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THE CAUCHY PROBLEM FOR THE INFINITESIMAL MODEL IN THE REGIME OF SMALL VARIANCE

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We study the asymptotic behavior of solutions of the Cauchy problem associated to a quantitative genetics model with a sexual mode of reproduction. It combines trait-dependent mortality and a nonlinear integral reproduction operator, *the infinitesimal model*. A parameter describes the standard deviation between the offspring and the mean parental traits. We show that under mild assumptions upon the mortality rate m, when the deviations are small, the solutions stay close to a Gaussian profile with small variance, uniformly in time. Moreover, we characterize accurately the dynamics of the mean trait in the population. Our study extends previous results on the existence and uniqueness of stationary solutions for the model. It relies on perturbative analysis techniques with a sharp description of the correction from the Gaussian profile.

A list of symbols can be found on page 1348.

1. Introduction

We investigate solutions $f_\varepsilon \in L^1(\mathbb{R}_+ \times \mathbb{R})$ of the Cauchy problem

$$\begin{cases} \varepsilon^2 \partial_t f_\varepsilon(t,z) + m(z) f_\varepsilon(t,z) = \mathcal{B}_\varepsilon(f_\varepsilon)(t,z), & t>0,\ z\in\mathbb{R}, \\ f_\varepsilon(0,z) = f_\varepsilon^0(z), \end{cases} \tag{$P_t f_\varepsilon$}$$

where $\mathcal{B}_\varepsilon(f)$ is the following nonlinear, homogeneous mixing operator associated with the infinitesimal model of [Fisher 1918], see also [Barton et al. 2017] for a modern perspective:

$$\mathcal{B}_\varepsilon(f)(z) := \frac{1}{\varepsilon\sqrt{\pi}} \iint_{\mathbb{R}^2} \exp\left[-\frac{1}{\varepsilon^2}\left(z - \frac{z_1+z_2}{2}\right)^2\right] f(z_1) \frac{f(z_2)}{\int_{\mathbb{R}} f(z_2')\,dz_2'}\,dz_1\,dz_2. \tag{1-1}$$

This problem originates from quantitative genetics in the context of evolutionary biology. The variable z denotes a phenotypic trait, f_ε is the distribution of the population with respect to z and m is the trait-dependent mortality rate.

The mixing operator $\mathcal{B}_\varepsilon$ models the inheritance of quantitative traits in the population, under the assumption of a sexual mode of reproduction. As formulated in (1-1), it is assumed that offspring traits are distributed normally around the mean of the parental traits $\frac{1}{2}(z_1+z_2)$, with a constant variance, here $\frac{1}{2}\varepsilon^2$. We are interested in the evolutionary dynamics resulting in the selection of well-fitted (low mortality) individuals, i.e., the concentration of the distribution around some dominant traits with standing variance.

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In theoretical evolutionary biology, a broad literature deals with this model to describe sexual reproduction; see e.g., [Barfield et al. 2011; Barton et al. 2017; Bulmer 1980; Cotto and Ronce 2014; Huisman and Tufto 2012; Roughgarden 1972; Slatkin 1970; Slatkin and Lande 1976; Tufto 2000; Turelli 2017; Turelli and Barton 1994].

We are interested in the asymptotic behavior of the trait distribution f_ε as ε^2 vanishes. It is expected that the profile concentrates around some particular traits under the influence of selection.

The asymptotic description of concentration around some particular trait(s) has been extensively investigated for various linear operators $\mathcal{B}_\varepsilon$ associated with asexual reproduction such as, for instance, the diffusion operator $f_\varepsilon(t,z)+\varepsilon^2\Delta f_\varepsilon(t,z)$, or the convolution operator $(1/\varepsilon)K(z/\varepsilon)*f_\varepsilon(t,z)$ where K is a probability kernel with unit variance; see [Barles and Perthame 2007; Barles et al. 2009; Diekmann et al. 2005; Lorz et al. 2011; Perthame 2007] for the earliest investigations and [Bouin et al. 2018; Méléard and Mirrahimi 2015; Mirrahimi 2020] for the case of fat-tailed kernels K. In those linear cases, the asymptotic analysis usually leads to a Hamilton–Jacobi equation after performing the Hopf–Cole transform $u_\varepsilon = -\varepsilon \log f_\varepsilon$. Those problems require a careful well-posedness analysis for uniqueness and convergence as $\varepsilon \to 0$; see [Barles et al. 2009; Calvez and Lam 2020; Mirrahimi and Roquejoffre 2016].

Much less is known about the operator $\mathcal{B}_\varepsilon$ defined by (1-1). From a mathematical viewpoint, in the field of probability theory, [Barton et al. 2017] derived the model from a microscopic framework. In [Mirrahimi and Raoul 2013; Raoul 2017], the authors deal with a different scaling than the current small variance assumption $\varepsilon^2 \ll 1$ and add a spatial structure in order to derive the celebrated Kirkpatrick and Barton system [1997].

Gaussian distributions will play a pivotal role in our analysis as they are left-invariant by the infinitesimal operator $\mathcal{B}_\varepsilon$; see [Mirrahimi and Raoul 2013; Turelli and Barton 1994]. In [Calvez et al. 2019], the authors studied special *stationary* solutions, having the form

$$\exp\left[\frac{\lambda_\varepsilon t}{\varepsilon^2}\right]F_\varepsilon(z), \quad \text{with } F_\varepsilon(z)=\frac{1}{\varepsilon\sqrt{2\pi}}\exp\left[-\frac{(z-z_*)^2}{2\varepsilon^2}-U_\varepsilon^s(z)\right].$$

In this paper we tackle the Cauchy problem $(P_t f_\varepsilon)$, and we hereby look for solutions that are close to Gaussian distributions uniformly in time of the form

$$f_\varepsilon(t,z)=\frac{1}{\varepsilon\sqrt{2\pi}}\exp\left[\frac{\lambda(t)}{\varepsilon^2}-\frac{(z-z_*(t))^2}{2\varepsilon^2}-U_\varepsilon(t,z)\right]. \tag{1-2}$$

The scalar function λ measures the growth (or decay according to its sign) of the population. The mean of the Gaussian density, z_*, is also the trait at which the population concentrates when $\varepsilon \to 0$. The pair (λ, z_*) will be determined by the analysis at all times. It is somehow related to invariant properties of the operator $\mathcal{B}_\varepsilon$. The function U_ε measures the deviation from the Gaussian profile induced by the selection function m. It is a cornerstone of our analysis that U_ε is Lipschitz continuous with respect to z, uniformly in t and ε. Plugging the transformation (1-2) into $(P_t f_\varepsilon)$ yields the following equivalent problem:

$$\begin{aligned}-\varepsilon^2\partial_t U_\varepsilon(t,z)+\dot{\lambda}(t)+(z-z_*(t))\dot{z}_*(t)+m(z)&\\ =I_\varepsilon(U_\varepsilon)(t,z)\exp[U_\varepsilon(t,z)-2U_\varepsilon(t,\bar{z}(t))+U_\varepsilon(t,z_*(t))],&\end{aligned} \tag{P_tU_ε}$$

where $\bar{z}(t)$ is the midpoint between z and $z_*(t)$:

$$\bar{z}(t) = \tfrac{1}{2}(z + z_*(t)),$$

and the functional I_ε is defined by

$$I_\varepsilon(U_\varepsilon)(t,z) = \frac{\iint_{\mathbb{R}^2} \exp\bigl[-\frac{1}{2}y_1y_2 - \frac{3}{4}(y_1^2+y_2^2) + 2U_\varepsilon(t,\bar{z}) - U_\varepsilon(t,\bar{z}+\varepsilon y_1) - U_\varepsilon(t,\bar{z}+\varepsilon y_2)\bigr]\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}} \exp\bigl[-\frac{1}{2}y^2 + U_\varepsilon(t,z_*) - U_\varepsilon(t,z_*+\varepsilon y)\bigr]\,dy}. \tag{1-3}$$

This functional is the residual shape of the infinitesimal operator (1-1) after suitable transformations. It was first introduced in the formal analysis of [Garnier et al. 2022] and in the study of the corresponding stationary problem in [Calvez et al. 2019]. The Lipschitz continuity of U_ε is pivotal here as it ensures that $I_\varepsilon(U_\varepsilon) \to 1$ when $\varepsilon \to 0$. Thus for small ε, we expect that $(P_t f_\varepsilon)$ is well approximated by the following problem:

$$\dot{\lambda}(t) + (z - z_*(t))\dot{z}_*(t) + m(z) = \exp[U_0(t,z) - 2U_0(t,\bar{z}(t)) + U_0(t,z_*(t))]. \tag{1-4}$$

Interestingly, this characterizes the dynamics of $(\lambda(t), z_*(t))$. By differentiating (1-4) and evaluating at the point $z = z_*(t)$, then simply evaluating (1-4) at $z = z_*(t)$, we find the following pair of relationships:

$$\dot{z}_*(t) + m'(z_*(t)) = 0, \tag{1-5}$$

$$\dot{\lambda}(t) + m(z_*(t)) = 1. \tag{1-6}$$

Then, a more compact way to write the limit problem for $\varepsilon = 0$ is

$$M(t,z) = \exp[U_0(t,z) - 2U_0(t,\bar{z}(t)) + U_0(t,z_*(t))], \tag{P_tU_0}$$

with the notation

$$M(t,z) := 1 + m(z) - m(z_*(t)) - m'(z_*(t))(z - z_*(t)). \tag{1-7}$$

It follows from (1-6) and (1-5) that

$$M(t,z_*(t)) = 1, \qquad \partial_z M(t,z_*(t)) = 0. \tag{1-8}$$

An explicit solution of (P_tU_0) exists under the form of an infinite series:

$$V^*(t,z) := \sum_{k\geq 0} 2^k \log(M(t, z_*(t) + 2^{-k}(z - z_*(t)))). \tag{1-9}$$

This formula is obtained by noticing a recursive relation on the first derivative of $\partial_z U_0$, as in Section 2.2 of [Calvez et al. 2019]. The same recursive argument is used here in Section 7G. Interestingly, this series is convergent thanks to the relationships of (1-8). The function V^* is a solution of (P_tU_0), but not the only one. There are two degrees of freedom when solving (P_tU_0), since adding any affine function to U_0 leaves the right-hand side unchanged. Therefore, a general expression of solutions is the following, where the scalar functions p_0 and q_0 are arbitrary:

$$U_0(t,z) = p_0(t) + q_0(t)(z - z_*(t)) + V^*(t,z). \tag{1-10}$$

We have foreseen that the Lipschitz regularity of U_ε was the way to guarantee that $I_\varepsilon(U_\varepsilon) \to 1$ as $\varepsilon \to 0$. As a matter of fact, an important part of [Calvez et al. 2019] is dedicated to proving such regularity for U_ε^s, the solution of the stationary problem

$$\lambda_\varepsilon^s + m(z) = I_\varepsilon(U_\varepsilon^s)(z) \exp\bigl[U_\varepsilon^s(z) - 2U_\varepsilon^s\bigl(\tfrac{1}{2}(z+z_*^s)\bigr) + U_\varepsilon^s(z_*^s)\bigr], \quad z \in \mathbb{R}. \tag{PU_ε stat}$$

The authors introduced an appropriate functional space controlling Lipschitz bound. They were then able to show the existence of U_ε^s and its (local) uniqueness in that space. They also proved that U_ε^s was converging when $\varepsilon \to 0$ towards solutions of (P_tU_0); see Figure 1 for a schematic comparison of the scope of the present article compared to previous work.

Here, to tackle the nonstationary problem (P_tU_ε), we make the following assumptions of asymptotic growth on the selection function m, when $|z| \to \infty$.

Assumption 1.1. We suppose that the function m is a $\mathcal{C}^5(\mathbb{R})$ function, bounded below. We define the scalar function z_* as the gradient flow

$$\dot{z}_*(t) = -m'(z_*(t)), \quad t > 0, \tag{1-11}$$

associated to a prescribed initial data $z_*(0)$. Next, we make the following assumptions:

- We suppose that $z_*(0)$ lies next to a nondegenerate *local* minimum of m, denoted by z_*^s, such that

$$z_*(t) \xrightarrow[t\to\infty]{} z_*^s. \tag{1-12}$$

- We also require that there exists a uniform positive lower bound on M:

$$\inf_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} M(t,z) > 0. \tag{1-13}$$

- We make growth assumptions on M in the following way:

$$\text{for } k = 1,2,3,4,5, \quad (1+|z-z_*|)^\alpha \frac{\partial_z^k M(t,z)}{M(t,z)} \in L^\infty(\mathbb{R}_+\times\mathbb{R}), \tag{1-14}$$

for some $0 < \alpha < 1$, the same as in Definition 1.2.

- We make a final assumption upon the behavior of m at infinity, roughly that it has superlinear growth, uniformly in time:

$$\limsup_{z\to\infty}\left|\frac{M(t,\bar{z})}{M(t,z)}\right| := a < \frac{1}{2}, \quad \limsup_{z\to\infty}\left|\frac{\partial_z M(t,\bar{z})}{\partial_z M(t,z)}\right| < \infty. \tag{1-15}$$

The first assumption on m and z_* guarantees the following local convexity property, at least for times t large enough:

$$\exists\, \mu_0 > 0,\ \ \exists\, t_0 > 0,\ \text{ such that } \forall t \geq t_0,\ \ m''(z_*(t)) \geq \mu_0. \tag{1-16}$$

Any convex function m with at least quadratic growth at infinity will satisfy Assumption 1.1, without restriction on the initial data. This type of fitness landscape is fairly standard in the asexual models, e.g., the Fisher geometrical model [Fisher 1999; Martin and Roques 2016] assumes a quadratic fitness function. However, our analysis also encompasses different scenarios with possibly multiple optima, the limiting condition is the positivity of M. This corresponds to a global assumption on the behavior of z_* and m, that reduces the choice of $z_*(0)$.[1] The relationship (1-14) corresponds to algebraic decay assumptions for M, and accordingly, it holds true if m behaves like any (at least quadratic) polynomial function as $|z| \to +\infty$ (as well as (1-15)). The shape of the selection function, even far from the optimum, changes the qualitative behavior of a population; see [Osmond and Klausmeier 2017]. A detailed discussion on the behavior of the solution whether our assumptions are satisfied or not is carried out in Section 9 with some numerical simulations displayed.

The purpose of this work is to rigorously prove the convergence of the solution of $(P_t U_\varepsilon)$ towards a particular solution of $(P_t U_0)$. Given the general shape of U_0, see (1-10), it is natural to decompose U_ε by separating the affine part from the rest:

$$U_\varepsilon(t,z) = p_\varepsilon(t) + q_\varepsilon(t)(z - z_*(t)) + V_\varepsilon(t,z). \tag{1-17}$$

We require accordingly that at all times $t > 0$,

$$V_\varepsilon(t, z_*) = \partial_z V_\varepsilon(t, z_*) = 0,$$

which is another way of saying that the pair $(p_\varepsilon, q_\varepsilon)$ tunes the affine part of U_ε. The pair $(q_\varepsilon, V_\varepsilon)$ is the main unknown of this problem. It is expected that V_ε converges to V^* when $\varepsilon \to 0$. Our analysis will be able to determine the limit of q_ε even if it cannot be identified by the problem at $\varepsilon = 0$. Indeed, in $(P_t U_0)$, the linear part q_0 can be any constant. Our limit candidate for q_ε is q^*, that we define as the solution of the differential equation

$$\dot{q}^*(t) = -m''(z_*(t))q^*(t) + \tfrac{1}{2}m^{(3)}(z_*(t)) - 2m''(z_*(t))m'(z_*(t)), \tag{1-18}$$

corresponding to an initial value of $q^*(0)$. Moreover we define p^* as the function which satisfies, for a given $p^*(0)$,

$$\dot{p}^*(t) = -m'(z_*(t))q^*(t) + m''(z_*(t)). \tag{1-19}$$

These expressions for p^* and q^* are obtained formally by canceling same order (in ε) terms when differentiating $(P_t U_\varepsilon)$ and looking at the main terms when ε is very small. More precisely, we must also evaluate the differentiated problem at $z = z^*$. Thus, those expressions are somehow linked to the formulas for λ and z_* in (1-5) and (1-6). Note that differentiating and evaluating at $z = z_*$ the problem for $\varepsilon > 0$ will be our strategy of proof to tackle the convergence of p_ε and q_ε, in Sections 5A and 5B. Before detailing these technical points, let us note that the function

$$U^*(t,z) := p^*(t) + q^*(t)(z - z_*(t)) + V^*(t,z) \tag{1-20}$$

will be our candidate for the limit of U_ε when $\varepsilon \to 0$. The problem for V_ε equivalent to the problem $(P_t U_\varepsilon)$, using (1-17), is

$$M(t,z) - \varepsilon^2(\dot{p}_\varepsilon(t) + \dot{q}_\varepsilon(t)(z - z_*(t)) + m'(z_*(t))q_\varepsilon(t)) - \varepsilon^2 \partial_t V_\varepsilon(t,z)$$
$$= \mathcal{I}_\varepsilon(q_\varepsilon, V_\varepsilon)(t,z) \exp[V_\varepsilon(t,z) - 2V_\varepsilon(t, \bar{z}(t)) + V_\varepsilon(t, z_*(t))]. \tag{$P_t V_\varepsilon$}$$

[1] M is structurally positive, based on the formulation of $(P_t U_0)$. The uniform lower bound in (1-13) is mainly technical.

One can notice that thanks to cancellations the functional $I_\varepsilon(U_\varepsilon)$ does not depend on p_ε, which explains for the most part why we focus upon $(q_\varepsilon, V_\varepsilon)$. We choose to write $\mathcal{I}_\varepsilon(q_\varepsilon, V_\varepsilon)(t,z) := I_\varepsilon(U_\varepsilon)(t,z)$ as a functional of both unknowns because we will study variations in both directions. One of the main difficulties to prove the link between $(P_t V_\varepsilon)$ and $(P_t U_0)$ is that, formally, the terms with the time derivatives in q_ε and V_ε vanish when $\varepsilon \to 0$. This makes our study belong to the class of singular limit problems.

Before stating our main result we need to define appropriate functional spaces. We first define a reference space $\mathcal{E}$, similar to the one introduced in [Calvez et al. 2019] for the study of the stationary problem $(PU_\varepsilon \text{ stat})$. However, compared to that case we will need more precise controls, which is why we introduce a subspace $\mathcal{F}$ with more stringent conditions.

Definition 1.2 (functional spaces). We define $\alpha < 2 - \ln 3/\ln 2$ such that $\alpha \in (0,1)$ along with the weight function φ_α:

$$\varphi_\alpha(t,z) := (1 + |z - z_*(t)|)^\alpha.$$

The corresponding functional space $\mathcal{E}$ is given by

$$\mathcal{E} = \left\{ v \in \mathcal{C}^3(\mathbb{R}_+ \times \mathbb{R}) \;\middle|\; \forall t > 0,\ v(t, z_*(t)) = \partial_z v(t, z_*(t)) = 0 \right\} \\ \cap \left\{ v \in \mathcal{C}^3(\mathbb{R}_+ \times \mathbb{R}) \;\middle|\; |\partial_z v(t,z)|,\ \varphi_\alpha(t,z)|\partial_z^2 v(t,z)|,\ \varphi_\alpha(t,z)|\partial_z^3 v(t,z)| \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \right\}$$

equipped with the norm

$$\|v\|_{\mathcal{E}} = \max\Big\{ \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} |\partial_z v(t,z)|,\ \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} (\varphi_\alpha(t,z)|\partial_z^2 v(t,z)|),\ \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} (\varphi_\alpha(t,z)|\partial_z^3 v(t,z)|) \Big\}.$$

We also define the subspace $\mathcal{F}$:

$$\mathcal{F} := \mathcal{E} \cap \left\{ v \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}) \;\middle|\; |2v(t, \bar{z}(t)) - v(t,z)|,\ \varphi_\alpha(t,z)|\partial_z v(t, \bar{z}(t)) - \partial_z v(t,z)| \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \right\},$$

and we associate to it the corresponding norm

$$\|v\|_{\mathcal{F}} = \max\Big\{ \|v\|_{\mathcal{E}},\ \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} (|2v(t,\bar{z}(t)) - v(t,z)|),\ \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} (\varphi_\alpha(t,z)|\partial_z v(t,\bar{z}(t)) - \partial_z v(t,z)|) \Big\}.$$

The condition on α exists for computational reasons, highlighted at the end of the discussion of Proposition 7.7. The threshold coincides with that of the stationary case; see [Calvez et al. 2019, (5.11)]. The weight function φ_α is another similar feature. Its role is mainly to have a uniform bound on the first derivative using previous estimates on further derivatives, for which we need α to be bounded. We refer to Section 7G for comments on the tuning of this parameter.

Since most of this paper is focused around the pair $(q_\varepsilon, V_\varepsilon) \in \mathbb{R} \times \mathcal{F}$, we will use the convenient notation $\|(q, V)\| := \max(|q|, \|V\|_{\mathcal{F}})$. Our main theorem is the following convergence result:

Theorem 1.3 (convergence). *There exist K_0, K_0' and $\varepsilon_0 > 0$ such that if we make the following assumptions on the initial condition, for all $\varepsilon \le \varepsilon_0$:*

$$\|V_\varepsilon(0,\cdot) - V^*(0,\cdot)\|_{\mathcal{F}} \le \varepsilon^2 K_0, \quad |q_\varepsilon(0) - q^*(0)| \le \varepsilon^2 K_0 \quad \textit{and} \quad |p_\varepsilon(0) - p^*(0)| \le \varepsilon^2 K_0,$$

then we have uniform estimates of the solutions of the Cauchy problem:

$$\sup_{t>0} \|V_\varepsilon - V^*\|_{\mathcal{F}} \le \varepsilon^2 K_0', \quad \sup_{t>0} |q_\varepsilon(t) - q^*(t)| \le \varepsilon^2 K_0' \quad \textit{and} \quad \sup_{t>0} |p_\varepsilon(t) - p^*(t)| \le \varepsilon^2 K_0',$$

where q^* *is the solution of* (1-18) *associated to* $q^*(0)$ *and* p^* *is the solution of* (1-19) *associated to* $p^*(0)$. *The function* V^* *is defined in* (1-9).

Therefore, as predicted, the limit of U_ε when $\varepsilon \to 0$ is the function

$$p^*(t) + q^*(t)(z - z_*(t)) + V^*(t, z).$$

Theorem 1.3 establishes the stability, with respect to ε and uniformly in time, of Gaussian distributions around the dynamics of the dominant trait driven by a gradient flow differential equation.

In [Calvez et al. 2019], a fixed-point argument was used to build solutions of the stationary problem (PU_ε stat) when $\varepsilon \ll 1$. Estimates were uniform in ε, in order to pass to the limit $\varepsilon \to 0$. As a matter of fact, their limit problem when $\varepsilon = 0$ [Calvez et al. 2019, Problem PU_0] is consistent with (1-4), without time dependency. However, their method of proof can no longer be applied in our case because the (singular) derivative in time of the Cauchy problem ($P_t f_\varepsilon$) breaks the structure that made the stationary problem equivalent to a fixed-point mapping. In fact, in the present article, ($P_t U_0$) and ($P_t U_\varepsilon$) are different in nature due to the fast time relaxation dynamics. This is one of the main difficulties of this work compared to [Calvez et al. 2019]. For this reason, we replace the fixed-point argument by a perturbative analysis. This program is schematized in Figure 1. We introduce the corrector terms κ_ε and W_ε, our aim is to bound them uniformly:

$$V_\varepsilon(t, z) = V^*(t, z) + \varepsilon^2 W_\varepsilon(t, z), \tag{1-21}$$

$$q_\varepsilon(t) = q^*(t) + \varepsilon^2 \kappa_\varepsilon(t). \tag{1-22}$$

The scalar q^*, perturbed by $\varepsilon^2\kappa_\varepsilon$, will tune further the affine part of the solution. The function W_ε measures the error made when approximating ($P_t U_\varepsilon$) by ($P_t U_0$). We choose not to perturb p_ε because we will realize in Section 5B that it can be straightforwardly deduced from the analysis.

This decomposition highlights a crucial part of our analysis, coming back to the initial ($P_t f_\varepsilon$). The main part (in ε) of the solution f_ε is quadratic (up to the transform (1-2)). This means that it does not belong to the space of the corrective term V_ε. After this main (quadratic) part of f_ε, of order $1/\varepsilon^2$, the corrective terms are much more precise for small ε: V^* is of order 1, while $\varepsilon^2 W_\varepsilon$ is of order ε^2. The objective of this article is to show that κ_ε and W_ε are uniformly bounded with respect to time and ε.

2. Heuristics and method of proof

For this section only, we focus on the function U_ε instead of V_ε to get a heuristic argument in favor of the decomposition (1-17) and some elements supporting Theorem 1.3. We will denote by R_ε the perturbation such that we look for solutions of ($P_t U_\varepsilon$) of the form

$$U_\varepsilon(t, z) = U^*(t, z) + \varepsilon^2 R_\varepsilon(t, z).$$

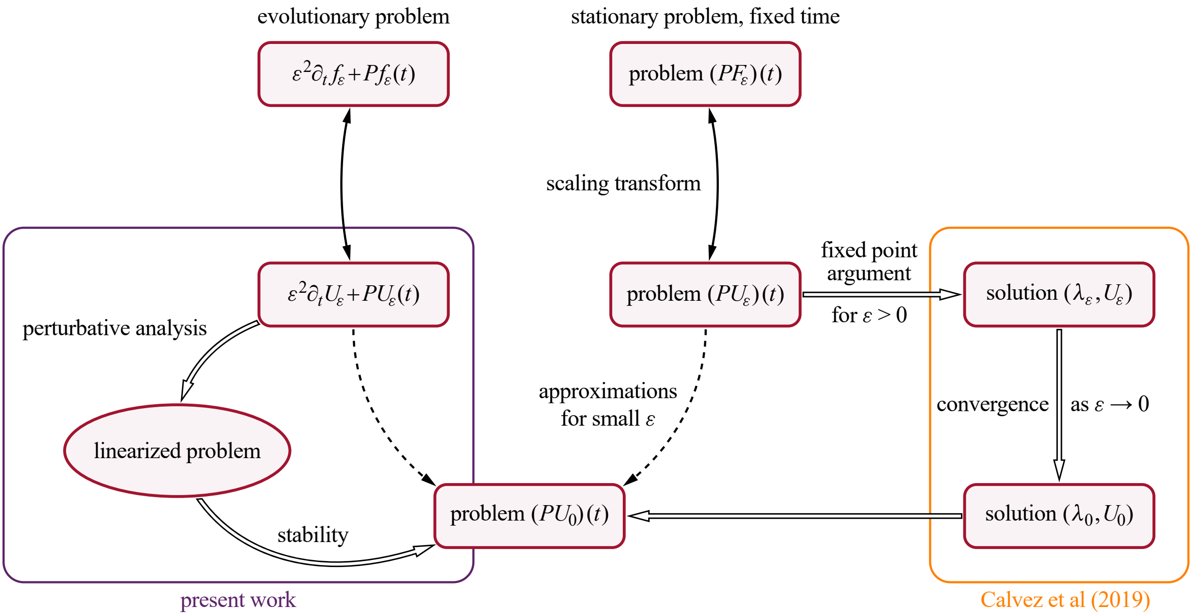

Figure 1. Scope of our paper compared to preceding work.

The function U^*, defined in (1-20), also solves $(P_t U_0)$. Plugging this perturbation into $(P_t U_\varepsilon)$ yields the following perturbed equation for R_ε:

$$\begin{aligned} M(t,z) - \varepsilon^2 \partial_t U^*(t,z) - \varepsilon^4 \partial_t R_\varepsilon(t,z) \\ = I_\varepsilon(U^* + \varepsilon^2 R_\varepsilon)(t,z) \\ \times \exp[U^*(t,z) - 2U^*(t,\bar{z}(t)) + U^*(t,z_*(t))] \exp[\varepsilon^2(R_\varepsilon(t,z) - 2R_\varepsilon(t,\bar{z}(t)) + R_\varepsilon(t,z_*(t)))]. \end{aligned}$$

By using $(P_t U_0)$, one gets that R_ε solves

$$\begin{aligned} M(t,z) - \varepsilon^2 \partial_t U^*(t,z) - \varepsilon^4 \partial_t R_\varepsilon(t,z) \\ = I_\varepsilon(U^* + \varepsilon^2 R_\varepsilon)(t,z) M(t,z) \exp[\varepsilon^2(R_\varepsilon(t,z) - 2R_\varepsilon(t,\bar{z}(t)) + R_\varepsilon(t,z_*(t)))]. \end{aligned}$$

To prove the boundedness of R_ε, a solution to this nonlinear equation, we shall linearize it and show a stability result on the linearized problem; see Theorem 7.1. We explain here the heuristics about the linearization. We have already said that I_ε is expected to converge to 1. Therefore by linearizing the exponential, a natural linearized equation when ε is small appears to be

$$\varepsilon^2 \partial_t \tilde{R}_\varepsilon(t,z) = M(t,z)(-\tilde{R}_\varepsilon(t,z) + 2\tilde{R}_\varepsilon(t,\bar{z}(t)) - \tilde{R}_\varepsilon(t,z_*(t))), \tag{2-1}$$

For clarity we denote by T the linear operator

$$T(R)(t,z) := M(t,z)(2R(t,\bar{z}(t)) - R(t,z) + R(t,z_*(t))).$$

We know precisely the eigenelements of this linear operator. The eigenvalue 0 has multiplicity two, the eigenspace consisting of affine functions. More generally one can get every eigenvalue by differentiating iteratively the operator T and evaluating at $z = z_*$. This corresponds to the following table:

Eigenvalue:	0	0	$-\frac{1}{2}$	$-\frac{3}{4}$	$\cdots$
Dual eigenvector:	$\delta_{z_*(t)}$	$\delta'_{z_*(t)}$	$\delta''_{z_*(t)}$	$\delta^{(3)}_{z_*(t)}$	$\cdots$

This explains why R_ε should be decomposed between affine parts and the rest, and, as a consequence, why this is also the case for the solution U_ε we are investigating. The scalars p_ε and q_ε of the decomposition (1-17) correspond to the projection of U_ε upon the eigenspace associated to the (double) eigenvalue 0. On the other hand, the rest is expected to remain uniformly bounded since the corresponding eigenvalues are negative, below $-\frac{1}{2}$.

Beyond the heuristics about the stability, this linear analysis also illustrates the discrepancy between V_ε and q_ε in Theorem 1.3. While V_ε is expected to relax to an explicit bounded value arbitrary quickly as $\varepsilon \to 0$ (fast dynamics), this is not true for q_ε, and its limit q^* solves a differential equation (slow dynamics):

$$\dot{q}^*(t) = -m''(z_*(t))q^*(t) + \tfrac{1}{2}m^{(3)}(z_*(t)) - 2m''(z_*(t))m'(z_*(t)).$$

One interpretation of this formula is that, for $\varepsilon > 0$, the second eigenvalue, which corresponds to the affine part, is not 0 as in the table above. Our intuition, given the equation above, is that it is of order $-\varepsilon^2 m''(z_*(t))$. We can guess that this explains why, in Section 8, we obtain directly with contraction arguments that the perturbation of V_ε is bounded (fast dynamics), while to show that the perturbation of q_ε is uniformly bounded, we must deal with an ODE that it solves. This "vanishing" but negative second eigenvalue could also explain why our analysis needs a uniform contraction argument for the affine part while it can be chosen freely at $\varepsilon = 0$; see $(P_t U_0)$.

The technique we will use in the following sections to bound W_ε in $\mathcal{F}$ will seem more natural in light of this formal analysis. The first step will be to work around z_*, the base point of the dual eigenelements in the table above. We derive uniform bounds up to the third derivative to estimate W_ε; see Theorem 7.1.

By plugging the expansions of (1-21) and (1-22) associated to the decomposition (1-17) and the logarithmic transform (1-2) into our original model $(P_t f_\varepsilon)$, we obtain the following main reference equation that we will study in the rest of this article:

$$\begin{aligned} M(t,z) - \varepsilon^2(\dot{p}_\varepsilon(t) + \dot{q}^*(t)(z-z_*) + m'(z_*)q^*(t) + \partial_t V^*(t,z)) - \varepsilon^4(\dot{\kappa}_\varepsilon(t)(z-z_*) + m'(z_*)\kappa_\varepsilon(t) + \partial_t W_\varepsilon(t,z)) \\ = M(t,z)\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)\exp[\varepsilon^2(W_\varepsilon(t,z) - 2W_\varepsilon(t,\bar{z}(t)) + W_\varepsilon(t,z_*(t)))]. \quad (2\text{-}2) \end{aligned}$$

Our main objective will be to linearize (2-2), in order to deduce the boundedness of the unknowns, $(\kappa_\varepsilon W_\varepsilon)$, by working on the linear part of the equations. We will need to investigate different scales (in ε) to capture the different behaviors of each contribution.

We will pay attention to the remaining terms. We will use the classical notation $O(1)$ and $O(\varepsilon)$, and we will write $\|(\kappa_\varepsilon, W_\varepsilon)\| O(\varepsilon)$ to illustrate when the constants of $O(\varepsilon)$ depend on $(\kappa_\varepsilon, W_\varepsilon)$. We also define a refinement of the classical notation $O(\varepsilon)$:

Definition 2.1 ($O^*(\varepsilon^\beta)$). For $\beta \in \mathbb{N}$, we say that a function $g(\varepsilon, t, z)$ is such that $g(\varepsilon, t, z) = O^*(\varepsilon^\beta)$ if there exists ε^* such that for all $\varepsilon \leq \varepsilon^*$,

$$|g(\varepsilon, t, z)| \leq C^* \varepsilon^\beta,$$

and the constants ε^* and C^* depend only on the pair (q^*, V^*).

More generally, when we write $O(\varepsilon)$, the constants involved may a priori depend upon the pair $(\kappa_\varepsilon, W_\varepsilon)$. Our intent is to make the dependency of the constants clear when we linearize. This will prove to be a crucial point when we go back to the nonlinear problem (2-2). We will see that all the terms that do not have a sufficient order in ε, to be negligible, will be $O^*(1)$, and therefore uniformly bounded independently of $(\kappa_\varepsilon, W_\varepsilon)$. A key point of our analysis is to segregate those terms when doing the linearization.

The rest of the paper is organized as follows:

- First we prove some properties upon the reference pair (q^*, V^*) around which all the terms of (2-2) are linearized.
- A key part of our perturbative analysis is to be able to linearize $\mathcal{I}_\varepsilon$, which we do in Section 4 thanks to careful estimates upon the directional derivatives.
- We derive an equation on κ_ε in Section 5A, and later a linear equation for the approximation W_ε, as well as derivations for all of its derivatives in Section 6, while controlling precisely the error terms.
- We next show the boundedness of the solutions of the linear problem in the space $\mathcal{F}$, see Section 7, mainly through maximum principles and a dyadic division of the space to take into account the nonlocal behavior of the infinitesimal operator. This is the content of Theorem 7.1.
- Finally, we tackle the proof of Theorem 1.3 in Section 8, using contraction arguments deduced from the previous section.
- To conclude, in Section 9 we discuss some of our assumptions made in Assumption 1.1, illustrated by some numerical simulations.

3. Preliminary results: estimates of $\mathcal{I}_\varepsilon^*$ and V^*

3A. *Control of* (q^*, V^*). Before tackling the main difficulties of this article, we first state some controls on the function V^*, the solution of $(P_t U_0)$. Most of them use the explicit expression of (1-9) and were proved in [Calvez et al. 2019]. To be able to measure this function we introduce another functional space, with more constraints.

Definition 3.1 (subspace $\mathcal{E}^*$). We define $\mathcal{E}^*$ as the following subspace of $\mathcal{E}$:

$$\mathcal{E}^* := \mathcal{E} \cap \{v \in \mathcal{C}^5(\mathbb{R}_+ \times \mathbb{R}) \mid \varphi_\alpha(t, z)|\partial_z^4 v(t, z)|,\ \varphi_\alpha(t, z)|\partial_z^5 v(t, z)| \in L^\infty(\mathbb{R}_+ \times \mathbb{R})\},$$

and we equip it with the norm $\|\cdot\|_*$:

$$\|v\|_* = \max\Big(\|v\|_{\mathcal{E}}, \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \varphi_\alpha(t, z)|\partial_z^4 v(t, z)|, \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \varphi_\alpha(t, z)|\partial_z^5 v(t, z)|\Big).$$

Our intention with the successive definitions of the functional spaces is to be able to measure each term of the decomposition made in (1-21) as follows:

$$V_\varepsilon = V^* + \varepsilon^2 W_\varepsilon, \quad \text{with } V_\varepsilon \in \mathcal{E}, \quad V^* \in \mathcal{E}^* \quad \text{and} \quad W_\varepsilon \in \mathcal{F}.$$

The fact that $V^* \in \mathcal{E}^*$ is part of the claim of the following lemma:

Lemma 3.2 (properties of V^*). *The function V^* belongs to the space $\mathcal{E}^*$. Moreover,*

$$\partial_z^2 V^*(t, z_*) = 2m''(z_*) \quad \textit{and} \quad \partial_z^3 V^*(t, z_*) = \tfrac{4}{3} m^{(3)}(z_*). \tag{3-1}$$

Proof. Precise estimates of the summation operator that defines V^* in (1-9) are studied in [Calvez et al. 2019]. They can be applied there thanks to the decay assumptions about M; see (1-14). The only difference here is that a uniform bound for the fourth and fifth derivative are required. The proofs of those bounds rely solely upon the assumption made in (1-14), for the fourth and fifth derivative of M. This shows that $V^* \in \mathcal{E}^*$. Explicit computations based on (1-9) prove the relationships (3-1). □

A consequence of Lemma 3.2 is that since $m''(z_*(t)) > 0$ for $t > t_0$, thanks to (1-16), we have that V^* is locally convex around $z_*(t)$. However, we need more information about V^* than the space it belongs to. We will bound (q^*, V^*) independently of time. This is the content of the following result:

Proposition 3.3 (uniform bound on (q^*, V^*)). *There exists a constant K^* such that for $j = 0, 1, 2, 3$, we have*

$$\max(\|V^*\|_*, \|q^*\|_{L^\infty(\mathbb{R}_+)}, \|\partial_t \partial_z^j V^*\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}) \le K^*.$$

Proof. For the estimates upon V^* and $\partial_t V^*$, it is a direct consequence of the definition of $\mathcal{E}^*$ and the explicit formula (1-9). The technique to bound the sums is to distinguish between the small and large indices, as was detailed in [Calvez et al. 2019].

For q^*, one must look to (1-18). The boundedness of q^* is a straightforward consequence of the convexity of m at $z_*(t)$ for large times; see (1-16) and the convergence of z_* to bound the other terms. □

3B. *Estimates of $\mathcal{I}_\varepsilon^*$ and its derivatives.* We next define a notational shortcut for the functional $\mathcal{I}_\varepsilon$ introduced in (1-3), when it is evaluated at the reference pair (q^*, V^*):

$$\mathcal{I}_\varepsilon^* := \mathcal{I}_\varepsilon(q^*, V^*).$$

This section is devoted to getting precise estimates of this function. This will be crucial for the linearization of $\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)$ as can be seen on the full equation (2-2).

Proposition 3.4 (estimation of $\mathcal{I}_\varepsilon^*$). *We have that*

$$\mathcal{I}_\varepsilon^*(t, z) = 1 + O^*(\varepsilon^2),$$

where the constants of $O^*(\varepsilon^2)$ depend only on K^*, as introduced in Proposition 3.3 and as defined by Definition 2.1.

The proof involves exact Taylor expansions in ε. Very similar expansions were performed in Lemma 3.1 of [Calvez et al. 2019]. We adapt the method of proof here, since it will be used extensively throughout this article.

Proof of Proposition 3.4. We recall that by Proposition 3.3, $\max(|q^*|, \|V^*\|_*) \le K^*$, and, by definition,

$$\mathcal{I}_\varepsilon^*(t,z) = \frac{\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} \exp[-\varepsilon q^*(t)(y_1+y_2) + 2V^*(t,\bar z) - V^*(t,\bar z+\varepsilon y_1) - V^*(t,\bar z+\varepsilon y_2)]\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon q^*(t)y + V^*(t,z_*) - V^*(t,z_*+\varepsilon y)]\,dy}$$
$$:= \frac{N(t,z)}{D(t)},$$

where Q is the quadratic form appearing after the rescaling of the infinitesimal operator in (1-3):

$$Q(y_1,y_2) := \tfrac{1}{2}y_1y_2 + \tfrac{3}{4}(y_1^2+y_2^2).$$

This quadratic form will appear very frequently in what follows, mostly, as here, through the bivariate Gaussian distribution it defines. Once and for all, we state that a correct normalization of this Gaussian distribution is

$$\frac{1}{\sqrt{2}\pi}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}\,dy_1\,dy_2 = 1.$$

We start the estimates with the more complicated term, the numerator N. With an exact Taylor expansion inside the exponential, there exists generic $0<\xi_i<1$, for $i=1,2$, such that

$$N(t,z) = \frac{1}{\sqrt{2}\pi}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} \exp\Big[-\varepsilon q^*(t)(y_1+y_2) - \varepsilon(y_1+y_2)\partial_z V^*(t,\bar z)$$
$$- \tfrac{1}{2}\varepsilon^2\big(y_1^2\partial_z^2 V^*(t,\bar z+\varepsilon\xi_1 y_1) + y_2^2\partial_z^2 V^*(t,\bar z+\varepsilon\xi_2 y_2)\big)\Big]\,dy_1\,dy_2.$$

Moreover, we can write, for some $\theta=\theta(y_1,y_2)\in(0,1)$,

$$\exp[-\varepsilon P] = 1-\varepsilon P + \tfrac{1}{2}\varepsilon^2 P^2\exp[-\theta\varepsilon P], \quad \text{with}$$
$$P := (y_1+y_2)(q^*(t)+\partial_z V^*(t,\bar z)) + \tfrac{1}{2}\varepsilon\big(y_1^2\partial_z^2 V^*(t,\bar z+\varepsilon\xi_1 y_1) + y_2^2\partial_z^2 V^*(t,\bar z+\varepsilon\xi_2 y_2)\big),$$

such that

$$|P| \le K^*\big(|y_1|+|y_2|+\tfrac{1}{2}\varepsilon(y_1^2+y_2^2)\big). \tag{3-2}$$

Combining the expansions, we find that

$$N(t,z) = \frac{1}{\sqrt{2}\pi}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}\big(1-\varepsilon P+\tfrac{1}{2}\varepsilon^2P^2\exp[-\theta\varepsilon P]\big)\,dy_1\,dy_2$$
$$= 1 - \varepsilon\frac{1}{\sqrt{2}\pi}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}P\,dy_1\,dy_2 + \frac{\varepsilon^2}{2\sqrt{2}\pi}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}P^2\exp[-\theta\varepsilon P]\,dy_1\,dy_2. \tag{3-3}$$

The key part is the cancellation of the terms $O(\varepsilon)$ due to the symmetry of Q:

$$\frac{1}{\sqrt{2}\pi}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}(y_1+y_2)\,dy_1\,dy_2 = 0.$$

Therefore,

$$\frac{\varepsilon}{\sqrt{2\pi}}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} P\,dy_1\,dy_2 \\ = \frac{\varepsilon^2}{2\sqrt{2\pi}}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}(y_1^2\partial_z^2 V^*(t,\bar{z}+\varepsilon\xi_1 y_1)+y_2^2\partial_z^2 V^*(t,\bar{z})+\varepsilon\xi_2 y_2)\,dy_1\,dy_2,$$

and we get the estimate

$$\left|\frac{\varepsilon}{\sqrt{\pi}}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} P\,dy_1\,dy_2\right| \le \frac{\varepsilon^2}{2\sqrt{2\pi}}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}(y_1^2+y_2^2)K^*\,dy_1\,dy_2 \le O^*(\varepsilon^2).$$

Thanks to (3-2) it is easy to verify that the last term of (3-3) behaves similarly:

$$\frac{\varepsilon^2}{2\sqrt{2\pi}}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} P^2\exp[-\theta\varepsilon P]\,dy_1\,dy_2 = O^*(\varepsilon^2).$$

Indeed, it states that the term P is at most quadratic with respect to y_i, so $Q+\theta\varepsilon P$ is uniformly bounded below by a positive quadratic form for ε small enough. This shows that

$$N(t,z) = 1 + O^*(\varepsilon^2).$$

The denominator is easier. With the same arguments, using the Gaussian density, we find that

$$D(t) = 1 + O^*(\varepsilon^2).$$

Combining the estimates of N and D, we get the desired result. □

There exists a link between q^* and $\partial_z\mathcal{I}_\varepsilon^*(t,z_*)$, which is in fact the motivation behind the choice of q^*.

Proposition 3.5 (link between q^* and $\partial_z\mathcal{I}_\varepsilon^*(t,z_*)$).

$$\partial_z\mathcal{I}_\varepsilon^*(t,z_*(t)) = \varepsilon^2\big(m''(z_*(t))q^*(t)-\tfrac{1}{2}m^{(3)}(z_*(t))\big)+O^*(\varepsilon^4),$$

where the constants of $O^*(\varepsilon^4)$ only depend on K^*.

The proof of this result was the content of [Calvez et al. 2019, Lemma 3.1] and only requires that the pair (q^*, V^*) is uniformly bounded, as stated in Proposition 3.3. Its proof follows the same procedure of exact Taylor expansions as that of Proposition 3.4.

It will be useful to dispose of estimates of $\partial_z\mathcal{I}_\varepsilon^*$ not only at the point z_*. They are less precise, as stated in the following proposition:

Proposition 3.6 (estimates of the decay of the derivatives of $\mathcal{I}_\varepsilon^*$). *There exists a constant ε_* that depends only on K^* such that for all $\varepsilon \le \varepsilon*$, for $j = 1, 2, 3$,*

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \varphi_\alpha(t,z)|\partial_z^{(j)}\mathcal{I}_\varepsilon^*(t,z)| = O^*(\varepsilon^2).$$

To simplify notations, we introduce the following difference operator that appears in the integral $\mathcal{I}_\varepsilon$; see (1-3):

$$\begin{aligned}\mathcal{D}_\varepsilon(V)(Y,t,z) &:= V(t,\bar{z}) - \tfrac{1}{2}V(t,\bar{z}+\varepsilon y_1) - \tfrac{1}{2}V(t,\bar{z}+\varepsilon y_2), \quad \text{with } Y=(y_1,y_2),\\ \mathcal{D}^*_\varepsilon(V)(y,t) &:= V(t,z_*) - V(t,z_*+\varepsilon y).\end{aligned} \tag{3-4}$$

We will use the following technical lemma giving an estimate of the weight function against the derivatives of a given function.

Lemma 3.7 (influence of the weight function). *There exists a constant C such that for each ball B of $\mathcal{E}^*$ or $\mathcal{F}$, there exists ε_B such that for every $W \in B$, for every $y \in \mathbb{R}$ and $\varepsilon \le \varepsilon_B$, for $j = 2, 3, 4, 5$,*

$$\varphi_\alpha(t,z)|\partial_z^{(j)} W(t,\bar{z}(t)+\varepsilon y)| \le \begin{cases} C\|W\| & \text{if } |y| \le |z - z_*(t)|,\\ (1+|y|^\alpha)\|W\| & \text{otherwise,}\end{cases}$$

with $\|W\| = \|W\|_*$ or $\|W\|_\mathcal{F}$ depending on the case.

Proposition 3.6 is a prototypical result. It will be followed by a series of similar statements. Therefore, we propose two different proofs. In the first one, we write exact Taylor expansions. However, the formalism is heavy, which is why we propose next a formal argument, where the Taylor expansions are written without the exact remainder for the sake of clarity.

In the rest of this paper more complicated estimates will be proved, in the spirit of Proposition 3.6; see Proposition 4.1 and Lemma 4.8 for instance. The notations and formulas will be very long, so we shall only write the *formal* parts of the argument. However, it can all be made rigorous, as below.

Proof of Proposition 3.6. First, write the expression for the derivative, using our notation $\mathcal{D}_\varepsilon$ introduced in (3-4):

$$\begin{aligned}\partial_z \mathcal{I}^*_\varepsilon(t,z) &= \frac{\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} \exp[-\varepsilon q^*(y_1+y_2) + 2\mathcal{D}_\varepsilon(V^*)(Y,t,z)]\mathcal{D}_\varepsilon(\partial_z V^*)(Y,t,z)\,dy_1\,dy_2}{\sqrt{\pi}\int_\mathbb{R} e^{-|y|^2/2}\exp[-\varepsilon q^* y + \mathcal{D}^*_\varepsilon(V^*)(y,t)]\,dy}\\ &:= \frac{N(t,z)}{D(t)}.\end{aligned} \tag{3-5}$$

We only focus on the numerator. The denominator D can be handled similarly as in the proof of Proposition 3.4, where we show that it is essentially $1 + O^*(\varepsilon^2)$. We perform two Taylor expansions in the numerator N, namely,

$$\begin{aligned}2\mathcal{D}_\varepsilon(V^*)(Y,t,\bar{z}) &= -\varepsilon(y_1+y_2)\partial_z V^*(t,\bar{z}) - \tfrac{1}{2}\varepsilon^2(y_1^2\partial_z^2 V^*(t,\bar{z}+\varepsilon\xi_1 y_1) + y_2^2\partial_z^2 V^*(t,\bar{z}+\varepsilon\xi_2 y_2)),\\ \mathcal{D}_\varepsilon(\partial_z V^*)(Y,t,\bar{z}) &= -\tfrac{1}{2}\varepsilon(y_1+y_2)\partial_z^2 V^*(t,\bar{z}) - \tfrac{1}{4}\varepsilon^2(y_1^2\partial_z^3 V^*(t,\bar{z}+\varepsilon\xi_1 y_1) + y_2^2\partial_z^3 V^*(t,\bar{z}+\varepsilon\xi_2 y_2)),\end{aligned} \tag{3-6}$$

where the ξ_i denote some generic number such that $0 < \xi_i < 1$ for $i = 1, 2$. Moreover, we can write

$$\begin{aligned}&\exp[-\varepsilon P] = 1 - \varepsilon P \exp[-\theta\varepsilon P], \quad \text{with}\\ &\quad P := (y_1+y_2)(\partial_z V^*(t,\bar{z}) + q^*) + \tfrac{1}{2}(y_1^2\partial_z^2 V^*(t,\bar{z}+\varepsilon\xi_1 y_1) + y_2^2\partial_z^2 V^*(t,\bar{z}+\varepsilon\xi_2 y_2)),\end{aligned} \tag{3-7}$$

for some $\theta=\theta(y_1,y_2)\in(0,1)$. Combining the expansions, we find that

$$\begin{aligned}
&\varphi_\alpha(t,z)\partial_z\mathcal{I}_\varepsilon^*(t,z)\\
&=\frac{\varphi_\alpha(t,z)}{\sqrt{2}\pi}\iint_{\mathbb{R}^2}e^{-Q(y_1,y_2)}(1-\varepsilon P\exp[-\theta\varepsilon P])\\
&\qquad\times\bigl(-\tfrac12\varepsilon(y_1+y_2)\partial_z^2V^*(t,\bar z)-\tfrac14\varepsilon^2(y_1^2\partial_z^3V^*(t,\bar z+\varepsilon\xi_1y_1)+y_2^2\partial_z^3V^*(t,\bar z+\varepsilon\xi_2y_2))\bigr)\,dy_1\,dy_2.
\end{aligned}$$

Crucially, the $O(\varepsilon)$ contribution cancels due to the symmetry of Q, as already observed above:

$$\iint_{\mathbb{R}^2}e^{-Q(y_1,y_2)}(y_1+y_2)\,dy_1\,dy_2=0.$$

So, it remains that

$$\begin{aligned}
&\varphi_\alpha(t,z)N(t,z)\\
&\ =-\varepsilon^2\frac{\varphi_\alpha(t,z)}{4\sqrt{2}\pi}\iint_{\mathbb{R}^2}e^{-Q(y_1,y_2)}(y_1^2\partial_z^3V^*(t,\bar z+\varepsilon\xi_1y_1)+y_2^2\partial_z^3V^*(t,\bar z+\varepsilon\xi_2y_2))\,dy_1\,dy_2\\
&\quad+\varepsilon^2\frac{\varphi_\alpha(t,z)}{2\sqrt{2}\pi}\iint_{\mathbb{R}^2}e^{-Q(y_1,y_2)}P\exp[-\theta\varepsilon P](y_1+y_2)\partial_z^2V^*(t,\bar z)\,dy_1\,dy_2\\
&\quad+\varepsilon^3\frac{\varphi_\alpha(t,z)}{4\sqrt{2}\pi}\iint_{\mathbb{R}^2}e^{-Q(y_1,y_2)}P\exp[-\theta\varepsilon P](y_1^2\partial_z^3V^*(t,\bar z+\varepsilon\xi_1y_1)+y_2^2\partial_z^3V^*(t,\bar z+\varepsilon\xi_2y_2))\,dy_1\,dy_2.
\end{aligned}$$

If we forget about the weight in front of each term, clearly the last two contributions are uniform $O^*(\varepsilon)$ since $\varepsilon\le\varepsilon_*$ is small enough and V^* and q^* are uniformly bounded by K^*; see Proposition 3.3. The term P is at most quadratic with respect to y_i, see (3-7), so $Q+\theta\varepsilon P$ is uniformly bounded below by a positive quadratic form for ε small enough.

A difficulty is to add the weight to those estimates. To do so, we use Lemma 3.7, for each integral term appearing in the previous formula, because each time a term of the following form appears:

$$\varphi_\alpha(t,z)\partial_z^{(j)}V^*(t,\bar z+\varepsilon\xi_iy_i). \tag{3-8}$$

Since every ξ_i satisfies $0<\xi_i<1$, the bounds given by Lemma 3.7 ensure that each integral remains bounded by moments of the bivariate Gaussian defined by Q, as if there were no weight function. This concludes the proof of the first estimate of Proposition 3.6.

Bounding the quantity $\varphi_\alpha(t,z)|\partial_z^{(j)}\mathcal{I}_\varepsilon^*(t,z)|$ for $j=2,3$ follows the same steps, as seen here:

$$\begin{aligned}
&\partial_z^2\mathcal{I}_\varepsilon^*(t,z)\\
&\ =\frac{\iint_{\mathbb{R}^2}\exp[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(V^*)(Y,t,z)]\bigl(\mathcal{D}_\varepsilon(\partial_zV^*)^2+\frac12\mathcal{D}_\varepsilon(\partial_z^2V^*)\bigr)(Y,t,z)\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}}e^{-|y|^2/2}\exp[-\varepsilon q^*y+\mathcal{D}_\varepsilon^*(V^*)(y,t)]\,dy},
\end{aligned}$$

$$\partial_z^3\mathcal{I}_\varepsilon^*(t,z)=\frac{\begin{aligned}&\iint_{\mathbb{R}^2}\exp[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(V^*)(Y,t,z)]\\&\qquad\times\bigl(\mathcal{D}_\varepsilon(\partial_zV^*)^3+\tfrac32\mathcal{D}_\varepsilon(\partial_zV^*)\mathcal{D}_\varepsilon(\partial_z^2V^*)+\tfrac14\mathcal{D}_\varepsilon(\partial_z^3V^*)\bigr)(Y,t,z)\,dy_1\,dy_2\end{aligned}}{\sqrt{\pi}\int_{\mathbb{R}}e^{-|y|^2/2}\exp[-\varepsilon q^*y+\mathcal{D}_\varepsilon^*(V^*)(y,t)]\,dy}.$$

The motivation behind going up to the fifth derivative of V^* in Definition 3.1 lies in the terms

$$\tfrac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V^*) \quad \text{and} \quad \tfrac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V^*). \tag{3-9}$$

To gain an order ε^2 as needed in Proposition 3.6 for the estimates, one needs to go up by two orders in the Taylor expansions, which involve fourth and fifth derivatives. The importance of the order ε^2 will later appear in Proposition 4.2 and Section 7. □

We now propose a formal argument which is much simpler to read.

Formal proof of Proposition 3.6. We tackle the first derivative. We use the same notations as previously, see (3-5), and again focus on the numerator N. Formally,

$$N(t,z) = \iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} \exp[-\varepsilon(y_1+y_2)(q^* + \partial_z V^*(t,\bar{z})) + (y_1^2+y_2^2)O^*(\varepsilon^2)] \\ \times (-\varepsilon(y_1+y_2)\partial_z^2 V^*(t,\bar{z}) + (y_1^2+y_2^2)O^*(\varepsilon^2))\, dy_1\, dy_2.$$

Thanks to the linear approximation of the exponential, we find that

$$N(t,z) = \iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} (1-\varepsilon(y_1+y_2)(q^* + \partial_z V^*(t,\bar{z})) + (y_1^2+y_2^2)O^*(\varepsilon^2)) \\ \times (-\varepsilon(y_1+y_2)\partial_z^2 V^*(t,\bar{z}) + (y_1^2+y_2^2)O^*(\varepsilon^2))\, dy_1\, dy_2. \tag{3-10}$$

By sorting out the orders in ε, this can be rewritten as

$$N(t,z) = \varepsilon N_1 + O^*(\varepsilon^2).$$

By symmetry,

$$N_1 := -\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)} (\varepsilon(y_1+y_2)\partial_z^2 V^*(t,\bar{z}))\, dy_1\, dy_2 = 0.$$

To conclude, we notice that we can add the weight function to those estimates and make the same arguments as in the previous proof. □

Proof of Lemma 3.7. If $|z-z_*| \le 1$, then $1+|z-z_*| \le 2$, and the result is immediate by Definitions 3.1 and 1.2 of the adequate functional spaces. Therefore, one can suppose that $|z-z_*| > 1$. We first look at the regime $|y| \le |z-z_*|$. Then, by definition of the norms,

$$\begin{aligned}\varphi_\alpha(t,z)|\partial_z^{(j)} W(t,\bar{z}+\varepsilon y)| &\le 2\frac{|z-z_*|^\alpha}{|\bar{z}+\varepsilon y - z_*|^\alpha} |\bar{z}+\varepsilon y - z_*|^\alpha |\partial_z^{(j)} W(t,\bar{z}+\varepsilon y)| \\ &\le 2\frac{|z-z_*|^\alpha}{|\bar{z}+\varepsilon y - z_*|^\alpha} \|W\|.\end{aligned} \tag{3-11}$$

To bound the last quotient, we use the following inequality, that holds true because we are in the regime $|y| \le |z-z_*|$:

$$|\bar{z}+\varepsilon y - z_*| \ge -|\varepsilon y| + |\bar{z}-z_*| \ge \tfrac{1}{2}|z-z_*| - \varepsilon|z-z_*|.$$

This yields

$$2\frac{|z-z_*|}{|\bar{z}+\varepsilon y - z_*|} \le \frac{2}{1/2-\varepsilon}. \tag{3-12}$$

Bringing together (3-11) and (3-12), one gets Lemma 3.7 in the regime $|y| \le |z - z_*|$, on the condition that $\varepsilon < \frac{1}{2}$.

On the contrary, when $|z - z_*| \le |y|$, we have immediately that

$$(1 + |z - z_*|^\alpha) |\partial_z^{(j)} W(t, \bar{z} + \varepsilon y)| \le (1 + |y|^\alpha) \|W\|. \qquad \square$$

4. Linearization of $\mathcal{I}_\varepsilon$ and its derivatives

The first step to obtain a linearized equation on W_ε is to study the nonlinear terms of (2-2). A key point is the study of the functional $\mathcal{I}_\varepsilon$ defined in (1-3), which plays a major role in our study. We will show that it converges uniformly to 1, as we claimed in Section 1, and that its derivatives are uniformly small, with some decay for large z, similarly to what we proved for the function $\mathcal{I}_\varepsilon^*$ in the previous section. This will enable us to linearize $\mathcal{I}_\varepsilon$ and its derivatives in Propositions 4.2 and 4.5.

4A. ***Linearization of $\mathcal{I}_\varepsilon$.*** We first bound uniformly all the terms that appear during the linearization of $\mathcal{I}_\varepsilon$ by Taylor expansions. One starts by measuring the first order directional derivatives.

Proposition 4.1 (bounds on the directional derivatives of $\mathcal{I}_\varepsilon$). *For any ball B of $\mathbb{R} \times \mathcal{E}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$ we have, for all $(g, V) \in B$ and $H \in \mathcal{E}$:*

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} |\partial_g \mathcal{I}_\varepsilon(g, V)(t, z)| \le \|(g, V)\| O(\varepsilon^2) \tag{4-1}$$

and

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} |\partial_V \mathcal{I}_\varepsilon(g, V) \cdot H(t, z)| \le \|(g, V)\| \, \|H\|_{\mathcal{E}} O(\varepsilon^2). \tag{4-2}$$

Proof. As in the estimates of $\mathcal{I}_\varepsilon^*$ and its derivatives in the previous section, the argument to obtain the result will be to perform exact Taylor expansions. As explained before we will not pay attention to the exact remainders that can be handled exactly as before, and we refer to the proofs of Propositions 3.4 and 3.6 for the details. However, our computations will make clear the order ε^2 of (4-1) and (4-2). First, thanks to the derivation with respect to g, an order of ε is gained straightforwardly:

$$\begin{aligned}\partial_g \mathcal{I}_\varepsilon(g, V)(t, z) &\\ = -\varepsilon \Biggl(&\frac{\iint_{\mathbb{R}^2} \exp[-Q(y_1, y_2) - \varepsilon g(y_1 + y_2) + 2\mathcal{D}_\varepsilon(V)(Y, t, z)](y_1 + y_2)\, dy_1\, dy_2}{\sqrt{\pi} \int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y, t)]\, dy} \\ &\qquad - \mathcal{I}_\varepsilon(g, W)(t, z) \frac{\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y, t)] y\, dy}{\sqrt{\pi} \int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y, t)]\, dy} \Biggr).\end{aligned} \tag{4-3}$$

The common denominator is bounded:

$$\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y, t)]\, dy \ge \int_{\mathbb{R}} \exp\bigl[-\tfrac{1}{2}|y|^2 - 2\varepsilon |y| \|(g, V)\|\bigr]\, dy.$$

For the numerators, a supplementary order in ε is gained by symmetry of Q, as in other estimates; see Proposition 3.6 for instance. For the single integral we write

$$\begin{aligned}\int_{\mathbb{R}} e^{-|y|^2/2} y \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)]\,dy &\le \int_{\mathbb{R}} e^{-|y|^2/2} y \exp[-\varepsilon g y + 2\varepsilon |y| \|(g,V)\|]\,dy \\ &\le \int_{\mathbb{R}} e^{-|y|^2/2} y (1 - \varepsilon g y + O(\varepsilon)|y| \|(g,V)\|)\,dy.\end{aligned}$$

Finally,

$$\int_{\mathbb{R}} e^{-|y|^2/2} y \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)]\,dy \le \|(g,V)\| O(\varepsilon). \tag{4-4}$$

For the first numerator of (4-3), the computations work in the same way:

$$\begin{aligned}&\iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(Y,t,z)](y_1+y_2)\,dy_1\,dy_2 \\ &\qquad \le \iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2) + O(\varepsilon)(y_1+y_2)\|(g,V)\|](y_1+y_2)\,dy_1\,dy_2 \\ &\qquad \le \iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2)](1 + O(\varepsilon)(y_1+y_2)\|(g,V)\|)(y_1+y_2)\,dy_1\,dy_2 \le \|(g,V)\| O(\varepsilon).\end{aligned} \tag{4-5}$$

Therefore, combining (4-3)–(4-5) we have proven the bound upon the first derivative of $\mathcal{I}_\varepsilon$ in (4-1).

Concerning (4-2), one starts by writing the following formula for the Fréchet derivative:

$$\begin{aligned}&\partial_V \mathcal{I}_\varepsilon(g,V)\cdot H(t,z) \\ &= \frac{\iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(Y,t,z)] 2\mathcal{D}_\varepsilon(H)(Y,t,z)\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)]\,dy} \\ &\qquad - \mathcal{I}_\varepsilon(g,V)(t,z) \frac{\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)] \mathcal{D}_\varepsilon^*(H)(y,t)\,dy}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}.\end{aligned} \tag{4-6}$$

The claimed order ε^2 holds true, by similar symmetry arguments. For instance, when we do the Taylor expansions on the numerator of the first term of (4-6), we find that

$$\begin{aligned}&\iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(Y,t,z)] 2\mathcal{D}_\varepsilon(H)(Y,t,z)\,dy_1\,dy_2 \\ &= 2\iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2)](1 - \varepsilon(y_1+y_2)(g + \partial_z V(t,\bar z)) + O(\varepsilon^2)\|V\|_{\mathcal{E}}) \\ &\qquad\qquad \times (-\varepsilon(y_1+y_2)\partial_z H(t,\bar z) + O(\varepsilon^2)(y_1^2+y_2^2)\|H\|_{\mathcal{E}})\,dy_1\,dy_2 \\ &= -2\varepsilon \partial_z H(t,\bar z) \iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2)](y_1+y_2)\,dy_1\,dy_2 \\ &\qquad + \varepsilon^2 \partial_z H(t,\bar z)(g + \partial_z V(t,\bar z)) \iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2)](y_1+y_2)^2\,dy_1\,dy_2 + O(\varepsilon^2)\|H\|_{\mathcal{E}}\|(g,V)\| \\ &\le \|(g,V)\|\|H\|_{\mathcal{E}} O(\varepsilon^2).\end{aligned} \tag{4-7}$$

For the second term of (4-6), we also gain an order ε^2 when making Taylor expansions of $\mathcal{D}_\varepsilon^*(V)$, since $\partial_z H(t, z_*) = 0$:

$$\begin{aligned}\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)]\mathcal{D}_\varepsilon^*(H)(y,t)\,dy \\ = -\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + 2\varepsilon|y|\|(g,V)\|]y^2 O(\varepsilon^2)\|H\|_{\mathcal{E}}\,dy \\ = -\int_{\mathbb{R}} e^{-|y|^2/2}(1-\varepsilon g y + 2\varepsilon|y|\|(g,V)\|)y^2 O(\varepsilon^2)\|H\|_{\mathcal{E}}\,dy \le \|(g,V)\|\|H\|_{\mathcal{E}} O(\varepsilon^2).\end{aligned} \tag{4-8}$$

As before, the denominator of (4-6) has a uniform lower bound, therefore combining (4-6)–(4-8) concludes the proof. □

We have proven all the tools to linearize $\mathcal{I}_\varepsilon$ as follows, thanks to the previous estimates on the directional derivatives of $\mathcal{I}_\varepsilon$.

Proposition 4.2 (linearization of $\mathcal{I}_\varepsilon$). *For any ball B of $\mathbb{R}\times\mathcal{E}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$ we have, for all $(g, W) \in B$,*

$$\mathcal{I}_\varepsilon(q^* + \varepsilon^2 g, V^* + \varepsilon^2 W)(t,z) = \mathcal{I}_\varepsilon^*(t,z) + O(\varepsilon^3)\|(g,W)\| \tag{4-9}$$

$$= 1 + O^*(\varepsilon^2) + O(\varepsilon^3)\|(g,W)\|, \tag{4-10}$$

where $O(\varepsilon^3)$ only depends on the ball B.

Proof. We write an exact Taylor expansion

$$\mathcal{I}_\varepsilon(q^*+\varepsilon^2 g, V^*+\varepsilon^2 W) = \mathcal{I}_\varepsilon^* + \varepsilon^2(\partial_g \mathcal{I}_\varepsilon(q^*+\varepsilon^2\xi g, V^*+\varepsilon^2\xi W) + \partial_V \mathcal{I}_\varepsilon(q^*+\varepsilon^2\xi g, V^*+\varepsilon^2\xi W)\cdot W),$$

for some $0 < \xi < 1$. Therefore (4-9) is a direct application of Proposition 4.1 to $g' = q^* + \varepsilon^2\xi g$, $V = V^* + \varepsilon^2 \xi W$ and $H = W$. One deduces the estimation of (4-10) from Proposition 3.4. □

As a matter of fact, in (4-10), we have even shown an estimate $1 + O^*(\varepsilon^2) + O(\varepsilon^4)\|(g,W)\|$. However, we choose to reduce arbitrarily the order in ε for consistency reasons with further estimates of this article. It suffices for our purposes.

4B. ***Linearization of $\partial_z\mathcal{I}_\varepsilon$ and decay estimates.*** In order to prove Theorem 1.3, we need to uniformly bound $\|W_\varepsilon\|_{\mathcal{F}}$, and this implies L^∞ bounds of the derivatives of W_ε. To obtain those, our method is to work on the linearized equations they satisfy. Therefore, linearizing $\mathcal{I}_\varepsilon$ is not enough, we need to linearize $\partial_z^{(j)}\mathcal{I}_\varepsilon$ as well, for $j = 1, 2, 3$. For that purpose we need more details than previously about the nature of the negligible terms. More precisely, we need to know how it behaves relatively to the weight function of the space $\mathcal{E}$ and $\mathcal{F}$, that acts by definition upon the second and third derivatives. The objective of this section is to linearize $\partial_z^{(j)}\mathcal{I}_\varepsilon$ to obtain similar results to Proposition 4.2. We first prove the following estimates on the derivatives of $\mathcal{I}_\varepsilon$:

Proposition 4.3 (decay estimate of $\partial_z\mathcal{I}_\varepsilon$). *For any ball B of $\mathbb{R}\times\mathcal{E}$, there exists a constant ε_B that depends only on B such that for any pair (g, V) in B, for all $\varepsilon \le \varepsilon_B$:*

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \varphi_\alpha(t,z)|\partial_z\mathcal{I}_\varepsilon(g,V)(t,z)| \le \|(g,V)\|O(\varepsilon),$$

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \varphi_\alpha(t,z)|\partial_z^2\mathcal{I}_\varepsilon(g,V)(t,z)| \le \|(g,V)\|O(\varepsilon),$$

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \varphi_\alpha(t,z)|\partial_z^3\mathcal{I}_\varepsilon(g,V)(t,z)| \le \|(g,V)\|O(\varepsilon^\alpha)+\frac{1}{2^{1-\alpha}}\|\varphi_\alpha\partial_z^3V\|_\infty,$$

where all the $O(\varepsilon)$ depend only on the ball B, and $\|\varphi_\alpha\partial_z^3V\|_\infty = \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}}\varphi_\alpha(t,z)|\partial_z^3V(t,z)|$.

We are not able to propagate an order ε for all derivatives. There is a factor of order 0 in ε in the third one: $\|\varphi_\alpha\partial_z^3V\|_\infty/2^{1-\alpha}$. It will be dealt with using a contraction argument, since $2^{\alpha-1} < k(\alpha) < 1$; and $k(\alpha)$ plays the same role as in Theorem 7.1. This has to be put in parallel with [Calvez et al. 2019, Proposition 4.6].

Proof. We focus on the first derivative, the proof for the second derivative is straightforward to adapt:

$$\begin{aligned}
&\varphi_\alpha(t,z)\partial_z\mathcal{I}_\varepsilon(g,V)(t,z)\\
&\quad=\varphi_\alpha(t,z)\frac{\iint_{\mathbb{R}^2}\exp[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(V)(Y,t,z)]\mathcal{D}_\varepsilon(\partial_zV)(Y,t,z)\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}}e^{-|y|^2/2}\exp[-\varepsilon gy+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}.
\end{aligned}\tag{4-11}$$

As before, the following formal Taylor expansions hold true for the numerator, ignoring the weight in the first step:

$$\begin{aligned}
&\iint_{\mathbb{R}^2}\exp[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(V)(Y,t,z)]\mathcal{D}_\varepsilon(\partial_zV)(Y,t,z)\,dy_1\,dy_2\\
&\quad=\iint_{\mathbb{R}^2}\exp[-Q(y_1,y_2)](1-O(\varepsilon)(y_1+y_2)\|(g,V)\|)(-O(\varepsilon)(y_1+y_2)\|(g,V)\|)\,dy_1\,dy_2,\\
&\quad\le O(\varepsilon)\|(g,V)\|.
\end{aligned}\tag{4-12}$$

Meanwhile the denominator has a uniform lower bound:

$$\int_{\mathbb{R}}e^{-|y|^2/2}\exp[-\varepsilon gy+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy \ge \int_{\mathbb{R}}\exp\big[-\tfrac{1}{2}|y|^2-2\varepsilon|y|\|(g,V)\|\big]\,dy.$$

The estimate of (4-12) can be made rigorous as in the proof of Proposition 3.6, for instance. Moreover, one can add the weight to bound (4-11) thanks to Lemma 3.7, as explained in the proof of Proposition 3.6. Therefore, the proof of the first estimate of Proposition 4.3 is achieved.

For the second term of Proposition 4.3, involving the second derivative, the arguments and decomposition of the space are the same. We follow the same steps, arriving at the formula

$$\begin{aligned}
&\partial_z^2\mathcal{I}_\varepsilon(g,V)(t,z)\\
&\quad=\frac{\iint_{\mathbb{R}^2}\exp[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(V)(Y,t,z)]\big(\mathcal{D}_\varepsilon(\partial_zV)^2+\frac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2V)\big)(Y,t,z)\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}}e^{-|y|^2/2}\exp[-\varepsilon gy+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}.
\end{aligned}$$

Things are a little bit different for the third derivative, as can be seen in the following explicit formula:

$$\partial_z^3 \mathcal{I}_\varepsilon(t,z) = \frac{\begin{array}{r}\iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(Y,t,z)] \\ \times \big(\mathcal{D}_\varepsilon(\partial_z V)^3 + \frac{3}{2}\mathcal{D}_\varepsilon(\partial_z V)\mathcal{D}_\varepsilon(\partial_z^2 V) + \frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V)\big)(Y,t,z)\, dy_1\, dy_2\end{array}}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)]\, dy}. \tag{4-13}$$

All terms in this formula will provide an order ε exactly as before, except for the linear contribution of $\mathcal{D}_\varepsilon(\partial_z^3 V)$ since we lack a priori controls of the fourth derivative of V in $\mathcal{F}$. Therefore, for this term we proceed as follows:

$$\begin{aligned}
\varphi_\alpha(t,z)|\mathcal{D}_\varepsilon(\partial_z^3 V)(Y,t,z)| \\
&= (1+|z-z_*|)^\alpha \big|\partial_z^3 V(t,\bar z) - \tfrac{1}{2}\partial_z^3 V(t,\bar z+\varepsilon y_1) - \tfrac{1}{2}\partial_z^3 V(t,\bar z+\varepsilon y_2)\big| \\
&\le (1+|z-z_*|)^\alpha \big(|\partial_z^3 V(t,\bar z)| + \tfrac{1}{2}|\partial_z^3 V(t,\bar z+\varepsilon y_1)| + \tfrac{1}{2}|\partial_z^3 V(t,\bar z+\varepsilon y_2)|\big) \\
&\le 2^{\alpha+1}\|\varphi_\alpha \partial_z^3 V\|_\infty (1+\varepsilon^\alpha(|y_1|^\alpha + |y_2|^\alpha)).
\end{aligned} \tag{4-14}$$

For this computation, we used the following property of the weight function, which was also of crucial importance in [Calvez et al. 2019, Lemma 4.5]:

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \frac{\varphi_\alpha(t,z)}{\varphi_\alpha(t,\bar z)} \le 2^\alpha.$$

As a consequence, take $i = 1$ or 2. Then

$$\begin{aligned}
\varphi_\alpha(t,z)|\partial_z^3 V(\bar z+\varepsilon y_i)| &\le \frac{2^\alpha \varphi_\alpha(t,\bar z)}{(1+|\bar z+\varepsilon y_i - z_*|)^\alpha}\|\varphi_\alpha \partial_z^3 V\|_\infty \\
&\le 2^\alpha\left(1+\frac{|\varepsilon y_i|}{1+|\bar z+\varepsilon y_i - z_*|}\right)^\alpha \|\varphi_\alpha \partial_z^3 V\|_\infty \\
&\le 2^\alpha(1+\varepsilon^\alpha |y_i|^\alpha)\|\varphi_\alpha \partial_z^3 V\|_\infty.
\end{aligned}$$

We deduce that

$$\begin{aligned}
\varphi_\alpha(t,z)\frac{\iint_{\mathbb{R}^2} \exp[-Q(y_1,y_2) - \varepsilon g(y_1+y_2) + 2\mathcal{D}_\varepsilon(V)(Y,t,z)]\big(\frac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V)(Y,t,z)\big)\, dy_1\, dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y,t)]\, dy} \\
\le \frac{1}{2^{1-\alpha}}\|\varphi_\alpha \partial_z^3 V\|_\infty + O(\varepsilon^\alpha)\|(g,V)\|,
\end{aligned}$$

by subadditivity of $|\cdot|^\alpha$. This justifies (4-14). Once added to other estimates of the terms of (4-13), obtained by Taylor expansions of $\mathcal{D}_\varepsilon$ as before, we get the desired estimate. □

One can notice in the proof that the order $O(\varepsilon)$ is not the sharpest one can possibly get for the first derivative; see (4-12). However, it is sufficient for our purposes. We now detail the control upon the directional derivatives of $\mathcal{I}_\varepsilon$.

Proposition 4.4 (bound of the directional derivatives of $\mathcal{I}_\varepsilon$). *For any ball B of $\mathbb{R}\times\mathcal{E}$, there exists a constant ε_B that depends only on B such that for any pair (g, V) in B and any function $H \in \mathcal{E}$, for every $\varepsilon \le \varepsilon_B$:*

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} (\varphi_\alpha(t,z)|\partial_g\partial_z^{(j)}\mathcal{I}_\varepsilon(g,V)(t,z)|) \le O(\varepsilon)\|(g,V)\|_{\mathcal{E}}, \qquad j=1,2,3, \tag{4-15}$$

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} (\varphi_\alpha(t,z)|\partial_V\partial_z^{(j)}\mathcal{I}_\varepsilon(g,V)\cdot H(t,z)|) \le O(\varepsilon)\|H\|_{\mathcal{E}}, \qquad j=1,2, \tag{4-16}$$

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} (\varphi_\alpha(t,z)|\partial_V\partial_z^3\mathcal{I}_\varepsilon(g,V)\cdot H(t,z)|) \le O(\varepsilon^\alpha)\|H\|_{\mathcal{E}} + \frac{1}{2^{1-\alpha}}\|\varphi_\alpha\partial_z^3 H\|_\infty, \tag{4-17}$$

where the $O(\varepsilon)$ depend only on the ball B.

As for Proposition 4.3, in those estimates, the order of precision $O(\varepsilon)$ is not optimal and we could improve it without it being useful. We will not give the full proof for each estimate of this Proposition. However, we see that it follows the same pattern than in Proposition 4.3, and we will even use those results for the proof. In particular for the third derivative, it is not possible to completely recover an order ε, hence the term

$$\|\varphi_\alpha\partial_z^3 H\|_\infty/2^{1-\alpha}.$$

It comes from the *linear* part $\mathcal{D}_\varepsilon(\partial_z^3 V)$ that appears in $\partial_z^3\mathcal{I}_\varepsilon$, see (4-13). However, it does not prevent us from carrying our analysis since the factor $1/2^{1-\alpha}$ will be absorbed by a contraction argument; see Section 8.

Proof of Proposition 4.4. We first detail the proof of (4-15), because derivatives in g are somehow easier to estimate. The formula for the first derivative is:

$$\begin{aligned}
&\partial_g\partial_z\mathcal{I}_\varepsilon(g,V)(t,z)\\
&= -\varepsilon\Bigg(\frac{\iint_{\mathbb{R}^2}\exp[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(V)(Y,t,z)](y_1+y_2)\mathcal{D}_\varepsilon(\partial_z V)(Y,t,z)\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon g y+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}\\
&\qquad - \partial_z\mathcal{I}_\varepsilon(g,V)(t,z)\frac{\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon g y+\mathcal{D}_\varepsilon^*(V)(y,t)]y\,dy}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon g y+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}\Bigg).
\end{aligned}\tag{4-18}$$

The first term of this formula closely resembles the one for $\partial_z I_\varepsilon(g,V)$, with an additional factor $\varepsilon(y_1+y_2)$. We do not detail how to bound it, as it follows the same steps; see the work done following (4-11). For the second term we first use the following bound:

$$\frac{\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon g y+\mathcal{D}_\varepsilon^*(V)(y,t)]y\,dy}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon g y+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy} \le \frac{\int_{\mathbb{R}}\exp\big[-\frac{1}{2}|y|^2+2\varepsilon|y|\|(g,V)\|\big]y\,dy}{\int_{\mathbb{R}}\exp\big[-\frac{1}{2}|y|^2-2\varepsilon|y|\|(g,V)\|\big]\,dy}. \tag{4-19}$$

For ε sufficiently small that depends only on $\|(g,V)\|$ we deduce a uniform bound with moments of the Gaussian distribution. We then use the estimate from Proposition 4.3 on $\partial_z I_\varepsilon(g,V)$, which takes the weight into account, to conclude.

Every other estimate of Proposition 4.4 works along the same lines. We illustrate this with the second derivative in g and z:

$$\partial_g \partial_z^2 \mathcal{I}_\varepsilon(g,V)(t,z)$$
$$= -\varepsilon\left(\frac{\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}\exp[-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(V)(Y,t,z)](y_1+y_2)\big(\mathcal{D}_\varepsilon(\partial_z V)^2+\tfrac{1}{2}\mathcal{D}_\varepsilon(\partial_z^2 V)\big)(Y,t,z)\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon g y+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}\right.$$
$$\left. -\partial_z^2\mathcal{I}_\varepsilon(g,V)\frac{\int_{\mathbb{R}} e^{-|y|^2/2} y\exp[-\varepsilon g y+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}{\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon g y+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}\right). \tag{4-20}$$

This is very close to $\partial_z^2 I_\varepsilon$ that has already been estimated in Proposition 4.3, and therefore the same arguments as before hold.

The structure is different for the derivatives in V, as can be seen for $\partial_V\partial_z\mathcal{I}_\varepsilon(g,V)\cdot H$:

$$\partial_V\partial_z\mathcal{I}_\varepsilon(g,V)\cdot H(t,z)$$
$$= \frac{\iint_{\mathbb{R}^2}\exp[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(V)(Y,t,z)](2\mathcal{D}_\varepsilon(\partial_z V)\mathcal{D}_\varepsilon(H)+\mathcal{D}_\varepsilon(\partial_z H))(Y,t,z)\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon g y+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}$$
$$-\partial_z\mathcal{I}_\varepsilon(g,V)(t,z)\frac{\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon g y+\mathcal{D}_\varepsilon^*(V)(y,t)]\mathcal{D}_\varepsilon^*(H)(y,t)\,dy}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon g y+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}. \tag{4-21}$$

The second term can still be bounded using Proposition 4.3 and estimate (4-19) along with the following immediate result:

$$|\mathcal{D}_\varepsilon^*(V)(y,t)| \le \varepsilon|y|\|V\|_{\mathcal{E}}.$$

For the first term, we must do Taylor expansions of $2\mathcal{D}_\varepsilon(\partial_z V)\mathcal{D}_\varepsilon(H)+\mathcal{D}_\varepsilon(\partial_z H)$ to control them with the weight. One ends up with moments of the multidimensional Gaussian distribution just as in all the previous proofs. For instance,

$$\begin{aligned}2\varphi_\alpha(t,z)|\mathcal{D}_\varepsilon(\partial_z V)\mathcal{D}_\varepsilon(H)|(t,z) &\le \varphi_\alpha(t,z)|\mathcal{D}_\varepsilon(\partial_z V)(t,z)|O(\varepsilon)(|y_1|+|y_2|)\|H\|_{\mathcal{E}}\\ &\le O(\varepsilon)(|y_1|+|y_2|+|y_1|^{\alpha+1}+|y_2|^{1+\alpha})(|y_1|+|y_2|)\|H\|_{\mathcal{E}}\|V\|_{\mathcal{E}}.\end{aligned}$$

The same method holds for the second derivative in V and z.

The estimate of the third derivative in g and z is similar to the previous computations with the following formula:

$$\partial_g\partial_z^3\mathcal{I}_\varepsilon(t,z)$$
$$= \frac{\begin{aligned}&-\varepsilon\iint_{\mathbb{R}^2}\exp[-Q(y_1,y_2)-\varepsilon g(y_1+y_2)+2\mathcal{D}_\varepsilon(V)(Y,t,z)]\\ &\qquad\times(y_1+y_2)\big(\mathcal{D}_\varepsilon(\partial_z V)^3+\tfrac{3}{2}\mathcal{D}_\varepsilon(\partial_z V)\mathcal{D}_\varepsilon(\partial_z^2 V)+\tfrac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 V)\big)(Y,t,z)\,dy_1\,dy_2\end{aligned}}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon g y+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}$$
$$-\varepsilon\partial_z^3\mathcal{I}_\varepsilon(t,z)\frac{\int_{\mathbb{R}} e^{-|y|^2/2} y\exp[-\varepsilon g y+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}{\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon g y+\mathcal{D}_\varepsilon^*(V)(y,t)]\,dy}. \tag{4-22}$$

However, to get the bound (4-17), things are a little bit different, because of the linear term of higher order, $\mathcal{D}_\varepsilon(\partial_z^3 H)$:

$$\partial_V \partial_z^3 \mathcal{I}_\varepsilon(g, V) \cdot H(t, z)$$
$$= \frac{\begin{aligned}&\iint_{\mathbb{R}^2} \exp[-Q(y_1, y_2) - \varepsilon g(y_1 + y_2) + 2\mathcal{D}_\varepsilon(V)(Y, t, z)] \\ &\qquad \times \Big(\mathcal{D}_\varepsilon(H)\big(2\mathcal{D}_\varepsilon(\partial_z V)^3 + 3\mathcal{D}_\varepsilon(\partial_z V)\mathcal{D}_\varepsilon(\partial_z^2 V) + \tfrac{1}{2}\mathcal{D}_\varepsilon(\partial_z^3 V)\big) + 3\mathcal{D}_\varepsilon(\partial_z H)\mathcal{D}_\varepsilon(\partial_z V)^2 \\ &\qquad\qquad + \tfrac{3}{2}(\mathcal{D}_\varepsilon(\partial_z V)\mathcal{D}_\varepsilon(\partial_z^2 H) + \mathcal{D}_\varepsilon(\partial_z H)\mathcal{D}_\varepsilon(\partial_z^2 V)) + \tfrac{1}{4}\mathcal{D}_\varepsilon(\partial_z^3 H)\Big)(Y, t, z)\, dy_1\, dy_2\end{aligned}}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y, t)]\, dy}$$
$$+ \partial_z^3 \mathcal{I}_\varepsilon(t, z) \frac{\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y, t)]\mathcal{D}_\varepsilon^*(H)(y, t)\, dy}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon g y + \mathcal{D}_\varepsilon^*(V)(y, t)]\, dy}.$$

We do not get an order ε from the linear part $\mathcal{D}_\varepsilon(\partial_z^3 H)$, since we do not control the fourth derivative in $\mathcal{E}$. We then proceed with arguments following (4-13) in the proof of Proposition 4.3. □

Thanks to those estimates we are able to write our main result for this part, which is a precise control of the linearization of the derivatives of $\mathcal{I}_\varepsilon$:

Proposition 4.5 (linearization with weight). *For any ball B of $\mathbb{R} \times \mathcal{E}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$ we have, for all $(g, W) \in B$:*

$$\partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2 g, V^* + \varepsilon^2 W)(t, z) = \partial_z \mathcal{I}_\varepsilon^*(t, z) + \frac{\|(g, W)\|}{\varphi_\alpha(t, z)} O(\varepsilon^3), \tag{4-23}$$
$$\partial_z^2 \mathcal{I}_\varepsilon(q^* + \varepsilon^2 g, V^* + \varepsilon^2 W)(t, z) = \partial_z^2 \mathcal{I}_\varepsilon^*(t, z) + \frac{\|(g, W)\|}{\varphi_\alpha(t, z)} O(\varepsilon^3), \tag{4-24}$$
$$\partial_z^3 \mathcal{I}_\varepsilon(q^* + \varepsilon^2 g, V^* + \varepsilon^2 W)(t, z) = \partial_z^3 \mathcal{I}_\varepsilon^*(t, z) + \frac{\varepsilon^2 \|\varphi_\alpha \partial_z^3 W\|_\infty}{2^{1-\alpha}\varphi_\alpha(t, z)} + \frac{\|(g, W)\|}{\varphi_\alpha(t, z)} O(\varepsilon^{2+\alpha}), \tag{4-25}$$

where the $O(\varepsilon^3)$ only depend on the ball B.

Proof. The methodology for (4-23)–(4-25) is the same. We detail for instance how to prove (4-23). One begins by writing the following exact Taylor expansion up to the second order:

$$\begin{aligned}&\partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2 g, V^* + \varepsilon^2 W)(t, z) \\ &\quad = \partial_z \mathcal{I}_\varepsilon^*(t, z) + \varepsilon^2(\partial_g \partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2 \xi g, V^* + \varepsilon^2 \xi W)(t, z) + \partial_V \partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2 \xi g, V^* + \varepsilon^2 \xi W) \cdot W(t, z)),\end{aligned}$$

with $0 < \xi < 1$. The result for (4-23) is then given by the directional decay estimates of Proposition 4.4 applied to $g' = q^* + \varepsilon^2 \xi g,\ V = V^* + \varepsilon^2 \xi W$ and $H = W$. □

Together with Proposition 3.6, we know exactly how $\partial_z^j I_\varepsilon$ behaves when ε is small:

$$\partial_z^{(j)} \mathcal{I}_\varepsilon(q^* + \varepsilon^2 g, V^* + \varepsilon^2 W)(t, z) = O^*(\varepsilon^2) + \frac{\|(g, W)\|}{\varphi_\alpha(t, z)} O(\varepsilon^3),$$

where $j = 1, 2$, and the behavior is only slightly different for $j = 3$.

4C. *Refined estimates of $\mathcal{I}_\varepsilon^*$ at $z = z_*$.* To conclude this section dedicated to estimates of $\mathcal{I}_\varepsilon$, we now show that our estimates above can be made much more precise when looking at the particular case of the function $\mathcal{I}_\varepsilon^*$ evaluated at the point z_*. In particular, we will gain information about the sign of the derivatives, that will prove crucial regarding the stability of κ_ε. This additional precision is similar to what was needed in the stationary case [Calvez et al. 2019, Lemma 3.1] where detailed expansions of $\mathcal{I}_\varepsilon$ were needed for the study of the affine part, there named γ_ε. We will find it convenient to use the following notations, as in [Calvez et al. 2019]:

Definition 4.6 (measure notations). We introduce the following measures:

$$dG_\varepsilon^*(Y,z,t) := \frac{G_\varepsilon^*(Y,t,z)}{\iint_{\mathbb{R}^2} G_\varepsilon^*(Y,t,z)\,dy_1\,dy_2} = \frac{\exp[-Q(y_1,y_2)-\varepsilon q^*(y_1+y_2)+2\mathcal{D}_\varepsilon(V^*)(Y,t,z)]}{\iint_{\mathbb{R}^2}\exp[-Q(y_1,y_2)-\varepsilon q^*(y_1+y_2)+2\mathcal{D}_\varepsilon(V^*)(Y,t,z)]\,dy_1\,dy_2}, \tag{4-26}$$

with $Y = (y_1, y_2)$, and

$$dN_\varepsilon^*(y,t) := \frac{N_\varepsilon^*(y,t)}{\int_{\mathbb{R}} N_\varepsilon^*(\cdot\,,t)} := \frac{\exp\bigl[-\frac{1}{2}|y|^2-\varepsilon q^* y+\mathcal{D}_\varepsilon^*(V^*)(y,t)\bigr]}{\int_{\mathbb{R}}\exp\bigl[-\frac{1}{2}|y|^2-\varepsilon q^* y+\mathcal{D}_\varepsilon^*(V^*)(y,t)\bigr]\,dy}. \tag{4-27}$$

Proposition 4.7 (uniform control of the directional derivatives of $\partial_z\mathcal{I}_\varepsilon^*$). *There exist a function of time R_ε^* such that for any ball B of $\mathcal{E}$, there exists a constant ε^* that depends only on K^*, that satisfies for all $\varepsilon \le \varepsilon^*$, for all $H \in \mathcal{E}$:*

$$\partial_g\partial_z\mathcal{I}_\varepsilon^*(t,z_*) = \varepsilon^2 R_\varepsilon^*(t) + O^*(\varepsilon^3) \quad \textit{and} \quad \partial_V\partial_z\mathcal{I}_\varepsilon^*\cdot H(t,z_*) = O^*(\varepsilon^2)\|H\|_{\mathcal{E}}, \tag{4-28}$$

where all the $O^*(\varepsilon^j)$ depend only on K^* defined in Proposition 3.3 and R_ε^* is given by the formula

$$R_\varepsilon^*(t) := m''(t,z_*)\iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)(y_1+y_2)^2\,dy_1\,dy_2. \tag{4-29}$$

So R_ε^* is uniformly bounded and there exists a constant R_0 and time t_0 such that $R_\varepsilon^* \ge R_0 > 0$ for all $t \ge t_0$.

The sign of R_ε^* is directly connected to the behavior of z_* we assumed in the introduction; see (1-16). The derivative in V admits a lower order in ε as in previous estimates; see (4-25) and (4-17) for instance. This lower order term will be absorbed by a contraction argument, see Section 8, once we have a definitive estimate of $\|W_\varepsilon\|_{\mathcal{F}}$; see estimate (8-2).

Proof of Proposition 4.7. First we focus on the bound of the first equation in (4-28). Similarly to (4-18), the explicit formula for the derivative is

$$\begin{aligned}
&\partial_g\partial_z\mathcal{I}_\varepsilon^*(t,z_*)\\
&:= -\varepsilon(I_1+I_2)\\
&= -\varepsilon\Biggl(\frac{\iint_{\mathbb{R}^2}\exp[-Q(y_1,y_2)-\varepsilon q^*(y_1+y_2)+2\mathcal{D}_\varepsilon(V^*)(Y,t,z_*)](y_1+y_2)\mathcal{D}_\varepsilon(\partial_z V^*)(Y,t,z_*)\,dy_1\,dy_2}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon q^* y+\mathcal{D}_\varepsilon^*(V^*)(y,t)]\,dy}\\
&\qquad\qquad - \partial_z\mathcal{I}_\varepsilon^*(t,z_*)\frac{\int_{\mathbb{R}} e^{-|y|^2/2} y\exp[-\varepsilon q^* y+\mathcal{D}_\varepsilon^*(V^*)(y,t)]\,dy}{\sqrt{\pi}\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-\varepsilon q^* y+\mathcal{D}_\varepsilon^*(V^*)(y,t)]\,dy}\Biggr).
\end{aligned} \tag{4-30}$$

Thanks to Proposition 4.4, we already know that $|\partial_z \mathcal{I}_\varepsilon^*(t,z_*)| = O^*(\varepsilon^2)$. Moreover, we bound uniformly the second term as follows:

$$\left| \frac{\int_{\mathbb{R}} e^{-|y|^2/2} y \exp[-\varepsilon q^* y + \mathcal{D}_\varepsilon^*(V^*)(y,t)]\, dy}{\sqrt{\pi} \int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon q^* y + \mathcal{D}_\varepsilon^*(V^*)(y,t)]\, dy} \right| \le \frac{\int_{\mathbb{R}} \exp\bigl[-\frac{1}{2}|y|^2 + 2\varepsilon K^* |y|\bigr] |y|\, dy}{\sqrt{\pi} \int_{\mathbb{R}} \exp\bigl[-\frac{1}{2}|y|^2 - 2\varepsilon K^* |y|\bigr]\, dy} \le O^*(1),$$

where K^* was defined in Proposition 3.3. This shows that $I_2 = O^*(\varepsilon^2)$. Therefore one can focus on I_1. In order to gather information about the sign of this quantity and not only get a bound in absolute value, we perform exact Taylor expansions of $\mathcal{D}_\varepsilon(\partial_z V^*)$. We divide I_1 by $\mathcal{I}_\varepsilon^*(t,z_*)$, and thanks to the definitions of (4-26) and (4-27) we get

$$\frac{I_1}{\mathcal{I}_\varepsilon^*(t,z_*)} = \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)(y_1+y_2)\mathcal{D}_\varepsilon(\partial_z V^*)(Y,t,z_*)\, dy_1\, dy_2.$$

As usual, we make Taylor expansions: there exists $0 < \xi_1, \xi_2 < 1$ such that

$$\frac{I_1}{\mathcal{I}_\varepsilon^*(t,z_*)} = \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)\Bigl(-\varepsilon\tfrac{1}{2}(y_1+y_2)^2\partial_z^2 V^*(t,z_*) - \tfrac{1}{4}\varepsilon^2 y_1^2(y_1+y_2)\partial_z^3 V^*(t,z_*+\varepsilon\xi_1 y_1) \\ - \tfrac{1}{4}\varepsilon^2 y_2^2(y_1+y_2)\partial_z^3 V^*(t,z_*+\varepsilon\xi_2 y_2)\Bigr)\, dy_1\, dy_2. \tag{4-31}$$

We next define R_ε^* as

$$\varepsilon\partial_z^2 V^*(t,z_*) \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)\tfrac{1}{2}(y_1+y_2)^2\, dy_1\, dy_2 =: \varepsilon R_\varepsilon^*(t),$$

with the following uniform bounds, that come from bounding by moments of a Gaussian distribution:

$$0 < R_0 \le R_\varepsilon^*(t), \quad \forall t \ge t_0.$$

Moreover, it is easy to see that R_ε^* is uniformly bounded. The next terms of (4-31) are of order superior to ε^2 and can be bounded uniformly by

$$\frac{1}{4}\varepsilon^2 \left| \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y,t,z_*)(y_1^2(y_1+y_2) + y_2^2(y_1+y_2))K^*\, dy_1\, dy_2 \right| \le O^*(\varepsilon^2).$$

Therefore one can rewrite (4-31) as

$$\frac{I_1}{\mathcal{I}_\varepsilon^*(t,z_*)} = -\varepsilon R_\varepsilon^*(t) + O^*(\varepsilon^2).$$

Thanks to Proposition 3.4, we recover a similar estimate for I_1:

$$I_1 = -\varepsilon R_\varepsilon^*(t) + O^*(\varepsilon^2).$$

Finally coming back to (4-30), we have shown that

$$\partial_g \partial_z \mathcal{I}_\varepsilon^*(t,z_*) = \varepsilon^2 R_\varepsilon^*(t) + O^*(\varepsilon^3).$$

This concludes the proof of the first estimate in (4-28). Next, we tackle the proof of the estimate upon the Fréchet derivative in (4-28), where, again, we first divide by $\mathcal{I}_\varepsilon^*(t, z_*)$:

$$\frac{\partial_V \partial_z \mathcal{I}_\varepsilon^* \cdot H(t, z_*)}{\mathcal{I}_\varepsilon^*(t, z_*)} = \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y, t, z_*)(\mathcal{D}_\varepsilon(\partial_z V^*)2\mathcal{D}_\varepsilon(H) + \mathcal{D}_\varepsilon(\partial_z H))(Y, t, z_*)\, dy_1\, dy_2 - \frac{\partial_z \mathcal{I}_\varepsilon^*(t, z_*)}{\mathcal{I}_\varepsilon^*(t, z_*)} \int_{\mathbb{R}} dN_\varepsilon^*(y, t)\mathcal{D}_\varepsilon^*(H)(y, t)\, dy. \tag{4-32}$$

Thanks to Propositions 3.6 and 3.4 and a uniform bound on $\mathcal{D}_\varepsilon^*(W)$, we have

$$\left| \frac{\partial_z \mathcal{I}_\varepsilon^*(t, z_*)}{\mathcal{I}_\varepsilon^*(t, z_*)} \int_{\mathbb{R}} dN_\varepsilon^*(y, t)\mathcal{D}_\varepsilon^*(H)(y, t)\, dy \right| \le O^*(\varepsilon^3)\|H\|_{\mathcal{E}}. \tag{4-33}$$

For the first term of (4-32), we first make a bound based on Taylor expansions of $\mathcal{D}_\varepsilon(H)$:

$$|\mathcal{D}_\varepsilon(H)(Y, t, z_*)| \le \tfrac{1}{2}\varepsilon^2(|y_1|^2 + |y_2|^2)\|H\|_{\mathcal{E}}.$$

The key element here is that since $\mathcal{D}_\varepsilon$ is evaluated at z_*, one gains an order in ε because $\partial_z H(t, z_*) = 0$, by definition of $\mathcal{E}$. Therefore, one gets

$$\left| \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y, t, z_*)(\mathcal{D}_\varepsilon(\partial_z V^*)2\mathcal{D}_\varepsilon(H))(Y, t, z_*)\, dy_1\, dy_2 \right| \le O^*(\varepsilon^3)\|H\|_{\mathcal{E}}, \tag{4-34}$$

where the additional order in ε is gained through a Taylor expansion of $\mathcal{D}_\varepsilon(\partial_z V^*)$. We finally tackle the last term of (4-32) we did not yet estimate, involving $\mathcal{D}_\varepsilon(\partial_z H)$. Based only on Taylor expansions in $\mathcal{E}$, we do not gain an order ε^3 as in the previous terms, which explains our estimate of order ε^2 in (4-32). Rather, we obtain, for some $0 < \xi < 1$,

$$\begin{aligned}\iint_{\mathbb{R}^2} & dG_\varepsilon^*(Y, t, z_*)\mathcal{D}_\varepsilon(\partial_z H)(Y, t, z_*)\, dy_1\, dy_2 \\ &= \varepsilon \frac{\partial_z^2 H(t, z_*)}{2} \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y, t, z_*)(y_1 + y_2)\, dy_1\, dy_2 \\ &\quad + \frac{\varepsilon^2}{4} \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y, t, z_*)(y_1^2 \partial_z^3 H(t, z_* + \varepsilon\xi y_1) + y_2^2 \partial_z^3 H(t, z_* + \varepsilon \xi y_2))\, dy_1\, dy_2.\end{aligned} \tag{4-35}$$

It is straightforward, based on multiple similar computations, to deduce that the first moment of dG_ε^* is zero at the leading order. Therefore,

$$\varepsilon \frac{\partial_z^2 H(t, z_*)}{2} \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y, t, z_*)(y_1 + y_2)\, dy_1\, dy_2 = \varepsilon \frac{\partial_z^2 H(t, z_*)}{2} O^*(\varepsilon) = O^*(\varepsilon^2)\|H\|_{\mathcal{E}}. \tag{4-36}$$

See for instance the proof of Proposition 3.4 for similar computations. In the second term of (4-35), we also cannot do better than an order in ε^2:

$$\begin{aligned}\frac{\varepsilon^2}{4} \iint_{\mathbb{R}^2} & dG_\varepsilon^*(Y, t, z_*)(y_1^2 \partial_z^3 H(t, z_* + \varepsilon\xi y_1) + y_2^2 \partial_z^3 H(t, z_* + \varepsilon\xi y_2))\, dy_1\, dy_2 \\ &\le \frac{\varepsilon^2 \|H\|_{\mathcal{E}}}{4} \iint_{\mathbb{R}^2} dG_\varepsilon^*(Y, t, z_*)(y_1^2 + y_2^2)\, dy_1\, dy_2 = O^*(\varepsilon^2)\|H\|_{\mathcal{E}}.\end{aligned}$$

Finally, by putting together (4-33)–(4-35) and finally (4-36), the second estimate of (4-28) is proven. □

The order ε^3 of the second equation in (4-28) will be crucial in our analysis around κ_ε, the perturbation of the linear part q_ε defined in (1-22). Next, we provide an accurate linearization of $\partial_z\mathcal{I}_\varepsilon$ compared to the one provided before in Proposition 4.5 and (4-23). This is possible thanks to an evaluation at $z = z_*$, and it will prove useful when tackling the perturbation of the linear part κ_ε. This is the content of the following lemma.

Lemma 4.8 (uniform control of the second Fréchet derivative of $\partial_z\mathcal{I}_\varepsilon$). *For any ball B of $\mathbb{R}\times\mathcal{E}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$ we have, for all $(g, W) \in B$, that*

$$\begin{aligned}\partial_z\mathcal{I}_\varepsilon(q^* + \varepsilon^2 g, V^* + \varepsilon^2 W)(t, z_*)\\ = \partial_z\mathcal{I}^*_\varepsilon(t, z_*) + \varepsilon^2(\partial_g\partial_z\mathcal{I}^*_\varepsilon(t, z_*)g + (\partial_V\partial_z\mathcal{I}^*_\varepsilon\cdot W)(t, z_*)) + O(\varepsilon^5)\|(g, W)\|.\end{aligned} \tag{4-37}$$

Proof. We write $f(p) := \partial_z\mathcal{I}_\varepsilon(q^* + pg, V^* + pW)(t, z)$. We recognize in formula (4-37) a Taylor expansion of f. Then, to prove the estimate of (4-37) it is sufficient to bound $f''(\varepsilon^2)$ uniformly:

$$f''(\varepsilon^2) \le O(\varepsilon)\|(g, W)\|.$$

The formula for f'' is very long, so for clarity we will denote by $A_\varepsilon(p)$ the numerator and by $B_\varepsilon(p)$ the denominator of $f(p)$, respectively, so that when we differentiate we have the structure

$$f''(p) = \frac{A''_\varepsilon(p)}{B_\varepsilon(p)} - 2\frac{A'_\varepsilon(p)B'_\varepsilon(p)}{B_\varepsilon(p)^2} - \frac{A_\varepsilon(p)B''_\varepsilon(p)}{B_\varepsilon(p)^2} + 2\frac{A_\varepsilon(p)B'_\varepsilon(p)^2}{B_\varepsilon(p)^3}. \tag{4-38}$$

The numerator is defined as

$$\begin{aligned}A_\varepsilon(p) := \iint_{\mathbb{R}^2} \exp[-Q(y_1, y_2) + 2\mathcal{D}_\varepsilon(V^* + pW)(Y, t, z_*) - \varepsilon(q^* + pg)(y_1 + y_2)]\\ \times \mathcal{D}_\varepsilon(\partial_z V^* + pW)(Y, t, z_*)\, dy_1\, dy_2,\end{aligned}$$

while the denominator reads

$$B_\varepsilon(p) := \int_{\mathbb{R}} e^{-|y|^2/2} \exp[-\varepsilon(q^* + pg)y + \mathcal{D}^*_\varepsilon(V^* + pW)(y, t)]\, dy.$$

Therefore we will divide each term by $\mathcal{I}^*_\varepsilon$ to simplify the notations. This will make the measures dG^*_ε, dN^*_ε, introduced in (4-26) and (4-27), appear. For instance,

$$\begin{aligned}&\frac{A_\varepsilon(p)}{\mathcal{I}^*_\varepsilon(t, z_*)B_\varepsilon(p)}\\ &:= \frac{\iint_{\mathbb{R}^2} dG^*_\varepsilon(Y, t, z_*)\exp[-\varepsilon pg(y_1 + y_2) + 2p\mathcal{D}_\varepsilon(W)(Y, t, z_*)](\mathcal{D}_\varepsilon(\partial_z V^* + p\partial_z V^*)(Y, t, z_*))\, dy_1\, dy_2}{\int_{\mathbb{R}} dN^*_\varepsilon(y, t)\exp[p\mathcal{D}^*_\varepsilon(W)(y, t) - \varepsilon pgy]\, dy}.\end{aligned}$$

We notice that any factor of the sum in (4-38) (divided by $\mathcal{I}^*_\varepsilon$) is a sum (and a product) of terms of the form

$$\frac{A^{(j)}_\varepsilon(p)B^{(k)}_\varepsilon(p)}{B_\varepsilon(p)\mathcal{I}^*_\varepsilon(t, z_*)} = \frac{A^{(j)}_\varepsilon(p)}{\mathcal{I}^*_\varepsilon(t, z_*)B_\varepsilon(p)}\,\frac{B^{(k)}_\varepsilon(p)}{B_\varepsilon(p)},$$

with $j = 0, 1, 2,\ k = 1, 2$ and the constraint $j + k = 2$. It is rather convenient to bound separately each of those terms. For instance, we deal with the second one:

$$\frac{A'_\varepsilon(p)B'_\varepsilon(p)}{B_\varepsilon(p)^2\mathcal{I}^*_\varepsilon(t, z_*)} = \frac{A'_\varepsilon(p)}{\mathcal{I}^*_\varepsilon(t, z_*)B_\varepsilon(p)}\frac{B'_\varepsilon(p)}{B_\varepsilon(p)}. \tag{4-39}$$

The first term of this product is

$$\begin{aligned}
&\frac{A'_\varepsilon(p)}{\mathcal{I}^*_\varepsilon(t, z_*)B_\varepsilon(p)}\\
&:= \frac{\iint_{\mathbb{R}^2} dG^*_\varepsilon(Y, t, z_*)\exp[2p\mathcal{D}_\varepsilon(W) - \varepsilon gp(y_1+y_2)]\mathcal{D}_\varepsilon(\partial_z W)\,dy_1\,dy_2}{\int_{\mathbb{R}} dN^*_\varepsilon(y, t)\exp[2\mathcal{D}^*_\varepsilon(W)(y, t) - \varepsilon gy]\,dy}\\
&\quad + \frac{\iint_{\mathbb{R}^2} dG^*_\varepsilon(Y, t, z_*)\exp[2p\mathcal{D}_\varepsilon(W) - \varepsilon gp(y_1+y_2)]2\mathcal{D}_\varepsilon(\partial_z V^* + p\partial_z W)(\mathcal{D}_\varepsilon(W) - \varepsilon g(y_1+y_2))\,dy_1\,dy_2}{\int_{\mathbb{R}} dN^*_\varepsilon(y, t)\exp[2\mathcal{D}^*_\varepsilon(W)(y, t) - \varepsilon gy]\,dy}.
\end{aligned}$$

The numerator and denominator can be bounded by estimating naively $\mathcal{D}_\varepsilon$:

$$\begin{aligned}
&\left|\frac{A'_\varepsilon(p)}{B_\varepsilon(p)\mathcal{I}^*_\varepsilon(t, z)}\right|\\
&\le \frac{\iint_{\mathbb{R}^2} dG^*_\varepsilon(Y, t, z_*)\exp[3\varepsilon\|(g, W)\|(|y_1|+|y_2|)]\varepsilon(|y_1|+|y_2|)\|(g, W)\|\,dy_1\,dy_2}{\int_{\mathbb{R}} dN^*_\varepsilon(y, t)\exp[-3\varepsilon\|(g, W)\||y|]\,dy}\\
&\quad + \frac{\iint_{\mathbb{R}^2} dG^*_\varepsilon(Y, t, z_*)\exp[3\varepsilon\|(g, W)\|(|y_1|+|y_2|)]\varepsilon^2(|y_1|+|y_2|)^2(3\|(g, W)\|+2K^*)3\|(g, W)\|\,dy_1\,dy_2}{\int_{\mathbb{R}} dN^*_\varepsilon(y, t)\exp[-3\varepsilon\|(g, W)\||y|]\,dy}.
\end{aligned}$$

Therefore, we only get moments of a Gaussian distribution, so the previous bound is in fact

$$\left|\frac{A'_\varepsilon(p)}{B_\varepsilon(p)\mathcal{I}^*_\varepsilon(t, z)}\right| \le O(\varepsilon)\|(g, W)\|. \tag{4-40}$$

With the exact same arguments but more convoluted formulas, one shows that

$$\left|\frac{A''_\varepsilon(p)}{B_\varepsilon(p)\mathcal{I}^*_\varepsilon(t, z)}\right| \le O(\varepsilon)\|(g, W)\|. \tag{4-41}$$

For the quotients of B in (4-38), we lose the structure of the measures dG^*_ε and dN^*_ε, but they are replaced by an actual Gaussian measure $\exp[-y^2/2]$. Therefore, with the same arguments as before, we bound the quotient by the moments of a Gaussian distribution. For instance,

$$\begin{aligned}
\left|\frac{B'_\varepsilon(p)}{B_\varepsilon(p)}\right| &= \left|\frac{\int_{\mathbb{R}} e^{-|y|^2/2}\exp[2\mathcal{D}^*_\varepsilon(V^* + pW) - \varepsilon(q^* + gp)y](2\mathcal{D}^*_\varepsilon(W) - \varepsilon gy)\,dy}{\int_{\mathbb{R}} e^{-|y|^2/2}\exp[2\mathcal{D}^*_\varepsilon(V^* + pW) - \varepsilon(q^* + gp)y]\,dy}\right|\\
&\le \frac{\int_{\mathbb{R}} e^{-|y|^2/2}\exp[3\varepsilon|y|K^* + 3\varepsilon\|(g, W)\||y|](3\varepsilon\|(g, W)\||y|)\,dy}{\int_{\mathbb{R}} e^{-|y|^2/2}\exp[-3\varepsilon K^*|y| - 3\varepsilon\|(g, W)\||y|]\,dy}\\
&\le O(\varepsilon)\|(g, W)\|.
\end{aligned} \tag{4-42}$$

Multiplying each term of (4-41) by (4-42) and then combining them yields the desired estimate result, given the separation of terms made in (4-38):

$$\left|\frac{f''(p)}{\mathcal{I}_\varepsilon^*(t,z)}\right| \le O(\varepsilon)\|(g,W)\|.$$

Thanks to Proposition 3.4, Lemma 4.8 is proven. □

5. Linearized equation for κ_ε, convergence of p_ε

5A. ***Uniform boundedness of κ_ε.*** Thanks to the estimates of the previous sections, all the useful tools to look at the perturbation κ_ε are made available. We recall that our final goal is to show that κ_ε is bounded as it is the perturbation from q^*; see (1-22). We show in this section that one gets an approximated ODE on κ_ε with good properties when linearizing; see Proposition 5.1. It is obtained by differentiating (2-2) and evaluating at $z = z_*$. This is exactly what suggested the spectral analysis of the formal linearized operator T in the table on page 1297. Now, thanks to our previous set of estimates from Section 4, we are able to carefully justify our linearization. Finally, the limit ODE we introduced for q^* in (1-18) will appear clearly when we do our analysis to balance contributions of smaller order.

To simplify expressions, we introduce the following alternative notations for all $t, z \in \mathbb{R}_+ \times \mathbb{R}$:

$$\Xi_\varepsilon(t,z) := W_\varepsilon(t,z) - 2W_\varepsilon(t,\bar{z}(t)). \tag{5-1}$$

Compared to previous sections, and for the rest of this article, we will work in the space $\mathcal{F}$ that is well suited to measure W_ε and build the linearization results, here for κ_ε. All our previous estimates that were established in $\mathcal{E}$ remain true in $\mathcal{F}$.

Proposition 5.1 (equation on κ_ε). *For any ball B of $\mathbb{R} \times \mathcal{F}$ there exists a constant ε_B that depends only on B such that if $(\kappa_\varepsilon, W_\varepsilon) \in B$ is a solution of (2-2), then for all $\varepsilon \le \varepsilon_B$, we have that κ_ε is a solution of the following ODE*:

$$-\dot{\kappa}_\varepsilon(t) = R_\varepsilon^*(t)\kappa_\varepsilon + O^*(1)\|W_\varepsilon\|_{\mathcal{F}} + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \tag{5-2}$$

where the $O(\varepsilon)$ depend only on B, and the R_ε^* are defined in Proposition 4.7.

Proof. As announced above, one starts by differentiating (2-2). This yields, with the notation Ξ_ε introduced in (5-1),

$$\begin{aligned}
&\partial_z M(t,z) - \varepsilon^2\dot{q}^*(t) - \varepsilon^2\partial_z\partial_t V^*(t,z) - \varepsilon^4\dot{\kappa}_\varepsilon(t) - \varepsilon^4\partial_z\partial_t W_\varepsilon(t,z)\\
&\quad = M(t,z)\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\\
&\qquad\qquad + \partial_z M(t,z)\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\\
&\qquad\qquad\qquad + \varepsilon^2 M(t,z)\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\partial_z\Xi_\varepsilon(t,z).
\end{aligned}$$

When we evaluate the expression at $z = z_*$, the last two terms vanish, since $\partial_z M(t,z_*) = \partial_z\Xi_\varepsilon(t,z_*) = 0$. Therefore, the equation becomes, since $\Xi_\varepsilon(t,z_*) = 0$ and $M(t,z_*) = 1$,

$$-\varepsilon^2\dot{q}^*(t) - \varepsilon^2\partial_z\partial_t V^*(t,z_*) - \varepsilon^4\dot{\kappa}_\varepsilon(t) - \varepsilon^4\partial_z\partial_t W_\varepsilon(t,z_*) = \partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z_*). \tag{5-3}$$

We then use directly the linearization result of Lemma 4.8 that we prepared for that purpose:

$$\partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2 \kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t, z_*) \\ = \partial_z \mathcal{I}_\varepsilon^*(t, z_*) + \varepsilon^2 (\partial_g \partial_z \mathcal{I}_\varepsilon^*(t, z_*)\kappa_\varepsilon + (\partial_V \partial_z \mathcal{I}_\varepsilon^* \cdot W_\varepsilon)(t, z_*)) + O(\varepsilon^5)\|(\kappa_\varepsilon, W_\varepsilon)\|. \quad (5\text{-}4)$$

We see that for most of the terms, we previously provided a careful estimate in Section 4. First, by Proposition 3.5,

$$\partial_z \mathcal{I}_\varepsilon^*(t, z_*) = \varepsilon^2 \big(m''(z_*) q^*(t) - \tfrac{1}{2} m^{(3)}(z_*)\big) + O^*(\varepsilon^4).$$

Plugging this into the asymptotic development of (5-4), we get the following:

$$\partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2 \kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t, z_*) \\ = \varepsilon^2 \big(m''(z_*(t)) q^*(t) - \tfrac{1}{2} m^{(3)}(z_*(t))\big) \\ + \varepsilon^2 (\partial_g \partial_z \mathcal{I}_\varepsilon^*(t, z_*)\kappa_\varepsilon + \partial_V \partial_z \mathcal{I}_\varepsilon^* \cdot W_\varepsilon(t, z_*)) + O^*(\varepsilon^4) + O(\varepsilon^5)\|(\kappa_\varepsilon, W_\varepsilon)\|.$$

Combining this with Proposition 4.7 where we got precise estimates at the point z_*, we complete the expansion of $\partial_z \mathcal{I}_\varepsilon$:

$$\partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2 \kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t, z_*) \\ = \varepsilon^2 \big(m''(z_*) q^*(t) - \tfrac{1}{2} m^{(3)}(z_*)\big) + \varepsilon^4 R_\varepsilon^*(t)\kappa_\varepsilon + O^*(\varepsilon^4)\|W_\varepsilon\|_{\mathcal{F}} + O^*(\varepsilon^4) + O(\varepsilon^5)\|(\kappa_\varepsilon, W_\varepsilon)\|.$$

When we turn back to (5-3), we have shown at this point the following relationship:

$$-\varepsilon^2 \dot{q}^*(t) - \varepsilon^2 \partial_z \partial_t V^*(t, z_*) - \varepsilon^4 \dot{\kappa}_\varepsilon(t) - \varepsilon^4 \partial_z \partial_t W_\varepsilon(t, z_*) \\ = \varepsilon^2 \big(m''(z_*) q^*(t) - \tfrac{1}{2} m^{(3)}(z_*)\big) + \varepsilon^4 R_\varepsilon^*(t)\kappa_\varepsilon + O^*(\varepsilon^4)\|W_\varepsilon\|_{\mathcal{F}} + O^*(\varepsilon^4) + O(\varepsilon^5)\|(\kappa_\varepsilon, W_\varepsilon)\|. \quad (5\text{-}5)$$

To get a stable equation on κ_ε, the terms of order ε^2 must cancel out. This is precisely the role played by the dynamics of q^* defined in (1-18). To see it, we just rewrite a term of (5-5), using $\partial_z V^*(t, z_*) = 0$ and Lemma 3.2:

$$\partial_z \partial_t V^*(t, z_*) = m'(z_*) \partial_z^2 V^*(t, z_*) = 2m'(z_*) m''(z_*).$$

Therefore, we recognize that by the definition of q^* in (1-18), the following terms cancel:

$$\varepsilon^2 \big(\dot{q}^*(t) + m''(z_*) q^*(t) - \tfrac{1}{2} m^{(3)}(z_*) + 2m''(z_*) m'(z_*)\big) = 0.$$

We then rewrite the second term of (5-5) of order ε^4:

$$\partial_z \partial_t W_\varepsilon(t, z_*) = m'(z_*) \partial_z^2 W_\varepsilon(t, z_*) = O^*(1)\|W_\varepsilon\|_{\mathcal{F}}.$$

Finally, we deduce from (5-5) the following relationship:

$$-\dot{\kappa}_\varepsilon(t) = R_\varepsilon^*(t)\kappa_\varepsilon + O^*(1)\|W_\varepsilon\|_{\mathcal{F}} + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|.$$

We have proven (5-2). $\square$

In this ODE solved by κ_ε, each term plays a separate part. First the function R^*_ε is what guarantees the stability of κ_ε because it is positive for large times. The other terms come from our perturbative analysis methodology. The term $O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|$ measures the error made when linearizing to obtain the ODE, and it ensures that it is of superior order in ε except for the part that comes from the reference point of our linearization: $O^*(1)$. Interestingly, there is also an error term that is not of superior order when linearizing, $O^*(1)\|W_\varepsilon\|_{\mathcal{F}}$, but what saves our contraction argument of Section 8 is that this term only involves W_ε, which we can bound independently, see Section 7.

5B. ***Equation on p_ε.*** We did not perturb the number p_ε as we did for $(q_\varepsilon, V_\varepsilon)$ since it can be straightforwardly computed from our reference equation (2-2). Given the spectral decomposition in the table on page 1297 in the heuristics of Section 2, it is consistent to evaluate (2-2) at $z = z_*$ to gain the necessary information about p_ε. This yields

$$1 - \varepsilon^2(\dot{p}_\varepsilon(t) + m'(z_*)q^*(t)) - \varepsilon^4 m'(z_*)\kappa_\varepsilon(t) = \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t, z_*). \tag{5-6}$$

Thanks to Propositions 3.3 and 4.2, and as long as κ_ε is bounded (which we will show in Section 8),

$$\varepsilon^2(\dot{p}_\varepsilon(t) + m'(z_*(t))q^*(t)) = O(\varepsilon^2).$$

In this last equation, the order of precision is not enough to recover the equation on p^* when $\varepsilon \to 0$. The problem is that the linearization of $\mathcal{I}_\varepsilon$ made in (4-10) is a little too rough. Coming back to Proposition 3.4, we make the more precise estimate

$$\mathcal{I}^*_\varepsilon(t, z_*) = 1 - \tfrac{1}{2}\varepsilon^2\partial_z^2 V^*(t, z_*) + O^*(\varepsilon^4). \tag{5-7}$$

The proof of this result is a direct adaptation of that of Proposition 3.4, by making Taylor expansions up to the fourth derivative of V^*, as made possible by the introduction of $\mathcal{E}^*$; see Definition 3.1. This involves computing the moments of the Gaussian distribution $\exp[-Q]$:

$$\frac{1}{\sqrt{2\pi}}\iint_{\mathbb{R}^2} e^{-Q(y_1,y_2)}(y_1^2 + y_2^2)\,dy_1\,dy_2 = \frac{1}{2}. \tag{5-8}$$

By plugging (5-7) into (5-6), and using (4-9), we find that

$$\dot{p}_\varepsilon(t) + m'(z_*)q^*(t) = \tfrac{1}{2}\partial_z^2 V^*(t, z_*) + O(\varepsilon^2) = m''(z_*) + O(\varepsilon^2). \tag{5-9}$$

We used (3-1) for the last equality. From (5-9), the convergence of p_ε towards p^* defined by (1-19), stated in Theorem 1.3, is straightforward.

6. Linearization results

We finally tackle the complete linearization of (2-2). A preview was given when we studied the equation on κ_ε, however it was local since we had beforehand evaluated at $z_*(t)$. Here, we will provide global (in space) results.

6A. ***Linearization of W_ε.*** A first step is to control the function Ξ_ε, which, we recall, is a byproduct of W_ε, introduced in (5-1).

Lemma 6.1 (control of Ξ_ε). *For any ball B of $\mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$, if $W_\varepsilon \in B$, then Ξ_ε defined in (5-1) satisfies*

$$\exp[\varepsilon^2 \Xi_\varepsilon(t,z)] = 1 + \varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|W_\varepsilon\|_{\mathcal{F}},$$

where $O(\varepsilon^4)$ depends only on the ball B.

Proof. By the choice of the norm in $\mathcal{F}$ and in the setting of $W_\varepsilon \in B$ we have the following uniform control for all t, z:

$$|\Xi_\varepsilon(t,z)| \le \|W_\varepsilon\|_{\mathcal{F}}.$$

Then, by performing an exact Taylor expansion, there exists $0 < \xi < 1$ such that

$$\exp[\varepsilon^2 \Xi_\varepsilon(t,z)] = 1 + \varepsilon^2 \Xi_\varepsilon(t,z) + \tfrac{1}{2}\varepsilon^4 \Xi_\varepsilon(t,z)^2 \exp[\varepsilon^2 \xi\, \Xi_\varepsilon(t,z)].$$

To conclude we uniformly bound the rest for $\varepsilon^2 \le 1/\|W_\varepsilon\|_{\mathcal{F}}$:

$$\left|\tfrac{1}{2}\varepsilon^4 \Xi_\varepsilon(t,z)^2 \exp[\varepsilon^2 \xi\, \Xi_\varepsilon(t,z)]\right| \le \tfrac{1}{2}\varepsilon^4 \|W_\varepsilon\|_{\mathcal{F}}^2. \qquad \square$$

This first result is prototypical of the tools we will employ to linearize the problem (2-2) solved by $(\kappa_\varepsilon, W_\varepsilon)$. We now write the linearized problem satisfied by W_ε.

Proposition 6.2 (linearization for W_ε). *For any ball B of $\mathbb{R} \times \mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$, any pair $(\kappa_\varepsilon, W_\varepsilon) \in B$, a solution of (2-2), satisfies the estimate*

$$-\varepsilon^2 \partial_t W_\varepsilon(t,z) = M(t,z)(\Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|), \tag{6-1}$$

where $O(\varepsilon)$ depends only on B.

Proof. One starts from (2-2):

$$\begin{aligned} M(t,z) - \varepsilon^2(\dot p_\varepsilon(t) + m'(z_*)q^*(t) + \dot q^*(t)(z-z_*) + \partial_t V^*(t,z)) - \varepsilon^4(\dot\kappa_\varepsilon(t)(z-z_*) + m'(z_*)\kappa_\varepsilon(t) + \partial_t W_\varepsilon(t,z)) \\ = M(t,z)\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z) \exp[\varepsilon^2 \Xi_\varepsilon(t,z)]. \end{aligned} \tag{6-2}$$

Thanks to Lemma 6.1 and Proposition 4.2, where we linearized $\mathcal{I}_\varepsilon$, and the term in Ξ_ε, one can expand the right-hand side as follows:

$$\begin{aligned} &M(t,z)\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z) \exp[\varepsilon^2 \Xi_\varepsilon(t,z)] \\ &\quad = M(t,z)(1 + O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|)(1 + \varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ &\quad = M(t,z) + \varepsilon^2 M(t,z)\Xi_\varepsilon(t,z) + M(t,z)(O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|). \end{aligned} \tag{6-3}$$

The left-hand side of (6-2) is a little bit more involved. We will use our previous work on $(p_\varepsilon, \kappa_\varepsilon)$. First, thanks to (5-6) which states the relationship satisfied by p_ε, we have

$$-\varepsilon^2(\dot p_\varepsilon(t) + m'(z_*)q^*(t)) - \varepsilon^4 \kappa_\varepsilon m'(z_*) = 1 - \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t, z_*).$$

We then use Proposition 4.2 involving the linearization of $\mathcal{I}_\varepsilon$ to get that

$$-\varepsilon^2(\dot{p}_\varepsilon(t)+m'(z_*)q^*(t))-\varepsilon^4\kappa_\varepsilon m'(z_*)=O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|. \tag{6-4}$$

From Proposition 3.3, we have the following uniform bound:

$$|\partial_t V^*(t,z)|\le K^*. \tag{6-5}$$

Thanks to our preliminary work on κ_ε, and more precisely (5-5), we know that

$$\dot{q}^*(t)+\varepsilon^2\dot{\kappa}_\varepsilon(t)=O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|.$$

Therefore, the affine terms are comparable to M, since M is a superlinear function that admits a uniform lower bound by hypothesis; see (1-13):

$$\left|\frac{(\dot{q}^*(t)+\varepsilon^2\dot{\kappa}_\varepsilon(t))(z-z_*)}{M(t,z)}\right|=O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|. \tag{6-6}$$

When adding up the estimates of (6-5) and (6-6), we have shown that

$$\begin{aligned}-\varepsilon^2(\dot{p}_\varepsilon(t)+m'(z_*)q^*(t)+\dot{q}^*(t)(z-z_*)+\partial_t V^*(t,z))-\varepsilon^4(\dot{\kappa}_\varepsilon(t)(z-z_*)+m'(z_*(t))\kappa_\varepsilon(t)+\partial_t W_\varepsilon(t,z))\\ =M(t,z)(O^*(\varepsilon^2)+O(\varepsilon^3))\|(\kappa_\varepsilon, W_\varepsilon)\|-\varepsilon^4\partial_t W_\varepsilon(t,z).\end{aligned} \tag{6-7}$$

We have divided by M the relationships (6-4) and (6-5), which is possible thanks to the uniform lower bound of M.

Finally, when putting together (6-6) and (6-3) in (6-2), the terms M cancel each other, and we find (6-1) by factoring out ε^2. □

One can notice the similarity between what we just proved rigorously and the heuristics made in (2-1). From this result one can straightforwardly deduce an approximated linear equation satisfied by Ξ_ε.

Corollary 6.3 (linearization in $\Xi_\varepsilon(t,z)$). *For any ball B of $\mathbb{R}\times\mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon\le\varepsilon_B$, any pair $(\kappa_\varepsilon, W_\varepsilon)\in B$ satisfies the estimate*

$$\varepsilon^2\partial_t\Xi_\varepsilon(t,z)=M(t,z)\left(2\frac{M(t,\bar{z}(t))}{M(t,z)}\Xi_\varepsilon(t,\bar{z})-\Xi_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|\right), \tag{6-8}$$

where $O(\varepsilon)$ depends only on B.

Remark 6.4. • The reader may notice that the computation of $\partial_t\Xi_\varepsilon$ yields a parasite term $\varepsilon^2\dot{z}_*\partial_z\Xi_\varepsilon(t,\bar{z})$ not dealt with by (6-1). However, this is a lower order term since it satisfies

$$\varepsilon^2\dot{z}_*(t)\partial_z\Xi_\varepsilon(t,z)=O(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|. \tag{6-9}$$

• Under the same assumption as Corollary 6.3, W_ε also satisfies the following linear equation:

$$-\varepsilon^2\partial_t W_\varepsilon(t,z)=M(t,z)(\Xi_\varepsilon(t,z)+O(1)).$$

However, in Section 7, we will study the stability of the solution of the linear problem. We will see that one needs precise estimates about the structure of the nonlinear negligible terms, which explains the more detailed estimate (6-1) and is the purpose of all our previous sections.

6B. *Linearization of $\partial_z W_\varepsilon$.* The computations for $\partial_z W_\varepsilon$ are slightly more complex because of the differentiation of the triple product in the right-hand side (2-2). However, the key point is that when we linearize $\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2\kappa_\varepsilon)$ the derivatives of $\mathcal{I}_\varepsilon$ are negligible in ε. Therefore the intuitive linearized problem for $\partial_z W_\varepsilon$, given by the derivation of the linearized equation for W_ε, actually holds true. This is the content of the following proposition:

Proposition 6.5 (linearization in $\partial_z W_\varepsilon$). *For any ball B of $\mathbb{R} \times \mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$, any pair $(\kappa_\varepsilon, W_\varepsilon) \in B$ solution of (2-2) satisfies the following estimate*:

$$-\varepsilon^2\partial_t\partial_z W_\varepsilon(t,z) = M(t,z)\left(\partial_z \Xi_\varepsilon(t,z) + \frac{O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right) + \partial_z M(t,z)(\Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|), \quad (6\text{-}10)$$

where $O(\varepsilon)$ depends only on B.

Proof. One starts by differentiating (2-2) as in the proof of Proposition 5.1 to highlight κ_ε. This yields

$$\begin{aligned}
&\partial_z M(t,z) - \varepsilon^2\dot{q}^*(t) - \varepsilon^2\partial_z\partial_t V^*(t,z) - \varepsilon^4\dot{\kappa}_\varepsilon(t) - \varepsilon^4\partial_z\partial_t W_\varepsilon(t,z)\\
&\quad = M(t,z)\partial_z\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\\
&\qquad + \partial_z M(t,z)\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\\
&\qquad + \varepsilon^2 M(t,z)\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\partial_z\Xi_\varepsilon(t,z).
\end{aligned}$$

However, contrary to the case where we were studying $\dot{\kappa}_\varepsilon$, we will not evaluate at z_*. We introduce the notations R_i corresponding to each of the three terms of the right-hand side of the previous equation. We will linearize each R_i starting with R_1, which we estimate thanks to Proposition 4.5 and Lemma 6.1, paired with the estimate of Proposition 3.6:

$$\begin{aligned}
R_1 &:= \partial_z\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)M(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\\
&= M(t,z)\left(\partial_z\mathcal{I}_\varepsilon^*(t,z) + \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1 + \varepsilon^2\Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\
&= M(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1 + \varepsilon^2\Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|).
\end{aligned}$$

Therefore, the final contribution of R_1 is

$$R_1 = M(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right). \quad (6\text{-}11)$$

Next, one looks at R_2. Thanks to Proposition 4.2,

$$\begin{aligned}
R_2 &:= \partial_z M(t,z)\mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\\
&= \partial_z M(t,z)(1 + O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|)(1 + \varepsilon^2\Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\
&= \partial_z M(t,z) + \varepsilon^2\partial_z M(t,z)\Xi_\varepsilon(t,z) + \partial_z M(t,z)(O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|). \quad (6\text{-}12)
\end{aligned}$$

We finally tackle R_3 with the same techniques, using Proposition 4.2 and Lemma 6.1:

$$\begin{aligned} R_3 &:= \varepsilon^2 M(t,z)\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\partial_z\Xi_\varepsilon(t,z) \\ &= \varepsilon^2 M(t,z)\partial_z\Xi_\varepsilon(t,z)(1+O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ &= \varepsilon^2 M(t,z)+M(t,z)\frac{O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}. \end{aligned} \tag{6-13}$$

In that last estimate, we chose to write $O^*(\varepsilon^4)$ as a regular $O(\varepsilon^4)$. Coming back to our initial problem, when we assemble (6-11)–(6-13), we obtain

$$\begin{aligned} &\partial_z M(t,z)-\varepsilon^2\dot q^*(t)-\varepsilon^2\partial_z\partial_t V^*(t,z)-\varepsilon^4\dot\kappa_\varepsilon(t)-\varepsilon^4\partial_z\partial_t W_\varepsilon(t,z) \\ &\quad=\partial_z M(t,z)+\varepsilon^2\partial_z M(t,z)(\Xi_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ &\qquad\qquad+\varepsilon^2 M(t,z)\left(\partial_z\Xi_\varepsilon(t,z)+\frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right). \end{aligned} \tag{6-14}$$

We now deal with the left-hand side of (6-14). First, the terms $\partial_z M(t,z)$ on each side cancel. Next, using the ODE that defines q^* in (1-18), our linearized equation on $\dot\kappa_\varepsilon$ stated in (5-2) and finally our bound of $\partial_t V^*$ made in Proposition 3.3, we find that

$$-\varepsilon^2(\dot q^*(t)+\partial_z\partial_t V^*(t,z)+\varepsilon^2\dot\kappa_\varepsilon(t))=O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|. \tag{6-15}$$

Finally, if we divide by M, the following estimate holds true since $\alpha<1$:

$$\left|\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{M(t,z)}\right|\le\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}.$$

Plugging this into (6-14), and dividing each side by ε^2, we therefore recover the relationship we wanted to prove:

$$\begin{aligned} -\varepsilon^2\partial_t\partial_z W_\varepsilon(t,z)&=M(t,z)\left(\partial_z\Xi_\varepsilon(t,z)+\frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right) \\ &\qquad+\partial_z M(t,z)(\Xi_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|). \qquad\square \end{aligned}$$

We deduce straightforwardly a linearization result upon the quantity $\partial_z\Xi_\varepsilon$.

Corollary 6.6 (linearization for $\partial_z\Xi_\varepsilon(t,z)$). *For any ball B of $\mathbb{R}\times\mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon\le\varepsilon_B$, any pair $(\kappa_\varepsilon, W_\varepsilon)\in B$, a solution of* (2-2), *satisfies the following estimate*:

$$\begin{aligned} \varepsilon^2\partial_t\partial_z\Xi_\varepsilon(t,z)&=M(t,z)\left(\frac{M(t,\bar z)}{M(t,z)}\partial_z\Xi_\varepsilon(t,\bar z)-\partial_z\Xi_\varepsilon(t,z)+\frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right) \\ &\qquad+\partial_z M(t,z)\left(\frac{\partial_z M(t,\bar z)}{\partial_z M(t,z)}\Xi_\varepsilon(t,\bar z)-\Xi_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|\right), \end{aligned}$$

where the $O(\varepsilon)$ depends only on B.

6C. *Linearization of $\partial_z^2 W_\varepsilon(t,z)$.* We now tackle the linearized equation for $\partial_z^2 W_\varepsilon$.

Proposition 6.7 (linearization for $\partial_z^2 W_\varepsilon$). *For any ball B of $\mathbb{R}\times\mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$, any pair $(\kappa_\varepsilon, W_\varepsilon) \in B$, a solution of (2-2), satisfies the following estimate*:

$$-\varepsilon^2\partial_z^2\partial_t W_\varepsilon(t,t)$$
$$= \partial_z^2 M(t,z)(\Xi_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|)+2\partial_z M(t,z)\left(\partial_z\Xi_\varepsilon(t,z)+\frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)$$
$$+M(t,z)\left(\partial_z^2\Xi_\varepsilon(t,z)+\frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right), \quad (6\text{-}16)$$

where the $O(\varepsilon)$ depend only on B.

In the next sections, we choose to write the second derivative $\partial_z^2\Xi_\varepsilon(t,z)$ in full,

$$\partial_z^2 W_\varepsilon(t,z) - \tfrac{1}{2}\partial_z^2 W_\varepsilon(t,\bar{z}),$$

as the factor $\frac{1}{2}$ will be the key to ensure the uniform boundedness of $\partial_z^2 W_\varepsilon$; see Section 7.

Proof. We start by differentiating (2-2) twice. This yields

$$\partial_z^2 M(t,z) - \varepsilon^2\partial_z^2\partial_t V^*(t,z) - \varepsilon^4\partial_z^2\partial_t W_\varepsilon(t,z) = R_1+R_2+R_3+R_4+R_5+R_6,$$

with the following notations:

$$R_1 := \partial_z^2\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z)M(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)],$$
$$R_2 := 2\partial_z M(t,z)\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)],$$
$$R_3 := 2M(t,z)\varepsilon^2\partial_z\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\partial_z\Xi_\varepsilon(t,z),$$
$$R_4 := \mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z)\partial_z^2 M(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)],$$
$$R_5 := 2\varepsilon^2\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z)\partial_z M(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)]\partial_z\Xi_\varepsilon(t,z),$$

and finally,

$$R_6 := \varepsilon^2 M(t,z)\mathcal{I}_\varepsilon(q^*+\varepsilon^2\kappa_\varepsilon, V^*+\varepsilon^2 W_\varepsilon)(t,z)\exp[\varepsilon^2\Xi_\varepsilon(t,z)](\varepsilon^2\partial_z\Xi_\varepsilon(t,z)^2+\partial_z^2\Xi_\varepsilon(t,z)).$$

We will estimate each term separately, starting with R_1, for which we apply Proposition 4.5, Lemma 6.1 and Proposition 3.6:

$$R_1 = M(t,z)\left(\partial_z^2\mathcal{I}_\varepsilon^*(t,z)+\frac{O(\varepsilon^3)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon,W_\varepsilon)\|)$$
$$= M(t,z)\left(\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon,W_\varepsilon)\|).$$

Therefore, the final estimate of R_1 is

$$R_1 = M(t,z)\left(\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right). \quad (6\text{-}17)$$

Next, for the term R_2 we use Propositions 4.5 and 3.6 and find that

$$\begin{aligned}R_2 &= 2\left(\partial_z \mathcal{I}_\varepsilon^*(t,z) + \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)\partial_z M(t,z)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\ &= 2\partial_z M(t,z)\left(\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|).\end{aligned}$$

We can simplify this expression as

$$R_2 = \partial_z M(t,z)\left(\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right). \tag{6-18}$$

The term R_3 will not contribute at the order ε^2, because of Proposition 3.6, and $|\partial_z \Xi_\varepsilon(t,z)| \le \|W_\varepsilon\|_{\mathcal{F}}$:

$$\begin{aligned}R_3 &= 2\varepsilon^2 M(t,z)\partial_z \Xi_\varepsilon(t,z)\left(\frac{O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\ &= \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)} M(t,z).\end{aligned} \tag{6-19}$$

For R_4, the zeroth order terms are more entangled. With Proposition 4.2 and Lemma 6.1 we have

$$\begin{aligned}R_4 &= \partial_z^2 M(t,z)(1+O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\ &= \partial_z^2 M(t,z) + \varepsilon^2\partial_z^2 M(t,z)(\Xi_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|).\end{aligned} \tag{6-20}$$

We see in R_4 the appearance of the term $\varepsilon^2\partial_z^2 M(t,z)\Xi_\varepsilon(t,z)$ which is also in (6-16), and so it is a good opportunity to do at first a summary of the computations when adding (6-17)–(6-20):

$$\begin{aligned}&R_1+R_2+R_3+R_4\\ &\quad= \partial_z^2 M(t,z) + \varepsilon^2\partial_z^2 M(t,z)(\Xi_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\ &\qquad + \varepsilon^2 M(t,z)\frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)} + \varepsilon^2\partial_z M(t,z)\frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}.\end{aligned} \tag{6-21}$$

We continue the estimations by looking at R_5, and thanks to Proposition 4.2 we have

$$\begin{aligned}R_5 &= 2\varepsilon^2\partial_z M(t,z)\partial_z\Xi_\varepsilon(t,z)(1+O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\ &= 2\varepsilon^2\partial_z M(t,z)\partial_z\Xi_\varepsilon(t,z) + \varepsilon^2\partial_z M(t,z)\frac{O^*(\varepsilon)+O(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}.\end{aligned} \tag{6-22}$$

Finally, we tackle the last term, R_6, with Proposition 4.2:

$$\begin{aligned}R_6 &= \varepsilon^2 M(t,z)(1+O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\ &\qquad\times\left(\frac{O(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)} + \partial_z^2\Xi_\varepsilon(t,z)\right)\\ &= \varepsilon^2 M(t,z)\partial_z^2\Xi_\varepsilon(t,z) + \varepsilon^2 M(t,z)\frac{O(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}.\end{aligned} \tag{6-23}$$

Thanks to those last two estimates, (6-22) and (6-23), that we add with the previous result of (6-21), we obtain for the full equation

$$\begin{aligned}
&\partial_z^2 M(t,z) - \varepsilon^2 \partial_z^2 \partial_t V^*(t,z) - \varepsilon^4 \partial_z^2 \partial_t W_\varepsilon(t,z) \\
&\quad = \partial_z^2 M(t,z) + \varepsilon^2 \partial_z^2 M(t,z)(\Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\
&\qquad\qquad + 2\varepsilon^2 \partial_z M(t,z)\left(\partial_z \Xi_\varepsilon(t,z) + \frac{O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right) \\
&\qquad\qquad\qquad + \varepsilon^2 M(t,z)\left(\partial_z^2 \Xi_\varepsilon(t,z) + \frac{O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right).
\end{aligned}$$

Thanks to Proposition 3.3 we know that $\|\varepsilon^2 \partial_z^2 \partial_t V^*(t,z)\|_\infty \leq O^*(\varepsilon^2)$. Then,

$$\begin{aligned}
&-\varepsilon^4 \partial_z^2 \partial_t W_\varepsilon(t,t) \\
&\quad = \varepsilon^2 \partial_z^2 M(t,z)(\Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\
&\qquad\qquad + 2\varepsilon^2 \partial_z M(t,z)\left(\partial_z \Xi_\varepsilon(t,z) + \frac{O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right) \\
&\qquad\qquad\qquad + \varepsilon^2 M(t,z)\left(\partial_z^2 \Xi_\varepsilon(t,z) + \frac{O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right),
\end{aligned}$$

which proves (6-16) after dividing by ε^2. □

6D. *Linearization of $\partial_z^3 W_\varepsilon(t,z)$.* Our last linearized equation is the one for $\partial_z^3 W_\varepsilon$, and we proceed with the same technique, with slightly more complex formulas.

Proposition 6.8 (linearization in $\partial_z^3 W_\varepsilon$). *For any ball B of $\mathbb{R} \times \mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \leq \varepsilon_B$, any pair $(\kappa_\varepsilon, W_\varepsilon) \in B$, a solution of (2-2), satisfies the following estimate*:

$$\begin{aligned}
-\varepsilon^2 \partial_t \partial_z^3 W_\varepsilon(t,z) &= \partial_z^3 M(t,z)(\Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\
&\quad + 3\partial_z^2 M(t,z)\left(\partial_z \Xi_\varepsilon(t,z) + \frac{O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right) \\
&\quad + 3\partial_z M(t,z)\left(\partial_z^2 \Xi_\varepsilon(t,z) + \frac{O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right) \\
&\quad + M(t,z)\left(\partial_z^3 \Xi_\varepsilon(t,z) + \frac{\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty}{2^{1-\alpha}\varphi_\alpha(t,z)} + \frac{O^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right), \qquad (6\text{-}24)
\end{aligned}$$

where the $O(\varepsilon)$ depend only on B.

Proof of Proposition 6.7. We start, as ever, by differentiating (2-2), but now three times. This yields, for the right-hand side, ten terms:

$$\partial_z^3 M(t,z) - \varepsilon^2 \partial_z^3 \partial_t V^*(t,z) - \varepsilon^4 \partial_z^3 \partial_t W_\varepsilon(t,t) = R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_8 + R_9 + R_{10}, \quad (6\text{-}25)$$

with the following notations:

$$R_1 := \partial_z^3 \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t, z) M(t, z) \exp[\varepsilon^2 \Xi_\varepsilon(t, z)],$$
$$R_2 := 3\partial_z^2 \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t, z) \partial_z M(t, z) \exp[\varepsilon^2 \Xi_\varepsilon(t, z)],$$
$$R_3 := 3\varepsilon^2 \partial_z^2 \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t, z) M(t, z) \exp[\varepsilon^2 \Xi_\varepsilon(t, z)] \partial_z \Xi_\varepsilon(t, z),$$
$$R_4 := 6\varepsilon^2 \partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t, z) \partial_z M(t, z) \exp[\varepsilon^2 \Xi_\varepsilon(t, z)] \partial_z \Xi_\varepsilon(t, z),$$
$$R_5 := 3\partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t, z) \partial_z^2 M(t, z) \exp[\varepsilon^2 \Xi_\varepsilon(t, z)],$$

and, moreover,

$$R_6 := 3\varepsilon^2 \partial_z \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t, z) M(t, z) \exp[\varepsilon^2 \Xi_\varepsilon(t, z)](\varepsilon^2 \partial_z \Xi_\varepsilon(t, z)^2 + \partial_z^2 \Xi_\varepsilon(t, z)),$$
$$R_7 := 3\varepsilon^2 \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t, z) \partial_z M(t, z) \exp[\varepsilon^2 \Xi_\varepsilon(t, z)](\varepsilon^2 \partial_z \Xi_\varepsilon(t, z)^2 + \partial_z^2 \Xi_\varepsilon(t, z)),$$
$$R_8 := 3\varepsilon^2 \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t, z) \partial_z^2 M(t, z) \exp[\varepsilon^2 \Xi_\varepsilon(t, z)] \partial_z \Xi_\varepsilon(t, z),$$
$$R_9 := \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t, z) \partial_z^3 M(t, z) \exp[\varepsilon^2 \Xi_\varepsilon(t, z)].$$

The last term corresponds to the third derivative of the exponential term $\exp[\varepsilon^2 \Xi_\varepsilon]$:

$$R_{10} := \varepsilon^2 \mathcal{I}_\varepsilon(q^* + \varepsilon^2\kappa_\varepsilon, V^* + \varepsilon^2 W_\varepsilon)(t, z) M(t, z) \exp[\varepsilon^2 \Xi_\varepsilon(t, z)] \times (\varepsilon^4 \partial_z \Xi_\varepsilon(t, z)^3 + 3\varepsilon^2 \partial_z \Xi_\varepsilon(t, z) \partial_z^2 \Xi_\varepsilon(t, z) + \partial_z^3 \Xi_\varepsilon(t, z)).$$

We first tackle R_1. We use the linearization of the third derivative of $\mathcal{I}_\varepsilon$ in Proposition 4.5 to find that

$$\begin{aligned} R_1 &= M(t, z)\left(\partial_z^3 \mathcal{I}_\varepsilon^*(t, z) + \frac{\varepsilon^2 \|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty}{2^{1-\alpha}\varphi_\alpha(t, z)} + \frac{O(\varepsilon^{2+\alpha})\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t, z)}\right)(1 + \varepsilon^2 \Xi_\varepsilon(t, z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ &= \varepsilon^2 M(t, z)\left(\frac{\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty}{2^{1-\alpha}\varphi_\alpha(t, z)} + \frac{O^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t, z)}\right)(1 + \varepsilon^2 \Xi_\varepsilon(t, z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|). \end{aligned}$$

We end up with the estimate

$$R_1 = \varepsilon^2 M(t, z)\left(\frac{\|\partial_z^3 W_\varepsilon\|_\infty}{2^{1-\alpha}\varphi_\alpha(t, z)} + \frac{O^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t, z)}\right). \tag{6-26}$$

For R_2, with Proposition 4.5 we have

$$\begin{aligned} R_2 &= 3\partial_z M(t, z)\left(\partial_z^2 \mathcal{I}_\varepsilon^*(t, z) + \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t, z)}\right)(1 + \varepsilon^2 \Xi_\varepsilon(t, z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ &= 3\partial_z M(t, z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t, z)}\right)(1 + \varepsilon^2 \Xi_\varepsilon(t, z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|). \end{aligned}$$

We can simplify this expression to

$$R_2 = \varepsilon^2 \partial_z M(t, z)\left(\frac{O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t, z)}\right). \tag{6-27}$$

For R_3, we get

$$\begin{aligned} R_3 &= 3\varepsilon^2 M(t,z)\partial_z \Xi_\varepsilon(t,z)\left(\partial_z^2 \mathcal{I}_\varepsilon^*(t,z) + \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ &= 3\varepsilon^2 M(t,z)\partial_z \Xi_\varepsilon(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|). \end{aligned}$$

We can simplify roughly this expression to

$$R_3 = \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)} M(t,z). \tag{6-28}$$

For R_4 one has very similarly

$$\begin{aligned} R_4 &= 6\varepsilon^2 \partial_z M(t,z)\partial_z \Xi_\varepsilon(t,z)\left(\partial_z \mathcal{I}_\varepsilon^*(t,z) + \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ &= 6\varepsilon^2 \partial_z M(t,z)\partial_z \Xi_\varepsilon(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|). \end{aligned}$$

We can simplify this expression to

$$R_4 = \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)} \partial_z M(t,z). \tag{6-29}$$

The expression for R_5 still follows the same road:

$$\begin{aligned} R_5 &= 3\partial_z^2 M(t,z)\left(\partial_z \mathcal{I}_\varepsilon^*(t,z) + \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ &= 3\partial_z^2 M(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)(1+\varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|). \end{aligned}$$

This last expression can be shortened to

$$R_5 = 3\varepsilon^2 \partial_z^2 M(t,z) \frac{O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}. \tag{6-30}$$

For R_6, the expression is a little more involved due to the second derivative of the exponential:

$$\begin{aligned} R_6 = \varepsilon^2 M(t,z)\left(\frac{O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)&(1+\varepsilon^2 \Xi_\varepsilon(t,z) + O(\varepsilon^4)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\ &\times \left(\frac{O(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)} + \partial_z^2 \Xi_\varepsilon(t,z)\right). \end{aligned}$$

We eventually shorten R_6 to

$$R_6 = 3M(t,z) \frac{O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}. \tag{6-31}$$

If we bring together all of our previous estimates in (6-26)–(6-31), we obtain that

$$R_1+R_2+R_3+R_4+R_5+R_6$$
$$=\varepsilon^2 M(t,z)\left(\frac{O^*(1)+O(\varepsilon^\alpha)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)+\varepsilon^2\partial_z M(t,z)\left(\frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)$$
$$+\varepsilon^2\partial_z^2 M(t,z)\left(\frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right)+\frac{\varepsilon^2\|\varphi_\alpha\partial_z^3 W_\varepsilon\|_\infty}{2^{1-\alpha}\varphi_\alpha(t,z)}M(t,z). \tag{6-32}$$

In that first round of estimates, we have shown that all the contributions of the terms with the derivatives of $\mathcal{I}_\varepsilon$ do not appear when linearizing because they are of high order in ε. Therefore, the most meaningful contribution will now appear, because $\mathcal{I}_\varepsilon$ now contributes mainly as 1 and no longer vanishes.

We start with R_7:

$$R_7=3\varepsilon^2\partial_z M(t,z)(1+O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon,W_\varepsilon)\|)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon,W_\varepsilon)\|)$$
$$\times\left(\frac{O(\varepsilon^2)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}+\partial_z^2\Xi_\varepsilon(t,z)\right),$$

which can be rewritten as

$$R_7=3\varepsilon^2\partial_z M(t,z)(1+O^*(\varepsilon)+O(\varepsilon^2)\|(\kappa_\varepsilon,W_\varepsilon)\|)\left(\partial_z^2\Xi_\varepsilon(t,z)+\frac{O(\varepsilon^2)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right).$$

Finally, for R_7,

$$R_7=3\varepsilon^2\partial_z M(t,z)\partial_z^2\Xi_\varepsilon(t,z)+\partial_z M(t,z)\left(\frac{O^*(\varepsilon^3)+O(\varepsilon^4)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right). \tag{6-33}$$

For R_8, the following estimates hold true:

$$R_8=3\varepsilon^2\partial_z^2 M(t,z)\partial_z\Xi_\varepsilon(t,z)(1+O^*(\varepsilon)+O(\varepsilon^2)\|(\kappa_\varepsilon,W_\varepsilon)\|)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon,W_\varepsilon)\|).$$

Therefore,

$$R_8=3\varepsilon^2\partial_z^2 M(t,z)\partial_z\Xi_\varepsilon(t,z)+\partial_z^2 M(t,z)\left(\frac{O^*(\varepsilon^3)+O(\varepsilon^4)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right). \tag{6-34}$$

For the last two terms, the derivatives up to the third order appear. The simplest is given by R_9:

$$R_9=\partial_z^3 M(t,z)(1+O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon,W_\varepsilon)\|)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon,W_\varepsilon)\|)$$
$$=\partial_z^3 M(t,z)+\varepsilon^2\partial_z^3 M(t,z)(\Xi_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|). \tag{6-35}$$

At last, for the term R_{10}, we have

$$R_{10}=\varepsilon^2 M(t,z)(1+O^*(\varepsilon^2)+O(\varepsilon^3)\|(\kappa_\varepsilon,W_\varepsilon)\|)(1+\varepsilon^2\Xi_\varepsilon(t,z)+O(\varepsilon^4)\|(\kappa_\varepsilon,W_\varepsilon)\|)$$
$$\times\left(\frac{O(\varepsilon^2)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}+\partial_z^3\Xi_\varepsilon(t,z)\right). \tag{6-36}$$

This is shortened to

$$R_{10}=\varepsilon^2 M(t,z)\partial_z^3\Xi_\varepsilon(t,z)+\varepsilon^2 M(t,z)\frac{O(\varepsilon^2)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)}. \tag{6-37}$$

We now add every estimate, starting from (6-32) and including (6-33)–(6-37) to obtain

$$
\begin{aligned}
\sum_{j=1}^{10} R_j = \partial_z^3 M(t,z) &+ \varepsilon^2 \partial_z^3 M(t,z)(\Xi_\varepsilon(t,z) + O^*(\varepsilon^2) + O(\varepsilon^3)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\
&+ 3\varepsilon^2 \partial_z^2 M(t,z)\left(\partial_z \Xi_\varepsilon(t,z) + \frac{O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right) \\
&+ 3\varepsilon^2 \partial_z M(t,z)\left(\partial_z^2 \Xi_\varepsilon(t,z) + \frac{O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right) \\
&+ \varepsilon^2 M(t,z)\left(\partial_z^3 \Xi_\varepsilon(t,z) + \frac{\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty}{2^{1-\alpha}\varphi_\alpha(t,z)} + \frac{O^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}\right). \qquad \text{(6-38)}
\end{aligned}
$$

To conclude the proof, we deal with the left-hand side of (6-25) as in the linearization of the second derivative, noticing that the terms $\partial_z^3 M$ cancel on each side. □

7. Stability of the linearized equations

Building upon the series of linear approximations, we can study the stability of W_ε in the space $\mathcal{F}$. The first result is to control the different terms of $\mathcal{F}$ in the norm $\|\cdot\|_{\mathcal{F}}$; see Definition 1.2. The weight function introduced in the definition of $\mathcal{E}$ is meant to control the behavior at infinity.

Theorem 7.1 (stability analysis). *For any ball B of $\mathbb{R} \times \mathcal{F}$, there exists a constant ε_B that depends only on B such that for all $\varepsilon \le \varepsilon_B$, any pair $(\kappa_\varepsilon, W_\varepsilon) \in B$, a solution of (2-2), satisfies the following bounds*:

$$
\begin{aligned}
\|\Xi_\varepsilon\|_\infty &\le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \\
\|\partial_z W_\varepsilon\|_\infty &\le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \\
\|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty &\le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \\
\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty &\le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \\
\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty &\le O_0^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\| + k(\alpha)\|W_\varepsilon\|_{\mathcal{F}},
\end{aligned}
$$

where $O_0^*(1) = \max(O^*(1), O(1)\|W_\varepsilon(0,\cdot)\|_{\mathcal{F}})$ and $k(\alpha) < 1$ is a uniform constant.

The proof of this theorem is quite intricate and will be divided into several subsections. The plan is as follows:

- First, we focus on a fixed ball around $z_*(t)$. The first step is to get bounds only on a small time interval on this ball, and the second step is to propagate this bound uniformly in time, locally in space.

- Next, we propagate this bound on the whole space by successively dividing it into growing balls B_n and dyadic rings D_n centered around z_*; see the definitions in (7-2) and (7-3).

The main arguments are the maximum principle coupled with a suitable division of the space that accounts for the nonlocal nature of the infinitesimal operator. The purpose of this dyadic decomposition in rings is to obtain a decay of the norm with respect to the radius of the ring.

7A. ***Division of the space into a ball surrounded by growing balls and dyadic rings.*** Let us first consider a time T_*. Then for all times t and s such that $0 \le t, s \le T_*$, the inequality

$$|z_*(t) - z_*(s)| \le \sup_{s \ge 0} |m'(z_*(s))| T_* := r_*$$

holds true, and the supremum is finite because z_* lives in a bounded domain uniquely determined by m and $z_*(0)$; see (1-5).

We slightly expand this ball by a constant r_0 to be defined later, and define the ball

$$B_0 := \{z : |z - z_*(0)| \le r_0 + r_*\}.$$

Our intention behind this choice is that the ball B_0 satisfies the following property:

$$\forall t \le T_*, \quad \forall z \in B_0, \quad |z - \bar{z}(t)| = \tfrac{1}{2}|z - z_*(t)| = \tfrac{1}{2}|z - z_*(0) + z_*(0) - z_*(t)| \le \tfrac{1}{2} r_0 + r_*. \tag{7-1}$$

We recall that $\bar{z}(t) := \frac{1}{2}(z + z_*(t))$. We will split the rest of the space around B_0 into successive balls. The first ball is defined as $B_1 = \{z : |z - z_*(0)| \le 2r_0 + r_*\}$. It contains B_0, and more importantly, it satisfies for every $t \le T_*$ the following identity on the middle point:

$$|\bar{z}(t) - z_*(0)| = \left|\tfrac{1}{2}(z + z_*(t)) - z_*(0)\right| \le \left|\tfrac{1}{2}(z - z_*(0))\right| + \left|\tfrac{1}{2}(z_*(0) - z_*(t))\right| \le r_0 + r_*.$$

This shows that for any $z \in B_1$ and time $t \le T_*$, the corresponding middle point $\bar{z}(t)$ lies in B_0. More generally, the following lemma holds true if we define, for $n \ge 2$,

$$B_n := \{z : |z - z_*(0)| \le 2^n r_0 + r_*\}. \tag{7-2}$$

Lemma 7.2 (middle point property). *For every time* $0 \le t \le T_*$,

$$\forall n \ge 1, \quad \forall z \in B_n, \quad \bar{z}(t) \in B_{n-1}.$$

The proof will also feature prominently the dyadic rings D_n, defined as

$$D_n := \{2^{n-1} r_0 + r_* \le |z - z_*(0)| \le 2^n r_0 + r_*\}, \tag{7-3}$$

with the convention that $D_0 = B_0$. Note that D_n (a subset of B_n) is the set such that $B_{n-1} \cup D_n = B_n$; see Figure 2. On the rings, we will need the following notations:

$$a_n := \sup_{(t,z) \in \mathbb{R}_+ \times D_n} \left| \frac{M(t, \bar{z})}{M(t, z)} \right|, \quad b_n := \sup_{(t,z) \in \mathbb{R}_+ \times D_n} \left| \frac{\partial_z M(t, \bar{z})}{\partial_z M(t, z)} \right|. \tag{7-4}$$

From the asymptotic hypothesis made in (1-15) on the quotient of M, the sequence a_n is bounded and satisfies $a_n \to a < \frac{1}{2}$ as $n \to \infty$. The sequence b_n is uniformly bounded.

Notations for this section. We will denote by $\|\cdot\|_\infty^n$ the L^∞ norm on $\mathbb{R}_+ \times B_n$:

$$\text{for } n \ge 0, \quad \|\cdot\|_\infty^n := \sup_{(t,z) \in \mathbb{R}_+ \times B_n} |\cdot|. \tag{7-5}$$

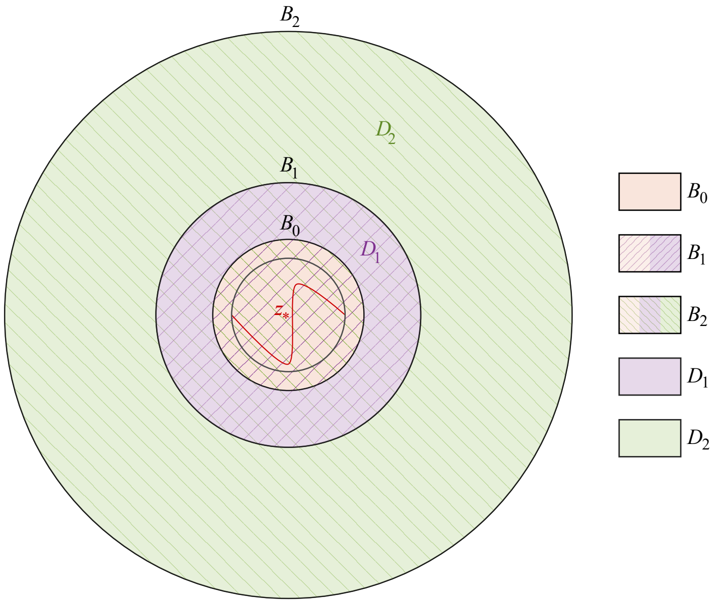

Figure 2. The division of the space into successive dyadic balls and rings.

7B. ***Local bounds on B_0.*** The first step of the proof of Theorem 7.1 consists in getting uniform bounds (in time) on the ball B_0. The estimates on the third derivative are dealt with slightly differently, and are thus delayed to Section 7F.

Proposition 7.3 (local bounds). *For a convenient choice of T^* and r_0 introduced above, and made explicit in (7-7), there exists a constant ε_B that depends only on B, such that with the conditions of Theorem 7.1, W_ε satisfies, for $\varepsilon \le \varepsilon_B$:*

$$\begin{aligned}
\|\Xi_\varepsilon\|_\infty^0 &\le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|,\\
\|\partial_z W_\varepsilon\|_\infty^0 &\le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|,\\
\|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^0 &\le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|,\\
\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^0 &\le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|,
\end{aligned}$$

where $O_0^*(1) = \max(O^*(1), O^*(1)\|W_\varepsilon(0,\cdot)\|_{\mathcal{F}})$.

To prove this *local* bound, i.e., in the ball B_0, one must start with the higher order derivative to build a contraction argument. Estimates of the lower order derivatives are then successively deduced by integration. Clearly, our argument for the third derivative is more technical because it involves a lot of terms through the linearized approximation made in Proposition 6.8. Therefore, for reasons of clarity, third derivatives are left out of Proposition 7.3, we will deal with them, locally and on the balls, in Proposition 7.7. We present here our argument on the simpler derivatives up to order two, and we refer to Section 7F for the generalization of the method to the third derivative. Interestingly, to prove the nonlocal estimates on the balls, we will proceed in the reverse way by first dealing with the lower order derivatives.

Proof of Proposition 7.3. By the derivation of the linearized equation in Proposition 6.7, W_ε satisfies the following, see (5-1):

$$\begin{aligned}\varepsilon^2\partial_t\partial_z^2 W_\varepsilon(t,z) = -\partial_z^2 M(t,z)(W_\varepsilon(t,z)-2W_\varepsilon(t,\bar z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|)\\ -2\partial_z M(t,z)(\partial_z W_\varepsilon(t,z)-\partial_z W_\varepsilon(t,\bar z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|)\\ +M(t,z)\big(\tfrac12\partial_z^2 W_\varepsilon(t,\bar z)-\partial_z^2 W_\varepsilon(t,z)+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|\big).\end{aligned}$$

We will use the maximum principle on the ball B_0. The key point is that on this ball, all other factors are controlled by $\|\partial_z^2 W_\varepsilon\|_\infty$. To compare all those terms with $\partial_z^2 W_\varepsilon$, we perform Taylor expansions with respect to the space variable. First, thanks to (7-1), for any $z\in B_0$ we write

$$\partial_z W_\varepsilon(t,\bar z)-\partial_z W_\varepsilon(t,z)\le\left(\frac{r_0}{2}+r_*\right)\|\partial_z^2 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)}.$$

Similarly, there exists $\xi\in(z,\bar z)$ and $\xi'\in(z_*,\bar z)$ such that

$$\begin{aligned}\Xi_\varepsilon(t,z)&=W_\varepsilon(t,z)-2W_\varepsilon(t,\bar z)+W_\varepsilon(t,z_*)\\ &=\left(\frac{z-z_*}{2}\right)\partial_z W(t,\bar z)+\frac12\frac{(z-z_*)^2}{4}\partial_z^2 W(t,\xi)-\left(\frac{z-z_*}{2}\right)\partial_z W(\bar z)+\frac12\frac{(z-z_*)^2}{4}\partial_z^2 W(\xi')\\ &\le\frac14\left(\frac{r_0}{2}+r_*\right)^2\|\partial_z^2 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)}.\end{aligned}\tag{7-6}$$

Moreover, by the hypothesis made in (1-14) on M, for $j=1,2$,

$$\sup_{(t,z)\in\mathbb{R}_+\times B_0}\left|\frac{\partial_z^{(j)}M(t,z)}{M(t,z)}\right|\le O^*(1).$$

Thanks to those a priori bounds, when we evaluate (6-16) at the maximum point of $\partial_z^2 W_\varepsilon$ on B_0, we get

$$\begin{aligned}&\varepsilon^2\partial_t[\|\partial_z^2 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)}]\\ &\quad\le M(t,z)\Big(\tfrac12\|\partial_z^2 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)}-\|\partial_z^2 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)}\\ &\qquad+O^*(1)\big(\tfrac14\big(\tfrac12 r_0+r_*\big)^2+\tfrac12 r_0+r_*\big)\|\partial_z^2 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)}+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|\Big).\end{aligned}$$

The *crucial* step is that we choose T^* and r^* so small, so that

$$O^*(1)\big(\tfrac14\big(\tfrac12 r_0+r_*\big)^2+\tfrac12 r_0+r_*\big)\le\tfrac14.\tag{7-7}$$

The consequence is that

$$\varepsilon^2\partial_t[\|\partial_z^2 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)}]\le M(t,z)\big(-\tfrac14\|\partial_z^2 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)}+O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|\big).$$

The function $M(t,z)$ admits a lower bound. Therefore, we can apply the maximum principle, on the ball B_0, and get

$$\|\partial_z^2 W_\varepsilon\|_{L^\infty([0,T^*]\times B_0)}\le\max(O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|,\|\partial_z^2 W_\varepsilon(0,\cdot)\|_{L^\infty(B_0)}).$$

We now detail how to propagate this bound uniformly in time. One can renew every previous estimate on each interval $I_k := [kT_*, (k+1)T_*]$. By going over the same steps, we notice that the only argument that changes for different k is the center of the ball B_0 around z_*, but interestingly not its radius; see (7-7). Every other estimate is the same and is independent of k. Therefore, since the condition (7-7) is uniform in time ($O^*(1)$ does not depend on time), once the radius is chosen small enough depending only on K^*, see (7-7), we can repeat recursively the estimates on each interval I_k. Considering all $k \in \mathbb{N}$, we have therefore proven that

$$\|\partial_z^2 W_\varepsilon\|_\infty^0 \le \max(O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \|\partial_z^2 W_\varepsilon(0,\cdot)\|_{L^\infty(B_0)}) \le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|. \quad (7\text{-}8)$$

We will use this estimate as the starting point in order to prove the rest of Proposition 7.3. First, notice that adding the weight function φ_α is straightforward, since it is uniformly bounded on B_0:

$$\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^0 \le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|.$$

Next, taking advantage of the fact that both W_ε and $\partial_z W_\varepsilon$ vanish at z^*, we write

$$|\partial_z W_\varepsilon(t,z)| = \left|\int_{z_*(t)}^{z} \partial_z^2 W_\varepsilon(t,z')\,dz'\right| \le (r_0 + 2r_*)\|\partial_z^2 W_\varepsilon\|_\infty^0.$$

As a consequence, using again the expansion of (7-6),

$$|\Xi_\varepsilon(t,z)| = |2W_\varepsilon(t,\bar z(t)) - W_\varepsilon(t,z)| \le \tfrac14\left(\tfrac12 r_0 + r_*\right)^2 \|\partial_z^2 W_\varepsilon\|_\infty^0.$$

Similarly, we get a uniform bound on $\partial_z \Xi_\varepsilon$. Combining those estimates with the first estimate in (7-8), which comes from the maximum principle, the proof of Proposition 7.3 is concluded. □

7C. *Bound on the balls*: Ξ_ε. We will now propagate those bounds beyond the small ball. It is very important to keep the level of precision of $O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|$, to which we will add some decay property due to the specific shape of the rings D_n.

Proposition 7.4 (in the balls, Ξ_ε). *There exists a constant ε_B that depends only on B such that with the conditions of Theorem 7.1, W_ε satisfies, for $\varepsilon \le \varepsilon_B$,*

$$\|\Xi_\varepsilon\|_\infty^n \le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \quad (7\text{-}9)$$

for all $n \ge 1$.

Proof. The starting point of the analysis is the linearized equation given by Corollary 6.3. For $t \in \mathbb{R}_+$, $n \ge 1$, take z in the ball B_n defined previously. We know that

$$\varepsilon^2 \partial_t \Xi_\varepsilon(t,z) = M(t,z)\left(2\frac{M(t,\bar z)}{M(t,z)}\Xi_\varepsilon(t,\bar z) - \Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|\right).$$

One can multiply by $\operatorname{sign}(\Xi_\varepsilon)$ this equality to bound the absolute value; it is important to keep the minus sign on the right-hand side. We get

$$\varepsilon^2 \partial_t |\Xi_\varepsilon(t,z)| \le M(t,z)\left(2\frac{M(t,\bar z)}{M(t,z)}|\Xi_\varepsilon(t,\bar z)| - |\Xi_\varepsilon(t,z)| + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|\right).$$

Then, from Lemma 7.2, we know that the middle point $\bar{z}$ is in the smaller ball B_{n-1}, and so we have the following estimate:

$$\varepsilon^2 \partial_t |\Xi_\varepsilon(t,z)| \le M(t,z)\left(2\frac{M(t,\bar{z})}{M(t,z)}\|\Xi_\varepsilon\|_\infty^{n-1} - \Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|\right).$$

To obtain precise bounds from this inequality, we shall discuss whether the maximum point of Ξ_ε on the ball D_n is reached inside the ring D_{n-1}, defined in (7-3), or not. If it is the case, we obtain a sharper estimate than if it is not the case.

- Suppose that the maximum point that reaches $\|\Xi_\varepsilon\|_\infty^n$ belongs to the ring D_n. We can then control the quotient of M by the sequence a_n defined in (7-4). Moreover, M admits a uniform lower bound by (1-13), thus, we can apply the maximum principle to get

$$\|\Xi_\varepsilon\|_\infty^n \le \max(2a_n\|\Xi_\varepsilon\|_\infty^{n-1} + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \|\Xi_\varepsilon(0,\cdot)\|_{L^\infty(B_n)}). \tag{7-10}$$

We first notice that for all $n \in \mathbb{N}$,

$$\|\Xi_\varepsilon(0,\cdot)\|_{L^\infty(B_n)} \le O_0^*(1).$$

Therefore, from (7-10),

$$\|\Xi_\varepsilon\|_\infty^n \le 2a_n\|\Xi_\varepsilon\|_\infty^{n-1} + O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|. \tag{7-11}$$

Here lies the motivation behind the introduction of the notation $O_0^*(1)$. It allows us to take into account the initial data and to make recursive estimates that were a priori not possible with (7-10).

- Before going further, we now assume that the maximum point that reaches $\|\Xi_\varepsilon\|_\infty^n$ is outside the ring D_n, in $B_n \setminus D_n = B_{n-1}$. In that case, the estimate of (7-4) is not helpful, as we would need to define $\tilde{a}_n$ to be the supremum over B_n, but then this sequence would not give a contraction factor as in (7-10). Therefore, we simply write for this case

$$\|\Xi_\varepsilon\|_\infty^n \le \|\Xi_\varepsilon\|_\infty^{n-1}. \tag{7-12}$$

- The combination of (7-11) and (7-12) yields

$$\|\Xi_\varepsilon\|_\infty^n \le \max(2a_n\|\Xi_\varepsilon\|_\infty^{n-1} + O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|,\ \|\Xi_\varepsilon\|_\infty^{n-1}). \tag{7-13}$$

This inequality guarantees that the sequence $(\|\Xi_\varepsilon\|_\infty^n)_n$ is uniformly bounded. Heuristically, on the right-hand side of (7-13), the geometric part of the maximum satisfies $2a_n \to 2a < 1$ when $n \to \infty$, thanks to (1-15), therefore it ensures a contraction, while the other part of the maximum yields at worst a bound by the term $n = 0$.

We detail more rigorously the steps as it will serve as a model for future proofs. We assume without loss of generality that $2a_n < \theta$ for all $n \in \mathbb{N}$, with, for instance the factor $\theta := a + \frac{1}{2}$, such that $2a < \theta < 1$. We know this is true, but for a finite number of terms, by (7-4). For this handful of terms, we do not need a contraction argument, since the bound (7-9) follows from a finite number of iterations of (7-13). Let f_n be the function $f_n(\xi) = \max(\xi, 2a_n\xi + C)$, and a sequence ξ_n be such that $\xi_{n+1} \le f_n(\xi_n)$. We will then show that for all $n \in \mathbb{N}$,

$$\xi_n \leq \max\left(\xi_0, \frac{C}{1-\theta}\right). \tag{7-14}$$

The proof is done by induction, the initial step is obvious. If we now assume that the inequality holds true for a certain $n \in \mathbb{N}$, we get

$$\xi_{n+1} \leq f_n(\xi_n) \leq \max(\xi_n, 2a_n\xi_n + C).$$

If the previous max is ξ_n, then we immediately deduce by the induction hypothesis the following:

$$\xi_{n+1} \leq \xi_n \leq \max\left(\xi_0, \frac{C}{1-\theta}\right).$$

Otherwise,

$$\xi_{n+1} \leq 2a_n \max\left(\xi_0, \frac{C}{1-\theta}\right) + C.$$

We once again discuss where the maximum point is reached. If it is ξ_0, then we end up with

$$\xi_{n+1} \leq 2a_n\xi_0 + C \leq (2a_n - \theta)\xi_0 + \xi_0 \leq \xi_0.$$

Similarly, if it is not ξ_0,

$$\xi_{n+1} \leq \frac{2a_n C}{1-\theta} + C = \frac{(2a_n - \theta)C + C}{1-\theta} \leq \frac{C}{1-\theta}.$$

Therefore, we have shown that in all cases,

$$\xi_{n+1} \leq \max\left(\xi_0, \frac{C}{1-\theta}\right),$$

which proves (7-14). We conclude, given the bound (7-13), that

$$\|\Xi_\varepsilon\|_\infty^n \leq \max\left(\frac{O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{1-\theta}, \|\Xi_\varepsilon\|_\infty^0\right) \leq O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|. \qquad \square$$

All the remaining proofs of this section will follow this blueprint.

7D. *Bound on the balls*: $\partial_z \Xi_\varepsilon$. We now state a similar result for $\partial_z \Xi_\varepsilon$. We will see the appearance of the weight function φ_α in the estimates. It slightly worsens the expressions but the strategy deployed to prove Proposition 7.4 will still works.

Proposition 7.5 (in the balls, $\partial_z \Xi_\varepsilon$). *There exists a constant ε_B that depends only on B such that upon the condition of Theorem 7.1, W_ε satisfies for $\varepsilon \leq \varepsilon_B$*

$$\|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^n \leq O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|,$$

for $n \geq 1$.

Proof. The proof is similar to the bound on Ξ_ε, but we have to take the weight function into account. We start with the linear equation satisfied by $\partial_t\partial_z\Xi_\varepsilon$ in Corollary 6.6. It yields, for $z \in B_n$ and $t \in \mathbb{R}_+$,

$$\begin{aligned}\varepsilon^2\partial_t|\partial_z\Xi_\varepsilon(t,z)| \\ \le M(t,z)\Biggl(\frac{M(t,\bar z)}{M(t,z)}|\partial_z\Xi_\varepsilon(t,\bar z)| - |\partial_z\Xi_\varepsilon(t,z)| + \frac{O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|}{\varphi_\alpha(t,z)} \\ + \frac{O^*(1)}{\varphi_\alpha(t,z)}\biggl(\frac{\partial_z M(t,\bar z)}{\partial_z M(t,z)}\|\Xi_\varepsilon\|_\infty^{n-1} + \|\Xi_\varepsilon\|_\infty^n + O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|\biggr)\Biggr).\end{aligned} \tag{7-15}$$

In the second factor, thanks to (1-14), we used that

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}}\left(\varphi_\alpha(t,z)\left|\frac{\partial_z M(t,z)}{M(t,z)}\right|\right) \le O^*(1).$$

To take into account the weight function, we make the following computation:

$$\partial_t[\varphi_\alpha\partial_z\Xi_\varepsilon](t,z) = \varphi_\alpha(t,z)\partial_t\partial_z\Xi_\varepsilon(t,z) + \partial_z\Xi_\varepsilon(t,z)\partial_t\varphi_\alpha(t,z).$$

First,

$$\partial_z\Xi_\varepsilon(t,z)\partial_t\varphi_\alpha(t,z) = \alpha\partial_z\Xi_\varepsilon(t,z)\frac{m'(z_*)\operatorname{sign}(z-z_*)}{(1+|z-z_*|)^{1-\alpha}} = O^*(1)\partial_z\Xi_\varepsilon(t,z), \tag{7-16}$$

and therefore,

$$\varepsilon^2\partial_z\Xi_\varepsilon(t,z)\partial_t\varphi_\alpha(t,z) = O^*(\varepsilon^2)\|(\kappa_\varepsilon,W_\varepsilon)\|. \tag{7-17}$$

By multiplying (7-15) by φ_α and taking into account (7-17), we deduce that

$$\begin{aligned}\varepsilon^2\partial_t[\varphi_\alpha|\partial_z\Xi_\varepsilon|](t,z) \\ \le M(t,z)\Biggl(-\varphi_\alpha(t,z)|\partial_z\Xi_\varepsilon(t,z)| + \frac{M(t,\bar z)}{M(t,z)}\varphi_\alpha(t,z)|\partial_z\Xi_\varepsilon(t,\bar z)| + O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\| \\ + O^*(1)\biggl(\frac{\partial_z M(t,\bar z)}{\partial_z M(t,z)}\|\Xi_\varepsilon\|_\infty^{n-1} + \|\Xi_\varepsilon\|_\infty^n + O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|\biggr) + \frac{O^*(\varepsilon^2)\|(\kappa_\varepsilon,W_\varepsilon)\|}{M(t,z)}\Biggr),\end{aligned}$$

As in the previous proof, to obtain sharp bounds from the maximum principle, we discuss whether the maximum point of $\varphi_\alpha\partial_z\Xi_\varepsilon$ on B_n is reached on the subset D_n or not. *We now assume that it is the case.*

We can then use the sequences a_n and b_n defined in (7-4) to control the right-hand side of (7-19). Moreover, with Proposition 7.4 we can estimate the terms involving Ξ_ε on the balls. We then find that

$$\begin{aligned}\varepsilon^2\partial_t[\varphi_\alpha|\partial_z\Xi_\varepsilon|](t,z) \\ \le M(t,z)\Biggl(-\varphi_\alpha(t,z)|\partial_z\Xi_\varepsilon(t,z)| + a_n\left|\frac{\varphi_\alpha(t,z)}{\varphi_\alpha(t,\bar z)}\right|\|\varphi_\alpha\partial_z\Xi_\varepsilon\|_\infty^{n-1} + b_n(O_0^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\|) \\ + O_0^*(1)+O(\varepsilon)\|(\kappa_\varepsilon,W_\varepsilon)\| + \frac{O^*(\varepsilon^2)\|(\kappa_\varepsilon,W_\varepsilon)\|}{M(t,z)}\Biggr).\end{aligned}$$

The weight function was chosen precisely to satisfy the scaling estimate

$$\sup_{\mathbb{R}_+\times\mathbb{R}}\left|\frac{\varphi_\alpha(t,z)}{\varphi_\alpha(t,\bar{z})}\right| \le 2^\alpha. \tag{7-18}$$

Since the function $1/M$ has a uniform upper bound, and the sequence b_n is uniformly bounded, we finally conclude that

$$\begin{aligned}\varepsilon^2\partial_t[\varphi_\alpha|\partial_z\Xi_\varepsilon|](t,z)&\\ \le M(t,z)(-\varphi_\alpha(t,z)|\partial_z\Xi_\varepsilon(t,z)| &+ 2^\alpha a_n\|\varphi_\alpha\partial_z\Xi_\varepsilon\|_\infty^{n-1} + O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|).\end{aligned} \tag{7-19}$$

The maximum principle applied to (7-19) gives

$$\|\varphi_\alpha\partial_z\Xi_\varepsilon\|_\infty^n \le \max(2^\alpha a_n\|\varphi_\alpha\partial_z\Xi_\varepsilon\|_\infty^{n-1} + O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \|\varphi_\alpha\partial_z\Xi_\varepsilon(0,\cdot)\|_\infty^n).$$

Notice that for all $n\in\mathbb{N}$,

$$\|\varphi_\alpha\partial_z\Xi_\varepsilon(0,\cdot)\|_{L^\infty(B_n)} \le \|W_\varepsilon(0,\cdot)\|_{\mathcal{F}} \le O_0^*(1).$$

Therefore, we obtain finally, in the case where the maximum point of $\varphi_\alpha\partial_z\Xi_\varepsilon$ on B_n is reached on the subset D_n,

$$\|\varphi_\alpha\partial_z\Xi_\varepsilon\|_\infty^n \le 2^\alpha a_n\|\varphi_\alpha\partial_z\Xi_\varepsilon\|_\infty^{n-1} + O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|. \tag{7-20}$$

When this is not the case, we will only state that

$$\|\varphi_\alpha\partial_z\Xi_\varepsilon\|_\infty^n \le \|\varphi_\alpha\partial_z\Xi_\varepsilon\|_\infty^{n-1}. \tag{7-21}$$

Combining (7-20) and (7-21), we eventually conclude that

$$\|\varphi_\alpha\partial_z\Xi_\varepsilon\|_\infty^n \le \max(2^\alpha a_n\|\varphi_\alpha\partial_z\Xi_\varepsilon\|_\infty^{n-1} + O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|, \|\varphi_\alpha\partial_z\Xi_\varepsilon\|_\infty^{n-1}). \tag{7-22}$$

This implies that the sequence $(\|\varphi_\alpha\partial_z\Xi_\varepsilon\|_\infty^n)_n$ is a contraction, using the same recursive arguments as in the previous proof. Indeed, by hypothesis, $2^\alpha a_n \le 2^\alpha a < 1$, but for a finite number of terms, which gives for instance a contraction factor $\theta := a + \frac{1}{2}$, such that $2a < \theta < 1$. The second part of the maximum in (7-22) does not perturb the contraction part, and we deduce that

$$\|\varphi_\alpha\partial_z\Xi_\varepsilon\|_\infty^n \le \max\left(\frac{O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{1-\theta}, \|\varphi_\alpha\partial_z\Xi_\varepsilon\|_\infty^0\right) \le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|.$$

The second inequality uses the local bounds of the ball B_0 made in Proposition 7.3. □

7E. *Bound on the balls*: $\partial_z^2 W_\varepsilon$. We now make a similar statement about the second derivative.

Proposition 7.6 (in the balls, $\partial_z^2 W_\varepsilon$). *There exists a constant ε_B that depends only on B such that with the condition of Theorem 7.1, W_ε satisfies, for $\varepsilon \le \varepsilon_B$,*

$$\|\varphi_\alpha\partial_z^2 W_\varepsilon\|_\infty^n \le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|,$$

for $n \ge 1$.

Proof. We proceed as in the proof of Proposition 7.5. We already know a linearized approximation for $\partial_z^2 W_\varepsilon$, thanks to (6-16) in Proposition 6.7. Taking the weight into account, one finds that $\varphi_\alpha \partial_z^2 W_\varepsilon$ satisfies the following:

$$\begin{aligned}
&\varepsilon^2 \partial_t[\varphi_\alpha(t,z)\partial_z^2 W_\varepsilon](t,z)\\
&= -\partial_z^2 M(t,z)\varphi_\alpha(t,z)(\Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\
&\qquad - 2\partial_z M(t,z)(\varphi_\alpha(t,z)\partial_z \Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\
&\qquad\qquad + M(t,z)\big(\tfrac{1}{2}\varphi_\alpha(t,z)\partial_z^2 W_\varepsilon(t,\bar{z}) - \varphi_\alpha(t,z)\partial_z^2 W_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|\big)\\
&\qquad\qquad\qquad + O(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|. \qquad (7\text{-}23)
\end{aligned}$$

The last term comes from the same computation of $\partial_t \varphi_\alpha$ as that made in (7-17). To continue, we first use the following uniform controls of (1-14):

$$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \left(\varphi_\alpha(t,z)\left|\frac{\partial_z^2 M(t,z)}{M(t,z)}\right|\right) \le O^*(1) \quad \text{and} \quad \sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \left|\frac{\partial_z M(t,z)}{M(t,z)}\right| \le O^*(1).$$

We also need the scaling estimate of the weight function, stated in (7-18). Then, we can bound the right-hand side of (7-23) after factorizing by M, for $t \in \mathbb{R}_+$ and $z \in B_n$:

$$\begin{aligned}
&\varepsilon^2 \partial_t[\varphi_\alpha(t,z)|\partial_z^2 W_\varepsilon|](t,z)\\
&\le M(t,z)\bigg(-\varphi_\alpha(t,z)|\partial_z^2 W_\varepsilon(t,z)| + \frac{1}{2^{1-\alpha}}\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^{n-1} + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|\\
&\qquad + O^*(1)(\|\Xi_\varepsilon\|_\infty^n + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|)\\
&\qquad\qquad + O^*(1)(\|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^n + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|)\bigg).
\end{aligned}$$

Plugging in the controls on the balls of Ξ_ε and $\partial_z \Xi_\varepsilon$ from Propositions 7.4 and 7.5, we have

$$\begin{aligned}
&\varepsilon^2 \partial_t[\varphi_\alpha(t,z)|\partial_z^2 W_\varepsilon|](t,z)\\
&\qquad \le M(t,z)\Big(-\varphi_\alpha(t,z)|\partial_z^2 W_\varepsilon(t,z)| + \frac{1}{2^{1-\alpha}}\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^{n-1} + O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|\Big).
\end{aligned}$$

The function $M(t,z)$ admits a positive lower bound by (1-13). We apply the maximum principle to get

$$\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^n \le \max\Big(\frac{1}{2^{1-\alpha}}\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^{n-1} + O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|,\ \|\varphi_\alpha \partial_z^2 W_\varepsilon(0,\cdot)\|_\infty^n\Big).$$

As earlier, we can get rid of the initial data in the maximum by stating that for all $n \in \mathbb{N}$,

$$\|\varphi_\alpha \partial_z^2 W_\varepsilon(0,\cdot)\|_\infty^n \le \|W_\varepsilon(0,\cdot)\|_{\mathcal{F}} \le O_0^*(1). \qquad (7\text{-}24)$$

Then, the recursive arguments are somehow a little easier in that case than in the proofs of Propositions 7.4 and 7.5 since the geometric term, $2^{\alpha-1}$, does not depend on n. Hence we have

$$\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^n \le \frac{1}{2^{1-\alpha}}\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^{n-1} + O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|.$$

Because $2^{\alpha-1} < 1$, we immediately get that

$$\|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^n \le \max\left(\frac{O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{1 - 2^{\alpha-1}}, \|\varphi_\alpha \partial_z^2 W_\varepsilon\|_\infty^0\right) \le O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|. \qquad \square$$

7F. *Local and on-the-balls bound for $\partial_z^3 W_\varepsilon$.* We dedicate this section to the study of $\partial_z^3 W_\varepsilon$ since it does not exactly fit the mold of the previous estimates due to the additional factor $\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty / 2^{1-\alpha}$ in the linearized equation in Proposition 6.8.

• We highlight the difference by first proving the initial bound on the local ball B_0. We write the linear equation solved by $\varphi_\alpha \partial_z^3 W_\varepsilon$:

$$\begin{aligned}
-\varepsilon^2 \partial_t [\varphi_\alpha \partial_z^3 W_\varepsilon](t,z)& \\
= \varphi_\alpha(t,z)\partial_z^3 M(t,z)&(\Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\
&+ 3\partial_z^2 M(t,z)(\varphi_\alpha(t,z)\partial_z \Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\
&+ 3\partial_z M(t,z)(\varphi_\alpha(t,z)\partial_z^2 \Xi_\varepsilon(t,z) + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\
&+ M(t,z)\left(\varphi_\alpha(t,z)\partial_z^3 \Xi_\varepsilon(t,z) + \frac{\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty}{2^{1-\alpha}} + O^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|\right) \\
&- \varepsilon^2 \partial_z^3 W_\varepsilon(t,z)\partial_t \varphi_\alpha(t,z).
\end{aligned} \tag{7-25}$$

First, one finds that

$$\varepsilon^2 \partial_z^3 \Xi_\varepsilon(t,z)\partial_t \varphi_\alpha(t,z) = O^*(\varepsilon^2)\|(\kappa_\varepsilon, W_\varepsilon)\|.$$

We recall that Ξ_ε, $\partial_z \Xi_\varepsilon$ and $\partial_z^2 \Xi_\varepsilon$ were all uniformly bounded on B_0, with the weight, in Proposition 7.3. Moreover, from (1-13), for $j = 1, 2$,

$$\begin{aligned}
\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \left|\frac{\partial_z^{(j)} M(t,z)}{M(t,z)}\right| &\le O^*(1), \\
\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}} \left(\varphi_\alpha(t,z)\left|\frac{\partial_z^3 M(t,z)}{M(t,z)}\right|\right) &\le O^*(1).
\end{aligned} \tag{7-26}$$

Finally,

$$\varphi_\alpha(t,z)|\partial_z^3 W_\varepsilon(t,\bar{z})| \le \frac{2^\alpha}{4}|\varphi_\alpha(t,\bar{z})\partial_z^3 W_\varepsilon(t,\bar{z})|.$$

When plugging all of this into (7-25), we obtain, by evaluating at the point of maximum on B_0,

$$\begin{aligned}
&\varepsilon^2 \partial_t \|\varphi_\alpha(t,\cdot)\partial_z^3 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)} \\
&\quad \le M(t,z)\bigg(-\|\varphi_\alpha(t,\cdot)\partial_z^3 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)} + \frac{1}{2^{2-\alpha}}\|\varphi_\alpha(t,\cdot)\partial_z^3 W_\varepsilon(t,\cdot)\|_{L^\infty(B_0)} + \frac{\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty}{2^{1-\alpha}} \\
&\qquad\qquad + O_0^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|\bigg).
\end{aligned}$$

Since there is a positive lower bound of M, we recognize a contraction argument on the ball B_0, and for bounded times $0 < t \le T^*$,

$$\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_{L^\infty([0,T^*]\times B_0)} \le \max\Big(\Big(\frac{1}{1-2^{\alpha-2}}\Big)\Big(O_0^*(1)+O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\| + \frac{1}{2^{1-\alpha}}\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty\Big), \|\varphi_\alpha(0,\cdot)\partial_z^3 W_\varepsilon(0,\cdot)\|_{L^\infty(B_0)}\Big).$$

Therefore, since the initial data is controlled by $O_0^*(1)$, we may write

$$\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_{L^\infty([0,T^*]\times B_0)} \le O_0^*(1)+O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\| + \frac{2^{\alpha-1}}{1-2^{\alpha-2}}\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty.$$

As explained in Section 7A, we can now repeat the procedure on each interval of time

$$I_k := [kT_*, (k+1)T_*],$$

and end up with a bound uniform in time on the ball B_0:

$$\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^0 \le O_0^*(1)+O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\| + \frac{2^{\alpha-1}}{1-2^{\alpha-2}}\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty. \tag{7-27}$$

- We now proceed to propagate this bound on the balls, starting again from (7-25) and using the maximum principle. For any $t \in \mathbb{R}_+$ and $z \in B_n$, we have

$$\begin{aligned}
&\varepsilon^2 \partial_t[\varphi_\alpha |\partial_z^3 W_\varepsilon|](t,z) \\
&\le M(t,z)\Bigg(-\varphi_\alpha(t,z)|\partial_z^3 W_\varepsilon(t,z)| + \frac{1}{2^{2-\alpha}}\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^{n-1} + \frac{1}{2^{1-\alpha}}\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty \\
&\qquad + O_0^*(1)+O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\| + \left|\varphi_\alpha(t,z)\frac{\partial_z^3 M(t,z)}{M(t,z)}\right|(\|\Xi_\varepsilon\|_\infty^n + O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\
&\qquad + 3\left|\frac{\partial_z^2 M(t,z)}{M(t,z)}\right|(\|\varphi_\alpha \partial_z \Xi_\varepsilon\|_\infty^n + O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|) \\
&\qquad + 3\left|\frac{\partial_z M(t,z)}{M(t,z)}\right|(\|\varphi_\alpha \partial_z^2 \Xi_\varepsilon\|_\infty^n + O^*(1)+O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|)\Bigg).
\end{aligned}$$

We will use once more our hypothesis (1-14), in the form stated in (7-26). We also need all our previous estimates on the balls: Propositions 7.4, 7.5 and 7.6. We then obtain

$$\begin{aligned}
&\varepsilon^2 \partial_t[\varphi_\alpha |\partial_z^3 W_\varepsilon|](t,z) \\
&\le M(t,z)\Big(-\varphi_\alpha(t,z)|\partial_z^3 W_\varepsilon(t,z)| + \frac{1}{2^{2-\alpha}}\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^{n-1} + \frac{1}{2^{1-\alpha}}\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty + O_0^*(1)+O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|\Big).
\end{aligned}$$

We recall that the term $\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty$ is a global control on the whole space $\mathbb{R}$ and not localized on the balls. By applying the maximum principle, one gets

$$\begin{aligned}
&\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^n \\
&\le \max\Big(\frac{1}{2^{2-\alpha}}\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^{n-1} + \frac{1}{2^{1-\alpha}}\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty + O_0^*(1)+O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|,\ \|\varphi_\alpha(0,\cdot)\partial_z^3 W_\varepsilon(0,\cdot)\|_\infty^n\Big).
\end{aligned}$$

We can absorb the initial data in the term $O_0^*(1)$ to deduce that

$$\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^n \leq \frac{1}{2^{2-\alpha}} \|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^{n-1} + \frac{1}{2^{1-\alpha}} \|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty + O_0^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|.$$

This sequence is bounded, because its ratio satisfies $2^{\alpha-2} < 1$. Hence,

$$\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^n \leq \max\left(\frac{O_0^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\|}{1-2^{\alpha-2}} + \frac{2^{\alpha-1}}{1-2^{\alpha-2}} \|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty, \|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^0\right). \tag{7-28}$$

We define $k(\alpha)$ as follows:

$$k(\alpha) := \frac{2^{\alpha-1}}{1-2^{\alpha-2}},$$

and from (7-28) we finally conclude, taking the local bound (7-27) into account, that

$$\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^n \leq O_0^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\| + k(\alpha)\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty.$$

We have therefore proven the following proposition:

Proposition 7.7 (in the rings, $\partial_z^3 W_\varepsilon$). *There exists a constant ε_B that depends only on B such that with the condition of Theorem 7.1, W_ε satisfies, for $\varepsilon \leq \varepsilon_B$,*

$$\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty^n \leq O_0^*(1) + O(\varepsilon^\alpha)\|(\kappa_\varepsilon, W_\varepsilon)\| + k(\alpha)\|\varphi_\alpha \partial_z^3 W_\varepsilon\|_\infty,$$

for $n \geq 1$, with

$$0 < k(\alpha) := \frac{2^{\alpha-1}}{1-2^{\alpha-2}} < 1. \tag{7-29}$$

The scalar $k(\alpha)$ is a contraction factor, only upon the condition that

$$\alpha < 2 - \frac{\ln 3}{\ln 2} \approx 0.415. \tag{7-30}$$

We made that assumption prospectively when we introduced $\mathcal{E}$ in Definition 1.2. Beside (7-30), another reason for which α cannot be taken too large is that it worsens the contraction estimate $\varphi_\alpha(t,z) \leq 2^\alpha \varphi_\alpha(t,\bar{z})$; see (7-18).

7G. *Conclusion: proof of Theorem 7.1.* All our previous estimates from Propositions 7.4, 7.5, 7.6 and 7.7 are uniform in n, and therefore apply to the whole space. Therefore, so far, every bound of Theorem 7.1 has been proved except for the one upon $\partial_z W_\varepsilon$. A proof by recursion on the balls could be adapted from that of Proposition 7.5, starting from the linearized equation of Proposition 6.5. We propose here a more concise argument on the whole space, that uses instead the result of Proposition 7.5.

For all times $t > 0$ and $z \in \mathbb{R}$,

$$\begin{aligned} |\partial_z \Xi_\varepsilon(t,z)| &= |\partial_z W_\varepsilon(t,z) - \partial_z W_\varepsilon(t,\bar{z})| \\ &\leq \frac{O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|}{\varphi_\alpha(t,z)}. \end{aligned} \tag{7-31}$$

This estimate is helpful for large z, given the weight function. To control uniformly on the whole space, let $h \in \mathbb{R}$ and define N_h^0 as the lowest index such that $z_*(t) + 2^{-k}|h| \in B_0$ for all $k > N_h^0$. From Section 7A we know that $N_h^0 \geq \lceil \ln(|h|/r_0)/\ln(2) \rceil$. Then, by iterating (7-31), we get

$$|\partial_z W_\varepsilon(t, z_* + h)| \leq |\partial_z W_\varepsilon(t, z_* + 2^{-(N_0^h+1)}h)| + (O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|) \sum_{k=0}^{N_h^0} \frac{1}{\varphi_\alpha(t, z_* + 2^{-k}h)}.$$

Given the control of $\partial_z W_\varepsilon$ on B_0 from Proposition 7.3 and the explicit form of the weight φ_α, we deduce that

$$\partial_z W_\varepsilon(t, z_* + h) \leq O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\| + (O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|) \sum_{k=0}^{N_h^0} \frac{2^{\alpha k}}{|h|^\alpha}. \tag{7-32}$$

This series has a finite number of terms (roughly $\log(|h|)$), by definition of N_0^h, and otherwise it would not be converging. Indeed, since $2^{N_0^h}/|h| \leq 1/r_0$, this sum is uniformly bounded:

$$\sum_{k=0}^{N_h^0} \frac{2^{\alpha k}}{|h|^\alpha} \leq O^*(1) \frac{2^{\alpha N_h^0}}{|h|^\alpha} \leq O^*(1).$$

Plugging this into (7-32), we have shown, as stated in Theorem 7.1, that

$$\|\partial_z W_\varepsilon\|_\infty \leq O_0^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\|. \qquad \square$$

One should note that the weight function φ_α plays a crucial role here, by "compensating" for the diverging terms. Namely, if $\alpha = 0$, the previous argument crumbles starting as early as estimate (7-31). This shows that the weight φ_α is necessary to ensure uniform Lipschitz bounds of W_ε.

8. Proof of Theorem 1.3

We now prove the main result of this paper, that is the boundedness of $(\kappa_\varepsilon, W_\varepsilon)$ in $\mathbb{R} \times \mathcal{F}$. We first suppose that there exists K_0 such that

$$|\kappa_\varepsilon(0)| \leq K_0 \quad \text{and} \quad \|W_\varepsilon(0, \cdot)\|_{\mathcal{F}} \leq K_0, \tag{8-1}$$

and we look to prove that

$$|\kappa_\varepsilon| \leq K_0' \quad \text{and} \quad \|W_\varepsilon\|_{\mathcal{F}} \leq K_0',$$

with K_0' to be determined by the analysis.

Theorem 7.1 grants precise bounds of W_ε, as long as there exists K such that $\|(\kappa_\varepsilon, W_\varepsilon)\| \leq K$. More precisely, there exists a constant C_0^* that depends only on K_0 and K^* and a constant C_K' that depends only K such that

$$\|W_\varepsilon\|_{\mathcal{F}} \leq C_0^* + C_K' \varepsilon^\alpha K + k(\alpha)\|W_\varepsilon\|_{\mathcal{F}}.$$

Therefore, up to renaming the constants,

$$\|W_\varepsilon\|_{\mathcal{F}} \leq \frac{C_0^* + C_K' \varepsilon^\alpha K}{1 - k(\alpha)} \leq C_0^* + C_K' \varepsilon^\alpha K. \tag{8-2}$$

Clearly, a crucial part of this contraction was to ensure that $k(\alpha) < 1$. We can now work on κ_ε. We go back to Proposition 5.1, from which we learned that κ_ε solves

$$-\dot{\kappa}_\varepsilon(t) = R_\varepsilon^*(t)\kappa_\varepsilon + O^*(1) + O(\varepsilon)\|(\kappa_\varepsilon, W_\varepsilon)\| + O^*(1)\|W_\varepsilon\|_{\mathcal{F}}. \tag{8-3}$$

Thanks to our previous contraction argument, we have an estimate of $\|W_\varepsilon\|_{\mathcal{F}}$, and we can get rid of this last term of order 0. We can plug the estimate (8-2) into (8-3) to finally conclude the argument on κ_ε.

Since R_ε^* is a positive function that admits, for $t \geq t_0$, a uniform lower bound R_0, per Proposition 4.7, it is straightforward from (8-2) and (8-3), and our subsequent bounds, that there exists C_0^* and C_K' such that, for all time t,

$$|\kappa_\varepsilon(t)| \leq C_0^* + C_K'\varepsilon^\alpha K. \tag{8-4}$$

Coupled with (8-2), those are the stability results we needed on the pair $(\kappa_\varepsilon, W_\varepsilon)$. We now set a scalar K_0' such that

$$K_0' \geq 2C_0^*. \tag{8-5}$$

Then, choose ε_0 in the following way:

$$\varepsilon_0 := \left(\frac{1}{2C_0'}\right)^{1/\alpha},$$

where C_0' is the constant in (8-2) and (8-4) corresponding to the choice made in (8-5) of the size of the ball K_0'. Then for $\varepsilon \leq \varepsilon_0$, starting from an initial data that satisfies (8-1), the bound is propagated in time, and furthermore

$$\|W_\varepsilon\|_{\mathcal{F}} \leq K_0', \quad |\kappa_\varepsilon| \leq K_0'.$$

Since $V_\varepsilon = V^* + \varepsilon^2 W_\varepsilon$ and $q_\varepsilon = q^* + \varepsilon^2\kappa_\varepsilon$, Theorem 1.3 is proven. □

9. Numerical simulations and discussion

In this section we display some numerical simulations showing the behavior of the solution of the Cauchy problem for positive ε, and we will provide insight into the structural assumption we made in (1-13).

***Influence of the condition* (1-13).** A first example for our study is to consider a quadratic selection function, as depicted in Figure 3. In that case, according to Theorem 1.3, starting from any initial data $z_*(0)$, the solution f_ε stays close to a Gaussian density with variance ε^2. In addition, its mean z_* converges to the unique minimum of m when the time is large.

Our framework encompasses more general selection functions with multiple local minima, such as the one depicted in Figure 4. The condition in (1-13) restricts somehow the position of those minima. If one assumes that z_* starts from a local minimum, that is $m'(z_*(0)) = 0$, then this condition implies that the selective difference between minima must be inferior to 1: $m(z_*(0)) - m(z_{\mathrm{opt}}) < 1$. We recover the structural condition under which the analysis for the stationary case was performed; see [Calvez et al. 2019].

The selection function depicted in Figure 4, coupled with $z_*(0)$, satisfies the condition (1-13). Then as stated by Theorem 1.3 the population density f_ε concentrates around the local minimum, according to the gradient flow dynamics of Assumption 1.1.

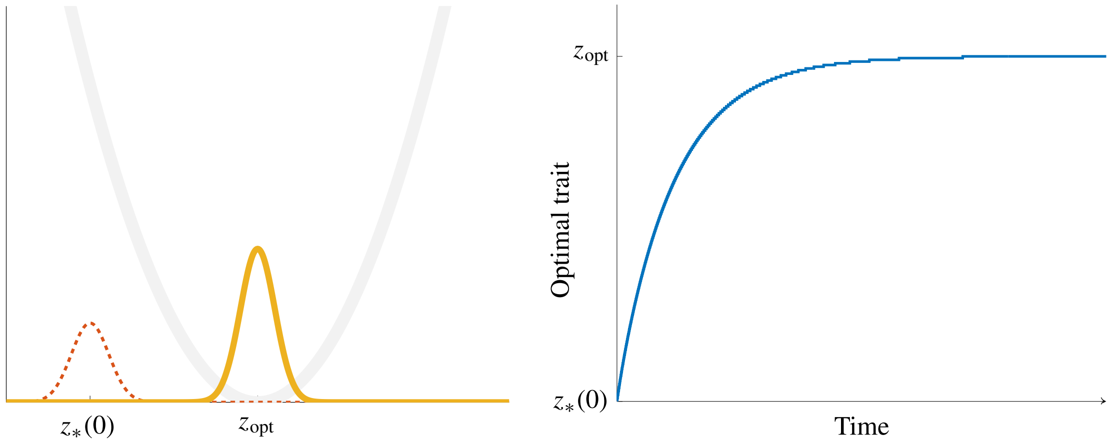

Figure 3. On the left, in dotted red, the initial data $f_\varepsilon(0,\cdot)$, and in orange, the distribution f_ε after a long time. In the background the selection function m with a global optimum z_{opt}. On the right, the trajectory of the dominant trait z_*.

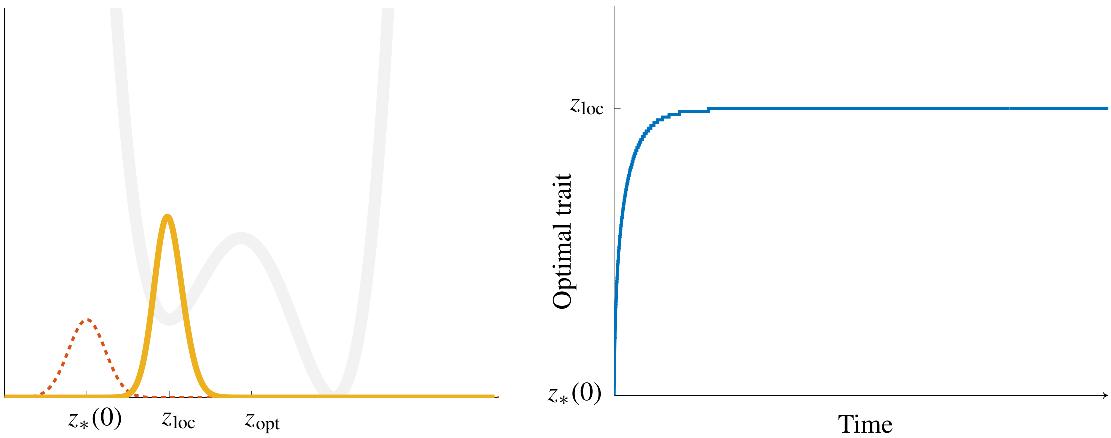

Figure 4. On the left, in dotted red, the initial data $f_\varepsilon(0,\cdot)$, and in orange, the distribution f_ε after a long time. In the background the selection function m with a global optimum z_{opt} and a local optimum z_{loc}. On the right, the trajectory of the dominant trait z_*. The function M is uniformly positive.

A case not taken into account by our methodology is when (1-13) is not satisfied at all times. This is the case if the slopes of the lines between local and global minima are too sharp. For instance, this is true in the case of Figure 5. Interestingly, what is observed is a critical behavior. The solution will first concentrate around the first local minimum before jumping sharply in the attraction basin of the global minimum; see the right-hand picture of Figure 5.

Under this model it would seem that the population will concentrate around the global minimum of selection if it is much better than the other selective optima. Interestingly, the value of the local maximum in between the two minima, that could act as an obstacle between the two convex selection valleys, do not

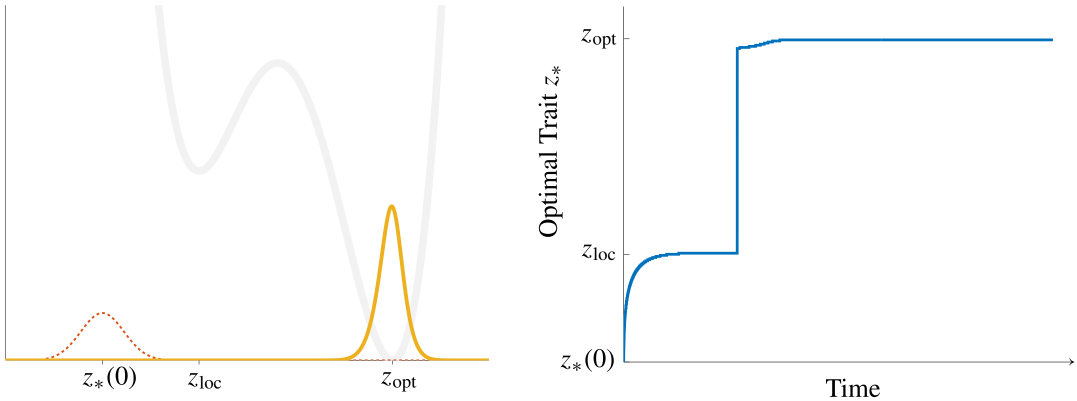

Figure 5. On the left, in dotted red, the initial data $f_\varepsilon(0,\cdot)$, and in orange, the distribution f_ε after a long time. In the background the selection function m with a global optimum z_{opt} and a local optimum z_{loc}. On the right, the trajectory of the dominant trait z_*. The function M is *not* uniformly positive.

appear to play a role. On the other hand, if the global minimum is not much better than a local minima, in the sense that each of them falls under the regime of (1-13), the population can concentrate around this local minimum.

Influence of the sign of q_ε. We introduced the scalar q_ε in (1-17) as part of the decomposition of U_ε between the affine parts and the rest of the function, which we later justified by heuristics on the linearized problem; see the table on page 1297. We can propose a different interpretation of this scalar, related to the Gaussian distribution.

The logarithmic transform (1-2) coupled with the decomposition (1-17) can be rewritten as the following transform on the solution of $(P_t f_\varepsilon)$:

$$f_\varepsilon(t,z)=\frac{1}{\varepsilon\sqrt{2\pi}}\exp\left[\frac{\lambda(t)-\varepsilon^2p_\varepsilon(t)+\varepsilon^4q_\varepsilon(t)^2}{\varepsilon^2}-\frac{(z-(z_*(t)-\varepsilon^2q_\varepsilon(t)))^2}{2\varepsilon^2}-V_\varepsilon(t,z)\right]. \tag{9-1}$$

Therefore one can see that q_ε is the correction to the mean of the Gaussian distribution at the next order in ε. Its sign corresponds to the sign of the error made on the mean of the Gaussian distribution. If q_ε is positive, the correction of z_* lies on its left. This is consistent with the following reasoning on the limit value $q^*=\lim_{\varepsilon\to 0}q_\varepsilon$, defined in (1-18). For clarity, suppose that z_* does not depend on time, that is the regime of the stationary case. Then from (1-18), we find an explicit value for q^*, which coincides with [Calvez et al. 2019, (3.2)]:

$$q^*=\frac{m^{(3)}(z_*)}{2m''(z_*)}.$$

By local convexity of m around z_*, see (1-12), the sign of q^* is the same as the sign of $m^{(3)}(z_*)$. Therefore, if this scalar is positive, selection leans the profile towards the left, which has better selective values than the right, since it is flatter. Therefore, we recover what we deduced from (9-1); the sign of q_ε is linked to the skewness of the selection function m around z_*.

List of symbols

U_ε	Perturbation to solve the Cauchy problem $(P_t f_\varepsilon)$	(1-2)
m	Selection function	Assumption 1.1
I_ε	Residual shape of the infinitesimal operator	(1-3)
$z_*(t)$	Dominant trait in the population	(1-11)
$\bar{z}(t)$	$\frac{1}{2}(z+z_*(t))$	$(P_t U_\varepsilon)$
M	$1+m(z)-m(z_*(t))-m'(z_*(t))(z-z_*(t))$	$(P_t U_0)$
$p_\varepsilon, q_\varepsilon, V_\varepsilon$	$U_\varepsilon(t,z)=p_\varepsilon(t)+q_\varepsilon(t)(z-z_*(t))+V_\varepsilon(t,z)$	(1-17)
$\mathcal{I}_\varepsilon$	I_ε, but as a function of *two* variables: $\mathcal{I}_\varepsilon(q_\varepsilon, V_\varepsilon)=I_\varepsilon(U_\varepsilon)$	$(P_t V_\varepsilon)$
p^*, q^*, V^*	Limits of our unknowns when $\varepsilon\to 0$	(1-19), (1-18), (1-9)
U^*	Limit of U_ε, $U^*(t,z)=p^*(t)+q^*(t)(z-z_*(t))+V^*(t,z)$	(1-20)
R_ε	Decomposition $U_\varepsilon=U^*+\varepsilon^2R_\varepsilon$	Section 2
$(\kappa_\varepsilon, W_\varepsilon)$	$q_\varepsilon=q^*+\varepsilon^2\kappa_\varepsilon$, $V_\varepsilon=V^*+\varepsilon^2 W_\varepsilon$ perturbations	(1-21), (1-22)
$\mathcal{E}, \lVert\cdot\rVert_{\mathcal{E}}$	Functional space to measure V_ε, and its norm	Definition 1.2
$\mathcal{F}, \lVert\cdot\rVert_{\mathcal{F}}$	Functional space to measure W_ε, and its norm	Definition 1.2
φ_α	Weight function $\varphi_\alpha(t,z)=(1+\lvert z-z_*(t)\rvert)^\alpha$	Definition 1.2
$\lVert(g,W)\rVert$	$\max(\lvert g\rvert, \lVert W\rVert_{\mathcal{F}})$ or $\max(\lvert g\rvert, \lVert W\rVert_{\mathcal{E}})$	Definition 2.1
$\mathcal{E}^*, \lVert\cdot\rVert_*$	Functional space to measure V^*, and its norm	Definition 3.1
K^*	Uniform bound of V^* in $\mathcal{E}^*$	Proposition 3.3
$O^*(\varepsilon)$	Special negligible term $O(\varepsilon)$	Definition 2.1
$\mathcal{I}_\varepsilon^*$	$\mathcal{I}_\varepsilon(q^*, V^*)$	Proposition 3.4
$\partial_g\mathcal{I}_\varepsilon$, $\partial_V\mathcal{I}_\varepsilon$	Fréchet derivatives w.r.t. the first and second variables	Section 4
$Q(y_1,y_2)$	Quadratic form: $\frac{1}{2}y_1y_2+\frac{3}{4}(y_1^2+y_2^2)$	(1-3)
Y	Refers to the pair of traits (y_1,y_2)	(3-4)
$\mathcal{D}_\varepsilon(V)(Y,t,z)$	$V(t,\bar{z})-\frac{1}{2}V(t,\bar{z}+\varepsilon y_1)-\frac{1}{2}V(t,\bar{z}+\varepsilon y_2)$	(3-4)
$\mathcal{D}_\varepsilon^*(V)(y,t)$	$V(t,z_*)-V(t,z_*+\varepsilon y)$	(3-4)
$\lVert W\rVert_\infty$	$\sup_{(t,z)\in\mathbb{R}_+\times\mathbb{R}}\lvert W(t,z)\rvert$	
$dG_\varepsilon^*, dN_\varepsilon^*$	Probability densities	Section 4C, (4-26), (4-27)
$\Xi_\varepsilon(t,z)$	Notation shortcut for $W_\varepsilon(t,z)-2W_\varepsilon(t,\bar{z})$	(5-1)
$O_0^*(1)$	Dependency upon the initial data at $t=0$	Theorem 7.1
B_0	"Small" ball around z_*	Figure 2, Section 7A, Proposition 7.3
r_*, T_*	Parameters in the definition of B_0	Section 7A, (7-7)
B_n, D_n	Successive balls and dyadic rings	Section 7: (7-2), (7-3), Figure 2
$(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$	Controls of M on the rings D_n	(7-4)
$\lVert W\rVert_\infty^n$	L^∞ norm on $\mathbb{R}_+\times B_n$, for $n\geq 0$	(7-5)

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