mathsf{X}}_{\boldsymbol{\rho}}\boldsymbol{\rho}).$$

From this and Hölder's inequality for operators, we deduce that if $K^2 \in \dot{H}_w^{2s-d,1}$ with $s \in \left(\frac{1}{2}d, d\right]$, then

$$\|\mathsf{X}_{\boldsymbol{\rho}}\boldsymbol{\rho}_2\|_p \le \|\boldsymbol{\rho}_2\|_q \|\mathsf{X}_{\boldsymbol{\rho}}\|_2 \le h^{d/q'} \|\boldsymbol{\rho}_2\|_{\mathcal{L}^q} \,\mathrm{Tr}(\tilde{\mathsf{X}}_{\boldsymbol{\rho}}\boldsymbol{\rho})^{1/2},$$

which by (38) for K^2 leads exactly to (39a).

- *Proof of inequality* (39b). We use the fact that $\|\mathsf{X}_{\boldsymbol{\rho}}\boldsymbol{\rho}_2\|_p \le \|\mathsf{X}_{\boldsymbol{\rho}}\boldsymbol{\rho}_2\|_2$ for any $p \ge 2$ and then we use (39a) for $p = 2$ to get

$$\|\mathsf{X}_{\boldsymbol{\rho}}\boldsymbol{\rho}_2\|_{\mathcal{L}^p} \le h^{(d/2-d/p')} \|\mathsf{X}_{\boldsymbol{\rho}}\boldsymbol{\rho}_2\|_{\mathcal{L}^2} \le C h^{s+d(1/p-1/2)} \|K^2\|_{\dot{H}_w^{2s-d,1}}^{1/2} \| |\boldsymbol{p}|^{a/2}\boldsymbol{\rho}\|_{\mathcal{L}^2} \|\boldsymbol{\rho}\|_{\mathcal{L}^\infty}.$$

The use of the nonsemiclassical inequality $\|\mathsf{X}_{\boldsymbol{\rho}}\boldsymbol{\rho}_2\|_p \le \|\mathsf{X}_{\boldsymbol{\rho}}\boldsymbol{\rho}_2\|_2$ explains the deterioration of the rate, which might not be optimal. □

When $a < 0$, we have similar bounds using moments in x instead of moments in $\boldsymbol{p}$.

Proposition 5.3. *Let $a < 0$ and $K(x) = |x|^{|a|}$. Then for any positive operator $\boldsymbol{\rho}$,*

$$\mathrm{Tr}(\mathsf{X}_{\boldsymbol{\rho}}) \le C h^d \| |x|^{|a|/2}\boldsymbol{\rho}\|_{\mathcal{L}^2}^2. \tag{41}$$

Moreover, for any $p \in [1,\infty]$, there exists a constant $C > 0$ such that for any operator $\boldsymbol{\rho}_2$,

$$\|\mathsf{X}_{\boldsymbol{\rho}}\boldsymbol{\rho}_2\|_{\mathcal{L}^p} \leq C h^d \||x|^{|a|/2}\boldsymbol{\rho}\|_{\mathcal{L}^2}\|\boldsymbol{\rho}_2\|_{\mathcal{L}^q} \quad \text{when } p \in [1,2), \tag{42a}$$

$$\|\mathsf{X}_{\boldsymbol{\rho}}\boldsymbol{\rho}_2\|_{\mathcal{L}^p} \leq C h^{d(1/p+1/2)} \||x|^{|a|/2}\boldsymbol{\rho}\|_{\mathcal{L}^2}\|\boldsymbol{\rho}_2\|_{\mathcal{L}^\infty} \quad \text{when } p \in [2,\infty], \tag{42b}$$

where $q = (2p)/(2-p) \in [2,\infty)$ when $p < 2$ and the constants C depend only on a and d.

Proof. The proof of (41) follows simply by writing

$$\iint_{\mathbb{R}^{2d}} K(x-y)|\boldsymbol{\rho}(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y \leq C \iint_{\mathbb{R}^{2d}} (|x|^{|a|} + |y|^{|a|})|\boldsymbol{\rho}(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y$$

and observing that the right-hand side is exactly the right-hand side of (41). The two other inequalities follow by taking K^2 instead of K and using Hölder's inequality as in the proof of Proposition 5.3. $\square$

Proof of Theorem 1.6. We proceed as in the proof of Theorem 1.1 and consider the one-parameter group of unitary transformations $\mathcal{U}_t$ generated by the Hartree–Fock Hamiltonian, i.e.,

$$i\hbar\partial_t \mathcal{U}_t = H_{\mathrm{HF}}(t)\mathcal{U}_t,$$

and compute

$$i\hbar\partial_t(\mathcal{U}_t^*(\boldsymbol{\rho} - \boldsymbol{\rho}_\hbar^W(f))\mathcal{U}_t) = \mathcal{U}_t^*[K*(\rho-\rho_f), \boldsymbol{\rho}_\hbar^W(f)]\mathcal{U}_t + \mathcal{U}_t^* B_t \mathcal{U}_t - \mathcal{U}_t^*[\mathsf{X}_{\boldsymbol{\rho}}, (\boldsymbol{\rho} - \boldsymbol{\rho}_\hbar^W(f))]\mathcal{U}_t.$$

Using Duhamel's formula and taking the $\mathcal{L}^p$-norm using the fact that $\mathcal{U}_t$ is a unitary operator, we obtain

$$\|\boldsymbol{\rho} - \boldsymbol{\rho}_\hbar^W(f)\|_{\mathcal{L}^p} \leq \|\boldsymbol{\rho}^{\mathrm{in}} - \boldsymbol{\rho}_f^{\mathrm{in}}\|_{\mathcal{L}^p} + \frac{1}{\hbar}\int_0^t \|[K*(\rho-\rho_f), \boldsymbol{\rho}_\hbar^W(f)]\|_{\mathcal{L}^p}\,\mathrm{d}s \\ + \frac{1}{\hbar}\int_0^t \|B_s\|_{\mathcal{L}^p}\,\mathrm{d}s + \frac{1}{\hbar}\int_0^t \|[\mathsf{X}_{\boldsymbol{\rho}}, (\boldsymbol{\rho} - \boldsymbol{\rho}_\hbar^W(f))]\|_{\mathcal{L}^p}\,\mathrm{d}s. \tag{43}$$

The second and third terms on the right-hand side in (43) can be bounded as in Theorem 1.1. As for the fourth term, we use Proposition 5.1.

More precisely, when $K(x) = \pm|x|^{-a}$ with $a \in \left[0, \frac{1}{2}d\right)$, using (39a) or (39b) yields

$$\frac{1}{\hbar}\|[\mathsf{X}_{\boldsymbol{\rho}}, (\boldsymbol{\rho} - \boldsymbol{\rho}_\hbar^W(f))]\|_{\mathcal{L}^p} \leq C h^{\tilde{s}-1}\|K^2\|_{\dot{H}_w^{d-2a,1}}^{1/2}\||\boldsymbol{p}|^{a/2}\boldsymbol{\rho}\|_{\mathcal{L}^2}\|\boldsymbol{\rho} - \boldsymbol{\rho}_\hbar^W(f)\|_{\mathcal{L}^p} \\ \leq C h^{\tilde{s}-1}\|K^2\|_{\dot{H}_w^{d-2a,1}}^{1/2}\||\boldsymbol{p}|^{a/2}\boldsymbol{\rho}\|_{\mathcal{L}^2}(\|\boldsymbol{\rho}\|_{\mathcal{L}^p} + \|\boldsymbol{\rho}_\hbar^W(f)\|_{\mathcal{L}^p}),$$

with $\tilde{s} = d - a - d(1/2 - 1/p)_+$. When $\tilde{s} \geq 2$, this does not change the order of the rate of convergence $O(h)$. When $\tilde{s} < 2$ (i.e., for high values of a and p), the contribution of the exchange term becomes bigger than that of the second term on the right-hand side of (43), thus leading to a rate of convergence of the order $O(h^{\tilde{s}-1})$.

When $K(x) = \pm|x|^{-a}$ with $a \in (-1, 0)$, we use (42a) or (39b) to get bounds in terms of $\||x|^{|a|/2}\boldsymbol{\rho}\|_{\mathcal{L}^2}$ instead of $\||\boldsymbol{p}|^{|a|/2}\boldsymbol{\rho}\|_{\mathcal{L}^2}$.

When $K(x) = \pm\ln(|x|)$, we write $K(x) \leq C_\varepsilon(|x|^\varepsilon + |x|^{-\varepsilon})$ and use both types of inequalities to get bounds with $\|(|x|^{\varepsilon/2} + |\boldsymbol{p}|^{\varepsilon/2})\boldsymbol{\rho}\|_{\mathcal{L}^2}$ instead.

For all the choices of K, when $p = 1$, we can therefore conclude that

$$\|\boldsymbol{\rho} - \boldsymbol{\rho}_\hbar^W(f)\|_{\mathcal{L}^1} \leq (\|\boldsymbol{\rho}^{\mathrm{in}} - \boldsymbol{\rho}_f^{\mathrm{in}}\|_{\mathcal{L}^1} + C_0(t)\hbar + C_1(t)\hbar^{s-1})e^{\lambda(t)}. \tag{44}$$

When $p \in (1, \mathfrak{b})$, we proceed as in the proof of (12) (the Hartree case) and use (44) to get

$$\|\boldsymbol{\rho} - \boldsymbol{\rho}_\hbar^W(f)\|_{\mathcal{L}^p} \leq \|\boldsymbol{\rho}^{\mathrm{in}} - \boldsymbol{\rho}_f^{\mathrm{in}}\|_{\mathcal{L}^p} + C(t)(\|\boldsymbol{\rho}^{\mathrm{in}} - \boldsymbol{\rho}_f^{\mathrm{in}}\|_{\mathcal{L}^1} + \hbar + \hbar^{\tilde{s}-1})e^{\lambda(t)}. \tag{45}$$

Moreover, when $p \in [\mathfrak{b}, \infty)$, again as in the Hartree case, we proceed as in the proof of (13). Following the exact same argument, $\|\boldsymbol{\rho} - \boldsymbol{\rho}_\hbar^W(f)\|_{\mathcal{L}^\infty}$ is bounded uniformly in $\hbar$ as soon as $\boldsymbol{\rho}^{\mathrm{in}} \in \mathcal{L}^\infty$ and $f \in W^{2\lfloor d/2\rfloor+2,\infty}$. Hence,

$$\|\boldsymbol{\rho} - \boldsymbol{\rho}_\hbar^W(f)\|_{\mathcal{L}^p} \leq C(t)(\|\boldsymbol{\rho}^{\mathrm{in}} - \boldsymbol{\rho}_f^{\mathrm{in}}\|_1^{1-\theta} + C_0\hbar^{p/q} + C_1\hbar^{(\tilde{s}-1)p/q})e^{\lambda(t)}.$$

In particular, if $\|\boldsymbol{\rho}^{\mathrm{in}} - \boldsymbol{\rho}_f^{\mathrm{in}}\|_1 \leq C\hbar$, we get for the Hilbert–Schmidt norm ($p = 2$) a convergence rate of $\hbar^{(3/4-\varepsilon)\min\{1,s-1\}}$. □

Appendix A: Propagation of weighted Sobolev norms for Vlasov equation

The existence of global smooth solutions and the propagation of regularity is a classical result for the Vlasov–Poisson equation. It can be deduced starting from the works of Pfaffelmoser [1992] or Lions and Perthame [1991], which imply the boundedness of the force field, so that any solution with compact support in the phase space will remain compactly supported at any time. Other general results concerning the propagation of regularity can be found in the more recent work by Han-Kwan [2019] or in Appendix A in the work by the second author [Saffirio 2020b]. In our case, we need the boundedness of the solutions of the Vlasov equation in weighted Sobolev norms, and we will see that we can propagate norms of the form $W_n^{\sigma,\infty}(\mathbb{R}^{2d})$. We prefer to work in the framework of [Lions and Perthame 1991], which allows us to have noncompactly supported solutions which are very interesting physically, since they include for example Gaussian distributions of velocities. Moreover, compactly supported solutions are perhaps less pertinent in the context of quantum mechanics. Furthermore, the proof here follows a completely Eulerian point of view. The result of this section is the following.

Proposition A.1. *Let $K = 1/|x|^a$ with $a \in (-1, d-2]$ and let $(n, \sigma, n_1) \in \mathbb{N}^3$ be such that $n > d$ and $n_1 > d/(\mathfrak{b} - 1)$ with $\mathfrak{b} = d/(a+1)$. Let $f \geq 0$ be a solution of the Vlasov equation (1) with initial data $f^{\mathrm{in}} \in W_n^{\sigma,\infty}(\mathbb{R}^{2d})$ satisfying*

$$\iint_{\mathbb{R}^{2d}} f^{\mathrm{in}} |\xi|^{n_1}\, \mathrm{d}x\, \mathrm{d}\xi < \infty.$$

Then the following regularity estimates hold:

$$f \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, W_n^{\sigma,\infty}(\mathbb{R}^{2d})), \tag{46a}$$

$$\nabla^\sigma \rho_f \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^\infty). \tag{46b}$$

If in addition $f^{\mathrm{in}} \in H_k^\sigma(\mathbb{R}^{2d})$ for some $k \in \mathbb{R}_+$, then

$$f \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, H_k^\sigma(\mathbb{R}^{2d})).$$

The proof works in two steps. We first explain in the next lemma how to get control of the regularity as soon as ρ_f is uniformly bounded. Then we finish the proof of the theorem by proving that this assumption on ρ_f holds.

Lemma A.2. *Let f be a solution of the Vlasov equation (1) as in Proposition A.1 with $\sigma \geq 1$ and assume moreover that*

$$\rho_f \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^\infty \cap L^1). \tag{47}$$

Then the regularity estimates (46a) and (46b) hold.

Proof. For clarity, we first start with the case $\sigma = 1$ for which we present a detailed proof, and we will then explain how to modify the proof to get higher regularity estimates. We follow the strategy explained in the course notes [Golse 2013].

Case 1: $\sigma = 1$. Define the transport operator $\mathsf{T} := \xi \cdot \nabla_x + E \cdot \nabla_\xi$. Then we have

$$\partial_t(\nabla_x f) = -\mathsf{T}\nabla_x f - \nabla E \cdot \nabla_\xi f, \tag{48a}$$

$$\partial_t(\nabla_\xi f) = -\mathsf{T}\nabla_\xi f - \nabla_x f. \tag{48b}$$

To simplify the computations, recall that $\mathsf{T}^* = -\mathsf{T}$ and $\mathsf{T}(uv) = u\mathsf{T}(v) + \mathsf{T}(u)v$. Hence, by writing $m_n := 1 + |\xi|^{np} + |x|^{np}$ and using the notation $u^p := |u|^{p-1}u$, we have

$$\iint_{\mathbb{R}^{2d}} \mathsf{T}(u) \cdot u^{p-1} m_n = -\iint_{\mathbb{R}^{2d}} u \cdot \mathsf{T}(u^{p-1}) m_n + |u|^p \mathsf{T}(m_n). \tag{49}$$

However, noticing that

$$\begin{aligned} u \cdot (\mathsf{T}(u^{p-1})) = u \cdot (|u|^{p-2}\mathsf{T}(u) + (p-2)(\mathsf{T}(u)\cdot u)u^{p-3}) &= u^{p-1} \cdot \mathsf{T}(u) + (p-2)(\mathsf{T}(u)\cdot u)|u|^{p-2} \\ &= (p-1)u^{p-1} \cdot \mathsf{T}(u), \end{aligned}$$

we can simplify (49) as

$$-p\iint_{\mathbb{R}^{2d}} \mathsf{T}(u) \cdot u^{p-1} m_n = \iint_{\mathbb{R}^{2d}} |u|^p \mathsf{T}(m_n). \tag{50}$$

Now define

$$M_x := \iint_{\mathbb{R}^{2d}} |\nabla_x f|^p m_n \quad \text{and} \quad M_\xi := \iint_{\mathbb{R}^{2d}} |\nabla_\xi f|^p m_n.$$

Then using (48a) and (50) for $u = \nabla_x f$ leads to

$$\frac{\mathrm{d}M_x}{\mathrm{d}t} = -p\iint_{\mathbb{R}^{2d}} (\nabla_x f)^{p-1} \cdot (\mathsf{T}\nabla_x f + \nabla E \cdot \nabla_\xi f) m_n \leq \iint_{\mathbb{R}^{2d}} |\nabla_x f|^p \mathsf{T}(m_n) + \|\nabla E\|_{L^\infty}(M_\xi + (p-1)M_x),$$

where we used Young's inequality for products, $pab^{p-1} \leq a^p + (p-1)b^p$, to get the second term. In the same way, using (48b) and taking $u = \nabla_\xi f$ yields

$$\frac{\mathrm{d}M_\xi}{\mathrm{d}t} \leq \iint_{\mathbb{R}^{2d}} |\nabla_\xi f|^p \mathsf{T}(m_n) + (M_x + (p-1)M_\xi).$$

Then again by Young's inequality for products,

$$\mathsf{T}(m_n) = np(E \cdot \xi^{np-1} + \xi \cdot x^{np-1}) \leq np(\|E\|_{L^\infty} + 1)m_n.$$

Thus, for $M_{x,\xi} := M_x + M_\xi$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} M_{x,\xi} \le p(n\|E\|_{L^\infty} + 1 + \|\nabla E\|_{L^\infty}) M_{x,\xi}.$$

However, since we know that $\rho_f \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^\infty \cap L^1)$ by assumption, we also get the following control on $\|E\|_{L^\infty}$:

$$\|E\|_{L^\infty} \le C(\|\rho_f\|_{L^\infty} + \|\rho_f\|_{L^1}) \le C_t$$

for some function of time C_t locally bounded on $\mathbb{R}_+$. To control ∇E, we can use the integral Young's inequality if ∇K is less singular than the Coulomb potential (i.e., if $a < d-2$), and if $a = 1$, then we use a singular integral estimate in the spirit of that in [Beale et al. 1984] which can be found in the course notes [Golse 2013] and can be written as

$$\|\nabla E\|_{L^\infty} \le C(1 + M_0 + \|\rho_f\|_{L^\infty} \ln(1 + \|\nabla \rho_f\|_{L^\infty})) \le C_t(1 + \ln(1 + \|\nabla \rho_f\|_{L^\infty})) =: J(t).$$

Combining these bounds we arrive at $\frac{\mathrm{d}}{\mathrm{d}t} M_{x,\xi} \le p(1+n) J(t) M_{x,\xi}$, which by Grönwall's lemma implies

$$M^{1/p}_{x,\xi}(t) \le M^{1/p}_{x,\xi}(0) e^{(1+n)\int_0^t J}.$$

Now, since $M^{1/p}_{x,\xi}$ is equivalent to $\|f\|_{W^{1,p}_n(\mathbb{R}^{2d})}$ in the sense that each one is bounded above by the other up to a multiplicative constant, letting $p \to \infty$, we obtain

$$\|f\|_{W^{1,\infty}_n(\mathbb{R}^{2d})} \le \|f^{\mathrm{in}}\|_{W^{1,\infty}_n(\mathbb{R}^{2d})} e^{(1+n)\int_0^t J}. \tag{51}$$

However, since $n > d$, we have

$$|\nabla \rho_f| \le \int_{\mathbb{R}^d} |\nabla_x f| \,\mathrm{d}\xi \le C_{d,n} \|f\|_{W^{1,\infty}_n(\mathbb{R}^{2d})}, \tag{52}$$

where $C_{d,n} = \int_{\mathbb{R}^d} \langle \xi \rangle^{-n} \,\mathrm{d}\xi < \infty$. Combining the two inequalities (51) and (52) and the fact that $e^{J(t)} \ge 1$, we deduce that

$$\begin{aligned} J(t) &\le C_t + C_t \ln((1 + C_{d,n}\|f^{\mathrm{in}}\|_{W^{1,\infty}_n(\mathbb{R}^{2d})}) e^{(1+n)\int_0^t J}) \\ &\le C_t + C_t \ln(1 + C_{d,n}\|f^{\mathrm{in}}\|_{W^{1,\infty}_n(\mathbb{R}^{2d})}) + C_t(1+n)\int_0^t J. \end{aligned}$$

Hence, by Grönwall's lemma,

$$J(t) \le J(0) + \frac{1 + \ln(1 + C_{d,n}\|f^{\mathrm{in}}\|_{W^{1,\infty}_n(\mathbb{R}^{2d})})}{n+1} \frac{e^{C_t(1+n)t}}{1+n}.$$

We then deduce the bounds on $\|f\|_{W^{1,\infty}_n(\mathbb{R}^{2d})}$ and $\nabla \rho_f$ by (51) and (52).

Case 2: $\sigma > 1$. We give details for $\sigma = 2$. The generalization to $\sigma \ge 2$ follows in the same way. In the case $\sigma = 2$, (48b) and (48a) become

$$\partial_t(\nabla^2_\xi f) + \mathsf{T}\nabla^2_\xi f = -2\nabla_x \nabla_\xi f \quad \text{and} \quad \partial_t(\nabla^2_x f) + \mathsf{T}\nabla^2_x f = -2\nabla E \cdot \nabla_\xi \nabla_x f - \nabla^2 E \cdot \nabla_\xi f.$$

Moreover, the mixed derivative of order two solves

$$\partial_t(\nabla_x \nabla_\xi f) + \mathsf{T}\nabla_x \nabla_\xi f = -\nabla_x^2 f - \nabla E \cdot \nabla_\xi^2 f.$$

We define the quantities

$$M_{xx} := \iint_{\mathbb{R}^{2d}} |\nabla_x^2 f|^p m_n \,\mathrm{d}x\,\mathrm{d}\xi, \quad M_{\xi\xi} := \iint_{\mathbb{R}^{2d}} |\nabla_\xi^2 f|^p m_n \,\mathrm{d}x\,\mathrm{d}\xi, \quad M_{x\xi} := \iint_{\mathbb{R}^{2d}} |\nabla_x \nabla_\xi f|^p m_n \,\mathrm{d}x\,\mathrm{d}\xi,$$

and compute their time derivatives, using Young's inequality for products, the bound on $\mathsf{T}(m_n)$ as in Case 1 and the fact that $p > 1$:

$$\begin{aligned}
\frac{\mathrm{d}M_{\xi\xi}}{\mathrm{d}t} &\le p(n\|E\|_{L^\infty} + n + 2)M_{\xi\xi} + 2pM_{x\xi}, \\
\frac{\mathrm{d}M_{x\xi}}{\mathrm{d}t} &\le p(n\|E\|_{L^\infty} + n + 1 + \|\nabla E\|_{L^\infty})M_{x\xi} + pM_{xx} + p\|\nabla E\|_{L^\infty} M_{\xi\xi}, \\
\frac{\mathrm{d}M_{xx}}{\mathrm{d}t} &\le p(n\|E\|_{L^\infty} + n + 2\|\nabla E\|_{L^\infty} + \|\nabla^2 E\|_{L^\infty})M_{xx} + 2p\|\nabla E\|_{L^\infty} M_{x\xi} + p\|\nabla^2 E\|_{L^\infty} M_\xi,
\end{aligned}$$

where M_ξ is defined and bounded as in Case 1. Thus, for $M_2 := M_{xx} + M_{x\xi} + M_{\xi\xi}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} M_2 \le Cp(n\|E\|_{L^\infty} + n + 2 + 2\|\nabla E\|_{L^\infty} + \|\nabla^2 E\|_{L^\infty})M_2.$$

We proved in Case 1 that $\|E\|_{L^\infty}$ and $\|\nabla E\|_{L^\infty}$ are bounded. To control $\nabla^2 E$, we proceed analogously to Case 1. More generally, we can bound $\nabla^\sigma E$ by $\nabla_x^\sigma f$. This leads, by Grönwall's lemma, to

$$M_2^{1/p}(t) \le M_2^{1/p}(0) e^{C_t}, \tag{53}$$

for some positive time-dependent constant $C_t > 0$. Now, since $M_2^{1/p}$ is equivalent to $\|f\|_{W_n^{2,p}(\mathbb{R}^{2d})}$ (with the exact same meaning given in Case 1), letting $p \to \infty$, we obtain

$$\|f\|_{W_n^{2,\infty}(\mathbb{R}^{2d})} \le \|f^{\mathrm{in}}\|_{W_n^{2,\infty}(\mathbb{R}^{2d})} e^{C_t}.$$

The general case $\sigma > 1$ can be handled analogously by defining

$$M_\sigma := \iint |\nabla^\sigma f|^p m_n \,\mathrm{d}x\,\mathrm{d}\xi,$$

where $\sigma = |\boldsymbol{\sigma}|$ stands for the sum of the components of the multi-index $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots)$. □

Proof of Proposition A.1. It just remains to prove that assumption (47) holds. First notice that the method used in [Lions and Perthame 1991, Theorem 1] actually works for any $a \in (-1, d-2]$ since the Coulomb potential is decomposed in two parts of the form $\nabla K \in L^{3/2,\infty} \cap L^1 + W^{2,\infty}$. This proves that the n_1 moments can be propagated, which implies that $\rho_f \in L^p$ for $p = 1 + n_1/d$ by the kinetic interpolation inequality. Then, by Young's inequality, since $n_1 > d/(\mathfrak{b} - 1)$, we deduce that

$$E \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^\infty).$$

Finally, as proved in [Lafleche 2019, Corollary 5.1], this bound combined with the initial assumption $f \in L^\infty(1+|\xi|^n)$ is sufficient to control $\|\rho_f\|_{L^\infty}$ and gives

$$\|\rho_f(t)\|_{L^\infty} \leq C\left(1+\int_0^t \|E(s)\|_{L^\infty}\,\mathrm{d}s\right),$$

which implies (47) so that we can apply the lemma. Then once we know the $W^{s,\infty}_n(\mathbb{R}^{2d})$-norm is bounded at any time, if the $H^\sigma_k(\mathbb{R}^{2d})$-norm is also initially bounded, we can again use (53) but with $p=2$ and then bound the terms involving E and $\nabla_x f$ by the $W^{\sigma,\infty}_n(\mathbb{R}^{2d})$-norm. Again we conclude the proof using Grönwall's lemma. □

Appendix B: Operator identities

Here we list some formulas for operators which are used in this paper. First, if A and B are self-adjoint, then we have

$$\|AB\|_p = \|BA\|_p, \tag{54}$$

which follows from the fact that the singular values are the same for an operator and its adjoint [Simon 2005, (1.3)]. Then we shall remember Hölder's inequality for operators [Simon 2005, Theorem 2.8], which says that for any bounded operators A and B and any $(p,q,r)\in[1,\infty]^3$ such that $1/p=1/q+1/r$,

$$\|AB\|_p \leq \|A\|_q\|B\|_r. \tag{Hölder}$$

The second important inequality is the Araki–Lieb–Thirring inequality [Araki 1990, Theorem 1] which states that for any operators $A, B \geq 0$ and any $(q,r)\in[1,\infty)\times\mathbb{R}_+$, the following inequality is true:

$$\mathrm{Tr}((BAB)^{qr}) \leq \mathrm{Tr}((B^qA^qB^q)^r).$$

Replacing A by A^2 and observing that $|AB|^2 = BA^2B$, this can be rewritten as

$$\|AB\|_{qr}^q \leq \|A^qB^q\|_r. \tag{55}$$

These inequalities show that regrouping operators together in Schatten norms increases the value of the norm, while *mixing* them will lower the value. In the same spirit, for any $A, B\geq 0$, $p\geq 1$ and $r\geq 0$, the following mixing inequality holds:

$$\|B^rAB\|_p \leq \|AB^{r+1}\|_p. \tag{56}$$

Proof of inequality (56). By (Hölder)'s inequality, we have

$$\|B^rAB\|_p \leq \|B^rA^{r/(r+1)}\|_{((r+1)/r)p}\|A^{1/(r+1)}B\|_{(r+1)p}.$$

Now, by the cyclicity property (54) and by (55), we get

$$\|B^rA^{r/(r+1)}\|_{((r+1)/r)p} \leq \|AB^{r+1}\|_p^{r/(r+1)} \quad \text{and} \quad \|A^{1/(r+1)}B\|_{(r+1)p} \leq \|AB^{r+1}\|_p^{1/(r+1)},$$

which yield the result. □

References

[Amour et al. 2013a] L. Amour, M. Khodja, and J. Nourrigat, "The classical limit of the Heisenberg and time-dependent Hartree–Fock equations: the Wick symbol of the solution", *Math. Res. Lett.* **20**:1 (2013), 119–139. MR Zbl

[Amour et al. 2013b] L. Amour, M. Khodja, and J. Nourrigat, "The semiclassical limit of the time dependent Hartree–Fock equation: the Weyl symbol of the solution", *Anal. PDE* **6**:7 (2013), 1649–1674. MR Zbl

[Araki 1990] H. Araki, "On an inequality of Lieb and Thirring", *Lett. Math. Phys.* **19**:2 (1990), 167–170. MR Zbl

[Athanassoulis et al. 2011] A. Athanassoulis, T. Paul, F. Pezzotti, and M. Pulvirenti, "Strong semiclassical approximation of Wigner functions for the Hartree dynamics", *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **22**:4 (2011), 525–552. MR Zbl

[Bach et al. 2016] V. Bach, S. Breteaux, S. Petrat, P. Pickl, and T. Tzaneteas, "Kinetic energy estimates for the accuracy of the time-dependent Hartree–Fock approximation with Coulomb interaction", *J. Math. Pures Appl.* (9) **105**:1 (2016), 1–30. MR Zbl

[Bardos and Degond 1985] C. Bardos and P. Degond, "Global existence for the Vlasov–Poisson equation in 3 space variables with small initial data", *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **2**:2 (1985), 101–118. MR Zbl

[Bardos et al. 2000] C. Bardos, F. Golse, and N. J. Mauser, "Weak coupling limit of the N-particle Schrödinger equation", *Methods Appl. Anal.* **7**:2 (2000), 275–293. MR Zbl

[Bardos et al. 2002] C. Bardos, L. Erdős, F. Golse, N. Mauser, and H.-T. Yau, "Derivation of the Schrödinger–Poisson equation from the quantum N-body problem", *C. R. Math. Acad. Sci. Paris* **334**:6 (2002), 515–520. MR Zbl

[Beale et al. 1984] J. T. Beale, T. Kato, and A. Majda, "Remarks on the breakdown of smooth solutions for the 3-D Euler equations", *Comm. Math. Phys.* **94**:1 (1984), 61–66. MR Zbl

[Benedikter et al. 2014] N. Benedikter, M. Porta, and B. Schlein, "Mean-field evolution of fermionic systems", *Comm. Math. Phys.* **331**:3 (2014), 1087–1131. MR Zbl

[Benedikter et al. 2016a] N. Benedikter, V. Jakšić, M. Porta, C. Saffirio, and B. Schlein, "Mean-field evolution of fermionic mixed states", *Comm. Pure Appl. Math.* **69**:12 (2016), 2250–2303. MR Zbl

[Benedikter et al. 2016b] N. Benedikter, M. Porta, C. Saffirio, and B. Schlein, "From the Hartree dynamics to the Vlasov equation", *Arch. Ration. Mech. Anal.* **221**:1 (2016), 273–334. MR Zbl

[Bergh and Löfström 1976] J. Bergh and J. Löfström, *Interpolation spaces: an introduction*, Grundlehren der Math. Wissenschaften **223**, Springer, 1976. MR Zbl

[Boulkhemair 1999] A. Boulkhemair, "L^2 estimates for Weyl quantization", *J. Funct. Anal.* **165**:1 (1999), 173–204. MR Zbl

[Castella 1997] F. Castella, "L^2 solutions to the Schrödinger–Poisson system: existence, uniqueness, time behaviour, and smoothing effects", *Math. Models Methods Appl. Sci.* **7**:8 (1997), 1051–1083. MR Zbl

[Chen et al. 2011] L. Chen, J. O. Lee, and B. Schlein, "Rate of convergence towards Hartree dynamics", *J. Stat. Phys.* **144**:4 (2011), 872–903. MR Zbl

[Chen et al. 2018] L. Chen, J. O. Lee, and J. Lee, "Rate of convergence toward Hartree dynamics with singular interaction potential", *J. Math. Phys.* **59**:3 (2018), art. id. 031902. MR Zbl

[Desvillettes et al. 2015] L. Desvillettes, E. Miot, and C. Saffirio, "Polynomial propagation of moments and global existence for a Vlasov–Poisson system with a point charge", *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **32**:2 (2015), 373–400. MR Zbl

[Dobrushin 1979] R. L. Dobrushin, "Vlasov equations", *Funktsional. Anal. i Prilozhen.* **13**:2 (1979), 48–58. In Russian; translated in *Funct. Anal. Appl.* **13**:2 (1979), 115–123. MR Zbl

[Elgart et al. 2004] A. Elgart, L. Erdős, B. Schlein, and H.-T. Yau, "Nonlinear Hartree equation as the mean field limit of weakly coupled fermions", *J. Math. Pures Appl.* (9) **83**:10 (2004), 1241–1273. MR Zbl

[Erdős and Yau 2001] L. Erdős and H.-T. Yau, "Derivation of the nonlinear Schrödinger equation from a many body Coulomb system", *Adv. Theor. Math. Phys.* **5**:6 (2001), 1169–1205. MR Zbl

[Figalli et al. 2012] A. Figalli, M. Ligabò, and T. Paul, "Semiclassical limit for mixed states with singular and rough potentials", *Indiana Univ. Math. J.* **61**:1 (2012), 193–222. MR Zbl

[Fröhlich and Knowles 2011] J. Fröhlich and A. Knowles, "A microscopic derivation of the time-dependent Hartree–Fock equation with Coulomb two-body interaction", *J. Stat. Phys.* **145**:1 (2011), 23–50. MR Zbl

[Fröhlich et al. 2009] J. Fröhlich, A. Knowles, and S. Schwarz, "On the mean-field limit of bosons with Coulomb two-body interaction", *Comm. Math. Phys.* **288**:3 (2009), 1023–1059. MR Zbl

[Gasser et al. 1998] I. Gasser, R. Illner, P. A. Markowich, and C. Schmeiser, "Semiclassical, $t \to \infty$ asymptotics and dispersive effects for Hartree–Fock systems", *RAIRO Modél. Math. Anal. Numér.* **32**:6 (1998), 699–713. MR Zbl

[Ginibre and Velo 1980] J. Ginibre and G. Velo, "On a class of nonlinear Schrödinger equations with nonlocal interaction", *Math. Z.* **170**:2 (1980), 109–136. MR Zbl

[Ginibre and Velo 1985] J. Ginibre and G. Velo, "The global Cauchy problem for the nonlinear Schrödinger equation revisited", *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **2**:4 (1985), 309–327. MR Zbl

[Golse 2013] F. Golse, "Mean field kinetic equations", course notes, École Polytechnique, 2013, available at https://tinyurl.com/golsepoly.

[Golse and Paul 2017] F. Golse and T. Paul, "The Schrödinger equation in the mean-field and semiclassical regime", *Arch. Ration. Mech. Anal.* **223**:1 (2017), 57–94. MR Zbl

[Golse and Paul 2019] F. Golse and T. Paul, "Empirical measures and quantum mechanics: applications to the mean-field limit", *Comm. Math. Phys.* **369**:3 (2019), 1021–1053. MR Zbl

[Golse et al. 2016] F. Golse, C. Mouhot, and T. Paul, "On the mean field and classical limits of quantum mechanics", *Comm. Math. Phys.* **343**:1 (2016), 165–205. MR Zbl

[Golse et al. 2018] F. Golse, T. Paul, and M. Pulvirenti, "On the derivation of the Hartree equation from the N-body Schrödinger equation: uniformity in the Planck constant", *J. Funct. Anal.* **275**:7 (2018), 1603–1649. MR Zbl

[Graffi et al. 2003] S. Graffi, A. Martinez, and M. Pulvirenti, "Mean-field approximation of quantum systems and classical limit", *Math. Models Methods Appl. Sci.* **13**:1 (2003), 59–73. MR Zbl

[Grillakis et al. 2010] M. G. Grillakis, M. Machedon, and D. Margetis, "Second-order corrections to mean field evolution of weakly interacting bosons, I", *Comm. Math. Phys.* **294**:1 (2010), 273–301. MR Zbl

[Han-Kwan 2019] D. Han-Kwan, "On propagation of higher space regularity for nonlinear Vlasov equations", *Anal. PDE* **12**:1 (2019), 189–244. MR Zbl

[Holding and Miot 2018] T. Holding and E. Miot, "Uniqueness and stability for the Vlasov–Poisson system with spatial density in Orlicz spaces", pp. 145–162 in *Mathematical analysis in fluid mechanics: selected recent results* (Marseille, 2017), edited by R. Danchin et al., Contemp. Math. **710**, Amer. Math. Soc., Providence, RI, 2018. MR Zbl

[Hunt 1966] R. A. Hunt, "On $L(p,\,q)$ spaces", *Enseign. Math.* (2) **12**:4 (1966), 249–276. MR Zbl

[Iordanskiĭ 1961] S. V. Iordanskiĭ, "The Cauchy problem for the kinetic equation of plasma", *Trudy Mat. Inst. Steklov.* **60** (1961), 181–194. In Russian; translated as pp. 351–363 in *Twelve papers on analysis and applied mathematics*, Amer. Math. Soc. Transl. (2) **35**, Amer. Math. Soc., Providence, RI, 1964. MR Zbl

[Kuz 2015] E. Kuz, "Rate of convergence to mean field for interacting bosons", *Comm. Partial Differential Equations* **40**:10 (2015), 1831–1854. MR Zbl

[Lafleche 2019] L. Lafleche, "Propagation of moments and semiclassical limit from Hartree to Vlasov equation", *J. Stat. Phys.* **177**:1 (2019), 20–60. MR Zbl

[Lafleche 2021] L. Lafleche, "Global semiclassical limit from Hartree to Vlasov equation for concentrated initial data", *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **38**:6 (2021), 1739–1762. MR Zbl

[Lieb and Loss 2001] E. H. Lieb and M. Loss, *Analysis*, 2nd ed., Grad. Stud. in Math. **14**, Amer. Math. Soc., Providence, RI, 2001. MR Zbl

[Lions and Paul 1993] P.-L. Lions and T. Paul, "Sur les mesures de Wigner", *Rev. Mat. Iberoam.* **9**:3 (1993), 553–618. MR Zbl

[Lions and Perthame 1991] P.-L. Lions and B. Perthame, "Propagation of moments and regularity for the 3-dimensional Vlasov–Poisson system", *Invent. Math.* **105**:2 (1991), 415–430. MR Zbl

[Loeper 2006] G. Loeper, "Uniqueness of the solution to the Vlasov–Poisson system with bounded density", *J. Math. Pures Appl.* (9) **86**:1 (2006), 68–79. MR Zbl

[Markowich and Mauser 1993] P. A. Markowich and N. J. Mauser, "The classical limit of a self-consistent quantum-Vlasov equation in 3D", *Math. Models Methods Appl. Sci.* **3**:1 (1993), 109–124. MR Zbl

[Miot 2016] E. Miot, "A uniqueness criterion for unbounded solutions to the Vlasov–Poisson system", *Comm. Math. Phys.* **346**:2 (2016), 469–482. MR Zbl

[Mitrouskas et al. 2019] D. Mitrouskas, S. Petrat, and P. Pickl, "Bogoliubov corrections and trace norm convergence for the Hartree dynamics", *Rev. Math. Phys.* **31**:8 (2019), art. id. 1950024. MR Zbl

[Narnhofer and Sewell 1981] H. Narnhofer and G. L. Sewell, "Vlasov hydrodynamics of a quantum mechanical model", *Comm. Math. Phys.* **79**:1 (1981), 9–24. MR

[Pallard 2012] C. Pallard, "Moment propagation for weak solutions to the Vlasov–Poisson system", *Comm. Partial Differential Equations* **37**:7 (2012), 1273–1285. MR Zbl

[Pallard 2014] C. Pallard, "Space moments of the Vlasov–Poisson system: propagation and regularity", *SIAM J. Math. Anal.* **46**:3 (2014), 1754–1770. MR Zbl

[Petrat 2017] S. Petrat, "Hartree corrections in a mean-field limit for fermions with Coulomb interaction", *J. Phys. A* **50**:24 (2017), art. id. 244004. MR Zbl

[Petrat and Pickl 2016] S. Petrat and P. Pickl, "A new method and a new scaling for deriving fermionic mean-field dynamics", *Math. Phys. Anal. Geom.* **19**:1 (2016), art. id. 3. MR Zbl

[Pezzotti and Pulvirenti 2009] F. Pezzotti and M. Pulvirenti, "Mean-field limit and semiclassical expansion of a quantum particle system", *Ann. Henri Poincaré* **10**:1 (2009), 145–187. MR Zbl

[Pfaffelmoser 1992] K. Pfaffelmoser, "Global classical solutions of the Vlasov–Poisson system in three dimensions for general initial data", *J. Differential Equations* **95**:2 (1992), 281–303. MR Zbl

[Pickl 2011] P. Pickl, "A simple derivation of mean field limits for quantum systems", *Lett. Math. Phys.* **97**:2 (2011), 151–164. MR Zbl

[Porta et al. 2017] M. Porta, S. Rademacher, C. Saffirio, and B. Schlein, "Mean field evolution of fermions with Coulomb interaction", *J. Stat. Phys.* **166**:6 (2017), 1345–1364. MR Zbl

[Rodnianski and Schlein 2009] I. Rodnianski and B. Schlein, "Quantum fluctuations and rate of convergence towards mean field dynamics", *Comm. Math. Phys.* **291**:1 (2009), 31–61. MR Zbl

[Saffirio 2018] C. Saffirio, "Mean-field evolution of fermions with singular interaction", pp. 81–99 in *Macroscopic limits of quantum systems* (Munich, 2017), edited by D. Cadamuro et al., Springer Proc. Math. Stat. **270**, Springer, 2018. Zbl

[Saffirio 2020a] C. Saffirio, "From the Hartree equation to the Vlasov–Poisson system: strong convergence for a class of mixed states", *SIAM J. Math. Anal.* **52**:6 (2020), 5533–5553. MR Zbl

[Saffirio 2020b] C. Saffirio, "Semiclassical limit to the Vlasov equation with inverse power law potentials", *Comm. Math. Phys.* **373**:2 (2020), 571–619. MR Zbl

[Simon 2005] B. Simon, *Trace ideals and their applications*, 2nd ed., Math. Surv. Monogr. **120**, Amer. Math. Soc., Providence, RI, 2005. MR Zbl

[Spohn 1981] H. Spohn, "On the Vlasov hierarchy", *Math. Methods Appl. Sci.* **3**:4 (1981), 445–455. MR Zbl

[Ukai and Okabe 1978] S. Ukai and T. Okabe, "On classical solutions in the large in time of two-dimensional Vlasov's equation", *Osaka Math. J.* **15**:2 (1978), 245–261. MR Zbl

[Yafaev 1999] D. Yafaev, "Sharp constants in the Hardy–Rellich inequalities", *J. Funct. Anal.* **168**:1 (1999), 121–144. MR Zbl

Received 4 May 2020. Revised 20 Sep 2021. Accepted 19 Oct 2021.

LAURENT LAFLECHE: lafleche@math.utexas.edu
Department of Mathematics, The University of Texas at Austin, Austin, TX, United States

CHIARA SAFFIRIO: chiara.saffirio@unibas.ch
Department of Mathematics and Computer Science, University of Basel, Basel, Switzerland

ANALYSIS AND PDE
Vol. 16 (2023), No. 4, pp. 927–996
DOI: 10.2140/apde.2023.16.927

MARSTRAND–MATTILA RECTIFIABILITY CRITERION FOR 1-CODIMENSIONAL MEASURES IN CARNOT GROUPS

Andrea Merlo

In this paper, we show that the flatness of tangents of 1-codimensional measures in Carnot groups implies $C^1_{\mathbb{G}}$-rectifiability. As applications we prove a criterion for intrinsic Lipschitz rectifiability of finite perimeter sets in general Carnot groups and we show that measures with $(2n+1)$-density in the Heisenberg groups $\mathbb{H}^n$ are $C^1_{\mathbb{H}^n}$-rectifiable, providing the first non-Euclidean extension of Preiss's rectifiability theorem.

Introduction

In Euclidean spaces the following rectifiability criterion, known as the Marstrand–Mattila rectifiability theorem, is available. It was first proved by J. M. Marstrand [1961] for $m = 2$ and $n = 3$, later extended by P. Mattila [1975] to every $m \leq n$ and eventually strengthened by D. Preiss [1987].

Theorem 1. *Suppose ϕ is a Radon measure on $\mathbb{R}^n$ and let $m \in \{1, \dots, n-1\}$. Then the following are equivalent*:

(i) *ϕ is absolutely continuous with respect to the m-dimensional Hausdorff measure $\mathcal{H}^m$, and ϕ-almost all of $\mathbb{R}^n$ can be covered with countably many m-dimensional Lipschitz surfaces.*

(ii) *ϕ satisfies the following two conditions for ϕ-almost every $x \in \mathbb{R}^n$*:

  (a) $0 < \Theta^m_*(\phi, x) \leq \Theta^{m,*}(\phi, x) < \infty$.

  (b) $\mathrm{Tan}_m(\phi, x) \subseteq \{\lambda \mathcal{H}^m \llcorner V : \lambda > 0,\ V \in \mathrm{Gr}(n, m)\}$, *where the set of tangent measures* $\mathrm{Tan}_m(\phi, x)$ *is introduced in Definition 1.24.*

The rectifiability of a measure, namely that (i) of Theorem 1 holds, is a global property and as such it is usually very difficult to verify in applications. Rectifiability criteria serve the purpose of characterizing such global properties with local ones, which are usually conditions on the *density* and on the *tangents* of the measure. Most of the more basic criteria impose condition (iia) and the existence of an affine plane $V(x)$, depending only on the point x, on which at small scales the support of the measure is squeezed on around x. The difference between these various elementary criteria relies on how one defines *squeezed on*; for an example see Theorem 15.19 of [Mattila 1995]. However, the existence of just one plane approximating the measure at small scales may be still too difficult to prove in many applications and this is where Theorem 1 comes into play. The Marstrand–Mattila rectifiability criterion says that even if we allow a priori the approximating plane to rotate at different scales, the density hypothesis (iia) guarantees a posteriori this cannot happen almost everywhere.

MSC2020: primary 28A75, 53C17; secondary 22E25, 49Q15.
Keywords: Marstrand–Mattila rectifiability criterion, Preiss's rectifiability theorem, Carnot groups.

It is well known that if a Carnot group $\mathbb{G}$ has Hausdorff dimension $\mathfrak{Q}$, then it is $(\mathfrak{Q}-1)$-purely unrectifiable in the sense of Federer; see for instance Theorem 1.2 of [Magnani 2004]. Despite this geometric irregularity, in the foundational paper [Franchi et al. 2001], B. Franchi, F. Serra Cassano and R. Serapioni introduced the new notion of $C^1_{\mathbb{G}}$-rectifiability in Carnot groups; see Definition 1.34. This definition allowed them to establish De Giorgi's rectifiability theorem for finite perimeter sets in the Heisenberg groups $\mathbb{H}^n$.

Theorem 2 [Franchi et al. 2001, Corollary 7.6]. *Suppose $\Omega \subseteq \mathbb{H}^n$ is a finite perimeter set. Then its reduced boundary $\partial^*_{\mathbb{H}}\Omega$ is $C^1_{\mathbb{H}^n}$-rectifiable.*

It is not hard to see that an open set with smooth boundary is of finite perimeter in $\mathbb{H}^n$, but there are finite perimeter sets in $\mathbb{H}^1$ whose boundary is a fractal from an Euclidean perspective; see for instance [Kirchheim and Serra Cassano 2004]. This means that the Euclidean and $C^1_{\mathbb{G}}$-rectifiability are not equivalent.

The main goal of this paper is to establish a 1-codimensional analogue of Theorem 1 in Carnot groups.

Theorem 3. *Suppose ϕ is a Radon measure on $\mathbb{G}$. Then the following are equivalent*:

(i) *ϕ is absolutely continuous with respect to the $(\mathfrak{Q}-1)$-dimensional Hausdorff measure $\mathcal{H}^{\mathfrak{Q}-1}$, and ϕ-almost all of $\mathbb{G}$ can be covered by countably many $C^1_{\mathbb{G}}$-surfaces.*

(ii) *ϕ satisfies the following two conditions for ϕ-almost every $x \in \mathbb{G}$*:

  (a) $0 < \Theta^{\mathfrak{Q}-1}_*(\phi, x) \le \Theta^{\mathfrak{Q}-1,*}(\phi, x) < \infty$.

  (b) $\mathrm{Tan}_{\mathfrak{Q}-1}(\phi, x)$ *is contained in $\mathfrak{M}$, the family of Haar measures of the elements of* $\mathrm{Gr}(\mathfrak{Q}-1)$, *the* 1*-codimensional homogeneous subgroups of* $\mathbb{G}$.

While the fact that (i) implies (ii) follows from [Vittone et al. 2022, Lemma 3.4 and Corollary 3.6], for instance, the reverse implication is the subject of this work. Besides the already mentioned importance for the applications, Theorem 1 is also relevant because it establishes that $C^1_{\mathbb{G}}$-rectifiability is characterized in the same way as the Euclidean one, and this is the main motivation behind the definition of $\mathscr{P}$-rectifiable measures, given in Definition 4.5. Our main application of Theorem 3 is the proof of the first extension of Preiss's rectifiability theorem outside the Euclidean spaces, which is obtained by combining Theorem 3 with [Merlo 2022, Theorem 1.2]:

Theorem 4. *Suppose ϕ is a Radon measure on the Heisenberg group $\mathbb{H}^n$ such that for ϕ-almost every $x \in \mathbb{H}^n$, we have*

$$0 < \Theta^{2n+1}(\phi, x) := \lim_{r \to 0} \frac{\phi(B(x, r))}{r^{2n+1}} < \infty,$$

where $B(x, r)$ are the metric balls relative to the Koranyi metric. Then ϕ is absolutely continuous with respect to $\mathcal{H}^{2n+1}$, and ϕ-almost all of $\mathbb{H}^n$ can be covered with countably many $C^1_{\mathbb{H}^n}$-regular surfaces.

Finally, an easy adaptation of the arguments used to prove Theorem 3 also provides the following rectifiability criterion for finite perimeter sets in arbitrary Carnot groups. Theorem 5 asserts that if the tangent measures to the boundary of a finite perimeter set are sufficiently close to vertical hyperplanes, then the boundary can be covered by countably many intrinsic Lipschitz graphs.

Theorem 5. *There exists an $\varepsilon_{\mathbb{G}} > 0$ such that if $\Omega \subseteq \mathbb{G}$ is a finite perimeter set for which*

$$\limsup_{r\to 0} d_{x,r}(|\partial\Omega|_{\mathbb{G}}, \mathfrak{M}) := \limsup_{r\to 0} \inf_{\nu\in\mathfrak{M}} \frac{W_1(|\partial\Omega|_{\mathbb{G}} \llcorner B(x,r), \nu \llcorner B(x,r))}{r^{\mathfrak{Q}}} \le \varepsilon_{\mathbb{G}},$$

for $|\partial\Omega|_{\mathbb{G}}$-almost every $x \in \mathbb{G}$, where W_1 is the 1-Wasserstein distance, then $|\partial\Omega|_{\mathbb{G}}$-almost all of $\mathbb{G}$ can be covered with countably many intrinsic Lipschitz graphs.

Before giving an overview of the strategy of the proof, we briefly compare our setting to the Euclidean one and explain why Theorem 3 is so hard won. For the sake of discussion, let us put ourselves in a simplified situation. Assume E is a compact subset of a Carnot group $\mathbb{G} = (\mathbb{R}^n, *)$ such that

(α) there exists an $\eta_1 \in \mathbb{N}$ such that $\eta_1^{-1} r^{\mathfrak{Q}-1} \le \mathcal{H}^{\mathfrak{Q}-1}(E \cap B(x,r)) \le \eta_1 r^{\mathfrak{Q}-1}$ for any $0 < r < \operatorname{diam}(E)$ and any $x \in E$, and

(β) the functions $x \mapsto d_{x,r}(\mathcal{H}^{\mathfrak{Q}-1} \llcorner E, \mathfrak{M})$ converge uniformly to 0 on E as r goes to 0.

Proving that the set E is $C^1_{\mathbb{G}}$-rectifiable is (roughly) equivalent to constructing a plane $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ and a V-intrinsic Lipschitz graph Γ such that $\mathcal{H}^{\mathfrak{Q}-1}(P_V(E \cap \Gamma)) > 0$, where intrinsic Lipschitz graphs are introduced in Definition 1.36 and P_V is the splitting projection on V introduced in 1.10. With this in mind, it is easy to see that the difficulty one has to face when trying to prove Theorem 3 is twofold. On the one hand intrinsic Lipschitz graphs are not Lipschitz in almost any sense of the word as their natural parametrization is Hölder continuous, both from the Euclidean and the intrinsic perspective. On the other hand, splitting projections P_V are just (intrinsic) Hölder continuous maps. This latter complication means that there is no a priori reason for which measure, or even dimension, should be preserved by the projections or the parametrizations. This is indeed the case already in Heisenberg groups $\mathbb{H}^n$, and for further details we refer the reader to [Balogh et al. 2012; 2013].[1]

Unfortunately, the classical approaches to the proof of Theorem 1 all rely on the ideas H. Federer used to prove his celebrated projection theorem, see for instance [Federer 1969, §3.3], and these arguments all *crucially* exploit the fact that orthogonal projections are *Lipschitz*; see [De Lellis 2008; Mattila 1975; 1995; Preiss 1987]. We remark that even in Carnot groups, in some particular cases and for high codimensions, splitting projections are Lipschitz homomorphisms and thus the classical machinery works, although with some highly nontrivial complications; see [Antonelli and Merlo 2022a; 2022b].

This unavoidable technical obstruction of the Hölderianity of intrinsic Lipschitz graphs and of projections implies that, at low codimension, we need to seek a completely different approach. The first pillar of the alternative approach we pursue is the observation, encapsulated in Proposition 1.18, that despite the lack of *metric regularity*, one can still nicely control the measure of the projection of a 1-codimensional set. The other will be combining the classical ideas from [Mattila 1975] with quantitative techniques of [David and Semmes 1993a]. We present here a survey on the strategy of the proof of our main result, Theorem 3, in the simplified hypotheses (α) and (β) for E, that from now on should be considered standing throughout the section.

[1]One could attempt to use *metric projections* instead, however one quickly realizes that in some simple cases, like the Heisenberg groups $\mathbb{H}^n$, splitting projections and metric projections coincide.

The cryptic condition (β) can be reformulated, thanks to Propositions 2.6 and 2.7 in the following more geometric way. For any $\epsilon > 0$ there is a $\mathfrak{r}(\epsilon) > 0$ such that for $\mathcal{H}^{\mathfrak{Q}-1}$-almost any $x \in E$ and any $0 < \rho < \mathfrak{r}(\epsilon)$ there is a plane $V(x, \rho) \in \mathrm{Gr}(\mathfrak{Q} - 1)$, depending on both the point x and the scale ρ, for which

$$E \cap B(x, \rho) \subseteq \{y \in \mathbb{G} : \mathrm{dist}(y, x * V(x, \rho)) \leq \epsilon\rho\}, \tag{1}$$

$$B(y, \epsilon\rho) \cap E \neq \varnothing \quad \text{for any } y \in B\big(x, \tfrac{1}{2}\rho\big) \cap x * V(x, \rho). \tag{2}$$

In Euclidean spaces if a Borel set E satisfies (α), (1) and (2) it is said to be *weakly linear approximable*.[2] The condition (1) says that at small scales E is squeezed on the plane $x * V(x, \rho)$, while (2) implies that inside $B(x, \rho)$ any point of $x * V(x, \rho)$ is very close to E; see Figure 1 on page 931.

Proposition 1 shows that if at some point x the set E has also big projections on some plane W, i.e., (3) holds, then around x the set E is almost a W-intrinsic Lipschitz surface.

Proposition 1. *Let $k > 10\eta_1^2$ and $\omega > 0$. Suppose further that $x \in E$ and $\rho > 0$ are such that*

(i) $d_{x,k\rho}(\mathcal{H}^{\mathfrak{Q}-1} \llcorner E, \mathfrak{M}) \leq \omega$,

(ii) *there exists a plane $W \in \mathrm{Gr}(\mathfrak{Q} - 1)$ such that*

$$(\rho/k)^{\mathfrak{Q}-1} \leq \mathcal{H}^{\mathfrak{Q}-1} \llcorner W(P_W(B(x, \rho) \cap E)), \tag{3}$$

where P_W is the splitting projection on W; see Definition 1.10.

If k^{-1} and ω are suitably small with respect to η_1, there exists an $\alpha = \alpha(\eta_1, k, \omega) > 0$ with the following property. For any $z \in E \cap B(x, \rho)$ and any $y \in B\big(x, \frac{1}{8}k\rho\big) \cap E$ for which $10\omega\rho \leq d(z, y) \leq \frac{1}{2}k\rho$, we have that y is contained in the cone $zC_W(\alpha)$, which is introduced in Definition 1.13.

We remark that thanks to our assumption (β) on E, hypothesis (i) of the above proposition is satisfied almost everywhere on E whenever $\rho < \tilde{\mathfrak{r}}(\omega)$, where $\tilde{\mathfrak{r}}(\omega)$ is suitably small and depends only on ω. Let us explain some of the ideas of the proof of Proposition 1. If the plane W is almost orthogonal to $V(x, \rho)$ (the element of $\mathrm{Gr}(\mathfrak{Q} - 1)$ for which (1) and (2) are satisfied by E at x at scale ρ), we would have that the projection of E on W would be too small and in contradiction with (3); see Figure 2 on page 931.

If the constants k^{-1} and ω are chosen suitably small with respect to η_1 it is possible to show not only that the planes $V(x, \rho)$ and W are not orthogonal but that they must be at a very small angle indeed. In particular, this means that inside $B(x, \rho)$ the plane $x * V(x, \rho)$ must be very close to $x * W$. So close in fact that it can be proved that $E \cap B(x, \rho)$ is contained in a $2\omega\rho$-neighborhood $W_{2\omega\rho}$ of W. This implies that $z, y \in W_{2\omega\rho}$, and since W and $V(x, \rho)$ are at a small angle, it is possible to show that $\mathrm{dist}(y, zW) \leq 4\omega\rho$. Furthermore, by assumption on y, z we have $d(z, y) > 10\omega\rho$ and thus we infer that $\mathrm{dist}(y, zW) \leq 5d(y, z)$. This implies in particular that $y \in zC_W\big(\frac{2}{5}\big)$.

The second step towards the proof of the main result is to show that at any point x of E and for any $\rho > 0$ sufficiently small there is a plane $W_{x,\rho} \in \mathrm{Gr}(\mathfrak{Q} - 1)$ on which E has big projections.

---

[2]The reader might notice that our definition of weakly linearly approximable sets does not coincide with that which can be commonly found in the literature; see for instance [Balogh et al. 2012, Definition 5.4], [De Lellis 2008, Section 5] and [Mattila 1995, Definition 15.7]. However, the assumption (α) on the AD-regularity of E makes our definition equivalent to all the others.

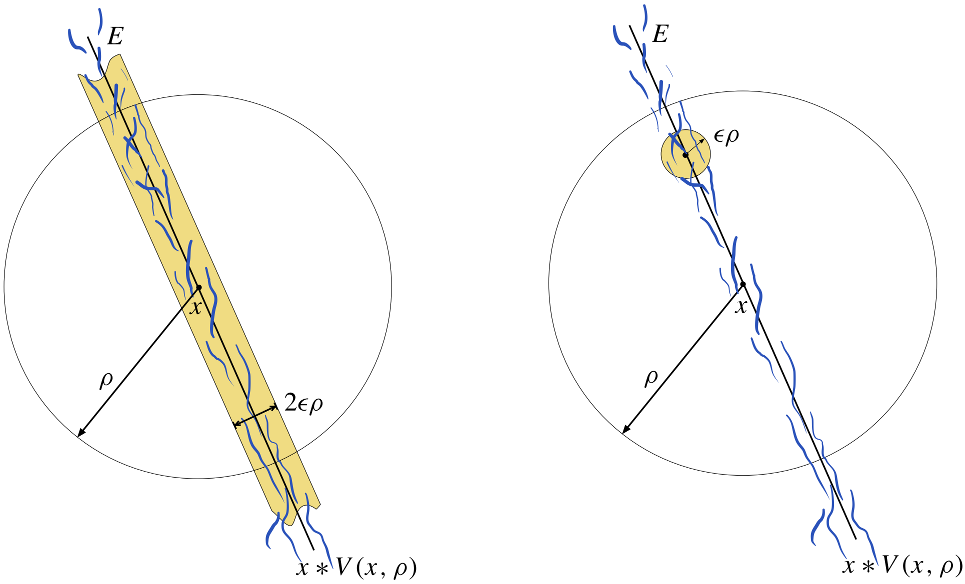

Figure 1. On the left we see that (1) implies that at the scale ρ the set E (collection of blue wavy lines) is contained in a narrow strip of size $2\epsilon\rho$ (shaded yellow) around $x * V(x, \rho)$. On the right we see that (2) implies that any ball centered on the plane $x * V(x, \rho)$ inside $B\left(x, \frac{1}{2}\rho\right)$ and of radius $\epsilon\rho$ (shaded yellow) must meet E.

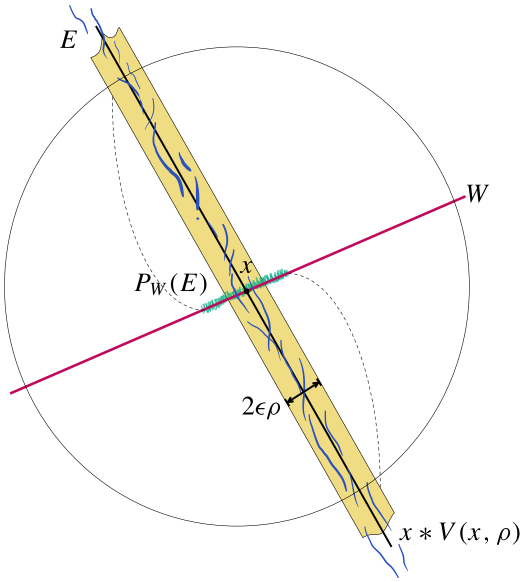

Figure 2. The weak linear approximability of E implies that $E \cap B(x, \rho)$ is contained inside $V_{\omega\rho}$, an $\omega\rho$-neighborhood of the plane $V(x, \rho)$. If $V(x, \rho)$ and W (a red line) are almost orthogonal, i.e., the Euclidean scalar product of their normals is very small, it can be shown that the projection $P_W(E)$ on W of $V_{\omega\rho} \cap B(x, \rho)$ has $\mathcal{H}^{\mathfrak{Q}-1}$-measure smaller than $(\omega\rho)^{\mathfrak{Q}-1}$.

Theorem 6. *There is an $\eta_2 \in \mathbb{N}$ such that for $\mathcal{H}^{\mathfrak{Q}-1}$-almost every $x \in E$ and $\rho > 0$ sufficiently small there is a plane $W_{x,\rho} \in \mathrm{Gr}(\mathfrak{Q}-1)$ for which*

$$\mathcal{H}^{\mathfrak{Q}-1}(P_{W_{x,\rho}}(E \cap B(x,\rho))) \geq \eta_2^{-1}\rho^{\mathfrak{Q}-1}. \tag{4}$$

We now briefly explain the ideas behind the proof of Theorem 6. Fix two parameters $\eta_3 \in \mathbb{N}$ and $\omega > 0$ such that $\omega < 1/\eta_3^{\mathfrak{Q}(\mathfrak{Q}+1)}$ and for which

$$\begin{aligned} B_+ &:= B(\delta_{10\eta_3^{-1}}(\mathfrak{n}(W_{x,\rho})), \eta_3^{-1}) \subseteq \{y \in B(0,1) : \langle y, \mathfrak{n}(W_{x,\rho})\rangle > \omega\},\\ B_- &:= B_+ * \delta_{20\eta_3^{-1}}(\mathfrak{n}(W_{x,\rho})^{-1}) \subseteq \{y \in B(0,1) : \langle y, \mathfrak{n}(W_{x,\rho})\rangle < -\omega\}, \end{aligned}$$

where the δ_λ are the intrinsic dilations introduced in (5) and $\mathfrak{n}(W_{x,\rho}) \in V_1$ is the Euclidean normal of $W_{x,\rho}$. Thanks to assumption (1) on E, for any $0 < \rho < \mathfrak{r}(\omega)$ we have that

$$E \cap B(x,\rho) \subseteq \{y \in B(x,\rho) : \mathrm{dist}(y, x * V(x,\rho)) \leq \omega\rho\}.$$

In particular, thanks to the assumptions on η_3 and ω we infer that $E \cap x\delta_\rho B_+ = \varnothing = E \cap x\delta_\rho B_-$. Let $W_{x,\rho} := V(x,\rho)$, and for any $z \in x\delta_\rho B_+$ define the curve

$$\gamma_z(t) := z\delta_{20\eta_3^{-1}t}(\mathfrak{n}(W_{x,\rho})^{-1}),$$

as t varies in $[0,1]$. The curve γ_z must intersect $W_{x,\rho}$ at the point $P_{W_{x,\rho}}(z)$ since $\gamma_z(1) \in x\delta_\rho B_-$, and as a consequence we have the inclusion $\gamma_z([0,1]) \subseteq P_{W_{x,\rho}}^{-1}(P_{W_{x,\rho}}(z))$. Since conditions (1) and (2) heuristically say that E almost coincides with the plane $x * W_{x,\rho}$ inside $B(x,\rho)$ and it has very few holes, most of the curves γ_z should intersect the set E too.

More precisely, we prove that if some γ_z does not intersect E, there is a small ball U_z centered at some $q \in E$ such that $\gamma_z \cap U_z \neq \varnothing$. It is clear that, defining the set

$$F := E \cup \bigcup_{\substack{z \in x\delta_r B_+\\ \gamma_z \cap E = \varnothing}} U_z,$$

we have $P_{W_{x,r}}(x\delta_r B_+) \subseteq P_{W_{x,r}}(F)$. So, intuitively speaking adding these balls U_z allows us to close the holes of E. An easy computation proves that $\mathcal{H}^{\mathfrak{Q}-1}(P_{W_{x,r}}(x\delta_r B_+)) \geq r^{\mathfrak{Q}-1}/\eta_3^{\mathfrak{Q}-1}$, and thus in order to be able to conclude the proof of (4) we should have some control over the size of the projection of the balls U_z. This control is achievable thanks to (2) (see Proposition 2.27 and Theorem 2.28), and in particular we are able to show that

$$\mathcal{H}^{\mathfrak{Q}-1}\Bigg(P_{W_{x,r}}\Bigg(\bigcup_{\substack{z \in x\delta_r B_+\\ \gamma_z \cap E = \varnothing}} U_z\Bigg)\Bigg) \leq \omega r^{\mathfrak{Q}-1}.$$

This implies that E satisfies the big projection properties, i.e., (4) holds with $\eta_2 := 2\eta_3^{\mathfrak{Q}-1}$. This part of the argument is rather delicate and technical. For the details we refer to the proof of Theorem 2.28.

The third step towards the proof of Theorem 3 is achieved in Section 2D, where we prove the following:

Theorem 7. *There exists an intrinsic Lipschitz graph Γ such that $\mathcal{H}^{\mathfrak{Q}-1}(E \cap \Gamma) > 0$.*

The strategy we employ to prove the above theorem is the following. We know that at $\mathcal{H}^{\mathfrak{Q}-1}$-almost every point of $x \in E$ there exists a plane $W_{x,\rho}$ such that $\mathcal{H}^{\mathfrak{Q}-1}(P_{W_{x,\rho}}(E \cap B(x,\rho))) \geq \eta_2^{-1}\rho^{\mathfrak{Q}-1}$. For any $x \in E$ at which the previous inequality holds, we let $\mathscr{B}$ be the points $y \in B(x,\rho)$ for which there is a scale $s \in (0,\rho)$ for which $W_{y,s}$ is almost orthogonal to $W_{x,\rho}$. Choosing the angle between $W_{y,s}$ and $W_{x,\rho}$ sufficiently big it is possible to prove that the projection of $\mathscr{B}$ on $W_{x,\rho}$ is smaller than $\frac{1}{2}\eta_2^{-1}\rho^{\mathfrak{Q}-1}$. This follows from the intuitive idea that if $y \in \mathscr{B}$, the set $E \cap B(y,s)$ is contained in a narrow strip that is almost orthogonal to $W_{x,\rho}$ inside $B(y,s)$ and thus its projection on $W_{x,\rho}$ has very small $\mathcal{H}^{\mathfrak{Q}-1}$-measure. On the other hand, Proposition 1.18 tells us that $\mathcal{S}^{\mathfrak{Q}-1}\llcorner V(P_{W_{x,\rho}}(E\cap B(x,\rho)\setminus\mathscr{B})) \leq 2c(V)\mathcal{S}^{\mathfrak{Q}-1}(E\cap B(x,\rho)\setminus\mathscr{B})$, and this allows us to infer that there are many points $z \in B(x,\rho) \cap E$ for which $W_{z,s}$ is contained in a (potentially large) fixed cone with axis $W_{x,\rho}$ for any $0 < s < \rho$. This uniformity on the scales allows us to infer thanks to Proposition 1 that $E \cap B(x,\rho) \setminus \mathscr{B}$ is an intrinsic Lipschitz graph.

Since the property (β) is stable for the restriction-to-a-subset operation and for the sake of discussion we can assume that (α) is also, Theorem 7 implies by means of a classical argument that $\mathcal{H}^{\mathfrak{Q}-1}$-almost all of E can be covered with intrinsic Lipschitz graphs.

Therefore, we are reduced to seeing how we can improve the regularity of the surfaces Γ_i covering E from intrinsic Lipschitz to $C^1_{\mathbb{G}}$. Since the blowups of $\mathcal{H}^{\mathfrak{Q}-1}\llcorner E$ are almost everywhere flat, the locality of the tangents, i.e., Proposition 1.27, implies that the blowups of the measures $\mathcal{H}^{\mathfrak{Q}-1}\llcorner\Gamma_i$ are flat as well, where we recall that a measure is said to be flat if it is the Haar measure of a 1-codimensional homogeneous subgroup of $\mathbb{G}$. Furthermore, since intrinsic Lipschitz graphs can be extended to boundaries of sets of finite perimeter, see Theorem 1.38, they have an associated normal vector field $\mathfrak{n}_i$. Therefore, for $\mathcal{H}^{\mathfrak{Q}-1}$-almost every $x \in \Gamma_i$, the elements of $\mathrm{Tan}_{\mathfrak{Q}-1}(\mathcal{H}^{\mathfrak{Q}-1}\llcorner\Gamma_i, x)$ are also the perimeter measures of sets with constant horizontal normal $\mathfrak{n}_i(x)$; see Propositions B.12, B.13, and B.16. The above argument shows that on the one hand the $\mathrm{Tan}_{\mathfrak{Q}-1}(\mathcal{H}^{\mathfrak{Q}-1}\llcorner\Gamma_i, x)$ are flat measures and on the other if seen as the boundary of finite perimeter sets, they must have constant horizontal normal coinciding with $\mathfrak{n}_i(x)$ almost everywhere. Therefore, for $\mathcal{H}^{\mathfrak{Q}-1}$-almost every $x \in E \cap \Gamma_i$, the set $\mathrm{Tan}_{\mathfrak{Q}-1}(\mathcal{H}^{\mathfrak{Q}-1}\llcorner\Gamma_i, x)$ must be contained in the family of Haar measures of the 1-codimensional subgroup orthogonal to $\mathfrak{n}_i(x)$. The fact that $E \cap \Gamma_i$ is covered with countably many $C^1_{\mathbb{G}}$-surfaces follows by means of the rigidity of the tangents discussed above and a Whitney-type theorem, which is obtained in Appendix B with an adaptation of the arguments of [Franchi and Serapioni 2016].

Structure of the paper

In Section 1 we recall some well-known facts about Carnot groups and Radon measures. Section 2 is divided in four parts. The main results of Section 2A are Propositions 2.6 and 2.7, which allow us to interpret the flatness of tangents in a more geometric way. Section 2B is devoted to the proof of Proposition 2.11, which is roughly Theorem 6. Section 2C is the technical core of this work and the main result proved in it is Theorem 2.28, which codifies the fact that the flatness of tangents implies big projections on planes. Finally, in Section 2D we put together the results of the previous three subsections to prove Theorem 2.30, which asserts that for any Radon measure satisfying condition (ii) of Theorem 3, there is an intrinsic Lipschitz graph of positive ϕ-measure. In Section 3 we prove that measures with almost-flat tangents and which are asymptotically AD-regular are intrinsic rectifiable, and we will use this in Section 4 to prove Theorem 4.2. In Section 4 we prove Theorem 4.1, which is the main result of the paper, Theorem 4.2 and their consequences. In Appendix A we construct the dyadic cubes that are needed in Section 2 and in Appendix B we recall some well-known facts about finite perimeter sets in Carnot groups and intrinsic Lipschitz graphs whose surface measures have flat tangents.

1. Preliminaries

This preliminary section is divided into four subsections. In Subsections 1A and 1B we introduce the setting, fix notations and prove some basic facts on splitting projections and intrinsic cones. In Section 1C we recall some well-known facts on Radon measures and their blowups and finally in Section 1D we introduce the two main notions of 1-codimensional rectifiable sets available in Carnot groups.

1A. *Carnot groups.* In this subsection we briefly introduce some notations on Carnot groups that we will extensively use throughout the paper. For a detailed account on Carnot groups and sub-Riemannian geometry we refer to [Serra Cassano 2016].

We recall that a *positive grading* of a Lie algebra $\mathfrak{g}$ is a direct-sum decomposition $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$, for some integer $s \geq 1$, where $V_s \neq 0$ and $[V_1, V_j] \subseteq V_{j+1}$ for all integers $j \in \{1, \ldots, s\}$ and where we set $V_{s+1} = 0$. A positive grading is said to be a *stratification* if $[V_1, V_j] = V_{j+1}$ for all $j \in \{1, \ldots, s\}$. We also recall that the first layer V_1 of a stratification is usually referred to as the *horizontal layer*.

A Carnot group $\mathbb{G}$ of step s is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a stratification $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$. Throughout the paper we denote by n the topological dimension of $\mathfrak{g}$, by n_j the dimension of V_j and by h_j the number $\sum_{i=1}^{j} n_i$.

Furthermore, we let $\pi_i : \mathbb{G} \to V_i$ be the projection maps on the i-th layer of the Lie algebra V_i. We shall remark that more often than not, we will shorten the notation to $v_i := \pi_i v$.

The exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ is a global diffeomorphism from $\mathfrak{g}$ to $\mathbb{G}$. Hence, if we choose a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$, any $p \in \mathbb{G}$ can be written in a unique way as $p = \exp(p_1 X_1 + \cdots + p_n X_n)$. This means that we can identify any $p \in \mathbb{G}$ with the n-tuple $(p_1, \ldots, p_n) \in \mathbb{R}^n$ and the group $\mathbb{G}$ itself with $\mathbb{R}^n$ endowed with $*$, the operation determined by the Campbell–Hausdorff formula. From now on, we will always assume that $\mathbb{G} = (\mathbb{R}^n, *)$ and, as a consequence, that the exponential map exp acts as the identity.

The stratification of $\mathfrak{g}$ carries with it a natural family of dilations $\delta_\lambda : \mathfrak{g} \to \mathfrak{g}$, which are Lie algebra automorphisms of $\mathfrak{g}$ and are defined by

$$\delta_\lambda(v_1, \ldots, v_s) = (\lambda v_1, \lambda^2 v_2, \ldots, \lambda^s v_s), \tag{5}$$

where $v_i \in V_i$. The stratification of the Lie algebra $\mathfrak{g}$ naturally induces a grading on each of its homogeneous Lie subalgebras $\mathfrak{h}$, that is,

$$\mathfrak{h} = V_1 \cap \mathfrak{h} \oplus \cdots \oplus V_s \cap \mathfrak{h}. \tag{6}$$

Furthermore, note that since the exponential map acts as the identity, the Lie algebra automorphisms δ_λ are also group automorphisms of $\mathbb{G}$.

Definition 1.1 (homogeneous subgroups). A subgroup V of $\mathbb{G}$ is said to be *homogeneous* if it is a Lie subgroup of $\mathbb{G}$ that is invariant under the dilations δ_λ for any $\lambda > 0$.

Thanks to Lie's theorem and the fact that exp acts as the identity map, homogeneous Lie subgroups of $\mathbb{G}$ are in bijective correspondence through exp with the Lie subalgebras of $\mathfrak{g}$ that are invariant under the dilations δ_λ. Therefore, homogeneous subgroups in $\mathbb{G}$ are identified with the Lie subalgebras of $\mathfrak{g}$ (that in particular are vector subspaces of $\mathbb{R}^n$) that are invariant under the intrinsic dilations δ_λ.

For any nilpotent Lie algebra $\mathfrak{h}$ with stratification $W_1 \oplus \cdots \oplus W_{\bar{s}}$, we define its *homogeneous dimension*

$$\dim_{\mathrm{hom}}(\mathfrak{h}) := \sum_{i=1}^{\bar{s}} i \cdot \dim(W_i).$$

Thanks to (6) we infer that, if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, we have $\dim_{\mathrm{hom}}(\mathfrak{h}) := \sum_{i=1}^{s} i \cdot \dim(\mathfrak{h} \cap V_i)$. It is a classical fact that the Hausdorff dimension[3] with respect to any left-invariant homogeneous metric (see Definition 1.3) of a nilpotent, connected and simply connected Lie group coincides with the homogeneous dimension $\dim_{\mathrm{hom}}(\mathfrak{h})$ of its Lie algebra. Therefore, the above discussion implies that if $\mathfrak{h}$ is a vector subspace of $\mathbb{R}^n$ which is also an α-dimensional homogeneous subgroup of $\mathbb{G}$, we have

$$\alpha = \sum_{i=1}^{s} i \cdot \dim(\mathfrak{h} \cap V_i) = \dim_{\mathrm{hom}}(\mathfrak{h}). \tag{7}$$

Definition 1.2. Let $\mathfrak{Q} := \dim_{\mathrm{hom}}(\mathfrak{g})$, and for any $m \in \{1, \dots, \mathfrak{Q} - 1\}$ we define the m-dimensional Grassmannian of $\mathbb{G}$, denoted by $\mathrm{Gr}(m)$, as the family of all homogeneous subgroups V of $\mathbb{G}$ of Hausdorff dimension m.

Furthermore, thanks to (7) and some easy algebraic considerations that we omit, one deduces that for the elements of $\mathrm{Gr}(\mathfrak{Q} - 1)$ the following identities hold:

$$\dim(V \cap V_1) = n_1 - 1 \qquad \text{and} \qquad \dim(V \cap V_i) = \dim(V_i), \quad \text{for any } i = 2, \dots, s. \tag{8}$$

Thanks to (8), we infer that for any $V \in \mathrm{Gr}(\mathfrak{Q} - 1)$ there exists a $\mathfrak{n}(V) \in V_1$ such that

$$V = \mathscr{V} \oplus V_2 \oplus \cdots \oplus V_s,$$

where $\mathscr{V} := \{w \in V_1 : \langle \mathfrak{n}(V), w \rangle = 0\}$. In the following we will denote by $\mathfrak{N}(V)$ the 1-dimensional homogeneous subgroup generated by the horizontal vector $\mathfrak{n}(V)$. We shall remark that the above discussion implies that the elements of $\mathrm{Gr}(\mathfrak{Q} - 1)$ are hyperplanes in $\mathbb{R}^n$ whose normals lie in V_1. It is not hard to see that the converse holds too and that the elements of $\mathrm{Gr}(\mathfrak{Q} - 1)$ are normal subgroups of $\mathbb{G}$.

For any $p \in \mathbb{G}$, we define the left translation $\tau_p : \mathbb{G} \to \mathbb{G}$ as

$$q \mapsto \tau_p q := p * q.$$

[3]For a definition of Hausdorff dimension, see for instance [Mattila 1995, Definition 4.8].

As already remarked above, we assume without loss of generality that the group operation $*$ is determined by the Campbell–Hausdorff formula, and therefore it has the form

$$p * q = p + q + \mathscr{Q}(p, q) \quad \text{for all } p, q \in \mathbb{R}^n,$$

where $\mathscr{Q} = (\mathscr{Q}_1, \dots, \mathscr{Q}_s) : \mathbb{R}^n \times \mathbb{R}^n \to V_1 \oplus \cdots \oplus V_s$, and the $\mathscr{Q}_i$s have the following properties. For any $i = 1, \dots, s$ and any $p, q \in \mathbb{G}$ we have

(i) $\mathscr{Q}_i(\delta_\lambda p, \delta_\lambda q) = \lambda^i \mathscr{Q}_i(p, q)$,

(ii) $\mathscr{Q}_i(p, q) = -\mathscr{Q}_i(-q, -p)$,

(iii) $\mathscr{Q}_1 = 0$ and $\mathscr{Q}_i(p, q) = \mathscr{Q}_i(p_1, \dots, p_{i-1}, q_1, \dots, q_{i-1})$.

Thus, we can represent the product $*$ more precisely as

$$p * q = (p_1 + q_1, p_2 + q_2 + \mathscr{Q}_2(p_1, q_1), \dots, p_s + q_s + \mathscr{Q}_s(p_1, \dots, p_{s-1}, q_1, \dots, q_{s-1})).$$

Definition 1.3. A metric $d : \mathbb{G} \times \mathbb{G} \to \mathbb{R}$ is said to be homogeneous and left-invariant if for any $x, y \in \mathbb{G}$ we have

(i) $d(\delta_\lambda x, \delta_\lambda y) = \lambda d(x, y)$ for any $\lambda > 0$,

(ii) $d(\tau_z x, \tau_z y) = d(x, y)$ for any $z \in \mathbb{G}$.

Throughout the paper, if not otherwise stated, we will endow the group $\mathbb{G}$ with the following homogeneous and left-invariant metric:

Definition 1.4. For any $g \in \mathbb{G}$, we let

$$\|g\| := \max\{\epsilon_1|g_1|, \epsilon_2|g_2|^{1/2}, \dots, \epsilon_s|g_s|^{1/s}\},$$

where $\epsilon_1 = 1$ and $\epsilon_2, \dots, \epsilon_s$ are suitably small parameters depending only on the group $\mathbb{G}$. For the proof that $\|\cdot\|$ is a left-invariant, homogeneous norm on $\mathbb{G}$ for a suitable choice of $\epsilon_2, \dots, \epsilon_s$, we refer to Section 5 of [Franchi et al. 2003]. Furthermore, we define

$$d(x, y) := \|x^{-1} * y\|,$$

and let $B(x, r) := \{z \in \mathbb{G} : d(x, z) < r\}$ be the open metric ball relative to the distance d centered at x at radius $r > 0$.

Remark 1.5. Fix an orthonormal basis $\mathcal{E} := \{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ such that

$$e_j \in V_i, \quad \text{whenever } h_i \le j < h_{i+1}. \tag{9}$$

From the definition of the metric d, it immediately follows that the ball $B(0, r)$ is contained in the box

$$\mathrm{Box}_{\mathcal{E}}(0, r) := \{p \in \mathbb{R}^n : \text{for any } i = 1, \dots, s \text{ whenever } |\langle p, e_j\rangle| \le r^i/\epsilon_i \text{ for any } h_i \le j < h_{j+1}\}.$$

Definition 1.6. For any $0 \le \alpha \le \mathfrak{Q}$, we define the spherical Hausdorff measure to be the Carathéodory completion of the exterior measure that acts on Borel sets $A \subseteq \mathbb{G}$ as

$$\mathcal{S}^\alpha(A) := \sup_{\delta>0} \inf\left\{\sum_{j=1}^\infty r_j^\alpha : A \subseteq \bigcup_{j=1}^\infty \overline{B(x_j, r_j)},\ r_j \le \delta\right\}.$$

In the following definition, we introduce a family of measures that will be of great relevance throughout the paper.

Definition 1.7 (flat measures). For any $m \in \{1, \dots, \mathfrak{Q}-1\}$ the set of m-dimensional flat measures $\mathfrak{M}(m)$ is defined as

$$\mathfrak{M}(m) := \{\lambda \mathcal{S}^m \llcorner V : \text{for some } \lambda > 0 \text{ and } V \in \operatorname{Gr}(m)\}. \tag{10}$$

In order to simplify notation in the following we let $\mathfrak{M} := \mathfrak{M}(\mathfrak{Q}-1)$.

The following proposition gives a representation of $(\mathfrak{Q}-1)$-flat measures, which will come in handy later on.

Proposition 1.8. *For any $V \in \operatorname{Gr}(\mathfrak{Q}-1)$ we have $\mathcal{S}^{\mathfrak{Q}-1} \llcorner V = \beta^{-1} \mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner V$, where $\beta := \mathcal{H}^{n-1}_{\mathrm{eu}}(B(0,1) \cap V)$ and β does not depend on V.*

Proof. Let $E := \{z \in \mathbb{G} : \langle z_1, \mathfrak{n}(V)\rangle < 0\}$ and let ∂E be the perimeter measure of E; see Definition B.4. Either by direct computation or thanks to identity (2.8) in [Ambrosio et al. 2009], it can be proven that $\partial E = \mathfrak{n}(V)\mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner V$. On the other hand, since the reduced boundary $\partial^* E = V$ of E is a $C^1_{\mathbb{G}}$-surface, see Definition 1.34, thanks to Theorem 4.1 of [Magnani 2017] we conclude that

$$\beta(\|\cdot\|, \mathfrak{n}(V))\mathcal{S}^{\mathfrak{Q}-1} \llcorner V = |\partial E|_{\mathbb{G}} = \mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner V,$$

where $\beta(\|\cdot\|, \mathfrak{n}(V)) := \max_{z \in B(0,1)} \mathcal{H}^{n-1}_{\mathrm{eu}}(B(z,1) \cap V)$. Since $B(0,1)$ is convex as a subset of $\mathbb{R}^n$, [Magnani 2017, Theorem 5.2] implies that

$$\beta(\|\cdot\|, \mathfrak{n}(V)) = \mathcal{H}^{n-1}_{\mathrm{eu}}(B(0,1) \cap V).$$

Finally note that the right-hand side of the above identity does not depend on V since $B(0,1)$ is invariant under rotations of the first layer V_1. □

The above proposition has the following useful consequence:

Proposition 1.9. *A function $\varphi : \mathbb{G} \to \mathbb{R}$ is said to be radially symmetric if there is a profile function $g : [0, \infty) \to \mathbb{R}$ such that $\varphi(x) = g(\|x\|)$. For any $V \in \operatorname{Gr}(\mathfrak{Q}-1)$ and any radially symmetric, positive function φ we have*

$$\int \varphi \, d\mathcal{S}^{\mathfrak{Q}-1} \llcorner V = (\mathfrak{Q}-1) \int s^{\mathfrak{Q}-2} g(s)\, ds.$$

Proof. The thesis of the proposition is trivially satisfied for indicator functions of balls. The general result follows by the monotone convergence theorem. □

1B. ***Cones and splitting projections.*** For any $V \in \mathrm{Gr}(\mathfrak{Q}-1)$, the group $\mathbb{G}$ can be written as a semidirect product of V and $\mathfrak{N}(V)$, i.e.,

$$\mathbb{G} = V \rtimes \mathfrak{N}(V). \tag{11}$$

In this subsection we adapt some of the results on projections from Subsection 2.2.2 of [Franchi and Serapioni 2016] to the case in which splitting of $\mathbb{G}$ is given by (11).

Definition 1.10 (splitting projections). For any $g \in \mathbb{G}$, there are two unique elements $P_V g \in V$ and $P_{\mathfrak{N}(V)} g \in \mathfrak{N}(V)$ such that

$$g = P_V g * P_{\mathfrak{N}(V)} g.$$

The following result is a particular case of [Franchi and Serapioni 2016, Proposition 2.17].

Proposition 1.11. *For any $V \in \mathrm{Gr}(\mathfrak{Q}-1)$, we let*

$$A_2 g_2 := g_2 - \mathscr{Q}_2(\pi_{\mathscr{V}} g_1, \pi_{\mathfrak{n}(V)} g_1),$$
$$A_i g_i := g_i - \mathscr{Q}_i(\pi_{\mathscr{V}} g_1, A_2 g_2, \dots, A_{i-1} g_{i-1}, \pi_{\mathfrak{n}(V)} g_1, 0, \dots, 0), \quad \textit{whenever } i = 3, \dots, s,$$

where $\pi_{\mathfrak{n}(V)} g_1 := \langle g_1, \mathfrak{n}(V)\rangle \mathfrak{n}(V)$ and $\pi_{\mathscr{V}} g_1 = g_1 - \pi_{\mathfrak{n}(V)} g_1$. With these definitions, the projections P_V and $P_{\mathfrak{N}(V)}$ have the following expressions in coordinates:

$$P_V g = (\pi_{\mathscr{V}} g_1, A_2 g_2, \dots, A_s g_s) \quad \textit{and} \quad P_{\mathfrak{N}(V)} g = (\pi_{\mathfrak{n}(V)} g_1, 0, \dots, 0).$$

Furthermore, for any $x, y \in \mathbb{G}$, the above representations and the fact that V is a normal and homogeneous subgroup of $\mathbb{G}$ imply:

(i) $P_V(x * y) = x * P_V y * P_{\mathfrak{N}(V)} x^{-1}$,

(ii) $P_{\mathfrak{N}(V)}(x * y) = P_{\mathfrak{N}(V)}(x) * P_{\mathfrak{N}(V)}(y) = P_{\mathfrak{N}(V)}(x) + P_{\mathfrak{N}(V)}(y)$,

where here the symbol $+$ has to be interpreted as the sum of vectors.

Remark 1.12. Throughout the paper the reader should always keep in mind that the projections P_V are *not* Lipschitz maps and, as stated in the introduction, this is the major source of the technical problems we have to overcome in order to prove our main result, Theorem 4.1.

The splitting projections allow us to give the following intrinsic notion of cone:

Definition 1.13. For any $\alpha > 0$ and $V \in \mathrm{Gr}(\mathfrak{Q}-1)$, we define the cone $C_V(\alpha)$ as

$$C_V(\alpha) := \{w \in \mathbb{G} : \|P_{\mathfrak{N}(V)}(w)\| \le \alpha \|P_V(w)\|\}.$$

The next proposition is very useful, since one of the major difficulties when dealing with geometric problems in Carnot groups is that $d(x, y) \approx |x - y|^{1/s}$ if x and y are not suitably chosen. However, Proposition 1.14 shows that if $y \notin x C_V(\alpha)$, then $d(x, y)$ is bi-Lipschitz equivalent to the Euclidean distance $|x - y|$.

Proposition 1.14. *For any $x, y \in \mathbb{G}$ for which $x^{-1} y \notin C_V(\alpha)$ for some $\alpha > 0$ and $V \in \mathrm{Gr}(\mathfrak{Q}-1)$, we have*

$$d(x, y) \le \Lambda(\alpha) |\pi_1(x^{-1} y)|, \quad \textit{where } \Lambda(\alpha) := (1 + \alpha^{-1}).$$

Proof. For any $\alpha, \beta > 0$ define

$$C(\alpha) := \bigcup_{V \in \mathrm{Gr}(\mathfrak{Q}-1)} (\mathbb{G} \setminus C_V(\alpha)) \quad \text{and} \quad D(\beta) := \{x \in \mathbb{G} : \|x\| \le \beta |\pi_1(x)|\}.$$

Now let us prove that $C(\alpha) \subseteq D(\Lambda(\alpha))$. For any $w \in C(\alpha)$ there exists a $W \in \mathrm{Gr}(\mathfrak{Q}-1)$ such that $\|P_{\mathfrak{N}(W)}(w)\| > \alpha \|P_W(w)\|$ and, in particular,

$$\|w\| \le \|P_W(w)\| + \|P_{\mathfrak{N}(W)}(w)\| \le (1+\alpha^{-1})\|P_{\mathfrak{N}(W)}(w)\| = (1+\alpha^{-1})|\pi_{\mathfrak{n}(W)}(\pi_1 w)| \le (1+\alpha^{-1})|\pi_1(w)|,$$

where the only identity in the equation above comes from the choice of the metric and Proposition 1.11. This concludes the proof of the inclusion $C(\alpha) \subseteq D(\Lambda(\alpha))$.

Since $x^{-1}y \notin C_V(\alpha)$, then $x^{-1}y \in C(\alpha)$ and hence $d(x, y) = \|x^{-1}y\| \le (1+\alpha^{-1})|\pi_1(x^{-1}y)|$, which concludes the proof of the proposition. □

The following proposition allows us to precisely quantify the distance of a point $g \in \mathbb{G}$ from a plane $V \in \mathrm{Gr}(\mathfrak{Q}-1)$.

Proposition 1.15. *For any $V \in \mathrm{Gr}(\mathcal{Q}-1)$ and any $g \in \mathbb{G}$ we have $\mathrm{dist}(P_{\mathfrak{N}(V)}g, V) = |\pi_{\mathfrak{n}(V)}g_1|$ and, in particular, $\mathrm{dist}(g, V) = |\pi_{\mathfrak{n}(V)}g_1|$. In addition, for any $g \in \mathbb{G}$ we have*

$$\|P_V(g)\| \le 2\|g\|. \tag{12}$$

Proof. First of all, we note that

$$\mathrm{dist}(P_{\mathfrak{N}(V)}g, V) \le d(P_{\mathfrak{N}(V)}g, 0) = |\pi_{\mathfrak{n}(V)}g_1|, \tag{13}$$

where the last identity above comes from Proposition 1.11 and the definition of the metric. In addition, once again thanks to the definition of the metric, we have

$$\mathrm{dist}(P_{\mathfrak{N}(V)}(g), V) = \inf_{v \in V} \|P_{\mathfrak{N}(V)}(g)^{-1} * v\| \ge \inf_{v \in V} |-\pi_{\mathfrak{n}(V)}g_1 + v_1| = |\pi_{\mathfrak{n}(V)}g_1|. \tag{14}$$

Putting together (13) and (14) we conclude the proof of the identity $\mathrm{dist}(P_{\mathfrak{N}(V)}(g), V) = |\pi_{\mathfrak{n}(V)}g_1|$. Thanks to this, we conclude that

$$\begin{aligned}\mathrm{dist}(g, V) = \inf_{v \in V} d(g, v) &= \inf_{v \in V} d(P_V g * P_{\mathfrak{N}(V)}g, v) \\ &= \inf_{v \in V} d(P_{\mathfrak{N}(V)}g, P_V g^{-1} * v) = \mathrm{dist}(P_{\mathfrak{N}(V)}g, V) = |\pi_{\mathfrak{n}(V)}g_1|,\end{aligned}$$

proving the second claimed identity. In order to conclude the proof of (12) we just note that

$$\|P_V(g)\| = \|g * P_{\mathfrak{N}(V)}g^{-1}\| \le \|g\| + \|P_{\mathfrak{N}(V)}g\| = \|g\| + |\pi_{\mathfrak{n}(V)}g_1| \le \|g\| + |g_1| \le 2\|g\|,$$

where the second identity above comes from the definition of the norm and Proposition 1.11. □

The following result is the analogue of [Franchi and Serapioni 2016, Proposition 2.12] where $\mathbb{M} := V$ and $\mathbb{H} := \mathfrak{N}(V)$.

Proposition 1.16. *For any $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ and any $g \in \mathbb{G}$ we have*

$$\tfrac{1}{3}(\|P_{\mathfrak{N}(V)}g\| + \|P_V g\|) \le \|g\| \le \|P_{\mathfrak{N}(V)}g\| + \|P_V g\|. \tag{15}$$

Proof. The right-hand side of (15) follows directly from the triangular inequality. Furthermore, thanks to Propositions 1.11 and 1.15 we deduce on the one hand that $\|P_{\mathfrak{N}(V)}(g)\| = |\pi_{\mathfrak{n}(V)}(g_1)| \le \|g\|$ and on the other that $\|P_V g\| \le 2\|g\|$. The first inequality in (15) follows from combining these two inequalities. □

The following proposition allows us to estimate the distance of parallel 1-codimensional planes.

Proposition 1.17. *Let $x, y \in \mathbb{G}$ and $V \in \mathrm{Gr}(\mathfrak{Q}-1)$. Defining*

$$\mathrm{dist}(xV, yV) := \max\Big\{\sup_{v\in V} \mathrm{dist}(xv, yV), \sup_{v\in V} \mathrm{dist}(yv, xV)\Big\},$$

we have

(i) $\mathrm{dist}(xV, yV) = \mathrm{dist}(x, yV) = \mathrm{dist}(y, xV) = |\pi_{\mathfrak{n}(V)}(\pi_1(x^{-1}y))|$,

(ii) $\mathrm{dist}(u, xV) \le \mathrm{dist}(u, yV) + \mathrm{dist}(xV, yV)$, *for any $u \in \mathbb{G}$.*

Proof. For any $v \in V$ we have

$$\mathrm{dist}(xv, yV) = \inf_{w\in V} \mathrm{dist}(xv, yw) = \inf_{w\in V} d(x, y(y^{-1}xv^{-1}x^{-1}y)w) = \inf_{w\in V} d(x, yw) = \mathrm{dist}(x, yV),$$

where the second last identity comes from the fact that $v^* := y^{-1}xv^{-1}x^{-1}y \in V$ and from the transitivity of the translation by v^* on V. Therefore, we have $\sup_{v\in V} \mathrm{dist}(xv, yV) = \mathrm{dist}(x, yV)$ and thus by Proposition 1.15 we infer that

$$\begin{aligned}\mathrm{dist}(xV, yV) &= \max\{\mathrm{dist}(x, yV), \mathrm{dist}(y, xV)\} = \max\{|\pi_{\mathfrak{n}(V)}(\pi_1(y^{-1}x))|, |\pi_{\mathfrak{n}(V)}(\pi_1(x^{-1}y))|\}\\ &= |\pi_{\mathfrak{n}(V)}(\pi_1(x^{-1}y))| = \mathrm{dist}(x, yV) = \mathrm{dist}(y, xV),\end{aligned}$$

where the last identity comes from interchanging x and y and exploiting the symmetry of the definition of $\mathrm{dist}(xV, yV)$. In order to prove (ii), let w^* be the element of V for which $\mathrm{dist}(u, yV) = d(u, yw^*)$ and note that

$$\begin{aligned}\mathrm{dist}(u, xV) &= \inf_{v\in V} d(u, xv) \le d(u, yw^*) + \inf_{v\in V} d(yw^*, xv) = \mathrm{dist}(u, yV) + \inf_{v\in V} d(yw^*, xv)\\ &= \mathrm{dist}(u, yV) + \mathrm{dist}(yw^*, xV) \le \mathrm{dist}(u, yV) + \mathrm{dist}(xV, yV).\end{aligned}$$ □

The following result is a direct consequence of [Franchi and Serapioni 2016, Proposition 2.2]. The bound (16) can be obtained with the same argument used by V. Chousionis, K. Fässler and T. Orponen to prove [Chousionis et al. 2019, Lemma 3.6]. In particular, (16) will play the role of a surrogate for the Lipschitzianity of projections. The proof is omitted.

Proposition 1.18. *For any $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ there is a constant $1 \le c(V) \le \mathcal{S}^{\mathfrak{Q}-1}(B(0,2)\cap V) =: C_1$ such that for any $p \in \mathbb{G}$ and any $r > 0$ we have*

$$\mathcal{S}^{\mathfrak{Q}-1} \llcorner V(P_V(B(p,r))) = c(V) r^{\mathfrak{Q}-1}.$$

Furthermore, for any Borel set $A \subseteq \mathbb{G}$ for which $\mathcal{S}^{\mathfrak{Q}-1}(A) < \infty$, we have

$$\mathcal{S}^{\mathfrak{Q}-1} \llcorner V(P_V(A)) \le 2c(V)\mathcal{S}^{\mathfrak{Q}-1}(A). \tag{16}$$

1C. *Densities and tangents of Radon measures.* In this subsection we briefly recall some facts and notations about Radon measures on Carnot groups and their blowups.

Definition 1.19. If ϕ is a Radon measure on $\mathbb{G}$, we define

$$\Theta^m_*(\phi, x) := \liminf_{r\to 0} \frac{\phi(B(x,r))}{r^m} \quad \text{and} \quad \Theta^{m,*}(\phi, x) := \limsup_{r\to 0} \frac{\phi(B(x,r))}{r^m},$$

and say that $\Theta^m_*(\phi, x)$ and $\Theta^{m,*}(\phi, x)$ are the lower and upper m-densities of ϕ at the point $x \in \mathbb{G}$, respectively.

Definition 1.20 (weak convergence of measures). A sequence of Radon measures $\{\mu_i\}_{i\in\mathbb{N}}$ is said to be weakly converging in the sense of measures to some Radon measure ν if, for any continuous functions with compact support $f \in \mathcal{C}_c$, we have

$$\int f\, d\mu_i \to \int f\, d\nu.$$

Throughout the paper, we denote such convergence with the symbol $\mu_i \rightharpoonup \nu$.

Definition 1.21. For any pair of Radon measures ϕ and ψ and any compact set $K \subseteq \mathbb{G}$ we let

$$F_K(\phi, \psi) := \sup\left\{\left|\int f\, d\phi - \int f\, d\psi\right| : f \in \mathrm{Lip}_1^+(K)\right\}, \tag{17}$$

where $\mathrm{Lip}_1^+(K)$ is the set of nonnegative 1-Lipschitz functions whose support is contained in K. Furthermore, if $K = \overline{B(x,r)}$, we shorten the notation to $F_{x,r}(\phi, \psi) := F_{\overline{B(x,r)}}(\phi, \psi)$.

The next lemma is an elementary fact about Radon measures. We omit its proof.

Lemma 1.22. *If ϕ is a Radon measure on $\mathbb{G}$, for any $x \in \mathbb{G}$ there are at most countably many radii $R > 0$ for which $\phi(\partial B(x, R)) > 0$.*

The following proposition allows us to characterize the weak convergence of measures by means of the convergence to 0 of the functionals F_K.

Proposition 1.23. *Assume that $\{\mu_i\}_{i\in\mathbb{N}}$ is a sequence of Radon measures and let μ be a Radon measure on $\mathbb{G}$. Then the following are equivalent*:

(i) $\mu_i \rightharpoonup \mu$.

(ii) $\lim_{i\to\infty} F_K(\mu_i, \mu) = 0$ *for any compact set $K \subseteq \mathbb{G}$.*

Proof. The proof can be achieved with an argument similar to the Euclidean one; see for instance [Preiss 1987, Proposition 1.11]. □

Definition 1.24 (tangent measures). Let ϕ be a Radon measure on $\mathbb{G}$. For any $x \in \mathbb{G}$ and any $r > 0$, we define $T_{x,r}\phi$ to be the Radon measure for which

$$T_{x,r}\phi(B) = \phi(x\delta_r(B)), \quad \text{for any Borel set } B \subseteq \mathbb{G}.$$

For any $m \in \{1, \dots, \mathfrak{Q}\}$ define $\mathrm{Tan}_m(\phi, x)$, *the set of the m-dimensional tangent measures to ϕ at x*, as the collection of Radon measures ν for which there is an infinitesimal sequence $\{r_i\}_{i\in\mathbb{N}}$ such that $r_i^{-m} T_{x,r}\phi \rightharpoonup \nu$.

Proposition 1.25. *Let ϕ be a Radon measure, $\nu \in \mathrm{Tan}_m(\phi, x)$ and $\{r_i\}_{i\in\mathbb{N}}$ an infinitesimal sequence such that $r_i^{-m} T_{x,r_i}\phi \rightharpoonup \nu$. Then, if $y \in \mathrm{supp}(\nu)$, there exists a sequence $\{z_i\}_{i\in\mathbb{N}} \subseteq \mathrm{supp}(\phi)$ such that $\delta_{1/r_i}(x^{-1}z_i) \to y$.*

Proof. A simple argument by contradiction yields the claim. The proof follows verbatim its Euclidean analogue; see for instance the proof of [De Lellis 2008, Proposition 3.4]. □

Proposition 1.26. *Suppose ϕ is a Radon measure on $\mathbb{G}$ such that*

$$0 < \Theta^m_*(\phi, x) \le \Theta^{m,*}(\phi, x) < \infty, \quad \textit{for } \phi\textit{-almost every } x \in \mathbb{G}.$$

Then $\mathrm{Tan}_m(\phi, x) \neq \varnothing$ *for ϕ-almost every $x \in \mathbb{G}$.*

Proof. This is an immediate consequence of the local uniform boundedness of the rescaled measures $T_{x,r}\phi$ together with the compactness of measures. See Proposition [Preiss 1987, Proposition 1.12]. □

The following result is the analogue of [De Lellis 2008, Proposition 3.12], which establishes the locality of tangents in the Euclidean space. This proposition is of capital importance since it will ensure that restricting and multiplying a measure with flat tangents by a density will yield a measure still having flat tangents.

Proposition 1.27 (locality of the tangents). *In the hypothesis of Proposition 1.26, for any nonnegative $\rho \in L^1(\phi)$ we have $\mathrm{Tan}_m(\rho\phi, x) = \rho(x)\,\mathrm{Tan}_m(\phi, x)$ for ϕ-almost every $x \in \mathbb{G}$.*

Proof. First of all, let us note that ϕ is locally asymptotically doubling. Indeed,

$$\begin{aligned}\limsup_{r\to 0} \frac{\phi(B(x, 2r))}{\phi(B(x, r))} &\le \limsup_{r\to 0} \frac{\phi(B(x, 2r))}{(2r)^m} \frac{2^m r^m}{\phi(B(x, r))} \\ &\le \frac{2^m \Theta^{m,*}(\phi, x)}{\Theta^m_*(\phi, x)} < \infty, \qquad \text{for } \phi\text{-almost every } x \in \mathbb{G}.\end{aligned} \tag{18}$$

Thanks to [Heinonen et al. 2015, Theorem 3.4.3], we know that the *Lebesgue differentiation theorem* holds for ϕ; see [Heinonen et al. 2015, page 77]. In particular, the argument that proves the equivalent of this result in Euclidean spaces, see for instance the aforementioned [De Lellis 2008, Proposition 3.12], can be applied verbatim to ϕ. □

Proposition 1.28. *Suppose ϕ is a Radon measure supported on a compact set K such that for ϕ-almost every $x \in \mathbb{G}$ we have*

$$0 < \Theta^{\mathfrak{Q}-1}_*(\phi, x) \le \Theta^{\mathfrak{Q}-1,*}(\phi, x) < \infty.$$

Then, for any $\vartheta, \gamma \in \mathbb{N}$, the set $E^\phi(\vartheta, \gamma) := \{x \in K : \vartheta^{-1} r^{\mathfrak{Q}-1} \le \phi(B(x, r)) \le \vartheta r^{\mathfrak{Q}-1}$ for any $0 < r < 1/\gamma\}$ is compact.

Proof. Since K is compact, in order to verify that $E^{\phi}(\vartheta,\gamma)$ is compact, it suffices to prove that it is closed. If $E^{\phi}(\vartheta,\gamma)$ is empty or finite, there is nothing to prove. So, suppose there is a sequence $\{x_i\}_{i\in\mathbb{N}}\subseteq E^{\phi}(\vartheta,\gamma)$ converging to some $x\in K$. Fix an $0<r<1/\gamma$ and assume that $\delta>0$ is so small that $r+\delta<1/\gamma$. Therefore, if $d(x,x_i)<\delta$ and $r-d(x,x_i)>0$, we have

$$\begin{aligned}\vartheta^{-1}(r-d(x,x_i))^{\mathfrak{Q}-1}&\le\phi(B(x_i,r-d(x,x_i)))\le\phi(B(x,r))\\&\le\phi(B(x_i,r+d(x,x_i)))\le\vartheta(r+d(x,x_i))^{\mathfrak{Q}-1}.\end{aligned}$$

Taking the limit as i goes to ∞, we see that $x\in E^{\phi}(\vartheta,\gamma)$. □

Proposition 1.29. *With the hypothesis of Proposition 1.28, for any $\vartheta,\gamma,\mu,\nu\in\mathbb{N}$, the set*

$$\mathscr{E}^{\phi}_{\vartheta,\gamma}(\mu,\nu)=\{x\in E^{\phi}(\vartheta,\gamma):(1-1/\mu)\phi(B(x,r))\le\phi(B(x,r)\cap E^{\phi}(\vartheta,\gamma))\text{ for any }0<r<1/\nu\}$$

is compact.

Proof. If $\mathscr{E}^{\phi}_{\vartheta,\gamma}(\mu,\nu)$ is empty or finite, there is nothing to prove. Furthermore, since by Proposition 1.28 we know that the sets $E^{\phi}(\vartheta,\gamma)$ are compact, in order to prove our claim it is sufficient to show that $\mathscr{E}^{\phi}_{\vartheta,\gamma}(\mu,\nu)$ is closed in $E^{\phi}(\vartheta,\gamma)$. Take a sequence $\{y_i\}_{i\in\mathbb{N}}\subseteq\mathscr{E}^{\phi}_{\vartheta,\gamma}(\mu,\nu)$ converging to some $y\in E^{\phi}(\vartheta,\gamma)$. Fix an $0<r<1/\nu$ and a $\delta\in\left(0,\frac{1}{4}\right)$ and let $i_0(\delta)\in\mathbb{N}$ be such that for any $i\ge i_0(\delta)$ we have $d(y,y_i)<\delta r$. These choices imply that

$$(1-1/\mu)\phi(B(y_i,r-2d(y,y_i)))\le\phi(B(y_i,r-2d(y,y_i))\cap E^{\phi}(\vartheta,\gamma))\le\phi(B(y,r)\cap E^{\phi}(\vartheta,\gamma)).$$

Note that the sequence of functions $f_i(z):=\chi_{B(y_i,r-2d(y,y_i))}(z)$ converges pointwise ϕ-almost everywhere to $\chi_{B(y,r)}(z)$. This is due to the fact that, for any $i\ge i_0(\delta)$, on the one hand we have $\operatorname{supp}(f_i)\Subset B(y,r)$ and on the other the functions f_i are equal to 1 on $B(y,r(1-3\delta))$. Thus, the dominated convergence theorem implies that

$$(1-1/\mu)\phi(B(y,r))=\lim_{i\to\infty}(1-1/\mu)\phi(B(y_i,r-2d(y,y_i)))\le\phi(B(y,r)\cap E^{\phi}(\vartheta,\gamma)).$$

Since r was arbitrarily chosen in $(0,1/\nu)$, this shows that $y\in\mathscr{E}_{\vartheta,\gamma}(\mu,\nu)$, concluding the proof. □

Proposition 1.30. *With the hypothesis of Proposition 1.28, for any $0<\epsilon<\frac{1}{10}$ there are $\vartheta_0,\gamma_0\in\mathbb{N}$ such that for any $\vartheta\ge\vartheta_0$, $\gamma\ge\gamma_0$ and $\mu\in\mathbb{N}$ there is a $\nu=\nu(\vartheta,\gamma,\mu)\in\mathbb{N}$ such that*

$$\phi(K\setminus\mathscr{E}^{\phi}_{\vartheta,\gamma}(\mu,\nu))\le\epsilon\phi(K).\tag{19}$$

Proof. The proof is an elementary application of the Lebesgue differentiation theorem that can be found in [Heinonen et al. 2015, page 77]. □

The following result allows us to compare the measure ϕ when restricted to $E^{\phi}(\vartheta,\gamma)$ with the spherical Hausdorff measure. Since the proof is a well-known application of the Lebesgue differentiation theorem that can be found in [Heinonen et al. 2015, page 77], of [Franchi et al. 2015, Theorem 3.1] and the mutual absolute continuity of the spherical and centered Hausdorff measures, see for instance [Franchi et al. 2015], we choose to leave it to the reader.

Proposition 1.31. *Let ϕ be a Radon measure and suppose further that there are $0 < \delta_1 \le \delta_2 < \infty$ such that*

$$\delta_1 \le \Theta^m_*(\phi, x) \le \Theta^{m,*}(\phi, x) \le \delta_2, \quad \textit{for } \phi\textit{-almost every } x \in E.$$

Then $\delta_1 \mathcal{S}^m \llcorner E \le \phi \llcorner E \le \delta_2 2^m \mathcal{S}^m \llcorner E$ and in particular, for any $\vartheta, \gamma \in \mathbb{N}$, we have

$$\vartheta^{-1} \mathcal{S}^{\mathfrak{Q}-1} \llcorner E^{\phi}(\vartheta, \gamma) \le \phi \llcorner E^{\phi}(\vartheta, \gamma) \le \vartheta 2^{\mathfrak{Q}-1} \mathcal{S}^{\mathfrak{Q}-1} \llcorner E^{\phi}(\vartheta, \gamma).$$

The following result will be used in the proof of the very important Proposition 2.4. It establishes the natural request that if a sequence of planes V_i in $\mathrm{Gr}(\mathfrak{Q}-1)$ converges in the Grassmannian to some plane $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ (i.e., the normals converge as vectors in V_1), then the surface measures on the V_i converge weakly to the surface measure on V.

Proposition 1.32. *Suppose that $\{V(i)\}_{i \in \mathbb{N}}$ is a sequence of planes in $\mathrm{Gr}(\mathfrak{Q}-1)$ such that $\mathfrak{n}(V(i)) \to \mathfrak{n}$ for some $\mathfrak{n} \in V_1$. Then there exists a $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ such that $\mathfrak{n}(V) = \mathfrak{n}$ and $\mathcal{S}^{\mathfrak{Q}-1} \llcorner V(i) \rightharpoonup \mathcal{S}^{\mathfrak{Q}-1} \llcorner V$.*

Proof. For any continuous function of compact support, $f \in \mathcal{C}_c$, we have thanks to Proposition 1.8 that

$$\lim_{i\to\infty} \int f \, d\mathcal{S}^{\mathfrak{Q}-1} \llcorner V(i) - \int f \, d\mathcal{S}^{\mathfrak{Q}-1} \llcorner V = \lim_{i\to\infty} \beta^{-1} \biggl(\int f \, d\mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner V(i) - \int f \, d\mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner V \biggr) = 0, \quad (20)$$

where the last identity comes from the fact that $\mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner V(i) \rightharpoonup \mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner V$. □

1D. *Rectifiable sets in Carnot groups.* In this subsection we recall the two main notions of rectifiability in Carnot groups that will be extensively used throughout the paper. First of all, let us recall the definitions of horizontal vector fields and horizontal distributions.

Definition 1.33. Let $e_1, \dots, e_{n_1}$ be an orthonormal basis of V_1 with respect to the Euclidean scalar product. For any $i = 1, \dots, n_1$ and any $x \in \mathbb{G}$ we let $X_i(x) := \partial_t (x * \delta_t(e_i))|_{t=0}$ and say that the map $X_i : \mathbb{G} \cong \mathbb{R}^n \to \mathbb{R}^n$ so defined is the i-th *horizontal vector field*. Furthermore, we define the *horizontal distribution* of $\mathbb{G}$ to be the following n_1-dimensional distribution of planes in $\mathbb{R}^n$:

$$H\mathbb{G}(x) := \mathrm{span}\{X_1(x), \dots, X_{n_1}(x)\}.$$

Finally, for any open set Ω in $\mathbb{G}$ we denote by $\mathcal{C}^1_0(\Omega, H\mathbb{G})$ the sections of $H\mathbb{G}$ of class $\mathcal{C}^1$ with support contained in Ω.

The definition of regular surfaces we are about to give is reminiscent of the characterization of smooth surfaces in the Euclidean spaces through the local inversion theorem. Heuristically speaking, a $C^1_{\mathbb{G}}$-surface is a set that is transverse to $H\mathbb{G}$ and whose sections with $H\mathbb{G}$ are C^1-surfaces.

Definition 1.34 ($C^1_{\mathbb{G}}$-surfaces). We say that a closed set $C \subseteq \mathbb{G}$ is a $C^1_{\mathbb{G}}$-*surface* if there exists a continuous function $f : \mathbb{G} \to \mathbb{R}$ such that $C = f^{-1}(0)$ and whose horizontal distributional gradient $\nabla_{\mathbb{G}} f := (X_1 f, \dots, X_{n_1} f)$ can be represented by a continuous, never-vanishing section of $H\mathbb{G}$.

Remark 1.35. Thanks to [Serra Cassano 2016, Corollary 4.27], if C is a $C^1_{\mathbb{G}}$-regular surface, then $\mathcal{S}^{\mathfrak{Q}-1} \llcorner C$ is σ-finite.

The second notion of regular surface we give in this subsection is inspired by the characterization of Lipschitz graphs through cones.

Definition 1.36 (intrinsic Lipschitz graphs). Let $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ and E be a Borel subset of V. A function $f : E \to \mathfrak{N}(V)$ is said to be *intrinsic Lipschitz* if there exists an $\alpha > 0$ such that for any $v \in E$ we have

$$\mathrm{gr}(f) := \{wf(w) : w \in E\} \subseteq vf(v)C_V(\alpha).$$

A Borel set $A \subseteq \mathbb{G}$ is said to be a *V-intrinsic Lipschitz graph*, or simply an *intrinsic Lipschitz graph*, if there is an intrinsic Lipschitz function $f : E \subseteq V \to \mathfrak{N}(V)$ such that $A = \mathrm{gr}(f)$.

Proposition 1.37. *Suppose E is a Borel subset of $\mathbb{G}$ and assume there is a plane $W \in \mathrm{Gr}(\mathfrak{Q}-1)$ and an $\alpha > 0$ such that for any $w \in E$ we have $E \subseteq wC_W(\alpha)$. Then E is contained in an intrinsic Lipschitz graph.*

Proof. Thanks to the assumption on E, for any $w_1, w_2 \in E$ we have $w_1^{-1}w_2 \in C_W(\alpha)$. This implies that for any $v \in P_W(E)$ there exists a unique $w \in E$ such that $P_W(w) = v$, otherwise we would have $w_1^{-1}w_2 \in \mathfrak{N}(W)$.

Let $f : P_W(E) \to \mathfrak{N}(V)$ be the map associating every $w \in P_W(E)$ to the only element in its preimage $P_W^{-1}(w)$. With this definition we have that the set $\mathrm{gr}(f) := \{vf(v) : v \in P_W^{-1}(E)\}$ coincides with E and this shows that f is an intrinsic Lipschitz function since $\mathrm{gr}(f) \subseteq vf(v)C_W(\alpha)$ for any $v \in E$. □

The following extension theorem is of capital importance for us:

Theorem 1.38 [Vittone 2012, Theorem 3.4]. *Suppose $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ and let $f : E \to \mathfrak{N}(V)$ be an intrinsic Lipschitz function. Then there is an intrinsic Lipschitz function $\tilde{f} : V \to \mathfrak{N}(V)$ such that $f(v) = \tilde{f}(v)$ for any $v \in E$.*

The following result is an immediate consequence of Theorem 1.38:

Proposition 1.39. *If $f : E \subseteq V \to \mathfrak{N}(V)$ is an intrinsic Lipschitz function, then $\mathcal{S}^{\mathfrak{Q}-1} \llcorner \mathrm{gr}(f)$ is σ-finite.*

Proof. Theorem 1.38 together with [Franchi and Serapioni 2016, Theorem 3.9] immediately implies that $\mathcal{S}^{\mathfrak{Q}-1}(\mathrm{gr}(f) \cap B(0, R)) < \infty$ for any $R > 0$. □

From the notions of $C^1_{\mathbb{G}}$-surfaces and of intrinsic Lipschitz surfaces rise the two following definitions of rectifiability:

Definition 1.40. A Borel set $A \subseteq \mathbb{G}$ of finite $\mathcal{S}^{\mathfrak{Q}-1}$-measure is said to be

(i) *$C^1_{\mathbb{G}}$-rectifiable* if there are countably many $C^1_{\mathbb{G}}$-surfaces Γ_i such that

$$\mathcal{S}^{\mathfrak{Q}-1}\Big(A \setminus \bigcup_{i\in\mathbb{N}} \Gamma_i\Big) = 0,$$

(ii) *intrinsic rectifiable* if there are countably many intrinsic Lipschitz graphs Γ_i such that

$$\mathcal{S}^{\mathfrak{Q}-1}\Big(A \setminus \bigcup_{i\in\mathbb{N}} \Gamma_i\Big) = 0.$$

The following proposition is an adaptation of the well-known fact that Borel sets can be written in an essentially unique way, as the union of a rectifiable and a purely unrectifiable set.

Proposition 1.41 (decomposition theorem). *Suppose $\mathscr{F}$ is a family of Borel sets in $\mathbb{G}$ for which $\mathcal{S}^{\mathfrak{Q}-1}\llcorner C$ is σ-finite for any $C \in \mathscr{F}$. Then, for any Borel set $E \subseteq \mathbb{G}$ such that $\mathcal{S}^{\mathfrak{Q}-1}(E) < \infty$, there are two Borel sets E^u, $E^r \subseteq E$ such that*

(i) $E^u \cup E^r = E$,

(ii) *E^r is contained in a countable union of elements of $\mathscr{F}$,*

(iii) *$\mathcal{S}^{\mathfrak{Q}-1}(E^u \cap C) = 0$ for any $C \in \mathscr{F}$.*

Such a decomposition is unique up to $\mathcal{S}^{\mathfrak{Q}-1}$-null sets, i.e., if F^u and F^r are Borel sets satisfying the three properties listed above, we have $\mathcal{S}^{\mathfrak{Q}-1}(E^r \triangle F^r) = \mathcal{S}^{\mathfrak{Q}-1}(E^u \triangle F^u) = 0$.

Proof. The proof follows verbatim the argument of [De Lellis 2008, Theorem 5.7]. □

Corollary 1.42. *For any Borel set $E \subseteq \mathbb{G}$ such that $\mathcal{S}^{\mathfrak{Q}-1}(E) < \infty$, there are two Borel sets E^u, $E^r \subseteq E$ such that*

(i) $E^u \cup E^r = E$,

(ii) *there are countably many intrinsic Lipschitz functions $f_i : V_i \to \mathfrak{N}(V_i)$, where $V_i \in \mathrm{Gr}(\mathfrak{Q}-1)$, whose graphs cover $\mathcal{S}^{\mathfrak{Q}-1}$-almost all of E^r,*

(iii) *$\mathcal{S}^{\mathfrak{Q}-1}(E^u \cap C) = 0$ for any C-intrinsic Lipschitz graph.*

Proof. Thanks to Proposition 1.39 we know that every intrinsic Lipschitz graph is $\mathcal{S}^{\mathfrak{Q}-1}$-$\sigma$-finite. If we choose $\mathscr{F}$ in the statement of Proposition 1.41 to be the family of all intrinsic Lipschitz graphs of $\mathbb{G}$, we get two sets E^u and E^r whose union is the whole set E, such that E^u has $\mathcal{S}^{\mathfrak{Q}-1}$-null intersection with every intrinsic Lipschitz graph and E_r can be covered by countably many graphs of intrinsic Lipschitz functions $f_i : E_i \subseteq V_i \to \mathfrak{N}(V_i)$. The conclusion follows from Theorem 1.38. □

2. The support of 1-codimensional measures with flat tangents is intrinsic rectifiable

Throughout this section *we assume ϕ to be a fixed Radon measure on* $\mathbb{G}$ whose support is a compact set K and such that for ϕ-almost every $x \in \mathbb{G}$ we have

(H1) $0 < \Theta_*^{\mathfrak{Q}-1}(\phi, x) \leq \Theta^{\mathfrak{Q}-1,*}(\phi, x) < \infty$,

(H2) $\mathrm{Tan}_{\mathfrak{Q}-1}(\phi, x) \subseteq \mathfrak{M}$, where $\mathfrak{M}$ is the family of 1-codimensional flat measures from Definition 1.7.

The main goal of this section is to prove the following:

Theorem 2.1. *There is an intrinsic Lipschitz graph Γ such that $\phi(\Gamma) > 0$.*

The strategy we employ in order to prove Theorem 2.1 is divided into four parts: First of all in Section 2A we show that hypotheses (H1) and (H2) on ϕ imply that for ϕ-almost any $x \in K$ and $r > 0$ sufficiently small, there is a plane $V_{x,r}$ for which K as a set is very close in the Hausdorff distance to $V_{x,r}$. In Section 2B we prove that if $K \cap B(x, r)$ has a big projection on some plane W, then W is very close to $V_{x,r}$ and there exists an $\alpha > 0$ such that for any $y, z \in B(x, r) \cap K$ for which $d(y, z) \geq \mathrm{dist}(W, V_{x,r})r$, we have $z \in yC_W(\alpha)$. Section 2C is the technical core of this section, and its main result, Theorem 2.28, shows that for ϕ-almost any $x \in K$ we have that the set $B(x, r) \cap K$ has a big projection on $V_{x,r}$. Finally, in Section 2D, making use of the results of the previous subsections, we construct the wanted ϕ-positive intrinsic Lipschitz graph.

2A. *Geometric implications of flat tangents.* In this subsection we reformulate the hypothesis (H2) on ϕ in more geometric terms. In order to obtain such a reformulation, we need a way to pass from the purely pointwise information on the flatness of tangents to a more local understanding of the measure ϕ at small scales. In the following Definition 2.2, we introduce two functionals on Radon measures that will be used for this precise objective. These functionals can be considered the Carnot analogue of the functional $d(\cdot\,,\mathfrak{M})$ of Section 2 of [Preiss 1987].

Definition 2.2. For any $x\in\mathbb{G}$ and any $r>0$ we define the functionals

$$d_{x,r}(\phi,\mathfrak{M}):=\inf_{\substack{\Theta>0\\ V\in\mathrm{Gr}(\mathfrak{Q}-1)}}\frac{F_{x,r}(\phi,\Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV)}{r^{\mathfrak{Q}}}\quad\text{and}\quad \tilde d_{x,r}(\phi,\mathfrak{M}):=\inf_{\substack{\Theta>0,\ z\in\mathbb{G}\\ V\in\mathrm{Gr}(\mathfrak{Q}-1)}}\frac{F_{x,r}(\phi,\Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner zV)}{r^{\mathfrak{Q}}},$$

where $F_{x,r}$ was introduced in (17).

In the following proposition we summarize some useful properties of the functionals $d_{x,r}$ and $\tilde d_{x,r}$.

Proposition 2.3. *The functionals $d_{x,r}(\cdot\,,\mathfrak{M})$ and $\tilde d_{x,r}(\cdot\,,\mathfrak{M})$ satisfy the following properties*:

(i) *For any $x\in\mathbb{G}$, $k>0$ and $r>0$, we have $d_{x,kr}(\phi,\mathfrak{M})=d_{0,k}(r^{-(\mathfrak{Q}-1)}T_{x,r}\phi,\mathfrak{M})$.*

(ii) *For any $r>0$, the function $x\mapsto d_{x,r}(\phi,\mathfrak{M})$ is continuous.*

(iii) *For any $x,y\in\mathbb{G}$ and $r,s>0$ for which $B(y,s)\subseteq B(x,r)$, we have $(s/r)^{\mathfrak{Q}}\tilde d_{y,s}(\phi,\mathfrak{M})\le\tilde d_{x,r}(\phi,\mathfrak{M})$.*

(iv) *For any $x\in\mathbb{G}$ and any $s\le r$, we have $(s/r)^{\mathfrak{Q}}d_{x,s}(\phi,\mathfrak{M})\le d_{x,r}(\phi,\mathfrak{M})$.*

Proof. It is immediate to see that f belongs to $\mathrm{Lip}_1^+(\overline{B(x,kr)})$ if and only if there is a $g\in\mathrm{Lip}_1^+(\overline{B(0,k)})$ such that $f(z)=rg(\delta_{1/r}(x^{-1}z))$. This implies that

$$\begin{aligned}\frac{1}{(kr)^{\mathfrak{Q}}}&\Big(\int f\,d\phi-\Theta\int f\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV\Big)\\ &=\frac{1}{k^{\mathfrak{Q}}r^{\mathfrak{Q}-1}}\Big(\int g(\delta_{1/r}(x^{-1}z))\,d\phi(z)-\Theta\int g(\delta_{1/r}(x^{-1}z))\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV\Big)\\ &=\frac{1}{k^{\mathfrak{Q}}}\Big(\int g(z)\,d\frac{T_{x,r}\phi}{r^{Q-1}}(z)-\Theta\int g(z)\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner V\Big),\end{aligned}$$

and this concludes the proof of (i). To show that the map $x\mapsto d_{x,r}(\phi,\mathfrak{M})$ is continuous, we prove the following stronger fact. There exists a constant $\tilde C$ depending only on $\mathbb{G}$ such that for any $x,y\in\mathbb{G}$ with $d(x,y)<1$ we have

$$|d_{x,r}(\phi,\mathfrak{M})-d_{y,r}(\phi,\mathfrak{M})|\le\tilde C(\mathbb{G})\frac{2(r+2)d(x,y)^{1/s}}{r^{\mathfrak{Q}}}\phi(B(x,r+d(x,y))).\tag{21}$$

In order to prove (21), for any $\epsilon>0$ we let $\Theta^*>0$ and $V^*\in\mathrm{Gr}(\mathfrak{Q}-1)$ be such that

$$\Big|\int f\,d\frac{T_{y,r}\phi}{r^{\mathfrak{Q}-1}}-\Theta^*\int f\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner V^*\Big|\le d_{y,r}(\phi,\mathfrak{M})+\epsilon,\quad\text{for any }f\in\mathrm{Lip}_1^+(\overline{B(0,1)}).$$

Furthermore, by definition of $d_{y,r}$ we can find an $f^*\in \mathrm{Lip}_1^+(\overline{B(0,1)})$ such that

$$d_{x,r}(\phi,\mathfrak{M})-\epsilon\le\left|\int f^*\,d\frac{T_{x,r}\phi}{r^{\mathfrak{Q}-1}}-\Theta^*\int f^*\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner V_*\right|.$$

This choice of f^*, Θ^* and V^* implies that

$$\begin{aligned}
&d_{x,r}(\phi,\mathfrak{M})-d_{y,r}(\phi,\mathfrak{M})\\
&\le\left|\int f^*\,d\frac{T_{x,r}\phi}{r^{\mathfrak{Q}-1}}-\Theta^*\int f^*\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner V^*\right|-\left|\int f^*\,d\frac{T_{y,r}\phi}{r^{\mathfrak{Q}-1}}-\Theta^*\int f^*\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner V^*\right|+2\epsilon\\
&\le\left|\int f^*\,d\frac{T_{x,r}\phi}{r^{\mathfrak{Q}-1}}-\int f^*\,d\frac{T_{y,r}\phi}{r^{\mathfrak{Q}-1}}\right|+2\epsilon\le r^{-(\mathfrak{Q}-1)}\int|f^*(\delta_{1/r}(x^{-1}w))-f^*(\delta_{1/r}(y^{-1}w))|\,d\phi(w)+2\epsilon\\
&\le r^{-\mathfrak{Q}}\int_{B(x,r+d(x,y))}d(x^{-1}w,y^{-1}w)\,d\phi(x)+2\epsilon\\
&\le r^{-\mathfrak{Q}}\big(d(x,y)+C(G)\big(d(x,y)^{1/s}(r+d(x,y))^{(s-1)/s}+d(x,y)^{(s-1)/s}(r+d(x,y))^{1/s}\big)\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\phi(B(x,r+d(x,y)))+2\epsilon,
\end{aligned}$$

where the last inequality comes from [Franchi and Serapioni 2016, Proposition 2.13] together with the constant $C(\mathbb{G})$. Interchanging x and y, the bound (21) is proved thanks to the arbitrariness of ϵ. Finally, statements (iii) and (iv) follow directly from the definitions. □

The following proposition allows us to rephrase the rather geometric condition on ϕ, the flatness of the tangents, into a more malleable functional-analytic condition that is the ϕ-almost everywhere convergence of the functions $x\mapsto d_{x,kr}(\phi,\mathfrak{M})$ to 0. We omit the straightforward proof.

Proposition 2.4. *Assume μ is a Radon measure on $\mathbb{G}$ such that $0<\Theta^{\mathfrak{Q}-1}(\mu,x)<\infty$ for μ-almost every $x\in\mathbb{G}$. Then the following are equivalent*:

(i) $\lim_{r\to 0}d_{x,kr}(\mu,\mathfrak{M})=0$ *for μ-almost every $x\in\mathbb{G}$ and any $k>0$.*

(ii) $\mathrm{Tan}_{\mathfrak{Q}-1}(\mu,x)\subseteq\mathfrak{M}$ *for μ-almost every $x\in\mathbb{G}$.*

Notation 2.5. Throughout Section 2 we let $0<\varepsilon_1<\frac{1}{10}$ be a fixed constant. Proposition 1.30 yields two natural numbers $\vartheta,\gamma\in\mathbb{N}$, that from now on we consider fixed, such that $\phi(K\setminus E^{\phi}(\vartheta,\gamma))\le\varepsilon_1\phi(K)$. We can assume without loss of generality, again thanks to Proposition 1.30, that ϑ and γ have the further property that for any $\mu\ge 4\vartheta$ there is a $\nu\in\mathbb{N}$ for which

$$\phi(K\setminus\mathscr{E}^{\phi}_{\vartheta,\gamma}(\mu,\nu))\le\varepsilon_1\phi(K).$$

For future convenience, we define $\eta:=1/\mathfrak{Q}$ and let

$$\delta_{\mathbb{G}}(\vartheta):=\min\left\{\frac{1}{2^{4(\mathfrak{Q}+1)}\vartheta},\frac{\eta^{\mathfrak{Q}+1}(1-\eta)^{\mathfrak{Q}^2-1}}{(32\vartheta)^{\mathfrak{Q}+1}}\right\}.$$

Eventually, if $d_{x,r}(\phi\llcorner E^{\phi}(\vartheta,\gamma),\mathfrak{M})+d_{x,r}(\phi,\mathfrak{M})\le\delta$ for some $0<\delta<\delta_{\mathbb{G}}(\vartheta)$, we define $\Pi_\delta(x,r)$ to be the subset of planes $V\in\mathrm{Gr}(\mathfrak{Q}-1)$ for which there exists a $\Theta>0$ such that

$$F_{x,r}(\phi\llcorner E^{\phi}(\vartheta,\gamma),\Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV)+F_{x,r}(\phi,\Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV)\le 2\delta r^{\mathfrak{Q}}.\tag{22}$$

The following two propositions are the main results of this subsection. They are so relevant since they give a more geometric interpretation of the condition we call *flatness of the tangents* and in particular tell us that $E^\phi(\vartheta,\gamma)$ is in essence a weakly linearly approximable set. For a discussion on how this will play a role in the proof of the main result of this work, we refer to the Introduction.

Proposition 2.6. *Let $x\in E^\phi(\vartheta,\gamma)$ be such that $\tilde{d}_{x,r}(\phi,\mathfrak{M})\leq\delta$ for some $\delta<\delta_{\mathbb{G}}(\vartheta)$ and $0<r<1/\gamma$. Then, for every $V\in\mathrm{Gr}(\mathfrak{Q}-1)$ for which there is a $z\in\mathbb{G}$ and a $\Theta>0$ such that $F_{x,r}(\phi,\Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner zV)\leq 2\delta r^{\mathfrak{Q}}$, we have*

$$\sup_{w\in E^\phi(\vartheta,\gamma)\cap B(x,r/4)}\frac{\mathrm{dist}(w,xV)}{r}\leq 2^{2+3/\mathfrak{Q}}\vartheta^{1/\mathfrak{Q}}\delta^{1/\mathfrak{Q}}=:C_2(\vartheta)\delta^{1/\mathfrak{Q}}.$$

Proof. Since $g(w):=\min\{\mathrm{dist}(w,B(x,r)^c),\mathrm{dist}(w,zV)\}$ belongs to $\mathrm{Lip}_1^+(\overline{B(x,r)})$, we deduce that

$$2\delta r^{\mathfrak{Q}}\geq\int g(w)\,d\phi(w)-\Theta\int g(w)\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner zV=\int g(w)\,d\phi(w)\geq\int_{B(x,r/2)}\min\bigl\{\tfrac{1}{2}r,\mathrm{dist}(w,zV)\bigr\}\,d\phi(w).$$

Suppose that y is a point in $\overline{B(x,r/4)}\cap E^\phi(\vartheta,\gamma)$ furthest from zV, and let $D=\mathrm{dist}(y,zV)$. If $D\geq\frac{1}{8}r$, this would imply that

$$\begin{aligned}2\delta r^{\mathfrak{Q}}&\geq\int_{B(x,r/2)}\min\bigl\{\tfrac{1}{2}r,\mathrm{dist}(w,zV)\bigr\}\,d\phi(w)\\&\geq\int_{B(y,r/16)}\min\bigl\{\tfrac{1}{2}r,\mathrm{dist}(w,zV)\bigr\}\,d\phi(w)\geq\tfrac{1}{16}r\phi\bigl(B\bigl(y,\tfrac{1}{16}r\bigr)\bigr)\geq\frac{r^{\mathcal{Q}}}{\vartheta 16^{\mathfrak{Q}}},\end{aligned}$$

which is not possible thanks to the choice of δ. This implies that $D<\frac{1}{8}r$ and as a consequence, we have

$$\begin{aligned}2\delta r^{\mathfrak{Q}}&\geq\int_{B(x,r/2)}\min\bigl\{\tfrac{1}{2}r,\mathrm{dist}(w,zV)\bigr\}\,d\phi(w)\\&\geq\int_{B(y,D/2)}\min\bigl\{12r,\mathrm{dist}(w,zV)\bigr\}\,d\phi(w)\geq\tfrac{1}{2}D\phi\bigl(B\bigl(y,\tfrac{1}{2}D\bigr)\bigr)\geq\vartheta^{-1}\bigl(\tfrac{1}{2}D\bigr)^{\mathfrak{Q}},\end{aligned}\tag{23}$$

where the second inequality comes from the fact that $B\bigl(y,\frac{1}{2}D\bigr)\subseteq B\bigl(x,\frac{1}{2}r\bigr)$. This implies, thanks to (23), that

$$\sup_{w\in E^\phi(\vartheta,\gamma)\cap B(x,r/4)}\frac{\mathrm{dist}(w,zV)}{r}\leq\frac{D}{r}\leq 2^{1+3/\mathfrak{Q}}\vartheta^{1/\mathfrak{Q}}\delta^{1/\mathfrak{Q}}=\tfrac{1}{2}C_2(\vartheta)\delta^{1/\mathfrak{Q}}.$$

Furthermore, since $x\in E^\phi(\vartheta,\gamma)$, we also infer that $\mathrm{dist}(x,zV)/r\leq\frac{1}{2}C_2(\vartheta)\delta^{1/\mathfrak{Q}}$. Therefore, thanks to Proposition 1.17, we conclude that

$$\sup_{w\in E^\phi(\vartheta,\gamma)\cap B(x,r/4)}\frac{\mathrm{dist}(w,xV)}{r}\leq\sup_{w\in E^\phi(\vartheta,\gamma)\cap B(x,r/4)}\frac{\mathrm{dist}(w,zV)+\mathrm{dist}(xV,zV)}{r}\leq C_2(\vartheta)\delta^{1/\mathfrak{Q}}.\qquad\square$$

Proposition 2.7. *Let $x\in E^\phi(\vartheta,\gamma)$ and $0<r<1/\gamma$ be such that for some $0<\delta<\delta_{\mathbb{G}}(\vartheta)$ we have*

$$d_{x,r}(\phi,\mathfrak{M})+d_{x,r}(\phi\llcorner E^\phi(\vartheta,\gamma),\mathfrak{M})\leq\delta.\tag{24}$$

Then for any $V\in\Pi_\delta(x,r)$ and any $w\in B\bigl(x,\frac{1}{2}r\bigr)\cap xV$ we have $E^\phi(\vartheta,\gamma)\cap B(w,\delta^{1/(\mathfrak{Q}+1)}r)\neq\varnothing$.

Proof. By the definition of $\Pi_\delta(x,r)$ (see Notation 2.5), for any $V\in\Pi_\delta(x,r)$, where here we choose $\delta:=2^{-\mathfrak{Q}^2-\mathfrak{Q}}\varepsilon_2$, there exists a $\Theta>0$ such that

$$\frac{F_{x,r}(\phi,\Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV)+F_{x,r}(\phi\llcorner E^\phi(\vartheta,\gamma),\Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV)}{r^{\mathfrak{Q}}}\leq 2\delta. \tag{25}$$

Therefore, defining $g(x):=\min\{\operatorname{dist}(x,B(0,1)^c),\eta\}$, we infer that

$$\begin{aligned}\vartheta^{-1}(1-\eta)^{\mathfrak{Q}-1}\eta r^{\mathfrak{Q}}-\Theta\eta r^{\mathfrak{Q}}&\leq \eta r\phi(B(x,(1-\eta)r))-\eta r\Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV(B(x,r))\\&\leq\int rg(\delta_{1/r}(x^{-1}z))\,d\phi(z)-\Theta\int rg(\delta_{1/r}(x^{-1}z))\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV\leq 2\delta r^{\mathfrak{Q}},\end{aligned}$$

where the last inequality above comes from (25) and the fact that $rg(\delta_{1/r}(x^{-1}\cdot))\in\operatorname{Lip}_1^+(\overline{B(x,r)})$. Simplifying and rearranging the above chain of inequalities we infer that

$$\Theta\geq\vartheta^{-1}(1-\eta)^{\mathfrak{Q}-1}-2\delta/\eta\geq(2\vartheta)^{-1}(1-\eta)^{\mathfrak{Q}-1}=(2\vartheta)^{-1}(1-1/\mathfrak{Q})^{\mathfrak{Q}-1},$$

where the first inequality comes from the choice of δ and the last equality from that of $\eta=1/\mathfrak{Q}$; see Notation 2.5. Since the function $\mathfrak{Q}\mapsto(1-1/\mathfrak{Q})^{\mathfrak{Q}-1}$ is decreasing and $\lim_{\mathfrak{Q}\to\infty}(1-1/\mathfrak{Q})^{\mathfrak{Q}-1}=1/e$, we infer that $\Theta\geq\frac{1}{2}\vartheta e$. Suppose that $\delta^{1/(\mathfrak{Q}+1)}<\lambda<\frac{1}{2}$ and assume that we can find a $w\in xV\cap B\big(x,\frac{1}{2}r\big)$ such that $\phi(B(w,\lambda r)\cap E^\phi(\vartheta,\gamma))=0$. This would imply that

$$\begin{aligned}&\Theta\eta(1-\eta)^{\mathfrak{Q}-1}\lambda^{\mathfrak{Q}}r^{\mathfrak{Q}}\\&\quad=\Theta\eta\lambda r\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV(B(w,(1-\eta)\lambda r))\\&\quad\leq\Theta\int\lambda rg(\delta_{1/\lambda r}(w^{-1}z))\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV(z)\\&\quad=\Theta\int\lambda rg(\delta_{1/\lambda r}(w^{-1}z))\,d\mathcal{S}^{\mathfrak{Q}-1}\llcorner xV(z)-\int\lambda rg(\delta_{1/\lambda r}(w^{-1}z))\,d\phi\llcorner E^\phi(\vartheta,\gamma)(z)\leq 2\delta r^{\mathfrak{Q}},\end{aligned}\tag{26}$$

where the inequality on the middle line is a consequence of the fact that, thanks to the precise choice of g, we have $g=\eta$ on $B(0,1-\eta)$, whereas the last inequality comes from the choice of Θ, V, the fact that $\lambda rg(\delta_{1/\lambda r}(w^{-1}\cdot))\in\operatorname{Lip}_1^+(\overline{B(x,r)})$ and the constraint on $\phi\llcorner E^\phi(\vartheta,\gamma)$ given by (25). Thanks to (26), the choice of λ and the fact that $\frac{1}{4}e\vartheta<\Theta$, we have that

$$\frac{\delta^{\mathfrak{Q}/(\mathfrak{Q}+1)}}{4e\vartheta}\eta(1-\eta)^{\mathfrak{Q}-1}<\Theta\lambda^{\mathfrak{Q}}\eta(1-\eta)^{\mathfrak{Q}-1}\leq 2\delta.$$

However, a few algebraic computations that we omit show that the above inequality chain is in contradiction with the choice of $\delta<\delta_{\mathbb{G}}(\vartheta)$. □

2B. *Construction of cones complementing* supp(ϕ) *in case it has big projections on planes.* This subsection is devoted to the proof of Proposition 2.11, which tells us that if the measure ϕ is well approximated inside a ball $B(x,r)$ by some plane V and if there exists some *other* plane W on which the $\mathcal{S}^{\mathfrak{Q}-1}$-measure of the projection $P_W(\operatorname{supp}(\phi)\cap B(x,r))$ is comparable with $r^{\mathfrak{Q}-1}$, then at scales comparable with r the set $\operatorname{supp}(\phi)\cap B(x,r)$ is a W-intrinsic Lipschitz surface. In other words, we can find an $\alpha > 0$ such that

$$y \in zC_W(\alpha) \quad \text{whenever } y, z \in B(x, r) \cap \operatorname{supp}(\phi) \text{ and } d(z, y) \gtrsim r.$$

Before proceeding with the statement and the proof of Proposition 2.11, we fix some notation that will be extensively used throughout the rest of the paper.

Notation 2.8. Throughout this paragraph we assume that ψ is a Radon measure on $\mathbb{G}$ supported on a compact set K such that $0 < \Theta_*^{\mathfrak{Q}-1}(\psi, x) \leq \Theta^{\mathfrak{Q}-1,*}(\psi, x) < \infty$ for ψ-almost every $x \in \mathbb{G}$ and that $\sigma \in \mathbb{N}$ is a fixed positive natural number. First of all, let us define the following two numbers:

$$\zeta(\sigma) := 2^{-50\mathfrak{Q}}\sigma^{-2} \quad \text{and} \quad N(\sigma) := \lfloor -4\log(\zeta(\sigma)) \rfloor + 40(\mathfrak{Q}+1).$$

Secondly, we let

$$\begin{aligned} C_3(\sigma) &:= 2^{20}(n_1 - 1)C_2(\sigma)^2, & C_4(\sigma) &:= 2^{24\mathfrak{Q}}\sigma, \\ C_5(\sigma) &:= C_4(\sigma)(32\zeta(\sigma)^{-2})^{\mathfrak{Q}-1}, & C_6(\sigma) &:= 2^{2\log C_4(\sigma)/(\mathfrak{Q}-1)+N(\sigma)+6}\zeta(\sigma)^{-2}. \end{aligned}$$

Finally, we introduce six further new constants that depend only on σ. Although we could avoid giving an explicit expression for such constants, we choose nonetheless to make them explicit for the following reasons: First of all, having their values helps keep their interactions in proofs under control, getting more precise statements. Secondly, fixing these constants once and for all, we avoid the practice of choosing them *large enough* when necessary. In doing so we hope to help the reader not to get distracted with the problem of whether these choices were legitimate or not.

For the sake of readability, we choose not to make the dependence on σ of the numbers N, ζ and the constants $C_1, \dots, C_6$ explicit in the following. In addition, in the forthcoming definitions, we choose to suppress any dependence on σ in the right-hand side of the expression. We let:

(i)
$$A_0(\sigma) := 2\max\left\{\log(C_6) + C_6, \left\lfloor \frac{2\log_2 C_4}{N(\mathfrak{Q}-1)} \right\rfloor + 1, \frac{7\log 2 - 2\log\zeta}{N\log 2 - 2}\right\};$$

(ii)
$$k(\sigma) := 80^{N+8}\zeta^{-2}A_0^4(1 + e^{8NA_0^2}) \quad \text{and} \quad 0 < R < 2^{-(N+11)}\zeta^2 k;$$

(iii)
$$\varepsilon_{\mathbb{G}}(\sigma) := \min\left\{2^{-20}, \frac{2^{2\mathfrak{Q}-n-18}\beta\prod_{j=2}^{s}\epsilon_i^{n_i}}{(A_0 k)^{\mathfrak{Q}-1}C_5^2}\right\},$$

where β is the constant introduced in Proposition 1.8, the n_i are the topological dimensions of the i-th layer, V_i, of the Lie algebra $\mathfrak{g}$ and the ϵ_i are the structure constants used to construct the metric; see Definition 1.4;

(iv)
$$\varepsilon_2(\sigma) := \min\left\{\frac{\delta_{\mathbb{G}}}{4}, \frac{\varepsilon_{\mathbb{G}}}{(2^{20}C_2^2C_5^2A_0k)(1 + 36kR^{-1})}, \frac{k-20}{20C_2k}, \frac{1}{2A_0^2C_3 + 2A_0kC_2C_4e^{8NA_0^2}}\right\}^{\mathfrak{Q}^2+\mathfrak{Q}}$$

with $\delta_{\mathbb{G}} = \delta_{\mathbb{G}}(\sigma)$ and $C_2 = C_2(\sigma)$;

(v)
$$\varepsilon_3(\sigma) := \frac{1}{2^{2\mathfrak{Q}}C_4^2C_5^2(A_0C_6)^{\mathfrak{Q}-1}}.$$

Since in the rest of Section 2 we make extensive use of the dyadic cubes whose existence is stated in Appendix A, we recall here some of the notation. For any $\xi, \tau \in \mathbb{N}$ for which $\psi(E^{\psi}(\vartheta, \gamma)) > 0$, we denote by $\Delta^{\psi}(\xi, \tau)$ the family of dyadic cubes relative to ψ and to the parameters ξ and τ yielded by Theorem A.2. Furthermore, for any compact subset κ of $E^{\psi}(\xi, \tau)$ and $l \in \mathbb{N}$ we let

$$\Delta^{\psi}(\kappa; \xi, \tau, l) := \{Q \in \Delta^{\psi}(\xi, \tau) : Q \cap \kappa \neq \varnothing \text{ and } Q \in \Delta^{\psi}_j(\xi, \tau) \text{ for some } j \geq l\},$$

where $\Delta^{\psi}_j(\xi, \tau)$ is the j-th layer of cubes; see Theorem A.2. Finally, for any $Q \in \Delta^{\psi}(E^{\psi}(\xi, \tau); \xi, \tau, 1)$, we define

$$\alpha(Q) := \tilde{d}_{\mathfrak{c}(Q), 2k \operatorname{diam} Q}(\psi, \mathfrak{M}) + \tilde{d}_{\mathfrak{c}(Q), 2k \operatorname{diam} Q}(\psi \llcorner E^{\psi}(\xi, \tau), \mathfrak{M}),$$

where $\mathfrak{c}(Q) \in Q$ is the center of the cube Q; see Theorem A.2 (v).

Eventually, we recall for the reader's sake some standard nomenclature on dyadic cubes: for any pair of dyadic cubes $Q_1, Q_2 \in \Delta^{\psi}(\xi, \tau)$,

(i) if $Q_1 \subseteq Q_2$, then Q_2 is said to be an *ancestor* of Q_1 and Q_1 a *subcube* of Q_2,

(ii) if Q_2 is the smallest cube for which $Q_1 \subsetneq Q_2$, then Q_2 is said to be the *parent* of Q_1 and Q_1 the *child* of Q_2.

Notation 2.9. If not otherwise stated, in order to simplify notation throughout Section 2 we will always denote by $\Delta := \Delta^{\phi}(\vartheta, \gamma)$ the family of dyadic cubes constructed in Theorem A.2 relative to the measure ϕ, which was fixed at the beginning of this Section, and to the parameters ϑ and γ, fixed in Notation 2.5. Furthermore, we let

$$E(\vartheta, \gamma) := E^{\phi}(\vartheta, \gamma), \quad \mathscr{E}(\mu, \nu) := \mathscr{E}^{\phi}_{\vartheta,\gamma}(\mu, \nu) \quad \text{and} \quad \Delta(\kappa, l) := \Delta^{\phi}(\kappa; \vartheta, \gamma, l).$$

Finally, if the dependence on σ of the constants introduced above is *not* specified, we will always assume that $\sigma = \vartheta$, where once again ϑ is the one natural number fixed in Notation 2.5.

Remark 2.10. For any compact subset κ of $E(\vartheta, \gamma)$, we let $\mathcal{M}(\kappa, l)$ be the set of maximal cubes of $\Delta(\kappa, l)$ ordered by inclusion. The elements of $\mathcal{M}(\kappa, l)$ are pairwise disjoint and enjoy the following properties:

(i) For any $Q \in \Delta(\kappa, l)$ there is a cube $Q_0 \in \mathcal{M}(\kappa, l)$ such that $Q \subseteq Q_0$.

(ii) If $Q_0 \in \mathcal{M}(\kappa, l)$ and there exists some $Q' \in \Delta(\kappa, l)$ for which $Q_0 \subseteq Q'$, then $Q_0 = Q'$.

The proof of the following proposition is inspired by the argument employed in proving [David and Semmes 1993b, Lemma 2.19] and its counterpart in the first Heisenberg group $\mathbb{H}^1$ [Chousionis et al. 2019, Lemma 3.8].

Proposition 2.11. *Let $\iota \in \mathbb{N}$ be such that $\iota > N^{-1}(5 + \log_2(4k))$, and suppose that Q is a cube in $\Delta(E(\vartheta, \gamma), \iota)$ satisfying the two following conditions*:

(i) $\tilde{d}_{\mathfrak{c}(Q), 4k \operatorname{diam} Q}(\phi \llcorner E(\vartheta, \gamma), \mathfrak{M}) + \tilde{d}_{\mathfrak{c}(Q), 4k \operatorname{diam} Q}(\phi, \mathfrak{M}) \leq \varepsilon_2$.

(ii) *There exists a plane $W \in \operatorname{Gr}(\mathfrak{Q} - 1)$ such that*

$$\frac{\operatorname{diam} Q^{\mathfrak{Q}-1}}{4C_5^2 A_0^{\mathfrak{Q}-1}} \leq \mathcal{S}^{\mathfrak{Q}-1} \llcorner W(P_W[\mathfrak{c}(Q)^{-1}(Q \cap E(\vartheta, \gamma))]). \tag{27}$$

Let $x \in E(\vartheta, \gamma) \cap Q$ and $y \in B\big(x, \frac{1}{8}(k-1)\operatorname{diam} Q\big) \cap E(\vartheta, \gamma)$ be two points for which

$$R \operatorname{diam} Q \leq d(x, y) \leq 2^{N+6}\zeta^{-2} R \operatorname{diam} Q. \tag{28}$$

Then, for any $\alpha > (\zeta^2 \varepsilon_{\mathbb{G}}/(6 \cdot 2^{8+N} R^{-1} k))^{-1} =: \alpha_0$, we have $y \in x C_W(\alpha)$.

Remark 2.12. Thanks to the definition of R and k, we have

$$2^{(N+6)}\zeta^{-2} R < 2^{(N+6)}\zeta^{-2} \cdot 2^{-(N+11)}\zeta^2 k = \tfrac{1}{32}k < \tfrac{1}{8}(k-1).$$

This implies that $B(x, 2^{N+6}\zeta^{-2} R \operatorname{diam} Q) \Subset B\big(x, \frac{1}{8}(k-1)\operatorname{diam} Q\big)$, and thus the requested inequality $d(x, y) \geq R \operatorname{diam} Q$ is compatible with the fact that y is chosen in $B\big(x, \frac{1}{8}(k-1)\operatorname{diam} Q\big)$.

Proof of Proposition 2.11. Suppose by contradiction there are two points $x, y \in E(\vartheta, \gamma)$ satisfying the hypothesis of the proposition such that $y \notin x C_W(\alpha)$ for some $\alpha > \alpha_0$. This implies, since the cone $C_W(\alpha)$ is closed by definition, that we have $\pi_1(x^{-1}y) \neq 0$. Furthermore, Proposition 1.14 along with (28) yields

$$\operatorname{diam} Q \leq R^{-1} d(x, y) \leq R^{-1}\Lambda(\alpha)|\pi_1(x^{-1}y)| \leq R^{-1}\Lambda(1)|\pi_1(x^{-1}y)| = 2R^{-1}|\pi_1(x^{-1}y)|, \tag{29}$$

where the last inequality comes from the fact that Λ (the function yielded by Proposition 1.14) is decreasing and from the last identity from the very definition of the function Λ. Let $\rho := \operatorname{diam}(Q)$ and note that Proposition 2.3 (iii) and the fact that $B(x, 4(k-1)\rho) \subseteq B(\mathfrak{c}(Q), 4k\rho)$ imply that

$$\begin{aligned}\tilde{d}_{x,4(k-1)\rho}(\phi, \mathfrak{M}) &+ \tilde{d}_{x,4(k-1)\rho}(\phi \llcorner E(\vartheta, \gamma), \mathfrak{M}) \\ &\leq \left(\frac{k}{(k-1)}\right)^{\mathfrak{Q}} (\tilde{d}_{\mathfrak{c}(Q),4k\rho}(\phi, \mathfrak{M}) + \tilde{d}_{\mathfrak{c}(Q),4k\rho}(\phi \llcorner E(\vartheta, \gamma), \mathfrak{M})) \leq 2^{\mathfrak{Q}}\varepsilon_2.\end{aligned}$$

In addition, we also have that $4(k-1)\rho < 1/\gamma$; indeed,

$$4(k-1)\rho = 4(k-1)\operatorname{diam}(Q) \leq 4(k-1) \cdot 2^{-N\iota+5}/\gamma < 1/\gamma,$$

where the first inequality above comes from Theorem A.2 and the last one from the choice of ι.

Therefore, thanks to Proposition 2.6 and the fact that $2^{\mathfrak{Q}}\varepsilon_2 \leq \delta_{\mathbb{G}}(\vartheta)$, we infer that there exists a plane $V \in \operatorname{Gr}(\mathfrak{Q}-1)$, *that we consider fixed throughout the proof*, such that

$$\sup_{w \in E(\vartheta,\gamma) \cap B(x,(k-1)\rho)} \frac{\operatorname{dist}(w, xV)}{4(k-1)\rho} \leq 2C_2\varepsilon_2^{1/\mathfrak{Q}}. \tag{30}$$

Since $y \in E(\vartheta, \gamma) \cap B(x, (k-1)\rho)$, we deduce from (30) that

$$\operatorname{dist}(y, xV) \leq 8(k-1)C_2\varepsilon_2^{1/\mathfrak{Q}}\rho. \tag{31}$$

In this paragraph we prove that if there exists a point $v \in V$ such that $v_1 \neq 0$ and $|\pi_1(P_W v)| \leq \theta|v_1|$ for some $0 < \theta < 1$, then

$$|\langle \mathfrak{n}(V), \mathfrak{n}(W)\rangle| \leq \theta/\sqrt{1-\theta^2}. \tag{32}$$

We note that the assumptions on v_1 imply that

$$|v_1|^2 - \langle \mathfrak{n}(W), v_1\rangle^2 = |v_1 - \langle \mathfrak{n}(W), v_1\rangle \mathfrak{n}(W)|^2 = |\pi_{\mathscr{W}} v_1|^2 = |\pi_1(P_W v)|^2 \leq \theta^2|v_1|^2, \tag{33}$$

where $\pi_{\mathscr{W}}$ is the projection in V_1 onto $W \cap V_1$ that was defined in Proposition 1.11. By means of a few omitted algebraic manipulations of (33), we conclude that $\sqrt{1-\theta^2}|v_1| \le |\langle \mathfrak{n}(W), v_1\rangle|$. Finally, since $\langle \mathfrak{n}(V), v_1\rangle = 0$, thanks to (33) and the Cauchy–Schwartz inequality, we have

$$\begin{aligned}\theta|v_1| &\ge |\langle \pi_{\mathscr{W}} v_1, \mathfrak{n}(V)\rangle| = |\langle v_1 - \langle \mathfrak{n}(W), v_1\rangle \mathfrak{n}(W), \mathfrak{n}(V)\rangle| \\ &= |\langle \mathfrak{n}(W), v_1\rangle\langle \mathfrak{n}(V), \mathfrak{n}(W)\rangle| \ge \sqrt{1-\theta^2}|v_1|\,|\langle \mathfrak{n}(V), \mathfrak{n}(W)\rangle|. \end{aligned} \tag{34}$$

It is immediate to see that (34) is equivalent to (32), proving the claim.

Given $x, y \in E(\vartheta, \gamma)$ and $V, W \in \mathrm{Gr}(\mathfrak{Q}-1)$ as above, in this paragraph using the counterassumption $x^{-1}y \notin C_W(\alpha)$ we construct a $v \in V$ with $v_1 \neq 0$ that satisfies the bound $|\pi_1(P_W v)| \le \theta|v_1|$ for a suitably small θ. Since $y \notin xC_W(\alpha)$, thanks to Proposition 1.11 we have

$$|\pi_1(P_W(x^{-1}y))| \le \|P_W(x^{-1}y)\| < \alpha^{-1}\|P_{\mathfrak{n}(W)}(x^{-1}y)\| = \alpha^{-1}|\langle \mathfrak{n}(W), \pi_1(x^{-1}y)\rangle| \le \alpha^{-1}|\pi_1(x^{-1}y)|.$$

Defined v to be the point of V for which $d(y, xv) = \mathrm{dist}(y, xV)$, thanks to (31) and the fact that $y \in B\big(x, \frac{1}{8}(k-1)\rho\big)$ we have

$$\|v\| \le d(xv, y) + d(y, x) \le \mathrm{dist}(y, xV) + \tfrac{1}{8}((k-1)\rho) \le \big(8C_2\varepsilon_2^{1/\mathfrak{Q}} + \tfrac{1}{8}\big)k\rho < (k-1)\rho,$$

where the last inequality comes from the choice of ε_2. Furthermore, thanks to (29) and (31) we have

$$\begin{aligned}\tfrac{1}{4}R\rho \le \big(\tfrac{1}{2}R - 8C_2k\varepsilon_2^{1/\mathfrak{Q}}\big)\rho &\le |\pi_1(x^{-1}y)| - d(y, xv) \le |\pi_1(x^{-1}y)| - |\pi_1(y^{-1}xv)| \\ &\le |\pi_1(x^{-1}y) - \pi_1(y^{-1}xv)| = |v_1|, \end{aligned} \tag{35}$$

and where the first inequality above, comes from the choice of ε_2. Let us prove that v satisfies the inequality

$$|\pi_1(P_W v)| \le 4R^{-1}k(16C_2\varepsilon_2^{1/\mathfrak{Q}} + 2^{6+N}\zeta^{-2}\alpha^{-1})|v_1|. \tag{36}$$

Since $x^{-1}y \notin C_W(\alpha)$, thanks to Proposition 1.11 we have

$$\begin{aligned}|\pi_1(P_W(v))| &\le |\pi_1(P_W(v)) - \pi_1(P_W(x^{-1}y))| + |\pi_1(P_W(x^{-1}y))| \\ &\le |\pi_1(P_W(y^{-1}xv))| + \|P_W(x^{-1}y)\| \le |\pi_1(P_W(y^{-1}xv))| + \alpha^{-1}\|P_{\mathfrak{N}(W)}(x^{-1}y)\| \\ &\le \|P_W(y^{-1}xv)\| + \alpha^{-1}|\pi_1(x^{-1}y)| \le \|P_W(y^{-1}xv)\| + 2^{6+N}\zeta^{-2}R\alpha^{-1}\rho, \end{aligned} \tag{37}$$

where the last inequality of the last line above comes from (28). Proposition 1.15 together with (31), (35) and (37) implies that

$$\begin{aligned}|\pi_1(P_W(v))| &\overset{(37)}{\le} \|P_W(y^{-1}xv)\| + 2^{6+N}\zeta^{-2}R\alpha^{-1}\rho \le 2\|y^{-1}xv\| + 2^{6+N}\zeta^{-2}R\alpha^{-1}\rho \\ &= 2\,\mathrm{dist}(y, xV) + 2^{6+N}\zeta^{-2}R\alpha^{-1}\rho \overset{(31)}{\le} (16C_2(k-1)\varepsilon_2^{1/\mathfrak{Q}} + 2^{6+N}\zeta^{-2}R\alpha^{-1})\rho \\ &\overset{(35)}{\le} 4R^{-1}k(16C_2\varepsilon_2^{1/\mathfrak{Q}} + 2^{6+N}\zeta^{-2}\alpha^{-1})|v_1| =: \theta(\alpha, \varepsilon_2)|v_1|. \end{aligned} \tag{38}$$

Thanks to the choice of the constants α_0, ε_2, R and k together with some elementary algebraic computations that we omit, it is possible to prove that $\sqrt{1-\theta(\alpha, \varepsilon_2)^2} \ge \frac{1}{2}$. Since $|\pi_1(P_W(v))| \le \theta(\alpha, \varepsilon_2)|\pi_1(v)|$, we deduce thanks to (32) that

$$|\langle \mathfrak{n}(V), \mathfrak{n}(W)\rangle| \leq \frac{\theta(\alpha,\varepsilon_2)}{\sqrt{1-\theta(\alpha,\varepsilon_2)^2}} \leq 2\theta(\alpha,\varepsilon_2). \tag{39}$$

Let us take a step back and let us examine what we have shown so far. Starting from the absurd hypothesis $y^{-1}x \in C_W(\alpha)$ we have shown that there is a nonnull $v \in V$ with $|\pi_1(P_W v)| \leq \theta(\alpha,\varepsilon_2)|v|$. This can be alternatively read as the fact that the normals $\mathfrak{n}(V)$ and $\mathfrak{n}(W)$ of V and W respectively are almost orthogonal. However, one should expect this orthogonality to be incompatible with (27).

Let us prove that (39) is in contradiction with (27). Choose some $z \in B\big(x, \frac{1}{8}(k-1)\rho\big) \cap E(\vartheta,\gamma)$ and note that

$$\begin{aligned}
|\langle \mathfrak{n}(V), \pi_1(P_W(x^{-1}z))\rangle| &= |\langle \mathfrak{n}(V), \pi_{\mathscr{W}}(z_1 - x_1)\rangle| \\
&\leq |\langle \mathfrak{n}(V), z_1 - x_1\rangle| + |\langle \mathfrak{n}(V), \pi_{\mathfrak{n}(W)}(z_1 - x_1)\rangle| \\
&\leq |\langle \mathfrak{n}(V), z_1 - x_1\rangle| + |\langle \mathfrak{n}(V), \mathfrak{n}(W)\rangle|\,|\langle z_1 - x_1, \mathfrak{n}(W)\rangle| \\
&\leq \|P_{\mathfrak{N}(V)}(x^{-1}z)\| + d(x,z)|\langle \mathfrak{n}(V), \mathfrak{n}(W)\rangle| \\
&= \operatorname{dist}(z, xV) + d(x,z)|\langle \mathfrak{n}(V), \mathfrak{n}(W)\rangle|,
\end{aligned} \tag{40}$$

where the last identity comes from Proposition 1.15. Inequalities (30), (39), (40) and the choice of z imply that

$$\begin{aligned}
|\langle \mathfrak{n}(V), \pi_1(P_W(x^{-1}z))\rangle| &\overset{(40)}{\leq} \operatorname{dist}(z, xV) + d(x,z)|\langle \mathfrak{n}(V), \mathfrak{n}(W)\rangle| \\
&\overset{(30),(39)}{\leq} 8C_2k\varepsilon_2^{1/\mathfrak{Q}}\rho + 2\theta(\alpha,\varepsilon_2)d(x,z) \leq 8C_2k\varepsilon_2^{1/\mathfrak{Q}}\rho + 2\theta(\alpha,\varepsilon_2)k\rho.
\end{aligned} \tag{41}$$

Furthermore, defining $\mathfrak{n} := \pi_{\mathscr{W}}(\mathfrak{n}(V))$, it is immediate to see from (39) that $|\mathfrak{n} - \mathfrak{n}(V)| \leq 2\theta(\alpha,\varepsilon_2)$, which yields thanks to the triangular inequality and Proposition 1.15 the bound

$$\begin{aligned}
|\langle \mathfrak{n}, \pi_1(P_W(x^{-1}z))\rangle| &\leq |\langle \mathfrak{n}(V), \pi_1(P_W(x^{-1}z))\rangle| + |\mathfrak{n} - \mathfrak{n}(V)|\,|\pi_1(P_W(x^{-1}z))| \\
&\leq |\langle \mathfrak{n}(V), \pi_1(P_W(x^{-1}z))\rangle| + |\mathfrak{n} - \mathfrak{n}(V)|\,\|P_W(x^{-1}z)\| \\
&\overset{(41)}{\leq} (8C_2k\varepsilon_2^{1/\mathfrak{Q}}\rho + 2\theta(\alpha,\varepsilon_2)k\rho) + 4\theta(\alpha,\varepsilon_2)k\rho \leq 8(C_2\varepsilon_2^{1/\mathfrak{Q}} + 3\theta(\alpha,\varepsilon_2))k\rho.
\end{aligned} \tag{42}$$

For the sake of notation, we introduce the set

$$S := \{w \in W : |\langle \mathfrak{n}, w_1\rangle| \leq 8(C_2\varepsilon_2^{1/\mathfrak{Q}} + 3\theta(\alpha,\varepsilon_2))k\rho\}.$$

The bound (42) implies that the projection of $x^{-1}E(\vartheta,\gamma) \cap B\big(0, \frac{1}{8}(k-1)\rho\big)$ on W is contained in S, which is a very narrow strip around $V \cap W$ inside W. Furthermore, we recall that from Proposition 1.15 we have

$$P_W\big(B\big(0, \tfrac{1}{8}(k-1)\rho\big)\big) \subseteq B\big(0, 2\cdot\tfrac{1}{8}(k-1)\rho\big). \tag{43}$$

Finally, putting together (42) and (43), we deduce that

$$\begin{aligned}
P_W\big(x^{-1}E(\vartheta,\gamma) \cap B\big(0, \tfrac{1}{8}(k-1)\rho\big)\big) &\subseteq P_W(x^{-1}E(\vartheta,\gamma)) \cap P_W\big(B\big(0, \tfrac{1}{8}(k-1)\rho\big)\big) \\
&\subseteq S \cap B\big(0, \tfrac{1}{4}(k-1)\rho\big).
\end{aligned} \tag{44}$$

Completing $\{\mathfrak{n}(W), \mathfrak{n}\}$ to an orthonormal basis $\mathcal{E} := \{\mathfrak{n}(W), \mathfrak{n}/|\mathfrak{n}|, e_3, \dots, e_n\}$ of $\mathbb{R}^n$ satisfying (9), thanks to Remark 1.5 we have

$$S \cap B\big(0, 2 \cdot \tfrac{1}{8}(k-1)\rho\big) \subseteq S \cap \mathrm{Box}_{\mathcal{E}}\big(0, \tfrac{1}{4}k\rho\big). \tag{45}$$

The above inclusion together with Tonelli's theorem yields

$$\begin{aligned}
&\mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner W\big(P_W\big(x^{-1}E(\vartheta,\gamma) \cap B\big(0, \tfrac{1}{8}(k-1)\rho\big)\big)\big)\\
&\qquad\le \mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner W\big(S \cap B\big(0, \tfrac{1}{4}(k-1)\rho\big)\big) \le \mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner W\big(S \cap \mathrm{Box}_{\mathcal{E}}\big(0, \tfrac{1}{4}k\rho\big)\big)\\
&\qquad= 16(C_2\varepsilon_2^{1/\mathfrak{Q}} + 3\theta(\alpha,\varepsilon_2))k\rho \cdot 2^{n-2} \prod_{i=2}^{s} \epsilon_i^{-n_i} \big(\tfrac{1}{4}k\rho\big)^{\mathfrak{Q}-2}\\
&\qquad= 2^{n-2\mathfrak{Q}+8} \prod_{i=2}^{s} \epsilon_i^{-n_i} (C_2\varepsilon_2^{1/\mathfrak{Q}} + 3\theta(\alpha,\varepsilon_2))(k\rho)^{\mathfrak{Q}-1}.
\end{aligned} \tag{46}$$

The inclusion (44), the bound (46), Proposition 1.8 and the definition of A_0 finally imply that

$$\begin{aligned}
&\mathcal{S}^{\mathfrak{Q}-1} \llcorner W\big(P_W\big(x^{-1}E(\vartheta,\gamma) \cap B\big(0, \tfrac{1}{8}(k-1)\rho\big)\big)\big)\\
&\qquad\le \mathcal{S}^{\mathfrak{Q}-1} \llcorner W\big(S \cap B\big(0, \tfrac{1}{4}k\rho\big)\big) = \beta^{-1}\mathcal{H}^{n-1}_{\mathrm{eu}} \llcorner W\big(S \cap B\big(0, \tfrac{1}{4}k\rho\big)\big)\\
&\qquad\le \beta^{-1} 2^{n-2\mathfrak{Q}+8} \prod_{i=2}^{s} \epsilon_i^{-n_i} (C_2\varepsilon_2^{1/\mathfrak{Q}} + 3\theta(\alpha,\varepsilon_2))(k\rho)^{\mathfrak{Q}-1}\\
&\qquad= 2^{-10} C_5^{-2} \varepsilon_{\mathbb{G}}^{-1} A_0^{-(\mathfrak{Q}-1)} (C_2\varepsilon_2^{1/\mathfrak{Q}} + 3\theta(\alpha,\varepsilon_2))\rho^{\mathfrak{Q}-1},
\end{aligned} \tag{47}$$

where β is the constant introduced in Proposition 1.8 and where the last identity comes from the definitions of $\varepsilon_{\mathbb{G}}$ and A_0; see Notation 2.8. Furthermore, since $\mathcal{S}^{\mathfrak{Q}-1} \llcorner W(P_W(p * E) = \mathcal{S}^{\mathfrak{Q}-1} \llcorner W(P_W(E))$ for any measurable set E in $\mathbb{G}$, see for instance the proof in [Franchi and Serapioni 2016, Proposition 2.2], we deduce that

$$\begin{aligned}
&\mathcal{S}^{\mathfrak{Q}-1} \llcorner W\big(P_W\big(x^{-1}E(\vartheta,\gamma) \cap B\big(0, \tfrac{1}{8}(k-1)\rho\big)\big)\big)\\
&\qquad= \mathcal{S}^{\mathfrak{Q}-1} \llcorner W\big(P_W\big(\mathfrak{c}(Q)^{-1}E(\vartheta,\gamma) \cap B\big(\mathfrak{c}(Q)^{-1}x, \tfrac{1}{8}(k-1)\rho\big)\big)\big).
\end{aligned}$$

Thanks to the choice of k and the fact that $x \in Q$, we infer that $B(0,\rho) \subseteq B\big(\mathfrak{c}(Q)^{-1}x, \tfrac{1}{8}(k-1)\rho\big)$. Together with (27), this allows us to deduce that

$$\begin{aligned}
&\mathcal{S}^{\mathfrak{Q}-1} \llcorner W\big(P_W\big(x^{-1}E(\vartheta,\gamma) \cap B\big(0, \tfrac{1}{8}(k-1)\rho\big)\big)\big)\\
&\ge \mathcal{S}^{\mathfrak{Q}-1} \llcorner W(P_W(\mathfrak{c}(Q)^{-1}E(\vartheta,\gamma) \cap B(0,\rho))) \ge \mathcal{S}^{\mathfrak{Q}-1} \llcorner W(P_W(\mathfrak{c}(Q)^{-1}(E(\vartheta,\gamma) \cap Q))) \ge \frac{\rho^{\mathfrak{Q}-1}}{4C_5^2 A_0^{\mathfrak{Q}-1}}.
\end{aligned} \tag{48}$$

Putting together (47) and (48) we conclude that

$$2^8 \varepsilon_{\mathbb{G}} \le (C_2\varepsilon_2^{1/\mathfrak{Q}} + 3\theta(\alpha,\varepsilon_2)) = C_2\varepsilon_2^{1/\mathfrak{Q}} + 12R^{-1}k(2C_2\varepsilon_2^{1/\mathfrak{Q}} + 2^{6+N}\zeta^{-2}\alpha^{-1}).$$

The choice of ε_2 and α imply, with some algebraic computations that we omit, that the above inequality is false, showing that the assumption $y \notin xC_W(\alpha)$ is false. We have reached a contradiction, proving the proposition. ☐

2C. ***Flat tangents imply big projections.*** We recall that the measure ϕ is supposed to be supported on a compact set K and that for ϕ-almost every $x \in \mathbb{G}$ we assume that $0 < \Theta_*^{\mathfrak{Q}-1}(\phi, x) \leq \Theta^{\mathfrak{Q}-1,*}(\phi, x) < \infty$ and $\mathrm{Tan}_{\mathfrak{Q}-1}(\phi, x) \subseteq \mathfrak{M}$. This subsection is devoted to the proof of the following result, which asserts that hypothesis (ii) of Proposition 2.11 is satisfied by the measure $\phi \llcorner E(\vartheta, \gamma)$.

Theorem 2.13. *There exists a compact subset C of $E(\vartheta, \gamma)$ having big measure inside $E(\vartheta, \gamma)$ such that for any cube Q of sufficiently small diameter for which $(1-\varepsilon_3)\phi(Q) \leq \phi(Q \cap C)$, there exists a plane $\Pi(Q) \in \mathrm{Gr}(\mathfrak{Q}-1)$ such that*

$$\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q)}(Q \cap C)) \geq \frac{\operatorname{diam} Q^{\mathfrak{Q}-1}}{2A_0^{\mathfrak{Q}-1}}.$$

The compact set C will be constructed in Proposition 2.14 while the scale below which the thesis of Theorem 2.13 is known to hold will be determined in Lemma 2.16 together with the plane $\Pi(Q)$. The reader can find the precise statement of the above result in Theorem 2.28.

In the following it will be useful to reduce to a compact subset C of $E(\vartheta, \gamma)$ where the distance of ϕ from planes is uniformly small below a fixed scale.

Proposition 2.14. *For any $\mu \geq 4\vartheta$, there exists a $\nu \in \mathbb{N}$, a compact subset C of $\mathscr{E}(\mu, \nu)$ and an $\iota_0 \in \mathbb{N}$ such that*

(i) $\phi(K \setminus C) \leq 2\varepsilon_1 \phi(K)$,

(ii) $d_{x,4kr}(\phi, \mathfrak{M}) + d_{x,4kr}(\phi \llcorner E(\vartheta, \gamma), \mathfrak{M}) \leq 2^{-\mathfrak{Q}^2-\mathfrak{Q}}\varepsilon_2$ *for any $x \in C$ and any $0 < r \leq 2^{-\iota_0 N+5}/\gamma$.*

Proof. Since by assumption $\mathrm{Tan}_{\mathfrak{Q}-1}(\phi, x) \subseteq \mathfrak{M}$ for ϕ-almost every $x \in \mathbb{G}$, thanks to Proposition 2.4 we infer that the functions $f_r(x) := d_{x,4kr}(\phi, \mathfrak{M})$ converge ϕ-almost everywhere to 0 on K as r goes to 0. Thanks to Proposition 1.27, the same line of reasoning implies also that $f_r^{\vartheta,\gamma}(x) := d_{x,4kr}(\phi \llcorner E(\vartheta, \gamma), \mathfrak{M})$ converges ϕ-almost everywhere to 0 on $E(\vartheta, \gamma)$. Proposition 2.3 and Severini–Egoroff's theorem yield a compact subset C of $\mathscr{E}(\mu, \nu)$ such that $\phi(E(\vartheta, \gamma) \setminus C) \leq \varepsilon_1 \phi(E(\vartheta, \gamma))$ and such that the functions $x \mapsto d_{x,4kr}(\phi, \mathfrak{M}) + d_{x,4kr}(\phi \llcorner E(\vartheta, \gamma), \mathfrak{M})$ converge uniformly to 0 on C as r goes to 0. This directly implies both (i) and (ii) thanks to the choice of ϑ and γ. □

Notation 2.15. From now on *we consider the integer $\mu \geq 4C_4\vartheta$ and the compact set C and the natural numbers ν and ι_0 yielded by Proposition 2.14 to be fixed*. Furthermore, we define $\iota := \max\{\iota_0, \nu\}$.

The following lemma rephrases Propositions 2.6 and 2.7 into the language of dyadic cubes.

Lemma 2.16. *For any cube $Q \in \Delta(C, \iota)$ we have $\alpha(Q) \leq \varepsilon_2$. Furthermore, there exists a plane $\Pi(Q) \in \mathrm{Gr}(\mathfrak{Q}-1)$ for which*

(i) $$\sup_{w \in E(\vartheta,\gamma) \cap B(\mathfrak{c}(Q), k \operatorname{diam} Q/2)} \frac{\operatorname{dist}(w, \mathfrak{c}(Q)\Pi(Q))}{2k \operatorname{diam} Q} \leq C_2 \varepsilon_2^{1/\mathfrak{Q}}, \quad \textit{and}$$

(ii) *for any $w \in B\big(\mathfrak{c}(Q), \tfrac{1}{2}k \operatorname{diam} Q\big) \cap \mathfrak{c}(Q)\Pi(Q)$ we have $E(\vartheta, \gamma) \cap B(w, 3kC_2\varepsilon_2^{1/(\mathfrak{Q}+1)} \operatorname{diam} Q) \neq \varnothing$.*

Proof. Let $Q \in \Delta(C, \iota)$, fix an $x \in Q \cap C$ and define $\rho := \operatorname{diam} Q$. Thanks to Proposition 2.14 we know that

$$d_{x,4kr}(\phi, \mathfrak{M}) + d_{x,4kr}(\phi \llcorner E(\vartheta, \gamma), \mathfrak{M}) \leq 2^{-\mathfrak{Q}^2-\mathfrak{Q}}\varepsilon_2, \tag{49}$$

for any $r \leq 2^{-\iota N+5}/\gamma$. Thanks to Theorem A.2 (ii) we have that $\rho \leq 2^{-\iota N+5}/\gamma$ and thus by Proposition 2.3 we infer that

$$\begin{gathered} \tilde{d}_{\mathfrak{c}(Q),2k\rho}(\phi, \mathfrak{M}) \leq 2^{\mathfrak{Q}}\tilde{d}_{x,4k\rho}(\phi, \mathfrak{M}) \leq 2^{-\mathfrak{Q}^2}\varepsilon_2, \\ \tilde{d}_{\mathfrak{c}(Q),2k\rho}(\phi \llcorner E(\vartheta, \gamma), \mathfrak{M}) \leq 2^{\mathfrak{Q}}\tilde{d}_{x,4k\rho}(\phi \llcorner E(\vartheta, \gamma), \mathfrak{M}) \leq 2^{-\mathfrak{Q}^2}\varepsilon_2. \end{gathered} \tag{50}$$

The bounds in (50) together with Proposition 2.6 imply that $\alpha(Q) \leq 2^{-\mathfrak{Q}^2+1}\varepsilon_2 \leq \varepsilon_2$.

The proof of the second part of the statement is a little more delicate. Since C is a subset of $\mathscr{E}_{\vartheta,\gamma}(\mu, \nu)$, thanks to the choice of μ and ι, by Proposition A.5 we have that $\mathfrak{c}(Q) \in E(\vartheta, \gamma) \cap B(x, \rho)$. Let us choose $\Pi(Q) \in \Pi_\delta(x, 2k\rho)$, where $\delta := 2^{-\mathfrak{Q}^2-\mathfrak{Q}}$, and note that Propositions 1.17 (i), (ii) and 2.6 imply that for any $w \in E(\vartheta, \gamma) \cap B\big(\frac{1}{2}k\rho\big)$ we have

$$\begin{aligned} \operatorname{dist}(w, \mathfrak{c}(Q)V) &\leq \operatorname{dist}(w, xV) + \operatorname{dist}(xV, \mathfrak{c}(Q)V) = \operatorname{dist}(w, xV) + \operatorname{dist}(\mathfrak{c}(Q), xV) \\ &\leq 2 \cdot 2k\rho \cdot C_2(2^{-\mathfrak{Q}^2-\mathfrak{Q}}\varepsilon_2)^{1/\mathfrak{Q}} \leq 2k\rho \cdot 2^{-\mathfrak{Q}} \cdot C_2\varepsilon_2^{1/\mathfrak{Q}}, \end{aligned} \tag{51}$$

where the last inequality comes from (49). This concludes the proof of (i).

Let us move to the proof of (ii). For any $V \in \Pi_\delta(x, 2k\rho)$ and any $w \in B\big(0, \frac{1}{2}k\rho\big) \cap V$, we define

$$w^* := x^{-1}c(Q)wP_{\mathfrak{N}(V)}(\mathfrak{c}(Q)^{-1}x) = P_{\mathfrak{N}(V)}(\mathfrak{c}(Q)^{-1}x)^{-1}P_V(\mathfrak{c}(Q)^{-1}x)^{-1}wP_{\mathfrak{N}(V)}(\mathfrak{c}(Q)^{-1}x) \in V.$$

With a few computations that we omit, it is not difficult to see that

$$d(\mathfrak{c}(Q)w, xw^*) = \|P_{\mathfrak{N}(V)}(\mathfrak{c}(Q)^{-1}x)\| = \operatorname{dist}(\mathfrak{c}(Q), xV) \leq 2^{-(\mathfrak{Q}-2)}k\rho C_2\varepsilon_2^{1/\mathfrak{Q}}, \tag{52}$$

where the second identity follows from Proposition 1.17 and the last inequality from the second last inequality in (51). Thanks to the definition of w^*, the triangle inequality, Proposition 1.15 and the fact that $d(\mathfrak{c}(Q), x) \leq \rho$, the norm of w^* can be estimated as

$$\|w^*\| \leq 2\|P_{\mathfrak{N}(V)}(\mathfrak{c}(Q)^{-1}x)\| + \|P_V(\mathfrak{c}(Q)^{-1}x)\| + \|w\| \leq 2\rho + 2\rho + \tfrac{1}{2}k\rho < k\rho. \tag{53}$$

Thanks to inequalities (49) and (53) and Proposition 2.7, we infer that

$$B(xw^*, 2k\rho \cdot 2^{-\mathfrak{Q}}\varepsilon_2^{1/(\mathfrak{Q}+1)}) \cap E(\vartheta, \gamma) \neq \varnothing.$$

Finally, since $2^{1-\mathfrak{Q}} < C_2$, thanks to (52) we conclude that

$$E(\vartheta, \gamma) \cap B(\mathfrak{c}(Q)w, 3k\rho C_2\varepsilon_2^{1/(\mathfrak{Q}+1)}) \supseteq E(\vartheta, \gamma) \cap B(xw^*, 2k\rho \cdot 2^{-\mathfrak{Q}}\varepsilon_2^{1/(\mathfrak{Q}+1)}) \neq \varnothing. \qquad \square$$

The arguments we will use in the rest of the subsection to prove Proposition 2.18 through Theorem 2.28 follow from an adaptation of the techniques found in Chapter 2, §2 of [David and Semmes 1993a]. The first of such adaptations is the following definition, which is a way of saying that two cubes are close both in metric and in size terms:

Definition 2.17 (neighbor cubes). Let $A := 4A_0^2$ and let $Q_j \in \Delta_{i_j}^{\phi}(\vartheta,\gamma)$ be two cubes with $j = 1, 2$.[4] We say that Q_1 and Q_2 are *neighbors* if

$$\underline{\mathrm{dist}}(Q_1, Q_2) := \inf_{x\in Q_1, y\in Q_2} d(x, y) \underset{\text{(I)}}{\leq} A(\operatorname{diam} Q_1 + \operatorname{diam} Q_2) \quad \text{and} \quad |i_1 - i_2| \underset{\text{(II)}}{\leq} A.$$

Furthermore, in the following (for the sake of notation), for any $Q \in \Delta(C, \iota)$ we let

$$\mathfrak{n}(Q) := \mathfrak{n}(\Pi(Q)),$$

where $\Pi(Q) \in \mathrm{Gr}(\mathfrak{Q} - 1)$ is the plane yielded by Lemma 2.16.

Finally, two planes $V, W \in \mathrm{Gr}(\mathfrak{Q} - 1)$ are said to have *compatible orientations* if their normals $\mathfrak{n}(V), \mathfrak{n}(W) \in V_1$ are chosen in such a way that $\langle \mathfrak{n}(V), \mathfrak{n}(W)\rangle > 0$. By extension, we will say that two cubes $Q_1, Q_2 \in \Delta(C, \iota)$ have *compatible orientations* themselves if $\Pi(Q_1)$ and $\Pi(Q_2)$ are chosen to have compatible orientations.

Proposition 2.18. *Suppose that $Q_j \in \Delta_{i_j}^{\phi}(\vartheta,\gamma)$ for $j = 1, 2$. Then the following hold*:

(i) *If Q_1 is the parent of Q_2, then Q_1 and Q_2 are neighbors.*

(ii) *If Q_1 and Q_2 are neighbors for any nonnegative integer $k \leq \min\{i_1, i_2\}$, then their ancestors $\tilde{Q}_1 \in \Delta_{i_1-k}^{\phi}(\vartheta,\gamma)$ and $\tilde{Q}_2 \in \Delta_{i_2-k}^{\phi}(\vartheta,\gamma)$ are neighbors.*

(iii) *If $Q_1, Q_2 \in \Delta(E(\vartheta,\gamma), 1)$ are neighbors, then* $|\log(\operatorname{diam} Q_1/\operatorname{diam} Q_2)| \leq 2AN$.

Proof. Let us prove (i). Since $Q_2 \subseteq Q_1$, we have that (I) of Definition 2.17 follows immediately. On the other hand, since Q_1 is the parent of Q_2, Proposition A.4 implies that

$$|i_1 - i_2| \leq \lfloor 2\log_2 C_4/N(\mathfrak{Q} - 1)\rfloor + 1 \leq 4A_0^2 = A,$$

where the second inequality comes from the choice of A_0 (see Notation 2.8) and this proves (II) of Definition 2.17. In order to prove (ii), we first note that $|(i_1 - k) - (i_2 - k)| = |i_1 - i_2| \leq A$ and secondly that

$$\underline{\mathrm{dist}}(\tilde{Q}_1, \tilde{Q}_2) \leq \underline{\mathrm{dist}}(Q_1, Q_2) \leq A(\operatorname{diam} Q_1 + \operatorname{diam} Q_2) \leq A(\operatorname{diam} \tilde{Q}_1 + \operatorname{diam} \tilde{Q}_2).$$

In order to prove (iii), we just need to note that thanks to Theorem A.2 (ii), (v) we infer that

$$\left|\log \frac{\operatorname{diam} Q_1}{\operatorname{diam} Q_2}\right| \leq \left|\log \frac{2^{-Ni_1+5}/\gamma}{\zeta^2 2^{-Ni_2-1}/\gamma}\right| = (N|i_2 - i_1| + 6)\log 2 - 2\log\zeta \leq \log(C_6) \leq 8A_0^2 N = 2AN,$$

where the two last inequalities come from the choice of C_6 and A_0. □

Remark 2.19. If $Q \in \Delta(C, \iota)$ then $\mathfrak{c}(Q) \in E(\vartheta,\gamma)$ thanks to the choices of μ and ι in Notation 2.15 and Proposition A.5.

Remark 2.20. Note that if $Q_1, Q_2 \in \Delta(E(\vartheta,\gamma), 1)$ are neighbors, Proposition 2.18 (iii) implies that

$$e^{-2AN} \operatorname{diam} Q_2 \leq \operatorname{diam} Q_1 \leq e^{2AN} \operatorname{diam} Q_2.$$

[4]The symbol $\Delta_{i_j}^{\phi}(\vartheta,\gamma)$ denotes the i_j-th layer of dyadic cubes; see Theorem A.2.

Remark 2.20 explicitly tells us that if two cubes $Q_1, Q_2 \in \Delta(C, \iota)$ are neighbors, then they have comparable diameters which are in turn comparable with the distance of their diameters. The information we have on the measure, by means of Lemma 2.16, tells us that ϕ is well approximated by two planes V_1 and V_2 inside the balls $B_1 := B(\mathfrak{c}(Q_1), k\,\mathrm{diam}(Q_1))$ and $B_2 := B(\mathfrak{c}(Q_2), k\,\mathrm{diam}(Q_2))$, respectively. However, since we have chosen k in such a way that $k \gg A \approx \mathrm{dist}(\mathfrak{c}(Q_1), \mathfrak{c}(Q_2))/\,\mathrm{diam}(Q_1)$, the balls B_1 and B_2 have a big overlap while having approximately the same size. Hence, the planes V_1 and V_2 are in essence approximating the same portion of the measure and as a consequence they must be almost the same plane. This heuristic argument is formalized in the following:

Proposition 2.21. *Suppose that $Q_1, Q_2 \in \Delta(C, \iota)$ are two neighbor cubes. Then*

$$(1 - C_3\varepsilon_2^{2/(\mathfrak{Q}+1)})^{1/2} = (1 - 2^{20}(n_1 - 1)C_2^2\varepsilon_2^{2/(\mathfrak{Q}+1)})^{1/2} \le |\langle \mathfrak{n}(Q_1), \mathfrak{n}(Q_2)\rangle|.$$

Proof. Thanks to the definition of k, we have

$$A(\mathrm{diam}\, Q_1 + \mathrm{diam}\, Q_2) \le 2A \max\{\mathrm{diam}\, Q_1, \mathrm{diam}\, Q_2\} \le \tfrac{1}{4}k \max\{\mathrm{diam}\, Q_1, \mathrm{diam}\, Q_2\}.$$

Without loss of generality we can assume that $\mathrm{diam}\, Q_2 \le \mathrm{diam}\, Q_1$. Therefore, since the cubes Q_1 and Q_2 are supposed to be neighbors, we deduce that

$$\underline{\mathrm{dist}}(Q_1, Q_2) \le A(\mathrm{diam}\, Q_1 + \mathrm{diam}\, Q_2) \le \tfrac{1}{4}k\,\mathrm{diam}\, Q_1. \tag{54}$$

This implies that for any $z \in Q_1$, we have

$$\begin{aligned} \mathrm{dist}(z, Q_2) &\le \mathrm{diam}\, Q_1 + \inf_{y \in Q_1} \mathrm{dist}(y, Q_2) = \mathrm{diam}\, Q_1 + \underline{\mathrm{dist}}(Q_1, Q_2) \\ &\le \big(\tfrac{1}{4}k + 1\big)\,\mathrm{diam}\, Q_1 < \big(\tfrac{1}{2}k - 1\big)\,\mathrm{diam}\, Q_1. \end{aligned} \tag{55}$$

Inequality (55) implies that for any $z \in Q_1$ we have $Q_2 \subseteq B\big(z, \tfrac{1}{2}k\,\mathrm{diam}\, Q_1\big)$. This, together with Lemma 2.16 (i), implies that for any $w \in E(\vartheta, \gamma) \cap Q_2$ we have

$$\mathrm{dist}(w, \mathfrak{c}(Q_1)\Pi(Q_1)) \le 2C_2\varepsilon_2^{1/\mathfrak{Q}}k\,\mathrm{diam}\, Q_1. \tag{56}$$

We now claim that $B_2 := \big\{u \in \mathbb{G} : \mathrm{dist}(u, Q_2) \le \tfrac{1}{20}k\,\mathrm{diam}\, Q_2\big\} \subseteq B\big(\mathfrak{c}(Q_1), \tfrac{1}{2}k\,\mathrm{diam}\, Q_1\big)$. In order to prove this inclusion, let $u \in B_2$ and note that

$$\begin{aligned} &\mathrm{dist}(u, \mathfrak{c}(Q_1)) \\ &\quad\le \inf_{w \in Q_2} (d(u, w) + d(w, \mathfrak{c}(Q_1))) \le \inf_{w \in Q_2} d(u, w) + \mathrm{diam}\, Q_1 + \underline{\mathrm{dist}}(Q_1, Q_2) + \mathrm{diam}\, Q_2 \\ &\quad\underset{u \in B_2}{\le} \tfrac{1}{20}k\,\mathrm{diam}\, Q_2 + \mathrm{diam}\, Q_1 + \underline{\mathrm{dist}}(Q_1, Q_2) + \mathrm{diam}\, Q_2 \le \tfrac{1}{10}(3k + 20)\,\mathrm{diam}\, Q_1 < \tfrac{1}{2}k\,\mathrm{diam}\, Q_1, \end{aligned} \tag{57}$$

where the second last inequality comes from (54) and the assumption that Q_1 is the cube with the biggest diameter. Inequality (57) concludes the proof of the inclusion $B_2 \subseteq B\big(\mathfrak{c}(Q_1), \tfrac{1}{2}k\,\mathrm{diam}\, Q_1\big)$. The inclusion just proved, together with Remark 2.20, the fact that $Q_1, Q_2 \in \Delta(E(\vartheta, \gamma), \iota)$ and Lemma 2.16 (i), implies that for any $u \in E(\vartheta, \gamma) \cap B_2$ we have

$$\mathrm{dist}(u, \mathfrak{c}(Q_1)\Pi(Q_1)) \le 2C_2\varepsilon_2^{1/\mathfrak{Q}}k\,\mathrm{diam}\, Q_1 \le 2C_2e^{2NA}\varepsilon_2^{1/\mathfrak{Q}}k\,\mathrm{diam}\, Q_2. \tag{58}$$

Furthermore, thanks to Remark 2.19 we have $\mathfrak{c}(Q_2) \in B_2 \cap E(\vartheta, \gamma)$. Therefore, by Proposition 1.17 for any $u \in B_2 \cap E(\vartheta, \gamma)$ we conclude that

$$\begin{aligned}\mathrm{dist}(u, \mathfrak{c}(Q_2)\Pi(Q_1))&\\ &\leq \mathrm{dist}(u, \mathfrak{c}(Q_1)\Pi(Q_1)) + \mathrm{dist}(\mathfrak{c}(Q_2)\Pi(Q_1), \mathfrak{c}(Q_1)\Pi(Q_1))\\ &= \mathrm{dist}(u, \mathfrak{c}(Q_1)\Pi(Q_1)) + \mathrm{dist}(\mathfrak{c}(Q_2), \mathfrak{c}(Q_1)\Pi(Q_1)) \overset{(58)}{\leq} 4C_2 e^{2NA}\varepsilon_2^{1/\mathfrak{Q}} k \,\mathrm{diam}\, Q_2. \end{aligned} \tag{59}$$

Thanks to Lemma 2.16 (ii), we deduce that for any $y \in B\big(\mathfrak{c}(Q_2), \frac{1}{40}k \,\mathrm{diam}\, Q_2\big) \cap \mathfrak{c}(Q_2)\Pi(Q_2)$ there exists some $w(y)$ in $E(\vartheta, \gamma) \cap B(y, 3kC_2\varepsilon_2^{1/(\mathfrak{Q}+1)} \,\mathrm{diam}\, Q_2)$. Since by definition $\varepsilon_2 \leq ((k-20)/20C_2k)^{\mathfrak{Q}+1}$, we have

$$\begin{aligned}\mathrm{dist}(w(y), Q_2) &\leq \inf_{p \in Q_2} d(w(y), y) + d(y, \mathfrak{c}(Q_2)) + d(\mathfrak{c}(Q_2), p)\\ &\leq 3kC_2\varepsilon_2^{1/(\mathfrak{Q}+1)} \,\mathrm{diam}\, Q_2 + \tfrac{1}{40}k \,\mathrm{diam}\, Q_2 + \mathrm{diam}\, Q_2 \leq \tfrac{1}{20}k \,\mathrm{diam}\, Q_2, \end{aligned} \tag{60}$$

where the last inequality comes from the choice of k. Inequality (60) implies that $w(y) \in B_2$, and thanks to (59) we infer that

$$\mathrm{dist}(w(y), \mathfrak{c}(Q_2)\Pi(Q_1)) \leq 4C_2 e^{2NA}\varepsilon_2^{1/\mathfrak{Q}} k \,\mathrm{diam}\, Q_2.$$

Summing up, for any $y \in B\big(\mathfrak{c}(Q_2), \frac{1}{40}k \,\mathrm{diam}\, Q_2\big) \cap \mathfrak{c}(Q_2)\Pi(Q_2)$, we have

$$\begin{aligned}\mathrm{dist}(y, \mathfrak{c}(Q_2)\Pi(Q_1)) &\leq d(y, w(y)) + \mathrm{dist}(w(y), \mathfrak{c}(Q_2)\Pi(Q_1))\\ &\leq 3C_2\varepsilon_2^{1/(\mathfrak{Q}+1)} k \,\mathrm{diam}\, Q_2 + 4C_2 e^{2NA}\varepsilon_2^{1/\mathfrak{Q}} k \,\mathrm{diam}\, Q_2\\ &\leq (3C_2 + 4C_2 e^{2NA}\varepsilon_2^{1/(\mathfrak{Q}(\mathfrak{Q}+1))})\varepsilon_2^{1/(\mathfrak{Q}+1)} k \,\mathrm{diam}\, Q_2 \leq 6C_2\varepsilon_2^{1/(\mathfrak{Q}+1)} k \,\mathrm{diam}\, Q_2, \end{aligned} \tag{61}$$

where the last inequality comes from the choice of ε_2 and a few elementary algebraic computations that we omit. Furthermore, inequality (61) and Proposition 1.15 imply that

$$|\langle \pi_1(\mathfrak{c}(Q_2)^{-1}y), \mathfrak{n}(Q_1)\rangle| = \|P_{\mathfrak{n}(Q_1)}(\mathfrak{c}(Q_2)^{-1}y)\| = \mathrm{dist}(y, \mathfrak{c}(Q_2)\Pi(Q_1)) \leq 6C_2\varepsilon_2^{1/(\mathfrak{Q}+1)} k \,\mathrm{diam}\, Q_2. \tag{62}$$

Suppose $\{v_i\}_{i=1,\dots,n_1-1}$ are the orthonormal vectors of the first layer V_1 spanning the orthogonal complement of $\mathfrak{n}(Q_2)$ inside V_1, and let $y_j := \mathfrak{c}(Q_2)\delta_{k \,\mathrm{diam}\, Q_2/80}(v_j)$. Then, from inequality (62), we deduce that

$$\begin{aligned}1 = |\langle \mathfrak{n}(Q_1), \mathfrak{n}(Q_2)\rangle|^2 + \sum_{j=1}^{n_1-1} |\langle v_j, \mathfrak{n}(Q_1)\rangle|^2 &= |\langle \mathfrak{n}(Q_1), \mathfrak{n}(Q_2)\rangle|^2 + \sum_{j=1}^{n_1-1} \frac{|\langle \pi_1(\mathfrak{c}(Q_2)^{-1}y_j), \mathfrak{n}(Q_1)\rangle|^2}{(k \,\mathrm{diam}\, Q_2/80)^2}\\ &\leq |\langle \mathfrak{n}(Q_1), \mathfrak{n}(Q_2)\rangle|^2 + 2^{20}(n_1-1)C_2^2\varepsilon_2^{2/(\mathfrak{Q}+1)}. \qquad \square\end{aligned}$$

Proposition 2.22. *Let $Q_1, Q_2 \in \Delta(C, \iota)$ be neighbor cubes and suppose that $\Pi(Q_1)$ and $\Pi(Q_2)$, the planes yielded by Lemma 2.16, are chosen with compatible orientations. Then*

$$|\mathfrak{n}(Q_1) - \mathfrak{n}(Q_2)| \leq 2\sqrt{C_3}\varepsilon_2^{1/(\mathfrak{Q}+1)}. \tag{63}$$

Furthermore, denote by $\tilde{Q}_1$ and $\tilde{Q}_2$ the parent cubes of Q_1 and Q_2, respectively, and assume that the planes $\Pi(\tilde{Q}_1)$ and $\Pi(\tilde{Q}_2)$ have compatible orientations with $\Pi(Q_1)$ and $\Pi(Q_2)$, respectively. Then the $\Pi(Q_i)$ have compatible orientations if and only if the planes $\Pi(\tilde{Q}_i)$ do.

Proof. Since Q_1 and Q_2 are neighbors and have compatible orientations, by definition, $\langle\mathfrak{n}(Q_1),\mathfrak{n}(Q_2)\rangle\geq 0$. Thanks to Proposition 2.21 we infer that

$$|\mathfrak{n}(Q_1)-\mathfrak{n}(Q_2)|^2=2-2\langle\mathfrak{n}(Q_1),\mathfrak{n}(Q_2)\rangle\leq 2-2(1-C_3\varepsilon_2^{2/(\mathfrak{Q}+1)})^{1/2}\leq 2\sqrt{C_3}\varepsilon_2^{1/(\mathfrak{Q}+1)},$$

and (63) is proved. Let us move to the second part of the proposition. Thanks to Proposition 2.18, the pairs $\tilde{Q}_1$ and $\tilde{Q}_2$, Q_1 and $\tilde{Q}_1$, and Q_2 and $\tilde{Q}_2$ are neighbors as well. Therefore Proposition 2.21 implies that

$$\begin{aligned}\langle\mathfrak{n}(\tilde{Q}_1),\mathfrak{n}(\tilde{Q}_2)\rangle&=\langle\mathfrak{n}(Q_1),\mathfrak{n}(Q_2)\rangle+\langle\mathfrak{n}(\tilde{Q}_1)-\mathfrak{n}(Q_1),\mathfrak{n}(Q_2)\rangle+\langle\mathfrak{n}(\tilde{Q}_1),\mathfrak{n}(\tilde{Q}_2)-\mathfrak{n}(Q_2)\rangle\\&\geq(1-C_3\varepsilon_2^{2/(\mathfrak{Q}+1)})^{1/2}-4\sqrt{C_3}\varepsilon_2^{1/(\mathfrak{Q}+1)}\geq\tfrac{1}{10}.\end{aligned}$$

Conversely, if $\Pi(\tilde{Q}_1)$ and $\Pi(\tilde{Q}_2)$ have the same orientation, the same line of reasoning yields that the planes $\Pi(Q_1)$ and $\Pi(Q_2)$ have compatible orientations as well. □

Proposition 2.23. *It is possible to fix an orientation on the planes* $\{\Pi(Q):Q\in\Delta(C,\iota)\}$ *in such a way that*

$$|\mathfrak{n}(Q_1)-\mathfrak{n}(Q_2)|\leq\tfrac{1}{10},$$

whenever $Q_1,Q_2\in\Delta(C,\iota)$ *are neighbors and are contained in the same maximal cube* $Q_0\in\mathcal{M}(C,\iota)$*, where the set* $\mathcal{M}(C,\iota)$ *was introduced in Remark 2.10.*

Proof. Suppose $Q_i\in\Delta^{\phi}_{j_i}(\vartheta,\gamma)$ for $i=1,2$, and assume without loss of generality that $j_1\leq j_2$. Fix the normal of the plane $\Pi(Q_0)$, and determine the normals of all other planes $\Pi(Q)$ as Q varies in $\Delta(C,\iota)$ by demanding that the orientation of the cube Q is compatible with that of $\tilde{Q}$, its parent cube.

If $Q_1=Q_0$, let us consider the finite sequence $\{\tilde{Q}_i\}_{i=1,\dots,M}$ of ancestors of Q_2 for which $\tilde{Q}_1=Q_2$, $\tilde{Q}_M=Q_0$ and such that $\tilde{Q}_{i+1}$ is the parent of $\tilde{Q}_i$. Then the scalar product between $\mathfrak{n}(Q_0)$ and $\mathfrak{n}(Q_2)$ can be estimated as

$$\langle\mathfrak{n}(Q_0),\mathfrak{n}(Q_2)\rangle\geq\langle\mathfrak{n}(\tilde{Q}_2),\mathfrak{n}(Q_2)\rangle-\sum_{i=2}^{M}|\mathfrak{n}(\tilde{Q}_i)-\mathfrak{n}(\tilde{Q}_{i+1})|\geq(1-C_3\varepsilon_2^{2/(\mathfrak{Q}+1)})-2\sqrt{C_3}M\varepsilon_2^{1/(\mathfrak{Q}+1)},\quad(64)$$

where the last inequality comes from Propositions 2.21 and 2.22 and the fact that the orientation of $\tilde{Q}_i$ and $\tilde{Q}_{i+1}$ were chosen to be compatible. Since Q_0 and Q_2 were assumed to be neighbors, from Definition 2.17 (II) it follows that $M\leq A$ and thus, thanks to (64) and the choice of ε_2, we have

$$\langle\mathfrak{n}(Q_0),\mathfrak{n}(Q_2)\rangle\geq(1-C_3\varepsilon_2^{2/(\mathfrak{Q}+1)})-2\sqrt{C_3}A\varepsilon_2^{1/(\mathfrak{Q}+1)}>0.$$

This proves the statement if one of the cubes is Q_0. The proof of the general case can be obtained with the following argument. Thanks to Proposition 2.22, we know that the orientations of the planes $\Pi(Q_1)$ and $\Pi(Q_2)$ are compatible if and only if the orientations of $\Pi(\tilde{Q}_1)$ and $\Pi(\tilde{Q}_2)$, the planes relative to their parent cubes $\tilde{Q}_1$ and $\tilde{Q}_2$, are compatible.[5] Thus, taking the parents of the parents and so on, one can reduce to the case in which one of the cubes is Q_0. □

[5]Note that this is the case, since by construction we enforced that every element in $\Delta(C,\iota)$ has a compatible orientation with its parent cube.

Definition 2.24. For each cube $Q \in \Delta(C, \iota)$, we let

$$\begin{aligned} G_\pm(Q) &:= \mathfrak{c}(Q)\{u \in B(0, A_0 \operatorname{diam} Q) : \pm\langle \pi_1 u, \mathfrak{n}(Q)\rangle > A_0^{-1} \operatorname{diam} Q\} \\ &= \{u \in B(\mathfrak{c}(Q), A_0 \operatorname{diam} Q) : \pm\langle \pi_1 u - \pi_1(\mathfrak{c}(Q)), \mathfrak{n}(Q)\rangle > A_0^{-1} \operatorname{diam} Q\} \end{aligned}$$

and $G(Q) = G_+(Q) \cup G_-(Q)$. Furthermore, for any $\overline{Q} \in \mathcal{M}(C, \iota)$ we let

$$\mathfrak{G}_\pm(\overline{Q}) := \bigcup_{\substack{Q \in \Delta(C,\iota) \\ Q \subseteq \overline{Q}}} G_\pm(Q) \quad \text{and} \quad \mathfrak{G}(\overline{Q}) := \bigcup_{\substack{Q \in \Delta(C,\iota) \\ Q \subseteq \overline{Q}}} G(Q).$$

For any Q in the set $G(Q)$, there is a ball B with radius comparable with $\operatorname{diam}(Q)$ from which a tubular neighborhood T of the plane $\Pi(Q)$ has been subtracted. The following lemma tells us that our choice of parameters is sufficient to get the inclusion $B \cap E(\vartheta, \gamma) \subseteq T$:

Lemma 2.25. *For any cube Q of $\Delta(C, \iota)$ and any $x \in G(Q)$, we have*

$$\tfrac{1}{2} A_0^{-1} \operatorname{diam} Q \underset{(A)}{\leq} \operatorname{dist}(x, E(\vartheta, \gamma)) \underset{(B)}{\leq} A_0 \operatorname{diam} Q. \tag{65}$$

Proof. Since $A_0 \leq \frac{1}{4}k$, if we let $z \in E(\vartheta, \gamma)$ be the point realizing the minimum distance of x from $E(\vartheta, \gamma)$, we deduce that

$$d(x, z) = \operatorname{dist}(x, E(\vartheta, \gamma)) \leq d(x, \mathfrak{c}(Q)) \leq A_0 \operatorname{diam} Q, \tag{66}$$

where the first inequality above comes from the fact that $\mathfrak{c}(Q) \in E(\vartheta, \gamma)$ (see Remark 2.19) and the last inequality comes from the very definition of $G(Q)$. Note that inequality (66) proves (65) (B). Furthermore, since $1 + A_0 < \frac{1}{2}k$, the bound (66) also implies that $z \in B\big(\mathfrak{c}(Q), \frac{1}{2}k \operatorname{diam} Q\big) \cap E(\vartheta, \gamma)$ and thus, thanks to Lemma 2.16 (i), we deduce that

$$\operatorname{dist}(z, \mathfrak{c}(Q)\Pi(Q)) \leq 2C_2 \varepsilon_2^{1/\mathfrak{Q}} k \operatorname{diam} Q. \tag{67}$$

Let w be an element of $\Pi(Q)$ satisfying the identity $d(z, \mathfrak{c}(Q)w) = \operatorname{dist}(z, \mathfrak{c}(Q)\Pi(Q))$, and note that

$$\begin{aligned} \operatorname{dist}(x, E(\vartheta, \gamma)) = \operatorname{dist}(x, z) &\geq d(x, \mathfrak{c}(Q)w) - d(\mathfrak{c}(Q)w, z) \\ &\geq \operatorname{dist}(\mathfrak{c}(Q)^{-1}x, \Pi(Q)) - \operatorname{dist}(z, \mathfrak{c}(Q)\Pi(Q)) \\ &\underset{(67)}{\geq} |\langle \mathfrak{n}(Q), \pi_1(\mathfrak{c}(Q)^{-1}x)\rangle| - 2C_2\varepsilon_2^{1/\mathfrak{Q}} k \operatorname{diam} Q \\ &\geq A_0^{-1} \operatorname{diam} Q - 2C_2\varepsilon_2^{1/\mathfrak{Q}} k \operatorname{diam} Q \geq \tfrac{1}{2} A_0^{-1} \operatorname{diam} Q, \end{aligned} \tag{68}$$

where the second last inequality used the fact that $x \in G(Q)$ and the last inequality used the choice of ε_2 and A_0. □

The following is a disconnection result for $\mathfrak{G}(\overline{Q})$. It tells us that $\mathfrak{G}_+(\overline{Q})$ and $\mathfrak{G}_-(\overline{Q})$ can be regarded as two *sides* of $\mathfrak{G}(\overline{Q})$ in the same way that $G_+(Q)$ and $G_-(Q)$ are the two sides of $G(Q)$. The intuitive idea for which this phenomenon occurs is the following. First, if $Q_1, Q_2 \in \Delta(C, \iota)$ are two cubes contained in Q such that $G_+(Q_1) \cap G_-(Q_2) \neq \varnothing$, then Lemma 2.25 implies that Q_1 and Q_2 must be neighbors.

Since Q_1 and Q_2 are neighbors, the approximating planes $\Pi(Q_1)$ and $\Pi(Q_2)$ are very close thanks to Proposition 2.21. In particular, $G_+(Q_1)$ and $G_-(Q_2)$ are in essence on opposite sides of a plane and thus they cannot intersect, resulting in a contradiction.

Lemma 2.26. *For any $\overline{Q} \in \mathcal{M}(C, \iota)$ we have that the sets $\mathfrak{G}_\pm(\overline{Q})$ are open and $\mathfrak{G}_+(\overline{Q}) \cap \mathfrak{G}_-(\overline{Q}) = \varnothing$.*

Proof. The fact that the $\mathfrak{G}_\pm(\overline{Q})$ are open sets follows immediately from the definitions of the $G_\pm(Q)$. Suppose that $\mathfrak{G}_+(\overline{Q}) \cap \mathfrak{G}_-(\overline{Q}) \neq \varnothing$. Then we can find two cubes $Q_1, Q_2 \in \Delta(C, \iota)$ contained in $\overline{Q}$ such that $G_+(Q_1) \cap G_-(Q_2) \neq \varnothing$ and let x be a point of intersection. In order to fix notations, we also suppose that $Q_i \in \Delta^{\phi}_{j_i}(\vartheta, \gamma)$ for $i = 1, 2$. Thanks to the definition of $G_\pm(Q)$, we immediately deduce that

$$B(\mathfrak{c}(Q_1), A_0 \operatorname{diam} Q_1) \cap B(\mathfrak{c}(Q_2), A_0 \operatorname{diam} Q_2) \neq \varnothing. \tag{69}$$

This in particular implies that $\underline{\operatorname{dist}}(Q_1, Q_2) \le 2A_0(\operatorname{diam} Q_1 + \operatorname{diam} Q_2)$. Therefore, since $2A_0 \le A$, we have that Q_1 and Q_2 satisfy condition (I) of Definition 2.17. Furthermore, since by construction $x \in G_+(Q_1) \cap G_-(Q_2)$, Lemma 2.25 implies that

$$\frac{\operatorname{diam} Q_1}{2A_0} \le \operatorname{dist}(x, E(\vartheta, \gamma)) \le A_0 \operatorname{diam} Q_1 \quad \text{and} \quad \frac{\operatorname{diam} Q_2}{2A_0} \le \operatorname{dist}(x, E(\vartheta, \gamma)) \le A_0 \operatorname{diam} Q_2. \tag{70}$$

Putting together the bounds in (70), we infer that

$$(2A_0^2)^{-1} \le \frac{\operatorname{diam} Q_1}{\operatorname{diam} Q_2} \le 2A_0^2. \tag{71}$$

Thanks to (71) and Theorem A.2 (ii), (v) we have that

$$(2A_0^2)^{-1} \le \frac{\operatorname{diam} Q_1}{\operatorname{diam} Q_2} \le \frac{2^{-j_1 N+5}/\gamma}{\zeta^2 2^{-j_2 N-1}/\gamma} \quad \text{and} \quad \frac{\zeta^2 2^{-j_1 N-1}/\gamma}{2^{-j_2+5}/\gamma} \le \frac{\operatorname{diam} Q_1}{\operatorname{diam} Q_2} \le 2A_0^2. \tag{72}$$

Finally, thanks to the bounds in (72) together with some computations that we omit, we deduce that

$$|j_2 - j_1| \le \frac{\log(2^7 \zeta^{-2} A_0^2)}{N \log 2} \le \log A_0,$$

where the last inequality comes from the choice of A_0. Since $A_0 \ge 2$, we infer that $|j_2 - j_1| \le A$, proving condition (II) of Definition 2.17. This concludes the proof that Q_1 and Q_2 are neighbors.

Now that we know that Q_1 and Q_2 are neighbors, (69) together with Proposition 2.18 (iii) implies that

$$\begin{aligned} d(\mathfrak{c}(Q_1), \mathfrak{c}(Q_2)) &\le d(\mathfrak{c}(Q_1), x) + d(x, \mathfrak{c}(Q_2)) \le A_0(\operatorname{diam} Q_1 + \operatorname{diam} Q_2) \\ &\le A_0(1 + e^{2NA}) \operatorname{diam} Q_2 < \tfrac{1}{2} k \operatorname{diam} Q_2, \end{aligned} \tag{73}$$

where the last inequality comes from the choice of k and of A. Since by (73) and Remark 2.19, we have $\mathfrak{c}(Q_2) \in E(\vartheta, \gamma) \cap B\big(\mathfrak{c}(Q_1), \tfrac{1}{2} k \operatorname{diam} Q_2\big)$, thanks to Lemma 2.16 (i) and Remark 2.20, we deduce that

$$\operatorname{dist}(\mathfrak{c}(Q_2), \mathfrak{c}(Q_1)\Pi(Q_1)) \le 2C_2 k \varepsilon_2^{1/\mathfrak{Q}} \operatorname{diam} Q_1 \le 2C_2 k e^{2NA} \varepsilon_2^{1/\mathfrak{Q}} \operatorname{diam} Q_2. \tag{74}$$

Furthermore, since Q_1 and Q_2 are neighbors, we infer by Proposition 2.22 that

$$|\mathfrak{n}(Q_1) - \mathfrak{n}(Q_2)| \le 2C_3 \varepsilon_2^{1/(\mathfrak{Q}+1)},$$

and this in turn implies that

$$\begin{aligned}
&\langle \pi_1(\mathfrak{c}(Q_1)^{-1}x), \mathfrak{n}(Q_1)\rangle \\
&\qquad = \langle \pi_1(\mathfrak{c}(Q_2)^{-1}x), \mathfrak{n}(Q_2)\rangle + \langle \pi_1(\mathfrak{c}(Q_2)^{-1}x), \mathfrak{n}(Q_1)-\mathfrak{n}(Q_2)\rangle + \langle \pi_1(\mathfrak{c}(Q_1)^{-1}\mathfrak{c}(Q_2)), \mathfrak{n}(Q_1)\rangle \\
&\qquad \leq -A_0^{-1}\operatorname{diam} Q_2 + |\pi_1(\mathfrak{c}(Q_2)^{-1}x)||\mathfrak{n}(Q_1)-\mathfrak{n}(Q_2)| + \operatorname{dist}(\mathfrak{c}(Q_2), \mathfrak{c}(Q_1)\Pi(Q_1)) \\
&\qquad \leq -A_0^{-1}\operatorname{diam} Q_2 + A_0 \operatorname{diam} Q_2 \cdot 2C_3\varepsilon_2^{1/(\mathfrak{Q}+1)} + 2C_2ke^{2NA}\varepsilon_2^{1/\mathfrak{Q}}\operatorname{diam} Q_2,
\end{aligned} \tag{75}$$

where third line above comes from the fact that $x \in G_-(Q_2)$ and the bound on $|\mathfrak{n}(Q_1)-\mathfrak{n}(Q_2)|$ discussed above while the last inequality follows from (74). The chain of inequalities in (75) and the definition of A imply that

$$\langle \pi_1(\mathfrak{c}(Q_1)^{-1}x), \mathfrak{n}(Q_1)\rangle \leq (-A_0^{-1} + A_0C_3\varepsilon_2^{1/(\mathfrak{Q}+1)} + C_2ke^{8NA_0^2}\varepsilon_2^{1/\mathfrak{Q}})\operatorname{diam} Q_2 \leq 0, \tag{76}$$

where the last inequality comes from the definition of ε_2 and some algebraic computations that we omit. This contradicts the fact that $x \in G_+(Q_1)$, proving that the assumption that $\mathfrak{G}(\overline{Q})_+ \cap \mathfrak{G}_-(\overline{Q}) \neq \varnothing$ was absurd. □

Let us take a step back and explain what the set $\mathfrak{G}(\overline{Q})$ is. Starting from a measure ϕ with flat blowups, in this section we constructed a countable family of pairs $(\mathfrak{c}(Q), \Pi(Q))$, parametrized by the cubes in $\Delta(C,\iota)$ inside $\overline{Q}$, of points of $\operatorname{supp}(\phi)$ and planes that are a good approximation of ϕ around $\mathfrak{c}(Q)$ at the scale $\operatorname{diam} Q$. From this family of pointed planes we built $\mathfrak{G}(\overline{Q})$, which should be imagined as the complement of the union of very thin tubular neighborhoods of the disks $\mathfrak{c}(Q)\Pi(Q) \cap B(\mathfrak{c}(Q), \operatorname{diam} Q)$. So, since the planes $\Pi(Q)$ are very efficiently approximating ϕ one should expect that $\phi(\mathfrak{G}(\overline{Q})) \approx 0$, allowing us to regard $\mathfrak{G}(\overline{Q})^c$ as an extension of $\operatorname{supp}(\phi)$ inside the ball $B(\mathfrak{c}(\overline{Q}), \operatorname{diam} \overline{Q})$. An extension, however, that can ultimately be considered and treated as a countable union of planes. The next proposition shows that $\operatorname{supp}(\phi)$ is quite dense inside $\mathfrak{G}(\overline{Q})^c$.

Proposition 2.27. *Let $\overline{Q} \in \mathcal{M}(C,\iota)$ and define*

$$I(\overline{Q}) := \bigcup_{\substack{Q\in\Delta(C,\iota)\\ Q\subseteq\overline{Q}}} B(\mathfrak{c}(Q), (A_0-2)\operatorname{diam} Q).$$

In addition, for any $x \in I(\overline{Q})$ we let

$$d(x) := \inf_{\substack{Q\in\Delta(C,\iota)\\ Q\subseteq\overline{Q}}} \operatorname{dist}(x, Q) + \operatorname{diam} Q. \tag{77}$$

Then $\operatorname{dist}(x, E(\vartheta,\gamma)) \leq 4A_0^{-1}d(x)$ *whenever $x \in I(\overline{Q}) \setminus \mathfrak{G}(\overline{Q})$.*

Proof. Fix some $x \in I(\overline{Q}) \setminus \mathfrak{G}(\overline{Q})$, and let $Q \subseteq \overline{Q}$ be a cube of $\Delta(C,\iota)$ such that

$$\operatorname{dist}(x, Q) + \operatorname{diam} Q \leq \tfrac{4}{3}d(x). \tag{78}$$

Let Q' be an ancestor of Q in $\Delta(C,\iota)$, possibly Q itself. Since $x \notin \mathfrak{G}(\overline{Q})$, then $x \notin G(Q')$ and, thanks to Proposition 1.15, we have

$$\operatorname{dist}(x, \mathfrak{c}(Q')\Pi(Q')) = |\langle \pi_1(\mathfrak{c}(Q')^{-1}x), \mathfrak{n}(Q')\rangle| \leq A_0^{-1}\operatorname{diam} Q', \tag{79}$$

where the last inequality is true provided that $\operatorname{dist}(x, \mathfrak{c}(Q')) < A_0 \operatorname{diam} Q'$. Since $x \in I(\overline{Q})$, there must exist some $\tilde{Q} \in \Delta(C, \iota)$ such that $\tilde{Q} \subseteq \overline{Q}$ and $x \in B(\mathfrak{c}(\tilde{Q}), (A_0 - 2) \operatorname{diam} \tilde{Q})$. This implies that

$$\operatorname{dist}(x, \mathfrak{c}(\overline{Q})) \leq d(x, \mathfrak{c}(\tilde{Q})) + d(\mathfrak{c}(\tilde{Q}), \mathfrak{c}(\overline{Q})) \leq (A_0 - 2) \operatorname{diam} \tilde{Q} + \operatorname{diam} \overline{Q} < A_0 \operatorname{diam} \overline{Q}. \tag{80}$$

Therefore the inequality $\operatorname{dist}(x, \mathfrak{c}(\overline{Q})) < A_0 \operatorname{diam} \overline{Q}$ is verified and hence (79) holds for $Q' = \overline{Q}$. Let $Q \subseteq Q_0 \subseteq \overline{Q}$ be the smallest cube in $\Delta(C, \iota)$ for which $\operatorname{dist}(x, \mathfrak{c}(Q_0)) < A_0 \operatorname{diam} Q_0$ holds.

Let $w \in \Pi(Q_0)$ be the point for which $d(x, \mathfrak{c}(Q_0)w) = \operatorname{dist}(x, \mathfrak{c}(Q_0)\Pi(Q_0))$, and note that the choice of Q_0 and the bound (79) imply that

$$\begin{aligned} \|w\| = \operatorname{dist}(\mathfrak{c}(Q_0)w, \mathfrak{c}(Q_0)) &\leq d(\mathfrak{c}(Q_0)w, x) + d(x, \mathfrak{c}(Q_0)) \\ &\leq \operatorname{dist}(x, \mathfrak{c}(Q_0)\Pi(Q_0)) + A_0 \operatorname{diam} Q_0 \\ &\leq A_0^{-1} \operatorname{diam} Q_0 + A_0 \operatorname{diam} Q_0 \leq 2A_0 \operatorname{diam} Q_0 < \tfrac{1}{2} k \operatorname{diam} Q_0, \end{aligned} \tag{81}$$

where the last inequality comes from the choice of A_0 and k made in Notation 2.8. Since $Q_0 \in \Delta(C, \iota)$, thanks to inequality (81) we have $\mathfrak{c}(Q_0)w \in B\big(\mathfrak{c}(Q_0), \tfrac{1}{2} k \operatorname{diam} Q_0\big)$ and thus Lemma 2.16 (ii) implies that $E(\vartheta, \gamma) \cap B(\mathfrak{c}(Q_0)w, 3kC_2\varepsilon_2^{1/(\mathfrak{Q}+1)} \operatorname{diam} Q_0) \neq \varnothing$. Therefore, since by definition of Q_0 the bound (79) holds with $Q' = Q_0$, we have

$$\begin{aligned} \operatorname{dist}(x, E(\vartheta, \gamma)) &\leq d(x, \mathfrak{c}(Q_0)w) + \operatorname{dist}(\mathfrak{c}(Q_0)w, E(\vartheta, \gamma)) \\ &= d(x, \mathfrak{c}(Q_0)\Pi(Q_0)) + \operatorname{dist}(\mathfrak{c}(Q_0)w, E(\vartheta, \gamma)) \\ &\leq A_0^{-1} \operatorname{diam} Q_0 + 3kC_2\varepsilon_2^{1/(\mathfrak{Q}+1)} \operatorname{diam} Q_0 \leq 2A_0^{-1} \operatorname{diam} Q_0, \end{aligned} \tag{82}$$

where the last inequality comes from the choice of ε_2.

If $Q_0 = Q$, then (78) implies that $\operatorname{dist}(x, E(\vartheta, \gamma)) \leq 2A_0^{-1} \operatorname{diam} Q_0 \leq 4A_0^{-1} d(x)$. Otherwise, let Q_1 be the child of Q_0 that contains Q. Thanks to the minimality of Q_0, we have $\operatorname{dist}(x, \mathfrak{c}(Q_1)) \geq A_0 \operatorname{diam} Q_1$, and thus

$$\begin{aligned} \operatorname{dist}(x, Q_1) \geq d(x, \mathfrak{c}(Q_1)) - \operatorname{diam} Q_1 &\geq (A_0 - 1) \operatorname{diam} Q_1 \\ &\geq \frac{A_0 - 1}{C_6} \operatorname{diam} Q_0 \geq \operatorname{diam} Q_0, \end{aligned} \tag{83}$$

where the second last inequality above follows from Proposition A.4 and the fact that Q_0 is the parent of Q_1, whereas the last inequality comes from the choice of A_0. Eventually, thanks to (78), (82), (83) and the fact that $Q \subseteq Q_1$, we deduce that

$$\begin{aligned} \operatorname{dist}(x, E(\vartheta, \gamma)) &\overset{(82)}{\leq} 2A_0^{-1} \operatorname{diam} Q_0 \overset{(83)}{\leq} 2A_0^{-1} \operatorname{dist}(x, Q_1) \\ &\leq 2A_0^{-1} \operatorname{dist}(x, Q) \overset{(78)}{\leq} 4A_0^{-1} d(x). \end{aligned}$$ □

The following is the main result of this subsection. Theorem 2.28 transforms the qualitative information on the relationship between $\mathfrak{G}(Q)^c$ and $\operatorname{supp}(\phi)$ yielded by Proposition 2.27 into a quantitative one, i.e., the bound on projections given in (84). The proof of the theorem reduces to constructing, for any (suitable) cube Q, a family of balls $\{B_i\}_{i \in \mathbb{N}}$ with the two following properties: First, the projection on $\Pi(Q)$ of $\operatorname{supp}(\phi) \cup \bigcup_i B_i$ contains an open set with measure comparable with $\operatorname{diam} Q^{\mathfrak{Q}-1}$. Second, the sum of the radii of the balls B_i is small and in particular the projection on planes of the set $\bigcup_i B_i$ has small measure compared to diam $Q^{\mathfrak{Q}-1}$. The construction of the balls B_i, that the reader may imagine centered at points of $\mathfrak{G}(Q)^c$, relies on the one hand on the previously discussed fact that the set $\mathfrak{G}(\overline{Q})^c$ can be regarded as a countable union of disks and on the other, that the holes of supp(ϕ), seen as a subset of $\mathfrak{G}(\overline{Q})^c$, are really small and patching them does not require too much measure.

Theorem 2.28. *For any cube $Q \in \Delta(C, \iota)$ such that $(1-\varepsilon_3)\phi(Q) \le \phi(Q \cap C)$, we have*

$$\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q)}(Q \cap C)) \ge \frac{\operatorname{diam} Q^{\mathfrak{Q}-1}}{2A_0^{\mathfrak{Q}-1}}. \tag{84}$$

Proof. Let $Q_0 \in \Delta(C, \iota)$ be such that $(1-\varepsilon_3)\phi(Q_0) \le \phi(Q_0 \cap C)$, and define

$$F(Q_0) := C \cap Q_0 \cup \bigcup_{Q \in \mathscr{I}(Q_0)} B(\mathfrak{c}(Q), 2C_6 \operatorname{diam} Q),$$

where $\mathscr{I}(Q_0)$ is a family of maximal cubes $Q \in \Delta(E(\vartheta, \gamma), \iota)$ such that $Q \subseteq Q_0$ and $Q \notin \Delta(C, \iota)$. As a first step, we estimate the size of the projection of the balls $\bigcup_{Q \in \mathscr{I}(Q_0)} B(\mathfrak{c}(Q), C_6 \operatorname{diam} Q)$. Thanks to Proposition 1.18 we have

$$\mathcal{S}^{\mathfrak{Q}-1}\Bigg(P_{\Pi(Q)}\bigg(\bigcup_{Q \in \mathscr{I}(Q_0)} B(\mathfrak{c}(Q), 2C_6 \operatorname{diam} Q)\bigg)\Bigg) \le 2^{\mathfrak{Q}-1} c(\Pi(Q_0)) C_6^{\mathfrak{Q}-1} \sum_{Q \in \mathscr{I}(Q_0)} \operatorname{diam} Q^{\mathfrak{Q}-1}. \tag{85}$$

We now need to estimate the sum in the right-hand side of (85). Since the cubes in $\mathscr{I}(Q_0)$ are disjoint and they are contained in $\Delta(E(\vartheta, \gamma), \iota)$, thanks to Remark A.3 and the fact that $(1-\varepsilon_3)\phi(Q_0) \le \phi(Q_0 \cap C)$, we deduce that

$$\begin{aligned} C_5^{-1} \sum_{Q \in \mathscr{I}(Q_0)} \operatorname{diam} Q^{\mathfrak{Q}-1} &\le \sum_{Q \in \mathscr{I}(Q_0)} \phi(Q) = \phi\bigg(\bigcup_{Q \in \mathscr{I}(Q_0)} Q\bigg) \\ &\le \phi(Q_0 \setminus C) \le \varepsilon_3 \phi(Q_0) \le \varepsilon_3 C_5 \operatorname{diam} Q_0^{\mathfrak{Q}-1}. \end{aligned} \tag{86}$$

Putting together (85) and (86), we conclude that

$$\begin{aligned} \mathcal{S}^{\mathfrak{Q}-1}\Bigg(P_{\Pi(Q)}\bigg(\bigcup_{Q \in \mathscr{I}(Q_0)} B(\mathfrak{c}(Q), 2C_6 \operatorname{diam} Q)\bigg)\Bigg) &\le 2^{\mathfrak{Q}-1} c(\Pi(Q_0)) C_5^2 \varepsilon_3 C_6^{\mathfrak{Q}-1} \operatorname{diam} Q_0^{\mathfrak{Q}-1} \\ &\le \frac{c(\Pi(Q_0))}{2A_0^{\mathfrak{Q}-1}} \operatorname{diam} Q_0^{\mathfrak{Q}-1}, \end{aligned} \tag{87}$$

where the last inequality comes from the choice of ε_3; see Notation 2.8.

In this first part of the proof of the theorem we have constructed the family of balls B_i mentioned in the introductory paragraph to the statement of the theorem and we have also proved the second necessary property of the B_i, that is the smallness of the measure of their projection. The rest of the proof will be devoted to proving that supp$(\phi) \cup \bigcup_i B_i$ has big projections.

More precisely, the next step in the proof of the theorem is to show that

$$\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(F(Q_0))) \geq \frac{c(\Pi(Q_0))\operatorname{diam} Q_0^{\mathfrak{Q}-1}}{A_0^{\mathfrak{Q}-1}}. \tag{88}$$

In order to ease notations in the following we let $x = \mathfrak{c}(Q_0)\delta_{10A_0^{-1}\operatorname{diam} Q_0}(\mathfrak{n}(Q_0))$ and define

$$B_+ := B(x, A_0^{-1}\operatorname{diam} Q_0) \quad \text{and} \quad B_- := B(x, A_0^{-1}\operatorname{diam} Q_0)\delta_{20A_0^{-1}\operatorname{diam} Q_0}(\mathfrak{n}(Q_0)^{-1}).$$

Before proceeding further with the proof of (88), we give a brief outline of what we are going to do, hoping to help the reader keep track of the purpose of each computation. As a first step towards the proof of (88), we prove that B_+ and B_- are contained in $G_+(Q_0)$ and $G_-(Q_0)$, respectively. Note that this implies that B_+ and B_- are each on one side of the plane $\Pi(Q_0)$. Let $\overline{Q}$ be the element of $\mathcal{M}(C, \iota)$ containing Q_0 and recall that by Lemma 2.26, $\mathfrak{G}_+(\overline{Q})$ and $\mathfrak{G}_-(\overline{Q})$ are disjoint open sets. This implies in particular that for any horizontal line parallel to the normal of the plane $\Pi(Q_0)$ with starting point in B_+ and end point in B_-, we can find a y in such a segment belonging to the complement of $\mathfrak{G}(Q_0)$. Our final step in the proof of (88) is to show that y belongs to $F(Q_0)$, thus proving the inclusion $P_{\Pi(Q_0)}(B_+) \subseteq P_{\Pi(Q_0)}(F(Q_0))$ and in turn our claim.

Let us proceed with the proof of (88). We will prove that $B_+ \subseteq G_+(Q_0)$ and $B_- \subseteq G_-(Q_0)$ separately, since the computations differ.

Let us begin with the proof of the inclusion $B_+ \subseteq G_+(Q_0)$. For any $\Delta \in \mathbb{G}$ such that $\|\Delta\| \leq A_0^{-1}\operatorname{diam} Q_0$, we have

$$d(\mathfrak{c}(Q_0), x\Delta) = \|\delta_{10A_0^{-1}\operatorname{diam} Q_0}(\mathfrak{n}(Q_0))\Delta\| \leq 11A_0^{-1}\operatorname{diam} Q_0 \leq A_0 \operatorname{diam} Q_0. \tag{89}$$

In addition, the choices of x and Δ imply that

$$\begin{aligned}\langle \pi_1(\mathfrak{c}(Q_0)^{-1}x\Delta), \mathfrak{n}(Q_0)\rangle &= \langle \pi_1(\delta_{10A_0^{-1}\operatorname{diam} Q_0}(\mathfrak{n}(Q_0))\Delta), \mathfrak{n}(Q_0)\rangle \\ &= 10A_0^{-1}\operatorname{diam} Q_0 + \langle \pi_1\Delta, \mathfrak{n}(Q_0)\rangle \geq 9A_0^{-1}\operatorname{diam} Q_0. \end{aligned}\tag{90}$$

The bounds (89) and (90) together with the definitions of B_+ and $G_+(Q_0)$ finally imply that $B_+ \subseteq G_+(Q_0)$.

Let us prove that $B_- \subseteq G_-(Q_0)$. Similar to the previous case, for any $\|\Delta\| \leq A_0^{-1}\operatorname{diam} Q_0$, we have

$$\begin{aligned} d(\mathfrak{c}(Q_0), x\Delta\delta_{20A_0^{-1}\operatorname{diam} Q_0}(\mathfrak{n}(Q_0)^{-1})) &= \|\delta_{10A_0^{-1}\operatorname{diam} Q_0}(\mathfrak{n}(Q_0))\Delta\delta_{20A_0^{-1}\operatorname{diam} Q_0}(\mathfrak{n}(Q_0)^{-1})\| \\ &\leq 31A_0^{-1}\operatorname{diam} Q_0 \leq A_0 \operatorname{diam} Q_0. \end{aligned}\tag{91}$$

Once again, the choices of x and Δ imply that

$$\begin{aligned} &\langle \pi_1(\mathfrak{c}(Q_0)^{-1}x\Delta\delta_{20A_0^{-1}\operatorname{diam} Q_0}(\mathfrak{n}(Q_0)^{-1})), \mathfrak{n}(Q_0)\rangle \\ &\qquad = \langle \pi_1(\delta_{10A_0^{-1}\operatorname{diam} Q_0}(\mathfrak{n}(Q_0))\Delta\delta_{20A_0^{-1}\operatorname{diam} Q_0}(\mathfrak{n}(Q_0)^{-1})), \mathfrak{n}(Q_0)\rangle \\ &\qquad = -10A_0^{-1}\operatorname{diam} Q_0 + \langle \pi_1\Delta, \mathfrak{n}(Q_0)\rangle \leq -9A_0^{-1}\operatorname{diam} Q_0. \end{aligned}\tag{92}$$

The bounds (91) and (92) together with the definitions of B_- and $G_-(Q_0)$ show that $B_- \subseteq G_-(Q_0)$.

Now that we have shown that B_+ and B_- lie on different sides of $\Pi(Q_0)$, we construct horizontal curves parallel to $\mathfrak{n}(Q_0)$ joining B_+ and B_- and we show that each one of these lines intersect $F(Q_0)$.

First of all, let $\overline{Q}$ be the unique cube in $\mathcal{M}(C,\iota)$ containing Q_0. Thanks to Lemma 2.26 we know that the sets $\mathfrak{G}_+(\overline{Q})$ and $\mathfrak{G}_-(\overline{Q})$ are disconnected. With this in mind, for any $a \in B_+$ we define the curve $\gamma_a : [0,1] \to \mathbb{G}$ as

$$\gamma_a(t) := a\delta_{20A_0^{-1}\operatorname{diam} Q_0 t}(\mathfrak{n}(Q_0)^{-1}).$$

By the definition of B_-, it is immediate to see that $\gamma_a(1) \in B_-$. On the other hand, since $\gamma_a(0) \in B_+$ and the image of γ_a is connected, we infer that γ_a must meet the complement of $\mathfrak{G}(\overline{Q})$ at $y = \gamma_a(s)$ for some $s \in (0,1)$.

We now prove that $y \in F(Q_0)$. First, we estimate the distance of y from $\mathfrak{c}(Q_0)$ as

$$\begin{aligned}
d(y,\mathfrak{c}(Q_0)) &\le d(a\delta_{20A_0^{-1}\operatorname{diam} Q_0 s}(\mathfrak{n}(Q_0)^{-1}),\mathfrak{c}(Q_0)) \le d(a,\mathfrak{c}(Q_0)) + 20A_0^{-1}\operatorname{diam} Q_0 s\\
&\le d(x,\mathfrak{c}(Q_0)) + d(x,a) + 20A_0^{-1}\operatorname{diam} Q_0 s\\
&\le 10A_0^{-1}\operatorname{diam} Q_0 + A_0^{-1}\operatorname{diam} Q_0 + 20A_0^{-1}\operatorname{diam} Q_0 s\\
&\le 40A_0^{-1}\operatorname{diam} Q_0 < (A_0-2)\operatorname{diam} Q_0,
\end{aligned} \tag{93}$$

where the inequality in the third line comes from the definition of x and the fact that $a \in B(x, A_0^{-1}\operatorname{diam} Q_0)$. The above computation together with the fact that $\overline{Q}$ is an ancestor of Q_0 shows that $y \in I(\overline{Q})$. In addition, we have that

$$\begin{aligned}
\operatorname{dist}(y, E(\vartheta,\gamma)\setminus Q_0) &\ge \operatorname{dist}(\mathfrak{c}(Q_0), E(\vartheta,\gamma)\setminus Q_0) - d(y,\mathfrak{c}(Q_0))\\
&\ge 64^{-1}\zeta^2\operatorname{diam} Q_0 - 40A_0^{-1}\operatorname{diam} Q_0 \ge 100A_0^{-1}\operatorname{diam} Q_0,
\end{aligned} \tag{94}$$

where the first inequality in the last line above comes from the second last inequality of (93), Remark 2.19 and Theorem A.2 (v), while the last inequality follows from the choice of A_0. From (93) and (94) we deduce that

$$\operatorname{dist}(y, E(\vartheta,\gamma)\setminus Q_0) \overset{(94)}{\ge} 100A_0^{-1}\operatorname{diam} Q_0 \overset{(93)}{>} d(y,\mathfrak{c}(Q_0)) \ge \operatorname{dist}(y, Q_0\cap E(\vartheta,\gamma)), \tag{95}$$

where the last inequality comes from the fact that $\mathfrak{c}(Q_0)$ belongs to $E(\vartheta,\gamma)$; see Remark 2.19. Therefore, if $z \in E(\vartheta,\gamma)$ is the point of minimal distance of y from $E(\vartheta,\gamma)$, (95) implies that $z \in Q_0 \cap E(\vartheta,\gamma)$. Furthermore, since by assumption $y \notin \mathfrak{G}(\overline{Q})$ and by (93) we have $y \in I(\overline{Q})$, Proposition 2.27 implies that

$$d(z,y) = \operatorname{dist}(y, E(\vartheta,\gamma)) \le 4A_0^{-1}d(y) < \tfrac{1}{10}d(y), \tag{96}$$

where the last inequality can be strict only if $d(y) > 0$. The definition of the function d, see (77), implies further by (96) that

$$d(z) \ge d(y) - d(z,y) > \tfrac{9}{10}d(y), \tag{97}$$

where last inequality is strict only if $d(y) > 0$. Summing up what we know so far about z is that it must be contained in $Q_0 \cap E(\vartheta,\gamma)$, however (97) implies that z cannot be contained in a cube $Q \in \Delta(C,\iota)$ with $\operatorname{diam} Q \le \frac{9}{10}d(y)$.

On the one hand, if $d(y)=0$, the bound (96) implies that $d(y,z)=0$ and thus since $E(\vartheta,\gamma)$ is compact we have $y=z\in E(\vartheta,\gamma)$. This implies in particular that

$$y\in E(\vartheta,\gamma)\cap Q_0\subseteq C\cap Q_0\cup\bigcup_{Q\in\mathscr{I}(Q_0)}Q\subseteq C\cap Q_0\cup\bigcup_{Q\in\mathscr{I}(Q_0)}B(\mathfrak{c}(Q),2C_6\operatorname{diam}Q)=F(Q_0).$$

If, on the other hand, $d(y)>0$, we will now show that $y\in F(Q_0)$. We claim that there is a cube $Q_1\in\Delta(C,\iota)$, contained in Q_0 and possibly coinciding with Q_0 itself, such that

(a) $z\in Q_1$ and for any cube $Q\in\Delta(C,\iota)$ contained in Q_1 we have $z\notin Q$,

(b) $\operatorname{diam}Q_1\geq\frac{9}{10}d(y)$,

(c) there exists a $\tilde{Q}\in\mathscr{I}(Q_0)$, that is a child of Q_1 for which $z\in\tilde{Q}$.

Let us verify that such a cube Q_1 exists. Since $z\in Q_0$, for any cube $Q\in\Delta(C,\iota)$ such that $Q\subseteq Q_0$ and $z\in Q$ we have

$$\tfrac{9}{10}d(y)\leq d(z)\leq\operatorname{dist}(z,Q)+\operatorname{diam}Q=\operatorname{diam}Q,\tag{98}$$

where the first inequality above comes from (97) and the second from the definition of d. Let Q_1 be the smallest cube of $\Delta(C,\iota)$ containing z, and note that for any cube $Q\subseteq Q_1$ belonging to $\Delta(C,\iota)$ we have that $z\notin Q$. This proves (a) and (b). In order to prove (c), we note that any ancestor of Q_1 in $\Delta(E(\vartheta,\gamma),\iota)$ must be contained in $\Delta(C,\iota)$. Furthermore, since the condition $\operatorname{diam}Q_1\geq\frac{9}{10}d(y)$ implies that $z\in E(\vartheta,\gamma)\setminus C$, we infer that there must exist a cube $\tilde{Q}$ in $\mathscr{I}(Q_0)$ for which $z\in\tilde{Q}$. Such a cube must be a child of Q_1 otherwise the maximality of $\tilde{Q}$ would be contradicted.

Let us use (a), (b) and (c) to conclude the proof of the theorem. Items (a), (b) and inequality (96) imply that

$$\operatorname{dist}(y,Q_1)\overset{\text{(a)}}{\leq}d(y,z)\overset{(96)}{\leq}\tfrac{1}{10}d(y)\overset{\text{(b)}}{\leq}\tfrac{1}{9}\operatorname{diam}Q_1.\tag{99}$$

Therefore, Proposition A.4 together with (c) and (99) implies that

$$\begin{aligned}d(\mathfrak{c}(\tilde{Q}),y)\leq d(\mathfrak{c}(\tilde{Q}),z)+d(z,y)&\leq\operatorname{diam}\tilde{Q}+\tfrac{1}{9}\operatorname{diam}Q_1\\&\leq\operatorname{diam}\tilde{Q}+\tfrac{1}{9}C_6\operatorname{diam}\tilde{Q}<2C_6\operatorname{diam}\tilde{Q}.\end{aligned}\tag{100}$$

The bound (100) finally proves that $y\in F(Q_0)$ thanks to the fact that $\tilde{Q}\in\mathscr{I}(Q_0)$ by (c). Summing up, this shows that for any $a\in B_+$, the curve γ_a meets the set $F(Q_0)$ somewhere.

In turn, this shows that $F(Q_0)$ has big projections. Indeed,

$$\begin{aligned}\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(F(Q_0)))&\geq\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(B(x,A_0^{-1}\operatorname{diam}Q_0)))\\&=c(\Pi(Q_0))A_0^{-(\mathfrak{Q}-1)}\operatorname{diam}Q_0^{\mathfrak{Q}-1},\end{aligned}\tag{101}$$

where the first inequality comes from the fact that the images of the curves γ_a are contained in $P_{\Pi(Q_0)}^{-1}(a)$ for any $a\in B_+$ and the last identity comes from Proposition 1.18. This concludes the proof of the main step of the proof, which was to verify the validity of (88).

In order to conclude the proof of the theorem we just need to put (87) together with (101) to get

$$\begin{aligned}\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(Q_0\cap C)) &\geq \mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(F(Q_0)))-\mathcal{S}^{\mathfrak{Q}-1}\Bigg(P_{\Pi(Q_0)}\Bigg(\bigcup_{Q\in\mathscr{I}(Q_0)} B(\mathfrak{c}(Q),C_6\operatorname{diam} Q_0)\Bigg)\Bigg)\\ &\overset{(87),(101)}{\geq} \frac{c(\Pi(Q_0))}{2A_0^{\mathfrak{Q}-1}}\operatorname{diam} Q_0^{\mathfrak{Q}-1}\geq\frac{\operatorname{diam} Q_0^{\mathfrak{Q}-1}}{2A_0^{\mathfrak{Q}-1}},\end{aligned}$$

where the last inequality comes from the fact that $c(\Pi(Q_0))\geq 1$; see Proposition 1.18. □

2D. *Construction of the ϕ-positive intrinsic Lipschitz graph.* This subsection is devoted to the proof of the main result of Section 2, Theorem 2.1, which we restate here for the reader's convenience:

Theorem 2.1. *There is an intrinsic Lipschitz graph Γ such that $\phi(\Gamma)>0$.*

We outline the proof of Theorem 2.1 here: For a fixed cube $Q\in\mathcal{M}(C,\iota)$, we prove that the family $\mathcal{B}(Q)$ of the maximal subcubes of Q having small projection on $\Pi(Q)$, thanks to Theorem 2.28, is small in measure. Therefore, we can find a cube $Q'\in\Delta(C,\iota)\setminus\mathcal{B}(Q)$ that is contained in Q and for which any subcube $\tilde{Q}$ of Q' has big projections on $\Pi(Q)$. This independence on the scales, thanks to Proposition 2.11, implies that $C\cap Q$ is a $\Pi(Q)$-intrinsic Lipschitz graph.

Proposition 2.29. *Define $\varepsilon_4:=\min\{\varepsilon_1,(32\vartheta C_1C_5A_0^{\mathfrak{Q}-1})^{-1}\}$. There exists a compact set $C_1\subseteq C$ and an $\iota_1\in\mathbb{N}$ such that*

(i) $\phi(C\setminus C_1)\leq\varepsilon_4\phi(C)$,

(ii) *whenever $Q\in\Delta(C_1,\iota_1)$ we have $\big(1-\frac{1}{32}\varepsilon_3\big)\phi(Q)\leq\phi(Q\cap C)$.*

Proof. First of all, we prove that the set $\Delta(C,\iota)$ is a $\phi\llcorner C$ Vitali relation. It is immediate to see that the family $\Delta(C,\iota)$ is a fine covering of C. Furthermore, let E be a Borel set contained in C and suppose $\mathcal{A}\subseteq\Delta(C,\iota)$ is a fine covering of E. Defining $\mathcal{A}^*:=\{Q\in\mathcal{A}:Q \text{ is maximal}\}$, the identity $\bigcup_{Q\in\mathcal{A}}Q=\bigcup_{Q\in\mathcal{A}^*}Q$ is trivially satisfied and thus the family $\mathcal{A}^*$ is still a covering of E. The maximality of the elements of $\mathcal{A}^*$ implies that they are pairwise disjoint and thus $\Delta(C,\iota)$ is a ϕ-Vitali relation in the sense of [Federer 1969, §2.8.16]. Therefore, thanks to [Federer 1969, Theorem 2.9.11], we deduce that

$$\lim_{Q\to x}\frac{\phi(C\cap Q)}{\phi(Q)}=1, \tag{102}$$

for ϕ-almost every $x\in C$. For any $j\in\mathbb{N}$, define the functions $f_j(x):=\phi(C\cap Q_j(x))/\phi(Q_j(x))$, where $Q_j(x)$ is the unique cube of the generation $\Delta_j^{\phi}(\vartheta,\gamma)$ containing x. Identity (102) implies that $\lim_{j\to\infty}f_j(x)=1$ for ϕ-almost every $x\in C$ and thus, the Severini–Egoroff theorem concludes that we can find a compact subset C_1 of C such that $\phi(C\setminus C_1)\leq\varepsilon_4\phi(C)$ and $f_j(x)$ converges uniformly to 1 on C_1. This proves (i) and (ii) at once. □

Theorem 2.30. *Let C_1 be the compact set from Proposition 2.29. Then there exists a cube $Q'\in\Delta(C_1,2\iota_1)$ such that $Q'\cap C_1$ is an intrinsic Lipschitz graph of positive ϕ-measure.*

Proof. For any $Q_0 \in \mathcal{M}(C_1, 2\iota_1)$, Theorem 2.28 and Proposition 2.29 imply that

$$\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(Q_0 \cap C)) \geq \frac{\operatorname{diam} Q_0^{\mathfrak{Q}-1}}{2A_0^{\mathfrak{Q}-1}}. \tag{103}$$

Therefore, for any $Q_0 \in \mathcal{M}(C_1, 2\iota_1)$ we let $\mathcal{B}(Q_0)$ be the family of the maximal cubes $Q \in \Delta(C_1, 2\iota_1)$ contained in Q_0 for which

$$\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(E(\vartheta, \gamma) \cap Q)) < \frac{\operatorname{diam} Q^{\mathfrak{Q}-1}}{4C_5^2 A_0^{\mathfrak{Q}-1}}, \tag{104}$$

and we define $\mathscr{B}(Q_0) := \bigcup_{Q \in \mathcal{B}(Q_0)} Q$.

The first step of the proof of the theorem is to show that the projection of $\mathscr{B}(Q_0)$ has small measure, or more precisely, that

$$\phi(C \cap [Q_0 \setminus \mathscr{B}(Q_0)]) > \frac{\phi(Q_0)}{8\vartheta C_1 C_5 A_0^{\mathfrak{Q}-1}}, \quad \text{for any } Q_0 \in \mathcal{M}(C_1, 2\iota_1). \tag{105}$$

Throughout this paragraph *we shall assume that $Q_0 \in \mathcal{M}(C_1, 2\iota_1)$ is fixed*. The maximality of the elements of $\mathcal{B}(Q_0)$ implies that they are pairwise disjoint and since by definition we have $Q \cap E(\vartheta, \gamma) \neq \varnothing$, for any $Q \in \mathcal{B}(Q_0)$ Remark A.3 yields

$$\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(E(\vartheta, \gamma) \cap Q)) < \frac{\operatorname{diam} Q^{\mathfrak{Q}-1}}{4C_5^2 A_0^{\mathfrak{Q}-1}} \leq \frac{\phi(Q)}{4C_5 A_0^{\mathfrak{Q}-1}}. \tag{106}$$

Thanks to the fact that $C \subseteq E(\vartheta, \gamma)$, Propositions 1.18 and 1.31 allow us to infer that

$$\phi(C \cap [Q_0 \setminus \mathscr{B}(Q_0)]) \geq \frac{\mathcal{S}^{\mathfrak{Q}-1}(C \cap [Q_0 \setminus \mathscr{B}(Q_0)])}{\vartheta} \geq \frac{\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(C \cap [Q_0 \setminus \mathscr{B}(Q_0)]))}{2c(\Pi(Q_0))\vartheta}. \tag{107}$$

On the other hand, thanks to (103) we conclude that

$$\begin{aligned}
\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(C \cap [Q_0 \setminus \mathscr{B}(Q_0)])) &\geq \mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(C \cap Q_0)) - \mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(E(\vartheta, \gamma) \cap \mathscr{B}(Q_0))) \\
&\geq \frac{\operatorname{diam} Q_0^{\mathfrak{Q}-1}}{2A_0^{\mathfrak{Q}-1}} - \sum_{Q \in \mathcal{B}(Q_0)} \mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(E(\vartheta, \gamma) \cap Q))).
\end{aligned} \tag{108}$$

Since by definition $Q_0 \cap E(\vartheta, \gamma) \neq \varnothing$, Remark A.3, (106), (108) and the fact that the cubes in $\mathcal{B}(Q_0)$ are disjoint imply that

$$\begin{aligned}
\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0)}(C \cap [Q_0 \setminus \mathscr{B}(Q_0)])) &\geq \frac{\phi(Q_0)}{2C_5 A_0^{\mathfrak{Q}-1}} - \frac{1}{4C_5 A_0^{\mathfrak{Q}-1}} \sum_{Q \in \mathcal{B}(Q_0)} \phi(Q) \\
&= \frac{\phi(Q_0)}{2C_5 A_0^{\mathfrak{Q}-1}} - \frac{1}{4C_5 A_0^{\mathfrak{Q}-1}} \phi(\mathscr{B}(Q_0)).
\end{aligned} \tag{109}$$

Putting together (107) and (109), we eventually deduce that

$$2c(\Pi(Q_0))\vartheta\phi(C\cap[Q_0\setminus\mathscr{B}(Q_0)])\geq\frac{\phi(Q_0)}{2C_5A_0^{\mathfrak{Q}-1}}-\frac{1}{4C_5A_0^{\mathfrak{Q}-1}}\phi(\mathscr{B}(Q_0))$$
$$=\frac{\phi(Q_0)}{4C_5A_0^{\mathfrak{Q}-1}}+\frac{1}{4C_5A_0^{\mathfrak{Q}-1}}\phi(Q_0\setminus\mathscr{B}(Q_0)),\tag{110}$$

where the last equality above follows from the inclusion $\mathcal{B}(Q_0)\subseteq Q_0$. Inequality (110) together with the fact that $c(\Pi(Q_0))\leq C_1$, see Proposition 1.18, immediately implies (105).

Now that (105) is proved, the second step in the proof is to construct a cube $Q'\in\Delta(C_1,2\iota_1)$ disjoint from $\bigcup_{Q_0\in\mathcal{M}(C_1,2\iota_1)}\mathscr{B}(Q_0)$ such that $\phi(C_1\cap Q')>0$. Every subcube of Q' contained in $\Delta(C_1,2\iota_1)$ thus enjoys a big projections property, and this is what in the end allows us to prove that $C_1\cap Q'$ is contained in an intrinsic Lipschitz graph. Since the elements of $\mathcal{M}(C_1,2\iota_1)$ are pairwise disjoint and their union covers C_1, we infer that

$$\phi\Bigg(C_1\setminus\bigcup_{Q_0\in\mathcal{M}(C_1,2\iota_2)}\mathscr{B}(Q_0)\Bigg)$$
$$=\phi\Bigg(\bigcup_{Q_0\in\mathcal{M}(C_1,2\iota_1)}C_1\cap[Q_0\setminus\mathscr{B}(Q_0)]\Bigg)=\sum_{Q_0\in\mathcal{M}(C_1,2\iota_1)}\phi(C_1\cap[Q_0\setminus\mathscr{B}(Q_0)])$$
$$\geq\sum_{Q_0\in\mathcal{M}(C_1,2\iota_1)}\phi(C\cap[Q_0\setminus\mathscr{B}(Q_0)])-\phi((C\setminus C_1)\cap Q_0)$$
$$\overset{(106)}{\geq}\Bigg(\sum_{Q_0\in\mathcal{M}(C_1,2\iota_1)}\frac{\phi(Q_0)}{8\vartheta C_1C_5A_0^{\mathfrak{Q}-1}}\Bigg)-\phi(C\setminus C_1)\geq\frac{\phi(C_1)}{8\vartheta C_1C_5A_0^{\mathfrak{Q}-1}}-\varepsilon_4\phi(C).\tag{111}$$

Therefore, the choice of ε_4, Proposition 2.29 and (111) imply that

$$\phi\Bigg(C_1\setminus\bigcup_{Q_0\in\mathcal{M}(C_1,2\iota_2)}\mathscr{B}(Q_0)\Bigg)\geq\frac{1-\varepsilon_4}{8\vartheta C_1C_5A_0^{\mathfrak{Q}-1}}\phi(C)-\varepsilon_4\phi(C)\geq\frac{\phi(C)}{16\vartheta C_1C_5A_0^{\mathfrak{Q}-1}}.\tag{112}$$

Inequality (112) implies that there must exist a cube $Q_0'\in\mathcal{M}(C_1,2\iota_1)$ such that $\phi\big(C_1\setminus\bigcup_{Q\in\mathcal{B}(Q_0')}Q\big)>0$. Defining $\mathscr{G}$ to be the set of maximal cubes in $\Delta(C_1,2\iota_1)\setminus\mathcal{B}(Q_0')$ contained in Q_0', we can find at least a cube $Q'\in\mathscr{G}$ for which $\phi(C_1\cap Q')>0$. Furthermore, thanks to the maximality of the elements in $\mathcal{B}(Q_0')$ and the fact that $Q'\cap\mathscr{B}(Q_0')=\varnothing$, we also deduce that any subcube of Q' cannot satisfy (104).

In the final step of the proof we show that $C_1\cap Q'$ is contained in an intrinsic Lipschitz graph. Indeed, we claim that

$$x_1^{-1}x_2\in C_{\Pi(Q_0')}(2\alpha_0),\qquad\text{for any }\ x_1,x_2\in C_1\cap Q',\tag{113}$$

where α_0 was defined in Proposition 2.11. Fix $x_1,x_2\in C_1\cap Q'$, and note that there exists a unique $j\in\mathbb{N}$ such that

$$R\gamma^{-1}2^{-jN+5}\leq d(x_1,x_2)\leq R\gamma^{-1}2^{-(j-1)N+5}.\tag{114}$$

For $i = 1, 2$ we let Q_{x_i} be the unique cubes in the j-th layer of cubes $\Delta_j^{\phi}(\vartheta, \gamma)$ for which $x_i \in Q_{x_i}$. Suppose $Q' \in \Delta_{\bar{j}}^{\phi}(\vartheta, \gamma)$ and note that Theorem A.2 (iv) and (114) imply that

$$R\gamma^{-1}2^{-jN+5} \leq d(x_1, x_2) \leq \operatorname{diam} Q' \leq \gamma^{-1}2^{-\bar{j}N+5}. \tag{115}$$

The chain of inequalities (115) implies that $\bar{j} \leq j$ and thus by Theorem A.2 (i) we infer that $Q_{x_i} \subseteq Q'$ for $i = 1, 2$. Furthermore, thanks to Theorem A.2 (ii) and (v), for $i = 1, 2$ we have

$$\begin{aligned} R \operatorname{diam} Q_{x_i} \leq R\gamma^{-1}2^{-jN+5} \leq d(x_1, x_2) &\leq R\gamma^{-1}2^{-(j-1)N+5} \\ &= 2^{N+6}\gamma^{-1}R2^{-jN-1} \leq 2^{N+6}\zeta^{-2}R \operatorname{diam} Q_{x_i}, \end{aligned} \tag{116}$$

since by construction $Q_{x_1} \in \Delta(\vartheta, \gamma)$. In addition to this, since as already remarked $Q_{x_i} \in \Delta(C_1, 2\iota_1)$, Lemma 2.16 implies that $\alpha(Q_{x_i}) \leq \varepsilon_2$ for $i = 1, 2$. Furthermore, the construction of Q' ensures that for any cube $Q \in \Delta(C_1, 2\iota_1)$ contained in Q', we have

$$\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q_0')}(E(\vartheta, \gamma) \cap Q)) \geq \frac{\operatorname{diam} Q^{\mathfrak{Q}-1}}{4C_5^2 A_0^{\mathfrak{Q}-1}}. \tag{117}$$

This proves that the hypotheses of Proposition 2.11 are satisfied and thus $x_1 \in x_2 C_{\Pi(Q_0')}(2\alpha_0)$. Finally, $C_1 \cap Q'$ is proved to be contained in an intrinsic Lipschitz graph by means of Proposition 1.37. ☐

Remark 2.31. Note that the proof of Theorem 2.1 is an immediate consequence of Theorem 2.30.

3. The support of 1-codimensional measures with almost-flat tangents is intrinsic rectifiable

A careful examination of the arguments of Section 2 shows that in order to prove Theorem 2.1, we never fully exploited the fact that ϕ-almost everywhere we have $\operatorname{Tan}_{\mathfrak{Q}-1}(\phi, x) \subseteq \mathfrak{M}$. Indeed, we used the flatness of tangents just to show that there exists a set C with large ϕ-measure on which the 1-Wasserstein distance between ϕ and some flat measure — below a certain (uniform on C) scale — is smaller than some fixed constant, which in the specific case of Section 2 is in essence ε_2. See for instance Proposition 2.14 and Lemma 2.16. This quantified closeness to flat measures is sufficient to construct the cones that yield the intrinsic rectifiability property of the set C. This is a typical phenomena occurring even in Euclidean spaces that has been observed explicitly in [David and Semmes 1993a, §II.2.1 Remark 2.5] and less explicitly in [Preiss 1987, Lemma 5.2].

In this section we aim to show how to modify the arguments of Section 2 in order to prove the intrinsic rectifiability of asymptotically AD-regular measures with almost flat tangents.

Throughout this section *we let $\delta \in \mathbb{N}$ be a fixed natural number and ψ be a fixed Radon measure on $\mathbb{G}$* whose support is a compact set K and such that for ψ-almost every $x \in \mathbb{G}$ we have

(H1′) $\delta^{-1} \leq \Theta_*^{\mathfrak{Q}-1}(\psi, x) \leq \Theta^{\mathfrak{Q}-1,*}(\psi, x) \leq \delta$,

(H2′) $\limsup_{r\to 0} d_{x,r}(\psi, \mathfrak{M}) < 4^{-(\mathfrak{Q}+1)^2}\varepsilon_2(2\delta)$.

In the following *we will make extensive use of constants, parameters and sets introduced in Notation 2.8 specializing them for the measure ψ*. For clarity, we stress if not explicitly mentioned throughout this section *we will always assume that $\sigma := 2\delta$*.

The first step in the understanding of the structure of ψ is to show that for any $k > 0$ the limit $\limsup_{r\to 0} d_{x,kr}(\psi,\mathfrak{M})$ can be read as the maximum distance from flat measures among all the elements of $\mathrm{Tan}_{\mathfrak{Q}-1}(\psi,x)$ inside $B(0,k)$:

Proposition 3.1. *For ψ-almost all $x\in\mathbb{G}$ and any $k>0$ we have*

$$\limsup_{r\to 0} d_{x,kr}(\psi,\mathfrak{M}) = \sup\{d_{0,k}(\nu,\mathfrak{M}) : \nu\in \mathrm{Tan}_{\mathfrak{Q}-1}(\psi,x)\}.$$

Proof. Fix a point $x\in K$ where $\mathrm{Tan}_{\mathfrak{Q}-1}(\psi,x)\neq\varnothing$ and where assumptions (H1′) and (H2′) hold. Recall that this choice of x can be made without loss of generality thanks to Proposition 1.26. Suppose $\{r_i\}_{i\in\mathbb{N}}$ is an infinitesimal sequence such that $\lim_{i\to\infty} d_{x,kr_i}(\psi,x) = \limsup_{r\to 0} d_{x,kr}(\psi,x)$ and assume up to nonrelabeled subsequences that there exists a $\nu\in\mathrm{Tan}_{\mathfrak{Q}-1}(\psi,x)$ such that

$$r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi \rightharpoonup \nu.$$

As a first step let us prove that $\limsup_{r\to 0} d_{x,kr}(\psi,\mathfrak{M}) \le d_{0,k}(\nu,\mathfrak{M})$. For any $0<\eta<1$ we let $\Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner V$ be an element of $\mathfrak{M}$ such that $F_{0,k}(\nu,\Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner V)/k^{\mathfrak{Q}} \le d_{0,k}(\nu,\mathfrak{M})+\eta$. With this choice, thanks to the triangle inequality, we infer that

$$\begin{aligned}
\limsup_{i\to\infty} d_{0,k}(r_i^{-(\mathfrak{Q}-1)}T_{x,r_i}\psi,\mathfrak{M}) &\le \limsup_{i\to\infty} \frac{F_{0,k}(r_i^{-(\mathfrak{Q}-1)}T_{x,r_i}\psi,\Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner V)}{k^{\mathfrak{Q}}}\\
&\le \limsup_{i\to\infty} \frac{F_{0,k}(r_i^{-(\mathfrak{Q}-1)}T_{x,r_i}\psi,\nu)+F_{0,k}(\nu,\Theta\mathcal{S}^{\mathfrak{Q}-1}\llcorner V)}{k^{\mathfrak{Q}}}\\
&\le d_{0,k}(\nu,\mathfrak{M})+\eta, \qquad (118)
\end{aligned}$$

where the last inequality comes from the choice of Θ and V and Proposition 1.23. The arbitrariness of η concludes the proof of the first claim.

As a second and final step of the proof, fix a $\mu\in\mathrm{Tan}_{\mathfrak{Q}-1}(\psi,x)$ and show that $\limsup_{r\to 0} d_{x,kr}(\psi,\mathfrak{M})\ge d_{0,k}(\mu,\mathfrak{M})$. Since $\mu\in\mathrm{Tan}_{\mathfrak{Q}-1}(\psi,x)$, we can find an infinitesimal sequence $\{r_i\}_{i\in\mathbb{N}}$ such that

$$r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi \rightharpoonup \mu.$$

Furthermore, for any $0<\eta<2^{-(\mathfrak{Q}+1)}(\delta^{-1}-2^{-\mathfrak{Q}}\varepsilon_2(2\delta))$ and any $i\in\mathbb{N}$ there exists a $\Theta_i>0$ and a $V_i\in\mathrm{Gr}(\mathfrak{Q}-1)$ such that

$$\frac{F_{0,k}(r_i^{-(\mathfrak{Q}-1)}T_{x,r_i}\psi,\Theta_i\mathcal{S}^{\mathfrak{Q}-1}\llcorner V_i)}{k^{\mathfrak{Q}}} \le d_{0,k}(r_i^{-(\mathfrak{Q}-1)}T_{x,r_i}\psi,\mathfrak{M})+\eta = d_{x,kr_i}(\psi,\mathfrak{M})+\eta,$$

where the last identity above comes from Proposition 2.3 (i).

Our next task is to show that there exists a compact subinterval I of $(0,\infty)$ such that $\{\Theta_i\}_{i\in\mathbb{N}}\subseteq I$. Thanks to assumption (H2′) on ψ, there exists an $i_0\in\mathbb{N}$ such that we have $d_{x,kr_i}(\psi,\mathfrak{M})\le 4^{-(\mathfrak{Q}+1)^2}\varepsilon_2(2\delta)$, for any $i \geq i_0$. This implies for any $i \geq i_0$ that

$$\left|\int g(w)\, d\frac{T_{x,r_i}\psi(w)}{r_i^{\mathfrak{Q}-1}} - \Theta_i \int g(w)\, d\mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i(w)\right| \leq F_{0,k}(r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi, \Theta_i \mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i) \leq 4^{-(\mathfrak{Q}+1)^2} \varepsilon_2(2\delta) k^{\mathfrak{Q}} + \eta k^{\mathfrak{Q}}, \tag{119}$$

where $g(x) := \max\{k - d(0,x), 0\}$. Thanks to the definition of g and to (119) we infer that

$$\begin{aligned}\Theta_i 2^{-\mathfrak{Q}} k^{\mathfrak{Q}} - \frac{k\psi(B(x,kr_i))}{r_i^{\mathfrak{Q}-1}} &\leq \Theta_i \int_{B(0,k/2)} g(w)\, d\mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i(w) - \int_{B(0,k)} k\, \frac{dT_{x,r_i}\psi(w)}{r_i^{\mathfrak{Q}-1}} \\ &\leq \left|\Theta_i \int g(w)\, d\mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i(w) - \int g(w) \frac{dT_{x,r_i}\psi(w)}{r_i^{\mathfrak{Q}-1}}\right| \leq 4^{-(\mathfrak{Q}+1)^2} \varepsilon_2(2\delta) k^{\mathfrak{Q}} + \eta k^{\mathfrak{Q}}. \end{aligned} \tag{120}$$

On the other hand, a similar argument shows that

$$\begin{aligned}\frac{k}{2}\frac{\psi(B(x,kr_i/2))}{r_i^{\mathfrak{Q}-1}} - \Theta_i k^{\mathfrak{Q}} &= \int_{B(0,k/2)} \frac{k}{2} \frac{dT_{x,r_i}\psi(y)}{r_i^{\mathfrak{Q}-1}} - \Theta_i \int g(y)\, d\mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i(y) \\ &\leq \left|\int g(y) \frac{dT_{x,r_i}\psi(y)}{r_i^{\mathfrak{Q}-1}} - \Theta_i \int g(y)\, d\mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i(y)\right| \leq 4^{-(\mathfrak{Q}+1)^2} \varepsilon_2(2\delta) k^{\mathfrak{Q}} + \eta k^{\mathfrak{Q}}. \end{aligned} \tag{121}$$

Rearranging inequality (120) and dividing both sides by $\left(\frac{1}{2}k\right)^{\mathfrak{Q}}$, thanks to the choice of x and to the arbitrariness of i, we have

$$\limsup_{i\to\infty} \Theta_i \leq 2^{\mathfrak{Q}} \limsup_{i\to\infty} \frac{\psi(B(x,kr_i))}{(kr_i)^{\mathfrak{Q}-1}} + 2^{-(\mathfrak{Q}+1)} \varepsilon_2(2\delta) + 2^{\mathfrak{Q}}\eta \leq 2^{\mathfrak{Q}}(\delta+1) + 2^{-(\mathfrak{Q}+1)} \varepsilon_2(2\delta), \tag{122}$$

where the second last inequality comes from the fact that (H1′) is satisfied at x and the last inequality from the fact that $\eta < 1$.

Similarly, rearranging inequality (121) and dividing both sides by $k^{\mathfrak{Q}}$, thanks to the arbitrariness of i, we infer that

$$2^{-\mathfrak{Q}}\delta^{-1} \leq \frac{\Theta_*^{\mathfrak{Q}-1}(\psi,x)}{2^{\mathfrak{Q}}} \leq \frac{1}{2^{\mathfrak{Q}}} \liminf_{i\to\infty} \frac{\psi(B(x,kr_i/2))}{(kr_i/2)^{\mathfrak{Q}-1}} \leq \liminf_{i\to\infty} \Theta_i + 4^{-(\mathfrak{Q}+1)^2} \varepsilon_2(2\delta) + \eta. \tag{123}$$

On the other hand, (123) and the choice of η imply that

$$0 < 2^{-(\mathfrak{Q}+1)}(\delta^{-1} - 2^{-\mathfrak{Q}} \varepsilon_2(2\delta)) \leq 2^{-\mathfrak{Q}}(\delta^{-1} - 2^{-\mathfrak{Q}} \varepsilon_2(2\delta)) - \eta \leq \liminf_{i\to\infty} \Theta_i, \tag{124}$$

where the first inequality comes from the choice of $\varepsilon_2(2\delta)$ and the second inequality from that of η. The bounds (122) and (124) together imply that up to taking a nonrelabeled subsequence of $\{\Theta_i\}_{i\in\mathbb{N}}$ we can assume that the Θ_i converge to some $\Theta \in [2^{-(\mathfrak{Q}+1)}(\delta^{-1} - 2^{-\mathfrak{Q}} \varepsilon_2(2\delta)), 2^{\mathfrak{Q}}(\delta+1) + 2^{-(\mathfrak{Q}+1)} \varepsilon_2(2\delta)]$.

Without loss of generality, we can assume that there exists a $V \in \mathrm{Gr}(\mathfrak{Q}-1)$ such that $\mathfrak{n}(V_i) \to \mathfrak{n}(V)$. Since under such an assumption Proposition 1.32 implies that $\Theta_i \mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i \rightharpoonup \Theta \mathcal{S}^{\mathfrak{Q}-1} \llcorner V$, the triangle inequality implies for any $i \in \mathbb{N}$ that

$$
\begin{aligned}
&d_{0,k}(\mu, \mathfrak{M}) \\
&\quad\le \frac{F_{0,k}(\mu, r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi) + F_{0,k}(r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi, \Theta_i \mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i) + F_{0,k}(\Theta_i \mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i, \Theta \mathcal{S}^{\mathfrak{Q}-1} \llcorner V)}{k^{\mathfrak{Q}}} \\
&\quad\le \frac{F_{0,k}(\mu, r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi)}{k^{\mathfrak{Q}}} + d_{0,k}(r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi, \mathfrak{M}) + \eta + \frac{F_{0,k}(\Theta_i \mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i, \Theta \mathcal{S}^{\mathfrak{Q}-1} \llcorner V)}{k^{\mathfrak{Q}}}.
\end{aligned}
$$

Finally, thanks to the arbitrariness of i and of η and to Proposition 1.23, we infer that

$$d_{0,k}(\mu, \mathfrak{M}) \le \limsup_{i \to \infty} d_{0,k}(r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}\psi, \mathfrak{M}). \qquad \square$$

The following result is the analogue of Proposition 2.14 for ψ as it serves the same purpose, i.e., find a compact subset $\tilde{C}$ of K in such a way that $\psi \llcorner \tilde{C}$ is essentially an AD-regular measure and the functions $x \mapsto d_{x,4kr}(\psi, \mathfrak{M})$ have small supremum norms on $\tilde{C}$ provided r is small enough.

Proposition 3.2. *There exist an $\tilde{\iota}_0 \in \mathbb{N}$ and a $\tilde{\gamma} \in \mathbb{N}$ such that for any $\mu \ge 8C_4(2\delta)\delta$ we can find a $\nu \in \mathbb{N}$ and a compact set $\tilde{C} \subseteq \mathscr{E}^{\psi}_{2\delta,\tilde{\gamma}}(\mu, \nu)$ such that*

(i) $\psi(K \setminus \tilde{C}) \le 2\varepsilon_1 \psi(K)$,

(ii) $d_{x,4k(2\delta)r}(\psi, \mathfrak{M}) + d_{x,4k(2\delta)r}(\psi \llcorner E^{\psi}(2\delta, \tilde{\gamma}), \mathfrak{M}) \le 4^{-\mathfrak{Q}(\mathfrak{Q}+1)}\varepsilon_2(2\delta)$ *for any* $0 < r < 2^{-\tilde{\iota}_0 N(2\delta)+5}/\tilde{\gamma}$ *and any* $x \in \tilde{C}$,

where $\varepsilon_2(2\delta)$ is the constant introduced in Notation 2.8 and ε_1 is chosen in the same way as it was in Notation 2.5.[6]

Proof. First of all, thanks to Propositions 1.28 and 1.30 we can find a $\tilde{\gamma} \in \mathbb{N}$ and a $\nu \in \mathbb{N}$ such that $\psi(K \setminus \mathscr{E}^{\psi}_{2\delta,\tilde{\gamma}}(\mu, \nu)) \le \varepsilon_1 \psi(K)$. Let us now prove that

$$\limsup_{r \to 0} d_{x,4k(2\delta)r}(\psi \llcorner E^{\psi}(2\delta, \tilde{\gamma}), \mathfrak{M}) \le 4^{-(\mathfrak{Q}+1)^2}\varepsilon_2(2\delta), \quad \text{for } \psi\text{-almost every } x \in E^{\psi}(2\delta, \tilde{\gamma}).$$

Recall that for ψ-almost every $x \in E^{\psi}(2\delta, \tilde{\gamma})$, we have that $\mathrm{Tan}_{\mathfrak{Q}-1}(\psi \llcorner E^{\psi}(2\delta, \tilde{\gamma}), x) = \mathrm{Tan}_{\mathfrak{Q}-1}(\psi, x)$. Thanks to this, Proposition 3.1 yields

$$
\begin{aligned}
\limsup_{r \to 0} d_{x,4k(2\delta)r}(\psi \llcorner E^{\psi}(2\delta, \tilde{\gamma}), \mathfrak{M}) &\le \limsup_{r \to 0} \frac{F_{x,4k(2\delta)r}(\psi \llcorner E^{\psi}(2\delta, \tilde{\gamma}), \psi)}{(4k(2\delta)r)^{\mathfrak{Q}}} + d_{x,4k(2\delta)r}(\psi, \mathfrak{M}) \\
&= \limsup_{r \to 0} d_{x,4k(2\delta)r}(\psi, \mathfrak{M}) \le 4^{-(\mathfrak{Q}+1)^2}\varepsilon_2(2\delta),
\end{aligned}
$$

for ψ-almost every $x \in E^{\psi}(2\delta, \tilde{\gamma})$, where the identity in the last line comes from hypothesis (H1′) and the Lebesgue differentiation theorem of [Heinonen et al. 2015, page 77]. Therefore, for ψ-almost every

[6] The reader should notice that the objects and symbols introduced in Notation 2.5 were specific to the measure ϕ. However, ε_1 was just required to be a positive real number smaller than $1/10$.

$x \in E^\psi(2\delta, \tilde\gamma)$ there exists an $r(x) > 0$ such that for every $0 < r < r(x)$,

$$d_{x,4k(2\delta)r}(\psi, \mathfrak{M}) + d_{x,4k(2\delta)r}(\psi \llcorner E^\psi(2\delta, \tilde\gamma), \mathfrak{M}) \le 4^{-\mathfrak{Q}(\mathfrak{Q}+1)}\varepsilon_2(2\delta).$$

For any $j \in \mathbb{N}$, let us define $E_j := \{x \in \mathscr{E}^\psi_{2\delta,\tilde\gamma}(\mu, \nu) : r(x) > 1/j\}$ and show that the E_j are Borel sets. Thanks to Proposition 2.3 (ii), the map $x \mapsto d_{x,r}(\psi, \mathfrak{M}) + d_{x,r}(\psi \llcorner E^\psi(2\delta, \tilde\gamma), \mathfrak{M})$ is continuous and thus for any $r > 0$ the set $\Omega_r := \{y \in \mathscr{E}^\psi_{2\delta,\tilde\gamma}(\mu, \nu) : d_{y,r}(\psi, \mathfrak{M}) + d_{y,r}(\psi \llcorner E^\psi(2\delta, \tilde\gamma), \mathfrak{M}) < 4^{-\mathfrak{Q}(\mathfrak{Q}+1)}\varepsilon_2(2\delta)\}$ is relatively open in $\mathscr{E}^\psi_{2\delta,\tilde\gamma}(\mu, \nu)$. In particular, if $x \in \Omega_r$ for any $r \in (0, 1/j) \cap \mathbb{Q}$ we have $r(x) > 1/j$ thanks to Proposition 2.3 (iv) and hence $x \in E_j$. On the other hand, if $x \in E_j$ then obviously $x \in \Omega_r$ for any $0 < r < 1/j$. Since $\mathscr{E}^\psi_{2\delta,\tilde\gamma}(\mu, \nu)$ is compact, this shows that the sets E_j are G_δ and thus Borel. Let us note that since ψ-almost every $x \in \mathscr{E}^\psi_{2\delta,\tilde\gamma}(\mu, \nu)$ is contained in some E_j, thanks to the existence of $r(x)$, we infer that

$$\psi\Big(\mathscr{E}^\psi_{2\delta,\tilde\gamma}(\mu, \nu) \setminus \bigcup_{j\in\mathbb{N}} E_j\Big) = 0. \tag{125}$$

Finally, (125) together with the measurability of the nested sets E_j implies that we can find a $j \in \mathbb{N}$ big enough and a compact set $\tilde C$ contained in E_j satisfying items (i) and (ii). □

As in the case of Proposition 2.14, one can impose slightly different conditions on the measure and obtain a family of cubes satisfying the same thesis as Lemma 2.16. From here on we will employ all the notations introduced in Notation 2.8.

Proposition 3.3. *Fixing $\mu \ge 8C_4(2\delta)\delta$ if $\tilde\gamma, \tilde\iota_0, \nu \in \mathbb{N}$ and $\tilde C \Subset \mathscr{E}^\psi_{2\delta,\tilde\gamma}(\mu, \nu)$ are the natural numbers and the compact set yielded by Proposition 3.2, respectively, and defining $\tilde\iota := \max\{\tilde\iota_0, \nu\}$ for any cube $Q \in \Delta^\psi(\tilde C; 2\delta, \tilde\gamma, \tilde\iota)$, we have that $\alpha(Q) \le \varepsilon_2(2\delta)$ and for any such cube Q there is a plane $\Pi(Q) \in \mathrm{Gr}(\mathfrak{Q} - 1)$ for which*

(i) $$\sup_{w \in E^\psi(2\delta,\tilde\gamma)\cap B(\mathfrak{c}(Q),k(2\delta)\operatorname{diam} Q/2)} \frac{\operatorname{dist}(w, \mathfrak{c}(Q)\Pi(Q))}{2k(2\delta)\operatorname{diam} Q} \le C_2(2\delta)\varepsilon_2(2\delta)^{1/\mathfrak{Q}}, \quad \textit{and}$$

(ii) *for any $w \in B\big(\mathfrak{c}(Q), \frac{1}{2}k(2\delta)\operatorname{diam} Q\big) \cap \mathfrak{c}(Q)\Pi(Q)$ we have*

$$E^\psi(2\delta, \tilde\gamma) \cap B(w, 3k(2\delta)C_2(2\delta)\varepsilon_2(2\delta)^{1/(\mathfrak{Q}+1)}\operatorname{diam} Q) \ne \varnothing.$$

Proof. Thanks to Proposition 3.2, we can find a $\tilde\gamma \in \mathbb{N}$ and a compact set $\tilde C$ contained in $E^\psi(2\delta, \gamma)$ such that

(i) $\psi(K \setminus \tilde C) \le 2\varepsilon_1\psi(K)$, where ε_1 was introduced in Notation 2.5,

(ii) $d_{x,4k(2\delta)r}(\psi, \mathfrak{M}) + d_{x,4k(2\delta)r}(\psi \llcorner E^\psi(2\delta, \gamma), \mathfrak{M}) \le 4^{-\mathfrak{Q}(\mathfrak{Q}+1)}\varepsilon_2(2\delta)$ for any $0 < r < 2^{-\tilde\iota_0 N(2\delta)+5}/\gamma$ and any $x \in \tilde C$.

Thus, if $\Delta^\psi(2\delta, \tilde\gamma)$ is the family of dyadic cubes relative to the parameters $2\delta, \tilde\gamma$ and the measure ψ yielded by Theorem A.2, one can prove that the cubes of $\Delta^\psi(\tilde C; 2\delta, \tilde\gamma, \tilde\iota)$ satisfy (i) and (ii) by using verbatim the argument we employed in the proof of Lemma 2.16. □

As remarked at the beginning of this section, the arguments we used to prove Propositions 2.21 and 2.27, Lemmas 2.25 and 2.26 and Theorem 2.28 just relied on the possibility of proving Lemma 2.16 for the measure ϕ. Proposition 3.3 is the counterpart of Lemma 2.16 for the measure ψ where ϑ has been substituted by 2δ, γ by $\tilde{\gamma}$, and so on. Therefore, repeating the proofs of Section 2C for ψ and its associated parameters and compact set $\tilde{C}$, one can show the following:

Theorem 3.4. *For any cube $Q \in \Delta^{\psi}(\tilde{C}; 2\delta, \tilde{\gamma}, \tilde{\iota})$ such that $(1-\varepsilon_3(2\delta))\phi(Q) \leq \phi(Q \cap \tilde{C})$, we have*

$$\mathcal{S}^{\mathfrak{Q}-1}(P_{\Pi(Q)}(Q \cap \tilde{C})) \geq \frac{\operatorname{diam} Q^{\mathfrak{Q}-1}}{2A_0^{\mathfrak{Q}-1}}.$$

Remark 3.5. Similar to what we did in Proposition 2.29, we can construct a compact subset $\tilde{C}_1$ of $\tilde{C}$ and an $\tilde{\iota}_1 \in \mathbb{N}$ satisfying (i) and (ii) of Proposition 2.29, provided ε_3 is substituted with $\varepsilon_3(2\delta)$, ε_4 with $\varepsilon_4(2\delta) := \min\{\varepsilon_1, (64\delta C_1 C_5(2\delta) A_0^{\mathfrak{Q}-1}(2\delta))^{-1}\}$ and $\Delta(C, \iota)$ with $\Delta^{\psi}(\tilde{C}; 2\delta, \gamma, \tilde{\iota}_1)$.

The above remark allows us to construct the ψ-positive intrinsic Lipschitz graph that will be used to prove Theorem 4.2 in Section 4.

Theorem 3.6. *Let $\tilde{C}_1$ be as in Remark 3.5. Then there exists a cube $Q' \in \Delta^{\psi}(\tilde{C}_1; 2\delta, \gamma, 2\tilde{\iota}_1)$ such that $Q' \cap \tilde{C}_1$ is an intrinsic Lipschitz graph of positive ψ-measure.*

Proof. Thanks to Propositions 3.2, 3.3, Remark 3.5 and Theorem 3.4, the argument we used to prove Theorem 2.30 can be applied here verbatim. □

4. Conclusions and discussion of the results

In this section we use the main result of Section 2, i.e., Theorem 2.1, to deduce a number of consequences. First of all we prove the main result of the paper, Theorem 4.1, which is a 1-codimensional extension of the Marstrand–Mattila rectifiability criterion to general Carnot groups. Secondly, we provide in Corollary 4.3 a rigidity result for finite perimeter sets in Carnot groups: we are able to show that if locally a finite perimeter set is not too far from its natural tangent plane, then its boundary is an *intrinsic rectifiable set*; see Definition 1.40. Eventually, we use Theorem 4.1 to prove a 1-codimensional version of Preiss's rectifiability theorem in the Heisenberg groups $\mathbb{H}^n$.

4A. ***Main results.*** In this subsection we finally conclude the proof of the main results of this work.

Theorem 4.1. *Suppose ϕ is a Radon measure on $\mathbb{G}$ and let $\tilde{d}(\cdot,\cdot)$ be a left-invariant, homogeneous distance on $\mathbb{G}$. Assume further that for ϕ-almost all $x \in \mathbb{G}$ we have*

(i) $$0 < \liminf_{r\to 0} \frac{\phi(\tilde{B}(x,r))}{r^{\mathfrak{Q}-1}} \leq \limsup_{r\to 0} \frac{\phi(\tilde{B}(x,r))}{r^{\mathfrak{Q}-1}} < \infty,$$

where $\tilde{B}(x,r)$ is the ball relative to the metric $\tilde{d}$ centered at x of radius $r > 0$,

(ii) $\operatorname{Tan}_{\mathfrak{Q}-1}(\phi, x) \subseteq \mathfrak{M}$, *where $\mathfrak{M}$ is the family of 1-codimensional flat measures from Definition 1.7.*

Then ϕ is absolutely continuous with respect to $\mathcal{S}^{\mathfrak{Q}-1}$ and ϕ-almost all of $\mathbb{G}$ can be covered with countably many $C^1_{\mathbb{G}}$-hypersurfaces.

Proof. Since $\tilde{d}$ is bi-Lipschitz equivalent to d, see for instance Corollary 5.15 in [Bonfiglioli et al. 2007], hypothesis (i) implies that

$$0 < \Theta_*^{\mathfrak{Q}-1}(\phi, x) \le \Theta^{\mathfrak{Q}-1,*}(\phi, x) < \infty, \tag{126}$$

for ϕ-almost every $x \in \mathbb{G}$. For any $\vartheta, \gamma, R \in \mathbb{N}$ we define

$$E(\vartheta, \gamma, R) := \{x \in \overline{B(0, R)} : \vartheta^{-1} r^{\mathfrak{Q}-1} \le \phi(B(x, r)) \le \vartheta r^{\mathfrak{Q}-1} \text{ for any } 0 < r < 1/\gamma\}.$$

It is possible to prove, with the same arguments used in the proof of Proposition 1.28, that the $E(\vartheta, \gamma, R)$ are compact sets and

$$\phi\biggl(\mathbb{G} \setminus \bigcup_{\vartheta,\gamma,R} E(\vartheta, \gamma, R)\biggr) = 0. \tag{127}$$

Thus, if A is an $\mathcal{S}^{\mathfrak{Q}-1}$-null Borel set, Proposition 1.31 yields

$$\phi(A) \le \sum_{\vartheta,\gamma,R\in\mathbb{N}} \phi(A \cap E(\vartheta, \gamma, R)) \le \sum_{\vartheta,\gamma,R\in\mathbb{N}} \vartheta 2^{\mathfrak{Q}-1} \mathcal{S}^{\mathfrak{Q}-1}(A \cap E(\vartheta, \gamma, R)) = 0.$$

The above computation proves that ϕ is absolutely continuous with respect to $\mathcal{S}^{\mathfrak{Q}-1}$ and just to fix notations we let $\rho \in L^1(\mathcal{S}^{\mathfrak{Q}-1})$ be such that $\phi = \rho\mathcal{S}^{\mathfrak{Q}-1}$.

As a second step, we show that ϕ-almost all of $\mathbb{G}$ can be covered with countably many intrinsic Lipschitz graphs. Assume by contradiction there are $\vartheta, \gamma, R \in \mathbb{N}$ for which we can find a subset of $E(\vartheta, \gamma, R)$ of positive ϕ-measure that we denote by $E(\vartheta, \gamma, R)^u$ (following the notations of Corollary 1.42) and that has $\mathcal{S}^{\mathfrak{Q}-1}$-null intersection with any intrinsic Lipschitz graph. Thanks to Corollary 2.9.11 of [Federer 1969] it is immediate to see that

$$\vartheta^{-1} \le \Theta_*^{\mathfrak{Q}-1}(\phi \llcorner E(\vartheta, \gamma, R)^u, x) \le \Theta^{\mathfrak{Q}-1,*}(\phi \llcorner E(\vartheta, \gamma, R)^u, x) \le \vartheta,$$

for ϕ-almost every $x \in E(\vartheta, \gamma, R)^u$. Further, from Proposition 1.27, for ϕ-almost every $x \in E(\vartheta, \gamma, R)^u$, we infer that $\mathrm{Tan}_{\mathfrak{Q}-1}(\phi \llcorner E(\vartheta, \gamma, R)^u, x) \subseteq \mathfrak{M}$. And since its hypothesis is satisfied, Theorem 2.1 implies that there exists an intrinsic Lipschitz graph Γ such that $\phi(\Gamma \cap E(\vartheta, \gamma, R)^u) > 0$. However, this is not possible since Proposition 1.31 would yield

$$0 < \phi(\Gamma \cap E(\vartheta, \gamma, R)^u) \le \vartheta 2^{\mathfrak{Q}-1} \mathcal{S}^{\mathfrak{Q}-1}(E(\vartheta, \gamma, R)^u \cap \Gamma),$$

and this contradicts the fact that $E(\vartheta, \gamma, R)$ intersects in a $\mathcal{S}^{\mathfrak{Q}-1}$-null set every intrinsic Lipschitz graph.

Up to this point we have shown that, for any choice of ϑ, γ, R, we have that $\mathcal{S}^{\mathfrak{Q}-1}$-almost all of the sets $E(\vartheta, \gamma, R)$ are covered by countably many intrinsic Lipschitz graphs. Furthermore, since $\phi \ll \mathcal{S}^{\mathfrak{Q}-1}$, thanks to (127) we conclude that ϕ-almost all of $\mathbb{G}$ can be covered by countably many intrinsic Lipschitz graphs. This concludes the first part of the proof of the theorem.

So far we have shown that we can find countably many intrinsic Lipschitz graphs that cover ϕ-almost all of $\mathbb{G}$. Since by Remark B.7 we know that intrinsic Lipschitz graphs are boundaries of finite perimeter sets, if $\mathbb{G}$ is a group where boundaries of finite perimeter sets are $C^1_{\mathbb{G}}$-rectifiable, the proof of the proposition would be completed here. In the moment of writing some broad families of Carnot groups where

De Giorgi's rectifiability theorem is known to hold include step 2 groups (see [Franchi et al. 2003]), groups of type * (see [Marchi 2014]) and groups of diamond type (see [Le Donne and Moisala 2021]).

In this paragraph, we assume that ϑ, γ, $R \in \mathbb{N}$ are fixed. Thanks to Proposition 1.31 we infer that $\mathcal{S}^{\mathfrak{Q}-1} \llcorner E(\vartheta, \gamma, R)$ is mutually absolutely continuous with respect to $\phi \llcorner E(\vartheta, \gamma)$ and in particular that

$$\vartheta^{-1} \leq \rho(x) \leq \vartheta 2^{\mathfrak{Q}-1} \quad \text{for } \mathcal{S}^{\mathfrak{Q}-1}\text{-almost every } x \in E(\vartheta, \gamma, R).$$

Let $\{\gamma_i\}_{i\in\mathbb{N}}$ be the sequence of intrinsic Lipschitz functions $\gamma_i : W_i \to \mathfrak{N}(W_i)$ for which

$$\phi\biggl(E(\vartheta, \gamma, R) \setminus \bigcup_{i\in\mathbb{N}} \mathrm{gr}(\gamma_i)\biggr) = 0,$$

and let $E_i := \mathrm{epi}(\gamma_i)$ be the epigraph of the function γ_i which is defined in (142). Since $\mathcal{S}^{\mathfrak{Q}-1} \llcorner \mathrm{gr}(\gamma_i)$ and $|\partial E_i|_{\mathbb{G}}$[7] are asymptotically doubling measures by [Franchi and Serapioni 2016, Theorem 3.9] and Theorems B.6 and B.8, respectively, we deduce thanks to Proposition 1.27 that for ϕ-almost every $x \in E(\vartheta, \gamma, R) \cap \mathrm{gr}(\gamma_i)$ we have

$$\begin{aligned}\mathfrak{M} \supseteq \mathrm{Tan}_{\mathfrak{Q}-1}(\phi \llcorner E(\vartheta, \gamma, R) \cap \mathrm{gr}(\gamma_i), x) &= \rho(x)\, \mathrm{Tan}_{\mathfrak{Q}-1}(\mathcal{S}^{\mathfrak{Q}-1} \llcorner \mathrm{gr}(\gamma_i), x) \\ &= \rho(x)\mathfrak{d}(x)\, \mathrm{Tan}_{\mathfrak{Q}-1}(|\partial E_i|_{\mathbb{G}}, x),\end{aligned} \tag{128}$$

where $\mathfrak{d}$ is the density yielded by Remark B.7. Finally, Proposition B.16 implies that

$$\mathrm{Tan}_{\mathfrak{Q}-1}(\phi \llcorner E(\vartheta, \gamma, R) \cap \mathrm{gr}(\gamma_i), x) \subseteq \rho(x)\mathfrak{d}(x)\{\lambda \mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i(x) : \lambda \in [L_{\mathbb{G}}^{-1}, l_{\mathbb{G}}^{-1}]\}, \tag{129}$$

for ϕ-almost every $x \in E(\vartheta, \gamma, R) \cap \mathrm{gr}(\gamma_i)$, where $V_i(x) \in \mathrm{Gr}(\mathfrak{Q}-1)$ is the plane orthogonal to $\mathfrak{n}_{E_i}(x)$, the generalized inward normal introduced in Definition B.4, and the constants $l_{\mathbb{G}}$ and $L_{\mathbb{G}}$ are those yielded by Theorem B.6. We now prove that (129) implies that for $\mathcal{S}^{\mathfrak{Q}-1}$-almost every $x \in \mathrm{gr}(\gamma_i) \cap E(\vartheta, \gamma, R)$ and every $\alpha > 0$ we have

$$\lim_{r\to 0} \frac{\mathcal{S}^{\mathfrak{Q}-1}(\mathrm{gr}(\gamma_i) \cap E(\vartheta, \gamma, R) \cap B(x, r) \setminus x X_{V_i(x)}(\alpha))}{r^{\mathfrak{Q}-1}} = 0, \tag{130}$$

where $X_{V_i(x)}(\alpha) := \{w \in \mathbb{G} : \mathrm{dist}(w, V_i(x)) \leq \alpha \|w\|\}$. Thanks to (128) and (129), for $\mathcal{S}^{\mathfrak{Q}-1}$-almost every $x \in \mathrm{gr}(\gamma_i) \cap E(\vartheta, \gamma, R)$ and any sequence $r_j \to 0$, there exists a $\lambda > 0$ for which

$$\frac{T_{x,r} \mathcal{S}^{\mathfrak{Q}-1} \llcorner E(\vartheta, \gamma, R) \cap \mathrm{gr}(\gamma_i)}{r_j^{\mathfrak{Q}-1}} \rightharpoonup \lambda \mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i(x). \tag{131}$$

The convergence in (131) implies that

$$\begin{aligned}\lim_{i\to\infty} &\frac{\mathcal{S}^{\mathfrak{Q}-1} \llcorner \mathrm{gr}(\gamma_i) \cap E(\vartheta, \gamma, R)(B(x, r_j) \setminus x X_{V_i(x)}(\alpha))}{r_j^{\mathfrak{Q}-1}} \\ &= \lim_{i\to\infty} \frac{T_{x,r_j}(\mathcal{S}^{\mathfrak{Q}-1} \llcorner \mathrm{gr}(\gamma_i) \cap E(\vartheta, \gamma, R))(B(0, 1) \setminus X_{V_i(x)}(\alpha))}{r_j^{\mathfrak{Q}-1}} \\ &= \lambda(\mathcal{S}^{\mathfrak{Q}-1} \llcorner V_i(x))(B(0, 1) \setminus X_{V_i(x)}(\alpha)) = 0,\end{aligned} \tag{132}$$

[7]With $|\partial E_i|_{\mathbb{G}}$ we denote as usual the perimeter measure associated to E_i.

where the second last identity above comes from the fact that $\mathcal{S}^{\mathfrak{Q}-1}(V_i(x)\cap\partial B(0,1)\setminus X_{V_i(x)}(\alpha))=0$ and [De Lellis 2008, Proposition 2.7].

Proposition B.17 and (130) together imply that each one of the intrinsic Lipschitz graphs $\mathrm{gr}(\gamma_i)\cap E(\vartheta,\gamma,R)$ can be covered $\mathcal{S}^{\mathfrak{Q}-1}$-almost all with $C^1_{\mathbb{G}}$-surfaces. In particular this shows that for any ϑ,γ,R the set $E(\vartheta,\gamma,R)$ can be covered $\mathcal{S}^{\mathfrak{Q}-1}$-almost all, and thus ϕ-almost all, by countably many $C^1_{\mathbb{G}}$-surfaces. This, the arbitrariness of $\vartheta,\ \gamma,\ R\in\mathbb{N}$ and (127) conclude the proof of the theorem. □

The following theorem trades off the regularity of tangents, which are assumed only to be close enough to flat measures, with a strengthened hypothesis on the $(\mathfrak{Q}-1)$-density of ϕ.

Theorem 4.2. *Suppose ϕ is a Radon measure on $\mathbb{G}$ and let $\tilde{d}(\cdot,\cdot)$ be a left-invariant, homogeneous distance on $\mathbb{G}$. If there exists a $\delta\in\mathbb{N}$ such that*

$$\delta^{-1}<\liminf_{r\to 0}\frac{\phi(\tilde{B}(x,r))}{r^{\mathfrak{Q}-1}}\leq\limsup_{r\to 0}\frac{\phi(\tilde{B}(x,r))}{r^{\mathfrak{Q}-1}}<\delta\quad\textit{for } \phi\textit{-almost every } x\in\mathbb{G},\tag{133}$$

where $\tilde{B}(x,r)$ is the ball relative to the metric $\tilde{d}$ centered at x of radius $r>0$, then we can find an $\varepsilon(\delta,\tilde{d})>0$ such that, if

$$\limsup_{r\to 0} d_{x,r}(\phi,\mathfrak{M})\leq\varepsilon(\delta,\tilde{d})\quad\textit{for } \phi\textit{-almost every } x\in\mathbb{G},$$

then ϕ is absolutely continuous with respect to $\mathcal{S}^{\mathfrak{Q}-1}$, and ϕ-almost all of $\mathbb{G}$ can be covered with countably many intrinsic Lipschitz surfaces.

Proof. The first step in the proof is to note that since the metric $\tilde{d}$ and d are bi-Lipschitz equivalent, there exists a constant $\mathfrak{c}>1$, which we can assume without loss of generality to be a natural number, such that

$$(\mathfrak{c}\delta)^{-1}<\liminf_{r\to 0}\frac{\phi(B(x,r))}{r^{\mathfrak{Q}-1}}\leq\limsup_{r\to 0}\frac{\phi(B(x,r))}{r^{\mathfrak{Q}-1}}<\mathfrak{c}\delta\quad\text{for } \phi\text{-almost every } x\in\mathbb{G}.$$

If we let $\varepsilon(\delta,\tilde{d}):=4^{-\mathfrak{Q}(\mathfrak{Q}+1)}\varepsilon_2(\mathfrak{c}\delta)$ then the verbatim repetition of the first part of the argument used to prove Theorem 4.1, where instead of Theorem 2.1 we make use of Theorem 3.6, proves the claim. □

An immediate consequence of Theorem 4.2 is the following:

Corollary 4.3. *Let $\vartheta_{\mathbb{G}}:=\max\{l_{\mathbb{G}}^{-1},L_{\mathbb{G}}\}$, where $l_{\mathbb{G}}$ and $L_{\mathbb{G}}$ are the constants yielded by Theorem B.6, and suppose $\Omega\subseteq\mathbb{G}$ is a finite perimeter set such that*

$$\limsup_{r\to 0} d_{x,r}(|\partial\Omega|_{\mathbb{G}},\mathfrak{M})\leq\varepsilon(\vartheta_{\mathbb{G}},d)\quad\textit{for } |\partial\Omega|_{\mathbb{G}}\textit{-almost every } x\in\mathbb{G},$$

where $\varepsilon(\vartheta_G,d)$ is the constant yielded by Theorem 4.2 and d is the metric introduced in Definition 1.4. Then $|\partial\Omega|_{\mathbb{G}}$-almost all of $\mathbb{G}$ can be covered with countably many intrinsic Lipschitz surfaces.

Proof. Theorem B.6 implies that $l_{\mathbb{G}}<\Theta_*^{\mathfrak{Q}-1}(|\partial\Omega|_{\mathbb{G}},x)\leq\Theta^{\mathfrak{Q}-1,*}(|\partial\Omega|_{\mathbb{G}},x)<L_{\mathbb{G}}$ for ϕ-almost every $x\in\mathbb{G}$. Theorem 4.2 directly implies the statement. □

As mentioned at the beginning of this section, the main application of Theorem 4.1 is an extension of Preiss's rectifiability theorem to 1-codimensional measures in $\mathbb{H}^n$.

Theorem 4.4. *Suppose d is the Koranyi metric in $\mathbb{H}^n$ and ϕ is a Radon measure on $\mathbb{H}^n$ such that*

$$0<\Theta^{2n+1}(\phi,x):=\lim_{r\to 0}\frac{\phi(B(x,r))}{r^{2n+1}}<\infty,\quad \text{for } \phi\text{-almost every } x\in\mathbb{H}^n. \tag{134}$$

Then ϕ is absolutely continuous with respect to $\mathcal{S}^{2n+1}$, and ϕ-almost all of $\mathbb{H}^n$ can be covered with $C^1_{\mathbb{H}^n}$-surfaces.

Proof. Thanks to Theorem 1.2 of [Merlo 2022], the almost sure existence of the limit in (134) implies that $\operatorname{Tan}(\phi,x)\subseteq\mathfrak{M}$, for ϕ-almost every $x\in\mathbb{G}$. Thanks to Theorem 4.1, this proves the claim. □

4B. *Discussion of the results.* Theorem 4.1 shows that $C^1_{\mathbb{G}}$-rectifiability in Carnot groups can be characterized by the same conditions on the densities and on the tangents as the Lipschitz rectifiability in Euclidean spaces. With this in mind we introduce the following two definitions:

Definition 4.5 ($\mathscr{P}$-rectifiable measures). Suppose that ϕ is a Radon measure on some Carnot group $\mathbb{G}$ endowed with a left-invariant and homogeneous metric d, and let m be a positive integer. We say that ϕ is $\mathscr{P}_m$-rectifiable if

(i) $0<\Theta^m_*(\phi,x)\le\Theta^{m,*}(\phi,x)<\infty$, for ϕ-almost every $x\in\mathbb{G}$,

(ii) $\operatorname{Tan}_m(\phi,x)\subseteq\{\lambda\mu_x:\lambda>0\}$, for ϕ-almost every $x\in\mathbb{G}$, where μ_x is some Radon measure on $\mathbb{G}$.

Remark 4.6. It was already remarked by P. Mattila [2005] that Definition 4.5 may be considered the correct notion of rectifiability in $\mathbb{H}^1$; see the last paragraph of that work.

Remark 4.7. Instead of condition (ii) of Definition 4.5, we can assume without loss of generality that $\mu_x=\mathcal{H}^m\llcorner V(x)$ for some $V(x)\in\operatorname{Gr}(m)$, where $\operatorname{Gr}(m)$ is the family of m-dimensional homogeneous subgroups of $\mathbb{G}$ introduced in Definition 1.7. This is due to Theorem 3.2 of [Mattila 2005] and Theorem 3.6 of [Onishchik 1993]: the former result tells us that μ_x must be the Haar measure of a closed, dilation-invariant subgroup of $\mathbb{G}$ and the latter that such subgroup is actually a Lie subgroup.

Definition 4.8 ($\mathscr{P}^*$-rectifiable measures). Suppose that ϕ is a Radon measure on some Carnot group $\mathbb{G}$ endowed with a left-invariant and homogeneous metric d, and let m be a positive integer. We say that ϕ is $\mathscr{P}^*_m$-rectifiable if

(i) $0<\Theta^m_*(\phi,x)\le\Theta^{m,*}(\phi,x)<\infty$, for ϕ-almost every $x\in\mathbb{G}$,

(ii) $\operatorname{Tan}_m(\phi,x)\subseteq\mathfrak{M}(m)$, for ϕ-almost every $x\in\mathbb{G}$.

The difference between Definitions 4.5 and 4.8 is that in the former the tangent to ϕ is the same plane at every scale, while in the latter the tangents are planes that may vary at different scales. Although there is no a priori reason for which these definition should be equivalent in general, we see that our main result, Theorem 4.1, may be rewritten as follows:

Theorem 4.9. *Suppose ϕ is a Radon measure on $\mathbb{G}$. Then the following are equivalent*:

(i) *ϕ is $\mathscr{P}_{\mathfrak{Q}-1}$-rectifiable.*

(ii) *ϕ is $\mathscr{P}^*_{\mathfrak{Q}-1}$-rectifiable.*

(iii) *ϕ is absolutely continuous with respect to $\mathcal{H}^{\mathfrak{Q}-1}$, and ϕ-almost all of $\mathbb{G}$ can be covered with countably many $C^1_{\mathbb{G}}$-hypersurfaces.*

The notion of $\mathscr{P}$-rectifiable measures is also relevant since in different contexts it appears to imply the right notion of rectifiability. This is summarized in the following theorem, which is an immediate consequence of the Euclidean Marstrand–Mattila rectifiability criterion and Theorem 4.1:

Theorem 4.10. *The following two statements hold*:

(i) *A Radon measure ϕ on $\mathbb{R}^n$ is $\mathscr{P}_m$-rectifiable if and only if it is Euclidean m-rectifiable*;

(ii) *A Radon measure ϕ on $\mathbb{G}$ is $\mathscr{P}_{\mathfrak{Q}-1}$-rectifiable if and only if it is a 1-codimensional $C^1_{\mathbb{G}}$-rectifiable measure.*

In [Mattila et al. 2010], P. Mattila, F. Serra Cassano and R. Serapioni proved in Theorems 3.14 and 3.15 that whenever a good notion of regular surface is available in the Heisenberg group, provided the tangents are selected carefully (see Definition 2.16 of the aforementioned work), a $\mathscr{P}_m$-rectifiable measure is also rectifiable with respect to the family of regular surfaces of the right dimension. However, because of the algebraic structure of the group $\mathbb{H}^n$, there is not an a priori (known) good notion of regular surface that includes the vertical line $\mathcal{V} := \{(0, 0, t) : t \in \mathbb{R}\}$. For this reason the uniform measure $\mathcal{S}^2 \llcorner \mathcal{V}$ is considered to be *nonrectifiable* from the standpoint of [Mattila et al. 2010]. Up to this point Haar measures of not complemented homogeneous subgroups (like the vertical line $\mathcal{V}$ in $\mathbb{H}^1$) were considered nonrectifiable and thus prevented a possible extension of Preiss's theorem to low dimension even in $\mathbb{H}^1$. This was already remarked in [Chousionis and Tyson 2015]. On the other hand, we have the following theorem:

Theorem 4.11. *Let ϕ be a Radon measure on $\mathbb{H}^1$ such that for ϕ-almost every $x \in \mathbb{H}^1$ we have*

$$0 < \Theta^2(\phi, x) := \lim_{r\to 0} \frac{\phi(B(x, r))}{r^2} < \infty,$$

where $B(x, r)$ are the metric balls with respect to the Koranyi metric. Then ϕ is $\mathscr{P}_2$-rectifiable.

Proof. This follows from Proposition 2.2 of [Merlo 2022] and Theorem 1.4 of [Chousionis et al. 2020]. □

As remarked in the previous paragraph, to our knowledge, there is not a good candidate of rectifiability in Carnot groups in the literature for which the density problem may have a positive answer. On the other hand, Theorems 4.4, 4.10 and 4.11 encourage us to state the density problem in Carnot groups in the following way:

Density Problem. *Suppose ϕ is a Radon measure on the Carnot group $\mathbb{G}$. Then there exists a left-invariant distance d on $\mathbb{G}$ such that the following are equivalent*:

(i) *There exists an $\alpha > 0$ such that for ϕ-almost every $x \in \mathbb{G}$ we have*

$$0 < \Theta^\alpha(\phi, x) := \lim_{r\to 0} \frac{\phi(B(x, r))}{r^\alpha} < \infty.$$

(ii) *$\alpha \in \{0, \dots, \mathfrak{Q}\}$, and ϕ is $\mathscr{P}_\alpha$- rectifiable.*

Neither one of the implications of the formulation of the density problem has an easy solution. In [Antonelli and Merlo 2022a], the current author and G. Antonelli proved the implication (ii) $\Rightarrow$ (i) of the Density Problem when the tangent measures to ϕ are supported on complemented subgroups.

Furthermore, as already observed in [Merlo 2022], if d is a left-invariant distance coming from a polynomial norm on $\mathbb{G}$ with the same argument used in [Kirchheim and Preiss 2002] and later on in [Chousionis and Tyson 2015], it is possible to show that if (i) in the Density Problem holds, then $\alpha \in \mathbb{N}$. In $\mathbb{R}^n$ this implies, thanks to Theorem 3.1 of [Ahmadi et al. 2019], that there is an open and dense set Ω in the space of norms (with the distance induced by the Hausdorff distance of the unit balls) for which, for any $\|\cdot\| \in \Omega$, Marstrand's theorem holds.

Appendix A. Dyadic cubes

Throughout this section *we assume ϕ to be a fixed Radon measure on the Carnot group* $\mathbb{G}$, supported on a compact set K, and such that

$$0 < \liminf_{r\to 0} \frac{\phi(B(x,r))}{r^{\mathfrak{Q}-1}} \leq \limsup_{r\to 0} \frac{\phi(B(x,r))}{r^{\mathfrak{Q}-1}} < \infty, \quad \text{for } \phi\text{-almost every } x \in \mathbb{G}. \tag{135}$$

There are many constructions in the literature of such dyadic cubes for Radon measures both in Euclidean and in (rather general) metric spaces; see for instance [Christ 1990]. In this section we state the existence of a family of dyadic cubes for ϕ, we list their properties and we prove a number of consequences.

Throughout this Appendix, *we will always assume that ξ and τ are two fixed natural numbers* such that $\phi(E^\phi(\xi,\tau)) > 0$, where the set $E^\phi(\xi,\tau)$ was defined in Proposition 1.28.

Definition A.1. For any subset A of $\mathbb{G}$ and any $\delta > 0$, we let

$$\partial(A,\delta) := \{u \in A : \mathrm{dist}(u, K \setminus A) \leq \delta\} \cup \{u \in K \setminus A : \mathrm{dist}(u, A) \leq \delta\},$$

where we recall that K is the compact set supporting the measure ϕ.

For the rest of this subsection, we simplify the expressions of the constants introduced in Notation 2.8 to

$$N := N(\xi), \quad \zeta := \zeta(\xi), \quad C_4 := C_4(\xi), \quad C_5 := C_5(\xi), \quad C_6 := C_6(\xi).$$

The construction of the dyadic cubes for the measure ϕ under the hypothesis (135) can be performed with a very similar approach to that employed for AD-regular measures in [David 1991, Appendix 1]. However, since (135) is a weaker condition than the AD-regularity, the construction needs some tweaks. For the sake of completeness we recall that a dyadic lattice for general Radon measures in the Euclidean spaces was constructed in [David and Mattila 2000, Section 3] and that such proof still follows pretty closely the argument of [David 1991, Appendix 1].

In order to adapt the construction in [David 1991], one reduces to discussing the properties of those cubes that intersect the set $E^\phi(\xi,\tau)$, where the measure ϕ behaves locally as an AD-regular measure; see items (iii) and (v) of Theorem A.2 where a uniform bound on the lower density of the measure is crucially exploited. Items (i) and (ii) hold by construction while (iv) can be seen as a fancy way of saying that since ϕ is a radon measure, almost every sphere has null measure. For a complete construction of these cubes we refer to the version of this paper that can be found in the arXiv [Merlo 2020, Subsection A.3].

Theorem A.2. *There are disjoint partitions $\{\Delta_j^\phi(\xi,\tau)\}_{j\in\mathbb{N}}$, usually called **layers**, of K having the following properties*:

(i) *If $j\leq j'$, $Q\in\Delta_j^\phi(\xi,\tau)$ and $Q'\in\Delta_{j'}^\phi(\xi,\tau)$, then either Q contains Q' or $Q\cap Q'=\varnothing$.*

(ii) *If $Q\in\Delta_j^\phi(\xi,\tau)$, we have* $\mathrm{diam}(Q)\leq 2^{-Nj+5}/\tau$.

(iii) *If $Q\in\Delta_j^\phi(\xi,\tau)$ and $Q\cap E^\phi(\xi,\tau)\neq\varnothing$, then $C_4^{-1}(2^{-Nj}/\tau)^{\mathfrak{Q}-1}\leq\phi(Q)\leq C_4(2^{-Nj}/\tau)^{\mathfrak{Q}-1}$.*

(iv) *If $Q\in\Delta_j^\phi(\xi,\tau)$, we have $\phi(\partial(Q,\zeta^2 2^{-Nj}/\tau))\leq C_4\zeta(2^{-Nj}/)^{\mathfrak{Q}-1}$.*

(v) *If $Q\in\Delta_j^\phi(\xi,\tau)$ and $Q\cap E^\phi(\xi,\tau)\neq\varnothing$, there exists a $\mathfrak{c}(Q)\in Q$ such that $B(\mathfrak{c}(Q),\zeta^2 2^{-Nj-1}/\tau)\subseteq Q$.*

We define $\Delta^\phi(\xi,\tau):=\bigcup\{Q:Q\in\Delta_j^\phi(\xi,\tau)$ for some $j\in\mathbb{N}\}$ and call it the family of all dyadic cubes.

Remark A.3. Part (iii) of Theorem A.2 can be rephrased in the following useful way. Recalling that $C_5(\xi)=C_4(32\zeta^{-2})^{\mathfrak{Q}-1}$ and putting together Theorem A.2 (ii), (iii) and (v) we infer that

(iii)′ if $Q\cap E^\phi(\xi,\tau)\neq\varnothing$, then $C_5^{-1}\,\mathrm{diam}\,Q^{\mathfrak{Q}-1}\leq\phi(Q)\leq C_5\,\mathrm{diam}\,Q^{\mathfrak{Q}-1}$.

The families of cubes yielded by Theorem A.2 may have the annoying property that for a fixed cube $Q\in\Delta_j^\phi(\xi,\tau)$, the only subcube of Q in the layer $\Delta_{j+1}^\phi(\xi,\tau)$ contained in Q is just Q itself. The following proposition shows that this is not much of a problem for the cubes intersecting $E^\phi(\xi,\tau)$.

Proposition A.4. *Recall that given two cubes $Q_1,Q_2\in\Delta^\phi(\xi,\tau)$, if Q_2 is the smallest cube for which $Q_1\subsetneq Q_2$, then Q_2 is said to be the **parent** of Q_1.*

Suppose $Q^*\in\Delta_j^\phi(\xi,\tau)$ is the parent of some cube $Q\in\Delta_{j+\kappa}^\phi(\xi,\tau)$ such that $Q\cap E^\phi(\xi,\tau)\neq\varnothing$. Then

$$\kappa<\lfloor 2\log_2 C_4/N(\mathfrak{Q}-1)\rfloor+1\quad\textit{and}\quad\frac{\mathrm{diam}\,Q^*}{\mathrm{diam}\,Q}\leq C_6.$$

Proof. Suppose $\tilde{Q}$ is the ancestor of the cube Q contained in the layer $\Delta_{j'}^\phi(\xi,\tau)$ for some j' for which $j'-j\geq\lfloor 2\log_2 C_4/N(\mathfrak{Q}-1)\rfloor+1$. Then $\tilde{Q}\cap E^\phi(\xi,\tau)\neq\varnothing$, and thanks to Theorem A.2 (i) and (iii), we infer that

$$\begin{aligned}\phi(\tilde{Q}\setminus Q)=\phi(\tilde{Q})-\phi(Q)&\geq C_4^{-1}\left(\frac{2^{-jN}}{\tau}\right)^{\mathfrak{Q}-1}-C_4\left(\frac{2^{-j'N}}{\tau}\right)^{\mathfrak{Q}-1}\\&=C_4^{-2}\left(\frac{2^{-jN}}{\tau}\right)^{\mathfrak{Q}-1}(1-C_4^2 2^{-(j'-j)N(\mathfrak{Q}-1)})>0,\end{aligned}\tag{136}$$

where the last inequality above comes from the choice of $j'-j$. It is immediate to see that inequality (136) implies that Q is strictly contained in $\tilde{Q}$. Therefore, the parent cube of Q must be contained in some $\Delta_{j'-\kappa}^\phi(\xi,\tau)$ with $0\leq\kappa<\lfloor 2\log C_4/N(\mathfrak{Q}-1)\rfloor+1$. Hence, thanks to Theorem A.2 (v), we infer that

$$\begin{aligned}\mathrm{diam}\,Q^*\leq 2^{-Nj+5}/\tau=2^{N\kappa+6}\zeta^{-2}\cdot\zeta^2 2^{-N(j+\kappa)-1}/\tau&\leq 2^{N\kappa+6}\zeta^{-2}\,\mathrm{diam}\,Q\\&\leq 2^{2\log C_4/(\mathfrak{Q}-1)+N+6}\zeta^{-2}\,\mathrm{diam}\,Q=C_6\,\mathrm{diam}\,Q.\end{aligned}$$

□

The following result tells us that item (v) of Theorem A.2 in some cases can be strengthened to assuming that the center of the cube $\mathfrak{c}(Q)$ is contained in $E^\phi(\xi,\tau)$.

Proposition A.5. *Assume $\mu\in\mathbb{N}$ is such that $\mu\geq 4C_4\xi$. Then, for any cube $Q\in\Delta^\phi(\mathscr{E}^\phi_{\xi,\tau}(\mu,\nu);\xi,\tau,\nu)$, we can find a $\mathfrak{c}(Q)\in E^\phi(\xi,\tau)\cap Q$ such that*

$$B\big(\mathfrak{c}(Q),\tfrac{1}{64}\zeta^2\operatorname{diam} Q\big)\cap K\subseteq Q.$$

Remark A.6. Recall that the set $\mathscr{E}^\phi_{\xi,\tau}(\mu,\nu)$ was introduced in Proposition 1.29 and $\Delta^\phi(\kappa;\xi,\tau,\nu)$ in Notation 2.8.

Proof. In order to prove the proposition it suffices to show that

$$E^\phi(\xi,\tau)\cap Q\setminus\partial\big(Q,\tfrac{1}{32}\zeta^2\operatorname{diam} Q\big)\neq\varnothing. \tag{137}$$

In order to fix ideas, we let $j\geq\nu$ be such that $Q\in\Delta^\phi_j(\xi,\tau)$ and note that since $Q\cap E^\phi(\xi,\tau)\neq\varnothing$, thanks to Theorem A.2 (ii), (iii) and (iv), we have

$$\begin{aligned}
&\phi\big(E^\phi(\xi,\tau)\cap Q\setminus\partial\big(Q,\tfrac{1}{32}\zeta^2\operatorname{diam} Q\big)\big)\\
&\qquad\geq\ \phi(E^\phi(\xi,\tau)\cap Q)-\phi\big(\partial\big(Q,\tfrac{1}{32}\zeta^2\operatorname{diam} Q\big)\big)\overset{\text{A.2 (ii)}}{\geq}\phi(E^\phi(\xi,\tau)\cap Q)-\phi(\partial(Q,\zeta^2 2^{-jN}/\tau))\\
&\qquad\overset{\text{A.2 (iv)}}{\geq}\phi(E^\phi(\xi,\tau)\cap Q)-C_4\zeta(2^{-jN}/\tau)^{\mathfrak{Q}-1}=\phi(Q)-\phi(Q\setminus E^\phi(\xi,\tau))-C_4\zeta(2^{-jN}/\tau)^{\mathfrak{Q}-1}\\
&\qquad\overset{\text{A.2 (iii)}}{\geq}\phi(Q)-\phi(Q\setminus E^\phi(\xi,\tau))-C_4^2\zeta\phi(Q).
\end{aligned}\tag{138}$$

Since $Q\in\Delta^\phi(\mathscr{E}^\phi_{\xi,\tau}(\mu,\nu);\xi,\tau,\nu)$, we have $\operatorname{diam} Q\leq 2^{-N\nu+5}/\tau$ and there exists a $w\in\mathscr{E}^\phi_{\xi,\tau}(\mu,\nu)\cap Q$. Therefore, the definition of $\mathscr{E}^\phi_{\xi,\tau}(\mu,\nu)$ and Theorem A.2 (iii) imply that

$$\begin{aligned}
\phi(Q\setminus E^\phi(\xi,\tau))\leq\phi(B(w,2^{-jN+5}/\tau)\setminus E^\phi(\xi,\tau))&\leq\mu^{-1}\phi(B(w,2^{-jN+5}/\tau))\\
&\leq\mu^{-1}\xi(2^{-jN+5}/\tau)^{\mathfrak{Q}-1}\leq C_4\mu^{-1}\xi\phi(Q).
\end{aligned}\tag{139}$$

Putting together (138) and (139), we conclude that

$$\phi(E^\phi(\xi,\tau)\cap Q\setminus\partial(Q,\zeta^2\operatorname{diam} Q))\geq(1-C_4\mu^{-1}\xi-C_4^2\zeta)\phi(Q)\geq\tfrac{1}{4}\phi(Q),$$

where the last inequality follows from the fact that $C_4^2\zeta=2^{48\mathfrak{Q}}\xi^2\cdot 2^{-50\mathfrak{Q}}\xi^{-2}\leq\frac{1}{2}$ and $C_4\mu^{-1}\xi\leq\frac{1}{4}$. This proves (137) and in turn the proposition. □

Appendix B. Finite perimeter sets in Carnot groups

Throughout this second appendix if not otherwise stated, *we will always endow $\mathbb{G}$ with the box metric d introduced in Definition 1.4.*

Finite perimeter sets and their blow ups. In this subsection we recall the definitions of functions of bounded variation and finite perimeter sets, and we collect from various papers some results that will be useful throughout the paper.

Definition B.1. We say that a function $f : \mathbb{G} \to \mathbb{R}$ is of *local bounded variation* if $f \in L^1_{\mathrm{loc}}(\mathbb{G})$ and

$$\|\nabla_{\mathbb{G}} f\|(\Omega) := \sup\left\{\int_\Omega f(x)\,\mathrm{div}_{\mathbb{G}}\,\varphi(x)\,dx : \varphi \in \mathcal{C}^1_0(\Omega, H\mathbb{G}),\ |\varphi(x)| \le 1\right\} < \infty,$$

for any bounded open set $\Omega \subseteq \mathbb{G}$, where $\mathrm{div}_{\mathbb{G}}\,\varphi := \sum_{i=1}^{n_1} X_i\varphi_i$ and where $X_1, \dots, X_{n_1}$ are the vector fields introduced in Definition 1.33. We denote by $\mathrm{BV}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$ the set of all functions of locally bounded variation. As usual a Borel set $E \subseteq \mathbb{G}$ is said to be of *finite perimeter* if χ_E is of bounded variation.

The following result is a classical application of Riesz's representation theorem:

Theorem B.2. *If f is a function of bounded variation, then $\|\nabla_{\mathbb{G}} f\|$ is a Radon measure on $\mathbb{G}$. Moreover, there exists a $\|\nabla_{\mathbb{G}} f\|$-measurable horizontal section $\sigma_f : \mathbb{G} \to H\mathbb{G}$ such that $|\sigma_f(x)| = 1$ for $\|\nabla_{\mathbb{G}} f\|$-almost every $x \in \mathbb{G}$ and for any open set Ω we have*

$$\int_\Omega f(x)\,\mathrm{div}_{\mathbb{G}}\,\varphi(x)\,dx = \int_\Omega \langle \varphi, \sigma_f\rangle\, d\|\nabla_{\mathbb{G}} f\|, \quad \textit{for every } \varphi \in \mathcal{C}^1_0(\Omega, H\mathbb{G}).$$

As in the Euclidean spaces functions of bounded variation are compactly embedded in L^1.

Theorem B.3 [Franchi et al. 2003, Theorem 2.16]. *The set* $\mathrm{BV}_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$ *is compactly embedded in* $L^1_{\mathrm{loc}}(\mathbb{G})$.

Definition B.4. If $E \subseteq \mathbb{G}$ is a Borel set of locally finite perimeter, we let $|\partial E|_{\mathbb{G}} := \|\nabla_{\mathbb{G}}\chi_E\|$. Furthermore, we call the horizontal vector $\mathfrak{n}_E(x) := \sigma_{\chi_E}(x)$ the *generalized horizontal inward $\mathbb{G}$-normal to ∂E*. Finally, we define the *reduced boundary* $\partial^*_{\mathbb{G}} E$ to be the set of those $x \in \mathbb{G}$ for which

(i) $|\partial E|_{\mathbb{G}}(B(x, r)) > 0$ for any $r > 0$,

(ii) $\lim_{r\to 0} \fint_{B(x,r)} \mathfrak{n}_E\, d|\partial E|_{\mathbb{G}}$ exists,

(iii) $\lim_{r\to 0} \left|\fint_{B(x,r)} \mathfrak{n}_E\, d|\partial E|_{\mathbb{G}}\right|_{\mathbb{R}^{n_1}} = 1$.

The following lemma on the scaling of the perimeter will come in handy later on.

Lemma B.5. *Assume E is a set of finite perimeter in $\mathbb{G}$ and let $x \in \mathbb{G}$ and $r > 0$. Then*

$$|\partial(\delta_{1/r}(x^{-1}E))|_{\mathbb{G}} = r^{-(\mathfrak{Q}-1)} T_{x,r}|\partial E|_{\mathbb{G}}.$$

Proof. For any $\varphi \in C^1_0(\mathbb{G}, H\mathbb{G})$, any $x \in \mathbb{G}$ and any $r > 0$, defining $\tilde\varphi(z) := \varphi(\delta_{1/r}(x^{-1}z))$, we have the identity

$$\mathrm{div}_{\mathbb{G}}\,\tilde\varphi(z) = r^{-1}\,\mathrm{div}_{\mathbb{G}}\,\varphi(\delta_{1/r}(x^{-1}z)). \tag{140}$$

This, indeed, is due to the fact that

$$\begin{aligned} X_j\tilde\varphi_j(z) &:= \lim_{h\to 0} \frac{\tilde\varphi_j(z\delta_h(e_j)) - \tilde\varphi_j(z)}{h} \\ &= \lim_{h\to 0} \frac{\varphi_j(\delta_{1/r}(x^{-1}z\delta_h(e_j))) - \varphi_j(\delta_{1/r}(x^{-1}z))}{h} = r^{-1} X_j\varphi_j(\delta_{1/r}(x^{-1}z)). \end{aligned}$$

Thanks to identity (140) and the fact that the Lebesgue measure is a Haar measure for $\mathbb{G}$, we infer that

$$\int \chi_{\delta_{1/r}(x^{-1}E)}(y)\,\mathrm{div}_{\mathbb{G}}\,\varphi(y)\,dy = r^{-\mathfrak{Q}} \int \chi_E\,\mathrm{div}_{\mathbb{G}}\,\varphi(\delta_{1/r}(x^{-1}y))\,dy = r^{-(\mathfrak{Q}-1)} \int \chi_E(y)\,\mathrm{div}_{\mathbb{G}}\,\tilde\varphi(y)\,dy.$$

It is not hard to see that $\varphi \in C^1_0(\Omega, H\mathbb{G})$ if and only if $\tilde{\varphi} \in C^1_0(x\delta_r\Omega, H\mathbb{G})$, and thus for any open set Ω we have

$$|\partial(\delta_{1/r}(x^{-1}E))|_{\mathbb{G}}(\Omega) = r^{-(\mathfrak{Q}-1)}|\partial E|_{\mathbb{G}}(x\delta_r\Omega) = r^{-(\mathfrak{Q}-1)}T_{x,r}|\partial E|_{\mathbb{G}}(\Omega). \qquad \square$$

Theorem B.6 [Ambrosio et al. 2009, Theorem 4.16]. *Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter. Then $|\partial E|_{\mathbb{G}}$ is asymptotically doubling, and more precisely the following holds. For $|\partial E|_{\mathbb{G}}$-almost every $x \in \mathbb{G}$ there exists an $\bar{r}(x) > 0$ such that*

$$l_{\mathbb{G}} r^{\mathfrak{Q}-1} \le |\partial E|_{\mathbb{G}}(B(x,r)) \le L_{\mathbb{G}} 2^{-(\mathfrak{Q}-1)} r^{\mathfrak{Q}-1}, \quad \textit{for any } r \in (0, \bar{r}(x)), \tag{141}$$

where the constants $l_{\mathbb{G}}$ and $L_{\mathbb{G}}$ depend only on $\mathbb{G}$ and the metric d and $|\partial E|_{\mathbb{G}}$ is concentrated on $\partial^*_{\mathbb{G}}E$, i.e., $|\partial E|_{\mathbb{G}}(\mathbb{G} \setminus \partial^*_{\mathbb{G}}E) = 0$.

Remark B.7. Proposition 1.31 and Theorem B.6 imply that $l_{\mathbb{G}}\mathcal{S}^{\mathfrak{Q}-1} \llcorner \partial^*_{\mathbb{G}}E \le |\partial E|_{\mathbb{G}} \le L_{\mathbb{G}}\mathcal{S}^{\mathfrak{Q}-1} \llcorner \partial^*_{\mathbb{G}}E$. Therefore, the measures $\mathcal{S}^{\mathfrak{Q}-1} \llcorner \partial^*_{\mathbb{G}}E$ and $|\partial E|_{\mathbb{G}}$ are mutually absolutely continuous. In particular there exists a $\mathfrak{d} \in L^1(|\partial E|_{\mathbb{G}})$ such that

$$\mathcal{S}^{\mathfrak{Q}-1} \llcorner \partial^*_{\mathbb{G}}E = \mathfrak{d}|\partial E|_{\mathbb{G}},$$

and for $|\partial E|_{\mathbb{G}}$-almost every $x \in \mathbb{G}$ we have $L^{-1}_{\mathbb{G}} \le \mathfrak{d}(x) \le l^{-1}_{\mathbb{G}}$.

Theorem B.8 [Franchi and Serapioni 2016, Theorem 3.9]. *If $f : V \to \mathfrak{N}(V)$ is an intrinsic Lipschitz map, the epigraph of f,*

$$\operatorname{epi}(f) := \{v * \delta_t(\mathfrak{n}(V)) : t < \langle \pi_1 f(v), \mathfrak{n}(V)\rangle\}, \tag{142}$$

is a set with locally finite $\mathbb{G}$-perimeter.

Since the topological boundary of $\operatorname{epi}(f)$ coincides with $\operatorname{gr}(f)$, thanks to [Franchi and Serapioni 2016, Theorem 3.9], we infer that $|\partial \operatorname{epi}(f)|_{\mathbb{G}}(\mathbb{G} \setminus \partial^*_{\mathbb{G}} \operatorname{epi}(f)) = |\partial \operatorname{epi}(f)|_{\mathbb{G}}(\operatorname{gr}(f) \setminus \partial^*_{\mathbb{G}} \operatorname{epi}(f)) = 0$. In particular, thanks to Remark B.7, we deduce the following proposition:

Proposition B.9. $\mathcal{S}^{\mathfrak{Q}-1}(\operatorname{gr}(f) \setminus \partial^*_{\mathbb{G}} \operatorname{epi}(f)) = 0$.

It is convenient to associate a normal vector field to the graph of every intrinsic Lipschitz function $f : V \to \mathfrak{N}(V)$.

Definition B.10. For any intrinsic Lipschitz function $f : V \to \mathfrak{N}(V)$, we denote by $\mathfrak{n}_f : \partial^*_{\mathbb{G}} \operatorname{epi}(f) \to H\mathbb{G}$ the inward inner $\mathbb{G}$-normal of $\operatorname{epi}(f)$.

Tangents measures versus tangent sets to finite perimeter sets. In this subsection we connect the notion of tangent sets to finite perimeter sets, which is extensively used in the theory of finite perimeter sets, to the notion of tangent measures. This will help us to prove that if the perimeter measure associated to the boundary of a finite perimeter set has flat tangents, then it has a unique tangent that coincides with the plane in $\operatorname{Gr}(\mathfrak{Q}-1)$ orthogonal to the normal.

Definition B.11 (tangent sets). Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter and assume $x \in \partial^*_{\mathbb{G}}E$. We denote by $\operatorname{Tan}(E, x)$ the limit points in the topology of the local convergence in measure of the sets $\{\delta_{1/r}(x^{-1}E)\}_{r>0}$ as $r \to 0$.

For a proof of the following proposition, we refer to [Ambrosio et al. 2009] and in particular to Proposition 5.3.

Proposition B.12. *If E is a set of finite perimeter, for $\mathcal{S}^{\mathfrak{Q}-1}$-almost every $x \in \partial^*_{\mathbb{G}} E$ we have*

(i) $\mathrm{Tan}(E, x) \neq \varnothing$,

(ii) *the elements of* $\mathrm{Tan}(E, x)$ *are sets of locally finite perimeter sets*,

(iii) *for any $F \in \mathrm{Tan}(E, x)$, that $\mathfrak{n}_F(y) = \mathfrak{n}_E(x)$ for $|\partial F|_{\mathbb{G}}$-almost every $y \in \mathbb{G}$.*

The following proposition is a characterization of the tangent measures of perimeter measures.

Proposition B.13. *If E is a set of locally finite perimeter, for $|\partial E|_{\mathbb{G}}$-almost every $x \in \partial^*_{\mathbb{G}} E$ we have the following*:

(i) *If $\{r_i\}_{i\in\mathbb{N}}$ is an infinitesimal sequence such that $\delta_{1/r_i}(x^{-1}E)$ converges locally in measure to some Borel set L, then L is a finite perimeter set and $r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}|\partial E|_{\mathbb{G}} \rightharpoonup |\partial L|_{\mathbb{G}}$. In particular, if $L \in \mathrm{Tan}(E, x)$, then $|\partial L|_{\mathbb{G}} \in \mathrm{Tan}_{\mathfrak{Q}-1}(|\partial E|_{\mathbb{G}}, x)$.*

(ii) *If $\nu \in \mathrm{Tan}_{\mathfrak{Q}-1}(|\partial E|_{\mathbb{G}}, x)$, then there is an $L \in \mathrm{Tan}(E, x)$ such that $\nu = |\partial L|_{\mathbb{G}}$.*

Proof. Let us first prove (i). From now on, thanks to Proposition B.12, we can assume without loss of generality that x is a fixed point where properties (i), (ii) and (iii) of Proposition B.12 hold. Fix now an open and bounded set Ω of $\mathbb{G}$ and note that, defining $E_i := \delta_{1/r_i}(x^{-1}E)$, we have

$$\|\chi_{E_i}\|_{L^1(\Omega)} + \|\nabla_{\mathbb{G}}\chi_{E_i}\|(\Omega) \leq \mathcal{L}^n(\Omega) + r_i^{-(\mathfrak{Q}-1)}|\partial E|_{\mathbb{G}}(x\delta_{r_i}\Omega). \tag{143}$$

The above bound implies that χ_{E_i} is a compact sequence in $L^1(\Omega)$ thanks to Theorems B.3 and B.6 and thus the sets E_i converge in $L^1(\Omega)$ to some locally finite perimeter set E which must coincide $\mathcal{L}^n$-almost everywhere with L inside Ω, by the uniqueness of the limit in measure. This implies in particular that for any $\varphi \in \mathcal{C}^1_0(\Omega, H\mathbb{G})$ we have

$$\begin{aligned}\lim_{i\to 0}\int_\Omega \langle \varphi, \mathfrak{n}_{E_i}\rangle\, d|\partial E_i|_{\mathbb{G}} &= \lim_{i\to 0}\int \chi_{E_i\cap\Omega}(y)\,\mathrm{div}_{\mathbb{G}}\,\varphi(y)\,dy \\ &= \int \chi_{L\cap\Omega}(y)\,\mathrm{div}_{\mathbb{G}}\,\varphi(y)\,dy = \int_\Omega \langle\varphi, \mathfrak{n}_L\rangle\, d|\partial L|_{\mathbb{G}}.\end{aligned} \tag{144}$$

The above identity (144) implies in particular that $\mathfrak{n}_{E_i}|\partial E_i|_{\mathbb{G}} \llcorner \Omega \rightharpoonup \mathfrak{n}_L|\partial L|_{\mathbb{G}} \llcorner \Omega$. However, the arbitrariness of Ω and the well-known fact that the weak convergence implies the convergence of the total variations implies that $|\partial E_i|_{\mathbb{G}} \rightharpoonup |\partial L|_{\mathbb{G}}$. The second part of the statement of (i) follows immediately from Lemma B.5.

We now prove (ii). We can assume without loss of generality that $x = 0$ satisfy the thesis of Theorem B.6 and that $\{r_i\}$ is an infinitesimal sequence such that

$$r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}|\partial E|_{\mathbb{G}} \rightharpoonup \nu \in \mathrm{Tan}_{\mathfrak{Q}-1}(|\partial E|_{\mathbb{G}}, x).$$

Now let $E_i := \delta_{1/r_i}(E)$, so that $|\partial E_i|_{\mathbb{G}} = r_i^{\mathfrak{Q}-1} T_{0,r_i}|\partial E|_{\mathbb{G}}$. For any open and bounded set Ω we can find an $R > 0$ such that $\Omega \subseteq B(0, R)$. Therefore, thanks to Theorem B.3, we have

$$|\partial(\delta_{1/r_i}(x^{-1}E))|_{\mathbb{G}}(\Omega) \leq |\partial(\delta_{1/r_i}(x^{-1}E))|_{\mathbb{G}}(B(0,R)) = r_i^{-(\mathfrak{Q}-1)} T_{x,r_i}|\partial E|_{\mathbb{G}}(B(0,R)) = \frac{|\partial E|_{\mathbb{G}}(B(x, Rr_i))}{r_i^{\mathfrak{Q}-1}}.$$

Since we assumed that Theorem B.6 holds at x, we have

$$\limsup_{i\to\infty}|\partial(\delta_{1/r}(x^{-1}E))|_{\mathbb{G}}(\Omega)\le\limsup_{i\to\infty}\frac{|\partial E|_{\mathbb{G}}(B(x,Rr_i))}{r_i^{\mathfrak{Q}-1}}\le L_{\mathbb{G}}R^{\mathfrak{Q}-1}.$$

Thus, thanks to Theorem B.3, the sequence $\{\delta_{1/r_i}(x^{-1}E)\}_{i\in\mathbb{N}}$ is precompact in $L^1_{\mathrm{loc}}(\mathbb{G})$ and since we assumed $\delta_{1/r_i}(x^{-1}E)$ converges locally in measure to L, we have that $\delta_{1/r_i}(x^{-1}E)$ converges in $L^1_{\mathrm{loc}}(\mathbb{G})$ to L. In particular, thanks to Theorem 2.17 of [Franchi et al. 2003], we infer that L is of local finite perimeter. Thus, by definition of the tangent sets, we have $L\in\mathrm{Tan}(E,0)$, and thanks to item (i), we conclude that $r_i^{-(\mathfrak{Q}-1)}T_{0,r_i}|\partial E|_{\mathbb{G}}\rightharpoonup|\partial L|_{\mathbb{G}}$. Thanks to the uniqueness of the limit we conclude that $|\partial L|_{\mathbb{G}}=\nu$. □

Proposition B.14. *If E is an open set of finite perimeter in $\mathbb{G}$, for $\mathcal{S}^{\mathfrak{Q}-1}$-almost any $x\in\partial E$ and any $L\in\mathrm{Tan}(E,x)$ we have $\mathcal{L}^n(L\setminus\mathrm{int}(L))=0$. In particular, the measures $|\partial L|_{\mathbb{G}}$ and $|\partial(\mathrm{int}(L))|_{\mathbb{G}}$ coincide on Borel sets.*

Proof. This proposition follows for instance from Proposition B.12 and [Bellettini and Le Donne 2021, Theorem 1.1]. □

Remark B.15. Let $V_\pm:=\{w\in\mathbb{G}:\pm\langle\mathfrak{n}(V),w\rangle>0\}$. Thanks to (2.8) in [Ambrosio et al. 2009], it is immediate to see that $V_\pm$ are open sets of locally finite perimeter in $\mathbb{G}$ and that $\partial V_\pm=\mp\mathfrak{n}(V)\mathcal{H}^{n-1}_{\mathrm{eu}}\llcorner V$. This implies that the horizontal normal of each of the half spaces determined by V coincides, up to a sign, $|\partial V_\pm|_{\mathbb{G}}$-almost everywhere with $\mathfrak{n}(V)$.

Proposition B.16. *Let $V\in\mathrm{Gr}(\mathfrak{Q}-1)$ and $f:V\to\mathfrak{N}(V)$ be an intrinsic Lipschitz function. Suppose that E is a compact subset of $\mathrm{gr}(\gamma)$ such that*

$$\mathrm{Tan}_{\mathfrak{Q}-1}(|\partial\,\mathrm{epi}(f)|_{\mathbb{G}},x)\subseteq\mathfrak{M},\quad \textit{for } |\partial\,\mathrm{epi}(f)|_{\mathbb{G}}\textit{-almost every } x\in E.$$

Then for $|\partial\,\mathrm{epi}(f)|_{\mathbb{G}}$-almost every $x\in E$, we have

$$\mathrm{Tan}_{\mathfrak{Q}-1}(|\partial\,\mathrm{epi}(f)|_{\mathbb{G}},x)\subseteq\{\lambda\mathcal{S}^{\mathfrak{Q}-1}\llcorner V(x):\lambda\in[L_{\mathbb{G}}^{-1},l_{\mathbb{G}}^{-1}]\},$$

where $V(x)\in\mathrm{Gr}(\mathfrak{Q}-1)$ is the plane orthogonal to $\mathfrak{n}_f(x)$, which is the normal to $\mathrm{gr}(f)$ introduced in Definition B.10, and where the constants $l_{\mathbb{G}}$ and $L_{\mathbb{G}}$ were introduced in Theorem B.6.

Proof. Proposition B.13, the asymptotic AD-regularity of the perimeter and Lebesgue's differentiation theorem at [Heinonen et al. 2015, page 77] imply that for $\mathcal{S}^{\mathfrak{Q}-1}$-almost every $x\in\partial^*_{\mathbb{G}}\,\mathrm{epi}(f)\cap E$ and for every $L\in\mathrm{Tan}(\mathrm{epi}(f),x)$ we have

$$|\partial L|_{\mathbb{G}}=\lambda\mathcal{S}^{\mathfrak{Q}-1}\llcorner V_{L,x}\quad\text{for some } V_{L,x}\in\mathrm{Gr}(\mathfrak{Q}-1)\ \text{and}\ \lambda>0. \tag{145}$$

Furthermore Remark B.7, Proposition 1.8 and a simple computation that we omit, imply that $\lambda\in[l_{\mathbb{G}},L_{\mathbb{G}}]$.

Fix now an $x\in\partial^*\,\mathrm{epi}(f)\cap E$ at which (145) holds and that satisfies the thesis of Proposition B.12, and let $L\in\mathrm{Tan}(\mathrm{epi}(f),x)$. Thanks to these choices, L is a finite perimeter set with constant horizontal normal and Proposition B.9 and (145) tell us that its topological boundary must coincide up to $\mathcal{S}^{\mathfrak{Q}-1}$-null sets with the plane $V_{L,x}$. Therefore, since by Proposition B.14 we can assume without loss of generality that L is an open set, we conclude that L must coincide with one of the two half-spaces determined by $V_{L,x}$. This implies however, thanks to Remark B.15, that

$$\mathfrak{n}(V_{L,x})=\mathfrak{n}_L(y)\quad \text{for } \mathcal{S}^{\mathfrak{Q}-1}\text{-almost every } y\in\partial L. \tag{146}$$

Furthermore, Proposition B.12 (iii) and (146) imply that $\mathfrak{n}(V_{L,x})=\mathfrak{n}_L(y)=\mathfrak{n}_f(x)$ for $\mathcal{S}^{\mathfrak{Q}-1}$-almost all $y\in\partial L$. This shows however that for $\mathcal{S}^{\mathfrak{Q}-1}$-almost all $x\in\mathrm{gr}(f)\cap E$, every element of $\mathrm{Tan}(\mathrm{epi}(f),x)$ is a half-space whose boundary is the plane orthogonal to $\mathfrak{n}_f(x)$ and Proposition B.13 concludes the proof. □

Proposition B.17. *Suppose E is a compact subset of V and let $\gamma: E\subseteq V\to\mathfrak{N}(V)$ be an intrinsic Lipschitz function such that for $\mathcal{S}^{\mathfrak{Q}-1}$-almost every $x\in E$ there exists a plane $V_\gamma(x)\in\mathrm{Gr}(\mathfrak{Q}-1)$ for which*

$$\lim_{r\to 0}\frac{\mathcal{S}^{\mathfrak{Q}-1}(\mathrm{gr}(\gamma)\cap B(x\gamma(x),r)\setminus x\gamma(x)X_{V_\gamma(x\gamma(x))}(\alpha))}{r^{\mathfrak{Q}-1}}=0 \tag{147}$$

whenever $\alpha>0$, and where $X_{V_\gamma(x\gamma(x))}(\alpha):=\{w\in\mathbb{G}:\mathrm{dist}(w,V(x\gamma(x)))\le\alpha\|w\|\}$. Then $\mathrm{gr}(\gamma)$ can be covered with countably many $C^1_{\mathbb{G}}$-surfaces.

Proof. Since the graph map $x\mapsto x*\gamma(x)$ is continuous, let us notice that the set $\mathrm{gr}(\gamma)$ is compact and for any $i\in\mathbb{N}$ let us define the sets

$$A_i:=\{x\in\mathrm{gr}(\gamma):\text{(147) holds at } x \text{ and } \mathcal{S}^{\mathcal{Q}-1}(B(x,r)\cap\mathrm{gr}(\gamma))\ge 2^{-1}L_{\mathbb{G}}^{-1}l_{\mathbb{G}}r^{\mathfrak{Q}-1} \text{ for any } 0<r<1/i\}.$$

As a first step in the proof, we show that the A_i are $\mathcal{S}^{\mathcal{Q}-1}\llcorner\mathrm{gr}(\gamma)$-measurable. It is immediate to see that if we show that the set

$$\tilde{A}_i:=\{x\in\mathrm{gr}(\gamma):\mathcal{S}^{\mathcal{Q}-1}(B(x,r)\cap\mathrm{gr}(\gamma))\ge 2^{-1}L_{\mathbb{G}}^{-1}l_{\mathbb{G}}r^{\mathfrak{Q}-1} \text{ for any } 0<r<1/i\}$$

is closed, the measurability of A_i immediately follows since (147) holds on a set of full $\mathcal{S}^{\mathcal{Q}-1}\llcorner\mathrm{gr}(\gamma)$-measure. Since $\mathrm{gr}(\gamma)$ is closed, to prove the closedness of $\tilde{A}_i$ it is sufficient to show that if a sequence $\{x_j\}_{j\in\mathbb{N}}\subseteq\tilde{A}_i$ converges to some $x\in\mathrm{gr}(\gamma)$, then $x\in\tilde{A}_i$. So, let $0<r<1/i$ and note that if $d(x,x_j)<r$ we have

$$2^{-1}L_{\mathbb{G}}^{-1}l_{\mathbb{G}}(r-d(x,x_j))^{\mathfrak{Q}-1}\le\mathcal{S}^{\mathcal{Q}-1}\llcorner\mathrm{gr}(\gamma)(B(x_j,r-d(x,x_j)))\le\mathcal{S}^{\mathcal{Q}-1}\llcorner\mathrm{gr}(\gamma)(B(x,r)).$$

The arbitrariness of j implies that for any $0<r<1/i$ we have $\mathcal{S}^{\mathcal{Q}-1}\llcorner\mathrm{gr}(\gamma)(B(x,r))\ge 2^{-1}L_{\mathbb{G}}^{-1}l_{\mathbb{G}}r^{\mathfrak{Q}-1}$, proving that $x\in\tilde{A}_i$.

We now prove that the sets A_i cover $\mathcal{S}^{\mathcal{Q}-1}$-almost all $\mathrm{gr}(\gamma)$. Thanks to Theorem 1.38 we can extend γ to an intrinsic Lipschitz function $\tilde{\gamma}:V\to\mathfrak{N}(V)$. Recall now that $\mathrm{gr}(\tilde{\gamma})$ is the boundary of the set of locally finite perimeter $\mathrm{epi}(\tilde{\gamma})$. Thanks to Theorem B.6, this implies that for $|\partial\,\mathrm{epi}(\tilde{\gamma})|_{\mathbb{G}}$-almost every $x\in\mathbb{G}$ there exists a $\bar{r}(x)>0$ such that for any $0<r<\bar{r}(x)$ we have

$$L_{\mathbb{G}}\mathcal{S}^{\mathcal{Q}-1}\llcorner\mathrm{gr}(\tilde{\gamma})(B(x,r))\ge|\partial\,\mathrm{epi}(\tilde{\gamma})|_{\mathbb{G}}(B(x,r))\ge l_{\mathbb{G}}r^{\mathfrak{Q}-1},$$

where the first inequality above comes from Remark B.7. In addition, thanks to [Franchi and Serapioni 2016, Theorem 3.9], [Heinonen et al. 2015, Theorem 3.4.3] and to the Lebesgue differentiation theorem that can be found in [Heinonen et al. 2015, page 77], we deduce that

$$\Theta^{\mathfrak{Q}-1}_*(\mathcal{S}^{\mathfrak{Q}-1}\llcorner\mathrm{gr}(\gamma),x)=\Theta^{\mathfrak{Q}-1}_*(\chi_{\mathrm{gr}(\gamma)}\mathcal{S}^{\mathfrak{Q}-1}\llcorner\mathrm{gr}(\tilde\gamma),x)=\Theta^{\mathfrak{Q}-1}_*(\mathcal{S}^{\mathfrak{Q}-1}\llcorner\mathrm{gr}(\tilde\gamma),x)\geq L_{\mathbb{G}}^{-1}l_{\mathbb{G}},\tag{148}$$

for $\mathcal{S}^{\mathfrak{Q}-1}\llcorner\mathrm{gr}(\gamma)$-almost every $x\in\mathbb{G}$. From (148), we infer that for $\mathcal{S}^{\mathfrak{Q}-1}\llcorner\mathrm{gr}(\gamma)$-almost every $x\in\mathbb{G}$ there exists an $r(x)>0$ such that $\mathcal{S}^{\mathfrak{Q}-1}(B(x,r)\cap\mathrm{gr}(\gamma))\geq 2^{-1}L_{\mathbb{G}}^{-1}l_{\mathbb{G}}r^{\mathfrak{Q}-1}$ for any $0<r<r(x)$. Therefore, if $r(x)>1/i$ and (147) holds at x, then $x\in A_i$ and this concludes the proof of the fact that $\mathcal{S}^{\mathfrak{Q}-1}\big(\mathrm{gr}(\gamma)\setminus\bigcup_{i\in\mathbb{N}}A_i\big)=0$.

For any $i,j\in\mathbb{N}$ and any $x\in A_i$, we let

$$\rho_{i,j}(x):=\sup\bigg\{\frac{|\langle\mathfrak{n}_\gamma(x),\pi_1(x^{-1}y)\rangle|}{d(x,y)}:y\in A_i\text{ and }0<d(x,y)<1/j\bigg\}.$$

We remark that the functions $\rho_{i,j}$ are measurable for any $i,j\in\mathbb{N}$. Indeed, on the one hand the function $(x,y)\mapsto|\langle\mathfrak{n}_\gamma(x),\pi_1(x^{-1}y)\rangle|/d(x,y)$ is $\mathcal{S}^{\mathfrak{Q}-1}\llcorner\mathrm{gr}(\gamma)$-measurable since it is the quotient of two $\mathcal{S}^{\mathfrak{Q}-1}\llcorner\mathrm{gr}(\gamma)$-measurable functions. On the other, since $\mathbb{G}$ is separable, it is immediate to see that $\rho_{i,j}$ can be rewritten as the supremum on y over a countable subset of $B(x,\delta)\cap A_i$ showing that $\rho_{i,j}$ is indeed measurable. We want to prove that for any $i\in\mathbb{N}$ and any $x\in A_i$ we have

$$\lim_{j\to\infty}\rho_{i,j}(x)=0.\tag{149}$$

Assume by contradiction this is not the case and that there exists an $i\in\mathbb{N}$ and a $z\in A_i$ for which (149) fails. Then there is a $0<\mathfrak{c}\leq 1$ and an increasing sequence of natural numbers $\{j_k\}_{k\in\mathbb{N}}$ such that for any $k\in\mathbb{N}$ there is a $y_k\in A_i$ for which $y_k\in B(z,1/j_k)$ and $|\langle\mathfrak{n}_\gamma(z),\pi_1(z^{-1}y_k)\rangle|>\mathfrak{c}d(z,y_k)$. Thanks to Proposition 1.15, we infer that $y_i\notin zX_{V_\gamma(z)}\big(\tfrac12\mathfrak{c}\big)$; indeed,

$$\mathrm{dist}(V_\gamma(z),z^{-1}y_k)=|\langle\mathfrak{n}_\gamma(z),\pi_1(z^{-1}y_k)\rangle|>\mathfrak{c}d(z,y_k).\tag{150}$$

We now claim that for any $k\in\mathbb{N}$ we have

$$B\big(y_k,\tfrac14\mathfrak{c}d(z,y_k)\big)\subseteq B(z,2d(z,y_k))\setminus zX_{V_\gamma(z)}\big(\tfrac14\mathfrak{c}\big).\tag{151}$$

In order to prove the inclusion (151) we fix a $k\in\mathbb{N}$ and let $w:=y_kv$ for some $v\in B\big(y_k,\tfrac18\mathfrak{c}d(z,y_k)\big)$. With these choices Proposition 1.15 and the triangle inequality imply that

$$\begin{aligned}\mathrm{dist}(V_\gamma(z),z^{-1}w)=|\langle\mathfrak{n}_\gamma(z),\pi_1(z^{-1}w)\rangle|&\geq|\langle\mathfrak{n}_\gamma(z),\pi_1(z^{-1}y_k)\rangle|-|\langle\mathfrak{n}_\gamma(z),\pi_1(y_k^{-1}w)\rangle|\\&\geq\mathfrak{c}d(z,y_k)-d(y_k,w)\geq\mathfrak{c}d(z,w)-(1+\mathfrak{c})d(y_k,w).\end{aligned}\tag{152}$$

Furthermore, thanks to the choice of w we have

$$d(y_k,w)=\|v\|\leq\tfrac14\mathfrak{c}d(z,y_k)\leq\tfrac14\mathfrak{c}d(z,w)+\tfrac14\mathfrak{c}d(y_k,w),\tag{153}$$

$$d(z,w)\leq d(z,y_k)+d(y_k,w)\leq d(z,y_k)+\|v\|\leq\big(1+\tfrac18\mathfrak{c}\big)d(z,y_k)\leq 2d(z,y_k).\tag{154}$$

From (152) we infer in particular that $(4/\mathfrak{c}-1)d(y_k,w)\leq d(z,w)$. This implies in particular that

$$\mathrm{dist}(V_\gamma(z),z^{-1}w)\overset{(152)}{\geq}\mathfrak{c}d(z,w)-(1+\mathfrak{c})d(y_k,w)\geq\mathfrak{c}d(z,w)-\frac{1+\mathfrak{c}}{4/\mathfrak{c}-1}d(z,w)\geq\tfrac14\mathfrak{c}d(z,w),\tag{155}$$

where the last inequality comes from the fact that $\mathfrak{c} \le 1$. The inclusion (151) follows immediately from the above bound and (154). Therefore, (151) implies that

$$\begin{aligned}\limsup_{r\to 0} &\frac{\mathcal{S}^{\mathfrak{Q}-1}(\mathrm{gr}(\gamma)\cap B(z,r)\setminus zX_{V_\gamma(z)}(\mathfrak{c}/8))}{r^{\mathfrak{Q}-1}}\\ &\ge \lim_{k\to\infty}\frac{\mathcal{S}^{\mathfrak{Q}-1}(\mathrm{gr}(\gamma)\cap B(z,2d(z,y_k))\setminus zX_{V_\gamma(z)}(\mathfrak{c}/8))}{(2d(z,y_k))^{\mathfrak{Q}-1}}\\ &\ge \lim_{k\to\infty}\frac{\mathcal{S}^{\mathfrak{Q}-1}(\mathrm{gr}(\gamma)\cap B(y_k,\mathfrak{c}d(z,y_k)/8))}{2^{\mathfrak{Q}-1}d(z,y_k)^{\mathfrak{Q}-1}} \ge \lim_{k\to\infty}\frac{L_{\mathbb{G}}^{-1}l_{\mathbb{G}}(\mathfrak{c}d(z,y_k)/8)^{\mathfrak{Q}-1}}{2^{\mathfrak{Q}}d(z,y_k)} = \frac{l_{\mathbb{G}}}{2L_{\mathbb{G}}}\left(\frac{\mathfrak{c}}{16}\right)^{\mathfrak{Q}-1}, \qquad (156)\end{aligned}$$

where the second last inequality comes from the fact that $y_k \in A_i$ for any k and that $\frac{1}{8}\mathfrak{c}d(z,y_k) < 1/i$ definitely. However, since by construction (147) holds at any point of A_i, (156) is in contradiction with (147) and thus (149) must hold at any $x \in A_i$. Define f_i to be the function identically 0 on A_i and for any $\iota \in \mathbb{N}$ we let $K_i(\iota)$ be a compact subset of A_i for which

(i) $\mathcal{S}^{\mathfrak{Q}-1}(A_i \setminus K_i(\iota)) \le 1/\iota$,

(ii) $\mathfrak{n}_\gamma$ is continuous on $K_i(\iota)$,

(iii) $\rho_{i,j}$ converges uniformly to 0 on $K_i(\iota)$.

The existence of $K_i(\iota)$ is implied by Lusin's theorem and Severini–Egoroff's theorem. Thanks to Whitney's extension theorem, see for instance Theorem 5.2 in [Franchi et al. 2003], we infer that we can find a $C^1_{\mathbb{G}}$-function such that $f_{i,\iota}|_K = 0$ and $\nabla_{\mathbb{H}} f_{i,\iota}(x) = \mathfrak{n}_\gamma(x)$ for any $x \in K_i(\iota)$. This implies that A_i, and thus $\mathrm{gr}(\gamma)$, can be covered $\mathcal{S}^{\mathfrak{Q}-1}$-almost all with $C^1_{\mathbb{G}}$-surfaces. □

Acknowledgements

I am deeply grateful to Roberto Monti, without whose guidance and constant encouragement I would have never been able to tackle the density problem in the Carnot setting. I would like to also thank Joan Verdera, who invited me to Barcelona during February of 2019 allowing me to have many fruitful conversations with him and with Xavier Tolsa. I would also like to thank Gioacchino Antonelli, Giacomo Del Nin, Carmelo Puliatti and Davide Vittone for many valuable suggestions and corrections to the early versions of this work. Finally, I am deeply grateful to the anonymous referees whose careful reading found a number of typos and inaccuracies and helped me to significantly improve the exposition.

References

[Ahmadi et al. 2019] A. A. Ahmadi, E. de Klerk, and G. Hall, "Polynomial norms", *SIAM J. Optim.* **29**:1 (2019), 399–422. MR Zbl

[Ambrosio et al. 2009] L. Ambrosio, B. Kleiner, and E. Le Donne, "Rectifiability of sets of finite perimeter in Carnot groups: existence of a tangent hyperplane", *J. Geom. Anal.* **19**:3 (2009), 509–540. MR Zbl

[Antonelli and Merlo 2022a] G. Antonelli and A. Merlo, "On rectifiable measures in Carnot groups: existence of density", *J. Geom. Anal.* **32**:9 (2022), art. id. 239. MR Zbl

[Antonelli and Merlo 2022b] G. Antonelli and A. Merlo, "On rectifiable measures in Carnot groups: Marstrand–Mattila rectifiability criterion", *J. Funct. Anal.* **283**:1 (2022), art. id. 109495. MR Zbl

[Balogh et al. 2012] Z. M. Balogh, K. Fässler, P. Mattila, and J. T. Tyson, "Projection and slicing theorems in Heisenberg groups", *Adv. Math.* **231**:2 (2012), 569–604. MR Zbl

[Balogh et al. 2013] Z. M. Balogh, E. Durand-Cartagena, K. Fässler, P. Mattila, and J. T. Tyson, "The effect of projections on dimension in the Heisenberg group", *Rev. Mat. Iberoam.* **29**:2 (2013), 381–432. MR Zbl

[Bellettini and Le Donne 2021] C. Bellettini and E. Le Donne, "Sets with constant normal in Carnot groups: properties and examples", *Comment. Math. Helv.* **96**:1 (2021), 149–198. MR Zbl

[Bonfiglioli et al. 2007] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer, 2007. MR Zbl

[Chousionis and Tyson 2015] V. Chousionis and J. T. Tyson, "Marstrand's density theorem in the Heisenberg group", *Bull. Lond. Math. Soc.* **47**:5 (2015), 771–788. MR Zbl

[Chousionis et al. 2019] V. Chousionis, K. Fässler, and T. Orponen, "Intrinsic Lipschitz graphs and vertical β-numbers in the Heisenberg group", *Amer. J. Math.* **141**:4 (2019), 1087–1147. MR Zbl

[Chousionis et al. 2020] V. Chousionis, V. Magnani, and J. T. Tyson, "On uniform measures in the Heisenberg group", *Adv. Math.* **363** (2020), art. id. 106980. MR Zbl

[Christ 1990] M. Christ, "A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral", *Colloq. Math.* **60**:2 (1990), 601–628. MR Zbl

[David 1991] G. David, *Wavelets and singular integrals on curves and surfaces*, Lecture Notes in Math. **1465**, Springer, 1991. MR Zbl

[David and Mattila 2000] G. David and P. Mattila, "Removable sets for Lipschitz harmonic functions in the plane", *Rev. Mat. Iberoam.* **16**:1 (2000), 137–215. MR Zbl

[David and Semmes 1993a] G. David and S. Semmes, *Analysis of and on uniformly rectifiable sets*, Math. Surv. Monogr. **38**, Amer. Math. Soc., Providence, RI, 1993. MR Zbl

[David and Semmes 1993b] G. David and S. Semmes, "Quantitative rectifiability and Lipschitz mappings", *Trans. Amer. Math. Soc.* **337**:2 (1993), 855–889. MR Zbl

[De Lellis 2008] C. De Lellis, *Rectifiable sets, densities and tangent measures*, Eur. Math. Soc., Zürich, 2008. MR Zbl

[Federer 1969] H. Federer, *Geometric measure theory*, Grundl. Math. Wissen. **153**, Springer, 1969. MR Zbl

[Franchi and Serapioni 2016] B. Franchi and R. P. Serapioni, "Intrinsic Lipschitz graphs within Carnot groups", *J. Geom. Anal.* **26**:3 (2016), 1946–1994. MR Zbl

[Franchi et al. 2001] B. Franchi, R. Serapioni, and F. Serra Cassano, "Rectifiability and perimeter in the Heisenberg group", *Math. Ann.* **321**:3 (2001), 479–531. MR Zbl

[Franchi et al. 2003] B. Franchi, R. Serapioni, and F. Serra Cassano, "On the structure of finite perimeter sets in step 2 Carnot groups", *J. Geom. Anal.* **13**:3 (2003), 421–466. MR Zbl

[Franchi et al. 2015] B. Franchi, R. P. Serapioni, and F. Serra Cassano, "Area formula for centered Hausdorff measures in metric spaces", *Nonlinear Anal.* **126** (2015), 218–233. MR Zbl

[Heinonen et al. 2015] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. T. Tyson, *Sobolev spaces on metric measure spaces: an approach based on upper gradients*, New Math. Monogr. **27**, Cambridge Univ. Press, 2015. MR Zbl

[Kirchheim and Preiss 2002] B. Kirchheim and D. Preiss, "Uniformly distributed measures in Euclidean spaces", *Math. Scand.* **90**:1 (2002), 152–160. MR Zbl

[Kirchheim and Serra Cassano 2004] B. Kirchheim and F. Serra Cassano, "Rectifiability and parameterization of intrinsic regular surfaces in the Heisenberg group", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **3**:4 (2004), 871–896. MR Zbl

[Le Donne and Moisala 2021] E. Le Donne and T. Moisala, "Semigenerated Carnot algebras and applications to sub-Riemannian perimeter", *Math. Z.* **299**:3-4 (2021), 2257–2285. MR Zbl

[Magnani 2004] V. Magnani, "Unrectifiability and rigidity in stratified groups", *Arch. Math. (Basel)* **83**:6 (2004), 568–576. MR Zbl

[Magnani 2017] V. Magnani, "A new differentiation, shape of the unit ball, and perimeter measure", *Indiana Univ. Math. J.* **66**:1 (2017), 183–204. MR Zbl

[Marchi 2014] M. Marchi, "Regularity of sets with constant intrinsic normal in a class of Carnot groups", *Ann. Inst. Fourier (Grenoble)* **64**:2 (2014), 429–455. MR Zbl

[Marstrand 1961] J. M. Marstrand, "Hausdorff two-dimensional measure in 3-space", *Proc. Lond. Math. Soc.* (3) **11** (1961), 91–108. MR Zbl

[Mattila 1975] P. Mattila, "Hausdorff m regular and rectifiable sets in n-space", *Trans. Amer. Math. Soc.* **205** (1975), 263–274. MR Zbl

[Mattila 1995] P. Mattila, *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*, Cambridge Stud. Adv. Math. **44**, Cambridge Univ. Press, 1995. MR Zbl

[Mattila 2005] P. Mattila, "Measures with unique tangent measures in metric groups", *Math. Scand.* **97**:2 (2005), 298–308. MR Zbl

[Mattila et al. 2010] P. Mattila, R. Serapioni, and F. Serra Cassano, "Characterizations of intrinsic rectifiability in Heisenberg groups", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **9**:4 (2010), 687–723. MR Zbl

[Merlo 2020] A. Merlo, "Marstrand–Mattila rectifiability criterion for 1-codimensional measures in Carnot groups", preprint, 2020. Zbl arXiv 2007.03236

[Merlo 2022] A. Merlo, "Geometry of 1-codimensional measures in Heisenberg groups", *Invent. Math.* **227**:1 (2022), 27–148. MR Zbl

[Onishchik 1993] A. L. Onishchik (editor), *Lie groups and Lie algebras, I*, Encycl. Math. Sci. **20**, Springer, 1993. MR Zbl

[Preiss 1987] D. Preiss, "Geometry of measures in $\mathbb{R}^n$: distribution, rectifiability, and densities", *Ann. of Math.* (2) **125**:3 (1987), 537–643. MR Zbl

[Serra Cassano 2016] F. Serra Cassano, "Some topics of geometric measure theory in Carnot groups", pp. 1–121 in *Geometry, analysis and dynamics on sub-Riemannian manifolds, I* (Marseille, 2014), edited by D. Barilari et al., Eur. Math. Soc., Zürich, 2016. MR Zbl

[Vittone 2012] D. Vittone, "Lipschitz surfaces, perimeter and trace theorems for BV functions in Carnot–Carathéodory spaces", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **11**:4 (2012), 939–998. MR Zbl

[Vittone et al. 2022] A. Julia, S. Nicolussi Golo, and D. Vittone, "Area of intrinsic graphs and coarea formula in Carnot groups", *Math. Z.* **301**:2 (2022), 1369–1406. MR Zbl

Received 16 Dec 2020. Revised 14 Sep 2021. Accepted 28 Oct 2021.

ANDREA MERLO: andrea.merlo@ehu.eus
Departamento de Matemáticas, Universidad del País Vasco, Leioa, Spain

ANALYSIS AND PDE
Vol. 16 (2023), No. 4, pp. 997–1032
DOI: 10.2140/apde.2023.16.997

FINITE-TIME BLOWUP FOR A NAVIER–STOKES MODEL EQUATION FOR THE SELF-AMPLIFICATION OF STRAIN

EVAN MILLER

We consider a model equation for the Navier–Stokes strain equation which has the same identity for enstrophy growth and a number of the same regularity criteria as the full Navier–Stokes strain equation, and is also an evolution equation on the same constraint space. We prove finite-time blowup for this model equation, which shows that the identity for enstrophy growth and the strain constraint space are not sufficient on their own to guarantee global regularity for Navier–Stokes. The mechanism for the finite-time blowup of this model equation is the self-amplification of strain, which is consistent with recent research suggesting that strain self-amplification, not vortex stretching, is the main mechanism behind the turbulent energy cascade. Because the strain self-amplification model equation is obtained by dropping certain terms from the full Navier–Stokes strain equation, we will also prove a conditional blowup result for the full Navier–Stokes equation involving a perturbative condition on the terms neglected in the model equation.

1. Introduction

The incompressible Navier–Stokes equation is one of the fundamental equations of fluid mechanics. Although it is over 150 years old, much about its solutions, including the global existence of smooth solutions, remains unknown. The Navier–Stokes equation is given by

$$\begin{aligned} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p &= 0, \\ \nabla \cdot u &= 0, \end{aligned} \tag{1-1}$$

where $u \in \mathbb{R}^3$ is the velocity and p is the pressure. The first equation is a statement of Newton's second law, $F = ma$, where $\partial_t u + (u \cdot \nabla) u$ gives the acceleration in the Lagrangian frame, Δu describes the viscous forces due to the internal friction of the fluid, and $-\nabla p$ describes the force due to the pressure. The second equation, the divergence-free constraint, comes from the conservation of mass. We will note that p is not an independently evolving function, but is determined entirely by u by convolution with the Poisson kernel,

$$p = (-\Delta)^{-1} \sum_{i,j=1}^{3} \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j}.$$

It is possible to state the incompressible Navier–Stokes equation without giving any reference to pressure at all by making use of the Helmholtz projection onto the space of divergence-free vector fields,

MSC2020: 35Q30.
Keywords: Navier–Stokes, finite-time blowup.

yielding the equation

$$\partial_t u - \Delta u + P_{\mathrm{df}} \nabla \cdot (u \otimes u) = 0. \tag{1-2}$$

Note that we have used the fact that $\nabla \cdot (u \otimes u) = (u \cdot \nabla)u$ because $\nabla \cdot u = 0$, and the fact that the Helmholtz decomposition implies that $P_{\mathrm{df}}(\nabla p) = 0$.

The first major advances towards a rigorous mathematical understanding of the Navier–Stokes equation came in the seminal paper by Leray [1934]. For all initial data $u^0 \in L^2_{\mathrm{df}}$, Leray proved the global-in-time existence of weak solutions, in the sense of integrating against smooth test functions, satisfying the energy inequality, which states that for all $t > 0$,

$$\tfrac{1}{2}\|u(t)\|_{L^2}^2 + \int_0^t \|u(\tau)\|_{\dot{H}^1}^2 \,\mathrm{d}\tau \le \tfrac{1}{2}\|u^0\|_{L^2}^2.$$

Unfortunately, while such solutions are well suited to study in the sense that global-in-time existence is guaranteed for all finite energy initial data, they are not known to be either smooth or unique, leaving major problems for the well-posedness theory.

The lack of a uniqueness and regularity theory for Leray weak solutions led Fujita and Kato to develop the notion of mild solutions, which satisfy (1-2) in the sense of Duhamel's formula. Unlike Leray's weak solutions, mild solutions must be both smooth and unique. Fujita and Kato [1964] proved the local-in-time existence, uniqueness, and smoothness of mild solutions for initial data in $\dot{H}^1$, with the time of existence bounded below uniformly in the $\dot{H}^1$ norm.

Theorem 1.1. *There exists an absolute constant $C > 0$ such that for all initial data $u^0 \in \dot{H}^1_{\mathrm{df}}$, there exists $T_{\max} \ge C/\|u^0\|_{\dot{H}^1}^4$ and a unique mild solution to the Navier–Stokes equation $u \in C([0, T_{\max}); \dot{H}^1_{\mathrm{df}})$. Furthermore, we have the higher regularity $u \in C^\infty((0, T_{\max}) \times \mathbb{R}^3)$. If in addition we have $u^0 \in H^1_{\mathrm{df}}$, then the energy inequality holds with equality; that is for all $0 < t < T_{\max}$,*

$$\tfrac{1}{2}\|u(t)\|_{L^2}^2 + \int_0^t \|u(\tau)\|_{\dot{H}^1}^2 \,\mathrm{d}\tau = \tfrac{1}{2}\|u^0\|_{L^2}^2.$$

We will note that because mild solutions are smooth and unique, the initial value problem for mild solutions of the Navier–Stokes equation is locally well posed in $\dot{H}^1$ — and also in a number of larger spaces; however, it is not known to be globally well posed. Whether the Navier–Stokes equation has global smooth solutions or admits smooth solutions that blowup in finite time is one of the biggest open problems in PDEs and one of the *Millennium Problems* put forward by the Clay Mathematics Institute [Fefferman 2006].

The main difficulty is that the only bounds that are available on the growth of solutions are the bounds in $L^\infty_t L^2_x$ and $L^2_t \dot{H}^1_x$ due to the energy equality, and these bounds are not enough to guarantee the global existence of smooth solutions because the energy equality is supercritical with respect to the invariant rescaling of the Navier–Stokes equation. The solution set of the Navier–Stokes equation is preserved under the rescaling

$$u^\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t),$$

for all $\lambda > 0$. This means that is not enough to control the $L_t^\infty L_x^2$- or $L_t^2 \dot{H}_x^1$-norms of u, which are supercritical in terms of scaling; in order to guarantee global regularity, we need to control a scale critical norm. Ladyzhenskaya [1967], Prodi [1959], and Serrin [1962] independently proved a family of scale critical regularity criteria, which state that if $T_{\max} < +\infty$ and $2/p + 3/q = 1$ with $3 < q \leq +\infty$, then

$$\int_0^{T_{\max}} \|u(t)\|_{L^q}^p \, \mathrm{d}t = +\infty.$$

Escauriaza, Seregin and Sv̌erák [Escauriaza et al. 2003] extended this result to the endpoint case $q = 3$. They proved that if $T_{\max} < +\infty$, then

$$\limsup_{t \to T_{\max}} \|u(t)\|_{L^3} = +\infty.$$

Recently, Tao [2021] further extended this regularity criterion giving a quantitative lower bound on the rate of the blowup of the L^3-norm. This result is very slightly supercritical — in fact triple logarithmic — with respect to scaling, and is the first supercritical regularity criterion for the Navier–Stokes equation.

Two crucially important objects for the study of the Navier–Stokes equation are the strain, which is the symmetric gradient of the velocity, $S = \nabla_{\mathrm{sym}} u$, with $S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, and the vorticity, which is a vector that represents the antisymmetric part of the velocity and is given by $\omega = \nabla \times u$. Physically, the strain describes how a parcel of the fluid is deformed, while the vorticity describes how a parcel of the fluid is rotated.

Taking the curl of (1-1), we find the evolution equation for ω is given by

$$\partial_t \omega - \Delta \omega + (u \cdot \nabla)\omega - S\omega = 0.$$

Taking the symmetric gradient of (1-1), we find the evolution equation for S is given by,

$$\partial_t S - \Delta S + (u \cdot \nabla) S + S^2 + \tfrac{1}{4}\omega \otimes \omega - \tfrac{1}{4}|\omega|^2 I_3 + \mathrm{Hess}(p) = 0. \tag{1-3}$$

We will note that the vorticity equation is invariant under the rescaling

$$\omega^\lambda(x,t) = \lambda^2 \omega(\lambda x, \lambda^2 t),$$

and the strain equation is invariant under the rescaling

$$S^\lambda(x,t) = \lambda^2 S(\lambda x, \lambda^2 t).$$

The extra factor of λ comes from the fact that both ω and S scale like ∇u.

The vorticity has been studied fairly exhaustively for its role in the dynamics of the Navier–Stokes equation. For instance, the Beale–Kato–Majda regularity criterion [Beale et al. 1984], which holds for smooth solutions of both the Euler and Navier–Stokes equations, states that if $T_{\max} < +\infty$, then

$$\int_0^{T_{\max}} \|\omega(\cdot, t)\|_{L^\infty} \, \mathrm{d}t = +\infty.$$

Chae and Choe [1999] proved a regularity criterion on two components of vorticity that has a geometric significance, guaranteeing that the blowup must be fully three dimensional. They showed that if a smooth solution of the Navier–Stokes equation blows up in finite time $T_{\max} < +\infty$, then for all $3/2 < q < +\infty$ and $2/p + 3/q = 2$,

$$\int_0^{T_{\max}} \|e_3 \times \omega(\cdot, t)\|_{L^q}^p \, \mathrm{d}t = +\infty. \tag{1-4}$$

The fixed direction condition in this regularity criterion was recently loosened by the author in [Miller 2021]. In another key result involving vorticity, Constantin and Fefferman [1993] proved that the direction of the vorticity must vary rapidly in regions where the vorticity is large if there is finite-time blowup. There are many other results involving vorticity, far too many to list here.

The strain equation has been investigated much less thoroughly, but can provide some insights that do not follow as clearly from the vorticity equation. We will refer to the evolution equation for S in (1-3) as the Navier–Stokes strain equation. This equation is an evolution equation on the constraint space L^2_{st}, the space of strain matrices, which replaces the divergence-free constraint for the Navier–Stokes and vorticity equations. We define L^2_{st} as follows.

Definition 1.2. Define $L^2_{\mathrm{st}} \subset L^2(\mathbb{R}^3; \mathbb{S}^{3\times 3})$ by

$$L^2_{\mathrm{st}} = \{\nabla_{\mathrm{sym}} u : u \in \dot{H}^1, \nabla \cdot u = 0\}.$$

The role of this constraint space in the evolution equation (1-3) was examined by the author in [Miller 2020]. One geometric restriction on the matrices $S \in L^2_{\mathrm{st}}$ is that they must be trace-free because

$$\operatorname{tr}(S) = \nabla \cdot u = 0.$$

Furthermore, in that paper, the author proved that Hessians and scalar multiples of the identity matrix must be in the orthogonal compliment of L^2_{st}.

Proposition 1.3. *For all $f \in \dot{H}^2(\mathbb{R}^3)$ and for all $g \in L^2(\mathbb{R}^3)$, we have $\operatorname{Hess}(f), gI_3 \in (L^2_{\mathrm{st}})^\perp$. That is for all $S \in L^2_{\mathrm{st}}$,*

$$\langle \operatorname{Hess}(f), S\rangle = 0 \quad \textit{and} \quad \langle gI_3, S\rangle = 0.$$

For sufficiently smooth solutions to the Navier–Stokes strain equation, $\frac{1}{4}|\omega|^2$, $\operatorname{Hess}(p) \in L^2$, so we can conclude that the terms $\frac{1}{4}|\omega|^2 I_3$ and $\operatorname{Hess}(p)$ are orthogonal to the constraint space, or in other words that $\frac{1}{4}|\omega|^2 I_3, \operatorname{Hess}(p) \in (L^2_{\mathrm{st}})^\perp$. This means that the Navier–Stokes strain equation can be expressed in terms of the projection onto L^2_{st} as

$$\partial_t S - \Delta S + P_{\mathrm{st}}\big((u \cdot \nabla)S + S^2 + \tfrac{1}{4}\omega \otimes \omega\big) = 0. \tag{1-5}$$

This is analogous to defining the Navier–Stokes equation without any reference to ∇p by using the Helmholtz projection onto the space of divergence-free vector fields in (1-2). We will use (1-5) to define mild solutions to the Navier–Stokes strain equation in Section 3.

It is not actually necessary to separately prove the existence of mild solutions to the strain equation, as it is straightforward to reduce this problem to the existence of mild solutions of the Navier–Stokes equation. The author proved the equivalence of these formulations in [Miller 2020].

Proposition 1.4. *A velocity field $u \in C([0, T_{\max}); \dot{H}^1_{\mathrm{df}})$ is a mild solution of the Navier–Stokes equation if and only if $S \in C([0, T_{\max}); L^2_{\mathrm{st}})$ is a mild solution to the Navier–Stokes strain equation, where $S = \nabla_{\mathrm{sym}} u$ and $u = -2\,\mathrm{div}(-\Delta)^{-1} S$.*

The strain evolution equation is extremely useful because it allows us to prove a simplified identity for enstrophy growth, which can equivalently be defined in terms of the square of the L^2-norm of S, ω, or ∇u based on an isometry proved by the author in [Miller 2020].

Proposition 1.5. *For all $-\frac{3}{2} < \alpha < \frac{3}{2}$ and for all $S \in \dot{H}^\alpha_{\mathrm{st}}$,*

$$\|S\|^2_{\dot{H}^\alpha} = \tfrac{1}{2}\|\omega\|^2_{\dot{H}^\alpha} = \tfrac{1}{2}\|\nabla u\|^2_{\dot{H}^\alpha}.$$

Remark 1.6. We should note here that the factor of $\frac{1}{2}$ in Proposition 1.5 is entirely an artifact of how the vorticity is defined. The vorticity is a vector representation of the antisymmetric part of ∇u, with

$$A = \frac{1}{2}\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix},$$

where A is the antisymmetric part of ∇u given by $A_{ij} = \frac{1}{2}(\partial_i u_j - \partial_j u_i)$. From this identity we can see that

$$\|S\|^2_{\dot{H}^\alpha} = \|A\|^2_{\dot{H}^\alpha},$$

so the isometry in Proposition 1.5 tells us that all the Hilbert norms of the symmetric and antisymmetric parts of the gradient of a divergence-free vector field are equal.

Definition 1.7. Based on the isometry in Proposition 1.5, we will define the enstrophy of a solution to the Navier–Stokes equation, which can be equivalently expressed as

$$E(t) = \|S(t)\|^2_{L^2} = \tfrac{1}{2}\|\omega(t)\|^2_{L^2} = \tfrac{1}{2}\|\nabla u(t)\|^2_{L^2},$$

and the energy of a solution of the Navier–Stokes equation, which can be equivalently expressed as

$$K(t) = \|S(t)\|^2_{\dot{H}^{-1}} = \tfrac{1}{2}\|\omega(t)\|^2_{\dot{H}^{-1}} = \tfrac{1}{2}\|u(t)\|^2_{L^2}.$$

Remark 1.8. The energy equality for smooth solutions of the Navier–Stokes equation can be stated in terms of energy and enstrophy as

$$K(t) + 2\int_0^t E(\tau)\,\mathrm{d}\tau = K_0.$$

Enstrophy is a very important quantity because Theorem 1.1 states that a smooth solution of the Navier–Stokes equation must exist locally in time for initial data in $u^0 \in \dot{H}^1$. This implies that enstrophy controls regularity, because as long as enstrophy remains bounded on some time interval, a smooth solution can be continued to some later time.

The standard estimate for enstrophy growth is given in terms of nonlocal interaction of the vorticity and the strain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\omega(t)\|^2_{L^2} = -\|\omega\|^2_{\dot{H}^1} + \langle S, \omega \otimes \omega\rangle.$$

This is a nonlocal identity because S can be determined in terms of ω by a nonlocal, zeroth order pseudo-differential operator, with $S = \nabla_{\mathrm{sym}} \nabla \times (-\Delta)^{-1}\omega$. Using the isometry in Proposition 1.5 and the evolution equations for both the strain and the vorticity, this identity can be drastically simplified, with the nonlocal term replaced by a term involving only the determinant of S.

Proposition 1.9. *Suppose $u \in C([0, T_{\max}); \dot{H}^1_{\mathrm{df}})$ is a mild solution of the Navier–Stokes equation. Note that this is equivalent to assuming that $S \in C([0, T_{\max}); L^2_{\mathrm{st}})$ is a mild solution to the Navier–Stokes strain equation. Then for all $0 < t < T_{\max}$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|S(t)\|_{L^2}^2 = -2\|S\|_{\dot{H}^1}^2 - \frac{4}{3}\int \operatorname{tr}(S^3) = -2\|S\|_{\dot{H}^1}^2 - 4\int \det(S).$$

This identity was first proven by Neustupa and Penel [2001; 2005]. The analogous result without the dissipation term $-2\|S\|_{\dot{H}^1}^2$ was later proven independently by Chae [2006] in the context of smooth solutions of the Euler equation using similar methods to Neustupa and Penel. This identity was also proven using the evolution equation for the strain, a different approach to that of Neustupa and Penel, by the author in [Miller 2020]. The identity in Proposition 1.9 directly implies a family of scale-invariant regularity criteria in terms of the positive part of the middle eigenvalue of S.

Theorem 1.10. *Suppose $u \in C([0, T_{\max}); \dot{H}^1_{\mathrm{df}})$ is a mild solution of the Navier–Stokes equation, or equivalently that $S \in C([0, T_{\max}); L^2_{\mathrm{st}})$ is a mild solution to the Navier–Stokes strain equation. Let $\lambda_1(x,t) \le \lambda_2(x,t) \le \lambda_3(x,t)$ be the eigenvalues of $S(x,t)$, and let $\lambda_2^+(x,t) = \max\{0, \lambda_2(x,t)\}$. Then for all $3/q + 2/p = 2$ and $3/2 < q \le +\infty$, there exists $C_q > 0$ depending only on q such that for all $0 < t < T_{\max}$,*

$$\|S(t)\|_{L^2}^2 \le \|S^0\|_{L^2}^2 \exp\biggl(C_q \int_0^t \|\lambda_2^+(\tau)\|_{L^q}^p \,\mathrm{d}\tau\biggr).$$

In particular, if $T_{\max} < +\infty$, then

$$\int_0^{T_{\max}} \|\lambda_2^+(t)\|_{L^q}^p \,\mathrm{d}t = +\infty.$$

This regularity criterion was first proven by Neustupa and Penel [2001; 2005; 2018]. It was also proven independently by the author in [Miller 2020]. Note that because $\operatorname{tr}(S) = 0$, this regularity criterion significantly restricts the geometry of any finite-time blowup for the Navier–Stokes equation: any blowup must be driven by unbounded planar stretching and axial compression, with the strain having two positive eigenvalues and one very negative eigenvalue.

There are many other conditional regularity results, which guarantee the regularity of solutions as long as some scale critical quantity remains finite, including regularity criteria involving the derivative in just one direction $\partial_3 u$ [Kukavica and Ziane 2007], and involving just one velocity direction u_3 [Chemin and Zhang 2016; Chemin et al. 2017]. For a more thorough, but by no means exhaustive, treatment of regularity criteria for the Navier–Stokes equation, see Chapter 11 in [Lemarié-Rieusset 2016].

In this paper we will take the opposite approach. We will prove finite-time blowup for solutions of the Navier–Stokes equation with a fairly broad set of initial data, assuming that a certain scale invariant quantity related to the structure of the nonlinearity remains small. We will do this first by considering a model equation for the Navier–Stokes strain equation and proving finite-time blowup for solutions of this model equation, and then by viewing the actual Navier–Stokes strain equation as a perturbation of the model equation.

In order to do this, we will drop the advection and the vorticity terms from the evolution equation (1-5) entirely, along with a piece of the S^2 term so that the enstrophy growth identity in Proposition 1.9 still holds. We will show that

$$\langle S, \omega \otimes \omega \rangle = -4 \int \det(S) = -\tfrac{4}{3} \langle S^2, S \rangle$$

and

$$\langle (u \cdot \nabla) S, S \rangle = 0,$$

and therefore

$$\big\langle P_{\mathrm{st}}\big((u \cdot \nabla) S + \tfrac{1}{3} S^2 + \tfrac{1}{4} \omega \otimes \omega\big), S \big\rangle = 0.$$

Using this identity, we can rewrite the full Navier–Stokes strain equation as

$$\partial_t S - \Delta S + \tfrac{2}{3} P_{\mathrm{st}}(S^2) + P_{\mathrm{st}}\big((u \cdot \nabla) S + \tfrac{1}{3} S^2 + \tfrac{1}{4} \omega \otimes \omega\big) = 0.$$

Dropping the term $P_{\mathrm{st}}\big((u \cdot \nabla) S + \tfrac{1}{3} S^2 + \tfrac{1}{4} \omega \otimes \omega\big)$ from the evolution equation, our strain model equation will be given by

$$\partial_t S - \Delta S + \tfrac{2}{3} P_{\mathrm{st}}(S^2) = 0. \tag{1-6}$$

We will refer to (1-6) as the strain self-amplification model equation because it isolates the interaction of the strain with itself, discarding the nonlocal interaction with the vorticity and the effects of advection. In the model equation, we are dropping a combination of terms that are orthogonal to S in L^2, while keeping the two terms that contribute to the evolution in time of the L^2-norm to first order. We will also show that for solutions of the strain self-amplification model equation we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|S(t)\|_{L^2}^2 = -2 \|S\|_{\dot{H}^1}^2 - 4 \int \det(S),$$

so the strain self-amplification model equation does in fact have the same identity for enstrophy growth as the Navier–Stokes equation, and consequently has a regularity criterion for λ_2^+ in the critical Lebesgue spaces $L_t^p L_x^q$ entirely analogous to the regularity criterion for the Navier–Stokes equation in Theorem 1.10.

Solutions of this model equation blowup in finite time for a fairly wide range of initial conditions.

Theorem 5.3. *Suppose $S \in C([0, T_{\max}); H_{\mathrm{st}}^1)$ is a mild solution of the strain self-amplification model equation such that*

$$-3 \|S^0\|_{\dot{H}^1}^2 - 4 \int \det(S^0) > 0.$$

Then for all $0 < t < T_{\max}$,

$$E(t) > \frac{E_0}{(1 - r_0 t)^2}, \tag{1-7}$$

where

$$r_0 = \frac{-3 \|S^0\|_{\dot{H}^1}^2 - 4 \int \det(S^0)}{2 \|S^0\|_{L^2}^2}.$$

Note in particular that this implies

$$T_{\max} \leq \frac{2\|S^0\|_{L^2}^2}{-3\|S^0\|_{L^2}^2 - 4\int \det(S^0)}.$$

Furthermore, for all $2/p + 3/q = 2$ *and* $3/2 < q \leq +\infty$,

$$\int_0^{T_{\max}} \|\lambda_2^+(t)\|_{L^q}^p \, \mathrm{d}t = +\infty.$$

Remark 1.11. The key to the proof of Theorem 5.3, which is the main result of this paper, is a $\frac{3}{2}$ lower bound on the rate of enstrophy growth for a wide range of initial conditions. In particular, we will show that if

$$g_0 := \frac{-3\|S^0\|_{\dot{H}^1}^2 - 4\int \det(S^0)}{\|S^0\|_{L^2}^3} > 0,$$

then for all $0 < t < T_{\max}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) > g_0 E(t)^{3/2},$$

which immediately yields estimate (1-7) in Theorem 5.3.

Remark 1.12. Theorem 5.3 shows that the regularity criterion in Theorem 1.10, which guarantees the existence of smooth solutions of the Navier–Stokes equation so long as $\lambda_2^+ \in L_t^p L_x^q$, is not enough to guarantee the global existence of smooth solutions to the Navier–Stokes equation just by making use of the constraint space. For solutions of the strain self-amplification model equation, which is an evolution equation on L^2_{st} (the constraint space), λ_2^+ becomes unbounded in this whole family of scale critical spaces. The regularity criterion on λ_2^+ implies that the blowup for the Navier–Stokes equation must be characterized by unbounded planar stretching and axial compression, corresponding to a strain matrix with two positive eigenvalues and one very negative eigenvalue in turbulent regions. One physical example of such a structure in turbulent fluids is two colliding jets. The blowup result for the strain self-amplification model equation shows that a blowup with these features is possible within the relevant constraint space.

Because we chose our strain self-amplification model equation (1-6) by dropping some terms from the full strain equation, we can prove a new conditional blowup result for the full Navier–Stokes equation, by viewing the actual strain equation as a perturbation of the strain self-amplification model equation.

Theorem 6.1. *Suppose* $u \in C([0, T_{\max}); H^2_{\mathrm{df}})$ *is a mild solution of the Navier–Stokes equation such that*

$$f_0 := -3\|S^0\|_{\dot{H}^1}^2 - 4\int \det(S^0) > 0,$$

and for all $0 < t < T_{\max}$,

$$\frac{\left\| P_{\mathrm{st}}\left((u\cdot\nabla)S + \frac{1}{3}S^2 + \frac{1}{4}\omega\otimes\omega\right)(\cdot\,,t)\right\|_{L^2}}{\left\|\left(-\Delta S + P_{\mathrm{st}}\left(\frac{1}{2}(u\cdot\nabla)S + \frac{5}{6}S^2 + \frac{1}{8}\omega\otimes\omega\right)\right)(\cdot\,,t)\right\|_{L^2}} \leq 2.$$

Then there is finite-time blowup with

$$T_{\max} < T_* := \frac{-E_0 + \sqrt{E_0^2 + f_0 K_0}}{f_0},$$

where K_0 *and* E_0 *are taken as in Definition 1.7 and* f_0 *is as defined above.*

Remark 1.13. Theorem 6.1 quantifies how close solutions of the Navier–Stokes strain equation have to be to solving the model equation in order to be guaranteed to blowup in finite time. This result is — to the knowledge of the author — the first of its kind. There are many results stating that if some scale invariant quantity is finite, then solutions of the Navier–Stokes equation must be smooth, such as the aforementioned Ladyzhenskaya–Prodi–Serrin and Beale–Kato–Majda regularity criteria. Theorem 6.1 is the first result to say that, for some set of initial data, if a scale invariant quantity remains small enough for the history of the solution, there must be blowup in finite time.

Remark 1.14. The mechanism for blowup proposed in Theorem 6.1 for the Navier–Stokes equation is also consistent with research on the turbulent energy cascade. Very recently, Carbone and Bragg [2020] showed both theoretically and numerically that strain self-amplification is a more important factor in the average turbulent energy cascade than vortex stretching. This gave a concrete statement to a line of inquiry on the turbulent energy cascade begun by Tsinober [2001]. The turbulent energy cascade is directly tied to the Navier–Stokes regularity problem, as finite-time blowup requires a transfer of energy to arbitrarily small scales, so this suggests that the self-amplification of strain is the most likely potential mechanism for the finite-time blowup of solutions of the Navier–Stokes equation. The conditional blowup result in this paper gives a quantitative estimate on the structure of the nonlinearity that will lead to finite-time blowup for the Navier–Stokes equation via the self-amplification of strain if it is maintained by the dynamics.

Remark 1.15. Turbulent solutions of the Navier–Stokes equation are, almost by definition, difficult to impossible to write down in closed form. This poses a significant barrier to proving the existence of smooth solutions of the Navier–Stokes equation that blowup in finite time: if finite-time blowup solutions do in fact exist, it will still almost certainly not be possible to give a negative answer to the Navier–Stokes regularity problem by providing a counterexample in closed form. Any progress on the Navier–Stokes regularity problem in the direction of proving the existence of finite-time blowup will likely require an interplay of analysis and numerics. Theorem 6.1 provides a quantitative criterion that could guide further numerical work searching for possible blowup solutions.

We cannot show that there are any solutions of Navier–Stokes equation which satisfy the perturbative condition in Theorem 6.1 up until T^*. If we could, then this would solve the Navier–Stokes regularity problem by implying the existence of finite-time blowup. We can, however, use scaling arguments to prove that this condition is satisfied for short times for some solutions of the Navier–Stokes equation.

Theorem 6.3. *There exists a mild solution of the Navier–Stokes equation $u \in C([0, T_{\max}); H^3_{\mathrm{df}})$ and $\epsilon > 0$ such that*

$$-3\|S^0\|^2_{\dot{H}^1} - 4\int \det(S^0) > 0,$$

and for all $0 \leq t < \epsilon$,

$$\frac{\left\| P_{\mathrm{st}}\big((u\cdot\nabla)S + \frac{1}{3}S^2 + \frac{1}{4}\omega\otimes\omega\big)(\cdot\,,t)\right\|_{L^2}}{\left\|\big(-\Delta S + P_{\mathrm{st}}\big(\frac{1}{2}(u\cdot\nabla)S + \frac{5}{6}S^2 + \frac{1}{8}\omega\otimes\omega\big)\big)(\cdot\,,t)\right\|_{L^2}} \leq 2.$$

Remark 1.16. In this paper we have taken the viscosity to be $\nu = 1$. For the Navier–Stokes regularity problem, we can fix the viscosity to be $\nu = 1$ without loss of generality because it is equivalent up to rescaling the Navier–Stokes regularity problem for arbitrary $\nu > 0$. It is useful, however, to see how the blowup results that we will prove scale with respect to the viscosity parameter $\nu > 0$. If we take the viscosity to be $\nu > 0$, then the Navier–Stokes equation is now given by

$$\partial_t u - \nu \Delta u + P_{\mathrm{df}} \nabla \cdot (u \otimes u) = 0,$$

and the strain self-amplification model equation is given by

$$\partial_t S - \nu \Delta S + \tfrac{2}{3} P_{\mathrm{st}}(S^2) = 0.$$

In this case, the condition

$$-3\|S^0\|_{\dot{H}^1}^2 - 4\int \det(S^0) > 0$$

in Theorems 5.3 and 6.1 is replaced with the condition

$$-3\nu\|S^0\|_{\dot{H}^1}^2 - 4\int \det(S^0) > 0.$$

Likewise the condition

$$\frac{\left\| P_{\mathrm{st}}\big((u\cdot\nabla)S + \frac{1}{3}S^2 + \frac{1}{4}\omega\otimes\omega\big)(\cdot\,,t)\right\|_{L^2}}{\left\|\big(-\Delta S + P_{\mathrm{st}}\big(\frac{1}{2}(u\cdot\nabla)S + \frac{5}{6}S^2 + \frac{1}{8}\omega\otimes\omega\big)\big)(\cdot\,,t)\right\|_{L^2}} \leq 2$$

in Theorems 6.1 and 6.3 is replaced by

$$\frac{\left\| P_{\mathrm{st}}\big((u\cdot\nabla)S + \frac{1}{3}S^2 + \frac{1}{4}\omega\otimes\omega\big)(\cdot\,,t)\right\|_{L^2}}{\left\|\big(-\nu\Delta S + P_{\mathrm{st}}\big(\frac{1}{2}(u\cdot\nabla)S + \frac{5}{6}S^2 + \frac{1}{8}\omega\otimes\omega\big)\big)(\cdot\,,t)\right\|_{L^2}} \leq 2.$$

Remark 1.17. We should note in particular this means that if $S^0 \in H^1_{\mathrm{st}}$ and $-\int \det(S^0) > 0$, then for all

$$0 < \nu < \nu^0 := \frac{-4\int \det(S^0)}{3\|S^0\|_{\dot{H}^1}^2},$$

the strain self-amplification model equation with viscosity ν blows up in finite time. This implies that the blowup for the strain self-amplification model equation is generic at sufficiently large Reynolds number, subject only to the geometric sign constraint on initial data, i.e., $-\int \det(S^0) > 0$. This suggests that the self-amplification of strain is likely the driving factor behind possible blowup for the full Navier–Stokes equation, and any depletion of nonlinearity preventing finite-time blowup must come from the effects of advection and the nonlocal interaction of strain and vorticity.

This also means that finite-time blowup may occur for the strain self-amplification model equation even in simplified geometric settings where blowup is ruled out for the full Navier–Stokes equation. We will show that there is finite-time blowup for the strain self-amplification model equation even when restricted to axisymmetric, swirl-free solutions. This contrasts strongly with the Navier–Stokes equation where there is global regularity for arbitrarily large initial data in the axisymmetric, swirl-free case. There are also axisymmetric, swirl-free solutions of the Navier–Stokes equation that satisfy the perturbative condition for short times, as in Theorem 6.3. Such solutions cannot, of course, satisfy the conditions of Theorem 6.1 because they cannot blowup in finite time, and hence the perturbative condition can only be satisfied for short times in such cases.

Remark 1.18. Because such a wide range of initial data lead to finite-time blowup for the strain self-amplification model equation, the set of initial data for which there is finite-time blowup for this model equation is too broad a set to consider as possible candidates for finite-time blowup for the Navier–Stokes equation. While initial data that blowup in finite time are ubiquitous at high Reynolds number, subject only to a sign constraint on the integral of the determinant of the strain, this does not necessarily mean that blowup itself is generic. There could be certain structures or scaling laws that emerge as the blowup time is approached for any blowup solution; further study is needed.

One possible avenue for further work would be to allow the dynamics of the strain self-amplification model equation to select candidates for blowup for the full Navier–Stokes equation. Consider a solution of the strain self-amplification model equation that is not axisymmetric and swirl free, and that blows up in finite time $T_{\max} < +\infty$. If we take $S(\cdot, T_{\max} - \epsilon)$ for some $0 < \epsilon \ll T_{\max}$ as our initial data for the full Navier–Stokes equation, then this would be a very natural candidate for blowup if blowup does in fact occur for the full Navier–Stokes equation. To consider such an approach, more detailed study of the qualitative features of blowup solutions of the strain self-amplification model equation is needed. At present, essentially all we know about such solutions is a lower bound on the growth of enstrophy and that λ_2^+ blows up in the scale-critical $L_t^p L_x^q$ spaces.

In Section 2, we will discuss the relationship between our results and previous results for simplified model equations for Navier–Stokes. In Section 3, we will define a number of the spaces used in our analysis and give precise definitions of mild solutions. In Section 4, we will develop the local well-posedness theory for the strain self-amplification model equation, including proving global well-posedness for small initial data, and scale critical regularity criteria in terms of λ_2^+ and in terms of two vorticity components. In Section 5, we will prove Theorem 5.3, demonstrating the existence of finite-time blowup for solutions of the strain self-amplification model equation, and will prove a number of properties about the set of initial data satisfying the hypothesis of this theorem. Finally in Section 6, we will prove Theorem 6.1, the conditional blowup result for the full Navier–Stokes equation when a perturbative condition is satisfied by the history of the solution, and further show that this perturbative condition is satisfied for short times for some solutions of the Navier–Stokes equation.

2. Relationship to previous literature

There are a number of previous results that prove blowup for simplified model equations for Navier–Stokes with the hope of elucidating possibilities of extending this to the full Navier–Stokes equation. Montgomery-Smith [2001] introduced a scalar toy model equation, replacing the first order pseudo-differential operator $P_{\mathrm{df}}\nabla\cdot$ by $-(-\Delta)^{1/2}$ and replacing the quadratic term $u \otimes u$ by u^2, giving the scalar equation

$$\partial_t u - \Delta u - (-\Delta)^{1/2}(u^2) = 0,$$

and proved the existence of finite-time blowup solutions for this equation. This blowup result was extended by Gallagher and Paicu [2009] to a model equation on the space of divergence-free vector fields by adjusting the Fourier symbol of the first order pseudo differential operator. However, while Gallagher and Paicu's model equation is an evolution equation on natural constraint space, the space of divergence-free vector fields, neither of these model equations respects the energy equality, and so both are still quite far from the actual fluid equations. They are nonetheless important in that they establish that it is not possible to prove global regularity for the Navier–Stokes equation using heat semigroup methods alone.

Tao [2016] improved on these earlier blowup results by introducing a Fourier space averaged Navier–Stokes model equation. His model equation is given by

$$\partial_t u - \Delta u + \tilde{B}(u, u) = 0, \tag{2-1}$$

where $\tilde{B}(u, u)$ is a Fourier space averaged version of $P_{\mathrm{df}} \nabla \cdot (u \otimes u)$. This equation is an improvement over the previous results because $\tilde{B}$ is constructed so that

$$\langle \tilde{B}(u, u), u \rangle = 0,$$

so Tao's model equation (2-1) respects the energy equality with, for all $0 < t < T_{\max}$,

$$\tfrac{1}{2}\|u(t)\|_{L^2}^2 + \int_0^t \|u(\tau)\|_{\dot{H}^1}^2 \,\mathrm{d}\tau = \tfrac{1}{2}\|u^0\|_{L^2}^2,$$

while also exhibiting finite-time blowup. The operator $\tilde{B}$ also has some of the same harmonic analysis bounds as those found for full Navier–Stokes equation, in particular,

$$\|\tilde{B}(u, u)\|_{L^2} \le C \|u\|_{L^4} \|\nabla u\|_{L^4}. \tag{2-2}$$

The fact that there are finite-time blowup solutions to Tao's model equation shows that if there is global regularity for solutions of the Navier–Stokes equation with arbitrary smooth initial data, the proof will require more than the energy equality and the standard harmonic analysis techniques. New a priori bounds are needed. We will note in particular that the bound in (2-2) implies that Tao's model equation respects the Ladyzhenskaya–Prodi–Serrin regularity criterion, that is if $T_{\max} < +\infty$ for a solution u of (2-1), then for all $2/p + 3/q = 1$ and $3 < q \le +\infty$,

$$\int_0^{T_{\max}} \|u\|_{L^q}^p = +\infty.$$

While the Tao model equation respects the energy equality and some of the structure of the velocity equation, it does not respect the structure of the vorticity or strain equations. In particular, Tao's model does not respect — or at least has not been shown to respect — the identity for enstrophy growth in Proposition 1.9, the regularity criterion on λ_2^+ in Theorem 1.10, or the regularity criterion on two components of the vorticity in (1-4). The finite-time blowup result for the strain self-amplification model equation is an advance on Tao's model equation if the Navier–Stokes regularity problem is considered from the point of view of enstrophy growth. The model equation considered here, unlike Tao's model equation, does not respect the energy equality; however, from a mathematical point of view, the energy equality is less fundamental to the Navier–Stokes regularity problem than the identity for enstrophy growth because energy does not control regularity. Blowup for the Navier–Stokes equation in finite time is equivalent to the blowup of enstrophy in finite time, so mathematically it is very significant that we are able to show blowup for an evolution equation on L^2_{st} that respects the identity for enstrophy growth in Proposition 1.9. In summary, Tao's model equation reflects more of the structure of the velocity formulation of the Navier–Stokes regularity problem, while the strain self-amplification model equation reflects more of the structure of both the strain and vorticity formulations of the regularity problem.

The strain self-amplification model equation is the first model equation of possible Navier–Stokes blowup that respects regularity criteria for the Navier–Stokes equation based not just on size, but on geometric structure as well. It is straightforward to show that the strain self-amplification, Montgomery-Smith, Gallagher–Paicu, and Tao model equations all respect the Ladyzhenskaya–Prodi–Serrin regularity criterion on the size of u, but the strain self-amplification model equation also respects the regularity criterion on λ_2^+ in Theorem 1.10. This implies as a corollary that the strain self-amplification model equation must respect the regularity criterion on two vorticity components proved by Chae and Choe as well. This suggests it captures significantly more of the geometry of potential Navier–Stokes blowup than any of the previous model equations, at least as far as deformation and vorticity are concerned.

Theorem 6.1 shows that the local part of the nonlinearity of the strain evolution equation tends to lead to finite-time blowup for a wide range of initial conditions, so there must be finite-time blowup for the Navier–Stokes equation similar to the blowup for the model equation for the self-amplification of strain unless the vorticity and advection terms act to deplete this nonlinearity and prevent blowup. This is consistent with a number of previous works for model equations related to the Navier–Stokes and Euler equations that suggest that advection plays a regularizing role. For instance, there are a number of previous works on the Constantin–Lax–Majda [Constantin et al. 1985] and De Gregorio [1990] 1D models for the vorticity equation which showed that advection may have a regularizing effect [Córdoba et al. 2005; Elgindi and Jeong 2020; Jia et al. 2019]. Theorem 5.3, which states that finite-time blowup occurs for a wide range of initial data for the strain self-amplification model equation, extends the analysis of the regularizing role of advection from 1D models that do not respect the structure of the constraint space to a 3D model that does respect the structure of the constraint space.

There is also previous research on model equations for the axisymmetric Navier–Stokes and Euler equations which preserve more of the structures of three dimensional fluid mechanics than the Constantin–Lax–Majda or De Gregorio models. These model equations also show that advection plays a regularizing role [Hou and Lei 2009; Hou et al. 2018]. Furthermore, there has been research on the possible role of advection in the depletion of nonlinearity related to its interaction with the pressure in the growth of subcritical L^q-norms of u [Tran and Yu 2015]. Theorem 6.1 is entirely novel, however, because it is the first perturbative, finite-time blowup result related to the possible role of nonlinear depletion by advection. The previous results were either heuristic or numerical; in contrast, Theorem 6.1 provides a quantitative condition guaranteeing blowup as long as the terms which could potentially deplete the nonlinear self-amplification of strain are small enough relative to strain self-amplification.

Finally, we should mention that very recently, Elgindi [2021] and Elgindi, Ghoul, and Masmoudi [Elgindi et al. 2021] proved finite-time blowup for a class of $C^{1,\alpha}(\mathbb{R}^3)$ solutions of the Euler equation that conserve energy. While the question of blowup for smooth solutions to the Euler equation remains open, this represents an enormous step forward in providing an example of classical solutions to the Euler equation that blowup in finite time. The blowup solutions of the Euler equation constructed in [Elgindi 2021; Elgindi et al. 2021] are axisymmetric and swirl free, and are closely related to an example of finite-time blowup that we will construct for the strain self-amplification model equation. We will discuss this further in Section 5.

3. Definitions

We begin by defining the homogeneous and inhomogeneous Hilbert spaces.

Definition 3.1. For all $s \in \mathbb{R}$, let $H^s(\mathbb{R}^3)$ be the Hilbert space with norm

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}^3} (1+4\pi^2|\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi = \|(1+4\pi^2|\xi|^2)^{s/2}\hat{f}\|_{L^2}^2,$$

and for all $-\frac{3}{2} < s < \frac{3}{2}$, let $\dot{H}^s(\mathbb{R}^3)$ be the homogeneous Hilbert space with norm

$$\|f\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^3} (2\pi|\xi|)^{2s} |\hat{f}(\xi)|^2 \, d\xi = \|(2\pi|\xi|)^s \hat{f}\|_{L^2}^2.$$

Note that when referring to $H^s(\mathbb{R}^3)$, $\dot{H}^s(\mathbb{R}^3)$, or $L^p(\mathbb{R}^3)$, we will often omit the $\mathbb{R}^3$ for brevity. All Hilbert and Lebesgue norms are taken over $\mathbb{R}^3$ unless otherwise specified. Furthermore, $\mathbb{S}^{3\times 3}$ will refer to the space of three by three symmetric matrices:

$$\mathbb{S}^{3\times 3} = \left\{ \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix} : a, b, c, d, e, f \in \mathbb{R} \right\}.$$

We now define the subspaces of divergence-free vector fields and strain matrices in Hilbert spaces.

Definition 3.2. For all $s \in \mathbb{R}$, define $H^s_{\mathrm{df}} \subset H^s(\mathbb{R}^3; \mathbb{R}^3)$ by

$$H^s_{\mathrm{df}} = \{u \in H^s(\mathbb{R}^3; \mathbb{R}^3) : \xi \cdot \hat{u}(\xi) = 0 \text{ almost everywhere } \xi \in \mathbb{R}^3\}.$$

For all $-\frac{3}{2} < s < \frac{3}{2}$, define $\dot{H}^s_{\mathrm{df}} \subset \dot{H}^s(\mathbb{R}^3; \mathbb{R}^3)$ by

$$\dot{H}^s_{\mathrm{df}} = \{u \in \dot{H}^s(\mathbb{R}^3; \mathbb{R}^3) : \xi \cdot \hat{u}(\xi) = 0 \text{ almost everywhere } \xi \in \mathbb{R}^3\}.$$

Definition 3.3. For all $s \in \mathbb{R}$, define $H^s_{\mathrm{st}} \subset H^s(\mathbb{R}^3; \mathbb{S}^{3\times 3})$ by

$$H^s_{\mathrm{st}} = \{\nabla_{\mathrm{sym}}(-\Delta)^{-1/2} u : u \in H^s_{\mathrm{df}}\}.$$

For all $-\frac{3}{2} < s < \frac{3}{2}$, define $\dot{H}^s_{\mathrm{st}} \subset \dot{H}^s(\mathbb{R}^3; \mathbb{S}^{3\times 3})$ by

$$\dot{H}^s_{\mathrm{st}} = \{\nabla_{\mathrm{sym}}(-\Delta)^{-1/2} u : u \in \dot{H}^s_{\mathrm{df}}\}.$$

Definition 3.4. For all $1 < q < +\infty$, define L^q_{st} by

$$L^q_{\mathrm{st}} = \{S \in L^q(\mathbb{R}^3; \mathbb{S}^{3\times 3}) : \operatorname{tr}(S) = 0, \; S = \nabla_{\mathrm{sym}}(-\Delta)^{-1}(-2\operatorname{div}(S))\}.$$

Remark 3.5. We will note that we have already defined L^2_{st} in the introduction, so we now have two definitions of L^2_{st}. These definitions are equivalent, as was proven by the author in [Miller 2020]. The key reason for this is that, just as the vorticity can be inverted to obtain the velocity, with

$$u = \nabla \times (-\Delta)^{-1}\omega,$$

so too can the strain be inverted to obtain the velocity, with

$$u = -2\,\mathrm{div}(-\Delta)^{-1}S.$$

This means that for all $S \in L^2_{\mathrm{st}}$,

$$S = \nabla_{\mathrm{sym}} u \quad \iff \quad u = -2\,\mathrm{div}(-\Delta)^{-1}S.$$

This implies the condition in Definition 3.4 in the case $q = 2$ is equivalent to the condition in Definition 1.2. See [Miller 2020] for more details. We will also note that Definition 3.4 is well defined because the operator $-2\nabla_{\mathrm{sym}}\,\mathrm{div}(-\Delta)^{-1}$ is a bounded linear operator mapping $L^q \to L^q$ for all $1 < q < +\infty$. This follows from the boundedness of the Riesz transform $R = \nabla(-\Delta)^{-1/2}$ because

$$-2\nabla_{\mathrm{sym}}\,\mathrm{div}(-\Delta)^{-1}S = -2R_{\mathrm{sym}}R \cdot S.$$

We will also define axisymmetric, swirl-free vector fields and strain matrices.

Definition 3.6. Begin by letting

$$r = \sqrt{x_1^2 + x_2^2}, \quad z = x_3, \quad e_r = \frac{1}{r}(x_1, x_2, 0), \quad e_z = (0, 0, 1).$$

We will say that $u \in \dot{H}^1_{\mathrm{df}}$ is an axisymmetric, swirl-free vector field if

$$u(x) = u_r(r, z)e_r + u_z(r, z)e_z.$$

Note that the divergence-free condition can be expressed in this case by

$$\nabla \cdot u = \partial_r u_r + \frac{1}{r}u_r + \partial_z u_z = 0.$$

We will say that $S \in L^2_{\mathrm{st}}$ is an axisymmetric, swirl-free strain matrix if

$$S = \nabla_{\mathrm{sym}} u,$$

where $u \in \dot{H}^1_{\mathrm{df}}$ is an axisymmetric, swirl-free vector field.

We conclude this section by providing the precise definitions for mild solutions of the Navier–Stokes equation, the Navier–Stokes strain equation, and the strain self-amplification model equation.

Definition 3.7. A velocity field $u \in C([0, T_{\max}); \dot{H}^1_{\mathrm{df}})$ is a mild solution to the Navier–Stokes equation if it satisfies (1-2) in the sense of Duhamel's formula, that is, if for all $0 < t < T_{\max}$,

$$u(t) = e^{t\Delta}u^0 - \int_0^t e^{\tau\Delta} P_{\mathrm{df}} \nabla \cdot (u \otimes u)(t - \tau)\, d\tau.$$

Note that $e^{t\Delta}$ is defined in terms of convolution with the heat kernel

$$G(x,t)=\frac{1}{(4\pi t)^{3/2}}\exp\left(-\frac{|x|^2}{4t}\right),$$

so that

$$e^{t\Delta}f=G(\cdot\,,t)*f.$$

Remark 3.8. $T_{\max}$ is the maximal time of existence for a mild solution. If there is a mild solution globally in time for some initial data $u^0\in\dot{H}^1_{\mathrm{df}}$, then $T_{\max}=+\infty$, and if there is not a mild solution globally in time, then $T_{\max}<+\infty$ is the blowup time when the solution becomes singular.

Definition 3.9. A strain matrix $S\in C([0,T_{\max});L^2_{\mathrm{st}})$ is a mild solution to the Navier–Stokes strain equation if it satisfies (1-5) in the sense of Duhamel's formula, that is, if for all $0<t<T_{\max}$,

$$S(t)=e^{t\Delta}S^0-\int_0^t e^{\tau\Delta}P_{\mathrm{st}}\big((u\cdot\nabla)S+S^2+\tfrac{1}{4}\omega\otimes\omega\big)(t-\tau)\,d\tau,$$

with $u=(-\Delta)^{-1}(-2\operatorname{div}(S))$ and $\omega=\nabla\times u$.

Definition 3.10. $S\in C([0,T_{\max});L^2_{\mathrm{st}})$ is a mild solution to the strain self-amplification model equation (1-6) if S satisfies this equation in the sense of Duhamel's formula, that is, for all $0<t<T_{\max}$,

$$S(t)=e^{t\Delta}S^0-\frac{2}{3}\int_0^t e^{\tau\Delta}P_{\mathrm{st}}(S^2)(t-\tau)\,d\tau.$$

4. Some properties of the strain self-amplification model equation

We begin this section by considering the local-in-time existence of mild solutions to the strain self-amplification model equation.

Theorem 4.1. *Let $C=\left(\frac{3}{32}\|g\|_{L^2}\right)^4$, where $g(x)=\exp(-|x|^2/4)/(2\pi)^{3/2}$. For all $S^0\in L^2_{\mathrm{st}}$, there exists a unique mild solution to the strain self-amplification model equation, $S\in C([0,T_{\max});L^2_{\mathrm{st}})$, with $T_{\max}\ge C/\|S^0\|^4_{L^2}$. Furthermore, $S\in C((0,T_{\max});H^\infty)$ and is therefore smooth for all positive times up until possible blowup.*

Proof. The proof of Theorem 4.1 is essentially the same as the proof of local existence of mild solutions for the Navier–Stokes equation introduced by Kato and Fujita. It will be based on a Banach fixed point argument.

We begin by fixing

$$T<\frac{C}{\|S^0\|^4_{L^2}}.$$

Note that this implies

$$\tfrac{32}{3}\|g\|_{L^2}\|S^0\|_{L^2}T^{1/4}<1.$$

Define the map $W:C([0,T];L^2_{\mathrm{st}})\to C([0,T];L^2_{\mathrm{st}})$ by

$$W[M](t)=e^{t\Delta}S^0+\int_0^t e^{\tau\Delta}P_{\mathrm{st}}(M^2(t-\tau))\,d\tau.$$

Note that S being a mild solution of the heat equation is equivalent to S being a fixed point of this map with $W[S]=S$.

We will first show that if $\|M\|_{C([0,T];L^2)} \leq 2\|S^0\|_{L^2}$, then $\|W[M]\|_{C([0,T];L^2)} \leq 2\|S^0\|_{L^2}$. Recall that

$$e^{t\Delta} f = G(\,\cdot\,,t) * f,$$

where

$$G(x,t) = t^{-3/2} g(t^{-1/2}x).$$

Therefore we can compute that

$$\|G(\,\cdot\,,t)\|_{L^2} = \|g\|_{L^2} t^{-3/4}.$$

Applying Young's inequality for convolutions we find that for all $0 \leq t \leq T$,

$$\begin{aligned}
\|W[M](t)\|_{L^2} &\leq \|S^0\|_{L^2} + \frac{2}{3}\int_0^t \|P_{\mathrm{st}}(e^{\tau\Delta})M^2)\|_{L^2}\,\mathrm{d}\tau \leq \|S^0\|_{L^2} + \frac{2}{3}\int_0^t \|G(\,\cdot\,,t)\|_{L^2}\|M^2(t-\tau)\|_{L^1}\,\mathrm{d}\tau \\
&\leq \|S^0\|_{L^2} + \frac{2}{3}\|M^2\|_{C([0,T];L^1)}\int_0^t \|g\|_{L^2}\tau^{-3/4}\,\mathrm{d}\tau \leq \|S^0\|_{L^2} + \frac{8}{3}\|M\|^2_{C([0,T];L^2)}\|g\|_{L^2} t^{1/4} \\
&\leq \|S^0\|_{L^2} + \frac{8}{3}\|M\|^2_{C([0,T];L^2)}\|g\|_{L^2} T^{1/4}.
\end{aligned}$$

Using the fact that $\|M\|_{C([0,T];L^2)} \leq 2\|S^0\|_{L^2}$ and recalling that $\frac{32}{3}\|g\|_{L^2}\|S^0\|_{L^2}T^{1/4} < 1$, we can see that

$$\tfrac{8}{3}\|M\|^2_{C([0,T];L^2)}\|g\|_{L^2}T^{1/4} \leq \tfrac{32}{3}\|S^0\|^2_{L^2}\|g\|_{L^2}T^{1/4} \leq \|S^0\|_{L^2}.$$

This implies that

$$\|W[M]\|_{C([0,T];L^2)} \leq 2\|S^0\|_{L^2}.$$

Therefore W is an automorphism on the closed ball

$$B = \{M \in C([0,T];\, L^2_{\mathrm{st}}) : \|M\|_{C([0,T];L^2)} \leq 2\|S^0\|_{L^2}\}.$$

We will now show that W is a contraction mapping on B. Fix $M, Q \in B$. Using Young's convolution inequality as above we can compute that for all $0 \leq t \leq T$,

$$\begin{aligned}
\|W[M](t) - W[Q](t)\|_{L^2} &= \frac{1}{3}\left\| P_{\mathrm{st}} \int_0^t e^{t\Delta}((M+Q)(M-Q) + (M-Q)(M+Q)) \right\|_{L^2} \\
&\leq \frac{8}{3}\|g\|_{L^2} t^{1/4}\|M+Q\|_{C([0,T];L^2)}\|M-Q\|_{C([0,T];L^2)} \\
&\leq \frac{8}{3}\|g\|_{L^2} t^{1/4}(\|M\|_{C([0,T];L^2)} + \|Q\|_{C([0,T];L^2)})\|M-Q\|_{C([0,T];L^2)} \\
&\leq \frac{32}{3}\|g\|_{L^2}\|S^0\|_{L^2}T^{1/4}\|M-Q\|_{C([0,T];L^2)}.
\end{aligned}$$

Letting

$$r = \tfrac{32}{3}\|g\|_{L^2}\|S^0\|_{L^2}T^{1/4} < 1,$$

we find that

$$\|W[M] - W[Q]\|_{C([0,T];L^2)} \leq r\|M-Q\|_{C([0,T];L^2)}.$$

Note that B is a complete metric space, so by the Banach fixed point theorem, we can conclude that there exists a unique $S \in B \subset C([0,T]; L^2_{\mathrm{st}})$ such that

$$W[S] = S.$$

This implies that there is a unique, mild solution with initial data in $S^0 \in L^2_{\mathrm{st}}$ locally in time. Note that the higher regularity $S \in C((0,T]; H^\infty)$ is a result of the smoothing due to the heat kernel, but we will not go through the details of that here. This higher regularity follows from a bootstrapping argument that is essentially the same as the argument in the case of the Navier–Stokes equation given in [Fujita and Kato 1964]. □

We will now prove a useful proposition giving an identity for the determinant of 3×3, symmetric, trace-free matrices.

Proposition 4.2. *Suppose $M \in \mathbb{S}^{3\times 3}$ is a 3×3 symmetric matrix such that* $\operatorname{tr}(M) = 0$. *Then*

$$\operatorname{tr}(M^3) = 3\det(M).$$

Proof. Every symmetric matrix is diagonalizable over $\mathbb{R}$, so let $\lambda_1 \le \lambda_2 \le \lambda_3$ be the eigenvalues of M. Using the trace-free condition we can see that

$$\operatorname{tr}(M) = \lambda_1 + \lambda_2 + \lambda_3 = 0.$$

Therefore we can compute that

$$\begin{aligned}\operatorname{tr}(M^3) = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = (-\lambda_1 - \lambda_2)^3 + \lambda_1^3 + \lambda_2^3 &= -3\lambda_1^2\lambda_2 - 3\lambda_1\lambda_2^2 \\ &= 3(-\lambda_1 - \lambda_2)\lambda_1\lambda_2 = 3\lambda_1\lambda_2\lambda_3 = 3\det(M).\end{aligned}$$ □

Using this proposition, we will show that the strain self-amplification model equation has the same identity for enstrophy growth as the Navier–Stokes strain equation.

Proposition 4.3. *Suppose $S \in C([0,T_{\max}); L^2_{\mathrm{st}})$ is a mild solution to the strain self-amplification model equation. Then for all $0 < t < T_{\max}$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|S(\cdot,t)\|_{L^2}^2 = -2\|S\|_{\dot{H}^1}^2 - \frac{4}{3}\int \operatorname{tr}(S^3) = -2\|S\|_{\dot{H}^1}^2 - 4\int \det(S).$$

Proof. Taking the derivative in time of the L^2-norm, we plug equation (1-6) into the strain self-amplification model, finding

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|S(\cdot,t)\|_{L^2}^2 = 2\langle \partial_t S, S\rangle &= -2\Big\langle -\Delta S + \frac{2}{3}P_{\mathrm{st}}(S^2), S\Big\rangle = -2\|S\|_{\dot{H}^1}^2 - \frac{4}{3}\langle P_{\mathrm{st}}(S^2), S\rangle \\ &= -2\|S\|_{\dot{H}^1}^2 - \frac{4}{3}\langle S^2, S\rangle = -2\|S\|_{\dot{H}^1}^2 - \frac{4}{3}\int \operatorname{tr}(S^3) = -2\|S\|_{\dot{H}^1}^2 - 4\int \det(S),\end{aligned}$$

where we have used the fact that S belongs to L^2_{st} to drop the projection P_{st} and the fact that S is symmetric to compute the inner product, and have finally applied Proposition 4.2. □

In fact, the vortex stretching and the integral of the determinant of the strain can be related in a general way as follows. This will be useful in showing the term we dropped in the model equation does not contribute to enstrophy growth.

Proposition 4.4. *For all $S \in L^3_{\mathrm{st}}$,*

$$-4\int \det(S) = \langle S; \omega \otimes \omega\rangle,$$

where $u = -2\,\mathrm{div}(-\Delta)^{-1}S$ and $\omega = \nabla \times u$. In particular, this implies that

$$\left\langle S, P_{\mathrm{st}}\left(\tfrac{1}{3}S^2 + \tfrac{1}{4}\omega\otimes\omega\right)\right\rangle = 0.$$

Proof. The first step of the proof will be to show that

$$\int \mathrm{tr}((\nabla u)^3) = 0. \tag{4-1}$$

We begin by recalling that by definition,

$$S = \nabla_{\mathrm{sym}}(-\Delta)^{-1}(-2\,\mathrm{div}(S)),$$

and so we can see that $S = \nabla_{\mathrm{sym}} u$. We may conclude that

$$\nabla u = -2\nabla\,\mathrm{div}(-\Delta)^{-1}S.$$

Using the boundedness of the Riesz transform from $L^3 \to L^3$, this implies that $\nabla u \in L^3$, so the integral in (4-1) is absolutely convergent.

Using the divergence-free condition we note that

$$\sum_{i=1}^3 \frac{\partial u_i}{\partial x_i} = \nabla\cdot u = 0.$$

Therefore for all $u \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ with $\nabla\cdot u = 0$, we can integrate by parts — without worrying about boundary terms because of compact support — finding

$$\begin{aligned}\int \mathrm{tr}((\nabla u)^3) &= \sum_{i,j,k=1}^3 \int \frac{\partial u_j}{\partial x_i}\frac{\partial u_k}{\partial x_j}\frac{\partial u_i}{\partial x_k} = -\sum_{i,j,k=1}^3 \int u_j \frac{\partial^2 u_k}{\partial x_i \partial x_j}\frac{\partial u_i}{\partial x_k} \\ &= \sum_{i,j,k=1}^3 \int u_j \frac{\partial u_k}{\partial x_i}\frac{\partial^2 u_i}{\partial x_j \partial x_k} = -\sum_{i,j,k=1}^3 \int \frac{\partial u_j}{\partial x_k}\frac{\partial u_k}{\partial x_i}\frac{\partial u_i}{\partial x_j} = -\int \mathrm{tr}((\nabla u)^3) = 0.\end{aligned}$$

Because $C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ is dense in $L^3(\mathbb{R}^3;\mathbb{R}^3)$, this suffices to guarantee that for all $\nabla u \in L^3$ with $\nabla\cdot u = 0$,

$$\int \mathrm{tr}((\nabla u)^3) = 0.$$

We know that $\nabla u = S + A$. Using the fact that S is symmetric and A is antisymmetric, and that all antisymmetric matrices are trace-free, we compute

$$\mathrm{tr}((\nabla u)^3) = \mathrm{tr}(S^3) + 3\,\mathrm{tr}(SA^2).$$

Recall from the introduction that

$$A=\frac{1}{2}\begin{pmatrix}0 & \omega_3 & -\omega_2\\ -\omega_3 & 0 & \omega_1\\ \omega_2 & -\omega_1 & 0\end{pmatrix},$$

and we can compute that

$$A^2=\tfrac{1}{4}\omega\otimes\omega-\tfrac{1}{4}|\omega|^2 I_3.$$

Therefore we find

$$3\operatorname{tr}(SA^2)=\tfrac{3}{4}(S:\omega\otimes\omega)+\tfrac{3}{4}|\omega|^2\operatorname{tr}(S)=\tfrac{3}{4}(S:\omega\otimes\omega).$$

Applying Proposition 4.2, we find that

$$\operatorname{tr}(S^3)=3\det(S).$$

Therefore we find

$$\operatorname{tr}((\nabla u)^3)=3\det(S)+\tfrac{3}{4}(S:\omega\otimes\omega).$$

Integrating this equality over $\mathbb{R}^3$ we find

$$\langle S;\,\omega\otimes\omega\rangle+4\int\det(S)=\frac{4}{3}\int\Big(3\det(S)+\frac{3}{4}(S:\omega\otimes\omega)\Big)=\frac{4}{3}\int\operatorname{tr}((\nabla u)^3)=0.$$

Finally we compute

$$\Big\langle P_{\mathrm{st}}\Big(\frac{1}{3}S^2+\frac{1}{4}\omega\otimes\omega\Big);\,S\Big\rangle=\Big\langle\frac{1}{3}S^2+\frac{1}{4}\omega\otimes\omega;\,S\Big\rangle=\frac{1}{4}\langle S;\,\omega\otimes\omega\rangle+\frac{1}{3}\int\operatorname{tr}(S^3)=\frac{1}{4}\langle S;\,\omega\otimes\omega\rangle+\int\det(S)=0,$$

and this completes the proof. The author would like to thank the anonymous referee from an earlier version of [Miller 2020] for this observation. □

Using this result, we will observe that the term we have dropped from the Navier–Stokes strain equation to obtain our strain self-amplification model equation is orthogonal to S with respect to the L^2-inner product.

Corollary 4.5. *Suppose $S\in H^1_{\mathrm{st}}$ with $S=\nabla_{\mathrm{sym}}u$ and $\omega=\nabla\times u$. Then*

$$\big\langle P_{\mathrm{st}}\big((u\cdot\nabla)S+\tfrac{1}{3}S^2+\tfrac{1}{4}\omega\otimes\omega\big);\,S\big\rangle=0.$$

Proof. We begin by observing that $H^1\hookrightarrow\dot{H}^{1/2}\hookrightarrow L^3$, and so clearly $S\in L^3_{\mathrm{st}}$. Applying Proposition 4.4 we see that

$$\big\langle P_{\mathrm{st}}\big(\tfrac{1}{3}S^2+\tfrac{1}{4}\omega\otimes\omega\big);\,S\big\rangle=0.$$

Next we use the divergence-free condition, $\nabla\cdot u=0$, and the fact that we have sufficient regularity to integrate by parts to compute

$$\langle P_{\mathrm{st}}((u\cdot\nabla)S);\,S\rangle=\langle(u\cdot\nabla)S;\,S\rangle=-\langle S;\,(u\cdot\nabla)S\rangle=0.$$ □

Note that this means the term $P_{\mathrm{st}}\big((u\cdot\nabla)S+\tfrac{1}{3}S^2+\tfrac{1}{4}\omega\otimes\omega\big)$ does not contribute to enstrophy growth, so when we write the Navier–Stokes strain equation as

$$\partial_t S-\Delta S+\tfrac{2}{3}P_{\mathrm{st}}(S^2)+P_{\mathrm{st}}\big((u\cdot\nabla)S+\tfrac{1}{3}S^2+\tfrac{1}{4}\omega\otimes\omega\big)=0,$$

only the terms $-\Delta S$ and $\frac{2}{3}P_{\text{st}}(S^2)$ contribute to enstrophy growth. This is the justification for studying the dynamics of enstrophy growth using a model equation that drops the term $P_{\text{st}}\big((u\cdot\nabla)S+\frac{1}{3}S^2+\frac{1}{4}\omega\otimes\omega\big)$, retaining only the terms that actually contribute to the growth of enstrophy.

The strain self-amplification model equation, like the Navier–Stokes strain equation, is invariant under the rescaling

$$S^\lambda(s,t)=\lambda^2 S(\lambda x,\lambda^2 t).$$

We will now show the existence of global smooth solutions of the strain self-amplification model equation with small initial data in the critical Hilbert space $\dot{H}^{-1/2}$.

Theorem 4.6. *Suppose $S^0\in L^2_{\text{st}}\cap\dot{H}^{-1/2}_{\text{st}}$ and*

$$\|S^0\|_{\dot{H}^{-1/2}}<\frac{3\sqrt{3}}{4\sqrt{2}}\pi.$$

Then there exists a unique, global smooth solution to the strain self-amplification model equation $S\in C([0,+\infty);L^2_{\text{st}})$, that is $T_{\max}=+\infty$.

Proof. We begin by observing there must be a smooth solution $S\in C((0,T_{\max});L^2_{\text{st}})$, for some $T_{\max}>0$. We will consider the growth of the $\dot{H}^{-1/2}$-norm on this time interval. We will use the fractional Sobolev inequality proven in [Lieb 1983; Lieb and Loss 1997]. For all $f\in L^{3/2}(\mathbb{R}^3)$,

$$\|f\|_{\dot{H}^{-1/2}}\le\frac{1}{2^{1/6}\pi^{1/3}}\|f\|_{L^{3/2}},$$

and for all $g\in L^3(\mathbb{R}^3)$,

$$\|g\|_{L^3}\le\frac{1}{2^{1/6}\pi^{1/3}}\|g\|_{\dot{H}^{1/2}}.$$

Applying both fractional Sobolev inequalities we find that

$$\begin{aligned}\frac{d}{dt}\|S(t)\|^2_{\dot{H}^{-1/2}}&=-2\|S\|^2_{\dot{H}^{1/2}}-\frac{4}{3}\langle(-\Delta)^{-1/2}S,S^2\rangle\le-2\|S\|^2_{\dot{H}^{1/2}}+\frac{4}{3}\|(-\Delta)^{-1/2}S\|_{\dot{H}^{1/2}}\|S^2\|_{\dot{H}^{-1/2}}\\&\le-2\|S\|^2_{\dot{H}^{1/2}}+\frac{4}{3}\frac{1}{2^{1/6}\pi^{1/3}}\|S\|_{\dot{H}^{-1/2}}\|S^2\|_{L^{3/2}}\le-2\|S\|^2_{\dot{H}^{1/2}}+\frac{4}{3}\frac{1}{2^{1/6}\pi^{1/3}}\|S\|_{\dot{H}^{-1/2}}\|S\|^2_{L^3}\\&\le-2\|S\|^2_{\dot{H}^{1/2}}+\frac{4}{3}\frac{1}{2^{1/2}\pi}\|S\|_{\dot{H}^{-1/2}}\|S\|^2_{\dot{H}^{1/2}}\le 2\|S\|^2_{\dot{H}^{1/2}}\Big(-1+\frac{\sqrt{2}}{3\pi}\|S\|_{\dot{H}^{-1/2}}\Big).\end{aligned}$$

From this bound on the growth of the $\dot{H}^{-1/2}$-norm it is clear that if

$$\|S(t)\|_{\dot{H}^{-1/2}}<\frac{3\pi}{\sqrt{2}},$$

then

$$\frac{d}{dt}\|S(t)\|_{\dot{H}^{-1/2}}<0.$$

We know that

$$\|S^0\|_{\dot{H}^{-1/2}}<\frac{3\sqrt{3}}{4\sqrt{2}}\pi<\frac{3\pi}{\sqrt{2}},$$

so we can conclude that for all $0\le t<T_{\max}$,

$$\|S(t)\|_{\dot{H}^{-1/2}}<\frac{3\sqrt{3}}{4\sqrt{2}}\pi.$$

To finish the proof we will need to consider bounds on the enstrophy growth in terms of the $\dot{H}^{-1/2}$-norm. In addition to the fractional sharp Sobolev inequality, we will also make use of the ordinary sharp Sobolev inequality [Sobolev 1963; Talenti 1976], which states that for all $f \in L^6(\mathbb{R}^3)$,

$$\|f\|_{L^6} \leq \frac{1}{\sqrt{3}}\left(\frac{2}{\pi}\right)^{2/3}\|f\|_{\dot{H}^1}.$$

Applying the Sobolev inequality, the fractional Sobolev inequality, Hölder's inequality, and the product rule to the identity for enstrophy growth in Proposition 4.3, we find

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|S(t)\|_{L^2}^2 &= -2\|S\|_{\dot{H}^1}^2 - \frac{4}{3}\langle S, S^2\rangle \leq -2\|S\|_{\dot{H}^1}^2 + \frac{4}{3}\|S\|_{\dot{H}^{-1/2}}\|S^2\|_{\dot{H}^{1/2}} \\
&= -2\|S\|_{\dot{H}^1}^2 + \frac{4}{3}\|S\|_{\dot{H}^{-1/2}}\|\nabla(S^2)\|_{\dot{H}^{-1/2}} \leq -2\|S\|_{\dot{H}^1}^2 + \frac{4}{3}\frac{1}{2^{1/6}\pi^{1/3}}\|S\|_{\dot{H}^{-1/2}}\|\nabla(S^2)\|_{L^{3/2}} \\
&\leq -2\|S\|_{\dot{H}^1}^2 + \frac{4}{3}\frac{1}{2^{1/6}\pi^{1/3}}\|S\|_{\dot{H}^{-1/2}}2\|\nabla S\|_{L^2}\|S\|_{L^6} \\
&\leq -2\|S\|_{\dot{H}^1}^2 + \frac{8}{3}\frac{1}{2^{1/6}\pi^{1/3}}\frac{1}{\sqrt{3}}\left(\frac{2}{\pi}\right)^{2/3}\|S\|_{\dot{H}^{-1/2}}\|S\|_{\dot{H}^1}^2 = 2\|S\|_{\dot{H}^1}^2\left(-1 + \frac{4\sqrt{2}}{3\sqrt{3}\pi}\|S\|_{\dot{H}^{-1/2}}\right).
\end{aligned}$$

We have already shown that for all $0 \leq t < T_{\max}$,

$$\frac{4\sqrt{2}}{3\sqrt{3}\pi}\|S(t)\|_{\dot{H}^{-1/2}} < 1,$$

so for all $0 \leq t < T_{\max}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|S(t)\|_{L^2}^2 \leq 0.$$

This implies that for all $0 \leq t < T_{\max}$,

$$\|S(t)\|_{L^2} \leq \|S^0\|_{L^2}.$$

We know from Theorem 4.1 that for all $0 \leq t < T_{\max}$,

$$T_{\max} - t > \frac{C}{\|S(t)\|_{L^2}^4}.$$

This means that if $T_{\max} < +\infty$, then

$$\lim_{t \to T_{\max}} \|S(t)\|_{L^2} = +\infty.$$

We know that for all $0 \leq t < T_{\max}$ we have $\|S(t)\|_{L^2} \leq \|S^0\|_{L^2}$, so we can conclude that $T_{\max} = +\infty$. □

Remark 4.7. We will note that the assumption $S \in \dot{H}^{-1/2} \cap L^2$ is not actually necessary; it is sufficient to have small initial data in $\dot{H}^{-1/2}$ to guarantee global regularity with no assumption that $S^0 \in L^2$. However, dropping this assumption makes the proof a little more technical, and, more importantly, the whole point of a strain self-amplification model equation is to model enstrophy growth, so if our solution is not in L^2 the model does not mean very much.

Likewise, some of the other results in this section are not optimal: for example, it should be straightforward to prove the local existence of mild, smooth solutions with initial data in $B_{p,\infty}^{-2+3/p}$, for $2 \leq p < +\infty$, without too much difficulty. Because the strain self-amplification model equation is adapted specifically to study L^2 solutions however, getting local existence or small data results down to the largest scale critical spaces is not particularly useful or illuminating.

We will now prove that because the strain self-amplification model equation has the same identity for enstrophy growth as the Navier–Stokes equation, it also has a regularity criterion on the positive part of the middle eigenvalue of the strain matrix that is precisely the same as the analogous result for the Navier–Stokes equation, Theorem 1.10.

Theorem 4.8. *Suppose $S \in C([0, T_{\max}); L^2_{\mathrm{st}})$ is a mild solution to the strain self-amplification model equation. Let $\lambda_1(x,t) \leq \lambda_2(x,t) \leq \lambda_3(x,t)$ be the eigenvalues of $S(x,t)$, and let $\lambda_2^+(x,t) = \max\{0, \lambda_2(x,t)\}$. Then for all $3/q + 2/p = 2$ and $3/2 < q \leq +\infty$, there exists $C_q > 0$ depending only on q such that for all $0 < t < T_{\max}$,*

$$\|S(t)\|_{L^2}^2 \leq \|S^0\|_{L^2}^2 \exp\Big(C_q \int_0^t \|\lambda_2^+(\tau)\|_{L^q}^p \,\mathrm{d}\tau\Big). \tag{4-2}$$

In particular, if $T_{\max} < +\infty$, then

$$\int_0^{T_{\max}} \|\lambda_2^+(t)\|_{L^q}^p \,\mathrm{d}t = +\infty.$$

Proof. We know from Theorem 4.1 that if $T_{\max} < +\infty$, then

$$\lim_{t \to T_{\max}} \|S(t)\|_{L^2}^2 = +\infty,$$

so it suffices to prove estimate (4-2). Because $\operatorname{tr}(S) = 0$, we know that $\lambda_1 \leq 0$ and $\lambda_3 \geq 0$, and therefore

$$-\lambda_1\lambda_3 \geq 0.$$

We can compute from the identity for enstrophy growth in Proposition 4.3 that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|S(t)\|_{L^2}^2 &= -2\|S\|_{\dot{H}^1}^2 - 4\int \det(S) = -2\|S\|_{\dot{H}^1}^2 + 4\int (-\lambda_1\lambda_3)\lambda_2 \\ &\leq -2\|S\|_{\dot{H}^1}^2 + 4\int (-\lambda_1\lambda_3)\lambda_2^+ \leq -2\|S\|_{\dot{H}^1}^2 + 2\int \lambda_2^+|S|^2 \leq C_q\|\lambda_2^+\|_{L^q}^p\|S\|_{L^2}^2,\end{aligned}$$

after applying Hölder's inequality, the Sobolev inequality, and Young's inequality. This computation is precisely the same as the one done in the proof of the regularity criterion on λ_2^+ in [Miller 2020], so we refer the reader there for more details on these steps. Applying Grönwall's inequality, we find for all $0 < t < T_{\max}$,

$$\|S(t)\|_{L^2}^2 \leq \|S^0\|_{L^2} \exp\Big(C_q \int_0^t \|\lambda_2^+(\tau)\|_{L^q}^p \,\mathrm{d}\tau\Big). \qquad \square$$

This regularity criterion means that there must be unbounded planar stretching in the scale critical $L_t^p L_x^q$ spaces in order for finite-time blowup to occur. The strength of the strain formulation of the Navier–Stokes regularity problem means that not only does the strain self-amplification model equation respect geometric regularity criteria in terms of the strain; it also respects the regularity criterion on two components of the vorticity proven for the full Navier–Stokes equation by Chae and Choe [1999].

Corollary 4.9. *Suppose $S \in C([0, T_{\max}); L^2_{\mathrm{st}})$ is a mild solution to the strain self-amplification model equation. Let $\omega = \nabla \times (-\Delta)^{-1}(-2\,\mathrm{div}(S))$ be the vorticity associated with the strain S. Then for all $3/q + 2/p = 2$ and $3/2 < q < +\infty$, there exists $C_q > 0$ depending only on q such that for all $0 < t < T_{\max}$,*

$$\|S(t)\|_{L^2}^2 \leq \|S^0\|_{L^2}^2 \exp\Bigl(C_q \int_0^t \|e_3 \times \omega(\tau)\|_{L^q}^p \,\mathrm{d}\tau\Bigr).$$

In particular, if $T_{\max} < +\infty$, then

$$\int_0^{T_{\max}} \|e_3 \times \omega(t)\|_{L^q}^p \,\mathrm{d}t = +\infty.$$

Proof. We know that $\mathrm{tr}(S) = 0$ and $\lambda_1 \leq \lambda_2 \leq \lambda_3$, so λ_2 is the smallest eigenvalue of S in magnitude. This implies for all unit vectors $v \in \mathbb{R}^3$, that $|\lambda_2| \leq |Sv|$. Consequently we can see that for all $x \in \mathbb{R}^3$,

$$|\lambda_2| \leq |Se_3|.$$

Next we observe that

$$\begin{aligned} 2Se_3 &= \nabla u_3 + \partial_3 u, \\ e_3 \times \omega &= \nabla u_3 - \partial_3 u. \end{aligned}$$

We can see that ∇u_3 is a gradient and that $\nabla \cdot \partial_3 u = 0$, and so using the Helmholtz projections unto the spaces of divergence-free vector fields and gradients, we can see that

$$\begin{aligned} \nabla u_3 &= P_{gr}(e_3 \times \omega), \\ \partial_3 u &= -P_{\mathrm{df}}(e_3 \times \omega). \end{aligned}$$

The boundedness of the Helmholtz decomposition then implies that for all $1 < q < +\infty$,

$$\|\lambda_2\|_{L^q} \leq \|Se_3\|_{L^q} \leq \tfrac{1}{2}\|\nabla u_3\|_{L^q} + \tfrac{1}{2}\|\partial_3 u\|_{L^q} \leq C_q \|e_3 \times \omega\|_{L^q}.$$

The result then follows as an immediate corollary of Theorem 4.8. □

5. Finite-time blowup for the strain self-amplification model equation

In this section, we will prove the existence of finite-time blowup for the strain self-amplification model equation. We begin by proving a nonlinear differential inequality giving a lower bound on the rate of enstrophy growth that is sufficient to guarantee finite-time blowup for some initial data.

Proposition 5.1. *Suppose $S \in C([0, T_{\max}); H^1_{\mathrm{st}})$ is a mild solution of the strain self-amplification model equation. Then for all $0 \leq t < T_{\max}$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \geq g_0 E(t)^{3/2},$$

where

$$g_0 = \frac{-3\|S^0\|_{\dot{H}^1}^2 - 4\int \det(S^0)}{\|S^0\|_{L^2}^3}.$$

Proof. We will begin by letting

$$g(t) = \frac{-3\|S(\cdot,t)\|_{\dot{H}^1}^2 - 4\int \det(S(\cdot,t))}{\|S(\cdot,t)\|_{L^2}^3} = \frac{-3\|S\|_{\dot{H}^1}^2 - \frac{4}{3}\int \operatorname{tr}(S^3)}{E(t)^{3/2}}.$$

Differentiating g, we find that for all $0 < t < T_{\max}$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} g(t) &= \frac{6\left\| -\Delta S + \frac{2}{3} P_{\mathrm{st}}(S^2) \right\|_{L^2}^2}{E(t)^{3/2}} - \frac{3}{2} \frac{\left(-3\|S\|_{\dot{H}^1}^2 - \frac{4}{3}\int \operatorname{tr}(S^3)\right)\left(-2\|S\|_{\dot{H}^1}^2 - \frac{4}{3}\int \operatorname{tr}(S^3)\right)}{E(t)^{5/2}} \\
&= \frac{6\left\| -\Delta S + \frac{2}{3} P_{\mathrm{st}}(S^2) \right\|_{L^2}^2}{\|S\|_{L^2}^3} - \frac{3}{2} \frac{\left(-3\|S\|_{\dot{H}^1}^2 - \frac{4}{3}\int \operatorname{tr}(S^3)\right)\left(-2\|S\|_{\dot{H}^1}^2 - \frac{4}{3}\int \operatorname{tr}(S^3)\right)}{\|S\|_{L^2}^5} \\
&\geq \frac{6\left\| -\Delta S + \frac{2}{3} P_{\mathrm{st}}(S^2) \right\|_{L^2}^2}{\|S\|_{L^2}^3} - \frac{3}{2} \frac{\left(-2\|S\|_{\dot{H}^1}^2 - \frac{4}{3}\int \operatorname{tr}(S^3)\right)^2}{\|S\|_{L^2}^5} \\
&= \frac{6}{\|S\|_{L^2}^5}\left(\|S\|_{L^2}^2 \left\| -\Delta S + \frac{2}{3} P_{\mathrm{st}}(S^2) \right\|_{L^2}^2 - \left(-\|S\|_{\dot{H}^1}^2 - \frac{2}{3}\int \operatorname{tr}(S^3) \right)^2 \right).
\end{aligned}$$

Applying Young's inequality, we find

$$-\|S\|_{\dot{H}^1}^2 - \frac{2}{3}\int \operatorname{tr}(S^3) = -\left\langle -\Delta S + \frac{2}{3} P_{\mathrm{st}}(S^2), S \right\rangle \leq \left\| \Delta S + \frac{2}{3} P_{\mathrm{st}}(S^2) \right\|_{L^2} \|S\|_{L^2},$$

and so we can conclude that for all $0 < t < T_{\max}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) \geq 0.$$

Therefore, we can conclude that for all $0 < t < T_{\max}$,

$$g(t) \geq g_0.$$

Finally, we observe that for all $0 < t < T_{\max}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = -2\|S\|_{\dot{H}^1}^2 - 4\int \det(S) \geq -3\|S\|_{\dot{H}^1}^2 - 4\int \det(S) = g(t)E(t)^{3/2} \geq g_0 E(t)^{3/2}, \tag{5-1}$$

and this completes the proof. □

Remark 5.2. Note that as long as S is not the trivial solution — as long as $\|S\|_{\dot{H}^1}^2 > 0$ — then the inequality in (5-1) is strict and therefore for all $0 < t < T_{\max}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) > g_0 E(t)^{3/2}.$$

This differential inequality is sufficient to guarantee finite-time blowup for any solution with initial data such that $g_0 > 0$. We will now prove Theorem 5.3, which is restated here for the reader's convenience.

Theorem 5.3. *Suppose $S \in C([0, T_{\max}); H_{\mathrm{st}}^1)$ is a mild solution of the strain self-amplification model equation such that*

$$-3\|S^0\|_{\dot{H}^1}^2 - 4\int \det(S^0) > 0.$$

Then for all $0 < t < T_{\max}$,

$$E(t) > \frac{E_0}{(1-r_0 t)^2},$$

where

$$r_0 = \frac{-3\|S^0\|_{\dot{H}^1}^2 - 4\int \det(S^0)}{2\|S^0\|_{L^2}^2}.$$

Note in particular that this implies

$$T_{\max} \leq \frac{2\|S^0\|_{L^2}^2}{-3\|S^0\|_{L^2}^2 - 4\int \det(S^0)}.$$

Furthermore, for all $2/p + 3/q = 2$ *and* $3/2 < q \leq +\infty$,

$$\int_0^{T_{\max}} \|\lambda_2^+(t)\|_{L^q}^p \, \mathrm{d}t = +\infty.$$

Proof. The main argument of the proof will be integrating the differential inequality in Proposition 5.1. Applying Proposition 5.1, we can see that for all $0 < t < T_{\max}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) > g_0 E(t)^{3/2}, \tag{5-2}$$

where $g_0 > 0$ by hypothesis. Applying the chain rule and (5-2), for all $0 < t < T_{\max}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(-E(t)^{-1/2}) = \tfrac{1}{2} E(t)^{-3/2} \frac{\mathrm{d}}{\mathrm{d}t} E(t) > \tfrac{1}{2} g_0,$$

Integrating this differential inequality, we find that for all $0 < t < T_{\max}$,

$$E_0^{-1/2} - E(t)^{-1/2} > \tfrac{1}{2} g_0 t.$$

This implies that

$$E(t)^{-1/2} < E_0^{-1/2} - \tfrac{1}{2} g_0 t,$$

and therefore that

$$E(t) > \frac{1}{\left(E_0^{-1/2} - \frac{1}{2} g_0 t\right)^2}.$$

Multiplying the numerator and denominator by E_0, we find that for all $0 < t < T_{\max}$,

$$E(t) > \frac{E_0}{\left(1 - \frac{1}{2} g_0 E_0^{1/2} t\right)^2}.$$

It is easy to check that

$$r_0 = \tfrac{1}{2} g_0 E_0^{1/2},$$

so we have now established that for all $0 < t < T_{\max}$,

$$E(t) > \frac{E_0}{(1-r_0 t)^2}.$$

Furthermore, this clearly implies that

$$T_{\max} \leq \frac{1}{r_0} = \frac{2\|S^0\|_{L^2}^2}{-3\|S^0\|_{L^2}^2 - 4\int \det(S^0)}.$$

Finally, applying Theorem 4.8, we conclude that for all $2/p + 3/q = 2$ and $3/2 < q \leq +\infty$,

$$\int_0^{T_{\max}} \|\lambda_2^+(t)\|_{L^q}^p \, \mathrm{d}t = +\infty. \qquad \square$$

Next we will show that the set of initial data satisfying the hypothesis of Theorem 5.3 is nonempty and bounded below in $\dot{H}^{-1/2}$. We will also show that λ_2^+ is bounded below in $L^{3/2}$ for all S in this set. First we will need to perform a few calculations related to the determinant of the strain.

Proposition 5.4. *There exists $S \in H^1_{\mathrm{st}}$, axisymmetric and swirl-free, such that*

$$-\int_{\mathbb{R}^3} \det(S) > 0.$$

Note that we will say that S is axisymmetric and swirl free if $S = \nabla_{\mathrm{sym}} u$, where u is an axisymmetric, swirl-free, divergence-free vector field.

Proof. We begin by taking $u \in H^2_{\mathrm{df}}$ using axisymmetric coordinates, letting

$$u(x) = (r - 2rz^2)\exp(-r^2 - z^2)e_r + (-2z + 2r^2 z)\exp(-r^2 - z^2)e_z.$$

We will observe that

$$u(x) = \left((1 - 2x_3^2)\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} + (-2x_3 + 2(x_1^2 + x_2^2)x_3)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)\exp(-(x_1^2 + x_2^2 + x_3^2)),$$

and so not only do we have $u \in H^2$, but we have the stronger result that u must be in the Schwartz class of smooth functions, which have, along with all their derivatives, faster than polynomial decay at infinity. Taking the divergence of u we find that

$$\nabla \cdot u = \left(\partial_r + \frac{1}{r}\right)u_r + \partial_z u_z = ((2 - 4z^2 - 2r^2 + 4r^2 z^2) + (-2 + 2r^2 + 4z^2 - 4r^2 z^2))\exp(-r^2 - z^2) = 0,$$

as required, so that $u \in H^2_{\mathrm{df}}$. Taking the curl of u we find that

$$\begin{aligned}\omega = (\partial_z u_r - \partial_r u_z)e_\theta &= ((-4rz - 2rz + 4rz^3) - (4rz + 4rz - 4r^3 z))\exp(-r^2 - z^2)e_\theta \\ &= (-14rz + 4rz^3 + 4r^3 z)\exp(-r^2 - z^2)e_\theta.\end{aligned}$$

Next we will observe that the gradient can be represented in axisymmetric coordinates as

$$\nabla = \frac{1}{r}e_\theta \partial_\theta + e_r \partial_r + e_z \partial_z.$$

Using this representation and recalling that

$$e_r = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix},$$

we can see that

$$\partial_\theta e_r = e_\theta.$$

This means we can compute

$$\operatorname{tr}(S(e_\theta \otimes e_\theta)) = \operatorname{tr}(\nabla u(e_\theta \otimes e_\theta)) = \frac{u_r}{r} = (1-2z^2)\exp(-r^2-z^2).$$

Applying Proposition 4.4 we find that

$$\begin{aligned}
-\int \det(S) = \frac{1}{4}\langle S; \omega \otimes \omega\rangle &= \frac{1}{4}\int_{\mathbb{R}^3} \operatorname{tr}(S(e_\theta \otimes e_\theta))(x)|\omega(x)|^2 \,\mathrm{d}x \\
&= \frac{1}{4}\int_0^\infty\int_{-\infty}^\infty 2\pi r(1-2z^2)(-14rz+4rz^3+4r^3z)^2 \exp(-3r^2-3z^2)\,\mathrm{d}z\,\mathrm{d}r \\
&= \pi\int_0^\infty\int_0^\infty r(1-2z^2)(-14rz+4rz^3+4r^3z)^2 \exp(-3r^2-3z^2)\,\mathrm{d}z\,\mathrm{d}r \\
&= 4\pi\int_0^\infty\int_0^\infty r^3z^2(1-2z^2)(-7+2z^2+2r^2)^2 \exp(-3r^2-3z^2)\,\mathrm{d}z\,\mathrm{d}r,
\end{aligned}$$

using the fact that the integrand is even in z. Making the substitution $v = z^2$ and $w = r^2$, we find that

$$-\int \det(S) = \pi\int_0^\infty\int_0^\infty w\sqrt{v}(1-2v)(-7+2v+2w)^2 \exp(-3w-3v)\,\mathrm{d}v\,\mathrm{d}w = \frac{8\pi^{3/2}}{81\sqrt{3}}.$$

Therefore we can conclude that there exists $S \in H^1_{\mathrm{st}}$, axisymmetric and swirl-free, such that

$$-\int \det(S) > 0. \qquad \square$$

Theorem 5.5. *Let the set of initial data satisfying the hypotheses of Theorem 5.3, $\Gamma_{\mathrm{blowup}} \subset H^1_{\mathrm{st}}$, be given by*

$$\Gamma_{\mathrm{blowup}} = \left\{ S \in H^1_{\mathrm{st}} : -3\|S\|^2_{\dot{H}^1} - 4\int \det(S) > 0 \right\}.$$

Then Γ_{blowup} is nonempty.

Proof. Take any $S \in H^1_{\mathrm{st}}$ such that

$$-\int \det(S) > 0.$$

We know such an S must exist from Proposition 5.4. If we multiply such an $S \in H^1_{\mathrm{st}}$ by a sufficiently large constant we will end up with an element of Γ_{blowup}. In particular, we compute

$$\lim_{m\to+\infty}\left(-3\|mS\|^2_{\dot{H}^1} - 4\int \det(mS)\right) = \lim_{m\to+\infty}\left(-3m^2(\|S\|^2_{\dot{H}^1}) + 4m^3\left(-\int \det(S)\right)\right) = +\infty.$$

Therefore we may conclude that for all $S \in H^1_{\mathrm{st}}$ such that $-\int \det(S) > 0$ and for sufficiently large $m > 0$, we have $mS \in \Gamma_{\mathrm{blowup}}$. $\square$

Remark 5.6. Note that near the origin, the velocity corresponding to finite-time blowup for the strain self-amplification model equation from Proposition 5.4 and Theorem 5.5 has a very similar geometric structure to the $C^{1,\alpha}$ finite-time blowup solution to the Euler equation from [Elgindi 2021; Elgindi et al. 2021].

Both involve planar stretching and axial compression near the origin. In particular, approximating the velocity in Proposition 5.4 near the origin by the first order Taylor polynomial, we have

$$u(x) \approx re_r - 2ze_z.$$

Remark 5.7. The fact the Γ_{blowup} is nonempty means that the condition in Theorem 5.3 is satisfied for some initial data, and so we can conclude that there must exist solutions of the strain self-amplification model equation that blowup in finite time. In addition to knowing the Γ_{blowup} is nonempty, we also know that Γ_{blowup} is bounded below in $\dot{H}^{-1/2}$ because Theorem 4.6 states that there is global regularity for solutions of the strain self-amplification model equation with small initial data in $\dot{H}^{-1/2}$, and Theorem 5.3 requires that all of the solutions with initial data in Γ_{blowup} must blowup in finite time. This can also be shown directly by computation using the relevant Sobolev embeddings along with Hölder's inequality. In addition, we have a lower bound on the amount of planar stretching for $S \in \Gamma_{\text{blowup}}$ in the form of a lower bound on λ_2^+ in the scale critical Lebesgue space.

Proposition 5.8. *For all $S \in \Gamma_{\text{blowup}}$,*

$$\|\lambda_2^+\|_{L^{3/2}} > \frac{9}{2}\left(\frac{\pi}{2}\right)^{4/3}.$$

Proof. We will prove the contrapositive. Suppose $S \in H^1_{\text{st}}$ with

$$\|\lambda_2^+\|_{L^{3/2}} \leq \frac{9}{2}\left(\frac{\pi}{2}\right)^{4/3}.$$

We will begin by observing that because $\operatorname{tr}(S) = 0$, we have $\lambda_1 \leq 0$ and $\lambda_3 \geq 0$ because three positive (respectively negative) eigenvalues would violate the trace-free condition. This implies that $-\lambda_1\lambda_3 \geq 0$. Therefore, we can compute

$$-\det(S) = (-\lambda_1\lambda_3)\lambda_2 \leq (-\lambda_1\lambda_3)\lambda_2^+ \leq \tfrac{1}{2}(\lambda_1^2 + \lambda_2^2)\lambda_2^+ \leq \tfrac{1}{2}\lambda_2^+|S|^2.$$

Applying this estimate, Hölder's inequality, and the Sobolev inequality, we find that

$$\begin{aligned} -3\|S\|_{\dot{H}^1}^2 - 4\int \det(S) \leq -3\|S\|_{\dot{H}^1}^2 + 2\int \lambda_2^+|S|^2 \leq -3\|S\|_{\dot{H}^1}^2 + 2\|\lambda_2^+\|_{L^{3/2}}\|S\|_{L^6}^2 \\ \leq -3\|S\|_{\dot{H}^1}^2 + \frac{2}{3}\left(\frac{2}{\pi}\right)^{4/3}\|\lambda_2^+\|_{L^{3/2}}\|S\|_{\dot{H}^1}^2 \leq 0. \end{aligned}$$

Therefore we can see that $S \notin \Gamma_{\text{blowup}}$, and this completes the proof. □

6. A perturbative blowup condition for the full Navier–Stokes equation

In this section, we will prove a perturbative condition for blowup, and we will also show that this perturbative condition is satisfied at least for short times. We begin by proving Theorem 6.1, which is restated for the reader's convenience.

Theorem 6.1. *Suppose $u \in C([0, T_{\max}); H^2_{\text{df}})$ is a mild solution of the Navier–Stokes equation such that*

$$f_0 := -3\|S^0\|_{\dot{H}^1}^2 - 4\int \det(S^0) > 0,$$

and for all $0 < t < T_{\max}$,

$$\frac{\left\| P_{\mathrm{st}}\big((u\cdot\nabla)S+\frac{1}{3}S^2+\frac{1}{4}\omega\otimes\omega\big)(\,\cdot\,,t)\right\|_{L^2}}{\left\|\big(-\Delta S+P_{\mathrm{st}}\big(\frac{1}{2}(u\cdot\nabla)S+\frac{5}{6}S^2+\frac{1}{8}\omega\otimes\omega\big)\big)(\,\cdot\,,t)\right\|_{L^2}} \leq 2. \tag{6-1}$$

Then there is finite-time blowup with

$$T_{\max} < T_* := \frac{-E_0+\sqrt{E_0^2+f_0K_0}}{f_0},$$

where K_0 *and* E_0 *are taken as in Definition 1.7 and* f_0 *is as defined above.*

Proof. We will begin by letting

$$f(t) = -3\|S(\,\cdot\,,t)\|_{\dot{H}^1}^2 - 4\int \det(S)(\,\cdot\,,t).$$

We know that $\operatorname{tr}(S)=0$ and that therefore $\det(S)=\frac{1}{3}\int\operatorname{tr}(S^3)$. Therefore we can see that

$$f(t) = -3\|S\|_{\dot{H}^1}^2 - \frac{4}{3}\int\operatorname{tr}(S^3).$$

Differentiating f, we find that for all $0<t<T_{\max}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) = 6\big\langle \Delta S+\tfrac{2}{3}P_{\mathrm{st}}(S^2),\, -\Delta S+P_{\mathrm{st}}\big((u\cdot\nabla)S+S^2+\tfrac{1}{4}\omega\otimes\omega\big)\big\rangle.$$

Observe that for any $M,\, Q\in L^2$,

$$\langle M, M+Q\rangle = \big\|M+\tfrac{1}{2}Q\big\|_{L^2}^2-\tfrac{1}{4}\|Q\|_{L^2}^2,$$

and so letting $M=-\Delta S+\frac{2}{3}P_{\mathrm{st}}(S^2)$ and $Q=P_{\mathrm{st}}\big((u\cdot\nabla)S+\frac{1}{3}S^2+\frac{1}{4}\omega\otimes\omega\big)$, we find that for all $0<t<T_{\max}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) = 6\big\|-\Delta S+P_{\mathrm{st}}\big(\tfrac{1}{2}(u\cdot\nabla)S+\tfrac{5}{6}S^2+\tfrac{1}{4}\omega\otimes\omega\big)\big\|_{L^2}^2-\tfrac{3}{2}\big\|P_{\mathrm{st}}\big((u\cdot\nabla)S+\tfrac{1}{3}S^2+\tfrac{1}{4}\omega\otimes\omega\big)\big\|_{L^2}^2.$$

Applying the perturbative condition (6-1), we find that for all $0<t<T_{\max}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t)\geq 0,$$

and therefore for all $0<t<T_{\max}$,

$$f(t)\geq f_0.$$

Using the identity for enstrophy growth we find that for all $0<t<T_{\max}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -2\|S\|_{\dot{H}^1}^2-4\int\det(S) > -3\|S\|_{\dot{H}^1}^2-4\int\det(S) = f(t)\geq f_0.$$

Integrating this differential inequality we find that for all $0<t<T_{\max}$,

$$E(t) > E_0+f_0t,$$

and integrating this lower bound for enstrophy growth, we find that for all $0<t<T_{\max}$,

$$\int_0^t E(\tau)\,\mathrm{d}\tau > E_0t+\tfrac{1}{2}f_0t^2.$$

Now suppose towards a contradiction that $T_{\max} \geq T_*$. Using the definition

$$T_* = \frac{-E_0 + \sqrt{E_0^2 + f_0 K_0}}{f_0},$$

we find that

$$\int_0^{T_*} E(t)\,\mathrm{d}t > E_0 T_* + \tfrac{1}{2} f_0 T_*^2 = \tfrac{1}{2} K_0.$$

However this contradicts the bound from the energy equality, which requires that

$$\int_0^{T_*} E(t)\,\mathrm{d}t \leq \tfrac{1}{2} K_0.$$

Therefore we may conclude that $T_{\max} < T_*$, and this complete the proof. □

We cannot show that the perturbative condition (6-1) is satisfied up until T_* — if we could this would resolve the Navier–Stokes regularity problem by proving the existence of finite-time blowup. We can, however, show that it is satisfied for short times. The first step will be to show that it holds at the level of initial data.

Proposition 6.2. *There exists $S \in H^2_{\mathrm{st}} \cap \dot{H}^{-1}_{\mathrm{st}}$ such that*

$$-3\|S\|^2_{\dot{H}^1} - 4\int \det(S) > 0 \tag{6-2}$$

and

$$\frac{\left\| P_{\mathrm{st}}\big((u\cdot\nabla)S + \frac{1}{3}S^2 + \frac{1}{4}\omega\otimes\omega\big)\right\|_{L^2}}{\left\| -\Delta S + P_{\mathrm{st}}\big(\frac{1}{2}(u\cdot\nabla)S + \frac{5}{6}S^2 + \frac{1}{8}\omega\otimes\omega\big)\right\|_{L^2}} < 2. \tag{6-3}$$

Proof. Begin by taking $M \in H^2_{\mathrm{st}} \cap \dot{H}^{-1}_{\mathrm{st}}$ such that

$$-3\|M\|^2_{\dot{H}^1} - 4\int \det(M) > 0$$

and $Q \in H^2_{\mathrm{st}} \cap \dot{H}^{-1}_{\mathrm{st}}$ not identically zero. For all $\lambda > 0$, let

$$Q^\lambda(x) = Q(\lambda x) \quad \text{and} \quad S^\lambda = M + Q^\lambda.$$

It is a simple computation to observe that

$$\|Q^\lambda\|_{\dot{H}^1} = \lambda^{-1/2}\|Q\|_{\dot{H}^1} \quad \text{and} \quad \|Q^\lambda\|_{L^3} = \lambda^{-1}\|Q\|_{L^3}.$$

Therefore we can see that

$$\lim_{\lambda\to+\infty} Q^\lambda = 0$$

in both $\dot{H}^1$ and L^3. This implies that

$$\lim_{\lambda\to+\infty} -3\|S^\lambda\|^2_{\dot{H}^1} - 4\int \det(S^\lambda) = -3\|M\|^2_{\dot{H}^1} - 4\int \det(M) > 0,$$

so S^λ satisfies (6-2) for sufficiently large $\lambda > 0$.

Now take

$$v = -2\,\mathrm{div}(-\Delta)^{-1}M \quad \text{and} \quad w = -2\,\mathrm{div}(-\Delta)^{-1}Q.$$

Note that we then have

$$M = \nabla_{\mathrm{sym}} v \quad \text{and} \quad Q = \nabla_{\mathrm{sym}} w.$$

Likewise we will take

$$a = \nabla \times v \quad \text{and} \quad b = \nabla \times w.$$

Finally we will let

$$w^\lambda(x) = \lambda^{-1} w(\lambda x) \quad \text{and} \quad b^\lambda(x) = b(\lambda x),$$

noting that this implies

$$Q^\lambda = \nabla_{\mathrm{sym}} w^\lambda \quad \text{and} \quad b^\lambda = \nabla \times w^\lambda.$$

Going back to our linear combination, we can see that

$$u^\lambda = v + w^\lambda \quad \text{and} \quad \omega^\lambda = a + b^\lambda.$$

Applying the triangle inequality we can see that

$$\begin{aligned}
&\left\| P_{\mathrm{st}}\big((u^\lambda \cdot \nabla) S^\lambda + \tfrac{1}{3}(S^\lambda)^2 + \tfrac{1}{4}\omega^\lambda \otimes \omega^\lambda\big)\right\|_{L^2} \\
&\qquad \le \left\| P_{\mathrm{st}}\big((v\cdot\nabla)M + \tfrac{1}{3}M^2 + \tfrac{1}{4}a\otimes a\big)\right\|_{L^2} + \left\| P_{\mathrm{st}}\big((w^\lambda\cdot\nabla)Q^\lambda + \tfrac{1}{3}(Q^\lambda)^2 + \tfrac{1}{4}b^\lambda\otimes b^\lambda\big)\right\|_{L^2} \\
&\qquad\qquad + \left\| P_{\mathrm{st}}\big((w^\lambda\cdot\nabla)M + (v\cdot\nabla)Q^\lambda + \tfrac{1}{3}(Q^\lambda M + MQ^\lambda) + \tfrac{1}{4}(b^\lambda\otimes a + a\otimes b^\lambda)\big)\right\|_{L^2},
\end{aligned}$$

and applying Hölder's inequality and our scaling laws from above, we can conclude

$$\begin{aligned}
&\left\| P_{\mathrm{st}}\big((u^\lambda \cdot \nabla) S^\lambda + \tfrac{1}{3}(S^\lambda)^2 + \tfrac{1}{4}\omega^\lambda \otimes \omega^\lambda\big)\right\|_{L^2} \\
&\le \left\| P_{\mathrm{st}}\big((v\cdot\nabla)M + \tfrac{1}{3}M^2 + \tfrac{1}{4}a\otimes a\big)\right\|_{L^2} + \lambda^{-3/2}\left\| P_{\mathrm{st}}\big((w\cdot\nabla)Q + \tfrac{1}{3}Q^2 + \tfrac{1}{4}b\otimes b\big)\right\|_{L^2} + \lambda^{-1}\|w\|_{L^\infty}\|\nabla M\|_{L^2} \\
&\qquad + \lambda^{-1/2}\|v\|_{L^\infty}\|\nabla Q\|_{L^2} + \tfrac{2}{3}\lambda^{-3/4}\|M\|_{L^4}\|Q\|_{L^4} + \tfrac{1}{2}\lambda^{-3/4}\|a\|_{L^4}\|b\|_{L^4}.
\end{aligned} \tag{6-4}$$

Likewise we may compute that

$$\begin{aligned}
&\left\| -\Delta S^\lambda + P_{\mathrm{st}}\big(\tfrac{1}{2}(u^\lambda \cdot \nabla) S^\lambda + \tfrac{5}{6}(S^\lambda)^2 + \tfrac{1}{8}\omega^\lambda \otimes \omega^\lambda\big)\right\|_{L^2} \\
&\qquad \ge \lambda^{1/2}\|-\Delta Q\|_{L^2} - \lambda^{-3/2}\left\| P_{\mathrm{st}}\big(\tfrac{1}{2}(w\cdot\nabla)Q + \tfrac{5}{6}Q^2 + \tfrac{1}{8}b\otimes b\big)\right\|_{L^2} \\
&\qquad\quad - \left\| -\Delta M + P_{\mathrm{st}}\big(\tfrac{1}{2}(v\cdot\nabla)M + \tfrac{5}{6}M^2 + \tfrac{1}{8}a\otimes a\big)\right\|_{L^2} - \tfrac{5}{3}\lambda^{-3/4}\|M\|_{L^4}\|Q\|_{L^4} \\
&\qquad\quad - \tfrac{1}{4}\lambda^{-3/4}\|a\|_{L^4}\|b\|_{L^4} - \tfrac{1}{2}\lambda^{-1/2}\|v\|_{L^\infty}\|\nabla Q\|_{L^2} - \tfrac{1}{2}\lambda^{-1}\|w\|_{L^\infty}\|\nabla M\|_{L^2}.
\end{aligned} \tag{6-5}$$

Putting together the inequalities in (6-4) and (6-5), we find that

$$\lim_{\lambda\to+\infty} \frac{\left\| P_{\mathrm{st}}\big((u^\lambda \cdot \nabla) S^\lambda + \tfrac{1}{3}(S^\lambda)^2 + \tfrac{1}{4}\omega^\lambda \otimes \omega^\lambda\big)\right\|_{L^2}}{\left\| -\Delta S^\lambda + P_{\mathrm{st}}\big(\tfrac{1}{2}(u^\lambda \cdot \nabla) S^\lambda + \tfrac{5}{6}(S^\lambda)^2 + \tfrac{1}{8}\omega^\lambda \otimes \omega^\lambda\big)\right\|_{L^2}} = 0,$$

and so in particular for sufficiently large $\lambda > 0$,

$$\frac{\left\| P_{\mathrm{st}}\big((u^\lambda \cdot \nabla) S^\lambda + \tfrac{1}{3}(S^\lambda)^2 + \tfrac{1}{4}\omega^\lambda \otimes \omega^\lambda\big)\right\|_{L^2}}{\left\| -\Delta S^\lambda + P_{\mathrm{st}}\big(\tfrac{1}{2}(u^\lambda \cdot \nabla) S^\lambda + \tfrac{5}{6}(S^\lambda)^2 + \tfrac{1}{8}\omega^\lambda \otimes \omega^\lambda\big)\right\|_{L^2}} < 2. \qquad \square$$

Now that we have established that the perturbative condition (6-1) can hold for initial data, it is straightforward to show that it can hold for at least short times by continuity. This result is Theorem 6.3, which is restated for the reader's convenience.

Theorem 6.3. *There exists a mild solution of the Navier–Stokes equation $u \in C([0, T_{\max}); H^3_{\mathrm{df}})$ and $\epsilon > 0$ such that*

$$-3\|S^0\|^2_{\dot{H}^1} - 4\int \det(S^0) > 0,$$

and for all $0 \le t < \epsilon$,

$$\frac{\big\|P_{\mathrm{st}}\big((u\cdot\nabla)S + \frac{1}{3}S^2 + \frac{1}{4}\omega\otimes\omega\big)(\cdot\,,t)\big\|_{L^2}}{\big\|\big(-\Delta S + P_{\mathrm{st}}\big(\frac{1}{2}(u\cdot\nabla)S + \frac{5}{6}S^2 + \frac{1}{8}\omega\otimes\omega\big)\big)(\cdot\,,t)\big\|_{L^2}} < 2.$$

Proof. Fix initial data $S^0 \in H^2_{\mathrm{st}} \cap \dot{H}^{-1}_{\mathrm{st}}$ such that

$$-3\|S^0\|^2_{\dot{H}^1} - 4\int \det(S^0) > 0$$

and

$$\frac{\big\|P_{\mathrm{st}}\big((u^0\cdot\nabla)S^0 + \frac{1}{3}(S^0)^2 + \frac{1}{4}\omega^0\otimes\omega^0\big)\big\|_{L^2}}{\big\|-\Delta S^0 + P_{\mathrm{st}}\big(\frac{1}{2}(u^0\cdot\nabla)S^0 + \frac{5}{6}(S^0)^2 + \frac{1}{8}\omega^0\otimes\omega^0\big)\big\|_{L^2}} < 2,$$

where $S^0 = \nabla_{\mathrm{sym}} u^0$ and $\omega^0 = \nabla\times u^0$. Note that $u^0 \in H^3_{\mathrm{df}}$ by definition and is given by $u^0 = -2\,\mathrm{div}(-\Delta)^{-1}S^0$. Let $u \in C([0, T_{\max}); H^3_{\mathrm{df}})$ be the unique mild solution of the Navier–Stokes equation with initial data u^0. Next we will let

$$c(t) = \big\|P_{\mathrm{st}}\big((u\cdot\nabla)S + \tfrac{1}{3}S^2 + \tfrac{1}{4}\omega\otimes\omega\big)(\cdot\,,t)\big\|_{L^2} - 2\big\|\big(-\Delta S + P_{\mathrm{st}}\big(\tfrac{1}{2}(u\cdot\nabla)S + \tfrac{5}{6}S^2 + \tfrac{1}{8}\omega\otimes\omega\big)\big)(\cdot\,,t)\big\|_{L^2}.$$

The fact that $u \in C([0, T_{\max}); H^3_{\mathrm{df}})$ immediately implies that $c \in C([0, T_{\max}))$. We also know by hypothesis that

$$c(0) < 0,$$

so by continuity there must exist $\epsilon > 0$ such that for all $0 \le t < \epsilon$,

$$c(t) < 0.$$

This completes the proof. □

Remark 6.4. The key to the proof of Proposition 6.2 and Theorem 6.3 rests on the fact that we can add a perturbative term which is small in both $\dot{H}^1$ and L^3, leaving (6-2) essentially unaffected, but which is very large in $\dot{H}^2$, making the denominator in (6-3) as large as we like. The key is to add a perturbative term that is supported at very high Fourier modes, but with a scaling chosen so that the perturbation remains small in $\dot{H}^1 \cap L^3$.

Remark 6.5. There are axisymmetric, swirl-free initial data that satisfy Theorem 6.3. To see this, in the context of Proposition 6.2, take M as in Proposition 5.4 and Q to be an arbitrary axisymmetric, swirl-free strain matrix and the result follows. In this case, however, we know that the perturbative condition can only be satisfied for short times because there is global regularity for axisymmetric, swirl-free solutions of the Navier–Stokes equation. There is something in the geometry of axisymmetric swirl-free solutions that, when coupled with the dynamics of the equation, guarantees the perturbative condition will fail after short times.

Corollary 6.6. *Suppose $u \in C([0,+\infty); H^2_{\mathrm{df}})$ is an axisymmetric, swirl-free, mild solution of the Navier–Stokes equation such that*

$$f_0 := -3\|S^0\|^2_{\dot{H}^1} - 4\int \det(S^0) > 0.$$

Then there exists $0 < t < T^* := (-E_0 + \sqrt{E_0^2 + f_0 K_0})/f_0$ such that

$$\frac{\left\| P_{\mathrm{st}}\big((u\cdot\nabla)S + \frac{1}{3}S^2 + \frac{1}{4}\omega\otimes\omega\big)(\cdot,t)\right\|_{L^2}}{\left\|\big(-\Delta S + P_{\mathrm{st}}\big(\frac{1}{2}(u\cdot\nabla)S + \frac{5}{6}S^2 + \frac{1}{8}\omega\otimes\omega\big)\big)(\cdot,t)\right\|_{L^2}} > 2.$$

Proof. Ladyzhenskaya [1968b; 1968a] first proved global regularity for solutions of the Navier–Stokes equation with swirl-free, axisymmetric initial data. The corollary follows immediately from Theorem 6.1 and the global regularity of axisymmetric, swirl-free solutions of the Navier–Stokes equation because if the perturbative condition from Theorem 6.1 was satisfied up until T^*, then there must be finite-time blowup, which we know cannot occur. □

Acknowledgements

This publication was supported in part by the Fields Institute for Research in the Mathematical Sciences while the author was in residence during the Fall 2020 semester. Its contents are solely the responsibility of the author and do not necessarily represent the official views of the Fields Institute. This material is based upon work supported by the National Science Foundation under grant no. DMS-1440140 while the author participated in a program that was hosted by the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2021 semester. The material in this publication is also based on research conducted by the author while a PhD student at the University of Toronto and a postdoctoral fellow at McMaster University.

References

[Beale et al. 1984] J. T. Beale, T. Kato, and A. Majda, "Remarks on the breakdown of smooth solutions for the 3-D Euler equations", *Comm. Math. Phys.* **94**:1 (1984), 61–66. MR Zbl

[Carbone and Bragg 2020] M. Carbone and A. D. Bragg, "Is vortex stretching the main cause of the turbulent energy cascade?", *J. Fluid Mech.* **883** (2020), art. id. R2. MR Zbl

[Chae 2006] D. Chae, "On the spectral dynamics of the deformation tensor and new a priori estimates for the 3D Euler equations", *Comm. Math. Phys.* **263**:3 (2006), 789–801. MR Zbl

[Chae and Choe 1999] D. Chae and H.-J. Choe, "Regularity of solutions to the Navier–Stokes equation", *Electron. J. Differential Equations* **1999** (1999), art. id. 05. MR Zbl

[Chemin and Zhang 2016] J.-Y. Chemin and P. Zhang, "On the critical one component regularity for 3-D Navier–Stokes systems", *Ann. Sci. École Norm. Sup.* (4) **49**:1 (2016), 131–167. MR Zbl

[Chemin et al. 2017] J.-Y. Chemin, P. Zhang, and Z. Zhang, "On the critical one component regularity for 3-D Navier–Stokes system: general case", *Arch. Ration. Mech. Anal.* **224**:3 (2017), 871–905. MR Zbl

[Constantin and Fefferman 1993] P. Constantin and C. Fefferman, "Direction of vorticity and the problem of global regularity for the Navier–Stokes equations", *Indiana Univ. Math. J.* **42**:3 (1993), 775–789. MR Zbl

[Constantin et al. 1985] P. Constantin, P. D. Lax, and A. Majda, "A simple one-dimensional model for the three-dimensional vorticity equation", *Comm. Pure Appl. Math.* **38**:6 (1985), 715–724. MR Zbl

[Córdoba et al. 2005] A. Córdoba, D. Córdoba, and M. A. Fontelos, "Formation of singularities for a transport equation with nonlocal velocity", *Ann. of Math.* (2) **162**:3 (2005), 1377–1389. MR Zbl

[De Gregorio 1990] S. De Gregorio, "On a one-dimensional model for the three-dimensional vorticity equation", *J. Statist. Phys.* **59**:5-6 (1990), 1251–1263. MR Zbl

[Elgindi 2021] T. Elgindi, "Finite-time singularity formation for $C^{1,\alpha}$ solutions to the incompressible Euler equations on $\mathbb{R}^3$", *Ann. of Math.* (2) **194**:3 (2021), 647–727. MR Zbl

[Elgindi and Jeong 2020] T. M. Elgindi and I.-J. Jeong, "On the effects of advection and vortex stretching", *Arch. Ration. Mech. Anal.* **235**:3 (2020), 1763–1817. MR Zbl

[Elgindi et al. 2021] T. M. Elgindi, T.-E. Ghoul, and N. Masmoudi, "On the stability of self-similar blow-up for $C^{1,\alpha}$ solutions to the incompressible Euler equations on $\mathbb{R}^3$", *Camb. J. Math.* **9**:4 (2021), 1035–1075. MR Zbl

[Escauriaza et al. 2003] L. Escauriaza, G. A. Seregin, and V. Šverák, "$L_{3,\infty}$-solutions of Navier–Stokes equations and backward uniqueness", *Uspekhi Mat. Nauk* **58**:2(350) (2003), 3–44. In Russian; translated in *Russian Math. Surv.* **58**:2 (2003), 211–250. MR Zbl

[Fefferman 2006] C. L. Fefferman, "Existence and smoothness of the Navier–Stokes equation", pp. 57–67 in *The millennium prize problems*, edited by J. Carlson et al., Clay Math. Inst., Cambridge, MA, 2006. MR Zbl

[Fujita and Kato 1964] H. Fujita and T. Kato, "On the Navier–Stokes initial value problem, I", *Arch. Ration. Mech. Anal.* **16** (1964), 269–315. MR Zbl

[Gallagher and Paicu 2009] I. Gallagher and M. Paicu, "Remarks on the blow-up of solutions to a toy model for the Navier–Stokes equations", *Proc. Amer. Math. Soc.* **137**:6 (2009), 2075–2083. MR Zbl

[Hou and Lei 2009] T. Y. Hou and Z. Lei, "On the stabilizing effect of convection in three-dimensional incompressible flows", *Comm. Pure Appl. Math.* **62**:4 (2009), 501–564. MR Zbl

[Hou et al. 2018] T. Y. Hou, P. Liu, and F. Wang, "Global regularity for a family of 3D models of the axi-symmetric Navier–Stokes equations", *Nonlinearity* **31**:5 (2018), 1940–1954. MR Zbl

[Jia et al. 2019] H. Jia, S. Stewart, and V. Sverak, "On the De Gregorio modification of the Constantin–Lax–Majda model", *Arch. Ration. Mech. Anal.* **231**:2 (2019), 1269–1304. MR Zbl

[Kukavica and Ziane 2007] I. Kukavica and M. Ziane, "Navier–Stokes equations with regularity in one direction", *J. Math. Phys.* **48**:6 (2007), art. id. 065203. MR Zbl

[Ladyzhenskaya 1967] O. A. Ladyzhenskaya, "Uniqueness and smoothness of generalized solutions of Navier–Stokes equations", *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*) **5** (1967), 169–185. In Russian; translated in *Sem. Math. Steklov Math. Inst., Leningrad* **5** (1969), 60–67. MR Zbl

[Ladyzhenskaya 1968a] O. A. Ladyzhenskaya, "Certain nonlinear problems of the theory of continuous media", pp. 560–573 in *Proceedings of the International Congress of Mathematicians* (Moscow, 1966), edited by I. G. Petrovskiĭ, Izdat. "Mir", Moscow, 1968. In Russian. MR Zbl

[Ladyzhenskaya 1968b] O. A. Ladyzhenskaya, "Unique global solvability of the three-dimensional Cauchy problem for the Navier–Stokes equations in the presence of axial symmetry", *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*) **7** (1968), 155–177. In Russian. MR Zbl

[Lemarié-Rieusset 2016] P. G. Lemarié-Rieusset, *The Navier–Stokes problem in the 21st century*, CRC Press, Boca Raton, FL, 2016. MR Zbl

[Leray 1934] J. Leray, "Sur le mouvement d'un liquide visqueux emplissant l'espace", *Acta Math.* **63**:1 (1934), 193–248. MR Zbl

[Lieb 1983] E. H. Lieb, "Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities", *Ann. of Math.* (2) **118**:2 (1983), 349–374. MR Zbl

[Lieb and Loss 1997] E. H. Lieb and M. Loss, *Analysis*, Grad. Stud. in Math. **14**, Amer. Math. Soc., Providence, RI, 1997. MR Zbl

[Miller 2020] E. Miller, "A regularity criterion for the Navier–Stokes equation involving only the middle eigenvalue of the strain tensor", *Arch. Ration. Mech. Anal.* **235**:1 (2020), 99–139. MR Zbl

[Miller 2021] E. Miller, "A locally anisotropic regularity criterion for the Navier–Stokes equation in terms of vorticity", *Proc. Amer. Math. Soc. Ser. B* **8** (2021), 60–74. MR Zbl

[Montgomery-Smith 2001] S. Montgomery-Smith, "Finite time blow up for a Navier–Stokes like equation", *Proc. Amer. Math. Soc.* **129**:10 (2001), 3025–3029. MR Zbl

[Neustupa and Penel 2001] J. Neustupa and P. Penel, "Anisotropic and geometric criteria for interior regularity of weak solutions to the 3D Navier–Stokes equations", pp. 237–265 in *Mathematical fluid mechanics*: *recent results and open questions*, edited by J. Neustupa and P. Penel, Birkhäuser, Basel, 2001. MR Zbl

[Neustupa and Penel 2005] J. Neustupa and P. Penel, "Regularity of a weak solution to the Navier–Stokes equation in dependence on eigenvalues and eigenvectors of the rate of deformation tensor", pp. 197–212 in *Trends in partial differential equations of mathematical physics* (Óbidos, Portugal, 2003), edited by J. F. Rodrigues et al., Progr. Nonlinear Differential Equations Appl. **61**, Birkhäuser, Basel, 2005. MR Zbl

[Neustupa and Penel 2018] J. Neustupa and P. Penel, "On regularity of a weak solution to the Navier–Stokes equations with the generalized Navier slip boundary conditions", *Adv. Math. Phys.* **2018** (2018), art. id. 4617020. MR Zbl

[Prodi 1959] G. Prodi, "Un teorema di unicità per le equazioni di Navier–Stokes", *Ann. Mat. Pura Appl.* (4) **48** (1959), 173–182. MR Zbl

[Serrin 1962] J. Serrin, "On the interior regularity of weak solutions of the Navier–Stokes equations", *Arch. Ration. Mech. Anal.* **9** (1962), 187–195. MR Zbl

[Sobolev 1963] S. L. Sobolev, *Applications of functional analysis in mathematical physics*, Transl. Math. Monogr. **7**, Amer. Math. Soc., Providence, RI, 1963. MR Zbl

[Talenti 1976] G. Talenti, "Best constant in Sobolev inequality", *Ann. Mat. Pura Appl.* (4) **110** (1976), 353–372. MR Zbl

[Tao 2016] T. Tao, "Finite time blowup for an averaged three-dimensional Navier–Stokes equation", *J. Amer. Math. Soc.* **29**:3 (2016), 601–674. MR Zbl

[Tao 2021] T. Tao, "Quantitative bounds for critically bounded solutions to the Navier–Stokes equations", pp. 149–193 in *Nine mathematical challenges*: *an elucidation* (Pasadena, CA, 2019), edited by A. Kechris et al., Proc. Sympos. Pure Math. **104**, Amer. Math. Soc., Providence, RI, 2021. MR Zbl

[Tran and Yu 2015] C. V. Tran and X. Yu, "Depletion of nonlinearity in the pressure force driving Navier–Stokes flows", *Nonlinearity* **28**:5 (2015), 1295–1306. MR Zbl

[Tsinober 2001] A. Tsinober, *An informal introduction to turbulence*, Fluid Mech. Appl. **63**, Kluwer, Dordrecht, Germany, 2001. MR Zbl

Received 31 Jan 2021. Revised 21 Jul 2021. Accepted 16 Sep 2021.

EVAN MILLER: emiller@msri.org
Department of Mathematics, University of British Columbia, Vancouver, Canada

ANALYSIS AND PDE
Vol. 16 (2023), No. 4, pp. 1033–1060
DOI: 10.2140/apde.2023.16.1033

EIGENVALUE BOUNDS FOR SCHRÖDINGER OPERATORS WITH RANDOM COMPLEX POTENTIALS

OLEG SAFRONOV

We consider the Schrödinger operator perturbed by a random complex-valued potential. For this operator, we consider its eigenvalues situated in the unit disk. We obtain an estimate on the rate of accumulation of these eigenvalues to the positive half-line.

1. Introduction and main results

We study the behavior of eigenvalues of the operator $H = -\Delta + V$ acting on a Hilbert space $L^2(\mathbb{R}^d)$, where $d \geq 3$. The potential V is assumed to be a complex-valued function of the form

$$V(x) = \sum_{n\in\mathbb{Z}^d} \omega_n v_n \chi(x-n), \qquad v_n \in \mathbb{C}, \quad x \in \mathbb{R}^d,$$

where the ω_n are independent random variables taking values in the interval $[-1, 1]$ and χ is the characteristic function of the unit cube $[0, 1)^d$.

The probability space in our theorems is the set Σ of all infinite sequences $\omega = \{\omega_n\}_{n\in\mathbb{Z}^d}$. The probability measure is defined on Σ as the infinite product of corresponding measures on intervals $[-1, 1]$. Since ω_n can be viewed as a function on Σ whose value is equal to the n-th coordinate of ω, its expectation $\mathbb{E}[\omega_n]$ can be viewed as an integral over Σ. We impose the condition

$$\mathbb{E}[\omega_n] = 0$$

on ω_n guaranteeing oscillations of V. The coefficients v_n do not have to be real.

To formulate the main result, we set

$$\widetilde{V}(x) = \sum_{n\in\mathbb{Z}^d} |v_n|\chi(x-n).$$

Note that $\widetilde{V}$ is a nonnegative function such that $|V| \leq \widetilde{V}$.

Theorem 1.1. *Let $d \geq 3$, let $R_0 > 0$ and let $1 < \nu < q < 2$. Then the eigenvalues λ_j of the operator $-\Delta + V$ satisfy*

$$\mathbb{E}\Biggl[\sum_{|\lambda_j|<R_0^2} \operatorname{Im}\sqrt{\lambda_j}\,|\lambda_j|^{(q-1)/2}\Biggr] \leq C|R_0|^{q-\nu}\biggl(\int_{\mathbb{R}^d} |\widetilde{V}(x)|^p\,dx\biggr)^2, \tag{1.1}$$

MSC2020: 35P15, 35Q40, 47A75, 47B80, 47F05.
Keywords: Schrödinger operators, complex potentials, eigenvalue bounds.

with

$$p = \frac{d}{2} + \frac{d-q}{2(d-2)}. \tag{1.2}$$

It is assumed that $\operatorname{Im}\sqrt{\lambda_j} \geq 0$. *The constant* C *in* (1.1) *depends only on* d, ν *and* q.

Theorem 1.1 is a particular case of the following statement, which has rather complicated looking conditions imposed on the parameters.

Theorem 1.2. *Let* $d \geq 3$, *and let* $R_0 > 0$. *Assume that the parameters* $\varkappa$ *and* p *obey the conditions*

$$\frac{\varkappa}{2p} + \frac{d-1}{2} < \varkappa < \frac{d+1}{2},$$

and

$$\max\{2, \varkappa\} \leq p < \min\left\{2\varkappa, \frac{d\varkappa}{2\varkappa - 1}\right\}.$$

Assume also that $\widetilde{V} \in L^p(\mathbb{R}^d)$. *Then the eigenvalues* λ_j *of the operator* $-\Delta + V$ *satisfy*

$$\mathbb{E}\Bigg[\sum_{|\lambda_j| \leq R_0^2} \operatorname{Im}\sqrt{\lambda_j}|\lambda_j|^{(q-1)/2}\Bigg] \leq C|R_0|^{q-2p-p(d-1)/\varkappa}\left(\int_{\mathbb{R}^d} |\widetilde{V}(x)|^p\,dx\right)^2, \quad q > 2p - \frac{p(d-1)}{\varkappa}. \tag{1.3}$$

It is assumed that $\operatorname{Im}\sqrt{\lambda_j} \geq 0$. *If* $\varkappa = \frac{1}{2}(d+1)$, *then* (1.3) *holds with* $p = \varkappa$. *The constant* C *in* (1.3) *depends only on* d, p, $\varkappa$ *and* q.

It is known that, if $v_n \in \mathbb{R}$, the eigenvalues λ_j obey the Lieb–Thirring estimate (see [Helffer and Robert 1990; Laptev and Weidl 2000; Lieb and Thirring 1976])

$$\sum_j |\lambda_j|^\gamma \leq C \int_{\mathbb{R}^d} |V(x)|^{d/2+\gamma}\,dx, \quad V = \overline{V}, \;\; d \geq 3, \;\; \gamma \geq 0. \tag{1.4}$$

Theorem 1.1 allows one to consider real potentials V for which the right-hand side of (1.4) is infinite, while the left-hand side is finite almost surely. Indeed, let $1 < 2\gamma = q < d/(d-1)$. Then the parameter p in (1.2) satisfies the inequality

$$p > \tfrac{1}{2}d + \gamma. \tag{1.5}$$

Similar results for real random potentials $V = \overline{V}$ were obtained by the author and Vainberg in [Safronov and Vainberg 2008]. However, there is a big difference between Theorem 1.1 and the results of that earlier work, since the only point of accumulation of eigenvalues of the operator H considered there is the point $\lambda = 0$. When one studies complex-valued potentials, the fact that the eigenvalues λ_j might accumulate to points other than $\lambda = 0$ should not be excluded. Examples of decaying complex potentials V such that eigenvalues of $H = -\Delta + V$ accumulate to points of the positive real line $\mathbb{R}_+$ are constructed in [Bögli 2017]. Because of the difference between the cases of real and complex potentials, it would be more appropriate to ask what new information Theorem 1.1 provides compared to [Frank 2018; Frank and Sabin 2017], rather than realize that this theorem does not follow from the Lieb–Thirring estimate even in the selfadjoint case.

The related result of [Frank and Sabin 2017] says that there is a constant C that depends on d, p and γ such that

$$\sum_j \operatorname{dist}(\lambda_j, \mathbb{R}_+)|\lambda_j|^{\gamma-1} \le C\left(\int_{\mathbb{R}^d} |V|^p\,dx\right)^{2\gamma/(2p-d)}, \tag{1.6}$$

under conditions on γ and p implying that $p < \gamma + \frac{1}{2}d$. One can now refer to (1.5) to conclude that our results do give new information about the distribution of eigenvalues in the complex plane.

The same conclusion could be made by an analysis of the results of [Frank 2018], where the eigenvalues in the disk

$$\mathbb{D}_V = \left\{ z \in \mathbb{C} : |z|^{p-d/2} \le C_{p,d} \int |V|^p\,dx \right\}$$

are considered separately from the rest of the eigenvalues; here $p > \frac{1}{2}d$. R. Frank [2018] proves that under some restrictions on p,

$$\left(\sum_{\lambda_j \in \mathbb{D}_V} \operatorname{dist}(\lambda_j, \mathbb{R}_+)^\gamma \right)^\sigma \le C \int_{\mathbb{R}^d} |V|^p\,dx, \tag{1.7}$$

for γ equal to either p or $2p - d + \varepsilon$. The constants $C > 0$ and $\sigma > 0$ depend only on d and p in the first case but also on $\varepsilon > 0$ in the second. In its turn, $\varepsilon > 0$ belongs to the interval whose size depends on p. The observation we make is that $p < \gamma + \frac{1}{2}d$ in (1.7). On the other hand, in deterministic results, p simply can not be larger than $\gamma + \frac{1}{2}d$.

Theorem 1.1 gives information about the eigenvalues of H situated in a finite disk about the origin. The behavior of the eigenvalues outside of this disk is described below.

Theorem 1.3. *Let $d \ge 3$, let $R > 0$ and let $1 < \nu < q < 2$. Then the eigenvalues λ_j of the operator $-\Delta + V$ satisfy*

$$\mathbb{E}\left[\sum_{|\lambda_j| \ge R^2} \frac{\operatorname{Im}\sqrt{\lambda_j}\,(|\lambda_j| - R^2)}{|\lambda_j| R} \right] \le C|R|^{-\nu} \left(\int_{\mathbb{R}^d} |\widetilde{V}(x)|^p\,dx \right)^2,$$

with

$$p = \frac{d}{2} + \frac{d-q}{2(d-2)}.$$

It is assumed that $\operatorname{Im}\sqrt{\lambda_j} \ge 0$. *The constant C in (1.1) depends only on d, ν and q.*

According to Theorem 1.3, the condition $\widetilde{V} \in L^p$ implies that, for any $R > 0$,

$$\sum_{|\lambda_j| \ge R^2} |\operatorname{Im}\sqrt{\lambda_j}| < \infty \tag{1.8}$$

almost surely. Eigenvalues of H outside a finite disk about the point $z = 0$ were also studied in [Frank 2018]. However, in the theorems of that work the radius R of the disk depends on V. Moreover, when $d \ge 3$, these theorems guarantee convergence of $\sum_{|\lambda_j| \ge R^2} |\operatorname{Im}\lambda_j|^\alpha |\lambda_j|^{-\beta}$ for some $\alpha > 1$ and $\beta > 0$ rather than convergence of the series (1.8).

Theorem 1.3 immediately implies the following assertion.

Corollary 1.4. *Let $d \geq 3$, let $R > 0$ and let $1 < \nu < q < 2$. Then the eigenvalues λ_j of the operator $-\Delta + V$ satisfy*

$$\mathbb{E}\Big[\sum_{R^2 \leq |\lambda_j| \leq 2R^2} \operatorname{Im}\sqrt{\lambda_j}\,|\lambda_j|^{(\nu-1)/2}\Big] \leq C\Big(\int_{\mathbb{R}^d} |\widetilde{V}(x)|^p\,dx\Big)^2,$$

with

$$p = \frac{d}{2} + \frac{d-q}{2(d-2)}.$$

It is assumed that $\operatorname{Im}\sqrt{\lambda_j} \geq 0$. *The constant C in (1.1) depends only on d, ν and q.*

We also mention the article [Frank 2018] because Theorem 1.1 of that paper deals with the question about the shape of the domain containing all eigenvalues of H. In particular, it implies that the imaginary part of an eigenvalue tends to zero as the real part tends to infinity (in a quantitative way) once $V \in L^p$ with $p > \frac{1}{2}(d+1)$. Despite a vague visual resemblance of Corollary 1.4 to such a theorem, it does not give new information about the region containing all eigenvalues of H.

The next statement is an improvement of Theorem 1.1 for $3 \leq d \leq 5$ and $R_0 \leq 1$.

Theorem 1.5. *Let $3 \leq d \leq 5$, and let $0 < R_0 \leq 1$. Assume that τ_1 satisfies*

$$0 \leq \bigg(\Big(\frac{d}{2} + \frac{(\eta-1)(d+1)}{7d} + \frac{d-\eta}{2(d-2)}\Big) - 2\bigg)\tau_1 \leq \frac{(\nu-1)(d+1)}{7d},$$

with η and ν such that $1 < \nu < \eta < 2$. If $d = 3$, then we assume additionally that $8\nu + 9\eta < 26$. Let p, q and r be the numbers defined by

$$p = \frac{d}{7\tau_1}, \quad \frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2} \quad \text{and} \quad \frac{1}{r} = \frac{1-\theta}{2p} + \frac{\theta}{2},$$

where θ is the solution of the equation

$$\tau_1(1-\theta) + \frac{\theta}{2}\Big(\frac{d}{2} + \frac{d-\eta}{2(d-2)}\Big) = 1.$$

Then the eigenvalues λ_j of the operator $-\Delta + V$ satisfy

$$\mathbb{E}\Big[\sum_{|\lambda_j| \leq R_0^2} \operatorname{Im}\sqrt{\lambda_j}\,|\lambda_j|^{(\sigma-1)/2}\Big] \leq C_{\tau_1,\sigma}|R_0|^{\sigma-\theta q\nu/2}\Big(\int_{\mathbb{R}^d} |\widetilde{V}(x)|^r\,dx\Big)^{2q/r}, \quad \sigma > \tfrac{1}{2}\theta q \nu.$$

Besides its dependence on d, the constant $C_{\tau_1,\sigma}$ in this inequality depends on a choice of the parameters τ_1 and σ.

Theorem 1.5 gives new information about eigenvalues of H. Even in the case $V = \overline{V}$, this theorem does not follow from the Lieb–Thirring estimates. It turns into Theorem 1.1 for dimensions $3 \leq d \leq 5$ once we set $\tau_1 = 0$. On the other hand, since it allows us to consider ratios σ/r smaller than ratios q/p allowed by Theorem 1.1, Theorem 1.5 is an improvement of Theorem 1.1 for dimensions $3 \leq d \leq 5$ and the values of the parameter $R_0 < 1$.

One of the difficulties we encountered in this paper is that our statements can not be derived by taking expectations in the inequalities obtained by Borichev, Golinskii and Kupin [Borichev et al. 2009]. The reason is that operators of the Birman–Schwinger type we are dealing with might have different properties for different ω. This difficulty was overcome through an application of the Joukowski transform to a half-plane with a removed semidisk and consecutive integration with respect to the radius.

Eigenvalue bounds for Schrödinger operators with complex potentials have been studied for a long time. First of all, one should mention the related work of B. Pavlov, who found sharp conditions on V guaranteeing that H has only finitely many eigenvalues in $\mathbb{C} \setminus \mathbb{R}_+$. In particular, this is true for the one dimensional operator on the half-line $\mathbb{R}_+$ (see [Pavlov 1966]) if

$$|V(x)| \le Ce^{-c\sqrt{x}}, \quad \forall x \in \mathbb{R}_+,$$

for some constants C and $c > 0$.

In 2001, E. B. Davies posed a question whether the estimate

$$|\lambda| \le \frac{1}{4}\left(\int_{\mathbb{R}} |V(x)|\,dx\right)^2, \quad d = 1,$$

that he and his collaborators established for any nonreal eigenvalue λ of H (see [Abramov et al. 2001; Davies and Nath 2002]) can be extended to higher dimensions. This question was nicely handled by R. Frank [2011]. It was shown that, if $0 < \gamma \le \frac{1}{2}$ and $d \ge 2$, then there is a positive constant $C_{\gamma,d}$ such that

$$|\lambda|^\gamma \le C_{\gamma,d} \int_{\mathbb{R}^d} |V(x)|^{d/2+\gamma}\,dx, \tag{1.9}$$

for any eigenvalue of H in $\mathbb{C} \setminus \mathbb{R}_+$. The technique of [Frank 2011] was further developed and combined with some complex analysis in [Frank and Sabin 2017], where the authors already give the estimate (1.6) on the rate of accumulation of eigenvalues to the positive half-line $\mathbb{R}_+$. Another bound of this type is the inequality (1.7) established in [Frank 2018].

Note also, that if one only considers eigenvalues outside of a cone

$$\Gamma_\varepsilon = \{z \in \mathbb{C} : \operatorname{Re} z \ge 0,\ |\operatorname{Im} z| \le \varepsilon \operatorname{Re} z\}$$

(here $\varepsilon > 0$), then the Lieb–Thirring bound holds for these eigenvalues (see [Frank et al. 2006]):

$$\sum_{\lambda_j \notin \Gamma_\varepsilon} |\lambda_j|^\gamma \le C_{\gamma,d,\varepsilon} \int_{\mathbb{R}^d} |V(x)|^{d/2+\gamma}\,dx, \quad \gamma \ge 1.$$

While we do not intend to describe all results related to the theory of operators with complex-valued potentials, we would like to mention the articles [Briet et al. 2021; Cuenin 2017; Cuenin et al. 2014; Demuth et al. 2009; Demuth and Katriel 2008; Hansmann 2011; 2017; Korotyaev 2020; Korotyaev and Laptev 2018; Korotyaev and Safronov 2020; Laptev and Safronov 2009; Pavlov 1967] in addition to those already mentioned, all of which could be viewed as valuable contributions in this area.

2. Preliminaries

Everywhere below, $\mathfrak{S}_p$ denotes the class of compact operators K obeying

$$\|K\|_{\mathfrak{S}_p}^p = \operatorname{Tr}(K^*K)^{p/2} < \infty, \qquad p > 1.$$

Note that if $K \in \mathfrak{S}_p$ for some $p > 1$, then $K \in \mathfrak{S}_q$ for $q > p$ and $\|K\|_q \le \|K\|_p$.

Let z_j be the eigenvalues of a compact operator $K \in \mathfrak{S}_n$ where $n \in \mathbb{N} \setminus \{0\}$. We define the n-th determinant of $I + K$ as

$$\det_n(I+K) = \prod_j (1+z_j) \exp\biggl(\sum_{m=1}^{n-1} \frac{(-1)^m z_j^m}{m}\biggr), \qquad n \ge 2,$$
$$\det(I+K) = \prod_j (1+z_j), \qquad n = 1.$$

There exists a constant $C_n > 0$ depending only on n such that

$$|\det_n(I+X)| \le e^{C_n \|X\|_{\mathfrak{S}_n}^n}, \qquad \forall X \in \mathfrak{S}_n.$$

Moreover, we have the following statement; see Proposition 2.1 of [Korotyaev and Safronov 2020].

Proposition 2.1. *Let $n \ge 2$. Then for any $n - 1 \le p \le n$, there exists a constant $C_{p,n} > 0$ depending only on p and n such that*

$$|\det_n(I+X)| \le e^{C_{p,n} \|X\|_{\mathfrak{S}_p}^p}, \qquad \forall X \in \mathfrak{S}_p. \tag{2.1}$$

The way the eigenvalue bounds are obtained in [Korotyaev and Safronov 2020] uses applications of the following abstract result.

Theorem 2.2. *Let H_0 be a selfadjoint operator on a Hilbert space $\mathfrak{H}$. Let W_1 and W_2 be two bounded operators on $\mathfrak{H}$, and let $V = W_2 W_1$. Assume that the function*

$$\mathbb{C}_+ \;\ni\; z \mapsto W_1(H_0 - z)^{-1} W_2 \;\in\; \mathfrak{S}_p, \qquad 1 \le p < \infty,$$

is analytic in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and continuous up to the real line $\mathbb{R}$. Assume also that

$$\|W_1(H_0 - z)^{-1} W_2\|_{\mathfrak{S}_p}^p = o\biggl(\frac{1}{|z|}\biggr), \qquad \text{as } |z| \to \infty. \tag{2.2}$$

Then the eigenvalues λ_j of $H_0 + V$ in $\mathbb{C}_+$ satisfy

$$\sum_j \operatorname{Im} \lambda_j \le C_p \int_{-\infty}^{\infty} \|W_1(H_0 - \lambda - i0)^{-1} W_2\|_{\mathfrak{S}_p}^p \, d\lambda, \tag{2.3}$$

where C_p depends only on the parameter p.

Proof. The proof of this statement relies on Jensen's inequality for zeros of an analytic function, which is (also) justified in Proposition 3.11 of [Korotyaev and Safronov 2020]. ☐

Proposition 2.3. *Let $a(z)$ be an analytic function on $\mathbb{C}_+$ satisfying the condition*

$$a(z) = 1 + o\left(\frac{1}{|z|}\right) \quad \textit{as } |z| \to \infty.$$

Assume that for some $\gamma > 0$,

$$\ln|a(\lambda + i\gamma)| \le f(\lambda), \quad \forall \lambda \in \mathbb{R}.$$

Then zeros of $a(z)$ situated above the line $\operatorname{Im} z = \gamma$ satisfy the inequality

$$\sum_j (\operatorname{Im} \lambda_j - \gamma)_+ \le \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\lambda)\, d\lambda. \tag{2.4}$$

The statement also holds for $\gamma = 0$, if $a(z)$ is continuous up to the real line $\mathbb{R}$.

The bound (2.3) follows from (2.1) and the estimate (2.4) with $\gamma = 0$ once we set

$$a(z) = \det_n(I + W_1(H_0 - z)^{-1} W_2)$$

and

$$f(\lambda) = C_{p,n} \| W_1(H_0 - \lambda - i0)^{-1} W_2 \|_{\mathfrak{S}_p}^p.$$

According to the Birman–Schwinger principle, z is an eigenvalue of $H_0 + V$ if and only if $a(z) = 0$ (multiplicities coincide). This completes the proof of Theorem 2.2. □

One of the tools used in the present paper is an interpolation. Interpolation has been also used to prove Theorem 1.2 of [Korotyaev and Safronov 2020], which can be generalized and formulated as follows.

Theorem 2.4. *Let (Ω, μ) be a space with an σ-finite measure μ such that $L^2(\Omega, \mu)$ is separable. Let H_0 be a selfadjoint operator on the Hilbert space $L^2(\Omega, \mu)$. Assume that the integral kernel of the operator e^{-itH_0} satisfies the estimate*

$$|e^{-itH_0}(x, y)| \le \frac{C}{t^{\varkappa}}, \quad \forall t > 0, \ \ \forall x, y \in \Omega,$$

for some $\varkappa > 0$. Let $V \in L^p(\Omega, \mu) \cap L^\infty(\Omega, \mu)$ for $p > \varkappa$ such that $p \ge 1$. Assume also that (2.2) holds for all W_1 and W_2 that belong to a class of functions dense in $L^{2p}(\Omega, \mu)$. Then eigenvalues of the operator $H = H_0 + V$ satisfy

$$\sum_j |\operatorname{Im} \lambda_j|^r \le C_{p,r} \left(\int_\Omega |V(x)|^p \, d\mu \right)^{r/p - \varkappa},$$

for any $r > \max\{2(p - \varkappa), 1\}$.

The proof of this result is a counterpart of the proof of Theorem 1.2 from [Korotyaev and Safronov 2020], with the only differences being that the value of the parameter $\varkappa$ in Theorem 1.2 of that work is $\frac{3}{2}$ and $\Omega = \mathbb{R}^3$. However, one can consider different $\varkappa$ as well as spaces Ω which are different from $\mathbb{R}^d$. Especially interesting are spaces of fractional dimensions for which $2\varkappa$ is not an integer.

Another object that we will work with is the operator

$$X(k) = |V|^{1/2}(-\Delta - z)^{-1} V (-\Delta - z)^{-1} V |V|^{-1/2}, \quad z = k^2, \ \ k \in \mathbb{C}_+.$$

If V is a bounded compactly supported function, then $X(k)$ is a trace class operator for $d \le 3$, and $X(k) \in \mathfrak{S}_p$ for $p > \frac{1}{4}d$ and $d \ge 4$. In this case, we set

$$D_n(k) = \det_n(I - X(k)), \quad n > \tfrac{1}{4}d, \ n \in \mathbb{N}.$$

Proposition 2.5. *Let V be a compactly supported function on $\mathbb{R}^d$. If a point $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ is an eigenvalue of $H = -\Delta + V$, then $D_n(k) = 0$ for $k = \sqrt{\lambda}$. The algebraic multiplicity of the eigenvalue λ does not exceed the multiplicity of the root of the function $D_n(\cdot)$.*

Proof. According to the Birman–Schwinger principle, a point λ is an eigenvalue of H if and only if -1 is an eigenvalue of $|V|^{1/2}(-\Delta - \lambda)^{-1} V |V|^{-1/2}$. Therefore, 1 is an eigenvalue of $X(k_0)$ with $k_0^2 = \lambda$. On the other hand, if 1 is an eigenvalue of $X(k_0)$, then $D_n(k_0) = 0$.

The statement about the multiplicity follows from the fact that an isolated eigenvalue of H whose multiplicity m is larger than 1 can be turned into m simple eigenvalues by an arbitrarily small perturbation of finite rank (which does not have to be a function). For any $\varepsilon > 0$ there is a finite rank operator K_ε such that $\|K_\varepsilon\| < \varepsilon$ and that all eigenvalues of $-\Delta + K_\varepsilon + V$ near λ are simple. Define now the function

$$d_\varepsilon(k) = \det_n(I - |V|^{1/2}(-\Delta + K_\varepsilon - z)^{-1} V(-\Delta + K_\varepsilon - z)^{-1} V |V|^{-1/2}),$$

analytic in the neighborhood of $k_0 = \sqrt{\lambda}$ for sufficiently small $\varepsilon > 0$. In this neighborhood of the point k_0, we have $d_\varepsilon(k) \to D_n(k)$ uniformly, as $\varepsilon \to 0$. Since the function $d_\varepsilon(k)$ has at least m zeros near k_0, the multiplicity of the zero of the function $D_n(k)$ at $k = k_0$ can not be smaller than m by the argument principle. □

3. Large values of Re ζ without projections

The following proposition gives an important estimate for the integral kernel of $(-\Delta - z)^{-\zeta}$.

Proposition 3.1. *Let $d \ge 2$, and let $\frac{1}{2}(d-1) \le \operatorname{Re}\zeta \le \frac{1}{2}(d+1)$. The integral kernel of the operator $(-\Delta - z)^{-\zeta}$ satisfies the estimate*

$$|(-\Delta - z)^{-\zeta}(x, y)| \le \beta e^{\alpha(\operatorname{Im}\zeta)^2} |k|^{(d-1)/2 - \operatorname{Re}\zeta} |x - y|^{\operatorname{Re}\zeta - (d+1)/2}, \tag{3.1}$$

for $z \notin \mathbb{R}_+$. The positive constants β and α in this inequality depend only on d and $\operatorname{Re}\zeta$.

The proof of this proposition, as well as related references, can be found in [Frank and Sabin 2017].

Everywhere below, we use the notation $\chi_l(x) = \chi(x - l)$, where $l \in \mathbb{Z}^d$.

Corollary 3.2. *Let $\frac{1}{2}(d-1) \le \operatorname{Re}\zeta < \frac{1}{2}(d+1)$, where $d \ge 2$. Let $2 \le r < 2d/(2\operatorname{Re}\zeta - 1)$. Suppose that W is a function of the form*

$$W(x) = \sum_{n \in \mathbb{Z}^d} w_n \chi(x - n), \quad w_n \in \mathbb{C}, \ x \in \mathbb{R}^d.$$

Then

$$\|W(-\Delta - z)^{-\zeta} \chi_l\|_{\mathfrak{S}_2} \le \beta e^{\alpha(\operatorname{Im}\zeta)^2} |k|^{(d-1)/2 - \operatorname{Re}\zeta} \|W\|_r, \tag{3.2}$$

for $z \notin \mathbb{R}_+$. The positive constants β and α in this inequality depend only on d and $\operatorname{Re}\zeta$. *If* $\operatorname{Re}\zeta = \frac{1}{2}(d+1)$ *and $d \geq 2$, then* (3.2) *holds with $r = 2$.*

Proof. It follows from (3.1) that

$$\|W(-\Delta - z)^{-\zeta}\chi_l\|_{\mathfrak{S}_2}^2 \leq C e^{2\alpha(\operatorname{Im}\zeta)^2}|k|^{(d-1)-2\operatorname{Re}\zeta} \sum_{n\in\mathbb{Z}^d}(|n-l|+1)^{2\operatorname{Re}\zeta-(d+1)}|w_n|^2.$$

A simple application of Hölder's inequality leads to (3.2). □

We need to turn (3.2) into a similar estimate for the $\mathfrak{S}_4$-norm of the operator corresponding to smaller values of $\operatorname{Re}\zeta$. For that purpose, we employ the inequality

$$\|W(-\Delta - z)^{-\zeta}\chi_l\| \leq \beta e^{\alpha(\operatorname{Im}\zeta)^2}\|W\|_\infty, \tag{3.3}$$

for $\operatorname{Re}\zeta = 0$.

By interpolation we obtain the following proposition from (3.2) and (3.3).

Proposition 3.3. *Let $\frac{1}{2}(d-1) \leq \varkappa < \frac{1}{2}(d+1)$, where $d \geq 2$. Let $2 \leq r < 2d/(2\varkappa - 1)$. Suppose that W is a function of the form*

$$W(x) = \sum_{n\in\mathbb{Z}^d} w_n\chi(x-n), \quad w_n \in \mathbb{C}, \quad x \in \mathbb{R}^d.$$

Then, for any $\operatorname{Re}\zeta = \tau \in (0, \varkappa]$ *and $z \notin \mathbb{R}_+$,*

$$\|W(-\Delta - z)^{-\zeta}\chi_l\|_{\mathfrak{S}_{2\varkappa/\tau}} \leq \beta e^{\alpha(\operatorname{Im}\zeta)^2}|k|^{((d-1)/(2\varkappa)-1)\tau}\|W\|_{r\varkappa/\tau}. \tag{3.4}$$

The positive constants β and α in this inequality depend only on d and τ. If $\varkappa = \frac{1}{2}(d+1)$ and $d \geq 2$, then (3.4) *holds with $r = 2$.*

Proof. Indeed, let $\operatorname{Re}\zeta_0 = \tau$, and let

$$A = \Omega|A|$$

be the polar decomposition of the operator

$$A = |W|^{\zeta_0/\tau}(-\Delta - z)^{-\zeta_0}\chi_l.$$

Consider the function

$$f(\zeta) = e^{\alpha\zeta^2}\operatorname{Tr}(|W|^{\zeta/\tau}(-\Delta - z)^{-\zeta}\chi_l|A|^{(2\varkappa-\zeta+i\operatorname{Im}\zeta_0)/\tau}\Omega^*).$$

If $\operatorname{Re}\zeta = 0$, then

$$|f(\zeta)| \leq C_1\|A\|_{\mathfrak{S}_{2\varkappa/\tau}}^{2\varkappa/\tau}.$$

If $\operatorname{Re}\zeta = \varkappa$, then

$$|f(\zeta)| \leq C_2|k|^{(d-1)/2-\varkappa}\|A\|_{\mathfrak{S}_{2\varkappa/\tau}}^{\varkappa/\tau}\|W\|_{r\varkappa/\tau}^{\varkappa/\tau}.$$

Consequently, by the three lines lemma,

$$|f(\zeta_0)| \leq C|k|^{\theta((d-1)/2-\varkappa)}\|W\|_{r\varkappa/\tau}^{\theta\varkappa/\tau}\|A\|_{\mathfrak{S}_{2\varkappa/\tau}}^{(2-\theta)\varkappa/\tau}, \quad \theta = \tau/\varkappa.$$

Put differently,

$$|e^{\alpha\zeta_0^2}|\|A\|^{2\varkappa/\tau}_{\mathfrak{S}_{2\varkappa/\tau}} \le C|k|^{\theta((d-1)/2-\varkappa)}\|W\|^{\theta\varkappa/\tau}_{r\varkappa/\tau}\|A\|^{(2-\theta)\varkappa/\tau}_{\mathfrak{S}_{2\varkappa/\tau}}, \quad \theta = \tau/\varkappa.$$

The latter inequality implies (3.4). □

In particular, once we set $r\varkappa/\tau = 4$, we obtain the following.

Corollary 3.4. *Let $\frac{1}{2}(d-1) \le \varkappa < \frac{1}{2}(d+1)$, where $d \ge 2$. Suppose that W is a function of the form*

$$W(x) = \sum_{n\in\mathbb{Z}^d} w_n \chi(x-n), \quad w_n \in \mathbb{C}, \ \ x \in \mathbb{R}^d.$$

Then

$$\|W(-\Delta - z)^{-\zeta}\chi_l\|_{\mathfrak{S}_4} \le \beta e^{\alpha(\operatorname{Im}\zeta)^2}|k|^{((d-1)/(2\varkappa)-1)\operatorname{Re}\zeta}\|W\|_4, \tag{3.5}$$

for any $\frac{1}{2}\varkappa \le \operatorname{Re}\zeta < \min\{\varkappa, d\varkappa/(4\varkappa-2)\}$ and $z \notin \mathbb{R}_+$. The positive constants β and α in this inequality depend only on d and $\operatorname{Re}\zeta$. If $\varkappa = \frac{1}{2}(d+1)$ and $d \ge 2$, then (3.5) holds with $\operatorname{Re}\zeta = \frac{1}{2}\varkappa$.

Let us now consider the operator

$$\mathfrak{X}(\zeta) = e^{\alpha_0\zeta^2} W(-\Delta - z)^{-\zeta} V(-\Delta - z)^{-\zeta} W,$$

where W is a *fixed* function independent of ω. The proof of the following proposition is based on the fact that $\mathbb{E}[\omega_n] = 0$.

Proposition 3.5. *Let $\frac{1}{2}(d-1) \le \varkappa < \frac{1}{2}(d+1)$, where $d \ge 2$. Let $\frac{1}{2}\varkappa \le \operatorname{Re}\zeta < \min\{\varkappa, d\varkappa/(4\varkappa-2)\}$. Assume that $\widetilde{V} \in L^2(\mathbb{R}^d)$, $W \in L^4(\mathbb{R}^d)$ and $\alpha_0 > 2\alpha$. Then*

$$(\mathbb{E}(\|\mathfrak{X}(\zeta)\|^2_{\mathfrak{S}_2}))^{1/2} \le C_{\operatorname{Re}\zeta} e^{(2\alpha-\alpha_0)(\operatorname{Im}\zeta)^2}|k|^{((d-1)/\varkappa-2)\operatorname{Re}\zeta}\|\widetilde{V}\|_2\|W\|_4^2. \tag{3.6}$$

If $\varkappa = \frac{1}{2}(d+1)$ and $d \ge 2$, then (3.6) holds with $\operatorname{Re}\zeta = \frac{1}{2}\varkappa$.

Proof. Obviously,

$$\mathbb{E}(\|\mathfrak{X}(\zeta)\|^2_{\mathfrak{S}_2}) = \mathbb{E}(\operatorname{Tr}\mathfrak{X}(\zeta)^*\mathfrak{X}(\zeta)) \le e^{2\alpha_0\operatorname{Re}\zeta^2}\sum_{l\in\mathbb{Z}^d}|v_l|^2\|W(-\Delta-z)^{-\zeta}\chi_l\|^2_{\mathfrak{S}_4}\|\chi_l(-\Delta-z)^{-\zeta}W\|^2_{\mathfrak{S}_4}.$$

Together with Corollary 3.4, this implies (3.6). □

Corollary 3.6. *Let $\frac{1}{2}(d-1) \le \varkappa < \frac{1}{2}(d+1)$, where $d \ge 2$. Let $\frac{1}{2}\varkappa \le \operatorname{Re}\zeta < \min\{\varkappa, d\varkappa/(4\varkappa-2)\}$. Assume that $\widetilde{V} \in L^2(\mathbb{R}^d)$, $W = \widetilde{V}^{1/2}$ and $\alpha_0 > 2\alpha$. Then*

$$(\mathbb{E}(\|\mathfrak{X}(\zeta)\|^2_{\mathfrak{S}_2}))^{1/2} \le C_{\operatorname{Re}\zeta} e^{(2\alpha-\alpha_0)(\operatorname{Im}\zeta)^2}|k|^{((d-1)/\varkappa-2)\operatorname{Re}\zeta}\|\widetilde{V}\|_2^2. \tag{3.7}$$

If $\varkappa = \frac{1}{2}(d+1)$ and $d \ge 2$, then (3.7) holds with $\operatorname{Re}\zeta = \frac{1}{2}\varkappa$.

4. An estimate for the square of the Birman–Schwinger operator

According to the observations that we made, if $W = \sqrt{\widetilde{V}}$, then $\mathfrak{X}(\zeta)$ is a function that obeys (3.7) for some rather large values of $\operatorname{Re}\zeta$, and it also obeys

$$\|\mathfrak{X}(\zeta)\| \le C\|\widetilde{V}\|_\infty^2,$$

for $\operatorname{Re}\zeta = 0$. To obtain our first result about eigenvalues, we can interpolate between these two cases. Let

$$\widetilde{X}(k) = W(-\Delta - z)^{-1}V(-\Delta - z)^{-1}W, \quad z = k^2, \ k \in \mathbb{C}_+,$$

where W is a fixed function independent of ω. What follows is the result of the interpolation (which does not work for $d = 2$).

Proposition 4.1. *Let $\frac{1}{2}(d-1) \le \varkappa < \frac{1}{2}(d+1)$, where $d \ge 3$. Let*

$$\max\{2, \varkappa\} \le p < \min\left\{2\varkappa, \frac{d\varkappa}{2\varkappa - 1}\right\}. \tag{4.1}$$

Let $W = \widetilde{V}^{1/2}$. Assume that $\widetilde{V} \in L^p(\mathbb{R}^d)$. Then

$$(\mathbb{E}(\|\widetilde{X}(k)\|_{\mathfrak{S}_p}^p))^{1/p} \le C|k|^{(d-1)/\varkappa - 2}\|\widetilde{V}\|_p^2. \tag{4.2}$$

If $\varkappa = \frac{1}{2}(d+1)$ and $d \ge 3$, then (4.2) holds with $p = \varkappa$.

Proof. Note that $X(k) = \mathfrak{X}(1)$. The logic of interpolation says that (4.2) holds for p defined as

$$p = 2/\theta, \quad \text{for } \theta \text{ such that } 1 = \theta\tau,$$

where $\frac{1}{2}\varkappa \le \tau < \min\{\varkappa, d\varkappa/(4\varkappa - 2)\}$. Of course, this interpolation works only if $\tau > 1$, which is impossible for $d = 2$. Observe that, with this notation, $p = 2\tau$.

Let

$$X(k) = \Omega|X(k)|$$

be the polar decomposition of the operator $X(k)$. Consider the function

$$f(\zeta) = e^{\alpha_0\zeta^2}\mathbb{E}(\operatorname{Tr}(|W|^\zeta(-\Delta - z)^{-\zeta}V_\zeta(-\Delta - z)^{-\zeta}|W|^\zeta|X(k)|^{2\tau-\zeta}\Omega^*)),$$

where

$$V_\zeta(x) := \sum_n \omega_n|v_n|^\zeta e^{i\arg v_n}\chi(x - n).$$

If $\operatorname{Re}\zeta = 0$, then

$$|f(\zeta)| \le C_1\mathbb{E}(\|X(k)\|_{\mathfrak{S}_{2\tau}}^{2\tau}).$$

If $\operatorname{Re}\zeta = \tau$, then

$$|f(\zeta)| \le C_2|k|^{((d-1)/\varkappa-2)\tau}(\mathbb{E}(\|X(k)\|_{\mathfrak{S}_{2\tau}}^{2\tau}))^{1/2}\|\widetilde{V}\|_{2\tau}^{2\tau}.$$

Consequently, by the three lines lemma,

$$|f(1)| \le C|k|^{(d-1)/\varkappa-2}\|\widetilde{V}\|_{2\tau}^2(\mathbb{E}(\|X(k)\|_{\mathfrak{S}_{2\tau}}^{2\tau}))^{1-1/(2\tau)}.$$

Put differently,

$$\mathbb{E}(\|X(k)\|_{\mathfrak{S}_{2\tau}}^{2\tau}) \le C|k|^{(d-1)/\varkappa-2}\|\widetilde{V}\|_{2\tau}^2(\mathbb{E}(\|X(k)\|_{\mathfrak{S}_{2\tau}}^{2\tau}))^{1-1/(2\tau)}.$$

The latter inequality implies (4.2) because $2\tau = p$. □

Now we can formulate and prove the following result.

Theorem 4.2. *Let $d \ge 3$, and let $1 < \nu < q < 2$. Assume that $W = |V|^{1/2}$. Then*

$$\mathbb{E}(\|X(k)\|_{\mathfrak{S}_p}^p) \le C|k|^{-\nu}\|\widetilde{V}\|_p^{2p}, \tag{4.3}$$

for p defined by

$$p = \frac{d(d-1)-q}{2(d-2)} = \frac{d}{2} + \frac{d-q}{2(d-2)}. \tag{4.4}$$

Proof. Observe that the assumption $\nu < q < 2$ leads to the inequalities

$$\frac{d+1}{2} < p < \frac{d(d-1)-\nu}{2(d-2)}. \tag{4.5}$$

We will show that the conditions of Proposition 4.1 are fulfilled for the parameter $\varkappa$ defined by

$$\varkappa = \frac{(d-1)p}{2p-\nu}.$$

The latter relation simply means that

$$\nu = \left(2 - \frac{(d-1)}{\varkappa}\right)p. \tag{4.6}$$

Consequently, (4.3) follows from (4.2). The second inequality in (4.5) implies

$$\varkappa > \frac{d(d-1)-\nu}{2(d-\nu)} > \frac{d-1}{2}, \tag{4.7}$$

while the first inequality in (4.5) combined with the condition $\nu < 2$ implies

$$\varkappa < \frac{d+1}{2}.$$

One can also see that the first inequality in (4.7) is equivalent to the estimate

$$p = \frac{\varkappa\nu}{2\varkappa-(d-1)} < \frac{d\varkappa}{2\varkappa-1}.$$

Finally, note that when $d \ge 3$, the condition $p < 2\varkappa$ follows from the fact that $\nu + q > 2$. □

5. Proof of Theorem 1.1

We will work with the function

$$d(z) = \det_n(I - X(k)), \quad n = [p] + 1,$$

where z is related to k via the Joukowski mapping

$$z = \frac{R}{k} + \frac{k}{R}, \quad R > 0,$$

which maps the set $\{k \in \mathbb{C} : \operatorname{Im} k > 0,\ |k| > R\}$ onto the upper half-plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Rather standard arguments lead to the estimate

$$\sum_j \operatorname{Im} z_j \le C \int_{-\infty}^{\infty} \ln|d(z)|\, dz, \tag{5.1}$$

where the z_j are the zeros of the function $d(z)$ situated in the upper half-plane $\mathbb{C}_+$. In fact, (5.1) could be established in the same way as Jensen's inequality for zeros of an analytic function on a unit disk. In (5.1) we assume that V is compactly supported. The relation (5.1) leads to the estimate

$$\sum_j \left(\frac{|k_j|^2 - R^2}{|k_j|^2 R}\right)_+ \operatorname{Im} k_j \le C\left(\int_{-\infty}^{\infty} \|X(k)\|_{\mathfrak{S}_p}^p \left(\frac{1}{R} - \frac{R}{k^2}\right)_+ dk + \int_0^{\pi} \|X(R\cdot e^{i\theta})\|_{\mathfrak{S}_p}^p \sin\theta\, d\theta\right).$$

Taking the expectation we obtain

$$\begin{aligned}\mathbb{E}\left[\sum_j \frac{\operatorname{Im} k_j (|k_j|^2 - R^2)_+}{|k_j|^2 R}\right] \\ \le C\left(\int_{-\infty}^{\infty} \mathbb{E}[\|X(k)\|_{\mathfrak{S}_p}^p]\left(\frac{1}{R} - \frac{R}{k^2}\right)_+ dk + \int_0^{\pi} \mathbb{E}[\|X(R\cdot e^{i\theta})\|_{\mathfrak{S}_p}^p] \sin\theta\, d\theta\right).\end{aligned} \tag{5.2}$$

Due to Theorem 4.2, the latter inequality leads to

$$\mathbb{E}\left[\sum_j \frac{\operatorname{Im} k_j (|k_j|^2 - R^2)_+}{|k_j|^2 R}\right] \le C|R|^{-\nu}\|\widetilde{V}\|_p^{2p}. \tag{5.3}$$

Now, suppose that we consider only the eigenvalues $\lambda_j = k_j^2$ that satisfy the inequality

$$|k_j| \le R_0.$$

Multiplying (5.3) by R^{q-1} and integrating with respect to R from 0 to R_0, we obtain

$$\mathbb{E}\left[\sum_{|k_j| \le R_0} \operatorname{Im} k_j |k_j|^{q-1}\right] \le C|R_0|^{q-\nu}\|\widetilde{V}\|_p^{2p}, \quad q > \nu. \tag{5.4}$$

This implies Theorem 1.1. □

Theorem 1.2 can be proved in the same way. The only difference is that one needs to use Proposition 4.1 instead of Theorem 4.2.

Note also that (5.3) implies Theorem 1.3.

6. Operators of the Birman–Schwinger type

Let a, b and V be functions on $\mathbb{R}^d$. Define

$$A_\zeta = |a|^\zeta F V_\zeta F^* |b|^\zeta,$$

where F is the unitary Fourier transform operator. For any complex number z, we understand V_z as the sum

$$V_z(x) := \sum_n \omega_n |v_n|^z e^{i \arg v_n} \chi(x-n).$$

Note that the operator A_ζ can be viewed as a sum over the lattice $\mathbb{Z}^d$:

$$A_\zeta = \sum_{n\in\mathbb{Z}^d} A_{\zeta,n}, \tag{6.1}$$

where

$$A_{\zeta,n} = \omega_n |a|^\zeta F |v_n|^\zeta e^{i \arg v_n} \chi(\cdot - n) F^* |b|^\zeta.$$

We will show that while A_ζ might not be bounded at some points ω, it is still a compact operator almost surely if a, b and $\widetilde{V}$ are in L^2. We remind the reader that $\widetilde{V}$ was defined as the function

$$\widetilde{V}(x) = \sum_n |v_n| \chi(x-n).$$

Remark. Operators of the form $aFWF^*b$ do not have to be bounded for all a, b and W from L^2. Indeed, let

$$W(x) = (|x|+1)^{-s}, \quad \text{with } \tfrac{1}{2}d < s < \tfrac{2}{3}d,$$

and let

$$a(\xi) = b(\xi) = \begin{cases} |\xi|^{-3s/4} & \text{if } |\xi| \le 1, \\ 0 & \text{if } |\xi| > 1. \end{cases}$$

If $aFWF^*b$ was bounded, the operator $T = aF\sqrt{W}$ would be bounded as well. The latter is not true, simply because $T\psi \notin L^2$ for $\psi = W$ (the singularity of $T\psi$ at zero is $|\xi|^{3s/4-d}$).

Proposition 6.1. *Let $a \in L^2$, $b \in L^2$ and $\widetilde{V} \in L^2$. Let also $p \ge 2$. Then the sum (6.1) with* $\operatorname{Re}\zeta = 2/p$ *converges almost surely in $\mathfrak{S}_p$. Moreover,*

$$(\mathbb{E}[\|A_\zeta\|_{\mathfrak{S}_p}^p])^{1/p} \le (2\pi)^{-2d/p} \|a\|_2^{2/p} \|b\|_2^{2/p} \|\widetilde{V}\|_2^{2/p}, \quad \operatorname{Re}\zeta = 2/p. \tag{6.2}$$

Proof. We are going to prove (6.2) for one point ζ_0 such that $\operatorname{Re}\zeta_0 = 2/p$. For that purpose, we define the operator $K(\omega) = |A_{\zeta_0}|^{p/2}$. Then, obviously,

$$\beta := \mathbb{E}(\|K\|_{\mathfrak{S}_2}^2) = \mathbb{E}[\|A_{\zeta_0}\|_{\mathfrak{S}_p}^p].$$

Let $\Omega = \Omega(\omega)$ be the partially isometric operator appearing in the polar decomposition

$$A_{\zeta_0} = \Omega(\omega)|A_{\zeta_0}|.$$

We introduce the analytic function

$$f(\zeta) = \mathbb{E}[\operatorname{Tr} A_\zeta |K|^{2-\zeta} |K|^{i \operatorname{Im} \zeta_0} \Omega^*],$$

which will be treated by the three lines lemma. Since $\|A_\zeta\| \le 1$ for $\operatorname{Re}\zeta = 0$, and $\||K|^{i \operatorname{Im}\zeta_0}\Omega^*\| \le 1$, we obtain that

$$|f(\zeta)| \le \beta, \quad \text{for } \operatorname{Re}\zeta = 0. \tag{6.3}$$

On the other hand,

$$|f(\zeta)| \le (2\pi)^{-d} \beta^{1/2} \|\widetilde{V}\|_2 \|a\|_2 \|b\|_2, \quad \text{for } \operatorname{Re}\zeta = 1, \tag{6.4}$$

by an analogue of Hölder's inequality valid for Schatten classes. Indeed, for $\operatorname{Re}\zeta = 1$,

$$|f(\zeta)|^2 \le \mathbb{E}[\|A_\zeta\|_{\mathfrak{S}_2}^2] \cdot \mathbb{E}[\|K\|_{\mathfrak{S}_2}^2],$$

and

$$\mathbb{E}[\|A_\zeta\|_{\mathfrak{S}_2}^2] = \mathbb{E}[\operatorname{Tr} A_\zeta^* A_\zeta] = \sum_{n \in \mathbb{Z}^d} \mathbb{E}[\operatorname{Tr} A_{\zeta,n}^* A_{\zeta,n}] \le (2\pi)^{-2d} \|\widetilde{V}\|_2^2 \|a\|_2^2 \|b\|_2^2.$$

Using the three lines lemma, we obtain from (6.3) and (6.4) that

$$|f(\zeta)| \le (2\pi)^{-d \operatorname{Re}\zeta} \beta^{1-\operatorname{Re}\zeta/2} \|\widetilde{V}\|_2^{\operatorname{Re}\zeta} \|a\|_2^{\operatorname{Re}\zeta} \|b\|_2^{\operatorname{Re}\zeta}.$$

Note now that $f(\zeta_0) = \beta$. Consequently,

$$\beta^{1/p} \le (2\pi)^{-2d/p} \|\widetilde{V}\|_2^{2/p} \|a\|_2^{2/p} \|b\|_2^{2/p}. \qquad \square$$

Corollary 6.2. *Let T be a random operator of the form*

$$T = |a| F V F^* |b|,$$

with

$$V(x) := \sum_n \omega_n v_n \chi(x-n).$$

Let $a \in L^p$, $b \in L^p$, $v_n \in \ell^p$ and $p \ge 2$. Then

$$(\mathbb{E}[\|T\|_{\mathfrak{S}_p}^p])^{1/p} \le (2\pi)^{-2d/p} \|a\|_p \|b\|_p \|\widetilde{V}\|_p.$$

Proof. Observe that the functions $|a|^{p/2}$, $|b|^{p/2}$ and $\widetilde{V}^{p/2}$ belong to L^2. Therefore, according to the proposition, the $\mathfrak{S}_p$-norm of the operator

$$\widetilde{K} = |a|^{p\zeta/2} F V_{p\zeta/2} F^* |b|^{p\zeta/2}$$

obeys the inequality

$$(\mathbb{E}[\|\widetilde{K}\|_{\mathfrak{S}_p}^p])^{1/p} \le (2\pi)^{-2d/p} \||a|^{p/2}\|_2^{2/p} \||b|^{p/2}\|_2^{2/p} \|\widetilde{V}^{p/2}\|_2^{2/p}, \quad \operatorname{Re}\zeta = 2/p. \qquad \square$$

The following result is a very well-known bound obtained by E. Seiler and B. Simon [Seiler and Simon 1975]. Moreover, the reader can easily prove it using standard interpolation.

Proposition 6.3. *Let a and W be two functions from $L^p(\mathbb{R}^d)$ with $p \geq 2$. Let T be the operator*

$$T = aFW,$$

where F is the operator of the Fourier transform. Then

$$\|T\|_{\mathfrak{S}_p} \leq (2\pi)^{-d/p}\|a\|_p\|W\|_p, \quad p \geq 2.$$

Corollary 6.4. *Let $q \geq p \geq 2$. Let T be a random operator of the form*

$$T = |a|FVF^*|b|,$$

with

$$V(x) := \sum_n \omega_n v_n \chi(x-n).$$

Let $a \in L^p$, $b \in L^q$ and $v_n \in \ell^p$. Then

$$(\mathbb{E}[\|T\|^q_{\mathfrak{S}_p}])^{1/q} \leq (2\pi)^{-d/p-d/q}\|a\|_p\|b\|_q\|\widetilde{V}\|_p.$$

Proof. According to Proposition 6.3,

$$\|T\|_{\mathfrak{S}_p} \leq (2\pi)^{-d/p}\|a\|_p\|b\|_\infty\|\widetilde{V}\|_p, \quad p \geq 2.$$

On the other hand, according to Corollary 6.2,

$$(\mathbb{E}[\|T\|^p_{\mathfrak{S}_p}])^{1/p} \leq (2\pi)^{-2d/p}\|a\|_p\|b\|_p\|\widetilde{V}\|_p.$$

It remains to interpolate between the two cases. For that purpose, we introduce the function

$$f(\zeta) = \mathbb{E}[(\operatorname{Tr} K^p)^{(1+q-p)(1-\zeta)/p+\zeta(p-1)(q-p)/p^2}\operatorname{Tr}|a|FVF^*|b|^{q\zeta/p}K^{p-1}\Omega^*],$$

where $K = ||a|FVF^*|b||$ and Ω is the partially isometric operator appearing in the polar decomposition

$$|a|FVF^*|b| = \Omega K.$$

For convenience, we write

$$\beta := \mathbb{E}[(\operatorname{Tr} K^p)^{q/p}].$$

If $\operatorname{Re}\zeta = 0$, then by Hölder's inequality,

$$|f(\zeta)| \leq (2\pi)^{-d/p}\beta\|a\|_p\|\widetilde{V}\|_p.$$

If $\operatorname{Re}\zeta = 1$, then

$$|f(\zeta)| \leq \mathbb{E}[(\operatorname{Tr} K^p)^{(p-1)(q-p)/p^2}\||a|FVF^*|b|^{q/p}\|_{\mathfrak{S}_p}(\operatorname{Tr} K^p)^{(p-1)/p}],$$

which leads to

$$|f(\zeta)| \leq \beta^{1-1/p}(2\pi)^{-2d/p}\|a\|_p\|b\|_q^{q/p}\|\widetilde{V}\|_p.$$

Observe also that

$$f(p/q) = \beta.$$

Thus by the three lines lemma,

$$\beta \leq \beta^{1-1/q}(2\pi)^{-d/p-d/q}\|a\|_p\|b\|_q\|\widetilde{V}\|_p. \qquad \square$$

7. Large values of $\operatorname{Re}\zeta$

Let $0 < R \le 1$. Let $\chi_{0,k}$ be the characteristic function of the ball

$$\mathfrak{B} = \left\{\xi \in \mathbb{R}^d : |\xi| \le \frac{2|k|}{R}\right\},$$

and let $\chi_{1,k} = 1 - \chi_{0,k}$ be the characteristic function of its complement

$$\mathbb{R}^d \setminus \mathfrak{B} = \left\{\xi \in \mathbb{R}^d : |\xi| > \frac{2|k|}{R}\right\}.$$

We introduce the operators

$$P_{n,k} = F\chi_{n,k}F^*,$$

which are the spectral projections of $-\Delta$ corresponding to the intervals $[0, 4|k|^2/R^2]$ and $(4|k|^2/R^2, \infty)$.

Besides depending on the properties of $(-\Delta - z)^{-\zeta}$, the arguments of this paper also rely on the properties of the operators $P_{n,k}(-\Delta - z)^{-\zeta}$ for different values of ζ. In this section, we discuss relatively large values of $\operatorname{Re}\zeta$. The following proposition gives an important estimate for the integral kernel of $P_{n,k}(-\Delta - z)^{-\zeta}$.

Proposition 7.1. *Let $R \le 1$. Let $d \ge 2$, and let $\frac{1}{2}(d-1) < \operatorname{Re}\zeta \le \frac{1}{2}(d+1)$. The integral kernel of the operator $P_{j,k}(\Delta - z)^{-\zeta}$ satisfies the estimate*

$$|P_{j,k}(-\Delta - z)^{-\zeta}(x, y)| \le \beta e^{\alpha(\operatorname{Im}\zeta)^2}|k|^{(d-1)/2-\operatorname{Re}\zeta}|x-y|^{\operatorname{Re}\zeta-(d+1)/2}, \tag{7.1}$$

for $z \notin \mathbb{R}_+$ and $j = 0, 1$. The positive constants β and α in this inequality depend only on d and $\operatorname{Re}\zeta$.

Proof. Due to Proposition 3.1, it is sufficient to prove only one of the inequalities (7.1). Let us first estimate the integrals

$$I_n = \int_{2^n|k|<R|\xi|<2^{n+1}|k|} \frac{e^{i\xi(x-y)}\,d\xi}{(|\xi|^2-k^2)^\zeta} = -|x-y|^{-2}\int_{2^n|k|<R|\xi|<2^{n+1}|k|} \frac{\Delta_\xi e^{i\xi(x-y)}\,d\xi}{(|\xi|^2-k^2)^\zeta}$$
$$= |x-y|^{-2}\int_{\mathbb{S}_{2^{n+1}|k|/R}\cup\mathbb{S}_{2^n|k|/R}} \frac{\pm i(x-y)\xi e^{i\xi(x-y)}\,dS_\xi}{|\xi|(|\xi|^2-k^2)^\zeta}$$
$$- \zeta|x-y|^{-2}\int_{2^n|k|<R|\xi|<2^{n+1}|k|} \frac{2i\xi(x-y)e^{i\xi(x-y)}\,d\xi}{(|\xi|^2-k^2)^{\zeta+1}}, \tag{7.2}$$

for $n \ge 1$. We will show that

$$|I_n| \le \beta e^{\alpha(\operatorname{Im}\zeta)^2}(2^n|k|/R)^{(d-1)/2-\operatorname{Re}\zeta}|x-y|^{\operatorname{Re}\zeta-(d+1)/2}, \tag{7.3}$$

for some $\beta > 0$ and $\alpha > 0$. A priori,

$$|I_n| \le C_d e^{2\pi|\operatorname{Im}\zeta|}(2^n|k|/R)^{d-2\operatorname{Re}\zeta}, \tag{7.4}$$

but the representation (7.2) leads to

$$|I_n| \le C_d e^{2\pi|\operatorname{Im}\zeta|}(2^n|k|/R)^{d-2\operatorname{Re}\zeta-1}|x-y|^{-1}. \tag{7.5}$$

The first estimate (7.4) implies (7.3) for $2^n|k||x-y|<R$, because in this case,

$$|I_n| \le C_d e^{2\pi|\mathrm{Im}\,\zeta|}(2^n|k|/R)^{d-2\,\mathrm{Re}\,\zeta}(2^n|k||x-y|/R)^{\mathrm{Re}\,\zeta-(d+1)/2}.$$

The second inequality (7.5) implies (7.3) for $2^n|k||x-y|\ge R$, because $\frac{1}{2}(d+1)-\mathrm{Re}\,\zeta\le 1$ and, therefore,

$$(2^n|k|/R)^{d-2\,\mathrm{Re}\,\zeta-1}|x-y|^{-1}\le(2^n|k|/R)^{d-2\,\mathrm{Re}\,\zeta+\mathrm{Re}\,\zeta-(d+1)/2}|x-y|^{\mathrm{Re}\,\zeta-(d+1)/2}.$$

The estimates (7.3) imply (7.1) for $j=1$, because

$$P_{1,k}(-\Delta-z)^{-\zeta}(x,y)=(2\pi)^{-d}\sum_{n=1}^{\infty}I_n. \qquad \square$$

Corollary 7.2. *Let $\frac{1}{2}(d-1)<\mathrm{Re}\,\zeta<\frac{1}{2}(d+1)$, where $d\ge 2$. Let $2\le r<2d/(2\,\mathrm{Re}\,\zeta-1)$. Suppose that W is a function of the form*

$$W(x)=\sum_{n\in\mathbb{Z}^d}w_n\chi(x-n),\qquad w_n\in\mathbb{C},\quad x\in\mathbb{R}^d.$$

Then

$$\|WP_{j,k}(-\Delta-z)^{-\zeta}\chi_l\|_{\mathfrak{S}_2}\le\beta e^{\alpha(\mathrm{Im}\,\zeta)^2}|k|^{(d-1)/2-\mathrm{Re}\,\zeta}\|W\|_r, \tag{7.6}$$

for $z\notin\mathbb{R}_+$ and $j=0,1$. The positive constants β and α in this inequality depend only on d and $\mathrm{Re}\,\zeta$. If $\mathrm{Re}\,\zeta=\frac{1}{2}(d+1)$ and $d\ge 2$, then (7.6) holds with $r=2$.

Proof. It follows from (7.1) that

$$\|WP_{j,k}(-\Delta-z)^{-\zeta}\chi_l\|^2_{\mathfrak{S}_2}\le Ce^{2\alpha(\mathrm{Im}\,\zeta)^2}|k|^{(d-1)-2\,\mathrm{Re}\,\zeta}\sum_{n\in\mathbb{Z}^d}(|n-l|+1)^{2\,\mathrm{Re}\,\zeta-(d+1)}|w_n|^2.$$

A simple application of Hölder's inequality leads to (7.6). $\square$

On the other hand, we have the inequality

$$\|WP_{j,k}(-\Delta-z)^{-\zeta}\chi_l\|\le\beta e^{\alpha(\mathrm{Im}\,\zeta)^2}\|W\|_\infty, \tag{7.7}$$

for $\mathrm{Re}\,\zeta=0$.

By interpolation, we obtain the following from (7.6) and (7.7).

Proposition 7.3. *Let $\frac{1}{2}(d-1)<\varkappa<\frac{1}{2}(d+1)$, where $d\ge 2$. Let $2\le r<2d/(2\varkappa-1)$. Suppose that W is a function of the form*

$$W(x)=\sum_{n\in\mathbb{Z}^d}w_n\chi(x-n),\qquad w_n\in\mathbb{C},\quad x\in\mathbb{R}^d.$$

Then, for any $\mathrm{Re}\,\zeta=\tau\in(0,\varkappa)$, $z\notin\mathbb{R}_+$ and $j=0,1$,

$$\|WP_{j,k}(-\Delta-z)^{-\zeta}\chi_l\|_{\mathfrak{S}_{2\varkappa/\tau}}\le\beta e^{\alpha(\mathrm{Im}\,\zeta)^2}|k|^{((d-1)/(2\varkappa)-1)\tau}\|W\|_{r\varkappa/\tau}. \tag{7.8}$$

The positive constants β and α in this inequality depend only on d and τ. If $\varkappa=\frac{1}{2}(d+1)$ and $d\ge 2$, then (7.8) holds with $r=2$.

Proof. Indeed, let $\operatorname{Re}\zeta_0 = \tau$, and let
$$A = \Omega|A|$$
be the polar decomposition of the operator
$$A = |W|^{\zeta_0/\tau} P_{j,k}(-\Delta - z)^{-\zeta_0}\chi_l.$$
Consider the function
$$f(\zeta) = e^{\alpha\zeta^2}\operatorname{Tr}(|W|^{\zeta/\tau} P_{j,k}(-\Delta - z)^{-\zeta}\chi_l |A|^{(2\varkappa-\zeta+i\operatorname{Im}\zeta_0)/\tau}\Omega^*).$$
If $\operatorname{Re}\zeta = 0$, then
$$|f(\zeta)| \le C_1\|A\|^{2\varkappa/\tau}_{\mathfrak{S}_{2\varkappa/\tau}}.$$
If $\operatorname{Re}\zeta = \varkappa$, then
$$|f(\zeta)| \le C_2|k|^{(d-1)/2-\varkappa}\|A\|^{\varkappa/\tau}_{\mathfrak{S}_{2\varkappa/\tau}}\|W\|^{\varkappa/\tau}_{r\varkappa/\tau}.$$
Consequently, by the three lines lemma,
$$|f(\zeta_0)| \le C|k|^{\theta((d-1)/2-\varkappa)}\|W\|^{\theta\varkappa/\tau}_{r\varkappa/\tau}\|A\|^{(2-\theta)\varkappa/\tau}_{\mathfrak{S}_{2\varkappa/\tau}}, \quad \theta = \tau/\varkappa.$$
Put differently,
$$|e^{\alpha\zeta_0^2}|\|A\|^{2\varkappa/\tau}_{\mathfrak{S}_{2\varkappa/\tau}} \le C|k|^{\theta((d-1)/2-\varkappa)}\|W\|^{\theta\varkappa/\tau}_{r\varkappa/\tau}\|A\|^{(2-\theta)\varkappa/\tau}_{\mathfrak{S}_{2\varkappa/\tau}}, \quad \theta = \tau/\varkappa.$$
The latter inequality implies (7.8), and the proof is completed. □

In particular, once we set $r\varkappa/\tau = 4$, we obtain the following.

Corollary 7.4. *Let $\frac{1}{2}(d-1) < \varkappa < \frac{1}{2}(d+1)$, where $d \ge 2$. Suppose that W is a function of the form*
$$W(x) = \sum_{n\in\mathbb{Z}^d} w_n\chi(x-n), \quad w_n \in \mathbb{C}, \quad x \in \mathbb{R}^d.$$
Then
$$\|WP_{j,k}(-\Delta - z)^{-\zeta}\chi_l\|_{\mathfrak{S}_4} \le \beta e^{\alpha(\operatorname{Im}\zeta)^2}|k|^{((d-1)/(2\varkappa)-1)\operatorname{Re}\zeta}\|W\|_4, \tag{7.9}$$
for any $\frac{1}{2}\varkappa \le \operatorname{Re}\zeta < \min\{\varkappa, d\varkappa/(4\varkappa-2)\}$, $z \notin \mathbb{R}_+$ and $j = 0, 1$. The positive constants β and α in this inequality depend only on d and $\operatorname{Re}\zeta$. If $\varkappa = \frac{1}{2}(d+1)$ and $d \ge 2$, then (7.9) holds with $\operatorname{Re}\zeta = \frac{1}{2}\varkappa$.

We will now discuss the properties of the random operators
$$X_{n,m}(\zeta) = e^{\alpha_0\zeta^2}(WP_{n,k}(-\Delta - z)^{-\zeta}V(-\Delta - z)^{-\zeta}P_{m,k}W).$$
Here W is a *fixed* function which does not depend on ω.

Proposition 7.5. *Let $\frac{1}{2}(d-1) < \varkappa < \frac{1}{2}(d+1)$, where $d \ge 2$. Let $\frac{1}{2}\varkappa \le \operatorname{Re}\zeta < \min\{\varkappa, d\varkappa/(4\varkappa-2)\}$. Assume that $\widetilde{V} \in L^2(\mathbb{R}^d)$, $W \in L^4(\mathbb{R}^d)$ and $\alpha_0 > 2\alpha$. Then*
$$(\mathbb{E}(\|X_{n,m}(\zeta)\|^2_{\mathfrak{S}_2}))^{1/2} \le C_{\operatorname{Re}\zeta}e^{(2\alpha-\alpha_0)(\operatorname{Im}\zeta)^2}|k|^{((d-1)/\varkappa-2)\operatorname{Re}\zeta}\|\widetilde{V}\|_2\|W\|_4^2. \tag{7.10}$$
If $\varkappa = \frac{1}{2}(d+1)$ and $d \ge 2$, then (7.10) holds with $\operatorname{Re}\zeta = \frac{1}{2}\varkappa$.

Proof. Obviously,

$$\mathbb{E}(\|X_{n,m}(\zeta)\|^2_{\mathfrak{S}_2}) = \mathbb{E}(\operatorname{Tr} X_{n,m}(\zeta)^* X_{n,m}(\zeta))$$
$$\le e^{2\alpha_0 \operatorname{Re}\zeta^2} \sum_{l\in\mathbb{Z}^d} |v_l|^2 \|W P_{n,k}(-\Delta - z)^{-\zeta}\chi_l\|^2_{\mathfrak{S}_4} \|\chi_l(-\Delta - z)^{-\zeta} P_{m,k} W\|^2_{\mathfrak{S}_4}.$$

Together with Corollary 7.4, this implies (7.10). □

We will also study the spectral properties of the operator

$$Y(\zeta) = X_{0,0}(\zeta) + X_{0,1}(\zeta) + X_{1,0}(\zeta).$$

Corollary 7.6. *Let* $\frac{1}{2}(d-1) < \varkappa < \frac{1}{2}(d+1)$, *where* $d \ge 2$. *Let* $\frac{1}{2}\varkappa \le \operatorname{Re}\zeta < \min\{\varkappa, d\varkappa/(4\varkappa - 2)\}$. *Assume that* $\widetilde{V} \in L^2(\mathbb{R}^d)$, $W = \widetilde{V}^{1/2}$ *and* $\alpha_0 > 2\alpha$. *Then*

$$(\mathbb{E}(\|Y(\zeta)\|^2_{\mathfrak{S}_2}))^{1/2} \le C_{\operatorname{Re}\zeta} e^{(2\alpha-\alpha_0)(\operatorname{Im}\zeta)^2} |k|^{((d-1)/\varkappa - 2)\operatorname{Re}\zeta} \|\widetilde{V}\|^2_2. \tag{7.11}$$

If $\varkappa = \frac{1}{2}(d+1)$ *and* $d \ge 2$, *then* (7.11) *holds with* $\operatorname{Re}\zeta = \frac{1}{2}\varkappa$.

8. Small values of Re ζ

The notations we use in this section are the same as in the previous one. In particular, the projections $P_{n,k}$ are the same as before. As was mentioned, the arguments of this paper rely on the properties of the operators $P_{n,k}(-\Delta - z)^{-\zeta}$ for different values of ζ. In this section, we discuss the case $0 \le \operatorname{Re}\zeta < 1$.

In the next two propositions, we discuss the properties of the random operators

$$X_{n,m}(\zeta) = e^{\alpha_0\zeta^2}(W P_{n,k}(-\Delta - z)^{-\zeta} V (-\Delta - z)^{-\zeta} P_{m,k} W),$$

for $\operatorname{Re}\zeta = \frac{1}{2}\gamma$ and $0 < \gamma < \frac{3}{2}$. Here W is a *fixed* function which does not depend on ω. The value of the parameter α_0 should be sufficiently large as in Corollary 7.6.

Later, we will also study the spectral properties of the operator

$$Y(\zeta) = X_{0,0}(\zeta) + X_{0,1}(\zeta) + X_{1,0}(\zeta).$$

However, the terms in this representation will be studied separately. A this point, we do not discuss $X_{1,1}(\zeta)$ at all.

Proposition 8.1. *Let* $d \ge 2$. *Let* $z \in \mathbb{C} \setminus \mathbb{R}_+$, *and let* $2 \le 2p < 3/\gamma$. *Assume that* $0 < R \le 1$. *If* $\operatorname{Re}\zeta = \frac{1}{2}\gamma$, $W \in L^{4p}$ *and* $\widetilde{V} \in L^{2p}$, *then* $X_{0,0}(\zeta) \in \mathfrak{S}_p$ *almost surely. Moreover,*

$$\mathbb{E}(\|X_{0,0}(\zeta)\|^p_{\mathfrak{S}_p})^{1/p} \le C_{p,\gamma} e^{-\alpha_0|\operatorname{Im}\zeta|^2/2} \left(\frac{|k|}{R}\right)^{3d/(2p)-2\gamma} \|\widetilde{V}\|_{2p} \|W\|^2_{4p}. \tag{8.1}$$

Proof. This statement follows from Corollary 6.2 and Proposition 6.3. If $r = \frac{1}{2}q = 2p$, then $1/r + 2/q = 1/p$. Moreover, since

$$X_{0,0}(\zeta) = e^{\alpha_0\zeta^2}(W(-\Delta - z)^{-\zeta/3} P_{0,k}(-\Delta - z)^{-2\zeta/3} V (-\Delta - z)^{-2\zeta/3} P_{0,k}(-\Delta - z)^{-\zeta/3} W),$$

we obtain the estimate

$$\begin{aligned}&\|\widetilde{X}_{0,0}(\zeta)\|_p\\&\le |e^{\alpha_0\zeta^2}|\cdot\|W(-\Delta-z)^{-\zeta/3}P_{0,k}\|_q\|\widetilde{P}_{0,k}(-\Delta-z)^{-2\zeta/3}V(-\Delta-z)^{-2\zeta/3}P_{0,k}\|_r\|P_{0,k}(-\Delta-z)^{-\zeta/3}W\|_q.\end{aligned}$$

It remains to realize that

$$\begin{aligned}\left(\int_{\mathbb{R}^d}\frac{\chi_{0,k}\,d\xi}{|(|\xi|^2-z)^{2\zeta/3}|^r}\right)^{2/r} &\le \left(\int_{|\xi|<2|k|}\frac{d\xi}{|(|\xi|^2-z)^{2\zeta/3}|^r}\right)^{2/r}+c_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\left(\int_{|\xi|<2|k|/R}\frac{d\xi}{|\xi|^{2\gamma r/3}}\right)^{2/r}\\&\le C_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\left(\frac{|k|}{R}\right)^{2(d-2r\gamma/3)/r}=C_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\left(\frac{|k|}{R}\right)^{d/p-4\gamma/3}, \qquad \gamma r<3,\end{aligned}$$

while a similar argument shows that

$$\left(\int_{\mathbb{R}^d}\frac{\chi_{0,k}\,d\xi}{|(|\xi|^2-z)^{\zeta/3}|^q}\right)^{2/q}\le \widetilde{C}_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\left(\frac{|k|}{R}\right)^{2(d-q\gamma/3)/q}=\widetilde{C}_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\left(\frac{|k|}{R}\right)^{d/(2p)-2\gamma/3}. \qquad \square$$

Proposition 8.2. *Let* $2\le d\le 5$. *Let* $z\in\mathbb{C}\setminus\mathbb{R}_+$, *and let* $2\le 2p<3/\gamma$. *Assume that* $4p\gamma>d$ *and* $0<R\le 1$. *If* $\operatorname{Re}\zeta=\frac{1}{2}\gamma$, $W\in L^{4p}$ *and* $\widetilde{V}\in L^{2p}$, *then* $X_{0,1}(\zeta)\in\mathfrak{S}_p$ *for all* ω. *Moreover,*

$$\|X_{0,1}(\zeta)\|_{\mathfrak{S}_p}\le C_{p,\gamma}e^{-\alpha_0|\operatorname{Im}\zeta|^2/2}\left(\frac{|k|}{R}\right)^{d/p-2\gamma}\|\widetilde{V}\|_{2p}\|W\|_{4p}^2. \tag{8.2}$$

Proof. Since

$$X_{0,1}(\zeta)=e^{\alpha_0\zeta^2}(W(-\Delta-z)^{-\zeta/3}P_{0,k}(-\Delta-z)^{-2\zeta/3}VP_{1,k}(-\Delta-z)^{-\zeta}W),$$

we obtain the estimate

$$\|X_{0,1}(\zeta)\|_p\le|e^{\alpha_0\zeta^2}|\cdot\|W(-\Delta-z)^{-\zeta/3}P_{0,k}\|_{4p}\|P_{0,k}(-\Delta-z)^{-2\zeta/3}V\|_{2p}\|P_{1,k}(-\Delta-z)^{-\zeta}W\|_{4p}.$$

It remains to realize that

$$\left(\int_{\mathbb{R}^d}\frac{\chi_{0,k}\,d\xi}{|(|\xi|^2-z)^{2\zeta/3}|^{2p}}\right)^{1/(2p)}\le\widetilde{C}_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\left(\frac{|k|}{R}\right)^{d/(2p)-2\gamma/3},$$

while

$$\left(\int_{\mathbb{R}^d}\frac{\chi_{0,k}\,d\xi}{|(|\xi|^2-z)^{\zeta/3}|^{4p}}\right)^{1/(4p)}\le\widetilde{C}_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\left(\frac{|k|}{R}\right)^{d/(4p)-\gamma/3}.$$

Finally,

$$\left(\int_{\mathbb{R}^d}\frac{\chi_{1,k}\,d\xi}{|(|\xi|^2-z)^{\zeta}|^{4p}}\right)^{1/(4p)}\le 2e^{c|\operatorname{Im}\zeta|}\left(\int_{|\xi|>2|k|/R}\frac{d\xi}{\left(\frac{3}{4}|\xi|^2\right)^{2\gamma p}}\right)^{1/(4p)}\le\widetilde{C}_{p,\gamma}e^{c|\operatorname{Im}\zeta|}\left(\frac{|k|}{R}\right)^{d/(4p)-\gamma}. \qquad \square$$

Let us now talk about the operator $Y(\zeta)$. The study of this operator must be harder compared to the study of $X_{1,1}(\zeta)$ simply because $P_{1,k}(-\Delta-z)^{-\zeta}$ is bounded uniformly in z while this is not true about $P_{0,k}(-\Delta-z)^{-\zeta}$.

Corollary 8.3. *Let $2 \le d \le 5$. Let $|k| \ge R$ where $0 < R \le 1$. Let also $W = \sqrt{\widetilde{V}}$. Assume that $2 \le 2p < 3/\gamma$ and $4p\gamma > d$. If $\operatorname{Re}\zeta = \frac{1}{2}\gamma$ and $\widetilde{V} \in L^{2p}$, then*

$$\mathbb{E}(\|Y(\zeta)\|_{\mathfrak{S}_p}^p)^{1/p} \le C_{p,\gamma} e^{-\alpha_0|\operatorname{Im}\zeta|^2/2} \left(\frac{|k|}{R}\right)^{3d/(2p)-2\gamma} \|\widetilde{V}\|_{2p}^2.$$

In particular, we can set $p = 1$ and prove the following statement.

Proposition 8.4. *Let $2 \le d \le 5$. Let $|k| \ge R$ where $0 < R \le 1$. Let also $W = \sqrt{\widetilde{V}}$. Assume that*

$$\tfrac{1}{8}d < \tfrac{1}{2}\gamma = \operatorname{Re}\zeta < \tfrac{3}{4}.$$

Then

$$\mathbb{E}(\|Y(\zeta)\|_{\mathfrak{S}_1}) \le C_{\operatorname{Re}\zeta} e^{-\alpha_0|\operatorname{Im}\zeta|^2/2} \left(\frac{|k|}{R}\right)^{3d/2-4\operatorname{Re}\zeta} \|\widetilde{V}\|_2^2.$$

9. Another interpolation between small and large values of Re ζ

Let us recall two theorems that hold for the operator

$$Y(\zeta) = X_{0,0}(\zeta) + X_{0,1}(\zeta) + X_{1,0}(\zeta),$$

with $W = \widetilde{V}^{1/2}$. By small values of $\operatorname{Re}\zeta$ we mean the values that are considered in Corollary 8.3, which states that, for any $p \ge 1$ and $d/(8p) < \operatorname{Re}\zeta < 3/(4p)$,

$$\mathbb{E}(\|Y(\zeta)\|_{\mathfrak{S}_p}^p)^{1/p} \le C_{\operatorname{Re}\zeta,p} e^{-\alpha_0|\operatorname{Im}\zeta|^2/2} \left(\frac{|k|}{R}\right)^{3d/(2p)-4\operatorname{Re}\zeta} \|\widetilde{V}\|_{2p}^2. \tag{9.1}$$

In this corollary, we had to assume that $2 \le d \le 5$ and $|k| \ge R$, where $0 < R \le 1$. One should also not forget that our assumptions about $\gamma = 2\operatorname{Re}\zeta$ imply that $\operatorname{Re}\zeta < \frac{3}{4}$.

In the next result, we only replace $4\operatorname{Re}\zeta$ by $d/(2p)$ in the right-hand side of (9.1).

Theorem 9.1. *Let $2 \le d \le 5$. Let $W = \widetilde{V}^{1/2}$. Let*

$$0 < \operatorname{Re}\zeta < \tfrac{3}{4}.$$

Assume that

$$\frac{d}{8\operatorname{Re}\zeta} < p < \frac{3}{4\operatorname{Re}\zeta}, \qquad p \ge 1,$$

and $0 < R \le 1$. Then

$$\mathbb{E}(\|Y(\zeta)\|_{\mathfrak{S}_p}^p)^{1/p} \le C_{\operatorname{Re}\zeta,p} e^{-\alpha_0|\operatorname{Im}\zeta|^2/2} \left(\frac{|k|}{R}\right)^{d/p} \|\widetilde{V}\|_{2p}^2,$$

for $|k| \ge R$.

For the sake of simplicity, we choose

$$p = \frac{d}{7\operatorname{Re}\zeta}.$$

In this case, because of the assumption $p \ge 1$ that we made, we have to assume that

$$0 < \operatorname{Re}\zeta \le \tfrac{1}{7}d.$$

Note that $\frac{1}{7}d < \frac{3}{4}$. Thus, we can formulate the following assertion.

Corollary 9.2. *Let $2 \leq d \leq 5$. Let $0 < \operatorname{Re}\zeta \leq \frac{1}{7}d$ and let $p = d/(7\operatorname{Re}\zeta)$. Assume that $0 < R \leq 1$. Then*

$$\mathbb{E}(\|Y(\zeta)\|_{\mathfrak{S}_p}^p)^{1/p} \leq C_{\operatorname{Re}\zeta,p} e^{-\alpha_0 |\operatorname{Im}\zeta|^2/2} \left(\frac{|k|}{R}\right)^{d/p} \|\widetilde{V}\|_{2p}^2,$$

for $|k| \geq R$.

By the large values of $\operatorname{Re}\zeta$ we mean the values appearing in Corollary 7.6. We will use only a simpler version of this result.

Theorem 9.3. *Let $d \geq 3$. Let $1 < \nu < \eta < 2$. Let*

$$2\operatorname{Re}\zeta = \frac{d}{2} + \frac{d-\eta}{2(d-2)}. \tag{9.2}$$

Assume that $V \in L^2(\mathbb{R}^d)$ and $\alpha_0 > 2\alpha$. Then

$$(\mathbb{E}(\|Y(\zeta)\|_{\mathfrak{S}_2}^2))^{1/2} \leq C_{\operatorname{Re}\zeta} e^{(2\alpha-\alpha_0)(\operatorname{Im}\zeta)^2} |k|^{-\nu/2} \|\widetilde{V}\|_2^2. \tag{9.3}$$

Proof. For $\operatorname{Re}\zeta$ defined in (9.2), the assumption $\nu < \eta < 2$ leads to the inequalities

$$\frac{d+1}{2} < 2\operatorname{Re}\zeta < \frac{d(d-1)-\nu}{2(d-2)}. \tag{9.4}$$

Let us now introduce the parameter $\varkappa$, setting

$$\varkappa = \frac{2(d-1)\operatorname{Re}\zeta}{4\operatorname{Re}\zeta - \nu}.$$

The latter relation simply means that

$$\nu = \left(2 - \frac{(d-1)}{\varkappa}\right) 2\operatorname{Re}\zeta. \tag{9.5}$$

Thus (9.3) coincides with (7.11). Let us check that all conditions of Corollary 7.6 are fulfilled. The second inequality in (9.4) implies

$$\varkappa > \frac{d(d-1)-\nu}{2(d-\nu)} > \frac{d-1}{2}, \tag{9.6}$$

while the first inequality in (9.4) combined with the condition $\nu < 2$ implies that

$$\varkappa < \frac{d+1}{2}.$$

One can also see that the first inequality in (9.6) is equivalent to the estimate

$$2\operatorname{Re}\zeta = \frac{\varkappa\nu}{2\varkappa-(d-1)} < \frac{d\varkappa}{2\varkappa-1}.$$

Finally, note that when $d \geq 3$, the condition $\operatorname{Re}\zeta < \varkappa$ follows from the fact that $\nu + \eta > 2$. Consequently, Corollary 7.6 implies Theorem 9.3. □

We interpolate between Corollary 9.2 and Theorem 9.3.

Theorem 9.4. *Let $3 \le d \le 5$. Assume that τ_1 satisfies*

$$0 \le \left(\left(\frac{d}{2} + \frac{(\eta-1)(d+1)}{7d} + \frac{d-\eta}{2(d-2)}\right) - 2\right)\tau_1 \le \frac{(\nu-1)(d+1)}{7d}, \tag{9.7}$$

with η and ν such that $1 < \nu < \eta < 2$. If $d = 3$, then we assume additionally that $8\nu + 9\eta < 26$. Let p, q and r be the numbers defined by

$$p = \frac{d}{7\tau_1}, \quad \frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2} \quad \text{and} \quad \frac{1}{r} = \frac{1-\theta}{2p} + \frac{\theta}{2}, \tag{9.8}$$

where θ is the solution of the equation

$$\tau_1(1-\theta) + \frac{\theta}{2}\left(\frac{d}{2} + \frac{d-\eta}{2(d-2)}\right) = 1. \tag{9.9}$$

Then

$$(\mathbb{E}(\|Y(1)\|^q_{\mathfrak{S}_q}))^{1/q} \le C_q \left(\frac{|k|}{R}\right)^{d(1-\theta)/p} |k|^{-\theta\nu/2} \|\widetilde{V}\|_r^2, \tag{9.10}$$

for $|k| \ge R$ and $0 < R \le 1$.

Proof. Observe that

$$\tau_1 < \begin{cases} \dfrac{2(\nu-1)(d+1)}{7(d-3)d} \le \dfrac{d}{7} & \text{if } d > 3, \\ \dfrac{8(\nu-1)}{21(2-\eta)} \le \dfrac{d}{7} & \text{if } 8\nu + 9\eta < 26 \text{ and } d = 3. \end{cases}$$

In both cases, τ_1 obeys

$$0 < \tau_1 \le \tfrac{1}{7}d.$$

Consider $Y(\zeta)$ for ζ running over the strip

$$\tau_1 \le \operatorname{Re}\zeta \le \frac{d}{4} + \frac{d-\eta}{4(d-2)}.$$

Since we have some information about the values of this function on the boundary of the strip, we obtain (9.10) by interpolation between Corollary 9.2 and Theorem 9.3. □

Remark. We need to explain why the parameters were selected as described in Theorem 9.4. The work with perturbation determinants requires convergence of integrals of the form

$$\int_\varepsilon^\infty \mathbb{E}(\|Y(1)\|^q_{\mathfrak{S}_q})\,dk, \qquad \varepsilon > 0,$$

so we need the parameters to satisfy the condition

$$\frac{qd(1-\theta)}{p} - \frac{q\theta\nu}{2} < -1,$$

which is equivalent to the inequality

$$\tau_1(1-\theta) < \frac{\theta\nu}{14} - \frac{1}{7q} = \frac{\theta(\nu-1)}{14} - \frac{(1-\theta)\tau_1}{d},$$

implying that

$$\tau_1(1-\theta) < \frac{\theta(\nu-1)(d+1)}{14d}.$$

The latter can be written differently as

$$1-\frac{\theta}{2}\left(\frac{d}{2}+\frac{d-\eta}{2(d-2)}\right) < \frac{\theta(\nu-1)(d+1)}{14d}.$$

In other words,

$$2 < \theta\left(\frac{d}{2}+\frac{(\nu-1)(d+1)}{7d}+\frac{d-\eta}{2(d-2)}\right). \tag{9.11}$$

The condition that θ is large can be converted into an inequality showing that τ_1 is small. The relation (9.11) is satisfied if

$$\left(\left(\frac{d}{2}+\frac{(\nu-1)(d+1)}{7d}+\frac{d-\eta}{2(d-2)}\right)-2\right)\tau_1 < \frac{(\nu-1)(d+1)}{7d}.$$

Since $\eta > \nu$, this condition is obviously fulfilled if (9.7) holds.

In the next statement, we estimate the remainder $X_{1,1}(\zeta)$ for $\zeta = 1$.

Theorem 9.5. *Let $p > \frac{3}{4}d \geq 2$, and let $\zeta = 1$. Then*

$$\mathbb{E}[\|X_{1,1}(\zeta)\|^p_{\mathfrak{S}_{p/2}}]^{1/p} \leq C\left(\frac{|k|}{R}\right)^{-4}\|\widetilde{V}\|_p^2.$$

Proof. In this theorem, we deal with the operator

$$W(-\Delta-z)^{-1}P_{1,k}V(-\Delta-z)^{-1}P_{1,k}W.$$

On the one hand, we see that

$$\mathbb{E}[\|(-\Delta-z)^{-2/3}P_{1,k}V(-\Delta-z)^{-2/3}P_{1,k}\|^p_{\mathfrak{S}_p}]^{1/p} \leq C\left(\int_{|\xi|>2|k|/R}||\xi|^2-z|^{-2p/3}\,d\xi\right)^{2/p}\|\widetilde{V}\|_p,$$

which implies the inequality

$$\mathbb{E}[\|(-\Delta-z)^{-2/3}P_{1,k}V(-\Delta-z)^{-2/3}P_{1,k}\|^p_{\mathfrak{S}_p}]^{1/p} \leq C\left(\frac{|k|}{R}\right)^{-8/3}\|\widetilde{V}\|_p, \quad p > \tfrac{3}{4}d.$$

On the other hand,

$$\|W(-\Delta-z)^{-1/3}P_{1,k}\|^2_{\mathfrak{S}_{2p}} \leq C\left(\frac{|k|}{R}\right)^{-4/3}\|\widetilde{V}\|_p, \quad p > \tfrac{3}{4}d.$$

Consequently,

$$\mathbb{E}[\|W(-\Delta-z)^{-1}P_{1,k}V(-\Delta-z)^{-1}P_{1,k}W\|^p_{\mathfrak{S}_{p/2}}]^{1/p} \leq C\left(\frac{|k|}{R}\right)^{-4}\|\widetilde{V}\|_p^2, \quad p > \tfrac{3}{4}d. \qquad \square$$

The next statement follows by Hölder's inequality.

Corollary 9.6. *Let $q > \frac{3}{8}d \geq 1$, and let $\zeta = 1$. Then*

$$\mathbb{E}[\|X_{1,1}(\zeta)\|_{\mathfrak{S}_q}^q]^{1/q} \leq C\left(\frac{|k|}{R}\right)^{-4} \|\widetilde{V}\|_{2q}^2.$$

Surprisingly, q in (9.8) satisfies the inequality $q > \frac{3}{8}d \geq 1$. Thus, we obtain the following result.

Theorem 9.7. *Let $3 \leq d \leq 5$. Assume that τ_1 satisfies (9.7) with η and ν such that $1 < \nu < \eta < 2$. If $d = 3$, then we assume additionally that $8\nu + 9\eta < 26$. Let p, q and r be the numbers defined by*

$$p = \frac{d}{7\tau_1}, \quad \frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2} \quad \text{and} \quad \frac{1}{r} = \frac{1-\theta}{2p} + \frac{\theta}{2}, \tag{9.12}$$

where θ is the solution of the equation

$$\tau_1(1-\theta) + \frac{\theta}{2}\left(\frac{d}{2} + \frac{d-\eta}{2(d-2)}\right) = 1. \tag{9.13}$$

Then

$$(\mathbb{E}(\|X(k)\|_{\mathfrak{S}_q}^q))^{1/q} \leq C_q\left[\left(\frac{|k|}{R}\right)^{d(1-\theta)/p} |k|^{-\theta\nu/2} + \left(\frac{|k|}{R}\right)^{-4}\right] \|\widetilde{V}\|_r^2,$$

for $|k| \geq R$ and $0 < R \leq 1$.

10. Proof of Theorem 1.5

Again, we work with the function

$$d(z) = \det_n(I - X(k)), \quad n = [q] + 1,$$

where z is related to k via the Joukowski mapping

$$z = \frac{R}{k} + \frac{k}{R}, \quad R > 0.$$

Standard arguments allow us to rewrite (5.2) with p replaced by q as

$$\mathbb{E}\left[\sum_j \frac{\operatorname{Im} k_j(|k_j|^2 - R^2)_+}{|k_j|^2 R}\right] \leq C\left(\int_{-\infty}^{\infty} \mathbb{E}[\|X(k)\|_{\mathfrak{S}_q}^q]\left(\frac{1}{R} - \frac{R}{k^2}\right)_+ dk + \int_0^\pi \mathbb{E}[\|X(R\cdot e^{i\theta})\|_{\mathfrak{S}_q}^q] \sin\theta \, d\theta\right),$$

where the k_j are defined as square roots of eigenvalues of H. Due to Theorem 9.7, the latter inequality yields

$$\mathbb{E}\left[\sum_j \frac{\operatorname{Im} k_j(|k_j|^2 - R^2)_+}{|k_j|^2 R}\right] \leq C|R|^{-\theta q\nu/2} \|\widetilde{V}\|_r^{2q}. \tag{10.1}$$

Now, suppose that we consider only the eigenvalues $\lambda_j = k_j^2$ that satisfy the inequality

$$|k_j| \leq R_0.$$

Multiplying (10.1) by $R^{\sigma-1}$ and integrating with respect to R from 0 to R_0, we obtain

$$\mathbb{E}\left[\sum_{|k_j| \leq R_0} \operatorname{Im} k_j |k_j|^{\sigma-1}\right] \leq C|R_0|^{\sigma-\theta q\nu/2} \|\widetilde{V}\|_r^{2q}, \quad \sigma > \tfrac{1}{2}\theta q\nu. \qquad \square$$

References

[Abramov et al. 2001] A. A. Abramov, A. Aslanyan, and E. B. Davies, "Bounds on complex eigenvalues and resonances", *J. Phys. A* **34**:1 (2001), 57–72. MR Zbl

[Bögli 2017] S. Bögli, "Schrödinger operator with non-zero accumulation points of complex eigenvalues", *Comm. Math. Phys.* **352**:2 (2017), 629–639. MR Zbl

[Borichev et al. 2009] A. Borichev, L. Golinskii, and S. Kupin, "A Blaschke-type condition and its application to complex Jacobi matrices", *Bull. Lond. Math. Soc.* **41**:1 (2009), 117–123. MR Zbl

[Briet et al. 2021] P. Briet, J.-C. Cuenin, L. Golinskii, and S. Kupin, "Lieb–Thirring inequalities for an effective Hamiltonian of bilayer graphene", *J. Spectr. Theory* **11**:3 (2021), 1145–1178. MR Zbl

[Cuenin 2017] J.-C. Cuenin, "Eigenvalue bounds for Dirac and fractional Schrödinger operators with complex potentials", *J. Funct. Anal.* **272**:7 (2017), 2987–3018. MR Zbl

[Cuenin et al. 2014] J.-C. Cuenin, A. Laptev, and C. Tretter, "Eigenvalue estimates for non-selfadjoint Dirac operators on the real line", *Ann. Henri Poincaré* **15**:4 (2014), 707–736. MR Zbl

[Davies and Nath 2002] E. B. Davies and J. Nath, "Schrödinger operators with slowly decaying potentials", *J. Comput. Appl. Math.* **148**:1 (2002), 1–28. MR Zbl

[Demuth and Katriel 2008] M. Demuth and G. Katriel, "Eigenvalue inequalities in terms of Schatten norm bounds on differences of semigroups, and application to Schrödinger operators", *Ann. Henri Poincaré* **9**:4 (2008), 817–834. MR Zbl

[Demuth et al. 2009] M. Demuth, M. Hansmann, and G. Katriel, "On the discrete spectrum of non-selfadjoint operators", *J. Funct. Anal.* **257**:9 (2009), 2742–2759. MR Zbl

[Frank 2011] R. L. Frank, "Eigenvalue bounds for Schrödinger operators with complex potentials", *Bull. Lond. Math. Soc.* **43**:4 (2011), 745–750. MR Zbl

[Frank 2018] R. L. Frank, "Eigenvalue bounds for Schrödinger operators with complex potentials, III", *Trans. Amer. Math. Soc.* **370**:1 (2018), 219–240. MR Zbl

[Frank and Sabin 2017] R. L. Frank and J. Sabin, "Restriction theorems for orthonormal functions, Strichartz inequalities, and uniform Sobolev estimates", *Amer. J. Math.* **139**:6 (2017), 1649–1691. MR Zbl

[Frank et al. 2006] R. L. Frank, A. Laptev, E. H. Lieb, and R. Seiringer, "Lieb–Thirring inequalities for Schrödinger operators with complex-valued potentials", *Lett. Math. Phys.* **77**:3 (2006), 309–316. MR Zbl

[Hansmann 2011] M. Hansmann, "An eigenvalue estimate and its application to non-selfadjoint Jacobi and Schrödinger operators", *Lett. Math. Phys.* **98**:1 (2011), 79–95. MR Zbl

[Hansmann 2017] M. Hansmann, "Some remarks on upper bounds for Weierstrass primary factors and their application in spectral theory", *Complex Anal. Oper. Theory* **11**:6 (2017), 1467–1476. MR Zbl

[Helffer and Robert 1990] B. Helffer and D. Robert, "Riesz means of bound states and semiclassical limit connected with a Lieb–Thirring's conjecture", *Asymptot. Anal.* **3**:2 (1990), 91–103. MR Zbl

[Korotyaev 2020] E. Korotyaev, "Trace formulas for Schrödinger operators with complex potentials on a half line", *Lett. Math. Phys.* **110**:1 (2020), 1–20. MR Zbl

[Korotyaev and Laptev 2018] E. Korotyaev and A. Laptev, "Trace formulae for Schrödinger operators with complex-valued potentials on cubic lattices", *Bull. Math. Sci.* **8**:3 (2018), 453–475. MR Zbl

[Korotyaev and Safronov 2020] E. Korotyaev and O. Safronov, "Eigenvalue bounds for Stark operators with complex potentials", *Trans. Amer. Math. Soc.* **373**:2 (2020), 971–1008. MR Zbl

[Laptev and Safronov 2009] A. Laptev and O. Safronov, "Eigenvalue estimates for Schrödinger operators with complex potentials", *Comm. Math. Phys.* **292**:1 (2009), 29–54. MR Zbl

[Laptev and Weidl 2000] A. Laptev and T. Weidl, "Sharp Lieb–Thirring inequalities in high dimensions", *Acta Math.* **184**:1 (2000), 87–111. MR Zbl

[Lieb and Thirring 1976] E. H. Lieb and W. E. Thirring, "Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities", pp. 269–303 in *Studies in mathematical physics*, edited by E. H. Lieb et al., Princeton Univ. Press, 1976. Zbl

[Pavlov 1966] B. S. Pavlov, "The nonself-adjoint Schrödinger operator", pp. 102–132 in *Problems of mathematical physics*, *I*: *Spectral theory and wave processes*, edited by M. S. Birman, Leningrad Univ. Press, 1966. In Russian; translated as pp. 87–114 in *Topics in mathematical physics, I*: *Spectral theory and wave processes*, Consultants Bureau, New York, 1967. MR

[Pavlov 1967] B. S. Pavlov, "The non-selfadjoint Schrödinger operator, II", pp. 133–157 in *Problems of mathematical physics*, *II*: *Spectral theory and problems in diffraction*, edited by M. S. Birman, Leningrad Univ. Press, 1967. In Russian; translated as pp. 111–134 in *Topics in mathematical physics*, *II*: *Spectral theory and problems in diffraction*, Consultants Bureau, New York, 1968. MR

[Safronov and Vainberg 2008] O. Safronov and B. Vainberg, "Estimates for negative eigenvalues of a random Schrödinger operator", *Proc. Amer. Math. Soc.* **136**:11 (2008), 3921–3929. MR Zbl

[Seiler and Simon 1975] E. Seiler and B. Simon, "Bounds in the Yukawa$_2$ quantum field theory: upper bound on the pressure, Hamiltonian bound and linear lower bound", *Comm. Math. Phys.* **45**:2 (1975), 99–114. MR

Received 22 Feb 2021. Revised 10 Aug 2021. Accepted 26 Oct 2021.

OLEG SAFRONOV: osafrono@uncc.edu
Mathematics and Statistics, The University of North Carolina at Charlotte, Charlotte, NC, United States

ANALYSIS AND PDE
Vol. 16 (2023), No. 4, pp. 1061–1088
DOI: 10.2140/apde.2023.16.1061

CARLESON MEASURE ESTIMATES FOR CALORIC FUNCTIONS AND PARABOLIC UNIFORMLY RECTIFIABLE SETS

SIMON BORTZ, JOHN HOFFMAN, STEVE HOFMANN,
JOSÉ LUIS LUNA GARCÍA AND KAJ NYSTRÖM

Let $E \subset \mathbb{R}^{n+1}$ be a parabolic uniformly rectifiable set. We prove that every bounded solution u to

$$\partial_t u - \Delta u = 0 \quad \text{in } \mathbb{R}^{n+1} \setminus E$$

satisfies a Carleson measure estimate condition. An important technical novelty of our work is that we develop a corona domain approximation scheme for E in terms of regular Lip(1/2, 1) graph domains. This scheme has an analogous elliptic version which improves on the known results in that setting.

1. Introduction

For more than forty years, there has been significant interest in quantitative estimates for solutions of (linear) elliptic and parabolic partial differential equations in the absence of smoothness. In this area of research, the lack of smoothness presents itself in the structure or regularity of the coefficients of the operator, or in the geometry of the domain. Recently, sustained efforts in this area have provided characterizations of quantitative geometric notions (e.g., uniform rectifiability) in terms of quantitative estimates for harmonic functions [Garnett et al. 2018; Hofmann et al. 2016] and a geometric characterization of the L^p-solvability of the Dirichlet problem [Azzam et al. 2020]. This paper concerns the parabolic analogue of [Hofmann et al. 2016] and overcomes the substantial difficulty introduced by the distinguished time direction and the anisotropic scaling. To deal with this difficulty, we are forced to build appropriate approximating domains with better properties than would be enjoyed by the parabolic analogues of the chord-arc domains constructed in that paper. In particular, our construction improves on that of [Hofmann et al. 2016], even in the elliptic setting. We shall discuss these issues in more detail momentarily.

We shall prove the following.

Theorem 1.1 (a Carleson measure estimate for bounded caloric functions). *Let $n \geq 2$. Let $E \subset \mathbb{R}^{n+1}$ be a set which is uniformly rectifiable in the parabolic sense. Then for any solution to $(\partial_t - \Delta_X)u = 0$ in $\mathbb{R}^{n+1} \setminus E$ with $u \in L^\infty(\mathbb{R}^{n+1} \setminus E)$,*

$$\sup_{(t,X)\in E, r>0} r^{-n-1} \iint_{B((t,X),r)} |\nabla u|^2 \delta(s, Y)\, dY\, ds \leq C\|u\|^2_{L^\infty(E^c)}, \tag{1.2}$$

where $\delta(s, Y) := \operatorname{dist}((s, Y), E)$ and C depends only on the dimension and the parabolic uniformly rectifiable constants for E.

MSC2020: primary 28A75; secondary 28A78, 35K10, 42B37.
Keywords: parabolic equations, parabolic uniform rectifiability, Carleson measures.

Here and below, $\operatorname{dist}((s, Y), E)$ is the parabolic distance from (s, Y) to the given set E, and the ball $B((t, X), r)$ is defined with respect to the parabolic metric; see (2.1) and (2.2) below.

In the case that Ω is an open set, the following holds.

Theorem 1.3. *Let $n \geq 2$. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set for which $\partial\Omega$ is uniformly rectifiable in the parabolic sense. Then for any solution to $(\partial_t - \Delta_X)u = 0$ in Ω with $u \in L^\infty(\Omega)$,*

$$\sup_{(t,X)\in E, r>0} r^{-n-1} \iint_{B((t,X),r)\cap\Omega} |\nabla u|^2 \delta(s, Y)\, dY\, ds \leq C\|u\|^2_{L^\infty(\Omega)},$$

where C depends only the dimension and the parabolic uniformly rectifiable constants for $\partial\Omega$. Here, $\delta(s, Y) := \operatorname{dist}((s, Y), \partial\Omega)$, the parabolic distance to $\partial\Omega$.

The notion of parabolic uniform rectifiability was introduced in [Hofmann et al. 2003; 2004] and is defined below, but we first provide some context here. Through the works of Hofmann, Lewis, Murray and Silver [Hofmann 1995; 1997; Hofmann and Lewis 1996; 2005; Lewis and Murray 1995; Lewis and Silver 1988], it was shown that the *good* parabolic graphs for parabolic singular integrals and parabolic potential theory are regular $\operatorname{Lip}(1/2, 1)$ graphs, that is, graphs which are $\operatorname{Lip}(1/2, 1)$ (in time-space coordinates) and which possess extra regularity in time in the sense that a (nonlocal) half-order time derivative of the defining function of the graph is in the space of functions of parabolic bounded mean oscillation. This is in contrast to the elliptic setting, where one often views Lipschitz graphs as the *good* graphs for singular integrals and potential theory (because of [Coifman et al. 1982; 1983; Coifman and Semmes 1991; Dahlberg 1977; David 1991]), and where the BMO estimate for the gradient is an automatic consequence of Rademacher's theorem and the inclusion of L^∞ in BMO. The definition of parabolic uniform rectifiability in [Hofmann et al. 2003; 2004] is given in terms of parabolic β *numbers*,[1] but we do not work with this definition directly here. Instead, we work with an equivalent notion of parabolic uniform rectifiability in terms of the existence of appropriate corona decompositions recently established in [Bortz et al. 2023; 2022a]. However, it is worth remarking that the graph of a $\operatorname{Lip}(1/2, 1)$ function is parabolic uniformly rectifiable if and only if the function has a half-order time derivative in parabolic BMO. In contrast to the case of *elliptic* uniform rectifiability, which has reached a state of maturity that includes numerous interesting characterizations, this is not the case for parabolic uniform rectifiability. In fact, beyond [Hofmann et al. 2003; 2004], the only correct and more systematic studies of parabolic uniformly rectifiable sets can be found in [Bortz et al. 2023; 2022a].[2] In these works, parabolic uniform rectifiability is characterized in terms of a bilateral coronization by regular $\operatorname{Lip}(1/2, 1)$ graphs (Lemma 2.14), and this characterization is the starting point for the analysis in this paper. In general there are many interesting open problems in this and related areas, and it should be emphasized that parabolic uniform rectifiability is *significantly* different to its elliptic counterpart; see [Bortz et al. 2022a, Observation 4.19].

[1] These β numbers can be traced back to the work of P. Jones [1990].

[2] There are works of J. Rivera-Noriega in this area, but these articles have significant gaps or no proofs. Some of these gaps are outlined in [Bortz et al. 2023; 2022a; 2022b].

To give an idea of the methods involved in the proof of Theorem 1.1, the primary novelty of our work is a *corona domain approximation scheme* (Proposition 3.25) in terms of regular Lip($1/2$, 1) graph *domains*. This is in contrast to the (elliptic) NTA domain approximations produced in [Hofmann et al. 2016] for uniformly rectifiable sets. In fact, our proof here carries over without modification to the elliptic setting,[3] providing an (improved) approximation by Lipschitz domains. In [Hofmann et al. 2016] the authors use Whitney cubes to construct these NTA domains using *dyadic sawtooths* and exploiting an elliptic bilateral corona decomposition. The heuristic in the elliptic setting is that these *sawtooth domains* inherit many essential properties of the original boundary. In contrast, in the parabolic setting the analogous constructions do not necessarily inherit even the most basic properties. One of the most readily apparent difficulties in the parabolic setting comes from the fact that the natural *lower dimensional parabolic measure* can easily fail to see relatively nice sets. In particular, given a cube (with respect to the standard coordinates) in $\mathbb{R}^{n+1}$, two of the faces (those orthogonal to the time axis) have zero natural *parabolic surface measure*, which says that, not only does the boundary of a cube fail to be uniformly rectifiable in the parabolic sense, it fails even to have the Ahlfors–David regularity property. The method outlined in this paper circumvents this difficulty by *lifting* the graphs in the parabolic bilateral corona decomposition (Lemma 2.14) in a manner that respects the stopping time regimes and thereby produces the graph domains rather directly. We also point out that, while the analogous elliptic results (in [Hofmann et al. 2016]) proceed along the lines of *extrapolation of Carleson measures* it was later seen in [Hofmann et al. 2019] that this was unnecessary and a more direct approach is available. Therefore, upon proving Proposition 3.25, the proof of Theorem 1.1 proceeds as in [Hofmann et al. 2019].

Let us provide some motivation for the estimate in Theorem 1.1. As remarked above, (elliptic) uniform rectifiability has been characterized by various properties of harmonic functions and among these characterizations is the elliptic version of the Carleson measure estimate in Theorem 1.1; see [Garnett et al. 2018; Hofmann et al. 2016]. We therefore expect that the estimate in Theorem 1.1 is a significant step in characterizing parabolic uniform rectifiability by properties of caloric functions. We suspect that additional considerations and conditions will need to be made, as was the case for nonsymmetric operators in the elliptic setting [Azzam et al. 2022], in the converse, *free-boundary* direction due to the lack of self-adjointness of the heat operator. In domains that are sufficiently nice topologically, the estimate in Theorem 1.1 (and its elliptic analogue) is also intimately tied to the solvability of the L^p-Dirichlet boundary value problem in the parabolic setting [Dindoš et al. 2017; Genschaw and Hofmann 2020] (see [Kenig et al. 2000; 2016] and related work in [Dindoš et al. 2011; Hofmann and Le 2018; Zhao 2018] for the elliptic theory). Indeed, in the case of regular Lip($1/2$, 1) graph domains it is known that estimate (1.2) for bounded null-solutions to general parabolic operators of the form $\mathcal{L} = \partial_t - \operatorname{div}_X A\nabla_X$ is equivalent to the solvability of the L^p-Dirichlet boundary value problem for some $p > 1$ [Dindoš et al. 2017] (boundary value problem means the data is prescribed on the lateral boundary). In fact, merely assuming parabolic Ahlfors–David regularity[4] and a *backwards thickness condition* (also of Ahlfors–David regular type), the solvability of the L^p-Dirichlet boundary value problem is implied by a stronger estimate where the

[3] Except that the technical Lemma 3.24 is no longer needed.

[4] In particular, without assuming that the domain is the region above a regular Lip($1/2$, 1) graph.

L^∞-norm on the right-hand side of (1.2) is replaced by the (boundary) BMO norm of the data; see [Genschaw and Hofmann 2020]. This stronger estimate is unlikely to hold[5] in the present setting due to the lack of (nontangential) accessibility to the boundary.

The rest of this paper is organized as follows. In Section 2 we introduce the notions and notation used throughout the paper. In Section 3 we construct approximating domains, each adapted to a particular stopping time regime in the parabolic bilateral corona decomposition, Lemma 2.14. In Section 4 we prove the main theorems of the paper (Theorems 1.1 and 1.3) using the constructions produced in Section 3. In Section 5 we discuss some possible extensions of the results here.

2. Preliminaries

In this paper, we work in $\mathbb{R}^{n+1}$ identified with $\mathbb{R}\times\mathbb{R}^n = \{(t, X) : t \in \mathbb{R},\ X \in \mathbb{R}^n\}$ and $n \in \mathbb{N},\ n \geq 2$.[6] We use the notation

$$\operatorname{dist}(A, B) := \inf_{(t,X)\in A,(s,Y)\in B} |X-Y| + |t-s|^{1/2} \tag{2.1}$$

to denote the parabolic distance between A and B, with $A, B \subseteq \mathbb{R}^{n+1}$. We also use the notation $B((t, X), r)$ for the parabolic ball centered at (t, X) with radius $r > 0$, that is,

$$B((t, X), r) := \{(s, Y) : \operatorname{dist}((t, X),\ (s, Y)) < r\}. \tag{2.2}$$

Given $E \subset \mathbb{R}^{n+1}$ we let $\operatorname{diam}(E)$ denote the diameter, or parabolic diameter, defined with respect to the parabolic metric.

Definition 2.3 (parabolic Hausdorff measure). Given $s > 0$ we let $\mathcal{H}^s_p$ denote the s-dimensional parabolic Hausdorff measure. More specifically, for a set $E \subset \mathbb{R}^{n+1}$ and $\epsilon > 0$ we define

$$\mathcal{H}^s_{p,\epsilon}(E) := \inf\Big\{\sum_i \operatorname{diam}(E_i)^s : E \subseteq \bigcup_i E_i,\ \operatorname{diam}(E_i) \leq \epsilon\Big\},$$
$$\mathcal{H}^s_p(E) := \lim_{\epsilon\to 0^+} \mathcal{H}^s_{p,\epsilon}(E) = \limsup_{\epsilon\to 0^+} \mathcal{H}^s_{p,\epsilon}(E).$$

The following family of planes will be important in this work.

Definition 2.4 (t-independent planes). We say that an n-dimensional plane P in $\mathbb{R}^{n+1}$ is t-independent if it contains a line in the t-direction. Equivalently, if $\vec{\nu}$ is the normal vector to P, then $\vec{\nu}\cdot(1, \vec{0}) = 0$.

The following local energy (Caccioppoli) inequality holds for solutions to the heat equation.

Lemma 2.5 (Caccioppoli inequality). *Let $B = B((t, X), r)$, and suppose that $u \in (1+\alpha)B$ is a solution to $(\partial_t - \Delta_X)u = 0$ for some $\alpha > 0$. Then*

$$\int_B |\nabla_X u(t, X)|^2\, dX\, dt \lesssim r^{-2} \int_{(1+\alpha)B} |u|^2\, dX\, dt,$$

where the implicit constant depends on the dimension and α.

[5]The elliptic analogue does not hold (in general) in the complement of uniformly rectifiable set.

[6]We apologize for the departure from the usual notation (X, t), but we will often be working with graphs and it is convenient to have the *last* variable as the graph variable.

Definition 2.6 (Ahlfors–David regular). We say $E \subset \mathbb{R}^{n+1}$ is (parabolic) Ahlfors–David regular, written E *is* ADR, if it is closed and there exists a constant $C > 0$ such that

$$C^{-1}r^{n+1} \le \mathcal{H}_p^{n+1}(B((t,X),r)\cap E) \le Cr^{n+1}, \quad \forall (t,X)\in E, \quad r\in(0,\operatorname{diam}(E)).$$

We will call the C of Definition 2.6 the Ahlfors–David regularity constant and if a particular constant depends on the Ahlfors–David regularity constant, we will say that the constant *depends on* ADR. We will sometimes write $\sigma := \mathcal{H}_p^{n+1}|_E$ to denote the *surface measure* on E. (The underlying set defining σ will always be clear from the context.)

An ADR set E can be viewed as a space of homogeneous type, $(E, \operatorname{dist}, \sigma)$, with homogeneous dimension $n+1$. All such sets have a nice filtration, which we will refer to as the *dyadic cubes* on E.

Lemma 2.7 [Christ 1990; David and Semmes 1991; 1993; Hytönen and Kairema 2012; Hytönen and Martikainen 2012]. *Assume that $E \subset \mathbb{R}^{n+1}$ is* (*parabolic*) ADR *in the sense of Definition 2.6 with constant C. Then E admits a parabolic dyadic decomposition in the sense that there exist constants $a_0 > 0$, $\gamma > 0$, and $c_* < \infty$, such that for each $k \in \mathbb{Z}$ there exists a collection of Borel sets, $\mathbb{D}_k$, which we will call* (*dyadic*) *cubes, such that*

$$\mathbb{D}_k := \{Q_j^k \subset E : j \in \mathfrak{I}_k\},$$

where $\mathfrak{I}_k$ denotes some (*countable*) *index set depending on k, with the decomposition satisfying*

(i) $E = \cup_j Q_j^k$, *for each $k \in \mathbb{Z}$.*

(ii) *If $m \ge k$, then either $Q_i^m \subset Q_j^k$ or $Q_i^m \cap Q_j^k = \varnothing$.*

(iii) *For each (j,k) and each $m < k$, there is a unique i such that $Q_j^k \subset Q_i^m$.*

(iv) $\operatorname{diam}(Q_j^k) \le c_* 2^{-k}$.

(v) *Each Q_j^k contains $E \cap B((t_j^k, Z_j^k), a_0 2^{-k})$ for some $(t_j^k, Z_j^k) \in E$.*

(vi) $E(\{(t,Z)\in Q_j^k : \operatorname{dist}((t,Z), E\setminus Q_j^k) \le \varrho 2^{-k}\}) \le c_*\varrho^\gamma E(Q_j^k)$, *for all k, j and for all $\varrho \in (0,\alpha)$.*

Remark 2.8. We denote by $\mathbb{D} = \mathbb{D}(E)$ the collection of all Q_j^k, i.e.,

$$\mathbb{D} := \bigcup_k \mathbb{D}_k.$$

Given a cube $Q \in \mathbb{D}$, we set

$$\mathbb{D}_Q := \{Q' \in \mathbb{D} : Q' \subseteq Q\}.$$

For a dyadic cube $Q \in \mathbb{D}_k$, we let $\ell(Q) := 2^{-k}$, and we refer to this quantity as the size or side-length of Q. Evidently, $\ell(Q) \sim \operatorname{diam}(Q)$ with constant of comparison depending at most on n and C. Note that (iv) and (v) of Lemma 2.7 imply that for each cube $Q \in \mathbb{D}_k$, there is a point $(t_Q, X_Q) \in E$ and a ball $B((t_Q, X_Q), r)$ such that $r \approx 2^{-k} \approx \operatorname{diam}(Q)$ and

$$E \cap B((t_Q, X_Q), r) \subset Q \subset E \cap B((t_Q, X_Q), Cr), \tag{2.9}$$

for some uniform constant C. We shall refer to the point (t_Q, X_Q) as the center of Q. Given a dyadic cube $Q \subset E$ and $K > 1$, we define the K *dilate* of Q as

$$KQ := \{(t, X) \in E : \operatorname{dist}((t, X), E) < (K-1)\operatorname{diam}(Q)\}. \tag{2.10}$$

Throughout the paper we assume that E is uniformly rectifiable in the parabolic sense. We nominally define this notion in language that will be meaningful to those intimately familiar with the work of David and Semmes, but we will not discuss and introduce all the relevant terminology (the interested reader may consult [Bortz et al. 2022a, Definition 4]), as it will not be used in the present work. In fact, the reader can safely ignore the following definition, as parabolic uniform rectifiability is equivalent to the existence of a bilateral corona decomposition [Bortz et al. 2023, Theorem 3.3] (see Lemma 2.14 below) and the latter is the formulation of parabolic uniform rectifiability that we will actually use throughout the paper.

Definition 2.11 (uniformly rectifiable in the parabolic sense (P-UR)). We say a set $E \subset \mathbb{R}^{n+1}$ is uniformly rectifiable in the parabolic sense (P-UR) if E is ADR and satisfies the $(2, 2)$ geometric lemma with respect to t-independent planes and the measure $\mathcal{H}_p^{n+1}$; see [Bortz et al. 2022a, Definition 4.1].[7] We say that a constant depends on P-UR if it depends on the ADR and Carleson measure constant in the definition of the $(2, 2)$ geometric lemma (with respect to t-independent planes and the measure $\mathcal{H}_p^{n+1}$).

In order to state the bilateral corona decomposition, we need to define regular $\mathrm{Lip}(1/2, 1)$ graphs and coherent subsets of dyadic cubes.

Definition 2.12 (regular $\mathrm{Lip}(1/2, 1)$ graphs). We say that Γ is a regular $\mathrm{Lip}(1/2, 1)$ graph if there exists a t-independent plane P and a function $\psi : P \to P^{\perp}$ such that

$$\Gamma = \{(p, \psi(p)) : p \in P\},$$

where, upon identifying P with $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} = \{(t, x') : t \in \mathbb{R},\ x' \in \mathbb{R}^{n-1}\}$, there exist constants b_1 and b_2 such that ψ has the following two properties:

- ψ is a $\mathrm{Lip}(1/2, 1)$ function with constant bounded by b_1, that is

$$|\psi(t, x') - \psi(s, y')| \le b_1(|x' - y'| + |t - s|^{1/2}), \quad \forall (t, x'),\ (s, y') \in \mathbb{R}^n.$$

- ψ has a half-order time derivative in parabolic-BMO with parabolic-BMO norm bounded by b_2, that is,

$$\|D_t^{1/2}\psi\|_{\text{P-BMO}(\mathbb{R}^n)} \le b_2,$$

where P-BMO is the space of bounded mean oscillation with respect to parabolic balls (or cubes) and $D_t^{1/2}\psi(t, x')$ denotes the half-order time derivative. The half-order time derivative of ψ can be defined by the Fourier transform or by

$$D_t^{1/2}\psi(t, x') := \hat{c}\ \text{p.v.}\int_{\mathbb{R}} \frac{\psi(s, x') - \psi(t, x')}{|s - t|^{3/2}}\, dt, \quad \forall t \in \mathbb{R},\ \forall x' \in \mathbb{R}^{n-1},$$

where $\hat{c}$ is an appropriate constant.

[7] In [Bortz et al. 2022a], a different measure was used in place of $\mathcal{H}_p^{n+1}$, but these measures are equivalent when the set E is P-UR (with respect to either measure). See [Bortz et al. 2023, Corollary B.2].

Definition 2.13 (coherency [David and Semmes 1993]). Suppose E is a d-dimensional ADR set with dyadic cubes $\mathbb{D}(E)$. Let $\boldsymbol{S} \subset \mathbb{D}(E)$. We say that $\boldsymbol{S}$ is *coherent* if the following conditions hold:

(a) $\boldsymbol{S}$ contains a unique maximal element $Q(\boldsymbol{S})$ which contains all other elements of $\boldsymbol{S}$ as subsets.

(b) If Q belongs to $\boldsymbol{S}$ and if $Q \subset \widetilde{Q} \subset Q(\boldsymbol{S})$, then $\widetilde{Q} \in \boldsymbol{S}$.

(c) Given a cube $Q \in \boldsymbol{S}$, either all of its children belong to $\boldsymbol{S}$, or none of them do.

We say that $\boldsymbol{S}$ is *semicoherent* if only conditions (a) and (b) hold.

The following is the bilateral corona decomposition.

Lemma 2.14 [Bortz et al. 2023, Theorem 3.3]. *Suppose that $E \subset \mathbb{R}^{n+1}$ is* P-UR. *Given any positive constant $\eta \ll 1$ and $K := \eta^{-1}$, there are constants $C_\eta = C_\eta(\eta, n,$ ADR, P-UR$)$ and $b_2 = b_2(n,$ ADR, P-UR$)$ and a disjoint decomposition $\mathbb{D}(E) = \mathcal{G} \cup \mathcal{B}$ satisfying the following properties*:

(1) *The **good** collection $\mathcal{G}$ is further subdivided into disjoint **stopping time regimes**, $\mathcal{G} = \cup_{\boldsymbol{S}^* \in \mathcal{S}} \boldsymbol{S}^*$ such that each such regime $\boldsymbol{S}^*$ is coherent.*

(2) *The **bad** cubes, as well as the maximal cubes $Q(\boldsymbol{S}^*)$ satisfy a Carleson packing condition*:

$$\sum_{Q' \subset Q,\, Q' \in \mathcal{B}} \sigma(Q') + \sum_{\boldsymbol{S}^* : Q(\boldsymbol{S}^*) \subset Q} \sigma(Q(\boldsymbol{S}^*)) \le C_\eta \sigma(Q), \quad \forall Q \in \mathbb{D}(E).$$

(3) *For each $\boldsymbol{S}^*$, there is a regular* $\mathrm{Lip}(1/2, 1)$ *graph $\Gamma_{\boldsymbol{S}^*}$, where the function defining the graph has* $\mathrm{Lip}(1/2, 1)$ *constant at most η (that is, $b_1 \le \eta$) and whose half-order time derivative has* P-BMO *norm bounded by b_2, such that, for every $Q \in \boldsymbol{S}^*$,*

$$\sup_{(t,X) \in K\,Q} \operatorname{dist}((t, X), \Gamma_{\boldsymbol{S}^*}) + \sup_{(s,Y) \in B_Q^* \cap \Gamma_{\boldsymbol{S}^*}} \operatorname{dist}((s, Y), E) < \eta \operatorname{diam}(Q), \tag{2.15}$$

where $B_Q^* := B(x_Q, K \operatorname{diam}(Q))$.

Remark 2.16. Notice that if $\boldsymbol{S}$ is any coherent subregime of $\boldsymbol{S}^*$,[8] then item (3) holds for every $Q \in \boldsymbol{S}$. Also, note that below we may insist that K is large, but this should be interpreted as taking η small.

Definition 2.17 (Whitney cubes and Whitney regions). Given an ADR set $E \subset \mathbb{R}^{n+1}$ we let $\mathcal{W}(E^c)$ be the standard (parabolic) Whitney decomposition of E^c, that is, $\mathcal{W}(E^c) = \{I_i\}$ is a collection of closed parabolic dyadic cubes[9] with disjoint interiors, $\cup_{\mathcal{W}(E^c)} I_i = E^c$, and for each $I \in \mathcal{W}(E^c)$,

$$4 \operatorname{diam}(I) \le \operatorname{dist}(4I, E) \le \operatorname{dist}(I, E) \le 100 \operatorname{diam}(I).$$

(A similar construction can be found in Lemma 3.24 below). For $\eta \ll 1 \ll K$ and $Q \in \mathbb{D}(E)$, we define

$$\mathcal{W}_Q(\eta, K) = \{I \in \mathcal{W}(E^c) : \eta^{1/4} \operatorname{diam}(Q) \le \operatorname{dist}(I, E) \le \operatorname{dist}(I, Q) \le K^{1/4} \operatorname{diam}(Q)\}$$

and

$$\mathcal{W}_Q^*(\eta, K) = \{I \in \mathcal{W}(E^c) : \eta^4 \operatorname{diam}(Q) \le \operatorname{dist}(I, E) \le \operatorname{dist}(I, Q) \le K \operatorname{diam}(Q)\}.$$

[8] This means $\boldsymbol{S} \subseteq \boldsymbol{S}^*$ and $\boldsymbol{S}$ satisfies the coherency conditions in Definition 2.13.

[9] This means cubes from the collection of parabolic cubes in $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ with vertices at the lattice points $2^{2k}\mathbb{Z} \times 2^k\mathbb{Z}^n$, for each $k \in \mathbb{Z}$.

Comparing volumes, we see that $\#\mathcal{W}_Q \le C(n, \eta, K)$ (we use the notation $\#A$ to denote the cardinality of a finite set A). For $\eta \ll 1 \ll K$ and $Q \in \mathbb{D}(E)$, we set

$$U_Q(\eta, K) = \bigcup_{I \in \mathcal{W}_Q(\eta, K)} I \quad \text{and} \quad U_Q^*(\eta, K) = \bigcup_{I \in \mathcal{W}_Q^*(\eta, K)} I.$$

Remark 2.18. The reader may readily verify that the Whitney regions U_Q and U_Q^* have bounded overlaps, that is,

$$\sum_{Q \in \mathbb{D}(E)} 1_{U_Q}(t, X) + \sum_{Q \in \mathbb{D}(E)} 1_{U_Q^*}(t, X) \lesssim 1, \quad \forall (t, X) \in \mathbb{R}^{n+1},$$

where the implicit constant depends on the dimension, ADR, η, and K.

3. Domain approximation in stopping time regimes

In this section we assume that E has a bilateral corona decomposition and we fix $\boldsymbol{S}$, a coherent subregime of a stopping time regime $\boldsymbol{S}^*$ in the bilateral corona decomposition (by Remark 2.16 the same estimates hold for $\boldsymbol{S}$). Our goal is to construct a family of graphs that approximate the set E well in the sense of Lemma 2.14 (3) but have the additional property that they lie *above* (or on) the set E at the scale and location of the maximal cube $Q_{\boldsymbol{S}}$. Other important properties of the construction will also be established including containment properties with respect to the Whitney regions defined above (see Definition 2.17). In the sequel we will often insist on further smallness of η depending on dimension and the ADR constant for E. Compared to [Hofmann et al. 2016], the constructions outlined in this section are the main novelties of this paper.

Let $Q_{\boldsymbol{S}} := Q(\boldsymbol{S})$ be the maximal cube in the coherent subregime under consideration. Recall that $\boldsymbol{S} \subseteq \boldsymbol{S}^*$ and that there exists a regular Lip(1/2, 1) graph, $\Gamma_{\boldsymbol{S}^*}$, such that Lemma 2.14 (3) holds for $\boldsymbol{S}^*$ and hence also for $\boldsymbol{S}$. Without loss of generality we may assume that the t-independent plane over which $\Gamma := \Gamma_{\boldsymbol{S}^*}$ is defined is the plane $\mathbb{R}^n \times \{0\}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be the regular Lip(1/2, 1) function that defines $\Gamma_{\boldsymbol{S}^*}$, that is,

$$\Gamma := \Gamma_{\boldsymbol{S}^*} = \{(t, x', f(t, x')) : (t, x') \in \mathbb{R}^n\}.$$

We define the $\mathbb{R}^{n+1}$-valued function

$$F(t, x') = (t, x', f(t, x')).$$

Inspired by [David and Semmes 1991], we define the *stopping time distance* $d : \mathbb{R}^{n+1} \to \mathbb{R}$ by[10]

$$d_{\boldsymbol{S}}[(t, X)] = \inf_{Q \in \boldsymbol{S}} [\operatorname{dist}((t, X), Q) + \operatorname{diam}(Q)].$$

Given $\alpha \in \left[\frac{7}{8}, \frac{31}{32}\right]$ we introduce

$$g_\alpha(t, x') := f(t, x') + \eta^\alpha d[F(t, x')] \quad \text{and} \quad G_\alpha(t, x') := (t, x', g_\alpha(t, x')).$$

As $\alpha \in \left[\frac{7}{8}, \frac{31}{32}\right]$, below we will drop the subscript α and all constants will be independent of α. As we have fixed $\boldsymbol{S}$, we will also drop the subscript $\boldsymbol{S}$ from $d_{\boldsymbol{S}}$.

[10]Note that we take the stopping time distance in the *subregime*.

We first prove that g is Lip($1/2$, 1).

Lemma 3.1. *If $\eta^{7/8} \leq \frac{1}{2}$, then g is a* Lip($1/2$, 1) *function with constant less than $3\eta^\alpha$, and the function*

$$G(t, x') := (t, x', g(t, x'))$$

satisfies

$$\tfrac{1}{2} d[F(t, x')] \leq d[G(t, x')] \leq 2d[F(t, x')].$$

Proof. Note first that d is Lip($1/2$, 1) (on $\mathbb{R}^{n+1}$) with constant no more than 1, that is,

$$|d[(t, X)] - d[(s, Y)]| \leq \operatorname{dist}((t, X), (s, Y)).$$

This follows from the fact that d is the infimum of nonnegative Lip($1/2$, 1) functions with constant 1. Using this we see that

$$\begin{aligned} |g(t, x') - g(s, y')| &\leq |f(t, x') - f(s, y')| + \eta^\alpha |d[(t, x', f(t, x'))] - d[(s, y', f(s, y'))]| \\ &\leq \eta[|t - s|^{1/2} + |x' - y'|] + \eta^\alpha [|t - s|^{1/2} + |x' - y'| + |f(t, x') - f(s, y')|] \\ &\leq 3\eta^\alpha [|t - s|^{1/2} + |x' - y'|]. \end{aligned}$$

To deduce the inequalities involving $d[G(t, x')]$ and $d[F(t, x')]$ we consider two cases. If $d[F(t, x')] = 0$, then $G(t, x') = F(t, x')$ so that $d[G(t, x')] = 0$. Otherwise, $d[F(t, x')] > 0$, and using that d is Lip($1/2$, 1) with constant 1, we have

$$|d[F(t, x')] - d[G(t, x')]| \leq \operatorname{dist}(F(t, x'), G(t, x')) = |f(t, x') - g(t, x')| \leq \eta^\alpha d[F(t, x')] \leq \tfrac{1}{2} d[F(t, x')].$$

From this we easily obtain

$$\tfrac{1}{2} d[F(t, x')] \leq d[G(t, x')] \leq 2d[F(t, x')]. \qquad \square$$

We will use the following elementary lemma several times.

Lemma 3.2. *If Γ' is the graph of a* Lip($1/2$, 1) *function φ with* Lip($1/2$, 1)*-norm less than $\frac{1}{2}$, then*

$$\tfrac{1}{2} |x_n - \varphi(t, x')| \leq \operatorname{dist}((t, X), \Gamma') \leq |x_n - \varphi(t, x')|,$$

for all $(t, X) = (t, x', x_n)$.

Proof. The inequality on the right-hand side is trivial. To prove the inequality on the left-hand side, we can, after a translation, assume that $(t, x', \varphi(t, x')) = (0, 0, 0)$. Furthermore, we can without loss of generality assume that $x_n \geq 0$ (the case $x_n < 0$ is treated in the same way). Then $|x_n - \varphi(0, 0)| = x_n$. If $(s, y') \in \mathbb{R}^n$ satisfies $|y'| + |s|^{1/2} > x_n$, then

$$\operatorname{dist}((t, X), (s, y', \varphi(s, y'))) \geq |y'| + |s|^{1/2} \geq x_n.$$

If $(s, y') \in \mathbb{R}^n$ satisfies $|y'| + |s|^{1/2} \leq x_n$, then $|\varphi(s, y')| \leq \frac{1}{2} x_n$ and hence

$$\operatorname{dist}((t, X), (s, y', \varphi(s, y'))) \geq |x_n - \varphi(s, y')| \geq \left(1 - \tfrac{1}{2}\right) x_n = \tfrac{1}{2} x_n.$$

These estimates prove the lemma. $\square$

We will need the following properties of the stopping time distance.

Lemma 3.3. *Let $A > 1$. If $(t, X) \in \mathbb{R}^{n+1}$ satisfies $0 < 2d[(t, X)] \le A \operatorname{diam}(Q_{\boldsymbol{S}})$, then there exists $Q^* \in \boldsymbol{S}$ such that*

$$\operatorname{dist}((t, X), Q^*) \le 2d[(t, X)] \le A \operatorname{diam}(Q^*) \le C_{n,\mathrm{ADR}} d[(t, X)]. \tag{3.4}$$

If $d[(t, X)] = 0$, then there exists, for every $\epsilon \in (0, A \operatorname{diam}(Q_{\boldsymbol{S}}))$, $Q_\epsilon \in \boldsymbol{S}$ such that

$$\operatorname{dist}((t, X), Q_\epsilon) \le \epsilon < A \operatorname{diam}(Q_\epsilon) \le C_{n,\mathrm{ADR}} \epsilon. \tag{3.5}$$

Proof. We start with proving (3.4). By definition there exists $Q \in \boldsymbol{S}$ such that

$$\operatorname{dist}((t, X), Q) + \operatorname{diam}(Q) \le 2d[(t, X)].$$

Let $Q^* \in \boldsymbol{S}$ be the smallest cube satisfying $Q \subseteq Q^* \subseteq Q_{\boldsymbol{S}}$ such that

$$A \operatorname{diam}(Q^*) \ge 2d[(t, X)]. \tag{3.6}$$

Such a cube exists because $Q_{\boldsymbol{S}}$ is a *candidate*. Notice that since Q^* contains Q, $\operatorname{dist}((t, X), Q^*) \le \operatorname{dist}((t, X), Q) \le 2d[(t, X)]$ which proves the first inequality in (3.4). The second inequality in (3.4) holds by the choice of Q^*. To see that the last inequality holds, we first note that if $Q^* = Q$, then $\operatorname{diam}(Q^*) = \operatorname{diam}(Q) \le 2d[(t, X)]$ and we are done. Otherwise, the child of Q^* containing Q, namely Q', fails to satisfy (3.6) and hence

$$A \operatorname{diam}(Q^*) \lesssim_{n,\mathrm{ADR}} A \operatorname{diam}(Q') \le 2d[(t, X)].$$

Since $A > 1$, we have that $\operatorname{diam}(Q^*) \lesssim_{n,\mathrm{ADR}} d[(t, X)]$ (with the implicit constant independent of A). This proves (3.4).

To verify (3.5), note that by definition there exists $Q \in \boldsymbol{S}$ such that

$$\operatorname{dist}((t, X), Q) + \operatorname{diam}(Q) \le \epsilon \le A \operatorname{diam}(Q_{\boldsymbol{S}}).$$

This allows us to repeat the argument above to produce Q_ϵ. □

Lemma 3.7. *If $(t, X) \in B\big((t_{Q_{\boldsymbol{S}}}, X_{Q_{\boldsymbol{S}}}), \frac{1}{4} K \operatorname{diam}(Q_{\boldsymbol{S}})\big) \cap E$ with $(t, X) = (t, x', x_n)$, then*

$$\operatorname{dist}((t, X), \Gamma) \lesssim \eta d[(t, X)] \quad \textit{and} \quad |x_n - f(t, x')| \lesssim \eta d[(t, X)].$$

Here the implicit constants depend only on the dimension and ADR.

Proof. The second inequality follows from the first and Lemma 3.2. If $d[(t, X)] = 0$, then Lemma 3.3 gives that for $n \in \mathbb{N}$, we have $(t, X) \in K Q_{1/n}$ for some $Q_{1/n} \in \boldsymbol{S}$ with $\operatorname{diam}(Q_{1/n}) \approx 1/n$. Then using Lemma 2.14(3) we have $\operatorname{dist}((t, X), \Gamma) \lesssim 1/n$ for all n and hence $(t, X) \in \Gamma$. This proves the lemma in the case $d[(t, X)] = 0$.

Now assume $d[(t, X)] > 0$ and note that $2d[(t, X)] < (K-1) \operatorname{diam}(Q_{\boldsymbol{S}})$ if $K > 6$. Applying Lemma 3.3, there exists Q^* such that

$$\operatorname{dist}((t, X), Q^*) \le (K-1) \operatorname{diam}(Q^*) \lesssim d[(t, X)].$$

Then Lemma 2.14 (3) gives the desired inequality

$$\operatorname{dist}((t,X),\Gamma) \le \eta \operatorname{diam}(Q^*) \lesssim \eta d[(t,X)]. \qquad \square$$

Let

$$\Gamma^+ := \{(t,x',g(t,x')) : (t,x') \in \mathbb{R}^n\}$$

denote the graph of g. We first prove that we did not lose too much by modifying f and that, in fact, E lies below Γ^+ (near Q_S).

Lemma 3.8. *If* $(t,X) \in B\big((t_{Q_S}, X_{Q_S}), \frac{1}{4}K \operatorname{diam}(Q_S)\big) \cap E$ *with* $(t,X) = (t,x',x_n)$, *then*

(a) $\frac{1}{8}\eta^\alpha d[(t,X)] \le \operatorname{dist}((t,X),\Gamma^+) \le 3\eta^\alpha d[(t,X)]$, *and*

(b) $x_n \le g(t,x') - \frac{1}{4}\eta^\alpha d[(t,X)]$.

Proof. If $d[(t,X)] = 0$, then $d[F(t,x')] = 0$ by Lemma 3.7, and

$$(t,X) = (t,x',f(t,x')) = (t,x',g(t,x')).$$

This implies (a) and (b).

Assume $d[(t,X)] > 0$. Lemma 3.7 yields the estimate

$$|x_n - f(t,x')| \le C\eta d[(t,X)]. \tag{3.9}$$

If $C\eta < \frac{1}{2}$, then following the lines of the proof of Lemma 3.1, we have that

$$\tfrac{1}{2}d[(t,X)] \le d[F(t,x')] \le 2d[(t,X)]. \tag{3.10}$$

Thus, by definition of g,

$$g(t,x') - x_n = \eta^\alpha d[F(t,x')] + (f(t,x') - x_n) \ge \tfrac{1}{2}\eta^\alpha d[(t,X)] - C\eta d[(t,X)] \ge \tfrac{1}{4}\eta^\alpha d[(t,X)],$$

provided $C\eta \le \frac{1}{4}\eta^\alpha$. This proves (b), and when combined with Lemma 3.2, it gives the lower bound in (a). To verify the upper bound in (a), we use (3.9) and (3.10) to write

$$|g(t,x') - x_n| \le \eta^\alpha d[F(t,x')] + |f(t,x') - x_n| \le 2\eta^\alpha d[(t,X)] + C\eta d[(t,X)] \le 3\eta^\alpha d[(t,X)]. \qquad \square$$

We remind the reader that we have previously defined certain Whitney regions (see Definition 2.17). We now investigate how these Whitney regions interact with the graphs we are constructing. First we need to see how they interact with the original graph Γ. As in the elliptic setting [Hofmann et al. 2016], we have the following.

Lemma 3.11. *If* $Q \in \boldsymbol{S}$ *and* $I \in \mathcal{W}_Q$, *then* I *is either above or below* Γ (*it does not meet* Γ). *Moreover, we have the estimate*

$$\operatorname{dist}(I,\Gamma) \ge \eta^{1/2} \operatorname{diam}(Q).$$

Proof. The first statement, about the cubes being above or below the graph, follows from the estimate. Suppose for the sake of contradiction that there exists $I \in \mathcal{W}_Q$, $Q \in \boldsymbol{S}$ such that $\operatorname{dist}(I,\Gamma) < \eta^{1/2} \operatorname{diam}(Q)$, and let $(s, Y) \in \Gamma$ be such that $\operatorname{dist}((s, Y), I) \le \eta^{1/2} \operatorname{diam}(Q)$. By construction, $\operatorname{dist}((t, Z), (t_Q, X_Q)) \lesssim K^{1/4} \operatorname{diam}(Q)$ for all $(t, Z) \in I$ and hence

$$\operatorname{dist}((s, Y), (t_Q, X_Q)) \le \eta^{1/2} \operatorname{diam}(Q) + C K^{1/4} \operatorname{diam}(Q) \lesssim K^{1/4} \operatorname{diam}(Q).$$

By Lemma 2.14 (3), $\operatorname{dist}((s, Y), E) \le \eta \operatorname{diam}(Q)$. Choosing $(t_0, Z_0) \in I$ such that $\operatorname{dist}((t_0, Z_0), (s, Y)) = \operatorname{dist}((s, Y), I) \le \eta^{1/2} \operatorname{diam}(Q)$, we have that

$$\begin{aligned} \operatorname{dist}(I, E) &\le \operatorname{dist}((t_0, Z_0), (s, Y)) + \operatorname{dist}((s, Y), E) \\ &\le \eta^{1/2} \operatorname{diam}(Q) + \eta \operatorname{diam}(Q) \le 2\eta^{1/2} \operatorname{diam}(Q) < \eta^{1/4} \operatorname{diam}(Q), \end{aligned}$$

provided $\eta^{1/4} < \frac{1}{2}$. This violates the assumption that $I \in \mathcal{W}_Q$. □

In light of Lemma 3.11, for $Q \in \boldsymbol{S}$ we have that $\mathcal{W}_Q = \mathcal{W}_Q^+ \cup \mathcal{W}_Q^-$, where $\mathcal{W}_Q^+$ is the collection of Whitney cubes above Γ and $\mathcal{W}_Q^-$ is the collection of Whitney cubes below Γ. We then define

$$U_Q^\pm := \bigcup_{I \in \mathcal{W}_Q^\pm} I.$$

The following lemma says that U_Q^+ still lies above Γ^+ and, when $(t, X) \in U_Q^+$, the distance from (t, X) to Γ^+ is roughly the distance to E.

Lemma 3.12. *Let $Q \in \boldsymbol{S}$. If η is sufficiently small, then U_Q^+ lies above Γ^+ and*

$$x_n - g(t, x') \ge \tfrac{1}{2} \operatorname{dist}((t, X), \Gamma), \quad \forall (t, X) = (t, x', x_n) \in U_Q^+. \tag{3.13}$$

Moreover,

$$\operatorname{dist}((t, X), \Gamma^+) \approx \operatorname{dist}((t, X), E), \quad (t, X) \in U_Q^+, \tag{3.14}$$

where the implicit constants depend on dimension, ADR, η, *and* K.

Proof. Recall that $\eta = K^{-1}$. Let $(t, X) = (t, x', x_n) \in I$ for some $I \in \mathcal{W}_Q^+$. As $\operatorname{dist}((t, X), (t_Q, X_Q)) \lesssim K^{1/4} \operatorname{diam}(Q)$ and $Q \in \boldsymbol{S}$, we have

$$\operatorname{dist}((t, X), \Gamma) \lesssim (K^{1/4} + \eta) \operatorname{diam}(Q) \lesssim K^{1/4} \operatorname{diam}(Q). \tag{3.15}$$

Using Lemma 3.2,

$$|(t, X) - F(t, x')| \lesssim K^{1/4} \operatorname{diam}(Q),$$

and therefore $\operatorname{dist}(F(t, x'), Q) \lesssim K^{1/4} \operatorname{diam}(Q)$. It follows that $d[F(t, x')] \lesssim K^{1/4} \operatorname{diam}(Q)$, and using Lemma 3.11,

$$x_n - f(t, x') \ge \operatorname{dist}((t, X), \Gamma) \ge \eta^{1/2} \operatorname{diam}(Q) \gtrsim \eta^{1/2} K^{-1/4} d[F(t, x')] \approx \eta^{3/4} d[F(t, x')].$$

By the definition of g and the fact that $\alpha \ge \frac{7}{8}$, Lemma 3.2 implies that

$$\begin{aligned} x_n - g(t, x') = x_n - f(t, x') - \eta^\alpha d[F(t, x')] &\ge \tfrac{1}{2}(x_n - f(t, x')) \\ &\ge \tfrac{1}{2} \operatorname{dist}((t, X), \Gamma) \ge \tfrac{1}{2}\eta^{1/2} \operatorname{diam}(Q), \end{aligned} \tag{3.16}$$

where the next-to-last inequality yields (3.13), and where we have used Lemma 3.11 in the last step. In particular, U_Q^+ lies above Γ^+. Using (3.15) and the last inequality in (3.16), and then the properties of the Whitney cubes in $\mathcal{W}_Q$, we have

$$\operatorname{dist}((t,X),\Gamma) \approx_{\eta,K} \operatorname{diam}(Q) \approx_{\eta,K} \operatorname{dist}((t,X),E).$$

Combining (3.16) and the last displayed estimate and then using Lemma 3.2, we obtain

$$\operatorname{dist}((t,X),\Gamma^+) \geq \tfrac{1}{2}(x_n - g(t,x')) \geq \tfrac{1}{4}\operatorname{dist}((t,X),\Gamma) \approx_{\eta,K} \operatorname{dist}((t,X),E)$$

and

$$\operatorname{dist}((t,X),\Gamma^+) \leq x_n - g(t,x') \leq x_n - f(t,x') \leq 2\operatorname{dist}((t,X),\Gamma) \approx \operatorname{dist}((t,X),E). \qquad \square$$

We also require that close to $Q_{\mathbf{S}}$, the region above Γ^+ shall be contained in a collection of Whitney regions associated to $Q \in \mathbf{S}$. This can be done using the Whitney regions U_Q^*.

Lemma 3.17. *Suppose $(t,X) = (t,x',x_n)$ satisfies $x_n > g(t,x')$ and*

$$(t,X) \in B\big((t_{Q_{\mathbf{S}}}, X_{Q_{\mathbf{S}}}), \tfrac{1}{32}K\operatorname{diam}(Q_{\mathbf{S}})\big).$$

Then

$$\operatorname{dist}((t,X),E) \geq \operatorname{dist}((t,X),\Gamma^+), \tag{3.18}$$

and there exists $Q^* \in \mathbf{S}$ such that $(t,X) \in U_{Q^*}^*$.

Proof. Let (t,X) be as above. By Lemma 3.8, we see that $d[(t,X)] \neq 0$. To prove (3.18), we note that if (s,Y) is the closest point to (t,X) in E, then

$$\operatorname{dist}((t,X),(s,Y)) \leq \operatorname{dist}((t,X),(t_{Q_{\mathbf{S}}}, X_{Q_{\mathbf{S}}})) < \tfrac{1}{32}K\operatorname{diam}(Q_{\mathbf{S}}).$$

Thus $\operatorname{dist}((s,Y),(t_{Q_{\mathbf{S}}}, X_{Q_{\mathbf{S}}})) < \frac{1}{16}K\operatorname{diam}(Q_{\mathbf{S}})$, so in particular,

$$(s,Y) \in B\big((t_{Q_{\mathbf{S}}}, X_{Q_{\mathbf{S}}}), \tfrac{1}{4}K\operatorname{diam}(Q_{\mathbf{S}})\big) \cap E.$$

By Lemma 3.8 (b), (s,Y) lies below Γ^+ and hence the line segment between (s,Y) and (t,X) meets Γ^+. This proves (3.18).

To prove the existence of $Q^* \in \mathbf{S}$ such that $(t,X) \in U_{Q^*}^*$, we break the proof into cases.

<u>Case 1</u>: $x_n - g(t,x') \geq \eta^3 d[(t,X)]$. In this case, by (3.18) and Lemma 3.2,

$$\operatorname{dist}((t,X),E) \geq \operatorname{dist}((t,X),\Gamma^+) \geq \tfrac{1}{2}\eta^3 d[(t,X)].$$

Since $d[(t,X)] \leq \big(\frac{1}{32}K + 1\big)\operatorname{diam}(Q_{\mathbf{S}}) < \big(\frac{1}{2}(K-1)\big)\operatorname{diam}(Q_{\mathbf{S}})$, we use Lemma 3.3 to produce Q^* with

$$\operatorname{dist}((t,X),Q^*) \leq (K-1)\operatorname{diam}(Q^*) \lesssim_{n,\mathrm{ADR}} d[(t,X)].$$

Thus,

$$(t,X) \in B((t_{Q^*}, X_{Q^*}), K\operatorname{diam}(Q^*))$$

and

$$\operatorname{dist}((t,X),E) \gtrsim \eta^3 d[(t,X)] \gtrsim \eta^3 (K-1)\operatorname{diam}(Q^*) \approx \eta^2 \operatorname{diam}(Q),$$

where the implicit constants depend on the dimension and ADR. Letting $I \in \mathcal{W}$ be such that $(t, X) \in I$, it follows that $I \in \mathcal{W}^*_Q$, provided η is sufficiently small.

Case 2: $x_n - g(t, x') \le \eta^3 d[(t, X)]$. In this case, note that

$$\operatorname{dist}((t, X), G(t, x')) \le x_n - g(t, x') \le \eta^3 d[(t, X)]. \tag{3.19}$$

Thus, since d is Lipschitz with norm 1 with respect to $\operatorname{dist}(\,\cdot\,)$, we have, for $\eta^3 < \frac{1}{2}$,

$$\tfrac{1}{2} d[(t, X)] \le d[G(t, x')] \le 2d[(t, X)]. \tag{3.20}$$

In particular, $d[G(t, x')] > 0$. Notice then that

$$d[G(t, x')] \le 2d[(t, X)] \le \left(\tfrac{1}{16} K + 2\right) \operatorname{diam}(Q_S) \le \tfrac{1}{8}(K-1) \operatorname{diam}(Q_S),$$

provided that K is large enough, and Lemma 3.3 then yields $Q^* \in \boldsymbol{S}$ such that

$$\operatorname{dist}(G(t, x'), Q^*) \le \tfrac{1}{4}(K-1) \operatorname{diam}(Q^*) \approx d[G(t, x')]. \tag{3.21}$$

Combining the latter estimate with (3.19) and (3.20), we see that

$$(t, X) \in B\left((t_{Q^*}, X_{Q^*}), \tfrac{1}{2}(K-1) \operatorname{diam}(Q^*)\right).$$

Claim 3.22. *For η chosen small enough ($\eta^2 < \frac{1}{2}$ will suffice at this stage),*

$$\operatorname{dist}(G(t, x'), E) \ge \eta^2 d[G(t, x')].$$

Taking the claim for granted momentarily, by (3.19), (3.20), and (3.21), we have

$$\begin{aligned}\operatorname{dist}((t, X), E) &\ge \operatorname{dist}(G(t, x'), E) - \operatorname{dist}((t, X), G(t, x')) \\ &\ge \operatorname{dist}(G(t, x'), E) - \eta^3 d[(t, X)] \gtrsim \eta^2 d[(t, X)] \\ &\approx \eta^2 [G(t, x')] \gtrsim \eta^2 (K-1) \operatorname{diam}(Q^*) \approx \eta \operatorname{diam}(Q^*),\end{aligned}$$

and the lemma is proved. It remains to prove Claim 3.22.

Proof of Claim 3.22. Let $(s, Y) = (s, y', y_n) \in E$ be such that

$$\operatorname{dist}(G(t, x'), (s, Y)) = \operatorname{dist}(G(t, x'), E).$$

Assume, for the sake of obtaining a contradiction, that

$$\operatorname{dist}(G(t, x'), (s, Y)) < \eta^2 d[G(t, x')].$$

Then for $\eta^2 < \frac{1}{2}$, since d is Lipschitz with norm 1 with respect to $\operatorname{dist}(\,\cdot\,)$, we have

$$d[G(t, y')] \le 2d[(s, Y)].$$

Hence, under the current assumption that $\operatorname{dist}(G(t, x'), (s, Y)) < \eta^2 d[G(t, x')]$,

$$|y' - x'| + |t - s|^{1/2} < 2\eta^2 d[(s, Y)].$$

Since g is $\mathrm{Lip}(1/2, 1)$ with constant $3\eta^\alpha$ (in particular less than 1),

$$\mathrm{dist}(G(t, x'), G(s, y')) \lesssim \eta^2 d[(s, Y)].$$

Thus,

$$|y_n - g(s, y')| = \mathrm{dist}((s, Y), G(s, y')) \lesssim \eta^2 d[(s, Y)],$$

which contradicts the conclusion of Lemma 3.8, provided that

$$(s, Y) \in B\big((t_{Q_S}, X_{Q_S}), \tfrac{1}{4} K \,\mathrm{diam}(Q_S)\big). \tag{3.23}$$

Indeed, the latter is true, as we now show. Recall that by hypothesis

$$(t, X) \in B\big((t_{Q_S}, X_{Q_S}), \tfrac{1}{32} K \,\mathrm{diam}(Q_S)\big).$$

Moreover, in the scenario of Case 2,

$$\mathrm{dist}((t, X), G(t, x')) = x_n - g(t, x') < \eta^3 d[(t, X)] \le \mathrm{diam}(Q_S),$$

and therefore

$$\mathrm{dist}(G(t, x'), (t_{Q_S}, X_{Q_S})) < \tfrac{1}{16} K \,\mathrm{diam}(Q_S).$$

Since (s, Y) is the closest point on E to $G(t, x')$, it must be that

$$(s, Y) \in B\big((t_{Q_S}, X_{Q_S}), \tfrac{1}{8} K \,\mathrm{diam}(Q_S)\big).$$

In particular, (3.23) holds, and this proves the claim. □

Our next goal is to produce a regular version of the graphs we have constructed above. The vehicle for this regularization is the following lemma.

Lemma 3.24. *Let* $h : \mathbb{R}^n \to \mathbb{R}$, $h(t, x') \ge 0$, *be a* $\mathrm{Lip}(1/2, 1)$ *function with* $\mathrm{Lip}(1/2, 1)$ *constant* (*at most*) 1. *There exists a function* $H : \mathbb{R}^n \to \mathbb{R}$ *such that*

(1) $c_1 h(t, x') \le H(t, x') \le c_2 h(t, x')$ *for all* $(t, x') \in \mathbb{R}^n$.

(2) *If* $Z = \{(t, x') : h(t, x') = 0\}$, *then*

$$h(t, x')^{2m-1} |\partial_t^m H(t, x')| + h(t, x')^{m-1} |\nabla_{x'}^m H(t, x')| \le c_{n,m}, \quad \forall (t, x') \in Z^c, \quad m \in \mathbb{N}.$$

(3) $H \in \mathrm{Lip}(1/2, 1)$ *with constant less than* c_3.

Here c_1, c_2, *and* c_3 *depend on the dimension alone and* $c_{n,m}$ *depends on the dimension and* m. *Moreover*, H *enjoys the estimate*

$$\|D_t^{1/2} H\|_{\text{P-BMO}} \le c_4,$$

where c_4 *depends only on the dimension.*

The proof has many *standard* elements (if one knows where to look), but is a little lengthy. The proof can be found in the Appendix.

Now we are ready to create our regularized graph. Let $h(t,x') := \frac{1}{2}d[F(t,x')]$, and let $H(t,x')$ be the function provided by Lemma 3.24.[11] We define two functions

$$\psi^{\pm}_{\eta,\boldsymbol{S}}(t,x) := f(t,x) \pm \eta^{15/16} H(t,x').$$

We hope that it is clear to the reader that the function

$$g^{-}_{\alpha}(t,x') := f(t,x') - \eta^{\alpha} d[F(t,x')]$$

has properties analogous to those of $g_\alpha(t,x')$ except that g^-_α is below f and E, that the cubes in $\mathcal{W}^-_Q$ are below the graph of g^-_α, etc. We next deduce that $\psi^{\pm}_{\eta,\boldsymbol{S}}$ has the same properties as the functions g_α (and g^-_α), enumerated below.

Proposition 3.25. *Let E be uniformly rectifiable in the parabolic sense. Let $\mathbb{D}(E) = \mathcal{G} \cup \mathcal{B}$, $\mathcal{G} = \cup_{\boldsymbol{S}^* \in \mathcal{S}} \boldsymbol{S}^*$, and $\{\Gamma_{\boldsymbol{S}^*}\}_{\boldsymbol{S}^* \in \mathcal{S}}$ be the bilateral corona decomposition of E given by Lemma 2.14, with constants $\eta \ll 1$, $K = \eta^{-1}$, and b_2. Let M_0 be the constant from Lemma 4.1 below, with $\tilde{b}_1 = 2$ and $\tilde{b}_2 = 1 + b_2$. If η is sufficiently small, depending only on the dimension and* ADR, *then the following holds.*

Let $\boldsymbol{S}^* \in \mathcal{S}$. Then for every coherent subregime $\boldsymbol{S}$ of $\boldsymbol{S}^*$, there is a t-independent plane $P_{\boldsymbol{S}}$ and two regular parabolic graphs $\Gamma^{\pm}_{\boldsymbol{S}}$ over $P_{\boldsymbol{S}}$ given by the functions $\psi^{\pm}_{\eta,\boldsymbol{S}}$, with

$$\|\psi^{\pm}_{\eta,\boldsymbol{S}}\|_{\mathrm{Lip}(1/2,1)} \le C_n \eta^{15/16} \quad \textit{and} \quad \|D^{1/2}_t \psi^{\pm}_{\eta,\boldsymbol{S}}\|_{\text{P-BMO}} \le (1+b_2),$$

with the following properties (in the coordinates given by $P_{\boldsymbol{S}} \oplus P^{\perp}_{\boldsymbol{S}}$):

(1) *If $Q \in \boldsymbol{S}$, then $\mathcal{W}_Q$ has a disjoint decomposition $\mathcal{W}_Q = \mathcal{W}^+_Q \cup \mathcal{W}^-_Q$, and if we let $U^{\pm}_Q := \cup_{I \in \mathcal{W}^{\pm}_Q} I$, then*

$$U^{\pm}_Q \subseteq B((t_{Q_S}, X_{Q_S}), K^{3/4}\operatorname{diam}(Q_{\boldsymbol{S}})) \cap \{\pm x_n > \pm \psi^{\pm}_{\eta,\boldsymbol{S}}(t,x')\}.$$

Here the notation $\{\pm x_n > \pm \psi^{\pm}_{\eta,\boldsymbol{S}}(t,x')\}$ means

$$\{(t,X) = (t,x',x_n) : \pm x_n > \pm \psi^{\pm}_{\eta,\boldsymbol{S}}(t,x')\}.$$

In particular,

$$\bigcup_{Q \in S} U^{\pm}_Q \subseteq B((t_{Q_S}, X_{Q_S}), K^{3/4}\operatorname{diam}(Q_{\boldsymbol{S}})) \cap \{\pm x_n > \pm \psi^{\pm}_{\eta,\boldsymbol{S}}(t,x')\}.$$

(2) *If $(t,X) \in \cup_{Q \in \boldsymbol{S}} U^{\pm}_Q$, then*

$$\operatorname{dist}((t,X),E) \approx_\eta \operatorname{dist}((t,X),\Gamma^{\pm}_{\boldsymbol{S}}).$$

(3) *If $(t,X) \in B\big((t_{Q_S}, X_{Q_S}), \frac{1}{32}K\operatorname{diam}(Q_{\boldsymbol{S}})\big) \cap \{\pm x_n > \pm \psi^{\pm}_{\eta,\boldsymbol{S}}(t,x')\}$, then*

$$\operatorname{dist}((t,X),E) \ge \operatorname{dist}((t,X),\Gamma^{\pm}_{\boldsymbol{S}}).$$

(4) $B\big((t_{Q_S}, X_{Q_S}), \frac{1}{32}K\operatorname{diam}(Q_{\boldsymbol{S}})\big) \cap \{\pm x_n > \pm \psi^{\pm}_{\eta,\boldsymbol{S}}(t,x')\} \subseteq \cup_{Q \in \boldsymbol{S}} U^*_Q$.

[11] See the proof of Lemma 3.1, from which one can easily deduce that $d[F(t,x')]$ has $\mathrm{Lip}(1/2,1)$ norm less than $1+\eta$ and hence $h(t,x')$ has $\mathrm{Lip}(1/2,1)$-norm less than 1. This allows one to apply Lemma 3.24.

(5) *There exist* $(t_S^\pm, X_S^\pm) \in \Gamma_S^\pm$ *such that*

$$B((t_{Q_S}, X_{Q_S}), M_0 K^{3/4} \operatorname{diam}(Q_S)) \subset B((t_S^\pm, X_S^\pm), K^{7/8} \operatorname{diam}(Q_S)) \tag{3.26}$$

and

$$B((t_S^\pm, X_S^\pm), K^{7/8} \operatorname{diam}(Q_S)) \subset B\big((t_{Q_S}, X_{Q_S}), \tfrac{1}{32} K \operatorname{diam}(Q_S)\big). \tag{3.27}$$

Proof. Let $\psi_{\eta,S}^\pm$ be as constructed before the statement of the proposition. Both $\psi_{\eta,S}^\pm$ are $\operatorname{Lip}(1/2, 1)$ with constant less than $C\eta^{15/16}$ because f is $\operatorname{Lip}(1/2, 1)$ with constant less than η and H has $\operatorname{Lip}(1/2, 1)$-norm less than $c_3 = c_3(n)$; see Lemma 3.24. Similarly,

$$\|D_t^{1/2}\psi_{\eta,S}^\pm\|_{\text{P-BMO}} \le \|D_t^{1/2} f\|_{\text{P-BMO}} + \eta^{15/16}\|D_t^{1/2} H\|_{\text{P-BMO}} \le \|D_t^{1/2} f\|_{\text{P-BMO}} + c_4\eta^{15/16} < b_2+1, \tag{3.28}$$

provided η is sufficiently small. We define $\Gamma_S^\pm$ to be the graphs of $\psi_{\eta,S}^\pm$, respectively.

We claim that

$$g_{7/8}(t, x') \ge \psi_{\eta,S}^+(t, x') \ge g_{31/32}(t, x') \tag{3.29}$$

and

$$g_{7/8}^-(t, x') \le \psi_{\eta,S}^-(t, x') \le g_{31/32}^-(t, x'). \tag{3.30}$$

Indeed, these inequalities are a result of the fact that $\eta^{7/8} \gg \eta^{15/16} \gg \eta^{31/32}$ when η is very small along with the properties of H in relation to d. For example,

$$\psi_{\eta,S}^+(t, x') - g_{31/32}(t, x') = \eta^{15/16} H(t, x') - \eta^{31/32} d[F(t, x')].$$

Using Lemma 3.24 we have

$$d[F(t, x')] = 2h(t, x') \approx_n H(t, x').$$

Since the constants are independent of η the second inequality in (3.29) follows. The other inequalities are treated similarly.

With (3.29)–(3.30) at hand, properties (1) and (2) can be deduced directly from Lemmas 3.11 and 3.12. Note that to prove property (2), we observe that $(t, X) \in U_Q$ implies that (t, X) is above the graphs of *both* $g_{7/8}$ and $g_{31/32}$. Similarly, properties (3) and (4) can be deduced from (3.29) (or (3.30)) and Lemma 3.17: to prove (3) and (4) in, e.g., the context of $\psi_{\eta,S}^+(t, x')$, we simply observe that if (t, X) is above $\psi_{\eta,S}^+(t, x')$, then it is also above $g_{31/32}$.

To prove (5), let (s, Y) be the closest point on Γ_S to (t_{Q_S}, X_{Q_S}) and observe from Lemma 2.14 (3) that

$$\operatorname{dist}((t_{Q_S}, X_{Q_S}), (s, Y)) \le \eta \operatorname{diam}(Q_S).$$

As $(s, Y) = (s, y', y_n) = F(s, y')$, we have that

$$H(s, y') \approx_n d[F(s, y')] \le \operatorname{diam}(Q_S) + \eta \operatorname{diam}(Q_S) < 2 \operatorname{diam}(Q_S),$$

where we have used the properties of H given by Lemma 3.24. Then by definition,

$$\operatorname{dist}((t_{Q_S}, X_{Q_S}), (s, y', \psi_{\eta,S}^\pm(s, y'))) \le \operatorname{dist}((t_{Q_S}, X_{Q_S}), (s, Y)) + \eta^{15/16} H(s, y') \lesssim_n \eta^{15/16} \operatorname{diam}(Q_S).$$

Setting $(t_S^\pm, X_S^\pm) = (s, y', \psi_{\eta,S}^\pm(s, y'))$ and taking η sufficiently small (and hence K sufficiently large), we arrive at (5). □

4. Carleson measure estimates: Proof of Theorems 1.1 and 1.3

Before we get into the details of proving Theorems 1.1 and 1.3, we point out that the domains we produced in Proposition 3.25 support (a local version of) the Carleson measure estimate.

Lemma 4.1 [Hofmann and Lewis 2005, Lemma A.2]. *Let $\tilde{b}_1$, $\tilde{b}_2$ be fixed nonnegative constants. Let $\varphi(t, x')$ be a regular* $\mathrm{Lip}(1/2, 1)$ *function, with* $\mathrm{Lip}(1/2, 1)$ *constant $\tilde{b}_1$, such that $\|D_t^{1/2}\varphi\|_{\text{P-BMO}} \le \tilde{b}_2$. Let*

$$\Omega^+ = \{(t, X) = (t, x', x_n) : x_n > \varphi(t, x')\}.$$

Then there exist $M_0 = M_0(n, \tilde{b}_1, \tilde{b}_2) > 1$ and $c_5 = c_5(n, \tilde{b}_1, \tilde{b}_2)$, such that if u is a bounded solution to $(\partial_t - \Delta_x)u = 0$ in

$$\Omega^+((t_0, X_0), M_0 r) := B((t_0, X_0), M_0 r) \cap \Omega^+,$$

for some $(t_0, X_0) \in \partial\Omega$, then

$$\iint_{B((t_0,X_0),r)\cap\Omega^+} |\nabla u(s, Y)|^2 \tilde{\delta}(s, Y)\, dY\, ds \le c_5 r^{n+1} \|u\|^2_{L^\infty(\Omega^+((t_0,X_0),M_0 r))}. \tag{4.2}$$

Here $\tilde{\delta}(s, Y) = \mathrm{dist}((s, Y), \partial\Omega^+)$. An analogous statement holds for bounded solutions to $(\partial_t - \Delta_x)u = 0$ in

$$\Omega^-((t_0, X_0), M_0 r) := B((t_0, X_0), M_0 r) \cap \Omega^-,$$

where

$$\Omega^- = \{(t, X) = (t, x', x_n) : x_n < \varphi(t, x')\}.$$

Proof. The lemma is a consequence of [Hofmann and Lewis 2005, Lemma A.2], henceforth abbreviated [HL, A.2]. However, to reduce the proof of the lemma to [HL, A.2] one has to note two things. First, by using the parabolic version of the Dahlberg–Kenig–Stein pullback, the operator $(\partial_t - \Delta)$ is transformed to an operator of the form treated in [HL, A.2] in the upper half-space. Furthermore, $\Omega^+((t_0, X_0), M_0 r)$ is transformed into a region containing a Carleson region of size roughly $M_0 r$. Second, while stated for solutions in the upper half-space, [HL, A.2] uses only that u is a solution in a Carleson region. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let E be uniformly rectifiable in the parabolic sense and let u be a bounded solution to $(\partial_t - \Delta_X)u = 0$ in E^c. We may assume that $\|u\|_{L^\infty(E^c)} \neq 0$ since the conclusion of Theorem 1.1 holds trivially if $\|u\|_{L^\infty(E^c)} = 0$. Let

$$v := \frac{u}{\|u\|_{L^\infty(E^c)}}.$$

Then $\|v\|_{L^\infty(E^c)} = 1$ and it clearly suffices to prove the theorem with v in place of u.

For each $Q \in \mathbb{D}(E)$ we set

$$\beta_Q = \iint_{U_Q} |\nabla_X v|^2 \delta(s, Y)\, dY\, ds.$$

We first reduce the proof of the theorem to a statement concerning the β_Q.

Claim 4.3. *If there exists C (independent of v) such that*

$$\sum_{Q \subseteq Q_0} \beta_Q \leq C\sigma(Q_0), \quad \forall Q_0 \in \mathbb{D}(E), \tag{4.4}$$

then there exists C' such that

$$\sup_{(t,X)\in E, r>0} r^{-n-1} \iint_{B((t,X),r)} |\nabla_X v|^2 \delta(s,Y)\, dY\, ds \leq C'. \tag{4.5}$$

In particular, to prove the theorem it is enough to verify (4.4)

Sketch of Proof of Claim 4.3. To prove that (4.4) implies (4.5), we select a collection $\{Q_0^i\}_i \subset \mathbb{D}(E)$ such that, for each i, $\operatorname{diam}(Q_0^i) \approx \kappa r$, and such that the collection has uniformly bounded cardinality depending only on n and ADR. Furthermore, $B((t,X),r) \cap E \subset \cup_i Q_0^i$. Choosing κ large enough depending only on allowable parameters, we have that $B((t,X),r) \setminus E \subset \cup_i \cup_{Q \subset Q_0^i} U_Q$. We can then apply (4.4) to each $Q_0 = Q_0^i$. We omit the routine details. □

We have now reduced everything to the setting of our dyadic machinery and we are almost ready to begin employing the constructions in Proposition 3.25. Notice that these constructions are only likely to be helpful when bounding a β_Q when Q is a *good* cube. That is why the following claim is important when handling the *bad* cubes.

Claim 4.6. *There exists a constant A depending only the dimension, K, η, and* ADR *such that*

$$\beta_Q \leq A\sigma(Q).$$

Sketch of Proof of Claim 4.6. The claim follows readily from Lemma 2.5 and ADR as in [Hofmann et al. 2016]. We omit the details. □

We now prove (4.4). Fix $Q_0 \in \mathbb{D}(E)$. If $Q_0 \in \boldsymbol{S}^*$ for some $\boldsymbol{S}^* \in \mathcal{S}$ we let $\boldsymbol{S} = \boldsymbol{S}^* \cap \mathbb{D}_{Q_0}$ and note that $\boldsymbol{S}$ is a coherent subregime of $\boldsymbol{S}^*$ with maximal cube Q_0. $\mathbb{D}_{Q_0}$ has the disjoint decomposition

$$\mathbb{D}_{Q_0} = \{Q \in \mathcal{B} : Q \subseteq Q_0\} \cup \Bigg(\bigcup_{\boldsymbol{S}^* : Q(\boldsymbol{S}^*) \subset Q_0} \boldsymbol{S}^* \Bigg) \cup \boldsymbol{S}, \tag{4.7}$$

where $\boldsymbol{S} = \varnothing$ if Q_0 is not in a stopping time regime (i.e., if Q_0 is a bad cube). By Lemma 2.14 (2) and Claim 4.6,

$$\sum_{\substack{Q \subseteq Q_0 \\ Q \in \mathcal{B}}} \beta_Q \leq C \sum_{\substack{Q \subseteq Q_0 \\ Q \in \mathcal{B}}} \sigma(Q) \leq C C_{\eta,K} \sigma(Q_0). \tag{4.8}$$

Let us suppose, for the moment, that we can show that

$$\sum_{Q \in \boldsymbol{S}^*} \beta_Q \leq C\sigma(Q(\boldsymbol{S}^*)), \tag{4.9}$$

for all $\boldsymbol{S}^*$ such that $Q(\boldsymbol{S}^*) \subset Q_0$, and that

$$\sum_{Q \in \boldsymbol{S}} \beta_Q \leq C\sigma(Q(\boldsymbol{S})) = C\sigma(Q_0), \tag{4.10}$$

if Q_0 is in some stopping time regime. Then, by Lemma 2.14 (2),

$$\sum_{Q\in S}\beta_Q+\sum_{S^*:Q(S^*)\subset Q_0}\sum_{Q\in S^*}\beta_Q\lesssim\sigma(Q_0)+\sum_{S^*:Q(S^*)\subseteq Q_0}\sigma(Q(S^*))\lesssim(C_{\eta,K}+1)\sigma(Q_0).$$

Combining this estimate with (4.8) and using the decomposition of $\mathbb{D}_{Q_0}$ in (4.7) proves (4.4) and hence the theorem. Thus it suffices to verify (4.9) and (4.10). In the following we only prove (4.9) as the only change needed when proving (4.10) is to change S^* to S.

To prove (4.9) we use Proposition 3.25. Fix S^* such that $Q(S^*)\subset Q$, and let $\psi^{\pm}_{\eta,S^*}$ be the functions from Proposition 3.25. Then by Proposition 3.25 (1), if $Q\in S^*$, then $U_Q=U_Q^+\cup U_Q^-$ and so $\beta_Q=\beta_Q^++\beta_Q^-$, where

$$\beta_Q^{\pm}:=\iint_{U_Q^{\pm}}|\nabla_X v|^2\delta(s,Y)\,dY\,ds.$$

Clearly it is enough to show that

$$\sum_{Q\in S^*}\beta_Q^{\pm}\le C\sigma(Q(S^*)).$$

We prove the estimate for the sum of the β_Q^+ leaving the straightforward modification needed to handle the sum of the β_Q^- to the interested reader. Moreover, since the $\{U_Q\}$, and hence the $\{U_Q^{\pm}\}$, have bounded overlap it is enough to prove that

$$\iint_{\cup_{Q\in S^*}U_Q^+}|\nabla_X v|^2\delta(s,Y)\,dY\,ds\le C\sigma(Q(S^*)).\tag{4.11}$$

Let

$$\begin{aligned}\Omega^+_{S^*}&=B((t^+_{S^*},X^+_{S^*}),\,K^{7/8}\operatorname{diam}(Q_{S^*}))\cap\{x_n>\psi^+_{\eta,S^*}(t,x')\},\\ \widetilde{\Omega}^+_{S^*}&=B((t_{Q_{S^*}},X_{Q_{S^*}}),\,K^{3/4}\operatorname{diam}(Q_{S^*}))\cap\{x_n>\psi^+_{\eta,S^*}(t,x')\},\\ \widehat{\Omega}^+_{S^*}&=B((t_{Q_{S^*}},X_{Q_{S^*}}),\,M_0K^{3/4}\operatorname{diam}(Q_{S^*}))\cap\{x_n>\psi^+_{\eta,S^*}(t,x')\},\end{aligned}$$

where we recall that we use the coordinates $P_{S^*}\oplus P^{\perp}_{S^*}$ and that the notation $\{x_n>\psi^+_{\eta,S^*}(t,x')\}$ means

$$\{(t,x',x_n):x_n>\psi^+_{\eta,S^*}(t,x')\}.$$

We note that $\widehat{\Omega}^+_{S^*}\subset\Omega^+_{S^*}$ by (3.26). Proposition 3.25 (4) and (3.27) ensure that $\Omega^+_{S^*}$ is an open subset of E^c and hence v is a solution in $\Omega^+_{S^*}$. Applying Lemma 4.1 we have

$$\iint_{\widetilde{\Omega}^+_{S^*}}|\nabla_X v|^2\tilde{\delta}(s,Y)\,dY\,ds\lesssim\operatorname{diam}(Q(S^*))^{n+1}\approx\sigma(Q(S^*)),\tag{4.12}$$

where $\tilde{\delta}(s,Y)=\operatorname{dist}((s,Y),\Gamma^+_{S^*})$ and $\Gamma^+_{S^*}$ is the graph of $\psi^+_{S^*}$. Note that if $Q\in S^*$, we have $U_Q^+\in\widetilde{\Omega}^+_{S^*}$ by Proposition 3.25 (1). Moreover, by Proposition 3.25 (2), we have $\tilde{\delta}(s,Y)\approx\delta(s,Y)$ in $\cup_{Q\in S^*}U_Q^+$. Thus,

$$\begin{aligned}\iint_{\cup_{Q\in S^*}U_Q^+}|\nabla_X v|^2\delta(s,Y)\,dY\,ds&\approx\iint_{\cup_{Q\in S^*}U_Q^+}|\nabla_X v|^2\tilde{\delta}(s,Y)\,dY\,ds\\ &\le\iint_{\widetilde{\Omega}^+_S}|\nabla_X v|^2\tilde{\delta}(s,Y)\,dY\,ds\lesssim\sigma(Q(S^*)),\end{aligned}\tag{4.13}$$

where we used (4.12) in the last inequality. This proves (4.11), and the proof of the theorem is complete. □

The proof of Theorem 1.3 is nearly identical, the only difference being that in this case one needs to take $\mathcal{W}(\Omega)$, a Whitney decomposition of Ω, instead of $\mathcal{W}(E^c)$. Modulo the following remark we leave the details to the interested reader.

Remark 4.14. As in the elliptic setting, Theorem 1.3 does *not* require the corkscrew condition. On the other hand, the converse of the *elliptic* version of Theorem 1.3 [Garnett et al. 2018; Azzam et al. 2022] requires the additional assumption of interior corkscrews. Note that when carrying out the proof of Theorem 1.3, without the corkscrew assumption it may be the case that the Whitney regions U_Q are empty for some cubes $Q \in \mathbb{D}(\partial\Omega)$, but this does not affect the analysis above.

5. Further remarks

In this section we make some remarks concerning possible extensions and consequences of Theorem 1.1 and the constructions in Proposition 3.25. These extensions and consequences can be proved, or, we expect that they can be proved, largely using the tools already developed in the elliptic setting. Again, we believe that the main novelty of this paper is the approximation scheme, that is, Proposition 3.25.

The first observation is that solutions to the heat equation are (locally) smooth and that t-derivatives of solutions are, in fact, solutions. This allows one to produce a Caccioppoli-type inequality for the t-derivative which, in turn, allows one to improve the Carleson measure estimate in Theorems 1.1 and 1.3 to one that includes the t-derivative. In particular, under the same hypotheses as Theorem 1.1, the estimate

$$\sup_{(t,X)\in E, r>0} r^{-n-1} \iint_{B((t,X),r)} (|\nabla u|^2 + \delta(s,Y)^2 |\partial_s u|^2)\delta(s,Y)\, dY\, ds \le C\|u\|^2_{L^\infty(E^c)} \tag{5.1}$$

holds with a constant C depending only the dimension and the parabolic uniformly rectifiable constants for E.

The second observation is that the proof of Theorem 1.1 uses essentially three properties of u:

(i) $u \in L^\infty(E^c)$,

(ii) the Caccioppoli's inequality of Lemma 2.5, and

(iii) the local square function estimate stated in Lemma 4.1.

If one wants to extend the validity of Theorem 1.1 to more general parabolic equations in divergence form,

$$\mathcal{L} = \partial_t - \operatorname{div}_X A(t,X)\nabla_X,$$

where A is an $n \times n$ uniformly elliptic matrix, then some regularity conditions on the coefficients need to be imposed in order to guarantee property (iii). A natural sufficient condition is the parabolic analogue of the *Kenig–Pipher condition*.[12] More specifically, this means that A satisfies

$$|\nabla_X A(s,Y)|\delta(s,Y),\ |\partial_t A(s,Y)|\delta^2(s,Y) \in L^\infty(\mathbb{R}^{n+1} \setminus E),$$

[12] In fact, in [Hofmann and Lewis 2005, Lemma A.2], a slightly more general class of coefficients is permitted.

where $\delta(s, Y) = \operatorname{dist}((s, Y), E)$, and that there exists a constant M such that

$$\sup_{(t,X)\in E, r>0} r^{-n-1} \iint_{B((t,X),r)} |\nabla_X A(s, Y)|^2 \delta(s, Y)\, dY\, ds \le M,$$
$$\sup_{(t,X)\in E, r>0} r^{-n-1} \iint_{B((t,X),r)} |\partial_t A(s, Y)|^2 \delta^3(s, Y)\, dY\, ds \le M. \tag{5.2}$$

In particular, our results apply to this class of coefficients.

A final observation is that it seems likely that some form of ϵ-approximability [Hofmann et al. 2016; 2019] should hold in this parabolic setting along with the corresponding quantitative Fatou theorem [Bortz and Hofmann 2020]. In fact, it may be more reasonable to use the dyadic constructions from [Hofmann et al. 2016] in proving these results. Indeed, our construction here would provide some of the necessary initial estimates (Theorem 1.1), but it seems easier to deduce (parabolic) BV estimates using dyadic cubes. We also mention that it is natural to use the estimate (5.1) and to prove ϵ-approximability via a Carleson measure estimate which includes the t-derivative of the approximator. Note that this estimate was not included in [Rivera-Noriega 2003] and therefore the proof used in [Rivera-Noriega 2003, Proposition 4.3] is valid only if one works with a vertical version of the nontangential counting function $\mathcal{N}$ (or by a spatially nontangential version based on time-slice cones with t fixed), rather than a fully nontangential version.

Appendix: Proof of Lemma 3.24

The idea is to follow Stein's construction of the regularized distance [Stein 1970, Chapter VI, §1 & §2] and to combine this with ideas from some of the estimates produced in [David and Semmes 1991].

Proof of Lemma 3.24. Let $d := n + 1$ and $Z = \{(t, x') : h(t, x') = 0\}$. We note that Z is closed since h is continuous. We set $H(t, x') = 0$ for all $(t, x') \in Z$.

We need to define H off of Z, and following [David and Semmes 1991], we begin by producing a Whitney-type decomposition of Z^c with respect to h.[13] For each $(t, x') \in Z^c$, we let $I_{(t,x')}$ be the largest closed (parabolic) dyadic cube containing (t, x') satisfying

$$\operatorname{diam}(I_{(t,x')}) \le \tfrac{1}{20} \inf_{(\tau,z')\in I_{(t,x')}} h(\tau, z').$$

Recall that the diameter is defined with respect to the parabolic metric. To see that such a cube exists, set $r = \frac{1}{2} h(t, x')$ and note, as h is $\mathrm{Lip}(1/2, 1)$ with constant 1, that $h(\tau, z') \ge r$ in $B((t, x'), r)$. Therefore, any cube which contains (t, x') and which has diameter less than $\frac{1}{20} r$ is a *candidate* for $I_{(t,x')}$. We conduct this construction for each $(t, x') \in Z^c$, and we enumerate the resulting maximal cubes (without repetition) as $\{I_i\}_{i\in\Lambda}$. We note that

$$10 \operatorname{diam}(I_i) \le h(t, x') \le 60 \operatorname{diam}(I_i), \quad \forall (t, x') \in 10 I_i, \tag{A.1}$$

where κI is the parabolic dilation of I by a factor of κ. Indeed, if $(t, x') \in 10 I_i$, then (t, x') is at most a (parabolic) distance of $10 \operatorname{diam}(I_i)$ from a point in I_i, and hence, using the selection criterion for I_i and

[13] In contrast to the usual Whitney decomposition, in which h is the distance to a closed set, the present version remains valid even in the case that Z is empty.

the Lip(1/2, 1) condition for h,

$$h(t,x') \geq \min_{(\tau,z')\in I_i} h(\tau,z') - 10\,\mathrm{diam}(I_i) \geq 10\,\mathrm{diam}(I_i).$$

To prove the upper bound in (A.1), we note that if I is the parent of I_i, then I fails the selection criteria. Hence there exists $(\tau,z') \in I$ such that $h(\tau,z') < 20\,\mathrm{diam}(I) = 40\,\mathrm{diam}(I_i)$, and as h is Lip(1/2, 1) with constant 1 and $\mathrm{dist}((\tau,z'),(t,x')) \leq 20\,\mathrm{diam}(I_i)$, it follows that

$$h(t,x') \leq h(\tau,z') + 20\,\mathrm{diam}(I_i) \leq 60\,\mathrm{diam}(I_i).$$

Using (A.1), we have that

$$\tfrac{1}{6}\,\mathrm{diam}(I_j) \leq \mathrm{diam}(I_i) \leq 6\,\mathrm{diam}(I_j) \tag{A.2}$$

whenever $10I_i \cap 10_j \neq \varnothing$. By comparing volumes, it follows that the $\{10I_j\}$ have bounded overlap, with a constant depending on the dimension alone, that is,

$$\sum_{i\in\Lambda} 1_{I_i}(t,x') \leq N, \qquad \forall (t,x') \in \mathbb{R}^n, \tag{A.3}$$

where $N = N(n)$.

Let $Q_0 = \left\{(t,x') \in \mathbb{R}^n : |x'|_\infty \leq \frac{1}{2},\ |t| < \frac{1}{4}\right\}$ be the unit parabolic cube in $\mathbb{R}^n$. Let $\varphi \in C_0^\infty(3Q_0)$, with $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on $2Q_0$. Clearly the Lip(1/2, 1) constant of φ is bounded. For each $i \in \Lambda$, let (t_i, x_i') be the center of I_i and $\ell(I_i)$ be the parabolic side length of I_i, that is, $\ell(I_i) = \frac{1}{2}r_i$ and

$$I_i = \{(t,x') : |x - x_i|_\infty < r_i,\ |t - t_i| < r_i^2\}.$$

For $i \in \Lambda_i$ we set

$$\varphi_i(t,x') = \varphi\left(\frac{t - t_i}{\ell(I_i)^2}, \frac{x - x_i}{\ell(I_i)}\right).$$

Then $0 \leq \varphi_i \leq 1$, φ_i is supported in $3I_i$, $\varphi_i \equiv 1$ on $2I_i$, φ_i is Lip(1/2, 1) with constant less than $\ell(I_i)^{-1} \approx_n \mathrm{diam}(I_i)^{-1}$, and (on all of $\mathbb{R}^n$),

$$\ell(I_i)^{2m}|\partial_t^m \varphi_i| + \ell(I_i)|\nabla_{x'}\varphi_i| \approx_{n,m} \mathrm{diam}(I_i)^{2m}|\partial_t^m \varphi_i| + \mathrm{diam}(I_i)|\nabla_{x'}\varphi_i| \lesssim \tilde{c}_{n,m}.$$

We now define

$$H(t,x') := \sum_{i\in\Lambda} \mathrm{diam}(I_i)\varphi_i(t,x').$$

Using (A.1) we see that $3I_i$ does not meet Z for any $i \in \Lambda$, and hence $H(t,x') = 0$ for all $(t,x') \in Z$. By construction, if $(t,x') \in Z^c$ then $(t,x') \in I_j$ for some $j \in \Lambda$, and as $\varphi_j(t,x') = 1$, using also (A.1), we have that

$$H(t,x') \geq \mathrm{diam}(I_j) \geq \tfrac{1}{60}h(t,x').$$

This proves the lower bound in (1). To prove the upper bound in (1) we have, by (A.2) and (A.3),

$$H(t,x') \leq \sum_{i:3I_i\cap 3I_j\neq\varnothing} \mathrm{diam}(Q_j) \leq 6N\,\mathrm{diam}(Q_i) \leq \tfrac{3}{5}Nh(t,x'),$$

where we used (A.1) in the last inequality. Having proved (1), we see that the proof of (2) is very similar. For instance, using the bounds for the t-derivatives of φ_i, if $(t,x')\in Z^c$, then $(t,x')\in I_j$ for some j, and hence

$$|\partial_t^m H(t,x')| \le \tilde{c}_{n,m} \sum_{i:3I_i\cap 3I_j\neq\varnothing} \operatorname{diam}(I_i)\operatorname{diam}(I_i)^{-2m} \lesssim CN \operatorname{diam}(I_j)^{-2m+1} \approx h(t,x')^{-2m+1}.$$

The bound for $|\nabla_{x'}^m H|$ has a similar proof. Finally, to see that H is $\operatorname{Lip}(1/2,1)$, we have

$$\begin{aligned}|H(t,x')-H(s,y')| &\le \sum_{i:(t,x')\in 3I_i} \operatorname{diam}(I_i)|\varphi_i(t,x')-\varphi(s,y')| + \sum_{i:(s,y')\in 3I_i} \operatorname{diam}(I_i)|\varphi_i(t,x')-\varphi(s,y')| \\ &\le 2c'N[|t-s|^{1/2}+|x'-y'|],\end{aligned}$$

where we used that φ_i is a $\operatorname{Lip}(1/2,1)$ function with constant $c'\operatorname{diam}(I_i)^{-1}$.

Now we get to the heart of the matter, that is, proving the half-order in time regularity of H (this part is not in Stein's book, but rather draws a great deal of inspiration from [David and Semmes 1991]). By the results in [Hofmann et al. 2003, pp. 370–373], it suffices to verify the Carleson measure estimate

$$\tilde{\nu}(s,y',\rho) \le c_4'\rho^{n+1}, \quad \forall (s,y')\in\mathbb{R}^n, \quad \rho>0, \tag{A.4}$$

where

$$\tilde{\nu}(s,y',\rho) := \int_0^\rho \iint_{B((s,y'),\rho)} \hat{\gamma}(\tau,z',r)^2\, d\sigma(\tau,z')\,\frac{dr}{r},$$

where $d\sigma(\tau,z')=\sqrt{1+|\nabla_{z'}H(\tau,z')|}\,dz'\,dt$ and

$$\hat{\gamma}(\tau,z',r) := \inf_L \left[r^{-d} \iint_{B((\tau,z'),r)} \left(\frac{H(t,x')-L(x')}{r}\right)^2 d\sigma(t,x')\right]^{1/2},$$

where the infimum is taken over all affine functions L of x' only, and we recall that $d=n+1$.

The idea behind the proof of the estimate (A.4) is as follows. If the scale r is large with respect to $h(\tau,z')$, then H is well approximated by just the linear function 0, If the scale is small with respect to $h(\tau,z')$, then, necessarily, $h(\tau,z')>0$ and H is flat (below the scale r) by the derivative estimates (2) and therefore we can approximate H by its *x'-tangent plane*.

Now let us begin the proof of (A.4). Fix $(s,y')\in\mathbb{R}^n$ and $\rho>0$. Set $h_\rho(t,x') := \min\{\frac{1}{60}h(t,x'),\rho\}$, and write

$$\begin{aligned}\tilde{\nu}(s,y',\rho) &= \int_0^\rho \iint_{B((s,y'),\rho)} \hat{\gamma}(\tau,z',r)^2 d\sigma(\tau,z')\,\frac{dr}{r} \\ &= \iint_{B((s,y'),\rho)} \int_{h_\rho(\tau,z')}^{\rho} \hat{\gamma}(\tau,z',r)^2\,\frac{dr}{r}\,d\sigma(\tau,z') + \iint_{B((s,y'),\rho)} \int_0^{h_\rho(\tau,z')} \hat{\gamma}(\tau,z',r)^2\,\frac{dr}{r}\,d\sigma(\tau,z') \\ &= T_1+T_2.\end{aligned}$$

Let us handle term T_2 first. For (τ,z') and r in the domain of integration, $r>0$ and $h(\tau,z')\ge 60r$. In particular, $(\tau,z')\in I_j$ for some $j\in\Lambda$, and for all such j it holds that $I_j\cap B((s,y'),\rho)\neq\varnothing$ and $r\le\operatorname{diam}(I_j)$

(by the right-hand estimate in (A.1)). Thus we have

$$T_2 \le \sum_{j\in\widetilde{\Lambda}} \iint_{I_j\cap B((s,y'),\rho)} \int_0^{\min\{\operatorname{diam}(I_j),\rho\}} \hat{\gamma}(\tau,z',r)^2\,\frac{dr}{r}\,d\sigma(\tau,z'), \tag{A.5}$$

where $\widetilde{\Lambda} = \{j\in\Lambda : B((s,y'),\rho)\cap I_j \ne \varnothing\}$. Fix $j\in\widetilde{\Lambda}$, $(\tau,z')\in I_j$, and $r\le \operatorname{diam}(I_j)$. Using the affine function

$$L_{(\tau,z')}(x') = H(\tau,z') + \nabla_{z'}H(\tau,z')\cdot(x'-z'),$$

we find by Taylor's theorem, Lemma 3.24 (2) (already proved above), and (A.1) that

$$\begin{aligned}\hat{\gamma}(\tau,z',r)^2 &\le r^{-d}\iint_{B((\tau,z'),r)} \left(\frac{|H(t,x')-L_{(\tau,z')}(x')|}{r}\right)^2 d\sigma(t,x')\\ &\lesssim r^2 \sup_{B(\boldsymbol{x}_{I_j},2\operatorname{diam}(I_j))} [|\partial_t H|^2 + |\nabla^2 H|]^2 \lesssim r^2 \sup_{B(\boldsymbol{x}_{I_j},2\operatorname{diam}(I_j))} h^{-2}(t,x')\\ &\lesssim r^2\operatorname{diam}(I_j)^{-2} \lesssim r^2(\min\{\operatorname{diam}(I_j),\rho\})^{-2},\end{aligned} \tag{A.6}$$

where $\boldsymbol{x}_{I_j} = (t_{I_j}, x'_{I_j})$ is the center of I_j. Using (A.6) and (A.5) we obtain

$$\begin{aligned}T_2 &\lesssim \sum_{j\in\widetilde{\Lambda}} \iint_{I_j\cap B((s,y'),\rho)} \int_0^{\min\{\operatorname{diam}(I_j),\rho\}} r^2(\min\{\operatorname{diam}(I_j),\rho\})^{-2}\,\frac{dr}{r}\,d\sigma(\tau,z')\\ &\lesssim \sum_{j\in\widetilde{\Lambda}} \iint_{I_j\cap B((s,y'),\rho)} 1\,d\sigma(\tau,z') \lesssim \sum_{j\in\widetilde{\Lambda}} \sigma(I_j\cap B((s,y'),\rho)) \lesssim \sigma(B((s,y'),\rho)) \lesssim \rho^d,\end{aligned}$$

as desired.

Having obtained the desired bound for T_2, we turn our attention to T_1. For $(\tau,z')\in\mathbb{R}^n$ and $r>0$, set

$$\Lambda'(\tau,z',r) := \{i \in I_i\cap B((\tau,z'),r) \ne \varnothing\}.$$

Note that in term T_1, we have $r\in\left(\frac{1}{60}h(\tau,z'),\rho\right)$, so that $h(\hat{\tau},\hat{z}') < 61r$ for all $(\hat{\tau},\hat{z}')\in B((\tau,z'),r)$ because h has Lip$(1/2,1)$ constant 1. Hence, by (A.1), we have $\operatorname{diam}(I_i)\le 7r\le 7\rho$ for all $i\in\Lambda'(\tau,z',r)$. In particular, since $(\tau,z')\in B((s,y'),\rho)$,

$$\bigcup_{i\in\Lambda'(\tau,z',r)} I_i \subset B((s,y'),10\rho). \tag{A.7}$$

For $(\tau,z')\in B((s,y'),\rho)$, with $r\in\left(\frac{1}{60}h(\tau,z'),\rho\right)$, we plug $L=0$ into the definition of $\hat{\gamma}$ and use property (1) (which we have already established) along with (A.1) to see that

$$\begin{aligned}\hat{\gamma}(\tau,z',r)^2 &\le r^{-d}\iint_{B((\tau,z'),r)}\left(\frac{H(t,x')}{r}\right)^2 d\sigma(t,x') \lesssim \sum_{i\in\Lambda'(\tau,z',r)} r^{-d}\iint_{I_i}\operatorname{diam}(I_i)^2 r^{-2}\,d\sigma(t,x')\\ &\lesssim \sum_{i\in\Lambda'(\tau,z',r)}\left(\frac{\operatorname{diam}(I_i)}{r}\right)^{d+2} \lesssim \sum_{i\in\Lambda'(\tau,z',r)}\left(\frac{\operatorname{diam}(I_i)}{r}\right)^{d+1},\end{aligned} \tag{A.8}$$

where we used the fact that $\operatorname{diam}(I_i) \lesssim r$ in the estimate on the last line. Thus, if we let

$$\Lambda_0 := \{i \in \Lambda : I_i \subset B((s, y'), 10\rho),\ \operatorname{diam}(I_i) \le 7\rho\},$$

then using (A.7), the definition of $\Lambda'(\tau, z', r)$, and again using the fact that $\operatorname{diam}(I_i) \le 7r$ for $i \in \Lambda'(\tau, z', r)$, we obtain

$$\begin{aligned} T_1 &\le \iint_{B((s,y'),\rho)} \int_{h(\tau,z')/60}^{\rho} \sum_{i\in\Lambda'(\tau,z',r)} \operatorname{diam}(I_i)^{d+1} \frac{dr}{r^{d+2}}\, d\sigma(\tau, z') \\ &\lesssim \sum_{i\in\Lambda_0} \operatorname{diam}(I_i)^{d+1} \int_{\operatorname{diam}(I_i)/7}^{\rho} \int_{\operatorname{dist}((\tau,z'),I_i)<r} 1\, d\sigma(\tau, z') \frac{dr}{r^{d+2}} \\ &\lesssim \sum_{i\in\Lambda_0} \operatorname{diam}(I_i)^{d+1} \int_{\operatorname{diam}(I_i)/7}^{\infty} \frac{dr}{r^2} \lesssim \sum_{i\in\Lambda_0} \operatorname{diam}(I_i)^{d} \lesssim \rho^d. \end{aligned}$$

This yields the desired Carleson measure estimate and concludes the proof of the lemma. □

Acknowledgements

The authors Hoffman, Hofmann, and Luna Garcia were partially supported by NSF grants DMS-1664047 and DMS-2000048. Nyström was partially supported by grant 2017-03805 from the Swedish research council (VR).

References

[Azzam et al. 2020] J. Azzam, S. Hofmann, J. M. Martell, M. Mourgoglou, and X. Tolsa, "Harmonic measure and quantitative connectivity: geometric characterization of the L^p-solvability of the Dirichlet problem", *Invent. Math.* **222**:3 (2020), 881–993. MR Zbl

[Azzam et al. 2022] J. Azzam, J. Garnett, M. Mourgoglou, and X. Tolsa, "Uniform rectifiability, elliptic measure, square functions, and ε-approximability via an ACF monotonicity formula", *Int. Math. Res. Not.* **2022** (online publication June 2022).

[Bortz and Hofmann 2020] S. Bortz and S. Hofmann, "Quantitative Fatou theorems and uniform rectifiability", *Potential Anal.* **53**:1 (2020), 329–355. MR Zbl

[Bortz et al. 2022a] S. Bortz, J. Hoffman, S. Hofmann, J. L. Luna-Garcia, and K. Nyström, "Coronizations and big pieces in metric spaces", *Ann. Inst. Fourier (Grenoble)* **72**:5 (2022), 2037–2078. MR Zbl

[Bortz et al. 2022b] S. Bortz, J. Hoffman, S. Hofmann, J. L. Luna-Garcia, and K. Nyström, "On big pieces approximations of parabolic hypersurfaces", *Ann. Fenn. Math.* **47**:1 (2022), 533–571. MR Zbl

[Bortz et al. 2023] S. Bortz, J. Hoffman, S. Hofmann, J. L. Luna-Garcia, and K. Nyström, "Corona decompositions for parabolic uniformly rectifiable sets", *J. Geom. Anal.* **33**:3 (2023), art. id. 96. MR Zbl

[Christ 1990] M. Christ, "A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral", *Colloq. Math.* **60**:2 (1990), 601–628. MR Zbl

[Coifman and Semmes 1991] R. Coifman and S. Semmes, "L^2 estimates in nonlinear Fourier analysis", pp. 79–95 in *Harmonic analysis* (Sendai, Japan, 1990), edited by S. Igari, Springer, 1991. MR Zbl

[Coifman et al. 1982] R. R. Coifman, A. McIntosh, and Y. Meyer, "L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes", *Ann. of Math.* (2) **116**:2 (1982), 361–387. MR Zbl

[Coifman et al. 1983] R. R. Coifman, G. David, and Y. Meyer, "La solution des conjectures de Calderón", *Adv. Math.* **48**:2 (1983), 144–148. MR Zbl

[Dahlberg 1977] B. E. J. Dahlberg, "Estimates of harmonic measure", *Arch. Ration. Mech. Anal.* **65**:3 (1977), 275–288. MR Zbl

[David 1991] G. David, *Wavelets and singular integrals on curves and surfaces*, Lecture Notes in Math. **1465**, Springer, 1991. MR Zbl

[David and Semmes 1991] G. David and S. Semmes, *Singular integrals and rectifiable sets in $\mathbb{R}^n$: beyond Lipschitz graphs*, Astérisque **193**, Soc. Math. France, Paris, 1991. MR Zbl

[David and Semmes 1993] G. David and S. Semmes, *Analysis of and on uniformly rectifiable sets*, Math. Surv. Monogr. **38**, Amer. Math. Soc., Providence, RI, 1993. MR Zbl

[Dindoš et al. 2011] M. Dindoš, C. Kenig, and J. Pipher, "BMO solvability and the A_∞ condition for elliptic operators", *J. Geom. Anal.* **21**:1 (2011), 78–95. MR Zbl

[Dindoš et al. 2017] M. Dindoš, S. Petermichl, and J. Pipher, "BMO solvability and the A_∞ condition for second order parabolic operators", *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **34**:5 (2017), 1155–1180. MR Zbl

[Garnett et al. 2018] J. Garnett, M. Mourgoglou, and X. Tolsa, "Uniform rectifiability from Carleson measure estimates and ε-approximability of bounded harmonic functions", *Duke Math. J.* **167**:8 (2018), 1473–1524. MR Zbl

[Genschaw and Hofmann 2020] A. Genschaw and S. Hofmann, "A weak reverse Hölder inequality for caloric measure", *J. Geom. Anal.* **30**:2 (2020), 1530–1564. MR Zbl

[Hofmann 1995] S. Hofmann, "A characterization of commutators of parabolic singular integrals", pp. 195–210 in *Fourier analysis and partial differential equations* (Miraflores de la Sierra, Spain, 1992), edited by J. García-Cuerva et al., CRC, Boca Raton, FL, 1995. MR Zbl

[Hofmann 1997] S. Hofmann, "Parabolic singular integrals of Calderón-type, rough operators, and caloric layer potentials", *Duke Math. J.* **90**:2 (1997), 209–259. MR Zbl

[Hofmann and Le 2018] S. Hofmann and P. Le, "BMO solvability and absolute continuity of harmonic measure", *J. Geom. Anal.* **28**:4 (2018), 3278–3299. MR Zbl

[Hofmann and Lewis 1996] S. Hofmann and J. L. Lewis, "L^2 solvability and representation by caloric layer potentials in time-varying domains", *Ann. of Math.* (2) **144**:2 (1996), 349–420. MR Zbl

[Hofmann and Lewis 2005] S. Hofmann and J. L. Lewis, "The L^p Neumann problem for the heat equation in non-cylindrical domains", *J. Funct. Anal.* **220**:1 (2005), 1–54. MR Zbl

[Hofmann et al. 2003] S. Hofmann, J. L. Lewis, and K. Nyström, "Existence of big pieces of graphs for parabolic problems", *Ann. Fenn. Math.* **28**:2 (2003), 355–384. MR Zbl

[Hofmann et al. 2004] S. Hofmann, J. L. Lewis, and K. Nyström, "Caloric measure in parabolic flat domains", *Duke Math. J.* **122**:2 (2004), 281–346. MR Zbl

[Hofmann et al. 2016] S. Hofmann, J. M. Martell, and S. Mayboroda, "Uniform rectifiability, Carleson measure estimates, and approximation of harmonic functions", *Duke Math. J.* **165**:12 (2016), 2331–2389. MR Zbl

[Hofmann et al. 2019] S. Hofmann, J. M. Martell, and S. Mayboroda, "Transference of scale-invariant estimates from Lipschitz to non-tangentially accessible to uniformly rectifiable domains", preprint, 2019. arXiv 1904.13116

[Hytönen and Kairema 2012] T. Hytönen and A. Kairema, "Systems of dyadic cubes in a doubling metric space", *Colloq. Math.* **126**:1 (2012), 1–33. MR Zbl

[Hytönen and Martikainen 2012] T. Hytönen and H. Martikainen, "Non-homogeneous Tb theorem and random dyadic cubes on metric measure spaces", *J. Geom. Anal.* **22**:4 (2012), 1071–1107. MR Zbl

[Jones 1990] P. W. Jones, "Rectifiable sets and the traveling salesman problem", *Invent. Math.* **102**:1 (1990), 1–15. MR Zbl

[Kenig et al. 2000] C. Kenig, H. Koch, J. Pipher, and T. Toro, "A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations", *Adv. Math.* **153**:2 (2000), 231–298. MR Zbl

[Kenig et al. 2016] C. Kenig, B. Kirchheim, J. Pipher, and T. Toro, "Square functions and the A_∞ property of elliptic measures", *J. Geom. Anal.* **26**:3 (2016), 2383–2410. MR Zbl

[Lewis and Murray 1995] J. L. Lewis and M. A. M. Murray, *The method of layer potentials for the heat equation in time-varying domains*, Mem. Amer. Math. Soc. **545**, Amer. Math. Soc., Providence, RI, 1995. MR Zbl

[Lewis and Silver 1988] J. L. Lewis and J. Silver, "Parabolic measure and the Dirichlet problem for the heat equation in two dimensions", *Indiana Univ. Math. J.* **37**:4 (1988), 801–839. MR Zbl

[Rivera-Noriega 2003] J. Rivera-Noriega, "Absolute continuity of parabolic measure and area integral estimates in non-cylindrical domains", *Indiana Univ. Math. J.* **52**:2 (2003), 477–525. MR Zbl

[Stein 1970] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Math. Ser. **30**, Princeton Univ. Press, 1970. MR Zbl

[Zhao 2018] Z. Zhao, "BMO solvability and A_∞ condition of the elliptic measures in uniform domains", *J. Geom. Anal.* **28**:2 (2018), 866–908. MR Zbl

Received 29 Mar 2021. Revised 23 Sep 2021. Accepted 26 Oct 2021.

SIMON BORTZ: sbortz@ua.edu
Department of Mathematics, University of Alabama, Tuscaloosa, AL, United States

JOHN HOFFMAN: jlh82b@mail.missouri.edu
Department of Mathematics, University of Missouri at Columbia, Columbia, MO, United States

STEVE HOFMANN: hofmanns@missouri.edu
Department of Mathematics, University of Missouri at Columbia, Columbia, MO, United States

JOSÉ LUIS LUNA GARCÍA: lunagaj@mcmaster.ca
Department of Mathematics & Statistics, McMaster University, Hamilton, Canada

KAJ NYSTRÖM: kaj.nystrom@math.uu.se
Department of Mathematics, Uppsala University, Uppsala, Sweden

Guidelines for Authors

Authors may submit manuscripts in PDF format on-line at the Submission page at msp.org/apde.

Originality. Submission of a manuscript acknowledges that the manuscript is original and and is not, in whole or in part, published or under consideration for publication elsewhere. It is understood also that the manuscript will not be submitted elsewhere while under consideration for publication in this journal.

Language. Articles in APDE are usually in English, but articles written in other languages are welcome.

Required items. A brief abstract of about 150 words or less must be included. It should be self-contained and not make any reference to the bibliography. If the article is not in English, two versions of the abstract must be included, one in the language of the article and one in English. Also required are keywords and subject classifications for the article, and, for each author, postal address, affiliation (if appropriate), and email address.

Format. Authors are encouraged to use LaTeX but submissions in other varieties of TeX, and exceptionally in other formats, are acceptable. Initial uploads should be in PDF format; after the refereeing process we will ask you to submit all source material.

References. Bibliographical references should be complete, including article titles and page ranges. All references in the bibliography should be cited in the text. The use of BibTeX is preferred but not required. Tags will be converted to the house format, however, for submission you may use the format of your choice. Links will be provided to all literature with known web locations and authors are encouraged to provide their own links in addition to those supplied in the editorial process.

Figures. Figures must be of publication quality. After acceptance, you will need to submit the original source files in vector graphics format for all diagrams in your manuscript: vector EPS or vector PDF files are the most useful.

Most drawing and graphing packages (Mathematica, Adobe Illustrator, Corel Draw, MATLAB, etc.) allow the user to save files in one of these formats. Make sure that what you are saving is vector graphics and not a bitmap. If you need help, please write to graphics@msp.org with details about how your graphics were generated.

White space. Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Proofs. Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

Analysis & PDE

Volume 16 No. 4 2023

Strong semiclassical limits from Hartree and Hartree–Fock to Vlasov–Poisson equations 891
LAURENT LAFLECHE and CHIARA SAFFIRIO

Marstrand–Mattila rectifiability criterion for 1-codimensional measures in Carnot groups 927
ANDREA MERLO

Finite-time blowup for a Navier–Stokes model equation for the self-amplification of strain 997
EVAN MILLER

Eigenvalue bounds for Schrödinger operators with random complex potentials 1033
OLEG SAFRONOV

Carleson measure estimates for caloric functions and parabolic uniformly rectifiable sets 1061
SIMON BORTZ, JOHN HOFFMAN, STEVE HOFMANN, JOSÉ LUIS LUNA GARCÍA and KAJ NYSTRÖM