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STRONG ILL-POSEDNESS FOR SQG IN CRITICAL SOBOLEV SPACES

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We prove that the inviscid surface quasigeostrophic (SQG) equations are strongly ill-posed in critical Sobolev spaces: there exists an initial data $H^2(\mathbb{T}^2)$ without any solutions in $L^\infty_t H^2$. Moreover, we prove strong critical norm inflation for C^∞-smooth data. Our proof is robust and extends to give similar ill-posedness results for the family of modified SQG equations which interpolate the SQG with the two-dimensional incompressible Euler equations.

1. Introduction

1A. ***Main results.*** We are concerned with the Cauchy problem for the inviscid surface quasigeostrophic (SQG) equations on $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$,

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^\perp (-\Delta)^{-1/2} \theta. \end{cases} \tag{SQG}$$

Our first main result shows that *strong norm inflation* occurs for the solution map of (SQG) in $H^2(\mathbb{T}^2)$ with C^∞-smooth solutions.

Theorem A (strong norm inflation). *For any $\epsilon, \delta, A > 0$, there exists $\theta_0 \in C^\infty(\mathbb{T}^2)$ satisfying*

$$\|\theta_0\|_{H^2 \cap W^{1,\infty}} < \epsilon$$

such that the unique local-in-time smooth solution θ to (SQG) *with initial data θ_0 exists on $[0, \delta^*]$ for some $0 < \delta^* \le \delta$ and satisfies*

$$\sup_{t \in [0,\delta^*]} \|\theta(t, \cdot)\|_{H^2} > A.$$

The above result implies that the solution operator defined from $H^2 \cap C^\infty$ to H^2 by $\theta_0 \mapsto \theta(t)$ for any $t > 0$ cannot be continuous at the trivial solution. On the other hand, the following result shows that actually it is impossible to define the solution operator from H^2 to $L^\infty_t H^2$.

Theorem B (nonexistence). *For any $\epsilon > 0$, there exists $\theta_0 \in H^2 \cap W^{1,\infty}(\mathbb{T}^2)$ satisfying*

$$\|\theta_0\|_{H^2 \cap W^{1,\infty}} < \epsilon$$

such that there is ***no*** *solution to* (SQG) *with initial data θ_0 belonging to $L^\infty([0, \delta]; H^2(\mathbb{T}^2))$ with any $\delta > 0$.*

Remark 1.1. We give a few remarks relevant to the statements above.

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• With a rather straightforward modification of our proof, the space H^2 in Theorems A and B can be replaced with $W^{1+2/p,p}$, with *any* $p > 1$. Later, we shall sketch the proof in the endpoint case $p = \infty$. Moreover, the domain $\mathbb{T}^2$ can be replaced with $\mathbb{R}^2$ or bounded domains having symmetry axes.

• The initial data for which nonexistence occur can be given explicitly; see (4-4) below.

• The arguments we present can be adapted to prove ill-posedness for the case of *modified* (and *logarithmically regularized*) SQG equations; see Section 1C below.

1B. ***Well-posedness theory for SQG.*** To put the above ill-posedness results into context, let us briefly recall the well-posedness theory for the SQG equation. Depending on the regularity of the solutions considered, one has the following categories:

• Strong solutions: local existence and uniqueness. Using the Kato–Ponce commutator estimate [1988], one obtains the a priori estimate

$$\frac{d}{dt}\|\theta\|_{H^s} \le C\|\nabla u\|_{L^\infty}\|\theta\|_{H^s}$$

for a solution of (SQG), which allows one to close $\|\theta(t)\|_{H^s} \lesssim \|\theta_0\|_{H^s}$ for $t \lesssim \|\theta_0\|_{H^s}^{-1}$ once $s > 2$, using that $\|\nabla u\|_{L^\infty} \lesssim \|\theta\|_{H^s}$. Similarly, H^s can be replaced with $W^{s,p}$, as long as $s > 1 + 2/p$. Based on this a priori estimate, one can prove local existence and uniqueness of a strong solution in the class $L^\infty_t W^{s,p}$ with $s > 1 + 2/p$. On the other hand, note that the borderline inequality $\|\nabla u\|_{L^\infty} \lesssim \|\theta\|_{H^2}$ fails; this makes the Sobolev space H^2 (and similarly $W^{1+2/p,p}$) *critical* for local well-posedness. This space is also *scaling-critical*: the critical norm is left-invariant under the transformation

$$\theta(t,x) \mapsto \lambda^{-1}\theta(t,\lambda x), \quad u(t,x) \mapsto \lambda^{-1}u(t,\lambda x). \tag{1-1}$$

While not much is known for long-time dynamics of (SQG), see a recent breakthrough of [He and Kiselev 2021] for a construction of smooth initial data with Sobolev norms growing at least exponentially for all times. Moreover, existence of traveling-wave solutions [Li 2009; Cao et al. 2023] and rotating solutions [Hassainia and Hmidi 2015; de la Hoz et al. 2016; Castro et al. 2016] are known.

• Weak solutions: global existence. Global existence of L^p-weak solutions is known, thanks to [Resnick 1995; Marchand 2008; Bae and Granero-Belinchón 2015]. While such solutions are in general expected to be nonunique, see [Córdoba et al. 2018] for a uniqueness result for patches. On the other hand, for "very" weak solutions, nonuniqueness has been established; see [Buckmaster et al. 2019; Cheng et al. 2021; Isett and Ma 2021]. Note the gap of regularity between week and strong solutions.

• Ill-posedness in $W^{1,\infty}$: To the best of our knowledge, the only critical space ill-posedness result concerning (SQG) is the one given in [Elgindi and Masmoudi 2020] for $W^{1,\infty}$, where a powerful general method for proving ill-posedness of active scalar systems in L^∞-type spaces is developed. To be precise, in Section 9.2 of that work the authors show that there exist smooth steady states $\bar\theta$ and a sequence of perturbations $\tilde\theta_0^{(\epsilon)}$ ($\epsilon \to 0^+$) so that the associated (SQG) solution $\theta^{(\epsilon)}$ with data $\bar\theta + \tilde\theta_0^{(\epsilon)}$ satisfies

$$\|\theta^{(\epsilon)}(0,\cdot) - \bar\theta\|_{W^{1,\infty}} < \epsilon, \qquad \sup_{0<t<\epsilon}\|\theta^{(\epsilon)}(t,\cdot) - \bar\theta\|_{W^{1,\infty}} > c,$$

where $c > 0$ depends only on $\bar{\theta}$. It is very interesting to note that the authors use well-posedness in *critical Besov spaces* with summability index 1. Such Besov well-posedness theory goes back to the pioneering work [Vishik 1999]. Our result (which applies in the $W^{1,\infty}$ case as well) basically says that one can take $\bar{\theta} \equiv 0$ and replace c by ϵ^{-1}. On the other hand, one can restore well-posedness in $W^{1,\infty}$ by assuming some *rotational symmetry* and anisotrophic Hölder regularity [Elgindi and Jeong 2020b].

The current work settles the issue of *strong ill-posedness* of (SQG) at critical Sobolev spaces, and we believe that this could be a first step in understanding the dynamics of "slightly" supercritical and subcritical solutions (e.g., evolution of H^s-data with $|s-2| \ll 1$), thereby bridging the gap between the theory of weak and strong solutions. Indeed, in the very recent work [Elgindi 2021] on singularity formation for the three-dimensional Euler equations, one of the key steps was to understand precisely the mechanism of C^1-ill-posedness. Closing this section, let us mention some interesting works which seem contradictory to our main results:

- Miura [2006] proved that the fractionally dissipative SQG system

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + (-\Delta)^{\beta} \theta = 0, \\ u = \nabla^{\perp} (-\Delta)^{-1/2} \theta \end{cases} \tag{1-2}$$

is actually *well-posed* in the critical Sobolev space $H^{2-2\beta}$ for all $\beta > 0$ (for data of any size), and this seems to suggest H^2 well-posedness of the inviscid system by taking $\beta \to 0$! See [Li 2021; Jolly et al. 2021; 2022] for related recent advances.

- An invariant measure defined on $H^2(\mathbb{T}^2)$ which guarantees global well-posedness in $L_t^{\infty} H^2$ for any initial data in the support of the measure was constructed in [Földes and Sy 2021]. The data in Theorem B certainly does not belong to the support of such a measure.

1C. ***Generalized SQG equations.*** In the recent years, there has been significant interest in the study of so-called *generalized* SQG equations, given by

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^{\perp} P(\Lambda) \theta, \end{cases} \tag{1-3}$$

where $P(\Lambda)$ is some Fourier multiplier, with $\Lambda = (-\Delta)^{1/2}$. Two distinguished cases are $P(\Lambda) = \Lambda^{-1}$ (SQG) and $P(\Lambda) = \Lambda^{-2}$ (two-dimensional incompressible Euler). Of particular interest is the case of α-SQG systems given by $P(\Lambda) = \Lambda^{-\alpha}$, with $1 \le \alpha \le 2$, which interpolates the SQG and two-dimensional Euler equations. The L^2-based critical Sobolev space is then given by $H^{3-\alpha}$, and let us point out that the methods developed in the current work can handle the entire range $1 \le \alpha \le 2$ without any essential change in the proof, after deriving a generalized version of the "key lemma"; see the Appendix. One could consider even more general symbols such as $P(\Lambda) = \Lambda^{-\alpha} \log^{-\gamma}(10 + \Lambda)$, with $\gamma > 0$, which give rise to the so-called *logarithmically regularized* systems [Chae and Wu 2012; Chae et al. 2011; Dong and Li 2010]. It is known that if the power of the logarithm is sufficiently large, then one can restore well-posedness in $H^{3-\alpha}$ [Chae and Wu 2012], but at this point it is more appropriate to regard a logarithmically singularized Sobolev space to be critical. Indeed, one can see from our proof that there is a “logarithmic” room[1] in the arguments and therefore the same proof can cover same ill-posedness results in the slightly logarithmically regularized systems. We shall not dwell on this issue any further.

1D. ***Critical space ill-posedness for Euler.*** It should be emphasized that the strong Sobolev ill-posedness statements, Theorems A and B, were first established in the groundbreaking works [Bourgain and Li 2015; 2021] for the case of two- and three-dimensional Euler equations, respectively. Further developments, including the current work, seem to have been strongly inspired by these papers. Recently, Kwon [2021] settled the problem of strong ill-posedness in H^1 for logarithmically regularized (strictly speaking, powers of the log less than or equal to $\frac{1}{2}$) two-dimensional Euler equations, nicely complementing previous H^1 well-posedness from [Chae and Wu 2012]. On the other hand, much simpler proofs of H^1 ill-posedness for two-dimensional Euler, which also shows continuous-in-time degeneration of the solution in Sobolev spaces, have appeared in [Elgindi and Jeong 2017; Jeong and Yoneda 2021]. Some details of these simplified arguments will be given in the next section.

2. Ingredients of the proof

The purpose of this section is to sketch the main ingredients of the proof. Several key ideas have already appeared in earlier works establishing ill-posedness in the Euler case; we briefly review those in Section 2A. Additional difficulties arising in the (generalized) SQG case and new ideas are covered then in Section 2B.

2A. ***Strategy in the Euler case.*** In this section, let us give an overview of the ill-posedness proof in the two-dimensional Euler case. We recall that in $\mathbb{T}^2$ the Euler equations are given by

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ u = \nabla^\perp (-\Delta)^{-1} \omega. \end{cases} \tag{Euler}$$

In terms of ω, the critical L^2-based Sobolev space is $H^1(\mathbb{T}^2)$; indeed, $\omega \in H^1$ barely fails to guarantee $\nabla u \in L^\infty$, which is necessary to close the a priori estimate in H^1.

Choice of data for Euler. As a starting point of discussion, we present an interesting identity observed by T. Elgindi:

$$\frac{d}{dt}(\|\partial_2 \omega\|_{L^2}^2 - \|\partial_1 \omega\|_{L^2}^2) = \frac{1}{2} \int_{\mathbb{T}^2} \partial_1 u_1 ((\partial_2 \omega)^2 + (\partial_1 \omega)^2) + \omega \, \partial_1 \omega \, \partial_2 \omega \, \mathrm{d}x. \tag{2-1}$$

For $\omega_0 \in L^\infty$, Yudovich theory provides a unique global solution in $L^\infty([0, \infty) \times \mathbb{T}^2)$, and note that the last term in (2-1) cannot contribute to a large growth of the H^1-norm in a small time interval. Therefore, to prove existence of an $H^1 \cap L^\infty$-initial data ω_0 which “escapes” H^1 instantaneously, the goal would be to find $\omega_0 \in H^1 \cap L^\infty$ such that

$$\int_0^t \int_{\mathbb{T}^2} \partial_1 u_1 (\partial_2 \omega)^2 \, \mathrm{d}x = +\infty \tag{2-2}$$

[1] Strictly speaking, some power of the logarithm.

for *any* $t > 0$, where ω is the Yudovich solution with data ω_0 and $u = \nabla^{\perp}(-\Delta)^{-1}\omega$. In particular, it is necessary that at the initial time we have

$$\int_{\mathbb{T}^2} \partial_1 u_{0,1} (\partial_2 \omega_0)^2 \, \mathrm{d}x = +\infty. \tag{2-3}$$

The choice in [Elgindi and Jeong 2017] was

$$\omega_0(x) \simeq \frac{x_1 x_2}{|x|^2} |\ln |x||^{-\alpha}, \quad |x| \ll 1, \tag{2-4}$$

since then [Bahouri and Chemin 1994; Denisov 2015b]

$$\partial_1 u_{0,1}(x) \simeq |\ln |x||^{1-\alpha}, \quad |x| \ll 1, \tag{2-5}$$

which in particular guarantees (2-3) for a range of $\alpha > 0$.

Hyperbolic flow. Geometrically, vorticity which is positive on the first quadrant and odd with respect to both axes (as in (2-4)) induces velocity which is stretching in the x_1-direction and contracting in the other, which leads to squeezing of the vorticity near the x_1-axis and growth of H^1-norm. This so-called "hyperbolic flow scenario" has been used to produce Euler solutions with gradient growth; see [Kiselev and Šverák 2014; Zlatoš 2015; Xu 2016; Denisov 2009; 2015a; 2015b; Elgindi and Jeong 2019; 2020a; Choi and Jeong 2021]. Flattening of the vorticity level sets in such a flow configuration was studied in detail in [Zlatoš 2018; Jeong 2021].

Regularization effect. The main task is then to ensure that the velocity field, for a small time interval, retains its logarithmic divergence near the origin: indeed, instantaneous blow-up of the H^1-norm is not too difficult to see for the passive transport equation

$$\partial_t \omega + u_0 \cdot \nabla \omega = 0,$$

by solving the equation along the flow generated by u_0. When one tries to replace u_0 by u, a fundamental difficulty arises: anisotropic stretching of the vorticity regularizes the velocity. Indeed, rather involved computations in [Elgindi and Jeong 2023] suggest the asymptotics $\|\nabla u(t)\|_{L^\infty} \lesssim t^{-1}$, which is barely nonintegrable in time; this indicates that it could be a very delicate problem to verify (2-2). This upper bound of t^{-1} can be seen for instance by solving the passive problem above and recalculating the associated velocity at later times.

Key lemma and Lagrangian approach. Towards the goal of obtaining a *lower bound* on the velocity gradient $|\nabla u(t)| \gtrsim t^{-1}$, one needs to have a robust way of estimating the velocity gradient and proving some "stability" of the initial data. Regarding the former, the celebrated *key lemma* of Kiselev and Šverák asserts that (stated roughly)

$$\frac{u_1(x)}{x_1} \simeq \int_{[x_1,1]\times[x_2,1]} \frac{y_1 y_2}{|y|^4} \omega(y) \, \mathrm{d}y \tag{2-6}$$

for $\omega \in L^\infty$ with odd-odd symmetry. Note that $u_1 = 0$ for $x_1 = 0$ by symmetry, so that the left-hand side is an approximation of $\partial_1 u_1$. The lower bound of the form (2-6) has proven to be extremely powerful in establishing growth of the vorticity [Kiselev and Šverák 2014; Zlatoš 2015; Xu 2016; Kiselev et al. 2016;

Gancedo and Patel 2021; Elgindi 2021; He and Kiselev 2021]. It is interesting to note that [Bourgain and Li 2015] independently derived similar lower bounds. Next, regarding the issue of showing stability of the data, the key observation is the hierarchy of vortex dynamics expressed in (2-6): the vorticity around a point x is being affected mainly by the vorticity supported in $|y| \geq 2|x|$. This suggests that the chunk of vorticity supported far away from the origin is more stable, thereby contributing to the right-hand side of (2-6) for a longer time interval, to squeeze the vorticity closer to the origin. The proof of such stability and squeezing phenomena should be done in the Lagrangian variable, using the transport formulas

$$\omega(t,x) = \omega_0(\Phi_t^{-1}x), \quad \nabla\omega(t,x) = \nabla\omega_0(\Phi_t^{-1}(x))\nabla\Phi_t^{-1}(x),$$

where Φ_t is the flow map at time t. In the actual ill-posedness proofs, Lagrangian versions of the formula (2-1) are used.

2B. ***Difficulties in the SQG case.*** Overall, the strategy of the ill-posedness proof in the SQG case is similar to that explained in the above for two-dimensional Euler. Roughly speaking, the initial data is now modified to be

$$\theta_0 \simeq \frac{x_1x_2}{|x|}|\ln|x||^{-\alpha}, \quad |x| \ll 1,$$

which is odd-odd and nonnegative in the first quadrant. The associated SQG velocity then satisfies the asymptotics (2-5) with strong hyperbolicity near the origin, which should stretch θ near the x_1-axis. The issue is whether such a stretching effect is sufficiently strong to remove θ from the critical Sobolev space it started from. Let us now explain some main differences with the Euler case and new ideas employed to handle those.

While the equation for θ in (SQG) is simply the transport equation exactly as in the two-dimensional Euler case, probably the most significant difference is that while the L^∞-norm is the common strongest conservation law, it is critical for two-dimensional Euler but one order weaker for SQG. Furthermore, there is global well-posedness for two-dimensional Euler with $\omega_0 \in L^\infty$ [Yudovich 1963], and the associated sharp estimates given by Yudovich theory have been very useful in understanding the dynamics.[2] On the other hand, the corresponding quantity in the SQG case, $\|\nabla\theta\|_{L^\infty}$, blows up together with $\|\theta\|_{H^2}$.

It seems that the only way to handle this issue is to rely entirely on a contradiction argument — we *assume* that there is an $L^\infty([0,T]; H^2)$-solution, and then prove that, for any $t > 0$, the H^2-norm of the solution must be actually infinite. The whole point in this contradiction argument is that we can use the hypothetical H^2-bound to control the solution, an idea originated in [Bourgain and Li 2015]. Again, the difficulty in the SQG case is that this hypothetical H^2 control is the only useful bound, whereas in the Euler case one has both H^1 and L^∞ control. Fortunately, it turns out that having an H^2-bound guarantees that the velocity is log-Lipschitz, which implies in particular *uniqueness* in the class $L_t^\infty H^2$ (this guarantees propagation in time of odd-odd symmetry and nonnegativity) and existence of the flow map. That is, an $L_t^\infty H^2$-solution is Lagrangian, and therefore we can apply transport formulas to understand the dynamics.

[2] Even in the three-dimensional Euler case, [Bourgain and Li 2021] actually carefully identifies a class of initial data for which $\omega \in L^\infty$ propagates locally in time. Then, one can prove and utilize estimates similar to Yudovich's in three dimensions.

Under the contradiction hypothesis, the main part of the argument is to derive and apply a version of the key lemma adapted to the SQG case. Series technical difficulties appear; to begin with, in the remainder estimate of the key lemma (see estimates (3-1) and (3-2) in Lemma 3.2) we are only allowed to use $\theta \in H^2$. As a consequence, the remainder term blows up super-logarithmically (the power $\frac{3}{2}$ in (3-2)) as the point x approaches the axes, whereas only logarithmic errors are allowed in the nonexistence proof. It seems that the only way to overcome this issue is to track carefully the geometry of the support of θ in time so that the problematic remainder term disappears. To achieve this, we replace θ_0 with a disjoint union of dyadic "bubbles" satisfying the same asymptotics as $|x| \to 0$ (see (4-4) below) and obtain detailed information on the location of these bubbles for an interval of time inductively, starting from the largest one. Such refined information appears in technical Claims I, II and III in the proof. In the context of controlling bubbles, another significant difference with Euler is that the "self-interaction" of a bubble is *not* a bounded term anymore. To overcome this issue we need to track the location of the "top point" of each bubble, which is the slowest point but does not suffer from self-interactions.

Closing this section, we remark that the versions of the key lemma derived in this work should be useful in improving previous growth results for the active scalar equations, as we handle the remainder term only with the critical quantity.

2C. ***Organization of the paper.*** The rest of this paper is organized as follows. The main technical tool, which we shall refer to as the key lemma, is stated and proved in Section 3. After that, the proofs of Theorems B and A are given in Sections 4 and 5, respectively.

3. The key lemma

To begin with, we recall the famous Hardy inequality.

Lemma 3.1 (Hardy's inequality). *Let f be a smooth function defined on the interval $(0, l)$ that vanishes in a neighborhood of $x = 0$. Then we have for any $l \in [0, 1]$*

$$\|x^{-1} f(x)\|_{L^2(0,l)} \le 2\|\partial f(x)\|_{L^2(0,l)}, \quad \|x^{-2} f(x)\|^2_{L^2(0,l)} \le 2\|\partial^2 f(x)\|^2_{L^2(0,l)}.$$

Proof. By the fundamental theorem of calculus and the assumption for f, we see

$$\int_0^l \frac{f(x)^2}{x^2}\,\mathrm{d}x = -\frac{1}{l} f(l)^2 + 2\int_0^l \frac{f(x)}{x}\partial f(x)\,\mathrm{d}x \le 2\int_0^l \frac{f(x)}{x}\partial f(x)\,\mathrm{d}x.$$

Using Hölder's inequality, we have

$$\int_0^l \frac{f(x)^2}{x^2}\,\mathrm{d}x \le 4\int_0^l |\partial f(x)|^2\,\mathrm{d}x.$$

Similarly, we have

$$\int_0^l \frac{f(x)^2}{x^4}\,\mathrm{d}x = -\frac{1}{3l^3} f(l)^2 + \frac{2}{3}\int_0^l \frac{f(x)}{x^3}\partial f(x)\,\mathrm{d}x \le \frac{2}{3}\int_0^l \frac{f(x)}{x^2}\frac{\partial f(x)}{x}\,\mathrm{d}x \le \frac{1}{2}\int_0^l \frac{\partial f(x)^2}{x^2}\,\mathrm{d}x.$$

Applying the above estimate, we obtain

$$\int_0^l \frac{f(x)^2}{x^4}\,\mathrm{d}x \le 2\int_0^l |\partial^2 f(x)|^2\,\mathrm{d}x. \qquad \square$$

We shall now state and prove the key lemma. For convenience, we shall normalize the SQG Biot–Savart law in such a way that

$$u(t,x)=\sum_{n\in\mathbb{Z}^2}\int_{[-1,1)^2}\frac{(x-(y+2n))^{\perp}}{|x-(y+2n)|^3}\theta(t,y)\,\mathrm{d}y.$$

Lemma 3.2. *We impose the following assumptions on* $\theta\in H^2$:

- θ *is odd with respect to both axes, i.e.,* $\theta(x)=-\theta(\bar{x})=\theta(-x)=-\theta(\tilde{x})$, *where* $\bar{x}:=(x_1,-x_2)$ *and* $\tilde{x}:=(-x_1,x_2)$.
- θ *vanishes near the axis; to be precise, for any* $x\neq(0,0)$ *satisfying either* $x_1=0$ *or* $x_2=0$, *there exists an open neighborhood of* x *such that* θ *vanishes.*

Then, for any x *satisfying* $|x|<\frac{1}{4}$ *and* $x_1>x_2>0$, *we have*

$$\left|\frac{u_1(x)}{x_1}-12\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right|\le B_1(x) \tag{3-1}$$

and

$$\left|\frac{u_2(x)}{x_2}+12\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right|\le\left(1+\log\frac{x_1}{x_2}\right)B_2(x)+\left(1+\log\frac{x_1}{x_2}\right)^{3/2}B_3(x), \tag{3-2}$$

where $Q(x):=[2x_1,1]\times[0,1]$ *and* B_1, B_2, B_3 *satisfy*

$$|B_1(x)|+|B_2(x)|\le C(\|\nabla^2\theta\|_{L^2([0,1]^2)}+\|\theta\|_{L^\infty([0,1]^2)})$$

and

$$|B_3(x)|\le C(\|\nabla^2\theta\|_{L^2(R(x))}+\|y_2^{-1}\partial_1\theta(y)\|_{L^2(R(x))}),\qquad R(x):=[x_1/2,2x_1]\times[2x_2,1].$$

Remark 3.3. We clearly have that $\|y_2^{-1}\partial_1\theta(y)\|_{L^2(R(x))}\le 2\|\nabla^2\theta\|_{L^2([0,1]^2)}$.

Proof. We fix a point $x=(x_1,x_2)$ satisfying the assumptions of the lemma. After a symmetrization, we have

$$u(x)=\sum_{n\in\mathbb{Z}^2}\int_{[0,1]^2}\left(\frac{(x-(y+2n))^{\perp}}{|x-(y+2n)|^3}-\frac{(x-(\tilde{y}+2n))^{\perp}}{|x-(\tilde{y}+2n)|^3}+\frac{(x-(-y+2n))^{\perp}}{|x-(-y+2n)|^3}-\frac{(x-(\bar{y}+2n))^{\perp}}{|x-(\bar{y}+2n)|^3}\right)\theta(y)\,\mathrm{d}y. \tag{3-3}$$

Estimate of u_1. We consider

$$I_1(n):=-\int_{[0,1]^2}\left(\frac{x_2-(y_2+2n_2)}{|x-(y+2n)|^3}-\frac{x_2-(y_2+2n_2)}{|x-(\tilde{y}+2n)|^3}\right)\theta(y)\,\mathrm{d}y,$$

$$I_2(n):=-\int_{[0,1]^2}\left(\frac{x_2-(-y_2+2n_2)}{|x-(-y+2n)|^3}-\frac{x_2-(-y_2+2n_2)}{|x-(\bar{y}+2n)|^3}\right)\theta(y)\,\mathrm{d}y$$

so that from (3-3)

$$u_1(x)=\sum_{n\in\mathbb{Z}^2}(I_1(n)+I_2(n)).$$

We think of the cases $n=0$ and $n\neq 0$ separately. For $n\neq 0$, we see that

$$|I_1(n)+I_1(\tilde n)|\le O(|n|^{-4})\|\theta\|_{L^\infty([0,1]^2)}x_1,\quad |I_2(n)+I_2(\tilde n)|\le O(|n|^{-4})\|\theta\|_{L^\infty([0,1]^2)}x_1.$$

Therefore,

$$\Bigl|\sum_{n\in\mathbb{Z}^2\setminus\{0\}}(I_1(n)+I_2(n))\Bigr|\le Cx_1\|\theta\|_{L^\infty([0,1]^2)}. \tag{3-4}$$

We now estimate the case of $n=0$. Using

$$\frac{1}{A^3}-\frac{1}{B^3}=\frac{(B^2-A^2)(A^2+AB+B^2)}{A^3B^3(A+B)}, \tag{3-5}$$

we have

$$I_1(0)=-4x_1\int_{[0,1]^2}\frac{y_1(x_2-y_2)(|x-y|^2+|x-y|\,|x-\tilde y|+|x-\tilde y|^2)}{|x-y|^3|x-\tilde y|^3(|x-y|+|x-\tilde y|)}\theta(y)\,\mathrm{d}y.$$

Noting that $[0,1]^2=Q(x)\cup[0,2x_1]\times[2x_1,1]\cup[0,2x_1]^2$, we estimate the integral for each set.

(i) Suppose $y\in Q(x)$. In this case we can show that

$$\tfrac14|y|\le|x-y|\le|y|,\quad \tfrac12|y|\le|x-\tilde y|\le 2|y| \tag{3-6}$$

because the first inequality comes from

$$|x-y|^2\ge|x_1-y_1|^2\ge\tfrac14 y_1^2\ge\tfrac18|y|^2,\quad y_1\ge y_2,$$

and

$$|x-y|^2=|x_1-y_1|^2+|x_2-y_2|^2\ge\tfrac14 y_1^2+\tfrac14 y_2^2,\quad y_1\le y_2.$$

The goal is to prove that

$$-\int_{Q(x)}\frac{y_1(x_2-y_2)(|x-y|^2+|x-y|\,|x-\tilde y|+|x-\tilde y|^2)}{|x-y|^3|x-\tilde y|^3(|x-y|+|x-\tilde y|)}\theta(y)\,\mathrm{d}y=:J$$

satisfies

$$\Bigl|J-\frac32\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\Bigr|\le C\|\nabla^2\theta\|_{L^2([0,1]^2)}. \tag{3-7}$$

We separate $J=J_1+J_2$, where

$$J_1:=\int_{Q(x)}\frac{y_1y_2(|x-y|^2+|x-y|\,|x-\tilde y|+|x-\tilde y|^2)}{|x-y|^3|x-\tilde y|^3(|x-y|+|x-\tilde y|)}\theta(y)\,\mathrm{d}y,$$

$$J_2:=-\int_{Q(x)}\frac{y_1x_2(|x-y|^2+|x-y|\,|x-\tilde y|+|x-\tilde y|^2)}{|x-y|^3|x-\tilde y|^3(|x-y|+|x-\tilde y|)}\theta(y)\,\mathrm{d}y.$$

Using (3-6), we may estimate

$$|J_2|\le C|x|\int_{Q(x)}\frac{1}{|y|^2}\frac{|\theta(y)|}{|y|^2}\,\mathrm{d}y.$$

Note that by Hölder's inequality,

$$|x|\int_{Q(x)}\frac{1}{|y|^2}\frac{|\theta(y)|}{|y|^2}\,\mathrm{d}y\le|x|\Bigl(\int_{2x_1}^\infty\frac{1}{r^3}\,\mathrm{d}r\Bigr)^{1/2}\||y|^{-2}\theta(y)\|_{L^2(Q(x))}\le C\||y|^{-2}\theta(y)\|_{L^2([0,1]^2)}.$$

Then with the Hardy's inequality we have

$$|J_2| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.$$

On the other hand, regarding J_1, we shall show that

$$\left| J_1 - \frac{3}{2}\int_{Q(x)} \frac{y_1 y_2}{|y|^5}\theta(y)\,\mathrm{d}y \right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.$$

For this purpose we simply write $J_1 = J_{11} + J_{12} + J_{13}$, where

$$J_{11} = \int_{Q(x)} \frac{y_1 y_2 |x-y|^2}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y,$$
$$J_{12} = \int_{Q(x)} \frac{y_1 y_2 |x-y|\,|x-\tilde{y}|}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y,$$
$$J_{13} = \int_{Q(x)} \frac{y_1 y_2 |x-\tilde{y}|^2}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y$$

and show that

$$\left| J_{1k} - \frac{1}{2}\int_{Q(x)} \frac{y_1 y_2}{|y|^5}\theta(y)\,\mathrm{d}y \right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}$$

for each $k = 1, 2, 3$. We supply the proof only for the case $k = 1$, since the others can be treated similarly. We directly compute

$$J_{11} - \frac{1}{2}\int_{Q(x)} \frac{y_1 y_2}{|y|^5}\theta(y)\,\mathrm{d}y = \int_{Q(x)} y_1 y_2 \frac{2|y|^5 - |x-y|\,|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}{2|y|^5|x-y|\,|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y.$$

We rewrite the numerator as

$$\begin{aligned} &2|y|^5 - |x-y|\,|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|) \\ &\quad = (|y|^2 - |x-y|^2)|y|^3 + |x-y|^2(|y|^3 - |x-\tilde{y}|^3) + (|y| - |x-y|)|y|^4 + |x-y|(|y|^4 - |x-\tilde{y}|^4), \end{aligned}$$

and further rewriting

$$|y| - |x-y| = \frac{|y|^2 - |x-y|^2}{|y| + |x-y|}, \quad |y|^3 - |x-\tilde{y}|^3 = \frac{(|y|^2 - |x-\tilde{y}|^2)(|y|^2 + |y|\,|x-\tilde{y}| + |x-\tilde{y}|^2)}{|y| + |x-\tilde{y}|},$$

we see using (3-6) that

$$\bigl|2|y|^5 - |x-y|\,|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)\bigr| \le C|x|\,|y|^4.$$

Then, we can infer that

$$\left| J_{11} - \frac{1}{2}\int_{Q(x)} \frac{y_1 y_2}{|y|^5}\theta(y)\,\mathrm{d}y \right| \le C|x| \int_{Q(x)} \frac{1}{|y|^2}\frac{|\theta(y)|}{|y|^2}\,\mathrm{d}y \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.$$

Collecting the estimates for J_1 and J_2 gives (3-7).

(ii) Suppose $y \in [0, 2x_1] \times [2x_1, 1]$. In this case, using $y_1 \le y_2$, we can see that

$$\tfrac{1}{2}y_2 \le |x-y| \le 2y_2, \quad \tfrac{1}{2}y_2 \le |x-\tilde{y}| \le 2y_2.$$

Thus, Hölder's inequality and Hardy's inequality imply that

$$\begin{aligned}\left|-\int_{[0,2x_1]\times[2x_1,1]} \frac{y_1(x_2-y_2)(|x-y|^2+|x-y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y\right| \\ \le C\int_{[0,2x_1]\times[2x_1,1]} \frac{1}{y_2}\frac{\theta(y)}{y_2^2}\,\mathrm{d}y \le C\left(\int_{[0,2x_1]\times[2x_1,1]}\frac{\mathrm{d}y}{y_2^2}\right)^{1/2}\|y_2^{-2}\theta(y)\|_{L^2([0,1]^2)} \\ \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}. \end{aligned} \tag{3-8}$$

(iii) Suppose $y\in[0,2x_1]^2$. Thanks to $\theta(y_1,0)=0$, using integration by parts gives

$$\begin{aligned}-\int_{[0,2x_1]^2} \frac{y_1(x_2-y_2)(|x-y|^2+|x-y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y \\ = \frac{1}{4x_1}\int_{[0,2x_1]^2}\left(\frac{1}{|x-y|}-\frac{1}{|x-\tilde{y}|}\right)\partial_2\theta(y)\,\mathrm{d}y \\ -\frac{1}{4x_1}\int_0^{2x_1}\left(\frac{1}{|(x_1-y_1,x_2-2x_1)|}-\frac{1}{|(x_1+y_1,x_2-2x_1)|}\right)\theta(y_1,2x_1)\,\mathrm{d}y_1.\end{aligned}$$

By Hölder's inequality, we estimate the second integral as

$$\left|-\frac{1}{x_1}\int_0^{2x_1}\left(\frac{1}{|(x_1-y_1,x_2-2x_1)|}-\frac{1}{|(x_1+y_1,x_2-2x_1)|}\right)\theta(y_1,2x_1)\,\mathrm{d}y_1\right| \le Cx_1^{-1}\|\theta\|_{L^\infty([0,2x_1]^2)}.$$

We notice that since θ vanishes near the axis, we have

$$\begin{aligned}\|\theta\|_{L^\infty([0,2x_1]^2)} &\le \sup_{y_1\in[0,2x_1]}\int_0^{2x_1}|\partial_2\theta(y_1,y_2)|\,\mathrm{d}y_2 \le (2x_1)^{1/2}\left\|\sup_{y_1\in[0,2x_1]}|\partial_2\theta(y_1,\cdot\,)|\right\|_{L^2(0,2x_1)} \\ &\le 2x_1\|\partial_1\partial_2\theta\|_{L^2([0,2x_1]^2)}.\end{aligned} \tag{3-9}$$

Thus, we have

$$\left|-\frac{1}{x_1}\int_0^{2x_1}\left(\frac{1}{|(x_1-y_1,x_2-2x_1)|}-\frac{1}{|(x_1+y_1,x_2-2x_1)|}\right)\theta(y_1,2x_1)\,\mathrm{d}y_1\right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.$$

Calculating the first integral with Hölder's inequality, we see that

$$\begin{aligned}\left|\frac{1}{x_1}\int_{[0,2x_1]^2}\left(\frac{1}{|x-y|}-\frac{1}{|x-\tilde{y}|}\right)\partial_2\theta(y)\,\mathrm{d}y\right| &\le \frac{2}{x_1}\int_{[0,2x_1]^2}\frac{|\partial_2\theta(y)|}{|x-y|}\,\mathrm{d}y \\ &\le \frac{2}{x_1}\left(\int_0^{4x_1} r^{-1/3}\,\mathrm{d}r\right)^{3/4}\|\partial_2\theta\|_{L^4([0,2x_1]^2)} \\ &\le Cx_1^{-1/2}\|\nabla\theta\|_{L^4([0,2x_1]^2)}.\end{aligned}$$

The Gagliardo–Nirenberg interpolation inequality implies

$$x_1^{-1/2}\|\nabla\theta\|_{L^4([0,2x_1]^2)} \le Cx_1^{-1/2}\|\nabla^2\theta\|^{3/4}_{L^2([0,2x_1]^2)}\|\theta\|^{1/4}_{L^2([0,2x_1]^2)} + Cx_1^{-2}\|\theta\|_{L^2([0,2x_1]^2)},$$

where the constant $C>0$ is independent of x_1. Applying Hardy's inequality to it, we have

$$x_1^{-1/2}\|\partial_2\theta\|_{L^4([0,2x_1]^2)} \le C\|\nabla^2\theta\|_{L^2([0,2x_1]^2)},$$

and hence it follows that

$$\left|\frac{1}{x_1}\int_{[0,2x_1]^2}\left(\frac{1}{|x-y|}-\frac{1}{|x-\tilde{y}|}\right)\partial_2\theta(y)\,\mathrm{d}y\right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.$$

Combining the above estimates, we obtain

$$\left|-\int_{[0,2x_1]^2}\frac{y_1(x_2-y_2)(|x-y|^2+|x-y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.$$

We collect the estimates for each region and deduce that

$$\left|\frac{I_1(0)}{x_1}-6\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.$$

We can estimate

$$I_2(0)=4x_1\int_{[0,1]^2}\frac{y_1(x_2+y_2)(|x+y|^2+|x+y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x+y|^3|x-\bar{y}|^3(|x+y|+|x-\bar{y}|)}\theta(y)\,\mathrm{d}y$$

similarly to $I_1(0)$, resulting in the bound

$$\left|\frac{I_2(0)}{x_1}-6\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}. \tag{3-10}$$

We omit the details. Thus we have (3-1).

Estimate of u_2. Now we estimate u_2. Note that

$$u_2(x)=\sum_{n\in\mathbb{Z}^2}(I_3(n)+I_4(n)),$$

where

$$I_3(n):=\int_{[0,1]^2}\left(\frac{x_1-(y_1+2n_1)}{|x-(y+2n)|^3}-\frac{x_1-(y_1+2n_1)}{|x-(\bar{y}+2n)|^3}\right)\theta(y)\,\mathrm{d}y,$$

$$I_4(n):=\int_{[0,1]^2}\left(\frac{x_1-(-y_1+2n_1)}{|x-(-y+2n)|^3}-\frac{x_1-(-y_1+2n_1)}{|x-(\tilde{y}+2n)|^3}\right)\theta(y)\,\mathrm{d}y.$$

Since we can similarly see that

$$\left|\sum_{n\in\mathbb{Z}^2\setminus\{0\}}(I_3(n)+I_4(n))\right| \le Cx_2\|\theta\|_{L^\infty([0,1]^2)},$$

it suffices to estimate for $n=0$. Using (3-5), we have

$$I_3(0)=4x_2\int_{[0,1]^2}\frac{y_2(x_1-y_1)(|x-y|^2+|x-y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x-y|^3|x-\bar{y}|^3(|x-y|+|x-\bar{y}|)}\theta(y)\,\mathrm{d}y.$$

We divide the domain into four regions as $[0,1]^2=Q(x)\cup[0,x_1/2]\times[0,2x_1]\cup[x_1/2,2x_1]\times[0,2x_1]\cup[0,2x_1]\times[2x_1,1]$ and estimate the integral in each region.

(i) Suppose $y\in Q(x)$. In this case, we note by $\frac{1}{4}y_1^2+y_2^2\le|x_1-y_1|^2+|x_2+y_2|^2$ that

$$\tfrac{1}{2}|y|\le|x-\bar{y}|\le 2|y|.$$

Recalling (3-6) holds, we can prove similarly to (3-7)

$$\left|\int_{Q(x)} \frac{y_2(x_1-y_1)(|x-y|^2+|x-y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x-y|^3|x-\bar{y}|^3(|x-y|+|x-\bar{y}|)}\theta(y)\,\mathrm{d}y + \frac{3}{2}\int_{Q(x)} \frac{y_1 y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.$$

(ii) Suppose $y \in [0, 2x_1] \times [2x_1, 1]$. It follows that

$$\tfrac{1}{2}y_2 \le |x-y| \le 2y_2, \quad y_2 \le |x-\bar{y}| \le 2y_2.$$

Hence, we can show

$$\left|\int_{[0,2x_1]\times[2x_1,1]} \frac{y_2(x_1-y_1)(|x-y|^2+|x-y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x-y|^3|x-\bar{y}|^3(|x-y|+|x-\bar{y}|)}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}$$

in a way similar to (3-8).

(iii) Suppose $y \in [0, x_1/2] \times [0, 2x_1]$. This implies

$$\tfrac{1}{2}x_1 \le |x-y| \le 4x_1, \quad \tfrac{1}{2}x_1 \le |x-\bar{y}| \le 4x_1,$$

with $y_2 \le 2x_1$ we have

$$\begin{aligned}\left|\int_{[0,x_1/2]\times[0,2x_1]} \frac{y_2(x_1-y_1)(|x-y|^2+|x-y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x-y|^3|x-\bar{y}|^3(|x-y|+|x-\bar{y}|)}\theta(y)\,\mathrm{d}y\right| \\ \le Cx_1^{-1}\int_{[0,x_1/2]\times[0,2x_1]} \frac{\theta(y)}{y_2^2}\,\mathrm{d}y \le Cx_1^{-1}\left(\int_{[0,x_1/2]\times[0,2x_1]} 1\,\mathrm{d}y\right)^{1/2} \|y_2^{-2}\theta(y)\|_{L^2([0,1]^2)} \\ \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.\end{aligned}$$

(iv) Suppose $y \in [x_1/2, 2x_1] \times [0, 2x_1]$. We claim that

$$\int_{[x_1/2,2x_1]\times[0,2x_1]} \frac{y_2(x_1-y_1)(|x-y|^2+|x-y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x-y|^3|x-\bar{y}|^3(|x-y|+|x-\bar{y}|)}\theta(y)\,\mathrm{d}y =: K$$

satisfies

$$|K| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right) + C(\|\nabla^2\theta\|_{L^2(R(x))} + \|y_2^{-1}\partial_1\theta(y)\|_{L^2(R(x))})\left(1+\log\frac{x_1}{x_2}\right)^{3/2}. \tag{3-11}$$

Using integration by parts, we take the decomposition $K = K_1 + K_2 + K_3$, where

$$\begin{aligned} K_1 &:= -\frac{1}{4x_2}\int_{[x_1/2,2x_1]\times[0,2x_1]} \left(\frac{1}{|x-y|} - \frac{1}{|x-\bar{y}|}\right)\partial_1\theta(y)\,\mathrm{d}y, \\ K_2 &:= \frac{1}{4x_2}\int_0^{2x_1} \left(\frac{1}{|(x_1, x_2-y_2)|} - \frac{1}{|(x_1, x_2+y_2)|}\right)\theta(2x_1, y_2)\,\mathrm{d}y_2, \\ K_3 &:= -\frac{1}{4x_2}\int_0^{2x_1} \left(\frac{1}{|(x_1/2, x_2-y_2)|} - \frac{1}{|(x_1/2, x_2+y_2)|}\right)\theta(x_1/2, y_2)\,\mathrm{d}y_2. \end{aligned}$$

With (3-9) we may estimate K_2 as

$$|K_2| = \left| \int_0^{2x_1} \frac{y_2\theta(2x_1, y_2)}{|(x_1, x_2-y_2)||(x_1, x_2+y_2)|(|(x_1, x_2-y_2)|+|(x_1, x_2+y_2)|)} \,\mathrm{d}y_2 \right|$$
$$\leq Cx_1^{-2} \int_0^{2x_1} \theta(2x_1, y_2)\,\mathrm{d}y_2 \leq Cx_1^{-1} \|\theta\|_{L^\infty([0,2x_1]^2)} \leq C\|\nabla^2\theta\|_{L^2([0,1]^2)}. \tag{3-12}$$

Similarly, we obtain

$$|K_3| \leq C\|\nabla^2\theta\|_{L^2([0,1]^2)}.$$

Noting that

$$K_1 = \int_{[x_1/2, 2x_1]\times[0,2x_1]} \frac{-y_2\partial_1\theta(y)}{|x-y||x-\bar{y}|(|x-y|+|x-\bar{y}|)}\,\mathrm{d}y,$$

we set $K_1 = K_{11} + K_{12}$, where

$$K_{11} := \int_{[x_1/2, 2x_1]\times[0,2x_2]} \frac{-y_2\partial_1\theta(y)}{|x-y||x-\bar{y}|(|x-y|+|x-\bar{y}|)}\,\mathrm{d}y,$$
$$K_{12} := \int_{[x_1/2, 2x_1]\times[2x_2,2x_1]} \frac{-y_2\partial_1\theta(y)}{|x-y||x-\bar{y}|(|x-y|+|x-\bar{y}|)}\,\mathrm{d}y.$$

From $|x-\bar{y}| \geq x_2 + y_2 \geq y_2$ we have

$$|K_{11}| \leq C \int_0^{2x_2} \frac{\sup_{y_1\in[x_1/2,2x_1]} |\partial_1\theta(y_1, y_2)|}{x_2+y_2} \int_0^{2x_1} \frac{1}{|x-y|}\,\mathrm{d}y_1\,\mathrm{d}y_2,$$
$$|K_{12}| \leq C \int_{2x_2}^{2x_1} \frac{\sup_{y_1\in[x_1/2,2x_1]} |\partial_1\theta(y_1, y_2)|}{x_2+y_2} \int_0^{2x_1} \frac{1}{|x-y|}\,\mathrm{d}y_1\,\mathrm{d}y_2.$$

By the Gagliardo–Nirenberg interpolation inequality with $y_2 \leq 2x_1$, we can see that

$$y_2^{-1/2} \sup_{y_1\in[x_1/2,2x_1]} |\partial_1\theta(y_1, y_2)| \leq C(\|\partial_1^2\theta(\,\cdot\,, y_2)\|_{L^2(x_1/2,2x_1)} + y_2^{-1}\|\partial_1\theta(\,\cdot\,, y_2)\|_{L^2(x_1/2,2x_1)}),$$

where the constant $C > 0$ does not depend on x_1. On the other hand,

$$\int_0^{2x_1} \frac{1}{|x-y|}\,\mathrm{d}y_1 = \int_0^{x_1} \frac{2}{\sqrt{\tau^2+(x_2-y_2)^2}}\,\mathrm{d}\tau$$
$$= 2\log\bigl(x_1 + \sqrt{x_1^2+(x_2-y_2)^2}\bigr) - 2\log|x_2-y_2| \leq C\log\left(1+\frac{x_1}{|x_2-y_2|}\right). \tag{3-13}$$

Hence, with $y_2 \leq 2x_1$ and Hölder's inequality, we infer that

$$|K_{11}| \leq C(\|\nabla^2\theta\|_{L^2([0,1]^2)} + \|y_2^{-1}\partial_1\theta(y)\|_{L^2([0,1]^2)}) \left\{ \int_0^{2x_2} \frac{1}{x_2+y_2} \left|\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\right|^2 \mathrm{d}y_2 \right\}^{1/2},$$
$$|K_{12}| \leq C(\|\nabla^2\theta\|_{L^2(R(x))} + \|y_2^{-1}\partial_1\theta(y)\|_{L^2(R(x))}) \left\{ \int_{2x_2}^{2x_1} \frac{1}{x_2+y_2} \left|\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\right|^2 \mathrm{d}y_2 \right\}^{1/2}.$$

Using Hardy's inequality and that

$$\begin{aligned}\int_0^{2x_2} \frac{1}{x_2+y_2}\left|\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\right|^2 \mathrm{d}y_2 &\le \frac{1}{x_2}\int_0^{2x_2}\left(\log\frac{2x_1}{|x_2-y_2|}\right)^2 \mathrm{d}y_2 \\ &= \frac{2}{x_2}\int_0^{x_2}\left(\log\frac{2x_1}{x_2-y_2}\right)^2 \mathrm{d}y_2 \le C\left(1+\log\frac{x_1}{x_2}\right)^2,\end{aligned}$$

we obtain

$$|K_{11}| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right).$$

By the estimate

$$\begin{aligned}\int_{2x_2}^{2x_1} \frac{1}{x_2+y_2}\left|\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\right|^2 \mathrm{d}y_2 &\le \int_{2x_2}^{2x_1}\frac{1}{y_2-x_2}\left(\log\frac{3x_1}{y_2-x_2}\right)^2 \mathrm{d}y_2 \\ &= \frac{1}{3}\left(\log\frac{3x_1}{x_2}\right)^3 - \frac{1}{3}\left(\log\frac{3x_1}{2x_1-x_2}\right)^3 \le C\left(1+\log\frac{x_1}{x_2}\right)^3,\end{aligned}$$

we have

$$|K_{12}| \le C(\|\nabla^2\theta\|_{L^2(R(x))} + \|y_2^{-1}\partial_1\theta(y)\|_{L^2(R(x))})\left(1+\log\frac{x_1}{x_2}\right)^{3/2}.$$

This implies

$$|K_1| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right) + C(\|\nabla^2\theta\|_{L^2(R(x))} + \|y_2^{-1}\partial_1\theta(y)\|_{L^2(R(x))})\left(1+\log\frac{x_1}{x_2}\right)^{3/2},$$

and collecting the estimates for K_1, K_2, and K_3, we obtain (3-11). Therefore, we arrive at

$$\begin{aligned}&\left|\frac{I_3(0)}{x_2} + 6\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \\ &\quad\le C\|\nabla^2\theta\|_{L^2([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right) + C(\|\nabla^2\theta\|_{L^2(R(x))} + \|y_2^{-1}\partial_1\theta(y)\|_{L^2(R(x))})\left(1+\log\frac{x_1}{x_2}\right)^{3/2}.\end{aligned}$$

Using (3-5), we can estimate

$$I_4(0) = -4x_2\int_{[0,1]^2}\frac{y_2(x_1+y_1)(|x+y|^2+|x+y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x+y|^3|x-\tilde{y}|^3(|x+y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y,$$

similarly to $I_3(0)$. Hence we have (3-2), and this completes the proof. □

Lemma 3.4. *Let θ satisfy the assumptions in Lemma 3.2. Then, for any x satisfying $|x| < \frac{1}{4}$ and $x_1 > x_2 > 0$, we have*

$$\left|\frac{u_1(x)}{x_1} - 12\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le B_4(x) \tag{3-14}$$

and

$$\left|\frac{u_2(x)}{x_2} + 12\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le \left(1+\log\frac{x_1}{x_2}\right)B_5(x) + \left(1+\log\frac{x_1}{x_2}\right)^2 B_6(x), \tag{3-15}$$

where B_4, B_5, B_6 satisfy

$$|B_4(x)| + |B_5(x)| \le C(\|\nabla\theta\|_{L^\infty([0,1]^2)} + \|\theta\|_{L^\infty([0,1]^2)}), \quad |B_6(x)| \le C\|\nabla\theta\|_{L^\infty(R(x))}.$$

Proof. We follow the proof of Lemma 3.2. To obtain (3-14), we recall (3-4) and have

$$|u_1(x)| = \left|\sum_{n\in\mathbb{Z}^2} (I_1(n) + I_2(n))\right| \le Cx_1\|\theta\|_{L^\infty([0,1]^2)} + I_1(0) + I_2(0).$$

We estimate

$$I_1(0) = -4x_1 \int_{[0,1]^2} \frac{y_1(x_2-y_2)(|x-y|^2 + |x-y|\,|x-\tilde{y}| + |x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y$$

for each set $Q(x)$, $[0, 2x_1]\times[2x_1, 1]$, and $[0, 2x_1]^2$.

(i) Suppose $y \in Q(x)$. Using the notation J_1 and J_2 in Lemma 3.2, it suffices to obtain

$$\left| J_1 - \frac{3}{2}\int_{Q(x)} \frac{y_1 y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| + |J_2| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}. \tag{3-16}$$

We already showed that

$$|J_2| \le C|x|\int_{Q(x)} \frac{1}{|y|^3}\frac{|\theta(y)|}{|y|}\,\mathrm{d}y.$$

By (3-6) and Hölder's inequality, we have

$$|x|\int_{Q(x)} \frac{1}{|y|^3}\frac{|\theta(y)|}{|y|}\,\mathrm{d}y \le |x|\left(\int_{|x|}^{\infty}\frac{1}{r^2}\,\mathrm{d}r\right)\||y|^{-1}\theta(y)\|_{L^\infty(Q(x))} \le \||y|^{-1}\theta(y)\|_{L^\infty([0,1]^2)}.$$

Since θ vanishes near the axis, it follows

$$|J_2| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}.$$

Letting $J_1 = J_{11} + J_{12} + J_{13}$ as in the proof of Lemma 3.2, we can prove that

$$\left| J_{1k} - \frac{1}{2}\int_{Q(x)} \frac{y_1 y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le C|x|\int_{Q(x)}\frac{1}{|y|^3}\frac{|\theta(y)|}{|y|}\,\mathrm{d}y \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}$$

for each $k = 1, 2, 3$. Therefore, (3-16) is obtained.

(ii) Suppose $y \in [0, 2x_1]\times[2x_1, 1]$. In (3-8) we observed that

$$\left|\int_{[0,2x_1]\times[2x_1,1]} \frac{y_1(x_2-y_2)(|x-y|^2 + |x-y|\,|x-\tilde{y}| + |x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y\right| C\int_{[0,2x_1]\times[2x_1,1]}\frac{1}{y_2^2}\frac{\theta(y)}{y_2}\,\mathrm{d}y.$$

Since

$$\int_{[0,2x_1]\times[2x_1,1]}\frac{1}{y_2^2}\frac{\theta(y)}{y_2}\,\mathrm{d}y \le C\left(\int_{[0,2x_1]\times[2x_1,1]}\frac{1}{y_2^2}\,\mathrm{d}y\right)\|y_2^{-1}\theta(y)\|_{L^\infty([0,1]^2)} \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}, \tag{3-17}$$

we have

$$\left| -\int_{[0,2x_1]\times[2x_1,1]} \frac{y_1(x_2-y_2)(|x-y|^2 + |x-y|\,|x-\tilde{y}| + |x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}.$$

(iii) Suppose $y \in [0, 2x_1]^2$. We recall that

$$
\begin{aligned}
&-\int_{[0,2x_1]^2} \frac{y_1(x_2-y_2)(|x-y|^2+|x-y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y \\
&\quad = \frac{1}{4x_1}\int_{[0,2x_1]^2}\Big(\frac{1}{|x-y|}-\frac{1}{|x-\tilde{y}|}\Big)\partial_2\theta(y)\,\mathrm{d}y \\
&\qquad\qquad -\frac{1}{4x_1}\int_0^{2x_1}\Big(\frac{1}{|(x_1-y_1,x_2-2x_1)|}-\frac{1}{|(x_1+y_1,x_2-2x_1)|}\Big)\theta(y_1,2x_1)\,\mathrm{d}y_1.
\end{aligned}
$$

Using Hölder's inequality, we have

$$
\begin{aligned}
\left|\frac{1}{x_1}\int_{[0,2x_1]^2}\Big(\frac{1}{|x-y|}-\frac{1}{|x-\tilde{y}|}\Big)\partial_2\theta(y)\,\mathrm{d}y\right| &\le \frac{2}{x_1}\Big(\int_{[0,2x_1]^2}\frac{1}{|x-y|}\,\mathrm{d}y\Big)\|\partial_2\theta\|_{L^\infty([0,1]^2)} \\
&\le C\|\nabla\theta\|_{L^\infty([0,2x_1]^2)}.
\end{aligned}
$$

From Hölder's inequality and the mean value theorem, it follows

$$
\left|-\frac{1}{x_1}\int_0^{2x_1}\Big(\frac{1}{|(x_1-y_1,x_2-2x_1)|}-\frac{1}{|(x_1+y_1,x_2-2x_1)|}\Big)\theta(y_1,2x_1)\,\mathrm{d}y_1\right| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}.
$$

Therefore, we obtain

$$
\left|-\int_{[0,2x_1]^2}\frac{y_1(x_2-y_2)(|x-y|^2+|x-y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.
$$

Combining the above estimates, it follows that

$$
\left|\frac{I_1(0)}{x_1}-6\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}.
$$

In a similar way, we can show that

$$
I_2(0)=4x_1\int_{[0,1]^2}\frac{y_1(x_2+y_2)(|x+y|^2+|x+y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x+y|^3|x-\bar{y}|^3(|x+y|+|x-\bar{y}|)}\theta(y)\,\mathrm{d}y
$$

satisfies

$$
\left|\frac{I_2(0)}{x_2}-6\int_{R(2x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}.
$$

We omit the details. Thus we have (3-14).

To estimate u_2, we start with

$$
|u_2(x)| = \left|\sum_{n\in\mathbb{Z}^2}(I_3(n)+I_4(n))\right| \le Cx_2\|\theta\|_{L^\infty([0,1]^2)} + I_3(0)+I_4(0).
$$

To estimate

$$
I_3(0)=4x_2\int_{[0,1]^2}\frac{y_2(x_1-y_1)(|x-y|^2+|x-y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x-y|^3|x-\bar{y}|^3(|x-y|+|x-\bar{y}|)}\theta(y)\,\mathrm{d}y,
$$

we consider $[0,1]^2 = Q(x)\cup[0,x_1/2]\times[0,2x_1]\cup[x_1/2,2x_1]\times[0,2x_1]\cup[0,2x_1]\times[2x_1,1]$ and estimate the integral in each region.

(i) Suppose $y \in Q(x)$. In this case, recalling that (3-6) holds, we can prove, proceeding much as for (3-16),

$$\left| \int_{Q(x)} \frac{y_2(x_1-y_1)(|x-y|^2+|x-y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x-y|^3|x-\bar{y}|^3(|x-y|+|x-\bar{y}|)} \theta(y)\,\mathrm{d}y + \frac{3}{2}\int_{Q(x)} \frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y \right| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}. \tag{3-18}$$

(ii) Suppose $y \in [0, 2x_1] \times [2x_1, 1]$. Since in this case we have

$$\tfrac{1}{2}y_2 \le |x-y| \le 2y_2, \quad y_2 \le |x-\bar{y}| \le 2y_2,$$

with (3-17) we can show that

$$\left| \int_{[0,2x_1]\times[2x_1,1]} \frac{y_2(x_1-y_1)(|x-y|^2+|x-y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x-y|^3|x-\bar{y}|^3(|x-y|+|x-\bar{y}|)} \theta(y)\,\mathrm{d}y \right| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}.$$

(iii) Suppose $y \in [0, x_1/2] \times [0, 2x_1]$. This implies

$$\tfrac{1}{2}x_1 \le |x-y| \le 4x_1, \quad \tfrac{1}{2}x_1 \le |x-\bar{y}| \le 4x_1,$$

we can see that

$$\left| \int_{[0,x_1/2]\times[0,2x_1]} \frac{y_2(x_1-y_1)(|x-y|^2+|x-y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x-y|^3|x-\bar{y}|^3(|x-y|+|x-\bar{y}|)} \theta(y)\,\mathrm{d}y \right| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}.$$

(iv) Suppose $y \in [x_1/2, 2x_1] \times [0, 2x_1]$. Recalling the notation K_1, K_2, and K_3 in the proof of Lemma 3.2, we claim

$$|K_1| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right) + C\|\nabla\theta\|_{L^\infty(R(x))}\left(1+\log\frac{x_1}{x_2}\right)^2 \tag{3-19}$$

and

$$|K_2| + |K_3| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}. \tag{3-20}$$

As in (3-12), we have from the mean value theorem that

$$|K_2| \le Cx_1^{-1}\|\theta\|_{L^\infty([0,2x_1]^2)} \le C\|\nabla\theta\|_{L^\infty([0,1]^2)},$$

and similarly,

$$|K_3| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}.$$

Hence, (3-20) follows. We recall $K_1 = K_{11} + K_{12}$, where

$$K_{11} = \int_{[x_1/2,2x_1]\times[0,2x_2]} \frac{-y_2\partial_1\theta(y)}{|x-y|\,|x-\bar{y}|(|x-y|+|x-\bar{y}|)}\,\mathrm{d}y,$$

$$K_{12} = \int_{[x_1/2,2x_1]\times[2x_2,2x_1]} \frac{-y_2\partial_1\theta(y)}{|x-y|\,|x-\bar{y}|(|x-y|+|x-\bar{y}|)}\,\mathrm{d}y.$$

From Hölder's inequality and (3-13), we can deduce that

$$|K_{11}| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)} \int_0^{2x_2} \frac{1}{x_2+y_2}\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\mathrm{d}y_2,$$

$$|K_{12}| \le C\|\nabla\theta\|_{L^\infty(R(x))} \int_{2x_2}^{2x_1} \frac{1}{x_2+y_2}\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\mathrm{d}y_2.$$

Since

$$\int_0^{2x_2} \frac{1}{x_2+y_2} \log\left(1+\frac{x_1}{|x_2-y_2|}\right) \mathrm{d}y_2 \le \frac{1}{x_2}\int_0^{2x_2} \log \frac{2x_1}{|x_2-y_2|}\,\mathrm{d}y_2 \le C\left(1+\log\frac{x_1}{x_2}\right),$$
$$\int_{2x_2}^{2x_1} \frac{1}{x_2+y_2} \log\left(1+\frac{x_1}{|x_2-y_2|}\right) \mathrm{d}y_2 \le \int_{2x_2}^{2x_1} \frac{1}{y_2-x_2} \log \frac{3x_1}{y_2-x_2}\,\mathrm{d}y_2 \le C\left(1+\log\frac{x_1}{x_2}\right)^2,$$

it follows

$$|K_{11}| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right), \quad |K_{12}| \le C\|\nabla\theta\|_{L^\infty(R(x))}\left(1+\log\frac{x_1}{x_2}\right)^2.$$

This shows (3-19). Combining the estimates, we arrive at

$$\left|\frac{I_3(0)}{x_2} + 6\int_{Q(x)} \frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right) + C\|\nabla\theta\|_{L^\infty(R(x))}\left(1+\log\frac{x_1}{x_2}\right)^2.$$

Using (3-5), we can estimate

$$I_4(0) = -4x_2\int_{[0,1]^2} \frac{y_2(x_1+y_1)(|x+y|^2+|x+y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x+y|^3|x-\tilde{y}|^3(|x+y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y,$$

similarly to $I_3(0)$. Hence we have (3-15), and this completes the proof. □

4. Nonexistence

In this section, we prove Theorem B. We begin with a simple uniqueness result which in particular guarantees that the hypothetical solution in $L^\infty_t H^2$ satisfies the same symmetries with the initial data.

Proposition 4.1. *Given $\theta_0 \in H^2$ and $T > 0$, there exists at most one solution to* (SQG) *belonging to $L^\infty([0, T]; H^2)$ with initial data θ_0.*

Proof. The proof can be given by simply adapting the inequalities derived in [Yudovich 1963; 1995]. This statement can be found in [Azzam and Bedrossian 2015] as well. □

Proof of Theorem B. For convenience, we shall divide the proof into several parts.

Part 1: velocity and flow map: an $L^\infty_t H^2$-solution is Lagrangian. Assume that we are given a solution to (SQG) satisfying

$$\sup_{t\in[0,T]} \|\theta(t,\cdot)\|_{H^2} \le M.$$

Then, by the Sobolev embedding, $u = \nabla^\perp(-\Delta)^{-1/2}\theta$ satisfies

$$\sup_{t\in[0,T]} (\|\nabla u(t,\cdot)\|_{\mathrm{BMO}} + \|u(t,\cdot)\|_{W^{1,1}}) \le C \sup_{t\in[0,T]} \|u(t,\cdot)\|_{H^2} \le CM,$$

with some absolute constant $C > 0$. In particular, u is log-Lipschitz: for any $x, y \in \mathbb{T}^2$, we have

$$\sup_{t\in[0,T]} |u(t,x)-u(t,y)| \le CM|x-y| \ln\left(10+\frac{1}{|x-y|}\right). \tag{4-1}$$

On the time interval $[0, T]$, we consider the flow map $\Phi(t, \cdot) : \mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$\begin{cases} \dfrac{d}{dt}\Phi(t, x) = u(t, \Phi(t, x)), \\ \Phi(0, x) = x. \end{cases} \tag{4-2}$$

It is well known that under the estimate (4-1), there is a unique solution to the ODE (4-2) for any $x \in \mathbb{T}^2$ [Majda and Bertozzi 2002; Marchioro and Pulvirenti 1994]. The solution Φ satisfies the estimate

$$|x - y|^{\exp(CMt)} \le |\Phi(t, x) - \Phi(t, y)| \le |x - y|^{\exp(-CMt)} \tag{4-3}$$

for some absolute constant $C > 0$, uniformly in $x, y \in \mathbb{T}^2$ satisfying $|x - y| < \frac{1}{2}$ and $t \in [0, T]$. We have the representation

$$\theta(t, \Phi(t, x)) = \theta_0(x).$$

The estimate (4-3) shows that, for each $t \in [0, T]$, $\Phi(t, \cdot)$ is a Hölder continuous homeomorphism $\mathbb{T}^2 \to \mathbb{T}^2$, and we denote the inverse map by Φ_t^{-1}. Then, with this notation, we have

$$\theta(t, x) = \theta_0(\Phi_t^{-1}(x)).$$

The inverse map Φ_t^{-1} is again Hölder continuous. As an immediate consequence, we have that if θ_0 is an odd function with respect to both axes and satisfies

$$\mathrm{supp}(\theta_0) \cup \{x : x_1 = 0 \text{ or } x_2 = 0\} \subset \{(0, 0)\},$$

then the same properties are satisfied by $\theta(t, \cdot)$, as long as $\theta \in L^\infty([0, t]; H^2)$. Indeed, the uniqueness assertion from Proposition 4.1 guarantees that $\theta(t, \cdot)$ is odd with respect to both axes. Furthermore, Hölder continuity of the flow map and its inverse ensures that $\theta(t, \cdot)$ vanishes near the axes, possibly except at the origin. Therefore, the last assumption in Lemma 3.2 is satisfied.

Part 2: choice of initial data. We fix some smooth bump function $\phi : \mathbb{R}^2 \to \mathbb{R}_{\ge 0}$ satisfying the following properties:

- ϕ is C^∞-smooth and radial.
- ϕ is supported in $B_0\big(\frac{1}{8}\big)$ and $\phi = 1$ in $B_0\big(\frac{1}{32}\big)$.

Then, we define

$$\theta_0 := \sum_{n=n_0}^{\infty} n^{-\alpha} \theta_{0,\mathrm{loc}}^{(n)} \tag{4-4}$$

for some $\frac{1}{2} < \alpha < \frac{3}{4}$, where

$$\theta_{0,\mathrm{loc}}^{(n)}(x) := 4^{-n} \phi(4^n(x_1 - 4^{-n-1}, x_2 - 4^{-n-2}))$$

for $x \in [0, 1]^2$. The precise value of α will be determined later, but for now let us just mention that it will be taken slightly larger than $\frac{1}{2}$. Next, let us extend each of $\theta_{0,\mathrm{loc}}^{(n)}$ (and similarly θ_0) to $\mathbb{T}^2$ as an odd function with respect to both axes. Note that by taking $n_0 \ge 1$ sufficiently large in a way depending only on $\epsilon > 0$, we can guarantee that

$$\|\theta_0\|_{H^2 \cap W^{1,\infty}(\mathbb{T}^2)} < \epsilon.$$

Towards a contradiction, we shall assume that there exists $M > 0$ and $T > 0$ such that

$$\sup_{t\in[0,T]} \|\theta(t)\|_{\dot{H}^2(\mathbb{T}^2)} \le M. \tag{4-5}$$

For simplicity, we shall assume that $M \ge 1$. Observe that the assumptions in the key lemma (Lemma 3.2) are satisfied by θ_0. Recalling the discussion above, we have that $\theta(t,\cdot)$ is odd with respect to both axes and vanishes near the axes, except at the origin.

Part 3: preliminary bounds on the solution. Let us remark in advance that in the following proof we shall take $T > 0$ to be smaller, if necessary, to satisfy $T \le c/M$ for some small absolute constant $c > 0$. We shall begin with a simple result:

Lemma 4.2. *Assume that θ is a solution satisfying* (4-5) *with initial data* (4-4). *Then, by redefining T to satisfy $T \le c/M$ if necessary, we have*

$$\theta(t, y) = 0, \quad 0 \le y_1 \le y_2,\ 0 \le t \le T.$$

Proof. Since $\theta(t,\Phi(t,x)) = \theta_0(x)$, to prove the claim, it suffices to show that, for $x \in \operatorname{supp}(\theta_0)\backslash\{(0,0)\}$, $\Phi_2(t,x) \le \Phi_1(t,x)$ for $0 \le t \le T$. Let us fix some $x \in \operatorname{supp}(\theta_0)\backslash\{(0,0)\}$. Then, from the choice of initial data, we have $2x_2 \le x_1$. From continuity in time of the flow map, there exists some $0 < T^* \le T$ such that $\Phi_2(t,x) < \Phi_1(t,x)$ for $0 \le t < T^*$. Then, on this time interval, key lemma is applicable for $\Phi(t,x)$ and we compute

$$\begin{aligned}\frac{d}{dt}\left(\frac{\Phi_2(t,x)}{\Phi_1(t,x)}\right) &= \frac{\Phi_2(t,x)}{\Phi_1(t,x)}\left(\frac{u_2(t,\Phi(t,x))}{\Phi_2(t,x)} - \frac{u_1(t,\Phi(t,x))}{\Phi_1(t,x)}\right)\\ &\le C\frac{\Phi_2(t,x)}{\Phi_1(t,x)}(|B_1(\Phi(t,x))| + |B_2(\Phi(t,x))| + |B_3(\Phi(t,x))|) \le CM\frac{\Phi_2(t,x)}{\Phi_1(t,x)}.\end{aligned}$$

Therefore, we actually obtain

$$\frac{\Phi_2(t,x)}{\Phi_1(t,x)} \le \frac{1}{2}\exp(CMt) < \frac{3}{4}$$

on $t \in [0, T^*]$, as long as $T^* \le c/M$ for $c > 0$ depending only on C. This bootstrap procedure allows us to get $\Phi_2/\Phi_1 < \frac{3}{4}$ uniformly in $x \in \operatorname{supp}(\theta_0)\backslash\{(0,0)\}$ by the time $\min\{T, c/M\} = T$. □

The above lemma guarantees that on $[0,T]$, the key lemma is applicable to points in $\operatorname{supp}(\theta(t,\cdot))$. Next, let us set $\Omega_n := \operatorname{supp}(\theta^{(n)}_{0,\mathrm{loc}}) \cap \{x \in \mathbb{T}^2 : x_1 > x_2 > 0\}$ and prove that, by reducing $c > 0$ if necessary, the bubbles $\{\Phi(t,\Omega_n)\}_{n\ge n_0}$ are "well-ordered" with respect to the x_1-axis for $t \in [0,T]$ with $T \le c/M$.

Claim I. *We have*

$$\sup_{x\in\Omega_n} \Phi_1(t,x) \le 2\inf_{x\in\Omega_n} \Phi_1(t,x) \quad \textit{and} \quad 2\sup_{x\in\Omega_{n+1}} \Phi_1(t,x) \le \inf_{x\in\Omega_n} \Phi_1(t,x) \tag{4-6}$$

uniformly for all $n \ge n_0$ and $t \in [0,T]$, by reducing T to satisfy $T \le c/(1+M)$ for some small absolute constant $c > 0$.

For simplicity we let

$$\widehat{\Phi}^n_j(t) := \sup_{x\in\Omega_n} \Phi_j(t,x)$$

for $j=1,2$. We can prove the Claim I inductively in n, using the key lemma, which gives

$$\left|\frac{u_1(t,x)}{x_1}-12\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(t,y)\,\mathrm{d}y\right|\le CM.$$

In the proof, we shall take $T>0$ smaller several times, but in a way which is independent of n. To begin with, for $x\in\Omega_{n_0}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\Phi_1(t,x)\ge -CM.$$

Thus,

$$\log\Phi_1(t,x)-\log x_1\ge -CMt.$$

We also have

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\widehat{\Phi}_1^{n_0}(t)-\frac{\mathrm{d}}{\mathrm{d}t}\log\Phi_1(t,x)\le 2CM,$$

and thus,

$$\log\widehat{\Phi}_1^{n_0}(t)-\log\Phi_1(t,x)\le 2CMt+(\log\hat{x}_1^{n_0}-\log x_1).$$

Since $\hat{x}_1^{n_0}/x_1<\frac{3}{2}$, we can take $T>0$ sufficiently small such that

$$2CMT+(\log\hat{x}_1^{n_0}-\log x_1)\le\log 2,$$

which implies that

$$\widehat{\Phi}_1^{n_0}(t)\le 2\inf_{x\in\Omega_{n_0}}\Phi_1(t,x)$$

for all $t\in[0,T]$. Indeed, it suffices to take $T=c/(1+M)$ with a small absolute constant $c>0$. To show

$$2\widehat{\Phi}_1^{n_0+1}(t)\le\inf_{x\in\Omega_{n_0}}\Phi_1(t,x)\tag{4-7}$$

for all $t\in[0,T]$, we use the notation

$$\widehat{\Psi}_1^{n}(t):=\sup_{n\le m}\widehat{\Phi}_1^{m}(t).$$

Then, for $x\in\Omega_{n_0}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\Phi_1(t,x)-\frac{\mathrm{d}}{\mathrm{d}t}\log\widehat{\Psi}_1^{n_0+1}(t)\ge -12\int_{\Omega_{n_0}}\frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5}\theta_0(y)\,\mathrm{d}y-2CM.$$

From the above estimates, it follows

$$\int_{\Omega_{n_0}}\frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5}\theta_0(y)\,\mathrm{d}y\le\left(\sup_{x\in\Omega_{n_0}}\frac{x_1}{\Phi_1(t,x)}\right)^3\int_{\Omega_{n_0}}\frac{\theta_0(y)}{y_1^3}\,\mathrm{d}y\le C_0e^{3CMT}.$$

Using it, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\Phi_1(t,x)-\frac{\mathrm{d}}{\mathrm{d}t}\log\widehat{\Psi}_1^{n_0+1}(t)\ge -12C_0e^{3CMT}-2CM$$

and

$$\log\Phi_1(t,x)-\log\widehat{\Psi}_1^{n_0+1}(t)\ge -12C_0e^{3CMT}t-2CMt+(\log x_1-\log\hat{x}_1^{n_0+1}).$$

Since $x_1/\hat{x}_1^{n_0+1}>2$, we can take $T>0$ sufficiently small such that

$$-12C_0e^{3CMT}T-2CMT+(\log x_1-\log\hat{x}_1^{n_0+1})\ge\log 2.$$

Hence, with $\widehat{\Psi}_1^{n_0+1}(t) \geq \widehat{\Phi}_1^{n_0+1}(t)$, we can obtain (4-7). Now let $x \in \Omega_{n_0+1}$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \Phi_1(t,x) \geq -CM.$$

Thus,

$$\log \Phi_1(t,x) - \log x_1 \geq -CMt.$$

Since $\hat{x}_1^{n_0+1}/x_1$ takes the same value as in the previous case, we see that

$$2CMT + (\log \hat{x}_1^{n_0+1} - \log x_1) \leq \log 2,$$

and therefore, we have

$$\widehat{\Phi}_1^{n_0+1}(t) \leq 2 \inf_{x\in\Omega_{n_0+1}} \Phi_1(t,x)$$

for all $t \in [0,T]$. Note that by (4-7),

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \Phi_1(t,x) - \frac{\mathrm{d}}{\mathrm{d}t} \log \widehat{\Psi}_1^{n_0+2}(t) \geq -12 \int_{\Omega_{n_0+1}} \frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5} \theta_0(y)\,\mathrm{d}y - 2CM.$$

With the above estimates, we have

$$\int_{\Omega_{n_0+1}} \frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5} \theta_0(y)\,\mathrm{d}y \leq \left(\sup_{x\in\Omega_{n_0+1}} \frac{x_1}{\Phi_1(t,x)} \right)^{n_0+2} \int_{\Omega_{n_0+1}} \frac{\theta_0(y)}{y_1^{n_0+2}}\,\mathrm{d}y \leq C_0 e^{3CMT}.$$

Using it, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \Phi_1(t,x) - \frac{\mathrm{d}}{\mathrm{d}t} \log \widehat{\Psi}_1^{n_0+2}(t) \geq -12C_0 e^{3CMT} - 2CM$$

and

$$\log \Phi_1(t,x) - \log \widehat{\Psi}_1^{n_0+2}(t) \geq -12C_0 e^{3CMT} t - 2CMt + (\log x_1 - \log \hat{x}_1^{n_0+2}).$$

Since $x_1/\hat{x}_1^{n_0+2} > 2$ is the same value as in the previous case, it follows

$$-12C_0 e^{3CMT} T - 2CMT + (\log x_1 - \log \hat{x}_1^{n_0+2}) \geq \log 2$$

and

$$2\widehat{\Phi}_1^{n_0+2}(t) \leq \inf_{x\in\Omega_{n_0+1}} \Phi_1(t,x)$$

for all $t \in [0,T]$. Repeating this argument, one can finish the proof of Claim I. □

Claim II. *There exists $T > 0$ and $C > 0$ such that*

$$\log \frac{\widehat{\Phi}_2^n(T)}{\hat{x}_2^n} \leq -10 \sum_{n_0 \leq j \leq n-1} \int_0^T \int_{\Omega_j} \frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5} \theta_0(y)\,\mathrm{d}y\,\mathrm{d}t + CMT$$

uniformly for all $n > n_0$.

Recall that

$$\frac{u_2(x)}{x_2} \leq -12 \int_{Q(x)} \frac{y_1 y_2}{|y|^5} \theta(t,y)\,\mathrm{d}y + CM\left(\log \frac{x_1}{x_2}\right)$$

if $\theta(y)=0$ for y satisfying $x_1/2 \le y_1 \le 2x_1$ and $2x_2 \le y_2 \le 1$. According to the order of the bubbles, for $x \in \Omega_n$, we have

$$\int_{Q(\Phi(t,x))} \frac{y_1 y_2}{|y|^5}\theta(t,y)\,\mathrm{d}y = \sum_{n_0\le j\le n-1} \int_{\Omega_j} \frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5}\theta_0(y)\,\mathrm{d}y.$$

And note that

$$\sup_{2\Phi_2(t,x)\ge \widehat{\Phi}_2^n(t)} CM\left(\log\frac{\Phi_1(t,x)}{\Phi_2(t,x)}\right) \le CM\left(\log\frac{2\widehat{\Phi}_1^n(t)}{\widehat{\Phi}_2^n(t)}\right).$$

Thus, we can see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\log\frac{\widehat{\Phi}_2^n(t)}{\hat{x}_2^n} &\le -12\sum_{n_0\le j\le n-1}\int_{\Omega_j}\frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5}\theta_0(y)\,\mathrm{d}y + CM\log\frac{2\widehat{\Phi}_1^n(t)}{\widehat{\Phi}_2^n(t)}\\ &\le -12\sum_{n_0\le j\le n-1}\int_{\Omega_j}\frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5}\theta_0(y)\,\mathrm{d}y + CM\log\frac{\widehat{\Phi}_1^n(t)}{\hat{x}_1^n} - CM\log\frac{\widehat{\Phi}_2^n(t)}{\hat{x}_2^n} + CM\end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{CMt}\log\frac{\widehat{\Phi}_2^n(t)}{\hat{x}_2^n}\right) \le e^{CMt}\left(-12\sum_{n_0\le j\le n-1}\int_{\Omega_j}\frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5}\theta_0(y)\,\mathrm{d}y + CM\left(\log\frac{\widehat{\Phi}_1^n(t)}{\hat{x}_1^n}+1\right)\right).$$

It suffices to bound the time integral of the right-hand side. Note that from

$$\log\frac{\widehat{\Phi}_1^n(t)}{\hat{x}_1^n} \le 12\sum_{n_0\le j\le n-1}\int_0^t\int_{\Omega_j}\frac{\Phi_1(\tau,y)\Phi_2(\tau,y)}{|\Phi(\tau,y)|^5}\theta_0(y)\,\mathrm{d}y\,\mathrm{d}\tau + CMt,$$

we obtain

$$CM\int_0^t \log\frac{\widehat{\Phi}_1^n(\tau)}{\hat{x}_1^n}\,\mathrm{d}\tau \le 12CMt\sum_{n_0\le j\le n-1}\int_0^t\int_{\Omega_j}\frac{\Phi_1(\tau,y)\Phi_2(\tau,y)}{|\Phi(\tau,y)|^5}\theta_0(y)\,\mathrm{d}y\,\mathrm{d}\tau + (CMt)^2.$$

Therefore, using Grönwall's inequality on the quantity $\log(\widehat{\Phi}_2^n(t)/\hat{x}_2^n)$ and taking $T>0$ small depending only on CM, we can complete the proof of Claim II. □

Part 4: almost invariant timescales. We shall write $\Phi(t,\Omega_n)\sim\Omega_n$ if

$$\mathrm{supp}(\Phi(t,\Omega_n)) \subset B((4^{-n-1},4^{-n-2}),4^{-n-1}).$$

Here, $B((4^{-n-1},4^{-n-2}),4^{-n-1})$ denotes the ball of radius 4^{-n-1} centered at $(4^{-n-1},4^{-n-2})$. Recall from the definition of initial data that

$$\Omega_n = B((4^{-n-1},4^{-n-2}),2^{-1}4^{-n-1}).$$

An immediate consequence of $\Phi(t,\Omega_n)\sim\Omega_n$ is that once we define

$$I_n(t) = \int_{\Omega_n}\frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5}\theta_0(y)\,\mathrm{d}y,$$

we have

$$I_n(t) \ge a_0 I_n(0)$$

for some absolute constant a_0. The following claim gives the sharp bound on the "almost invariant" timescale for each bubble.

Claim III. *For all $n \geq n_0$, we have*

$$\Phi(t, \Omega_n) \sim \Omega_n \quad \text{for } 0 \leq t \leq \min\left\{T, \frac{c}{M + \sum_{j=n_0}^{n-1} j^{-\alpha}}\right\} =: T_n,$$

with some constants $c, C > 0$ independent of n.

This claim can be proved easily with an induction in n. In the base case $n = n_0$, we simply note that, for $x \in \Phi(t, \Omega_{n_0})$,

$$\left|\frac{u_j(t, x)}{x_j}\right| \leq CM$$

from which the claim follows in this case. Assume that Claim III holds for all $n < n_0 + k$ for some $k \geq 1$. Note that using the key lemma and the induction hypothesis, we have for $x \in \Phi(t, \Omega_{n_0+k})$ that

$$\left|\frac{u_j(t, x)}{x_j}\right| \leq C\biggl(M + \sum_{\ell=0}^{k-1} I_{n_0+\ell}\biggr), \quad 0 \leq t \leq T_{n_0+k-1}.$$

A simple application of Gronwall's inequality gives Claim III. □

We have proven that the n-th bubble remains almost invariant for T_n, which is bounded from below by

$$T_n \geq \frac{c_0}{\sum_{j=n_0}^{n-1} j^{-\alpha}} \geq \frac{(1-\alpha)c_0}{n^{1-\alpha}}$$

for all $n \geq N$ with some large N depending only on M, T. Now, we observe that

$$\int_0^{T_n} I_n(t)\, \mathrm{d}t \gtrsim T_n I_n(0) \gtrsim \frac{1}{n},$$

with constants independent of n, recalling that $I_n(0) \gtrsim n^{-\alpha}$. (We shall take α close to $\frac{1}{2}$.) Hence, summation gives

$$\sum_{k=\ell}^{n} I_k(0) T_k \geq c_0\left(\frac{1}{\ell} + \cdots + \frac{1}{n}\right) \geq \log\left(\frac{n}{\ell}\right)^{c_0} \tag{4-8}$$

for some absolute constant $c_0 > 0$, as long as $\ell > N$.

Part 5: norm inflation and conclusion the proof. We are now in a position to complete the proof. For each $\ell > N$ and $n \gg \ell$ (so that $\log(n/\ell)^{c_0} \gg M$), we can bound for $x \in \Omega_n$

$$\log \frac{\widehat{\Phi}_2^n(T_\ell)}{\hat{x}_2^n} \leq CM - 10 \sum_{k=\ell}^{n} I_k(0) T_k \leq \log\left(\frac{n}{\ell}\right)^{-c_0}.$$

In other words, we have the growth

$$\frac{\hat{x}_2^n}{\widehat{\Phi}_2^n(T_\ell)} \geq \left(\frac{n}{\ell}\right)^{c_0}. \tag{4-9}$$

Now, we can write the solution in the form

$$\theta = \sum_{n=n_0}^{\infty} n^{-\alpha} \theta^{(n)}, \quad \theta^{(n)}(t, \Phi(t, x)) = \theta_{0,\mathrm{loc}}^{(n)}(x),$$

so that the support of $\theta^{(n)}$ is disjoint from each other. We now take $t = T_\ell$. Since $\theta^{(n)}(T_\ell, \cdot) = 1$ in a region of area $\gtrsim 4^{-2n}$ and $\theta^{(n')} = 0$ for $n' \neq n$ in that region, with Hardy's inequality and (4-9), we obtain that

$$\|\theta^{(n)}(T_\ell)\|_{\dot{H}^2}^2 \gtrsim \left(\frac{n}{\ell}\right)^{4c_0}.$$

This estimate holds for all sufficiently large n. Then

$$\|\theta(T_\ell)\|_{H^2}^2 \geq \sum_{n \geq n_0} n^{-2\alpha} \|\theta^{(n)}(T_\ell)\|_{\dot{H}^2}^2 \gtrsim \ell^{-4c_0} \sum_{n \gg \ell} n^{4c_0 - 2\alpha}.$$

In the last inequality, since $c_0 > 0$ is an absolute constant, and we could have chosen $\alpha = \frac{1}{2} + c_0$. This gives a contradiction to $\|\theta(T_\ell)\|_{H^2} < \infty$ since $\sum_{n \gg \ell} n^{-1+c_0} = \infty$. □

Remark 4.3. The nonexistence of the solution in $W^{1,\infty}$ is obtained similarly. We define the initial data θ_0 with (4-4) for some $0 < \alpha < \frac{1}{4}$ and repeat the above process with Lemma 3.4 instead of Lemma 3.2. Then we can have an absolute constant $c_0' > 0$ with (4-9). Since this implies

$$\|\theta^{(n)}(T_\ell)\|_{\dot{W}^{1,\infty}} \gtrsim \left(\frac{n}{\ell}\right)^{c_0'},$$

it follows that

$$\|\theta(T_\ell)\|_{W^{1,\infty}} \geq n^{-\alpha} \|\theta^{(n)}(T_\ell)\|_{\dot{W}^{1,\infty}} \gtrsim \ell^{-c_0'} n^{c_0' - \alpha}.$$

Therefore, taking $\alpha = c_0'/2$, we complete the proof.

5. Norm inflation for smooth data

We establish Theorem A in this section, by proving a quantitative norm inflation result for data obtained by truncating the data used in the proof of Theorem B.

Proposition 5.1 (quantitative norm inflation). *We consider the C^∞-smooth initial data*

$$\theta_0^{(N)} := \sum_{n=n_0}^{N} n^{-\alpha} \theta_{0,\mathrm{loc}}^{(n)}, \tag{5-1}$$

where ϕ, α, n_0 are the same as in (4-4). *Then, there exists $N_0 \geq 1$ depending only on ϕ, n_0 such that, for all $N \geq N_0$, the unique local in time C^∞-solution $\theta^{(N)}$ to (SQG) with initial data $\theta_0^{(N)}$ exists on the time interval* $[0, T^*]$ *for some $0 < T^* \leq T_N$ and satisfies*

$$\|\theta_0^{(N)}\|_{H^2 \cap W^{1,\infty}} \leq \epsilon, \qquad \sup_{t \in [0, T^*]} \|\theta^{(N)}(t)\|_{H^2} > M_N, \tag{5-2}$$

where

$$M_N := \frac{c_0}{2} \ln N, \quad T_N := \frac{1}{M_N \ln M_N}, \tag{5-3}$$

with $c_0 > 0$ from (4-8).

Proof. We shall establish the proposition with a contradiction argument: let $0 < T^* \leq +\infty$ be the lifespan of the smooth solution associated with the initial data $\theta_0^{(N)}$ and assume that

$$\|\theta^{(N)}\|_{L^\infty([0, \min\{T^*, T_N\}]; H^2)} \leq M_N.$$

Under this contradiction hypothesis, we can actually prove that $T^* > T_N$, so that

$$\|\theta^{(N)}\|_{L^\infty([0,T_N];H^2)} \le M_N. \tag{5-4}$$

This is simply because the H^2-norm gives a blow-up criterion. To illustrate this point, we estimate the H^3-norm of $\theta := \theta^{(N)}$ on $[0, T_N]$:[3] from the equation for $\Delta\theta$

$$\partial_t \Delta\theta + u\cdot\nabla\Delta\theta + \Delta u\cdot\nabla\theta + 2\sum_{i=1,2}\partial_i u\cdot\nabla\partial_i\theta = 0,$$

we estimate for $j = 1, 2$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_j\Delta\theta\|_{L^2}^2 \le C(\|\nabla u\|_{L^\infty} + \|\nabla\theta\|_{L^\infty})\|\partial_j\Delta\theta\|_{L^2}^2 + C\|\theta\|_{H^2}\|\theta\|_{H^3}^2.$$

Here, we have used L^4 boundedness of the Riesz operator $\theta \mapsto u$ to bound

$$\|\nabla^2 u\|_{L^4} + \|\nabla^2\theta\|_{L^4} \le C\|\theta\|_{H^2}^{1/2}\|\theta\|_{H^3}^{1/2}.$$

Next, we use the logarithmic Sobolev inequality

$$\|\nabla\theta\|_{L^\infty} \le C\|\theta\|_{H^2}\log\left(10 + \frac{\|\theta\|_{H^3}}{\|\theta\|_{H^2}}\right)$$

and

$$\|\nabla u\|_{L^\infty} \le C\|u\|_{H^2}\log\left(10 + \frac{\|u\|_{H^3}}{\|u\|_{H^2}}\right) \le C\|\theta\|_{H^2}\log\left(10 + \frac{\|\theta\|_{H^3}}{\|\theta\|_{H^2}}\right)$$

(we have used the *lower bound* $\|u\|_{H^2} \ge C\|\theta\|_{H^2}$). Lastly, using $\|\theta\|_{H^2} \le M_N$, we may deduce the a priori estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{H^3}^2 \le CM_N\log(10 + \|\theta\|_{H^3})\|\theta\|_{H^3}^2,$$

which shows that the H^3-norm of θ must remain finite up to $t = T_N$. Higher norms of θ can be similarly controlled, so that the solution θ remains C^∞-smooth up to $t = T_N$.

In the following argument, $N_0 \gg n_0$ will be taken to be sufficiently large (but in a way depending only on a few absolute constants) whenever it becomes necessary. Recall that we are assuming $N \ge N_0$. The following argument is mainly a repetition of the proof of Theorem B above. For convenience, let us fix

$$\ell_N := M_N^3. \tag{5-5}$$

Then, note from the definition of M_N in (5-3) that $n_0 \ll \ell_N \ll N$. Here and in the following, we write $A \ll B$ if $A/B \to 0$ as $N \to \infty$, where A and B are some positive expressions involving N.

Observe that the solution θ defined on $[0, T_N]$ satisfies the properties stated in Lemma 4.2 and Claims I, II, III on the *entire* time interval $[0, T_N]$ (by taking N_0 larger if necessary), simply because we have

$$T_N \ll \frac{1}{M_N}$$

[3]For simplicity, from now on we shall refrain from writing out the dependence of the solution θ in N.

from our choice of T_N in (5-3). As in the above, we write the solution in the form

$$\theta = \sum_{n=n_0}^{N} n^{-\alpha}\theta^{(n)}, \quad \theta^{(n)}(t, \Phi(t,x)) = \theta^{(n)}_{0,\mathrm{loc}}(x),$$

and $\theta^{(n)}$ will be referred to as the n-th bubble. Then, for any $\ell_N \le k \le N$, we have that the invariant timescale T_k for the k-th bubble satisfies

$$T_k \le T_N \quad \text{and} \quad T_k \gtrsim \frac{1}{M_N + \sum_{j=n_0}^{k-1} j^{-\alpha}} \gtrsim k^{\alpha-1}.$$

We have used that α is close to $\frac{1}{2}$. Now we consider the values of n satisfying

$$n \ge C\ell_N \exp(c_0^{-1} M_N) \tag{5-6}$$

for a sufficiently large absolute constant $C > 0$. Then, at $t = T_{\ell_N}$, we obtain much as before

$$\|\theta^{(n)}(T_{\ell_N})\|^2_{\dot H^2} \gtrsim \left(\frac{n}{\ell_N}\right)^{2c_0}$$

whenever $n \le N$ satisfies (5-6). Hence

$$\|\theta(T_{\ell_N})\|^2_{H^2} \gtrsim \sum_{n=1+\lfloor C\ell_N \exp(c_0^{-1}M_N)\rfloor}^{N} n^{-2\alpha}\left(\frac{n}{\ell_N}\right)^{2c_0} \gtrsim \ell_N^{-2c_0} N^{1-2\alpha+2c_0} \gg M_N^2,$$

recalling the definitions of M_N and ℓ_N. We have used that $N \gg \ell_N \exp(c_0^{-1}M_N)$ to derive the last inequality. In particular, for all sufficiently large N, we obtain

$$\|\theta(T_{\ell_N})\|_{H^2} > M_N,$$

which is a contradiction. □

Appendix: Key lemma for generalized SQG

We provide a version of the "key lemma" for generalized SQG equations (1-3) with $1 < \alpha < 2$.

Lemma A.1. *Let θ satisfy the assumptions in Lemma 3.2, and let x satisfy $|x| < \frac{1}{4}$ and $x_1 > x_2 > 0$. Then, $u = \nabla^\perp \Lambda^{-\alpha}\theta$ satisfies*

$$\left|\frac{u_1(x)}{x_1} - 4(4-\alpha)\int_{Q(2x)} \frac{y_1 y_2}{|y|^{6-\alpha}}\theta(y)\,\mathrm{d}y\right| \le B_7(x) \tag{A-1}$$

and

$$\left|\frac{u_2(x)}{x_2} + 4(4-\alpha)\int_{Q(2x)} \frac{y_1 y_2}{|y|^{6-\alpha}}\theta(y)\,\mathrm{d}y\right| \le \left(1 + \log\frac{x_1}{x_2}\right)B_8(x) + \left(1 + \log\frac{x_1}{x_2}\right)^{(5-2\alpha)/2} B_9(x), \tag{A-2}$$

where B_7, B_8, B_9 satisfy

$$|B_7(x)| + |B_8(x)| \le C(\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)} + \|\theta\|_{L^\infty([0,1]^2)})$$

and

$$|B_9(x)| \le C(\|\nabla^{3-\alpha}\theta\|_{L^2(S(x))} + \|y_2^{-1}\partial_1\theta(y)\|_{L^2(S(x))}), \quad S(x) := [x_1/2, 4x_1] \times [4x_2, 1].$$

Our proof will be brief, since the structure of the proof is similar to the SQG case. Unfortunately, this argument cannot be specialized to give the lemma in the SQG case, since the case $\alpha = 1$ is critical (being an integer) in some sense.

Proof. We fix a point $x = (x_1, x_2)$ satisfying the assumptions of the lemma. We write

$$u(x) = \sum_{n\in\mathbb{Z}^2} \int_{[0,1]^2} \left(\frac{(x-(y+2n))^\perp}{|x-(y+2n)|^{4-\alpha}} - \frac{(x-(\tilde{y}+2n))^\perp}{|x-(\tilde{y}+2n)|^{4-\alpha}} + \frac{(x-(-y+2n))^\perp}{|x-(-y+2n)|^{4-\alpha}} - \frac{(x-(\bar{y}+2n))^\perp}{|x-(\bar{y}+2n)|^{4-\alpha}} \right) \theta(y)\,\mathrm{d}y.$$

We estimate u_1 first. We introduce

$$I_1(n) := -\int_{[0,1]^2} \left(\frac{x_2-(y_2+2n_2)}{|x-(y+2n)|^{4-\alpha}} - \frac{x_2-(y_2+2n_2)}{|x-(\tilde{y}+2n)|^{4-\alpha}} \right) \theta(y)\,\mathrm{d}y,$$
$$I_2(n) := -\int_{[0,1]^2} \left(\frac{x_2-(-y_2+2n_2)}{|x-(-y+2n)|^{4-\alpha}} - \frac{x_2-(-y_2+2n_2)}{|x-(\bar{y}+2n)|^{4-\alpha}} \right) \theta(y)\,\mathrm{d}y$$

and we see

$$u_1(x) = \sum_{n\in\mathbb{Z}^2} (I_1(n) + I_2(n)).$$

In the case of $n \neq 0$, we have

$$|I_1(n) + I_1(\tilde{n})| + |I_2(n) + I_2(\tilde{n})| \le O(|n|^{-5+\alpha}) \|\theta\|_{L^\infty([0,1]^2)} x_1;$$

hence,

$$\left| \sum_{n\in\mathbb{Z}^2\setminus\{0\}} (I_1(n) + I_2(n)) \right| \le C x_1 \|\theta\|_{L^\infty([0,1]^2)}. \tag{A-3}$$

For $n = 0$, we estimate first

$$I_1(0) = -\int_{[0,1]^2} \left(\frac{x_2-y_2}{|x-y|^{4-\alpha}} - \frac{x_2-y_2}{|x-\tilde{y}|^{4-\alpha}} \right) \theta(y)\,\mathrm{d}y.$$

Using $[0,1]^2 = Q(2x) \cup [0, 4x_1] \times [4x_1, 1] \cup [0, 4x_1]^2$, we estimate the integral for each set.

(i) Suppose $y \in Q(2x)$. In this case, we note

$$-\tfrac{5}{8}|y|^2 \le |x|^2 - 2x\cdot y \le |x|^2 - 2x\cdot\tilde{y} \le \tfrac{5}{8}|y|^2$$

from $-|x|^2 + 2x\cdot y \le 3|x|^2 + \frac{1}{4}|y|^2 \le \frac{3}{8}y_1^2 + \frac{1}{4}|y|^2$ and $|x|^2 + 2x_1y_1 \le \frac{5}{8}y_1^2$. Hence, it holds

$$\left| \frac{|x|^2}{|y|^2} - \frac{2x\cdot y}{|y|^2} \right| \le \frac{5}{8}, \quad \left| \frac{|x|^2}{|y|^2} - \frac{2x\cdot\tilde{y}}{|y|^2} \right| \le \frac{5}{8}.$$

Then, using the Taylor series expansion

$$\frac{1}{(t+1)^{(4-\alpha)/2}} = 1 - \frac{4-\alpha}{2}t + \frac{(4-\alpha)(6-\alpha)}{8}t^2 g(t), \quad -1 < t < 1, \tag{A-4}$$

where g is an analytic function on $(-1,1)$ with $g(0)=1$, we can verify

$$\begin{aligned}\frac{x_2-y_2}{|x-y|^{4-\alpha}}-\frac{x_2-y_2}{|x-\tilde{y}|^{4-\alpha}} &= \frac{x_2-y_2}{|y|^{4-\alpha}}\left(\frac{|x|^2}{|y|^2}-\frac{2x\cdot y}{|y|^2}+1\right)^{-(4-\alpha)/2}-\frac{x_2-y_2}{|y|^{4-\alpha}}\left(\frac{|x|^2}{|y|^2}-\frac{2x\cdot\tilde{y}}{|y|^2}+1\right)^{-(4-\alpha)/2}\\ &= \frac{x_2-y_2}{|y|^{4-\alpha}}\left[2(4-\alpha)\frac{x_1y_1}{|y|^2}+\frac{(4-\alpha)(6-\alpha)}{8}\left\{h\left(\frac{|x|^2}{|y|^2}-\frac{2x\cdot y}{|y|^2}\right)-h\left(\frac{|x|^2}{|y|^2}-\frac{2x\cdot\tilde{y}}{|y|^2}\right)\right\}\right],\end{aligned}$$

where $h(t):=t^2g(t)$. We set

$$f(\tau)=h\left(\frac{|x|^2}{|y|^2}-\frac{2x\cdot(y+(\tau-1)(y-\tilde{y}))}{|y|^2}\right),\quad 0\le\tau\le 1,$$

so that

$$\frac{x_2-y_2}{|x-y|^{4-\alpha}}-\frac{x_2-y_2}{|x-\tilde{y}|^{4-\alpha}}=2(4-\alpha)\frac{x_1y_1(x_2-y_2)}{|y|^{6-\alpha}}+\frac{(4-\alpha)(6-\alpha)}{8}\frac{x_2-y_2}{|y|^{4-\alpha}}(f(1)-f(0)).$$

The mean value theorem and (3-6) imply

$$\begin{aligned}|f(1)-f(0)| &= |f'(\tau)|\\ &= \left|-8\frac{x_1y_1}{|y|^2}\left(\frac{|x|^2}{|y|^2}-\frac{2x\cdot(y+(\tau-1)(y-\tilde{y}))}{|y|^2}\right)g\left(\frac{|x|^2}{|y|^2}-\frac{2x\cdot(y+(\tau-1)(y-\tilde{y}))}{|y|^2}\right)\right.\\ &\qquad\left.-4\frac{x_1y_1}{|y|^2}\left(\frac{|x|^2}{|y|^2}-\frac{2x\cdot(y+(\tau-1)(y-\tilde{y}))}{|y|^2}\right)^2 g'\left(\frac{|x|^2}{|y|^2}-\frac{2x\cdot(y+(\tau-1)(y-\tilde{y}))}{|y|^2}\right)\right|\\ &\le Cx_1\frac{|x|}{|y|^2}.\end{aligned}$$

Thus, we have

$$\frac{x_2-y_2}{|x-y|^{4-\alpha}}-\frac{x_2-y_2}{|x-\tilde{y}|^{4-\alpha}}=-2(4-\alpha)x_1\frac{y_1y_2}{|y|^{6-\alpha}}+x_1|x|O\left(\frac{1}{|y|^{5-\alpha}}\right)$$

and

$$\begin{aligned}&\left|-\frac{1}{x_1}\int_{Q(2x)}\left(\frac{x_2-y_2}{|x-y|^{4-\alpha}}-\frac{x_2-y_2}{|x-\tilde{y}|^{4-\alpha}}\right)\theta(y)\,\mathrm{d}y-2(4-\alpha)\int_{Q(2x)}\frac{y_1y_2}{|y|^{6-\alpha}}\theta(y)\,\mathrm{d}y\right|\\ &\qquad\le C|x|\int_{Q(2x)}\frac{1}{|y|^2}\frac{|\theta(y)|}{|y|^{3-\alpha}}\,\mathrm{d}y\\ &\qquad\le C\||y|^{-(3-\alpha)}\theta\|_{L^2([0,1]^2)}.\end{aligned}$$

For $y_1\ge y_2$, using Lemma 3.1 and [Zhang 2006, Theorem 3.1], we obtain

$$\begin{aligned}\||y|^{-(3-\alpha)}\theta(y)\|_{L^2([0,1]^2)} &\le \||y_1|^{-(2-\alpha)}y_2^{-1}\theta\|_{L^2([0,1]^2)}\\ &\le C\||y|^{-(2-\alpha)}\partial_2\theta(y)\|_{L^2([0,1]^2)}\le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.\end{aligned}$$

Similarly, we can deduce for $y_1\le y_2$ that

$$\||y|^{-(3-\alpha)}\theta(y)\|_{L^2([0,1]^2)}\le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.$$

(ii) Suppose $y \in [0, 4x_1] \times [4x_1, 1]$. In this case, we set $f(\tau) = |x - (y - (1-\tau)(y - \tilde{y}))|^{4-\alpha}$ for $0 \le \tau \le 1$ to see

$$\left| -\frac{1}{x_1} \int_{[0,4x_1]\times[4x_1,1]} \left(\frac{x_2-y_2}{|x-y|^{4-\alpha}} - \frac{x_2-y_2}{|x-\tilde{y}|^{4-\alpha}} \right) \theta(y)\,\mathrm{d}y \right| \le \frac{1}{x_1} \int_{[0,4x_1]\times[4x_1,1]} \frac{|x_2-y_2|\,|f(1)-f(0)|}{|x-y|^{4-\alpha}|x-\tilde{y}|^{4-\alpha}} |\theta(y)|\,\mathrm{d}y.$$

The mean value theorem implies

$$\begin{aligned} f(1) - f(0) = f'(\tau) &= \frac{4-\alpha}{2}(-4y_1(x_1+y_1) + 8\tau y_1^2)|\tau(x-y) + (1-\tau)(x-\tilde{y})|^{2-\alpha} \\ &= -\frac{4-\alpha}{2} 4y_1(\tau(x_1-y_1) + (1-\tau)(x_1+y_1))|\tau(x-y) + (1-\tau)(x-\tilde{y})|^{2-\alpha}. \end{aligned}$$

Applying $y_1 \le 4x_1$ and

$$\tfrac{1}{2}y_2 \le |x-y| \le 2y_2, \quad \tfrac{1}{2}y_2 \le |x-\tilde{y}| \le 2y_2,$$

we obtain

$$\frac{|x_2-y_2|\,|f(1)-f(0)|}{|x-y|^{4-\alpha}|x-\tilde{y}|^{4-\alpha}} \le C\frac{x_1}{y_2^{4-\alpha}}.$$

Thus, it follows

$$\begin{aligned} &\left| -\frac{1}{x_1} \int_{[0,4x_1]\times[4x_1,1]} \left(\frac{x_2-y_2}{|x-y|^{4-\alpha}} - \frac{x_2-y_2}{|x-\tilde{y}|^{4-\alpha}} \right) \theta(y)\,\mathrm{d}y \right| \\ &\qquad \le C \int_{[0,4x_1]\times[4x_1,1]} \frac{1}{y_2} \frac{\theta(y)}{y_2^{3-\alpha}}\,\mathrm{d}y \\ &\qquad \le C\| |y|^{-(3-\alpha)}\theta(y)\|_{L^2([0,1]^2)} \le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}. \end{aligned} \tag{A-5}$$

(iii) Suppose $y \in [0, 4x_1]^2$. Due to $\theta(y_1, 0) = 0$, using integration by parts gives

$$\begin{aligned} &-\frac{1}{x_1} \int_{[0,4x_1]^2} \left(\frac{x_2-y_2}{|x-y|^{4-\alpha}} - \frac{x_2-y_2}{|x-\tilde{y}|^{4-\alpha}} \right) \theta(y)\,\mathrm{d}y \\ &\quad = \frac{1}{(2-\alpha)x_1} \int_{[0,4x_1]^2} \left(\frac{1}{|x-y|^{2-\alpha}} - \frac{1}{|x-\tilde{y}|^{2-\alpha}} \right) \partial_2\theta(y)\,\mathrm{d}y \\ &\qquad - \frac{1}{(2-\alpha)x_1} \int_0^{4x_1} \left(\frac{1}{|(x_1-y_1, x_2-4x_1)|^{2-\alpha}} - \frac{1}{|(x_1+y_1, x_2-4x_1)|^{2-\alpha}} \right) \theta(y_1, 4x_1)\,\mathrm{d}y_1. \end{aligned}$$

By Hölder's inequality we estimate the second integral as

$$\begin{aligned} &\left| -\frac{1}{(2-\alpha)x_1} \int_0^{4x_1} \left(\frac{1}{|(x_1-y_1, x_2-4x_1)|^{2-\alpha}} - \frac{1}{|(x_1+y_1, x_2-4x_1)|^{2-\alpha}} \right) \theta(y_1, 4x_1)\,\mathrm{d}y_1 \right| \\ &\qquad \le Cx_1^{-(3-\alpha)} \int_0^{4x_1} y_1^{2-\alpha} |y_1^{-(2-\alpha)}\theta(y_1, 4x_1)|\,\mathrm{d}y_1 \le Cx_1^{-1/2} \|y_1^{-(2-\alpha)}\theta(y_1, 4x_1)\|_{L^2(0,4x_1)}. \end{aligned}$$

Since θ vanishes near the axis, it follows

$$|\theta(y_1, 4x_1)| \le \int_0^{4x_1} |\partial_2\theta(y_1, \tau)|\,\mathrm{d}\tau \le (4x_1)^{1/2} \|\partial_2\theta(y_1, \cdot\,)\|_{L^2(0,4x_1)}$$

for all $y_1 \in [0, 4x_1]$. Then, with [Zhang 2006, Theorem 3.1], we obtain

$$\left|-\frac{1}{(2-\alpha)x_1}\int_0^{2x_1}\left(\frac{1}{|(x_1-y_1, x_2-2x_1)|^{2-\alpha}}-\frac{1}{|(x_1+y_1, x_2-2x_1)|^{2-\alpha}}\right)\theta(y_1, 2x_1)\,\mathrm{d}y_1\right| \\ \le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.$$

On the other hand, using Hölder's inequality, we have

$$\left|\frac{1}{(2-\alpha)x_1}\int_{[0,4x_1]^2}\left(\frac{1}{|x-y|^{2-\alpha}}-\frac{1}{|x-\tilde{y}|^{2-\alpha}}\right)\partial_2\theta(y)\,\mathrm{d}y\right| \\ \le \frac{C}{x_1}\int_{[0,4x_1]^2}\frac{|\partial_2\theta(y)|}{|x-y|^{2-\alpha}}\,\mathrm{d}y \le \frac{C}{x_1}\left(\int_0^{8x_1} r^{-(3-2\alpha)}\,\mathrm{d}r\right)^{1/2}\|\partial_2\theta\|_{L^2([0,4x_1]^2)} \\ \le Cx_1^{-(2-\alpha)}\|\nabla\theta\|_{L^2([0,4x_1]^2)}.$$

Therefore, we can deduce

$$\left|\frac{1}{(2-\alpha)x_1}\int_{[0,4x_1]^2}\left(\frac{1}{|x-y|^{2-\alpha}}-\frac{1}{|x-\tilde{y}|^{2-\alpha}}\right)\partial_2\theta(y)\,\mathrm{d}y\right| \le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.$$

Combining the above estimates, we obtain

$$\left|-\frac{1}{x_1}\int_{[0,2x_1]^2}\left(\frac{x_2-y_2}{|x-y|^{4-\alpha}}-\frac{x_2-y_2}{|x-\tilde{y}|^{4-\alpha}}\right)\theta(y)\,\mathrm{d}y\right| \le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.$$

We collect the estimates for each region and deduce that

$$\left|\frac{I_1(0)}{x_1}-2(4-\alpha)\int_{Q(x)}\frac{y_1y_2}{|y|^{6-\alpha}}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.$$

Similarly, we can show

$$\left|\frac{I_2(0)}{x_1}-2(4-\alpha)\int_{Q(x)}\frac{y_1y_2}{|y|^{6-\alpha}}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.$$

We omit the details. Thus we have (A-1).

Now we estimate u_2. Note that

$$u_2(x) = \sum_{n\in\mathbb{Z}^2}(I_3(n)+I_4(n)),$$

where

$$I_3(n) := \int_{[0,1]^2}\left(\frac{x_1-(y_1+2n_1)}{|x-(y+2n)|^{4-\alpha}}-\frac{x_1-(y_1+2n_1)}{|x-(\bar{y}+2n)|^{4-\alpha}}\right)\theta(y)\,\mathrm{d}y,$$
$$I_4(n) := \int_{[0,1]^2}\left(\frac{x_1-(-y_1+2n_1)}{|x-(-y+2n)|^{4-\alpha}}-\frac{x_1-(-y_1+2n_1)}{|x-(\tilde{y}+2n)|^{4-\alpha}}\right)\theta(y)\,\mathrm{d}y.$$

Since we can similarly see that

$$\left|\sum_{n\in\mathbb{Z}^2\setminus\{0\}}(I_3(n)+I_4(n))\right| \le Cx_2\|\theta\|_{L^\infty([0,1]^2)},$$

it suffices to estimate for $n = 0$. We estimate $I_3(0)$ by dividing the domain into four regions as $[0, 1]^2 = Q(2x) \cup [0, x_1/2] \times [0, 4x_1] \cup [x_1/2, 4x_1] \times [0, 4x_1] \cup [0, 4x_1] \times [4x_1, 1]$.

(i) Suppose $y \in Q(2x)$. Then we can see

$$\left|\frac{|x|^2}{|y|^2} - \frac{2x \cdot \bar{y}}{|y|^2}\right| \le \frac{5}{8}$$

by $|x|^2 + 2x_2 y_2 \le |x|^2 + 4x_2^2 + \frac{1}{4}y_2^2 \le \frac{3}{8}y_1^2 + \frac{1}{4}y_2^2$. With (A-4) we can prove

$$\left|\frac{1}{x_2}\int_{Q(2x)}\left(\frac{x_1-y_1}{|x-y|^{4-\alpha}} - \frac{x_1-y_1}{|x-\bar{y}|^{4-\alpha}}\right)\theta(y)\,\mathrm{d}y + 2(4-\alpha)\int_{Q(2x)}\frac{y_1 y_2}{|y|^{6-\alpha}}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.$$

(ii) Suppose $y \in [0, 4x_1] \times [4x_1, 1]$. Then it follows

$$\tfrac{1}{2}y_2 \le |x-y| \le 2y_2, \quad y_2 \le |x-\bar{y}| \le 2y_2.$$

Using it, we can show

$$\left|\frac{1}{x_2}\int_{[0,4x_1]\times[4x_1,1]}\left(\frac{x_1-y_1}{|x-y|^{4-\alpha}} - \frac{x_1-y_1}{|x-\bar{y}|^{4-\alpha}}\right)\theta(y)\,\mathrm{d}y\right| \le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}$$

in a way similar to how we obtained (A-5).

(iii) Suppose $y \in [0, x_1/2] \times [0, 4x_1]$. We set

$$f(\tau) = |x - (y - (1-\tau)(y-\bar{y}))|^{4-\alpha}, \quad 0 \le \tau \le 1.$$

Then, we can see

$$\left|\frac{1}{x_2}\int_{[0,x_1/2]\times[0,4x_1]}\left(\frac{x_1-y_1}{|x-y|^{4-\alpha}} - \frac{x_1-y_1}{|x-\bar{y}|^{4-\alpha}}\right)\theta(y)\,\mathrm{d}y\right| \le \frac{1}{x_2}\int_{[0,x_1/2]\times[0,4x_1]}\frac{|x_1-y_1|\,|f(1)-f(0)|}{|x-y|^{4-\alpha}|x-\bar{y}|^{4-\alpha}}|\theta(y)|\,\mathrm{d}y.$$

Since the mean value theorem implies

$$f(1) - f(0) = f'(\tau) = -\frac{4-\alpha}{2}4y_2(\tau(x_2-y_2) + (1-\tau)(x_2+y_2))|\tau(x-y) + (1-\tau)(x-\tilde{y})|^{2-\alpha},$$

with $y_2 \le 4x_1$ and

$$\tfrac{1}{2}x_1 \le |x-y| \le 8x_1, \quad \tfrac{1}{2}x_1 \le |x-\bar{y}| \le 8x_1,$$

we have

$$\frac{|x_1-y_1|\,|f(1)-f(0)|}{|x-y|^{4-\alpha}|x-\bar{y}|^{4-\alpha}} \le C\frac{x_2}{x_1^{4-\alpha}}.$$

Thus, we can obtain

$$\begin{aligned}\left|\frac{1}{x_2}\int_{[0,x_1/2]\times[0,4x_1]}\left(\frac{x_1-y_1}{|x-y|^{4-\alpha}} - \frac{x_1-y_1}{|x-\bar{y}|^{4-\alpha}}\right)\theta(y)\,\mathrm{d}y\right| &\le \frac{C}{x_1^{4-\alpha}}\int_{[0,x_1/2]\times[0,4x_1]}|\theta(y)|\,\mathrm{d}y \\ &\le C\||y|^{-(3-\alpha)}\theta(y)\|_{L^2([0,1]^2)} \le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1^2])}.\end{aligned}$$

(iv) Suppose $y \in [x_1/2, 4x_1] \times [0, 4x_1]$. Integration by parts and $\theta(0, y_2) = 0$ give

$$\begin{aligned}\frac{1}{x_2}\int_{[x_1/2,4x_1]\times[0,4x_1]}\left(\frac{x_1-y_1}{|x-y|^{4-\alpha}}-\frac{x_1-y_1}{|x-\bar{y}|^{4-\alpha}}\right)\theta(y)\,\mathrm{d}y \\ = -\frac{1}{(2-\alpha)x_2}\int_{[x_1/2,4x_1]\times[0,4x_1]}\left(\frac{1}{|x-y|^{2-\alpha}}-\frac{1}{|x-\bar{y}|^{2-\alpha}}\right)\partial_1\theta(y)\,\mathrm{d}y, \\ +\frac{1}{(2-\alpha)x_2}\int_0^{4x_1}\left(\frac{1}{|(3x_1, x_2-y_2)|^{2-\alpha}}-\frac{1}{|(3x_1, x_2+y_2)|^{2-\alpha}}\right)\theta(4x_1, y_2)\,\mathrm{d}y_2 \\ -\frac{1}{(2-\alpha)x_2}\int_0^{4x_1}\left(\frac{1}{|(x_1/2, x_2-y_2)|^{2-\alpha}}-\frac{1}{|(x_1/2, x_2+y_2)|^{2-\alpha}}\right)\theta(x_1/2, y_2)\,\mathrm{d}y_2.\end{aligned}$$

To estimate the second integral on the right-hand side first, we set

$$f(\tau) = |(3x_1, x_2 - (y_2 - 2(1-\tau)y_2))|^{2-\alpha}, \quad 0 \le \tau \le 1,$$

so that

$$\begin{aligned}\left|\frac{1}{(2-\alpha)x_2}\int_0^{4x_1}\left(\frac{1}{|(3x_1, x_2-y_2)|^{2-\alpha}}-\frac{1}{|(3x_1, x_2+y_2)|^{2-\alpha}}\right)\theta(4x_1, y_2)\,\mathrm{d}y_2\right| \\ \le \frac{C}{x_2}\int_0^{4x_1}\frac{|f(1)-f(0)|}{|(3x_1, x_2-y_2)|^{2-\alpha}|(3x_1, x_2+y_2)|^{2-\alpha}}|\theta(4x_1, y_2)|\,\mathrm{d}y_2.\end{aligned}$$

Using the mean value theorem

$$f(1) - f(0) = f'(\tau) = \frac{2-\alpha}{2}4y_2(\tau(x_2-y_2)+(1-\tau)(x_2+y_2))|(3x_1, x_2-(y_2-2(1-\tau)y_2))|^{-\alpha},$$

we have

$$\frac{|f(1)-f(0)|}{|(3x_1, x_2-y_2)|^{2-\alpha}|(3x_1, x_2+y_2)|^{2-\alpha}} \le C\frac{x_2}{x_1^{3-\alpha}}.$$

With the simple inequality

$$|\theta(4x_1, y_2)| \le \int_0^{4x_1}|\partial_1\theta(\tau, y_2)|\,\mathrm{d}\tau \le (4x_1)^{1/2}\|\partial_1\theta(\,\cdot\,, y_2)\|_{L^2(0,4x_1)},$$

we obtain

$$\begin{aligned}\left|\frac{1}{(2-\alpha)x_2}\int_0^{4x_1}\left(\frac{1}{|(3x_1, x_2-y_2)|^{2-\alpha}}-\frac{1}{|(3x_1, x_2+y_2)|^{2-\alpha}}\right)\theta(4x_1, y_2)\,\mathrm{d}y_2\right| \\ \le \frac{C}{x_1^{2-\alpha}}\|\partial_1\theta\|_{L^2([0,4x_1]^2)} \le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.\end{aligned}$$

Similarly, we can show

$$\left|\frac{1}{(2-\alpha)x_2}\int_0^{4x_1}\left(\frac{1}{|(x_1/2, x_2-y_2)|^{2-\alpha}}-\frac{1}{|(x_1/2, x_2+y_2)|^{2-\alpha}}\right)\theta(x_1/2, y_2)\,\mathrm{d}y_2\right| \le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.$$

We omit the details. Now, consider $f(\tau) = |x - (y - (1-\tau)(y-\bar{y}))|^{2-\alpha}$, $0 \le \tau \le 1$, and note that

$$\begin{aligned}\left|\frac{1}{(2-\alpha)x_2}\int_{[x_1/2,4x_1]\times[0,4x_1]}\left(\frac{1}{|x-y|^{2-\alpha}}-\frac{1}{|x-\bar{y}|^{2-\alpha}}\right)\partial_1\theta(y)\,\mathrm{d}y\right| \\ \le \frac{C}{x_2}\int_{[x_1/2,4x_1]\times[0,4x_1]}\frac{|f(1)-f(0)|}{|x-y|^{2-\alpha}|x-\bar{y}|^{2-\alpha}}|\partial_1\theta(y)|\,\mathrm{d}y.\end{aligned}$$

Since the mean value theorem gives

$$f(1)-f(0)=f'(\tau)=\frac{2-\alpha}{2}4y_2(\tau(x_2-y_2)+(1-\tau)(x_2+y_2))|\tau(x-y)+(1-\tau)(x-\bar{y})|^{-\alpha},$$

it follows

$$\frac{|f(1)-f(0)|}{|x-y|^{2-\alpha}|x-\bar{y}|^{2-\alpha}}\le\frac{Cx_2}{|x-y|^{2-\alpha}|x-\bar{y}|}.$$

Hence,

$$\left|\frac{1}{(2-\alpha)x_2}\int_{[x_1/2,4x_1]\times[0,4x_1]}\left(\frac{1}{|x-y|^{2-\alpha}}-\frac{1}{|x-\bar{y}|^{2-\alpha}}\right)\partial_1\theta(y)\,\mathrm{d}y\right| \le C\int_{[x_1/2,4x_1]\times[0,4x_1]}\frac{1}{|x-y|^{2-\alpha}|x-\bar{y}|}|\partial_1\theta(y)|\,\mathrm{d}y.$$

By Fubini's theorem and Hölder's inequality, we have

$$\begin{aligned}\int_{[x_1/2,4x_1]\times[0,4x_1]}&\frac{1}{|x-y|^{2-\alpha}|x-\bar{y}|}|\partial_1\theta(y)|\,\mathrm{d}y\\ &=\int_0^{4x_1}\frac{1}{x_2+y_2}\int_{x_1/2}^{4x_1}\frac{1}{|x-y|^{2-\alpha}}|\partial_1\theta(y)|\,\mathrm{d}y_1\,\mathrm{d}y_2\\ &\le\int_0^{4x_1}\frac{1}{x_2+y_2}\|\partial_1\theta\|_{L^{1/(\alpha-1)}(x_1/2,4x_1)}\left(\int_0^{4x_1}\frac{1}{|x-y|}\,\mathrm{d}y_1\right)^{2-\alpha}\mathrm{d}y_2\\ &\le C\int_0^{4x_1}\frac{1}{x_2+y_2}\|\partial_1\theta\|_{L^{1/(\alpha-1)}(x_1/2,4x_1)}\left|\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\right|^{2-\alpha}\mathrm{d}y_2.\end{aligned}$$

The Gagliardo–Nirenberg interpolation inequality and $y_2\le 4x_1$ yield

$$y_2^{-1/2}\|\partial_1\theta(\,\cdot\,,y_2)\|_{L^{1/(\alpha-1)}(x_1/2,4x_1)}\le C(\|\partial_1^{3-\alpha}\theta(\,\cdot\,,y_2)\|_{L^2(x_1/2,4x_1)}+y_2^{-(2-\alpha)}\|\partial_1\theta(\,\cdot\,,y_2)\|_{L^2(x_1/2,4x_1)}).$$

Then, we can have

$$\begin{aligned}\int_0^{4x_1}&\frac{1}{x_2+y_2}\|\partial_1\theta\|_{L^{1/(\alpha-1)}(x_1/2,4x_1)}\left(\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\right)^{2-\alpha}\mathrm{d}y_2\\ &\le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}\left(\int_0^{4x_2}\frac{1}{x_2+y_2}\left|\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\right|^{2(2-\alpha)}\mathrm{d}y_2\right)^{1/2}\\ &\quad+C(\|\nabla^{3-\alpha}\theta\|_{L^2(S(x))}+\|y_2^{-(2-\alpha)}\partial_1\theta\|_{L^2(S(x))})\left(\int_{4x_2}^{4x_1}\frac{1}{x_2+y_2}\left|\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\right|^{2(2-\alpha)}\mathrm{d}y_2\right)^{1/2}.\end{aligned}$$

As estimating K_{11} and K_{12}, we can show

$$\begin{aligned}\int_0^{4x_2}\frac{1}{x_2+y_2}\left|\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\right|^{2(2-\alpha)}\mathrm{d}y_2&\le C\left(1+\log\frac{x_1}{x_2}\right)^2,\\ \int_{4x_2}^{4x_1}\frac{1}{x_2+y_2}\left|\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\right|^{2(2-\alpha)}\mathrm{d}y_2&\le C\left(1+\log\frac{x_1}{x_2}\right)^{5-2\alpha}.\end{aligned}$$

This implies

$$\begin{aligned}&\left|\frac{1}{(2-\alpha)x_2}\int_{[x_1/2,4x_1]\times[0,4x_1]}\left(\frac{1}{|x-y|^{2-\alpha}}-\frac{1}{|x-\bar{y}|^{2-\alpha}}\right)\partial_1\theta(y)\,\mathrm{d}y\right|\\ &\le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right)+C(\|\nabla^{3-\alpha}\theta\|_{L^2(S(x))}+\|y_2^{-(2-\alpha)}\partial_1\theta\|_{L^2(S(x))})\left(1+\log\frac{x_1}{x_2}\right)^{(5-2\alpha)/2}.\end{aligned}$$

Collecting the above estimates gives

$$\left|\frac{I_3(0)}{x_2}+2(4-\alpha)\int_{Q(x)}\frac{y_1y_2}{|y|^{6-\alpha}}\theta(y)\,\mathrm{d}y\right|$$
$$\leq C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right)+C(\|\nabla^{3-\alpha}\theta\|_{L^2(S(x))}+\|y_2^{-(2-\alpha)}\partial_1\theta\|_{L^2(S(x))})\left(1+\log\frac{x_1}{x_2}\right)^{(5-2\alpha)/2},$$

and we can similarly obtain

$$\left|\frac{I_4(0)}{x_2}+2(4-\alpha)\int_{Q(x)}\frac{y_1y_2}{|y|^{6-\alpha}}\theta(y)\,\mathrm{d}y\right|$$
$$\leq C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right)+C(\|\nabla^{3-\alpha}\theta\|_{L^2(S(x))}+\|y_2^{-(2-\alpha)}\partial_1\theta\|_{L^2(S(x))})\left(1+\log\frac{x_1}{x_2}\right)^{(5-2\alpha)/2}.$$

Hence we have (3-2), and this completes the proof. □

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Around the same time, Cordoba and Martinez-Zoroa [2022] proved similar strong ill-posedness results for SQG in $\mathbb{R}^2$. They proved ill-posedness for Sobolev spaces below H^2 as well. We point out that in the case of $\mathbb{R}^2$ (unlike $\mathbb{T}^2$), it is not too difficult to pass from norm inflation to nonexistence since one can keep adding “bubbles” which gives growth further away from previous ones.

References

[Azzam and Bedrossian 2015] J. Azzam and J. Bedrossian, “Bounded mean oscillation and the uniqueness of active scalar equations”, *Trans. Amer. Math. Soc.* **367**:5 (2015), 3095–3118. MR Zbl

[Bae and Granero-Belinchón 2015] H. Bae and R. Granero-Belinchón, “Global existence for some transport equations with nonlocal velocity”, *Adv. Math.* **269** (2015), 197–219. MR Zbl

[Bahouri and Chemin 1994] H. Bahouri and J.-Y. Chemin, “Équations de transport relatives á des champs de vecteurs non-lipschitziens et mécanique des fluides”, *Arch. Ration. Mech. Anal.* **127**:2 (1994), 159–181. MR Zbl

[Bourgain and Li 2015] J. Bourgain and D. Li, “Strong ill-posedness of the incompressible Euler equation in borderline Sobolev spaces”, *Invent. Math.* **201**:1 (2015), 97–157. MR Zbl

[Bourgain and Li 2021] J. Bourgain and D. Li, “Strong ill-posedness of the 3D incompressible Euler equation in borderline spaces”, *Int. Math. Res. Not.* **2021**:16 (2021), 12155–12264. MR Zbl

[Buckmaster et al. 2019] T. Buckmaster, S. Shkoller, and V. Vicol, “Nonuniqueness of weak solutions to the SQG equation”, *Comm. Pure Appl. Math.* **72**:9 (2019), 1809–1874. MR Zbl

[Cao et al. 2023] D. Cao, G. Qin, W. Zhan, and C. Zou, “Existence and stability of smooth traveling circular pairs for the generalized surface quasi-geostrophic equation”, *Int. Math. Res. Not.* **2023**:6 (2023), 4761–4804. MR Zbl

[Castro et al. 2016] A. Castro, D. Córdoba, and J. Gómez-Serrano, “Existence and regularity of rotating global solutions for the generalized surface quasi-geostrophic equations”, *Duke Math. J.* **165**:5 (2016), 935–984. MR Zbl

[Chae and Wu 2012] D. Chae and J. Wu, "Logarithmically regularized inviscid models in borderline Sobolev spaces", *J. Math. Phys.* **53**:11 (2012), art. id. 115601. MR Zbl

[Chae et al. 2011] D. Chae, P. Constantin, and J. Wu, "Inviscid models generalizing the two-dimensional Euler and the surface quasi-geostrophic equations", *Arch. Ration. Mech. Anal.* **202**:1 (2011), 35–62. MR Zbl

[Cheng et al. 2021] X. Cheng, H. Kwon, and D. Li, "Non-uniqueness of steady-state weak solutions to the surface quasi-geostrophic equations", *Comm. Math. Phys.* **388**:3 (2021), 1281–1295. MR Zbl

[Choi and Jeong 2021] K. Choi and I.-J. Jeong, "On vortex stretching for anti-parallel axisymmetric flows", preprint, 2021. arXiv 2110.09079

[Córdoba and Martínez-Zoroa 2022] D. Córdoba and L. Martínez-Zoroa, "Non existence and strong ill-posedness in C^k and Sobolev spaces for SQG", *Adv. Math.* **407** (2022), art. id. 108570. MR Zbl

[Córdoba et al. 2018] A. Córdoba, D. Córdoba, and F. Gancedo, "Uniqueness for SQG patch solutions", *Trans. Amer. Math. Soc. Ser. B* **5** (2018), 1–31. MR Zbl

[Denisov 2009] S. A. Denisov, "Infinite superlinear growth of the gradient for the two-dimensional Euler equation", *Discrete Contin. Dyn. Syst.* **23**:3 (2009), 755–764. MR Zbl

[Denisov 2015a] S. A. Denisov, "The centrally symmetric V-states for active scalar equations: two-dimensional Euler with cut-off", *Comm. Math. Phys.* **337**:2 (2015), 955–1009. MR Zbl

[Denisov 2015b] S. A. Denisov, "Double exponential growth of the vorticity gradient for the two-dimensional Euler equation", *Proc. Amer. Math. Soc.* **143**:3 (2015), 1199–1210. MR Zbl

[Dong and Li 2010] H. Dong and D. Li, "On the 2D critical and supercritical dissipative quasi-geostrophic equation in Besov spaces", *J. Differential Equations* **248**:11 (2010), 2684–2702. MR Zbl

[Elgindi 2021] T. Elgindi, "Finite-time singularity formation for $C^{1,\alpha}$ solutions to the incompressible Euler equations on $\mathbb{R}^3$", *Ann. of Math.* (2) **194**:3 (2021), 647–727. MR Zbl

[Elgindi and Jeong 2017] T. M. Elgindi and I.-J. Jeong, "Ill-posedness for the incompressible Euler equations in critical Sobolev spaces", *Ann. PDE* **3**:1 (2017), art. id. 7. MR Zbl

[Elgindi and Jeong 2019] T. M. Elgindi and I.-J. Jeong, "Finite-time singularity formation for strong solutions to the axi-symmetric 3D Euler equations", *Ann. PDE* **5**:2 (2019), art. id. 16. MR Zbl

[Elgindi and Jeong 2020a] T. M. Elgindi and I.-J. Jeong, "On singular vortex patches, II: Long-time dynamics", *Trans. Amer. Math. Soc.* **373**:9 (2020), 6757–6775. MR Zbl

[Elgindi and Jeong 2020b] T. M. Elgindi and I.-J. Jeong, "Symmetries and critical phenomena in fluids", *Comm. Pure Appl. Math.* **73**:2 (2020), 257–316. MR Zbl

[Elgindi and Jeong 2023] T. Elgindi and I.-J. Jeong, *On singular vortex patches, I: Well-posedness issues*, Mem. Amer. Math. Soc. **1400**, Amer. Math. Soc., Providence, RI, 2023. MR Zbl

[Elgindi and Masmoudi 2020] T. M. Elgindi and N. Masmoudi, "L^∞ ill-posedness for a class of equations arising in hydrodynamics", *Arch. Ration. Mech. Anal.* **235**:3 (2020), 1979–2025. MR Zbl

[Földes and Sy 2021] J. Földes and M. Sy, "Invariant measures and global well posedness for the SQG equation", *Arch. Ration. Mech. Anal.* **241**:1 (2021), 187–230. MR Zbl

[Gancedo and Patel 2021] F. Gancedo and N. Patel, "On the local existence and blow-up for generalized SQG patches", *Ann. PDE* **7**:1 (2021), art. id. 4. MR Zbl

[Hassainia and Hmidi 2015] Z. Hassainia and T. Hmidi, "On the V-states for the generalized quasi-geostrophic equations", *Comm. Math. Phys.* **337**:1 (2015), 321–377. MR Zbl

[He and Kiselev 2021] S. He and A. Kiselev, "Small-scale creation for solutions of the SQG equation", *Duke Math. J.* **170**:5 (2021), 1027–1041. MR Zbl

[de la Hoz et al. 2016] F. de la Hoz, Z. Hassainia, and T. Hmidi, "Doubly connected V-states for the generalized surface quasi-geostrophic equations", *Arch. Ration. Mech. Anal.* **220**:3 (2016), 1209–1281. MR Zbl

[Isett and Ma 2021] P. Isett and A. Ma, "A direct approach to nonuniqueness and failure of compactness for the SQG equation", *Nonlinearity* **34**:5 (2021), 3122–3162. MR Zbl

[Jeong 2021] I.-J. Jeong, "Loss of regularity for the 2D Euler equations", *J. Math. Fluid Mech.* **23**:4 (2021), art. id. 95. MR Zbl

[Jeong and Yoneda 2021] I.-J. Jeong and T. Yoneda, "Enstrophy dissipation and vortex thinning for the incompressible 2D Navier–Stokes equations", *Nonlinearity* **34**:4 (2021), 1837–1853. MR

[Jolly et al. 2021] M. S. Jolly, A. Kumar, and V. R. Martinez, "On the existence, uniqueness, and smoothing of solutions to the generalized SQG equations in critical Sobolev spaces", *Comm. Math. Phys.* **387**:1 (2021), 551–596. MR Zbl

[Jolly et al. 2022] M. S. Jolly, A. Kumar, and V. R. Martinez, "On local well-posedness of logarithmic inviscid regularizations of generalized SQG equations in borderline Sobolev spaces", *Commun. Pure Appl. Anal.* **21**:1 (2022), 101–120. MR Zbl

[Kato and Ponce 1988] T. Kato and G. Ponce, "Commutator estimates and the Euler and Navier–Stokes equations", *Comm. Pure Appl. Math.* **41**:7 (1988), 891–907. MR Zbl

[Kiselev and Šverák 2014] A. Kiselev and V. Šverák, "Small scale creation for solutions of the incompressible two-dimensional Euler equation", *Ann. of Math.* (2) **180**:3 (2014), 1205–1220. MR Zbl

[Kiselev et al. 2016] A. Kiselev, L. Ryzhik, Y. Yao, and A. Zlatoš, "Finite time singularity for the modified SQG patch equation", *Ann. of Math.* (2) **184**:3 (2016), 909–948. MR Zbl

[Kwon 2021] H. Kwon, "Strong ill-posedness of logarithmically regularized 2D Euler equations in the borderline Sobolev space", *J. Funct. Anal.* **280**:7 (2021), art. id. 108822. MR Zbl

[Li 2009] D. Li, "Existence theorems for the 2D quasi-geostrophic equation with plane wave initial conditions", *Nonlinearity* **22**:7 (2009), 1639–1651. MR Zbl

[Li 2021] D. Li, "Optimal Gevrey regularity for supercritical quasi-geostrophic equations", preprint, 2021. arXiv 2106.12439

[Majda and Bertozzi 2002] A. J. Majda and A. L. Bertozzi, *Vorticity and incompressible flow*, Cambridge Texts Appl. Math. **27**, Cambridge Univ. Press, 2002. MR Zbl

[Marchand 2008] F. Marchand, "Existence and regularity of weak solutions to the quasi-geostrophic equations in the spaces L^p or $\dot{H}^{-1/2}$", *Comm. Math. Phys.* **277**:1 (2008), 45–67. MR Zbl

[Marchioro and Pulvirenti 1994] C. Marchioro and M. Pulvirenti, *Mathematical theory of incompressible nonviscous fluids*, Appl. Math. Sci. **96**, Springer, 1994. MR Zbl

[Miura 2006] H. Miura, "Dissipative quasi-geostrophic equation for large initial data in the critical Sobolev space", *Comm. Math. Phys.* **267**:1 (2006), 141–157. MR Zbl

[Resnick 1995] S. G. Resnick, *Dynamical problems in non-linear advective partial differential equations*, Ph.D. thesis, University of Chicago, 1995, available at https://www.proquest.com/docview/304242616.

[Vishik 1999] M. Vishik, "Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type", *Ann. Sci. École Norm. Sup.* (4) **32**:6 (1999), 769–812. MR Zbl

[Xu 2016] X. Xu, "Fast growth of the vorticity gradient in symmetric smooth domains for 2D incompressible ideal flow", *J. Math. Anal. Appl.* **439**:2 (2016), 594–607. MR Zbl

[Yudovich 1963] V. I. Yudovich, "Non-stationary flows of an ideal incompressible fluid", *Zh. Vychisl. Mat. Mat. Fiz.* **3** (1963), 1032–1066. In Russian; translated in *USSR Computat. Math. Math. Phys.* **3**:6 (1963), 1407–1456. MR

[Yudovich 1995] V. I. Yudovich, "Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid", *Math. Res. Lett.* **2**:1 (1995), 27–38. MR Zbl

[Zhang 2006] J. Zhang, "Extensions of Hardy inequality", *J. Inequal. Appl.* **2006** (2006), art. id. 69379. MR Zbl

[Zlatoš 2015] A. Zlatoš, "Exponential growth of the vorticity gradient for the Euler equation on the torus", *Adv. Math.* **268** (2015), 396–403. MR Zbl

[Zlatoš 2018] A. Zlatoš, "On the rate of merging of vorticity level sets for the 2D Euler equations", *J. Nonlinear Sci.* **28**:6 (2018), 2329–2341. MR Zbl

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