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JUAN CARLOS CANTERO, JOAN MATEU,
JOAN OROBITG AND JOAN VERDERA

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We prove the persistence of boundary smoothness of vortex patches for a nonlinear transport equation in \mathbb{R}^n with velocity field given by convolution of the density with an odd kernel, homogeneous of degree $-(n-1)$ and of class $C^2(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$. This allows the velocity field to have nontrivial divergence. The quasigeostrophic equation in \mathbb{R}^3 and the Cauchy transport equation in the plane are examples.

1. Introduction

The vorticity form of the Euler equation in the plane is

$$\begin{aligned}\partial_t \omega(x, t) + v(x, t) \cdot \nabla \omega(x, t) &= 0, \\ v(x, t) &= (\nabla^\perp N * \omega(\cdot, t))(x), \\ \omega(x, 0) &= \omega_0(x),\end{aligned}\tag{1}$$

where $x \in \mathbb{R}^2$, $t \in \mathbb{R}$, $N = \frac{1}{2\pi} \log |x|$ is the fundamental solution of the laplacian in the plane, $\nabla^\perp N$ is a rotation of ∇N of 90° in the counterclockwise direction and ω_0 is the initial vorticity. A deep result of Yudovich [1963] asserts that the vorticity equation is well-posed in L_c^∞ , the measurable bounded functions with compact support. A vortex patch is the special weak solution of (1) when the initial condition is the characteristic function of a bounded domain D_0 . Since the vorticity equation is a transport equation, vorticity is conserved along trajectories and thus $\omega(x, t) = \chi_{D_t}(x)$ for some domain D_t . A challenging problem, raised in the eighties, was to show that boundary smoothness persists for all times. Specifically, if D_0 has boundary of class $C^{1+\gamma}$, $0 < \gamma < 1$, then one would like D_t to have boundary of the same class for all times. This was viewed as a 2-dimensional problem which featured some of the main difficulties of the regularity problem for the Euler equation in \mathbb{R}^3 . It was conjectured, on the basis of numerical simulations, that the boundary of D_t could become of infinite length in finite time [Majda 1986]. Chemin [1993] proved that boundary regularity persists for all times using paradifferential calculus, and Bertozzi and Constantin [1993] found shortly after a minimal beautiful proof based on methods of classical analysis with a geometric flavor.

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The vortex patch problem was considered for the aggregation equation with newtonian kernel in higher dimensions in [Bertozzi et al. 2016]. The equation is

$$\begin{aligned} \partial_t \rho(x, t) + \operatorname{div}(\rho(x, t)v(x, t)) &= 0, \\ v(x, t) &= -(\nabla N * \rho(\cdot, t))(x), \\ \rho(x, 0) &= \rho_0(x), \end{aligned} \tag{2}$$

$x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. In [Bertozzi et al. 2012] a well-posedness theory in L_c^∞ was developed, following the path of [Yudovich 1963; 2002, Theorem 8.1]. When the initial condition is the characteristic function of a bounded domain, one calls the unique weak solution a vortex patch, as for the vorticity equation. One proves in [Bertozzi et al. 2016] that if the boundary of D_0 is of class $C^{1+\gamma}$, $0 < \gamma < 1$, then the solution of (2) with initial condition $\rho_0 = \chi_{D_0}$ is of the form

$$\rho(x, t) = \frac{1}{1-t} \chi_{D_t}(x), \quad x \in \mathbb{R}^n, \quad 0 \leq t < 1,$$

where D_t is a $C^{1+\gamma}$ domain for all $t < 1$. The restriction to times less than 1 obeys a blow-up phenomenon studied in [Bertozzi et al. 2012]. Hence the preceding result is the analog of Chemin’s theorem for the aggregation equation. See [Bae and Kelliher 2021] for a more general result concerning striated regularity.

After a change in the time scale the aggregation equation for vortex patches becomes the nonlinear transport equation

$$\begin{aligned} \partial_t \rho(x, t) + v(x, t) \cdot \nabla \rho(x, t) &= 0, \\ v(x, t) &= -(\nabla N * \rho(\cdot, t))(x), \\ \rho(x, 0) &= \chi_{D_0}(x), \end{aligned} \tag{3}$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}$, where N is the fundamental solution of the laplacian in \mathbb{R}^n and D_0 is a bounded domain. In this formulation one proves in [Bertozzi et al. 2016] that if D_0 is of class $C^{1+\gamma}$, then there is a solution of (3) of the form $\chi_{D_t}(x)$ with D_t a domain of class $C^{1+\gamma}$. To the best of our knowledge there is no well-posedness theory in L_c^∞ for (3), for a general initial condition in L_c^∞ . However, if the initial condition is the characteristic function of a domain D_0 , not necessarily smooth, one has existence and uniqueness for the transport equation (3). For existence, solve the equation (2) with initial condition $\rho_0(x) = \chi_{D_0}(x)$. Then the unique solution has the form $\rho(x, t) = (1/(1-t))\chi_{D_t}(x)$ and hence, after changing the time scale as in [Bertozzi et al. 2012], one obtains a solution for (3) which is a vortex patch. For uniqueness, we resort to an argument which combines results of [Clop et al. 2016a; 2016b] to prove that each weak solution of (3) in L_c^∞ is lagrangian and so a vortex patch. Changing the time scale, one obtains a weak solution of (2), which is unique.

The proof follows the scheme of [Bertozzi and Constantin 1993] and overcomes difficulties related to the fact that the velocity field has a nonzero divergence and to the higher-dimensional context. The reader can consult [Bertozzi et al. 2016] for connections with the existing literature and for references to models leading to various aggregation equations.

This paper originated from an attempt to deeply understand the role of the kernel that gives the velocity field. For the aggregation equation the kernel is $-\nabla N$ and for the vorticity equation in the plane the kernel is a rotation of 90° of ∇N . These are odd kernels, smooth off the origin and homogeneous of

degree $-(n - 1)$. We wondered what would happen for the Cauchy kernel

$$\frac{1}{2\pi z} = L(\nabla N), \quad \text{with } L(x, y) = (x, -y), \quad z = (x, y) \in \mathbb{R}^2 = \mathbb{C}.$$

Although apparently there is no model leading to the nonlinear transport equation given by the Cauchy kernel, from the mathematical perspective the question makes sense. We then embarked in the study of the nonlinear transport equation

$$\begin{aligned} \partial_t \rho(z, t) + v(z, t) \cdot \nabla \rho(z, t) &= 0, \\ v(z, t) &= \left(\frac{1}{2\pi z} * \rho(\cdot, t) \right)(z), \\ \rho(z, 0) &= \chi_{D_0}(z), \end{aligned} \tag{4}$$

where $z = (x, y)$ is the complex variable and D_0 is a bounded domain with $C^{1+\gamma}$ boundary, $0 < \gamma < 1$. A first remark is that apparently there does not exist a well-posedness theory in L_c^∞ for the equation above, but this does not prevent the study of smooth vortex patches, as a particular subclass of L_c^∞ enjoying a bit of smoothness.

To grasp what could be expected we looked at an initial datum which is the characteristic function of the domain enclosed by an ellipse

$$D_0 = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right\}.$$

We proved that there exists a weak solution of (4) of the form $\rho(z, t) = \chi_{D_t}(z)$, with D_t the domain enclosed by an ellipse with semiaxes $a(t)$ and $b(t)$ collapsing to a segment on the horizontal axis as $t \rightarrow \infty$.

A key remark is that (4) is not rotation invariant. Fix an angle $0 < \theta < \frac{\pi}{2}$ and consider as initial domain the set enclosed by a tilted ellipse

$$D_0 = e^{i\theta} \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right\}.$$

As before we find a weak solution of (4) of the form $\rho(z, t) = \chi_{D_t}(z)$, with D_t the domain enclosed by an ellipse with semiaxes $a(t)$ and $b(t)$ forming an angle $\theta(t)$ with the horizontal axis. The evolution is different according to whether $0 < \theta \leq \frac{\pi}{4}$ or $\frac{\pi}{4} < \theta < \frac{\pi}{2}$. Under the assumption that $a_0 > b_0$, in the case $0 < \theta \leq \frac{\pi}{4}$ the semiaxis $a(t)$ increases as $t \rightarrow \infty$ to a positive number a_∞ , $b(t)$ decreases to 0 and $\theta(t)$ decreases to a positive angle θ_∞ . Hence D_t collapses into an interval on a line forming a positive angle with the horizontal axis. If $\frac{\pi}{4} < \theta < \frac{\pi}{2}$, then for small times $a(t)$ decreases and $b(t)$ increases, so that the ellipse at time t tends initially to become a circle. This happens until a critical time is reached after which $a(t)$ increases and $b(t)$ decreases. The angle $\theta(t)$ decreases for all positive times and at some point it becomes $\frac{\pi}{4}$; after that one falls into the regime of the first case and the domain D_t collapses as $t \rightarrow \infty$, into a segment on a line which forms a positive angle with the horizontal axis. The case $a_0 < b_0$ is similar and can be reduced to the previous situation by conjugation (symmetry with respect to the horizontal axis).

Detailed proofs of the results just described can be found in Section 7. What they show is that the behavior of vortex patches for the Cauchy transport equation can be much more complicated than for the vorticity or aggregation equations. This is also easily understood if one looks at the divergence of the vector field in (4). If ∂ and $\bar{\partial}$ denote respectively the derivatives with respect to the z - and \bar{z} -variables, then we get

$$2 \bar{\partial} v(z, t) = \rho(z, t)$$

and

$$2 \operatorname{div} v(z, t) = -\frac{1}{\pi} \text{p.v.} \int \frac{1}{(z-w)^2} \rho(w, t) dA(w) = \mathbf{B}(\rho(\cdot, t))(z),$$

where \mathbf{B} is the Beurling transform, one of the basic Calderón–Zygmund operators in the plane. Here dA is 2-dimensional Lebesgue measure. The divergence of v is given by

$$\begin{aligned} \operatorname{div} v &= \Re(2 \operatorname{div} v) = -\text{p.v.} \frac{1}{\pi} \int \Re\left(\frac{1}{(z-w)^2}\right) \rho(w, t) dA(w) \\ &= -\text{p.v.} \frac{1}{\pi} \left(\frac{x^2 - y^2}{|z|^4} \star \rho(\cdot, t)\right)(z). \end{aligned}$$

The last convolution is a Calderón–Zygmund operator (a second-order Riesz transform) and so it does not map bounded functions into bounded functions. The most one can say a priori on the divergence of the velocity field is that it is a BMO function in the plane, provided the density $\rho(\cdot, t)$ is a bounded function. It is a well-known fact, already used in [Bertozzi and Constantin 1993; Chemin 1993], that if D is a domain with boundary of class $C^{1+\gamma}$, then an even Calderón–Zygmund operator applied to χ_D is a bounded function. Thus we indeed expect $\operatorname{div} v$ to be bounded. Nevertheless, the expression of the divergence of the field in terms of a Calderón–Zygmund operator applied to the density is potentially difficult to handle.

We have succeeded in proving that there exists a weak solution of (4) of the form χ_{D_t} with D_t a domain with boundary of class $C^{1+\gamma}$ for all times $t \in \mathbb{R}$. This weak solution is unique in the class of characteristic functions of $C^{1+\gamma}$ domains.

The Cauchy kernel belongs to a wider class for which the preceding well-posedness theorem holds. We refer to the class of kernels in \mathbb{R}^n which are odd, homogeneous of degree $-(n-1)$ and of class $C^2(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$. Interesting examples of such kernels are those of the form $L(\nabla N)$, where L is a linear mapping from \mathbb{R}^n into itself and N is the fundamental solution of the laplacian in \mathbb{R}^n . They are harmonic off the origin. In particular in \mathbb{R}^3 one can take $L(x_1, x_2, x_3) = (-x_2, x_1, 0)$. The corresponding field is divergence-free and the associated equation is the well-known quasigeostrophic equation. See [García et al. 2022] for recent results on rotating vortex patches for the quasigeostrophic equation.

Our main result is the following.

Theorem. *Let $k : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n$ be an odd function, homogeneous of degree $-(n-1)$ and of class $C^2(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$. Let D_0 be a bounded domain with boundary of class $C^{1+\gamma}$, $0 < \gamma < 1$. Then the nonlinear transport equation*

$$\begin{aligned} \partial_t \rho(x, t) + v(x, t) \cdot \nabla \rho(x, t) &= 0, \\ v(x, t) &= (k \star \rho(\cdot, t))(x), \\ \rho(x, 0) &= \chi_{D_0}(x), \end{aligned} \tag{5}$$

$x \in \mathbb{R}^n, t \in \mathbb{R}$, has a weak solution of the form

$$\rho(x, t) = \chi_{D_t}(x), \quad x \in \mathbb{R}^n, t \in \mathbb{R},$$

with D_t a bounded domain with boundary of class $C^{1+\gamma}$.

This solution is unique in the class of characteristic functions of domains with boundary of class $C^{1+\gamma}$.

For the notion of weak solution see [Majda and Bertozzi 2002, Chapter 8].

A remark on the special case in which the kernel k is divergence-free is in order. In this case, in particular for the quasigeostrophic equation, one has well-posedness in L_c^∞ . Existence can be proved following closely the argument in [Majda and Bertozzi 2002, Chapter 8] for the vorticity equation (for the smooth case see [Cantero 2021]). For uniqueness one resorts to [Nieto et al. 2001] whenever the kernel has the special form $L(\nabla N)$ with L a linear map from \mathbb{R}^n into itself. Indeed, in that work uniqueness in L_c^∞ is proven for the continuity equation in higher dimensions with velocity field given by convolution with $\pm \nabla N$. The changes needed to take care of the case $L(\nabla N)$ are straightforward. If k is divergence-free and satisfies the general hypothesis stated in the theorem, then one appeals to [Crippa and Stefani 2021], where uniqueness is proved for lagrangian solutions, and to [Clop et al. 2016a; 2016b], in which one shows that a weak solution is lagrangian.

The paper is organized as follows. In the next section we present an outline of the proof, in which only a few facts are proven. The other sections are devoted to presenting complete proofs of our results. Section 3 is devoted to an auxiliary result. In Section 4 an appropriate defining function for the patch at time t is constructed. Section 5 deals with the material derivative of the gradient of the defining function and its expression in terms of differences of commutators. In Section 6 we estimate the differences of commutators in the Hölder norm on the boundary via Whitney's extension theorem. Domains enclosed by ellipses as initial patches for the Cauchy transport equation are studied in Section 7 and the unexpected phenomena that turn up along the vortex patch evolution are described in detail. Finally, there is an Appendix on the existence of principal values of singular integrals in a very special context.

Constants will be denoted by C , mostly without an explicit reference to innocuous parameters, and may be different at different occurrences. If D is a domain with smooth boundary $\sigma = \sigma_{\partial D}$ denotes the surface measure on ∂D and when there is no confusion possible we omit the subscript. The exterior unit normal vector to ∂D at the point x is denoted by $\vec{n}(x) = (n_1(x), \dots, n_n(x))$, without explicit reference to the boundary.

2. Outline of the proof

The proof follows the general scheme devised in [Bertozzi and Constantin 1993]. There are serious obstructions caused by the fact that the field is not divergence-free and we will explain below how to confront them. The reader will find useful to consult [Bertozzi and Constantin 1993; Bertozzi et al. 2016].

2.1. The contour dynamics equation. Assume that one has a weak solution of (5) of the form $\rho(x, t) = \chi_{D_t}(x)$, D_t being a bounded domain of class $C^{1+\gamma}$ for t in some interval $[0, T]$. The field $v(\cdot, t)$ is Lipschitz. This is due to the fact that our kernel has homogeneity $-(n-1)$ and so ∇v is given by a matrix whose entries are even convolution Calderón–Zygmund operators applied to the characteristic function of D_t plus, possibly, a constant multiple of such a characteristic function (coming from a delta function at the origin). Since D_t has boundary of class $C^{1+\gamma}$ all entries of the matrix ∇v are functions in $L^\infty(\mathbb{R}^n)$ [Bertozzi and Constantin 1993]. Thus the equation of particle trajectories (the flow mapping)

$$\frac{dX(\alpha, t)}{dt} = v(X(\alpha, t), t), \quad X(\alpha, 0) = \alpha, \quad (6)$$

has a unique solution and $X(\cdot, t)$ is a bilipschitz mapping of \mathbb{R}^n into itself, $0 \leq t \leq T$. Indeed one has the usual estimate

$$\|\nabla X(\cdot, t)\|_\infty \leq \exp \int_0^t \|\nabla v(\cdot, s)\|_\infty ds. \tag{7}$$

Since k is homogeneous of degree $-(n - 1)$ and smooth off the origin we have

$$k = \partial_1(x_1k) + \partial_2(x_2k) + \dots + \partial_n(x_nk), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{0\}. \tag{8}$$

This follows straightforwardly from Euler’s theorem on homogeneous functions.

Assume that $\rho(x, t) = \chi_{D_t}(x)$ is a weak solution of the general equation (5). The velocity field is

$$\begin{aligned} v(\cdot, t) &= \chi_{D_t} \star k = \chi_{D_t} \star (\partial_1(x_1k) + \dots + \partial_n(x_nk)) \\ &= \partial_1 \chi_{D_t} \star (x_1k) + \dots + \partial_n \chi_{D_t} \star (x_nk) \\ &= -n_1 d\sigma_{\partial D_t} \star (x_1k) - \dots - n_n d\sigma_{\partial D_t} \star (x_nk). \end{aligned}$$

Thus

$$\begin{aligned} v(x, t) &= - \sum_{j=1}^n \int_{\partial D_t} (x_j - y_j)k(x - y)n_j(y) d\sigma_{\partial D_t}(y) \\ &= - \int_{\partial D_t} k(x - y)\langle x - y, \vec{n}(y) \rangle d\sigma_{\partial D_t}(y), \quad x \in \mathbb{R}^n. \end{aligned} \tag{9}$$

The next step is to set $x = X(\alpha, t)$ and to make the change of variables $y = X(\beta, t)$ in the preceding surface integral. To do this conveniently let $T_1(\beta), \dots, T_{n-1}(\beta)$ be an orthonormal basis of the tangent space to ∂D_0 at the point $\beta \in \partial D_0$ and let $DX(\cdot, t)$ be the differential of $X(\cdot, t)$ as a differentiable mapping from ∂D_0 into \mathbb{R}^n . The vectors $DX(\beta, t)(T_j(\beta))$ are tangent to ∂D_t at the point $X(\beta, t)$ for $1 \leq j \leq n - 1$. Hence the vector

$$\bigwedge_{j=1}^{n-1} DX(\beta, t)(T_j(\beta)) \tag{10}$$

is orthogonal to ∂D_t at the point $X(\beta, t)$ and a different choice of the orthonormal basis $T_j(\beta)$, $1 \leq j \leq n - 1$, has the effect of introducing a \pm sign in front of (10). We may choose the $T_j(\beta)$ so that $\vec{n}(\beta), T_1(\beta), \dots, T_{n-1}(\beta)$ gives the standard orientation of \mathbb{R}^n . Substituting the expression (9) for the velocity field in (6) and making the change of variables $y = X(\beta, t)$ we get

$$\begin{aligned} \frac{d}{dt} X(\alpha, t) &= v(X(\alpha, t), t) \\ &= - \int_{\partial D_0} k(X(\alpha, t) - X(\beta, t)) \left\langle X(\alpha, t) - X(\beta, t), \bigwedge_{j=1}^{n-1} DX(\beta, t)(T_j(\beta)) \right\rangle d\sigma_{\partial D_0}(\beta). \end{aligned}$$

Let $X : \partial D_0 \rightarrow \mathbb{R}^n$ be a mapping of class $C^{1+\gamma}$ such that for some constant $\mu > 0$

$$|X(\alpha) - X(\beta)| \geq \frac{1}{\mu} |\alpha - \beta|, \quad \alpha, \beta \in \partial D_0. \tag{11}$$

In other words $X \in C^{1+\gamma}(\partial D_0, \mathbb{R}^n)$, X is bilipschitz onto the image and μ is a Lipschitz constant for the inverse mapping.

Define a mapping $F(X) : \partial D_0 \rightarrow \mathbb{R}^n$ by

$$F(X)(\alpha) = - \int_{\partial D_0} k(X(\alpha) - X(\beta)) \left\langle X(\alpha) - X(\beta), \bigwedge_{j=1}^{n-1} DX(\beta)(T_j(\beta)) \right\rangle d\sigma_{\partial D_0}(\beta). \tag{12}$$

The contour dynamics equation (CDE) is

$$\begin{aligned} \frac{dX(\alpha, t)}{dt} &= F(X(\cdot, t))(\alpha), \quad \alpha \in \partial D_0, \\ X(\cdot, 0) &= I, \end{aligned}$$

where I denotes the identity mapping on ∂D_0 .

We conclude that if there exists a weak solution of the type we are looking for, then the flow restricted to ∂D_0 is a solution of the CDE.

To proceed in the reverse direction, we need some preparation. Let Ω be the open set in the Banach space $C^{1+\gamma}(\partial D_0, \mathbb{R}^n)$ consisting of those $X \in C^{1+\gamma}(\partial D_0, \mathbb{R}^n)$ satisfying (11) for some $\mu > 0$. The set Ω is open in $C^{1+\gamma}(\partial D_0, \mathbb{R}^n)$ and the CDE can be thought of as an ODE in the open set Ω . We want to show that a solution $X(\cdot, t)$ to the CDE in an interval $(-T, T)$ provides a weak solution of the nonlinear transport equation (5). Clearly $X(\cdot, t)$ maps ∂D_0 onto a $(n-1)$ -dimensional hypersurface S_t . The goal now is to identify an open set D_t with boundary S_t . If we add the hypothesis that ∂D_0 is connected, and hence a connected $(n-1)$ -dimensional hypersurface of class $C^{1+\gamma}$, then the analog of the Jordan curve theorem holds [Guillemin and Pollack 1974, p. 89]. Then the complement of ∂D_0 in \mathbb{R}^n has only one bounded connected component which is D_0 . In the same vein, the complement of S_t has only one bounded connected component, which we denote by D_t , so that the boundary of D_t is S_t . The definition of D_t is less direct if we drop the assumption that ∂D_0 is connected. We proceed as follows. Let S_t^j , $1 \leq j \leq m$, be the connected components of S_t . Denote by U_t^j the bounded connected component of the complement of S_t^j in \mathbb{R}^n . Among the U_t^j there is one, say U_t^1 , that contains all the others. This is so at time $t = 0$ because D_0 is connected and this property is preserved by the flow $X(\cdot, t)$. We set $D_t = U_t^1 \setminus (\bigcup_{j=2}^m \bar{U}_t^j)$, so that the boundary of D_t is S_t .

Indeed, as the reader may have noticed, it is not necessary to assume that D_0 is connected in our theorem. It can be any bounded open set with $C^{1+\gamma}$ boundary. Then the argument we have just described is applied to each connected component.

Define a velocity field by

$$v(x, t) = (k \star \chi_{D_t})(x), \quad x \in \mathbb{R}^n, \quad t \in (-T, T). \tag{13}$$

Since D_t has boundary of class $C^{1+\gamma}$, the field $v(\cdot, t)$ is Lipschitz for each $t \in (-T, T)$ and the equation of the flow (6) has a unique solution which is a bilipschitz mapping of \mathbb{R}^n onto itself whose restriction to ∂D_0 is the solution of the CDE we were given. Thus $X(D_0, t) = D_t$ and χ_{D_t} is a weak solution of the nonlinear transport equation (5).

2.2. The local theorem. As a first step we solve the CDE locally in time. For this we look at the CDE as an ODE in the open set Ω of the Banach space $C^{1+\gamma}(\partial D_0, \mathbb{R}^n)$. To show local existence and uniqueness we apply the Picard theorem. First one has to check that $F(X) \in C^{1+\gamma}(\partial D_0, \mathbb{R}^n)$ for each $X \in \Omega$. After

taking a derivative in α in (12) one gets a p.v. integral on ∂D_0 , which defines a Calderón–Zygmund operator (not of convolution type) with respect to the underlying measure $d\sigma_{\partial D_0}$, acting on a function satisfying a Hölder condition of order γ . The result is again a Hölder function of the same order, since one shows that Calderón–Zygmund operators of the type one gets preserve Hölder spaces. In a second step one needs to prove that $F(X)$ is locally a Lipschitz function of the variable X or, equivalently, that the differential $DF(X)$ of F at the point $X \in \Omega$ is locally bounded in X . Again one has to estimate operators of Calderón–Zygmund type with respect Hölder spaces of order γ . These estimates, subtle at some points, are proved in full detail in [Bertozzi et al. 2016] for the kernel $k = -\nabla N$. The variations needed to cover the present situation are minor and are left to the reader. It is important that, as in [Bertozzi et al. 2016], the time interval on which the local solution exists depends continuously only on the dimension n , the kernel k , the diameter of D_0 , the $(n-1)$ -dimensional surface measure of ∂D_0 and the constant $q(D_0)$ determining the $C^{1+\gamma}$ character of ∂D_0 , whose definition we discuss below.

Let D be a bounded domain with boundary of class $C^{1+\gamma}$. Then there exists a defining function of class $C^{1+\gamma}$, that is, a function $\varphi \in C^{1+\gamma}(\mathbb{R}^n)$, such that $D = \{x \in \mathbb{R}^n : \varphi(x) < 0\}$ and $\nabla\varphi(x) \neq 0$ if $\varphi(x) = 0$. We set

$$q(D) = \inf \left\{ \frac{\|\nabla\varphi\|_{\gamma, \partial D}}{|\nabla\varphi|_{\inf}} : \varphi \text{ a defining function of } D \text{ of class } C^{1+\gamma} \right\}, \tag{14}$$

where $|\nabla\varphi(x)| = \sqrt{\sum_{j=1}^n \partial_j\varphi(x)^2}$,

$$\begin{aligned} \|\nabla\varphi\|_{\gamma, \partial D} &= \sup \left\{ \frac{|\nabla\varphi(x) - \nabla\varphi(y)|}{|x - y|^\gamma} : x, y \in \partial D, x \neq y \right\}, \\ |\nabla\varphi|_{\inf} &= \inf\{|\nabla\varphi(x)| : \varphi(x) = 0\}. \end{aligned}$$

There is here an important variation with respect to [Bertozzi and Constantin 1993; Bertozzi et al. 2016]: the Hölder seminorm of order γ of $\nabla\varphi$ is taken in those papers in the whole of \mathbb{R}^n . For reasons that will become clear later on we need to restrict our attention to the boundary of D and this requires finer estimates.

2.3. Global existence: a priori estimates. Assume that the maximal time of existence for the solution $X(\cdot, t)$ of the CDE is T . By this we mean that $X(\cdot, t)$ is defined for $t \in (-T, T)$ but cannot be extended to a larger interval. We want to prove that $T = \infty$. For that it suffices to prove that for some constant $C = C(T)$ one has

$$\text{diam}(D_t) + \sigma(\partial D_t) + q(D_t) \leq C, \quad t \in (-T, T). \tag{15}$$

If the preceding inequality holds, then we take $t_0 < T$ close enough to T so that after the application of the existence and uniqueness theorem for the CDE to the domain D_{t_0} at time t_0 we get an interval of existence for the solution which goes beyond T (the same argument applies to the lower extreme $-T$).

To obtain (15) we look for a priori estimates in terms of $\|\nabla v\|_\infty$. For $\text{diam}(D_t)$ and $\sigma(\partial D_t)$ this is straightforward in view of (7). The core of the paper is the a priori estimate of $q(D_t)$, which we get by

constructing an appropriate defining function $\Phi(\cdot, t)$ for D_t satisfying

$$|\nabla\Phi(\cdot, t)|_{\text{inf}} \geq |\nabla\varphi_0|_{\text{inf}} \exp\left(-C_n \int_0^t \|\nabla v(\cdot, s)\|_\infty ds\right), \quad t > 0, \tag{16}$$

$$\|\nabla\Phi(\cdot, t)\|_{\gamma, \partial D_t} \leq \|\nabla\varphi_0\|_{\gamma, \partial D_0} \exp\left(C_n \int_0^t (1 + \|\nabla v(\cdot, s)\|_\infty) ds\right), \quad t > 0. \tag{17}$$

As it was pointed out in [Bertozzi et al. 2016] if one transports a defining function φ_0 of D_0 by $\varphi_t = \varphi_0 \circ X^{-1}(\cdot, t)$, then $\nabla\varphi_t$ may have jumps at the boundary of D_t for $t \neq 0$ and so φ_t is not necessarily differentiable. In [Bertozzi et al. 2016] one shows that

$$\lim_{D_t \ni y \rightarrow x} \nabla\varphi(y, t) = \lim_{D_t \ni y \rightarrow x} \det \nabla X^{-1}(y, t) \frac{|\nabla\varphi_0(X^{-1}(x, t))|}{\det D(x)} \vec{n}(x), \quad x \in \partial D_t, \tag{18}$$

$$\lim_{\mathbb{R}^n \setminus \bar{D}_t \ni y \rightarrow x} \nabla\varphi(y, t) = \lim_{\mathbb{R}^n \setminus \bar{D}_t \ni y \rightarrow x} \det \nabla X^{-1}(y, t) \frac{|\nabla\varphi_0(X^{-1}(x, t))|}{\det D(x)} \vec{n}(x), \quad x \in \partial D_t, \tag{19}$$

where $X^{-1}(\cdot, t)$ is the inverse mapping of $X(\cdot, t)$ and $D(x)$ is the differential at x of the restriction of $X^{-1}(\cdot, t)$ to ∂D_t , as a differentiable mapping from ∂D_t onto ∂D_0 . Define

$$\Phi(x, t) = \begin{cases} 0, & x \in \partial D_t, \\ \det \nabla X(X^{-1}(x, t), t) \varphi(x, t), & x \notin \partial D_t. \end{cases} \tag{20}$$

We show in Section 4 that $\Phi(x, t)$ is a defining function of D_t of class $C^{1+\gamma}$.

The definition of Φ yields a formula for its material derivative $D/(Dt) = \partial_t + v \cdot \nabla$, namely,

$$\frac{D\Phi}{Dt} = \text{div}(v) \Phi. \tag{21}$$

Taking gradient in the preceding identity one gets

$$\frac{D(\nabla\Phi)}{Dt} = \nabla(\text{div}(v)) \Phi + \text{div}(v) \nabla\Phi - (\nabla v)^t (\nabla\Phi), \tag{22}$$

where $(\nabla v)^t$ stands for the transpose of the matrix ∇v . The right-hand side of (22) can be split into two terms which behave differently. The first is $\nabla(\text{div}(v)) \Phi$ and the second $\text{div}(v) \nabla\Phi - (\nabla v)^t (\nabla\Phi)$. We prove that the second term is a finite sum of differences of commutators, which can be shown, with some effort, to have the right estimates. The first term does not combine with others to yield a commutator and because of that we call it the solitary term. A priori it is the most singular term on the right-hand side of (22), since it contains second-order derivatives of v . We show that the solitary term extends continuously to ∂D_t by 0 and so it can be ignored at the price of working only on the boundary of D_t for all t .

To prove that the solitary term extends continuously to the boundary by 0 we need a recent result of [Vasin 2017] whose statement is as follows. Let T be a convolution homogeneous even Calderón–Zygmund operator of the type

$$T(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} L(x - y) f(y) dy = \lim_{\epsilon \rightarrow 0} \int_{|y-x|>\epsilon} L(x - y) f(y) dy, \tag{23}$$

where L is an even kernel, homogeneous of degree $-n$, satisfying the smoothness condition $L \in C^1(\mathbb{R}^n \setminus \{0\})$ and the cancellation property $\int_{|x|=1} L(x) d\sigma(x) = 0$. The function f is in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and the principal value integral (23) is defined a.e. on \mathbb{R}^n . Vasin’s result states that if D is a bounded domain with boundary of class $C^{1+\gamma}$ then

$$|\nabla T(\chi_D)(x)| \operatorname{dist}(x, \partial D)^{1-\gamma} \leq C, \quad x \in D \cup (\mathbb{R}^n \setminus \bar{D}), \tag{24}$$

where the constant C depends only on n, γ and the constants giving the smoothness of ∂D . We provide a proof of (24) in Section 3 for completeness.

One applies (24) to the second derivatives of the velocity field $v = k \star \chi_{D_t}$ with t fixed. One has in the distributions sense

$$\partial_j k = \text{p.v. } \partial_j k + \vec{c}_j \delta_0, \quad \text{with } \vec{c}_j = \int_{|\xi|=1} k(\xi) \xi_j d\sigma(\xi), \tag{25}$$

and so

$$(\partial_j v)(x) = (\text{p.v. } \partial_j k \star \chi_{D_t})(x) + \vec{c}_j \chi_{D_t}(x), \quad x \in D_t \cup (\mathbb{R}^n \setminus \bar{D}_t),$$

and, taking a second derivative,

$$(\partial_l \partial_j v)(x) = \partial_l (\text{p.v. } \partial_j k \star \chi_{D_t})(x), \quad x \in D_t \cup (\mathbb{R}^n \setminus \bar{D}_t), \quad 1 \leq l, j \leq n. \tag{26}$$

By (24) applied to the operator T associated with the kernel $L = \partial_j k$

$$|(\partial_l \partial_j v)(x)| \operatorname{dist}(x, \partial D_t)^{1-\gamma} \leq C(t), \quad x \in D_t \cup (\mathbb{R}^n \setminus \bar{D}_t), \quad 1 \leq l, j \leq n, \tag{27}$$

where $C(t)$ depends on n, γ and the constants related to the smoothness of ∂D_t . This implies that the solitary term has limit 0 at the boundary of ∂D_t , coming from the complement, because $|\Phi(x, t)|$ is comparable to $\operatorname{dist}(x, \partial D_t)$ as x approaches ∂D_t ($\Phi(\cdot, t)$ is continuously differentiable and vanishes on the boundary but the gradient does not).

It is worth remarking that if each component of the kernel k is harmonic off the origin, then (24) can be obtained readily from the fact that $T(\chi_D)$ satisfies a Hölder condition of order γ in D , which is the main lemma of [Mateu et al. 2009].

From (22) at boundary points, and thus without the solitary term, one gets straightforwardly (16). Thus the a priori estimate of $q(D_t)$ is reduced to (17).

We turn now our attention to the second term in (22). We prove that the i -th component of the vector $\operatorname{div}(v) \nabla \Phi - (\nabla v)^t (\nabla \Phi)$ evaluated at the point $x \in \mathbb{R}^n$ is a sum of $n - 1$ terms, each of which is a difference of two commutators. In fact, the i -th component is

$$\sum_{j \neq i} \text{p.v.} \int_{D_t} \partial_j k_j(x - y) (\partial_i \Phi(x) - \partial_i \Phi(y)) dy - \text{p.v.} \int_{D_t} \partial_i k_j(x - y) (\partial_j \Phi(x) - \partial_j \Phi(y)) dy. \tag{28}$$

It is crucial here that we obtain differences of commutators, which provides eventually an extra cancellation.

In [Bertozi and Constantin 1993] it was shown that the Hölder seminorm of order γ of each commutator in (28) can be estimated by $C_n (1 + \|\nabla v(\cdot, t)\|_\infty) \|\nabla \Phi(\cdot, t)\|_{\gamma, \mathbb{R}^n}$. This is not enough in our situation, because of the presence of the factor $\|\nabla \Phi(\cdot, t)\|_{\gamma, \mathbb{R}^n}$, which should be replaced by a boundary quantity like $\|\nabla \Phi(\cdot, t)\|_{\gamma, \partial D_t}$.

To obtain the correct estimate we transform the j -th term in (28) into a difference of two boundary commutators:

$$\text{p.v.} \int_{\partial D_t} k_j(x-y)(\partial_i \Phi(x) - \partial_i \Phi(y))n_j(y) d\sigma(y) - \text{p.v.} \int_{\partial D_t} k_j(x-y)(\partial_j \Phi(x) - \partial_j \Phi(y))n_i(y) d\sigma(y). \quad (29)$$

It is worth emphasizing here that it is not true that the commutator

$$\text{p.v.} \int_{D_t} \partial_j k_j(x-y)(\partial_i \Phi(x) - \partial_i \Phi(y)) dy$$

equals

$$\text{p.v.} \int_{\partial D_t} k_j(x-y)(\partial_i \Phi(x) - \partial_i \Phi(y))n_j(y) d\sigma(y).$$

What is true is that the difference of two commutators in the j -th term of (28) equals the difference of two commutators in (29). There is some magic here in arranging all terms so that certain hidden cancellation takes place. To get the right estimates on the boundary commutators one cannot adapt [Bertozzi and Constantin 1993, Lemma 7.3, p. 26] to the underlying measure $d\sigma$ on ∂D_t , because this would give a constant of the type

$$C_t = \sup_{x \in \partial D_t} \sup_{r > 0} \frac{\sigma(B(x, r))}{r^{n-1}},$$

which can be estimated by the Lipschitz constant of $X(\cdot, t)$, namely, $\exp \int_0^t \|\nabla v(\cdot, s)\|_\infty ds$. This exponential constant is by far too large.

One needs to replace the standard bound $C_n (1 + \|\nabla v(\cdot, t)\|_\infty) \|\nabla \Phi(\cdot, t)\|_{\gamma, \mathbb{R}^n}$ for a “solid” commutator of the type (28) by $C_n (1 + \|\nabla v(\cdot, t)\|_\infty) \|\nabla \Phi(\cdot, t)\|_{\gamma, \partial D_t}$. Here we have used the term solid commutator to indicate that the integration is on D_t with respect to n -dimensional Lebesgue measure as opposed to a boundary commutator in which the integration is on the boundary of ∂D_t with respect to surface measure σ . To get the estimate in terms of $\|\nabla \Phi(\cdot, t)\|_{\gamma, \partial D_t}$ we resort to the difference-of-commutators structure, which allows us to appeal to Whitney’s extension theorem, the reason being that one can switch between a difference of boundary commutators and a difference of solid commutators via the divergence theorem. The final outcome is (17).

Of course for those cases in which the kernel is divergence-free, the quasigeostrophic equation in particular, one does not need the boundary commutators and getting the commutator formula (28) suffices to complete the proof as in [Bertozzi and Constantin 1993]. Indeed in these cases the transported defining function is already a genuine defining function, since the gradient has no jump according (18) and (19), or appealing to a regularization argument, as in [Radu 2022].

To complete the proof from the a priori estimates is a standard reasoning. One needs a logarithmic inequality for $\|\nabla v(\cdot, t)\|_\infty$, which is a consequence of the boundedness of $T(\chi_D)$ for an even smooth convolution Calderón–Zygmund operator T and a domain D with boundary of class $C^{1+\gamma}$, and of the particular form of the constant. One obtains

$$\|\nabla v(\cdot, t)\|_\infty \leq \frac{C_n}{\gamma} \left(1 + \log^+ \left(|D_t|^{1/n} \frac{\|\nabla \Phi\|_{\gamma, \partial D_t}}{|\nabla \Phi|_{\inf}} \right) \right), \quad (30)$$

where C_n is a dimensional constant and $|D|$ stands for the n -dimensional Lebesgue measure of the measurable set D . The novelty in inequality (30) is that $\|\nabla\Phi\|_{\gamma,\partial D_t}$ is now replacing the larger constant $\|\nabla\Phi\|_{\gamma,\mathbb{R}^n}$ which appears in [Bertozzi and Constantin 1993; Bertozzi et al. 2016, Corollary 6.3] in dealing with the corresponding inequality. This follows from a scrutiny of the constants that appear along the proof and an application of the implicit differentiation formula.

Inserting (16) and (17) in (30) one gets, for a dimensional constant C ,

$$\|\nabla v(\cdot, t)\|_\infty \leq C + C \int_0^t (1 + \|\nabla v(\cdot, s)\|_\infty) ds,$$

which yields, by Gronwall,

$$\|\nabla v(x, t)\|_\infty \leq C e^{Ct}, \quad -T < t < T,$$

and this completes the proof of (15).

The reader may have observed that it is not strictly necessary for the proof to use the quantity $q(D_t)$, defined in (14). Nevertheless, it is the canonical quantity to take into consideration and helps to make some statements clearer. We will use it again in Section 4.

3. An auxiliary result

The result we are referring to is the following and can be found in [Vasin 2017].

Lemma. *Let $D \subset \mathbb{R}^n$ be a bounded domain with boundary of class $C^{1+\gamma}$, $0 < \gamma < 1$, and L an even kernel in $C^1(\mathbb{R}^n \setminus 0)$, homogeneous of degree $-n$. Then*

$$|\nabla(L \star \chi_D)(x)| \operatorname{dist}(x, \partial D)^{1-\gamma} \leq C, \quad x \in \mathbb{R}^n \setminus \partial D,$$

where C is a constant depending only on D .

Proof. Placing the gradient on the characteristic function of D we obtain

$$\nabla(L \star \chi_D) = L \star (-\vec{n} d\sigma_{\partial D}).$$

Fix $x \in D$ and set $d = d(x) = \operatorname{dist}(x, \partial D)$. By the divergence theorem

$$(L \star \vec{n} d\sigma_{\partial D})(x) = (L \star \vec{n} d\sigma_{\partial B(x,d)})(x) - \int_{D \setminus B(x,d)} \nabla L(x-y) dy.$$

Now

$$(L \star \vec{n} d\sigma_{\partial B(x,d)})(x) = \int_{|y-x|=d} L(x-y)\vec{n}(y) d\sigma(y) = \int_{|z|=d} L(z)\vec{n}(z) d\sigma(z)$$

and the last integral clearly vanishes, owing to the oddness of $L(z)\vec{n}(z)$. Thus

$$\nabla(L \star \chi_D)(x) = -(L \star \vec{n} d\sigma_{\partial D})(x) = \int_{D \setminus B(x,d)} \nabla L(x-y) dy,$$

and

$$\operatorname{dist}(x, \partial D) |\nabla(L \star \chi_D)(x)| = d \left| \int_{D \setminus B(x,d)} \nabla L(x-y) dy \right| \leq d \int_{D \setminus B(x,d)} \frac{C}{|y-x|^{n+1}} dy \leq C.$$

Therefore in proving the lemma one can assume that $d \leq \frac{1}{2}r_0$, where $r_0 = r_0(D)$ has the property that, given a point p in the boundary of D , $B(p, 2r_0) \cap D$ is the set of points in $B(p, 2r_0)$ lying below the graph of a $C^{1+\gamma}$ function defined on the tangent hyperplane through p .

We assume, without loss of generality, that 0 is the closest point of ∂D to x and that the tangent hyperplane to ∂D at 0 is $\{x \in \mathbb{R}^n : x_n = 0\}$. We also assume that $D \cap B(0, 2r_0) = \{x \in \mathbb{R}^n : x_n < \varphi(x')\}$, where $x' = (x_1, \dots, x_{n-1})$, $\varphi \in C^{1+\gamma}(B'(0, 2r_0))$, $B'(0, 2r_0) = \{x' \in \mathbb{R}^{n-1} : |x'| < 2r_0\}$. In particular,

$$|\varphi(x')| \leq \|\nabla\varphi\|_{\gamma, B'(0, 2r_0)} |x'|^{1+\gamma}, \quad x' \in B'(0, 2r_0).$$

We clearly have

$$\int_{D \setminus B(x, d)} \nabla L(x - y) dy = \int_{(D \setminus B(x, d)) \cap B(0, r_0)} \nabla L(x - y) dy + \int_{D \cap B^c(0, r_0)} \nabla L(x - y) dy.$$

The second term above is easy to estimate:

$$\left| \int_{D \cap B^c(0, r_0)} \nabla L(x - y) dy \right| \leq \int_{B^c(0, r_0)} \frac{dy}{|y|^{n+1}} \leq \frac{C}{r_0}.$$

For the first term one uses the fact that if H is a halfspace then $L \star \chi_H$ vanishes on H . This follows from the fact that the preceding statement is true for balls instead of halfspaces [Mateu et al. 2009] and a straightforward limiting argument. Then one has

$$\begin{aligned} \int_{(D \setminus B(x, d)) \cap B(0, r_0)} \nabla L(x - y) dy &= \int_{(D \setminus B(x, d)) \cap B(0, r_0)} \nabla L(x - y) dy - \int_{H_-} \nabla L(x - y) dy \\ &= \int_{(D \setminus H_-) \cap B(0, r_0)} \nabla L(x - y) dy \\ &\quad - \int_{(H_- \setminus (D \cup B(x, d))) \cap B(0, r_0)} \nabla L(x - y) dy - \int_{H_- \cap B^c(0, r_0)} \nabla L(x - y) dy, \end{aligned}$$

and the last term is estimated as we did above with D in place of H_- . The remaining two terms are tangential and they are treated similarly. For the first we set

$$\int_{(D \setminus H_-) \cap B(0, r_0)} \nabla L(x - y) dy = \int_{(D \setminus H_-) \cap B(0, 2d)} \nabla L(x - y) dy + \int_{(D \setminus H_-) \cap (B(0, r_0) \setminus B(0, 2d))} \nabla L(x - y) dy.$$

Since for $x \in D \setminus H_-$ one has $|y - x| \geq d$, we get

$$\begin{aligned} \left| \int_{(D \setminus H_-) \cap B(0, 2d)} \nabla L(x - y) dy \right| &\leq \frac{C}{d^{n+1}} |(D \setminus H_-) \cap B(0, 2d)| \\ &\leq \frac{C}{d^{n+1}} \int_0^{2d} \rho^{n-1} \sigma \{\theta \in S^{n-1} : \rho\theta \in D \setminus H_-\} d\rho \\ &\leq \frac{C}{d^{n+1}} \int_0^{2d} \rho^{n-1+\gamma} d\rho = C d^{\gamma-1}. \end{aligned}$$

Finally

$$\begin{aligned} \left| \int_{(D \setminus H_-) \cap (B(0, r_0) \setminus B(0, 2d))} \nabla L(x - y) dy \right| &\leq C \int_{(D \setminus H_-) \cap (B(0, r_0) \setminus B(0, 2d))} \frac{1}{|y|^{n+1}} dy \\ &\leq C \int_0^{2d} \frac{1}{\rho^{n+1}} \rho^{n-1+\gamma} d\rho = C d^{\gamma-1}. \end{aligned} \quad \square$$

It is an interesting fact that the preceding lemma implies the main lemma in [Mateu et al. 2009], which states that under the hypothesis of Vasin’s lemma the function $L \star \chi_D$ satisfies a Hölder condition of order γ on D and on $\mathbb{R}^n \setminus \bar{D}$. Incidentally, it is worth mentioning that this result has been proved independently by various authors at different times and with various degrees of generality. We are grateful to M. Lanza de Cristoforis for bringing to our attention the oldest reference we are aware of, namely, the paper of Carlo Miranda [1965].

We give an account of the proof of this fact only for the statement concerning D . In the exterior of D one applies similar arguments.

Take two points x and y in D . Let $d = \text{dist}(x, \partial D)$ be the distance from x to the boundary. As before, we assume, without loss of generality, that 0 is the closest point of ∂D to x and that the tangent hyperplane to ∂D at 0 is $\{x \in \mathbb{R}^n : x_n = 0\}$. We can also assume that $D \cap B(0, 2r_0) = \{x \in \mathbb{R}^n : x_n < \varphi(x')\}$, where $x' = (x_1, \dots, x_{n-1})$, $\varphi \in C^{1+\gamma}(B'(0, 2r_0))$, $B'(0, 2r_0) = \{x' \in \mathbb{R}^{n-1} : |x'| < 2r_0\}$. Then

$$|\varphi(x')| \leq \|\nabla \varphi\|_{\gamma, B'(0, 2r_0)} |x'|^{1+\gamma}, \quad x' \in B'(0, 2r_0). \tag{31}$$

As in [Mateu et al. 2009] we can reduce matters to the case in which $d \leq \frac{1}{2}r_0$, because otherwise we resort to the smoothness of $L \star \chi_D$ on the domain $\{z \in D : \text{dist}(z, \partial D) > \frac{1}{2}r_0\}$.

Let K be the closed cone with aperture 45° and axis the negative x_n -axis. That is

$$K = \{x \in \mathbb{R}^n : -\sqrt{2}x_n \geq |x|\}.$$

We say that x and y are in nontangential position if $x, y \in K$. Otherwise they are in tangential position.

Assume first that $x, y \in D$ are in nontangential position and distinguish two cases. The first is $y \in B(0, 2d) \setminus B(0, d)$. Apply the mean value theorem on an arc contained in $K \cap (B(0, 2d) \setminus B(0, d))$ of length comparable to $|y - x|$. One gets

$$|f(y) - f(x)| \leq C \sup\{\text{dist}(\xi, \partial D)^{\gamma-1} : \xi \in K \cap (B(0, 2d) \setminus B(0, d))\} |y - x|. \tag{32}$$

We claim that there exists an absolute constant c_0 with $0 < c_0 < 1$ satisfying

$$\text{dist}(\xi, \partial D) \geq c_0 |\xi_n|, \quad \xi \in K \cap B(0, r_0), \tag{33}$$

provided r_0 is small enough. Let $p \in \partial D$ be such that $|\xi - p| = \text{dist}(\xi, \partial D)$. Since $p = (p', p_n)$ is on the graph of φ we have, by (31), $|p_n| \leq C |p'|^{1+\epsilon} \leq C r_0^\epsilon |p'|$. Thus

$$\begin{aligned} |\xi_n| &\leq |\xi_n - p_n| + |p_n| \leq |\xi - p| + C r_0^\epsilon |p'| \\ &\leq |\xi - p| + C r_0^\epsilon (|p' - \xi'| + |\xi'|) \\ &\leq |\xi - p| (1 + C r_0^\epsilon) + C r_0^\epsilon |\xi_n|, \end{aligned}$$

where in the last inequality we used that $|\xi| \leq \sqrt{2}|\xi_n|$, $\xi \in K$. Taking r_0 so small that $C r_0^\epsilon \leq \frac{1}{2}$ we obtain

$$|\xi_n| \leq 2(1 + C r_0^\epsilon)|\xi - p| = 2(1 + C r_0^\epsilon) \text{dist}(\xi, \partial D).$$

Indeed, the constant C is the previous string of inequalities is $\sqrt{2} \|\nabla \varphi\|_{\gamma, B'(0, 2r_0)}$, which also depends on r_0 . But this is not an obstruction because it decreases with r_0 .

Therefore, by (32),

$$\begin{aligned} |f(y) - f(x)| &\leq C (c_0|\xi_n|)^{\gamma-1} |y - x| \leq C c_0^{\gamma-1} d^{\gamma-1} |y - x|^{1-\gamma} |y - x|^\gamma \\ &\leq C c_0^{\gamma-1} d^{\gamma-1} (3d)^{1-\gamma} |y - x|^\gamma = C |y - x|^\gamma. \end{aligned}$$

Let us turn our attention to the case $y \in K \cap B^c(0, 2d)$. Note that there exists an absolute constant $C_0 > 1$ such that

$$|y - x| \leq C_0 |y_n - x_n|, \quad y \in K \cap B^c(0, 2d).$$

Apply the fundamental theorem of calculus on the interval with endpoints x and y and estimate the gradient of f by a constant times the distance to the boundary raised to the power $\gamma - 1$. By (33) we obtain

$$\begin{aligned} |f(y) - f(x)| &\leq C \int_0^1 \text{dist}(x + t(y - x), \partial D)^{\gamma-1} |y - x| dt \\ &\leq C c_0^{\gamma-1} \int_0^1 |x_n + t(y_n - x_n)|^{\gamma-1} |y - x| dt \\ &= C \frac{|y - x|}{|y_n - x_n|} \int_0^{|y_n - x_n|} (d + \tau)^{\gamma-1} d\tau \\ &= C C_0 ((d + |y_n - x_n|)^\gamma - d^\gamma) \leq C |y_n - x_n|^\gamma \leq C |y - x|^\gamma, \end{aligned}$$

as desired.

We are left with the case in which x and y are in tangential position, that is, $y \in D \cap (\mathbb{R}^n \setminus K)$. In [Mateu et al. 2009] there is a reduction argument to the nontangential case, which we now reproduce for completeness. Take a point $p \in \partial D$ with $|y - p| = \text{dist}(y, \partial D)$ and let \vec{N} be the exterior unit normal vector to ∂D at p . We will take r_0 so small that \vec{N} is very close to the exterior unit normal vector \vec{n} to ∂D at 0. Then the ray $y - t\vec{N}$, $t > 0$, will intersect K at some point y_0 and the pairs x, y_0 and y, y_0 will be in tangential position. Let us seek a condition on t so that $y - t\vec{N} \in K$, that is, so that

$$|y - t\vec{N}| \leq \sqrt{2}|\langle y - t\vec{N}, \vec{n} \rangle|. \tag{34}$$

Here $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^n . Since $|\langle y - t\vec{N}, \vec{n} \rangle| \geq t\langle \vec{N}, \vec{n} \rangle - |y|$ and $|y - t\vec{N}| \leq |y| + t$, a sufficient condition for (34) is

$$(1 + \sqrt{2})|y| \leq t(\sqrt{2}\langle \vec{N}, \vec{n} \rangle - 1).$$

Take r_0 small enough so that $\sqrt{2}\langle \vec{N}, \vec{n} \rangle - 1 \geq (\sqrt{2} - 1)/2$. A simpler sufficient condition for (34) is

$$|y| \leq c_0 t, \quad \text{with } c_0 = \frac{1}{2} \frac{\sqrt{2} - 1}{\sqrt{2} + 1}.$$

Define t_0 by $|y| = c_0 t_0$ and then set $y_0 = y - t_0 \vec{N}$. By construction, $y_0 \in K \cap D$ and the pairs x, y_0 and y, y_0 are in nontangential position. Hence we only have to check that

$$|y - y_0| \leq C_0 |x - y|. \tag{35}$$

We have $c_0 |y - y_0| = c_0 t_0 = |y|$. On the other hand, the condition $y \notin K$ is exactly $|y| < \sqrt{2} |y'|$, and clearly $|x - y| \geq |y'|$. Therefore (35) holds with an absolute constant C_0 .

4. The defining function for D_t

In this section we prove that the function Φ defined by (20) is a defining function of D_t of class $C^{1+\gamma}$. Our assumption now is that the CDE has a solution $X(\cdot, t)$ for t in an interval $(-T; T)$ and that D_t is the domain with $\partial D_t = X(\partial D_0, t)$ which has been defined in Section 2.1. The field defined by (13) has a flow map (6) whose restriction to ∂D_0 is precisely the solution of the CDE.

Taking the gradient in (20) we get, for $x \notin \partial D_t$,

$$\nabla \Phi(x, t) = \det \nabla X(X^{-1}(x, t), t) \nabla \varphi(x, t) + \nabla(\det \nabla X(X^{-1}(x, t), t)) \varphi(x, t). \tag{36}$$

In [Bertozi et al. 2016, Section 8] it was shown that $\nabla X^{-1}(\cdot, t)$ satisfies a Hölder condition of order γ on the open set $\mathbb{R}^n \setminus \partial D_t$ (but may have jumps at ∂D_t). What remains to be proved is that $\nabla \Phi(\cdot, t)$ extends continuously to ∂D_t . This is straightforward for the first term in the right-hand side of (36), just by the jump formulas (18) and (19). We have

$$\lim_{\mathbb{R}^n \setminus \partial D_t \ni y \rightarrow x} \det \nabla X(X^{-1}(y, t), t) \nabla \varphi(y, t) = \frac{|\nabla \varphi_0(X^{-1}(x, t))|}{\det D(x)} \vec{n}(x),$$

where $D(x)$ is the differential at $x \in \partial D_t$ of $X^{-1}(\cdot, t)$ viewed as a differentiable mapping from ∂D_t into ∂D_0 .

The second term in the right-hand side of (36) tends to 0 as x approaches a point in ∂D_t . Proving this requires some work. For the sake of simplicity of notation let us consider positive times t less than T . Since $X(\cdot, t)$ is a continuously differentiable function of t with values in the Banach space $C^{1+\gamma}(\partial D_0, \mathbb{R}^n)$, the constants $q(D_s)$ determining the $C^{1+\gamma}$ smoothness of the boundary of D_s are uniformly bounded for $0 \leq s \leq t$. Hence

$$\|\nabla v(\cdot, s)\|_\infty \leq C(t), \quad 0 \leq s \leq t, \tag{37}$$

$$\|\nabla v(\cdot, s)\|_{\gamma, D_s} + \|\nabla v(\cdot, s)\|_{\gamma, \mathbb{R}^n \setminus \bar{D}_s} \leq C(t), \quad 0 \leq s \leq t, \tag{38}$$

where $C(t)$ denotes here and in the sequel a positive constant depending on t but not on $s \in [0, t]$. Inequality (37) follows from the fact, already mentioned, that standard even convolution Calderón–Zygmund operators are bounded on characteristic functions of $C^{1+\gamma}$ domains with bounds controlled by the constants giving the smoothness of the domain (see, for instance, (30)). Inequality (38) has appeared in the literature several times with various degrees of generality, as we mentioned in the previous section, where a complete proof was presented. In [Mateu et al. 2009] the reader will find another accessible proof independent of Vasin’s lemma. The constants are not logarithmic, but this is not relevant here. The

statement is that if T is an even smooth (of class C^1) convolution homogeneous Calderón–Zygmund operator and D is a domain with boundary of class $C^{1+\gamma}$, $0 < \gamma < 1$, then $T(\chi_D)$ satisfies a Hölder condition of order γ in D and in $\mathbb{R}^n \setminus \bar{D}$.

As we said in Section 2 one applies (24) to the second-order derivatives of the field v to conclude that

$$|\partial_j \partial_k v(x, s)| \leq C(t) \operatorname{dist}(x, \partial D_s)^{\gamma-1}, \quad x \notin \partial D_s, \quad 0 \leq s \leq t, \quad 1 \leq j, k \leq n. \tag{39}$$

See (26) and (27).

Combining (7), the analogous inequality with $\nabla X(\cdot, t)$ replaced by $\nabla X^{-1}(\cdot, t)$ and (37) we get

$$C(t)^{-1} \leq \|\nabla X(\cdot, s)\|_\infty \leq C(t), \quad 0 \leq s \leq t. \tag{40}$$

Therefore $X(\cdot, s)$ is a bilipschitz homeomorphism of \mathbb{R}^n and consequently, for all $\alpha \in R^n$,

$$C(t)^{-1} \operatorname{dist}(\alpha, \partial D_0) \leq \operatorname{dist}(X(\alpha, s), \partial D_s) \leq C(t) \operatorname{dist}(\alpha, \partial D_0), \quad 0 \leq s \leq t. \tag{41}$$

Now let us turn to the second term in the right-hand side of (36)

$$II(x) = \nabla(\det \nabla X(X^{-1}(x, t), t))\varphi(x, t) = \varphi_0(\alpha) \nabla_x J(\alpha, t), \tag{42}$$

where we have set $x = X(\alpha, t)$ and $J(\alpha, t) = \det \nabla X(\alpha, t)$. The jacobian satisfies

$$\frac{d}{dt} J(\alpha, t) = \operatorname{div} v(X(\alpha, t), t) J(\alpha, t)$$

and so

$$J(\alpha, t) = \exp \int_0^t \operatorname{div} v(X(\alpha, s), s) ds.$$

Hence $\nabla_x J(\alpha, t)$ is

$$\left(\exp \int_0^t \operatorname{div} v(X(\alpha, s), s) ds \right) \left(\int_0^t \operatorname{div}((\nabla v)^t(X(\alpha, s), s)) \nabla X(\alpha, s) ds \right) \nabla X^{-1}(x, t), \tag{43}$$

where the divergence of a matrix is the vector with components the divergence of rows. Combining (37), (39), (40), (41), (42) and (43) we get

$$\begin{aligned} |II(x)| &\leq C(t) |\varphi_0(\alpha)| \int_0^t \operatorname{dist}(X(\alpha, s), \partial D_s)^{\gamma-1} ds \\ &\leq C(t) |\varphi_0(\alpha)| \operatorname{dist}(\alpha, \partial D_0)^{\gamma-1} \leq C(t) \operatorname{dist}(\alpha, \partial D_0)^\gamma. \end{aligned}$$

If $\operatorname{dist}(x, \partial D_t) \rightarrow 0$ then $\operatorname{dist}(\alpha, \partial D_0) \rightarrow 0$ and thus $II(x) \rightarrow 0$.

5. The commutators

The material derivative $D/(Dt) = \partial_t + v \cdot \nabla$ of the defining function of the previous section is

$$\frac{D}{Dt} \Phi(x) = \frac{d}{dt} (J(\alpha, t) \varphi_0(\alpha)) = \operatorname{div} v(X(\alpha, t), t) J(\alpha, t) \varphi_0(\alpha) = \operatorname{div} v(x, t) \Phi(x, t),$$

which proves (21). Taking derivatives in the equation above and rearranging terms one obtains

$$\frac{D}{Dt} \nabla \Phi = \nabla(\operatorname{div} v) \Phi + (\operatorname{div} v) \nabla \Phi - (\nabla v)^t (\nabla \Phi). \tag{44}$$

The first term tends to 0 at the boundary of D_t , by (39). This section is devoted to proving that the second term in the right-hand side, namely $(\operatorname{div} v)\nabla\Phi - (\nabla v)^t(\nabla\Phi)$, is a sum of $n - 1$ terms, each of which is a difference of boundary commutators. It clearly suffices to prove that each coordinate is a sum of $n - 1$ differences of boundary commutators. We present the details for the first coordinate, which is

$$\partial_2 v_2 \partial_1 \Phi - \partial_1 v_2 \partial_2 \Phi + \dots + \partial_n v_n \partial_1 \Phi - \partial_1 v_n \partial_n \Phi. \tag{45}$$

Let us work with the first term $\partial_2 v_2 \partial_1 \Phi - \partial_1 v_2 \partial_2 \Phi$. The others are treated similarly. The preceding expression is evaluated at (x, t) with $x \in \partial D_t$. To lighten the notation we set $D = D_t$, so that t is fixed, and $\chi = \chi_{D_t}$. Recall that $v(\cdot, t) = k \star \chi$ and so

$$v_j(\cdot, t) = k_j \star \chi, \quad 1 \leq j \leq n.$$

By (25) we have in the distributions sense

$$\partial_j v_j(\cdot, t) = \partial_j k_j \star \chi = \text{p.v. } \partial_j k_j \star \chi + c_j \chi, \quad 1 \leq j \leq n,$$

where $c_j = \int_{|\xi|=1} k_j(\xi)\xi_j d\sigma(\xi)$. Thus

$$\partial_2 v_2(\cdot, t) \partial_1 \Phi(\cdot, t) = (\partial_2 k_2 \star \chi)(\cdot) \partial_1 \Phi(\cdot, t) = \text{p.v.}(\partial_2 k_2 \star \chi) \partial_1 \Phi + c_2 \chi \partial_1 \Phi$$

and

$$\partial_2 k_2 \star (\chi \partial_1 \Phi) = \text{p.v. } \partial_2 k_2 \star (\chi \partial_1 \Phi) + c_2 \chi \partial_1 \Phi,$$

which yields

$$\partial_2 v_2(\cdot, t) \partial_1 \Phi(\cdot, t) = \text{p.v.}(\partial_2 k_2 \star \chi) \partial_1 \Phi - \text{p.v. } \partial_2 k_2 \star (\chi \partial_1 \Phi) + \partial_2 k_2 \star (\chi \partial_1 \Phi). \tag{46}$$

Similarly

$$\partial_1 v_2(\cdot, t) \partial_2 \Phi(\cdot, t) = \text{p.v.}(\partial_1 k_2 \star \chi) \partial_2 \Phi - \text{p.v. } \partial_1 k_2 \star (\chi \partial_2 \Phi) + \partial_1 k_2 \star (\chi \partial_2 \Phi). \tag{47}$$

Since $\chi \partial_j \Phi = \partial_j(\chi \Phi)$, $1 \leq j \leq n$, we have

$$\partial_2 k_2 \star (\chi \partial_1 \Phi) = \partial_1 k_2 \star (\chi \partial_2 \Phi),$$

and subtracting (47) from (46) yields

$$\begin{aligned} &\partial_2 v_2(\cdot, t) \partial_1 \Phi(\cdot, t) - \partial_1 v_2(\cdot, t) \partial_2 \Phi(\cdot, t) \\ &= \text{p.v.}(\partial_2 k_2 \star \chi) \partial_1 \Phi - \text{p.v. } \partial_2 k_2 \star (\chi \partial_1 \Phi) - (\text{p.v.}(\partial_1 k_2 \star \chi) \partial_2 \Phi - \text{p.v. } \partial_1 k_2 \star (\chi \partial_2 \Phi)), \end{aligned} \tag{48}$$

which is the difference of two solid commutators. Here we are using the term ‘‘solid’’ to indicate that the integration is taken with respect to n -dimensional Lebesgue measure. Our next task is to bring the solid commutators to the boundary.

Formula (48) is an identity between distributions and is not a priori obvious that the principal value integrals exist at boundary points. The same can be said about the principal values on the boundary which appear in the calculation below. That they do exist in our context is a routine argument, which we postpone to the [Appendix](#).

Let $x \in \partial D$. Given $\epsilon > 0$ set $D_\epsilon = D \setminus \overline{B(x, \epsilon)}$. By the divergence theorem

$$\begin{aligned} (\text{p.v. } \partial_2 k_2 \star (\chi \partial_1 \Phi))(x) &= \lim_{\epsilon \rightarrow 0} \int_{D_\epsilon} \partial_2 k_2(x-y) \partial_1 \Phi(y) dy \\ &= - \lim_{\epsilon \rightarrow 0} \int_{\partial D_\epsilon} k_2(x-y) \partial_1 \Phi(y) n_2(y) d\sigma(y) + \int_D k_2(x-y) \partial_{21} \Phi(y) dy \\ &= - \text{p.v.} \int_{\partial D} k_2(x-y) \partial_1 \Phi(y) n_2(y) d\sigma(y) + \int_D k_2(x-y) \partial_{12} \Phi(y) dy \\ &\quad + \lim_{\epsilon \rightarrow 0} \int_{\partial B(x, \epsilon) \cap D} k_2(x-y) \partial_1 \Phi(y) n_2(y) d\sigma_\epsilon(y), \end{aligned}$$

where σ_ϵ is the surface measure on $\partial B(x, \epsilon)$. We do not need to compute explicitly the term

$$\lim_{\epsilon \rightarrow 0} \int_{\partial B(x, \epsilon) \cap D} k_2(x-y) \partial_1 \Phi(y) n_2(y) d\sigma_\epsilon(y),$$

nor to worry about the second-order derivative of Φ which has appeared, because they will eventually cancel out (a routine regularization argument takes care of the actual presence of the second derivatives of Φ).

We turn now to the computation of $(\text{p.v. } \partial_{12} N \star \chi)(x)$. We have

$$\begin{aligned} (\text{p.v. } \partial_2 k_2 \star \chi)(x) &= \lim_{\epsilon \rightarrow 0} \int_{D_\epsilon} \partial_2 k_2(x-y) dy \\ &= - \lim_{\epsilon \rightarrow 0} \int_{\partial D_\epsilon} k_2(x-y) n_2(y) d\sigma(y) \\ &= - \text{p.v.} \int_{\partial D} k_2(x-y) n_2(y) d\sigma(y) + \lim_{\epsilon \rightarrow 0} \int_{\partial B(x, \epsilon) \cap D} k_2(x-y) n_2(y) d\sigma_\epsilon(y). \end{aligned}$$

Therefore

$$\begin{aligned} &(\text{p.v. } \partial_2 k_2 \star (\chi \partial_1 \Phi))(x) - (\text{p.v. } \partial_2 k_2 \star \chi)(x) \partial_1 \Phi(x) \\ &= \text{p.v.} \int_{\partial D} k_2(x-y) n_2(y) d\sigma(y) \partial_1 \Phi(x) - \text{p.v.} \int_{\partial D} k_2(x-y) \partial_1 \Phi(y) n_2(y) d\sigma(y) \\ &\quad + \int_D k_2(x-y) \partial_{21} \Phi(y) dy, \quad (49) \end{aligned}$$

since

$$\lim_{\epsilon \rightarrow 0} \int_{\partial B(x, \epsilon) \cap D} k_2(x-y) (\partial_1 \Phi(y) - \partial_1 \Phi(x)) d\sigma_\epsilon(y) = 0,$$

because k_2 is homogeneous of order $-(n-1)$ and $\partial_1 \Phi$ is continuous at x . The conclusion is that the solid commutator in the left-hand side of (49) is a boundary commutator plus an additional term involving second-order derivatives of Φ . This term will disappear soon and in the final formulas no second derivatives of Φ are present, so that the $C^{1+\gamma}$ condition on Φ is enough.

Proceeding in a similar way we find

$$\begin{aligned} &(\text{p.v. } \partial_1 k_2 \star (\chi \partial_2 \Phi))(x) - (\text{p.v. } \partial_1 k_2 \star \chi)(x) \partial_2 \Phi(x) \\ &= \text{p.v.} \int_{\partial D} k_2(x-y) n_1(y) d\sigma(y) \partial_2 \Phi(x) - \text{p.v.} \int_{\partial D} k_2(x-y) \partial_2 \Phi(y) n_1(y) d\sigma(y) \\ &\quad + \int_D k_2(x-y) \partial_{12} \Phi(y) dy, \quad (50) \end{aligned}$$

Subtracting (50) from (49) we see that the difference of the two solid commutators in (48) is exactly, on the boundary of D , a difference of boundary commutators. Hence $((\operatorname{div} v)I - (\nabla v)^t)(\nabla \Phi)$ is a sum of $n - 1$ terms, each being a difference of two vector-valued boundary commutators.

6. Hölder estimate of differences of boundary commutators

We keep the notation of the previous section $D = D_t$, $\nabla \Phi = \nabla \Phi(\cdot, t)$, with t fixed. Our goal is to estimate the Hölder seminorm of order γ on ∂D of the difference of two boundary commutators. For instance,

$$DB(x) := \text{p.v.} \int_{\partial D} k_2(x - y) \partial_2 \Phi(y) n_1(y) d\sigma(y) - \text{p.v.} \int_{\partial D} k_2(x - y) n_1(y) d\sigma(y) \partial_2 \Phi(x) - \left(\text{p.v.} \int_{\partial D} k_2(x - y) \partial_1 \Phi(y) n_2(y) d\sigma(y) - \text{p.v.} \int_{\partial D} k_2(x - y) n_2(y) d\sigma(y) \partial_1 \Phi(x) \right).$$

The general case follows immediately by the same arguments. The strategy consists in exploiting the fact that $DB(x)$ is also, for $x \in \partial D$, a difference $DS(x)$ of two solid commutators, as we checked in the previous section. That is, $DB(x)$ for $x \in \partial D$ is identical to

$$DS(x) = DS(\Phi)(x) := (\text{p.v.} \partial_2 k_2 \star (\chi \partial_1 \Phi) - (\text{p.v.} \partial_2 k_2 \star \chi) \partial_1 \Phi) - (\partial_1 k_2 \star (\chi \partial_2 \Phi) - (\partial_1 k_2 \star \chi) \partial_2 \Phi).$$

By [Bertozzi and Constantin 1993, Corollary, p. 24, and Lemma, p. 26], estimating each commutator separately, we have $\|DS\|_{\gamma, \mathbb{R}^n} \leq C_n \|\nabla v(\cdot, t)\|_\infty \|\nabla \Phi\|_{\gamma, \mathbb{R}^n}$, which is not good enough, because we need $\|\nabla \Phi\|_{\gamma, \partial D}$ in place of $\|\nabla \Phi\|_{\gamma, \mathbb{R}^n}$.

We now consider the jet

$$(0, \partial_1 \Phi, \dots, \partial_n \Phi)$$

on ∂D . By Whitney’s extension theorem [Stein 1970, Chapter VI, p. 177] there exists Ψ of class $C^{1+\gamma}(\mathbb{R}^n)$ such that $\Psi = 0$ and $\nabla \Psi = \nabla \Phi$ on ∂D , satisfying

$$\|\nabla \Psi\|_{\gamma, \mathbb{R}^n} \leq C_n \left(\|\nabla \Phi\|_{\gamma, \partial D} + \sup \left\{ \frac{|\nabla \Phi(x) \cdot (y - x)|}{|y - x|^{1+\gamma}} : y \neq x, y, x \in \partial D \right\} \right). \tag{51}$$

This precise estimate is not stated explicitly in Stein’s book but it follows from the proof. We claim that

$$\sup \left\{ \frac{|\nabla \Phi(x) \cdot (y - x)|}{|y - x|^{1+\gamma}} : y \neq x, y, x \in \partial D \right\} \leq 2^{(1+\gamma)/2} \|\nabla \Phi\|_{\gamma, \partial D}. \tag{52}$$

We postpone the proof of the claim and we complete the estimate of $\|DB\|_{\gamma, \partial D}$.

The extension Ψ of the jet $(0, \partial_1 \Phi, \dots, \partial_n \Phi)$ on ∂D , given by Whitney’s extension theorem, satisfies, in view of (51) and (52),

$$\|\nabla \Psi\|_{\gamma, \mathbb{R}^n} \leq C_{n,\gamma} \|\nabla \Phi\|_{\gamma, \partial D}.$$

Since $\nabla \Psi = \nabla \Phi$ on ∂D , the differences of solid commutators $DS(\Phi)$ and $DS(\Psi)$ are equal on ∂D . Thus

$$\begin{aligned} \|DB\|_{\gamma, \partial D} &= \|DS(\Psi)\|_{\gamma, \partial D} \leq \|DS(\Psi)\|_{\gamma, \mathbb{R}^n} \\ &\leq C_n (\|\nabla v(\cdot, t)\|_\infty + 1) \|\nabla \Psi\|_{\gamma, \mathbb{R}^n} \leq C_n (\|\nabla v(\cdot, t)\|_\infty + 1) \|\nabla \Phi\|_{\gamma, \partial D}. \end{aligned}$$

This can be used to prove the a priori estimate (17) as in [Bertozzi and Constantin 1993].

We turn now to the proof of the claim (52). Fix a point $x \in \partial D$. Assume without loss of generality that $x = 0$ and $\nabla\Phi(0) = (0, \dots, 0, \partial_n\Phi(0))$, $\partial_n\Phi(0) > 0$. Define $\delta = \delta(x)$ by

$$\delta^{-\gamma} = 2 \frac{\|\nabla\Phi\|_{\gamma, \partial D}}{|\nabla\Phi(0)|}.$$

This choice of δ implies that the normal vector $\nabla\Phi(y)$ remains for $y \in B(0, \delta) \cap \partial D$ in the ball $B(\nabla\Phi(0), |\nabla\Phi(0)|/2)$. Indeed

$$|\nabla\Phi(y) - \nabla\Phi(0)| \leq \|\nabla\Phi\|_{\gamma, \partial D} \delta^\gamma = \frac{|\nabla\Phi(0)|}{2}.$$

Then given $y \in B(0, \delta) \cap \partial D$, the tangent hyperplane to ∂D at y forms an angle less than 30° with the horizontal plane and thus ∂D is the graph of a function $y_n = \varphi(y'_n)$ which satisfies a Lipschitz condition with constant less than 1. Here we have used the standard notation $y = (y', y_n)$, $y' = (y_1, \dots, y_{n-1})$. The function φ is defined in the open set U which is the projection of $B(0, \delta) \cap \partial D$ into \mathbb{R}^{n-1} defined by $y \rightarrow y'$. By the implicit function theorem φ is of class $C^{1+\gamma}$ in its domain.

Note that for each $y \in \partial D \cap B(0, \delta)$, the segment $\{t y' : 0 \leq t \leq 1\}$ is contained in U , as an elementary argument shows. The mean value theorem on that segment for the function $t \rightarrow \varphi(t y')$ yields

$$\frac{|\nabla\Phi(0) \cdot y|}{|y|^{1+\gamma}} = \frac{|\nabla\Phi(0)| |\varphi(y')|}{|y|^{1+\gamma}} \leq \frac{|\nabla\Phi(0)|}{|y|^{1+\gamma}} \sup\{|\nabla\varphi(z')| : z' \in U, |z'| \leq |y'|\} |y'|.$$

By implicit differentiation

$$\partial_j \varphi(z') = - \frac{\partial_j \Phi(z', \varphi(z'))}{\partial_n \Phi(z', \varphi(z'))}, \quad 1 \leq j \leq n-1,$$

and so, recalling that $\partial_j \Phi(0) = 0$, $1 \leq j \leq n-1$, and that $z = (z', \varphi(z'))$,

$$|\nabla\varphi(z')| \leq \frac{\|\nabla\Phi\|_{\gamma, \partial D} |z|^\gamma}{|\partial_n \Phi(z)|} \leq \frac{2}{|\nabla\Phi(0)|} \|\nabla\Phi\|_{\gamma, \partial D} 2^{\gamma/2} |z'|^\gamma, \quad |z'| \leq |y'|,$$

because

$$\begin{aligned} |\partial_n \Phi(z)| &\geq |\partial_n \Phi(0)| - |\partial_n \Phi(z) - \partial_n \Phi(0)| \\ &\geq |\nabla\Phi(0)| - \|\nabla\Phi\|_{\gamma, \partial D} \delta^\gamma = \frac{|\nabla\Phi(0)|}{2} \end{aligned}$$

and

$$|z| = (|z'|^2 + \varphi(z')^2)^{1/2} \leq \sqrt{2} |z'|.$$

Thus

$$\frac{|\nabla\Phi(0) \cdot y|}{|y|^{1+\gamma}} \leq 2^{1+\gamma/2} \|\nabla\Phi\|_{\gamma, \partial D}, \quad y \in \partial D \cap B(0, \delta).$$

If $y \in \partial D \setminus B(0, \delta)$,

$$\frac{|\nabla\Phi(0) \cdot y|}{|y|^{1+\gamma}} \leq \frac{|\nabla\Phi(0)|}{|y|^\gamma} \leq \frac{|\nabla\Phi(0)|}{\delta^\gamma} = 2 \|\nabla\Phi\|_{\gamma, \partial D},$$

which completes the proof of (52).

7. Domains enclosed by ellipses as Cauchy patches

In this section we consider the transport equation in the plane given by the Cauchy kernel

$$\begin{aligned} \partial_t \rho(z, t) + v(z, t) \cdot \nabla \rho(z, t) &= 0, \\ v(z, t) &= \left(\frac{1}{\pi z} * \rho(\cdot, t) \right)(z), \\ \rho(z, 0) &= \chi_{D_0}(z), \end{aligned} \tag{53}$$

$z = x + iy \in \mathbb{C} = \mathbb{R}^2$ and $t \in \mathbb{R}$. Note that we have changed the normalization of the velocity field in (4) by a factor of 2.

We take the initial patch to be the domain enclosed by an ellipse

$$E_0 = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{a_0^2} + \frac{y^2}{b_0^2} < 1 \right\}.$$

We will show that the solution provided by the theorem is of the form $\chi_{E_t}(z)$, with

$$E_t = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{a(t)^2} + \frac{y^2}{b(t)^2} < 1 \right\}$$

and

$$a(t) = a_0 \frac{(a_0 + b_0) e^{2t}}{b_0 + a_0 e^{2t}}, \quad t \in \mathbb{R}, \tag{54}$$

$$b(t) = b_0 \frac{(a_0 + b_0)}{b_0 + a_0 e^{2t}}, \quad t \in \mathbb{R}. \tag{55}$$

As $t \rightarrow \infty$, $a(t) \rightarrow a_0 + b_0$ and $b(t) \rightarrow 0$, so that the ellipse at time t degenerates into the segment $[-(a_0 + b_0), a_0 + b_0]$ as $t \rightarrow +\infty$ and into the segment $i[-(a_0 + b_0), a_0 + b_0]$ on the vertical axis as $t \rightarrow -\infty$.

Since (53) is not rotation invariant, one has to consider also the case of an initial patch given by the domain enclosed by a tilted ellipse

$$E(a, b, \theta) = e^{i\theta} \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right\}.$$

In this case the straight line containing the semiaxis of length a makes an angle θ with the horizontal axis and we take $0 < \theta < \frac{\pi}{2}$.

Assume that the initial patch is $E_0 = E(a_0, b_0, \theta_0)$. Then we will show that the solution given by the theorem is χ_{E_t} with $E_t = E(a(t), b(t), \theta(t))$, where $a(t)$, $b(t)$ and $\theta(t)$ are the unique solutions of the system

$$\begin{aligned} a'(t) &= \frac{2}{a_0 + b_0} a(t) b(t) \cos(2\theta(t)), \\ b'(t) &= -\frac{2}{a_0 + b_0} a(t) b(t) \cos(2\theta(t)), \\ \theta'(t) &= -\frac{2}{a_0 + b_0} \frac{a(t) b(t)}{a(t) - b(t)} \sin(2\theta(t)), \end{aligned} \tag{56}$$

with initial conditions $a(0) = a_0$, $b(0) = b_0$, $\theta(0) = \theta_0$.

We start the proof by assuming that the patch D_t of the weak solution provided by the theorem is indeed E_t . Let $z(t)$ be the trajectory of the particle that at time 0 is at $z(0) \in \partial E_0$. Then

$$\frac{dz}{dt} = v(z(t), t), \quad z(0) \in \partial E_0, \tag{57}$$

and v is the velocity field of (53). It is a well-known fact that v can be explicitly computed [Hmidi et al. 2015]. One has

$$v(z, t) = \left(\frac{1}{\pi z} \star \chi_{E_t} \right)(z) = \bar{z} - q(t) e^{-2\theta(t)} z, \quad z \in E_t, \quad q(t) = \frac{a(t) - b(t)}{a(t) + b(t)}. \tag{58}$$

Indeed in [Hmidi et al. 2015] only the case $\theta(t) = 0$ is dealt with, but the general case follows easily from the behavior under rotations of a convolution with the Cauchy kernel.

To lighten the notation we do not stress the dependence on t and write $a = a(t)$, $b = b(t)$, $\theta = \theta(t)$, $q = q(t)$, $z = z(t) = x(t) + iy(t) = x + iy$. The condition $z(t) \in \partial E_t$ is equivalent to $e^{-i\theta(t)} z(t) \in \partial E(a(t), b(t), 0)$, which is

$$\frac{(x \cos(\theta) + y \sin(\theta))^2}{a^2} + \frac{(-x \sin(\theta) + y \cos(\theta))^2}{b^2} = 1,$$

and can also be written more concisely as

$$\frac{\langle z, e^{i\theta} \rangle^2}{a^2} + \frac{\langle z, i e^{i\theta} \rangle^2}{b^2} = 1. \tag{59}$$

Here we have denoted by $\langle u, v \rangle$ the scalar product of the vectors u and v . Now proceed as follows. Take the derivative in (59) with respect to t and then replace $z'(t)$ by the expression of the field given by (58). We get an equation containing a, b, θ and z , which determines $z(t)$, the solution of the CDE. This equation is

$$0 = \frac{\langle z, e^{i\theta} \rangle}{a^2} (\langle z, e^{i\theta} i \theta' - q e^{3i\theta} \rangle + \langle \bar{z}, e^{i\theta} \rangle) - \frac{a'}{a^3} \langle z, e^{i\theta} \rangle^2 + \frac{\langle z, i e^{i\theta} \rangle}{b^2} (\langle z, -e^{i\theta} \theta' - i q e^{3i\theta} \rangle + \langle \bar{z}, i e^{i\theta} \rangle) - \frac{b'}{b^3} \langle z, i e^{i\theta} \rangle^2. \tag{60}$$

Evaluate at $z = z(t) = a(t)e^{i\theta(t)}$ (which is a vertex of the ellipse at time t). One gets the equation

$$a' = 2 \frac{a b}{a + b} \cos(2\theta). \tag{61}$$

Evaluating at the other vertex of the ellipse at time t , that is, at $z = z(t) = b(t) i e^{i\theta(t)}$, yields

$$b' = -2 \frac{a b}{a + b} \cos(2\theta). \tag{62}$$

Adding (61) and (62) we see that $a + b$ is constant, then equal to $a_0 + b_0$. Thus we have the first two equations in (56).

Before getting the third equation let us solve the case in which the initial ellipse has axes parallel to the coordinate axes ($\theta_0 = 0$). In this case set $\theta(t) = 0$, $t \in \mathbb{R}$. Replacing in (62) a by $a_0 + b_0 - b$ and solving we get (55) and then (54).

Now take the domain $E_t = E(a(t), b(t), 0)$, the vector field

$$v(z, t) = \left(\frac{1}{\pi z} \star \chi_{E_t} \right)(z) = \bar{z} - q(t)z, \quad z \in E_t,$$

and the flow

$$\frac{dz}{dt} = \bar{z} - q(t)z, \quad z(0) \in \partial E_0,$$

The preceding system is

$$\frac{dx(t)}{dt} = \frac{2b(t)}{a_0 + b_0}x(t), \quad \frac{dy(t)}{dt} = -\frac{2a(t)}{a_0 + b_0}y(t).$$

Then the flow map is linear on E_0 and given by a diagonal matrix. Hence the flow preserves the coordinate axes and maps ∂E_0 into an ellipse with axes parallel to the coordinate axes enclosing a domain \tilde{E}_t . But (61) and (62) say exactly that the vertices of ∂E_t belong to $\partial \tilde{E}_t$. Thus $\tilde{E}_t = E_t$ and so χ_{E_t} is the unique weak solution of the Cauchy transport equation in the class of characteristic functions of $C^{1+\gamma}$ domains.

Let us now go back to the general case and obtain a third equation involving θ' . Impose that the intersection of the ellipse ∂E_t with the positive real axis belongs to the image of ∂E_0 under the flow. In other words replace $z(t)$ in (60) by

$$\left(\left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right)^{-1/2}, 0 \right).$$

After a lengthy computation one gets

$$\theta' = -\frac{2}{a_0 + b_0} \frac{ab}{a - b} \sin(2\theta), \tag{63}$$

provided $a \neq b$.

We know claim that the system (56) has a unique solution defined for all times $t \in \mathbb{R}$ provided $a_0 \neq b_0$. The case $a_0 = b_0$ corresponds to an initial disc and so to the case $\theta_0 = 0$, which has been discussed before. Consider the open set

$$\Omega = \left\{ (a, b, \theta) \in \mathbb{R}^3 : a > 0, b > 0, a \neq b \text{ and } 0 < \theta < \frac{\pi}{2} \right\}.$$

Clearly a unique solution of the system exists locally in time for any initial condition $(a_0, b_0, \theta_0) \in \Omega$, because the function giving the system is C^∞ in Ω . We claim that this solution exists for all times. Assume that the maximal interval of existence is $(-T, T)$ for some $0 < T < \infty$. By the first two equations of the system (56) $|a'|$ and $|b'|$ are bounded above by $2(a_0 + b_0)$ and hence the limits $\lim_{t \rightarrow T} a(t) = a(T)$ and $\lim_{t \rightarrow T} b(t) = b(T)$ exist. We also have

$$\left| \frac{a'}{a} \right| \leq 2 \quad \text{and} \quad \left| \frac{b'}{b} \right| \leq 2$$

and so

$$0 < a_0 e^{-2T} \leq a(T) \leq a_0 e^{2T} \quad \text{and} \quad 0 < b_0 e^{-2T} \leq b(T) \leq b_0 e^{2T}.$$

Note that $\theta'(t)$ cannot vanish. Otherwise, by (63), $\theta(t) = 0$ for some t , and in this case we have already checked that the system can be solved for all times. Hence θ' has constant sign. When $a_0 > b_0$, the

function θ decreases, and if $a_0 < b_0$, the function θ increases. In any case we have that there exists

$$\theta(T) = \lim_{t \rightarrow T} \theta(t).$$

We cannot have $\theta(T) = 0$ or $\theta(T) = \frac{\pi}{2}$ because we have solved the equation in these cases for all times. For the same reason we cannot have $a(T) = b(T)$. Therefore $(a(T), b(T), \theta(T)) \in \Omega$ and we can solve the system past T , which is a contradiction.

We proceed now to prove that the domain $E_t = E(a(t), b(t), \theta(t))$ enclosed by the ellipse provided by the solution of (56) yields the weak solution χ_{E_t} of the transport equation (53) with initial condition $D_0 = E_0$. We consider the field (58) and the trajectory (57) of a particle initially at the boundary point $z(0) \in \partial E_0$. Since the velocity field is linear in E_t the flow is a linear function of $z(0) \in E_0$. Thus the initial ellipse ∂E_0 is mapped into an ellipse $\partial \tilde{E}_t$ enclosing \tilde{E}_t , the image of E_0 under the flow map. To show that χ_{E_t} is a weak solution of the Cauchy transport equation we only need to ascertain that $E_t = \tilde{E}_t$. But the three equations of (56) simply mean that the vertices of E_t and the intersection of E_t with the horizontal axis are in the image of ∂E_0 under the flow map. It is now a simple matter to realize that there is only one ellipse centered at the origin containing those three points.

A surprising result arises when examining the asymptotic behavior as $t \rightarrow \infty$ of the weak solution of the Cauchy transport equation (53) when the initial condition is $E(a_0, b_0, \theta_0)$, with $a_0 \neq b_0$ and $\theta_0 > 0$. We know that the solution of the system (56) never leaves the open set Ω . In particular $a(t) - b(t)$ has a definite sign determined by the initial condition. Assume for definiteness that $a_0 - b_0 > 0$, so that $a(t) - b(t) > 0$, $t \in \mathbb{R}$, and hence $\theta(t)$ is a decreasing function. Then the limit $\theta_\infty = \lim_{t \rightarrow \infty} \theta(t)$ exists. The system (56) readily yields that the function $(a - b) \sin(2\theta)$ has vanishing derivative, so that

$$(a(t) - b(t)) \sin(2\theta(t)) = (a_0 - b_0) \sin(2\theta_0), \quad t \in \mathbb{R}. \tag{64}$$

Thus $(a_0 + b_0) \sin(2\theta(t)) \geq (a(t) - b(t)) \sin(2\theta(t)) = (a_0 - b_0) \sin(2\theta_0)$ and taking limits

$$\sin(2\theta_\infty) \geq \frac{a_0 - b_0}{a_0 + b_0} \sin(2\theta_0) > 0,$$

which means that the limit angle θ_∞ is positive. In other words, the axes of the ellipses at time t do not approach the coordinate axes.

Assume that $0 < \theta_0 \leq \frac{\pi}{4}$. Since $\theta(t)$ decreases, $0 < 2\theta(t) < \frac{\pi}{2}$, $t > 0$, which implies that $a(t)$ increases and $b(t)$ decreases. By (62)

$$b(t) = b_0 \exp \int_0^t -\frac{2}{a_0 + b_0} a(s) \cos(2\theta(s)) ds \leq b_0 \exp\left(-\frac{2a_0}{a_0 + b_0} \cos(2\theta_0) t\right),$$

and so $b_\infty = 0$, provided $\theta_0 < \frac{\pi}{4}$. If $\theta_0 = \frac{\pi}{4}$ we break the integral above into two pieces, the first between 0 and 1 and the second between 1 and t . We get, for some constant C independent of t ,

$$b(t) \leq C \exp\left(-\frac{2a_0}{a_0 + b_0} \cos(2\theta(1)) (t - 1)\right), \quad t > 1,$$

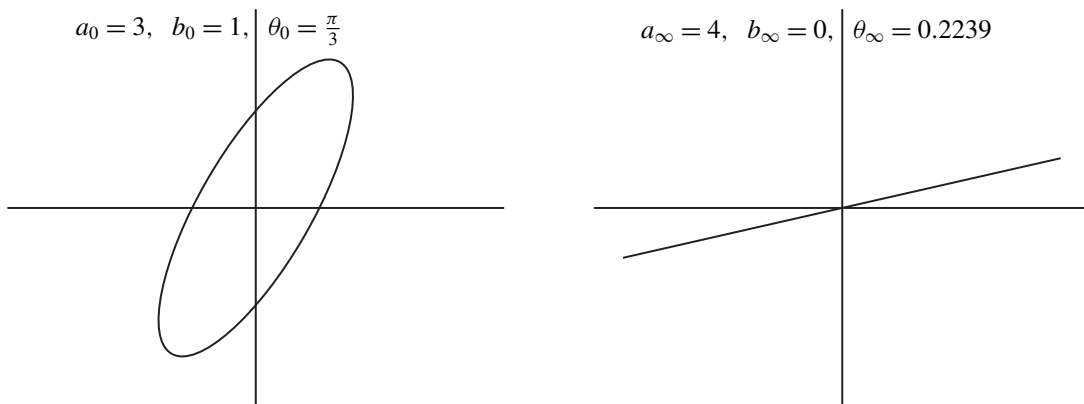


Figure 1. The initial ellipse and the final segment.

which again yields $b_\infty = 0$. By (64)

$$\sin(2\theta_\infty) = \frac{a_0 - b_0}{a_0 + b_0} \sin(2\theta_0), \tag{65}$$

which determines the limit angle in terms of the initial data.

Let us turn now to the case $\frac{\pi}{4} < \theta_0 < \frac{\pi}{2}$. In Figure 1 one can see the initial ellipse and the final segment. In view of the first two equations of the system (56), at least for a short time $a(t)$ decreases and $b(t)$ increases. If one has $\frac{\pi}{4} \leq \theta_\infty$, then $\cos(2\theta(t)) < 0$ for $t > 0$ and $a(t)$ decreases and $b(t)$ increases for all times. Integrating the third equation in (56) we obtain

$$\tan(\theta(t)) = \tan(\theta_0) \exp\left(\frac{-4}{a_0 + b_0} \int_0^t \frac{a(s)b(s)}{a(s) - b(s)} ds\right) \leq \tan(\theta_0) \exp\left(-\frac{4b_0^2}{a_0^2 - b_0^2} t\right).$$

Letting $t \rightarrow \infty$ we get $\tan(\theta_\infty) = 0$, which is impossible. Hence $\theta_\infty < \frac{\pi}{4}$. Then for some t_0 we have $\theta(t_0) < \frac{\pi}{4}$, which brings us into the previous case, in particular to the expression (65) for the limiting angle θ_∞ .

Arguing similarly with $t \rightarrow -\infty$ we get (65) with $\sin(2\theta_\infty)$ replaced by $\sin(2\theta_{-\infty})$, where $\theta_{-\infty} = \lim_{t \rightarrow -\infty} \theta(t)$. Thus $\theta_{-\infty} = \frac{\pi}{2} - \theta_\infty$.

The case $a_0 < b_0$ is reduced to $a_0 > b_0$ by taking conjugates (symmetry with respect to the horizontal axis). Indeed, (53) is invariant by taking conjugates, as a simple computation shows. If one has $a_0 < b_0$ and an angle θ_0 , the symmetric ellipse has semiaxes $A_0 = b_0$, $B_0 = a_0$ and angle $\theta'_0 = \frac{\pi}{2} - \theta_0$.

Appendix: Existence of principal values

The first fact we prove in this section is the following.

Lemma. *Let D be a bounded domain with boundary of class $C^{1+\gamma}$, $0 < \gamma < 1$. Let $L : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ be an even kernel, continuous on $\mathbb{R}^n \setminus \{0\}$, homogeneous of degree $-n$, which satisfies cancellation property*

$\int_{|\xi|=1} L(\xi) d\sigma(\xi) = 0$. Then for each $x \in \partial D$ the principal value

$$(\text{p.v. } L \star \chi_D)(x) = \lim_{\epsilon \rightarrow 0} \int_{\{y \in D: |y-x| > \epsilon\}} L(x-y) dy$$

exists.

Proof. Without loss of generality assume that $x = 0$, that the tangent hyperplane to ∂D at 0 is $\{y \in \mathbb{R}^n : y_n = 0\}$ and that $r_0 > 0$ is so small that there exists a function

$$\varphi \in C^{1+\gamma}(B'(0, 2r_0)), \quad B'(0, 2r_0) = \{y \in \mathbb{R}^n : |y'| < 2r_0\}, \quad y' = (y_1, \dots, y_{n-1}),$$

such that $D \cap B(0, r_0) = \{y \in B(0, r_0) : y_n < \varphi(y')\}$.

For $0 < r$ set

$$S_r = \{y \in \mathbb{R}^n : |y| = r\}, \quad S_r^+ = \{y \in S_r : y_n > 0\} \quad \text{and} \quad S_r^- = \{y \in S_r : y_n < 0\}.$$

Since L is even,

$$0 = \int_{S_r} L(y) d\sigma(y) = \int_{S_r^+} L(y) d\sigma(y) + \int_{S_r^-} L(y) d\sigma(y) = 2 \int_{S_r^-} L(y) d\sigma(y).$$

Set $H_- = \{y \in \mathbb{R}^n : y_n < 0\}$. For $0 < \delta < \epsilon < r_0$ we then have

$$\begin{aligned} - \int_{\{y \in D: |y| > \epsilon\}} L(y) dy + \int_{\{y \in D: |y| > \delta\}} L(y) dy \\ = \int_{\{y \in \mathbb{R}^n: \delta < |y| < \epsilon\} \cap (D \setminus H_-)} L(y) dy - \int_{\{y \in \mathbb{R}^n: \delta < |y| < \epsilon\} \cap (H_- \setminus D)} L(y) dy. \end{aligned}$$

The tangential domains $(D \setminus H_-) \cap B(0, \epsilon)$ and $(H_- \setminus D) \cap B(0, \epsilon)$ are very small. Indeed,

$$\begin{aligned} \left| \int_{\{y \in \mathbb{R}^n: \delta < |y| < \epsilon\} \cap (D \setminus H_-)} L(y) dy \right| &\leq \int_{\delta}^{\epsilon} \frac{1}{\rho^n} \rho^{n-1} \sigma\{\theta \in S^{n-1} : \rho\theta \in D \setminus H_-\} d\rho \\ &\leq C \int_{\delta}^{\epsilon} \rho^{-1+\gamma} d\rho \leq \frac{C}{\gamma} \epsilon^{\gamma}. \end{aligned}$$

One obtains in the same way

$$\left| \int_{\{y \in \mathbb{R}^n: \delta < |y| < \epsilon\} \cap (H_- \setminus D)} L(y) dy \right| \leq \frac{C}{\gamma} \epsilon^{\gamma},$$

and so the proof is complete. □

The second result is the following.

Lemma. Let D be a bounded domain with boundary of class $C^{1+\gamma}$, $0 < \gamma < 1$. Let $K : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ be an odd kernel of class $C^1(\mathbb{R}^n \setminus \{0\})$, homogeneous of degree $-(n-1)$. Let φ be a function defined on ∂D satisfying a Hölder condition of some positive order on ∂D . Then for each $x \in \partial D$ and each $1 \leq j \leq n$ the principal value

$$(\text{p.v. } K \star \varphi n_j d\sigma)(x) = \lim_{\epsilon \rightarrow 0} \int_{\{y \in \partial D: |y-x| > \epsilon\}} K(x-y) \varphi(y) n_j(y) d\sigma(y)$$

exists.

Proof. It is easy to get rid of φ . Indeed

$$\begin{aligned} & \int_{\{y \in \partial D : \epsilon < |y-x|\}} K(x-y)\varphi(y)n_j(y) d\sigma(y) \\ &= \int_{\{y \in \partial D : \epsilon < |y-x|\}} K(x-y)(\varphi(y) - \varphi(x))n_j(y) d\sigma(y) + \varphi(x) \int_{\{y \in \partial D : \epsilon < |y-x|\}} K(x-y)n_j(y) d\sigma(y) \end{aligned}$$

and the first integral in the right-hand side tends as $\epsilon \rightarrow 0$ to the absolutely convergent integral

$$\int_{\partial D} K(x-y)(\varphi(y) - \varphi(x))n_j(y) d\sigma(y).$$

Hence we can assume that φ is identically 1.

We can also assume, as in the proof of the previous lemma, that $x = 0$, the tangent hyperplane to ∂D at 0 is $\{y \in \mathbb{R}^n : y_n = 0\}$ and the domain D inside $B(0, \epsilon)$ is exactly $\{y \in B(0, \epsilon) : y_n < \varphi(y')\}$. By the divergence theorem

$$\begin{aligned} \int_{\{y \in \partial D : \epsilon < |y|\}} K(-y)n_j(y) d\sigma(y) &= - \int_{\{y \in D : \epsilon < |y|\}} \partial_j K(y) dy + \int_{\{y \in D : |y|=\epsilon\}} K(y)n_j(y) d\sigma(y) \\ &= -I + II, \end{aligned}$$

where in the last identity one is defining I and II .

To apply the previous lemma to I we need to check that $\partial_j K(y)$, which is continuous off the origin, even and homogeneous of degree $-n$, and has vanishing integral on the unit sphere. By the divergence theorem

$$\int_{1 < |y| < 2} \partial_j K(y) dy = \int_{|y|=2} K(y) n_j(y) d\sigma(y) - \int_{|y|=1} K(y) n_j(y) d\sigma(y),$$

which is 0, since the two integrals over the spheres are the same by homogeneity. Hence, changing to polar coordinates,

$$0 = \int_{1 < |y| < 2} \partial_j K(y) dy = \log 2 \int_{|\theta|=1} K(\theta) d\sigma(\theta),$$

which takes care of I .

For the term II , set, as before, $H_- = \{y \in \mathbb{R}^n : y_n < 0\}$. We then have

$$\begin{aligned} \int_{\{y \in D : |y|=\epsilon\}} K(y)n_j(y) d\sigma(y) &= \int_{\{y \in D \setminus H_- : |y|=\epsilon\}} K(y) n_j(y) d\sigma(y) \\ &+ \int_{\{y \in H_- : |y|=\epsilon\}} K(y) n_j(y) d\sigma(y) - \int_{\{y \in H_- \setminus D : |y|=\epsilon\}} K(y) n_j(y) d\sigma(y). \end{aligned}$$

The first and third terms tend to 0 with ϵ , because the domains of integration are tangential. Indeed,

$$\sigma(\partial B(0, \epsilon) \cap (D \setminus H_-)) + \sigma(\partial B(0, \epsilon) \cap (H_- \setminus D)) \leq C \epsilon^{n-1+\gamma}$$

and so the absolute value of the first and third terms can be estimated by $C \epsilon^\gamma$.

It only remains to note that the second term is independent of ϵ , by homogeneity. □

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JUAN CARLOS CANTERO: canteroguardeno@gmail.com

Departament de Matemàtiques, Universitat Autònoma de Barcelona, Barcelona, Catalonia

JOAN MATEU: joan.mateu@uab.cat

Departament de Matemàtiques, Universitat Autònoma de Barcelona and Centre de Recerca Matemàtica, Barcelona, Catalonia

JOAN OROBITG: joan.orobitg@uab.cat

Departament de Matemàtiques, Universitat Autònoma de Barcelona and Centre de Recerca Matemàtica, Barcelona, Catalonia

JOAN VERDERA: joan.verdera@uab.cat

Departament de Matemàtiques, Universitat Autònoma de Barcelona and Centre de Recerca Matemàtica, Barcelona, Catalonia

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