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LONG TIME SOLUTIONS FOR QUASILINEAR HAMILTONIAN PERTURBATIONS OF SCHRÖDINGER AND KLEIN–GORDON EQUATIONS ON TORI

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We consider quasilinear, Hamiltonian perturbations of the cubic Schrödinger and of the cubic (derivative) Klein–Gordon equations on the d-dimensional torus. If $\epsilon \ll 1$ is the size of the initial datum, we prove that the lifespan of solutions is *strictly* larger than the local existence time ϵ^{-2}. More precisely, concerning the Schrödinger equation we show that the lifespan is at least of order $O(\epsilon^{-4})$, and in the Klein–Gordon case we prove that the solutions exist at least for a time of order $O(\epsilon^{-8/3^-})$ as soon as $d \geq 3$. Regarding the Klein–Gordon equation, our result presents novelties also in the case of semilinear perturbations: we show that the lifespan is at least of order $O(\epsilon^{-10/3^-})$, improving, for cubic nonlinearities and $d \geq 4$, the general results of Delort (*J. Anal. Math.* **107** (2009), 161–194) and Fang and Zhang (*J. Differential Equations* **249**:1 (2010), 151–179).

1. Introduction

This paper is concerned with the study of the lifespan of solutions of two classes of quasilinear, Hamiltonian equations on the d-dimensional torus $\mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$, $d \geq 1$. We study quasilinear perturbations of the Schrödinger and Klein–Gordon equations.

The Schrödinger equation we consider is

$$\begin{cases} \mathrm{i}\partial_t u + \Delta u - V * u + [\Delta(h(|u|^2))]h'(|u|^2)u - |u|^2 u = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{NLS}$$

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where $\mathbb{C} \ni u := u(t,x)$, $x \in \mathbb{T}^d$, $d \geq 1$, $V(x)$ is a real-valued potential even with respect to x, $h(x)$ is a function in $C^\infty(\mathbb{R};\mathbb{R})$ such that $h(x) = O(x^2)$ as $x \to 0$. The initial datum u_0 has small size and belongs to the Sobolev space $H^s(\mathbb{T}^d)$ (see (3-2)) with $s \gg 1$.

We examine also the Klein–Gordon equation

$$\begin{cases} \partial_{tt}\psi - \Delta\psi + m\psi + f(\psi) + g(\psi) = 0, \\ \psi(0,x) = \psi_0, \\ \partial_t\psi(0,x) = \psi_1, \end{cases} \tag{KG}$$

where $\mathbb{R} \ni \psi := \psi(t,x)$, $x \in \mathbb{T}^d$, $d \geq 1$ and $m > 0$. The initial data (ψ_0, ψ_1) have small size and belong to the Sobolev space $H^s(\mathbb{T}^d) \times H^{s-1}(\mathbb{T}^d)$ for some $s \gg 1$. The nonlinearity $f(\psi)$ has the form

$$f(\psi) := -\sum_{j=1}^{d} \partial_{x_j}(\partial_{\psi_{x_j}} F(\psi, \nabla\psi)) + (\partial_\psi F)(\psi, \nabla\psi), \tag{1-1}$$

where $F(y_0, y_1, \dots, y_d) \in C^\infty(\mathbb{R}^{d+1}, \mathbb{R})$, and has a zero of order at least 5 at the origin. The nonlinear term $g(\psi)$ has the form

$$g(\psi) = (\partial_{y_0} G)(\psi, \Lambda_{\mathrm{KG}}^{1/2}\psi) + \Lambda_{\mathrm{KG}}^{1/2}(\partial_{y_1} G)(\psi, \Lambda_{\mathrm{KG}}^{1/2}\psi), \tag{1-2}$$

where $G(y_0, y_1) \in C^\infty(\mathbb{R}^2;\mathbb{R})$ is a homogeneous polynomial of degree 4 and Λ_{KG} is the operator

$$\Lambda_{\mathrm{KG}} := (-\Delta + m)^{1/2}, \tag{1-3}$$

defined by linearity as

$$\Lambda_{\mathrm{KG}} e^{\mathrm{i} j\cdot x} = \Lambda_{\mathrm{KG}}(j) e^{\mathrm{i} j\cdot x}, \qquad \Lambda_{\mathrm{KG}}(j) = \sqrt{|j|^2 + m} \quad \text{for all } j \in \mathbb{Z}^d. \tag{1-4}$$

Historical introduction for (NLS). Quasilinear Schrödinger equations of the specific form (NLS) appear in many domains of physics like plasma physics and fluid mechanics [Litvak and Sergeev 1978; Porkolab and Goldman 1976], quantum mechanics [Hasse 1980], and condensed matter theory [Makhankov and Fedyanin 1984]. They are also important in the study of Kelvin waves in the superfluid turbulence [Laurie et al. 2010]. Equations of the form (NLS) posed in the *Euclidean space* have received the attention of many mathematicians. The first result, concerning the local well-posedness, is due to Poppenberg [2001] in the one-dimensional case. This has been generalized by Colin [2002] to any dimension. A more general class of equations is considered in the pioneering work by Kenig, Ponce and Vega [Kenig et al. 2004]. These results of local well-posedness have been recently optimized, in terms of regularity of the initial condition, by Marzuola, Metcalfe and Tataru [Marzuola et al. 2021]. Existence of standing waves has been studied in [Colin 2003; Colin and Jeanjean 2004]. The global well-posedness was established by de Bouard, Hayashi and Saut [de Bouard et al. 1997] in dimensions 2 and 3 for small data. This proof is based on dispersive estimates and the energy method. New ideas have been introduced in studying the global well-posedness for other quasilinear equations on the *Euclidean space*. Here the aforementioned tools are combined with *normal form* reductions. We quote [Ionescu and Pusateri 2015; 2018] for the water-waves equation in two dimensions.

Very little is known when (NLS) is posed on a *compact manifold*. The first local well-posedness results on the circle are given in the work by Baldi, Haus and Montalto [Baldi et al. 2018] and in [Feola and Iandoli 2019]. Recently these results have been generalized to the case of tori of any dimension in [Feola and Iandoli 2022]. Except these local existence results, nothing is known concerning the long time behavior of the solutions. The problem of *global existence/blow-up* is completely open. In the aforementioned paper [de Bouard et al. 1997] they use the dispersive character of the flow of the linear Schrödinger equation. This property is not present on compact manifolds: the solutions of the linear Schrödinger equation do not decay when the time goes to infinity. However in the one-dimensional case in [Feola and Iandoli 2020; 2021] it is proven that *small* solutions of quasilinear Schrödinger equations exist for long, but finite, times. In these works two of us exploit the fact that quasilinear Schrödinger equations may be reduced to constant coefficients through a *paracomposition* generated by a diffeomorphism of the circle. This powerful tool has been used for the same purpose by other authors in the context of water-waves equations, firstly by Berti and Delort [2018] in a *nonresonant* regime, and secondly by Berti, Feola and Pusateri [Berti et al. 2023; 2021b] and Berti, Feola and Franzoi [Berti et al. 2021a] in the *resonant* case. We also mention that this feature has been used in other contexts for the same equations; for instance Feola and Procesi [2015] proved the existence of a large set of *quasiperiodic* (and hence globally defined) solutions when the problem is posed on the circle. This "reduction to constant coefficients" is a peculiarity of one-dimensional problems; in higher dimensions new ideas have to be introduced. For quasilinear equations on tori of dimension 2 we quote the paper about long-time solutions for water-waves problem in [Ionescu and Pusateri 2019], where a different *normal form* analysis was presented.

***Historical introduction for* (KG).** The local existence for (KG) is classical and we refer to [Kato 1975]. Many analyses have been done for *global/long time* existence.

When the equation is posed on the Euclidean space we have global existence for small and localized data in [Delort 2016; Stingo 2018]; here the authors use dispersive estimates on the linear flow combined with quasilinear normal forms.

For (KG) on compact manifolds we quote [Delort 2012; 2015] on $\mathbb{S}^d$ and [Delort and Szeftel 2004] on $\mathbb{T}^d$. The results obtained, in terms of length of the lifespan of solutions, are stronger in the case of the spheres. More precisely, in the case of spheres the authors show the following: if m in (KG) is chosen outside of a set of zero Lebesgue measure, then for any natural number N, any initial condition of size ϵ (small depending on N) produces a solution whose lifespan is at least of magnitude ϵ^{-N}. In the case of tori in [Delort and Szeftel 2004] they consider a quasilinear equation, vanishing quadratically at the origin and they prove that the lifespan of solutions is of order ϵ^{-2} if the initial condition has size ϵ small enough. The differences between the two results are due to the different behaviors of the eigenvalues of the square root of the Laplace–Beltrami operator on $\mathbb{S}^d$ and $\mathbb{T}^d$. The difficulty on the tori is a consequence of the fact that the set of differences of eigenvalues of $\sqrt{-\Delta_{\mathbb{T}^d}}$ is dense in $\mathbb{R}$ if $d \geq 2$; this does not happen in the case of spheres. A more general set of manifolds where this does not happen is the Zoll manifolds; in this case we quote Delort and Szeftel [2006] and Bambusi, Delort, Grébert and Szeftel [Bambusi et al. 2007] for semilinear Klein–Gordon equations. For semilinear Klein–Gordon equations on tori we have the results of [Delort 2009; Fang and Zhang 2010]. In [Delort 2009] the author proves that if the nonlinearity is vanishing at order $k+1$ at zero then any initial datum of small size ϵ produces a solution whose lifespan is at least of magnitude $\epsilon^{-k(1+2/d)}$, up to a logarithmic loss. In [Fang and Zhang 2010] the authors obtain a time $O(\epsilon^{-k(3/2)^-})$. We improve these results; see Theorems 4 and 3, when $k=2$.

Statement of the main results. The aim of this paper is to prove, in the spirit of [Ionescu and Pusateri 2019], that we may go beyond the trivial time of existence, given by the local well-posedness theorem, which is ϵ^{-2} since we are considering equations vanishing cubically at the origin and initial conditions of size ϵ.

In order to state our main theorem for (NLS) we need to make some hypotheses on the potential V. We consider potentials having the form

$$V(x)=(2\pi)^{-d/2}\sum_{\xi\in\mathbb{Z}^d}\widehat{V}(\xi)e^{\mathrm{i}\xi\cdot x},$$
$$\widehat{V}(\xi)=\frac{x_\xi}{4(1+|\xi|^2)^{m/2}},\qquad x_\xi\in\left[-\tfrac{1}{2},\tfrac{1}{2}\right]\subset\mathbb{R},\ m\in\mathbb{N},\ m>\tfrac{1}{2}d. \tag{1-5}$$

We endow the set $\mathcal{O}:=\left[-\frac{1}{2},\frac{1}{2}\right]^{\mathbb{Z}^d}$ with the standard probability measure on product spaces. This choice of the function defining the convolution potential is standard [Faou and Grébert 2013; Bambusi and Grébert 2006]; essentially one needs that the Fourier coefficients decay at a certain rate and that the function $V(x)$ depends on some free parameters x_ξ. Our main theorem is the following.

Theorem 1 (long-time existence for NLS). *Consider (NLS) with $d\geq 2$. There exists $\mathcal{N}\subset\mathcal{O}$ having zero Lebesgue measure such that if x_ξ in (1-5) is in $\mathcal{O}\setminus\mathcal{N}$, we have the following. There exists $s_0=s_0(d,m)\gg 1$ such that for any $s\geq s_0$ there are constants $c_0>0$ and $\epsilon_0>0$ such that for any $0<\epsilon\leq\epsilon_0$ we have the following. If $\|u_0\|_{H^s}<\frac{1}{4}\epsilon$, there exists a unique solution of the Cauchy problem (NLS) such that*

$$u(t,x)\in C^0([0,T);H^s(\mathbb{T}^d)),\qquad \sup_{t\in[0,T)}\|u(t,\cdot)\|_{H^s}\leq\epsilon,\qquad T\geq c_0\epsilon^{-4}. \tag{1-6}$$

In the one-dimensional case we do not need any external parameter and we shall prove the following theorem.

Theorem 2. *Consider (NLS) with $V\equiv 0$ and $d=1$. There exists $s_0\gg 1$ such that for any $s\geq s_0$ there are constants $c_0>0$ and $\epsilon_0>0$ such that for any $0<\epsilon\leq\epsilon_0$ we have the following. If $\|u_0\|_{H^s}<\frac{1}{4}\epsilon$, there exists a unique solution of the Cauchy problem (NLS) such that*

$$u(t,x)\in C^0([0,T);H^s(\mathbb{T}^d)),\qquad \sup_{t\in[0,T)}\|u(t,\cdot)\|_{H^s}\leq\epsilon,\qquad T\geq c_0\epsilon^{-4}. \tag{1-7}$$

These are, to the best of our knowledge, the firsts results of this kind for quasilinear Schrödinger equations posed on compact manifolds of dimension greater than 1.

Our main theorem regarding the problem (KG) is the following.

Theorem 3 (long-time existence for KG). *Consider (KG) with $d\geq 2$. There exists $\mathcal{N}\subset[1,2]$ having zero Lebesgue measure such that if $m\in[1,2]\setminus\mathcal{N}$ we have the following. There exists $s_0=s_0(d)\gg 1$ such that for any $s\geq s_0$ the following holds. For any $\delta>0$ there exists $\epsilon_0=\epsilon_0(s,m,\delta)>0$ such that for any $0<\epsilon\leq\epsilon_0$ and any initial data $(\psi_0,\psi_1)\in H^{s+1/2}(\mathbb{T}^d)\times H^{s-1/2}(\mathbb{T}^d)$ such that*

$$\|\psi_0\|_{H^{s+1/2}}+\|\psi_1\|_{H^{s-1/2}}\leq\tfrac{1}{32}\epsilon,$$

there exists a unique solution of the Cauchy problem (KG) *such that*

$$\begin{aligned}
&\psi(t,x) \in C^0([0,T); H^{s+1/2}(\mathbb{T}^d)) \cap C^1([0,T); H^{s-1/2}(\mathbb{T}^d)),\\
&\sup_{t\in[0,T)} (\|\psi(t,\cdot)\|_{H^{s+1/2}} + \|\partial_t\psi(t,\cdot)\|_{H^{s-1/2}}) \le \epsilon, \quad T \ge \epsilon^{-\mathtt{a}+\delta},
\end{aligned} \tag{1-8}$$

where $\mathtt{a} = 3$ *if* $d = 2$ *and* $\mathtt{a} = \frac{8}{3}$ *if* $d \ge 3$.

The time of existence in (1-8) is intimately connected with the lower bounds on the four waves interactions given in Section 2B. More precisely the time of existence is larger then $\epsilon^{-2-2/\beta}$ with β given in Proposition 2.2. This is the reason for the difference between the result in $d = 2$ (where $\beta = 2^+$) and $d \ge 3$ (where $\beta = 3^+$). We do not know if this result is sharp; this is an open problem. Despite this fact, Theorem 2 improves the general result in [Delort 2009; Fang and Zhang 2010] in the particular case of cubic nonlinearities in the following sense. First of all we can consider more general equations containing derivatives in the nonlinearity (with "small" quasilinear term). Furthermore, adapting our proof to the semilinear case (i.e., when $f = 0$ in (KG) and (1-1) and G in (1-2) does not depend on y_1), we obtain the better time of existence $\epsilon^{-10/3^-}$ for any $d \ge 4$. Indeed, in this case, the time of existence is $\epsilon^{-2-4/\beta}$ with β as above. This is the content of the next theorem.

Theorem 4. *Consider* (KG) *with* $f = 0$ *and* g *independent of* y_1. *Then the result of* Theorem 3 *holds true*, *replacing* $\mathtt{a} = 3$ *and* $\mathtt{a} = \frac{8}{3}$ *with* $\mathtt{a} = 4$ *and* $\mathtt{a} = \frac{10}{3}$ *respectively.*

Comments on the results. We begin by discussing the (NLS) case. Our method covers also more general cubic terms. For instance we could replace the term $|u|^2u$ with $g(|u|^2)u$, where $g(\cdot)$ is any analytic function vanishing at the origin and having a primitive $G' = g$. We preferred not to write the paper in the most general case since the nonlinearity $|u|^2u$ is a good representative for the aforementioned class and allows us to avoid complicating the notation further. We also remark that we consider a class of potentials V more general than the one we used in [Feola and Iandoli 2020; 2021] and more similar to the one used in [Bambusi and Grébert 2006] in a semilinear context.

Secondly, we remark that, beside the mathematical interest, it would be very interesting, from a physical point of view, to be able to deal with the case $h(\tau) \sim \tau$ with $\tau \sim 0$. Indeed, for instance, if we choose $h(\tau) = \sqrt{1+\tau} - 1$, the respective equation (NLS) models the self-channeling of a high-power, ultra-short laser pulse in matter; see [Borovskiĭ and Galkin 1993]. Unfortunately we need in our estimates $h(\tau) \sim \tau^{1+\sigma}$ with $\sigma > 0$. More precisely we need the purely quasilinear part of the equation $[\Delta(h(|u|^2))]h'(|u|^2)u$ to be *smaller* ($O(\epsilon^{3+4\sigma})$, $\epsilon \ll 1$) than the semilinear one ($O(\epsilon^3)$). At present we are not able to perform a normal form analysis which is able to reduce the size of the purely quasilinear part. Whence, if such a quasilinear term were $O(\epsilon^3)$, then the time of existence we are able to obtain would not be better than $O(\epsilon^{-2})$. Since h has to be smooth, this leads to $h(\tau) \sim \tau^2$, $\tau \sim 0$.

Also in the (KG) case we are not able to deal with the interesting case of cubic quasilinear term. This is the reason why we require that the nonlinearity f in (1-1) has a zero of order at least 4 at the origin.

We introduce the following notation: given $j_1, \dots, j_p \in \mathbb{R}^+$, $p \ge 2$, we define

$$\max_i\{j_1, \dots, j_p\} = i\text{-th largest among } j_1, \dots, j_p. \tag{1-9}$$

We use normal forms (the same strategy is used for (NLS) as well) and therefore small divisors problems arise. The small divisors, coming from the four waves interaction, are of the form

$$\Lambda_{\mathrm{KG}}(\xi-\eta-\zeta)-\Lambda_{\mathrm{KG}}(\eta)+\Lambda_{\mathrm{KG}}(\zeta)-\Lambda_{\mathrm{KG}}(\xi), \tag{1-10}$$

with Λ_{KG} defined in (1-4). In this case we prove the lower bound (see (1-9))

$$\begin{aligned} &|\Lambda_{\mathrm{KG}}(\xi-\eta-\zeta)-\Lambda_{\mathrm{KG}}(\eta)+\Lambda_{\mathrm{KG}}(\zeta)-\Lambda_{\mathrm{KG}}(\xi)| \\ &\qquad\qquad \gtrsim \max{}_2\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}^{-N_0}\max\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}^{-\beta} \end{aligned} \tag{1-11}$$

for almost any value of the mass m in the interval $[1,2]$ and where β is any real number in the open interval $(3,4)$. The second factor in the right-hand side of the above inequality represents a loss of derivatives when dividing by the quantity (1-10) which may be transformed in a loss of length of the lifespan through partition of frequencies. This is an extra difficulty, compared with the (NLS) case (for which lower bounds without loss have been proved in [Faou and Grébert 2013]), which makes the problem challenging already in a semilinear setting. The estimate (1-11) with $\beta\in(3,4)$ has been already obtained in [Fang and Zhang 2010]. We provide here a different and simpler proof, in the particular case of four waves interaction, which does not use the theory of subanalytic functions. We also quote Bernier, Faou and Grébert [Bernier et al. 2020] who use a control of the small divisors involving only the largest index (and not $\max_2$ as in (1-11)). They obtained, in the semilinear case, the control of the Sobolev norm for a time $T\sim\epsilon^{-\mathtt{a}}$, with $\mathtt{a}$ arbitrarily large, but assuming that the initial datum satisfies $\|\psi_0\|_{H^{s'+1/2}}+\|\psi_1\|_{H^{s'-1/2}}<c_0\epsilon$ for some $s'\equiv s'(\mathtt{a})>s$, i.e., allowing a loss of regularity.

Ideas of the proof. In our proof we shall use a *quasilinear normal forms/modified energies* approach; this seems to be the only successful one in order to improve the time of existence implied by the local theory. We recall, indeed, that on $\mathbb{T}^d$ the dispersive character of the solutions is absent. Moreover, the lack of conservation laws and the quasilinear nature of the equation prevent the use of *semilinear* techniques as done by Bambusi and Grébert [2006] and Bambusi, Delort, Grébert and Szeftel [Bambusi et al. 2007].

The most important feature of (NLS) and (KG), for our purposes, is their Hamiltonian structure. This property guarantees some key cancellations in the *energy estimates* that will be explained later on in this introduction.

Equation (NLS) may be indeed rewritten as

$$\partial_t u=-\mathrm{i}\nabla_{\bar u}\mathscr{H}_{\mathrm{NLS}}(u,\bar u)=\mathrm{i}(\Delta u-V*u-p(u)),$$

where $\nabla_{\bar u}:=\frac12(\nabla_{\mathrm{Re}(u)}+\mathrm{i}\nabla_{\mathrm{Im}(u)})$, ∇ denotes the L^2-gradient, and the Hamiltonian function $\mathscr{H}_{\mathrm{NLS}}$ and the nonlinearity p are

$$\begin{gathered} \mathscr{H}_{\mathrm{NLS}}(u,\bar u):=\int_{\mathbb{T}^d}|\nabla u|^2+(V*u)\bar u+P(u,\nabla u)\,dx, \\ P(u,\nabla u):=\tfrac12(|\nabla(h(|u|^2))|^2+|u|^4),\qquad p(u):=(\partial_{\bar u}P)(u,\nabla u)-\sum_{j=1}^d\partial_{x_j}(\partial_{\bar u_{x_j}}P)(u,\nabla u). \end{gathered} \tag{1-12}$$

Equation (KG) is Hamiltonian as well. Thanks to (1-1), (1-2) we have that (KG) can be written as

$$\begin{cases}\partial_t\psi=\partial_\phi\mathscr{H}_{\mathrm{KG}}(\psi,\phi)=\phi,\\ \partial_t\phi=-\partial_\psi\mathscr{H}_{\mathrm{KG}}(\psi,\phi)=-\Lambda_{\mathrm{KG}}^2\psi-f(\psi)-g(\psi),\end{cases}\tag{1-13}$$

where $\mathscr{H}_{\mathrm{KG}}(\psi,\phi)$ is the Hamiltonian

$$\mathscr{H}_{\mathrm{KG}}(\psi,\phi)=\int_{\mathbb{T}^d}\tfrac{1}{2}\phi^2+\tfrac{1}{2}(\Lambda_{\mathrm{KG}}^2\psi)\psi+F(\psi,\nabla\psi)+G(\psi,\Lambda_{\mathrm{KG}}^{1/2}\psi)\,dx.\tag{1-14}$$

We describe below our strategy in the case of the (NLS) equation. The strategy for (KG) is similar.

In [Feola and Iandoli 2022] we proved an energy estimate, without any assumption of smallness on the initial condition, for a more general class of equations. This energy estimate, on (NLS) with small initial datum, would read

$$E(t)-E(0)\lesssim\int_0^t\|u(\tau,\cdot)\|_{H^s}^2E(\tau)\,d\tau,\tag{1-15}$$

where $E(t)\sim\|u(t,\cdot)\|_{H^s}^2$. An estimate of this kind implies, by a standard bootstrap argument, that the lifespan of the solutions is of order at least $O(\epsilon^{-2})$, where ϵ is the size of the initial condition. To increase the time to $O(\epsilon^{-4})$ one would like to show the improved inequality

$$E(t)-E(0)\lesssim\int_0^t\|u(\tau,\cdot)\|_{H^s}^4E(\tau)\,d\tau.\tag{1-16}$$

Our main goal is to obtain such an estimate.

Paralinearization of (NLS). The first step is the paralinearization, à la [Bony 1981], of the equation as a system of the variables $(u,\bar{u})$; see Proposition 4.2. We rewrite (NLS) as a system of the form (compare with (4-12))

$$\partial_tU=-\mathrm{i}E\big((-\Delta+V*)U+\mathscr{A}_2(U)U+\mathscr{A}_1(U)U\big)+X_{H_4}(U)+R(U),\qquad E:=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\qquad U:=\begin{bmatrix}u\\ \bar{u}\end{bmatrix},$$

where $\mathscr{A}_2(U)$ is a 2×2 self-adjoint matrix of *paradifferential operators* of order 2 (see (4-11), (4-10)), $\mathscr{A}_1(U)$ is a self-adjoint, diagonal matrix of paradifferential operators of order 1 (see (4-12), (4-10)). This algebraic configuration of the matrices (in particular the fact that $\mathscr{A}_1(U)$ is diagonal) is a consequence of the Hamiltonian structure of the equation. The summand X_{H_4} is the cubic term (coming from the paralinearization of $|u|^2u$, see (4-13)) and $\|R(U)\|_{H^s}$ is bounded from above by $\|U\|_{H^s}^7$ for s large enough.

Both the matrices $\mathscr{A}_2(U)$ and $\mathscr{A}_1(U)$ vanish when U goes to 0. Since we assume that the function h, appearing in (NLS), vanishes quadratically at zero, as a consequence of (4-10), we have

$$\|\mathscr{A}_2(U)\|_{\mathscr{L}(H^s;H^{s-2})},\ \|\mathscr{A}_1(U)\|_{\mathscr{L}(H^s;H^{s-1})}\lesssim\|U\|_{H^s}^6,$$

where by $\mathscr{L}(X;Y)$ we denote the space of linear operators from X to Y. We also remark that the summand X_{H_4} is a *Hamiltonian vector field* with Hamiltonian function $H_4(u)=\int_{\mathbb{T}^d}|u|^4\,dx$.

Diagonalization of the second-order operator. The matrix of paradifferential operators $\mathscr{A}_2(U)$ is not diagonal; therefore the first step, in order to be able to get at least the weak estimate (1-15), is to diagonalize the system at the maximum order. This is possible since, because of the smallness assumption, the operator $E(-\Delta+\mathscr{A}_2(U))$ is locally elliptic. In Section 6A1 we introduce a new unknown $W=\Phi_{\rm NLS}(U)U$, where $\Phi_{\rm NLS}(U)$ is a parametrix built from the matrix of the eigenvectors of $E(-\Delta+\mathscr{A}_2(U))$; see (6-4), (6-2). The system in the new coordinates reads

$$\partial_t W=-\mathrm{i}E\big((-\Delta+V*)U+\mathscr{A}_2^{(1)}(U)W+\mathscr{A}_1^{(1)}(U)W\big)+X_{H_4}(W)+R^{(1)}(U),$$

where both $\mathscr{A}_2^{(1)}(U)$, $\mathscr{A}_1^{(1)}(U)$ are diagonal, see (6-11), and where $\|R^{(1)}(U)\|_{H^s}\lesssim\|U\|_{H^s}^7$ for s large enough. We note also that the cubic vector field X_{H_4} remains the same because the map $\Phi_{\rm NLS}(U)$ is equal to the identity plus a term vanishing at order 6 at zero; see (6-5).

Diagonalization of the cubic vector field. In the second step, in Section 6A2, we diagonalize the cubic vector field X_{H_4}. It is fundamental for our purposes to preserve the Hamiltonian structure of this cubic vector field in this diagonalization procedure. In view of this we perform a (approximately) *symplectic* change of coordinates generated from the Hamiltonian in (5-3) and (5-2) (note that this is not the case for the diagonalization at order 2). Actually the simplecticity of this change of coordinates is one of the most delicate points in our paper. The entire Section 5 is devoted to this. This diagonalization is implemented in order to simplify a *low-high* frequencies analysis. More precisely we prove that the cubic vector field may be conjugated to a diagonal one modulo a smoothing remainder. The diagonal part shall cancel out in the energy estimate due to a symmetrization argument based on its Hamiltonian character. As a consequence the time of existence shall be completely determined by the smoothing reminder. Since this remainder is smoothing, the contribution coming from high frequencies is already "small"; therefore the normal form analysis involves only the *low modes*. This will be explained later on in this introduction.

We explain the result of this diagonalization. We define a new variable $Z=\Phi_{\mathscr{B}_{\rm NLS}}(W)$, see (6-20), and we obtain the new diagonal system (compare with (6-22))

$$\partial_t Z=-\mathrm{i}E\big((-\Delta+V*)Z+\mathscr{A}_2^{(1)}(U)Z+\mathscr{A}_1^{(1)}(U)Z\big)+X_{\mathrm{H}_4}(Z)+R_5^{(2)}(U), \tag{1-17}$$

where the new vector field $X_{\mathrm{H}_4}(Z)$ is still Hamiltonian, with Hamiltonian function defined in (6-25), and it is equal to a skew-selfadjoint and diagonal matrix of bounded paradifferential operators modulo smoothing reminders; see (6-23). Here $R_5^{(2)}(U)$ satisfies the quintic estimates (6-24).

Introduction of the energy norm. Once we achieve the diagonalization of the system, we introduce an *energy norm* which is equivalent to the Sobolev one. Assume for simplicity $s=2n$, with n a natural number. Thanks to the smallness condition on the initial datum, we prove in Section 7A1 that

$$\|(-\Delta\mathbb{1}+\mathscr{A}_2(U)+\mathscr{A}_1(U))^{s/2}f\|_{L^2}\sim\|f\|_{H^s}$$

for any function f in $H^s(\mathbb{T}^d)$. Therefore by setting[1]

$$Z_n:=[E(-\Delta\mathbb{1}+\mathscr{A}_2(U)+\mathscr{A}_1(U))]^{s/2}Z,$$

[1]To be precise, the definition of $Z_n=(z_n,\bar{z}_n)$ in 7A1 is slightly different than the one presented here, but they coincide modulo smoothing corrections. For simplicity of notation, and in order to avoid technicalities, in this introduction we presented it in this way.

we are reduced to studying the L^2 norm of the function Z_n. This is done in Lemma 7.2. Since the system is now diagonalized, we write the scalar equation, see Lemma 7.3, solved by z_n

$$\partial_t z_n = -\mathrm{i} T_{\mathscr{L}} z_n - \mathrm{i} V * z_n - \Delta^n X^+_{\mathrm{H}_4}(Z) + R_n(U),$$

where we denote by $T_{\mathscr{L}}$ the element on the diagonal of the self-adjoint operator $-\Delta \mathbb{1} + \mathscr{A}_2(U) + \mathscr{A}_1(U)$; see (7-1), (3-6). $X^+_{\mathrm{H}_4}(Z)$ is the first component of the Hamiltonian vector field $X_{\mathrm{H}_4}(Z)$ and $R_n(U)$ is a bounded remainder satisfying the quintic estimate (7-12).

Cancellations and normal forms. At this point, still in Lemma 7.3, we split the Hamiltonian vector field $X^+_{\mathrm{H}_4} = X^{+,\mathrm{res}}_{\mathrm{H}_4} + X^{+,\perp}_{\mathrm{H}_4}$, where $X^{+,\mathrm{res}}_{\mathrm{H}_4}$ is the *resonant* part; see (3-84) and (3-83). The first important fact, which is an effect of the Hamiltonian- and Gauge-preserving structure, is that the resonant term $\Delta^n X^{+,\mathrm{res}}_{\mathrm{H}_4}$ does not give any contribution to the energy estimates. This key cancellation may be interpreted as a consequence of the fact that the *super actions*

$$I_p := \sum_{j \in \mathbb{Z}^d, |j| = p} |\hat{z}(j)|^2, \quad p \in \mathbb{N}, \; Z := \begin{bmatrix} z \\ \bar{z} \end{bmatrix},$$

where $\hat{z}$ is defined in (3-1), are prime integrals of the resonant Hamiltonian vector field $X^{+,\mathrm{res}}_{\mathrm{H}_4}(Z)$ in the spirit[2] of [Faou et al. 2013]. This is the content of Lemma 7.4, more specifically (7-16).

We are left with the study of the term $-\Delta^n X^{+,\perp}_{\mathrm{H}_4}$. In Lemma 7.3 we prove $-\Delta^n X^{+,\perp}_{\mathrm{H}_4} = B^{(1)}_n(Z) + B^{(2)}_n(Z)$, where $B^{(1)}_n(Z)$ does not contribute to energy estimates and $B^{(2)}_n(Z)$ is smoothing, gaining one space derivative; see (7-11) and Lemma 3.7. The cancellation for $B^{(1)}_n(Z)$ is again a consequence of the Hamiltonian structure and it is proven in Lemma 7.4, more specifically (7-17).

Summarizing we obtain the energy estimate (see (3-3))

$$\frac{1}{2}\frac{d}{dt}\|z_n(t)\|^2_{L^2} = \mathrm{Re}(-\mathrm{i} T_{\mathscr{L}} z_n, z_n)_{L^2} + \mathrm{Re}(-\mathrm{i} V * z_n, z_n)_{L^2} \tag{1-18}$$
$$+ \mathrm{Re}(R_n(U), z_n)_{L^2} \tag{1-19}$$
$$+ \mathrm{Re}(-\Delta^n X^{+,\mathrm{res}}_{\mathrm{H}_4}(Z), z_n)_{L^2} \tag{1-20}$$
$$+ \mathrm{Re}(B^{(1)}_n(Z), z_n)_{L^2} \tag{1-21}$$
$$+ \mathrm{Re}(B^{(2)}_n(Z), z_n)_{L^2}. \tag{1-22}$$

The right-hand side in (1-18) equals zero because $\mathrm{i}T_{\mathscr{L}}$ is skew-self-adjoint and the Fourier coefficients of V in (1-5) are real-valued. The term (1-19) is bounded from above by $\|z_n\|_{L^2}\|U\|^5_{H^s}$; (1-20) equals zero thanks to (7-16); the summand (1-21) equals zero as well because of (7-17). Setting $E(t) = \|z_n(t)\|^2_{L^2}$, the only term which is still not good in order to obtain an estimate of the form (1-16) is (1-22).

In order to improve the time of existence we need to reduce the size of this new term (1-22) by means of *normal forms/integration by parts*. Our aim is to prove that

$$\int_0^t \mathrm{Re}(B^{(2)}_n(Z), z_n)_{L^2}(\sigma)\, d\sigma \lesssim \epsilon^2 \tag{1-23}$$

[2]More generally, this cancellation can be viewed as a consequence of the commutation of the linear flow with the resonant part of the nonlinear perturbation which is a key of the Birkhoff normal form theory; see for instance [Grébert 2007].

as long as $t \lesssim \epsilon^{-4}$ and $\|z_n\|_{L^2} \lesssim \epsilon$. The thesis follows from this fact by using a classical bootstrap argument. Let us set $\mathscr{B}_{\mathrm{NLS}}(\sigma) := \mathrm{Re}(B_n^{(2)}(Z), z_n)_{L^2}(\sigma)$. The term $\mathscr{B}_{\mathrm{NLS}}$ may be expressed as (see Proposition 7.5)

$$\mathscr{B}_{\mathrm{NLS}} \sim \sum_{\xi,\eta,\zeta \in \mathscr{R}^c} \langle \xi \rangle^{2n} b(\xi,\eta,\zeta)\hat{z}(\xi-\eta-\zeta)\hat{\bar{z}}(\eta)\hat{z}(\zeta)\hat{\bar{z}}(-\xi); \tag{1-24}$$

the sum is restricted to the set of *nonresonant indexes*, see (3-83), and the coefficients satisfy

$$|b(\xi,\eta,\zeta)| \lesssim \frac{\langle \xi \rangle^{2n}}{\max_1(\langle \xi-\eta-\zeta\rangle, \langle \eta \rangle, \langle \zeta \rangle)},$$

where the constant depends on $\max_2(\langle \xi-\eta-\zeta\rangle, \langle \eta \rangle, \langle \zeta \rangle)$ and where we have defined the Japanese bracket $\langle \xi \rangle := \sqrt{1+|\xi|^2}$ for $\xi \in \mathbb{R}^d$. We fix $N \in \mathbb{R}^+$ and we let $\mathscr{B}_{\mathrm{NLS}} := \mathscr{B}_{\mathrm{NLS},\le N} + \mathscr{B}_{\mathrm{NLS},>N}$, where $\mathscr{B}_{\mathrm{NLS},\le N}$ is as in (1-24) with the sum restricted to those indexes such that $\max_1(\langle \xi-\eta-\zeta\rangle, \langle \eta \rangle, \langle \zeta \rangle) \le N$. It is easy to show (see Lemma 7.7) that $\int_0^t \|\mathscr{B}_{\mathrm{NLS},>N}(\sigma)\|_{H^s}\, d\sigma \lesssim tN^{-1}\|z\|_{H^s}^4$. This is due to the fact that the coefficients $b(\xi,\eta,\zeta)$ are decaying. Let us analyze the contribution given by $\mathscr{B}_{\mathrm{NLS},\le N}$.

We define the operator Λ_{NLS} as the Fourier multiplier acting on periodic functions as

$$\Lambda_{\mathrm{NLS}} e^{\mathrm{i}\xi\cdot x} = \Lambda_{\mathrm{NLS}}(\xi) e^{\mathrm{i}\xi\cdot x}, \quad \Lambda_{\mathrm{NLS}} \in \mathbb{R}, \ \Lambda_{\mathrm{NLS}}(\xi) := |\xi|^2 + \widehat{V}(\xi), \ \xi \in \mathbb{Z}^d, \tag{1-25}$$

where $\widehat{V}(\xi)$ are the *real* Fourier coefficients of the convolution potential $V(x)$ given in (1-5). Recalling (1-17), we have

$$\partial_t \hat{z}(\xi) = -\mathrm{i}\Lambda_{\mathrm{NLS}}(\xi)\hat{z}(\xi) + \widehat{Q}(\xi),$$

where $Q := -\mathrm{i}T_\Sigma z + X^+_{\mathrm{H}_4}(z) + R_5^{(2)}$, with T_Σ a paradifferential operator (see (3-6)) with symbol Σ, which is real, of order 2 and homogeneity 6 in z, and $R_5^{(2)}$ is a quintic reminder. We set $\hat{g}(\xi) := e^{\mathrm{i}t\Lambda_{\mathrm{NLS}}(\xi)}\hat{z}(\xi)$ and we obtain

$$\begin{aligned}\int_0^t \mathscr{B}_{\mathrm{NLS},\le N}(\sigma)\,d\sigma \\ \sim \int_0^t \sum_{\xi,\eta,\zeta\in\mathscr{R}^c} \mathbb{1}_{\{\max\{\langle \xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}\le N\}} b(\xi,\eta,\zeta) e^{-\mathrm{i}\sigma\omega_{\mathrm{NLS}}(\xi,\eta,\zeta)} \hat{g}(\xi-\eta-\zeta)\hat{\bar{g}}(\eta)\hat{g}(\zeta)\hat{\bar{g}}(-\xi)\langle\xi\rangle^{2n}\,d\sigma,\end{aligned}$$

with ω_{NLS} defined in (2-1). Integrating by parts in σ, we obtain

$$\begin{aligned}\int_0^t \mathscr{B}_{\mathrm{NLS},\le N}(\sigma)\,d\sigma \sim \int_0^t &(\mathscr{T}_<[z,\bar{z},z], T_{\langle\xi\rangle^{2n}}(\partial_t+\mathrm{i}\Lambda_{\mathrm{NLS}})z)_{L^2}\,d\sigma \\ &+ \int_0^t (\mathscr{T}_<[(\partial_t+\mathrm{i}\Lambda_{\mathrm{NLS}})z,\bar{z},z], T_{\langle\xi\rangle^{2n}}z)_{L^2}\,d\sigma \\ &+ \int_0^t (\mathscr{T}_<[z,\bar{z},(\partial_t+\mathrm{i}\Lambda_{\mathrm{NLS}})z], T_{\langle\xi\rangle^{2n}}z)_{L^2}\,d\sigma \\ &+ \int_0^t (\mathscr{T}_<[z,\overline{(\partial_t+\mathrm{i}\Lambda_{\mathrm{NLS}})z},z], T_{\langle\xi\rangle^{2n}}z)_{L^2}\,d\sigma + O(\|u\|^4_{H^s}),\end{aligned} \tag{1-26}$$

where $\mathscr{T}_<(z_1,z_2,z_3)$ is the multilinear form whose Fourier coefficient is

$$\widehat{\mathscr{T}_<}(\xi) = \frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d} \mathtt{t}_<(\xi,\eta,\zeta)\hat{z}_1(\xi-\eta-\zeta)\hat{z}_2(\eta)\hat{z}_3(\zeta), \quad \mathtt{t}_<(\xi,\eta,\zeta) = \frac{-1}{\mathrm{i}\omega_{\mathrm{NLS}}(\xi,\eta,\zeta)} b(\xi,\eta,\zeta).$$

The denominators ω_{NLS} are never dangerous since we have very good lower bounds on them; see Proposition 2.1 (see also Lemma 7.7). Let us consider, for instance, the first term in the right-hand side of (1-26). We have

$$\int_0^t (\mathscr{T}_{<}[z,\bar{z},z], T_{\langle\xi\rangle^{2n}}(\partial_t+\mathrm{i}\Lambda_{\mathrm{NLS}})z)_{L^2}(\sigma)\,d\sigma$$
$$=\int_0^t (T_{\langle\xi\rangle^2}\mathscr{T}_{<}[z,\bar{z},z], -T_{\langle\xi\rangle^{2n-2}}\mathrm{i}T_{\Sigma}z)_{L^2}(\sigma)\,d\sigma+\int_0^t \big(\mathscr{T}_{<}[z,\bar{z},z], T_{\langle\xi\rangle^{2n}}(X^{+}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z)+R_5^{(2,+)}(U))\big)_{L^2}(\sigma)\,d\sigma.$$

The first term may be estimated by the Cauchy–Schwarz inequality obtaining

$$\int_0^t \|\mathscr{T}_{<}(z,\bar{z},z)\|_{H^2}(\sigma)\|T_{\Sigma}z\|_{H^{s-2}}(\sigma)\,d\sigma. \tag{1-27}$$

Since Σ is a symbol of order 2 and homogeneity 6, the second factor is bounded from above by ϵ^6 as soon as $\|z(\sigma)\|_{H^s}\lesssim\epsilon$. Since $\mathscr{T}_{<}$ is supported on frequencies lower than N, the $\langle\xi\rangle^2$ symbol of H^2 norm, multiplied by the coefficients $b(\xi,\eta,\zeta)$ of the first term in (1-27), provides a factor N (see Lemma 7.7 for details); since it has homogeneity 4, we have also a factor ϵ^4 as soon as $\|z(\sigma)\|_{H^s}\lesssim\epsilon$. We eventually bound (1-27) by $tN\epsilon^{10}$. Analogously, the second term in (1-27) may be bounded from above by $t\epsilon^6$.

Recalling the contribution given by $\mathscr{B}_{\mathrm{NLS},>N}$, we can bound $\int_0^t\mathscr{B}_{\mathrm{NLS}}(\sigma)\,d\sigma$ from above by $t(\epsilon^4N^{-1}+\epsilon^{10}N+\epsilon^6)+\epsilon^4$. Choosing $N=\epsilon^{-2}$ we immediately note that the last quantity stays of size ϵ^2 as soon as $t\lesssim\epsilon^{-4}$.

As said before the strategy for (KG) is similar except for the control of the small divisors (1-11).

We summarize the plan concerning (KG) focusing on the main differences with respect to (NLS). In Section 4B we paralinearize the equation obtaining, after passing to the complex variables (4-24), the system of equations of order 1 (4-44). In Section 6B we diagonalize the system: the operator of order 1 is treated in Proposition 6.11 and that of order zero in Proposition 6.13. As done for (NLS), we diagonalize the operator of order zero paying attention to preserve its Hamiltonian structure. We consider the function z solving (6-48) and we define the new variable $z_n:=\langle D\rangle^n z$, where $\langle D\rangle$ is the Fourier multiplier having symbol $\langle\xi\rangle$. We want to bound the L^2-norm of the variable z_n, which solves (7-41). The evolution of the L^2-norm is studied in Proposition 7.10. From this proposition we understand that, in order to improve the energy estimates, we need to perform a normal form on the nonresonant term $\mathscr{B}$ in (7-55), which has coefficients decaying as in (7-56). We proceed as done in the (NLS) case. We fix $N\in\mathbb{R}^+$ and we split $\mathscr{B}$ in two pieces, one supported for frequencies such that $\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}\le N$ and the other for $\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}>N$. The contribution to the energy estimate of the second one is $tN^{-1}\epsilon^4$. Again in this point we exploit the smoothing property in (7-55). We focus on the part of $\mathscr{B}$ coming from small frequencies. We perform in Proposition 7.12 an integration by parts in the same spirit as done in the (NLS) case; see (7-74). When integrating by parts, the small denominators $\omega_{\mathrm{KG}}(\xi,\eta,\zeta)$ appear. In this case we do not have nice bounds as in the (NLS) case, indeed we only know that $|\omega_{\mathrm{KG}}(\xi,\eta,\zeta)|\gtrsim\max_1\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}^{-\beta}$, where β is bigger than 3 in dimension $d\ge3$ and it is bigger than 2 in dimension $d=2$. Hence such divisors give an extra factor N^{β} in the energy estimates (recall that we are dealing with the case of small frequencies $\le N$). After the integration by parts one has to use (7-39). The term $\tilde{a}_2^+(x,\xi)\Lambda_{\mathrm{KG}}(\xi)$ therein has homogeneity 3 and order 1, so that its contribution to the energy estimates in (7-75) is $tN^\beta\epsilon^7$. Indeed the unboundedness of Λ_{KG} is compensated for by the coefficients of $\mathscr{B}$, which gain one derivative. The vector field $X^+_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z)$ has homogeneity 3 and has no loss of derivatives, so that its contribution to (7-76) is $tN^{\beta-1}\epsilon^6$ (the "-1" comes from the coefficients of $\mathscr{B}$). The contribution of the remainder in (7-39) is negligible. We have one last term which is the one coming from the boundary term of the integration by parts which is bounded by $N^{\beta-1}\epsilon^4$. Summarizing we have obtained (compare with (7-63))

$$\left|\int_0^t \mathscr{B}(\sigma)\,d\sigma\right| \lesssim t(\epsilon^7N^\beta+\epsilon^6N^{\beta-1}+\epsilon^4N^{-1})+\epsilon^4N^{\beta-1},$$

where the term $\epsilon^4N^{-1}t$ is coming from the high frequencies of $\mathscr{B}$. Choosing $N:=\epsilon^{-2/\beta}$ we note that the right-hand side of the above inequality is controlled by ϵ^2 as soon as $t\lesssim\epsilon^{-2(1+1/\beta)}$, which is the time announced just after the statement of Theorem 3.

We explain the role of the parameter $\mathtt{a}$ in Theorem 4. In the semilinear case we have $f=0$ and g independent of y_1 in (KG), so there are no derivatives in the nonlinearity. When we pass to the system of order 1 in (4-44), one has $\mathscr{A}_1\equiv 0$ and that the cubic term $X_{\mathscr{H}^{(4)}_{\mathrm{KG}}}(U)$ may be decomposed as a paradifferential operator of order -1 plus a trilinear reminder whose coefficients have the better (compared to the quasilinear case (7-56)) decay $\max_1(\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle)^{-2}$ (see Remark 4.6). We perform the integration by parts as in the quasilinear case. Here we do not have the contribution coming from $\tilde{a}_2^+(x,\xi)\Lambda_{\mathrm{KG}}(\xi)$ (because this term equals zero in the semilinear case), which was ϵ^7N^β. Moreover the contribution of the cubic semilinear term is $\epsilon^6N^{\beta-2}$ (instead of $\epsilon^6N^{\beta-1}$ as before), thanks to the better decay of the coefficients in the cubic reminder. The high-frequency part is also smaller and it gives $N^{-2}\epsilon^6$, instead of $N^{-1}\epsilon^6$. One eventually obtains $\left|\int_0^t\mathscr{B}(\sigma)\,d\sigma\right|\lesssim t(\epsilon^6N^{\beta-2}+\epsilon^4N^{-2})+\epsilon^4N^{\beta-2}$. If one chooses $N=\epsilon^{-2/\beta}$ one can bound the previous quantity as soon as $t\lesssim\epsilon^{-(2+4/\beta)}$, which means $t\lesssim\epsilon^{-10/3^-}$ when $d\geq 3$ and $t\lesssim\epsilon^{-4^-}$ if $d=2$.

2. Small divisors

As pointed out in the Introduction the proofs our main theorems are based on a normal form approach. As a consequence we shall deal with small divisors problems. This section is devoted to establishing suitable lower bounds for generic (in a probabilistic way) choices of the parameters (x_ξ in (1-5) for (NLS) and m in (1-4) for (KG)), except for indices for which the small divisor is identically zero.

2A. *Nonresonance conditions for* (NLS). In the following proposition we give lower bounds for the small divisors arising from the normal form for (NLS).

Proposition 2.1. *Consider the phase $\omega_{\mathrm{NLS}}(\xi,\eta,\zeta)$ defined as*

$$\omega_{\mathrm{NLS}}(\xi,\eta,\zeta):=\Lambda_{\mathrm{NLS}}(\xi-\eta-\zeta)-\Lambda_{\mathrm{NLS}}(\eta)+\Lambda_{\mathrm{NLS}}(\zeta)-\Lambda_{\mathrm{NLS}}(\xi),\quad (\xi,\eta,\zeta)\in\mathbb{Z}^{3d}, \tag{2-1}$$

where Λ_{NLS} is in (1-25) and the potential V is in (1-5). We have the following:

(i) *Let $d \geq 2$. There exists $\mathcal{N} \subset \mathcal{O}$ with zero Lebesgue measure such that, for any $(x_i)_{i \in \mathbb{Z}^d} \in \mathcal{O} \setminus \mathcal{N}$, there exist $\gamma > 0$, $N_0 := N_0(d, m) > 0$ ($m > \frac{1}{2}d$ see (1-5)) such that for any $(\xi, \eta, \zeta) \notin \mathscr{R}$ (see (3-83)) one has*

$$|\omega_{\mathrm{NLS}}(\xi, \eta, \zeta)| \geq \gamma \max_2\{|\xi - \eta - \zeta|, |\eta|, |\zeta|\}^{-N_0}. \tag{2-2}$$

(ii) *Let $d = 1$ and assume that $V \equiv 0$. Then one has $|\omega_{\mathrm{NLS}}(\xi, \eta, \zeta)| \gtrsim 1$ unless*

$$\xi = \zeta, \quad \eta = \xi - \eta - \zeta \quad \textit{or} \quad \xi = \xi - \eta - \zeta, \quad \eta = \zeta, \quad \xi, \eta, \zeta \in \mathbb{Z}. \tag{2-3}$$

Proof. Item (i) follows by Proposition 2.8 in [Faou and Grébert 2013]. Item (ii) is classical. □

2B. *Nonresonance conditions for* (KG). Recall the symbol $\Lambda_{\mathrm{KG}}(j)$ in (1-4). We shall prove the following important proposition.

Proposition 2.2 (nonresonance conditions). *Consider the phase $\omega^{\vec{\sigma}}_{\mathrm{KG}}(\xi, \eta, \zeta)$ defined as*

$$\omega^{\vec{\sigma}}_{\mathrm{KG}}(\xi, \eta, \zeta) := \sigma_1 \Lambda_{\mathrm{KG}}(\xi - \eta - \zeta) + \sigma_2 \Lambda_{\mathrm{KG}}(\eta) + \sigma_3 \Lambda_{\mathrm{KG}}(\zeta) - \Lambda_{\mathrm{KG}}(\xi), \quad (\xi, \eta, \zeta) \in \mathbb{Z}^{3d}, \tag{2-4}$$

where $\vec{\sigma} := (\sigma_1, \sigma_2, \sigma_3) \in \{\pm\}^3$ and Λ_{KG} is in (1-4). Let $0 < \sigma \ll 1$ and set $\beta := 2 + \sigma$ if $d = 2$, and $\beta := 3 + \sigma$ if $d \geq 3$. There exists $\mathscr{C}_\beta \subset [1, 2]$ with Lebesgue measure 1 such that, for any $m \in \mathscr{C}_\beta$, there exist $\gamma > 0$, $N_0 := N_0(d, m) > 0$ such that for any $(\xi, \eta, \zeta) \notin \mathscr{R}$ (see (3-83)) one has

$$|\omega^{\vec{\sigma}}_{\mathrm{KG}}(\xi, \eta, \zeta)| \geq \gamma \max_2\{|\xi - \eta - \zeta|, |\eta|, |\zeta|\}^{-N_0} \max\{|\xi - \eta - \zeta|, |\eta|, |\zeta|\}^{-\beta}. \tag{2-5}$$

The case $d = 2$ follows by Theorem 2.1.1 in [Delort 2009]; the rest of this subsection is devoted to the proof of Proposition 2.2 in the case $d \geq 3$. Throughout this subsection, in order to lighten the notation, we shall write $\Lambda_{\mathrm{KG}}(j) \rightsquigarrow \Lambda_j$ for any $j \in \mathbb{Z}^d$ and $d \geq 3$. The main ingredient is the following.

Proposition 2.3. *Let $4 > \beta > 3$. There exist $\alpha > 0$ and $\mathscr{C}_\beta \subset [1, 2]$ a set of Lebesgue measure 1 and for $m \in \mathscr{C}_\beta$ there exists $\kappa(m) > 0$ such that*

$$|\sigma_1 \Lambda_{j_1} + \sigma_2 \Lambda_{j_2} + \sigma_3 \Lambda_{j_3} + \sigma_4 \Lambda_{j_4}| \geq \frac{\kappa(m)}{|j_3|^\alpha |j_1|^\beta} \tag{2-6}$$

for all $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \{-1, 1\}$, $j_1, j_2, j_3, j_4 \in \mathbb{Z}^d$ satisfying $|j_1| \geq |j_2| \geq |j_3| \geq |j_4|$ and $\sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 + \sigma_4 j_4 = 0$, except when $\sigma_1 = \sigma_4 = -\sigma_2 = -\sigma_3$ and $|j_1| = |j_2| \geq |j_3| = |j_4|$.

The Proposition 2.3 implies Proposition 2.2. Its proof is done in three steps.

Step 1: control with respect to the highest index.

Lemma 2.4. *There exist $\nu > 0$ and $\mathscr{M}_\nu \subset [1, 2]$ a set of Lebesgue measure 1 and for $m \in \mathscr{M}_\nu$ there exists $\gamma(m) > 0$ such that*

$$|\sigma_1 \Lambda_{j_1} + \sigma_2 \Lambda_{j_2} + \sigma_3 \Lambda_{j_3} + \sigma_4 \Lambda_{j_4}| \geq \gamma(m) |j_1|^{-\nu} \tag{2-7}$$

for all $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \{-1, 1\}$, $j_1, j_2, j_3, j_4 \in \mathbb{Z}^d$ satisfying $|j_1| \geq |j_2| \geq |j_3| \geq |j_4|$, except when $\sigma_1 = \sigma_4 = -\sigma_2 = -\sigma_3$ and $|j_1| = |j_2| \geq |j_3| = |j_4|$.

The proof of this lemma is standard and follows the line of Theorem 6.5 in [Bambusi 2003]; see also [Bambusi and Grébert 2006; Eliasson et al. 2016]. We briefly repeat the steps.

Let us assume that $j_1, j_2, j_3, j_4 \in \mathbb{Z}^d$ satisfy $|j_1| > |j_2| > |j_3| > |j_4|$. First of all, by reasoning as in Lemma 3.2 in [Eliasson et al. 2016], one can deduce the following.

Lemma 2.5. *Consider the matrix D whose entry at place (p, q) is given by $(d^p/dm^p)\Lambda_{j_q}$, $p, q = 1, \dots, 4$. The modulus of the determinant of D is bounded from below: one has $|\det(D)| \geq C|j_1|^{-\mu}$, where $C > 0$ and $\mu > 0$ are universal constants.*

From Lemma 3.3 in [Eliasson et al. 2016] we learn:

Lemma 2.6. *Let $u^{(1)}, \dots, u^{(4)}$ be four independent vectors in $\mathbb{R}^4$ with $\|u^{(i)}\|_{\ell^1} \leq 1$. Let $w \in \mathbb{R}^4$ be an arbitrary vector. Then there exist $i \in [1, \dots, 4]$ such that $|u^{(i)} \cdot w| \geq C\|w\|_{\ell^1} \det(u^{(1)}, \dots, u^{(4)})$.*

Let us define

$$\psi_{\mathrm{KG}}(m) = \sigma_1\Lambda_{j_1}(m) + \sigma_2\Lambda_{j_2}(m) + \sigma_3\Lambda_{j_3}(m) + \sigma_4\Lambda_{j_4}(m).$$

Combining Lemmas 2.5 and 2.6 we deduce the following.

Corollary 2.7. *For any $m \in [1, 2]$ there exists an index $i \in [1, \dots, 4]$ such that*

$$\left|\frac{d^i\psi_{\mathrm{KG}}}{dm^i}(m)\right| \geq C|j_1|^{-\mu}.$$

Now we need the following result (see Lemma B.1 in [Eliasson 2002]):

Lemma 2.8. *Let $\mathfrak{g}(x)$ be a C^{n+1}-smooth function on the segment $[1, 2]$ such that*

$$|\mathfrak{g}'|_{C^n} = \beta \quad \text{and} \quad \max_{1\leq k\leq n}\min_x |\partial^k\mathfrak{g}(x)| = \sigma.$$

Then

$$\operatorname{meas}(\{x : |\mathfrak{g}(x)| \leq \rho\}) \leq C_n\Big(\frac{\beta}{\sigma} + 1\Big)\Big(\frac{\rho}{\sigma}\Big)^{1/n}.$$

Define

$$\mathscr{E}_j(\kappa) := \{m \in [1, 2] : |\sigma_1\Lambda_{j_1} + \sigma_2\Lambda_{j_2} + \sigma_3\Lambda_{j_3} + \sigma_4\Lambda_{j_4}| \leq \kappa|j_1|^{-\nu}\}.$$

By combining Corollary 2.7 and Lemma 2.8 we get

$$\operatorname{meas}(\mathscr{E}_j(\kappa)) \leq C|j_1|^{\mu}(\kappa|j_1|^{\mu-\nu})^{1/4} \leq C\kappa^{1/4}|j_1|^{(5\mu-\nu)/4}. \tag{2-8}$$

Define

$$\mathscr{E}(\kappa) = \bigcup_{|j_1|>|j_2|>|j_3|>|j_4|} \mathscr{E}_j(\kappa),$$

and set $\nu = 5\mu + 4(4d+1)$. Then (2-8) implies $\operatorname{meas}(\mathscr{E}(\kappa)) \leq C\kappa^{1/4}$. Then taking $m \in \bigcup_{\kappa>0}([1, 2]\backslash\mathscr{E}(\kappa))$ we obtain (2-7) for any $|j_1| > |j_2| > |j_3| > |j_4|$. Furthermore $\bigcup_{\kappa>0}([1, 2]\backslash\mathscr{E}(\kappa))$ has measure 1. Now if for instance $|j_1| = |j_2|$ then we are left with a small divisor of the type $|2\Lambda_{j_1} + \sigma_3\Lambda_{j_3} + \sigma_4\Lambda_{j_4}|$ or $|\Lambda_{j_3} + \sigma_4\Lambda_{j_4}|$, i.e., involving two or three frequencies. So following the same line we can also manage this case.

Step 2: control with respect to the third-highest index. In this step we show that small divisors can be controlled by a smaller power of $|j_1|$, even if it means transferring part of the weight to $|j_3|$.

Proposition 2.9. *Let $4 > \beta > 3$. There exists $\mathcal{N}_\beta \subset [1,2]$ a set of Lebesgue measure 1 and for $m \in \mathcal{N}_\beta$ there exists $\kappa(m) > 0$ such that*

$$|\Lambda_{j_1} - \Lambda_{j_2} + \sigma_3 \Lambda_{j_3} + \sigma_4 \Lambda_{j_4}| \geq \frac{\kappa(m)}{|j_3|^{2d+6}|j_1|^\beta}$$

for all $\sigma_3, \sigma_4 \in \{-1, +1\}$, for all $j_1, j_2, j_3, j_4 \in \mathbb{Z}^d$ satisfying $|j_1| > |j_2| \geq |j_3| > |j_4|$, the momentum condition $j_1 - j_2 + \sigma_3 j_3 + \sigma_4 j_4 = 0$ and

$$|j_1| \geq J(\kappa, |j_3|) := \left(\frac{C}{\kappa}\right)^{1/(4-\beta)} |j_3|^{(2d+11)/(4-\beta)},$$

where C is a universal constant.

We begin with two elementary lemmas:

Lemma 2.10. *Let $\sigma = \pm 1$, $j, k \in \mathbb{Z}^d$, with $|j| > |k| > 0$ and $|j| \geq 8$, and $[1,2] \ni m \mapsto \mathfrak{g}(m)$ a C^1 function satisfying $|\mathfrak{g}'(m)| \leq 1/(10|j|^3)$ for $m \in [1,2]$. For all $\kappa > 0$ there exists $\mathcal{D} \equiv \mathcal{D}(j, k, \sigma, \kappa, \mathfrak{g}) \subset [1,2]$ such that for $m \in \mathcal{D}$*

$$|\Lambda_j + \sigma \Lambda_k - \mathfrak{g}(m)| \geq \kappa$$

and

$$\operatorname{meas}([1,2] \setminus \mathcal{D}) \leq 10\kappa |j|^3.$$

Proof. Let $f(m) = \Lambda_j + \sigma \Lambda_k - \mathfrak{g}(m)$. In the case $\sigma = -1$, which is the worst, we have

$$f'(m) = \frac{1}{2}\left(\frac{1}{\sqrt{|j|^2+m}} - \frac{1}{\sqrt{|k|^2+m}}\right) - \mathfrak{g}'(m) = \frac{|k|^2 - |j|^2}{2(\sqrt{|j|^2+m} + \sqrt{|k|^2+m})\sqrt{|j|^2+m}\sqrt{|k|^2+m}} - \mathfrak{g}'(m).$$

We want to estimate $|f'(m)|$ from above. By using that $4(|j|^2+2)^{3/2} \leq 5|j|^3$ for $|j| \geq 8$ we get

$$|f'(m)| \geq \frac{1}{5|j|^3} - \frac{1}{10|j|^3} \geq \frac{1}{10|j|^3}.$$

In the case $\sigma = 1$, the same bound holds true. Then we conclude by a standard argument that

$$\operatorname{meas}\{m \in [1,2] : |f(m)| \leq \kappa\} \leq 10\kappa |j|^3. \qquad \square$$

Lemma 2.11. *Let $j, k \in \mathbb{Z}^d$, with $|j| \geq |k|$ and $|j - k| \leq |j|^{1/2}$. Then*

$$\Lambda_j - \Lambda_k = \frac{(j, j-k)}{|j|} + \mathfrak{g}(|j|, |j-k|, (j-k, j), m) + O\left(\frac{|j-k|^5}{|j|^4}\right) \tag{2-9}$$

for some explicit rational function $\mathfrak{g}$.

Furthermore

$$|\partial_m \mathfrak{g}(|j|, |j-k|, (j, j-k), m)| \leq \frac{1}{2|j|^{3/2}}, \tag{2-10}$$

$$|\mathfrak{g}(|j|, |j-k|, (j, j-k), m)| \leq \frac{3|j-k|^2}{|j|} \tag{2-11}$$

uniformly with respect to $j, k \in \mathbb{Z}^d$ with $|j| \geq |k|$, $|j-k| \leq |j|^{1/2}$ and $|j|$ large enough.

Proof. By Taylor expansion we have for $|j|$ large

$$\Lambda_j = |j|\Big(1+\frac{m}{|j|^2}\Big)^{1/2} = |j| + \frac{m}{2|j|} - \frac{m^2}{8|j|^3} + O\Big(\frac{1}{|j|^5}\Big)$$

and

$$\begin{aligned}\Lambda_k &= |j|\Big(1+\frac{2(k-j,j)+|j-k|^2+m}{|j|^2}\Big)^{1/2}\\ &= |j| + \frac{2(k-j,j)+|j-k|^2+m}{2|j|} - \frac{(2(k-j,j)+|j-k|^2+m)^2}{8|j|^3}\\ &\quad + \frac{3}{48}\frac{(2(k-j,j)+|j-k|^2+m)^3}{|j|^5} - \frac{15}{16}\frac{1}{4!}\frac{(2(k-j,j)+|j-k|^2+m)^4}{|j|^7} + O\Big(\frac{|j-k|^5}{|j|^4}\Big),\end{aligned}$$

which leads to (2-9) where (we use that $|(k-j,j)| \le |j-k|\,|j|$ and $|j-k| \le |j|^{1/2}$)

$$\mathfrak{g}(x,y,z,m) = \frac{-y^2}{2x} + \frac{(-2z+y^2+m)^2-m^2}{8x^3} + \frac{3}{48}\frac{8z^3-12z^2(y^2+m)}{x^5} + \frac{1}{4!}\frac{15}{16}\frac{16z^4}{x^7}. \qquad \square$$

We are now in position to prove the main result of this subsection.

Proof of Proposition 2.9. We want to control the small divisor

$$\Delta = \Lambda_{j_1} - \Lambda_{j_2} + \sigma_3\Lambda_{j_3} + \sigma_4\Lambda_{j_4}.$$

Let $\mathfrak{g}$ be the rational function introduced in Lemma 2.11. We write

$$\Delta = \sigma_3\Lambda_{j_3} + \sigma_4\Lambda_{j_4} + \frac{(j_1, j_1-j_2)}{|j_1|} + \mathfrak{g}(|j_1|, |j_1-j_2|, (j_1, j_1-j_2), m) + O\Big(\frac{|j_1-j_2|^5}{|j_1|^4}\Big).$$

Remember that by assumption $j_1 - j_2 + \sigma_3 j_3 + \sigma_4 j_4 = 0$ and in particular $|j_1-j_2| \le 2|j_3|$.

Fix $\gamma > 0$. Choosing

$$\kappa = \frac{\gamma}{|j_3|^{2d+6}|j_1|^{\beta}}$$

in Lemma 2.10 and assuming $2|j_3| \le |j_1|^{1/2}$ we have by Lemmas 2.10 and 2.11

$$|\Delta| \ge \frac{\gamma}{|j_3|^{2d+6}|j_1|^{\beta}} - C\frac{|j_3|^5}{|j_1|^4} \ge \frac{\gamma}{2|j_3|^{2d+6}|j_1|^{\beta}}$$

as soon as[3]

$$|j_1| \ge \Big(\frac{C}{\gamma}\Big)^{1/(4-\beta)} |j_3|^{(2d+11)/(4-\beta)} =: J(\gamma, |j_3|) \ge 5|j_3|^3$$

(where C is an universal constant) and $m \in \mathscr{D}(j_3, j_4, \sigma, \kappa, \sigma_3\,\mathfrak{g}(|j_1|, |j_1-j_2|, (j_1, j_1-j_2), \cdot))$ (the set $\mathscr{D}$ is defined in Lemma 2.10 and we set $\sigma = \sigma_3\sigma_4$). Then defining

$$\begin{aligned}\mathscr{C}(\gamma, j_3, j_4, \sigma_3, \sigma_4) := \Big\{ m \in [1,2] : |\Delta| \ge \frac{\gamma}{2|j_3|^{2d+6}|j_1|^{\beta}} \text{ for all } (j_1, j_2)\\ \text{such that } |j_1| \ge \max(|j_2|, J(\gamma, |j_3|)),\ \ j_1 - j_2 + \sigma_3 j_3 + \sigma_4 j_4 = 0\Big\},\end{aligned}$$

[3]Note that this estimate implies $|\partial_m \mathfrak{g}(|j_1|, [j_1-j_2|, (j_1-j_2, j_1), m)| \le 1/(2|j_1|^{3/2}) \le 1/(10|j_3|^3)$ and thus Lemma 2.10 applies.

we have

$$\mathscr{C}(\gamma, j_3, j_4, \sigma_3, \sigma_4) = \bigcap_{\mathfrak{g}} \mathscr{D}\left(j_3, j_4, \sigma, \frac{\gamma}{|j_3|^{2d+6}|j_1|^{\beta}}, \sigma_3 \mathfrak{g}(|j_1|, |j_1 - j_2|, (j_1, j_1 - j_2), \cdot)\right),$$

where the intersection is taken over all functions $\mathfrak{g}$ generated by $(j_1, j_2) \in (\mathbb{Z}^d)^2$ such that

$$|j_1| \geq \max(|j_2|, J(\gamma, |j_3|))$$

and $j_1 - j_2 + \sigma_3 j_3 + \sigma_4 j_4 = 0$. Thus by Lemma 2.10

$$\begin{aligned}\operatorname{meas}([1, 2] \setminus \mathscr{C}(\gamma, j_3, j_4, \sigma_3, \sigma_4))& \\ \leq \sum_{n \geq 1} \frac{10\gamma}{|j_3|^{2d+3} n^{\beta/2}} \#\{(|j_1|, |\sigma_3 j_3 + \sigma_4 j_4|, (j_1, \sigma_3 j_3 + \sigma_4 j_4)) &: j_1 \in \mathbb{Z}^d,\ |j_1|^2 = n\}.\end{aligned}$$

The scalar product $(j_1, \sigma_3 j_3 + \sigma_4 j_4))$ takes only integer values smaller than $2|j_1||j_3|$. Then, since $\beta > 3$, we get

$$\operatorname{meas} \mathscr{C}(\gamma, j_3, j_4, \sigma_3, \sigma_4) \leq \frac{20\gamma}{|j_3|^{2d+2}} \sum_{n \geq 1} \frac{1}{n^{(\beta-1)/2}} \leq C_\beta \frac{\gamma}{|j_3|^{2d+2}}.$$

Then it remains to define

$$\mathscr{N}_\beta = \bigcup_{\gamma > 0} \bigcap_{\substack{(j_3, j_4) \in (\mathbb{Z}^d)^2 \\ |j_4| \leq |j_3| \\ \sigma_3, \sigma_4 \in \{-1, 1\}}} \mathscr{C}(\gamma, j_3, j_4, \sigma_3, \sigma_4)$$

to conclude the proof. □

<u>Step 3</u>: *Proof of Proposition 2.3.* We are now in a position to prove Proposition 2.3. Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \{-1, 1\}$, $j_1, j_2, j_3, j_4 \in \mathbb{Z}^d$ satisfying $|j_1| \geq |j_2| \geq |j_3| \geq |j_4|$ and $\sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 + \sigma_4 j_4 = 0$. If $\sigma_1 = \sigma_2$, then, since $|j_1| \geq |j_2| \geq |j_3| \geq |j_4|$, we conclude that the associated small divisor cannot be small except if $\sigma_1 = \sigma_2 = -\sigma_3 = -\sigma_4$. Then we have to control $|\Lambda_{j_1} + \Lambda_{j_2} - \Lambda_{j_3} - \Lambda_{j_4}|$ knowing that $|j_1| \geq |j_2| \geq |j_3| \geq |j_4|$. We first notice that if $|j_1|^2 \leq |j_3|^2 + 1$, then we can conclude using Lemma 2.4 that (2-6) is satisfied with $\alpha = \nu$ for $m \in \mathscr{M}_\nu$. On the other hand if $|j_1|^2 \geq |j_3|^2 + 1$ then

$$\Lambda_{j_1} + \Lambda_{j_2} - \Lambda_{j_3} - \Lambda_{j_4} \geq \Lambda_{j_1} - \Lambda_{j_3} \geq \frac{\Lambda_{j_1}^2 - \Lambda_{j_3}^2}{\Lambda_{j_1} + \Lambda_{j_3}} \geq \frac{1}{2\sqrt{|j_3|^2 + 2}},$$

which implies (2-6). Thus we can assume $\sigma_1 = -\sigma_2$ and we can apply Proposition 2.9, which implies the control (2-6) for $m \in \mathscr{N}_\beta$ with $\alpha = 2d + 3$ under the additional constraint $|j_1| \geq J(\gamma(m), |j_3|)$. Now if $|j_1| \leq J(\gamma(m), |j_3|)$, we can apply Lemma 2.4 to obtain that there exists $\nu > 0$ and full measure set $\mathscr{M}_\nu$ such that for $m \in \mathscr{M}_\nu \cap \mathscr{N}_\beta := \mathscr{C}_\beta$ there exists $\kappa(m) > 0$ such that

$$|\sigma_1 \Lambda_{j_1} + \sigma_2 \Lambda_{j_2} + \sigma_3 \Lambda_{j_3} + \sigma_4 \Lambda_{j_4}| \geq \frac{\kappa(m)}{|j_1|^\nu} \geq \frac{\kappa(m)}{J(\gamma(m), |j_3|)^\nu} = C \frac{\kappa(m)\gamma(m)^{4-\beta}}{|j_3|^\alpha},$$

with $\alpha = \nu(2d + 8)/(4 - \beta)$ which, of course, implies (2-6).

3. Functional setting

We denote by $H^s(\mathbb{T}^d;\mathbb{C})$ (respectively $H^s(\mathbb{T}^d;\mathbb{C}^2)$) the usual Sobolev space of functions $\mathbb{T}^d \ni x \mapsto u(x) \in \mathbb{C}$ (resp. $\mathbb{C}^2$). We expand a function $u(x)$, $x \in \mathbb{T}^d$, in Fourier series as

$$u(x) = \frac{1}{(2\pi)^{d/2}} \sum_{n\in\mathbb{Z}^d} \hat{u}(n) e^{\mathrm{i}n\cdot x}, \qquad \hat{u}(n) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} u(x) e^{-\mathrm{i}n\cdot x}\, dx. \tag{3-1}$$

We set $\langle j\rangle := \sqrt{1+|j|^2}$ for $j \in \mathbb{Z}^d$. We endow $H^s(\mathbb{T}^d;\mathbb{C})$ with the norm

$$\|u(\cdot)\|_{H^s}^2 := \sum_{j\in\mathbb{Z}^d} \langle j\rangle^{2s} |\hat{u}(j)|^2. \tag{3-2}$$

For $U = (u_1, u_2) \in H^s(\mathbb{T}^d;\mathbb{C}^2)$ we set $\|U\|_{H^s} = \|u_1\|_{H^s} + \|u_2\|_{H^s}$. Moreover, for $r \in \mathbb{R}^+$, we denote by $B_r(H^s(\mathbb{T}^d;\mathbb{C}))$ (resp. $B_r(H^s(\mathbb{T}^d;\mathbb{C}^2))$) the ball of $H^s(\mathbb{T}^d;\mathbb{C})$ (resp. $H^s(\mathbb{T}^d;\mathbb{C}^2)$) with radius r centered at the origin. We shall also write the norm in (3-2) as $\|u\|_{H^s}^2 = (\langle D\rangle^s u, \langle D\rangle^s u)_{L^2}$, where $\langle D\rangle e^{\mathrm{i}j\cdot x} = \langle j\rangle e^{\mathrm{i}j\cdot x}$ for any $j \in \mathbb{Z}^d$, and $(\cdot,\cdot)_{L^2}$ denotes the standard complex L^2-scalar product

$$(u, v)_{L^2} := \int_{\mathbb{T}^d} u \cdot \bar{v}\, dx \quad \text{for all } u, v \in L^2(\mathbb{T}^d;\mathbb{C}). \tag{3-3}$$

Notation. We shall use the notation $A \lesssim B$ to denote $A \le CB$, where C is a positive constant depending on parameters fixed once for all, for instance d and s. We will emphasize by writing $\lesssim_q$ when the constant C depends on some other parameter q.

Basic paradifferential calculus. We follow the notation of [Feola and Iandoli 2022]. We introduce the symbols we shall use in this paper. We shall consider symbols $\mathbb{T}^d \times \mathbb{R}^d \ni (x,\xi) \to a(x,\xi)$ in the spaces $\mathcal{N}_s^m$, $m, s \in \mathbb{R}$, $s \ge 0$, defined by the norms

$$|a|_{\mathcal{N}_s^m} := \sup_{|\alpha|+|\beta|\le s} \sup_{\xi\in\mathbb{R}} \langle\xi\rangle^{-m+|\beta|} \|\partial_\xi^\beta \partial_x^\alpha a(x,\xi)\|_{L^\infty}. \tag{3-4}$$

The constant $m \in \mathbb{R}$ indicates the *order* of the symbols, while s denotes its differentiability. Let $0 < \epsilon < \frac{1}{2}$ and consider a smooth function $\chi : \mathbb{R} \to [0,1]$,

$$\chi(\xi) = \begin{cases} 1 & \text{if } |\xi| \le \frac{5}{4}, \\ 0 & \text{if } |\xi| \ge \frac{8}{5} \end{cases} \quad \text{and define} \quad \chi_\epsilon(\xi) := \chi\left(\frac{|\xi|}{\epsilon}\right). \tag{3-5}$$

For a symbol $a(x,\xi)$ in $\mathcal{N}_s^m$ we define its (Weyl) quantization as

$$T_a h := \frac{1}{(2\pi)^d} \sum_{j\in\mathbb{Z}^d} e^{\mathrm{i}j\cdot x} \sum_{k\in\mathbb{Z}^d} \chi_\epsilon\left(\frac{|j-k|}{\langle j+k\rangle}\right) \hat{a}\left(j-k, \frac{j+k}{2}\right) \hat{h}(k), \tag{3-6}$$

where $\hat{a}(\eta,\xi)$ denotes the Fourier transform of $a(x,\xi)$ in the variable $x \in \mathbb{T}^d$. Moreover the definition of the operator T_a is independent of the choice of the cut-off function χ_ϵ up to smoothing terms; this will be proved later in Lemma 3.1.

Notation. Given a symbol $a(x,\xi)$, we shall also write

$$T_a[\,\cdot\,] := \mathrm{Op}^{\mathrm{BW}}(a(x,\xi))[\,\cdot\,] \tag{3-7}$$

to denote the associated paradifferential operator. In the notation B stands for Bony and W for Weyl.

We now collect some fundamental properties of paradifferential operators on tori. The results are similar to the ones given in [Feola and Iandoli 2022]. One could also look at [Berti et al. 2021c] for recent improvements.

Lemma 3.1. *The following hold*:

(i) *Let* $m_1, m_2 \in \mathbb{R}$, $s > \frac{1}{2}d$, $s \in \mathbb{N}$ *and* $a \in \mathcal{N}_s^{m_1}$, $b \in \mathcal{N}_s^{m_2}$. *One has*

$$|ab|_{\mathcal{N}_s^{m_1+m_2}} + |\{a,b\}|_{\mathcal{N}_{s-1}^{m_1+m_2-1}} \lesssim |a|_{\mathcal{N}_s^{m_1}} |b|_{\mathcal{N}_s^{m_2}}, \tag{3-8}$$

where

$$\{a,b\} := \sum_{j=1}^{d} ((\partial_{\xi_j} a)(\partial_{x_j} b) - (\partial_{x_j} a)(\partial_{\xi_j} b)). \tag{3-9}$$

(ii) *Let* $s_0 > d$, $s_0 \in \mathbb{N}$, $m \in \mathbb{R}$ *and* $a \in \mathcal{N}_{s_0}^m$. *Then, for any* $s \in \mathbb{R}$, *one has*

$$\|T_a h\|_{H^{s-m}} \lesssim |a|_{\mathcal{N}_{s_0}^m} \|h\|_{H^s} \quad \textit{for all } h \in H^s(\mathbb{T}^d;\mathbb{C}). \tag{3-10}$$

(iii) *Let* $s_0 > d$, $s_0 \in \mathbb{N}$, $m \in \mathbb{R}$, $\rho \in \mathbb{N}$, *and* $a \in \mathcal{N}_{s_0+\rho}^m$. *For* $0 < \epsilon_2 \le \epsilon_1 < \frac{1}{2}$ *and any* $h \in H^s(\mathbb{T}^d;\mathbb{C})$, *we define*

$$R_a h := \frac{1}{(2\pi)^d} \sum_{j\in\mathbb{Z}^d} e^{\mathrm{i} j\cdot x} \sum_{k\in\mathbb{Z}^d} (\chi_{\epsilon_1} - \chi_{\epsilon_2})\Big(\frac{|j-k|}{\langle j+k\rangle}\Big) \hat{a}\Big(j-k, \frac{j+k}{2}\Big) \hat{h}(k), \tag{3-11}$$

where $\chi_{\epsilon_1}, \chi_{\epsilon_2}$ *are as in* (3-5). *Then one has*

$$\|R_a h\|_{H^{s+\rho-m}} \lesssim \|h\|_{H^s} |a|_{\mathcal{N}_{\rho+s_0}^m} \quad \textit{for all } h \in H^s(\mathbb{T}^d;\mathbb{C}). \tag{3-12}$$

Proof. (i) For any $|\alpha| + |\beta| \le s$ we have

$$\partial_x^\alpha \partial_\xi^\beta (a(x,\xi) b(x,\xi)) = \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} C_{\alpha,\beta} (\partial_x^{\alpha_1}\partial_\xi^{\beta_1} a)(x,\xi)(\partial_x^{\alpha_2}\partial_\xi^{\beta_2} b)(x,\xi)$$

for some combinatorial coefficients $C_{\alpha,\beta} > 0$. Then, recalling (3-4),

$$\|(\partial_x^{\alpha_1}\partial_\xi^{\beta_1} a)(x,\xi)(\partial_x^{\alpha_2}\partial_\xi^{\beta_2} b)(x,\xi)\|_{L^\infty} \lesssim_{\alpha,\beta} |a|_{\mathcal{N}_s^{m_1}} |b|_{\mathcal{N}_s^{m_2}} \langle\xi\rangle^{m_1+m_2-|\beta|}.$$

This implies (3-8) for the product ab. Inequality (3-8) for the symbol $\{a,b\}$ follows similarly using (3-9).

(ii) First of all notice that, since $a \in \mathcal{N}_{s_0}^m$, $s_0 \in \mathbb{N}$, we have (recall (3-4))

$$\|a(\,\cdot\,,\xi)\|_{H^{s_0}} \lesssim \langle\xi\rangle^m |a|_{\mathcal{N}_{s_0}^m} \quad \text{for all } \xi \in \mathbb{Z}^d,$$

which implies

$$|\hat{a}(j,\xi)| \lesssim \langle\xi\rangle^m |a|_{\mathcal{N}_{s_0}^m} \langle j\rangle^{-s_0} \quad \text{for all } j, \xi \in \mathbb{Z}^d. \tag{3-13}$$

Moreover, since $0<\epsilon<\frac{1}{2}$ we note that, for $\xi,\eta\in\mathbb{Z}^d$,

$$\chi_\epsilon\Big(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\Big)\neq 0 \quad\Longrightarrow\quad \begin{cases}(1-\tilde\epsilon)|\xi|\le(1+\tilde\epsilon)|\eta|,\\(1-\tilde\epsilon)|\eta|\le(1+\tilde\epsilon)|\xi|,\end{cases} \tag{3-14}$$

where $0<\tilde\epsilon<\frac{4}{5}$, and hence $\langle\xi+\eta\rangle\sim\langle\xi\rangle$. Therefore

$$\begin{aligned}
\|T_ah\|^2_{H^{s-m}} &\lesssim \sum_{\xi\in\mathbb{Z}^d}\langle\xi\rangle^{2(s-m)}\Big|\sum_{\eta\in\mathbb{Z}^d}\chi_\epsilon\Big(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\Big)\hat a\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)\hat h(\eta)\Big|^2 && \text{(by (3-2))}\\
&\lesssim \sum_{\xi\in\mathbb{Z}^d}\langle\xi\rangle^{-2m}\Big(\sum_{\eta\in\mathbb{Z}^d}\frac{\langle\xi\rangle^m}{\langle\xi-\eta\rangle^{s_0}}|\hat h(\eta)|\langle\eta\rangle^s\Big)^2|a|^2_{\mathcal{N}^m_{s_0}} && \text{(by (3-13), (3-14))},\\
&\lesssim |a|^2_{\mathcal{N}^m_{s_0}}\sum_{\xi\in\mathbb{Z}^d}\Big(\sum_{\eta\in\mathbb{Z}^d}\Big|\hat h(\eta)\langle\eta\rangle^s\frac{1}{\langle\xi-\eta\rangle^{s_0}}\Big|\Big)^2\\
&\lesssim |a|^2_{\mathcal{N}^m_{s_0}}\|\hat h(\xi)\langle\xi\rangle^s\star\langle\xi\rangle^{-s_0}\|^2_{\ell^2(\mathbb{Z}^d)}\le|a|^2_{\mathcal{N}^m_{s_0}}\|\hat h(\xi)\langle\xi\rangle^s\|^2_{\ell^2(\mathbb{Z}^d)}\|\langle\xi\rangle^{-s_0}\|^2_{\ell^1(\mathbb{Z}^d)}\\
&\lesssim \|h\|^2_{H^s}|a|^2_{\mathcal{N}^m_{s_0}},
\end{aligned}\tag{3-15}$$

where we denote by $\star$ the convolution between sequences, and in the penultimate inequality we used the Young inequality for sequences and in the last one that $\langle\xi\rangle^{-s_0}$ is in $\ell^1(\mathbb{Z}^d)$ since $s_0>d$.

(iii) Notice that the set of ξ,η such that $(\chi_{\epsilon_1}-\chi_{\epsilon_2})(|\xi-\eta|/\langle\xi+\eta\rangle)=0$ contains the set such that

$$|\xi-\eta|\ge\tfrac{8}{5}\epsilon_1\langle\xi+\eta\rangle\quad\text{or}\quad|\xi-\eta|\le\tfrac{5}{4}\epsilon_2\langle\xi+\eta\rangle.$$

Therefore $(\chi_{\epsilon_1}-\chi_{\epsilon_2})(|\xi-\eta|/\langle\xi+\eta\rangle)\neq0$ implies

$$\tfrac{5}{4}\epsilon_2\langle\xi+\eta\rangle\le|\xi-\eta|\le\tfrac{8}{5}\epsilon_1\langle\xi+\eta\rangle. \tag{3-16}$$

For $\xi\in\mathbb{Z}^d$ we denote by $\mathscr{A}(\xi)$ the set of $\eta\in\mathbb{Z}^d$ such that (3-16) holds. Moreover (reasoning as in (3-13)), since $a\in\mathcal{N}^m_{s_0+\rho}$, we have

$$|\hat a(j,\xi)|\lesssim\langle\xi\rangle^m|a|_{\mathcal{N}^m_{s_0+\rho}}\langle j\rangle^{-s_0-\rho}\quad\text{for all } j,\xi\in\mathbb{Z}^d. \tag{3-17}$$

To estimate the remainder in (3-11) we reason as in (3-15). By (3-16) and setting $\rho=s-s_0$ we have

$$\begin{aligned}
\|R_ah\|^2_{H^{s+\rho-m}} &\lesssim \sum_{\xi\in\mathbb{Z}^d}\langle\xi\rangle^{2(s+\rho-m)}\Big|(\chi_{\epsilon_1}-\chi_{\epsilon_2})\Big(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\Big)\hat a\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)\hat h(\eta)\Big|^2 && \text{(by (3-2))}\\
&\lesssim \sum_{\xi\in\mathbb{Z}^d}\langle\xi\rangle^{-2m}\Big(\sum_{\eta\in\mathscr{A}(\xi)}\frac{\langle\xi-\eta\rangle^\rho\langle\xi+\eta\rangle^m}{\langle\xi-\eta\rangle^{\rho+s_0}}|\hat h(\eta)|\langle\eta\rangle^s\Big)^2|a|^2_{\mathcal{N}^m_{s_0+\rho}} && \text{(by (3-17))}\\
&\lesssim \|\hat h(\xi)\langle\xi\rangle^s\star\langle\xi\rangle^{-s_0}\|^2_{\ell^2(\mathbb{Z}^d)}|a|^2_{\mathcal{N}^m_{\rho+s_0}}\\
&\lesssim \|\hat h(\xi)\langle\xi\rangle^s\|^2_{\ell^2(\mathbb{Z}^d)}\|\langle\xi\rangle^{-s_0}\|^2_{\ell^1(\mathbb{Z}^d)}|a|^2_{\mathcal{N}^m_{\rho+s_0}}\lesssim\|h\|^2_{H^s}|a|^2_{\mathcal{N}^m_{\rho+s_0}},
\end{aligned}\tag{3-18}$$

where we denote by $\star$ the convolution between sequences, and in the penultimate inequality we used the Young inequality for sequences and in the last one we used that $\langle\xi\rangle^{-s_0}$ is in $\ell^1(\mathbb{Z}^d)$ since $s_0>d$. ☐

Proposition 3.2 (composition). *Fix $s_0 > d$, $s_0 \in \mathbb{N}$, and $m_1, m_2 \in \mathbb{R}$. For $a \in \mathcal{N}^{m_1}_{s_0+2}$ and $b \in \mathcal{N}^{m_2}_{s_0+2}$ we have* (*recall* (3-9))

$$T_a \circ T_b = T_{ab} + R_1(a,b), \qquad T_a \circ T_b = T_{ab} + \frac{1}{2\mathrm{i}} T_{\{a,b\}} + R_2(a,b), \tag{3-19}$$

where $R_j(a,b)$ are remainders satisfying, for any $s \in \mathbb{R}$,

$$\|R_j(a,b)h\|_{H^{s-m_1-m_2+j}} \lesssim \|h\|_{H^s} |a|_{\mathcal{N}^{m_1}_{s_0+j}} |b|_{\mathcal{N}^{m_2}_{s_0+j}}. \tag{3-20}$$

Moreover, if $a, b \in H^{\rho+s_0}(\mathbb{T}^d;\mathbb{C})$ are functions (*independent of $\xi \in \mathbb{R}^n$*) *then, for all $s \in \mathbb{R}$,*

$$\|(T_a T_b - T_{ab})h\|_{H^{s+\rho}} \lesssim \|h\|_{H^s} \|a\|_{H^{\rho+s_0}} \|b\|_{H^{\rho+s_0}}. \tag{3-21}$$

Proof. We start by proving (3-21). For $\xi, \theta, \eta \in \mathbb{Z}^d$ we define

$$r_1(\xi,\theta,\eta) := \chi_\epsilon\left(\frac{|\xi-\theta|}{\langle \xi+\theta\rangle}\right)\chi_\epsilon\left(\frac{|\theta-\eta|}{\langle \theta+\eta\rangle}\right), \qquad r_2(\xi,\eta) := \chi_\epsilon\left(\frac{|\xi-\eta|}{\langle \xi+\eta\rangle}\right). \tag{3-22}$$

Recalling (3-6) and that a, b are functions we have

$$\begin{aligned} R_0 h &:= (T_a T_b - T_{ab})h, \\ \widehat{(R_0 h)}(\xi) &= (2\pi)^{-3d/2} \sum_{\eta,\theta \in \mathbb{Z}^d} (r_1 - r_2)(\xi,\theta,\eta)\hat{a}(\xi-\theta)\hat{b}(\theta-\eta)\hat{h}(\eta). \end{aligned} \tag{3-23}$$

Let us define the sets

$$D := \{(\xi,\theta,\eta) \in \mathbb{Z}^{3d} : (r_1 - r_2)(\xi,\theta,\eta) = 0\}, \tag{3-24}$$

$$A := \left\{(\xi,\theta,\eta) \in \mathbb{Z}^{3d} : \frac{|\xi-\theta|}{\langle \xi+\theta\rangle} \le \frac{5\epsilon}{4}, \ \frac{|\xi-\eta|}{\langle \xi+\eta\rangle} \le \frac{5\epsilon}{4}, \ \frac{|\theta-\eta|}{\langle \theta+\eta\rangle} \le \frac{5\epsilon}{4}\right\}, \tag{3-25}$$

$$B := \left\{(\xi,\theta,\eta) \in \mathbb{Z}^{3d} : \frac{|\xi-\theta|}{\langle \xi+\theta\rangle} \ge \frac{8\epsilon}{5}, \ \frac{|\xi-\eta|}{\langle \xi+\eta\rangle} \ge \frac{8\epsilon}{5}, \ \frac{|\theta-\eta|}{\langle \theta+\eta\rangle} \ge \frac{8\epsilon}{5}\right\}. \tag{3-26}$$

We note that

$$D \supseteq A \cup B \quad \Longrightarrow \quad D^c \subseteq A^c \cap B^c.$$

Let $(\xi,\theta,\eta) \in D^c$ and assume in particular that $(\xi,\theta,\eta) \in \operatorname{Supp}(r_1) := \overline{\{(\xi,\theta,\eta) : r_1 \neq 0\}}$. Then, reasoning as in (3-14), we can note that

$$|\xi - \eta| \le \epsilon\langle \xi+\eta\rangle \quad \text{and} \quad \langle\xi\rangle \sim \langle\eta\rangle. \tag{3-27}$$

Notice also that $(\xi,\theta,\eta) \in \operatorname{Supp}(r_2)$ implies (3-27) as well. The rough idea of the proof is based on the fact that, if $(\xi,\theta,\eta) \in D^c$, then there are *at least three* equivalent frequencies among $\xi, \xi-\theta, \theta-\eta, \eta$; therefore (3-23) restricted to $(\xi,\theta,\eta) \in D^c$ is a regularizing operator. We need to estimate

$$\|R_0 h\|^2_{H^{s+\rho}} \lesssim \sum_{\xi \in \mathbb{Z}^d} \left(\sum^{*}_{\eta,\theta} |\hat{a}(\xi-\theta)||\hat{b}(\theta-\eta)||\hat{h}(\eta)|\langle\xi\rangle^{s+\rho}\right)^2 = I + II + III,$$

where $\sum^*_{\eta,\theta}$ denotes the sum over indexes satisfying (3-27), the term I denotes the sum on indexes satisfying also $|\xi-\theta| > c\epsilon|\xi|$, II denotes the sum on indexes satisfying also $|\eta-\theta| > c\epsilon|\eta|$ for some $0 < c \ll 1$

and III is defined by the difference. We estimate the term I. Using (3-27) and $|\xi-\theta|>c\epsilon|\xi|$ we get

$$\begin{aligned} I &\lesssim \sum_{\xi\in\mathbb{Z}^d}\Big(\sum_{\eta,\theta}^{*}|\hat{a}(\xi-\theta)||\hat{b}(\theta-\eta)||\hat{h}(\eta)|\langle\eta\rangle^s\langle\xi-\theta\rangle^\rho\Big)^2 \\ &\lesssim \|\,|\hat{h}(\xi)|\langle\xi\rangle^s \star |\hat{a}(\xi)|\langle\xi\rangle^\rho \star |\hat{b}(\xi)|\,\|^2_{\ell^2(\mathbb{Z}^d)} \\ &\lesssim \||\hat{h}(\xi)|\langle\xi\rangle^s\|^2_{\ell^2(\mathbb{Z}^d)}\||\hat{a}(\xi)|\langle\xi\rangle^\rho\|^2_{\ell^1(\mathbb{Z}^d)}\||\hat{b}(\xi)|\|^2_{\ell^1(\mathbb{Z}^d)} \\ &\lesssim \|h\|^2_{H^s}\|a\|^2_{H^{s_0+\rho}}\|b\|^2_{H^{s_0}}, \end{aligned}$$

where in the last inequality we used Cauchy–Schwarz and $s_0>d>\frac{1}{2}d$.

Reasoning similarly one obtains $II\lesssim\|h\|^2_{H^s}\|a\|^2_{H^{s_0}}\|b\|^2_{H^{s_0+\rho}}$. The sum III is restricted to indexes satisfying (3-27) and $|\xi-\theta|\le c\epsilon|\xi|$, $|\eta-\theta|\le c\epsilon|\eta|$. For $c\ll 1$ small enough these restrictions imply that $(\xi,\eta,\zeta)\in A$, which is a contradiction since $(\xi,\eta,\zeta)\in D^c\subseteq A^c$.

Let us check (3-20). We prove that

$$T_a\circ T_b=T_{ab}+\frac{1}{2\mathrm{i}}T_{\{a,b\}}+R_2(a,b),\quad \|R_2(a,b)h\|_{H^{s-m_1-m_2+2}}\lesssim\|h\|_{H^s}|a|_{\mathcal{N}^{m_1}_{s_0+2}}|b|_{\mathcal{N}^{m_2}_{s_0+2}}. \tag{3-28}$$

First of all we note that

$$(\widehat{T_aT_bh})(\xi)=\frac{1}{(\sqrt{2\pi})^{3d}}\sum_{\eta,\theta\in\mathbb{Z}^d}r_1(\xi,\theta,\eta)\hat{a}\Big(\xi-\theta,\frac{\xi+\theta}{2}\Big)\hat{b}\Big(\theta-\eta,\frac{\theta+\eta}{2}\Big)\hat{h}(\eta), \tag{3-29}$$

$$(\widehat{T_{ab}h})(\xi)=\frac{1}{(\sqrt{2\pi})^{3d}}\sum_{\eta,\theta\in\mathbb{Z}^d}r_2(\xi,\eta)\hat{a}\Big(\xi-\theta,\frac{\xi+\eta}{2}\Big)\hat{b}\Big(\theta-\eta,\frac{\xi+\eta}{2}\Big)\hat{h}(\eta), \tag{3-30}$$

$$\begin{aligned}\frac{1}{2\mathrm{i}}(\widehat{T_{\{a,b\}}h})(\xi)={}&\frac{1}{2\mathrm{i}(\sqrt{2\pi})^{3d}}\sum_{\eta,\theta\in\mathbb{Z}^d}r_2(\xi,\eta)\widehat{(\partial_\xi a)}\Big(\xi-\theta,\frac{\xi+\eta}{2}\Big)\cdot\widehat{(\partial_x b)}\Big(\theta-\eta,\frac{\xi+\eta}{2}\Big)\hat{h}(\eta)\\ &-\frac{1}{2\mathrm{i}(\sqrt{2\pi})^{3d}}\sum_{\eta,\theta\in\mathbb{Z}^d}r_2(\xi,\eta)\widehat{(\partial_x a)}\Big(\xi-\theta,\frac{\xi+\eta}{2}\Big)\cdot\widehat{(\partial_\xi b)}\Big(\theta-\eta,\frac{\xi+\eta}{2}\Big)\hat{h}(\eta).\end{aligned} \tag{3-31}$$

In the formulas above we used the notation $\partial_x=(\partial_{x_1},\dots,\partial_{x_d})$, similarly for ∂_ξ. We remark that we can substitute the cut-off function r_2 in (3-30), (3-31) with r_1 up to smoothing remainders. This follows because one can treat the cut-off function $r_1(\xi,\theta,\eta)-r_2(\xi,\eta)$ as done in the proof of (3-21). Write $\xi+\theta=\xi+\eta+(\theta-\eta)$. By Taylor expanding the symbols at $\xi+\eta$, we have

$$\begin{aligned}\hat{a}\Big(\xi-\theta,\frac{\xi+\theta}{2}\Big)={}&\hat{a}\Big(\xi-\theta,\frac{\xi+\eta}{2}\Big)+\widehat{(\partial_\xi a)}\Big(\xi-\theta,\frac{\xi+\eta}{2}\Big)\cdot\frac{\theta-\eta}{2}\\ &+\frac{1}{4}\sum_{j,k=1}^{d}\int_0^1(1-\sigma)\widehat{(\partial_{\xi_j\xi_k}a)}\Big(\xi-\theta,\frac{\xi+\eta}{2}+\sigma\frac{\theta-\eta}{2}\Big)(\theta_j-\eta_j)(\theta_k-\eta_k)\,d\sigma.\end{aligned} \tag{3-32}$$

Similarly one obtains

$$\begin{aligned}\hat{b}\Big(\theta-\eta,\frac{\theta+\eta}{2}\Big)={}&\hat{b}\Big(\theta-\eta,\frac{\xi+\eta}{2}\Big)+\widehat{(\partial_\xi b)}\Big(\theta-\eta,\frac{\xi+\eta}{2}\Big)\cdot\frac{\theta-\xi}{2}\\ &+\frac{1}{4}\sum_{j,k=1}^{d}\int_0^1(1-\sigma)\widehat{(\partial_{\xi_j\xi_k}b)}\Big(\theta-\eta,\frac{\xi+\eta}{2}+\sigma\frac{\theta-\xi}{2}\Big)(\theta_j-\xi_j)(\theta_k-\xi_k)\,d\sigma.\end{aligned} \tag{3-33}$$

By (3-32), (3-33) we deduce that

$$
\begin{aligned}
&T_a T_b h - T_{ab} h - \frac{1}{2\mathrm{i}} T_{\{a,b\}} h = \sum_{p=1}^{6} R_p h, \\
&\widehat{(R_p h)}(\xi) := \frac{1}{(\sqrt{2\pi})^{3d}} \sum_{\eta,\theta \in \mathbb{Z}^d} r_1(\xi,\theta,\eta) g_p(\xi,\theta,\eta) \hat{h}(\eta),
\end{aligned}
\tag{3-34}
$$

where the symbols g_i are defined as

$$
g_1 := -\frac{1}{4} \sum_{j,k=1}^{d} \int_0^1 (1-\sigma) (\widehat{\partial_{x_k x_j} a}) \Big(\xi-\theta, \frac{\xi+\eta}{2}\Big) (\widehat{\partial_{\xi_k \xi_j} b}) \Big(\theta-\eta, \frac{\xi+\eta}{2} + \sigma \frac{\theta-\xi}{2}\Big)\, d\sigma, \tag{3-35}
$$

$$
g_2 := -\frac{1}{4} \sum_{j,k=1}^{d} \int_0^1 (1-\sigma) (\widehat{\partial_{\xi_k \xi_j} a}) \Big(\xi-\theta, \frac{\xi+\eta}{2} + \sigma \frac{\theta-\eta}{2}\Big) (\widehat{\partial_{x_k x_j} b}) \Big(\theta-\eta, \frac{\xi+\eta}{2}\Big)\, d\sigma, \tag{3-36}
$$

$$
g_3 := \frac{1}{4} \sum_{j,k=1}^{d} (\widehat{\partial_{x_j} \partial_{\xi_k} a}) \Big(\xi-\theta, \frac{\xi+\eta}{2}\Big) (\widehat{\partial_{x_k} \partial_{\xi_j} b}) \Big(\theta-\eta, \frac{\xi+\eta}{2}\Big), \tag{3-37}
$$

$$
g_4 := -\frac{1}{8\mathrm{i}} \sum_{j,k,p=1}^{d} \int_0^1 (1-\sigma) (\widehat{\partial_{x_k x_j \xi_p} a}) \Big(\xi-\theta, \frac{\xi+\eta}{2}\Big) (\widehat{\partial_{x_p \xi_k \xi_j} b}) \Big(\theta-\eta, \frac{\xi+\eta}{2} + \sigma \frac{\theta-\xi}{2}\Big)\, d\sigma, \tag{3-38}
$$

$$
g_5 := -\frac{1}{8\mathrm{i}} \sum_{j,k,p=1}^{d} \int_0^1 (1-\sigma) (\widehat{\partial_{\xi_k \xi_j x_p} a}) \Big(\xi-\theta, \frac{\xi+\eta}{2} + \sigma \frac{\theta-\eta}{2}\Big) (\widehat{\partial_{\xi_p x_k x_j} b}) \Big(\theta-\eta, \frac{\xi+\eta}{2}\Big)\, d\sigma, \tag{3-39}
$$

$$
\begin{aligned}
g_6 := \frac{1}{16} \sum_{j,k,p,q=1}^{d} \iint_0^1 (1-\sigma_1)(1-\sigma_2) (\widehat{\partial_{\xi_j \xi_k x_p x_q} a}) \Big(\xi-\theta, \frac{\xi+\eta}{2} + \sigma_1 \frac{\theta-\eta}{2}\Big) \\
\times (\widehat{\partial_{\xi_p \xi_q x_j x_k} b}) \Big(\theta-\eta, \frac{\xi+\eta}{2} + \sigma_2 \frac{\theta-\xi}{2}\Big)\, d\sigma_1\, d\sigma_2.
\end{aligned}
\tag{3-40}
$$

We prove the estimate (3-20) (with $j=2$) on each term of the sum in (3-34). First of all we note that $r_1(\xi,\theta,\eta) \neq 0$ implies

$$
(\theta,\eta) \in \left\{ \frac{|\xi-\theta|}{\langle \xi+\theta \rangle} \leq \frac{8}{5}\epsilon \right\} \cap \left\{ \frac{|\theta-\eta|}{\langle \theta+\eta \rangle} \leq \frac{8}{5}\epsilon \right\} =: \mathscr{B}(\xi), \quad \xi \in \mathbb{Z}^d. \tag{3-41}
$$

Moreover we note that

$$
(\theta,\eta) \in \mathscr{B}(\xi) \quad \Longrightarrow \quad |\xi| \lesssim |\theta|,\ |\theta| \lesssim |\eta|,\ |\eta| \lesssim |\xi|. \tag{3-42}
$$

We now study the term $R_3 h$ in (3-34) depending on $g_3(\xi,\theta,\eta)$ in (3-37). We need to bound from above, for any $j,k = 1,\dots,d$, the $H^{s-m_1-m_2+2}$-Sobolev norm (see (3-41)) of a term like

$$
\begin{aligned}
\widehat{F}_{j,k}(\xi) &:= \sum_{(\theta,\eta) \in \mathscr{B}(\xi)} (\widehat{\partial_{x_j} \partial_{\xi_k} a}) \Big(\xi-\theta, \frac{\xi+\eta}{2}\Big) (\widehat{\partial_{x_k} \partial_{\xi_j} b}) \Big(\theta-\eta, \frac{\xi+\eta}{2}\Big) \hat{h}(\eta) \\
&= \sum_{\eta \in \mathbb{Z}^d} \hat{c}_{j,k} \Big(\xi-\eta, \frac{\xi+\eta}{2}\Big) \hat{h}(\eta),
\end{aligned}
\tag{3-43}
$$

where we let

$$\hat{c}_{j,k}(p,\zeta) := \sum_{\ell\in\mathbb{Z}^d} (\widehat{\partial_{x_j}\partial_{\xi_k}a})(p-\ell,\zeta)(\widehat{\partial_{x_k}\partial_{\xi_j}b})(\ell,\zeta)1_{\mathscr{C}(p,\zeta)}, \qquad p,\zeta\in\mathbb{Z}^d,$$
$$\mathscr{C}(p,\zeta) := \left\{\ell\in\mathbb{Z}^d : \frac{|p-\ell|}{\langle 2\zeta+\ell\rangle} \le \frac{8}{5}\epsilon\right\} \cap \left\{\ell\in\mathbb{Z}^d : \frac{|\ell|}{\langle \ell-p+2\zeta\rangle} \le \frac{8}{5}\epsilon\right\}$$

and $1_{\mathscr{C}(p,\zeta)}$ is the characteristic function of the set $\mathscr{C}(p,\zeta)$. Reasoning as in (3-42), we can deduce that for $\ell\in\mathscr{C}(p,\zeta)$ one has

$$|2\zeta| \lesssim \tfrac{1}{2}|2\zeta+p|. \tag{3-44}$$

Indeed $\ell\in\mathscr{C}(p,\zeta)$ implies $(\theta,\eta)\in\mathscr{B}(\xi)$ by setting

$$2\xi = 2\zeta+p, \quad 2\theta = 2\ell+2\zeta-p, \quad 2\eta = 2\zeta-p. \tag{3-45}$$

Hence (3-44) follows by (3-42) by observing that $2\zeta=\xi+\eta$. Using that $a\in\mathscr{N}^{m_1}_{s_0+2}$, $b\in\mathscr{N}^{m_2}_{s_0+2}$ and reasoning as in (3-13), we deduce

$$|\hat{c}_{j,k}(p,\zeta)| \lesssim \langle\zeta\rangle^{m_1+m_2-2}\langle p\rangle^{-s_0}|a|_{\mathscr{N}^{m_1}_{s_0+2}}|b|_{\mathscr{N}^{m_2}_{s_0+2}}. \tag{3-46}$$

By (3-43), (3-42), (3-2), we get

$$\begin{aligned}
\|F_{j,k}\|^2_{H^{s-m_1-m_2+2}} &\lesssim \sum_{\xi\in\mathbb{Z}^d}\langle\xi\rangle^{-2m_1-2m_2+2}\left(\sum_{\eta\in\mathbb{Z}^d}\left|\hat{c}_{j,k}\left(\xi-\eta,\frac{\xi+\eta}{2}\right)\right||\hat{h}(\eta)|\langle\eta\rangle^s\right)^2 \\
&\lesssim |a|^2_{\mathscr{N}^{m_1}_{s_0+2}}|b|^2_{\mathscr{N}^{m_2}_{s_0+2}}\sum_{\xi\in\mathbb{Z}^d}\left(\sum_{\eta\in\mathbb{Z}^d}|\hat{h}(\eta)|\langle\eta\rangle^s\frac{1}{\langle\xi-\eta\rangle^{s_0}}\right)^2 \quad \text{(by (3-46), (3-44), (3-45))} \\
&\lesssim |a|^2_{\mathscr{N}^{m_1}_{s_0+2}}|b|^2_{\mathscr{N}^{m_2}_{s_0+2}}\||\hat{h}(\xi)|\langle\xi\rangle^s\star\langle\xi\rangle^{-s_0}\|_{\ell^2(\mathbb{Z}^d)} \\
&\lesssim \|h\|^2_{H^s}|a|^2_{\mathscr{N}^{m_1}_{s_0+2}}|b|^2_{\mathscr{N}^{m_2}_{s_0+2}},
\end{aligned}$$

where in the last step we used the Young inequality for sequences, the Cauchy–Schwarz inequality and that $\langle\xi\rangle^{-s_0}$ is in $\ell^1(\mathbb{Z}^d)$ if $s_0>d$. Since the estimate above holds for any $j,k=1,\dots,d$, we may absorb the remainder R_3h in (3-34) in $R_2(a,b)h$ satisfying (3-28). One can deal with the other terms g_1,g_2,g_4,g_5,g_6 similarly. □

Lemma 3.3. *Fix $s_0>\frac{1}{2}d$ and let $f,g,h\in H^s(\mathbb{T};\mathbb{C})$ for $s\ge s_0$. Then*

$$fgh = T_{fg}h + T_{gh}f + T_{fh}g + \mathscr{R}(f,g,h), \tag{3-47}$$

where

$$\begin{aligned}
\widehat{\mathscr{R}(f,g,h)}(\xi) &= \frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d} a(\xi,\eta,\zeta)\hat{f}(\xi-\eta-\zeta)\hat{g}(\eta)\hat{h}(\zeta), \\
|a(\xi,\eta,\zeta)| &\lesssim_\rho \frac{\max_2(|\xi-\eta-\zeta|,|\eta|,|\zeta|)^\rho}{\max(|\xi-\eta-\zeta|,|\eta|,|\zeta|)^\rho} \quad \textit{for all } \rho\ge 0.
\end{aligned} \tag{3-48}$$

Remark 3.4. An estimate of the form (3-48) implies that the function $(f,g,h)\mapsto\mathscr{R}(f,g,h)$ defines a continuous trilinear form on $H^s\times H^s\times H^s$ with values in $H^{s+\rho}$ as soon as $s>\rho+\frac{1}{2}d$. This will be proved in Lemma 3.7.

Proof. We start by proving the following claim: the term

$$T_{fg}h-\sum_{\xi\in\mathbb{Z}^d}e^{\mathrm{i}\xi\cdot x}\sum_{\eta,\zeta\in\mathbb{Z}^d}\chi_\epsilon\left(\frac{|\xi-\eta-\zeta|+|\eta|}{\langle\zeta\rangle}\right)\hat{f}(\xi-\eta-\zeta)\hat{g}(\eta)\hat{h}(\zeta)$$

is a remainder of the form (3-48). By (3-6) this is actually true with coefficients $a(\xi,\eta,\zeta)$ of the form

$$a(\xi,\eta,\zeta):=\chi_\epsilon\left(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\right)-\chi_\epsilon\left(\frac{|\xi-\eta-\zeta|+|\eta|}{\langle\zeta\rangle}\right).$$

In order to prove this, we consider the following partition of the unity:

$$\Theta_\epsilon(\xi,\eta,\zeta):=1-\chi_\epsilon\left(\frac{|\xi-\eta-\zeta|+|\zeta|}{\langle\eta\rangle}\right)-\chi_\epsilon\left(\frac{|\eta|+|\zeta|}{\langle\xi-\eta-\zeta\rangle}\right)-\chi_\epsilon\left(\frac{|\xi-\eta-\zeta|+|\eta|}{\langle\zeta\rangle}\right). \tag{3-49}$$

Then we can write

$$\begin{aligned}a(\xi,\eta,\zeta)=\left(\chi_\epsilon\left(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\right)-1\right)\chi_\epsilon\left(\frac{|\xi-\eta-\zeta|+|\eta|}{\langle\zeta\rangle}\right)+\chi_\epsilon\left(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\right)\chi_\epsilon\left(\frac{|\xi-\eta-\zeta|+|\zeta|}{\langle\eta\rangle}\right)\\+\chi_\epsilon\left(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\right)\chi_\epsilon\left(\frac{|\eta|+|\zeta|}{\langle\xi-\eta-\zeta\rangle}\right)+\chi_\epsilon\left(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\right)\Theta_\epsilon(\xi,\eta,\zeta).\end{aligned} \tag{3-50}$$

Using (3-5) one can prove that each summand in the right-hand side of the equation above is nonzero only if $\max_2(|\xi-\eta-\zeta|,|\eta,||\zeta|)\sim\max_1(|\xi-\eta-\zeta|,|\eta,||\zeta|)$. This implies that each summand defines a smoothing remainder as in (3-48). A similar property holds also for $T_{gh}f$ and $T_{fh}g$. At this point we write

$$\begin{aligned}fgh=\sum_{\xi\in\mathbb{Z}^d}e^{\mathrm{i}\xi\cdot x}\sum_{\eta,\zeta\in\mathbb{Z}^d}\Bigg[\Theta_\epsilon(\xi,\eta,\zeta)+\chi_\epsilon\left(\frac{|\xi-\eta-\zeta|+|\zeta|}{\langle\eta\rangle}\right)\\+\chi_\epsilon\left(\frac{|\eta|+|\zeta|}{\langle\xi-\eta-\zeta\rangle}\right)+\chi_\epsilon\left(\frac{|\xi-\eta-\zeta|+|\eta|}{\langle\zeta\rangle}\right)\Bigg]\hat{f}(\xi-\eta-\zeta)\hat{g}(\eta)\hat{h}(\zeta).\end{aligned}$$

One concludes by using the claim at the beginning of the proof. □

Matrices of symbols and operators. Let us consider the subspace $\mathcal{U}$ defined as

$$\mathcal{U}:=\{(u^+,u^-)\in L^2(\mathbb{T}^d;\mathbb{C})\times L^2(\mathbb{T}^d;\mathbb{C}):u^+=\bar{u}^-\}. \tag{3-51}$$

Throughout the paper we shall deal with matrices of linear operators acting on $H^s(\mathbb{T}^d;\mathbb{C}^2)$ preserving the subspace $\mathcal{U}$. Consider two operators R_1,R_2 acting on $C^\infty(\mathbb{T}^d;\mathbb{C})$. We define the operator $\mathfrak{F}$ acting on $C^\infty(\mathbb{T}^d;\mathbb{C}^2)$ as

$$\mathfrak{F}:=\begin{bmatrix}R_1 & R_2\\ \bar{R}_2 & \bar{R}_1\end{bmatrix}, \tag{3-52}$$

where the linear operators $\bar{R}_i[\,\cdot\,]$, $i=1,2$, are defined by the relation $\bar{R}_i[v]:=\overline{R_i[\bar{v}]}$. We say that an operator of the form (3-52) is *real-to-real*. It is easy to note that real-to-real operators preserve $\mathcal{U}$ in (3-51). Consider now a symbol $a(x,\xi)$ of order m and set $A:=T_a$. Using (3-6) one can check that

$$\bar{A}[h]=\overline{A[\bar{h}]}\quad\Longrightarrow\quad\bar{A}=T_{\tilde{a}},\quad\tilde{a}(x,\xi)=\overline{a(x,-\xi)}, \tag{3-53}$$

$$\text{(adjoint)}\quad(Ah,v)_{L^2}=(h,A^*v)_{L^2}\quad\Longrightarrow\quad A^*=T_{\bar{a}}. \tag{3-54}$$

By (3-54) we deduce that the operator A is self-adjoint with respect to the scalar product (3-3) if and only if the symbol $a(x,\xi)$ is real-valued. We need the following definition. Consider two symbols $a, b \in \mathcal{N}_s^m$ and the matrix

$$A := A(x,\xi) := \begin{pmatrix} a(x,\xi) & b(x,\xi) \\ \overline{b(x,-\xi)} & \overline{a(x,-\xi)} \end{pmatrix}.$$

Define the operator (recall (3-7))

$$M := \mathrm{Op}^{\mathrm{BW}}(A(x,\xi)) := \begin{pmatrix} \mathrm{Op}^{\mathrm{BW}}(a(x,\xi)) & \mathrm{Op}^{\mathrm{BW}}(b(x,\xi)) \\ \mathrm{Op}^{\mathrm{BW}}(\overline{b(x,-\xi)}) & \mathrm{Op}^{\mathrm{BW}}(\overline{a(x,-\xi)}) \end{pmatrix}. \tag{3-55}$$

The matrix of paradifferential operators defined above has the following properties:

- *Real-to-real-ness* : by (3-53) we have that the operator M in (3-55) has the form (3-52); hence it is real-to-real.
- *Self-adjointness*: using (3-54) the operator M in (3-55) is self-adjoint with respect to the scalar product on (3-51)

$$(U,V)_{L^2} := \int_{\mathbb{T}^d} U \cdot \overline{V}\, dx, \quad U = \begin{bmatrix} u \\ \bar{u} \end{bmatrix}, \; V = \begin{bmatrix} v \\ \bar{v} \end{bmatrix}. \tag{3-56}$$

if and only if

$$a(x,\xi) = \overline{a(x,\xi)}, \quad b(x,-\xi) = b(x,\xi). \tag{3-57}$$

Nonhomogeneous symbols. In this paper we deal with symbols satisfying (3-4) which depend nonlinearly on an extra function $u(t,x)$ (which in the application will be a solution either of (NLS) or (KG)). We are interested in providing estimates of the seminorms (3-4) in terms of the Sobolev norms of the function u.

We recall classical tame estimates for composition of functions; we refer to [Baldi 2013] (see also [Taylor 2000]). A function $f : \mathbb{T}^d \times B_R \to \mathbb{C}$, where $B_R := \{y \in \mathbb{R}^m : |y| < R\}$, $R > 0$, induces the composition operator (Nemytskii)

$$\tilde{f}(u) := f(x, u(x), Du(x), \dots, D^p u(x)), \tag{3-58}$$

where $D^k u(x)$ denote the derivatives ∂_x^α of order $|\alpha| = k$ (the number m of y-variables depends on p, d).

Lemma 3.5. *Fix $\gamma > 0$ and assume that $f \in C^\infty(\mathbb{T}^d \times B_R; \mathbb{R})$. Then, for any $u \in H^{\gamma+p}$ with $\|u\|_{W^{p,\infty}} < R$, one has*

$$\|\tilde{f}(u)\|_{H^\gamma} \le C\|f\|_{C^\gamma}(1 + \|u\|_{H^{\gamma+p}}), \tag{3-59}$$

$$\Big\| \tilde{f}(u+h) - \sum_{n=0}^{N} \frac{1}{n!} \partial_u^n \tilde{f}[h, \dots, h] \Big\|_{H^\gamma} \le C\|h\|_{W^{p,\infty}}^N (\|h\|_{H^\gamma} + \|h\|_{W^{p,\infty}} \|u\|_{H^{\gamma+p}}) \tag{3-60}$$

for any $h \in H^{\gamma+p}$ with $\|h\|_{W^{p,\infty}} < \frac{1}{2}R$ and where $C > 0$ is a constant depending on γ and the norm $\|u\|_{W^{p,\infty}}$.

Consider a function $F(y_0, y_1, \dots, y_d)$ in $C^\infty(\mathbb{C}^{d+1}; \mathbb{R})$ in the *real* sense; i.e., F is C^∞ as function of $\mathrm{Re}(y_i)$, $\mathrm{Im}(y_i)$. Assume that F has a zero of order at least $p+2 \in \mathbb{N}$ at the origin. Consider a symbol $f(\xi)$, independent of $x \in \mathbb{T}^d$, such that $|f|_{\mathcal{N}_s^m} \le C < +\infty$ for some constant C. Let us define the symbol

$$a(x,\xi) := (\partial_{z_j^\alpha z_k^\beta} F)(u, \nabla u) f(\xi), \qquad z_j^\alpha := \partial_{x_j}^\alpha u^\sigma,\ z_k^\beta := \partial_{x_k}^\beta u^{\sigma'} \tag{3-61}$$

for some $j,k = 1,\dots,d$, $\alpha,\beta \in \{0,1\}$ and $\sigma,\sigma' \in \{\pm\}$, where we use the notation $u^+ = u$ and $u^- = \bar{u}$.

Lemma 3.6. *Fix $s_0 > \frac{1}{2}d$. For $u \in B_R(H^{s+s_0+1}(\mathbb{T}^d;\mathbb{C}))$ with $0 < R < 1$, we have*

$$|a|_{\mathcal{N}_s^m} \lesssim \|u\|^p_{H^{s+s_0+1}},$$

where a is the symbol in (3-61). Moreover, the map $h \to (\partial_u a)(u;x,\xi)h$ is a $\mathbb{C}$-linear map from H^{s+s_0+1} to $\mathbb{C}$ and satisfies

$$|(\partial_u a)h|_{\mathcal{N}_s^m} \lesssim \|h\|_{H^{s+s_0+1}} \|u\|^{p-1}_{H^{s+s_0+1}}.$$

The same holds for $\partial_{\bar{u}} a$. Moreover if the symbol a does not depend on ∇u, then the same results are true with $s_0 + 1 \rightsquigarrow s_0$.

Proof. It follows from Lemma 3.5. □

Trilinear operators. Throughout the paper we shall deal with trilinear operators on the Sobolev spaces. We shall adopt a combination of notation introduced in [Berti and Delort 2018; Ionescu and Pusateri 2019]. In particular we are interested in studying properties of operators of the form

$$\begin{aligned} &Q = Q[u_1,u_2,u_3] : (C^\infty(\mathbb{T}^d;\mathbb{C}))^3 \to C^\infty(\mathbb{T}^d;\mathbb{C}),\\ &\widehat{Q}(\xi) = \frac{1}{(2\pi)^d} \sum_{\eta,\zeta\in\mathbb{Z}^d} q(\xi,\eta,\zeta)\hat{u}_1(\xi-\eta-\zeta)\hat{u}_2(\eta)\hat{u}_3(\zeta) \quad \text{for all } \xi \in \mathbb{Z}^d, \end{aligned} \tag{3-62}$$

where $q(\xi,\eta,\zeta) \in \mathbb{C}$ for any $\xi,\eta,\zeta \in \mathbb{Z}^d$. We now prove that, under certain conditions on the coefficients, the operators of the form (3-62) extend as continuous maps on the Sobolev spaces.

Lemma 3.7. *Let $\mu \ge 0$ and $m \in \mathbb{R}$. Assume that for any $\xi,\eta,\zeta \in \mathbb{Z}^d$ one has*

$$|q(\xi,\eta,\zeta)| \lesssim \frac{\max_2\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^\mu}{\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^m}. \tag{3-63}$$

Then, for $s \ge s_0 > \frac{1}{2}d + \mu$, the map Q in (3-62) with coefficients satisfying (3-63) extends as a continuous map form $(H^s(\mathbb{T}^d;\mathbb{C}))^3$ to $H^{s+m}(\mathbb{T}^d;\mathbb{C})$. Moreover one has

$$\|Q(u_1,u_2,u_3)\|_{H^{s+m}} \lesssim \sum_{i=1}^3 \|u_i\|_{H^s} \prod_{i\ne k} \|u_k\|_{H^{s_0}}. \tag{3-64}$$

Proof. By (3-2) we have

$$\begin{aligned} &\|Q(u_1,u_2,u_3)\|^2_{H^{s+m}}\\ &\quad\le \sum_{\xi\in\mathbb{Z}^d} \langle\xi\rangle^{2(s+m)} \Big(\sum_{\eta,\zeta\in\mathbb{Z}^d} |q(\xi,\eta,\zeta)||\hat{u}_1(\xi-\eta-\zeta)||\hat{u}_2(\eta)||\hat{u}_3(\zeta)| \Big)^2\\ &\quad\lesssim \sum_{\xi\in\mathbb{Z}^d} \Big(\sum_{\eta,\zeta\in\mathbb{Z}^d} \langle\xi\rangle^s \max_2\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^\mu |\hat{u}_1(\xi-\eta-\zeta)||\hat{u}_2(\eta)||\hat{u}_3(\zeta)| \Big)^2 \quad \text{(by (3-63))}\\ &\quad:= I + II + III, \end{aligned} \tag{3-65}$$

where I, II, III are the terms in (3-65) which are supported respectively on indexes such that

$$\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}=\langle\xi-\eta-\zeta\rangle,$$
$$\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}=\langle\eta\rangle,$$
$$\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}=\langle\zeta\rangle.$$

Consider for instance the term III. By using the Young inequality for sequences we deduce

$$III\lesssim\|(\langle p\rangle^{\mu}\hat{u}_1(p))*(\langle\eta\rangle^{\mu}\hat{u}_2(\eta))*(\langle\zeta\rangle^{s}\hat{u}_3(\zeta))\|_{\ell^2}\lesssim\|u_1\|_{H^{s_0}}\|u_2\|_{H^{s_0}}\|u_3\|_{H^{s}},$$

which is (3-64). The bounds of I and II are similar. □

In the following lemma we shall prove that a class of "paradifferential" trilinear operators, having some decay on the coefficients, satisfies the hypothesis of the previous lemma.

Lemma 3.8. *Let $\mu\geq 0$ and $m\in\mathbb{R}$, $m\geq 0$. Consider a trilinear map Q as in (3-62) with coefficients satisfying*

$$q(\xi,\eta,\zeta)=f(\xi,\eta,\zeta)\chi_\epsilon\left(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\right),\quad |f(\xi,\eta,\zeta)|\lesssim\frac{|\xi-\zeta|^{\mu}}{\langle\zeta\rangle^{m}}\tag{3-66}$$

for any $\xi,\eta,\zeta\in\mathbb{Z}^d$ and $0<\epsilon\ll 1$. Then the coefficients $q(\xi,\eta,\zeta)$ satisfy (3-63) with $\mu\rightsquigarrow\mu+m$.

Proof. First of all we write $q(\xi,\eta,\zeta)=q_1(\xi,\eta,\zeta)+q_2(\xi,\eta,\zeta)$ with

$$q_1(\xi,\eta,\zeta)=f(\xi,\eta,\zeta)\chi_\epsilon\left(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\right)\chi_\epsilon\left(\frac{|\xi-\eta-\zeta|+|\eta|}{\langle\zeta\rangle}\right),\tag{3-67}$$

$$q_2(\xi,\eta,\zeta)=f(\xi,\eta,\zeta)\chi_\epsilon\left(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\right)\left[\chi_\epsilon\left(\frac{|\xi-\eta-\zeta|+|\zeta|}{\langle\eta\rangle}\right)+\chi_\epsilon\left(\frac{|\eta|+|\zeta|}{\langle\xi-\eta-\zeta\rangle}\right)+\Theta_\epsilon(\xi,\eta,\zeta)\right],\tag{3-68}$$

where $\Theta_\epsilon(\xi,\eta,\zeta)$ is defined in (3-49). Recalling (3-5) one can check that if $q_1(\xi,\eta,\zeta)\neq 0$ then $|\xi-\eta-\zeta|+|\eta|\ll|\zeta|\sim|\xi|$. Together with the bound on $f(\xi,\eta,\zeta)$ in (3-66) we deduce that the coefficients in (3-67) satisfy (3-63). The coefficients in (3-68) satisfy (3-63) because of the support of the cut off function in (3-5). □

Hamiltonian formalism in complex variables. Given a Hamiltonian function $H:H^1(\mathbb{T}^d;\mathbb{C}^2)\to\mathbb{R}$, its Hamiltonian vector field has the form

$$X_H(U):=-\mathrm{i}J\nabla H(U)=-\mathrm{i}\begin{pmatrix}\nabla_{\bar{u}}H(U)\\-\nabla_u H(U)\end{pmatrix},\quad J=\begin{bmatrix}0&1\\-1&0\end{bmatrix},\ U=\begin{bmatrix}u\\\bar{u}\end{bmatrix}.\tag{3-69}$$

Indeed one has

$$dH(U)[V]=-\Omega(X_H(U),V)\quad\text{for all } U=\begin{bmatrix}u\\\bar{u}\end{bmatrix},\ V=\begin{bmatrix}v\\\bar{v}\end{bmatrix},\tag{3-70}$$

where Ω is the nondegenerate symplectic form

$$\Omega(U,V)=-\int_{\mathbb{T}^d}U\cdot\mathrm{i}JV\,dx=-\int_{\mathbb{T}^d}\mathrm{i}(u\bar{v}-\bar{u}v)\,dx.\tag{3-71}$$

The Poisson brackets between two Hamiltonians H, G are defined as

$$\{G, H\} := \Omega(X_G, X_H) \overset{(3\text{-}71)}{=} -\int_{\mathbb{T}^d} \mathrm{i} J \nabla G \cdot \nabla H \, dx = -\mathrm{i} \int_{\mathbb{T}^d} \nabla_u H \nabla_{\bar{u}} G - \nabla_{\bar{u}} H \nabla_u G \, dx. \tag{3-72}$$

The nonlinear commutator between two Hamiltonian vector fields is given by

$$[X_G, X_H](U) = dX_G(U)[X_H(U)] - dX_H(U)[X_G(U)] = -X_{\{G,H\}}(U). \tag{3-73}$$

Hamiltonian formalism in real variables. Given a Hamiltonian function $H_{\mathbb{R}} : H^1(\mathbb{T}^d; \mathbb{R}^2) \to \mathbb{R}$, its Hamiltonian vector field has the form

$$X_{H_{\mathbb{R}}}(\psi, \phi) := J \nabla H_{\mathbb{R}}(\psi, \phi) = \begin{pmatrix} \nabla_\phi H_{\mathbb{R}}(\psi, \phi) \\ -\nabla_\psi H_{\mathbb{R}}(\psi, \phi) \end{pmatrix}, \tag{3-74}$$

where J is in (3-69). Indeed one has

$$dH_{\mathbb{R}}(\psi, \phi)[h] = -\widetilde{\Omega}(X_{H_{\mathbb{R}}}(\psi, \phi), h) \quad \text{for all } \begin{bmatrix} \psi \\ \phi \end{bmatrix}, \; h = \begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix}, \tag{3-75}$$

where $\widetilde{\Omega}$ is the nondegenerate symplectic form

$$\widetilde{\Omega}\left(\begin{bmatrix} \psi_1 \\ \phi_1 \end{bmatrix}, \begin{bmatrix} \psi_2 \\ \phi_2 \end{bmatrix} \right) := \int_{\mathbb{T}^d} \begin{bmatrix} \psi_1 \\ \phi_1 \end{bmatrix} \cdot J^{-1} \begin{bmatrix} \psi_2 \\ \phi_2 \end{bmatrix} dx = \int_{\mathbb{T}^d} -(\psi_1 \phi_2 - \phi_1 \psi_2) \, dx. \tag{3-76}$$

We introduce the complex symplectic variables

$$\begin{pmatrix} u \\ \bar{u} \end{pmatrix} = \mathscr{C} \begin{pmatrix} \psi \\ \phi \end{pmatrix} := \frac{1}{\sqrt{2}} \begin{pmatrix} \Lambda_{\mathrm{KG}}^{1/2} \psi + \mathrm{i} \Lambda_{\mathrm{KG}}^{-1/2} \phi \\ \Lambda_{\mathrm{KG}}^{1/2} \psi - \mathrm{i} \Lambda_{\mathrm{KG}}^{-1/2} \phi \end{pmatrix}, \quad \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \mathscr{C}^{-1} \begin{pmatrix} u \\ \bar{u} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Lambda_{\mathrm{KG}}^{-1/2}(u + \bar{u}) \\ -\mathrm{i} \Lambda_{\mathrm{KG}}^{1/2}(u - \bar{u}) \end{pmatrix}, \tag{3-77}$$

where Λ_{KG} is in (1-3). The symplectic form in (3-76) transforms, for

$$U = \begin{bmatrix} u \\ \bar{u} \end{bmatrix}, \quad V = \begin{bmatrix} v \\ \bar{v} \end{bmatrix},$$

into $\Omega(U, V)$ where Ω is in (3-71). In these coordinates the vector field $X_{H_{\mathbb{R}}}$ in (3-74) assumes the form X_H as in (3-69) with $H := H_{\mathbb{R}} \circ \mathscr{C}^{-1}$.

We now study some algebraic properties enjoyed by the Hamiltonian functions previously defined. Let us consider a homogeneous Hamiltonian $H : H^1(\mathbb{T}^d; \mathbb{C}^2) \to \mathbb{R}$ of degree 4 of the form

$$H(U) = (2\pi)^{-d} \sum_{\xi, \eta, \zeta \in \mathbb{Z}^d} \mathtt{h}_4(\xi, \eta, \zeta) \hat{u}(\xi - \eta - \zeta) \hat{\bar{u}}(\eta) \hat{u}(\zeta) \hat{\bar{u}}(-\xi), \quad U = \begin{bmatrix} u \\ \bar{u} \end{bmatrix}, \tag{3-78}$$

for some coefficients $\mathtt{h}_4(\xi, \eta, \zeta) \in \mathbb{C}$ such that

$$\begin{aligned} \mathtt{h}_4(\xi, \eta, \zeta) &= \mathtt{h}_4(-\eta, -\xi, \zeta) = \mathtt{h}_4(\xi, \eta, \xi - \eta - \zeta), \\ \mathtt{h}_4(\xi, \eta, \zeta) &= \overline{\mathtt{h}_4(\zeta, \eta + \zeta - \xi, \xi)} \quad \text{for all } \xi, \eta, \zeta \in \mathbb{Z}^d. \end{aligned} \tag{3-79}$$

By (3-79) one can check that the Hamiltonian H is real-valued and symmetric in its entries. Recalling (3-69) we have that its Hamiltonian vector field can be written as

$$\begin{pmatrix} -\mathrm{i}\nabla_{\bar{u}} H(U) \\ \mathrm{i}\nabla_u H(U) \end{pmatrix} = \begin{pmatrix} X_H^+(U) \\ \overline{X_H^+(U)} \end{pmatrix}, \tag{3-80}$$

$$\widehat{X_H^+(U)}(\xi) = (2\pi)^{-d} \sum_{\eta,\zeta\in\mathbb{Z}^d} f(\xi,\eta,\zeta)\hat{u}(\xi-\eta-\zeta)\hat{\bar{u}}(\eta)\hat{u}(\zeta), \tag{3-81}$$

where the coefficients $f(\xi,\eta,\zeta)$ have the form

$$f(\xi,\eta,\zeta) = -2\mathrm{i}\mathtt{h}_4(\xi,\eta,\zeta), \quad \xi,\eta,\zeta\in\mathbb{Z}^d. \tag{3-82}$$

We need the following definition.

Definition 3.9 (resonant set). We define the following set of *resonant* indexes:

$$\mathscr{R} := \{(\xi,\eta,\zeta)\in\mathbb{Z}^{3d} : |\xi|=|\zeta|,\ |\eta|=|\xi-\eta-\zeta|\} \cup \{(\xi,\eta,\zeta)\in\mathbb{Z}^{3d} : |\xi|=|\xi-\eta-\zeta|,\ |\eta|=|\zeta|\}. \tag{3-83}$$

Consider the vector field in (3-81) with Hamiltonian H defined in (3-78). We define the field $X_H^{+,\mathrm{res}}(U)$ by

$$\widehat{X_H^{+,\mathrm{res}}}(\xi) = (2\pi)^{-d} \sum_{\eta,\zeta\in\mathbb{Z}^d} f^{(\mathrm{res})}(\xi,\eta,\zeta)\hat{u}(\xi-\eta-\zeta)\hat{\bar{u}}(\eta)\hat{u}(\zeta), \tag{3-84}$$

where

$$f^{(\mathrm{res})}(\xi,\eta,\zeta) := f(\xi,\eta,\zeta)\mathtt{1}_{\mathscr{R}}(\xi,\eta,\zeta), \tag{3-85}$$

where $\mathtt{1}_{\mathscr{R}}$ is the characteristic function of the set $\mathscr{R}$ and f is defined in (3-82).

In the next lemma we prove a fundamental cancellation.

Lemma 3.10. *For $n\geq 0$ one has* (*recall* (3-2))

$$\operatorname{Re}(\langle D\rangle^n X_H^{+,\mathrm{res}}(U), \langle D\rangle^n u)_{L^2} \equiv 0. \tag{3-86}$$

Proof. Using (3-83)–(3-85) one can check that

$$\widehat{X_H^{+,\mathrm{res}}}(\xi) = (2\pi)^{-d} \sum_{(\eta,\zeta)\in\mathscr{R}(\xi)} \mathscr{F}(\xi,\eta,\zeta)\hat{u}(\xi-\eta-\zeta)\hat{\bar{u}}(\eta)\hat{u}(\zeta),$$

with $\mathscr{R}(\xi) := \{(\eta,\zeta)\in\mathbb{Z}^{2d} : |\xi|=|\zeta|,\ |\eta|=|\xi-\eta-\zeta|\}$ for $\xi\in\mathbb{Z}^d$, and

$$\mathscr{F}(\xi,\eta,\zeta) := f(\xi,\eta,\zeta) + f(\xi,\eta,\xi-\eta-\zeta). \tag{3-87}$$

By an explicit computation we have

$$\begin{aligned} &\operatorname{Re}(\langle D\rangle^s X_H^{+,\mathrm{res}}(U), \langle D\rangle^s u)_{L^2} \\ &\quad = (2\pi)^{-d} \sum_{\xi\in\mathbb{Z}^d,(\eta,\zeta)\in\mathscr{R}(\xi)} \langle\xi\rangle^{2s}[\mathscr{F}(\xi,\eta,\zeta) + \overline{\mathscr{F}(\zeta,\zeta+\eta-\xi,\xi)}]\hat{u}(\xi-\eta-\zeta)\hat{\bar{u}}(\eta)\hat{u}(\zeta)\hat{\bar{u}}(-\xi). \end{aligned}$$

By (3-87), (3-82) and using the symmetries (3-79) we have $\mathscr{F}(\xi,\eta,\zeta) + \overline{\mathscr{F}(\zeta,\zeta+\eta-\xi,\xi)} = 0$. □

Remark 3.11. Throughout the paper we shall deal with general Hamiltonian functions of the form

$$H(W)=(2\pi)^{-d}\sum_{\substack{\sigma_1,\sigma_2,\sigma_3,\sigma_4\in\{\pm\}\\ \xi,\eta,\zeta\in\mathbb{Z}^d}}\mathtt{h}^{\sigma_1,\sigma_2,\sigma_3,\sigma_4}(\xi,\eta,\zeta)\widehat{u^{\sigma_1}}(\xi-\eta-\zeta)\widehat{u^{\sigma_2}}(\eta)\widehat{u^{\sigma_3}}(\zeta)\widehat{u^{\sigma_4}}(-\xi),$$

where we use the notation

$$\widehat{u^{\sigma}}(\,\cdot\,)=\hat{u}(\,\cdot\,)\quad\text{if }\sigma=+\qquad\text{and}\qquad\widehat{u^{\sigma}}(\,\cdot\,)=\hat{\bar{u}}(\,\cdot\,)\quad\text{if }\sigma=-. \tag{3-88}$$

However, by the definition of the resonant set (3-83), we can note that the resonant vector field has still the form (3-84) and it depends only on the monomials in the Hamiltonian $H(U)$ which are *gauge*-invariant, i.e., of the form (3-78).

4. Paradifferential formulation of the problems

In this section we rewrite the equations in a paradifferential form by means of the paralinearization formula (à la [Bony 1981]). In Section 4A we deal with the problem (NLS) and in Section 4B we deal with (KG).

4A. ***Paralinearization of the NLS.*** In the following we paralinearize (NLS), with respect to the variables $(u,\bar{u})$. We recall that (NLS) may be rewritten as (1-12) and we define $\widetilde{P}(u):=P(u,\nabla u)-\frac{1}{2}|u|^4=\frac{1}{2}|\nabla h(|u|^2)|^2$. We set

$$\tilde{p}(u):=(\partial_{\bar{u}}\widetilde{P})(u,\nabla u)-\sum_{j=1}^{d}\partial_{x_j}(\partial_{\bar{u}_{x_j}}\widetilde{P})(u,\nabla u). \tag{4-1}$$

Lemma 4.1. *Fix $s_0>\frac{1}{2}d$ and $0\le\rho<s-s_0,\ s\ge s_0$. Consider $u\in H^s(\mathbb{T}^d;\mathbb{C})$. Then we have*

$$\begin{aligned}\tilde{p}(u)&=T_{\partial_{u\bar{u}}\widetilde{P}}[u]+T_{\partial_{\bar{u}\bar{u}}\widetilde{P}}[\bar{u}] &(4\text{-}2)\\ &\quad+\sum_{j=1}^{d}(T_{\partial_{\bar{u}u_{x_j}}\widetilde{P}}[u_{x_j}]+T_{\partial_{\bar{u}\bar{u}_{x_j}}\widetilde{P}}[\bar{u}_{x_j}])-\sum_{j=1}^{d}\partial_{x_j}(T_{\partial_{u\bar{u}_{x_j}}\widetilde{P}}[u]+T_{\partial_{\bar{u}\bar{u}_{x_j}}\widetilde{P}}[\bar{u}]) &(4\text{-}3)\\ &\quad-\sum_{j=1}^{d}\partial_{x_j}(T_{\partial_{\bar{u}_{x_j}u_{x_j}}\widetilde{P}}[u_{x_j}]+T_{\partial_{\bar{u}_{x_j}\bar{u}_{x_j}}\widetilde{P}}[\bar{u}_{x_j}])+R(u), &(4\text{-}4)\end{aligned}$$

where $R(u)$ is a remainder satisfying

$$\|R(u)\|_{H^{s+\rho}}\lesssim C\|u\|_{H^s}^7 \tag{4-5}$$

for some constant $C>0$ depending on s, s_0.

Proof. By using the Bony paralinearization formula, see [Bony 1981; Métivier 2008; Taylor 2000], and passing to the Weyl quantization we obtain

$$\begin{aligned}\tilde{p}(u)&=T_{\partial_{u\bar{u}}\widetilde{P}}[u]+T_{\partial_{\bar{u}\bar{u}}\widetilde{P}}[\bar{u}] &(4\text{-}6)\\ &\quad+\sum_{j=1}^{d}(T_{\partial_{\bar{u}u_{x_j}}\widetilde{P}}[u_{x_j}]+T_{\partial_{\bar{u}\bar{u}_{x_j}}\widetilde{P}}[\bar{u}_{x_j}])-\sum_{j=1}^{d}\partial_{x_j}(T_{\partial_{u\bar{u}_{x_j}}\widetilde{P}}[u]+T_{\partial_{\bar{u}\bar{u}_{x_j}}\widetilde{P}}[\bar{u}]) &(4\text{-}7)\\ &\quad-\sum_{j=1}^{d}\partial_{x_j}\sum_{k=1}^{d}(T_{\partial_{\bar{u}_{x_j}u_{x_k}}\widetilde{P}}[u_{x_k}]+T_{\partial_{\bar{u}_{x_j}\bar{u}_{x_k}}\widetilde{P}}[\bar{u}_{x_k}])+R(u), &(4\text{-}8)\end{aligned}$$

where $R(u)$ satisfies the estimate (4-5) since $h(x)\sim x^2$ for $x\sim 0$. The first term in (4-8) is equal to the first in (4-4) because $\partial_{\bar u_{x_j}u_{x_k}}\widetilde{P}=\frac{1}{2}\partial_{\bar u_{x_j}u_{x_k}}|\nabla h(|u|^2)|^2=0$ if $j\neq k$. $\square$

We shall use the following notation throughout the rest of the paper:

$$U:=\begin{bmatrix}u\\ \bar u\end{bmatrix},\quad E:=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad \mathbb{1}:=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \mathrm{diag}(b):=b\mathbb{1},\quad b\in\mathbb{C}. \tag{4-9}$$

Define the *real* symbols

$$\begin{aligned} a_2(x)&:=[h'(|u|^2)]^2|u|^2,\quad b_2(x):=[h'(|u|^2)]^2u^2,\\ \vec a_1(x)\cdot\xi&:=[h'(|u|^2)]^2\sum_{j=1}^d \mathrm{Im}(u\bar u_{x_j})\xi_j,\quad \xi=(\xi_1,\dots,\xi_d).\end{aligned}\tag{4-10}$$

We define also the matrix of functions

$$A_2(x):=A_2(U;x):=\begin{bmatrix}a_2(U;x)&b_2(U;x)\\ \overline{b_2(U;x)}&a_2(U;x)\end{bmatrix}=\begin{bmatrix}a_2(x)&b_2(x)\\ \overline{b_2(x)}&a_2(x)\end{bmatrix},\tag{4-11}$$

with $a_2(x)$ and $b_2(x)$ defined in (4-10).

Proposition 4.2 (paralinearization of NLS). *Equation (NLS) is equivalent to the system*

$$\dot U=-\mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}((\mathbb{1}+A_2(x))|\xi|^2)U-\mathrm{i}E\,V*U-\mathrm{i}\,\mathrm{Op}^{\mathrm{BW}}(\mathrm{diag}(\vec a_1(x)\cdot\xi))U+X_{\mathscr{H}^{(4)}_{\mathrm{NLS}}}(U)+R(U),\tag{4-12}$$

where V is the convolution potential in (1-5), the matrix $A_2(x)$ is the one in (4-11), the symbol $\vec a_1(x)\cdot\xi$ is in (4-10) and the vector field $X_{\mathscr{H}^{(4)}_{\mathrm{NLS}}}(U)$ is defined as

$$X_{\mathscr{H}^{(4)}_{\mathrm{NLS}}}(U)=-\mathrm{i}E\left[\mathrm{Op}^{\mathrm{BW}}\left(\begin{bmatrix}2|u|^2&u^2\\ \bar u^2&2|u|^2\end{bmatrix}\right)U+Q_3(U)\right].\tag{4-13}$$

The seminorms of the symbols satisfy the estimates

$$\begin{aligned}|a_2|_{\mathcal{N}_p^0}+|b_2|_{\mathcal{N}_p^0}&\lesssim\|u\|^6_{H^{p+s_0}}\quad \text{for all } p+s_0\le s,\ p\in\mathbb{N},\\ |\vec a_1\cdot\xi|_{\mathcal{N}_p^1}&\lesssim\|u\|^6_{H^{p+s_0+1}}\quad \text{for all } p+s_0+1\le s,\ p\in\mathbb{N},\end{aligned}\tag{4-14}$$

where we have chosen $s_0>d$. The remainder $Q_3(U)$ has the form $(Q_3^+(U),\overline{Q_3^+(U)})^T$ and

$$\widehat{Q_3^+}(\xi)=(2\pi)^{-d}\sum_{\eta,\zeta\in\mathbb{Z}^d}\mathtt{q}(\xi,\eta,\zeta)\hat u(\xi-\eta-\zeta)\hat{\bar u}(\eta)\hat u(\zeta)\tag{4-15}$$

for some $\mathtt{q}(\xi,\eta,\zeta)\in\mathbb{C}$. The coefficients of Q_3^+ satisfy

$$|\mathtt{q}(\xi,\eta,\zeta)|\lesssim\frac{\max_2\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^\rho}{\max\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^\rho}\quad \text{for all } \rho\ge 0.\tag{4-16}$$

The remainder $R(U)$ has the form $(R^+(U),\overline{R^+(U)})^T$. Moreover, for any $s>2d+2$, we have the estimates

$$\|R(U)\|_{H^s}\lesssim\|U\|^7_{H^s},\quad \|Q_3(U)\|_{H^{s+2}}\lesssim\|U\|^3_{H^s}.\tag{4-17}$$

Proof. By Lemma 3.3 the cubic term $|u|^2u$ in (NLS) is equal to $2T_{|u|^2}u + T_{u^2}\bar{u} + \mathscr{R}(u,u,\bar{u})$. Setting $Q_3^+(U) = \mathscr{R}(u,u,\bar{u})$, we get (4-15) by (3-48). The second estimate in (4-17) is a consequence of Lemma 3.7 applied with $\rho = \mu = m = 2$.

We now deal with the remaining quasilinear term $\tilde{p}(u)$ defined in (4-1). We start by noting that

$$\partial_{x_j} := \mathrm{Op}^{\mathrm{BW}}(\mathrm{i}\xi_j), \qquad j = 1, \dots d, \tag{4-18}$$

and that the quantization of a symbol $a(x)$ is given by $\mathrm{Op}^{\mathrm{BW}}(a(x))$. We also remark that the symbols appearing in (4-2), (4-3) and (4-4) can be estimated (in the norm $|\cdot|_{\mathcal{N}_s^m}$) by using Lemma 3.6. Consider now the first paradifferential term in (4-4). We have, for any $j = 1, \dots, d$,

$$\partial_{x_j} T_{\partial_{\bar{u}_{x_j} u_{x_j}} \widetilde{P}} \partial_{x_j} u = \mathrm{Op}^{\mathrm{BW}}(\mathrm{i}\xi_j) \circ \mathrm{Op}^{\mathrm{BW}}(\partial_{\bar{u}_{x_j} u_{x_j}} \widetilde{P}) \circ \mathrm{Op}^{\mathrm{BW}}(\mathrm{i}\xi_j) u.$$

By applying Proposition 3.2 and recalling the Poisson bracket in (3-9), we deduce

$$\mathrm{Op}^{\mathrm{BW}}(\mathrm{i}\xi_j) \circ \mathrm{Op}^{\mathrm{BW}}(\partial_{\bar{u}_{x_j} u_{x_j}} \widetilde{P}) \circ \mathrm{Op}^{\mathrm{BW}}(\mathrm{i}\xi_j) = \mathrm{Op}^{\mathrm{BW}}(-\xi_j^2 \partial_{\bar{u}_{x_j} u_{x_j}} \widetilde{P}) \tag{4-19}$$

$$+ \mathrm{Op}^{\mathrm{BW}}\Big(\frac{\mathrm{i}}{2}\xi_j \partial_{x_j}(\partial_{\bar{u}_{x_j} u_{x_j}} \widetilde{P}) - \frac{\mathrm{i}\xi_j}{2} \partial_{x_j}(\partial_{\bar{u}_{x_j} u_{x_j}} \widetilde{P})\Big) \tag{4-20}$$

$$+ \widetilde{R}_j^{(1)}(u) + \widetilde{R}_j^{(2)}(u), \tag{4-21}$$

where

$$\widetilde{R}_j^{(1)}(u) := \mathrm{Op}^{\mathrm{BW}}\big(-\tfrac{1}{4}\partial_{x_j x_j}(\partial_{\bar{u}_{x_j} u_{x_j}} \widetilde{P})\big)$$

and $\widetilde{R}_j^{(2)}(u)$ is some bounded operator. More precisely, using (3-20), (3-10) and the estimates given by Lemma 3.6, we have, for all $h \in H^s(\mathbb{T}^d; \mathbb{C})$,

$$\|\widetilde{R}_j^{(2)}(u)h\|_{H^s} \le C\|h\|_{H^s}\|u\|_{H^s}^6, \qquad \|\widetilde{R}_j^{(1)}(u)h\|_{H^s} \le C\|h\|_{H^s}\|u\|_{H^{2s_0+3}}^6 \tag{4-22}$$

for some constant $C > 0$ and $s_0 \ge d+1$, $s_0 \in \mathbb{N}$. We set

$$\widetilde{R}(u) := \sum_{j=1}^{d} (\widetilde{R}_j^{(1)}(u) + \widetilde{R}_j^{(2)}(u)).$$

Then

$$\begin{aligned}
-\sum_{j=1}^{d} \partial_{x_j} T_{\partial_{\bar{u}_{x_j} u_{x_j}} \widetilde{P}} \partial_{x_j} u
&= \mathrm{Op}^{\mathrm{BW}}\Big(\sum_{j=1}^{d} \xi_j^2 \partial_{\bar{u}_{x_j} u_{x_j}} \widetilde{P}\Big) + \widetilde{R}(u) - \frac{\mathrm{i}}{2}\mathrm{Op}^{\mathrm{BW}}\Big(\sum_{j=1}^{d}(-\xi_j \partial_{x_j}(\partial_{\bar{u}_{x_j} u_{x_j}} \widetilde{P}) + \xi_j \partial_{x_j}(\partial_{\bar{u}_{x_j} u_{x_j}} \widetilde{P}))\Big) \\
&= \mathrm{Op}^{\mathrm{BW}}(a_2(x)|\xi|^2) + \widetilde{R}(u) + \frac{\mathrm{i}}{2}\mathrm{Op}^{\mathrm{BW}}\Big(\sum_{j=1}^{d} \xi_j \partial_{x_j}((\partial_{\bar{u}_{x_j} u_{x_j}} \widetilde{P}) - (\partial_{\bar{u}_{x_j} u_{x_j}} \widetilde{P}))\Big) \quad \text{(by (4-10))} \\
&= \mathrm{Op}^{\mathrm{BW}}(a_2(x)|\xi|^2) + \widetilde{R}(u),
\end{aligned}$$

where we used the symmetry of the matrix $\partial_{\overline{\nabla u}\nabla u} \widetilde{P}$ (recall $\widetilde{P}$ is real) and that

$$\partial_{\bar{u}_{x_j} u_{x_j}} \widetilde{P}(u) = \tfrac{1}{2}\partial_{\bar{u}_{x_j} u_{x_j}} |\nabla h(|u|^2)|^2 \overset{(4\text{-}10)}{=} a_2(x).$$

By performing similar explicit computations on the other summands in (4-2)-(4-4) we get (4-12), (4-11) with symbols in (4-10). By the discussion above we deduce that the remainder $R(U)$ in (4-12) satisfies the bound (4-17). $\square$

Remark 4.3. • The cubic term $X_{\mathscr{H}^{(4)}_{\mathrm{NLS}}}(U)$ in (4-13) is the Hamiltonian vector field of the Hamiltonian function

$$\mathscr{H}^{(4)}_{\mathrm{NLS}}(U) := \frac{1}{2}\int_{\mathbb{T}^d} |u|^4\,dx, \quad X_{\mathscr{H}^{(4)}_{\mathrm{NLS}}}(U) = -\mathrm{i}|u|^2\begin{bmatrix} u \\ \bar{u}\end{bmatrix}. \tag{4-23}$$

• The operators

$$\mathrm{Op}^{\mathrm{BW}}((\mathbb{1}+A_2(x))|\xi|^2), \quad \mathrm{Op}^{\mathrm{BW}}(\mathrm{diag}(\vec{a}_1(x)\cdot\xi)), \quad \mathrm{Op}^{\mathrm{BW}}\left(\begin{bmatrix} 2|u|^2 & u^2 \\ \bar{u}^2 & 2|u|^2\end{bmatrix}\right)$$

are self-adjoint thanks to (3-57) and (4-10).

4B. ***Paralinearization of the KG.*** In this section we rewrite (KG) as a paradifferential system. This is the content of Proposition 4.7. Before stating this result we need some preliminaries. In particular in Lemma 4.4 below we analyze some properties of the cubic terms in (KG). Define the *real* symbols

$$\begin{aligned} a_2(x,\xi) &:= a_2(u;x,\xi) := \sum_{j,k=1}^{d} (\partial_{\psi_{x_j}\psi_{x_k}}F)(\psi,\nabla\psi)\xi_j\xi_k, \quad \psi = \frac{\Lambda_{\mathrm{KG}}^{-1/2}}{\sqrt{2}}(u+\bar{u}), \\ a_0(x,\xi) &:= a_0(u;x,\xi) := \tfrac{1}{2}(\partial_{y_1y_1}G)(\psi,\Lambda_{\mathrm{KG}}^{1/2}\psi) + (\partial_{y_1y_0}G)(\psi,\Lambda_{\mathrm{KG}}^{1/2}\psi)\Lambda_{\mathrm{KG}}^{-1/2}(\xi). \end{aligned} \tag{4-24}$$

We define also the matrices of symbols

$$\mathscr{A}_1(x,\xi) := \mathscr{A}_1(u;x,\xi) := \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1\end{bmatrix}\Lambda_{\mathrm{KG}}^{-2}(\xi)a_2(u;x,\xi), \tag{4-25}$$

$$\mathscr{A}_0(x,\xi) := \mathscr{A}_0(u;x,\xi) := \begin{bmatrix} 1 & 1 \\ 1 & 1\end{bmatrix}a_0(u;x,\xi), \tag{4-26}$$

and the Hamiltonian function

$$\mathscr{H}^{(4)}_{\mathrm{KG}}(U) := \int_{\mathbb{T}^d} G(\psi,\Lambda_{\mathrm{KG}}^{1/2}\psi)\,dx, \tag{4-27}$$

with G the function appearing in (1-14). First of all we study some properties of the vector field of the Hamiltonian $\mathscr{H}^{(4)}_{\mathrm{KG}}$.

Lemma 4.4. *We have*

$$X_{\mathscr{H}^{(4)}_{\mathrm{KG}}}(U) = -\mathrm{i}J\nabla\mathscr{H}^{(4)}_{\mathrm{KG}}(U) = -\mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}(\mathscr{A}_0(x,\xi))U + Q_3(u), \tag{4-28}$$

with $\mathscr{A}_0$ in (4-26). The remainder $Q_3(u)$ has the form $(Q_3^+(u), \overline{Q_3^+(u)})^T$ and (recall (3-88))

$$\widehat{Q_3^+}(\xi) = \frac{1}{(2\pi)^d}\sum_{\substack{\sigma_1,\sigma_2,\sigma_3\in\{\pm\} \\ \eta,\zeta\in\mathbb{Z}^d}} \mathtt{q}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)\widehat{u^{\sigma_1}}(\xi-\eta-\zeta)\widehat{u^{\sigma_2}}(\eta)\widehat{u^{\sigma_3}}(\zeta) \tag{4-29}$$

for some $\mathtt{q}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)\in\mathbb{C}$. The coefficients of Q_3^+ satisfy

$$|\mathtt{q}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)| \lesssim \frac{\max_2\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}}{\max\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}} \tag{4-30}$$

for any $\sigma_1, \sigma_2, \sigma_3 \in \{\pm\}$. *Finally, for* $s > 2d+1$, *we have*

$$|a_0|_{\mathcal{N}_p^0} \lesssim \|u\|^2_{H^{p+s_0}}, \qquad p+s_0 \le s, \quad s_0 > d, \tag{4-31}$$

$$\|X_{\mathscr{H}^{(4)}_{\mathrm{KG}}}(U)\|_{H^s} \lesssim \|u\|^3_{H^s}, \qquad \|Q_3(u)\|_{H^{s+1}} \lesssim \|u\|^3_{H^s}, \tag{4-32}$$

$$\|d_U X_{\mathscr{H}^{(4)}_{\mathrm{KG}}}(U)[h]\|_{H^s} \lesssim \|u\|^2_{H^s}\|h\|_{H^s} \quad \textit{for all } h \in H^s(\mathbb{T}^d;\mathbb{C}^2). \tag{4-33}$$

Proof. By an explicit computation and using (1-2) we get

$$X_{\mathscr{H}^{(4)}_{\mathrm{KG}}}(U) = (X^+_{\mathscr{H}^{(4)}_{\mathrm{KG}}}(U), \overline{X^+_{\mathscr{H}^{(4)}_{\mathrm{KG}}}(U)})^T, \quad X^+_{\mathscr{H}^{(4)}_{\mathrm{KG}}}(U) = -\mathrm{i}\frac{\Lambda_{\mathrm{KG}}^{-1/2}}{\sqrt{2}} g(\psi).$$

The function g is a homogeneous polynomial of degree 3. Hence, by using Lemma 3.3, we obtain

$$\mathrm{i}X^+_{\mathscr{H}^{(4)}_{\mathrm{KG}}}(U) = A_0 + A_{-1/2} + A_{-1} + Q^{-\rho}(u), \tag{4-34}$$

where

$$A_0 := \tfrac12 \mathrm{Op}^{\mathrm{BW}}(\partial_{y_1y_1}G(\psi, \Lambda_{\mathrm{KG}}^{1/2}\psi))[u+\bar u], \tag{4-35}$$

$$A_{-1/2} := \tfrac12 \mathrm{Op}^{\mathrm{BW}}(\partial_{y_1y_0}G(\psi, \Lambda_{\mathrm{KG}}^{1/2}\psi))[\Lambda_{\mathrm{KG}}^{-1/2}(u+\bar u)] + \tfrac12 \Lambda_{\mathrm{KG}}^{-1/2}\mathrm{Op}^{\mathrm{BW}}(\partial_{y_1y_0}G(\psi, \Lambda_{\mathrm{KG}}^{1/2}\psi))[u+\bar u], \tag{4-36}$$

$$A_{-1} := \tfrac12 \Lambda_{\mathrm{KG}}^{-1/2}\mathrm{Op}^{\mathrm{BW}}(\partial_{y_0y_0}G(\psi, \Lambda_{\mathrm{KG}}^{1/2}\psi))[\Lambda_{\mathrm{KG}}^{-1/2}(u+\bar u)], \tag{4-37}$$

and $Q^{-\rho}$ is a cubic smoothing remainder of the form (3-48) whose coefficients satisfy the bound (4-30). The symbols of the paradifferential operators have the form (using that G is a polynomial)

$$(\partial_{kj}G)\left(\frac{\Lambda_{\mathrm{KG}}^{-1/2}(u+\bar u)}{\sqrt2}, \frac{u+\bar u}{\sqrt2}\right) = (2\pi)^{-d}\sum_{\sigma_1,\sigma_2\in\{\pm\}}\sum_{\xi\in\mathbb{Z}^d} e^{\mathrm{i}\xi\cdot x}\sum_{\eta\in\mathbb{Z}^d} \mathrm{g}^{\sigma_1,\sigma_2}_{k,j}(\xi,\eta)\widehat{u^{\sigma_1}}(\xi-\eta)\widehat{u^{\sigma_2}}(\eta), \tag{4-38}$$

where $k, j \in \{y_0, y_1\}$ and where the coefficients $\mathrm{g}^{\sigma_1,\sigma_2}_{k,j}(\xi,\eta) \in \mathbb{C}$ satisfy $|\mathrm{g}^{\sigma_1,\sigma_2}_{k,j}(\xi,\eta)| \lesssim 1$.

We claim that the term in (4-37) is a cubic remainder of the form (4-29) with coefficients satisfying (4-30). By (3-6) we have

$$\begin{aligned}\widehat{A}_{-1}(\xi) &= \frac{1}{2(2\pi)^d}\sum_{\zeta\in\mathbb{Z}^d,\sigma\in\{\pm\}} \widehat{\partial_{y_0y_0}G}(\xi-\zeta)\Lambda_{\mathrm{KG}}^{-1/2}(\xi)\Lambda_{\mathrm{KG}}^{-1/2}(\zeta)\chi_\epsilon\left(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\right)\widehat{u^\sigma}(\xi)\\ &= \frac{1}{2(2\pi)^d}\sum_{\substack{\sigma_1,\sigma_2,\sigma\in\{\pm\}\\ \eta,\zeta\in\mathbb{Z}^d}} \mathrm{g}^{\sigma_1,\sigma_2}_{y_0,y_0}(\xi-\zeta,\eta)\Lambda_{\mathrm{KG}}^{-1/2}(\xi)\Lambda_{\mathrm{KG}}^{-1/2}(\zeta)\\ &\qquad\qquad \times\chi_\epsilon\left(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\right)\widehat{u^{\sigma_1}}(\xi-\eta-\zeta)\widehat{u^{\sigma_2}}(\eta)\widehat{u^\sigma}(\zeta) \quad (\text{by (4-38)}),\end{aligned}$$

which implies that A_{-1} has the form (4-29) with coefficients

$$\mathtt{a}^{\sigma_1,\sigma_2,\sigma_3}_{-1}(\xi,\eta,\zeta) = \tfrac12 \mathrm{g}^{\sigma_1,\sigma_2}_{y_0,y_0}(\xi-\zeta,\eta)\Lambda_{\mathrm{KG}}^{-1/2}(\xi)\Lambda_{\mathrm{KG}}^{-1/2}(\zeta)\chi_\epsilon\left(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\right). \tag{4-39}$$

By Lemma 3.8 we have that the coefficients in (4-39) satisfy (4-30). This proves the claim for the operator A_{-1}. We now study the term in (4-36). We remark that, by Proposition 3.2 (see the composition formula (3-19)), we have $A_{-1/2} = \mathrm{Op}^{\mathrm{BW}}(\Lambda_{\mathrm{KG}}^{-1/2}(\xi)\partial_{y_0y_1}G)$ up to a smoothing operator of order $-\frac32$.

Actually to prove that such a remainder has the form (4-29) with coefficients (4-30) it is more convenient to compute the composition operator explicitly. In particular, recalling (3-6), we get

$$A_{-1/2} = \mathrm{Op}^{\mathrm{BW}}(\Lambda_{\mathrm{KG}}^{-1/2}(\xi)\partial_{y_0 y_1} G) + R_{-1}, \tag{4-40}$$

where

$$\widehat{R}_{-1}(\xi) = (2\pi)^{-d} \sum_{\substack{\sigma_1,\sigma_2,\sigma\in\{\pm\}\\ \eta,\zeta\in\mathbb{Z}^d}} \mathtt{r}^{\sigma_1,\sigma_2,\sigma}(\xi-\eta-\zeta,\eta,\zeta)\widehat{u^{\sigma_1}}(\xi-\eta-\zeta)\widehat{u^{\sigma_2}}(\eta)\widehat{u^{\sigma}}(\zeta),$$

$$\mathtt{r}^{\sigma_1,\sigma_2,\sigma}(\xi-\eta-\zeta,\eta,\zeta) = \frac{1}{2}\mathtt{g}_{y_0,y_1}^{\sigma_1,\sigma_2}(\xi-\zeta,\eta)\chi_\epsilon\Big(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)\Big[\Lambda_{\mathrm{KG}}^{-1/2}(\xi)+\Lambda_{\mathrm{KG}}^{-1/2}(\zeta)-2\Lambda_{\mathrm{KG}}^{-1/2}\Big(\frac{\xi+\zeta}{2}\Big)\Big].$$

We note that

$$\Lambda_{\mathrm{KG}}^{-1/2}(\xi) = \Lambda_{\mathrm{KG}}^{-1/2}\Big(\frac{\xi+\zeta}{2}\Big) - \frac{1}{2}\int_0^1 \Lambda_{\mathrm{KG}}^{-3/2}\Big(\frac{\xi+\zeta}{2}+\tau\frac{\xi-\zeta}{2}\Big)\,d\tau.$$

Then we deduce

$$\Big|\Lambda_{\mathrm{KG}}^{-1/2}(\xi)+\Lambda_{\mathrm{KG}}^{-1/2}(\zeta)-2\Lambda_{\mathrm{KG}}^{-1/2}\Big(\frac{\xi+\zeta}{2}\Big)\Big| \lesssim |\xi|^{-3/2}+|\zeta|^{-3/2}.$$

Again by Lemma 3.8 one can conclude that $\mathtt{r}^{\sigma_1,\sigma_2,\sigma}(\xi-\eta-\zeta,\eta,\zeta)$ satisfies (4-30). By (4-40), (4-35), (4-37) and recalling the definition of $a_0(x,\xi)$ in (4-24), we obtain (4-28). The bound (4-32) for Q_3 follows by (4-30) and Lemma 3.7. Moreover the bound (4-31) follows by Lemma 3.6 recalling that $G(\psi,\Lambda_{\mathrm{KG}}^{1/2}\psi)\sim O(u^4)$. Then the bound (4-32) for $X_{\mathscr{H}_{\mathrm{KG}}^{(4)}}$ follows by Lemma 3.1. Let us prove (4-33). By differentiating (4-28) we get

$$d_U X_{\mathscr{H}_{\mathrm{KG}}^{(4)}}(U)[h] = -\mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}(\mathscr{A}_0(x,\xi))h - \mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}(d_U\mathscr{A}_0(x,\xi)h)U + d_U Q_3(u)[h]. \tag{4-41}$$

The first summand in (4-41) satisfies (4-33) by Lemma 3.1 and (4-31). Moreover using (4-38) and (4-24) one can check that

$$|d_U\mathscr{A}_0(x,\xi)h|_{\mathcal{N}_p^0} \lesssim \|u\|_{H^{p+s_0}}\|h\|_{H^{p+s_0}}, \qquad p+s_0\le s.$$

Then the second summand in (4-41) verifies the bound (4-33) again by Lemma 3.1. The estimate on the third summand in (4-41) follows by (4-29), (4-30) and Lemma 3.7. □

Remark 4.5. We remark that the symbol $a_0(x,\xi)$ in (4-24) is homogeneous of degree 2 in the variables $u,\bar{u}$. In particular, by (4-38), we have

$$\begin{aligned} a_0(x,\xi) &= (2\pi)^{-d/2}\sum_{p\in\mathbb{Z}^d} e^{\mathrm{i}p\cdot x}\hat{a}_0(p,\xi),\\ \hat{a}_0(p,\xi) &= (2\pi)^{-d}\sum_{\substack{\sigma_1,\sigma_2\in\{\pm\}\\ \eta\in\mathbb{Z}^d}} a_0^{\sigma_1,\sigma_2}(p,\eta,\xi)\widehat{u^{\sigma_1}}(p-\eta)\widehat{u^{\sigma_2}}(\eta),\\ a_0^{\sigma_1,\sigma_2}(p,\eta,\xi) &:= \tfrac{1}{2}\mathtt{g}_{y_1,y_1}^{\sigma_1,\sigma_2}(p,\eta)+\mathtt{g}_{y_0,y_1}^{\sigma_1,\sigma_2}(p,\eta)\Lambda_{\mathrm{KG}}^{-1/2}(\xi). \end{aligned} \tag{4-42}$$

Moreover one has $|a_0^{\sigma_1,\sigma_2}(p,\eta,\xi)|\lesssim 1$. Since the symbol $a_0(x,\xi)$ is real-valued, one can check that

$$a_0^{\sigma_1,\sigma_2}(p,\eta,\xi) = \overline{a_0^{-\sigma_1,-\sigma_2}(-p,-\eta,\xi)} \quad \text{for all } \xi,p,\eta\in\mathbb{Z}^d,\ \sigma_1,\sigma_2\in\{\pm\}. \tag{4-43}$$

Remark 4.6. Consider the special case when the function G in (1-2) is independent of y_1. Following the proof of Lemma 4.4 one can obtain the formula (4-28) with symbol $a_0(x,\xi)$ of order -1 given by (see (4-37))

$$a_0(x,\xi) := \tfrac{1}{2}\partial_{y_0 y_0} G(\psi)\Lambda_{\mathrm{KG}}^{-1}(\xi).$$

The remainder Q_3 would satisfy (4-30) with better denominator $\max\{\langle \xi-\eta-\zeta\rangle, \langle\eta\rangle, \langle\zeta\rangle\}^2$.

The main result of this section is the following.

Proposition 4.7 (paralinearization of KG). *The system (1-13) is equivalent to*

$$\dot{U} = -\mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}\big((\mathbb{1}+\mathscr{A}_1(x,\xi))\Lambda_{\mathrm{KG}}(\xi)\big)U + X_{\mathscr{H}^{(4)}_{\mathrm{KG}}}(U) + R(u), \tag{4-44}$$

where

$$U := \begin{bmatrix} u \\ \bar{u} \end{bmatrix} := \mathscr{C}\begin{bmatrix} \psi \\ \phi \end{bmatrix}$$

(see (3-77)), $\mathscr{A}_1(x,\xi)$ is in (4-25), and $X_{\mathscr{H}^{(4)}_{\mathrm{KG}}}(U)$ is the Hamiltonian vector field of (4-27). The operator $R(u)$ has the form $(R^+(u), \overline{R^+(u)})^T$. Moreover we have

$$|\mathscr{A}_1|_{\mathscr{N}_p^0} + |a_2|_{\mathscr{N}_p^2} + \lesssim \|u\|^3_{H^{p+s_0+1}} \quad \textit{for all } p+s_0+1 \le s,\ \ p\in\mathbb{N}, \tag{4-45}$$

where we have chosen $s_0 > d$. Finally there is $\mu > 0$ such that, for any $s > 2d+\mu$, the remainder $R(u)$ satisfies

$$\|R(u)\|_{H^s} \lesssim \|u\|^4_{H^s}. \tag{4-46}$$

Proof. First of all we note that system (1-13) in the complex coordinates (3-77) reads

$$\partial_t u = -\mathrm{i}\Lambda_{\mathrm{KG}} u - \mathrm{i}\frac{\Lambda_{\mathrm{KG}}^{-1/2}}{\sqrt{2}}(f(\psi)+g(\psi)), \qquad \psi = \frac{\Lambda_{\mathrm{KG}}^{-1/2}(u+\bar{u})}{\sqrt{2}}, \tag{4-47}$$

with $f(\psi)$, $g(\psi)$ in (1-1), (1-2). The term $(-\mathrm{i}/\sqrt{2})\Lambda_{\mathrm{KG}}^{-1/2}g(\psi)$ is the first component of the vector field $X_{\mathscr{H}^{(4)}_{\mathrm{KG}}}(U)$, which was studied in Lemma 4.4. By using the Bony paralinearization formula (see [Bony 1981; Métivier 2008; Taylor 2000]), passing to the Weyl quantization and (1-1) we get

$$f(\psi) = -\sum_{j,k=1}^{d} \partial_{x_j}\circ \mathrm{Op}^{\mathrm{BW}}((\partial_{\psi_{x_j}\psi_{x_k}}F)(\psi,\nabla\psi))\circ\partial_{x_k}\psi \tag{4-48}$$

$$+\sum_{j=1}^{d}[\mathrm{Op}^{\mathrm{BW}}((\partial_{\psi\psi_{x_j}}F)(\psi,\nabla\psi)), \partial_{x_j}]\psi + \mathrm{Op}^{\mathrm{BW}}((\partial_{\psi\psi}F)(\psi,\nabla\psi))\psi + R^{-\rho}(\psi), \tag{4-49}$$

where $R^{-\rho}(\psi)$ satisfies $\|R^{-\rho}(\psi)\|_{H^{s+\rho}} \lesssim \|\psi\|^4_{H^s}$ for any $s \ge s_0 > d+\rho$. By Lemma 3.6, and recalling that $F(\psi,\nabla\psi)\sim O(\psi^5)$, we have

$$|\partial_{\psi_{x_k}\psi_{x_j}}F|_{\mathscr{N}_p^0} + |\partial_{\psi\psi_{x_j}}F|_{\mathscr{N}_p^0} + |\partial_{\psi\psi}F|_{\mathscr{N}_p^0} \lesssim \|\psi\|^3_{H^{p+s_0+1}}, \qquad p+s_0+1\le s, \tag{4-50}$$

where $s_0 > d$. Recall that $\partial_{x_j} = \mathrm{Op}^{\mathrm{BW}}(\xi_j)$. Then, by Proposition 3.2, we have

$$[\mathrm{Op}^{\mathrm{BW}}(\partial_{\psi\psi_{x_j}}F), \partial_{x_j}]\psi = \mathrm{Op}^{\mathrm{BW}}(-\mathrm{i}\{\partial_{\psi\psi_{x_j}}F, \xi_j\})\psi + Q(\psi),$$

with (see (3-20)) $\|Q(\psi)\|_{H^{s+1}} \lesssim |\partial_{\psi\psi_{x_j}} F|_{\mathcal{N}^0_{s_0+2}} \|\psi\|_{H^s}$. Then by (3-8), (4-50) and (3-10) (see Lemma 3.1 and Proposition 3.2) we deduce that the terms in (4-49) can be absorbed in a remainder satisfying (4-46) with $s \gg 2d$ large enough. We now consider the right-hand side of (4-48). We have

$$-\partial_{x_j} \circ \mathrm{Op}^{\mathrm{BW}}((\partial_{\psi_{x_j}\psi_{x_k}} F)(\psi, \nabla\psi)) \circ \partial_{x_k} = \mathrm{Op}^{\mathrm{BW}}(\xi_j)\, \mathrm{Op}^{\mathrm{BW}}((\partial_{\psi_{x_j}\psi_{x_k}} F)(\psi, \nabla\psi))\, \mathrm{Op}^{\mathrm{BW}}(\xi_k).$$

By using again Lemma 3.1 and Proposition 3.2 we get

$$f(\psi) = \mathrm{Op}^{\mathrm{BW}}(a_2(x,\xi))\psi + \widetilde{R}(\psi), \tag{4-51}$$

where a_2 is in (4-24) and $\widetilde{R}(\psi)$ is a remainder satisfying (4-46). The symbol $a_2(x,\xi)$ satisfies (4-45) by (4-50). Moreover

$$\frac{1}{\sqrt{2}}\Lambda_{\mathrm{KG}}^{-1/2} f(\psi) = \frac{1}{\sqrt{2}}\Lambda_{\mathrm{KG}}^{-1/2} f\left(\frac{\Lambda_{\mathrm{KG}}^{-1/2}(u+\bar{u})}{\sqrt{2}}\right) \overset{(4\text{-}51)}{=} \frac{1}{2}\mathrm{Op}^{\mathrm{BW}}(a_2(x,\xi)\Lambda_{\mathrm{KG}}^{-1}(\xi))[u+\bar{u}] \tag{4-52}$$

up to remainders satisfying (4-46). Here we used Proposition 3.2 to study the composition operator $\Lambda_{\mathrm{KG}}^{-1/2}\mathrm{Op}^{\mathrm{BW}}(a_2(x,\xi))\Lambda_{\mathrm{KG}}^{-1/2}$. By the discussion above and formula (4-47) we deduce (4-44). □

Remark 4.8. In the semilinear case, i.e., when $f = 0$ and g does not depend on y_1 (see (1-1), (1-2)), equation (4-44) reads

$$\dot{U} = -\mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}(\mathbb{1}\Lambda_{\mathrm{KG}}(\xi))U + X_{\mathscr{H}^{(4)}_{\mathrm{KG}}}(U),$$

where the vector field $X_{\mathscr{H}^{(4)}_{\mathrm{KG}}}$ has the particular structure described in Remark 4.6.

5. Approximately symplectic maps

5A. ***Paradifferential Hamiltonian vector fields.*** In this section we shall construct some approximately symplectic changes of coordinates which will be important for the diagonalization procedure of Section 6.

Define the frequency localization

$$S_\xi w := \sum_{k\in\mathbb{Z}^d} \hat{w}(k)\chi_\epsilon\left(\frac{|k|}{\langle\xi\rangle}\right)e^{\mathrm{i}k\cdot x}, \quad \xi \in \mathbb{Z}^d, \tag{5-1}$$

for some $0 < \epsilon < 1$, where χ_ϵ is defined in (3-5). Consider the matrix of symbols

$$B_{\mathrm{NLS}}(W; x, \xi) := B_{\mathrm{NLS}}(x,\xi) := \begin{pmatrix} 0 & b_{\mathrm{NLS}}(x,\xi) \\ \overline{b_{\mathrm{NLS}}(x,-\xi)} & 0 \end{pmatrix}, \quad b_{\mathrm{NLS}}(x,\xi) = \tilde{\chi}(\xi)w^2\frac{1}{2|\xi|^2}, \tag{5-2}$$

where $\tilde{\chi}(\xi)$ is a $C^\infty(\mathbb{R};\mathbb{R}^+)$ function equal to 0 if $|\xi| \le \frac{1}{4}$ and 1 if $|\xi| \ge \frac{1}{2}$. Define also the Hamiltonian function

$$\mathscr{B}_{\mathrm{NLS}}(W) := \frac{1}{2}\int_{\mathbb{T}^d} \mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}(B_{\mathrm{NLS}}(S_\xi W; x, \xi))W \cdot \overline{W}\, dx, \tag{5-3}$$

where $S_\xi W := (S_\xi w, S_\xi \bar{w})^T$. The presence of truncation on the high modes (S_ξ) will be decisive in obtaining Lemma 5.1 (see comments in the proof of this lemma).

Analogously we define the following. Consider the matrix of symbols

$$B_{\mathrm{KG}}(W;x,\xi):=B_{\mathrm{KG}}(x,\xi):=\begin{pmatrix}0 & b_{\mathrm{KG}}(x,\xi)\\ \overline{b_{\mathrm{KG}}(x,-\xi)} & 0\end{pmatrix},\quad b_{\mathrm{KG}}(W;x,\xi)=\frac{a_0(x,\xi)}{2\Lambda_{\mathrm{KG}}(\xi)},\tag{5-4}$$

with $a_0(x,\xi)$ in (4-24) and Λ_{KG} in (1-4), and define the Hamiltonian function

$$\mathscr{B}_{\mathrm{KG}}(W):=\frac{1}{2}\int_{\mathbb{T}^d}\mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}(B_{\mathrm{KG}}(S_\xi W;x,\xi))W\cdot\overline{W}\,dx,\tag{5-5}$$

where $S_\xi W:=(S_\xi w,S_\xi\bar{w})^T$, where S_ξ is in (5-1).

In this section we study some properties of the maps generated by the Hamiltonians $\mathscr{B}_{\mathrm{NLS}}(W)$ in (5-3) and $\mathscr{B}_{\mathrm{KG}}(W)$ in (5-5). In the next lemma we show that their Hamiltonian vector fields are given by $\mathrm{Op}^{\mathrm{BW}}(B_{\mathrm{NLS}}(W;x,\xi))W$ and $\mathrm{Op}^{\mathrm{BW}}(B_{\mathrm{KG}}(W;x,\xi))W$ respectively, modulo smoothing remainders. More precisely we have the following.

Lemma 5.1. *Consider the Hamiltonian function $\mathscr{B}(W)$ equal to $\mathscr{B}_{\mathrm{NLS}}$ in (5-3) or $\mathscr{B}_{\mathrm{KG}}$ in (5-5). One has that the Hamiltonian vector field of $\mathscr{B}(W)$ has the form*

$$X_{\mathscr{B}}(W)=-\mathrm{i}J\nabla\mathscr{B}(W)=\mathrm{Op}^{\mathrm{BW}}(B(W;x,\xi))W+Q_{\mathscr{B}}(W),\tag{5-6}$$

where $Q_{\mathscr{B}}(W)$ is a smoothing remainder of the form $(Q^+_{\mathscr{B}}(W),\overline{Q^+_{\mathscr{B}}(W)})^T$ and the symbol $B(W;x,\xi)$ is respectively equal to $B_{\mathrm{NLS}}(W;x,\xi)$ in (5-2) or $B_{\mathrm{KG}}(W;x,\xi)$ in (5-4). In particular the cubic remainder $Q_{\mathscr{B}}(W)$ has the form

$$(\widehat{Q^+_{\mathscr{B}}(W)})(\xi)=\frac{1}{(2\pi)^d}\sum_{\substack{\sigma_1,\sigma_2,\sigma_3\in\{\pm\}\\ \eta,\zeta\in\mathbb{Z}^d}}\mathtt{q}^{\sigma_1,\sigma_2,\sigma_3}_{\mathscr{B}}(\xi,\eta,\zeta)\widehat{w^{\sigma_1}}(\xi-\eta-\zeta)\widehat{w^{\sigma_2}}(\eta)\widehat{w^{\sigma_3}}(\zeta),\quad \xi\in\mathbb{Z}^d,\tag{5-7}$$

where $\mathtt{q}^{\sigma_1,\sigma_2,\sigma_3}_{\mathscr{B}}(\xi,\eta,\zeta)\in\mathbb{C}$ satisfy, for any $\xi,\eta,\zeta\in\mathbb{Z}^d$, a bound like (3-48). In the case that $\mathscr{B}=\mathscr{B}_{\mathrm{NLS}}$ we have $\sigma_1=+$, $\sigma_2=-$, $\sigma_3=+$. Moreover, for $s>\frac{1}{2}d+\rho$, we have

$$\|d_W^kQ_{\mathscr{B}}(W)[h_1,\dots,h_k]\|_{H^{s+\rho}}\lesssim\|w\|_{H^s}^{3-k}\prod_{i=1}^k\|h_i\|_{H^s}\quad \textit{for all } h_i\in H^s(\mathbb{T}^d;\mathbb{C}^2),\ i=1,2,3,\tag{5-8}$$

for $k=0,1,2,3$. Moreover, for any $s>2d+2$, one has

$$\|d_W^kX_{\mathscr{B}_{\mathrm{NLS}}}(W)[h_1,\dots,h_k]\|_{H^{s+2}}\lesssim\|w\|_{H^s}^{3-k}\prod_{i=1}^k\|h_i\|_{H^s}\quad \textit{for all } h_i\in H^s(\mathbb{T}^d;\mathbb{C}^2),\ i=1,2,3,\tag{5-9}$$

$$\|d_W^kX_{\mathscr{B}_{\mathrm{KG}}}(W)[h_1,\dots,h_k]\|_{H^{s+1}}\lesssim\|w\|_{H^s}^{3-k}\prod_{i=1}^k\|h_i\|_{H^s}\quad \textit{for all } h_i\in H^s(\mathbb{T}^d;\mathbb{C}^2),\ i=1,2,3,\tag{5-10}$$

with $k=0,1,2,3$.

Proof. We prove the statement in the case $\mathscr{B}=\mathscr{B}_{\mathrm{NLS}}$; the other case is similar. Using the formulas (5-2), (5-3) we obtain $\mathscr{B}_{\mathrm{NLS}}(W)=-G_1(W)-G_2(W)$ with

$$G_1(W):=-\frac{\mathrm{i}}{2}\int_{\mathbb{T}^d}\mathrm{Op}^{\mathrm{BW}}(b_{\mathrm{NLS}}(S_\xi w))\bar{w}\bar{w}\,dx,\quad G_2(W):=\frac{\mathrm{i}}{2}\int_{\mathbb{T}^d}\mathrm{Op}^{\mathrm{BW}}(\overline{b_{\mathrm{NLS}}(S_\xi w)})ww\,dx,$$

where we recall (5-1). By (5-2) we obtain that $\nabla_{\bar{w}} G_1(W) = -\mathrm{i}\,\mathrm{Op}^{\mathrm{BW}}(b_{\mathrm{NLS}}(S_\xi w))\bar{w}$. We compute the gradient with respect $\bar{w}$ of the term $G_2(W)$. We have

$$\begin{aligned}
d_{\bar{w}} G_2(W)(\bar{h}) &= \frac{\mathrm{i}}{2}\int_{\mathbb{T}^d} \mathrm{Op}^{\mathrm{BW}}\Big(S_\xi(\bar{w}) S_\xi(\bar{h}) \frac{1}{|\xi|^2}\tilde{\chi}(\xi)\Big) ww\, dx \\
&= \frac{\mathrm{i}}{2}\frac{1}{(2\pi)^d} \sum_{\xi,\eta,\zeta\in\mathbb{Z}^d} \widehat{S_{(\xi+\zeta)/2}(\bar{w})}(\xi-\eta-\zeta)\widehat{S_{(\xi+\zeta)/2}(\bar{h})}(\eta)\hat{w}(\zeta) \\
&\qquad\qquad \times \frac{4}{|\zeta+\xi|^2}\tilde{\chi}\Big(\frac{\zeta+\xi}{2}\Big)\chi_\epsilon\Big(\frac{|\xi-\zeta|}{\langle \xi+\zeta\rangle}\Big)\hat{w}(-\xi) \quad (\text{by (3-6)}) \\
&= 2\mathrm{i}\frac{1}{(2\pi)^d} \sum_{\xi,\eta,\zeta\in\mathbb{Z}^d} \tilde{\chi}\Big(\frac{\zeta+\xi}{2}\Big)\frac{1}{|\zeta+\xi|^2}\chi_\epsilon\Big(\frac{|\xi-\zeta|}{\langle \xi+\zeta\rangle}\Big)\chi_\epsilon\Big(\frac{2|\xi-\eta-\zeta|}{\langle \xi+\zeta\rangle}\Big)\chi_\epsilon \\
&\qquad\qquad \times \Big(\frac{2|\eta|}{\langle \xi+\zeta\rangle}\Big)\hat{\bar{w}}(\xi-\eta-\zeta)\hat{\bar{h}}(\eta)\hat{w}(\zeta)\hat{w}(-\xi) \quad (\text{by (5-1)}) \\
&= 2\mathrm{i}\frac{1}{(2\pi)^d} \sum_{\eta\in\mathbb{Z}^d}\hat{\bar{h}}(-\eta) \sum_{\xi,\zeta\in\mathbb{Z}^d} \tilde{\chi}\Big(\frac{\zeta+\xi}{2}\Big)\frac{1}{|\zeta+\xi|^2}\chi_\epsilon\Big(\frac{|\xi-\zeta|}{\langle \xi+\zeta\rangle}\Big) \\
&\qquad\qquad \times \chi_\epsilon\Big(\frac{2|\xi+\eta-\zeta|}{\langle \xi+\zeta\rangle}\Big)\chi_\epsilon\Big(\frac{2|\eta|}{\langle \xi+\zeta\rangle}\Big)\hat{\bar{w}}(\xi+\eta-\zeta)\hat{w}(\zeta)\hat{w}(-\xi).
\end{aligned}$$

Recalling (3-69) and the computations above, after some changes of variables in the summations, we obtain

$$X_{\mathscr{B}_{\mathrm{NLS}}}(W) = \mathrm{Op}^{\mathrm{BW}}(B_{\mathrm{NLS}}(S_\xi W; x, \xi))W + R_1(W),$$

where the remainder $R_1(W)$ has the form $(R_1^+(W), \overline{R_1^+(W)})^T$, where (recall (3-5))

$$\begin{aligned}
\widehat{(R_1^+(W))}(\xi) &= \frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d} r_1(\xi,\eta,\zeta)\hat{w}(\xi-\eta-\zeta)\hat{\bar{w}}(\eta)\hat{w}(\zeta), \quad \xi\in\mathbb{Z}^d, \\
r_1(\xi,\eta,\zeta) &= -\frac{2}{|2\zeta-\xi+\eta|^2}\tilde{\chi}\Big(\frac{2\zeta-\xi+\eta}{2}\Big)\chi_\epsilon\Big(\frac{|\eta-\xi|}{\langle 2\zeta-\xi+\eta\rangle}\Big)\chi_\epsilon\Big(\frac{2|\xi|}{\langle \xi-\eta-2\zeta\rangle}\Big)\chi_\epsilon\Big(\frac{2|\eta|}{\langle \xi-\eta-2\zeta\rangle}\Big).
\end{aligned}$$

One can check, for $0<\epsilon<1$ small enough, $|\xi|+|\eta| \ll |\xi-\eta-\zeta| \sim |\zeta|$. Therefore the coefficient $r_1(\xi,\eta,\zeta)$ satisfies (3-48). Here we really need the truncation operator S_ξ: if you don't insert it in the definition of $\mathscr{B}_{\mathrm{NLS}}$ (see (5-3)) then R_1 is not a regularizing operator. Furthermore this truncation does not affect the leading term: Define the operator

$$R_2(W) = \begin{pmatrix} R_2^+(W) \\ \overline{R_2^+(W)} \end{pmatrix} := \mathrm{Op}^{\mathrm{BW}}\big(B_{\mathrm{NLS}}(S_\xi W; x, \xi) - B_{\mathrm{NLS}}(W; x, \xi)\big)W.$$

We are going to prove that R_2 is also a regularizing operator. By an explicit computation using (3-6), (5-1) and (5-2) one can check that

$$\begin{aligned}
\widehat{(R_2^+(W))}(\xi) &= \frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d} r_2(\xi,\eta,\zeta)\hat{w}(\xi-\eta-\zeta)\hat{\bar{w}}(\eta)\hat{w}(\zeta), \quad \xi\in\mathbb{Z}^d, \\
r_2(\xi,\eta,\zeta) &= -\frac{1}{|\xi+\zeta|^2}\tilde{\chi}\Big(\frac{\xi+\zeta}{2}\Big)\chi_\epsilon\Big(\frac{|\xi-\zeta|}{\langle \xi+\zeta\rangle}\Big)\Big(1-\chi_\epsilon\Big(\frac{|\xi-\eta-\zeta|}{\langle \xi+\zeta\rangle}\Big)\chi_\epsilon\Big(\frac{|\eta|}{\langle \xi+\zeta\rangle}\Big)\Big).
\end{aligned}$$

We write $1\cdot r_2(\xi,\eta,\zeta)$ and we use the partition of the unity in (3-49). Hence using (3-5) one can check that each summand satisfies the bound in (3-48). Therefore the operator $Q_G := R_1+R_2$ has the form (5-7) and (5-6) is proved. The estimates (5-8) follow by Lemma 3.7. We note that

$$d_W\big(\mathrm{Op}^{\mathrm{BW}}(B_{\mathrm{NLS}}(W;x,\xi))W\big)[h] = \mathrm{Op}^{\mathrm{BW}}(B_{\mathrm{NLS}}(W;x,\xi))h + \mathrm{Op}^{\mathrm{BW}}(d_W B_{\mathrm{NLS}}(W;x,\xi)[h])W.$$

Then the estimates (5-9) with $k=0,1$ follow by using (5-8), the explicit formula of $B(W;x,\xi)$ in (5-2) and Lemma 3.1. Reasoning similarly one can prove (5-9) with $k=2,3$. □

In the next proposition we define the changes of coordinates generated by the Hamiltonian vector fields $X_{\mathscr{B}_{\mathrm{NLS}}}$ and $X_{\mathscr{B}_{\mathrm{KG}}}$ and we study their properties as maps on Sobolev spaces.

Proposition 5.2. *For any $s\geq s_0>2d+2$ there is $r_0>0$ such that for $0\leq r\leq r_0$ and*

$$W=\begin{bmatrix} w\\ \bar{w}\end{bmatrix}\in B_r(H^s(\mathbb{T}^d;\mathbb{C}^2))$$

the following holds. Define

$$Z := \Phi_{\mathscr{B}_\star}(W) := W + X_{\mathscr{B}_\star}(W), \tag{5-11}$$

where $\star\in\{\mathrm{NLS},\mathrm{KG}\}$ (recall (5-3), (5-5)). Then one has

$$\|Z\|_{H^s}\leq \|w\|_{H^s}(1+C\|w\|^2_{H^s}) \tag{5-12}$$

for some $C>0$ depending on s, and

$$W = Z - X_{\mathscr{B}_\star}(Z) + r(w), \tag{5-13}$$

where

$$\|r(w)\|_{H^s}\lesssim \|w\|^5_{H^s}. \tag{5-14}$$

Proof. By (5-11) we can write

$$W = Z - X_{\mathscr{B}_\star}(W) = Z - X_{\mathscr{B}_\star}(Z) + [X_{\mathscr{B}_\star}(W) - X_{\mathscr{B}_\star}(Z)].$$

By using estimates (5-9) or (5-10) one can deduce that $X_{\mathscr{B}_\star}(W)-X_{\mathscr{B}_\star}(Z)$ satisfies the bound (5-14). The bound (5-12) follows by Lemma 5.1. □

5B. *Conjugations.* Recalling (1-25) and (4-23) we set

$$\mathscr{H}^{(\leq 4)}_{\mathrm{NLS}}(W) := \mathscr{H}^{(2)}_{\mathrm{NLS}}(W) + \mathscr{H}^{(4)}_{\mathrm{NLS}}(W), \quad \mathscr{H}^{(2)}_{\mathrm{NLS}}(Z) := \int_{\mathbb{T}^d}\Lambda_{\mathrm{NLS}}z\cdot\bar{z}\,dx. \tag{5-15}$$

Analogously, recalling (4-27) and (1-4), we set

$$\mathscr{H}^{(\leq 4)}_{\mathrm{KG}}(W) := \mathscr{H}^{(2)}_{\mathrm{KG}}(W) + \mathscr{H}^{(4)}_{\mathrm{KG}}(W), \quad \mathscr{H}^{(2)}_{\mathrm{KG}}(Z) := \int_{\mathbb{T}^d}\Lambda_{\mathrm{KG}}z\cdot\bar{z}\,dx. \tag{5-16}$$

In the following lemma we study how the Hamiltonian vector fields $X_{\mathscr{H}^{(\leq 4)}_{\mathrm{NLS}}}(W)$ in (5-15) and $X_{\mathscr{H}^{(\leq 4)}_{\mathrm{KG}}}(W)$ in (5-16) transform under the change of variables given by the previous lemma.

Lemma 5.3. *Let $s_0 > 2d+4$. Then for any $s \geq s_0$ there is $r_0 > 0$ such that for all $0 < r \leq r_0$ and*

$$Z = \begin{bmatrix} z \\ \bar{z} \end{bmatrix} \in B_r(H^s(\mathbb{T}^d; \mathbb{C}^2))$$

the following holds. Consider the Hamiltonian $\mathscr{B}_\star$ with $\star \in \{\mathrm{NLS}, \mathrm{KG}\}$ (recall (5-3), (5-5)) and the Hamiltonian $\mathscr{H}_\star^{(\leq 4)}$ (see (5-15), (5-16)). Then

$$d_W \Phi_{\mathscr{B}_\star}(W)[X_{\mathscr{H}_\star^{(\leq 4)}}(W)] = X_{\mathscr{H}_\star^{(\leq 4)}}(Z) + [X_{\mathscr{B}_\star}(Z), X_{\mathscr{H}_\star^{(2)}}(Z)] + R_5(Z), \tag{5-17}$$

where the remainder R_5 satisfies

$$\|R_5(Z)\|_{H^s} \lesssim \|z\|_{H^s}^5, \tag{5-18}$$

and $[\,\cdot\,,\cdot\,]$ is the nonlinear commutator defined in (3-73).

Proof. We prove the statement in the case $\mathscr{B}_\star = \mathscr{B}_{\mathrm{NLS}}$ and $\mathscr{H}_\star^{(\leq 4)} = \mathscr{H}_{\mathrm{NLS}}^{(\leq 4)}$; the KG-case is similar. One can check that (5-17) follows by setting

$$\begin{aligned} R_5 := {} & d_W X_{\mathscr{B}_{\mathrm{NLS}}}(W)[X_{\mathscr{H}_{\mathrm{NLS}}^{(\leq 4)}}(W) - X_{\mathscr{H}_{\mathrm{NLS}}^{(\leq 4)}}(Z)] && \text{(5-19)} \\ & + (d_W X_{\mathscr{B}_{\mathrm{NLS}}}(W) - d_W X_{\mathscr{B}_{\mathrm{NLS}}}(Z))[X_{\mathscr{H}_{\mathrm{NLS}}^{(\leq 4)}}(Z)] && \text{(5-20)} \\ & + X_{\mathscr{H}_{\mathrm{NLS}}^{(\leq 4)}}(W) - X_{\mathscr{H}_{\mathrm{NLS}}^{(\leq 4)}}(Z) + d_W X_{\mathscr{H}_{\mathrm{NLS}}^{(\leq 4)}}(Z)[X_{\mathscr{B}_{\mathrm{NLS}}}(Z)] && \text{(5-21)} \\ & + [X_{\mathscr{B}_{\mathrm{NLS}}}(Z), X_{\mathscr{H}_{\mathrm{NLS}}^{(4)}}(Z)]. && \text{(5-22)} \end{aligned}$$

We are left to prove that R_5 satisfies (5-18). We start from the term in (5-19). First of all we note that

$$X_{\mathscr{H}_{\mathrm{NLS}}^{(\leq 4)}}(W) - X_{\mathscr{H}_{\mathrm{NLS}}^{(\leq 4)}}(Z) = -\mathrm{i}E\Lambda_{\mathrm{NLS}}(W-Z) + X_{\mathscr{H}_{\mathrm{NLS}}^{(4)}}(W) - X_{\mathscr{H}_{\mathrm{NLS}}^{(\leq 4)}}(Z),$$

where we used that $X_{\mathscr{H}_{\mathrm{NLS}}^{(2)}}(W) = -\mathrm{i}E\Lambda_{\mathrm{NLS}}W$. By Proposition 5.2, (4-23) and (5-9) we deduce that

$$\|X_{\mathscr{H}_{\mathrm{NLS}}^{(\leq 4)}}(W) - X_{\mathscr{H}_{\mathrm{NLS}}^{(\leq 4)}}(Z)\|_{H^s} \lesssim \|w\|_{H^s}^3.$$

Hence using again the bounds (5-9) we obtain

$$\|d_W X_{\mathscr{B}_{\mathrm{NLS}}}(W)[X_{\mathscr{H}_{\mathrm{NLS}}^{(\leq 4)}}(W) - X_{\mathscr{H}_{\mathrm{NLS}}^{(\leq)}}(Z)]\|_{H^s} \lesssim \|w\|_{H^s}^5.$$

Reasoning in the same way, using also (5-13), one can check that the terms in (5-20), (5-21), (5-22) satisfy the same quintic estimates. □

In the next lemma we study the structure of the cubic terms in the vector field in (5-17) in the NLS case.

Lemma 5.4. *Consider the Hamiltonian $\mathscr{B}_{\mathrm{NLS}}(W)$ in (5-3) and recall (4-23), (5-15). Then we have*

$$X_{\mathscr{H}_{\mathrm{NLS}}^{(4)}}(Z) + [X_{\mathscr{B}_{\mathrm{NLS}}}(Z), X_{\mathscr{H}_{\mathrm{NLS}}^{(2)}}(Z)] = -\mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}\begin{pmatrix} 2|z|^2 & 0 \\ 0 & 2|z|^2 \end{pmatrix} Z + Q_{\mathrm{H}_{\mathrm{NLS}}^{(4)}}(Z), \tag{5-23}$$

where the remainder $Q_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}$ *has the form* $Q_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z) = (Q^{+}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z), \overline{Q^{+}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z)})^{T}$ *and*

$$\widehat{(Q^{+}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z))}(\xi) = \frac{1}{(2\pi)^{d}} \sum_{\eta,\zeta\in\mathbb{Z}^{d}} \mathtt{q}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(\xi,\eta,\zeta)\hat{z}(\xi-\eta-\zeta)\hat{\bar{z}}(\eta)\hat{z}(\zeta), \quad \xi\in\mathbb{Z}^{d}, \tag{5-24}$$

with symbol satisfying

$$|\mathtt{q}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(\xi,\eta,\zeta)| \lesssim \frac{\max_{2}\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^{4}}{\max_{1}\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^{2}}. \tag{5-25}$$

Proof. We start by considering the commutator between $X_{\mathscr{B}_{\mathrm{NLS}}}$ and $X_{\mathscr{H}^{(2)}_{\mathrm{NLS}}}$. First of all notice that (see (5-6) and (5-2))

$$X_{\mathscr{B}_{\mathrm{NLS}}}(Z) = \begin{pmatrix} X^{+}_{\mathscr{B}_{\mathrm{NLS}}}(Z) \\ \overline{X^{+}_{\mathscr{B}_{\mathrm{NLS}}}(Z)} \end{pmatrix}, \quad X^{+}_{\mathscr{B}_{\mathrm{NLS}}}(Z) := \mathrm{Op}^{\mathrm{BW}}\left(\frac{z^{2}}{2|\xi|^{2}}\tilde{\chi}(\xi)\right)[\bar{z}] + Q^{+}_{\mathscr{B}_{\mathrm{NLS}}}(Z),$$

and hence (recall (3-6)), for $\xi\in\mathbb{Z}^{d}$,

$$\begin{aligned}\widehat{(X^{+}_{\mathscr{B}_{\mathrm{NLS}}}(Z))}(\xi) &= \frac{1}{(2\pi)^{d}} \sum_{\eta,\zeta\in\mathbb{Z}^{d}} \hat{z}(\xi-\eta-\zeta)\hat{\bar{z}}(\eta)\hat{z}(\zeta)\left[\frac{2}{|\xi+\eta|^{2}}\tilde{\chi}\left(\frac{\xi+\eta}{2}\right)\chi_{\epsilon}\left(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\right) + q_{\mathscr{B}_{\mathrm{NLS}}}(\xi,\eta,\zeta)\right],\end{aligned} \tag{5-26}$$

where $q_{\mathscr{B}_{\mathrm{NLS}}}(\zeta,\eta,\zeta)$ satisfies the bound in (3-48). Hence, by using formulas (1-25), (5-26), (3-73), one obtains

$$X_{\mathscr{H}^{(4)}_{\mathrm{NLS}}}(Z) + [X_{\mathscr{B}_{\mathrm{NLS}}}(Z), X_{\mathscr{H}^{(2)}_{\mathrm{NLS}}}(Z)] = \begin{pmatrix} \mathscr{C}^{+}(Z) \\ \overline{\mathscr{C}^{+}(Z)} \end{pmatrix},$$

$$\widehat{(\mathscr{C}^{+}(Z))}(\xi) = \frac{-1}{(2\pi)^{d}} \sum_{\eta,\zeta\in\mathbb{Z}^{d}} \mathrm{i}\mathtt{c}(\xi,\eta,\zeta)\hat{z}(\xi-\eta-\zeta)\hat{\bar{z}}(\eta)\hat{z}(\zeta),$$

where

$$\begin{aligned}\mathtt{c}(\xi,\eta,\zeta) = 1 + \left[\frac{2}{|\xi+\eta|^{2}}\tilde{\chi}\left(\frac{\xi+\eta}{2}\right)\chi_{\epsilon}\left(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\right) + q_{\mathscr{B}_{\mathrm{NLS}}}(\xi,\eta,\zeta)\right] \\ \times [\Lambda_{\mathrm{NLS}}(\xi-\eta-\zeta) - \Lambda_{\mathrm{NLS}}(\eta) + \Lambda_{\mathrm{NLS}}(\zeta) - \Lambda_{\mathrm{NLS}}(\xi)].\end{aligned} \tag{5-27}$$

We need to prove that this can be written as the right-hand side of (5-23). First we note that the term in (5-27),

$$q_{\mathscr{B}_{\mathrm{NLS}}}(\xi,\eta,\zeta)[\Lambda_{\mathrm{NLS}}(\xi-\eta-\zeta) - \Lambda_{\mathrm{NLS}}(\eta) + \Lambda_{\mathrm{NLS}}(\zeta) - \Lambda_{\mathrm{NLS}}(\xi)], \tag{5-28}$$

can be absorbed in $\mathtt{R}_{1}$ since (5-28) satisfies the same bound as in (5-25). Moreover, using (1-25) and (1-5), we have that the coefficients

$$\frac{2}{|\xi+\eta|^{2}}\tilde{\chi}\left(\frac{\xi+\eta}{2}\right)\chi_{\epsilon}\left(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\right)[\widehat{V}(\xi-\eta-\zeta) - \widehat{V}(\eta) + \widehat{V}(\zeta) - \widehat{V}(\xi)]$$

satisfy the bound in (5-25) by using also Lemma 3.8. Therefore the corresponding operator contributes to $\mathtt{R}_{1}$. The same holds for the operator corresponding to the coefficients

$$\frac{2}{|\xi+\eta|^{2}}\tilde{\chi}\left(\frac{\xi+\eta}{2}\right)\chi_{\epsilon}\left(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\right)[|\xi-\eta-\zeta|^{2} + |\zeta|^{2}].$$

We are left with the most relevant terms in (5-27) containing the highest frequencies η and ξ. We have

$$\frac{-2(|\xi|^2+|\eta|^2)}{|\xi+\eta|^2}\chi_\epsilon\Big(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\Big)\tilde{\chi}\Big(\frac{\xi+\eta}{2}\Big) = -\chi_\epsilon\Big(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\Big) - r_1(\xi,\eta,\zeta),$$

where

$$r_1(\xi,\eta,\zeta) = \Big(\tilde{\chi}\Big(\frac{\xi+\eta}{2}\Big)-1\Big)\chi_\epsilon\Big(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\Big) + \frac{|\xi-\eta|^2}{|\xi+\eta|^2}\tilde{\chi}\Big(\frac{\xi+\eta}{2}\Big)\chi_\epsilon\Big(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\Big).$$

Again we note that the coefficients $r_1(\xi,\eta,\zeta)$, using Lemma 3.8 and the definition of $\tilde{\chi}$ below (5-2), satisfy (5-25). Then it remains to study the operator $\mathscr{R}^+(Z)$ with

$$\widehat{(\mathscr{R}^+(Z))}(\xi) := -\frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d} \mathrm{i}\Big(1-\chi_\epsilon\Big(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\Big)\Big)\hat{z}(\xi-\eta-\zeta)\hat{\bar{z}}(\eta)\hat{z}(\zeta).$$

By formula (4-13) and (3-6), $\mathscr{R}^+(Z) = -\mathrm{i}\,\mathrm{Op}^{\mathrm{BW}}(2|z|^2)z + Q_3^+(U)$, where Q_3 satisfies (4-15), (4-16). □

In the next lemma we study the structure of the the cubic terms in the vector field in (5-17) in the KG case.

Lemma 5.5. *Consider the Hamiltonian $\mathscr{B}_{\mathrm{KG}}(W)$ in (5-5) and recall (4-27), (5-16). Then we have*

$$X_{\mathscr{H}^{(4)}_{\mathrm{KG}}}(Z) + [X_{\mathscr{B}_{\mathrm{KG}}}(Z), X_{\mathscr{H}^{(2)}_{\mathrm{KG}}}(Z)] = -\mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}(\mathrm{diag}(a_0(x,\xi)))Z + Q_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z), \tag{5-29}$$

where the symbol $a_0(x,\xi) = a_0(u,x,\xi)$ is as in (4-24) and the remainder $Q_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z)$ has the form $(Q^+_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z), \overline{Q^+_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z)})^T$, with

$$\widehat{Q^+_{\mathrm{H}^{(4)}_{\mathrm{KG}}}}(\xi) = (2\pi)^{-d}\sum_{\substack{\sigma_1,\sigma_2,\sigma_3\in\{\pm\}\\ \eta,\zeta\in\mathbb{Z}^d}} \mathtt{q}^{\sigma_1,\sigma_2,\sigma_3}_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(\xi,\eta,\zeta)\widehat{z^{\sigma_1}}(\xi-\eta-\zeta)\widehat{z^{\sigma_2}}(\eta)\widehat{z^{\sigma_3}}(\zeta) \tag{5-30}$$

for some $\mathtt{q}^{\sigma_1,\sigma_2,\sigma_3}_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(\xi,\eta,\zeta)\in\mathbb{C}$ satisfying

$$|\mathtt{q}^{\sigma_1,\sigma_2,\sigma_3}_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(\xi,\eta,\zeta)| \lesssim \frac{\max_2\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^\mu}{\max\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}} \tag{5-31}$$

for some $\mu>1$.

Proof. Using (5-6) (with $\mathscr{B}=\mathscr{B}_{\mathrm{KG}}$) we can note that

$$[X_{\mathscr{B}_{\mathrm{KG}}}(Z), X_{\mathscr{H}^{(2)}_{\mathrm{KG}}}(Z)] = [\mathrm{Op}^{\mathrm{BW}}(B_{\mathrm{KG}}(Z;x,\xi)), X_{\mathscr{H}^{(2)}_{\mathrm{KG}}}(Z)] + R_2(Z), \tag{5-32}$$

where $R_2(Z) = (R_2^+(Z), \overline{R_2^+(Z)})^T$, with

$$\begin{aligned}\widehat{(R_2^+(Z))}(\xi) &= (2\pi)^{-d}\sum_{\substack{\sigma_1,\sigma_2,\sigma_3\in\{\pm\}\\ \eta,\zeta\in\mathbb{Z}^d}} \mathtt{r}_2^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)\widehat{z^{\sigma_1}}(\xi-\eta-\zeta)\widehat{z^{\sigma_2}}(\eta)\widehat{z^{\sigma_3}}(\zeta), \quad \xi\in\mathbb{Z}^d,\\ \mathtt{r}_2^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta) &:= \mathtt{q}^{\sigma_1,\sigma_2,\sigma_3}_{\mathscr{B}_{\mathrm{KG}}}(\xi,\eta,\zeta)[\sigma_1\Lambda_{\mathrm{KG}}(\xi-\eta-\zeta)+\sigma_2\Lambda_{\mathrm{KG}}(\eta)+\sigma_3\Lambda_{\mathrm{KG}}(\zeta)-\Lambda_{\mathrm{KG}}(\xi)],\end{aligned} \tag{5-33}$$

where the coefficients are defined in (5-7). The remainder R_2 has the form (5-30) and we have that the coefficients $\mathtt{r}_2^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)$ satisfy the bound (5-31). On the other hand, recalling (5-4), (3-73), we have

$$[\mathrm{Op}^{\mathrm{BW}}(B_{\mathrm{KG}}(Z;x,\xi)), X_{\mathscr{H}_{\mathrm{KG}}^{(2)}}(Z)] = R_3(Z)+R_4(Z), \quad R_j(Z) = \begin{pmatrix} R_j^+(Z) \\ \overline{R_j^+(Z)} \end{pmatrix}, \quad j=3,4, \tag{5-34}$$

where

$$R_3^+(Z) := \mathrm{Op}^{\mathrm{BW}}(b_{\mathrm{KG}}(Z;x,\xi))[\mathrm{i}\Lambda_{\mathrm{KG}}\bar{z}] + \mathrm{i}\Lambda_{\mathrm{KG}}\,\mathrm{Op}^{\mathrm{BW}}(b_{\mathrm{KG}}(Z;x,\xi))[\bar{z}], \tag{5-35}$$

$$R_4^+(Z) := \mathrm{Op}^{\mathrm{BW}}((d_Z b_{\mathrm{KG}})(z;x,\xi)[X_{\mathscr{H}_{\mathrm{KG}}^{(2)}}(Z)])[\bar{z}]. \tag{5-36}$$

By Remark 4.5 and (3-6) we get

$$\widehat{R_4^+}(\xi) = (2\pi)^{-d} \sum_{\sigma_1,\sigma_2\in\{\pm\}\eta,\zeta\in\mathbb{Z}^d} a_0^{\sigma_1,\sigma_2}\Big(\xi-\zeta,\eta,\frac{\xi+\zeta}{2}\Big)\frac{1}{2\Lambda_{\mathrm{KG}}((\xi+\zeta)/2)} \\ \times \chi_\epsilon\Big(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)[-\mathrm{i}\sigma_1\Lambda_{\mathrm{KG}}(\xi-\eta-\zeta)-\mathrm{i}\sigma_2\Lambda_{\mathrm{KG}}(\eta)]\widehat{z^{\sigma_1}}(\xi-\eta-\zeta)\widehat{z^{\sigma_2}}(\eta)\hat{\bar{z}}(\zeta).$$

Using the explicit form of the coefficients of R_4^+ and Lemma 3.8 one can conclude that the operator R_4^+ has the form (5-30) with coefficients satisfying (5-31). To summarize, by (5-32) and (5-34), we have obtained (recall also (4-28), (4-26))

$$\text{LHS of (5-29)} = \mathrm{Op}^{\mathrm{BW}}\begin{pmatrix} -\mathrm{i}a_0(x,\xi) & 0 \\ 0 & \mathrm{i}a_0(x,\xi)\end{pmatrix} Z + F_3(Z)+Q_3(Z)+R_2(Z)+R_4(Z), \tag{5-37}$$

where R_4 is in (5-36), R_2 is in (5-33), $Q_3(Z)$ is in (4-28) and

$$F_3(Z) = \begin{pmatrix} F_3^+(Z) \\ \overline{F_3^+(Z)} \end{pmatrix}, \quad F_3^+(Z) = -\mathrm{i}\,\mathrm{Op}^{\mathrm{BW}}(a_0(x,\xi))[\bar{z}] + R_3^+(Z), \tag{5-38}$$

where R_3^+ is in (5-35). By the discussion above and by Lemma 4.4 we have that the remainders R_2, R_4 and Q_3 have the form (5-30) with coefficients satisfying (5-31). To conclude the prove we need to show that F_3 has the same property. This will be a consequence of the choice of the symbol $b_{\mathrm{KG}}(W;x,\xi)$ in (5-4). Indeed, by (5-4), Remark 4.5, (5-38), (5-35), we have

$$\widehat{F_3^+}(\xi) = (2\pi)^{-d} \sum_{\substack{\sigma_1,\sigma_2\in\{\pm\} \\ \eta,\zeta\in\mathbb{Z}^d}} \mathtt{f}_3^{\sigma_1,\sigma_2,-}(\xi,\eta,\zeta)\widehat{z^{\sigma_1}}(\xi-\eta-\zeta)\widehat{z^{\sigma_2}}(\eta)\hat{\bar{z}}(\zeta),$$

where

$$\mathtt{f}_3^{\sigma_1,\sigma_2,-}(\xi,\eta,\zeta) := a_0^{\sigma_1,\sigma_2}\Big(\xi-\zeta,\eta,\frac{\xi+\zeta}{2}\Big)\mathrm{i}\Big[\frac{\Lambda_{\mathrm{KG}}(\xi)+\Lambda_{\mathrm{KG}}(\zeta)}{2\Lambda_{\mathrm{KG}}((\xi+\zeta)/2)}-1\Big]\chi_\epsilon\Big(\frac{|\xi-\zeta|}{\langle\zeta+\zeta\rangle}\Big). \tag{5-39}$$

By Taylor expanding the symbol $\Lambda_{\mathrm{KG}}(\xi)$ in (1-4) (see also Remark 4.5) one deduces that

$$\Big|a_0^{\sigma_1,\sigma_2}\Big(\xi-\zeta,\eta,\frac{\xi+\zeta}{2}\Big)\mathrm{i}\Big[\frac{\Lambda_{\mathrm{KG}}(\xi)+\Lambda_{\mathrm{KG}}(\zeta)}{2\Lambda_{\mathrm{KG}}((\xi+\zeta)/2)}-1\Big]\Big| \lesssim \frac{|\xi-\zeta|}{(\langle\xi\rangle+\langle\zeta\rangle)^{3/2}}.$$

Therefore, using Lemma 3.8, we have that the coefficients $\mathtt{f}_3^{\sigma_1,\sigma_2,-}(\xi,\eta,\zeta)$ in (5-39) satisfy (5-31). This implies (5-29). □

6. Diagonalization

6A. ***Diagonalization of the NLS.*** In this section we diagonalize the system (4-12). We first diagonalize the matrix $E(\mathbb{1}+A_2(x))$ in (4-12) by means of a change of coordinates as the ones made in the papers [Feola and Iandoli 2021; 2022]. After that we diagonalize the matrix of symbols of order 0 at homogeneity 3, by means of an approximately *symplectic* change of coordinates. Throughout the rest of the section we shall assume the following.

Hypothesis 6.1. We restrict the solution of (NLS) on the interval of times $[0, T)$, with T such that

$$\sup_{t\in[0,T)} \|u(t,x)\|_{H^s} \le \epsilon, \qquad \|u_0(x)\|_{H^s} \le \tfrac{1}{4}\epsilon.$$

Note that such a time $T > 0$ exists thanks to the local existence theorem in [Feola and Iandoli 2022].

6A1. *Diagonalization at order 2.* We consider the matrix $E(\mathbb{1}+A_2(x))$ in (4-12). We define

$$\lambda_{\mathrm{NLS}}(x) := \lambda_{\mathrm{NLS}}(U;x) := \sqrt{1+2|u|^2[h'(|u|^2)]^2}, \qquad a_2^{(1)}(x) := \lambda_{\mathrm{NLS}}(x) - 1, \tag{6-1}$$

and we note that $\pm\lambda_{\mathrm{NLS}}(x)$ are the eigenvalues of the matrix $E(\mathbb{1}+A_2(x))$. We denote by S matrix of the eigenvectors of $E(\mathbb{1}+A_2(x))$; more explicitly

$$\begin{aligned} S &= \begin{pmatrix} s_1 & s_2 \\ \bar{s}_2 & s_1 \end{pmatrix}, \qquad S^{-1} = \begin{pmatrix} s_1 & -s_2 \\ -\bar{s}_2 & s_1 \end{pmatrix}, \\ s_1(x) &:= \frac{1+|u|^2[h'(|u|^2)]^2 + \lambda_{\mathrm{NLS}}(x)}{\sqrt{2\lambda_{\mathrm{NLS}}(x)(1+[h'(|u|^2)]^2|u|^2+\lambda_{\mathrm{NLS}}(x))}}, \\ s_2(x) &:= \frac{-u^2[h'(|u|^2)]^2}{\sqrt{2\lambda_{\mathrm{NLS}}(x)(1+[h'(|u|^2)]^2|u|^2+\lambda_{\mathrm{NLS}}(x))}}. \end{aligned} \tag{6-2}$$

Since $\pm\lambda_{\mathrm{NLS}}(x)$ are the eigenvalues and $S(x)$ is the matrix of eigenvectors of $E(\mathbb{1}+A_2(x))$ we have

$$S^{-1}E(\mathbb{1}+A_2(x))S = E\,\mathrm{diag}(\lambda_{\mathrm{NLS}}(x)), \qquad s_1^2 - |s_2|^2 = 1, \tag{6-3}$$

where we have used the notation (4-9). In the lemma we estimate the seminorms of the symbols defined above.

Lemma 6.2. *Let $\mathbb{N} \ni s_0 > d$. The symbols $a_2^{(1)}$ defined in (6-1), $s_1 - 1$ and s_2 defined in (6-2) satisfy the following estimate*

$$|a_2^{(1)}|_{\mathcal{N}_p^0} + |s_1 - 1|_{\mathcal{N}_p^0} + |s_2|_{\mathcal{N}_p^0} \lesssim \|u\|^6_{H^{p+s_0}}, \qquad p + s_0 \le s, \ \ p \in \mathbb{N}.$$

Proof. The proof follows by using the estimate (4-14) on the symbols in (4-10), the fact that $h'(s) \sim s$ when $s \sim 0$, $\|u\|_s \ll 1$, and the explicit expression (6-1), (6-2). □

We now study how the system (4-12) transforms under the maps

$$\Phi_{\mathrm{NLS}} := \Phi_{\mathrm{NLS}}(U) := \mathrm{Op}^{\mathrm{BW}}(S^{-1}(U;x)), \qquad \Psi_{\mathrm{NLS}} := \Psi_{\mathrm{NLS}}(U) := \mathrm{Op}^{\mathrm{BW}}(S(U;x)). \tag{6-4}$$

Lemma 6.3. *Let*

$$U = \begin{bmatrix} u \\ \bar{u} \end{bmatrix}$$

be a solution of (4-12) *and assume* Hypothesis 6.1. *Then for any* $s \geq 2s_0 + 2$, $\mathbb{N} \ni s_0 > d$, *we have the following*:

(i) *One has the upper bound*

$$\begin{aligned} \|\Phi_{\mathrm{NLS}}(U)W\|_{H^s} + \|\Psi_{\mathrm{NLS}}(U)W\|_{H^s} &\leq \|W\|_{H^s}(1 + C\|u\|^6_{H^{2s_0}}), \\ \|(\Phi_{\mathrm{NLS}}(U) - \mathbb{1})W\|_{H^s} + \|(\Psi_{\mathrm{NLS}}(U) - \mathbb{1})W\|_{H^s} &\lesssim \|W\|_{H^s}\|u\|^6_{H^{2s_0}} \quad \textit{for all } W \in H^s(\mathbb{T}^d; \mathbb{C}), \end{aligned} \tag{6-5}$$

where the constant C *depends on* s.

(ii) *One has* $\Psi_{\mathrm{NLS}}(U) \circ \Phi_{\mathrm{NLS}}(U) = \mathbb{1} + R(u)$, *where* R *is a real-to-real remainder of the form* (3-52) *satisfying*

$$\|R(u)W\|_{H^{s+2}} \lesssim \|W\|_{H^s}\|u\|^6_{H^{2s_0+2}}. \tag{6-6}$$

The map $\mathbb{1} + R(u)$ *is invertible with inverse* $(\mathbb{1} + R(u))^{-1} := (1 + \widetilde{R}(u))$, *with* $\widetilde{R}(u)$ *of the form* (3-52) *and*

$$\|\widetilde{R}(u)W\|_{H^{s+2}} \lesssim \|W\|_{H^s}\|u\|^6_{H^{2s_0+2}}; \tag{6-7}$$

as a consequence the map Φ_{NLS} *is invertible and* $\Phi^{-1}_{\mathrm{NLS}} = (1 + \widetilde{R})\Psi_{\mathrm{NLS}}$ *with estimates*

$$\|\Phi^{-1}_{\mathrm{NLS}}(U)W\|_{H^s} \leq \|W\|_{H^s}(1 + C\|u\|^6_{H^{2s_0+2}}), \tag{6-8}$$

where the constant C *depends on* s.

(iii) *For any* $t \in [0, T)$, *one has* $\partial_t \Phi_{\mathrm{NLS}}(U)[\,\cdot\,] = \mathrm{Op}^{\mathrm{BW}}(\partial_t S^{-1}(U; x))$ *and*

$$|\partial_t S^{-1}(U; x)|_{\mathcal{N}^0_{s_0}} \lesssim \|u\|^6_{H^{2s_0+2}}, \qquad \|\partial_t \Phi_{\mathrm{NLS}}(U)V\|_{H^s} \lesssim \|W\|_{H^s}\|u\|^6_{H^{2s_0+2}}. \tag{6-9}$$

Proof. (i) The bounds (6-5) follow by (3-10) and Lemma 6.2.

(ii) We apply Proposition 3.2 to the maps in (6-4); in particular the first part of the item follows by using the expansion (3-21) and recalling that symbols $s_1(x)$ and $s_2(x)$ do not depend on ξ. Inequality (6-7) is obtained by Neumann series by using that (see Hypothesis 6.1) $\|u\|_{H^s} \ll 1$.

(iii) We note that $\partial_t s_1(x, \xi) = (\partial_u s_1)(u; x, \xi)[\dot{u}] + (\partial_{\bar{u}} s_1)(u; x, \xi)[\dot{\bar{u}}]$. Since u solves (4-12) and satisfies Hypothesis 6.1, then using Lemma 3.1 and (4-17) we deduce that $\|\dot{u}\|_{H^s} \lesssim \|u\|_{H^{s+2}}$. Hence the estimates (6-9) follow by direct inspection by using the explicit structure of the symbols s_1, s_2 in (6-2), Lemma 3.6 and (3-10). □

We are now in position to state the following proposition.

Proposition 6.4 (diagonalization at order 2). *Consider the system* (4-12) *and set*

$$W = \Phi_{\mathrm{NLS}}(U)U, \tag{6-10}$$

with $\Phi_{\rm NLS}$ defined in (6-4). Then W solves the equation

$$\dot W = -\mathrm{i}E\,\mathrm{Op}^{\rm BW}(\mathrm{diag}(1+a_2^{(1)}(U;x))|\xi|^2)W - \mathrm{i}E V * W \\ - \mathrm{i}\,\mathrm{Op}^{\rm BW}(\mathrm{diag}(\vec a_1^{(1)}(U;x)\cdot\xi))W + X_{\mathscr{H}^{(4)}_{\rm NLS}}(W) + R^{(1)}(U), \tag{6-11}$$

*where the vector field $X_{\mathscr{H}^{(4)}_{\rm NLS}}$ is defined in (4-13). The symbols $a_2^{(1)}$ and $\vec a_1^{(1)}\cdot\xi$ are **real**-valued and satisfy the estimates*

$$\begin{aligned} |a_2^{(1)}|_{\mathscr{N}_p^0} &\lesssim \|u\|^6_{H^{p+s_0}} && \textit{for all } p+s_0\le s,\ p\in\mathbb{N},\\ |\vec a_1^{(1)}\cdot\xi|_{\mathscr{N}_p^1} &\lesssim \|u\|^6_{H^{p+s_0+1}} && \textit{for all } p+s_0+1\le s,\ p\in\mathbb{N}, \end{aligned} \tag{6-12}$$

where we have chosen $s_0>d$. The remainder $R^{(1)}$ has the form $(R^{(1,+)},\overline{R^{(1,+)}})^T$. Moreover, for any $s>2d+2$, it satisfies the estimate

$$\|R^{(1)}(U)\|_{H^s} \lesssim \|U\|^7_{H^s}. \tag{6-13}$$

Proof. The function W defined in (6-10) satisfies

$$\begin{aligned} \dot W &= [\partial_t\Phi_{\rm NLS}(U)]U + \Phi_{\rm NLS}(U)\dot U \\ &= -\Phi_{\rm NLS}(U)\mathrm{i}E\,\mathrm{Op}^{\rm BW}((\mathbb{1}+A_2(U))|\xi|^2)\Psi_{\rm NLS}(U)W - \Phi_{\rm NLS}(U)\mathrm{i}EV*\Psi_{\rm NLS}(U)W && (6\text{-}14)\\ &\quad -\mathrm{i}\Phi_{\rm NLS}(U)\,\mathrm{Op}^{\rm BW}(\mathrm{diag}(\vec a_1(U)\cdot\xi))\Psi_{\rm NLS}(U)W && (6\text{-}15)\\ &\quad +\Phi_{\rm NLS}(U)X_{\mathscr{H}^{(4)}_{\rm NLS}}(U) && (6\text{-}16)\\ &\quad +\Phi_{\rm NLS}(U)R(U) + \mathrm{Op}^{\rm BW}(\partial_t S^{-1}(U))U && (6\text{-}17)\\ &\quad -\Phi_{\rm NLS}(U)\mathrm{i}[E\,\mathrm{Op}^{\rm BW}((\mathbb{1}+A_2(U))|\xi|^2) + \mathrm{Op}^{\rm BW}(\mathrm{diag}(\vec a_1\cdot\xi)) + EV*]\widetilde R(U)\Psi_{\rm NLS}(U)W, && (6\text{-}18) \end{aligned}$$

where we have used items (ii) and (iii) of Lemma 6.3.

We are going to analyze each term in the right-hand side of the equation above. Because of estimates (6-7), (6-5) (applied to the map $\Phi_{\rm NLS}$), Lemma 6.2 (applied to the symbols a_2, b_2 and $\vec a_1\cdot\xi$) and finally item (ii) of Lemma 3.1, we may absorb term (6-18) in the remainder $R^{(1)}(U)$ verifying (6-13). The term in (6-17) may be absorbed in $R^{(1)}(U)$ as well because of (4-17) and (6-5) for the first term and because of (6-9) and item (ii) of Lemma 3.1 for the second one.

We study the first term in (6-14). We recall (6-4) and (6-2), we apply Proposition 3.2 and we get, by direct inspection, that the new term, modulo contribution that may be absorbed in $R^{(1)}(U)$, is given by

$$-\mathrm{i}E\,\mathrm{Op}^{\rm BW}(\mathrm{diag}(\lambda_{\rm NLS}))W - 2\mathrm{i}\,\mathrm{Op}^{\rm BW}\big(\mathrm{diag}(\mathrm{Im}\{(s_2\bar b_2)\nabla s_1 + (s_1 b_2 + s_2(1+a_2))\nabla\bar s_2\}\cdot\xi)\big)W,$$

where by $\mathrm{Im}\{\vec b\}$, with $\vec b=(b_1,\dots,b_d)$, we denote the vector $(\mathrm{Im}(b_1),\dots,\mathrm{Im}(b_d))$. The second term in (6-14) is equal to $-\mathrm{i}EV*W$ modulo contributions to $R^{(1)}(U)$ thanks to (1-5) and (6-5).

Reasoning analogously one can prove that the term in (6-15) equals $-\mathrm{i}\,\mathrm{Op}^{\rm BW}(\mathrm{diag}(\vec a_1(U)\cdot\xi))W$, modulo contributions to $R^{(1)}(U)$. We are left with studying (6-16). First of all we note that $X_{\mathscr{H}^{(4)}_{\rm NLS}}(U) = -\mathrm{i}E|u|^2U$; then we write

$$X_{\mathscr{H}^{(4)}_{\rm NLS}}(U) = X_{\mathscr{H}^{(4)}_{\rm NLS}}(W) + X_{\mathscr{H}^{(4)}_{\rm NLS}}(U) - X_{\mathscr{H}^{(4)}_{\rm NLS}}(W).$$

Lemma 6.2 and Lemma 3.1(ii) (recall also (6-2)), imply $\|\Phi_{\rm NLS}(U)U-U\|_{H^s}\lesssim\|U\|^7_{H^s}$; therefore it is a contribution to $R^{(1)}(U)$. We have obtained $\Phi_{\rm NLS}(U)X_{\mathscr{H}^{(4)}_{\rm NLS}}(U)=X_{\mathscr{H}^{(4)}_{\rm NLS}}(W)$ modulo $R^{(1)}(U)$.

Summarizing we obtained (6-11) with symbols $a_2^{(1)}$ defined in (6-1) and

$$\vec{a}_1^{(1)}=\vec{a}_1+2\,\mathrm{Im}\{(s_2\bar{b}_2)\nabla s_1+(s_1b_2+s_2(1+a_2))\nabla\bar{s}_2\}\in\mathbb{R}, \tag{6-19}$$

with $\vec{a}_1$ in (4-10). □

6A2. *Diagonalization of cubic terms at order 0.* The aim of this section is to diagonalize the cubic vector field $X_{\mathscr{H}^{(4)}_{\rm NLS}}$ in (6-11) (see also (4-13)) up to smoothing remainder. In order to do this we will consider a change of coordinates which is *symplectic* up to high degree of homogeneity. We reason as follows.

Let

$$Z:=\begin{bmatrix}z\\ \bar{z}\end{bmatrix}:=\Phi_{\mathscr{B}_{\rm NLS}}(W):=W+X_{\mathscr{B}_{\rm NLS}}(W), \tag{6-20}$$

where $X_{\mathscr{B}_{\rm NLS}}$ is the Hamiltonian vector field of (5-3). We note that $\Phi_{\mathscr{B}_{\rm NLS}}$ is not symplectic; nevertheless it is close to the flow of $\mathscr{B}_{\rm NLS}(W)$, which is symplectic. The properties of $X_{\mathscr{B}_{\rm NLS}}$ and the estimates of $\Phi_{\mathscr{B}_{\rm NLS}}$ have been discussed in Lemma 5.1 and in Proposition 5.2.

Remark 6.5. Recall (6-10) and (6-20). One can note that, owing to Hypothesis 6.1, for $s>2d+2$, we have

$$\left(1-\tfrac{1}{100}\right)\|U\|_{H^s}\le\|W\|_{H^s}\le\left(1+\tfrac{1}{100}\right)\|U\|_{H^s},\quad \left(1-\tfrac{1}{100}\right)\|W\|_{H^s}\le\|Z\|_{H^s}\le\left(1+\tfrac{1}{100}\right)\|W\|_{H^s}. \tag{6-21}$$

This is a consequence of the estimates (6-5), (6-8), (5-12), (5-9), (5-14) tanking ϵ small enough depending on s.

We prove the following.

Proposition 6.6 (diagonalization at order 0). *Let $U=(u,\bar{u})$ be a solution of (4-12) and assume Hypothesis 6.1. Define $W:=\Phi_{\rm NLS}(U)U$, where $\Phi_{\rm NLS}(U)$ is the map in (6-4) given in Lemma 6.3. Then the function*

$$Z=\begin{bmatrix}z\\ \bar{z}\end{bmatrix}$$

defined in (6-20) satisfies (recall (1-25))

$$\begin{aligned}\partial_t Z=-\mathrm{i}E\Lambda_{\rm NLS}Z-\mathrm{i}E\,\mathrm{Op}^{\rm BW}(\mathrm{diag}(a_2^{(1)}(x)|\xi|^2))Z\\ -\mathrm{i}\,\mathrm{Op}^{\rm BW}(\mathrm{diag}(\vec{a}_1^{(1)}(x)\cdot\xi))Z+X_{\mathrm{H}^{(4)}_{\rm NLS}}(Z)+R_5^{(2)}(U),\end{aligned} \tag{6-22}$$

where $a_2^{(1)}(x)$, $\vec{a}_1^{(1)}(x)$ are the real-valued symbols appearing in Proposition 6.4, the cubic vector field $X_{\mathrm{H}^{(4)}_{\rm NLS}}(Z)$ has the form (see (5-23))

$$X_{\mathrm{H}^{(4)}_{\rm NLS}}(Z):=-\mathrm{i}E\,\mathrm{Op}^{\rm BW}\begin{pmatrix}2|z|^2&0\\0&2|z|^2\end{pmatrix}Z+Q_{\mathrm{H}^{(4)}_{\rm NLS}}(Z), \tag{6-23}$$

the remainder $Q_{\mathrm{H}^{(4)}_{\rm NLS}}$ is given by Lemma 5.4 and satisfies (5-24)–(5-25). The remainder $R_5^{(2)}(U)$ has the form $(R_5^{(2,+)},\overline{R_5^{(2,+)}})^T$. Moreover, for any $s>2d+4$,

$$\|R_5^{(2)}(U)\|_{H^s}\lesssim\|U\|^5_{H^s}. \tag{6-24}$$

The vector field $X_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z)$ *in* (6-23) *is Hamiltonian*; *i.e.*, (*see* (3-69), (3-72)) $X_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z) := -\mathrm{i}J\nabla\mathrm{H}^{(4)}_{\mathrm{NLS}}(Z)$, *with*

$$\mathrm{H}^{(4)}_{\mathrm{NLS}}(Z) := \mathscr{H}^{(4)}_{\mathrm{NLS}}(Z) - \{\mathscr{B}_{\mathrm{NLS}}(Z), \mathscr{H}^{(2)}_{\mathrm{NLS}}(Z)\}, \quad \mathscr{H}^{(2)}_{\mathrm{NLS}}(Z) = \int_{\mathbb{T}^d} \Lambda_{\mathrm{NLS}} z \cdot \bar{z}\, dx, \tag{6-25}$$

where $\mathscr{H}^{(4)}_{\mathrm{NLS}}$ *is in* (4-23), *and* $\mathscr{B}_{\mathrm{NLS}}$ *is in* (5-3), (5-2).

Proof. Recall (5-15). We have that (6-11) reads

$$\partial_t W = X_{\mathscr{H}^{(\le 4)}_{\mathrm{NLS}}}(W) - \mathrm{i}\,\mathrm{Op}^{\mathrm{BW}}(A(U; x, \xi))W + R^{(1)}(U),$$

where we set

$$A(U; x, \xi) := E\,\mathrm{diag}(a_2^{(1)}(U; x)|\xi|^2) + \mathrm{diag}(\vec{a}_1^{(1)}(U; x)\cdot\xi). \tag{6-26}$$

Hence by (6-20) we get

$$\begin{aligned}\partial_t Z = (d_W\Phi_{\mathscr{B}_{\mathrm{NLS}}}(W))[-\mathrm{i}\,\mathrm{Op}^{\mathrm{BW}}(A(U; x, \xi))W] + (d_W\Phi_{\mathscr{B}_{\mathrm{NLS}}})(W)[X_{\mathscr{H}^{(\le 4)}_{\mathrm{NLS}}}(W)] \\ + (d_W\Phi_{\mathscr{B}_{\mathrm{NLS}}})(W)[R^{(1)}(U)]. \end{aligned}\tag{6-27}$$

We study each summand separately. First of all we have

$$\|d_W\Phi_{\mathscr{B}_{\mathrm{NLS}}}(W)[R^{(1)}(U)]\|_{H^s} \overset{(5\text{-}9),(6\text{-}13)}{\lesssim} \|u\|^7_{H^s}(1 + \|w\|^2_{H^s}) \overset{(6\text{-}21)}{\lesssim} \|u\|^7_{H^s}. \tag{6-28}$$

Let us now analyze the first summand in the right-hand side of (6-27). We write

$$\begin{aligned}(d_W\Phi_{\mathscr{B}_{\mathrm{NLS}}}(W))[\mathrm{i}\,\mathrm{Op}^{\mathrm{BW}}(A(U; x, \xi))W] &= \mathrm{i}\,\mathrm{Op}^{\mathrm{BW}}(A(U; x, \xi))Z + P_1 + P_2, \\ P_1 &:= \mathrm{i}\,\mathrm{Op}^{\mathrm{BW}}(A(U; x, \xi))[W - Z], \\ P_2 &:= ((d_W\Phi_{\mathscr{B}_{\mathrm{NLS}}}(W)) - \mathbb{1})[\mathrm{i}\,\mathrm{Op}^{\mathrm{BW}}(A(U; x, \xi))W].\end{aligned}\tag{6-29}$$

Fix $s_0 > d$, we have, for $s \ge 2s_0 + 4$,

$$\|P_2\|_{H^s} \overset{(5\text{-}9)}{\lesssim} \|w\|^2_{H^s}\|\mathrm{Op}^{\mathrm{BW}}(A(U; x, \xi))W\|_{H^{s-2}} \overset{(6\text{-}12),(3\text{-}10),(6\text{-}21)}{\lesssim} \|u\|^9_{H^s}. \tag{6-30}$$

By (6-20), (5-9) we get $\|W - Z\|_{H^s} \lesssim \|w\|^3_{H^{s-2}}$. Therefore, by (6-29), (6-26), (6-12), (3-10) and (6-21) we get

$$\|P_1\|_{H^s} \lesssim \|u\|^6_{H^{2s_0+1}}\|W - Z\|_{H^{s+2}} \lesssim \|u\|^6_{H^{2s_0+1}}\|w\|^3_{H^s} \lesssim \|u\|^9_{H^s}. \tag{6-31}$$

The estimates (6-28), (6-30), (6-31) imply that the terms P_1, P_2 and $d_W\Phi_{\mathscr{B}_{\mathrm{NLS}}}(W)[R^{(1)}(U)]$ can be absorbed in a remainder satisfying (6-24). Finally we consider the second summand in (6-27). By Lemma 5.3 we deduce

$$d_W\Phi_{\mathscr{B}_{\mathrm{NLS}}}(W)[X_{\mathscr{H}^{(\le 4)}_{\mathrm{NLS}}}(W)] = X_{\mathscr{H}^{(\le 4)}_{\mathrm{NLS}}}(Z) + [X_{\mathscr{B}_{\mathrm{NLS}}}(Z), X_{\mathscr{H}^{(2)}_{\mathrm{NLS}}}(Z)] + R_5(Z),$$

where R_5 is a remainder satisfying the quintic estimate (5-18). By Lemma 5.4 we also have

$$X_{\mathscr{H}^{(\le 4)}_{\mathrm{NLS}}}(Z) + [X_{\mathscr{B}_{\mathrm{NLS}}}(Z), X_{\mathscr{H}^{(2)}_{\mathrm{NLS}}}(Z)] = -\mathrm{i}E\Lambda_{\mathrm{NLS}}Z + X_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z),$$

with $X_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}$ as in (6-23). Moreover it is Hamiltonian with Hamiltonian as in (6-25) by (5-23) and (3-73). □

Remark 6.7. The Hamiltonian function in (6-25) may be rewritten, up to symmetrizations, as in (3-78) with coefficients $\mathtt{h}_4(\xi,\eta,\zeta)$ satisfying (3-79). The coefficients of its Hamiltonian vector field have the form (3-82) (see also (3-81)). Moreover, by (6-23), (3-6), (5-23), (5-24), we deduce that

$$-2\mathrm{i}\mathtt{h}_4(\xi,\eta,\zeta)=-2\mathrm{i}\chi_\epsilon\left(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\right)+\mathtt{q}_{\mathtt{H}^{(4)}_{\mathrm{NLS}}}(\xi,\eta,\zeta). \tag{6-32}$$

6B. *Diagonalization of the KG.* In this section we diagonalize the system (4-44) up to a smoothing remainder. This will be done into two steps. We first diagonalize the matrix $E(\mathbb{1}+\mathscr{A}_1(x,\xi))$ in (4-44) by means of a change of coordinates similar to the one made in the previous section for the (NLS) case. After that we diagonalize the matrix of symbols of order 0 at homogeneity 3, by means of an *approximately symplectic* change of coordinates. Consider the Cauchy problem associated to (KG). Throughout the rest of the section we shall assume the following.

Hypothesis 6.8. We restrict the solution of (KG) on the interval of times $[0,T)$, with T such that

$$\sup_{t\in[0,T)}(\|\psi(t,\cdot)\|_{H^{s+1/2}}+\|\partial_t\psi(t,\cdot)\|_{H^{s-1/2}})\le\epsilon,\qquad \|\psi_0(\cdot)\|_{H^{s+1/2}}+\|\psi_1(\cdot)\|_{H^{s-1/2}}\le\tfrac{1}{32}\epsilon,$$

with $\psi(0,x)=\psi_0(x)$ and $(\partial_t\psi)(0,x)=\psi_1(x)$.

Note that such a T exists thanks to the local well-posedness proved in [Kato 1975].

Remark 6.9. Recall (3-77). Then one can note that

$$\tfrac14(\|\psi(t,\cdot)\|_{H^{s+1/2}}+\|\partial_t\psi(t,\cdot)\|_{H^{s-1/2}})\le\|u\|_{H^s}\le 2(\|\psi(t,\cdot)\|_{H^{s+1/2}}+\|\partial_t\psi(t,\cdot)\|_{H^{s-1/2}}).$$

6B1. *Diagonalization at order 1.* Consider the matrix of symbols (see (4-24), (4-25))

$$E(\mathbb{1}+\mathscr{A}_1(x,\xi)),\quad \mathscr{A}_1(x,\xi):=\begin{bmatrix}1&1\\1&1\end{bmatrix}\tilde a_2(x,\xi),\quad \tilde a_2(x,\xi):=\tfrac12\Lambda_{\mathrm{KG}}^{-2}(\xi)a_2(x,\xi). \tag{6-33}$$

Define

$$\lambda_{\mathrm{KG}}(x,\xi):=\sqrt{(1+\tilde a_2(x,\xi))^2-(\tilde a_2(x,\xi))^2},\quad \tilde a_2^+(x,\xi):=\lambda_{\mathrm{KG}}(x,\xi)-1. \tag{6-34}$$

Notice that the symbol $\lambda_{\mathrm{KG}}(x,\xi)$ is well-defined by taking $\|u\|_{H^s}\ll 1$ small enough. The matrix of eigenvectors associated to the eigenvalues of $E(\mathbb{1}+\mathscr{A}_1(x,\xi))$ is

$$\begin{gathered}S(x,\xi):=\begin{pmatrix}s_1(x,\xi)&s_2(x,\xi)\\s_2(x,\xi)&s_1(x,\xi)\end{pmatrix},\qquad S^{-1}(x,\xi):=\begin{pmatrix}s_1(x,\xi)&-s_2(x,\xi)\\-s_2(x,\xi)&s_1(x,\xi)\end{pmatrix},\\ s_1:=\frac{1+\tilde a_2+\lambda_{\mathrm{KG}}}{\sqrt{2\lambda_{\mathrm{KG}}(1+\tilde a_2+\lambda_{\mathrm{KG}})}},\qquad s_2:=\frac{-\tilde a_2}{\sqrt{2\lambda_{\mathrm{KG}}(1+\tilde a_2+\lambda_{\mathrm{KG}})}}.\end{gathered} \tag{6-35}$$

By a direct computation one can check that

$$S^{-1}(x,\xi)E(\mathbb{1}+\mathscr{A}_1(x,\xi))S(x,\xi)=E\,\mathrm{diag}(\lambda_{\mathrm{KG}}(x,\xi)),\quad s_1^2-|s_2|^2=1. \tag{6-36}$$

We shall study how the system (4-44) transforms under the maps

$$\Phi_{\mathrm{KG}}=\Phi_{\mathrm{KG}}(U)[\cdot]:=\mathrm{Op}^{\mathrm{BW}}(S^{-1}(x,\xi)),\quad \Psi_{\mathrm{KG}}=\Psi_{\mathrm{KG}}(U)[\cdot]:=\mathrm{Op}^{\mathrm{BW}}(S(x,\xi)). \tag{6-37}$$

Lemma 6.10. *Assume Hypothesis 6.8. We have the following*:

(i) *If $s_0 > d$, then*

$$|\tilde{a}_2^+|_{\mathcal{N}_p^0} + |\tilde{a}_2|_{\mathcal{N}_p^0} + |s_1 - 1|_{\mathcal{N}_p^0} + |s_2|_{\mathcal{N}_p^0} \lesssim \|u\|^3_{H^{p+s_0+1}}, \quad p + s_0 + 1 \le s. \tag{6-38}$$

(ii) *For any $s \in \mathbb{R}$ one has*

$$\|\Phi_{\mathrm{KG}}(U)V - V\|_{H^s} + \|\Psi_{\mathrm{KG}}(U)V - V\|_{H^s} \lesssim \|V\|_{H^s}\|u\|^3_{H^{2s_0+1}} \quad \text{for all } V \in H^s(\mathbb{T}^d; \mathbb{C}^2). \tag{6-39}$$

(iii) *One has $\Psi_{\mathrm{KG}}(U) \circ \Phi_{\mathrm{KG}}(U) = \mathbb{1} + Q(U)$, where Q is a real-to-real remainder satisfying*

$$\|Q(U)V\|_{H^{s+1}} \lesssim \|V\|_{H^s}\|u\|^3_{H^{2s_0+3}}. \tag{6-40}$$

(iv) *For any $t \in [0, T)$, one has $\partial_t \Phi_{\mathrm{KG}}(U)[\,\cdot\,] = \mathrm{Op}^{\mathrm{BW}}(\partial_t S^{-1}(x, \xi))$ and*

$$|\partial_t S^{-1}(x, \xi)|_{\mathcal{N}_{s_0}^0} \lesssim \|u\|^3_{H^{2s_0+3}}, \qquad \|\partial_t \Phi_{\mathrm{KG}}(U)V\|_{H^s} \lesssim \|V\|_{H^s}\|u\|^3_{H^{2s_0+3}}. \tag{6-41}$$

Proof. (i) Inequality (6-38) follows by (4-45) using the explicit formulas (6-35), (6-34).

(ii) This follows by using (6-38) and Lemma 3.1(ii).

(iii) By formula (3-19) in Proposition 3.2 one gets

$$\Psi_{\mathrm{KG}}(U) \circ \Phi_{\mathrm{KG}}(U) = \mathbb{1} + \mathrm{Op}^{\mathrm{BW}}\begin{pmatrix} 0 & \mathrm{i}\{s_1, s_2\} \\ -\mathrm{i}\{s_1, s_2\} & 0 \end{pmatrix} + R(s_1, s_2)$$

for some remainder satisfying (3-20) with $a \rightsquigarrow s_1$ and $b \rightsquigarrow s_2$. Therefore (6-40) follows by using (3-8), (3-10) and (6-38).

(iv) This is similar to the proof of Lemma 6.3(iii). □

Proposition 6.11 (diagonalization at order 1). *Consider the system (4-44) and set*

$$W = \Phi_{\mathrm{KG}}(U)U, \tag{6-42}$$

with Φ_{KG} defined in (6-37). Then W solves the equation (recall (4-9))

$$\partial_t W = -\mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}\big(\mathrm{diag}(1 + \tilde{a}_2^+(x, \xi))\Lambda_{\mathrm{KG}}(\xi)\big)W + X_{\mathscr{H}^{(4)}_{\mathrm{KG}}}(W) + R^{(1)}(u), \tag{6-43}$$

where the vector field $X_{\mathscr{H}^{(4)}_{\mathrm{KG}}}$ is defined in (4-28). The symbol $\tilde{a}_2^+$ is defined in (6-34). The remainder $R^{(1)}$ has the form $(R^{(1,+)}, \overline{R^{(1,+)}})^T$. Moreover, for any $s > 2d + \mu$, for some $\mu > 0$, it satisfies the estimate

$$\|R^{(1)}(u)\|_{H^s} \lesssim \|u\|^4_{H^s}. \tag{6-44}$$

Proof. By (6-42) and (4-44) we get

$$\begin{aligned}\partial_t W &= \Phi_{\mathrm{KG}}(U)\dot{U} + (\partial_t \Phi_{\mathrm{KG}}(U))[U] \\ &= -\mathrm{i}\Phi_{\mathrm{KG}}(U)\,\mathrm{Op}^{\mathrm{BW}}\big(E(\mathbb{1} + \mathscr{A}_1(x, \xi))\Lambda_{\mathrm{KG}}(\xi)\big)\Psi_{\mathrm{KG}}(U)W + \Phi_{\mathrm{KG}}(U)X_{\mathscr{H}^{(4)}_{\mathrm{KG}}}(U) \\ &\quad + \Phi_{\mathrm{KG}}(U)R(u) + (\partial_t \Phi_{\mathrm{KG}}(U))[U] \\ &\qquad + \mathrm{i}\Phi_{\mathrm{KG}}(U)\,\mathrm{Op}^{\mathrm{BW}}\big(E(\mathbb{1} + \mathscr{A}_1(x, \xi))(\xi)\big)Q(U)U,\end{aligned} \tag{6-45}$$

where we used items (ii), (iii) in Lemma 6.10. We study the first summand in the right-hand side of (6-45). By direct inspection, using Lemma 3.1 and Proposition 3.2 we get

$$
\begin{aligned}
-\mathrm{i}\Phi_{\mathrm{KG}}(U)\,\mathrm{Op}^{\mathrm{BW}}\big(E(\mathbb{1}+\mathscr{A}_1(x,\xi))\Lambda_{\mathrm{KG}}(\xi)\big)\Psi_{\mathrm{KG}}(U) &= -\mathrm{i}\,\mathrm{Op}^{\mathrm{BW}}(S^{-1}E(\mathbb{1}+\mathscr{A}_1(x,\xi))S)+R(u)\\
&= -\mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}(\mathrm{diag}(\lambda_{\mathrm{KG}}(x,\xi)))+R(u) \quad (\text{by (6-36)}),
\end{aligned}
$$

where $R(u)$ is a remainder satisfying (6-44). Thanks to the discussion above and (6-34) we obtain the highest-order term in (6-43). All the other summands in the right-hand side of (6-45) may be analyzed as done in the proof of Proposition 6.4 by using Lemma 6.10. □

6B2. *Diagonalization of cubic terms at order 0.* Above we showed that if the function U solves (4-44) then W in (6-42) solves (6-43). The cubic terms in the system (6-43) are the same as those in (4-44) and have the form (4-28). The aim of this section is to diagonalize the matrix of symbols of order zero $\mathscr{A}_0(x,\xi)$.

Let us define

$$
Z := \begin{bmatrix} z \\ \bar{z} \end{bmatrix} := \Phi_{\mathscr{B}_{\mathrm{KG}}}(W) := W + X_{\mathscr{B}_{\mathrm{KG}}}(W), \tag{6-46}
$$

where $X_{\mathscr{B}_{\mathrm{KG}}}$ is the Hamiltonian vector field of (5-5) and W is the function in (6-42). The properties of $X_{\mathscr{B}_{\mathrm{KG}}}$ and the estimates of $\Phi_{\mathscr{B}_{\mathrm{KG}}}$ have been discussed in Lemma 5.1 and in Proposition 5.2.

Remark 6.12. Recall (6-42) and (6-46). One can note that, owing to Hypothesis 6.8, for $s > 2d+3$, we have

$$
\big(1-\tfrac{1}{100}\big)\|U\|_{H^s} \le \|W\|_{H^s} \le \big(1+\tfrac{1}{100}\big)\|U\|_{H^s}, \quad \big(1-\tfrac{1}{100}\big)\|W\|_{H^s} \le \|Z\|_{H^s} \le \big(1+\tfrac{1}{100}\big)\|W\|_{H^s}. \tag{6-47}
$$

This is a consequence of the estimates (6-39), (6-40) (5-12), (5-10), (5-14) taking ϵ small enough.

Proposition 6.13 (diagonalization at order 0). *Let U be a solution of (4-44) and assume Hypothesis 6.8 (see also Remark 6.9). Then the function Z defined in (6-46), with W given in (6-42), satisfies*

$$
\partial_t Z = -\mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}\big(\mathrm{diag}(1+\tilde{a}_2^{+}(x,\xi))\Lambda_{\mathrm{KG}}(\xi)\big)Z + X_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z) + R_4^{(2)}(u), \tag{6-48}
$$

where $\tilde{a}_2^{+}(x,\xi)$ is the real-valued symbol in (6-34), the cubic vector field $X_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z)$ has the form

$$
X_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z) := -\mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}(\mathrm{diag}(a_0(x,\xi)))Z + Q_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z), \tag{6-49}
$$

the symbol $a_0(x,\xi)$ is as in (4-24), and the remainder $Q_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z)$ is the cubic remainder given in Lemma 5.5. The remainder $R_4^{(2)}(u)$ has the form $(R_4^{(2,+)}(u), \overline{R_4^{(2,+)}(u)})^T$. Moreover, for any $s > 2d+\mu$, for some $\mu > 0$, we have the estimate

$$
\|R_4^{(2)}(u)\|_{H^s} \lesssim \|u\|_{H^s}^4. \tag{6-50}
$$

Finally the vector field $X_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z)$ in (6-49) is Hamiltonian; i.e., $X_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z) := -\mathrm{i}J\nabla \mathrm{H}^{(4)}_{\mathrm{KG}}(Z)$ with

$$
\mathrm{H}^{(4)}_{\mathrm{KG}}(Z) := \mathscr{H}^{(4)}_{\mathrm{KG}}(Z) - \{\mathscr{B}_{\mathrm{KG}}(Z), \mathscr{H}^{(2)}_{\mathrm{KG}}(Z)\}, \quad \mathscr{H}^{(2)}_{\mathrm{KG}}(Z) = \int_{\mathbb{T}^d} \Lambda_{\mathrm{KG}} z \cdot \bar{z}\, dx, \tag{6-51}
$$

where $\mathscr{H}^{(4)}_{\mathrm{KG}}$ is in (4-27), and $\mathscr{B}_{\mathrm{KG}}$ is in (5-5), (5-4).

Proof. We recall (5-16) and we rewrite (6-43) as

$$\partial_t W = X_{\mathscr{H}^{(\le 4)}_{\mathrm{KG}}}(W) - \mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}(\tilde{a}_2^+(x,\xi)\Lambda_{\mathrm{KG}}(\xi))W + R^{(1)}(u).$$

Then, using (6-46), we get

$$\begin{aligned}\partial_t Z &= d_W\Phi_{\mathscr{B}_{\mathrm{KG}}}(W)[\partial_t W]\\ &= d_W\Phi_{\mathscr{B}_{\mathrm{KG}}}(W)[X_{\mathscr{H}^{(\le 4)}_{\mathrm{KG}}}(W)] \qquad (6\text{-}52)\\ &\quad + d_W\Phi_{\mathscr{B}_{\mathrm{KG}}}(W)[-\mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}(\mathrm{diag}(\tilde{a}_2^+(x,\xi)\Lambda_{\mathrm{KG}}(\xi)))W] \qquad (6\text{-}53)\\ &\quad + d_W\Phi_{\mathscr{B}_{\mathrm{KG}}}(W)[R^{(1)}(u)]. \qquad (6\text{-}54)\end{aligned}$$

By estimates (5-10) and (6-44) we have that the term in (6-54) can be absorbed in a remainder satisfying (6-50). Consider the term in (6-53). We write

$$\begin{aligned}(6\text{-}53) &= -\mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}(\mathrm{diag}(\tilde{a}_2^+(x,\xi)\Lambda_{\mathrm{KG}}(\xi)))Z + P_1 + P_2,\\ P_1 &:= -\mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}(\mathrm{diag}(\tilde{a}_2^+(x,\xi)\Lambda_{\mathrm{KG}}(\xi)))[W-Z],\\ P_2 &:= ((d_W\Phi_{\mathscr{B}_{\mathrm{KG}}}(W)) - \mathbb{1})[-\mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}(\mathrm{diag}(\tilde{a}_2^+(x,\xi)\Lambda_{\mathrm{KG}}(\xi)))W].\end{aligned} \qquad (6\text{-}55)$$

We have, for $s \ge 2s_0 + 2$,

$$\|P_2\|_{H^s} \overset{(5\text{-}10)}{\lesssim} \|u\|^2_{H^s}\|\mathrm{Op}^{\mathrm{BW}}(\tilde{a}_2^+(x,\xi)\Lambda_{\mathrm{KG}}(\xi))w\|_{H^{s-1}} \overset{(6\text{-}38),(3\text{-}10),(6\text{-}47)}{\lesssim} \|u\|^6_{H^s},$$

which implies (6-50). By (5-14) in Proposition 5.2 and estimate (5-10) we deduce $\|W-Z\|_{H^{s+1}} \lesssim \|u\|^3_{H^s}$. Hence using again (6-38), (3-10), (6-47) we get P_1 satisfies (6-50). It remains to discuss the structure of the term in (6-52). By Lemma 5.3 we obtain

$$d_W\Phi_{\mathscr{B}_{\mathrm{KG}}}(W)[X_{\mathscr{H}^{(\le 4)}_{\mathrm{KG}}}(W)] = X_{\mathscr{H}^{(\le 4)}_{\mathrm{KG}}}(Z) + [X_{\mathscr{B}_{\mathrm{KG}}}(Z), X_{\mathscr{H}^{(2)}_{\mathrm{KG}}}(Z)], \qquad (6\text{-}56)$$

modulo remainders that can be absorbed in $R_4^{(2)}$ satisfying (6-50). Then (6-56), (6-52)–(6-54) and the discussion above imply (6-48), where the cubic vector field has the form

$$X_{\mathtt{H}^{(4)}_{\mathrm{KG}}}(Z) = X_{\mathscr{H}^{(4)}_{\mathrm{KG}}}(Z) + [X_{\mathscr{B}_{\mathrm{KG}}}(Z), X_{\mathscr{H}^{(2)}_{\mathrm{KG}}}(Z)]. \qquad (6\text{-}57)$$

Using (3-73), (3-72), we conclude that $X_{\mathtt{H}^{(4)}_{\mathrm{KG}}}$ is the Hamiltonian vector field of $\mathtt{H}^{(4)}_{\mathrm{KG}}$ in (6-51). Equation (6-49) follows by Lemma 5.5. □

Remark 6.14. In view of Remarks 4.6 and 4.8, following the same proof as Proposition 6.13, in the semilinear case we obtain that (6-48) reads

$$\partial_t Z = -\mathrm{i}E\,\mathrm{Op}^{\mathrm{BW}}(\mathrm{diag}(\Lambda_{\mathrm{KG}}(\xi)))Z + X_{\mathtt{H}^{(4)}_{\mathrm{KG}}}(Z) + R_4^{(2)}(u),$$

where $X_{\mathtt{H}^{(4)}_{\mathrm{KG}}}$ has the form (6-49) with $a_0(x,\xi)$ a symbol of order -1 and $Q_{\mathtt{H}^{(4)}_{\mathrm{KG}}}$ a remainder of the form (5-30) with coefficients satisfying (5-31) with the better denominator $\max\{\langle\xi-\eta-\zeta\rangle, \langle\eta\rangle, \langle\zeta\rangle\}^2$.

7. Energy estimates

7A. ***Estimates for the NLS.*** In this section we prove a priori energy estimates on the Sobolev norms of the variable Z in (6-20). In Section 7A1 we introduce a convenient energy norm on $H^s(\mathbb{T}^d;\mathbb{C})$ which is equivalent to the classic H^s-norm. This is the content of Lemma 7.2. In Section 7A2, using the nonresonance conditions of Proposition 2.1, we provide bounds on the nonresonant terms appearing in the energy estimates. We deal with resonant interactions in Lemma 7.4.

7A1. *Energy norm.* Let us define the symbol

$$\mathscr{L}=\mathscr{L}(x,\xi):=|\xi|^2+\Sigma,\quad \Sigma=\Sigma(x,\xi):=a_2^{(1)}(x)|\xi|^2+\vec{a}_1^{(1)}(x)\cdot\xi, \tag{7-1}$$

where the symbols $a_2^{(1)}(x)$, $\vec{a}_1^{(1)}(x)$ are given in Proposition 6.4.

Lemma 7.1. *Assume Hypothesis 6.1 and let $\gamma>0$. Then for $\epsilon>0$ small enough we have the following*:

(i) *One has*

$$|\Sigma|_{\mathscr{N}^2_{s_0}}\le C\|u\|^6_{H^{2s_0+1}},\quad |(1+\mathscr{L})^\gamma-(|\xi|^2+1)^\gamma|_{\mathscr{N}^{2\gamma}_{s_0}}\lesssim_\gamma C\|u\|^6_{H^{2s_0+1}} \tag{7-2}$$

for some $C>0$ depending on s_0.

(ii) *For any $s\in\mathbb{R}$ and any $h\in H^s(\mathbb{T}^d;\mathbb{C})$, one has*

$$\begin{aligned}\|T_{\mathscr{L}^\gamma}h\|_{H^{s-2\gamma}}&\le\|h\|_{H^s}(1+C\|u\|^6_{H^{2s_0+1}}),\\ \|T_\Sigma h\|_{H^{s-2}}+\|T_{(1+\mathscr{L})^\gamma-(|\xi|^2+1)^\gamma}h\|_{H^{s-2\gamma}}&\lesssim_\gamma\|h\|_{H^s}\|u\|^6_{H^{2s_0+1}}\end{aligned} \tag{7-3}$$

for some $C>0$ depending on s and γ.

(iii) *For any $t\in[0,T)$ one has $|\partial_t\Sigma|_{\mathscr{N}^2_{s_0}}\lesssim\|u\|^6_{H^{2s_0+3}}$. Moreover*

$$\|(T_{\partial_t(1+\mathscr{L})^\gamma})h\|_{H^{s-2\gamma}}\lesssim_\gamma\|h\|_{H^s}\|u\|^6_{H^{2s_0+3}}\quad \textit{for all } h\in H^s(\mathbb{T}^d;\mathbb{C}). \tag{7-4}$$

(iv) *The operators $T_{\mathscr{L}}$, $T_{(1+\mathscr{L})^\gamma}$ are self-adjoint with respect to the L^2-scalar product (3-3).*

Proof. (i)–(ii) Inequalities (7-2) follow by using (7-1), the bounds (6-12) on the symbols $a_2^{(1)}$ and $\vec{a}_1^{(1)}\cdot\xi$; (7-3) follows by Lemma 3.1.

(iii) The bound on $\partial_t\Sigma$ follows by reasoning as in Lemma 6.3(iii) using the explicit formula of $a_2^{(1)}$ in (6-1) and the formula for $a_1^{(1)}\cdot\xi$ in (6-19) (see also (6-2)). Then (3-10) implies (7-4).

(iv) This follows by (3-54) since the symbol $\mathscr{L}$ in (7-1) is real-valued. □

In the following we shall construct the *energy norm*. By using this norm we are able to achieve the *energy estimates* on the previously diagonalized system. For $s\in\mathbb{R}$ we define

$$z_n:=T_{(1+\mathscr{L})^n}z,\quad Z_n=\begin{bmatrix}z_n\\ \bar{z}_n\end{bmatrix}:=T_{(1+\mathscr{L})^n}\mathbb{1}Z,\quad Z=\begin{bmatrix}z\\ \bar{z}\end{bmatrix},\quad n:=\tfrac{1}{2}s. \tag{7-5}$$

Lemma 7.2 (equivalence of the energy norm). *Assume Hypothesis 6.1 with $s>2d+4$. Then, for $\epsilon>0$ small enough enough, one has*

$$\left(1-\tfrac{1}{100}\right)\|z\|_{H^s}\le\|z_n\|_{L^2}\le\left(1+\tfrac{1}{100}\right)\|z\|_{H^s}. \tag{7-6}$$

Proof. Let $s=2n$. Then by (7-3) and (7-5) we have $\|z_n\|_{L^2}\le\|z\|_{H^s}(1+C\|u\|^6_{H^{2s_0+1}})\le 2\|z\|_{H^2}$, with $s_0>d$. Moreover

$$\|z\|_{H^s}=\|T_{(1+|\xi|^2)^n}z\|_{L^2}\overset{(7\text{-}3)}{\le}\|z_n\|_{L^2}+C\|z\|_{H^s}\|u\|^6_{H^{2s_0+1}},$$

which implies $(1-C\|u\|^6_{H^{2s_0+1}})\|z\|_{H^s}\le\|z_n\|_{L^2}$ for some constant C depending on s. The discussion above implies (7-6) by taking $\epsilon>0$ in Hypothesis 6.1 small enough. □

Recalling (6-22), (1-25) and (7-1) we have

$$(\partial_t+\mathrm{i}\Lambda_{\mathrm{NLS}})z=-\mathrm{i}T_\Sigma z+X^+_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z)+R_5^{(2,+)}(U),\qquad Z=\begin{bmatrix}z\\ \bar z\end{bmatrix},\tag{7-7}$$

where $X_{\mathrm{H}^{(4)}_4}$ is given in (6-23) (see also Remark 6.7) and $R_5^{(2,+)}$ is the remainder satisfying (6-24).

Lemma 7.3. *Fix $s>2d+4$ and recall (7-7). One has that the function z_n defined in (7-5) solves the problem*

$$\partial_t z_n=-\mathrm{i}T_{\mathscr{L}}z_n-\mathrm{i}V*z_n+T_{(1+|\xi|^2)^n}X^{+,\mathrm{res}}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z)+B_n^{(1)}(Z)+B_n^{(2)}(Z)+R_{5,n}(U),\tag{7-8}$$

where $X^{+,\mathrm{res}}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}$ is defined as in Definition 3.9,

$$\begin{aligned}\widehat{B_n^{(1)}(Z)}(\xi)&=\frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d}b^{(1)}(\xi,\eta,\zeta)\hat z(\xi-\eta-\zeta)\hat{\bar z}(\eta)\hat z_n(\zeta),\\ \widehat{B_n^{(2)}(Z)}(\xi)&=\frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d}b_n^{(2)}(\xi,\eta,\zeta)\hat z(\xi-\eta-\zeta)\hat{\bar z}(\eta)\hat z(\zeta),\end{aligned}\tag{7-9}$$

with

$$b^{(1)}(\xi,\eta,\zeta):=-2\mathrm{i}\chi_\epsilon\left(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\right)\mathbb{1}_{\mathscr{R}^c}(\xi,\eta,\zeta),\tag{7-10}$$

$$|b_n^{(2)}(\xi,\eta,\zeta)|\lesssim\frac{\langle\xi\rangle^{2n}\max_2\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}^4}{\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}}\mathbb{1}_{\mathscr{R}^c}(\xi,\eta,\zeta),\tag{7-11}$$

and where the remainder $R_{5,n}$ satisfies

$$\|R_{5,n}(U)\|_{L^2}\lesssim\|u\|^5_{H^s}.\tag{7-12}$$

Proof. Recalling (3-84) we define

$$X^{+,\perp}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z):=X^{+}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z)-X^{+,\mathrm{res}}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z).\tag{7-13}$$

By differentiating (7-5) and using (7-1) and (7-7) we get

$$\begin{aligned}\partial_t z_n&=T_{(1+\mathscr{L})^n}\partial_t z+T_{\partial_t(1+\mathscr{L})^n}z\\&=-\mathrm{i}T_{\mathscr{L}}z_n-\mathrm{i}T_{(1+\mathscr{L})^n}(V*z)+T_{(1+\mathscr{L})^n}X^+_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z)+T_{(1+\mathscr{L})^n}R_5^{(2,+)}(U)\\&\qquad+T_{\partial_t(1+\mathscr{L})^n}z-\mathrm{i}[T_{(1+\mathscr{L})^n},T_{\mathscr{L}}]z.\end{aligned}\tag{7-14}$$

By using Lemmas 3.1 and 7.1, Proposition 3.2, and (7-6), (6-21) one proves that the last summand gives a contribution to $R_{5,n}(U)$ satisfying (7-12). By using (7-4), (6-21), (6-24) we deduce that

$$\|T_{(1+\mathscr{L})^n}R_5^{(2,+)}(U)\|_{L^2}+\|T_{\partial_t(1+\mathscr{L})^n}z\|_{L^2}\lesssim\|u\|^5_{H^s}.$$

Secondly we write

$$\mathrm{i}T_{(1+\mathscr{L})^n}(V*z)=\mathrm{i}V*z_n+\mathrm{i}V*(T_{(1+|\xi|^2)^n-(1+\mathscr{L})^n}z)+\mathrm{i}T_{(1+\mathscr{L})^n-(1+|\xi|^2)^n}(V*z).$$

By (7-3), (6-21), and recalling (1-5) we conclude $\|T_{(1+\mathscr{L})^n}(V*z)-V*z_n\|_{L^2}\lesssim\|u\|^7_{H^s}$. We now study the third summand in (7-14). We have (see (7-13))

$$T_{(1+\mathscr{L})^n}X^{+}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z)=T_{(1+|\xi|^2)^n}X^{+,\mathrm{res}}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z)+T_{(1+|\xi|^2)n}X^{+,\perp}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z)+T_{(1+\mathscr{L})^n-(1+|\xi|^2)^n}X^{+}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z).$$

By (7-3), (6-23), (3-10), Lemma 3.7 and using the estimate (5-25), one obtains

$$\|T_{(1+\mathscr{L})^n-(1+|\xi|^2)^n}X^{+}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z)\|_{L^2}\lesssim\|u\|^9_{H^s}.$$

Recalling (6-32) and (7-13) we write

$$\begin{aligned}
&T_{(1+|\xi|^2)^n}X^{+,\perp}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z)=\mathscr{C}_1+\mathscr{C}_2+\mathscr{C}_3,\quad \widehat{\mathscr{C}}_i(\xi)=\frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d}\mathtt{c}_i(\xi,\eta,\zeta)\hat{z}(\xi-\eta-\zeta)\hat{\bar{z}}(\eta)\hat{z}(\zeta),\\
&\qquad \mathtt{c}_1(\xi,\eta,\zeta):=-2\mathrm{i}\chi_\epsilon\left(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\right)(1+|\zeta|^2)^n\mathbb{1}_{\mathscr{R}^c}(\xi,\eta,\zeta),\\
&\qquad \mathtt{c}_2(\xi,\eta,\zeta):=-2\mathrm{i}\chi_\epsilon\left(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\right)[(1+|\xi|^2)^n-(1+|\zeta|^2)^n]\mathbb{1}_{\mathscr{R}^c}(\xi,\eta,\zeta),\\
&\qquad \mathtt{c}_3(\xi,\eta,\zeta):=\mathtt{q}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(\xi,\eta,\zeta)(1+|\xi|^2)^n\mathbb{1}_{\mathscr{R}^c}(\xi,\eta,\zeta).
\end{aligned}\tag{7-15}$$

We now consider the operator $\mathscr{C}_1$ with coefficients $\mathtt{c}_1(\xi,\eta,\zeta)$. First of all we remark that it can be written as $\mathscr{C}_1=M(z,\bar{z},z)$, where M is a trilinear operator of the form (3-62). Moreover, setting

$$z_n=T_{(1+|\xi|^2)^n}z+h_n,\quad h_n:=T_{(1+\mathscr{L})^n-(1+|\xi|^2)^n}z,$$

we can write $\mathscr{C}_1=B_n^{(1)}(Z)-M(z,\bar{z},h_n)$, where $B_n^{(1)}$ has the form (7-9) with coefficients as in (7-10). Using that $|\mathtt{c}_1(\xi,\eta,\zeta)|\lesssim 1$, Lemma 3.7 (with $m=0$) and (7-3) we deduce that $\|M(z,\bar{z},h_n)\|_{L^2}\lesssim\|u\|^9_{H^s}$. Therefore this is a contribution to $R_{5,n}(U)$ satisfying (7-12). The discussion above implies formula (7-8) by setting $B_n^{(2)}$ as the operator of the form (7-9) with coefficients $b_n^{(2)}(\xi,\eta,\zeta):=\mathtt{c}_2(\xi,\eta,\zeta)+\mathtt{c}_3(\xi,\eta,\zeta)$. The coefficient $\mathtt{c}_3(\xi,\eta,\zeta)$ satisfies (7-11) by (5-25). For the coefficient $\mathtt{c}_2(\xi,\eta,\zeta)$ one has to apply Lemma 3.8 with $\mu=m=1$ and $f(\xi,\eta,\zeta):=((1+|\xi|^2)^n-(1+|\zeta|^2)^n)\langle\xi\rangle^{-2n}$. ☐

In the following lemma we prove a key cancellation due to the fact that the *super actions* are prime integrals of the resonant Hamiltonian vector field $X^{+,\mathrm{res}}_{\mathrm{H}_4}(Z)$ in the spirit of [Faou et al. 2013]. We also prove an important algebraic property of the operator $B_n^{(1)}$ in (7-8).

Lemma 7.4. *For any $n\geq 0$ we have*

$$\mathrm{Re}(T_{\langle\xi\rangle^n}X^{+,\mathrm{res}}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z),T_{\langle\xi\rangle^n}z)_{L^2}=0,\tag{7-16}$$

$$\mathrm{Re}(B_n^{(1)}(Z),z_n)_{L^2}=0,\tag{7-17}$$

where $X^{+,\mathrm{res}}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}$ is defined in Lemma 7.3 and $B_n^{(1)}$ in (7-9), (7-10).

Proof. Equation (7-16) follows by Lemma 3.10. Let us check (7-17). By an explicit computation using (3-3), (7-9) we get

$$\begin{aligned}\operatorname{Re}(B_n^{(1)}(Z), z_n)_{L^2} &= \frac{1}{(2\pi)^d}\sum_{\xi,\eta,\zeta\in\mathbb{Z}^d} b^{(1)}(\xi,\eta,\zeta)\hat{z}(\xi-\eta-\zeta)\hat{\bar{z}}(\eta)\hat{z}_n(\zeta)\hat{\bar{z}}_n(-\xi)\\ &\qquad + \frac{1}{(2\pi)^d}\sum_{\xi,\eta,\zeta\in\mathbb{Z}^d} \overline{b^{(1)}(\xi,\eta,\zeta)}\hat{\bar{z}}(-\xi+\eta+\zeta)\hat{z}(-\eta)\hat{\bar{z}}_n(-\zeta)\hat{z}_n(\xi)\\ &= \frac{1}{(2\pi)^d}\sum_{\xi,\eta,\zeta\in\mathbb{Z}^d} [b^{(1)}(\xi,\eta,\zeta)+\overline{b^{(1)}(\zeta,\zeta+\eta-\xi,\xi)}]\hat{z}(\xi-\eta-\zeta)\hat{\bar{z}}(\eta)\hat{z}_n(\zeta)\hat{\bar{z}}_n(-\xi).\end{aligned}$$

By (7-10) we have

$$b^{(1)}(\xi,\eta,\zeta)+\overline{b^{(1)}(\zeta,\zeta+\eta-\xi,\xi)} = 2\mathrm{i}\chi_\epsilon\Big(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)[1_{\mathscr{R}^c}(\xi,\eta,\zeta)-1_{\mathscr{R}^c}(\zeta,\zeta+\eta-\xi,\xi)] = 0,$$

where we used the form of the resonant set $\mathscr{R}$ in (3-83). □

We conclude the section with the following proposition.

Proposition 7.5. *Let $u(t,x)$ be a solution of (NLS) satisfying Hypothesis 6.1 and consider the function z_n in (7-5) (see also (6-20), (6-10)). Then, setting $s=2n>2d+4$ we have*

$$\frac{1}{2^{1/4}}\|u(t)\|_{H^s} \le \|z_n(t)\|_{L^2} \le 2^{1/4}\|u(t)\|_{H^s} \tag{7-18}$$

and

$$\partial_t\|z_n(t)\|_{L^2}^2 = \mathscr{B}(t)+\mathscr{B}_{>5}(t), \qquad t\in[0,T), \tag{7-19}$$

where:

- *The term $\mathscr{B}(t)$ has the form*

$$\begin{aligned}\mathscr{B}(t) &= \frac{2}{(2\pi)^d}\sum_{\xi,\eta,\zeta\in\mathbb{Z}^d}\langle\xi\rangle^{2n}\mathtt{b}(\xi,\eta,\zeta)\hat{z}(\xi-\eta-\zeta)\hat{\bar{z}}(\eta)\hat{z}(\zeta)\hat{\bar{z}}(-\xi),\\ \mathtt{b}(\xi,\eta,\zeta) &= b_n^{(2)}(\xi,\eta,\zeta)+\overline{b_n^{(2)}(\zeta,\zeta+\eta-\xi,\xi)}, \qquad \xi,\eta,\zeta\in\mathbb{Z}^d,\end{aligned}\tag{7-20}$$

where $b_n^{(2)}(\xi,\eta,\zeta)$ are the coefficients in (7-9), (7-11).

- *The term $\mathscr{B}_{>5}(t)$ satisfies*

$$|\mathscr{B}_{>5}(t)| \lesssim \|u\|_{H^s}^6, \qquad t\in[0,T). \tag{7-21}$$

Proof. The norm $\|z_n\|_{L^2}$ is equivalent to $\|u\|_{H^s}$ by using Lemma 7.2 and Remark 6.5. Using (7-8) we get

$$\tfrac12\partial_t\|z_n(t)\|_{L^2}^2 = \operatorname{Re}(T_{\langle\xi\rangle^{2n}}X^{+,\mathrm{res}}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z), z_n)_{L^2} \tag{7-22}$$
$$+\operatorname{Re}(-\mathrm{i}T_{\mathscr{L}}z_n, z_n)_{L^2}+\operatorname{Re}(B_n^{(1)}(Z), z_n)_{L^2}+\operatorname{Re}(-\mathrm{i}V*z_n, z_n)_{L^2} \tag{7-23}$$
$$+\operatorname{Re}(B_n^{(2)}(Z), z_n)_{L^2} \tag{7-24}$$
$$+\operatorname{Re}(R_{5,n}(Z), z_n)_{L^2}. \tag{7-25}$$

Recall that $T_{\mathscr{L}}$ is self-adjoint (see Lemma 7.1(iv)) and the convolution potential V has real Fourier coefficients. Then by using also Lemma 7.4 (see (7-17)) we deduce (7-23) $=0$. Moreover by the

Cauchy–Schwarz inequality and estimates (7-12), (7-6) and (6-21) we obtain that the term in (7-25) is bounded from above by $\|u\|_{H^s}^6$. Consider the terms in (7-22) and (7-24). Recalling (7-5) and (7-1) we write

$$\begin{aligned}\mathrm{Re}(T_{\langle\xi\rangle^{2n}} X^{+,\mathrm{res}}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z), z_n)_{L^2} &= \mathrm{Re}(T_{\langle\xi\rangle^{2n}} X^{+,\mathrm{res}}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z), T_{\langle\xi\rangle^{2n}} z)_{L^2} + \mathrm{Re}(T_{\langle\xi\rangle^{2n}} X^{+,\mathrm{res}}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z), T_{(1+\mathscr{L})^n-\langle\xi\rangle^{2n}} z)_{L^2}\\ &= \mathrm{Re}(T_{\langle\xi\rangle^{2n}} X^{+,\mathrm{res}}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z), T_{(1+\mathscr{L})^n-\langle\xi\rangle^{2n}} z)_{L^2} \qquad (\text{by (7-16)}).\end{aligned}$$

Moreover we write

$$\mathrm{Re}(B_n^{(2)}(Z), z_n)_{L^2} = \mathrm{Re}(B_n^{(2)}(Z), T_{\langle\xi\rangle^{2n}} z)_{L^2} + \mathrm{Re}(B_n^{(2)}(Z), T_{(1+\mathscr{L})^n-\langle\xi\rangle^{2n}} z)_{L^2}.$$

Using the bound (7-3) in Lemma 7.1 to estimate the operator $T_{(1+\mathscr{L})^n-\langle\xi\rangle^{2n}}$ and Lemma 3.7 and (7-11) to estimate the operator $B_n^{(2)}(Z)$, we get

$$|\mathrm{Re}(T_{\langle\xi\rangle^{2n}} X^{+,\mathrm{res}}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z), T_{(1+\mathscr{L})^n-\langle\xi\rangle^{2n}} z)_{L^2}| + |\mathrm{Re}(B_n^{(2)}(Z), T_{(1+\mathscr{L})^n-\langle\xi\rangle^{2n}} z)_{L^2}| \lesssim \|u\|_{H^s}^{10},$$

which means that these remainders can be absorbed in the term $\mathscr{B}_{>5}(t)$. Then we set

$$\mathscr{B}(t) := 2\,\mathrm{Re}(B_n^{(2)}(Z), T_{\langle\xi\rangle^{2n}} z)_{L^2}.$$

Formulas (7-20) follow by an explicit computation using (7-9), (7-11). □

7A2. *Estimates of nonresonant terms.* In this subsection we provide estimates on the term $\mathscr{B}(t)$ appearing in (7-19). We state the main result of this section.

Proposition 7.6. *Let $N > 0$. Then there is $s_0 = s_0(N_0)$, where $N_0 > 0$ is given by Proposition 2.1, such that, if Hypothesis 6.1 holds with $s \geq s_0$, one has*

$$\left|\int_0^t \mathscr{B}(\sigma)\,d\sigma\right| \lesssim \|u\|_{L^\infty H^s}^{10} TN + \|u\|_{L^\infty H^s}^{6} T + \|u\|_{L^\infty H^s}^{4} TN^{-1} + \|u\|_{L^\infty H^s}^{4}, \tag{7-26}$$

where $\mathscr{B}(t)$ is in (7-20).

We need some preliminary results. We consider the trilinear maps

$$\mathscr{B}_i = \mathscr{B}_i[z_1, z_2, z_3], \quad \widehat{\mathscr{B}}_i(\xi) = \frac{1}{(2\pi)^d} \sum_{\eta,\zeta\in\mathbb{Z}^d} \mathtt{b}_i(\xi,\eta,\zeta)\hat{z}_1(\xi-\eta-\zeta)\hat{z}_2(\eta)\hat{z}_3(\zeta), \ i = 1, 2, \tag{7-27}$$

$$\mathscr{T}_< = \mathscr{T}_<[z_1, z_2, z_3], \quad \widehat{\mathscr{T}}_<(\xi) = \frac{1}{(2\pi)^d} \sum_{\eta,\zeta\in\mathbb{Z}^d} \mathtt{t}_<(\xi,\eta,\zeta)\hat{z}_1(\xi-\eta-\zeta)\hat{z}_2(\eta)\hat{z}_3(\zeta), \tag{7-28}$$

where

$$\mathtt{b}_1(\xi,\eta,\zeta) = \mathtt{b}(\xi,\eta,\zeta)1_{\{\max\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}\leq N\}}, \tag{7-29}$$

$$\mathtt{b}_2(\xi,\eta,\zeta) = \mathtt{b}(\xi,\eta,\zeta)1_{\{\max\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}> N\}}, \tag{7-30}$$

$$\mathtt{t}_<(\xi,\eta,\zeta) = \frac{-1}{\mathrm{i}\omega_{\mathrm{NLS}}(\xi,\eta,\zeta)}\mathtt{b}_1(\xi,\eta,\zeta), \tag{7-31}$$

where $\mathtt{b}(\xi,\eta,\zeta)$ are the coefficients in (7-20), and ω_{NLS} is the phase in (2-1). We remark that if $(\xi,\eta,\zeta) \in \mathscr{R}$ (see Definition 3.9) then the coefficients $\mathtt{b}(\xi,\eta,\zeta)$ are equal to zero (see (7-20), (7-9), (7-11)). Therefore, since ω_{NLS} is nonresonant (see Proposition 2.1), the coefficients in (7-31) are well-defined. We now prove an abstract results on the trilinear maps introduced in (7-27)–(7-28).

Lemma 7.7. *One has that, for* $s = 2n > \frac{1}{2}d + 4$,

$$\|\mathscr{B}_2[z_1, z_2, z_3]\|_{L^2} \lesssim N^{-1} \sum_{i=1}^{3} \|z_i\|_{H^s} \prod_{i \neq k} \|z_k\|_{H^{d/2+4+\epsilon}} \quad \text{for all } \epsilon > 0. \tag{7-32}$$

There is $s_0(N_0) > 0$ ($N_0 > 0$ *given by Proposition 2.1*) *such that for* $s \geq s_0(N_0)$ *one has*

$$\|\mathscr{T}_<[z_1, z_2, z_3]\|_{H^p} \lesssim N \sum_{i=1}^{3} \|z_i\|_{H^{s+p-2}} \prod_{i \neq k} \|z_k\|_{H^{s_0}}, \qquad p \in \mathbb{N}, \tag{7-33}$$

$$\|\mathscr{T}_<[z_1, z_2, z_3]\|_{L^2} \lesssim \sum_{i=1}^{3} \|z_i\|_{H^s} \prod_{i \neq k} \|z_k\|_{H^{s_0}}. \tag{7-34}$$

Proof. Using (7-30), (7-20), (7-11) we get

$$\begin{aligned}\|\mathscr{B}_2[z_1, z_2, z_3]\|_{L^2}^2 &\lesssim \sum_{\xi \in \mathbb{Z}^d} \Big(\sum_{\eta, \zeta \in \mathbb{Z}^d} |\mathtt{b}_2(\xi, \eta, \zeta)| |\hat{z}_1(\xi - \eta - \zeta)| |\hat{z}_2(\eta)| |\hat{z}_3(\zeta)| \Big)^2 \\ &\lesssim N^{-2} \sum_{\xi \in \mathbb{Z}^d} \Big(\sum_{\eta, \zeta \in \mathbb{Z}^d} \langle \xi \rangle^{2n} \max_2 \{|\xi - \eta - \zeta|, |\eta|, |\zeta|\}^4 |\hat{z}_1(\xi - \eta - \zeta)| |\hat{z}_2(\eta)| |\hat{z}_3(\zeta)| \Big)^2.\end{aligned}$$

Then, by reasoning as in the proof of Lemma 3.7, one obtains (7-32). Let us prove the bound (7-33) for $p = 0$; the others are similar. Using (7-31), (2-2), (7-20), (7-11) we have

$$\begin{aligned}\|\mathscr{T}_<[z_1, z_2, z_3]\|_{L^2}^2 &\lesssim \sum_{\xi \in \mathbb{Z}^d} \Big(\sum_{\eta, \zeta \in \mathbb{Z}^d} |\mathtt{t}_<(\xi, \eta, \zeta)| |\hat{z}_1(\xi - \eta - \zeta)| |\hat{z}_2(\eta)| |\hat{z}_3(\zeta)| \Big)^2 \\ &\lesssim_\gamma N^2 \sum_{\xi \in \mathbb{Z}^d} \Big(\sum_{\eta, \zeta \in \mathbb{Z}^d} \frac{\langle \xi \rangle^{2n} \max_2 \{|\xi - \eta - \zeta|, |\eta|, |\zeta|\}^{N_0+4}}{\max_1 \{|\xi - \eta - \zeta|, |\eta|, |\zeta|\}^2} |\hat{z}_1(\xi - \eta - \zeta)| |\hat{z}_2(\eta)| |\hat{z}_3(\zeta)| \Big)^2.\end{aligned}$$

Again, reasoning as in the proof of Lemma 3.7, one obtains (7-33). Inequality (7-34) follows similarly. □

Proof of Proposition 7.6. By (7-27), (7-29), (7-30) and recalling the definition of $\mathscr{B}$ in (7-20), we can write

$$\int_0^t \mathscr{B}(\sigma)\, d\sigma = \int_0^t (\mathscr{B}_1[z, \bar{z}, z], T_{\langle \xi \rangle^{2n}} z)_{L^2}\, d\sigma + \int_0^t (\mathscr{B}_2[z, \bar{z}, z], T_{\langle \xi \rangle^{2n}} z)_{L^2}\, d\sigma. \tag{7-35}$$

By Lemma 7.7 we have

$$\Big| \int_0^t (\mathscr{B}_2[z, \bar{z}, z], T_{\langle \xi \rangle^{2n}} z)_{L^2}\, d\sigma \Big| \overset{(7\text{-}32)}{\lesssim} N^{-1} \int_0^t \|z\|_{H^s}^4\, d\sigma \overset{(6\text{-}21)}{\lesssim} N^{-1} \int_0^t \|u\|_{H^s}^4\, d\sigma. \tag{7-36}$$

Consider now the first summand in the right-hand side of (7-35). We claim that we have the identity

$$\begin{aligned}\int_0^t (\mathscr{B}_1[z, \bar{z}, z], T_{\langle \xi \rangle^{2n}} z)_{L^2}\, d\sigma = &\int_0^t (\mathscr{T}_<[z, \bar{z}, z], T_{\langle \xi \rangle^{2n}} (\partial_t + \mathrm{i}\Lambda_{\mathrm{NLS}}) z)_{L^2}\, d\sigma \\ &+ \int_0^t (\mathscr{T}_<[(\partial_t + \mathrm{i}\Lambda_{\mathrm{NLS}}) z, \bar{z}, z], T_{\langle \xi \rangle^{2n}} z)_{L^2}\, d\sigma \\ &+ \int_0^t (\mathscr{T}_<[z, \bar{z}, (\partial_t + \mathrm{i}\Lambda_{\mathrm{NLS}}) z], T_{\langle \xi \rangle^{2n}} z)_{L^2}\, d\sigma \\ &+ \int_0^t (\mathscr{T}_<[z, \overline{(\partial_t + \mathrm{i}\Lambda_{\mathrm{NLS}}) z}, z], T_{\langle \xi \rangle^{2n}} z)_{L^2}\, d\sigma + O(\|u\|_{H^s}^4).\end{aligned} \tag{7-37}$$

We use the claim, postponing its proof. Consider the first summand in the right-hand side of (7-37). Using the self-adjointness of $T_{\langle\xi\rangle^2}$ and (7-7) we write

$$(\mathscr{T}_{<}[z,\bar{z},z],T_{\langle\xi\rangle^{2n}}(\partial_t+\mathrm{i}\Lambda_{\mathrm{NLS}})z)_{L^2}$$
$$=(T_{\langle\xi\rangle^2}\mathscr{T}_{<}[z,\bar{z},z],-T_{\langle\xi\rangle^{2n-2}}\mathrm{i}T_{\Sigma}z)_{L^2}+\big(\mathscr{T}_{<}[z,\bar{z},z],T_{\langle\xi\rangle^{2n}}(X^{+}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z)+R_5^{(2,+)}(U))\big)_{L^2}.$$

We estimate the first summand in the right-hand side by means of the Cauchy–Schwarz inequality, (7-33) with $p=2$ and (7-3); analogously we estimate the second summand by means of the Cauchy–Schwarz inequality, (7-34), (6-23) and (6-24), obtaining

$$\left|\int_0^t(\mathscr{T}_{<}[z,\bar{z},z],T_{\langle\xi\rangle^{2n}}(\partial_t+\mathrm{i}\Lambda_{\mathrm{NLS}})z)_{L^2}\,d\sigma\right|\le\int_0^t\|u(\sigma)\|^{10}_{H^s}N+\|u(\sigma)\|^6_{H^s}\,d\sigma.$$

The other terms in (7-37) are estimated in a similar way. We eventually obtain (7-26).

We now prove the claim (7-37). Recalling (7-7) we have

$$\partial_t\hat{z}(\xi)=-\mathrm{i}\Lambda_{\mathrm{NLS}}(\xi)\hat{z}(\xi)+\widehat{\mathscr{Q}}(\xi),\quad \xi\in\mathbb{Z}^d,\qquad \mathscr{Q}:=-\mathrm{i}T_{\Sigma}z+X^{+}_{\mathrm{H}^{(4)}_{\mathrm{NLS}}}(Z)+R_5^{(2,+)}(U).$$

We define $\hat{g}(\xi):=e^{\mathrm{i}t\Lambda_{\mathrm{NLS}}(\xi)}\hat{z}(\xi)$ for all $\xi\in\mathbb{Z}^d$. One can note that $\hat{g}(\xi)$ satisfies

$$\partial_t\hat{g}(\xi)=e^{\mathrm{i}t\Lambda_{\mathrm{NLS}}(\xi)}\widehat{\mathscr{Q}}(\xi)=e^{\mathrm{i}t\Lambda_{\mathrm{NLS}}(\xi)}(\partial_t+\mathrm{i}\Lambda_{\mathrm{NLS}})\hat{z}(\xi)\quad\text{for all } \xi\in\mathbb{Z}^d. \tag{7-38}$$

According to this notation and using (7-27) and (2-1) we have

$$\int_0^t(\mathscr{B}_1[z,\bar{z},z],T_{\langle\xi\rangle^{2n}}z)_{L^2}\,d\sigma$$
$$=\int_0^t\sum_{\xi,\eta,\zeta\in\mathbb{Z}^d}\frac{1}{(2\pi)^d}\mathtt{b}_1(\xi,\eta,\zeta)e^{-\mathrm{i}\sigma\omega_{\mathrm{NLS}}(\xi,\eta,\zeta)}\hat{g}(\xi-\eta-\zeta)\hat{\bar{g}}(\eta)\hat{g}(\zeta)\hat{\bar{g}}(-\xi)\langle\xi\rangle^{2n}\,d\sigma.$$

By integrating by parts in σ and using (7-38) one gets (7-37) with

$$O(\|u\|^4_{H^s})=(\mathscr{T}_{<}[z(t),\bar{z}(t),z(t)],T_{\langle\xi\rangle^{2n}}z(t))_{L^2}-(\mathscr{T}_{<}[z(0),\bar{z}(0),z(0)],T_{\langle\xi\rangle^{2n}}z(0))_{L^2}.$$

The remainder above is bounded from above by $\|u\|^4_{L^\infty H^s}$ using Cauchy–Schwarz and (7-34). □

7B. *Estimates for the KG.* In this section we provide a priori energy estimates on the variable Z solving (6-48). This implies similar estimates on the solution U of the system (4-44) thanks to the equivalence (6-47). In Section 7B1 we introduce an equivalent energy norm and we provide a first energy inequality. This is the content of Proposition 7.10. Then in Section 7B2 we give improved bounds on the nonresonant terms.

7B1. *First energy inequality.* We recall that the system (6-48) is diagonal up to smoothing terms plus some higher degree of homogeneity remainder. Hence, for simplicity, we pass to the scalar equation

$$\partial_t z+\mathrm{i}\Lambda_{\mathrm{KG}}z=-\mathrm{i}\,\mathrm{Op}^{\mathrm{BW}}(\tilde{a}_2^{+}(x,\xi)\Lambda_{\mathrm{KG}}(\xi))z+X^{+}_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z)+R_4^{(2,+)}(u), \tag{7-39}$$

where (recall (6-49)) $X^{+}_{\mathtt{H}^{(4)}_{\mathrm{KG}}}(Z) = -\mathrm{i}\,\mathrm{Op}^{\mathrm{BW}}(a_0(x,\xi))z + Q^{+}_{\mathtt{H}^{(4)}_{\mathrm{KG}}}(Z)$. For $n \in \mathbb{R}$ we define

$$z_n := \langle D\rangle^n z, \quad Z_n = \begin{bmatrix} z_n \\ \bar{z}_n \end{bmatrix} := \mathbb{1}\langle D\rangle^n Z, \quad Z = \begin{bmatrix} z \\ \bar{z} \end{bmatrix}. \tag{7-40}$$

Lemma 7.8. *Fix $n := n(d) \gg 1$ large enough and recall (7-39). One has that the function z_n defined in (7-40) solves the problem*

$$\partial_t z_n = -\mathrm{i}\,\mathrm{Op}^{\mathrm{BW}}\big((1+\tilde{a}_2^{+}(x,\xi))\Lambda_{\mathrm{KG}}(\xi)\big)z_n + \langle D\rangle^n X^{+,\mathrm{res}}_{\mathtt{H}^{(4)}_{\mathrm{KG}}}(Z) + B_n^{(1)}(Z) + B_n^{(2)}(Z) + R_{4,n}(U), \tag{7-41}$$

where the resonant vector field $X^{+,\mathrm{res}}_{\mathtt{H}^{(4)}_{\mathrm{KG}}}$ is defined as in Definition 3.9 (see also Remark 3.11), the cubic terms $B_n^{(i)}$, $i = 1, 2$, have the form

$$\widehat{B_n^{(1)}(Z)}(\xi) = \frac{1}{(2\pi)^d} \sum_{\substack{\sigma_1,\sigma_2\in\{\pm\} \\ \eta,\zeta\in\mathbb{Z}^d}} b_1^{\sigma_1,\sigma_2}(\xi,\eta,\zeta)\widehat{z^{\sigma_1}}(\xi-\eta-\zeta)\widehat{z^{\sigma_2}}(\eta)\hat{z}_n(\zeta), \tag{7-42}$$

$$\widehat{B_n^{(2)}(Z)}(\xi) = \frac{1}{(2\pi)^d} \sum_{\substack{\sigma_1,\sigma_2,\sigma_3\in\{\pm\} \\ \eta,\zeta\in\mathbb{Z}^d}} b_{2,n}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)\widehat{z^{\sigma_1}}(\xi-\eta-\zeta)\widehat{z^{\sigma_2}}(\eta)\widehat{z^{\sigma_3}}(\zeta), \tag{7-43}$$

with (recall Remark 4.5)

$$b_1^{\sigma_1,\sigma_2}(\xi,\eta,\zeta) := -\mathrm{i}a_0^{\sigma_1,\sigma_2}\Big(\xi-\zeta,\eta,\frac{\xi+\zeta}{2}\Big)\chi_\epsilon\Big(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)1_{\mathscr{R}^c}(\xi,\eta,\zeta), \tag{7-44}$$

$$|b_{2,n}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)| \lesssim \frac{\langle\xi\rangle^n \max_2\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}^\mu}{\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}}1_{\mathscr{R}^c}(\xi,\eta,\zeta) \tag{7-45}$$

for some $\mu > 1$. The remainder satisfies

$$\|R_{4,n}(U)\|_{L^2} \lesssim \|u\|^4_{H^n}. \tag{7-46}$$

Proof. Recalling the definition of resonant vector fields in Definition 3.9 we set

$$X^{+,\perp}_{\mathtt{H}^{(4)}_{\mathrm{KG}}}(Z) := X^{+}_{\mathtt{H}^{(4)}_{\mathrm{KG}}}(Z) - X^{+,\mathrm{res}}_{\mathtt{H}^{(4)}_{\mathrm{KG}}}(Z), \tag{7-47}$$

which represents the nonresonant terms in the cubic vector field of (7-39). By differentiating in t (7-40) and using (7-39) we get

$$\begin{aligned} \partial_t z_n = -\mathrm{i}\,\mathrm{Op}^{\mathrm{BW}}&\big((1+\tilde{a}_2^{+}(x,\xi))\Lambda_{\mathrm{KG}}(\xi)\big)z_n + \langle D\rangle^n X^{+,\mathrm{res}}_{\mathtt{H}^{(4)}_{\mathrm{KG}}}(Z) \\ &-\mathrm{i}[\langle D\rangle^n, \mathrm{Op}^{\mathrm{BW}}((1+\tilde{a}_2^{+}(x,\xi))\Lambda_{\mathrm{KG}}(\xi))]z &\text{(7-48)} \\ &+\langle D\rangle^n X^{+,\perp}_{\mathtt{H}^{(4)}_{\mathrm{KG}}}(Z) &\text{(7-49)} \\ &+\langle D\rangle^n R_4^{(2,+)}(u). &\text{(7-50)} \end{aligned}$$

We analyze each summand above separately. By estimate (6-50) we deduce $\|(7\text{-}50)\|_{L^2} \lesssim \|u\|^4_{H^n}$. Let us now consider the commutator term in (7-48). By Lemma 3.1, Proposition 3.2 and the estimate on the seminorm of the symbol $\tilde{a}_2^{+}(x,\xi)$ in (6-38), we obtain that $\|(7\text{-}48)\|_{L^2} \lesssim \|u\|^3_{H^n}\|z\|_{H^n} \lesssim \|u\|^4_{H^n}$; we have used also (6-47). The term in (7-49) is the most delicate. By (6-49) and (7-47) (recall also Remark 4.5 and (3-6))

$$\langle D\rangle^n X^{+,\perp}_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z) = B_n^{(1)}(Z) + \mathscr{C}_1 + \mathscr{C}_2, \tag{7-51}$$

with $B_n^{(1)}(Z)$ as in (7-42) and coefficients as in (7-44), the term $\mathscr{C}_1$ has the form

$$\begin{gathered}\widehat{\mathscr{C}}_1(\xi) = \frac{1}{(2\pi)^d} \sum_{\substack{\sigma_1,\sigma_2\in\{\pm\}\\ \eta,\zeta\in\mathbb{Z}^d}} \mathtt{c}_1^{\sigma_1,\sigma_2}(\xi,\eta,\zeta)\widehat{z^{\sigma_1}}(\xi-\eta-\zeta)\widehat{z^{\sigma_2}}(\eta)\hat{z}(\zeta),\\ \mathtt{c}_1^{\sigma_1,\sigma_2}(\xi,\eta,\zeta) = -\mathrm{i}a_0^{\sigma_1,\sigma_2}\Big(\xi-\zeta,\eta,\frac{\xi+\zeta}{2}\Big)\chi_\epsilon\Big(\frac{|\xi-\zeta|}{\langle\xi+\zeta\rangle}\Big)[\langle\xi\rangle^n - \langle\zeta\rangle^n]1_{\mathscr{R}^c}(\xi,\eta,\zeta),\end{gathered} \tag{7-52}$$

and the term $\mathscr{C}_2$ has the form (7-43) with coefficients (see (5-30))

$$\mathtt{c}_2^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta) := \mathtt{q}^{\sigma_1,\sigma_2,\sigma_3}_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(\xi,\eta,\zeta)\langle\xi\rangle^n 1_{\mathscr{R}^c}(\xi,\eta,\zeta). \tag{7-53}$$

In order to conclude the proof we need to show that the coefficients in (7-52), (7-53) satisfy the bound (7-45). This is true for the coefficients in (7-53) thanks to the bound (5-31). Moreover notice that

$$|\langle\xi\rangle^n - \langle\zeta\rangle^n| \lesssim |\xi-\zeta| \max\{\langle\xi\rangle,\langle\zeta\rangle\}^{n-1}.$$

Then the coefficients in (7-52) satisfy (7-45) by using Remark 4.5 and Lemma 3.8. □

Remark 7.9. In view of Remarks 4.6, 4.8, 6.14 if (KG) is semilinear then the symbol $\tilde{a}_2^+$ in (7-41) is equal to zero and the coefficients $b_{2,n}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)$ in (7-43) satisfy the bound (7-45) with the better denominator $\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^2$.

In view of Lemma 7.8 we deduce the following.

Proposition 7.10. *Let $\psi(t,x)$ be a solution of (KG) satisfying Hypothesis 6.8 and consider the function z_n in (7-40) (see also (6-46), (6-42)). Then, setting $s=n=n(d)\gg 1$ we have $\|z_n\|_{L^2}\sim\|\psi\|_{H^{s+1/2}}+\|\dot{\psi}\|_{H^{s-1/2}}$ and*

$$\partial_t\|z_n(t)\|^2_{L^2} = \mathscr{B}(t) + \mathscr{B}_{>4}(t), \qquad t\in[0,T), \tag{7-54}$$

where:

- *The term $\mathscr{B}(t)$ has the form*

$$\mathscr{B}(t) = \sum_{\substack{\sigma_1,\sigma_2,\sigma_3\in\{\pm\}\\ \xi,\eta,\zeta\in\mathbb{Z}^d}} \langle\xi\rangle^{2n}\mathtt{b}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)\widehat{z^{\sigma_1}}(\xi-\eta-\zeta)\widehat{z^{\sigma_2}}(\eta)\widehat{z^{\sigma_3}}(\zeta)\hat{\bar{z}}(-\xi), \tag{7-55}$$

where $\mathtt{b}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)\in\mathbb{C}$ satisfy, for $\xi,\eta,\zeta\in\mathbb{Z}^d$,

$$|\mathtt{b}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)| \lesssim \frac{\max_2\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}^\mu}{\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}} 1_{\mathscr{R}^c}(\xi,\eta,\zeta) \tag{7-56}$$

for some $\mu>1$.

- *The term $\mathscr{B}_{>5}(t)$ satisfies*

$$|\mathscr{B}_{>4}(t)| \lesssim \|u\|^5_{H^s}, \qquad t\in[0,T). \tag{7-57}$$

Proof. The equivalence between $\|z_n\|_{L^2}$ and $\|\psi\|_{H^{s+1/2}}+\|\dot\psi\|_{H^{s-1/2}}$ follows by Remarks 6.12 and 6.9. By using (7-41) we get

$$\tfrac12\partial_t\|z_n(t)\|_{L^2}^2=\mathrm{Re}\big(-\mathrm{i}\,\mathrm{Op}^{\mathrm{BW}}((1+\tilde a_2^+(x,\xi))\Lambda_{\mathrm{KG}}(\xi))z_n,z_n\big)_{L^2} \tag{7-58}$$
$$+\mathrm{Re}(\langle D\rangle^n X^{+,\mathrm{res}}_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z),z_n)_{L^2} \tag{7-59}$$
$$+\mathrm{Re}(B_n^{(1)}(Z),z_n)_{L^2} \tag{7-60}$$
$$+\mathrm{Re}(B_n^{(2)}(Z),z_n)_{L^2} \tag{7-61}$$
$$+\mathrm{Re}(R_{4,n}(Z),z_n)_{L^2}. \tag{7-62}$$

By (6-34), (6-33) and (4-24) we have that the symbol $(1+\tilde a_2^+(x,\xi))\Lambda_{\mathrm{KG}}(\xi)$ is real-valued. Hence the operator $\mathrm{i}\,\mathrm{Op}^{\mathrm{BW}}((1+\tilde a_2^+(x,\xi))\Lambda_{\mathrm{KG}}(\xi))$ is skew-self-adjoint. We deduce (7-58)$\equiv 0$. By Lemma 3.10 (see also Remark 3.11) we have (7-59) $\equiv 0$. We also have (7-60) $\equiv 0$; to see this one can reason as done in the proof of Lemma 7.4, by using Remark 4.5, in particular (4-43). By formula (7-43) and estimates (7-45) we have that the term in (7-61) has the form (7-55) with coefficients satisfying (7-56). By the Cauchy–Schwarz inequality and estimate (7-46) we get that the term in (7-62) satisfies the bound (7-57). □

Remark 7.11. In view of Remark 7.9, if (KG) is semilinear, then the coefficients $\mathtt{b}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)$ of the energy in (7-55) satisfy the bound (7-56) with the better denominator $\max_1\{\langle\xi-\eta-\zeta\rangle,\langle\eta\rangle,\langle\zeta\rangle\}^2$.

7B2. *Estimates of nonresonant terms.* In Proposition 7.10 we provided a precise structure of the term $\mathscr{B}(t)$ of degree 4 in (7-54). In this section we show that, actually, $\mathscr{B}(t)$ satisfies better bounds with respect to a general quartic multilinear map by using that it is *nonresonant*. We state the main result of this section.

Proposition 7.12. *Let $N>0$ and let β be as in Proposition 2.2. Then there is $s_0=s_0(N_0)$, where $N_0>0$ is given by Proposition 2.2, such that, if Hypothesis 6.8 holds with $s\ge s_0$, one has*

$$\left|\int_0^t\mathscr{B}(\sigma)\,d\sigma\right|\lesssim\|u\|^6_{L^\infty H^s}TN^{\beta-1}+\|u\|^7_{L^\infty H^s}N^\beta T+\|u\|^4_{L^\infty H^s}TN^{-1}+N^{\beta-1}\|u\|^4_{L^\infty H^s}, \tag{7-63}$$

where $\mathscr{B}(t)$ is in (7-55).

We first introduce some notation. Let $\vec\sigma:=(\sigma_1,\sigma_2,\sigma_3)\in\{\pm\}^3$ and consider the trilinear maps

$$\mathscr{B}_i^{\vec\sigma}=\mathscr{B}_i^{\vec\sigma}[z_1,z_2,z_3],\quad \widehat{\mathscr{B}_i^{\vec\sigma}}(\xi)=\frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d}\langle\xi\rangle^s\mathtt{b}_i^{\vec\sigma}(\xi,\eta,\zeta)\widehat{z^{\sigma_1}}_1(\xi-\eta-\zeta)\widehat{z^{\sigma_2}}_2(\eta)\widehat{z^{\sigma_3}}_3(\zeta), \tag{7-64}$$

$$\mathscr{T}_<^{\vec\sigma}=\mathscr{T}_<^{\vec\sigma}[z_1,z_2,z_3],\quad \widehat{\mathscr{T}_<^{\vec\sigma}}(\xi)=\frac{1}{(2\pi)^d}\sum_{\eta,\zeta\in\mathbb{Z}^d}\langle\xi\rangle^s\mathtt{t}_<^{\vec\sigma}(\xi,\eta,\zeta)\hat z_1^{\sigma_1}(\xi-\eta-\zeta)\hat z_2^{\sigma_2}(\eta)\widehat{z^{\sigma_3}}_3(\zeta), \tag{7-65}$$

where

$$\mathtt{b}_1^{\vec\sigma}(\xi,\eta,\zeta)=\mathtt{b}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)1_{\{\max\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}\le N\}}, \tag{7-66}$$
$$\mathtt{b}_2^{\vec\sigma}(\xi,\eta,\zeta)=\mathtt{b}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)1_{\{\max\{|\xi-\eta-\zeta|,|\eta|,|\zeta|\}> N\}}, \tag{7-67}$$
$$\mathtt{t}_<^{\vec\sigma}(\xi,\eta,\zeta)=\frac{-1}{\mathrm{i}\omega^{\vec\sigma}_{\mathrm{KG}}(\xi,\eta,\zeta)}\mathtt{b}_1^{\vec\sigma}(\xi,\eta,\zeta), \tag{7-68}$$

where $\mathtt{b}^{\sigma_1,\sigma_2,\sigma_3}(\xi,\eta,\zeta)$ are the coefficients in (7-55), and $\omega_{\mathrm{KG}}^{\vec{\sigma}}$ is the phase in (2-4). We remark that if $(\xi,\eta,\zeta)\in\mathscr{R}$ (see Definition 3.9) then the coefficients $\mathtt{b}(\xi,\eta,\zeta)$ are equal to zero (see (7-55), (7-43), (7-45)). Therefore, since $\omega_{\mathrm{KG}}^{\vec{\sigma}}$ is nonresonant (see Proposition 2.2), the coefficients in (7-68) are well-defined. We now state an abstract result on the trilinear maps introduced in (7-64)–(7-65).

Lemma 7.13. *Let $\mu>1$ as in (7-56). One has that, for $s>\frac{1}{2}d+\mu$,*

$$\|\mathscr{B}_2^{\vec{\sigma}}[z_1,z_2,z_3]\|_{L^2}\lesssim N^{-1}\sum_{i=1}^{3}\|z_i\|_{H^s}\prod_{i\neq k}\|z_k\|_{H^{d/2+\mu+\epsilon}} \tag{7-69}$$

for any $\vec{\sigma}\in\{\pm\}^3$ and any $\epsilon>0$. There is $s_0(N_0)>0$ ($N_0>0$ given by Proposition 2.2) such that for $s\geq s_0(N_0)$ one has

$$\|\mathscr{T}_{<}[z_1,z_2,z_3]\|_{H^p}\lesssim N^{\beta}\sum_{i=1}^{3}\|z_i\|_{H^{s+p-1}}\prod_{i\neq k}\|z_k\|_{H^{s_0}},\qquad p\in\mathbb{N}, \tag{7-70}$$

$$\|\mathscr{T}_{<}[z_1,z_2,z_3]\|_{L^2}\lesssim N^{\beta-1}\sum_{i=1}^{3}\|z_i\|_{H^s}\prod_{i\neq k}\|z_k\|_{H^{s_0}}, \tag{7-71}$$

where β is defined in Proposition 2.2.

Proof. The proof is similar to that of Lemma 7.7. One has to use Proposition 2.2 instead of Proposition 2.1 to estimate the small divisors. □

Remark 7.14. In view of Remark 7.11, if (KG) is semilinear we may improve (7-69) and (7-71) with

$$\|\mathscr{B}_2^{\vec{\sigma}}[z_1,z_2,z_3]\|_{L^2}\lesssim N^{-2}\sum_{i=1}^{3}\|z_i\|_{H^s}\prod_{i\neq k}\|z_k\|_{H^{d/2+\mu+\epsilon}},$$

$$\|\mathscr{T}_{<}[z_1,z_2,z_3]\|_{L^2}\lesssim N^{\beta-2}\sum_{i=1}^{3}\|z_i\|_{H^s}\prod_{i\neq k}\|z_k\|_{H^{s_0}}.$$

We are now in a position to prove Proposition 7.12.

Proof of Proposition 7.12. By (7-64), (7-66), (7-67), and recalling the definition of $\mathscr{B}$ in (7-55), we can write

$$\int_0^t\mathscr{B}(\tau)\,d\tau=\sum_{\vec{\sigma}\in\{\pm\}^3}\int_0^t(\mathscr{B}_1^{\vec{\sigma}}[z,z,z],\langle D\rangle^s z)_{L^2}\,d\tau+\sum_{\vec{\sigma}\in\{\pm\}^3}\int_0^t(\mathscr{B}_2^{\vec{\sigma}}[z,z,z],\langle D\rangle^s z)_{L^2}\,d\tau. \tag{7-72}$$

By Lemma 7.13 we have

$$\left|\int_0^t(\mathscr{B}_2^{\vec{\sigma}}[z,z,z],\langle D\rangle^s z)_{L^2}\,d\sigma\right|\overset{(7\text{-}69)}{\lesssim}N^{-1}\int_0^t\|z\|_{H^s}^4\,d\tau\overset{(6\text{-}47)}{\lesssim}N^{-1}\int_0^t\|u\|_{H^s}^4\,d\tau. \tag{7-73}$$

Consider now the first summand in the right-hand side of (7-72). Integrating by parts as done in the proof of Proposition 7.6 we have

$$\begin{aligned}\int_0^t (\mathscr{B}_1^{\vec{\sigma}}[z,\bar{z},z], \langle D\rangle^s z)_{L^2}\,d\tau = \int_0^t (\mathscr{T}_<^{\vec{\sigma}}[z,z,z], \langle D\rangle^s(\partial_t + \mathrm{i}\Lambda_{\mathrm{KG}})z)_{L^2}\,d\tau \\ + \int_0^t (\mathscr{T}_<^{\vec{\sigma}}[(\partial_t+\mathrm{i}\Lambda_{\mathrm{KG}})z,z,z], \langle D\rangle^s z)_{L^2}\,d\tau \\ + \int_0^t (\mathscr{T}_<^{\vec{\sigma}}[z,\bar{z},(\partial_t+\mathrm{i}\Lambda_{\mathrm{KG}})z], \langle D\rangle^s z)_{L^2}\,d\tau \\ + \int_0^t (\mathscr{T}_<^{\vec{\sigma}}[z,(\partial_t+\mathrm{i}\Lambda_{\mathrm{KG}})z,z], \langle D\rangle^s z)_{L^2}\,d\tau + R,\end{aligned} \tag{7-74}$$

where

$$R = (\mathscr{T}_<^{\vec{\sigma}}[z(t),z(t),z(t)], \langle D\rangle^s z(t))_{L^2} - (\mathscr{T}_<^{\vec{\sigma}}[z(0),z(0),z(0)], \langle D\rangle^s z(0))_{L^2}.$$

The remainder R above is bounded from above by $N^\beta\|u\|^4_{L^\infty H^s}$ using Cauchy–Schwarz and (7-70). Let us now consider the first summand in the right-hand side of (7-74). Using that the operator $\langle D\rangle$ is self-adjoint and recalling (7-39) we have

$$\begin{aligned}(\mathscr{T}_<^{\vec{\sigma}}[z,z,z], \langle D\rangle^s(\partial_t+\mathrm{i}\Lambda_{\mathrm{KG}})z)_{L^2} &= (\langle D\rangle\mathscr{T}_<^{\vec{\sigma}}[z,z,z], \langle D\rangle^{s-1}(\partial_t+\mathrm{i}\Lambda_{\mathrm{KG}})z)_{L^2} \\ &= (\langle D\rangle\mathscr{T}_<^{\vec{\sigma}}[z,z,z], \langle D\rangle^{s-1}\mathrm{Op}^{\mathrm{BW}}(-\mathrm{i}\tilde{a}_2^+(x,\xi)\Lambda_{\mathrm{KG}}(\xi))z)_{L^2} \qquad (7\text{-}75) \\ &\quad + (\mathscr{T}_<^{\vec{\sigma}}[z,z,z], \langle D\rangle^s(X^+_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z) + R_4^{(2,+)}(u)))_{L^2}. \qquad (7\text{-}76)\end{aligned}$$

By the Cauchy–Schwarz inequality, estimate (7-70) with $p=1$, estimate (6-38) on the seminorm of the symbol $\tilde{a}_2^+(x,\xi)$, Lemma 3.1 and the equivalence (6-47), we get $|(7\text{-}75)| \lesssim \|u\|^7_{H^s}N^\beta$. Consider the term in (7-76). First of all notice that, by (4-31) and Lemma 3.1, and by (5-31) and Lemma 3.7, the field $X_{\mathrm{H}^{(4)}_{\mathrm{KG}}}(Z)$ in (6-49) satisfies the same estimates (4-32) as the field $X_{\mathscr{H}^{(4)}_{\mathrm{KG}}}$. Therefore, using (7-71) and (6-50), we obtain $|(7\text{-}76)| \lesssim \|u\|^6_{H^s}N^{\beta-1}$. Using that (see Hypothesis 6.8) $\|u\|_{H^s} \ll 1$, we conclude that the first summand in the right-hand side of (7-74) is bounded from above by $N^\beta\int_0^t\|u(\tau)\|^7\,d\tau + N^{\beta-1}\int_0^t\|u(\tau)\|^6\,d\tau$. The other terms in (7-74) are estimated in a similar way. We eventually obtain (7-63). □

Remark 7.15. In view of Remarks 4.6, 4.8, 6.14, 7.9, 7.11 and 7.14, if (KG) is semilinear we have the better (with respect to (7-63)) estimate

$$\left|\int_0^t \mathscr{B}(\sigma)\,d\sigma\right| \lesssim \|u\|^6_{L^\infty H^s}TN^{\beta-2} + \|u\|^4_{L^\infty H^s}TN^{-2} + N^{\beta-2}\|u\|^4_{L^\infty H^s}. \tag{7-77}$$

8. Proof of the main results

In this section we conclude the proof of our main theorems.

Proof of Theorem 1. Consider (NLS) and let u_0 be as in the statement of Theorem 1. By the result in [Feola and Iandoli 2022] we have that there is $T>0$ and a unique solution $u(t,x)$ of (NLS) with $V\equiv 0$

such that Hypothesis 6.1 is satisfied. To recover the result when $V \neq 0$ one can argue as done in [Feola and Iandoli 2019]. Consider a potential V as in (1-5), with $\vec{x} \in \mathscr{O} \setminus \mathscr{N}$, where $\mathscr{N}$ is the zero measure set given in Proposition 2.1. We claim that we have the following a priori estimate: Fix any $0 < N$. Then for any $t \in [0, T)$, with T as in Hypothesis 6.1, one has

$$\|u(t)\|_{H^s}^2 \leq 2\|u_0\|_{H^s}^2 + C(\|u\|_{L^\infty H^s}^{10} TN + \|u\|_{L^\infty H^s}^6 T + \|u\|_{L^\infty H^s}^4 TN^{-1} + \|u\|_{L^\infty H^s}^4) \tag{8-1}$$

for some $C > 0$ depending on s. To prove the claim we reason as follows. By Proposition 4.2 we have that (NLS) is equivalent to the system (4-12). By Propositions 6.4, 6.6 and Lemma 7.3 we can construct a function z_n with $2n = s$ such that if $u(t, x)$ solves the (NLS) then z_n solves (7-8). Moreover by Proposition 7.5 we have the equivalence (7-18), and we deduce

$$\|u(t)\|_{H^s}^2 \leq 2^{1/2}\|z_n(t)\|_{L^2}^2 \leq 2\|u_0\|_{H^s}^2 + 2\left|\int_0^t \mathscr{B}(\sigma)\, d\sigma\right| + 2\left|\int_0^t \mathscr{B}_{>5}(\sigma)\, d\sigma\right|. \tag{8-2}$$

Propositions 7.5 and 7.6 apply; therefore, by (7-26) and (7-21), we obtain (8-1). The thesis of Theorem 1 follows from the following lemma.

Lemma 8.1 (main bootstrap). *Let $u(t, x)$ be a solution of (NLS) with $t \in [0, T)$ and initial condition $u_0 \in H^s(\mathbb{T}^d; \mathbb{C})$. Then, for $s \gg 1$ large enough, there exist $\epsilon_0, c_0 > 0$ such that, for any $0 < \epsilon \leq \epsilon_0$, if*

$$\|u_0\|_{H^s} \leq \tfrac{1}{4}\epsilon, \qquad \sup_{t\in[0,T)} \|u(t)\|_{H^s} \leq \epsilon, \qquad T \leq c_0\epsilon^{-4}, \tag{8-3}$$

then we have the improved bound $\sup_{t\in[0,T)} \|u(t)\|_{H^s} \leq \frac{1}{2}\epsilon$.

Proof. For ϵ small enough the bound (8-1) holds true, and we fix $N := \epsilon^{-3}$. Therefore, there is $C = C(s) > 0$ such that, for any $t \in [0, T)$,

$$\begin{aligned}
\|u(t)\|_{H^s}^2 &\leq 2\|u_0\|_{H^s}^2 + C(\|u\|_{L^\infty H^s}^4 + \|u\|_{L^\infty H^s}^{10} T\epsilon^{-3} + \|u\|_{L^\infty H^s}^6 T + \|u\|_{L^\infty H^s}^4 T\epsilon^3) \\
&\leq \tfrac{1}{8}\epsilon^2 + C(\epsilon^4 + 2\epsilon^7 T + \epsilon^6 T) \qquad \text{(by (8-3))} \\
&\leq \tfrac{1}{4}\epsilon^2\big(\tfrac{1}{2} + 4C(\epsilon^4 + 2\epsilon c_0 + c_0)\big) \leq \tfrac{1}{4}\epsilon^2,
\end{aligned} \tag{8-4}$$

where in the last inequality we have chosen c_0 and ϵ sufficiently small. This implies the thesis. □

Proof of Theorem 2. One has to follow almost word by word the proof of Theorem 1. The only difference relies on the estimates on the small divisors, which in this case are given by Proposition 2.1(ii).

Proof of Theorem 3. Consider (KG) and let (ψ_0, ψ_1) be as in the statement of Theorem 3. Let $\psi(t, x)$ be a solution of (KG) satisfying the condition in Hypothesis 6.8. By Proposition 4.7, recall (3-77), the function

$$U := \begin{bmatrix} u \\ \bar{u} \end{bmatrix}$$

solves (4-12) with initial condition

$$u_0 = \frac{1}{\sqrt{2}}(\Lambda_{\mathrm{KG}}^{1/2}\psi_0 + \mathrm{i}\Lambda_{\mathrm{KG}}^{-1/2}\psi_1).$$

Moreover, by Hypothesis 6.8 and Remark 6.9 one has $\|u_0\|_{H^s} \le \frac{1}{16}\epsilon$. By Remark 6.9, in order to get (1-8), we have to show that the bound $\sup_{t\in[0,T)} \|u\|_{H^s} \le \frac{1}{4}\epsilon$ holds for time $T \gtrsim \epsilon^{-3^-}$ if $d=2$ and $T \gtrsim \epsilon^{-8/3^-}$ if $d \ge 3$. Fix β as in Proposition 2.2 and let $m \in \mathscr{C}_\beta$. By Propositions 6.11, 6.13 and Lemma 7.8 we can construct a function z_n with $n=s$ such that if $\psi(t,x)$ solves (KG) then z_n solves (7-41). By Proposition 7.10 and Remark 6.12 we get

$$\|u(t)\|_{H^s}^2 \le 2^{1/2}\|z_n(t)\|_{L^2}^2 \le 2\|u_0\|_{H^s}^2 + 2\left|\int_0^t \mathscr{B}(\sigma)\,d\sigma\right| + 2\left|\int_0^t \mathscr{B}_{>5}(\sigma)\,d\sigma\right|. \tag{8-5}$$

Propositions 7.10 and 7.12 apply, therefore, by (7-63) and (7-57), we obtain the following a priori estimate: Fix any $0<N$. Then for any $t\in[0,T)$, with T as in Hypothesis 6.8, one has

$$\|u(t)\|_{H^s}^2 \le 2\|u_0\|_{H^s}^2 + C\big(\|u\|_{L^\infty H^s}^6 T N^{\beta-1} + \|u\|_{L^\infty H^s}^7 T N^{\beta} + \|u\|_{L^\infty H^s}^6 T + \|u\|_{L^\infty H^s}^4 T N^{-1} + N^{\beta-1}\|u\|_{L^\infty H^s}^4\big) \tag{8-6}$$

for some $C>0$ depending on s. The thesis of Theorem 3 follows from the lemma below.

Lemma 8.2 (main bootstrap). *Let $u(t,x)$ be a solution of (4-44) with $t\in[0,T)$ and initial condition $u_0\in H^s(\mathbb{T}^d;\mathbb{C})$. Define $\mathtt{a}=3$ if $d=2$ and $\mathtt{a}=\frac{8}{3}$ if $d\ge 3$. Then, for $s\gg 1$ large enough and any $\delta>0$, there exists $\epsilon_0=\epsilon_0(d,s,m,\delta)>0$ such that, for any $0<\epsilon\le\epsilon_0$, if*

$$\|u_0\|_{H^s} \le \tfrac{1}{16}\epsilon, \qquad \sup_{t\in[0,T)} \|u(t)\|_{H^s} \le \tfrac{1}{4}\epsilon, \qquad T \le \epsilon^{-\mathtt{a}+\delta}, \tag{8-7}$$

then we have the improved bound $\sup_{t\in[0,T)} \|u(t)\|_{H^s} \le \frac{1}{8}\epsilon$.

Proof. We start with $d\ge 3$. For ϵ small enough the bound (8-6) holds true. Let $\delta>0$ and $0<\sigma\ll\delta$. Define

$$\beta := 3+\sigma, \qquad N := \epsilon^{-2/(3+\sigma)}. \tag{8-8}$$

By (8-6), (8-7), (8-8), there is $C=C(s)>0$ such that, for any $t\in[0,T)$,

$$\|u(t)\|_{H^s}^2 \le 2\tfrac{1}{16^2}\epsilon^2 + C\epsilon^2\epsilon^{2/(3+\sigma)} + 2CT\epsilon^2(\epsilon^3+\epsilon^{2+2/(3+\sigma)}) \le \tfrac{1}{64}\epsilon^2, \tag{8-9}$$

where in the last inequality we have chosen ϵ sufficiently small and we used the choice of T in (8-7) and that $\sigma\ll\delta$. This implies the thesis for $d\ge 3$. In the case $d=2$ the proof is similar setting $\beta=2+\sigma$ and $N=\epsilon^{-2/(2+\sigma)}$. □

Proof of Theorem 4. Using Remarks 4.6, 4.8, 6.14, 7.9, 7.11, 7.14, 7.15 one deduces the result by reasoning as in the proof of Theorem 3 and using in particular the estimate (7-77).

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