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ON A FAMILY OF FULLY NONLINEAR INTEGRODIFFERENTIAL OPERATORS: FROM FRACTIONAL LAPLACIAN TO NONLOCAL MONGE–AMPÈRE

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We introduce a new family of intermediate operators between the fractional Laplacian and the nonlocal Monge–Ampère introduced by Caffarelli and Silvestre that are given by infimums of integrodifferential operators. Using rearrangement techniques, we obtain representation formulas and give a connection to optimal transport. Finally, we consider a global Poisson problem prescribing data at infinity, and prove existence, uniqueness, and $C^{1,1}$-regularity of solutions in the full space.

1. Introduction

Integro-differential equations arise in the study of stochastic processes with jumps, such as Lévy processes. A classical elliptic integrodifferential operator is the fractional Laplacian

$$\Delta^s u(x_0) = c_{n,s}\,\mathrm{PV}\int_{\mathbb{R}^n} (u(x_0+x)-u(x_0))\frac{1}{|x|^{n+2s}}\,dx, \qquad s\in(0,1),$$

which can be understood as an infinitesimal generator of a stable Lévy process. These types of processes are very well studied in probability, and their generators may be given by

$$L_K u(x_0) = \int_{\mathbb{R}^n} (u(x_0+x)-u(x_0)-x\cdot\nabla u(x_0))K(x)\,dx,$$

where the kernel K is a nonnegative function satisfying some integrability condition.

Recently, there has been significant interest in studying linear and nonlinear integrodifferential equations from the analytical point of view. In particular, extremal operators like

$$Fu(x_0) = \inf_{K\in\mathcal{K}} L_K u(x_0) \tag{1-1}$$

play a fundamental role in the regularity theory. See [Caffarelli and Silvestre 2009; 2011a; 2011b; Ros-Oton and Serra 2016]. The above equation is an example of a fully nonlinear equation that appears in optimal control problems and stochastic games [Krylov 1980; Nisio 1988]. The infimum in (1-1) is taken over a family of admissible kernels $\mathcal{K}$ that depends on the applications. In fact, nonlocal Monge–Ampère equations have been developed in the form (1-1) for some choice of $\mathcal{K}$ [Caffarelli and Charro 2015; Caffarelli and Silvestre 2016; Guillen and Schwab 2012].

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The Monge–Ampère equation arises in several problems in analysis and geometry, such as the mass transportation problem and the prescribed Gaussian curvature problem [De Philippis and Figalli 2014]. The classical equation prescribes the determinant of the Hessian of some convex function u:

$$\det(D^2u) = f.$$

In the literature, there are different nonlocal versions of the Monge–Ampère operator that Guillen and Schwab [2012], Caffarelli and Charro [2015], and Caffarelli and Silvestre [2016] have considered. Maldonado and Stinga [2017] have also given a nonlocal linearized Monge–Ampère equation. These definitions are motivated by the following property: if B is a positive definite symmetric matrix, then

$$n\det(B)^{1/n} = \inf_{A\in\mathcal{A}} \operatorname{tr}(A^TBA), \tag{1-2}$$

where

$$\mathcal{A} = \{A \in M_n : A > 0,\ \det(A) = 1\}$$

and M_n is the set of $n \times n$ matrices. If a convex function u is C^2 at a point x_0, then, by the previous identity with $B = D^2u(x_0)$, we may write the Monge–Ampère operator as a concave envelope of linear operators. It follows that

$$n\det(D^2u(x_0))^{1/n} = \inf_{A\in\mathcal{A}} \Delta[u\circ A](A^{-1}x_0).$$

Caffarelli and Charro [2015] study a fractional version of $\det(D^2u)^{1/n}$, replacing the Laplacian by the fractional Laplacian in the previous identity. More precisely,

$$\begin{aligned}\mathcal{D}^s u(x_0) &= \inf_{A\in\mathcal{A}} \Delta^s[u\circ A](A^{-1}x_0)\\ &= c_{n,s}\inf_{A\in\mathcal{A}} \mathrm{PV}\int_{\mathbb{R}^n} \frac{u(x_0+x)-u(x_0)}{|A^{-1}x|^{n+2s}}\,dx,\end{aligned}$$

where $s \in (0,1)$ and $c_{n,s} \approx 1-s$ as $s \to 1$; see also [Guillen and Schwab 2012]. A different approach based on geometric considerations was given by Caffarelli and Silvestre [2016]. In fact, the authors consider kernels whose level sets are volume preserving transformations of the fractional Laplacian kernel. Namely,

$$\mathrm{MA}^s u(x_0) = c_{n,s}\inf_{K\in\mathcal{K}_n^s}\int_{\mathbb{R}^n} (u(x_0+x)-u(x_0)-x\cdot\nabla u(x_0))K(x)\,dx,$$

where the infimum is taken over the family

$$\mathcal{K}_n^s = \{K : \mathbb{R}^n \to \mathbb{R}_+ : |\{x\in\mathbb{R}^n : K(x) > r^{-n-2s}\}| = |B_r| \text{ for all } r > 0\}. \tag{1-3}$$

Notice that $|A^{-1}x|^{-n-2s} \in \mathcal{K}_n^s$ for any $A \in \mathcal{A}$. Therefore,

$$\mathrm{MA}^s u(x_0) \le \mathcal{D}^s u(x_0) \le \Delta^s u(x_0).$$

Moreover, both $\mathrm{MA}^s u$ and $\mathcal{D}^s u$ converge to $\det(D^2u)^{1/n}$, up to some constant, as $s \to 1$.

In this paper, we introduce a new family of operators of the form

$$\inf_{K\in\mathcal{K}_k^s}\int_{\mathbb{R}^n}(u(x_0+x)-u(x_0)-x\cdot\nabla u(x_0))K(x)\,dx \tag{1-4}$$

for any integer $1\le k<n$, which arises from imposing certain geometric conditions on the kernels. Moreover, we will see that

$$|y|^{-n-2s}\in\mathcal{K}_1^s\subset\mathcal{K}_k^s\subset\mathcal{K}_n^s\quad\text{for } 1<k<n,$$

and thus, this family will be monotone decreasing, and bounded from above by the fractional Laplacian and from below by the Caffarelli–Silvestre nonlocal Monge–Ampère.

The paper is organized as follows. In Section 2, we construct the family of admissible kernels $\mathcal{K}_k^s$ and give the precise definition of our operators for $C^{1,1}$-functions. We introduce in Section 3 the basic tools from the theory of rearrangements necessary for our goals. In Section 4, we study the infimum in (1-4) and obtain a representation formula, provided some condition on the level sets is satisfied (see Theorem 4.1). We also study the limit as $s\to 1$ and give a connection to optimal transport. The Hölder continuity of $\mathcal{F}_k^s u$ is proved in Section 5, following similar geometric techniques from [Caffarelli and Silvestre 2016]. In Section 6, we consider a global Poisson problem prescribing data at infinity, and introduce a new definition of our operators for functions that are merely continuous and convex. We show existence of solutions via Perron's method and $C^{1,1}$-regularity in the full space by constructing appropriate barriers. Finally, we discuss some future directions in Section 7.

2. Construction of kernels

Let us start with the construction of the family of admissible kernels. Notice that any kernel K in $\mathcal{K}_n^s$, defined in (1-3), will have the same distribution function as the kernel of the fractional Laplacian, since, for any $r>0$,

$$\{x\in\mathbb{R}^n:|x|^{-n-2s}>r^{-n-2s}\}=B_r.$$

Geometrically, this means that the level sets of K are deformations in *any* direction of $\mathbb{R}^n$ of the level sets of $|x|^{-n-2s}$, preserving the n-dimensional volume.

In view of this approach, a natural way of finding an intermediate family of operators between the nonlocal Monge–Ampère and the fractional Laplacian is to consider kernels whose level sets are deformations that preserve the k-dimensional Hausdorff measure $\mathcal{H}^k$, with $1\le k<n$, of the restrictions of balls in $\mathbb{R}^n$ to hyperplanes generated by $\{e_i\}_{i=1}^k$.

We define the set of admissible kernels as follows.

Definition 2.1. We say that $K\in\mathcal{K}_k^s$ if, for all $z\in\mathbb{R}^{n-k}$ and all $r>0$,

$$\mathcal{H}^k(\{y\in\mathbb{R}^k:K(y,z)>r^{-n-2s}\})=\begin{cases}\mathcal{H}^k(B_{(r^2-|z|^2)^{1/2}}) & \text{if } |z|<r,\\ 0 & \text{if } |z|\ge r,\end{cases} \tag{2-1}$$

where $B_{(r^2-|z|^2)^{1/2}}$ is the ball in $\mathbb{R}^k$ of radius $(r^2-|z|^2)^{1/2}$.

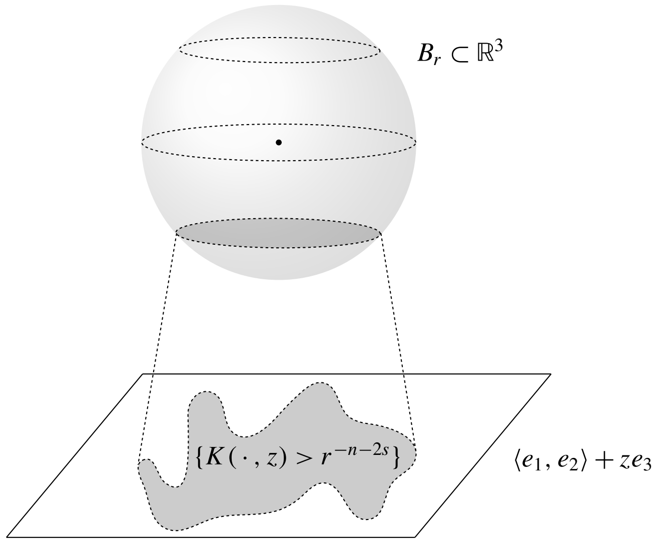

Figure 1. Area-preserving deformation in $\mathbb{R}^3$.

In Figure 1 we illustrate condition (2-1) for $k = 2$ and $n = 3$. Note that, for $k = n$, we recover the definition of $\mathcal{K}^s_n$. Moreover, $|x|^{-n-2s} \in \mathcal{K}^s_k$ for all k.

Proposition 2.2. *Let* $1 \le k < n$. *Then* $\mathcal{K}^s_k \subset \mathcal{K}^s_{k+1} \subseteq \mathcal{K}^s_n$.

Proof. Let $K \in \mathcal{K}^s_k$. Fix any $z \in \mathbb{R}^{n-k-1}$ and $r > 0$. Then

$$\begin{aligned}
\mathcal{H}^{k+1}(\{y \in \mathbb{R}^{k+1} : K(y,z) > r^{-n-2s}\}) &= \int_{\mathbb{R}^{k+1}} \chi_{\{y\in\mathbb{R}^{k+1}:K(y,z)>r^{-n-2s}\}}(y)\,dy \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^k} \chi_{\{(w,t)\in\mathbb{R}^k\times\mathbb{R}:K(w,t,z)>r^{-n-2s}\}}(w,t)\,dw \right) dt \\
&= \int_{\mathbb{R}} \mathcal{H}^k(\{w \in \mathbb{R}^k : K(w,t,z) > r^{-n-2s}\})\,dt \equiv \mathrm{I}.
\end{aligned}$$

If $|z| \ge r$, then for any $t \in \mathbb{R}$, we have that $(t,z) \in \mathbb{R}^{n-k}$, with $|(t,z)| > r$. Therefore, by (2-1), it follows that $\mathrm{I} = 0$. If $|z| < r$, then

$$\begin{aligned}
\mathrm{I} &= \int_{\mathbb{R}} \mathcal{H}^k(B_{(r^2-t^2-|z|^2)^{1/2}})\,dt = \omega_k \int_{-(r^2-|z|^2)^{1/2}}^{(r^2-|z|^2)^{1/2}} (r^2 - t^2 - |z|^2)^{k/2}\,dt \\
&= \omega_k (r^2-|z|^2)^{k/2} \int_{-(r^2-|z|^2)^{1/2}}^{(r^2-|z|^2)^{1/2}} \left(1 - \left(\frac{t}{(r^2-|z|^2)^{1/2}}\right)^2\right)^{k/2} dt \\
&= \omega_k (r^2-|z|^2)^{(k+1)/2} \int_{-1}^{1} (1-\sigma^2)^{k/2}\,d\sigma = \frac{\pi^{k/2}}{\Gamma\left(\frac{1}{2}k+1\right)} \frac{\pi^{1/2}\Gamma\left(\frac{1}{2}k+1\right)}{\Gamma\left(\frac{1}{2}(k+1)+1\right)} (r^2-|z|^2)^{(k+1)/2} \\
&= \omega_{k+1}(r^2-|z|^2)^{(k+1)/2} = \mathcal{H}^{k+1}(B_{(r^2-|z|^2)^{1/2}}),
\end{aligned}$$

where $\omega_l = \mathcal{H}^l(B_1) = \pi^{l/2}/\Gamma(l/2+1)$ and $B_{(r^2-|z|^2)^{1/2}}$ is the ball of radius $(r^2-|z|^2)^{1/2}$ in $\mathbb{R}^{k+1}$. ☐

Definition 2.3. A function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be $C^{1,1}$ at the point x_0, and we write $u \in C^{1,1}(x_0)$, if there is a vector $p \in \mathbb{R}^n$, a radius $\rho > 0$, and a constant $C > 0$ such that

$$|u(x_0 + x) - u(x_0) - x \cdot p| \le C|x|^2 \quad \text{for all } x \in B_\rho.$$

We denote by $[u]_{C^{1,1}(x_0)}$ the minimum constant for which this property holds, among all admissible vectors p and radii ρ.

Definition 2.4. Let $s \in \left(\frac{1}{2}, 1\right)$ and $1 \le k < n$. For any $u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0)$, we define

$$\mathcal{F}_k^s u(x_0) = c_{n,s} \inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^n} (u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0)) K(x)\, dx,$$

where $\mathcal{K}_k^s$ is the set of kernels satisfying (2-1) and $c_{n,s}$ is the constant in Δ^s.

As an immediate consequence of Proposition 2.2, we obtain that the operators are ordered.

Corollary 2.5. *Let $s \in \left(\frac{1}{2}, 1\right)$ and $1 \le k < n$. Then, for any $u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0)$,*

$$\mathrm{MA}^s u(x_0) \le \mathcal{F}_k^s u(x_0) \le \Delta^s u(x_0).$$

Moreover, $\{\mathcal{F}_k^s\}_{k=1}^{n-1}$ is monotone decreasing.

The regularity condition on u in Definition 2.4 allows us to compute $\mathcal{F}_k^s u$ at the point x_0 in the classical sense. To obtain a finite number, we need to impose two extra conditions:

(1) An integrability condition at infinity:

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{(1 + |x|)^{n+2s}}\, dx < \infty. \tag{P_1}$$

(2) A convexity condition at x_0:

$$\tilde{u}(x) \equiv u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0) \ge 0 \quad \text{for all } x \in \mathbb{R}^n. \tag{P_2}$$

Proposition 2.6. *If $u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0)$ and u satisfies (P_1) and (P_2), then*

$$0 \le \mathcal{F}_k^s u(x_0) < \infty.$$

Proof. Let $\rho > 0$ be as in Definition 2.3. Then

$$\begin{aligned}
0 \le \mathcal{F}_k^s u(x_0) &\le \int_{\mathbb{R}^n} (u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0)) \frac{1}{|x|^{n+2s}}\, dx \\
&\le \int_{B_\rho} \frac{[u]_{C^{1,1}(x_0)} |x|^2}{|x|^{n+2s}}\, dx + \int_{\mathbb{R}^n \setminus B_\rho(x_0)} \frac{|u(x)|}{|x - x_0|^{n+2s}}\, dx \\
&\qquad + |u(x_0)| \int_{\mathbb{R}^n \setminus B_\rho} \frac{1}{|x|^{n+2s}}\, dx + |\nabla u(x_0)| \int_{\mathbb{R}^n \setminus B_\rho} \frac{|x|}{|x|^{n+2s}}\, dx \\
&\le C(s, \rho)(|u(x_0)| + |\nabla u(x_0)| + [u]_{C^{1,1}(x_0)}) \\
&\qquad + \frac{1 + |x_0| + \rho}{\rho} \int_{\mathbb{R}^n} \frac{|u(x)|}{(1 + |x|)^{n+2s}}\, dx < \infty, \qquad \text{since } s \in \left(\tfrac{1}{2}, 1\right).
\end{aligned}$$

□

We point out that if u is not convex at x_0, then the infimum could be $-\infty$. We show this result in the next proposition.

Proposition 2.7. *Let $u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0)$. Assume that u satisfies (P_1). If there exists $\bar{x} \in \mathbb{R}^n$ with $\bar{x} = (\bar{y}, 0)$ and $\bar{y} \in \mathbb{R}^k$ such that*

$$\tilde{u}(\bar{x}) = u(x_0 + \bar{x}) - u(x_0) - \bar{x} \cdot \nabla u(x_0) < 0,$$

then $\mathcal{F}_k^s u(x_0) = -\infty$.

Proof. Let $K(x) = |x - \bar{x}|^{-n-2s}$. For any $r > 0$ and $z \in \mathbb{R}^{n-k}$, if $|z| < r$, then

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : K(y, z) > r^{-n-2s}\}) = \mathcal{H}^k(\{y \in \mathbb{R}^k : |y - \bar{y}|^2 + |z|^2 < r^2\}) = \mathcal{H}^k(B_{(r^2-|z|^2)^{1/2}}).$$

Also, the measure is clearly zero if $|z| \geq r$. Therefore, $K \in \mathcal{K}_k^s$. It follows that

$$\begin{aligned}\mathcal{F}_k^s u(x_0) &\leq \int_{\mathbb{R}^n} \tilde{u}(x)|x - \bar{x}|^{-n-2s}\,dx \\ &= \int_{B_\varepsilon(\bar{x})} \tilde{u}(x)|x - \bar{x}|^{-n-2s}\,dx + \int_{\mathbb{R}^n \setminus B_\varepsilon(\bar{x})} \tilde{u}(x)|x - \bar{x}|^{-n-2s}\,dx \equiv \mathrm{I} + \mathrm{II}.\end{aligned}$$

Since $u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0)$, we have that $\tilde{u}$ is continuous. Hence, given that $\tilde{u}(\bar{x}) < 0$, it follows that $\tilde{u}(x) < 0$ for all $x \in B_\varepsilon(\bar{x})$ for some $\varepsilon > 0$. Moreover, since $K \notin L^1(B_\varepsilon(\bar{x}))$, we have that $\mathrm{I} = -\infty$. Arguing similarly as in the proof of Proposition 2.6, we see that $\mathrm{II} < \infty$. Therefore,

$$\mathcal{F}_k^s u(x_0) = -\infty. \qquad \square$$

Remark 2.8. The operators $\mathcal{F}_k^s$ are not rotation invariant. This is because, for simplicity, in the construction of the family of admissible kernels $\mathcal{K}_k^s$ we chose the first k vectors from the canonical basis of $\mathbb{R}^n$. In general, we may take any subset of k unitary vectors, $\tau = \{\tau_i\}_{i=1}^k$, and replace the first condition on (2-1) by

$$\mathcal{H}^k(\{y \in \langle\tau\rangle^\perp : K(y + z\tau) > r^{-n-2s}\}) = \mathcal{H}^k(B_{(r^2-|z|^2)^{1/2}}) \tag{2-2}$$

for all $z \in \langle\tau\rangle$ and $r > 0$, where $\langle\tau\rangle$ denotes the span of $\{\tau_i\}_{i=1}^k$ and $\langle\tau\rangle^\perp$ the orthogonal subspace to $\langle\tau\rangle$. Let $\mathrm{SO}(n)$ be the group of $n \times n$ rotation matrices. Since $\tau_i = Ae_i$ for some $A \in \mathrm{SO}(n)$, it follows that any kernel K_τ satisfying (2-2) can be written as $K_\tau = K \circ A$, where K satisfies (2-1). Therefore, to make the operators rotation invariant, one possibility is to take the infimum over all possible rotations. Namely,

$$\inf_{A \in \mathrm{SO}(n)} \inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^n} \tilde{u}(x) K(Ax)\,dx.$$

To focus on the main ideas, we will not explore this operator in this work.

3. Rearrangements and measure-preserving transformations

We introduce some definitions and preliminary results regarding rearrangements of nonnegative functions. For more detailed information, see for instance [Baernstein 2019; Bennett and Sharpley 1988].

Definition 3.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative measurable function. We define the decreasing rearrangement of f as the function f^* defined on $[0, \infty)$ and given by

$$f^*(t) = \sup\{\lambda > 0 : |\{x \in \mathbb{R}^n : f(x) > \lambda\}| > t\},$$

and the increasing rearrangement of f as the function f_* defined on $[0, \infty)$ and given by

$$f_*(t) = \inf\{\lambda > 0 : |\{x \in \mathbb{R}^n : f(x) \le \lambda\}| > t\}.$$

We use the convention that $\inf \varnothing = \infty$.

Proposition 3.2. *Let $f,\ g : \mathbb{R}^n \to \mathbb{R}$ be nonnegative measurable functions. Then*

$$\int_0^\infty f_*(t) g^*(t)\, dt \le \int_{\mathbb{R}^n} f(x) g(x)\, dx \le \int_0^\infty f^*(t) g^*(t)\, dt.$$

The upper bound is the classical Hardy–Littlewood inequality. For the proof, see [Bennett and Sharpley 1988, Theorem 2.2] or [Baernstein 2019, Corollary 2.16]. For the sake of completeness, we give the proof of the lower bound.

Proof. For $j \ge 1$, let $f_j = f|_{B_j}$ and $g_j = g|_{B_j}$, where B_j denotes the ball of radius j centered at 0 in $\mathbb{R}^n$. By [Baernstein 2019, Corollary 2.18], it follows that

$$\int_0^{|B_j|} (f_j)_*(t)(g_j)^*(t)\, dt \le \int_{B_j} f_j(x) g_j(x)\, dx.$$

Since $f,\ g \ge 0$, we get

$$\int_{B_j} f_j(x) g_j(x)\, dx \le \int_{\mathbb{R}^n} f(x) g(x)\, dx.$$

Note that, for any $t \in [0, |B_j|]$, we have

$$\{\lambda > 0 : |\{x \in B_j : f_j(x) \le \lambda\}| > t\} \subset \{\lambda > 0 : |\{x \in \mathbb{R}^n : f(x) \le \lambda\}| > t\}.$$

Hence $(f_j)_*(t) \ge f_*(t)$ and

$$\int_0^{|B_j|} (f_j)_*(t)(g_j)^*(t)\, dt \ge \int_0^{|B_j|} f_*(t)(g_j)^*(t)\, dt.$$

Moreover, $g_j \nearrow g$ pointwise on $\mathbb{R}^n$. Then by [Baernstein 2019, Proposition 1.39], we have $(g_j)^* \nearrow g^*$ pointwise on $[0, \infty)$ as $j \to \infty$. By the monotone convergence theorem, we get

$$\lim_{j\to\infty} \int_0^{|B_j|} f_*(t)(g_j)^*(t)\, dt = \int_0^\infty f_*(t) g^*(t)\, dt.$$

Combining the previous estimates, we conclude that

$$\int_0^\infty f_*(t) g^*(t)\, dt \le \int_{\mathbb{R}^n} f(x) g(x)\, dx. \qquad \square$$

Definition 3.3. We say that a measurable function $\psi : \mathbb{R}^l \to \mathbb{R}^m$ is a measure-preserving transformation, or *measure-preserving*, if, for any measurable set E in $\mathbb{R}^m$,

$$\mathcal{H}^l(\psi^{-1}(E)) = \mathcal{H}^m(E).$$

Lemma 3.4. *If $\psi : \mathbb{R}^l \to \mathbb{R}^m$ is measure-preserving, then, for any measurable function $f : \mathbb{R}^m \to \mathbb{R}$ and any measurable set E in $\mathbb{R}^m$,*

$$\int_E f(y)\,dy = \int_{\psi^{-1}(E)} f(\psi(z))\,dz.$$

An important result by Ryff [1970] provides a sufficient condition for which we can recover a function given its decreasing/increasing rearrangement, by means of a measure-preserving transformation.

Theorem 3.5 (Ryff's theorem). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative measurable function. If $\lim_{t\to\infty} f^*(t)$ equals zero, then there exists a measure-preserving transformation $\sigma : \operatorname{supp}(f) \to \operatorname{supp}(f^*)$ such that*

$$f = f^* \circ \sigma$$

almost everywhere on the support of f. Similarly, if $\lim_{t\to\infty} f_*(t) = \infty$, then $f = f_* \circ \sigma$.

We will call a measure-preserving transformation σ satisfying Ryff's theorem a *Ryff's map*.

Remark 3.6. In general, σ is not invertible. Furthermore, there may not exist a measure-preserving transformation ψ such that $f^* = f \circ \psi$.

As a consequence of Ryff's theorem, we obtain a representation formula for the admissible kernels. We write $\omega_k = \mathcal{H}^k(B_1)$.

Lemma 3.7. *Let $K \in \mathcal{K}_k^s$. Fix $z \in \mathbb{R}^{n-k}$ and use the notation $K_z(y) = K(y,z)$. Then*

$$K_z^*(t) = ((\omega_k^{-1}t)^{2/k} + |z|^2)^{-(n+2s)/2}.$$

In particular, there exists a measure-preserving transformation $\sigma_z : \operatorname{supp}(K_z) \to (0,\infty)$ such that

$$K(y,z) = K_z^*(\sigma_z(y)) \quad \textit{for a.e. } y \in \operatorname{supp}(K_z).$$

Proof. Fix $z \in \mathbb{R}^{n-k}$. Then

$$\begin{aligned} K_z^*(t) &= \sup\{\lambda > 0 : \mathcal{H}^k(\{y \in \mathbb{R}^k : K(y,z) > \lambda\}) > t\} \\ &= \sup\{\lambda < |z|^{-n-2s} : \mathcal{H}^k(B_{(\lambda^{-2/(n+2s)} - |z|^2)^{1/2}}) > t\} \\ &= \sup\{\lambda < |z|^{-n-2s} : \omega_k(\lambda^{-2/(n+2s)} - |z|^2)^{k/2} > t\} \\ &= \sup\{\lambda < |z|^{-n-2s} : \lambda^{-2/(n+2s)} > (\omega_k^{-1}t)^{2/k} + |z|^2\} = ((\omega_k^{-1}t)^{2/k} + |z|^2)^{-(n+2s)/2}. \end{aligned}$$

Moreover, $\lim_{t\to\infty} K_z^*(t) = 0$. Therefore, the result follows from Theorem 3.5. □

In view of Definition 3.1, we introduce the symmetric rearrangement of a function in $\mathbb{R}^n$ with respect to the first k variables as follows. Fix $k \in \mathbb{N}$ with $1 \le k < n$. Given $x \in \mathbb{R}^n$, we write $x = (y,z)$ with $y \in \mathbb{R}^k$ and $z \in \mathbb{R}^{n-k}$. Furthermore, for z fixed, we call f_z the restriction of f to $\mathbb{R}^k$. Namely, $f_z(y) = f(y,z)$.

Definition 3.8. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative measurable function. We define the k-symmetric *decreasing* rearrangement of f as the function $f^{*,k} : \mathbb{R}^n \to [0,\infty]$ given by

$$f^{*,k}(x) = f_z^*(\omega_k|y|^k),$$

and the k-symmetric *increasing* rearrangement as the function $f_{*,k} : \mathbb{R}^n \to [0,\infty]$ given by

$$f_{*,k}(x) = (f_z)_*(\omega_k|y|^k).$$

When $k = n$, we obtain the usual symmetric rearrangement.

Remark 3.9. (1) Notice that $f^{*,k}$ and $f_{*,k}$ are radially symmetric and monotone decreasing/increasing, with respect to y. In the literature, this type of symmetrization is also known as the Steiner symmetrization [Baernstein 2019, Chapter 6].

(2) By Lemma 3.7, we see that any kernel $K \in \mathcal{K}_k^s$ satisfies

$$K^{*,k}(x) = |x|^{-n-2s} \quad \text{for } x \neq 0. \tag{3-1}$$

4. Analysis of $\mathcal{F}_k^s$

Our main goal of this section is to study the infimum in the definition of the operator

$$\mathcal{F}_k^s u(x_0) = c_{n,s} \inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^n} \tilde{u}(x) K(x)\, dx,$$

where $\tilde{u}(x) = u(x_0+x) - u(x_0) - x \cdot \nabla u(x_0)$. Throughout the section, we assume that $u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0)$ and that u satisfies properties (P_1) and (P_2), so that $0 \leq \mathcal{F}_k^s u(x_0) < \infty$.

Analysis of the infimum. We will study the following cases:

Case 1. For all $\lambda > 0$ and $z \in \mathbb{R}^{n-k}$,

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda\}) < \infty.$$

Case 2. There exists some $\lambda_0 > 0$ such that, for all $z \in \mathbb{R}^{n-k}$,

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda\}) \begin{cases} < \infty & \text{for } 0 < \lambda < \lambda_0, \\ = \infty & \text{for } \lambda \geq \lambda_0. \end{cases}$$

Case 3. For all $\lambda > 0$ and $z \in \mathbb{R}^{n-k}$,

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda\}) = \infty.$$

In Case 1, when all of the level sets of $\tilde{u}$ have finite measure, we show that the infimum is attained at some kernel whose level sets depend on the measure-preserving transformation that rearranges the level sets of $\tilde{u}$. More precisely:

Theorem 4.1. *Suppose that, for all $\lambda > 0$ and $z \in \mathbb{R}^{n-k}$,*

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda\}) < \infty.$$

Then, for any $z \in \mathbb{R}^{n-k}$, there exists a measure-preserving transformation $\sigma_z : \mathbb{R}^k \to [0, \infty)$ such that

$$\mathcal{F}_k^s u(x_0) = c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \frac{\tilde{u}(y, z)}{((\omega_k^{-1} \sigma_z(y))^{2/k} + |z|^2)^{(n+2s)/2}}\, dy\, dz.$$

In particular, the infimum is attained.

Remark 4.2. Observe that if $\tilde{u}(\cdot\,, z)$ is constant in some set of positive measure, then the kernel where the infimum is attained is not unique since the integral is invariant under any measure-preserving rearrangement of K within this set; see [Ryff 1970].

Before we give the proof of Theorem 4.1, we need a lemma regarding the k-symmetric increasing rearrangement of $\tilde{u}$. By Definition 3.8, this is given by the expression

$$\tilde{u}_{*,k}(y,z) = \inf\{\lambda > 0 : \mathcal{H}^k(\{w \in \mathbb{R}^k : \tilde{u}(w,z) \le \lambda\}) > \omega_k |y|^k\}.$$

Lemma 4.3. *Fix $z \in \mathbb{R}^{n-k}$. If $\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}(y,z) \le \lambda\}) < \infty$ for all $\lambda > 0$, then*

$$\lim_{|y|\to\infty} \tilde{u}_{*,k}(y,z) = \infty.$$

Proof. Assume there exists $M > 0$ independent of λ such that

$$\mathcal{H}^k(\{w \in \mathbb{R}^k : \tilde{u}(w,z) \le \lambda\}) \le M \quad \text{for all } \lambda > 0. \tag{4-1}$$

Then, for any $y \in \mathbb{R}^k$ with $\omega_k |y|^k > M$, we have that

$$\tilde{u}_{*,k}(y,z) = \infty,$$

since $\inf \varnothing = \infty$. If (4-1) does not hold, then there must be an increasing sequence $\{M_\lambda\}_{\lambda>0}$ with $M_\lambda \to \infty$ as $\lambda \to \infty$ such that

$$\mathcal{H}^k(\{w \in \mathbb{R}^k : \tilde{u}(w,z) \le \lambda\}) = M_\lambda.$$

Then, for any $M > 0$, there exists $\Lambda = \Lambda(M) > 0$ such that $M_\lambda > M$ for all $\lambda > \Lambda$. Since M_λ is monotone increasing, we can assume without loss of generality that $M_\Lambda \le M$. Otherwise, we take Λ to be the minimum for which this property holds. Also, $\Lambda(M)$ is monotone increasing, and $\Lambda(M) \to \infty$ as $M \to \infty$. In particular,

$$\inf\{\lambda > 0 : M_\lambda > M\} \ge \Lambda(M) \to \infty \quad \text{as } M \to \infty.$$

Then, for any $K > 0$, there exists $M > 0$ such that

$$\inf\{\lambda > 0 : M_\lambda > M\} \ge K.$$

Therefore, for any $y \in \mathbb{R}^k$ with $\omega_k |y|^k > M$, we have

$$\tilde{u}_{*,k}(y,z) = \inf\{\lambda > 0 : M_\lambda > \omega_k |y|^k\} \ge \inf\{\lambda > 0 : M_\lambda > M\} \ge K.$$

We conclude that

$$\lim_{|y|\to\infty} \tilde{u}_{*,k}(y,z) = \infty. \qquad \square$$

Proof of Theorem 4.1. Since u is convex at x_0, we have that $\tilde{u}(y,z) \ge 0$. Moreover,

$$\mathcal{F}_k^s u(x_0) = c_{n,s} \inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y,z) K(y,z)\, dy\, dz.$$

Fix $z \in \mathbb{R}^{n-k}$ and consider the functions $f(y) = \tilde{u}(y,z)$ and $g(y) = K(y,z)$. Since

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}(y,z) \le \lambda\}) < \infty$$

for any $\lambda > 0$, then by Lemma 4.3 we have

$$\lim_{t\to\infty} f_*(t) = \lim_{|y|\to\infty} f_{*,k}(x) = \infty,$$

with $f_{*,k}(x) = \tilde{u}_{*,k}(y, z)$ and $f_{*,k}(x) = f_*(\omega_k|y|^k)$. By Ryff's theorem (Theorem 3.5), there exists a measure-preserving transformation $\sigma_z : \mathbb{R}^k \to [0, \infty)$ depending on z such that

$$\tilde{u}(y, z) = f_*(\sigma_z(y)) \tag{4-2}$$

for all $y \in \operatorname{supp} \tilde{u}(\,\cdot\,, z) \subseteq \mathbb{R}^k$.

Let $K(y, z) = ((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2)^{-(n+2s)/2}$. For any $r > |z|$, we have

$$\begin{aligned}\mathcal{H}^k(\{y \in \mathbb{R}^k : K(y, z) > r^{-n-2s}\}) &= \mathcal{H}^k(\{y \in \mathbb{R}^k : ((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2)^{-(n+2s)/2} > r^{-n-2s}\})\\ &= \mathcal{H}^k(\{y \in \mathbb{R}^k : \sigma_z(y) < \omega_k(r^2 - |z|^2)^{k/2}\})\\ &= \mathcal{H}^k(\sigma_z^{-1}((0, \omega_k(r^2 - |z|^2)^{k/2}))) = \mathcal{H}^1((0, \omega_k(r^2 - |z|^2)^{k/2}))\\ &= \omega_k(r^2 - |z|^2)^{k/2} = \mathcal{H}^k(B_{(r^2-|z|^2)^{k/2}}),\end{aligned}$$

since σ_k is measure-preserving (see Definition 3.3). Then $K \in \mathcal{K}_k^s$, and thus

$$\mathcal{F}_k^s u(x_0) \le c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \frac{\tilde{u}(y, z)}{((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2)^{(n+2s)/2}}\, dy\, dz.$$

To prove the reverse inequality, let $K \in \mathcal{K}_k^s$. Applying Proposition 3.2, we see that

$$\int_{\mathbb{R}^k} \tilde{u}(y, z) K(y, z)\, dy \ge \int_0^\infty f_*(t) g^*(t)\, dt = \int_{\mathbb{R}^k} f_*(\sigma_z(y)) g^*(\sigma_z(y))\, dy = \int_{\mathbb{R}^k} \tilde{u}(y, z) g^*(\sigma_z(y))\, dy$$

by Lemma 3.4 and (4-2). Moreover, by the definition of rearrangements,

$$g^*(\sigma_z(y)) = \sup\{\lambda > 0 : \mathcal{H}^k(\{w \in \mathbb{R}^k : K(w, z) > \lambda\}) > \sigma_z(y)\} = K^{*,k}(\tilde{y}, z),$$

with $\omega_k|\tilde{y}|^k = \sigma_z(y)$. By (3-1), we get

$$g^*(\sigma_z(y)) = (|\tilde{y}|^2 + |z|^2)^{-(n+2s)/2} = ((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2)^{-(n+2s)/2}.$$

Hence integrating over all $z \in \mathbb{R}^{n-k}$ and taking the infimum over all kernels $K \in \mathcal{K}_k^s$, we conclude that

$$\mathcal{F}_k^s u(x) = c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \frac{\tilde{u}(y, z)}{((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2)^{(n+2s)/2}}\, dy\, dz. \qquad \square$$

Remark 4.4. A natural question that arises from this result is whether there exists a measure-preserving transformation $\varphi_z : \mathbb{R}^k \to \mathbb{R}^k$ such that

$$|\varphi_z(y)| = (\omega_k^{-1}\sigma_z(y))^{1/k}.$$

In that case, we would have that the infimum is attained at a kernel K such that

$$K(y, z) = |\phi(y, z)|^{-n-2s},$$

where $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is measure-preserving with $\phi(y, z) = (\varphi_z(y), z)$.

Recall that Ryff's theorem gives a representation of a function f in terms of its increasing rearrangement f_*, that is, $f = f_* \circ \sigma$ with $\sigma : \mathbb{R}^k \to \mathbb{R}$ measure-preserving. If this result were also true for the *symmetric* increasing rearrangement, given by $f_\#(x) = f_*(\omega_k|x|^k)$, then there would exist a measure-preserving transformation $\varphi : \mathbb{R}^k \to \mathbb{R}^k$ such that $f = f_\# \circ \psi$. In particular,

$$f(x) = f_\#(\varphi(x)) = f_*(\omega_k|\varphi(x)|^k) = f_*(\sigma(x)).$$

Hence it seems reasonable that $\omega_k|\varphi(x)|^k = \sigma(x)$. As far as we know, this is an open problem.

As an immediate consequence of Theorem 4.1, we obtain the following representation of the function $\mathcal{F}_k^s u$ in terms of the k-symmetric increasing rearrangement of $\tilde{u}$.

Corollary 4.5. *Under the assumptions of Theorem 4.1, we have*

$$\mathcal{F}_k^s u(x_0) = \Delta^s \tilde{u}_{*,k}(0).$$

Proof. Note that $\tilde{u}_{*,k}(0) = 0$ since $\tilde{u}(0) = 0$. Therefore, using the same notation as in the proof of Theorem 4.1, we showed that

$$\begin{aligned}
\mathcal{F}_k^s u(x_0) &= c_{n,s} \int_{\mathbb{R}^{n-k}} \int_0^\infty f_*(t) g^*(t)\, dt\, dz = \omega_k c_{n,s} \int_{\mathbb{R}^{n-k}} \int_0^\infty f_*(\omega_k r^k) g^*(\omega_k r^k) r^{k-1}\, dr\, dz \\
&= c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} f_*(\omega_k|y|^k) g^*(\omega_k|y|^k)\, dy\, dz = c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}_{*,k}(y,z) K^{*,k}(y,z)\, dy\, dz \\
&= c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \frac{\tilde{u}_{*,k}(y,z)}{(|y|^2+|z|^2)^{(n+2s)/2}}\, dy\, dz = \Delta^s \tilde{u}_{*,k}(0). \qquad \square
\end{aligned}$$

From the previous result and the fact that the family of operators $\{\mathcal{F}_k\}_{k=1}^{n-1}$ is monotone decreasing, we see that the fractional Laplacian of the k-symmetric rearrangements are ordered at the origin.

Corollary 4.6. *Suppose we are under the assumptions of Theorem 4.1. Then*

$$\Delta^s \tilde{u}_{*,k+1}(0) \le \Delta^s \tilde{u}_{*,k}(0).$$

Next we treat Case 2.

Theorem 4.7. *Suppose that there exists some $\lambda_0 > 0$ such that, for all $z \in \mathbb{R}^{n-k}$,*

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}(y,z) \le \lambda\}) \begin{cases} < \infty & \text{for } 0 < \lambda < \lambda_0, \\ = \infty & \text{for } \lambda \ge \lambda_0. \end{cases}$$

Then there exists a kernel $K_0 \in \mathcal{K}_k^s$ with supp $K_0(\cdot, z) \subseteq \{y \in \mathbb{R}^k : \tilde{u}(y,z) \le \lambda_0\}$ *such that*

$$\mathcal{F}_k^s u(x_0) = c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y,z) K_0(y,z)\, dy\, dz.$$

In particular, the infimum is attained.

Proof. Fix $z \in \mathbb{R}^{n-k}$. For $j \ge 1$, define the set

$$A_j(z) = \Big\{ y \in \mathbb{R}^k : \tilde{u}(y,z) \le \lambda_0 - \frac{1}{j} \Big\}.$$

For simplicity, we drop the notation of z. We have that $\mathcal{H}^k(A_j) < \infty$, $A_j \subseteq A_{j+1}$, and

$$A_\infty = \bigcup_{j=1}^\infty A_j = \{y \in \mathbb{R}^k : \tilde{u}(y,z) < \lambda_0\}.$$

Observe that if $K \in \mathcal{K}_k^s$, then

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : K(y,z) > 0\}) = \lim_{r \to 0} \mathcal{H}^k(\{y \in \mathbb{R}^k : K(y,z) > r\}) = \infty.$$

Hence we need to distinguish two subcases:

Case 2A. Assume that $\mathcal{H}^k(A_\infty) = \infty$. Let $K \in \mathcal{K}_k^s$ and $v_j = \tilde{u}\chi_{A_j}$. By Proposition 3.2,

$$\int_{A_j} \tilde{u}(y,z) K(y,z)\, dy = \int_{\mathbb{R}^k} v_j(y,z) K(y,z)\, dy \geq \int_0^\infty (v_j)_*(t) K^*(t)\, dt.$$

By Lemma 3.4, for any measure-preserving transformation $\sigma : \mathbb{R}^k \to [0,\infty)$, it follows that

$$\int_0^\infty (v_j)_*(t) K^*(t)\, dt = \int_{\mathbb{R}^k} (v_j)_*(\sigma(y)) K^*(\sigma(y))\, dy.$$

By Ryff's theorem (Theorem 3.5), there exists $\sigma_j : A_j \to [0, \mathcal{H}^k(A_j)]$ measure-preserving such that $v_j = (v_j)_* \circ \sigma_j$ in A_j. Therefore,

$$\int_{A_j} \tilde{u}(y,z) K(y,z)\, dy \geq \int_{A_j} \tilde{u}(y,z) K^*(\sigma_j(y))\, dy. \tag{4-3}$$

We claim that $\sigma_{j+1}(y) \leq \sigma_j(y)$, for all $y \in A_j$. Indeed, since $A_j \subseteq A_{j+1}$, we have

$$\begin{cases} v_j(y) = v_{j+1}(y) & \text{for all } y \in A_j, \\ v_j(y) \leq v_{j+1}(y) & \text{for all } y \in A_{j+1} \setminus A_j. \end{cases}$$

In particular,

$$(v_{j+1})_*(\sigma_{j+1}(y)) = (v_j)_*(\sigma_j(y)) \leq (v_{j+1})_*(\sigma_j(y)) \quad \text{for all } \ y \in A_j.$$

Since $(v_{j+1})_*$ is monotone increasing, we must have

$$\sigma_{j+1}(y) \leq \sigma_j(y) \quad \text{for all } y \in A_j.$$

Therefore, there exists $\sigma_\infty : A_\infty \to [0,\infty)$ measure-preserving such that

$$\sigma_\infty(y) = \lim_{j \to \infty} \sigma_j(y).$$

Define the kernel K_0 as

$$K_0(y,z) = ((\omega_k^{-1}\sigma_\infty(y))^{k/2} + |z|^2)^{-(n+2s)/2} \chi_{A_\infty}(y).$$

Since $\mathcal{H}^k(A_\infty) = \infty$, we have that $K_0 \in \mathcal{K}_k^s$. Furthermore, we note that $K_0(y,z) = K_0^*(\sigma_\infty(y))$ and $\operatorname{supp} K_0(\,\cdot\,,z) = \overline{A_\infty} = \{y \in \mathbb{R}^k : \tilde{u}(y,z) \leq \lambda_0\}$ for all $y \in A_\infty$. Then by Fatou's lemma, Lemma 3.7, and (4-3), we get

$$\begin{aligned} \int_{\mathbb{R}^k} \tilde{u}(y,z) K_0(y,z)\, dy &= \int_{A_\infty} \tilde{u}(y,z) K_0^*(\sigma_\infty(y))\, dy \leq \liminf_{j\to\infty} \int_{A_j} \tilde{u}(y,z) K_0^*(\sigma_j(y))\, dy \\ &= \liminf_{j\to\infty} \int_{A_j} \tilde{u}(y,z) K^*(\sigma_j(y))\, dy \leq \int_{\mathbb{R}^k} \tilde{u}(y,z) K(y,z)\, dy \end{aligned}$$

for any $K \in \mathcal{K}_k^s$. Integrating over z and taking the infimum over all kernels K, we conclude the result.

Case 2B. Assume that $\mathcal{H}^k(A_\infty) < \infty$. Set $A = \{y \in \mathbb{R}^k : \tilde{u}(y,z) = \lambda_0\}$. Then

$$\mathcal{H}^k(A) = \infty, \tag{4-4}$$

since $\{y \in \mathbb{R}^k : \tilde{u}(y,z) \le \lambda_0\} = A_\infty \cup A$. Fix $\varepsilon > 0$ and define

$$v_\varepsilon(y,z) = \tilde{u}(y,z)\chi_{A_\infty}(y) + \max\{\lambda_0, (\lambda_0+\varepsilon)\phi(y,z)\}\chi_A(y),$$

with $\phi(y,z) = 1 - e^{-|y|^2-|z|^2}$. Note that $0 < \phi \le 1$, $\phi(y,z) \to 1$ as $|(y,z)| \to \infty$, and $\phi(y,z) \approx |y|^2 + |z|^2$ as $|(y,z)| \to 0$. Also, $\{v_\varepsilon\}_{\varepsilon>0}$ is a monotone increasing sequence and

$$\begin{aligned}\lim_{\varepsilon\to 0} v_\varepsilon(y,z) &= \tilde{u}(y,z)\chi_{A_\infty}(y) + \max\Big\{\lambda_0, \lim_{\varepsilon\to 0}(\lambda_0+\varepsilon)\phi(y,z)\Big\}\chi_A(y)\\ &= \tilde{u}(y,z)\chi_{A_\infty}(y) + \max\{\lambda_0, \lambda_0\phi(y,z)\}\chi_A(y) = \tilde{u}(y,z)\chi_{A_\infty\cup A}(y).\end{aligned} \tag{4-5}$$

For any $j \in \mathbb{N}$ with $j > 1/\varepsilon$, consider the set

$$B_j^\varepsilon(z) = \Big\{y \in \mathbb{R}^k : v_\varepsilon(y,z) \le \lambda_0 + \varepsilon - \frac{1}{j}\Big\}.$$

Then $B_j^\varepsilon \subseteq B_{j+1}^\varepsilon$ and $B_\infty^\varepsilon = \bigcup_{j>1/\varepsilon} B_j^\varepsilon = \{y \in \mathbb{R}^k : v_\varepsilon(y,z) < \lambda_0 + \varepsilon\}$. Moreover, we have

$$\mathcal{H}^k(B_j^\varepsilon) \le \mathcal{H}^k(A_\infty) + \mathcal{H}^k\Big(\Big\{y \in A : \max\{\lambda_0, (\lambda_0+\varepsilon)\phi(y,z)\} \le \lambda_0 + \varepsilon - \frac{1}{j}\Big\}\Big). \tag{4-6}$$

Choose $R > 0$ large enough (depending on ε, j, λ_0, and z) that

$$(\lambda_0+\varepsilon)e^{-R^2-|z|^2} < \frac{1}{j}.$$

Then $(\lambda_0+\varepsilon)\phi(y,z) > \lambda_0 + \varepsilon - 1/j > \lambda_0$ for all $y \in B_R^c$, and thus

$$\mathcal{H}^k\Big(\Big\{y \in A \cap B_R^c : \max\{\lambda_0, (\lambda_0+\varepsilon)\phi(y,z)\} \le \lambda_0 + \varepsilon - \frac{1}{j}\Big\}\Big) = 0. \tag{4-7}$$

By (4-6) and (4-7), we see that

$$\mathcal{H}^k(B_j^\varepsilon(z)) \le \mathcal{H}^k(A_\infty) + \mathcal{H}^k(A \cap B_R) < \infty.$$

Furthermore, $A \subseteq B_\infty^\varepsilon$, and thus, by (4-4), we get

$$\mathcal{H}^k(B_\infty^\varepsilon) \ge \mathcal{H}^k(A) = \infty.$$

In particular, v_ε satisfies the assumptions of Case 2A, so there exists $K_\varepsilon \in \mathcal{K}_k^s$ defined by

$$K_\varepsilon(y,z) = ((\omega_k^{-1}\sigma_\varepsilon(y))^{k/2} + |z|^2)^{-(n+2s)/2}\chi_{B_\infty^\varepsilon}(y), \tag{4-8}$$

with $\sigma_\varepsilon : B_\infty^\varepsilon \to [0,\infty)$ measure-preserving, depending on v_ε, such that

$$\inf_{K\in\mathcal{K}_k^s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} v_\varepsilon(y,z)K(y,z)\,dy\,dz = \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} v_\varepsilon(y,z)K_\varepsilon(y,z)\,dy\,dz. \tag{4-9}$$

Finally, we need to pass to the limit. First, we prove that $\{\sigma_\varepsilon\}_{\varepsilon>0}$ is monotone decreasing. Indeed, let $V_\varepsilon = \{y \in \mathbb{R}^k : v_\varepsilon(y,z) = \tilde{u}(y,z)\}$. In particular, $A_\infty \subseteq V_\varepsilon \subseteq A_\infty \cup A$. Also, $V_{\varepsilon_2} \subseteq V_{\varepsilon_1}$ for any $\varepsilon_1 \le \varepsilon_2$. By Ryff's theorem, recall that

$$v_{\varepsilon_1}(y,z) = (v_{\varepsilon_1})_*(\sigma_{\varepsilon_1}(y)) \quad \text{and} \quad v_{\varepsilon_2}(y,z) = (v_{\varepsilon_2})_*(\sigma_{\varepsilon_2}(y)).$$

Since $v_{\varepsilon_2}(y,z) = v_{\varepsilon_1}(y,z)$ for all $y \in V_{\varepsilon_2}$ and $v_{\varepsilon_1}(y,z) \le v_{\varepsilon_2}(y,z)$ for all $y \in \mathbb{R}^k$, we see that

$$(v_{\varepsilon_2})_*(\sigma_{\varepsilon_2}(y)) = (v_{\varepsilon_1})_*(\sigma_{\varepsilon_1}(y)) \le (v_{\varepsilon_2})_*(\sigma_{\varepsilon_1}(y)) \quad \text{for all } \ y \in V_{\varepsilon_2}.$$

Since $(v_{\varepsilon_2})_*$ is monotone increasing, we must have that $\sigma_{\varepsilon_2}(y) \le \sigma_{\varepsilon_1}(y)$ for all $y \in V_{\varepsilon_2}$. Hence there exists $\sigma_0 : B_\infty \to [0,\infty)$ measure-preserving such that

$$\sigma_0(y) = \lim_{\varepsilon\to 0} \sigma_\varepsilon(y),$$

where $B_\infty = \bigcap_{\varepsilon>0} B_\infty^\varepsilon = \{y \in \mathbb{R}^k : \tilde{u}(y,z) \le \lambda_0\} = A_\infty \cup A$. In particular, the sequence of kernels $\{K_\varepsilon\}_{\varepsilon>0}$ is monotone decreasing. Define

$$K_0(y,z) = \lim_{\varepsilon\to 0} K_\varepsilon(y,z). \tag{4-10}$$

By (4-8) and (4-10), we have

$$K_0(y,z) = ((\omega_k^{-1}\sigma_0(y))^{k/2} + |z|^2)^{-(n+2s)/2}\chi_{B_\infty}(y).$$

Moreover, $K_0 \in \mathcal{K}_k^s$ since $K_\varepsilon \in \mathcal{K}_k^s$, and, for any $r > 0$, it follows that

$${}^k(D_0(r)) = \lim_{\varepsilon\to 0} \mathcal{H}^k(D_\varepsilon(r)),$$

where $D_\varepsilon(r) = \{y \in \mathbb{R}^k : K_\varepsilon(y,z) > r^{-(n+2s)}\}$.

Finally, using (4-5), (4-9), (4-10), and the monotone convergence theorem, we get

$$\begin{aligned}
\int_{\mathbb{R}^{n-k}}\int_{\mathbb{R}^k} \tilde{u}(y,z)K_0(y,z)\,dy\,dz &= \int_{\mathbb{R}^{n-k}}\int_{\mathbb{R}^k} \lim_{\varepsilon\to 0}(v_\varepsilon(y,z)K_\varepsilon(y,z))\,dy\,dz \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^{n-k}}\int_{\mathbb{R}^k} v_\varepsilon(y,z)K_\varepsilon(y,z)\,dy\,dz \\
&= \lim_{\varepsilon\to 0}\inf_{K\in\mathcal{K}_k^s}\int_{\mathbb{R}^{n-k}}\int_{\mathbb{R}^k} v_\varepsilon(y,z)K(y,z)\,dy\,dz \\
&\le \inf_{K\in\mathcal{K}_k^s}\int_{\mathbb{R}^{n-k}}\int_{\mathbb{R}^k} \bigl(\lim_{\varepsilon\to 0} v_\varepsilon(y,z)\bigr)K(y,z)\,dy\,dz \\
&= \inf_{K\in\mathcal{K}_k^s}\int_{\mathbb{R}^{n-k}}\int_{\mathbb{R}^k} \tilde{u}(y,z)(K(y,z)\chi_{A_\infty\cup A}(y))\,dy\,dz \\
&= \inf_{K\in\mathcal{K}_k^s}\int_{\mathbb{R}^{n-k}}\int_{\mathbb{R}^k} \tilde{u}(y,z)K(y,z)\,dy\,dz.
\end{aligned}$$

The last equality follows from the observation that, since

$$\widetilde{\mathcal{K}}_k^s = \{K \in \mathcal{K}_k^s : \operatorname{supp} K(\,\cdot\,,z) \subseteq A_\infty \cup A\} \subseteq \mathcal{K}_k^s,$$

the infimum over all kernels in $\mathcal{K}_k^s$ is less than or equal to the infimum over $\widetilde{\mathcal{K}}_k^s$. Moreover, the reverse inequality holds trivially. □

Finally, we deal with Case 3, that is, when all of the level sets of $\tilde{u}$ have infinite measure. In particular, notice that

$$\tilde{u}_{*,k}(x) = 0 \quad \text{for all } x \in \mathbb{R}^n.$$

This is the only case where the infimum is not attained. Indeed, we prove in the following theorem that the infimum is equal to zero.

Theorem 4.8. *Suppose that, for all $\lambda > 0$ and $z \in \mathbb{R}^{n-k}$,*

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda\}) = \infty.$$

Then $\mathcal{F}_k^s u(x_0) = 0$.

Proof. From (P_2), we have that $\mathcal{F}_k^s u(x_0) \geq 0$. To prove the reverse inequality, it is enough to find a sequence of kernels $\{K_\varepsilon\}_{\varepsilon>0} \subset \mathcal{K}_k^s$ such that

$$\liminf_{\varepsilon \to 0} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y, z) K_\varepsilon(y, z)\, dy\, dz = 0. \tag{4-11}$$

Fix $\varepsilon > 0$ and $z \in \mathbb{R}^{n-k}$. For any $j \geq 0$, we define the set

$$U_j(z) = \{y \in \mathbb{R}^k : \tilde{u}(y, z) < \varepsilon 2^{-j(n+2s)} e^{-|z|^2}\}.$$

Note that $U_{j+1} \subseteq U_j$. Also, by assumption, with $\lambda = \varepsilon 2^{-j(1+2s)} e^{-|z|^2}$, we have that

$$\mathcal{H}^k(U_j) = \infty \quad \text{for all } j \geq 0.$$

We will construct $K_\varepsilon \in \mathcal{K}_k^s$ by describing first where to locate each level set of the form

$$\begin{aligned} A_{-1} \equiv A_{-1}(z) &= \{y \in \mathbb{R}^k : 0 < K_\varepsilon(y, z) \leq 1\}, \\ A_j \equiv A_j(z) &= \{y \in \mathbb{R}^k : 2^{j(n+2s)} < K_\varepsilon(y, z) \leq 2^{(j+1)(n+2s)}\} \quad \text{for } j \geq 0. \end{aligned}$$

Recall that $K \in \mathcal{K}_k^s$ if, for all $r > 0$, we have

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : K(y, z) > r^{-(n+2s)}\}) = \mathcal{H}^k(\{y \in \mathbb{R}^k : (|y|^2 + |z|^2)^{-(n+2s)/2} > r^{-(n+2s)}\}).$$

In view of this definition, we define the sets

$$\begin{aligned} B_{-1} \equiv B_{-1}(z) &= \{y \in \mathbb{R}^k : 0 < (|y|^2 + |z|^2)^{-(n+2s)/2} \leq 1\}, \\ B_j \equiv B_j(z) &= \{y \in \mathbb{R}^k : 2^{j(n+2s)} < (|y|^2 + |z|^2)^{-(n+2s)/2} \leq 2^{(j+1)(n+2s)}\} \quad \text{for } j \geq 0. \end{aligned}$$

Note that

$$\begin{cases} \mathcal{H}^k(A_{-1}) = \mathcal{H}^k(B_{-1}) = \infty, \\ \mathcal{H}^k(A_j) = \mathcal{H}^k(B_j) < \infty \quad \text{for all } j \geq 0. \end{cases}$$

More precisely, for $j \geq 0$, if $|z| < 2^{-(j+1)} < 2^{-j}$, then

$$\mathcal{H}^k(A_j) = \mathcal{H}^k(B_{(2^{-2j}-|z|^2)^{1/2}}) - \mathcal{H}^k(B_{(2^{-2(j+1)}-|z|^2)^{1/2}}) = \omega_k (2^{-2j} - |z|^2)^{k/2} - \omega_k (2^{-2(j+1)} - |z|^2)^{k/2} \leq \omega_k 2^{-kj}.$$

If $2^{-(j+1)} \leq |z| < 2^{-j}$, then

$$\mathcal{H}^k(A_j) = \mathcal{H}^k(B_{(2^{-2j}-|z|^2)^{1/2}}) = \omega_k (2^{-2j} - |z|^2)^{k/2} \leq \omega_k \left(\tfrac{3}{4}\right)^{k/2} 2^{-kj}.$$

If $|z| \geq 2^{-j} > 2^{-(j+1)}$, then

$$\mathcal{H}^k(A_j) = 0.$$

Therefore, $\mathcal{H}^k(A_j) \leq c2^{-kj}$, where $c > 0$ only depends on k. It follows that

$$\mathcal{H}^k\biggl(\bigcup_{j=0}^{\infty} A_j\biggr) = \sum_{j=0}^{\infty} \mathcal{H}^k(A_j) \leq c \sum_{j=0}^{\infty} 2^{-jk} < \infty. \tag{4-12}$$

For any $i \geq 0$, let $\mathcal{D}_i$ be the collection of all dyadic closed cubes of the form

$$[m2^{-i}, (m+1)2^{-i}]^k = [m2^{-i}, (m+1)2^{-i}] \times \cdots \times [m2^{-i}, (m+1)2^{-i}].$$

Note that if $Q \in \mathcal{D}_i$, then $l(Q) = 2^{-i}$, where $l(Q)$ denotes the side length of the cube Q. For any $j \geq 0$, since U_j is an open set, by a standard covering argument, we have that there exists a family of dyadic cubes $\mathcal{F}_j$ such that

$$U_j = \bigcup_{Q \in \mathcal{F}_j} Q$$

satisfying the following properties:

(1) For any $Q \in \mathcal{F}_j$, there exists some $i \geq 0$ such that $Q \in \mathcal{D}_i$.

(2) $\mathrm{Int}(Q) \cap \mathrm{Int}(\widetilde{Q})\,\mathrm{Int}(\widetilde{Q}) = \varnothing$ for any $Q, \widetilde{Q} \in \mathcal{F}_j$ with $Q \neq \widetilde{Q}$.

(3) If $x \in Q \in \mathcal{F}_j$, then Q is the maximal dyadic cube contained in U_j that contains x.

Analogously, for the sets B_j with $j \geq -1$, there exists a family of dyadic cubes $\widetilde{\mathcal{F}}_j$ satisfying properties (1)–(3) such that

$$\mathrm{Int}(B_j) = \bigcup_{Q \in \widetilde{\mathcal{F}}_j} Q.$$

Note that $\widetilde{\mathcal{F}}_j \cap \widetilde{\mathcal{F}}_{j+1} = \varnothing$ since $B_j \cap B_{j+1} = \varnothing$.

We will construct the sets A_j by properly translating the dyadic cubes partitioning the sets B_j into U_j. In particular, we will prove that

$$\begin{cases} A_0 = T_0(B_0) \subset U_0, \\ A_j = T_j(B_j) \subset U_j \setminus \bigcup_{i=0}^{j-1} A_i \quad \text{for all } j \geq 1, \\ A_{-1} = T_{-1}(B_{-1}) \subset U_0 \setminus \bigcup_{i=0}^{\infty} A_i, \end{cases}$$

for some *translation* mappings $T_j : \widetilde{\mathcal{F}}_j \to \mathcal{F}_j$ to be determined.

We start with the case $j = 0$. For any $i \geq 0$, write

$$m_i = \mathcal{H}^0(\mathcal{F}_0 \cap \mathcal{D}_i) \quad \text{and} \quad n_i = \mathcal{H}^0(\widetilde{\mathcal{F}}_0 \cap \mathcal{D}_i),$$

where $\mathcal{H}^0(E)$ is equal to the cardinal of the set E. Note that $m_i,\ n_i \in \mathbb{Z}^+ \cup \{\infty\}$.

We will recursively place B_0 into U_0. First, fix $i = 0$. If $m_0 \geq n_0$, then, for any $\widetilde{Q} \in \widetilde{\mathcal{F}}_0 \cap \mathcal{D}_0$, there exists some $\tau \in \mathbb{R}^k$ and some $Q \in \mathcal{F}_0 \cap \mathcal{D}_0$ such that $Q = \widetilde{Q} + \tau$. Then define

$$T_0 : \widetilde{\mathcal{F}}_0 \cap \mathcal{D}_0 \to \mathcal{F}_0 \cap \mathcal{D}_0, \quad T_0(\widetilde{Q}) = Q. \tag{4-13}$$

Moreover, we can define T_0 to be one-to-one since $m_0 \geq n_0$, and we can always choose a different Q for each $\widetilde{Q}$. Note that there are p_0 cubes in $\mathcal{F}_0 \cap D_0$ with $p_0 = m_0 - n_0$ that have not been used. Hence for all of these cubes, divide each side in half, so that each cube gives rise to 2^k cubes with side length 2^{-1}. Call this collection of new cubes $\mathcal{Q} = \{Q_l\}_{l=1}^{2^{kp_0}} \subset \mathcal{D}_1$ and add them to the family $\mathcal{F}_0 \cap \mathcal{D}_1$. Namely, we replace $\mathcal{F}_0 \cap \mathcal{D}_1$ by $(\mathcal{F}_0 \cap \mathcal{D}_1) \cup \mathcal{Q}$.

If $m_0 < n_0$, then take q_0 cubes in $\widetilde{\mathcal{F}}_0 \cap \mathcal{D}_0$ with $q_0 = n_0 - m_0$ and divide each side in half. Call this collection of new cubes $\widetilde{\mathcal{Q}} = \{\widetilde{Q}_l\}_{l=1}^{2^{kq_0}} \subset \mathcal{D}_1$. Then, we replace $\widetilde{\mathcal{F}}_0$ by $\widehat{\mathcal{F}}_0$, where

$$\begin{aligned}
\widehat{\mathcal{F}}_0 \cap \mathcal{D}_0 &= (\widetilde{\mathcal{F}}_0 \setminus \widetilde{\mathcal{Q}}) \cap \mathcal{D}_0, \\
\widehat{\mathcal{F}}_0 \cap \mathcal{D}_1 &= (\widetilde{\mathcal{F}}_0 \cup \widetilde{\mathcal{Q}}) \cap \mathcal{D}_1, \\
\widehat{\mathcal{F}}_0 \cap \mathcal{D}_i &= \widetilde{\mathcal{F}}_0 \cap \mathcal{D}_i \quad \text{for all } i \geq 2.
\end{aligned}$$

If $\hat{n}_0 = \mathcal{H}^0(\widehat{\mathcal{F}} \cap \mathcal{D}_0)$, then $m_0 = \hat{n}_0$. Hence, by the same argument as in the previous case, we find T_0 as in (4-13). For $i \geq 1$, we can repeat the same process until we run out of cubes from $\widetilde{\mathcal{F}}_0$ (or the modified family). We know the process will end since $\mathcal{H}^k(B_0) < \mathcal{H}^k(U_0)$. When this happens, we will have constructed a one-to-one mapping $T_0 : \widetilde{\mathcal{F}}_0 \to \mathcal{F}_0$, since $\widetilde{\mathcal{F}}_0 = \bigcup_{i=0}^\infty \widetilde{\mathcal{F}}_0 \cap \mathcal{D}_i$ and $\mathcal{F}_0 = \bigcup_{i=0}^\infty \mathcal{F}_0 \cap \mathcal{D}_i$. Then define

$$A_0 = T_0(B_0) \subset U_0.$$

Iterating this process, we find a sequence of translation mappings $\{T_j\}_{j=0}^\infty$ with $T_j : \widetilde{F}_j \to \mathcal{F}_j$ and a sequence of disjoint sets $\{A_j\}_{j=0}^\infty$ such that

$$A_j = T_j(B_j) \subset U_j \setminus \bigcup_{i=0}^{j-1} A_i.$$

The case $j = -1$ is somewhat special since $\mathcal{H}^k(A_{-1}) = \mathcal{H}^k(B_{-1}) = \infty$. We will see that

$$A_{-1} = T_{-1}(B_{-1}) \subset U_0 \setminus \bigcup_{i=0}^{\infty} A_i.$$

This is possible because $\mathcal{H}^k\big(U_0 \setminus \bigcup_{i=0}^\infty A_i\big) = \infty$ using (4-12). Indeed, we can write

$$\{y \in \mathbb{R}^k : 0 < K_\varepsilon(y, z) \leq 1\} = \bigcup_{j=0}^{\infty} \{2^{-(j+1)(n+2s)} < K_\varepsilon(y, z) \leq 2^{-j(n+2s)}\}.$$

Now write

$$C_j = \{2^{-(j+1)(n+2s)} < (|y|^2 + |z|^2)^{-(n+2s)/2} \leq 2^{-j(n+2s)}\} \quad \text{for } j \geq 0.$$

Then $B_{-1} = \bigcup_{j=0}^\infty C_j$ with $\mathcal{H}^k(C_j) < \infty$ for all $j \geq 0$. Hence, instead of partitioning all of B_{-1} into dyadic cubes, we partition each of its disjoint components C_j. Arguing as before, we place them into $U_0 \setminus \bigcup_{i=0}^\infty A_i$ recursively, according to the following scheme:

$$\begin{cases} T_{-1}^0(C_0) \subset U_0 \setminus \bigcup_{i=0}^\infty A_i, \\ T_{-1}^j(C_j) \subset U_0 \setminus \big(\bigcup_{i=0}^\infty A_i \cup \bigcup_{i=0}^{j-1} C_i\big) \quad \text{for } j \geq 1, \end{cases}$$

where T^j_{-1} is defined as before. At the end of this process, we find a translation map T_{-1} defined by $T_{-1}(Q) = T^j_{-1}(Q)$ for $Q \in C_j$. Therefore, we define

$$A_{-1} = T_{-1}(B_{-1}).$$

Lastly, let $y \in \mathbb{R}^k = A_{-1} \cup \big(\bigcup_{j=0}^{\infty} A_j\big)$. In particular, there exists some $j \geq -1$ such that $y \in A_j$. Furthermore, recall that $A_j = T_j(B_j)$, where T_j is a one-to-one and onto translation map. Hence there exists a unique $w \in B_j$ such that $y = T_j(w) = w + \tau$ for some $\tau \in \mathbb{R}^k$. Let $T_z : \mathbb{R}^k \to \mathbb{R}^k$ be given by $T_z(y) = w$. Note that T_z is measure-preserving. Then we define the kernel

$$K_\varepsilon(y, z) = (|T_z(y)|^2 + |z|^2)^{-(n+2s)/2}.$$

We have

$$\int_{\mathbb{R}^k} \tilde{u}(y, z) K_\varepsilon(y, z)\, dy = \int_{A_{-1}} \tilde{u}(y, z) K_\varepsilon(y, z)\, dy + \sum_{j=0}^{\infty} \int_{A_j} \tilde{u}(y, z) K_\varepsilon(y, z)\, dy \equiv \mathrm{I} + \mathrm{II}.$$

For I, we use that $\tilde{u}(y, z) \leq \varepsilon e^{-|z|^2}$, since $A_{-1} \subset U_0$. Then by Lemmas 3.7 and 3.4,

$$\begin{aligned} \mathrm{I} \leq \varepsilon e^{-|z|^2} \int_{\{0 < K_\varepsilon(y,z) \leq 1\}} K_\varepsilon(y, z)\, dy &= \varepsilon e^{-|z|^2} \int_{\{0 < |\sigma_z(y)|^{-n-2s} \leq 1\}} |\sigma_z(y)|^{-n-2s}\, dy \\ &= \varepsilon e^{-|z|^2} \int_{\{|y| \geq 1\}} |y|^{-n-2s}\, dy = C\varepsilon e^{-|z|^2}, \end{aligned}$$

where $C > 0$ depends only on n and s. For II, we use that $\tilde{u}(y, z) \leq \varepsilon 2^{-j(n+2s)} e^{-|z|^2}$, since $A_j \subset U_j$ and $K_\varepsilon(y, z) \leq 2^{(j+1)(n+2s)}$ in A_j, by definition. Then

$$\mathrm{II} \leq \varepsilon e^{-|z|^2} \sum_{j=0}^{\infty} 2^{-j(n+2s)} 2^{(j+1)(n+2s)} \mathcal{H}^k(A_j) \leq c\varepsilon e^{-|z|^2} 2^{n+2s} \sum_{j=0}^{\infty} 2^{-kj} \leq C\varepsilon e^{-|z|^2},$$

where $C > 0$ depends only on $n,\ s$, and k.

Integrating over z, we see that

$$\int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y, z) K_\varepsilon(y, z)\, dy\, dz \leq C\varepsilon \int_{\mathbb{R}^{n-k}} e^{-|z|^2}\, dz \leq \widetilde{C}\varepsilon.$$

Letting $\varepsilon \to 0$, we conclude (4-11). □

Limit as s → 1. Let $u \in C^2(\mathbb{R}^n)$. We define $\mathrm{MA}_k u$ as the Monge–Ampère operator acting on u with respect to the first k variables, that is,

$$\mathrm{MA}_k u(x) = k(\det((u_{ij}(x))_{1 \leq i, j \leq k}))^{1/k},$$

with $D^2 u(x) = (u_{ij}(x))_{1 \leq i, j \leq n}$. We define $\Delta_{n-k} u$ as the Laplacian of u with respect to the last $n - k$ variables, that is,

$$\Delta_{n-k} u(x) = \sum_{i=k+1}^{n} u_{ii}(x).$$

Then under some *special* conditions,

$$\lim_{s\to 1} \mathcal{F}_k^s u(x) = \mathrm{MA}_k u(x) + \Delta_{n-k} u(x). \tag{4-14}$$

In particular, the operators in the family $\{\mathcal{F}_k^s\}_{k=1}^{n-1}$ can be understood as nonlocal analogs of concave second order elliptic operators, which are decomposed into a Monge–Ampère operator restricted to $\mathbb{R}^k$ and a Laplacian restricted to $\mathbb{R}^{n-k}$.

Indeed, by Corollary 4.5, we have $\mathcal{F}_k^s u(x) = \Delta^s \tilde{u}_{*,k}(0)$. Since the k-symmetric rearrangement does not depend on s and $\Delta^s \to \Delta$ as $s \to 1$, passing to the limit we see that

$$\lim_{s\to 1} \mathcal{F}_k^s u(x) = \Delta \tilde{u}_{*,k}(0).$$

Suppose that $\tilde{u}_{*,k}(y,z) = \tilde{u}(\varphi_z^{-1}(y), z)$, where $\varphi_z : \mathbb{R}^k \to \mathbb{R}^k$ is an invertible measure-preserving transformation with $\varphi_z(0) = 0$ and

$$\omega_k |\varphi_z(y)|^{1/k} = \sigma_z(y).$$

Recall that σ_z is given in Theorem 4.1 (see also Remark 4.4). In this case,

$$\Delta \tilde{u}_{*,k}(0) = \Delta_y \tilde{u}(\varphi_z^{-1}(y), z) + \Delta_z \tilde{u}(\varphi_z^{-1}(y), z)|_{(y,z)=(0,0)}. \tag{4-15}$$

For the first term, we use

$$\mathrm{MA}_k u(x) = \inf_{\psi\in\Psi} \Delta(\tilde{u}\circ\psi)(0),$$

where $\Psi = \{\psi : \mathbb{R}^k \to \mathbb{R}^k$ measure-preserving such that $\psi(0) = 0\}$, and the fact that the infimum is attained when $\tilde{u}\circ\psi$ is a radially symmetric increasing function [Caffarelli and Silvestre 2016]. Hence

$$\Delta_y \tilde{u}(\varphi_z^{-1}(y), z)|_{(y,z)=(0,0)} = \mathrm{MA}_k u(x). \tag{4-16}$$

For the second term, write $\phi(y,z) = (\varphi_z^{-1}(y), z)$ and compute

$$\Delta_z(\tilde{u}\circ\phi)(0) = \mathrm{tr}(D_z\phi(0)^T D_z^2 \tilde{u}(\phi(0)) D_z\phi(0)) + \nabla_z \tilde{u}(\phi(0))^T \cdot \Delta_z \phi(0).$$

Recall that $\phi(0) = 0$ and $\tilde{u}(y,z) = u(x+(y,z)) - u(x) - \nabla_y u(x)\cdot y - \nabla_z u(x)\cdot z$. Then

$$\nabla_z \tilde{u}(\phi(0)) = 0, \quad D_z^2 \tilde{u}(\phi(0)) = D_z^2 u(x), \quad \text{and} \quad D_z\phi(0) = (0, I_{n-k}),$$

where I_{n-k} denotes the identity matrix in M_{n-k}. Therefore,

$$\Delta_z \tilde{u}(\varphi_z^{-1}(y), z)|_{(y,z)=(0,0)} = \Delta_z(\tilde{u}\circ\phi)(0) = \mathrm{tr}(D_z^2 u(x)) = \Delta_{n-k} u(x). \tag{4-17}$$

Combining (4-15)–(4-17) we conclude (4-14).

Connection to optimal transport. In Corollary 4.5 we obtained a representation of the function $\mathcal{F}_k^s u$ in terms of the k-symmetric increasing rearrangement. Using this representation, we find an equivalent expression of $\mathcal{F}_k^s u$ that can be understood from the viewpoint of optimal transport.

Theorem 4.9. *Suppose we are under the assumptions of Theorem 4.1. Then, for any* $z \in \mathbb{R}^{n-k}$, $z \neq 0$, *there exists an invertible map* $\varphi_z : \mathbb{R}^k \to \mathbb{R}^k$ *such that*

$$\mathcal{F}_k^s u(x) = c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \frac{\tilde{u}(\varphi_z^{-1}(y), z)}{(|y|^2 + |z|^2)^{(n+2s)/2}} \, dy \, dz. \tag{4-18}$$

Moreover, if $\sigma_z : \mathbb{R}^k \to [0, \infty)$ *is the Ryff's map given in Theorem 4.1, then* φ_z *is measure-preserving if and only if*

$$\omega_k |\varphi_z(y)|^k = \sigma_z(y) \quad \text{for a.e. } y \in \mathbb{R}^k. \tag{4-19}$$

The key tool to prove Theorem 4.9 is Brenier–McCann's theorem, a very well-known result in the theory of optimal transport [Brenier 1991; McCann 1995]. We state it here in the form that we will use it.

Theorem 4.10. *Let* $f, g \in L^1(\mathbb{R}^k)$. *Assume that*

$$\|f\|_{L^1(\mathbb{R}^k)} = \|g\|_{L^1(\mathbb{R}^k)}.$$

Then there exists a convex function $\psi : \mathbb{R}^k \to \mathbb{R}$ *whose gradient* $\nabla\psi$ *pushes forward* $f\,dy$ *to* $g\,dy$. *Namely, for any measurable function* h *in* $\mathbb{R}^k$,

$$\int_{\mathbb{R}^k} h(y) g(y) \, dy = \int_{\mathbb{R}^k} h(\nabla\psi(y)) f(y) \, dy. \tag{4-20}$$

Moreover, $\nabla\psi : \mathbb{R}^k \to \mathbb{R}^k$ *is invertible and unique.*

In the literature, $\nabla\psi$ is known as the (optimal) transport map.

Proof of Theorem 4.9. Fix $z \in \mathbb{R}^{n-k}$, $z \neq 0$, and consider $f_z, g_z \in L^1(\mathbb{R}^k)$ given by

$$f_z(y) = (|y|^2 + |z|^2)^{-(n+2s)/2} \quad \text{and} \quad g_z(y) = ((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2)^{-(n+2s)/2},$$

where $\sigma_z : \mathbb{R}^k \to [0, \infty)$ is given in Theorem 4.1. Note that

$$\begin{aligned} \|f\|_{L^1(\mathbb{R}^k)} &= \int_{\mathbb{R}^k} ((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2)^{-(n+2s)/2} \, dy \\ &= k\omega_k \int_0^\infty (r^2 + |z|^2)^{-(n+2s)/2} r^{k-1} \, dr \\ &= \int_{\mathbb{R}^k} (|y|^2 + |z|^2)^{-(n+2s)/2} \, dy = \|g\|_{L^1(\mathbb{R}^k)}, \end{aligned}$$

since σ_z is measure-preserving. By Theorem 4.10, there exists a convex function $\psi_z : \mathbb{R}^k \to \mathbb{R}$ (depending on z) whose gradient $\nabla\psi_z$ pushes forward $f_z\,dy$ to $g_z\,dy$. Moreover, $\nabla\psi_z$ is invertible and unique. Write $\varphi_z = (\nabla\psi_z)^{-1}$. Using (4-20) with $h(y) = \tilde{u}(y, z)$, we see that

$$\int_{\mathbb{R}^k} \frac{\tilde{u}(y, z)}{((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2)^{(n+2s)/2}} \, dy = \int_{\mathbb{R}^k} \frac{\tilde{u}(\varphi_z^{-1}(y), z)}{(|y|^2 + |z|^2)^{(n+2s)/2}} \, dy. \tag{4-21}$$

Integrating over $z \in \mathbb{R}^{n-k}$, we obtain (4-18).

It remains to show that φ_z is measure-preserving if and only if (4-19) holds. Indeed, for any measurable set $E \subset \mathbb{R}^k$, we have

$$\begin{aligned}\mathcal{H}^k(\varphi_z^{-1}(E)) = \int_{\varphi_z^{-1}(E)} dy &= \int_{\varphi_z^{-1}(E)} \frac{(|y|^2+|z|^2)^{(n+2s)/2}}{(|y|^2+|z|^2)^{(n+2s)/2}}\,dy \\ &= \int_{\varphi_z^{-1}(E)} \frac{(|\varphi_z(\varphi_z^{-1}(y))|^2+|z|^2)^{(n+2s)/2}}{(|y|^2+|z|^2)^{(n+2s)/2}}\,dy \\ &= \int_E \frac{(|\varphi_z(y)|^2+|z|^2)^{(n+2s)/2}}{((\omega_k^{-1}\sigma_z(y))^{2/k}+|z|^2)^{(n+2s)/2}}\,dy,\end{aligned}$$

where the last equality follows from (4-21) with $h(y) = (|\varphi_z(y)|^2+|z|^2)^{(n+2s)/2}\chi_E(y)$. Therefore,

$$\mathcal{H}^k(\varphi_z^{-1}(E)) = \mathcal{H}^k(E)$$

if and only if $\omega_k|\varphi_z(y)|^k = \sigma_z(y)$ for a.e. $y \in \mathbb{R}^k$. □

5. Regularity of $\mathcal{F}_k^s u$

Given $x_0 \in \mathbb{R}^n$, we define the sections

$$D_{x_0}u(t) = \{x \in \mathbb{R}^n : u(x) - u(x_0) - (x-x_0)\cdot\nabla u(x_0) \le t\} \quad \text{for } t > 0.$$

Our main regularity result is the following.

Theorem 5.1. *Let $s \in \left(\frac{1}{2}, 1\right)$ and $1 \le k < n$. Let $u \in C^{1,1}(\mathbb{R}^n)$ be convex. Fix $x_0 \in \mathbb{R}^n$ and $r_0,\ \varepsilon > 0$. Suppose that $\Lambda = \sup_{x\in B_{r_0}(x_0)} \operatorname{diam}(D_x u(\varepsilon)) < \infty$ and $M = \sup_{x\in B_{r_0}(x_0)} \mathcal{F}_k^s u(x) < \infty$. Then we have $\mathcal{F}_k^s u \in C^{0,1-s}(\overline{B_r(x_0)})$ with $r < \min\{r_0/4,\ \Lambda,\ \varepsilon/(8\Lambda)\}$ and*

$$[\mathcal{F}_k^s]_{C^{0,1-s}(\overline{B_r(x_0)})} \le C_0[u]_{C^{1,1}(\mathbb{R}^n)}$$

for some constant $C_0 > 0$ depending only on $n,\ k,\ s,\ \varepsilon,\ \Lambda$, and M.

This theorem will be a consequence of the next proposition.

Proposition 5.2. *Fix $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$. Suppose that $\Lambda = \operatorname{diam}(D_{x_0}u(\varepsilon)) < \infty$ and $[u]_{C^{1,1}(\mathbb{R}^n)} \le 1$. Then, for any $x_1 \in B_r(x_0)$ with $r \le \varepsilon/(4\Lambda)$, we have*

$$\mathcal{F}_k^s u(x_1) - \mathcal{F}_k^s u(x_0) \le C\Lambda^{1-s}|x_1-x_0|^{1-s} + \frac{4\Lambda}{\varepsilon}|x_1-x_0|\mathcal{F}_k^s u(x_0)$$

for some $C > 0$ depending only on $n,\ k$, and s.

First, we prove Theorem 5.1.

Proof of Theorem 5.1. Without loss of generality, we may assume that $[u]_{C^{1,1}(\mathbb{R}^n)} \le 1$. Otherwise, we consider $u/[u]_{C^{1,1}(\mathbb{R}^n)}$. Let $r < \min\{r_0/4,\ \Lambda,\ \varepsilon/(8\Lambda)\}$. It is enough to show that

$$[\mathcal{F}_k^s]_{C^{0,1-s}(\overline{B_r(x_0)})} \le C_0 \tag{5-1}$$

for some constant $C_0 > 0$ depending only on $n,\ k,\ s,\ \varepsilon,\ \Lambda$, and M.

Let $x_1,\ x_2 \in \overline{B_r(x_0)}$. Then $x_2 \in \overline{B_{2r}(x_1)} \subset B_{r_0}(x_0)$, since $4r < r_0$. Moreover, $\operatorname{diam}(D_{x_1}u(\varepsilon)) \le \Lambda < \infty$. Hence, applying Proposition 5.2 to u and $B_{2r}(x_1)$ in place of $B_r(x_0)$, we get

$$\mathcal{F}_k^s u(x_2) - \mathcal{F}_k^s u(x_1) \le C\Lambda^{1-s}|x_2 - x_1|^{1-s} + \frac{4\Lambda}{\varepsilon}|x_2 - x_1|\mathcal{F}_k^s u(x_1) \le C_0|x_2 - x_1|^{1-s},$$

where $C_0 = C\Lambda^{1-s} + 4\Lambda^{1+s}M/(\varepsilon 2^s)$. Since x_1 and x_2 are arbitrary, we conclude (5-1). □

Before we prove Proposition 5.2, we need several preliminary results.

Lemma 5.3. *If f is monotone increasing, then*

$$\int_0^\infty f(r)\omega(r)\,dr = \int_0^\infty\int_{\mu_f(t)}^\infty \omega(r)\,dr\,dt,$$

with $\mu_f(t) = |\{r > 0 : f(r) \le t\}|$.

Proof. By Fubini's theorem, we have

$$\int_0^\infty\int_{\mu_f(t)}^\infty \omega(r)\,dr\,dt = \int_0^\infty \omega(r)\int_{\{r>\mu_f(t)\}} dt\,dr.$$

Since f is monotone increasing, $r > \mu_f(t)$ if and only if $t < f(r)$. Therefore,

$$\int_{\{r>\mu_f(t)\}} dt = \int_0^{f(r)} dt = f(r).$$ □

Proposition 5.4. *Let $x \in \mathbb{R}^n$. Under the assumptions of Corollary 4.5,*

$$\mathcal{F}_k^s u(x) = c_{n,s}\int_0^\infty\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\biggl(\frac{\mu_x u(t,z)^{1/k}}{|z|}\biggr)\,dz\,dt,$$

where $\mu_x u(t,z) = \omega_k^{-1}\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}_x(y,z) \le t\})$ and

$$W(\rho) = k\omega_k\int_\rho^\infty \frac{r^{k-1}}{(1+r^2)^{(n+2s)/2}}\,dr. \tag{5-2}$$

Proof. By Corollary 4.5, we have that

$$\begin{aligned}\mathcal{F}_k^s u(x) = \Delta^s\tilde{u}_{*,k}(0) &= c_{n,s}\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n+2s}}\biggl(\int_{\mathbb{R}^k} \frac{\tilde{u}_{*,k}(y,z)}{(||z|^{-1}y|^2+1)^{(n+2s)/2}}\,dy\biggr)\,dz\\ &= c_{n,s}\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}}\biggl(k\omega_k\int_0^\infty v(|z|r,z)\frac{r^{k-1}}{(r^2+1)^{(n+2s)/2}}\,dr\biggr)\,dz,\end{aligned}$$

where $v(r,z) = \tilde{u}_{*,k}(y,z)$ for $|y| = r$.

Next we apply Lemma 5.3 to $f(r) = v(|z|r,z)$ and $\omega(r) = k\omega_k r^{k-1}(r^2+1)^{-(n+2s)/2}$. Note that since v is the k-symmetric increasing rearrangement of $\tilde{u}$, we have

$$\mu_f(t) = \frac{1}{|z|}|\{r > 0 : v(r,z) < t\}| = \frac{\omega_k^{-1/k}}{|z|}\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}(y,z) < t\})^{1/k} = \frac{1}{|z|}\mu_x u(t,z)^{1/k}.$$

Therefore,

$$k\omega_k \int_0^\infty v(|z|r, z)\frac{r^{k-1}}{(r^2+1)^{(n+2s)/2}}\,dr = \int_0^\infty \biggl(k\omega_k \int_{\mu_x u(t,z)^{1/k}/|z|}^\infty \frac{r^{k-1}}{(r^2+1)^{(n+2s)/2}}\,dr\biggr)\,dt = \int_0^\infty W\biggl(\frac{\mu_x u(t,z)^{1/k}}{|z|}\biggr)\,dt,$$

where W is given in (5-2). By Fubini's theorem, we conclude that

$$\mathcal{F}_k^s u(x) = c_{n,s}\int_0^\infty\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\biggl(\frac{\mu_x u(t,z)^{1/k}}{|z|}\biggr)\,dz\,dt. \qquad \square$$

Lemma 5.5. *Suppose we are under the assumptions of Proposition 5.2. Let* $x_1 \in \overline{B_r(x_0)}$ *and* $d = |x_1 - x_0|$. *The following hold*:

(a) *If* $t \in (2\Lambda d, \varepsilon]$, *then* $D_{x_0}u(t-2\Lambda d) \subset D_{x_1}u(t)$.

(b) *If* $t \in (\varepsilon, \infty)$, *then* $D_{x_0}u(t-2\Lambda dt/\varepsilon) \subset D_{x_1}u(t)$.

Proof. First we prove (a). Fix $t \in (2\Lambda d, \varepsilon]$, and let $x \in D_{x_0}u(t-2\Lambda d)$. Then

$$u(x) - u(x_0) - (x-x_0)\cdot\nabla u(x_0) \le t - 2\Lambda d. \tag{5-3}$$

Using (5-3), convexity, and $[u]_{C^{1,1}(\mathbb{R}^n)} \le 1$, we see that

$$\begin{aligned} u(x) - u(x_1) - (x-x_1)\cdot\nabla u(x_1) &= u(x) - u(x_0) - (x-x_0)\cdot\nabla u(x_0) \\ &\quad - (u(x_1) - u(x_0) - (x_1-x_0)\cdot\nabla u(x_0)) \\ &\quad + (x-x_1)\cdot(\nabla u(x_0) - \nabla u(x_1)) \\ &\le t - 2\Lambda d + |x-x_1|d. \end{aligned}$$

Moreover, $x \in D_{x_0}u(\varepsilon)$, since $t \le \varepsilon$, and thus,

$$|x-x_1| \le |x-x_0| + |x_0-x_1| \le \Lambda + d \le 2\Lambda.$$

Therefore, $x \in D_{x_1}u(t)$.

Next we prove (b). Fix $t \in (\varepsilon, \infty)$, and let $x \in D_{x_0}u(t-2\Lambda dt/\varepsilon)$. By the previous computation, we have that

$$u(x) - u(x_1) - (x-x_1)\cdot\nabla u(x_1) \le t - 2\Lambda dt/\varepsilon + (|x-x_0| + \Lambda)d. \tag{5-4}$$

To control $|x-x_0|$, the distance from x to x_0, we need to estimate the diameter of $D_{x_0}u(t)$. We take $y \in D_{x_0}u(t) \setminus D_{x_0}u(\varepsilon)$ and let z be in the intersection of $\partial D_{x_0}u(\varepsilon)$ and the line segment joining x_0 and y. Then there is some $\lambda > 1$ such that $y - x_0 = \lambda(z-x_0)$. By convexity of u,

$$u(z) \le \frac{\lambda-1}{\lambda}u(x_0) + \frac{1}{\lambda}u(y).$$

Therefore,

$$\begin{aligned} \lambda\varepsilon &= \lambda(u(z) - u(x_0) - (z-x_0)\cdot\nabla u(x_0)) \\ &\le (\lambda-1)u(x_0) + u(y) - \lambda u(x_0) - (y-x_0)\cdot\nabla u(x_0) = u(y) - u(x_0) - (y-x_0)\cdot\nabla u(x_0) \le t, \end{aligned}$$

so $\lambda \le t/\varepsilon$. By convexity, we have that $D_{x_0}u(t) \subset x_0 + (t/\varepsilon)(D_{x_0}u(\varepsilon) - x_0)$. It follows that

$$\operatorname{diam} D_{x_0}u(t) \le \frac{t}{\varepsilon} \operatorname{diam} D_{x_0}u(\varepsilon) = \frac{\Lambda t}{\varepsilon}.$$

Hence $|x - x_0| \le \Lambda t/\varepsilon$, and, by (5-4), we get

$$u(x) - u(x_1) - (x - x_1) \cdot \nabla u(x_1) \le t - \frac{2\Lambda d t}{\varepsilon} + \Big(\frac{\Lambda t}{\varepsilon} + \Lambda\Big) d \le t,$$

which means that $x \in D_{x_1}u(t)$. □

We are ready to give the proof of Proposition 5.2.

Proof of Proposition 5.2. Let $x_1 \in B_r(x_0)$ with $r \le \varepsilon/(4\Lambda)$, and write $d = |x_0 - x_1|$. We will estimate $\mathcal{F}_k^s u(x_1)$ using Proposition 5.4:

$$\mathcal{F}_k^s u(x_1) = c_{n,s} \int_0^\infty \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\bigg(\frac{\mu_{x_1} u(t,z)^{1/k}}{|z|}\bigg)\, dz\, dt.$$

In view of Lemma 5.5, we separate the above integral into terms I + II + III by dividing the integral with respect to t into three parts as follows:

$$\mathrm{I}: \ t \in (0, 2\Lambda d], \quad \mathrm{II}: \ t \in (2\Lambda d, \varepsilon], \quad \mathrm{III}: \ t \in (\varepsilon, \infty).$$

Let us start with I. Since $u \in C^{1,1}(\mathbb{R}^n)$ with $[u]_{C^{1,1}(\mathbb{R}^n)} \le 1$, we have

$$\mu_{x_1} u(t,z) \ge (t - |z|^2)_+^{k/2}.$$

Hence, using that $W(\rho)$ is monotone decreasing, we get

$$W\bigg(\frac{\mu_{x_1} u(t,z)^{1/k}}{|z|}\bigg) \le W\bigg(\Big(\frac{t}{|z|^2} - 1\Big)_+^{1/2}\bigg).$$

Therefore,

$$\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\bigg(\frac{\mu_{x_1} u(t,z)^{1/k}}{|z|}\bigg) dz \le \int_{\{|z|<t^{1/2}\}} \frac{1}{|z|^{n-k+2s}} W\bigg(\Big(\frac{t}{|z|^2} - 1\Big)^{1/2}\bigg) dz + W(0) \int_{\{|z|>t^{1/2}\}} \frac{1}{|z|^{n-k+2s}}\, dz \equiv I_1 + I_2.$$

Note that $W(0) = C(n,k,s) < \infty$. Then

$$I_2 \lesssim \int_{t^{1/2}}^\infty \frac{1}{\rho^{n-k+2s}} \rho^{n-k-1}\, d\rho \approx t^{-s}.$$

For I_1, we make the change of variables $w = z/t^{1/2}$. We see that

$$I_1 = \int_{\{|w|<1\}} \frac{1}{t^{(n-k+2s)/2}|w|^{n-k+2s}} W\bigg(\Big(\frac{1}{|w|^2} - 1\Big)^{1/2}\bigg) t^{(n-k)/2}\, dw \approx \frac{1}{t^s} \int_0^1 \frac{1}{\rho^{1+2s}} W\bigg(\Big(\frac{1}{\rho^2} - 1\Big)^{1/2}\bigg) d\rho.$$

Note that if $0 < \rho \le \frac{1}{2}$, then

$$\Big(\frac{1}{\rho^2} - 1\Big)^{1/2} \ge \frac{1}{\sqrt{2}\rho}.$$

Hence

$$W\left(\left(\frac{1}{\rho^2}-1\right)^{1/2}\right) \le W\left(\frac{1}{\sqrt{2}\rho}\right) = \int_{1/(\sqrt{2}\rho)}^{\infty} \frac{r^{k-1}}{(1+r^2)^{(n+2s)/2}}\,dr \lesssim \rho^{n-k+2s}.$$

Therefore,

$$I_1 \lesssim t^{-s}\int_0^{1/2} \frac{1}{\rho^{1+2s}}\rho^{n-k+2s}\,d\rho + t^{-s}W(0)\int_{1/2}^1 \frac{1}{\rho^{1+2s}}\,d\rho \approx t^{-s},$$

since $n-k>0$. We conclude that

$$\begin{aligned} \mathrm{I} = c_{n,s}\int_0^{2\Lambda d}\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_1}u(t,z)^{1/k}}{|z|}\right) dz\,dt &\lesssim \int_0^{2\Lambda d} t^{-s}\,dt \\ &\approx (2\Lambda d)^{1-s} = (2\Lambda)^{1-s}|x_1-x_0|^{1-s}. \end{aligned}$$

Next we estimate the integral for $t \in (2\Lambda d, \varepsilon]$. To this end, we use Lemma 5.5 (a) to get

$$D_{x_0}u(t-2\Lambda d) \subset D_{x_1}u(t).$$

In particular, for any $z \in \mathbb{R}^{n-k}$ fixed, we have

$$\{y \in \mathbb{R}^k : \tilde{u}_{x_0}(y,z) \le t-2\Lambda d\} \subset \{y \in \mathbb{R}^k : \tilde{u}_{x_1}(y,z) \le t\}.$$

Hence $\mu_{x_0}(t-2\Lambda d, z) \le \mu_{x_1}(t,z)$, which yields

$$\begin{aligned} \mathrm{II} &= c_{n,s}\int_{2\Lambda d}^{\varepsilon}\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_1}u(t,z)^{1/k}}{|z|}\right) dz\,dt \\ &\le c_{n,s}\int_0^{\varepsilon-2\Lambda d}\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_0}u(t,z)^{1/k}}{|z|}\right) dz\,dt. \end{aligned}$$

Finally, we estimate the integral for $t \in [\varepsilon, \infty)$. By Lemma 5.5 (b),

$$D_{x_0}u\left(t-\frac{2\Lambda dt}{\varepsilon}\right) \subset D_{x_1}u(t).$$

Hence $\mu_{x_0}u(t-2\Lambda dt/\varepsilon, z) \le \mu_{x_1}u(t,z)$, and

$$\begin{aligned} \mathrm{III} &= c_{n,s}\int_{\varepsilon}^{\infty}\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_1}u(t,z)^{1/k}}{|z|}\right) dz\,dt \\ &\lesssim \int_{\varepsilon}^{\infty}\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_0}u(t-2\Lambda dt/\varepsilon,z)^{1/k}}{|z|}\right) dz\,dt \\ &= \frac{1}{1-2\Lambda d/\varepsilon}\int_{\varepsilon-2\Lambda d}^{\infty}\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_0}u(t,z)^{1/k}}{|z|}\right) dz\,dt. \end{aligned}$$

Note that

$$\mathrm{II}+\mathrm{III} \le \frac{c_{n,s}}{1-2\Lambda d/\varepsilon}\int_0^{\infty}\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_0}u(t,z)^{1/k}}{|z|}\right) dz\,dt = \frac{\varepsilon}{\varepsilon-2\Lambda d}\mathcal{F}_k^s u(x_0).$$

Therefore, we conclude that

$$\begin{aligned}\mathcal{F}_k^s u(x_1) - \mathcal{F}_k^s u(x_0) &\le C\Lambda^{1-s}|x_1 - x_0|^{1-s} + \left(\frac{\varepsilon}{\varepsilon - 2\Lambda d} - 1\right)\mathcal{F}_k^s u(x_0) \\ &\le C\Lambda^{1-s}|x_1 - x_0|^{1-s} + \frac{4\Lambda}{\varepsilon}|x_1 - x_0|\mathcal{F}_k^s u(x_0),\end{aligned}$$

since $d < r \le \varepsilon/(4\Lambda)$, and thus, $\varepsilon - 2\Lambda d \ge \varepsilon/2$. □

6. A global Poisson problem

We consider the following Poisson problem in the full space:

$$\begin{cases}\mathcal{F}_k^s u = u - \varphi & \text{in } \mathbb{R}^n, \\ (u-\varphi)(x) \to 0 & \text{as } |x| \to \infty,\end{cases} \tag{6-1}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is nonnegative, smooth, and strictly convex. Furthermore, we ask that φ behaves asymptotically at infinity as a cone ϕ, that is,

$$\lim_{|x|\to\infty} (\varphi - \phi)(x) = 0. \tag{6-2}$$

Similar problems have been studied for nonlocal Monge–Ampère operators [Caffarelli and Charro 2015; Caffarelli and Silvestre 2016].

We will prove the following theorem.

Theorem 6.1. *There exists a unique solution u to* (6-1) *such that $u \in C^{1,1}(\mathbb{R}^n)$ with*

$$[u]_{C^{1,1}(\mathbb{R}^n)} \le [\varphi]_{C^{1,1}(\mathbb{R}^n)}.$$

To define the notion of a solution, we introduce a natural pointwise definition of $\mathcal{F}_k^s u$ for functions u that are merely continuous.

Definition 6.2. Let $u \in C^0(\mathbb{R}^n)$.

(a) We say that a linear function $l(y) = y \cdot p + b$, with $p \in \mathbb{R}^n$ and $b \in \mathbb{R}$, is a supporting plane of u at a point x if $l(x) = u(x)$ and $l(y) \le u(y)$ for all $y \in \mathbb{R}^n$.

(b) We define the subdifferential of u at a point x as the set $\partial u(x)$ of all vectors $p \in \mathbb{R}^n$ such that $l(y) = y \cdot p + b$ is a supporting plane of u at x for some $b \in \mathbb{R}$.

Definition 6.3. Let $u \in C^0(\mathbb{R}^n)$ be a convex function. For $x_0 \in \mathbb{R}^n$, we define

$$\mathcal{F}_k^s u(x_0) = c_{n,s} \sup_{p \in \partial u(x_0)} \inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^n} (u(x_0 + x) - u(x_0) - x \cdot p) K(x)\, dx.$$

Remark 6.4. Note that if $u \in C^{1,1}(x_0)$, then $\partial u(x_0) = \{\nabla u(x_0)\}$, and the previous definition coincides with Definition 2.4.

The following properties of $\mathcal{F}_k^s u$ will be useful for our purposes. The proof is analogous to the one in [Caffarelli and Silvestre 2016], so we omit it here.

Lemma 6.5. *Let $u,\ v \in C^0(\mathbb{R}^n)$ be convex functions. The following hold*:

(a) (*homogeneity*) *For any $\lambda > 0$,*

$$\mathcal{F}_k^s(\lambda u) = \lambda \mathcal{F}_k^s u.$$

(b) (*monotonicity*) *Assume that $u(x_0) = v(x_0)$ and $u(x) \geq v(x)$ for all $x \in \mathbb{R}^n$. Then*

$$\mathcal{F}_k^s u(x_0) \geq \mathcal{F}_k^s v(x_0).$$

(c) (*concavity*) *For any $x \in \mathbb{R}^n$,*

$$\mathcal{F}_k^s\big(\tfrac{1}{2}(u+v)\big)(x) \geq \tfrac{1}{2}(\mathcal{F}_k^s u(x) + \mathcal{F}_k^s v(x)).$$

(d) (*lower semicontinuity*) *Assume that $u \in C^{1,1}(\mathbb{R}^n)$. Then*

$$\mathcal{F}_k^s u(x_0) \leq \liminf_{x \to x_0} \mathcal{F}_k^s u(x).$$

Definition 6.6. Let $u \in C^0(\mathbb{R}^n)$ be a convex function. We say that u is a subsolution to $\mathcal{F}_k^s u = u - \varphi$ in $\mathbb{R}^n$ if

$$\mathcal{F}_k^s u(x_0) \geq u(x_0) - \varphi(x_0) \quad \text{for all } x_0 \in \mathbb{R}^n.$$

Similarly, u is a supersolution if

$$\mathcal{F}_k^s u(x_0) \leq u(x_0) - \varphi(x_0) \quad \text{for all } x_0 \in \mathbb{R}^n.$$

We say that u is a solution if it is both a subsolution and a supersolution.

Lemma 6.7. *If u and v are subsolutions*, *then* $\max\{u, v\}$ *is a subsolution.*

Proof. Let $w = \max\{u, v\}$. Then w is continuous and convex. Fix $x_0 \in \mathbb{R}^n$. Without loss of generality, we may assume that $u(x_0) \geq v(x_0)$. Then $w(x_0) = u(x_0)$ and $w(x) \geq u(x)$ for any $x \in \mathbb{R}^n$. By monotonicity (see Lemma 6.5), we have

$$\mathcal{F}_k^s w(x_0) \geq \mathcal{F}_k^s u(x_0) \geq u(x_0) - \varphi(x_0) = w(x_0) - \varphi(x_0).$$

Hence w is a subsolution. □

We will show existence and uniqueness of solutions to (6-1) using Perron's method. The key ingredients are the comparison principle and the existence of a subsolution (lower barrier) and a supersolution (upper barrier). We state this in the following proposition. We omit the proof since it is similar to that in [Caffarelli and Silvestre 2016].

Proposition 6.8. *Consider the equation $\mathcal{F}_k^s u = u - \varphi$ in $\mathbb{R}^n$. The following hold*:

(a) (*comparison principle*) *Let u and v be a subsolution and supersolution*, *respectively. Assume that $u \leq v$ in $\mathbb{R}^n \setminus \Omega$ for some bounded domain $\Omega \subset \mathbb{R}^n$. Then $u \leq v$ in $\mathbb{R}^n$.*

(b) (*lower barrier*) *The function φ is a subsolution.*

(c) (*upper barrier*) *The function $\varphi + w$ is a supersolution*, *where $w = (I - \Delta^s)^{-1}\Delta^s \varphi$. In particular*, *$w(x) \leq C(1+|x|)^{1-2s}$ for some $C > 0$.*

An immediate consequence of the comparison principle is the uniqueness of solutions.

Lemma 6.9 (uniqueness). *There exists at most one solution to* (6-1).

Proof. Suppose by means of contradiction that there exist two functions $u,\ v \in C^0(\mathbb{R}^n)$, with $u \neq v$, satisfying (6-1). Then $|u(x) - v(x)| \to 0$ as $|x| \to \infty$. Hence, for any $\varepsilon > 0$, there exists a compact set $\Omega_\varepsilon \in \mathbb{R}^n$, depending on ε, such that

$$v(x) - \varepsilon \leq u(x) \leq v(x) + \varepsilon \quad \text{for all } x \in \mathbb{R}^n \setminus \Omega_\varepsilon.$$

Moreover, for any $x_0 \in \mathbb{R}^n$, the function $v + \varepsilon$ satisfies

$$\mathcal{F}_k^s(v + \varepsilon)(x_0) = v(x_0) - \varphi(x_0) < (v(x_0) + \varepsilon) - \varphi(x_0).$$

Therefore, v is a supersolution and, by the comparison principle, it follows that $u \leq v + \varepsilon$ in $\mathbb{R}^n$. Similarly, we see that $v - \varepsilon$ is a subsolution and $u \geq v - \varepsilon$ in $\mathbb{R}^n$. Hence

$$\|u - v\|_{L^\infty(\mathbb{R}^n)} \leq \varepsilon,$$

and letting $\varepsilon \to 0$, we get $u = v$ in $\mathbb{R}^n$, which is a contradiction. □

To prove existence of a solution, we define

$$u(x) = \sup_{v \in \mathcal{S}} v(x), \tag{6-3}$$

where $\mathcal{S}$ is the set of admissible subsolutions given by

$$\mathcal{S} = \{v \in C^{0,1}(\mathbb{R}^n) : v \text{ a subsolution, } \varphi \leq v \leq \varphi + w, \text{ and } [v]_{C^{0,1}(\mathbb{R}^n)} \leq [\varphi]_{C^{0,1}(\mathbb{R}^n)}\}.$$

Note that $\mathcal{S} \neq \varnothing$ since $\varphi \in \mathcal{S}$, and the supremum is finite since $v \leq \varphi + w$ for any $v \in \mathcal{S}$. Moreover, u is convex and Lipschitz with

$$[u]_{C^{0,1}(\mathbb{R}^n)} \leq [\varphi]_{C^{0,1}(\mathbb{R}^n)}.$$

From $\varphi \leq u \leq \varphi + w$ and the upper bound for w in Proposition 6.8, it follows that

$$0 \leq (u - \varphi)(x) \leq w(x) \leq C(1 + |x|)^{1-2s} \to 0$$

as $|x| \to \infty$, since $1 - 2s < 0$.

Proposition 6.10. *The function u given in* (6-3) *is $C^{1,1}(\mathbb{R}^n)$ with*

$$[u]_{C^{1,1}(\mathbb{R}^n)} \leq [\varphi]_{C^{1,1}(\mathbb{R}^n)}.$$

Proof. We will show that, for any $x_0,\ x_1 \in \mathbb{R}^n$,

$$0 \leq u(x_0 + x_1) - u(x_0 - x_1) - 2u(x_0) \leq [\varphi]_{C^{1,1}(\mathbb{R}^n)} |x_1|^2.$$

Indeed, the lower bound follows from convexity of u. Hence we only need to prove the upper bound. Write $M = [\varphi]_{C^{1,1}(\mathbb{R}^n)}$. Then

$$\varphi(x_0 + x_1) - \varphi(x_0 - x_1) - M|x_1|^2 \leq 2\varphi(x_0). \tag{6-4}$$

Take any $v \in \mathcal{S}$ and fix $x_1 \in \mathbb{R}^n$. Define

$$\hat{v}(x_0) = \tfrac{1}{2}(v(x_0+x_1)+v(x_0-x_1)-M|x_1|^2) \quad \text{for } x_0 \in \mathbb{R}^n.$$

We claim that $\hat{v}$ is a subsolution to $\mathcal{F}_k^s u = u - \varphi$ in $\mathbb{R}^n$. Indeed, since $\mathcal{F}_k^s$ is homogeneous of degree 1, concave, and translation-invariant (see Lemma 6.5), we have

$$\begin{aligned}
\mathcal{F}_k^s \hat{v}(x_0) &= \mathcal{F}_k^s\big(\tfrac{1}{2}v(x_0+x_1)+\tfrac{1}{2}v(x_0-x_1)\big)\\
&\geq \tfrac{1}{2}\mathcal{F}_k^s v(x_0+x_1)+\tfrac{1}{2}\mathcal{F}_k^s v(x_0-x_1)\\
&\geq \tfrac{1}{2}(v(x_0+x_1)-\varphi(x_0+x_1)+v(x_0-x_1)-\varphi(x_0-x_1))\\
&= \tfrac{1}{2}(v(x_0+x_1)-v(x_0-x_1)-M|x_1|^2)-\tfrac{1}{2}(\varphi(x_0+x_1)+\varphi(x_0-x_1)-M|x_1|^2)\\
&\geq \hat{v}(x_0)-\varphi(x_0).
\end{aligned}$$

Moreover, using that $v \leq \varphi + w$, we get

$$\hat{v}(x_0) \leq \tfrac{1}{2}(\varphi(x_0+x_1)+\varphi(x_0-x_1)-M|x_1|^2)+\tfrac{1}{2}(w(x_0+x_1)+w(x_0-x_1)).$$

By (6-4) and the upper bound of w in Proposition 6.8 (c), we see that

$$\hat{v}(x_0)-\varphi(x_0) \leq \tfrac{1}{2}C(1+|x_0+x_1|^{1-2s})+\tfrac{1}{2}C(1+|x_0-x_1|^{1-2s}) \to 0$$

as $|x_0| \to \infty$ with x_1 fixed, since $1-2s<0$. Then, for all $\varepsilon>0$, there is some compact set Ω_ε, depending on ε and x_1, such that

$$\hat{v}(x_0)-\varepsilon \leq \varphi(x_0) \quad \text{for all } x_0 \in \mathbb{R}^n \setminus \Omega_\varepsilon.$$

Consider $\hat{v}_\varepsilon = \max\{\hat{v}-\varepsilon,\ \varphi\}$. Then $\hat{v}_\varepsilon$ is a subsolution, since the maximum of subsolutions is a subsolution (see Lemma 6.7). Also, $\hat{v}_\varepsilon = \varphi \leq \varphi + w$ in $\mathbb{R}^n \setminus \Omega_\varepsilon$, and $\varphi + w$ is a supersolution by Proposition 6.8 (c). Applying the comparison principle, we get $\varphi \leq \hat{v}_\varepsilon \leq \varphi + w$. Moreover, $[\hat{v}_\varepsilon]_{C^{0,1}(\mathbb{R}^n)} \leq [\varphi]_{C^{0,1}(\mathbb{R}^n)}$. Therefore, $\hat{v}_\varepsilon \in \mathcal{S}$.

Since $u(x_0)=\sup_{v\in\mathcal{S}} v(x_0)$, it follows that $u(x_0) \geq \hat{v}_\varepsilon(x_0) \geq \hat{v}(x_0)-\varepsilon$. Letting $\varepsilon \to 0$, we conclude that, for any $v \in \mathcal{S}$ and $x_0,\ x_1 \in \mathbb{R}^n$,

$$u(x_0) \geq \tfrac{1}{2}(v(x_0+x_1)+v(x_0-x_1)-M|x_1|^2). \tag{6-5}$$

Finally, by definition of supremum, for any $\delta>0$ and $x_0,\ x_1 \in \mathbb{R}^n$, there exist $v_1,\ v_2 \in \mathcal{S}$ such that $u(x_0+x_1)-\delta < v_1(x_0+x_1)$ and $u(x_0-x_1)-\delta < v_2(x_0-x_1)$. Let $v=\max\{v_1,\ v_2\}$. Then using (6-5) for this v, we get

$$u(x_0) \geq \tfrac{1}{2}(u(x_0+x_1)-\delta+u(x_0-x_1)-\delta-M|x_1|^2).$$

Letting $\delta \to 0$, we conclude that

$$u(x_0+x_1)-u(x_0-x_1)-2u(x_0) \leq [\varphi]_{C^{1,1}(\mathbb{R}^n)}|x_1|^2. \qquad \square$$

To complete the proof of Theorem 6.1, it remains to see that u is a solution. Hence, we need to show that u is both a subsolution and a supersolution. We will prove these results in the next two propositions.

Lemma 6.11. *For any $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, the set*

$$D_{x_0}u(\varepsilon) = \{x \in \mathbb{R}^n : u(x) - u(x_0) - (x - x_0) \cdot \nabla u(x_0) \le \varepsilon\}$$

is compact.

Proof. Let $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$. Without loss of generality, we may assume that $x_0 = 0$. Let l be the supporting plane of u at 0, that is, $l(x) = u(0) + x \cdot \nabla u(0)$. Clearly, $D_{x_0}u(\varepsilon)$ is closed. Hence we only need to show that it is bounded. Recall that

$$\phi(x) < \varphi(x) \le u(x) \quad \text{for all } x \in \mathbb{R}^n, \tag{6-6}$$

where ϕ is a cone. Note that the strict inequality in (6-6) follows from the strict convexity of φ. Moreover, by (6-1) and (6-2) we have

$$\lim_{|x|\to\infty} (u - \phi)(x) = 0.$$

Therefore, $D_{x_0}u(\varepsilon) \subset \{\phi < l + \varepsilon\}$. We claim that

$$\lim_{|x|\to\infty} (\phi - l)(x) = \infty. \tag{6-7}$$

If this condition holds, then, for all $M > 0$, there exists $R > 0$ such that

$$\phi(x) - l(x) > M \quad \text{for all } |x| > R.$$

Choosing $M = \varepsilon$, we have $\{\phi < l + \varepsilon\} \subset B_R$ for some R depending on ε. Hence the set $D_{x_0}u(\varepsilon)$ is bounded.

To prove the claim, we distinguish two cases. If $u(0) = 0$, then u attains an absolute minimum at 0, so $\nabla u(0) = 0$. In particular, $l(x) = 0$ for all $x \in \mathbb{R}^n$, and thus (6-7) is clearly satisfied. Hence it remains to show the claim when

$$u(0) > 0.$$

We will prove it by contradiction. If (6-7) is not true, then there exists a sequence of points $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}^n$ such that $|x_j| \to \infty$ as $j \to \infty$ and

$$\lim_{j\to\infty} (\phi - l)(x_j) < \infty.$$

Using that ϕ is continuous and homogeneous of degree 1, and letting $j \to \infty$, we get

$$\frac{\phi(x_j)}{|x_j|} - \frac{l(x_j)}{|x_j|} = \phi\left(\frac{x_j}{|x_j|}\right) - \frac{u(0)}{|x_j|} - \frac{x_j}{|x_j|} \cdot \nabla u(0) \to \phi(e) - D_e u(0) = 0,$$

where $x_j/|x_j| \to e$, up to a subsequence. Therefore, $\phi(e) = D_e u(0)$. For any $\lambda > 0$, we have

$$l(\lambda e) = u(0) + \lambda e \cdot \nabla u(0) = u(0) + \lambda \phi(e) = u(0) + \phi(\lambda e).$$

Since l is a supporting plane of u, we know that $u(x) \ge l(x)$ for all $x \in \mathbb{R}^n$, and thus,

$$u(\lambda e) \ge l(\lambda e) = \phi(\lambda e) + u(0).$$

Letting $\lambda \to \infty$, we see that

$$0 = \lim_{\lambda\to\infty} (u - \phi)(\lambda e) \ge u(0) > 0,$$

which is a contradiction. □

Proposition 6.12 (u is a subsolution). *The function u given in* (6-3) *satisfies*

$$\mathcal{F}_k^s u(x_0) \geq u(x_0) - \varphi(x_0) \quad \textit{for all } x_0 \in \mathbb{R}^n.$$

Proof. By Proposition 6.10, we know that $u \in C^{1,1}(\mathbb{R}^n)$. Without loss of generality, we may assume that $[u]_{C^{1,1}(\mathbb{R}^n)} = 1$. Otherwise, consider $u/[u]_{C^{1,1}(\mathbb{R}^n)}$.

Let $x_0 \in \mathbb{R}^n$. Then the quadratic polynomial

$$P(x) = u(x_0) + \nabla u(x_0) \cdot (x - x_0) + |x - x_0|^2$$

touches u from above at x_0. Moreover, we may assume that P touches u strictly from above at x_0. If not, we replace P by $P + \varepsilon|x - x_0|^2$ with $\varepsilon > 0$ small.

Fix $\delta > 0$. Then there exists $h > 0$, with $h \to 0$ as $\delta \to 0$, such that

$$P(x) - u(x) \geq h > 0 \quad \text{for all } x \in \mathbb{R}^n \setminus B_\delta(x_0).$$

Since $u(x) = \sup_{v \in \mathcal{S}} v(x)$ and $v \in \mathcal{S}$ is uniformly continuous, there is a monotone sequence $\{v_j\}_{j=1}^\infty \subset \mathcal{S}$ such that $v_j \to u$ uniformly in compact subsets of $\mathbb{R}^n$. In particular, there exists $j_0 \geq 1$, depending on h, such that, for all $j > j_0$,

$$u(x) - h < v_j(x) \quad \text{for all } x \in \overline{B_\delta(x_0)}. \tag{6-8}$$

Write $v = v_j$ for some $j > j_0$. It follows that

$$\begin{cases} P - v \geq h & \text{in } \mathbb{R}^n \setminus B_\delta(x_0), \\ P - v < P - u + h & \text{in } B_\delta(x_0). \end{cases}$$

Let $d = \inf_{\mathbb{R}^n}(P - v)$. Then $d = P(x_1) - v(x_1)$ for some $x_1 \in \overline{B_h(x_0)}$ with $0 \leq d < h$, and

$$\begin{cases} P(x_1) - d = v(x_1), \\ P(x) - d \geq v(x) & \text{for all } x \in \mathbb{R}^n. \end{cases}$$

Hence $P - d$ is a quadratic polynomial that touches v from above at x_1. In particular, since v is convex, v has a unique supporting plane l at x_1, so $\partial v(x_1) = \{\nabla l\}$.

Let $\tau \geq 0$ be such that $l + \tau$ is the supporting plane of u at some point x_2. Note that x_2 approaches x_0 as h goes to 0, and thus, there exists some $r = r(h) > 0$ such that $r \to 0$ as $h \to 0$ and $x_2 \in B_r(x_0)$. Furthermore, since $l(x_1) + d = v(x_1) + d = P(x_1) \geq u(x_1)$, then $\tau \leq d < h$ (see Figure 2).

Fix $\varepsilon > 0$. By Lemma 6.11, we have that $D_{x_0}u(\varepsilon)$ is bounded, so $\Lambda = \operatorname{diam} D_{x_0}u(\varepsilon) < \infty$. Choose δ sufficiently small that $r < \varepsilon/(4\Lambda)$. Then by Proposition 5.2,

$$\mathcal{F}_k^s u(x_2) \leq \mathcal{F}_k^s u(x_0) + C\Lambda^{1-s}|x_2 - x_0|^{1-s} + \frac{4\Lambda}{\varepsilon}\mathcal{F}_k^s u(x_0)|x_2 - x_0| \leq \mathcal{F}_k^s u(x_0) + C(r), \tag{6-9}$$

where $C(r) \to 0$ as $r \to 0$. Next we will show that

$$\mathcal{F}_k^s v(x_1) - C\tau^{1-s} \leq \mathcal{F}_k^s u(x_2) \tag{6-10}$$

for some constant $C > 0$ depending only on n, k, and s. Since $\partial v(x_1) = \{\nabla l\}$ we have $v \in C^{1,1}(x_1)$, and using Proposition 5.4 we get

$$\mathcal{F}_k^s v(x_1) = c_{n,s} \int_0^\infty \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_1} v(t,z)^{1/k}}{|z|}\right) dz\, dt,$$

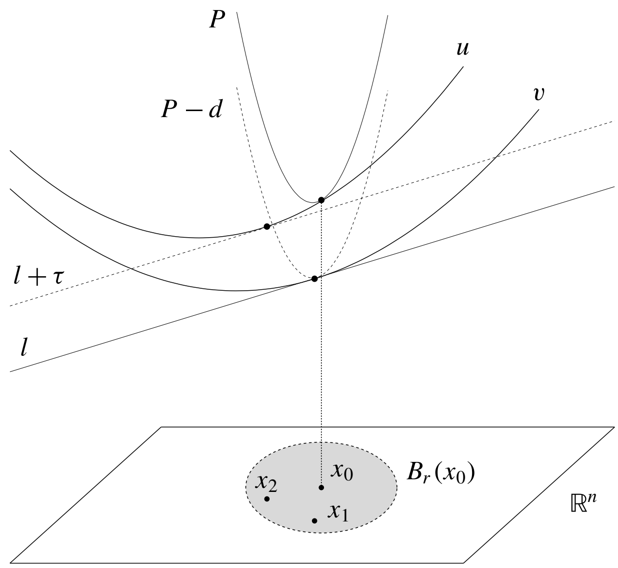

Figure 2. Geometry involved in the proof of Proposition 6.12.

where $\mu_x v(t,z) = \omega_k^{-1}\mathcal{H}^k(\{y\in\mathbb{R}^k : \tilde{v}_x(y,z)\le t\})$ and W is the monotone decreasing function given in (5-2). Observe that since $v\le u$, the supporting plane of v at x_1 is l, and the supporting plane of u at x_2 is $l+\tau$. Then, for any $t>0$, it follows that

$$D_{x_2}u(t)=\{u-(l+\tau)\le t\}\subseteq\{v-l\le t+\tau\}=D_{x_1}v(t+\tau).$$

In particular, $\mu_{x_2}u(t,z)\le\mu_{x_1}v(t+\tau,z)$ for any $z\in\mathbb{R}^{n-k}$. Therefore,

$$W(\mu_{x_2}u(t,z))\ge W(\mu_{x_1}v(t+\tau,z)),$$

which yields

$$\begin{aligned}\mathcal{F}_k^s u(x_2) &\ge c_{n,s}\int_\tau^\infty\int_{\mathbb{R}^{n-k}}\frac{1}{|z|^{n-k+2s}}W\biggl(\frac{\mu_{x_1}v(t,z)^{1/k}}{|z|}\biggr)\,dz\,dt\\ &=\mathcal{F}_k^s v(x_1)-c_{n,s}\int_0^\tau\int_{\mathbb{R}^{n-k}}\frac{1}{|z|^{n-k+2s}}W\biggl(\frac{\mu_{x_1}v(t,z)^{1/k}}{|z|}\biggr)\,dz\,dt\\ &\ge\mathcal{F}_k^s v(x_1)-C\tau^{1-s},\end{aligned}$$

where the last inequality follows from the fact that $\mu_{x_1}v(t,z)\ge C(t-|z|^2)_+^{k/2}$ and W is monotone decreasing.

Combining (6-9) and (6-10), using that v is a subsolution, and using (6-8), we get

$$\mathcal{F}_k^s u(x_0)+C(r)\ge\mathcal{F}_k^s v(x_1)-C\tau^{1-s}\ge v(x_1)-\varphi(x_1)-C\tau^{1-s}>u(x_1)-h-\varphi(x_1)-C\tau^{1-s}.$$

Letting $\delta\to 0$, it follows that $h\to 0$, $C(r)\to 0$, $\tau\to 0$, and $x_1\to x_0$. By continuity of u and φ, we conclude the result. $\square$

Proposition 6.13 (u is a supersolution). *The function u given in* (6-3) *satisfies*

$$\mathcal{F}_k^s u(x_0) \le u(x_0) - \varphi(x_0) \quad \textit{for all } x_0 \in \mathbb{R}^n.$$

Proof. Assume the statement is false. Then there exists some $x_0 \in \mathbb{R}^n$ such that

$$\mathcal{F}_k^s u(x_0) > u(x_0) - \varphi(x_0).$$

Without loss of generality, we may assume that $u(x_0) = 0$ and $\nabla u(x_0) = 0$. Otherwise, consider $v(x) = u(x) - u(x_0) - (x - x_0) \cdot \nabla u(x_0)$. Then there exists some $\delta > 0$ such that

$$\mathcal{F}_k^s u(x_0) \ge -\varphi(x_0) + \delta. \tag{6-11}$$

Fix $\varepsilon > 0$ and let $u^\varepsilon(x) = \max\{u(x),\ \varepsilon\}$. We will show that, for ε sufficiently small, u^ε is an admissible subsolution, and thus reach a contradiction with u being the largest subsolution. Indeed, u^ε is convex and $u^\varepsilon \in C^{0,1}(\mathbb{R}^n)$ with $[u^\varepsilon]_{C^{0,1}(\mathbb{R}^n)} \le [\varphi]_{C^{0,1}(\mathbb{R}^n)}$. Moreover, note that $u^\varepsilon(x) = u(x)$ for x large. Hence, once we show that u^ε is a subsolution, it will follow from the comparison principle that $\varphi \le u^\varepsilon \le \varphi + w$.

If $x \in \{u_\varepsilon = u\}$, then $u_\varepsilon(x) = u(x)$ and $u_\varepsilon \ge u$ in $\mathbb{R}^n$. By monotonicity (Lemma 6.5),

$$\mathcal{F}_k^s u^\varepsilon(x) \ge \mathcal{F}_k^s u(x) \ge u(x) - \varphi(x) = u^\varepsilon(x) - \varphi(x),$$

since u is a subsolution, by Proposition 6.12.

If $x \in \{u^\varepsilon > u\}$, then $u^\varepsilon(x) = \varepsilon$ and $\partial u^\varepsilon(x) = \{0\}$. In particular,

$$\mathcal{F}_k^s u^\varepsilon(x) = \mathcal{F}_k^s u^\varepsilon(x_0). \tag{6-12}$$

Moreover, for any $t > 0$, we have $D_{x_0} u^\varepsilon(t) = \{u^\varepsilon - \varepsilon \le t\} = \{u \le t + \varepsilon\} = D_{x_0} u(t + \varepsilon)$. Therefore, in view of Proposition 5.4, we get

$$\mathcal{F}_k^s u^\varepsilon(x_0) = \mathcal{F}_k^s u(x_0) - \int_0^\varepsilon \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_0} u(t,z)^{1/k}}{|z|}\right) dz\, dt \ge \mathcal{F}_k^s u(x_0) - C\varepsilon^{1-s}, \tag{6-13}$$

since $u \in C^{1,1}(\mathbb{R}^n)$ and $\mu_{x_0} u(t,z) \ge (t - |z|^2)_+^{k/2}$.

Combining (6-11)–(6-13), we see that

$$\begin{aligned} \mathcal{F}_k^s u^\varepsilon(x) = \mathcal{F}_k^s u^\varepsilon(x_0) \ge \mathcal{F}_k^s u(x_0) - C\varepsilon^{1-s} &\ge -\varphi(x_0) + \delta - C\varepsilon^{1-s} \\ &= u^\varepsilon(x) - \varphi(x) + (\varphi(x) - \varphi(x_0) + \delta - C\varepsilon^{1-s} - \varepsilon), \end{aligned}$$

since $u^\varepsilon(x) = \varepsilon$. We need the term inside the parenthesis to be nonnegative. Hence it remains to control $\varphi(x) - \varphi(x_0)$. Since φ is smooth,

$$|\varphi(x) - \varphi(x_0)| \le [\varphi]_{C^{0,1}(\mathbb{R}^n)} |x - x_0|.$$

We distinguish two cases. If $\{u = 0\} = \{x_0\}$, then $|x - x_0| \le d_\varepsilon \to 0$ as $\varepsilon \to 0$. Hence, choosing ε sufficiently small, we see that

$$\varphi(x) - \varphi(x_0) + \delta - C\varepsilon^{1-s} - \varepsilon \ge \delta - [\varphi]_{C^{0,1}(\mathbb{R}^n)} d_\varepsilon - C\varepsilon^{1-s} - \varepsilon \ge 0.$$

Therefore, $u^\varepsilon \in \mathcal{S}$, which contradicts $u^\varepsilon(x_0) > u(x_0) = \sup_{v \in \mathcal{S}} v(x_0) \ge u^\varepsilon(x_0)$.

Suppose now that $\{u=0\}$ contains more than one point. By compactness of $\{u=0\}$ and continuity of φ, there exists some $x_1 \in \{u=0\}$ where φ attains its maximum. Then

$$\mathcal{F}_k^s u(x_1) = \mathcal{F}_k^s u(x_0) \geq u(x_0) - \varphi(x_0) + \delta \geq u(x_1) - \varphi(x_1) + \delta.$$

Moreover, by convexity of $\{u=0\}$ (since $u \geq \varphi \geq 0$) and φ, we must have that $x_1 \in \partial\{u=0\}$. Hence there exists $\{x_j\}_{j=2}^{\infty} \subset \{u>0\}$ such that $x_j \to x_1$ and u is strictly convex at x_j. Namely, there is a supporting plane that touches u only at x_j.

By continuity of u, there exists some $j_0 \geq 2$ such that

$$u(x_1) > u(x_j) - \tfrac{1}{4}\delta \quad \text{for all } j > j_0.$$

By continuity of φ, there exists some $j_1 \geq 2$ such that

$$\varphi(x_1) < \varphi(x_j) + \tfrac{1}{4}\delta \quad \text{for all } j > j_1.$$

By lower semicontinuity of $\mathcal{F}_k^s u$, up to a subsequence, there exists some $j_2 \geq 2$ such that

$$\mathcal{F}_k^s u(x_j) > \mathcal{F}_k^s u(x_1) - \tfrac{1}{4}\delta \quad \text{for all } j > j_2.$$

Let $J > \max\{j_0, j_1, j_2\}$. Then

$$\mathcal{F}_k^s u(x_J) > \mathcal{F}_k^s u(x_1) - \tfrac{1}{4}\delta \geq u(x_1) - \varphi(x_1) + \tfrac{3}{4}\delta > u(x_J) - \varphi(x_J) + \tfrac{1}{4}\delta,$$

and we can repeat the previous argument, replacing x_0 by x_J. We conclude that

$$\mathcal{F}_k^s u(x_0) \leq u(x_0) - \varphi(x_0) \quad \text{for all } x_0 \in \mathbb{R}^n. \qquad \square$$

7. Future directions

As mentioned in the introduction, the main idea to define a nonlocal analog to the Monge–Ampère operator is to write it as a concave envelope of linear operators. More precisely,

$$n \det(D^2u(x))^{1/n} = \inf_{M \in \mathcal{M}} \operatorname{tr}(M D^2 u(x)),$$

where $\mathcal{M} = \{M \in \mathcal{S}^n : M > 0,\ \det(M) = 1\}$ and $\mathcal{S}^n$ is the set of $n \times n$ symmetric matrices. Note that this identity is equivalent to the one given in (1-2) taking $M = AA^T$ and $B = D^2u(x)$, since $\operatorname{tr}(A^T B A) = \operatorname{tr}(AA^T B)$. In fact, this extremal property does not only hold for $n \det(B)^{1/n}$ with $B \in \mathcal{S}^n$ and $B > 0$. If $\lambda = (\lambda_1, \dots, \lambda_n)$, where λ_i are the eigenvalues of B, then the function f defined on $\Gamma = \{\lambda \in \mathbb{R}^n : \lambda_i > 0 \text{ for all } i = 1, \dots, n\}$ and given by

$$f(\lambda) = n\left(\prod_{i=1}^n \lambda_i\right)^{1/n} = n \det(B)^{1/n}$$

is differentiable, concave, and homogeneous of degree 1. In general, if f satisfies these conditions in an open convex set Γ in $\mathbb{R}^n$, then

$$f(\lambda) = \inf_{\mu \in \Gamma} \{f(\mu) + \nabla f(\mu) \cdot (\lambda - \mu)\} = \inf_{\mu \in \Gamma} \nabla f(\mu) \cdot \lambda,$$

where the second identity follows by Euler's theorem. Therefore,

$$f(\lambda) = \inf_{M \in \mathcal{M}_f} \operatorname{tr}(MB),$$

where $\mathcal{M}_f = \{M \in \mathcal{S}^n : \lambda(M) \in \nabla f(\Gamma)\}$, $\nabla f(\Gamma) = \{\nabla f(\mu) : \mu \in \Gamma\}$, and $\lambda(M)$ are the eigenvalues of M.

For instance, the k-Hessian functions introduced by Caffarelli, Nirenberg, and Spruck in [Caffarelli et al. 1985] satisfy these conditions and, in fact, fractional analogs have been recently studied by Wu [2019]. It would be interesting to explore fractional analogs to a wider class of fully nonlinear concave operators, like the ones mentioned above.

We remark that the 1-Hessian is equal to the Laplacian, and the n-Hessian is equal to the Monge–Ampère operator. Moreover, for $1 < k < n$, we obtain an intermediate *discrete* family between these operators. In view of this observation, a natural question of finding a *continuous* family connecting the Laplacian with the Monge–Ampère operator arises. Here we suggest possible families that smoothly connect these two operators and pass through the k-Hessians, in some sense. Indeed, let $\alpha \in (0, 1]^n$ and write $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For $\lambda \in \mathbb{R}^n_+$, we consider the functions

$$f_\alpha(\lambda) = \left(\sum_{\sigma \in S} \lambda_{\sigma(1)}^{\alpha_1} \cdots \lambda_{\sigma(n)}^{\alpha_n} \right)^{1/|\alpha|},$$

where S is the set of all cyclic permutations of $\{1, \dots, n\}$. Observe that, for any $1 \le k \le n$, if $\alpha = \sum_{i \in \mathcal{I}} e_i$ with $|\mathcal{I}| = k$, then f_α is precisely the k-Hessian function. Consider any smooth simple curve $\gamma : [0, 1] \to (0, 1]^n$ such that

(1) $\gamma(0) = e_i$ for some $1 \le i \le n$,

(2) $\gamma(t_k) = \sum_{i \in \mathcal{I}_k} e_i$ with $|\mathcal{I}_k| = k$ and $0 < t_k < t_{k+1} < 1$ for all $1 < k < n$, and

(3) $\gamma(1) = (1, \dots, 1)$.

Then the family $\{f_\alpha\}_{\alpha \in \operatorname{Im}(\gamma)}$ is as we described. In particular, fractional analogs of these functions would give a continuous family from the fractional Laplacian to the nonlocal Monge–Ampère. We will study this problem in a forthcoming paper.

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