ANALYSIS & PDE

Volume 16 No. 5 2023

PRITAM GANGULY, RAMESH MANNA AND SUNDARAM THANGAVELU

AN EXTENSION PROBLEM, TRACE HARDY AND HARDY'S INEQUALITIES FOR THE ORNSTEIN–UHLENBECK OPERATOR

msp

ANALYSIS AND PDE
Vol. 16 (2023), No. 5, pp. 1205–1244
DOI: 10.2140/apde.2023.16.1205

AN EXTENSION PROBLEM, TRACE HARDY AND HARDY'S INEQUALITIES FOR THE ORNSTEIN–UHLENBECK OPERATOR

PRITAM GANGULY, RAMESH MANNA AND SUNDARAM THANGAVELU

We study an extension problem for the Ornstein–Uhlenbeck operator $L = -\Delta + 2x \cdot \nabla + n$, and we obtain various characterisations of the solution of the same. We use a particular solution of that extension problem to prove a trace Hardy inequality for L from which Hardy's inequality for fractional powers of L is obtained. We also prove an isometry property of the solution operator associated to the extension problem. Moreover, new $L^p - L^q$ estimates are obtained for the fractional powers of the Hermite operator.

1. Introduction and the main results

It is said that analysts are obsessed with inequalities. The usefulness of various weighted and unweighted inequalities in applications to problems in differential geometry, quantum mechanics, partial differential equations, etc., have made this a very attractive area of research. Hardy's inequality is one such inequality which finds its origin in an old paper of G. H. Hardy [1919] written more than a hundred years ago; see also [Hardy 1920]. In recent years, this has been intensively studied in different settings and various contexts. For a historical review of Hardy's inequality, we refer the reader to [Kufner et al. 2007].

We begin by recalling the classical Hardy's inequality which states that, given $f \in C_0^\infty(\mathbb{R}^n)$,

$$\frac{1}{4}(n-2)^2 \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^2}\, dx \le \int_{\mathbb{R}^n} |\nabla f(x)|^2\, dx, \quad n \ge 3,$$

where ∇ denotes the gradient in $\mathbb{R}^n$. This can be rephrased as follows in terms of the Euclidean Laplacian $\Delta := \sum_{j=1}^n \partial^2/\partial x_j^2$:

$$\frac{1}{4}(n-2)^2 \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^2}\, dx \le \langle (-\Delta) f, f \rangle,$$

MSC2020: primary 26D10, 35J15; secondary 26A33, 33C45, 35A23, 43A80.
Keywords: extension problem, trace Hardy inequality, Hardy's inequality, Ornstein–Uhlenbeck operator.

which has been generalised to fractional powers of the Laplacian. In fact, for $0 < s < \frac{1}{2}n$ and $f \in C_0^\infty(\mathbb{R}^n)$, we have

$$4^s \frac{\Gamma\big(\frac{1}{4}(n+2s)\big)^2}{\Gamma\big(\frac{1}{4}(n-2s)\big)^2} \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^{2s}}\, dx \le \langle (-\Delta)^s f, f \rangle. \tag{1-1}$$

The constant appearing on the left-hand side is known to be sharp (see [Beckner 2012; Yafaev 1999] for instance), but the equality is never achieved. Frank et al. [2008] used a ground state representation to give a new proof of (1-1) when $0 < s < \min\{1, \frac{1}{2}n\}$, improving the previous results. On the other hand, replacing the homogeneous weight $|x|^{-2s}$ by a nonhomogeneous weight we have the following version of Hardy's inequality:

$$4^s \frac{\Gamma\big(\frac{1}{4}(n+2s)\big)}{\Gamma\big(\frac{1}{4}(n-2s)\big)} \rho^{2s} \int_{\mathbb{R}^n} \frac{|f(x)|^2}{(\rho^2+|x|^2)^{2s}}\, dx \le \langle (-\Delta)^s f, f \rangle, \quad \rho > 0, \tag{1-2}$$

where the constant is sharp and the equality is achieved for the functions $(\rho^2 + |x|^2)^{-(n-2s)/2}$. See [Boggarapu et al. 2019, Remark 2.6] for a proof of inequality (1-2). Note that proving such an inequality for fractional powers depends on how one views this type of operator. In fact, there are several ways of obtaining fractional powers of the Laplacian. Caffarelli and Silvestre [2007] first studied an extension problem associated to the Laplacian on $\mathbb{R}^n$ and obtained the fractional power as a mapping which takes Dirichlet data to the Neumann data. Motivated by this work, [Boggarapu et al. 2019] studied the extension problem in a more general setting of sums of squares of vector fields on certain stratified Lie groups. They used a solution of that extension problem to prove a trace Hardy inequality from which Hardy's inequality is obtained. Because of its several interesting features, the study of extension problems for various operators has received considerable attention in recent times, see e.g., [Roncal and Thangavelu 2020b; Stinga and Torrea 2010], etc.

Inspired by [Frank et al. 2015], Roncal and Thangavelu [2020a] considered a modified extension problem for the sub-Laplacian on the H-type groups which gives conformally invariant fractional powers of the sub-Laplacian, and they proved Hardy's inequality for the same. In this regard, we would also like to mention that Garofalo and Tralli [2021] recently used an extension problem for the heat operator associated to the sub-Laplacian on the H-type groups to study the usual and conformal fractional powers of the sub-Laplacian. See also [Garofalo and Tralli 2023] by the same authors in this direction.

Although this fractional Hardy-type inequality has been studied extensively in the setting of Euclidean harmonic analysis, not much has been studied in the framework of Gaussian harmonic analysis. As we know that the role of the Laplacian in Gaussian harmonic analysis is played by the Ornstein–Uhlenbeck operator defined by $\tilde{L} := -\Delta + 2x \cdot \nabla$, it is therefore natural to ask for such a fractional Hardy inequality for this operator. It is also convenient to work with $L := -\Delta + 2x \cdot \nabla + n$ instead of $\tilde{L}$. In fact, from the mathematical point of view, L is very closely related to the Hermite operator; see (1-3) below. Later in this article, this relationship will be discussed and exploited in some of our studies. Because of its various applications in probability theory, stochastic calculus, etc., the study of the Ornstein–Uhlenbeck operator experienced a lot of developments in the last couple of decades. We refer the reader to the book of Urbina-Romero [2019] in this regard.

Our aim in this article is to establish Hardy and trace Hardy inequalities for fractional powers of the Ornstein–Uhlenbeck operator L. Recall that $L = -\Delta + 2x \cdot \nabla + n$ can be defined on the Gaussian L^2 space: $L^2(\gamma) = L^2(\mathbb{R}^n, \gamma(x)\,dx)$ with $\gamma(x) = \pi^{-n/2} e^{-|x|^2}$ is a positive self-adjoint operator. We observe that $\sum_{j=1}^n \partial_j^* \partial_j = -\Delta + 2x \cdot \nabla$, where $\partial_j = \partial/\partial x_j$ and $\partial_j^* = 2x_j - \partial_j$ is its adjoint on $L^2(\gamma)$. The relation between L and the Hermite operator $H = -\Delta + |x|^2$ is given by

$$M_\gamma L M_\gamma^{-1} = H, \quad \text{where } M_\gamma f(x) = \gamma(x)^{1/2} f(x). \tag{1-3}$$

Hardy's inequality for the fractional powers H^s of the Hermite operator has been studied in [Ciaurri et al. 2018]. Here H^s is defined by spectral theory as

$$H^s = \sum_{k=0}^{\infty} (2k+n)^s P_k,$$

where $(2k+n)$, $k \in \mathbb{N}$ are the eigenvalues of H on $L^2(\mathbb{R}^n)$ and P_k is the orthogonal projection of $L^2(\mathbb{R}^n)$ onto the finite-dimensional eigenspace corresponding to the eigenvalue $(2k+n)$. However, there is another natural candidate for fractional powers of H, and hence of L, which will be treated here.

To motivate the new definition of fractional powers, denoted by H_s, it is better to recall the conformally invariant fractional powers of the sub-Laplacian $\mathcal{L}$ on the Heisenberg group $\mathbb{H}^n$. The connection between $\mathcal{L}$ and H is given by the relation $\pi_\lambda(\mathcal{L} f) = \pi_\lambda(f) H(\lambda)$, where the π_λ are the Schrödinger representations of $\mathbb{H}^n$ and $H(\lambda) = -\Delta + \lambda^2 |x|^2$. The spectral decomposition of $H(\lambda)$ is given by

$$H(\lambda) = \sum_{k=0}^{\infty} (2k+n)|\lambda| P_k(\lambda).$$

The conformally invariant fractional powers of $\mathcal{L}$ are then defined, for $0 < s < (n+1)$, by the relation

$$\pi_\lambda(\mathcal{L}_s f) = \pi_\lambda(f) \sum_{k=0}^{\infty} (2|\lambda|)^s \frac{\Gamma\big(\frac{1}{2}(2k+n+1+s)\big)}{\Gamma\big(\frac{1}{2}(2k+n+1-s)\big)} P_k(\lambda).$$

The operator on the right-hand side which multiplies $\pi_\lambda(f)$ is the alternate candidate for fractional powers of $H(\lambda)$, which we denote by $H(\lambda)_s$. By defining $Q_k = M_\gamma^{-1} P_k M_\gamma$, the spectral decomposition of L becomes $L = \sum_{k=0}^\infty (2k+n) Q_k$, and hence the fractional powers we are interested in are given by

$$L_s f(x) = \sum_{k=0}^{\infty} 2^s \frac{\Gamma\big(\frac{1}{2}(2k+n+1+s)\big)}{\Gamma\big(\frac{1}{2}(2k+n+1-s)\big)} Q_k f(x).$$

Along with L we also consider $U = \frac{1}{2} L$ and the associated fractional powers

$$U_s f(x) = \sum_{k=0}^{\infty} 2^s \frac{\Gamma\big(\frac{1}{2}(k+n/2+1+s)\big)}{\Gamma\big(\frac{1}{2}(k+n/2+1-s)\big)} Q_k f(x).$$

For these operators, we prove the inequality in the following theorem. Letting A be either L or U, we define the trace norm of a suitable function $u(x,\rho)$ on $\mathbb{R}^n \times [0,\infty)$ as

$$a_s(A,u)^2 = \int_0^\infty \int_{\mathbb{R}^n} \big(|\nabla_A u(x,\rho)|^2 + \big(\tfrac{1}{2}n + \tfrac{1}{4}\rho^2\big) u(x,\rho)^2 \big) \rho^{1-2s} \, d\gamma(x)\, d\rho,$$

where

$$\nabla_U u := (2^{-1/2}\partial_1 u, 2^{-1/2}\partial_2 u, \dots, 2^{-1/2}\partial_n u, \partial_\rho u)$$

and ∇_L is defined without the scaling factor $2^{-1/2}$.

Theorem 1.1 (general trace Hardy inequality). *Let $0 < s < 1$, and let A be either L or U. Suppose $\phi \in L^2(\gamma)$ is a real-valued function in the domain of A_s such that $\phi^{-1}A_s\phi$ is locally integrable. Then for any real-valued function $u(x,\rho)$ from the space $C_0^2([0,\infty), C_b^2(\mathbb{R}^n))$ we have*

$$a_s(A,u)^2 \geq 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}\int_{\mathbb{R}^n} u(x,0)^2 \frac{A_s\phi(x)}{\phi(x)}\,d\gamma(x).$$

It would be nice if we could choose a function ϕ so that $A_s\phi$ can be calculated explicitly. It turns out that for $A = U$ we can do that. Indeed, with such a choice of ϕ we can prove an explicit trace Hardy inequality from which Hardy's inequality can be deduced.

Theorem 1.2 (Hardy's inequality for U_s). *Let $0 < s < 1$. Assume that $f \in L^2(\gamma)$ with $U_s f \in L^2(\gamma)$. Then for every $\rho > 0$ we have*

$$\langle U_s f, f\rangle_{L^2(\gamma)} \geq (2\rho)^s \frac{\Gamma\big(\frac{1}{2}(n/2+1+s)\big)}{\Gamma\big(\frac{1}{2}(n/2+1-s)\big)}\int_{\mathbb{R}^n}\frac{f(x)^2}{(\rho+|x|^2)^s} w_s(\rho+|x|^2)\,d\gamma(x)$$

for an explicit $w_s(t) \geq 1$. The inequality is sharp, and equality is attained for

$$f(x) = \sqrt{2}\frac{2^{-(n/2+1-s)/2}}{\Gamma\big(\frac{1}{2}(n/2+1-s)\big)} e^{|x|^2/2}(\rho+|x|^2)^{-(n/2+1-s)/2} K_{(n/2+1-s)/2}(\rho+|x|^2),$$

where K_μ denotes the Macdonald's function.

We remark that since $w_s(t) \geq 1$, we have the following inequality which is slightly weaker:

$$\langle U_s f, f\rangle_{L^2(\gamma)} \geq (2\rho)^s \frac{\Gamma\big(\frac{1}{2}(n/2+1+s)\big)}{\Gamma\big(\frac{1}{2}(n/2+1-s)\big)}\int_{\mathbb{R}^n}\frac{f(x)^2}{(\rho+|x|^2)^s}\,d\gamma(x). \tag{1-4}$$

However, written in this form, we do not yet know if the constant appearing in the above inequality is sharp or not. Observe that the constant we have obtained is analogous to the sharp constant in the Euclidean case; see (1-2). It is worth pointing out that Hardy's inequality for the pure fractional powers U^s can be deduced from Theorem 1.2. Indeed, writing $R_s := U_s U^{-s}$, we see that R_s is a bounded operator on $L^2(\gamma)$ and its operator norm is given by

$$\|R_s\|_{\mathrm{op}} = \sup_{k\geq 0}\big(\tfrac{1}{2}(k+n/2)\big)^{-s}\frac{\Gamma\big(\frac{1}{2}(k+n/2+1+s)\big)}{\Gamma\big(\frac{1}{2}(k+n/2+1-s)\big)}.$$

To estimate this norm we use the fact that $x^{\beta-\alpha}\Gamma(x+\alpha)/\Gamma(x+\beta) \leq (x+\beta)/(x+\alpha)$ for $\alpha > 0$ (see [Roncal and Thangavelu 2016]), which gives the estimate

$$\big(\tfrac{1}{2}(k+n/2)\big)^{-s}\frac{\Gamma\big(\frac{1}{2}(k+n/2+1+s)\big)}{\Gamma\big(\frac{1}{2}(k+n/2+1-s)\big)} \leq \frac{2k+n+2(1-s)}{2k+n+2(1+s)}.$$

The right-hand side of the above inequality being an increasing function of k, we obtain $\|R_s\|_{\mathrm{op}} \leq 1$. Using this, Hardy's inequality for U^s reads as follows:

Corollary 1.3. *Let $0 < s < 1$. Assume that $f \in L^2(\gamma)$ with $U^s f \in L^2(\gamma)$. Then for any $\rho > 0$ we have*

$$\langle U^s f, f\rangle_{L^2(\gamma)} \geq (2\rho)^s \frac{\Gamma\big(\frac{1}{2}(n/2+1+s)\big)}{\Gamma\big(\frac{1}{2}(n/2+1-s)\big)} \int_{\mathbb{R}^n} \frac{f(x)^2}{(\rho+|x|^2)^s}\, d\gamma(x).$$

As a consequence of Hardy's inequality with nonhomogeneous weight, we obtain a Heisenberg-type uncertainty principle for the fractional powers of the Ornstein–Uhlenbeck operator. Indeed, an application of the Cauchy–Schwarz inequality yields

$$\int_{\mathbb{R}^n} |f(x)|^2\, d\gamma(x) \leq \left(\int_{\mathbb{R}^n} |f(x)|^2(\rho+|x|^2)^s\, d\gamma(x)\right)^{1/2} \left(\int_{\mathbb{R}^n} \frac{f(x)^2}{(\rho+|x|^2)^s}\, d\gamma(x)\right)^{1/2},$$

which along with Theorem 1.2 gives the following:

Corollary 1.4. *For any $f \in L^2(\gamma)$ with $U_s f \in L^2(\gamma)$, we have*

$$\left(\int_{\mathbb{R}^n} |f(x)|^2(\rho+|x|^2)^s\, d\gamma(x)\right)\langle U_s f, f\rangle_{L^2(\gamma)} \geq (2\rho)^s \frac{\Gamma\big(\frac{1}{2}(n/2+1+s)\big)}{\Gamma\big(\frac{1}{2}(n/2+1-s)\big)} \left(\int_{\mathbb{R}^n} |f(x)|^2\, d\gamma(x)\right)^2.$$

We must mention that one can use the L^2-boundedness of $U_s L^{-s}$ along with the inequality for U_s to derive an inequality for L^s. Indeed, the operator norm of $\mathfrak{R}_s := U_s L^{-s}$ is given by

$$\|\mathfrak{R}_s\|_{\mathrm{op}} = \sup_{k\geq 0} 2^s(2k+n)^{-s} \frac{\Gamma\big(\frac{1}{2}(k+n/2+1+s)\big)}{\Gamma\big(\frac{1}{2}(k+n/2+1-s)\big)},$$

which can be estimated as above to get $\|\mathfrak{R}_s\|_{\mathrm{op}} \leq 2^{-s}$. The fact that $\|\mathfrak{R}_s\|_{\mathrm{op}}\langle L^s f, f\rangle_{L^2(\gamma)} \geq \langle U_s f, f\rangle_{L^2(\gamma)}$ together with this estimate yields the following:

Theorem 1.5 (Hardy's inequality for L^s). *Let $0 < s < 1$. Assume that $f \in L^2(\gamma)$ with $L^s f \in L^2(\gamma)$. Then for any $\rho > 0$ we have*

$$\langle L^s f, f\rangle_{L^2(\gamma)} \geq (4\rho)^s \frac{\Gamma\big(\frac{1}{2}(n/2+1+s)\big)}{\Gamma\big(\frac{1}{2}(n/2+1-s)\big)} \int_{\mathbb{R}^n} \frac{f(x)^2}{(\rho+|x|^2)^s}\, d\gamma(x).$$

The main ingredient in proving the above mentioned trace Hardy and Hardy's inequality for fractional powers of L is a solution of the extension problem for L:

$$\Big(-L + \partial_\rho^2 + \frac{1-2s}{\rho}\partial_\rho - \frac{1}{4}\rho^2\Big)u(x,\rho) = 0, \quad u(x,0) = f(x). \tag{1-5}$$

As we will see later, a solution of the above partial differential equation will play a very crucial role for our purpose. The second theme of this article is the study of general solutions of the extension problem for L under consideration. In fact, we prove a characterisation of the solution when the initial data is a tempered distribution. In order to state the result we need to introduce some more notations which will be briefly described here. More details can be found in Section 3. We introduce the following two operators. For any distribution f for which $M_\gamma f$ is tempered, we define

$$S_\rho^1 f(x) := \frac{\big(\frac{1}{2}\rho^2\big)^{(s-1)/2}}{\Gamma(s)} \sum_{k=0}^{\infty} \Gamma\big(\tfrac{1}{2}(2k+n+s+1)\big) W_{-(k+n/2),\, s/2}\big(\tfrac{1}{2}\rho^2\big) Q_k f(x),$$

and for any function g for which $Q_k g$ has enough decay as a function of k, we define

$$S_\rho^2 g(x) := \left(\tfrac{1}{2}\rho^2\right)^{(s-1)/2} \sum_{k=0}^{\infty} M_{-(k+n/2),\, s/2}\left(\tfrac{1}{2}\rho^2\right) Q_k g(x),$$

where $W_{-(k+n/2),\, s/2}$ and $M_{-(k+n/2),\, s/2}$ are Whittaker functions.

In view of the asymptotic properties of the Whittaker functions stated in Lemma 3.2, it follows that the series defining $S_\rho^1 f$ converges for any tempered distribution $M_\gamma f$. Moreover, if we take g from $H^2_{\gamma,\rho}(\mathbb{R}^n)$, which is the image of $L^2(\mathbb{R}^n, \gamma)$ under the semigroup $e^{-\rho\sqrt{L}}$, then the series defining $S_\rho^2 g$ also converges and defines a smooth function. With these notations we prove the following characterisation:

Theorem 1.6. *Let f be a distribution such that $M_\gamma f$ is tempered. Then any function $u(x, \rho)$ for which $M_\gamma u(x, \rho)$ is tempered in x is a solution of the extension problem* (1-5) *with initial condition f if and only if $u(x, \rho) = S_\rho^1 f(x) + S_\rho^2 g(x)$ for some $g \in \bigcap_{t>0} H^2_{\gamma,t}(\mathbb{R}^n)$.*

We also prove another characterisation of the solution of the extension problem in terms of its holomorphic extendability. In order to state this we need to introduce some more notations. For any $t, \delta > 0$ we consider the positive weight function

$$w_t^\delta(x, y) = \frac{1}{\Gamma(\delta)} \int_{\mathbb{R}^n} e^{-2ux} \left(1 - \frac{|u|^2 + |y|^2}{t^2}\right)_+^{\delta-1} e^{-(|u|^2+|y|^2)}\, du.$$

For any $\rho > 0$, we let $H^2(\Omega_\rho, w_\rho^{2s})$ stand for the weighted Bergman space consisting of holomorphic functions on the tube domain $\Omega_\rho := \{z = x + iy \in \mathbb{C}^n : |y| < \rho\}$ belonging to $L^2(\Omega_\rho, w_\rho^{2s})$. Also for $m \in \mathbb{R}$, let $W_H^m(\mathbb{R}^n)$ stand for the Sobolev space associated to the Hermite operator H. This is a Hilbert space in which the norm is given by

$$\|f\|^2_{W_H^m} := \sum_{k=0}^{\infty} (2k+n)^{2m} \|P_k f\|_2^2.$$

Theorem 1.7. *A solution of the extension problem* (1-5) *is of the form $u(x, \rho) = S_\rho^1 f(x)$ for some distribution f such that $M_\gamma f \in W_H^{m_n}(\mathbb{R}^n)$, where $2m_n = -\frac{1}{4}(2n+1)$, if and only if for every $\rho > 0$, $M_\gamma u(\cdot\,, \rho)$ extends holomorphically to $\Omega_{\rho/2}$ and satisfies the uniform estimate*

$$\int_{\Omega_{\rho/2}} |M_\gamma u(z, \rho)|^2 w_{\rho/2}^{2s}(z)\, dz \le C\rho^{n-1/2}$$

for all $0 < \rho \le 1$.

We conclude the introduction by describing the plan of the paper. In Section 2, we study an extension problem for the Ornstein–Uhlenbeck operator. We provide two representations of solutions and their equivalence. In Section 3, we prove several characterisations of the solution of the extension problem under consideration. Using the results obtained in Section 2, we prove the trace Hardy and Hardy's inequality in Section 4. Then in Section 5, we prove an isometry property of the solution to the extension problem. Finally we end our discussion by proving an inequality of Hardy–Littlewood–Sobolev type for the fractional powers of the Hermite operator in Section 6.

2. The extension problem for the Ornstein–Uhlenbeck operator and fractional powers

The extension problem. Our strategy to prove Hardy's inequality for L_s is via the trace Hardy inequality which in turn requires the study of the following extension problem for the operator L:

$$\Big(-L+\partial_\rho^2+\frac{1-2s}{\rho}\partial_\rho-\frac{1}{4}\rho^2\Big)u(x,\rho)=0,\quad u(x,0)=f(x). \tag{2-1}$$

If u is a solution of the above problem, it follows that $v(x,\rho)=M_\gamma u(x,\rho)$ solves the problem

$$\Big(-H+\partial_\rho^2+\frac{1-2s}{\rho}\partial_\rho-\frac{1}{4}\rho^2\Big)v(x,\rho)=0,\quad v(x,0)=M_\gamma f(x). \tag{2-2}$$

A solution of the above problem can be obtained in terms of the solution of an extension problem for the sub-Laplacian on the Heisenberg group.

Let $\mathcal{L}$ be the sub-Laplacian on the Heisenberg group $\mathbb{H}^n$. Then a solution of the extension problem

$$\Big(-\mathcal{L}+\partial_\rho^2+\frac{1-2s}{\rho}\partial_\rho+\frac{1}{4}\rho^2\partial_t^2\Big)w(z,t,\rho)=0,\quad w(z,t,0)=f(z,t)$$

is given by $w(z,t,\rho)=\rho^{2s}f*\Phi_{s,\rho}(z,t)$, see [Roncal and Thangavelu 2020a], where $\Phi_{s,\rho}$ is an explicit function given by

$$\Phi_{s,\rho}(z,a)=\frac{2^{-(n+1+s)}}{\pi^{n+1}\Gamma(s)}\Gamma\big(\tfrac{1}{2}(n+1+s)\big)^2\big(\big(\tfrac{1}{4}\rho^2+\tfrac{1}{4}|z|^2\big)^2+a^2\big)^{-(n+1+s)/2}.$$

If we let π stand for the Schrödinger representation of $\mathbb{H}^n$ on $L^2(\mathbb{R}^n)$, then we have the following result.

Theorem 2.1. *For any $f\in L^2(\gamma)$ the function $v(x,\rho)$ defined by the equation*

$$v(x,\rho)=\rho^{2s}\int_{\mathbb{H}^n}\Phi_{s,\rho}(g)\pi(g)^*M_\gamma f(x)\,dg$$

solves the extension problem for the Hermite operator with initial condition $M_\gamma f$. Consequently, the extension problem for L is solved by $u(x,\rho)=e^{|x|^2/2}v(x,\rho)$.

Proof. For any X from the Heisenberg Lie algebra $\mathfrak{h}^n$ viewed as a left-invariant vector field on $\mathbb{H}^n$, we can easily check that

$$\pi(X)\int_{\mathbb{H}^n}\varphi(g)\pi(g)^*f(x)\,dg=-\int_{\mathbb{H}^n}X\varphi(g)\pi(g)^*f(x)\,dg.$$

This leads to

$$H\int_{\mathbb{H}^n}\varphi(g)\pi_\lambda(g)^*f(x)\,dg=\int_{\mathbb{H}^n}\mathcal{L}\varphi(g)\pi_\lambda(g)^*f(x)\,dg,$$

and consequently, as

$$\rho^{2s}\mathcal{L}\Phi_{s,\rho}(g)=\Big(\partial_\rho^2+\frac{1-2s}{\rho}\partial_\rho+\frac{1}{4}\rho^2\partial_t^2\Big)\rho^{2s}\Phi_{s,\rho}(g)=0,$$

we obtain

$$\Big(-H+\partial_\rho^2+\frac{1-2s}{\rho}\partial_\rho-\frac{1}{4}\rho^2\Big)\Big(\rho^{2s}\int_{\mathbb{H}^n}\Phi_{s,\rho}(g)\pi(g)^*f(x)\,dg\Big)=0.$$

To check that $v(x,\rho)$ satisfies the initial condition, we make the change of variables $(z,t)\to(\rho z,\rho^2 t)$, so that

$$v(x,\rho)=\int_{\mathbb{H}^n}\Phi_{s,1}(z,t)\pi(\rho z,\rho^2 t)^* M_\gamma f(x)\,dz\,dt.$$

Since $\pi(\rho z,\rho^2 t)M_\gamma f$ converges to $M_\gamma f$ in $L^2(\mathbb{R}^n)$, we obtain $v(x,\rho)\to M_\gamma f$ as $\rho\to 0$ in view of the fact that $\int_{\mathbb{H}^n}\Phi_{s,1}(g)\,dg=1$. This completes the proof of the theorem. □

There is yet another convenient way of representing the solution of the extension problem for L. If we let $k_{t,s}(\rho)=(\sinh t)^{-s-1}e^{-(\coth t)\rho^2/4}$, then it is known that this function satisfies the equation

$$\partial_t k_{t,s}(\rho)=\Big(\partial_\rho^2+\frac{1+2s}{\rho}\partial_\rho-\frac{1}{4}\rho^2\Big)k_{t,s}(\rho).$$

Theorem 2.2. *For $f\in L^p(\gamma)$ with $1\le p\le\infty$, a solution of the extension problem for L is given by*

$$u(x,\rho)=\frac{4^{-s}}{\Gamma(s)}\rho^{2s}\int_0^\infty k_{t,s}(\rho)e^{-tL}f(x)\,dt. \tag{2-3}$$

Moreover, as $\rho\to 0$, the solution $u(\cdot,\rho)$ converges to f in $L^p(\gamma)$ for any $1\le p<\infty$.

Proof. That u solves the extension problem follows easily from the fact that $e^{-tL}f(x)$ solves the heat equation associated to L, i.e., $-Le^{-tL}f(x)=\partial_t e^{-tL}f(x)$, and the definition of $k_{t,s}(\rho)$. Indeed, we have

$$-Lu(x,\rho)=\frac{4^{-s}}{\Gamma(s)}\rho^{2s}\int_0^\infty k_{t,s}(\rho)\partial_t v(x,t)\,dt,$$

which after an integration by parts in the t variable yields

$$Lu(x,\rho)=\frac{4^{-s}}{\Gamma(s)}\rho^{2s}\int_0^\infty \partial_t k_{t,s}(\rho)v(x,t)\,dt.$$

Since $k_{t,s}(\rho)$ is the heat kernel associated to the operator $\big(\partial_\rho^2+\frac{1+2s}{\rho}\partial_\rho-\frac14\rho^2\big)$, we have

$$Lu(x,\rho)=\frac{4^{-s}}{\Gamma(s)}\rho^{2s}\Big(\partial_\rho^2+\frac{1+2s}{\rho}\partial_\rho-\frac{1}{4}\rho^2\Big)\int_0^\infty k_{t,s}(\rho)e^{-tL}f(x)\,dt.$$

Finally, an easy calculation shows that for any function $v(\rho)$ one has

$$\Big(\partial_\rho^2+\frac{1-2s}{\rho}\partial_\rho-\frac{1}{4}\rho^2\Big)(\rho^{2s}v(\rho))=\rho^{2s}\Big(\partial_\rho^2+\frac{1+2s}{\rho}\partial_\rho-\frac{1}{4}\rho^2\Big)v(\rho),$$

and hence it follows that $u(x,\rho)$ solves the extension problem.

Now to prove the $L^p(\gamma)$ convergence of the solution to the initial condition, we make use of the fact that e^{-tL} is a contraction semigroup on every $L^p(\gamma)$ and $e^{-tL}f$ converges to f in $L^p(\gamma)$ as $t\to 0$. We first make a change of variables $t\to\rho^2 t$ to get

$$u(x,\rho)=\frac{4^{-s}}{\Gamma(s)}\rho^{2s+2}\int_0^\infty k_{\rho^2 t,s}(\rho)e^{-\rho^2 tL}f(x)\,dt.$$

Note that

$$\rho^{2s+2}k_{\rho^2t,s}(\rho)=\rho^{2s+2}(\sinh\rho^2t)^{-s-1}e^{-(\coth\rho^2t)\rho^2/4}$$
$$=t^{-s-1}\left(\frac{\rho^2t}{\sinh\rho^2t}\right)^{s+1}e^{-(\coth\rho^2t)\rho^2t/(4t)}\quad\to\quad t^{-s-1}e^{-1/(4t)}\qquad\text{as }\rho\to 0.\tag{2-4}$$

Here we have used the facts that $(\sinh y)/y\to 1$ and $y\coth y\to 1$ as $y\to 0$. Also we see that $t^{-s-1}e^{-1/(4t)}\in L^1(0,\infty)$, and an easy calculation yields

$$\int_0^\infty t^{-s-1}e^{-1/(4t)}\,dt=4^s\Gamma(s).$$

Now using this result we can write, for any $x\in\mathbb{R}^n$,

$$\begin{aligned}u(x,\rho)-f(x)&=\frac{4^{-s}}{\Gamma(s)}\rho^{2s+2}\int_0^\infty k_{\rho^2t,\,s}(\rho)e^{-\rho^2tL}f(x)\,dt-\frac{4^{-s}}{\Gamma(s)}\int_0^\infty t^{-s-1}e^{-1/(4t)}f(x)\,dt\\&=\frac{4^{-s}}{\Gamma(s)}\int_0^\infty(\rho^{2s+2}k_{\rho^2t,\,s}(\rho)-t^{-s-1}e^{-1/(4t)})e^{-\rho^2tL}f(x)\,dt\\&\qquad+\frac{4^{-s}}{\Gamma(s)}\int_0^\infty t^{-s-1}e^{-1/(4t)}(e^{-\rho^2tL}f(x)-f(x))\,dt.\end{aligned}$$

Therefore, using Minkowski's integral inequality and the fact that $\|e^{-\rho^2tL}f\|_{L^p(\gamma)}\le\|f\|_{L^p(\gamma)}$, we have

$$\begin{aligned}\|u(\,\cdot\,,\rho)-f\|_{L^p(\gamma)}&\le\frac{4^{-s}}{\Gamma(s)}\int_0^\infty|\rho^{2s+2}k_{\rho^2t,\,s}(\rho)-t^{-s-1}e^{-1/(4t)}|\|f\|_{L^p(\gamma)}\,dt\\&\qquad+\frac{4^{-s}}{\Gamma(s)}\int_0^\infty t^{-s-1}e^{-1/(4t)}\|e^{-\rho^2tL}f-f\|_{L^p(\gamma)}\,dt.\end{aligned}\tag{2-5}$$

Note that using the asymptotics of the sine and cotangent hyperbolic functions, we have

$$|\rho^{2s+2}k_{\rho^2t,\,s}(\rho)-t^{-s-1}e^{-1/(4t)}|\le C\rho^{2s+2}e^{-\rho^2t(s+1)}+t^{-s-1}e^{-1/(4t)}:=h_\rho(t),\quad t>M.\tag{2-6}$$

It is not hard to see that for every $\rho>0$, we have $h_\rho\in L^1$ and $\lim_{\rho\to0}\int_M^\infty h_\rho(t)\,dt=\int_M^\infty h(t)\,dt$, and also as $\rho\to 0$ we have $h_\rho(t)\to t^{-s-1}e^{-1/(4t)}=:h(t)$ pointwise. Hence by the generalised dominated convergence theorem (DCT) we have

$$\int_M^\infty|\rho^{2s+2}k_{\rho^2t,s}(\rho)-t^{-s-1}e^{-1/(4t)}|\|f\|_{L^p(\gamma)}\,dt\to 0\quad\text{as }\rho\to 0.$$

Now see that, similar to (2-4), one can show the function $h_\rho(t)$ goes to a finite limit as $t\to 0$, so there is no singularity of h_ρ at 0. Hence it is easy to see that

$$\int_0^M|\rho^{2s+2}k_{\rho^2t,s}(\rho)-t^{-s-1}e^{-1/(4t)}|\|f\|_{L^p(\gamma)}\,dt$$

goes to zero as $\rho\to 0$. Hence it follows that the first integral in the right-hand side of (2-5) goes to zero. Also the integrand of the second integral is bounded above by an integrable function of t. Indeed,

$$t^{-s-1}e^{-1/(4t)}\|e^{-\rho^2tL}f-f\|_{L^p(\gamma)}\le 2t^{-s-1}e^{-1/(4t)}\|f\|_{L^p(\gamma)}.$$

Hence by DCT the second integral goes to zero as $\rho \to 0$. Therefore we have

$$u(\,\cdot\,,\rho) = \frac{4^{-s}}{\Gamma(s)}\rho^{2s+2}\int_0^\infty k_{\rho^2 t,\,s}(\rho)e^{-\rho^2 tL} f\,dt \to f \quad \text{in } L^p(\gamma) \qquad \text{as } \rho \to 0. \qquad \square$$

We have thus given two representations for solutions of the extension problem, and we now claim they are the same. This is not obvious and needs a proof. It is convenient to work with the functions

$$\varphi_{s,\delta}(z,a) = \left(\left(\delta + \tfrac{1}{4}|z|^2\right)^2 + a^2\right)^{-(n+1+s)/2},$$

in terms of which we can express $\Phi_{s,\rho}$ as follows: with $\delta = \frac{1}{4}\rho^2$,

$$\Phi_{s,\rho}(z,a) = \frac{2^{-(n+1+s)}}{\pi^{n+1}\Gamma(s)}\Gamma\left(\tfrac{1}{2}(n+1+s)\right)^2\varphi_{s,\delta}(z,a). \tag{2-7}$$

For a function $\varphi(z,t)$ on $\mathbb{H}^n$ we let $\varphi^\lambda(z)$ denote the inverse Fourier transform of φ in the t variable. Thus

$$\varphi^\lambda_{s,\delta}(z) = \int_{-\infty}^\infty \varphi_{s,\delta}(z,t)e^{i\lambda t}\,dt.$$

This is a radial function on $\mathbb{C}^n$ and hence has an expansion in terms of the Laguerre functions:

$$\varphi^\lambda_k(z) = L_k^{n-1}\left(\tfrac{1}{2}|\lambda|\,|z|^2\right)e^{-|\lambda|\,|z|^2/4}. \tag{2-8}$$

We let $c^\lambda_{k,\delta}(s)$ be the coefficients defined by

$$\varphi^\lambda_{s,\delta}(z) = (2\pi)^{-n}|\lambda|^n \sum_{k=0}^\infty c^\lambda_{k,\delta}(s)\varphi^\lambda_k(z). \tag{2-9}$$

These coefficients are given in terms of the auxiliary function $L(a,b,c)$ defined for $a, b \in \mathbb{R}_+$ and $c \in \mathbb{R}$ as follows:

$$L(a,b,c) = \int_0^\infty e^{-a(2x+1)}x^{b-1}(1+x)^{-c}\,dx. \tag{2-10}$$

The following proposition expresses the $c^\lambda_{k,\delta}(s)$ in terms of L; see [Cowling and Haagerup 1989].

Proposition 2.3 (Cowling–Haagerup). *For any $\delta > 0$ and $0 < s < \frac{1}{2}(n+1)$, we have*

$$c^\lambda_{k,\delta}(s) = \frac{(2\pi)^{n+1}|\lambda|^s}{\Gamma\left(\frac{1}{2}(n+1+s)\right)^2} L\left(\delta|\lambda|,\, \tfrac{1}{2}(2k+n+1+s),\, \tfrac{1}{2}(2k+n+1-s)\right).$$

Using this proposition we can compute the explicit formula for the group Fourier transform of $\Phi_{s,\rho}(g)$ on $\mathbb{H}^n$. Let $P_k(\lambda)$ stand for the projections associated to $H(\lambda) = -\Delta + \lambda^2|x|^2$. Then making use of the fact that

$$\int_{\mathbb{C}^n} \varphi^\lambda_k(z)\pi_\lambda(z,0)\,dz = (2\pi)^{-n}|\lambda|^{-n}P_k(\lambda),$$

we obtain the following formula: with $\delta = \frac{1}{4}\rho^2$, as before,

$$\int_{\mathbb{H}^n} \Phi_{s,\rho}(g)\pi_\lambda(g)^*\,dg = \frac{2^{-(n+1+s)}}{\pi^{n+1}\Gamma(s)}\Gamma\left(\tfrac{1}{2}(n+1+s)\right)^2 \sum_{k=0}^\infty c^\lambda_{k,\delta}(s)P_k(\lambda).$$

As the projections associated to L are given by $Q_k = M_\gamma^{-1} P_k M_\gamma$, we see that the solution defined in Theorem 2.1 is given by

$$u(x, \rho) = \frac{2^{-(n+1+s)}}{\pi^{n+1}\Gamma(s)} \Gamma\big(\tfrac{1}{2}(n+1+s)\big)^2 \rho^{2s} \sum_{k=0}^{\infty} c^1_{k,\delta}(s) Q_k f(x).$$

Therefore, in order to prove our claim, we only need to check that

$$\frac{4^{-s}}{\Gamma(s)} \rho^{2s} \int_0^\infty k_{t,s}(\rho) e^{-tL} f(x)\, dt = \frac{2^{-(n+1+s)}}{\pi^{n+1}\Gamma(s)} \Gamma\big(\tfrac{1}{2}(n+1+s)\big)^2 \rho^{2s} \sum_{k=0}^{\infty} c^1_{k,\delta}(s) Q_k f(x),$$

where $\delta = \frac{1}{4}\rho^2$. Equivalently, we need to check that

$$\int_0^\infty k_{t,s}(\rho) e^{-t(2k+n)}\, dt = L\big(\tfrac{1}{4}\rho^2, \tfrac{1}{2}(2k+n+1+s), \tfrac{1}{2}(2k+n+1-s)\big).$$

In order to compute the above integral, we make the change of variable $\coth t = 2z+1$ and note that $-(\sinh^2 t)^{-1}\, dt = 2\, dz$ and $\sinh t = (2z(2z+2))^{-1/2}$. We get

$$\begin{aligned}
&\int_0^\infty (\sinh t)^{-s-1} e^{-(\coth t)\rho^2/4} e^{-t(2k+n)}\, dt \\
&\qquad = 2\int_0^\infty (2z(2z+2))^{(s-1)/2} e^{-(2z+1)\rho^2/4} \Big(\frac{2z+2}{2z}\Big)^{-(2k+n)/2} dz \\
&\qquad = 2\int_0^\infty e^{-(2z+1)\rho^2/4} (2z)^{[(s-1)+(2k+n)]/2} (2z+2)^{-[(1-s)+(2k+n)]/2}\, dz \\
&\qquad = 2^s \int_0^\infty e^{-(2z+1)\rho^2/4} (z)^{[(s+1)+(2k+n)]/2-1} (z+1)^{-[(1-s)+(2k+n)]/2}\, dz \\
&\qquad = 2^s L\big(\tfrac{1}{4}\rho^2, \tfrac{1}{2}(2k+n+1+s), \tfrac{1}{2}(2k+n+1-s)\big).
\end{aligned}$$

This proves our claim that Theorems 2.1 and 2.2 define the same solution of the extension problem.

The above proof also shows that the function $u(x,\rho)$ defined by the integral (using U in place of L)

$$u(x, \rho) = \frac{4^{-s}}{\Gamma(s)} \rho^{2s} \int_0^\infty k_{t,s}(\rho) e^{-tU} f(x)\, dt,$$

solves the extension problem for U and the following expansion for the solution u is valid.

Proposition 2.4. *For $0 < s < \frac{1}{2}(n+1)$ and $f \in L^2(\gamma)$, the solution of the extension problem associated to U is given by*

$$u(\cdot, \rho) = \frac{2^{-s}}{\Gamma(s)} \rho^{2s} \sum_{k=0}^{\infty} L\big(\tfrac{1}{4}\rho^2, \tfrac{1}{2}(k+n/2+1+s), \tfrac{1}{2}(k+n/2+1-s)\big) Q_k f. \tag{2-11}$$

We let $T_{s,\rho}$ stand for the solution operator which takes f into the solution $u(x,\rho)$ of the extension problem. Thus

$$T_{s,\rho} f(x) = \frac{4^{-s}}{\Gamma(s)} \rho^{2s} \int_0^\infty k_{t,s}(\rho) e^{-tL} f(x)\, dt,$$

which is also given by the expansion in the above proposition. In what follows we make use of the transformation property

$$\frac{(2\lambda)^a}{\Gamma(a)} L(\lambda, a, b) = \frac{(2\lambda)^b}{\Gamma(b)} L(\lambda, b, a) \tag{2-12}$$

satisfied by the L function for all admissible values of (a, b, c); see Cowling and Haagerup [1989].

Fractional powers of the operators L and U. In what follows let A stand for either L or U. Note that the associated eigenvalues λ_k are given by $(2k+n)$ and $\left(k+\frac{1}{2}n\right)$, respectively. The above representation of the solution of the extension problem allows us to define A_s as the Neumann boundary data associated to the extension problem. More precisely we have the following result:

Theorem 2.5. *Assume that* $0 < s < 1$. *Let* $f \in L^2 \cap L^p(\gamma)$ *with* $1 \le p < \infty$ *be such that* $A_s f \in L^p(\gamma)$. *Then the solution of the extension problem* $u(x, \rho) = T_{s,\rho} f(x)$ *satisfies*

$$\lim_{\rho \to 0} \rho^{1-2s} \partial_\rho u(x, \rho) = -2^{1-2s} \frac{\Gamma(1-s)}{\Gamma(s)} A_s f,$$

where the convergence is understood in the $L^p(\gamma)$ *sense.*

Proof. The expansion of $T_{s,\rho} f$ given in Proposition 2.4 and the transformation property (2-12) of the L function allows us to verify the identity

$$\rho^{2s} T_{-s,\rho}(A_s f)(x) = \frac{4^s \Gamma(s)}{\Gamma(-s)} T_{s,\rho} f(x), \tag{2-13}$$

which when expanded reads as

$$\frac{4^s \Gamma(s)}{\Gamma(-s)} u(x, \rho) = \frac{4^s}{\Gamma(-s)} \int_0^\infty (\sinh t)^{s-1} e^{-(\coth t)\rho^2/4} e^{-tA} A_s f(x)\, dt.$$

Differentiating with respect to ρ and multiplying both sides by $-\rho^{1-2s}$, we get

$$-\rho^{1-2s} \partial_\rho u(x, \rho) = \frac{1}{2\Gamma(s)} \rho^{2(1-s)} \int_0^\infty (\sinh t)^{s-1} (\coth t) e^{-(\coth t)\rho^2/4} e^{-tA} A_s f(x)\, dt.$$

Now we make the change of variable $t \to t\rho^2$ to get

$$\begin{aligned} -\rho^{1-2s} \partial_\rho u(x, \rho) &= \frac{1}{2\Gamma(-s)} \rho^{4-2s} \int_0^\infty (\sinh(t\rho^2))^{s-1} \coth(t\rho^2) e^{-\coth(t\rho^2)\rho^2/4} e^{-t\rho^2 A} A_s f(x)\, dt \\ &= \frac{1}{2\Gamma(-s)} \int_0^\infty t^{s-2} \left(\frac{\sinh(t\rho^2)}{t\rho^2}\right)^{s-1} \coth(t\rho^2)(t\rho^2) e^{-\coth(t\rho^2)\rho^2/4} e^{-t\rho^2 A} A_s f(x)\, dt. \end{aligned}$$

Under the extra assumption that $A_s f \in L^p(\gamma)$ with $1 \le p < \infty$, we know that $\lim_{\rho \to 0} e^{-\rho^2 t A} A_s f = A_s f$, in $L^p(\gamma)$. So as $\rho \to 0$, we can argue as in the proof of Theorem 2.2 to obtain

$$\lim_{\rho \to 0} (-\rho^{1-2s} \partial_\rho u(x, \rho)) = \frac{1}{2\Gamma(s)} A_s f \left(\int_0^\infty t^{s-2} e^{-1/(4t)}\, dt \right).$$

Computing the last integral and simplifying we obtain

$$\lim_{\rho \to 0} (\rho^{1-2s} \partial_\rho u(x, \rho)) = -2^{(1-2s)} \frac{\Gamma(1-s)}{\Gamma(s)} A_s f. \qquad \square$$

3. Characterisations of solutions of the extension problem

In this section we prove several characterisations of solutions of the extension problem for L. Recall that the extension problem for L reads as

$$\Big(-L+\partial_\rho^2+\frac{1-2s}{\rho}\partial_\rho-\frac{1}{4}\rho^2\Big)u(x,\rho)=0,\quad u(x,0)=f(x).$$

Now given $\alpha\in\mathbb{N}^n$ and $\rho>0$ we define the Fourier–Hermite coefficients associated to the expansion in terms of the normalised Hermite polynomials H_α as

$$\tilde{u}(\alpha,\rho):=\int_{\mathbb{R}^n}u(x,\rho)H_\alpha(x)\,d\gamma(x).$$

Now letting $v_\alpha(\rho):=\tilde{u}(\alpha,2\sqrt{\rho})$, we see that

$$(-(2|\alpha|+n)+\rho\partial_\rho^2+(1-s)\partial_\rho-\rho)v_\alpha(\rho)=0,\quad v_\alpha(0)=(f,H_\alpha)_{L^2(\gamma)}.$$

Again if we write $v_\alpha(\rho)=e^{-\rho}g_\alpha(2\rho)$, then it can be easily checked that the above equation becomes

$$rg_\alpha''(r)+(1-s-r)g_\alpha'(r)-\tfrac{1}{2}(2|\alpha|+n+1-s)g_\alpha(r)=0,$$

where $r=2\rho$. Now we let $g_\alpha(r)=r^sh_\alpha(r)$, which leads to

$$rh_\alpha''(r)+(1+s-r)h_\alpha'(r)-\tfrac{1}{2}(2|\alpha|+n+1+s)h_\alpha(r)=0. \tag{3-1}$$

Note that this is in the form of Kummer's equation: $xh''(x)+(b-x)h'-ah(x)=0$. The solutions of Kummer's equation are given by the functions $M(a,b,x)$ and $V(a,b,x)$, which are known as the confluent hypergeometric functions. The function M, given by $M(a,b,x)=\sum_{m=0}^{\infty}((a)_m/(b)_m m!)x^m$, is analytic, and

$$V(a,b,x)=\frac{\pi}{\sin\pi b}\bigg(\frac{M(a,b,x)}{\Gamma(1+a-b)\Gamma(b)}-x^{1-b}\frac{M(1+a-b,2-b,x)}{\Gamma(a)\Gamma(2-b)}\bigg),\quad x>0.$$

Also, V has the integral representation given by

$$V(a,b,x)=\frac{1}{\Gamma(a)}\int_0^\infty e^{-tx}t^{a-1}(1+t)^{b-a-1}\,dt,\quad x>0.$$

For more details, see for instance [Abramowitz and Stegun 1964, Chapter 13] and also [Frank et al. 2015, Lemma 5.2].

Finally, writing $\mu=\frac{1}{2}s$ and $\kappa=|\alpha|+\frac{1}{2}n$, performing another substitution $w_\alpha(r)=e^{-1/2r}r^{1/2+\mu}h_\alpha(r)$, transforms (3-1) to

$$w_\alpha''(r)+\bigg(-\frac{1}{4}-\frac{\kappa}{r}+\frac{1/4-\mu^2}{r^2}\bigg)w_\alpha(r)=0, \tag{3-2}$$

which is in the form of a Whittaker equation. This warrants the following lemma which describes the properties of solutions of Whittaker equations.

Lemma 3.1 [Olver and Maximon 2010]. *Let $\kappa \in \mathbb{R}$ and $-2\mu \notin \mathbb{N}$. The two linearly independent solutions of the ordinary differential equation*

$$w''(x) + \left(-\frac{1}{4} + \frac{\kappa}{x} + \frac{1/4 - \mu^2}{x^2}\right) w(x) = 0$$

are given by the functions $M_{\kappa,\mu}(x)$ and $W_{\kappa,\mu}(x)$, where

$$M_{\kappa,\mu}(x) = e^{-x/2} x^{1/2+\mu} \sum_{p=0}^{\infty} \frac{1/2 + \mu - \kappa}{(1+2\mu)_p \, p!} x^p,$$

and when 2μ is not an integer,

$$W_{\kappa,\mu}(x) = \frac{\Gamma(-2\mu)}{\Gamma(1/2 - \mu - \kappa)} M_{\kappa,\mu}(x) + \frac{\Gamma(+2\mu)}{\Gamma(1/2 + \mu - \kappa)} M_{\kappa,-\mu}(x). \tag{3-3}$$

Moreover, we have the following asymptotic properties of these Whittaker functions:

For large x,

$$M_{\kappa,\mu}(x) \sim \frac{\Gamma(1+2\mu)}{\Gamma(1/2 + \mu - \kappa)} e^{x/2} x^{-\kappa}, \quad \mu - \kappa \neq -\tfrac{1}{2}, -\tfrac{3}{2}, \dots \quad \text{and} \quad W_{\kappa,\mu}(x) \sim e^{-x/2} x^{\kappa}. \tag{3-4}$$

Also as $x \to 0$ we have

$$M_{\kappa,\mu}(x) = x^{\mu+1/2}(1 + O(x)), \quad 2\nu \neq -1, -2, -3, \dots, \tag{3-5}$$

$$W_{\kappa,\mu}(x) = \frac{\Gamma(2\mu)}{\Gamma(1/2 + \mu - \kappa)} x^{1/2-\mu} + \frac{\Gamma(-2\mu)}{\Gamma(1/2 - \mu - \kappa)} x^{1/2+\mu} + O(x^{3/2-\mu}), \quad 0 < \mu < \tfrac{1}{2}. \tag{3-6}$$

In view of the above lemma, generic solutions of (3-2) are given by

$$w_\alpha(r) = C_1(|\alpha|) M_{-(|\alpha|+n/2),\, s/2}(r) + C_2(|\alpha|) W_{-(|\alpha|+n/2),\, s/2}(r).$$

But we know $v_\alpha(\rho) = e^{-\rho} g_\alpha(2\rho) = e^{-\rho}(2\rho)^s h_\alpha(\rho) = e^{-\rho}(2\rho)^s e^{\rho/2} \rho^{-1/2-\mu} w_\alpha(\rho)$, and by definition $v_\alpha(\rho) = \tilde{u}(\alpha, 2\sqrt{\rho})$. Hence we have

$$\tilde{u}(\alpha, \rho) = \left(\tfrac{1}{2}\rho^2\right)^{(s-1)/2} \left(C_1(|\alpha|) W_{-(|\alpha|+n/2),\, s/2}\left(\tfrac{1}{2}\rho^2\right) + C_2(|\alpha|) M_{-(|\alpha|+n/2),\, s/2}\left(\tfrac{1}{2}\rho^2\right) \right). \tag{3-7}$$

The initial condition on the solution along with the behaviour of the Whittaker functions stated in the previous lemma allows us to conclude that

$$C_1(|\alpha|) = \frac{1}{\Gamma(s)} \Gamma\left(\tfrac{1}{2}(2k + n + s + 1)\right) (f, H_\alpha)_{L^2(\gamma)}.$$

Thus the solution of the extension problem can be written as a sum of two functions, namely

$$\left(\tfrac{1}{2}\rho^2\right)^{(s-1)/2} \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \Gamma\left(\tfrac{1}{2}(2k + n + s + 1)\right) W_{-(k+n/2),\, s/2}\left(\tfrac{1}{2}\rho^2\right) Q_k f,$$

$$\left(\tfrac{1}{2}\rho^2\right)^{(s-1)/2} \sum_{\alpha \in \mathbb{N}^n} C_2(|\alpha|) M_{-(|\alpha|+n/2),\, s/2}\left(\tfrac{1}{2}\rho^2\right) H_\alpha(x).$$

The second series above converges under some decay conditions on the coefficients $C_2(|\alpha|)$ as we will see soon. We make use of these considerations in the proof of Theorem 3.3 below.

To proceed further with our description of solutions of the extension problem, we need the following asymptotic properties of the Whittaker functions appearing in the above expressions for large values of the parameter k.

Lemma 3.2. *For any $\rho \in (0, \infty)$, we have the following asymptotic properties, as k tends to infinity*:

$$\left(\tfrac{1}{2}\rho^2\right)^{(s-1)/2} M_{-(k+n/2),\,s/2}\left(\tfrac{1}{2}\rho^2\right) \sim (\rho)^{s-1/2}(\sqrt{2k+n})^{-s-1/2} \exp\left(2(2k+n)\zeta\left(\frac{\rho^2}{4(2k+n)}\right)^{1/2}\right), \quad (3\text{-}8)$$

$$\left(\tfrac{1}{2}\rho^2\right)^{(s-1)/2} W_{-(k+n/2),\,s/2}\left(\tfrac{1}{2}\rho^2\right) \sim \frac{(\rho\sqrt{2k+n})^{s-1/2}}{\Gamma\left(\frac{1}{2}(2k+n+1+s)\right)} \exp\left(-2(2k+n)\zeta\left(\frac{\rho^2}{4(2k+n)}\right)^{1/2}\right), \quad (3\text{-}9)$$

where $2\sqrt{\zeta(x)} = \sqrt{x+x^2} + \ln(\sqrt{x} + \sqrt{x+1})$ *for* $x > 0$.

Proof. For large values of κ and for any $x \in (0, \infty)$, the following asymptotic properties can be found in [Olver and Maximon 2010, 13.21.6, 13.21.7]:

$$M_{-\kappa,\mu}(4\kappa x) = \frac{2\Gamma(2\mu+1)}{\kappa^{\mu-1/2}}\left(\frac{x\zeta(x)}{1+x}\right)^{1/4} I_{2\mu}(4\kappa\zeta(x)^{1/2})(1+O(\kappa^{-1})), \quad (3\text{-}10)$$

$$W_{-\kappa,\mu}(4\kappa x) = \frac{\sqrt{8/\pi}\, e^{\kappa}}{\kappa^{\kappa-1/2}}\left(\frac{x\zeta(x)}{1+x}\right)^{1/4} K_{2\mu}(4\kappa\zeta(x)^{1/2})(1+O(\kappa^{-1})), \quad (3\text{-}11)$$

where $I_{2\mu}$ is the modified Bessel function of the first kind and $K_{2\mu}$ denotes the Macdonald function of order 2μ. Taking $x = \frac{1}{2}\rho^2$, $\kappa = k + \frac{1}{2}n$ and $\mu = \frac{1}{2}s$, for large values of k, from (3-10) we have

$$M_{-(k+n/2),\,s/2}(x) = \frac{2\Gamma(2\mu+1)}{(k+n/2)^{\mu-1/2}}\left(\frac{x\zeta(x/2(2k+n))}{2(2k+n)+x}\right)^{1/4} I_{2\mu}\left(2(2k+n)\zeta\left(\frac{x}{2(2k+n)}\right)^{1/2}\right)(1+O(k^{-1})).$$

Recall that the modified Bessel function of the first kind has the following asymptotic property:

$$I_{2\mu}(x) \sim \frac{1}{\sqrt{2\pi x}}e^{x} \quad \text{when } x \text{ is real and } x \to \infty. \quad (3\text{-}12)$$

But it is easy to see that $2(2k+n)\zeta(x/(2(2k+n)))^{1/2}$ goes to infinity as $k \to \infty$, which by the above asymptotic property yields

$$I_{2\mu}\left(2(2k+n)\zeta\left(\frac{x}{2(2k+n)}\right)^{1/2}\right) \sim \left(2(2k+n)\zeta\left(\frac{x}{2(2k+n)}\right)^{1/2}\right)^{-1/2} \exp\left(2(2k+n)\zeta\left(\frac{x}{2(2k+n)}\right)^{1/2}\right), \quad (3\text{-}13)$$

valid for large values of k. It can be easily checked that for any $x > 0$ and large k,

$$\left(\frac{1}{4}\right)^{1/4}\left(\frac{x}{2k+n}\right)^{1/4} \le \left(\frac{x}{2(2k+n)+x}\right)^{1/4} \le \left(\frac{3}{4}\right)^{1/4}\left(\frac{x}{2k+n}\right)^{1/4}. \quad (3\text{-}14)$$

This, along with (3-13), proves the result for the function $M_{-(k+n/2),\,s/2}$.

We obtain the asymptotic property for the other function similarly: for large k, from (3-11) we have

$$W_{-(k+n/2),\,s/2}(x) = \frac{\sqrt{8/\pi}\,e^{k+n/2}}{(k+n/2)^{k+n/2-1/2}}\left(\frac{x\zeta(x/(2(2k+n)))}{2(2k+n)+x}\right)^{1/4} K_{2\mu}\left(2(2k+n)\zeta\left(\frac{x}{2(2k+n)}\right)^{1/2}\right)(1+O(k^{-1})).$$

Now the Macdonald's function $K_{2\mu}(z)$ has the following asymptotic property:

$$K_{2\mu}(x) \sim \sqrt{\pi/(2x)}\,e^{-x}, \quad \text{when } x \text{ is real and } x \to \infty. \tag{3-15}$$

Again for the same reason as above, as $k \to \infty$, using (3-15) we have

$$K_{2\mu}\left(2(2k+n)\zeta\left(\frac{x}{2(2k+n)}\right)^{1/2}\right) \sim \left(2(2k+n)\zeta\left(\frac{x}{2(2k+n)}\right)^{1/2}\right)^{-1/2} \exp\left(-2(2k+n)\zeta\left(\frac{x}{2(2k+n)}\right)^{1/2}\right). \tag{3-16}$$

Using Stirling's formula, $\Gamma(x) = \sqrt{2\pi}\,x^{x-1/2}e^{-x}e^{\theta(x)/12x}$ for $0 < \theta(x) < 1$ which is true for $x > 0$, see [Ahlfors 1953], we have

$$\frac{\Gamma\big(\frac{1}{2}(2k+n+1+s)\big)e^{(k+n/2)}}{(k+n/2)^{(k+n/2)-1/2}} = \frac{\Gamma\big(\frac{1}{2}(2k+n+1+s)\big)}{e^{-\theta(k+n/2)/6(2k+n)}\Gamma\big(\frac{1}{2}(2k+n)\big)} \sim \big(\tfrac{1}{2}(2k+n)\big)^{(1+s)/2},$$

as $k \to \infty$. This observation along with (3-14) and the asymptotic property (3-16) yields

$$\Gamma\big(\tfrac{1}{2}(2k+n+1+s)\big)W_{-(k+n/2),\,s/2}(x) \sim (2k+n)^{s/2-1/4}x^{1/4}\exp\left(-2(2k+n)\zeta\left(\frac{x}{2(2k+n)}\right)^{1/2}\right). \qquad \square$$

Remark. It can be easily checked that for large κ the following inequality is valid for any $x > 0$:

$$\tfrac{1}{2}\sqrt{x\kappa} \le \kappa\sqrt{\zeta(x/\kappa)} \le \tfrac{3}{2}\sqrt{x\kappa}, \tag{3-17}$$

which can be used to further simplify the exponential part in the above estimates.

The analysis preceding Lemma 3.2 motivates us to define the following two operators. Given a distribution f such that $M_\gamma f$ is a tempered distribution, we define

$$S^1_\rho f = \frac{\big(\frac{1}{2}\rho^2\big)^{(s-1)/2}}{\Gamma(s)} \sum_{k=0}^{\infty} \Gamma\big(\tfrac{1}{2}(2k+n+s+1)\big)W_{-(k+n/2),\,s/2}\big(\tfrac{1}{2}\rho^2\big)Q_k f. \tag{3-18}$$

Recall that h is a tempered distribution on $\mathbb{R}^n$ if and only if the Hermite coefficients satisfy the estimate $|(h, \Phi_\alpha)| \le C(2|\alpha|+n)^m$ for some integer m. So $M_\gamma f$ being a tempered distribution, its Hermite coefficients have at most polynomial growth, and consequently $Q_k f$ has polynomial growth in k. So because of the exponential decay in (3-9), the above series defining $S^1_\rho f$ converges uniformly. Consequently, in view of (3-7), $S^1_\rho f$ defines a solution of the extension problem.

For the other solution of the Whittaker equation we define the operator S^2_ρ for nice functions g by

$$S^2_\rho g = \big(\tfrac{1}{2}\rho^2\big)^{(s-1)/2} \sum_{k=0}^{\infty} M_{-(k+n/2),\,s/2}\big(\tfrac{1}{2}\rho^2\big)Q_k g. \tag{3-19}$$

It is not hard to see that as the Whittaker function $M_{-(k+n/2),\,s/2}\big(\frac{1}{2}\rho^2\big)$ has exponential growth as $k \to \infty$, $Q_k g$ must have enough decay for the series in (3-19) to converge. This encourages us to determine a condition on the function g so that the projections $Q_k g$ have enough decay. Now as can be seen in the above lemma, the function $M_{-(k+n/2),\,s/2}\big(\frac{1}{2}\rho^2\big)$ is growing like $e^{c\rho\sqrt{2k+n}}$ for large values of k which leads us to consider the image of $L^2(\gamma)$ under the semigroup $e^{-tL^{1/2}}$, which we denote by $H^2_{\gamma,t}(\mathbb{R}^n)$. Clearly, if $g \in \bigcap_{t>0} H^2_{\gamma,t}(\mathbb{R}^n)$, the series in (3-19) converges and defines a smooth function. But in view of the connection between L and the Hermite operator H, we note that a function g is in $H^2_{\gamma,t}(\mathbb{R}^n)$ if and only if $ge^{-|\cdot|^2/2}$ is in the image of $L^2(\mathbb{R}^n)$ under the Poisson semigroup $e^{-tH^{1/2}}$. Let us write $H^2_t(\mathbb{R}^n) := e^{-tH^{1/2}}(L^2(\mathbb{R}^n))$. We are ready to prove the following characterisation for the solution of the extension problem.

Theorem 3.3. *Let f be a distribution such that $M_\gamma f$ is tempered. Then any function $u(x,\rho)$ for which $M_\gamma u(x,\rho)$ is tempered in x is a solution of the extension problem* (1-5) *with initial condition f if and only if $u(x,\rho) = S^1_\rho f(x) + S^2_\rho g(x)$ for some $g \in \bigcap_{t>0} H^2_{\gamma,t}(\mathbb{R}^n)$.*

Proof. First suppose $u(x,\rho) = S^1_\rho f(x) + S^2_\rho g(x)$ for some g such that $g \in \bigcap_{t>0} H^2_{\gamma,t}(\mathbb{R}^n)$. Consequently, for every $t > 0$, we have $\|Q_k g\|^2_{L^2(\gamma)} \le Ce^{-2t\sqrt{2k+n}}$ for large k. So the expression (3-19) defining $S^2_\rho g$ is well defined and solves the extension problem.

Now since $M_\gamma f$ is a tempered distribution, as mentioned above, the Fourier–Hermite coefficients associated to Hermite polynomials of f satisfy

$$|\tilde{f}(\alpha)| = |(f, H_\alpha)_{L^2(\gamma)}| \le C(2|\alpha|+n)^m \quad \text{for some integer } m.$$

But in view of the fact that $\sum_{|\alpha|=k} 1 = (k+n-1)!/(k!(n-1)!) \le C(2k+n)^{n-1}$, we must have that $\|Q_k f\|^2_{L^2(\gamma)} \le C(2k+n)^{2m+n-1}$. Now the asymptotic property (3-8) in Lemma 3.2 along with estimate (3-17) gives

$$\big(\tfrac{1}{2}\rho^2\big)^{(s-1)/2}\Gamma\big(\tfrac{1}{2}(2k+n+1+s)\big)W_{-(k+n/2),\,s/2}\big(\tfrac{1}{2}\rho^2\big) \le (\rho\sqrt{2k+n})^{s-1/2}e^{-\rho\sqrt{2k+n}/2},$$

which allows us to conclude that

$$\sum_{k=0}^{\infty}\Big(\Gamma\big(\tfrac{1}{2}(2k+n+1+s)\big)W_{-(k+n/2),\,s/2}\big(\tfrac{1}{2}\rho^2\big)\Big)^2(2k+n)^{2m+n-1} < \infty.$$

Consequently, $S^1_\rho f$ make sense and hence solves the extension problem. Now we observe that an easy calculation yields

$$\frac{\big(\frac{1}{2}\rho^2\big)^{(s-1)/2}}{\Gamma(s)}\Gamma\big(\tfrac{1}{2}(2k+n+1+s)\big)W_{-(k+n/2),\,s/2}\big(\tfrac{1}{2}\rho^2\big) = \frac{2^{-s}}{\Gamma(s)}\rho^{2s}L\big(\tfrac{1}{4}\rho^2, \tfrac{1}{2}(2|\alpha|+n+1+s), \tfrac{1}{2}(2|\alpha|+n+1-s)\big), \tag{3-20}$$

which together with the expression (3-18) yields that $S^1_\rho f$ is in the form (2-3) and, as discussed in the previous subsection, this converges to f as $\rho \to 0$. Also note that from the asymptotic property in (3-5), we have $\big(\frac{1}{2}\rho^2\big)^{(s-1)/2}M_{-(k+n/2),\,s/2}\big(\frac{1}{2}\rho^2\big)$ approaches zero as $\rho \to 0$. So $S^2_\rho g \to 0$ as $\rho \to 0$. Therefore $u = S^1_\rho f + S^2_\rho g$ solves the extension problem with initial condition f.

Conversely, suppose $u(x, \rho)$ is a solution of the extension equation (2-1) with initial condition f whose Fourier–Hermite coefficients associated to the Hermite polynomials have tempered growth. Then as discussed in the beginning of this subsection we have

$$\tilde{u}(\alpha, \rho) = \left(\tfrac{1}{2}\rho^2\right)^{(s-1)/2}\left(C_1(|\alpha|)W_{-(k+n/2),\, s/2}\left(\tfrac{1}{2}\rho^2\right) + C_2(|\alpha|)M_{-(k+n/2),\, s/2}\left(\tfrac{1}{2}\rho^2\right)\right).$$

Now using $\tilde{u}(\alpha, 0) = (f, H_\alpha)$ and the behaviour of $\left(\tfrac{1}{2}\rho^2\right)^{(s-1)/2} W_{-(k+n/2),\, s/2}\left(\tfrac{1}{2}\rho^2\right)$ near $\rho = 0$, see (3-5), we have

$$C_1(|\alpha|) = \frac{\Gamma\left(\tfrac{1}{2}(2|\alpha| + n + 1 + s)\right)}{\Gamma(s)} (f, H_\alpha)_{L^2(\gamma)}.$$

Also since $M_\gamma u(x, \rho)$ is tempered, $\tilde{u}(\alpha, \rho)$ has at most polynomial growth in $|\alpha|$. But estimate (3-17) along with the asymptotic property (3-8) yields

$$\left(\tfrac{1}{2}\rho^2\right)^{(s-1)/2} M_{-(k+n/2),\, s/2}\left(\tfrac{1}{2}\rho^2\right) \le C(\rho)^{s-1/2}(\sqrt{2k+n})^{-s-1/2} e^{3\rho\sqrt{2k+n}/2}$$

for large k. Hence we must have $C_2(|\alpha|)$ decaying as $e^{-3\rho\sqrt{2|\alpha|+n}/2}$ for every $\rho > 0$. So let us take $g = \sum_{\alpha\in\mathbb{N}^n} C_2(|\alpha|)H_\alpha$. Then the function g satisfies $\|Q_k g\|^2_{L^2(\gamma)} \le Ce^{-3\rho\sqrt{2k+n}/2}$ for every $\rho > 0$. This ensures that $g \in H^2_{\gamma,3\rho/2}(\mathbb{R}^n)$ for every $\rho > 0$, which completes the proof. □

Remark. For any $\rho > 0$, the space $H^2_\rho(\mathbb{R}^n)$ has an interesting characterisation. It is well known that any g from this space has a holomorphic extension to the tube domain $\Omega_\rho = \{z = x + iy \in \mathbb{C}^n : |y| < \rho\}$ in $\mathbb{C}^n$ which belongs to $L^2(\Omega_\rho, w_\rho)$ for an explicit positive weight function w_ρ given by

$$w_\rho(z) = (\rho^2 - |y|^2)^{n/2} \frac{J_{n/2-1}(2i(\rho^2 - |x|^2)^{1/2}|x|)}{(2i(\rho^2 - |x|^2)|x|)^{n/2-1}}, \quad z = x + iy \in \mathbb{C}^n,$$

where $J_{n/2-1}$ denotes the Bessel function of order $(n/2 - 1)$. We denote this weighted Bergman space by $H^2_\rho(\mathbb{C}^n)$. Thangavelu [2010] proved that for any holomorphic function F on Ω_ρ,

$$\int_{\Omega_\rho} |F(z)|^2 w_\rho(z)\, dz = c_n \sum_{k=0}^{\infty} \|P_k f\|^2 \frac{k!(n-1)!}{(k+n-1)!} L_k^{n-1}(-2\rho^2)e^{\rho^2}, \tag{3-21}$$

where f is the restriction of F to $\mathbb{R}^n$. In view of this identity we see that $g \in H^2_\rho(\mathbb{R}^n)$ if and only if the function $M_\gamma g$ extends holomorphically to Ω_ρ and belongs to $H^2_\rho(\mathbb{C}^n)$. We refer the reader to [Thangavelu 2010] for more details in this regard. From this observation we infer that the condition $g \in \bigcap_{t>0} H^2_{\gamma,t}(\mathbb{R}^n)$ in the above theorem can be replaced by the requirement that $M_\gamma g$ extends holomorphically and belongs to $\bigcap_{t>0} H^2_t(\mathbb{C}^n)$.

We also have the following characterisation of the solution $u(x, \rho)$ when $M_\gamma u(x, \rho)$ has tempered growth in both the variables.

Theorem 3.4. *Suppose $u(x, \rho)$ is a solution of the extension problem* (2-1), *where $M_\gamma u$ is tempered* (*in both variables*). *Then $u = S^1_\rho f$ for some $f \in L^p(\gamma)$ if and only if* $\sup_{\rho>0} \|u(\cdot, \rho)\|_{L^p(\gamma)} \le C$.

Proof. Suppose $f \in L^p(\gamma)$, and let $u = S_\rho^1 f$. Then, as mentioned earlier,

$$u(x,\rho) = \frac{4^{-s}}{\Gamma(s)}\rho^{2s}\int_0^\infty k_{t,s}(\rho)e^{-tL}f(x)\,dt.$$

Now since e^{-tL} is a contraction semigroup on $L^p(\gamma)$, we have

$$\|u(\cdot,\rho)\|_{L^p(\gamma)} \le \|f\|_{L^p(\gamma)}\frac{4^{-s}}{\Gamma(s)}\rho^{2s}\int_0^\infty k_{t,s}(\rho)\,dt.$$

Proceeding in a similar way as before, one can easily see that

$$\int_0^\infty k_{t,s}(\rho)\,dt = 2^{s+1}L\big(\tfrac14\rho^2, \tfrac12(1+s), \tfrac12(1-s)\big).$$

So we have

$$\|u(\cdot,\rho)\|_{L^p(\gamma)} \le C_s\|f\|_{L^p(\gamma)}\rho^{2s}L\big(\tfrac14\rho^2, \tfrac12(1+s), \tfrac12(1-s)\big).$$

Now we make use of an estimate for the L function, see [Roncal and Thangavelu 2020b, p. 18], to get

$$\|u(\cdot,\rho)\|_{L^p(\gamma)} \le C_s\|f\|_{L^p(\gamma)}\rho^{2s}\Gamma(s)\big(\tfrac12\rho^2\big)^{-s}e^{-\rho^2/4} = 2\|f\|_{L^p(\gamma)}e^{-\rho^2/4},$$

which gives the required boundedness.

Conversely, let $\sup_{\rho>0}\|u(\cdot,\rho)\|_{L^p(\gamma)} \le C$. This condition allows us to extract a subsequence ρ_j along which $u(\cdot,\rho)$ converges weakly to a function $f \in L^p(\gamma)$. Letting ρ go to zero along ρ_j, from (3-7) we have

$$\tilde u(\alpha,\rho)\big(\tfrac12\rho^2\big)^{(s-1)/2} = \frac{\Gamma\big(\tfrac12(2|\alpha|+n+1+s)\big)}{\Gamma(s)}(f,H_\alpha)_{L^2(\gamma)}W_{-(k+n/2),\,s/2}\big(\tfrac12\rho^2\big) + C_2(|\alpha|)M_{-(k+n/2),\,s/2}\big(\tfrac12\rho^2\big).$$

Now as $\rho \to \infty$ we have

$$M_{-(k+n/2),\,s/2}\big(\tfrac12\rho^2\big) \sim \frac{\Gamma(1+2\mu)}{\Gamma(1/2+\mu+(k+n/2))}e^{\rho^2/4}\rho^{(2k+n)}.$$

But it is given that $\tilde u(\alpha,\rho)$ has polynomial growth in the ρ variable, so we must have $C_2(\alpha) = 0$, and hence we are done. □

Now we turn our attention to the holomorphic extendability of solutions of the extension problem under consideration. To motivate what we plan to do, we first recall a result about holomorphic extendability of solutions of the following extension problem for the Laplacian on $\mathbb{R}^n$:

$$\Big(\Delta + \partial_\rho^2 + \frac{1-s}{\rho}\partial_\rho\Big)u(x,\rho) = 0, \quad u(x,0) = f(x), \quad x \in \mathbb{R}^n, \quad \rho > 0.$$

After the remarkable work of Caffarelli and Silvestre [2007], this problem has been extensively studied in the literature. See, for example, the work of Stinga and Torrea [2010]. It is known that for $f \in L^2(\mathbb{R}^n)$, the function $u(x,\rho) = \rho^s f * \varphi_{s,\rho}(x)$, where $\varphi_{s,\rho}$ is the generalised Poisson kernel given by

$$\varphi_{s,\rho}(x) = \pi^{-n/2}\frac{\Gamma\big(\tfrac12(n+s)\big)}{|\Gamma(s)|}(\rho^2+|x|^2)^{-(n+s)/2}, \quad x \in \mathbb{R}^n,$$

is a solution of the extension problem. Recently in [Roncal and Thangavelu 2020b], the authors proved that a necessary and sufficient condition for the solution of the above problem to be of the form $u(x,\rho) = \rho^s f * \varphi_{s,\rho}(x)$ for some $f \in L^2(\mathbb{R}^n)$ is that $u(\cdot,\rho)$ extends holomorphically to the tube domain Ω_ρ in $\mathbb{C}^n$, belongs to a weighted Bergman space $B_s(\Omega_\rho)$ and satisfies the uniform estimate $\|u(\cdot,\rho)\|_{B_s} \le C$ for all $\rho > 0$, where the norm $\|\cdot\|_{B_s}$ is given by

$$\|F\|_{B_s}^2 := \rho^{-n} \int_{\Omega_\rho} |F(x+iy)|^2 \left(1 - \frac{|y|^2}{\rho^2}\right)_+^{s-1} dx\,dy.$$

Our aim in the rest of this section is to prove an analogous result for the extension problem we considered for the Ornstein–Uhlenbeck operator L. In order to do so, we require the following Gutzmer's formula for the Hermite expansions. In order to state the same, we need to introduce some more notations.

Let $\mathrm{Sp}(n,\mathbb{R})$ denote the symplectic group consisting of $2n \times 2n$ real matrices which preserves the symplectic form $[(x,u),(y,v)] = (u\cdot y - v\cdot x)$ on $\mathbb{R}^{2n}$ with determinant 1. Recall that $O(2n,\mathbb{R})$ stands for the orthogonal group, and let $K := \mathrm{Sp}(n,\mathbb{R}) \cap O(2n,\mathbb{R})$. For a complex matrix $\sigma = a+ib$, it is known that σ is unitary if and only if the matrix $\sigma_A := \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ belongs to the group K which yields a one to one correspondence between K and the unitary group $U(n)$. A proof of this can be found in [Folland 1989]. We let $\sigma\cdot(x,u)$ stand for the action of σ_A on (x,u), which clearly has a natural extension to $\mathbb{C}^n \times \mathbb{C}^n$. Also given $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$, let $\pi(x,u)$ be the unitary operator acting on $L^2(\mathbb{R}^n)$ defined by

$$\pi(x,u)\phi(\xi) = e^{i(x\cdot\xi + x\cdot u/2)}\phi(\xi+u), \quad \xi \in \mathbb{R}^n.$$

Clearly for $(z,w) \in \mathbb{C}^n \times C^n$, as long as ϕ is holomorphic, $\pi(z,w)\phi(\xi)$ makes perfect sense. Also note that Laguerre functions of type $(n-1)$, defined earlier in (2-8), can be considered as a function on $\mathbb{R}^n \times \mathbb{R}^n$ which can be holomorphically extended to $\mathbb{C}^n \times \mathbb{C}^n$ as follows:

$$\varphi_k(z,w) := L_k^{n-1}\big(\tfrac{1}{2}(z^2+w^2)\big)e^{-(z^2+w^2)/4}, \quad z, w \in \mathbb{C}^n.$$

We have the following very useful identity proved in [Thangavelu 2008]:

Theorem 3.5 (Gutzmer's formula). *For a holomorphic function f on $\mathbb{C}^n$, we have*

$$\int_{\mathbb{R}^n}\int_K |\pi(\sigma\cdot(z,w))f(\xi)|^2\,d\sigma\,d\xi = e^{(u\cdot y - v\cdot x)} \sum_{k=0}^{\infty} \frac{k!\,(n-1)!}{(k+n-1)!}\varphi_k(2iy, 2iv)\|P_k f\|_2^2,$$

where $z = x+iy,\ w = u+iv \in \mathbb{C}^n$.

We use this to prove the following result:

Proposition 3.6. *Let $\delta > 0$. For a holomorphic function F on Ω_t, we have the identity*

$$\int_{\mathbb{R}^n}\int_{|y|<t} |F(x+iy)|^2 w_t^\delta(x,y)\,dx\,dy = C_n \sum_{k=0}^{\infty} \|P_k f\|_2^2 \frac{\Gamma(k+1)\Gamma(n+\delta)}{\Gamma(k+n+\delta)} L_k^{n+\delta-1}(-2t^2)t^{2n},$$

where f denotes the restriction of F to $\mathbb{R}^n$ and the weight $w_t^\delta > 0$ is given by

$$w_t^\delta(x,y) = \frac{1}{\Gamma(\delta)} \int_{\mathbb{R}^n} e^{-2u\cdot x}\left(1 - \frac{|u|^2+|y|^2}{t^2}\right)_+^{\delta-1} e^{-(|u|^2+|y|^2)}\,du.$$

Proof. Let F be holomorphic in the tube domain $\Omega_t = \{z = x+iy : |y| < t\}$ of $\mathbb{C}^n$. Now since the Lebesgue measure is rotationally invariant, $(1-(|u|^2+|y|^2)/t^2)_+^{\delta-1} e^{-(|u|^2+|y|^2)}\, dy\, du$ is a rotation-invariant measure. So, using Gutzmer's formula, we have

$$\int_{\mathbb{R}^{2n}}\biggl(\int_{\mathbb{R}^n} |\pi(iy, iv)F(\xi)|^2\, d\xi\biggr)\biggl(1-\frac{|u|^2+|y|^2}{t^2}\biggr)_+^{\delta-1} e^{-(|u|^2+|y|^2)}\, dy\, du$$
$$= c_n \sum_{k=0}^{\infty} \|P_k f\|_2^2 \frac{k!\,(n-1)!}{(k+n-1)!} \int_{\mathbb{R}^{2n}} \varphi_k(2iy, 2iu)\biggl(1-\frac{|u|^2+|y|^2}{t^2}\biggr)_+^{\delta-1} e^{-(|u|^2+|y|^2)}\, dy\, du. \tag{3-22}$$

Integrating in polar coordinates, the integral on the right-hand side becomes

$$\int_{\mathbb{R}^{2n}} \varphi_k(2iy, 2iu)\biggl(1-\frac{|u|^2+|y|^2}{t^2}\biggr)_+^{\delta-1} e^{-(|u|^2+|y|^2)}\, dy\, du = \omega_{2n}\int_0^\infty L_k^{n-1}(-2r^2)\biggl(1-\frac{r^2}{t^2}\biggr)_+^{\delta-1} r^{2n-1}\, dr.$$

Now using a change of variable $r \to rt$ followed by another change of variable $r \to \sqrt{r}$ in the integral in the right-hand side of the above equation, we have

$$\int_0^\infty L_k^{n-1}(-2r^2)\biggl(1-\frac{r^2}{t^2}\biggr)_+^{\delta-1} r^{2n-1}\, dr = \frac{1}{2}t^{2n}\int_0^1 L_k^{n-1}(r(-2t^2))(1-r)^{\delta-1} r^{n-1}\, dr.$$

By making use of the following identity (see [Szegő 1967]),

$$L_k^{\alpha}(t) = \frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha-\beta)\Gamma(k+\beta+1)}\int_0^1 (1-r)^{\alpha-\beta-1} r^{\beta} L_k^{\beta}(rt)\, dr,$$

the above yields

$$\frac{1}{\Gamma(\delta)}\int_{\mathbb{R}^{2n}} \varphi_k(2iy, 2iu)\biggl(1-\frac{|u|^2+|y|^2}{t^2}\biggr)_+^{\delta-1} e^{-(|u|^2+|y|^2)}\, dy\, du$$
$$= \frac{1}{2}t^{2n}\omega_{2n}\frac{\Gamma(k+n)}{\Gamma(k+n+\delta)} L_k^{n+\delta-1}(-2t^2). \tag{3-23}$$

Now we simplify the left-hand side of (3-22):

$$\begin{aligned}
&\frac{1}{\Gamma(\delta)}\int_{\mathbb{R}^{2n}}\biggl(\int_{\mathbb{R}^n} |\pi(iy, iv)F(\xi)|^2\, d\xi\biggr)\biggl(1-\frac{|u|^2+|y|^2}{t^2}\biggr)_+^{\delta-1} e^{-(|u|^2+|y|^2)}\, dy\, du\\
&\quad= \frac{1}{\Gamma(\delta)}\int_{\mathbb{R}^{2n}}\biggl(\int_{\mathbb{R}^n} |e^{i(iy\cdot\xi+iy\cdot iu/2)}F(\xi+iu)|^2\, d\xi\biggr)\biggl(1-\frac{|u|^2+|y|^2}{t^2}\biggr)_+^{\delta-1} e^{-(|u|^2+|y|^2)}\, dy\, du\\
&\quad= \frac{1}{\Gamma(\delta)}\int_{\mathbb{R}^{2n}}\biggl(\int_{\mathbb{R}^n} |e^{-2y\cdot\xi}F(\xi+iu)|^2\, d\xi\biggr)\biggl(1-\frac{|u|^2+|y|^2}{t^2}\biggr)_+^{\delta-1} e^{-(|u|^2+|y|^2)}\, dy\, du\\
&\quad= \int_{\mathbb{R}^{2n}} |F(\xi+iu)|^2\biggl(\frac{1}{\Gamma(\delta)}\int_{\mathbb{R}^n} e^{-2y\cdot\xi}\biggl(1-\frac{|u|^2+|y|^2}{t^2}\biggr)_+^{\delta-1} e^{-(|u|^2+|y|^2)}\, dy\biggr)\, d\xi\, du\\
&\quad= \int_{\mathbb{R}^{2n}} |F(\xi+iu)|^2 w_t^{\delta}(u, \xi)\, d\xi\, du.
\end{aligned}$$

Now, when $|u| \geq t$, we see that $(1-(|u|^2+|y|^2)/t^2)_+^{\delta-1}=0$ for all $y \in \mathbb{R}^n$. Thus,

$$\int_{\mathbb{R}^{2n}} |F(\xi+iu)|^2 w_t^\delta(u,\xi)\,d\xi\,du = \int_{\mathbb{R}^n}\int_{|u|<t} |F(\xi+iu)|^2 w_t^\delta(u,\xi)\,d\xi\,du.$$

Finally, we have

$$\int_{\mathbb{R}^n}\int_{|u|<t} |F(\xi+iu)|^2 w_t^\delta(u,\xi)\,d\xi\,du = c_n \sum_{k=0}^\infty \|P_k f\|_2^2 \frac{\Gamma(k+1)\Gamma(n+\delta)}{\Gamma(k+n+\delta)} L_k^{n+\delta-1}(-2t^2)t^{2n}. \qquad \square$$

For $s>0$, we consider the following positive weight function $\widetilde{w}_\rho(k)$ on $\mathbb{N}$ given by the sequence

$$\left(\tfrac{1}{2}\rho^2\right)^{s-1}\left(\Gamma\left(\tfrac{1}{2}(2k+n+1+s)\right)W_{-(k+n/2),\,s/2}\left(\tfrac{1}{2}\rho^2\right)\right)^2 \frac{\Gamma(k+1)\Gamma(n+2s)}{\Gamma(k+n+2s)} L_k^{n+2s-1}\left(-\tfrac{1}{2}\rho^2\right).$$

We define $W_\rho^s(\mathbb{R}^n)$ to be the space of all tempered distributions f for which

$$\|f\|_{s,\rho}^2 := \sum_{k=0}^\infty \widetilde{w}_\rho(k)\|P_k f\|_2^2 < \infty.$$

Remark. For $r<0$, the following asymptotic property of Laguerre functions is well known (see [Szegő 1967, Theorem 8.22.3]) and is valid for large k, for $r \leq -c$ and for $c>0$:

$$L_k^\alpha(r) = \frac{1}{2\sqrt{\pi}} e^{r/2}(-r)^{-\alpha/2-1/4} k^{\alpha/2-1/4} e^{2\sqrt{-kr}}(1+O(k^{-1/2})). \tag{3-24}$$

The asymptotic property (3-9) together with (3-17) gives

$$\left(\tfrac{1}{2}\rho^2\right)^{s-1}\left(\Gamma\left(\tfrac{1}{2}(2k+n+1+s)\right)W_{-(k+n/2),\,s/2}\left(\tfrac{1}{2}\rho^2\right)\right)^2 \leq c_1(\rho\sqrt{2k+n})^{2s-1}e^{-\rho\sqrt{2k+n}},$$

and from (3-24) we have

$$L_k^{n+2s-1}\left(-\tfrac{1}{2}\rho^2\right) \leq c e^{\rho^2/4}\rho^{-n-2s+1/2}(2k+n)^{(n+2s-1)/2-1/4}e^{\rho\sqrt{2k+n}}.$$

Now using the fact that $\Gamma(k+1)\Gamma(n+2s)/\Gamma(k+n+2s) \sim (2k+n)^{-(n+2s-1)}$, we have

$$\widetilde{w}_\rho(k) \leq c_1 e^{\rho^2/4}(\rho^2(2k+n))^{-(2n+1)/4}.$$

On the other hand, using (3-9) and (3-24), for large k, we have

$$\widetilde{w}_\rho(k) \geq c_2 e^{\rho^2/4}(\rho^2(2k+n))^{-(2n+1)/4}e^{-\psi_\rho(k)},$$

where $\psi_\rho(k) = 4(2k+n)\zeta(\rho^2/(4(2k+n)))^{1/2} - \rho\sqrt{2k}$. It can be checked that for $0<\rho\leq 1$, the function $\psi_\rho(k)$ is decreasing in k, whence $\psi_\rho(k) \leq c$ for some constant c depending on ρ. So finally we have

$$c_2 e^{\rho^2/4}(\rho^2(2k+n))^{-(2n+1)/4} \leq \widetilde{w}_\rho(k) \leq c_1 e^{\rho^2/4}(\rho^2(2k+n))^{-(2n+1)/4}. \tag{3-25}$$

By letting $m_n = -\frac{1}{8}(2n+1)$, we clearly see that $f \in W_\rho^s(\mathbb{R}^n)$ if and only if $f \in W_H^{m_n}(\mathbb{R}^n)$ whenever $0<\rho\leq 1$. Here $W_H^m(\mathbb{R}^n)$ denotes the Hermite Sobolev spaces.

In view of the connection between the operators H and L, to prove Theorem 1.7 it suffices to prove the following characterisation for the solution of the extension problem for H. Note that the extension problem for the Hermite operator H we are talking about reads as

$$\Big(-H+\partial_\rho^2+\frac{1-2s}{\rho}\partial_\rho-\frac{1}{4}\rho^2\Big)u(x,\rho)=0,\quad u(x,0)=f(x).$$

For $\rho>0$, let T_ρ stand for the operator defined for reasonable f by

$$T_\rho f(x):=\big(\tfrac{1}{2}\rho^2\big)^{(s-1)/2}\frac{1}{\Gamma(s)}\sum_{k=0}^{\infty}\Gamma\big(\tfrac{1}{2}(2k+n+s+1)\big)W_{-(k+n/2),\,s/2}\big(\tfrac{1}{2}\rho^2\big)P_kf(x).$$

Using similar reasoning as in the case of L, we point out that for a tempered distribution f, the above expression makes sense and solves the extension problem for H. Moreover, in view of the relation $Q_k=M_\gamma^{-1}P_kM_\gamma$, we have $T_\rho f=M_\gamma^{-1}S_\rho^1M_\gamma^{-1}f$. Thus Theorem 1.7 easily follows from the following:

Theorem 3.7. *A solution of the extension problem for H is of the form $u(x,\rho)=T_\rho f(x)$ for some $f\in W_H^{m_n}(\mathbb{R}^n)$ if and only if for every $\rho>0$, $u(\cdot\,,\rho)$ extends holomorphically to $\Omega_{\rho/2}$ and satisfies the estimate*

$$\int_{\Omega_{\rho/2}}|u(z,\rho)|^2w_{\rho/2}^{2s}(z)\,dz\le C\rho^{n-1/2},\tag{3-26}$$

for all $0<\rho\le 1$.

Proof. First suppose $u(x,\rho)=T_\rho f(x)$ for some f such that $f\in W_H^{m(s)}(\mathbb{R}^n)$. So clearly

$$u(x,\rho)=\frac{\big(\tfrac{1}{2}\rho^2\big)^{(s-1)/2}}{\Gamma(s)}\sum_{k=0}^{\infty}\Gamma\big(\tfrac{1}{2}(2k+n+s+1)\big)W_{-(k+n/2),\,s/2}\big(\tfrac{1}{2}\rho^2\big)P_kf(x).$$

But the Hermite function $\Phi_\alpha(x)=H_\alpha(x)e^{-|x|^2/2}$ has holomorphic extension to $\mathbb{C}^n$. Let $\Phi_k(z,w):=\sum_{|\alpha|=k}\Phi_\alpha(z)\Phi_\alpha(w)$. Then using the estimate (see [Thangavelu 2010])

$$|\Phi_k(z,\bar z)|\le C(y)(2k+n)^{3(n-1)/4}e^{2\sqrt{2k+n}|y|}$$

along with the asymptotic property (3-9), we conclude that the series

$$\sum_{k=0}^{\infty}\Gamma\big(\tfrac{1}{2}(2k+n+s+1)\big)W_{-(k+n/2),\,s/2}\big(\tfrac{1}{2}\rho^2\big)P_kf(z)$$

converges uniformly over compact subsets of $\Omega_{\rho/2}$ and hence defines a holomorphic function in the domain $\Omega_{\rho/2}$. Now noting that

$$\|P_ku(\cdot\,,\rho)\|_2^2=\rho^{2s-2}c_s^2\Big(\Gamma\big(\tfrac{1}{2}(2k+n+s+1)\big)W_{-(k+n/2),\,s/2}\big(\tfrac{1}{2}\rho^2\big)\Big)^2\|P_kf\|_2^2,$$

in view of Proposition 3.6 we obtain

$$\int_{\mathbb{R}^n}\int_{|y|<\rho/2}|u(x+iy,\rho)|^2w_{\rho/2}^{2s}(x,y)\,dx\,dy=c_n\rho^{2n}\sum_{k=0}^{\infty}\widetilde{w}_\rho(k)\|P_kf\|_2^2.$$

But in view of (3-25),

$$\|f\|_{s,\rho}^2 \le C e^{\rho^2/4} \rho^{-(2n+1)/2} \sum_{k=0}^{\infty} (2k+n)^{2m_n} \|P_k f\|_2^2,$$

which gives

$$\int_{\mathbb{R}^n} \int_{|y|<\rho/2} |u(x+iy,\rho)|^2 w_{\rho/2}^{2s}(x,y)\,dx\,dy \le C e^{\rho^2/4} \rho^{n-1/2} \|f\|_{W_H^{m_n}}^2,$$

proving the first part of the theorem.

Conversely, let $u(z,\rho)$ be holomorphic on $\Omega_{\rho/2}$ for every $\rho > 0$ satisfying the estimate (3-26). Let g_ρ be a tempered distribution such that

$$P_k u(\,\cdot\,,\rho) = \big(\tfrac{1}{2}\rho^2\big)^{(s-1)/2} \Gamma\big(\tfrac{1}{2}(2k+n+s+1)\big) W_{-(k+n/2),\,s/2}\big(\tfrac{1}{2}\rho^2\big) P_k g_\rho. \tag{3-27}$$

Now for $0 < \rho \le 1$, using (3-25) we have

$$\|g_\rho\|_{W_H^{m_n}}^2 \le C e^{-\rho^2/4} \rho^{(2n+1)/2} \sum_{k=0}^{\infty} \widetilde{w}_\rho(k) \|P_k g_\rho\|_2^2.$$

Note that using Proposition 3.6 we obtain

$$\int_{\Omega_{\rho/2}} |u(z,\rho)|^2 w_{\rho/2}^{2s}(z)\,dz = c_n \rho^{2n} \sum_{k=0}^{\infty} \widetilde{w}_\rho(k) \|P_k g_\rho\|_2^2,$$

which by the hypothesis yields $\|g_\rho\|_{W_H^{m_n}}^2 \le C$ for all $0 < \rho \le 1$. Now by the Banach–Alaoglu theorem, we choose a sequence $\{\rho_m\}$ going to 0 such that g_{ρ_m} converges weakly in $W_H^{m_n}(\mathbb{R}^n)$ as $k \to \infty$. Let f be the weak limit in this case. Now given $\varphi \in S(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} u(x,\rho_m)\varphi(x)\,dx = \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} P_k u(x,\rho_m) \overline{P_k \varphi(x)}\,dx.$$

But using (3-27), the above integral equals

$$\sum_{k=0}^{\infty} \big(\tfrac{1}{2}\rho_m^2\big)^{(s-1)/2} \Gamma\big(\tfrac{1}{2}(2|\alpha|+n+s+1)\big) W_{-(|\alpha|+n/2),\,s/2}\big(\tfrac{1}{2}\rho_m^2\big) \int_{\mathbb{R}^n} P_k g_{\rho_m}(x) \overline{P_k \varphi(x)}\,dx.$$

This allows us to conclude that $u(\,\cdot\,,\rho_m)$ converges to f in the sense of distribution. Now under the assumption that u solves the extension problem for H, the exact same argument as in the beginning of this subsection gives

$$\hat{u}(\alpha,\rho) = \big(\tfrac{1}{2}\rho^2\big)^{(s-1)/2} \big(C_1(|\alpha|) W_{-(|\alpha|+n/2),\,s/2}\big(\tfrac{1}{2}\rho^2\big) + C_2(|\alpha|) M_{-(|\alpha|+n/2),\,s/2}\big(\tfrac{1}{2}\rho^2\big)\big),$$

where $\hat{u}(\alpha,\rho)$ denotes the Hermite coefficients. But the estimate (3-26) gives

$$\sum_{k=0}^{\infty} \big(C_2(k) M_{-(k+n/2),\,s/2}\big(\tfrac{1}{2}\rho^2\big)\big)^2 L_k^{n+2s-1}\big(-\tfrac{1}{2}\rho^2\big) \le C(\rho).$$

But since both $M_{-(k+n/2),\,s/2}\big(\tfrac{1}{2}\rho^2\big)$ and $L_k^{n+2s-1}\big(-\tfrac{1}{2}\rho^2\big)$ have exponential growth in k (see (3-8) and (3-24)), the above inequality forces $C_2(k)$ to be zero. Now, as $u(\,\cdot\,,\rho_m)$ converges to f and as ρ_m tends to zero, $\big(\tfrac{1}{2}\rho_m^2\big)^{(s-1)/2} W_{-(k+n/2),\,s/2}\big(\tfrac{1}{2}\rho_m^2\big)$ goes to a constant $\Gamma(s)/\Gamma\big(\tfrac{1}{2}(2k+n+s+1)\big)$ (see (3-5)), and the theorem follows. □

4. Trace Hardy and Hardy's inequality

Trace Hardy inequality. We prove the following trace Hardy inequality only for the operator U as the case of L is similar. We shall work with the gradient on $\mathbb{R}^n \times [0, \infty)$ defined by

$$\nabla_U u := (2^{-1/2}\partial_1 u, 2^{-1/2}\partial_2 u, \dots, 2^{-1/2}\partial_n u, \partial_\rho u).$$

We also let $P_s(\partial_x, \partial_\rho) = \big(-U + \partial_\rho^2 + \frac{1-2s}{\rho}\partial_\rho - \frac{1}{4}\rho^2\big)$ stand for the extension operator.

Lemma 4.1. *Let u and v be two real-valued functions on $\mathbb{R}^n \times [0, \infty)$ such that $u, v \in C_0^2([0, \infty), C^2(\mathbb{R}^n))$. Then for $0 < s < 1$ we have*

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n} \left| \nabla_U u(x, \rho) - \frac{u(x, \rho)}{v(x, \rho)} \nabla_U v(x, \rho) \right|^2 \rho^{1-2s} \, d\gamma(x) \, d\rho \\ = \int_0^\infty \int_{\mathbb{R}^n} \big(|\nabla_U u(x, \rho)|^2 + \big(\tfrac{1}{2}n + \tfrac{1}{4}\rho^2\big) u(x, \rho)^2 \big) \rho^{1-2s} \, d\gamma(x) \, d\rho \\ + \int_0^\infty \int_{\mathbb{R}^n} \frac{u(x, \rho)^2}{v(x, \rho)} (P_s(\partial_x, \partial_\rho) v(x, \rho)) \rho^{1-2s} \, d\gamma(x) \, d\rho \\ + \int_{\mathbb{R}^n} \frac{u(x, 0)^2}{v(x, 0)} \lim_{\rho \to 0} (\rho^{1-2s} \partial_\rho v)(x, \rho) \, d\gamma(x). \qquad (4\text{-}1)\end{aligned}$$

Proof. For any $1 \le j \le n$, we consider the integral

$$\int_{\mathbb{R}^n} \left(\partial_j u - \frac{u}{v} \partial_j v \right)^2 d\gamma(x) = \int_{\mathbb{R}^n} \left((\partial_j u)^2 - 2\frac{u}{v} \partial_j u \partial_j v + \frac{u^2}{v^2} (\partial_j v)^2 \right) d\gamma(x). \qquad (4\text{-}2)$$

Now by the definition of adjoint we get

$$\int_{\mathbb{R}^n} \frac{u}{v} \partial_j u \partial_j v \, d\gamma(x) = \int_{\mathbb{R}^n} u \partial_j^* \left(\frac{u}{v} \partial_j v \right) d\gamma(x).$$

Using the fact that $\partial_j^* = 2x_j - \partial_j$ on $L^2(\gamma)$, we have

$$u \partial_j^* \left(\frac{u}{v} \partial_j v \right) = 2x_j \frac{u^2}{v} \partial_j v - \frac{u}{v} \partial_j u \partial_j v - u^2 \partial_j \left(\frac{1}{v} \partial_j v \right),$$

which together with the above equation yields

$$\begin{aligned} 2 \int_{\mathbb{R}^n} \frac{u}{v} \partial_j u \partial_j v \, d\gamma(x) &= \int_{\mathbb{R}^n} \left(2x_j \frac{u^2}{v} \partial_j v - u^2 \partial_j \left(\frac{1}{v} \partial_j v \right) \right) d\gamma(x) \\ &= \int_{\mathbb{R}^n} \left(2x_j \frac{u^2}{v} \partial_j v - \frac{u^2}{v} \partial_j^2 v + \frac{u^2}{v^2} (\partial_j v)^2 \right) d\gamma(x). \end{aligned}$$

Hence we have

$$\int_{\mathbb{R}^n} \left(\frac{u^2}{v^2} (\partial_j v)^2 - 2\frac{u}{v} \partial_j u \partial_j v \right) d\gamma(x) = -\int_{\mathbb{R}^n} \frac{u^2}{v} \partial_j^* \partial_j v \, d\gamma(x).$$

Similarly, for any $x \in \mathbb{R}^n$ one can obtain

$$\int_0^\infty \left(\frac{u^2}{v^2}(\partial_\rho v)^2 - 2\frac{u}{v}\partial_\rho u \partial_\rho v\right)\rho^{1-2s}\,d\rho = \int_0^\infty \frac{u^2}{v^2}\partial_\rho(\rho^{1-2s}\partial_\rho v)\,d\rho + \frac{u(x,0)^2}{v(x,0)}\lim_{\rho\to 0}(\rho^{1-2s}\partial_\rho v)(x,\rho).$$

Multiplying both side of (4-2) by $\frac{1}{2}$ and summing over j we get the required result. □

Theorem 4.2 (general trace Hardy inequality). *Let $0 < s < 1$. Suppose $\phi \in L^2(\gamma)$ is a real-valued function in the domain of U_s such that $\phi^{-1}U_s\phi$ is locally integrable. Then for any real-valued function $u(x,\rho)$ from the space $C_0^2([0,\infty), C_b^2(\mathbb{R}^n))$ we have*

$$\int_0^\infty \int_{\mathbb{R}^n} \left(|\nabla_U u(x,\rho)|^2 + \left(\tfrac{1}{2}n + \tfrac{1}{4}\rho^2\right)u(x,\rho)^2\right)\rho^{1-2s}\,d\gamma(x)\,d\rho \ge C_{n,s}\int_{\mathbb{R}^n} u(x,0)^2 \frac{L_s\phi(x)}{\phi(x)}\,d\gamma(x).$$

Proof. To prove this result, we make use of Lemma 4.1. Since the left-hand side of (4-1) is always nonnegative, we have, for $0 < s < 1$,

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n} &\left(|\nabla_U u(x,\rho)|^2 + \left(\tfrac{1}{2}n + \tfrac{1}{4}\rho^2\right)u(x,\rho)^2\right)\rho^{1-2s}\,d\gamma(x)\,d\rho \\ &\ge -\int_0^\infty \int_{\mathbb{R}^n} \frac{u(x,\rho)^2}{v(x,\rho)}(P_s(\partial_x,\partial_\rho)v(x,\rho))\rho^{1-2s}\,d\gamma(x)\,d\rho \\ &\qquad - \int_{\mathbb{R}^n}\frac{u(x,0)^2}{v(x,0)}\lim_{\rho\to 0}(\rho^{1-2s}\partial_\rho v)(x,\rho)\,d\gamma(x). \end{aligned} \tag{4-3}$$

Now we take

$$v(x,\rho) = \frac{4^{-s}}{\Gamma(s)}\rho^{2s}\int_0^\infty k_{t,s}(\rho)e^{-tL}\phi(x)\,dt.$$

Then v solves the extension equation (2-1), i.e., $P_s(\partial_x,\partial_\rho)v = 0$ and $v(x,0) = \phi(x)$. Then from (4-3), we have

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n} &\left(|\nabla_U u(x,\rho)|^2 + \left(\tfrac{1}{2}n + \tfrac{1}{4}\rho^2\right)u(x,\rho)^2\right)\rho^{1-2s}\,d\gamma(x)\,d\rho \\ &\ge -\int_{\mathbb{R}^n}\frac{u(x,0)^2}{v(x,0)}\lim_{\rho\to 0}(\rho^{1-2s}\partial_\rho v)(x,\rho)\,d\gamma(x). \end{aligned} \tag{4-4}$$

In view of the above, we need to solve the extension problem for U with a given initial condition ϕ. Since

$$-\lim_{\rho\to 0}\rho^{1-2s}\partial_\rho u(x,\rho) = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}U_s\phi,$$

we get the desired inequality. □

Corollary 4.3. *Let $0 < s < 1$ and $f \in L^2(\gamma)$ with $U_s f \in L^2(\gamma)$. Then we have*

$$\langle U_s f, f\rangle_{L^2(\gamma)} \ge \int_{\mathbb{R}^n} f^2(x)\frac{U_s\phi}{\phi}\,d\gamma(x)$$

for any real-valued ϕ in the domain of U_s.

Proof. When u itself solves the extension problem with initial condition f, the proof of Lemma 4.1 shows that the left-hand side of the trace Hardy inequality reduces to $\langle U_s f, f\rangle_{L^2(\gamma)}$. □

Hardy's inequality from trace Hardy. In this subsection we construct a suitable function ϕ so that $(U_s\phi)/\phi$ simplifies. In order to do so, let us quickly recall some basic facts about Laguerre functions. Let $\alpha > -1$ and $k \in \mathbb{N}$. The Laguerre polynomial of degree k and type α, which we denote by $L_k^\alpha(x)$, is a solution of the ordinary differential equation

$$xy''(x) + (\alpha+1-x)y'(x) + ky(x) = 0,$$

whose explicit expression is given by

$$L_k^\alpha(x) = \sum_{j=0}^{k} \frac{\Gamma(k+\alpha+1)}{\Gamma(k-j+1)\Gamma(j+\alpha+1)} \frac{(-x)^j}{j!}. \tag{4-5}$$

Recall that the Laguerre functions of type $(n-1)$ are given by

$$\varphi_k^{n-1}(r) = L_k^{n-1}\big(\tfrac{1}{2}r^2\big)e^{-r^2/4}, \quad r \geq 0.$$

For more details about such functions we refer the reader to [Thangavelu 1993, Chapter 1]. Now given $s, \rho > 0$, we consider the function $\phi_{s,\rho}$ which is defined in terms of Laguerre polynomials as follows:

$$\phi_{s,\rho}(x) = \sum_{m=0}^{\infty} C_{2m,\rho}(s) L_m^{n/2-1}(|x|^2) = e^{|x|^2/2} \sum_{m=0}^{\infty} C_{2m,\rho}(s)\varphi_m^{n/2-1}(\sqrt{2}|x|),$$

where the coefficients are given in terms of the L function as

$$C_{k,\rho}(s) = \frac{2\pi}{\Gamma\big(\frac{1}{2}(n/2+1+s)\big)^2} L\big(\rho, \tfrac{1}{4}(2k+n) + \tfrac{1}{2}(1+s), \tfrac{1}{4}(2k+n) + \tfrac{1}{2}(1-s)\big).$$

In the following lemma we show how these functions are related via the fractional power of the operator under study.

Lemma 4.4. *For $-1 < s < 1$, we have*

$$U_s\phi_{-s,\rho} = \frac{\Gamma\big(\frac{1}{2}(n/2+1+s)\big)^2}{\Gamma\big(\frac{1}{2}(n/2+1-s)\big)^2} (4\rho)^s \phi_{s,\rho}. \tag{4-6}$$

Proof. Let us take two radial functions g and h on $\mathbb{R}^n$ such that

$$g(x) = \pi^{n/2} e^{|x|^2/2} h(x),$$

where $h \in L^2(\mathbb{R}^n)$. Moreover, we choose h in such a way that the Laguerre coefficients

$$R_m^{n/2-1}(h) = 2\frac{\Gamma(m+1)}{\Gamma(m+n/2)} \int_0^\infty h(r) L_m^{n/2-1}(r^2) e^{-r^2/2} r^{n-1}\, dr$$

are nonzero. By our choice of h and definition of g, it is not hard to see that $Q_k g(x) = e^{|x|^2/2} P_k h(x)$. Also, since h is radial, using a result proved in [Thangavelu 1993, Theorem 3.4.1] we have

$$P_k h(x) = \begin{cases} 0 & \text{if } k = 2m+1, \\ R_m^{n/2-1}(h) L_m^{n/2-1}(|x|^2) e^{-|x|^2/2} & \text{if } k = 2m. \end{cases} \tag{4-7}$$

Now using the definition of the Laguerre function along with the fact that $R_m^{n/2-1}(h) \neq 0$, we see that

$$\begin{aligned}\phi_{s,\rho}(x) &= e^{|x|^2/2} \sum_{m=0}^{\infty} \frac{2\pi}{\Gamma(n/2+1+s)^2} L\big(\rho, \tfrac{1}{4}(4m+n)+\tfrac{1}{2}(1+s), \tfrac{1}{4}(4m+n)+\tfrac{1}{2}(1-s)\big)\varphi_m^{n/2-1}(\sqrt{2}|x|) \\ &= e^{|x|^2/2} \sum_{m=0}^{\infty} C_{2m,\rho}(s)(R_m^{n/2-1}(h))^{-1} R_m^{n/2-1}(h) L_m^{n/2}(|x|^2) e^{|x|^2/2}.\end{aligned}$$

But observation (4-7) and the fact that

$$Q_k g(x) = e^{|x|^2/2} P_k h(x)$$

transform the above equation into

$$\phi_{s,\rho}(x) = \sum_{k=0}^{\infty} C_{k,\rho}(s)(R_{\lfloor k/2\rfloor}^{n/2-1}(h))^{-1} Q_k g(x). \tag{4-8}$$

Hence using the definition of U_s we have

$$U_s \phi_{-s,\rho} = \sum_{k=0}^{\infty} C_{k,\rho}(-s) 2^s \frac{\Gamma\big(\frac{1}{4}(2k+n)+\frac{1}{2}(1+s)\big)}{\Gamma\big(\frac{1}{4}(2k+n)+\frac{1}{2}(1-s)\big)} (R_{\lfloor k/2\rfloor}^{n/2-1}(h))^{-1} Q_k g. \tag{4-9}$$

But in view of the transformation property (2-12), we have

$$\begin{aligned}&C_{k,\rho}(-s) \\ &= \frac{2\pi}{\Gamma\big(\frac{1}{2}(n/2+1-s)\big)^2} L\big(\rho, \tfrac{1}{4}(2k+n)+\tfrac{1}{2}(1-s), \tfrac{1}{4}(2k+n)+\tfrac{1}{2}(1+s)\big) \\ &= \frac{2\pi}{\Gamma\big(\frac{1}{2}(n/2+1-s)\big)^2} (2\rho)^s \frac{\Gamma\big(\frac{1}{4}(2k+n)+\frac{1}{2}(1-s)\big)}{\Gamma\big(\frac{1}{4}(2k+n)+\frac{1}{2}(1+s)\big)} L\big(\rho, \tfrac{1}{4}(2k+n)+\tfrac{1}{2}(1+s), \tfrac{1}{4}(2k+n)+\tfrac{1}{2}(1-s)\big) \\ &= \frac{\Gamma\big(\frac{1}{2}(n/2+1+s)\big)^2}{\Gamma\big(\frac{1}{2}(n/2+1-s)\big)^2} (2\rho)^s \frac{\Gamma\big(\frac{1}{4}(2k+n)+\frac{1}{2}(1-s)\big)}{\Gamma\big(\frac{1}{4}(2k+n)+\frac{1}{2}(1+s)\big)} C_{k,\rho}(s).\end{aligned} \tag{4-10}$$

Hence from (4-9) we obtain

$$U_s \phi_{-s,\rho} = \frac{\Gamma\big(\frac{1}{2}(n/2+1+s)\big)^2}{\Gamma\big(\frac{1}{2}(n/2+1-s)\big)^2} (4\rho)^s \phi_{s,\rho}. \qquad \square$$

Now in the rest of the section we will calculate $\phi_{s,\rho}$ almost explicitly in terms of the Macdonald's function K_ν, defined for $z > 0$ by the integral

$$K_\nu(z) := 2^{-\nu-1} z^\nu \int_0^\infty e^{-t-z^2/(4t)} t^{-\nu-1}\, dt.$$

Proposition 4.5. *Let* $0 < s < 1$ *and* $\rho > 0$. *Then we have*

$$\phi_{s,\rho}(x) = 2 \frac{\sqrt{\pi}\, 2^{-(n/2+1+s)/2}}{\sqrt{2\pi}\,\Gamma\big(\frac{1}{2}(n/2+1+s)\big)} e^{|x|^2/2} (\rho+|x|^2)^{-(n/2+1+s)/2} K_{(n/2+1+s)/2}(\rho+|x|^2). \tag{4-11}$$

Proof. First we note the following formula proved in [Ciaurri et al. 2018, Lemma 3.8]:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda t}((\rho+r^2)^2+t^2)^{(\alpha+2+s)/2}\,dt = |\lambda|^{\alpha+1}\sum_{k=0}^{\infty} c_{k,\rho}^{\lambda}(s)\varphi_k^{\alpha}(\sqrt{(2|\lambda|)}r), \tag{4-12}$$

where the coefficients $c_{k,\rho}^{\lambda}(s)$ are given by

$$c_{k,\rho}^{\lambda}(s) = \frac{2\pi\,|\lambda|^s}{\Gamma\big(\frac{1}{2}(\alpha+2+s)\big)^2} L\big(\rho|\lambda|,\, \tfrac{1}{4}(4k+2\alpha+2)+\tfrac{1}{2}(1+s),\, \tfrac{1}{4}(4k+2\alpha+2)+\tfrac{1}{2}(1-s)\big).$$

This holds for any $\lambda \neq 0$ and $\alpha > -\frac{1}{2}$. In particular, taking $\alpha = \frac{1}{2}n - 1$ and $\lambda = 1$ in (4-12), we have

$$\phi_{s,\rho}(x) = e^{|x|^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it}((\rho+|x|^2)^2+t^2)^{-(n/2+1+s)/2}\,dt. \tag{4-13}$$

The right-hand side of the above equation can be computed in terms of the Macdonald's function K_ν. Now we make use of the formula (see [Prudnikov et al. 1986, p. 390])

$$\int_0^{\infty} \frac{\cos br}{(r^2+z^2)^{\delta}}\,dr = \left(\frac{2z}{b}\right)^{1/2-\delta}\frac{\sqrt{\pi}}{\Gamma(\delta)}K_{1/2-\delta}(bz), \tag{4-14}$$

which is valid for $b > 0$ and $\Re\delta, \Re z > 0$. This gives

$$\int_{-\infty}^{\infty} e^{it}((\rho+|x|^2)^2+t^2)^{-(n/2+1+s)/2}\,dt = 2\frac{\sqrt{\pi}\,2^{-(n/2+1+s)/2}}{\Gamma\big(\frac{1}{2}(n/2+1+s)\big)}(\rho+|x|^2)^{-(n/2+1+s)/2}K_{-(n/2+1+s)/2}(\rho+|x|^2). \tag{4-15}$$

Now using the fact that $K_\nu = K_{-\nu}$, we obtain

$$\phi_{s,\rho}(x) = 2\frac{\sqrt{\pi}\,2^{-(n/2+1+s)/2}}{\sqrt{2\pi}\,\Gamma\big(\frac{1}{2}(n/2+1+s)\big)}e^{|x|^2/2}(\rho+|x|^2)^{-(n/2+1+s)/2}K_{(n/2+1+s)/2}(\rho+|x|^2), \tag{4-16}$$

proving the proposition. □

We are now ready to prove Theorem 1.2. For the convenience of the reader we state the theorem here as well.

Theorem 4.6. *Let $0 < s < 1$. Assume that $f \in L^2(\gamma)$ such that $U_s f \in L^2(\gamma)$. Then for every $\rho > 0$ we have*

$$\langle U_s f, f\rangle_{L^2(\gamma)} \geq (2\rho)^s \frac{\Gamma\big(\frac{1}{2}(n/2+1+s)\big)}{\Gamma\big(\frac{1}{2}(n/2+1-s)\big)}\int_{\mathbb{R}^n}\frac{f(x)^2}{(\rho+|x|^2)^s}w_s(\rho+|x|^2)\,d\gamma(x)$$

for an explicit $w_s(t) \geq 1$. The inequality is sharp, and equality is attained for $f(x) = \phi_{-s,\rho}(x)$.

Proof. Taking $\phi = \phi_{-s,\rho}$ in 4.5, in view of Lemma 4.4 we have

$$\frac{U_s\phi}{\phi} = \frac{\Gamma\big(\frac{1}{2}(n/2+1+s)\big)^2}{\Gamma\big(\frac{1}{2}(n/2+1-s)\big)^2}(4\rho)^s\frac{\phi_{s,\rho}}{\phi_{-s,\rho}}.$$

Now we use Proposition 4.5 to simplify the right-hand side of the above equation. Note that

$$\frac{\phi_{s,\rho}}{\phi_{-s,\rho}} = \frac{\Gamma\big(\frac{1}{2}(n/2+1-s)\big)}{\Gamma\big(\frac{1}{2}(n/2+1+s)\big)} 2^{-s} (\rho+|x|^2)^{-s} \frac{K_{(n/2+1+s)/2}(\rho+|x|^2)}{K_{(n/2+1-s)/2}(\rho+|x|^2)}. \tag{4-17}$$

Let

$$w_s(t) := \frac{K_{(n/2+1+s)/2}(t)}{K_{(n/2+1-s)/2}(t)}, \quad t > 0.$$

Using the fact that $K_\nu(t)$ is an increasing function of ν for $t > 0$, we note that $w_s(t) \geq 1$, for all $t > 0$, and

$$\frac{U_s\phi}{\phi} = 2^s \rho^s \frac{\Gamma\big(\frac{1}{2}(n/2+1-s)\big)}{\Gamma\big(\frac{1}{2}(n/2+1+s)\big)} (\rho+|x|^2)^{-s} w_s(\rho+|x|^2).$$

Hence the required inequality follows from Corollary 4.3.

To see that equality holds for $f(x) = \phi_{-s,\rho}(x)$, using Lemma 4.4 we note that

$$\langle U_s\phi_{-s,\rho}, \phi_{-s,\rho}\rangle_{L^2(\gamma)} = \frac{\Gamma\big(\frac{1}{2}(n/2+1+s)\big)^2}{\Gamma\big(\frac{1}{2}(n/2+1-s)\big)^2} (4\rho)^s \int_{\mathbb{R}^n} \phi_{-s,\rho}(x)^2 \frac{\phi_{s,\rho}(x)}{\phi_{-s,\rho}(x)}\, d\gamma(x).$$

We finish by noting that (4-17) allows us to write the above as

$$\langle U_s\phi_{-s,\rho}, \phi_{-s,\rho}\rangle_{L^2(\gamma)} = (2\rho)^s \int_{\mathbb{R}^n} \frac{\phi_{-s,\rho}(x)^2}{(\rho+|x|^2)^s} w_s(\rho+|x|^2)\, d\gamma(x). \qquad \square$$

5. Isometry property for the solution of the extension problem

In this section we prove an isometry property of the solution operator associated to the extension problem for the Ornstein–Uhlenbeck operator under consideration. Such a property has been studied in the context of the extension problem for the Laplacian on $\mathbb{R}^n$ and for the sub-Laplacian on $\mathbb{H}^n$ in [Möllers et al. 2016]. See also the work of Roncal and Thangavelu [2020a], where they proved a similar result in the context of H-type groups.

We consider the Gaussian Sobolev space $\mathcal{H}^s_\gamma(\mathbb{R}^n)$ defined via the relation $f \in \mathcal{H}^s_\gamma(\mathbb{R}^n)$ if and only if $L_{s/2} f \in L^2(\gamma)$, where $L_{s/2}$ is the fractional power under consideration. Instead of $\|L_{s/2} f\|_2$, we use the equivalent norm for this space which is given by

$$\|f\|^2_{(s)} := \langle L_s f, f\rangle_{L^2(\gamma)} = \sum_{\alpha\in\mathbb{N}^n} 2^s \frac{\Gamma\big(\frac{1}{2}(2|\alpha|+n)+\frac{1}{2}(1+s)\big)}{\Gamma\big(\frac{1}{2}(2|\alpha|+n)+\frac{1}{2}(1-s)\big)} |\langle f, H_\alpha\rangle_{L^2(\gamma)}|^2.$$

Recall that the H_α are the normalised Hermite polynomials on $\mathbb{R}^n$ forming an orthonormal basis for $L^2(\gamma)$. As the solution of the extension equation (2-1) is a function of ρ^2, it can be thought of as a function of $(x, y) \in \mathbb{R}^{n+2}$ that is radial in y. Thus it makes sense to define $P_s f(x, y) = u(x, \sqrt{2}|y|)$, where $u(x, \rho)$ is the solution of the extension equation (2-1) given by (2-3). We can now consider $P_s f(x, y)$ as an element of $L^2(\mathbb{R}^{n+2}, \gamma)$. For $(\alpha, j) \in \mathbb{N}^n \times \mathbb{N}^2$, we let

$$H_{\alpha,j}(x, y) := H_\alpha(x) H_j(y), \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^2,$$

where the H_j are two-dimensional Hermite polynomials. Then $P_s f(x, y)$ can be expanded in terms of $H_{\alpha,j}(x, y)$. We will show that P_s takes $\mathcal{H}^s_\gamma(\mathbb{R}^n)$ into $\mathcal{H}^{s+1}(\mathbb{R}^{n+2})$. We equip $\mathcal{H}^{s+1}_\gamma(\mathbb{R}^{n+2})$ with a different but equivalent norm. For $u \in \mathcal{H}^{s+1}_\gamma(\mathbb{R}^{n+2})$, we define

$$\|u\|^2_{(1,s)} = \sum_{(\alpha,j)\in\mathbb{N}^n\times\mathbb{N}^2} 2^{s+1} \frac{\Gamma\big(\frac{1}{2}(2|\alpha|+2|j|+n+1)+\frac{1}{2}(1+(1+s))\big)}{\Gamma\big(\frac{1}{2}(2|\alpha|+2|j|+n+1)+\frac{1}{2}(1-(1+s))\big)} |\langle u^j, H_\alpha\rangle_{L^2(\gamma)}|^2,$$

where for any $j \in \mathbb{N}^2$ we let

$$u^j(x) := \int_{\mathbb{R}^2} u(x, y) H_j(y) e^{-|y|^2/2}\, dy.$$

Equipped with this norm we denote the space $\mathcal{H}^{s+1}_\gamma(\mathbb{R}^{n+2})$ by $\widetilde{\mathcal{H}}^{s+1}_\gamma(\mathbb{R}^{n+2})$.

Theorem 5.1. *For $0 < s < n$, the function $P_s : \mathcal{H}^s_\gamma(\mathbb{R}^n) \to \widetilde{\mathcal{H}}^{s+1}_\gamma(\mathbb{R}^{n+2})$ is a constant multiple of an isometry, i.e., $\|P_s f\|_{(1,s)} = C_{n,s}\|f\|_{(s)}$ for all $f \in \mathcal{H}^s_\gamma(\mathbb{R}^n)$.*

Proof. We have

$$P_s f(x, y) = \sum_{k=0}^{\infty} \frac{2^{-s}}{\Gamma(s)} (\sqrt{2}|y|)^{2s} L\big(\tfrac{1}{2}|y|^2, \tfrac{1}{2}(2k+n)+\tfrac{1}{2}(1+s), \tfrac{1}{2}(2k+n)+\tfrac{1}{2}(1-s)\big) Q_k f.$$

Now from (2-13) we note that

$$\begin{aligned} P_s f(x, y) &= T_{-s,\sqrt{2}|y|}(L_s f)(x) \\ &= \sum_{k=0}^{\infty} \frac{4^s}{\Gamma(-s)} L\big(\tfrac{1}{2}|y|^2, \tfrac{1}{2}(2k+n)+\tfrac{1}{2}(1-s), \tfrac{1}{2}(2k+n)+\tfrac{1}{2}(1+s)\big) \frac{\Gamma\big(\frac{1}{2}(2k+n)+\frac{1}{2}(1+s)\big)}{\Gamma\big(\frac{1}{2}(2k+n)+\frac{1}{2}(1-s)\big)} Q_k f. \end{aligned}$$

Now writing $a := \frac{1}{2}(2k+n)+\frac{1}{2}(1+s)$ and $b := \frac{1}{2}(2k+n)+\frac{1}{2}(1-s)$, we expand $L\big(\frac{1}{2}|y|^2, a, b\big)$ in terms of Hermite polynomials. In order to do this, we use Mehler's formula (see [Urbina-Romero 2019, Chapter 1]) for two-dimensional normalised Hermite polynomials:

$$\sum_{j\in\mathbb{N}^2} H_j(x) H_j(y) r^{|j|} = (1-r^2)^{-1} \exp\bigg(-\frac{r^2(|x|^2+|y|^2)}{1-r^2} - \frac{2rx\cdot y}{1-r^2}\bigg).$$

In view of the definition of the L function, we have

$$L\big(\tfrac{1}{2}|y|^2, a, b\big) = e^{-|y|^2/2} \int_0^\infty e^{-t|y|^2} t^{a-1}(1+t)^{-b}\, dt.$$

Now taking $r^2 = t/(1+t)$ in the above Mehler's formula, we have

$$e^{-t|y|^2} = (1+t)^{-1} \sum_{j\in\mathbb{N}^2} H_j(0) H_j(y) \Big(\frac{t}{1+t}\Big)^{|j|/2},$$

which yields

$$L\big(\tfrac{1}{2}|y|^2, a, b\big) = e^{-|y|^2/2} \sum_{j\in\mathbb{N}^2} H_j(0)H_j(y) \int_0^\infty t^{a+|j|/2-1}(1+t)^{-b-|j|/2-1}\,dt$$
$$= e^{-|y|^2/2} \sum_{j\in\mathbb{N}^2} H_j(0)H_j(y) \frac{\Gamma(a+|j|/2)\Gamma(b-a+1)}{\Gamma(b+|j|/2+1)}.$$

Here the second equality follows from the formula

$$\int_0^\infty (1+t)^{-b} t^{a-1}\,dt = \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)}. \tag{5-1}$$

Finally, writing $P_s f(x, y) = v(x, y)$ and using the above observations, we have

$$v(x,y) = c_s e^{-|y|^2/2} \sum_{(\alpha,j)\in\mathbb{N}^n\times\mathbb{N}^2} H_j(0)H_j(y) \frac{\Gamma(a+|j|/2)\Gamma(b-a+1)}{\Gamma(b+|j|/2+1)} \frac{\Gamma\big(\frac{1}{2}(2|\alpha|+n)+\frac{1}{2}(1+s)\big)}{\Gamma\big(\frac{1}{2}(2|\alpha|+n)+\frac{1}{2}(1-s)\big)} \langle f, H_\alpha\rangle_{L^2(\gamma)} H_\alpha(x),$$

where $c_s := 4^s/\Gamma(-s)$. Now note that for any $j \in \mathbb{N}^2$ we obtain

$$v^j(x) = c_s \sum_{\alpha\in\mathbb{N}^n} H_j(0) \frac{\Gamma(a+|j|/2)\Gamma(b-a+1)}{\Gamma(b+|j|/2+1)} \frac{\Gamma\big(\frac{1}{2}(2|\alpha|+n)+\frac{1}{2}(1+s)\big)}{\Gamma\big(\frac{1}{2}(2|\alpha|+n)+\frac{1}{2}(1-s)\big)} \langle f, H_\alpha\rangle_{L^2(\gamma)} H_\alpha(x),$$

which yields

$$\langle v^j, H_\alpha\rangle_{L^2(\gamma)} = c_s H_j(0) \frac{\Gamma(a+|j|/2)\Gamma(b-a+1)}{\Gamma(b+|j|/2+1)} \frac{\Gamma\big(\frac{1}{2}(2|\alpha|+n)+\frac{1}{2}(1+s)\big)}{\Gamma\big(\frac{1}{2}(2|\alpha|+n)+\frac{1}{2}(1-s)\big)} \langle f, H_\alpha\rangle_{L^2(\gamma)}.$$

As shown in [Urbina-Romero 2019], for any $k \in \mathbb{N}$ and for one-dimensional Hermite polynomials we have

$$H_{2k+1}(0) = 0 \quad \text{and} \quad (H_{2k}(0))^2 = \frac{2^{-2k}\Gamma(2k+1)}{\Gamma(k+1)^2}.$$

But making use of the formula $\Gamma(2z) = (2\pi)^{-1/2} 2^{2z-1/2}\Gamma(z)\Gamma\big(z+\frac{1}{2}\big)$, we obtain

$$(H_{2k}(0))^2 = \frac{1}{\sqrt{\pi}} \frac{\Gamma(k+1/2)}{\Gamma(k+1)}.$$

Hence, for $j = (j_1, j_2) \in \mathbb{N}^n$, we have

$$(H_{2j}(0))^2 = \frac{1}{\pi} \frac{\Gamma(j_1+1/2)\Gamma(j_2+1/2)}{\Gamma(j_1+1)\Gamma(j_2+1)}.$$

With these things in hand we proceed to calculate $\|v\|^2_{(1,s)}$, which is given by a constant multiple of

$$\sum_{k=0}^\infty \sum_{j\in\mathbb{N}^2} \Bigg(\frac{\Gamma\big(\frac{1}{2}(2k+2|j|+n+1)+\frac{1}{2}(1+(1+s))\big)}{\Gamma\big(\frac{1}{2}(2k+2|j|+n+1)+\frac{1}{2}(1-(1+s))\big)} \times \left| H_j(0) \frac{\Gamma(a+|j|/2)\Gamma(b-a+1)}{\Gamma(b+|j|/2+1)} \frac{\Gamma\big(\frac{1}{2}(2k+n)+\frac{1}{2}(1+s)\big)}{\Gamma\big(\frac{1}{2}(2k+n)+\frac{1}{2}(1-s)\big)} \right|^2 \|Q_k f\|^2 \Bigg),$$

where $\|Q_k f\|^2 = \sum_{|\alpha|=k} |\langle f, H_\alpha\rangle_{L^2(\gamma)}|^2$. Now we have already noted the fact that $H_{2k+1}(0) = 0$. In what follows both j_1 and j_2 should be even. Using the values of a and b we have

$$\sum_{j=(j_1,j_2)\in\mathbb{N}^2} \frac{\Gamma\big(\frac{1}{2}(2k+2|j|+n+1)+\frac{1}{2}(1+(1+s))\big)}{\Gamma\big(\frac{1}{2}(2k+2|j|+n+1)+\frac{1}{2}(1-(1+s))\big)} \bigg(H_j(0)\frac{\Gamma(a+|j|/2)\Gamma(b-a+1)}{\Gamma(b+|j|/2+1)}\bigg)^2$$
$$= \frac{\Gamma(s+1)^2}{\pi} \sum_{j\in\mathbb{N}^2} \frac{\Gamma\big(\frac{1}{2}(2k+2|j|+n+1)+\frac{1}{2}(1-(1+s))\big)}{\Gamma\big(\frac{1}{2}(2k+2|j|+n+1)+\frac{1}{2}(1+(1+s))\big)} \frac{\Gamma(j_1+1/2)\Gamma(j_2+1/2)}{\Gamma(j_1+1)\Gamma(j_2+1)}.$$

In order to simplify this further we make use of some properties of Hypergeometric functions. We start by recalling that

$$F(\delta,\beta,\eta,z) = \sum_{k=0}^{\infty} \frac{(\delta)_k(\beta)_k}{(\eta)_k k!} z^k = \frac{\Gamma(\eta)}{\Gamma(\delta)\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\delta+k)\Gamma(\beta+k)}{\Gamma(\eta+k)\Gamma(k+1)} z^k.$$

Here we will be using the following property proved in [Olver and Maximon 2010]:

$$\frac{\Gamma(\eta)\Gamma(\eta-\delta-\beta)}{\Gamma(\eta-\delta)\Gamma(\eta-\beta)} = F(\delta,\beta,\eta,1) = \frac{\Gamma(\eta)}{\Gamma(\delta)\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\delta+k)\Gamma(\beta+k)}{\Gamma(\eta+k)\Gamma(k+1)}.$$

That is,

$$\sum_{k=0}^{\infty} \frac{\Gamma(\delta+k)\Gamma(\beta+k)}{\Gamma(\eta+k)\Gamma(k+1)} = \frac{\Gamma(\beta)\Gamma(\eta-\delta-\beta)\Gamma(\delta)}{\Gamma(\eta-\delta)\Gamma(\eta-\beta)}, \quad \text{provided } \Re(\eta-\delta-\beta) > 0. \tag{5-2}$$

Taking $\delta = \frac{1}{2}(2k+2j_2+n+1-s)$, $\beta = \frac{1}{2}$ and $\eta = \frac{1}{2}(2k+2j_2+n+3+s)$ in the above formula, we have

$$\sum_{j_1=0}^{\infty} \frac{\Gamma(\delta+j_1)\Gamma(\beta+j_1)}{\Gamma(\eta+j_1)\Gamma(j_1+1)} = \frac{\Gamma(s+1/2)\Gamma(1/2)}{\Gamma(s+1)} \frac{\Gamma\big(\frac{1}{2}(2k+2j_2+n+1-s)\big)}{\Gamma\big(\frac{1}{2}(2k+2j_2+n+2+s)\big)}.$$

This gives

$$\sum_{j\in\mathbb{N}^2} \frac{\Gamma\big(\frac{1}{2}(2k+2|j|+n+1)+\frac{1}{2}(1-(1+s))\big)}{\Gamma\big(\frac{1}{2}(2k+2|j|+n+1)+\frac{1}{2}(1+(1+s))\big)} \frac{\Gamma(j_1+1/2)\Gamma(j_2+1/2)}{\Gamma(j_1+1)\Gamma(j_2+1)}$$
$$= \frac{\Gamma(s+1/2)\Gamma(1/2)}{\Gamma(s+1)} \sum_{j_2=0}^{\infty} \frac{\Gamma\big(\frac{1}{2}(2k+n+1-s)+j_2\big)\Gamma(j_2+1/2)}{\Gamma\big(\frac{1}{2}(2k+n+2+s)+j_2\big)\Gamma(j_2+1)}$$
$$= \frac{\Gamma(s+1/2)\Gamma(1/2)}{\Gamma(s+1)} \frac{\Gamma\big(\frac{1}{2}(2k+n+1-s)\big)}{\Gamma\big(\frac{1}{2}(2k+n+1+s)\big)} \frac{\Gamma(s)\Gamma(1/2)}{\Gamma(s+1/2)}$$
$$= \frac{\Gamma(1/2)^2}{s} \frac{\Gamma\big(\frac{1}{2}(2k+n+1-s)\big)}{\Gamma\big(\frac{1}{2}(2k+n+1+s)\big)}.$$

Therefore, we have

$$\begin{aligned}\|v\|^2_{(1,s)} &= c_s^2\Gamma(s+1)^2\frac{2\Gamma(1/2)^2}{\pi s}\sum_{k=0}^{\infty}2^s\frac{\Gamma\big(\frac{1}{2}(2k+n+1-s)\big)}{\Gamma\big(\frac{1}{2}(2k+n+1+s)\big)}\|Q_k f\|^2\\ &= c_{n,s}\sum_{\alpha\in\mathbb{N}^n}2^s\frac{\Gamma\big(\frac{1}{2}(2|\alpha|+n+1-s)\big)}{\Gamma\big(\frac{1}{2}(2|\alpha|+n+1+s)\big)}|\langle f, H_\alpha\rangle_{L^2(\gamma)}|^2 = c_{n,s}\|f\|^2_{(s)}.\end{aligned}$$ □

6. Hardy–Littlewood–Sobolev inequality for H_s

In this section we are interested in the Hardy–Littlewood–Sobolev inequality for the fractional powers H_s. For the Laplacian on $\mathbb{R}^n$ and the sub-Laplacian on $\mathbb{H}^n$, such inequalities with sharp constants are known. Let us recall the inequality for the sub-Laplacian $\mathcal{L}$ on $\mathbb{H}^n$. Letting $q = 2(n+1)/(n+1-s)$, the Hardy–Littlewood–Sobolev inequality for $\mathcal{L}_s$ (see [Branson et al. 2013; Frank and Lieb 2012]) reads as

$$\frac{\Gamma\big(\frac{1}{2}(1+n+s)\big)^2}{\Gamma\big(\frac{1}{2}(1+n-s)\big)^2}w_{2n+1}^{s/(n+1)}\bigg(\int_{\mathbb{H}^n}|g(z,w)|^q\,dz\,dw\bigg)^{2/q} \le \langle\mathcal{L}_s g, g\rangle. \tag{6-1}$$

We first find an integral representation of H_{-s} using the integral representation of fractional powers of the sub-Laplacian, $\mathcal{L}_{-s}$. The integral kernel of $\mathcal{L}_{-s}$ is given by $c_{n,s}|(z,t)|^{-Q+2s}$ as shown in [Roncal and Thangavelu 2016]. Here $|(z,t)| := (|z|^4+t^2)^{1/4}$ denotes the Koranyi norm on the Heisenberg group and $Q = 2n+2$ is its homogeneous dimension. We consider the Schrödinger representation π_λ of $\mathbb{H}^n$ whose action on the representation space $L^2(\mathbb{R}^n)$ is given by

$$\pi_\lambda(z,t)\phi(\xi) = e^{i\lambda t}e^{i\lambda(x\cdot\xi+x\cdot y/2)}\phi(\xi+y).$$

The Fourier transform of a function $f \in L^1(\mathbb{H}^n)$ is the operator-valued function defined on the set of all nonzero real numbers, $\mathbb{R}^*$, given by

$$\hat{f}(\lambda) = \int_{\mathbb{H}^n} f(z,t)\pi_\lambda(z,t)\,dz\,dt.$$

The action of the Fourier transform on a function of the form $\mathcal{L}$ is well known and is given by $\widehat{\mathcal{L}f}(\lambda) = \hat{f}(\lambda)H(\lambda)$, where $H(\lambda)$ is the scaled Hermite operator. In view of this, it can be easily checked that

$$d\pi_\lambda(m(\mathcal{L})) = m(H(\lambda)), \tag{6-2}$$

where $d\pi_\lambda$ stands for the derived representation corresponding to π_λ. We refer the reader to [Thangavelu 1998] for more details in this regard. Recall that the fractional power $\mathcal{L}_{-s}$ is defined as follows (see [Roncal and Thangavelu 2016]):

$$\mathcal{L}_{-s}f(z,t) := (2\pi)^{-n-1}\int_{-\infty}^{\infty}\bigg(\sum_{k=0}^{\infty}(2|\lambda|)^{-s}\frac{\Gamma\big(\frac{1}{2}(2k+n)+\frac{1}{2}(1-s)\big)}{\Gamma\big(\frac{1}{2}(2k+n)+\frac{1}{2}(1+s)\big)}f^\lambda *_\lambda \varphi_k^\lambda(z)\bigg)e^{-i\lambda t}|\lambda|^n\,d\lambda.$$

So we have $d\pi_\lambda(\mathcal{L}_{-s}) = H(\lambda)_{-s}$. In particular, for $\lambda = 1$, using spectral decomposition, we have

$$H_{-s} f = \sum_{k=0}^{\infty} 2^{-s} \frac{\Gamma\big(\frac{1}{2}(2k+n) + \frac{1}{2}(1-s)\big)}{\Gamma\big(\frac{1}{2}(2k+n) + \frac{1}{2}(1+s)\big)} P_k f.$$

Now it is not hard to see that

$$H_{-s}(f e^{-|\cdot|^2/2})(x) = e^{-|x|^2/2} \sum_{k=0}^{\infty} 2^{-s} \frac{\Gamma\big(\frac{1}{2}(2k+n) + \frac{1}{2}(1-s)\big)}{\Gamma\big(\frac{1}{2}(2k+n) + \frac{1}{2}(1+s)\big)} Q_k f(x). \tag{6-3}$$

Hence from the definition of L_{-s} we have

$$H_{-s}(f e^{-|\cdot|^2/2})(x) = e^{-|x|^2/2} L_{-s} f(x). \tag{6-4}$$

In this section, we prove an analogue of (6-1) for the operator H_{-s}. We first study $L^p - L^q$ mapping properties of the operator H_{-s}.

In view of relation (6-2) we have

$$H_{-s} f(\xi) = c_{n,s} \int_{\mathbb{H}^n} |(z,t)|^{-Q+2s} \pi_1(z,t) f(\xi)\, dz\, dt.$$

Using the definition of π_1 and writing $z = x + iy$, we obtain

$$\begin{aligned} H_{-s} f(\xi) &= c_{n,s} \int_{\mathbb{H}^n} ((|x|^2 + |y|^2)^2 + t^2)^{-(n+1-s)/2} e^{it} e^{i(x\cdot\xi + x\cdot y/2)} f(\xi + y)\, dx\, dy\, dt \\ &= c_{n,s} \int_{\mathbb{H}^n} ((|x|^2 + |\eta - \xi|^2)^2 + t^2)^{-(n+1-s)/2} e^{it} e^{i(x\cdot\xi + x\cdot\eta)/2} f(\eta)\, dx\, d\eta\, dt \\ &= \int_{\mathbb{R}^n} K_H^s(\xi,\eta) f(\eta)\, d\eta, \end{aligned}$$

where the kernel K_H^s is defined by

$$K_H^s(\xi,\eta) = c_{n,s} \int_{\mathbb{R}^n\times\mathbb{R}} ((|x|^2 + |\eta-\xi|^2)^2 + t^2)^{-(n+1-s)/2} e^{it} e^{i(x\cdot\xi + x\cdot\eta)/2}\, dx\, dt. \tag{6-5}$$

Taking the modulus and then a change of variables leads to

$$|K_H^s(\xi,\eta)| \le c_{n,s} \int_{\mathbb{R}^n} (|x|^2 + |\eta - \xi|^2)^{-n+s}\, dx.$$

Now again a change of variable $x \to x|\xi - \eta|$ yields

$$|K_H^s(\xi,\eta)| \le C_{n,s} |\xi - \eta|^{-n+2s}. \tag{6-6}$$

It is a routine matter to check the following $L^p - L^q$-boundedness property; see e.g., [Grafakos 2009, Theorem 6.1.3]. In fact, for $1 < p < q < \infty$ with $1/p - 1/q = 2s/n$, we get

$$\|H_{-s} f\|_{L^q} \le C_{n,s}(p) \|f\|_{L^p}. \tag{6-7}$$

Nevertheless, in the following theorem, we obtain a better estimate for the kernel, improving the $L^p - L^q$ estimates mentioned above.

Theorem 6.1. *For any $1 \le p \le q < \infty$ with $1/p - 1/q \le 1$, there exists a constant $C_{n,s}(p)$ such that for all $f \in L^p(\mathbb{R}^n)$, the inequality $\|H_{-s} f\|_{L^q} \le C_{n,s}(p)\|f\|_{L^p}$ holds.*

Proof. In view of the formula stated in (4-14), from (6-5) we have

$$K_H^s(\xi,\eta) := 2c_{n,s}\frac{\sqrt{\pi}\,2^{-(n/2+1+s)/2}}{\Gamma\big(\frac{1}{2}(n+1-s)\big)}\int_{\mathbb{R}^n}(|x|^2+|\eta-\xi|^2)^{-(n+1-s)/2}K_{-(n+1-s)/2}(|x|^2+|\eta-\xi|^2)e^{ix\cdot(\eta+\xi)/2}\,dx.$$

Now we use the integral representation of K_ν to simplify the above integral giving the kernel as

$$K_\nu(z) = 2^{-\nu-1}z^\nu\int_0^\infty e^{-t-z^2/(4t)}t^{-\nu-1}\,dt. \tag{6-8}$$

A simple change of variables shows that

$$z^\nu K_\nu(z) = 2^{\nu-1}\int_0^\infty e^{-t-z^2/(4t)}t^{\nu-1}\,dt = z^\nu K_{-\nu}(z).$$

Thus

$$\begin{aligned}(|x|^2+|\eta-\xi|^2)^{-(n+1-s)/2}&K_{-(n+1-s)/2}(|x|^2+|\eta-\xi|^2)\\ &= (|x|^2+|\eta-\xi|^2)^{-(n+1-s)/2}K_{(n+1-s)/2}(|x|^2+|\eta-\xi|^2),\end{aligned}$$

leading to the formula

$$(|x|^2+|\eta-\xi|^2)^{-(n+1-s)/2}K_{-(n+1-s)/2}(|x|^2+|\eta-\xi|^2) = 2^{-\nu-1}\int_0^\infty e^{-t-z^2/(4t)}t^{-\nu-1}\,dt,$$

where $\nu = \frac{1}{2}(n+1-s)$ and $z = |x|^2+|\xi-\eta|^2$. Writing $a := \frac{1}{2}(\xi+\eta)$, we estimate the integral

$$\int_{\mathbb{R}^n} e^{-(|x|^2+r^2)^2/(4t)}e^{ix\cdot a}\,dx,$$

where we have let $r = |\xi-\eta|$. First note that

$$\int_{\mathbb{R}^n} e^{-(|x|^2+r^2)^2/(4t)}e^{ix\cdot a}\,dx = e^{-r^4/(4t)}\int_{\mathbb{R}^n} e^{ix\cdot a}e^{-2r^2|x|^2/(4t)}e^{-|x|^4/(4t)}\,dx.$$

Let φ stand for the Fourier transform of the function $e^{-|x|^4/4}$. So the above integral is bounded by

$$e^{-r^4/(4t)}\left(\frac{t}{r^2}\right)^{n/2}t^{n/4}\int_{\mathbb{R}^n}\varphi(t^{1/4}(a-y))e^{-t|y|^2/(2r^2)}\,dy,$$

which is bounded by (after making a change of variables and using $|\varphi(\xi)| \le C$)

$$e^{-r^4/(4t)}t^{n/4},$$

and $K_H^s(\xi,\eta)$ is bounded by

$$\int_0^\infty e^{-t}e^{-r^4/(4t)}t^{-(n+2-2s)/4-1}\,dt = r^{-(n+2-2s)/2}K_{(n+2-2s)/4}(r^2).$$

Finally we have

$$|K_H^s(\xi,\eta)| \le C|\xi-\eta|^{-(n+2-2s)/2}K_{(n+2-2s)/4}(|\xi-\eta|^2) =: G(\xi-\eta). \tag{6-9}$$

Now we see that

$$|H_{-s}f(\xi)| \le C|f| * G(\xi), \quad \forall \xi \in \mathbb{R}^n. \tag{6-10}$$

Now note that for $r \ge 1$, integrating in polar coordinates, we have

$$\int_{\mathbb{R}^n} G(x)^r \, dx = c_n \int_0^\infty (t^{-(n+2-2s)/2} K_{(n+2-2s)/4}(t^2))^r t^{n-1} \, dt.$$

Using the facts that $K_\nu(z) \sim z^{-1/2}e^{-z}$ for large z and near the origin $z^{-\nu}K_\nu(z)$ is bounded, we conclude that the above integral is finite. Now in view of Young's inequality we have

$$\||f| * G\|_q \le \|f\|_p \|G\|_r, \quad \text{where } \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}. \tag{6-11}$$

But this is true for any $r \ge 1$. Hence we are done. □

As a corollary to Theorem 6.1 we have the following analogue of (6-1).

Corollary 6.2. *For $q = 2n/(n-s)$, $0 < s < n$, we have the inequality*

$$C_{n,s}\left(\int_{\mathbb{R}^n} |f(x)|^q \, dx\right)^{2/q} \le \langle H_s f, f\rangle, \tag{6-12}$$

where $C_{n,s}$ is some constant depending only on n and s.

Proof. Replacing s by $\frac{1}{2}s$ and putting $p = 2$ in the above theorem, we have

$$\|H_{-s/2}f\|_q^2 \le c_{n,s}\|f\|_2^2, \tag{6-13}$$

where $q = 2n/(n-s)$. Now in the above inequality substituting f by $H_{s/2}f$ we have

$$\left(\int_{\mathbb{R}^n} |f(x)|^q \, dx\right)^{2/q} \le c_{n,s}\langle H_{s/2}f, H_{s/2}f\rangle.$$

But in view of Stirling's formula for the gamma function we know that $H_{s/2}^2$ and H_s differ by a bounded operator on $L^2(\mathbb{R}^n)$. Hence the result follows. □

Corollary 6.3 (Hardy's inequality for H_s). *Let $0 < s < 1$. Assume that $f \in L^2(\mathbb{R}^n)$ such that $H_s f \in L^2(\mathbb{R}^n)$. Then we have*

$$\langle H_s f, f\rangle_{L^2(\mathbb{R}^n)} \ge c_{n,s}\int_{\mathbb{R}^n} \frac{f(x)^2}{(1+|x|^2)^s} \, dx.$$

Proof. Given $f \in L^2(\mathbb{R}^n)$, in view of Holder's inequality we have

$$\int_{\mathbb{R}^n} \frac{f(x)^2}{(1+|x|^2)^s} \, dx \le A(n,s)\left(\int_{\mathbb{R}^n} |f(x)|^q \, dx\right)^{2/q}, \tag{6-14}$$

where

$$q = \frac{2n}{n-s}, \quad A(n,s) := \left(\int_{\mathbb{R}^n} (1+|x|^2)^{-sq'} \, dx\right)^{1/q'} \quad \text{and} \quad \frac{1}{q'} = 1 - \frac{2n-2s}{2n} = \frac{s}{n}.$$

Hence the result follows from the previous corollary. □

As a consequence of this we have a version of Hardy's inequality for L_s:

Corollary 6.4. *Let* $0 < s < 1$. *Assume that* $f \in L^2(\gamma)$ *such that* $L_s f \in L^2(\gamma)$. *Then we have*

$$\langle L_s f, f\rangle_{L^2(\gamma)} \geq c_{n,s} \int_{\mathbb{R}^n} \frac{f(x)^2}{(1+|x|^2)^s}\, d\gamma(x).$$

Proof. Let $f \in L^2(\gamma)$. It is easy to see that $g(x) := f(x)e^{-|x|^2/2} \in L^2(\mathbb{R}^n)$. By Corollary 6.3 we have

$$\langle H_s g, g\rangle_{L^2(\mathbb{R}^n)} \geq c_{n,s} \int_{\mathbb{R}^n} \frac{g(x)^2}{(1+|x|^2)^s}\, dx.$$

Also from the spectral decomposition we see that

$$H_s g(x) = H_s(f e^{-|\cdot|^2/2})(x) = e^{-|x|^2/2} L_s f(x),$$

which gives $\langle H_s g, g\rangle_{L^2(\mathbb{R}^n)} = \langle L_s f, f\rangle_{L^2(\gamma)}$. Hence the result follows. □

Remark. Frank and Lieb proved in [Frank et al. 2008] that the constant appearing in the left-hand side of the Hardy–Littlewood–Sobolev inequality (6-1) for the sub-Laplacian on the Heisenberg group is sharp. It would be interesting to see the sharp constant in the analogous inequality (6-12), which we have proved for the Hermite operator.

Acknowledgements

The authors are grateful to the anonymous reviewer for the valuable comments and suggestions. Ganguly is supported by an Integrated Ph.D. scholarship from the Indian Institute of Science. Manna is thankful for financial support from DST-INSPIRE [DST/INSPIRE/04/2019/001914] and R(IA)CVR-PDF/2020/224. Thangavelu is supported by a J. C. Bose Fellowship from the Department of Science and Technology, Government of India.

References

[Abramowitz and Stegun 1964] M. Abramowitz and I. A. Stegun (editors), *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Nat. Bureau Standards Appl. Math. Ser. **55**, US Govt. Printing Office, Washington, DC, 1964. Reprinted by Dover, New York, 1974. MR Zbl

[Ahlfors 1953] L. V. Ahlfors, *Complex analysis*: *an introduction to the theory of analytic functions of one complex variable*, McGraw-Hill, New York, 1953. MR Zbl

[Beckner 2012] W. Beckner, "Pitt's inequality and the fractional Laplacian: sharp error estimates", *Forum Math.* **24**:1 (2012), 177–209. MR Zbl

[Boggarapu et al. 2019] P. Boggarapu, L. Roncal, and S. Thangavelu, "On extension problem, trace Hardy and Hardy's inequalities for some fractional Laplacians", *Commun. Pure Appl. Anal.* **18**:5 (2019), 2575–2605. MR Zbl

[Branson et al. 2013] T. P. Branson, L. Fontana, and C. Morpurgo, "Moser–Trudinger and Beckner–Onofri's inequalities on the CR sphere", *Ann. of Math.* (2) **177**:1 (2013), 1–52. MR Zbl

[Caffarelli and Silvestre 2007] L. Caffarelli and L. Silvestre, "An extension problem related to the fractional Laplacian", *Comm. Partial Differential Equations* **32**:7-9 (2007), 1245–1260. MR Zbl

[Ciaurri et al. 2018] Ó. Ciaurri, L. Roncal, and S. Thangavelu, "Hardy-type inequalities for fractional powers of the Dunkl–Hermite operator", *Proc. Edinb. Math. Soc.* (2) **61**:2 (2018), 513–544. MR Zbl

[Cowling and Haagerup 1989] M. Cowling and U. Haagerup, "Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one", *Invent. Math.* **96**:3 (1989), 507–549. MR Zbl

[Folland 1989] G. B. Folland, *Harmonic analysis in phase space*, Ann. of Math. Stud. **122**, Princeton Univ. Press, 1989. MR Zbl

[Frank and Lieb 2012] R. L. Frank and E. H. Lieb, "Sharp constants in several inequalities on the Heisenberg group", *Ann. of Math.* (2) **176**:1 (2012), 349–381. MR Zbl

[Frank et al. 2008] R. L. Frank, E. H. Lieb, and R. Seiringer, "Hardy–Lieb–Thirring inequalities for fractional Schrödinger operators", *J. Amer. Math. Soc.* **21**:4 (2008), 925–950. MR Zbl

[Frank et al. 2015] R. L. Frank, M. d. M. González, D. D. Monticelli, and J. Tan, "An extension problem for the CR fractional Laplacian", *Adv. Math.* **270** (2015), 97–137. MR Zbl

[Garofalo and Tralli 2021] N. Garofalo and G. Tralli, "Feeling the heat in a group of Heisenberg type", *Adv. Math.* **381** (2021), art. id. 107635. MR Zbl

[Garofalo and Tralli 2023] N. Garofalo and G. Tralli, "A heat equation approach to intertwining", *J. Anal. Math.* **149**:1 (2023), 113–134. MR

[Grafakos 2009] L. Grafakos, *Modern Fourier analysis*, 2nd ed., Grad. Texts in Math. **250**, Springer, 2009. MR Zbl

[Hardy 1919] G. H. Hardy, "Notes on some points in the integral calculus, LI: On Hilbert's double series theorem and some connected theorems concerning the convergence of infinite series and integrals", *Messenger Math.* **48** (1919), 107–112. Zbl

[Hardy 1920] G. H. Hardy, "Note on a theorem of Hilbert", *Math. Z.* **6**:3-4 (1920), 314–317. MR Zbl

[Kufner et al. 2007] A. Kufner, L. Maligranda, and L.-E. Persson, *The Hardy inequality*: *about its history and some related results*, Vydavatelský Servis, Pilsen, Czech Republic, 2007. MR Zbl

[Möllers et al. 2016] J. Möllers, B. Ørsted, and G. Zhang, "On boundary value problems for some conformally invariant differential operators", *Comm. Partial Differential Equations* **41**:4 (2016), 609–643. MR Zbl

[Olver and Maximon 2010] F. W. J. Olver and L. C. Maximon, "Bessel functions", pp. 215–286 in *NIST handbook of mathematical functions*, edited by F. W. J. Olver et al., US Dept. Commerce, Washington, DC, 2010. MR

[Prudnikov et al. 1986] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and series*, *I*: *Elementary functions*, Gordon & Breach, New York, 1986. MR Zbl

[Roncal and Thangavelu 2016] L. Roncal and S. Thangavelu, "Hardy's inequality for fractional powers of the sublaplacian on the Heisenberg group", *Adv. Math.* **302** (2016), 106–158. MR Zbl

[Roncal and Thangavelu 2020a] L. Roncal and S. Thangavelu, "An extension problem and trace Hardy inequality for the sub-Laplacian on H-type groups", *Int. Math. Res. Not.* **2020**:14 (2020), 4238–4294. MR Zbl

[Roncal and Thangavelu 2020b] L. Roncal and S. Thangavelu, "Holomorphic extensions of eigenfunctions on NA groups", preprint, 2020. arXiv 2005.09894

[Stinga and Torrea 2010] P. R. Stinga and J. L. Torrea, "Extension problem and Harnack's inequality for some fractional operators", *Comm. Partial Differential Equations* **35**:11 (2010), 2092–2122. MR Zbl

[Szegő 1967] G. Szegő, *Orthogonal polynomials*, 3rd ed., Amer. Math. Soc. Colloq. Publ. **23**, Amer. Math. Soc., Providence, RI, 1967. MR

[Thangavelu 1993] S. Thangavelu, *Lectures on Hermite and Laguerre expansions*, Math. Notes **42**, Princeton Univ. Press, 1993. MR Zbl

[Thangavelu 1998] S. Thangavelu, *Harmonic analysis on the Heisenberg group*, Progr. Math. **159**, Birkhäuser, Boston, 1998. MR Zbl

[Thangavelu 2008] S. Thangavelu, "An analogue of Gutzmer's formula for Hermite expansions", *Studia Math.* **185**:3 (2008), 279–290. MR Zbl

[Thangavelu 2010] S. Thangavelu, "On the unreasonable effectiveness of Gutzmer's formula", pp. 199–217 in *Harmonic analysis and partial differential equations* (El Escorial, Spain, 2008), edited by P. Cifuentes et al., Contemp. Math. **505**, Amer. Math. Soc., Providence, RI, 2010. MR Zbl

[Urbina-Romero 2019] W. Urbina-Romero, *Gaussian harmonic analysis*, Springer, 2019. MR Zbl

[Yafaev 1999] D. Yafaev, “Sharp constants in the Hardy–Rellich inequalities”, *J. Funct. Anal.* **168**:1 (1999), 121–144. MR Zbl

Received 17 Dec 2020. Revised 19 Sep 2021. Accepted 28 Oct 2021.

PRITAM GANGULY: pritam1995.pg@gmail.com, pritamg@math.upb.de
Current address: *Institut für Mathematik, Universität Paderborn, Paderborn, Germany*
Department of Mathematics, Indian Institute of Science, Bangalore, India

RAMESH MANNA: rameshmanna@niser.ac.in
School of Mathematical Sciences, National Institute of Science Education and Research, Bhubaneswar, an OCC of Homi Bhabha National Institute, Jatni, 752050, India

SUNDARAM THANGAVELU: veluma@iisc.ac.in
Department of Mathematics, Indian Institute of Science, Bangalore, India

Analysis & PDE
msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2023 is US $405/year for the electronic version, and $630/year (+$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 16 No. 5 2023