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THE REGULARITY OF THE STRESS FIELD
FOR ELLIPTIC EQUATIONS IN DIVERGENCE FORM**



A GENERAL NOTION OF UNIFORM ELLIPTICITY AND THE REGULARITY OF THE STRESS FIELD FOR ELLIPTIC EQUATIONS IN DIVERGENCE FORM

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For solutions of $\operatorname{Div}(DF(Du)) = f$ we show that the quasiconformality of $z \mapsto DF(z)$ is the key property leading to the Sobolev regularity of the stress field $DF(Du)$, in relation with the summability of f . This class of nonlinearities encodes in a general way the notion of uniform ellipticity and encompasses all known instances where the stress field is known to be Sobolev regular. We provide examples showing the optimality of this assumption and present two applications: a nonlinear Cordes condition for equations in divergence form and some partial results on the $C^{p'}$ conjecture.

1. Introduction

In this work we are interested in Sobolev regularity results for the often-called “stress field” $DF(Du)$ corresponding to solutions of

$$\operatorname{div}(DF(Du)) = f, \quad (1-1)$$

seen as the Euler–Lagrange equation for the energy functional

$$J(w, \Omega) = \int_{\Omega} F(Dw) + f w \, dx, \quad (1-2)$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, $f \in L^m(\Omega)$ for some $m > 1$, and $F \in C^1(\mathbb{R}^N)$ is a strictly convex function obeying a suitable local form of uniform ellipticity condition. Questions regarding regularity of the stress field recently gained increasing interest as a basic tool to attack further regularity and locality properties of solutions to divergence form equations; see, e.g., [Avelin et al. 2018; Balci et al. 2020; Breit et al. 2018; 2022; Ciruolo et al. 2020; Colombo and Figalli 2014; Kuusi and Mingione 2013]. Despite its usefulness, however, most of the results are constrained to special kinds of nonlinearities of either p -Laplacian-type or having Uhlenbeck structure. Simple nonlinearities, such as

$$F(z) = |z - z_0|^p + |z + z_0|^p, \quad 1 < p < 2, \quad z_0 \neq 0, \quad (1-3)$$

do not fall within most of the currently available regularity theory. Since we focus on the stress field instead of Du itself, we briefly justify this point of view, recalling the general situation for functionals of the calculus of variations and their local minimizers.

Let u solve (1-1). If $F \in C^2(\mathbb{R}^N)$ is uniformly elliptic, i.e.,

$$\lambda |\xi|^2 \leq (D^2 F(z) \xi, \xi) \leq \Lambda |\xi|^2 \quad \text{for all } z, \xi \in \mathbb{R}^N,$$

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it is a classical result that $f \in L^2_{\text{loc}}(\Omega)$ if and only if $u \in W^{2,2}_{\text{loc}}(\Omega)$, and in this case the regularity of Du and $V = DF(Du)$ coincide, since $Du = DF^{-1}(V)$ and DF is bi-Lipschitz. Regarding $W^{2,m}$ -regularity for $m \neq 2$, it is clear that if $u \in W^{2,m}_{\text{loc}}(\Omega)$, then $DF(Du) \in W^{1,m}_{\text{loc}}(\Omega)$, and thus $f \in L^m_{\text{loc}}(\Omega)$. Conversely, suppose $f \in L^m_{\text{loc}}(\Omega)$; differentiating (1-1) gives

$$\operatorname{div}(D^2F(Du)Dv_k) = \partial_k f, \quad v_k = \partial_k u, \quad k = 1, \dots, N.$$

If $f \in L^m_{\text{loc}}(\Omega)$ for $m > N$, then Du is Hölder continuous by the De Giorgi–Nash theorem; so, freezing the (now continuous) coefficients of the matrix $D^2F(Du)$, we can apply the Calderón–Zygmund theorem to obtain that $u \in W^{2,m}_{\text{loc}}(\Omega)$, i.e., $DF(Du) \in W^{1,m}_{\text{loc}}(\Omega)$.

Next, consider the p -Poisson equation

$$\Delta_p u = f, \quad p > 1, \tag{1-4}$$

corresponding to the integrand $F(z) = |z|^p/p$. In this case F satisfies

$$\lambda |z|^{p-2} |\xi|^2 \leq (D^2F(z)\xi, \xi) \leq \Lambda |z|^{p-2} |\xi|^2,$$

and the Sobolev regularity of Du is much more involved. We are aware of only one result giving second-order Sobolev regularity of u from an L^m assumption on f : in the nondegenerate case $p \in (1, 2]$, that $u \in W^{2,p}(\mathbb{R}^N)$ if $f \in L^{p'}(\mathbb{R}^N)$ ($1/p + 1/p' = 1$, as usual) was proved in [Simon 1978] for global solutions and in [de Thélin 1982] for local ones.

Through difference quotients and Caccioppoli inequalities one can usually infer Sobolev regularity of Du from Sobolev regularity of f . In this framework, [Cellina 2017; Mercuri et al. 2016] treat the case when $p > 2$ is near uniform ellipticity, proving, respectively, $u \in W^{2,2}_{\text{loc}}(\Omega)$ for $2 \leq p < 3$ and $u \in W^{2,m}_{\text{loc}}(\Omega)$ for any m , as long as $p - 2$ is sufficiently small. The postulated regularity for f is $f \in W^{1,2}_{\text{loc}}(\Omega)$ in [Cellina 2017] and $f \in W^{1,m}_{\text{loc}}(\Omega)$ for $m > N$ in [Mercuri et al. 2016].

For $p > 2$ and only assuming L^m regularity of f in (1-4), the best results available prove *fractional* differentiability of Du ; see [Mingione 2007; 2010; Miśkiewicz 2018; Savaré 1998; Simon 1978]. The main idea of [Kuusi and Mingione 2013; Mingione 2007; 2010], which goes back to [Uhlenbeck 1977], is to study the regularity properties of the field

$$\mathcal{V} = |Du|^{(p-2)/2} Du \tag{1-5}$$

and deduce from the latter suitable regularity of Du . This approach is nowadays widespread, but still failed to produce estimates in terms of the Lebesgue norm of f paralleling the second-order Calderón–Zygmund theory depicted above in the nondegenerate case.

An alternative route is to consider the regularity of the stress field

$$V = |Du|^{p-2} Du,$$

which arises as an interesting object per se in a variety of situations, e.g., in the framework of nonconvex variational problems [Carstensen and Müller 2002] and in the dual formulation of traffic congestion problems; see [Brasco et al. 2010]. In particular, the applicability of DiPerna–Lions theory in the latter is tied to the Sobolev regularity of the stress field of the dual problem, which was the main concern

of [Brasco et al. 2010] for a very degenerate functional of the form (1-2). When f is Sobolev regular, variants of Caccioppoli inequalities and difference quotient methods have been used in the cited works to obtain Sobolev regularity of V ; see also [Damascelli and Sciunzi 2004].

For less regular f the seminal paper is [Lou 2008], treating the case $f \in L^m$ with $m \geq \max\{2, N/p\}$. In more recent years, the regularity of the stress field has been the object of fruitful investigations, also thanks to the fact that it seems to provide more natural estimates than (1-5). Starting from [Diening et al. 2012], a first-order nonlinear Calderón–Zygmund theory for the p -Laplacian problem with right-hand side in divergence form

$$\operatorname{div}(|Du|^{p-2} Du) = \operatorname{div} G$$

is nowadays well-developed, showing the principle that the divergence operator can be “canceled out” to get estimates for $|Du|^{p-2} Du$ in terms of G in the same space. We refer to [Balci et al. 2020; Breit et al. 2018; 2022] for this line of research, but let us remark that the order of differentiability for V considered in these works is always less than 1.

Indeed, regarding the second-order Calderón–Zygmund theory (i.e., full Sobolev regularity for V), much less is known. A natural conjecture for solutions of (1-4) via the same principle would be

$$f \in L^m_{\text{loc}}(\Omega) \iff V \in W^{1,m}_{\text{loc}}(\Omega), \tag{1-6}$$

which is actually false for $p \simeq 1$. The case $m = +\infty$, $p > 2$, corresponds to the well-known $C^{p'}$ conjecture, which will be discussed later, proved in the plane in [Araújo et al. 2017]. The endpoint $m = 1$ of (1-6) is considered by [Avelin et al. 2018], where, e.g., for $f \in L^1_{\text{loc}}(\Omega)$, it is proved that $V \in W^{1-\varepsilon,1}$ for all $\varepsilon > 0$ (actually, f can be a general Radon measure in [Avelin et al. 2018]).

The only positive result of the type (1-6) (beyond the close one in [Lou 2008]) concerns the Hilbertian case $m = 2$, which has been recently proved in [Cianchi and Mazya 2018] for equations having *Uhlenbeck structure*, i.e., of the form

$$\operatorname{div}(a(|Du|) Du) = f \quad (\text{or } \operatorname{div} G). \tag{1-7}$$

The role of this structural assumption is in fact the main motivation of this work: indeed, (with the exception of [Mingione 2007; Avelin et al. 2018]), the higher-order Calderón–Zygmund theorem exposed so far is restricted to equations of the form (1-7) and in this case it can be actually extended to systems; see [Breit et al. 2018; 2022; Cianchi and Mazya 2019; Balci et al. 2022]. As L^2 -theory seems to be the basic step to deal with the general problem (1-6), it is worth investigating *to what extent the Uhlenbeck structure is necessary to develop such a theory* and whether the general nonlinear problem (1-1) enjoys the same Sobolev regularity for its stress field $V = DF(Du)$.

It turns out that such regularity holds true when the map $z \mapsto DF(z)$ is *quasiconformal*. A quasiconformal map $G : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a homeomorphism belonging to $W^{1,N}_{\text{loc}}(\mathbb{R}^N)$ such that, for some finite K ,

$$|\lambda_{\max}(DG(z))|^N \leq K |\det DG(z)| \tag{1-8}$$

almost everywhere, with $\lambda_{\max}(DG)$ being the maximum singular value of DG ; we refer to [Martin 2014] for a short modern survey on quasiconformal mappings. The main outcome of our results is that quasiconformality of DF in (1-1) is a natural and robust notion of uniform ellipticity, flexible enough to encompass anisotropic examples such as (1-3), still allowing a reasonable regularity theory.

Quasiconformal maps which are gradients of convex functions (or being, more generally, monotone) have been systematically studied in [Kovalev and Maldonado 2005; Kovalev 2007]; convex potentials of quasiconformal mappings are called *quasiuniformly convex* functions.

Definition 1.1. A convex $F \in C^1(\mathbb{R}^N)$ is called K -quasiuniformly convex for some $K \in [1, +\infty)$ if it is not affine, $DF \in W_{loc}^{1,1}(\mathbb{R}^N)$, and

$$\lambda_{\max}(D^2F(z)) \leq K \lambda_{\min}(D^2F(z)) \quad \text{for a.e. } z \in \mathbb{R}^N, \tag{1-9}$$

where $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote the minimum and maximum eigenvalues of M .

For the most part, this will be the main assumption on F for the study of (1-1), and we will use the acronym “q.u.c.” for “quasiuniformly convex”. Clearly, (1-9) and (1-8) for $G = DF$ are equivalent (but (1-9) implies (1-8) with a constant K^{N-1}); in Proposition 2.3 we will show that any q.u.c. function is strictly convex and of (p, q) -growth. In Section 3C we will discuss concrete examples but, in the meantime, we note that $z \mapsto |z|^p$ is q.u.c. for any $p > 1$, and the sum of q.u.c. functions is q.u.c.; hence (1-3) is q.u.c.

1A. Main results. We now present our main results, referring to the appropriate theorems in the following sections for more complete statements. The first one concerns local minimizers $u \in W_{loc}^{1,p}(\Omega)$ for the functional J in (1-2), i.e., those u obeying $J(u) < +\infty$ and $J(u, B) \leq J(u + w, B)$ for all $B \Subset \Omega$ and $w \in W_0^{1,p}(B)$.

Theorem 1.2 (Theorem 3.3). *Let $\Omega \subseteq \mathbb{R}^N$ be open, $F \in C^1(\mathbb{R}^N)$ be K -q.u.c., and $f \in L^2(\Omega) \cap W^{-1,p'}(\Omega)$ for $p = 1 + 1/K$. Then any local minimizer $u \in W_{loc}^{1,p}(\Omega)$ of J satisfies*

$$\|DF(Du)\|_{W^{1,2}(B_R)} \leq C(1 + \|f\|_{L^2(B_{2R})} + \|F(Du)\|_{L^1(B_{2R})}^K)$$

for some $C = C(K, N, R)$ and all $B_{4R} \subseteq \Omega$.

Remark 1.3. Let us briefly discuss the main assumptions. Other comments may be found in Remark 3.4.

(1) *Assumptions on F .* The Sobolev regularity assumption $DF \in W_{loc}^{1,1}(\mathbb{R}^N)$ is necessary, as shown in Example 3.5, where a strictly convex, radial $F \in C^1(\mathbb{R}^N)$ obeying (1-9) is constructed in such a way that the stress field of a suitable solution to (1-1) with $f = 0$ is not absolutely continuous.

The q.u. convexity condition fails in simple examples such as the orthotropic p -Laplacian related to the integrand

$$F(z) = \sum_{i=1}^N |z_i|^p.$$

We remark that Giaquinta’s example [1987] in \mathbb{R}^6 is a local minimizer of an analytic functional whose integrand is not q.u.c. and such that the stress field is in $W_{loc}^{1,s}(\mathbb{R}^N)$ only for $s < \frac{5}{4}$. Playing around with examples of similar structure suggests intricate interplays between the maximal Sobolev regularity of the stress field and the possibly nonstandard structure of DF ; hence, it is not clear what to expect from functionals with non-quasiconformal gradient mapping.

In Example 3.6 we discuss integrands of the form $F(z) = F(|z|)$, while Examples 3.7 and 3.8 investigate more general anisotropic functionals.

- *Assumptions on f .* The hypothesis $f \in W^{-1,p'}(\Omega)$ has been made for expository reasons and the relation between p and K will be derived in [Proposition 2.3](#). On one hand, our results are mostly local in nature and therefore it suffices to require $f \in L^2_{\text{loc}}(\Omega) \cap W^{-1,p'}_{\text{loc}}(\Omega)$, meaning with the latter the intersection of the spaces $W^{-1,p'}(\Omega_n)$ on an exhausting sequence of open relatively compact $\Omega_n \uparrow \Omega$. For $p \geq 2N/(N+2)$, the condition $f \in W^{-1,p'}_{\text{loc}}(\Omega)$ automatically follows from Sobolev embedding and $f \in L^2_{\text{loc}}(\Omega)$; for $p < 2N/(N+2)$, $L^2(\Omega)$ is not embedded in $W^{-1,p'}(\Omega)$, destroying the variational framework we chose to be in. One should then resort to the notion of *approximable solutions* (briefly described in the last section), in order to deal with those cases. We refer to [\[Alberico et al. 2019\]](#) for a comprehensive theory of approximable solutions in the anisotropic framework. Anyway, in terms of summability, $f \in W^{-1,p'}(\Omega)$ is implied by $f \in L^{(p^*)'}(\Omega)$, which is a weaker summability than the one in [\[Lou 2008\]](#).

- *The exponent p .* As will be clear from the proof, the exponent $p = 1 + 1/K$ is not the only possible choice. Any p satisfying

$$|z|^p \leq C(F(z) + 1), \quad z \in \mathbb{R}^N,$$

will serve the purpose of proving [Theorem 1.2](#), and the previous estimate holds true for $p = 1 + 1/K$ and F K -q.u.c., thanks to [Proposition 2.3](#) below. In [Example 3.6](#) we will see that higher choices of p are sometimes feasible. Clearly, the variational condition $f \in W^{-1,p'}(\Omega)$ is weaker for higher p 's.

We will give two applications of [Theorem 1.2](#). The first one deals with nonlinear Cordes conditions for variational problems. Cordes conditions usually refer to L^m -theory for elliptic equations in nondivergence form with measurable coefficients, namely, to solutions of

$$\sum_{ij=1}^N a_{ij} D_{ij}u = f, \tag{1-10}$$

with measurable coefficients $a_{ij} : \Omega \subseteq \mathbb{R}^N \rightarrow \mathbb{R}$ satisfying

$$\lambda |\xi|^2 \leq \sum_{ij=1}^N a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2$$

for all $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$. Under these measurability assumptions alone, there is no hope in general for the Calderón–Zygmund inequality

$$\|u\|_{W^{2,m}(\Omega)} \leq C_m \|f\|_{L^m(\Omega)} \quad \text{for all } m > 1. \tag{1-11}$$

Some regularity has to be assumed on a_{ij} in order to obtain (1-11) for all $m > 1$ (VMO regularity suffices; see [\[Chiarenza et al. 1991\]](#)). Roughly stated, a Cordes condition for (1-10) with discontinuous a_{ij} says that (1-11) holds if either m is sufficiently near 2, or Λ/λ is sufficiently near 1. A similar situation takes place for nonlinear equations in divergence form.

Theorem 1.4 (Theorem 4.3). *Let F obey the assumptions of [Theorem 1.2](#) and $f \in L^m(\Omega) \cap W^{-1,p'}(\Omega)$ for some $m > 1$. Then any local minimizer $u \in W^{1,p}(\Omega)$ for J in (1-2) is such that $DF(Du) \in W^{1,m}_{\text{loc}}(\Omega)$ and*

$$\|DF(Du)\|_{W^{1,m}(B_R)} \leq C(\|f\|_{L^m(B_{2R})} + \|DF(Du)\|_{L^m(B_{2R})})$$

for $C = C(K, N, m, R)$ and all $B_{4R} \subseteq \Omega$, in either of the following cases:

- (i) $K \leq K_0$, with $K_0 = K_0(N, m) > 1$.
- (ii) $|m - 2| \leq \delta_0$ for $\delta_0 = \delta_0(K, N) > 0$.

As a second application we will derive some partial results pertaining the $C^{p'}$ conjecture, which states that any solution of the p -Poisson equation with bounded right-hand side is $C^{p'}$ regular if $p > 2$ (hereafter we use the notation $C^\gamma = C^{[\gamma], \gamma - [\gamma]}$, where $[\gamma]$ is the integer part of γ).

Theorem 1.5 (Corollary 4.5, Theorem 4.7, and Corollary 4.8). *Let $u : \Omega \rightarrow \mathbb{R}$ be an approximable solution of (1-4).*

- (i) *If $f \in L^\infty(\Omega)$, then for all sufficiently small $|p - 2|$ it holds $u \in C^{2 - \alpha_p}(\Omega)$, where $\alpha_p = c(N)|p - 2| > 0$.*
- (ii) *For any $m, p > 1$, if u and Ω have cylindrical symmetry, then $|Du|^{p-2} Du \in W_{loc}^{1,m}(\Omega)$ whenever $f \in L_{loc}^m(\Omega)$. In particular, any cylindrical approximable solution of (1-4) belongs to $C^{\min\{p', 2\} - \varepsilon}(\Omega)$ for any $\varepsilon > 0$.*

Remark 1.6. • Item (i) confirms the validity of the $C^{p'}$ conjecture near uniform ellipticity, and is inspired by [Mercuri et al. 2016]. Here, however, we obtain an explicit rate of the Hölder exponent and, more substantially, we do not require Sobolev regularity on f . When $f \in L^\infty(\Omega)$, the usual notion of weak solution suffices.

• Point (ii) gives a weak form of the $C^{p'}$ conjecture (namely, $u \in C^{p'-}$) in the class of cylindrical solutions for any $p \geq 2$, with a different approach than the one of [Araújo et al. 2018]. We need the notion of approximable solutions as in general f may fail to belong to $W^{-1,p'}(\Omega)$ for small $m > 1$, destroying the variational setting. For details on such a notion we refer to Section 4B.

By a cylindrical solution we mean a function of the form $u(x) = v(|x'|)$, $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $k \leq N$. It is worth noticing that the domain Ω may not contain the origin, in which case the approach of [Araújo et al. 2018] cannot be easily applied.

1B. Outline of the proofs. Consider, as a first step, a smooth compactly supported solution u of

$$\operatorname{div} DF(Du) = f \quad \text{in } \mathbb{R}^N,$$

with F and f smooth. Our starting point is the well-known identity

$$\|DV\|_{L^2(\mathbb{R}^N)}^2 = \|\operatorname{div} V\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{2} \|\operatorname{curl} V\|_{L^2(\mathbb{R}^N)}^2 \quad \text{for all } V \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N), \tag{1-12}$$

where $\operatorname{curl} V = DV - DV^t$. Applying (1-12) to the stress field $V = DF(Du)$, we are reduced to estimate $\operatorname{curl} V$.

The main observation is that, in the smooth setting, DV is of special type, namely

$$DV = D^2F(Du) D^2u,$$

where $D^2F(Du)$ is a symmetric positive definite matrix and D^2u is symmetric. An elementary lemma shows that any matrix of the form

$$X = PS, \quad P \text{ symmetric positive definite, } S \text{ symmetric,}$$

satisfies

$$|X - X^t|_2^2 \leq 2 \left(1 - \frac{\lambda_{\min}}{\lambda_{\max}} \right)^2 |X|_2^2, \tag{1-13}$$

where λ_{\min} and λ_{\max} are, respectively, the minimum and maximum eigenvalues of P . Thus the curl term in (1-12) can be reabsorbed to the left if $\lambda_{\max} \leq K \lambda_{\min}$ holds a.e. for the matrix D^2F , giving

$$\|DV\|_{L^2(\mathbb{R}^N)}^2 \leq K^2 \|f\|_{L^2(\mathbb{R}^N)}^2$$

for $V = DF(Du)$ in the smooth, global setting.

In order to prove [Theorem 1.2](#) we have to localize the estimate and to suitably build smooth approximating problems. We regularize the integrand, the source, and the boundary data through convolution but, in order to have strongly elliptic problems, we would like to add a small multiple of a strongly elliptic functional. Since we do not want to alter the q.u. convexity constant, the only viable choice is to add small multiples of $|z|^2$ to the regularized integrands. This is a quite unnatural choice if F is not of standard quadratic growth, and it forces some interplay between the regularization parameters. Here, the explicit a priori Lipschitz estimate for the corresponding solutions taken from [\[Bousquet and Brasco 2016\]](#) plays a key role.

The main step of the proof of [Theorem 1.4](#) is to represent the solutions of

$$\begin{cases} \operatorname{div} V = f, \\ \operatorname{curl} V = G \end{cases}$$

in \mathbb{R}^N through Riesz transforms and generalize (1-12) to the L^m -setting as

$$\|DV\|_{L^m(\mathbb{R}^N)} \leq C(m, N) (\|\operatorname{div} V\|_{L^m(\mathbb{R}^N)} + \|\operatorname{curl} V\|_{L^m(\mathbb{R}^N)}).$$

Thus, inequality (1-13) does the trick in case (i) of [Theorem 1.4](#), allowing reabsorption of the curl term for K sufficiently near 1. A standard Riesz interpolation argument, together with a careful choice of the norms involved, allows proving case (ii).

Finally, point (i) of [Theorem 1.5](#) is an immediate consequence of point (i) of [Theorem 1.4](#), while point (ii) stems from this observation: if u exhibits cylindrical symmetry, then the stress field $a(|Du|) Du$ is irrotational; by the Helmholtz decomposition, it can be locally represented as the gradient of a solution of $\Delta v = f \in L^m$, so the standard Calderón–Zygmund theorem applies.

1C. Structure of the paper. In [Section 2](#) we recall some functional inequalities and properties about quasiuniform convexity. In [Section 3A](#) we develop our basic estimate in the smooth setting; [Section 3B](#) is devoted to the proof of [Theorem 1.2](#), where the main approximation procedure, used also later, is described; [Section 3C](#) contains the relevant examples depicted above. In [Section 4](#) we focus on the applications: first we treat the Cordes conditions, and finally we collect the partial results pertaining the $C^{p'}$ conjecture.

Notations. • The euclidean norm of a vector $v \in \mathbb{R}^N$ is denoted by $|v|$ and (v, w) denotes the scalar product. By Ω we denote a bounded open subset of \mathbb{R}^N , while B_r denotes a ball of radius r not necessarily centered at the origin. Similarly, B denotes a ball with unspecified center and radius; if $B = B_r(x_0)$, then we set $\lambda B = B_{\lambda r}(x_0)$.

- For a $N \times N$ matrix $A = (a_{ij})$, its transpose is denoted by A^t ; on such matrices we consider the Frobenius norm

$$|A|_2 = \left(\sum_{i,j=1}^N |a_{ij}|^2 \right)^{1/2},$$

arising from the scalar product $(A, B)_2 = \text{Tr}(A B^t)$. For $v, w \in \mathbb{R}^N$, $v \otimes w$ is the matrix with entries $(v_i w_j)$, while $v \wedge w = v \otimes w - w \otimes v$; I denotes the identity matrix, while Id the identity function of \mathbb{R}^N ; O_N is the orthogonal group of $N \times N$ matrices such that $A A^t = A^t A = I$; if A is symmetric, $\lambda_{\min}(A)$, $\lambda_{\max}(A)$ are its minimum and its maximum eigenvalues.

- If a domain of integration is missing, it means that we are integrating on \mathbb{R}^N . We also set for brevity $\|f\|_m = \|f\|_{L^m(\mathbb{R}^N)}$. Finally, we sometimes will use the L^m “norm” also for $m \in (0, 1)$, when it is only positively 1-homogeneous.
- As is customary, we indicate with $W^{-1,p'}(\Omega)$ the dual space of $W_0^{1,p}(\Omega)$.

2. Preliminaries

2A. Functional inequalities. For $0 < \theta < m$, $m > 1$, starting from the inequality

$$\int_{B_1} |v|^m dx \leq \int_{B_1} |Dv|^m dx + C(N, m, \theta) \left(\int_{B_1} |v|^\theta dx \right)^{m/\theta} \tag{2-1}$$

(obtained by a standard compactness argument), and replicating, with the obvious modifications, the proof of [Cianchi and Mazya 2018, equation (5.4)], we get the following functional inequality.

Lemma 2.1. *Let $m > 1$, $0 < \theta < m$, and $R \leq r < s < 2R$. There exists $C = C(N, m, \theta)$ such that for any $v \in W^{1,m}(B_s \setminus B_r)$ and any $\delta \in (0, 1)$ it holds*

$$\int_{B_s \setminus B_r} |v|^m dx \leq \delta^m R^m \int_{B_s \setminus B_r} |Dv|^m dx + \frac{C}{(\delta^N (s-r) R^{N-1})^{(m-\theta)/\theta}} \left(\int_{B_s \setminus B_r} |v|^\theta dx \right)^{m/\theta}.$$

The following lemma is a straightforward generalization of the well-known identity

$$\int |DV|_2^2 dx = \frac{1}{2} \int |\text{curl } V|_2^2 dx + \int (\text{div } V)^2 dx, \tag{2-2}$$

valid for $V \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$.

Lemma 2.2. *If $V \in C^2(\mathbb{R}^N, \mathbb{R}^N)$, then for any $\varphi \in C_c^2(\mathbb{R}^N)$ it holds*

$$\begin{aligned} \int \varphi^2 |DV|_2^2 dx &= \frac{1}{2} \int \varphi^2 |\text{curl } V|_2^2 dx + \int \varphi^2 (\text{div } V)^2 dx \\ &\quad + \int [2(D\varphi^2, V) \text{div } V + (D^2\varphi^2, V \otimes V)_2] dx. \end{aligned} \tag{2-3}$$

Proof. Write, through parallelogram identity,

$$|DV|_2^2 = \frac{1}{4} |DV - DV^t|_2^2 + \frac{1}{4} |DV + DV^t|_2^2,$$

then multiply by φ^2 and integrate to obtain

$$\begin{aligned} \int \varphi^2 |DV|_2^2 dx &= \int \frac{\varphi^2}{4} |\operatorname{curl} V|_2^2 dx + \int \frac{\varphi^2}{4} \sum_{ij} (D_j V^i + D_i V^j)^2 dx \\ &= \int \frac{\varphi^2}{4} |\operatorname{curl} V|_2^2 dx + \int \frac{\varphi^2}{2} |DV|_2^2 dx + \int \frac{\varphi^2}{2} \sum_{ij} D_i V^j D_j V^i dx. \end{aligned} \tag{2-4}$$

The last term is computed integrating by parts twice: for any $i, j = 1, \dots, N$

$$\begin{aligned} \int \varphi^2 D_i V^j D_j V^i dx &= - \int D_i \varphi^2 V^j D_j V^i dx - \int \varphi^2 V^j D_{ij} V^i dx \\ &= \int D_{ij} \varphi^2 V^j V^i dx + \int D_i \varphi^2 V^i D_j V^j dx + \int D_j \varphi^2 V^j D_i V^i dx + \int \varphi^2 D_j V^j D_i V^i dx, \end{aligned}$$

so that summing over i, j gives

$$\int \varphi^2 \sum_{ij} D_i V^j D_j V^i dx = \int [\varphi^2 (\operatorname{div} V)^2 + 2 (D\varphi^2, V) \operatorname{div} V + (D^2 \varphi^2, V \otimes V)_2] dx.$$

Inserting this formula into (2-4) yields (2-3). □

2B. Quasiuniform convexity. In this section we show that the q.u. convexity condition provides, in a unified way, many of the properties that the usual integrands of the calculus of variations satisfy.

Let us begin by observing that the gradient of a convex F is defined a.e. and belongs to $BV_{\text{loc}}(\mathbb{R}^N)$; see [Alberti and Ambrosio 1999]. Accordingly, the second derivative of F can be decomposed into an absolutely continuous part, a jump part, and a Cantor part. If $F \in C^1(\mathbb{R}^N)$ the jump part vanishes; hence by requiring that $F \in C^1(\mathbb{R}^N) \cap W_{\text{loc}}^{2,1}(\mathbb{R}^N)$ we are actually excluding that $D^2 F$ has a Cantor part.

Now we discuss in detail some consequences of the q.u. convexity condition; although condition (iv) of Proposition 2.3 will be not used in the sequel, we prove it for the sake of completeness.

Proposition 2.3 (properties of K -quasiuniformly convex functions). *Let F be a K -quasiuniformly convex function. Then:*

- (i) DF is K^{N-1} -quasiconformal; hence $C^{1/K}(\mathbb{R}^N)$.
- (ii) F is strictly convex and of (p, q) -growth, i.e., there exists $C = C(N, K, F) > 0$ and $1 < p < q < +\infty$ such that

$$C^{-1}|z|^p - C \leq F(z) \leq C(|z|^q + 1), \tag{2-5}$$

$$C^{-1}|z|^{p-1} - C \leq |DF(z)| \leq C(|z|^{q-1} + 1) \tag{2-6}$$

for all $z \in \mathbb{R}^N$. More precisely, one can take $p = 1 + 1/K$ and $q = 1 + K$.

- (iii) If $\varphi \in C_c^\infty(\mathbb{R}^N; [0, +\infty))$, then $F * \varphi$ is K -q.u.c.

(iv) *Its Moreau–Yoshida regularization*

$$F_\delta(z) = \inf_{y \in \mathbb{R}^N} \left\{ F(y) + \frac{1}{2\delta} |y - z|^2 \right\} \tag{2-7}$$

is K -*q.u.c.*

Proof. (i) By the Alexandrov theorem DF is differentiable a.e., and [Väisälä 1971, Theorem 32.3] ensures that $DF \in W_{\text{loc}}^{1,N}(\mathbb{R}^N)$. Since (1-8) and (1-9) are equivalent up to changing the constants, u is quasiuniformly convex in the sense of [Kovalev and Maldonado 2005]. In particular Theorem 3.1 of that work shows that DF is K^{N-1} -quasiconformal. The regularity statement holds for any quasiconformal mapping; see [Martin 2014, Theorem 2.14].

(ii) The strict convexity of F follows from [Kovalev and Maldonado 2005, Lemma 3.2]. If z_0 is the unique minimum point for F we can consider $F(z_0 + \cdot) - F(z_0)$, so there is no loss in generality assuming $DF(0) = 0$, $F(0) = 0$. Let $G = DF$, which then is K^{N-1} -quasiconformal. By [Martin 2014, Theorem 2.14]

$$|G(z)| \leq C(N, K) \sup_{y \in B_1} |G(y)| |z|^{1/K}, \quad z \in B_1.$$

Since G^{-1} is still K^{N-1} quasiconformal, it obeys a similar estimate, proving the lower bound

$$|G(z)| \geq C(N, K, G) |z|^K, \quad z \in B_1.$$

Finally, the inversion

$$G^*(x) = \frac{G(x/|x|^2)}{|G(x/|x|^2)|^2}$$

is again K^{N-1} quasiconformal on B_1 , so that the previous estimates are transferred to the outside of B_1 as

$$C^{-1} |z|^{1/K} \leq |G(z)| \leq C |z|^K, \quad |z| \geq 1, \tag{2-8}$$

where $C = C(N, K, G)$. For $G = DF$, $p = 1 + 1/K$, $q = 1 + K$, we thus obtained (2-6). Moreover, by [Kovalev 2007, Lemma 18], DF is δ -convex for some $\delta = \delta(N, K) > 0$, meaning that

$$(DF(z) - DF(y), z - y) \geq \delta |DF(z) - DF(y)| |z - y| \quad \text{for all } z, y \in \mathbb{R}^N. \tag{2-9}$$

Using (2-9) and (2-6), we get

$$F(z) = \int_0^1 (DF(tz), z) dt \geq \delta \int_0^1 |DF(tz)| |z| dt \geq \frac{\delta}{pC} |z|^p - C |z| \geq \frac{\delta}{pC} |z|^p - C,$$

by sufficiently increasing C in the last inequality. This produces the lower bound in (2-5), while the upper bound follows from (2-6) alone through a similar calculation.

(iii) Let $\lambda_{\min}(z) = \lambda_{\min}(D^2F(z))$, where z is a second-order differentiability point for F . From the representation

$$\lambda_{\min}(z) := \inf\{(D^2F(z)\xi, \xi) : \xi \in D\},$$

where D is a fixed countable dense subset of \mathbb{S}^{N-1} , we infer that λ_{\min} is measurable and in $L^1_{\text{loc}}(\mathbb{R}^N)$. Then, for any $z \in \mathbb{R}^N$ and $\xi \in D$,

$$\begin{aligned} (D^2 F * \varphi(z) \xi, \xi) &= \int \varphi(z - y) (D^2 F(y) \xi, \xi) dy \\ &\geq \int \varphi(z - y) \lambda_{\min}(y) |\xi|^2 dx =: \tilde{\lambda}_{\min}(z) |\xi|^2, \end{aligned}$$

while

$$\int \varphi(z - y) (D^2 F(y) \xi, \xi) dy \leq \int \varphi(z - y) K \lambda_{\min}(y) |\xi|^2 dy = K \tilde{\lambda}_{\min}(z) |\xi|^2,$$

implying the claim.

(iv) Recall that the minimum in (2-7) is attained at a unique point $P_\delta(z)$ satisfying

$$P_\delta(z) + \delta DF(P_\delta(z)) = z, \quad DF_\delta(z) = DF(P_\delta(z)), \tag{2-10}$$

and the so-defined function $P_\delta = (\text{Id} + \delta DF)^{-1}$ is 1-Lipschitz and a homeomorphism of \mathbb{R}^N , since $F \in C^1(\mathbb{R}^N)$. Let E be the set of points where DF fails to be differentiable. The map $\text{Id} + \delta DF$ is the gradient of a K -quasiuniformly convex function, and hence by point (i) is quasiconformal and satisfies the Lusin (N) property, i.e., it sends null-measure sets to null-measure sets; see [Väisälä 1971]. Thus, $P_\delta^{-1}(E)$ has zero measure. Moreover, since P_δ is Lipschitz continuous, Rademacher’s theorem ensures that the set M where P_δ is not differentiable has zero measure. We will prove (1-9) at any point

$$z \notin M \cup P_\delta^{-1}(E),$$

the latter set having zero measure. Indeed, at any such point z we have that DF is differentiable at $P_\delta(z)$ and P_δ is differentiable at z . The chain rule applied to (2-10) then gives

$$(I + \delta D^2 F(P_\delta(z))) DP_\delta(z) = I, \quad D^2 F_\delta(z) = D^2 F(P_\delta(z)) DP_\delta(z),$$

which yields

$$D^2 F_\delta(z) = D^2 F(P_\delta(z))(I + \delta D^2 F(P_\delta(z)))^{-1}. \tag{2-11}$$

Let the eigenvalues of $D^2 F(P_\delta(z))$ be $\lambda_{\min} = \lambda_1 \leq \dots \leq \lambda_N = \lambda_{\max}$. The matrices $D^2 F(P_\delta(z))$ and $(I + \delta D^2 F(P_\delta(z)))^{-1}$ have the same basis of eigenvectors, with eigenvalues λ_i and $(1 + \delta \lambda_i)^{-1}$ respectively. Hence (2-11) implies that $D^2 F_\delta(z)$ has eigenvalues $\lambda_i / (1 + \delta \lambda_i)$. As $t \mapsto t / (1 + \delta t)$ is increasing, its minimum and maximum eigenvalues are

$$\lambda_{\delta, \min} := \frac{\lambda_{\min}}{1 + \delta \lambda_{\min}}, \quad \lambda_{\delta, \max} := \frac{\lambda_{\max}}{1 + \delta \lambda_{\max}},$$

which obey $\lambda_{\delta, \max} \leq K \lambda_{\delta, \min}$ as long as $\lambda_{\max} \leq K \lambda_{\min}$. □

Due to the previous proposition, we will denote henceforth by p and q the powers of the lower and upper bounds, respectively, for a given K -q.u.c. function F .

2C. Extensions. We conclude with a couple of tools which will be occasionally used in the following.

Lemma 2.4. *Let $F \in C^1(B_R)$ be a nonnegative, strictly convex function, and $\sigma \in (0, 1)$.*

- (i) *There exists a strictly convex $\tilde{F} \in C^1(\mathbb{R}^N)$ such that $\tilde{F}|_{B_{\sigma R}} = F$ and, for some C, α depending on F, R, N, σ , it holds*

$$|z|^2 \leq \alpha(\tilde{F}(z) + 1), \quad C^{-1}|z| - C \leq |D\tilde{F}(z)| \leq C(|z| + 1). \tag{2-12}$$

- (ii) *If F is K -q.u.c. in B_R as per Definition 1.1, and for some $\varepsilon > 0$ it holds $\lambda_{\min}(z) \geq \varepsilon$ in $B_R \setminus B_{\sigma R}$, in addition to (2-12) \tilde{F} can be chosen to be \tilde{K} -q.u.c., with $\tilde{K} = \tilde{K}(F, R, N, \sigma, \varepsilon)$.*

Proof. To prove (i), let $\tau = (1 + \sigma)/2$, choose a radial cut-off function $\eta \in C_c^\infty(B_R; [0, 1])$ such that $\eta \equiv 1$ in $B_{\tau R}$, and define

$$\tilde{F}(z) = \eta(z) F(z) + (1 - \eta(z)) \frac{|z|^2}{2} + C(|z| - \sigma R)_+^2,$$

where $C > 0$ is a constant to be chosen. Clearly $\tilde{F} \in C^1(\mathbb{R}^N)$ and obeys (2-12), so it remains to show that F is strictly convex for a suitable C . To this aim, set

$$A(z) := \frac{1}{2} D^2(|z| - \sigma R)^2 = \frac{\sigma R}{|z|} \frac{z}{|z|} \otimes \frac{z}{|z|} + \left(1 - \frac{\sigma R}{|z|}\right) I,$$

whose eigenvalues are 1 and $1 - \sigma R/|z|$. In particular, A is nonnegative definite in $B_R \setminus B_{\sigma R}$, and its eigenvalues are uniformly bounded below in $\mathbb{R}^N \setminus B_{\tau R}$ by a positive constant; since \tilde{F} agrees with F in $B_{\sigma R}$, it follows that \tilde{F} is strictly convex in $B_{\tau R}$ and in $\mathbb{R}^N \setminus B_R$. A straightforward computation yields

$$D^2 \tilde{F} = \eta D^2 F + M + 2CA,$$

$$M := (1 - \eta) I + D\eta \otimes DF + DF \otimes D\eta - 2D\eta \otimes z + \left(F - \frac{|z|^2}{2}\right) D^2 \eta$$

a.e. outside $B_{\sigma R}$, and we can choose C so that

$$\left(1 - \frac{\sigma}{\tau}\right) C = \max_{z \in B_R} |M(z)|_2,$$

ensuring

$$\lambda_{\min}(D^2 \tilde{F}) \geq \eta \lambda_{\min}(D^2 F) + \left(1 - \frac{\sigma}{\tau}\right) C \quad \text{in } B_R \setminus B_{\tau R}.$$

Summing up, \tilde{F} is globally strictly convex by an elementary argument.

To prove (ii), let $\tilde{\lambda}_{\min}(z)$ and $\tilde{\lambda}_{\max}(z)$ denote the minimum and maximum eigenvalues of $D^2 \tilde{F}(z)$, and $\lambda_{\min}, \lambda_{\max}$ those for $D^2 F$. It holds

$$\varepsilon \leq \lambda_{\min} \leq \tilde{\lambda}_{\min} \leq \tilde{\lambda}_{\max} \leq 3C + \lambda_{\max}$$

in $B_{\tau R} \setminus B_{\sigma R}$, so that from (1-9) we get

$$\frac{\tilde{\lambda}_{\max}}{\tilde{\lambda}_{\min}} \leq \frac{\tilde{\lambda}_{\max}}{\lambda_{\min}} \leq \frac{3C}{\lambda_{\min}} + K \leq \frac{3C}{\varepsilon} + K.$$

In $\mathbb{R}^N \setminus B_{\tau R}$ it holds

$$(1 - \sigma/\tau) C + \eta \lambda_{\min} \leq \tilde{\lambda}_{\min} \leq \tilde{\lambda}_{\max} \leq 3 C + \eta \lambda_{\max},$$

so that

$$\frac{\tilde{\lambda}_{\max}}{\tilde{\lambda}_{\min}} \leq \frac{3 C + K \eta \lambda_{\min}}{(1 - \sigma/\tau) C + \eta \lambda_{\min}} \leq \frac{3 \tau}{\tau - \sigma} + K.$$

Since $\tilde{F} = F$ on $B_{\sigma R}$ and (1-9) holds for F there, the claim is proved. □

3. Divergence form, quasiconformal equations

3A. The smooth setting. The core of our approach lies in the following elementary observation.

Lemma 3.1. *Let $X = PS$, where P and S are symmetric $N \times N$ matrices and P is positive definite with minimum and maximum eigenvalues λ_{\min} and λ_{\max} . Then*

$$|X - X^t|_2^2 \leq 2 \frac{(1 - \lambda_{\min}/\lambda_{\max})^2}{1 + (\lambda_{\min}/\lambda_{\max})^2} |X|_2^2. \tag{3-1}$$

Proof. Inequality (3-1) is invariant under rotations; thus, without loss of generality, we can suppose $P_{ij} = \lambda_i \delta_{ij}$ with $0 < \lambda_1 \leq \dots \leq \lambda_N$. Then from $X = PS$ we get

$$X_{ij} = \lambda_i S_{ij},$$

so that

$$|X - X^t|_2^2 = \sum_{ij} |X_{ji} - X_{ij}|^2 = 2 \sum_{i < j} |X_{ji} - X_{ij}|^2 = 2 \sum_{i < j} |\lambda_j S_{ji} - \lambda_i S_{ij}|^2,$$

and from the symmetry of S we conclude

$$|X - X^t|_2^2 = 2 \sum_{i < j} S_{ij}^2 (\lambda_j - \lambda_i)^2. \tag{3-2}$$

Similarly, we have

$$|X|_2^2 \geq \sum_{i < j} (X_{ij}^2 + X_{ji}^2) = \sum_{i < j} S_{ij}^2 (\lambda_i^2 + \lambda_j^2). \tag{3-3}$$

Let

$$\varphi(t) = \frac{(1 - t)^2}{1 + t^2},$$

which is decreasing in $[0, 1]$, and observe that for $j > i$ we have $\lambda_i/\lambda_j \in [0, 1]$. Therefore

$$(\lambda_j - \lambda_i)^2 = \frac{(\lambda_j - \lambda_i)^2}{\lambda_j^2 + \lambda_i^2} (\lambda_j^2 + \lambda_i^2) = \varphi\left(\frac{\lambda_i}{\lambda_j}\right) (\lambda_j^2 + \lambda_i^2) \leq \varphi\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right) (\lambda_j^2 + \lambda_i^2).$$

Inserting this estimate in (3-3) and recalling (3-2) we get

$$|X - X^t|_2^2 \leq 2 \varphi\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right) \sum_{i < j} S_{ij}^2 (\lambda_i^2 + \lambda_j^2) = 2 \varphi\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right) |X|_2^2. \tag{3-4} \quad \square$$

Theorem 3.2. *Let $u \in C^2(B_{2R})$ solve*

$$\operatorname{div}(DF(Du)) = f \quad \text{in } B_{2R}$$

for a K -q.u.c. $F \in C^2(\mathbb{R}^N)$, and let

$$V(x) = DF(Du(x)).$$

Then for any $\theta \in (0, 2]$ there exist $C = C(N, K, \theta)$ and $C_R = C(N, K, \theta, R)$ such that

$$\|V\|_{W^{1,2}(B_R)} \leq C \|f\|_{L^2(B_{2R})} + C_R \|V\|_{L^\theta(B_{2R})}. \tag{3-4}$$

Proof. For any $\varepsilon > 0$ let

$$F_\varepsilon(z) = F(z) + \varepsilon \frac{|z|^2}{2}, \quad f_\varepsilon = f + \varepsilon \Delta u,$$

so that D^2F_ε is symmetric and positive definite. It holds

$$\lambda_{\min}(D^2F_\varepsilon(z)) = \lambda_{\min}(D^2F(z)) + \varepsilon, \quad \lambda_{\max}(D^2F_\varepsilon(z)) = \lambda_{\max}(D^2F(z)) + \varepsilon,$$

so that if (1-9) holds for F , it does so for D^2F_ε as well, with the same constant K .

Clearly u solves

$$\operatorname{div}(DF_\varepsilon(Du)) = f_\varepsilon$$

in B_{2R} . Letting

$$V_\varepsilon = DF_\varepsilon(Du),$$

it holds

$$DV_\varepsilon = D^2F_\varepsilon(Du) D^2u,$$

where the first matrix is symmetric positive definite and the second one is symmetric. We thus apply Lemma 3.1 to the matrix

$$P = D^2F_\varepsilon(Du), \quad S = D^2u, \quad X = DV_\varepsilon = PS.$$

Recall that $D^2F_\varepsilon(Du)$ satisfies (1-9) with constant K , whence

$$\frac{(\lambda_{\max}(D^2F_\varepsilon(Du)) - \lambda_{\min}(D^2F_\varepsilon(Du)))^2}{\lambda_{\max}^2(D^2F_\varepsilon(Du)) + \lambda_{\min}^2(D^2F_\varepsilon(Du))} \leq \frac{(K-1)^2}{K^2+1}.$$

From (3-1) we get

$$|\operatorname{curl} V_\varepsilon|_2^2 \leq 2 \frac{(K-1)^2}{K^2+1} |DV_\varepsilon|_2^2. \tag{3-5}$$

For any r, s with $R \leq r < s \leq 2R$, fix $\varphi \in C_c^\infty(B_s, [0, 1])$ such that

$$\varphi|_{B_r} \equiv 1, \quad |D\varphi| \leq \frac{C}{s-r}, \quad |D^2\varphi| \leq \frac{C}{(s-r)^2}. \tag{3-6}$$

This will allow us to consider φV_ε as defined on the whole \mathbb{R}^N , so that (2-3) holds true. The stipulated properties of φ ensure that

$$\int 2(D\varphi^2, V_\varepsilon) f_\varepsilon + (D^2\varphi^2, V_\varepsilon \otimes V_\varepsilon)_2 dx \leq \frac{C}{(s-r)^2} \int_{B_s \setminus B_r} |V_\varepsilon|^2 dx + C \int_{B_{2R}} f_\varepsilon^2 dx,$$

where we used the Schwartz inequality on the first term and $s \leq 2R$. Using also (3-5) to control the curl term of Lemma 2.2 yields

$$\int \varphi^2 |DV_\varepsilon|_2^2 dx \leq \left(1 - \frac{1}{K}\right)^2 \int \varphi^2 |DV_\varepsilon|_2^2 dx + \frac{C}{(s-r)^2} \int_{B_s \setminus B_r} |V_\varepsilon|^2 dx + C \int_{B_{2R}} f_\varepsilon^2 dx.$$

We let $\varepsilon \rightarrow 0$ and bring the first term on the right to the left-hand side; recalling that $\varphi \equiv 1$ on B_r , we obtain

$$\int_{B_r} |DV|_2^2 dx \leq \frac{C_K}{(s-r)^2} \int_{B_s \setminus B_r} |V|^2 + C_K \int_{B_{2R}} f^2 dx \tag{3-7}$$

for any $R \leq r < s \leq 2R$. We next proceed as in [Cianchi and Mazya 2018]: by Lemma 2.1 with $m = 2$ and

$$\delta = \frac{s-r}{2\sqrt{C_K}R},$$

we get

$$\frac{C_K}{(s-r)^2} \int_{B_s \setminus B_r} |V|^2 dx \leq \frac{1}{4} \int_{B_s \setminus B_r} |DV|_2^2 dx + \frac{C_K R^{(2-\theta)/\theta}}{(s-r)^{2+(2-\theta)/\theta(N+1)}} \left(\int_{B_s \setminus B_r} |V|^\theta dx \right)^{2/\theta},$$

which inserted into (3-7) gives, for all $R \leq r < s \leq 2R$,

$$\int_{B_r} |DV|_2^2 dx \leq \frac{1}{4} \int_{B_s} |DV|_2^2 dx + C_K \int_{B_{2R}} f^2 dx + \frac{C_K R^{(2-\theta)/\theta}}{(s-r)^{2+(2-\theta)/\theta(N+1)}} \left(\int_{B_s \setminus B_r} |V|^\theta dx \right)^{2/\theta}.$$

A standard iteration lemma (see [Giaquinta 1983, Lemma 3.1, Chapter 5]) improves the latter to

$$\int_{B_R} |DV|_2^2 dx \leq C_K \int_{B_{2R}} f^2 dx + \frac{C_K}{R^{2+(2-\theta)/\theta N}} \left(\int_{B_{2R}} |V|^\theta dx \right)^{2/\theta},$$

which is the desired estimate on the derivative of V . In order to control $\|V\|_{L^2(B_R)}^2$, we invoke the rescaled form of (2-1) which, in conjunction with the previous estimate, completes the proof of (3-4). □

3B. Local minimizers. For a bounded $\Omega \subseteq \mathbb{R}^N$ we let

$$J(w, \Omega) = \int_{\Omega} F(Dw) dx + \int_{\Omega} fw dx$$

whenever the two integrands are in $L^1(\Omega)$, sometimes omitting the dependence on Ω when this causes no confusion. We will consider J under p -coercivity assumptions on F and for $f \in W^{-1,p'}(\Omega)$, so that it is well-defined on $W^{1,p}(\Omega)$.

Recall that $u \in W_{loc}^{1,p}(\Omega)$ is a local minimizer for J in $W^{1,p}(\Omega)$ if, for any $B \Subset \Omega$,

$$J(u, B) = \inf\{J(w, B) : w \in u + W_0^{1,p}(B)\}. \tag{3-8}$$

Theorem 3.3. *Let $F \in C^1(\mathbb{R}^N)$ be a q.u.c. function and $q > p > 1$ be given in Proposition 2.3(ii). For $f \in L^2(\Omega) \cap W^{-1,p'}(\Omega)$, let u be a local minimizer u of J in Ω . Then, for any ball B such that $4B \subseteq \Omega$ it holds $DF(Du) \in W^{1,2}(B)$,*

$$\|DF(Du)\|_{L^2(B)} \leq C(1 + \|f\|_{L^2(2B)} + \|F(Du)\|_{L^1(2B)}^{(q-1)/p}) \tag{3-9}$$

for $C = C(K, N, B) > 0$, and for any $\theta \in (0, 2]$

$$\|DF(Du)\|_{W^{1,2}(B)} \leq C(\|f\|_{L^2(2B)} + \|DF(Du)\|_{L^\theta(2B)}) \tag{3-10}$$

for $C = C(K, N, B, \theta) > 0$. Moreover, u satisfies the Euler–Lagrange equation

$$\int_{\Omega} (DF(Du), D\varphi) dx = \int_{\Omega} f\varphi dx \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{3-11}$$

Proof. Let $\text{Argmin}(F) = \{z_0\}$. By considering $\tilde{F}(z) = F(z + z_0) - F(z_0)$ and $\tilde{u}(x) = u(x) - (z_0, x)$ and noting that $f(\cdot)(z_0, \cdot) \in L^1_{\text{loc}}(\Omega)$, it is readily checked that \tilde{u} turns out to be a local minimizer of

$$\tilde{J}(w, \Omega) = \int_{\Omega} \tilde{F}(Dw) + fw dx.$$

Hence, hereafter we suppose $F(z) \geq F(0) = 0$. Finally, recalling Proposition 2.3(ii), we know that F is strictly convex and

$$C^{-1}|z|^p - C \leq F(z) \leq C(|z|^q + 1), \quad |DF(z)| \leq C(|z|^{q-1} + 1), \tag{3-12}$$

so that u is the unique minimizer locally, with respect to its own boundary values. We split the proof into several steps.

Step 1: approximating problems. Fix $\varphi \in C_c^\infty(B_1, [0, +\infty))$ such that $\|\varphi\|_1 = 1$ and let $\varphi_\varepsilon(x) = \varepsilon^{-N} \varphi(x/\varepsilon)$. For $B \Subset \Omega$ and $n \in \mathbb{N}$, let $f_n = f * \varphi_{1/n}$ and, for $\varepsilon_n, \mu_n \rightarrow 0^+$ to be chosen,

$$J_n(w) = \int_B F * \varphi_{\varepsilon_n}(Dw) + \frac{\mu_n}{2} |Dw|^2 + f_n w dx.$$

Set

$$\text{Lip}_\psi(B) = \{w \in \text{Lip}(\bar{B}) : w = \psi \text{ on } \partial B\}.$$

According to [Stampacchia 1963, Theorem 9.2], there is a solution $v_n \in \text{Lip}_{u*\varphi_{1/n}}(B)$ of

$$J_n(v_n) = \inf\{J_n(w) : w \in \text{Lip}_{u*\varphi_{1/n}}(B)\},$$

since $u * \varphi_{1/n}$ is smooth on ∂B (thus satisfying the bounded slope condition) and also f_n is smooth. Moreover, for any fixed n there is no Lavrentiev gap for J_n ; see [Bousquet et al. 2014, p. 5923]. Hence v_n also solves

$$J_n(v_n) = \inf\{J_n(w) : w \in u * \varphi_{1/n} + W_0^{1,p}(B)\}.$$

Step 2: determining the parameters. For any choice of ε_n, μ_n , the integrand

$$F_n(z) := F * \varphi_{\varepsilon_n}(z) + \frac{\mu_n}{2} |z|^2$$

is μ_n -uniformly convex; hence [Bousquet and Brasco 2016, Theorem 4.1] ensures the existence of constants A_n (depending only on B and $\|f_n\|_\infty$, as well as on the regularity of $u * \varphi_{1/n}$, but not on ε_n, μ_n), such that

$$\text{Lip}(v_n) \leq \frac{A_n}{\mu_n} =: L_n. \tag{3-13}$$

Without loss of generality, we can assume $L_n \geq 1$. We first choose $\mu_n \downarrow 0$ so that

$$\lim_n \mu_n \int_B |Du * \varphi_{1/n}|^2 dx = 0, \quad \lim_n \mu_n^{p-1} A_n^{2-p} = 0, \tag{3-14}$$

and observe that L_n is independent of ε_n . Then we choose ε_n : define the numbers

$$M_n = 1 + \sup_B |Du * \varphi_{1/n}| + L_n.$$

Since $\varphi_\varepsilon * F \rightarrow F$ in $C^1_{\text{loc}}(\mathbb{R}^N)$ as $\varepsilon \downarrow 0$, we can pick $(\varepsilon_n) \subseteq (0, 1)$, $\varepsilon_n \downarrow 0$, so that

$$\|F * \varphi_{\varepsilon_n} - F\|_{C^1(B_{M_n})} \leq \frac{1}{n}. \tag{3-15}$$

Clearly, it still holds

$$F_n \rightarrow F \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^N). \tag{3-16}$$

Step 3: the limsup inequality. Testing the minimality of v_n against the admissible function $u * \varphi_{1/n}$ gives

$$J_n(v_n) \leq J_n(u * \varphi_{1/n}).$$

Owing to $u * \varphi_{1/n} \rightarrow u$ in $W^{1,p}(B)$ and (3-14), one has

$$\overline{\lim}_n J_n(u * \varphi_{1/n}) = \int_B f u dx + \overline{\lim}_n \int_B F * \varphi_{\varepsilon_n}(Du * \varphi_{1/n}) dx. \tag{3-17}$$

To estimate the last integral, we use (3-15) to get

$$\int_B F * \varphi_{\varepsilon_n}(Du * \varphi_{1/n}) dx \leq \frac{|B|}{n} + \int_B F(Du * \varphi_{1/n}) dx.$$

The vector-valued Jensen inequality then leads to

$$\begin{aligned} \overline{\lim}_n \int_B F * \varphi_{\varepsilon_n}(Du * \varphi_{1/n}) dx &\leq \overline{\lim}_n \int_B F(Du * \varphi_{1/n}) dx \\ &\leq \overline{\lim}_n \int_{(1+1/n)B} \varphi_{1/n} * F(Du) dx = \int_B F(Du) dx. \end{aligned}$$

Inserting the latter into (3-17) we conclude

$$\overline{\lim}_n J_n(v_n) \leq J(u). \tag{3-18}$$

Step 4: convergence of (v_n) to u . From (3-18) we have that $(J_n(v_n))$ is bounded. By Jensen's inequality $F \leq F * \varphi_{\varepsilon_n}$ so that, through (3-12), for some constant $C = C(N, F, B) > 0$ we have

$$\begin{aligned} J_n(v_n) &\geq \int_B F(Dv_n) dx + \int_B f_n v_n dx \\ &\geq \frac{\|Dv_n\|_{L^p(B)}^p}{C} - C - \|f_n\|_{W^{-1,p'}(B)} \|D(v_n - u * \varphi_{1/n})\|_{L^p(B)} + \int_B f_n u * \varphi_{1/n} dx \\ &\geq \frac{\|Dv_n\|_{L^p(B)}^p}{C} - C - \|f_n\|_{W^{-1,p'}(B)} (\|Dv_n\|_{L^p(B)} + \|Du * \varphi_{1/n}\|_{L^p(B)}) + \int_B f_n u * \varphi_{1/n} dx. \end{aligned}$$

Since $f_n \rightarrow f$ in $W^{-1,p'}(B)$ and $u * \varphi_{1/n} \rightarrow u$ in $W^{1,p}(B)$, by (3-18) we deduce that (Dv_n) is bounded in $L^p(B)$. Moreover, by Poincaré’s inequality,

$$\begin{aligned} \|v_n\|_{L^p(B)} &\leq \|v_n - u * \varphi_{1/n}\|_{L^p(B)} + \|u * \varphi_{1/n}\|_{L^p(B)} \\ &\leq C(\|D(v_n - u * \varphi_{1/n})\|_{L^p(B)} + \|u * \varphi_{1/n}\|_{L^p(B)}) \\ &\leq C(\|Dv_n\|_{L^p(B)} + \|u * \varphi_{1/n}\|_{W^{1,p}(B)}), \end{aligned}$$

so that (v_n) is bounded in $L^p(B)$ as well. Therefore the sequence (v_n) is bounded in $W^{1,p}(B)$, and hence possesses a (not relabeled) subsequence weakly convergent to some $v \in W^{1,p}(B)$; actually, it is readily checked that $v \in u + W_0^{1,p}(B)$. We claim that

$$\lim_n \mu_n \int_B |Dv_n|^2 dx = 0.$$

Indeed, this is obvious by Hölder’s inequality when $p \geq 2$, while if $p < 2$ we use (3-13) and (3-14) to infer

$$\mu_n \int_B |Dv_n|^2 dx \leq \mu_n L_n^{2-p} \int_B |Dv_n|^p dx \leq \mu_n^{p-1} A_n^{2-p} \int_B |Dv_n|^p dx \rightarrow 0,$$

where we used the boundedness of (Dv_n) in $L^p(B)$. Thus

$$\lim_n \int_B \frac{\mu_n}{2} |Dv_n|^2 + f_n v_n dx = \int_B f v dx. \tag{3-19}$$

The functional

$$w \mapsto \int_B F(Dw) dx$$

is weakly lower semicontinuous in $W^{1,p}(B)$, whence (again by the Jensen inequality)

$$\begin{aligned} J(v) &\leq \varliminf_n \left[\int_B F(Dv_n) dx + \int_B f v_n dx \right] \\ &\leq \varliminf_n \left[\int_B F * \varphi_{\varepsilon_n}(Dv_n) dx + \frac{\mu_n}{2} \int_B |Dv_n|^2 + f_n v_n dx \right] = \varliminf_n J_n(v_n). \end{aligned} \tag{3-20}$$

Coupling the latter with (3-18) we get $J(v) \leq J(u)$, implying $v = u$ by the strict convexity of J . In particular we obtain, up to subsequences,

$$Dv_n \rightharpoonup Du \quad \text{in } L^p(B), \tag{3-21}$$

and from (3-20), (3-18) we infer $J_n(v_n) \rightarrow J(u)$. Subtracting (3-19) we get

$$\int_B F * \varphi_{\varepsilon_n}(Dv_n) dx \rightarrow \int_B F(Du) dx,$$

which, thanks to (3-15), implies

$$\int_B F(Dv_n) dx \rightarrow \int_B F(Du) dx. \tag{3-22}$$

Step 5: uniform Sobolev bound on $DF(Dv_n)$. By Proposition 2.3(iii), and the beginning of the proof of Theorem 3.2, F_n satisfies (1-9) with the same constant K ; since $F_n \in C^3(\mathbb{R}^N)$, (1-9) actually holds

everywhere. Moreover, standard regularity theory ensures that $v_n \in C^2(B)$, so we can apply [Theorem 3.2](#), and in particular [\(3-4\)](#), to obtain, for any $\theta \in (0, 2]$ and $r = \frac{1}{2}, 1$,

$$\|DF_n(Dv_n)\|_{W^{1,2}(rB/2)} \leq C (\|f_n\|_{L^2(rB)} + \|DF_n(Dv_n)\|_{L^\theta(rB)}). \tag{3-23}$$

The first term on the right is clearly bounded by a multiple of $\|f\|_{L^2(B)}$. For the second one, we let p, q be given in [\(3-12\)](#) and choose $\bar{\theta} = \min\{p/(q-1), 1\}$. By [\(3-12\)](#) we get

$$|DF(z)|^{\bar{\theta}} \leq C (|z|^{\bar{\theta}(q-1)} + 1) \leq C (|z|^p + 1)^{\bar{\theta}(q-1)/p} \leq C (F(z) + 1)^{\bar{\theta}(q-1)/p}. \tag{3-24}$$

Using [\(3-15\)](#) and [\(3-24\)](#) we obtain

$$\begin{aligned} & \int_B |DF_n(Dv_n)|^{\bar{\theta}} dx \\ & \leq \int_B |DF(Dv_n)|^{\bar{\theta}} dx + n^{-\bar{\theta}} |B| + \mu_n^{\bar{\theta}} \int_B |Dv_n|^{\bar{\theta}} dx \\ & \leq C \int_B (F(Dv_n) + 1)^{\bar{\theta}(q-1)/p} dx + n^{-\bar{\theta}} |B| + \mu_n^{\bar{\theta}} \|Dv_n\|_{L^p(B)}^{\bar{\theta}} |B|^{1-\bar{\theta}/p} \\ & \leq C |B|^{1-\bar{\theta}(q-1)/p} \left(\int_B (F(Dv_n) + 1) dx \right)^{\bar{\theta}(q-1)/p} + n^{-\bar{\theta}} |B| + \mu_n^{\bar{\theta}} \|Dv_n\|_{L^p(B)}^{\bar{\theta}} |B|^{1-\bar{\theta}/p}. \end{aligned} \tag{3-25}$$

The first integral is bounded by [\(3-22\)](#) and the remaining terms vanish when $n \rightarrow \infty$, so

$$\overline{\lim}_n \|DF_n(Dv_n)\|_{L^{\bar{\theta}}(B)} \leq C \left(\int_B (F(Du) + 1) dx \right)^{(q-1)/p}. \tag{3-26}$$

Thanks to [\(3-23\)](#) for $r = 1$, [\(3-26\)](#) implies the Sobolev bound

$$\overline{\lim}_n \|DF_n(Dv_n)\|_{W^{1,2}(B/2)} \leq C. \tag{3-27}$$

Step 6: passage to the limit. Let $B' = \frac{1}{2}B$ and

$$V_n = DF_n(Dv_n).$$

Thanks to [\(3-27\)](#), (V_n) is bounded in $W^{1,2}(B')$; hence we can pick a subsequence satisfying

$$V_n \rightarrow V \quad \text{weakly in } W^{1,2}(B'), \text{ strongly in } L^2(B'), \text{ and pointwise a.e. in } B', \tag{3-28}$$

for a suitable $V \in W^{1,2}(B')$.

Each F_n is strictly convex and superlinear by construction; hence DF_n is a homeomorphism of \mathbb{R}^N . Moreover, by [\(3-16\)](#) we know that $DF_n \rightarrow DF$ locally uniformly. According to a theorem by Arens (see [\[Dijkstra 2005\]](#) for a modern exposition), this implies that $DF_n^{-1} \rightarrow DF^{-1}$ locally uniformly. Since $V_n \rightarrow V$ pointwise a.e., we infer that

$$Dv_n = DF_n^{-1}(V_n) \rightarrow DF^{-1}(V) \quad \text{pointwise a.e.,}$$

which, coupled with [\(3-21\)](#), allows the identification $Du = DF^{-1}(V)$. Therefore, $V_n \rightarrow DF(Du)$ in B' in all the senses prescribed in [\(3-28\)](#).

By considering $4B$ instead of B , estimate (3-9) follows from (3-23) and (3-26), as long as $4B \subseteq \Omega$. Let $B'' = \frac{1}{4}B$. By Lebesgue’s dominated convergence theorem $\|V_n\|_{L^\theta(B')} \rightarrow \|V\|_{L^\theta(B')}$; hence, exploiting also the lower semicontinuity of the $W^{1,2}(B'')$ norm, we can pass to the limit in (3-23) with $r = \frac{1}{2}$. Again considering $4B$ instead of B yields (3-10). Finally, the validity of (3-11) can be checked only on balls B such that $4B \subseteq \Omega$, by a standard partition of unity argument. If B is such a ball, we can pass to the limit in the Euler Lagrange equations for the approximating problems constructed as before in $4B$, and since $DF_n(Dv_n) \rightarrow DF(Du)$, $f_n \rightarrow f$ strongly in $L^2(B)$, we get (3-11). \square

Remark 3.4. The previous theorem has an immediate consequence. The class of q.u.c. functionals is a (proper) subclass of the so-called *functionals with (p, q) -growth*, i.e., those obeying (2-5), (2-6). For example, the integrand

$$F(z) = |z_1|^p + |z_2|^q, \quad z = (z_1, z_2) \in \mathbb{R}^2, \quad p, q > 1,$$

is of (p, q) -growth but not q.u.c., even if $p = q$. Given a local minimizer of a convex functional of general (p, q) -growth, the a priori regularity on Du is just $Du \in L^p_{loc}(\Omega)$, and the first step towards higher regularity is showing that actually $Du \in L^q_{loc}(\Omega)$. For $2 \leq p \leq N$, this is to be expected only when $q < p(N + 2)/N$; see [Giaquinta 1987; Esposito et al. 1999]. For the subclass of q.u.c. integrands, from $DF(Du) \in W^{1,2}_{loc}(\Omega)$, we infer by Sobolev’s embedding that $DF(Du) \in L^{2^*}_{loc}(\Omega)$. Since $|DF(z)| \lesssim |z|^{p-1}$, it holds $Du \in L^{2^*(p-1)}_{loc}(\Omega)$; hence for such class of integrands we obtain the condition $q \leq 2^*(p - 1)$, which gives a larger range if $p \geq 2$. This range may not be optimal in the q.u.c. class, but on one hand it shows the advantages of considering the stress field instead of (1-5), while on the other hand it holds for any $f \in L^2_{loc}(\Omega) \cap W^{-1,p'}(\Omega)$. It is quite possible, in light of the results of [Beck and Mingione 2020], that, for f having a sufficiently high degree of summability, minimizers for q.u.c. integrands are automatically Lipschitz continuous, regardless of the largeness of the ratio q/p , a fact that, if true, would bypass completely the higher integrability issue for the gradient in the q.u.c. class. This is actually the case for functionals with Uhlenbeck structure; see [Cianchi and Mazya 2011].

3C. Examples.

Example 3.5 (on the assumption $F \in W^{2,1}$). In this example we show that, in order to obtain Sobolev regularity of $DF(Du)$, it is not sufficient to require that condition (1-9) holds at almost every point, but that Sobolev regularity of DF is a necessary assumption.

Let $N = 2$. For any ball $B \Subset \{(x, y) \in \mathbb{R}^2 : x > 0\}$, consider the smooth function

$$u(x, y) = \arctan(y/x).$$

We claim that, for any (not necessarily convex) C^2 function $F : \mathbb{R} \rightarrow \mathbb{R}$, u solves

$$\operatorname{div} DF(Du) = 0 \quad \text{in } B, \tag{3-29}$$

where here and in what follows we make the identification $F(z) = F(|z|)$. Letting $z = (x, y)$, $z^\perp = (-y, x)$, it holds

$$Du(z) = \frac{z^\perp}{|z|^2}, \quad D^2u(z) = \frac{1}{|z|^4} \begin{pmatrix} 2xy & y^2 - x^2 \\ y^2 - x^2 & -2xy \end{pmatrix},$$

while

$$DF(w) = F'(|w|) \frac{w}{|w|}, \quad D^2F(w) = F''(|w|) \frac{w}{|w|} \otimes \frac{w}{|w|} + \frac{F'(|w|)}{|w|} \left(I - \frac{w}{|w|} \otimes \frac{w}{|w|} \right).$$

An elementary computation then yields

$$D^2F(Du) D^2u(z) = \frac{1}{|z|^4} \begin{pmatrix} xy(F''(1/|z|)+F'(1/|z|)|z|) & F''(1/|z|)y^2-F'(1/|z|)|z|x^2 \\ F'(1/|z|)|z|y^2-F''(1/|z|x^2) & -xy(F''(1/|z|)+F'(1/|z|)|z|) \end{pmatrix},$$

which has zero trace, proving the claim.

Now, let $h : [0, 1] \rightarrow [0, 1]$ denote the Cantor function. By abuse of notation, we still denote by h its extension to the whole \mathbb{R} defined as

$$h(t) = k + h(t - k) \quad \text{if } k \leq t < k + 1, \quad k \in \mathbb{Z},$$

and we also denote by \mathcal{C} the periodic extension of the Cantor set to the whole \mathbb{R} . Consider

$$F(|w|) := \frac{|w|^2}{2} + H(|w|), \quad H(t) := \int_0^t h(\tau) d\tau,$$

which is a strictly convex C^1 function with quadratic growth. Clearly, F can be approximated in C^1 by a sequence $\{F_n\}$ of smooth radial functions, so that we can pass to the limit into the corresponding weak formulations of (3-29) to obtain that u solves

$$\operatorname{div}(DF(Du)) = 0 \quad \text{weakly in } B.$$

However,

$$DF(Du(z)) = \frac{z^\perp}{|z|^2} + h(|z|^{-1}) \frac{z^\perp}{|z|}$$

is not even absolutely continuous in B , since its distributional derivative has a Cantor part concentrated on $\{z : 1/|z| \in \mathcal{C}\}$, which has zero measure.

Since $h'(t) = 0$ in the classical sense for a.e. $t \in \mathbb{R}$, it is readily verified that F obeys (1-9) with $K = 1$ at every point of

$$\mathbb{R}^N \setminus \mathcal{C}_{\text{rad}}, \quad \mathcal{C}_{\text{rad}} = \{z \in \mathbb{R}^N : |z| \in \mathcal{C}\},$$

thus almost everywhere. Notice that DF is of bounded variation but does not belong to $W_{\text{loc}}^{1,1}(\mathbb{R}^N)$, since its derivative has a Cantor part concentrated on \mathcal{C}_{rad} .

Example 3.6 (Uhlenbeck structure). For divergence form equations having the Uhlenbeck structure

$$\operatorname{div}(a(|Du|) Du) = f \tag{3-30}$$

we recover the local regularity result of [Cianchi and Mazya 2018, Theorem 2.1], under the additional assumption $f \in W^{-1,p'}(\Omega)$ (see the second point in Remark 1.3 in this respect). Here, the exponent p is related to the function a as follows. Define F by

$$F(z) = \int_0^{|z|} t a(t) dt$$

to get $DF(z) = a(|z|)z$. If $a \in C^1(0, +\infty)$, it holds

$$D^2F(z) = a(|z|)I + |z|a'(|z|)\frac{z}{|z|} \otimes \frac{z}{|z|},$$

possessing the eigenvector $z/|z|$ with eigenvalue $a(|z|) + |z|a'(|z|)$, while its orthogonal eigenspace is relative to the unique eigenvalue $a(|z|)$. The equation is elliptic if and only if both eigenvalues are nonnegative, and in order to bound the ratio between them we look at

$$K = \sup_{t>0} \max \left\{ \frac{a(t)}{a(t) + ta'(t)}, \frac{a(t) + ta'(t)}{a(t)} \right\}.$$

It is readily checked that if

$$i_a = \inf_{t>0} \frac{t a'(t)}{a(t)}, \quad s_a = \sup_{t>0} \frac{t a'(t)}{a(t)},$$

then

$$K = \max \left\{ \frac{1}{1 + i_a}, 1 + s_a \right\},$$

so that the q.u. convexity condition (1-9) is equivalent to the common requirement

$$-1 < i_a \leq s_a < \infty, \tag{3-31}$$

which is the one used, e.g., in [Cianchi and Mazya 2018]. In our framework, the exponent p is given by Proposition 2.3 and turns out to be

$$p = 1 + \frac{1}{K} = \min \left\{ 2 + i_a, \frac{s_a + 2}{s_a + 1} \right\}.$$

In the model case $a(z) = |z|^{p-2}$, which corresponds to the p -Poisson equation, we directly have $i_a = s_a = p - 2$, so that the previous exponent is actually p . In the general case $i_a < i_s$, the exponent p can be improved, since it holds (see [Cianchi and Mazya 2011, Proposition 2.15])

$$F(z) \geq a(1) \frac{|z|^{2+i_a}}{2+i_a}, \quad |z| \geq 1,$$

so that one can take $p = 2 + i_a > 1$ by (3-31) (see the third point in Remark 1.3). Indeed, for $p = 2 + i_a$ and $f \in W^{-1,p'}(\Omega)$, J is also coercive on $W^{1,p}(\Omega)$ when supplemented with reasonable boundary conditions. The variational treatment of (3-30) in standard Sobolev spaces is thus justified if one is not looking for optimal rearrangement invariant estimates.

Example 3.7 (anisotropic examples). In [Ciraolo et al. 2020; Antonini et al. 2022; Cozzi et al. 2014; 2016] anisotropic equations whose principal part arises as the Euler–Lagrange equation of

$$\int_{\Omega} G(H(Du)) \, dx$$

are considered, where $H \in C^2(\mathbb{R}^N \setminus \{0\}, \mathbb{R}_+)$ is a convex, positively 1-homogeneous function and $G \in C^2(\mathbb{R}_+, \mathbb{R}_+)$ is an increasing, strictly convex function of p -growth. Clearly, H is fully determined

by the unit “ball”

$$B_H = \{z \in \mathbb{R}^N : H(z) < 1\} \ni 0,$$

and any open, bounded, convex B with $0 \in B$ will uniquely determine such an H through its Minkowski functional. Notice that in general H may not even be a norm, due to the possible lack of symmetry of B_H .

The more general ellipticity assumption in the cited works reads as follows: H is said to be *uniformly convex* if the principal curvatures of ∂B_H are bounded from below by a positive constant. From [Cozzi et al. 2014, Appendix A], the uniform ellipticity of H amounts to

$$(H(z) D^2 H(z) \eta, \eta) \geq \delta |\eta|^2 \quad \text{for all } z \in \mathbb{R}^N \setminus \{0\}, \eta \in DH(z)^\perp,$$

for some $\delta > 0$ (which is actually equivalent to the same type of inequality for $\eta \in z^\perp$). Under this assumption various results can be proved, and in particular the Sobolev regularity of the associated stress field is treated in [Ciraolo et al. 2020; Antonini et al. 2022] as a stepping-stone to more general results. Here we show that anisotropic functionals of this kind fall within our general framework of q.u.c. functionals.

To this end, set $F(z) = G(H(z))$ so that

$$D^2 F(z) = G''(H(z)) DH(z) \otimes DH(z) + \frac{G'(H(z))}{H(z)} H(z) D^2 H(z)$$

for $z \neq 0$, and notice that both

$$DH(z) \otimes DH(z) \quad \text{and} \quad H(z) D^2 H(z)$$

are 0-homogeneous. Inspecting the proof of [Cozzi et al. 2014, Appendix A], we see that for any $z, \xi \in \mathbb{R}^N \setminus \{0\}$ it either holds

$$(DH(z) \otimes DH(z) \xi, \xi) \geq \lambda_1 |\xi|^2$$

or

$$(H(z) D^2 H(z) \xi, \xi) \geq \lambda_2 |\xi|^2$$

(with $\lambda_i = \lambda_i(H) > 0$), while altogether

$$(H(z) D^2 H(z) \xi, \xi) \leq \Lambda |\xi|^2, \quad (DH(z) \otimes DH(z) \xi, \xi) \leq \Lambda |\xi|^2$$

for some $\Lambda = \Lambda(H)$. It follows that the minimum eigenvalue of $D^2 F(z)$ is bounded from below by

$$\min \left\{ \lambda_1 G''(H(z)), \lambda_2 \frac{G'(H(z))}{H(z)} \right\},$$

while its maximum eigenvalue is bounded from above by

$$\Lambda \left(G''(H(z)) + \frac{G'(H(z))}{H(z)} \right).$$

Therefore F is K -q.u.c. for

$$K = \frac{\Lambda}{\min\{\lambda_1, \lambda_2\}} \sup_{t \in \mathbb{R}_+} \max \left\{ 1 + \frac{G'(t)}{G''(t)t}, 1 + \frac{G''(t)t}{G'(t)} \right\}.$$

The finiteness of the latter is compatible with the standard Uhlenbeck example: for $a(t) = G'(t)/t$, a straightforward calculation shows that

$$\sup_{t \in \mathbb{R}_+} \max \left\{ \frac{G'(t)}{G''(t)t}, \frac{G''(t)t}{G'(t)} \right\} < +\infty \iff -1 < i_a \leq s_a < +\infty.$$

Example 3.8 (nonstandard anisotropic growth). It is straightforward to check that if F_1 and F_2 are q.u.c. C^1 functions satisfying (1-9) with constants K_1 and K_2 respectively, then $F_1 + F_2$ is $\max\{K_1, K_2\}$ -q.u.c. This observation allows to consider anisotropic Euler–Lagrange equations arising from integrands of the form

$$F(z) = \sum_{i=1}^M |A_i(z - z_i)|^{p_i}, \quad z_i \in \mathbb{R}^N, \quad A_i \geq I, \quad p_i > 1 \quad \text{for } i = 1, \dots, M.$$

As a more common example of anisotropic functionals, consider the weak solution of the Dirichlet problem

$$\begin{cases} \Delta_p u + D_1(|D_1 u|^{q-2} D_1 u) = f & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases}$$

for a ball B , $f \in L^\infty(B)$ and $q > p > 2$. This corresponds to the unique minimizer $u \in W_0^{1,p}(B)$ of

$$J(w) = \int_B \frac{1}{p} |Dw|^p + \frac{1}{q} |D_1 w|^q + f w \, dx,$$

which is globally Lipschitz continuous, since

$$F(z) = \frac{1}{p} |z|^p + \frac{1}{q} |z_1|^q$$

is uniformly p -convex outside B_1 (see [Bousquet and Brasco 2016]), i.e.,

$$(D^2 F(z) \xi, \xi) \geq c_p (1 + |z|^2)^{(p-2)/2} |\xi|^2, \quad |z| \geq 1.$$

Moreover,

$$D^2 F(z) = |z|^{p-2} \left(I + (p-2) \frac{z}{|z|} \otimes \frac{z}{|z|} \right) + (q-1) |z_1|^{q-2} e_1 \otimes e_1,$$

so that the minimum and maximum eigenvalues of $D^2 F(z)$ satisfy

$$\lambda_{\min}(z) \geq |z|^{p-2}, \quad \lambda_{\max}(z) \leq (p-1) |z|^{p-2} + (q-1) |z|^{q-2}.$$

Notice that the integrand F is *not* q.u.c. globally on \mathbb{R}^N , but on the range of Du we have $|z|^{q-2} \leq C |z|^{p-2}$ for C depending on $\text{Lip } u$, leading to

$$\lambda_{\max}(z)/\lambda_{\min}(z) \leq p-1 + C(q-1).$$

Hence Theorem 3.3 applies thanks to the local nature of (1-9) described in Lemma 2.4. Notice the role of the assumption $f \in L^\infty(B)$ (which could be weakened, but not down to $L^2(B)$, see [Beck and Mingione 2020]) and of the smooth boundary condition $u \in W_0^{1,p}(B)$: they provide the Lipschitz regularity of u , which in turn allows to employ Lemma 2.4 and to consider u as a minimizer of a functional with q.u.c. integrand on the whole \mathbb{R}^N .

4. Applications

4A. Cordes-type conditions. We start with a generalization of (2-2). In this section, given a matrix field $M : \Omega \rightarrow \mathbb{R}^N \otimes \mathbb{R}^N$, its L^m -norm will be computed with respect to the Frobenius norm of $M = (m_{ij})$, i.e.,

$$\|M\|_m = \left(\int |M(x)|_2^m dx \right)^{1/m}, \quad |M(x)|_2^2 = \sum_{i,j=1}^N |m_{ij}(x)|^2.$$

The proofs of this section make use of some tools from harmonic analysis; for an introductory exposition on this topic, the reader can consult [Duoandikoetxea 2001].

Lemma 4.1. *Let $V \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$ and, for $m > 1$, set $\hat{m} = \max\{m, m/(m - 1)\} \geq 2$. Then*

$$\|DV\|_m \leq N^2(\hat{m} - 1)(\|\operatorname{div} V\|_m + \|\operatorname{curl} V\|_m). \tag{4-1}$$

Proof. Let R_j be the Riesz transform, defined as the Fourier multiplier with symbol $-i \xi_j/|\xi|$; see [Duoandikoetxea 2001, p. 76]. Then it holds

$$D_h V_k = -R_h R_k \operatorname{div} V - \sum_{j=1}^N R_h R_j \operatorname{curl}_{kj} V, \tag{4-2}$$

which follows, taking the Fourier transform (denoted by $g \mapsto \hat{g}$), from the identity

$$i \xi_h \hat{V}_k = \frac{\xi_h \xi_k}{|\xi|^2} \sum_{j=1}^N i \xi_j \hat{V}_j + \sum_{j=1}^N \frac{\xi_h \xi_j}{|\xi|^2} (i \xi_j \hat{V}_k - i \xi_k \hat{V}_j).$$

The second-order Riesz transform has $L^m - L^m$ norm $\hat{m} - 1$ on diagonal terms and $(\hat{m} - 1)/2$ on off-diagonal terms, i.e.,

$$\begin{aligned} \|R_h R_k g\|_m &\leq \frac{\hat{m} - 1}{2} \|g\|_m, \quad h \neq k, \\ \|R_h^2 g\|_m &\leq (\hat{m} - 1) \|g\|_m; \end{aligned}$$

see [Bañuelos and Méndez-Hernández 2003, Theorem 2.4]. Thus from (4-2) we get

$$\|D_h V_k\|_m \leq (\hat{m} - 1) \left(\|\operatorname{div} V\|_m + \sum_{j=1}^N \|\operatorname{curl}_{kj} V\|_m \right).$$

We sum over $k = 1, \dots, N$ and use the Hölder inequality to get

$$\sum_{k,j=1}^N \|\operatorname{curl}_{kj} V\|_m \leq N^{2(1-1/m)} \left(\int \sum_{k,j=1}^N |\operatorname{curl}_{kj} V|^m \right)^{1/m}.$$

Since

$$\sum_{k,j=1}^N |\operatorname{curl}_{kj} V|^m \leq N^{2-m} \left(\sum_{k,j=1}^N |\operatorname{curl}_{kj} V|^2 \right)^{m/2},$$

we obtain

$$\sum_{k,j=1}^N \|\operatorname{curl}_{kj} V\|_m \leq N \|\operatorname{curl} V\|_m.$$

Summing also over $h = 1, \dots, N$ we finally get

$$\|DV\|_m \leq \sum_{h,k=1}^N \|D_h V_k\|_m \leq (\hat{m} - 1) N^2 (\|\operatorname{div} V\|_m + \|\operatorname{curl} V\|_m). \quad \square$$

Remark 4.2. Estimate (4-1) is very rough in its dependence on N . It is a common feature of L^m -bounds on Riesz transform that they do not depend on the dimension of the euclidean space. Up to our knowledge, optimal L^m estimates for the operator $D \operatorname{curl}^{-1}$ (let alone for the resolvent operator of the div-curl system) are not known. It is also not optimal as $m \rightarrow 2$; compare it with (2-2) in the case $m = 2$.

Theorem 4.3. *Let F obey the assumptions of Theorem 3.3, in particular (1-9) with a constant $K \geq 1$ and p, q given in Proposition 2.3. Let furthermore $m > 1$ and $f \in L^m(\Omega) \cap W^{-1,p'}(\Omega)$. Then any local minimizer $u \in W_{\text{loc}}^{1,p}(\Omega)$ for J given in (3-8) is such that $DF(Du) \in W_{\text{loc}}^{1,m}(\Omega)$ and satisfies the estimates*

$$\|DF(Du)\|_{L^m(B)} \leq C(1 + \|f\|_{L^m(2B)} + \|F(Du)\|_{L^1(2B)}^{(q-1)/p}), \quad (4-3)$$

$$\|DF(Du)\|_{W^{1,m}(B)} \leq C(\|f\|_{L^m(2B)} + \|DF(Du)\|_{L^m(2B)}) \quad (4-4)$$

in each of the following cases:

- (1) $K \leq K_0$, with $K_0 > 1$ depending on N and m , with $C = C(N, m, B)$.
- (2) $|m - 2| \leq \delta_0$ for $\delta_0 > 0$ depending on N and K , with $C = C(N, K, B)$.

Proof. Given B such that $4B \Subset \Omega$, we follow the first four steps of the proof of Theorem 3.3 to find $u_n \in C^\infty(B)$, $f_n \in C^\infty(B)$, $F_n \in C^\infty(\mathbb{R}^N)$ and $v_n \in C^2(B)$ such that

- (i) $u_n \rightarrow u$ in $W^{1,p}(B)$, $f_n \rightarrow f$ in $L^m(B) \cap W^{-1,p'}(B)$,
- (ii) F_n is K -q.u.c.,
- (iii) $v_n \rightarrow u$ in $W^{1,p}(B)$, $v_n \rightarrow u$ in $L^p(B)$, and v_n solves

$$\operatorname{div}(DF_n(Dv_n)) = f_n.$$

- (iv) $\|F_n(Dv_n)\|_{L^1(B)} \rightarrow \|F(Du)\|_{L^1(B)}$.

We then proceed as in Theorem 3.2, in order to find a uniform bound for $DF_n(Dv_n)$ in $W^{1,m}(B')$, $B' = \frac{1}{2}B$. To simplify the notation, we omit for the moment the dependence of v, F , and f on n .

Let $V = DF(Dv)$ and observe that the assumptions of Lemma 3.1 hold true for the matrix $X = DV = D^2F(Dv) D^2v$; hence, pointwise,

$$|\operatorname{curl} V|_2 \leq \sqrt{2} e(K) |DV|_2, \quad e(K) = 1 - \frac{1}{K}. \quad (4-5)$$

Suppose $B = B_{2R}$ and, for any $R \leq r < s \leq 2R$, fix $\varphi \in C_c^\infty(B_s; [0, 1])$ as in (3-6).

We split the proof in two cases, according to the situations under consideration in the two statements. For the first assertion, we apply (4-1) to the field $W := \varphi V$ to get

$$\begin{aligned} \|\varphi DV\|_m &\leq \|DW\|_m + \|D\varphi \otimes V\|_m \\ &\leq N^2 (\hat{m} - 1) (\|\operatorname{div} W\|_m + \|\operatorname{curl} W\|_m) + \frac{C}{s-r} \|V\|_{L^m(B_s \setminus B_r)} \end{aligned}$$

for any $m > 1$. From

$$\operatorname{div} W = \varphi f + (V, D\varphi), \quad \operatorname{curl} W = \varphi \operatorname{curl} V + V \wedge D\varphi$$

we thus infer

$$\|\varphi DV\|_m \leq N^2(\hat{m} - 1)(\|\varphi \operatorname{curl} V\|_m + \|\varphi f\|_m) + \frac{C}{s-r} \|V\|_{L^m(B_s \setminus B_r)}. \tag{4-6}$$

We then let $K_0 = K_0(N, m) > 1$ satisfying

$$\sqrt{2}N^2(\hat{m} - 1) \epsilon(K_0) < 1 \tag{4-7}$$

so that, if $K \leq K_0$, the curl term in (4-6) can be reabsorbed on the left. Thanks to the properties (3-6) of φ we deduce

$$\|DV\|_{L^m(B_r)} \leq C \|f\|_{L^m(B_R)} + \frac{C}{s-r} \|V\|_{L^m(B_s \setminus B_r)}, \quad R \leq r < s \leq 2R, \tag{4-8}$$

for a constant $C = C(N, m)$.

Regarding the second assertion, we consider the linear operator $T(f, G) = DV$, where V solves

$$\begin{cases} \operatorname{div} V = f, \\ \operatorname{curl} V = \sqrt{2} G, \end{cases} \quad f \in C_c^\infty(\mathbb{R}^N), \quad G \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N \wedge \mathbb{R}^N),$$

which, as already noted, is represented in terms of Riesz transforms as

$$T(f, G)_{kh} = -R_h R_k f - \sqrt{2} \sum_{j=1}^N R_h R_j G_{kj}.$$

Estimate (4-1) implies that T has an extension $T : X_m \rightarrow Y_m$, where

$$X_m := L^m(\mathbb{R}^N) \times L^m(\mathbb{R}^N; \mathbb{R}^N \wedge \mathbb{R}^N), \quad Y_m := L^m(\mathbb{R}^N; \mathbb{R}^N \otimes \mathbb{R}^N),$$

with the norms

$$\|(f, G)\|_m = \left(\int |f|^m + |G|_2^m dx \right)^{1/m}, \quad \|M\|_m = \left(\int |M|_2^m dx \right)^{1/m}$$

on X_m and Y_m , respectively. For the complex interpolation spaces it holds

$$[X_{m_1}, X_{m_2}]_\theta = X_m, \quad \frac{1}{m} = \frac{1-\theta}{m_1} + \frac{\theta}{m_2}, \quad \theta \in [0, 1],$$

with equality of norms, and the same holds for the Y_m . On the other hand, Lemma 2.2 ensures that, with respect to these norms,

$$\|T\|_{\mathcal{L}(X_2, Y_2)} = 1.$$

Fix $\bar{m}' < 2 < \bar{m}$. The Riesz–Thorin interpolation theorem [Duoandikoetxea 2001, Theorem 1.19] yields

$$\|T\|_{\mathcal{L}(X_m, Y_m)} \leq \|T\|_{\mathcal{L}(X_{\bar{m}}, Y_{\bar{m}})}^\theta, \quad \frac{1}{m} = \frac{1-\theta}{2} + \frac{\theta}{\bar{m}},$$

for any $2 \leq m \leq \bar{m}$, and a similar estimate holds also for $\bar{m}' \leq m \leq 2$. We infer that there exist $\eta : [\bar{m}', \bar{m}] \rightarrow [0, +\infty)$ such that

$$\lim_{m \rightarrow 2} \eta(m) = 0 \tag{4-9}$$

and for all $m \in [\bar{m}', \bar{m}]$ it holds

$$\|T\|_{\mathcal{L}(X_m, Y_m)} \leq 1 + \eta(m).$$

To complete the choice of δ_0 in the second case, we proceed as in the proof of (4-6), setting $W = \varphi V$ and using the previous operator norm estimate, to get, for $\eta = \eta(m)$,

$$\begin{aligned} \|\varphi DV\|_m &\leq \|DW\|_m + \|V \otimes D\varphi\|_m \\ &\leq \|T(\operatorname{div} W, 2^{-1/2} \operatorname{curl} W)\|_m + \|V \otimes D\varphi\|_m \\ &\leq (1 + \eta) \|(\operatorname{div} W, 2^{-1/2} \operatorname{curl} W)\|_m + \|V \otimes D\varphi\|_m \\ &\leq (1 + \eta) (\|\varphi f, \varphi 2^{-1/2} \operatorname{curl} V\|_m + \|((V, D\varphi), V \wedge D\varphi)\|_m) + \|V \otimes D\varphi\|_m \\ &\leq \frac{1+\eta}{\sqrt{2}} \|\varphi \operatorname{curl} V\|_m + C \|\varphi f\|_m + \frac{C}{s-r} \|V\|_{L^m(B_s \setminus B_r)} \\ &\stackrel{(4-5)}{\leq} (1 + \eta) e(K) \|\varphi DV\|_m + C \|f\|_{L^m(B_R)} + \frac{C}{s-r} \|V\|_{L^m(B_s \setminus B_r)}. \end{aligned}$$

Since $e(K) < 1$, thanks to (4-9) we can choose $\delta_0 = \delta_0(K, N)$ in such a way that

$$(1 + \eta(m)) e(K) < 1 \quad \text{for all } m \in [2 - \delta_0, 2 + \delta_0],$$

which again gives estimate (4-8) with a constant $C = C(N, K)$.

Proceeding as in the proof of Theorem 3.2, we see that estimate (4-8) improves to

$$\|V\|_{W^{1,m}(B_R)} \leq C \|f\|_{L^m(B_{2R})} + C_{R,\theta} \|V\|_{L^\theta(B_{2R})}, \quad \theta \in (0, m], \tag{4-10}$$

which therefore holds uniformly for all $V_n = DF_n(Dv_n)$ constructed at the beginning. If p, q are given in Proposition 2.3, point (ii), we set $\bar{\theta} = \min\{p/(q - 1), m\}$ and proceed as in (3-25) to get a uniform bound on $\|V_n\|_{L^{\bar{\theta}}(B_{2R})}$. Thanks to (4-10), the latter in turn implies the compactness of the V_n in $L^m(B_R)$. The rest of the proof of Theorem 3.3 follows verbatim, providing estimates (4-3) and (4-4) in both the stated cases. We omit the details. \square

Remark 4.4. Closely inspecting the previous proof yields the following asymptotic estimates. The constant K_0 goes to 1 as $m \rightarrow \infty$ or $m \rightarrow 1$. Similarly, (4-3), (4-4) hold true for $m \in (a(K), b(K))$ with $a(K) \rightarrow 1$ and $b(K) \rightarrow +\infty$ as $K \rightarrow 1$.

We made no attempt to obtain optimal estimates for K_0 and δ_0 as $m \rightarrow 2$, mainly due to the roughness of estimate (4-1) outlined in Remark 4.2. Thus, we do not recover Theorem 3.3 by simply letting $m \rightarrow 2$ in the previous statement.

4B. On the $C^{p'}$ conjecture. An immediate corollary of the Cordes condition proved in the previous section is the following one.

Corollary 4.5. Any weak solution $u \in W_{\text{loc}}^{1,p}(\Omega)$ of $\Delta_p u = f \in L_{\text{loc}}^\infty(\Omega)$ belongs to $C^{2-\alpha}(\Omega)$, where $\alpha = \alpha(N, p) \leq C(N) |p - 2|$, provided $|p - 2| < 1/(2N^3)$.

Proof. Recall that $z \mapsto |z|^p$ is K -q.u.c. with constant

$$K_p = \max\{p - 1, 1/(p - 1)\} = \begin{cases} p - 1 & \text{if } p \geq 2, \\ 1/(p - 1) & \text{if } p \in (1, 2). \end{cases}$$

We apply the Cordes estimates of the previous theorem, so that $|Du|^{p-2} Du \in W_{loc}^{1,m}(\Omega)$ for any $m \geq 2$ such that (see (4-7))

$$\sqrt{2}N^2(m-1)(1-1/K_p) < 1. \tag{4-11}$$

Given p , we let

$$m_p = \frac{1}{2N^2|p-2|}.$$

It is readily checked that, for all p such that $|p-2| < 1/(2N^3)$, inequality (4-11) holds for m_p and moreover $m_p > N$. By the Morrey embedding we thus have that $|Du|^{p-2} Du \in C^{1-N/m_p}(B)$.

The map $\Psi : \mathbb{R}^N \rightarrow \mathbb{R}^N$, defined as

$$\Psi(y) = \begin{cases} |y|^{(2-p)/(p-1)} y & \text{if } y \neq 0, \\ 0 & \text{if } y = 0 \end{cases} \tag{4-12}$$

is the inverse of the similarly defined map $z \mapsto |z|^{p-2}z$, for which the well-known inequalities (see the last chapter of [Lindqvist 2019])

$$(|z_1|^{p-2}z_1 - |z_2|^{p-2}z_2, z_1 - z_2) \geq \begin{cases} 2^{2-p} |z_1 - z_2|^p & \text{if } p \geq 2, \\ (p-1)|z_1 - z_2|^2(1 + |z_1|^2 + |z_2|^2)^{(p-2)/2} & \text{if } 1 < p < 2 \end{cases}$$

hold true. By the Schwartz inequality we deduce

$$||z_1|^{p-2}z_1 - |z_2|^{p-2}z_2| \geq \begin{cases} 2^{2-p} |z_1 - z_2|^{p-1} & \text{if } p \geq 2, \\ c_M |z_1 - z_2| & \text{if } 1 < p < 2 \text{ and } |z_1| + |z_2| \leq M, \end{cases}$$

which means that Ψ is globally $1/(p-1)$ -Hölder continuous if $p \geq 2$ and locally Lipschitz continuous if $1 < p < 2$. In the second case, we observe that $f \in L_{loc}^\infty(\Omega)$ implies that $Du \in L_{loc}^\infty(\Omega)$; hence Ψ is Lipschitz continuous on the range of Du . In both cases we thus have

$$Du \in C^{\alpha_p}(\Omega), \quad \alpha_p = \begin{cases} \frac{1}{p-1} \left(1 - \frac{N}{m_p}\right) = \frac{1-2N^3(p-2)}{p-1} & \text{if } p \geq 2, \\ 1-2N^3(2-p) & \text{if } 1 < p < 2, \end{cases}$$

giving the claim. □

Remark 4.6. A similar conclusion can be drawn for $W_{loc}^{1,p}(\Omega)$ local minimizers of $J(\cdot, \Omega)$ when F is K -q.u.c. and $f \in L_{loc}^\infty(\Omega)$. Indeed, DF is K^{N-1} quasiconformal; hence so is DF^{-1} . In particular, DF^{-1} is $1/K$ -Hölder continuous and the α -Hölder regularity of $DF(Du)$ translates to α/K -Hölder regularity for Du . The dependence of the Hölder exponent of Du from K turns out to be $1 - c_N(K-1)$ for K sufficiently near 1.

Consider now a solution u of the inhomogeneous elliptic equation with Uhlenbeck structure

$$\operatorname{div}(a(|Du|) Du) = f \in L_{loc}^m(\Omega), \quad m > 1, \tag{4-13}$$

where $a : (0, +\infty) \rightarrow (0, +\infty)$ is $C^1(0, +\infty)$ and satisfies the ellipticity condition

$$-1 < i_a \leq s_a < +\infty.$$

Solutions of (4-13) with f merely in $L^2_{\text{loc}}(\Omega)$ are to be meant in a generalized sense and, as in [Cianchi and Mazya 2018], we will use the notion of *approximable solutions* (or SOLA, solution obtained as limit of approximation) for (4-13): u is an approximable solution for (4-13) in Ω if $a(|Du|) Du \in L^1_{\text{loc}}(\Omega)$, (4-13) holds in the distributional sense in Ω , and there exists a sequence $(f_n) \subseteq C^\infty_c(\Omega)$ and corresponding weak solutions u_n of (4-13) in Ω with right-hand side f_n such that

$$f_n \rightarrow f \quad \text{in } L^m_{\text{loc}}(\Omega), \quad u_n \rightarrow u \quad \text{and} \quad Du_n \rightarrow Du \quad \text{a.e. in } \Omega,$$

and

$$\lim_n \int_{\Omega'} a(|Du_n|) |Du_n| dx = \int_{\Omega'} a(|Du|) |Du| dx$$

for all $\Omega' \Subset \Omega$. Notice that u may fail to belong to $W^{1,1}_{\text{loc}}(\Omega)$, but rather falls into the larger space

$$\mathcal{T}^{1,1}_{\text{loc}}(\Omega) = \{v : T_k v \in W^{1,1}_{\text{loc}}(\Omega) \text{ for all } k > 0\}, \quad T_k v = \max\{-k, \min\{v, k\}\},$$

for which a pointwise notion of Du is well-defined almost everywhere. Whenever $f \in W^{-1,p'}(\Omega)$, with $p = 2 + i_a$ (see the discussion at the end of Example 3.6), weak and SOLA solutions coincide, and in particular u belongs to the relevant Orlicz–Sobolev space. For the existence and uniqueness theory of SOLA we refer to [Cianchi and Mazya 2017].

We say that $u \in \mathcal{T}^{1,1}_{\text{loc}}(\Omega)$ is a cylindrical solution of (4-13) if there exists a point $x_0 \in \mathbb{R}^N$ and a k -dimensional vector subspace $V \subseteq \mathbb{R}^N$ with corresponding orthogonal projection $\pi_V : \mathbb{R}^N \rightarrow V$ such that

$$u(x) = v(|\pi_V(x - x_0)|)$$

for some $v : I \rightarrow \mathbb{R}$ with $I \subseteq [0, +\infty)$ open and $\Omega \subseteq \{x \in V : |x| \in I\} \times V^\perp$. In other terms, a cylindrical function *only* depends on the distance from some vector subspace.

In order to study the regularity properties of a cylindrical solution of (4-13), we first perform some straightforward reductions. It is clear that we can assume

$$V = \{x \in \mathbb{R}^N : x_i \equiv 0 \text{ for all } i = k + 1, \dots, N\}, \quad \Omega = A \times \mathbb{R}^{N-k},$$

with $A \subseteq \mathbb{R}^k$ invariant by the action of the orthogonal group O_k on \mathbb{R}^k . Actually, by the structure of (4-13), we can directly suppose that $k = N$, reducing to the case of radial solutions on a radial (meaning, invariant by O_N) domain Ω .

Theorem 4.7. *Let $a \in C^1((0, +\infty); (0, +\infty))$ satisfy (3-31) and $f \in L^m(\Omega)$ for some $m > 1$, where Ω is a radial domain. If u is a radial approximable solution of*

$$\text{div}(a(|Du|) Du) = f$$

in Ω then, for any $m > 1$ and B_R such that $B_{2R} \Subset \Omega$, it holds

$$\|a(|Du|) Du\|_{W^{1,m}(B_{R/2})} \leq C_{m,R} (\|f\|_{L^m(B_{2R})} + \|a(|Du|) Du\|_{L^1(B_{2R})}). \tag{4-14}$$

Proof. The field $V = a(|Du|) Du \in L^1_{\text{loc}}(\Omega)$ is the pointwise limit of the fields $V_k = a(|DT_k u|) DT_k u$ for $k \rightarrow +\infty$, which satisfy

$$T \circ V_k = V_k \circ T \quad \text{for all } T \in O_N, \quad |(V_k(x), x)| = |V(x)| |x|; \tag{4-15}$$

hence it satisfies these as well. We extend V and f as zero outside Ω (thus keeping the previous properties for V) and let $V_\varepsilon = V * \varphi_\varepsilon$, $f_\varepsilon = f * \varphi_\varepsilon$, where φ_ε is a standard radial convolution kernel supported in B_ε . We claim that V_ε obeys (4-15). By changing variables and using the radially of φ_ε , it is readily checked that V_ε satisfies $T \circ V_\varepsilon = V_\varepsilon \circ T$ for all $T \in O_N$. Thus, in order to check the second condition in (4-15) it suffices to prove it at a point $x = r e_1$, where

$$V_\varepsilon(r e_1) = \int V(y) \varphi_\varepsilon(|r e_1 - y|) dy.$$

The integrand above is odd with respect to the reflections $y_k \mapsto -y_k$, $k = 2, \dots, N$, which implies that $V_\varepsilon(r e_1)$ is parallel to e_1 , and this concludes the proof of (4-15) for V_ε . It follows that for

$$h_\varepsilon(x) := (V_\varepsilon(x), x/|x|^2) \in C^\infty(\mathbb{R}^N \setminus \{0\})$$

it holds, with a slight abuse of notation, $h_\varepsilon(x) = h_\varepsilon(|x|)$ and

$$V_\varepsilon(x) = h_\varepsilon(|x|) x. \tag{4-16}$$

Given a radial subdomain $\Omega' \Subset \Omega$ and using Fubini's theorem, we have

$$\begin{aligned} \int_{\Omega'} (V_\varepsilon, D\psi) dx &= \int_{\Omega'} (V, \varphi_\varepsilon * D\psi) dx = \int_{\Omega'} (V, D(\psi * \varphi_\varepsilon)) dx \\ &= - \int_{\Omega'} f \psi * \varphi_\varepsilon dx = - \int_{\Omega'} f_\varepsilon \psi dx; \end{aligned}$$

thus V_ε satisfies $\operatorname{div} V_\varepsilon = f_\varepsilon$ weakly (and thus strongly) in Ω' for all sufficiently small $\varepsilon > 0$. From (4-16) we compute, for $x \neq 0$,

$$DV_\varepsilon(x) = h_\varepsilon(|x|) I + |x| h'_\varepsilon(|x|) \frac{x}{|x|} \otimes \frac{x}{|x|},$$

which is a symmetric matrix, so that

$$\operatorname{curl} V_\varepsilon = 0 \quad \text{in } \Omega'.$$

By the Poincaré lemma, for any $B_{2R} \subseteq \Omega'$ and sufficiently small $\varepsilon > 0$, we thus have $V_\varepsilon = Dv_\varepsilon$ for some $v_\varepsilon \in C^2(B_{2R})$, satisfying weakly $\Delta v_\varepsilon = f_\varepsilon$. For $R < r < s < 2R$ we choose cut-off functions as in (3-6) and suppose, without loss of generality, that v_ε has zero mean in B_s . By the standard Calderón–Zygmund estimates (see [Duoandikoetxea 2001, Theorem 5.1]) and Poincaré's inequality, it holds

$$\begin{aligned} \|D^2 v_\varepsilon\|_{L^m(B_r)}^m &\leq C_m \left(\|f_\varepsilon\|_{L^m(B_{2R})}^m + \frac{1}{(s-r)^m} \|Dv_\varepsilon\|_{L^m(B_s)}^m + \frac{1}{(s-r)^{2m}} \|v_\varepsilon\|_{L^m(B_s)}^m \right) \\ &\leq C_m \|f_\varepsilon\|_{L^m(B_{2R})}^m + C_m \left(\frac{1}{(s-r)^m} + \frac{R^m}{(s-r)^{2m}} \right) \|Dv_\varepsilon\|_{L^m(B_s)}^m. \end{aligned}$$

Hence, we can proceed as in the final part of the proof of Theorem 3.2 to improve the latter to

$$\|V_\varepsilon\|_{W^{1,m}(B_{R/2})} \leq C_m \|f_\varepsilon\|_{L^m(B_{2R})} + C_{m,R} \|V_\varepsilon\|_{L^1(B_{2R})}.$$

Since $V_\varepsilon \rightarrow V$ in $L^1(B_R)$, we obtain the claimed estimate (4-14) by lower semicontinuity. □

Corollary 4.8. *Let $u \in W^{1,p}(B_R)$ be a cylindrical weak solution of*

$$\Delta_p u = f \in L^\infty(B_R).$$

Then

- (1) *If $p \geq 2$, $u \in C_{\text{loc}}^{p'-\varepsilon}(B_R)$ for all $\varepsilon > 0$, and also for $\varepsilon = 0$ if furthermore $f \in C_{\text{Dini}}^0(B_R)$.*
- (2) *If $p \leq 2$, $u \in C_{\text{loc}}^{2-\varepsilon}(B_R)$ for all $\varepsilon > 0$, and also for $\varepsilon = 0$ if furthermore $f \in C_{\text{Dini}}^0(B_R)$.*

Proof. According to [Theorem 4.7](#), the field $V = |Du|^{p-2} Du$ belongs to $W_{\text{loc}}^{1,m}(B_R)$ for any $m > 1$; hence, by the Morrey embedding, it lies in $C^{1-\varepsilon}(B_R)$ for any $\varepsilon > 0$. Similarly, $V \in \text{Lip}_{\text{loc}}(B_R)$ if f is Dini continuous, being the gradient of a solution of $\Delta v = f \in C_{\text{Dini}}^0(B_R)$. We conclude through the properties of the map Ψ in [\(4-12\)](#), as in the proof of [Corollary 4.5](#). \square

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