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Combining Elliott, Gong, Lin and Niu's result and Castillejos and Evington's result, we see that if A is a simple separable nuclear monotracial C*-algebra, then $A \otimes \mathcal{W}$ is isomorphic to $\mathcal{W}$, where $\mathcal{W}$ is the Razak–Jacelon algebra. In this paper, we give another proof of this. In particular, we show that if $\mathcal{D}$ is a simple separable nuclear monotracial M_{2^∞}-stable C*-algebra which is KK-equivalent to $\{0\}$, then $\mathcal{D}$ is isomorphic to $\mathcal{W}$ without considering tracial approximations of C*-algebras with finite nuclear dimension. Our proof is based on Matui and Sato's technique, Schafhauser's idea in his proof of the Tikuisis–White–Winter theorem and properties of Kirchberg's central sequence C*-algebra $F(\mathcal{D})$ of $\mathcal{D}$. Note that some results for $F(\mathcal{D})$ are based on Elliott, Gong, Lin and Niu's stable uniqueness theorem. Also, we characterize $\mathcal{W}$ by using properties of $F(\mathcal{W})$. Indeed, we show that a simple separable nuclear monotracial C*-algebra D is isomorphic to $\mathcal{W}$ if and only if D satisfies the following properties:

(i) For any $\theta \in [0,1]$, there exists a projection p in $F(D)$ such that $\tau_{D,\omega}(p) = \theta$.

(ii) If p and q are projections in $F(D)$ such that $0 < \tau_{D,\omega}(p) = \tau_{D,\omega}(q)$, then p is Murray–von Neumann equivalent to q.

(iii) There exists an injective homomorphism from D to $\mathcal{W}$.

1. Introduction

The Razak–Jacelon algebra $\mathcal{W}$ is a certain simple separable nuclear monotracial C*-algebra which is KK-equivalent to $\{0\}$. Note that such a C*-algebra must be stably projectionless; that is, $\mathcal{W} \otimes M_n(\mathbb{C})$ has no nonzero projections for any $n \in \mathbb{N}$. In particular, every stably projectionless C*-algebra is nonunital. Jacelon [2013] constructed $\mathcal{W}$ as an inductive limit C*-algebra of Razak's building blocks [2002]. We can regard $\mathcal{W}$ as a stably finite analogue of the Cuntz algebra $\mathcal{O}_2$. In particular, $\mathcal{W}$ is expected to play a central role in the classification theory of simple separable nuclear stably projectionless C*-algebras as $\mathcal{O}_2$ played in the classification theory of Kirchberg algebras; see, for example, [Rørdam 2002; Gabe 2020]. We refer the reader to [Elliott et al. 2020a; 2020b; Gong and Lin 2020] for recent progress in the classification of simple separable nuclear stably projectionless C*-algebras. Note that there exist many interesting examples of simple stably projectionless C*-algebras. See, for example, [Connes 1982; Elliott 1996; Kishimoto 1999; Kishimoto and Kumjian 1996; 1997; Robert 2012].

Combining Elliott, Gong, Lin and Niu's result [Elliott et al. 2020a] and Castillejos and Evington's result [2020] (see also [Castillejos et al. 2021]), we see that if A is a simple separable nuclear monotracial

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C*-algebra, then $A \otimes \mathcal{W}$ is isomorphic to $\mathcal{W}$. This can be considered as a Kirchberg–Phillips-type absorption theorem [2000] for $\mathcal{W}$. In this paper, we give another proof of this. In our proof, we do not consider tracial approximations of C*-algebras with finite nuclear dimension. Also, we mainly consider abstract settings and do not use any classification theorem based on inductive limit structures of $\mathcal{W}$ other than Razak's classification theorem [2002]. (Actually, we need Razak's classification theorem only for $\mathcal{W} \otimes M_{2^\infty} \cong \mathcal{W}$.) We obtain a Kirchberg–Phillips-type absorption theorem for $\mathcal{W}$ as a corollary of the following theorem.

Theorem 6.1. *Let $\mathcal{D}$ be a simple separable nuclear monotracial M_{2^∞}-stable* C**-algebra which is KK-equivalent to $\{0\}$. Then $\mathcal{D}$ is isomorphic to $\mathcal{W}$.*

Our proof of the theorem above is based on Matui and Sato's technique [2012; 2014a; 2014b], Schafhauser's idea [2020a] (see also [Schafhauser 2020b]) in his proof of the Tikuisis–White–Winter theorem [Tikuisis et al. 2017] and properties of Kirchberg's central sequence C*-algebra $F(\mathcal{D})$ of $\mathcal{D}$.

Matui and Sato's technique enables us to show that certain (relative) central sequence C*-algebras have strict comparison. Note that a key concept in their technique is property (SI). This concept was introduced in [Sato 2009; 2010].

Borrowing Schafhauser's idea, we show that if $\mathcal{D}$ is a simple separable nuclear monotracial (M_{2^∞}-stable) C*-algebra which is KK-equivalent to $\{0\}$, then there exist "trace-preserving" homomorphisms from $\mathcal{D}$ to ultrapowers B^ω of certain C*-algebras B. Combining this and a uniqueness result for approximate homomorphisms from $\mathcal{D}$, we obtain an existence result, that is, existence of homomorphisms from $\mathcal{D}$ to certain C*-algebras. Schafhauser's arguments are based on extension theory (or KK-theory) and Elliott and Kucerovsky's result [2001] with a correction by Gabe [2016]. Hence Schafhauser's arguments are suitable for our purpose, that is, a study of C*-algebras which are KK-equivalent to $\{0\}$.

We studied properties of $F(\mathcal{W})$ in [Nawata 2019; 2021] by using the stable uniqueness theorem in [Elliott et al. 2020a]. In particular, we showed that $F(\mathcal{W})$ has many projections and satisfies a certain comparison theory for projections. By these properties and Connes' 2×2 matrix trick, we can show that every trace-preserving endomorphism of $\mathcal{W}$ is approximately inner. (Note that Jacelon [2013, Corollary 4.6] showed this result as an application of Razak's results [2002].) This argument is a traditional argument in the theory of operator algebras; see [Connes 1976]. In this paper, we remark that arguments in [Nawata 2019; 2021] work for a simple separable nuclear monotracial M_{2^∞}-stable C*-algebra $\mathcal{D}$ which is KK-equivalent to $\{0\}$. Also, we characterize $\mathcal{W}$ by using these properties of $F(\mathcal{W})$. Indeed, we show the following theorem.

Theorem 6.4. *Let D be a simple separable nuclear monotracial* C**-algebra. Then D is isomorphic to $\mathcal{W}$ if and only if D satisfies the following properties*:

(i) *For any $\theta \in [0, 1]$, there exists a projection p in $F(D)$ such that $\tau_{D,\omega}(p) = \theta$.*

(ii) *If p and q are projections in $F(D)$ such that $0 < \tau_{D,\omega}(p) = \tau_{D,\omega}(q)$, then p is Murray–von Neumann equivalent to q.*

(iii) *There exists an injective homomorphism from D to $\mathcal{W}$.*

This paper is organized as follows. In Section 2, we collect notation, definitions and some results. In particular, we recall Matui and Sato's technique. In Section 3, we introduce the property W, which is a key property for uniqueness results. Also, we remark that arguments in [Nawata 2019; 2021] work for more general settings. In Section 4, we show uniqueness results. First, we show that if D has property W, then every trace-preserving endomorphism of D is approximately inner. Secondly, we consider a uniqueness theorem for approximate homomorphisms from a simple separable nuclear monotracial M_{2^∞}-stable C*-algebra $\mathcal{D}$ which is KK-equivalent to $\{0\}$ for an existence result in Section 5. In Section 5, we show an existence result by borrowing Schafhauser's idea. In Section 6, we show the main results in this paper.

2. Preliminaries

In this section we shall collect notation, definitions and some results. We refer the reader to [Blackadar 2006; Pedersen 1979] for basics of operator algebras.

For a C*-algebra A, we denote by A_+ the sets of positive elements of A and by $A^\sim$ the unitization algebra of A. Note that if A is unital, then $A = A^\sim$. For $a, b \in A_+$, we say that a is *Murray–von Neumann equivalent to* b, written $a \sim b$, if there exists an element z in A such that $z^*z = a$ and $zz^* = b$. Note that $\sim$ is an equivalence relation by [Pedersen 1998, Theorem 3.5]. For $a, b \in A$, we denote by $[a, b]$ the commutator $ab - ba$. For a subset F of A and $\varepsilon > 0$, we say that a completely positive (c.p.) map $\varphi : A \to B$ is (F, ε)-*multiplicative* if

$$\|\varphi(ab) - \varphi(a)\varphi(b)\| < \varepsilon$$

for any $a, b \in F$. Let $\mathcal{Z}$ and M_{2^∞} denote the Jiang–Su algebra and the CAR algebra, respectively. We say a C*-algebra A is *monotracial* if A has a unique tracial state and no unbounded traces. In the case where A is monotracial, we denote by τ_A the unique tracial state on A unless otherwise specified.

2A. *Razak–Jacelon algebra $\mathcal{W}$.* The *Razak–Jacelon algebra* $\mathcal{W}$ is a certain simple separable nuclear monotracial C*-algebra which is KK-equivalent to $\{0\}$. In [Jacelon 2013], $\mathcal{W}$ is constructed as an inductive limit C*-algebra of Razak's building blocks. By Razak's classification theorem [2002], $\mathcal{W}$ is M_{2^∞}-stable, and hence $\mathcal{W}$ is $\mathcal{Z}$-stable. In this paper, we do not assume any classification theorem for $\mathcal{W}$ other than Razak's classification theorem.

2B. *Kirchberg's central sequence* C*-*algebras.* We shall recall the definition of Kirchberg's central sequence C*-algebras [2006]. Fix a free ultrafilter ω on $\mathbb{N}$. For a C*-algebra B, put

$$c_\omega(B) := \big\{\{x_n\}_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, B) \mid \lim_{n\to\omega} \|x_n\| = 0\big\}, \quad B^\omega := \ell^\infty(\mathbb{N}, B)/c_\omega(B).$$

We denote by $(x_n)_n$ a representative of an element in B^ω. Let A be a C*-subalgebra of B^ω. Set

$$\mathrm{Ann}(A, B^\omega) := \{(x_n)_n \in B^\omega \cap A' \mid (x_n)_n a = 0 \text{ for any } a \in A\}.$$

Then $\mathrm{Ann}(A, B^\omega)$ is a closed ideal of $B^\omega \cap A'$. Define a (*relative*) *central sequence* C*-*algebra* $F(A, B)$ of $A \subseteq B^\omega$ by

$$F(A, B) := B^\omega \cap A' / \mathrm{Ann}(A, B^\omega).$$

We identify B with the C*-subalgebra of B^ω consisting of equivalence classes of constant sequences. In the case $A = B$, we denote $F(B, B)$ by $F(B)$ and call it the *central sequence* C*-*algebra of* B. If A is σ-unital, then $F(A, B)$ is unital by [Kirchberg 2006, Proposition 1.9]. Indeed, let $s = (s_n)_n$ be a strictly positive element in $A \subseteq B^\omega$. Since we have $\lim_{k\to\infty} s^{1/k}s = s$, taking a suitable sequence $\{k(n)\}_{n\in\mathbb{N}} \subset \mathbb{N}$, we obtain $s' = (s_n^{1/k(n)})_n \in B^\omega$ such that $s's = s$. Then it is easy to see that $s' \in B^\omega \cap A'$ and $[s'] = 1$ in $F(A, B)$. Note that the inclusion $B \subset B^\sim$ induces an isomorphism from $F(A, B)$ onto $F(A, B^\sim)$ because we have $[xs'] = [x]$ in $F(A, B^\sim)$ for any $x \in (B^\sim)^\omega \cap A'$.

Let τ_B be a tracial state on B. Define $\tau_{B,\omega} : B^\omega \to \mathbb{C}$ by $\tau_{B,\omega}((x_n)_n) = \lim_{n\to\omega} \tau_B(x_n)$ for any $(x_n)_n \in B^\omega$. Since ω is an ultrafilter, it is easy to see that $\tau_{B,\omega}$ is a well-defined tracial state on B^ω. The following proposition is a relative version of [Nawata 2019, Proposition 2.1].

Proposition 2.1. *Let B be a* C*-*algebra with a faithful tracial state τ_B, and let A be a* C*-*subalgebra of B^ω. Assume that $\tau_{B,\omega}|_A$ is a state. Then $\tau_{B,\omega}((x_n)_n) = 0$ for any $(x_n)_n \in \mathrm{Ann}(A, B^\omega)$.*

Proof. Let $\{h_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for A. Since $\tau_{B,\omega}|_A$ is a state, we have $\lim \tau_{B,\omega}(h_\lambda) = 1$. The rest of proof is same as the proof of [Nawata 2019, Proposition 2.1]. □

By the proposition above, if $\tau_{B,\omega}|_A$ is a state, then $\tau_{B,\omega}$ induces a tracial state on $F(A, B)$. We denote it by the same symbol $\tau_{B,\omega}$ for simplicity.

2C. *Invertible elements in unitization algebras.* Let $\mathrm{GL}(A^\sim)$ denote the set of invertible elements in $A^\sim$. The following proposition is trivial if $1_{A^\sim} = 1_{B^\sim}$.

Proposition 2.2. *Let $A \subseteq B$ be an inclusion of* C*-*algebras. Then* $\mathrm{GL}(A^\sim) \subset \overline{\mathrm{GL}(B^\sim)}$.

Proof. Let $x \in \mathrm{GL}(A^\sim)$. There exists $\varepsilon_0 > 0$ such that for any $0 \le \varepsilon < \varepsilon_0$ we have $x + \varepsilon 1_{A^\sim} \in \mathrm{GL}(A^\sim)$ because $\mathrm{GL}(A^\sim)$ is open. Since $\mathrm{Sp}_A(x) \cup \{0\} = \mathrm{Sp}_B(x) \cup \{0\}$, we have $x + \varepsilon 1_{B^\sim} \in \mathrm{GL}(B^\sim)$ for any $0 < \varepsilon < \varepsilon_0$. Therefore $x \in \overline{\mathrm{GL}(B^\sim)}$. □

The following corollary is an immediate consequence of the proposition above.

Corollary 2.3. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of* C*-*algebras with $A_n \subseteq A_{n+1}$, and let $A = \overline{\bigcup_{n=1}^\infty A_n}$. If $A_n \subseteq \overline{\mathrm{GL}(A_n^\sim)}$ for any $n \in \mathbb{N}$, then $A \subseteq \overline{\mathrm{GL}(A^\sim)}$.*

The following proposition is well known if B is unital. See, for example, the proof of [Schafhauser 2020a, Proposition 3.2].

Proposition 2.4. *Let B be a* C*-*algebra with $B \subseteq \overline{\mathrm{GL}(B^\sim)}$. Then $B^\omega \subseteq \overline{\mathrm{GL}((B^\omega)^\sim)}$.*

Proof. We shall show only the case where B is nonunital. Let $(x_n)_n \in B^\omega$. Because of $B \subseteq \overline{\mathrm{GL}(B^\sim)}$, there exists $(z_n)_n \in (B^\sim)^\omega$ such that $z_n \in \mathrm{GL}(B^\sim)$ for any $n \in \mathbb{N}$ and $(x_n)_n = (z_n)_n$ in $(B^\sim)^\omega$. For any $n \in \mathbb{N}$, put $u_n := z_n(z_n^* z_n)^{-1/2}$. Then u_n is a unitary element and $z_n = u_n(z_n^* z_n)^{1/2}$. Note that we have $(x_n)_n = (u_n)_n(x_n^* x_n)_n^{1/2}$. For any $n \in \mathbb{N}$, there exist $y_n \in B$ and $\lambda_n \in \mathbb{C}$ such that $u_n = y_n + \lambda_n 1_{B^\sim}$ and $|\lambda_n| = 1$ because u_n is a unitary element in $B^\sim$. Since ω is an ultrafilter, there exists $\lambda_0 \in \mathbb{C}$ such that $\lim_{n\to\omega} \lambda_n = \lambda_0$. Hence

$$(u_n)_n = (y_n)_n + \lambda_0 1_{(B^\omega)^\sim} \in (B^\omega)^\sim.$$

Since

$$((y_n)_n + \lambda_0 1_{(B^\omega)^\sim})((x_n^* x_n)_n^{1/2} + \varepsilon 1_{(B^\omega)^\sim}) \to (x_n)_n$$

as $\varepsilon \to 0$, we have $(x_n)_n \in \overline{\mathrm{GL}((B^\omega)^\sim)}$. □

Note that if B has almost stable rank 1 (see [Robert 2016] for the definition), then $B \subseteq \overline{\mathrm{GL}(B^\sim)}$. Also, if B is unital, then $B \otimes \mathbb{K} \subseteq \overline{\mathrm{GL}((B \otimes \mathbb{K})^\sim)}$, where $\mathbb{K}$ is the C*-algebra of compact operators on an infinite-dimensional separable Hilbert space.

2D. ***Matui and Sato's technique.*** We shall review Matui and Sato's technique [2012; 2014a; 2014b]. Let B be a monotracial C*-algebra, and let A be a simple separable nuclear monotracial C*-subalgebra of B^ω. Assume that τ_B is faithful and $\tau_{B,\omega}|_A$ is a state. Consider the Gelfand–Naimark–Segal (GNS) representation π_{τ_B} of B associated with τ_B, and put

$$M := \ell^\infty(\mathbb{N}, \pi_{\tau_B}(B)'')/\big\{\{x_n\}_{n\in\mathbb{N}} \mid \tilde{\tau}_{B,\omega}((x_n^* x_n)_n) := \lim_{n\to\omega} \tilde{\tau}_B(x_n^* x_n) = 0\big\},$$

where $\tilde{\tau}_B$ is the unique normal extension of τ_B on $\pi_{\tau_B}(B)''$. Note that M is a von Neumann algebraic ultrapower of $\pi_{\tau_B}(B)''$ and $\tilde{\tau}_{B,\omega}$ is a faithful normal tracial state on M. Since B is monotracial, $\pi_{\tau_B}(B)''$ is a finite factor, and hence M is also a finite factor. Define a homomorphism ϱ from B^ω to M by $\varrho((x_n)_n) = (\pi_{\tau_B}(x_n))_n$. Kaplansky's density theorem implies that ϱ is surjective. Moreover, [Matui and Sato 2014a, Theorem 3.1] (see also [Kirchberg and Rørdam 2014, Theorem 3.3]) implies that the restriction ϱ on $B^\omega \cap A'$ is a surjective homomorphism onto $M \cap \varrho(A)'$.

Proposition 2.5. *With notation as above, $M \cap \varrho(A)'$ is a finite factor.*

Proof. Note that $\tilde{\tau}_{B,\omega}$ is the unique tracial state on M since M is a finite factor. It is enough to show that $M \cap \varrho(A)'$ is monotracial. Let τ be a tracial state on $M \cap \varrho(A)'$. Since we assume that $\tau_{B,\omega}|_A$ is a state, we see that if A is unital, then $\varrho(1_A) = 1_M$. Hence ϱ can be extended to a unital homomorphism $\varrho^\sim$ from $A^\sim$ to M, and $M \cap \varrho(A)' = M \cap \varrho^\sim(A^\sim)'$. By [Bosa et al. 2019, Lemma 3.21], there exists a positive element a in $A^\sim$ such that $\tilde{\tau}_{B,\omega}(\varrho^\sim(a)) = 1$ and $\tau(x) = \tilde{\tau}_{B,\omega}(\varrho^\sim(a)x)$ for any $x \in M \cap \varrho(A)'$. Since A is monotracial,

$$\tau(x) = \tilde{\tau}_{B,\omega}(\varrho^\sim(a)x) = \tilde{\tau}_{B,\omega}(\varrho^\sim(a))\tilde{\tau}_{B,\omega}(x) = \tilde{\tau}_{B,\omega}(x).$$

Indeed, let x_0 be a positive contraction in $M \cap \varrho(A)'$. For any $a \in A$, define $\tau'(a) := \tilde{\tau}_{B,\omega}(\varrho(a)x_0)$. Then τ' is a tracial positive linear functional on A. Since A is monotracial and $\tau_{B,\omega}|_A$ is a tracial state on A, there exists a positive number t such that $\tau'(a) = t\,\tau_{B,\omega}(a)$ for any $a \in A$. Note that if $\{h_n\}_{n\in\mathbb{N}}$ is an approximate unit for A, then $t = \lim_{n\to\infty} \tau'(h_n)$. On the other hand, we have

$$\begin{aligned}|\tilde{\tau}_{B,\omega}(x_0) - \tau'(h_n)| &= |\tilde{\tau}_{B,\omega}((1-\varrho(h_n))x_0)| = |\tilde{\tau}_{B,\omega}((1-\varrho(h_n))^{1/2}x_0(1-\varrho(h_n))^{1/2})| \\ &\le |\tilde{\tau}_{B,\omega}(1-\varrho(h_n))| = |1-\tau_{B,\omega}(h_n)| \to 0\end{aligned}$$

as $n \to \infty$. Hence $t = \tilde{\tau}_{B,\omega}(x_0)$, and $\tilde{\tau}_{B,\omega}(\varrho(a)x_0) = \tilde{\tau}_{B,\omega}(\varrho(a))\tilde{\tau}_{B,\omega}(x_0)$ for any $a \in A$. It is easy to see that this implies $\tilde{\tau}_{B,\omega}(\varrho^\sim(a)x) = \tilde{\tau}_{B,\omega}(\varrho^\sim(a))\tilde{\tau}_{B,\omega}(x)$ for any $a \in A^\sim$ and $x \in M \cap \varrho(A)'$. Therefore we have $\tau(x) = \tilde{\tau}_{B,\omega}(x)$ for any $x \in M \cap \varrho(A)'$. Consequently, $M \cap \varrho(A)'$ is monotracial. □

For $a, b \in A_+$, we say that a is *Cuntz smaller than* b, written $a \precsim b$, if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of A such that $\|x_n^* b x_n - a\| \to 0$. A monotracial C^*-algebra B is said to have *strict comparison* if, for any $k \in \mathbb{N}$, $a, b \in M_k(B)_+$ with $d_{\tau_B\otimes \mathrm{Tr}_k}(a) < d_{\tau_B\otimes \mathrm{Tr}_k}(b)$ implies $a \precsim b$, where Tr_k is the unnormalized trace on $M_k(\mathbb{C})$ and $d_{\tau_B\otimes \mathrm{Tr}_k}(a) = \lim_{n\to\infty} \tau_B \otimes \mathrm{Tr}_k(a^{1/n})$. Using [Nawata 2013, Lemma 5.7], essentially the same proofs as [Matui and Sato 2012, Theorem 1.1; 2014a, Lemma 3.2] show the following proposition. See also the proof of [Nawata 2021, Lemma 3.6].

Proposition 2.6. *Let B be a monotracial C^*-algebra, and let A be a simple separable non-type-I nuclear monotracial C^*-subalgebra of B^ω. Assume that τ_B is faithful, $\tau_{B,\omega}|_A$ is a state and B has strict comparison. Then B has property (SI) relative to A; that is, for any positive contractions a and b in $B^\omega \cap A'$ satisfying*

$$\tau_{B,\omega}(a) = 0 \quad \textit{and} \quad \inf_{m\in\mathbb{N}} \tau_{B,\omega}(b^m) > 0,$$

there exists an element s in $B^\omega \cap A'$ such that $s^*s = a$ and $bs = s$.

By Proposition 2.1, ϱ induces a surjective homomorphism from $F(A, B)$ to $M \cap \varrho(A)'$. We denote it by the same symbol ϱ for simplicity. Using Propositions 2.5 and 2.6, essentially the same proofs as [Matui and Sato 2014a, Proposition 3.3; 2014b, Proposition 4.8] show the following proposition. See also the proof of [Nawata 2021, Proposition 3.8].

Proposition 2.7. *Let B be a monotracial C^*-algebra, and let A be a simple separable non-type-I nuclear monotracial C^*-subalgebra of B^ω. Assume that τ_B is faithful, $\tau_{B,\omega}|_A$ is a state and B has strict comparison. Then $F(A, B)$ is monotracial and has strict comparison. Furthermore, if a and b are positive elements in $F(A, B)$ satisfying $d_{\tau_{B,\omega}}(a) < d_{\tau_{B,\omega}}(b)$, then there exists an element r in $F(A, B)$ such that $r^*br = a$.*

3. Property W

In this section we shall introduce the property W, which is a key property in Section 4.

Definition 3.1. Let D be a simple separable nuclear monotracial C^*-algebra. We say that D has *property W* if $F(D)$ satisfies the following properties:

(i) For any $\theta \in [0, 1]$, there exists a projection p in $F(D)$ such that $\tau_{D,\omega}(p) = \theta$.

(ii) If p and q are projections in $F(D)$ such that $0 < \tau_{D,\omega}(p) = \tau_{D,\omega}(q)$, then p is Murray–von Neumann equivalent to q.

By arguments in [Nawata 2019; 2021], we see that if $\mathcal{D}$ is a simple separable nuclear monotracial M_{2^∞}-stable C^*-algebra which is KK-equivalent to $\{0\}$, then $\mathcal{D}$ has property W. We shall give a sketch of a proof for reader's convenience and show a slight generalization (or a relative version).

In this section, we assume that $\mathcal{D}$ is a simple separable nuclear monotracial M_{2^∞}-stable C^*-algebra which is KK-equivalent to $\{0\}$ and B is a simple monotracial C^*-algebra with strict comparison and $B \subseteq \overline{\mathrm{GL}(B^\sim)}$. Let Φ be a homomorphism from $\mathcal{D}$ to B^ω such that $\tau_{\mathcal{D}} = \tau_{B,\omega} \circ \Phi$. By the Choi–Effros lifting theorem, there exists a sequence $\{\Phi_n\}_{n\in\mathbb{N}}$ of contractive c.p. maps from $\mathcal{D}$ to B such that

$\Phi(x) = (\Phi_n(x))_n$ for any $x \in \mathcal{D}$. Since we assume $\tau_{\mathcal{D}} = \tau_{B,\omega} \circ \Phi$, we have $\tau_{B,\omega}|_{\Phi(\mathcal{D})}$ is a state. Hence $\tau_{B,\omega}$ is the unique tracial state on $F(\Phi(\mathcal{D}), B)$ by Proposition 2.7. The following proposition is analogous to [Nawata 2019, Proposition 4.2; 2021, Proposition 2.6].

Proposition 3.2. (i) *For any $N \in \mathbb{N}$, there exists a unital homomorphism from $M_{2^N}(\mathbb{C})$ to $F(\Phi(\mathcal{D}), B)$.*

(ii) *For any $\theta \in [0, 1]$, there exists a projection p in $F(\Phi(\mathcal{D}), B)$ such that $\tau_{B,\omega}(p) = \theta$.*

(iii) *Let h be a positive element in $F(\Phi(\mathcal{D}), B)$ such that $d_{\tau_{B,\omega}}(h) > 0$. For any $\theta \in [0, d_{\tau_{B,\omega}}(h))$, there exists a nonzero projection p in $\overline{hF(\Phi(\mathcal{D}), B)h}$ such that $\tau_{B,\omega}(p) = \theta$.*

Proof. (i) Since $\mathcal{D}$ is isomorphic to $\mathcal{D} \otimes M_{2^\infty} = \mathcal{D} \otimes \bigotimes_{n\in\mathbb{N}} M_{2^N}(\mathbb{C})$, an argument similar to that in the proof of Proposition 4.2 in [Nawata 2019], henceforth abbreviated [N19], shows that there exists a family $\{(e_{ij,m})_m\}_{i,j=1}^{2^N}$ of contractions in $\mathcal{D}^\omega \cap \mathcal{D}'$ such that

$$\left(\sum_{\ell=1}^{2^N} e_{\ell\ell,m}x\right)_m = x \quad \text{and} \quad (e_{ij,m}e_{kl,m}x)_m = (\delta_{jk}e_{il,m}x)_m$$

for any $1 \le i, j, k, l \le 2^N$ and $x \in \mathcal{D}$. Note that we have

$$\lim_{m\to\omega} \|([\Phi_n(e_{ij,m}), \Phi_n(x)])_n\| = 0, \quad \lim_{m\to\omega} \left\| \left(\sum_{\ell=1}^{2^N} \Phi_n(e_{\ell\ell,m})\Phi_n(x) - \Phi_n(x)\right)_n \right\| = 0$$

and

$$\lim_{m\to\omega} \|((\Phi_n(e_{ij,m})\Phi_n(e_{kl,m}) - \delta_{jk}\Phi_n(e_{il,m}))\Phi_n(x))_n\| = 0$$

for any $1 \le i, j, k, l \le 2^N$ and $x \in \mathcal{D}$. Hence, for any finite subset $F \subset \mathcal{D}$ and $\varepsilon > 0$, there exists a family of $\{(\Phi_n(e_{ij,(F,\varepsilon)}))_n\}_{i,j=1}^{2^N}$ of contractions in B^ω such that

$$\lim_{n\to\omega} \|[\Phi_n(e_{ij,(F,\varepsilon)}), \Phi_n(x)]\| < \varepsilon, \quad \lim_{n\to\omega} \left\| \sum_{\ell=1}^{2^N} \Phi_n(e_{\ell\ell,(F,\varepsilon)})\Phi_n(x) - \Phi_n(x) \right\| < \varepsilon$$

and

$$\lim_{n\to\omega} \|(\Phi_n(e_{ij,(F,\varepsilon)})\Phi_n(e_{kl,(F,\varepsilon)}) - \delta_{jk}\Phi_n(e_{il,(F,\varepsilon)}))\Phi_n(x)\| < \varepsilon$$

for any $1 \le i, j, k, l \le 2^N$ and $x \in F$. Let $\{F_m\}_{m\in\mathbb{N}}$ be an increasing sequence of finite subsets in $\mathcal{D}$ such that $\mathcal{D} = \overline{\bigcup_{m\in\mathbb{N}} F_m}$. We can find a sequence $\{X_m\}_{m\in\mathbb{N}}$ of elements in ω such that $X_{m+1} \subset X_m$, $\bigcap_{m\in\mathbb{N}} X_m = \varnothing$, and, for any $n \in X_m$,

$$\|[\Phi_n(e_{ij,(F_m,1/m)}), \Phi_n(x)]\| < \frac{1}{m}, \quad \left\| \sum_{\ell=1}^{2^N} \Phi_n(e_{\ell\ell,(F_m,1/m)})\Phi_n(x) - \Phi_n(x) \right\| < \frac{1}{m}$$

and

$$\|(\Phi_n(e_{ij,(F_m,1/m)})\Phi_n(e_{kl,(F_m,1/m)}) - \delta_{jk}\Phi_n(e_{il,(F_m,1/m)}))\Phi_n(x)\| < \frac{1}{m}$$

for any $1 \le i, j, k, l \le 2^N$ and $x \in F_m$. For any $1 \le i, j \le 2^N$, put

$$E_{ij,n} := \begin{cases} 0 & \text{if } n \notin X_1, \\ \Phi_n(e_{ij,(F_m,1/m)}) & \text{if } n \in X_m \setminus X_{m+1} \quad (m \in \mathbb{N}). \end{cases}$$

Then we have $(E_{ij,n})_n \in B^\omega \cap \Phi(\mathcal{D})'$,

$$\sum_{\ell=1}^{2^N}[(E_{\ell\ell,n})_n] = 1 \quad \text{and} \quad [(E_{ij,n})_n][(E_{kl,n})_n] = \delta_{jk}[(E_{il,n})_n]$$

in $F(\Phi(\mathcal{D}), B)$ for any $1 \le i, j, k, l \le 2^N$. Therefore there exists a unital homomorphism from $M_{2^N}(\mathbb{C})$ to $F(\Phi(\mathcal{D}), B)$.

(ii) Since $\mathcal{D}$ is isomorphic to $\mathcal{D} \otimes M_{2^\infty} = \mathcal{D} \otimes \bigotimes_{n\in\mathbb{N}} M_{2^\infty}$, an argument similar to that in the proof of [N19, Proposition 4.2] shows that there exists a positive contraction $(p_m)_m$ in $\mathcal{D}^\omega \cap \mathcal{D}$ such that $((p_m^2 - p_m)x)_m = 0$ for any $x \in \mathcal{D}$ and $\tau_{\mathcal{D},\omega}((p_m)_m) = \theta$. By an argument similar to that above, we obtain a projection p in $F(\Phi(\mathcal{D}), B)$ such that $\tau_{B,\omega}(p) = \theta$.

(iii) Using Proposition 2.7 instead of [N19, Proposition 4.1], we obtain the conclusion by the same argument as in the proof of [N19, Proposition 4.2]. □

The proposition above and the same arguments as in [N19, Section 4] show the following corollary.

Corollary 3.3 ((cf. [N19, Proposition 4.8])). *Let p and q be projections in $F(\Phi(\mathcal{D}), B)$ such that $\tau_{B,\omega}(p) < 1$. Then p and q are Murray–von Neumann equivalent if and only if p and q are unitarily equivalent.*

Since we assume $B \subseteq \overline{\mathrm{GL}(B^\sim)}$, we obtain the following proposition by the same argument as in the proof of [N19, Proposition 4.9].

Proposition 3.4. *Let u be a unitary element in $F(\Phi(\mathcal{D}), B)$. Then there exists a unitary element w in $(B^\sim)^\omega \cap \Phi(\mathcal{D})'$ such that $u = [w]$.*

There exists a homomorphism ρ from $F(\Phi(\mathcal{D}), B) \otimes \mathcal{D}$ to B^ω such that

$$\rho([(x_n)_n] \otimes a) = (x_n \Phi_n(a))_n$$

for any $[(x_n)_n] \in F(\Phi(\mathcal{D}), B)$ and $a \in \mathcal{D}$. For a projection p in $F(\Phi(\mathcal{D}), B)$, put

$$B_p^\omega := \overline{\rho(p \otimes s) B^\omega \rho(p \otimes s)},$$

where s is a strictly positive element in $\mathcal{D}$. Define a homomorphism σ_p from $\mathcal{D}$ to B_p^ω by $\sigma_p(a) := \rho(p \otimes a)$ for any $a \in \mathcal{D}$. Since B has strict comparison, we see that if p is a projection in $F(\Phi(\mathcal{D}), B)$ such that $\tau_{B,\omega}(p) > 0$, then σ_p is (L, N)-full for some maps L and N by [N19, Lemma 3.5 and Proposition 3.7]. (We refer the reader to [N19, Section 3] for details of the (L, N)-fullness.) Therefore [N19, Proposition 3.3] implies the following theorem. We may regard this theorem as a variant of Elliott, Gong, Lin and Niu's stable uniqueness theorem [Elliott et al. 2020a, Corollary 3.15]; see also [Elliott and Niu 2016, Corollary 8.16]. Note that [N19, Proposition 3.3] is also based on the results in [Elliott and Kucerovsky 2001; Gabe 2016; Dadarlat and Eilers 2001; 2002].

Theorem 3.5. *Let Ω be a compact metrizable space. For any finite subsets $F_1 \subset C(\Omega)$, $F_2 \subset \mathcal{D}$ and $\varepsilon > 0$, there exist finite subsets $G_1 \subset C(\Omega)$, $G_2 \subset \mathcal{D}$, $m \in \mathbb{N}$ and $\delta > 0$ such that the following holds. Let p be a projection in $F(\Phi(\mathcal{D}), B)$ such that $\tau_{B,\omega}(p) > 0$. For any contractive $(G_1 \odot G_2, \delta)$-multiplicative* *maps $\psi_1, \psi_2 : C(\Omega)\otimes\mathcal{D}\to B_p^{\omega}$, there exist a unitary element u in $M_{m^2+1}(B_p^{\omega})^{\sim}$ and $z_1, z_2, \dots, z_m\in\Omega$ such that*

$$\Big\| u\Big(\psi_1(f\otimes b)\oplus\overbrace{\bigoplus_{k=1}^{m} f(z_k)\rho(p\otimes b)\oplus\cdots\oplus\bigoplus_{k=1}^{m} f(z_k)\rho(p\otimes b)}^{m}\Big)u^* - \psi_2(f\otimes b)\oplus\overbrace{\bigoplus_{k=1}^{m} f(z_k)\rho(p\otimes b)\oplus\cdots\oplus\bigoplus_{k=1}^{m} f(z_k)\rho(p\otimes b)}^{m}\Big\| < \varepsilon$$

for any $f\in F_1$ and $b\in F_2$.

Using Proposition 2.7, Proposition 3.2 and Corollary 3.3 instead of Propositions 4.1, 4.2, and 4.8 of [N19], the same proof as [N19, Lemma 5.1] shows the following lemma.

Lemma 3.6. *Let Ω be a compact metrizable space, and let F be a finite subset of $C(\Omega)$ and $\varepsilon>0$. Suppose that ψ_1 and ψ_2 are unital homomorphisms from $C(\Omega)$ to $F(\Phi(\mathcal{D}), B)$ such that $\tau_{B,\omega}\circ\psi_1=\tau_{B,\omega}\circ\psi_2$. Then there exist a projection $p\in F(\Phi(\mathcal{D}), B)$, (F,ε)-multiplicative unital c.p. maps ψ_1' and ψ_2' from $C(\Omega)$ to $pF(\Phi(\mathcal{D}), B)p$, a unital homomorphism σ from $C(\Omega)$ to $(1-p)F(\Phi(\mathcal{D}), B)(1-p)$ with finite-dimensional range and a unitary element $u\in F(\Phi(\mathcal{D}), B)$ such that*

$$0<\tau_{B,\omega}(p)<\varepsilon,\quad \|\psi_1(f)-(\psi_1'(f)+\sigma(f))\|<\varepsilon,\quad \|\psi_2(f)-u(\psi_2'(f)+\sigma(f))u^*\|<\varepsilon$$

for any $f\in F$.

The following lemma is essentially the same as [N19, Theorem 5.2] and [Nawata 2021, Theorem 5.2].

Lemma 3.7. *Let Ω be a compact metrizable space, and let F_1 be a finite subset of $C(\Omega)$ and F_2 a finite subset of $\mathcal{D}$, and let $\varepsilon>0$. Then there exist mutually orthogonal positive elements $h_1, h_2, \dots, h_l$ in $C(\Omega)$ of norm 1 such that the following holds. If ψ_1 and ψ_2 are unital homomorphisms from $C(\Omega)$ to $F(\Phi(\mathcal{D}), B)$ such that*

$$\tau_{B,\omega}(\psi_1(h_i))>0,\quad 1\le\forall i\le l,\qquad \text{and}\qquad \tau_{B,\omega}\circ\psi_1=\tau_{B,\omega}\circ\psi_2,$$

then there exists a unitary element u in $(B^{\omega})^{\sim}$ such that

$$\|u\rho(\psi_1(f)\otimes a)u^*-\rho(\psi_2(f)\otimes a)\|<\varepsilon$$

for any $f\in F_1$, $a\in F_2$.

Proof. Take positive elements $h_1, h_2, \dots, h_l$ in $C(\Omega)$ in the same way as in the proof of [N19, Theorem 5.2]. Let ψ_1 and ψ_2 be unital homomorphisms from $C(\Omega)$ to $F(\Phi(\mathcal{D}), B)$ such that $\tau_{B,\omega}(\psi_1(h_i))>0$ for any $1\le i\le l$ and $\tau_{B,\omega}\circ\psi_1=\tau_{B,\omega}\circ\psi_2$. Define homomorphisms Ψ_1 and Ψ_2 from $C(\Omega)\otimes\mathcal{D}$ to B^{ω} by

$$\Psi_1 := \rho\circ(\psi_1\otimes \mathrm{id}_{\mathcal{D}})\quad\text{and}\quad \Psi_2 := \rho\circ(\psi_2\otimes \mathrm{id}_{\mathcal{D}}).$$

Note that there exists $\nu>0$ such that $\tau_{B,\omega}(\psi_1(h_i))\ge\nu$ for any $1\le i\le l$. Using Proposition 3.4, Theorem 3.5 and Lemma 3.6 instead of Corollaries 4.10, 3.8 and Lemma 5.1 in [N19], the same argument as in the proof of [N19, Theorem 5.2] shows that there exists a unitary element u in $(B^{\omega})^{\sim}$ such that

$$\|u\Psi_1(f\otimes a)u^* - \Psi_2(f\otimes a)\| < \varepsilon$$

for any $f\in F_1,\ a\in F_2$. Therefore we obtain the conclusion. □

The following theorem is a generalization of [N19, Theorem 5.3]. See also [N19, Theorem 5.3].

Theorem 3.8. *Let N_1 and N_2 be normal elements in $F(\Phi(\mathcal{D}),B)$ such that $\mathrm{Sp}(N_1)=\mathrm{Sp}(N_2)$ and $\tau_{B,\omega}(f(N_1))>0$ for any $f\in C(\mathrm{Sp}(N_1))_+\setminus\{0\}$. Then there exists a unitary element u in $F(\Phi(\mathcal{D}),B)$ such that $uN_1u^*=N_2$ if and only if $\tau_{B,\omega}(f(N_1))=\tau_{B,\omega}(f(N_2))$ for any $f\in C(\mathrm{Sp}(N_1))$.*

Proof. It is enough to show the "if" part because the "only if" part is obvious. Let $\Omega:=\mathrm{Sp}(N_1)=\mathrm{Sp}(N_2)$, and define unital homomorphisms ψ_1 and ψ_2 from $C(\Omega)$ to $F(\Phi(\mathcal{D}),B)$ by $\psi_1(f):=f(N_1)$ and $\psi_2(f):=f(N_2)$ for any $f\in C(\Omega)$. By the Choi–Effros lifting theorem, there exist sequences of unital c.p. maps $\{\psi_{1,n}\}_{n\in\mathbb{N}}$ and $\{\psi_{2,n}\}_{n\in\mathbb{N}}$ from $C(\Omega)$ to $B^{\sim}$ such that $\psi_1(f)=[(\psi_{1,n}(f))_n]$ and $\psi_2(f)=[(\psi_{2,n}(f))_n]$ for any $f\in C(\Omega)$. Let $F_1:=\{1,\iota\}\subset C(\Omega)$, where ι is the identity function on Ω, that is, $\iota(z)=z$ for any $z\in\Omega$, and let $\{F_{2,m}\}_{m\in\mathbb{N}}$ be an increasing sequence of finite subsets in $\mathcal{D}$ such that $\mathcal{D}=\overline{\bigcup_{m\in\mathbb{N}}F_{2,m}}$. For any $m\in\mathbb{N}$, applying Lemma 3.7 to F_1, $F_{2,m}$ and $1/m$, we obtain mutually orthogonal positive elements $h_{1,m},h_{2,m},\ldots,h_{l(m),m}$ in $C(\Omega)$ of norm 1. Since we have

$$\tau_{B,\omega}(\psi_1(h_{i,m}))>0,\quad 1\le\forall i\le l(m),\qquad\text{and}\qquad \tau_{B,\omega}\circ\psi_1=\tau_{B,\omega}\circ\psi_2$$

by the assumption, Lemma 3.7 implies that there exists a unitary element $(u_{m,n})_n$ in $(B^{\omega})^{\sim}$ such that

$$\|(u_{m,n})_n\rho(\psi_1(f)\otimes a)(u_{m,n}^*)_n-\rho(\psi_2(f)\otimes a)\|<\frac{1}{m}$$

for any $f\in F_1,\ a\in F_{2,m}$. By the definition of ρ, we have

$$\lim_{n\to\omega}\|u_{m,n}\psi_{1,n}(f)\Phi_n(a)u_{m,n}^*-\psi_{2,n}(f)\Phi_n(a)\|<\frac{1}{m}$$

for any $f\in F_1,\ a\in F_{2,m}$. Therefore we inductively obtain a decreasing sequence $\{X_m\}_{m\in\mathbb{N}}$ of elements in ω such that $\bigcap_{m\in\mathbb{N}}X_m=\varnothing$, and, for any $n\in X_m$,

$$\|u_{m,n}\psi_{1,n}(f)\Phi_n(a)u_{m,n}^*-\psi_{2,n}(f)\Phi_n(a)\|<\frac{1}{m}$$

for any $f\in F_1,\ a\in F_{2,m}$. Set

$$u_n:=\begin{cases}1 & \text{if } n\notin X_1,\\ u_{m,n} & \text{if } n\in X_m\setminus X_{m+1}\quad (m\in\mathbb{N}).\end{cases}$$

Then we have

$$\lim_{n\to\omega}\|u_n\Phi_n(a)u_n^*-\Phi_n(a)\|=0,\quad \lim_{n\to\omega}\|u_n\psi_{1,n}(\iota)\Phi_n(a)u_n^*-\psi_{2,n}(\iota)\Phi_n(a)\|=0$$

for any $a\in\mathcal{D}$. Therefore, $(u_n)_n\in(B^{\sim})^{\omega}\cap\Phi(\mathcal{D})'$ and $[(u_n)_n]N_1[(u_n)_n]^*=N_2$ in $F(\Phi(\mathcal{D}),B)$. Since $[(u_n)_n]$ is a unitary element in $F(\Phi(\mathcal{D}),B)$, we obtain the conclusion. □

The following corollary is an immediate consequence of the theorem above.

Corollary 3.9 (cf. [Nawata 2021, Corollary 5.4]). *Let p and q be projections in $F(\Phi(\mathcal{D}), B)$ such that $0 < \tau_{B\omega}(p) < 1$. Then p and q are unitarily equivalent if and only if $\tau_{B,\omega}(p) = \tau_{B,\omega}(q)$.*

The corollary above and the same argument as in the proof of [Nawata 2021, Corollary 5.5] show the following theorem.

Theorem 3.10. *Let p and q be projections in $F(\Phi(\mathcal{D}), B)$ such that $0 < \tau_{B,\omega}(p) \leq 1$. Then p and q are Murray–von Neumann equivalent if and only if $\tau_{B,\omega}(p) = \tau_{B,\omega}(q)$.*

By Proposition 3.2 and applying the theorem above to $B = \mathcal{D}$ and $\Phi = \mathrm{id}_{\mathcal{D}}$, we obtain the following corollary.

Corollary 3.11. *Let $\mathcal{D}$ be a simple separable nuclear monotracial M_{2^∞}-stable* C*-*algebra which is *KK*-equivalent to $\{0\}$. Then $\mathcal{D}$ has property W.*

4. Uniqueness theorem

In this section we shall show that if D has property W, then every trace-preserving endomorphism of D is approximately inner. Furthermore, we shall consider a uniqueness theorem for approximate homomorphisms from a simple separable nuclear monotracial M_{2^∞}-stable C*-algebra $\mathcal{D}$ which is *KK*-equivalent to $\{0\}$ for an existence theorem in Section 5.

Let D be a simple separable nuclear monotracial C*-algebra, and let φ be a trace-preserving endomorphism of D. Define a homomorphism Φ from D to $M_2(D)$ by

$$\Phi(a) := \begin{pmatrix} a & 0 \\ 0 & \varphi(a) \end{pmatrix}$$

for any $a \in D$. Since φ is trace-preserving, we see that $\tau_{M_2(D),\omega}|_{\Phi(D)}$ is a state. Hence $\tau_{M_2(D),\omega}$ is a tracial state on $F(\Phi(D), M_2(D))$. (See Proposition 2.1.) Define homomorphisms ι_{11} and ι_{22} from $F(D)$ to $F(\Phi(D), M_2(D))$ by

$$\iota_{11}([(x_n)_n]) := \left[\left(\begin{pmatrix} x_n & 0 \\ 0 & 0 \end{pmatrix}\right)_n\right] \quad \text{and} \quad \iota_{22}([(x_n)_n]) := \left[\left(\begin{pmatrix} 0 & 0 \\ 0 & \varphi(x_n) \end{pmatrix}\right)_n\right]$$

for any $[(x_n)_n]$ in $F(D)$. It is easy to see that ι_{11} and ι_{22} are well-defined. Put $p := \iota_{11}(1)$ and $q := \iota_{22}(1)$. Note that p and q are projections in $F(\Phi(D), M_2(D))$ and if $\{h_n\}_{n\in\mathbb{N}}$ is an approximate unit for D, then

$$p = \left[\left(\begin{pmatrix} h_n & 0 \\ 0 & 0 \end{pmatrix}\right)_n\right] \quad \text{and} \quad q = \left[\left(\begin{pmatrix} 0 & 0 \\ 0 & \varphi(h_n) \end{pmatrix}\right)_n\right].$$

It can be easily checked that ι_{11} is an isomorphism from $F(D)$ onto $pF(\Phi(D), M_2(D))p$.

Lemma 4.1. *Let D be a simple separable nuclear monotracial* C*-*algebra with property W. Then D is M_{2^∞}-stable, and hence D is $\mathcal{Z}$-stable.*

Proof. Since D has property W, there exists a projection p in $F(D)$ such that $\tau_{D,\omega}(p) = \frac{1}{2}$. Moreover, p is Murray–von Neumann equivalent to $1 - p$. Hence there exists a unital homomorphism from $M_2(\mathbb{C})$ to $F(D)$. By Corollary 1.13 and Proposition 4.11 in [Kirchberg 2006] (see [Blackadar et al. 1992, Proposition 2.12] for the pioneering work), D is M_{2^∞}-stable. □

The lemma above implies that if D has property W, then D has strict comparison and $D \subseteq \overline{\mathrm{GL}(D^{\sim})}$ by [Rørdam 2004a; Robert 2016]. Furthermore, $F(\Phi(D), M_2(D))$ is monotracial and has strict comparison by Proposition 2.7. The following lemma is related to [Nawata 2021, Lemma 6.2].

Lemma 4.2. *With notation as above, if D has property W, then p is Murray–von Neumann equivalent to q in $F(\Phi(D), M_2(D))$.*

Proof. For any $m \in \mathbb{N}$, there exists a projection q_m in $F(D)$ such that $\tau_{D,\omega}(q_m) = 1 - 1/m$ because D has property W. Proposition 2.7 implies that there exists a contraction r_m in $F(\Phi(D), M_2(D))$ such that $r_m^* p r_m = \iota_{22}(q_m)$. By a diagonal argument, we see that there exist a projection q' in $F(D)$ and a contraction r in $F(\Phi(D), M_2(D))$ such that $\tau_{D,\omega}(q') = 1$ and $r^* pr = \iota_{22}(q')$. Note that $\iota_{22}(q')$ is Murray–von Neumann equivalent to prr^*p. There exists a projection p' in $F(D)$ such that $\iota_{11}(p') = prr^*p$ and $\tau_{D,\omega}(p') = 1$ because ι_{11} is an isomorphism from $F(D)$ onto $pF(\Phi(D), M_2(D))p$. Since D has property W, there exist v_1 and v_2 in $F(D)$ such that $v_1^* v_1 = 1$, $v_1 v_1^* = p'$, $v_2^* v_2 = 1$ and $v_2 v_2^* = q'$. Therefore we have

$$p = \iota_{11}(1) \sim \iota_{11}(p') = prr^*p \sim r^*pr = \iota_{22}(q') \sim \iota_{22}(1) = q. \qquad \square$$

The following theorem is one of the main theorems in this section.

Theorem 4.3. *Let D be a simple separable nuclear monotracial* C**-algebra with property W, and let φ be a trace-preserving endomorphism of D. Then φ is approximately inner.*

Proof. By Lemma 4.2, there exists a contraction V in $F(\Phi(D), M_2(D))$ such that

$$V^*V = \left[\left(\begin{pmatrix} h_n & 0 \\ 0 & 0 \end{pmatrix}\right)_n\right] \quad \text{and} \quad VV^* = \left[\left(\begin{pmatrix} 0 & 0 \\ 0 & \varphi(h_n) \end{pmatrix}\right)_n\right],$$

where $\{h_n\}_{n\in\mathbb{N}}$ is an approximate unit for D. It can be easily checked that there exists an element $(v_n)_n$ in D^ω such that

$$V = \left[\left(\begin{pmatrix} 0 & 0 \\ v_n & 0 \end{pmatrix}\right)_n\right],$$

and we have

$$(v_n x)_n = (\varphi(x)v_n)_n, \quad (v_n^* v_n x)_n = x \quad \text{and} \quad (v_n v_n^* \varphi(x))_n = \varphi(x)$$

for any $x \in D$. Since $(v_n x)_n = (\varphi(x)v_n)_n$ and $(\varphi(x)v_n v_n^*)_n = \varphi(x)$, we have $(v_n x v_n^*)_n = \varphi(x)$ for any $x \in D$. Because of $D \subseteq \overline{\mathrm{GL}(D^{\sim})}$, we may assume that v_n is an invertible element in $D^{\sim}$ for any $n \in \mathbb{N}$. (See the proof of Proposition 2.4.) For any $n \in \mathbb{N}$, let $u_n := v_n(v_n^* v_n)^{-1/2}$. Then u_n is a unitary element in $D^{\sim}$. Since $(v_n^* v_n x)_n = x$, we have $(u_n x)_n = (v_n (v_n^* v_n)^{-1/2} x)_n = (v_n x)_n$ for any $x \in D$. Therefore

$$\varphi(x) = (v_n x v_n^*)_n = (u_n x v_n^*)_n = (u_n (v_n x^*)^*)_n = (u_n (u_n x^*)^*)_n = (u_n x u_n^*)_n$$

for any $x \in D$. Consequently, φ is approximately inner. $\square$

Let $\mathcal{D}$ be a simple separable nuclear monotracial M_{2^∞}-stable C*-algebra which is KK-equivalent to $\{0\}$. In the rest of this section, we shall consider a uniqueness theorem for approximate homomorphisms from $\mathcal{D}$ to certain C*-algebras. Let B be a simple monotracial C*-algebra with strict comparison, $B \subseteq \overline{\mathrm{GL}(B^{\sim})}$ and $M_2(B) \subseteq \overline{\mathrm{GL}(M_2(B)^{\sim})}$, and let φ and ψ be homomorphisms from $\mathcal{D}$ to B^ω such that

$\tau_{\mathcal{D}} = \tau_{B,\omega} \circ \varphi = \tau_{B,\omega} \circ \psi$. By the Choi–Effros lifting theorem, there exist sequences of contractive c.p. maps φ_n and ψ_n from $\mathcal{D}$ to B such that $\varphi(a) = (\varphi_n(a))_n$ and $\psi(a) = (\psi_n(a))_n$ for any $a \in \mathcal{D}$. Define a homomorphism Φ from $\mathcal{D}$ to $M_2(B)^\omega$ by

$$\Phi(a) := \left(\begin{pmatrix} \varphi_n(a) & 0 \\ 0 & \psi_n(a) \end{pmatrix}\right)_n$$

for any $a \in \mathcal{D}$. Since $\tau_{\mathcal{D}} = \tau_{B,\omega} \circ \varphi = \tau_{B,\omega} \circ \psi$, we know $\tau_{M_2(B),\omega}|_{\Phi(\mathcal{D})}$ is a state. Hence $\tau_{M_2(B),\omega}$ is a tracial state on $F(\Phi(\mathcal{D}), M_2(B))$ as above. Since $\mathcal{D}$ is separable, there exist elements $(s_n)_n$ and $(t_n)_n$ in B^ω such that $[(s_n)_n] = 1$ in $F(\varphi(\mathcal{D}), B)$ and $[(t_n)_n] = 1$ in $F(\psi(\mathcal{D}), B)$ by arguments in Section 2B. Put

$$p := \left[\left(\begin{pmatrix} s_n & 0 \\ 0 & 0 \end{pmatrix}\right)_n\right] \quad \text{and} \quad q := \left[\left(\begin{pmatrix} 0 & 0 \\ 0 & t_n \end{pmatrix}\right)_n\right]$$

in $F(\Phi(\mathcal{D}), M_2(B))$. It is easy to see that p and q are projections in $F(\Phi(\mathcal{D}), M_2(B))$ such that $\tau_{M_2(B),\omega}(p) = \tau_{M_2(B),\omega}(q) = \frac{1}{2}$. Theorem 3.10 implies that p is Murray–von Neumann equivalent to q. Therefore we obtain the following theorem by an argument similar to that in the proof of Theorem 4.3.

Theorem 4.4. *Let $\mathcal{D}$ be a simple separable nuclear monotracial M_{2^∞}-stable* C**-algebra which is KK-equivalent to $\{0\}$ and B a simple monotracial* C**-algebra with strict comparison, $B \subseteq \overline{\mathrm{GL}(B^\sim)}$ and $M_2(B) \subseteq \overline{\mathrm{GL}(M_2(B)^\sim)}$. If φ and ψ are homomorphisms from $\mathcal{D}$ to B^ω such that $\tau_{\mathcal{D}} = \tau_{B,\omega} \circ \varphi = \tau_{B,\omega} \circ \psi$, then there exists a unitary element u in $(B^\sim)^\omega$ such that $\varphi(a) = u\psi(a)u^*$ for any $a \in \mathcal{D}$.*

The following corollary is an immediate consequence of the theorem above.

Corollary 4.5. *Let $\mathcal{D}$ be a simple separable nuclear monotracial M_{2^∞}-stable* C**-algebra which is KK-equivalent to $\{0\}$ and B a simple monotracial* C**-algebra with strict comparison, $B \subseteq \overline{\mathrm{GL}(B^\sim)}$ and $M_2(B) \subseteq \overline{\mathrm{GL}(M_2(B)^\sim)}$. If φ and ψ are trace-preserving homomorphisms from $\mathcal{D}$ to B, then φ is approximately unitarily equivalent to ψ.*

Remark 4.6. If B is a simple separable exact monotracial $\mathcal{Z}$-stable C*-algebra, then B has strict comparison, $B \subseteq \overline{\mathrm{GL}(B^\sim)}$ and $M_2(B) \subseteq \overline{\mathrm{GL}(M_2(B)^\sim)}$ by [Rørdam 2004a; Robert 2016].

The following corollary is also an immediate consequence of Theorem 4.4.

Corollary 4.7. *Let $\mathcal{D}$ be a simple separable nuclear monotracial M_{2^∞}-stable* C**-algebra which is KK-equivalent to $\{0\}$ and B a simple monotracial* C**-algebra with strict comparison, $B \subseteq \overline{\mathrm{GL}(B^\sim)}$ and $M_2(B) \subseteq \overline{\mathrm{GL}(M_2(B)^\sim)}$. For any finite subset $F \subset \mathcal{D}$ and $\varepsilon > 0$, there exist a finite subset $G \subset \mathcal{D}$ and $\delta > 0$ such that the following holds. If φ and ψ are (G, δ)-multiplicative maps from $\mathcal{D}$ to B such that*

$$|\tau_B(\varphi(a)) - \tau_{\mathcal{D}}(a)| < \delta \quad \textit{and} \quad |\tau_B(\psi(a)) - \tau_{\mathcal{D}}(a)| < \delta$$

for any $a \in G$, then there exists a unitary element u in $B^\sim$ such that

$$\|\varphi(a) - u\psi(a)u^*\| < \varepsilon$$

for any $a \in F$.

5. Existence theorem

In this section, we assume that $\mathcal{D}$ is a simple separable nuclear monotracial M_{2^∞}-stable C*-algebra which is KK-equivalent to $\{0\}$ and B is a simple separable exact monotracial $\mathcal{Z}$-stable C*-algebra. We shall show that there exists a trace-preserving homomorphism from $\mathcal{D}$ to B. Many arguments in this section are motivated by Schafhauser's proof [2020a] (see also [Schafhauser 2020b]) of the Tikuisis–White–Winter theorem [Tikuisis et al. 2017].

The following lemma is related to [Kirchberg and Phillips 2000, Lemma 2.2].

Lemma 5.1. *Let $\mathcal{D}$ be a simple separable nuclear monotracial M_{2^∞}-stable* C**-algebra which is KK-equivalent to $\{0\}$ and B a simple separable exact monotracial $\mathcal{Z}$-stable* C**-algebra. If there exists a homomorphism φ from $\mathcal{D}$ to B^ω such that $\tau_{B,\omega}\circ\varphi=\tau_{\mathcal{D}}$, then there exists a trace-preserving homomorphism from $\mathcal{D}$ to B.*

Proof. By the Choi–Effros lifting theorem, there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of contractive c.p. maps from $\mathcal{D}$ to B such that $\varphi(a)=(\varphi_n(a))_n$ for any $a\in\mathcal{D}$. Let $\{F_m\}_{m\in\mathbb{N}}$ be an increasing sequence of finite subsets in $\mathcal{D}$ such that $\mathcal{D}=\overline{\bigcup_{m\in\mathbb{N}}F_m}$. For any $m\in\mathbb{N}$, applying Corollary 4.7 to F_m and $1/2^m$, we obtain a finite subset G_m of $\mathcal{D}$ and $\delta_m>0$. We may assume that $G_m\subset G_{m+1}$, $\delta_m>\delta_{m+1}$ for any $m\in\mathbb{N}$ and $\lim_{m\to\infty}\delta_m=0$. Since we have

$$\lim_{n\to\omega}\|\varphi_n(ab)-\varphi_n(a)\varphi_n(b)\|=0\quad\text{and}\quad\lim_{n\to\omega}|\tau_B(\varphi_n(a))-\tau_{\mathcal{D}}(a)|=0$$

for any $a,b\in\mathcal{D}$, there exists a subsequence $\{\varphi_{n(m)}\}_{m\in\mathbb{N}}$ of $\{\varphi_n\}_{n\in\mathbb{N}}$ such that

$$\|\varphi_{n(m)}(ab)-\varphi_{n(m)}(a)\varphi_{n(m)}(b)\|<\delta_m\quad\text{and}\quad|\tau_B(\varphi_{n(m)}(a))-\tau_{\mathcal{D}}(a)|<\delta_m$$

for any $a,b\in G_m$. Corollary 4.7 implies that for any $m\in\mathbb{N}$, there exists a unitary element u_m in $B^\sim$ such that

$$\|\varphi_{n(m)}(a)-u_m\varphi_{n(m+1)}(a)u_m^*\|<\frac{1}{2^m}$$

for any $a\in F_m$. Therefore it can easily be checked that the limit

$$\lim_{m\to\infty}u_1u_2\cdots u_{m-1}\varphi_{n(m)}(a)u_{m-1}^*\cdots u_2^*u_1^*$$

exists for any $a\in\mathcal{D}$. Define $\psi(a):=\lim_{m\to\infty}u_1u_2\cdots u_{m-1}\varphi_{n(m)}(a)u_{m-1}^*\cdots u_2^*u_1^*$ for any $a\in\mathcal{D}$. Then ψ is a trace-preserving homomorphism from $\mathcal{D}$ to B. □

By the lemma above, it is enough to show that there exists a homomorphism φ from $\mathcal{D}$ to B^ω such that $\tau_{B,\omega}\circ\varphi=\tau_{\mathcal{D}}$. Borrowing Schafhauser's idea [2020a], we shall show this. By arguments in Section 2D, there exists the extension

$$\eta:0\longrightarrow J\longrightarrow B^\omega\stackrel{\varrho}{\longrightarrow}M\longrightarrow 0,$$

where M is a von Neumann algebraic ultrapower of $\pi_{\tau_B}(B)''$ and

$$J=\ker\varrho=\{(x_n)_n\in B^\omega\mid\tilde{\tau}_{B,\omega}((x_n^*x_n)_n)=0\}.$$

Note that J is known as the trace kernel ideal. Also, M is a II_1-factor because B is infinite-dimensional (which is implied by $\mathcal{Z}$-stability) and monotracial. Since $\mathcal{D}$ is monotracial and nuclear, $\pi_{\tau_{\mathcal{D}}}(\mathcal{D})''$ is the injective II_1-factor. Hence there exists a unital homomorphism from $\pi_{\tau_{\mathcal{D}}}(\mathcal{D})''$ to M (see, for example, [Takesaki 2003, Chapter XIV, Proposition 2.15]). In particular, there exists a trace-preserving homomorphism Π from $\mathcal{D}$ to M. Consider the pullback extension

$$\begin{array}{ccccccccc} \Pi^*\eta: & 0 \longrightarrow & J & \longrightarrow & E & \xrightarrow{\hat{\varrho}} & \mathcal{D} & \longrightarrow 0 \\ & & \| & & \downarrow \widehat{\Pi} & & \downarrow \Pi & \\ \eta: & 0 \longrightarrow & J & \longrightarrow & B^\omega & \xrightarrow{\varrho} & M & \longrightarrow 0 \end{array}$$

where $E = \{(a, x) \in \mathcal{D} \oplus B^\omega \mid \Pi(a) = \varrho(x)\}$, $\hat{\varrho}((a, x)) = a$ and $\widehat{\Pi}((a, x)) = x$ for any $(a, x) \in E$. If we could show that $\Pi^*\eta$ is a split extension with a cross section γ, then $\widehat{\Pi} \circ \gamma$ is a homomorphism from $\mathcal{D}$ to B^ω such that $\tau_{B,\omega} \circ \widehat{\Pi} \circ \gamma = \tau_{\mathcal{D}}$. But we were unable to show this, immediately. Note that we need to consider a separable extension in order to use KK-theory and some results in [Elliott and Kucerovsky 2001; Gabe 2016]. We shall construct a suitable separable extension η_0 by Blackadar's technique [2006, Section II.8.5].

We shall recall some definitions and some results in [Elliott and Kucerovsky 2001; Gabe 2016]. An extension $0 \to I \to C \to A \to 0$ is said to be *purely large* if, for any $x \in C \setminus I$, $\overline{xIx^*}$ contains a stable C^*-subalgebra which is full in I. Note that $\overline{xIx^*} = \overline{xx^*Ixx^*} = I \cap \overline{xCx^*}$. By [Gabe 2016, Theorem 2.1] (see also [Elliott and Kucerovsky 2001, Corollary 16]), if A is nonunital and I is stable, then a separable extension $0 \to I \to C \to A \to 0$ is nuclear-absorbing if and only if it is purely large.

Lemma 5.2. *With notation as above, suppose that there exist separable* C^**-subalgebras* $J_0 \subset J$, $B_0 \subset B^\omega$ *and* $M_0 \subset M$ *such that* J_0 *is stable*,

$$\eta_0 : 0 \longrightarrow J_0 \longrightarrow B_0 \xrightarrow{\varrho|_{B_0}} M_0 \longrightarrow 0$$

is a purely large extension and $\Pi(\mathcal{D}) \subset M_0$. *Then there exists a homomorphism* φ *from* $\mathcal{D}$ *to* B^ω *such that* $\tau_{B,\omega} \circ \varphi = \tau_{\mathcal{D}}$.

Proof. Consider the pullback extension

$$\begin{array}{ccccccccc} \Pi^*\eta_0: & 0 \longrightarrow & J_0 & \longrightarrow & E_0 & \xrightarrow{\hat{\varrho}} & \mathcal{D} & \longrightarrow 0 \\ & & \| & & \downarrow \widehat{\Pi} & & \downarrow \Pi & \\ \eta_0: & 0 \longrightarrow & J_0 & \longrightarrow & B_0 & \xrightarrow{\varrho} & M_0 & \longrightarrow 0 \end{array}$$

where $E_0 = \{(a, x) \in \mathcal{D} \oplus B_0 \mid \Pi(a) = \varrho(x)\}$, $\hat{\varrho}((a, x)) = a$ and $\widehat{\Pi}((a, x)) = x$ for any $(a, x) \in E_0$. Since η_0 is purely large, it can be easily checked that $\Pi^*\eta_0$ is purely large. Hence $\Pi^*\eta_0$ is nuclear-absorbing by [Gabe 2016, Theorem 2.1]. Because $\mathcal{D}$ is KK-equivalent to $\{0\}$ and nuclear, we have $\mathrm{Ext}(\mathcal{D}, J_0) = \{0\}$, and hence $[\Pi^*\eta_0] = 0$ in $\mathrm{Ext}(\mathcal{D}, J_0)$. Therefore there exists a (nuclear) split extension η' such that $\Pi^*\eta_0 \oplus \eta'$ is a split extension. Since $\Pi^*\eta_0$ is nuclear-absorbing, $\Pi^*\eta_0$ is strongly unitarily equivalent to $\Pi^*\eta_0 \oplus \eta'$, and hence $\Pi^*\eta_0$ is a split extension. Let γ_0 be a cross section of $\Pi^*\eta_0$, and define $\varphi := \widehat{\Pi} \circ \gamma_0$. Then φ is the desired homomorphism. □

A key result in the proof of the pure largeness is the following characterization of stable C*-algebras.

Theorem 5.3 [Hjelmborg and Rørdam 1998; Rørdam 2004b, Theorem 2.2]. *Let A be a σ-unital* C**-algebra. Then A is stable if and only if, for any $a \in A_+$ and $\varepsilon > 0$, there exist positive elements a' and c in A such that $\|a - a'\| \leq \varepsilon$, $a' \sim c$ and $\|ac\| \leq \varepsilon$.*

Before we construct a separable extension η_0, we shall consider properties of η.

Proposition 5.4. *With notation as above, let b be a positive element in $B^{\omega} \setminus J$.*

(i) *For any positive element a in $\overline{bJb}$, there exists a positive element c in $\overline{bJb}$ such that $a \sim c$ and $ac = 0$.*

(ii) *For any positive element a in J and $\varepsilon > 0$, there exist a positive element d in $\overline{bJb}$ and an element r in J such that $\|r^* dr - a\| < \varepsilon$.*

(iii) *For any element x in B^{ω} and $\varepsilon > 0$, there exists an element y in $\mathrm{GL}((B^{\omega})^{\sim})$ such that $\|x - y\| < \varepsilon$.*

For the proof of the proposition above, we need some lemmas. For a positive element $a \in A$ and $\varepsilon > 0$, we denote by $(a - \varepsilon)_+$ the element $f(a)$ in A, where $f(t) = \max\{0, t - \varepsilon\}$, $t \in \mathrm{Sp}(a)$. The same proof as in [Rørdam 1992, Proposition 2.4] shows the following lemma. See also [Pedersen 1987, Corollary 8].

Lemma 5.5. *Let A be a* C**-algebra with $A \subseteq \overline{\mathrm{GL}(A^{\sim})}$, and let a and b be positive elements in A. Then a is Cuntz smaller than b if and only if, for any $\varepsilon > 0$, there exists a unitary element u in $A^{\sim}$ such that $u(a - \varepsilon)_+ u^* \in \overline{bAb}$.*

The following lemma can be regarded as an application of the construction of $\mathcal{Z}$.

Lemma 5.6. *Let A be a monotracial $\mathcal{Z}$-stable* C**-algebra. For any $\theta \in \left(0, \frac{1}{2}\right)$, there exist positive elements d and d' in A such that $dd' = 0$ and $d_{\tau_A}((d - \varepsilon)_+) = d_{\tau_A}((d' - \varepsilon)_+) = (1 - \varepsilon)\theta$ for any $0 \leq \varepsilon \leq 1$.*

Proof. Let μ be the Lebesgue measure on $[0, 1]$, and define a tracial state τ_0 on $C([0, 1])$ by $\tau_0(f) := \int_{[0,1]} f \, d\mu$ for any $f \in C([0, 1])$. By [Rørdam 2004a, Theorem 2.1(i)], there exists a unital homomorphism ψ from $C([0, 1])$ to $\mathcal{Z}$ such that $\tau_0 = \tau_{\mathcal{Z}} \circ \psi$. Define f and g in $C([0, 1])$ by

$$f(t) := \begin{cases} \frac{2}{\theta} t & \text{if } t \in \left[0, \frac{\theta}{2}\right], \\ -\frac{2}{\theta} t + 2 & \text{if } t \in \left(\frac{\theta}{2}, \theta\right], \\ 0 & \text{if } t \in (\theta, 1] \end{cases} \quad \text{and} \quad g(t) := \begin{cases} 0 & \text{if } t \in [0, \theta], \\ \frac{2}{\theta} t - 2 & \text{if } t \in \left(\theta, \frac{3\theta}{2}\right], \\ -\frac{2}{\theta} t + 4 & \text{if } t \in \left(\frac{3\theta}{2}, 2\theta\right], \\ 0 & \text{if } t \in (2\theta, 1]. \end{cases}$$

Note that for any $0 \leq \varepsilon \leq 1$, we have

$$(f - \varepsilon)_+(t) = \begin{cases} 0 & \text{if } t \in \left[0, \frac{\varepsilon\theta}{2}\right], \\ \frac{2}{\theta} t - \varepsilon & \text{if } t \in \left(\frac{\varepsilon\theta}{2}, \frac{\theta}{2}\right], \\ -\frac{2}{\theta} t + 2 - \varepsilon & \text{if } t \in \left(\frac{\theta}{2}, \theta - \frac{\varepsilon\theta}{2}\right], \\ 0 & \text{if } t \in \left(\theta - \frac{\varepsilon\theta}{2}, 1\right], \end{cases}$$

$$(g - \varepsilon)_+(t) = \begin{cases} 0 & \text{if } t \in \left[0, \theta + \frac{\varepsilon\theta}{2}\right], \\ \frac{2}{\theta} t - 2 - \varepsilon & \text{if } t \in \left(\theta + \frac{\varepsilon\theta}{2}, \frac{3\theta}{2}\right], \\ -\frac{2}{\theta} t + 4 - \varepsilon & \text{if } t \in \left(\frac{3\theta}{2}, 2\theta - \frac{\varepsilon\theta}{2}\right], \\ 0 & \text{if } t \in \left(2\theta - \frac{\varepsilon\theta}{2}, 1\right]. \end{cases}$$

Hence $d_{\tau_0}((f-\varepsilon)_+) = d_{\tau_0}((g-\varepsilon)_+) = (1-\varepsilon)\theta$. Let s be a strictly positive element in A, and put

$$d := s \otimes \psi(f) \quad \text{and} \quad d' := s \otimes \psi(g)$$

in $A \otimes \mathcal{Z} \cong A$. Then d and d' are desired positive elements in A. □

Lemma 5.7. *Let A be a simple separable exact monotracial $\mathcal{Z}$-stable* C*-*algebra, and let b be a* (*nonzero*) *positive element in A. For any $\theta \in (0, d_{\tau_A}(b)/2)$, there exist positive elements e and e' in $\overline{bAb}$ such that $ee' = 0$ and $d_{\tau_A}(e) = d_{\tau_A}(e') > \theta$.*

Proof. By Lemma 5.6, there exist contractions d and d' in A such that $dd' = 0$ and $\theta < d_{\tau_A}(d) = d_{\tau_A}(d') < d_{\tau_A}(b)/2$. Furthermore, we may assume that there exists $\varepsilon > 0$ such that $d_{\tau_A}((d-\varepsilon)_+) = d_{\tau_A}((d'-\varepsilon)_+) > \theta$. Since A has strict comparison and $d_{\tau_A}(d+d') = d_{\tau_A}(d) + d_{\tau_A}(d') < d_{\tau_A}(b)$, Lemma 5.5 implies that there exists a unitary element u in $A^{\sim}$ such that $u(d+d'-\varepsilon)_+u^* \in \overline{bAb}$. Note that $(d+d'-\varepsilon)_+ = (d-\varepsilon)_+ + (d'-\varepsilon)_+$ because of $dd' = 0$. Put

$$e := u(d-\varepsilon)_+u^* \quad \text{and} \quad e' := u(d'-\varepsilon)_+u^*.$$

Then e and e' are desired positive elements. □

Proof of Proposition 5.4. (i) We may assume $\|a\| = 1$ and $\|b\| = 1$. Since $b \notin J$, we have $\tau_{B,\omega}(b) > 0$. Take a representative $(b_n)_n$ of b such that $\|b_n\| = 1$ for any $n \in \mathbb{N}$, and choose $\varepsilon_0 > 0$ such that $\tau_{B,\omega}(b) - \varepsilon_0 > 0$. Since we have

$$\lim_{n\to\omega} d_{\tau_B}(b_n) \geq \lim_{n\to\omega} \tau_B(b_n) = \tau_{B,\omega}(b),$$

there exists an element $X_1 \in \omega$ such that, for any $n \in X_1$,

$$d_{\tau_B}(b_n) > \tau_{B,\omega}(b) - \varepsilon_0.$$

By an argument similar to that in the proof of [Sato 2010, Lemma 3.2], we see that there exists a representative $(a_n)_n$ of a such that $a_n \in \overline{b_nBb_n}$ and $\|a_n\| = 1$ for any $n \in \mathbb{N}$ and $\lim_{n\to\omega} d_{\tau_B}(a_n) = 0$ because of $a \in \overline{(b_n)_nJ(b_n)_n}$. Hence there exists an element $X_2 \in \omega$ such that for any $n \in X_2$,

$$d_{\tau_B}(a_n) < \frac{\tau_{B,\omega}(b) - \varepsilon_0}{2}.$$

Note that we have $d_{\tau_B}(a_n) < d_{\tau_B}(b_n)/2$ for any $n \in X_1 \cap X_2$. Hence Lemma 5.7 implies that for any $n \in X_1 \cap X_2$, there exist positive elements e_n and e'_n in $\overline{b_nBb_n}$ such that $e_ne'_n = 0$ and $d_{\tau_B}(e_n) = d_{\tau_B}(e'_n) > d_{\tau_B}(a_n)$. Since $\overline{b_nBb_n}$ has strict comparison and $\overline{b_nBb_n} \subseteq \mathrm{GL}(\overline{b_nBb_n}^{\sim})$ by [Rørdam 2004a; Robert 2016], Lemma 5.5 shows that for any $n \in X_1 \cap X_2$, there exist unitary elements u_n and v_n in $\overline{b_nBb_n}^{\sim}$ such that

$$u_n(a_n - 1/n)_+u_n^* \in \overline{e_nBe_n} \quad \text{and} \quad v_n(a_n - 1/n)_+v_n^* \in \overline{e'_nBe'_n}.$$

Note that $(a_n - 1/n)_+u_n^*v_n(a_n - 1/n)_+ = 0$ for any $n \in X_1 \cap X_2$. Define $z = (z_n)_n$ and $c = (c_n)_n$ in B^{ω} by

$$z_n := \begin{cases} 0 & \text{if } n \notin X_1 \cap X_2, \\ u_n^*v_n(a_n - 1/n)_+^{1/2} & \text{if } n \in X_1 \cap X_2 \end{cases}$$

and

$$c_n := \begin{cases} 0 & \text{if } n \notin X_1 \cap X_2, \\ u_n^* v_n (a_n - 1/n)_+ v_n^* u_n & \text{if } n \in X_1 \cap X_2. \end{cases}$$

It is easy to see that $z, c \in \overline{bB^\omega b}$, $z^*z = a$, $zz^* = c$ and $ac = 0$. Since $\overline{bJb}$ is a closed ideal in $\overline{bB^\omega b}$ and $a \in \overline{bJb}$, we know z and c are elements in $\overline{bJb}$. Therefore we obtain the conclusion.

(ii) Note that B^ω has strict comparison; see, for example, [Bosa et al. 2019, Lemma 1.23]. Since $a \in J$ and $b \notin J$, we have $d_{\tau_{B,\omega}}(a^{1/5}) = 0$ and $d_{\tau_{B,\omega}}(b) > 0$. Hence there exists a sequence $\{s_N\}_{N\in\mathbb{N}}$ in B^ω such that $\lim_{N\to\infty} \|s_N^* b s_N - a^{1/5}\| = 0$. Let $d_N := b s_N a^{1/5} s_N^* b$ and $r_N := s_N a^{1/5}$ for any $N \in \mathbb{N}$. Then we have $d_N \in \overline{bJb}$, $r_N \in J$ for any $N \in \mathbb{N}$ and

$$r_N^* d_N r_N = a^{1/5} s_N^* b s_N a^{1/5} s_N^* b s_N a^{1/5} \to a$$

as $N \to \infty$. Therefore we obtain the conclusion.

(iii) Since B is a simple monotracial $\mathcal{Z}$-stable C*-algebra, $B \subseteq \overline{\mathrm{GL}(B^\sim)}$ by [Rørdam 2004a; Robert 2016]. Therefore we obtain the conclusion by Proposition 2.4. □

If B is unital, then the following lemma is a well-known consequence of Proposition 2.4 and Blackadar's technique [2006, Proposition II.8.5.4].

Lemma 5.8. *With notation as above, let S be a separable subset of B^ω. Then there exists a separable* C*-*algebra A such that $S \subseteq A \subset B^\omega$ and $A \subseteq \overline{\mathrm{GL}(A^\sim)}$.*

Proof. We shall show only the case where B is nonunital. Let A_1 be the C*-subalgebra of B^ω generated by S. Since A_1 is separable, there exists a countable dense subset $\{x_k \mid k \in \mathbb{N}\}$ of A_1. By Proposition 5.4(iii), for any $k, m \in \mathbb{N}$, there exist $y_{k,m} \in B^\omega$ and $\lambda_{k,m} \in \mathbb{C} \setminus \{0\}$ such that

$$\|x_k - (y_{k,m} + \lambda_{k,m} 1_{(B^\omega)^\sim})\| < \frac{1}{m}$$

and $y_{k,m} + \lambda_{k,m} 1_{(B^\omega)^\sim} \in \mathrm{GL}((B^\omega)^\sim)$. Let A_2 be the C*-subalgebra of B^ω generated by A_1 and $\{y_{k,m} \mid k, m \in \mathbb{N}\}$. Then we have $A_1 \subseteq \overline{\mathrm{GL}(A_2^\sim)}$. Indeed, we have $y_{k,m} + \lambda_{k,m} 1_{A_2^\sim} \in \mathrm{GL}(A_2^\sim)$ for any $k, m \in \mathbb{N}$ because of $\mathrm{Sp}_{A_2}(y_{k,m}) \cup \{0\} = \mathrm{Sp}_{B^\omega}(y_{k,m}) \cup \{0\}$ and $\lambda_{k,m} \neq 0$. Since $A_1 = \overline{\{x_k \mid k \in \mathbb{N}\}}$ and

$$\begin{aligned} \|x_k - (y_{k,m} + \lambda_{k,m} 1_{(A_2)^\sim})\| &= \|1_{A_2^\sim} x_k - 1_{A_2^\sim}(y_{k,m} + \lambda_{k,m} 1_{(B^\omega)^\sim})\| \\ &\leq \|x_k - (y_{k,m} + \lambda_{k,m} 1_{(B^\omega)^\sim})\| < \frac{1}{m} \end{aligned}$$

for any $k, m \in \mathbb{N}$, we have $A_1 \subseteq \overline{\mathrm{GL}(A_2^\sim)}$. Repeating this process, we obtain a sequence $\{A_n\}_{n\in\mathbb{N}}$ of separable C*-subalgebras of B^ω such that $A_n \subseteq A_{n+1}$ and $A_n \subseteq \overline{\mathrm{GL}(A_{n+1}^\sim)}$ for any $n \in \mathbb{N}$. Put $A := \overline{\bigcup_{n=1}^\infty A_n}$. Since $A_n \subseteq \overline{\mathrm{GL}(A_{n+1}^\sim)} \subseteq \overline{\mathrm{GL}(A^\sim)}$ for any $n \in \mathbb{N}$ by Proposition 2.2, we have $A \subseteq \overline{\mathrm{GL}(A^\sim)}$. Therefore A is the desired separable C*-algebra. □

The following lemma is also based on Blackadar's technique.

Lemma 5.9. *With notation as above, let $\{b_k \mid k \in \mathbb{N}\}$ be a countable subset of $B^\omega \setminus J$ and S a separable subset of B^ω. Then there exists a separable* C*-*algebra A such that $\{b_k \mid k \in \mathbb{N}\} \cup S \subseteq A \subset B^\omega$ and $\overline{b_k(A \cap J)b_k}$ is full in $A \cap J$ for any $k \in \mathbb{N}$.*

Proof. Let A_1 be the C*-subalgebra of B^ω generated by $\{b_k \mid k \in \mathbb{N}\}$ and S. Since A_1 is separable, there exists a countable dense subset $\{a_l \mid l \in \mathbb{N}\}$ of $(A_1 \cap J)_+$. By Proposition 5.4(ii), for any $k, l, m \in \mathbb{N}$, there exist $d_{k,l,m} \in \overline{b_k J b_k}_+$ and $r_{k,l,m} \in J$ such that

$$\|r^*_{k,l,m} d_{k,l,m} r_{k,l,m} - a_l\| < \frac{1}{m}.$$

Let A_2 be the C*-subalgebra of B^ω generated by A_1 and $\{d_{k,l,m}, r_{k,l,m} \mid k, l, m \in \mathbb{N}\}$. Then we have $A_1 \cap J \subseteq \overline{(A_2 \cap J) b_k (A_2 \cap J) b_k (A_2 \cap J)}$ for any $k \in \mathbb{N}$ because $A_1 \cap J$ is generated by $\{a_l \mid l \in \mathbb{N}\}$. Repeating this process, we obtain a sequence $\{A_n\}_{n \in \mathbb{N}}$ of separable C*-subalgebras of B^ω such that $A_n \subseteq A_{n+1}$ and $A_n \cap J \subseteq \overline{(A_{n+1} \cap J) b_k (A_{n+1} \cap J) b_k (A_{n+1} \cap J)}$ for any $k, n \in \mathbb{N}$. Put $A := \overline{\bigcup_{n=1}^\infty A_n}$. Since we have $A \cap J = \overline{\bigcup_{n=1}^\infty (A_n \cap J)}$, we see that A is the desired separable C*-algebra. □

By Lemmas 5.8 and 5.9, [Blackadar 2006, Proposition II.8.5.3] implies the following lemma.

Lemma 5.10. *With notation as above, let $\{b_k \mid k \in \mathbb{N}\}$ be a countable subset of $B^\omega \setminus J$ and S a separable subset of B^ω. Then there exists a separable C*-algebra A such that $\{b_k \mid k \in \mathbb{N}\} \cup S \subseteq A \subset B^\omega$, $A \subseteq \overline{\mathrm{GL}(A^\sim)}$ and $\overline{b_k (A \cap J) b_k}$ is full in $A \cap J$ for any $k \in \mathbb{N}$.*

We shall construct the separable extension η_0 of Lemma 5.2.

Since ϱ is surjective and $\mathcal{D}$ is separable, there exists a separable subset S_0 of B^ω such that $\overline{\varrho(S_0)} = \Pi(\mathcal{D})$. Applying Lemma 5.8 to S_0, we obtain a separable C*-algebra B_1 such that $S_0 \subseteq B_1 \subset B^\omega$ and $B_1 \subseteq \overline{\mathrm{GL}(B_1^\sim)}$. Since B_1 is separable, there exist a countable subset $\{a_{1,m} \mid m \in \mathbb{N}\}$ of $(B_1 \cap J)_+$ and a countable subset $\{b_{1,k} \mid k \in \mathbb{N}\}$ of B_{1+} such that

$$\overline{\{a_{1,m} \mid m \in \mathbb{N}\}} = (B_1 \cap J)_+ \quad \text{and} \quad \overline{\{b_{1,k} \mid k \in \mathbb{N}\}} = B_{1+}.$$

Put $T_1 := \{(k,l) \in \mathbb{N} \times \mathbb{N} \mid (b_{1,k} - 1/l)_+ \notin J\}$. Applying Proposition 5.4(i) to $(b_{1,k} - 1/l)_+ a_{1,m} (b_{1,k} - 1/l)_+$ for any $(k,l) \in T_1$ and $m \in \mathbb{N}$, there exist a positive element $c_{1,1,(k,l),m}$ and an element $z_{1,1,(k,l),m}$ in $\overline{(b_{1,k} - 1/l)_+ J (b_{1,k} - 1/l)_+}$ such that

$$(b_{1,k} - 1/l)_+ a_{1,m} (b_{1,k} - 1/l)_+ c_{1,1,(k,l),m} = 0,$$
$$z^*_{1,1,(k,l),m} z_{1,1,(k,l),m} = (b_{1,k} - 1/l)_+ a_{1,m} (b_{1,k} - 1/l)_+,$$
$$z_{1,1,(k,l),m} z^*_{1,1,(k,l),m} = c_{1,1,(k,l),m}.$$

Let $S_2 := B_1 \cup \{c_{1,1,(k,l),m}, z_{1,1,(k,l),m} \mid (k,l) \in T_1, m \in \mathbb{N}\}$. Applying Lemma 5.10 to $\{(b_{1,k} - 1/l)_+ \mid (k,l) \in T_1\}$ and S_2, we obtain a separable C*-algebra B_2 such that

$$B_1 \cup \{c_{1,1,(k,l),m}, z_{1,1,(k,l),m} \mid (k,l) \in T_1, m \in \mathbb{N}\} \subseteq B_2 \subset B^\omega,$$

$B_2 \subseteq \overline{\mathrm{GL}(B_2^\sim)}$ and $\overline{(b_{1,k} - 1/l)_+ (B_2 \cap J) (b_{1,k} - 1/l)_+}$ is full in $B_2 \cap J$ for any $(k,l) \in T_1$. In the same way as above, there exist a countable subset $\{a_{2,m} \mid m \in \mathbb{N}\}$ of $(B_2 \cap J)_+$ and a countable subset $\{b_{2,k} \mid k \in \mathbb{N}\}$ of B_{2+} such that

$$\overline{\{a_{2,m} \mid m \in \mathbb{N}\}} = (B_2 \cap J)_+ \quad \text{and} \quad \overline{\{b_{2,k} \mid k \in \mathbb{N}\}} = B_{2+},$$

and we put $T_2 := \{(k,l) \in \mathbb{N}\times\mathbb{N} \mid (b_{2,k}-1/l)_+ \notin J\}$. Applying Proposition 5.4(i) to $(b_{i,k}-1/l)_+ a_{2,m} \times (b_{i,k}-1/l)_+$ for any $1 \le i \le 2$, $(k,l) \in T_i$ and $m \in \mathbb{N}$, there exist a positive element $c_{2,i,(k,l),m}$ and an element $z_{2,i,(k,l),m}$ in $\overline{(b_{i,k}-1/l)_+ J (b_{i,k}-1/l)_+}$ such that

$$(b_{i,k}-1/l)_+ a_{2,m} (b_{i,k}-1/l)_+ c_{2,i,(k,l),m} = 0,$$
$$z^*_{2,i,(k,l),m} z_{2,i,(k,l),m} = (b_{i,k}-1/l)_+ a_{2,m} (b_{i,k}-1/l)_+,$$
$$z_{2,i,(k,l),m} z^*_{2,i,(k,l),m} = c_{2,i,(k,l),m}.$$

Let $S_3 := B_2 \cup \{c_{2,i,(k,l),m}, z_{2,i,(k,l),m} \mid 1 \le i \le 2, (k,l) \in T_i, m \in \mathbb{N}\}$. Applying Lemma 5.10 to $\{(b_{i,k}-1/l)_+ \mid 1 \le i \le 2, (k,l) \in T_i\}$ and S_3, we obtain a separable C*-algebra B_3 such that

$$B_2 \cup \{c_{2,i,(k,l),m}, z_{2,i,(k,l),m} \mid 1 \le i \le 2, (k,l) \in T_i, m \in \mathbb{N}\} \subseteq B_3 \subset B^{\omega},$$

$B_3 \subseteq \overline{\mathrm{GL}(B_3^{\sim})}$ and $\overline{(b_{i,k}-1/l)_+ (B_3 \cap J)(b_{i,k}-1/l)_+}$ is full in $B_3 \cap J$ for any $1 \le i \le 2$ and $(k,l) \in T_i$. Repeating this process, for any $n \in \mathbb{N}$, we obtain

$$B_n \subset B^{\omega}, \quad \{a_{n,m} \mid m \in \mathbb{N}\} \subset (B_n \cap J)_+, \quad \{b_{n,k} \mid k \in \mathbb{N}\} \subset B_{n+},$$
$$T_n \subset \mathbb{N}\times\mathbb{N}, \quad \{c_{n,i,(k,l),m}, z_{n,i,(k,l),m} \mid 1 \le i \le n, (k,l) \in T_i, m \in \mathbb{N}\}$$

such that B_n is separable,

$$B_n \subseteq B_{n+1}, \quad B_n \subseteq \overline{\mathrm{GL}(B_n^{\sim})}, \quad \overline{\{a_{n,m} \mid m \in \mathbb{N}\}} = (B_n \cap J)_+,$$
$$\overline{\{b_{n,k} \mid k \in \mathbb{N}\}} = B_{n+}, \quad T_n = \{(k,l) \in \mathbb{N}\times\mathbb{N} \mid (b_{n,k}-1/l)_+ \notin J\},$$
$$c_{n,i,(k,l),m},\ z_{n,i,(k,l),m} \in \overline{(b_{i,k}-1/l)_+ (B_{n+1} \cap J)(b_{i,k}-1/l)_+},$$
$$(b_{i,k}-1/l)_+ a_{n,m} (b_{i,k}-1/l)_+ c_{n,i,(k,l),m} = 0,$$
$$z^*_{n,i,(k,l),m} z_{n,i,(k,l),m} = (b_{i,k}-1/l)_+ a_{n,m} (b_{i,k}-1/l)_+,$$
$$z_{n,i,(k,l),m} z^*_{n,i,(k,l),m} = c_{n,i,(k,l),m}$$

and $\overline{(b_{i,k}-1/l)_+ (B_{n+1} \cap J)(b_{i,k}-1/l)_+}$ is full in $B_{n+1} \cap J$ for any $1 \le i \le n$ and $(k,l) \in T_i$. Define

$$B_0 := \overline{\bigcup_{n=1}^{\infty} B_n}, \quad J_0 := B_0 \cap J \quad \text{and} \quad M_0 := \varrho(B_0).$$

Then

$$\eta_0 : 0 \longrightarrow J_0 \longrightarrow B_0 \stackrel{\varrho}{\longrightarrow} M_0 \longrightarrow 0$$

is a separable extension and $\Pi(\mathcal{D}) \subseteq M_0$. Corollary 2.3 implies $B_0 \subseteq \overline{\mathrm{GL}(B_0^{\sim})}$ since we have $B_n \subseteq \overline{\mathrm{GL}(B_n^{\sim})}$ for any $n \in \mathbb{N}$. Furthermore, for any $i \in \mathbb{N}$ and $(k,l) \in T_i$, $\overline{(b_{i,k}-1/l)_+ J_0 (b_{i,k}-1/l)_+}$ is full in J_0 by a similar argument as in the proof of Lemma 5.9. Note that, for any $n_0 \in \mathbb{N}$,

$$J_{0+} = \overline{\bigcup_{n=n_0}^{\infty} \{a_{n,m} \mid m \in \mathbb{N}\}} \quad \text{and} \quad B_{0+} = \overline{\bigcup_{n=n_0}^{\infty} \{b_{n,k} \mid k \in \mathbb{N}\}}.$$

We shall show that J_0 is stable and η_0 is purely large.

Proof of the stability of J_0. Let $a \in J_{0+} \setminus \{0\}$ and $\varepsilon > 0$. Set

$$\varepsilon' := \min\left\{\frac{\varepsilon}{2\|a\|},\ \sqrt{\frac{\varepsilon}{2}},\ \varepsilon\right\}.$$

Since B_0 is separable, there exists an approximate unit $\{h_n\}_{n\in\mathbb{N}}$ for B_0. Note that $h_n \notin J$ for sufficiently large n because of $M_0 \neq \{0\}$. Hence there exists $N \in \mathbb{N}$ such that $h_N \notin J$ and $\|h_N a h_N - a\| < \varepsilon'/2$. Since $B_{0+} = \overline{\bigcup_{n=1}^{\infty}\{b_{n,k} \mid k \in \mathbb{N}\}}$, for any $l \in \mathbb{N}$, there exist $n(l)$ and $k(l)$ in $\mathbb{N}$ such that

$$\|h_N - b_{n(l),k(l)}\| < \frac{1}{l}.$$

Note that $(b_{n(l),k(l)} - 1/l)_+ \to h_N$ as $l \to \infty$ because we have

$$\|h_N - (b_{n(l),k(l)} - 1/l)_+\| \le \|h_N - b_{n(l),k(l)}\| + \|b_{n(l),k(l)} - (b_{n(l),k(l)} - 1/l)_+\| < \frac{2}{l}.$$

Hence there exists $l_0 \in \mathbb{N}$ such that $(b_{n(l_0),k(l_0)} - 1/l_0)_+ \notin J$, that is, $(k(l_0), l_0) \in T_{n(l_0)}$ and

$$\|a - (b_{n(l_0),k(l_0)} - 1/l_0)_+ a (b_{n(l_0),k(l_0)} - 1/l_0)_+\| < \frac{\varepsilon'}{2}.$$

Since $J_{0+} = \overline{\bigcup_{n=n(l_0)}^{\infty}\{a_{n,m} \mid m \in \mathbb{N}\}}$, there exist $n_0 \ge n(l_0)$ and $m_0 \in \mathbb{N}$ such that

$$\|a - a_{n_0,m_0}\| < \frac{\varepsilon'}{2\|b_{n(l_0),k(l_0)}\|^2}.$$

Put $a' := (b_{n(l_0),k(l_0)} - 1/l_0)_+ a_{n_0,m_0} (b_{n(l_0),k(l_0)} - 1/l_0)_+$. Then

$$\|a - a'\| < \varepsilon' \le \varepsilon.$$

By construction of B_0 and J_0, there exist

$$z = z_{n_0,n(l_0),(k(l_0),l_0),m_0},\ \ c = c_{n_0,n(l_0),(k(l_0),l_0),m_0} \in J_0$$

such that $a'c = 0$, $z^*z = a'$ and $zz^* = c$. Hence $a' \sim c$ and

$$\|ac\| = \|ac - a'c\| \le \|a - a'\|\|c\| = \|a - a'\|\|a'\| < \varepsilon'(\|a\| + \varepsilon') \le \varepsilon.$$

Therefore J_0 is stable by Hjelmborg and Rørdam's characterization (Theorem 5.3). □

Proof of the pure largeness of η_0. Let $x \in B_0 \setminus J_0$. Note that we have $xx^* \notin J$. Since $B_{0+} = \overline{\bigcup_{n=1}^{\infty}\{b_{n,k} \mid k \in \mathbb{N}\}}$, for any $l \in \mathbb{N}$, there exist $n(l)$ and $k(l)$ in $\mathbb{N}$ such that

$$\|xx^* - b_{n(l),k(l)}\| < \frac{1}{2l}.$$

By an argument similar to that in the proof of stability of J_0, there exists $l_0 \in \mathbb{N}$ such that $(b_{n(l_0),k(l_0)} - 1/l_0)_+ \notin J$, that is, $(k(l_0), l_0) \in T_{n(l_0)}$. On the other hand, [Kirchberg and Rørdam 2002, Lemma 2.2] implies that $(b_{n(l_0),k(l_0)} - 1/2l_0)_+$ is Cuntz smaller than xx^*. Since we have $B_0 \subseteq \overline{\mathrm{GL}(B_0^{\sim})}$, there exists a unitary element u in $B_0^{\sim}$ such that

$$u(b_{n(l_0),k(l_0)} - 1/l_0)_+ u^* = u((b_{n(l_0),k(l_0)} - 1/2l_0)_+ - 1/2l_0)_+ u^* \in \overline{xx^* B_0 xx^*} = \overline{xB_0x^*}$$

by Lemma 5.5. Put

$$C := u\overline{(b_{n(l_0),k(l_0)} - 1/l_0)_+ J_0 (b_{n(l_0),k(l_0)} - 1/l_0)_+}u^* \subseteq \overline{xJ_0x^*}.$$

Then C is full in J_0 because $\overline{(b_{n(l_0),k(l_0)} - 1/l_0)_+ J_0 (b_{n(l_0),k(l_0)} - 1/l_0)_+}$ is full in J_0. We shall show that C is stable. Let $a \in C_+ \setminus \{0\}$ and $\varepsilon > 0$. Set

$$\varepsilon' := \min\left\{ \frac{\varepsilon}{2\|a\|}, \sqrt{\frac{\varepsilon}{2}}, \varepsilon \right\}.$$

By the definition of C and $J_{0+} = \overline{\bigcup_{n=n(l_0)}^{\infty} \{a_{n,m} \mid m \in \mathbb{N}\}}$, there exist $n_0 \geq n(l_0)$ and $m_0 \in \mathbb{N}$ such that

$$\|a - u(b_{n(l_0),k(l_0)} - 1/l_0)_+ a_{n_0,m_0} (b_{n(l_0),k(l_0)} - 1/l_0)_+ u^*\| < \varepsilon' \leq \varepsilon.$$

Put $a' = u(b_{n(l_0),k(l_0)} - 1/l_0)_+ a_{n_0,m_0} (b_{n(l_0),k(l_0)} - 1/l_0)_+ u^* \in C$, then $\|a - a'\| < \varepsilon' \leq \varepsilon$. By construction of B_0 and J_0, there exist elements

$$z_{n_0,n(l_0),(k(l_0),l_0),m_0},\ c_{n_0,n(l_0),(k(l_0),l_0),m_0}$$

in $\overline{(b_{n(l_0),k(l_0)} - 1/l_0)_+ J_0 (b_{n(l_0),k(l_0)} - 1/l_0)_+}$ such that

$$u^*a'uc_{n_0,n(l_0),(k(l_0),l_0),m_0} = 0, \quad z^*_{n_0,n(l_0),(k(l_0),l_0),m_0} z_{n_0,n(l_0),(k(l_0),l_0),m_0} = u^*a'u$$

and

$$z_{n_0,n(l_0),(k(l_0),l_0),m_0} z^*_{n_0,n(l_0),(k(l_0),l_0),m_0} = c_{n_0,n(l_0),(k(l_0),l_0),m_0}.$$

Put $c := uc_{n_0,n(l_0),(k(l_0),l_0),m_0}u^*$. It is easy to see that $c \in C$, $a'c = 0$ and

$$c \sim c_{n_0,n(l_0),(k(l_0),l_0),m_0} \sim u^*a'u \sim a' \quad \text{in } B_0.$$

Since C is a hereditary C*-subalgebra of B_0 and $a', c \in C$, we see that a' is Murray–von Neumann equivalent to c in C. Therefore, the same argument as in the proof of stability of J_0 shows $\|ac\| < \varepsilon$, and C is stable. Consequently, η_0 is a purely large extension. □

Therefore we obtain the following lemma.

Lemma 5.11. *With notation as above, there exist separable* C**-subalgebras $J_0 \subset J$, $B_0 \subset B^\omega$ and $M_0 \subset M$ such that J_0 is stable,*

$$\eta_0 : 0 \longrightarrow J_0 \longrightarrow B_0 \xrightarrow{\varrho|_{B_0}} M_0 \longrightarrow 0$$

is a purely large extension and $\Pi(\mathcal{D}) \subset M_0$.

Consequently, we obtain the following theorem by Lemma 5.1, Lemma 5.2 and the lemma above.

Theorem 5.12. *Let $\mathcal{D}$ be a simple separable nuclear monotracial M_{2^∞}-stable* C**-algebra which is KK-equivalent to $\{0\}$ and B a simple separable exact monotracial $\mathcal{Z}$-stable* C**-algebra. Then there exists a trace-preserving homomorphism from $\mathcal{D}$ to B.*

Remark 5.13. Actually, we need not assume that $\mathcal{D}$ is M_{2^∞}-stable in the theorem above. Indeed, define a homomorphism φ from $\mathcal{D}$ to $\mathcal{D} \otimes M_{2^\infty}$ by $\varphi(a) = a \otimes 1$. Then φ is a trace-preserving homomorphism from $\mathcal{D}$ to $\mathcal{D} \otimes M_{2^\infty}$. By the theorem above, there exists a trace-preserving homomorphism ψ from $\mathcal{D} \otimes M_{2^\infty}$ to B. Then $\psi \circ \varphi$ is a trace-preserving homomorphism from $\mathcal{D}$ to B.

The following corollary is an immediate consequence of the theorem above.

Corollary 5.14. *Let B a simple separable exact monotracial $\mathcal{Z}$-stable* C*-*algebra. Then there exists a trace-preserving homomorphism from $\mathcal{W}$ to B.*

The injective II_1-factor can embed unitally into every II_1-factor. Hence the following question is natural and interesting.

Question 5.15. (1) Let B be a simple monotracial infinite-dimensional C*-algebra. Does there exist a trace-preserving homomorphism from $\mathcal{W}$ to B?

(2) Let B be a simple non-type-I C*-algebra. Does there exist a (nonzero) homomorphism from $\mathcal{W}$ to B?

Note that Dadarlat, Hirshberg, Toms and Winter [Dadarlat et al. 2009] showed that there exists a unital simple separable nuclear infinite-dimensional C*-algebra B such that $\mathcal{Z}$ does not embed unitally into B.

6. Characterization of $\mathcal{W}$

In this section we shall show that if $\mathcal{D}$ is a simple separable nuclear monotracial M_{2^∞}-stable C*-algebra which is KK-equivalent to $\{0\}$, then $\mathcal{D}$ is isomorphic to $\mathcal{W}$. Also, we shall characterize $\mathcal{W}$ by using properties of $F(\mathcal{W})$.

Theorem 6.1. *Let $\mathcal{D}$ be a simple separable nuclear monotracial M_{2^∞}-stable* C*-*algebra which is KK-equivalent to $\{0\}$. Then $\mathcal{D}$ is isomorphic to $\mathcal{W}$.*

Proof. By Theorem 5.12 and Corollary 5.14, there exist trace-preserving homomorphisms φ and ψ from $\mathcal{D}$ to $\mathcal{W}$ and from $\mathcal{W}$ and $\mathcal{D}$, respectively. Since $\mathcal{D}$ and $\mathcal{W}$ have property W by Corollary 3.11, Theorem 4.3 implies that $\psi \circ \varphi$ and $\varphi \circ \psi$ are approximately inner. Therefore $\mathcal{D}$ is isomorphic to $\mathcal{W}$ by Elliott's approximate intertwining argument [Elliott 1993]; see also [Rørdam 2002, Corollary 2.3.4]. □

The following corollary is an immediate consequence of the theorem above.

Corollary 6.2. (i) *If A is a simple separable nuclear monotracial* C*-*algebra, then $A \otimes \mathcal{W}$ is isomorphic to $\mathcal{W}$. In particular, $\mathcal{W} \otimes \mathcal{W}$ is isomorphic to $\mathcal{W}$.*

(ii) *For any nonzero positive element h in $\mathcal{W}$, $\overline{h\mathcal{W}h}$ is isomorphic to $\mathcal{W}$.*

Following the definition in [Lin and Ng 2023], we say that a C*-algebra A is *$\mathcal{W}$-embeddable* if there exists an injective homomorphism from A to $\mathcal{W}$.

Lemma 6.3. *Let A be a monotracial $\mathcal{W}$-embeddable* C*-*algebra. Then there exists a trace-preserving homomorphism from A to $\mathcal{W}$.*

Proof. By the assumption, there exists an injective homomorphism φ from A to $\mathcal{W}$. Let s be a strictly positive element in A. (Note that A is separable because A is $\mathcal{W}$-embeddable.) Since φ is injective, $\varphi(s)$ is a nonzero positive element. Corollary 6.2 implies that there exists an isomorphism Φ from $\overline{\varphi(s)\mathcal{W}\varphi(s)}$ onto $\mathcal{W}$. Note that φ can be regarded as a homomorphism from A to $\overline{\varphi(s)\mathcal{W}\varphi(s)}$. Define $\psi := \Phi\circ\varphi$. Then ψ is a trace-preserving homomorphism from A to $\mathcal{W}$. □

The following theorem is a characterization of $\mathcal{W}$.

Theorem 6.4. *Let D be a simple separable nuclear monotracial* C*-*algebra. Then D is isomorphic to $\mathcal{W}$ if and only if D has property W and is $\mathcal{W}$-embeddable, that is, D satisfies the following properties*:

(i) *For any $\theta\in[0,1]$, there exists a projection p in $F(D)$ such that $\tau_{D,\omega}(p)=\theta$.*

(ii) *If p and q are projections in $F(D)$ such that $0<\tau_{D,\omega}(p)=\tau_{D,\omega}(q)$, then p is Murray–von Neumann equivalent to q.*

(iii) *There exists an injective homomorphism from D to $\mathcal{W}$.*

Proof. The "only if" part is obvious by Corollary 3.11. We shall show the "if" part. Since D is $\mathcal{W}$-embeddable, there exists a trace-preserving homomorphism φ from D to $\mathcal{W}$ by Lemma 6.3. Lemma 4.1 implies that D is $\mathcal{Z}$-stable because D has property W. Hence there exists a trace-preserving homomorphism ψ from $\mathcal{W}$ to D by Corollary 5.14. The rest of proof is same as the proof of Theorem 6.1. □

We think that every simple separable nuclear monotracial C*-algebra with property W ought to be $\mathcal{W}$-embeddable. Note that every simple separable nuclear monotracial C*-algebra with property W is stably projectionless by [Kirchberg 2006, Remark 2.13] and an argument similar to that in the proof of [Nawata 2019, Corollary 5.9]. Hence an affirmative answer to the following question, which can be regarded as an analogue of Kirchberg's embedding theorem [Kirchberg and Phillips 2000], would imply this.

Question 6.5. Let A be a simple separable exact stably projectionless monotracial C*-algebra. Assume that τ_A is amenable. Is A $\mathcal{W}$-embeddable?

Note that we need to assume that τ_A is amenable because $\pi_{\tau_{\mathcal{W}}}(\mathcal{W})''$ is the injective II_1-factor.

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