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THE PRESCRIBED CURVATURE PROBLEM FOR ENTIRE HYPERSURFACES IN MINKOWSKI SPACE

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We prove three results in this paper: First, we prove, for a wide class of functions $\varphi \in C^2(\mathbb{S}^{n-1})$ and $\psi(X, \nu) \in C^2(\mathbb{R}^{n+1} \times \mathbb{H}^n)$, there exists a unique, entire, strictly convex, spacelike hypersurface $\mathcal{M}_u$ satisfying $\sigma_k(\kappa[\mathcal{M}_u]) = \psi(X, \nu)$ and $u(x) \to |x| + \varphi(x/|x|)$ as $|x| \to \infty$. Second, when $k = n-1, n-2$, we show the existence and uniqueness of an entire, k-convex, spacelike hypersurface $\mathcal{M}_u$ satisfying $\sigma_k(\kappa[\mathcal{M}_u]) = \psi(x, u(x))$ and $u(x) \to |x| + \varphi(x/|x|)$ as $|x| \to \infty$. Last, we obtain the existence and uniqueness of entire, strictly convex, downward translating solitons $\mathcal{M}_u$ with prescribed asymptotic behavior at infinity for σ_k curvature flow equations. Moreover, we prove that the downward translating solitons $\mathcal{M}_u$ have bounded principal curvatures.

1. Introduction

Let $\mathbb{R}^{n,1}$ be the Minkowski space with the Lorentzian metric

$$ds^2 = \sum_{i=1}^{n} dx_i^2 - dx_{n+1}^2.$$

In this paper, we will devote ourselves to the study of spacelike hypersurfaces with prescribed σ_k curvature in Minkowski space $\mathbb{R}^{n,1}$. Here, σ_k is the k-th elementary symmetric polynomial, i.e.,

$$\sigma_k(\kappa) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \kappa_{i_1} \cdots \kappa_{i_k}.$$

Any such hypersurface $\mathcal{M}$ can be written locally as a graph of a function $x_{n+1} = u(x)$, $x \in \mathbb{R}^n$, satisfying the spacelike condition

$$|Du| < 1. \tag{1-1}$$

More precisely, we focus on the equation

$$\sigma_k(\kappa[\mathcal{M}_u]) = \psi(X, \nu), \tag{1-2}$$

where $X = (x, u(x))$ is the position vector of $\mathcal{M}_u = \{(x, u(x)) \mid x \in \mathbb{R}^n\}$, $\nu = (Du, 1)/\sqrt{1 - |Du|^2}$ is the future-directed unit normal lying on the hyperboloid $\mathbb{H}^n$, and $\kappa[\mathcal{M}_u] = (\kappa_1, \ldots, \kappa_n)$ is the set of principal curvatures of $\mathcal{M}_u$. Thus (1-2) can be rewritten as

$$\sigma_k(\kappa[\mathcal{M}_u]) = \psi(x, u(x), Du). \tag{1-3}$$

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Notice that the functions ψ in the right-hand sides of (1-2) and (1-3) are different. Slightly extending the notation, we use the same symbol here.

The classical Minkowski problem asks for the construction of a strictly convex compact surface Σ whose Gaussian curvature is a given positive function $f(\nu(X))$, where $\nu(X)$ denotes the normal to Σ at X. This problem has been discussed by Nirenberg [1953], Pogorelov [1978], and Cheng and Yau [1976]. The general problem of finding strictly convex hypersurfaces with prescribed surface area measures is called the Christoffel–Minkowski problem. This type of problem can be reduced to a fully nonlinear equation of the form (1-2). It may be traced back to Aleksandrov [1942], who established the problem of prescribing zeroth curvature measure. The prescribed curvature measure problem in convex geometry has been extensively studied by Aleksandrov [1956], Pogorelov [1953], Guan, Lin, and Ma [Guan et al. 2009], and Guan, Li, and Li [Guan et al. 2012]. A more general form of the prescribed curvature measure problem can be expressed as (1-3). In particular, Guan, Ren, and Wang [Guan et al. 2015] solved this problem in Euclidean space for convex hypersurfaces. Other related studies and references about the Minkowski problem may be found in [Bakelman and Kantor 1974; Caffarelli et al. 1986; 1988; Guan and Guan 2002; Oliker 1984; Treibergs and Wei 1983].

In Minkowski space, there have been fruitful results on the prescribed curvature problem for spacelike entire hypersurfaces. In [Treibergs 1982] and [Choi and Treibergs 1990], the authors obtained the existence of entire hypersurfaces with constant mean curvature. Li [1995] then extended [Treibergs 1982] and proved the existence of constant Gauss curvature hypersurfaces with Gauss image a unit ball. The existence of constant Gauss curvature hypersurfaces with Gauss image the convex hull in B_1 of an arbitrary closed set $\mathcal{F}\subset\mathbb{S}^{n-1}$ was proved by Guan, Jian, and Schoen [Guan et al. 2006a] and Bayard and Schnürer [2009]. Later, [Bayard 2006] and [Bayard and Delanoë 2009] considered the prescribed scalar curvature problem for entire, spacelike hypersurfaces under different settings. More recently, the second and third authors showed the existence of entire, spacelike, constant σ_k curvature hypersurfaces in [Wang and Xiao 2022].

Our goal here is to construct entire, spacelike hypersurfaces satisfying (1-2) in Minkowski space. The main results of this paper follow.

The first result is to construct entire, strictly convex, spacelike hypersurfaces satisfying (1-2).

Theorem 1. *Suppose φ is a C^2 function defined on $\mathbb{S}^{n-1}$, i.e., $\varphi\in C^2(\mathbb{S}^{n-1})$, $\psi(X,\nu)\in C^2(\mathbb{R}^{n+1}\times\mathbb{H}^n)$ is a positive function, and $c_1\geqslant\psi(X,\nu)\geqslant c_2$ for some positive constants c_1, c_2. We further assume that $\psi_{x_{n+1}}\geqslant 0$ (or $\psi_u\geqslant 0$). If either $\psi^{-1/k}(X,\nu)$ is locally strictly convex with respect to X for any ν or ψ only depends on ν, then there exists a unique, entire, strictly convex, spacelike hypersurface $\mathcal{M}_u=\{(x,u(x))\mid x\in\mathbb{R}^n\}$ satisfying (1-2). Moreover, as $|x|\to\infty$,*

$$u(x)\to|x|+\varphi\left(\frac{x}{|x|}\right). \tag{1-4}$$

Remark 2. Indeed, from the proof of the C^2 global estimate Lemma 10, we can see that the assumption that $\psi(X,\nu)$ does not depend on X can be replaced by a weaker assumption; that is, $\psi^{-1/k}(X,\nu)$ is convex with respect to X, and the corresponding form $\psi(x,u,Du)$ does not depend on $|x|$.

Remark 3. In the proof, we only can see that the hypersurface $\mathcal{M}_u$ we constructed is convex. In order to say it's strictly convex, we need to apply the constant rank theorem (see [Guan et al. 2006b, Theorem 1.2; Wang and Xiao 2022, Theorem 27]) and the splitting theorem (see [Wang and Xiao 2022, Theorem 28]) to obtain that, if $\mathcal{M}_u$ has a degenerate point in the interior, then $\mathcal{M}_u = \mathcal{M}^l \times \mathbb{R}^{n-l}$, where $\mathcal{M}^l \subset \mathbb{R}^{l,1}$ is a strictly convex, spacelike hypersurface. This contradicts (1-4).

Before stating our second result, we need the following definition.

Definition 4. A C^2 regular hypersurface $\mathcal{M} \subset \mathbb{R}^{n,1}$ is k-convex if the principal curvatures of $\mathcal{M}$ at $X \in \mathcal{M}$ satisfy $\kappa[X] \in \Gamma_k$ for all $X \in \mathcal{M}$, where Γ_k is the Gårding cone

$$\Gamma_k = \{\kappa \in \mathbb{R}^n \mid \sigma_m(\kappa) > 0,\ m = 1, \dots, k\}.$$

Using the newly developed methods in [Ren and Wang 2019; 2023], we are able to generalize results in [Bayard 2006] to prove the following.

Theorem 5. *Suppose φ is some C^2 function defined on $\mathbb{S}^{n-1}$ and $\psi(x, u(x)) \in C^2(\mathbb{R}^{n+1})$ is a positive function satisfying $c_1 \geqslant \psi(x, u(x)) \geqslant c_2$ for $c_1, c_2 > 0$. We further assume that $k = n-1, n-2$ and $\psi_u \geqslant 0$. Then there exists a unique, k-convex, spacelike hypersurface $\mathcal{M}_u = \{(x, u(x)) \mid x \in \mathbb{R}^n\}$ satisfying*

$$\sigma_k(\kappa[\mathcal{M}_u]) = \psi(x, u(x)). \tag{1-5}$$

Moreover, as $|x| \to \infty$,

$$u(x) \to |x| + \varphi\left(\frac{x}{|x|}\right). \tag{1-6}$$

Remark 6. Notice that unlike in the strictly convex case (Theorem 1), in this theorem, we only prove the existence result for the case when ψ depends on x and $u(x)$ (ψ is independent of Du). This is because the proofs of Lemma 12 (C^2 boundary estimates for k-convex hypersurfaces) and Lemma 15 (C^1 local estimates for k-convex hypersurfaces) crucially rely on the fact that ψ is independent of Du.

Now, let's consider the σ_k curvature flow with a forcing term in Minkowski space:

$$\frac{dX}{dt} = -\left(\mathcal{C} - \frac{\sigma_k^{1/k}(\kappa[\mathcal{M}_u])}{\binom{n}{k}^{1/k}}\right)\nu, \tag{1-7}$$

where $\kappa[\mathcal{M}_u] \in \Gamma_k$. This can be rewritten as the equation for the height function u:

$$\frac{u_t}{\sqrt{1-|Du|^2}} = \frac{\sigma_k^{1/k}(\kappa[\mathcal{M}_u])}{\binom{n}{k}^{1/k}} - \mathcal{C}. \tag{1-8}$$

The downward translating soliton to (1-8) is of the form

$$u(x, t) = u(x) - t, \tag{1-9}$$

where $u(x)$ satisfies

$$\left(\frac{\sigma_k}{\binom{n}{k}}\right)^{1/k}(\kappa[\mathcal{M}_u]) = \mathcal{C} - \frac{1}{\sqrt{1-|Du|^2}}. \tag{1-10}$$

Equation (1-10) can be viewed as the "degenerate" type of (1-2). In this case, we prove the following.

Theorem 7. *Suppose φ is a C^2 function defined on $\mathbb{S}^{n-1}_{\tilde{\mathcal{C}}} := \{x \in \mathbb{R}^n \mid |x| = \tilde{\mathcal{C}}\}$, where $\tilde{\mathcal{C}} = \sqrt{1-(1/\mathcal{C})^2}$ and $\mathcal{C} > 1$ is a constant. There exists a unique, strictly convex solution $u : \mathbb{R}^n \to \mathbb{R}$ of (1-10) such that, as $|x| \to \infty$,*

$$u(x) \to \tilde{\mathcal{C}}|x| - \frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}\log|x| + \varphi\Big(\tilde{\mathcal{C}}\frac{x}{|x|}\Big). \tag{1-11}$$

Moreover, $\mathcal{M}_u = \{(x, u(x)) \mid x \in \mathbb{R}^n\}$ has bounded principal curvatures.

When $k = 1$, (1-10) has been studied in [Ju et al. 2010; Spruck and Xiao 2016]; when $k = 2$, (1-10) has been studied in [Bayard 2023].

Remark 8. Under our assumptions on ψ, we can see that the linearized operators of (1-2), (1-5), and (1-10) satisfy the maximum principle. Therefore, the uniqueness properties in Theorem 1, 5, and 7 follow from the maximum principle directly.

The rest of this paper is organized as follows. In Section 2, we introduce some basic formulas and notation. The solvability of (1-2) and (1-5) on a bounded domain (Dirichlet problem) is discussed in Section 3. We prove the local C^1 and C^2 estimates for solutions of (1-2) and (1-5) in Section 4. This leads to the completion of the proof of our first two main results, Theorems 1 and 5, in Section 5. Section 6 and Section 7 are devoted to Theorem 7. In particular, in Section 6, we study the radially symmetric solution to (1-10), this solution will be used to construct barrier functions in Section 7. We finish the proof of Theorem 7 in Section 7.

2. Preliminaries

In this paper, we will follow notation in [Wang and Xiao 2022]. For the readers convenience, we will include some basic notation and formulas in this section. For more details, one can refer to [Choi and Treibergs 1990; Li 1995]. Readers who are already familiar with calculations in Minkowski space can skip this section.

We first recall that the Minkowski space $\mathbb{R}^{n,1}$ is $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric

$$ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2.$$

Throughout this paper, $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^{n,1}$.

2.1. ***Vertical graphs in $\mathbb{R}^{n,1}$.*** A spacelike hypersurface $\mathcal{M}$ in $\mathbb{R}^{n,1}$ is a codimension 1 submanifold whose induced metric is Riemannian. Locally, $\mathcal{M}$ can be written as the graph of a function, i.e.,

$$\mathcal{M}_u = \{X = (x, u(x)) \mid x \in \mathbb{R}^n\},$$

satisfying the spacelike condition (1-1). We let $E = (0, \dots, 0, 1)$. Then the height function of $\mathcal{M}$ is $u(x) = -\langle X, E\rangle$. It's easy to see that the induced metric and second fundamental form of $\mathcal{M}$ are given by

$$g_{ij} = \delta_{ij} - D_{x_i}uD_{x_j}u, \quad 1 \leqslant i, j \leqslant n,$$

and

$$h_{ij} = \frac{u_{x_ix_j}}{\sqrt{1-|Du|^2}},$$

respectively, while the timelike unit normal vector field to $\mathcal{M}$ is

$$\nu = \frac{(Du, 1)}{\sqrt{1-|Du|^2}},$$

where $Du = (u_{x_1}, \dots, u_{x_n})$ and $D^2u = (u_{x_i x_j})$ denote the ordinary gradient and Hessian, respectively, of u. By a straightforward calculation, we have that the principle curvatures of $\mathcal{M}$ are eigenvalues of the symmetric matrix $A = (a_{ij})$ given by

$$a_{ij} = \frac{1}{w}\gamma^{ik} u_{kl} \gamma^{lj},$$

where $\gamma^{ik} = \delta_{ik} + u_i u_k/(w(1+w))$ and $w = \sqrt{1-|Du|^2}$. Note that (γ^{ij}) is invertible with inverse $(\gamma_{ij}) = \delta_{ij} - u_i u_j/(1+w)$, which is the square root of (g_{ij}).

Let $\mathcal{S}$ be the vector of $n \times n$ symmetric matrices and

$$\mathcal{S}_k = \{A \in \mathcal{S} \mid \lambda(A) \in \Gamma_k\},$$

where $\lambda(A) = (\lambda_1, \dots, \lambda_n)$ is the set of eigenvalues of A. Define a function F by

$$F(A) = \sigma_k(\lambda(A)), \quad A \in \mathcal{S}_k.$$

Then (1-3) can be written as

$$F\Big(\frac{1}{w}\gamma^{ik} u_{kl} \gamma^{lj}\Big) = \psi(x, u(x), Du). \tag{2-1}$$

Throughout this paper, we write

$$F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}(A) \quad \text{and} \quad F^{ij,kl} = \frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}}.$$

Now, let $\{\tau_1, \tau_2, \dots, \tau_n\}$ be a local orthonormal frame on $T\mathcal{M}$. We will use ∇ to denote the induced Levi-Civita connection on $\mathcal{M}$. For a function v on $\mathcal{M}$, we write $v_i = \nabla_{\tau_i} v$, $v_{ij} = \nabla_{\tau_i}\nabla_{\tau_j} v$, etc. In particular, we have

$$|\nabla u| = \sqrt{g^{ij} u_{x_i} u_{x_j}} = \frac{|Du|}{\sqrt{1-|Du|^2}}.$$

Using normal coordinates, we also need the following well-known fundamental equations for a hypersurface $\mathcal{M}$ in $\mathbb{R}^{n,1}$:

$$\begin{aligned} X_{ij} &= h_{ij}\nu && \text{(Gauss formula)},\\ (\nu)_i &= h_{ij}\tau_j && \text{(Weigarten formula)},\\ h_{ijk} &= h_{ikj} && \text{(Codazzi equation)},\\ R_{ijkl} &= -(h_{ik}h_{jl} - h_{il}h_{jk}) && \text{(Gauss equation)}, \end{aligned} \tag{2-2}$$

and the Ricci identity

$$h_{ijkl} = h_{ijlk} + h_{mj}R_{imlk} + h_{im}R_{jmlk} = h_{klij} - (h_{mj}h_{il} - h_{ml}h_{ij})h_{mk} - (h_{mj}h_{kl} - h_{ml}h_{kj})h_{mi}. \tag{2-3}$$

2.2. ***The Gauss map.*** Let $\mathcal{M}$ be an entire, strictly convex, spacelike hypersurface, and let $\nu(X)$ be the timelike unit normal vector to $\mathcal{M}$ at X. It's well known that the hyperbolic space $\mathbb{H}^n(-1)$ is canonically embedded in $\mathbb{R}^{n,1}$ as the hypersurface

$$\langle X, X\rangle = -1, \quad x_{n+1} > 0.$$

By translation parallel to the origin, we can regard $\nu(X)$ as a point in $\mathbb{H}^n(-1)$. In this way, we define the Gauss map

$$G : \mathcal{M} \to \mathbb{H}^n(-1), \quad X \mapsto \nu(X).$$

Next, let's consider the support function of $\mathcal{M}$. We write

$$v := \langle X, \nu\rangle = \frac{1}{\sqrt{1-|Du|^2}}\biggl(\sum_i x_i \frac{\partial u}{\partial x_i} - u\biggr).$$

Let $\{e_1, \dots, e_n\}$ be an orthonormal frame on $\mathbb{H}^n$. We will also write $\{e_1^*, \dots, e_n^*\}$ for the pull-back of e_i by the Gauss map G. Similarly to the convex geometry case, we write

$$\Lambda_{ij} = v_{ij} - v\delta_{ij},$$

which is the hyperbolic Hessian. Here the v_{ij} denote the covariant derivatives with respect to the hyperbolic metric.

Let $\overline{\nabla}$ be the connection of the ambient space. Then we have

$$X = \sum_i v_i e_i - v\nu$$

and

$$\overline{\nabla}_{e_j^*} X = \sum_k (e_j(v_k)e_k + v_k \overline{\nabla}_{e_j} e_k) - v_j \nu - v\overline{\nabla}_{e_j}\nu = \sum_k \Lambda_{kj} e_k.$$

Note also that

$$g_{ij} = \langle \overline{\nabla}_{e_i^*} X, \overline{\nabla}_{e_j^*} X\rangle = \sum_k \Lambda_{ik}\Lambda_{kj} \tag{2-4}$$

and

$$h_{ij} = \langle \overline{\nabla}_{e_i^*} X, \overline{\nabla}_{e_j}\nu\rangle = \Lambda_{ij}. \tag{2-5}$$

This implies that the eigenvalues of the hyperbolic Hessian are equal to the curvature radius of $\mathcal{M}$. Therefore, (1-2) can be written as

$$F(v_{ij} - v\delta_{ij}) = \frac{1}{\psi(X, \nu)}, \tag{2-6}$$

where $F(A) = (\sigma_n/\sigma_{n-k})(\lambda(A))$. Moreover, it is clear that

$$(\overline{\nabla}_{e_j}\overline{\nabla}_{e_i}\nu)^{\perp} = \delta_{ij}\nu, \tag{2-7}$$

which yields, for $k = 1, 2, \dots, n+1$,

$$\nabla_{e_j}\nabla_{e_i} x_k = x_k \delta_{ij}, \tag{2-8}$$

where x_k is the coordinate function.

2.3. *Legendre transform.* Suppose $\mathcal{M}$ is an entire, strictly convex, spacelike hypersurface. Then $\mathcal{M}$ is the graph of a convex function

$$x_{n+1} = -\langle X, E\rangle = u(x_1, \dots, x_n),$$

where $E = (0, \dots, 0, 1)$. We introduce the Legendre transform

$$\xi_i = \frac{\partial u}{\partial x_i}, \quad u^* = \sum x_i \xi_i - u.$$

Next, we calculate the first and second fundamental forms in terms of ξ_i. Since it is well known that

$$\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = \left(\frac{\partial^2 u^*}{\partial \xi_i \partial \xi_j}\right)^{-1},$$

we have that the first and the second fundamental forms can be rewritten as

$$g_{ij} = \delta_{ij} - \xi_i \xi_j \quad \text{and} \quad h_{ij} = \frac{u^{*ij}}{\sqrt{1-|\xi|^2}},$$

where (u^{*ij}) denotes the inverse matrix of (u^*_{ij}) and $|\xi|^2 = \sum_i \xi_i^2$. Now, let W be the Weingarten matrix of $\mathcal{M}$. Then

$$(W^{-1})_{ij} = \sqrt{1-|\xi|^2}\, g_{ik} u^*_{kj}.$$

From the discussion above, we can see that if $\mathcal{M}_u = \{(x, u(x)) \mid x \in \mathbb{R}^n\}$ is an entire, strictly convex, spacelike hypersurface satisfying $\sigma_k(\kappa[\mathcal{M}]) = \psi$, then the Legendre transform of u, denoted by u^*, satisfies

$$F(w^* \gamma^*_{ik} u^*_{kl} \gamma^*_{lj}) = \frac{\sigma_n}{\sigma_{n-k}}(\kappa^*[w^* \gamma^*_{ik} u^*_{kl} \gamma^*_{lj}]) = \frac{1}{\psi}. \tag{2-9}$$

Here, $w^* = \sqrt{1-|\xi|^2}$, and $(\gamma^*_{ij}) = \delta_{ij} - \xi_i \xi_j/(1+w^*)$ is the square root of the matrix (g_{ij}).

3. The Dirichlet problem

We will divide this section into two subsections. In the first subsection, we only consider the convex solution to (1-2). In the second subsection, we restrict ourselves to the cases when $k = n-1$ $(n \geqslant 3)$, $n-2$ $(n \geqslant 5)$, and we will consider the k-convex, spacelike solution to (1-5). When $k = 2$, this problem has been studied in [Bayard 2003; Urbas 2003].

3.1. *Dirichlet problem for $1 \leqslant k \leqslant n$.* Recall that in [Wang and Xiao 2022] we proved the following:

Lemma 9. *Let $\mathcal{F} \subset \mathbb{S}^{n-1}$, $\widetilde{F} = \mathrm{Conv}(\mathcal{F})$, and u^* be a solution of*

$$\begin{cases} \hat{F}(w^* \gamma^*_{ik} u^*_{kl} \gamma^*_{lj}) = \binom{n}{k}^{-1/k} & \text{in } \widetilde{F}, \\ u^* = \varphi & \text{on } \partial \widetilde{F}, \end{cases} \tag{3-1}$$

where $\hat{F}(w^* \gamma^*_{ik} u^*_{kl} \gamma^*_{lj}) = (\sigma_n/\sigma_{n-k})^{1/k}(\kappa^*[w^* \gamma^*_{ik} u^*_{kl} \gamma^*_{lj}])$. Then the Legendre transform of u^*, denoted by u, satisfies, when $x/|x| \in \mathcal{F}$,

$$u(x) - |x| \to -\varphi\left(\frac{x}{|x|}\right) \quad \text{uniformly as } |x| \to \infty. \tag{3-2}$$

Notice that the proof of the above lemma is independent of the equation that the function u^* satisfies. Therefore, adapting the above lemma to the settings in this paper, this lemma tells us that if a strictly convex function $u^* : B_1 \to \mathbb{R}$ satisfies $u^*(\xi) = -\varphi(\xi)$ for $\xi \in \partial B_1$, then the Legendre transform of u^*, denoted by u, satisfies $u(x) \to |x| + \varphi(x/|x|)$ as $|x| \to \infty$. Moreover, by [Wang and Xiao 2022, Theorem 4], there exist two solutions $\underline{u}$ and $\bar{u}$ such that

$$\sigma_k(\kappa[\mathcal{M}_{\underline{u}}]) = c_1, \quad \sigma_k(\kappa[\mathcal{M}_{\bar{u}}]) = c_2,$$

and, as $|x| \to \infty$,

$$\underline{u}(x) - |x|,\ \bar{u}(x) - |x| \to \varphi\left(\frac{x}{|x|}\right).$$

Here, the constants c_1, c_2 are the same as those in Theorem 1. Throughout this paper, we will denote the Legendre transforms of $\underline{u}$ and $\bar{u}$ by $\underline{u}^*$ and $\bar{u}^*$, respectively. It's easy to see that $\underline{u}^*$ and $\bar{u}^*$ are the super- and subsolutions of (2-9).

Combining the discussions above with Section 2, we conclude that in order to find an entire, strictly convex solution u of (1-3), we only need to solve the equation

$$\begin{cases} F(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}) = \psi^* & \text{in } B_1, \\ u^* = -\varphi & \text{on } \partial B_1, \end{cases} \tag{3-3}$$

where

$$\psi^*(\xi, u^*, Du^*) = \frac{1}{\psi(x, u, Du)} = \frac{1}{\psi(Du^*, \xi \cdot Du^* - u^*, \xi)}$$

and

$$F(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}) = \frac{\sigma_n}{\sigma_{n-k}}(\kappa^*[w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}]).$$

Note that, by our assumption in Theorem 1, we have

$$\psi^*_{u^*} = \frac{\psi_u}{\psi^2} \geqslant 0. \tag{3-4}$$

Thus, (3-3) possesses the maximum principle.

Notice that (3-3) is degenerate on ∂B_1. Therefore, we will consider the approximate equation

$$\begin{cases} F(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}) = \psi^* & \text{in } B_r, \\ u^* = \underline{u}^* & \text{on } \partial B_r, \end{cases} \tag{3-5}$$

where $0 < r < 1$.

By the continuity method, we know that, if we can obtain a prior estimates up to the second order, then we can show (3-5) has a unique, strictly convex solution u^{r*}. In view of the super- and subsolutions $\underline{u}^*$ and $\bar{u}^*$, the C^0 estimates are easy to obtain. The C^1 estimates can be derived by following the argument in Section 9.2 of [Ren et al. 2020]. The C^2 estimate on the boundary can be derived from Lemma 27 in [Ren et al. 2020] and the argument of Bo Guan [Guan 1999]. In the following, we only need to consider the global C^2 estimate.

Let $\mathcal{M}_u = \{(x, u(x)) \mid x \in \mathbb{R}^n\}$ be a strictly convex, spacelike hypersurface, $v = \langle X, \nu\rangle$ be the support function of $\mathcal{M}_u$, and u^* be the Legendre transform of u. From Sections 2.2 and 2.3, we know that $\lambda[v_{ij} - v\delta_{ij}] = \kappa^*[w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}]$. Therefore, studying the global C^2 estimate of (3-5) is equivalent to studying the global C^2 estimate of (2-6).

For our convenience, we will consider the equation

$$\hat{F}(\Lambda) = \left(\frac{\sigma_n}{\sigma_{n-k}}\right)^{1/k}(\Lambda) = \tilde{\psi}, \tag{3-6}$$

where $\Lambda = (\Lambda_{ij}) = (v_{ij} - v\delta_{ij})$, $\tilde{\psi} = \psi^{-1/k}(X, \nu)$, and the v_{ij} are the covariant derivatives with respect to the hyperbolic metric.

We will write $\lambda[\Lambda] = (\lambda_1, \lambda_2, \dots, \lambda_n)$ for the set of eigenvalues of the matrix Λ. We define the Riemann curvature tensor

$$R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}.$$

Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal frame on $\mathbb{H}^n$; we use the notation

$$R_{ijkl} = R(e_i, e_j)e_k \cdot e_l \quad \text{and} \quad R^l_{ijk} = g^{lp}R_{ijkp}.$$

Then the commutation formulas are

$$v_{ijk} - v_{ikj} = R^l_{jki}v_l \quad \text{and} \quad v_{ijkl} - v_{ijlk} = R^m_{kli}v_{jm} + R^m_{klj}v_{im}.$$

Note that, in hyperbolic space, we have

$$R_{ijkl} = g_{ik}g_{jl} - g_{il}g_{jk}.$$

Therefore, given an orthonormal frame on $\mathbb{H}^n$, we obtain the geometric formulas

$$\Lambda_{ijk} = \Lambda_{ikj} \quad \text{and} \quad \Lambda_{lkji} - \Lambda_{lkij} = v_{lkji} - v_{lkij} = -v_{lj}\delta_{ik} + v_{li}\delta_{jk} - v_{jk}\delta_{il} + v_{ik}\delta_{jl}. \tag{3-7}$$

Lemma 10. *Let v be the solution of (3-6) in a bounded domain $U \subset \mathbb{H}^n$. Denote the set of eigenvalues of $(v_{ij} - v\delta_{ij})$ by $\lambda[v_{ij} - v\delta_{ij}] = (\lambda_1, \dots, \lambda_n)$. Then*

$$\lambda_{\max} \leqslant \max\{C, \lambda|_{\partial U}\},$$

where $\lambda_{\max} = \max\{\lambda_1, \dots, \lambda_n\}$ and C is a positive constant only depending on U and $\tilde{\psi}$.

Proof. Set

$$M = \max_{P\in\bar{U}} \max_{\substack{|\xi|=1 \\ \xi\in T_P\mathbb{H}^n}} (\log \Lambda_{\xi\xi} + Nx_{n+1}),$$

where x_{n+1} is the coordinate function. Without loss of generality, we assume M is achieved at an interior point $P_0 \in U$ for some direction ξ_0. Chose an orthonormal frame $\{e_1, \dots, e_n\}$ around P_0 such that $e_1(P_0) = \xi_0$ and $\Lambda_{ij}(P_0) = \lambda_i\delta_{ij}$.

Now, let's consider the test function

$$\phi = \log \Lambda_{11} + Nx_{n+1}.$$

At its maximum point P_0, we have

$$0 = \phi_i = \frac{\Lambda_{11i}}{\Lambda_{11}} + N(x_{n+1})_i, \tag{3-8}$$

$$0 \geqslant \phi_{ii} = \frac{\Lambda_{11ii}}{\Lambda_{11}} - \frac{\Lambda^2_{11i}}{\Lambda^2_{11}} + N(x_{n+1})_{ii}. \tag{3-9}$$

Note that $(x_{n+1})_{ij} = x_{n+1}\delta_{ij}$; thus

$$\hat{F}^{ii}\phi_{ii} = \frac{\hat{F}^{ii}\Lambda_{11ii}}{\Lambda_{11}} - \frac{\hat{F}^{ii}\Lambda_{11i}^2}{\Lambda_{11}^2} + N x_{n+1}\sum_i \hat{F}^{ii}. \tag{3-10}$$

In view of (3-7),

$$\Lambda_{11ii} = \Lambda_{i11i} = \Lambda_{i1i1} + v_{ii} - v_{11} = \Lambda_{ii11} + \Lambda_{ii} - \Lambda_{11}.$$

This yields

$$\hat{F}^{ii}\Lambda_{11ii} = \hat{F}^{ii}\Lambda_{ii11} + \hat{F}^{ii}\Lambda_{ii} - \Lambda_{11}\sum_i \hat{F}^{ii}. \tag{3-11}$$

Differentiating (3-6) twice, we obtain

$$\hat{F}^{ii}\Lambda_{ii11} = -\hat{F}^{pq,rs}\Lambda_{pq1}\Lambda_{rs1} + \tilde{\psi}_{11} = -\hat{F}^{pp,qq}\Lambda_{pp1}\Lambda_{qq1} - \sum_{p\neq q}\frac{\hat{F}^{pp} - \hat{F}^{qq}}{\lambda_p - \lambda_q}\Lambda_{pq1}^2 + \tilde{\psi}_{11}. \tag{3-12}$$

By the concavity of $(\sigma_n/\sigma_{n-k})^{1/k}$, we can see that the first term on the right-hand side is nonnegative. Combining (3-10)–(3-12), we have

$$\begin{aligned}
\hat{F}^{ii}\phi_{ii} &\geqslant \frac{\tilde{\psi}_{11}}{\Lambda_{11}} - \frac{1}{\Lambda_{11}}\sum_{p\neq q}\frac{\hat{F}^{pp} - \hat{F}^{qq}}{\lambda_p - \lambda_q}\Lambda_{pq1}^2 - \frac{\hat{F}^{ii}\Lambda_{11i}^2}{\Lambda_{11}^2} + (N x_{n+1} - 1)\sum_i \hat{F}^{ii} \\
&\geqslant \frac{\tilde{\psi}_{11}}{\Lambda_{11}} + \frac{1}{\Lambda_{11}}\sum_{i\neq 1}\frac{\hat{F}^{ii} - \hat{F}^{11}}{\lambda_1 - \lambda_i}\Lambda_{11i}^2 - \frac{\hat{F}^{ii}\Lambda_{11i}^2}{\Lambda_{11}^2} + (N x_{n+1} - 1)\sum_i \hat{F}^{ii}.
\end{aligned} \tag{3-13}$$

We need an explicit expression of $\hat{F}^{ii}$. A straightforward calculation gives

$$k\hat{F}^{k-1}\hat{F}^{ii} = \frac{\sigma_n^{ii}\sigma_{n-k} - \sigma_n\sigma_{n-k}^{ii}}{\sigma_{n-k}^2}, \tag{3-14}$$

where $\sigma_l^{ii} = \partial\sigma_l/\partial\lambda_i$ for $1 \leqslant l \leqslant n$. We find that

$$\begin{aligned}
\sigma_n^{ii}\sigma_{n-k} - \sigma_n\sigma_{n-k}^{ii} &= \sigma_{n-1}(\lambda|i)(\lambda_i\sigma_{n-k-1}(\lambda|i) + \sigma_{n-k}(\lambda|i)) - \lambda_i\sigma_{n-1}(\lambda|i)\sigma_{n-k-1}(\lambda|i) \\
&= \sigma_{n-1}(\lambda|i)\sigma_{n-k}(\lambda|i).
\end{aligned}$$

Here and in the following, $\sigma_l(\lambda|a)$ and $\sigma_l(\lambda|ab)$ are the l-th elementary symmetric polynomials of $\lambda_1, \ldots, \lambda_n$ with $\lambda_a = 0$ and $\lambda_a = \lambda_b = 0$, respectively. It follows that

$$k\hat{F}^{k-1}\hat{F}^{ii} = \frac{\sigma_{n-1}(\lambda|i)\sigma_{n-k}(\lambda|i)}{\sigma_{n-k}^2}. \tag{3-15}$$

Therefore, we get

$$\begin{aligned}
k\hat{F}^{k-1}(\hat{F}^{ii} - \hat{F}^{11}) &= \frac{1}{\sigma_{n-k}^2}[\sigma_{n-1}(\lambda|i)\sigma_{n-k}(\lambda|i) - \sigma_{n-1}(\lambda|1)\sigma_{n-k}(\lambda|1)] \\
&= \frac{\sigma_{n-2}(\lambda|1i)}{\sigma_{n-k}^2}[\lambda_1\sigma_{n-k}(\lambda|i) - \lambda_i\sigma_{n-k}(\lambda|1)] \\
&= \frac{\sigma_{n-2}(\lambda|1i)(\lambda_1 - \lambda_i)}{\sigma_{n-k}^2}[(\lambda_1 + \lambda_i)\sigma_{n-k-1}(\lambda|1i) + \sigma_{n-k}(\lambda|1i)].
\end{aligned} \tag{3-16}$$

When $i \geqslant 2$, we can see that

$$k\hat{F}^{k-1}\left(\frac{\hat{F}^{ii}-\hat{F}^{11}}{\lambda_1-\lambda_i}-\frac{\hat{F}^{ii}}{\lambda_1}\right)=\frac{\sigma_{n-2}(\lambda\,|\,1i)}{\sigma_{n-k}^2}[(\lambda_1+\lambda_i)\sigma_{n-k-1}(\lambda\,|\,1i)+\sigma_{n-k}(\lambda\,|\,1i)-\sigma_{n-k}(\lambda\,|\,i)]$$
$$=\frac{\sigma_{n-2}(\lambda\,|\,1i)}{\sigma_{n-k}^2}\lambda_i\sigma_{n-k-1}(\lambda\,|\,1i)=\frac{\sigma_{n-1}(\lambda\,|\,1)}{\sigma_{n-k}^2}\sigma_{n-k-1}(\lambda\,|\,1i)>0. \tag{3-17}$$

Plugging (3-17) into (3-13), we obtain

$$\hat{F}^{ii}\phi_{ii}\geqslant\frac{\tilde{\psi}_{11}}{\Lambda_{11}}-\hat{F}^{11}\frac{\Lambda_{11i}^2}{\Lambda_{11}^2}+(Nx_{n+1}-1)\sum_i\hat{F}^{ii}=\frac{\tilde{\psi}_{11}}{\Lambda_{11}}-\hat{F}^{11}N^2(y_{n+1})_1^2+(Nx_{n+1}-1)\sum_i\hat{F}^{ii}. \tag{3-18}$$

Here, in the last equality, we have used (3-8).

Now, let's calculate $\tilde{\psi}_{11}$. We denote by $\overline{\nabla}$ the connection of the ambient space and by $\{e_1^*, e_2^*, \dots, e_n^*\}$ the pull back of $\{e_1, e_2, \dots, e_n\}$ via the Gauss map. Differentiating $\tilde{\psi}$ with respect to e_1 twice, we get

$$\tilde{\psi}_1=d_X\psi^{-1/k}(\overline{\nabla}_{e_1^*}X)+d_\nu\psi^{-1/k}(e_1) \tag{3-19}$$

and

$$\begin{aligned}\tilde{\psi}_{11}&=d_Xd_X\psi^{-1/k}(\overline{\nabla}_{e_1^*}X,\overline{\nabla}_{e_1^*}X)+d_X\psi^{-1/k}(\overline{\nabla}_{e_1}\overline{\nabla}_{e_1^*}X)\\&\qquad+2d_Xd_\nu\psi^{-1/k}(e_1,\overline{\nabla}_{e_1^*}X)+d_\nu d_\nu\psi^{-1/k}(e_1,e_1)+d_\nu\psi^{-1/k}(\overline{\nabla}_{e_1}e_1)\\&\geqslant c_0\Lambda_{11}^2+d_X\psi^{-1/k}\left(\overline{\nabla}_{e_1}\sum_k\Lambda_{k1}e_k\right)+2d_Xd_\nu\psi^{-1/k}\left(e_1,\sum_l\Lambda_{l1}e_l\right)\\&\qquad+d_\nu d_\nu\psi^{-1/k}(e_1,e_1)+d_\nu\psi^{-1/k}(\nu)\\&\geqslant c_0\Lambda_{11}^2+\sum_k d_X\psi^{-1/k}(\Lambda_{k11}e_k+\Lambda_{k1}\delta_{k1}\nu)-C\lambda_1-C\\&\geqslant c_0\Lambda_{11}^2+\sum_k\Lambda_{11k}d_X\psi^{-1/k}(e_k)-C\lambda_1-C,\end{aligned} \tag{3-20}$$

where the first inequality comes from the locally strict convexity assumption on $\psi^{-1/k}$, i.e., for any spacelike vector $\xi\in\mathbb{R}^{n,1}$,

$$d_Xd_X\psi^{-1/k}(\xi,\xi)\geqslant c_0|\xi|_E^2\geqslant c_0|\xi|_M^2.$$

Here $c_0>0$ is some constant depending on the defining domain, and $|\cdot|_E$ and $|\cdot|_M$ are the Euclidean norm and Minkowski norm, respectively. At the point P_0, in view of (3-8) and the assumption that $\psi_{x_{n+1}}\geqslant 0$, we derive

$$\begin{aligned}\frac{\tilde{\psi}_{11}}{\Lambda_{11}}&\geqslant c_0\lambda_1-N\sum_k(x_{n+1})_kd_X\psi^{-1/k}(e_k)-C-\frac{C}{\lambda_1}\\&=c_0\lambda_1+\frac{N}{k}\psi^{-1/k-1}d_X\psi(\nabla x_{n+1})-C-\frac{C}{\lambda_1}\\&=c_0\lambda_1+\frac{N}{k}\psi^{-1/k-1}d_X\psi\left(-\frac{\partial}{\partial x_{n+1}}+x_{n+1}\nu\right)-C-\frac{C}{\lambda_1}\end{aligned}$$

$$
\begin{aligned}
&= c_0\lambda_1 + \frac{N}{k}\psi^{-1/k-1} d_X\psi\left(|x|^2\frac{\partial}{\partial x_{n+1}} + x_{n+1}\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}\right) - C - \frac{C}{\lambda_1}\\
&= c_0\lambda_1 + \frac{N|x|^2}{k}\psi^{-1/k-1}\frac{\partial\psi}{\partial x_{n+1}} + \frac{N}{k}\psi^{-1/k-1}x_{n+1}\sum_{i=1}^n x_i\frac{\partial\psi}{\partial x_i} - C - \frac{C}{\lambda_1}\\
&\geqslant c_0\lambda_1 + \frac{N}{k}\psi^{-1/k-1}x_{n+1}\sum_{i=1}^n x_i\frac{\partial\psi}{\partial x_i} - C - \frac{C}{\lambda_1} \geqslant -C - \frac{C}{\lambda_1}.
\end{aligned}
\tag{3-21}
$$

Here, in the last inequality, we have assumed $\lambda_1 = \lambda_1(|\psi|_{C^2}) > 0$ is large at P_0. On the other hand, note that the functional $\hat{F}$ is concave and homogenous of degree 1. Therefore,

$$
\sum_i \hat{F}^{ii} = \hat{F}(\lambda) + \sum_i \hat{F}^{ii}(1-\lambda_i) \geqslant \hat{F}(1) = \binom{n}{k}^{-1/k}. \tag{3-22}
$$

Combining (3-18)–(3-22), we obtain

$$
0 \geqslant \hat{F}^{ii}\phi_{ii} \geqslant -C - \frac{C}{\lambda_1} - \frac{C}{\lambda_1}N^2(x_{n+1})_1^2 + (Nx_{n+1}-1)\binom{n}{k}^{-1/k}.
$$

Letting N and λ_1 be sufficiently large, we obtain a contradiction. This completes the proof of Lemma 10.

Notice that this is the only place we need the locally strict convexity assumption of $\psi^{-1/k}$ in Theorem 1. It's also clear that the above proof can be easily modified to the case when $\psi^{-1/k}$ is convex with respect to X and the corresponding $\psi(x, u(x), Du)$ does not depend on $|x|$ (see the second inequality in (3-21)), as stated in the Remark 2. Therefore, (3-5) is solvable when either $\psi^{-1/k}$ is locally strictly convex with respect to X or $\psi^{-1/k}$ is convex with respect to X and $\psi(x, u(x), Du(x))$ does not depend on $|x|$. □

3.2. *Dirichilet problem for $k = n-1, n-2$.*

Let $n \in \mathbb{N}$ and $\Omega_n := \{x \in \mathbb{R}^n \mid \underline{u}(x) = n\}$. We will consider the Dirichlet problem

$$
\begin{cases}
\sigma_k(\kappa[\mathcal{M}_u]) = \psi(x, u(x)) & \text{in } \Omega_n,\\
u = n & \text{on } \partial\Omega_n.
\end{cases}
\tag{3-23}
$$

Note that since $\underline{u}$ is strictly convex, Ω_n is strictly convex. It's easy to see that if u is a solution of (3-23), then $\underline{u} \leqslant u \leqslant \bar{u}$. Therefore, in order to find a k-convex solution u for (3-23), we only need to study the C^1 and C^2 estimates of u.

3.2.1. C^1 *estimate for* (3-23).

Lemma 11. *Let u be a solution of* (3-23), *then $|Du| < C < 1$. Here C is a constant depending on $|D\underline{u}|_{\bar{\Omega}_n}$ and ψ.*

Proof. Let $V = -\langle \nu, E\rangle = 1/\sqrt{1-|Du|^2}$, and consider the test function $\phi = \ln V + Ku$, where $K > 0$ is to be determined. If ϕ achieves its maximum at an interior point $P_0 \in \mathcal{M}_u$, then at this point, we may choose a normal coordinate $\{\tau_1, \dots, \tau_n\}$ such that $h_{ij} = \kappa_i\delta_{ij}$. Since at P_0 we have

$$
\phi_i = \frac{V_i}{V} + Ku_i = 0 \quad \text{and} \quad 0 \geqslant \phi_{ii} = \frac{V_{ii}}{V} - \frac{V_i^2}{V^2} + Ku_{ii},
$$

a straightforward calculation yields

$$0 \geqslant -\frac{\langle \nabla\sigma_k, E\rangle}{V} - \frac{\sigma_k^{ii}\kappa_i^2 u_i^2}{V^2} + Kk\psi V + \sigma_k^{ii}\kappa_i^2.$$

Note that $|\langle \nabla\sigma_k, E\rangle| \leqslant CV^2$, where C only depends on $|\psi|_{C^1}$. Choosing $K > C+1$, we have

$$-\frac{\langle \nabla\sigma_k, E\rangle}{V} - \frac{\sigma_k^{ii}\kappa_i^2 u_i^2}{V^2} + Kk\psi V + \sigma_k^{ii}\kappa_i^2 > 0.$$

This leads to a contradiction. □

3.2.2. *C^2 boundary estimates for (3-23).* Now, we will establish the C^2 boundary estimate. For our convenience, we will consider the solvability of the Dirichlet problem

$$\begin{cases} G(Du, D^2u) = \sigma_k\big(\frac{1}{w}\gamma^{ik}u_{kl}\gamma^{lj}\big) = \psi(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3-24}$$

where Ω is strictly convex. We will follow the idea of [Caffarelli et al. 1988].

Infinitesimal stretching. If u is a solution of (3-24), let $v(x) = t^{-1}u(tx)$, where $t > 0$. Then the principal curvatures of $\mathcal{M}_v$ satisfy $\kappa[\mathcal{M}_v(x)] = t\kappa[\mathcal{M}_u(tx)]$. Therefore,

$$G(Dv, D^2v) = t^k\psi(tx, u(tx)) = t^k\psi(tx, tv(x)). \tag{3-25}$$

We write $\dot v = (d/dt)v = -t^{-2}u(tx) + x\cdot Du(tx)$; when $t = 1$,

$$\dot v = x\cdot Du(x) - u(x).$$

Differentiating (3-25) with respect to t then evaluating at $t = 1$, we obtain

$$G^{ij}\partial_{ij}\dot v + G^s\partial_s\dot v = k\psi + \psi_z(v + \dot v) + x\psi_x.$$

Writing $L := G^{ij}\partial_{ij} + G^s\partial_s$, we have

$$L(x\cdot Du - u) = k\psi + \psi_z(u + x\cdot Du - u) + x\psi_x = k\psi + x\psi_x + \psi_z x\cdot Du. \tag{3-26}$$

Infinitesimal rotation in Minkowski space. It is well known that Lorentz boosts are isometries of $\mathbb{R}^{n,1}$. Keeping the coordinates $x' = (x_1, \dots, x_{n-1})$ fixed, we rotate in the (x_n, u) variables:

$$\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}\begin{bmatrix} x_n \\ u \end{bmatrix} = \begin{bmatrix} \cosh\theta x_n + \sinh\theta u \\ \cosh\theta u + \sinh\theta x_n \end{bmatrix}.$$

To the first order in θ, the image of $(x, u(x))$ under such a rotation is

$$(x', x_n + u(x)\theta, u(x) + x_n\theta).$$

Therefore, to the first order in θ, the image of

$$(x', x_n - u(x)\theta, u(x', x_n - u(x)\theta))$$

is $(x', x_n, u(x', x_n - u(x)\theta) + x_n\theta)$. Considering this image as the graph of the function

$$v(x) = u(x', x_n - u(x)\theta) + x_n\theta + \text{higher order in } \theta,$$

we have

$$\begin{aligned} G(Dv, D^2v) &= \psi(x', x_n - u(x)\theta, u(x', x_n - u(x)\theta)) + \text{higher order in } \theta \\ &= \psi(x', x_n - u(x)\theta, v(x) - x_n\theta) + \text{higher order in } \theta. \end{aligned}$$

Notice that $(dv/d\theta)|_{\theta=0} = x_n - u_n u$, so we obtain

$$G^{ij}\partial_{ij}(x_n - u_n u) + G^s\partial_s(x_n - u_n u) = \psi_n(-u(x)) + \psi_z(x_n - u_n u - x_n). \tag{3-27}$$

Thus, we conclude that

$$L(x_n - uu_n) = -u\psi_n - u_n u\psi_z. \tag{3-28}$$

Lemma 12. *Let u be a solution of (3-24), then $|D^2u| < C$ on $\partial\Omega$. Here C is a constant depending on Ω and ψ.*

Proof. For any $p \in \partial\Omega$, we suppose p is the origin and that the x_n-axis is the interior normal of $\partial\Omega$ at p. We may also assume the boundary near the origin p is represented by

$$x_n = \frac{1}{2}\sum_{\alpha=1}^{n-1}\lambda_\alpha x_\alpha^2 + O(|x'|^3), \qquad x' = (x_1, \dots, x_{n-1}),$$

where $\lambda_\alpha > 0$, $1 \leqslant \alpha \leqslant n-1$, are the principal curvatures of $\partial\Omega$ at the origin. Let $T_\alpha = \partial_\alpha + \lambda_\alpha(x_\alpha\partial_n - x_n\partial_\alpha)$. Note that $G^{ij}u_{ij\alpha} + G^s u_{s\alpha} = \psi_\alpha + \psi_z u_\alpha$. In view of the fact that (3-23) is invariant under rotation (see (3.1) in [Caffarelli et al. 1988]), we get

$$|LT_\alpha u| \leqslant C. \tag{3-29}$$

Moreover, it's easy to see we have $|T_\alpha u| \leqslant C|x'|^2$ on $\partial\Omega$ near the origin. In the following, we write $\Omega_\beta := \Omega \cap \{x_n < \beta\}$. Set

$$h = (x \cdot Du - u) - \frac{\delta}{\beta}(x_n - uu_n).$$

On $\partial\Omega \cap \partial\Omega_\beta$, note that $u = 0$, so we have $x \cdot Du \leqslant C_1|x'|^2$. This implies, on $\partial\Omega \cap \partial\Omega_\beta$,

$$h = x \cdot Du - \frac{\delta}{\beta}x_n \leqslant \left(C_1 - \frac{\delta}{\beta}a\right)|x'|^2, \tag{3-30}$$

where $a > 0$ depends on the principal curvatures of $\partial\Omega$. Notice that u is a spacelike function, so we suppose $|Du| \leqslant \theta_0$ in $\bar{\Omega}$ for some $\theta_0 \in (0, 1)$. Then we have $0 \leqslant -u \leqslant \theta_0\beta$ in Ω_β. Therefore, on $\{x_n = \beta\}$,

$$h = \beta u_n + \sum_{\alpha=1}^{n-1}x_\alpha u_\alpha - u + \frac{\delta}{\beta}uu_n - \delta \leqslant \beta\theta_0 + C\beta^{1/2} + \theta_0\beta + \theta_0^2\delta - \delta \leqslant C\beta^{1/2} + \delta(\theta_0 - 1) \tag{3-31}$$

with C being independent of β and δ. Moreover,

$$Lh = k\psi + x\psi_x + \psi_z x \cdot Du - \frac{\delta}{\beta}(-u\psi_n - u_n u\psi_z) \geqslant k\psi - C\beta^{1/2} - C\delta \geqslant \frac{k}{2}\psi, \tag{3-32}$$

where δ and β are small positive constants.

Now choose $A = A(\delta) > 0$ large enough that

$$Ah \leqslant -|T_\alpha u| \quad \text{on } \partial\Omega_\beta \qquad \text{and} \qquad LAh > |LT_\alpha u| \quad \text{in } \Omega_\beta.$$

By the maximum principle, we conclude that

$$Ah \pm T_\alpha u \leqslant 0 \quad \text{in } \bar{\Omega}_\beta.$$

On the other hand, we have $h(0) = T_\alpha u(0) = 0$. Therefore,

$$|\partial_n T_\alpha u(0)| \leqslant -Ah_n(0) \leqslant \frac{A\delta}{\beta},$$

which yields

$$|u_{n\alpha}(0)| \leqslant C. \tag{3-33}$$

Next, following the notation in Section 2.1, we write $a_{ij} = \frac{1}{w}\gamma^{ik}u_{kl}\gamma^{lj}$, where $w = \sqrt{1-|Du|^2}$ and $\gamma^{ik} = \delta_{ik} + u_i u_k/(w(1+w))$. A straightforward calculation yields, at the origin,

$$\begin{aligned} a_{\alpha\alpha} &= \frac{u_{\alpha\alpha}}{w} = -\frac{u_n\lambda_\alpha}{w}, & a_{\alpha n} &= \frac{u_{\alpha n}}{w^2} & &\text{for } 1 \leqslant \alpha \leqslant n-1, \\ a_{nn} &= \frac{u_{nn}}{w^3}, & a_{ij} &= 0 & &\text{for all other } 1 \leqslant i, \quad j \leqslant n. \end{aligned} \tag{3-34}$$

Since $\partial\Omega$ is smooth, we know there exists $r_0 > 0$ and $z_p = (0, \dots, 0, r_0)$ such that $B_{r_0}(z_p) \subset \Omega$ and $\bar{B}_{r_0}(z_p) \cap \partial\Omega = p$. Here $B_{r_0}(z_p)$ is a ball of radius r_0 centered at z_p. Let

$$\bar{u} = -\sqrt{R^2 + r_0^2} + \sqrt{R^2 + |x - z_p|^2},$$

where $x = (x_1, \dots, x_n)$ and $R > 0$ is a constant to be determined. A straightforward calculation yields

$$\sigma_k\Big(\frac{1}{w}\gamma^{ik}\bar{u}_{kl}\gamma^{lj}\Big) = \binom{n}{k}\frac{1}{R} < c_2$$

when $R = R(c_2) > 0$ is sufficiently large. Here c_2 is the lower bound for ψ defined in Theorem 5. Therefore, $\bar{u}$ is a supersolution of (3-24). By the strong maximum principal, we have $u < \bar{u}$ in $B_{r_0}(z_p)$. Applying the Hopf lemma, we obtain

$$\frac{r_0}{\sqrt{R^2 + r_0^2}} = -\bar{u}_n(p) < -u_n(p).$$

In view of (3-34) and [Trudinger 1995, (2.5)], (3-24) can be written as

$$\frac{1}{w^k}\bigg[\frac{1}{w^2}(-u_n)^{k-1}\sigma_{k-1}(\lambda)u_{nn} + P\bigg] = \psi,$$

where P depends on w, $u_{\alpha\beta}$, and $u_{\alpha n}$, which are bounded by some uniform constants depending on n, k, $\partial\Omega$, $\|u\|_{C^1(\bar{\Omega})}$, and $\lambda = (\lambda_1, \dots, \lambda_{n-1})$. Moreover, by our assumption that ψ is bounded, we obtain an upper bound for $u_{nn}(0)$. The lower bound for $u_{nn}(0)$ comes from the fact that $\mathcal{M}_u$ is k-convex, which implies $\sum_{i=1}^n a_{ii} > 0$.

Finally, since $p \in \partial\Omega$ is arbitrary, we get

$$|D^2u(x)| \leqslant C \quad \text{for any } x \in \partial\Omega. \qquad \square$$

3.2.3. *C^2 global estimate for (3-23).* Finally, we will prove the C^2 global estimate. In this subsubsection, for greater generality, we will assume $\psi = \psi(X, \nu)$.

Lemma 13. *Let u be a solution of (3-24) with $\psi = \psi(X, \nu)$, then*

$$|D^2 u| < \max\{C, \max_{\partial\Omega} |D^2 u|\}$$

on Ω. Here C is a constant depending on $|Du|_\Omega$ and ψ.

Proof. We consider the following test function whose form first appeared in [Guan et al. 2015]:

$$\phi = \log\log P - N\langle \nu, E\rangle.$$

Here, $P := \sum_l e^{\kappa_l}$, and N is a sufficiently large constant to be determined later.

We may assume that the maximum of ϕ is achieved at some point $P_0 \in \mathcal{M}_u$, where u is the solution of (3-24). Suppose $\{\tau_1, \tau_2, \dots, \tau_n\}$ is a normal coordinate near P_0 such that, at P_0,

$$h_{ij} = \kappa_i \delta_{ij} \quad \text{and} \quad \kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant \kappa_n.$$

Differentiating the function ϕ twice at P_0, we have

$$\phi_i = \frac{P_i}{P\log P} + N h_{ii} u_i = 0, \tag{3-35}$$

and

$$\begin{aligned}\phi_{ii} &= \frac{P_{ii}}{P\log P} - \frac{P_i^2}{P^2\log P} - \frac{P_i^2}{(P\log P)^2} - N h_{ii}^2\langle \nu, E\rangle + \sum_s N u_s h_{isi}\\ &= \frac{1}{P\log P}\left[\sum_l e^{\kappa_l} h_{llii} + \sum_l e^{\kappa_l} h_{lli}^2 + \sum_{p\neq q} \frac{e^{\kappa_p} - e^{\kappa_q}}{\kappa_p - \kappa_q} h_{pqi}^2 - \left(\frac{1}{P} + \frac{1}{P\log P}\right) P_i^2\right]\\ &\qquad - N h_{ii}^2\langle \nu, E\rangle + \sum_s N u_s h_{iis}.\end{aligned}$$

Contracting with σ_k^{ii}, we get

$$\begin{aligned}\sigma_k^{ii}\phi_{ii} = \frac{\sigma_k^{ii}}{P\log P}&\left[\sum_l e^{\kappa_l} h_{llii} + \sum_l e^{\kappa_l} h_{lli}^2 + \sum_{p\neq q} \frac{e^{\kappa_p} - e^{\kappa_q}}{\kappa_p - \kappa_q} h_{pqi}^2 - \left(\frac{1}{P} + \frac{1}{P\log P}\right) P_i^2\right]\\ &- N\sigma_k^{ii}\kappa_i^2\langle \nu, E\rangle + \sum_s N u_s \sigma_k^{ii} h_{iis}.\end{aligned} \tag{3-36}$$

At P_0, differentiating (1-2) twice yields

$$\sigma_k^{ii} h_{iil} = d_X\psi(\tau_l) + \kappa_l d_\nu \psi(\tau_l) \tag{3-37}$$

and

$$\sigma_k^{ii} h_{iill} + \sigma_k^{pq,rs} h_{pql} h_{rsl} \geqslant -C - C h_{11}^2 + \sum_s h_{sll} d_\nu \psi(\tau_s), \tag{3-38}$$

where C is some uniform constant only depending on ψ. Note that

$$h_{llii} = h_{iill} - h_{ii} h_{ll}^2 + h_{ii}^2 h_{ll}. \tag{3-39}$$

Inserting (3-38) and (3-39) into (3-36), we obtain

$$\begin{aligned}\sigma_k^{ii}\phi_{ii} \geqslant \frac{1}{P\log P}\bigg[\sum_l e^{\kappa_l}\Big(-C - C\kappa_1^2 - \sigma_k^{pq,rs}h_{pql}h_{rsl} + \sum_s h_{sll}d_\nu\psi(\tau_s)\Big) \\ + \sum_l \sigma_k^{ii}e^{\kappa_l}h_{lli}^2 + \sigma_k^{ii}\sum_{p\neq q}\frac{e^{\kappa_p}-e^{\kappa_q}}{\kappa_p-\kappa_q}h_{pqi}^2 - \Big(\frac{1}{P}+\frac{1}{P\log P}\Big)\sigma_k^{ii}P_i^2\bigg] \\ -N\sigma_k^{ii}\kappa_i^2\langle\nu, E\rangle + \sum_s N u_s\sigma_k^{ii}h_{sii} - \sigma_k^{ii}\kappa_i^2. \qquad (3\text{-}40)\end{aligned}$$

By (3-35) and (3-37), we have

$$\frac{1}{P\log P}\sum_s\sum_l e^{\kappa_l}h_{sll}d_\nu\psi(\tau_s) + \sum_s N u_s\sigma_k^{ii}h_{sii} \geqslant -C.$$

Now, for any constant $K > 1$, we write

$$\begin{aligned}A_i &= e^{\kappa_i}\Big[K(\sigma_k)_i^2 - \sum_{p\neq q}\sigma_k^{pp,qq}h_{ppi}h_{qqi}\Big],\\ B_i &= 2\sum_{l\neq i}\sigma_k^{ii,ll}e^{\kappa_l}h_{lli}^2, \qquad C_i = \sigma_k^{ii}\sum_l e^{\kappa_l}h_{lli}^2,\\ D_i &= 2\sum_{l\neq i}\sigma_k^{ll}\frac{e^{\kappa_l}-e^{\kappa_i}}{\kappa_l-\kappa_i}h_{lli}^2, \qquad E_i = \frac{1+\log P}{P\log P}\sigma_k^{ii}P_i^2.\end{aligned}$$

Combining

$$-\sum_l \sigma_k^{pq,rs}h_{pql}h_{rsl} = \sum_{p\neq q}\sigma_k^{pp,qq}h_{pql}^2 - \sum_{p\neq q}\sigma_k^{pp,qq}h_{ppl}h_{qql}$$

with (3-40), we get

$$\sigma_k^{ii}\phi_{ii} \geqslant \frac{1}{P\log P}\sum_i(A_i + B_i + C_i + D_i - E_i) + (-N\langle\nu, E\rangle - 1)\sigma_k^{ii}\kappa_i^2 - C\kappa_1. \qquad (3\text{-}41)$$

Claim 1. *For any given $0 < \varepsilon < \frac{1}{2}$, we let $\alpha = (1-2\varepsilon)/(1+\varepsilon)$. There exists a positive constant $\delta < \frac{1}{2}$ such that, for any $|\kappa_i| \leqslant \delta\kappa_1$, $1 \leqslant i \leqslant n$, if the constant K and the maximum principal curvature κ_1 are both sufficiently large, we have*

$$A_i + B_i + C_i + D_i - E_i - \frac{\alpha}{P\log P}\sigma_k^{ii}P_i^2 \geqslant 0.$$

Applying Lemma 6 in [Ren and Wang 2019], we can see that when K is chosen to be sufficiently large, we have $A_i \geqslant 0$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}P_i^2 &= e^{2\kappa_i}h_{iii}^2 + 2\sum_{l\neq i}e^{\kappa_i+\kappa_l}h_{iii}h_{lli} + \Big(\sum_{l\neq i}e^{\kappa_l}h_{lli}\Big)^2\\ &\leqslant e^{2\kappa_i}h_{iii}^2 + 2\sum_{l\neq i}e^{\kappa_i+\kappa_l}h_{iii}h_{lli} + (P - e^{\kappa_i})\sum_{l\neq i}e^{\kappa_l}h_{lli}^2. \qquad (3\text{-}42)\end{aligned}$$

Thus,

$$\begin{aligned} B_i + C_i + D_i - E_i - \frac{\alpha}{P\log P}\sigma_k^{ii}P_i^2 \\ \geqslant 2\sum_{l\neq i} e^{\kappa_l}\sigma_k^{ll,ii}h_{lli}^2 + 2\sum_{l\neq i}\frac{e^{\kappa_l}-e^{\kappa_i}}{\kappa_l-\kappa_i}\sigma_k^{ll}h_{lli}^2 - \frac{1+\alpha}{\log P}\sum_{l\neq i} e^{\kappa_l}\sigma_k^{ii}h_{lli}^2 + \frac{1+\alpha+\log P}{P\log P}\sum_{l\neq i} e^{\kappa_l+\kappa_i}\sigma_k^{ii}h_{lli}^2 \\ + e^{\kappa_i}\sigma_k^{ii}h_{iii}^2 - \frac{1+\alpha+\log P}{P\log P}e^{2\kappa_i}\sigma_k^{ii}h_{iii}^2 - 2\frac{1+\alpha+\log P}{P\log P}\sum_{l\neq i} e^{\kappa_i+\kappa_l}\sigma_k^{ii}h_{iii}h_{lli}. \end{aligned} \tag{3-43}$$

Let ε be equal to the ε_T in Lemma 12 of [Ren and Wang 2019]. Then we know there exists a positive constant $\delta < \varepsilon$ such that, when $|\kappa_i| < \delta\kappa_1$,

$$(2-\varepsilon)\sum_{l\neq i} e^{\kappa_l}\sigma_k^{ll,ii}h_{lli}^2 + (2-\varepsilon)\sum_{l\neq i}\frac{e^{\kappa_l}-e^{\kappa_i}}{\kappa_l-\kappa_i}\sigma_k^{ll}h_{lli}^2 - \frac{1+\alpha}{\log P}\sum_{l\neq i} e^{\kappa_l}\sigma_k^{ii}h_{lli}^2 \geqslant 0. \tag{3-44}$$

On the other hand, we have

$$\sum_{l\neq i,1} e^{\kappa_l+\kappa_i}\sigma_k^{ii}h_{lli}^2 - 2\sum_{l\neq i,1} e^{\kappa_i+\kappa_l}\sigma_k^{ii}h_{iii}h_{lli} \geqslant -\sum_{l\neq i,1} e^{\kappa_l+\kappa_i}\sigma_k^{ii}h_{iii}^2. \tag{3-45}$$

It follows that

$$\begin{aligned} B_i + C_i + D_i - E_i - \frac{\alpha}{P\log P}\sigma_k^{ii}P_i^2 \\ \geqslant \frac{1+\alpha+\log P}{P\log P}e^{\kappa_1+\kappa_i}\sigma_k^{ii}h_{11i}^2 + e^{\kappa_i}\sigma_k^{ii}h_{iii}^2 - \frac{1+\alpha+\log P}{P\log P}\sum_{l\neq 1} e^{\kappa_l+\kappa_i}\sigma_k^{ii}h_{iii}^2 \\ - 2\frac{1+\alpha+\log P}{P\log P}e^{\kappa_i+\kappa_1}\sigma_k^{ii}h_{iii}h_{11i} + \varepsilon e^{\kappa_1}\sigma_k^{11,ii}h_{11i}^2 + \varepsilon\frac{e^{\kappa_1}-e^{\kappa_i}}{\kappa_1-\kappa_i}\sigma_k^{11}h_{11i}^2. \end{aligned} \tag{3-46}$$

A straightforward calculation shows that, when κ_1 is very large, the following inequalities hold:

$$e^{\kappa_i}\sigma_k^{ii}h_{iii}^2 - \frac{1+\alpha+\log P}{P\log P}\sum_{l\neq 1} e^{\kappa_l+\kappa_i}\sigma_k^{ii}h_{iii}^2 \geqslant \left(\frac{e^{\kappa_1}}{P} - \frac{1+\alpha}{\log P}\right)e^{\kappa_i}\sigma_k^{ii}h_{iii}^2 \geqslant \frac{1}{n+1}e^{\kappa_i}\sigma_k^{ii}h_{iii}^2,$$

and

$$-2\frac{1+\alpha+\log P}{P\log P}e^{\kappa_i+\kappa_1}\sigma_k^{ii}|h_{iii}h_{11i}| \geqslant -\frac{3}{P}e^{\kappa_i+\kappa_1}\sigma_k^{ii}|h_{iii}h_{11i}| \geqslant -3e^{\kappa_i}\sigma_k^{ii}|h_{iii}h_{11i}|.$$

Moreover, it is easy to see that

$$e^{\kappa_1}\sigma_k^{11,ii}h_{11i}^2 + \frac{e^{\kappa_1}-e^{\kappa_i}}{\kappa_1-\kappa_i}\sigma_k^{11}h_{11i}^2 = e^{\kappa_i}\sigma_k^{11,ii}h_{11i}^2 + \frac{e^{\kappa_1}-e^{\kappa_i}}{\kappa_1-\kappa_i}\sigma_k^{ii}h_{11i}^2. \tag{3-47}$$

By the Taylor expansion, we have

$$\frac{e^{\kappa_1}-e^{\kappa_i}}{\kappa_1-\kappa_i}\sigma_k^{ii}h_{11i}^2 = e^{\kappa_i}\sum_{m\geqslant 1}\frac{(\kappa_1-\kappa_i)^{m-1}}{m!}\sigma_k^{ii}h_{11i}^2. \tag{3-48}$$

Combining the previous four formulas with (3-46), when κ_1 is sufficiently large and $|\kappa_i| < \delta\kappa_1$, we obtain

$$B_i + C_i + D_i - E_i - \frac{\alpha}{P\log P}\sigma_k^{ii} P_i^2 \geqslant e^{\kappa_i}\sigma_k^{ii}\left[\frac{1}{n+1}h_{iii}^2 - 3|h_{iii}h_{11i}| + \varepsilon\sum_{m\geqslant 1}\frac{(\kappa_1-\kappa_i)^{m-1}}{m!}h_{11i}^2\right] \geqslant 0.$$

Therefore, Claim 1 is proved.

Recalling Section 4 of [Ren and Wang 2019] and the proof of Theorem 14 in [Ren and Wang 2023], we know the following claim is true.

Claim 2. *Suppose $k = n-1$ $(n \geqslant 3)$ or $k = n-2$ $(n \geqslant 5)$. For any index $1 \leqslant i \leqslant n$, if the positive constant K and the maximum principal curvature κ_1 are both sufficiently large, we have*

$$A_i + B_i + C_i + D_i - E_i \geqslant 0.$$

By Claims 1 and 2, (3-41) becomes

$$0 \geqslant \sum_{|\kappa_i|<\delta\kappa_1} \frac{\alpha}{(P\log P)^2}\sigma_k^{ii} P_i^2 + (-N\langle \nu, E\rangle - 1)\sigma_k^{ii}\kappa_i^2 - C\kappa_1. \tag{3-49}$$

Here, the constant δ is the constant chosen in Claim 1. Choosing $N > 0$ such that

$$\sigma_k^{11}\kappa_1^2(-N\langle \nu, E\rangle - 1) - C\kappa_1 > 0,$$

we get a contradiction. Therefore, our desired estimate follows immediately. □

By Lemmas 11, 12, and 13, we conclude that, when $k = n-1,\ n-2$, the Dirichlet problem (3-23) admits a k-convex solution.

4. The local estimates

We will devote this section to establishing the local C^1 and C^2 estimates for the solution u of (1-3).

4.1. ***Local C^1 estimates.*** In this subsection, we will prove the local C^1 estimate. We will split it into two cases. In the first case, we will assume u is a convex solution of (1-2); in the second case, we will assume u is a k-convex solution of (1-5). Note that in both cases our results hold for $1 \leqslant k \leqslant n$.

For strictly convex, spacelike hypersurfaces, [Bayard and Schnürer 2009] proved the following local gradient estimate lemma.

Lemma 14 [Bayard and Schnürer 2009, Lemma 5.1]. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $u, \bar{u}, \Psi : \Omega \to \mathbb{R}^n$ be strictly spacelike. Assume that u is strictly convex and $u < \bar{u}$ in Ω. Also assume that, near $\partial\Omega$, we have $\Psi > \bar{u}$. Consider the set with $u > \Psi$. For every x in this set, we have the following gradient estimate for u*:

$$\frac{1}{\sqrt{1-|Du|^2}} \leqslant \frac{1}{u(x)-\Psi(x)}\cdot\sup_{\{u>\Psi\}}\frac{\bar{u}-\Psi}{\sqrt{1-|D\Psi|^2}}.$$

For k-convex, spacelike hypersurfaces, [Bayard 2006] proved a similar result when $k = 2$. In the following, we will extend it to all k. Our argument is a modification of that in [Bayard 2006]. We would also like to mention that the basic idea of this argument appeared in [Chou and Wang 2001].

Lemma 15. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $u, \bar{u}, \Psi : \Omega \to \mathbb{R}^n$ be strictly spacelike. Assume that $\mathcal{M}_u = \{(x, u(x)) \mid x \in \Omega\}$ is a k-convex hypersurface satisfying*

$$\sigma_k(\kappa[\mathcal{M}_u]) = \psi(x, u(x))$$

and $u \leqslant \bar{u}$ in Ω. Also assume that, near $\partial\Omega$, we have $\Psi > \bar{u}$. Consider the set with $u > \Psi$. For every x in this set, we have the following gradient estimate for u:

$$\frac{1}{\sqrt{1-|Du|^2}} \leqslant \left[\frac{1}{u(x)-\Psi(x)} \cdot \sup_{\{u>\Psi\}} (\bar{u}-\Psi)\right]^N C.$$

Here, $N = N(n, k)$ is a uniform constant only depending on n and k, and $C = C(\bar{u}-\Psi, |\Psi|_{C^2}, |\psi|_{C^1})$ is a uniform constant depending on the upper bound of $\bar{u}-\Psi$, $1/\sqrt{1-|D\Psi|^2}$, $D^2\Psi$, and $|\psi|_{C^1}$.

Proof. Consider the test function

$$\phi = (u-\Psi)^N(-\langle \nu, E\rangle),$$

where N is a large undetermined constant. Assume the function ϕ achieves its maximum at P. We may choose a local normal coordinate $\{\tau_1, \dots, \tau_n\}$ such that, at P, we have $h_{ij} = \kappa_i\delta_{ij}$. Differentiating ϕ twice at P, we have

$$\begin{aligned} 0 &= \frac{\phi_i}{\phi} = N\frac{u_i-\Psi_i}{u-\Psi} + \frac{h_{im}u_m}{-\langle \nu, E\rangle}, \\ 0 &\geqslant \frac{\phi_{ii}}{\phi} - \frac{\phi_i^2}{\phi^2} = N\frac{u_{ii}-\Psi_{ii}}{u-\Psi} - N\frac{(u_i-\Psi_i)^2}{(u-\Psi)^2} + \frac{\sum_m h_{im}^2(-\langle \nu, E\rangle) + \sum_m h_{imi}u_m}{-\langle \nu, E\rangle} - \frac{\left(\sum_m h_{im}u_m\right)^2}{(-\langle \nu, E\rangle)^2}. \end{aligned} \tag{4-1}$$

Contracting with σ_k^{ii}, we get

$$0 \geqslant \frac{\sigma_k^{ii}\phi_{ii}}{\phi} = N\frac{\sigma_k^{ii}u_{ii}-\sigma_k^{ii}\Psi_{ii}}{u-\Psi} - N\frac{\sigma_k^{ii}(u_i-\Psi_i)^2}{(u-\Psi)^2} + \sigma_k^{ii}\kappa_i^2 + \frac{\sigma_k^{ii}\sum_m h_{iim}u_m}{-\langle \nu, E\rangle} - \frac{\sigma_k^{ii}\kappa_i^2u_i^2}{(-\langle \nu, E\rangle)^2}. \tag{4-2}$$

Without loss of generality, we may assume that, at P,

$$u_1^2 \geqslant \frac{|\nabla u|^2}{n},$$

where ∇ is the Levi-Civita connection on $\mathcal{M}_u$. By (4-1), we have

$$\kappa_1 = \frac{N\langle \nu, E\rangle}{u-\Psi}\left(1-\frac{\Psi_1}{u_1}\right).$$

We may also assume $|\nabla u(P)|$ is sufficiently large that $|\Psi_1/u_1| < \frac{1}{2}$. Then, at P, we can see

$$\kappa_1 < \frac{N}{2}\frac{\langle \nu, E\rangle}{u-\Psi}. \tag{4-3}$$

Thus, if N is sufficiently large, κ_1 is negative and its norm is large. Using inequality (26) in [Lin and Trudinger 1994], we obtain

$$\sum_{i\geqslant 2}\sigma_k^{ii}\kappa_i^2 \geqslant \eta\sigma_k^{11}\kappa_1^2,$$

where η is a uniform constant only depending on n and k. Therefore,

$$\sigma_k^{ii}\kappa_i^2 - \frac{\sigma_k^{ii}\kappa_i^2u_i^2}{(-\langle\nu, E\rangle)^2} \geqslant \sum_{i\geqslant 2}\sigma_k^{ii}\kappa_i^2 - \left(1-\frac{1}{n}\right)\sum_{i\geqslant 2}\sigma_k^{ii}\kappa_i^2 \geqslant \frac{\eta}{n}\sigma_k^{11}\kappa_1^2 := \eta_0\sigma_k^{11}\kappa_1^2.$$

By (4-3), we get

$$\sigma_k^{ii}\kappa_i^2 - \frac{\sigma_k^{ii}\kappa_i^2u_i^2}{(-\langle\nu, E\rangle)^2} \geqslant \frac{\eta_0N^2}{4}\sigma_k^{11}\frac{(-\langle\nu, E\rangle)^2}{(u-\Psi)^2}. \tag{4-4}$$

Inserting (1-2) and (4-4) into (4-2) yields

$$\begin{aligned} 0 \geqslant N(u-\Psi)[\sigma_k^{ii}\kappa_i(-\langle\nu, E\rangle) - \sigma_k^{ii}\Psi_{ii}] - N\sigma_k^{ii}(u_i-\Psi_i)^2 \\ + (u-\Psi)^2\frac{\sum_m\psi_mu_m}{-\langle\nu, E\rangle} + \frac{\eta_0N^2}{4}\sigma_k^{11}(-\langle\nu, E\rangle)^2. \end{aligned} \tag{4-5}$$

Noticing that

$$\psi_m = \sum_{l=1}^n \psi_{x_l}\left\langle\tau_m, \frac{\partial}{\partial x_l}\right\rangle + \psi_u\langle-\tau_m, E\rangle,$$

we calculate

$$\frac{\sum_m\psi_mu_m}{-\langle\nu, E\rangle} \geqslant -C(1+\langle-\nu, E\rangle). \tag{4-6}$$

Combining (4-5) with (4-6), we get

$$\begin{aligned} 0 \geqslant -(n-k+1)N(\bar{u}-\Psi)\sigma_{k-1}|\nabla^2\Psi| - 2(n-k+1)N\sigma_{k-1}(|\nabla u|^2+|\nabla\Psi|^2) \\ - C(\bar{u}-\Psi)^2(1+\langle-\nu, E\rangle) + \frac{\eta_0N^2}{4}\sigma_k^{11}(-\langle\nu, E\rangle)^2. \end{aligned} \tag{4-7}$$

Notice that, when $\kappa_1 < 0$, we have

$$\sigma_{k-1} = \kappa_1\sigma_{k-2}(\kappa\,|\,1) + \sigma_{k-1}(\kappa\,|\,1) \leqslant \sigma_k^{11}.$$

Moreover, $-\langle\nu, E\rangle = \sqrt{1+|\nabla u|^2}$. With N sufficiently large in (4-7), we obtain the desired estimate. □

4.2. ***The Pogorelov-type local C^2 estimates.*** Recall that in [Wang and Xiao 2022] (see Lemma 24) we proved the Pogorelov-type local C^2 estimate for strictly convex, spacelike, constant σ_k curvature hypersurfaces. With small modifications, we can show the following.

Lemma 16. *Let u^{r*} be the solution of (3-5) and u^r be the Legendre transform of u^{r*}. For any given $s > 2C_0 + 1$, where $C_0 > \min\bar{u}$ is an arbitrary constant, let $r_s > 0$ be a positive number such that, when $r > r_s$, we have $u^r|_{\partial\Omega_r} > s$, where $\Omega_r = Du^{r*}(B_r)$. Let $\kappa_{\max}(x)$ be the largest principal curvature of $\mathcal{M}_{u^r}$ at x, where $\mathcal{M}_{u^r} = \{(x, u^r(x)) \mid x \in \Omega_r\}$. Then, for $r > r_s$, we have*

$$\max_{\mathcal{M}_{u^r}}(s-u^r)\kappa_{\max} \leqslant C. \tag{4-8}$$

Here, C depends on the local C^1 estimates of u^r and s.

In the rest of this subsection, we will establish the Pogorelov-type local C^2 estimates for the k-convex solution of (1-2), where $k = n-1$ $(n \geqslant 3)$, $n-2$ $(n \geqslant 5)$.

Lemma 17. *Let u^n be the k-convex solution of (3-23) with $\psi = \psi(X, \nu)$, where $k = n-1$ ($n \geqslant 3$), $n-2$ ($n \geqslant 5$). For any given $s > 1$, let $m > s$. Then $u^m|_{\partial\Omega_m} = m > s$. Let $\kappa_{\max}(x)$ be the largest principal curvature of $\mathcal{M}_{u^m}$ at x, where $\mathcal{M}_{u^m} = \{(x, u^m(x)) \mid x \in \Omega_m\}$. Then, for $m > s$, we have*

$$\max_{\mathcal{M}_{u^m}} (s - u^m)\kappa_{\max} \leqslant C.$$

Here, C depends on the local C^1 estimates of u^m and s.

Proof. In this proof, for our convenience when there is no confusion, we will drop the superscript on u^m. Now, on Ω_m, we consider the following test function whose form first appeared in [Guan et al. 2015]:

$$\phi = \beta \log(s - u) + \log\log P - N\langle \nu, E\rangle.$$

Here the function P is defined by

$$P = \sum_l e^{\kappa_l},$$

and β and N are constants to be determined later.

Letting $U_s = \{x \in \mathbb{R}^n \mid u(x) < s\}$, we may assume that the maximum of ϕ is achieved at $P_0 \in U_s$. Choose a local normal coordinate $\{\tau_1, \tau_2, \dots, \tau_n\}$ such that $h_{ij} = \kappa_i \delta_{ij}$ and $\kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant \kappa_n$ at P_0.

Differentiating the function ϕ twice at P_0, we get

$$\phi_i = -\frac{\beta u_i}{s - u} + \frac{P_i}{P \log P} + N h_{ii} u_i = 0 \tag{4-9}$$

and

$$\begin{aligned} 0 \geqslant \phi_{ii} &= \frac{P_{ii}}{P \log P} - \frac{P_i^2}{P^2 \log P} - \frac{P_i^2}{(P \log P)^2} + \frac{\beta h_{ii}\langle \nu, E\rangle}{s - u} - \frac{\beta u_i^2}{(s - u)^2} - N h_{ii}^2 \langle \nu, E\rangle + \sum_s N u_s h_{isi} \\ &= \frac{1}{P \log P}\left[\sum_l e^{\kappa_l} h_{llii} + \sum_l e^{\kappa_l} h_{lli}^2 + \sum_{p \neq q} \frac{e^{\kappa_p} - e^{\kappa_q}}{\kappa_p - \kappa_q} h_{pqi}^2 - \left(\frac{1}{P} + \frac{1}{P \log P}\right) P_i^2\right] \\ &\qquad + \frac{\beta h_{ii}\langle \nu, E\rangle}{s - u} - \frac{\beta u_i^2}{(s - u)^2} - N h_{ii}^2 \langle \nu, E\rangle + \sum_s N u_s h_{iis}. \end{aligned}$$

Contracting with σ_k^{ii}, we have

$$\begin{aligned} \sigma_k^{ii}\phi_{ii} = \frac{\sigma_k^{ii}}{P \log P}&\left[\sum_l e^{\kappa_l} h_{llii} + \sum_l e^{\kappa_l} h_{lli}^2 + \sum_{p \neq q} \frac{e^{\kappa_p} - e^{\kappa_q}}{\kappa_p - \kappa_q} h_{pqi}^2 - \left(\frac{1}{P} + \frac{1}{P \log P}\right) P_i^2\right] \\ &+ \frac{\beta \sigma_k^{ii} \kappa_i \langle \nu, E\rangle}{s - u} - \frac{\beta \sigma_k^{ii} u_i^2}{(s - u)^2} - N \sigma_k^{ii} \kappa_i^2 \langle \nu, E\rangle + \sum_s N u_s \sigma_k^{ii} h_{iis}. \end{aligned} \tag{4-10}$$

At P_0, differentiating (1-2) twice yields,

$$\sigma_k^{ii} h_{iil} = d_X \psi(\tau_l) + \kappa_l d_\nu \psi(\tau_l) \tag{4-11}$$

and

$$\sigma_k^{ii} h_{iill} + \sigma_k^{pq,rs} h_{pql} h_{rsl} \geqslant -C - C h_{11}^2 + \sum_s h_{sll} d_\nu \psi(\tau_s), \tag{4-12}$$

where C is some uniform constant. Note that

$$h_{llii} = h_{iill} - h_{ii}h_{ll}^2 + h_{ii}^2 h_{ll}. \tag{4-13}$$

Inserting (4-12) and (4-13) into (4-10), we obtain

$$\begin{aligned}\sigma_k^{ii}\phi_{ii} \geqslant \frac{1}{P\log P}\biggl[\sum_l e^{\kappa_l}\Bigl(-C - C\kappa_1^2 - \sigma_k^{pq,rs}h_{pql}h_{rsl} + \sum_s h_{sll}d_\nu\psi(\partial_s)\Bigr)\\ + \sum_l \sigma_k^{ii}e^{\kappa_l}h_{lli}^2 + \sigma_k^{ii}\sum_{p\neq q}\frac{e^{\kappa_p}-e^{\kappa_q}}{\kappa_p-\kappa_q}h_{pqi}^2 - \Bigl(\frac{1}{P}+\frac{1}{P\log P}\Bigr)\sigma_k^{ii}P_i^2\biggr]\\ + \frac{\beta k\sigma_k\langle\nu, E\rangle}{s-u} - \frac{\beta\sigma_k^{ii}u_i^2}{(s-u)^2} - N\sigma_k^{ii}\kappa_i^2\langle\nu, E\rangle + \sum_s Nu_s\sigma_k^{ii}h_{sii} - \sigma_k^{ii}\kappa_i^2.\end{aligned} \tag{4-14}$$

From (4-9) and (4-11), we deduce

$$\frac{1}{P\log P}\sum_j\sum_l e^{\kappa_l}h_{jll}d_\nu\psi(\tau_j) + \sum_j Nu_j\sigma_k^{ii}h_{sii} \geqslant \sum_l d_\nu\psi(\tau_l)\frac{\beta u_l}{s-u} - C.$$

For any constant $K>1$, write

$$A_i = e^{\kappa_i}\biggl[K(\sigma_k)_i^2 - \sum_{p\neq q}\sigma_k^{pp,qq}h_{ppi}h_{qqi}\biggr], \quad B_i = 2\sum_{l\neq i}\sigma_k^{ii,ll}e^{\kappa_l}h_{lli}^2,$$

$$C_i = \sigma_k^{ii}\sum_l e^{\kappa_l}h_{lli}^2, \quad D_i = 2\sum_{l\neq i}\sigma_k^{ll}\frac{e^{\kappa_l}-e^{\kappa_i}}{\kappa_l-\kappa_i}h_{lli}^2, \quad E_i = \frac{1+\log P}{P\log P}\sigma_k^{ii}P_i^2.$$

Note that

$$-\sum_l \sigma_k^{pq,rs}h_{pql}h_{rsl} = \sum_{p\neq q}\sigma_k^{pp,qq}h_{pql}^2 - \sum_{p\neq q}\sigma_k^{pp,qq}h_{ppl}h_{qql}.$$

Therefore, (4-14) becomes

$$\begin{aligned}\sigma_k^{ii}\phi_{ii} \geqslant \frac{1}{P\log P}\sum_i(A_i+B_i+C_i+D_i-E_i) + \frac{\beta k\sigma_k\langle\nu, E\rangle}{s-u} - \frac{\beta\sigma_k^{ii}u_i^2}{(s-u)^2}\\ + (-N\langle\nu, E\rangle - 1)\sigma_k^{ii}\kappa_i^2 + \sum_l d_\nu\psi(\tau_l)\frac{\beta u_l}{s-u} - C\kappa_1.\end{aligned} \tag{4-15}$$

Following the same argument as that in the proof of Lemma 13, from (4-15) we obtain

$$\begin{aligned}0 \geqslant \sum_{|\kappa_i|<\delta\kappa_1}\frac{\alpha}{(P\log P)^2}\sigma_k^{ii}P_i^2 + \frac{\beta k\sigma_k\langle\nu, E\rangle}{s-u} - \frac{\beta\sigma_k^{ii}u_i^2}{(s-u)^2}\\ + (-N\langle\nu, E\rangle - 1)\sigma_k^{ii}\kappa_i^2 + \sum_l d_\nu\psi(\tau_l)\frac{\beta u_l}{s-u} - C\kappa_1.\end{aligned} \tag{4-16}$$

Here, the constant δ is the same constant as the one chosen in Claim 1 of Lemma 13. Moreover, by (4-9),

$$-\frac{\beta\sigma_k^{ii}u_i^2}{(s-u)^2} \geqslant -\frac{\sigma_k^{ii}}{\beta}\biggl[2\Bigl(\frac{P_i}{P\log P}\Bigr)^2 + 2N^2u_i^2\kappa_i^2\biggr].$$

Choosing $\beta > 0$ such that $\alpha\beta > 2$, (4-16) implies

$$0 \geqslant \frac{\beta k \sigma_k \langle \nu, E\rangle}{s-u} - \sum_{|\kappa_i| \geqslant \delta\kappa_1} \frac{\beta \sigma_k^{ii} u_i^2}{(s-u)^2} + (-N\langle \nu, E\rangle - 1)\sigma_k^{ii}\kappa_i^2 + \sum_l d_\nu \psi(\tau_l) \frac{\beta u_l}{s-u} - C\kappa_1 - \sum_{|\kappa_i| < \delta\kappa_1} \frac{\sigma_k^{ii}}{\beta} 2N^2 u_i^2 \kappa_i^2. \tag{4-17}$$

Now, first choose $N > 0$ such that

$$\frac{1}{2} \sum_{|\kappa_i| \geqslant \delta\kappa_1} \sigma_k^{ii} \kappa_i^2 (-N\langle \nu, E\rangle - 1) - C\kappa_1 \geqslant 0.$$

Then choose $\beta = \beta(N)$ sufficiently large such that

$$\sum_{|\kappa_i| < \delta\kappa_1} \left(\sigma_k^{ii} \kappa_i^2 (-N\langle \nu, E\rangle - 1) - \frac{\sigma_k^{ii}}{\beta} 2N^2 u_i^2 \kappa_i^2 \right) \geqslant 0.$$

We deduce

$$\frac{\beta C}{s-u} + \sum_{|\kappa_i| \geqslant \delta\kappa_1} \frac{2\beta \sigma_k^{ii} u_i^2}{(s-u)^2} \geqslant \sum_{|\kappa_i| \geqslant \delta\kappa_1} \sigma_k^{ii} \kappa_i^2 (-N\langle \nu, E\rangle - 1). \tag{4-18}$$

If

$$\frac{C}{s-u} \geqslant \sum_{|\kappa_i| \geqslant \delta\kappa_1} \frac{2\beta \sigma_k^{ii} u_i^2}{(s-u)^2},$$

we get

$$\frac{2C\beta}{s-u} \geqslant \sigma_k^{11} \kappa_1^2 (-N\langle \nu, E\rangle - 1) \geqslant c_0 (N-1)\kappa_1,$$

which implies the desired estimate. If

$$\frac{C}{s-u} \leqslant \sum_{|\kappa_i| \geqslant \delta\kappa_1} \frac{2\beta \sigma_k^{ii} u_i^2}{(s-u)^2},$$

we let i_0 denote the index of the maximum value element of the set

$$\left\{ \frac{2\beta \sigma_k^{ii} u_i^2}{(s-u)^2} \,\middle|\, |\kappa_i| \geqslant \delta\kappa_1 \right\}.$$

Then, we obtain the following, which implies our desired estimate:

$$4n \frac{\beta \sigma_k^{i_0 i_0} u_{i_0}^2}{(s-u)^2} \geqslant \sigma_k^{i_0 i_0} \kappa_{i_0}^2 (-N\langle \nu, E\rangle - 1) \geqslant C(N-1)\sigma_k^{i_0 i_0} \delta^2 \kappa_1^2. \qquad \square$$

5. The prescribed curvature problem

We will prove Theorem 1 and 5 in this section.

Let's consider the proof of Theorem 1 first. Recall that in Section 3.1, we have solved the approximate Dirichlet problem (3-5) on B_r for $r < 1$. We will denote the strictly convex solution of (3-5) by u^{r*}. We further denote the Legendre transform of (B_r, u^{r*}) by (Ω_r, u^r), where $\Omega_r = Du^{r*}(B_r)$ is the domain of u^r. By Lemmas 19 and 20 in [Wang and Xiao 2022], we have

$$\underline{u} \leqslant u^r \leqslant \bar{u} \quad \text{in } \Omega_r. \tag{5-1}$$

In the following, we will write $\widetilde{\Omega}_r = D\underline{u}^*(B_r)$ for the domain of $\underline{u}_r := \underline{u}|_{\widetilde{\Omega}_r}$. It is not difficult to see that these domains are increasing, namely,

$$\widetilde{\Omega}_r \subset \widetilde{\Omega}_s \quad \text{for } r < s.$$

Moreover, by the choice of $\underline{u}$ in Section 3.1, we have

$$\underline{u}|_{\partial\widetilde{\Omega}_r} \to +\infty \quad \text{as } r \to 1.$$

Thus, by the comparison principle, we have

$$u_r|_{\partial\Omega_r} = [\xi \cdot Du_r^*(\xi) - u_r^*(\xi)]|_{\partial B_r} \geqslant [\xi \cdot D\underline{u}^*(\xi) - \underline{u}^*(\xi)]|_{\partial B_r} = \underline{u}|_{\partial\widetilde{\Omega}_r}. \tag{5-2}$$

From this we can see that, as $r \to 1$, $u_r|_{\partial\Omega_r} \to +\infty$. This in turn implies, for any compact set $\mathcal{K} \subset \mathbb{R}^n$, there exists a constant $c_{\mathcal{K}} = c(\mathcal{K}) < 1$ such that, when $r > c_{\mathcal{K}}$, $\Omega_r \supset \mathcal{K}$. Therefore, for any compact set $\mathcal{K} \subset \mathbb{R}^n$, we can apply Lemmas 14 and 16 to obtain uniform C^1 and C^2 bounds for u^r in $\mathcal{K}$.

More precisely, in order to obtain the local C^1 estimate, we introduce a new subsolution $\underline{u}_1$ of (1-2), where $\underline{u}_1$ satisfies

$$\sigma_k(\kappa_1, \dots, \kappa_n) = c_1 + 100$$

and, as $|x| \to \infty$,

$$\underline{u}_1 \to |x| + \varphi\left(\frac{x}{|x|}\right).$$

By the strong maximum principle, we have, when $x \in \mathbb{R}^n$,

$$\underline{u}_1(x) < \underline{u}(x).$$

Thus, for any compact convex domain $\mathcal{K}$, let

$$2\delta = \min_{\mathcal{K}}(\underline{u} - \underline{u}_1).$$

We define a strict spacelike function $\Psi = \underline{u}_1 + \delta$. Set $\mathcal{K}' = \{x \in \mathbb{R}^n \mid \Psi \leqslant \bar{u}\}$. Since, as $|x| \to \infty$, we have $\underline{u}_1 - \bar{u} \to 0$, we know that $\mathcal{K}'$ is a compact set only depending on $\mathcal{K}$. Applying Lemma 14, for any (Ω_r, u^r), if $\mathcal{K}' \subset \Omega_r$, we have the gradient estimate

$$\sup_{\mathcal{K}} \frac{1}{\sqrt{1 - |Du^r|^2}} \leqslant \frac{1}{\delta} \sup_{\mathcal{K}'} \frac{\bar{u} - \Psi}{\sqrt{1 - |D\Psi|^2}}.$$

Next, we want to show that, for any given compact set $\mathcal{K} \subset \mathbb{R}^n$, the set $\{|D^2u^r|\}$ is uniformly bounded in $\mathcal{K}$. Without loss of generality, let's consider any $B_R \subset \mathbb{R}^n$. Let $C_0 = \max_{B_R} \bar{u}$ and $s = 2C_0 + 1$ in Lemma 16. Set $U_s = \{x \in \mathbb{R}^n \mid \underline{u}(x) < s\}$. Then by our earlier discussion, it's easy to see that there exists $r_s > 0$ such that, when $r > r_s$, we have $\Omega_r \supset U_s$. Applying Lemma 16, we obtain, when $r > r_s$,

$$\sup_{B_R} \kappa_{\max}(M_{u^r}) \leqslant C.$$

Here C depends on the upper bound of $1/\sqrt{1 - |Du^r|^2}$ on $\overline{U}_s$, which is independent of r. Using the classical regularity theorem and convergence theorem, we conclude that (Ω_r, u^r) converges locally smoothly to an entire, smooth convex function u satisfying (1-2). In view of (5-1) and the asymptotic behavior of $\underline{u}$ and $\bar{u}$, we know that, as $|x| \to \infty$, we have $u \to |x| + \varphi(x/|x|)$. Moreover, by Remark 2, we also know that u is strictly convex. Therefore, its Gauss map image is B_1, i.e., $Du(\mathbb{R}^n) = B_1$.

Theorem 5 follows by replacing Lemmas 14 and 16 in the proof of Theorem 1 with Lemmas 15 and 17.

6. The radial downward translating soliton

We will now study the radially symmetric downward translating soliton. Recall that we say $\mathcal{M}_u$ is a downward translating soliton when its principal curvatures satisfy

$$\sigma_k(\kappa[\mathcal{M}_u]) = \binom{n}{k}\left(\mathcal{C} - \frac{1}{\sqrt{1-|Du|^2}}\right)^k, \tag{6-1}$$

where $\mathcal{C} > 1$ is a constant. We want to point out that in this section and the next, $\mathcal{C}$ is the fixed constant in (6-1). We also write

$$\tilde{\mathcal{C}} = \sqrt{1 - \frac{1}{\mathcal{C}^2}}$$

as in Theorem 7. The following theorem is a generalization of Theorem 1 in [Bayard 2023].

Theorem 18. *Let $\mathcal{C} > 1$ be a positive constant. Then there exists a strictly convex radial solution $u : \mathbb{R}^n \to \mathbb{R}$ of (6-1) satisfying*

$$|Du| \to \tilde{\mathcal{C}} \quad \text{as } |x| \to +\infty.$$

Moreover, $u(x)$ has the following asymptotic expansion as $|x| \to \infty$:

$$u(x) = \tilde{\mathcal{C}}|x| - \frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}\log|x| + c_0 + o(1) \tag{6-2}$$

for some constant $c_0 \in \mathbb{R}$. In particular, the radial solution u is unique up to the addition of a constant.

For radial solutions, we will reduce (6-1) to an ODE. Let $u = u(r)$ and $y = \partial u/\partial r$. Then a straightforward calculation yields

$$D_i u = y\frac{x_i}{|x|} \quad \text{and} \quad D^2_{ij}u = \frac{y}{|x|}\left(\delta_{ij} - \frac{x_i x_j}{|x|^2}\right) + y'\frac{x_i x_j}{|x|^2}.$$

Therefore,

$$\kappa[\mathcal{M}_u] = \frac{1}{\sqrt{1-y^2}}\left(\frac{y'}{1-y^2}, \frac{y}{r}, \dots, \frac{y}{r}\right),$$

and (6-1) becomes

$$\frac{1}{(1-y^2)^{k/2}}\frac{y^{k-1}}{r^{k-1}}\left(\frac{k}{n}\frac{y'}{1-y^2} + \frac{n-k}{n}\frac{y}{r}\right) = \left(\mathcal{C} - \frac{1}{\sqrt{1-y^2}}\right)^k. \tag{6-3}$$

By a small modification of the proof of Proposition 2.1 in [Bayard 2023], we obtain the following.

Proposition 19. *Under the hypotheses of Theorem 18, there exists a solution y of (6-3), which is defined on $[0, +\infty)$ and smooth on $(0, +\infty)$, such that*

$$y(0) = 0, \quad 0 \leqslant y < \tilde{\mathcal{C}}, \quad \lim_{r\to+\infty} y(r) = \tilde{\mathcal{C}}, \quad y'(0) = \mathcal{C} - 1, \quad \text{and} \quad y' > 0 \quad \text{on } [0, +\infty).$$

Moreover, as $r \to 0+$, we have

$$\kappa[\mathcal{M}_u(r)] \to (\mathcal{C} - 1)(1, 1, \dots, 1).$$

Since the proof is a small modification of the proof of Proposition 2.1 in [Bayard 2023], we skip it here. Now, let's study the asymptotic behavior of y.

Proposition 20. *Let y be the solution of (6-3). Then y has the following asymptotic expansion as $r \to \infty$:*

$$y(r) = \tilde{\mathcal{C}} - \frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}\frac{1}{r} + O\left(\frac{1}{r^2}\right).$$

Proof. By Proposition 19, we may assume

$$y(r) = \tilde{\mathcal{C}} - \frac{z}{r}. \tag{6-4}$$

Then we have

$$\sqrt{1-y^2} - \frac{1}{\mathcal{C}} = \frac{1-1/\mathcal{C}^2 - y^2}{\sqrt{1-y^2}+1/\mathcal{C}} = \frac{z}{r}A(r), \quad \text{where } A(r) = \frac{\sqrt{1-1/\mathcal{C}^2}+y}{\sqrt{1-y^2}+1/\mathcal{C}}. \tag{6-5}$$

Differentiating (6-4) then substituting it into (6-3), we get

$$\frac{k}{n}\frac{y^{k-1}}{1-y^2}\left(-\frac{z'}{r^k} + \frac{z}{r^{k+1}}\right) + \frac{n-k}{n}\frac{y^k}{r^k} = \mathcal{C}^k\left(\sqrt{1-y^2} - \frac{1}{\mathcal{C}}\right)^k. \tag{6-6}$$

By (6-5), (6-6) can be simplified as

$$\frac{k}{n}\frac{y^{k-1}}{1-y^2}\left(-z' + \frac{z}{r}\right) + \frac{n-k}{n}y^k = \mathcal{C}^k z^k A^k(r).$$

Thus, we obtain

$$z' = -B(r)z^k + C(r), \tag{6-7}$$

where

$$B(r) = \mathcal{C}^k\frac{n}{k}\frac{1-y^2}{y^{k-1}}A^k(r) \quad \text{and} \quad C(r) = \frac{z}{r} + \frac{n-k}{k}y(1-y^2). \tag{6-8}$$

Applying Proposition 19, we can see that

$$\lim_{r\to+\infty} B(r) = \frac{n}{k}\mathcal{C}^{2k-2}\tilde{\mathcal{C}} \quad \text{and} \quad \lim_{r\to+\infty} C(r) = \frac{n-k}{k}\frac{1}{\mathcal{C}^2}\tilde{\mathcal{C}}.$$

Here, we have used $\lim_{r\to\infty}(z/r) = 0$, which is a direct consequence of Proposition 19. The next lemma is a generalization of Proposition A.2 in [Bayard 2023].

Lemma 21. *Assume $z : (0, +\infty) \to \mathbb{R}$ is a positive solution of the equation*

$$z' = -A(r)z^k + B(r),$$

where $A, B : (0, \infty) \to \mathbb{R}$ are continuous functions such that

$$\lim_{r\to+\infty} A(r) = A_0 > 0 \quad \textit{and} \quad \lim_{r\to+\infty} B(r) = B_0 > 0.$$

Then

$$\lim_{r\to+\infty} z(r) = \sqrt[k]{\frac{B_0}{A_0}}.$$

Proof. In order to prove this lemma, we only need to prove the following claim.

Claim 3. *Assume $z : (0, +\infty) \to \mathbb{R}$ is a positive solution of the equation*

$$z' = A_0 z^k + B_0,$$

with $A_0 < 0$ and $B_0 > 0$ constants. Then

$$\lim_{r\to\infty} z(r) = \left(-\frac{B_0}{A_0}\right)^{1/k}.$$

If this claim is true, following the same argument as Proposition A.2 in [Bayard 2023], we can prove Lemma 21. We will prove this claim below.

Without loss of generality, let's consider the positive solution of the equation

$$z' = B - z^k \tag{6-9}$$

instead. We will show that

$$\lim_{r\to\infty} z(r) = B^{1/k}. \tag{6-10}$$

First, since z is a positive solution of (6-9), let's assume $0 < z(r_0) = z_0 < B^{1/k}$. Then we have $z_0 < z(r) < B^{1/k}$ on (r_0, ∞). Writing $z_1 = B^{1/k}$, we get

$$z^k - B = (z - z_1)(z^{k-1} + z^{k-2} z_1 + \cdots + z_1^{k-1}).$$

Therefore, (6-9) can be written as

$$-dr = \left[\frac{A_1}{z - z_1} + \frac{Q_{k-2}(z)}{z^{k-1} + z^{k-2} z_1 + \cdots + z_1^{k-1}}\right] dz, \tag{6-11}$$

where $A_1 = z_1^{1-k}/k$ and $Q_{k-2}(z)$ is a polynomial of degree $k-2$. It's easy to see that

$$Q_{k-2}(z) = -A_1 z^{k-2} + Q(k-3)(z)$$

and $Q_{k-3}(z)$ is a polynomial of degree $k-3$. Integrating (6-11) from r_0 to r yields

$$\begin{aligned} -r + r_0 = A_1 \ln\left|\frac{z(r) - z_1}{z_0 - z_1}\right| - \int_{z_0}^{z(r)} \frac{A_1 z^{k-2}}{z^{k-1} + z^{k-2} z_1 + \cdots + z_1^{k-1}}\, dz \\ + \int_{z_0}^{z(r)} \frac{Q_{k-3}(z)}{z^{k-1} + z^{k-2} z_1 + \cdots + z_1^{k-1}}\, dz. \end{aligned} \tag{6-12}$$

Notice that, as $r \to \infty$, the left-hand side of (6-12) goes to $-\infty$, while

$$-\int_{z_0}^{z(r)} \frac{A_1 z^{k-2}}{z^{k-1} + z^{k-2} z_1 + \cdots + z_1^{k-1}}\, dz \geqslant -A_1 \ln\left|\frac{z_1}{z_0}\right|$$

and

$$\left|\int_{z_0}^{z(r)} \frac{Q_{k-3}(z)}{z^{k-1} + z^{k-2} z_1 + \cdots + z_1^{k-1}}\, dz\right|$$

is bounded. Therefore, $\lim_{r\to\infty} z(r) = z_1 = B^{1/k}$. We similarly prove the case when $z(r_0) = z_0 > z_1$. $\square$

From Lemma 21 and (6-7), we conclude

$$\lim_{r\to+\infty} z(r) = \frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}.$$

We further assume

$$z(r) = \frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}} + \frac{w(r)}{r}.$$

Inserting it into (6-7), we get

$$w' = -D(r)w + F(r),$$

where

$$D(r) = B(r)\sum_{i=1}^{k}\binom{k}{i}\left(\frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}\right)^{k-i}\left(\frac{w}{r}\right)^{i-1}$$

and

$$F(r) = r\left(C(r) - \frac{B(r)}{\mathcal{C}^{2k}}\frac{n-k}{n}\right) + \frac{w}{r}.$$

Notice that $\lim_{r\to+\infty}(w/r) = 0$ and $D(r)$ has a uniform positive lower bound. In the following, we want to find a positive upper bound for $F(r)$. Using the expressions (6-8) for $B(r)$ and $C(r)$, we obtain

$$\begin{aligned} F(r) &= \frac{w}{r} + z + \frac{n-k}{k}\frac{1-y^2}{y^{k-1}} r\left[y^k - \left(\frac{A(r)}{\mathcal{C}}\right)^k\right] \\ &= \frac{w}{r} + z + \frac{n-k}{k}\frac{1-y^2}{y^{k-1}} r\left(y - \frac{A(r)}{\mathcal{C}}\right)\sum_{i=1}^{k} y^{k-i}\left(\frac{A(r)}{\mathcal{C}}\right)^{i-1}. \end{aligned} \tag{6-13}$$

Therefore, we only need to show $r(y - A(r)/\mathcal{C})$ is bounded as $r\to\infty$. By (6-5), we have

$$\begin{aligned} r\left(y - \frac{A(r)}{\mathcal{C}}\right) &= r\left(y - \frac{1}{\mathcal{C}}\frac{\sqrt{1-1/\mathcal{C}^2}+y}{\sqrt{1-y^2}+1/\mathcal{C}}\right) \\ &= \frac{r(y\sqrt{1-y^2} - (1/\mathcal{C})\sqrt{1-1/\mathcal{C}^2})}{\sqrt{1-y^2}+1/\mathcal{C}}. \end{aligned} \tag{6-14}$$

Combining (6-14) with the expression for y and (6-5), we can derive

$$\begin{aligned} y\sqrt{1-y^2} - \frac{1}{\mathcal{C}}\sqrt{1-\frac{1}{\mathcal{C}^2}} &= \left(\sqrt{1-\frac{1}{\mathcal{C}^2}} - \frac{z}{r}\right)\left(\frac{1}{\mathcal{C}} + \frac{zA(r)}{r}\right) - \frac{1}{\mathcal{C}}\sqrt{1-\frac{1}{\mathcal{C}^2}} \\ &= \frac{z}{r}\left(-\frac{1}{\mathcal{C}} + A(r)\sqrt{1-\frac{1}{\mathcal{C}^2}}\right) - \frac{z^2A(r)}{r^2}. \end{aligned} \tag{6-15}$$

From (6-14), (6-15), and Lemma 21, we conclude that $r(y - A(r)/\mathcal{C})$ is uniformly bounded from above. Thus, $F(r)$ has an uniform upper bound. Applying Proposition A.3 in [Bayard 2023], we obtain a uniform upper bound for w. $\square$

It's not hard to see that Theorem 18 follows from Propositions 19 and 20.

7. The existence results

In this section we will prove Theorem 7. First, we want to prove the following existence theorem.

Proposition 22. *Suppose φ is a C^2 function defined on $\mathbb{S}^{n-1}_{\tilde{C}} := \{x \in \mathbb{R}^n \mid |x| = \tilde{C}\}$, where $\tilde{C} = \sqrt{1-(1/C)^2}$. There exists a unique, strictly convex solution $u : \mathbb{R}^n \to \mathbb{R}$ of (1-10) such that, as $|x| \to \infty$,*

$$u(x) \to \tilde{C}|x| - \frac{1}{C^2}\sqrt[k]{\frac{n-k}{n}}\log|x| + \varphi\left(\tilde{C}\frac{x}{|x|}\right). \tag{7-1}$$

7.1. ***Constructing barriers.*** We first construct the barrier functions of (1-10). Following the ideas of [Spruck and Xiao 2016; Treibergs 1982], we denote the radial solution of (1-10) by $z_0^k(|x|)$, whose asymptotic expansion satisfies (6-2) with $c_0 = 0$. Let

$$p_i(\tilde{C}y) = D\varphi(\tilde{C}y) + (-1)^{i+1}2M\tilde{C}y, \quad i = 1, 2,$$

for any $y \in \mathbb{S}^{n-1}$. Set

$$z_i^k(x, y) = \varphi(\tilde{C}y) - p_i(\tilde{C}y)\cdot\tilde{C}y + z_0^k(|x + p_i(\tilde{C}y)|) \quad \text{for all } x \in \mathbb{R}^n, \quad y \in \mathbb{S}^{n-1}.$$

Then

$$q_1^k(x) = \sup_{y\in\mathbb{S}^{n-1}} z_1^k(x, y)$$

is a subsolution of (1-10) and

$$q_2^k = \inf_{y\in\mathbb{S}^{n-1}} z_2^k(x, y)$$

is a supersolution of (1-10). Moreover, $q_1^k(x) \leqslant q_2^k(x)$, and, when $|x| \to +\infty$, we have

$$q_i^k(x) \to \tilde{C}|x| - \frac{1}{C^2}\sqrt[k]{\frac{n-k}{n}}\log|x| + \varphi\left(\tilde{C}\frac{x}{|x|}\right), \quad i = 1, 2.$$

7.2. ***The Dirichlet problem.*** First, let's solve (1-10) for the case $k = n$. For any $t > \min_{\mathbb{R}^n} q_2^n$, we let

$$\partial\Omega_t = \{x \in \mathbb{R}^n \mid q_1^n(x) < t < q_2^n(x)\}$$

and Ω_t be a smooth, strictly convex domain in $\mathbb{R}^n$. Consider the Dirichlet problem

$$\begin{cases} \sigma_n^{1/n}(\kappa(\mathcal{M}_{u_t})) = C + \langle \nu, E\rangle & \text{in } \Omega_t, \\ u_t = t & \text{on } \partial\Omega_t. \end{cases} \tag{7-2}$$

By a small modification of [Delanoë 1990], we know that there exists a unique solution u_t of (7-2). Then, applying the local C^1 and C^2 estimates obtained in [Bayard and Schnürer 2009], we conclude that there exists a subsequence $\{u_{t_i}\}_{i=1}^{\infty}$ ($t_i \to \infty$ as $i \to \infty$) that converges to an entire, strictly convex solution u of (1-10) for $k = n$. Moreover, it's easy to see that $u(x)$ satisfies the desired asymptotic behavior as $|x| \to \infty$. From now on, we will denote this solution by u^n. We will also denote the Legendre transform of u^n by u^{n*}.

Next, we consider the case when $k < n$. We denote the Legendre transform of z_0^k by $(z_0^k)^*$; that is,

$$(z_0^k)^*(\tau) = r\cdot\frac{\partial z_0^k}{\partial r} - z_0^k(r), \quad \text{where } \tau = \frac{\partial z_0^k}{\partial r}.$$

Using the asymptotic expansion of z_0 derived in Section 6, we know

$$(z_0^k)^*(\tau) = \frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}(\log r - 1) + O\Big(\frac{1}{r}\Big).$$

Writing its principal part as

$$(\tilde{z}_0^k)^*(\tau) = \frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}(\log r(\tau) - 1),$$

it is clear that $(\tilde{z}_0^k)^*$ is unbounded in $B_{\tilde{\mathcal{C}}}$.

To make sure our solution is convex, we consider the dual Dirichlet problem on B_τ for any $\tau < \tilde{\mathcal{C}}$:

$$\begin{cases} \hat{F}(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}) = \dfrac{\binom{n}{k}^{-1/k}}{\mathcal{C} - 1/\sqrt{1-|\xi|^2}} & \text{in } B_\tau, \\ u^* = u^{n*} + (z_0^k)^* - (z_0^n)^* & \text{on } \partial B_\tau. \end{cases} \tag{7-3}$$

Here, we have

$$w^* = \sqrt{1-|\xi|^2}, \quad \gamma^*_{ij} = \delta_{ij} - \frac{\xi_i\xi_j}{1+w^*}, \quad u^*_{kl} = \frac{\partial^2 u}{\partial\xi_k\partial\xi_l}, \quad \hat{F}(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}) = \bigg(\frac{\sigma_n}{\sigma_{n-k}}(\kappa^*[w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}])\bigg)^{1/k},$$

and $\kappa^*[w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}] = (\kappa^*_1, \dots, \kappa^*_n)$ is the set of eigenvalues of the matrix $(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj})$. The solvability of (7-3) has been established in Section 3. Therefore, by standard PDE theorems, in order to prove Proposition 22, we only need to obtain local C^1 and C^2 estimates for the translating soliton equation (1-10). In order to do so, we will need the following lemma.

Lemma 23. *Let $u^{\tau*}$ be a solution to (7-3) and u^τ be the Legendre transform of $u^{\tau*}$. Then, for any $x \in Du^{\tau*}(B_\tau)$, we have $q_1^k(x) \leqslant u^\tau(x) \leqslant q_2^k(x)$.*

Proof. Without causing confusion we shall drop the superscript τ in the proof. We only need to prove that

$$z_1^k(x, y) \leqslant u(x) \leqslant z_2^k(x, y)$$

for any $x \in Du^{\tau*}(B_\tau)$ and $y \in \mathbb{S}^{n-1}$. This is equivalent to proving

$$(z_2^k)^*(\xi, y) \leqslant u^*(\xi) \leqslant (z_1^k)^*(\xi, y)$$

for any $\xi \in B_\tau$ and $y \in \mathbb{S}^{n-1}$. Since we have

$$\begin{aligned} (z_i^k)^*(\xi, y) &= (z_0^k)^*(|\xi|) - p_i(\tilde{\mathcal{C}}y)\cdot\xi - \varphi(\tilde{\mathcal{C}}y) + p_i(\tilde{\mathcal{C}}y)\cdot\tilde{\mathcal{C}}y \\ &= (z_0^k)^*(|\xi|) - (z_0^n)^*(|\xi|) + (z_i^n)^*(\xi, y) \end{aligned} \tag{7-4}$$

and

$$(z_2^n)^*(\xi, y) < u^{n*}(\xi) < (z_1^n)^*(\xi, y),$$

we obtain, on ∂B_τ,

$$(z_2^k)^*(\xi, y) \leqslant u^*(\xi) \leqslant (z_1^k)^*(\xi, y).$$

By the comparison principle, we finish the proof. ∎

7.3. *Local C^1 and C^2 estimates.* Similar to Lemma 14, we have the following local C^1 estimate lemma for translating solitons.

Lemma 24. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $u, \bar{u}, \Psi : \Omega \to \mathbb{R}^n$ be strictly $\mathcal{C}$-spacelike, i.e.,*

$$|Du|,\ |D\bar{u}|,\ |D\Psi| < \tilde{\mathcal{C}}.$$

Assume that u is strictly convex and $u \leqslant \bar{u}$ in Ω. Also assume that, near $\partial\Omega$, we have $\Psi > \bar{u}$. Consider the set with $u > \Psi$. For every x in that set, we have the following gradient estimate for u:

$$\frac{1}{\sqrt{\tilde{\mathcal{C}}^2 - |Du|^2}} \leqslant \frac{1}{u(x) - \Psi(x)} \cdot \sup_{\{u>\Psi\}} \frac{\bar{u} - \Psi}{\sqrt{\tilde{\mathcal{C}}^2 - |D\psi|^2}}.$$

Since the proof is the same as the proof of Lemma 5.1 in [Bayard and Schnürer 2009], we skip it here.

We now construct Ψ. Following the argument in Section 4 of [Bayard 2023], let

$$\Psi(x) = -A_0 + \tilde{\mathcal{C}}\sqrt{1 + |x|^2}.$$

It is clear that, when $|x|$ is sufficiently large, we have $\Psi(x) > q_2(x)$. On the other hand, for any compact set $\mathcal{K} \subset \mathbb{R}^n$, we can always choose A_0 large enough that $\Psi(x) < q_1(x)$ in $\mathcal{K}$. Applying Lemma 24 we obtain that, for any $\mathcal{K} \subset \mathbb{R}^n$ and any strictly convex function $q_1(x) < u(x) < q_2(x)$ satisfying (1-10), whose domain of definition contains $\mathcal{K}$, there exists a local C^1 bound $C_{\mathcal{K}}$ for $u(x)$ in $\mathcal{K}$ that only depends on $\mathcal{K}$.

Using the idea of [Wang and Xiao 2022], we can prove the following Pogorelov-type local C^2 estimate for translating solitons.

Lemma 25. *Let u be the solution of (1-10) defined on Ω. For any given $s > \min_{\mathbb{R}^n} u(x) + 1$, suppose $u|_{\partial\Omega} > s$. Let $\kappa_{\max}(x)$ be the largest principal curvature of $\mathcal{M}_u = \{(x, u(x)) \mid x \in \Omega\}$ at x. Then we have*

$$\max_{\mathcal{M}_u} (s - u)\kappa_{\max} \leqslant C_1.$$

Here, C_1 only depends on the local C^1 estimate of u. More specifically, C_1 depends on the lower bound of $\mathcal{C} + \langle \nu, E \rangle$.

Following the argument in Section 5, we complete the proof of Proposition 22.

7.4. *Proof of Theorem 7.* In this subsection, we will prove that the hypersurface $\mathcal{M}_u$ constructed in Proposition 22 has bounded principal curvatures. This completes the proof of Theorem 7. For our convenience, in the following, we will drop the superscript k, and the updated configuration z_0^k now becomes z_0.

Suppose u is a strictly convex solution of (1-10) and u^* is the Legendre transform of u. Then u^* satisfies

$$\hat{F}(w^* \gamma^*_{ik} u^*_{kl} \gamma^*_{lj}) = \frac{\binom{n}{k}^{-1/k}}{\mathcal{C} - 1/\sqrt{1 - |\xi|^2}} \quad \text{in } B_{\tilde{\mathcal{C}}}. \tag{7-5}$$

We also denote the Legendre transform of z_0 by z_0^*; that is,

$$z_0^*(\tau) = r \cdot \frac{\partial z_0}{\partial r} - z_0(r), \quad \text{where } \tau = \frac{\partial z_0}{\partial r}.$$

Using the asymptotic expansion of z_0 derived in Section 6, we know

$$z_0^*(\tau) = \frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}(\log r - 1) + O\left(\frac{1}{r}\right).$$

Writing its principal part as

$$\tilde{z}_0^*(\tau) = \frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}(\log r(\tau) - 1),$$

it is clear that $\tilde{z}_0^*(\tau)$ is unbounded in $B_{\tilde{\mathcal{C}}}$.

Lemma 26. *Let u^* and $\tilde{z}_0^*$ be defined as above. Then we have*

$$\lim_{\xi\to\xi_0}(u^*(\xi) - \tilde{z}_0^*(|\xi|)) = -\varphi(\xi_0) \quad \textit{for any } \xi_0 \in \partial B_{\tilde{\mathcal{C}}}, \quad \xi \in B_{\tilde{\mathcal{C}}}. \tag{7-6}$$

Proof. We use the auxiliary functions $z_i(x, y)$, $i = 1, 2$, constructed in Section 7.1. It's easy to see that

$$z_1(x, y) < u(x) < z_2(x, y) \quad \text{for any } x \in \mathbb{R}^n, \quad y \in \mathbb{S}^{n-1}.$$

By the strict convexity of $z_i(x, y)$, we have

$$z_2^*(\xi, y) < u^*(\xi) < z_1^*(\xi, y) \quad \text{for any } \xi \in B_{\tilde{\mathcal{C}}}, \quad y \in \mathbb{S}^{n-1}. \tag{7-7}$$

Notice that

$$z_i^*(\xi, y) = z_0^*(|\xi|) - p_i(\tilde{\mathcal{C}}y)\cdot\xi - \varphi(\tilde{\mathcal{C}}y) + p_i(\tilde{\mathcal{C}}y)\cdot\tilde{\mathcal{C}}y.$$

Therefore, letting $\tilde{\mathcal{C}}y = \xi_0$ and $\xi \to \xi_0$, we get

$$z_i(\xi, \tilde{\mathcal{C}}^{-1}\xi_0) - z_0^*(|\xi|) \to -\varphi(\xi_0).$$

This together with (7-7) yields (7-6). □

Now we let

$$\partial = \xi_i\frac{\partial}{\partial\xi_j} - \xi_j\frac{\partial}{\partial\xi_i}$$

be the angular derivative. Similar to Section 10 in [Ren et al. 2020], we obtain following lemmas.

Lemma 27. *Let u^* be the solution of (7-5). Then $|\partial u^*|$ is bounded above by a constant depending on $|\varphi|_{C^1}$, and $\partial^2 u^*$ is bounded above by a constant depending on $|\varphi|_{C^2}$.*

Proof. Noticing that $\partial|\xi|^2 = 0$, we have that the angular derivative of the right-hand side of (7-5) is zero. Therefore, following the proof of Lemmas 29 and 30 in [Ren et al. 2020], we have

$$F^{ij}w^*\gamma_{ik}^*(\partial(u^* - \tilde{z}_0^*))_{kl}\gamma_{lj}^* = 0 \quad \text{and} \quad F^{ij}w^*\gamma_{ik}^*(\partial^2(u^* - \tilde{z}_0^*))_{kl}\gamma_{lj}^* \geqslant 0.$$

In view of (7-6) and the maximum principle, we obtain the desired estimates. □

Lemma 28. *Let u^* be the solution of (7-5). There is a positive constant b such that*

$$\sqrt{\tilde{\mathcal{C}}^2 - |\xi|^2}\,|\partial^2 u^*| < b.$$

Proof. We consider $u^* - \tilde{z}_0^*$, which has C^0 bound on $B_{\tilde{\mathcal{C}}}$. Since $\partial^2 u^* = \partial^2(u^* - \tilde{z}_0^*)$, the rest of the proof is the same as that of Lemma 5.3 in [Li 1995]. □

Lemma 29. *Suppose $a_0 < r < \tilde{C}$ for some $a_0 \in (0, \tilde{C})$ and $\mathbb{S}^{n-1}(r) = \{\xi \in \mathbb{R}^n \mid \sum \xi_i^2 = r^2\}$. For any point $\hat{\xi} \in \mathbb{S}^{n-1}(r)$, there is a function*

$$\bar{u}_0^* = z_0^* + b_1\xi_1 + \cdots + b_n\xi_n + b$$

such that

$$\bar{u}_0^*(\hat{\xi}) = u^*(\hat{\xi})$$

and

$$\bar{u}_0^*(\hat{\xi}) > u^*(\xi) \quad \textit{for any } \xi \in \mathbb{S}^{n-1}(r) \setminus \{\hat{\xi}\}.$$

Here, $b_1, \ldots, b_n$ are constants depending on $\hat{\xi}$, and b is a positive constant independent of $\hat{\xi}$ and r.

Proof. The proof is almost the same as the proof of Lemma 5.4 in [Li 1995]. We only need to replace u, $\bar{u}$, and $-\bar{k}\sqrt{1-|x|^2}$ by $u^* - \tilde{z}_0^*$, $\bar{u}_0^* - \tilde{z}_0^*$, and $z_0^* - \tilde{z}_0^*$, respectively, in Li's proof. □

Similarly, we can prove the following lemma analogous to Lemma 5.5 in [Li 1995].

Lemma 30. *Suppose $a_0 < r < \tilde{C}$ for some $a_0 \in (0, \tilde{C})$ and $\mathbb{S}^{n-1}(r) = \{\xi \in \mathbb{R}^n \mid \sum \xi_i^2 = r^2\}$. For any point $\hat{\xi} \in \mathbb{S}^{n-1}(r)$, there is a function*

$$\underline{u}_0^* = z_0^* + a_1\xi_1 + \cdots + a_n\xi_n - a$$

such that

$$\underline{u}_0^*(\hat{\xi}) = u^*(\hat{\xi})$$

and

$$\underline{u}_0^*(\hat{\xi}) < u^*(\xi) \quad \textit{for any } \xi \in \mathbb{S}^{n-1}(r) \setminus \{\hat{\xi}\}.$$

Here, $a_1, \ldots, a_n$ and a are constants depending on $\hat{\xi}$, $a > 0$, and $a\sqrt{\tilde{C}^2 - |\hat{\xi}|^2} < C_1$, where C_1 is a positive constant only depending on $|\varphi|_{C^2}$.

Using Lemmas 29 and 30 we can show the following.

Lemma 31. *Let u be the solution of (1-10) and u^* be the Legendre transform of u. There are positive constants $d_2 > d_1$ such that*

$$0 < d_1 \leqslant u(\tilde{C}^2 - |Du|^2) \leqslant d_2. \tag{7-8}$$

Here, d_2 depends on $|u|_{C^0(\Omega)}$, and $\Omega = \{x \in \mathbb{R}^n \mid |Du| \leqslant a_0\}$.

Proof. We modify the proof of Li [1995]. We first consider the lower bound. For any $\hat{\xi} \in \mathbb{S}^{n-1}(r)$, using Lemma 29, we have

$$u^*(\hat{\xi}) = \bar{u}_0^*(\hat{\xi}) \qquad \text{and} \qquad u^*(\xi) < \bar{u}_0^*(\xi) \quad \text{for } \xi \in \mathbb{S}^{n-1}(r) \setminus \{\hat{\xi}\}.$$

Thus, using that $\bar{u}_0^*$ is a supersolution, we get $u^*(\xi) < \bar{u}_0^*(\xi)$ in B_r. Therefore, at $\hat{\xi}$, we get

$$u(\hat{x}) = \hat{\xi} \cdot Du^* - u^* > \hat{\xi} \cdot D\bar{u}_0^* - \bar{u}_0^* = z_0(\hat{r}) - b,$$

where we assume $\hat{x} = Du^*(\hat{\xi})$ and $z_0'(\hat{r}) := \partial z_0/\partial r(\hat{r}) = |\hat{\xi}|$. Thus, at $\hat{x}$, we have

$$u(\tilde{C}^2 - |Du|^2) > z_0(\hat{r})(\tilde{C}^2 - |z_0'(\hat{r})|^2) - b(\tilde{C}^2 - |\hat{\xi}|^2). \tag{7-9}$$

Using the asymptotic behavior of z_0, we have

$$z_0(\tilde{\mathcal{C}}^2-|z_0'|^2)=\left[\tilde{\mathcal{C}}r-\frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}\log r+O\left(\frac{1}{r}\right)\right]\left[\tilde{\mathcal{C}}^2-\left(\tilde{\mathcal{C}}-\frac{1}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}\frac{1}{r}+O\left(\frac{1}{r^2}\right)\right)^2\right]=2\frac{\tilde{\mathcal{C}}^2}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}+o(1)$$

We write

$$2c_0=2\frac{\tilde{\mathcal{C}}^2}{\mathcal{C}^2}\sqrt[k]{\frac{n-k}{n}}.$$

Therefore, by (7-9), we obtain

$$u(\tilde{\mathcal{C}}^2-|Du|^2)>\tfrac{1}{2}c_0$$

for r sufficiently close to $\tilde{\mathcal{C}}$. We further assume $r>a_0$, since for $r<a_0$, without loss of generality, we can assume $u\geqslant 1$. Therefore,

$$u(\tilde{\mathcal{C}}^2-|\hat{\xi}|^2)\geqslant\tilde{\mathcal{C}}^2-a_0^2.$$

Thus, we obtain the uniform lower bound. For the upper bound, we apply a similar argument. For r sufficiently close to $\tilde{\mathcal{C}}$ and still assuming $r\geqslant a_0$, we have

$$u(\tilde{\mathcal{C}}^2-|Du|^2)<z_0(\hat{r})(\tilde{\mathcal{C}}^2-|z_0'(\hat{r})|^2)+a(\tilde{\mathcal{C}}^2-|\hat{\xi}|^2)\leqslant 3c_0+C_1\tilde{\mathcal{C}}.$$

We have obtained a uniform upper bound. □

Finally, we are ready to adapt the ideas in [Li 1995; Ren et al. 2020] to estimate the principal curvatures of $\mathcal{M}_u$.

Proposition 32. *Let u be the solution of (1-10). Then the hypersurface $\mathcal{M}_u=\{(x,u(x))\mid x\in\mathbb{R}^n\}$ has bounded principal curvatures.*

Proof. We will establish a Pogorelov-type interior estimate. For any $s>0$, consider

$$\phi=e^{-s/(s-u)}[u(\mathcal{C}+\langle\nu,E\rangle)]^{-N}P_m^{1/m},$$

where $P_m=\sum_j\kappa_j^m$ and $m,N>0$ are constants to be determined later. Without loss of generality, we also assume $u\geqslant 1$ in $\mathbb{R}^n$. It's easy to see that ϕ achieves its local maximum at an interior point of $U_s=\{x\in\mathbb{R}^n\mid u(x)<s\}$; we will assume this point is x_0. We can choose a local normal coordinate $\{\tau_1,\dots,\tau_n\}$ such that, at x_0, we have $h_{ij}=\kappa_i\delta_{ij}$ and $\kappa_1\geqslant\kappa_2\geqslant\cdots\geqslant\kappa_n$.

Differentiating $\log\phi$ at x_0, we get

$$\frac{\phi_i}{\phi}=\frac{\sum_j\kappa_j^{m-1}h_{jji}}{P_m}-N\frac{h_{ii}\langle\tau_i,E\rangle}{\mathcal{C}+\langle\nu,E\rangle}-N\frac{u_i}{u}-\frac{su_i}{(s-u)^2}=0 \tag{7-10}$$

and

$$\begin{aligned}\frac{\phi_{ii}}{\phi}-\frac{\phi_i^2}{\phi^2}=\frac{1}{P_m}\Bigg[&\sum_j\kappa_j^{m-1}h_{jjii}+(m-1)\sum_j\kappa_j^{m-2}h_{jji}^2+\sum_{p\neq q}\frac{\kappa_p^{m-1}-\kappa_q^{m-1}}{\kappa_p-\kappa_q}h_{pqi}^2\Bigg]\\&-\frac{m}{P_m^2}\Bigg(\sum_j\kappa_j^{m-1}h_{jji}\Bigg)^2-N\sum_l h_{ili}\frac{\langle\tau_l,E\rangle}{\mathcal{C}+\langle\nu,E\rangle}+Nh_{ii}^2\frac{-\langle\nu,E\rangle}{\mathcal{C}+\langle\nu,E\rangle}\\&+Nh_{ii}^2\frac{u_i^2}{(\mathcal{C}+\langle\nu,E\rangle)^2}+N\frac{h_{ii}\langle\nu,E\rangle}{u}+N\frac{u_i^2}{u^2}+s\frac{h_{ii}\langle\nu,E\rangle}{(s-u)^2}-2s\frac{u_i^2}{(s-u)^3}\leqslant 0.\end{aligned} \tag{7-11}$$

By (1-10), we derive

$$\sigma_k^{ii}h_{iij}=\binom{n}{k}k(\mathcal{C}+\langle\nu,E\rangle)^{k-1}(-h_{jj}u_j)$$

and

$$\begin{aligned}\sigma_k^{ii}h_{iijj}&=-\sigma_k^{pq,rs}h_{pqj}h_{rsj}+\binom{n}{k}k(k-1)(\mathcal{C}+\langle\nu,E\rangle)^{k-2}h_{jj}^2u_j^2\\&\qquad+\binom{n}{k}k(\mathcal{C}+\langle\nu,E\rangle)^{k-1}\Big(-\sum_l h_{jjl}u_l+h_{jj}^2\langle\nu,E\rangle\Big)\\&\geqslant-\sigma_k^{pq,rs}h_{pqj}h_{rsj}+\binom{n}{k}k(\mathcal{C}+\langle\nu,E\rangle)^{k-1}\Big(-\sum_l h_{jjl}u_l\Big)-K_0(\mathcal{C}+\langle\nu,E\rangle)^{k-1}\kappa_1^2,\end{aligned}\tag{7-12}$$

where $K_0=K_0(n,k,\mathcal{C})>0$ is a constant depending on $n,\ k$, and $\mathcal{C}$. Recall that, in Minkowski space,

$$h_{jjii}=h_{iijj}+h_{ii}^2h_{jj}-h_{ii}h_{jj}^2.$$

Thus,

$$\sigma_k^{ii}h_{jjii}=\sigma_k^{ii}h_{iijj}+\sigma_k^{ii}h_{ii}^2h_{jj}-\sigma_k^{ii}h_{ii}h_{jj}^2\geqslant\sigma_k^{ii}h_{iijj}-k\binom{n}{k}(\mathcal{C}+\langle\nu,E\rangle)^kh_{jj}^2.\tag{7-13}$$

Combining (7-13) with (7-11), we obtain

$$\begin{aligned}0\geqslant\sigma_k^{ii}\frac{\phi_{ii}}{\phi}&=\frac{\sigma_k^{ii}}{P_m}\Big[\sum_j\kappa_j^{m-1}h_{jjii}+(m-1)\sum_j\kappa_j^{m-2}h_{jji}^2+\sum_{p\neq q}\frac{\kappa_p^{m-1}-\kappa_q^{m-1}}{\kappa_p-\kappa_q}h_{pqi}^2\Big]\\&\quad-\frac{m\sigma_k^{ii}}{P_m^2}\Big(\sum_j\kappa_j^{m-1}h_{jji}\Big)^2-N\sigma_k^{ii}\sum_l h_{ili}\frac{\langle\tau_l,E\rangle}{(\mathcal{C}+\langle\nu,E\rangle)}+N\sigma_k^{ii}h_{ii}^2\frac{-\langle\nu,E\rangle}{\mathcal{C}+\langle\nu,E\rangle}\\&\quad+N\sigma_k^{ii}h_{ii}^2\frac{u_i^2}{(\mathcal{C}+\langle\nu,E\rangle)^2}+N\sigma_k^{ii}\frac{h_{ii}\langle\nu,E\rangle}{u}+N\sigma_k^{ii}\frac{u_i^2}{u^2}+s\frac{\sigma_k^{ii}h_{ii}\langle\nu,E\rangle}{(s-u)^2}-2s\frac{\sigma_k^{ii}u_i^2}{(s-u)^3}\\&\geqslant-K_0(\mathcal{C}+\langle\nu,E\rangle)^{k-1}\kappa_1+\sum_i(A_i+B_i+C_i+D_i-E_i)+\binom{n}{k}k(\mathcal{C}+\langle\nu,E\rangle)^{k-1}\frac{-\sum_{j,l}h_{jjl}\kappa_j^{m-1}u_l}{P_m}\\&\quad-Nk\binom{n}{k}(\mathcal{C}+\langle\nu,E\rangle)^{k-2}\sum_l\kappa_lu_l^2+N\sigma_k^{ii}\kappa_i^2\frac{-\langle\nu,E\rangle}{\mathcal{C}+\langle\nu,E\rangle}+N\sigma_k^{ii}h_{ii}^2\frac{u_i^2}{(\mathcal{C}+\langle\nu,E\rangle)^2}\\&\quad+N\sigma_k^{ii}\frac{h_{ii}\langle\nu,E\rangle}{u}+N\sigma_k^{ii}\frac{u_i^2}{u^2}+s\frac{\sigma_k^{ii}h_{ii}\langle\nu,E\rangle}{(s-u)^2}-2s\frac{\sigma_k^{ii}u_i^2}{(s-u)^3}.\end{aligned}\tag{7-14}$$

Here,

$$A_i=\frac{\kappa_i^{m-1}}{P_m}\Big[K(\sigma_k)_i^2-\sum_{p,q}\sigma_k^{pp,qq}h_{ppi}h_{qqi}\Big]\quad\text{for some constant } K>1,$$

$$B_i=\frac{2\kappa_j^{m-1}}{P_m}\sum_j\sigma_k^{jj,ii}h_{jji}^2,\qquad C_i=\frac{m-1}{P_m}\sigma_k^{ii}\sum_j\kappa_j^{m-2}h_{jji}^2,$$

$$D_i=\frac{2\sigma_k^{jj}}{P_m}\sum_{j\neq i}\frac{\kappa_j^{m-1}-\kappa_i^{m-1}}{\kappa_j-\kappa_i}h_{jji}^2,\qquad E_i=\frac{m\sigma_k^{ii}}{P_m^2}\Big(\sum_j\kappa_j^{m-1}h_{jji}\Big)^2.$$

By Lemmas 8 and 9 and Corollary 10 in [Li et al. 2016], we can assume the following claim holds.

Claim 4. *There exist two small positive constants δ and $\eta < 1$. If $\kappa_k \leqslant \delta\kappa_1$, we have*

$$\sum_i A_i + B_i + C_i + D_i - \left(1 + \frac{\eta}{m}\right) E_i \geqslant 0, \tag{7-15}$$

where $m > 0$ is sufficiently large.

If (7-15) doesn't hold, we would have $\kappa_k > \delta\kappa_1$. Since $\sigma_k \leqslant \binom{n}{k}\mathcal{C}^k$, we get

$$\delta^{k-1}\kappa_1^k \leqslant \kappa_1\kappa_2\cdots\kappa_k \leqslant \sigma_k \leqslant \binom{n}{k}\mathcal{C}^k.$$

Since this gives an upper bound for κ_1 at x_0 directly, we would be done. Therefore, we assume (7-15) holds. Plugging (7-15) into (7-14) yields

$$\begin{aligned} 0 \geqslant & -K_0(\mathcal{C} + \langle \nu, E\rangle)^{k-1}\kappa_1 + \eta\frac{\sigma_k^{ii}}{P_m^2}\left(\sum_j \kappa_j^{m-1}h_{jji}\right)^2 - k\binom{n}{k}(\mathcal{C} + \langle \nu, E\rangle)^{k-1}|\nabla u|^2\left(\frac{N}{u} + \frac{s}{(s-u)^2}\right) \\ & + N\sigma_k^{ii}\kappa_i^2\frac{-\langle \nu, E\rangle}{\mathcal{C} + \langle \nu, E\rangle} + N\sigma_k^{ii}h_{ii}^2\frac{u_i^2}{(\mathcal{C} + \langle \nu, E\rangle)^2} + N\sigma_k^{ii}\frac{h_{ii}\langle \nu, E\rangle}{u} \\ & + N\sigma_k^{ii}\frac{u_i^2}{u^2} + s\frac{\sigma_k^{ii}h_{ii}\langle \nu, E\rangle}{(s-u)^2} - 2s\frac{\sigma_k^{ii}u_i^2}{(s-u)^3}. \end{aligned} \tag{7-16}$$

From (7-10), we obtain

$$\begin{aligned} \left(\frac{\sum_j \kappa_j^{m-1}h_{jji}}{P_m}\right)^2 = N^2\frac{\kappa_i^2 u_i^2}{(\mathcal{C} + \langle \nu, E\rangle)^2} + N^2\frac{u_i^2}{u^2} + \frac{s^2u_i^2}{(s-u)^4} - 2N^2\frac{\kappa_i u_i^2}{u(\mathcal{C} + \langle \nu, E\rangle)} \\ - 2Ns\frac{\kappa_i u_i^2}{(\mathcal{C} + \langle \nu, E\rangle)(s-u)^2} + 2Ns\frac{u_i^2}{u(s-u)^2}. \end{aligned} \tag{7-17}$$

Inserting (7-17) into (7-16), we derive

$$\begin{aligned} 0 \geqslant & -K_0(\mathcal{C}+\langle \nu, E\rangle)^{k-1}\kappa_1 + \eta\frac{s^2\sigma_k^{ii}u_i^2}{(s-u)^4} + N(N\eta+1)\sigma_k^{ii}\kappa_i^2\frac{u_i^2}{(\mathcal{C}+\langle \nu, E\rangle)^2} - 2N^2\eta\frac{\sigma_k^{ii}\kappa_i u_i^2}{u(\mathcal{C}+\langle \nu, E\rangle)} \\ & -2Ns\eta\frac{\sigma_k^{ii}\kappa_i u_i^2}{(\mathcal{C}+\langle \nu, E\rangle)(s-u)^2} + 2Ns\eta\frac{\sigma_k^{ii}u_i^2}{u(s-u)^2} + N\sigma_k^{ii}\frac{h_{ii}\langle \nu, E\rangle}{u} + N(\eta N+1)\sigma_k^{ii}\frac{u_i^2}{u^2} + s\frac{\sigma_k^{ii}h_{ii}\langle \nu, E\rangle}{(s-u)^2} \\ & -2s\frac{\sigma_k^{ii}u_i^2}{(s-u)^3} - k\binom{n}{k}(\mathcal{C}+\langle \nu, E\rangle)^{k-1}|\nabla u|^2\left(\frac{N}{u} + \frac{s}{(s-u)^2}\right) + N\sigma_k^{ii}\kappa_i^2\frac{-\langle \nu, E\rangle}{\mathcal{C}+\langle \nu, E\rangle}. \end{aligned} \tag{7-18}$$

It's clear that

$$|\nabla u| = \frac{|Du|}{\sqrt{1-|Du|^2}} < -\langle \nu, E\rangle \leqslant \mathcal{C}. \tag{7-19}$$

We also notice that, for any $1 \leqslant i \leqslant n$, we have $\sigma_k^{ii}\kappa_i \leqslant \binom{n}{k}\mathcal{C}^k$ (no summation). By a simple calculation, we get, when $N > 1/\eta^2$,

$$\eta\frac{s^2\sigma_k^{ii}u_i^2}{(s-u)^4} + 2Ns\eta\frac{\sigma_k^{ii}u_i^2}{u(s-u)^2} - 2s\frac{\sigma_k^{ii}u_i^2}{(s-u)^3} \geqslant 0. \tag{7-20}$$

Moreover, applying Lemma 31, we know there exist two positive constants $\tilde{d}_2 > \tilde{d}_1 > 0$ such that

$$\tilde{d}_1 \leqslant u(\mathcal{C} + \langle \nu, E \rangle) \leqslant \tilde{d}_2. \tag{7-21}$$

Therefore, for $N > 1/\eta^2$ sufficiently large, combining (7-19)–(7-21) with (7-18) yields

$$\begin{aligned} 0 \geqslant &-K_0(\mathcal{C} + \langle \nu, E \rangle)^{k-1}\kappa_1 - \frac{2N^2}{\tilde{d}_1}|\nabla u|^2\sigma_k^{ii}\kappa_i - 2Ns\frac{|\nabla u|^2\sigma_k^{ii}\kappa_i}{(\mathcal{C} + \langle \nu, E \rangle)(s-u)^2} \\ &- N\mathcal{C}\sigma_k^{ii}\kappa_i - \mathcal{C}\sigma_k^{ii}\kappa_i\frac{s}{(s-u)^2} - k\mathcal{C}^2\binom{n}{k}(\mathcal{C} + \langle \nu, E \rangle)^{k-1}\frac{s}{(s-u)^2} \\ &- k\binom{n}{k}\mathcal{C}^2(\mathcal{C} + \langle \nu, E \rangle)^{k-1}N + N\frac{c_0\sigma_k\kappa_1}{\mathcal{C} + \langle \nu, E \rangle}. \end{aligned}$$

It's easy to see that the above inequality yields, at x_0,

$$\kappa_1 \leqslant K(N, \mathcal{C}, \tilde{d}_1)\frac{s^2}{(s-u)^2}.$$

Therefore, in U_s, by (7-21), we have

$$\phi \leqslant K(N, \mathcal{C}, \tilde{d}_1)e^{-s/(s-u)}\frac{s^2}{(s-u)^2}.$$

Note that, for any $t \in [0, s]$,

$$\varphi(t) = e^{-s/(s-t)}\frac{s^2}{(s-t)^2} \leqslant 4e^{-2}.$$

We obtain, at any point $x \in U_s$,

$$\phi \leqslant K(N, \mathcal{C}, \tilde{d}_1). \tag{7-22}$$

Now, for any $x \in \mathbb{R}^n$, we can choose $s > 0$ large enough that $x \in U_{s/2}$. Then, by (7-22) and (7-21), we conclude that

$$\kappa_1(x) \leqslant K(N, \mathcal{C}, \tilde{d}_1, \tilde{d}_2).$$

Since x is arbitrary, we have finished proving Proposition 32. □

Theorem 7 follows from Propositions 22 and 32 immediately.

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ANISOTROPIC MICROPOLAR FLUIDS SUBJECT TO A UNIFORM MICROTORQUE: THE STABLE CASE

ANTOINE REMOND-TIEDREZ AND IAN TICE

We study a three-dimensional, incompressible, viscous, micropolar fluid with anisotropic microstructure on a periodic domain. Subject to a uniform microtorque, this system admits a unique nontrivial equilibrium. We prove that when the microstructure is inertially oblate (i.e., pancake-like) this equilibrium is nonlinearly asymptotically stable.

Our proof employs a nonlinear energy method built from the natural energy dissipation structure of the problem. Numerous difficulties arise due to the dissipative-conservative structure of the problem. Indeed, the dissipation fails to be coercive over the energy, which itself is weakly coupled in the sense that, while it provides estimates for the fluid velocity and microstructure angular velocity, it only provides control of two of the six components of the microinertia tensor. To overcome these problems, our method relies on a delicate combination of two distinct tiers of energy-dissipation estimates, together with transport-like advection-rotation estimates for the microinertia. When combined with a quantitative rigidity result for the microinertia, these allow us to deduce the existence of global-in-time decaying solutions near equilibrium.

This paper, together with the companion paper [Remond-Tiedrez and Tice 2021], provides a sharp nonlinear stability criterion for an anisotropic micropolar fluid subject to a uniform microtorque. The companion paper is concerned with the unstable regime; we tackle the stable regime here.

Note to the reader: The introduction of Section 1 serves as a "shortest path" to the main result recorded in Theorem 1.2, providing the necessary physical and mathematical background to appropriately state the main result. For a more detailed discussion of the problem and the strategy employed to prove nonlinear stability, we direct the reader's attention to Section 2.

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1. Introduction

1A. ***Brief description of the model.*** Following the tradition of generalized continuum mechanics dating back to the Cosserat brothers [Cosserat and Cosserat 1909], micropolar fluids were introduced by Eringen [1966]. The theory of micropolar fluids extends classical continuum mechanics by taking into account the effects due to the microstructure present in the continuum. For viscous, incompressible fluids, this results in a model coupling the Navier–Stokes equations to an evolution equation for the rigid microstructure present at every point in the continuum. This theory has been used to describe aerosols and colloidal suspensions, such as those appearing in biological fluids [Maurya 1985], blood flow [Bég et al. 2008; Mekheimer and El Kot 2008; Ramkissoon 1985], lubrication [Allen and Kline 1971; Bayada and Łukaszewicz 1996; Rajasekhar Nicodemus and Sharma 2012] and the lubrication of human joints [Sinha et al. 1982], as well as liquid crystals [Eringen 1993; Gay-Balmaz et al. 2013; Lhuillier and Rey 2004] and ferromagnetic fluids [Nochetto et al. 2016].

We now provide a brief description of the model, introducing new terminology and concepts only if they are necessary to formulate the main result. For a thorough discussion of the model and for its careful derivation, see [Remond-Tiedrez and Tice 2021, Section 2] and [Remond-Tiedrez 2020, Chapter 1], respectively.

The state of a three-dimensional micropolar fluid at a point in space-time is described by the following variables: the fluid's velocity is a vector $u \in \mathbb{R}^3$, the fluid's pressure is a scalar $p \in \mathbb{R}$, the microstructure's angular velocity is a vector $\omega \in \mathbb{R}^3$, and the microstructure's moment of inertia is a positive definite symmetric matrix $J \in \mathbb{R}^{3\times 3}$ which is called the *microinertia tensor*. Here we study *homogeneous* micropolar fluids, meaning that the microstructures at any two points of the fluid are identical up to a proper rotation. Equivalently, this means that the microinertia tensors at any two points of the fluid are equal up to conjugation (by that same rotation). Note that the shape of the microstructure determines the microinertia tensor, but the converse fails since the same microinertia tensor may be achieved by microstructures of differing shapes.

We restrict our attention to problems in which the microinertia plays a significant role, and so in this paper we only consider *anisotropic* micropolar fluids. This means that the microinertia is not isotropic, or in other words that J has at least two distinct eigenvalues. To be precise, we study micropolar fluids whose microstructure has an *inertial axis of symmetry*. That is to say there are physical constants $\lambda, \nu > 0$ which depend on the microstructure such that, at every point, J is a symmetric matrix with spectrum $\{\lambda, \lambda, \nu\}$. Studying microstructures with an inertial axis of symmetry may be viewed as the intermediate case between the isotropic case where the microinertia has a repeated eigenvalue of multiplicity three and the "fully" anisotropic case where the microinertia has three distinct eigenvalues.

The equations governing the motion of a micropolar fluid in the periodic spatial domain $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ subject to an external microtorque τe_3 are

$$\begin{cases} \partial_t u + (u\cdot\nabla)u = \left(\mu + \tfrac{1}{2}\kappa\right)\Delta u + \kappa\nabla\times\omega - \nabla p & \text{on } (0,T)\times\mathbb{T}^3, \quad \text{(1-1a)}\\ \nabla\cdot u = 0 & \text{on } (0,T)\times\mathbb{T}^3, \quad \text{(1-1b)}\\ J(\partial_t\omega + (u\cdot\nabla)\omega) + \omega\times J\omega & \\ \qquad = \kappa\nabla\times u - 2\kappa\omega + (\tilde{\alpha}-\tilde{\gamma})\nabla(\nabla\cdot\omega) + \tilde{\gamma}\Delta\omega + \tau e_3 & \text{on } (0,T)\times\mathbb{T}^3, \quad \text{(1-1c)}\\ \partial_t J + (u\cdot\nabla)J = [\Omega, J] & \text{on } (0,T)\times\mathbb{T}^3, \quad \text{(1-1d)} \end{cases}$$

where $[\,\cdot\,,\cdot\,]$ denotes the commutator between two matrices, $\tilde{\alpha} = \alpha + \frac{4}{3}\beta$ and $\tilde{\gamma} = \beta + \gamma$, where μ, κ, α, β, and γ are nonnegative physical viscosity constants, τ denotes the magnitude of the microtorque, and Ω is the 3×3 antisymmetric matrix identified with ω via the identity $\Omega v = \omega \times v$ for every $v \in \mathbb{R}^3$.

We are considering the situation in which external forces are absent and the external microtorque is constant, namely equal to τe_3 for some constant $\tau > 0$. Note that the choice of e_3 as the direction of the microtorque is made without loss of generality since the equations of motion are equivariant under proper rotations. More precisely, if (u, p, ω, J) is a solution of (1-1) then, for any $\mathcal{R} \in \mathrm{SO}(3)$, $(u, p, \mathcal{R}\omega, \mathcal{R}J\mathcal{R}^T)$ is a solution of (1-1) *provided* that the microtorque τe_3 is replaced by $\tau \mathcal{R} e_3$.

We can motivate the choice to have no external forces and a constant microtorque in two ways. On one hand it is reminiscent of certain chiral active fluids constituted of self-spinning particles which continually drive energy into the system [Banerjee et al. 2017], as our constant microtorque does. On the other hand this choice of an external force-microtorque pair is motivated by the lack of analytical results on anisotropic micropolar fluids. As a first foray into the world of anisotropic micropolar fluids, it is natural to look for the simplest external force-microtorque pair which gives rise to nontrivial equilibria for the angular velocity ω and the microinertia J. The simplest such external force-microtorque pair is precisely $(0, \tau e_3)$.

The equilibrium and its stability. Let us now turn our attention to the aforementioned equilibrium. Subject to a constant and uniform microtorque, the unique equilibrium of the system is the following: the fluid's velocity is quiescent ($u_{\mathrm{eq}} = 0$), the pressure is null ($p_{\mathrm{eq}} = 0$), the angular velocity is aligned with the microtorque ($\omega_{\mathrm{eq}} = \tau/(2\kappa)$), and the inertial axis of symmetry of the microstructure is aligned with the microtorque such that the microinertia is $J_{\mathrm{eq}} = \mathrm{diag}(\lambda, \lambda, \nu)$.

A physically motivated heuristic suggests that the stability of the equilibrium depends on the microstructure, and more precisely that the equilibrium is stable if $\nu > \lambda$ and unstable if $\lambda > \nu$. This heuristic explanation is based on the analysis of the energy associated with the system and with a comparison with the ODE describing the rotation of a damped rigid body subject to an external torque. While we defer to the companion paper [Remond-Tiedrez and Tice 2021] for a detailed discussion of this heuristic, the core of the argument based on the analysis of the energy can be seen from the energy-dissipation relation recorded later in this paper. In particular, the energy recorded in (1-3) below (which then appears first in a rigorous setting in Proposition 4.9) only remains positive-definite when $\nu > \lambda$, which suggests that this may characterize the stable regime.

In the former case where $\nu > \lambda$ we say that the microstructure is *inertially oblate*, or pancake-like, and in the latter case where $\lambda > \nu$ we say that the microstructure is *inertially oblong*, or rod-like. This nomenclature is justified by the following fact. For rigid bodies with an axis of symmetry and a uniform mass density, the terms "oblate", which essentially means that the body is shorter along its axis of symmetry than it is wide across it, and "inertially oblate" describe the same thing (and similarly for the terms "oblong" and "inertially oblong"). Examples of inertially oblong and oblate rigid bodies are provided in Figure 1.

In the companion paper [Remond-Tiedrez and Tice 2021] we prove the instability of inertially oblong microstructures. In this paper we prove the asymptotic stability of inertially oblate microstructures in Theorem 1.2. In particular, combining the main results of these two papers produces a *sharp* nonlinear stability criterion, recorded in Theorem 1.4.

This rigid body is inertially oblong if $h^2 > 6r^2$. This rigid body is inertially oblate if $h^2 < 6r^2$.

Figure 1. Two rigid bodies with uniform density which possess an inertial axis of symmetry.

1B. ***A brief summary of techniques and difficulties.*** The main thrust of this paper is to prove that if the microstructure is inertially oblate then the equilibrium is nonlinearly asymptotically stable with almost exponential decay to equilibrium. Here "almost exponential" means that the rate of decay is algebraic and grows unboundedly as further smallness and regularity assumptions are imposed on the initial data. In order to provide context for our main result and to motivate the presence of the various functionals used in it, we will now attempt a quick overview of the difficulties associated to (1-1) and our techniques for dealing with them. A more detailed discussion is presented in Section 2.

As in many viscous fluid problems, the system (1-1) is of mixed dissipative-conservative type, with some of the unknowns having dissipation mechanisms and others not. This manifests as the mixed parabolic-hyperbolic structure of the PDEs. Such systems usually have a physical dissipation functional, $\mathfrak{D}$, that couples to a physical energy functional, $\mathfrak{E}$, via the energy-dissipation relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{E} + \mathfrak{D} = 0. \tag{1-2}$$

We won't need the precise form of $\mathfrak{E}$ and $\mathfrak{D}$ for our problem here, so we don't state them precisely, but they can be found in (2.8) of [Remond-Tiedrez and Tice 2021]. Our technique for analyzing the problem is based on higher-regularity versions of this structure, and since differentiating linearizes the PDEs, it's actually the linearized versions, $\mathfrak{E}_{\text{lin}}$ and $\mathfrak{D}_{\text{lin}}$, that are most relevant in our discussion. Indeed, the questions of if and how the unknowns appear in the linearized versions of $\mathfrak{E}_{\text{lin}}$ and $\mathfrak{D}_{\text{lin}}$ become paramount.

In general energy-dissipation relations, if we have a bound $\mathfrak{E}_{\text{lin}} \leqslant C\mathfrak{D}_{\text{lin}}$, then the dissipation is said to be *coercive*, and we expect to be able to prove the exponential decay of $\mathfrak{E}_{\text{lin}}$ via a linearized version of (1-2) and a Gronwall argument. However, if this inequality does not hold, we say the dissipation fails to be coercive, and the decay of solutions is no longer obvious. As we will see below, the latter holds for our problem, system (1-1).

Without coercivity, the role of the energy becomes more complicated. On the one hand, more terms in the energy means more a priori control, but on the other hand it means more things that the dissipation may fail to control, further complicating a proof of decay. If all of the unknowns appear in $\mathfrak{E}_{\text{lin}}$ we say that there is *strong coupling*, and otherwise we say that there is *weak coupling*. Based on the above relation to the dissipation, it may seem that weak coupling is preferable, but this is only true from the point of view of exploiting the dissipation for decay information.

In the case of strong coupling we can estimate all of the system's unknowns at the same time, with the same derivative counts. In the case of weak coupling only some of the unknowns appear in the energy, and the remaining quantities must be estimated in other ways. This typically entails exploiting some sort of conservative hyperbolic structure that scales differently in terms of derivative counts than the main energy-dissipation part. At the linear level this isn't a problem because the weak coupling actually leads to a decoupling of these parts of the linearized problem, and we just get different estimates for each part. However, the essential difficulty with weak coupling comes at the nonlinear level, where the scaling mismatch can make dealing with the high-regularity interaction terms extremely delicate.

Let's now focus this general discussion onto the specifics of the problem (1-1). In this case, if (v, θ, K) denotes the linearization of (u, ω, J) about the equilibrium, then

$$\mathfrak{E}_{\mathrm{lin}} = \int_{\mathbb{T}^3} \frac{1}{2}|v|^2 + \frac{1}{2}J\theta\cdot\theta + \frac{\tilde{\tau}^2}{\nu-\lambda}\frac{1}{2}|a|^2, \tag{1-3}$$

and

$$\mathfrak{D}_{\mathrm{lin}} = \int_{\mathbb{T}^3} \frac{\mu}{2}|\mathbb{D}v|^2 + 2\kappa\left|\frac{1}{2}\nabla\times v - \theta\right|^2 + \alpha|\nabla\cdot\theta|^2 + \frac{\beta}{2}|\mathbb{D}^0\theta|^2 + 2\gamma|\nabla\times\theta|^2, \tag{1-4}$$

where $a = (K_{13}, K_{23})$, and $\mathbb{D}$ and $\mathbb{D}^0$ denote twice the symmetric part and twice the traceless symmetric part of the gradient, respectively, and are defined precisely below (2-4). From these expressions it's clear that coercivity fails and that we have weak coupling. Indeed, only two of the six components of the symmetric matrix $K \in \mathbb{R}^{3\times 3}$ appear in $\mathfrak{E}_{\mathrm{lin}}$ (see Section 2C below for a more detailed discussion of the special role played by a). To estimate the entirety of K we are forced to appeal to the advection-rotation equation (1-1d) and its linearization. This is a hyperbolic equation coupling to both u and ω, but at different levels of regularity, which already reveals potentially problematic mismatches with energy-dissipation estimates. On the plus side, we can readily obtain L^q-based estimates from (1-1d) for values of q other than 2. On the down side, the estimates provided at the highest level of regularity are quite bad, as they grow linearly in time, which makes using them globally in time a delicate proposition.

Our strategy for getting around these problems is to employ a version of the two-tier energy method introduced in [Guo and Tice 2013a; 2013b] to handle the viscous surface wave problem, which is a strongly coupled problem with coercivity failure. Roughly, the idea behind this scheme of a priori estimates is that control of high-regularity terms (the high tier) can be synthesized with decay estimates of low-regularity terms (the low tier) to simultaneously overcome coercivity and interaction difficulties and prove the existence of global-in-time algebraically decaying solutions. The two-tier method is a strategy and not a black box, so it must be adapted to the particulars of each problem. In our case, due to the weak coupling, the complicated structure of the hyperbolic equation for K, and troubles in interfacing with the local existence theory, this requires significant work.

To see how decay information can be recovered in the two-tier scheme, consider the following. The energy-dissipation structure at low regularity will tell us that (assuming that the nonlinear interactions are brought under control)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\mathrm{low}} + \frac{1}{2}\mathcal{D}_{\mathrm{low}} \leqslant 0, \tag{1-5}$$

where $\mathcal{E}_{\text{low}}$ and $\mathcal{D}_{\text{low}}$ are low-regularity energy and dissipation functionals built from $\mathfrak{E}_{\text{lin}}$ and $\mathfrak{D}_{\text{lin}}$, respectively. To be concrete,

$$\mathcal{E}_{\text{low}} \sim \|(u,\theta)\|_{H^2}^2 + \|a\|_{H^2}^2 + \|\partial_t(u,\theta)\|_{L^2}^2 + \|\partial_t a\|_{L^2}^2, \tag{1-6}$$

where we have introduced the perturbative angular velocity $\theta = \omega - \omega_{\text{eq}}$. The exact form of $\mathcal{D}_{\text{low}}$ is not relevant here; all that matters is that $\mathcal{E}_{\text{low}} \lesssim \mathcal{E}_{\text{high}}^{1-\sigma}\mathcal{D}_{\text{low}}^{\sigma}$ for some high-regularity energy functional $\mathcal{E}_{\text{high}}$ and some $\sigma \in (0,1)$ which behaves as $\sigma \sim (\text{high}-\text{low})/(\text{high}-\text{low}+1)$. Here "low" and "high" are placeholders for regularities indices precisely measuring the regularity of the solution at each level. Crucially, this observation may be combined with (1-5), *provided* that $\mathcal{E}_{\text{high}}$ is bounded, to deduce the algebraic decay of $\mathcal{E}_{\text{low}}$ at a rate proportional to $(\text{high}-\text{low})$. Note that this is precisely almost exponential decay since the growing rate of decay is dependent on the regularity of the solution. More concretely, for some nonnegative integer M, we can write $\mathcal{E}_{\text{high}}$ and $\mathcal{D}_{\text{high}}$ as

$$\mathcal{E}_{\text{high}} = \sum_{j=0}^{M} \|\partial_t^j(u,\theta,a)\|_{H^{2M-2j}}^2 + \|(K,\partial_t K,\partial_t^2 K)\|_{H^{2M-3}}^2 + \sum_{j=3}^{M}\|\partial_t^j K\|_{H^{2M-2j+2}}^2 \tag{1-7}$$

and

$$\mathcal{D}_{\text{high}} = \sum_{j=0}^{M} \|\partial_t^j(u,\theta)\|_{H^{2M-2j+1}}^2 + \sum_{j=0}^{3}\|\partial_t^j a\|_{H^{2M-j-1}}^2 + \sum_{j=4}^{M}\|\partial_t^j a\|_{H^{2M-2j+3}}^2, \tag{1-8}$$

where we have introduced the perturbative microinertia $K = J - J_{\text{eq}}$ and where $\mathcal{D}_{\text{high}}$ is a high-regularity dissipation functional whose integral in time will remain bounded. Remarkably, although $\mathcal{E}_{\text{low}}$ provides no direct control of K except its components in $a = (K_{13}, K_{23})$, a special algebraic identity for symmetric matrices with spectrum $\{\lambda,\lambda,\nu\}$ leads to a quantitative rigidity result that will allow us to obtain decay information about all of K from a alone.

We have now witnessed the first key idea of the two-tier energy method: the decay of the low-level energy is intimately tied to the boundedness of the high-level energy. In the above sketch this dependence only goes one way, but in practice it also goes the other way since the transport estimates for K at the highest derivative count result in an upper bound that grows linearly in time (see Section 2 for a more thorough discussion). This warrants the introduction of the last functional we need in order to state the main result. We define $\mathcal{F}_{\text{high}}$ to contain all terms for which the only control we have is growing in time, namely

$$\mathcal{F}_{\text{high}} = \|K\|_{H^{2M+1}}^2 + \|\partial_t K\|_{H^{2M}}^2 + \|\partial_t^2 K\|_{H^{2M-1}}^2. \tag{1-9}$$

1C. *Statement of the main result.* We first introduce the global assumptions at play throughout this paper.

Definition 1.1 (global assumptions). We assume that the initial microinertia J_0 has an inertial axis of symmetry and is inertially oblate, i.e., for every $x \in \mathbb{T}^3$ the spectrum of $J_0(x)$ is $\{\lambda,\lambda,\nu\}$, where $\nu > \lambda > 0$. We also assume that the initial velocity u_0 has average zero and that the viscosity constants μ, κ, α, β, and γ are strictly positive.

Note that the assumption that u_0 has average zero is justified by the invariance of (1-1) under Galilean transformations $u(t,y) \mapsto u(t, y + t\bar{u}) - \bar{u}$ for any constant $\bar{u} \in \mathbb{R}$. We may now state the main result. A more precise form of this result is found in Theorem 7.6.

Theorem 1.2 (nonlinear asymptotic stability and decay). *Suppose the global assumptions of Definition 1.1 hold, and let*

$$X_{\mathrm{eq}} = (u_{\mathrm{eq}}, \omega_{\mathrm{eq}}, J_{\mathrm{eq}}) = \Big(0, \frac{\tau}{2\kappa} e_3, \operatorname{diag}(\lambda, \lambda, \nu)\Big) \quad \text{and} \quad p_{\mathrm{eq}} = 0$$

be the equilibrium solution of (1-1). For every integer $M \geqslant 4$ there exists $\eta, C > 0$ such that solutions to system (1-1) exist globally in time for every initial condition in the η-ball defined by

$$\|(u_0, \omega_0 - \omega_{\mathrm{eq}})\|^2_{H^{2M}} + \|J_0 - J_{\mathrm{eq}}\|^2_{H^{2M+1}} < \eta.$$

Moreover, the solutions satisfy the estimate

$$\sup_{t \geqslant 0} \mathcal{E}_{\mathrm{low}}(t)(1+t)^{2M-2} + \mathcal{E}_{\mathrm{high}}(t) + \frac{\mathcal{F}_{\mathrm{high}}(t)}{(1+t)} + \int_0^\infty \mathcal{D}_{\mathrm{high}}(s)\, ds \leqslant C(\|(u_0, \omega_0 - \omega_{\mathrm{eq}}\|^2_{H^{2M}} + \|J_0 - J_{\mathrm{eq}}\|^2_{H^{2M+1}}).$$

Recall that the functionals present on the left-hand side are defined in (1-6)–(1-9).

Note that in Theorem 1.2 the pressure has disappeared from consideration in the estimates. This is because the pressure only plays an auxiliary role in the problem and may be eliminated altogether from (1-1a) by projection onto the space of divergence-free vector fields.

At face value Theorem 1.2 only provides us with decay of u, θ, and a in terms of the norms appearing in $\mathcal{E}_{\mathrm{low}}$. However, we may interpolate between $\mathcal{E}_{\mathrm{low}}$ and $\mathcal{E}_{\mathrm{high}}$ to obtain decay estimates on intermediate norms of u, θ, and a. Algebraic identities may then be used to show that, if $\|K\|_{L^\infty}$ is sufficiently small, $|K| \lesssim |a|$ pointwise, from which we may deduce the decay of K. Interpolation can then once again allow us to obtain decay of higher-order norms, in this case obtaining decay of higher-order norms of K. However, the endpoint estimate at the highest derivative count now involves $\mathcal{F}_M$, which may be growing in time. This causes the decay rates of K to be slightly slower than the decay rates of a.

The precise decay rates are recorded in Corollary 1.3 below (which is proved at the end of Section 7). Note that this corollary only records the decay of the unknowns and their first time derivative. The decay rates of higher-order temporal derivatives can then be established by differentiating (1-1); however, since they are not necessary for our purposes here, we omit them. Crucially, with these decay rates in hand we deduce that Theorem 1.2 is indeed a proof of *asymptotic stability*.

Corollary 1.3 (decay rates). *Under the hypotheses of Theorem 1.2, the global solution (u, θ, K) satisfies*

$$\sup_{t \geqslant 0} \Big(\sup_{0 \leqslant s \leqslant 2M+1} \|K(t)\|^2_{H^s} (1+t)^{2M-4-s(2M-3)/(2M+1)} + \sup_{0 \leqslant s \leqslant 2M} \|\partial_t K(t)\|^2_{H^s} (1+t)^{2M-4-s(2M-3)/(2M)} + \sup_{2 \leqslant s \leqslant 2M} (\|(u, \theta, a)(t)\|^2_{H^s} + \|\partial_t (u, \theta, a)(t)\|^2_{H^{s-2}})(1+t)^{2M-s} \Big) \lesssim \|(u_0, \theta_0, K_0)\|^2_{H^{2M}} + \|K_0\|^2_{H^{2M+1}}.$$

Sharp nonlinear stability criterion. We may combine the main result of this paper, namely Theorem 1.2, with the decay rates of Corollary 1.3 and the main result of [Remond-Tiedrez and Tice 2021] to deduce a sharp nonlinear stability criterion recorded in Theorem 1.4 below. In order to formulate Theorem 1.4 in a clean, way we define appropriate spaces, namely

$$\mathcal{H}_0 = H^{2M}(\mathbb{T}^3;\mathbb{R}^3)\times H^{2M}(\mathbb{T}^3;\mathbb{R}^3)\times H^{2M+1}(\mathbb{T}^3;\mathrm{Sym}(3)),$$
$$\mathcal{H}_{\mathrm{s}} = H^{2M}(\mathbb{T}^3;\mathbb{R}^3)\times H^{2M}(\mathbb{T}^3;\mathbb{R}^3)\times H^{2M-4/(2M-3)}(\mathbb{T}^3;\mathrm{Sym}(3)),$$
$$\mathcal{H}_{\mathrm{as}} = H^{2M-\varepsilon}(\mathbb{T}^3;\mathbb{R}^3)\times H^{2M-\varepsilon}(\mathbb{T}^3;\mathbb{R}^3)\times H^{2M-4/(2M-3)-\varepsilon}(\mathbb{T}^3;\mathrm{Sym}(3)),$$

where $\varepsilon > 0$ may be taken to be arbitrarily small. We may now state the sharp nonlinear stability criterion.

Theorem 1.4. *Let $X_{\mathrm{eq}} = (u_{\mathrm{eq}}, \omega_{\mathrm{eq}}, J_{\mathrm{eq}}) = (0, (\tau/(2\kappa))e_3, \mathrm{diag}(\lambda,\lambda,\nu))$ be the equilibrium solution of (1-1).*

- *If the microstructure is inertially oblong ($\lambda > \nu$) then the equilibrium is nonlinearly unstable in L^2.*
- *If the microstructure is inertially oblate ($\nu > \lambda$) then the equilibrium is nonlinearly $\mathcal{H}_{\mathrm{s}}$-stable in $\mathcal{H}_0$ and nonlinearly asymptotically $\mathcal{H}_{\mathrm{as}}$-stable in $\mathcal{H}_0$.*

The notions of nonlinear stability and instability above are those familiar from dynamical systems. *Nonlinear instability* in L^2 means that there exists a radius $\delta > 0$ and a sequence of initial data $\{X_n^0\}_{n=0}^{\infty}$ which converge to X_{eq} in L^2 such that the solutions to (1-1) starting from X_n^0 exit the δ-ball about X_{eq} in finite time (which depends on n). *Nonlinear $\mathcal{H}_{\mathrm{s}}$-stability in $\mathcal{H}_0$* means that for every $\varepsilon > 0$ there exists a δ-ball about X_{eq} in $\mathcal{H}_0$ in which (1-1) is globally well-posed and such that solutions remain in the ε-ball about X_{eq} in $\mathcal{H}_{\mathrm{s}}$ for all time. *Nonlinear asymptotic $\mathcal{H}_{\mathrm{as}}$-stability in $\mathcal{H}_0$* means that nonlinear stability holds and that, moreover, solutions in that δ-ball about X_{eq} in $\mathcal{H}_0$ converge to X_{eq} in $\mathcal{H}_{\mathrm{as}}$ as time $t \to \infty$.

1D. *Previous work.* The continuum mechanics community has actively and extensively studied micropolar fluids over the past fifty years. While an exhaustive literature review is beyond the scope of this paper, we highlight the mathematics literature here. To the best of our knowledge, current mathematical results only consider *isotropic* microstructure, which means that the microinertia J is a scalar multiple of the identity. In particular, when a micropolar fluid is isotropic the precession term $\omega \times J\omega$ which appears in (1-1c) now vanishes and (1-1d), which governs the dynamics of the microinertia, is trivially satisfied. Note that in two dimensions the microinertia is a scalar, such that all two-dimensional micropolar fluids are isotropic.

The results known about isotropic micropolar fluids follow the pattern of what is known about viscous fluids. In two dimensions global well-posedness holds [Łukaszewicz 2001], and quantitative rates of decay are obtained in [Dong and Chen 2009]. In three dimensions, where well-posedness was first discussed by Galdi and Rionero [1977], weak solutions were constructed globally in time by Łukaszewicz [1990], who also proved that strong solutions are unique [Łukaszewicz 1989]. More recent work has established global well-posedness for small data in critical Besov spaces [Chen and Miao 2012] and in the space of pseudomeasures [Ferreira and Villamizar-Roa 2007], and a blow-up criterion was derived in [Yuan 2010]. There is also a body of work dedicated to the study of partially inviscid limits taking one or more of the viscosity coefficients to zero. We refer to [Dong and Zhang 2010] for an illustrative example.

Various extensions of the model of incompressible micropolar fluids presented here have been considered. These extensions treated the compressible case [Liu and Zhang 2016], and coupled the system to heat transfer [Kalita et al. 2019; Tarasińska 2006] and to magnetic fields [Ahmadi and Shahinpoor 1974; Rojas-Medar 1997]. Again, to the best of our knowledge all of these works consider *isotropic* micropolar fluids.

As mentioned above, we employ a two-tier nonlinear energy method as our scheme of a priori estimates. This was originally used by Guo and Tice [2013a; 2013b] in the analysis of the viscous surface wave problem, where the conservative variable is the free surface function, which is strongly coupled. This technique was also used to deal with the strongly coupled mass density for the compressible surface wave problem by Jang, Tice, and Wang [Jang et al. 2016]. Two-tier schemes have also proved useful in magnetohydrodynamic (MHD) problems without magnetic viscosity, where the magnetic field is the conservative unknown and is weakly coupled according to our above classification: see, for instance, the works of Ren, Wu, Ziang, and Zhang [Ren et al. 2014], Abidi and Zhang [2017], Tan and Wang [2018], and Wang [2019]. The weak coupling of our present problem is more complicated than in these MHD results since some of the components of K, namely $a = (K_{13}, K_{23})$, are strongly coupled, which means that K cannot be conveniently "integrated out" by solving for it in terms of the other unknowns.

2. Strategy and difficulties

In this section we describe the various obstructions to proving a stability result like Theorem 1.2 and we discuss our strategy to overcome them. Since we study the nonlinear stability of a nontrivial equilibrium it is natural to change variables and use perturbative unknowns. This is done in Section 2A.

In Sections 2B and 2C we then discuss the two main obstructions, namely the lack of a spectral gap and the weak coupling. In a nutshell, the difficulties are as follows. The lack of a spectral gap leads to a failure of coercivity. The key in overcoming that is to prove a θ-coercivity estimate. On top of that, weak coupling means that, even with θ-coercivity, we cannot immediately deduce decay of all the unknowns (and so this comes after the fact via an algebraic identity and interpolation).

We conclude in Sections 2D–2G with a discussion of the various moving pieces of our proof of Theorem 1.2. The centerpiece of our proof is the scheme of a priori estimates introduced in Section 2D. Section 2E describes the local well-posedness theory and Section 2F discusses how to "glue" the local well-posedness theory and the a priori estimates by means of a continuation argument. Finally, Section 2G explains how to synthesize the various pieces of the proof in order to deduce global well-posedness and decay, and hence asymptotic stability.

2A. ***Perturbative formulation and overall strategy.*** Since we study the nonlinear stability of (1-1) about the equilibrium $(u_{\text{eq}}, p_{\text{eq}}, \omega_{\text{eq}}, J_{\text{eq}}) = (0, 0, (\tau/(2\kappa))e_3, \operatorname{diag}(\lambda, \lambda, \nu))$, we naturally seek to write this system in terms of the perturbative variables $(u, p, \theta, K) = (u, p, \omega, J) - (u_{\text{eq}}, p_{\text{eq}}, \omega_{\text{eq}}, J_{\text{eq}})$. We may then write (1-1) equivalently as

$$
\begin{cases}
\partial_t u + u\cdot\nabla u = \big(\mu + \tfrac{1}{2}\kappa\big)\Delta u + \kappa\nabla\times\theta - \nabla p & \text{on } (0,T)\times\mathbb{T}^n, \qquad \text{(2-1a)}\\
\nabla\cdot u = 0 & \text{on } (0,T)\times\mathbb{T}^n, \qquad \text{(2-1b)}\\
(J_{\text{eq}}+K)(\partial_t\theta + u\cdot\nabla\theta) + (\omega_{\text{eq}}+\theta)\times(J_{\text{eq}}+K)(\omega_{\text{eq}}+\theta) & \\
\qquad\qquad = \kappa\nabla\times u - 2\kappa\theta + (\tilde\alpha - \tilde\gamma)\nabla(\nabla\cdot\theta) + \tilde\gamma\Delta\theta & \text{on } (0,T)\times\mathbb{T}^n, \qquad \text{(2-1c)}\\
\partial_t K + u\cdot\nabla K = [\Omega_{\text{eq}} + \Theta, J_{\text{eq}} + K]. & \text{on } (0,T)\times\mathbb{T}^n, \qquad \text{(2-1d)}
\end{cases}
$$

where recall that $a = (K_{12}, K_{13})$. This is the system that will be studied in this paper, and there are two important remarks to make about (2-1).

The first remark is that the pressure plays a different role from the other unknowns since it is essentially the Lagrange multiplier associated with the incompressibility constraint. We will therefore remove the pressure from consideration by the usual trick of projecting (2-1a) onto the space of divergence-free vector fields. This is done using the Leray projection $\mathbb{P}_L$, which on the three-dimensional torus takes the simple form $\mathbb{P}_L = -\mathrm{curl} \circ \Delta^{-1} \circ \mathrm{curl}$. We may then deduce from (2-1a) that

$$\partial_t u + \mathbb{P}_L(u \cdot \nabla u) = \left(\mu + \tfrac{1}{2}\kappa\right)\Delta u + \kappa \nabla \times \theta. \tag{2-2}$$

This equation will often be useful, in particular when it comes to the local well-posedness theory where it is convenient to view (2-1b)–(2-1d) and (2-2) as an ODE.

The second remark builds off of the fact that, as hinted at in Section 1B above and as discussed in more detail in Section 2C below, a is a component of K which plays a particularly important role. It will therefore be crucial, when performing energy estimates, to read off from (2-1d) the equation governing the dynamics of a, namely

$$\partial_t a + u \cdot \nabla a = -(\nu - \lambda)\bar{\theta}^\perp + (\bar{K} - K_{33} I_2)\bar{\theta}^\perp + \frac{\tau}{2\kappa} a^\perp + \theta_3 a^\perp, \tag{2-3}$$

where

$$\bar{\theta} = (\theta_1, \theta_2), \qquad v^\perp = (-v_2, v_1) \ \text{ for any } \ v \in \mathbb{R}^2, \qquad \text{and} \qquad \bar{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}.$$

To conclude this section we note that it is often useful to consider an alternative formulation of (2-1) involving the stress tensor T and the couple-stress tensor M. Note that, just as the classical stress tensor T encodes a fluid's response to forces, the couple-stress tensor M encodes a micropolar fluid's response to torques acting on the microstructure. These tensors are given by

$$T = \mu \mathbb{D} u + \kappa \,\mathrm{ten}\left(\tfrac{1}{2}\nabla \times u - \omega\right) - pI \quad \text{and} \quad M = \alpha(\nabla \cdot \omega) I + \beta \mathbb{D}^0 \omega + \gamma \,\mathrm{ten}\, \nabla \times \omega. \tag{2-4}$$

Here $\mathbb{D}v$ denotes (twice) the symmetric part of the derivative of a vector field v, i.e., $\mathbb{D}v = \nabla v + \nabla v^T$, and $\mathbb{D}^0 v$ denotes its trace-free part, i.e., $\mathbb{D}^0 v = \mathbb{D}v - (2/n)(\nabla \cdot v) I$. We may then formulate (2-1) equivalently as

$$\begin{cases} \partial_t u + u \cdot \nabla u = (\nabla \cdot T)(u, p, \theta) & \text{on } (0, T) \times \mathbb{T}^n, \quad \text{(2-5a)} \\ \nabla \cdot u = 0 & \text{on } (0, T) \times \mathbb{T}^n, \quad \text{(2-5b)} \\ (J_{\mathrm{eq}} + K)(\partial_t \theta + u \cdot \nabla \theta) + (\omega_{\mathrm{eq}} + \theta) \times (J_{\mathrm{eq}} + K)(\omega_{\mathrm{eq}} + \theta) & \\ \qquad = 2\,\mathrm{vec}\, T(u, p, \theta) + (\nabla \cdot M)(\theta) & \text{on } (0, T) \times \mathbb{T}^n, \quad \text{(2-5c)} \\ \partial_t K + u \cdot \nabla K = [\Omega_{\mathrm{eq}} + \Theta, J_{\mathrm{eq}} + K]. & \text{on } (0, T) \times \mathbb{T}^n. \quad \text{(2-5d)} \end{cases}$$

This formulation is particularly convenient when it comes to identifying the energy-dissipation relation since it makes it clear which terms contribute to the energy and which contribute to the dissipation. To be precise, we see that the dissipation comes precisely from the stress and couple-stress tensors since

$$\int_{\mathbb{T}^3} T : (\Theta - \nabla u) + M : \nabla \theta = \int_{\mathbb{T}^3} \frac{\mu}{2} |\mathbb{D} u|^2 + 2\kappa \left| \frac{1}{2} \nabla \times u - \theta \right|^2 + \alpha |\nabla \cdot \theta|^2 + \frac{\beta}{2} |\mathbb{D}^0 \theta|^2 + 2\gamma |\nabla \times \omega|^2, \tag{2-6}$$

where the right-hand side denotes the dissipation $D(u, \theta)$. In particular note that the dissipation does not provide any control over the perturbative microinertia K.

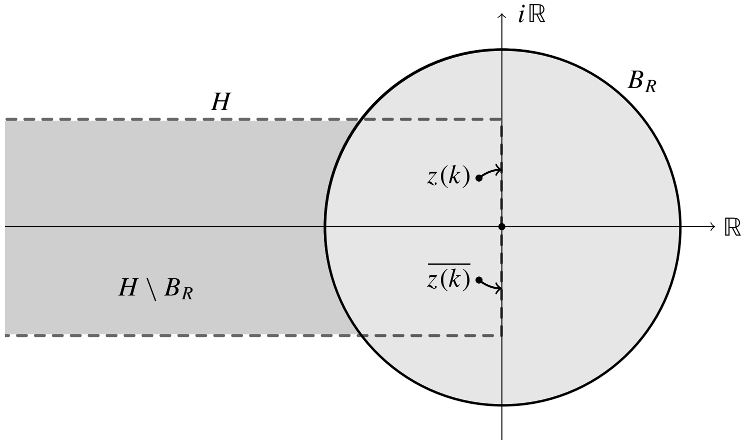

Figure 2. A pictorial summary of the argument carried out in Section 2B, which proves the lack of a spectral gap.

2B. ***Lack of spectral gap.*** In this section we describe the first of the two main obstructions in proving the asymptotic stability of the equilibrium, namely the *provable* absence of a spectral gap. To prove that the linearization of (2-1) about the equilibrium does not have a spectral gap we may leverage the careful spectral analysis carried out in the companion paper [Remond-Tiedrez and Tice 2021], where the instability of inertially oblong microstructure is established.

In order to describe this spectral analysis we must first recall the block structure of the linearization. Upon linearizing (2-1c)–(2-1d) and (2-2) and premultiplying the linearization of (2-1c) by J_{eq}^{-1}, we see that the resulting linear operator may be written in block form as $\mathcal{B} \oplus (\tau/(2\kappa))[R, \cdot\,] \oplus 0$. Here the first block acts on the variables (u, θ, a), the second block acts on $\overline{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$, and the third block acts on K_{33}. Crucially, the last two blocks have trivial dynamics since $(\tau/(2\kappa))[R, \cdot\,]$ is a three-dimensional antisymmetric operator which gives rise to one trivial mode and two conjugate oscillatory modes. The linear stability of (2-1c)–(2-1d) and (2-2) is therefore dictated by the spectrum of $\mathcal{B}$.

In practice we study the spectrum of the symbol $\hat{\mathcal{B}}(k)$, $k \in (2\pi\mathbb{Z})^3$, of $\mathcal{B}$. Note that the torus is rescaled for convenience since this particular scaling means that $\widehat{\nabla}(k) = ik$. A careful spectral analysis (the details of which can be found in [Remond-Tiedrez and Tice 2021]) then allows us to prove the following. For any $k \in (2\pi\mathbb{Z})^3$, the spectrum of $\hat{\mathcal{B}}(k)$ is contained in the half-slab $H = \{z \in \mathbb{C} : \operatorname{Re} z \leqslant 0 \text{ and } |\operatorname{Im} z| \leqslant C\}$ for some constant $C > 0$. In particular we may find a radius $R > C$ and a cutoff $k_* > 0$ such that if $|k| > k_*$ then there are precisely three eigenvalues of $\hat{\mathcal{B}}(k)$ in $H \cap B_R(0)$: zero (which is associated with the incompressibility constraint) and a conjugate pair of eigenvalues $z(k)$ and $\bar{z}(k)$. Crucially, this pair of eigenvalues satisfies $\operatorname{Re} z(k) \to 0$ as $|k| \to \infty$. This analysis, summarized pictorially in Figure 2, proves that $\mathcal{B}$, and hence the linearization itself, does not admit a spectral gap.

At the nonlinear level, the manifestation of the lack of a spectral gap is a failure of coercivity, meaning that an estimate of the form $\mathcal{E} \leqslant \mathcal{D}$ is out of reach. To overcome this, we prove a θ-coercivity estimate, which takes the form $\mathcal{E} \lesssim \mathcal{D}^\theta$ for some $\theta \in (0, 1)$ and leads to algebraic decay. The implementation of this is discussed in more detail in Section 2D1 below.

2C. *Weak coupling.* In this section we discuss the second major obstruction to proving the stability of the equilibrium, namely the so-called *weak coupling* of the parabolic part of the problem to its hyperbolic part.

Recall that as discussed in Section 1B, the term "weak coupling" describes the fact that only *some* of the unknowns appear in the energy. The remaining unknowns are solely controlled by conservative hyperbolic-type estimates. As seen in (1-3), here the weak coupling manifests itself at the level of the perturbative microinertia K since only two of its components, denoted by $a = (K_{13}, K_{23})$, appear in the energy. The reason why only a appears in the energy is due to the precession term $\omega \times J\omega$ from (1-1c). More precisely, when writing the precession term in perturbative form, as is done in (2-1c), we notice the appearance of the term $\omega_{\text{eq}} \times J\omega_{\text{eq}} = (\tau/(2\kappa))^2(-a_2, a_1)$.

At first glance, this means that we only expect decay of a and not of the remaining components of K. This is how our scheme of a priori estimates is built (see Section 2D below for more details). This means that the nonlinear estimates are particularly delicate. Indeed, since some terms appearing in the nonlinear interactions are not assumed to decay — at any level of regularity — it follows that we must be very careful about playing off these nondecaying terms against terms that decay sufficiently fast.

That being said, it must also be noted that, independent of our scheme of a priori estimates, a minor miracle of linear algebra occurs. This allows us to prove a *quantitative rigidity* result: if $\|K\|_{L^\infty} \leqslant \nu - \lambda$, then $|K| \leqslant 2|a|$ pointwise. In particular, we may then deduce the L^2 decay of K and then bootstrap via interpolation to obtain the decay of norms of K at higher regularity. Note that this "minor miracle" is recorded in Proposition 7.7.

Crucially, since the decay of K can be recovered a posteriori via algebraic identities, we do not incorporate it into our scheme of a priori estimates. Indeed, doing so would not strengthen the final statement of the theorem, and while it would give us another lever to pull when performing nonlinear estimates, it would further complicate our scheme of a priori estimates since the numerology of the precise decay rates of K and $\partial_t K$ is not particularly pleasant.

2D. *A priori estimates.* In this section we discuss the a priori estimates, which are carried out in Section 4. As mentioned previously, a fundamental observation about the problem at hand is that it is of *mixed* type. On one hand, the equations driving the dynamics of the velocity u and the (perturbative) angular velocity θ are parabolic. On the other hand, the equation driving the dynamics of the (perturbative) microinertia K is hyperbolic. Having made this fundamental observation, the two questions we seek to answer are the following:

(1) What kind of decay does the linearized problem possess?

(2) How can we massage the nonlinear structure to push this decay through to the nonlinear problem?

We will address the decay of the linearized problem in Section 2D1 and turn our attention to the nonlinear effects in Section 2D2. Throughout this discussion, we will underscore how the four pieces of the a priori estimates, namely (a) closing the energy estimates at the low regularity, (b) closing the energy estimates at the high regularity, (c) deriving advection-rotation estimates for K, and (d) obtaining the decay of intermediate norms, are related to one another. This is also summarized pictorially in Figure 3.

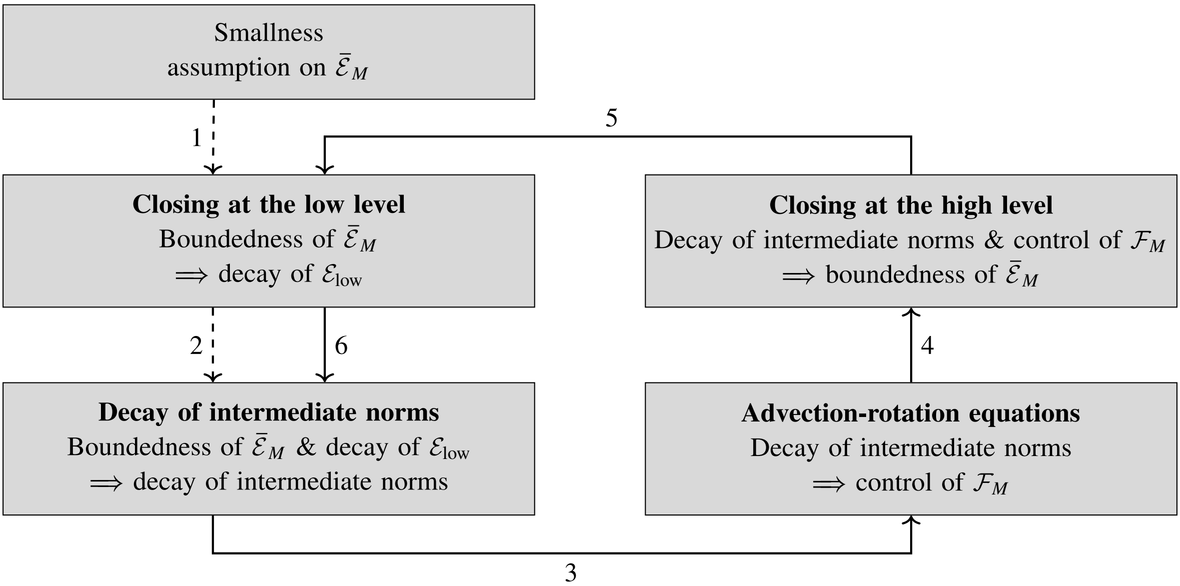

Figure 3. Pictorial summary of how the various pieces of the a priori estimates depend on one another. The arrows indicate the steps taken to close our scheme of a priori estimates — see Theorem 4.34 for details. In the first pass all estimates obtained are in terms of the smallness parameter; this is indicated by the dashed arrows. In the second pass all estimates obtained are in terms of the initial conditions; this is indicated by the solid arrows. Note that the seventh step of Theorem 4.34 is omitted here since it plays an essential role in the propagation of the estimates over time but is not essential for their closure.

Note that throughout this discussion we will use various versions of the energy and the dissipation. Their precise forms may be found in Section 3 below. At first pass, the following heuristics may be useful for the reader: $\bar{\mathcal{E}}$ and $\overline{\mathcal{D}}$ denote the energy and dissipation functionals that naturally arise when performing the energy estimates, whereas $\mathcal{E}$ and $\mathcal{D}$ denote *improved* versions of these functionals.

2D1. *Linear analysis.* θ-coercivity and the two-tier energy structure. We begin our discussion with the analysis of the linearized system and how it leads to almost-exponential decay. We will emphasize that it naturally gives rise to the aforementioned two-tier energy structure where the decay of the low-level energy is tied to the boundedness of the high-level energy.

The starting point is the energy-dissipation relation, which tells us that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathcal{E}}_{\text{low}} + \overline{\mathcal{D}}_{\text{low}} = 0. \tag{2-7}$$

Note that the mixed parabolic-hyperbolic structure already manifests itself here: the energy $\bar{\mathcal{E}}_{\text{low}}$ is a function of (u, θ, a), whereas the dissipation $\overline{\mathcal{D}}_{\text{low}}$ is only a function of (u, θ). To derive any decay estimate from (2-7) we need the dissipation to control the energy in some fashion, which at first glance is out of reach due to the absence of a in the dissipation.

Given the structure of the equations, this gap between the energy and the dissipation may not be fully closed. In particular, note that this energy-dissipation gap is consistent with the lack of spectral gap discussed in Section 2B. To partially close this energy-dissipation gap we *improve* the dissipation, i.e., leverage auxiliary estimates for a to see that

$$\overline{\mathcal{D}}_{\text{low}}(u,\theta) \gtrsim \mathcal{D}_{\text{low}}(u,\theta,a).$$

With this improvement in hand there is hope for the dissipation to control the energy. More precisely, what we can show is that

$$\bar{\mathcal{E}}_{\text{low}} \lesssim \bar{\mathcal{E}}_M^{1-\theta}\mathcal{D}_{\text{low}}^{\theta} \quad \text{for } \theta = \frac{2M-2}{2M-1} \in (0,1), \tag{2-8}$$

where $\bar{\mathcal{E}}_M$ is a high-regularity counterpart to the low-regularity energy $\bar{\mathcal{E}}_{\text{low}}$. When (2-8) holds we say that the dissipation is θ*-coercive* over the energy.

Crucially, this θ-coercivity estimate is only useful if we know that the high-regularity energy $\bar{\mathcal{E}}_M$ remains bounded. Thankfully this is immediate from the high-regularity version of the energy-dissipation relation, which reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathcal{E}}_M + \overline{\mathcal{D}}_M = 0.$$

The nonnegativity of $\overline{\mathcal{D}}_M$ then tells us that $\bar{\mathcal{E}}_M(t) \leqslant \bar{\mathcal{E}}_M(0)$.

Combining the θ-coercivity estimate at low regularity, the boundedness of the high-regularity energy, and a nonlinear Gronwall argument allows us to deduce the decay of the low-regularity energy. Indeed, we have that, for some $C > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathcal{E}}_{\text{low}} + \frac{C\bar{\mathcal{E}}_{\text{low}}^{1/\theta}}{\bar{\mathcal{E}}_M(0)^{1/\theta-1}} \leqslant 0, \quad \text{and hence } \bar{\mathcal{E}}_{\text{low}}(t) \lesssim \frac{\bar{\mathcal{E}}_M(0)}{(1+t)^{2M-2}}. \tag{2-9}$$

It is important to make two remarks here. First we note that as shown above the almost-exponential decay is *not* a nonlinear effect. It is the best rate of decay we can expect given the structure of the linearized problem. Indeed, similar θ-coercivity estimates appear in the viscous surface wave problem, where algebraic decay rates of the form recorded above are known to be sharp (see [Tice and Zbarsky 2020]).

Second we recall that, as mentioned previously, the two-tier energy structure is significantly more intricate for the nonlinear problem since in that case the decay of $\bar{\mathcal{E}}_{\text{low}}$ and the boundedness of $\bar{\mathcal{E}}_M$ are *interdependent* on one another, whereas here in this linear setting only the decay of the low-regularity energy is predicated on the boundedness of the high-regularity energy.

2D2. *Nonlinear effects.* Decay of intermediate norms. We begin our discussion of the nonlinear effects with an description of how the decay of the low-level energy and the boundedness of the high-level energy lead to the (slower) decay of intermediate norms. While this is not, in essence, a nonlinear feature, it is crucial in order to wrest some of the nonlinear effects under control, as described further below in this section. We note that this interpolation argument is carried out precisely in Section 4E.

Interpolation theory tells us that if the low-regularity energy decays as in (2-9) and the high-regularity energy is bounded by its initial value, then

$$\overline{\mathcal{K}}_I(t) \lesssim \frac{\bar{\mathcal{E}}_M(0)}{(1+t)^{2M-2I}} \quad \text{for } 1 \leqslant I \leqslant M, \tag{2-10}$$

where $\overline{\mathcal{K}}_I = \|(u,\theta,a)(t)\|^2_{H^{2I}} + \|(\partial_t u, \partial_t \theta, \partial_t a)(t)\|^2_{H^{2I-2}}$ is the sum of the (squared) norms for which we expect decay. Note that since only (u,θ,a) and $\partial_t(u,\theta,a)$ appear in $\bar{\mathcal{E}}_{\text{low}}$, these are also the only terms appearing in $\overline{\mathcal{K}}_I$. In other words, (2-10) does not yield any decay information on higher temporal derivatives, a fact which will be important later.

Controlling K: advection-rotation estimates. We continue our discussion of the nonlinear effect with an explanation of why energy estimates are not sufficient to close the scheme of a priori estimates. Then we discuss how the advection-rotation estimates which give us control over K give rise to a dichotomy between "good" terms and "bad" terms. The advection-rotation estimates for K are carried out in Section 4A and culminate in Proposition 4.8.

Energy estimates are not sufficient to close the scheme of a priori estimates for a simple reason: they produce interactions which are out of control due to the absence of K in the dissipation. Indeed, suppose that instead of using the equation which governs the dynamics of a in the energy estimates we used the equation which governs the dynamics of the full perturbative microinertia K. Schematically, we could then obtain an energy-dissipation relation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathcal{E}}(u,\theta,K) + \overline{\mathcal{D}}(u,\theta) = \overline{\mathcal{I}}(u,\theta,K),$$

where $\overline{\mathcal{I}}$ denotes the interaction terms. However, as described in Section 2D1, even after improving the dissipation we can only wrest a under control, and not K. This is due to the fact that only a appears in the equation governing the dynamics of θ, and this is where our auxiliary estimates for a begin. Ultimately, as described previously, this is because K only appears in that equation through the precession term, and we have the identity $\omega_{\text{eq}} \times K\omega_{\text{eq}} = (\tau/(2\kappa))^2 \tilde{a}^{\perp}$.

This lack of dissipative control over K is fatal when it comes to gaining control over the interaction terms. More precisely, when taking α many derivatives we see that one of the interaction terms is

$$\int_{\mathbb{T}^3} \partial^\alpha([\Theta, K]) : \partial^\alpha K =: \mathcal{I}^\alpha. \tag{2-11}$$

We cannot hope to control this term, since we are after an estimate that would allow us to absorb the interaction term into the dissipation, provided the energy is small, i.e., an estimate of the form $|\mathcal{I}^\alpha| \lesssim \mathcal{E}^{1/2}\mathcal{D}$. In light of this inability to close the scheme of a priori estimates by solely relying on energy estimates, we turn our attention to the equation governing the dynamics of K. This is essentially a reminder that since the problem is of mixed parabolic-hyperbolic type, we cannot build a complete scheme of a priori estimates leveraging only the parabolic structure of the problem (i.e., the structure that gives rise to the energy-dissipation relation) and must also take into account the hyperbolic structure embedded in the equation governing the dynamics of K.

The equation satisfied by K is an advection-rotation equation since it involves both advective effects due to the velocity u and rotational effects due to the (perturbative) angular velocity θ. The fundamental observation is the following: if v is divergence-free, A is antisymmetric, and S is symmetric, then (provided all unknowns are sufficiently regular)

$$\int_{\mathbb{T}^3} (\partial_t + v\cdot\nabla - [A,\cdot\,])S : S = 0.$$

This leads to the following L^p estimate. If S solves $(\partial_t + v\cdot\nabla - [A,\cdot\,])S = F$ for some forcing F, then
$$\|S(t)\|_{L^p} \leqslant \|S(0)\|_{L^p} + \int_0^t \|\mathrm{Sym}(F)(s)\|_{L^p}\,ds.$$
This immediately grants us control over L^p norms of K. To gain control over K in H^k we must couple this L^p estimate with high-low estimates. In particular, provided that some low-regularity norms decay sufficiently fast (which, as discussed in Section 2D1, is expected), we may combine such high-low estimates with the L^p estimate above to deduce that
$$\|K(t)\|_{H^k} \lesssim \|K(0)\|_{H^k} + \int_0^t \|(u,\theta)(s)\|_{H^k}\,ds.$$

Crucially, there are only two ways in which we can control $\int_0^t \|(u,\theta)\|_{H^k}$: (1) through the boundedness of $\int_0^t \overline{\mathcal{D}}_M$ and (2) through the decay of the intermediate norms in $\overline{\mathcal{K}}_I$. Comparing (1) and (2), the following trade-off comes to light: (1) gives us control of K at a higher regularity, at the cost of an upper bound growing in time. Indeed, on one hand, it follows from (1) and the Cauchy–Schwarz inequality that, for all $k \leqslant 2M+1$,
$$\int_0^t \|(u,\theta)\|_{H^k} \leqslant t\left(\fint_0^t \|(u,\theta)\|_{H^k}^2\right)^{1/2} \leqslant (t\overline{\mathcal{D}}_M)^{1/2}.$$
On the other hand, combining (2) and the decay of the intermediate norms of (2-10) tells us that, for $k \leqslant 2M-3$,
$$\int_0^t \|(u,\theta)\|_{H^k} \lesssim \int_0^t \frac{\bar{\mathcal{E}}_M(0)}{(1+s)^{M-k/2}}\,ds \lesssim \bar{\mathcal{E}}_M(0).$$

Note that this trade-off is only at play when two or fewer temporal derivatives hit K. This is because control of time derivatives of K does not come from advection-rotation estimates. Instead, it comes from applying derivatives to the equation satisfied by K. For the sake of exposition let us discuss this process under the assumption that K solves the linearized equation
$$\partial_t K = [\Omega_{\mathrm{eq}}, K] + [\Theta, J_{\mathrm{eq}}].$$
To control $\partial_t^j K$ we apply $j-1$ temporal derivatives to the equation. The crux of the argument is this: since $\partial_t^{j-1}\theta$ is controlled through the high-regularity energy $\bar{\mathcal{E}}_M$ in the space $H^{2M-2j+2}$, we see that this derivative count is below $2M-3$, i.e., $2M-2j+2 \leqslant 2M-3$ precisely when $j \geqslant 3$. So indeed this trade-off only concerns the first two temporal derivatives of K.

The practical implication of this trade-off is the following dichotomy between "good" and "bad" terms. If we seek to control K or one of its time derivatives in H^k for $k \leqslant 2M-3$, then we are dealing with a "good" term which is bounded in time. If we seek to control K, $\partial_t K$, or $\partial_t^2 K$ in H^k for $k > 2M-3$, then we are dealing with a "bad" term for which the only bound we have is growing in time.

Note that this distinction is by no means purely academic: nonlinear interaction terms appear that require us to control K (and its temporal derivatives) at high regularity, and for example it is critical to be able to control K in H^{2M+1} due to interactions of the form $\int_{\mathbb{T}^3} (\partial^\alpha K)\theta\cdot\partial^\alpha a$ when $|\alpha| = 2M$. Since we seek an upper bound of the form $\mathcal{E}^{1/2}\mathcal{D}$ even though K is not in the dissipation and a is only controlled dissipatively up to H^{2M-1} (this is precisely the manifestation of the lack of coercivity), we must integrate by parts, which requires control of K in H^{2M+1}.

As a concluding note regarding the advection-rotation estimates, it is essential to remember that this control of K is conditioned on the decay of intermediate norms. This is precisely what informs how the advection-rotation estimates fit in the overall scheme of a priori estimates.

Closing the energy estimates at the low level. We continue our discussion of the nonlinear effects and sketch how to close the energy estimates at the low level. The key observation here is that we may proceed as we did in the linear case (discussed in Section 2D1), with two differences. Note that the closure of the energy estimates at the low level is done in Proposition 4.20, which combines all the pieces from Section 4C.

The first difference is that the microinertia appears as a weight in the energy. This is readily addressed by the propagation in time of the spectrum of the microinertia since then $\int_{\mathbb{T}^3} J\theta \cdot \theta \asymp \int_{\mathbb{T}^3} |\theta|^2$. The second difference is that nonlinear interactions appear on the right-hand side of the energy-dissipation relation of (2-7). As was the case in the linear setting of Section 2D1, we leverage the boundedness of the high-level energy, which is used here to control these interactions. We may then deduce the decay of $\mathcal{E}_{\text{low}}$ as in (2-9).

Crucially, this decay is once again (as was the case in the linear analysis) predicated on the boundedness of the high-level energy $\bar{\mathcal{E}}_M$. However, by contrast with the linear case, it is very delicate to ensure that the high-level energy remains bounded in the nonlinear setting. This is discussed in detail below.

Closing the energy estimates at the high level. We near the end of our discussion of the a priori estimates and provide a sketch of how to close the energy estimates at the high level, noting in particular the difficulties that arise due to the presence of K, and describing how to overcome these challenges. This is carried out rigorously in Section 4D, leading up to the closure of the energy estimates at the high level in Proposition 4.29.

The fundamental principles used to close the energy estimates at the high level are the same as those used to close the estimates at the low level: improve the dissipation and control the interactions. However, difficulties arise due to the presence of K and the fact that, as discussed above, the only control we have over K, $\partial_t K$, and $\partial_t^2 K$ at regularity above $2M-3$ is growing in time.

To be precise, let us write the energy-dissipation relation at the high level as

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathcal{E}}_M + \overline{\mathcal{D}}_M = \overline{\mathcal{I}}_M, \tag{2-12}$$

where $\overline{\mathcal{I}}_M$ denotes the interactions. Immediately, when improving the dissipation and controlling the interactions, "bad" terms from the advection-rotation estimates for K appear. Since the upper bound on these bad terms is growing in time, our only hope that their appearance does not break the scheme of a priori estimates is that they may be counterbalanced by terms which decay in time. The decay of intermediate norms therefore plays an essential role in the closure of the energy estimates at the high level. With this careful balancing act in mind, between "bad" terms involving K and terms decaying sufficiently fast, the estimates establishing the improvement of the dissipation $\overline{\mathcal{D}}_M$ and the control of the interactions $\overline{\mathcal{I}}_M$ can be shown to take the form

$$\mathcal{D}_M \lesssim \overline{\mathcal{D}}_M + \overline{\mathcal{K}}_2 \mathcal{F}_M \quad \text{and} \quad |\overline{\mathcal{I}}_M| \lesssim \mathcal{E}_M^{1/2} \mathcal{D}_M + \mathcal{K}_{\text{low}}^{1/2} \mathcal{F}_M^{1/2} \mathcal{D}_M^{1/2}, \tag{2-13}$$

where $\mathcal{K}_{\text{low}}$ contains all the terms whose decay is needed to counteract the potential growth of $\mathcal{F}_M$.

A particular subtlety, which is worth pointing out, arises when identifying $\mathcal{K}_{\text{low}}$. Indeed, it turns out that $\mathcal{K}_{\text{low}} = \overline{\mathcal{K}}_2 + \|\partial_t^2\theta\|_{L^2}^2$, where the appearance of $\partial_t^2\theta$ is ultimately due to the interaction of the commutator with $J\partial_t$. The first term, $\overline{\mathcal{K}}_2$, is immediately known to decay from the decay of intermediate norms discussed at the beginning of Section 2D2. The decay of $\partial_t^2\theta$ is not quite so immediate since the norms in $\overline{\mathcal{K}}_I$ only involve one temporal derivative. We are therefore required to perform an auxiliary estimate for $\partial_t^2\theta$, which hinges on the structure of the equation governing the dynamics of θ and the propagation in time of the spectrum of the microinertia J, to establish that $\partial_t^2\theta$ decays when $\overline{\mathcal{K}}_2$ decays.

Having established (2-13), the heuristic which guides our next step is, as discussed above, that the decay of $\mathcal{K}_{\text{low}}$ will balance out the potential growth of $\mathcal{F}_M$. It turns out that establishing such a result rigorously can only be carried out in a time-integrated fashion. We end up proving that

$$\int_0^t \overline{\mathcal{K}}_2\mathcal{F}_M \lesssim \alpha\left(1+\int_0^t \overline{\mathcal{D}}_M\right) \quad \text{and} \quad \int_0^t \mathcal{K}_{\text{low}}^{1/2}\mathcal{F}_M^{1/2}\mathcal{D}_M^{1/2} \lesssim \alpha\int_0^t \overline{\mathcal{D}}_M, \tag{2-14}$$

where $\alpha > 0$ — which depends on the initial conditions and the decay of intermediate norms — can be made small. Crucially, the estimates above are obtained by leveraging the control of $\mathcal{F}_M$ afforded to us by the advection-rotation estimates for K. This shows that closing the energy estimates at the high level is a delicate affair which relies directly on two of the other three pieces of our scheme of a priori estimates: the decay of intermediate norms and the advection-rotation estimates for K.

To conclude it suffices to combine (2-13) and (2-14) with the energy-dissipation relation (2-12), from which we deduce the boundedness of the high-level energy $\bar{\mathcal{E}}_M$.

Synthesis. We conclude our discussion of the a priori estimates with a brief note on how to put all the pieces together. Each of the four pieces of the a priori estimates, namely closing the energy estimates at the low regularity, closing the energy estimates at the high regularity, deriving advection-rotation estimates for K, and obtaining the decay of intermediate norms, depends on one or more of the other. A careful assembly is therefore required to ensure that the argument does not end up being circular. This is summarized pictorially in Figure 3 on page 53 and done carefully in Section 4F, culminating in the main a priori estimates result recorded in Theorem 4.34.

The key insight is to kick off the scheme of a priori estimates by assuming the smallness of the solution. From there we can take two passes at the estimates: in the first pass we use the smallness assumption to ensure that all the pieces of our scheme of a priori estimates are in play, and in the second pass we obtain structured estimates where the smallness parameter disappears from the estimates and all the estimates obtained are in terms of the initial data.

2E. ***Local well-posedness.*** In this section we discuss the local well-posedness. In a nutshell, the key question is, how much of the nonlinear structure do we keep in order to be able to obtain good estimates on the sequence of approximate solutions? The local-posedness theory is developed in Section 5, whose main take-away is Theorem 5.24.

Strategy. We will produce solutions locally in time via a Galerkin scheme. We will (1) solve a sequence of approximate problems on finite-dimensional subspaces of the solution space, (2) obtain uniform estimates on the sequence of approximate solutions, and (3) pass to the limit by compactness. Since the domain is the torus, it is natural to approximate by cutting off at the first n Fourier modes. More precisely, writing $W_n \subseteq L^2$ such that $f \in W_n$ if and only if $\hat{f}(k) = 0$ for all $|k| > n$, we are looking to solve the approximate problem

$$\widetilde{T}_n(K)\partial_t Z = LZ + P_n\mathcal{N}(Z) \quad \text{for } Z = (u, \theta, K) \in W_n, \tag{2-15}$$

where L is a linear operator with constant coefficients, $\mathcal{N}$ accounts for the nonlinearities, and $\widetilde{T}_n(K)$ is an appropriate approximation of $I_3 \oplus (J_{\text{eq}} + K) \oplus I_{3\times 3}$, namely $\widetilde{T}_n(K) := I_3 \oplus (J_{\text{eq}} + P_n \circ K) \oplus I_{3\times 3}$ where $(P_n \circ K)\theta := P_n(K\theta)$ for every $\theta \in L^2$, for P_n denoting the L^2-orthogonal projection onto W_n.

<u>A subtle point</u>. Due to the presence of $\widetilde{T}_n(K)$ we will need to invert $J_{\text{eq}} + P_n \circ K$. Whilst fairly straightforward to do, this must nonetheless be done carefully since we are no longer merely inverting the matrix $J_{\text{eq}} + K$ pointwise, but rather we are inverting the *operator* $J_{\text{eq}} + P_n \circ K$ as an operator from W_n to itself. The corresponding results are recorded in Section 5A, where we also obtain H^k-to-H^k bounds on $\widetilde{T}_n(K)^{-1}$.

<u>Nonlinear structure</u>. Constructing a sequence of approximate solutions solving (2-15) is easy; however, we run into trouble when looking for estimates of the approximate solutions. The issue is that in (2-15) we have stripped away the nonlinear structure of the problem which helps us by providing good energy estimates.

To make this idea precise let us compare the two systems below. Both systems are cartoon versions of (2-15), where we neglect the velocity u, dismiss the dissipative contributions, and omit the projection P_n. Note that we write $J = J_{\text{eq}} + K$ and $\omega = \omega_{\text{eq}} + \theta$. We consider

$$(1)\ \begin{cases} J\partial_t\theta = f_1, \\ \partial_t K = F_2, \end{cases} \qquad \text{and} \qquad (2)\ \begin{cases} (J(\partial_t + u\cdot\nabla) + \omega\times J)\partial_t\theta = f_1, \\ (\partial_t + u\cdot\nabla)K = [\Omega, J]. \end{cases} \tag{2-16}$$

The energy associated with both systems is

$$\frac{1}{2}\int_{\mathbb{T}^3} J\theta\cdot\theta + \frac{1}{2}\int_{\mathbb{T}^3} |K|^2,$$

however, the interaction terms differ. To be precise, the issue is this: when taking α many derivatives the first system gives rise to an interaction of the form $I^\alpha = \int_{\mathbb{T}^3} \partial^\alpha([\Theta, K])\partial^\alpha\theta\cdot\partial^\alpha\theta$. However, this only grants us control of θ and K in $H^{|\alpha|}$, which is not sufficient to control I^α. Crucially, this interaction is not present when performing energy estimates with the second system. The moral of the story is that some nonlinear structure is optional while some is not. In particular, it is essential to keep the full nonlinear advection-rotation equation satisfied by K.

<u>A final wrinkle</u>. In the cartoon (2-16) above we brazenly dismissed any mention of the projection P_n. Of course, since we seek to frame the approximate problem as an ODE on the finite-dimensional space where only finitely many Fourier modes are nonzero and since that space is not closed under multiplication, the nonlinearities of (2-16) will require the presence of projections. However, this must be done carefully. Due to the fact that some nonlinear structure must be kept in the approximate problem (as discussed above), it turns out that we need to approximate K by using (schematically) twice as many Fourier modes as are used for the velocities u and θ.

Discrepancy in the energies. It is important here to note that the energies of the local well-posedness differ from those of the main scheme of a priori estimates. Schematically, these energies are of the form

$$E_{\mathrm{loc}} \sim \|(u,\theta,K)\|_{H^{2M}}^2 \quad \text{and} \quad E_{\mathrm{ap}} \sim \|(u,\theta,a)\|_{H^{2M}}^2 + \|K\|_{H^{2M-3}}^2, \tag{2-17}$$

respectively, where for simplicity we have omitted norms involving temporal derivatives.

In order to explain this discrepancy recall that, as discussed in Section 2D2 above, energy estimates are not sufficient to close the a priori estimates due to the absence of K from the dissipation and the appearance of interactions terms as in (2-11). We are thus led to employ advection-rotation estimates to control K in the scheme of a priori estimates, which means that we control K in H^{2M-3} and not H^{2M} (as would be the case when employing energy estimates).

However, when it comes to the local well-posedness theory, the interaction I^α of (2-11) is harmless since it can be estimated as $|I^\alpha| \lesssim E_{\mathrm{loc}}^3$. Such an estimate would be fatal for the a priori estimates since it cannot be absorbed into the dissipation but it is harmless locally in time since it is amenable to a nonlinear Gronwall argument.

The consequence of this discrepancy is that some additional work is required in order to ensure that the a priori estimates and the local well-posedness theory "glue" together nicely. This is discussed next in Section 2F.

2F. *Continuation argument.* In this section we discuss the continuation argument whose purpose is to allow us to glue together the a priori estimates and the local well-posedness theory. This gluing is nontrivial, in the sense that it requires a new set of estimates, precisely because of the mismatch between the energies used for the local well-posedness and the energies used for the a priori estimates (as discussed at the end of Section 2E above). The gluing is carried out in Section 6, where the key continuation argument it leads to is recorded in Theorem 6.13.

In order to justify the necessity of this additional set of estimates let us consider E_{ap} and E_{loc} defined as in (2-17) as cartoons of the energies used in the a priori estimates and in the local well-posedness theory, respectively. In particular note that for the sake of exposition we have omitted any mention of norms controlling temporal derivatives of the unknowns. Let us also consider the following cartoons of the a priori estimates and of the local well-posedness (which are simplified to the point of technical inaccuracy, but remain informative nonetheless)

$$\sup_{0\leqslant t\leqslant T} E_{\mathrm{ap}}(t) \leqslant \delta \quad\implies\quad \sup_{0\leqslant t\leqslant T} E_{\mathrm{ap}}(t) + \frac{\|K(t)\|_{H^{2M}}^2}{1+t} \leqslant C_1 E_{\mathrm{ap}}(0) \tag{AP}$$

$$\text{and} \quad \sup_{0\leqslant t\leqslant T} E_{\mathrm{loc}}(t) \leqslant \rho(E_{\mathrm{loc}}(0)), \tag{LWP}$$

where $\rho : (0,\infty) \to (0,\infty)$ is a strictly increasing function vanishing asymptotically at zero (whose appearance comes from the nonlinear Gronwall argument used in the local well-posedness theory). Note that here $\|K\|_{H^{2M}}^2$ is a placeholder for the "bad" terms comprising $\mathcal{F}_M$ (whose appearance is discussed in detail in Section 2D2). To glue the a priori estimates and the local well-posedness theory it suffices to fulfill the following goal.

Goal: If (AP) holds on the time interval $[0, T]$, find a sufficiently small timescale $\tau > 0$ such that (AP) continues to hold on the interval $[0, T+\tau]$.

Difficulty: For τ small enough the local well-posedness theory guarantees that we can always continue the solution from $[0, T]$ to $[0, T+\tau]$. The crux of the argument is therefore to ensure that the smallness hypothesis of (AP) remains satisfied on $[0, T+\tau]$. However, the growth of the bad term $\|K\|^2_{H^{2M}}$ in (AP) renders this impossible. Indeed, combining (AP) and (LWP) tells us that

$$\sup_{T\leqslant t\leqslant T+\tau} E_{\mathrm{ap}}(t) \leqslant \sup_{T\leqslant t\leqslant T+\tau} C_2 E_{\mathrm{loc}}(t) \leqslant C_2\rho(E_{\mathrm{loc}}(T)) \leqslant C_2\rho(C_3(1+T)E_{\mathrm{ap}}(0)),$$

and we cannot guarantee that the right-hand side be small independently of the time horizon T.

Solution: The remedy is to prove an estimate of the form

$$\sup_{T\leqslant t\leqslant T+\tau} E_{\mathrm{ap}}(t) \leqslant \tilde{\rho}(E_{\mathrm{ap}}(T)) \tag{E}$$

for $\tau > 0$ sufficiently small, where $\tilde{\rho}$ is another strictly increasing function which vanishes asymptotically at zero. We may then couple (E) to (AP) to deduce that

$$\sup_{0\leqslant t\leqslant T+\tau} E_{\mathrm{ap}}(t) \leqslant \tilde{\rho}(E_{\mathrm{ap}}(T)) \leqslant \tilde{\rho}(C_1 E_{\mathrm{ap}}(0)) \leqslant \delta,$$

provided the initial condition is sufficiently small. Note that this estimate is referred to in the sequel as a *reduced energy estimate* since it estimates the "reduced" unknown (u, θ, a), in contrast with the "full" unknown (u, θ, K). In practice, performing the estimate (E) relies on the same fundamental estimates as those used to prove (AP), with one fundamental difference: whereas (AP) relies on the smallness of the energy, (E) relies instead on the smallness of the timescale on which it holds.

2G. ***Global well-posedness and decay.*** In this section we discuss how to put together all the pieces of the puzzle to deduce the main result of Theorem 7.6. This is carried out in Section 7. In a nutshell, the local well-posedness developed in Section 5 couples to the a priori estimates of Section 4 to produce a solution which lives in the small energy regime, at which point the continuation argument recorded in Section 6 kicks in to tell us that the solution lives in the small energy regime globally in time.

The only subtlety in this process comes from coupling the local well-posedness theory to the a priori estimates. Indeed, the estimates provided by the local well-posedness theory are not quite strong enough to invoke the a priori estimates, due to insufficient control over K. To close that gap we rely on an auxiliary estimate for K, which is recorded in Lemma 7.1.

3. Notation

For the reader's convenience we record here the notation used in this paper.

Throughout, the unknown Z comprises all perturbative variables, i.e., $Z = (u, \theta, K)$, while $Y = (u, \theta, a)$ comprises all variables that are proved to decay.

The constant $\tilde{\tau}$ is defined to be $\tilde{\tau} = \tau/(2\kappa)$. It is omnipresent in the paper since it is equal to the magnitude of the angular velocity at equilibrium ω_{eq}.

Now we record some notation from linear algebra.

- For any vectors a and b, we denote by $a \otimes b$ the matrix acting via $(a \otimes b)v = (b \cdot v)a$ for any vector v.
- For any $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ and any $w \in \mathbb{R}^2$,

$$\bar{v} := (v_1, v_2), \quad \tilde{w} := (w_1, w_2, 0), \quad \bar{v}^\perp := (-v_2, v_1), \quad \text{and} \quad \tilde{w}^\perp := (-w_2, w_1, 0).$$

In other words: $v \mapsto \bar{v}$ is the projection onto the $e_1 - e_2$ plane, $w \mapsto \tilde{w}$ is its canonical right inverse, and the superscript $\perp$ denotes a $\frac{\pi}{2}$ (counterclockwise) rotation in the $e_1 - e_2$ plane.

- ten and vec denote the canonical identification of $\mathbb{R}^3$ with the space of antisymmetric 3×3 matrices using the cross product, and vice versa. To be precise, $(\operatorname{ten} a)v = a \times v$ and $(\operatorname{vec} A) \times v = Av$ for any vectors $a, v \in \mathbb{R}^3$ and any 3×3 antisymmetric matrix A.
- $\operatorname{Sym}(n)$ denotes the space of real symmetric $n \times n$ matrices. Moreover, for any matrix M, we denote by $\operatorname{Sym}(M)$ its symmetric part, i.e., $\operatorname{Sym}(M) := \frac{1}{2}(M + M^T)$.
- Given two linear maps $L_1 : V_1 \to W_1$ and $L_2 : V_2 \to W_2$, the linear map $L_1 \oplus L_2 : V_1 \times V_2 \to W_1 \times W_2$ is defined as $(L_1 \oplus L_2)(v_1, v_2) := (L_1 v_1, L_2 v_2)$ for every $v_1 \in V_1$ and $v_2 \in V_2$.

We now record the various functionals present throughout the paper. In order to do so, we first introduce parabolic norms.

- For any multi-index $\alpha \in \mathbb{N}^{1+3}$ we define the *parabolic count of derivatives* $|\alpha|_P$ to be $|\alpha|_P = 2\alpha_0 + |\bar{\alpha}|$, where we have written $\alpha = (\alpha_0, \bar{\alpha}) \in \mathbb{N} \times \mathbb{N}^3$.
- For $i, j, k \in \mathbb{N}$ satisfying $0 \leqslant i \leqslant j \leqslant \frac{1}{2}k$, we define the parabolic norms

$$\|f\|_{P^k}^2 = \sum_{|\alpha|_P \leqslant k} \|\partial^\alpha f\|_{L^2}^2, \quad \|f\|_{P^k_j}^2 = \sum_{\substack{|\alpha|_P \leqslant k \\ \alpha_0 \leqslant j}} \|\partial^\alpha f\|_{L^2}^2, \quad \text{and} \quad \|f\|_{P^k_{i,j}}^2 = \sum_{\substack{|\alpha|_P \leqslant k \\ i \leqslant \alpha_0 \leqslant j}} \|\partial^\alpha f\|_{L^2}^2. \tag{3-1}$$

First we record some energy and energy-like functionals. For any nonnegative integer M we define

$$\tilde{\mathcal{E}}_M = \sum_{|\alpha|_P \leqslant 2M} \frac{1}{2}\int_{\mathbb{T}^3} |\partial^\alpha u|^2 + \frac{1}{2}\int_{\mathbb{T}^3} J \partial^\alpha \theta \cdot \partial^\alpha \theta + \frac{\tilde{\tau}^2}{\nu - \lambda}\frac{1}{2}\int_{\mathbb{T}^3} |\partial^\alpha a|^2 \quad \text{and} \quad \bar{\mathcal{E}}_M = \|(u, \theta, a)\|_{P^{2M}}^2. \tag{3-2}$$

In particular, when $M = 1$ we define

$$\tilde{\mathcal{E}}_{\text{low}} = \tilde{\mathcal{E}}_1, \quad \bar{\mathcal{E}}_{\text{low}} = \bar{\mathcal{E}}_1, \quad \text{and} \quad \mathcal{E}_{\text{low}} = \bar{\mathcal{E}}_{\text{low}} + \|\partial_t a\|_{H^1}^2 + \|\partial_t^2 a\|_{L^2}^2. \tag{3-3}$$

When $M \geqslant 3$ we define

$$\mathcal{E}_M^{(K)} = \|K\|_{H^{2M-3}}^2 + \|\partial_t K\|_{H^{2M-3}}^2 + \|\partial_t^2 K\|_{H^{2M-3}}^2 + \sum_{j=3}^{M} \|\partial_t^j K\|_{H^{2M-2i+2}}^2, \tag{3-4}$$

$$\mathcal{E}_M = \bar{\mathcal{E}}_M + \mathcal{E}_M^{(K)}, \quad \text{and} \quad \mathcal{F}_M = \|K\|_{H^{2M+1}}^2 + \|\partial_t K\|_{H^{2M}}^2 + \|\partial_t^2 K\|_{H^{2M-2}}^2. \tag{3-5}$$

We also define the intermediate energy functionals

$$\overline{\mathcal{K}}_I = \|(u, \theta, a)\|_{P_1^{2I}}^2 \quad \text{and} \quad \mathcal{K}_{\text{low}} = \overline{\mathcal{K}}_2 + \|\partial_t^2 \theta\|_{L^2}^2. \tag{3-6}$$

We now record the dissipation functionals. The dissipation is given by

$$D(u,\theta)=\int_{\mathbb{T}^3}\frac{\mu}{2}|\mathbb{D}u|^2+2\kappa\left|\frac{1}{2}\nabla\times u-\theta\right|^2+\alpha|\nabla\cdot\theta|^2+\frac{\beta}{2}|\mathbb{D}^0\theta|^2+2\gamma|\nabla\times\theta|^2, \tag{3-7}$$

and we define

$$\overline{\mathcal{D}}_M=\|(u,\theta)\|^2_{P^{2M+1}},\quad \mathcal{D}^a_M=\sum_{j=0}^{3}\|\partial_t^j a\|^2_{H^{2M-j-1}}+\sum_{j=4}^{M}\|\partial_t^j a\|^2_{H^{2M-2i+3}},\quad\text{and}\quad \mathcal{D}_M=\overline{\mathcal{D}}_M+\mathcal{D}^a_M. \tag{3-8}$$

When $M=1$ we also define

$$\overline{\mathcal{D}}_{\text{low}}=\overline{\mathcal{D}}_1\quad\text{and}\quad \mathcal{D}_{\text{low}}=\overline{\mathcal{D}}_{\text{low}}+\|a\|^2_{H^1}+\|\partial_t a\|^2_{L^2}. \tag{3-9}$$

Finally, we write the interaction terms as

$$\overline{\mathcal{I}}_I:=\sum_{|\alpha|_P\leqslant 2M}\mathcal{I}^\alpha\quad\text{and}\quad \overline{\mathcal{I}}_{\text{low}}:=\overline{\mathcal{I}}_1 \tag{3-10}$$

for $\mathcal{I}^\alpha$ as in Lemma 4.10.

4. A priori estimates

In this section we develop the scheme of a priori estimates central to the stability result proven in this paper. We begin with advection-rotation estimates for K in Section 4A and then turn our attention to energy estimates in Sections 4B–4D. More precisely, in Section 4B we identify the energy-dissipation structure of the problem and use it in Sections 4C and 4D to close the energy estimates at the low and high level, respectively. We then record in Section 4E the interpolation result giving us decay of intermediate norms provided both the low and high-level energies are controlled. We conclude this section by putting all the pieces of the scheme of a priori estimates together in Section 4F.

4A. ***Advection-rotation estimates for K.*** In this section we record the advection-rotation estimates we may derive for K based on the advection-rotation equation (2-1d). The culmination of this section is Proposition 4.8, which synthesizes the estimates obtained in this section. We begin with L^p estimates for the advection-rotation operator encountered in (2-1d) which are foundational for all other advection-rotation estimates obtained here.

Proposition 4.1 (L^p estimates for advection-rotation equations). *Let $T>0$ be a finite time horizon, and let $1\leqslant p<\infty$. Let v be a continuously differentiable vector field on $[0,T)\times\mathbb{T}^n$, let M be a continuous matrix field on $[0,T)\times\mathbb{T}^n$, and let $F\in L^\infty([0,T);L^p(\mathbb{T}^n;\mathbb{R}^{n\times n}))$. If $S\in L^\infty([0,T);L^p(\mathbb{T}^n;\mathrm{Sym}(n)))$ is a distributional solution of*

$$(\partial_t+u\cdot\nabla-[M,\cdot])S=F\quad\textit{on }(0,T)\times\mathbb{T}^n\textit{ and }S(t=0)=S_0$$

for some $S_0\in L^p$, then it satisfies the estimate

$$\|S\|_{L^\infty_T L^p}\leqslant\exp\left(\int_0^t\frac{1}{p}\|(\nabla\cdot v)(s)\|_{L^\infty}\,ds\right)\|S_0\|_{L^p}+\int_0^t\exp\left(\int_0^s\frac{1}{p}\|(\nabla\cdot v)(r)\|_{L^\infty}\,dr\right)\|\mathrm{Sym}(F)(s)\|_{L^p}\,ds.$$

Proof. The fundamental idea behind this estimate is the following formal computation. First we compute, in light of Lemma A.5, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|S\|_{L^p} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{T}^n}|S|^p\right)^{1/p} = \frac{1}{p}\left(\int_{\mathbb{T}^n}|S|^p\right)^{1/p-1}\left(\int_{\mathbb{T}^n} p|S|^{p-2}S : F - \int_{\mathbb{T}^n} p|S|^{p-2}S : (u\cdot\nabla)S\right).$$

Now observe that, on one hand,

$$-p\int_{\mathbb{T}^n}|S|^{p-2}S : (u\cdot\nabla)S = -\int_{\mathbb{T}^n}(u\cdot\nabla)|S|^p = \int_{\mathbb{T}^n}(\nabla\cdot u)|S|^p,$$

whilst on the other hand, since $p'(1-p) = p$,

$$\int_{\mathbb{T}^n}|S|^{p-2}S : F \leqslant \int_{\mathbb{T}^n}|S|^{p-1}|\mathrm{Sym}(F)| \leqslant \left(\int_{\mathbb{T}^n}|S|^p\right)^{1/p'}\left(\int_{\mathbb{T}^n}|\mathrm{Sym}(F)|^p\right)^{1/p}.$$

So finally we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|S\|_{L^p} \leqslant \frac{1}{p}\left(\int_{\mathbb{T}^n}|S|^p\right)^{1/p-1}\|\nabla\cdot u\|_{L^\infty}\left(\int_{\mathbb{T}^n}|S|^p\right) + \left(\int_{\mathbb{T}^n}|S|^p\right)^{1/p-1}\left(\int_{\mathbb{T}^n}|S|^p\right)^{1/p'}\|\mathrm{Sym}(F)\|_{L^p},$$

from which the claim would follow upon performing a Gronwall argument.

To make this computation precise it suffices to use standard approximation techniques from the theory of L^p estimates for transport equations. For example, we may approximate $s \mapsto |s|^p$ by nonnegative C^1 functions in a monotone fashion and approximate S_0 and F by continuously differentiable functions. The computation above then holds rigorously at the level of the approximation, and we may pass to the limit using standard tools from measure theory. □

With the L^p estimates above in hand we derive L^∞ bounds on both K and ∇K. These bounds are used to control low-order terms appearing later in this section when we seek to parlay the L^p estimates above into H^k estimates for K.

Lemma 4.2 (L^∞ estimate for K). *Suppose that K solves (2-1d) for some given u and θ. Then it satisfies the estimate*

$$\|K(t)\|_{L^\infty} \lesssim \|K(0)\|_{L^\infty} + \int_0^t \|\bar{\theta}(s)\|_{L^\infty}\,ds.$$

Proof. Since $[\Omega_{\mathrm{eq}}, J_{\mathrm{eq}}] = 0$ (see Lemma A.6), we write (2-1d) as $\partial_t K + u\cdot\nabla K = [\Omega_{\mathrm{eq}} + \Theta, K] + [\Theta, J_{\mathrm{eq}}]$. It then follows from Proposition 4.1 that, for any $1 < p < \infty$,

$$\|K(t)\|_{L^p} \leqslant \|K(0)\|_{L^p} + \int_0^t \|[\Theta(s), J_{\mathrm{eq}}]\|_{L^p}\,ds.$$

Note that Lemma A.6 tells us that

$$[\Theta, J_{\mathrm{eq}}] = -(\nu-\lambda)\begin{pmatrix} 0 & \bar{\theta}^{\perp} \\ (\bar{\theta}^{\perp})^T & 0 \end{pmatrix},$$

from which we deduce that the Frobenius norm of this commutator is $|[\Theta, J_{\mathrm{eq}}]| = \sqrt{2(\nu-\lambda)}|\bar{\theta}|$. Since $\|\cdot\|_{L^p} \leqslant \|\cdot\|_{L^\infty}$, we may conclude that

$$\|K(t)\|_{L^p} \leqslant \|K(0)\|_{L^p} + \int_0^t \sqrt{2(\nu-\lambda)}\|\bar{\theta}(s)\|_{L^\infty}\,ds \lesssim \|K(0)\|_{L^p} + \int_0^t \|\bar{\theta}(s)\|_{L^\infty}\,ds.$$

The claim holds upon taking $p \to \infty$. □

Lemma 4.3 (L^∞ estimate for ∇K). *Suppose that K solves (2-1d) for some given u and θ. Then ∇K satisfies the estimate*

$$\|\nabla K(t)\|_{L^\infty} \lesssim \exp\left(\int_0^t \|\nabla u(r)\|_{L^\infty}\, dr\right)\left(\|\nabla K(0)\|_{L^\infty} + \int_0^t (1+\|K(s)\|_{L^\infty})\|\nabla\theta(s)\|_{L^\infty}\, ds\right).$$

Proof. Since K solves (2-1d), we see that $\partial_i K$ solves

$$(\partial_t + u\cdot\nabla - [\Omega_{\mathrm{eq}}+\Theta,\cdot\,])\partial_i K = [\partial_i\Theta, J_{\mathrm{eq}}+K] - \partial_i u\cdot\nabla K.$$

We note that the L^p norm of the right-hand side can be estimated in the following way:

$$\|(\mathrm{RHS})\|_{L^p} \leqslant \|[\partial_i\Theta, J]\|_{L^p} + \|\partial_i u\cdot\nabla K\|_{L^p} \lesssim \|\nabla\theta\|_{L^p}(1+\|K\|_{L^\infty}) + \|\nabla u\|_{L^\infty}\|\nabla K\|_{L^p}.$$

Proposition 4.1 therefore tells us that

$$\|\nabla K(t)\|_{L^p} \lesssim \exp\left(\int_0^t \|\nabla u(r)\|_{L^\infty}\, dr\right)\left(\|\nabla K(0)\|_{L^p} + \int_0^t (1+\|K(s)\|_{L^\infty})\|\nabla\theta(s)\|_{L^p}\, ds\right),$$

from which the result follows upon first recalling that $\|\cdot\|_{L^p} \leqslant \|\cdot\|_{L^\infty}$ and then taking $p\to\infty$. □

We now move towards estimates of K and its time derivatives in H^k. We begin with estimating K.

Lemma 4.4 (H^k estimate for K). *Suppose that K solves (2-1d) for some given u and θ. Then, for any $k\in\mathbb{N}$, it satisfies the estimate*

$$\|K(t)\|_{H^k} \lesssim \exp\left(\int_0^t \|\nabla u\|_{L^\infty} + \|\theta\|_{L^\infty}\right)\left(\|K(0)\|_{H^k} + \int_0^t (1+\|K\|_{L^\infty}+\|\nabla K\|_{L^\infty})\|(u,\theta)\|_{H^k}\right).$$

Proof. Since K solves (2-1d), we know, for any multi-index α with length $|\alpha|=k$, that $\partial^\alpha K$ solves

$$(\partial_t + u\cdot\nabla - [\Omega_{\mathrm{eq}}+\Theta,\cdot\,])\partial^\alpha K = [\partial^\alpha\Theta, J_{\mathrm{eq}}] + [u\cdot\nabla, \partial^\alpha]K - [[\Theta,\cdot\,],\partial^\alpha]K.$$

Applying Lemmas A.6 and B.3 then tells us that the right-hand side may be estimated as follows:

$$\begin{aligned}\|(RHS)\|_{L^2} &\leqslant \|[\partial^\alpha\Theta, J_{\mathrm{eq}}]\|_{L^2} + \|[u\cdot\nabla,\partial^\alpha]K\|_{L^2} + \|[[\Theta,\cdot\,],\partial^\alpha]K\|_{L^2}\\ &\lesssim \|\partial^\alpha\bar\theta\|_{L^2} + (\|\nabla u\|_{L^\infty}+\|\theta\|_{L^\infty})\|K\|_{H^k} + (\|K\|_{L^\infty}+\|\nabla K\|_{L^\infty})(\|u\|_{H^k}+\|\theta\|_{H^k}).\end{aligned}$$

Summing over $|\alpha|$ and appealing to Proposition 4.1 then yields the claim. □

Once K is under control we can read off estimates on $\partial_t K$ from (2-1d). The resulting estimate is recorded below.

Lemma 4.5 (H^k estimates for $\partial_t K$). *Suppose that K solves (2-1d) for some given u and θ. Then, for any $k\in\mathbb{N}$, we have that $\partial_t K$ satisfies the estimate*

$$\|\partial_t K\|_{H^k} \lesssim \|K\|_{H^k} + (\|u\|_{L^\infty}+\|\theta\|_{L^\infty})\|K\|_{H^{k+1}} + (1+\|K\|_{L^\infty}+\|\nabla K\|_{L^\infty})(\|u\|_{H^k}+\|\theta\|_{H^k}).$$

Proof. This follows immediately from using the high-low estimates of Corollary B.2 to estimate the quadratic terms in (2-1d). □

We continue establishing estimates on K and its time derivatives by taking a time derivative of (2-1d) and thus reading off an estimate for $\partial_t^2 K$, which is recorded below.

Lemma 4.6 (H^k estimates for $\partial_t^2 K$). *Suppose that K solves (2-1d) for some given u and θ. Then, for any $k \in \mathbb{N}$, we have that $\partial_t K$ satisfies the estimate*

$$\|\partial_t^2 K\|_{H^k} \lesssim \|\partial_t K\|_{H^k} + (1 + \|K\|_{L^\infty} + \|\nabla K\|_{L^\infty} + \|\partial_t K\|_{L^\infty} + \|\nabla \partial_t K\|_{L^\infty})(\|(u,\theta)\|_{H^k} + \|\partial_t(u,\theta)\|_{H^k}) + (\|(u,\theta)\|_{L^\infty} + \|\partial_t(u,\theta)\|_{L^\infty})(\|K\|_{H^{k+1}} + \|\partial_t K\|_{H^{k+1}}).$$

Proof. As in Lemma 4.5, this follows from the high-low estimates of Corollary B.2 upon noticing that $\partial_t^2 K$ solves

$$\partial_t^2 K = [\Omega_{\text{eq}}, \partial_t K] + [\partial_t \Theta, J_{\text{eq}}] + [\partial_t \Theta, K] + [\Theta, \partial_t K] - \partial_t u \cdot \nabla K - u \cdot \nabla \partial_t K. \qquad \square$$

We conclude our sequence of estimates on K and its temporal derivatives with an estimate on K when an arbitrary number of temporal derivatives are applied.

Lemma 4.7 (H^k estimates for $\partial_t^j K$). *Suppose that K solves (2-1d) for some given u and θ. Then, for any $k \in \mathbb{N}$ with $k > \frac{1}{2}n$ and any $j \geqslant 1$,*

$$\|\partial_t^j K\|_{H^k} \lesssim \|\partial_t^{j-1} K\|_{H^k} + \|\partial_t^{j-1}\theta\|_{H^k} + \sum_{l=0}^{j-1}(\|\partial_t^l u\|_{H^k}^2 + \|\partial_t^l \theta\|_{H^k}^2 + \|\partial_t^l K\|_{H^{k+1}}^2).$$

Proof. The proof is immediate upon noting that taking $j-1$ time derivatives of (2-1d) results in

$$\partial_t^j K = [\Omega_{\text{eq}}, \partial_t^{j-1} K] + [\partial_t^{j-1}\Theta, J_{\text{eq}}] + \sum_{l=0}^{j-1}([\partial_t^{(j-1)-l}\Theta, \partial_t^l K] - (\partial_t^{(j-1)-l} u \cdot \nabla)\partial_t^l K)$$

and recalling that H^k is a Banach algebra precisely when $k > \frac{1}{2}n$. $\square$

Having obtained estimates for K and its time derivatives above, we may now synthesize the results of this section in Proposition 4.8. Note that, as discussed in Section 2D, this proposition is one of the four building blocks of the scheme of a priori estimates. Recall that the functionals $\bar{\mathcal{E}}_M$, $\mathcal{E}_M^{(K)}$, $\mathcal{F}_M$, $\overline{\mathcal{K}}_I$, and $\overline{\mathcal{D}}_M$ are defined in (3-2), (3-4), (3-5), (3-6), and (3-8), respectively.

Proposition 4.8 (advection-rotation estimates for K). *Let $M \geqslant 3$ be an integer and suppose that, for some time horizon $T > 0$ and some universal constant $\overline{C} > 0$,*

$$\sup_{1 \leqslant I \leqslant M} \sup_{0 \leqslant t \leqslant T} \overline{\mathcal{K}}_I(t)(1+t)^{2M-2I} + \bar{\mathcal{E}}_M(t) + \int_0^t \overline{\mathcal{D}}_M(s)\,ds =: C_0 \leqslant \overline{C} < \infty. \tag{4-1}$$

Then, for every $0 \leqslant t \leqslant T$,

$$\mathcal{E}_M^{(K)}(t)^{1/2} \lesssim P_f \quad \text{and} \quad \mathcal{F}_M^{1/2}(t) \lesssim \mathcal{F}_M^{1/2}(0) + (1+P_e)\int_0^t \overline{\mathcal{D}}_M^{1/2}(s)\,ds + (1+P_f)\overline{\mathcal{K}}_M^{1/2}(t),$$

where the constants appearing in these two estimates depend on $\overline{C}$, and where

$$P_e := P(C_0^{1/2}, \mathcal{E}_M^{(K)}(0)^{1/2}) \quad \text{and} \quad P_f := P(C_0^{1/2}, \mathcal{E}_M^{(K)}(0)^{1/2}, \mathcal{F}_M(0)^{1/2})$$

for P — which may differ in each instance — a polynomial with nonnegative coefficients which vanishes at zero.

Proof. The strategy of the proof is as follows. The target estimates on $\mathcal{E}_M^{(K)}$ and $\mathcal{F}_M$ follow from putting together the advection-rotation estimates of Section 4A and appropriately leveraging the decay afforded to us by (4-1). The key idea is that the potential growth of terms controlled by $\mathcal{F}_M$ may be offset by the decay of terms appearing in $\overline{\mathcal{K}}_I$.

We begin by recording, in Step 1, elementary estimates which are consequences of (4-1) and can be used to control some of the time integrals appearing in the advection-rotation estimates of Section 4A. We then obtain L^∞ bounds on K and ∇K in Step 2 and deduce estimates on K, $\partial_t K$, $\partial_t^2 K$, and higher-order temporal derivatives in Steps 3–6. We conclude in Step 7 by recording how to perform the synthesis of Steps 1–6 and read off the desired estimates of $\mathcal{E}_M$ and $\mathcal{F}_M$.

Before we begin the proof in earnest, we fix some notation. For $x_1, \ldots, x_n \geqslant 0$, we denote by $P(x_1, \ldots, x_n)$ a polynomial of $(x_1, \ldots, x_n)$ which may change from line to line and has the following properties: it vanishes at zero and it has nonnegative coefficients. In particular, we write

$$P_e := P(C_0^{1/2}, \mathcal{E}_M^{(K)}(0)^{1/2}) \quad \text{and} \quad P_f := P(C_0^{1/2}, \mathcal{E}_M^{(K)}(0)^{1/2}, \mathcal{F}_M(0)^{1/2}).$$

Step 1: Preliminary estimates. We begin by recording some elementary estimates which are consequences of (4-1), such as estimates on time integrals of the functionals $\overline{\mathcal{K}}_I$. First, note that, for any $1 \leqslant I \leqslant M-2$,

$$\int_0^t \|(u,\theta)(s)\|_{H^{2I}}\, ds \lesssim \int_0^t \frac{C_0^{1/2}}{(1+s)^{M-I}}\, ds \lesssim C_0^{1/2}.$$

By interpolation, we note that similar estimates also hold for H^k norms of u and θ when k is odd. Indeed, observe first that, for any odd k satisfying $3 \leqslant k \leqslant 2M-1$, if we write $k = 2I+1$ for some $1 \leqslant I \leqslant M-1$ then we have the following bounds, pointwise-in-time:

$$\|(u,\theta)(t)\|_{H^k} \lesssim \|(u,\theta)(t)\|_{H^{k-1}}^{1/2} \|(u,\theta)(t)\|_{H^{k+1}}^{1/2} \lesssim \frac{C_0^{1/4}}{(1+t)^{(M-I)/2}} \frac{C_0^{1/4}}{(1+t)^{(M-I-1)/2}} = \frac{C_0^{1/2}}{(1+t)^{M-k/2}}.$$

Therefore, if $k \leqslant 2M-3$, we may deduce the time-integrated bound

$$\int_0^t \|(u,\theta)(s)\|_{H^k}\, ds \lesssim C_0^{1/2}. \tag{4-2}$$

Since the functionals $\overline{\mathcal{K}}_I$ which appear in (4-1) also involve temporal derivatives, we may proceed in the same way to deduce that, for any $2 \leqslant k \leqslant 2M-5$,

$$\int_0^t \|\partial_t(u,\theta)(s)\|_{H^k} \lesssim C_0^{1/2}. \tag{4-3}$$

The final preliminary estimate, before we being estimating K, has to do with exponential factors that arise in Lemmas 4.2 and 4.3. In light of (4-1) and (4-2) we see that, for some constants $C > 0$ which may change from line to line,

$$\exp\left(\int_0^t \|(\nabla u,\theta)(s)\|_{L^\infty}\, ds\right) \leqslant \exp\left(C\int_0^t \|(u,\theta)\|_{H^3}\right) \leqslant \exp(C_2 C_0^{1/2}) \lesssim 1, \tag{4-4}$$

where recall that, as in the statement of the proposition, the constants implied by the notation "$\lesssim$" may depend on $\overline{C}$.

Step 2: L^∞ estimates on K and ∇K. We are now ready to record the first estimates on K, which are L^∞ estimates on K and ∇K coming from Lemmas 4.2 and 4.3. We deduce from Lemma 4.2 the fact that $M \geqslant 3$ and from (4-1) that

$$\|K\|_{L^\infty} \lesssim \|K(0)\|_{H^2} + \int_0^t \|\theta\|_{H^2} \lesssim \mathcal{E}_M^{(K)}(0)^{1/2} + C_0^{1/2} \lesssim P_e. \tag{4-5}$$

Similarly, we deduce from Lemma 4.3 the fact that $M \geqslant 3$ and from (4-2), (4-4), and (4-5) that

$$\|\nabla K\|_{L^\infty} \lesssim \|K(0)\|_{H^3} + (1+P_e)\int_0^t \|\theta\|_{H^3} \lesssim \mathcal{E}_M^{(K)}(0)^{1/2} + (1+P_e)C_0^{1/2} \lesssim P_e. \tag{4-6}$$

Step 3: Estimating K. We are now ready to use Lemma 4.4 to estimate K. Combining Lemma 4.4 with (4-4)–(4-6) tells us that

$$\|K(t)\|_{H^{2M-3}} \lesssim \|K(0)\|_{H^{2M-3}} + (1+P_e)\int_0^t \|(u,\theta)\|_{H^{2M-3}}.$$

Using (4-2) allows us to conclude that

$$\|K\|_{H^{2M-3}} \lesssim \mathcal{E}_M^{(K)}(0)^{1/2} + (1+P_e)C_0^{1/2} \lesssim P_e. \tag{4-7}$$

Combining Lemma 4.4 with (4-4)–(4-6) also tells us that

$$\|K\|_{H^{2M+1}} \lesssim \|K(0)\|_{H^{2M+1}} + (1+P_e)\int_0^t \|(u,\theta)\|_{H^{2M+1}} \lesssim \mathcal{F}_M(0)^{1/2} + (1+P_e)\int_0^t \overline{\mathcal{D}}_M^{1/2}. \tag{4-8}$$

Step 4: Estimating $\partial_t K$. We now estimate $\partial_t K$ using Lemma 4.5. Combining Lemma 4.5 with (4-1) and (4-5)–(4-7) tells us that

$$\begin{aligned}\|\partial_t K\|_{H^{2M-3}} &\lesssim \|K\|_{H^{2M-3}} + \|(u,\theta)\|_{L^\infty}\|K\|_{H^{2M-2}} + (1+P_e)\|(u,\theta)\|_{H^{2M-3}} \\ &\lesssim P_e + \|(u,\theta)\|_{L^\infty}\|K\|_{H^{2M-2}}.\end{aligned} \tag{4-9}$$

The trick now lies in controlling the term $\|(u,\theta)\|_{L^\infty}\|K\|_{H^{2M-3}}$ by playing off the decay of $\|(u,\theta)\|_{L^\infty}$ against the (potential) growth of $\|K\|_{H^{2M-2}}$. Using (4-1) and (4-8) we see that

$$\|(u,\theta)\|_{L^\infty}\|K\|_{H^{2M-2}} \lesssim \frac{C_0^{1/2}}{(1+t)^{M-1}}\left(\mathcal{F}_M(0)^{1/2} + (1+P_e)\int_0^t \overline{\mathcal{D}}_M^{1/2}\right).$$

Note that by applying Cauchy–Schwarz to $\int \overline{\mathcal{D}}_M$, we see that, by virtue of (4-1),

$$\int_0^t \overline{\mathcal{D}}_M^{1/2} = t \fint_0^t \overline{\mathcal{D}}_M^{1/2} \leqslant t\left(\fint_0^t \overline{\mathcal{D}}_M\right)^{1/2} = t^{1/2}\left(\int_0^t \overline{\mathcal{D}}_M\right)^{1/2} \leqslant C_0^{1/2} t^{1/2}. \tag{4-10}$$

Therefore, since $M \geqslant 2$,

$$\|(u,\theta)\|_{L^\infty}\|K\|_{H^{2M-2}} \lesssim C_0^{1/2}(\mathcal{F}_M(0)^{1/2} + (1+P_e)C_0^{1/2})\frac{1+t^{1/2}}{(1+t)^{M-1}} \lesssim P_f \frac{(1+t)^{1/2}}{(1+t)^{M-1}} \lesssim P_f. \tag{4-11}$$

So finally, putting (4-9) and (4-11) together, we see that

$$\|\partial_t K\|_{H^{2M-3}} \lesssim P_f. \tag{4-12}$$

We now seek to control $\partial_t K$ in H^{2M}, i.e., through $\mathcal{F}_M$. This is slightly easier than controlling K in H^{2M-3} as is done above since now we do not have to deal with "decay-growth" interactions. Combining Lemma 4.5 with (4-1), (4-5), (4-6), and (4-8) shows that

$$\begin{aligned}\|\partial_t K\|_{H^{2M}} &\lesssim \|K\|_{H^{2M}} + \|(u,\theta)\|_{L^\infty}\|K\|_{H^{2M+1}} + (1+P_e)\|(u,\theta)\|_{H^{2M}} \\ &\lesssim (1+C_0^{1/2})\left(\mathcal{F}_M(0)^{1/2} + (1+P_e)\int_0^t \overline{\mathcal{D}}_M^{1/2}\right) + (1+P_e)C_0^{1/2} \\ &\lesssim \mathcal{F}_M^{1/2}(0) + (1+P_e)\int_0^t \overline{\mathcal{D}}_M^{1/2} + (1+P_e)\overline{\mathcal{K}}_M^{1/2}(t). \end{aligned} \tag{4-13}$$

Step 5: Estimating $\partial_t^2 K$. We now use Lemma 4.6 to control $\partial_t^2 K$. Lemma 4.6 tells us that

$$\begin{aligned}\|\partial_t^2 K\|_{H^{2M-3}} \lesssim \|\partial_t K\|_{H^{2M-3}} + (\|(u,\theta)\|_{L^\infty} + \|\partial_t(u,\theta)\|_{L^\infty})(\|K\|_{H^{2M-2}} + \|\partial_t K\|_{H^{2M-2}}) \\ + (1 + \|(K,\nabla K)\|_{L^\infty} + \|\partial_t(K,\nabla K)\|_{L^\infty})(\|(u,\theta)\|_{H^{2M-3}} + \|\partial_t(u,\theta)\|_{H^{2M-3}}).\end{aligned} \tag{4-14}$$

In particular, (4-12) allows us to control $\partial_t K$ and $\partial_t \nabla K$ in L^∞ since $M \geqslant 3$, and hence

$$\|\partial_t(K,\nabla K)\|_{L^\infty} \lesssim \|\partial_t K\|_{H^3} \lesssim \|\partial_t K\|_{H^{2M-3}} \lesssim P_f. \tag{4-15}$$

As in the estimate of $\partial_t K$ in H^{2M-3}, the subtlety now lies in estimating the decay-growth interaction. In light of (4-1), (4-8), (4-10), (4-13), and the fact that $M \geqslant 3$,

$$\begin{aligned}&(\|(u,\theta)\|_{L^\infty} + \|\partial_t(u,\theta)\|_{L^\infty})(\|K\|_{H^{2M-2}} + \|\partial_t K\|_{H^{2M-2}}) \\ &\qquad\lesssim \frac{C_0^{1/2}}{(1+t)^{M-2}}\left(P_f + (1+P_e)\int_0^t \overline{\mathcal{D}}_M^{1/2}\right) \lesssim C_0^{1/2}(1+P_f)C_0^{1/2}\frac{1+t^{1/2}}{(1+t)^{M-2}} \\ &\qquad\lesssim P_f \frac{(1+t)^{1/2}}{(1+t)^{M-2}} \lesssim P_f. \end{aligned} \tag{4-16}$$

So finally, combining (4-1), (4-5), (4-6), (4-12), (4-15), and (4-16) tells us that

$$\|\partial_t^2 K\|_{H^{2M-3}} \lesssim P_f. \tag{4-17}$$

We now seek to control $\partial_t^2 K$ in H^{2M-2}. We may put together Lemma 4.6, (4-1), (4-5), (4-6), (4-8), (4-13), and (4-15) to see that

$$\begin{aligned}\|\partial_t^2 K\|_{H^{2M-2}} &\lesssim \|\partial_t K\|_{H^{2M-2}} + (\|(u,\theta)\|_{L^\infty} + \|\partial_t(u,\theta)\|_{L^\infty})(\|K\|_{H^{2M-1}} + \|\partial_t K\|_{H^{2M-1}}) \\ &\qquad + (1 + \|(K,\nabla K)\|_{L^\infty} + \|\partial_t(K,\nabla K)\|_{L^\infty})(\|(u,\theta)\|_{H^{2M-2}} + \|\partial_t(u,\theta)\|_{H^{2M-2}}) \\ &\lesssim (1+C_0^{1/2})(\|K\|_{H^{2M-1}} + \|\partial_t K\|_{H^{2M-1}}) + (1+P_f)\overline{\mathcal{K}}_M^{1/2} \\ &\lesssim \mathcal{F}_M^{1/2}(0) + (1+P_e)\int_0^t \overline{\mathcal{D}}_M^{1/2} + (1+P_f)\overline{\mathcal{K}}_M^{1/2}. \end{aligned} \tag{4-18}$$

Step 6: Estimating $\partial_t^j K$ for $j \geqslant 3$. We conclude this proof by obtaining control over $\partial_t^j K$ when $j \geqslant 3$. We proceed by induction, relying on Lemma 4.7 for both the base case and the induction step, and we will show that

$$\|\partial_t^j K\|_{H^{2M-2j+2}} \lesssim P_f \quad \text{for every } 3 \leqslant j \leqslant M. \tag{4-19}$$

Note that the hypotheses of Lemma 4.7 are always satisfied here since $2M-2j+2 \geqslant 2 > \frac{3}{2}$ when $j \leqslant M$. We begin with the base case. By Lemma 4.7, (4-1), (4-7), (4-12), and (4-17) we obtain that

$$\begin{aligned}\|\partial_t^3 K\|_{H^{2M-4}} &\lesssim \|\partial_t^2 K\|_{H^{2M-4}} + \|\partial_t^2 \theta\|_{H^{2M-4}} + \sum_{l=0}^{2} (\|\partial_t^l (u,\theta)\|^2_{H^{2M-4}} + \|\partial_t^l (u,\theta)\|^2_{H^{2M-3}}) \\ &\lesssim P_f + C_0^{1/2} + (C_0 + P_f) \lesssim P_f.\end{aligned}$$

We may now proceed with the induction step. Suppose that there is some $3 \leqslant j < M$ such that

$$\|\partial_t^l K\|_{H^{2M-2l+2}} \lesssim P_f \quad \text{for every } 3 \leqslant l \leqslant j. \tag{4-20}$$

Then, by Lemma 4.7, (4-1), (4-7), (4-12), (4-17), and (4-20) we see that

$$\begin{aligned}\|\partial_t^{j+1} K\|_{H^{2M-2j}} &\lesssim \|\partial_t^j K\|_{H^{2M-2j}} + \|\partial_t^j \theta\|_{H^{2M-2j}} + \sum_{l=0}^{j} (\|\partial_t^l (u,\theta)\|^2_{H^{2M-2j}} + \|\partial_t^l K\|^2_{H^{2M-2j+1}}) \\ &\lesssim P_f + C_0^{1/2} + \|(u,\theta)\|^2_{P^{2M}} + P_f^2 \lesssim P_f.\end{aligned}$$

This proves that the induction step holds, from which (4-19) follows.

Step 7: Synthesis. We combine (4-7), (4-12), (4-17), and (4-19) to deduce the bound on $\mathcal{E}_M^{(K)}$, and we combine (4-8), (4-13), and (4-18) to deduce the bound on $\mathcal{F}_M$. □

4B. *Energy-dissipation structure.* In this section we identify the energy-dissipation structure of the problem and record some related auxiliary results, such as the precise form of the interactions, a comparison result for the various versions of the energy, and a coercivity estimate for the dissipation. Since the dissipation D will appear frequently throughout this section we recall that it is defined in (3-7). We begin with the energy-dissipation relation.

Proposition 4.9 (generic energy-dissipation relation). *Let the stress tensors T and M be as defined in (2-4) and suppose that (v, q, θ, b) solves*

$$\begin{cases} (\partial_t + u\cdot\nabla)v = (\nabla\cdot T)(v,q,\theta) + f, \\ \nabla\cdot v = 0, \\ J(\partial_t + u\cdot\nabla)\theta + (\omega\times J)\theta + \tilde{\tau}^2\tilde{b}^\perp + \theta\times J\omega_{\mathrm{eq}} = 2\operatorname{vec} T(v,q,\theta) + (\nabla\cdot M)(\theta) + g, \\ (\partial_t + u\cdot\nabla)b = -(\nu-\lambda)\bar{\theta}^\perp + \omega_3 b^\perp + h, \end{cases} \tag{4-21}$$

where (u, ω, J) are given and satisfy

$$\begin{cases} \nabla\cdot u = 0, & \text{(4-22a)} \\ (\partial_t + u\cdot\nabla)J = [\Omega, J], & \text{(4-22b)} \end{cases}$$

and where f, g, and h are given. Then the following energy-dissipation relation holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{T}^3} \frac{1}{2}|v|^2 + \frac{1}{2}J\theta\cdot\theta + \frac{\tilde{\tau}^2}{\nu-\lambda}\frac{1}{2}|b|^2\right) + D(v,\theta) = \int_{\mathbb{T}^3} f\cdot v + g\cdot\theta + \frac{\tilde{\tau}^2}{\nu-\lambda}h\cdot b. \tag{4-23}$$

Proof. We multiply by the unknowns and integrate by parts: since u and v are divergence-free,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}\frac{1}{2}|v|^2 = \int_{\mathbb{T}^3}(\partial_t + u\cdot\nabla)v\cdot v = \int_{\mathbb{T}^3}(\nabla\cdot T)\cdot v + \int_{\mathbb{T}^3} f\cdot v = -\int_{\mathbb{T}^3} T : \nabla v + \int_{\mathbb{T}^3} f\cdot v. \tag{4-24}$$

Similarly, using the incompressibility of u, (4-22b), and Lemma A.4, we see that

$$\tfrac{1}{2}(\partial_t + u\cdot\nabla)J\theta\cdot\theta = \tfrac{1}{2}[\Omega, J]\theta\cdot\theta = (\omega\times J)\theta\cdot\theta,$$

and from the fact that $\theta\times J\omega_{\mathrm{eq}}\cdot\theta = 0$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}\tfrac{1}{2}J\theta\cdot\theta = \int_{\mathbb{T}^3}\tfrac{1}{2}(\partial_t+u\cdot\nabla)J\theta\cdot\theta + \int_{\mathbb{T}^3}J(\partial_t+u\cdot\nabla)\theta\cdot\theta = \int_{\mathbb{T}^3}T:\Omega - M:\nabla\omega + g\cdot\theta + \tilde{\tau}^2 b\cdot\bar{\theta}^{\perp}. \tag{4-25}$$

Finally, we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}\tfrac{1}{2}|b|^2 = \int_{\mathbb{T}^3}(\partial_t + u\cdot\nabla)b\cdot b = -\int_{\mathbb{T}^3}(\nu-\lambda)\bar{\theta}^{\perp}\cdot b + \int_{\mathbb{T}^3}h\cdot b. \tag{4-26}$$

To conclude it suffices to add (4-24), (4-25), and $\tilde{\tau}^2/(\nu-\lambda)$ times (4-26) and observe that

$$\int_{\mathbb{T}^3}T(v,q,\theta):(\Theta-\nabla v) + M(\theta):(\nabla\theta) = D(v,\theta).$$

This equation follows from the identities $(\operatorname{vec} M)\cdot v = \frac{1}{2}M : \operatorname{ten} v$ and $\operatorname{Skew}(\nabla v) = \frac{1}{2}\operatorname{ten}\nabla\times v$ and the fact that $\mathbb{R}^{n\times n}$ may be orthogonally decomposed with respect to the Frobenius inner product as $\mathbb{R}^{n\times n}\cong\mathbb{R}I\oplus\operatorname{Dev}(n)\oplus\operatorname{Skew}(n)$, where $\operatorname{Dev}(n)$ denotes the set of trace-free symmetric $n\times n$ matrices and $\operatorname{Skew}(n)$ denotes the set of antisymmetric $n\times n$ matrices.

To conclude we add (4-24), (4-25), and $\tilde{\tau}^2/(\nu-\lambda)$ times (4-26) to obtain (4-23). □

Having established the precise form of the energy-dissipation relation we now record the specific form of the interactions. Lemma 4.10 is a necessary precursor to the interactions estimates of Sections 4C and 4D.

Lemma 4.10 (recording the form of the interactions). *If (u, p, θ, K) solves (2-1) then, for any multi-index $\alpha\in\mathbb{N}^{1+3}$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{T}^3}\frac{1}{2}|\partial^\alpha u|^2 + \frac{1}{2}J\partial^\alpha\theta\cdot\partial^\alpha\theta + \frac{\tilde{\tau}^2}{\nu-\lambda}\frac{1}{2}|\partial^\alpha a|^2\right) + D(\partial^\alpha u, \partial^\alpha\theta) = \mathcal{I}^\alpha, \tag{4-27}$$

where

$$\begin{aligned}\mathcal{I}^\alpha = \int_{\mathbb{T}^3}[u\cdot\nabla, \partial^\alpha]u\cdot\partial^\alpha u + \int_{\mathbb{T}^3}[J\partial_t, \partial^\alpha]\theta\cdot\partial^\alpha\theta + \int_{\mathbb{T}^3}[J(u\cdot\nabla), \partial^\alpha]\theta\cdot\partial^\alpha\theta \\ + \int_{\mathbb{T}^3}[\omega\times J, \partial^\alpha]\theta\cdot\partial^\alpha\theta - \int_{\mathbb{T}^3}[J\omega_{\mathrm{eq}}\times, \partial^\alpha]\theta\cdot\partial^\alpha\theta + \int_{\mathbb{T}^3}[u\cdot\nabla, \partial^\alpha]a\cdot\partial^\alpha a \\ + \int_{\mathbb{T}^3}[\omega_3 R, \partial^\alpha]a\cdot\partial^\alpha a + \int_{\mathbb{T}^3}\partial^\alpha((\bar{K}-K_{33}I_2)\bar{\theta}^{\perp})\cdot\partial^\alpha a =: \mathcal{I}_1^\alpha + \cdots + \mathcal{I}_8^\alpha.\end{aligned}$$

Proof. The first order of business is to write (2-1) in the form of Proposition 4.9. In order to do this we note that (2-1) can be written using the stress tensor T and the couple stress tensor M as (2-5). In light of (2-3) we therefore see that (u, p, θ, a) solves

$$\begin{cases}(\partial_t + u\cdot\nabla)u = (\nabla\cdot T)(u,p,\theta),\\ \nabla\cdot u = 0,\\ J(\partial_t+u\cdot\nabla)\theta + (\omega\times J)\theta + \tilde{\tau}^2\tilde{b}^{\perp} + \theta\times J\omega_{\mathrm{eq}} = 2\operatorname{vec}T(u,p,\theta) + (\nabla\cdot M)(\theta),\\ (\partial_t+u\cdot\nabla)a = -(\nu-\lambda)\bar{\theta}^{\perp} + \omega_3 a^{\perp} + (\bar{K}-K_{33}I_2)\bar{\theta}^{\perp},\end{cases} \tag{4-28}$$

subject to $(\partial_t + u\cdot\nabla)J = [\Omega, J]$, where $J = J_{\mathrm{eq}} + K$ and $\omega = \omega_{\mathrm{eq}} + \theta$. Note in particular that the full precession $\omega \times J\omega$ is present in the third equation of (4-28) since $\omega_{\mathrm{eq}} \times J_{\mathrm{eq}}\omega_{\mathrm{eq}} = 0$ and $\omega_{\mathrm{eq}} \times K\omega_{\mathrm{eq}} = \tilde{\tau}^2\tilde{a}^\perp$ such that indeed

$$\omega \times J\theta + \tilde{\tau}^2\tilde{b}^\perp + \theta \times J\omega_{\mathrm{eq}} = \omega \times J\omega.$$

We may now apply derivatives to (4-28) and use Proposition 4.9. This tells us

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{T}^3} \frac{1}{2}|\partial^\alpha u|^2 + \frac{1}{2}J\partial^\alpha\theta\cdot\partial^\alpha\theta + \frac{\tilde{\tau}^2}{\nu-\lambda}\frac{1}{2}|\partial^\alpha a|^2\right) + D(\partial^\alpha u, \partial^\alpha\theta) = \int_{\mathbb{T}^3} f^\alpha\cdot\partial^\alpha u + g^\alpha\cdot\partial^\alpha\theta + h^\alpha\cdot\partial^\alpha a \tag{4-29}$$

for

$$f^\alpha = [u\cdot\nabla, \partial^\alpha]u, \quad g^\alpha = [J\partial_t, \partial^\alpha]\theta + [J(u\cdot\nabla), \partial^\alpha]\theta + [\omega\times J, \partial^\alpha]\theta - [J\omega_{\mathrm{eq}}\times, \partial^\alpha]\theta,$$

and

$$h^\alpha = [u\cdot\nabla, \partial^\alpha]a + [\theta_3 R, \partial^\alpha]a + \partial^\alpha((\bar{K} - K_{33}I_2)\bar{\theta}^\perp),$$

where $R = e_2\otimes e_1 - e_1\otimes e_2 \in \mathbb{R}^{2\times 2}$ is the (counterclockwise) $\frac{\pi}{2}$ rotation in $\mathbb{R}^2$, such that (4-29) is precisely (4-27), as desired. □

We now record two auxiliary results related to the energy-dissipation structure of the problem. First we record a precise comparison of various versions of the energy, and then we record the coercivity of the dissipation over H^1. Recall that $\bar{\mathcal{E}}_M$ and $\tilde{\mathcal{E}}_M$ are defined in (3-2).

Lemma 4.11 (comparison of the different versions of the energy). *There exist constants $c_E, C_E > 0$ such that, for every time horizon $T > 0$, if*

$$\sup_{0\leqslant t<T} \|(u,\theta)(t)\|_{H^3} + \|J(t)\|_{H^3} + \|\partial_t(u,\theta)(t)\|_{H^2} + \|\partial_t J(t)\|_{H^2} < \infty \tag{4-30}$$

then, for any nonnegative integer M, we have that $c_E\bar{\mathcal{E}}_M \leqslant \tilde{\mathcal{E}}_M \leqslant C_E\bar{\mathcal{E}}_M$ on $[0,T)$.

Proof. It is crucial here to remember the global assumption according to which the spectrum of $J_0(x)$ is $\{\lambda,\lambda,\nu\}$, where $\nu > \lambda > 0$, for every $x\in\mathbb{T}^3$. The key observation is then that we may combine the assumption (4-30) and Proposition A.3 to deduce that, for every $(t,x)\in[0,T)\times\mathbb{T}^3$, the spectrum of $J(t,x)$ is $\{\lambda,\lambda,\nu\}$. Therefore,

$$\lambda\int_{\mathbb{T}^3}|\theta|^2 \leqslant \int_{\mathbb{T}^3} J\theta\cdot\theta \leqslant \nu\int_{\mathbb{T}^3}|\theta|^2,$$

and the claim follows upon picking $c_E = \frac{1}{2}\min(1,\lambda,\tilde{\tau}^2/(\nu-\lambda))$ and $C_E = \frac{1}{2}\max(1,\nu,\tilde{\tau}^2/(\nu-\lambda))$. □

We now record the coercivity of the dissipation over H^1 in Lemma 4.12 below. Note that this lemma is copied from Lemma 4.9 of the companion paper [Remond-Tiedrez and Tice 2021], and so we omit the proof.

Lemma 4.12 (coercivity of the dissipation). *There exists a universal constant $C_D > 0$ such that, for every $(u,\theta)\in H^1$ where u has average zero, $D(u,\theta)\geqslant C_D\|(u,\theta)\|_{H^1}$.*

4C. ***Closing the estimates at the low level.*** We now turn our attention to the second of the four building blocks of the scheme of a priori estimates: closing the energy estimates at the low level. This is carried out in this section and culminates in Proposition 4.20. In the remainder of this section we proceed as follows: first we derive auxiliary estimates of a and use them to improve the low-level energy and dissipation, then we estimate the low-level interactions, and finally we record the θ-coercivity central to the algebraic decay of the energy at the low level. We recall that the energy, dissipation, and interaction functionals at the low level, which will appear throughout this section, are defined in (3-3), (3-9), and (3-10), respectively. The various versions of the high-level energy are defined in (3-2) and (3-5).

We begin with auxiliary estimates for a intended to improve the low-level energy and dissipation. The strategy is simple: we use the appearance of a in the conservation of angular momentum (2-1c) to control a via the dissipation, then use (2-3) for $\partial_t a$ to bootstrap the control of a to new or better control of $\partial_t a$ in the dissipation and the energy, respectively, and finally use the time-differentiated equation for $\partial_t a$ in order to control $\partial_t^2 a$ energetically if the additional assumption (2) of Proposition 4.15 holds.

Lemma 4.13 (auxiliary estimate for a). *Suppose that (2-1c) holds. For any $k \in \mathbb{N}$ and any $s > \frac{3}{2}$ such that $s \geqslant k$, we have the estimate*

$$\|a\|_{H^k} \lesssim (1 + \|K\|_{H^s} + \|(u,\theta)\|_{H^s} + \|K\|_{H^s}\|(u,\theta)\|_{H^s})(\|\theta\|_{H^{k+2}} + \|\partial_t\theta\|_{H^k} + \|u\|_{H^{k+1}}).$$

Proof. This estimate follows from isolating a in (2-1c) and using Lemma B.4 to estimate the nonlinearities. To isolate a in (2-1c) we use the facts that $\omega_{\mathrm{eq}} \times J_{\mathrm{eq}}\omega_{\mathrm{eq}} = 0$ and $\omega_{\mathrm{eq}} \times K\omega_{\mathrm{eq}} = \tilde{\tau}^2\tilde{a}^{\perp}$ to rewrite the precession term as

$$(\omega_{\mathrm{eq}} + \theta) \times (J_{\mathrm{eq}} + K)(\omega_{\mathrm{eq}} + \theta) = (\omega_{\mathrm{eq}} + \theta) \times (J_{\mathrm{eq}} + K)\theta + \theta \times (J_{\mathrm{eq}} + K)\omega_{\mathrm{eq}} + \tilde{\tau}^2\tilde{a}^{\perp}.$$

We may then write (2-1c) as

$$\tilde{\tau}^2\tilde{a}^{\perp} = -(J_{\mathrm{eq}} + K)(\partial_t\theta + u\cdot\nabla\theta) - (\omega_{\mathrm{eq}} + \theta) \times (J_{\mathrm{eq}} + K)\theta - \theta \times (J_{\mathrm{eq}} + K)\omega_{\mathrm{eq}} + \kappa\nabla \times u - 2\kappa\theta + (\tilde{\alpha} - \tilde{\gamma})\nabla(\nabla\cdot\theta) + \tilde{\gamma}\Delta\theta. \qquad \square$$

We continue obtaining auxiliary estimates for a by obtaining an estimate for its first two time derivatives.

Lemma 4.14 (auxiliary estimate for $\partial_t a$ and $\partial_t^2 a$). *Suppose that (2-3) holds. For any $k \in \mathbb{N}$ and any $s > \frac{3}{2}$ such that $s \geqslant k$, we have the estimates*

$$\|\partial_t a\|_{H^k} \lesssim (1 + \|(u,\theta)\|_{H^s})\|a\|_{H^{k+1}} + (1 + \|K\|_{H^s})\|\theta\|_{H^s}$$

and

$$\|\partial_t^2 a\|_{H^k} \lesssim \|\partial_t(u,\theta)\|_{H^s}\|a\|_{H^{k+1}} + (1 + \|(u,\theta)\|_{H^s})\|\partial_t a\|_{H^{k+1}} + \|\partial_t K\|_{H^s}\|\theta\|_{H^k} + (1 + \|K\|_{H^s})\|\partial_t\theta\|_{H^k}.$$

Proof. The first estimate follows as in Lemma 4.13 from isolating $\partial_t a$ in (2-3) and using Lemma B.4 to estimate the quadratic terms. The second estimate follows from differentiating (2-3) in time and then proceeding as in Lemma 4.13, namely isolating $\partial_t^2 a$ and using Lemma B.4. $\square$

With these auxiliary estimates for a and its first two temporal derivatives in hand we may now improve the low-level energy and dissipation.

Proposition 4.15 (improvement of the low-level energy and dissipation). *Let $T > 0$ be a time horizon. Consider the assumptions*

(1) $\sup_{0 \leqslant t < T} \|(u, \theta)(t)\|_{H^3} + \|K(t)\|_{H^3} \leqslant \overline{C} < \infty$ *and*
(2) $\sup_{0 \leqslant t < T} \|\partial_t (u, \theta)(t)\|_{H^2} + \|\partial_t K(t)\|_{H^2} \leqslant \overline{C} < \infty$.

Then we have the following estimates, where the constants implicit in "$\lesssim$" may depend on $\overline{C}$. If (1) holds then $\bar{\mathcal{E}}_{\text{low}} \gtrsim \|\partial_t a\|_{H^1}^2$ and $\overline{\mathcal{D}}_{\text{low}} \gtrsim \|a\|_{H^1}^2 + \|\partial_t a\|_{L^2}^2$. If (1) and (2) hold then $\bar{\mathcal{E}}_{\text{low}} \gtrsim \|\partial_t^2 a\|_{L^2}^2$.

Proof. Assumption (1) and Lemma 4.13 tell us that $\|a\|_{H^1}^2 \lesssim \|(u, \theta)\|_{H^3}^2 + \|\partial_t \theta\|_{H^1}^2 \lesssim \overline{\mathcal{D}}_{\text{low}}$. Then we may use (1), Lemma 4.14, and the previous estimate to see that

$$\|\partial_t a\|_{L^2}^2 \lesssim \|a\|_{H^1}^2 + \|\theta\|_{L^2}^2 \lesssim \overline{\mathcal{D}}_{\text{low}}$$

and

$$\|\partial_t a\|_{H^1}^2 \lesssim \|a\|_{H^2}^2 + \|\theta\|_{H^1}^2 \lesssim \bar{\mathcal{E}}_{\text{low}}.$$

Finally, if both assumptions (1) and (2) hold then we may use Lemma 4.14 again to see that

$$\|\partial_t^2 a\|_{L^2}^2 \lesssim \|a\|_{H^1}^2 + \|\partial_t a\|_{H^1}^2 + \|\theta\|_{L^2}^2 + \|\partial_t \theta\|_{L^2}^2 \lesssim \bar{\mathcal{E}}_{\text{low}}. \qquad \square$$

We now turn our attention to the low-level interactions and record their estimates here. Note that they may be estimated in a simpler way for the sole purpose of closing the energy estimates at the low level, but by doing the estimates slightly more carefully as done below we can also use them when we study the local well-posedness theory (in Section 5).

Lemma 4.16 (careful estimates of the low-level interactions). *Recall that $\overline{\mathcal{I}}_{\text{low}}$ is defined in (3-10) for $\mathcal{I}^\alpha$ as in Lemma 4.10. The following estimate holds*:

$$|\overline{\mathcal{I}}_{\text{low}}| \lesssim (\|(u, \theta)\|_{P^2} + \|a\|_{P^3} + (1 + \|(u, \theta)\|_{P^2})(\|K\|_{H^3} + \|\partial_t K\|_{L^\infty}))\mathcal{D}_{\text{low}}.$$

Proof. Recall that, at the low level, $|\alpha|_P \leqslant 2$. In particular, if $\beta + \gamma = \alpha$ and $\beta > 0$ then $(\partial^\alpha, \partial^\beta, \partial^\gamma)$ corresponds to one of five possible cases: $(\partial_x^2, \partial_x^2, 0)$, $(\partial_x^2, \partial_x, \partial_x)$, $(\partial_x, \partial_x, 0)$, $(\partial_t, \partial_t, 0)$, and $(0, 0, 0)$, where ∂_x^k indicates a derivative ∂^α for a purely spatial multi-index $\alpha \in \mathbb{N}^3$ of length $|\alpha| = k$. Note that we have the bound $\|K\|_{L^\infty} + \|\nabla K\|_{L^\infty} + \|\nabla^2 K\|_{L^6} \lesssim \|K\|_{H^3}$ such that, for any $|\alpha|_P \leqslant 2$,

$$\|\partial^\alpha K\|_{L^6} \lesssim \|K\|_{H^3} + \|\partial_t K\|_{L^\infty}. \tag{4-31}$$

Recall from Lemma 4.10 that $\overline{\mathcal{I}}_{\text{low}} = \sum_{|\alpha|_P \leqslant 2} \sum_{i=1}^{8} \mathcal{I}_i^\alpha$. In light of (4-31) we may estimate $\mathcal{I}_1^\alpha$–$\mathcal{I}_5^\alpha$ and $\mathcal{I}_7^\alpha$ easily, obtaining

$$|\mathcal{I}_1^\alpha| \lesssim \|u\|_{P^2}\mathcal{D}_{\text{low}}, \quad |\mathcal{I}_2^\alpha| \lesssim (\|K\|_{H^3} + \|\partial_t K\|_{L^\infty})\mathcal{D}_{\text{low}}, \quad |\mathcal{I}_3^\alpha| \lesssim (1 + \|K\|_{H^3} + \|\partial_t K\|_{L^\infty})\|u\|_{P^2}\mathcal{D}_{\text{low}},$$
$$|\mathcal{I}_4^\alpha| \lesssim (1 + \|\theta\|_{P^2})(\|K\|_{H^3} + \|\partial_t K\|_{L^\infty})\mathcal{D}_{\text{low}} + (1 + \|K\|_{H^3})\|\theta\|_{P^2}\mathcal{D}_{\text{low}},$$
$$|\mathcal{I}_5^\alpha| \lesssim (\|K\|_{H^3} + \|\partial_t K\|_{L^\infty})\mathcal{D}_{\text{low}}, \quad \text{and} \quad |\mathcal{I}_7^\alpha| \lesssim \|a\|_{P^3}\mathcal{D}_{\text{low}}.$$

The only two terms requiring particularly delicate care are $\mathcal{I}_6$ and $\mathcal{I}_8$, due to the presence of $\partial_x^2 a$. We provide the details on how to estimate these two interactions below.

Estimating $\mathcal{I}_6^\alpha$. Recall that

$$\mathcal{I}_6^\alpha = -\sum_{\substack{\beta+\gamma=\alpha\\ \beta>0}} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} (\partial^\beta u \cdot \nabla \partial^\gamma a) \cdot \partial^\alpha a.$$

The difficulty lies in $\partial^\alpha = \partial_x^2$ since then two copies of $\partial_x^2 a$ may appear and we only control a dissipatively in H^1. We thus split into two cases, emphasizing that only the first case is somewhat troublesome and requires particular care. In the first case we consider $|\bar{\alpha}| = 2$ and $|\bar{\beta}| = |\bar{\gamma}| = 1$ and proceed by interpolation:

$$\left| \int_{\mathbb{T}^3} (\partial_x u \cdot \nabla \partial_x a) \cdot \partial_x^2 a \right| \leqslant \|\partial_x u\|_{L^\infty} \|\partial_x^2 a\|_{L^2}^2 \lesssim \|u\|_{H^3} \|a\|_{H^2}^2 \lesssim \|u\|_{H^3} (\|a\|_{H^1}^{1/2} \|a\|_{H^3}^{1/2})^2 \lesssim \|a\|_{H^3} \mathcal{D}_{\text{low}}.$$

In the second case we consider either $|\bar{\alpha}| = |\bar{\beta}| = 2$ or $|\bar{\alpha}| \leqslant 1$. Either way, since $\beta > 0$ we deduce that $\gamma = 0$, and hence $\beta = \alpha$. The estimate is then immediate:

$$\left| \int_{\mathbb{T}^3} (\partial^\alpha u \cdot \nabla a) \cdot \partial^\alpha a \right| \leqslant \|\partial^\alpha u\|_{L^6} \|\nabla a\|_{L^2} \|\partial^\alpha a\|_{L^6} \lesssim \|u\|_{P^3} \|a\|_{H^1} \|a\|_{P^3} \lesssim \|a\|_{P^3} \mathcal{D}_{\text{low}}.$$

Estimating $\mathcal{I}_8^\alpha$. Recall that

$$\mathcal{I}_8^\alpha = -\sum_{\beta+\gamma=\alpha} \int_{\mathbb{T}^3} \partial^\beta (\bar{K} - K_{33} I_2) \partial^\gamma \bar{\theta}^\perp \cdot \partial^\alpha a \sim \sum_{\beta+\gamma=\alpha} \int_{\mathbb{T}^3} (\partial^\beta K) \partial^\gamma \theta \cdot \partial^\alpha a,$$

where the left-hand side is the precise form of the interaction and the right-hand side is its schematic form which we will work with henceforth. The difficulty lies in $\partial^\alpha = \partial_x^2$ since we have no dissipative control over K and only control a dissipatively in H^1. We must therefore integrate by parts to reduce the term $\partial_x^2 a$ to $\partial_x a$. Now we split into two cases. As in the consideration of $\mathcal{I}_6^\alpha$ above, only the first case is somewhat troublesome.

In the first case we consider $|\bar{\alpha}| = 2$. We integrate by parts and estimate each term by hand. We write

$$\sum_{\beta+\gamma=2} \int_{\mathbb{T}^3} (\partial_x^\beta K) \partial_x^\gamma \theta \cdot \partial_x^2 a = \sum_{\beta+\gamma=2} \int_{\mathbb{T}^3} (\partial_x^{\beta+1} K)(\partial_x^\gamma \theta) \cdot \partial_x a + \int_{\mathbb{T}^3} (\partial_x^\beta K)(\partial_x^{\gamma+1} \theta) \cdot \partial^\alpha a =: \mathrm{I} + \mathrm{II},$$

where

$$\begin{aligned} |\mathrm{I}| &\leqslant \left| \int_{\mathbb{T}^3} (\partial_x K)(\partial_x^2 \theta) \partial_x a \right| + \left| \int_{\mathbb{T}^3} (\partial_x^2 K)(\partial_x \theta) \cdot \partial_x a \right| + \left| \int_{\mathbb{T}^3} (\partial_x^3 K) \theta \cdot \partial_x a \right| \\ &\leqslant (\|\partial_x K\|_{L^\infty} \|\partial_x^2 \theta\|_{L^6} + \|\partial_x^2 K\|_{L^6} \|\partial_x \theta\|_{L^\infty} + \|\partial_x^3 K\|_{L^2} \|\theta\|_{L^\infty}) \|\partial_x a\|_{L^2} \\ &\lesssim \|K\|_{H^3} \|\theta\|_{H^3} \|a\|_{H^1} \lesssim \|K\|_{H^3} \mathcal{D}_{\text{low}} \end{aligned}$$

and

$$|\mathrm{I}| \leqslant \sum_{\beta+\gamma=2} \|\partial_x^\beta K\|_{L^\infty} \|\partial_x^{\gamma+1} \theta\|_{L^2} \|\partial_x a\|_{L^2} \lesssim \|K\|_{H^3} \|\theta\|_{H^3} \|a\|_{H^1} \lesssim \|K\|_{H^3} \mathcal{D}_{\text{low}}.$$

In the second case we consider $|\bar{\alpha}| \leqslant 1$. Note that, as was noted above when considering $\mathcal{I}_6^\alpha$, it then follows from the constraint $\beta > 0$ that $\gamma = 0$ and $\beta = \alpha$. The estimate is then immediate:

$$\begin{aligned} |\mathrm{II}| \leqslant \|\partial^\alpha K\|_{L^2} \|\theta\|_{L^\infty} \|\partial^\alpha a\|_{L^2} &\lesssim (\|K\|_{H^3} + \|\partial_t K\|_{L^\infty}) \|\theta\|_{H^2} (\|a\|_{H^1} + \|\partial_t a\|_{L^2}) \\ &\lesssim (\|K\|_{H^3} + \|\partial_t K\|_{L^\infty}) \mathcal{D}_{\text{low}}. \end{aligned}$$

□

In particular, for our purposes here it suffices to control the low-level interactions in the following way.

Corollary 4.17 (control of the low-level interactions). *If $M \geqslant 3$ and $\mathcal{E}_M \leqslant 1$ then $|\bar{\mathcal{I}}_{\text{low}}| \lesssim \mathcal{E}_M^{1/2} \mathcal{D}_{\text{low}}$.*

Proof. Recall that $\mathcal{E}_M \gtrsim \|(u, \theta, a)\|_{P^{2M}}^2 + \|K\|_{H^{2M-3}}^2 + \|\partial_t K\|_{H^{2M-3}}^2$. In particular, since $M \geqslant 3$ and $\mathcal{E}_M \leqslant 1$, we may deduce the claim from Lemma 4.16. □

We now turn our attention to the last piece needed to close the energy estimates at the low level, namely the θ-coercivity estimate recorded below. In particular note that below, $\theta \uparrow 1$ as $M \uparrow \infty$.

Lemma 4.18 (θ-coercivity). *If $M \geqslant 2$ then $\bar{\mathcal{E}}_{\text{low}} \lesssim \bar{\mathcal{E}}_M^{1-\theta} \mathcal{D}_{\text{low}}^{\theta}$, where $\theta = (2M-2)/(2M-1)$.*

Proof. Since the low-level dissipation controls every term in the low-level energy *except* $\|a\|_{H^2}^2$, we rely on an interpolation estimate to control that term using the low-level dissipation and the *high-level* energy. More precisely, recall that

$$\begin{cases} \bar{\mathcal{E}}_{\text{low}} = \|(u, \theta)\|_{H^2}^2 + \|a\|_{H^2}^2 + \|\partial_t (u, \theta)\|_{L^2}^2 + \|\partial_t a\|_{L^2}^2, \\ \mathcal{D}_{\text{low}} = \|(u, \theta)\|_{H^3}^2 + \|a\|_{H^1}^2 + \|\partial_t (u, \theta)\|_{H^1}^2 + \|\partial_t a\|_{H^1}^2. \end{cases}$$

So let us write $\bar{\mathcal{E}}_{\text{low}} - \|a\|_{H^2}^2 =: \mathcal{E}_{\text{good}}$. Then

$$\mathcal{E}_{\text{good}} \lesssim \mathcal{D}_{\text{low}} \qquad \text{and} \qquad \|a\|_{H^2}^2 \lesssim \|a\|_{H^1}^{2\theta} \|a\|_{H^{2M}}^{2(1-\theta)}, \quad \text{where } \theta = \frac{2M-2}{2M-1}.$$

So finally, since $M \geqslant 2$, we note that $\bar{\mathcal{E}}_M \gtrsim \mathcal{D}_{\text{low}}$, and hence we may conclude that, for θ as above,

$$\bar{\mathcal{E}}_{\text{low}} = \mathcal{E}_{\text{good}} + \|a\|_{H^2}^2 \lesssim \mathcal{D}_{\text{low}} + \mathcal{D}_{\text{low}}^{\theta} \bar{\mathcal{E}}_M^{1-\theta} \lesssim \mathcal{D}_{\text{low}}^{\theta} \bar{\mathcal{E}}_M^{1-\theta}. \qquad \square$$

In light of the θ-coercivity result above we now record a particular instance of the Bihari lemma which applies to the low-level energy. This is recorded here in order to streamline the proof of Proposition 4.20 in which we close the energy estimates at the low level.

Lemma 4.19. *Suppose that the function $y : [0, \infty) \to [0, \infty)$ is continuously differentiable such that $y' + C y^{1/\theta} \alpha_0^{1-1/\theta} \leqslant 0$ on $[0, \infty)$ for some $\alpha_0, C > 0$ and $\theta \in (0, 1)$. Then*

$$y(t) \leqslant \alpha_0 \left(\left(\frac{\alpha_0}{y(0)} \right)^{1/\beta} + \widetilde{C} t \right)^{-\beta} \quad \text{for } \beta := \left(\frac{1}{\theta} - 1 \right)^{-1} > 0 \text{ and } \widetilde{C} = C \left(\frac{1}{\theta} - 1 \right) > 0.$$

In particular, note that $\beta \uparrow +\infty$ if $\theta \uparrow 1$.

Proof. Integrating in time tells us that

$$y(t) + \int_0^t C y(s)^{1/\theta} \alpha_0^{1-1/\theta} \, ds \leqslant y(0).$$

We apply the Bihari lemma (Lemma B.5) with $f(x) = C x^{1/\theta} \alpha_0^{1-1/\theta}$. Using the notation in Bihari's lemma, we compute $F(x) = (\alpha_0 / x)^{1/\beta} / \widetilde{C}$ and $F^{-1}(x) = \alpha_0 (\widetilde{C} x)^{-\beta}$, from which the claim follows. □

We conclude this section with Proposition 4.20, which performs the synthesis of the results proved in this section in order to close the energy estimates at the low level. Recall that, as discussed in Section 2D, this is one of the four building blocks of the scheme of a priori estimates.

Proposition 4.20 (closing the energy estimates at the low level). *Let $M \geqslant 3$ be an integer. There exist $0 < \delta_{\text{low}}, \delta_{\text{low}}^* \leqslant 1$, and $C_L > 0$ such that the following holds: for any time horizon $T > 0$ and any $0 < \delta \leqslant \delta_{\text{low}}$, if $\sup_{0 \leqslant t \leqslant T} \mathcal{E}_M(t) \leqslant \delta_{\text{low}}^*$ and $\sup_{0 \leqslant t \leqslant T} \bar{\mathcal{E}}_M(t) \leqslant \delta$ then $\sup_{0 \leqslant t \leqslant T} \mathcal{E}_{\text{low}}(t) (1+t)^{2M-2} \leqslant C_L \delta$.*

Proof. The strategy of the proof is as follows. We combine the energy-dissipation relation, the control of the interactions, and the improvement of the dissipation to see that $(\mathrm{d}/\mathrm{d}t)\tilde{\mathcal{E}}_{\text{low}} + \mathcal{D}_{\text{low}} \leqslant 0$. This differential inequality is coupled with the θ-coercivity and the improvement of the low-level energy to deduce the result.

More precisely, recall that, by Lemmas 4.10 and 4.12, $(\mathrm{d}/\mathrm{d}t)\tilde{\mathcal{E}}_{\text{low}} + \overline{\mathcal{D}}_{\text{low}} \lesssim \overline{\mathcal{I}}_{\text{low}}$. Since

$$\sup_{0\leqslant t\leqslant T} \|(u,\theta)\|_{H^3}^2 + \|J\|_{H^3}^2 + \|\partial_t(u,\theta)\|_{H^2}^2 + \|\partial_t J\|_{H^2}^2 \leqslant \sup_{0\leqslant t\leqslant T} \mathcal{E}_M(t) \leqslant 1, \tag{4-32}$$

it follows from Proposition 4.15 and Corollary 4.17 that $\overline{\mathcal{D}}_{\text{low}} \gtrsim \mathcal{D}_{\text{low}}$ and $|\overline{\mathcal{I}}_{\text{low}}| \lesssim \mathcal{E}_M^{1/2}\mathcal{D}_{\text{low}}$. Therefore, there exists $C_{ED} > 0$ such that $(\mathrm{d}/\mathrm{d}t)\tilde{\mathcal{E}}_{\text{low}} + \mathcal{D}_{\text{low}} \lesssim C_{ED}\mathcal{E}_M^{1/2}\mathcal{D}_{\text{low}}$. In particular, if $\delta_{\text{low}}^* > 0$ is chosen sufficiently small to ensure that $C_{ED}(\delta_{\text{low}}^*)^{1/2} \leqslant \frac{1}{2}$ then $(\mathrm{d}/\mathrm{d}t)\tilde{\mathcal{E}}_{\text{low}} + \frac{1}{2}\mathcal{D}_{\text{low}} \leqslant 0$. Now note that, as a consequence of (4-32), Proposition A.3 and Lemma 4.11 tell us that

$$\tfrac{1}{2}c_E\bar{\mathcal{E}}_{\text{low}} \leqslant \tilde{\mathcal{E}}_{\text{low}} \leqslant \tfrac{1}{2}C_E\bar{\mathcal{E}}_{\text{low}}. \tag{4-33}$$

We may combine this with Lemma 4.18 to deduce, for $\theta = (2M-2)/(2M-1)$, that $\tilde{\mathcal{E}}_{\text{low}} \lesssim \bar{\mathcal{E}}_{\text{low}} \lesssim \bar{\mathcal{E}}_M^{1-\theta}\mathcal{D}_{\text{low}}^{\theta}$, and hence there exists a constant $C > 0$ such that $(\mathrm{d}/\mathrm{d}t)\tilde{\mathcal{E}}_{\text{low}} + C\tilde{\mathcal{E}}_{\text{low}}^{1/\theta}\delta^{1-1/\theta} \leqslant 0$. We deduce from (4-32) that θ and J are sufficiently regular for $t \mapsto \tilde{\mathcal{E}}_{\text{low}}(t)$ to be continuously differentiable. Applying Lemma 4.19 thus tells us that, for $0 \leqslant t \leqslant T$,

$$\tilde{\mathcal{E}}_{\text{low}}(t) \lesssim \delta\left(\left(\frac{\delta}{\tilde{\mathcal{E}}_{\text{low}}(0)}\right)^{1/\beta} + \widetilde{C}t\right)^{-\beta}$$

for some $\widetilde{C} > 0$ and for $\beta = (1/\theta - 1)^{-1} = 2M - 2$. Using (4-33) once again we note that

$$\tilde{\mathcal{E}}_{\text{low}}(0) \leqslant \tfrac{1}{2}C_E\bar{\mathcal{E}}_{\text{low}}(0) \leqslant \tfrac{1}{2}C_E\delta,$$

and hence $\delta/\tilde{\mathcal{E}}_{\text{low}}(0) \geqslant 2/C_E$ such that $\tilde{\mathcal{E}}_{\text{low}} \lesssim \delta(1+t)^{-(2M-2)}$. To conclude this step, note that combining (4-32) and Proposition 4.15 tells us that $\mathcal{E}_{\text{low}} \lesssim \bar{\mathcal{E}}_{\text{low}}$, and hence, in light of (4-33), we deduce that $\mathcal{E}_{\text{low}}(t)(1+t)^{2M-2} \lesssim \delta$ for $0 \leqslant t \leqslant T$. □

4D. ***Closing the estimates at the high level.*** In this section we consider the third of the four building blocks of the scheme of a priori estimates and close the energy estimates at the high level. This section is structured similarly, but not identically, to Section 4C, where we close the energy estimates at the *low* level. The differences are due to the fact that at the high level the improvements to the dissipation and the estimates of the interactions only hold in a time-integrated sense, and not pointwise in time as was the case at the low level. This means that by contrast with the low level, where the auxiliary estimates relied on product estimates (see Lemma B.4), here at the high level the auxiliary estimates rely instead on high-low estimates (see Corollary B.2). Recall that the functionals $\mathcal{E}_M$ and $\mathcal{F}_M$, $\overline{\mathcal{K}}_I$ and $\mathcal{K}_{\text{low}}$, and $\overline{\mathcal{D}}_M$, which will be used throughout this section, are defined in (3-5), (3-6), and (3-8), respectively.

We begin with auxiliary estimates for a, which will allow for improvement of the high-level dissipation.

Lemma 4.21 (auxiliary estimate for a). *Suppose that* (2-1c) *holds. For any $k \in \mathbb{N}$ and any $s > \frac{3}{2}$ such that $s \geqslant k$, we have the estimate*

$$\|a\|_{H^k} \lesssim (\|K\|_{L^\infty} + \|\theta\|_{L^\infty} + \|\nabla\theta\|_{L^\infty} + \|u\|_{H^s} + \|\theta\|_{H^s})\|(u,\theta)\|_{P^{k+2}} + (\|u\|_{L^\infty} + \|\theta\|_{L^\infty} + \|\partial_t\theta\|_{L^\infty})\|K\|_{H^k}.$$

Proof. This estimate is similar to that of Lemma 4.13 since we begin by isolating a in (2-1c). However, we then use the high-low estimate of Corollary B.2 instead of the product estimate of Lemma B.4 to estimate the nonlinearities. Note that for cubic terms we perform high-low estimates in the following crude fashion: $\|fgh\|_{H^k} \lesssim \|f\|_{H^\infty}\|gh\|_{H^k} + \|f\|_{H^k}\|gh\|_{L^\infty}$, and then use Lemma B.4 and the fact that L^∞ is a Banach algebra. □

We continue our sequence of auxiliary estimates for a with an estimate on its time derivative.

Lemma 4.22 (auxiliary estimate for $\partial_t a$). *Suppose that* (2-3) *holds. For any $k \in \mathbb{N}$ and any $s > \frac{3}{2}$ such that $s \geqslant k$, we have the estimate*

$$\|\partial_t a\|_{H^k} \lesssim (1 + \|u\|_{H^s} + \|\theta\|_{H^s})\|a\|_{H^{k+1}} + (1 + \|K\|_{L^\infty})\|\theta\|_{H^k} + \|\theta\|_{L^\infty}\|K\|_{H^k}.$$

Proof. This follows immediately from isolating $\partial_t a$ in (2-3) and using Corollary B.2 and Lemma B.4. □

We conclude our sequence of auxiliary estimates on a with estimates on its higher-order temporal derivatives.

Lemma 4.23 (auxiliary estimate for $\partial_t^j a$). *Suppose that* (2-3) *holds. For any $k \in \mathbb{N}$, any $s > \frac{3}{2}$, and any $j \geqslant 1$, if $s \geqslant k$ then*

$$\|\partial_t^j a\|_{H^k} \lesssim \left(1 + \sum_{l=0}^{j-1}\|\partial_t^l(u,\theta)\|_{H^s}\right)\left(\sum_{l=0}^{j-1}\|\partial_t^l a\|_{H^{k+1}}\right) + \left(1 + \sum_{l=0}^{j-1}\|\partial_t^l K\|_{H^s}\right)\left(\sum_{l=0}^{j-1}\|\partial_t^l\theta\|_{H^k}\right).$$

Proof. This is immediate upon applying $j-1$ temporal derivatives to (2-3) and using Lemma B.4 to estimate the nonlinearities. □

With these various auxiliary estimates on a in hand we may now improve the high-level dissipation. Recall that $\mathcal{D}_M^a$ is defined in (3-8).

Proposition 4.24 (improvement of the dissipation at the high level). *If $M \geqslant 3$ and $\mathcal{E}_M \leqslant 1$ then*

$$\mathcal{D}_M^a \lesssim \overline{\mathcal{D}}_M^{1/2} + \|(u,\theta,\partial_t\theta)\|_{L^\infty}\mathcal{F}_M^{1/2}.$$

Proof. For simplicity, we will write $d := \overline{\mathcal{D}}_M^{1/2} + \|(u,\theta,\partial_t\theta)\|_{L^\infty}\mathcal{F}_M^{1/2}$ for the right-hand side of the inequality we are after. Since $M \geqslant 3$, since $H^2(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)$, and since $\mathcal{E}_M \leqslant 1$, Lemma 4.21 tells us that $\|a\|_{H^{2M-1}} \lesssim d$ and consequently Lemma 4.22 says that $\|\partial_t a\|_{H^{2M-2}} \lesssim d$. To see that

$$\sum_{j=2}^{3}\|\partial_t^j a\|_{H^{2M-j-1}} + \sum_{j=4}^{M}\|\partial_t^j a\|_{H^{2M-2j+3}} \lesssim d$$

and thus conclude the proof, it suffices to prove by induction that

$$\|\partial_t^j a\|_{H^{k(j)}} \lesssim d \quad \text{for } k(j) = \begin{cases} 2M - j - 1 & \text{if } j = 2 \text{ or } j = 3, \\ 2M - 2j + 3 & \text{if } j = 4, \dots, M. \end{cases}$$

This induction argument is immediate: the base cases $j = 0$ and $j = 1$ were taken care of above, and the induction step is precisely given by Lemma 4.23. □

We now turn our attention towards the control of the high-level interactions. First we record a technical lemma used to control derivatives of K.

Lemma 4.25. *If $\alpha \in \mathbb{N}^{1+3}$ satisfies $|\alpha|_P \leqslant 2M$ and $|\bar{\alpha}| \leqslant 2M-4$ then $\|\partial^\alpha K\|_{L^4} \lesssim \mathcal{E}_M^{1/2}$.*

Proof. We split the proof into two cases depending on the number of temporal derivatives hitting K. Suppose first that $\alpha_0 \leqslant 1$. Then the estimate is immediate: we have $\|\partial_x^{\bar{\alpha}} K\|_{L^4} \lesssim \|K\|_{H^{1+(2M-4)}} \lesssim \mathcal{E}_M^{1/2}$ and $\|\partial_t \partial_x^{\bar{\alpha}} K\|_{L^4} \lesssim \|\partial_t K\|_{H^{1+(2M-4)}} \lesssim \mathcal{E}_M^{1/2}$. Suppose now that $\alpha_0 \geqslant 2$. The estimate in this case follows from the fact that $\partial_t^j K$ is controlled at parabolic order $2M+1$ when $j \geqslant 2$. Therefore, since $1+|\alpha|_P-4 \leqslant 2M-3$, we may deduce that $\|\partial^\alpha K\|_{L^4} \lesssim \|\partial_t^2 K\|_{P^{1+|\alpha|_P-4}} \lesssim \mathcal{E}_M^{1/2}$. □

We may now state and prove the estimate of the high-level interactions. Recall that $\mathcal{D}_M$ and $\overline{\mathcal{I}}_M$ are defined in (3-8) and (3-10), respectively.

Proposition 4.26 (control of the high-order interactions). *Suppose that $M \geqslant 3$ and that $\mathcal{E}_M \leqslant 1$. Then $|\overline{\mathcal{I}}_M| \lesssim \mathcal{E}_M^{1/2}\mathcal{D}_M + \mathcal{K}_{\text{low}}^{1/2}\mathcal{F}_M^{1/2}\mathcal{D}_M^{1/2}$.*

Proof. Recall that the interactions are recorded in Lemma 4.10. There are three difficulties that manifest themselves here.

(1) a appears when hit with a full count of $2M$ spatial derivatives. This is troublesome because the lack of dissipative control of a in H^{2M} is precisely why coercivity fails. To handle this it will be necessary to integrate by parts. This issue manifests itself in $\mathcal{I}_6$ and $\mathcal{I}_8$.

(2) We have poorer spatial regularity control over $\partial_t^j K$ when j is small (i.e., $j = 0, 1$) than when it is large — this is due to the mixed hyperbolic-parabolic nature of the problem. There is no particularly clever workaround here besides simply breaking up the estimates into cases depending on the number of temporal derivatives hitting K and performing the estimates in each case. This manifests itself in $\mathcal{I}_2$–$\mathcal{I}_5$ and $\mathcal{I}_8$.

(3) In all but one of the interactions where $\mathcal{F}_M$ must be invoked, its possible growth is counteracted by the presence of $\overline{\mathcal{K}}_2$. However, in $\mathcal{I}_2$ this is not possible. Instead, we may only counteract the growth of $\mathcal{F}_M$ by $\|\partial_t^2\theta\|_{L^2}$. Since two temporal derivatives of θ are *not* controlled in the low-level energy, this term does not, at first pass, have any decay. Producing such decay will require the auxiliary estimate recorded in Lemma 4.31 in Section 4E below.

For the reader's sake, we briefly remark on which interaction terms are discussed in detail and why.

The details of the estimates of $\mathcal{I}_1$ are provided. This is a simple interaction to estimate but we do use "hands-on high-low estimates" which form the basis for all the other estimates of the interactions, and thus warrants a detailed discussion of $\mathcal{I}_1$. In particular, $\mathcal{I}_7$ is handled in exactly the same way.

The interactions $\mathcal{I}_2$–$\mathcal{I}_5$ are handled in essentially the same way. We only discuss $\mathcal{I}_2$ in detail since it has the additional wrinkle of requiring us to invoke $\|\partial_t^2\theta\|_{L^2}$ to counterbalance $\mathcal{F}_M$.

The last two interactions we discuss in detail are $\mathcal{I}_6$ and $\mathcal{I}_8$. Those are the most difficult interactions to control since they both require us to integrate by parts to get around the appearance of $\nabla_x^{2M} a$. Moreover, temporal derivatives of K appear in $\mathcal{I}_8$, which requires us to divide the estimates of that interaction into further subcases.

We now estimate the difficult interactions one by one.

Estimating $\mathcal{I}_1$ and $\mathcal{I}_7$. Recall that

$$\mathcal{I}_1 = \sum_{|\alpha|_P \leqslant 2M} \int_{\mathbb{T}^3} [u \cdot \nabla, \partial^\alpha] u \cdot \partial^\alpha u = - \sum_{|\alpha|_P \leqslant 2M} \sum_{\substack{\beta+\gamma=\alpha \\ \beta>0}} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} (\partial^\beta u \cdot \nabla \partial^\gamma u) \cdot \partial^\alpha u.$$

Therefore,

$$|\mathcal{I}_1| \lesssim \sum_{\substack{|\beta+\gamma|_P \leqslant 2M \\ |\gamma|_P \leqslant 2M-3}} \underbrace{\left| \int_{\mathbb{T}^3} (\partial^\beta u \cdot \nabla \partial^\gamma u) \cdot \partial^\alpha u \right|}_{\mathrm{I}} + \sum_{\substack{|\beta+\gamma|_P \leqslant 2M \\ |\beta|_P \leqslant 2M-2 \\ |\gamma|_P \leqslant 2M-1}} \underbrace{\left| \int_{\mathbb{T}^3} (\partial^\beta u \cdot \nabla \partial^\gamma u) \cdot \partial^\alpha u \right|}_{\mathrm{II}},$$

where

$$\begin{aligned} \mathrm{I} &\leqslant \|\partial^\beta u\|_{L^4} \|\nabla \partial^\gamma u\|_{L^\infty} \|\partial^{\beta+\gamma} u\|_{L^4} \lesssim \|\partial^\beta u\|_{H^1} \|\nabla \partial^\gamma u\|_{H^2} \|\partial^{\beta+\gamma} u\|_{H^1} \\ &\leqslant \|u\|_{P^{|\beta|_P+1}} \|u\|_{P^{|\gamma|_P+3}} \|u\|_{P^{|\beta+\gamma|_P+1}} \lesssim \mathcal{D}_M^{1/2} \mathcal{E}_M^{1/2} \mathcal{D}_M^{1/2} \end{aligned}$$

and

$$\mathrm{II} \leqslant \|\partial^\beta u\|_{L^\infty} \|\nabla \partial^\gamma u\|_{L^4} \|\partial^{\beta+\gamma} u\|_{L^4} \lesssim \|u\|_{P^{|\beta|_P+2}} \|u\|_{P^{|\gamma|_P+2}} \|u\|_{P^{|\beta+\gamma|_P+1}} \lesssim \mathcal{E}_M^{1/2} \mathcal{D}_M^{1/2} \mathcal{D}_M^{1/2}$$

such that $|\mathcal{I}_1| \leqslant \mathcal{E}_M^{1/2} \mathcal{D}_M$.

To control $\mathcal{I}_7$ we proceed in exactly the same way. Note that the presence of a in $\mathcal{I}_7$ is harmless since there is only at most one copy of $\nabla_x^{2M} a$ which appears, and hence there is no need to integrate by parts here. Hands-on high-low estimates very similar to those discussed in detail above therefore tell us that $|\mathcal{I}_7| \lesssim \mathcal{E}_M^{1/2} \mathcal{D}_M$.

Estimating $\mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4$, and $\mathcal{I}_5$. Recall that $\mathcal{I}_2$ is of particular importance since it is the only term that requires the incorporation of $\|\partial_t^2 \theta\|_{L^2}$ into the decaying functional $\mathcal{K}_{\mathrm{low}}$. We seek to estimate

$$\mathcal{I}_2 = \sum_{|\alpha|_P \leqslant 2M} \int_{\mathbb{T}^3} [J \partial_t, \partial^\alpha] \theta \cdot \partial^\alpha \theta = - \sum_{\substack{|\alpha|_P \leqslant 2M \\ \beta+\gamma=\alpha,\, \beta>0}} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} (\partial^\beta K)(\partial_t \partial^\gamma \theta) \cdot (\partial^\alpha \theta),$$

where we have used the fact that $\partial^\beta J = \partial^\beta K$ since $\beta > 0$ and J and K differ by a constant. There are two difficulties in handling this term, and so we split $\mathcal{I}_2$ accordingly as $\mathcal{I}_2 = \mathrm{I} + \mathrm{II}$.

(1) When few temporal derivatives hit K the only way we have of controlling a high number of spatial derivatives is through $\mathcal{F}_M$. Terms concerned by this issue are grouped in I.

(2) The "better" terms in II are estimated directly. However, due to the poorer spatial regularity of K and $\partial_t K$ relative to $\partial_t^j K$ for $j \geqslant 2$, we split the estimate of II into two pieces that are handled differently from one another.

To be precise, we write

$$-\mathcal{I}_2 = \sum_{\substack{|\alpha|_P \leqslant 2M \\ \beta+\gamma=\alpha,\, \beta>0}} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} (\partial^\beta K)(\partial_t \partial^\gamma \theta) \cdot (\partial^\alpha \theta) = \sum_{\substack{\cdots \\ |\bar\beta| \geqslant 2M-3}} \cdots + \sum_{\substack{\cdots \\ |\bar\beta| \leqslant 2M-4}} \cdots =: \mathrm{I} + \mathrm{II}.$$

Note that the condition $|\bar\beta| \geqslant 2M-3$ in the term I is coupled with the usual condition $|\beta|_P \leqslant 2M$, and thus requires that $\beta_0 = 0, 1$. In other words, only K and $\partial_t K$ appear in I.

First we estimate I. Two competing factors are at play here:

(1) $\partial^\beta K$, for $|\bar\beta| \geqslant 2M-3$, *must* be controlled in $\mathcal{F}_M$, and hence $\partial_t \partial^\gamma \theta$ *must* be controlled via decaying factors.

(2) Since only θ and $\partial_t\theta$ are controlled by $\mathcal{E}_{\mathrm{low}}$, the decay of $\partial_t^2\theta$ is obtained through Lemma 4.31, which only yields control of $\partial_t^2\theta$ in L^2. We thus have fairly poor control of $\partial_t^2\theta$ through decaying factors.

To carefully address this we split I into two:

$$\mathrm{I} = \sum_{\substack{\cdots \\ |\bar\beta|\geqslant 2M-2}} \cdots + \sum_{\substack{\cdots \\ |\bar\beta| = 2M-3}} \cdots =: \mathrm{I}_1 + \mathrm{I}_2.$$

To estimate I_1, we note that $|\gamma|_P = |\alpha|_P - |\beta|_P \leqslant |\alpha|_P - |\bar\beta| \leqslant 2M-(2M-2) = 2$. Therefore,

$$\begin{aligned} |\mathrm{I}_1| &\lesssim \sum_{\cdots} \|\partial^\beta K\|_{L^4} \|\partial_t\partial^\gamma\theta\|_{L^2} \|\partial^\alpha\theta\|_{L^4} \\ &\lesssim \max(\|K\|_{H^{2M+1}}, \|\partial_t K\|_{H^{2M-1}}) \max(\|\partial_t\theta\|_{H^2}, \|\partial_t^2\theta\|_{L^2}) \|\theta\|_{P^{2M+1}} \\ &\lesssim \mathcal{F}_M^{1/2} (\overline{\mathcal{K}}_2^{1/2} + \|\partial_t^2\theta\|_{L^2}) \mathcal{D}_M^{1/2} \lesssim \mathcal{F}_M^{1/2} \mathcal{K}_{\mathrm{low}}^{1/2} \mathcal{D}_M^{1/2}. \end{aligned}$$

To estimate I_2 we note that now $|\gamma|_P \leqslant 3$ such that, since $M \geqslant 3$,

$$|\mathrm{I}_2| \lesssim \sum_{\cdots} \|\partial^\beta K\|_{L^2} \|\partial_t\partial^\gamma\theta\|_{L^4} \|\partial^\alpha\theta\|_{L^4} \lesssim \max(\|K\|_{H^{2M-3}}, \|\partial_t K\|_{H^{2M-3}}) \|\theta\|_{P^6} \|\theta\|_{P^{2M+1}} \lesssim \mathcal{E}_M^{1/2} \mathcal{D}_M.$$

Second we estimate II. Due to the poorer spatial regularity of K and $\partial_t K$ relative to $\partial_t^j K$ when $j \geqslant 2$, we split II into two:

$$\mathrm{II} = \sum_{\substack{|\alpha|_P \leqslant 2M,\, |\bar\beta| \leqslant 2M-4 \\ \beta+\gamma=\alpha,\, \beta>0}} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} (\partial^\beta K)(\partial_t\partial^\gamma\theta)\cdot(\partial^\alpha\theta) = \sum_{\substack{\cdots \\ \beta_0 \leqslant 1}} \cdots + \sum_{\substack{\cdots \\ \beta_0 \geqslant 2}} \cdots =: \mathrm{II}_1 + \mathrm{II}_2.$$

Estimating II_1 is immediate upon recalling that $|\gamma|_P \leqslant 2M-1$ (since $\beta > 0$) and using Lemma 4.25:

$$|\mathrm{II}_1| \lesssim \sum_{\cdots} \|\partial^\beta K\|_{L^4} \|\partial_t\partial^\gamma\theta\|_{L^2} \|\partial^\alpha\theta\|_{L^4} \lesssim \mathcal{E}_M^{1/2} \|\theta\|_{P^{2+(2M-1)}} \|\theta\|_{P^{2M+1}} \lesssim \mathcal{E}_M^{1/2} \mathcal{D}_M^{1/2} \mathcal{D}_M^{1/2}.$$

Estimating II_2 relies on the crucial observation that $\|\partial_t^2 K\|_{P^{2M-3}} \lesssim \mathcal{E}_M^{1/2}$. The estimate for II_2 is then immediate:

$$|\mathrm{II}_2| \lesssim \sum_{\cdots} \|\partial^\beta K\|_{L^4} \|\partial_t\partial^\gamma\theta\|_{L^2} \|\partial^\alpha\theta\|_{L^4} \lesssim \|\partial_t^2 K\|_{P^{1-4+|\beta|_P}} \|\theta\|_{P^{2M+1}} \|\theta\|_{P^{2M+1}} \lesssim \mathcal{E}_M^{1/2} \mathcal{D}_M^{1/2} \mathcal{D}_M^{1/2}.$$

Putting it all together tells us that $|\mathcal{I}_2| \lesssim \mathcal{E}_M^{1/2} \mathcal{D}_M + \mathcal{F}_M^{1/2} \mathcal{K}_{\mathrm{low}}^{1/2} \mathcal{D}_M^{1/2}$.

We may proceed in a similar fashion to estimate $\mathcal{I}_3$, $\mathcal{I}_4$, and $\mathcal{I}_5$, splitting the interactions terms into cases depending on the number of temporal derivatives hitting K and using Lemma 4.25 where appropriate. Proceeding in this fashion we obtain that $|\mathcal{I}_3| + |\mathcal{I}_4| + |\mathcal{I}_5| \lesssim \mathcal{F}_M^{1/2} \overline{\mathcal{K}}_2^{1/2} \mathcal{D}_M^{1/2} + \mathcal{E}_M^{1/2} \mathcal{D}_M$.

<u>Estimating $\mathcal{I}_6$</u>. We seek to estimate

$$\mathcal{I}_6 = \sum_{|\alpha|_P \leqslant 2M} \int_{\mathbb{T}^3} [u\cdot\nabla, \partial^\alpha] a \cdot \partial^\alpha a = -\sum_{\substack{\beta+\gamma=\alpha \\ \beta>0}} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} (\partial^\beta u \cdot \nabla \partial^\gamma a) \cdot \partial^\alpha a.$$

However, recall that the failure of coercivity manifests itself precisely in the poor dissipative control over a. We thus treat the case where the multi-index α is purely spatial separately. This interaction is particularly troublesome when the derivatives are purely spatial *and* $|\gamma| = 2M-1$ (note that $|\gamma| = 2M$ is impossible since the conditions $\beta+\gamma=\alpha$ and $\beta > 0$ impose that $\gamma < \alpha$). In that case the interaction takes the (schematic) form

$$\int_{\mathbb{T}^3} (\partial_x u)(\partial_x^{2M} a)(\partial_x^{2M} a), \tag{4-34}$$

where ∂_x^k indicates a derivative ∂^α for a purely spatial multi-index $\alpha \in \mathbb{N}^3$ of length $|\alpha| = k$. This is out of reach of an estimate of the form $\mathcal{E}_M^{1/2}\mathcal{D}_M$ since we only control a in H^{2M-1} dissipatively. We thus treat this specific interaction (4-34) as a *subcase* of the case of purely spatial derivatives.

To summarize: (recall that $\alpha = (\alpha_0, \bar{\alpha}) \in \mathbb{N}^{1+3}$)

$$-\mathcal{I}_6 = \sum_{|\alpha|_P \leqslant 2M} \sum_{\substack{\beta+\gamma=\alpha \\ \beta>0}} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} (\partial^\beta u \cdot \nabla \partial^\gamma a) \cdot \partial^\alpha a = \sum_{\substack{|\alpha|_P \leqslant 2M \\ \alpha_0 = 0}} \sum_{\cdots} \cdots + \sum_{\substack{|\alpha|_P \leqslant 2M \\ \alpha_0 \geqslant 1}} \sum_{\cdots} \cdots =: \mathrm{I} + \mathrm{II},$$

where I corresponds to purely spatial derivatives and II corresponds to the remaining terms. We break up I further: (where now $\beta, \gamma \in \mathbb{N}^3$ and *not* $\mathbb{N}^{1+3}$ as above)

$$\begin{aligned} \mathrm{I} &= \sum_{\substack{|\beta|+|\gamma| \leqslant 2M \\ \beta > 0}} \binom{\beta+\gamma}{\beta} \int_{\mathbb{T}^3} (\partial^\beta u \cdot \nabla \partial^\gamma a) \cdot \partial^{\beta+\gamma} a \\ &= \sum_{\substack{|\beta|+|\gamma| \leqslant 2M \\ |\beta| = 1}} \binom{\beta+\gamma}{\beta} \int_{\mathbb{T}^3} (\partial^\beta u \cdot \nabla \partial^\gamma a) \cdot \partial^{\beta+\gamma} a + \sum_{\substack{|\beta|+|\gamma| \leqslant 2M \\ |\beta| > 1}} \binom{\beta+\gamma}{\beta} \int_{\mathbb{T}^3} (\partial^\beta u \cdot \nabla \partial^\gamma a) \cdot \partial^{\beta+\gamma} a =: \mathrm{I}_1 + \mathrm{I}_2, \end{aligned}$$

where I_1 consists of the most troublesome term. We also break up II further:

$$\mathrm{II} = \sum_{\substack{|\alpha|_P \leqslant 2M \\ \alpha_0 \geqslant 1}} \sum_{\substack{\beta+\gamma=\alpha \\ \beta>0}} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} (\partial^\beta u \cdot \nabla \partial^\gamma a) \cdot \partial^\alpha a = \sum_{\cdots} \sum_{\substack{\cdots \\ \gamma_0 = 0}} \cdots + \sum_{\cdots} \sum_{\substack{\cdots \\ \gamma_0 \geqslant 1}} \cdots =: \mathrm{II}_1 + \mathrm{II}_2.$$

Again, due to the poorer dissipative control of a compared with $\partial_t a$ and higher-order temporal derivatives of a, we must estimate II_1 carefully. Note that by "poorer control" we mean that we have control of spatial derivatives at a lower parabolic count. To be very clear, we control $\|a\|_{H^{2M-1}}$ and $\|\partial_t a\|_{H^{2M-2}}$ dissipatively, which means that we control a at a parabolic count of $2M-1$ and $\partial_t a$ at a parabolic count of $(2M-2)+2 = 2M$.

Estimating I_1. The key is to integrate by parts at the cost of having to invoke $\mathcal{F}$, which is possibly growing in time, to control $\nabla^{2M+1} a$. Then, where for every γ we pick i such that $\gamma \geqslant e_i$,

$$\begin{aligned} \mathrm{I}_1 &= \sum_{\substack{|\beta|=1 \\ |\gamma| \leqslant 2M-1}} \binom{\beta+\gamma}{\beta} \int_{\mathbb{T}^3} (\partial^\beta u \cdot \nabla \partial^\gamma a) \cdot \partial^{\beta+\gamma} a \\ &= -\sum_{\substack{|\beta|=1 \\ |\gamma| \leqslant 2M-1}} \binom{\beta+\gamma}{\gamma} \left(\int_{\mathbb{T}^3} (\partial^{\beta+e_i} u \cdot \nabla \partial^{\gamma-e_i} a) \cdot \partial^{\beta+\gamma} a + \int_{\mathbb{T}^3} (\partial^\beta u \cdot \nabla \partial^{\gamma-e_i} a) \cdot \partial^{\beta+\gamma+e_i} a \right). \end{aligned}$$

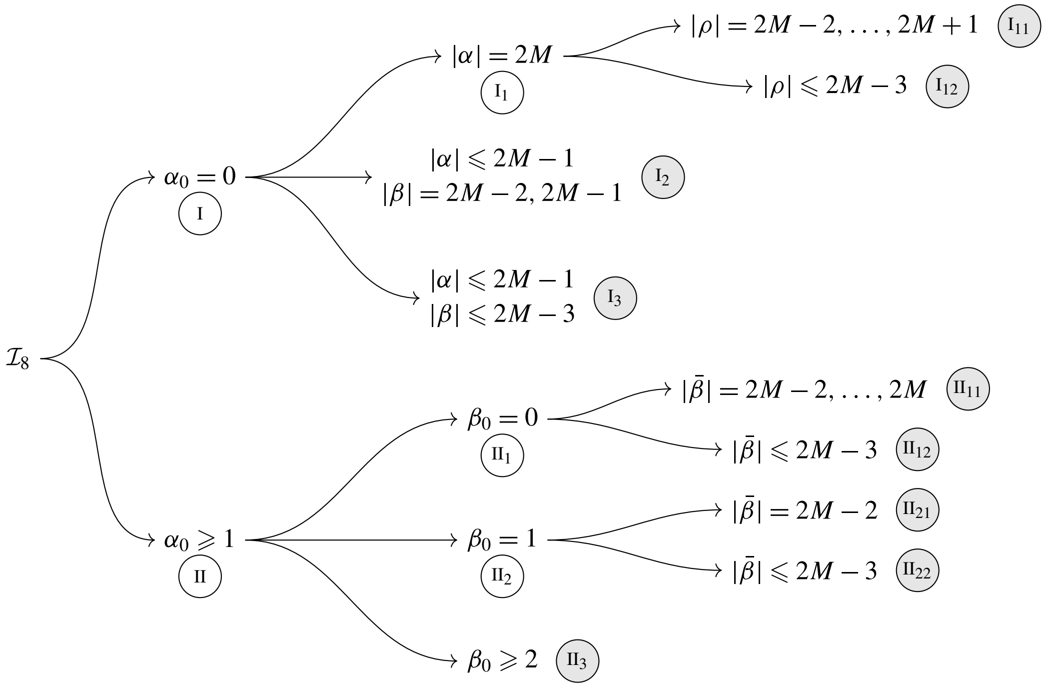

Figure 4. How the terms in the interaction $\mathcal{I}_8$ are broken up in order to be estimated.

Therefore,

$$
\begin{aligned}
|\mathrm{I}_1| &\lesssim \sum_{|\gamma| \leqslant 2M-1} \|\nabla^2 u\|_{L^\infty} \|\nabla \partial^{\gamma - e_i} a\|_{L^2} \|\nabla \partial^\gamma a\|_{L^2} + \|\nabla u\|_{L^\infty} \|\nabla \partial^{\gamma - e_i} a\|_{L^2} \|\nabla \partial^{\gamma + e_i} a\|_{L^2} \\
&\lesssim \mathcal{D}_2^{1/2} \mathcal{D}_M^{1/2} \mathcal{E}_M^{1/2} + \overline{\mathcal{K}}_2^{1/2} \mathcal{D}_M^{1/2} \mathcal{F}_M^{1/2}.
\end{aligned}
$$

Estimating I_2, II_1, and II_2. These terms are all estimated using standard "hands-on high-low estimates", which yield $|\mathrm{I}_2| + |\mathrm{II}_1| + |\mathrm{II}_2| \lesssim \mathcal{E}_M^{1/2} \mathcal{D}_M$. We have thus shown that

$$
|\mathcal{I}_6| \lesssim \mathcal{E}_M^{1/2} \mathcal{D}_M + \mathcal{F}_M^{1/2} \overline{\mathcal{K}}_2^{1/2} \mathcal{D}_M^{1/2}.
$$

Estimating $\mathcal{I}_8$. This is the trickiest interaction to estimate since it involves both $2M$ spatial derivatives of a and temporal derivatives of K. Recall that

$$
\mathcal{I}_8 = \sum_{\substack{|\alpha|_P \leqslant 2M \\ \beta + \gamma = \alpha}} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} \partial^\beta ((\overline{K} - K_{33} I_2) \bar\theta^\perp) \cdot \partial^\alpha a \sim \binom{\alpha}{\beta} \int_{\mathbb{T}^3} (\partial^\beta K)(\partial^\gamma \bar\theta) \cdot (\partial^\alpha a) =: \sum_{\substack{|\alpha|_P \leqslant 2M \\ \beta + \gamma = \alpha}} \mathcal{I}_8^{\alpha, \beta, \gamma},
$$

where the left-hand side is the precise form of the interaction and the right-hand side is its schematic form which we will now estimate. Note that, by contrast with *all* the other interactions, this one does *not* come from a commutator, and therefore there are no restrictions on β and γ besides the fact that $\beta + \gamma = \alpha$ (i.e., there is no restriction $\beta > 0$, or equivalently $\gamma < \alpha$).

To control $\mathcal{I}_8$ we will break it up into several pieces, as summarized pictorially in Figure 4.

More precisely, we now detail how we go about breaking up $\mathcal{I}_8$. First we separate terms for which the derivatives of a are purely spatial:

$$\mathcal{I}_8 = \sum_{\substack{|\alpha|_P \leqslant 2M \\ \beta+\gamma=\alpha}} \mathcal{I}_8^{\alpha,\beta,\gamma} = \sum_{\substack{\cdots \\ \alpha_0=0}} \mathcal{I}_8^{\alpha,\beta,\gamma} + \sum_{\substack{\cdots \\ \alpha_0 \geqslant 1}} \mathcal{I}_8^{\alpha,\beta,\gamma} =: \mathrm{I} + \mathrm{II}.$$

We now split I to account for two factors: the lack of dissipative control of $\partial_x^{2M} a$ and the poorer (i.e., through $\mathcal{F}_M$ and not $\mathcal{E}_M$) control of K when many spatial derivatives are applied to it. Recall that ∂_x^k indicates a derivative ∂^α for a purely spatial multi-index $\alpha \in \mathbb{N}^3$ of length $|\alpha| = k$.

$$\mathrm{I} = \sum_{\substack{\alpha_0=0 \\ |\alpha| \leqslant 2M \\ \beta+\gamma=\alpha}} \mathcal{I}_8^{\alpha,\beta,\gamma} = \sum_{\substack{\cdots \\ |\alpha|=2M}} \mathcal{I}_8^{\alpha,\beta,\gamma} + \sum_{\substack{\cdots \\ |\alpha| \leqslant 2M-1 \\ |\beta|=2M-2,2M-1}} \mathcal{I}_8^{\alpha,\beta,\gamma} + \sum_{\substack{\cdots \\ |\alpha| \leqslant 2M-1 \\ |\beta| \leqslant 2M-3}} \mathcal{I}_8^{\alpha,\beta,\gamma} =: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.$$

Estimating I_1 is the trickiest part of estimating $\mathcal{I}_8$ since we *must* control K via the energy and do *not* have control of $\partial_x^{2M} a$ via the dissipation. To get around this issue we integrate by parts (below, i is an index dependent of α chosen such that $\alpha_i \geqslant 0$, which may always be done since $\alpha \neq 0$):

$$\mathrm{I}_1 = - \sum_{\substack{\alpha_0=0,\ |\alpha|=2M \\ \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \left(\int_{\mathbb{T}^3} (\partial^{\beta+e_i} K)(\partial^\gamma \bar{\theta}) \cdot (\partial^{\alpha-e_i} a) + \int_{\mathbb{T}^3} (\partial^\beta K)(\partial^{\gamma+e_i} \bar{\theta}) \cdot (\partial^{\alpha-e_i} a) \right).$$

This allows us to split I_1 as

$$|\mathrm{I}_1| \lesssim \left| \sum_{\substack{|\pi|=2M-1 \\ |\rho+\sigma|=2M+1}} \int_{\mathbb{T}^3} (\partial^\rho K)(\partial^\sigma \bar{\theta}) \cdot (\partial^\pi a) \right| \leqslant \left| \sum_{\substack{\cdots \\ |\rho|=2M-2,\ldots,2M+1}} \cdots \right| + \left| \sum_{\substack{\cdots \\ |\rho| \leqslant 2M-3}} \cdots \right| =: |\mathrm{I}_{11}| + |\mathrm{I}_{12}|,$$

where π, ρ, and σ are *spatial* multi-indices, i.e., they belong to $\mathbb{N}^3$ and not $\mathbb{N}^{1+3}$. We now direct our attention to II (which is easier to handle than I since more temporal derivatives are involved). The key in the splitting here is that things get easier as more temporal derivatives of K are involved:

$$\mathrm{II} = \sum_{\substack{\alpha_0 \geqslant 1,\ |\alpha|_P \leqslant 2M \\ \beta+\gamma=\alpha}} \mathcal{I}_8^{\alpha,\beta,\gamma} = \sum_{\substack{\cdots \\ \beta_0=0}} \mathcal{I}_8^{\alpha,\beta,\gamma} + \sum_{\substack{\cdots \\ \beta_0=1}} \mathcal{I}_8^{\alpha,\beta,\gamma} + \sum_{\substack{\cdots \\ \beta_0 \geqslant 2}} \mathcal{I}_8^{\alpha,\beta,\gamma} =: \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3.$$

We finally split II_1 and II_2 further depending on the number of spatial derivatives hitting K (since this determines whether we estimate the factor involving K using $\mathcal{E}_M$ or $\mathcal{F}_M$):

$$\mathrm{II}_1 = \sum_{\substack{\alpha_0 \geqslant 1,\ \beta_0=0 \\ |\alpha|_P \leqslant 2M,\ \beta+\gamma=\alpha}} \mathcal{I}_8^{\alpha,\beta,\gamma} = \sum_{\substack{\cdots \\ |\bar{\beta}|=2M-2,\ldots,2M}} \mathcal{I}_8^{\alpha,\beta,\gamma} + \sum_{\substack{\cdots \\ |\bar{\beta}| \leqslant 2M-3}} \mathcal{I}_8^{\alpha,\beta,\gamma} =: \mathrm{II}_{11} + \mathrm{II}_{12}$$

and

$$\mathrm{II}_2 = \sum_{\substack{\alpha_0 \geqslant 1,\ \beta_0=1 \\ |\alpha|_P \leqslant 2M,\ \beta+\gamma=\alpha}} \mathcal{I}_8^{\alpha,\beta,\gamma} = \sum_{\substack{\cdots \\ |\bar{\beta}|=2M-2}} \mathcal{I}_8^{\alpha,\beta,\gamma} + \sum_{\substack{\cdots \\ |\bar{\beta}| \leqslant 2M-3}} \mathcal{I}_8^{\alpha,\beta,\gamma} =: \mathrm{II}_{21} + \mathrm{II}_{22}.$$

Having carefully split $\mathcal{I}_8$ into appropriate pieces, we now proceed to estimate each of these pieces. Note that due to this extensive subdivision of $\mathcal{I}_8$ into various pieces, most terms can be handled with similar techniques. The table below summarizes how each term is handled.

	I_{11}	I_{12}	I_2	I_3	II_{11}	II_{12}	II_{21}	II_{22}	III
direct			X	X			X	X	
hands-on high-low						X			
Corollary B.2	X	X							
special consideration					X				X

Direct estimates. The terms I_3, II_{21}, and II_{22} can be estimated directly:

$$|\mathrm{I}_3| \lesssim \sum_{\dots} \|\partial^\beta K\|_{L^2} \|\partial^\gamma \bar\theta\|_{L^\infty} \|\partial^\alpha a\|_{L^2} \lesssim \|K\|_{H^{2M-3}} \|\bar\theta\|_{H^{2M+1}} \|a\|_{H^{2M-1}} \lesssim \mathcal{E}_M^{1/2} \mathcal{D}_M^{1/2} \mathcal{D}_M^{1/2},$$
$$|\mathrm{II}_{21}| \lesssim \sum_{\dots} \|\partial_t \partial_x^{\bar\beta} K\|_{L^2} \|\theta\|_{L^\infty} \|\partial^\alpha a\|_{L^2} \lesssim \|\partial_t K\|_{H^{2M-2}} \|\theta\|_{H^2} \|\partial_t a\|_{P^{2M-2}} \lesssim \mathcal{F}_M^{1/2} \overline{\mathcal{K}}_1^{1/2} \mathcal{D}_M^{1/2},$$
$$|\mathrm{II}_{22}| \lesssim \sum_{\dots} \|\partial^\beta K\|_{L^2} \|\partial^\gamma \bar\theta\|_{L^\infty} \|\partial^\alpha a\|_{L^2} \lesssim \|\partial_t K\|_{H^{2M-3}} \|\bar\theta\|_{P^{2M}} \|\partial_t a\|_{P^{2M-2}} \lesssim \mathcal{E}_M^{1/2} \mathcal{D}_M^{1/2} \mathcal{D}_M^{1/2}.$$

Similarly, I_2 can be split into precisely three terms which can all be estimated directly:

$$\mathrm{I}_2 = \sum_{\substack{\alpha_0=0,\ \alpha=\beta \\ |\alpha|=2M-1}} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} (\partial^\beta K)\bar\theta \cdot (\partial^\alpha a)$$
$$+ \sum_{\substack{\alpha_0=0,\ |\alpha|=2M-1 \\ |\beta|=2M-2,\ \gamma=\alpha-\beta,\ |\gamma|_P=1}} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} (\partial^\beta K)(\partial^\gamma \bar\theta) \cdot (\partial^\alpha a) + \sum_{\substack{\alpha_0=0,\ \alpha=\beta \\ |\alpha|=2M-2}} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} (\partial^\beta K)\bar\theta \cdot (\partial^\alpha a)$$

such that

$$|\mathrm{I}_2| \lesssim \|\nabla^{2M-1} K\|_{L^\infty} \|\bar\theta\|_{L^2} \|\nabla^{2M-1} a\|_{L^2} + \|\nabla^{2M-2} K\|_{L^\infty} \|\nabla \bar\theta\|_{L^2} \|\nabla^{2M-1} a\|_{L^2}$$
$$+ \|\nabla^{2M-2} K\|_{L^\infty} \|\bar\theta\|_{L^2} \|\nabla^{2M-2} a\|_{L^2} \lesssim \|K\|_{H^{2M+1}} \|\bar\theta\|_{H^1} \|a\|_{H^{2M-1}} \lesssim \mathcal{F}_M^{1/2} \overline{\mathcal{K}}_1^{1/2} \mathcal{D}_M^{1/2}.$$

Hands-on high-low estimates. II_{12} can be estimated using "hands-on high-low estimates". We split II_{12} as

$$\mathrm{II}_{12} = \sum_{\substack{\alpha_0 \geqslant 1,\ \beta_0=0,\ \beta+\gamma=\alpha \\ |\alpha|_P \leqslant 2M,\ |\bar\beta| \leqslant 2M-3}} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} (\partial^\beta K)(\partial^\gamma \bar\theta) \cdot (\partial^\alpha a) \leqslant \sum_{\substack{\dots \\ |\bar\beta| \leqslant 2M-5}} \underbrace{|\cdots|}_{(1)} + \sum_{\substack{\dots \\ |\gamma|_P \leqslant 2M-1}} \underbrace{|\cdots|}_{(2)},$$

where

$$(1)+(2) \leqslant \|\partial^{\bar\beta} K\|_{L^\infty} \|\partial^\gamma \bar\theta\|_{L^2} \|\partial^\alpha a\|_{L^2} + \|\partial^{\bar\beta} K\|_{L^2} \|\partial^\gamma \bar\theta\|_{L^\infty} \|\partial^\alpha a\|_{L^2} \lesssim \mathcal{E}_M^{1/2} \mathcal{D}_M.$$

Using Corollary B.2. The terms I_{11} and I_{12} can be estimated using Corollary B.2, which provides a way to use the Gagliardo–Nirenberg inequality to obtain bounds on products of derivatives in L^2. Indeed,

$$|\mathrm{I}_{11}| \lesssim \sum_{\substack{p=2M-1 \\ r=2M-2,\dots,2M+1 \\ r+s=2M+1}} C(p,r) \int_{\mathbb{T}^3} |\nabla^r K| |\nabla^s \bar\theta| |\nabla^p a| \lesssim \sum_{\dots} \| |\nabla^{r-(2M-2)} \nabla^{2M-2} K| |\nabla^s \bar\theta| \|_{L^2} \|\nabla^p a\|_{L^2}$$
$$\lesssim \sum_{\substack{r=2M-2,\dots,2M+1 \\ r+s=2M+1}} (\|\nabla^{2M-2} K\|_{L^\infty} \|\bar\theta\|_{H^{r+s-(2M-2)}} + \|\nabla^{2M-2} K\|_{H^{r+s-(2M-2)}} \|\bar\theta\|_{L^\infty}) \|a\|_{H^{2M-1}}$$
$$\lesssim (\|K\|_{H^{2M}} \|\bar\theta\|_{H^3} + \|K\|_{H^{2M+1}} \|\bar\theta\|_{H^2}) \|a\|_{H^{2M-1}} \lesssim \|K\|_{H^{2M+1}} \|\bar\theta\|_{H^3} \|a\|_{H^{2M-1}}$$

such that $|\mathrm{I}_{11}| \lesssim \mathcal{F}_M^{1/2} \overline{\mathcal{K}}_2^{1/2} \mathcal{D}_M^{1/2}$, and, similarly,

$$|\mathrm{I}_{12}| \leqslant \sum_{\substack{p=2M-1 \\ r+s=2M+1 \\ r\leqslant 2M-3}} C(p,r) \int_{\mathbb{T}^3} |\nabla^r K| |\nabla^s \bar{\theta}| |\nabla^p a| \lesssim \sum_{\cdots} \| |\nabla^r K| |\nabla^{s-4} \nabla^4 \bar{\theta}| \|_{L^2} \|\nabla^p a\|_{L^2} \lesssim \mathcal{E}_M^{1/2} \mathcal{D}_M.$$

<u>Special consideration</u>. Finally, we estimate II_{11} and II_3. Recall that

$$\mathrm{II}_{11} = \sum_{\substack{\alpha_0 \geqslant 1,\, \beta_0 = 0,\, |\alpha|_P \leqslant 2M \\ |\bar{\beta}| = 2M-2,\ldots,2M,\, \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} (\partial^\beta K)(\partial^\gamma \bar{\theta}) \cdot (\partial^\alpha \alpha).$$

There are two important observations to make here.

- Since $\alpha_0 \geqslant 1$ and $|\alpha|_P \leqslant 2M$, we control $\partial^\alpha a$ in $\mathcal{D}_M$ in the following way:

$$\|\partial^\alpha a\|_{L^2} \lesssim \|\partial_t a\|_{P^{2M-2}} \lesssim \mathcal{D}_M^{1/2}.$$

- We must use $\mathcal{F}_M$ to control $\partial^\beta K$ when $\beta_0 = 0$ and $|\bar{\beta}| \geqslant 2M-2$, so we therefore ask as little regularity as possible of $\bar{\theta}$ (to invoke $\mathcal{E}_I$ for the smallest possible I). We thus split the estimate depending on whether or not we can control $\partial^\beta K$ in L^∞ via $\mathcal{F}_M$.

We obtain

$$|\mathrm{II}_{11}| \lesssim \sum_{\substack{\cdots \\ |\bar{\beta}|=2M-2,2M-1}} \cdots + \sum_{\substack{\cdots \\ |\bar{\beta}|=2M}} \cdots \lesssim \sum_{\cdots} \|\partial^\beta K\|_{L^\infty} \|\partial^\gamma \bar{\theta}\|_{L^2} \|\partial^\alpha a\|_{L^2} + \sum_{\cdots} \|\partial^\beta K\|_{L^4} \|\bar{\theta}\|_{L^4} \|\partial^\alpha a\|_{L^2}$$
$$\lesssim \|K\|_{H^{2M+1}} \|\bar{\theta}\|_{P^2} \|\partial_t a\|_{P^{2M-2}} + \|K\|_{H^{2M+1}} \|\bar{\theta}\|_{H^1} \|\partial_t a\|_{P^{2M-2}} \lesssim \mathcal{F}_M^{1/2} \overline{\mathcal{K}}_1^{1/2} \mathcal{D}_M^{1/2}.$$

Now we estimate II_3. Recall that

$$\mathrm{II}_3 = \sum_{\substack{\alpha_0 \geqslant 1,\, |\alpha|_P \leqslant 2M \\ \beta_0 \geqslant 2,\, \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} (\partial^\beta K)(\partial^\gamma \bar{\theta}) \cdot (\partial^\alpha a).$$

The key observation is the following: when $\beta_0 \geqslant 2$ we control $\partial_t^{\beta_0} K$ at parabolic order $2M+1$. Consequently, if $|\beta|_P \leqslant 2M-1$ then we control $\partial^\beta K$ in L^∞ via $\mathcal{E}_M$ since

$$\|\partial^\beta K\|_{L^\infty} \lesssim \|\partial_t^2 K\|_{P^{2+|\beta|_P-4}} \lesssim \|\partial_t^2 K\|_{P^{2M-3}} \lesssim \mathcal{E}_M^{1/2}.$$

We may then estimate II_3 with the usual "hands-on high-low" estimates. We note that, since $|\alpha|_P \leqslant 2M$ and $\beta + \gamma = \alpha$, it follows from Corollary B.2 that, as long as $M \geqslant 1$, either $|\beta|_P \leqslant 2M-1$ or $|\gamma|_P \leqslant 2M-1$. Therefore,

$$|\mathrm{II}_3| \lesssim \sum_{\substack{\cdots \\ |\beta|_P \leqslant 2M-1}} \underbrace{|\cdots|}_{(1)} + \sum_{\substack{\cdots \\ |\gamma|_P \leqslant 2M-1}} \underbrace{|\cdots|}_{(2)},$$

where

$$(1) + (2) \leqslant \|\partial^\beta K\|_{L^\infty} \|\partial^\gamma \bar{\theta}\|_{L^2} \|\partial^\alpha a\|_{L^2} + \|\partial^\beta K\|_{L^2} \|\partial^\gamma \bar{\theta}\|_{L^\infty} \|\partial^\alpha a\|_{L^2} \lesssim \mathcal{E}_M^{1/2} \mathcal{D}_M.$$

Putting all these estimates together we see that we have obtained $|\mathcal{I}_8| \lesssim \mathcal{E}_M^{1/2} \mathcal{D}_M + \mathcal{F}_M^{1/2} \overline{\mathcal{K}}_2^{1/2} \mathcal{D}_M^{1/2}$. $\square$

As mentioned at the beginning of this section, at the high level both the improvement to the dissipation and the control of the interaction only allows us to close the energy estimates in a *time-integrated* fashion. This is because the closure of the estimates relies crucially on playing the potential growth of $\mathcal{F}_M$ against the decay of intermediate norms $\overline{\mathcal{K}}_I$ and $\mathcal{K}_{\text{low}}$. The next two results record precisely this balancing act between growth and decay. First we consider the growth-decay interactions arising from the improvement of the dissipation.

Lemma 4.27. *Suppose that $M \geqslant 3$ and that, for some time horizon $T > 0$,*

$$\sup_{1 \leqslant I \leqslant M} \sup_{0 \leqslant t \leqslant T} \overline{\mathcal{K}}_I(t)(1+t)^{2M-2I} =: C_0 < \infty \tag{4-35}$$

and, for every $0 \leqslant t \leqslant T$,

$$\mathcal{F}_M^{1/2}(t) \lesssim \alpha_0^{1/2} + \beta_0^{1/2} \int_0^t \overline{\mathcal{D}}_M^{1/2}(s)\,ds + \gamma_0^{1/2} \overline{\mathcal{K}}_M^{1/2}(t)$$

for some $\alpha_0,\ \beta_0,\ \gamma_0 > 0$. Then, for every $0 \leqslant t \leqslant T$,

$$\int_0^t \|(u,\theta,\partial_t\theta)\|_{L^\infty}^2 \mathcal{F}_M \lesssim \alpha_0 C_0 + (\beta_0+\gamma_0) C_0 \int_0^t \overline{\mathcal{D}}_M(s)\,ds.$$

Proof. First we note that by interpolating (4-35) we may obtain decay estimates for fractional Sobolev norms — this is very similar to what was done by interpolation in Step 1 of Proposition 4.8. Indeed, for any $s \in \mathbb{R}$ which satisfies $2 \leqslant s \leqslant 2M$, we may pick $\sigma = (2M-s)/(2M-2)$ and deduce that

$$\|(u,\theta)\|_{H^s}^2 \lesssim \|(u,\theta)\|_{H^2}^{2\sigma} \|(u,\theta)\|_{H^{2M}}^{2(1-\theta)} \leqslant C_0^\sigma (1+t)^{(2M-2)\sigma} C_0^{1-\sigma} = C_0(1+t)^{2M-s}$$

and, similarly, $\|\partial_t(u,\theta)\|_{H^s}^2 \lesssim C_0(1+t)^{2M-2-s}$. Crucially, $s = \frac{7}{4}$ satisfies both $s > \frac{3}{2}$, such that $H^{7/4}$ embeds continuously into L^∞, and $2M-2-s > 2$ (since $M \geqslant 3$) such that the resulting decaying bound is integrable. The term $\|(u,\theta,\partial_t\theta)\|_{L^\infty}^2$ may then be shown to decay fast enough, in the space $H^{7/4}$, to justify the estimate.

Using Cauchy–Schwarz on $\int \overline{\mathcal{D}}_M^{1/2}$ thus tells us that

$$\begin{aligned}
\int_0^t \|(u,\theta,\partial_t\theta)\|_{L^\infty}^2 \mathcal{F}_M &\lesssim \int_0^t \left(\frac{C_0}{(1+s)^{2M-2-7/4}}\right)\left(\alpha_0 + \beta_0\left(\int_0^s \overline{\mathcal{D}}_M^{1/2}(r)\,dr\right)^2 ds\right) + \gamma_0 C_0 \int_0^t \overline{\mathcal{D}}_M \\
&\lesssim \alpha_0 C_0 + \beta_0 C_0 \int_0^t \frac{s}{(1+s)^{2M-15/4}} \left(\int_0^s \overline{\mathcal{D}}_M(r)\,dr\right) ds + \gamma_0 C_0 \int_0^t \overline{\mathcal{D}}_M \\
&\lesssim \alpha_0 C_0 + (\beta_0+\gamma_0) C_0 \int_0^t \overline{\mathcal{D}}_M(s)\,ds. \qquad \square
\end{aligned}$$

Now we consider the growth-decay interactions arising from the control of the high-level interactions.

Lemma 4.28. *Suppose that $M \geqslant 4$, that*

$$\mathcal{F}_M^{1/2}(t) \lesssim \alpha_0^{1/2} + \beta_0^{1/2} \int_0^t \overline{\mathcal{D}}_M^{1/2}(s)\,ds + \gamma_0^{1/2} \overline{\mathcal{K}}_M^{1/2}(t)$$

for some $\alpha_0,\ \beta_0,\ \gamma_0 > 0$, and that $\mathcal{K}_{\text{low}}(t) \lesssim \min(\overline{\mathcal{D}}_2(t), C_0(1+t)^{-(2M-4)})$. Then

$$\int_0^t \mathcal{K}_{\text{low}}^{1/2}(s) \mathcal{F}_M^{1/2}(s) \mathcal{D}_M^{1/2}(s)\,ds \lesssim (\alpha_0^{1/2} + \beta_0^{1/2} C_0^{1/2} + \gamma_0^{1/2} C_0^{1/2}) \int_0^t \mathcal{D}_M(s)\,ds.$$

Proof. By applying Cauchy–Schwarz to $\int \mathcal{D}_M^{1/2}$ we deduce that

$$
\begin{aligned}
\int_0^t \mathcal{K}_{\mathrm{low}}^{1/2}(s)\mathcal{F}_M^{1/2}(s)\mathcal{D}_M^{1/2}(s)\,ds \\
&\lesssim \alpha_0^{1/2}\int_0^t \overline{\mathcal{D}}_2^{1/2}\mathcal{D}_M^{1/2} + \beta_0^{1/2}\int_0^t \frac{C_0^{1/2}}{(1+s)^{M-2}}\left(\int_0^s \overline{\mathcal{D}}_M^{1/2}(r)\,dr\right)\mathcal{D}_M^{1/2}(s)\,ds + \gamma_0^{1/2}C_0^{1/2}\int_0^t \mathcal{D}_M \\
&\lesssim (\alpha_0+\gamma_0^{1/2}C_0^{1/2})\int_0^t \mathcal{D}_M + \beta_0^{1/2}C_0^{1/2}\underbrace{\int_0^t \frac{\mathcal{D}_M^{1/2}(s)}{(1+s)^{M-2}}s^{1/2}\left(\int_0^t \overline{\mathcal{D}}_M(r)\,dr\right)^{1/2}ds}_{(\star)}\,.
\end{aligned}
$$

Employing the Cauchy–Schwarz inequality again and noting that $2M-5>1$ (since $M \geqslant 4$), we see that

$$
\begin{aligned}
(\star) &\lesssim \left(\int_0^t \frac{\mathcal{D}_M^{1/2}(s)}{(1+s)^{M-5/2}}\,ds\right)\left(\int_0^t \overline{\mathcal{D}}_M(s)\,ds\right)^{1/2} \\
&\leqslant \left(\int_0^t \frac{1}{(1+s)^{2M-5}}\,ds\right)^{1/2}\left(\int_0^t \mathcal{D}_M(s)\,ds\right) \lesssim \int_0^t \mathcal{D}_M(s)\,ds. \qquad \square
\end{aligned}
$$

We conclude this section with the third of the four building blocks of the scheme of a priori estimates and close the interactions at the high level. This is done in Proposition 4.29 which synthesizes the results of this section. In particular, recall that $\bar{\mathcal{E}}_M$, which appears in Proposition 4.29 below, is defined in (3-2).

Proposition 4.29 (closing the energy estimates at the high level). *Let $M \geqslant 4$ be an integer. There exist $\eta_M > 0$, $0<\delta_M \leqslant 1$, and $C_H > 0$ such that the following holds: for any time horizon $T>0$, any $0<\eta\leqslant\eta_M$, any $0<\delta\leqslant\delta_M$, and any $C>0$, if*

$$
\begin{cases}
(\mathcal{E}_M+\mathcal{F}_M)(0) \leqslant \eta, & \text{(4-36a)}\\
\sup_{0\leqslant t\leqslant T}\sup_{1\leqslant I\leqslant M}\overline{\mathcal{K}}_I(t)(1+t)^{2M-2I}+\mathcal{K}_{\mathrm{low}}(t)(1+t)^{2M-4}\leqslant\delta, & \text{(4-36b)}\\
\sup_{0\leqslant t\leqslant T}\mathcal{E}_M(t)\leqslant\delta, & \text{(4-36c)}\\
\mathcal{F}_M(t)\leqslant C\big(\mathcal{F}_M(0)+\big(\int_0^t\overline{\mathcal{D}}_M^{1/2}(s)\,ds\big)^2+\overline{\mathcal{K}}_M(t)\big) \quad \textit{for all } 0\leqslant t\leqslant T, & \text{(4-36d)}
\end{cases}
$$

then

$$
\sup_{0\leqslant t\leqslant T}\bar{\mathcal{E}}_M(t)+\int_0^t \mathcal{D}_M(s)\,ds \leqslant C_H(\mathcal{E}_M+\mathcal{F}_M)(0). \tag{4-37}
$$

Proof. The basic idea of the proof is that we want to go from the energy-dissipation of the problem, namely $(\mathrm{d}/\mathrm{d}t)\tilde{\mathcal{E}}_M+\overline{\mathcal{D}}_M\lesssim\overline{\mathcal{I}}_M$, to the more useful energy-dissipation relation $(\mathrm{d}/\mathrm{d}t)\tilde{\mathcal{E}}_M+C\mathcal{D}_M\leqslant 0$, where $C>0$ is a universal constant and where the nonnegativity of the improved dissipation $\mathcal{D}_M$ ensures the boundedness of the energy $\mathcal{E}_M$. This is done by controlling the interactions and improving the dissipation. However, both of these steps, which are performed precisely in Propositions 4.26 and 4.24, respectively, are delicate and lead to the appearance of terms that must be controlled in a time-integrated fashion — this control is recorded in Lemmas 4.27 and 4.28.

First we note that, in light of (4-36b) and (4-36d), Lemmas 4.27 and 4.28 tell us, respectively, that, since $\delta_M\leqslant 1$,

$$
\int_0^t \|(u,\theta,\partial_t\theta)\|_{L^\infty}^2\mathcal{F}_M \lesssim \mathcal{F}_M(0)+\delta\int_0^t\overline{\mathcal{D}}_M \tag{4-38}
$$

and

$$\int_0^t \mathcal{K}_{\mathrm{low}}^{1/2}\mathcal{F}_M^{1/2}\mathcal{D}_M^{1/2} \lesssim (\delta^{1/2}+\mathcal{F}_M^{1/2}(0)) \int_0^t \mathcal{D}_M. \tag{4-39}$$

We may now proceed with the energy estimates. Lemmas 4.10 and 4.12 tell us that

$$\tilde{\mathcal{E}}_M(t) + \int_0^t \overline{\mathcal{D}}_M(s)\,ds \lesssim \tilde{\mathcal{E}}_M(0) + \int_0^t \overline{\mathcal{I}}_M(s)\,ds. \tag{4-40}$$

Combining the fact that

$$\sup_{0\leqslant t\leqslant T} \|(u,\theta)\|_{H^3}^2 + \|J\|_{H^3}^2 + \|\partial_t(u,\theta)\|_{H^2}^2 + \|\partial_t J\|_{H^2}^2 \leqslant \sup_{0\leqslant t\leqslant T} \mathcal{E}_M(t) \leqslant 1$$

with Proposition A.3 and Lemma 4.11, we obtain that

$$\tilde{\mathcal{E}}_M \asymp \bar{\mathcal{E}}_M. \tag{4-41}$$

We may now use (4-41) and Proposition 4.24 first, then use (4-40), Proposition 4.26, and (4-36c) to see that

$$\begin{aligned}\bar{\mathcal{E}}_M(t) + \int_0^t \mathcal{D}_M &\lesssim \tilde{\mathcal{E}}_M(t) + \int_0^t \overline{\mathcal{D}}_M + \int_0^t \|(u,\theta,\partial_t\theta)\|_{L^\infty}^2 \mathcal{F}_M \\ &\lesssim \tilde{\mathcal{E}}_M(0) + \delta^{1/2}\int_0^t \mathcal{D}_M + \int_0^t \mathcal{K}_{\mathrm{low}}^{1/2}\mathcal{F}_M^{1/2}\mathcal{D}_M^{1/2} + \int_0^t \|(u,\theta,\partial_t\theta)\|_{L^\infty}^2 \mathcal{F}_M.\end{aligned}$$

Combining this with (4-38), (4-39), and (4-41) allows us to deduce that there exists $C_s > 0$ such that

$$\bar{\mathcal{E}}_M(t) + \int_0^t \mathcal{D}_M \leqslant C_s(\mathcal{E}_M + \mathcal{F}_M)(0) + C_s(\delta^{1/2} + \mathcal{F}_M^{1/2}(0)) \int_0^t \mathcal{D}_M.$$

In particular, if $\eta_M, \delta_M > 0$ are chosen sufficiently small to ensure that $C_s(\delta_M^{1/2} + \eta_M^{1/2}) \leqslant \frac{1}{2}$, then we may deduce (4-37). □

4E. *Decay of intermediate norms.* In this section we consider the last of the four building blocks of our scheme of a priori estimates and proceed with the interpolation argument required to obtain the decay of intermediate norms provided that both the low-level and high-level energies are controlled. This is supplemented by an auxiliary estimate for $\partial_t^2\theta$ whose purpose is to improve $\overline{\mathcal{K}}_2$ in order to control the term involving $\partial_t^2\theta$ which appears when controlling the high-order interactions — recall that this is discussed in detail in Section 2D. Note that the functionals $\bar{\mathcal{E}}_M$, $\bar{\mathcal{E}}_{\mathrm{low}}$, $\mathcal{E}_M$, and $\overline{\mathcal{K}}_I$ and $\mathcal{K}_{\mathrm{low}}$, which will be used throughout this section, are defined in (3-2), (3-3), (3-5), and (3-6), respectively. We begin with the interpolation argument.

Proposition 4.30 (decay of intermediate norms). *Suppose that there exists a time horizon $T > 0$, an integer $M \geqslant 2$, and a constant $C_0 > 0$ such that*

$$\sup_{0\leqslant t\leqslant T} \bar{\mathcal{E}}_{\mathrm{low}}(t)(1+t)^{2M-2} + \bar{\mathcal{E}}_M(t) \leqslant C_0. \tag{4-42}$$

Then there exists a constant $C_I > 0$ which depends on M and is universal otherwise such that we may estimate $\sup_{1\leqslant I\leqslant M} \sup_{0\leqslant t\leqslant T} \overline{\mathcal{K}}_I(t)(1+t)^{2M-2I} \leqslant C_I C_0$.

Proof. This estimate on intermediate norms follows from the interpolation of H^s spaces which says that, if $f \in H^l \cap H^h$, then $f \in H^i$ for any $l \leqslant i \leqslant h$, with the interpolation estimate $\|f\|_{H^i} \lesssim \|f\|_{H^l}^{\theta} \|f\|_{H^h}^{1-\theta}$, where $\theta = (h-i)/(h-l)$. Therefore, for $\theta = (M-I)/(M-1)$,

$$\begin{aligned}
\overline{\mathcal{K}}_I &\lesssim \|(u,\theta,a)\|_{H^2}^{2\theta} \|(u,\theta,a)\|_{H^{2M}}^{2(1-\theta)} + \|\partial_t(u,\theta,a)\|_{L^2}^{2\theta} \|\partial_t(u,\theta,a)\|_{H^{2M-2}}^{2(1-\theta)} \\
&\leqslant 2\big(\|(u,\theta,a)\|_{H^2}^2 + \|\partial_t(u,\theta,a)\|_{L^2}^2\big)^{\theta} \big(\|(u,\theta,a)\|_{H^{2M}}^2 + \|\partial_t(u,\theta,a)\|_{H^{2M-2}}^2\big)^{1-\theta} \\
&\lesssim \bar{\mathcal{E}}_{\mathrm{low}}^{\theta} \bar{\mathcal{E}}_M^{1-\theta} \lesssim \left(\frac{C_0}{(1+t)^{2M-2}}\right)^{\theta} C_0^{1-\theta} = \frac{C_0}{(1+t)^{2M-2I}}. \qquad \square
\end{aligned}$$

We now record an auxiliary estimate for $\partial_t^2\theta$ which will be used to deduce the decay of $\partial_t^2\theta$ when $\overline{\mathcal{K}}_2$ decays.

Lemma 4.31 (auxiliary estimate for $\partial_t^2\theta$). *Suppose that (2-1c) holds. Then, for $J = J_{\mathrm{eq}} + K$,*

$$\begin{aligned}
\|J\partial_t^2\theta\|_{L^2} \lesssim \|\partial_t a\|_{L^2} + \|(u,\theta)\|_{P^2} + (1 + \|K\|_{L^\infty} + \|\partial_t K\|_{L^\infty})\|\theta\|_{P^1} \\
+ (1 + \|K\|_{L^\infty} + \|\partial_t K\|_{L^\infty})(1 + \|(u,\theta)\|_{L^\infty})\|(u,\theta)\|_{P^2}.
\end{aligned}$$

Proof. This estimate follows immediately from differentiating (2-1c) in time. $\square$

We now improve the control afforded to us by $\overline{\mathcal{K}}_2$ so as to also control the term involving $\partial_t^2\theta$ which appears when controlling the high-level interactions.

Corollary 4.32 (improvement of $\overline{\mathcal{K}}_2$). *For any time horizon $T > 0$, if*

$$\sup_{0\leqslant t<T} \|(u,\theta)(t)\|_{H^3} + \|J(t)\|_{H^3} + \|\partial_t(u,\theta)\|_{H^2} + \|\partial_t J\|_{H^2} < \infty \tag{4-43}$$

and $\bar{\mathcal{E}}_3 \leqslant 1$ on $[0,T)$, then $\|\partial_t\theta\|_{L^2} \lesssim (1+\mathcal{E}_3^{1/2})(\|\partial_t a\|_{L^2} + \overline{\mathcal{K}}_2^{1/2})$ holds in $[0,T)$, where the constant implicit in "$\lesssim$" is independent of the time horizon T.

Proof. It is crucial to recall here the global assumption that the spectrum of $J_0(x)$ is equal to $\{\lambda,\lambda,\nu\}$, where $\nu > \lambda > 0$, for every $x \in \mathbb{T}^3$. The key observation is then that the assumption (4-43) combines with Proposition A.3 to tell us that $\|\partial_t^2\theta\|_{L^2} \leqslant \lambda^{-1}\|J\partial_t^2\theta\|_{L^2}$. The result then follows from Lemma 4.31. $\square$

To conclude this section we record the decay of $\mathcal{K}_{\mathrm{low}}$, which is the improved version of $\overline{\mathcal{K}}_2$ which also controls $\partial_t^2\theta$.

Corollary 4.33 (decay of $\mathcal{K}_{\mathrm{low}}$). *Suppose that there exists a time horizon $T > 0$, an integer $M \geqslant 2$, and a constant $C_0 > 0$ such that*

$$\sup_{0\leqslant t\leqslant T} \bar{\mathcal{E}}_{\mathrm{low}}(t)(1+t)^{2M-2} + \bar{\mathcal{E}}_M(t) \leqslant C_0 \leqslant 1 \quad \textit{and} \quad \sup_{0\leqslant t\leqslant T} \|J(t)\|_{H^3} + \|\partial_t J(t)\|_{H^2} < \infty.$$

Then $\sup_{0\leqslant t\leqslant T} \mathcal{K}_{\mathrm{low}}(t)(1+t)^{2M-4} \leqslant \widetilde{C}_I C_0$ for some constant $\widetilde{C}_I > 0$, which depends only on M and is universal otherwise.

Proof. This follows directly from combining Proposition 4.30 and Corollary 4.32. $\square$

4F. *Synthesis.* In this section we put together all four building blocks of our scheme of a priori estimates that we have constructed in Sections 4A–4E. This allows us to state and prove our main "a priori estimates" result in Theorem 4.34 below. Recall that the various energy and dissipation functionals encountered in the statement and the proof of the theorem below are defined in (3-2)–(3-8).

Theorem 4.34 (a priori estimates). *Let $M \geqslant 4$. There exist $\eta_{\text{ap}}, \delta_{\text{ap}}, C_{\text{ap}} > 0$ depending only on M such that if (u, p, θ, K) is a solution of (2-1) on the time interval $[0, T]$, for any $T > 0$, which satisfies the smallness conditions*

$$(\mathcal{E}_M + \mathcal{F}_M)(0) =: \eta_0 \leqslant \eta_{\text{ap}} \leqslant 1 \tag{4-44}$$

and

$$\sup_{0 \leqslant t \leqslant T} \mathcal{E}_M(t) + \int_0^T \overline{\mathcal{D}}_M \leqslant \delta_{\text{ap}} \leqslant 1, \tag{4-45}$$

then the following estimates hold:

$$\sup_{0 \leqslant t \leqslant T} \mathcal{E}_{\text{low}}(t)(1+t)^{2M-2} + \bar{\mathcal{E}}_M(t) + \mathcal{E}_M^a(t) + \frac{\mathcal{F}_M(t)}{1+t} + \int_0^T \mathcal{D}_M \leqslant C_{\text{ap}}(\bar{\mathcal{E}}_M + \mathcal{F}_M)(0)$$

and

$$\sup_{0 \leqslant t \leqslant T} \mathcal{E}_M^{(K)}(t) \leqslant C_{\text{ap}}(\mathcal{E}_M + \mathcal{F}_M)(0).$$

Proof. We take two passes at the estimates in this proof. During the first pass we obtain *unstructured* estimates, meaning that the estimates are in term of the smallness parameter and not the initial conditions. During the second pass we obtain *structured* estimates, meaning that the estimates are in terms of the initial conditions.

Both of these passes rely on the four key results we have proved in Section 4, namely Proposition 4.8 where we record the advection-rotation estimates for K, Proposition 4.20 where we close the energy estimates at the low level, Proposition 4.29 where we close the energy estimates at the high level, and Proposition 4.30 and Corollary 4.33 where we obtain the decay of the intermediate norms.

Before beginning the proof in earnest we record the smallness conditions which δ_{ap} and η_{ap} must satisfy

(1) $\delta_{\text{ap}} \leqslant \max(\delta_{\text{low}}, \delta_{\text{low}}^*)$ for δ_{low} and δ_{low}^* as in Proposition 4.20,

(2) $(1 + C_L)\delta_{\text{ap}} \leqslant 1$ for C_L as in Proposition 4.20,

(3) $(1 + (C_I + \widetilde{C}_I)(1 + C_L))\delta_{\text{ap}} \leqslant 1$ for C_I and $\widetilde{C}_I$ as in Proposition 4.30 and Corollary 4.33, respectively,

(4) $\eta_{\text{ap}} \leqslant \eta_M$ for η_M as in Proposition 4.29,

(5) $(C_I + \widetilde{C}_I)(1 + C_L)\delta_{\text{ap}} \leqslant \delta_M$ for δ_M as in Proposition 4.29,

(6) $\delta_{\text{ap}} \leqslant \delta_M$,

(7) $C_H \eta_{\text{ap}} \leqslant \delta_{\text{low}}$ for C_H as in Proposition 4.29,

(8) $(1 + C_L)C_H \eta_{\text{ap}} \leqslant 1$, and

(9) $(1 + (C_I + \widetilde{C}_I)(1 + C_L))C_H \eta_{\text{ap}} \leqslant 1$.

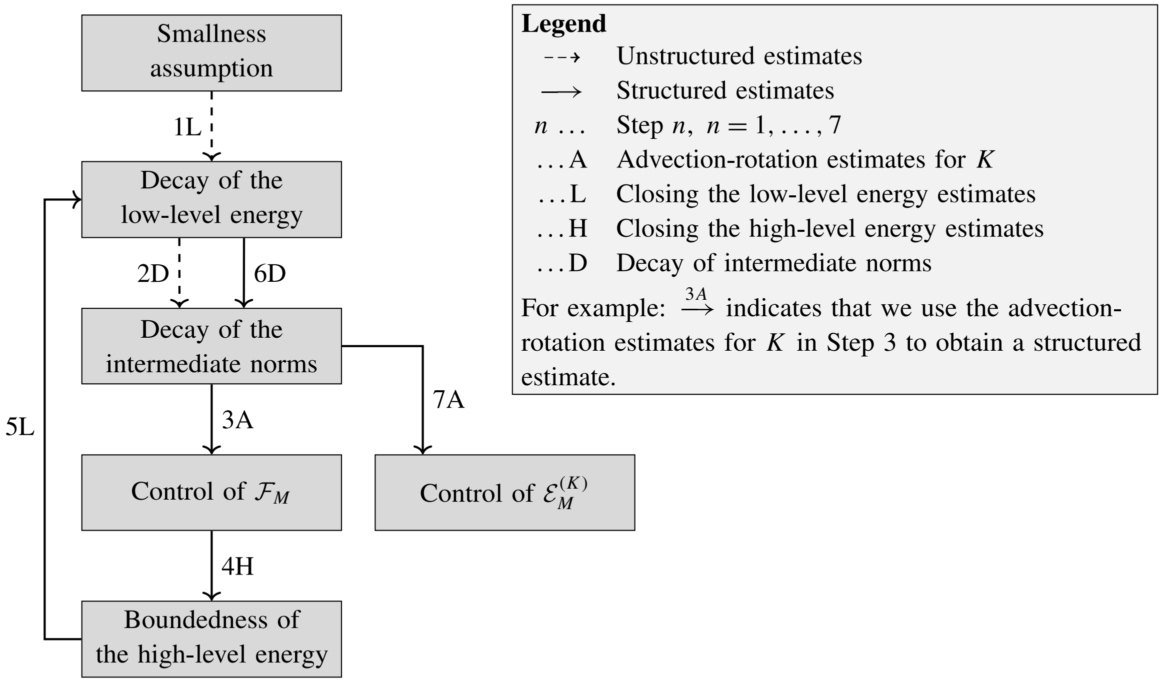

Figure 5. The strategy of the theorem for the main a priori estimates, namely Theorem 4.34.

To be very clear about the structure of the proof we break it up into seven steps. Note that our scheme of a priori estimates is also summarized diagrammatically in Figure 5.

Step 1: We close the energy estimates at the low level to deduce the unstructured decay of the low-level energy. We deduce from (4-45), smallness condition (1), and Proposition 4.20 that

$$\sup_{0 \leqslant t \leqslant T} \mathcal{E}_{\text{low}}(t)(1+t)^{2M-2} \leqslant C_L \delta_{\text{ap}}. \tag{4-46}$$

Step 2: We obtain the unstructured decay of the intermediate norms. Observe that

$$\sup_{0 \leqslant t \leqslant T} \|J(t)\|_{H^3}^2 + \|\partial_t J(t)\|_{H^2}^2 \leqslant \sup_{0 \leqslant t \leqslant T} \mathcal{E}_M(t) < \infty.$$

Combining this with smallness condition (2) and (4-46), Proposition 4.30 and Corollary 4.33 tell us that

$$\sup_{0 \leqslant t \leqslant T} \sup_{1 \leqslant I \leqslant M} \overline{\mathcal{K}}_I(t)(1+t)^{2M-2I} + \mathcal{K}_{\text{low}}(t)(1+t)^{2M-4} \leqslant (C_I + \widetilde{C}_I)(1+C_L)\delta_{\text{ap}}. \tag{4-47}$$

Step 3: We obtain our first structured estimate, using the advection-rotation estimates for K to wrest control over $\mathcal{F}_M$. We obtain from (4-44), (4-45), smallness condition (3), (4-47), and Proposition 4.8 that, for all $0 \leqslant t \leqslant T$,

$$\mathcal{F}_M(t) \lesssim \mathcal{F}_M(0) + \left(\int_0^t \overline{\mathcal{D}}_M^{1/2}(s)\, ds\right)^2 + \overline{\mathcal{K}}_M(t). \tag{4-48}$$

Step 4: We close the energy estimates at the high level to obtain the structured boundedness of the high-level energy (and the time-integrated control over the high-level dissipation). By virtue of (4-44), (4-45), smallness conditions (4)–(6), (4-47), and (4-48), we may apply Proposition 4.29, which tells us that

$$\sup_{0\leqslant t\leqslant T} \bar{\mathcal{E}}_M(t) + \int_0^t \mathcal{D}_M(s)\,ds \leqslant C_H(\mathcal{E}_M + \mathcal{F}_M)(0). \tag{4-49}$$

Step 5: We continue our second pass by obtaining structured versions of previously unstructured estimates. We close the energy estimates at the low level to deduce the structured decay of the low-level energy. Smallness condition (7), (4-49), and Proposition 4.20 show us that

$$\sup_{0\leqslant t\leqslant T} \mathcal{E}_{\text{low}}(t)(1+t)^{2M-2} \leqslant C_L C_H(\mathcal{E}_M + \mathcal{F}_M)(0). \tag{4-50}$$

Step 6: We revisit Step 2 and obtain the structured decay of intermediate norms. Proposition 4.30 and Corollary 4.33 tell us, in light of (4-44), smallness condition (8), (4-49), and (4-50), that

$$\sup_{0\leqslant t\leqslant T}\ \sup_{1\leqslant I\leqslant M} \overline{\mathcal{K}}_I(t)(1+t)^{2M-2I} + \mathcal{K}_{\text{low}}(t)(1+t)^{2M-4} \leqslant (C_I + \widetilde{C}_I)(1+C_L)C_H(\mathcal{E}_M + \mathcal{F}_M)(0). \tag{4-51}$$

Step 7: We conclude the proof by using the advection-rotation estimates to get the energetic terms involving K, i.e., $\mathcal{E}_M^{(K)}$, under control. We deduce from (4-44), smallness condition (9), (4-49), (4-51), and Proposition 4.8 that $\sup_{0\leqslant t\leqslant T} \mathcal{E}_M^{(K)}(t) \lesssim (\mathcal{E}_M + \mathcal{F}_M)(0)$, which concludes the proof. □

5. Local well-posedness

In this section we build a local well-posedness theory sufficient to prove the existence of solutions in the spaces where our a priori estimates apply. We employ a Galerkin scheme to construct a sequence of approximate solutions of (2-1), and this section is structured as follows. First we formulate appropriate approximate problems, then in Section 5A we treat in detail the matter of inverting the operator $J_{\text{eq}} + P_n \circ K$ which appears in the approximate problems (where P_n is a projection onto the subspaces where the approximate solutions live), we obtain various estimates on our sequence of solutions in Section 5B, and finally we produce local solutions via our Galerkin scheme in Section 5C.

Before writing down the approximate system we will solve, we must introduce the spaces in which we will solve it. We take V to be the subspace of L^2 defined as

$$V := \left\{ Z = (u,\theta,K) \in L^2(\mathbb{T}^3;\mathbb{R}^3\times\mathbb{R}^3\times \text{Sym}(3)) : \nabla\cdot u = 0 \text{ and } \fint_{\mathbb{T}^3} u = 0 \right\},$$

we define $\sigma = \sigma(\delta) > 0$ for any $\delta > 0$ such that, for any $K \in L^2(\mathbb{T}^3;\text{Sym}(3))$,

$$\text{if } \|K\|_{H^3} < \sigma(\delta) \text{ then } \|K\|_{L^\infty} < \tfrac{1}{2}\lambda \text{ and } \|\nabla K\|_{L^\infty} < \delta, \tag{5-1}$$

and we define

$$\mathcal{U}(\sigma) := \{Z = (u,\theta,K) \in V : \|K\|_{H^3} < \sigma\}. \tag{5-2}$$

Recall that $\lambda > 0$ is the smallest eigenvalue of J_0 (and hence of J) since our global assumption is that $J_0(x)$ has spectrum $\{\lambda,\lambda,\nu\}$ for every $x\in\mathbb{T}^3$, where $\nu > \lambda > 0$. To define the function spaces where the approximate solutions will live, we first define P_n to be the projection onto the Fourier modes with wavenumber at most n, i.e., $\widehat{P_n f}(k) = \mathbb{1}(|k| \leqslant n)\hat{f}(k)$. This allows us to define $\mathcal{P}_n : V \to V$ via

$$\mathcal{P}_n := P_n \oplus P_n \oplus P_{2n} \quad \text{and} \quad V_n := \mathcal{P}_n V = \{Z = (u, \theta, K) \in V : u, \theta \in \operatorname{Im} P_n \text{ and } K \in \operatorname{Im} P_{2n}\},$$

as well as

$$\mathcal{U}_n(\sigma) := \{Z_n = (u_n, \theta_n, K_n) \in V_n : \|K_n\|_{H^3} < \sigma\} = \mathcal{P}_n \mathcal{U}(\sigma).$$

Note that since the projection is performed in a symmetric fashion — i.e., its symbol $\widehat{\mathbb{P}}_n(k) = \mathbb{1}(|k| \leqslant n)$ is even — it maps real-valued spaces to real-valued spaces. We will produce solutions $(u_n, \theta_n, K_n) \in \mathcal{U}_n(\sigma)$ to the approximate system

$$\begin{cases} \partial_t u_n - (\nabla \cdot T)(u_n, \theta_n) = -P_n \mathbb{P}_L(u_n \cdot \nabla u_n), \\ \nabla \cdot u_n = 0, \\ (J_{\text{eq}} + P_n \circ K_n)\partial_t \theta_n + P_n((J_{\text{eq}} + K_n)(u_n \cdot \nabla)\theta_n) + P_n((\omega_{\text{eq}} + \theta_n) \times (J_{\text{eq}} + K_n)\theta_n) \\ \qquad\qquad + \tilde{\tau}^2 \tilde{a_n}^{\perp} + \theta \times J_{\text{eq}}\omega_{\text{eq}} - 2 \operatorname{vec} T(u_n, \theta_n) - (\nabla \cdot M)(\theta_n) = -P_n(\theta_n \times K_n \omega_{\text{eq}}), \\ \partial_t K_n - [\Theta_n, J_{\text{eq}}] - [\Omega_{\text{eq}}, K_n] = -P_{2n}(u_n \cdot \nabla K_n) + P_{2n}([\Theta_n, K_n]), \end{cases} \tag{5-3}$$

where $(P_n \circ K_n)v := P_n(K_n v)$ for any $v \in L^2(\mathbb{T}^3; \mathbb{R}^3)$.

When writing down the associated energy estimate further below we will need to distinguish between the variables that are viewed as *unknowns* and those that are viewed as enforcing the *constraints*. We will thus recapitulate the system above in the following form (in particular in order to fix notation regarding a compact way to write down the system above):

$$\begin{cases} \partial_t v_n - (\nabla \cdot T)(v_n, \phi_n) = f_1, \\ (J_{\text{eq}} + P_n \circ K_n)\partial_t \phi_n + P_n((J_{\text{eq}} + K_n)(u_n \cdot \nabla)\phi_n) + P_n((\omega_{\text{eq}} + \theta_n) \times (J_{\text{eq}} + K_n)\phi_n) \\ \qquad\qquad + \tilde{\tau}^2 \tilde{b}_n^{\perp} + \phi_n \times J_{\text{eq}}\omega_{\text{eq}} - 2 \operatorname{vec} T(v_n, \phi_n) - (\nabla \cdot M)(\phi_n) = f_2, \\ \partial_t H_n - [\Phi_n, J_{\text{eq}}] - [\Omega_{\text{eq}}, H_n] = F_3 \end{cases} \tag{5-4}$$

subject to the constraints

$$\begin{cases} \nabla \cdot u_n = 0, \\ \partial_t K_n + P_{2n}(u_n \cdot \nabla K_n) = P_{2n}([\Omega_{\text{eq}} + \Theta_n, J_{\text{eq}} + K_n]). \end{cases} \tag{5-5}$$

Here we view (v_n, ϕ_n, H_n) as the unknowns, where $b_n = ((H_n)_{12}, (H_n)_{13})$ and $\Phi_n = \operatorname{ten} \phi_n$, and (u_n, θ_n, K_n) as the variables enforcing the constraints. In particular, for $Z_n = (v_n, \phi_n, H_n)$, we have $W_n = (u_n, \theta_n, K_n)$ and, for $\mathcal{F} = (f_1, f_2, F_3)$, we may rewrite this form of the system as

$$\widetilde{T}_n(K_n)\partial_t Z_n - \mathcal{L}_{W_n,n} Z_n = \mathcal{F}$$

subject to the constraints (5-5), where $\widetilde{T}_n(K_n) := I_3 \oplus (J_{\text{eq}} + P_n \circ K_n) \oplus I_{3\times 3}$ and the operator $\mathcal{L}_{W_n,n}$ is given by $\mathcal{L}_{W_n,n} Z_n = (-(\nabla \cdot T)(v_n, \phi_n), (\star), -[\Phi_n, J_{\text{eq}}] - [\Omega_{\text{eq}}, H_n])$, where

$$(\star) = P_n((J_{\text{eq}} + K_n)(u_n \cdot \nabla)\phi_n) + P_n((\omega_{\text{eq}} + \theta_n) \times (J_{\text{eq}} + K_n)\phi_n) + \tilde{\tau}^2 \tilde{b}_n^{\perp} + \phi_n \times J_{\text{eq}}\omega_{\text{eq}} \\ - 2 \operatorname{vec} T(v_n, \phi_n) - (\nabla \cdot M)(\phi_n).$$

Since $\widetilde{T}_n$ only has one nontrivial block we will write $\widetilde{T}_n(K_n) = I_3 \oplus T_n(K_n) \oplus I_{3\times 3}$, where we define $T_n(K_n) := J_{\mathrm{eq}} + P_n \circ K_n$. This allows us to rewrite the approximate problem in the form

$$\widetilde{T}_n(K_n)\partial_t Z_n - \mathcal{L}_{Z_n,n} Z_n = N_n(Z_n) \quad \text{subject to } \nabla \cdot u_n = 0, \tag{5-6}$$

where

$$N_n(Z_n) = (-P_n \mathbb{P}_L(u_n \cdot \nabla u_n), -P_n(\theta_n \times K_n \omega_{\mathrm{eq}}), -P_{2n}(u_n \cdot \nabla K_n) + P_{2n}([\Theta_n, K_n])).$$

Note that in some situations it is helpful to decompose the linear operator $\mathcal{L}_{W_n,n}$ into its part that has constant coefficient and the remainder. More precisely, we write

$$\mathcal{L}_{W_n,n} = \mathcal{L}_0 + \overline{\mathcal{L}}_{W_n,n}, \tag{5-7}$$

where

$$\overline{\mathcal{L}}_{W_n,n} Z_n = (0, P_n((J_{\mathrm{eq}} + K_n)(u_n \cdot \nabla)\phi_n) + P_n((\omega_{\mathrm{eq}} + \theta_n) \times (J_{\mathrm{eq}} + K_n)\phi_n) - \omega_{\mathrm{eq}} \times J_{\mathrm{eq}}\theta_n, 0)$$

and

$$\mathcal{L}_0 Z_n = \begin{pmatrix} -(\nabla \cdot T)(v_n, \phi_n) \\ \tilde{\tau}^2 \tilde{b}_n^{\perp} + \phi_n \times J_{\mathrm{eq}}\omega_{\mathrm{eq}} + \omega_{\mathrm{eq}} \times J_{\mathrm{eq}}\theta_n - 2 \operatorname{vec} T(v_n, \phi_n) - (\nabla \cdot M)(\phi_n) \\ -[\Phi_n, J_{\mathrm{eq}}] - [\Omega_{\mathrm{eq}}, H_n] \end{pmatrix}.$$

5A. ***Inverting $T(K)$.*** In this section we deal carefully with the inversion of $T(K) = J_{\mathrm{eq}} + P \circ K$ and the smoothness of its inverse, and we obtain H^k-to-H^k bounds on the inverse. Note that in this section we will work in the generic framework where P is an L^2-orthogonal projection onto a finite-dimensional subspace of L^2 which is not necessarily V_n (and so P is not necessarily P_n). We begin by establishing the invertibility of $T(K)$.

Lemma 5.1 (invertibility of $T(K)$). *Let $V \subseteq L^2(\mathbb{T}^3, \mathbb{R}^3)$ be a finite-dimensional subspace and let P denote the L^2-orthogonal projection onto V. Let $K \in L^\infty(\mathbb{T}^3, \mathbb{R}^{3\times 3})$ be almost everywhere symmetric and satisfy $\|K\|_\infty < \frac{1}{2}\lambda$. Recall that λ is the repeated eigenvalue of the microinertia, as stated in the global assumptions of Definition 1.1. Then $T(K) := J_{\mathrm{eq}} + P \circ K$, where $(P \circ K)v := P(Kv)$ for every $v \in L^2(\mathbb{T}^3, \mathbb{R}^3)$, is, with respect to the L^2 inner product, a self-adjoint invertible operator on V. Moreover, we have the bound $\|T(K)^{-1}\|_{\mathcal{L}(V,V)} \leqslant 2/\lambda$.*

Proof. The self-adjointness of $T(K)$ follows from the symmetry of K. Indeed, for every $\theta, \phi \in V$,

$$(T(K)\theta, \phi)_{L^2} = ((J_{\mathrm{eq}} + P \circ K)\theta, \phi)_{L^2} = ((J_{\mathrm{eq}} + K)\theta, \phi)_{L^2} = (\theta, (J_{\mathrm{eq}} + K)\phi)_{L^2} = (\theta, T(K)\phi)_{L^2}.$$

The invertibility of $T(K)$ follows from the almost-everywhere invertibility of $J_{\mathrm{eq}} + K$. Indeed, note that since $T(K)$ is a self-adjoint operator it suffices to study the quadratic form that it generates in order to determine its spectrum. So we note that, for every $\theta \in V$,

$$(T(K)\theta, \theta)_{L^2} = ((J_{\mathrm{eq}} + P \circ K)\theta, \theta)_{L^2} = ((J_{\mathrm{eq}} + K)\theta, \theta)_{L^2} > \tfrac{1}{2}\lambda \|\theta\|_{L^2}^2,$$

and hence $\lambda_{\min}(T(K)) \geqslant \frac{1}{2}\lambda$. In particular, we deduce that $T(K)$ is an invertible operator from V to itself, and we have the bound $\|T(K)^{-1}\|_{\mathcal{L}(V,V)} \leqslant 2/\lambda$. ∎

Now that we know that $T(K)^{-1}$ is well-defined we verify that its dependence on K is smooth.

Lemma 5.2 (smoothness of $T(K)^{-1}$). *Let $V \subseteq L^2(\mathbb{T}^3, \mathbb{R}^3)$ and $W \subseteq L^2(\mathbb{T}^3, \mathrm{Sym}(\mathbb{R}^{3\times 3}))$ be finite-dimensional subspaces, let P denote the L^2-orthogonal projection onto V, let $\mathcal{U} := \{K \in W : \|K\|_\infty < \frac{1}{2}\lambda\}$ be the open L^∞-ball of radius $\frac{1}{2}\lambda$ in W, and let $T(K) := J_{\mathrm{eq}} + P \circ K$ for any $K \in \mathcal{U}$, where $(P \circ K)v := P(Kv)$ for every $v \in L^2(\mathbb{T}^3, \mathbb{R}^3)$. Then the map $\Phi : \mathcal{U} \to \mathcal{L}(V, V)$ defined by $\Phi(K) := T(K)^{-1}$ is smooth.*

Proof. The crucial observation here is that Φ may be written as the composition of $T : \mathcal{U} \to \mathcal{L}(V, V)$ and $\mathrm{inv} : \mathcal{GL}(V) \to \mathcal{GL}(V)$, where $\mathcal{GL}(V) := \{L \in \mathcal{L}(V, V) : L \text{ is invertible}\}$ and $\mathrm{inv}(L) := L^{-1}$ for every $L \in \mathcal{GL}(V)$. Note that it is precisely Lemma 5.1 which tells us that $T(\mathcal{U}) \subseteq \mathcal{GL}(V)$ such that $\Phi = \mathrm{inv} \circ T$ is indeed well-defined. All that remains to show is that both T and inv are smooth. The smoothness of inv is a well-known fact — see for example [Abraham et al. 1988]. To see that T is smooth note that, for every $K, H \in \mathcal{U}$, we have $T(K) - T(H) = P \circ (K - H)$. We deduce that T is affine and hence smooth. $\square$

We now turn our attention towards the establishment of H^k-to-H^k estimates on $T(K)^{-1}$. In order to do so we first define the operator M which will be useful when deriving formulae for the derivatives of $T(K)^{-1}$.

Definition 5.3. Let $V \subseteq L^2(\mathbb{T}^3, \mathbb{R}^3)$ be a finite-dimensional subspace and let P denote the L^2-orthogonal projection onto V. For any $K \in L^\infty(\mathbb{T}^3, \mathbb{R}^{3\times 3})$, let $T(K) := J_{\mathrm{eq}} + P \circ K$, where $(P \circ K)v := P(Kv)$ for any v in $L^2(\mathbb{T}^3, \mathbb{R}^3)$. We define, for any multi-indices $\alpha_1, \dots, \alpha_m \in \mathbb{N}^{3\times 3}$ with $k := \max|\alpha_i|$ and any $K \in W^{k,\infty}(\mathbb{T}^3; \mathbb{R}^{3\times 3})$ for which $T(K)$ is invertible,

$$M(\alpha_1, \dots, \alpha_m) := T(K)^{-1}(P \circ \partial^{\alpha_1} K)T(K)^{-1}(P \circ \partial^{\alpha_2} K)T(K)^{-1} \cdots T(K)^{-1}(P \circ \partial^{\alpha_m} K)T(K)^{-1}.$$

With the operator M in hand we may write down useful formulae for derivatives of $T(K)^{-1}$.

Lemma 5.4 (formula for the derivatives of $T(K)^{-1}$). *Let $\mathcal{U}$ and T be as in Lemma 5.2 and let M be as in Definition 5.3. For any multi-index $\alpha \in \mathbb{N}^{3\times 3}$ and any $K \in W^{|\alpha|,\infty}(T^3, \mathbb{R}^{3\times 3})$, we have the identity*

$$\partial^\alpha (T(K)^{-1}) = \sum_{k=1}^{|\alpha|} (-1)^k \sum_{\beta_1 + \cdots + \beta_k = \alpha} M(\beta_1, \dots, \beta_k)(K).$$

Proof. The fundamental observations are that taking a single derivative of the maps $K \mapsto T(K)$ and $K \mapsto T(K)^{-1}$ yields

$$\partial_i (T(K)) = P \circ \partial_i K \quad \text{and} \quad \partial_i (T(K)^{-1}) = -T(K)^{-1}(P \circ \partial_i K)T(K)^{-1}$$

(see Lemma 5.2 for analogous computations). Using these two identities we may deduce an identity for derivatives of M:

$$\begin{aligned}
&\partial_i (M(\alpha_1, \alpha_2, \dots, \alpha_m)(K)) \\
&\quad = M(\alpha_1 + e_i, \alpha_2, \dots, \alpha_m)(K) + \cdots + M(\alpha_1, \alpha_2, \dots, \alpha_m + e_i)(K) \\
&\qquad\qquad - M(e_i, \alpha_1, \alpha_2, \dots, \alpha_m)(K) - M(\alpha_1, e_i, \alpha_2, \dots, \alpha_m)(K) - \cdots - M(\alpha_1, \alpha_2, \dots, \alpha_m, e_i)(K).
\end{aligned}$$

The result then follows by induction. $\square$

In light of these formulae for derivatives of $T(K)^{-1}$ we may now conclude this section and obtain H^k-to-H^k bounds on $T(K)^{-1}$.

Lemma 5.5 (H^k bounds on $T(K)^{-1}$). *Let $\mathcal{U}$ and T be as in Lemma 5.2. For every $k \geqslant 2$ and every $K \in \mathcal{U} \cap H^{k+2}$, we have $\|T(K)^{-1}\|_{\mathcal{L}(H^k,H^k)} \lesssim \|K\|_{H^k} + \|K\|_{H^{k+2}}^k$.*

Proof. As a starting point, combining Lemmas 5.1 and 5.4 tells us that, for any multi-indices $\beta_1, \dots, \beta_m$ and for M as in Definition 5.3, if we write $\alpha := \beta_1 + \cdots + \beta_m$ and $k = |\alpha|$ then

$$\|M(\beta_1, \dots, \beta_m)(K)\|_{\mathcal{L}(L^2,L^2)} \leqslant \|T(K)^{-1}\|_{\mathcal{L}(L^2,L^2)}^{k+1} \prod_{i=1}^{m} \|\partial^{\beta_i} K\|_{L^\infty} \leqslant \left(\frac{2}{\lambda}\right)^{k+1} \|\nabla K\|_{W^{k-1,\infty}}^k \lesssim \|K\|_{H^{k+2}}^k$$

since, when $n = 3$, $H^2(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)$.

We may now combine this inequality with Lemma 5.4 to obtain L^2-to-L^2 bounds on $\partial^\beta(T(K)^{-1})$: for any multi-index $\beta \in \mathbb{N}^3$ with $|\beta| = l$,

$$\|\partial^\beta(T(K)^{-1})\|_{\mathcal{L}(L^2,L^2)} \leqslant \sum_{i=1}^{l} \sum_{\gamma_1+\cdots+\gamma_i=\beta} \|M(\gamma_1, \dots, \gamma_i)(K)\|_{\mathcal{L}(L^2,L^2)} \lesssim \|K\|_{H^{l+2}}^l.$$

We may now finally obtain H^k-to-H^k bounds on $T(K)^{-1}$. For any $v \in H^k(\mathbb{T}^3, \mathbb{R}^3)$,

$$\begin{aligned}\|T(K)^{-1}v\|_{H^k}^2 &= \sum_{|\alpha|\leqslant k} \|\partial^\alpha(T(K)^{-1}v)\|_{L^2}^2 \leqslant \sum_{|\alpha|\leqslant k} \sum_{\beta+\delta} \|\partial^\beta(T(K)^{-1})\|_{\mathcal{L}(L^2,L^2)}^2 \|\partial^\delta v\|_{L^2}^2 \\ &\lesssim \sum_{|\alpha|\leqslant k} \sum_{\beta+\delta=\alpha} \|K\|_{H^{|\beta|+2}}^{2|\beta|} \|v\|_{H^{|\delta|}}^2 \lesssim (\|K\|_{H^2}^2 + \|K\|_{H^{k+2}}^{2k}) \|v\|_{H^k}^2,\end{aligned}$$

where note that the last inequality follows by interpolation. □

5B. *Estimates for the approximate problem.* In this section we obtain two types of estimates; a priori estimates on the sequence of approximate solutions and estimates of the initial energy (which involves temporal derivatives) in terms of purely spatial norms.

Note that by contrast with the main scheme of a priori estimates built in Section 4, the a priori estimates here are almost exclusively centered around energy estimates (some advection-rotation estimates for K_n are present, but play an auxiliary role). This is because we are working locally in time and therefore can get away with "sloppier" estimates, in the sense that the nonlinear interactions need not be estimated in structured ways (e.g., as $|\mathcal{I}| \lesssim \sqrt{\mathcal{E}\mathcal{D}}$) such that cruder estimates (e.g., $|\mathcal{I}| \lesssim \mathcal{E}^{3/2}$) suffice.

This section is structured as follows. First we record some projected variants of the advection-rotations estimates, then we proceed with the energy estimates, and finally we turn our attention to estimates of the initial energy in terms of purely spatial norms.

As a precursor to H^k estimates for the projected advection-rotation operator appearing in the last equation of system (5-3), we first obtain an L^2 estimate. Note that (5-8) in the statement of Lemma 5.6 is equivalent to the last equation of system (5-3) for an appropriate definition of F. It is written in this slightly different form since it makes it clear which operator produces good L^2 estimates, and hence which operator must be kept on the left-hand side when taking derivatives and performing H^k estimates.

Lemma 5.6 (L^2 estimates for projected advection-rotation equations). *Let* $K_n \in L^2(\mathbb{T}^3; \mathrm{Sym}(3)) \cap \mathrm{Im}\, P_n$, *let* u *be divergence-free*, *and let* Θ *be antisymmetric. If* K_n *solves*

$$P_n \circ (\partial_t + u \cdot \nabla - [\Omega_{\mathrm{eq}} + \Theta, \cdot\,])K_n = F, \tag{5-8}$$

then $\frac{\mathrm{d}}{\mathrm{d}t}\|K_n\|_{L^2} \leqslant \|F\|_{L^2}$.

Proof. The key observation is that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|K_n\|_{L^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{T}^3}|K_n|^2\right)^{1/2} = \frac{1}{2}\left(\int_{\mathbb{T}^3}|K_n|^2\right)^{-1/2}\int_{\mathbb{T}^3} 2K_n : \partial_t K_n = \frac{(K_n, \partial_t K_n)_{L^2}}{\|K_n\|_{L^2}}, \tag{5-9}$$

where we may bound $(K_n, \partial_t K_n)_{L^2}$ via a simple energy estimate on (5-8). Indeed it follows from (5-8), the incompressibility of u, and Lemma A.5 that

$$\int_{\mathbb{T}^3} \partial_t K_n : K_n = \int_{\mathbb{T}^3} (\partial_t + u \cdot \nabla - [\Omega_{\mathrm{eq}} + \Theta, \cdot\,])K_n : K_n = \int_{\mathbb{T}^3} F : K_n. \tag{5-10}$$

Putting (5-9) and (5-10) together with the Cauchy–Schwarz inequality allows us to conclude. □

With this L^2 estimate in hand we may now derive H^k estimates for K_n.

Lemma 5.7 (H^k estimates for projected advection-rotation equations). *Let* $K_n \in L^2(\mathbb{T}^3; \mathrm{Sym}(3)) \cap \mathrm{Im}\, P_n$, *let* u *be divergence-free*, *and let* Θ *be antisymmetric. If* K_n *solves* $P_n \circ (\partial_t + u \cdot \nabla - [\Omega_{\mathrm{eq}} + \Theta, \cdot\,])K_n = 0$ *and satisfies* $\|K_n\|_\infty, \|\nabla K_n\|_\infty \lesssim 1$, *then*, *for every* $k \geqslant 0$,

$$\|K_n(t)\|_{H^k} \lesssim \exp\left(\int_0^t \|(u, \theta)(s)\|_{H^3}\, ds\right)\left(\|K_n(0)\|_{H^k} + \int_0^t \|(u, \theta)(s)\|_{H^k}\, ds\right).$$

Proof. Since P_n commutes with ∂^α and since $\|P_n\|_{\mathcal{L}(L^2, L^2)} \leqslant 1$, we may deduce that

$$\|[P_n \circ (u \cdot \nabla), \partial^\alpha]K_n\|_{L^2} = \|(P_n \circ [u \cdot \nabla, \partial^\alpha])K_n\|_{L^2} \leqslant \|[u \cdot \nabla, \partial^\alpha]K_n\|_{L^2},$$

and similarly

$$\|[P_n \circ [\Theta, \cdot\,], \partial]K_n\|_{L^2} \leqslant \|[[\Theta, \cdot\,], \partial^\alpha]K_n\|_{L^2}.$$

With these two commutator inequalities and Lemma 5.6 in hand we may proceed as in Lemma 4.4 to deduce the claim, keeping in mind that $\|K_n\|_{L^\infty}, \|\nabla K_n\|_{L^\infty} \lesssim 1$. □

We now turn our attention to the energy-dissipation structure of the approximate problem. We begin by defining appropriate versions of the energy.

Definition 5.8 (versions of the local energies). For $Z = (u, \theta, K)$, we define $E_{K,\mathrm{loc}}$, $\tilde{\mathcal{E}}_{M,\mathrm{loc}}$, and $\bar{\mathcal{E}}_{M,\mathrm{loc}}$ as follows:

$$E_{K,\mathrm{loc}}(u, \theta, K) := \frac{1}{2}\int_{\mathbb{T}^3}|u|^2 + \frac{1}{2}\int_{\mathbb{T}^3}(J_{\mathrm{eq}} + K)\theta \cdot \theta + \frac{1}{2}\frac{\tilde{\tau}^2}{\nu - \lambda}\int_{\mathbb{T}^3}|K|^2, \tag{5-11}$$

while

$$\tilde{\mathcal{E}}_{M,\mathrm{loc}}(Z) := \sum_{|\alpha|_P \leqslant 2M} E_{K,\mathrm{loc}}(\partial^\alpha Z) \quad \text{and} \quad \bar{\mathcal{E}}_{M,\mathrm{loc}}(Z) := \sum_{|\alpha|_P \leqslant 2M} \|\partial^\alpha Z\|_{L^2}^2. \tag{5-12}$$

We now precisely compare various versions of the energy. We emphasize that Lemma 5.9 differs from Lemma 4.11; the former is a consequence of *smallness*, whereas the latter is a consequence of *regularity*.

Lemma 5.9 (comparisons of the different versions of the local energies). *Let $\bar{\mathcal{E}}_{M,\mathrm{loc}}$ and $\tilde{\mathcal{E}}_{M,\mathrm{loc}}$ be defined as in Definition 5.8. There exist constants $\tilde{c}_E, \widetilde{C}_E > 0$ such that if $\|K\|_{L^\infty} < \frac{1}{2}\lambda$ then we have the estimate $\tilde{c}_E \bar{\mathcal{E}}_{M,\mathrm{loc}} \leqslant \tilde{\mathcal{E}}_{M,\mathrm{loc}} \leqslant \widetilde{C}_E \bar{\mathcal{E}}_{M,\mathrm{loc}}$.*

Proof. The key observation is that since the spectrum of J_{eq} is $\{\lambda, \lambda, \nu\}$, if $\|K\|_{L^\infty} < \frac{1}{2}\lambda$ then the spectrum of $J_{\mathrm{eq}} + K$ is contained in $\left(\frac{1}{2}\lambda, \nu + \frac{1}{2}\lambda\right)$. The claim then follows as in Lemma 4.11. □

We now record an elementary lemma which is crucial in deriving the energy-dissipation relation associated with the approximate system. Indeed, Lemma 5.10 below is precisely what justifies approximating K with twice as many Fourier modes as the other variables.

Lemma 5.10 (finite Fourier mode cut-off of products). *For any $M \in L^2(\mathbb{T}^3; \mathbb{R}^{3\times 3})$ and any L^2 vector field $v \in \operatorname{Im} P_n$, if $P_{2n} M = 0$ then $P_n(Mv) = 0$.*

Proof. Suppose that $P_{2n} M = 0$ and that $v \in V_n$. Then $\widehat{Mv}(k) = \sum_{|l|>2n} \widehat{M}(l)\hat{v}(k-l)$. In particular, if $|k| \leqslant n$ and $|l| > 2n$ then $|k-l| > n$ such that, since $v \in V_n$, we have that $\hat{v}_j(k-l) = 0$. This shows that $\widehat{Mv}(k) = 0$ for any $|k| \leqslant n$, i.e., indeed $P_n(Mv) = 0$. □

We are now equipped to state and prove the energy-dissipation relation associated with the approximate system. In particular, as discussed in more detail in Section 2E, note that in the approximate system considered below in Proposition 5.11 we use regular time derivatives for the unknowns v and H and an advective time derivative for ϕ. We could have used advective derivatives for v and H, but this formulation makes it more clear which nonlinear structure is optional and which is not. In particular, as discussed in Section 2E, the nonlinear structure in the equation governing the dynamics of ϕ is essential in order to obtain a good energy-dissipation relation.

Proposition 5.11 (generic energy-dissipation relation associated with the approximate system). *Suppose that the unknowns (v, ϕ, H) and b, where $b = (H_{12}, H_{13})$ and $\Phi = \operatorname{ten}\phi$, and the constraint variables (u, θ, K) satisfy*

$$\begin{cases} \partial_t v - (\nabla \cdot T)(v, \phi) = f_1, \\ (J_{\mathrm{eq}} + P_n \circ K)\partial_t \phi + P_n((J_{\mathrm{eq}} + K)(u \cdot \nabla)\phi) + P_n((\omega_{\mathrm{eq}} + \theta) \times (J_{\mathrm{eq}} + K)\phi) \\ \qquad\qquad + \tilde{\tau}^2 \tilde{b}^\perp + \phi \times J_{\mathrm{eq}}\omega_{\mathrm{eq}} - 2\operatorname{vec} T(v, \phi) - (\nabla \cdot M)(\phi) = f_3, \\ \partial_t H - [\Phi, J_{\mathrm{eq}}] - [\Omega_{\mathrm{eq}}, H] = F_3 \end{cases}$$

subject to the constraints

$$\begin{cases} \nabla \cdot u = 0, \\ \partial_t K + P_{2n}(u \cdot \nabla K) = P_{2n}([\Omega_{\mathrm{eq}} + \Theta, J_{\mathrm{eq}} + K]), \end{cases} \tag{5-13}$$

where $(v, \phi, H) \in V_n$ and $K \in \operatorname{Im} P_{2n}$. Then the following energy-dissipation identity holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{T}^3} \frac{1}{2}|v|^2 + \int_{\mathbb{T}^3} \frac{1}{2}(J_{\mathrm{eq}} + K)\phi\cdot\phi + \int_{\mathbb{T}^3} \frac{1}{2}\frac{\tilde{\tau}^2}{\nu - \lambda}|H|^2\right) + D(v, \phi) = \int_{\mathbb{T}^3} f_1 \cdot v + \int_{\mathbb{T}^3} f_2 \cdot \phi + \int_{\mathbb{T}^3} \check{\tau} F_3 : H,$$

where $\check{\tau} = \frac{1}{2}\tilde{\tau}^2/(\nu - \lambda)$. Recall that the dissipation D is defined in (3-7).

Proof. We begin by computing the time derivative of the kinetic energy due to u:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}\tfrac{1}{2}|v|^2=\int_{\mathbb{T}^3}(\partial_t v)\cdot v=\int_{\mathbb{T}^3}(\nabla\cdot T)(v,\phi)\cdot v+\int_{\mathbb{T}^3}f_1\cdot v=-\int_{\mathbb{T}^3}T(v,\phi):\nabla v+\int_{\mathbb{T}^3}f_1\cdot v.$$

We now compute the time derivative of the kinetic energy due to ω. In light of (5-13) we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}\tfrac{1}{2}(J_{\mathrm{eq}}+K)\phi\cdot\phi=\int_{\mathbb{T}^3}\tfrac{1}{2}((\partial_t+u\cdot\nabla)K)\phi\cdot\phi+\int_{\mathbb{T}^3}(J_{\mathrm{eq}}+K)(\partial_t+u\cdot\nabla)\phi\cdot\phi=:\mathrm{I}+\mathrm{II},$$

where we may combine the second constraint of (5-13) and Lemma 5.10, and use Lemma A.4 to see that

$$\begin{aligned}\mathrm{I}=\int_{\mathbb{T}^3}\tfrac{1}{2}P_n(((\partial_t+u\cdot\nabla)K)\phi)\cdot\phi=\int_{\mathbb{T}^3}\tfrac{1}{2}P_n([\Omega_{\mathrm{eq}}+\Theta,\,J_{\mathrm{eq}}+K]\phi)\cdot\phi=\int_{\mathbb{T}^3}\tfrac{1}{2}[\Omega_{\mathrm{eq}}+\Theta,\,J_{\mathrm{eq}}+K]\phi\cdot\phi\\=\int_{\mathbb{T}^3}(\omega_{\mathrm{eq}}+\theta)\times(J_{\mathrm{eq}}+K)\phi\cdot\phi=\int_{\mathbb{T}^3}P_n((\omega_{\mathrm{eq}}+\theta)\times(J_{\mathrm{eq}}+K)\phi)\cdot\phi,\end{aligned}$$

and where we may compute directly that

$$\mathrm{II}=\int_{\mathbb{T}^3}P_n((J_{\mathrm{eq}}+K)(\partial_t+u\cdot\nabla)\phi)\cdot\phi=\int_{\mathbb{T}^3}(J_{\mathrm{eq}}+P_n\circ K)\partial_t\phi+P_n((J_{\mathrm{eq}}+K)(u\cdot\nabla)\phi)\phi.$$

Adding I and II together therefore tells us that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}\tfrac{1}{2}(J_{\mathrm{eq}}+K)\phi\cdot\phi&=-\int_{\mathbb{T}^3}\tilde{\tau}^2\tilde{b}^{\perp}\cdot\phi+\int_{\mathbb{T}^3}2\,\mathrm{vec}\,T(v,\phi)\cdot\phi+\int_{\mathbb{T}^3}(\nabla\cdot M)(\phi)\cdot\phi+\int_{\mathbb{T}^3}f_2\cdot\phi\\&=\int_{\mathbb{T}^3}\tilde{\tau}^2 b\cdot\bar{\phi}^{\perp}+\int_{\mathbb{T}^3}T(v,\phi):\Phi-\int_{\mathbb{T}^3}M(\phi):\nabla\phi+\int_{\mathbb{T}^3}f_2\cdot\phi.\end{aligned}$$

Finally, we compute the energetic contribution from H. As a preliminary, note that we can deduce from Lemma A.6 that

$$[\Phi,\,J_{\mathrm{eq}}]=-(\nu-\lambda)\begin{pmatrix}0&\bar{\phi}^{\perp}\\(\bar{\phi}^{\perp})^T&0\end{pmatrix}.$$

We may therefore compute that, in light of the equation above and Lemma A.5,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}\tfrac{1}{2}|H|^2=\int_{\mathbb{T}^3}(\partial_t H):H&=\int_{\mathbb{T}^3}[\Phi,\,J_{\mathrm{eq}}]:H+\int_{\mathbb{T}^3}[\Omega_{\mathrm{eq}},\,H]:H+\int_{\mathbb{T}^3}F_3:H\\&=-(\nu-\lambda)\int_{\mathbb{T}^3}\bar{\phi}^{\perp}\cdot b+\int_{\mathbb{T}^3}F_3:H.\end{aligned}$$

To conclude we multiply this last identity by $\tilde{\tau}^2/(\nu-\lambda)$ and add it to the identities obtained above for the evolution of the different components of the kinetic energy. Upon noting that

$$\int_{\mathbb{T}^3}T(v,\phi):(\nabla v-\Phi)+\int_{\mathbb{T}^3}M(\phi):\nabla\phi=D(v,\phi)$$

(see Proposition 4.9 for details), we deduce the claim. □

With the energy-dissipation in hand, we now tackle the nonlinear interactions. We begin by recording the precise form of the interactions. Recall that $E_{K,\mathrm{loc}}$ and D are defined in (5-11) and (3-7), respectively.

Lemma 5.12 (recording the form of the local interactions). *Let $Z = (u, \theta, K) \in V_n$ solve (5-6). Then for every multi-index $\alpha \in \mathbb{N}^{1+3}$ we have that*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{K,\mathrm{loc}}(\partial^\alpha Z) + D(\partial^\alpha u, \partial^\alpha \theta) = \mathcal{N}^\alpha,$$

where

$$\begin{aligned}\mathcal{N}^\alpha = &-\int_{\mathbb{T}^3} \partial^\alpha (u \cdot \nabla u) \cdot \partial^\alpha u - \int_{\mathbb{T}^3} \partial^\alpha (\theta \times K\omega_{\mathrm{eq}}) \cdot \partial^\alpha \theta \\ &- \check{\tau} \int_{\mathbb{T}^3} \partial^\alpha (u \cdot \nabla K) : \partial^\alpha K + \check{\tau} \int_{\mathbb{T}^3} \partial^\alpha ([\Theta, K]) : \partial^\alpha K + \int_{\mathbb{T}^3} [K\partial_t, \partial^\alpha]\theta \cdot \partial^\alpha \theta \\ &- \int_{\mathbb{T}^3} [(J_{\mathrm{eq}} + K)(u \cdot \nabla), \partial^\alpha]\theta \cdot \partial^\alpha \theta - \int_{\mathbb{T}^3} [(\omega_{\mathrm{eq}} + \theta) \times (J_{\mathrm{eq}} + K), \partial^\alpha]\theta \cdot \partial^\alpha \theta\end{aligned}$$

for $\check{\tau} := \tilde{\tau}^2/(\nu - \lambda)$.

Proof. If Z solves (5-6) then, for any multi-index α, we have that $\partial^\alpha Z$ solves

$$\widetilde{T}_n(K)\partial_t \partial^\alpha Z - \mathcal{L}_{Z,n} \partial^\alpha Z = \partial^\alpha (N_n(Z)) + [\widetilde{T}_n(K)\partial_t, \partial^\alpha]Z - [\mathcal{L}_{Z,n}, \partial^\alpha]Z =: F_n^\alpha$$

subject to

$$\nabla \cdot u = 0 \quad \text{and} \quad \partial_t K + P_{2n}(u \cdot \nabla K) = P_{2n}([\Omega_{\mathrm{eq}} + \Theta, J_{\mathrm{eq}} + K]),$$

and hence Proposition 5.11 tells us that

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{K,\mathrm{loc}}(\partial^\alpha Z) + D(\partial^\alpha u, \partial^\alpha \theta) = \int_{\mathbb{T}^3} F_n^\alpha \cdot \mathfrak{C} \partial^\alpha Z =: \mathcal{N}^\alpha,$$

where $\mathfrak{C} = I_3 \oplus I_3 \oplus \check{\tau} I_{3\times 3}$. To compute $\mathcal{N}^\alpha$ it suffices to use the fact that P_n and ∂^α commute, to recall that $\widetilde{T}_n(K) = I_3 \oplus (J_{\mathrm{eq}} + P_n \circ K) \oplus I_{3\times 3}$, and to split $\mathcal{L}_{Z,n}$ into its part with constant coefficients and the remainder, as is done in (5-7). □

Having recorded the precise form of the interactions we estimate them.

Lemma 5.13 (estimates of the local interactions). *Let $M \geqslant 4$ be an integer and let $\mathcal{N} := \sum_{|\alpha|_P \leqslant 2M} \mathcal{N}^\alpha$ for $\mathcal{N}^\alpha$ as in Lemma 5.12. The following estimate holds*:

$$|\mathcal{N}| \lesssim \|\nabla K\|_{L^\infty} \|(u, \theta)\|_{P^{2M+1}}^2 + \|(u, \theta, K)\|_{P^{2M}}^3 + \|(u, \theta, K)\|_{P^{2M}}^4.$$

Proof. Let us write the terms in $\mathcal{N}$ as $\mathcal{N}_1^\alpha, \dots, \mathcal{N}_7^\alpha$, following the indexing of Lemma 5.12, such that

$$\mathcal{N} = \sum_{|\alpha|_P \leqslant 2M} \mathcal{N}_1^\alpha + \cdots + \mathcal{N}_7^\alpha. \tag{5-14}$$

We will estimate each of these seven terms in turn. First, however, we note that the interaction term $\mathcal{N}_5^\alpha = -\int_{\mathbb{T}^3} [K\partial_t, \partial^\alpha]\theta \cdot \partial^\alpha \theta$ bears a particular importance in this estimate. Indeed, due to the temporal derivative appearing in the commutator we must invoke a parabolic count of $2M + 1$ derivatives acting on θ, which gives rise to the term $\|\nabla K\|_{L^\infty} \|(u, \theta, K)\|_{P^{2M+1}}^2$ in the estimate. Most notably, $\mathcal{N}_5$ is the only interaction which requires us to invoke a parabolic count of $2M + 1$ derivatives.

We now go through the estimates of the interactions one by one — although, due to the great similarity in estimating many terms, we will only provide details for a few of the interactions.

Estimating $\mathcal{N}_1$. By applying the Leibniz rule we see that

$$-\mathcal{N}_1^\alpha = \sum_{\substack{\beta+\gamma=\alpha \\ |\alpha|_P \leqslant 2M}} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} (\partial^\beta u \cdot \nabla \partial^\gamma u) \cdot \partial^\alpha u,$$

where we have used the fact that u is divergence-free to deduce that

$$\int_{\mathbb{T}^3} (u \cdot \nabla \partial^\alpha u) \cdot \partial^\alpha u = -\frac{1}{2} \int_{\mathbb{T}^3} (\nabla \cdot u) |\partial^\alpha u|^2 = 0.$$

To estimate $\mathcal{N}_1^\alpha$ it then suffices to perform a "hands-on high-low" estimate. Since $M \geqslant 3$, we note that $(2M-2)+(2M-3) > 2M$, and hence either $|\beta|_P \leqslant 2M-2$ or $|\gamma|P \geqslant 2M-3$, so we may estimate

$$|\mathcal{N}_1^\alpha| \lesssim \sum_{\substack{\cdots \\ |\beta|_P \leqslant 2M-2}} \|\partial^\beta u\|_{L^\infty} \|\nabla \partial^\gamma u\|_{L^2} \|\partial^\alpha u\|_{L^2} + \sum_{\substack{\cdots \\ |\gamma|_P \leqslant 2M-3}} \|\partial^\beta u\|_{L^2} \|\nabla \partial^\gamma u\|_{L^\infty} \|\partial^\alpha u\|_{L^2} \lesssim \|u\|_{P^{2M}}^3.$$

Estimating $\mathcal{N}_2$, $\mathcal{N}_3$, and $\mathcal{N}_4$. We proceed as we did for $\mathcal{N}_1$ and obtain

$$|\mathcal{N}_2^\alpha| \lesssim \|K\|_{P^{2M}} \|\theta\|_{P^{2M}}^2, \quad |\mathcal{N}_3^\alpha| \lesssim \|K\|_{P^{2M}}^2 \|u\|_{P^{2M}}, \quad \text{and} \quad |\mathcal{N}_4^\alpha| \lesssim \|K\|_{P^{2M}}^2 \|\theta\|_{P^{2M}}.$$

Estimating $\mathcal{N}_5$. We split $\mathcal{N}_5^\alpha$ into two pieces:

$$\mathcal{N}_5^\alpha = \sum_{\substack{\beta+\gamma=\alpha \\ \beta>0}} \int_{\mathbb{T}^3} (\partial^\beta K)(\partial_t \partial^\gamma \theta) \cdot (\partial^\alpha \theta) = \sum_{\substack{\cdots \\ |\beta|=1}} \cdots + \sum_{\substack{\cdots \\ |\beta|=2}} \cdots =: \mathrm{I} + \mathrm{II},$$

where I is the *only* term in $\mathcal{N}$ that requires the use of $\|\nabla K\|_{L^\infty}$ since it unavoidably contains a parabolic count of derivatives of $2M+1$. Estimating I is immediate:

$$|\mathrm{I}| = \left| \sum_{i=1}^3 \int_{\mathbb{T}^3} (\partial_i K)(\partial_t^{\alpha-e_i} \theta) \cdot (\partial^\alpha \theta) \right| \lesssim \sum_i \|\nabla K\|_{L^\infty} \|\partial_t \partial^{\alpha-e_i} \theta\|_{L^2} \|\partial^\alpha \theta\|_{L^4} \lesssim \|\nabla K\|_{L^\infty} \|\theta\|_{P^{2M+1}}^2.$$

Estimating II can be done via "hands-on high-low" estimates very similar to those employed to control $\mathcal{N}_1^\alpha$. Since $M \geqslant 4$, we have $(2M-2)+(2M-4) > 2M$, and hence $|\beta|_P \leqslant 2M-2$ or $|\gamma|_P \leqslant 2M-4$, such that

$$|\mathrm{II}| \leqslant \sum_{\substack{\cdots \\ |\beta|_P \leqslant 2M-2}} \|\partial^\beta K\|_{L^\infty} \|\partial_t \partial^\gamma \theta\|_{L^2} \|\partial^\alpha \theta\|_{L^2} + \sum_{\substack{\cdots \\ |\gamma|_P \leqslant 2M-4}} \|\partial^\beta K\|_{L^2} \|\partial_t \partial^\gamma \theta\|_{L^\infty} \|\partial^\alpha \theta\|_{L^2} \lesssim \|K\|_{P^{2M}} \|\theta\|_{P^{2M}}^2.$$

Estimating $\mathcal{N}_6$. We split $\mathcal{N}_6$ into two pieces:

$$\begin{aligned} \mathcal{N}_6^\alpha &= \int_{\mathbb{T}^3} [J_{\mathrm{eq}}(u \cdot \nabla), \partial^\alpha] \theta \cdot \partial^\alpha \theta + \int_{\mathbb{T}^3} [K(u \cdot \nabla), \partial^\alpha] \theta \cdot \partial^\alpha \theta \\ &= \sum_{\substack{\beta+\gamma=\alpha \\ \beta>0}} \int_{\mathbb{T}^3} J_{\mathrm{eq}} (\partial^\beta u \cdot \nabla) \partial^\gamma \theta \cdot \partial^\alpha \theta + \sum_{\substack{\beta+\gamma+\delta=\alpha \\ \beta+\gamma>0}} \int_{\mathbb{T}^3} (\partial^\beta K)(\partial^\gamma u \cdot \nabla) \partial^\delta \theta \cdot \partial^\alpha \theta =: \mathrm{I} + \mathrm{II}. \end{aligned}$$

To control I we proceed as we did for $\mathcal{N}_1$ and obtain that $|\mathrm{I}| \lesssim \|(u,\theta)\|_{P^{2M}}^3$. To control II we proceed in a similar fashion, namely with "hands-on high-low" estimates. Since the interaction is quartic we will rely on Lemma B.6 in order to ensure that there are always at least two factors that have a sufficiently low derivative count.

More precisely, the key observation is that, since $\delta < \alpha$, all factors in II are controlled in L^2 via P^{2M}. To control II it therefore suffices to ensure that two of the four factors are controlled in L^∞ through P^{2M}. This occurs when

$$(1)\ |\beta| \leqslant |\alpha| - 2 \ \text{ for } \partial^\beta K, \quad (2)\ |\gamma| \leqslant |\alpha| - 2 \ \text{ for } \partial^\gamma u, \quad \text{and} \quad (3)\ |\delta| \leqslant |\alpha| - 3 \ \text{ for } \nabla \partial^\delta \theta.$$

Crucially, since $M \geqslant 3$, and hence $(2M-2) + (2M-3) > 2M$, Lemma B.6 tells us that at least two out of (1), (2), or (3) hold. We may then deduce that $|\mathrm{II}| \lesssim \|K\|_{P^{2M}} \|(u, \theta)\|_{P^{2M}}^3$. For example if (1) and (2) hold then we estimate the interaction as follows:

$$\left| \int_{\mathbb{T}^3} (\partial^\beta K)(\partial^\gamma u \cdot \nabla) \partial^\delta \theta \cdot \partial^\alpha \theta \right| \lesssim \|\partial^\beta K\|_{L^\infty} \|\partial^\gamma u\|_{L^\infty} \|\nabla \partial^\delta \theta\|_{L^2} \|\partial^\alpha \theta\|_{L^2} \lesssim \|K\|_{P^{2M}} \|(u, \theta)\|_{P^{2M}}^3.$$

Estimating $\mathcal{N}_7$. We proceed similarly to how we handled $\mathcal{N}_6$. We begin by splitting $\mathcal{N}_7$ into four pieces:

$$\begin{aligned}\mathcal{N}_7^\alpha &= \int_{\mathbb{T}^3} [\omega_{\mathrm{eq}} \times J_{\mathrm{eq}}, \partial^\alpha] \theta \cdot \partial^\alpha \theta + \int_{\mathbb{T}^3} [\omega_{\mathrm{eq}} \times K, \partial^\alpha] \theta \cdot \partial^\alpha \theta + \int_{\mathbb{T}^3} [\theta \times J_{\mathrm{eq}}, \partial^\alpha] \theta \cdot \partial^\alpha \theta + \int_{\mathbb{T}^3} [\theta \times K, \partial^\alpha] \theta \cdot \partial^\alpha \theta \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},\end{aligned}$$

where note that $\mathrm{I} = 0$ since $[\omega_{\mathrm{eq}} \times J_{\mathrm{eq}}, \partial^\alpha] = 0$. To estimate II and III we proceed as we did for $\mathcal{N}_1$ and obtain that

$$|\mathrm{II}| \lesssim \|K\|_{P^{2M}} \|\theta\|_{P^{2M}}^2 \quad \text{and} \quad |\mathrm{III}| \lesssim \|\theta\|_{P^{2M}}^3.$$

Finally, to estimate IV we proceed as we did for II of $\mathcal{N}_6$, namely using Lemma B.6 to split up the terms in a fashion amenable to "hands-on high-low estimates, and obtain that $|\mathrm{IV}| \lesssim \|K\|_{P^{2M}} \|\theta\|_{P^{2M}}^3$. □

Once the nonlinear interactions are controlled we may deduce the a priori energy estimates recorded in Lemma 5.14. Recall that $\tilde{\mathcal{E}}_{M,\mathrm{loc}}$ and $\overline{\mathcal{D}}_M$ are defined in (5-12) and (3-8), respectively.

Lemma 5.14 (local a priori energy estimates). *Suppose that $Z_n = (u_n, \theta_n, K_n) \in V_n$ solves (5-6) and satisfies $\|K_n\|_{L^\infty} < \frac{1}{2}\lambda$. For any integer $M \geqslant 4$, there exists $\delta_{\mathrm{ap}}^{\mathrm{loc}} > 0$ such that if $\|\nabla K_n\|_{L^\infty} < \delta_{\mathrm{ap}}^{\mathrm{loc}}$ then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{\mathcal{E}}_{M,\mathrm{loc}}(Z_n) + \tfrac{1}{2} \overline{\mathcal{D}}_M(u_n, \theta_n) \leqslant C_G (\tilde{\mathcal{E}}_{M,\mathrm{loc}}^{3/2}(Z_n) + \tilde{\mathcal{E}}_{M,\mathrm{loc}}^2(Z_n)). \tag{5-15}$$

Proof. The energy estimate of Proposition 5.11 combined with Lemma 4.12 tells us that, for $\mathcal{N}$ as in Lemma 5.13,

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{\mathcal{E}}_{M,\mathrm{loc}}(Z_n) + \overline{\mathcal{D}}_M(u_n, \theta_n) \lesssim \mathcal{N}.$$

We may combine with the estimate of $\mathcal{N}$ of Lemma 5.13 and with Lemma 5.9, since $\|K_n\|_{L^\infty} < \frac{1}{2}\lambda$, to deduce that there exists $C_{\mathrm{ap}}^{\mathrm{loc}} > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{\mathcal{E}}_{M,\mathrm{loc}} + \overline{\mathcal{D}}_M(u_n, \theta_n) \leqslant C_{\mathrm{ap}}^{\mathrm{loc}} \|\nabla K_n\|_{L^\infty} \overline{\mathcal{D}}_M(u_n, \theta_n) + \tilde{\mathcal{E}}_{M,\mathrm{loc}}^{3/2}(Z_n) + \tilde{\mathcal{E}}_{M,\mathrm{loc}}^2(Z_n).$$

So finally, if we pick $\delta_{\mathrm{ap}}^{\mathrm{loc}} > 0$ sufficiently small to ensure that $C_{\mathrm{ap}}^{\mathrm{loc}} \delta_{\mathrm{ap}}^{\mathrm{loc}} \leqslant \frac{1}{2}$, then we may conclude that there exists $C_G > 0$ such that (5-15) holds. □

To produce uniform bounds on the approximate solutions from the a priori estimates of Lemma 5.14 it suffices to couple it with a nonlinear Gronwall-type argument.

Lemma 5.15 (Bihari argument). *Suppose that, for some $T > 0$, the functions $e, d : [0, T) \to [0, \infty)$ are continuous and satisfy, for some $\alpha_0 > 0$ and $C > 0$,*

$$e'(t) + d(t) \leqslant Cf(e(t)) \quad \text{for every } 0 \leqslant t < T \text{ and } e(0) \leqslant \alpha_0,$$

where $f(x) := x^{3/2} + x^2$ for every $x \geqslant 0$. Then, for every $0 \leqslant t < \min(T, 2F(\alpha_0)/C)$,

$$e(t) \leqslant F^{-1}\bigl(F(\alpha_0) - \tfrac{1}{2}Ct\bigr) \quad \text{and} \quad \int_0^t d(s)\,ds \leqslant \alpha_0 + Ct(f \circ F)^{-1}\bigl(F(\alpha_0) - \tfrac{1}{2}Ct\bigr),$$

where $F(x) := 1/\sqrt{x} - \log(1 + 1/\sqrt{x})$ for every $x > 0$.

Proof. Bounding e follows from a standard nonlinear Gronwall argument; see for example [Boyer and Fabrie 2013]. The bound on d follows from integrating the differential inequality in time and the monotonicity of f. □

We may now state the first of the two main results of this section, obtaining uniform bounds on the approximate solutions. Recall that $\tilde{\mathcal{E}}_{M,\mathrm{loc}}$ and $\overline{\mathcal{D}}_M$ are defined in (5-12) and (3-8), respectively.

Corollary 5.16 (uniform a priori bounds on approximate solutions). *Let $M \geqslant 4$ be an integer, let $T > 0$ be some time horizon, and let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of solutions $Z_n = (u_n, \theta_n, K_n) \in V_n$ such that, for every $n \in \mathbb{N}$, we have that Z_n solves (5-6) and satisfies $\|K_n\|_\infty < \frac{1}{2}\lambda$ and $\|\nabla K_n\|_\infty < \delta_{\mathrm{ap}}^{\mathrm{loc}}$ for $\delta_{\mathrm{ap}}^{\mathrm{loc}}$ as in Lemma 5.14 on the time interval $[0, T)$. Then, for every $n \in \mathbb{N}$ and every $\alpha_0 > 0$, if $\tilde{\mathcal{E}}_{M,\mathrm{loc}}(Z_n(0)) \leqslant \alpha_0$ it follows that, for every $0 \leqslant t < \min(T, 2F(\alpha_0)/C)$,*

$$\tilde{\mathcal{E}}_{M,\mathrm{loc}}(Z_n(t)) \leqslant F^{-1}\bigl(F(\alpha_0) - \tfrac{1}{2}C_G t\bigr)$$

and

$$\int_0^t \overline{\mathcal{D}}_M(u_n(s), \theta_n(s))\,ds \leqslant \alpha_0 + C_G t(f \circ F^{-1})\bigl(F(\alpha_0) - \tfrac{1}{2}C_G t\bigr),$$

where $C_G > 0$ is as in Lemma 5.14, $f(x) := x^{3/2} + x^2$, and $F(x) := 1/\sqrt{x} - \log(1 + 1/\sqrt{x})$.

Proof. This follows immediately from combining Lemmas 5.14 and 5.15. □

We now turn our attention towards the second of the two main results of this section, namely controlling the initial energy (which involves temporal derivatives) exclusively in terms of spatial norms. In order to do so we first record the following estimates of the nonlinearities.

Lemma 5.17 (estimates of the nonlinearities for the approximate problem). *Let $n, j, k, M \in \mathbb{N}$, where $2 \leqslant j \leqslant M$, and let $Z = (u, \theta, K) \in L^2(\mathbb{T}^3; \mathbb{R}^3 \times \mathbb{R}^3 \times \mathrm{Sym}(3))$. The following estimates hold:*

(1) $\|\mathcal{L}_{Z,n} Z\|_{H^k} \lesssim \|Z\|_{H^{k+2}} + \|Z\|^2_{H^{k+1}}$.

(2) $\|N_n(Z)\|_{H^k} \lesssim \|Z\|^2_{H^{k+1}}$.

(3) $\|[K\partial_t, \partial_t^{j-1}]\|_{H^{2M-2j}} \lesssim \|Z\|^2_{P^{2M}_{j-1}}$.

(4) $\|\partial_t^{j-1}(N_n(Z))\|_{H^{2M-2j}} \lesssim \|Z\|^2_{P^{2M}_{j-1}}$.

(5) $\|[(J_{\mathrm{eq}} + K)(u \cdot \nabla), \partial_t^{j-1}]\theta\|_{H^{2M-2j}} \lesssim \|Z\|^2_{P^{2M}_{j-1}} + \|Z\|^3_{P^{2M}_{j-1}}$.

(6) $\|[(\omega_{\mathrm{eq}} + \theta) \times (J_{\mathrm{eq}} + K), \partial_t^{j-1}]\|_{H^{2M-2j}} \lesssim \|Z\|^2_{P^{2M}_{j-1}} + \|Z\|^3_{P^{2M}_{j-1}}$.

Proof. These estimates rely mostly on the fact that, for $s > \frac{3}{2}$, $H^s(\mathbb{T}^3)$ is a Banach algebra.

To obtain (1) we proceed as in the beginning of Section 5 and split $\mathcal{L}_{Z,n}$ into its part with constant coefficients and the remainder, writing $\mathcal{L}_{Z,n} = \mathcal{L}_0 + \overline{\mathcal{L}}_{Z,n}$. In particular, the estimate $\|\mathcal{L}_0 Z\|_{H^k} \lesssim \|Z\|_{H^{k+2}}$ is immediate. The estimate $\|\overline{\mathcal{L}}_{Z,n} Z\|_{H^k} \lesssim \|Z\|_{H^{k+1}} + \|Z\|^2_{H^{k+1}}$ follows from the fact that H^{k+2} is a Banach algebra and from Lemma B.4. Obtaining (2) follows in the same way.

Obtaining (3)–(6) follows a similar procedure, and we thus only provide the details for (3). Observe that if $a \leqslant j-1$ and $b \leqslant j-2$ then $2M-2a,\ 2M-2b-2 \geqslant 2M-2j+2$. Crucially, since $2M-2j+2 \geqslant 2$, we know that $H^{2M-2j+2}$ is a Banach algebra, and hence

$$\|\partial_t^a K \partial_t^{b+1}\theta\|_{H^{2M-2j}} \lesssim \|\partial_t^a K\|_{H^{2M-2j+2}} \|\partial_t^{b+1}\theta\|_{H^{2M-2j+2}} \lesssim \|\partial_t^a K\|_{H^{2M-2j+2}} \|\partial_t^{b+1}\theta\|_{H^{2M-2b+2}} \lesssim \|Z\|^2_{P^{2M}_{j-1}}.$$

So finally

$$\|[K\partial_t,\, \partial_t^{j-1}]\theta\|_{H^{2M-2j}} \lesssim \sum_{\substack{a+b=j-1\\ b<j-1}} \|\partial_t^a K \partial_t^{b+1}\theta\|_{H^{2M-2j}} \lesssim \|Z\|^2_{P^{2M}_{j-1}}. \qquad \square$$

We may now conclude this section with the second of the two main results of this section and bound the initial energy in terms of purely spatial norms.

Lemma 5.18 (bounds on the initial energy in terms of purely spatial norms). *Let $M \geqslant 0$ be an integer and $n \in \mathbb{N}$. Then there exist constants $C_{IC}, C_M > 0$ such that if $Z = (u, \theta, K)$ solves*

$$\widetilde{T}_n(K)\partial_t Z = \mathcal{L}_{Z,n} Z + N_n(Z) \tag{5-16}$$

and satisfies $\|K\|_{L^\infty} < \frac{1}{2}\lambda$ then $Z(t)$ satisfies, for every t for which it is defined,

$$\|Z(t)\|_{P^{2M}} \leqslant C_{IC}(\|Z(t)\|_{H^{2M}} + \|Z(t)\|^{C_M}_{H^{2M}}).$$

In particular, this holds when $t = 0$.

Proof. Suppose that Z solves (5-16). Applying $j-1$ temporal derivatives then tells us that

$$\widetilde{T}_n(K)\partial_t(\partial_t^{j-1} Z) = \mathcal{L}_{Z,n}(\partial_t^{j-1} Z) + [\widetilde{T}_n(K)\partial_t,\, \partial_t^{j-1}]Z - [\mathcal{L}_{Z,n},\, \partial_t^{j-1}]Z + \partial_t^{j-1}(N_n(Z)) =: F^j(Z),$$

where $[\widetilde{T}_n(K)\partial_t,\, \partial_t^{j-1}] = 0_3 \oplus P_n \circ [K\partial_t,\, \partial_t^{j-1}] \oplus 0_{3\times 3}$ and

$$[\mathcal{L}_{Z,n},\, \partial_t^{j-1}] = 0_3 \oplus P_n \circ ([(J_{\text{eq}} + K)(u\cdot\nabla),\, \partial_t^{j-1}] + [(\omega_{\text{eq}} + \theta) \times (J_{\text{eq}} + K),\, \partial_t^{j-1}]) \oplus 0_{3\times 3}.$$

Therefore,

$$\begin{aligned} F^j(Z) = \mathcal{L}_{Z,n}(\partial_t^{j-1} Z) + P_n([K_n\partial_t,\, \partial_t^{j-1}]\theta) - P_n([(J_{\text{eq}} + K)(u\cdot\nabla),\, \partial_t^{j-1}]\theta) \\ - P_n([(\omega_{\text{eq}} + \theta) \times (J_{\text{eq}} + K),\, \partial_t^{j-1}]\theta) + \partial_t^{j-1}(N_n(Z)) \end{aligned}$$

such that, by Lemma 5.17,

$$\|F^j\|_{H^{2M-2j}} \lesssim \|\partial_t^{j-1} Z\|_{H^{2M-2j+2}} + \|\partial_t^{j-1} Z\|^2_{H^{2M-2j+1}} + \|Z\|^2_{P^{2M}_{j-1}} + \|Z\|^3_{P^{2M}_{j-1}} \lesssim \|Z\|_{P^{2M}_{j-1}} + \|Z\|^3_{P^{2M}_{j-1}}. \tag{5-17}$$

In particular, we see that $\partial_t^j Z = \widetilde{T}(K)^{-1} F^j(Z)$. We now break into two cases, depending on whether $j \leqslant M-1$ or $j = M$. For $1 \leqslant j \leqslant M-1$, we have that $2M-2j \geqslant 2$, so combining Lemma 5.5 with (5-17) tells us that

$$\|\partial_t^j Z\|_{H^{2M-2j}} \lesssim (\|K\|_{H^{2M-2j}} + \|K\|^{2M-2j}_{H^{2M-2j+2}}) \|F^j\|_{H^{2M-2j}} \lesssim \|Z\|^2_{P^{2M}_{j-1}} + \|Z\|^{2M-2j+3}_{P^{2M}_{j-1}}. \tag{5-18}$$

For $j = M$, we may not apply Lemma 5.5, since $2M - 2j = 0 < 2$, but we do not need to, since in that case we are only after an L^2 bound, which Lemma 5.1 readily provides. Indeed, using Lemma 5.1 and (5-17) we see that

$$\|\partial_t^M Z\|_{L^2} \leqslant \frac{2}{\lambda}\|F^{M-1}(Z)\|_{L^2} \lesssim \|Z\|_{P^{2M}_{M-1}} + \|Z\|^3_{P^{2M}_{M-1}}. \tag{5-19}$$

Crucially, combining (5-18) and (5-19) tells us that, for $1 \leqslant j \leqslant M$,

$$\|Z\|_{P^{2M}_j} \asymp \|Z\|_{P^{2M}_{j-1}} + \|\partial_t^j Z\|_{H^{2M-2j}} \lesssim \|Z\|_{P^{2M}_{j-1}} + \|Z\|^{2M-2j+3}_{P^{2M}_{j-1}},$$

from which the claim follows by induction. □

5C. ***The Galerkin scheme.*** In this section we put together the Galerkin scheme that will produce solutions to (2-1) locally in time. We proceed in a standard manner, first producing local approximate solutions, then obtaining uniform estimates on the approximates sufficient to obtain a uniform lower bound on the time of existence and to pass to the limit by compactness. To conclude we pass to the limit and reconstruct the pressure. We begin by producing local approximate solutions.

Proposition 5.19 (producing local approximate solutions). *Let $\delta^{\mathrm{loc}}_{\mathrm{ap}} > 0$ be as in Lemma 5.14, pick some $0 < \sigma < \sigma(\delta^{\mathrm{loc}}_{\mathrm{ap}})$ as in (5-1), and let $\mathcal{U}(\sigma)$ be defined as in (5-2). For every $Z_0 = (u_0, \theta_0, K_0) \in \mathcal{U}(\sigma)$ and every $n \in \mathbb{N}$, there exists a maximal time of existence $T_n > 0$ and a unique solution $Z_n = (u_n, \theta_n, K_n)$ in $C^\infty([0, T_n); \mathcal{U}_n(\sigma))$ of*

$$\begin{cases} \widetilde{T}_n(K_n)\partial_t Z_n = \mathcal{L}_{Z_n,n} Z_n + N_n(Z_n), \\ Z_n(0) = \mathcal{P}_n Z_0. \end{cases} \tag{5-20}$$

Moreover, we have the following blow-up criterion: for any $T > 0$, if $\sup_{0\leqslant t\leqslant T}\|K_n(t)\|_{H^3} < \sigma$ then $T_n \geqslant T$.

Proof. The key is to write (5-20) as a finite-dimensional ODE in the standard form $\dot{x}(t) = f(x(t))$. Observe that by choice of $\sigma > 0$ and definition of $\mathcal{U}(\sigma)$, it follows from Lemma 5.1 that $\widetilde{T}_n(K_n)$ is invertible for any $Z_n \in \mathcal{U}_n(\sigma)$. The system (5-20) is thus equivalent to

$$\partial_t Z_n = \widetilde{T}_n(K_n)^{-1}(\mathcal{L}_{Z_n,n} Z_n + N_n(Z_n)) =: F_n(Z_n) \quad \text{and} \quad Z_n(0) = \mathbb{P}_n Z_0.$$

Since $Z_n \mapsto \mathcal{L}_{Z_n,n} Z_n + N_n(Z_n)$ is, up to the appearances of the projections P_n and $\mathbb{P}_L$, a polynomial in $(Z_n, \nabla Z_n, \nabla^2 Z_n)$, it follows from Lemma 5.2 and the equivalence of $H^s(\mathbb{T}^3)$ norms ($s \geqslant 0$) on V_n that $Z_n \mapsto F_n(Z_n)$ is a smooth map from $\mathcal{U}_n(\sigma)$ to V_n. Note that deducing that the image of F_n lies in V_n comes from the fact that the Leray projection $\mathbb{P}_L$ enforces the divergence-free condition and preserves the average of the velocity of u since $\hat{P}_L(0) = I$. By standard well-posedness theory for finite-dimensional ODEs we may now deduce the result, noting that the blow-up criterion follows from the definition of $\mathcal{U}(\sigma)$. □

We may now put together the local a priori estimates of Corollary 5.16 and the a priori projected advection-rotation estimates for K_n of Lemma 5.7 in order to deduce uniform bounds on the approximate solutions.

Proposition 5.20 (uniform bounds on approximate solutions and their intervals of existence). *Let $\sigma > 0$ and $\mathcal{U}(\sigma)$ be as in Proposition 5.19, let $C_K > 0$ be the constant implicit in the result of Lemma 5.7 when $k = 3$, let $M \geqslant 4$ be an integer, let $Z_0 = (u_0, \theta_0, K_0) \in \mathcal{U}(\sigma)$ with $\|Z_0\|_{H^{2M}}, \|K_0\|_{H^{2M+1}} < \infty$, and*

$$\|K_0\|_{H^3} < \sigma_* := \frac{\sigma}{2C_K}, \tag{5-21}$$

and let $(Z_n)_{n\in\mathbb{N}}$ be the sequence of approximate solutions obtained in Proposition 5.19, with corresponding maximal times of existence $(T_n)_{n\in\mathbb{N}}$. There exists $0 < T_{\mathrm{lwp}} \leqslant 1$ and there exist $\rho_e, \rho_d : (0,\infty) \to (0,\infty)$ and $\rho_f : (0,\infty)^2 \to (0,\infty)$ which are continuous, strictly increasing in each of their arguments, and asymptotically vanishing at zero such that $T_n \geqslant T_{\mathrm{lwp}}$ for all $n \in \mathbb{N}$ and

$$\begin{cases} \sup_{n\in\mathbb{N}} \sup_{0\leqslant j\leqslant M} \|\partial_t^j Z_n\|_{L^\infty([0,T_{\mathrm{lwp}}],H^{2M-2j})} \leqslant \rho_e(\|Z_0\|_{H^{2M}}), \\ \sup_{n\in\mathbb{N}} \sup_{0\leqslant j\leqslant M} \|(u_n,\theta_n)\|_{L^2([0,T_{\mathrm{lwp}}],H^{2M+1-2j})} \leqslant \rho_d(\|Z_0\|_{H^{2M}}), \\ \sup_{n\in\mathbb{N}} \|K_n\|_{L^\infty([0,T_{\mathrm{lwp}}],H^{2M+1})} \leqslant \rho_f(\|Z_0\|_{H^{2M}}, \|K_0\|_{H^{2M+1}}). \end{cases} \tag{5-22}$$

Moreover, $T_{\mathrm{lwp}} = \phi(\|Z_0\|_{H^{2M}})$, where ϕ is nonincreasing.

Proof. More precisely, let us define

$$\sigma_0 := 2\widetilde{C}_E C_{IC}^2 (\|Z_0\|_{H^{2M}}^2 + \|Z_0\|_{H^{2M}}^{2C_M}), \tag{5-23}$$

where $\widetilde{C}_E, C_{IC}, C_M > 0$ are as in Lemmas 5.9 and 5.18. We note that,

- by definition of $\mathcal{U}(\sigma)$ (and of $\mathcal{U}_n(\sigma)$), for every $n \in \mathbb{N}$, we have $\|K_n\|_{H^3} < \sigma$ on $[0, T_n)$, where $\sigma > 0$ is as in Proposition 5.19, and that,
- by Lemmas 5.9 and 5.18, $\tilde{\mathcal{E}}_{M,\mathrm{loc}}(Z_n(0)) \leqslant \sigma_0$.

We may thus use Corollary 5.16 to deduce that, for all $n \in \mathbb{N}$ and all $0 \leqslant t < \min(T_n, 2F(\sigma_0)/C_G)$,

$$\tilde{\mathcal{E}}_{M,\mathrm{loc}}(Z_n(t)) \leqslant F^{-1}\big(F(\sigma_0) - \tfrac{1}{2}C_G t\big)$$

and

$$\int_0^t \overline{\mathcal{D}}_M(u_n,\theta_n)(s)\,ds \leqslant \sigma_0 + C_G t (f\circ F^{-1})\big(F(\sigma_0) - \tfrac{1}{2}C_G t\big),$$

where recall that $\tilde{\mathcal{E}}_{M,\mathrm{loc}}$ and $\overline{\mathcal{D}}_M$ are defined in (5-12) and (3-8), respectively. In particular, if we pick $t = \frac{1}{2}(2F(\sigma_0)/C_G) =: \frac{1}{2}T_G$ then we have that $F^{-1}\big(F(\sigma_0) - \frac{1}{2}C_G t\big) = F^{-1}\big(\frac{1}{2}F(\sigma_0)\big)$, and hence, for every $n \in \mathbb{N}$ and every $0 \leqslant t \leqslant T_n \wedge \frac{1}{2}T_G \wedge 1$,

$$\|Z_n(t)\|_{P^{2M}}^2 = \bar{\mathcal{E}}_{M,\mathrm{loc}}(Z_n(t)) \leqslant \frac{1}{\tilde{c}_E}\tilde{\mathcal{E}}_{M,\mathrm{loc}}(Z_n(t)) \leqslant \frac{1}{\tilde{c}_E} F^{-1}\big(\tfrac{1}{2}F(\sigma_0)\big) =: \rho_e^2(\|Z_0\|_{H^{2M}}) \tag{5-24}$$

and, since $\fint_{\mathbb{T}^3} u = 0$, it follows from Lemma 4.12 that, for every $n \in \mathbb{N}$,

$$\begin{aligned} \int_0^{T_n\wedge\frac{1}{2}T_G\wedge 1} \|(u_n,\theta_n)(s)\|_{P^{2M+1}}^2\,ds &\leqslant C_D \int_0^{T_n\wedge\frac{1}{2}T_G\wedge 1} \overline{\mathcal{D}}_M(s)\,ds \\ &\leqslant C_D\big(\sigma_0 + C_G(f\circ F^{-1})\big(\tfrac{1}{2}F(\sigma_0)\big)\big) =: \rho_d^2(\|Z_0\|_{H^{2M}}). \end{aligned} \tag{5-25}$$

We may now appeal to the estimates for $\|K_n\|_{H^3}$ from Lemma 5.7 to obtain a lower bound on T_n which is uniform in n. Since $\|K_n(0)\|_{H^3} < \sigma$ and since $M \geqslant 2$, we know from Lemma 5.7 and (5-21) that, for any $n \in \mathbb{N}$ and any $0 \leqslant t \leqslant T_n \wedge \frac{1}{2}T_G \wedge 1$,

$$\sup_{0\leqslant s\leqslant t} \|K_n(s)\|_{H^3} \leqslant C_K e^{t\rho_e(\|Z_0\|_{H^{2M}})}\Big(\frac{\sigma}{2C_K} + t\rho_e(\|Z_0\|_{H^{2M}})\Big) =: \omega(t\rho_e(\|Z_0\|_{H^{2M}})),$$

where ω depends on σ. Crucially, observe that $\omega(0) = \frac{1}{2}\sigma$ and that ω is strictly increasing, so we see that, for $T_{\text{small}} := \omega^{-1}\big(\frac{2}{3}\sigma\big)/\rho_e(\|Z_0\|_{H^{2M}})$ and $\widetilde{T}_n := T_n \wedge \frac{1}{2}T_G \wedge T_{\text{small}} \wedge 1$,

$$\sup_{0\leqslant t\leqslant \widetilde{T}_n} \|K_n(t)\|_{H^3} \leqslant \omega(T_{\text{small}}\rho_e(\|Z_0\|_{H^{2M}})) \leqslant \tfrac{2}{3}\sigma.$$

Therefore, by the blow-up criterion, $T_n \geqslant \frac{1}{2}T_G \wedge T_{\text{small}} \wedge 1 =: T_{\text{lwp}}$ for every $n \in \mathbb{N}$.

Note that

$$T_{\text{lwp}} = \tfrac{1}{2}T_G \wedge T_{\text{small}} \wedge 1 = \frac{F(\sigma_0)}{C_G} \wedge \frac{\omega^{-1}\big(\frac{2}{3}\sigma\big)}{\rho_e(\|Z_0\|_{H^{2M}})} \wedge 1.$$

In light of (5-23) and the facts that F and ρ_e are strictly decreasing and strictly increasing, respectively, we deduce that T_{lwp} is nonincreasing with respect to $\|Z_0\|_{H^{2M}}$, as desired.

Finally we record the estimates on K obtained in Lemma 5.7. It follows from the energy-dissipation estimates (5-24) and (5-25) that

$$\sup_{0\leqslant t\leqslant T_{\text{lwp}}} \|K_n(t)\|_{H^{2M+1}} \leqslant C'_K e^{\rho_e T_{\text{lwp}}}(\|K_0\|_{H^{2M+1}} + \rho_d T_{\text{lwp}}) =: \rho_f(\|Z_0\|_{H^{2M}} + \|K_0\|_{H^{2M+1}}). \qquad \square$$

With these uniform bounds in hand we may move towards passing to the limit. First we record the following technical lemma which is essential in allowing us to pass to the limit.

Lemma 5.21. *Let $s \geqslant 0$ and $f \in L^2([0,T); H^s(\mathbb{T}^n))$ for $T > 0$. Then $\|(P_n - I)f\|_{L^2H^s} \to 0$ as $n \to \infty$.*

Proof. This follows immediately from Tonelli's theorem and the monotone convergence theorem. $\square$

We may now pass to the limit by compactness.

Proposition 5.22 (compactness and passage to the limit). *Let $\mathcal{U}(\sigma)$ be as in Proposition 5.19, let $M \geqslant 4$ be an integer, and let $Z_0 = (u_0, \theta_0, K_0) \in \mathcal{U}(\sigma)$ with $\|Z_0\|_{H^{2M}}$, $\|K_0\|_{H^{2M+1}} < \infty$, and $\|K_0\|_{H^3} < \sigma_*$ for $\sigma_* > 0$ as in Proposition 5.20. There exist $0 < T_{\text{lwp}} \leqslant 1$ and $Z = (u, \theta, K) \in C^2([0, T_{\text{lwp}}] \times \mathbb{T}^3)$ such that $Z(t,\cdot) \in \mathcal{U}(\sigma)$ for all $0 \leqslant t \leqslant T_{\text{lwp}}$ and Z solves (2-1) and (2-2). Moreover, Z satisfies the estimates*

$$\sup_{0\leqslant j\leqslant M} \|\partial_t^j Z\|_{L^\infty H^{2M-2j}} \leqslant \rho_e(\|Z_0\|_{H^{2M}}), \qquad \sup_{0\leqslant j\leqslant M} \|\partial_t^j (u,\theta)\|_{L^2 H^{2M-2j+1}} \leqslant \rho_d(\|Z_0\|_{H^{2M}}),$$
$$\|K\|_{L^\infty H^{2M+1}} \leqslant \rho_f(\|Z_0\|_{H^{2M}}, \|K_0\|_{H^{2M+1}})$$

for ρ_e, ρ_d, and ρ_f as in Proposition 5.20.

Proof. Let $(Z_n)_{n\in\mathbb{N}}$ and $(T_n)_{n\in\mathbb{N}}$ denote the approximate solutions and their times of existence as obtained in Proposition 5.19. Note that, as per Proposition 5.20, we know that $T_n \geqslant T_{\text{lwp}} > 0$ for some T_{lwp} which is independent of n. We also know from Proposition 5.20 that estimates (5-22) holds. It then follows from Banach–Alaoglu (i.e., weak-* compactness) that, up to a subsequence which we do not relabel,

$$\begin{cases} \partial_t^j Z_n \overset{*}{\rightharpoonup} \partial_t^j Z & \text{in } L^\infty H^{2M-2j} \quad \text{for every } 0 \leqslant j \leqslant M, \\ \partial_t^j (u_n, \theta_n) \rightharpoonup \partial_t^j (u, \theta) & \text{in } L^2 H^{2M-2j+1} \quad \text{for every } 0 \leqslant j \leqslant M, \\ K_n \overset{*}{\rightharpoonup} K & \text{in } L^\infty H^{2M+1} \end{cases} \tag{5-26}$$

for some $Z = (u, \theta, K)$ which satisfies, by weak and weak-* lower semicontinuity of the respective norms,

$$\sup_{0\leqslant j\leqslant M} \|\partial_t^j Z\|_{L^\infty H^{2M-2j}} \leqslant \|Z\|_{L^\infty P^{2M}} \leqslant \sqrt{\rho_e}, \qquad \sup_{0\leqslant j\leqslant M} \|\partial_t^j (u, \theta)\|_{L^2 H^{2M-2j+1}} \leqslant \|(u, \theta)\|_{L^2 P^{2M+1}} \leqslant \sqrt{\rho_d},$$
$$\|K\|_{L^\infty H^{2M+1}} \leqslant \sqrt{\rho_f}.$$

All that remains is passing to the limit, for which we omit the details since this is done with a standard application of the Aubin–Lions–Simon compactness theorem (see for example [Boyer and Fabrie 2013]) in combination with Lemma 5.21 and the fact that H^s is a Banach algebra when $s > \frac{3}{2}$. In particular, we can pass to the limit in the nonlinearities uniformly on $[0, T_{\text{lwp}}] \times \mathbb{T}^3$ such that the following limits hold in $C^0_{t,x}$:

$$P_n(K_n \partial_t \theta_n) \to K \partial_t \theta, \quad P_n((J_{\text{eq}} + K_n)(u_n \cdot \nabla)\theta_n) \to (J_{\text{eq}} + K)(u \cdot \nabla)\theta,$$
$$N_n(Z_n) \to N(Z), \quad \text{and} \quad P_n((\omega_{\text{eq}} + \theta_n) \times (J_{\text{eq}} + K_n)\theta_n) \to (\omega_{\text{eq}} + \theta) \times (J_{\text{eq}} + K)\theta. \qquad \square$$

The last step of our Galerkin scheme is to reconstruct the pressure and the initial condition.

Corollary 5.23 (reconstructing the pressure and the initial condition). *Under the assumptions found in Proposition 5.22, we know that $Z(0) = Z_0$ pointwise and that there exists $p \in L^2 H^{2M+1} \cap L^\infty P^{2M}_{M-1}$ such that Z and p solve (2-1).*

Proof. Recovering the initial condition is trivial. Since $Z_n(0) := P_n Z_0$ with $Z_0 \in H^2$, it follows directly from the weak convergence in Proposition 5.22 that $Z_n \to Z$ in $C^0 H^2$, and hence Lemma 5.21 tells us that

$$\|Z(0) - Z_0\|_{C^0_x} \lesssim \|Z(0) - Z_n(0)\|_{C^0_x} + \|Z_n(0) - Z_0\|_{C^0_x} \lesssim \|Z - Z_n\|_{C^0 H^2} + \|(P_n - I)Z_0\|_{H^2} \to 0$$

as $n \to \infty$.

We now reconstruct the pressure. We have split

$$\partial_t u + u \cdot \nabla u = \left(\mu + \tfrac{1}{2}\kappa\right)\Delta u - \kappa \nabla \times \omega - \nabla p$$

subject to $\nabla \cdot u = 0$ into two parts, namely

$$\partial_t u + \mathbb{P}_L(u \cdot \nabla u) = \left(\mu + \tfrac{1}{2}\kappa\right)u - \kappa \nabla \times \omega \quad \text{and} \quad (I - \mathbb{P}_L)(u \cdot \nabla u) = -\nabla p,$$

where $\mathbb{P}_L$ is the Leray projector, i.e., the L^2-orthogonal projection onto divergence-free vector fields given by $\hat{P}_L(k) = I - (k \otimes k)/|k|^2$ for every $k \in \mathbb{Z}^3$, where $((k \otimes k)/|k|^2)|_{k=0} := 0$. Then $I - \mathbb{P}_L = \nabla \Delta^{-1} \nabla \cdot$, and hence we may define $p := -\Delta^{-1} \nabla \cdot (u \cdot \nabla u)$. In particular, for $s > \frac{3}{2}$, we have the estimate $\|p\|_{H^{s+1}} \lesssim \|u\|_{H^s} \|u\|_{H^{s+1}}$ from which we deduce that, using standard "hands-on high-low estimates",

$$\|p\|_{L^2 P^{2M+1}} \lesssim \|u\|_{L^\infty P^{2M}} \|u\|_{L^2 P^{2M+1}} \quad \text{and} \quad \|p\|_{L^\infty P^{2M}_{M-1}} \lesssim \|u\|^2_{L^\infty P^{2M}}. \qquad \square$$

In conclusion, we have proved in this section the following local well-posedness result.

Theorem 5.24 (local well-posedness). *Let $M \geqslant 4$ be an integer, let $\delta_{\mathrm{ap}}^{\mathrm{loc}} > 0$ be as in Lemma 5.14, let $C_K > 0$ be the constant implicit in the result of Lemma 5.7 when $k = 3$, let $0 < \sigma < \sigma(\delta_{\mathrm{loc}}^{\mathrm{ap}})$, and let $\sigma_* := \sigma/(2C_K)$. Let $Z_0 = (u_0, \theta_0, K_0) \in L^2(\mathbb{T}^3; \mathbb{R}^3 \times \mathbb{R}^3 \times \mathrm{Sym}(3))$ such that*

$$\nabla \cdot u_0 = 0, \quad \fint_{\mathbb{T}^3} u_0 = 0, \quad \|K_0\|_{H^3} < \sigma_*, \quad \text{and} \quad \|Z_0\|_{H^{2M}}, \ \|K_0\|_{H^{2M+1}} < \infty.$$

There exist $0 < T_{\mathrm{lwp}} \leqslant 1$,

$$Z = (u, \theta, K) \in C^2([0, T_{\mathrm{lwp}}] \times \mathbb{T}^3; \mathbb{R}^3 \times \mathbb{R}^3 \times \mathrm{Sym}(3)),$$

and

$$p \in C^2([0, T_{\mathrm{lwp}}] \times \mathbb{T}^3; \mathbb{R})$$

such that Z and p form the unique strong solution of (2-1). Moreover, $T_{\mathrm{lwp}} = \phi(\|Z_0\|_{H^{2M}})$ for some nonincreasing function ϕ, and for every $0 \leqslant t \leqslant T_{\mathrm{lwp}}$, the solution satisfies

$$\nabla \cdot u(t, \cdot) = 0, \quad \fint_{\mathbb{T}^3} u(t, \cdot) = 0, \quad \text{and} \quad \|K(t, \cdot)\|_{H^3} < \sigma,$$

as well as the estimates

$$\sup_{0 \leqslant j \leqslant M} \|\partial_t^j Z\|_{L^\infty H^{2M-2j}} + \sup_{0 \leqslant j \leqslant M-1} \|\partial_t^j p\|_{L^\infty H^{2M-2j}} \leqslant \rho_e(\|Z_0\|_{H^{3M}}),$$

$$\sup_{0 \leqslant j \leqslant M} \|\partial_t^j (u, \theta)\|_{L^2 H^{2M-2j+1}} + \|\partial_t^j p\|_{L^2 H^{2M-2j+1}} \leqslant \rho_d(\|Z_0\|_{H^{2M}}),$$

$$\|K\|_{L^\infty H^{2M+1}} \leqslant \rho_f(\|Z_0\|_{H^{2M}}, \|K_0\|_{H^{2M+1}}),$$

where ρ_e, $\rho_d : (0, \infty) \to (0, \infty)$ and $\rho_f : (0, \infty)^2 \to (0, \infty)$ are continuous, strictly increasing in each of their arguments, and asymptotically vanishing at zero.

Proof. This follows from combining the various results of this section. Producing local approximate solutions is done using Proposition 5.19. We then employ Proposition 5.20 to obtain uniform bounds on the times of existence and the approximate solutions, which allows us to pass to the limit using Proposition 5.22. Finally we reconstruct the pressure and the initial condition using Corollary 5.23. Note that the uniqueness follows from Theorem A.5 of [Remond-Tiedrez and Tice 2021], which is recorded below in Theorem 5.25 for the reader's convenience. □

Theorem 5.25 (uniqueness). *Suppose that $(u_1, p_1, \omega_1, J_1)$ and $(u_2, p_2, \omega_2, J_2)$ are strong solutions of (1-1) on some common time interval $(0, T)$ such that they agree at time $t = 0$. If J_1 is uniformly positive-definite, p_i, $\partial_t(u_i, \omega_i, J_i) \in L_T^2 L^2$, (u_i, ω_i, J_i), $\nabla(u_i, \omega_i, J_i) \in L_T^\infty L^\infty$, and $\partial_t J_1$, $\partial_t \omega_2 \in L_T^\infty L^\infty$, then these solutions coincide on $(0, T)$. Note that here $L_T^p L^q$ denotes the space $L^p([0, T); L^q(\mathbb{T}^3))$.*

6. Continuation argument

In this section we derive the estimates necessary to "glue" the a priori estimates of Section 4 and the local well-posedness theory of Section 5. We begin with "reduced energy estimates" in Section 6A (whose purposed is detailed in Section 2F). We recall that while the a priori estimates of Section 4 rely on the smallness of the solution, the estimates here rely on the smallness of the time interval on which they hold.

Once we have these reduced energy estimates in hand we obtain supplementary estimates in Section 6B before recording a continuation argument in Section 6C. In some sense this continuation argument is the technical implementation of what was heuristically described as "gluing" the a priori estimates and the local well-posedness together.

6A. ***Local-in-time reduced energy estimates.*** In this section we derive the local-in-time reduced energy estimates. We follow a procedure familiar from Sections 4 and 5B: we first introduce appropriate notation, then record the relevant energy-dissipation relation and the precise form of the nonlinear interactions that arise, and finally we estimate these nonlinear interactions and close the reduced energy estimates.

Let us introduce compact notation that will be used throughout this section when developing the local-in-time reduced energy estimates. Considering the functions $Y = (v, \phi, b) : [0, T) \times \mathbb{T}^3 \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^2$, $W = (u, \theta, K) : [0, T) \times \mathbb{T}^3 \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathrm{Sym}(3)$, and $\mathcal{F} = (f_1, f_2, f_3) : [0, T) \times \mathbb{T}^3 \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^2$, we will write the system

$$\begin{cases} \partial_t v - (\nabla \cdot T)(v, \phi) = f_1, \\ (J_{\mathrm{eq}} + K)\partial_t \phi + (J_{\mathrm{eq}} + K)(u \cdot \nabla)\phi + (\omega_{\mathrm{eq}} + \theta) \times (J_{\mathrm{eq}} + K)\phi \\ \qquad\qquad + \tilde{\tau}^2 \tilde{b}^{\perp} + \phi \times J_{\mathrm{eq}}\omega_{\mathrm{eq}} - 2\,\mathrm{vec}\, T(v, \phi) + (\nabla \cdot M)(\phi) = f_2, \\ \partial_t b - \tilde{\tau} b^{\perp} + (\nu - \lambda)\bar{\phi}^{\perp} = f_3 \end{cases}$$

in a more compact form as $\overline{T}(K)\partial_t Y - \mathcal{L}_W Y = \mathcal{F}$, where $\overline{T}(K) = I_3 \oplus (J_{\mathrm{eq}} + K) \oplus I_2$ and the operator $\mathcal{L}_W$ is given by $\mathcal{L}_W Y = (-(\nabla \cdot T)(v, \phi), (\star), -\tilde{\tau} b^{\perp} + (\nu - \lambda)\bar{\phi}^{\perp})$ for

$$(\star) = (J_{\mathrm{eq}} + K)(u \cdot \nabla)\phi + (\omega_{\mathrm{eq}} + \theta) \times (J_{\mathrm{eq}} + K)\phi + \tilde{\tau}^2 \tilde{b}^{\perp} + \phi \times J_{\mathrm{eq}}\omega_{\mathrm{eq}} - 2\,\mathrm{vec}\, T(v, \phi) + (\nabla \cdot M)(\phi).$$

We also define the associated energy, namely

$$\mathbb{E}(Y; K) := \frac{1}{2}\int_{\mathbb{T}^3} |v|^2 + \frac{1}{2}\int_{\mathbb{T}^3} (J_{\mathrm{eq}} + K)\phi \cdot \phi + \frac{1}{2}\frac{\tilde{\tau}^2}{\nu - \lambda}\int_{\mathbb{T}^3} |b|^2, \tag{6-1}$$

and its counterpart summed up to a $2M$ count of parabolic derivatives, i.e.,

$$\tilde{\mathscr{E}}_{M,K}(Y) := \sum_{|\alpha|_P \leqslant 2M} \mathbb{E}(\partial^\alpha Y; K). \tag{6-2}$$

We now introduce notation used to write the full system in terms of the system introduced above. So let us define, for $p : [0, T) \times \mathbb{T}^3 \to \mathbb{R}$, $\Lambda(p) := (-\nabla p, 0, 0)$, and, for $Z = (u, \theta, K)$,

$$N(Z) = (-u \cdot \nabla u, -\theta \times K\omega_{\mathrm{eq}}, -u \cdot \nabla a + \theta_3 a^{\perp} + (\overline{K} - K_{33} I_2)\bar{\theta}^{\perp}). \tag{6-3}$$

We may then write the full system (2-1) as

$$\overline{T}(K)\partial_t Y - \mathcal{L}_Z Y = N(Z) + \Lambda(p) \tag{6-4}$$

subject to

$$\nabla \cdot u = 0 \quad \text{and} \quad \partial_t K + u \cdot \nabla K = [\Omega_{\mathrm{eq}} + \Theta, J_{\mathrm{eq}} + K].$$

Note that the form of $N_3(Z)$ in (6-3) comes from Lemma A.6 since, for $S = [\Omega_{\mathrm{eq}} + \Theta, J_{\mathrm{eq}} + K]$,

$$(S_{12}, S_{13}) = -(\nu - \lambda)\bar{\theta}^{\perp} + \tilde{\tau} a^{\perp} + (\overline{K} - K_{33} I_2)\bar{\theta}^{\perp} + \theta_3 a^{\perp}.$$

We now record a result akin to Lemma 5.9, precisely comparing various versions of the local energy.

Lemma 6.1 (comparisons of the different versions of the reduced energies). *Let $\tilde{\mathscr{E}}_{M,K}$ be defined as in (6-2). There exist constants $\tilde{c}_E, \widetilde{C}_E > 0$ such that if $\|K\|_{L^\infty} < \frac{1}{2}\lambda$ then $\tilde{c}_E \bar{\mathcal{E}}_M \leqslant \tilde{\mathscr{E}}_{M,K} \leqslant \widetilde{C}_E \bar{\mathcal{E}}_M$.*

Proof. This follows by choosing $\tilde{c}_E$ and $\widetilde{C}_E$ exactly as in Lemma 5.9. □

We now turn our attention to the energy-dissipation relation, which we record below.

Lemma 6.2 (generic energy-dissipation relation for the local-in-time reduced energy estimates). *Suppose that $Y = (v, \phi, b)$, $W = (u, \theta, K)$, and p satisfy $\overline{T}(K)\partial_t Y - \mathcal{L}_W Y = \mathcal{F} + \Lambda(p)$, where we define $\Lambda(p) =: (-\nabla p, 0, 0)$, subject to $\nabla \cdot u = \nabla \cdot v = 0$ and $\partial_t K + (u \cdot \nabla)K = [\Omega_{\text{eq}} + \Theta, J_{\text{eq}} + K]$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(Y; K) + D(u, \theta) = \int_{\mathbb{T}^3} \overline{\mathfrak{C}}\mathcal{F} \cdot Y, \tag{6-5}$$

where $\overline{\mathfrak{C}} := I_3 \oplus I_3 \oplus \tilde{\tau}^2/(\nu - \lambda) I_2$ and D is the usual dissipation, as given in (2-6).

Proof. This energy estimate is obtained in the same way as the energy estimate of Proposition 4.9. □

With the energy-dissipation relation in hand we may identify the precise forms of the nonlinear interactions in Lemma 6.3 below. Recall that the energy $\mathbb{E}$ is defined in (6-1).

Lemma 6.3 (recording the form of the interactions for the local-in-time reduced energy estimate). *Suppose that $Z = (u, \theta, K)$, where $a = (K_{12}, K_{12})$ and p solve (2-1). Then we have that, for every multi-index $\alpha \in \mathbb{N}^{1+3}$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(\partial^\alpha u, \partial^\alpha \theta, \partial^\alpha a; K) + D(\partial^\alpha u, \partial^\alpha \theta) = \overline{\mathcal{N}}^\alpha,$$

where, for $\check{\tau} := \tilde{\tau}^2/(\nu - \lambda)$,

$$\begin{aligned}\overline{\mathcal{N}}^\alpha = &\int_{\mathbb{T}^3} [K\partial_t, \partial^\alpha]\theta \cdot \partial^\alpha\theta - \int_{\mathbb{T}^3} [(J_{\text{eq}} + K)(u \cdot \nabla), \partial^\alpha]\theta \cdot \partial^\alpha\theta - \int_{\mathbb{T}^3} [(\omega_{\text{eq}} + \theta) \times (J_{\text{eq}} + K), \partial^\alpha]\theta \cdot \partial^\alpha\theta \\ &- \int_{\mathbb{T}^3} \partial^\alpha(u \cdot \nabla u) \cdot \partial^\alpha u - \int_{\mathbb{T}^3} \partial^\alpha(\theta \times K\omega_{\text{eq}}) \cdot \partial^\alpha\theta - \check{\tau}\int_{\mathbb{T}^3} \partial^\alpha(u \cdot \nabla a) \cdot \partial^\alpha a \\ &+ \check{\tau}\int_{\mathbb{T}^3} \partial^\alpha(\theta_3 a^\perp) \cdot \partial^\alpha a + \check{\tau}\int_{\mathbb{T}^3} \partial^\alpha((\overline{K} - K_{33}I_2)\bar{\theta}^\perp) \cdot \partial^\alpha a.\end{aligned} \tag{6-6}$$

Proof. In order to streamline the proof let us write $Y = (u, \theta, a)$. Applying a derivative ∂^α to (2-1) shows that $\partial^\alpha Y$ solves

$$\overline{T}(K)\partial_t\partial^\alpha Y - \mathcal{L}_Z\partial^\alpha Y = [\overline{T}(K)\partial_t, \partial^\alpha]Y - [\mathcal{L}_Z, \partial^\alpha]Y + \partial^\alpha(N(Z)) + \Lambda(p) =: F^\alpha + \Lambda(\partial^\alpha p) \tag{6-7}$$

subject to (2-1b) and (2-1d). We may thus apply Lemma 6.2 to deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(\partial^\alpha Y; K) + D(\partial^\alpha u, \partial^\alpha\theta) = \int_{\mathbb{T}^3} \overline{\mathfrak{C}}F^\alpha \cdot \partial^\alpha Y =: \overline{\mathcal{N}}^\alpha,$$

where $\overline{\mathfrak{C}} = I_3 \oplus I_3 \oplus \check{\tau} I_2$ is as in Lemma 6.2 and where

$$\overline{\mathcal{N}}^\alpha = \underbrace{\int_{\mathbb{T}^3} \overline{\mathfrak{C}}[\overline{T}(K)\partial_t, \partial^\alpha]Y \cdot \partial^\alpha Y}_{\overline{\mathcal{N}}^\alpha_{\mathrm{I}}} - \underbrace{\int_{\mathbb{T}^3} \overline{\mathfrak{C}}[\mathcal{L}_Z, \partial^\alpha]Y \cdot \partial^\alpha Y}_{-\overline{\mathcal{N}}^\alpha_{\mathrm{II}}} + \underbrace{\int_{\mathbb{T}^3} \overline{\mathfrak{C}}\partial^\alpha(N(Z)) \cdot \partial^\alpha Y}_{\overline{\mathcal{N}}^\alpha_{\mathrm{III}}}.$$

It now suffices to compute $\overline{\mathcal{N}}_{\mathrm{I}}^{\alpha}$, $\overline{\mathcal{N}}_{\mathrm{II}}^{\alpha}$, and $\overline{\mathcal{N}}_{\mathrm{III}}^{\alpha}$. Since $[\overline{T}(K)\partial_t, \partial^\alpha] = 0_3 \oplus [K\partial_t, \partial^\alpha] \oplus 0_2$, we see that

$$\overline{\mathcal{N}}_{\mathrm{I}}^{\alpha} = \int_{\mathbb{T}^3} [K\partial_t, \partial^\alpha]\theta \cdot \partial^\alpha \theta. \tag{6-8}$$

Now note that

$$[\mathcal{L}_Z, \partial^\alpha] = 0_3 \oplus ([(J_{\mathrm{eq}} + K)(u \cdot \nabla), \partial^\alpha] + [(\omega_{\mathrm{eq}} + \theta) \times (J_{\mathrm{eq}} + K), \partial^\alpha]) \oplus 0_2,$$

and hence

$$\overline{\mathcal{N}}_{\mathrm{II}}^{\alpha} = -\int_{\mathbb{T}^3} [(J_{\mathrm{eq}} + K)(u \cdot \nabla), \partial^\alpha]\theta \cdot \partial^\alpha \theta - \int_{\mathbb{T}^3} [(\omega_{\mathrm{eq}} + \theta) \times (J_{\mathrm{eq}} + K), \partial^\alpha]\theta \cdot \partial^\alpha \theta. \tag{6-9}$$

Finally, it follows immediately from the form of N that

$$\overline{\mathcal{N}}_{\mathrm{III}}^{\alpha} = -\int_{\mathbb{T}^3} \partial^\alpha (u \cdot \nabla a) \cdot \partial^\alpha u - \int_{\mathbb{T}^3} \partial^\alpha (\theta \times K\omega_{\mathrm{eq}}) \cdot \partial^\alpha \theta - \check{\tau} \int_{\mathbb{T}^3} \partial^\alpha (u \cdot \nabla a) \cdot \partial^\alpha a$$
$$+ \check{\tau} \int_{\mathbb{T}^3} \partial^\alpha (\theta_3 a^\perp) \cdot \partial^\alpha a + \check{\tau} \int_{\mathbb{T}^3} \partial^\alpha ((\overline{K} - K_{33} I_2)\bar{\theta}^\perp) \cdot \partial^\alpha a. \tag{6-10}$$

To conclude we sum (6-8)–(6-10) and obtain (6-6). □

We now estimate these interactions.

Lemma 6.4 (estimating the interactions for the local-in-time reduced energy estimates). *Let $M \geqslant 4$ be an integer and let $\overline{\mathcal{N}} = \sum_{|\alpha|_P \leqslant 2M} \overline{\mathcal{N}}^\alpha$ for $\overline{\mathcal{N}}^\alpha$ as in Lemma 6.3. The following estimate holds: for $Y = (u, \theta, a)$,*

$$|\overline{\mathcal{N}}| \lesssim \|\nabla K\|_{L^\infty} \|(u, \theta)\|_{P^{2M+1}}^2 + \|K\|_{P^{2M}} (\|Y\|_{P^{2M}}^2 + \|Y\|_{P^{2M}}^3) + \|Y\|_{P^{2M}}^3.$$

Proof. Let us write the terms in $\overline{\mathcal{N}}$ in order as $\overline{\mathcal{N}}^{\alpha_1}, \ldots, \overline{\mathcal{N}}^{\alpha_8}$. These interactions are either identical or very similar to the interactions $\mathcal{N}_i$, $i = 1, \ldots, 7$, estimated in Lemma 5.13. We will thus provide very few details here and instead point to the relevant portions of the proof of Lemma 5.13.

Identical interactions. Some terms in $\overline{\mathcal{N}}$ here are identical to terms in $\mathcal{N}$ in Lemma 5.13. The correspondence between these terms, and the ensuing estimates, are recorded below:

$$\begin{aligned}
\overline{\mathcal{N}}_1 &= \mathcal{N}_5, & |\overline{\mathcal{N}}_1| &\lesssim \|\nabla K\|_{L^\infty} \|\theta\|_{P^{2M+1}}^2 + \|K\|_{P^{2M}} \|\theta\|_{P^{2M}}^2, \\
\overline{\mathcal{N}}_2 &= \mathcal{N}_6, & |\overline{\mathcal{N}}_2| &\lesssim (1 + \|K\|_{P^{2M}}) \|(u, \theta)\|_{P^{2M}}^3, \\
\overline{\mathcal{N}}_3 &= \mathcal{N}_7, & |\overline{\mathcal{N}}_3| &\lesssim \|K\|_{P^{2M}} (\|\theta\|_{P^{2M}}^2 + \|\theta\|_{P^{2M}}^3) + \|\theta\|_{P^{2M}}^3, \\
\overline{\mathcal{N}}_4 &= \mathcal{N}_1, & |\overline{\mathcal{N}}_4| &\lesssim \|u\|_{P^{2M}}^3, \\
\overline{\mathcal{N}}_5 &= \mathcal{N}_2, & |\overline{\mathcal{N}}_5| &\lesssim \|K\|_{P^{2M}} \|\theta\|_{P^{2M}}^2.
\end{aligned}$$

Similar interactions. The terms $\overline{\mathcal{N}}_6$–$\overline{\mathcal{N}}_8$ are similar to $\mathcal{N}_1$ in Lemma 5.13, so proceeding similarly yields

$$|\overline{\mathcal{N}}_6| \lesssim \|u\|_{P^{2M}} \|a\|_{P^{2M}}^2, \quad |\overline{\mathcal{N}}_7| \lesssim \|\theta\|_{P^{2M}} \|a\|_{P^{2M}}^2, \quad \text{and} \quad |\overline{\mathcal{N}}_8| \lesssim \|K\|_{P^{2M}} \|\theta\|_{P^{2M}} \|a\|_{P^{2M}}. \qquad \square$$

We may now combine the energy-dissipation relation of Lemma 6.2 and the interactions estimates of Lemma 6.4 in order to derive a preliminary form of the reduced energy estimates. Recall that $\tilde{\mathscr{E}}_{M,K}$ and $\overline{\mathcal{D}}_M$ are defined in (6-2) and (3-8), respectively.

Lemma 6.5 (reduced a priori estimate). *There exist $\delta_r^{\mathrm{loc}} > 0$ and $C_G > 0$ such that if K satisfies $\|K\|_{L^\infty} < \frac{1}{2}\lambda$ and $\|\nabla K\|_{L^\infty} < \delta_r^{\mathrm{loc}}$, and $Y = (u, \theta, a)$ and $Z = (u, \theta, K)$, where $a = (K_{12}, K_{13})$, and p solve (2-1) then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathscr{E}}_{M,K}(Y) + \tfrac{1}{2}\overline{\mathcal{D}}_M(u,\theta) \leqslant C_G(1 + \|K\|_{P^{2M}})g(\tilde{\mathscr{E}}_{M,K}(Y)),$$

where $g(x) = x + x^{3/2}$ for every $x \geqslant 0$.

Proof. Lemma 6.3 tells us that $(\mathrm{d}/\mathrm{d}t)\mathbb{E}(\partial^\alpha Y; K) + D(\partial^\alpha u, \partial^\alpha \theta) = \overline{\mathcal{N}}^\alpha$ for any multi-index $\alpha \in \mathbb{N}^{1+3}$. We may thus sum over $|\alpha|_P \leqslant 2M$ and use Lemmas 4.12, 6.1, and 6.4 to deduce that, for some $C_1 > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathscr{E}}_{M,K}(Y) + \overline{\mathcal{D}}_M(u,\theta) \leqslant C_1\|\nabla K\|_{L^\infty}\overline{\mathcal{D}}_M(u,\theta) + C_1(1 + \|K\|_{P^{2M}})g(\tilde{\mathscr{E}}_{M,K}(Y)).$$

In particular, picking $\delta_r^{\mathrm{loc}} > 0$ sufficiently small to ensure that $C_1\delta_r^{\mathrm{loc}} \leqslant \frac{1}{2}$, we may deduce the result. □

The last tool required to derive the reduced energy estimates is a nonlinear Gronwall-type argument:

Lemma 6.6 (local-in-time Bihari argument). *Suppose that $e, d : [0, T) \to [0, \infty)$ for some $T > 0$ are continuously differentiable and satisfy, for some $C > 0$, that $e'(t) + d(t) \leqslant Cg(e(t))$ for every $0 < t < T$, where $g(x) =: x + x^{3/2}$ for every $x \geqslant 0$. Suppose, moreover, that there are some $0 \leqslant t_1 < t_2 < T$ and $\alpha_1 > 0$ such that $e(t_1) \leqslant \alpha_1$ and $t_2 - t_1 \leqslant \min(1, G(\alpha_1))/C$, where $G(x) := \log(1 + 1/\sqrt{x})$ for every $x \geqslant 0$. Then, for any $t_1 \leqslant t \leqslant t_2$,*

$$e(t) \leqslant G^{-1}\big(\tfrac{1}{2}G(\alpha_1)\big) \quad \text{and} \quad d(t) \leqslant \alpha_1 + (g \circ G^{-1})\big(\tfrac{1}{2}G(\alpha_1)\big).$$

Proof. Similar to Lemma 5.15, this result follows from a nonlinear Gronwall argument; see for example [Boyer and Fabrie 2013]. □

We now have in hand all the pieces necessary to prove the local-in-time reduced energy estimates. In particular, recall that both $\bar{\mathcal{E}}_M$ and $\overline{\mathcal{D}}_M$ (which are defined in Section 3) are functionals which depend only on u, θ, and a. This is precisely why this is called a *reduced* energy estimate. Note that the definitions of $\bar{\mathcal{E}}_M$ and $\overline{\mathcal{D}}_M$ may be found in (3-2) and (3-8), respectively.

Proposition 6.7 (local-in-time reduced energy estimate). *Let $\delta_r^{\mathrm{loc}} > 0$ be as in Lemma 6.5. There is a nonincreasing and continuous function $\phi_r : (0, \infty) \to (0, \infty)$ and a strictly decreasing and continuous function $\rho_r : (0, \infty) \to (0, \infty)$ which vanishes asymptotically at zero such that, for any $T > 0$ and any $Y = (u, \theta, a)$, $Z = (u, \theta, K)$, where $a = (K_{12}, K_{13})$, and p satisfying*

$$\sup_{0 \leqslant t < T} \|K(t)\|_{L^\infty} < \tfrac{1}{2}\lambda \quad \text{and} \quad \sup_{0 \leqslant t < T} \|\nabla K(t)\|_{L^\infty} < \delta_r^{\mathrm{loc}} \tag{6-11}$$

and solving (2-1), if $0 \leqslant t_1 < t_2 < T$ satisfy

$$t_2 - t_1 \leqslant \frac{\phi_r(\|Y(t_1)\|_{P^{2M}})}{1 + \sup_{0 \leqslant t \leqslant t_2}\|K(t)\|_{P^{2M}}} \tag{6-12}$$

then the following estimate holds on $[t_1, t_2]$:

$$\sup_{t_1 \leqslant t \leqslant t_2} \bar{\mathcal{E}}_M(t) + \int_{t_1}^{t_2} \overline{\mathcal{D}}_M(s)\,ds \leqslant \rho_r(\|Y(t_1)\|_{P^{2M}}).$$

Recall that $\bar{\mathcal{E}}_M$ and $\overline{\mathcal{D}}_M$ are defined in Section 3.

Proof. Let us begin by defining $C_K := \sup_{0\leqslant t\leqslant t_2} \|K(t)\|_{P^{2M}}$ and noting that, by virtue of (6-11), Lemma 6.1 tells us that $\tilde{\mathscr{E}}_{M,K}(Y(t_1)) \leqslant C_E \|Y(t_1)\|^2_{P^{2M}} =: \alpha_1$. Now Lemma 6.5 tells us that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathscr{E}}_{M,K}(Y) + \overline{\mathcal{D}}_M(u,\theta) \leqslant C_G(1+C_K)g(\alpha_1)$$

for $g(x) := x + x^{3/2}$. Therefore, for $G(x) := \log(1+1/\sqrt{x})$ as in Lemma 6.6 and

$$\phi_r(\alpha) := \frac{\min(1, G(C_E\alpha))}{C_G} \quad \text{for every } \alpha > 0,$$

(6-12) tells us that we may apply Lemma 6.6. Combining Lemma 6.6 with Lemma 6.1, we deduce that

$$\sup_{t_1\leqslant t\leqslant t_2} \bar{\mathcal{E}}_M(t) + \int_{t_1}^{t_2} \overline{\mathcal{D}}_M(s)\,ds \leqslant \frac{1}{c_e} G^{-1}\bigl(\tfrac{1}{2}G(\alpha_1)\bigr) + \alpha_1 + (g\circ G^{-1})\bigl(\tfrac{1}{2}G(\alpha_1)\bigr) =: \rho_r(\|Y(t_1)\|_{P^{2M}}). \quad \square$$

6B. *Supplementary estimates.* In this section we record supplementary estimates that are required to parlay the reduced energy estimates obtained in Section 6A above into a continuation argument (recorded in Section 6C) capable of gluing together the main a priori estimates of Section 4 and the local well-posedness theory of Section 5.

Many of the results in this section are variants of results obtained in Section 4 which no longer rely on any smallness assumption on the solution. Correspondingly, the bounds obtained are often polynomial (whereas they were linear when a smallness assumption was made). In particular, we will employ the functionals $\bar{\mathcal{E}}_M$ and $\bar{\mathcal{E}}_{\text{low}}$ several times, so we recall that their definitions may be found in (3-2) and (3-3), respectively. We begin by recording a result comparing two versions of the energy, where recall that $\tilde{\mathcal{E}}_M$ is also defined in (3-2).

Lemma 6.8 (comparisons of different versions of the energies under a smallness condition). *There exist constants $\tilde{c}_E, \widetilde{C}_E > 0$ such that if $\|K\|_{L^\infty} < \frac{1}{2}\lambda$ then $\tilde{c}_E\bar{\mathcal{E}}_M \leqslant \tilde{\mathcal{E}}_M \leqslant \widetilde{C}_E\bar{\mathcal{E}}_M$.*

Proof. This follows by choosing $\tilde{c}_E$ and $\widetilde{C}_E$ exactly as in Lemma 5.9. □

We now record an auxiliary L^∞ estimate for $\partial_t K$ which is necessary in order to control the low-level interactions.

Lemma 6.9 (L^∞ estimate for $\partial_t K$). *If K solves (2-1d) then $\|\partial_t K\|_{L^\infty} \lesssim \|\theta\|_{H^2} + (1+\|(u,\theta)\|_{H^2})\|K\|_{H^3}$.*

Proof. This follows from (2-1d), the fact that L^∞ is a Banach algebra, and the embedding $H^2 \hookrightarrow L^\infty$. □

With Lemma 6.9 in hand we may record the following reformulation of the control of the low-level interactions obtained in Lemma 4.16. We recall that $\mathcal{D}_{\text{low}}$ and $\overline{\mathcal{I}}_{\text{low}}$ are defined in (3-9) and (3-10), respectively.

Corollary 6.10 (careful estimates of the low-level interactions). *There is a polynomial P with nonnegative coefficients and which vanishes at zero such that $|\overline{\mathcal{I}}_{\text{low}}| \leqslant P(\|Y\|_{P^3}, \|K\|_{H^3})\mathcal{D}_{\text{low}}$. In particular, if $\|Y\|_{P^3} \leqslant 1$ and $\|K\|_{H^3} \leqslant 1$ then $|\overline{\mathcal{I}}_{\text{low}}| \lesssim (\|Y\|_{P^3} + \|K\|_{H^3})\mathcal{D}_{\text{low}}$.*

Proof. This follows immediately from combining Lemmas 4.16 and 6.9. □

We now turn our attention to a result similar to Proposition 4.30 where we obtain the decay of intermediate norms. The difference here is that the smallness assumption on $\mathcal{E}_M$ (present in Proposition 4.30) is replaced by a smallness assumption on $\|K\|_{H^3}$ since the latter is guaranteed to be small due to the space in which our local well-posedness theory produces solutions.

Proposition 6.11 (decay of low-level energy provided smallness of the reduced high-level energy). *Let $M \geqslant 3$ be an integer. There exist $\delta_I > 0$ and $C_I > 0$ such that, for every $T > 0$, if*

$$\sup_{0\leqslant t<T} \|K(t)\|_{H^3} \leqslant \delta_I \quad \text{and} \quad \sup_{0\leqslant t<T} \bar{\mathcal{E}}_M(t) =: \delta_0 \leqslant \tfrac{1}{2} \tag{6-13}$$

then

$$\sup_{0\leqslant t<T} \bar{\mathcal{E}}_{\mathrm{low}}(t)(1+t)^{2M-2} \leqslant C_I \delta_0. \tag{6-14}$$

Proof. The proof of this result employs the same strategy as the proof of Proposition 4.20 where we close the energy estimates at the low level, so we omit the details and only discuss how the proof of Proposition 4.20 must be modified to apply here. There are two key differences: (1) the low-level interactions are controlled by Corollary 6.10 (and not Corollary 4.17) because here we must clearly identify how K appears in the low-level interactions and (2) the different versions of the energy are compared using Lemma 6.8 (instead of Proposition A.3 and Lemma 4.11) since here we use the smallness of K, instead of the regularity of the solutions, to ensure the positive-definiteness of $J = J_{\mathrm{eq}} + K$. There is also a minor difference to take into account: there is no need here to improve the energy, so by contrast with Proposition 4.20 we do not need to appeal to Proposition 4.15. □

We conclude this section with auxiliary estimates for K which are a consequence of the advection-rotation estimates proved in Section 4A. Proposition 6.12 is therefore similar to Proposition 4.8 which performed the synthesis of the advection-rotation estimates proved in Section 4A. The key difference here is that there are no smallness assumptions being made, and as a result the bounds in both the hypotheses and the conclusion of Proposition 6.12 below are in terms of nonlinear functions of a smallness parameter. We note that the various energy and dissipation functionals used below are defined in (3-3)–(3-5) and (3-8).

Proposition 6.12 (auxiliary estimates for K). *Let $M \geqslant 3$ be an integer. If there is some time horizon $T > 0$ such that*

$$\sup_{0\leqslant t<T} \bar{\mathcal{E}}_{\mathrm{low}}(t)(1+t)^{2M-2} + \bar{\mathcal{E}}_M(t) + \int_0^T \bar{\mathcal{D}}_M(s)\, ds \leqslant \rho_{\mathrm{ap}}(\delta_0) \leqslant 1 \tag{6-15}$$

and

$$(\mathcal{E}_M + \mathcal{F}_M)(0) \leqslant \rho_0(\delta_0) \leqslant 1 \tag{6-16}$$

for some $\delta_0 \geqslant 0$ and some $\rho_{\mathrm{ap}}, \rho_0 : (0, \infty) \to (0, \infty)$ which are strictly increasing and vanish asymptotically at zero, then there exists $\rho_K : (0, \infty) \to (0, \infty)$ which is strictly increasing and vanishes asymptotically at zero such that

$$\sup_{0\leqslant t<T} \mathcal{E}_M^{(K)}(t) + \frac{\mathcal{F}_M(t)}{1+t} \leqslant \rho_K(\delta_0), \tag{6-17}$$

where ρ_K depends on ρ_{ap} and ρ_0. Moreover, if ρ_{ap} and ρ_0 are continuous then so is ρ_K.

Proof. In light of (6-15), Proposition 4.30 tells us that

$$\sup_{0\leqslant t<T}\ \sup_{1\leqslant I\leqslant M} \overline{\mathcal{K}}_I(t)(1+t)^{2M-2} \lesssim \rho_{\mathrm{ap}}(\delta_0). \tag{6-18}$$

Combining (6-15), (6-16), and (6-18), we use Proposition 4.8 to deduce that there exist $C_1, C_2 > 0$ such that, for every $0 \leqslant t < T$,

$$\mathcal{E}_M^{(K)}(t) \leqslant C_1(\rho_{\mathrm{ap}}(\delta_0) + \rho_0(\delta_0)) \tag{6-19}$$

and, also using Cauchy–Schwarz to deal with $\int \overline{\mathcal{D}}_M^{1/2}$,

$$\mathcal{F}_M(t) \leqslant C_2\biggl(\mathcal{F}_M(0) + \biggl(\int_0^t \overline{\mathcal{D}}_M^{1/2}(s)\,ds\biggr)^2 + \overline{\mathcal{K}}_M(t)\biggr) \leqslant C_2((1+t)\rho_0(\delta_0) + \rho_{\mathrm{ap}}(\delta_0)). \tag{6-20}$$

Combining (6-19) and (6-20) yields (6-17). □

6C. *Synthesis.* In this section we record the continuation argument which allows us to glue together the local well-posedness theory and our scheme of a priori estimates. Recall that $\mathcal{E}_{\mathrm{low}}$, $\mathcal{E}_M$ and $\mathcal{F}_M$, and $\mathcal{D}_M$ are defined in (3-3), (3-5), and (3-8), respectively.

Theorem 6.13 (continuation argument). *Let $M \geqslant 4$ be an integer. There exists $\eta_{\mathrm{cont}} > 0$ such that the following holds: for any finite time horizon $T > 0$, if we have a solution of (2-1) on $[0, T]$ whose initial condition satisfies*

$$(\mathcal{E}_M + \mathcal{F}_M)(0) =: \eta_0 \leqslant \eta_{\mathrm{cont}} \tag{6-21}$$

and which lives in the small energy regime, i.e.,

$$\sup_{0\leqslant t\leqslant T} \mathcal{E}_{\mathrm{low}}(t)(1+t)^{2M-2} + \mathcal{E}_M(t) + \frac{\mathcal{F}_M(t)}{1+t} + \int_0^T \mathcal{D}_M(s)\,ds \leqslant C_{\mathrm{ap}}\eta_0 \tag{6-22}$$

for $C_{\mathrm{ap}} > 0$ as in Theorem 4.34, then there exists a timescale $\tau > 0$ such that the solution can be uniquely continued on $[0, T+\tau]$ where it continues to live in the small energy regime, i.e., (6-22) holds with T replaced by $T+\tau$.

Proof. Step 1. We define the smallness parameter $\eta > 0$. We begin by picking $\sigma > 0$ small enough that

(1) $\sigma \leqslant \sigma(\delta_{\mathrm{ap}}^{\mathrm{loc}})$ for $\delta_{\mathrm{ap}}^{\mathrm{loc}}$ as in Lemma 5.14 and $\sigma = \sigma(\delta)$ as in the beginning of Section 5,

(2) $\sigma \leqslant \sigma(\delta_r^{\mathrm{loc}})$ for δ_r^{loc} as in Lemma 6.5 and $\sigma = \sigma(\delta)$ as in the beginning of Section 5,

(3) $\sigma \leqslant \delta_I$ for δ_I as in Proposition 6.12.

We then pick $\eta > 0$ sufficiently small to satisfy

(4) $\eta \leqslant \sigma_*^2/C_{\mathrm{ap}}$ for $\sigma_* = \sigma/(2C_K)$ as in Theorem 5.24,

(5) $\max(C_{\mathrm{ap}}\eta, \rho_r(\sqrt{C_{\mathrm{ap}}\eta})) \leqslant \frac{1}{2}$ for ρ_r as in Proposition 6.7,

(6) $\eta \leqslant 1$,

(7) $\rho_r(\sqrt{C_{\mathrm{ap}}\eta}) + \rho_K(\eta) \leqslant \frac{1}{2}\delta_{\mathrm{ap}}$ for δ_{ap} as in Theorem 4.34 and ρ_K as in Proposition 6.12, where ρ_K depends on ρ_0 and ρ_{ap} given by $\rho_0 := \mathrm{id}$ and $\rho_{\mathrm{ap}}(x) := \max(C_{\mathrm{ap}}x, \rho_r(\sqrt{C_{\mathrm{ap}}x}))$,

(8) $C_{\mathrm{ap}}\eta \leqslant \frac{1}{2}\delta_{\mathrm{ap}}$, and

(9) $\eta \leqslant \eta_{\mathrm{ap}}$ for η_{ap} as in Theorem 4.34.

In particular, note that choosing the parameter η in this way enforces the following implication: since $M \geqslant 3$, if (6-22) holds then

$$\sup_{0 \leqslant t \leqslant T} \|K(t)\|_{H^3}^2 \leqslant \sup_{0 \leqslant t \leqslant T} \mathcal{E}_M^{(K)}(t) \leqslant \sigma_*^2, \tag{6-23}$$

i.e., $\|K(t)\|_{H^3} < \sigma$ when the solution lives in the small energy regime.

Step 2. We now identify the timescale τ on which we may both (1) continue the solution and (2) obtain estimates on the continued solution. Correspondingly, there are two constraints on how large the timescale τ may be: (1) the first constraint comes from the local well-posedness theory of Theorem 5.24 and (2) the second constraint comes from the local-in-time reduced energy estimate of Proposition 6.7.

So let us define, for ϕ as in Theorem 5.24 and ϕ_r as in Proposition 6.7,

$$\tau_{\text{lwp}} := \phi(\sqrt{C_{\text{ap}}(2+T)\eta_0}), \quad \tau_r := \frac{\phi_r(\sqrt{C_{\text{ap}}\eta_0})}{1+\sqrt{C_{\text{ap}}(2+T)\eta_0}}, \quad \text{and} \quad \tau := \tfrac{1}{3}\min(\tau_{\text{lwp}}, \tau_r).$$

In particular, note that $2\tau < \min(\tau_{\text{lwp}}, \tau_r)$. Note also that, for every $0 \leqslant t \leqslant T$, by virtue of (6-22), if we write $Z = (u, \theta, K)$ and $Y = (u, \theta, a)$ then

$$\phi(\|Z(t)\|_{H^{2M}}) \geqslant \phi((\mathcal{E}_M + \mathcal{F}_M)^{1/2}(t)) \geqslant \tau_{\text{lwp}} \tag{6-24}$$

and

$$\frac{\phi_r(\|Y(t)\|_{P^{2M}})}{1+\sup_{0 \leqslant s \leqslant T}\|K(s)\|_{P^{2M}}} \geqslant \frac{\phi_r(\sqrt{C_{\text{ap}}\eta_0})}{1+\sup_{0 \leqslant s \leqslant T}(\mathcal{E}_M + \mathcal{F}_M)^{1/2}(s)} \geqslant \tau_r. \tag{6-25}$$

Step 3. Having identified the appropriate timescale τ we may now turn the crank of the local well-posedness theory. Feeding (6-23) and (6-24) into Theorem 5.24 using the initial condition $Z(T-\tau)$, where $Z = (u, \theta, K)$ as usual, we see that the solution may be uniquely continued to $[0, T+\tau]$ where, in light of (6-23), it satisfies

$$\sup_{T-\tau \leqslant t \leqslant T+\tau} \|K(t)\|_{H^3} < \sigma. \tag{6-26}$$

Step 4. We conclude by performing estimates on the solution on the time interval $[T-\tau, T+\tau]$ that ensure that the solution remains in the small energy regime of (6-22). In light of (6-25) and (6-26) we apply the local-in-time reduced energy estimate of Proposition 6.7 on the interval $[T-\tau, T+\tau]$ to obtain

$$\sup_{T-\tau \leqslant t \leqslant T+\tau} \bar{\mathcal{E}}_M(t) + \int_{T-\tau}^{T+\tau} \bar{\mathcal{D}}_M(s)\,ds \leqslant \rho_r(\sqrt{C_{\text{ap}}\eta_0}) \tag{6-27}$$

for ρ_r as in Proposition 6.7.

We now string together (6-27) and Propositions 6.11 and 6.12, which tells us that, in light of the smallness conditions (3)–(5),

$$\sup_{T-\tau \leqslant t \leqslant T+\tau} \bar{\mathcal{E}}_{\text{low}}(t)(1+t)^{2M-2} \leqslant C_I \max(C_{\text{ap}}\eta_0, \rho_r(\sqrt{C_{\text{ap}}\eta_0})),$$

where C_I is as in Proposition 6.11, and hence, in light of the smallness condition (6),

$$\sup_{T-\tau \leqslant t \leqslant T+\tau} \mathcal{E}_M^{(K)}(t) + \frac{\mathcal{F}_M(t)}{1+t} \leqslant \rho_K(\eta_0) \tag{6-28}$$

for ρ_K defined as in the smallness condition (7).

So finally, putting together (6-27) and (6-28) tells us that, in light of the smallness condition (7),

$$\sup_{T-\tau\leqslant t\leqslant T+\tau} \mathcal{E}_M(t)+\int_{T-\tau}^{T+\tau}\overline{\mathcal{D}}_M(s)\,ds\leqslant \tfrac{1}{2}\delta_{\mathrm{ap}}. \tag{6-29}$$

In light of the smallness condition (8) we may therefore combine (6-22) and (6-29) to see that

$$\sup_{0\leqslant t\leqslant T+\tau} \mathcal{E}_M(t)+\int_{0}^{T+\tau}\overline{\mathcal{D}}_M(s)\,ds\leqslant \delta_{\mathrm{ap}}. \tag{6-30}$$

To conclude we feed (6-30) into the a priori estimates of Theorem 4.34, which is legal in light of the smallness condition (9), and obtain

$$\sup_{0\leqslant t\leqslant T+\tau} \mathcal{E}_{\mathrm{low}}(t)(1+t)^{2M-2}+\mathcal{E}_M(t)+\frac{\mathcal{F}_M(t)}{1+t}+\int_0^{T+\tau}\mathcal{D}_M(s)\,ds\leqslant C_{\mathrm{ap}}\eta_0. \qquad \square$$

7. Global well-posedness and decay

In this section we put together the a priori estimates of Section 4, the local well-posedness of Section 5, and the continuation argument of Section 6 in order to obtain the main result of this paper, namely global well-posedness and decay about equilibrium. This is supplemented by a quantitative rigidity result which allows us to deduce decay of K.

In order to prove the main result, there are two auxiliary results that we need in addition to the results proved in Sections 4–6. The first one is the first part of Lemma 7.1, which accounts for the mismatch between the energies used for the local well-posedness and the a priori estimates. This result ensures that, close to time $t=0$, the local solution lives in the smallness regime to which the a priori estimates apply. The second one is Proposition 7.5 which allows us to control the initial energy, involving time derivatives, in terms of purely spatial norms. Note that this is reminiscent of Lemma 5.18 from the local well-posedness theory, which fulfilled a similar purpose for solutions of the approximate systems. In particular, the first part of Lemma 7.1 and Lemmas 7.2–7.4 only serve the purpose of leading up to Proposition 7.5.

We begin with Lemma 7.1 below. Note that Lemma 4.7 forms the crux of the argument in the first part of Lemma 7.1, as it does for similar estimates in Section 4. The difference here (by contrast with estimates recorded in Section 4) is that we do not make any smallness assumptions. Note that in the first part of Lemma 7.1 we only control (parts of) $\mathcal{E}_M^{(K)}$, whereas in the second part we control $\mathcal{F}_M$ as well.

Lemma 7.1 (auxiliary estimates for K). *Let $M\geqslant 2$ be an integer. There exists a constant $C_K>0$ such that if K solves (2-1d) then, for $Z=(u,\theta,K)$, we have the estimates*

$$\|\partial_t^2 K\|^2_{H^{2M-3}}+\sum_{j=3}^{M}\|\partial_t^j K\|^2_{H^{2M-2j+2}}\leqslant C_K\big(\|Z\|^2_{P^{2M}}+\|Z\|^{(2^{M+1})}_{P^{2M}}\big)$$

and

$$\|K\|^2_{H^{2M+1}}+\sum_{j=1}^{M}\|\partial_t^j K\|^2_{H^{2M-2j+2}}\leqslant C_K\big(\|Z\|^2_{P^{2M}}+\|K\|^2_{H^{2M+1}}+(\|Z\|^2_{P^{2M}}+\|K\|^2_{H^{2M+1}})^{(2^M)}\big).$$

Note that the summation on the left-hand side of the second inequality can be written more compactly as $\|K\|_{P^{2M+2}_{1,M}}$, which comes in handy in the sequel.

Proof. We begin with the first inequality. For any $j \geqslant 1$, Lemma 4.7 tells us that

$$\|\partial_t^j K\|_{H^{2M-2j+2}} \lesssim \|Z\|_{P^{2M}} + \|Z\|_{P^{2M}}^2 + \|\partial_t^{j-1} K\|_{H^{2M-2j+3}}^2. \tag{7-1}$$

We immediately deduce that, since $2M-3 = (2M-2\cdot 2+2)-1$,

$$\|\partial_t^2 K\|_{H^{2M-3}} \lesssim \|Z\|_{P^{2M-1}} + \|Z\|_{P^{2M-1}}^2 + \|\partial_t K\|_{H^{2M-2}}^2 \lesssim \|Z\|_{P^{2M}} + \|Z\|_{P^{2M}}^4$$

and

$$\|\partial_t^3 K\|_{H^{2M-4}} \lesssim \|Z\|_{P^{2M}} + \|Z\|_{P^{2M}}^2 + \|\partial_t^2 K\|_{H^{2M-3}}^2 \lesssim \|Z\|_{P^{2M}} + \|Z\|_{P^{2M}}^8.$$

To conclude we proceed by induction. Suppose that, for some $3 \leqslant j \leqslant M-1$,

$$\|\partial_t^j K\|_{H^{2M-2j+2}} \lesssim \|Z\|_{P^{2M}} + \|Z\|_{P^{2M}}^{(2^j)}.$$

Then, by (7-1),

$$\|\partial_t^{j+1} K\|_{H^{2M-2j}} \lesssim \|Z\|_{P^{2M}} + \|Z\|_{P^{2M}}^2 + \|\partial_t^j K\|_{H^{2M-2j+1}}^2 \lesssim \|Z\|_{P^{2M}} + \|Z\|_{P^{2M}}^{(2^{j+1})},$$

and so the claim follows by induction.

We now prove the second inequality. We observe that, since H^s is a Banach algebra when $s > \frac{3}{2}$, we may immediately deduce from (2-1d) that

$$\|\partial_t^j K\|_{H^{2M-2j+2}} \lesssim \|Z\|_{P_{j-1}^{2M}} + \|K\|_{P_{j-1}^{2M+1}} + (\|Z\|_{P_{j-1}^{2M}} + \|K\|_{P_{j-1}^{2M+1}})^2 \quad \text{for every } 1 \leqslant j \leqslant M.$$

For simplicity, we will write $p_a^b(x) := x^a + x^b$. The inequality above may thus be written as

$$\|\partial_t^j K\|_{H^{2M-2j+2}} \lesssim p_1^2(\|Z\|_{P_{j-1}^{2M}} + \|K\|_{P_{j-1}^{2M+1}}) \quad \text{for every } 1 \leqslant j \leqslant M. \tag{7-2}$$

This notation is particularly useful due to its behavior under composition. Indeed, we see immediately that $p_a^b \circ p_c^d \lesssim p_{ac}^{bd}$. The result now follows from iterating (7-2), and we may induct on $1 \leqslant j \leqslant M$ to show that

$$\|K\|_{H^{2M+1}} + \|K\|_{P_{1,j}^{2M+2}} \lesssim p_1^{2^j}(\|Z\|_{P^{2M}} + \|K\|_{H^{2M+1}}),$$

which proves the claim. □

We now turn our attention towards the control of the initial energy in terms of purely spatial norms. In order to do so, we first record H^k bounds on the inverse of $J_{\text{eq}} + K$ reminiscent of Lemma 5.5. However, such bounds are easier to obtain here since we do not have to deal with any projections, as was the case in Lemma 5.5. Note that in the lemma below we consider $K : \mathbb{T}^3 \to \text{Sym}(3)$ (i.e., there is no dependence in time). When K satisfies (2-1) (and is hence time-dependent), this means that the lemma below applies pointwise in time.

Lemma 7.2 (H^k bounds on $(J_{\text{eq}} + K)^{-1}$). *Suppose that* $J_{\text{eq}} = \text{diag}(\lambda, \lambda, \nu)$ *for* $\nu > \lambda > 0$ *and that* $K : \mathbb{T}^3 \to \text{Sym}(3)$ *satisfies* $\|K\|_{L^\infty} < \frac{1}{2}\lambda$. *Then* $J_{\text{eq}} + K$ *is pointwise invertible and, for every integer* $k \geqslant 1$,

(1) $\|(J_{\text{eq}} + K)^{-1}\|_{\mathcal{L}(L^2;L^2)} \leqslant 2/\lambda$ *and*

(2) $\|(J_{\text{eq}} + K)^{-1}\|_{\mathcal{L}(H^k;H^k)} \lesssim \|K\|_{H^{k+2}} + \|K\|_{H^{k+2}}^k$.

Proof. The strategy here is the same as that used when studying the inverse of $J_{\mathrm{eq}} + P_n \circ K$ in Lemmas 5.1, 5.4, and 5.5 of Section 5A when developing the local well-posedness theory.

First we show that $J_{\mathrm{eq}} + K$ is pointwise invertible. This follows from the fact that the quadratic form that it generates is pointwise positive-definite and indeed, for every $x \in \mathbb{T}^3$ and every $w \in \mathbb{R}^3$,

$$(J_{\mathrm{eq}} + K)(x)w \cdot w = J_{\mathrm{eq}} w \cdot w + K(x)w \cdot w > \lambda |w|^2 - \|K\|_{L^\infty} |w|^2 > \tfrac{1}{2}\lambda |w|^2. \tag{7-3}$$

Moreover, we may immediately deduce from (7-3) that $\|(J_{\mathrm{eq}} + K)^{-1}\|_{L^\infty} \leqslant 2/\lambda$ from which item (1) follows.

Now we establish formulae for derivatives of $(J_{\mathrm{eq}} + K)^{-1}$ reminiscent of the formulae of Lemma 5.4. Note that $\partial_i (J_{\mathrm{eq}} + K)^{-1} = -(J_{\mathrm{eq}} + K)^{-1} (\partial_i K)(J_{\mathrm{eq}} + K)^{-1}$, and hence for any multi-index $\alpha \in \mathbb{N}^3$,

$$\partial^\alpha (J_{\mathrm{eq}} + K) = \sum_{l=1}^{|\alpha|} (-1)^l \sum_{\alpha_1 + \cdots + \alpha_l = \alpha} \mathbb{M}(\alpha_1, \ldots, \alpha_l)(K),$$

where

$$\mathbb{M}(\alpha_1, \alpha_2, \ldots, \alpha_l)(K) := (J_{\mathrm{eq}} + K)^{-1} (\partial^{\alpha_1} K)(J_{\mathrm{eq}} + K)^{-1} (\partial^{\alpha_2} K) \cdots (\partial^{\alpha_l} K)(J_{\mathrm{eq}} + K)^{-1}.$$

The crux of the argument now lies in obtaining L^2-to-L^2 bounds on the operators $\mathbb{M}$. In light of the L^∞ bound on $(J_{\mathrm{eq}} + K)^{-1}$, we may proceed as in Lemma 5.5 and, for any $v \in L^2$ and for $k := \max |\alpha_i|$, estimate $\|\mathbb{M}(\alpha_1, \ldots, \alpha_l)(K)\|_{\mathcal{L}(L^2; L^2)} \lesssim \|K\|^l_{H^{k+2}}$.

We may now conclude the proof and obtain item (2) by proceeding once again as in Lemma 5.5. For any $k \geqslant 2$ and any $v \in H^k$, the L^2-to-L^2 bounds on $\mathbb{M}$ tells us that

$$\|(J_{\mathrm{eq}} + K)^{-1} v\|_{H^k} \lesssim (\|K\|_{H^{k+2}} + \|K\|^k_{H^{k+2}}) \|v\|_{H^k},$$

from which item (2) follows. □

We continue our progress towards Proposition 7.5 and record elementary estimates on the nonlinearities of the problem.

Lemma 7.3 (auxiliary estimates of the nonlinearity). *Let* $(\star)$ *denote any of the nonlinear terms in* (2-1c), (2-1d), *or* (2-2), *except* $K \partial_t \theta$. *Writing* $Z = (u, \theta, K)$, *we have that, for every* $j, k \in \mathbb{N}$ *with* $j \geqslant 1$,

$$(1)\ \|[K\partial_t, \partial_t^{j-1}]\theta\|_{H^k} \lesssim \|Z\|^2_{\mathcal{P}^{k+2j}_{j-1}} \quad \textit{and} \quad (2)\ \|\partial_t^{j-1}(\star)\|_{H^k} \lesssim \|Z\|_{\mathcal{P}^{k+2j}_{j-1}} + \|Z\|^3_{\mathcal{P}^{k+2j}_{j-1}}.$$

Proof. These estimates rely on the fact that H^s is a Banach algebra when $s > \frac{3}{2}$ and on the product estimates of Lemma B.4. We omit the details — see Lemma 5.17 for very similar estimates. □

The last result we need in order to prove Proposition 7.5 is reminiscent of Lemma 5.18. In Lemma 7.4 below we show that the parabolic norm of Z can be controlled by a purely spatial norm.

Lemma 7.4 (bounds on the parabolic norm by purely spatial norms). *Let* $M \geqslant 1$ *be an integer. There exists a constant* $C_M > 0$ *such that, for any time horizon* $T > 0$, *the following holds*: *if* $Z = (u, \theta, K)$ *solves* (2-1b)–(2-1d) *and* (2-2) *on* $[0, T]$ *and satisfies* $\sup_{0 \leqslant t \leqslant T} \|K(t)\|_{L^\infty} \leqslant \frac{1}{2}\lambda$, *then, for every* $0 \leqslant t \leqslant T$, *we have that* $\|Z(t)\|_{\mathcal{P}^{2M}} \lesssim \|Z(t)\|_{H^{2M}} + \|Z(t)\|^{C_M}_{H^{2M}}$. *In particular, this holds when* $t = 0$.

Proof. We proceed as in Lemma 5.18. We apply ∂_t^{j-1} to (2-1b)–(2-1d) and (2-2), invert $J_{\mathrm{eq}} + K$ (which is allowed as per Lemma 7.2), and deduce the following. On one hand, for $1 \leqslant j \leqslant M-1$, we may use Lemmas 7.2 and 7.3 to obtain that, using the notation $p_a^b(x) := x^a + x^b$ for $a < b$ and $x \geqslant 0$,

$$\begin{aligned}\|\partial_t^j Z\|_{H^{2M-2j}} &\lesssim (\|K\|_{H^{2M-2j+2}} + \|K\|_{H^{2M-2j+2}}^{2M-2j})(\|Z\|_{P_{j-1}^{2M}} + \|Z\|_{P_{j-1}^{2M}}^3) \\ &= p_1^{2M-2j}(\|K\|_{H^{2M}}) p_1^3(\|Z\|_{P_{j-1}^{2M}}) \lesssim p_1^{2M-2j+3}(\|Z\|_{P_{j-1}^{2M}}).\end{aligned} \tag{7-4}$$

On the other hand, for $j = M$, using Lemmas 7.2 and 7.3 tells us that

$$\|\partial_t^M Z\|_{L^2} \lesssim \|Z\|_{P_{M-1}^{2M}} + \|Z\|_{P_{M-1}^{2M}} = p_1^3(\|Z\|_{P_{j-1}^{2M}}). \tag{7-5}$$

Combining (7-4) and (7-5) and unpacking the definition of $\|\cdot\|_{P_j^k}$ we see that, for every $1 \leqslant j \leqslant M$,

$$\|Z\|_{P_j^{2M}} \asymp \|Z\|_{P_{j-1}^{2M}} + \|\partial_t^j Z\|_{H^{2M-2j}} \lesssim p_1^{2M-2j+3}(\|Z\|_{P_{j-1}^{2M}}). \tag{7-6}$$

Iterating this inequality yields

$$\|Z\|_{P^{2M}} = \|Z\|_{P_M^{2M}} \lesssim (p_1^3 \circ p_1^5 \circ \cdots \circ p_1^{2M-1})(\|Z\|_{P_0^{2M}}), \tag{7-7}$$

from which, since $\|Z\|_{P_0^{2M}} = \|Z\|_{H^{2M}}$, the claim follows. In particular, note that, as in Lemma 5.18, $C_M = \prod_{j=1}^M (2M - 2j + 3)$. □

We may now prove the second auxiliary result of this section required to prove Theorem 7.6 below. In Proposition 7.5 we prove that the initial energy may be controlled in terms of purely spatial norms. Recall that $\mathcal{E}_M$ and $\mathcal{F}_M$ are defined in (3-5).

Proposition 7.5 (control of the full energy by purely spatial norms). *Let $M \geqslant 1$ be an integer. There exist $C_s, C_M > 0$ such that, for any time horizon $T > 0$, the following holds: if $Z = (u, \theta, K)$ solves (2-1b)–(2-1d) and (2-2) on $[0, T]$ and satisfies $\sup_{0 \leqslant t \leqslant T} \|K(t)\|_{L^\infty} \leqslant \frac{1}{2}\lambda$, then, for every $0 \leqslant t \leqslant T$,*

$$\mathcal{E}_M + \mathcal{F}_M \leqslant C_s(\|Z\|_{H^{2M}}^2 + \|Z\|_{H^{2M}}^{2^{M+1}C_M} + \|K\|_{H^{2M+1}}^2 + \|K\|_{H^{2M+1}}^{2^{M+1}}).$$

In particular, this holds when $t = 0$.

Proof. We proceed in two steps. First we use Lemma 7.1 to show that $\mathcal{E}_M + \mathcal{F}_M$ can be controlled by $\|Z\|_{P^{2M}}$ and $\|K\|_{H^{2M+1}}$, then we use Lemma 7.4 to show that $\|Z\|_{P^{2M}}$ can be controlled by $\|Z\|_{H^{2M}}$.

Before we begin the proof in earnest, note that we may write

$$\mathcal{E}_M + \mathcal{F}_M \asymp \|Z\|_{P^{2M}}^2 + \|K\|_{H^{2M+1}}^2 + \|K\|_{P_{1,M}^{2M+2}}^2.$$

It then follows immediately from Lemma 7.1 that

$$\mathcal{E}_M + \mathcal{F}_M \lesssim \|Z\|_{P^{2M}}^2 + \|Z\|_{P^{2M}}^{2^{M+1}} + \|K\|_{H^{2M+1}}^2 + \|K\|_{H^{2M+1}}^{2^{M+1}}.$$

We may combine this inequality with Lemma 7.4 to conclude that indeed

$$\mathcal{E}_M + \mathcal{F}_M \lesssim \|Z\|_{H^{2M}}^2 + \|Z\|_{H^{2M}}^{2^{M+1}C_M} + \|K\|_{H^{2M+1}}^2 + \|K\|_{H^{2M+1}}^{2^{M+1}}$$

for $C_M > 0$ as in Lemma 7.4. □

We may now prove the main result of this paper. In order to do so, recall first that $\mathcal{E}_{\text{low}}$, $\mathcal{E}_M$ and $\mathcal{F}_M$, and $\mathcal{D}_M$ are defined in (3-3), (3-5), and (3-8), respectively.

Theorem 7.6 (global well-posedness and decay). *Let $M \geqslant 4$ be an integer and recall that the global assumptions of Definition 1.1 hold. There exist universal constants $\eta, C > 0$ depending only on M such that the following holds: for any $Z_0 = (u_0, \theta_0, K_0) \in L^2(\mathbb{T}^3; \mathbb{R}^3 \times \mathbb{R}^3 \times \operatorname{Sym}(3))$ satisfying*

$$\nabla \cdot u_0 = 0, \quad \fint_{\mathbb{T}^3} u_0 = 0, \quad \text{and} \quad \|Z_0\|^2_{H^{2M}} + \|K\|^2_{H^{2M+1}} < \eta,$$

there exists a unique strong solution (Z, p) of (2-1), where

$$Z = (u, \theta, K) \in C^2([0, \infty) \times \mathbb{T}^3; \mathbb{R}^3 \times \mathbb{R}^3 \times \operatorname{Sym}(3))$$

and

$$p \in C^2([0, \infty) \times \mathbb{T}^3; \mathbb{R}).$$

Moreover, the solution satisfies the estimate

$$\sup_{t \geqslant 0} \mathcal{E}_{\text{low}}(t)(1+t)^{2M-1} + \mathcal{E}_M(t) + \frac{\mathcal{F}_M}{1+t} + \int_0^\infty \mathcal{D}_M(s)\, ds \leqslant C(\|Z_0\|^2_{H^{2M}} + \|K_0\|^2_{H^{2M+1}}). \tag{7-8}$$

Proof. The strategy of the proof is as follows. Coupling the local well-posedness theory of Theorem 5.24 to the auxiliary estimate for K of Lemma 7.1, which allows us to account for the mismatch between the energies used for the local well-posedness and the energies used for the a priori estimates, we produce a solution locally in time on which we have enough control to invoke the a priori estimates of Theorem 4.34. This tells us that this (possibly very short-lived) solution lives in the small energy regime defined by (7-8). Continuing this solution globally in time then follows immediately from leveraging the continuation argument of Theorem 6.13.

We begin by defining the smallness parameter $\eta > 0$. We pick $0 < \eta \leqslant 1$ satisfying

(1) $\eta^{1/2} < \sigma(\delta_{\text{ap}}^{\text{loc}})$ for $\delta_{\text{ap}}^{\text{loc}}$ as in Lemma 6.5 and $\sigma = \sigma(\delta)$ as in the beginning of Section 5,

(2) $(2 + C_K) p_1^{2M}((\rho_e + \rho_d)(\eta)) < \delta_{\text{ap}}$ for δ_{ap} as in Theorem 4.34, C_K as in Lemma 7.1, ρ_e and ρ_d as in Theorem 5.24, and $p_1^{2M}(x) := x + x^{2M}$ for all $x \geqslant 0$,

(3) $C_s \eta \leqslant \eta_{\text{ap}}$ for C_s as in Proposition 7.5 and η_{ap} as in Theorem 4.34, and

(4) $C_s \eta \leqslant \eta_{\text{cont}}$ for η_{cont} as in Theorem 6.13.

We may now construct a solution locally in time which lives in the small energy regime, as defined by (7-8). In light of the smallness condition (1), which tells us that $\|K_0\|_{H^3} < \eta^{1/2} < \sigma(\delta_{\text{ap}}^{\text{loc}})$, the local well-posedness theory of Theorem 5.24 shows that there exists $T_{\text{lwp}} > 0$ and a strong solution

$$Z = (u, \theta, K) \in C^2([0, T_{\text{lwp}}] \times \mathbb{T}^3; \mathbb{R}^3 \times \mathbb{R}^3 \times \operatorname{Sym}(3))$$

and

$$p \in C^2([0, T_{\text{lwp}}] \times \mathbb{T}^3; \mathbb{R})$$

which satisfies

$$\sup_{0 \leqslant t \leqslant T_{\text{lwp}}} \|K(t)\|_{H^3} < \eta^{1/2} \tag{7-9}$$

and

$$\sup_{0 \leqslant t \leqslant T_{\mathrm{lwp}}} \|Z(t)\|_{P^{2M}}^2 + \int_0^{T_{\mathrm{lwp}}} \overline{\mathcal{D}}_M(s)\,ds \leqslant (\rho_e + \rho_d)(\|Z_0\|_{H^{2M}}), \tag{7-10}$$

where recall that $\overline{\mathcal{D}}_M$ is defined in (3-8). In particular, the auxiliary estimate for K of Lemma 7.1 tells us that, in light of (7-10) and recalling that $\mathcal{E}_M^{(K)}$ is defined in (3-4),

$$\begin{aligned}\sup_{0 \leqslant t \leqslant T_{\mathrm{lwp}}} \|Z(t)\|_{P^{2M}}^2 + \mathcal{E}_M^{(K)}(t) &\leqslant (1 + C_K) p_1^{2M}\Big(\sup_{0 \leqslant t \leqslant T_{\mathrm{lwp}}} \|Z(t)\|_{P^{2M}}\Big) \\ &\leqslant (1 + C_K) p_1^{2M}((\rho_e + \rho_d)(\|Z_0\|_{H^{2M}})). \end{aligned} \tag{7-11}$$

Putting (7-10) and (7-11) together tells us that, by virtue of the smallness condition (2),

$$\sup_{0 \leqslant t \leqslant T_{\mathrm{lwp}}} \mathcal{E}_M(t) + \int_0^{T_{\mathrm{lwp}}} \overline{\mathcal{D}}_M(s)\,ds \leqslant (2 + C_K) p_1^{2M}((\rho_e + \rho_d)(\|Z_0\|_{H^{2M}})) \leqslant \delta_{\mathrm{ap}}. \tag{7-12}$$

Note also that we may deduce from the smallness condition (3) and Proposition 7.5 that, since $\eta \leqslant 1$,

$$(\mathcal{E}_M + \mathcal{F}_M)(0) \leqslant C_s(\|Z_0\|_{H^{2M}}^2 + \|K_0\|_{H^{2M+1}}^2) \leqslant \eta_{\mathrm{ap}}. \tag{7-13}$$

Combining (7-12) and (7-13) allows us to use the a priori estimate of Theorem 4.34, from which we deduce that

$$\sup_{0 \leqslant t \leqslant T_{\mathrm{lwp}}} \mathcal{E}_{\mathrm{low}}(t)(1+t)^{2M-2} + \mathcal{E}_M(t) + \frac{\mathcal{F}_M(t)}{1+t} + \int_0^{T_{\mathrm{lwp}}} \mathcal{D}_M(s)\,ds \leqslant C_{\mathrm{ap}}(\mathcal{E}_M + \mathcal{F}_M)(0). \tag{7-14}$$

To conclude we employ a standard continuation argument revolving around the continuation argument of Theorem 6.13. Let us define, for any $T \in (0, \infty]$,

$$\mathcal{G}(T) := \sup_{0 \leqslant t \leqslant T} \mathcal{E}_{\mathrm{low}}(t)(1+t)^{2M-2} + \mathcal{E}_M(t) + \frac{\mathcal{F}_M(t)}{1+t} + \int_0^T \mathcal{D}_M(s)\,ds$$

which we use to define the maximal time of existence

$$T_{\max} := \sup\{T > 0 : \exists!\text{ strong solution on } [0, T] \text{ and } \mathcal{G}(T) \leqslant C_{\mathrm{ap}}(\mathcal{E}_M + \mathcal{F}_M)(0)\}.$$

By virtue of (7-14), we know that $T_{\max} > T_{\mathrm{lwp}} > 0$. Crucially, Theorem 6.13 tells us, in light of the smallness condition (4) and (7-13) and since $T_{\max} > 0$, that $T_{\max}$ cannot be finite. So indeed the solution exists globally in time and, since $T_{\max} = \infty$, we appeal to Proposition 7.5 one last time to deduce that

$$\mathcal{G}(\infty) \leqslant C_{\mathrm{ap}}(\mathcal{E}_M + \mathcal{F}_M)(0) \lesssim \|Z_0\|_{H^{2M}}^2 + \|K_0\|_{H^{2M+1}}^2,$$

i.e., indeed (7-8) holds. □

In order to deduce the decay of K from Theorem 7.6 above we need the quantitative rigidity estimate of Proposition 7.7 below. Note that the term *quantitative rigidity* is motivated by contrast with the following *qualitative rigidity* result: if $a = 0$ and $\|J - J_{\mathrm{eq}}\|_{L^\infty} < \nu - \lambda$ then $J = J_{\mathrm{eq}}$ (this can be seen by noticing that if $a = 0$ then J_{33} must be an eigenvalue of J, and it cannot be that $J_{33} = \lambda$ since that would contradict the condition $\|J - J_{\mathrm{eq}}\|_{L^\infty} < \nu - \lambda$).

Proposition 7.7 (quantitative rigidity). *Let $T > 0$ be a time horizon and suppose that*

$$\sup_{0\leqslant t\leqslant T} \|(u,\theta)(t)\|_{H^3} + \|J(t)\|_{H^3} + \|\partial_t(u,\theta)\|_{H^2}^2 + \|\partial_t J\|_{H^2}^2 < \infty. \tag{7-15}$$

If $\sup_{0\leqslant t\leqslant T}\|K\|_{L^\infty} \leqslant \nu-\lambda$ *then* $\sup_{0\leqslant t\leqslant T}\|K\|_{L^p} \leqslant 2\sup_{0\leqslant t\leqslant T}\|a\|_{L^p}$ *for any* $1\leqslant p\leqslant\infty$.

Proof. Since (7-15) holds we know from Proposition A.3 that $J(t,x)$ is pointwise symmetric with spectrum $\{\lambda,\lambda,\nu\}$. The key observation now is that we may therefore find a unit vector field $n(t,x)$ such that $J=\nu n\otimes n+\lambda(I-n\otimes n)$ pointwise (indeed we may simply take n to be the unit eigenvector of J corresponding to the eigenvalue ν). Writing $J_{\mathrm{eq}}=\nu e_3\otimes e_3+\lambda(I-e_3\otimes e_3)$, we may then compute

$$|J-J_{\mathrm{eq}}|^2=(\nu-\lambda)^2|n\otimes n-e_3\otimes e_3|^2=2(\nu-\lambda)^2(1-n_3^2)=2(\nu-\lambda)^2|\bar{n}|^2.$$

In particular, if $\|K\|_{L^\infty}\leqslant\nu-\lambda$ then we may deduce that $n_3^2\geqslant\frac{1}{2}$ pointwise. To conclude we note that since $J_{ij}=Je_j\cdot e_i$ we may compute that $a=(\nu-\lambda)n_3\bar{n}$. So finally

$$|K|^2=\frac{2|a|^2}{n_3^2}\leqslant 4|a|^2,$$

from which the claim follows. □

In light of this quantitative rigidity result we may deduce the decay of K, and hence $\partial_t K$, from the decay of a. As discussed in Section 1, this argument could be iterated further in order to derive the decay of higher-order temporal derivatives of K, but this is not done here since that decay is not used in the scheme of a priori estimates. Recall that $\overline{\mathcal{K}}_I$ is defined in (3-6).

Proposition 7.8 (rates of decay of K and $\partial_t K$). *Let $M\geqslant 3$ be an integer and let $T>0$ be a time horizon. There exists $C_1>0$ such that the following holds: if (u,θ,K) solves (2-1d) and satisfies*

$$C:=\sup_{0\leqslant t\leqslant T}\|K(t)\|_{L^2}^2(1+t)^{2M-4}+\overline{\mathcal{K}}_2(1+t)^{2M-2}+\frac{\mathcal{F}_M(t)}{1+t}<\infty, \tag{7-16}$$

then, for $j=0,1$,

$$\sup_{0\leqslant s\leqslant s_j}\sup_{0\leqslant t\leqslant T}\|\partial_t^j K\|_{H^s}^2(1+t)^{2M-4-(2M-3)/s_j}\leqslant C_1 C,$$

where $s_0=2M+1$ and $s_1=2M$.

Proof. We interpolate between the decay of $\|K\|_{L^2}^2$ and the growth of $\mathcal{F}_M$ in (7-16) to deduce the bounds on $\|K\|_{H^s}^2$ recorded here. To obtain the bounds on $\|\partial_t K\|_{H^s}^2$ we first use (2-1d) to read off the L^2 bound on $\partial_t K$ using Hölder's inequality and (7-16), and then interpolate between this L^2 bound and $\mathcal{F}_M$. □

Remark 7.9. Note that the estimates recorded above are not all decay estimates. To be precise, $\|K(t)\|_{H^s}$ decays when $s<2M-4/(2M-3)$, whereas $\|\partial_t K\|_{H^s}$ decays when $s<2M-1-3/(2M-3)$. In particular these regularity cut-offs approach $2M$ and $2M-1$, respectively, asymptotically from below as $M\to+\infty$.

We conclude this section by proving Corollary 1.3, which records the precise decay rates of the unknowns and their temporal derivatives.

Proof of Corollary 1.3. It suffices to combine Theorem 7.6 and Propositions 4.30, 7.7, and 7.8. □

Appendix A: Identities involving the microinertia

In this section we record various computations and identities involving the microinertia tensor J which are used throughout the paper.

We now record two lemmas that are used in the proof of Proposition A.3 below. This proposition is essential to our scheme of a priori estimates and is a fundamental feature of the micropolar fluid model. It shows that if the solution is regular enough, then the spectrum of the microinertia is propagated by the flow. First we record a well-known result showing that the advective derivative is simply a time derivative up to a change of variables using the flow map (i.e., with respect to Eulerian coordinates).

Lemma A.1 (calculus of advective derivatives). *Let $\eta \in C^2([0, T) \times \mathbb{R}^n; \mathbb{R}^n)$ be a flow map, i.e., for all $0 \leqslant t < T$, we have the C^1-diffeomorphism $\eta_t := \eta(t, \cdot)$ with velocity $u \in C^1([0, T) \times \mathbb{R}^n, \mathbb{R}^n)$ defined by $u(t, x) := \partial_t \eta(t, \eta_t^{-1}(x))$. Then $\partial_t(\det \nabla\eta) = ((\nabla \cdot u) \circ \eta) \det \nabla\eta$ and, for every $f \in C^1([0, T) \times \mathbb{R}^n; \mathbb{R})$, we have $\partial_t(f \circ \eta) = ((\partial_t + u \cdot \nabla) f) \circ \eta$, where, for any $g : [0, T) \times \mathbb{R}^n \to \mathbb{R}$, we write $g \circ \eta$ to denote the composition $(g \circ \eta)(t, x) := g(t, \eta(t, x))$.*

Proof. The first identity is the well-known Liouville theorem and the second identity follows from the first by the chain rule. □

We continue our progress towards a proof of Proposition A.3 below with an ODE result recorded in Lemma A.2. This lemma provides an equivalent characterization of the ODE satisfied by the microinertia (denoted by S in Lemma A.2) in Lagrangian coordinates in terms of the ODE satisfied by its rotation matrix (denoted by Q in Lemma A.2).

Lemma A.2 (two-sided integrating factors for ODEs with commutators). *Let $S, A \in C^1([0, T); \mathbb{R}^{n\times n})$ be time-dependent symmetric and antisymmetric matrices, respectively, and let S_0 be a fixed symmetric real $n \times n$ matrix. The following are equivalent:*

(1) *S solves the initial value problem $\partial_t S = [A, S]$ on $(0, T)$ and $S(0) = S_0$.*

(2) *There exists a time-dependent orthogonal matrix $Q \in C^1([0, T); O(n))$ such that $S = QS_0Q^T$ and Q solves the initial value problem $\partial_t Q = AQ$ on $(0, T)$ and $Q(0) = I$.*

Here $O(n)$ denotes the space of $n \times n$ real orthogonal matrices.

Proof. First we show that (2) $\Longrightarrow$ (1). If (2) holds then $\partial_t Q^T = (AQ)^T = -Q^T A$ and therefore,

$$\partial_t S = \partial_t QS_0Q^T + QS_0\partial_t Q^T = AQS_0Q^T - QS_0\partial_t Q^T A = [A, S].$$

Now we show that (1) $\Longrightarrow$ (2). Suppose that (1) holds and let us define $Q(t) := \exp\left(\int_0^t A(s)\, ds\right)$ such that Q solves the initial value problem of (2). Since (1) is a linear ODE it has a unique solution, so in order to show that $S = QS_0S^T$ it suffices to show that QS_0S^T is a solution of the initial value problem of (1). This follows immediately from the same computation as that which was carried out above in order to show that (2) $\Longrightarrow$ (1). □

We are now ready to prove Proposition A.3 which shows that if the velocity fields and the microinertia are sufficiently regular then the spectrum of the microinertia is propagated in time.

Proposition A.3 (persistence of the spectrum for solutions of advection-rotation equations). *Suppose that $u \in C^1([0, T) \times \mathbb{R}^n; \mathbb{R}^n)$ is divergence-free and consider Ω, $J \in C^1([0, T) \times \mathbb{R}^n; \mathbb{R}^{n\times n})$, where Ω is antisymmetric. If they satisfy*

$$\partial_t J + u \cdot \nabla J = [\Omega, J] \quad \text{and} \quad J(0, \cdot) = J_0$$

for some real $n \times n$ matrix J_0 then there exists a flow map $\eta \in C^2([0, T) \times \mathbb{R}^n; \mathbb{R}^n)$, where $\eta_t := \eta(t, \cdot)$ is a C^1-diffeomorphism for all $0 \leqslant t < T$, and there exists an Eulerian rotation map $R \in C^1([0, T) \times \mathbb{R}^n; O(n))$ such that

$$J = R(J_0 \circ \eta^{-1})R^T,$$

or, more precisely, $J(t, x) = R(t, x)J(t, \eta_t^{-1}(x))R^T(t, x)$. In particular, for every $(t, x) \in [0, T) \times \mathbb{R}^n$, if we write $y = \eta_t^{-1}(x)$ then $J_0(y)$ and $J(t, x)$ have the same spectrum.

Proof. The key ideas are that (1) by virtue of Lemma A.1, $\partial_t + u \cdot \nabla$ is nothing more than a time derivative up to a change of coordinates and (2) in light of Lemma A.2, solutions of $\partial_t = [\Omega, \cdot\,]$ are pointwise conjugate to their initial conditions by some rotation matrix with angular velocity Ω.

Step 1: We define the flow map η to be the solution of $\partial_t \eta = u \circ \eta$ with initial condition $\eta(t = 0) = \mathrm{id}$. As a consequence of u being divergence-free, it follows from Lemma A.1 that $\partial_t(\det \nabla\eta) = 0$, and hence $\det \nabla\eta = \det \nabla\eta(t = 0) \equiv 1$, so indeed η_t is invertible at all times t. Finally we deduce that η_t is a C^1-diffeomorphism for all times t from the fact that $\nabla(\eta^{-1}) = (\nabla\eta)^{-1} \circ \eta^{-1}$.

Step 2: Let us define $\mathcal{J}$ and Θ to be the Lagrangian counterparts of J and Ω, respectively, i.e., $\mathcal{J} := J \circ \eta$ and $\Theta := \Omega \circ \eta$. Then, by Lemma A.1,

$$\partial_t \mathcal{J} = \partial_t(J \circ \eta) = ((\partial_t + u \cdot \nabla)J) \circ \eta = [\Theta, \mathcal{J}] \quad \text{and} \quad \mathcal{J}(0, \cdot) = J_0 \circ \eta_0 = J_0.$$

So $\mathcal{J}$ solves $\partial_t \mathcal{J} = [\Theta, \mathcal{J}]$ with initial condition $\mathcal{J}(0, \cdot) = J_0$.

Step 3: We define the Lagrangian rotation map $Q(t, y) := \exp\left(\int_0^t \Theta(s, y)\, ds\right)$ such that, by Lemma A.2, $\mathcal{J} = QJ_0Q^T$. So finally, if we introduce the Eulerian rotation map $R := Q \circ \eta^{-1}$ we may conclude that $J = R(J_0 \circ \eta^{-1})R^T$. □

We now record some elementary identities which are useful throughout the paper. The first identity allows us to deal with the precession term appearing in the conservation of angular momentum when deriving energy-dissipation relations.

Lemma A.4. *Let A and S be $n \times n$ matrices which are antisymmetric and symmetric, respectively. Then $\frac{1}{2}[A, S] = \mathrm{Sym}(AS)$. In particular, if $n = 3$ and we let $a := \mathrm{vec}\, A$ then $\frac{1}{2}[A, S] = \mathrm{Sym}(a \times S)$.*

Proof. This is immediate: $\mathrm{Sym}(AS) = \frac{1}{2}(AS + S^TA^T) = \frac{1}{2}(AS - SA) = \frac{1}{2}[A, S]$. □

The second identity shows that one of the terms appearing in the conservation of microinertia (1-1d) is antisymmetric (as a map on the space of symmetric matrices), and hence does not contribute to energy or transport estimates.

Lemma A.5. *Let S and M be real $n \times n$ matrices such that S is symmetric. Then $[M, S] : S = 0$.*

Proof. The proof follows from the observation that $SM : S = M : S^TS = M : SS^T = MS : S$. □

Finally we record a detailed computation of the block form of $[\Omega, J]$, which comes in handy when reading off the equation governing the dynamics of a.

Lemma A.6 (block form of $[\Omega, J]$). *Let J be a symmetric 3×3 matrix written in $(2+1) \times (2+1)$ block form as*

$$J = \begin{pmatrix} \bar{J} & a \\ a^T & J_{33} \end{pmatrix},$$

and let $\Omega = \operatorname{ten} \omega$ for some $\omega \in \mathbb{R}^3$. Then we may write the commutator $[\Omega, J]$ in $(2+1) \times (2+1)$ block form as

$$[\Omega, J] = \begin{pmatrix} \omega_3 [R, \bar{J}] - (\bar{\omega}^\perp \otimes a + a \otimes \bar{\omega}^\perp) & (\bar{J} - J_{33} I_2)\bar{\omega}^\perp + \omega_3 a^\perp \\ ((\bar{J} - J_{33} I_2)\bar{\omega}^\perp + \omega_3 a^\perp)^T & 2a \cdot \bar{\omega}^\perp \end{pmatrix}, \tag{A-1}$$

where $R = e_2 \otimes e_1 - e_1 \otimes e_2 \in \mathbb{R}^{2\times 2}$ denotes the (counterclockwise) $\frac{\pi}{2}$ rotation in $\mathbb{R}^2$.

Proof. Note that we may write Ω in block form using the rotation matrix R as

$$\Omega = \begin{pmatrix} \omega_3 R & -\bar{\omega}^\perp \\ (\bar{\omega}^\perp)^\perp & 0 \end{pmatrix}.$$

We may then compute

$$\Omega J = \begin{pmatrix} \omega_3 R \bar{J} - \bar{\omega}^\perp \otimes a & \omega_3 a^\perp - J_{33} \bar{\omega}^\perp \\ (\bar{\omega}^\perp)^T \bar{J} & \bar{\omega}^\perp \cdot a \end{pmatrix}.$$

Since $J\Omega = -(\Omega J)^T$ we deduce that indeed (A-1) holds. □

Appendix B: Analytical results

In this section we record precise statements of well-known analytical results for the reader's convenience. First we record the Gagliardo–Nirenberg interpolation inequalities on bounded domains, which is crucial in several nonlinear estimates.

Theorem B.1 (Gagliardo–Nirenberg interpolation inequalities). *Let $u \in L^q(\mathbb{T}^n)$ with $\nabla^k u \in L^r(\mathbb{T}^n)$ such that*

$$\frac{1}{p} - \frac{l}{n} = \theta \frac{1}{q} + (1-\theta)\left(\frac{1}{r} - \frac{k}{n}\right) \quad \text{and} \quad (1-\theta)k \geqslant l \qquad \text{for some } 0 \leqslant \theta \leqslant 1.$$

Then $\nabla^l u \in L^p(\mathbb{T}^n)$ and we have the estimate $\|\nabla^l u\|_{L^p(\mathbb{T}^n)} \lesssim \|u\|^\theta_{L^q(\mathbb{T}^n)} \|u\|^{1-\theta}_{W^{k,r}(\mathbb{T}^n)}$.

Proof. This is a standard result. See for example Section 13.3 in [Leoni 2017] for a proof of this result on cubes which immediately implies the result on the torus. □

In practice the Gagliardo–Nirenberg interpolation inequality is used in the form recorded in Corollary B.2 throughout the paper. In particular, the second inequality recorded in Corollary B.2 is a high-low estimate which is central to our efforts to balance out terms that grow in time and terms that decay in time when designing our scheme of a priori estimates.

Corollary B.2 (estimate of interactions in L^2). *Let $f, g \in H^k(\mathbb{T}^n)$, and let α and β be multi-indices satisfying $|\alpha|+|\beta|=k$. Then we have the estimates*

$$\|(\partial^\alpha f)(\partial^\beta g)\|_{L^2} \lesssim \|f\|_{L^\infty}\|g\|_{H^k} + \|f\|_{H^k}\|g\|_{L^\infty}$$

and

$$\|fg\|_{H^k} \lesssim \|f\|_{L^\infty}\|g\|_{H^k} + \|f\|_{H^k}\|g\|_{L^\infty}.$$

Proof. The first estimate follows from the Gagliardo–Nirenberg inequality recorded in Theorem B.1. So let us define $\theta := |\beta|/k,\ 1/p := \frac{1}{2}(1-\theta)$, and $1/q := \frac{1}{2}\theta$. Then, by the Hölder and Gagliardo–Nirenberg inequalities,

$$\|(\partial^\alpha f)(\partial^\beta g)\|_{L^2} \leqslant \|\partial^\alpha f\|_{L^p}\|\partial^\beta g\|_{L^q} \lesssim \|f\|_{L^\infty}^{\theta}\|f\|_{H^k}^{1-\theta}\|g\|_{L^\infty}^{1-\theta}\|g\|_{H^k}^{\theta} \lesssim \|f\|_{L^\infty}\|g\|_{H^k} + \|f\|_{H^k}\|g\|_{L^\infty},$$

where we have used Young's inequality at the end, namely using the fact that, for any $x, y \geqslant 0$ and any $0 \leqslant \theta \leqslant 1$, we have $xy \leqslant \theta x^{1/\theta} + (1-\theta)y^{1/(1-\theta)}$. The second estimate then follows from the first by using the Leibniz rule. □

From Corollary B.2 we may deduce commutator estimates for transport and multiplication operators.

Lemma B.3 (commutator estimates for transport and multiplication operators). *Let $u \in H^k(\mathbb{T}^n; \mathbb{R}^n)$, let $f, g \in H^k(\mathbb{T}^n; \mathbb{R})$, and let $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$. Then*

$$\|[g, \partial^\alpha]f\|_{L^2} \lesssim \|g\|_{L^\infty}\|f\|_{H^k} + \|g\|_{H^k}\|f\|_{L^\infty}$$

and

$$\|[u\cdot\nabla, \partial^\alpha]f\|_{L^2} \lesssim \|\nabla u\|_{L^\infty}\|\nabla f\|_{H^{k-1}} + \|\nabla u\|_{H^{k-1}}\|\nabla f\|_{L^\infty}.$$

Proof. This follows from Corollary B.2 and the Leibniz rule. □

We conclude this section with other well-known analytical results. First, a product estimate in H^s spaces.

Lemma B.4 (product estimate). *Let $s > \frac{1}{2}n$ and let $0 \leqslant t \leqslant s$. There exists $C = C(s,t) > 0$ such that, for every $f \in H^s(\mathbb{T}^n)$ and every $g \in H^t(\mathbb{T}^n)$, we have $\|fg\|_{H^t} \leqslant C\|f\|_{H^s}\|g\|_{H^t}$. In other words, H^s is a continuous multiplier on H^t.*

Proof. The key observation is that H^t is the interpolation space of order t/s of the pair (L^2, H^s). Since $H^s \hookrightarrow L^\infty$ and H^s is a Banach algebra we know that $g \mapsto fg$ is bounded on both L^2 and H^s. The result then follows by interpolation. □

We also record a nonlinear Gronwall-type argument which is crucial in closing the energy estimates at the low level when developing the scheme of a priori estimates, in obtaining uniform bounds on the approximate solutions when building the local well-posedness, and in deriving the reduced energy estimates necessary to produce the continuation argument that glues the a priori and the local well-posedness together.

Lemma B.5 (Bihari's lemma). *Let $f : [0,\infty) \to [0,\infty)$ be nondecreasing and continuous such that $f > 0$ on $(0,\infty)$ and $\int_1^\infty 1/f < \infty$. Let F be the antiderivative of $-1/f$ which vanishes at $+\infty$. For every continuous function $y : [0,\infty) \to [0,\infty)$, if there exists $\alpha_0 > 0$ such that*

$$y(t) + \int_0^t f(y(s))\,ds \leqslant \alpha_0 \quad \text{for every } t \geqslant 0$$

then, for every $t \geqslant 0$, we have $y(t) \leqslant F^{-1}(t + F(\alpha_0))$.

Proof. This is proven in Lemma II.4.12 of [Boyer and Fabrie 2013]. □

We conclude this section with an elementary result which is very handy when it comes to ensuring that derivatives do not accumulate unduly on a single term in the nonlinear interactions.

Lemma B.6. *Let $x,\ y,\ z,\ C_x,\ C_y$, and C_z be real numbers such that $x, y, z \geqslant 0$. If*

$$x + y + z \leqslant \min(C_x + C_y, C_y + C_z, C_z + C_x),$$

then either

$$(1)\ x \leqslant C_x \text{ and } y \leqslant C_y, \quad (2)\ y \leqslant C_y \text{ and } z \leqslant C_z, \quad \text{or} \quad (3)\ z \leqslant C_z \text{ and } x \leqslant C_x.$$

Proof. This can be seen to be true by contraposition. The key observation is that

$$(1) \text{ or } (2) \text{ or } (3) \iff (x \leqslant C_x \text{ or } y \leqslant C_y) \text{ and } (y \leqslant C_y \text{ or } z \leqslant C_z) \text{ and } (z \leqslant C_z \text{ or } x \leqslant C_x).$$

We may then use this equivalence to rewrite the negation of the conclusion of the lemma and deduce that the contrapositive holds. □

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STRONG ILL-POSEDNESS FOR SQG IN CRITICAL SOBOLEV SPACES

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We prove that the inviscid surface quasigeostrophic (SQG) equations are strongly ill-posed in critical Sobolev spaces: there exists an initial data $H^2(\mathbb{T}^2)$ without any solutions in $L^\infty_t H^2$. Moreover, we prove strong critical norm inflation for C^∞-smooth data. Our proof is robust and extends to give similar ill-posedness results for the family of modified SQG equations which interpolate the SQG with the two-dimensional incompressible Euler equations.

1. Introduction

1A. *Main results.* We are concerned with the Cauchy problem for the inviscid surface quasigeostrophic (SQG) equations on $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$,

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^\perp (-\Delta)^{-1/2} \theta. \end{cases} \tag{SQG}$$

Our first main result shows that *strong norm inflation* occurs for the solution map of (SQG) in $H^2(\mathbb{T}^2)$ with C^∞-smooth solutions.

Theorem A (strong norm inflation). *For any $\epsilon, \delta, A > 0$, there exists $\theta_0 \in C^\infty(\mathbb{T}^2)$ satisfying*

$$\|\theta_0\|_{H^2 \cap W^{1,\infty}} < \epsilon$$

such that the unique local-in-time smooth solution θ to (SQG) with initial data θ_0 exists on $[0, \delta^*]$ for some $0 < \delta^* \le \delta$ and satisfies

$$\sup_{t \in [0,\delta^*]} \|\theta(t, \cdot)\|_{H^2} > A.$$

The above result implies that the solution operator defined from $H^2 \cap C^\infty$ to H^2 by $\theta_0 \mapsto \theta(t)$ for any $t > 0$ cannot be continuous at the trivial solution. On the other hand, the following result shows that actually it is impossible to define the solution operator from H^2 to $L^\infty_t H^2$.

Theorem B (nonexistence). *For any $\epsilon > 0$, there exists $\theta_0 \in H^2 \cap W^{1,\infty}(\mathbb{T}^2)$ satisfying*

$$\|\theta_0\|_{H^2 \cap W^{1,\infty}} < \epsilon$$

such that there is ***no*** *solution to (SQG) with initial data θ_0 belonging to $L^\infty([0, \delta]; H^2(\mathbb{T}^2))$ with any $\delta > 0$.*

Remark 1.1. We give a few remarks relevant to the statements above.

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• With a rather straightforward modification of our proof, the space H^2 in Theorems A and B can be replaced with $W^{1+2/p,p}$, with *any* $p > 1$. Later, we shall sketch the proof in the endpoint case $p = \infty$. Moreover, the domain $\mathbb{T}^2$ can be replaced with $\mathbb{R}^2$ or bounded domains having symmetry axes.

• The initial data for which nonexistence occur can be given explicitly; see (4-4) below.

• The arguments we present can be adapted to prove ill-posedness for the case of *modified* (and *logarithmically regularized*) SQG equations; see Section 1C below.

1B. *Well-posedness theory for SQG.* To put the above ill-posedness results into context, let us briefly recall the well-posedness theory for the SQG equation. Depending on the regularity of the solutions considered, one has the following categories:

• Strong solutions: local existence and uniqueness. Using the Kato–Ponce commutator estimate [1988], one obtains the a priori estimate

$$\frac{d}{dt}\|\theta\|_{H^s} \le C\|\nabla u\|_{L^\infty}\|\theta\|_{H^s}$$

for a solution of (SQG), which allows one to close $\|\theta(t)\|_{H^s} \lesssim \|\theta_0\|_{H^s}$ for $t \lesssim \|\theta_0\|_{H^s}^{-1}$ once $s > 2$, using that $\|\nabla u\|_{L^\infty} \lesssim \|\theta\|_{H^s}$. Similarly, H^s can be replaced with $W^{s,p}$, as long as $s > 1 + 2/p$. Based on this a priori estimate, one can prove local existence and uniqueness of a strong solution in the class $L^\infty_t W^{s,p}$ with $s > 1 + 2/p$. On the other hand, note that the borderline inequality $\|\nabla u\|_{L^\infty} \lesssim \|\theta\|_{H^2}$ fails; this makes the Sobolev space H^2 (and similarly $W^{1+2/p,p}$) *critical* for local well-posedness. This space is also *scaling-critical*: the critical norm is left-invariant under the transformation

$$\theta(t,x) \mapsto \lambda^{-1}\theta(t,\lambda x), \qquad u(t,x) \mapsto \lambda^{-1}u(t,\lambda x). \tag{1-1}$$

While not much is known for long-time dynamics of (SQG), see a recent breakthrough of [He and Kiselev 2021] for a construction of smooth initial data with Sobolev norms growing at least exponentially for all times. Moreover, existence of traveling-wave solutions [Li 2009; Cao et al. 2023] and rotating solutions [Hassainia and Hmidi 2015; de la Hoz et al. 2016; Castro et al. 2016] are known.

• Weak solutions: global existence. Global existence of L^p-weak solutions is known, thanks to [Resnick 1995; Marchand 2008; Bae and Granero-Belinchón 2015]. While such solutions are in general expected to be nonunique, see [Córdoba et al. 2018] for a uniqueness result for patches. On the other hand, for "very" weak solutions, nonuniqueness has been established; see [Buckmaster et al. 2019; Cheng et al. 2021; Isett and Ma 2021]. Note the gap of regularity between week and strong solutions.

• Ill-posedness in $W^{1,\infty}$: To the best of our knowledge, the only critical space ill-posedness result concerning (SQG) is the one given in [Elgindi and Masmoudi 2020] for $W^{1,\infty}$, where a powerful general method for proving ill-posedness of active scalar systems in L^∞-type spaces is developed. To be precise, in Section 9.2 of that work the authors show that there exist smooth steady states $\bar\theta$ and a sequence of perturbations $\tilde\theta_0^{(\epsilon)}$ ($\epsilon \to 0^+$) so that the associated (SQG) solution $\theta^{(\epsilon)}$ with data $\bar\theta + \tilde\theta_0^{(\epsilon)}$ satisfies

$$\|\theta^{(\epsilon)}(0,\cdot) - \bar\theta\|_{W^{1,\infty}} < \epsilon, \qquad \sup_{0<t<\epsilon}\|\theta^{(\epsilon)}(t,\cdot) - \bar\theta\|_{W^{1,\infty}} > c,$$

where $c > 0$ depends only on $\bar{\theta}$. It is very interesting to note that the authors use well-posedness in *critical Besov spaces* with summability index 1. Such Besov well-posedness theory goes back to the pioneering work [Vishik 1999]. Our result (which applies in the $W^{1,\infty}$ case as well) basically says that one can take $\bar{\theta} \equiv 0$ and replace c by ϵ^{-1}. On the other hand, one can restore well-posedness in $W^{1,\infty}$ by assuming some *rotational symmetry* and anisotrophic Hölder regularity [Elgindi and Jeong 2020b].

The current work settles the issue of *strong ill-posedness* of (SQG) at critical Sobolev spaces, and we believe that this could be a first step in understanding the dynamics of "slightly" supercritical and subcritical solutions (e.g., evolution of H^s-data with $|s-2| \ll 1$), thereby bridging the gap between the theory of weak and strong solutions. Indeed, in the very recent work [Elgindi 2021] on singularity formation for the three-dimensional Euler equations, one of the key steps was to understand precisely the mechanism of C^1-ill-posedness. Closing this section, let us mention some interesting works which seem contradictory to our main results:

- Miura [2006] proved that the fractionally dissipative SQG system

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + (-\Delta)^\beta \theta = 0, \\ u = \nabla^\perp (-\Delta)^{-1/2} \theta \end{cases} \tag{1-2}$$

is actually *well-posed* in the critical Sobolev space $H^{2-2\beta}$ for all $\beta > 0$ (for data of any size), and this seems to suggest H^2 well-posedness of the inviscid system by taking $\beta \to 0$! See [Li 2021; Jolly et al. 2021; 2022] for related recent advances.

- An invariant measure defined on $H^2(\mathbb{T}^2)$ which guarantees global well-posedness in $L^\infty_t H^2$ for any initial data in the support of the measure was constructed in [Földes and Sy 2021]. The data in Theorem B certainly does not belong to the support of such a measure.

1C. *Generalized SQG equations.* In the recent years, there has been significant interest in the study of so-called *generalized* SQG equations, given by

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^\perp P(\Lambda) \theta, \end{cases} \tag{1-3}$$

where $P(\Lambda)$ is some Fourier multiplier, with $\Lambda = (-\Delta)^{1/2}$. Two distinguished cases are $P(\Lambda) = \Lambda^{-1}$ (SQG) and $P(\Lambda) = \Lambda^{-2}$ (two-dimensional incompressible Euler). Of particular interest is the case of α-SQG systems given by $P(\Lambda) = \Lambda^{-\alpha}$, with $1 \le \alpha \le 2$, which interpolates the SQG and two-dimensional Euler equations. The L^2-based critical Sobolev space is then given by $H^{3-\alpha}$, and let us point out that the methods developed in the current work can handle the entire range $1 \le \alpha \le 2$ without any essential change in the proof, after deriving a generalized version of the "key lemma"; see the Appendix. One could consider even more general symbols such as $P(\Lambda) = \Lambda^{-\alpha} \log^{-\gamma}(10 + \Lambda)$, with $\gamma > 0$, which give rise to the so-called *logarithmically regularized* systems [Chae and Wu 2012; Chae et al. 2011; Dong and Li 2010]. It is known that if the power of the logarithm is sufficiently large, then one can restore well-posedness in $H^{3-\alpha}$ [Chae and Wu 2012], but at this point it is more appropriate to regard a logarithmically singularized Sobolev space to be critical. Indeed, one can see from our proof that there is a “logarithmic” room[1] in the arguments and therefore the same proof can cover same ill-posedness results in the slightly logarithmically regularized systems. We shall not dwell on this issue any further.

1D. ***Critical space ill-posedness for Euler.*** It should be emphasized that the strong Sobolev ill-posedness statements, Theorems A and B, were first established in the groundbreaking works [Bourgain and Li 2015; 2021] for the case of two- and three-dimensional Euler equations, respectively. Further developments, including the current work, seem to have been strongly inspired by these papers. Recently, Kwon [2021] settled the problem of strong ill-posedness in H^1 for logarithmically regularized (strictly speaking, powers of the log less than or equal to $\frac{1}{2}$) two-dimensional Euler equations, nicely complementing previous H^1 well-posedness from [Chae and Wu 2012]. On the other hand, much simpler proofs of H^1 ill-posedness for two-dimensional Euler, which also shows continuous-in-time degeneration of the solution in Sobolev spaces, have appeared in [Elgindi and Jeong 2017; Jeong and Yoneda 2021]. Some details of these simplified arguments will be given in the next section.

2. Ingredients of the proof

The purpose of this section is to sketch the main ingredients of the proof. Several key ideas have already appeared in earlier works establishing ill-posedness in the Euler case; we briefly review those in Section 2A. Additional difficulties arising in the (generalized) SQG case and new ideas are covered then in Section 2B.

2A. ***Strategy in the Euler case.*** In this section, let us give an overview of the ill-posedness proof in the two-dimensional Euler case. We recall that in $\mathbb{T}^2$ the Euler equations are given by

$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega = 0, \\ u = \nabla^{\perp}(-\Delta)^{-1}\omega. \end{cases} \tag{Euler}$$

In terms of ω, the critical L^2-based Sobolev space is $H^1(\mathbb{T}^2)$; indeed, $\omega\in H^1$ barely fails to guarantee $\nabla u\in L^\infty$, which is necessary to close the a priori estimate in H^1.

Choice of data for Euler. As a starting point of discussion, we present an interesting identity observed by T. Elgindi:

$$\frac{d}{dt}(\|\partial_2\omega\|_{L^2}^2 - \|\partial_1\omega\|_{L^2}^2) = \frac{1}{2}\int_{\mathbb{T}^2} \partial_1 u_1((\partial_2\omega)^2 + (\partial_1\omega)^2) + \omega\,\partial_1\omega\,\partial_2\omega\,\mathrm{d}x. \tag{2-1}$$

For $\omega_0\in L^\infty$, Yudovich theory provides a unique global solution in $L^\infty([0,\infty)\times\mathbb{T}^2)$, and note that the last term in (2-1) cannot contribute to a large growth of the H^1-norm in a small time interval. Therefore, to prove existence of an $H^1\cap L^\infty$-initial data ω_0 which “escapes” H^1 instantaneously, the goal would be to find $\omega_0\in H^1\cap L^\infty$ such that

$$\int_0^t\int_{\mathbb{T}^2} \partial_1 u_1(\partial_2\omega)^2\,\mathrm{d}x = +\infty \tag{2-2}$$

[1] Strictly speaking, some power of the logarithm.

for *any* $t > 0$, where ω is the Yudovich solution with data ω_0 and $u = \nabla^\perp(-\Delta)^{-1}\omega$. In particular, it is necessary that at the initial time we have

$$\int_{\mathbb{T}^2} \partial_1 u_{0,1}(\partial_2\omega_0)^2 \,\mathrm{d}x = +\infty. \tag{2-3}$$

The choice in [Elgindi and Jeong 2017] was

$$\omega_0(x) \simeq \frac{x_1 x_2}{|x|^2}|\ln|x||^{-\alpha}, \qquad |x| \ll 1, \tag{2-4}$$

since then [Bahouri and Chemin 1994; Denisov 2015b]

$$\partial_1 u_{0,1}(x) \simeq |\ln|x||^{1-\alpha}, \qquad |x| \ll 1, \tag{2-5}$$

which in particular guarantees (2-3) for a range of $\alpha > 0$.

Hyperbolic flow. Geometrically, vorticity which is positive on the first quadrant and odd with respect to both axes (as in (2-4)) induces velocity which is stretching in the x_1-direction and contracting in the other, which leads to squeezing of the vorticity near the x_1-axis and growth of H^1-norm. This so-called "hyperbolic flow scenario" has been used to produce Euler solutions with gradient growth; see [Kiselev and Šverák 2014; Zlatoš 2015; Xu 2016; Denisov 2009; 2015a; 2015b; Elgindi and Jeong 2019; 2020a; Choi and Jeong 2021]. Flattening of the vorticity level sets in such a flow configuration was studied in detail in [Zlatoš 2018; Jeong 2021].

Regularization effect. The main task is then to ensure that the velocity field, for a small time interval, retains its logarithmic divergence near the origin: indeed, instantaneous blow-up of the H^1-norm is not too difficult to see for the passive transport equation

$$\partial_t\omega + u_0 \cdot \nabla\omega = 0,$$

by solving the equation along the flow generated by u_0. When one tries to replace u_0 by u, a fundamental difficulty arises: anisotropic stretching of the vorticity regularizes the velocity. Indeed, rather involved computations in [Elgindi and Jeong 2023] suggest the asymptotics $\|\nabla u(t)\|_{L^\infty} \lesssim t^{-1}$, which is barely nonintegrable in time; this indicates that it could be a very delicate problem to verify (2-2). This upper bound of t^{-1} can be seen for instance by solving the passive problem above and recalculating the associated velocity at later times.

Key lemma and Lagrangian approach. Towards the goal of obtaining a *lower bound* on the velocity gradient $|\nabla u(t)| \gtrsim t^{-1}$, one needs to have a robust way of estimating the velocity gradient and proving some "stability" of the initial data. Regarding the former, the celebrated *key lemma* of Kiselev and Šverák asserts that (stated roughly)

$$\frac{u_1(x)}{x_1} \simeq \int_{[x_1,1]\times[x_2,1]} \frac{y_1 y_2}{|y|^4}\omega(y)\,\mathrm{d}y \tag{2-6}$$

for $\omega \in L^\infty$ with odd-odd symmetry. Note that $u_1 = 0$ for $x_1 = 0$ by symmetry, so that the left-hand side is an approximation of $\partial_1 u_1$. The lower bound of the form (2-6) has proven to be extremely powerful in establishing growth of the vorticity [Kiselev and Šverák 2014; Zlatoš 2015; Xu 2016; Kiselev et al. 2016;

Gancedo and Patel 2021; Elgindi 2021; He and Kiselev 2021]. It is interesting to note that [Bourgain and Li 2015] independently derived similar lower bounds. Next, regarding the issue of showing stability of the data, the key observation is the hierarchy of vortex dynamics expressed in (2-6): the vorticity around a point x is being affected mainly by the vorticity supported in $|y| \geq 2|x|$. This suggests that the chunk of vorticity supported far away from the origin is more stable, thereby contributing to the right-hand side of (2-6) for a longer time interval, to squeeze the vorticity closer to the origin. The proof of such stability and squeezing phenomena should be done in the Lagrangian variable, using the transport formulas

$$\omega(t,x) = \omega_0(\Phi_t^{-1}x), \quad \nabla\omega(t,x) = \nabla\omega_0(\Phi_t^{-1}(x))\nabla\Phi_t^{-1}(x),$$

where Φ_t is the flow map at time t. In the actual ill-posedness proofs, Lagrangian versions of the formula (2-1) are used.

2B. ***Difficulties in the SQG case.*** Overall, the strategy of the ill-posedness proof in the SQG case is similar to that explained in the above for two-dimensional Euler. Roughly speaking, the initial data is now modified to be

$$\theta_0 \simeq \frac{x_1 x_2}{|x|} |\ln|x||^{-\alpha}, \quad |x| \ll 1,$$

which is odd-odd and nonnegative in the first quadrant. The associated SQG velocity then satisfies the asymptotics (2-5) with strong hyperbolicity near the origin, which should stretch θ near the x_1-axis. The issue is whether such a stretching effect is sufficiently strong to remove θ from the critical Sobolev space it started from. Let us now explain some main differences with the Euler case and new ideas employed to handle those.

While the equation for θ in (SQG) is simply the transport equation exactly as in the two-dimensional Euler case, probably the most significant difference is that while the L^∞-norm is the common strongest conservation law, it is critical for two-dimensional Euler but one order weaker for SQG. Furthermore, there is global well-posedness for two-dimensional Euler with $\omega_0 \in L^\infty$ [Yudovich 1963], and the associated sharp estimates given by Yudovich theory have been very useful in understanding the dynamics.[2] On the other hand, the corresponding quantity in the SQG case, $\|\nabla\theta\|_{L^\infty}$, blows up together with $\|\theta\|_{H^2}$.

It seems that the only way to handle this issue is to rely entirely on a contradiction argument — we *assume* that there is an $L^\infty([0,T]; H^2)$-solution, and then prove that, for any $t > 0$, the H^2-norm of the solution must be actually infinite. The whole point in this contradiction argument is that we can use the hypothetical H^2-bound to control the solution, an idea originated in [Bourgain and Li 2015]. Again, the difficulty in the SQG case is that this hypothetical H^2 control is the only useful bound, whereas in the Euler case one has both H^1 and L^∞ control. Fortunately, it turns out that having an H^2-bound guarantees that the velocity is log-Lipschitz, which implies in particular *uniqueness* in the class $L_t^\infty H^2$ (this guarantees propagation in time of odd-odd symmetry and nonnegativity) and existence of the flow map. That is, an $L_t^\infty H^2$-solution is Lagrangian, and therefore we can apply transport formulas to understand the dynamics.

[2]Even in the three-dimensional Euler case, [Bourgain and Li 2021] actually carefully identifies a class of initial data for which $\omega \in L^\infty$ propagates locally in time. Then, one can prove and utilize estimates similar to Yudovich's in three dimensions.

Under the contradiction hypothesis, the main part of the argument is to derive and apply a version of the key lemma adapted to the SQG case. Series technical difficulties appear; to begin with, in the remainder estimate of the key lemma (see estimates (3-1) and (3-2) in Lemma 3.2) we are only allowed to use $\theta \in H^2$. As a consequence, the remainder term blows up super-logarithmically (the power $\frac{3}{2}$ in (3-2)) as the point x approaches the axes, whereas only logarithmic errors are allowed in the nonexistence proof. It seems that the only way to overcome this issue is to track carefully the geometry of the support of θ in time so that the problematic remainder term disappears. To achieve this, we replace θ_0 with a disjoint union of dyadic "bubbles" satisfying the same asymptotics as $|x| \to 0$ (see (4-4) below) and obtain detailed information on the location of these bubbles for an interval of time inductively, starting from the largest one. Such refined information appears in technical Claims I, II and III in the proof. In the context of controlling bubbles, another significant difference with Euler is that the "self-interaction" of a bubble is *not* a bounded term anymore. To overcome this issue we need to track the location of the "top point" of each bubble, which is the slowest point but does not suffer from self-interactions.

Closing this section, we remark that the versions of the key lemma derived in this work should be useful in improving previous growth results for the active scalar equations, as we handle the remainder term only with the critical quantity.

2C. ***Organization of the paper.*** The rest of this paper is organized as follows. The main technical tool, which we shall refer to as the key lemma, is stated and proved in Section 3. After that, the proofs of Theorems B and A are given in Sections 4 and 5, respectively.

3. The key lemma

To begin with, we recall the famous Hardy inequality.

Lemma 3.1 (Hardy's inequality). *Let f be a smooth function defined on the interval $(0,l)$ that vanishes in a neighborhood of $x=0$. Then we have for any $l \in [0,1]$*

$$\|x^{-1} f(x)\|_{L^2(0,l)} \le 2\|\partial f(x)\|_{L^2(0,l)}, \quad \|x^{-2} f(x)\|^2_{L^2(0,l)} \le 2\|\partial^2 f(x)\|^2_{L^2(0,l)}.$$

Proof. By the fundamental theorem of calculus and the assumption for f, we see

$$\int_0^l \frac{f(x)^2}{x^2}\,\mathrm{d}x = -\frac{1}{l} f(l)^2 + 2\int_0^l \frac{f(x)}{x}\partial f(x)\,\mathrm{d}x \le 2\int_0^l \frac{f(x)}{x}\partial f(x)\,\mathrm{d}x.$$

Using Hölder's inequality, we have

$$\int_0^l \frac{f(x)^2}{x^2}\,\mathrm{d}x \le 4\int_0^l |\partial f(x)|^2\,\mathrm{d}x.$$

Similarly, we have

$$\int_0^l \frac{f(x)^2}{x^4}\,\mathrm{d}x = -\frac{1}{3l^3} f(l)^2 + \frac{2}{3}\int_0^l \frac{f(x)}{x^3}\partial f(x)\,\mathrm{d}x \le \frac{2}{3}\int_0^l \frac{f(x)}{x^2}\frac{\partial f(x)}{x}\,\mathrm{d}x \le \frac{1}{2}\int_0^l \frac{\partial f(x)^2}{x^2}\,\mathrm{d}x.$$

Applying the above estimate, we obtain

$$\int_0^l \frac{f(x)^2}{x^4}\,\mathrm{d}x \le 2\int_0^l |\partial^2 f(x)|^2\,\mathrm{d}x. \qquad \square$$

We shall now state and prove the key lemma. For convenience, we shall normalize the SQG Biot–Savart law in such a way that

$$u(t,x) = \sum_{n\in\mathbb{Z}^2}\int_{[-1,1)^2} \frac{(x-(y+2n))^\perp}{|x-(y+2n)|^3}\theta(t,y)\,\mathrm{d}y.$$

Lemma 3.2. *We impose the following assumptions on $\theta \in H^2$:*

- *θ is odd with respect to both axes, i.e., $\theta(x) = -\theta(\bar{x}) = \theta(-x) = -\theta(\tilde{x})$, where $\bar{x} := (x_1, -x_2)$ and $\tilde{x} := (-x_1, x_2)$.*
- *θ vanishes near the axis; to be precise, for any $x \neq (0,0)$ satisfying either $x_1 = 0$ or $x_2 = 0$, there exists an open neighborhood of x such that θ vanishes.*

Then, for any x satisfying $|x| < \frac{1}{4}$ and $x_1 > x_2 > 0$, we have

$$\left|\frac{u_1(x)}{x_1} - 12\int_{Q(x)} \frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le B_1(x) \tag{3-1}$$

and

$$\left|\frac{u_2(x)}{x_2} + 12\int_{Q(x)} \frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le \left(1+\log\frac{x_1}{x_2}\right)B_2(x) + \left(1+\log\frac{x_1}{x_2}\right)^{3/2} B_3(x), \tag{3-2}$$

where $Q(x) := [2x_1, 1]\times[0,1]$ and B_1, B_2, B_3 satisfy

$$|B_1(x)| + |B_2(x)| \le C(\|\nabla^2\theta\|_{L^2([0,1]^2)} + \|\theta\|_{L^\infty([0,1]^2)})$$

and

$$|B_3(x)| \le C(\|\nabla^2\theta\|_{L^2(R(x))} + \|y_2^{-1}\partial_1\theta(y)\|_{L^2(R(x))}), \qquad R(x) := [x_1/2, 2x_1]\times[2x_2, 1].$$

Remark 3.3. We clearly have that $\|y_2^{-1}\partial_1\theta(y)\|_{L^2(R(x))} \le 2\|\nabla^2\theta\|_{L^2([0,1]^2)}$.

Proof. We fix a point $x = (x_1, x_2)$ satisfying the assumptions of the lemma. After a symmetrization, we have

$$u(x) = \sum_{n\in\mathbb{Z}^2}\int_{[0,1]^2}\left(\frac{(x-(y+2n))^\perp}{|x-(y+2n)|^3} - \frac{(x-(\tilde{y}+2n))^\perp}{|x-(\tilde{y}+2n)|^3} + \frac{(x-(-y+2n))^\perp}{|x-(-y+2n)|^3} - \frac{(x-(\bar{y}+2n))^\perp}{|x-(\bar{y}+2n)|^3}\right)\theta(y)\,\mathrm{d}y. \tag{3-3}$$

Estimate of u_1. We consider

$$I_1(n) := -\int_{[0,1]^2}\left(\frac{x_2-(y_2+2n_2)}{|x-(y+2n)|^3} - \frac{x_2-(y_2+2n_2)}{|x-(\tilde{y}+2n)|^3}\right)\theta(y)\,\mathrm{d}y,$$

$$I_2(n) := -\int_{[0,1]^2}\left(\frac{x_2-(-y_2+2n_2)}{|x-(-y+2n)|^3} - \frac{x_2-(-y_2+2n_2)}{|x-(\bar{y}+2n)|^3}\right)\theta(y)\,\mathrm{d}y$$

so that from (3-3)

$$u_1(x) = \sum_{n\in\mathbb{Z}^2}(I_1(n) + I_2(n)).$$

We think of the cases $n=0$ and $n\neq 0$ separately. For $n\neq 0$, we see that

$$|I_1(n)+I_1(\tilde{n})|\le O(|n|^{-4})\|\theta\|_{L^\infty([0,1]^2)}x_1,\quad |I_2(n)+I_2(\tilde{n})|\le O(|n|^{-4})\|\theta\|_{L^\infty([0,1]^2)}x_1.$$

Therefore,

$$\left|\sum_{n\in\mathbb{Z}^2\setminus\{0\}}(I_1(n)+I_2(n))\right|\le Cx_1\|\theta\|_{L^\infty([0,1]^2)}. \tag{3-4}$$

We now estimate the case of $n=0$. Using

$$\frac{1}{A^3}-\frac{1}{B^3}=\frac{(B^2-A^2)(A^2+AB+B^2)}{A^3B^3(A+B)}, \tag{3-5}$$

we have

$$I_1(0)=-4x_1\int_{[0,1]^2}\frac{y_1(x_2-y_2)(|x-y|^2+|x-y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y.$$

Noting that $[0,1]^2=Q(x)\cup[0,2x_1]\times[2x_1,1]\cup[0,2x_1]^2$, we estimate the integral for each set.

(i) Suppose $y\in Q(x)$. In this case we can show that

$$\tfrac{1}{4}|y|\le|x-y|\le|y|,\quad \tfrac{1}{2}|y|\le|x-\tilde{y}|\le 2|y| \tag{3-6}$$

because the first inequality comes from

$$|x-y|^2\ge|x_1-y_1|^2\ge\tfrac{1}{4}y_1^2\ge\tfrac{1}{8}|y|^2,\quad y_1\ge y_2,$$

and

$$|x-y|^2=|x_1-y_1|^2+|x_2-y_2|^2\ge\tfrac{1}{4}y_1^2+\tfrac{1}{4}y_2^2,\quad y_1\le y_2.$$

The goal is to prove that

$$-\int_{Q(x)}\frac{y_1(x_2-y_2)(|x-y|^2+|x-y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y=:J$$

satisfies

$$\left|J-\frac{3}{2}\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right|\le C\|\nabla^2\theta\|_{L^2([0,1]^2)}. \tag{3-7}$$

We separate $J=J_1+J_2$, where

$$J_1:=\int_{Q(x)}\frac{y_1y_2(|x-y|^2+|x-y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y,$$

$$J_2:=-\int_{Q(x)}\frac{y_1x_2(|x-y|^2+|x-y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y.$$

Using (3-6), we may estimate

$$|J_2|\le C|x|\int_{Q(x)}\frac{1}{|y|^2}\frac{|\theta(y)|}{|y|^2}\,\mathrm{d}y.$$

Note that by Hölder's inequality,

$$|x|\int_{Q(x)}\frac{1}{|y|^2}\frac{|\theta(y)|}{|y|^2}\,\mathrm{d}y\le|x|\left(\int_{2x_1}^\infty\frac{1}{r^3}\,\mathrm{d}r\right)^{1/2}\||y|^{-2}\theta(y)\|_{L^2(Q(x))}\le C\||y|^{-2}\theta(y)\|_{L^2([0,1]^2)}.$$

Then with the Hardy's inequality we have

$$|J_2| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.$$

On the other hand, regarding J_1, we shall show that

$$\left| J_1 - \frac{3}{2}\int_{Q(x)} \frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y \right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.$$

For this purpose we simply write $J_1 = J_{11}+J_{12}+J_{13}$, where

$$J_{11} = \int_{Q(x)} \frac{y_1y_2|x-y|^2}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y,$$

$$J_{12} = \int_{Q(x)} \frac{y_1y_2|x-y|\,|x-\tilde{y}|}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y,$$

$$J_{13} = \int_{Q(x)} \frac{y_1y_2|x-\tilde{y}|^2}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y$$

and show that

$$\left| J_{1k} - \frac{1}{2}\int_{Q(x)} \frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y \right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}$$

for each $k=1,2,3$. We supply the proof only for the case $k=1$, since the others can be treated similarly. We directly compute

$$J_{11} - \frac{1}{2}\int_{Q(x)} \frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y = \int_{Q(x)} y_1y_2\frac{2|y|^5 - |x-y|\,|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}{2|y|^5|x-y|\,|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y.$$

We rewrite the numerator as

$$\begin{aligned}
&2|y|^5 - |x-y|\,|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)\\
&\quad = (|y|^2-|x-y|^2)|y|^3 + |x-y|^2(|y|^3-|x-\tilde{y}|^3) + (|y|-|x-y|)|y|^4 + |x-y|(|y|^4-|x-\tilde{y}|^4),
\end{aligned}$$

and further rewriting

$$|y|-|x-y| = \frac{|y|^2-|x-y|^2}{|y|+|x-y|},\quad |y|^3-|x-\tilde{y}|^3 = \frac{(|y|^2-|x-\tilde{y}|^2)(|y|^2+|y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|y|+|x-\tilde{y}|},$$

we see using (3-6) that

$$\big|2|y|^5 - |x-y|\,|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)\big| \le C|x|\,|y|^4.$$

Then, we can infer that

$$\left| J_{11} - \frac{1}{2}\int_{Q(x)} \frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y \right| \le C|x|\int_{Q(x)} \frac{1}{|y|^2}\frac{|\theta(y)|}{|y|^2}\,\mathrm{d}y \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.$$

Collecting the estimates for J_1 and J_2 gives (3-7).

(ii) Suppose $y \in [0,2x_1]\times[2x_1,1]$. In this case, using $y_1 \le y_2$, we can see that

$$\tfrac{1}{2}y_2 \le |x-y| \le 2y_2,\quad \tfrac{1}{2}y_2 \le |x-\tilde{y}| \le 2y_2.$$

Thus, Hölder's inequality and Hardy's inequality imply that

$$\begin{aligned}\left|-\int_{[0,2x_1]\times[2x_1,1]} \frac{y_1(x_2-y_2)(|x-y|^2+|x-y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y\right| \\ \le C\int_{[0,2x_1]\times[2x_1,1]} \frac{1}{y_2}\frac{\theta(y)}{y_2^2}\,\mathrm{d}y \le C\biggl(\int_{[0,2x_1]\times[2x_1,1]}\frac{\mathrm{d}y}{y_2^2}\biggr)^{1/2}\|y_2^{-2}\theta(y)\|_{L^2([0,1]^2)} \\ \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.\end{aligned} \tag{3-8}$$

(iii) Suppose $y\in[0,2x_1]^2$. Thanks to $\theta(y_1,0)=0$, using integration by parts gives

$$\begin{aligned}-\int_{[0,2x_1]^2} \frac{y_1(x_2-y_2)(|x-y|^2+|x-y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y \\ =\frac{1}{4x_1}\int_{[0,2x_1]^2}\biggl(\frac{1}{|x-y|}-\frac{1}{|x-\tilde{y}|}\biggr)\partial_2\theta(y)\,\mathrm{d}y \\ -\frac{1}{4x_1}\int_0^{2x_1}\biggl(\frac{1}{|(x_1-y_1,x_2-2x_1)|}-\frac{1}{|(x_1+y_1,x_2-2x_1)|}\biggr)\theta(y_1,2x_1)\,\mathrm{d}y_1.\end{aligned}$$

By Hölder's inequality, we estimate the second integral as

$$\left|-\frac{1}{x_1}\int_0^{2x_1}\biggl(\frac{1}{|(x_1-y_1,x_2-2x_1)|}-\frac{1}{|(x_1+y_1,x_2-2x_1)|}\biggr)\theta(y_1,2x_1)\,\mathrm{d}y_1\right| \le Cx_1^{-1}\|\theta\|_{L^\infty([0,2x_1]^2)}.$$

We notice that since θ vanishes near the axis, we have

$$\begin{aligned}\|\theta\|_{L^\infty([0,2x_1]^2)} \le \sup_{y_1\in[0,2x_1]}\int_0^{2x_1}|\partial_2\theta(y_1,y_2)|\,\mathrm{d}y_2 \le (2x_1)^{1/2}\Bigl\|\sup_{y_1\in[0,2x_1]}|\partial_2\theta(y_1,\cdot)|\Bigr\|_{L^2(0,2x_1)} \\ \le 2x_1\|\partial_1\partial_2\theta\|_{L^2([0,2x_1]^2)}.\end{aligned} \tag{3-9}$$

Thus, we have

$$\left|-\frac{1}{x_1}\int_0^{2x_1}\biggl(\frac{1}{|(x_1-y_1,x_2-2x_1)|}-\frac{1}{|(x_1+y_1,x_2-2x_1)|}\biggr)\theta(y_1,2x_1)\,\mathrm{d}y_1\right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.$$

Calculating the first integral with Hölder's inequality, we see that

$$\begin{aligned}\left|\frac{1}{x_1}\int_{[0,2x_1]^2}\biggl(\frac{1}{|x-y|}-\frac{1}{|x-\tilde{y}|}\biggr)\partial_2\theta(y)\,\mathrm{d}y\right| &\le \frac{2}{x_1}\int_{[0,2x_1]^2}\frac{|\partial_2\theta(y)|}{|x-y|}\,\mathrm{d}y \\ &\le \frac{2}{x_1}\biggl(\int_0^{4x_1} r^{-1/3}\,\mathrm{d}r\biggr)^{3/4}\|\partial_2\theta\|_{L^4([0,2x_1]^2)} \\ &\le Cx_1^{-1/2}\|\nabla\theta\|_{L^4([0,2x_1]^2)}.\end{aligned}$$

The Gagliardo–Nirenberg interpolation inequality implies

$$x_1^{-1/2}\|\nabla\theta\|_{L^4([0,2x_1]^2)} \le Cx_1^{-1/2}\|\nabla^2\theta\|^{3/4}_{L^2([0,2x_1]^2)}\|\theta\|^{1/4}_{L^2([0,2x_1]^2)}+Cx_1^{-2}\|\theta\|_{L^2([0,2x_1]^2)},$$

where the constant $C>0$ is independent of x_1. Applying Hardy's inequality to it, we have

$$x_1^{-1/2}\|\partial_2\theta\|_{L^4([0,2x_1]^2)} \le C\|\nabla^2\theta\|_{L^2([0,2x_1]^2)},$$

and hence it follows that

$$\left|\frac{1}{x_1}\int_{[0,2x_1]^2}\left(\frac{1}{|x-y|}-\frac{1}{|x-\tilde{y}|}\right)\partial_2\theta(y)\,\mathrm{d}y\right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.$$

Combining the above estimates, we obtain

$$\left|-\int_{[0,2x_1]^2}\frac{y_1(x_2-y_2)(|x-y|^2+|x-y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.$$

We collect the estimates for each region and deduce that

$$\left|\frac{I_1(0)}{x_1}-6\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.$$

We can estimate

$$I_2(0)=4x_1\int_{[0,1]^2}\frac{y_1(x_2+y_2)(|x+y|^2+|x+y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x+y|^3|x-\bar{y}|^3(|x+y|+|x-\bar{y}|)}\theta(y)\,\mathrm{d}y$$

similarly to $I_1(0)$, resulting in the bound

$$\left|\frac{I_2(0)}{x_1}-6\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}. \tag{3-10}$$

We omit the details. Thus we have (3-1).

Estimate of u_2. Now we estimate u_2. Note that

$$u_2(x)=\sum_{n\in\mathbb{Z}^2}(I_3(n)+I_4(n)),$$

where

$$I_3(n):=\int_{[0,1]^2}\left(\frac{x_1-(y_1+2n_1)}{|x-(y+2n)|^3}-\frac{x_1-(y_1+2n_1)}{|x-(\bar{y}+2n)|^3}\right)\theta(y)\,\mathrm{d}y,$$
$$I_4(n):=\int_{[0,1]^2}\left(\frac{x_1-(-y_1+2n_1)}{|x-(-y+2n)|^3}-\frac{x_1-(-y_1+2n_1)}{|x-(\tilde{y}+2n)|^3}\right)\theta(y)\,\mathrm{d}y.$$

Since we can similarly see that

$$\left|\sum_{n\in\mathbb{Z}^2\setminus\{0\}}(I_3(n)+I_4(n))\right| \le Cx_2\|\theta\|_{L^\infty([0,1]^2)},$$

it suffices to estimate for $n=0$. Using (3-5), we have

$$I_3(0)=4x_2\int_{[0,1]^2}\frac{y_2(x_1-y_1)(|x-y|^2+|x-y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x-y|^3|x-\bar{y}|^3(|x-y|+|x-\bar{y}|)}\theta(y)\,\mathrm{d}y.$$

We divide the domain into four regions as $[0,1]^2=Q(x)\cup[0,x_1/2]\times[0,2x_1]\cup[x_1/2,2x_1]\times[0,2x_1]\cup[0,2x_1]\times[2x_1,1]$ and estimate the integral in each region.

(i) Suppose $y\in Q(x)$. In this case, we note by $\frac{1}{4}y_1^2+y_2^2\le|x_1-y_1|^2+|x_2+y_2|^2$ that

$$\tfrac{1}{2}|y|\le|x-\bar{y}|\le 2|y|.$$

Recalling (3-6) holds, we can prove similarly to (3-7)

$$\left|\int_{Q(x)} \frac{y_2(x_1-y_1)(|x-y|^2+|x-y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x-y|^3|x-\bar{y}|^3(|x-y|+|x-\bar{y}|)}\theta(y)\,dy+\frac{3}{2}\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,dy\right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.$$

(ii) Suppose $y\in[0,2x_1]\times[2x_1,1]$. It follows that

$$\tfrac{1}{2}y_2\le|x-y|\le 2y_2,\qquad y_2\le|x-\bar{y}|\le 2y_2.$$

Hence, we can show

$$\left|\int_{[0,2x_1]\times[2x_1,1]} \frac{y_2(x_1-y_1)(|x-y|^2+|x-y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x-y|^3|x-\bar{y}|^3(|x-y|+|x-\bar{y}|)}\theta(y)\,dy\right|\le C\|\nabla^2\theta\|_{L^2([0,1]^2)}$$

in a way similar to (3-8).

(iii) Suppose $y\in[0,x_1/2]\times[0,2x_1]$. This implies

$$\tfrac{1}{2}x_1\le|x-y|\le 4x_1,\qquad \tfrac{1}{2}x_1\le|x-\bar{y}|\le 4x_1,$$

with $y_2\le 2x_1$ we have

$$\begin{aligned}\left|\int_{[0,x_1/2]\times[0,2x_1]} \frac{y_2(x_1-y_1)(|x-y|^2+|x-y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x-y|^3|x-\bar{y}|^3(|x-y|+|x-\bar{y}|)}\theta(y)\,dy\right| &\\ \le Cx_1^{-1}\int_{[0,x_1/2]\times[0,2x_1]}\frac{\theta(y)}{y_2^2}\,dy \le Cx_1^{-1}\left(\int_{[0,x_1/2]\times[0,2x_1]}1\,dy\right)^{1/2}\|y_2^{-2}\theta(y)\|_{L^2([0,1]^2)}&\\ \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.&\end{aligned}$$

(iv) Suppose $y\in[x_1/2,2x_1]\times[0,2x_1]$. We claim that

$$\int_{[x_1/2,2x_1]\times[0,2x_1]} \frac{y_2(x_1-y_1)(|x-y|^2+|x-y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x-y|^3|x-\bar{y}|^3(|x-y|+|x-\bar{y}|)}\theta(y)\,dy=:K$$

satisfies

$$|K|\le C\|\nabla^2\theta\|_{L^2([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right)+C(\|\nabla^2\theta\|_{L^2(R(x))}+\|y_2^{-1}\partial_1\theta(y)\|_{L^2(R(x))})\left(1+\log\frac{x_1}{x_2}\right)^{3/2}. \tag{3-11}$$

Using integration by parts, we take the decomposition $K=K_1+K_2+K_3$, where

$$\begin{aligned}K_1&:=-\frac{1}{4x_2}\int_{[x_1/2,2x_1]\times[0,2x_1]}\left(\frac{1}{|x-y|}-\frac{1}{|x-\bar{y}|}\right)\partial_1\theta(y)\,dy,\\ K_2&:=\frac{1}{4x_2}\int_0^{2x_1}\left(\frac{1}{|(x_1,x_2-y_2)|}-\frac{1}{|(x_1,x_2+y_2)|}\right)\theta(2x_1,y_2)\,dy_2,\\ K_3&:=-\frac{1}{4x_2}\int_0^{2x_1}\left(\frac{1}{|(x_1/2,x_2-y_2)|}-\frac{1}{|(x_1/2,x_2+y_2)|}\right)\theta(x_1/2,y_2)\,dy_2.\end{aligned}$$

With (3-9) we may estimate K_2 as

$$|K_2| = \left| \int_0^{2x_1} \frac{y_2\theta(2x_1, y_2)}{|(x_1, x_2 - y_2)|\,|(x_1, x_2 + y_2)|(|(x_1, x_2 - y_2)| + |(x_1, x_2 + y_2)|)}\,\mathrm{d}y_2 \right|$$
$$\le C x_1^{-2} \int_0^{2x_1} \theta(2x_1, y_2)\,\mathrm{d}y_2 \le C x_1^{-1} \|\theta\|_{L^\infty([0,2x_1]^2)} \le C \|\nabla^2\theta\|_{L^2([0,1]^2)}. \tag{3-12}$$

Similarly, we obtain

$$|K_3| \le C \|\nabla^2\theta\|_{L^2([0,1]^2)}.$$

Noting that

$$K_1 = \int_{[x_1/2, 2x_1]\times[0,2x_1]} \frac{-y_2\partial_1\theta(y)}{|x-y|\,|x-\bar{y}|(|x-y| + |x-\bar{y}|)}\,\mathrm{d}y,$$

we set $K_1 = K_{11} + K_{12}$, where

$$K_{11} := \int_{[x_1/2, 2x_1]\times[0,2x_2]} \frac{-y_2\partial_1\theta(y)}{|x-y|\,|x-\bar{y}|(|x-y| + |x-\bar{y}|)}\,\mathrm{d}y,$$
$$K_{12} := \int_{[x_1/2, 2x_1]\times[2x_2,2x_1]} \frac{-y_2\partial_1\theta(y)}{|x-y|\,|x-\bar{y}|(|x-y| + |x-\bar{y}|)}\,\mathrm{d}y.$$

From $|x-\bar{y}| \ge x_2 + y_2 \ge y_2$ we have

$$|K_{11}| \le C \int_0^{2x_2} \frac{\sup_{y_1\in[x_1/2,2x_1]} |\partial_1\theta(y_1, y_2)|}{x_2 + y_2} \int_0^{2x_1} \frac{1}{|x-y|}\,\mathrm{d}y_1\,\mathrm{d}y_2,$$
$$|K_{12}| \le C \int_{2x_2}^{2x_1} \frac{\sup_{y_1\in[x_1/2,2x_1]} |\partial_1\theta(y_1, y_2)|}{x_2 + y_2} \int_0^{2x_1} \frac{1}{|x-y|}\,\mathrm{d}y_1\,\mathrm{d}y_2.$$

By the Gagliardo–Nirenberg interpolation inequality with $y_2 \le 2x_1$, we can see that

$$y_2^{-1/2} \sup_{y_1\in[x_1/2,2x_1]} |\partial_1\theta(y_1, y_2)| \le C(\|\partial_1^2\theta(\,\cdot\,, y_2)\|_{L^2(x_1/2,2x_1)} + y_2^{-1}\|\partial_1\theta(\,\cdot\,, y_2)\|_{L^2(x_1/2,2x_1)}),$$

where the constant $C > 0$ does not depend on x_1. On the other hand,

$$\int_0^{2x_1} \frac{1}{|x-y|}\,\mathrm{d}y_1 = \int_0^{x_1} \frac{2}{\sqrt{\tau^2 + (x_2 - y_2)^2}}\,\mathrm{d}\tau$$
$$= 2\log\bigl(x_1 + \sqrt{x_1^2 + (x_2 - y_2)^2}\bigr) - 2\log|x_2 - y_2| \le C \log\left(1 + \frac{x_1}{|x_2 - y_2|}\right). \tag{3-13}$$

Hence, with $y_2 \le 2x_1$ and Hölder's inequality, we infer that

$$|K_{11}| \le C(\|\nabla^2\theta\|_{L^2([0,1]^2)} + \|y_2^{-1}\partial_1\theta(y)\|_{L^2([0,1]^2)}) \left\{ \int_0^{2x_2} \frac{1}{x_2 + y_2} \left| \log\left(1 + \frac{x_1}{|x_2 - y_2|}\right) \right|^2 \mathrm{d}y_2 \right\}^{1/2},$$
$$|K_{12}| \le C(\|\nabla^2\theta\|_{L^2(R(x))} + \|y_2^{-1}\partial_1\theta(y)\|_{L^2(R(x))}) \left\{ \int_{2x_2}^{2x_1} \frac{1}{x_2 + y_2} \left| \log\left(1 + \frac{x_1}{|x_2 - y_2|}\right) \right|^2 \mathrm{d}y_2 \right\}^{1/2}.$$

Using Hardy's inequality and that

$$\int_0^{2x_2} \frac{1}{x_2+y_2}\left|\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\right|^2 \mathrm{d}y_2 \le \frac{1}{x_2}\int_0^{2x_2}\left(\log\frac{2x_1}{|x_2-y_2|}\right)^2 \mathrm{d}y_2 \\ = \frac{2}{x_2}\int_0^{x_2}\left(\log\frac{2x_1}{x_2-y_2}\right)^2 \mathrm{d}y_2 \le C\left(1+\log\frac{x_1}{x_2}\right)^2,$$

we obtain

$$|K_{11}| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right).$$

By the estimate

$$\int_{2x_2}^{2x_1} \frac{1}{x_2+y_2}\left|\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\right|^2 \mathrm{d}y_2 \le \int_{2x_2}^{2x_1}\frac{1}{y_2-x_2}\left(\log\frac{3x_1}{y_2-x_2}\right)^2 \mathrm{d}y_2 \\ = \frac{1}{3}\left(\log\frac{3x_1}{x_2}\right)^3 - \frac{1}{3}\left(\log\frac{3x_1}{2x_1-x_2}\right)^3 \le C\left(1+\log\frac{x_1}{x_2}\right)^3,$$

we have

$$|K_{12}| \le C(\|\nabla^2\theta\|_{L^2(R(x))} + \|y_2^{-1}\partial_1\theta(y)\|_{L^2(R(x))})\left(1+\log\frac{x_1}{x_2}\right)^{3/2}.$$

This implies

$$|K_1| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right) + C(\|\nabla^2\theta\|_{L^2(R(x))} + \|y_2^{-1}\partial_1\theta(y)\|_{L^2(R(x))})\left(1+\log\frac{x_1}{x_2}\right)^{3/2},$$

and collecting the estimates for K_1, K_2, and K_3, we obtain (3-11). Therefore, we arrive at

$$\left|\frac{I_3(0)}{x_2} + 6\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \\ \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right) + C(\|\nabla^2\theta\|_{L^2(R(x))} + \|y_2^{-1}\partial_1\theta(y)\|_{L^2(R(x))})\left(1+\log\frac{x_1}{x_2}\right)^{3/2}.$$

Using (3-5), we can estimate

$$I_4(0) = -4x_2\int_{[0,1]^2}\frac{y_2(x_1+y_1)(|x+y|^2+|x+y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x+y|^3|x-\tilde{y}|^3(|x+y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y,$$

similarly to $I_3(0)$. Hence we have (3-2), and this completes the proof. $\square$

Lemma 3.4. *Let θ satisfy the assumptions in Lemma 3.2. Then, for any x satisfying $|x| < \frac{1}{4}$ and $x_1 > x_2 > 0$, we have*

$$\left|\frac{u_1(x)}{x_1} - 12\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le B_4(x) \tag{3-14}$$

and

$$\left|\frac{u_2(x)}{x_2} + 12\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le \left(1+\log\frac{x_1}{x_2}\right)B_5(x) + \left(1+\log\frac{x_1}{x_2}\right)^2 B_6(x), \tag{3-15}$$

where B_4, B_5, B_6 satisfy

$$|B_4(x)|+|B_5(x)| \le C(\|\nabla\theta\|_{L^\infty([0,1]^2)} + \|\theta\|_{L^\infty([0,1]^2)}), \quad |B_6(x)| \le C\|\nabla\theta\|_{L^\infty(R(x))}.$$

Proof. We follow the proof of Lemma 3.2. To obtain (3-14), we recall (3-4) and have

$$|u_1(x)| = \left|\sum_{n\in\mathbb{Z}^2}(I_1(n)+I_2(n))\right| \le Cx_1\|\theta\|_{L^\infty([0,1]^2)} + I_1(0)+I_2(0).$$

We estimate

$$I_1(0) = -4x_1\int_{[0,1]^2}\frac{y_1(x_2-y_2)(|x-y|^2+|x-y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y$$

for each set $Q(x)$, $[0, 2x_1]\times[2x_1, 1]$, and $[0, 2x_1]^2$.

(i) Suppose $y \in Q(x)$. Using the notation J_1 and J_2 in Lemma 3.2, it suffices to obtain

$$\left|J_1 - \frac{3}{2}\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| + |J_2| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}. \tag{3-16}$$

We already showed that

$$|J_2| \le C|x|\int_{Q(x)}\frac{1}{|y|^3}\frac{|\theta(y)|}{|y|}\,\mathrm{d}y.$$

By (3-6) and Hölder's inequality, we have

$$|x|\int_{Q(x)}\frac{1}{|y|^3}\frac{|\theta(y)|}{|y|}\,\mathrm{d}y \le |x|\left(\int_{|x|}^{\infty}\frac{1}{r^2}\,\mathrm{d}r\right)\||y|^{-1}\theta(y)\|_{L^\infty(Q(x))} \le \||y|^{-1}\theta(y)\|_{L^\infty([0,1]^2)}.$$

Since θ vanishes near the axis, it follows

$$|J_2| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}.$$

Letting $J_1 = J_{11}+J_{12}+J_{13}$ as in the proof of Lemma 3.2, we can prove that

$$\left|J_{1k} - \frac{1}{2}\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le C|x|\int_{Q(x)}\frac{1}{|y|^3}\frac{|\theta(y)|}{|y|}\,\mathrm{d}y \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}$$

for each $k = 1, 2, 3$. Therefore, (3-16) is obtained.

(ii) Suppose $y \in [0, 2x_1]\times[2x_1, 1]$. In (3-8) we observed that

$$\left|\int_{[0,2x_1]\times[2x_1,1]}\frac{y_1(x_2-y_2)(|x-y|^2+|x-y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y\right| C\int_{[0,2x_1]\times[2x_1,1]}\frac{1}{y_2^2}\frac{\theta(y)}{y_2}\,\mathrm{d}y.$$

Since

$$\int_{[0,2x_1]\times[2x_1,1]}\frac{1}{y_2^2}\frac{\theta(y)}{y_2}\,\mathrm{d}y \le C\left(\int_{[0,2x_1]\times[2x_1,1]}\frac{1}{y_2^2}\,\mathrm{d}y\right)\|y_2^{-1}\theta(y)\|_{L^\infty([0,1]^2)} \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}, \tag{3-17}$$

we have

$$\left|-\int_{[0,2x_1]\times[2x_1,1]}\frac{y_1(x_2-y_2)(|x-y|^2+|x-y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}.$$

(iii) Suppose $y \in [0, 2x_1]^2$. We recall that

$$
\begin{aligned}
-\int_{[0,2x_1]^2} & \frac{y_1(x_2-y_2)(|x-y|^2+|x-y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y \\
&= \frac{1}{4x_1}\int_{[0,2x_1]^2}\left(\frac{1}{|x-y|}-\frac{1}{|x-\tilde{y}|}\right)\partial_2\theta(y)\,\mathrm{d}y \\
&\qquad -\frac{1}{4x_1}\int_0^{2x_1}\left(\frac{1}{|(x_1-y_1,x_2-2x_1)|}-\frac{1}{|(x_1+y_1,x_2-2x_1)|}\right)\theta(y_1,2x_1)\,\mathrm{d}y_1.
\end{aligned}
$$

Using Hölder's inequality, we have

$$
\begin{aligned}
\left|\frac{1}{x_1}\int_{[0,2x_1]^2}\left(\frac{1}{|x-y|}-\frac{1}{|x-\tilde{y}|}\right)\partial_2\theta(y)\,\mathrm{d}y\right| &\le \frac{2}{x_1}\left(\int_{[0,2x_1]^2}\frac{1}{|x-y|}\,\mathrm{d}y\right)\|\partial_2\theta\|_{L^\infty([0,1]^2)} \\
&\le C\|\nabla\theta\|_{L^\infty([0,2x_1]^2)}.
\end{aligned}
$$

From Hölder's inequality and the mean value theorem, it follows

$$
\left|-\frac{1}{x_1}\int_0^{2x_1}\left(\frac{1}{|(x_1-y_1,x_2-2x_1)|}-\frac{1}{|(x_1+y_1,x_2-2x_1)|}\right)\theta(y_1,2x_1)\,\mathrm{d}y_1\right| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}.
$$

Therefore, we obtain

$$
\left|-\int_{[0,2x_1]^2}\frac{y_1(x_2-y_2)(|x-y|^2+|x-y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x-y|^3|x-\tilde{y}|^3(|x-y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla^2\theta\|_{L^2([0,1]^2)}.
$$

Combining the above estimates, it follows that

$$
\left|\frac{I_1(0)}{x_1}-6\int_{Q(x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}.
$$

In a similar way, we can show that

$$
I_2(0)=4x_1\int_{[0,1]^2}\frac{y_1(x_2+y_2)(|x+y|^2+|x+y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x+y|^3|x-\bar{y}|^3(|x+y|+|x-\bar{y}|)}\theta(y)\,\mathrm{d}y
$$

satisfies

$$
\left|\frac{I_2(0)}{x_2}-6\int_{R(2x)}\frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}.
$$

We omit the details. Thus we have (3-14).

To estimate u_2, we start with

$$
|u_2(x)| = \left|\sum_{n\in\mathbb{Z}^2}(I_3(n)+I_4(n))\right| \le Cx_2\|\theta\|_{L^\infty([0,1]^2)}+I_3(0)+I_4(0).
$$

To estimate

$$
I_3(0)=4x_2\int_{[0,1]^2}\frac{y_2(x_1-y_1)(|x-y|^2+|x-y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x-y|^3|x-\bar{y}|^3(|x-y|+|x-\bar{y}|)}\theta(y)\,\mathrm{d}y,
$$

we consider $[0,1]^2 = Q(x)\cup[0,x_1/2]\times[0,2x_1]\cup[x_1/2,2x_1]\times[0,2x_1]\cup[0,2x_1]\times[2x_1,1]$ and estimate the integral in each region.

(i) Suppose $y \in Q(x)$. In this case, recalling that (3-6) holds, we can prove, proceeding much as for (3-16),

$$\left| \int_{Q(x)} \frac{y_2(x_1-y_1)(|x-y|^2+|x-y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x-y|^3|x-\bar{y}|^3(|x-y|+|x-\bar{y}|)} \theta(y)\,\mathrm{d}y + \frac{3}{2}\int_{Q(x)} \frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y \right| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}. \tag{3-18}$$

(ii) Suppose $y \in [0, 2x_1] \times [2x_1, 1]$. Since in this case we have

$$\tfrac{1}{2}y_2 \le |x-y| \le 2y_2, \quad y_2 \le |x-\bar{y}| \le 2y_2,$$

with (3-17) we can show that

$$\left| \int_{[0,2x_1]\times[2x_1,1]} \frac{y_2(x_1-y_1)(|x-y|^2+|x-y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x-y|^3|x-\bar{y}|^3(|x-y|+|x-\bar{y}|)} \theta(y)\,\mathrm{d}y \right| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}.$$

(iii) Suppose $y \in [0, x_1/2] \times [0, 2x_1]$. This implies

$$\tfrac{1}{2}x_1 \le |x-y| \le 4x_1, \quad \tfrac{1}{2}x_1 \le |x-\bar{y}| \le 4x_1,$$

we can see that

$$\left| \int_{[0,x_1/2]\times[0,2x_1]} \frac{y_2(x_1-y_1)(|x-y|^2+|x-y|\,|x-\bar{y}|+|x-\bar{y}|^2)}{|x-y|^3|x-\bar{y}|^3(|x-y|+|x-\bar{y}|)} \theta(y)\,\mathrm{d}y \right| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}.$$

(iv) Suppose $y \in [x_1/2, 2x_1] \times [0, 2x_1]$. Recalling the notation K_1, K_2, and K_3 in the proof of Lemma 3.2, we claim

$$|K_1| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right) + C\|\nabla\theta\|_{L^\infty(R(x))}\left(1+\log\frac{x_1}{x_2}\right)^2 \tag{3-19}$$

and

$$|K_2| + |K_3| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}. \tag{3-20}$$

As in (3-12), we have from the mean value theorem that

$$|K_2| \le Cx_1^{-1}\|\theta\|_{L^\infty([0,2x_1]^2)} \le C\|\nabla\theta\|_{L^\infty([0,1]^2)},$$

and similarly,

$$|K_3| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}.$$

Hence, (3-20) follows. We recall $K_1 = K_{11} + K_{12}$, where

$$K_{11} = \int_{[x_1/2,2x_1]\times[0,2x_2]} \frac{-y_2\partial_1\theta(y)}{|x-y|\,|x-\bar{y}|(|x-y|+|x-\bar{y}|)}\,\mathrm{d}y,$$
$$K_{12} = \int_{[x_1/2,2x_1]\times[2x_2,2x_1]} \frac{-y_2\partial_1\theta(y)}{|x-y|\,|x-\bar{y}|(|x-y|+|x-\bar{y}|)}\,\mathrm{d}y.$$

From Hölder's inequality and (3-13), we can deduce that

$$|K_{11}| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)} \int_0^{2x_2} \frac{1}{x_2+y_2}\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\mathrm{d}y_2,$$
$$|K_{12}| \le C\|\nabla\theta\|_{L^\infty(R(x))} \int_{2x_2}^{2x_1} \frac{1}{x_2+y_2}\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\mathrm{d}y_2.$$

Since

$$\int_0^{2x_2} \frac{1}{x_2+y_2} \log\left(1+\frac{x_1}{|x_2-y_2|}\right) \mathrm{d}y_2 \le \frac{1}{x_2}\int_0^{2x_2} \log\frac{2x_1}{|x_2-y_2|}\,\mathrm{d}y_2 \le C\left(1+\log\frac{x_1}{x_2}\right),$$
$$\int_{2x_2}^{2x_1} \frac{1}{x_2+y_2} \log\left(1+\frac{x_1}{|x_2-y_2|}\right) \mathrm{d}y_2 \le \int_{2x_2}^{2x_1} \frac{1}{y_2-x_2} \log\frac{3x_1}{y_2-x_2}\,\mathrm{d}y_2 \le C\left(1+\log\frac{x_1}{x_2}\right)^2,$$

it follows

$$|K_{11}| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right), \quad |K_{12}| \le C\|\nabla\theta\|_{L^\infty(R(x))}\left(1+\log\frac{x_1}{x_2}\right)^2.$$

This shows (3-19). Combining the estimates, we arrive at

$$\left|\frac{I_3(0)}{x_2} + 6\int_{Q(x)} \frac{y_1y_2}{|y|^5}\theta(y)\,\mathrm{d}y\right| \le C\|\nabla\theta\|_{L^\infty([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right) + C\|\nabla\theta\|_{L^\infty(R(x))}\left(1+\log\frac{x_1}{x_2}\right)^2.$$

Using (3-5), we can estimate

$$I_4(0) = -4x_2\int_{[0,1]^2} \frac{y_2(x_1+y_1)(|x+y|^2+|x+y|\,|x-\tilde{y}|+|x-\tilde{y}|^2)}{|x+y|^3|x-\tilde{y}|^3(|x+y|+|x-\tilde{y}|)}\theta(y)\,\mathrm{d}y,$$

similarly to $I_3(0)$. Hence we have (3-15), and this completes the proof. □

4. Nonexistence

In this section, we prove Theorem B. We begin with a simple uniqueness result which in particular guarantees that the hypothetical solution in $L^\infty_t H^2$ satisfies the same symmetries with the initial data.

Proposition 4.1. *Given $\theta_0 \in H^2$ and $T > 0$, there exists at most one solution to (SQG) belonging to $L^\infty([0,T]; H^2)$ with initial data θ_0.*

Proof. The proof can be given by simply adapting the inequalities derived in [Yudovich 1963; 1995]. This statement can be found in [Azzam and Bedrossian 2015] as well. □

Proof of Theorem B. For convenience, we shall divide the proof into several parts.

Part 1: velocity and flow map: an $L^\infty_t H^2$-solution is Lagrangian. Assume that we are given a solution to (SQG) satisfying

$$\sup_{t\in[0,T]} \|\theta(t,\cdot)\|_{H^2} \le M.$$

Then, by the Sobolev embedding, $u = \nabla^\perp(-\Delta)^{-1/2}\theta$ satisfies

$$\sup_{t\in[0,T]} (\|\nabla u(t,\cdot)\|_{\mathrm{BMO}} + \|u(t,\cdot)\|_{W^{1,1}}) \le C \sup_{t\in[0,T]} \|u(t,\cdot)\|_{H^2} \le CM,$$

with some absolute constant $C > 0$. In particular, u is log-Lipschitz: for any $x, y \in \mathbb{T}^2$, we have

$$\sup_{t\in[0,T]} |u(t,x)-u(t,y)| \le CM|x-y|\ln\left(10+\frac{1}{|x-y|}\right). \tag{4-1}$$

On the time interval $[0, T]$, we consider the flow map $\Phi(t, \cdot) : \mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$\begin{cases} \dfrac{d}{dt}\Phi(t, x) = u(t, \Phi(t, x)), \\ \Phi(0, x) = x. \end{cases} \tag{4-2}$$

It is well known that under the estimate (4-1), there is a unique solution to the ODE (4-2) for any $x \in \mathbb{T}^2$ [Majda and Bertozzi 2002; Marchioro and Pulvirenti 1994]. The solution Φ satisfies the estimate

$$|x - y|^{\exp(CMt)} \le |\Phi(t, x) - \Phi(t, y)| \le |x - y|^{\exp(-CMt)} \tag{4-3}$$

for some absolute constant $C > 0$, uniformly in $x, y \in \mathbb{T}^2$ satisfying $|x - y| < \frac{1}{2}$ and $t \in [0, T]$. We have the representation

$$\theta(t, \Phi(t, x)) = \theta_0(x).$$

The estimate (4-3) shows that, for each $t \in [0, T]$, $\Phi(t, \cdot)$ is a Hölder continuous homeomorphism $\mathbb{T}^2 \to \mathbb{T}^2$, and we denote the inverse map by Φ_t^{-1}. Then, with this notation, we have

$$\theta(t, x) = \theta_0(\Phi_t^{-1}(x)).$$

The inverse map Φ_t^{-1} is again Hölder continuous. As an immediate consequence, we have that if θ_0 is an odd function with respect to both axes and satisfies

$$\operatorname{supp}(\theta_0) \cup \{x : x_1 = 0 \text{ or } x_2 = 0\} \subset \{(0, 0)\},$$

then the same properties are satisfied by $\theta(t, \cdot)$, as long as $\theta \in L^\infty([0, t]; H^2)$. Indeed, the uniqueness assertion from Proposition 4.1 guarantees that $\theta(t, \cdot)$ is odd with respect to both axes. Furthermore, Hölder continuity of the flow map and its inverse ensures that $\theta(t, \cdot)$ vanishes near the axes, possibly except at the origin. Therefore, the last assumption in Lemma 3.2 is satisfied.

Part 2: choice of initial data. We fix some smooth bump function $\phi : \mathbb{R}^2 \to \mathbb{R}_{\ge 0}$ satisfying the following properties:

- ϕ is C^∞-smooth and radial.
- ϕ is supported in $B_0\left(\frac{1}{8}\right)$ and $\phi = 1$ in $B_0\left(\frac{1}{32}\right)$.

Then, we define

$$\theta_0 := \sum_{n=n_0}^{\infty} n^{-\alpha} \theta_{0,\mathrm{loc}}^{(n)} \tag{4-4}$$

for some $\frac{1}{2} < \alpha < \frac{3}{4}$, where

$$\theta_{0,\mathrm{loc}}^{(n)}(x) := 4^{-n} \phi(4^n (x_1 - 4^{-n-1}, x_2 - 4^{-n-2}))$$

for $x \in [0, 1]^2$. The precise value of α will be determined later, but for now let us just mention that it will be taken slightly larger than $\frac{1}{2}$. Next, let us extend each of $\theta_{0,\mathrm{loc}}^{(n)}$ (and similarly θ_0) to $\mathbb{T}^2$ as an odd function with respect to both axes. Note that by taking $n_0 \ge 1$ sufficiently large in a way depending only on $\epsilon > 0$, we can guarantee that

$$\|\theta_0\|_{H^2 \cap W^{1,\infty}(\mathbb{T}^2)} < \epsilon.$$

Towards a contradiction, we shall assume that there exists $M > 0$ and $T > 0$ such that

$$\sup_{t\in[0,T]} \|\theta(t)\|_{\dot{H}^2(\mathbb{T}^2)} \le M. \tag{4-5}$$

For simplicity, we shall assume that $M \ge 1$. Observe that the assumptions in the key lemma (Lemma 3.2) are satisfied by θ_0. Recalling the discussion above, we have that $\theta(t,\cdot)$ is odd with respect to both axes and vanishes near the axes, except at the origin.

Part 3: preliminary bounds on the solution. Let us remark in advance that in the following proof we shall take $T > 0$ to be smaller, if necessary, to satisfy $T \le c/M$ for some small absolute constant $c > 0$. We shall begin with a simple result:

Lemma 4.2. *Assume that θ is a solution satisfying (4-5) with initial data (4-4). Then, by redefining T to satisfy $T \le c/M$ if necessary, we have*

$$\theta(t, y) = 0, \quad 0 \le y_1 \le y_2,\ 0 \le t \le T.$$

Proof. Since $\theta(t,\Phi(t,x)) = \theta_0(x)$, to prove the claim, it suffices to show that, for $x \in \operatorname{supp}(\theta_0)\backslash\{(0,0)\}$, $\Phi_2(t,x) \le \Phi_1(t,x)$ for $0 \le t \le T$. Let us fix some $x \in \operatorname{supp}(\theta_0)\backslash\{(0,0)\}$. Then, from the choice of initial data, we have $2x_2 \le x_1$. From continuity in time of the flow map, there exists some $0 < T^* \le T$ such that $\Phi_2(t,x) < \Phi_1(t,x)$ for $0 \le t < T^*$. Then, on this time interval, key lemma is applicable for $\Phi(t,x)$ and we compute

$$\begin{aligned}\frac{d}{dt}\left(\frac{\Phi_2(t,x)}{\Phi_1(t,x)}\right) &= \frac{\Phi_2(t,x)}{\Phi_1(t,x)}\left(\frac{u_2(t,\Phi(t,x))}{\Phi_2(t,x)} - \frac{u_1(t,\Phi(t,x))}{\Phi_1(t,x)}\right)\\ &\le C\frac{\Phi_2(t,x)}{\Phi_1(t,x)}(|B_1(\Phi(t,x))| + |B_2(\Phi(t,x))| + |B_3(\Phi(t,x))|) \le CM\frac{\Phi_2(t,x)}{\Phi_1(t,x)}.\end{aligned}$$

Therefore, we actually obtain

$$\frac{\Phi_2(t,x)}{\Phi_1(t,x)} \le \frac{1}{2}\exp(CMt) < \frac{3}{4}$$

on $t \in [0, T^*]$, as long as $T^* \le c/M$ for $c > 0$ depending only on C. This bootstrap procedure allows us to get $\Phi_2/\Phi_1 < \frac{3}{4}$ uniformly in $x \in \operatorname{supp}(\theta_0)\backslash\{(0,0)\}$ by the time $\min\{T, c/M\} = T$. □

The above lemma guarantees that on $[0, T]$, the key lemma is applicable to points in $\operatorname{supp}(\theta(t,\cdot))$. Next, let us set $\Omega_n := \operatorname{supp}(\theta^{(n)}_{0,\mathrm{loc}}) \cap \{x \in \mathbb{T}^2 : x_1 > x_2 > 0\}$ and prove that, by reducing $c > 0$ if necessary, the bubbles $\{\Phi(t,\Omega_n)\}_{n\ge n_0}$ are "well-ordered" with respect to the x_1-axis for $t \in [0, T]$ with $T \le c/M$.

Claim I. *We have*

$$\sup_{x\in\Omega_n} \Phi_1(t,x) \le 2 \inf_{x\in\Omega_n} \Phi_1(t,x) \quad \text{and} \quad 2 \sup_{x\in\Omega_{n+1}} \Phi_1(t,x) \le \inf_{x\in\Omega_n} \Phi_1(t,x) \tag{4-6}$$

uniformly for all $n \ge n_0$ and $t \in [0, T]$, by reducing T to satisfy $T \le c/(1+M)$ for some small absolute constant $c > 0$.

For simplicity we let

$$\widehat{\Phi}^n_j(t) := \sup_{x\in\Omega_n} \Phi_j(t,x)$$

for $j = 1, 2$. We can prove the Claim I inductively in n, using the key lemma, which gives

$$\left| \frac{u_1(t,x)}{x_1} - 12 \int_{Q(x)} \frac{y_1 y_2}{|y|^5} \theta(t,y)\, \mathrm{d}y \right| \le CM.$$

In the proof, we shall take $T > 0$ smaller several times, but in a way which is independent of n. To begin with, for $x \in \Omega_{n_0}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \Phi_1(t,x) \ge -CM.$$

Thus,

$$\log \Phi_1(t,x) - \log x_1 \ge -CMt.$$

We also have

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \widehat{\Phi}_1^{n_0}(t) - \frac{\mathrm{d}}{\mathrm{d}t} \log \Phi_1(t,x) \le 2CM,$$

and thus,

$$\log \widehat{\Phi}_1^{n_0}(t) - \log \Phi_1(t,x) \le 2CMt + (\log \hat{x}_1^{n_0} - \log x_1).$$

Since $\hat{x}_1^{n_0}/x_1 < \frac{3}{2}$, we can take $T > 0$ sufficiently small such that

$$2CMT + (\log \hat{x}_1^{n_0} - \log x_1) \le \log 2,$$

which implies that

$$\widehat{\Phi}_1^{n_0}(t) \le 2 \inf_{x \in \Omega_{n_0}} \Phi_1(t,x)$$

for all $t \in [0, T]$. Indeed, it suffices to take $T = c/(1+M)$ with a small absolute constant $c > 0$. To show

$$2\widehat{\Phi}_1^{n_0+1}(t) \le \inf_{x \in \Omega_{n_0}} \Phi_1(t,x) \tag{4-7}$$

for all $t \in [0, T]$, we use the notation

$$\widehat{\Psi}_1^n(t) := \sup_{n \le m} \widehat{\Phi}_1^m(t).$$

Then, for $x \in \Omega_{n_0}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \Phi_1(t,x) - \frac{\mathrm{d}}{\mathrm{d}t} \log \widehat{\Psi}_1^{n_0+1}(t) \ge -12 \int_{\Omega_{n_0}} \frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5} \theta_0(y)\, \mathrm{d}y - 2CM.$$

From the above estimates, it follows

$$\int_{\Omega_{n_0}} \frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5} \theta_0(y)\, \mathrm{d}y \le \left(\sup_{x \in \Omega_{n_0}} \frac{x_1}{\Phi_1(t,x)} \right)^3 \int_{\Omega_{n_0}} \frac{\theta_0(y)}{y_1^3}\, \mathrm{d}y \le C_0 e^{3CMT}.$$

Using it, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \Phi_1(t,x) - \frac{\mathrm{d}}{\mathrm{d}t} \log \widehat{\Psi}_1^{n_0+1}(t) \ge -12 C_0 e^{3CMT} - 2CM$$

and

$$\log \Phi_1(t,x) - \log \widehat{\Psi}_1^{n_0+1}(t) \ge -12 C_0 e^{3CMT} t - 2CMt + (\log x_1 - \log \hat{x}_1^{n_0+1}).$$

Since $x_1/\hat{x}_1^{n_0+1} > 2$, we can take $T > 0$ sufficiently small such that

$$-12 C_0 e^{3CMT} T - 2CMT + (\log x_1 - \log \hat{x}_1^{n_0+1}) \ge \log 2.$$

Hence, with $\widehat{\Psi}_1^{n_0+1}(t) \geq \widehat{\Phi}_1^{n_0+1}(t)$, we can obtain (4-7). Now let $x \in \Omega_{n_0+1}$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \Phi_1(t,x) \geq -CM.$$

Thus,

$$\log \Phi_1(t,x) - \log x_1 \geq -CMt.$$

Since $\hat{x}_1^{n_0+1}/x_1$ takes the same value as in the previous case, we see that

$$2CMT + (\log \hat{x}_1^{n_0+1} - \log x_1) \leq \log 2,$$

and therefore, we have

$$\widehat{\Phi}_1^{n_0+1}(t) \leq 2 \inf_{x \in \Omega_{n_0+1}} \Phi_1(t,x)$$

for all $t \in [0,T]$. Note that by (4-7),

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \Phi_1(t,x) - \frac{\mathrm{d}}{\mathrm{d}t} \log \widehat{\Psi}_1^{n_0+2}(t) \geq -12 \int_{\Omega_{n_0+1}} \frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5} \theta_0(y)\,\mathrm{d}y - 2CM.$$

With the above estimates, we have

$$\int_{\Omega_{n_0+1}} \frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5} \theta_0(y)\,\mathrm{d}y \leq \left(\sup_{x \in \Omega_{n_0+1}} \frac{x_1}{\Phi_1(t,x)} \right)^{n_0+2} \int_{\Omega_{n_0+1}} \frac{\theta_0(y)}{y_1^{n_0+2}}\,\mathrm{d}y \leq C_0 e^{3CMT}.$$

Using it, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \Phi_1(t,x) - \frac{\mathrm{d}}{\mathrm{d}t} \log \widehat{\Psi}_1^{n_0+2}(t) \geq -12C_0 e^{3CMT} - 2CM$$

and

$$\log \Phi_1(t,x) - \log \widehat{\Psi}_1^{n_0+2}(t) \geq -12C_0 e^{3CMT} t - 2CMt + (\log x_1 - \log \hat{x}_1^{n_0+2}).$$

Since $x_1/\hat{x}_1^{n_0+2} > 2$ is the same value as in the previous case, it follows

$$-12C_0 e^{3CMT} T - 2CMT + (\log x_1 - \log \hat{x}_1^{n_0+2}) \geq \log 2$$

and

$$2\widehat{\Phi}_1^{n_0+2}(t) \leq \inf_{x \in \Omega_{n_0+1}} \Phi_1(t,x)$$

for all $t \in [0,T]$. Repeating this argument, one can finish the proof of Claim I. □

Claim II. *There exists $T > 0$ and $C > 0$ such that*

$$\log \frac{\widehat{\Phi}_2^n(T)}{\hat{x}_2^n} \leq -10 \sum_{n_0 \leq j \leq n-1} \int_0^T \int_{\Omega_j} \frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5} \theta_0(y)\,\mathrm{d}y\,\mathrm{d}t + CMT$$

uniformly for all $n > n_0$.

Recall that

$$\frac{u_2(x)}{x_2} \leq -12 \int_{Q(x)} \frac{y_1 y_2}{|y|^5} \theta(t,y)\,\mathrm{d}y + CM\left(\log \frac{x_1}{x_2}\right)$$

if $\theta(y) = 0$ for y satisfying $x_1/2 \le y_1 \le 2x_1$ and $2x_2 \le y_2 \le 1$. According to the order of the bubbles, for $x \in \Omega_n$, we have

$$\int_{Q(\Phi(t,x))} \frac{y_1 y_2}{|y|^5}\theta(t,y)\,\mathrm{d}y = \sum_{n_0 \le j \le n-1} \int_{\Omega_j} \frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5}\theta_0(y)\,\mathrm{d}y.$$

And note that

$$\sup_{2\Phi_2(t,x) \ge \widehat{\Phi}_2^n(t)} CM\left(\log\frac{\Phi_1(t,x)}{\Phi_2(t,x)}\right) \le CM\left(\log\frac{2\widehat{\Phi}_1^n(t)}{\widehat{\Phi}_2^n(t)}\right).$$

Thus, we can see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\log\frac{\widehat{\Phi}_2^n(t)}{\hat{x}_2^n} &\le -12\sum_{n_0 \le j \le n-1}\int_{\Omega_j}\frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5}\theta_0(y)\,\mathrm{d}y + CM\log\frac{2\widehat{\Phi}_1^n(t)}{\widehat{\Phi}_2^n(t)}\\ &\le -12\sum_{n_0 \le j \le n-1}\int_{\Omega_j}\frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5}\theta_0(y)\,\mathrm{d}y + CM\log\frac{\widehat{\Phi}_1^n(t)}{\hat{x}_1^n} - CM\log\frac{\widehat{\Phi}_2^n(t)}{\hat{x}_2^n} + CM\end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{CMt}\log\frac{\widehat{\Phi}_2^n(t)}{\hat{x}_2^n}\right) \le e^{CMt}\left(-12\sum_{n_0 \le j \le n-1}\int_{\Omega_j}\frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5}\theta_0(y)\,\mathrm{d}y + CM\left(\log\frac{\widehat{\Phi}_1^n(t)}{\hat{x}_1^n} + 1\right)\right).$$

It suffices to bound the time integral of the right-hand side. Note that from

$$\log\frac{\widehat{\Phi}_1^n(t)}{\hat{x}_1^n} \le 12\sum_{n_0 \le j \le n-1}\int_0^t\int_{\Omega_j}\frac{\Phi_1(\tau,y)\Phi_2(\tau,y)}{|\Phi(\tau,y)|^5}\theta_0(y)\,\mathrm{d}y\,\mathrm{d}\tau + CMt,$$

we obtain

$$CM\int_0^t \log\frac{\widehat{\Phi}_1^n(\tau)}{\hat{x}_1^n}\,\mathrm{d}\tau \le 12CMt\sum_{n_0 \le j \le n-1}\int_0^t\int_{\Omega_j}\frac{\Phi_1(\tau,y)\Phi_2(\tau,y)}{|\Phi(\tau,y)|^5}\theta_0(y)\,\mathrm{d}y\,\mathrm{d}\tau + (CMt)^2.$$

Therefore, using Grönwall's inequality on the quantity $\log(\widehat{\Phi}_2^n(t)/\hat{x}_2^n)$ and taking $T > 0$ small depending only on CM, we can complete the proof of Claim II. □

Part 4: almost invariant timescales. We shall write $\Phi(t,\Omega_n) \sim \Omega_n$ if

$$\mathrm{supp}(\Phi(t,\Omega_n)) \subset B((4^{-n-1}, 4^{-n-2}), 4^{-n-1}).$$

Here, $B((4^{-n-1}, 4^{-n-2}), 4^{-n-1})$ denotes the ball of radius 4^{-n-1} centered at $(4^{-n-1}, 4^{-n-2})$. Recall from the definition of initial data that

$$\Omega_n = B((4^{-n-1}, 4^{-n-2}), 2^{-1}4^{-n-1}).$$

An immediate consequence of $\Phi(t,\Omega_n) \sim \Omega_n$ is that once we define

$$I_n(t) = \int_{\Omega_n}\frac{\Phi_1(t,y)\Phi_2(t,y)}{|\Phi(t,y)|^5}\theta_0(y)\,\mathrm{d}y,$$

we have

$$I_n(t) \ge a_0 I_n(0)$$

for some absolute constant a_0. The following claim gives the sharp bound on the "almost invariant" timescale for each bubble.

Claim III. *For all $n \geq n_0$, we have*

$$\Phi(t, \Omega_n) \sim \Omega_n \quad \textit{for } 0 \leq t \leq \min\left\{T, \frac{c}{M + \sum_{j=n_0}^{n-1} j^{-\alpha}}\right\} =: T_n,$$

with some constants $c, C > 0$ independent of n.

This claim can be proved easily with an induction in n. In the base case $n = n_0$, we simply note that, for $x \in \Phi(t, \Omega_{n_0})$,

$$\left|\frac{u_j(t, x)}{x_j}\right| \leq CM$$

from which the claim follows in this case. Assume that Claim III holds for all $n < n_0 + k$ for some $k \geq 1$. Note that using the key lemma and the induction hypothesis, we have for $x \in \Phi(t, \Omega_{n_0+k})$ that

$$\left|\frac{u_j(t, x)}{x_j}\right| \leq C\left(M + \sum_{\ell=0}^{k-1} I_{n_0+\ell}\right), \quad 0 \leq t \leq T_{n_0+k-1}.$$

A simple application of Gronwall's inequality gives Claim III. □

We have proven that the n-th bubble remains almost invariant for T_n, which is bounded from below by

$$T_n \geq \frac{c_0}{\sum_{j=n_0}^{n-1} j^{-\alpha}} \geq \frac{(1-\alpha)c_0}{n^{1-\alpha}}$$

for all $n \geq N$ with some large N depending only on M, T. Now, we observe that

$$\int_0^{T_n} I_n(t)\, \mathrm{d}t \gtrsim T_n I_n(0) \gtrsim \frac{1}{n},$$

with constants independent of n, recalling that $I_n(0) \gtrsim n^{-\alpha}$. (We shall take α close to $\frac{1}{2}$.) Hence, summation gives

$$\sum_{k=\ell}^{n} I_k(0) T_k \geq c_0\left(\frac{1}{\ell} + \cdots + \frac{1}{n}\right) \geq \log\left(\frac{n}{\ell}\right)^{c_0} \tag{4-8}$$

for some absolute constant $c_0 > 0$, as long as $\ell > N$.

Part 5: norm inflation and conclusion the proof. We are now in a position to complete the proof. For each $\ell > N$ and $n \gg \ell$ (so that $\log(n/\ell)^{c_0} \gg M$), we can bound for $x \in \Omega_n$

$$\log \frac{\widehat{\Phi}_2^n(T_\ell)}{\hat{x}_2^n} \leq CM - 10 \sum_{k=\ell}^{n} I_k(0) T_k \leq \log\left(\frac{n}{\ell}\right)^{-c_0}.$$

In other words, we have the growth

$$\frac{\hat{x}_2^n}{\widehat{\Phi}_2^n(T_\ell)} \geq \left(\frac{n}{\ell}\right)^{c_0}. \tag{4-9}$$

Now, we can write the solution in the form

$$\theta = \sum_{n=n_0}^{\infty} n^{-\alpha} \theta^{(n)}, \quad \theta^{(n)}(t, \Phi(t, x)) = \theta_{0,\mathrm{loc}}^{(n)}(x),$$

so that the support of $\theta^{(n)}$ is disjoint from each other. We now take $t = T_\ell$. Since $\theta^{(n)}(T_\ell, \cdot) = 1$ in a region of area $\gtrsim 4^{-2n}$ and $\theta^{(n')} = 0$ for $n' \neq n$ in that region, with Hardy's inequality and (4-9), we obtain that

$$\|\theta^{(n)}(T_\ell)\|_{\dot{H}^2}^2 \gtrsim \left(\frac{n}{\ell}\right)^{4c_0}.$$

This estimate holds for all sufficiently large n. Then

$$\|\theta(T_\ell)\|_{H^2}^2 \geq \sum_{n \geq n_0} n^{-2\alpha} \|\theta^{(n)}(T_\ell)\|_{\dot{H}^2}^2 \gtrsim \ell^{-4c_0} \sum_{n \gg \ell} n^{4c_0 - 2\alpha}.$$

In the last inequality, since $c_0 > 0$ is an absolute constant, and we could have chosen $\alpha = \frac{1}{2} + c_0$. This gives a contradiction to $\|\theta(T_\ell)\|_{H^2} < \infty$ since $\sum_{n \gg \ell} n^{-1+c_0} = \infty$. $\square$

Remark 4.3. The nonexistence of the solution in $W^{1,\infty}$ is obtained similarly. We define the initial data θ_0 with (4-4) for some $0 < \alpha < \frac{1}{4}$ and repeat the above process with Lemma 3.4 instead of Lemma 3.2. Then we can have an absolute constant $c_0' > 0$ with (4-9). Since this implies

$$\|\theta^{(n)}(T_\ell)\|_{\dot{W}^{1,\infty}} \gtrsim \left(\frac{n}{\ell}\right)^{c_0'},$$

it follows that

$$\|\theta(T_\ell)\|_{W^{1,\infty}} \geq n^{-\alpha} \|\theta^{(n)}(T_\ell)\|_{\dot{W}^{1,\infty}} \gtrsim \ell^{-c_0'} n^{c_0' - \alpha}.$$

Therefore, taking $\alpha = c_0'/2$, we complete the proof.

5. Norm inflation for smooth data

We establish Theorem A in this section, by proving a quantitative norm inflation result for data obtained by truncating the data used in the proof of Theorem B.

Proposition 5.1 (quantitative norm inflation). *We consider the C^∞-smooth initial data*

$$\theta_0^{(N)} := \sum_{n=n_0}^{N} n^{-\alpha} \theta_{0,\mathrm{loc}}^{(n)}, \tag{5-1}$$

where ϕ, α, n_0 are the same as in (4-4). Then, there exists $N_0 \geq 1$ depending only on ϕ, n_0 such that, for all $N \geq N_0$, the unique local in time C^∞-solution $\theta^{(N)}$ to (SQG) with initial data $\theta_0^{(N)}$ exists on the time interval $[0, T^*]$ for some $0 < T^* \leq T_N$ and satisfies

$$\|\theta_0^{(N)}\|_{H^2 \cap W^{1,\infty}} \leq \epsilon, \qquad \sup_{t \in [0, T^*]} \|\theta^{(N)}(t)\|_{H^2} > M_N, \tag{5-2}$$

where

$$M_N := \frac{c_0}{2} \ln N, \qquad T_N := \frac{1}{M_N \ln M_N}, \tag{5-3}$$

with $c_0 > 0$ from (4-8).

Proof. We shall establish the proposition with a contradiction argument: let $0 < T^* \leq +\infty$ be the lifespan of the smooth solution associated with the initial data $\theta_0^{(N)}$ and assume that

$$\|\theta^{(N)}\|_{L^\infty([0, \min\{T^*, T_N\}]; H^2)} \leq M_N.$$

Under this contradiction hypothesis, we can actually prove that $T^* > T_N$, so that

$$\|\theta^{(N)}\|_{L^\infty([0,T_N];H^2)} \le M_N. \tag{5-4}$$

This is simply because the H^2-norm gives a blow-up criterion. To illustrate this point, we estimate the H^3-norm of $\theta := \theta^{(N)}$ on $[0, T_N]$:[3] from the equation for $\Delta\theta$

$$\partial_t \Delta\theta + u\cdot\nabla\Delta\theta + \Delta u\cdot\nabla\theta + 2\sum_{i=1,2}\partial_i u\cdot\nabla\partial_i\theta = 0,$$

we estimate for $j = 1, 2$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_j\Delta\theta\|_{L^2}^2 \le C(\|\nabla u\|_{L^\infty} + \|\nabla\theta\|_{L^\infty})\|\partial_j\Delta\theta\|_{L^2}^2 + C\|\theta\|_{H^2}\|\theta\|_{H^3}^2.$$

Here, we have used L^4 boundedness of the Riesz operator $\theta \mapsto u$ to bound

$$\|\nabla^2 u\|_{L^4} + \|\nabla^2\theta\|_{L^4} \le C\|\theta\|_{H^2}^{1/2}\|\theta\|_{H^3}^{1/2}.$$

Next, we use the logarithmic Sobolev inequality

$$\|\nabla\theta\|_{L^\infty} \le C\|\theta\|_{H^2}\log\left(10 + \frac{\|\theta\|_{H^3}}{\|\theta\|_{H^2}}\right)$$

and

$$\|\nabla u\|_{L^\infty} \le C\|u\|_{H^2}\log\left(10 + \frac{\|u\|_{H^3}}{\|u\|_{H^2}}\right) \le C\|\theta\|_{H^2}\log\left(10 + \frac{\|\theta\|_{H^3}}{\|\theta\|_{H^2}}\right)$$

(we have used the *lower bound* $\|u\|_{H^2} \ge C\|\theta\|_{H^2}$). Lastly, using $\|\theta\|_{H^2} \le M_N$, we may deduce the a priori estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{H^3}^2 \le CM_N\log(10 + \|\theta\|_{H^3})\|\theta\|_{H^3}^2,$$

which shows that the H^3-norm of θ must remain finite up to $t = T_N$. Higher norms of θ can be similarly controlled, so that the solution θ remains C^∞-smooth up to $t = T_N$.

In the following argument, $N_0 \gg n_0$ will be taken to be sufficiently large (but in a way depending only on a few absolute constants) whenever it becomes necessary. Recall that we are assuming $N \ge N_0$. The following argument is mainly a repetition of the proof of Theorem B above. For convenience, let us fix

$$\ell_N := M_N^3. \tag{5-5}$$

Then, note from the definition of M_N in (5-3) that $n_0 \ll \ell_N \ll N$. Here and in the following, we write $A \ll B$ if $A/B \to 0$ as $N \to \infty$, where A and B are some positive expressions involving N.

Observe that the solution θ defined on $[0, T_N]$ satisfies the properties stated in Lemma 4.2 and Claims I, II, III on the *entire* time interval $[0, T_N]$ (by taking N_0 larger if necessary), simply because we have

$$T_N \ll \frac{1}{M_N}$$

[3]For simplicity, from now on we shall refrain from writing out the dependence of the solution θ in N.

from our choice of T_N in (5-3). As in the above, we write the solution in the form

$$\theta = \sum_{n=n_0}^{N} n^{-\alpha}\theta^{(n)}, \quad \theta^{(n)}(t, \Phi(t,x)) = \theta^{(n)}_{0,\mathrm{loc}}(x),$$

and $\theta^{(n)}$ will be referred to as the n-th bubble. Then, for any $\ell_N \le k \le N$, we have that the invariant timescale T_k for the k-th bubble satisfies

$$T_k \le T_N \quad \text{and} \quad T_k \gtrsim \frac{1}{M_N + \sum_{j=n_0}^{k-1} j^{-\alpha}} \gtrsim k^{\alpha-1}.$$

We have used that α is close to $\frac{1}{2}$. Now we consider the values of n satisfying

$$n \ge C\ell_N \exp(c_0^{-1}M_N) \tag{5-6}$$

for a sufficiently large absolute constant $C > 0$. Then, at $t = T_{\ell_N}$, we obtain much as before

$$\|\theta^{(n)}(T_{\ell_N})\|^2_{\dot H^2} \gtrsim \left(\frac{n}{\ell_N}\right)^{2c_0}$$

whenever $n \le N$ satisfies (5-6). Hence

$$\|\theta(T_{\ell_N})\|^2_{H^2} \gtrsim \sum_{n=1+\lfloor C\ell_N \exp(c_0^{-1}M_N)\rfloor}^{N} n^{-2\alpha}\left(\frac{n}{\ell_N}\right)^{2c_0} \gtrsim \ell_N^{-2c_0} N^{1-2\alpha+2c_0} \gg M_N^2,$$

recalling the definitions of M_N and ℓ_N. We have used that $N \gg \ell_N \exp(c_0^{-1}M_N)$ to derive the last inequality. In particular, for all sufficiently large N, we obtain

$$\|\theta(T_{\ell_N})\|_{H^2} > M_N,$$

which is a contradiction. $\square$

Appendix: Key lemma for generalized SQG

We provide a version of the "key lemma" for generalized SQG equations (1-3) with $1 < \alpha < 2$.

Lemma A.1. *Let θ satisfy the assumptions in Lemma 3.2, and let x satisfy $|x| < \frac{1}{4}$ and $x_1 > x_2 > 0$. Then, $u = \nabla^\perp \Lambda^{-\alpha}\theta$ satisfies*

$$\left|\frac{u_1(x)}{x_1} - 4(4-\alpha)\int_{Q(2x)} \frac{y_1y_2}{|y|^{6-\alpha}}\theta(y)\,dy\right| \le B_7(x) \tag{A-1}$$

and

$$\left|\frac{u_2(x)}{x_2} + 4(4-\alpha)\int_{Q(2x)} \frac{y_1y_2}{|y|^{6-\alpha}}\theta(y)\,dy\right| \le \left(1+\log\frac{x_1}{x_2}\right)B_8(x) + \left(1+\log\frac{x_1}{x_2}\right)^{(5-2\alpha)/2} B_9(x), \tag{A-2}$$

where B_7, B_8, B_9 satisfy

$$|B_7(x)| + |B_8(x)| \le C(\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)} + \|\theta\|_{L^\infty([0,1]^2)})$$

and

$$|B_9(x)| \le C(\|\nabla^{3-\alpha}\theta\|_{L^2(S(x))} + \|y_2^{-1}\partial_1\theta(y)\|_{L^2(S(x))}), \quad S(x) := [x_1/2, 4x_1] \times [4x_2, 1].$$

Our proof will be brief, since the structure of the proof is similar to the SQG case. Unfortunately, this argument cannot be specialized to give the lemma in the SQG case, since the case $\alpha = 1$ is critical (being an integer) in some sense.

Proof. We fix a point $x = (x_1, x_2)$ satisfying the assumptions of the lemma. We write

$$u(x) = \sum_{n\in\mathbb{Z}^2}\int_{[0,1]^2}\biggl(\frac{(x-(y+2n))^\perp}{|x-(y+2n)|^{4-\alpha}} - \frac{(x-(\tilde{y}+2n))^\perp}{|x-(\tilde{y}+2n)|^{4-\alpha}} + \frac{(x-(-y+2n))^\perp}{|x-(-y+2n)|^{4-\alpha}} - \frac{(x-(\bar{y}+2n))^\perp}{|x-(\bar{y}+2n)|^{4-\alpha}}\biggr)\theta(y)\,\mathrm{d}y.$$

We estimate u_1 first. We introduce

$$I_1(n) := -\int_{[0,1]^2}\biggl(\frac{x_2-(y_2+2n_2)}{|x-(y+2n)|^{4-\alpha}} - \frac{x_2-(y_2+2n_2)}{|x-(\tilde{y}+2n)|^{4-\alpha}}\biggr)\theta(y)\,\mathrm{d}y,$$
$$I_2(n) := -\int_{[0,1]^2}\biggl(\frac{x_2-(-y_2+2n_2)}{|x-(-y+2n)|^{4-\alpha}} - \frac{x_2-(-y_2+2n_2)}{|x-(\bar{y}+2n)|^{4-\alpha}}\biggr)\theta(y)\,\mathrm{d}y$$

and we see

$$u_1(x) = \sum_{n\in\mathbb{Z}^2}(I_1(n)+I_2(n)).$$

In the case of $n \neq 0$, we have

$$|I_1(n)+I_1(\tilde{n})| + |I_2(n)+I_2(\tilde{n})| \le O(|n|^{-5+\alpha})\|\theta\|_{L^\infty([0,1]^2)}x_1;$$

hence,

$$\biggl|\sum_{n\in\mathbb{Z}^2\setminus\{0\}}(I_1(n)+I_2(n))\biggr| \le Cx_1\|\theta\|_{L^\infty([0,1]^2)}. \tag{A-3}$$

For $n = 0$, we estimate first

$$I_1(0) = -\int_{[0,1]^2}\biggl(\frac{x_2-y_2}{|x-y|^{4-\alpha}} - \frac{x_2-y_2}{|x-\tilde{y}|^{4-\alpha}}\biggr)\theta(y)\,\mathrm{d}y.$$

Using $[0, 1]^2 = Q(2x)\cup[0, 4x_1]\times[4x_1, 1]\cup[0, 4x_1]^2$, we estimate the integral for each set.

(i) Suppose $y \in Q(2x)$. In this case, we note

$$-\tfrac{5}{8}|y|^2 \le |x|^2 - 2x\cdot y \le |x|^2 - 2x\cdot\tilde{y} \le \tfrac{5}{8}|y|^2$$

from $-|x|^2+2x\cdot y \le 3|x|^2 + \frac{1}{4}|y|^2 \le \frac{3}{8}y_1^2 + \frac{1}{4}|y|^2$ and $|x|^2 + 2x_1y_1 \le \frac{5}{8}y_1^2$. Hence, it holds

$$\biggl|\frac{|x|^2}{|y|^2} - \frac{2x\cdot y}{|y|^2}\biggr| \le \frac{5}{8}, \qquad \biggl|\frac{|x|^2}{|y|^2} - \frac{2x\cdot\tilde{y}}{|y|^2}\biggr| \le \frac{5}{8}.$$

Then, using the Taylor series expansion

$$\frac{1}{(t+1)^{(4-\alpha)/2}} = 1 - \frac{4-\alpha}{2}t + \frac{(4-\alpha)(6-\alpha)}{8}t^2g(t), \quad -1 < t < 1, \tag{A-4}$$

where g is an analytic function on $(-1, 1)$ with $g(0) = 1$, we can verify

$$\begin{aligned}
&\frac{x_2 - y_2}{|x-y|^{4-\alpha}} - \frac{x_2 - y_2}{|x-\tilde{y}|^{4-\alpha}} \\
&\qquad = \frac{x_2 - y_2}{|y|^{4-\alpha}}\left(\frac{|x|^2}{|y|^2} - \frac{2x\cdot y}{|y|^2} + 1\right)^{-(4-\alpha)/2} - \frac{x_2 - y_2}{|y|^{4-\alpha}}\left(\frac{|x|^2}{|y|^2} - \frac{2x\cdot \tilde{y}}{|y|^2} + 1\right)^{-(4-\alpha)/2} \\
&\qquad = \frac{x_2 - y_2}{|y|^{4-\alpha}}\left[2(4-\alpha)\frac{x_1 y_1}{|y|^2} + \frac{(4-\alpha)(6-\alpha)}{8}\left\{h\left(\frac{|x|^2}{|y|^2} - \frac{2x\cdot y}{|y|^2}\right) - h\left(\frac{|x|^2}{|y|^2} - \frac{2x\cdot \tilde{y}}{|y|^2}\right)\right\}\right],
\end{aligned}$$

where $h(t) := t^2 g(t)$. We set

$$f(\tau) = h\left(\frac{|x|^2}{|y|^2} - \frac{2x\cdot(y + (\tau-1)(y-\tilde{y}))}{|y|^2}\right), \quad 0 \le \tau \le 1,$$

so that

$$\frac{x_2 - y_2}{|x-y|^{4-\alpha}} - \frac{x_2 - y_2}{|x-\tilde{y}|^{4-\alpha}} = 2(4-\alpha)\frac{x_1 y_1 (x_2 - y_2)}{|y|^{6-\alpha}} + \frac{(4-\alpha)(6-\alpha)}{8}\frac{x_2 - y_2}{|y|^{4-\alpha}}(f(1) - f(0)).$$

The mean value theorem and (3-6) imply

$$\begin{aligned}
|f(1) - f(0)| &= |f'(\tau)| \\
&= \left|-8\frac{x_1 y_1}{|y|^2}\left(\frac{|x|^2}{|y|^2} - \frac{2x\cdot(y+(\tau-1)(y-\tilde{y}))}{|y|^2}\right) g\left(\frac{|x|^2}{|y|^2} - \frac{2x\cdot(y+(\tau-1)(y-\tilde{y}))}{|y|^2}\right)\right. \\
&\qquad \left. -4\frac{x_1 y_1}{|y|^2}\left(\frac{|x|^2}{|y|^2} - \frac{2x\cdot(y+(\tau-1)(y-\tilde{y}))}{|y|^2}\right)^2 g'\left(\frac{|x|^2}{|y|^2} - \frac{2x\cdot(y+(\tau-1)(y-\tilde{y}))}{|y|^2}\right)\right| \\
&\le C x_1 \frac{|x|}{|y|^2}.
\end{aligned}$$

Thus, we have

$$\frac{x_2 - y_2}{|x-y|^{4-\alpha}} - \frac{x_2 - y_2}{|x-\tilde{y}|^{4-\alpha}} = -2(4-\alpha)x_1\frac{y_1 y_2}{|y|^{6-\alpha}} + x_1|x|\,O\left(\frac{1}{|y|^{5-\alpha}}\right)$$

and

$$\begin{aligned}
&\left|-\frac{1}{x_1}\int_{Q(2x)}\left(\frac{x_2 - y_2}{|x-y|^{4-\alpha}} - \frac{x_2 - y_2}{|x-\tilde{y}|^{4-\alpha}}\right)\theta(y)\,\mathrm{d}y - 2(4-\alpha)\int_{Q(2x)}\frac{y_1 y_2}{|y|^{6-\alpha}}\theta(y)\,\mathrm{d}y\right| \\
&\qquad\qquad \le C|x|\int_{Q(2x)}\frac{1}{|y|^2}\frac{|\theta(y)|}{|y|^{3-\alpha}}\,\mathrm{d}y \\
&\qquad\qquad \le C\||y|^{-(3-\alpha)}\theta\|_{L^2([0,1]^2)}.
\end{aligned}$$

For $y_1 \ge y_2$, using Lemma 3.1 and [Zhang 2006, Theorem 3.1], we obtain

$$\begin{aligned}
\||y|^{-(3-\alpha)}\theta(y)\|_{L^2([0,1]^2)} &\le \||y_1|^{-(2-\alpha)}y_2^{-1}\theta\|_{L^2([0,1]^2)} \\
&\le C\||y|^{-(2-\alpha)}\partial_2\theta(y)\|_{L^2([0,1]^2)} \le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.
\end{aligned}$$

Similarly, we can deduce for $y_1 \le y_2$ that

$$\||y|^{-(3-\alpha)}\theta(y)\|_{L^2([0,1]^2)} \le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.$$

(ii) Suppose $y \in [0,4x_1] \times [4x_1,1]$. In this case, we set $f(\tau) = |x-(y-(1-\tau)(y-\tilde{y}))|^{4-\alpha}$ for $0 \le \tau \le 1$ to see

$$\left| -\frac{1}{x_1} \int_{[0,4x_1]\times[4x_1,1]} \left(\frac{x_2-y_2}{|x-y|^{4-\alpha}} - \frac{x_2-y_2}{|x-\tilde{y}|^{4-\alpha}} \right) \theta(y)\,\mathrm{d}y \right| \le \frac{1}{x_1} \int_{[0,4x_1]\times[4x_1,1]} \frac{|x_2-y_2|\,|f(1)-f(0)|}{|x-y|^{4-\alpha}|x-\tilde{y}|^{4-\alpha}} |\theta(y)|\,\mathrm{d}y.$$

The mean value theorem implies

$$\begin{aligned} f(1)-f(0) = f'(\tau) &= \frac{4-\alpha}{2}(-4y_1(x_1+y_1)+8\tau y_1^2)|\tau(x-y)+(1-\tau)(x-\tilde{y})|^{2-\alpha} \\ &= -\frac{4-\alpha}{2} 4y_1(\tau(x_1-y_1)+(1-\tau)(x_1+y_1))|\tau(x-y)+(1-\tau)(x-\tilde{y})|^{2-\alpha}. \end{aligned}$$

Applying $y_1 \le 4x_1$ and

$$\tfrac{1}{2}y_2 \le |x-y| \le 2y_2, \quad \tfrac{1}{2}y_2 \le |x-\tilde{y}| \le 2y_2,$$

we obtain

$$\frac{|x_2-y_2|\,|f(1)-f(0)|}{|x-y|^{4-\alpha}|x-\tilde{y}|^{4-\alpha}} \le C\frac{x_1}{y_2^{4-\alpha}}.$$

Thus, it follows

$$\begin{aligned} &\left| -\frac{1}{x_1} \int_{[0,4x_1]\times[4x_1,1]} \left(\frac{x_2-y_2}{|x-y|^{4-\alpha}} - \frac{x_2-y_2}{|x-\tilde{y}|^{4-\alpha}} \right) \theta(y)\,\mathrm{d}y \right| \\ &\qquad \le C \int_{[0,4x_1]\times[4x_1,1]} \frac{1}{y_2} \frac{\theta(y)}{y_2^{3-\alpha}}\,\mathrm{d}y \\ &\qquad \le C\| |y|^{-(3-\alpha)}\theta(y)\|_{L^2([0,1]^2)} \le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}. \end{aligned} \tag{A-5}$$

(iii) Suppose $y \in [0,4x_1]^2$. Due to $\theta(y_1,0)=0$, using integration by parts gives

$$\begin{aligned} &-\frac{1}{x_1}\int_{[0,4x_1]^2} \left(\frac{x_2-y_2}{|x-y|^{4-\alpha}} - \frac{x_2-y_2}{|x-\tilde{y}|^{4-\alpha}} \right)\theta(y)\,\mathrm{d}y \\ &\quad = \frac{1}{(2-\alpha)x_1}\int_{[0,4x_1]^2} \left(\frac{1}{|x-y|^{2-\alpha}} - \frac{1}{|x-\tilde{y}|^{2-\alpha}} \right)\partial_2\theta(y)\,\mathrm{d}y \\ &\qquad - \frac{1}{(2-\alpha)x_1}\int_0^{4x_1} \left(\frac{1}{|(x_1-y_1,x_2-4x_1)|^{2-\alpha}} - \frac{1}{|(x_1+y_1,x_2-4x_1)|^{2-\alpha}} \right)\theta(y_1,4x_1)\,\mathrm{d}y_1. \end{aligned}$$

By Hölder's inequality we estimate the second integral as

$$\begin{aligned} &\left| -\frac{1}{(2-\alpha)x_1}\int_0^{4x_1} \left(\frac{1}{|(x_1-y_1,x_2-4x_1)|^{2-\alpha}} - \frac{1}{|(x_1+y_1,x_2-4x_1)|^{2-\alpha}} \right)\theta(y_1,4x_1)\,\mathrm{d}y_1 \right| \\ &\qquad \le Cx_1^{-(3-\alpha)}\int_0^{4x_1} y_1^{2-\alpha}|y_1^{-(2-\alpha)}\theta(y_1,4x_1)|\,\mathrm{d}y_1 \le Cx_1^{-1/2}\|y_1^{-(2-\alpha)}\theta(y_1,4x_1)\|_{L^2(0,4x_1)}. \end{aligned}$$

Since θ vanishes near the axis, it follows

$$|\theta(y_1,4x_1)| \le \int_0^{4x_1} |\partial_2\theta(y_1,\tau)|\,\mathrm{d}\tau \le (4x_1)^{1/2}\|\partial_2\theta(y_1,\cdot)\|_{L^2(0,4x_1)}$$

for all $y_1 \in [0, 4x_1]$. Then, with [Zhang 2006, Theorem 3.1], we obtain

$$\left| -\frac{1}{(2-\alpha)x_1} \int_0^{2x_1} \left(\frac{1}{|(x_1-y_1, x_2-2x_1)|^{2-\alpha}} - \frac{1}{|(x_1+y_1, x_2-2x_1)|^{2-\alpha}} \right) \theta(y_1, 2x_1)\, \mathrm{d}y_1 \right| \\ \leq C \|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.$$

On the other hand, using Hölder's inequality, we have

$$\begin{aligned} &\left| \frac{1}{(2-\alpha)x_1} \int_{[0,4x_1]^2} \left(\frac{1}{|x-y|^{2-\alpha}} - \frac{1}{|x-\tilde{y}|^{2-\alpha}} \right) \partial_2\theta(y)\, \mathrm{d}y \right| \\ &\qquad \leq \frac{C}{x_1} \int_{[0,4x_1]^2} \frac{|\partial_2\theta(y)|}{|x-y|^{2-\alpha}}\, \mathrm{d}y \leq \frac{C}{x_1} \left(\int_0^{8x_1} r^{-(3-2\alpha)}\, \mathrm{d}r \right)^{1/2} \|\partial_2\theta\|_{L^2([0,4x_1]^2)} \\ &\qquad \leq C x_1^{-(2-\alpha)} \|\nabla\theta\|_{L^2([0,4x_1]^2)}. \end{aligned}$$

Therefore, we can deduce

$$\left| \frac{1}{(2-\alpha)x_1} \int_{[0,4x_1]^2} \left(\frac{1}{|x-y|^{2-\alpha}} - \frac{1}{|x-\tilde{y}|^{2-\alpha}} \right) \partial_2\theta(y)\, \mathrm{d}y \right| \leq C \|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.$$

Combining the above estimates, we obtain

$$\left| -\frac{1}{x_1} \int_{[0,2x_1]^2} \left(\frac{x_2-y_2}{|x-y|^{4-\alpha}} - \frac{x_2-y_2}{|x-\tilde{y}|^{4-\alpha}} \right) \theta(y)\, \mathrm{d}y \right| \leq C \|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.$$

We collect the estimates for each region and deduce that

$$\left| \frac{I_1(0)}{x_1} - 2(4-\alpha) \int_{Q(x)} \frac{y_1 y_2}{|y|^{6-\alpha}} \theta(y)\, \mathrm{d}y \right| \leq C \|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.$$

Similarly, we can show

$$\left| \frac{I_2(0)}{x_1} - 2(4-\alpha) \int_{Q(x)} \frac{y_1 y_2}{|y|^{6-\alpha}} \theta(y)\, \mathrm{d}y \right| \leq C \|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.$$

We omit the details. Thus we have (A-1).

Now we estimate u_2. Note that

$$u_2(x) = \sum_{n \in \mathbb{Z}^2} (I_3(n) + I_4(n)),$$

where

$$\begin{aligned} I_3(n) &:= \int_{[0,1]^2} \left(\frac{x_1-(y_1+2n_1)}{|x-(y+2n)|^{4-\alpha}} - \frac{x_1-(y_1+2n_1)}{|x-(\bar{y}+2n)|^{4-\alpha}} \right) \theta(y)\, \mathrm{d}y, \\ I_4(n) &:= \int_{[0,1]^2} \left(\frac{x_1-(-y_1+2n_1)}{|x-(-y+2n)|^{4-\alpha}} - \frac{x_1-(-y_1+2n_1)}{|x-(\tilde{y}+2n)|^{4-\alpha}} \right) \theta(y)\, \mathrm{d}y. \end{aligned}$$

Since we can similarly see that

$$\left| \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} (I_3(n) + I_4(n)) \right| \leq C x_2 \|\theta\|_{L^\infty([0,1]^2)},$$

it suffices to estimate for $n = 0$. We estimate $I_3(0)$ by dividing the domain into four regions as $[0, 1]^2 = Q(2x) \cup [0, x_1/2] \times [0, 4x_1] \cup [x_1/2, 4x_1] \times [0, 4x_1] \cup [0, 4x_1] \times [4x_1, 1]$.

(i) Suppose $y \in Q(2x)$. Then we can see

$$\left| \frac{|x|^2}{|y|^2} - \frac{2x \cdot \bar{y}}{|y|^2} \right| \le \frac{5}{8}$$

by $|x|^2 + 2x_2 y_2 \le |x|^2 + 4x_2^2 + \frac{1}{4} y_2^2 \le \frac{3}{8} y_1^2 + \frac{1}{4} y_2^2$. With (A-4) we can prove

$$\left| \frac{1}{x_2} \int_{Q(2x)} \left(\frac{x_1 - y_1}{|x-y|^{4-\alpha}} - \frac{x_1 - y_1}{|x-\bar{y}|^{4-\alpha}} \right) \theta(y)\,\mathrm{d}y + 2(4-\alpha) \int_{Q(2x)} \frac{y_1 y_2}{|y|^{6-\alpha}} \theta(y)\,\mathrm{d}y \right| \le C \|\nabla^{3-\alpha} \theta\|_{L^2([0,1]^2)}.$$

(ii) Suppose $y \in [0, 4x_1] \times [4x_1, 1]$. Then it follows

$$\tfrac{1}{2} y_2 \le |x-y| \le 2y_2, \quad y_2 \le |x-\bar{y}| \le 2y_2.$$

Using it, we can show

$$\left| \frac{1}{x_2} \int_{[0,4x_1]\times[4x_1,1]} \left(\frac{x_1 - y_1}{|x-y|^{4-\alpha}} - \frac{x_1 - y_1}{|x-\bar{y}|^{4-\alpha}} \right) \theta(y)\,\mathrm{d}y \right| \le C \|\nabla^{3-\alpha} \theta\|_{L^2([0,1]^2)}$$

in a way similar to how we obtained (A-5).

(iii) Suppose $y \in [0, x_1/2] \times [0, 4x_1]$. We set

$$f(\tau) = |x - (y - (1-\tau)(y - \bar{y}))|^{4-\alpha}, \quad 0 \le \tau \le 1.$$

Then, we can see

$$\left| \frac{1}{x_2} \int_{[0,x_1/2]\times[0,4x_1]} \left(\frac{x_1 - y_1}{|x-y|^{4-\alpha}} - \frac{x_1 - y_1}{|x-\bar{y}|^{4-\alpha}} \right) \theta(y)\,\mathrm{d}y \right| \le \frac{1}{x_2} \int_{[0,x_1/2]\times[0,4x_1]} \frac{|x_1 - y_1|\,|f(1) - f(0)|}{|x-y|^{4-\alpha}|x-\bar{y}|^{4-\alpha}} |\theta(y)|\,\mathrm{d}y.$$

Since the mean value theorem implies

$$f(1) - f(0) = f'(\tau) = -\frac{4-\alpha}{2} 4y_2(\tau(x_2 - y_2) + (1-\tau)(x_2 + y_2))|\tau(x-y) + (1-\tau)(x - \tilde{y})|^{2-\alpha},$$

with $y_2 \le 4x_1$ and

$$\tfrac{1}{2} x_1 \le |x-y| \le 8x_1, \quad \tfrac{1}{2} x_1 \le |x-\bar{y}| \le 8x_1,$$

we have

$$\frac{|x_1 - y_1|\,|f(1) - f(0)|}{|x-y|^{4-\alpha}|x-\bar{y}|^{4-\alpha}} \le C \frac{x_2}{x_1^{4-\alpha}}.$$

Thus, we can obtain

$$\begin{aligned} \left| \frac{1}{x_2} \int_{[0,x_1/2]\times[0,4x_1]} \left(\frac{x_1 - y_1}{|x-y|^{4-\alpha}} - \frac{x_1 - y_1}{|x-\bar{y}|^{4-\alpha}} \right) \theta(y)\,\mathrm{d}y \right| &\le \frac{C}{x_1^{4-\alpha}} \int_{[0,x_1/2]\times[0,4x_1]} |\theta(y)|\,\mathrm{d}y \\ &\le C \| |y|^{-(3-\alpha)} \theta(y) \|_{L^2([0,1]^2)} \le C \|\nabla^{3-\alpha} \theta\|_{L^2([0,1^2])}. \end{aligned}$$

(iv) Suppose $y \in [x_1/2, 4x_1] \times [0, 4x_1]$. Integration by parts and $\theta(0, y_2) = 0$ give

$$\begin{aligned}\frac{1}{x_2}\int_{[x_1/2,4x_1]\times[0,4x_1]}&\left(\frac{x_1-y_1}{|x-y|^{4-\alpha}}-\frac{x_1-y_1}{|x-\bar{y}|^{4-\alpha}}\right)\theta(y)\,\mathrm{d}y\\ &= -\frac{1}{(2-\alpha)x_2}\int_{[x_1/2,4x_1]\times[0,4x_1]}\left(\frac{1}{|x-y|^{2-\alpha}}-\frac{1}{|x-\bar{y}|^{2-\alpha}}\right)\partial_1\theta(y)\,\mathrm{d}y,\\ &\quad+\frac{1}{(2-\alpha)x_2}\int_0^{4x_1}\left(\frac{1}{|(3x_1,x_2-y_2)|^{2-\alpha}}-\frac{1}{|(3x_1,x_2+y_2)|^{2-\alpha}}\right)\theta(4x_1,y_2)\,\mathrm{d}y_2\\ &\quad-\frac{1}{(2-\alpha)x_2}\int_0^{4x_1}\left(\frac{1}{|(x_1/2,x_2-y_2)|^{2-\alpha}}-\frac{1}{|(x_1/2,x_2+y_2)|^{2-\alpha}}\right)\theta(x_1/2,y_2)\,\mathrm{d}y_2.\end{aligned}$$

To estimate the second integral on the right-hand side first, we set

$$f(\tau) = |(3x_1, x_2-(y_2-2(1-\tau)y_2))|^{2-\alpha}, \quad 0 \le \tau \le 1,$$

so that

$$\begin{aligned}&\left|\frac{1}{(2-\alpha)x_2}\int_0^{4x_1}\left(\frac{1}{|(3x_1,x_2-y_2)|^{2-\alpha}}-\frac{1}{|(3x_1,x_2+y_2)|^{2-\alpha}}\right)\theta(4x_1,y_2)\,\mathrm{d}y_2\right|\\ &\qquad\le \frac{C}{x_2}\int_0^{4x_1}\frac{|f(1)-f(0)|}{|(3x_1,x_2-y_2)|^{2-\alpha}|(3x_1,x_2+y_2)|^{2-\alpha}}|\theta(4x_1,y_2)|\,\mathrm{d}y_2.\end{aligned}$$

Using the mean value theorem

$$f(1)-f(0) = f'(\tau) = \frac{2-\alpha}{2}4y_2(\tau(x_2-y_2)+(1-\tau)(x_2+y_2))|(3x_1,x_2-(y_2-2(1-\tau)y_2))|^{-\alpha},$$

we have

$$\frac{|f(1)-f(0)|}{|(3x_1,x_2-y_2)|^{2-\alpha}|(3x_1,x_2+y_2)|^{2-\alpha}} \le C\frac{x_2}{x_1^{3-\alpha}}.$$

With the simple inequality

$$|\theta(4x_1,y_2)| \le \int_0^{4x_1}|\partial_1\theta(\tau,y_2)|\,\mathrm{d}\tau \le (4x_1)^{1/2}\|\partial_1\theta(\cdot\,,y_2)\|_{L^2(0,4x_1)},$$

we obtain

$$\begin{aligned}&\left|\frac{1}{(2-\alpha)x_2}\int_0^{4x_1}\left(\frac{1}{|(3x_1,x_2-y_2)|^{2-\alpha}}-\frac{1}{|(3x_1,x_2+y_2)|^{2-\alpha}}\right)\theta(4x_1,y_2)\,\mathrm{d}y_2\right|\\ &\qquad\le \frac{C}{x_1^{2-\alpha}}\|\partial_1\theta\|_{L^2([0,4x_1]^2)} \le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.\end{aligned}$$

Similarly, we can show

$$\left|\frac{1}{(2-\alpha)x_2}\int_0^{4x_1}\left(\frac{1}{|(x_1/2,x_2-y_2)|^{2-\alpha}}-\frac{1}{|(x_1/2,x_2+y_2)|^{2-\alpha}}\right)\theta(x_1/2,y_2)\,\mathrm{d}y_2\right| \le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}.$$

We omit the details. Now, consider $f(\tau) = |x-(y-(1-\tau)(y-\bar{y}))|^{2-\alpha}$, $0 \le \tau \le 1$, and note that

$$\begin{aligned}&\left|\frac{1}{(2-\alpha)x_2}\int_{[x_1/2,4x_1]\times[0,4x_1]}\left(\frac{1}{|x-y|^{2-\alpha}}-\frac{1}{|x-\bar{y}|^{2-\alpha}}\right)\partial_1\theta(y)\,\mathrm{d}y\right|\\ &\qquad\le \frac{C}{x_2}\int_{[x_1/2,4x_1]\times[0,4x_1]}\frac{|f(1)-f(0)|}{|x-y|^{2-\alpha}|x-\bar{y}|^{2-\alpha}}|\partial_1\theta(y)|\,\mathrm{d}y.\end{aligned}$$

Since the mean value theorem gives

$$f(1)-f(0)=f'(\tau)=\frac{2-\alpha}{2}4y_2(\tau(x_2-y_2)+(1-\tau)(x_2+y_2))|\tau(x-y)+(1-\tau)(x-\bar{y})|^{-\alpha},$$

it follows

$$\frac{|f(1)-f(0)|}{|x-y|^{2-\alpha}|x-\bar{y}|^{2-\alpha}}\le\frac{Cx_2}{|x-y|^{2-\alpha}|x-\bar{y}|}.$$

Hence,

$$\begin{aligned}&\left|\frac{1}{(2-\alpha)x_2}\int_{[x_1/2,4x_1]\times[0,4x_1]}\left(\frac{1}{|x-y|^{2-\alpha}}-\frac{1}{|x-\bar{y}|^{2-\alpha}}\right)\partial_1\theta(y)\,\mathrm{d}y\right|\\&\qquad\le C\int_{[x_1/2,4x_1]\times[0,4x_1]}\frac{1}{|x-y|^{2-\alpha}|x-\bar{y}|}|\partial_1\theta(y)|\,\mathrm{d}y.\end{aligned}$$

By Fubini's theorem and Hölder's inequality, we have

$$\begin{aligned}&\int_{[x_1/2,4x_1]\times[0,4x_1]}\frac{1}{|x-y|^{2-\alpha}|x-\bar{y}|}|\partial_1\theta(y)|\,\mathrm{d}y\\&\qquad=\int_0^{4x_1}\frac{1}{x_2+y_2}\int_{x_1/2}^{4x_1}\frac{1}{|x-y|^{2-\alpha}}|\partial_1\theta(y)|\,\mathrm{d}y_1\,\mathrm{d}y_2\\&\qquad\le\int_0^{4x_1}\frac{1}{x_2+y_2}\|\partial_1\theta\|_{L^{1/(\alpha-1)}(x_1/2,4x_1)}\left(\int_0^{4x_1}\frac{1}{|x-y|}\,\mathrm{d}y_1\right)^{2-\alpha}\mathrm{d}y_2\\&\qquad\le C\int_0^{4x_1}\frac{1}{x_2+y_2}\|\partial_1\theta\|_{L^{1/(\alpha-1)}(x_1/2,4x_1)}\left|\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\right|^{2-\alpha}\mathrm{d}y_2.\end{aligned}$$

The Gagliardo–Nirenberg interpolation inequality and $y_2\le 4x_1$ yield

$$y_2^{-1/2}\|\partial_1\theta(\,\cdot\,,y_2)\|_{L^{1/(\alpha-1)}(x_1/2,4x_1)}\le C(\|\partial_1^{3-\alpha}\theta(\,\cdot\,,y_2)\|_{L^2(x_1/2,4x_1)}+y_2^{-(2-\alpha)}\|\partial_1\theta(\,\cdot\,,y_2)\|_{L^2(x_1/2,4x_1)}).$$

Then, we can have

$$\begin{aligned}&\int_0^{4x_1}\frac{1}{x_2+y_2}\|\partial_1\theta\|_{L^{1/(\alpha-1)}(x_1/2,4x_1)}\left(\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\right)^{2-\alpha}\mathrm{d}y_2\\&\quad\le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}\left(\int_0^{4x_2}\frac{1}{x_2+y_2}\left|\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\right|^{2(2-\alpha)}\mathrm{d}y_2\right)^{1/2}\\&\qquad+C(\|\nabla^{3-\alpha}\theta\|_{L^2(S(x))}+\|y_2^{-(2-\alpha)}\partial_1\theta\|_{L^2(S(x))})\left(\int_{4x_2}^{4x_1}\frac{1}{x_2+y_2}\left|\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\right|^{2(2-\alpha)}\mathrm{d}y_2\right)^{1/2}.\end{aligned}$$

As estimating K_{11} and K_{12}, we can show

$$\begin{aligned}\int_0^{4x_2}\frac{1}{x_2+y_2}\left|\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\right|^{2(2-\alpha)}\mathrm{d}y_2&\le C\left(1+\log\frac{x_1}{x_2}\right)^2,\\\int_{4x_2}^{4x_1}\frac{1}{x_2+y_2}\left|\log\left(1+\frac{x_1}{|x_2-y_2|}\right)\right|^{2(2-\alpha)}\mathrm{d}y_2&\le C\left(1+\log\frac{x_1}{x_2}\right)^{5-2\alpha}.\end{aligned}$$

This implies

$$\begin{aligned}&\left|\frac{1}{(2-\alpha)x_2}\int_{[x_1/2,4x_1]\times[0,4x_1]}\left(\frac{1}{|x-y|^{2-\alpha}}-\frac{1}{|x-\bar{y}|^{2-\alpha}}\right)\partial_1\theta(y)\,\mathrm{d}y\right|\\&\quad\le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right)+C(\|\nabla^{3-\alpha}\theta\|_{L^2(S(x))}+\|y_2^{-(2-\alpha)}\partial_1\theta\|_{L^2(S(x))})\left(1+\log\frac{x_1}{x_2}\right)^{(5-2\alpha)/2}.\end{aligned}$$

Collecting the above estimates gives

$$\left|\frac{I_3(0)}{x_2}+2(4-\alpha)\int_{Q(x)}\frac{y_1y_2}{|y|^{6-\alpha}}\theta(y)\,\mathrm{d}y\right|$$
$$\le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right)+C(\|\nabla^{3-\alpha}\theta\|_{L^2(S(x))}+\|y_2^{-(2-\alpha)}\partial_1\theta\|_{L^2(S(x))})\left(1+\log\frac{x_1}{x_2}\right)^{(5-2\alpha)/2},$$

and we can similarly obtain

$$\left|\frac{I_4(0)}{x_2}+2(4-\alpha)\int_{Q(x)}\frac{y_1y_2}{|y|^{6-\alpha}}\theta(y)\,\mathrm{d}y\right|$$
$$\le C\|\nabla^{3-\alpha}\theta\|_{L^2([0,1]^2)}\left(1+\log\frac{x_1}{x_2}\right)+C(\|\nabla^{3-\alpha}\theta\|_{L^2(S(x))}+\|y_2^{-(2-\alpha)}\partial_1\theta\|_{L^2(S(x))})\left(1+\log\frac{x_1}{x_2}\right)^{(5-2\alpha)/2}.$$

Hence we have (3-2), and this completes the proof. □

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Around the same time, Cordoba and Martinez-Zoroa [2022] proved similar strong ill-posedness results for SQG in $\mathbb{R}^2$. They proved ill-posedness for Sobolev spaces below H^2 as well. We point out that in the case of $\mathbb{R}^2$ (unlike $\mathbb{T}^2$), it is not too difficult to pass from norm inflation to nonexistence since one can keep adding “bubbles” which gives growth further away from previous ones.

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LARGE-SCALE REGULARITY FOR THE STATIONARY NAVIER–STOKES EQUATIONS OVER NON-LIPSCHITZ BOUNDARIES

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We address the large-scale regularity theory for the stationary Navier–Stokes equations in highly oscillating bumpy John domains. These domains are very rough, possibly with fractals or cusps, at the microscopic scale, but are amenable to the mathematical analysis of the Navier–Stokes equations. We prove a large-scale Calderón–Zygmund estimate, a large-scale Lipschitz estimate, and large-scale higher-order regularity estimates, namely, $C^{1,\gamma}$ and $C^{2,\gamma}$ estimates. These nice regularity results are inherited only at mesoscopic scales, and clearly fail in general at the microscopic scales. We emphasize that the large-scale $C^{1,\gamma}$ regularity is obtained by using first-order boundary layers constructed via a new argument. The large-scale $C^{2,\gamma}$ regularity relies on the construction of second-order boundary layers, which allows for certain boundary data with linear growth at spatial infinity. To the best of our knowledge, our work is the first to carry out such an analysis. In the wake of many works in quantitative homogenization, our results strongly advocate in favor of considering the boundary regularity of the solutions to fluid equations as a multiscale problem, with improved regularity at or above a certain scale.

1. Introduction

We consider the large-scale boundary regularity for the stationary Navier–Stokes equations

$$\begin{cases} -\Delta u^\varepsilon + \nabla p^\varepsilon = -u^\varepsilon \cdot \nabla u^\varepsilon & \text{in } B^\varepsilon_{1,+}, \\ \nabla \cdot u^\varepsilon = 0 & \text{in } B^\varepsilon_{1,+}, \\ u^\varepsilon = 0 & \text{on } \Gamma^\varepsilon_1, \end{cases} \tag{NS$^\varepsilon$}$$

in a domain with a rough bumpy boundary. The no-slip boundary condition is prescribed only on the lower part Γ^ε_1 of $\partial B^\varepsilon_{1,+}$. The boundary is rough in two aspects: (i) possible lack of regularity at the microscopic

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scale as the boundary may have fractals or inward cusps; (ii) bumpiness, i.e., the boundary is highly oscillating. The functions $u^\varepsilon = (u_1^\varepsilon(x), u_2^\varepsilon(x), u_3^\varepsilon(x)) \in \mathbb{R}^3$ and $p^\varepsilon = p^\varepsilon(x) \in \mathbb{R}$ denote respectively the velocity field and the pressure field of the fluid. The definitions of $B_{r,+}^\varepsilon$ and Γ_r^ε are given in Section 1D. We will show large-scale regularity estimates, including a Lipschitz estimate (see Theorem A in Section 1A), a $C^{1,\gamma}$ estimate (see Theorem B) and a $C^{2,\gamma}$ estimate (see Theorem C). These improved regularity results at large scales are generally false at small scales due to the roughness of the boundary. The tools developed in this paper enable us to decouple the large-scale regularity from the small-scale properties of the boundary. Therefore, our results (i.e., Theorems A, B and C) show that stationary incompressible Newtonian fluids are regular above the microscopic scale, regardless of the irregularity of surfaces at the microscopic scale.

Before going into the details of our results and of the mathematical analysis, let us give some more general perspectives. The study of fluids over rough boundaries plays a prominent role in the field of hydrodynamics, at least for three reasons.

First, rough, bumpy or corrugated surfaces are ubiquitous in nature and engineering. They appear at any scales from geophysics (see for instance [Narteau et al. 2001] for the fractal-like core-mantle boundary in the Earth) to zoology [Pu et al. 2016] and microfluidics [Waheed et al. 2016]. At the microstructure, the geometry may be anything from fractal to periodic and crenellated. No surface is perfectly smooth, and the lack of smoothness may actually enable us to resolve certain oddities, such as the no-collision paradox for a sphere dropped in a viscous fluid under the action of gravity [Smart and Leighton 1989; Joseph et al. 2001; Davis et al. 2003; Gérard-Varet and Hillairet 2012; Izard et al. 2014]. Moreover, certain roughness patterns are either selected by biological processes and environmental pressure such as scales of sharks for their drag reduction properties, or designed for industrial applications especially in aeronautics, microfluidics and for the transport of fluids in pipes [Pu et al. 2016; Dean and Bhushan 2010; Lee and Jang 2005].

Second, the study of roughness is strongly tied to the derivation of boundary conditions in fluid mechanics. The question of whether or not fluids slip over surfaces is still a matter of active debate. Experiments show that there is no universal answer and that the slip behavior depends a lot on the geometry and microstructure of the surface [Bocquet and Barrat 2007; Lauga et al. 2007]. A widespread idea is that roughness favors slip. To give one specific example where finding the most accurate boundary condition is critical, let us cite the field of glaciology. The assessment of various friction laws for the flow of a glacier over a rough bedrock is crucial in order to understand the speed of glacier discharge and eventually estimate the sea level rise as a result of global warming [Joughin et al. 2019; Minchew and Joughin 2020].

Third, the study of the impact of roughness on the behavior of fluids accompanied the development of turbulence research, as underlined in [Jiménez 2004]:

> Turbulent flows over rough walls have been studied since the early works of Hagen (1854) and Darcy (1857), who were concerned with pressure losses in water conduits. They have been important in the history of turbulence. Had those conduits not been fully rough, turbulence theory would probably have developed more slowly. The pressure loss in pipes only becomes independent of viscosity in the fully rough limit, and this independence was the original indication that something was amiss with laminar theory. Flows over smooth walls never become fully turbulent, and their theory is correspondingly harder.

Investigations of the effect of roughness on fluid flows span three distinct regimes. In the laminar regime, studies focus on the drag-reducing properties of roughness elements [Bechert and Bartenwerfer 1989; García-Mayoral and Jiménez 2011]. As for the onset of turbulence [Schultz and Flack 2007; Squire et al. 2016], there are some indications that roughness lowers the critical Reynolds number for the transition from the laminar to turbulent regime [Varnik et al. 2007]. In the fully turbulent regime, a similarity hypothesis for the flow over flat surfaces and for the flow over rough surfaces was put forward [Townsend 1956]. The extent to which such a universal law holds is still being disputed [Jiménez 2004; Castro 2007; Flack et al. 2007; Schlichting and Gersten 2017].

The three main directions raised above are reflected in the mathematical works. The literature is vast. Therefore we do not aim for exhaustivity.

First, there is an extensive body of works that deal with wall (or friction) laws, or in other words, effective or homogenized boundary conditions. One aims at replacing rough boundaries by fictitious, smooth or flat boundaries. In that line of research, it is well known that Navier-slip boundary conditions provide refined approximations for fluids above bumpy boundaries. Under some quantitative ergodicity assumptions, one can get error estimates. Historically, periodic roughness profiles were first looked at [Amirat and Simon 1997; Achdou et al. 1998; Jäger and Mikelić 2001; 2003]. Analysis of almost-periodic [Gérard-Varet and Masmoudi 2010] or random stationary ergodic [Gérard-Varet 2009; Basson and Gérard-Varet 2008] boundary oscillations was done more recently. Let us also mention a few works that address nonstationary fluids [Bucur et al. 2010; Higaki 2016], for which the analysis is less developed due to its inherent difficulties. We also point out that some authors attempted to justify boundary conditions arising in fluid mechanics starting from boundary conditions at the microscopic scale; see for instance [Casado-Díaz et al. 2003; Bucur et al. 2008; Bonnivard and Bucur 2012] for the derivation of the no-slip boundary condition from a perfect slip boundary condition at the microscale, or [Dalibard and Gérard-Varet 2011] for the computation of the homogenized effect starting from Navier-slip boundary conditions at the microscale.

A second topic is the study of the effect of roughness on singular limits. The topics of rotating fluids and of the homogenized effect of bumpiness on Ekman pumping has been studied in numerous papers [Gérard-Varet 2003; Gérard-Varet and Dormy 2006; Dalibard and Prange 2014; Dalibard and Gérard-Varet 2017]. The paper [Gérard-Varet et al. 2018] carries out an analysis of the vanishing viscosity limit in a specific scaling regime. There are also studies concerned with equations in singularly perturbed domains such as the Stokes equations in rough thin films [Chupin and Martin 2012] or water waves above a rough topography in the shallow regime [Craig et al. 2012].

Third, rough domains pose considerable numerical challenge. This aspect has certainly driven the development of wall laws in a model reduction perspective; see for instance [Achdou et al. 1998; Deolmi et al. 2015]. Other approaches are being elaborated, such as direct numerical simulations [Cardillo et al. 2013], lattice Boltzmann methods that are adapted to intricate geometries [Varnik et al. 2007] and large eddy simulations [Anderson and Meneveau 2011; Bonnivard and Suárez-Grau 2018] that in this context cause important parametrization issues of the small scales.

In this work, we tackle these questions from the angle of regularity theory. The following two general objectives in regularity theory motivate our results: identify building blocks describing the local behavior of solutions, and estimate the decay of certain excess quantities at various scales. We prove that fluids above bumpy boundaries, that are very rough at the microscopic scale, have improved regularity at large scales. Our results are in the spirit of large-scale regularity estimates pioneered in [Avellaneda and Lin 1987] for periodic homogenization, and later extended to stochastic homogenization; see for instance [Armstrong and Smart 2016; Armstrong and Mourrat 2016; Gloria et al. 2015; 2020] and [Armstrong et al. 2016] for the higher-order large-scale regularity theory. Our research program was started with the works [Kenig and Prange 2015; 2018] concerned with uniform regularity estimates above highly oscillating boundaries for elliptic equations. In [Higaki and Prange 2020], the large-scale Lipschitz and $C^{1,\gamma}$ estimates for the stationary Navier–Stokes equations were established above Lipschitz boundaries. A local Navier wall law was also obtained. Finally, let us also mention [Zhuge 2021], which deals with the large-scale regularity of elliptic equations above arbitrarily rough microstructures.

1A. ***Outline of the main results of the paper.*** We study the large-scale regularity for stationary incompressible viscous fluids modeled by the Stokes or Navier–Stokes equations, in domains that are very rough and bumpy at the microscale. Our results show that the large-scale regularity is completely independent of small-scale properties of the boundary.

Let us stress some novel aspects of our results. We refer to Section 1B for a further comparison with a few related works, and to Section 1C for an outline of the proofs.

First we consider John domains, whose boundaries allow for fractals and inward cusps. Hence, the boundaries considered in this paper get closer to the modeling of real boundaries found in nature, that in particular do not need to be graphs. John domains have in a broad sense the minimal properties for the analysis of incompressible fluids. Indeed, we rely on a Bogovskii operator in John domains to estimate the pressure. For precise definitions and a more complete discussion, we refer to Section 1D below.

Second, beyond the Lipschitz estimate, we prove higher-order $C^{1,\gamma}$ and $C^{2,\gamma}$ estimates for $\gamma \in [0, 1)$, as stated in Theorems B and C. These require the construction of boundary layer correctors, which is at the heart of the paper in Section 4; see Section 4B for the first-order boundary layers and Section 4C for the second-order boundary layers. As far as we know, the present work is the first to construct the second-order boundary layers with a linear growth in the direction tangential to the boundary. To make the analysis more tractable, we assume that the boundary is periodic for the structure result of second-order boundary layers; see Theorems 4.3 and 4.4. We are aware of [Barrenechea et al. 2002; Bresch and Milisic 2010], where a refined second-order approximation is constructed for the Stokes equations in a two-dimensional rough channel. However, the boundary layers considered in [Barrenechea et al. 2002; Bresch and Milisic 2010] only involve data spanned by linear and quadratic polynomials of the vertical variable, x_2 and x_2^2 in this two-dimensional case, which are bounded on the bumpy boundary. In our three-dimensional situation, the class of "no-slip Stokes polynomials" (see Section 4A) is much richer and involves boundary data with linear growth at spatial infinity.

Third, we provide explicit quantitative regularity estimates in the nonperturbative regime.

Fourth, in the vein of the seminal works [Avellaneda and Lin 1987; 1991] and of [Kenig et al. 2014; Gu and Zhuge 2019], we provide pointwise estimates for the large-scale decay of the velocity and pressure parts of the Green's function associated to the Stokes system in bumpy John half-spaces; see Section 1C and Appendix B. These estimates are pivotal to constructing the first-order boundary layers in Section 4B.

We now state the three main theorems of the paper.

Theorem A (large-scale Lipschitz regularity). *For all $\varepsilon \in \left(0, \frac{1}{2}\right)$, $L \in (0, \infty)$, $M \in (0, \infty)$ and $\delta \in (0, 1)$, the following statement holds. Let Ω be a bumpy John domain with constant L according to Definition 1.2 below. If $(u^\varepsilon, p^\varepsilon) \in H^1(B^\varepsilon_{1,+})^3 \times L^2(B^\varepsilon_{1,+})$ is a weak solution of (NS$^\varepsilon$) satisfying*

$$\left(\fint_{B^\varepsilon_{1,+}} |\nabla u^\varepsilon|^2\right)^{1/2} \leq M \tag{1-1}$$

(*the precise definition of the bumpy cube $B^\varepsilon_{r,+} = Q_r(0) \cap \Omega^\varepsilon$ can be found in Section 1D*). *Then, for any $r \in \left(\varepsilon, \frac{1}{2}\right)$,*

$$\left(\fint_{B^\varepsilon_{r,+}} |\nabla u^\varepsilon|^2\right)^{1/2} + \left(\fint_{B^\varepsilon_{r,+}} \left| p^\varepsilon - \fint_{B^\varepsilon_{1/2,+}} p^\varepsilon \right|^2\right)^{1/2} \leq C(M + M^{4+\delta}), \tag{1-2}$$

where the constant C is independent of ε, M and r, and depends on L and δ.

Notice that Theorem A, as well as the subsequent results, holds in the nonperturbative regime for arbitrarily large M in (1-1). This is due to the energy subcritical nature of the stationary Navier–Stokes equations, which makes it an easier problem than the nonstationary Navier–Stokes system. Note also that the powers of M in the right-hand side of (1-2) are explicit.

For higher-order $C^{1,\gamma}$ and $C^{2,\gamma}$ regularity results, we measure the oscillation of the solution with respect to modified polynomials that vanish on the bumpy boundary. These modified polynomials are polynomials of degree 1 and 2 that are corrected by the first-order and second-order boundary layers.

Theorem B (large-scale $C^{1,\gamma}$ regularity). *For all $\gamma \in [0, 1)$, $\varepsilon \in \left(0, \frac{1}{2}\right)$, $L \in (0, \infty)$, $M \in (0, \infty)$ and $\delta \in (0, 1)$, the following statement holds. Let Ω be a bumpy John domain with constant L according to Definition 1.2 below. If $(u^\varepsilon, p^\varepsilon) \in H^1(B^\varepsilon_{1,+})^3 \times L^2(B^\varepsilon_{1,+})$ is a weak solution of (NS$^\varepsilon$) satisfying (1-1), then, there exists a constant $\overline{P}_1$ (depending on p^ε) such that, for any $r \in \left(\varepsilon, \frac{1}{2}\right)$,*

$$\inf_{(w,\pi) \in \mathscr{Q}_1(\Omega)} \left\{ \frac{1}{r} \left(\fint_{B^\varepsilon_{r,+}} \left| u^\varepsilon - \varepsilon w\left(\frac{x}{\varepsilon}\right)\right|^2 \mathrm{d}x\right)^{1/2} + \left(\fint_{B^\varepsilon_{r,+}} \left| p^\varepsilon - \pi\left(\frac{x}{\varepsilon}\right) - \overline{P}_1\right|^2 \mathrm{d}x\right)^{1/2} \right\} \leq C r^\gamma (M + M^{4+2\gamma+\delta}), \tag{1-3}$$

where $\mathscr{Q}_1(\Omega)$ is the class of all solutions to the Stokes equations in a bumpy John half-space Ω with linear growth at infinity that vanish on $\partial\Omega$; see (5-1). The constant C is independent of ε, M and r, but depends on L, γ and δ.

The velocity estimate in (1-3) will be derived via a large-scale estimate of $|\nabla u^\varepsilon - \nabla w(x/\varepsilon)|$ and the Poincaré inequality; see Section 5A.

While Theorem B holds for arbitrary bumpy John half-spaces, for the next result, we work in periodic John domains. As we outlined above, the extra periodicity assumption makes the analysis of the second-order boundary layers more manageable.

Theorem C (large-scale $C^{2,\gamma}$ regularity). *For all $\gamma \in [0,1)$, $\varepsilon \in \left(0, \frac{1}{2}\right)$, $L \in (0,\infty)$, $M \in (0,\infty)$ and $\delta \in (0,1)$, the following statement holds. Let Ω be a periodic bumpy John domain with constant L according to Definition 1.3 below. If $(u^\varepsilon, p^\varepsilon) \in H^1(B^\varepsilon_{1,+})^3 \times L^2(B^\varepsilon_{1,+})$ is a weak solution of (NS$^\varepsilon$) satisfying (1-1), then, there exists a constant $\overline{P}_2$ (depending on p^ε) such that, for any $r \in \left(\varepsilon, \frac{1}{2}\right)$,*

$$\inf_{\substack{(w_1,q_1)\in\mathscr{Q}_1(\Omega)\\(w_2,q_2)\in\mathscr{Q}_2(\Omega)}} \left\{ \frac{1}{r}\left(⨍_{B^\varepsilon_{r,+}} \left| u^\varepsilon - \varepsilon w_1\left(\frac{x}{\varepsilon}\right) - \varepsilon^2 w_2\left(\frac{x}{\varepsilon}\right)\right|^2 \mathrm{d}x\right)^{1/2} + \left(⨍_{B^\varepsilon_{r,+}} \left| p^\varepsilon - \pi_1\left(\frac{x}{\varepsilon}\right) - \varepsilon\pi_2\left(\frac{x}{\varepsilon}\right) - \overline{P}_2\right|^2 \mathrm{d}x\right)^{1/2}\right\} \leq C r^{1+\gamma}(M + M^{6+2\gamma+\delta}), \quad (1\text{-}4)$$

where $\mathscr{Q}_1(\Omega)$ is used in Theorem B and $\mathscr{Q}_2(\Omega)$ is the class of all solutions to the Stokes equations in a periodic bumpy John half-space Ω, with quadratic growth at infinity, that vanish on $\partial\Omega$; see (5-2). The constant C is independent of ε, M and r, but depends on L, γ and δ.

We point out that the building blocks in $\mathscr{Q}_1(\Omega)$ and $\mathscr{Q}_2(\Omega)$ are defined through the first-order and second-order boundary layers and play roles of correctors of Stokes system in the bumpy John domain Ω. It turns out that the above three regularity results, Theorems A, B and C, hold also for the linear Stokes equations, with a linear dependence on the size M of the solutions in $\dot{H}^1(B^\varepsilon_{1,+})$. Therefore, these statements immediately imply the Liouville theorems for Stokes equations in bumpy John half-spaces with sublinear (see Corollary 3.1), subquadratic or subcubic growth (see Theorem 5.8).

1B. ***Comparison to two closely related works.*** To further underline the novelty of our work, let us compare our results to the ones of two tightly linked papers.

In [Higaki and Prange 2020], the first and second authors carried out the analysis of the large-scale Lipschitz and $C^{1,\gamma}$ regularity for the stationary Navier–Stokes system. The results there, similar to Theorems A and B here, hold outside the perturbative regime, that is, for arbitrarily large M in (1-1). The main differences between [Higaki and Prange 2020] and the present work are:

(1) In [Higaki and Prange 2020] the bumpy boundary is given by a Lipschitz graph without structure, while here we work in bumpy John domains, as defined in Definition 1.2, that are not necessarily graphs, without structure for the large-scale Lipschitz and $C^{1,\gamma}$ regularity.

(2) In [Higaki and Prange 2020] the analysis relies on a compactness method originating from [Avellaneda and Lin 1987] and the first-order boundary layer correctors are needed to prove the large-scale Lipschitz estimate in Theorem A, while here we resort to a quantitative method, which enables us to by-pass the use of the first-order boundary layers for the large-scale Lipschitz regularity; see Section 1C.

(3) In [Higaki and Prange 2020] no analysis of the higher-order large-scale regularity is carried out, while here we build the second-order boundary layer correctors that make it possible to prove Theorem C.

(4) In [Higaki and Prange 2020], no pressure estimate is established, while in the present paper, we establish the pressure estimates in all cases, following the strategy developed recently in [Gu and Zhuge 2022].

(5) In [Higaki and Prange 2020], the nonlinear estimates are not explicit, while here the dependence on M in (1-2), (1-3) and (1-4) is given as an explicit polynomial in M.

In [Zhuge 2021], the third author carried out an analysis of the large-scale Lipschitz regularity for linear elliptic equations in domains with arbitrary roughness at small scales and quantitative Reifenberg flatness at large scales. Hence, those domains are much rougher than the bumpy John domains considered here. We underline that the discrepancy in these assumptions on the domains comes from the fact that for incompressible Navier–Stokes equations, as opposed to elliptic equations, we have to estimate the pressure in terms of the velocity, which relies on a Bogovskii-type operator as in [Higaki and Prange 2020]; see Section 1C and Appendix A. To address this point we work in bumpy John domains defined by Definition 1.2.

1C. *Outline of the strategy for the proofs.* We now point to some essential ingredients and ideas for the proofs. We mainly focus on two aspects: the lack of smoothness at the microscopic scale, which requires several innovations, and the higher-order regularity, new even in smoother domains, which requires the construction of higher-order boundary layers.

Analysis in John domains. We perform the analysis in bumpy John domains, as defined in Definition 1.2. This type of domain is a good compromise between

- on the one hand a high level of arbitrariness of the boundary, which is not a graph, includes certain fractals or cusps, does not oscillate with any structure, and hence approaches better the properties genuinely rough physical surfaces found in real fluids,
- and on the other hand the possibility of being amenable to mathematical analysis, considering the fact already underlined above that we work with incompressible fluid models that involve estimating the pressure, rather than elliptic equations which can be studied in even rougher domains.

In John domains, we can rely on the Bogovskii operator of [Acosta et al. 2006], whose properties are summarized in Theorem A.1. This operator is required from the beginning of our analysis in Section 2A in order to prove a weak Caccioppoli inequality for the Stokes system (the usual Caccioppoli inequality seems not available in John domains), which then implies the reverse Hölder inequality (2-2), as a starting point of the large-scale regularity theory.

All the boundary estimates of this work are mesoscopic estimates in the sense that they involve averaged quantities smoothing out the possibly rough microscales. Although it is a direct consequence of the weak Caccioppoli inequality, notice that the reverse Hölder inequality (2-2) is a large-scale estimate. Indeed, going from the weak Caccioppoli inequality (A-6) to (2-2) uses the Poincaré inequality that holds in balls large enough, typically at a scale greater than ε. At scales smaller than ε, inward cusps of highly oscillating bumpy John domains may be seen, preventing Poincaré's inequality from holding.

In a nutshell: In the works [Kenig and Prange 2018; Higaki and Prange 2020], tools were developed, particularly for the analysis of the first-order boundary layer correctors, to handle bumpy domains with a boundary given by the graph of a Lipschitz function without structure. Here, the analysis in bumpy John domains requires us to push the techniques even further, to the limit, as it seems, of what is technically possible. There is one particular point, where we are completely unable to transfer the techniques used above Lipschitz graphs to the present context. Indeed, in [Kenig and Prange 2018; Higaki and Prange 2020] we used a domain decomposition method pioneered in [Gérard-Varet and Masmoudi 2010] to study the well-posedness of the Stokes system for the first-order boundary layer correctors. We do not manage to adapt this strategy, in particular the technique of local energy estimates in the bumpy channel, to our current situation. In this paper, we develop a different argument to construct the first-order boundary layers, based on the large-scale Lipschitz estimate proved as an a priori estimate. We will discuss this intricate point in more details shortly later.

Quantitative method for the large-scale regularity. We rely on a quantitative method for large-scale regularity, inspired by the Schauder's theory pioneered by [Armstrong and Smart 2016; Armstrong and Shen 2016; Shen 2017], the Calderón–Zygmund theory motivated by [Caffarelli and Peral 1998] and [Shen 2018, Chapter 4] and the pressure estimate developed in [Gu and Xu 2017; Gu and Zhuge 2019; 2022]. This method is based on a perturbation argument. The principle of this method is the following:

(1) Approximate the original rough problem by a smooth problem at any mesoscopic scale and obtain suboptimal quantitative estimates.

(2) Use the improved regularity of the approximate problem to get the scale-by-scale decay of excess quantities (measuring for instance, Hölder continuity, Lipschitz, $C^{1,\gamma}$, $C^{2,\gamma}$, or higher regularity) for the original rough problem, up to a small error.

(3) Conclude by a real-variable argument such as Theorem 2.5 or an iteration lemma such as Lemma 3.10, which are in some sense black boxes oblivious to the equations.

In the context of homogenization, the homogenized limit problem with constant coefficients is the approximate problem. Here, the approximate problem is a Stokes problem in a domain with a flat boundary. Both problems have improved regularity, in the sense that the solutions are basically as smooth as one wishes.

We remark that from a high-level point of view all the regularity estimates in this paper follow the above scheme. For the large-scale $W^{1,p}$ regularity stated in Theorem 2.4, item (1) above corresponds to Lemma 2.6, item (2) corresponds to the estimate (2-14) and item (3) corresponds to Theorem 2.5. For the proof of Lipschitz estimate in Theorem A, item (1) corresponds to Lemma 3.2, item (2) corresponds to Lemma 3.5 and item (3) corresponds to Lemma 3.10. The proofs of higher-order regularity estimates in Theorems B and C follow a similar scheme.

We point out that in our quantitative method the nonlinear term $u \otimes u$ will also be regarded as a perturbation added to the linear Stokes system. In order to establish the Lipschitz estimate, we use the large-scale Calderón–Zygmund estimate of Theorem 2.4 in combination with a large-scale Sobolev embedding stated in Theorem 2.7 to bootstrap the integrability of the nonlinear term. For $C^{1,\gamma}$ and $C^{2,\gamma}$ estimates, the Lipschitz estimate of u in Theorem A leads to the $O(r^2)$ smallness of the perturbation term $u \otimes u$ near the boundary, which guarantees the higher-order regularity for up to $C^{2,\gamma}$ with any $\gamma \in (0, 1)$.

Construction of boundary layers. As aforementioned, we develop a different argument to construct the first-order boundary layers. In fact, the large-scale Lipschitz regularity in Theorem A makes it possible to construct the velocity and pressure parts of the Green's function in bumpy John domains, and to estimate its decay at large scales. This is the purpose of Appendix B, where we prove estimates for the velocity part of the Green's function (see Proposition B.4), its derivatives (see Proposition B.3), and the pressure part of the Green's function (see Proposition B.5). These estimates are the key for our new proof of the existence of the first-order boundary layer correctors; see Theorem 4.1. In this way we are able to by-pass the difficulties posed by the method used in [Gérard-Varet and Masmoudi 2010; Dalibard and Prange 2014; Dalibard and Gérard-Varet 2017; Kenig and Prange 2018; Higaki and Prange 2020].

To the best of our knowledge, the present work is the first to carry out a thorough analysis of the second-order boundary layer correctors, allowing for linear growth of the boundary data in the tangential direction. Our key observation is an algebraic connection between the first-order and second-order boundary layers on the boundary, which allows us to use the first-order boundary layer correctors in an ansatz for the second-order boundary layers. Unlike the first-order boundary layers (which form a two-dimensional vector space), the space of second-order boundary layers is six-dimensional and needs three different ways of construction, based on the structures of the associated Stokes polynomials; see Sections 4A and 4C. For our analysis to go through, we also need some good quantitative convergence/decay of the first-order boundary layers away from the boundary. Hence we work in a periodic framework, according to Definition 1.3; but this is by no means an optimal assumption. Other structures, such as almost-periodic structures with a nonresonance condition, or random ergodic with quantitative decorrelation properties at large scales, would certainly be manageable.

The key outcome of Section 4 handling the construction of boundary layers is summarized in Propositions 4.6 and 4.7. They are then used in Section 5 to run the excess decay method for the higher-order regularity in Theorems B and C.

1D. *Notation and definitions.*

John domains. We first define John domains. These domains were introduced in [John 1961] and named after John in [Martio and Sarvas 1979].

Definition 1.1. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set and $\tilde{x} \in \Omega$. We say that Ω is a John domain (or a bounded John domain) with respect to $\tilde{x}$ and with constant L if, for any $y \in \Omega$, there exists a Lipschitz mapping $\rho : [0, |y-\tilde{x}|] \to \Omega$ with Lipschitz constant $L \in (0,\infty)$ such that $\rho(0) = y$, $\rho(|y-\tilde{x}|) = \tilde{x}$ and $\operatorname{dist}(\rho(t), \partial\Omega) \geq t/L$ for all $t \in [0, |y-\tilde{x}|]$.

Our analysis takes advantage of a key property of John domains, namely the existence of a right inverse of the divergence operator. Such an operator is usually called a Bogovskii operator; see Appendix A where we state the result of [Acosta et al. 2006].

Examples of John domains are: Lipschitz domains, NTA domains, domains with inward cusps or certain fractals such as Koch's snowflake. Notice that domains with outward cusps are not John domains. For our work, we generalize the above definition from bounded domains to a class of unbounded domains.

Definition 1.2. Let Ω be a domain containing the upper half-space of $\mathbb{R}^3$ and assume $\partial\Omega \subset \{-1 < x_3 < 0\}$. We say that Ω is a *bumpy John domain* (or a *bumpy John half-space*) with constants (L, K) if, for any $x \in \{x_3 = 0\}$ and any $R \geq 1$, there exists a bounded John domain $\Omega_R(x)$ with respect to $x_R = x + Re_d$

and with constant $L \in (0, \infty)$ according to Definition 1.1 such that

$$B_{R,+}(x) \subset \Omega_R(x) \subset B_{KR,+}(x), \tag{1-5}$$

where $B_{R,+}(x) = Q_R(x) \cap \Omega$. Here $Q_R(x)$, defined later, is a cube centered at x with side length $2R$.

The above definition guarantees that the constants of John domains are rescaling- and translation-invariant. This is a natural requirement as we are considering unbounded domains.

Definition 1.3. We say that Ω is a *periodic bumpy John domain* if the following hold:

(i) Ω is a John domain with constant (L, K).

(ii) Ω is $(2\pi\mathbb{Z})^2$-translation-invariant, namely $2\pi z + \Omega = \Omega$ for any $z \in \mathbb{Z}^2 \times \{0\}$.

For simplicity, we assume $K = 2$ in the whole paper. Otherwise, the constant in our main theorems will also depend on K.

Throughout the paper, we assume that Ω is a bumpy John domain satisfying Definition 1.2, or a periodic bumpy John domain satisfying Definition 1.3. We will always specify in case periodicity is needed. In fact, periodicity is used to construct the second-order boundary layer correctors in Section 4C and hence is also an assumption of Theorem C, Proposition 4.7, Section 5B and Theorem 5.8(ii).

Let $\Omega^\varepsilon := \varepsilon\Omega = \{x \in \mathbb{R}^3 \mid \varepsilon^{-1}x \in \Omega\}$. We refer to Ω^ε as a *highly oscillating bumpy John domain*. Note that

$$\partial\Omega^\varepsilon \subset \{x \in \mathbb{R}^3 \mid -\varepsilon < x_3 < 0\}. \tag{1-6}$$

A key fact about Ω^ε is that Ω^ε is still a John domain with the same constants as in Definition 1.2, as these constants are scale-invariant.

Throughout the paper, we use the notation

$$B^\varepsilon_{r,+} = \{x = (x', x_3) \in \mathbb{R}^3 \mid x' \in (-r, r)^2,\ x_3 < r\} \cap \Omega^\varepsilon,$$
$$\Gamma^\varepsilon_r = \{x = (x', x_3) \in \mathbb{R}^3 \mid x' \in (-r, r)^2\} \cap \partial\Omega^\varepsilon.$$

Since the boundary could be very rough at small scales, $B^\varepsilon_{r,+}$ and Γ^ε_r may have disconnected components. Fortunately, this will not cause any issue since the solutions will be extended naturally by zero across the boundary. We also define

$$\begin{gathered} Q_r = Q_r(0) = (-r, r)^3, \quad Q_r(y) = y + Q_r(0), \quad Q^\varepsilon_r = Q_r \cap \{x \in \mathbb{R}^3 \mid x_3 > -\varepsilon\}, \\ Q^\varepsilon_r(y) = Q_r(y) \cap \{x \in \mathbb{R}^3 \mid x_3 > -\varepsilon\} \quad \text{and} \quad Q_{r,+}(y) = Q_r(y) \cap \{x \in \mathbb{R}^3 \mid x_3 > 0\}. \end{gathered} \tag{1-7}$$

From the definition of $B^\varepsilon_{r,+}$, one has $B^\varepsilon_{r,+} \subset Q^\varepsilon_r$ and $|Q^\varepsilon_r \setminus B^\varepsilon_r| \le 4\varepsilon r^2$.

Weak solutions. We work in the framework of weak solutions of (NS$^\varepsilon$). A velocity/pressure pair $(u^\varepsilon, p^\varepsilon) \in H^1(B^\varepsilon_{1,+})^3 \times L^2(B^\varepsilon_{1,+})$ is said to be a weak solution to (NS$^\varepsilon$) if u^ε satisfies $\nabla \cdot u^\varepsilon = 0$ in the sense of distributions, $\psi u^\varepsilon \in H^1_0(Q_1)^3$ for any cut-off function $\psi \in C^\infty_0(Q_1)$, and the weak formulation

$$\int_{B^\varepsilon_{1,+}} \nabla u^\varepsilon \cdot \nabla\varphi - \int_{B^\varepsilon_{1,+}} p^\varepsilon(\nabla \cdot \varphi) = -\int_{B^\varepsilon_{1,+}} (u^\varepsilon \cdot \nabla u^\varepsilon) \cdot \varphi \tag{1-8}$$

for any $\varphi \in C_0^\infty(B^\varepsilon_{1,+})^3$. The Poincaré inequality is a fundamental tool in our paper. Since the weak solution vanishes on the lower boundary Γ^ε_1, we extend it to Q^ε_1 by zero across Γ^ε_1. This enables us to use for instance [Giaquinta and Martinazzi 2012, Proposition 3.15], to get that, for all fixed bumpy John domain Ω with constant $L \in (0,\infty)$ according to Definition 1.2, for all fixed $r \geq \varepsilon$, and for all $u \in H^1(B^\varepsilon_{r,+})$ such that $u = 0$ on Γ^ε_r,

$$\int_{B^\varepsilon_{r,+}} |u|^2 \leq Cr^2 \int_{B^\varepsilon_{r,+}} |\nabla u|^2, \tag{1-9}$$

where C is an absolute constant independent of ε and r. Notice that this estimate is only valid at scales $r \geq \varepsilon$. Indeed, below that scale the constant in (1-9) may degenerate because in particular of inward cusps at small scales.

Other frequently used notation. The notation C denotes a positive constant that varies from line to line, and may or may not be universal. Whenever needed, we make precise what the constant depends on. The notation $x \cdot y$ stands for $x_1y_1 + \cdots + x_Ny_N$ for vectors $x,\ y \in \mathbb{C}^N$. The notation $a \lesssim b$ (resp. $a \gtrsim b$) means that there exists a universal constant C such that $a \leq Cb$ (resp. $Ca \geq b$). The notation $a \approx b$ stands for $a \lesssim b$ and $a \gtrsim b$.

1E. ***Outline of the paper.*** Section 2 is devoted to the proof of the large-scale Calderón–Zygmund estimate stated in Theorem 2.4. We then use this result to bootstrap the regularity and obtain a large-scale Hölder estimate for the nonlinear term in the Navier–Stokes equations; see Theorem 2.8. In Section 3, we prove Theorem A. In Section 4 we construct the first-order and second-order boundary layer correctors. Theorems B and C are proved in Section 5. There are three appendices. Appendix A is devoted to the results related to Bogovskii's operator in John domains. Appendix B handles the construction and estimates for the Green's function associated to the Stokes system in bumpy John domains. Appendix C provides a proof for the iteration Lemma 3.10.

2. Estimates for the nonlinearity

The goal of this section is to obtain some regularity estimates for the nonlinearity $-u^\varepsilon \otimes u^\varepsilon$ for the Navier–Stokes equations. As usual, this follows from a bootstrap argument for the stationary Navier–Stokes equations. However, since there is no smoothness up to the boundary, we have to carry out a delicate large-scale bootstrap argument.

2A. ***Large-scale Calderón–Zygmund estimate.*** Assume $r \geq \varepsilon$. Let Ω be a bumpy John domain with constant L according to Definition 1.2. Let $(u^\varepsilon, p^\varepsilon) \in H^1(B^\varepsilon_{1,+})^3 \times L^2(B^\varepsilon_{1,+})$ be a weak solution of the linear Stokes system

$$\begin{cases} -\Delta u^\varepsilon + \nabla p^\varepsilon = \nabla \cdot F^\varepsilon & \text{in } B^\varepsilon_{1,+}, \\ \nabla \cdot u^\varepsilon = 0 & \text{in } B^\varepsilon_{1,+} \\ u^\varepsilon = 0 & \text{on } \Gamma^\varepsilon_1. \end{cases} \tag{2-1}$$

We extend u^ε and F^ε by zero to the whole of $Q_1 = Q_1(0)$; they are denoted again by $u^\varepsilon \in H^1(Q_1)^3$ and $F^\varepsilon \in L^2(Q_1)^{3\times 3}$ respectively. Note that we also have $\nabla u^\varepsilon = 0$ in $Q_1(0) \setminus B^\varepsilon_{1,+}$. For any $r \geq \varepsilon$ and

$Q_{16r}(y) \subset Q_1(0)$, Lemma A.4 and the Sobolev–Poincaré inequality imply that for any $\theta \in (0, 1)$

$$\left(\fint_{Q_r(y)} |\nabla u^\varepsilon|^2 \right)^{1/2} \leq \theta \left(\fint_{Q_{16r}(y)} |\nabla u^\varepsilon|^2 \right)^{1/2} + \frac{C}{\theta} \left(\fint_{Q_{16r}(y)} |\nabla u^\varepsilon|^{6/5} \right)^{5/6} + \left(\fint_{Q_{16r}(y)} |F^\varepsilon|^2 \right)^{1/2}. \tag{2-2}$$

Here the constant C depends only on L.

We refer to [Zhuge 2021, Lemma 2.2] for a similar proof of (2-2) in the case of elliptic equations. The John boundary condition for Stokes system results in additional difficulties as we only have a weak Caccioppoli inequality in Lemma A.4. Notice that this estimate holds only at large scales, namely, $r \geq \varepsilon$, because Lemma A.4 as well as the Sobolev–Poincaré inequality fail for $r \ll \varepsilon$ (inward cusps are allowed in John domains and these cusps can be seen at a scale less than ε). As a result, we are not able to derive the full-scale Gehring inequality (e.g., [Giaquinta 1983, Chapter V, Proposition 1.1] or [Bensoussan and Frehse 2002, Theorem 1.10]). Instead, we can show a large-scale Gehring inequality; see Lemma 2.2 below.

For $p \in [1, \infty)$, define the averaging operator

$$\mathcal{M}_t^p[g](x) = \left(\fint_{Q_t(x)} |g|^p \right)^{1/p}.$$

The important exponents for us are $p = \frac{6}{5}$ and $p = 2$. For convenience, sometimes we write $\mathcal{M}_t^2$ as $\mathcal{M}_t$ in Section 2B. The following lemma collects useful properties of $\mathcal{M}_t$.

Lemma 2.1. *For $p \in [1, \infty)$ and $g \in L^p(Q_1)$, we have the following properties*:

(i) *For $1 \leq p' \leq p < \infty$ and $Q_t(x) \subset Q_1$,*

$$\mathcal{M}_t^{p'}[g](x) \leq \mathcal{M}_t^p[g](x). \tag{2-3}$$

(ii) *For $0 < t_1 \leq t_2 < 1$ and $Q_{t_2}(x) \subset Q_1$,*

$$\mathcal{M}_{t_1}^p[g](x) \leq C \left(\frac{t_2}{t_1} \right)^{3/p} \mathcal{M}_{t_2}^p[g](x). \tag{2-4}$$

(iii) *For $0 < t \leq s$ with $Q_{s+t}(y) \subset Q_1$,*

$$\int_{Q_s(y)} |g|^p \leq C \int_{Q_s(y)} \mathcal{M}_t^p[g]^p \leq C \int_{Q_{s+t}(y)} |g|^p. \tag{2-5}$$

(iv) *For $0 < t_1 \leq t_2 \leq s$ with $Q_{s+t_1+t_2}(y) \subset Q_1$ and $q \in [p, \infty)$,*

$$\fint_{Q_s(y)} \mathcal{M}_{t_2}^p[g]^q \leq C \fint_{Q_{s+t_2}(y)} \mathcal{M}_{t_1}^p[g]^q. \tag{2-6}$$

(v) *For $0 < s \leq t$ with $Q_{s+t}(y) \subset Q_1$,*

$$\mathcal{M}_t^p[g](y) \leq C \fint_{Q_s(y)} \mathcal{M}_t^p[g]. \tag{2-7}$$

Here the constant C depends on p and p', but not on s, t, t_1 or t_2.

Using the averaging operator and Lemma 2.1, we can show a large-scale Gehring inequality (also known as a self-improving property or Meyers' estimate).

Lemma 2.2. *Let $L \in (0, \infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. There exists some $p_0 \in (2, \infty)$ so that for any $0 < r < 1$, $\varepsilon \le t \le 1$ with $Q_{3r+t}(y) \subset Q_1(0)$,*

$$\left(\fint_{Q_r(y)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^{p_0}\right)^{1/p_0} \le C\left(\fint_{Q_{3r}(y)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^2\right)^{1/2} + C\left(\fint_{Q_{3r}(y)} |\mathcal{M}_t^2[F^\varepsilon]|^{p_0}\right)^{1/p_0}, \tag{2-8}$$

where the constant C and the Lebesgue exponent p_0 depend only on L.

Proof. Assume first that $r \ge t$. Then by Lemma 2.1, we may rewrite (2-2) as

$$\begin{aligned}
&\left(\fint_{Q_r(y)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^2\right)^{1/2} \\
&\qquad \le C\left(\fint_{Q_{2r}(y)} |\nabla u^\varepsilon|^2\right)^{1/2} \\
&\qquad \le C\theta\left(\fint_{Q_{32r}(y)} |\nabla u^\varepsilon|^2\right)^{1/2} + \frac{C}{\theta}\left(\fint_{Q_{32r}(y)} |\nabla u^\varepsilon|^{6/5}\right)^{5/6} + C\left(\fint_{Q_{32r}(y)} |F^\varepsilon|^2\right)^{1/2} \\
&\qquad \le C\theta\left(\fint_{Q_{32r}(y)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^2\right)^{1/2} + \frac{C}{\theta}\left(\fint_{Q_{32r}(y)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^{6/5}\right)^{5/6} + C\left(\fint_{Q_{32r}(y)} |\mathcal{M}_t^2[F^\varepsilon]|^2\right)^{1/2}.
\end{aligned}$$

For $0 < r < t$, Lemma 2.1(v) implies

$$\|\mathcal{M}_t^2[\nabla u^\varepsilon]\|_{L^\infty(Q_r(y))} \le C \fint_{Q_{4r}(y)} \mathcal{M}_t^2[\nabla u^\varepsilon].$$

These imply that a weaker reverse Hölder inequality holds for all scales $r \in (0, 1)$ with $Q_{32r+t}(y) \subset Q_1(0)$. By a version of Gehring's inequality [Giaquinta 1983, Chapter V, Proposition 1.1] or [Bensoussan and Frehse 2002, Theorem 1.10], and choosing θ sufficiently small, there exists some $p_0 > 2$ such that for all $r \in (0, 1)$ with $Q_{32r+t}(y) \in Q_1(0)$,

$$\left(\fint_{Q_r(y)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^{p_0}\right)^{1/p_0} \le C\left(\fint_{Q_{32r}(y)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^2\right)^{1/2} + C\left(\fint_{Q_{32r}(y)} |\mathcal{M}_t^2[F^\varepsilon]|^2\right)^{1/2}. \tag{2-9}$$

To conclude the proof, we use a covering argument to adjust the size of cubes. By covering the cube $Q_{32r}(y)$ by a finite number of cubes $Q_r(y_i)$ and applying the last estimate in every $Q_r(y_i)$, we get the estimate

$$\left(\fint_{Q_{32r}(y)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^{p_0}\right)^{1/p_0} \le C\left(\fint_{Q_{96r}(y)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^2\right)^{1/2} + C\left(\fint_{Q_{96r}(y)} |\mathcal{M}_t^2[F^\varepsilon]|^2\right)^{1/2}$$

for $\varepsilon \le t \le 1$ and $Q_{96r+t}(y) \subset Q_1(0)$, at the price of a larger constant C than in (2-9). Replacing $32r$ by r, we obtain the desired estimate. □

Remark 2.3 (covering argument). The covering argument above to adjust the size of cubes should be a standard technique in analysis. Similar arguments may be used later in this paper.

The following theorem is a large-scale boundary Calderón–Zygmund estimate, or in other words, a large-scale boundary $W^{1,p}$ estimate, for the linear Stokes system.

Theorem 2.4. *For all $\varepsilon \in \left(0, \frac{1}{2}\right)$, $L \in (0, \infty)$ and $p \in (2, \infty)$ the following statement holds. Let Ω be a bumpy John domain with constant L according to Definition 1.2. Suppose $\varepsilon \le t \le r \le \frac{1}{2}$, $Q_{5r}(x) \subset Q_1(0)$ and $\mathcal{M}_t^2[F^\varepsilon] \in L^p(Q_{4r}(x))$. Then the weak solution u^ε to (2-1) satisfies*

$$\left(\fint_{Q_r(x)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^p\right)^{1/p} \le C\left(\fint_{Q_{4r}(x)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^2\right)^{1/2} + C\left(\fint_{Q_{4r}(x)} |\mathcal{M}_t^2[F^\varepsilon]|^p\right)^{1/p}, \tag{2-10}$$

where the constant C depends only on L and p.

The proof of Theorem 2.4 relies on a combination of a real-variable argument (see Theorem 2.5), and the quantitative approximation at sufficiently large scales s of the solution u^ε to the Stokes system in the bumpy domain by a solution to a Stokes problem in a flat domain (see Lemma 2.6).

We first state the real variable result. The following theorem is taken from [Shen 2018, Theorem 4.2.3], where it is stated for balls instead of cubes. Notice that we introduce some flexibility for the size of the cubes as in [Zhuge 2021, Theorem 2.6] and [Shen 2023, Theorem 4.1] to fit the cubes in Lemma 2.6.

Theorem 2.5 [Shen 2018, Theorem 4.2.3]. *Let $N > 1$, $0 < c_1 < 1$, $\kappa > 0$ and $\lambda > 2$. Let Q_0 be a cube in $\mathbb{R}^3$ and $\mathcal{F} \in L^2(\lambda Q_0)$. Let $q > 2$ and $f \in L^p(\lambda Q_0)$ for some $2 < p < q$. Suppose that for each cube $Q \subset 2Q_0$ with $|Q| \le c_1|Q_0|$, there exist two measurable functions F_Q and R_Q on $2Q$ such that $|\mathcal{F}| \le |F_Q| + |R_Q|$ on $2Q$, and*

$$\left(\fint_{2Q} |R_Q|^q\right)^{1/q} \le N\left(\fint_{\lambda Q} |\mathcal{F}|^2\right)^{1/2},$$
$$\left(\fint_{2Q} |F_Q|^2\right)^{1/2} \le \kappa\left(\fint_{\lambda Q} |\mathcal{F}|^2\right)^{1/2} + \left(\fint_{\lambda Q} |f|^2\right)^{1/2}.$$

There exists $\kappa_0 > 0$, depending on λ, p, q, c_1 and N, with the property that if $0 < \kappa < \kappa_0$, then $\mathcal{F} \in L^p(Q_0)$ and

$$\left(\fint_{Q_0} |\mathcal{F}|^p\right)^{1/p} \le C\left\{\left(\fint_{\lambda Q_0} |\mathcal{F}|^2\right)^{1/2} + \left(\fint_{\lambda Q_0} |f|^p\right)^{1/p}\right\},$$

where C depends on λ, p, q, c_1 and N.

We now turn to the approximation. Fix $t \ge \varepsilon$. To apply Theorem 2.5, we introduce an approximation of u^ε at all scales $s \ge t$. Fix $y \in \{-1 \le x_3 \le 1\}$. Let $Q_r^\varepsilon(y) = Q_r(y) \cap \{x_3 > -\varepsilon\}$. Let $s \ge t$ be fixed. By the coarea formula [Evans and Gariepy 2015, Theorem 3.11, page 139] and the fact that $\nabla u^\varepsilon \equiv 0$ below the bottom boundary we have

$$\int_{Q_{2s}^\varepsilon(y)} |\nabla u^\varepsilon|^2 \, \mathrm{d}x = \int_0^{2s} \int_{\partial Q_r^\varepsilon(y)} |\nabla u^\varepsilon|^2 \, \mathrm{d}\sigma_r \, \mathrm{d}r \ge \int_s^{2s} \int_{\partial Q_r^\varepsilon(y)} |\nabla u^\varepsilon|^2 \, \mathrm{d}\sigma_r \, \mathrm{d}r.$$

A contradiction argument then gives that there exists $t_0 \in [1, 2]$ such that

$$\left(\int_{\partial Q^\varepsilon_{t_0 s}(y)} |\nabla u^\varepsilon|^2\right)^{1/2} \le \frac{2}{s^{1/2}} \left(\int_{Q^\varepsilon_{2s}(y)} |\nabla u^\varepsilon|^2\right)^{1/2}. \tag{2-11}$$

Note that t_0 depends particularly on the specific solution u^ε. But this is harmless as t_0 is bounded uniformly in ε in $[1, 2]$. Now, we construct an approximation of u^ε in $Q_{t_0 s}(y)$ by considering the Stokes system

$$\begin{cases} -\Delta w_s + \nabla q_s = 0 & \text{in } Q^\varepsilon_{t_0 s}(y), \\ \nabla \cdot w_s = 0 & \text{in } Q^\varepsilon_{t_0 s}(y), \\ w_s = u^\varepsilon & \text{on } \partial Q^\varepsilon_{t_0 s}(y). \end{cases} \tag{2-12}$$

Since $w_s = 0$ on $\partial Q^\varepsilon_s(y) \cap \{x_3 = -\varepsilon\}$, we may extend the solution w_s naturally across this boundary. For our purpose, we need some regularity estimates for w_s. First of all, the energy estimate implies

$$\left(⨍_{Q^\varepsilon_{t_0 s}(y)} |\nabla w_s|^2\right)^{1/2} + \left(⨍_{Q^\varepsilon_{t_0 s}(y)} \left| q_s - ⨍_{Q^\varepsilon_{t_0 s}(y)} q_s \right|^2\right)^{1/2} \le C \left(⨍_{Q_{t_0 s}(y)} |\nabla u^\varepsilon|^2\right)^{1/2}. \tag{2-13}$$

Second, by the classical regularity theory for the Stokes system over a flat boundary, we have

$$\begin{aligned} \|\nabla w_s\|_{L^\infty(Q_{s/2}(y))} &\le C \left(⨍_{Q_s(y)} |\nabla w_s|^2\right)^{1/2} \\ &\le \frac{C}{s^{3/2}} \left(\int_{Q_{t_0 s}(y)} |\nabla u^\varepsilon|^2\right)^{1/2} = C t_0^{3/2} \left(⨍_{Q_{t_0 s}(y)} |\nabla u^\varepsilon|^2\right)^{1/2} \le C \left(⨍_{Q_{t_0 s}(y)} |\nabla u^\varepsilon|^2\right)^{1/2}. \end{aligned} \tag{2-14}$$

Finally, since $Q^\varepsilon_{t_0 s}(y)$ is a Lipschitz domain and because (2-11) implies $w_s|_{\partial Q^\varepsilon_{t_0 s}(y)} \in H^1(\partial Q^\varepsilon_{t_0 s}(y))^3$, it follows from [Fabes et al. 1988] that $(\nabla w_s)^* \in L^2(\partial Q^\varepsilon_{t_0 s}(y))$, where $(\nabla w_s)^*$ is the nontangential maximal function. More precisely, we have

$$\left(\int_{\partial Q^\varepsilon_{t_0 s}(y)} |(\nabla w_s)^*|^2\right)^{1/2} \le C \left(\int_{\partial Q^\varepsilon_{t_0 s}(y)} |\nabla u^\varepsilon|^2\right)^{1/2} \le \frac{C}{s^{1/2}} \left(\int_{Q^\varepsilon_{2s}(y)} |\nabla u^\varepsilon|^2\right)^{1/2}.$$

This yields,

$$\left(\int_{Q^\varepsilon_{t_0 s}(y)} |\nabla w_s|^3\right)^{1/3} \le \frac{C}{s^{1/2}} \left(\int_{Q^\varepsilon_{2s}(y)} |\nabla u^\varepsilon|^2\right)^{1/2}; \tag{2-15}$$

see [Wei and Zhang 2014, Lemma 3.3] and [Kenig et al. 2013, Remark 9.3]. The above higher integrability of w_s plays an important role in the following lemma.

Lemma 2.6. *Let $L \in (0, \infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. Let (w_s, q_s) be given as above. Then there exists $\sigma \in \left(0, \frac{1}{12}\right]$ such that, for any $\theta \in (0, 1)$, $\varepsilon \in (0, \theta]$, $s \in [\varepsilon/\theta, 1]$, $Q_{7s}(y) \subset Q_1(0)$,*

$$\left(⨍_{Q_s(y)} |\nabla u^\varepsilon - \nabla w_s|^2\right)^{1/2} \le C\theta^\sigma \left(⨍_{Q_{7s}(y)} |\nabla u^\varepsilon|^2\right)^{1/2} + C_\theta \left(⨍_{Q_{7s}(y)} |F^\varepsilon|^2\right)^{1/2}, \tag{2-16}$$

where C depends only on L, and C_θ depends on L, σ and θ.

Proof. We rely on the variational definition of the weak solutions of (2-12). First of all, by (2-12), we see that $u^\varepsilon - w_s \in H_0^1(Q_{t_0 s}^\varepsilon(y))^3$ and $\nabla\cdot(u^\varepsilon - w_s) = 0$, since u^ε has been extended by zero. Thus we can test (2-12) against $u^\varepsilon - w_s$ to obtain

$$\int_{Q_{t_0 s}^\varepsilon(y)} \nabla w_s \cdot \nabla(u^\varepsilon - w_s) = 0. \tag{2-17}$$

Let $\eta_{\varepsilon,+}$ be a smooth cut-off function so that $0 \le \eta_{\varepsilon,+} \le 1$, $\eta_{\varepsilon,+}(x) = 1$ if $x_3 > 2\varepsilon$, $\eta_{\varepsilon,+}(x) = 0$ if $x_3 < \varepsilon$, and $|\nabla\eta_{\varepsilon,+}| \le C\varepsilon^{-1}$. It is easy to verify $\psi := (u^\varepsilon - w_s)\eta_{\varepsilon,+}^2 \in H_0^1(B_{t_0 s,+}^\varepsilon(y))^3$, where $B_{t_0 s,+}^\varepsilon(y) := y + B_{t_0 s,+}^\varepsilon$. Testing (2-1) against ψ, we obtain

$$\begin{aligned}\int_{B_{t_0 s,+}^\varepsilon(y)} &\nabla u^\varepsilon \cdot \nabla((u^\varepsilon - w_s)\eta_{\varepsilon,+}^2) \\ &= \int_{B_{t_0 s,+}^\varepsilon(y)} (p^\varepsilon - \mathcal{P})((u^\varepsilon - w_s)\cdot 2\eta_{\varepsilon,+}\nabla\eta_{\varepsilon,+}) - \int_{B_{t_0 s,+}^\varepsilon(y)} F^\varepsilon \cdot \nabla((u^\varepsilon - w_s)\eta_{\varepsilon,+}^2)\end{aligned} \tag{2-18}$$

for any $\mathcal{P} \in \mathbb{R}$ (to be determined later). Combining (2-17) and (2-18) and using the fact $\nabla u^\varepsilon = 0$ in $Q_{t_0 s}^\varepsilon(y) \setminus B_{t_0 s,+}^\varepsilon(y)$, we arrive at

$$\begin{aligned}\int_{Q_{t_0 s}^\varepsilon(y)} &\nabla(u^\varepsilon - w_s)\cdot\nabla(u^\varepsilon - w_s) \\ &= \int_{B_{t_0 s,+}^\varepsilon(y)} \nabla u^\varepsilon \cdot \nabla((u^\varepsilon - w_s)(1 - \eta_{\varepsilon,+}^2)) \\ &\qquad + \int_{B_{t_0 s,+}^\varepsilon(y)} (p^\varepsilon - \mathcal{P})((u^\varepsilon - w_s)\cdot 2\eta_{\varepsilon,+}\nabla\eta_{\varepsilon,+}) - \int_{B_{t_0 s,+}^\varepsilon(y)} F^\varepsilon\cdot\nabla((u^\varepsilon - w_s)\eta_{\varepsilon,+}^2).\end{aligned} \tag{2-19}$$

Now, we are going to estimate the integrals on the right-hand side of the above equation. Note that $1 - \eta_{\varepsilon,+}^2$ and $\nabla\eta_{\varepsilon,+}$ are both supported in $\{-\varepsilon < x_3 \le 2\varepsilon\}$. Let $R_s^\varepsilon := Q_{t_0 s}(y) \cap \{-\varepsilon \le x_3 \le 2\varepsilon\}$ and $T_s^\varepsilon = Q_{t_0 s}(y) \cap \{0 \le x_3 \le 2\varepsilon\}$. Clearly, $|T_s^\varepsilon| \le |R_s^\varepsilon| \le C\varepsilon s^2$. To estimate the first integral, we use the Poincaré inequality applied in R_s^ε to obtain

$$\begin{aligned}\bigg|\int_{B_{t_0 s,+}^\varepsilon(y)} &\nabla u^\varepsilon \cdot \nabla((u^\varepsilon - w_s)(1 - \eta_{\varepsilon,+}^2))\bigg| \\ &\le \bigg(\int_{R_s^\varepsilon} |\nabla u^\varepsilon|^2\bigg)^{1/2}\bigg(\int_{R_s^\varepsilon} |\nabla((u^\varepsilon - w_s)(1 - \eta_{\varepsilon,+}^2))|^2\bigg)^{1/2} \\ &\le C\bigg(\int_{R_s^\varepsilon} |\nabla u^\varepsilon|^2\bigg)^{1/2}\bigg\{\bigg(\int_{R_s^\varepsilon} |\nabla(u^\varepsilon - w_s)|^2\bigg)^{1/2} + \varepsilon^{-1}\bigg(\int_{R_s^\varepsilon} |u^\varepsilon - w_s|^2\bigg)^{1/2}\bigg\} \\ &\le C\bigg(\int_{R_s^\varepsilon} |\nabla u^\varepsilon|^2\bigg)^{1/2}\bigg\{\bigg(\int_{R_s^\varepsilon} |\nabla(u^\varepsilon - w_s)|^2\bigg)^{1/2} + \bigg(\int_{R_s^\varepsilon} |\nabla(u^\varepsilon - w_s)|^2\bigg)^{1/2}\bigg\} \\ &\le C\bigg(\int_{R_s^\varepsilon} |\nabla u^\varepsilon|^2\bigg)^{1/2}\bigg(\int_{Q_{t_0 s}^\varepsilon(y)} |\nabla(u^\varepsilon - w_s)|^2\bigg)^{1/2}.\end{aligned} \tag{2-20}$$

The last integral of $\nabla(u^\varepsilon - w_s)$ in the above estimate will eventually be absorbed by the left-hand side of (2-19). The main difficulty in proceeding is to obtain a certain estimate of smallness for ∇u^ε over the thin strip R_s^ε. This can be done by using Lemma 2.2. In fact, if $\theta \in (0,1)$ and $s \geq \varepsilon/\theta$, Lemma 2.2 yields

$$\left(\fint_{Q_{2s}(y)} |\mathcal{M}^2_{\theta s}[\nabla u^\varepsilon]|^{p_0} \right)^{1/p_0} \leq C \left(\fint_{Q_{6s}(y)} |\mathcal{M}^2_{\theta s}[\nabla u^\varepsilon]|^2 \right)^{1/2} + C \left(\fint_{Q_{6s}(y)} |\mathcal{M}^2_{\theta s}[F^\varepsilon]|^{p_0} \right)^{1/p_0}.$$

Since for any $z \in Q_{6s}(y)$

$$\mathcal{M}^2_{\theta s}[F^\varepsilon](z) \leq C_\theta \left(\fint_{Q_{7s}(y)} |F^\varepsilon|^2 \right)^{1/2},$$

together with (2-5), we obtain

$$\left(\fint_{Q_{2s}(y)} |\mathcal{M}^2_{\theta s}[\nabla u^\varepsilon]|^{p_0} \right)^{1/p_0} \leq C \left(\fint_{Q_{7s}(y)} |\nabla u^\varepsilon|^2 \right)^{1/2} + C_\theta \left(\fint_{Q_{7s}(y)} |F^\varepsilon|^2 \right)^{1/2}.$$

It is important to notice that in the last inequality, C is independent of θ and C_θ depends on θ. By an argument similar to that in [Zhuge 2021], we can now estimate the right-hand side of (2-20) as

$$\begin{aligned} \left(\frac{1}{|Q_s(y)|} \int_{R_s^\varepsilon} |\nabla u^\varepsilon|^2 \right)^{1/2} &\leq C \left(\frac{\theta s}{s} \right)^{1/2 - 1/p_0} \left(\fint_{Q_{2s}(y)} |\mathcal{M}^2_{\theta s}[\nabla u^\varepsilon]|^{p_0} \right)^{1/p_0} \\ &\leq C\theta^\sigma \left(\fint_{Q_{7s}(y)} |\nabla u^\varepsilon|^2 \right)^{1/2} + C_\theta \left(\fint_{Q_{7s}(y)} |F^\varepsilon|^2 \right)^{1/2}, \end{aligned} \tag{2-21}$$

with some $\sigma \in \left(0, \frac{1}{2}\right)$. This is the desired estimate of ∇u^ε in R_s^ε. Later on we will insert it into (2-20) and then (2-19) to reach a conclusion.

Let us turn to the estimate of the second integral on the right-hand side of (2-19). Using Hölder's inequality and the Poincaré inequality, we have

$$\begin{aligned} \left| \int_{B^\varepsilon_{t_0 s,+}(y)} (p^\varepsilon - \mathcal{P})((u^\varepsilon - w_s) \cdot 2\eta_{\varepsilon,+} \nabla \eta_{\varepsilon,+}) \right| &\leq C\varepsilon^{-1} \left(\int_{T_s^\varepsilon} |p^\varepsilon - \mathcal{P}|^2 \right)^{1/2} \left(\int_{R_s^\varepsilon} |u^\varepsilon - w_s|^2 \right)^{1/2} \\ &\leq C \left(\int_{T_s^\varepsilon} |p^\varepsilon - \mathcal{P}|^2 \right)^{1/2} \left(\int_{R_s^\varepsilon} |\nabla (u^\varepsilon - w_s)|^2 \right)^{1/2}. \end{aligned} \tag{2-22}$$

Now, we pick

$$\mathcal{P} := \fint_{Q_{t_0 s,+}(y)} p^\varepsilon,$$

where $Q_{t_0 s,+}(y) = Q_{t_0 s}(y) \cap \{x_3 > 0\}$. Then the Bogovskii lemma applied in a Lipschitz domain $Q_{t_0 s,+}(y)$ implies

$$\begin{aligned} \left(\int_{T_s^\varepsilon} |p^\varepsilon - \mathcal{P}|^2 \right)^{1/2} &\leq \left(\int_{Q_{t_0 s,+}(y)} |p^\varepsilon - \mathcal{P}|^2 \right)^{1/2} \\ &\leq C \left(\int_{Q_{t_0 s,+}(y)} |\nabla u^\varepsilon|^2 \right)^{1/2} + C \left(\int_{Q_{t_0 s,+}(y)} |F^\varepsilon|^2 \right)^{1/2} \\ &\leq C \left(\int_{B^\varepsilon_{t_0 s,+}} |\nabla u^\varepsilon|^2 \right)^{1/2} + C \left(\int_{B^\varepsilon_{t_0 s,+}} |F^\varepsilon|^2 \right)^{1/2}. \end{aligned} \tag{2-23}$$

Unlike the previous argument, we want to gain the smallness for (2-24) below from

$$\left(\int_{R_s^\varepsilon} |\nabla (u^\varepsilon - w_s)|^2 \right)^{1/2} \leq \left(\int_{R_s^\varepsilon} |\nabla u^\varepsilon|^2 \right)^{1/2} + \left(\int_{R_s^\varepsilon} |\nabla w_s|^2 \right)^{1/2}.$$

The estimate for ∇u^ε over R_s^ε is given (2-21). On the other hand, by (2-15) and the Hölder inequality, we have

$$\begin{aligned}\left(\int_{R_s^\varepsilon}|\nabla w_s|^2\right)^{1/2} &\le |R_s^\varepsilon|^{1/6}\left(\int_{R_s^\varepsilon}|\nabla w_s|^3\right)^{1/3}\\ &\le C\frac{|R_s^\varepsilon|^{1/6}}{s^{1/2}}\left(\int_{Q_{2s}^\varepsilon(y)}|\nabla u^\varepsilon|^2\right)^{1/2}\le C\left(\frac{\varepsilon}{s}\right)^{1/6}\left(\int_{Q_{2s}^\varepsilon(y)}|\nabla u^\varepsilon|^2\right)^{1/2}.\end{aligned}$$

Inserting this into (2-22), we have

$$\begin{aligned}&\left|\int_{B_{t_0 s,+}^\varepsilon(y)}(p^\varepsilon-\mathcal{P})((u^\varepsilon-w_s)\cdot 2\eta_{\varepsilon,+}\nabla\eta_{\varepsilon,+})\right|\\ &\le C\left\{\left(\int_{B_{t_0 s,+}^\varepsilon}|\nabla u^\varepsilon|^2\right)^{1/2}+\left(\int_{B_{t_0 s,+}^\varepsilon}|F^\varepsilon|^2\right)^{1/2}\right\}\left\{\left(\int_{R_s^\varepsilon}|\nabla u^\varepsilon|^2\right)^{1/2}+\left(\frac{\varepsilon}{s}\right)^{1/6}\left(\int_{Q_{2s}^\varepsilon(y)}|\nabla u^\varepsilon|^2\right)^{1/2}\right\}. \quad (2\text{-}24)\end{aligned}$$

Finally, for the last integral of (2-19), by the Poincaré inequality, we have

$$\left|\int_{B_{t_0 s,+}^\varepsilon(y)}F^\varepsilon\cdot\nabla((u^\varepsilon-w_s)\eta_{\varepsilon,+}^2)\right|\le C\left(\int_{B_{t_0 s,+}^\varepsilon(y)}|F^\varepsilon|^2\right)^{1/2}\left(\int_{Q_{t_0 s}^\varepsilon(y)}|\nabla(u^\varepsilon-w_s)|^2\right)^{1/2}. \quad (2\text{-}25)$$

Now, (2-19) together with (2-20), (2-21), (2-24) and (2-25) gives

$$\left(\int_{Q_{t_0 s}^\varepsilon(y)}|\nabla(u^\varepsilon-w_s)|^2\right)^{1/2}\le C\left(\theta^\sigma+\left(\frac{\varepsilon}{s}\right)^{1/12}\right)\left(\int_{Q_{7s}(y)}|\nabla u^\varepsilon|^2\right)^{1/2}+C_\theta\left(\int_{Q_{7s}(y)}|F^\varepsilon|^2\right)^{1/2}. \quad (2\text{-}26)$$

Since we assumed $s\ge\varepsilon/\theta$, we have $\varepsilon/s\le\theta$. In view of $t_0\in[1,2]$, (2-26) divided by $|Q_s(y)|^{1/2}$ leads to (2-16). □

Proof of Theorem 2.4. We will first prove a slightly weaker version of (2-10) when $Q_{57r}(x)\subset Q_1(0)$. Then, (2-10) can be recovered thanks to a covering argument at the price of enlarging the constant by a numerical factor; see Remark 2.3 for more details. When $Q_{57r}(x)$ is far away from the boundary Γ_1^ε, the estimate (2-10) is a consequence of interior regularity. Hence it suffices to prove (2-10) when $Q_{57r}(x)\cap\Gamma_1^\varepsilon\ne\varnothing$. Note that this case can be reduced to the case when $x\in\{z_3=0\}$ by a covering argument as well as interior regularity. To apply Theorem 2.5 to $Q_0:=Q_r(x)$, $\lambda:=56$ and $\mathcal{F}:=\mathcal{M}_t^2[\nabla u^\varepsilon]$ in $Q_{56r}(x)$ with $x\in\{z_3=0\}$, we approximate u^ε in any cube $Q_s(y)$ contained in $Q_{2r}(x)$ for any scales for $s\ge\varepsilon/\theta$, where θ is as in Lemma 2.6. If $Q_s(y)$ is entirely contained in $\{z_3>0\}$, then the well-known interior estimate for the Stokes system applies. If $Q_s(y)$ is contained entirely in $\{z_3<-\varepsilon\}$, then trivially $u^\varepsilon\equiv 0$ in $Q_s(y)$. Hence, it suffices to focus on the typical boundary case $Q_s(y)$ with $y\in\{z_3=0\}$. Moreover, we assume $s<r/2$ so that $Q_{57s}(y)\subset Q_{57r}(x)\subset Q_1(0)$ whenever $Q_s(y)\subset Q_{2r}(x)$.

Now, for each $Q_s(y)$ with $y\in\{z_3=0\}$, we will discuss two cases.

Case 1: $s\ge 4t$. By (2-14) and (2-16), there exists w_s solving (2-12) and satisfying

$$\|\nabla w_s\|_{L^\infty(Q_{s/2}(y))}\le C\left(⨍_{Q_{2s}(y)}|\nabla u^\varepsilon|^2\right)^{1/2} \quad (2\text{-}27)$$

and

$$\left(\fint_{Q_s(y)} |\nabla u^\varepsilon - \nabla w_s|^2\right)^{1/2} \le C\theta^\sigma \left(\fint_{Q_{7s}(y)} |\nabla u^\varepsilon|^2\right)^{1/2} + \left(\fint_{Q_{7s}(y)} |C_\theta F^\varepsilon|^2\right)^{1/2}. \tag{2-28}$$

Note that the above estimate only holds for $s \ge \varepsilon/\theta$. Therefore, we will use Lemma 2.1 and replace ∇u^ε and ∇w_s by $\mathcal{M}_t^2[\nabla u^\varepsilon]$ and $\mathcal{M}_t^2[\nabla w_s]$, respectively. Precisely, the above two inequalities imply for $s \ge 4t \ge \varepsilon/\theta$,

$$\|\mathcal{M}_t^2[\nabla w_s]\|_{L^\infty(Q_{s/4}(y))} \le C\left(\fint_{Q_{2s}(y)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^2\right)^{1/2} \le C\left(\fint_{Q_{7s}(y)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^2\right)^{1/2}$$

and

$$\left(\fint_{Q_{s/4}(y)} |\mathcal{M}_t^2[\nabla u^\varepsilon] - \mathcal{M}_t^2[\nabla w_s]|^2\right)^{1/2} \le C\theta^\sigma \left(\fint_{Q_{7s}(y)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^2\right)^{1/2} + \left(\fint_{Q_{7s}(y)} |\mathcal{M}_t^2[C_\theta F^\varepsilon]|^2\right)^{1/2}.$$

<u>Case 2</u>: $0 < s < 4t$. In this case $\mathcal{M}_t^2[\nabla u^\varepsilon]$ itself satisfies some trivial estimate. Note that for any $z \in Q_{s/2}(y)$, as $Q_{s/2}(z) \subset Q_s(y)$, by Lemma 2.1(v),

$$\mathcal{M}_t^2[\nabla u^\varepsilon](z) \le C\left(\fint_{Q_{s/2}(z)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^2\right)^{1/2} \le C\left(\fint_{Q_s(y)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^2\right)^{1/2},$$

which yields

$$\begin{aligned}\|\mathcal{M}_t^2[\nabla u^\varepsilon]\|_{L^\infty(Q_{s/4}(y))} &\le \|\mathcal{M}_t^2[\nabla u^\varepsilon]\|_{L^\infty(Q_{s/2}(y))} \\ &\le C\left(\fint_{Q_s(y)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^2\right)^{1/2} \le C\left(\fint_{Q_{7s}(y)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^2\right)^{1/2}.\end{aligned}$$

This ends the study of the two cases. We now apply Theorem 2.5 with $\lambda := 56$, $Q_0 := Q_r(x)$, $q := \infty$, $\mathcal{F} := \mathcal{M}_t^2[\nabla u^\varepsilon]$ and $f := \mathcal{M}_t^2[C_\theta F^\varepsilon]$. Moreover,

$$F_Q = \begin{cases} \mathcal{M}_t^2[\nabla u^\varepsilon] - \mathcal{M}_t^2[\nabla w_s], & s \ge 4t, \\ 0, & 0 < s < 4t, \end{cases}$$

and

$$R_Q = \begin{cases} \mathcal{M}_t^2[\nabla w_s], & s \ge 4t, \\ \mathcal{M}_t^2[\nabla u^\varepsilon], & 0 < s < 4t. \end{cases}$$

For any given $p > 2$, we may choose θ sufficiently small with $C\theta^\sigma < \kappa_0$ so that the requirement of Theorem 2.5 is satisfied. Consequently, we arrive at

$$\left(\fint_{Q_r(x)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^p\right)^{1/p} \le C\left(\fint_{Q_{56r}(x)} |\mathcal{M}_t^2[\nabla u^\varepsilon]|^2\right)^{1/2} + C\left(\fint_{Q_{56r}(x)} |\mathcal{M}_t^2[C_\theta F^\varepsilon]|^p\right)^{1/p} \tag{2-29}$$

for all $\varepsilon \in (0, \theta(p))$ and $\varepsilon/(4\theta) \le t \le r$ and $Q_{57r}(x) \subset Q_1(0)$. Estimate (2-10) now follows by a covering argument (see the proof of Lemma 2.2) and Lemma 2.1 (in order to adjust the size of balls and relax the condition $t \ge (4\theta)$ to $t \ge \varepsilon$). To remove the smallness condition $\varepsilon \in (0, \theta(p))$, we observe that the case $\theta(p) \le \varepsilon \le t \le r \le \frac{1}{2}$ is trivial as the constant C is allowed to depend on p. □

2B. *Bootstrap argument.* In this subsection, we apply the large-scale Calderón–Zygmund estimate proved previously to study the regularity of the stationary Navier–Stokes equations (NS$^\varepsilon$). Note that in Theorem 2.4, F^ε is a general function. We will take advantage of the nonlinearity $F^\varepsilon = -u^\varepsilon \otimes u^\varepsilon$. As usual, the proof relies on a bootstrap argument.

Throughout this subsection, we set $F^\varepsilon = -u^\varepsilon \otimes u^\varepsilon$. To begin with, note that the Sobolev embedding theorem implies $F^\varepsilon \in L^3$, which yields $\mathcal{M}_t[F^\varepsilon] \in L^3$. Hence, (2-10) holds with $p = 3$. To further improve the large-scale regularity, we need to lift the regularity of F^ε from that of ∇u^ε.

For any $0 \le a < b \le \infty$, define a new maximal function

$$\mathcal{M}^1_{(a,b)}[g](x) = \sup_{a<t<b} \fint_{Q_t(x)} |g|.$$

Note that $\mathcal{M}^1_{(0,\infty)}$ is the usual Hardy–Littlewood maximal function. Clearly, by the L^p boundedness of the Hardy–Littlewood maximal function, $\mathcal{M}^1_{(a,b)}$ is uniformly bounded in L^p space for $p \in (1,\infty)$.

Fix $t > 0$. Define

$$K_q(r) = K_{q,t}(r) = \left(\fint_{Q_r(0)} \mathcal{M}^2_t[\nabla u^\varepsilon]^q \right)^{1/q}.$$

The following estimate is a sort of the large-scale Sobolev embedding theorem.

Theorem 2.7. *Let $L \in (0,\infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. Let $\varepsilon \le t \le r \le \frac{1}{7}$ and $F^\varepsilon = -u^\varepsilon \otimes u^\varepsilon$. Then for any $p > 3$ and any q satisfying*

$$\frac{1}{q} < \frac{1}{2p} + \frac{1}{3}, \tag{2-30}$$

we have

$$\left(\fint_{Q_r(0)} |\mathcal{M}_t[F^\varepsilon]|^p \right)^{1/p} \le C r^2 (K_q(5r))^2, \tag{2-31}$$

where the constant C depends only L, p and q.

Proof. Let $p > 3$ and q satisfy (2-30). Without loss of generality, we assume in addition $\frac{1}{2p} < \frac{1}{q}$. Let $x \in B^\varepsilon_{r,+}(0)$. We first estimate

$$\mathcal{M}_t[F^\varepsilon](x) = \left(\fint_{Q_t(x)} |F^\varepsilon|^2 \right)^{1/2} \le C \left(\fint_{Q_t(x)} |u^\varepsilon|^4 \right)^{1/2}.$$

Let $x = (x', x_3)$. We consider the cases $x_3 \ge t$ and $x_3 < t$ separately. Assume first $x_3 \ge t$ and let N be the natural number so that $2^{N-1} t < x_3 \le 2^N t$. Note that u^ε vanishes in a large portion of $Q_{2^{N+1}t}(x)$. By the triangle inequality and the Poincaré inequality, we have

$$\begin{aligned}
\left(\fint_{Q_t(x)} |u^\varepsilon|^4 \right)^{1/4} &\le \left(\fint_{Q_t(x)} \left| u^\varepsilon - \fint_{Q_{2t}(x)} u^\varepsilon \right|^4 \right)^{1/4} + \sum_{j=1}^N \left| \fint_{Q_{2^j t}(x)} u^\varepsilon - \fint_{Q_{2^{j+1}t}(x)} u^\varepsilon \right| + \left| \fint_{Q_{2^{N+1}t}(x)} u^\varepsilon \right| \\
&\le C \sum_{j=0}^N 2^{j+1} t \left(\fint_{Q_{2^{j+1}t}(x)} |\nabla u^\varepsilon|^2 \right)^{1/2}.
\end{aligned}$$

Now, let $\alpha \in \bigl(0, \min\{\frac{1}{3}, \frac{1}{q}\}\bigr)$ and write

$$\begin{aligned}
2^{j+1}t\biggl(\fint_{Q_{2^{j+1}t}(x)} |\nabla u^\varepsilon|^2\biggr)^{1/2} &\le C2^{j+1}t\biggl(\fint_{Q_{2^{j+1}t}(x)} \mathcal{M}_t[\nabla u^\varepsilon]^2\biggr)^{1/2} \\
&\le C2^{j+1}t\biggl(\fint_{Q_{2^{j+1}t}(x)} \mathcal{M}_t[\nabla u^\varepsilon]^q\biggr)^{1/q} \\
&\le C2^{j+1}t\biggl(\fint_{Q_{2^{j+1}t}(x)} \mathcal{M}_t[\nabla u^\varepsilon]^q\biggr)^{\alpha}\biggl(\fint_{Q_{2^{j+1}t}(x)} \mathcal{M}_t[\nabla u^\varepsilon]^q\biggr)^{1/q-\alpha} \\
&\le C(2^{j+1}t)^{1-3\alpha}\biggl(\int_{Q_{2^{j+1}t}(x)} \mathcal{M}_t[\nabla u^\varepsilon]^q\biggr)^{\alpha}\biggl(\fint_{Q_{2^{j+1}t}(x)} \mathcal{M}_t[\nabla u^\varepsilon]^q\biggr)^{1/q-\alpha} \\
&\le C(2^{j+1}t)^{1-3\alpha}r^{3\alpha}\biggl(\fint_{Q_{5r}(0)} \mathcal{M}_t[\nabla u^\varepsilon]^q\biggr)^{\alpha}\biggl(\fint_{Q_{2^{j+1}t}(x)} \mathcal{M}_t[\nabla u^\varepsilon]^q\biggr)^{1/q-\alpha}. \qquad (2\text{-}32)
\end{aligned}$$

Using the definition of K_q and $\mathcal{M}^1_{(2t,5r)}$, we obtain

$$2^{j+1}t\biggl(\fint_{Q_{2^{j+1}t}(x)} |\nabla u^\varepsilon|^2\biggr)^{1/2} \le C(2^{j+1}t)^{1-3\alpha}r^{3\alpha}(K_q(5r))^{\alpha q}\bigl(\mathcal{M}^1_{(2t,5r)}[\mathcal{M}_t[\nabla u^\varepsilon]^q](x)\bigr)^{1/q-\alpha}.$$

It follows that

$$\begin{aligned}
\biggl(\fint_{Q_t(x)} |u^\varepsilon|^4\biggr)^{1/4} &\le C\sum_{j=0}^{N}(2^{j+1}t)^{1-3\alpha}r^{3\alpha}(K_q(5r))^{\alpha q}\bigl(\mathcal{M}^1_{(2t,5r)}[\mathcal{M}_t[\nabla u^\varepsilon]^q](x)\bigr)^{1/q-\alpha} \\
&\le Cr(K_q(5r))^{\alpha q}\bigl(\mathcal{M}^1_{(2t,5r)}[\mathcal{M}_t[\nabla u^\varepsilon]^q](x)\bigr)^{1/q-\alpha},
\end{aligned}$$

which yields

$$\mathcal{M}_t[F^\varepsilon](x) \le Cr^2(K_q(5r))^{2\alpha q}\bigl(\mathcal{M}^1_{(2t,5r)}[\mathcal{M}_t[\nabla u^\varepsilon]^q](x)\bigr)^{2/q-2\alpha}. \qquad (2\text{-}33)$$

On the other hand, if $x_3 < t$, then $B_{2t}(x)$ has a relatively large portion not contained in Ω^ε. Thus, the Sobolev–Poincaré inequality implies

$$\biggl(\fint_{Q_t(x)} |u^\varepsilon|^4\biggr)^{1/4} \le C\biggl(\fint_{Q_{2t}(x)} |u^\varepsilon|^4\biggr)^{1/4} \le Ct\biggl(\fint_{Q_{2t}(x)} |\nabla u^\varepsilon|^2\biggr)^{1/2}.$$

Using the same argument as (2-32), we see that $\mathcal{M}_t[F^\varepsilon](x)$ has the same bound as (2-33) for $x_3 < t$.

Since by assumption, $\frac{1}{2p} < \frac{1}{q} < \frac{1}{2p} + \frac{1}{3}$ and $0 < \alpha < \min\{\frac{1}{3}, \frac{1}{q}\}$ is arbitrary, we may choose α so that $\frac{1}{q} > \frac{1}{2p} + \alpha$. This implies $p\bigl(\frac{2}{q} - 2\alpha\bigr) > 1$. Thus, using the $L^{p(2/q-2\alpha)}$ boundedness of the Hardy–Littlewood maximal function, we obtain

$$\begin{aligned}
\int_{Q_r(0)} |\mathcal{M}_t[F^\varepsilon](x)|^p\,\mathrm{d}x &\le Cr^{2p}(K_q(5r))^{2\alpha pq}\int_{Q_r(0)} (\mathcal{M}^1_{(2t,5r)}[\mathcal{M}_t[\nabla u^\varepsilon]^q](x))^{p(2/q-2\alpha)}\,\mathrm{d}x \\
&\le Cr^{2p}(K_q(5r))^{2\alpha pq}\int_{Q_{5r}(0)} (\mathcal{M}_t[\nabla u^\varepsilon](x))^{2p(1-\alpha q)}\,\mathrm{d}x.
\end{aligned}$$

Consequently,

$$\left(\fint_{Q_r(0)} |\mathcal{M}_t[F^\varepsilon]|^p\right)^{1/p} \le Cr^2(K_q(5r))^{2\alpha q}(K_{2p(1-\alpha q)}(5r))^{2(1-\alpha q)}.$$

Now, observe that we may choose $\alpha < \frac{1}{q} - \frac{1}{2p}$ but sufficiently close to $\frac{1}{q} - \frac{1}{2p}$. Then $q < 2p(1-\alpha q) \to q$ as α approaches $\frac{1}{q} - \frac{1}{2p}$. This implies

$$\left(\fint_{Q_r(0)} |\mathcal{M}_t[F^\varepsilon]|^p\right)^{1/p} \le Cr^2(K_{2p(1-2\alpha)}(5r))^2 \le Cr^2(K_{\hat{q}}(5r))^2 \tag{2-34}$$

for any $\hat{q} > q$, where we also used the fact that $K_m(r) \le K_n(r)$ for any $1 \le m \le n$. Finally, to recover the case with the exact exponent q, we may start with a $\tilde{q} < q$ still satisfying $\frac{1}{\tilde{q}} < \frac{1}{2p} + \frac{1}{3}$. Then (2-34) holds for any $\hat{q} > \tilde{q}$, which includes the case $\hat{q} = q$. This proves the desired estimate. □

Now, a bootstrap argument between (2-10) and (2-31) shows that both $\mathcal{M}_t^2[\nabla u^\varepsilon]$ and $\mathcal{M}_t^2[F^\varepsilon]$ are in L^p for any $p \ge 3$. In the following, we use this to prove a large-scale Hölder's estimate for F^ε, which plays an important role in the Lipschitz estimate in the next section.

Theorem 2.8. *Let $L \in (0, \infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. Let $\varepsilon \le t \le r \le \frac{1}{2}$. Let $M \ge 0$ be such that*

$$\left(\fint_{B^\varepsilon_{1,+}} |\nabla u^\varepsilon|^2\right)^{1/2} \le M.$$

For every $l > 3$ and $\delta > 0$ satisfying $l\delta < 6$, we have

$$\left(\fint_{Q_r} |\mathcal{M}_t[F^\varepsilon]|^3\right)^{1/3} \le Cr^{2-6/l}(M + M^{2(4-6/l+\delta)}), \tag{2-35}$$

where the constant C depends only on L, l and δ.

Proof. Note that, by an argument similar to that at the end of the proof of Theorem 2.4, we only have to prove (2-35) when $\varepsilon/N_0 \le t \le r \le 1/N_1$ for some $N_0, N_1 \ge 2$. Let $l > 3$ and $\delta > 0$ with $l\delta < 6$ be given and fixed for the proof. First of all, by the Sobolev embedding theorem, $\|F^\varepsilon\|_{L^3(Q_1)} \le CM^2$. This implies $\|\mathcal{M}_t[F^\varepsilon]\|_{L^3(Q_{1/2})} \le CM^2$. By Theorem 2.4,

$$\left(\fint_{Q_{1/8}} |\mathcal{M}_t[\nabla u^\varepsilon]|^3\right)^{1/3} \le C(M + M^2).$$

Then, applying Theorem 2.7, we obtain that, for any $3 \le p < \infty$,

$$\left(\fint_{Q_{1/40}} |\mathcal{M}_t[F^\varepsilon]|^p\right)^{1/p} \le C\left(\fint_{Q_{1/8}} \mathcal{M}_t[\nabla u^\varepsilon]^3\right)^{2/3} \le C(M + M^4).$$

Now, using Theorem 2.4 again combined with a covering argument, we derive from the last inequality that

$$\left(\fint_{Q_{1/80}} |\mathcal{M}_t[\nabla u^\varepsilon]|^p\right)^{1/p} \le C(M + M^4).$$

Now, let $p > l$. By the interpolation, we have

$$\left(\fint_{Q_{1/80}} |\mathcal{M}_t[\nabla u^\varepsilon]|^l\right)^{1/l} \le \left(\fint_{Q_{1/80}} |\mathcal{M}_t[\nabla u^\varepsilon]|^3\right)^{\theta/3}\left(\fint_{Q_{1/80}} |\mathcal{M}_t[\nabla u^\varepsilon]|^p\right)^{(1-\theta)/p}$$
$$\le C(M+M^2)^\theta(M+M^4)^{1-\theta} \le C(M+M^{4-2\theta}), \tag{2-36}$$

where

$$\frac{1}{l} = \frac{\theta}{3} + \frac{1-\theta}{p}.$$

For the given $\delta \in (0, 1)$, we want $4-2\theta = 4-\frac{6}{l}+\delta$. This implies $\theta = \frac{3}{l}-\frac{\delta}{2}$ and thus we may choose

$$p = \frac{6}{\delta}\Big(1-\frac{3}{l}+\frac{\delta}{2}\Big).$$

One can easily verify that $\theta \in (0, 1)$ by the assumption on l and δ. Consequently, we derive from (2-36) that

$$\left(\fint_{Q_{1/80}} |\mathcal{M}_t[\nabla u^\varepsilon]|^l\right)^{1/l} \le C(M+M^{4-6/l+\delta}).$$

Finally, we apply Theorem 2.7 to obtain for $r \le \frac{1}{400}$,

$$\left(\fint_{Q_r} |\mathcal{M}_t[F^\varepsilon]|^3\right)^{1/3} \le Cr^2\left(\fint_{Q_{5r}} |\mathcal{M}_t[\nabla u^\varepsilon]|^l\right)^{2/l} \le Cr^{2-6/l}\left(\int_{Q_{5r}} |\mathcal{M}_t[\nabla u^\varepsilon]|^l\right)^{2/l}$$
$$\le Cr^{2-6/l}(M+M^{2(4-6/l+\delta)}). \qquad \square$$

Note that if ∇u^ε itself is in L^p for $p > 3$, then Morrey's inequality implies that u^ε is $C^{0,1-3/p}$, which implies, since u^ε vanishes on the boundary,

$$\left(\fint_{Q_r} |F^\varepsilon|^3\right)^{1/3} \le Cr^{2-6/p},$$

where C depends on M, L and p. Hence, (2-35) is consistent with the usual Morrey estimate.

3. Large-scale Lipschitz estimate

In this section, we will establish the large-scale Lipschitz estimate of u^ε and the oscillation estimate of p^ε. We remark, for later use in Section 4B, that Theorem A implies the following Liouville theorem for the Stokes system

$$\begin{cases} -\Delta u + \nabla p = 0 & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3-1}$$

where Ω is a John domain in Definition 1.2. The proof of the following statement is standard.

Corollary 3.1. *Let Ω be a bumpy John domain according to Definition 1.2. Let (u, p) be a weak solution of (3-1). If*

$$\lim_{R\to\infty} \frac{1}{R}\left(\fint_{B_R(0)\cap\Omega} |u|^2\right)^{1/2} = 0,$$

then $u \equiv 0$ (hence p is constant).

3A. ***Set-up and approximation.*** First of all, we may write (NS$^\varepsilon$) as a linear Stokes system

$$\begin{cases} -\Delta u^\varepsilon + \nabla p^\varepsilon = \nabla \cdot F^\varepsilon & \text{in } B^\varepsilon_{1,+}, \\ \nabla \cdot u^\varepsilon = 0 & \text{in } B^\varepsilon_{1,+}, \\ u^\varepsilon = 0 & \text{on } \Gamma^\varepsilon_1, \end{cases} \tag{S$^\varepsilon$}$$

where $F^\varepsilon = -u^\varepsilon \otimes u^\varepsilon$. As in the classical regularity theory for Stokes system, we will use the large-scale $C^{0,\alpha}$ estimate of F^ε in Theorem 2.8 to prove the large-scale Lipschitz estimate. The proof is based on the excess decay method.

Similarly to the large-scale Calderón–Zygmund estimate of Theorem 2.4, we also need to approximate the Stokes system (S$^\varepsilon$) at all scales greater than ε. Fix $r \in \left[\varepsilon, \frac{1}{2}\right]$ and let (v_r, q_r) be the weak solution of the Stokes system

$$\begin{cases} -\Delta v_r + \nabla q_r = 0 & \text{in } Q^\varepsilon_{t_0 r}, \\ \nabla \cdot v_r = 0 & \text{in } Q^\varepsilon_{t_0 r}, \\ v_r = u^\varepsilon & \text{on } \partial Q^\varepsilon_{t_0 r}, \end{cases} \tag{S$_r$}$$

where we have automatically extended u^ε across the bottom boundary by zero-extension and t_0 is a constant in the interval $[1, 2]$ chosen analogously to those in (2-11) and (2-12). Note that (S$_r$) is a special case of (2-12) with $s = r$ and $y = 0$, which means the estimates (2-13)–(2-15) hold also for (v_r, q_r), in place of (w_s, q_s). The following lemma is an analogue of Lemma 2.6.

Lemma 3.2. *Let $L \in (0, \infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. Let $(u^\varepsilon, p^\varepsilon)$ and (v_r, q_r) be weak solutions of (S$^\varepsilon$) and (S$_r$), respectively. If $\varepsilon \in \left(0, \frac{1}{10}\right]$ and $r \in \left[2\varepsilon, \frac{1}{5}\right]$, then*

$$\left(\fint_{B^\varepsilon_{r,+}} |\nabla u^\varepsilon - \nabla v_r|^2 \right)^{1/2} + \left(\fint_{B^\varepsilon_{r/2,+}} \left| p^\varepsilon - q_r - \fint_{B^\varepsilon_{r/2,+}} (p^\varepsilon - q_r) \right|^2 \right)^{1/2} \\ \leq C \left(\frac{\varepsilon}{r} \right)^{1/12} \left(\fint_{B^\varepsilon_{5r,+}} |\nabla u^\varepsilon|^2 \right)^{1/2} + C \left(\fint_{Q_{4r}} |\mathcal{M}^2_\varepsilon [F^\varepsilon]|^3 \right)^{1/3}, \tag{3-2}$$

where C depends only on L.

Proof. Let us set $R^\varepsilon_r = Q^\varepsilon_{t_0 r} \cap \{-\varepsilon < x_3 \leq 2\varepsilon\}$. By examining the proof of Lemma 2.6, we obtain

$$\int_{B^\varepsilon_{t_0 r,+}} |\nabla (u^\varepsilon - v_r)|^2 \leq C \int_{R^\varepsilon_r} |\nabla u^\varepsilon|^2 + C \left(\frac{\varepsilon}{r} \right)^{1/6} \int_{B^\varepsilon_{2r,+}} |\nabla u^\varepsilon|^2 + C \int_{B^\varepsilon_{2r,+}} |F^\varepsilon|^2. \tag{3-3}$$

From Lemma 2.1(iii) and Theorem 2.4 with $p = 3$,

$$\left(\frac{1}{|B^\varepsilon_{r,+}|} \int_{R^\varepsilon_r} |\nabla u^\varepsilon|^2 \right)^{1/2} \leq C \left(\frac{1}{|B^\varepsilon_{r,+}|} \int_{R^\varepsilon_r} |\mathcal{M}^2_\varepsilon [\nabla u^\varepsilon]|^2 \right)^{1/2} \leq C \left(\frac{|R^\varepsilon_r|}{|B^\varepsilon_{r,+}|} \right)^{1/6} \left(\fint_{B^\varepsilon_{r,+}} |\mathcal{M}^2_\varepsilon [\nabla u^\varepsilon]|^3 \right)^{1/3} \\ \leq C \left(\frac{\varepsilon}{r} \right)^{1/6} \left\{ \left(\fint_{B^\varepsilon_{5r,+}} |\nabla u^\varepsilon|^2 \right)^{1/2} + \left(\fint_{Q_{4r}} |\mathcal{M}^2_\varepsilon [F^\varepsilon]|^3 \right)^{1/3} \right\}.$$

Inserting this into (3-3), we have

$$\left(\fint_{B^\varepsilon_{r,+}} |\nabla u^\varepsilon - \nabla v_r|^2 \right)^{1/2} \leq C \left(\frac{\varepsilon}{r} \right)^{1/12} \left(\fint_{B^\varepsilon_{5r,+}} |\nabla u^\varepsilon|^2 \right)^{1/2} + C \left(\fint_{Q_{4r}} |\mathcal{M}^2_\varepsilon [F^\varepsilon]|^3 \right)^{1/3}. \tag{3-4}$$

Next, we estimate the pressure by using the Bogovskii lemma. The issue is that, in general, $B^\varepsilon_{r,+}$ is not a John domain. By Definition 1.2, for $r \geq 2\varepsilon$, there exists a John domain Ω^ε_r with constant L satisfying

$$B^\varepsilon_{r/2,+} \subset \Omega^\varepsilon_r \subset B^\varepsilon_{r,+}.$$

Note that $(u^\varepsilon - v_r,\, p^\varepsilon - q_r)$ satisfies

$$-\Delta(u^\varepsilon - v_r) + \nabla(p^\varepsilon - q_r) = \nabla\cdot F^\varepsilon \quad \text{in } B^\varepsilon_{r,+}.$$

Thus, we may use the Bogovskii lemma in Ω^ε_r and (3-4) to obtain

$$\begin{aligned}\biggl(\fint_{\Omega^\varepsilon_r}\biggl|p^\varepsilon - q_r - \fint_{\Omega^\varepsilon_r}(p^\varepsilon - q_r)\biggr|^2\biggr)^{1/2} &\leq C\biggl(\fint_{\Omega^\varepsilon_r}|\nabla u^\varepsilon - \nabla v_r|^2\biggr)^{1/2} + C\biggl(\fint_{\Omega^\varepsilon_r}|F^\varepsilon|^2\biggr)^{1/2}\\ &\leq C\Bigl(\frac{\varepsilon}{r}\Bigr)^{1/12}\biggl(\fint_{B^\varepsilon_{5r,+}}|\nabla u^\varepsilon|^2\biggr)^{1/2} + C\biggl(\fint_{Q_{4r}}|\mathcal{M}^2_\varepsilon[F^\varepsilon]|^3\biggr)^{1/3},\end{aligned} \tag{3-5}$$

where we also used the fact $|\Omega^\varepsilon_r| \approx |B^\varepsilon_{r,+}|$. Using a well-known fact

$$\int_E\biggl|f - \fint_E f\biggr|^2 = \inf_{a\in\mathbb{R}}\int_E|f-a|^2 \quad \text{for any open set } E,$$

we derive

$$\begin{aligned}\biggl(\fint_{B^\varepsilon_{r/2,+}}\biggl|p^\varepsilon - q_r - \fint_{B^\varepsilon_{r/2,+}}(p^\varepsilon - q_r)\biggr|^2\biggr)^{1/2} &\leq \biggl(\fint_{B^\varepsilon_{r/2,+}}\biggl|p^\varepsilon - q_r - \fint_{\Omega^\varepsilon_r}(p^\varepsilon - q_r)\biggr|^2\biggr)^{1/2}\\ &\leq C\biggl(\fint_{\Omega^\varepsilon_r}\biggl|p^\varepsilon - q_r - \fint_{\Omega^\varepsilon_r}(p^\varepsilon - q_r)\biggr|^2\biggr)^{1/2}.\end{aligned} \tag{3-6}$$

Combining (3-4), (3-5) and (3-6), we obtain the desired estimate. □

Remark 3.3. The pressure estimate in John domains in the proof of Lemma 3.2 is a standard technique that we will frequently use throughout this paper. It allows us to transfer the pressure estimate to the estimates of ∇u^ε and F^ε.

3B. ***Excess decay.*** Let $\mathscr{P}_1 = \{(ax_3, bx_3, 0) \mid a, b \in \mathbb{R}\}$. Note that $\mathscr{P}_1$ consists of all the linear solutions (velocity component) of the Stokes equations in the whole space with the no-slip condition on $\{x_3 = 0\}$. These linear solutions are dubbed as *no-slip Stokes polynomials* of degree 1.

For a pair of functions $(w^\varepsilon, \pi^\varepsilon) \in H^1(B^\varepsilon_{r,+})^3 \times L^2(B^\varepsilon_{r,+})$, with $r \in (0,1]$, we set

$$H(w^\varepsilon, \pi^\varepsilon; \rho) = \inf_{P\in\mathscr{P}_1}\biggl(\fint_{B^\varepsilon_{\rho,+}}|\nabla w^\varepsilon - \nabla P|^2\biggr)^{1/2} + \sup_{s,t\in[1/16,1/4]}\biggl|\fint_{B^\varepsilon_{s\rho,+}}\pi^\varepsilon - \fint_{B^\varepsilon_{t\rho,+}}\pi^\varepsilon\biggr|, \quad \rho\in(0,r], \tag{3-7}$$

$$\Phi(w^\varepsilon, \pi^\varepsilon; \rho) = \biggl(\fint_{B^\varepsilon_{\rho,+}}|\nabla w^\varepsilon|^2\biggr)^{1/2} + \sup_{s,t\in[1/16,1/4]}\biggl|\fint_{B^\varepsilon_{s\rho,+}}\pi^\varepsilon - \fint_{B^\varepsilon_{t\rho,+}}\pi^\varepsilon\biggr|, \quad \rho\in(0,r]. \tag{3-8}$$

The quantity H can be dubbed as a zeroth-order excess quantity. In Section 5 we will consider higher-order excess quantities H_{1st} and H_{2nd} to address the large-scale $C^{1,\gamma}$ and $C^{2,\gamma}$ regularity.

Moreover, for a pair of functions $(w_r, \pi_r) \in H^1(Q_r^\varepsilon)^3 \times L^2(Q_r^\varepsilon)$ with $r \in (0, 1]$, we set

$$\widetilde{H}(w_r, \pi_r; \rho) = \inf_{P \in \mathscr{P}_1} \left(\fint_{Q_\rho^\varepsilon} |\nabla w_r - \nabla P|^2 \right)^{1/2} + \sup_{s,t \in [1/16,1/4]} \left| \fint_{Q_{s\rho}^\varepsilon} \pi_r - \fint_{Q_{t\rho}^\varepsilon} \pi_r \right|, \quad \rho \in (0, r]. \tag{3-9}$$

The following lemma states the comparability between $H(v_r, q_r; \theta r)$ and $\widetilde{H}(v_r, q_r; \theta r)$.

Lemma 3.4. *Let $L \in (0, \infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. Fix $\varepsilon \in \left(0, \frac{1}{4}\right]$, $r \in \left[\varepsilon, \frac{1}{4}\right]$ and let (v_r, q_r) satisfy (S_r). Then we have the following statements*:

(i) *For all $\theta \in (0, 1]$,*

$$H(v_r, q_r; \theta r) \le C \widetilde{H}(v_r, q_r; \theta r) + C\theta^{-1} \left(\frac{\varepsilon}{r} \right)^{1/2} \left(\fint_{Q_{\theta r}^\varepsilon} |\nabla v_r|^2 \right)^{1/2}. \tag{3-10}$$

(ii) *For all $\theta \in (0, 1]$,*

$$\widetilde{H}(v_r, q_r; \theta r) \le C H(v_r, q_r; 2\theta r) + C\theta^{-5/2} \left(\frac{\varepsilon}{r} \right) \left(\fint_{Q_{2\theta r}^\varepsilon} |\nabla v_r|^2 \right)^{1/2}. \tag{3-11}$$

Here C depends only on L.

Proof. (i) We first deal with v_r. Since $B_{\theta r,+}^\varepsilon \subset Q_{\theta r}^\varepsilon$ and $|B_{\theta r,+}^\varepsilon| \approx |Q_{\theta r}^\varepsilon|$, we have

$$\inf_{P \in \mathscr{P}_1} \left(\fint_{B_{\theta r,+}^\varepsilon} |\nabla v_r - \nabla P|^2 \right)^{1/2} \le C \inf_{P \in \mathscr{P}_1} \left(\fint_{Q_{\theta r}^\varepsilon} |\nabla v_r - \nabla P|^2 \right)^{1/2}.$$

On the other hand, the triangle inequality implies

$$\begin{aligned} &\sup_{s,t \in [1/16,1/4]} \left| \fint_{B_{\theta sr,+}^\varepsilon} q_r - \fint_{B_{\theta tr,+}^\varepsilon} q_r \right| \\ &\qquad \le \sup_{s,t \in [1/16,1/4]} \left| \fint_{Q_{\theta sr}^\varepsilon} q_r - \fint_{Q_{\theta tr}^\varepsilon} q_r \right| + 2 \sup_{\rho \in [1/16,1/4]} \left| \fint_{Q_{\theta \rho r}^\varepsilon} q_r - \fint_{B_{\theta \rho r,+}^\varepsilon} q_r \right|. \end{aligned} \tag{3-12}$$

Combining the above two inequalities, we obtain

$$H(v_r, q_r; \theta r) \le C \widetilde{H}(v_r, q_r; \theta r) + 2 \sup_{\rho \in [1/16,1/4]} \left| \fint_{Q_{\theta \rho r}^\varepsilon} q_r - \fint_{B_{\theta \rho r,+}^\varepsilon} q_r \right|. \tag{3-13}$$

Since $|Q_{\theta\rho r}^\varepsilon \setminus B_{\theta\rho r,+}^\varepsilon|$ is less than $C\varepsilon(\theta\rho r)^2$, a direct computation yields

$$\begin{aligned} \left| \fint_{Q_{\theta\rho r}^\varepsilon} q_r - \fint_{B_{\theta\rho r,+}^\varepsilon} q_r \right| &\le \left(\frac{1}{|B_{\theta\rho r,+}^\varepsilon|} - \frac{1}{|Q_{\theta\rho r}^\varepsilon|} \right) \left| \int_{Q_{\theta\rho r}^\varepsilon} \left(q_r - \fint_{Q_{\theta r}^\varepsilon} q_r \right) \right| + \frac{1}{|B_{\theta\rho r,+}^\varepsilon|} \left| \int_{Q_{\theta\rho r}^\varepsilon \setminus B_{\theta\rho r,+}^\varepsilon} \left(q_r - \fint_{Q_{\theta r}^\varepsilon} q_r \right) \right| \\ &\le \left(\frac{|Q_{\theta\rho r}^\varepsilon \setminus B_{\theta\rho r,+}^\varepsilon| \, |Q_{\theta r}^\varepsilon|^{1/2}}{|B_{\theta\rho r,+}^\varepsilon| \, |Q_{\theta\rho r}^\varepsilon|^{1/2}} + \frac{|Q_{\theta\rho r}^\varepsilon \setminus B_{\theta\rho r,+}^\varepsilon|^{1/2} |Q_{\theta r}^\varepsilon|^{1/2}}{|B_{\theta\rho r,+}^\varepsilon|} \right) \left(\fint_{Q_{\theta r}^\varepsilon} \left| q_r - \fint_{Q_{\theta r}^\varepsilon} q_r \right|^2 \right)^{1/2} \\ &\le C \left(\theta^{-1} \rho^{-5/2} \left(\frac{\varepsilon}{r} \right) + \theta^{-1/2} \rho^{-2} \left(\frac{\varepsilon}{r} \right)^{1/2} \right) \left(\fint_{Q_{\theta r}^\varepsilon} |\nabla v_r|^2 \right)^{1/2}, \end{aligned} \tag{3-14}$$

where we have applied the Hölder inequality in the second inequality and the Bogovskii lemma in $Q^{\varepsilon}_{\theta r}$ in the third inequality. Noting $\rho \geq \frac{1}{16}$ and using (3-13) and (3-14), we obtain the first inequality (3-10).

(ii) Let $P_* \in \mathscr{P}_1$ be such that

$$\left(\fint_{B^{\varepsilon}_{\theta r,+}} |\nabla v_r - \nabla P_*|^2\right)^{1/2} = \inf_{P\in\mathscr{P}_1}\left(\fint_{B^{\varepsilon}_{\theta r,+}} |\nabla v_r - \nabla P|^2\right)^{1/2}.$$

Since $v_r(x) - P_*(x+\varepsilon \boldsymbol{e}_3)$ is a weak solution to (S_r) with the same pressure q_r and $P_*(x+\varepsilon\boldsymbol{e}_3) = P_*(x) + \varepsilon(\nabla P_*)\boldsymbol{e}_3$, by the Bogovskii lemma in $Q^{\varepsilon}_{\theta r}$, we see that

$$\begin{aligned}\sup_{s,t\in[1/16,1/4]}\left|\fint_{Q^{\varepsilon}_{\theta sr,+}} q_r - \fint_{Q^{\varepsilon}_{\theta tr,+}} q_r\right| &\leq 2\sup_{\rho\in[1/16,1/4]}\left(\fint_{Q^{\varepsilon}_{\theta\rho r}}\left|q_r - \fint_{Q^{\varepsilon}_{\theta r}} q_r\right|^2\right)^{1/2}\\ &\leq C\left(\fint_{Q^{\varepsilon}_{\theta r}} |\nabla v_r - \nabla P_*|^2\right)^{1/2}. \qquad (3\text{-}15)\end{aligned}$$

Moreover, notice that $v_r - \widetilde{P}(x+\varepsilon\boldsymbol{e}_3)$ vanishes on $x_3 = -\varepsilon$, so that by the Caccioppoli inequality (see Lemma A.3 in the rectangular region $Q^{\varepsilon}_{2\theta r}$ translated by $\boldsymbol{e}_3$), we have

$$\begin{aligned}\left(\fint_{Q^{\varepsilon}_{\theta r}} |\nabla v_r - \nabla P_*|^2\right)^{1/2} &\leq \frac{C}{\theta r}\left(\fint_{Q^{\varepsilon}_{2\theta r}} |v_r - P_*(x+\varepsilon\boldsymbol{e}_3)|^2\,\mathrm{d}x\right)^{1/2}\\ &\leq \frac{C}{\theta r}\left(\fint_{B^{\varepsilon}_{2\theta r,+}} |v_r - P_*|^2\right)^{1/2} + C\theta^{-1}\left(\frac{\varepsilon}{r}\right)|\nabla P_*|\\ &\quad + \frac{C}{(\theta r)^{5/2}}\left\{\left(\int_{Q^{\varepsilon}_{2\theta r}\setminus B^{\varepsilon}_{2\theta r,+}} |v_r|^2\right)^{1/2} + \left(\int_{Q^{\varepsilon}_{2\theta r}\setminus B^{\varepsilon}_{2\theta r,+}} |P_*(x+\varepsilon\boldsymbol{e}_3)|^2\right)^{1/2}\right\}. \qquad (3\text{-}16)\end{aligned}$$

Now (3-15) and (3-16) combined with the Poincaré inequality imply

$$\begin{aligned}\widetilde{H}(v_r, q_r; \theta r) &\leq CH(v_r, q_r; 2\theta r) + C\theta^{-1}\left(\frac{\varepsilon}{r}\right)|\nabla P_*|\\ &\quad + \frac{C}{(\theta r)^{5/2}}\left\{\left(\int_{Q^{\varepsilon}_{2\theta r}\setminus B^{\varepsilon}_{2\theta r,+}} |v_r|^2\right)^{1/2} + \left(\int_{Q^{\varepsilon}_{2\theta r}\setminus B^{\varepsilon}_{2\theta r,+}} |P_*(x+\varepsilon\boldsymbol{e}_3)|^2\right)^{1/2}\right\}. \qquad (3\text{-}17)\end{aligned}$$

By the definition of P_*, we have

$$|\nabla P_*| \leq C\left(\fint_{B^{\varepsilon}_{2\theta r,+}} |\nabla v_r|^2\right)^{1/2} \leq C\left(\fint_{Q^{\varepsilon}_{2\theta r}} |\nabla v_r|^2\right)^{1/2}. \qquad (3\text{-}18)$$

Consequently,

$$\begin{aligned}&\frac{C}{(\theta r)^{5/2}}\left\{\left(\int_{Q^{\varepsilon}_{2\theta r}\setminus B^{\varepsilon}_{2\theta r,+}} |v_r|^2\right)^{1/2} + \left(\int_{Q^{\varepsilon}_{2\theta r}\setminus B^{\varepsilon}_{2\theta r,+}} |P_*(x+\varepsilon\boldsymbol{e}_3)|^2\right)^{1/2}\right\}\\ &\qquad\leq \frac{C}{(\theta r)^{5/2}}\left\{\varepsilon\left(\int_{Q^{\varepsilon}_{2\theta r}\setminus B^{\varepsilon}_{2\theta r,+}} |\nabla v_r|^2\right)^{1/2} + \varepsilon(\varepsilon\theta^2 r^2)^{1/2}|\nabla P_*|\right\}\\ &\qquad\leq C\left(\theta^{-5/2}\left(\frac{\varepsilon}{r}\right) + \theta^{-3/2}\left(\frac{\varepsilon}{r}\right)^{3/2}\right)\left(\fint_{Q^{\varepsilon}_{2\theta r}} |\nabla v_r|^2\right)^{1/2}. \qquad (3\text{-}19)\end{aligned}$$

Hence, we obtain (3-11) from (3-17) combined with (3-18) and (3-19). □

Lemma 3.5. *Let $L \in (0,\infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. Fix $\alpha \in (0,1]$ arbitrarily. For all $\varepsilon \in \big(0,\frac{1}{2}\big]$, $r \in \big[\varepsilon,\frac{1}{2}\big]$, $\theta \in \big(0,\frac{1}{8}\big]$, and (v_r,q_r) satisfying* (S_r),

$$\widetilde{H}(v_r,q_r;\theta r) \le C\bigg(\theta^\alpha + \theta^{-1}\Big(\frac{\varepsilon}{r}\Big)\bigg)\widetilde{H}(v_r,q_r;r), \tag{3-20}$$

where C depends only on L and α.

Proof. By the regularity of the Stokes equations in flat domains,

$$v_r \in C^{1,\alpha}(Q^\varepsilon_{r/2}), \quad q_r \in C^{0,\alpha}(Q^\varepsilon_{r/2}).$$

Let $\boldsymbol{e}_3 = (0,0,1)$. The boundary $C^{1,\alpha}$ estimate of v_r on $\{x_3 = -\varepsilon\}$ implies

$$|v_r(x) - v_r(-\varepsilon\boldsymbol{e}_3) - (x_3+\varepsilon)\partial_3 v_r(-\varepsilon\boldsymbol{e}_3)| \le C\frac{|x+\varepsilon\boldsymbol{e}_3|^{1+\alpha}}{r^\alpha}\bigg(\fint_{Q^\varepsilon_{r/2}}|\nabla v_r|^2\bigg)^{1/2}.$$

Note that $\partial_3 v_{r,3}(-\varepsilon\boldsymbol{e}_3) = 0$ by the condition $\nabla\cdot v_r = 0$. Thus, from $v_r(-\varepsilon\boldsymbol{e}_3) = 0$, there exists $\widetilde{P}(x) = (\partial_3 v_{r,1}(-\varepsilon\boldsymbol{e}_3), \partial_3 v_{r,2}(-\varepsilon\boldsymbol{e}_3), 0)x_3 \in \mathscr{P}_1$ and

$$|v_r(x) - \widetilde{P}(x+\varepsilon\boldsymbol{e}_3)| \le C\frac{|x_3+\varepsilon\boldsymbol{e}_3|^{1+\alpha}}{r^\alpha}\bigg(\fint_{Q^\varepsilon_{r/2}}|\nabla v_r|^2\bigg)^{1/2}$$

for all $x \in Q^\varepsilon_{\theta r}$. Since $v_r(x) - \widetilde{P}(x+\varepsilon\boldsymbol{e}_3)$ is a weak solution to (S_r) with the same pressure q_r, by the Caccioppoli inequality (see Lemma A.3) in $Q^\varepsilon_{2\theta r}$, we have

$$\begin{aligned}\bigg(\fint_{Q^\varepsilon_{\theta r}}|\nabla v_r - \nabla\widetilde{P}(x+\varepsilon\boldsymbol{e}_3)|^2\,\mathrm{d}x\bigg)^{1/2} &\le \frac{C}{\theta r}\bigg(\fint_{Q^\varepsilon_{2\theta r}}|v_r - \widetilde{P}(x+\varepsilon\boldsymbol{e}_3)|^2\,\mathrm{d}x\bigg)^{1/2}\\ &\le C\bigg(\theta^\alpha + \theta^{-1}\Big(\frac{\varepsilon}{r}\Big)\bigg)\bigg(\fint_{Q^\varepsilon_{r/2}}|\nabla v_r|^2\bigg)^{1/2}.\end{aligned}$$

Then the observation $\widetilde{P}(x+\varepsilon\boldsymbol{e}_3) = \widetilde{P}(x) + \varepsilon(\nabla\widetilde{P})\boldsymbol{e}_3$ yields

$$\bigg(\fint_{Q^\varepsilon_{\theta r}}|\nabla v_r - \nabla\widetilde{P}|^2\bigg)^{1/2} \le C\bigg(\theta^\alpha + \theta^{-1}\Big(\frac{\varepsilon}{r}\Big)\bigg)\bigg(\fint_{Q^\varepsilon_{r/2}}|\nabla v_r|^2\bigg)^{1/2}. \tag{3-21}$$

The $C^{0,\alpha}$ estimate of q_r implies

$$|q_r(x) - q_r(0)| \le C\frac{|x|^\alpha}{r^\alpha}\bigg(\fint_{Q^\varepsilon_{r/2}}\Big|q_r - \fint_{Q^\varepsilon_{r/2}} q_r\Big|^2\bigg)^{1/2}.$$

Then by the Bogovskii lemma in a Lipschitz domain $Q^\varepsilon_{r/2}$, we have

$$|q_r(x) - q_r(0)| \le C\frac{|x|^\alpha}{r^\alpha}\bigg(\fint_{Q^\varepsilon_{r/2}}|\nabla v_r|^2\bigg)^{1/2},$$

which results in

$$\sup_{s,t\in[1/16,1/4]}\bigg|\fint_{Q^\varepsilon_{\theta sr}} q_r - \fint_{Q^\varepsilon_{\theta tr}} q_r\bigg| \le C\theta^\alpha\bigg(\fint_{Q^\varepsilon_{r/2}}|\nabla v_r|^2\bigg)^{1/2}. \tag{3-22}$$

Hence on the one hand, by (3-21) and (3-22) we see that

$$\widetilde{H}(v_r, q_r; \theta r) \le C\left(\theta^\alpha + \theta^{-1}\left(\frac{\varepsilon}{r}\right)\right)\left(\fint_{Q_r^\varepsilon} |\nabla v_r|^2\right)^{1/2}. \tag{3-23}$$

On the other hand, since $v_r(x) - P(x + \varepsilon \boldsymbol{e}_3)$, for any $P \in \mathscr{P}_1$, is a weak solution to (S_r) with the same pressure q_r and $P(x+\varepsilon \boldsymbol{e}_3) = P(x) + \varepsilon(\nabla P)\boldsymbol{e}_3$, we may apply (3-23) to $v_r(x) - P(x+\varepsilon \boldsymbol{e}_3)$ and obtain

$$\widetilde{H}(v_r - P(x+\varepsilon \boldsymbol{e}_3), q_r; \theta r) \le C\left(\theta^\alpha + \theta^{-1}\left(\frac{\varepsilon}{r}\right)\right)\left(\fint_{Q_r^\varepsilon} |\nabla v_r - \nabla P|^2 \,dx\right)^{1/2}. \tag{3-24}$$

In particular, we may choose $P = P_*$ that minimizes

$$\left(\fint_{Q_r^\varepsilon} |\nabla v_r - \nabla P|^2\right)^{1/2}.$$

Then, it is clear that

$$\left(\fint_{Q_r^\varepsilon} |\nabla v_r - \nabla P_*|^2\right)^{1/2} \le \widetilde{H}(v_r, q_r; r). \tag{3-25}$$

Thus the estimate (3-20) follows from (3-24) with $P = P_*$ and (3-25). □

Lemma 3.6. *Let $L \in (0,\infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. Let $\alpha \in (0,1]$ be the number in Lemma 3.5. For all $\varepsilon \in \left(0, \frac{1}{4}\right]$, $r \in \left[\varepsilon, \frac{1}{4}\right]$, $\theta \in \left(0, \frac{1}{8}\right]$, and (v_r, q_r) satisfying (S_r),*

$$H(v_r, q_r; \theta r) \le C\theta^\alpha H(v_r, q_r; 2r) + C\theta^{-5/2}\left(\frac{\varepsilon}{r}\right)^{1/2}\left(\fint_{Q_{2r}} |\nabla v_r|^2\right)^{1/2}, \tag{3-26}$$

where C depends only on L and α.

Proof. The estimate (3-26) follows readily from Lemmas 3.4 and 3.5. □

Lemma 3.7. *Let $L \in (0,\infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. Let $(u^\varepsilon, p^\varepsilon)$ be a weak solution of (S^ε) and let $\alpha \in (0,1]$ be the number in Lemma 3.5. For all $\varepsilon \in \left(0, \frac{1}{32}\right]$, $r \in \left[2\varepsilon, \frac{1}{16}\right]$ and $\theta \in \left(0, \frac{1}{8}\right]$,*

$$H(u^\varepsilon, p^\varepsilon; \theta r) \le C\theta^\alpha H(u^\varepsilon, p^\varepsilon; 2r) + C\theta^{-3}\left(\frac{\varepsilon}{r}\right)^{1/12}\Phi(u^\varepsilon, p^\varepsilon; 16r) + C\theta^{-3}\left(\fint_{Q_{10r}} |\mathcal{M}_\varepsilon^2[F^\varepsilon]|^3\right)^{1/3}, \tag{3-27}$$

where C depends only on L and α.

Proof. The triangle inequality and Lemma 3.6 imply

$$\begin{aligned}
H(u^\varepsilon, p^\varepsilon; \theta r) &\le H(v_r, q_r; \theta r) + H(u^\varepsilon - v_r, p^\varepsilon - q_r; \theta r) \\
&\le C\theta^\alpha H(v_r, q_r; 2r) + H(u^\varepsilon - v_r, p^\varepsilon - q_r; \theta r) + C\theta^{-5/2}\left(\frac{\varepsilon}{r}\right)^{1/2}\left(\fint_{Q_{2r}^\varepsilon} |\nabla v_r|^2\right)^{1/2} \\
&\le C\theta^\alpha H(u^\varepsilon, p^\varepsilon; 2r) + C\theta^\alpha H(u^\varepsilon - v_r, p^\varepsilon - q_r; r) \\
&\qquad + H(u^\varepsilon - v_r, p^\varepsilon - q_r; \theta r) + C\theta^{-5/2}\left(\frac{\varepsilon}{r}\right)^{1/2}\left(\fint_{B_{4r,+}^\varepsilon} |\nabla u^\varepsilon|^2\right)^{1/2},
\end{aligned} \tag{3-28}$$

where in the last line the energy estimate of (S$_r$) is applied. By the definition of H, we find

$$\begin{aligned}&\theta^\alpha H(u^\varepsilon - v_r, p^\varepsilon - q_r; r) + H(u^\varepsilon - v_r, p^\varepsilon - q_r; \theta r)\\ &\qquad \le C(\theta^\alpha + \theta^{-3})\Big\{\Big(\fint_{B^\varepsilon_{r,+}} |\nabla u^\varepsilon - \nabla v_r|^2\Big)^{1/2} + \sup_{\rho\in[1/16,1/4]} \fint_{B^\varepsilon_{\rho r,+}} \Big| p^\varepsilon - q_r - \fint_{B^\varepsilon_{r/2,+}} (p^\varepsilon - q_r)\Big|\Big\}. \end{aligned}\tag{3-29}$$

The Poincaré inequality and Lemma 3.2 imply

$$\begin{aligned}&\Big(\fint_{B^\varepsilon_{r,+}} |\nabla u^\varepsilon - \nabla v_r|^2\Big)^{1/2} + \sup_{\rho\in[1/16,1/4]} \fint_{B^\varepsilon_{\rho r,+}} \Big| p^\varepsilon - q_r - \fint_{B^\varepsilon_{r/2,+}} (p^\varepsilon - q_r)\Big|\\ &\qquad \le C\Big\{\Big(\fint_{B^\varepsilon_{r,+}} |\nabla u^\varepsilon - \nabla v_r|^2\Big)^{1/2} + \Big(\fint_{B^\varepsilon_{r/2,+}} \Big| p^\varepsilon - q_r - \fint_{B^\varepsilon_{r/2,+}} (p^\varepsilon - q_r)\Big|^2\Big)^{1/2}\Big\}\\ &\qquad \le C\Big(\frac{\varepsilon}{r}\Big)^{1/12}\Big(\fint_{B^\varepsilon_{5r,+}} |\nabla u^\varepsilon|^2\Big)^{1/2} + C\Big(\fint_{Q_{4r}} |\mathcal{M}^2_\varepsilon[F^\varepsilon]|^3\Big)^{1/3}. \end{aligned}\tag{3-30}$$

Now from (3-28) to (3-30), we obtain the desired estimate (3-27) by the definition of Φ in (3-8). □

3C. *Iteration.* In the following two lemmas, we prove some properties of H and Φ needed when iterating (3-27).

Lemma 3.8. *Let $L \in (0, \infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. Let $(u^\varepsilon, p^\varepsilon)$ be a weak solution of (S$^\varepsilon$). There exists a function $h(r)$ defined on $\big[\varepsilon, \frac{1}{2}\big]$ such that*

$$h(r) \le C(H(u^\varepsilon, p^\varepsilon; r) + \Phi(u^\varepsilon, p^\varepsilon; r)), \tag{3-31}$$

$$\Phi(u^\varepsilon, p^\varepsilon; r) \le C(H(u^\varepsilon, p^\varepsilon; r) + h(r)), \tag{3-32}$$

$$\sup_{r_1, r_2 \in [r, 2r]} |h(r_1) - h(r_2)| \le C H(u^\varepsilon, p^\varepsilon; 2r). \tag{3-33}$$

Here C depends only on L. Notice that the function h depends on u^ε.

Proof. The proof is similar to [Gu and Zhuge 2022, Lemma 6.1] and hence we provide the outline of the proof. Let $P_r \in \mathscr{P}_1$ be such that

$$\Big(\fint_{B^\varepsilon_{r,+}} |\nabla u^\varepsilon - \nabla P_r|^2\Big)^{1/2} = \inf_{P\in\mathscr{P}_1} \Big(\fint_{B^\varepsilon_{r,+}} |\nabla u^\varepsilon - \nabla P|^2\Big)^{1/2}.$$

We define

$$h(r) = |\nabla P_r|, \quad r \in \big[\varepsilon, \tfrac{1}{2}\big].$$

Then the inequality (3-31) follows from

$$h(r) \le C\Big(\fint_{B^\varepsilon_{r,+}} |\nabla P_r|^2\Big)^{1/2} \le C(H(u^\varepsilon, p^\varepsilon; r) + \Phi(u^\varepsilon, p^\varepsilon; r))$$

and (3-32) is trivial by definition. For (3-33), we observe that for any $r_1, r_2 \in [r, 2r]$

$$|h(r_1) - h(r_2)| \le C\Big(\fint_{B^\varepsilon_{r,+}} |\nabla P_{r_1} - \nabla P_{r_2}|^2\Big)^{1/2} \le C H(u^\varepsilon, p^\varepsilon; 2r).$$

□

Lemma 3.9. *Let $L \in (0,\infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. Let $(u^\varepsilon, p^\varepsilon)$ be a weak solution of (S^ε). Then for $\varepsilon \in \left(0, \frac{1}{8}\right]$, $r \in \left[2\varepsilon, \frac{1}{4}\right]$,*

$$\sup_{\tau\in[r,2r]} \Phi(u^\varepsilon, p^\varepsilon; \tau) \le C\Phi(u^\varepsilon, p^\varepsilon; 2r) + \left(\fint_{Q_{2r}} |\mathcal{M}_\varepsilon^2[F^\varepsilon]|^3\right)^{1/3}, \tag{3-34}$$

where C depends only on L.

Proof. Let $\tau \in [r, 2r]$. A simple computation implies

$$\left(\fint_{B^\varepsilon_{\tau,+}} |\nabla u^\varepsilon|^2\right)^{1/2} \le C\left(\fint_{B^\varepsilon_{2r,+}} |\nabla u^\varepsilon|^2\right)^{1/2}. \tag{3-35}$$

For the pressure estimate, by a similar argument as in (3-6),

$$\begin{aligned}\sup_{s,t\in[1/16,1/4]}\left|\fint_{B^\varepsilon_{s\tau,+}} p^\varepsilon - \fint_{B^\varepsilon_{t\tau,+}} p^\varepsilon\right| &\le C\fint_{B^\varepsilon_{r/2,+}}\left|p^\varepsilon - \fint_{B^\varepsilon_{r/2,+}} p^\varepsilon\right| \\ &\le C\left\{\left(\fint_{B^\varepsilon_{r,+}} |\nabla u^\varepsilon|^2\right)^{1/2} + \left(\fint_{B^\varepsilon_{r,+}} |F^\varepsilon|^2\right)^{1/2}\right\}, \end{aligned}\tag{3-36}$$

where we need to assume $r \ge 2\varepsilon$. Then (3-34) follows from (3-35), (3-36) and Lemma 2.1(iii). □

We now state the iteration lemma. Its proof is given in Appendix C.

Lemma 3.10. *Let $H, \Phi, h : (0,1] \to [0,\infty)$ be nonnegative functions. Let $\varepsilon \in \left(0, \frac{1}{48}\right]$. Suppose that there exist positive constants C_0, B_0, α, β and $\theta \in \left(0, \frac{1}{8}\right]$ so that*

$$H(\theta r) \le \tfrac{1}{2}H(2r) + C_0\left(\left(\frac{\varepsilon}{r}\right)^\alpha \Phi(16r) + B_0 r^\beta\right), \quad r \in \left[\varepsilon, \tfrac{1}{16}\right], \tag{3-37a}$$

$$H(r) \le C_0\Phi(r), \quad r \in \left[\varepsilon, \tfrac{1}{2}\right], \tag{3-37b}$$

$$\sup_{\tau\in[r,2r]} \Phi(\tau) \le C_0(\Phi(2r) + B_0 r^\beta), \quad r \in \left[\varepsilon, \tfrac{1}{4}\right], \tag{3-37c}$$

$$h(r) \le C_0(H(r) + \Phi(r)), \quad r \in \left[\varepsilon, \tfrac{1}{2}\right], \tag{3-37d}$$

$$\Phi(r) \le C_0(H(r) + h(r)), \quad r \in \left[\varepsilon, \tfrac{1}{2}\right], \tag{3-37e}$$

$$\sup_{r_1,r_2\in[r,2r]} |h(r_1) - h(r_2)| \le C_0 H(2r), \quad r \in \left[\varepsilon, \tfrac{1}{4}\right]. \tag{3-37f}$$

Then,

$$\int_\varepsilon^{1/2} \frac{H(t)}{t}\,\mathrm{d}t + \sup_{r\in[\varepsilon,1/2]} \Phi(r) \le C\left(\Phi\left(\tfrac{1}{2}\right) + B_0\right), \tag{3-38}$$

where the constant C depends only on C_0, α, β and θ.

Proof of Theorem A. In the following proof, we actually only need to show (1-2) for the case $N_0\varepsilon \le r \le 1/N_1$ for some $N_0, N_1 \ge 2$. The case $\frac{1}{2} \ge r \ge 1/N_1$ follows trivially by enlarging the size of the cube and a standard pressure estimate (see Remark 3.3); the case $\varepsilon \le r \le N_0\varepsilon$ follows from the case $r = N_0\varepsilon$. From the previous lemmas, we can choose $N_0 = 4$ and $N_1 = 16$. Hence, we may assume without loss of generality that $r \in \left[4\varepsilon, \frac{1}{16}\right]$.

We apply Lemma 3.10 to $H(r) = H(u^\varepsilon, p^\varepsilon; r)$ and $\Phi(r) = \Phi(u^\varepsilon, p^\varepsilon; r)$. Choose θ sufficiently small so that we have $C\theta^\alpha \le \frac{1}{2}$ in (3-27) in Lemma 3.7. We need to verify the conditions in Lemma 3.10. Note that (3-37b) is obvious and (3-37d)–(3-37f) follow from Lemma 3.8. To verify (3-37a) from Lemma 3.7 and (3-37c) from Lemma 3.9 (with ε replaced by 4ε), it suffices to note that Theorem 2.8 implies

$$\left(\fint_{Q_r} |\mathcal{M}_\varepsilon^2[F^\varepsilon]|^3\right)^{1/3} \le C(M + M^{4+2\beta+2\delta}) r^\beta \tag{3-39}$$

for any $\beta \in (0, 2)$, $\delta \in (0, 1)$ with $\beta + \delta < 2$ and $r \in \left[\varepsilon, \frac{1}{2}\right]$. Hence, we may apply Lemma 3.10 with $B_0 = C(M + M^{4+2\beta+2\delta})$ to obtain

$$\begin{aligned}\int_{4\varepsilon}^{1/2} \frac{H(u^\varepsilon, p^\varepsilon; t)}{t}\, dt + \sup_{r\in[4\varepsilon,1/2]} \Phi(u^\varepsilon, p^\varepsilon; r) &\le C(\Phi(u^\varepsilon, p^\varepsilon; \,) + (M + M^{4+2\beta+2\delta}))\\ &\le C(M + M^{4+2\beta+2\delta}),\end{aligned} \tag{3-40}$$

where in the last inequality, we have used a standard pressure estimate (see Remark 3.3) to bound $\Phi\big(u^\varepsilon, p^\varepsilon; \frac{1}{2}\big)$ by $C(M + M^2)$. Hence, for $r \in \left[4\varepsilon, \frac{1}{16}\right]$,

$$\begin{aligned}\left(\fint_{B^\varepsilon_{r,+}} |\nabla u^\varepsilon|^2\right)^{1/2} &\le C\big(\Phi\big(u^\varepsilon, p^\varepsilon; \tfrac{1}{2}\big) + (M + M^{4+2\beta+2\delta})\big)\\ &\le C(M + M^{4+2\beta+2\delta}),\end{aligned}$$

which proves the desired estimate of the velocity u^ε.

Next, we give an estimate for the pressure. For $r \in \left[\varepsilon, \frac{1}{4}\right]$, we observe that

$$\left(\fint_{B^\varepsilon_{r,+}} |p^\varepsilon - \fint_{B^\varepsilon_{1/2,+}} p^\varepsilon|^2\right)^{1/2} \le \left(\fint_{B^\varepsilon_{r,+}} |p^\varepsilon - \fint_{B^\varepsilon_{r,+}} p^\varepsilon|^2\right)^{1/2} + \left|\fint_{B^\varepsilon_{r,+}} p^\varepsilon - \fint_{B^\varepsilon_{1/2,+}} p^\varepsilon\right|.$$

Using the technique as in (3-6) and by the Bogovskii lemma, the desired estimate of ∇u^ε just proved and (3-39), we have

$$\begin{aligned}\left(\fint_{B^\varepsilon_{r,+}} \left|p^\varepsilon - \fint_{B^\varepsilon_{r,+}} p^\varepsilon\right|^2\right)^{1/2} &\le C\left\{\left(\fint_{B^\varepsilon_{2r,+}} |\nabla u^\varepsilon|^2\right)^{1/2} + \left(\fint_{B^\varepsilon_{2r,+}} |F^\varepsilon|^2\right)^{1/2}\right\}\\ &\le C(M + M^{4+2\beta+2\delta}).\end{aligned}$$

On the other hand, let $N \in \mathbb{N}$ be such that $2^N r \in \left[\frac{1}{32}, \frac{1}{16}\right]$. Then

$$\left|\fint_{B^\varepsilon_{r,+}} p^\varepsilon - \fint_{B^\varepsilon_{1/2,+}} p^\varepsilon\right| \le \sum_{j=0}^{N-1} \left|\fint_{B^\varepsilon_{2^j r,+}} p^\varepsilon - \fint_{B^\varepsilon_{2^{j+1} r,+}} p^\varepsilon\right| + \left|\fint_{B^\varepsilon_{2^N r,+}} p^\varepsilon - \fint_{B^\varepsilon_{1/2,+}} p^\varepsilon\right|.$$

Now, observe that for each $j = 0, 1, \dots, N-1$,

$$\left|\fint_{B^\varepsilon_{2^j r,+}} p^\varepsilon - \fint_{B^\varepsilon_{2^{j+1} r,+}} p^\varepsilon\right| \le 4 \int_{2^{j+3} r}^{2^{j+4} r} \frac{1}{\tilde{r}} \sup_{s,t\in[1/16,1/4]} \left|\fint_{B^\varepsilon_{s\tilde{r},+}} p^\varepsilon - \fint_{B^\varepsilon_{t\tilde{r},+}} p^\varepsilon\right| d\tilde{r}.$$

Thus, (3-40) leads to

$$\sum_{j=0}^{N-1}\left|\fint_{B^\varepsilon_{2^j r,+}} p^\varepsilon - \fint_{B^\varepsilon_{2^{j+1} r,+}} p^\varepsilon\right| \le 4\int_\varepsilon^{1/2}\frac{1}{\tilde r}\sup_{s,t\in[1/16,1/4]}\left|\fint_{B^\varepsilon_{s\tilde r,+}} p^\varepsilon - \fint_{B^\varepsilon_{t\tilde r,+}} p^\varepsilon\right| d\tilde r$$
$$\le C(M+M^{4+2\beta+2\delta}).$$

Finally, by the same trick as in (3-6), we obtain

$$\left|\fint_{B^\varepsilon_{2^N r,+}} p^\varepsilon - \fint_{B^\varepsilon_{1/2,+}} p^\varepsilon\right| \le C\left\{\left(\fint_{B^\varepsilon_{1,+}}|\nabla u^\varepsilon|^2\right)^{1/2} + \left(\fint_{B^\varepsilon_{1,+}}|F^\varepsilon|^2\right)^{1/2}\right\}$$
$$\le C(M+M^2).$$

Summarizing up the above estimates, we obtain the desired estimate for the pressure p^ε. □

4. Boundary layers in bumpy John domains

As seen in the previous section, the no-slip Stokes polynomials of degree 1 (i.e., the basis of $\mathscr{P}_1$)

$$P^{(11)} = (x_3, 0, 0), \qquad P^{(12)} = (0, x_3, 0) \tag{4-1}$$

are the key ingredients for the large-scale Lipschitz estimate. Their trace on nonflat bumpy boundaries can be corrected by adding boundary layer correctors. Consequently, one obtains polynomial solutions of the Stokes equations in the bumpy John domains considered in this paper.

In Section 4A, we determine the no-slip Stokes polynomials of degree 2 by explicit computation. The boundary layer equations are introduced as well. Sections 4B and 4C are respectively devoted to the analysis of the first-order and the second-order boundary layer equations. The estimates for the Green's function, obtained in Appendix B using the large-scale Lipschitz estimate of Theorem A, play a fundamental role. We summarize the estimates for the boundary layers in Section 4D. These estimates are key to the theory of higher-order regularity in Section 5.

4A. ***No-slip Stokes polynomials.*** Let u be a solution of $-\Delta u+\nabla p=0$ and $\nabla\cdot u=0$ in $Q_{1,+}(0)$ and $u=0$ on $\partial\mathbb{R}^3_+\cap B_1(0)$. The real analyticity of u in $Q_{1/2,+}(0)$ is classical and well known; see [Masuda 1967; Giga 1983]. Here we want to identify the form of the no-slip Stokes polynomials of degree 2 of u at 0.

Let $P(x) = (P_1(x), P_2(x), P_3(x))$ be the no-slip Stokes polynomials of degree 2 of u at 0. First of all, since $u=0$ on $\partial\mathbb{R}^3_+$, then we must have

$$\begin{aligned} P_1(x) &= a_1x_3 + b_{11}x_1x_3 + b_{12}x_2x_3 + b_{13}x_3^2,\\ P_2(x) &= a_2x_3 + b_{21}x_1x_3 + b_{22}x_2x_3 + b_{23}x_3^2,\\ P_3(x) &= b_{31}x_1x_3 + b_{32}x_2x_3 + b_{33}x_3^2. \end{aligned} \tag{4-2}$$

The linear part is familiar. So let us concentrate on the quadratic part. Note that there are no terms x_1^2, x_2^2, or x_1x_2, because $u=0$ on the boundary. If there is no further restriction on u, then there are nine free variables b_{ij}, $1 \le i, j \le 3$, as shown in (4-2). If $\nabla \cdot u = 0$ in $Q_{1/2,+}(0)$, then we claim that P is also divergence-free. If this claim is true, then we must have

$$b_{11} + b_{22} + 2b_{33} = 0, \quad b_{31} = b_{32} = 0.$$

Because of this restriction on the coefficients, the dimension for the homogeneous no-slip Stokes polynomials of degree 2 becomes 6. We can find basis polynomials

$$\begin{aligned} P^{(21)} &= (x_2x_3, 0, 0), & P^{(22)} &= (x_3^2, 0, 0), \\ P^{(23)} &= (0, x_1x_3, 0), & P^{(24)} &= (0, x_3^2, 0), \\ P^{(25)} &= (-2x_1x_3, 0, x_3^2), & P^{(26)} &= (0, -2x_2x_3, x_3^2). \end{aligned} \tag{4-3}$$

Note that these polynomials are solutions to the stationary Stokes system with associated pressure $L^{(2j)}$ given by

$$\begin{gathered} L^{(2j)}(x) = 0 \quad \text{for } j = 1, 3, \\ L^{(22)}(x) = 2x_1, \quad L^{(24)}(x) = 2x_2, \\ L^{2j}(x) = 2x_3 \quad \text{for } j = 5, 6. \end{gathered} \tag{4-4}$$

Now, let us show the claim that P is divergence-free. Since $u = P + O(|x|^3)$, we have that $\nabla \cdot u = \nabla \cdot P + O(|x|^2) = 0$ in $\{x_3 \ge 0\} \cap B_{1/2}(0)$. Because of $\nabla \cdot P = C_0 + C_1 \cdot x$ for some $C_0 \in \mathbb{R}$ and $C_1 \in \mathbb{R}^3$, we see that $C_0 + C_1 \cdot x = O(|x|^2)$. Hence we must have $C_0 = 0$ and $C_1 = 0$; otherwise, it is easy to find a contradiction by taking $x = \delta C_1$ or $-\delta C_1$ for sufficiently small δ.

Similarly to the linear solution pairs $(P^{(1j)}, 0)$, the fundamental fact about the polynomial pairs constructed above is that $(P^{(2j)}, L^{(2j)})$ are quadratic solutions of Stokes equations in the upper half-space $\mathbb{R}^3_+$, namely

$$\begin{cases} -\Delta P^{(2j)} + \nabla L^{(2j)} = 0 & \text{in } \mathbb{R}^3_+, \\ \nabla \cdot P^{(2j)} = 0 & \text{in } \mathbb{R}^3_+, \\ P^{(2j)} = 0 & \text{on } \partial\mathbb{R}^3_+. \end{cases}$$

To study the $C^{1,\gamma}$ and $C^{2,\gamma}$ regularity of (NS$^\varepsilon$), the linear and quadratic solutions of Stokes equations in $\mathbb{R}^3_+$ are not enough. We need to construct linear and quadratic solutions in Ω which vanish on $\partial\Omega$, where Ω is a bumpy John half-space in the sense of Definition 1.2. These solutions will be constructed based on $(P^{(1j)}, 0)$ and $(P^{(2j)}, L^{(2j)})$. Observe that $P^{(ij)}$ does not vanish on $\partial\Omega$. Therefore we have to introduce new correctors, called boundary layers, in order to correct the boundary discrepancy on $\partial\Omega$. Precisely, we will show the existence of weak solutions (with corresponding sublinear or subquadratic growth) of the boundary layer equations

$$\begin{cases} -\Delta v + \nabla q = 0 & \text{in } \Omega, \\ \nabla \cdot v = 0 & \text{in } \Omega, \\ v + P^{(ij)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{BL$^{(j)}_{i\text{-th}}$}$$

where $i \in \{1, 2\}$. Here a couple $(v, q) \in H^1_{\mathrm{loc}}(\overline{\Omega})^3 \times L^2_{\mathrm{loc}}(\overline{\Omega})$ is said to be a weak solution of (BL$^{(j)}_{i\text{-th}}$) if it satisfies $\nabla \cdot v = 0$ in the sense of distributions, $\chi(v + P^{(ij)}) \in H^1_0(\Omega)^3$ for any $\chi \in C^\infty_0(\mathbb{R}^3)$, and the weak formulation

$$\int_\Omega \nabla v \cdot \nabla \phi - \int_\Omega q(\nabla \cdot \phi) = 0 \quad \text{for any } \phi \in C_0^\infty(\Omega)^3. \tag{4-5}$$

4B. ***First-order boundary layers.*** We consider the first-order boundary layer equations

$$\begin{cases} -\Delta v + \nabla q = 0 & \text{in } \Omega, \\ \nabla \cdot v = 0 & \text{in } \Omega, \\ v + P^{(1j)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathrm{BL}^{(j)}_{1\mathrm{st}}$}$$

for $j \in \{1, 2\}$. The solvability of $(\mathrm{BL}^{(j)}_{1\mathrm{st}})$ follows from the next statement.

Theorem 4.1. *Let $L \in (0, \infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. For $j \in \{1, 2\}$, there exists a unique weak solution $(v^{(1j)}, q^{(1j)}) \in H^1_{\mathrm{loc}}(\bar{\Omega})^3 \times L^2_{\mathrm{loc}}(\bar{\Omega})$ of $(\mathrm{BL}^{(j)}_{1\mathrm{st}})$ satisfying*

$$\sup_{\xi \in \mathbb{Z}^2} \int_{\Omega \cap (\xi + (0,1)^2) \times \mathbb{R}} (|\nabla v^{(1j)}|^2 + |q^{(1j)}|^2) \le C, \tag{4-6}$$

where the constant C depends only on L.

In [Higaki and Prange 2020] the well-posedness of the system $(\mathrm{BL}^{(j)}_{1\mathrm{st}})$ was proved over Lipschitz graphs by a domain decomposition method: coupling of the Stokes problem in a bumpy channel $\Omega \cap \{x_3 < 0\}$ with the Stokes problem in the flat half-space $\{x_3 > 0\}$ via a nonlocal Dirichlet-to-Neumann boundary condition at the interface $\{x_3 = 0\}$. We face considerable technical difficulties when trying to adapt this strategy to the case of bumpy John domains. Indeed, the local energy estimates in the bumpy channel require to estimate the pressure, or to work with divergence-free test functions. In either case, we need to construct a Bogovskii operator for a sequence of exhausting domains containing $\Omega \cap \{|x'| \le k,\ x_3 < 0\}$ with a constant uniform in k. The construction of the Bogovskii operator of Theorem A.1 by [Acosta et al. 2006] relies on connecting any point in the bumpy John domain to a fixed neighborhood of a reference point $\tilde{x}$. Such a procedure gives, for a slim domain such as $\Omega \cap \{|x'| \le k,\ x_3 < 0\}$, a constant in the estimate (A-1) that scales proportionately to the horizontal size k of the domain. We are unable to take advantage of the small vertical extent of the domain to provide a modified construction of the Bogovskii operator. This would be needed to carry out the downward iteration on the local energy estimates, also called Saint-Venant estimates, in [Higaki and Prange 2020].

Here we take advantage of the fact that we already proved large-scale Lipschitz estimates by the quantitative method, without relying on boundary layers as in [Higaki and Prange 2020]. Therefore, we develop a new strategy using the large-scale Lipschitz estimate to prove the existence of solutions to $(\mathrm{BL}^{(j)}_{1\mathrm{st}})$. We rely on the Green's kernel estimates proved in Appendix B. For $N \in \mathbb{R}$, let us set

$$\Omega_{\le N} := \Omega \cap \{z_3 \le N\}, \qquad \Omega_{\ge N} := \Omega \cap \{z_3 \ge N\}. \tag{4-7}$$

We also define $\Omega_{<N}$ and $\Omega_{>N}$ in a similar manner.

Proof of Theorem 4.1. We define $(v^{(1j)}, q^{(1j)})$ by (v, q), so as not to burden the notation.

Uniqueness. Let $P^{(1j)} = 0$ in $(\mathrm{BL}^{(j)}_{1\mathrm{st}})$. Then the Liouville-type result, Corollary 3.1, implies $v = 0$ in the class

$$\sup_{\xi\in\mathbb{Z}^2}\int_{\Omega\cap(\xi+(0,1)^2)\times\mathbb{R}} |\nabla v|^2 < \infty.$$

This implies $q = 0$ in the class (4-6) as well from the equations.

Existence. Step 1: lifting the boundary data. Let $\eta_-(x_3)$ be a smooth cut-off function such that

$$\begin{gathered}\eta_-(t) \text{ is smooth and nonnegative,}\\ \eta_-(t) = 1 \text{ if } t < 3 \text{ and } \eta_-(t) = 0 \text{ if } t > 4.\end{gathered} \tag{4-8}$$

By writing $w = v + \eta_- P^{(1j)}$, we see that w satisfies

$$\begin{cases} -\Delta w + \nabla q = F := -\Delta(\eta_- P^{(1j)}) & \text{in } \Omega,\\ \nabla\cdot w = 0 & \text{in } \Omega,\\ w = 0 & \text{on } \partial\Omega.\end{cases} \tag{4-9}$$

Notice that F is a bounded function supported in a slim channel $S := \{x \in \mathbb{R}^3 \mid 3 \le x_3 \le 4\}$. Thus, the problem is reduced to finding a weak solution of (4-9) satisfying

$$\sup_{\xi\in\mathbb{Z}^2}\int_{\Omega\cap(\xi+(0,1)^2)\times\mathbb{R}} (|\nabla w|^2 + |q|^2) \le C\|F\|^2_{L^\infty}. \tag{4-10}$$

We rely on the representation of w and q by the Green's kernel

$$w(x) = \int_\Omega G(x,y)F(y)\,dy, \qquad q(x) = \int_\Omega \Pi(x,y)\cdot F(y)\,dy.$$

Thanks to the properties of the Green's function (G,Π), it suffices to prove that ∇w and q are well-defined and satisfy the estimate (4-10).

In the following proof, we take the zero-extension of (G,Π) as is done in Appendix B.

Step 2: estimate on $\Omega_{\ge 8}$. For any $y \in S$ and $x \in \Omega_{\ge 8}$, by Proposition B.3(i)

$$|\nabla_x G(x,y)| \le \frac{C}{|x-y|^3}.$$

Then it follows from the Hölder inequality that

$$|\nabla w(x)| \le \int_S |\nabla_x G(x,y)||F(y)|\,dy \le \|F\|_{L^\infty}\int_S \frac{C}{|x-y|^3}\,dy \le \frac{C\|F\|_{L^\infty}}{x_3}.$$

A similar computation using Proposition B.5(i) gives the same bound for the pressure $q(x)$ with $x_3 \ge 8$. Consequently,

$$\sup_{\xi\in\mathbb{Z}^2}\int_{\Omega_{\ge 8}\cap(\xi+(0,1)^2)\times\mathbb{R}} (|\nabla w|^2 + |q|^2) \le C\|F\|^2_{L^\infty}. \tag{4-11}$$

Step 3: estimate on $\Omega_{\le 8}$. Fix $\xi \in \mathbb{Z}^2$ arbitrarily. For simplicity, we denote the cubes in $\mathbb{R}^3$ centered at $(\xi,0)$ by $Q_R(\xi) = (\xi,0) + (-R,R)^3$. We would like to estimate $|\nabla w|$ and $|q|$ in the cube $Q_8(\xi)$. Notice that $\Omega_{\le 8} \subset \bigcup_{\xi\in\mathbb{Z}^2\times\{0\}} Q_8(\xi)$ with finite overlaps.

Taking a cube $Q_{40}(\xi)$, we decompose F into two parts as $F = F\chi_{Q_{40}(\xi)} + F(1-\chi_{Q_{40}(\xi)})$. Correspondingly, we decompose (w, q) into singular and regular parts, namely $(w, q) = (w_{\rm sing}, q_{\rm sing}) + (w_{\rm reg}, q_{\rm reg})$, where

$$\begin{cases} w_{\rm sing}(x) = \int_\Omega G(x, y)F(y)\chi_{Q_{40}(\xi)}(y)\,\mathrm{d}y, \\ q_{\rm sing}(x) = \int_\Omega \Pi(x, y)\cdot F(y)\chi_{Q_{40}(\xi)}(y)\,\mathrm{d}y \end{cases}$$

and

$$\begin{cases} w_{\rm reg}(x) = \int_\Omega G(x, y)F(y)(1-\chi_{Q_{40}(\xi)}(y))\,\mathrm{d}y, \\ q_{\rm reg}(x) = \int_\Omega \Pi(x, y)\cdot F(y)(1-\chi_{Q_{40}(\xi)}(y))\,\mathrm{d}y. \end{cases}$$

To estimate the regular part $(w_{\rm reg}, q_{\rm reg})$ in $Q_8(\xi)$, we use (B-19) and (B-23) in Proposition B.3 to obtain

$$\left(\int_{Q_8(\xi)} |\nabla_x G(x, y)|^2\,\mathrm{d}x\right)^{1/2} \le \frac{C}{|(\xi, 0) - y|^3}$$

for any $y \in S \setminus Q_{40}(\xi)$. As a result,

$$\begin{aligned} \left(\int_{Q_8(\xi)} |\nabla w_{\rm reg}|^2\right)^{1/2} &\le \|F\|_{L^\infty} \int_{S\setminus Q_{40}(\xi)} \left(\int_{Q_8(\xi)} |\nabla_x G(x, y)|^2\,\mathrm{d}x\right)^{1/2} \mathrm{d}y \\ &\le \|F\|_{L^\infty} \int_{S\setminus Q_{40}(\xi)} \frac{C}{|(\xi, 0) - y|^3}\,\mathrm{d}y \le C\|F\|_{L^\infty}. \end{aligned} \tag{4-12}$$

Similarly, by using (B-38) and (B-39), we can derive the estimate of $q_{\rm reg}$,

$$\left(\int_{Q_8(\xi)} |q_{\rm reg}|^2\right)^{1/2} \le C\|F\|_{L^\infty}. \tag{4-13}$$

Next, we consider the singular part $(w_{\rm sing}, q_{\rm sing})$, which actually is a weak solution of

$$\begin{cases} -\Delta w_{\rm sing} + \nabla q_{\rm sing} = F\chi_{Q_{40}(\xi)} & \text{in } \Omega, \\ \nabla\cdot w_{\rm sing} = 0 & \text{in } \Omega, \\ w_{\rm sing} = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that the energy relation yields

$$\|\nabla w_{\rm sing}\|_{L^2(\Omega)} \le C\|F\|_{L^\infty}, \tag{4-14}$$

where C is independent of ξ. This gives the local L^2 boundedness of $w_{\rm sing}$ in the channel $\Omega_{\le 8}$. On the other hand, the argument in Step 2, using (B-38), implies that, for any $x_3 \ge 8$,

$$|q_{\rm sing}(x)| \le \frac{C\|F\|_{L^\infty}}{x_3}. \tag{4-15}$$

Let $\Omega_{20}(\xi)$ be the John domain given by Definition 1.2 satisfying

$$\Omega \cap Q_{20}(\xi) \subset \Omega_{20}(\xi) \subset \Omega \cap Q_{40}(\xi).$$

By the Bogovskii lemma and (4-14),

$$\left(\fint_{\Omega_{20}(\xi)} \left| q_{\rm sing} - \fint_{\Omega_{20}(\xi)} q_{\rm sing} \right|^2\right)^{1/2} \le C\|F\|_{L^\infty}. \tag{4-16}$$

On the other hand, let $Q_1^*(\xi) = (\xi, 10) + (-1, 1)^3$. By (4-15),

$$\left| \fint_{Q_1^*(\xi)} q_{\rm sing} \right| \le C \|F\|_{L^\infty}. \tag{4-17}$$

Since $Q_1^*(\xi) \subset \Omega_{20}(\xi)$, by a familiar argument and (4-16), we have

$$\begin{aligned} \left| \fint_{Q_1^*(\xi)} q_{\rm sing} - \fint_{\Omega_{20}(\xi)} q_{\rm sing} \right| &\le \fint_{Q_1^*(\xi)} \left| q_{\rm sing} - \fint_{\Omega_{20}(\xi)} q_{\rm sing} \right| \\ &\le C \fint_{\Omega_{20}(\xi)} \left| q_{\rm sing} - \fint_{\Omega_{20}(\xi)} q_{\rm sing} \right| \le C \|F\|_{L^\infty}. \end{aligned} \tag{4-18}$$

This, together with (4-17), implies

$$\left| \fint_{\Omega_{20}(\xi)} q_{\rm sing} \right| \le C \|F\|_{L^\infty}. \tag{4-19}$$

Now, combining (4-16) and (4-19), we obtain

$$\begin{aligned} \left(\fint_{\Omega \cap Q_8(\xi)} |q_{\rm sing}|^2 \right)^{1/2} &\le \left(\fint_{\Omega \cap Q_8(\xi)} \left| q_{\rm sing} - \fint_{\Omega_{20}(\xi)} q_{\rm sing} \right|^2 \right)^{1/2} + \left| \fint_{\Omega_{20}(\xi)} q_{\rm sing} \right| \\ &\le C \left(\fint_{\Omega_{20}(\xi)} \left| q_{\rm sing} - \fint_{\Omega_{20}(\xi)} q_{\rm sing} \right|^2 \right)^{1/2} + \left| \fint_{\Omega_{20}(\xi)} q_{\rm sing} \right| \le C \|F\|_{L^\infty}, \end{aligned} \tag{4-20}$$

with C independent of ξ.

Now, combining (4-12), (4-13), (4-14) and (4-20), we have

$$\sup_{\xi \in \mathbb{Z}^2} \int_{\Omega \cap Q_8(\xi)} (|\nabla w|^2 + |q|^2) \le C \|F\|_{L^\infty}^2. \tag{4-21}$$

Finally, the desired estimate (4-10) is a consequence of (4-11) and (4-21). □

4C. *Second-order periodic boundary layers.* Let $P^{(2j)}$ be a no-slip Stokes polynomial of degree 2. We consider the second-order boundary layer equations

$$\begin{cases} -\Delta v + \nabla q = 0, & x \in \Omega, \\ \nabla \cdot v = 0, & x \in \Omega, \\ v + P^{(2j)} = 0, & x \in \partial\Omega. \end{cases} \tag{${\rm BL}_{\rm 2nd}^{(j)}$}$$

Constructing solutions to (${\rm BL}_{\rm 2nd}^{(j)}$) for $j \in \{1, 3, 5, 6\}$ with subquadratic growth is much more involved than constructing solutions to (${\rm BL}_{\rm 1st}^{(j)}$) with sublinear growth. Indeed, for $j \in \{1, 3, 5, 6\}$, the boundary data $-P^{(2j)}$ in (${\rm BL}_{\rm 2nd}^{(j)}$) grows linearly in the tangential direction. Solutions to (${\rm BL}_{\rm 2nd}^{(j)}$) for $j \in \{1, 3, 5, 6\}$ are constructed using the first-order correctors solving (${\rm BL}_{\rm 1st}^{(j)}$); see below. For this construction we rely on convergence/decay properties of the first-order correctors away from the boundary. Hence, we analyze (${\rm BL}_{\rm 2nd}^{(j)}$) under periodicity assumptions. Periodicity ensures exponential convergence/decay away from the boundary.

Throughout this subsection, we assume Ω is a periodic bumpy John domain according to Definition 1.3. Consider the fundamental periodic domain

$$\Omega_{\rm p} = \Omega \cap (-\pi, \pi]^2 \times (-1, \infty).$$

We regard Ω_{p} as a submanifold of $\mathbb{T}^2 \times \mathbb{R}$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the flat torus. By definition, Ω_{p} is open and connected in $\mathbb{T}^2 \times \mathbb{R}$. Moreover, $\Omega_{\mathrm{p}} \cap \{x_3 < 2\}$ is diffeomorphic to a bounded John domain in $\mathbb{R}^3$. We thus have the Bogovskii operator on $\Omega_{\mathrm{p}} \cap \{x_3 < 2\}$. It is important to notice that there is a one-to-one correspondence between the functions in Ω_{p} and the $(2\pi\mathbb{Z})^2$-periodic functions in Ω. We say a function F defined in Ω is $(2\pi\mathbb{Z})^2$-periodic if $F(x) = F(x+z)$ for any $x \in \Omega$ and $z \in (2\pi\mathbb{Z})^2 \times \{0\}$. In other words, if $f \in L^2_{\mathrm{loc}}(\Omega_{\mathrm{p}})$, then there exists a locally L^2 function F defined in Ω so that $F(x) = f(\tilde{x})$, where $\tilde{x}$ is the representation in Ω_{p} so that $x - \tilde{x} \in (2\pi\mathbb{Z})^2 \times \{0\}$. In this sense and for convenience, we do not distinguish between F and f.

Denote by $L^2(\Omega_{\mathrm{p}})$ and $\widehat{H}^1_0(\Omega_{\mathrm{p}})$ the closure of $C_0^\infty(\Omega_{\mathrm{p}})$ under the norms

$$\|f\|_{L^2(\Omega_{\mathrm{p}})} := \left(\int_{\Omega_{\mathrm{p}}} |f|^2\right)^{1/2}, \quad \|f\|_{\widehat{H}^1(\Omega_{\mathrm{p}})} := \left(\int_{\Omega_{\mathrm{p}}} |\nabla f|^2\right)^{1/2}.$$

Clearly, $\widehat{H}^1_0(\Omega_{\mathrm{p}})$ is a Hilbert space with respect to the inner product $\langle \nabla f, \nabla g\rangle_{\Omega_{\mathrm{p}}}$. Here and below,

$$\langle f, g\rangle_{\Omega_{\mathrm{p}}} := \int_{\Omega_{\mathrm{p}}} f \cdot \bar{g},$$

where $\bar{g}$ denotes the complex conjugate of g. Let $\widehat{H}^1_{0,\sigma}(\Omega_{\mathrm{p}})$ be the subspace of $\widehat{H}^1_0(\Omega_{\mathrm{p}})^3$ that consists of all the divergence-free functions, namely, $\widehat{H}^1_{0,\sigma}(\Omega_{\mathrm{p}}) = \{f \in \widehat{H}^1_0(\Omega_{\mathrm{p}})^3 \mid \nabla \cdot f = 0\}$.

We now recall the Fourier series representation for the solutions of $(\mathrm{BL}^{(j)}_{1\mathrm{st}})$ on the flat half-space $\{x_3 > 0\}$. The same formulas are obtained in [Higaki and Prange 2020, Proposition 3] based on the periodic Poisson kernel. Note that paper uses the fact that the equations are imposed on a domain whose boundary is given by the graph, but a similar proof is valid if we utilize the zero extension of the functions.

Proposition 4.2. *Let $L \in (0,\infty)$ and Ω be a periodic bumpy John domain with constant L according to Definition 1.3. Then the weak solution $(v^{(1j)}, q^{(1j)})$ of $(\mathrm{BL}^{(j)}_{1\mathrm{st}})$ given by Theorem 4.1 satisfies the following*:

(i) *$(v^{(1j)}, q^{(1j)})$ is expanded in Fourier series in $\{x_3 > 0\}$ as*

$$\begin{aligned} v^{(1j)}(x) &= \hat{v}^{(1j)}_{(0,0)} + \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left(\hat{v}^{(1j)}_k + \begin{pmatrix} -ik \\ |k| \end{pmatrix} V^{(1j)}(k) x_3\right) e^{-|k|x_3} e^{ik\cdot x'}, \\ q^{(1j)}(x) &= \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} 2|k| V^{(1j)}(k) e^{-|k|x_3} e^{ik\cdot x'}, \end{aligned} \tag{4-22}$$

where $V^{(1j)}(k)$ is a scalar function of k defined by

$$V^{(1j)}(k) = \hat{v}^{(1j)}_{k,3} - i\frac{k}{|k|} \cdot (\hat{v}^{(1j)}_k)', \tag{4-23}$$

and moreover, $\hat{v}^{(1j)}_k$ is the Fourier series coefficient of $v^{(1j)}(x',0)$:

$$\hat{v}^{(1j)}_k = \frac{1}{(2\pi)^2} \int_{(-\pi,\pi)^2} v^{(1j)}(x',0) e^{-ik\cdot x'}\,\mathrm{d}x', \quad k \in \mathbb{Z}^2. \tag{4-24}$$

(ii) *The third component of $\hat{v}^{(1j)}_{(0,0)}$ is zero. Particularly, by setting*

$$\hat{v}^{(1j)}_{(0,0)} =: \alpha^{(1j)} = (\alpha^{(1j)}_1, \alpha^{(1j)}_2, 0),$$

we have the exponential convergence

$$|v^{(1j)}(x) - \alpha^{(1j)}| + |\nabla v^{(1j)}(x)| + |q(x)| \le C\|v^{(1j)}(\,\cdot\,,0)\|_{L^2((-\pi,\pi)^2)}\, e^{-x_3/2} \quad \textit{for } x_3 > 1. \tag{4-25}$$

Here C is a universal constant.

Construction of $v^{(2j)}$ for $j \in \{1,3,5,6\}$. We construct the second-order boundary layers $v^{(2j)}$ corresponding to $P^{(2j)} = P^{(2j)}(x)$ for $j \in \{1,3,5,6\}$. These boundary layers are solutions to ($\mathrm{BL}^{(j)}_{\mathrm{2nd}}$) with subquadratic growth; see Theorem 4.3. We begin with the case $j = 1$, where $P^{(21)}(x) = x_2x_3\boldsymbol{e}_3$. We recall that $(v^{(21)}, q^{(21)})$ solves

$$\begin{cases} -\Delta v + \nabla q = 0 & \text{in } \Omega, \\ \nabla \cdot v = 0 & \text{in } \Omega, \\ v + x_2x_3\boldsymbol{e}_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\mathrm{BL}^{(1)}_{\mathrm{2nd}}$}$$

The difficulty in the analysis of ($\mathrm{BL}^{(1)}_{\mathrm{2nd}}$) is that the boundary value is not periodic and has linear growth as $x_2 \to \infty$. We aim at eliminating the growth factor x_2 and at recovering the periodic structure. The key finding is the connection between the first-order and second-order boundary layers on the boundary, namely

$$v^{(21)} - x_2v^{(11)} = 0, \quad x \in \partial\Omega. \tag{4-26}$$

This observation is the basis of the ansatz for $v^{(21)}$. Recall that $v^{(11)}$ converges exponentially fast to the constant $\alpha^{(11)} \in \mathbb{R}^3$, when $x_3 \to \infty$ by the spectral gap near frequency 0 yielded by the periodicity; see (4-25). Hence the nondecaying divergence $\nabla \cdot (x_2v^{(11)}(x)) = v_2^{(11)}(x)$ can be corrected by adding a corrector $-\alpha_2^{(11)}x_3\eta_+(x_3)\boldsymbol{e}_3$. Here $\eta_+(\,\cdot\,)$ is a function on $\mathbb{R}$ satisfying

$$\begin{gathered} \eta_+(t) \text{ is smooth and nonnegative,} \\ \eta_+(t) = 0 \text{ if } t < \tfrac{1}{2} \text{ and } \eta_+(t) = 1 \text{ if } t > 1. \end{gathered} \tag{4-27}$$

Below, we also need the cut-off η_- defined in (4-8).

The following statement gives the existence and the structure of second-order boundary layers with subquadratic growth.

Theorem 4.3. *Let $L \in (0,\infty)$ and Ω be a periodic bumpy John domain with constant L according to Definition 1.3. There exists a weak solution $(v^{(21)}, q^{(21)}) \in H^1_{\mathrm{loc}}(\overline{\Omega})^3 \times L^2_{\mathrm{loc}}(\overline{\Omega})$ to ($\mathrm{BL}^{(1)}_{\mathrm{2nd}}$) decomposed as*

$$\begin{aligned} v^{(21)}(x) &= x_2v^{(11)}(x) - \alpha_2^{(11)}x_3\eta_+(x_3)\boldsymbol{e}_3 + R^{(21)}(x), \\ q^{(21)}(x) &= x_2q^{(11)}(x) + Q^{(21)}(x), \end{aligned} \tag{4-28}$$

where $(R^{(21)}, Q^{(21)}) \in \widehat{H}^1_0(\Omega_{\mathrm{p}})^3 \times L^2(\Omega_{\mathrm{p}})$. Moreover, we have

$$\|\nabla R^{(21)}\|_{L^2(\Omega_{\mathrm{p}})} + \|Q^{(21)}\|_{L^2(\Omega_{\mathrm{p}})} \le C, \tag{4-29}$$

where the constant C depends only on L.

Proof. We aim at proving the existence of $(R^{(21)}, Q^{(21)})$ and estimating it so that $(v^{(21)}, q^{(21)})$ defined by the right-hand sides in (4-28) gives a weak solution of ($\mathrm{BL}^{(1)}_{\mathrm{2nd}}$).

Existence: By the previous discussion, we begin with a formal examination of $x_2 v^{(11)}(x)-\alpha_2^{(11)} x_3\eta_+(x_3)\boldsymbol{e}_3$. First of all, it is easy to see

$$\nabla\cdot(x_2 v^{(11)}(x)-\alpha_2^{(11)}x_3\eta_+(x_3)\boldsymbol{e}_3)=v_2^{(11)}(x)-\alpha_2^{(11)}\eta_+(x_3)-\alpha_2^{(11)}x_3\partial_3\eta_+(x_3)$$

for all $x\in\Omega_{\mathrm{p}}$. Notice that the expression above simplifies for $x_3>1$:

$$\nabla\cdot(x_2 v^{(11)}(x)-\alpha_2^{(11)}x_3\eta_+(x_3)\boldsymbol{e}_3)=v_2^{(11)}(x)-\alpha_2^{(11)}.$$

Then, by Proposition 4.2, we get

$$v_2^{(11)}(x)-\alpha_2^{(11)}=\sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}}(\hat{v}_{k,2}^{(11)}-ik_2V^{(11)}(k)x_3)e^{-|k|x_3}e^{ik\cdot x'},\quad x_3>1.$$

This means that $x_2v^{(11)}(x)-\alpha_2^{(11)}x_3\eta_+(x_3)\boldsymbol{e}_3$ is not divergence-free. Thus, our next goal is to construct a function to correct the divergence for $x_3>1$. Define

$$\begin{aligned}d(x):=\sum_{k\in\mathbb{Z}^2,\,k_1\neq 0}&\left(\frac{1}{ik_1}(\hat{v}_{k,2}^{(11)}-ik_2V^{(11)}(k)x_3)\,e^{-|k|x_3}e^{ik\cdot x'}\right)\boldsymbol{e}_1\\&+\sum_{k_1=0,\,k_2\in\mathbb{Z}\setminus\{0\}}\left(\frac{1}{ik_2}(\hat{v}_{k,2}^{(11)}-ik_2V^{(11)}(k)x_3)\,e^{-|k|x_3}e^{ik\cdot x'}\right)\boldsymbol{e}_2,\end{aligned}\tag{4-30}$$

which is an element of $H^1(\Omega_{\mathrm{p},>0})^3$, where $\Omega_{\mathrm{p},>0}:=\Omega_{\mathrm{p}}\cap\{x_3>0\}$. Of course, there is no unique way to construct a right-inverse of the divergence such as d. We may extend $d(x)$ to the whole domain Ω_{p} by multiplying it by $\eta_+(x_3)$ and still correct the divergence of $x_2v^{(11)}(x)-\alpha_2^{(11)}x_3\eta_+(x_3)\boldsymbol{e}_3-d(x)\eta_+(x_3)$. To check the divergence condition, we calculate

$$\begin{aligned}D(x)&:=\nabla\cdot\big(x_2v^{(11)}(x)-\alpha_2^{(11)}x_3\eta_+(x_3)\boldsymbol{e}_3-d(x)\eta_+(x_3)\big)\\&=v_2^{(11)}(x)-\alpha_2^{(11)}\eta_+(x_3)-\eta_+(x_3)\nabla\cdot d(x)-(\alpha_2^{(11)}x_3+d_3(x))\partial_3\eta_+(x_3).\end{aligned}$$

Obviously, D is supported in $\Omega_{\mathrm{p},<2}:=\Omega_{\mathrm{p}}\cap\{x_3<2\}$, in which we can rely on the Bogovskii operator to find a right-inverse of the divergence. Let $A:=\int_{\Omega_{\mathrm{p},<2}}D$. Let $\chi_+(x_3)$ be a smooth cut-off function such that

$$\chi_+(x_3)=0\quad\text{if } x_3\le 0,\qquad\text{and}\qquad\chi_+(x_3)=(2\pi)^{-2}A\quad\text{for } x_3>1.\tag{4-31}$$

This implies $\partial_3\chi_+(x_3)$ is supported in $\Omega_{\mathrm{p},<2}$ and $\int_{\Omega_{\mathrm{p},<2}}\partial_3\chi_+=A$. It follows that

$$\int_{\Omega_{\mathrm{p},<2}}(D(x)-\nabla\cdot(\chi_+(x_3)\boldsymbol{e}_3))\,dx=\int_{\Omega_{\mathrm{p},<2}}(D(x)-\partial_3\chi_+(x_3))\,dx=0.$$

Hence, by Appendix A, there is a Bogovskii corrector $\mathbb{B}\in H_0^1(\Omega_{\mathrm{p},<2})^3$ such that

$$\nabla\cdot\mathbb{B}(x)=D(x)-\partial_3\chi_+(x_3),$$

and $\|\mathbb{B}\|_{H^1(\Omega_{\mathrm{p},<2})}\le C$, where C depends only on the John constant L of Ω. We extend $\mathbb{B}$ by zero to the whole domain Ω_{p} and denote it again by $\mathbb{B}\in H_0^1(\Omega_{\mathrm{p}})^3$. Let us combine the above correctors and define

$$\mathcal{C}(x)=-d(x)\eta_+(x_3)-\chi_+(x_3)\boldsymbol{e}_3-\mathbb{B}(x).$$

Note that $\mathcal{C} \in \widehat{H}^1_0(\Omega_{\mathrm{p}})^3$. In particular, $\|\nabla \mathcal{C}\|_{L^2(\Omega_{\mathrm{p}})} \le C$, where C depends only on the John constant L of Ω. By (4-25), the function $\mathcal{C}$ converges exponentially fast to $-(2\pi)^{-2}A$ as $x_3 \to \infty$, and its derivatives decay exponentially fast to 0 as $x_3 \to \infty$.

By the crucial cancellation

$$x_2(-\Delta v^{(11)}(x) + \nabla q^{(11)}(x)) = 0,$$

as well as the definition of $\mathcal{C}$, we see that the pair

$$x_2 v^{(11)}(x) - \alpha_2^{(11)} x_3 \eta_+(x_3) \boldsymbol{e}_3 + \mathcal{C}(x) \quad \text{and} \quad x_2 q^{(11)}(x)$$

is a weak solution to $(\mathrm{BL}^{(1)}_{2\mathrm{nd}})$ with an additional external force

$$f^{(21)}(x) = -2\partial_2 v^{(11)}(x) + q^{(11)}(x)\boldsymbol{e}_2 - \Delta(-\alpha_2^{(11)} x_3 \eta_+(x_3)\boldsymbol{e}_3 + \mathcal{C}(x)).$$

In order to cancel this source term, we consider

$$\begin{cases} -\Delta \mathcal{R}^{(21)} + \nabla \mathcal{Q}^{(21)} = -f^{(21)} & \text{in } \Omega_{\mathrm{p}}, \\ \nabla \cdot \mathcal{R}^{(21)} = 0 & \text{in } \Omega_{\mathrm{p}}, \\ \mathcal{R}^{(21)} = 0 & \text{on } \partial\Omega_{\mathrm{p}}. \end{cases} \tag{4-32}$$

The weak formulation of (4-32) is written as

$$\langle \nabla \mathcal{R}^{(21)}, \nabla \varphi \rangle_{\Omega_{\mathrm{p}}} = -\langle f^{(21)}, \varphi \rangle_{\Omega_{\mathrm{p}}}, \quad \varphi \in \widehat{H}^1_{0,\sigma}(\Omega_{\mathrm{p}}). \tag{4-33}$$

Next, we prove the unique existence of the weak solution of (4-33). By the integration by parts for $\Delta \mathcal{C}$,

$$\langle f^{(21)}, \varphi \rangle_{\Omega_{\mathrm{p}}} = -2\langle \partial_2 v^{(11)}, \varphi \rangle_{\Omega_{\mathrm{p}}} + \langle q^{(11)}, \varphi_2 \rangle_{\Omega_{\mathrm{p}}} + \alpha_2^{(11)} \langle \Delta(x_3 \eta_+(x_3)\boldsymbol{e}_3), \varphi \rangle_{\Omega_{\mathrm{p}}} + \langle \nabla \mathcal{C}, \nabla \varphi \rangle_{\Omega_{\mathrm{p}}}. \tag{4-34}$$

By the Poincaré inequality in $\Omega_{\mathrm{p},<2}$ and the Cauchy–Schwarz inequality in Ω_{p},

$$\begin{aligned} |\langle f^{(21)}, \varphi \rangle_{\Omega_{\mathrm{p}}}| &\le C\big(\|\nabla v^{(11)}\|_{L^2(\Omega_{\mathrm{p}})} + \|q^{(11)}\|_{L^2(\Omega_{\mathrm{p}})} + \|\Delta(x_3\eta_+(x_3))\|_{L^2(\Omega_{\mathrm{p}})} + \|\nabla \mathcal{C}\|_{L^2(\Omega_{\mathrm{p}})}\big)\|\nabla \varphi\|_{L^2(\Omega_{\mathrm{p}})} \\ &\quad + \left| \int_1^\infty \int_{(-\pi,\pi)^2} \partial_2 v^{(11)}(x)\varphi(x)\, \mathrm{d}x'\, \mathrm{d}x_3 \right| + \left| \int_1^\infty \int_{(-\pi,\pi)^2} q^{(11)}(x)\varphi_2(x)\, \mathrm{d}x'\, \mathrm{d}x_3 \right|. \end{aligned} \tag{4-35}$$

From Proposition 4.2 again, we have the representation formulas

$$\partial_2 v^{(11)}(y) = \partial_2 \Bigg(\sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} \bigg(\hat{v}_k^{(1j)} + \begin{pmatrix} -ik \\ |k| \end{pmatrix} V^{(1j)}(k) x_3 \bigg) e^{-|k|x_3} e^{ik\cdot x'} \Bigg),$$

$$q^{(11)}(y) = \partial_1 \Bigg(\sum_{k \in \mathbb{Z}^2,\, k_1 \neq 0} 2|k| V^{(1j)}(k) e^{-|k|x_3} \frac{e^{ik\cdot x'}}{ik_1} \Bigg) + \partial_2 \Bigg(\sum_{k_1 = 0,\, k_2 \in \mathbb{Z}\setminus\{0\}} 2|k| V^{(1j)}(k) e^{-|k|x_3} \frac{e^{ik\cdot x'}}{ik_2} \Bigg).$$

Thus, by integration by parts in x_1 and x_2, the last two integrals in (4-35) are bounded by $C\|\nabla \varphi\|_{L^2(\Omega_{\mathrm{p}})}$. Consequently, in view of (4-6) and (4-35), we obtain

$$|\langle f^{(21)}, \varphi \rangle_{\Omega_{\mathrm{p}}}| \le C\|\nabla \varphi\|_{L^2(\Omega_{\mathrm{p}})}. \tag{4-36}$$

Then, by the Riesz representation theorem, there is an element $\mathcal{R}^{(21)} \in \widehat{H}^1_{0,\sigma}(\Omega_{\mathrm{p}})$ solving (4-33) and satisfying $\|\nabla \mathcal{R}^{(21)}\|_{L^2(\Omega_{\mathrm{p}})} \le C$. The existence of the pressure $\mathcal{Q}^{(21)} \in L^2_{\mathrm{loc}}(\overline{\Omega}_{\mathrm{p}})$ can be proved by using the Bogovskii lemma. Finally, the existence of the remainder $(R^{(21)}, Q^{(21)})$ in (4-28) is proved if we set $R^{(21)} = \mathcal{C}^{(21)} + \mathcal{R}^{(21)}$ and $Q^{(21)} = \mathcal{Q}^{(21)}$. Let us emphasize that $(R^{(21)}, Q^{(21)})$ can be "unwrapped" (without changing the notation) as a $(2\pi\mathbb{Z})^2$-periodic function in Ω.

Estimate: The estimate of $R^{(21)}$ is clear. Hence we focus on the pressure term $Q^{(21)} = \mathcal{Q}^{(21)}$. Notice that since $\mathcal{Q}^{(21)}$ is locally L^2, it suffices to prove

$$\|\mathcal{Q}^{(21)}\|_{L^2(\Omega_{\mathrm{p},>3})} \leq C. \tag{4-37}$$

We apply the Fourier series expansion in the flat domain $\{x_3 > 3\}$. We decompose $\mathcal{R}^{(21)}$ and $\mathcal{Q}^{(21)}$ into $\mathcal{R}^{(21)} = w_1 + w_2$ and $\mathcal{Q}^{(21)} = r_1 + r_2$ (up to a constant), where (w_1, r_1) is a solution of

$$\begin{cases} -\Delta w_1 + \nabla r_1 = 0, & x_3 > 3, \\ \nabla \cdot w_1 = 0, & x_3 > 3, \\ w_1(x', 3) = \mathcal{R}^{(21)}(x', 3), \end{cases}$$

while (w_2, r_2) solves

$$\begin{cases} -\Delta w_2 + \nabla r_2 = -f^{(21)}, & x_3 > 3, \\ \nabla \cdot w_2 = 0, & x_3 > 3, \\ w_2(x', 3) = 0. \end{cases}$$

Using the periodicity of $\mathcal{R}^{(21)}(x', 3)$ in x', the solution (w_1, r_1) may be written by the Poisson kernel as in Proposition 4.2, which implies

$$\|r_1\|_{L^2(\Omega_{\mathrm{p},>3})} \leq C\|\nabla \mathcal{R}^{(21)}\|_{L^2(\Omega_{\mathrm{p}})} \leq C. \tag{4-38}$$

On the other hand, observe that the source term $-f^{(21)}$ is represented as

$$-f^{(21)}(x) = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} (\mathcal{F}_1(k) + \mathcal{F}_2(k) x_3) e^{-|k| x_3} e^{ik\cdot x'}, \qquad x_3 > 3,$$

where

$$|\mathcal{F}_1(k)| + |\mathcal{F}_2(k)| \leq C|k|^2 e^{3|k|} |\hat{v}_k^{(11)}|,$$

with $\hat{v}_k^{(1j)}$ defined in (4-24). Then a simple computation shows that

$$w_2(x) = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} (\mathcal{G}_1(k) + \mathcal{G}_2(k) x_3 + \mathcal{G}_3(k) x_3^2 + \mathcal{G}_4(k) x_3^3) e^{-|k| x_3} e^{ik\cdot x'},$$

$$r_2(x) = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} (\mathcal{G}_5(k) + \mathcal{G}_6(k) x_3 + \mathcal{G}_7(k) x_3^2 + \mathcal{G}_8(k) x_3^3) e^{-|k| x_3} e^{ik\cdot x'},$$

where

$$|k| \sum_{l=1}^{4} |\mathcal{G}_l(k)| + \sum_{l=5}^{8} |\mathcal{G}_l(k)| \leq C|k| e^{3|k|} |\hat{v}_k^{(11)}|.$$

Now it is easy to see that

$$\|r_2\|_{L^2(\Omega_{\mathrm{p},>3})} \leq C\|v^{(11)}(\,\cdot\,, 0)\|_{L^2((-\pi,\pi)^2)} \leq C. \tag{4-39}$$

From (4-38) and (4-39), we obtain (4-37). □

By a similar consideration, we can obtain the existence of $(v^{(2j)}, q^{(2j)})$ for $j \in \{3, 5, 6\}$, whose proofs are parallel to Theorem 4.3 and therefore omitted. Recall that η_+ (resp. η_-) is defined in (4-27) (resp. (4-8)).

Theorem 4.4. *Let $L \in (0, \infty)$ and Ω be a periodic bumpy John domain with constant L according to Definition 1.3. Let $j \in \{3, 5, 6\}$. There exists a weak solution $(v^{(2j)}, q^{(2j)}) \in H^1_{\mathrm{loc}}(\overline{\Omega})^3 \times L^2_{\mathrm{loc}}(\overline{\Omega})$ to $(\mathrm{BL}^{(j)}_{\mathrm{2nd}})$ decomposed as, when $j = 3$,*

$$\begin{aligned} v^{(23)}(x) &= x_1 v^{(12)}(x) - \alpha_1^{(12)} x_3 \eta_+(x_3) \boldsymbol{e}_3 + R^{(23)}(x), \\ q^{(23)}(x) &= x_1 q^{(12)}(x) + Q^{(23)}(x), \end{aligned} \tag{4-40}$$

when $j = 5$,

$$\begin{aligned} v^{(25)}(x) &= -2x_1 v^{(11)}(x) - x_3^2 \eta_-(x_3) \boldsymbol{e}_3 + 2\alpha_1^{(11)} x_3 \eta_+(x_3) \boldsymbol{e}_3 + R^{(25)}(x), \\ q^{(25)}(x) &= -2x_1 q^{(11)}(x) + Q^{(25)}(x), \end{aligned} \tag{4-41}$$

and when $j = 6$,

$$\begin{aligned} v^{(26)}(x) &= -2x_2 v^{(12)}(x) - x_3^2 \eta_-(x_3) \boldsymbol{e}_3 + 2\alpha_2^{(12)} x_3 \eta_+(x_3) \boldsymbol{e}_3 + R^{(26)}(x), \\ q^{(26)}(x) &= -2x_2 q^{(12)}(x) + Q^{(26)}(x), \end{aligned} \tag{4-42}$$

where $(R^{(2j)}, Q^{(2j)}) \in \widehat{H}^1_0(\Omega_{\mathrm{p}})^3 \times L^2(\Omega_{\mathrm{p}})$. Moreover, we have

$$\|\nabla R^{(2j)}\|_{L^2(\Omega_{\mathrm{p}})} + \|Q^{(2j)}\|_{L^2(\Omega_{\mathrm{p}})} \le C, \tag{4-43}$$

where the constant C depends only on L.

Construction of $v^{(22)}$ and $v^{(24)}$. The boundary layers corresponding to $P^{(22)}$ and $P^{(24)}$ can be constructed by using the Green's function. The fact that $P^{(22)}$ and $P^{(24)}$ only depend on the vertical variable x_3 and that there is no growth in the tangential variable x' makes the analysis much easier than for $P^{(2j)}$, $j \in \{1, 3, 5, 6\}$, studied above. The proof of the following proposition is almost identical to the one of Theorem 4.1. Notice that here we state Theorem 4.5 in the periodic case only for convenience. Indeed we use these correctors in combination with $(v^{(2j)}, q^{(2j)})$ for $j \in \{1, 3, 5, 6\}$ whose existence is stated in Theorems 4.3 and 4.4 in periodic bumpy John domains. However, the existence of $(v^{(2j)}, q^{(2j)})$ for $j \in \{2, 4\}$ can be proved in general bumpy John domains according to Definition 1.2.

Theorem 4.5. *Let $L \in (0, \infty)$ and Ω be a periodic bumpy John domain with constant L according to Definition 1.3. Let $j \in \{2, 4\}$. There exists a unique weak solution $(v^{(2j)}, q^{(2j)}) \in H^1_{\mathrm{loc}}(\overline{\Omega})^3 \times L^2_{\mathrm{loc}}(\overline{\Omega})$ to $(\mathrm{BL}^{(j)}_{\mathrm{2nd}})$ satisfying*

$$\|\nabla v^{(2j)}\|_{L^2(\Omega_{\mathrm{p}})} + \|q^{(2j)}\|_{L^2(\Omega_{\mathrm{p}})} \le C, \tag{4-44}$$

where the constant C depends only on L.

4D. ***Estimates of boundary layers.*** Before closing this section, we summarize the estimates of the boundary layers. The following propositions can be proved in a similar manner as in [Higaki and Prange 2020, Lemma 4] combined with a direct computation. The details are omitted here.

Proposition 4.6. *Let $L \in (0, \infty)$ and Ω be a bumpy John domain with constant L as in Definition 1.2. For $j \in \{1, 2\}$, let $(v^{(1j)}, q^{(1j)})$ the weak solution of $(\mathrm{BL}^{(j)}_{1\mathrm{st}})$ provided by Theorem 4.1. Then, for $r \in (\varepsilon, 1)$,*

$$\left(\fint_{B^\varepsilon_{r,+}} \left|(\nabla v^{(1j)})\left(\frac{x}{\varepsilon}\right)\right|^2 \mathrm{d}x\right)^{1/2} + \left(\fint_{B^\varepsilon_{r,+}} \left|q^{(1j)}\left(\frac{x}{\varepsilon}\right)\right|^2 \mathrm{d}x\right)^{1/2} \le C\left(\frac{\varepsilon}{r}\right)^{1/2}, \tag{4-45}$$

where C depends only on L.

Proposition 4.7. *Let $L \in (0, \infty)$ and Ω be a periodic bumpy John domain with constant L as in Definition 1.3. For $j \in \{1, \dots, 6\}$, let $(v^{(2j)}, q^{(2j)})$ the weak solution of $(\mathrm{BL}^{(j)}_{2\mathrm{nd}})$ provided by Theorem 4.3 or 4.4. Then, for $r \in (\varepsilon, 1)$,*

$$\frac{1}{r}\left(\fint_{B^\varepsilon_{r,+}} \left|\varepsilon(\nabla v^{(2j)})\left(\frac{x}{\varepsilon}\right)\right|^2 \mathrm{d}x\right)^{1/2} + \frac{1}{r}\left(\fint_{B^\varepsilon_{r,+}} \left|\varepsilon q^{(2j)}\left(\frac{x}{\varepsilon}\right)\right|^2 \mathrm{d}x\right)^{1/2} \le C\left(\frac{\varepsilon}{r}\right)^{1/2}, \tag{4-46}$$

where C depends only on L.

5. Higher-order regularity

5A. *Large-scale $C^{1,\gamma}$ estimate.* The goal of this subsection is to prove the large-scale $C^{1,\gamma}$ regularity stated in Theorem B. We will use the first-order boundary layers and modify the argument of the Lipschitz estimate.

Recall that $\mathscr{P}_1 = \mathrm{span}\{P^{(11)}, P^{(12)}\} = \{(ax_3, bx_3, 0) \mid a, b \in \mathbb{R}\}$. Let $\mathscr{P}_2 = \mathrm{span}\{P^{(2j)} \mid j = 1, 2, \dots, 6\}$. Let $\mathscr{S}_2 = \mathrm{span}\{(P^{(2j)}, L^{(2j)}) \mid j = 1, 2, \dots, 6\}$. Note that any element of $\mathscr{S}_2$ is a weak solution of the Stokes system in $\mathbb{R}^3$. Let $(v^{(1k)}, q^{(1k)})$, with $k = 1, 2$, and $(v^{(2j)}, q^{(2j)})$, with $j = 1, 2, \dots, 6$, be the first-order and second-order boundary layers, respectively. Then define

$$\mathscr{Q}_1(\Omega) = \mathrm{span}\{(P^{(1k)}, 0) + (v^{(1k)}, q^{(1k)}) \mid k = 1, 2\}, \tag{5-1}$$

$$\mathscr{Q}_2(\Omega) = \mathrm{span}\{(P^{(2j)}, L^{(2j)}) + (v^{(2j)}, q^{(2j)}) \mid j = 1, 2, \dots, 6\}. \tag{5-2}$$

Hence, $\mathscr{Q}_1(\Omega)$ (resp. $\mathscr{Q}_2(\Omega)$) is the vector space that contains all the "linear" (resp. "quadratic") solutions of the Stokes system in Ω vanishing on the boundary $\partial\Omega$; see the Liouville-type results at the end of this section stated in Theorem 5.8.

Remark 5.1. Note that the pressure part in estimate (1-3) of Theorem B is different from the Lipschitz estimate in which $\bar{P}$ is $\fint_{B^\varepsilon_{1/2,+}} p^\varepsilon$. Actually, in (1-3), $\bar{P}_1$ is the average of the corrected pressure over a small ball, i.e.,

$$\bar{P}_1 = \fint_{B^\varepsilon_{O(\varepsilon),+}} \left(p^\varepsilon - \pi\left(\frac{x}{\varepsilon}\right)\right) \mathrm{d}x$$

for some $(w, \pi) \in \mathscr{Q}_1(\Omega)$; see (5-15). This is reasonable since we are concerned with the $C^{0,\gamma}$ estimate of the pressure and $\bar{P}_1$ plays a role similar to the zeroth-order term in the Taylor expansion of the pressure, if the boundary is flat. We emphasize that $\bar{P}_1$ depends on ε. The point here is that $\bar{P}_1$ is independent of r.

The critical fact we are going to use is that any $(w, \pi) \in \mathscr{Q}_1(\Omega)$ is a solution of the Stokes system in Ω that vanishes on $\partial\Omega$. Hence, by rescaling, $(u^\varepsilon, p^\varepsilon) - (\varepsilon w(x/\varepsilon), \pi(x/\varepsilon))$ is still a weak solution with a no-slip boundary condition. This observation allows us to capture the regularity beyond the Lipschitz estimate. To this end, we define the first-order excess by

$$H_{\mathrm{1st}}(u^\varepsilon, p^\varepsilon; \rho) = \inf_{(w,\pi)\in\mathscr{Q}_1(\Omega)} \Biggl\{ \biggl(\fint_{B^\varepsilon_{\rho,+}} \Bigl| \nabla u^\varepsilon - \nabla\Bigl(\varepsilon w\Bigl(\frac{x}{\varepsilon}\Bigr)\Bigr) \Bigr|^2 \,\mathrm{d}x \biggr)^{1/2} + \sup_{s,t\in[1/16,1/4]} \biggl| \fint_{B^\varepsilon_{s\rho,+}} \Bigl(p^\varepsilon - \pi\Bigl(\frac{x}{\varepsilon}\Bigr)\Bigr)\,\mathrm{d}x - \fint_{B^\varepsilon_{t\rho,+}} \Bigl(p^\varepsilon - \pi\Bigl(\frac{x}{\varepsilon}\Bigr)\Bigr)\,\mathrm{d}x \biggr| \Biggr\}. \tag{5-3}$$

Recall that $(w, \pi) \in \mathscr{Q}_1(\Omega)$ means that

$$(w, \pi) = \sum_{k=1}^{2} \ell_k (P^{(1k)} + v^{(1k)}, q^{(1k)})$$

for some $\ell_1, \ell_2 \in \mathbb{R}$. We will also use the quantity $\Phi(u^\varepsilon, p^\varepsilon; \rho)$ defined in (3-8).

Lemma 5.2. *Let $L \in (0, \infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. Let $(u^\varepsilon, p^\varepsilon)$ be as in Theorem B, namely, a weak solution of (S^ε) in Section 3A satisfying (1-1). For all $\varepsilon \in \bigl(0, \frac{1}{32}\bigr]$, $r \in \bigl[2\varepsilon, \frac{1}{16}\bigr]$ and $\theta \in \bigl(0, \frac{1}{8}\bigr]$,*

$$H_{\mathrm{1st}}(u^\varepsilon, p^\varepsilon; \theta r) \le C\biggl(\theta + \theta^{-3}\Bigl(\frac{\varepsilon}{r}\Bigr)^{1/12}\biggr)\Phi(u^\varepsilon, p^\varepsilon; 16r) + C\theta^{-3}\biggl(\fint_{Q_{10r}} |\mathcal{M}^2_\varepsilon[F^\varepsilon]|^3\biggr)^{1/3}, \tag{5-4}$$

where C depends only on L.

Proof. First, we apply Lemma 3.7 with $\alpha = 1$

$$H(u^\varepsilon, p^\varepsilon; \theta r) \le C\biggl(\theta + \theta^{-3}\Bigl(\frac{\varepsilon}{r}\Bigr)^{1/12}\biggr)\Phi(u^\varepsilon, p^\varepsilon; 16r) + C\theta^{-3}\biggl(\fint_{Q_{10r}} |\mathcal{M}^2_\varepsilon[F^\varepsilon]|^3\biggr)^{1/3}, \tag{5-5}$$

where we also used the fact $H(\cdot\,, \cdot, 2r) \le \Phi(\cdot\,, \cdot, 2r) \le C\Phi(\cdot\,, \cdot, 16r)$. Let $P^* = \ell_1^* P^{(11)} + \ell_2^* P^{(12)} \in \mathscr{P}_1$ be the linear solution that minimizes $H(u^\varepsilon, p^\varepsilon; \theta r)$. Then (3-31) implies

$$\sum_{k=1}^{2} |\ell_k^*| \le C(H(u^\varepsilon, p^\varepsilon; \theta r) + \Phi(u^\varepsilon, p^\varepsilon; \theta r)) \le C\theta^{-3/2}\Phi(u^\varepsilon, p^\varepsilon; r). \tag{5-6}$$

By the definition of H_{1st} and H, one has

$$\begin{aligned} &H_{\mathrm{1st}}(u^\varepsilon, p^\varepsilon, \theta r) \\ &\le \biggl(\fint_{B^\varepsilon_{\theta r,+}} \Bigl| \nabla u^\varepsilon - \nabla\Bigl(\sum_{k=1}^{2} \ell_k^* \Bigl(P^{(1k)} + \varepsilon v^{(1k)}\Bigl(\frac{x}{\varepsilon}\Bigr)\Bigr)\Bigr)\Bigr|^2 \,\mathrm{d}x \biggr)^{1/2} \\ &\qquad + \sup_{s,t\in[1/16,1/4]} \biggl| \fint_{B^\varepsilon_{s\theta r,+}} \Bigl(p^\varepsilon - \sum_{k=1}^{2} \ell_k^* q^{(1k)}\Bigl(\frac{x}{\varepsilon}\Bigr)\Bigr)\,\mathrm{d}x - \fint_{B^\varepsilon_{t\theta r,+}} \Bigl(p^\varepsilon - \sum_{k=1}^{2} \ell_k^* q^{(1k)}\Bigl(\frac{x}{\varepsilon}\Bigr)\Bigr)\,\mathrm{d}x \biggr| \\ &\le H(u^\varepsilon, p^\varepsilon; \theta r) + \sum_{k=1}^{2} |\ell_k^*| \biggl(\fint_{B^\varepsilon_{\theta r,+}} \Bigl| (\nabla v^{(1k)})\Bigl(\frac{x}{\varepsilon}\Bigr)\Bigr|^2 \,\mathrm{d}x \biggr)^{1/2} \\ &\qquad + 2 \sup_{\rho\in[1/16,1/4]} \sum_{k=1}^{2} |\ell_k^*| \biggl| \fint_{B^\varepsilon_{\rho\theta r,+}} q^{(1k)}\Bigl(\frac{x}{\varepsilon}\Bigr)\,\mathrm{d}x - \fint_{B^\varepsilon_{\theta r/2,+}} q^{(1k)}\Bigl(\frac{x}{\varepsilon}\Bigr)\,\mathrm{d}x \biggr|. \end{aligned} \tag{5-7}$$

From Proposition 4.6, we have the estimate for the first-order boundary layers

$$\sum_{k=1}^{2}\left\{\left(\fint_{B^{\varepsilon}_{\theta r,+}}\left|(\nabla v)^{(1k)}\left(\frac{x}{\varepsilon}\right)\right|^2 dx\right)^{1/2}+\left(\fint_{B^{\varepsilon}_{\theta r,+}}\left|q^{(1k)}\left(\frac{x}{\varepsilon}\right)\right|^2 dx\right)^{1/2}\right\}\le C\theta^{-1/2}\left(\frac{\varepsilon}{r}\right)^{1/2}. \tag{5-8}$$

Inserting this into (5-7) and using (5-6) and (5-5), we obtain

$$\begin{aligned}H_{1\mathrm{st}}(u^\varepsilon,p^\varepsilon;\theta r)&\le H(u^\varepsilon,p^\varepsilon;\theta r)+C\theta^{-3}\left(\frac{\varepsilon}{r}\right)^{1/2}\Phi(u^\varepsilon,p^\varepsilon;r)\\&\le C\left(\theta+\theta^{-3}\left(\frac{\varepsilon}{r}\right)^{1/12}\right)\Phi(u^\varepsilon,p^\varepsilon;16r)+C\theta^{-3}\left(\fint_{Q_{10r}}|\mathcal{M}^2_\varepsilon[F^\varepsilon]|^3\right)^{1/3}.\end{aligned}$$

□

Proposition 5.3. *Let $L\in(0,\infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. Let $(u^\varepsilon,p^\varepsilon)$ be as in Theorem B. For any $\gamma\in[0,1)$, $\delta\in(0,1)$, $\varepsilon\in\left(0,\frac{1}{2}\right]$ and $r\in\left[\varepsilon,\frac{1}{2}\right]$,*

$$H_{1\mathrm{st}}(u^\varepsilon,p^\varepsilon;r)\le Cr^\gamma(M+M^{4+2\gamma+\delta}), \tag{5-9}$$

where C depends on L, γ and δ. Here M is the number given in Theorem B.

Proof. Note that it suffices to prove (5-9) for $r\in[N_0\varepsilon,1/N_1]$ with some absolute constant $N_0,N_1\ge 2$. The cases for $r\in\left(1/N_1,\frac{1}{2}\right]$ or $\varepsilon\ge 1/(N_0N_1)$ follow directly from the Bogovskii lemma and the Poincaré inequality. The case $r\in[\varepsilon,N_0\varepsilon]$ follows from the case $r=N_0\varepsilon$.

Firstly, using (3-39) with $\beta=\gamma+\delta$ (with $\delta\in\left(0,\frac{2-\gamma}{2}\right)$ being arbitrary), we have

$$\left(\fint_{Q_r}|\mathcal{M}^2_\varepsilon[F^\varepsilon]|^3\right)^{1/3}\le C_\delta(M+M^{4+2\gamma+4\delta})r^{\gamma+\delta}, \tag{5-10}$$

with C_δ depending on δ.

Since $\theta\in\left(0,\frac{1}{8}\right]$ in Lemma 5.2 is arbitrary, we can choose θ sufficiently small so that $C\theta\le\frac{1}{2}\left(\frac{\theta}{16}\right)^\gamma$ holds in (5-4). For such fixed θ, we can find $\varepsilon_0\in\left(0,\frac{1}{2}\right)$ depending on γ and θ such that the factor in (5-4) satisfies

$$C\theta^{-3}\left(\frac{\varepsilon}{r}\right)^{1/12}\le\frac{1}{2}\left(\frac{\theta}{16}\right)^\gamma,\quad r\in\left[\frac{\varepsilon}{\varepsilon_0},\frac{1}{16}\right],$$

in (5-4). Then by (5-10) and (5-4),

$$H_{1\mathrm{st}}(u^\varepsilon,p^\varepsilon;\theta r)\le\left(\frac{\theta}{16}\right)^\gamma\Phi(u^\varepsilon,p^\varepsilon;16r)+C_\delta(M+M^{4+2\gamma+4\delta})r^{\gamma+\delta}. \tag{5-11}$$

Now the key observation is that, for any $(w,\pi)\in\mathscr{Q}_1(\Omega)$, the pair $(U^\varepsilon,\Pi^\varepsilon)$ defined by

$$U^\varepsilon(x)=u^\varepsilon(x)-\varepsilon w\left(\frac{x}{\varepsilon}\right),\quad \Pi^\varepsilon(x)=p^\varepsilon(x)-\pi\left(\frac{x}{\varepsilon}\right)$$

is still a weak solution of the Stokes system (S^ε) in Section 3A. Therefore, the estimate (5-11) still holds if we replace $\Phi(u^\varepsilon,p^\varepsilon;16r)$ by $\Phi(U^\varepsilon,\Pi^\varepsilon;16r)$ for any $(w,\pi)\in\mathscr{Q}_1(\Omega)$. Then taking the infimum over all $(w,\pi)\in\mathscr{Q}_1(\Omega)$, we can further replace $\Phi(U^\varepsilon,\Pi^\varepsilon;16r)$ by $H_{1\mathrm{st}}(u^\varepsilon,p^\varepsilon;16r)$. Hence we obtain

$$H_{1\mathrm{st}}(u^\varepsilon,p^\varepsilon;\theta r)\le\left(\frac{\theta}{16}\right)^\gamma H_{1\mathrm{st}}(u^\varepsilon,p^\varepsilon;16r)+C_\delta(M+M^{4+2\gamma+4\delta})r^{\gamma+\delta}. \tag{5-12}$$

This is the first-order excess decay estimate for the $C^{1,\gamma}$ regularity of $(u^\varepsilon, p^\varepsilon)$. Note that we can eventually replace 4δ by δ in the right-hand side of (5-12) as $\delta \in (0,1)$ is arbitrary. Thus, by a simple iteration, we have that for $\varepsilon/\varepsilon_0 \le r \le \frac{\theta}{16}$,

$$H_{1\mathrm{st}}(u^\varepsilon, p^\varepsilon; r) \le r^\gamma (H_{1\mathrm{st}}(u^\varepsilon, p^\varepsilon; r_0) + C_\delta(M + M^{4+2\gamma+\delta}))$$

for some $r_0 \in \left[\frac{\theta}{16}, 1\right]$. Clearly, $H_{1\mathrm{st}}(u^\varepsilon, p^\varepsilon; r_0) \le \Phi(u^\varepsilon, p^\varepsilon; r_0)$. It remains to show

$$\Phi(u^\varepsilon, p^\varepsilon; r_0) \le C(M + M^2).$$

Indeed, since r_0 is comparable to 1, the above estimate follows directly from the Poincaré inequality and Bogovskii's lemma. □

The above theorem directly implies the $C^{1,\gamma}$ estimate for the velocity. To handle the pressure estimate in Theorem B, we need the following lemma.

Lemma 5.4. *Let $L \in (0,\infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. For a given $\rho > 0$, let $(\ell_1(\rho), \ell_2(\rho))$ be the pair of real numbers so that*

$$(w, \pi) = \sum_{k=1}^{2} \ell_k(\rho)(P^{(1k)} + v^{(1k)}, q^{(1k)})$$

minimizes $H_{1\mathrm{st}}(u^\varepsilon, p^\varepsilon; \rho)$. Then there exists a constant $\varepsilon_1 \in (0,1)$ so that for all $\varepsilon \in (0, \varepsilon_1]$ and $r \in \left[\varepsilon/\varepsilon_1, \frac{1}{2}\right]$,

$$\sup_{r_1, r_2 \in [r, 2r]} \sum_{k=1}^{2} |\ell_k(r_1) - \ell_k(r_2)| \le C \sup_{t \in [r, 2r]} H_{1\mathrm{st}}(u^\varepsilon, p^\varepsilon; t), \tag{5-13}$$

where C depends only on L.

Proof. By the definition of $H_{1\mathrm{st}}$, the triangle inequality and using that the matrices $\nabla P^{(1k)}$ are linearly independent over $\mathbb{R}$, if $r \le r_1, r_2 \le 2r$,

$$\begin{aligned}
\sum_{k=1}^{2} |\ell_k(r_1) - \ell_k(r_2)| &\le C\left(\fint_{B^\varepsilon_{r,+}} \left| \sum_{k=1}^{2} (\ell_k(r_1) - \ell_k(r_2)) \nabla P^{(1k)} \right|^2 \right)^{1/2} \\
&\le C\left(\fint_{B^\varepsilon_{r,+}} \left| \nabla \left(\sum_{k=1}^{2} (\ell_k(r_1) - \ell_k(r_2)) \left(P^{(1k)} + \varepsilon v^{(1k)}\left(\frac{x}{\varepsilon}\right)\right)\right) \right|^2 \mathrm{d}x \right)^{1/2} \\
&\qquad + C \sum_{k=1}^{2} |\ell_k(r_1) - \ell_k(r_2)| \left(\fint_{B^\varepsilon_{r,+}} \left| (\nabla v^{(1k)})\left(\frac{x}{\varepsilon}\right) \right|^2 \mathrm{d}x \right)^{1/2} \\
&\le C H_{1\mathrm{st}}(u^\varepsilon, p^\varepsilon; r_1) + C H_{1\mathrm{st}}(u^\varepsilon, p^\varepsilon; r_2) + C_0 \left(\frac{\varepsilon}{r}\right)^{1/2} \sum_{k=1}^{2} |\ell_k(r_1) - \ell_k(r_2)|,
\end{aligned}$$

where in the last inequality, we inserted u^ε and enlarged the domain from $B^\varepsilon_{r,+}$ to $B^\varepsilon_{r_i,+}$ with $i = 1, 2$, and applied Proposition 4.6. Now if $r \ge \varepsilon/\varepsilon_1$ for some small $\varepsilon_1 \in (0,1)$ so that $C_0 \varepsilon_1^{1/2} < \frac{1}{2}$, then

$$\sum_{k=1}^{2} |\ell_k(r_1) - \ell_k(r_2)| \le C \sum_{i=1}^{2} H_{1\mathrm{st}}(u^\varepsilon, p^\varepsilon; r_i).$$

This gives the desired estimate. □

Proof of Theorem B. Let $\varepsilon_1 \in (0, 1)$ be the number in Lemma 5.4. Note that it suffices to prove (1-3) when $\varepsilon \in (0, \varepsilon_1]$ and $r \in \left[\varepsilon/\varepsilon_1, \frac{1}{16}\right]$ as a familiar argument enables us to remove the smallness condition on ε and the restriction on r. The velocity estimate in (1-3) follows from the Poincaré inequality and (5-9). Hence, it suffices to estimate the pressure. Let $(\ell_1(\rho), \ell_2(\rho))$ be as in Lemma 5.4. For $r \in \left[\varepsilon/\varepsilon_1, \frac{1}{16}\right]$, let K be the integer so that $4^{-K}r \in [\varepsilon/\varepsilon_1, 4\varepsilon/\varepsilon_1)$. By the triangle inequality, the estimate of $q^{(1j)}$ in Proposition 4.6,

$$
\begin{aligned}
&\left| \fint_{B^\varepsilon_{4^{-K}r,+}} \left(p^\varepsilon - \sum_{k=1}^{2} \ell_k(4^{1-K}r) q^{(1k)}\left(\frac{x}{\varepsilon}\right)\right) \mathrm{d}x - \fint_{B^\varepsilon_{r,+}} \left(p^\varepsilon - \sum_{k=1}^{2} \ell_k(4r) q^{(1k)}\left(\frac{x}{\varepsilon}\right)\right) \mathrm{d}x \right| \\
&\quad \le \sum_{i=1}^{K} \left| \fint_{B^\varepsilon_{4^{i-K-1}r,+}} \left(p^\varepsilon - \sum_{k=1}^{2} \ell_k(4^{i-K}r) q^{(1k)}\left(\frac{x}{\varepsilon}\right)\right) \mathrm{d}x - \fint_{B^\varepsilon_{4^{i-K}r,+}} \left(p^\varepsilon - \sum_{k=1}^{2} \ell_k(4^{i-K+1}r) q^{(1k)}\left(\frac{x}{\varepsilon}\right)\right) \mathrm{d}x \right| \\
&\quad \le \sum_{i=1}^{K} \left(H_{1\mathrm{st}}(u^\varepsilon, p^\varepsilon; 4^{i-K+1}r) + \sum_{k=1}^{2} |\ell_k(4^{i-K+1}r) - \ell_k(4^{i-K}r)| \left(\frac{\varepsilon}{4^{i-K}r}\right)^{1/2} \right) \\
&\quad \le C \sum_{i=1}^{K} (4^{i-K+1}r)^\gamma (M + M^{4+2\gamma+\delta}) \\
&\quad \le C r^\gamma (M + M^{4+2\gamma+\delta}),
\end{aligned}
\tag{5-14}
$$

where we have used (5-9) and (5-13) in the third inequality. Define

$$
\bar{P}_1 = \fint_{B^\varepsilon_{\varepsilon/\varepsilon_1,+}} \left(p^\varepsilon - \sum_{k=1}^{2} \ell_k(4\varepsilon/\varepsilon_1) q^{(1k)}\left(\frac{x}{\varepsilon}\right)\right) \mathrm{d}x. \tag{5-15}
$$

Then by (5-14) and another use of (5-9) and (5-13), we have

$$
\left| \fint_{B^\varepsilon_{r,+}} \left(p^\varepsilon - \sum_{k=1}^{2} \ell_k(4r) q^{(1k)}\left(\frac{x}{\varepsilon}\right)\right) \mathrm{d}x - \bar{P}_1 \right| \le C r^\gamma (M + M^{4+2\gamma+\delta}).
$$

On the other hand, by Bogovskii's lemma applied to the John domain between $B^\varepsilon_{r,+}$ and $B^\varepsilon_{2r,+}$ given by Definition 1.2 and (5-10) with 4δ replaced by δ, we have

$$
\begin{aligned}
&\left(\fint_{B^\varepsilon_{r,+}} \left| p^\varepsilon - \sum_{k=1}^{2} \ell_k(4r) q^{(1k)}\left(\frac{x}{\varepsilon}\right) - \fint_{B^\varepsilon_{r,+}} \left(p^\varepsilon - \sum_{k=1}^{2} \ell_k(4r) q^{(1k)}\left(\frac{x}{\varepsilon}\right)\right) \mathrm{d}x \right|^2 \mathrm{d}x \right)^{1/2} \\
&\qquad \le C \left\{ \left(\fint_{B^\varepsilon_{2r,+}} \left| \nabla u^\varepsilon - \nabla \left(\sum_{k=1}^{2} \ell_k(4r) \left(P^{(1k)} + \varepsilon v^{(1k)}\left(\frac{x}{\varepsilon}\right)\right)\right)\right|^2 \mathrm{d}x \right)^{1/2} + \left(\fint_{B^\varepsilon_{8r,+}} |F^\varepsilon|^2 \right)^{1/2} \right\} \\
&\qquad \le C H_{1\mathrm{st}}(u^\varepsilon, p^\varepsilon; 4r) + C \left(\fint_{Q_{8r}} |\mathcal{M}^2_\varepsilon[F^\varepsilon]|^3 \right)^{1/3} \\
&\qquad \le C r^\gamma (M + M^{4+2\gamma+\delta}).
\end{aligned}
\tag{5-16}
$$

Combining the above two inequalities, we obtain the desired estimate in (1-3) for the pressure. □

5B. ***Large-scale $C^{2,\gamma}$ estimate over periodic boundaries.*** The goal of this subsection is to prove the large-scale $C^{2,\gamma}$ regularity stated in Theorem C. In this subsection, we assume Ω is a periodic bumpy John domain defined in Definition 1.3. The argument for $C^{2,\gamma}$ estimate is similar to the $C^{1,\gamma}$ estimate. Throughout, we assume $(w_1,\pi_1)\in\mathscr{Q}_1(\Omega)$ and $(w_2,q_2)\in\mathscr{Q}_2(\Omega)$. In other words, for some $\ell_{1k},\ell_{2j}\in\mathbb{R}$,

$$(w_1,\pi_1)=\sum_{k=1}^{2}\ell_{1k}(P^{(1k)}+v^{(1k)},q^{(1k)}),$$

$$(w_2,\pi_2)=\sum_{j=1}^{6}\ell_{2j}(P^{(2j)}+v^{(2j)},L^{(2j)}+q^{(2j)}).$$

It is important to observe that, by rescaling,

$$\left(\varepsilon w_1\left(\frac{x}{\varepsilon}\right)+\varepsilon^2 w_2\left(\frac{x}{\varepsilon}\right),\pi_1\left(\frac{x}{\varepsilon}\right)+\varepsilon\pi_2\left(\frac{x}{\varepsilon}\right)\right)$$

is a solution of the Stokes system in Ω^ε with the no-slip boundary condition on $\partial\Omega^\varepsilon$.

Define the second-order excess as

$$\begin{aligned}
&H_{2\mathrm{nd}}(u^\varepsilon,p^\varepsilon;\rho)\\
&\quad=\inf_{\substack{(w_1,q_1)\in\mathscr{Q}_1(\Omega)\\(w_2,q_2)\in\mathscr{Q}_2(\Omega)}}\left\{\left(\fint_{B^\varepsilon_{\rho,+}}\left|\nabla u^\varepsilon-\nabla\left(\varepsilon w_1\left(\frac{x}{\varepsilon}\right)+\varepsilon^2 w_2\left(\frac{x}{\varepsilon}\right)\right)\right|^2\mathrm{d}x\right)^{1/2}\right.\\
&\qquad\left.+\sup_{s,t\in[1/16,1/4]}\left|\fint_{B^\varepsilon_{s\rho,+}}\left(p^\varepsilon-\pi_1\left(\frac{x}{\varepsilon}\right)-\varepsilon\pi_2\left(\frac{x}{\varepsilon}\right)\right)\mathrm{d}x-\fint_{B^\varepsilon_{t\rho,+}}\left(p^\varepsilon-\pi_1\left(\frac{x}{\varepsilon}\right)-\varepsilon\pi_2\left(\frac{x}{\varepsilon}\right)\right)\mathrm{d}x\right|\right\}.
\end{aligned}$$

Lemma 5.5. *Let $L\in(0,\infty)$ and Ω be a bumpy periodic John domain with constant L according to Definition 1.3. Let $(u^\varepsilon,p^\varepsilon)$ be as in Theorem C, namely, a weak solution of (S^ε) in Section 3A satisfying (1-1). For all $\varepsilon\in\left(0,\frac{1}{32}\right]$, $r\in\left[2\varepsilon,\frac{1}{16}\right]$ and $\theta\in\left(0,\frac{1}{8}\right]$,*

$$H_{2\mathrm{nd}}(u^\varepsilon,p^\varepsilon;\theta r)\le C\left(\theta^2+\theta^{-3}\left(\frac{\varepsilon}{r}\right)^{1/12}\right)\Phi(u^\varepsilon,p^\varepsilon;16r)+C\theta^{-3}\left(\fint_{Q_{10r}}|\mathcal{M}^2_\varepsilon[F^\varepsilon]|^3\right)^{1/3},\tag{5-17}$$

where C depends only on L.

Proof. The proof follows from the strategy developed in Section 3, in particular from Lemma 3.2 to Lemma 3.7. Let (v_r,q_r) be the solution of the approximate problem (S_r). We will first use the $C^{2,1}$ estimate of $v_r=(v_{r,1},v_{r,2},v_{r,3})$ at the lower boundary $x_3=-\varepsilon$. Precisely, in view of no-slip Stokes polynomials defined in Section 4A, the $C^{2,1}$ estimate v_r gives

$$\left|v_r(x)-\sum_{k=1}^{2}\ell^*_{1k}P^{(1k)}(x+\varepsilon\boldsymbol{e}_3)-\sum_{j=1}^{6}\ell^*_{2j}P^{(2j)}(x+\varepsilon\boldsymbol{e}_3)\right|\le C\frac{|x+\varepsilon\boldsymbol{e}_3|^3}{r^2}\left(\fint_{Q^\varepsilon_r}|\nabla v_r|^2\right)^{1/2}\tag{5-18}$$

for $x\in Q^\varepsilon_{r/2}$, where we choose

$$\ell^*_{1k}=\frac{\partial v_{r,k}}{\partial x_3}(-\varepsilon\boldsymbol{e}_3)\quad\text{for }k=1,2$$

and

$$\ell_{21}^* = \frac{\partial^2 v_{r,1}}{\partial x_2 \partial x_3}(-\varepsilon \boldsymbol{e}_3), \quad \ell_{22}^* = \frac{1}{2}\frac{\partial^2 v_{r,1}}{\partial x_3^2}(-\varepsilon \boldsymbol{e}_3), \quad \ell_{23}^* = \frac{\partial^2 v_{r,2}}{\partial x_1 \partial x_3}(-\varepsilon \boldsymbol{e}_3),$$
$$\ell_{24}^* = \frac{1}{2}\frac{\partial^2 v_{r,2}}{\partial x_3^2}(-\varepsilon \boldsymbol{e}_3), \quad \ell_{25}^* = -\frac{1}{2}\frac{\partial^2 v_{r,1}}{\partial x_1 \partial x_3}(-\varepsilon \boldsymbol{e}_3), \quad \ell_{26}^* = -\frac{1}{2}\frac{\partial^2 v_{r,2}}{\partial x_2 \partial x_3}(-\varepsilon \boldsymbol{e}_3).$$

Moreover,

$$\sum_{k=1}^{2} |\ell_{1k}^*| + r\sum_{j=1}^{6} |\ell_{2j}^*| \le C\biggl(\fint_{Q_r^\varepsilon} |\nabla v_r|^2\biggr)^{1/2}. \tag{5-19}$$

Observe that

$$v_r^*(x) = v_r(x) - \sum_{k=1}^{2} \ell_{1k}^* P^{(1k)}(x+\varepsilon \boldsymbol{e}_3) - \sum_{j=1}^{6} \ell_{2j}^* P^{(2j)}(x+\varepsilon \boldsymbol{e}_3),$$
$$q_r^*(x) = q_r(x) - \sum_{j=1}^{6} \ell_{2j}^* L^{(2j)}(x+\varepsilon \boldsymbol{e}_3)$$

is a solution of the Stokes system in Q_r^ε with a no-slip condition on $x_3 = -\varepsilon$. Therefore, for any $\theta \in \bigl(0, \frac{1}{8}\bigr]$ and $r > \varepsilon$, it follows from (5-18) and the Caccioppoli inequality in rectangular region $Q_{2\theta r}^\varepsilon$ that

$$\biggl(\fint_{Q_{\theta r}^\varepsilon} |\nabla v_r^*|\biggr)^{1/2} \le \frac{C}{\theta r}\biggl(\fint_{Q_{2\theta r}^\varepsilon} |v_r^*|^2\biggr)^{1/2} \le C\biggl(\theta^2 + \theta^{-1}\Bigl(\frac{\varepsilon}{r}\Bigr)\biggr)\biggl(\fint_{Q_r^\varepsilon} |\nabla v_r|^2\biggr)^{1/2}. \tag{5-20}$$

Then (5-19) implies

$$\biggl(\fint_{Q_{\theta r}^\varepsilon} \biggl|\nabla v_r - \nabla\biggl(\sum_{k=1}^{2} \ell_{1k}^* P^{(1k)} + \sum_{j=1}^{6} \ell_{2j}^* P^{(2j)}\biggr)\biggr|^2\biggr)^{1/2} \le C\biggl(\theta^2 + \theta^{-1}\Bigl(\frac{\varepsilon}{r}\Bigr)\biggr)\biggl(\fint_{Q_r^\varepsilon} |\nabla v_r|^2\biggr)^{1/2}. \tag{5-21}$$

Next, to see the oscillation estimate for the pressure, applying Bogovskii's lemma to q_r^* and the Caccioppoli inequality to v_r^* (combined with (5-20)) in Lipschitz domains, we have

$$\sup_{s,t\in[1/16,1/4]} \biggl|\fint_{Q_{s\theta r}^\varepsilon} q_r^* - \fint_{Q_{t\theta r}^\varepsilon} q_r^*\biggr| \le C\biggl(\theta^2 + \theta^{-1}\Bigl(\frac{\varepsilon}{r}\Bigr)\biggr)\biggl(\fint_{Q_r^\varepsilon} |\nabla v_r|^2\biggr)^{1/2}. \tag{5-22}$$

Notice that $L^{(2j)}$ are linear functions. Thus, an application of (5-19) and the triangle inequality to (5-22) leads to

$$\sup_{s,t\in[1/16,1/4]} \biggl|\fint_{Q_{s\theta r}^\varepsilon} \biggl(q_r - \sum_{j=1}^{6} \ell_{2j}^* L^{(2j)}\biggr) - \fint_{Q_{t\theta r}^\varepsilon} \biggl(q_r - \sum_{j=1}^{6} \ell_{2j}^* L^{(2j)}\biggr)\biggr| \le C\biggl(\theta^2 + \theta^{-1}\Bigl(\frac{\varepsilon}{r}\Bigr)\biggr)\biggl(\fint_{Q_r^\varepsilon} |\nabla v_r|^2\biggr)^{1/2}. \tag{5-23}$$

This, combined with (5-21), gives

$$\begin{aligned}\biggl(\fint_{Q_{\theta r}^\varepsilon} \biggl|\nabla v_r - \nabla\biggl(\sum_{k=1}^{2} \ell_{1k}^* P^{(1k)} + \sum_{j=1}^{6} \ell_{2j}^* P^{(2j)}\biggr)\biggr|^2\biggr)^{1/2} \\ + \sup_{s,t\in[1/16,1/4]} \biggl|\fint_{Q_{s\theta r}^\varepsilon} \biggl(q_r - \sum_{j=1}^{6} \ell_{2j}^* L^{(2j)}\biggr) - \fint_{Q_{t\theta r}^\varepsilon} \biggl(q_r - \sum_{j=1}^{6} \ell_{2j}^* L^{(2j)}\biggr)\biggr| \\ \le C\biggl(\theta^2 + \theta^{-1}\Bigl(\frac{\varepsilon}{r}\Bigr)\biggr)\biggl(\fint_{Q_r^\varepsilon} |\nabla v_r|^2\biggr)^{1/2}.\end{aligned}$$

This is the key second-order excess estimate we need for (v_r, q_r) in Q^ε_r. To proceed, we follow the similar argument developed in Section 3. Precisely, using an analogue of Lemma 3.4 and taking the approximation estimate in Lemma 3.2, we can replace (v_r, q_r) by $(u^\varepsilon, p^\varepsilon)$ with new errors in u^ε and F^ε. Combined with the energy estimate for (S_r), we now have

$$\begin{aligned}\bigg(\fint_{B^\varepsilon_{r,+}}\bigg|\nabla u^\varepsilon-\nabla\bigg(\sum_{k=1}^{2}\ell^*_{1k}P^{(1k)}+\sum_{j=1}^{6}\ell^*_{2j}P^{(2j)}\bigg)\bigg|^2\bigg)^{1/2}&\\ +\sup_{s,t\in[1/16,1/4]}\bigg|\fint_{B^\varepsilon_{s\theta r,+}}\bigg(p^\varepsilon-\sum_{j=1}^{6}\ell^*_{2j}L^{(2j)}\bigg)-\fint_{B^\varepsilon_{t\theta r,+}}\bigg(p^\varepsilon-\sum_{j=1}^{6}\ell^*_{2j}L^{(2j)}\bigg)\bigg|&\\ \le C\bigg(\theta^2+\theta^{-3}\Big(\frac{\varepsilon}{r}\Big)^{1/12}\bigg)\bigg(\fint_{B^\varepsilon_{5r,+}}|\nabla u^\varepsilon|^2\bigg)^{1/2}+C\theta^{-3}\bigg(\fint_{Q_{4r}}|\mathcal{M}^2_\varepsilon[F^\varepsilon]|^3\bigg)^{1/3}.&\qquad(5\text{-}24)\end{aligned}$$

Next, we insert the boundary layers into the above inequality. By (5-8) and

$$\sum_{j=1}^{6}\bigg\{\frac{1}{r}\bigg(\fint_{B^\varepsilon_{\theta r,+}}\Big|\varepsilon(\nabla v^{(2j)})\Big(\frac{x}{\varepsilon}\Big)\Big|^2\,\mathrm{d}x\bigg)^{1/2}+\frac{1}{r}\bigg(\fint_{B^\varepsilon_{\theta r,+}}\Big|\varepsilon q^{(2j)}\Big(\frac{x}{\varepsilon}\Big)\Big|^2\,\mathrm{d}x\bigg)^{1/2}\bigg\}\le C\theta^{-1/2}\Big(\frac{\varepsilon}{r}\Big)^{1/2},$$

which follows from Proposition 4.7, we obtain from (5-24) and (5-19) along with the energy estimate for (S_r) that

$$\begin{aligned}\bigg(\fint_{B^\varepsilon_{\theta r,+}}\bigg|\nabla u^\varepsilon-\nabla\bigg(\sum_{k=1}^{2}\ell^*_{1k}\Big(P^{(1k)}+\varepsilon v^{(1k)}\Big(\frac{x}{\varepsilon}\Big)\Big)+\sum_{j=1}^{6}\ell^*_{2j}\Big(P^{(2j)}+\varepsilon^2 v^{(2j)}\Big(\frac{x}{\varepsilon}\Big)\Big)\bigg)\bigg|^2\,\mathrm{d}x\bigg)^{1/2}&\\ +\sup_{s,t\in[1/16,1/4]}\bigg|\fint_{B^\varepsilon_{s\theta r,+}}\bigg(p^\varepsilon-\sum_{k=1}^{2}\ell^*_{1k}q^{(1k)}\Big(\frac{x}{\varepsilon}\Big)-\sum_{j=1}^{6}\ell^*_{2j}\Big(L^{(2j)}+\varepsilon q^{(2j)}\Big(\frac{x}{\varepsilon}\Big)\Big)\bigg)\,\mathrm{d}x&\\ -\fint_{B^\varepsilon_{t\theta r,+}}\bigg(p^\varepsilon-\sum_{k=1}^{2}\ell^*_{1k}q^{(1k)}\Big(\frac{x}{\varepsilon}\Big)-\sum_{j=1}^{6}\ell^*_{2j}\Big(L^{(2j)}+\varepsilon q^{(2j)}\Big(\frac{x}{\varepsilon}\Big)\Big)\bigg)\,\mathrm{d}x\bigg|&\\ \le C\bigg(\theta^2+\theta^{-3}\Big(\frac{\varepsilon}{r}\Big)^{1/12}\bigg)\bigg(\fint_{B^\varepsilon_{5r,+}}|\nabla u^\varepsilon|^2\bigg)^{1/2}+C\theta^{-3}\bigg(\fint_{Q_{4r}}|\mathcal{M}^2_\varepsilon[F^\varepsilon]|^3\bigg)^{1/3}.&\qquad(5\text{-}25)\end{aligned}$$

In view of the definition of H_{2nd}, we arrive at

$$H_{\mathrm{2nd}}(u^\varepsilon, p^\varepsilon;\theta r)\le C\bigg(\theta^2+\theta^{-3}\Big(\frac{\varepsilon}{r}\Big)^{1/12}\bigg)\bigg(\fint_{B^\varepsilon_{5r,+}}|\nabla u^\varepsilon|^2\bigg)^{1/2}+C\theta^{-3}\bigg(\fint_{Q_{4r}}|\mathcal{M}^2_\varepsilon[F^\varepsilon]|^3\bigg)^{1/3},$$

which implies the desired estimate. □

Proposition 5.6. *Let $L\in(0,\infty)$ and Ω be a bumpy periodic John domain with constant L according to Definition 1.3. Let $(u^\varepsilon, p^\varepsilon)$ be as in Theorem C. For any $\gamma\in[0,1)$, $\delta\in(0,1)$, $\varepsilon\in\big(0,\frac{1}{2}\big]$ and $r\in\big[\varepsilon,\frac{1}{2}\big]$,*

$$H_{\mathrm{2nd}}(u^\varepsilon, p^\varepsilon;r)\le Cr^{1+\gamma}(M+M^{6+2\gamma+\delta}),\qquad(5\text{-}26)$$

where C depends on L, γ and δ. Here M is the number in Theorem C.

Proof. For any $\gamma \in [0, 1)$, we choose an arbitrary $\delta > 0$ small enough so that $\delta < \frac{1-\gamma}{2}$. Then applying (3-39) with $\beta = 1+\gamma+\delta$, we have

$$\left(\fint_{Q_r} |\mathcal{M}_\varepsilon^2[F^\varepsilon]|^3\right)^{1/3} \le C(M + M^{6+2\gamma+4\delta})r^{1+\gamma+\delta}.$$

Now, the rest of the proof is parallel to Proposition 5.3. We omit the details. □

The following lemma is parallel to Lemma 5.4.

Lemma 5.7. *Let $L \in (0, \infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. For a given $\rho > 0$, let $\ell_{1k}(\rho)$ and $\ell_{2j}(\rho)$ be the real numbers so that*

$$(w_1, \pi_1) = \sum_{k=1}^{2} \ell_{1k}(\rho)(P^{(1k)} + v^{(1k)}, q^{(1k)}),$$

$$(w_2, \pi_2) = \sum_{j=1}^{6} \ell_{2j}(\rho)(P^{(2j)} + v^{(2j)}, L^{(2j)} + q^{(2j)})$$

minimize $H_{2\mathrm{nd}}(u^\varepsilon, p^\varepsilon; \rho)$. Then there exists a constant $\varepsilon_2 \in (0, 1)$ so that for all $\varepsilon \in (0, \varepsilon_2]$ and $r \in \left[\varepsilon/\varepsilon_2, \frac{1}{2}\right]$,

$$\sup_{r_1,r_2\in[r,2r]} \sum_{k=1}^{2} |\ell_{1k}(r_1) - \ell_{1k}(r_2)| + \sup_{r_1,r_2\in[r,2r]} \sum_{j=1}^{6} r|\ell_{2j}(r_1) - \ell_{2j}(r_2)| \le C \sup_{t\in[r,2r]} H_{2\mathrm{nd}}(u^\varepsilon, p^\varepsilon; t), \quad (5\text{-}27)$$

where C depends only on L.

Proof. First, observe that, for any $a_k, b_j \in \mathbb{R}$,

$$\sum_{k=1}^{2} |a_k| + \sum_{j=1}^{6} |b_j| \le C\left(\fint_{B_1(0)\cap\{x_3>0\}} \left|\sum_{k=1}^{2} a_k P^{(1k)} + \sum_{j=1}^{6} b_j P^{(2j)}\right|^2\right)^{1/2}. \quad (5\text{-}28)$$

This inequality is true because $P^{(1k)}$ and $P^{(2j)}$ are all linearly independent polynomials. Recall that $P^{(1k)}$ are homogeneous linear functions and $P^{(2j)}$ are homogeneous quadratic functions. This means $P^{(1k)}(rx) = rP^{(1k)}(x)$ and $P^{(2j)}(rx) = r^2 P^{(2j)}(x)$. Fix $r_1, r_2 \in [r, 2r]$. Applying (5-28) with $a_k = \ell_{1k}(r_1) - \ell(r_2)$ and $b_j = r(\ell_{2j}(r_1) - \ell_{2j}(r_2))$, we have

$$\begin{aligned}
&\sum_{k=1}^{2} |\ell_{1k}(r_1) - \ell_{1k}(r_2)| + \sum_{j=1}^{6} r|\ell_{2j}(r_1) - \ell_{2j}(r_2)| \\
&\quad\le C\left(\fint_{B_1(0)\cap\{x_3>0\}} \left|\sum_{k=1}^{2} (\ell_{1k}(r_1) - \ell_{1k}(r_2))P^{(1k)} + \sum_{j=1}^{6} r(\ell_{2j}(r_1) - \ell_{2j}(r_2))P^{(2j)}\right|^2\right)^{1/2} \\
&\quad\le \frac{C}{r}\left(\fint_{B_r(0)\cap\{x_3>0\}} \left|\sum_{k=1}^{2} (\ell_{1k}(r_1) - \ell_{1k}(r_2))P^{(1k)} + \sum_{j=1}^{6} (\ell_{2j}(r_1) - \ell_{2j}(r_2))P^{(2j)}\right|^2\right)^{1/2} \\
&\quad\le C\left(\fint_{B_r(0)\cap\{x_3>0\}} \left|\sum_{k=1}^{2} (\ell_{1k}(r_1) - \ell_{1k}(r_2))\nabla P^{(1k)} + \sum_{j=1}^{6} (\ell_{2j}(r_1) - \ell_{2j}(r_2))\nabla P^{(2j)}\right|^2\right)^{1/2},
\end{aligned}$$

where the Poincaré inequality has been applied in the last line. Now, inserting u^ε, $v^{(1k)}(x/\varepsilon)$ and $v^{(2j)}(x/\varepsilon)$ into the right-hand side, and using the triangle inequality, we obtain

$$\begin{aligned}
&\sum_{k=1}^{2}|\ell_{1k}(r_1)-\ell_{1k}(r_2)|+\sum_{j=1}^{6}r|\ell_{2j}(r_1)-\ell_{2j}(r_2)| \\
&\quad\le C\biggl(\fint_{B^\varepsilon_{r,+}}\biggl|\nabla\biggl(\sum_{k=1}^{2}(\ell_{1k}(r_1)-\ell_{1k}(r_2))\Bigl(P^{(1k)}+\varepsilon v^{(1k)}\Bigl(\frac{x}{\varepsilon}\Bigr)\Bigr) \\
&\qquad\qquad+\sum_{j=1}^{6}(\ell_{2j}(r_1)-\ell_{2j}(r_2))\Bigl(P^{(2j)}+\varepsilon^2 v^{(2j)}\Bigl(\frac{x}{\varepsilon}\Bigr)\Bigr)\biggr)\biggr|^2\,\mathrm{d}x\biggr)^{1/2} \\
&\qquad\qquad\qquad+C\sum_{k=1}^{2}|\ell_{1k}(r_1)-\ell_{1k}(r_2)|\biggl(\fint_{B^\varepsilon_{r,+}}\Bigl|(\nabla v^{(1j)})\Bigl(\frac{x}{\varepsilon}\Bigr)\Bigr|^2\,\mathrm{d}x\biggr)^{1/2} \\
&\qquad\qquad\qquad\qquad+\frac{C}{r}\sum_{j=1}^{6}r|\ell_{2j}(r_1)-\ell_{2j}(r_2)|\biggl(\fint_{B^\varepsilon_{r,+}}\Bigl|\varepsilon(\nabla v^{(2j)})\Bigl(\frac{x}{\varepsilon}\Bigr)\Bigr|^2\,\mathrm{d}x\biggr)^{1/2} \\
&\quad\le CH_{\mathrm{2nd}}(u^\varepsilon,p^\varepsilon;r_1)+CH_{\mathrm{2nd}}(u^\varepsilon,p^\varepsilon;r_2) \\
&\qquad\qquad+C_1\Bigl(\frac{\varepsilon}{r}\Bigr)^{1/2}\sum_{j=1}^{2}|\ell_{1k}(r_1)-\ell_{1k}(r_2)|+C_2\Bigl(\frac{\varepsilon}{r}\Bigr)^{1/2}\sum_{k=1}^{6}r|\ell_{2j}(r_1)-\ell_{2j}(r_2)|,
\end{aligned}$$

where Proposition 4.7 is applied in the last inequality. Thus, if $r>\varepsilon/\varepsilon_2$ for some sufficiently small constant $\varepsilon_2\in(0,1)$ so that $C_1(\varepsilon/r)^{1/2}<\frac{1}{2}$ and $C_2(\varepsilon/r)^{1/2}<\frac{1}{2}$, then

$$\sum_{k=1}^{2}|\ell_{1k}(r_1)-\ell_{1k}(r_2)|+\sum_{j=1}^{6}r|\ell_{2j}(r_1)-\ell_{2j}(r_2)|\le C\sum_{i=1}^{2}H_{\mathrm{2nd}}(w^\varepsilon,\pi^\varepsilon;r_i).$$

This leads to the assertion. ☐

Proof of Theorem C. The estimate for the velocity is contained in (5-26). The estimate for pressure can be derived similarly as Theorem B. The details are left to the reader. ☐

5C. *Liouville-type results.* As an application of the construction of boundary layers and uniform regularity, a Liouville-type theorem for Stokes systems can be shown by the large-scale Lipschitz, $C^{1,\gamma}$ and $C^{2,\gamma}$ estimates. We point out that our large-scale regularity results hold also for the linear Stokes system, although with linear dependence on M in the right-hand sides of (1-2), (1-3) and (1-4). The proofs are simpler, using that the source term $F^\varepsilon=0$. To describe the Liouville-type theorem, consider the Stokes system in the entire Ω

$$\begin{cases}-\Delta u+\nabla p=0, & x\in\Omega,\\ \nabla\cdot u=0, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}\tag{5-29}$$

where Ω is a bumpy John domain according to Definition 1.2. Let $B_R=B_R(0)$. We state the Liouville-type theorem as follows. Its proof follows from a routine rescaling of the large-scale regularity estimates. Notice that this result complements Corollary 3.1 already stated above.

Theorem 5.8. *Let Ω be a bumpy John domain according to Definition 1.2. Let (u, p) be a weak solution of (5-29).*

(i) *If for some $\sigma \in (0, 1)$*

$$\liminf_{R\to\infty} \frac{1}{R^{1+\sigma}} \left(\fint_{B_{R,+}} |u|^2 \right)^{1/2} = 0,$$

then $(u, p) \in \mathscr{Q}_1(\Omega)$ (up to a constant for p).

(ii) *In addition, assume Ω is periodic bumpy John domain according to Definition 1.3. If for some $\sigma \in (0, 1)$,*

$$\liminf_{R\to\infty} \frac{1}{R^{2+\sigma}} \left(\fint_{B_{R,+}} |u|^2 \right)^{1/2} = 0,$$

then $(u, p) \in \mathscr{Q}_1(\Omega) + \mathscr{Q}_2(\Omega)$ (up to a constant for p).

Appendix A: Bogovskii's lemma and some applications

For a bounded open set $D \subset \mathbb{R}^3$ and $p \in (1, \infty)$, let

$$L^q_0(D) = \left\{ f \in L^q(D) \,\middle|\, \fint_D f = 0 \right\}.$$

Theorem A.1 [Acosta et al. 2006, Theorem 4.1]. *Let $\Omega \subset \mathbb{R}^3$ be a bounded John domain according to Definition 1.1 with constant L. There exists an operator $\mathbb{B} : L^q_0(\Omega) \to W^{1,q}_0(\Omega)^3$ satisfying*

$$\nabla \cdot \mathbb{B}[f] = f \quad \text{in } \Omega$$

and

$$\|\mathbb{B}[f]\|_{W^{1,q}(\Omega)} \le C\|f\|_{L^q(\Omega)}, \tag{A-1}$$

with C depending on L.

Lemma A.2. *Let Ω be a bounded John domain according to Definition 1.1. Set*

$$H^1_{0,\sigma}(\Omega) := \{u \in H^1_0(\Omega)^3 \mid \nabla \cdot u = 0 \text{ in } \Omega\}.$$

Let $f \in L^2(\Omega)^3$ and $F \in L^2(\Omega)^{3\times 3}$. If $u \in H^1(\Omega)^3$ is a weak solution of the Stokes equations in the sense

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f \cdot \varphi - \int_\Omega F \cdot \nabla \varphi, \quad \varphi \in H^1_{0,\sigma}(\Omega),$$

then there exists a function $p \in L^2(\Omega)$ unique up to a constant for which we have

$$\int_\Omega \nabla u \cdot \nabla \phi - \int_\Omega p(\nabla \cdot \phi) = \int_\Omega f \cdot \phi - \int_\Omega F \cdot \nabla \phi, \quad \phi \in H^1_0(\Omega)^3.$$

Namely, the pair (u, p) is a weak solution of the Stokes equations. Moreover,

$$\left\| p - \fint_\Omega p \right\|_{L^2(\Omega)} \le C\big(\|\nabla u\|_{L^2(\Omega)} + \operatorname{diam}(\Omega)\|f\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)}\big), \tag{A-2}$$

where $\operatorname{diam}(\Omega)$ *denotes the diameter of* Ω.

A direct application of Bogovskii's operator is the Caccioppoli inequality for the Stokes equations. Let $Q_{r,+} = Q_r \cap \{x_3 > 0\}$. Suppose (u, p) is a weak solution of

$$\begin{cases} -\Delta u + \nabla p = \nabla \cdot F & \text{in } Q_{2r,+}, \\ \nabla \cdot u = 0 & \text{in } Q_{2r,+}, \\ u = 0 & \text{on } Q_{2r} \cap \{x_3 = 0\}. \end{cases} \tag{A-3}$$

The following is the Caccioppoli inequality over flat boundaries whose proof is classical [Giaquinta and Modica 1982, Theorem 1.1] (the interior Caccioppoli inequality is similar).

Lemma A.3. *Let $F \in L^2(Q_{2r,+})^{3\times 3}$ and let $(u, p) \in H^1(Q_{2r,+})^3 \times L^2(Q_{2r,+})$ be a weak solution to (A-3). Then,*

$$\|\nabla u\|_{L^2(Q_{r,+})} \le C\Big(\frac{1}{r}\|u\|_{L^2(Q_{2r,+})} + \|F\|_{L^2(Q_{2r,+})}\Big), \tag{A-4}$$

where the constant C is independent of r.

Now, consider the Stokes equations over John boundaries

$$\begin{cases} -\Delta u^\varepsilon + \nabla p^\varepsilon = \nabla \cdot F^\varepsilon & \text{in } B^\varepsilon_{4r,+}, \\ \nabla \cdot u^\varepsilon = 0 & \text{in } B^\varepsilon_{4r,+}, \\ u^\varepsilon = 0 & \text{on } \Gamma^\varepsilon_{4r}. \end{cases} \tag{A-5}$$

Unfortunately, the Caccioppoli inequality in the form of (A-4) cannot be derived for the weak solution of (A-5) by the usual iteration argument (see e.g., [Giaquinta and Modica 1982, Lemma 0.5] or [Giaquinta 1983, Chapter V, Lemma 3.1]) due to the assumption that the John domain condition (after rescaling) holds only for scales $r \ge \varepsilon$. Actually, we only have a weaker Caccioppoli inequality valid for $r \ge \varepsilon$, which is sufficient for us to show a (large-scale) Meyers estimate.

Lemma A.4 (a weak Caccioppoli inequality). *Let $L \in (0, \infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. Let $\varepsilon \in \big(0, \frac{1}{2}\big]$ and $F^\varepsilon \in L^2(B^\varepsilon_{4r,+})^{3\times 3}$, and let $(u^\varepsilon, p^\varepsilon) \in H^1(B^\varepsilon_{4r,+})^3 \times L^2(B^\varepsilon_{4r,+})$ be a weak solution to (A-5) with $r \ge \varepsilon$. Then, for any $\theta \in (0, 1)$,*

$$\|\nabla u^\varepsilon\|_{L^2(B^\varepsilon_{r,+})} \le \theta \|\nabla u^\varepsilon\|_{L^2(B^\varepsilon_{4r,+})} + \frac{C}{\theta r}\|u^\varepsilon\|_{L^2(B^\varepsilon_{4r,+})} + C\|F^\varepsilon\|_{L^2(B^\varepsilon_{4r,+})}, \tag{A-6}$$

where the constant C depends only on L. In particular C is independent of θ, ε and r. Moreover, if $r \ge 4\varepsilon$, then by the standard interior Caccioppoli inequality and a covering argument as in the proof of Lemma 2.2, $B^\varepsilon_{4r,+}$ may be replaced by $B^\varepsilon_{2r,+}$ on the right-hand side of (A-6).

Proof. Let ϕ_r be a smooth cut-off function so that $\phi_r(x) = 1$ for $x \in Q_r$, $\phi(x) = 0$ for $x \notin Q_{2r}$ and $|\nabla\phi| \le C/r$. Integrating the first equation of (A-5) against $u^\varepsilon\phi^2$, we have

$$\int_{B^\varepsilon_{2r,+}} \nabla u^\varepsilon \cdot \nabla u^\varepsilon \phi^2 = -2\int_{B^\varepsilon_{2r,+}} \phi \nabla u^\varepsilon \cdot (\nabla\phi \otimes u^\varepsilon) - \int_{B^\varepsilon_{2r,+}} \nabla p^\varepsilon \cdot u^\varepsilon \phi^2 - \int_{B^\varepsilon_{2r,+}} F^\varepsilon \cdot \nabla(u^\varepsilon\phi^2). \tag{A-7}$$

The first and third terms on the right-hand side are routine. For the sake of completeness, let us give some more details for the third term. We have

$$\begin{aligned}\left|\int_{B^{\varepsilon}_{2r,+}} F^{\varepsilon}\cdot\nabla(u^{\varepsilon}\phi^2)\right| &\leq \left|\int_{B^{\varepsilon}_{2r,+}} F^{\varepsilon}\cdot\nabla u^{\varepsilon}\phi^2\right| + 2\left|\int_{B^{\varepsilon}_{2r,+}} \phi F^{\varepsilon}\cdot(\nabla\phi\otimes u^{\varepsilon})\right| \\ &\leq \|F^{\varepsilon}\|_{L^2(B^{\varepsilon}_{2r,+})}\|\nabla u^{\varepsilon}\phi\|_{L^2(B^{\varepsilon}_{2r,+})} + \frac{C}{r}\|F^{\varepsilon}\|_{L^2(B^{\varepsilon}_{2r,+})}\|u^{\varepsilon}\phi\|_{L^2(B^{\varepsilon}_{2r,+})}.\end{aligned}$$

We then use Young's inequality in both terms and absorb the term $\frac{1}{2}\|\nabla u^{\varepsilon}\phi\|_{L^2(B^{\varepsilon}_{2r,+})}$ in the left-hand side of the inequality (A-7). To deal with the pressure, by Definition 1.2 of the bumpy John domain Ω, we use the Bogovskii operator in a John domain $\Omega^{\varepsilon}_{2r}$ satisfying $B^{\varepsilon}_{2r,+}\subset\Omega^{\varepsilon}_{2r}\subset B^{\varepsilon}_{4r,+}$ and (A-2) to obtain

$$\left(\int_{\Omega^{\varepsilon}_{2r}}\left|p^{\varepsilon} - ⨍_{\Omega^{\varepsilon}_{2r}} p^{\varepsilon}\right|^2\right)^{1/2} \leq C(\|\nabla u^{\varepsilon}\|_{L^2(\Omega^{\varepsilon}_{2r})} + \|F^{\varepsilon}\|_{L^2(\Omega^{\varepsilon}_{2r})}).$$

Let $L = ⨍_{\Omega^{\varepsilon}_{2r}} p^{\varepsilon}$. Then, using the above estimate and $\nabla\cdot u^{\varepsilon} = 0$,

$$\begin{aligned}\left|\int_{B^{\varepsilon}_{2r,+}}\nabla p^{\varepsilon}\cdot u^{\varepsilon}\phi^2\right| &= \left|\int_{B^{\varepsilon}_{2r,+}}\nabla(p^{\varepsilon}-L)\cdot u^{\varepsilon}\phi^2\right| = \left|\int_{B^{\varepsilon}_{2r,+}}(p^{\varepsilon}-L)u^{\varepsilon}\cdot 2\phi\nabla\phi\right| \\ &\leq \frac{C}{r}(\|\nabla u^{\varepsilon}\|_{L^2(\Omega^{\varepsilon}_{2r})} + \|F^{\varepsilon}\|_{L^2(\Omega^{\varepsilon}_{2r})})\|u^{\varepsilon}\|_{L^2(B^{\varepsilon}_{2r,+})} \\ &\leq \theta^2\|\nabla u^{\varepsilon}\|^2_{L^2(B^{\varepsilon}_{4r,+})} + \frac{C}{\theta^2 r^2}\|u^{\varepsilon}\|^2_{L^2(B^{\varepsilon}_{4r,+})} + C\|F^{\varepsilon}\|^2_{L^2(B^{\varepsilon}_{4r,+})}\end{aligned}$$

for any $\theta\in(0,1)$. In view of (A-7), this gives the desired estimate by a standard argument. □

Appendix B: Large-scale estimates for the Green's function

This appendix is devoted to the study of the Green's function for the Stokes equations in a bumpy John half-space according to Definition 1.2. The large-scale estimates proved in Section 3 will be applied. The basic scheme is to derive estimates for the velocity part of the Green's function directly from the interior and large-scale boundary Lipschitz estimates. For this we follow the strategy pioneered in [Avellaneda and Lin 1987; 1991]. Then, we deduce the estimates for the pressure part of the Green's function from Bogovskii's lemma and the estimates for the velocity part.

We use $B_R(x) = Q_R(x)$ to denote the cube centered at x with side length $2R$. If the center is not important in the context, it is abbreviated as B_R. Throughout this appendix, $\Omega_{\leq N}$, $\Omega_{\geq N}$, $\Omega_{<N}$, and $\Omega_{>N}$ defined around (4-7) will be used. Moreover, let $\hat{x}$ denote the projection of $x\in\mathbb{R}^3$ on $\partial\mathbb{R}^3_+$.

B1. *Construction of the Green's function.* Let D be an open set in $\mathbb{R}^3$. Denote by $Y^{1,2}(D)$ the space of functions

$$\{u\in L^6(D)\mid\nabla u\in L^2(D)^3\} \tag{B-1}$$

equipped with the norm $\|u\|_{Y^{1,2}(D)} = \|u\|_{L^6(D)} + \|\nabla u\|_{L^2(D)}$. Let $Y^{1,2}_0(D)$ be the closure of $C^{\infty}_0(D)$ under $\|\cdot\|_{Y^{1,2}(D)}$. The closed subspace of $Y^{1,2}_0(D)^3$

$$\{u\in Y^{1,2}_0(D)^3\mid\nabla\cdot u = 0 \text{ in } D\} \tag{B-2}$$

is denoted by $Y^{1,2}_{0,\sigma}(D)$. Note that, when the Lebesgue measure of D is finite, we have $Y^{1,2}_0(D) = H^{1,2}_0(D)$ by the Sobolev inequality $\|u\|_{L^6(D)} \le C\|\nabla u\|_{L^2(D)}$ if $u \in C^\infty_0(D)$. Moreover, we see that $Y^{1,2}_{0,\sigma}(D)$ as well as $Y^{1,2}_0(D)^3$ is a Hilbert space with an inner product $\langle u, v\rangle = \int_D \nabla u \cdot \nabla v$.

Let Ω be a bumpy John domain with constant $L \in (0, \infty)$ according to the Definition 1.2. Based on similar proofs in [Hofmann and Kim 2007; Choi and Lee 2017] and using the large-scale Lipschitz estimate of Theorem A proved in Section 3, we can construct the Green's function $(G, \Pi) = (G(x, y), \Pi(x, y))$, which satisfies the following properties:

(i) For any $q \in \left[1, \frac{3}{2}\right)$, $G(\,\cdot\,, y) \in W^{1,q}_{0,\mathrm{loc}}(\overline{\Omega})^{3\times 3}$ and $G(\,\cdot\,, y) \in Y^{1,2}(\Omega \setminus B_r(y))^{3\times 3}$ for each $y \in \Omega$ and $r > 0$. Moreover, $\Pi(\,\cdot\,, y) \in L^2_{\mathrm{loc}}(\overline{\Omega \setminus B_r(y)})^{3\times 1}$ for each $y \in \Omega$ and $r > 0$.

(ii) $(G(\,\cdot\,, y), \Pi(\,\cdot\,, y))$ satisfies, for each $y \in \Omega$,

$$\int_\Omega \nabla G(\,\cdot\,, y) \cdot \nabla \phi - \int_\Omega \Pi(\,\cdot\,, y)(\nabla \cdot \phi) = \phi(y), \quad \phi \in C^\infty_0(\Omega)^3. \tag{B-3}$$

(iii) For all $f \in C^\infty_0(\Omega)^3$, if the function $(u, p) \in Y^{1,2}_{0,\sigma}(\Omega)^3 \times L^2_{\mathrm{loc}}(\overline{\Omega})$, with $p(x) \to 0$ as $x_3 \to \infty$, satisfies the Stokes equations in the sense

$$\int_\Omega \nabla u \cdot \nabla \phi - \int_\Omega p(\nabla \cdot \phi) = \int_\Omega f \cdot \phi, \quad \phi \in C^\infty_0(\Omega)^3, \tag{B-4}$$

then

$$u(x) = \int_\Omega G(x, y) f(y)\,\mathrm{d}y, \quad p(x) = \int_\Omega \Pi(x, y) \cdot f(y)\,\mathrm{d}y. \tag{B-5}$$

We describe how to obtain (G, Π) meeting properties (i)–(iii) above. The existence and basic estimates of the velocity component $G(x, y)$ follow from a similar argument as [Hofmann and Kim 2007, Theorem 4.1] by working in the Hilbert space $Y^{1,2}_{0,\sigma}(\Omega)$. In fact, there is $G(x, y)$ such that $u(x)$ defined in (B-5) belongs to $u \in Y^{1,2}_{0,\sigma}(\Omega)$ and is the unique solution of the Stokes equations in the sense

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f \cdot \varphi \quad \text{for any } \varphi \in Y^{1,2}_{0,\sigma}(\Omega). \tag{B-6}$$

Then, by using Lemma A.2 on each bounded John subdomain, one sees that there is a pressure $p \in L^2_{\mathrm{loc}}(\overline{\Omega})$ for which we have (B-4), uniquely determined under the condition $p(x) \to 0$ as $x_3 \to \infty$.

When constructing the pressure component $\Pi(x, y)$ in (B-5), we need a careful analysis since the domain is unbounded unlike in [Choi and Lee 2017]. Here the oscillation estimate of p will play a crucial role. For an open set E, define the oscillation of p in E by

$$\operatorname*{osc}_E p = \sup_{x, y \in E} |p(x) - p(y)|. \tag{B-7}$$

The following lemma shows a fundamental oscillation estimate for the pressure.

Lemma B.1. *Let $L \in (0, \infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. Then we have the following statements*:

(i) *Let $\overline{B}_R \subset \Omega$. If $-\Delta u + \nabla p = 0$ and $\nabla \cdot u = 0$ in B_R, then*

$$\operatorname*{osc}_{B_{R/2}} p \le C\left(\fint_{B_R} |\nabla u|^2\right)^{1/2}, \tag{B-8}$$

where C is a universal constant.

(ii) *Let $z \in \partial\mathbb{R}^3_+$ and let $R > 2$. If $-\Delta u + \nabla p = 0$ and $\nabla \cdot u = 0$ in $\Omega \cap B_R(z)$ and $u = 0$ on $\partial\Omega \cap B_R(z)$, then*

$$\operatorname*{osc}_{\Omega_{>2} \cap B_{R/2}(z)} p \le C\left(\fint_{\Omega \cap B_R(z)} |\nabla u|^2\right)^{1/2}, \tag{B-9}$$

where C depends on L and is independent of z and R.

Proof. The interior case (i) is classical and the proof is omitted. Let us prove the boundary case (ii). Since only the case where R is sufficiently large is nontrivial, we assume that $R > 32$. For any $x \in \Omega_{>2} \cap B_{R/2}(z)$, the mean value property of harmonic functions yields

$$p(x) = \fint_{\mathcal{B}_r(x)} p \quad \text{if } \mathcal{B}_r(x) := \{y \mid |y - x| < r\} \subset \Omega. \tag{B-10}$$

Here we assume $r = \frac{1}{2}x_3 \le \frac{1}{16}R$; hence $r > 1$. Note that if $x_3 > \frac{1}{8}R$, the oscillation can be handled by the interior estimate (B-8).

Recall that $\hat{x}$ is the projection of x on $\partial\mathbb{R}^3_+$. Let $\Omega_{4r}(\hat{x})$ be a John domain given by Definition 1.2 so that $\Omega \cap B_{4r}(\hat{x}) \subset \Omega_{4r}(\hat{x}) \subset \Omega \cap B_{8r}(\hat{x})$. Clearly $\mathcal{B}_r(x) \subset \Omega_{4r}(\hat{x})$, $B_{R/4}(\hat{x}) \subset B_{3R/4}(z)$ and $B_{R/2}(\hat{x}) \subset B_R(z)$. By (B-10) and the Bogovskii lemma,

$$\begin{aligned}\left|p(x) - \fint_{\Omega_{4r}(\hat{x})} p\right| &\le \fint_{\mathcal{B}_r(x)}\left|p - \fint_{\Omega_{4r}(\hat{x})} p\right| \le C \fint_{\Omega_{4r}(\hat{x})}\left|p - \fint_{\Omega_{4r}(\hat{x})} p\right| \\ &\le C\left(\fint_{B_{8r}(\hat{x})\cap\Omega} |\nabla u|^2\right)^{1/2} \le C\left(\fint_{B_{R/2}(\hat{x})\cap\Omega} |\nabla u|^2\right)^{1/2} \le C\left(\fint_{B_R(z)\cap\Omega} |\nabla u|^2\right)^{1/2},\end{aligned} \tag{B-11}$$

where we also used the Lipschitz estimate of u in the fourth inequality. Similarly, we have

$$\left|\fint_{B_{4r}(\hat{x})\cap\Omega} p - \fint_{\Omega_{4r}(\hat{x})} p\right| \le C\left(\fint_{B_R(z)\cap\Omega} |\nabla u|^2\right)^{1/2}.$$

On the other hand, by the pressure estimate for the Stokes system (an analogue of Theorem A with linear dependence on M),

$$\left|\fint_{B_{4r}(\hat{x})\cap\Omega} p - \fint_{B_{R/4}(\hat{x})\cap\Omega} p\right| \le C\left(\fint_{B_R(z)\cap\Omega} |\nabla u|^2\right)^{1/2}.$$

Similar to (B-11), because $B_{R/4}(\hat{x}) \subset B_{3R/4}(z)$, we obtain

$$\left|\fint_{B_{R/4}(\hat{x})\cap\Omega} p - \fint_{B_{3R/4}(z)\cap\Omega} p\right| \le C\left(\fint_{B_{2R}(z)\cap\Omega} |\nabla u|^2\right)^{1/2}.$$

Finally, combining the above estimates, we arrive at

$$\left| p(x) - \fint_{B_{3R/4}(z)\cap\Omega} p \right| \le C\left(\fint_{B_{2R}(z)\cap\Omega} |\nabla u|^2\right)^{1/2}. \tag{B-12}$$

Since $x \in \Omega_{>2} \cap B_{R/2}(z)$ is arbitrary, this implies the estimate (B-9) with $2R$ in the right-hand side instead of R. Then a covering argument using (B-8) yields the desired estimate (B-9). □

Remark B.2. The interior oscillation estimate holds also for $-\Delta u + \nabla p = f$ and $\nabla \cdot u = 0$ in B_R provided $f \in L^q(B_R)^3$ for some $q > 3$. Precisely, by classical Schauder theory,

$$\operatorname*{osc}_{B_{R/2}} p \le C\left(\fint_{B_R} |\nabla u|^2\right)^{1/2} + CR\left(\fint_{B_R} |f|^q\right)^{1/q},$$

where C is a universal constant.

Now, we are ready to construct $\Pi(x, y)$ and prove properties (i)–(iii) of (G, Π). For a given $f \in C_0^\infty(B_R(0)\cap\Omega)^3$ with $R > 32$, we consider the Stokes equations (B-4) with u given by (B-5) and the associated pressure $p \in L^2_{\mathrm{loc}}(\overline{\Omega})$. For $x \in \Omega_{>2}$ such that $|x| \ge 4R$, we set $r = \frac{1}{2}|x| \ge 2R$. Since f is supported in $B_R(0)\cap\Omega$, we have $-\Delta u + \nabla p = 0$ in $B_r(x)\cap\Omega$. Moreover, by an energy estimate using (B-6), we have

$$\left(\fint_{B_r(x)\cap\Omega} |\nabla u|^2\right)^{1/2} \le \frac{CR}{r^{3/2}} \|f\|_{L^2(B_R(0)\cap\Omega)}. \tag{B-13}$$

Therefore, Lemma B.1 and a covering argument imply the oscillation estimate of p, namely,

$$\operatorname*{osc}_{\Omega_{>2}\cap B_{2r}(0)\setminus \bar{B}_r(0)} p \le \frac{CR}{r^{3/2}} \|f\|_{L^2(B_R(0)\cap\Omega)}.$$

This further implies

$$\operatorname*{osc}_{\Omega_{>2}\setminus \overline{B_r(0)}} p \le \sum_{k=1}^{\infty} \operatorname*{osc}_{\Omega_{>2}\cap B_{2^k r}(0)\setminus \overline{B_{2^{k-1}r}(0)}} p \le \sum_{k=1}^{\infty} \frac{CR}{(2^{k-1}r)^{3/2}} \|f\|_{L^2(B_R(0)\cap\Omega)} \le \frac{CR}{r^{3/2}} \|f\|_{L^2(B_R(0)\cap\Omega)}. \tag{B-14}$$

This shows that $p(x)$ converges to a constant as $x \to \infty$. By the assumption that $p(x) \to 0$ as $x_3 \to \infty$, we know the limiting constant is zero. Hence, in view of (B-14), we derive

$$|p(x)| \le \frac{CR}{|x|^{3/2}} \|f\|_{L^2(B_R(0)\cap\Omega)} \tag{B-15}$$

for all $x \in \Omega_{>2}$ satisfying $|x| \ge 4R$. Moreover, by arguing in a similar manner as in Step 3 in the proof of Theorem 4.1 and using (B-15) instead of (4-15), we find that for sufficiently large $R' \ge R$,

$$\|p\|_{L^2(B_{R'}(0)\cap\Omega)} \le C(R')\|f\|_{L^2(B_R(0)\cap\Omega)}, \tag{B-16}$$

with a constant $C(R')$ depending on R'.

On the other hand, for x with either $x \in \Omega_{\le 2}$ or $|x| \ge 4R$, we can connect x to another point $\tilde{x} \in \Omega_{>2}$ with $|\tilde{x}| \ge 4R$ by a chain of a finite number of cubes $\{B_{r_i}(z_i) \mid i = 1, 2, \dots, N\}$ such that $B_{2r_i}(z_i) \subset \Omega$.

Using Remark B.2 on each $B_{r_i}(z_i)$, as well as (B-15) applied to $\tilde{x}$, we see that for any x, R

$$|p(x)| \le C_q(x, R)\|f\|_{L^q(B_R(0)\cap\Omega)} \tag{B-17}$$

provided $q > 3$, where $C_q(x, R)$ is a constant depending only on q, x, and R.

From (B-15) and (B-17), for each fixed $x \in \Omega$, the map $f \mapsto p(x)$ is a bounded linear functional on $L^q(B_R(0)\cap\Omega)^3$. By the Riesz representation theorem, there is a unique function $\Pi(x,\cdot) \in L^{q'}(B_R(0)\cap\Omega)^3$ with $q' \in \left[1, \frac{3}{2}\right)$, so that

$$p(x) = \int_{B_R(0)\cap\Omega} \Pi(x, y)\cdot f(y)\,\mathrm{d}y.$$

Note that the above $\Pi(x,\cdot)$ is only defined in $B_R(0)\cap\Omega$ for a fixed x. As x and R vary, we can obtain a family of such functions, which can be glued together by the uniqueness of p. Thus we have constructed a function $\Pi(x, y)$ defined in the entire $\Omega\times\Omega$ satisfying $\Pi(x,\cdot) \in L^{q'}_{\mathrm{loc}}(\bar{\Omega})^3$. To investigate the local integrability of $\Pi(\cdot,\cdot)$, let us fix $R > 1$ and define a functional $S(f, g)$ for smooth f, g supported in $B_R(0)\cap\Omega$ by

$$S(f, g) = \int_{B_R(0)\cap\Omega} p(x)g(x)\,\mathrm{d}x = \int_{B_R(0)\cap\Omega}\int_{B_R(0)\cap\Omega} (\Pi(x, y)\cdot f(y))g(x)\,\mathrm{d}y\,\mathrm{d}x.$$

From (B-16), by taking a sufficiently large $R' \ge R$, we see that

$$|S(f, g)| \le \|p\|_{L^2(B_R(0)\cap\Omega)}\|g\|_{L^2(B_R(0)\cap\Omega)} \le C(R')\|f\|_{L^2(B_R(0)\cap\Omega)}\|g\|_{L^2(B_R(0)\cap\Omega)}.$$

Hence S is a bounded functional on $L^2(B_R(0)\cap\Omega)^3 \times L^2(B_R(0)\cap\Omega)$, which implies that

$$\int_{B_R(0)\cap\Omega} \Pi(x,\cdot)g(x)\,\mathrm{d}x \quad \text{is in} \quad L^2(B_R(0)\cap\Omega)^3. \tag{B-18}$$

Now we can prove that (G, Π) satisfies properties (i)–(iii). Property (iii) is obvious from the arguments so far. Property (ii) follows from property (iii) combined with the Lebesgue differentiation theorem. Here we use the fact that, for all $\phi \in C_0^\infty(\Omega)^3$, the function of y

$$\int_\Omega \Pi(x, y)(\nabla\cdot\phi)(x)\,\mathrm{d}x$$

belongs to $L^2_{\mathrm{loc}}(\bar{\Omega})^3$ because of (B-18). The integrability of $\Pi(\cdot, y)$ in property (i) follows from the weak form (B-3). Consequently, we have constructed the Green's function (G, Π) meeting properties (i)-(iii).

We should point out that in the above argument for existence, the estimate, for example of $\Pi(x,\cdot)$, is very rough, especially when x is close to the boundary $\partial\Omega$. This is because the large-scale regularity of $\Pi(x,\cdot)$ is not taken into consideration. In the following, we obtain some more careful estimates of (G, Π) by studying (B-3).

B2. ***Large-scale estimates of the velocity component.*** For convenience, let $G(x, y)$ and $\Pi(x, y)$ be zero-extended for both x and y. Recall the symmetry $G(x, y) = G^t(y, x)$, where G^t is the transpose of G. Thus by definition, $G(x, y) = 0$ if either $x \in \partial\Omega$ or $y \in \partial\Omega$ and $x \ne y$. Denote by $\delta(x)$ the distance from x to $\partial\Omega$.

Notice that $\nabla_x G$ denotes the derivative of G with respect to the first variable, i.e.,

$$(\nabla_x G)(x, y) = (\nabla G(\cdot\,, y))(x) \quad \text{for all } (x, y) \in \Omega.$$

Similarly, $\nabla_y G$ denotes the derivative of G with respect to the second variable. The following estimates for the derivatives of G are crucial.

Proposition B.3. *Let $L \in (0, \infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. The velocity component $G(x, y)$ satisfies*:

(i) *For $x_3 > 2$ and $y_3 > 2$,*

$$|\nabla_x G(x, y)| \le C \min\left\{\frac{1}{|x-y|^2}, \frac{\delta(y)}{|x-y|^3}\right\}, \tag{B-19}$$

$$|\nabla_y G(x, y)| \le C \min\left\{\frac{1}{|x-y|^2}, \frac{\delta(x)}{|x-y|^3}\right\}. \tag{B-20}$$

(ii) *For $x_3 > 2$ and $y_3 < 2$ with $|x - y| > 32$,*

$$\left(\int_{B_1(y)} |\nabla_y G(x, z)|^2 \,\mathrm{d}z\right)^{1/2} \le C \min\left\{\frac{1}{|x-y|^2}, \frac{\delta(x)}{|x-y|^3}\right\}, \tag{B-21}$$

$$\left(\int_{B_1(y)} |\nabla_x G(x, z)|^2 \,\mathrm{d}z\right)^{1/2} \le \frac{C}{|x-y|^3}. \tag{B-22}$$

(iii) *For $x_3 < 2$ and $y_3 > 2$ with $|x - y| > 32$,*

$$\left(\int_{B_1(x)} |\nabla_x G(z, y)|^2 \,\mathrm{d}z\right)^{1/2} \le C \min\left\{\frac{1}{|x-y|^2}, \frac{\delta(y)}{|x-y|^3}\right\}, \tag{B-23}$$

$$\left(\int_{B_1(x)} |\nabla_y G(z, y)|^2 \,\mathrm{d}z\right)^{1/2} \le \frac{C}{|x-y|^3}. \tag{B-24}$$

Here C depends on L.

Notice that G and Π are zero-extended outside Ω. Therefore, the integrals above make sense even in the case when $B_1(x)$ or $B_1(y)$ intersect Ω^{c}. For the estimates concerned with the oscillation of the pressure, on the contrary, we make precise when the balls intersect the boundary; see for instance Lemma B.1.

Proof of Proposition B.3. Note that (ii) and (iii) are symmetric. While (i) is the interior estimate whose proof is similar to (ii) and (iii). Hence, we will only prove (ii). Since we are working on cubes, it is more convenient to define $R = |x-y|_\infty := \max_{1\le i\le 3} |x_i - y_i|$, which is comparable to the usual distance $|x-y|$. Recall that $(G(x, \cdot), \Pi(x, \cdot))$ is a weak solution of Stoke system in $\Omega \setminus \{x\}$. To show (B-21), we begin with the interior and boundary Lipschitz estimates for $G(x, \cdot)$,

$$\left(\fint_{B_1(y)} |\nabla_y G(x, z)|^2 \,\mathrm{d}z\right)^{1/2} \le C\left(\fint_{B_3(\hat{y})} |\nabla_y G(x, z)|^2 \,\mathrm{d}z\right)^{1/2} \le C\left(\fint_{B_{R/2}(\hat{y})} |\nabla_y G(x, z)|^2 \,\mathrm{d}z\right)^{1/2}, \tag{B-25}$$

where $\hat{y}$ is the projection of y on $\{y_3 = 0\}$.

To proceed, let $F \in L^2(B_{R/2}(\hat{y}) \cap \Omega)^{3\times 3}$ (zero-extended to the whole of Ω). Let (u, p) be the weak solution of

$$\begin{cases} -\Delta u + \nabla p = \nabla \cdot F & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{B-26}$$

Recall from (B-5) that

$$u(x) = \int_\Omega G(x, y)\nabla \cdot F(y)\, \mathrm{d}y = -\int_\Omega \nabla_y G(x, y) F(y)\, \mathrm{d}y. \tag{B-27}$$

The energy estimate implies

$$\int_\Omega |\nabla u|^2 \leq C \int_\Omega |F|^2. \tag{B-28}$$

Next, we estimate $|u(x)|$ and $|\nabla u(x)|$. Let $r = x_3$, which is comparable to $\delta(x)$ since $x_3 > 2$. We consider two cases: $r < \frac{1}{10}R$ or $r > \frac{1}{10}R$. If $r > \frac{1}{10}R$, since F is supported in $B_{R/2}(\hat{y}) \cap \Omega$ which does not intersect with $B_{R/10}(x)$, we can apply the interior Lipschitz estimate to u and (B-28)

$$|\nabla u(x)| \leq C\biggl(\fint_{B_{R/10}(x)} |\nabla u|^2\biggr)^{1/2} \leq C R^{-3/2}\biggl(\int_\Omega |F|^2\biggr)^{1/2}. \tag{B-29}$$

On the other hand, we apply the interior estimate, Sobolev embedding and (B-28) to obtain

$$|u(x)| \leq C\biggl(\fint_{B_{R/10}(x)} |u|^6\biggr)^{1/6} \leq C R^{-1/2}\biggl(\int_\Omega |\nabla u|^2\biggr)^{1/2} \leq C R^{-1/2}\biggl(\int_\Omega |F|^2\biggr)^{1/2}. \tag{B-30}$$

If $r < \frac{1}{10}R$, by the interior and boundary Lipschitz estimate

$$\begin{aligned} |\nabla u(x)| &\leq C\biggl(\fint_{B_r(x)} |\nabla u|^2\biggr)^{1/2} \leq C\biggl(\fint_{B_{2r}(\hat{x})} |\nabla u|^2\biggr)^{1/2} \\ &\leq C\biggl(\fint_{B_{R/5}(\hat{x})} |\nabla u|^2\biggr)^{1/2} \leq C R^{-3/2}\biggl(\int_\Omega |F|^2\biggr)^{1/2}. \end{aligned} \tag{B-31}$$

Moreover, using the Poincaré inequality and the boundary Lipschitz estimate, we have

$$\begin{aligned} |u(x)| &\leq C\biggl(\fint_{B_r(x)} |u|^2\biggr)^{1/2} \leq C\biggl(\fint_{B_{2r}(\hat{x})} |u|^2\biggr)^{1/2} \\ &\leq Cr\biggl(\fint_{B_{2r}(\hat{x})} |\nabla u|^2\biggr)^{1/2} \leq Cr\biggl(\fint_{B_{R/5}(\hat{x})} |\nabla u|^2\biggr)^{1/2} \leq C r R^{-3/2}\biggl(\int_\Omega |F|^2\biggr)^{1/2}. \end{aligned} \tag{B-32}$$

From the estimates (B-29) - (B-32), (B-27) and duality, we see that

$$\biggl(\fint_{B_{R/2}(\hat{y})\cap\Omega} |\nabla_x \nabla_y G(x, z)|^2\, \mathrm{d}z\biggr)^{1/2} \leq \frac{C}{R^3}, \tag{B-33}$$

$$\biggl(\fint_{B_{R/2}(\hat{y})\cap\Omega} |\nabla_y G(x, z)|^2\, \mathrm{d}z\biggr)^{1/2} \leq \frac{Cr}{R^3}. \tag{B-34}$$

Note that (B-25) and (B-34) combined lead to (B-21). To see (B-22), notice that $(\nabla_x G(x,y), \nabla_x \Pi(x,y))$ is a weak solution in $y \in \Omega \setminus \{x\}$. Thus, we may apply (B-33), Poincaré inequality and boundary Lipschitz estimate to obtain

$$\begin{aligned}\left(\fint_{B_1(y)} |\nabla_x G(x,z)|^2\,dz\right)^{1/2} &\le C\left(\fint_{B_3(\hat{y})} |\nabla_x G(x,z)|^2\,dz\right)^{1/2} \le C\left(\fint_{B_3(\hat{y})} |\nabla_y \nabla_x G(x,z)|^2\,dz\right)^{1/2} \\ &\le C\left(\fint_{B_{R/2}(\hat{y})} |\nabla_y \nabla_x G(x,z)|^2\,dz\right)^{1/2} \le \frac{C}{R^3}.\end{aligned}$$

The proof of (ii) thus is complete. □

Analogously, we can also show the estimates for G itself. The proof is left to the reader.

Proposition B.4. *Let $L \in (0,\infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. The velocity component $G(x,y)$ satisfies*:

(i) *For $x_3 > 2$ and $y_3 > 2$,*

$$|G(x,y)| \le C \min\left\{\frac{1}{|x-y|}, \frac{\delta(x)}{|x-y|^2}, \frac{\delta(y)}{|x-y|^2}, \frac{\delta(x)\delta(y)}{|x-y|^3}\right\}. \tag{B-35}$$

(ii) *For $x_3 > 2$ and $|x-y| > 32$,*

$$\left(\int_{B_1(y)} |G(x,z)|^2\,dz\right)^{1/2} \le C \min\left\{\frac{1}{|x-y|}, \frac{\delta(x)}{|x-y|^2}, \frac{\delta(y)+1}{|x-y|^2}, \frac{\delta(x)(\delta(y)+1)}{|x-y|^3}\right\}. \tag{B-36}$$

(iii) *For $y_3 > 2$ and $|x-y| > 32$,*

$$\left(\int_{B_1(x)} |G(z,y)|^2\,dz\right)^{1/2} \le C \min\left\{\frac{1}{|x-y|}, \frac{\delta(y)}{|x-y|^2}, \frac{\delta(x)+1}{|x-y|^2}, \frac{\delta(y)(\delta(x)+1)}{|x-y|^3}\right\}. \tag{B-37}$$

Here C depends on L.

B3. ***Large-scale estimates of the pressure component.*** The estimates of Π are stated as follows.

Proposition B.5. *Let $L \in (0,\infty)$ and Ω be a bumpy John domain with constant L according to Definition 1.2. The pressure component $\Pi(x,y)$ satisfies*:

(i) *For $x_3 > 2$ and $y_3 > 2$,*

$$|\Pi(x,y)| \le C \min\left\{\frac{1}{|x-y|^2}, \frac{\delta(y)}{|x-y|^3}\right\}. \tag{B-38}$$

(ii) *For $x_3 < 2$ and $y_3 > 2$ with $|x-y| > 32$,*

$$\left(\int_{B_1(x)} |\Pi(z,y)|^2\,dz\right)^{1/2} \le C \min\left\{\frac{1}{|x-y|^2}, \frac{\delta(y)}{|x-y|^3}\right\}. \tag{B-39}$$

(iii) *For $x_3 > 2$ and $y_3 < 2$ with $|x-y| > 32$,*

$$\left(\int_{B_1(y)} |\Pi(x,z)|^2\,dz\right)^{1/2} \le \frac{C}{|x-y|^3}. \tag{B-40}$$

Here C depends on L.

Proof. We will carry out a delicate oscillation estimate of the pressure originating from [Gu and Zhuge 2019]. We first consider the estimate (i), i.e., $x_3 > 2$ and $y_3 > 2$. Consider a point $w \in \Omega$ with $w \neq y$. Let $t = |w - y|_\infty$. We claim

$$\operatorname*{osc}_{B_{t/4}(w)\cap\Omega_{>2}} \Pi(\,\cdot\,, y) \leq C \min\left\{\frac{1}{t^2}, \frac{\delta(y)}{t^3}\right\}, \tag{B-41}$$

with C independent of t, w, and y. The operator osc is defined in (B-7).

We prove the above claim by considering different situations. If $w \in B_{y_3/2}(y)$, then $t < \frac{1}{2}y_3$ and $B_t(w) \subset \Omega$. By the interior pressure estimate (B-8) in Lemma B.1 and (B-19),

$$\operatorname*{osc}_{B_{t/4}(w)} \Pi(\,\cdot\,, y) \leq C\left(\fint_{B_{t/2}(w)} |\nabla_x G(z, y)|^2 \, dz\right)^{1/2} \leq C \min\left\{\frac{1}{t^2}, \frac{\delta(y)}{t^3}\right\}.$$

Next, if $w \notin B_{y_3/2}(y)$, we consider two subcases: (a) $|w_3| < \frac{1}{4}t$; (b) $|w_3| \geq \frac{1}{4}t$. Without loss of generality, we assume $t > \frac{1}{2}y_3 > 20$.

For the case (a), let $\hat{w}$ be the projection of w on $\partial\mathbb{R}^3_+$. Using the interior and boundary pressure estimates in John domains from Lemma B.1 combined with a covering argument,

$$\operatorname*{osc}_{B_{t/4}(w)\cap\Omega_{>2}} \Pi(\,\cdot\,, y) \leq C\left(\fint_{B_{t/2}(\hat{w})} |\nabla_x G(z, y)|^2 \, dz\right)^{1/2} \leq C \min\left\{\frac{1}{t^2}, \frac{\delta(y)}{t^3}\right\}, \tag{B-42}$$

where we have also used (B-19) and (B-23) in the second inequality.

Now, for the case (b), $B_{t/4}(w)$ may be decomposed as a union of a finite number of cubes $B_{t/16}(w_i)$, with $i = 1, 2, \ldots, K_0$, where K_0 is an absolute constant, so that $B_{t/8}(w_i)$ is contained in $\Omega_{>2}$. Thus,

$$\operatorname*{osc}_{B_{t/4}(w)} \Pi(\,\cdot\,, y) \leq \sum_{i=1}^{K_0} \operatorname*{osc}_{B_{t/16}(w_i)} \Pi(\,\cdot\,, y) \leq C \sum_{i=1}^{K_0} \left(\fint_{B_{t/8}(w_i)} |\nabla_x G(z, y)|^2 \, dz\right)^{1/2} \leq C \min\left\{\frac{1}{t^2}, \frac{\delta(y)}{t^3}\right\},$$

where we have used (B-19). Thus, the claim (B-41) is proved.

Now, by a covering argument, it is not difficult to see from (B-41) that, for any $r > 0$,

$$\operatorname*{osc}_{\Omega_{>2}\cap B_{2r}(y)\setminus\overline{B_r(y)}} \Pi(\,\cdot\,, y) = \operatorname*{osc}_{(B_{2r}(y)\cap\Omega_{>2})\setminus\overline{(B_r(y)\cap\Omega_{>2})}} \Pi(\,\cdot\,, y) \leq C \min\left\{\frac{1}{r^2}, \frac{\delta(y)}{r^3}\right\}.$$

Consequently,

$$\operatorname*{osc}_{\Omega_{>2}\setminus\overline{B_r(y)}} \Pi(\,\cdot\,, y) \leq \sum_{k=1}^{\infty} \operatorname*{osc}_{\Omega_{>2}\cap B_{2^k r}(y)\setminus\overline{B_{2^{k-1}r}(y)}} \leq C \min\left\{\frac{1}{r^2}, \frac{\delta(y)}{r^3}\right\}. \tag{B-43}$$

This means that for each y with $y_3 > 2$, there exists a function $\widehat{\Pi}(y)$ such that

$$\lim_{|x|\to\infty,\, x_3>2} \Pi(x, y) = \widehat{\Pi}(y).$$

This convergence is uniform on any compact set in $\{y_3 > 2\}$. We show that $\widehat{\Pi}(y) \equiv 0$. In fact, if $f \in C_0^\infty(\Omega)^3$, the pressure of the Stokes equations with the source f is given by

$$p(x) = \int_\Omega \Pi(x, y) \cdot f(y) \, dy.$$

By the definition of the Green's function, $p(x) \to 0$ holds as $|x_3| \to \infty$. It follows that

$$\int_{\Omega} \widehat{\Pi}(y) \cdot f(y) \, \mathrm{d}y = 0.$$

This holds for any $f \in C_0^\infty(\Omega_0)^3$, where Ω_0 is a bounded open set whose closure is contained in $\{y_3 > 2\}$. Thus we have $\widehat{\Pi}(y) \equiv 0$. Therefore, (B-43) implies (B-38) since r is arbitrary.

Next, we prove (ii). Let $x_3 < 2$, $y_3 > 2$, and $r := |x - y|_\infty$. Without loss of generality, it suffices to assume $r > 32$. For such $x = (x_1, x_2, x_3)$, we pick $\tilde{x} = (x_1, x_2, 3)$. Because $-1 < x_3 < 2$ and $|x - \tilde{x}|_\infty < 4$, we have $r - 4 \le |\tilde{x} - y|_\infty \le r + 4$ and hence by (i),

$$|\Pi(\tilde{x}, y)| \le C \min\left\{ \frac{1}{(r-4)^2}, \frac{\delta(y)}{(r-4)^3} \right\} \le C \min\left\{ \frac{1}{r^2}, \frac{\delta(y)}{r^3} \right\}. \tag{B-44}$$

Next, we consider

$$\left| \Pi(\tilde{x}, y) - \fint_{\Omega_3(\hat{x})} \Pi(\,\cdot\,, y) \right|,$$

where $\hat{x} = (x_1, x_2, 0)$ is the projection and $\Omega_3(\hat{x})$ is the John domain between $\Omega \cap B_3(\hat{x})$ and $\Omega \cap B_6(\hat{x})$ given by Definition 1.2. Following the argument in the proof of Lemma B.1, we can show

$$\left| \Pi(\tilde{x}, y) - \fint_{\Omega_3(\hat{x})} \Pi(\,\cdot\,, y) \right| \le C \left(\fint_{\Omega_{10}(\hat{x})} \left| \nabla_x G(z, y) \right|^2 \mathrm{d}z \right)^{1/2} \le C \min\left\{ \frac{1}{r^2}, \frac{\delta(y)}{r^3} \right\},$$

where we have used (B-23) as well as (B-19) combined with a covering argument and the fact that $\operatorname{dist}(\Omega_{10}(\hat{x}), y) \approx r$ in the last inequality. On the other hand, observe that $\Omega \cap B_1(x) \subset \Omega_3(\hat{x})$. Hence, by the Bogovskii lemma in $\Omega_3(\hat{x})$ and (B-23) with a covering argument,

$$\begin{aligned} \left(\fint_{B_1(x)} |\Pi(z, y) - \fint_{\Omega_3(\hat{x})} \Pi(\,\cdot\,, y)|^2 \, \mathrm{d}z \right)^{1/2} &\le C \left(\fint_{\Omega_3(\hat{x})} |\Pi(z, y) - \fint_{\Omega_3(\hat{x})} \Pi(\,\cdot\,, y)|^2 \, \mathrm{d}z \right)^{1/2} \\ &\le C \left(\fint_{\Omega_3(\hat{x})} |\nabla_x G(z, y)|^2 \, \mathrm{d}z \right)^{1/2} \le C \min\left\{ \frac{1}{r^2}, \frac{\delta(y)}{r^3} \right\}. \end{aligned} \tag{B-45}$$

Combining the estimates above, we obtain

$$\left(\fint_{B_1(x)} |\Pi(z, y)|^2 \, \mathrm{d}z \right)^{1/2} \le C \min\left\{ \frac{1}{r^2}, \frac{\delta(y)}{r^3} \right\}.$$

This proves (B-39).

Next, we use a duality method to prove (iii). Let $f \in C_0^\infty(B_1(y) \cap \Omega)^3$, zero-extended to Ω, and consider

$$\begin{cases} -\Delta u + \nabla p = f \chi_{B_1(y)} & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{B-46}$$

By definition, the solution (u, p) with finite energy can be represented by (B-5). Since we already know the estimate of $\nabla_x G$ (namely, (B-22)), we have

$$|\nabla u(x)| \le \frac{C}{|x - y|^3} \|f\|_{L^2(B_1(y))}$$

for $|x-y|_\infty > 4$ with $x_3 > 2$. By a familiar oscillation argument, we obtain

$$|p(x)| = \left| \int_\Omega \Pi(x,y)\cdot f(y)\,\mathrm{d}y \right| \le \frac{C}{|x-y|^3} \|f\|_{L^2(B_1(y))}. \tag{B-47}$$

This implies (B-40). □

Appendix C: Proof of the iteration lemma

Proof of Lemma 3.10. The proof is a variation of the one in [Zhuge 2021]. For fixed $r \in \left(\varepsilon, \frac{1}{16}\right)$, the assumption (3-37f) implies

$$\begin{aligned}\int_r^{1/8} \frac{h(t)}{t}\,\mathrm{d}t &\le \int_r^{1/8} \frac{h(2t)}{t}\,\mathrm{d}t + C_0 \int_r^{1/8} \frac{H(2t)}{t}\,\mathrm{d}t \\ &\le \int_{2r}^{1/4} \frac{h(t)}{t}\,\mathrm{d}t + C_0 \int_{2r}^{1/4} \frac{H(t)}{t}\,\mathrm{d}t,\end{aligned}$$

which, combined with (3-37b), (3-37d) and (3-37c), gives

$$\int_r^{2r} \frac{h(t)}{t}\,\mathrm{d}t \le \int_{1/8}^{1/4} \frac{h(t)}{t}\,\mathrm{d}t + C_0 \int_{2r}^{1/4} \frac{H(t)}{t}\,\mathrm{d}t \le C\left(\Phi\left(\tfrac{1}{2}\right) + B_0\right) + C_0 \int_r^{1/2} \frac{H(t)}{t}\,\mathrm{d}t.$$

Then from (3-37f) we have

$$\int_r^{2r} \frac{h(t)}{t}\,\mathrm{d}t \ge \int_r^{2r} \frac{h(r) - C_0 H(2t)}{t}\,\mathrm{d}t \ge \frac{h(r)}{4} - C_0 \int_r^{1/2} \frac{H(t)}{t}\,\mathrm{d}t.$$

Therefore for $r \in \left(\varepsilon, \frac{1}{16}\right)$, we find

$$h(r) \le C\left(\Phi\left(\tfrac{1}{2}\right) + B_0\right) + C \int_r^{1/2} \frac{H(t)}{t}\,\mathrm{d}t. \tag{C-1}$$

Let $\delta \in \left(0, \min\left\{\frac{\theta}{4}, \frac{1}{(16)^2}\right\}\right)$ be a small number to be determined later and let us set $\varepsilon_* = \delta^2$. We temporarily assume that $\varepsilon \in (0, \theta\varepsilon_*)$ in the following proof. From (3-37a) we have

$$\begin{aligned}\int_{\varepsilon/\delta}^{\delta} \frac{H(\theta t)}{t}\,\mathrm{d}t &\le \frac{1}{2}\int_{\varepsilon/\delta}^{\delta} \frac{H(2t)}{t}\,\mathrm{d}t + C_0\left(\int_{\varepsilon/\delta}^{\delta} \left(\frac{\varepsilon}{t}\right)^\alpha \frac{\Phi(16t)}{t}\,\mathrm{d}t + B_0 \int_{\varepsilon/\delta}^{\delta} t^{\beta-1}\,\mathrm{d}t\right) \\ &\le \frac{1}{2}\int_{\varepsilon/\delta}^{1/2} \frac{H(t)}{t}\,\mathrm{d}t + C_0\left(\int_{\varepsilon/\delta}^{\delta} \left(\frac{\varepsilon}{t}\right)^\alpha \frac{\Phi(16t)}{t}\,\mathrm{d}t + \beta^{-1}B_0\right).\end{aligned}$$

From (3-37e) and the estimate (C-1) for $h(r)$, we have

$$\begin{aligned}\int_{\varepsilon/\delta}^{\delta} \left(\frac{\varepsilon}{t}\right)^\alpha \frac{\Phi(16t)}{t}\,\mathrm{d}t &\le C_0 \int_{\varepsilon/\delta}^{\delta} \left(\frac{\varepsilon}{t}\right)^\alpha \frac{H(16t)+h(16t)}{t}\,\mathrm{d}t \\ &\le C_0\delta^\alpha \int_{16\varepsilon/\delta}^{16\delta} \frac{H(t)}{t}\,\mathrm{d}t + C\left(\int_{\varepsilon/\delta}^{\delta} \left(\frac{\varepsilon}{t}\right)^\alpha \frac{\mathrm{d}t}{t}\right)\left(\left(\Phi\left(\frac{1}{2}\right) + B_0\right) + \int_{16\varepsilon/\delta}^{1/2} \frac{H(t)}{t}\,\mathrm{d}t\right) \\ &\le (C_0 + C_1\alpha^{-1})\delta^\alpha \int_{\varepsilon/\delta}^{1/2} \frac{H(t)}{t}\,\mathrm{d}t + C_1\alpha^{-1}\delta^\alpha\left(\Phi\left(\frac{1}{2}\right) + B_0\right).\end{aligned}$$

Now let us choose δ sufficiently small depending on α, C_0 and C_1 so that

$$\frac{1}{2}+C_0(C_0+C_1\alpha^{-1})\delta^{\alpha}\le\frac{3}{4}.$$

Then we obtain

$$\int_{\theta\varepsilon/\delta}^{\theta\delta}\frac{H(t)}{t}\,\mathrm{d}t\le\frac{3}{4}\int_{\varepsilon/\delta}^{1/2}\frac{H(t)}{t}\,\mathrm{d}t+C\Big(\Phi\Big(\frac{1}{2}\Big)+B_0\Big),$$

and consequently, from $\varepsilon/\delta<\theta\delta$,

$$\int_{\theta\varepsilon/\delta}^{\theta\delta}\frac{H(t)}{t}\,\mathrm{d}t\le 3\int_{\theta\delta}^{1/2}\frac{H(t)}{t}\,\mathrm{d}t+C\Big(\Phi\Big(\frac{1}{2}\Big)+B_0\Big).$$

Therefore from (3-37b) and (3-37c) we have

$$\int_{\theta\varepsilon/\delta}^{1/2}\frac{H(t)}{t}\,\mathrm{d}t\le 4\int_{\theta\delta}^{1/2}\frac{H(t)}{t}\,\mathrm{d}t+C\Big(\Phi\Big(\frac{1}{2}\Big)+B_0\Big)\le C\Big(\Phi\Big(\frac{1}{2}\Big)+B_0\Big),\tag{C-2}$$

where we have used

$$\sup_{\theta\delta\le r\le 1/2}\Phi(r)\le C\Big(\Phi\Big(\frac{1}{2}\Big)+B_0\Big),\tag{C-3}$$

with some constant C independent of ε, which is proved by applying (3-37c) finitely many times. Hence, from $4\varepsilon<\theta\varepsilon/\delta$, the estimates (C-1) and (C-2) lead to, for $r\in\big(\theta\varepsilon/\delta,\frac{1}{16}\big)$,

$$h(r)\le C\Big(\Phi\Big(\frac{1}{2}\Big)+B_0\Big)+C\int_{\theta\varepsilon/\delta}^{1/2}\frac{H(t)}{t}\,\mathrm{d}t\le C\Big(\Phi\Big(\frac{1}{2}\Big)+B_0\Big).\tag{C-4}$$

For $r\in\big(\theta\varepsilon/\delta,\frac{1}{32}\big)$, from (3-37e), (C-2) and (C-4), we see that

$$\int_r^{2r}\frac{\Phi(t)}{t}\,\mathrm{d}t\le C_0\int_r^{2r}\frac{H(t)}{t}\,\mathrm{d}t+C_0\int_r^{2r}\frac{h(t)}{t}\,\mathrm{d}t\le C\Big(\Phi\Big(\frac{1}{2}\Big)+B_0\Big).$$

From this, using the following inequality valid for all fixed $r\in\big(2\varepsilon,\frac{1}{2}\big)$

$$\Phi(r)\le C(\Phi(t)+B_0t^{\beta}),\qquad t\in[r,2r],$$

which is a consequence of (3-37c), we find

$$\sup_{\theta\varepsilon/\delta\le r\le 1/32}\Phi(r)\le C\Big(\Phi\Big(\frac{1}{2}\Big)+B_0\Big).$$

Using repeatedly (3-37c) finitely many times, we have

$$\sup_{\varepsilon\le r\le 1/32}\Phi(r)\le C\Big(\Phi\Big(\frac{1}{2}\Big)+B_0\Big),\tag{C-5}$$

with a constant C independent of ε. On the other hand, (3-37b) and (C-5) imply

$$\int_{\varepsilon}^{\theta\varepsilon/\delta}\frac{H(t)}{t}\,\mathrm{d}t\le C_0\int_{\varepsilon}^{\theta\varepsilon/\delta}\frac{\Phi(t)}{t}\,\mathrm{d}t\le C\Big(\Phi\Big(\frac{1}{2}\Big)+B_0\Big).\tag{C-6}$$

Combining (C-2), (C-3), (C-5) and (C-6), we obtain the assertion (3-38), provided $\varepsilon\in(0,\theta\varepsilon_*)$. Finally, if $\varepsilon\in\big(\theta\varepsilon_*,\frac{1}{48}\big)$, (3-38) is trivial by applying (3-37b) and (3-37c) finitely many times. □

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ON A FAMILY OF FULLY NONLINEAR INTEGRODIFFERENTIAL OPERATORS: FROM FRACTIONAL LAPLACIAN TO NONLOCAL MONGE–AMPÈRE

LUIS A. CAFFARELLI AND MARÍA SORIA-CARRO

We introduce a new family of intermediate operators between the fractional Laplacian and the nonlocal Monge–Ampère introduced by Caffarelli and Silvestre that are given by infimums of integrodifferential operators. Using rearrangement techniques, we obtain representation formulas and give a connection to optimal transport. Finally, we consider a global Poisson problem prescribing data at infinity, and prove existence, uniqueness, and $C^{1,1}$-regularity of solutions in the full space.

1. Introduction

Integro-differential equations arise in the study of stochastic processes with jumps, such as Lévy processes. A classical elliptic integrodifferential operator is the fractional Laplacian

$$\Delta^s u(x_0) = c_{n,s}\,\mathrm{PV}\int_{\mathbb{R}^n} (u(x_0+x)-u(x_0))\frac{1}{|x|^{n+2s}}\,dx, \qquad s \in (0,1),$$

which can be understood as an infinitesimal generator of a stable Lévy process. These types of processes are very well studied in probability, and their generators may be given by

$$L_K u(x_0) = \int_{\mathbb{R}^n} (u(x_0+x)-u(x_0)-x\cdot\nabla u(x_0))K(x)\,dx,$$

where the kernel K is a nonnegative function satisfying some integrability condition.

Recently, there has been significant interest in studying linear and nonlinear integrodifferential equations from the analytical point of view. In particular, extremal operators like

$$Fu(x_0) = \inf_{K\in\mathcal{K}} L_K u(x_0) \tag{1-1}$$

play a fundamental role in the regularity theory. See [Caffarelli and Silvestre 2009; 2011a; 2011b; Ros-Oton and Serra 2016]. The above equation is an example of a fully nonlinear equation that appears in optimal control problems and stochastic games [Krylov 1980; Nisio 1988]. The infimum in (1-1) is taken over a family of admissible kernels $\mathcal{K}$ that depends on the applications. In fact, nonlocal Monge–Ampère equations have been developed in the form (1-1) for some choice of $\mathcal{K}$ [Caffarelli and Charro 2015; Caffarelli and Silvestre 2016; Guillen and Schwab 2012].

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The Monge–Ampère equation arises in several problems in analysis and geometry, such as the mass transportation problem and the prescribed Gaussian curvature problem [De Philippis and Figalli 2014]. The classical equation prescribes the determinant of the Hessian of some convex function u:

$$\det(D^2u) = f.$$

In the literature, there are different nonlocal versions of the Monge–Ampère operator that Guillen and Schwab [2012], Caffarelli and Charro [2015], and Caffarelli and Silvestre [2016] have considered. Maldonado and Stinga [2017] have also given a nonlocal linearized Monge–Ampère equation. These definitions are motivated by the following property: if B is a positive definite symmetric matrix, then

$$n\det(B)^{1/n} = \inf_{A\in\mathcal{A}} \operatorname{tr}(A^TBA), \tag{1-2}$$

where

$$\mathcal{A} = \{A \in M_n : A > 0,\ \det(A) = 1\}$$

and M_n is the set of $n\times n$ matrices. If a convex function u is C^2 at a point x_0, then, by the previous identity with $B = D^2u(x_0)$, we may write the Monge–Ampère operator as a concave envelope of linear operators. It follows that

$$n\det(D^2u(x_0))^{1/n} = \inf_{A\in\mathcal{A}} \Delta[u\circ A](A^{-1}x_0).$$

Caffarelli and Charro [2015] study a fractional version of $\det(D^2u)^{1/n}$, replacing the Laplacian by the fractional Laplacian in the previous identity. More precisely,

$$\begin{aligned}\mathcal{D}^s u(x_0) &= \inf_{A\in\mathcal{A}} \Delta^s[u\circ A](A^{-1}x_0)\\ &= c_{n,s}\inf_{A\in\mathcal{A}} \mathrm{PV}\int_{\mathbb{R}^n} \frac{u(x_0+x)-u(x_0)}{|A^{-1}x|^{n+2s}}\,dx,\end{aligned}$$

where $s\in(0,1)$ and $c_{n,s}\approx 1-s$ as $s\to 1$; see also [Guillen and Schwab 2012]. A different approach based on geometric considerations was given by Caffarelli and Silvestre [2016]. In fact, the authors consider kernels whose level sets are volume preserving transformations of the fractional Laplacian kernel. Namely,

$$\mathrm{MA}^s u(x_0) = c_{n,s}\inf_{K\in\mathcal{K}_n^s}\int_{\mathbb{R}^n}(u(x_0+x)-u(x_0)-x\cdot\nabla u(x_0))K(x)\,dx,$$

where the infimum is taken over the family

$$\mathcal{K}_n^s = \{K:\mathbb{R}^n\to\mathbb{R}_+ : |\{x\in\mathbb{R}^n : K(x) > r^{-n-2s}\}| = |B_r| \text{ for all } r>0\}. \tag{1-3}$$

Notice that $|A^{-1}x|^{-n-2s}\in\mathcal{K}_n^s$ for any $A\in\mathcal{A}$. Therefore,

$$\mathrm{MA}^s u(x_0) \le \mathcal{D}^s u(x_0) \le \Delta^s u(x_0).$$

Moreover, both $\mathrm{MA}^s u$ and $\mathcal{D}^s u$ converge to $\det(D^2u)^{1/n}$, up to some constant, as $s\to 1$.

In this paper, we introduce a new family of operators of the form

$$\inf_{K\in\mathcal{K}_k^s}\int_{\mathbb{R}^n}(u(x_0+x)-u(x_0)-x\cdot\nabla u(x_0))K(x)\,dx \tag{1-4}$$

for any integer $1\le k<n$, which arises from imposing certain geometric conditions on the kernels. Moreover, we will see that

$$|y|^{-n-2s}\in\mathcal{K}_1^s\subset\mathcal{K}_k^s\subset\mathcal{K}_n^s\quad\text{for } 1<k<n,$$

and thus, this family will be monotone decreasing, and bounded from above by the fractional Laplacian and from below by the Caffarelli–Silvestre nonlocal Monge–Ampère.

The paper is organized as follows. In Section 2, we construct the family of admissible kernels $\mathcal{K}_k^s$ and give the precise definition of our operators for $C^{1,1}$-functions. We introduce in Section 3 the basic tools from the theory of rearrangements necessary for our goals. In Section 4, we study the infimum in (1-4) and obtain a representation formula, provided some condition on the level sets is satisfied (see Theorem 4.1). We also study the limit as $s\to 1$ and give a connection to optimal transport. The Hölder continuity of $\mathcal{F}_k^s u$ is proved in Section 5, following similar geometric techniques from [Caffarelli and Silvestre 2016]. In Section 6, we consider a global Poisson problem prescribing data at infinity, and introduce a new definition of our operators for functions that are merely continuous and convex. We show existence of solutions via Perron's method and $C^{1,1}$-regularity in the full space by constructing appropriate barriers. Finally, we discuss some future directions in Section 7.

2. Construction of kernels

Let us start with the construction of the family of admissible kernels. Notice that any kernel K in $\mathcal{K}_n^s$, defined in (1-3), will have the same distribution function as the kernel of the fractional Laplacian, since, for any $r>0$,

$$\{x\in\mathbb{R}^n:|x|^{-n-2s}>r^{-n-2s}\}=B_r.$$

Geometrically, this means that the level sets of K are deformations in *any* direction of $\mathbb{R}^n$ of the level sets of $|x|^{-n-2s}$, preserving the n-dimensional volume.

In view of this approach, a natural way of finding an intermediate family of operators between the nonlocal Monge–Ampère and the fractional Laplacian is to consider kernels whose level sets are deformations that preserve the k-dimensional Hausdorff measure $\mathcal{H}^k$, with $1\le k<n$, of the restrictions of balls in $\mathbb{R}^n$ to hyperplanes generated by $\{e_i\}_{i=1}^k$.

We define the set of admissible kernels as follows.

Definition 2.1. We say that $K\in\mathcal{K}_k^s$ if, for all $z\in\mathbb{R}^{n-k}$ and all $r>0$,

$$\mathcal{H}^k(\{y\in\mathbb{R}^k:K(y,z)>r^{-n-2s}\})=\begin{cases}\mathcal{H}^k(B_{(r^2-|z|^2)^{1/2}}) & \text{if } |z|<r,\\ 0 & \text{if } |z|\ge r,\end{cases} \tag{2-1}$$

where $B_{(r^2-|z|^2)^{1/2}}$ is the ball in $\mathbb{R}^k$ of radius $(r^2-|z|^2)^{1/2}$.

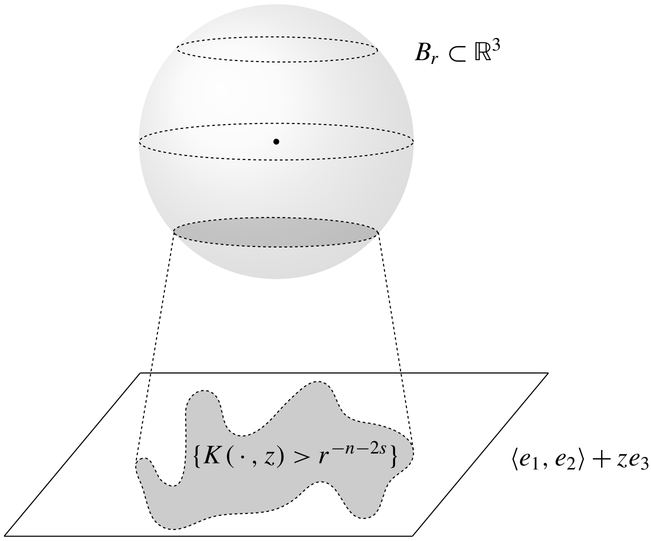

Figure 1. Area-preserving deformation in $\mathbb{R}^3$.

In Figure 1 we illustrate condition (2-1) for $k = 2$ and $n = 3$. Note that, for $k = n$, we recover the definition of $\mathcal{K}_n^s$. Moreover, $|x|^{-n-2s} \in \mathcal{K}_k^s$ for all k.

Proposition 2.2. *Let* $1 \le k < n$. *Then* $\mathcal{K}_k^s \subset \mathcal{K}_{k+1}^s \subseteq \mathcal{K}_n^s$.

Proof. Let $K \in \mathcal{K}_k^s$. Fix any $z \in \mathbb{R}^{n-k-1}$ and $r > 0$. Then

$$\begin{aligned}
\mathcal{H}^{k+1}(\{y \in \mathbb{R}^{k+1} : K(y, z) > r^{-n-2s}\}) &= \int_{\mathbb{R}^{k+1}} \chi_{\{y \in \mathbb{R}^{k+1} : K(y,z) > r^{-n-2s}\}}(y)\, dy \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^k} \chi_{\{(w,t) \in \mathbb{R}^k \times \mathbb{R} : K(w,t,z) > r^{-n-2s}\}}(w, t)\, dw \right) dt \\
&= \int_{\mathbb{R}} \mathcal{H}^k(\{w \in \mathbb{R}^k : K(w, t, z) > r^{-n-2s}\})\, dt \equiv \mathrm{I}.
\end{aligned}$$

If $|z| \ge r$, then for any $t \in \mathbb{R}$, we have that $(t, z) \in \mathbb{R}^{n-k}$, with $|(t, z)| > r$. Therefore, by (2-1), it follows that $\mathrm{I} = 0$. If $|z| < r$, then

$$\begin{aligned}
\mathrm{I} &= \int_{\mathbb{R}} \mathcal{H}^k(B_{(r^2-t^2-|z|^2)^{1/2}})\, dt = \omega_k \int_{-(r^2-|z|^2)^{1/2}}^{(r^2-|z|^2)^{1/2}} (r^2 - t^2 - |z|^2)^{k/2}\, dt \\
&= \omega_k (r^2 - |z|^2)^{k/2} \int_{-(r^2-|z|^2)^{1/2}}^{(r^2-|z|^2)^{1/2}} \left(1 - \left(\frac{t}{(r^2 - |z|^2)^{1/2}} \right)^2 \right)^{k/2} dt \\
&= \omega_k (r^2 - |z|^2)^{(k+1)/2} \int_{-1}^{1} (1 - \sigma^2)^{k/2}\, d\sigma = \frac{\pi^{k/2}}{\Gamma\left(\frac{1}{2}k + 1\right)} \frac{\pi^{1/2} \Gamma\left(\frac{1}{2}k + 1\right)}{\Gamma\left(\frac{1}{2}(k+1) + 1\right)} (r^2 - |z|^2)^{(k+1)/2} \\
&= \omega_{k+1} (r^2 - |z|^2)^{(k+1)/2} = \mathcal{H}^{k+1}(B_{(r^2-|z|^2)^{1/2}}),
\end{aligned}$$

where $\omega_l = \mathcal{H}^l(B_1) = \pi^{l/2}/\Gamma(l/2 + 1)$ and $B_{(r^2-|z|^2)^{1/2}}$ is the ball of radius $(r^2 - |z|^2)^{1/2}$ in $\mathbb{R}^{k+1}$. $\square$

Definition 2.3. A function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be $C^{1,1}$ at the point x_0, and we write $u \in C^{1,1}(x_0)$, if there is a vector $p \in \mathbb{R}^n$, a radius $\rho > 0$, and a constant $C > 0$ such that

$$|u(x_0 + x) - u(x_0) - x \cdot p| \le C|x|^2 \quad \text{for all } x \in B_\rho.$$

We denote by $[u]_{C^{1,1}(x_0)}$ the minimum constant for which this property holds, among all admissible vectors p and radii ρ.

Definition 2.4. Let $s \in \left(\frac{1}{2}, 1\right)$ and $1 \le k < n$. For any $u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0)$, we define

$$\mathcal{F}_k^s u(x_0) = c_{n,s} \inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^n} (u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0)) K(x)\, dx,$$

where $\mathcal{K}_k^s$ is the set of kernels satisfying (2-1) and $c_{n,s}$ is the constant in Δ^s.

As an immediate consequence of Proposition 2.2, we obtain that the operators are ordered.

Corollary 2.5. *Let $s \in \left(\frac{1}{2}, 1\right)$ and $1 \le k < n$. Then, for any $u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0)$,*

$$\mathrm{MA}^s u(x_0) \le \mathcal{F}_k^s u(x_0) \le \Delta^s u(x_0).$$

Moreover, $\{\mathcal{F}_k^s\}_{k=1}^{n-1}$ is monotone decreasing.

The regularity condition on u in Definition 2.4 allows us to compute $\mathcal{F}_k^s u$ at the point x_0 in the classical sense. To obtain a finite number, we need to impose two extra conditions:

(1) An integrability condition at infinity:

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{(1 + |x|)^{n+2s}}\, dx < \infty. \tag{P_1}$$

(2) A convexity condition at x_0:

$$\tilde{u}(x) \equiv u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0) \ge 0 \quad \text{for all } x \in \mathbb{R}^n. \tag{P_2}$$

Proposition 2.6. *If $u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0)$ and u satisfies (P_1) and (P_2), then*

$$0 \le \mathcal{F}_k^s u(x_0) < \infty.$$

Proof. Let $\rho > 0$ be as in Definition 2.3. Then

$$\begin{aligned}
0 \le \mathcal{F}_k^s u(x_0) &\le \int_{\mathbb{R}^n} (u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0)) \frac{1}{|x|^{n+2s}}\, dx \\
&\le \int_{B_\rho} \frac{[u]_{C^{1,1}(x_0)} |x|^2}{|x|^{n+2s}}\, dx + \int_{\mathbb{R}^n \setminus B_\rho(x_0)} \frac{|u(x)|}{|x - x_0|^{n+2s}}\, dx \\
&\qquad + |u(x_0)| \int_{\mathbb{R}^n \setminus B_\rho} \frac{1}{|x|^{n+2s}}\, dx + |\nabla u(x_0)| \int_{\mathbb{R}^n \setminus B_\rho} \frac{|x|}{|x|^{n+2s}}\, dx \\
&\le C(s, \rho)(|u(x_0)| + |\nabla u(x_0)| + [u]_{C^{1,1}(x_0)}) \\
&\qquad + \frac{1 + |x_0| + \rho}{\rho} \int_{\mathbb{R}^n} \frac{|u(x)|}{(1 + |x|)^{n+2s}}\, dx < \infty, \qquad \text{since } s \in \left(\tfrac{1}{2}, 1\right). \qquad \square
\end{aligned}$$

We point out that if u is not convex at x_0, then the infimum could be $-\infty$. We show this result in the next proposition.

Proposition 2.7. *Let $u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0)$. Assume that u satisfies (P_1). If there exists $\bar{x} \in \mathbb{R}^n$ with $\bar{x} = (\bar{y}, 0)$ and $\bar{y} \in \mathbb{R}^k$ such that*

$$\tilde{u}(\bar{x}) = u(x_0 + \bar{x}) - u(x_0) - \bar{x} \cdot \nabla u(x_0) < 0,$$

then $\mathcal{F}_k^s u(x_0) = -\infty$.

Proof. Let $K(x) = |x - \bar{x}|^{-n-2s}$. For any $r > 0$ and $z \in \mathbb{R}^{n-k}$, if $|z| < r$, then

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : K(y, z) > r^{-n-2s}\}) = \mathcal{H}^k(\{y \in \mathbb{R}^k : |y - \bar{y}|^2 + |z|^2 < r^2\}) = \mathcal{H}^k(B_{(r^2-|z|^2)^{1/2}}).$$

Also, the measure is clearly zero if $|z| \geq r$. Therefore, $K \in \mathcal{K}_k^s$. It follows that

$$\begin{aligned} \mathcal{F}_k^s u(x_0) &\leq \int_{\mathbb{R}^n} \tilde{u}(x)|x - \bar{x}|^{-n-2s}\, dx \\ &= \int_{B_\varepsilon(\bar{x})} \tilde{u}(x)|x - \bar{x}|^{-n-2s}\, dx + \int_{\mathbb{R}^n \setminus B_\varepsilon(\bar{x})} \tilde{u}(x)|x - \bar{x}|^{-n-2s}\, dx \equiv \mathrm{I} + \mathrm{II}. \end{aligned}$$

Since $u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0)$, we have that $\tilde{u}$ is continuous. Hence, given that $\tilde{u}(\bar{x}) < 0$, it follows that $\tilde{u}(x) < 0$ for all $x \in B_\varepsilon(\bar{x})$ for some $\varepsilon > 0$. Moreover, since $K \notin L^1(B_\varepsilon(\bar{x}))$, we have that $\mathrm{I} = -\infty$. Arguing similarly as in the proof of Proposition 2.6, we see that $\mathrm{II} < \infty$. Therefore,

$$\mathcal{F}_k^s u(x_0) = -\infty. \qquad \square$$

Remark 2.8. The operators $\mathcal{F}_k^s$ are not rotation invariant. This is because, for simplicity, in the construction of the family of admissible kernels $\mathcal{K}_k^s$ we chose the first k vectors from the canonical basis of $\mathbb{R}^n$. In general, we may take any subset of k unitary vectors, $\tau = \{\tau_i\}_{i=1}^k$, and replace the first condition on (2-1) by

$$\mathcal{H}^k(\{y \in \langle\tau\rangle^\perp : K(y + z\tau) > r^{-n-2s}\}) = \mathcal{H}^k(B_{(r^2-|z|^2)^{1/2}}) \tag{2-2}$$

for all $z \in \langle\tau\rangle$ and $r > 0$, where $\langle\tau\rangle$ denotes the span of $\{\tau_i\}_{i=1}^k$ and $\langle\tau\rangle^\perp$ the orthogonal subspace to $\langle\tau\rangle$. Let $\mathrm{SO}(n)$ be the group of $n \times n$ rotation matrices. Since $\tau_i = Ae_i$ for some $A \in \mathrm{SO}(n)$, it follows that any kernel K_τ satisfying (2-2) can be written as $K_\tau = K \circ A$, where K satisfies (2-1). Therefore, to make the operators rotation invariant, one possibility is to take the infimum over all possible rotations. Namely,

$$\inf_{A \in \mathrm{SO}(n)} \inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^n} \tilde{u}(x) K(Ax)\, dx.$$

To focus on the main ideas, we will not explore this operator in this work.

3. Rearrangements and measure-preserving transformations

We introduce some definitions and preliminary results regarding rearrangements of nonnegative functions. For more detailed information, see for instance [Baernstein 2019; Bennett and Sharpley 1988].

Definition 3.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative measurable function. We define the decreasing rearrangement of f as the function f^* defined on $[0, \infty)$ and given by

$$f^*(t) = \sup\{\lambda > 0 : |\{x \in \mathbb{R}^n : f(x) > \lambda\}| > t\},$$

and the increasing rearrangement of f as the function f_* defined on $[0, \infty)$ and given by

$$f_*(t) = \inf\{\lambda > 0 : |\{x \in \mathbb{R}^n : f(x) \leq \lambda\}| > t\}.$$

We use the convention that $\inf \varnothing = \infty$.

Proposition 3.2. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be nonnegative measurable functions. Then*

$$\int_0^\infty f_*(t) g^*(t)\, dt \leq \int_{\mathbb{R}^n} f(x) g(x)\, dx \leq \int_0^\infty f^*(t) g^*(t)\, dt.$$

The upper bound is the classical Hardy–Littlewood inequality. For the proof, see [Bennett and Sharpley 1988, Theorem 2.2] or [Baernstein 2019, Corollary 2.16]. For the sake of completeness, we give the proof of the lower bound.

Proof. For $j \geq 1$, let $f_j = f|_{B_j}$ and $g_j = g|_{B_j}$, where B_j denotes the ball of radius j centered at 0 in $\mathbb{R}^n$. By [Baernstein 2019, Corollary 2.18], it follows that

$$\int_0^{|B_j|} (f_j)_*(t)(g_j)^*(t)\, dt \leq \int_{B_j} f_j(x) g_j(x)\, dx.$$

Since $f, g \geq 0$, we get

$$\int_{B_j} f_j(x) g_j(x)\, dx \leq \int_{\mathbb{R}^n} f(x) g(x)\, dx.$$

Note that, for any $t \in [0, |B_j|]$, we have

$$\{\lambda > 0 : |\{x \in B_j : f_j(x) \leq \lambda\}| > t\} \subset \{\lambda > 0 : |\{x \in \mathbb{R}^n : f(x) \leq \lambda\}| > t\}.$$

Hence $(f_j)_*(t) \geq f_*(t)$ and

$$\int_0^{|B_j|} (f_j)_*(t)(g_j)^*(t)\, dt \geq \int_0^{|B_j|} f_*(t)(g_j)^*(t)\, dt.$$

Moreover, $g_j \nearrow g$ pointwise on $\mathbb{R}^n$. Then by [Baernstein 2019, Proposition 1.39], we have $(g_j)^* \nearrow g^*$ pointwise on $[0, \infty)$ as $j \to \infty$. By the monotone convergence theorem, we get

$$\lim_{j \to \infty} \int_0^{|B_j|} f_*(t)(g_j)^*(t)\, dt = \int_0^\infty f_*(t) g^*(t)\, dt.$$

Combining the previous estimates, we conclude that

$$\int_0^\infty f_*(t) g^*(t)\, dt \leq \int_{\mathbb{R}^n} f(x) g(x)\, dx. \qquad \square$$

Definition 3.3. We say that a measurable function $\psi : \mathbb{R}^l \to \mathbb{R}^m$ is a measure-preserving transformation, or *measure-preserving*, if, for any measurable set E in $\mathbb{R}^m$,

$$\mathcal{H}^l(\psi^{-1}(E)) = \mathcal{H}^m(E).$$

Lemma 3.4. *If $\psi : \mathbb{R}^l \to \mathbb{R}^m$ is measure-preserving, then, for any measurable function $f : \mathbb{R}^m \to \mathbb{R}$ and any measurable set E in $\mathbb{R}^m$,*

$$\int_E f(y)\,dy = \int_{\psi^{-1}(E)} f(\psi(z))\,dz.$$

An important result by Ryff [1970] provides a sufficient condition for which we can recover a function given its decreasing/increasing rearrangement, by means of a measure-preserving transformation.

Theorem 3.5 (Ryff's theorem). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative measurable function. If $\lim_{t\to\infty} f^*(t)$ equals zero, then there exists a measure-preserving transformation $\sigma : \operatorname{supp}(f) \to \operatorname{supp}(f^*)$ such that*

$$f = f^* \circ \sigma$$

almost everywhere on the support of f. Similarly, if $\lim_{t\to\infty} f_*(t) = \infty$, then $f = f_* \circ \sigma$.

We will call a measure-preserving transformation σ satisfying Ryff's theorem a *Ryff's map*.

Remark 3.6. In general, σ is not invertible. Furthermore, there may not exist a measure-preserving transformation ψ such that $f^* = f \circ \psi$.

As a consequence of Ryff's theorem, we obtain a representation formula for the admissible kernels. We write $\omega_k = \mathcal{H}^k(B_1)$.

Lemma 3.7. *Let $K \in \mathcal{K}_k^s$. Fix $z \in \mathbb{R}^{n-k}$ and use the notation $K_z(y) = K(y, z)$. Then*

$$K_z^*(t) = ((\omega_k^{-1} t)^{2/k} + |z|^2)^{-(n+2s)/2}.$$

In particular, there exists a measure-preserving transformation $\sigma_z : \operatorname{supp}(K_z) \to (0, \infty)$ such that

$$K(y, z) = K_z^*(\sigma_z(y)) \quad \textit{for a.e. } y \in \operatorname{supp}(K_z).$$

Proof. Fix $z \in \mathbb{R}^{n-k}$. Then

$$\begin{aligned}
K_z^*(t) &= \sup\{\lambda > 0 : \mathcal{H}^k(\{y \in \mathbb{R}^k : K(y, z) > \lambda\}) > t\} \\
&= \sup\{\lambda < |z|^{-n-2s} : \mathcal{H}^k(B_{(\lambda^{-2/(n+2s)} - |z|^2)^{1/2}}) > t\} \\
&= \sup\{\lambda < |z|^{-n-2s} : \omega_k(\lambda^{-2/(n+2s)} - |z|^2)^{k/2} > t\} \\
&= \sup\{\lambda < |z|^{-n-2s} : \lambda^{-2/(n+2s)} > (\omega_k^{-1} t)^{2/k} + |z|^2\} = ((\omega_k^{-1} t)^{2/k} + |z|^2)^{-(n+2s)/2}.
\end{aligned}$$

Moreover, $\lim_{t\to\infty} K_z^*(t) = 0$. Therefore, the result follows from Theorem 3.5. □

In view of Definition 3.1, we introduce the symmetric rearrangement of a function in $\mathbb{R}^n$ with respect to the first k variables as follows. Fix $k \in \mathbb{N}$ with $1 \le k < n$. Given $x \in \mathbb{R}^n$, we write $x = (y, z)$ with $y \in \mathbb{R}^k$ and $z \in \mathbb{R}^{n-k}$. Furthermore, for z fixed, we call f_z the restriction of f to $\mathbb{R}^k$. Namely, $f_z(y) = f(y, z)$.

Definition 3.8. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative measurable function. We define the k-symmetric *decreasing* rearrangement of f as the function $f^{*,k} : \mathbb{R}^n \to [0, \infty]$ given by

$$f^{*,k}(x) = f_z^*(\omega_k |y|^k),$$

and the k-symmetric *increasing* rearrangement as the function $f_{*,k} : \mathbb{R}^n \to [0, \infty]$ given by

$$f_{*,k}(x) = (f_z)_*(\omega_k |y|^k).$$

When $k = n$, we obtain the usual symmetric rearrangement.

Remark 3.9. (1) Notice that $f^{*,k}$ and $f_{*,k}$ are radially symmetric and monotone decreasing/increasing, with respect to y. In the literature, this type of symmetrization is also known as the Steiner symmetrization [Baernstein 2019, Chapter 6].

(2) By Lemma 3.7, we see that any kernel $K \in \mathcal{K}_k^s$ satisfies

$$K^{*,k}(x) = |x|^{-n-2s} \quad \text{for } x \neq 0. \tag{3-1}$$

4. Analysis of $\mathcal{F}_k^s$

Our main goal of this section is to study the infimum in the definition of the operator

$$\mathcal{F}_k^s u(x_0) = c_{n,s} \inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^n} \tilde{u}(x) K(x)\, dx,$$

where $\tilde{u}(x) = u(x_0+x) - u(x_0) - x \cdot \nabla u(x_0)$. Throughout the section, we assume that $u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0)$ and that u satisfies properties (P_1) and (P_2), so that $0 \leq \mathcal{F}_k^s u(x_0) < \infty$.

Analysis of the infimum. We will study the following cases:

<u>Case 1</u>. For all $\lambda > 0$ and $z \in \mathbb{R}^{n-k}$,

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda\}) < \infty.$$

<u>Case 2</u>. There exists some $\lambda_0 > 0$ such that, for all $z \in \mathbb{R}^{n-k}$,

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda\}) \begin{cases} < \infty & \text{for } 0 < \lambda < \lambda_0, \\ = \infty & \text{for } \lambda \geq \lambda_0. \end{cases}$$

<u>Case 3</u>. For all $\lambda > 0$ and $z \in \mathbb{R}^{n-k}$,

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda\}) = \infty.$$

In Case 1, when all of the level sets of $\tilde{u}$ have finite measure, we show that the infimum is attained at some kernel whose level sets depend on the measure-preserving transformation that rearranges the level sets of $\tilde{u}$. More precisely:

Theorem 4.1. *Suppose that, for all $\lambda > 0$ and $z \in \mathbb{R}^{n-k}$,*

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda\}) < \infty.$$

Then, for any $z \in \mathbb{R}^{n-k}$, there exists a measure-preserving transformation $\sigma_z : \mathbb{R}^k \to [0, \infty)$ such that

$$\mathcal{F}_k^s u(x_0) = c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \frac{\tilde{u}(y, z)}{((\omega_k^{-1} \sigma_z(y))^{2/k} + |z|^2)^{(n+2s)/2}}\, dy\, dz.$$

In particular, the infimum is attained.

Remark 4.2. Observe that if $\tilde{u}(\,\cdot\,, z)$ is constant in some set of positive measure, then the kernel where the infimum is attained is not unique since the integral is invariant under any measure-preserving rearrangement of K within this set; see [Ryff 1970].

Before we give the proof of Theorem 4.1, we need a lemma regarding the k-symmetric increasing rearrangement of $\tilde{u}$. By Definition 3.8, this is given by the expression

$$\tilde{u}_{*,k}(y,z) = \inf\{\lambda > 0 : \mathcal{H}^k(\{w \in \mathbb{R}^k : \tilde{u}(w,z) \le \lambda\}) > \omega_k |y|^k\}.$$

Lemma 4.3. *Fix $z \in \mathbb{R}^{n-k}$. If $\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}(y,z) \le \lambda\}) < \infty$ for all $\lambda > 0$, then*

$$\lim_{|y|\to\infty} \tilde{u}_{*,k}(y,z) = \infty.$$

Proof. Assume there exists $M > 0$ independent of λ such that

$$\mathcal{H}^k(\{w \in \mathbb{R}^k : \tilde{u}(w,z) \le \lambda\}) \le M \quad \text{for all } \lambda > 0. \tag{4-1}$$

Then, for any $y \in \mathbb{R}^k$ with $\omega_k|y|^k > M$, we have that

$$\tilde{u}_{*,k}(y,z) = \infty,$$

since $\inf \varnothing = \infty$. If (4-1) does not hold, then there must be an increasing sequence $\{M_\lambda\}_{\lambda>0}$ with $M_\lambda \to \infty$ as $\lambda \to \infty$ such that

$$\mathcal{H}^k(\{w \in \mathbb{R}^k : \tilde{u}(w,z) \le \lambda\}) = M_\lambda.$$

Then, for any $M > 0$, there exists $\Lambda = \Lambda(M) > 0$ such that $M_\lambda > M$ for all $\lambda > \Lambda$. Since M_λ is monotone increasing, we can assume without loss of generality that $M_\Lambda \le M$. Otherwise, we take Λ to be the minimum for which this property holds. Also, $\Lambda(M)$ is monotone increasing, and $\Lambda(M) \to \infty$ as $M \to \infty$. In particular,

$$\inf\{\lambda > 0 : M_\lambda > M\} \ge \Lambda(M) \to \infty \quad \text{as } M \to \infty.$$

Then, for any $K > 0$, there exists $M > 0$ such that

$$\inf\{\lambda > 0 : M_\lambda > M\} \ge K.$$

Therefore, for any $y \in \mathbb{R}^k$ with $\omega_k|y|^k > M$, we have

$$\tilde{u}_{*,k}(y,z) = \inf\{\lambda > 0 : M_\lambda > \omega_k|y|^k\} \ge \inf\{\lambda > 0 : M_\lambda > M\} \ge K.$$

We conclude that

$$\lim_{|y|\to\infty} \tilde{u}_{*,k}(y,z) = \infty. \qquad \square$$

Proof of Theorem 4.1. Since u is convex at x_0, we have that $\tilde{u}(y,z) \ge 0$. Moreover,

$$\mathcal{F}_k^s u(x_0) = c_{n,s} \inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y,z) K(y,z)\, dy\, dz.$$

Fix $z \in \mathbb{R}^{n-k}$ and consider the functions $f(y) = \tilde{u}(y,z)$ and $g(y) = K(y,z)$. Since

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}(y,z) \le \lambda\}) < \infty$$

for any $\lambda > 0$, then by Lemma 4.3 we have

$$\lim_{t\to\infty} f_*(t) = \lim_{|y|\to\infty} f_{*,k}(x) = \infty,$$

with $f_{*,k}(x) = \tilde{u}_{*,k}(y, z)$ and $f_{*,k}(x) = f_*(\omega_k |y|^k)$. By Ryff's theorem (Theorem 3.5), there exists a measure-preserving transformation $\sigma_z : \mathbb{R}^k \to [0, \infty)$ depending on z such that

$$\tilde{u}(y, z) = f_*(\sigma_z(y)) \tag{4-2}$$

for all $y \in \operatorname{supp} \tilde{u}(\,\cdot\,, z) \subseteq \mathbb{R}^k$.

Let $K(y, z) = ((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2)^{-(n+2s)/2}$. For any $r > |z|$, we have

$$\begin{aligned}
\mathcal{H}^k(\{y \in \mathbb{R}^k : K(y, z) > r^{-n-2s}\}) &= \mathcal{H}^k(\{y \in \mathbb{R}^k : ((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2)^{-(n+2s)/2} > r^{-n-2s}\}) \\
&= \mathcal{H}^k(\{y \in \mathbb{R}^k : \sigma_z(y) < \omega_k(r^2 - |z|^2)^{k/2}\}) \\
&= \mathcal{H}^k(\sigma_z^{-1}((0, \omega_k(r^2 - |z|^2)^{k/2}))) = \mathcal{H}^1((0, \omega_k(r^2 - |z|^2)^{k/2})) \\
&= \omega_k(r^2 - |z|^2)^{k/2} = \mathcal{H}^k(B_{(r^2-|z|^2)^{k/2}}),
\end{aligned}$$

since σ_k is measure-preserving (see Definition 3.3). Then $K \in \mathcal{K}_k^s$, and thus

$$\mathcal{F}_k^s u(x_0) \le c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \frac{\tilde{u}(y, z)}{((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2)^{(n+2s)/2}} \, dy \, dz.$$

To prove the reverse inequality, let $K \in \mathcal{K}_k^s$. Applying Proposition 3.2, we see that

$$\int_{\mathbb{R}^k} \tilde{u}(y, z) K(y, z) \, dy \ge \int_0^\infty f_*(t) g^*(t) \, dt = \int_{\mathbb{R}^k} f_*(\sigma_z(y)) g^*(\sigma_z(y)) \, dy = \int_{\mathbb{R}^k} \tilde{u}(y, z) g^*(\sigma_z(y)) \, dy$$

by Lemma 3.4 and (4-2). Moreover, by the definition of rearrangements,

$$g^*(\sigma_z(y)) = \sup\{\lambda > 0 : \mathcal{H}^k(\{w \in \mathbb{R}^k : K(w, z) > \lambda\}) > \sigma_z(y)\} = K^{*,k}(\tilde{y}, z),$$

with $\omega_k |\tilde{y}|^k = \sigma_z(y)$. By (3-1), we get

$$g^*(\sigma_z(y)) = (|\tilde{y}|^2 + |z|^2)^{-(n+2s)/2} = ((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2)^{-(n+2s)/2}.$$

Hence integrating over all $z \in \mathbb{R}^{n-k}$ and taking the infimum over all kernels $K \in \mathcal{K}_k^s$, we conclude that

$$\mathcal{F}_k^s u(x) = c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \frac{\tilde{u}(y, z)}{((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2)^{(n+2s)/2}} \, dy \, dz. \qquad \square$$

Remark 4.4. A natural question that arises from this result is whether there exists a measure-preserving transformation $\varphi_z : \mathbb{R}^k \to \mathbb{R}^k$ such that

$$|\varphi_z(y)| = (\omega_k^{-1}\sigma_z(y))^{1/k}.$$

In that case, we would have that the infimum is attained at a kernel K such that

$$K(y, z) = |\phi(y, z)|^{-n-2s},$$

where $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is measure-preserving with $\phi(y, z) = (\varphi_z(y), z)$.

Recall that Ryff's theorem gives a representation of a function f in terms of its increasing rearrangement f_*, that is, $f = f_* \circ \sigma$ with $\sigma : \mathbb{R}^k \to \mathbb{R}$ measure-preserving. If this result were also true for the *symmetric* increasing rearrangement, given by $f_\#(x) = f_*(\omega_k|x|^k)$, then there would exist a measure-preserving transformation $\varphi : \mathbb{R}^k \to \mathbb{R}^k$ such that $f = f_\# \circ \psi$. In particular,

$$f(x) = f_\#(\varphi(x)) = f_*(\omega_k|\varphi(x)|^k) = f_*(\sigma(x)).$$

Hence it seems reasonable that $\omega_k|\varphi(x)|^k = \sigma(x)$. As far as we know, this is an open problem.

As an immediate consequence of Theorem 4.1, we obtain the following representation of the function $\mathcal{F}_k^s u$ in terms of the k-symmetric increasing rearrangement of $\tilde{u}$.

Corollary 4.5. *Under the assumptions of Theorem 4.1, we have*

$$\mathcal{F}_k^s u(x_0) = \Delta^s \tilde{u}_{*,k}(0).$$

Proof. Note that $\tilde{u}_{*,k}(0) = 0$ since $\tilde{u}(0) = 0$. Therefore, using the same notation as in the proof of Theorem 4.1, we showed that

$$\begin{aligned}
\mathcal{F}_k^s u(x_0) &= c_{n,s} \int_{\mathbb{R}^{n-k}} \int_0^\infty f_*(t) g^*(t)\, dt\, dz = \omega_k c_{n,s} \int_{\mathbb{R}^{n-k}} \int_0^\infty f_*(\omega_k r^k) g^*(\omega_k r^k) r^{k-1}\, dr\, dz \\
&= c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} f_*(\omega_k|y|^k) g^*(\omega_k|y|^k)\, dy\, dz = c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}_{*,k}(y,z) K^{*,k}(y,z)\, dy\, dz \\
&= c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \frac{\tilde{u}_{*,k}(y,z)}{(|y|^2+|z|^2)^{(n+2s)/2}}\, dy\, dz = \Delta^s \tilde{u}_{*,k}(0). \qquad \square
\end{aligned}$$

From the previous result and the fact that the family of operators $\{\mathcal{F}_k\}_{k=1}^{n-1}$ is monotone decreasing, we see that the fractional Laplacian of the k-symmetric rearrangements are ordered at the origin.

Corollary 4.6. *Suppose we are under the assumptions of Theorem 4.1. Then*

$$\Delta^s \tilde{u}_{*,k+1}(0) \le \Delta^s \tilde{u}_{*,k}(0).$$

Next we treat Case 2.

Theorem 4.7. *Suppose that there exists some $\lambda_0 > 0$ such that, for all $z \in \mathbb{R}^{n-k}$,*

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}(y,z) \le \lambda\}) \begin{cases} < \infty & \text{for } 0 < \lambda < \lambda_0, \\ = \infty & \text{for } \lambda \ge \lambda_0. \end{cases}$$

Then there exists a kernel $K_0 \in \mathcal{K}_k^s$ with $\operatorname{supp} K_0(\cdot, z) \subseteq \{y \in \mathbb{R}^k : \tilde{u}(y,z) \le \lambda_0\}$ *such that*

$$\mathcal{F}_k^s u(x_0) = c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y,z) K_0(y,z)\, dy\, dz.$$

In particular, the infimum is attained.

Proof. Fix $z \in \mathbb{R}^{n-k}$. For $j \ge 1$, define the set

$$A_j(z) = \left\{ y \in \mathbb{R}^k : \tilde{u}(y,z) \le \lambda_0 - \frac{1}{j} \right\}.$$

For simplicity, we drop the notation of z. We have that $\mathcal{H}^k(A_j) < \infty$, $A_j \subseteq A_{j+1}$, and

$$A_\infty = \bigcup_{j=1}^\infty A_j = \{y \in \mathbb{R}^k : \tilde{u}(y,z) < \lambda_0\}.$$

Observe that if $K \in \mathcal{K}_k^s$, then

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : K(y,z) > 0\}) = \lim_{r\to 0} \mathcal{H}^k(\{y \in \mathbb{R}^k : K(y,z) > r\}) = \infty.$$

Hence we need to distinguish two subcases:

Case 2A. Assume that $\mathcal{H}^k(A_\infty) = \infty$. Let $K \in \mathcal{K}_k^s$ and $v_j = \tilde{u}\chi_{A_j}$. By Proposition 3.2,

$$\int_{A_j} \tilde{u}(y,z)K(y,z)\,dy = \int_{\mathbb{R}^k} v_j(y,z)K(y,z)\,dy \ge \int_0^\infty (v_j)_*(t)K^*(t)\,dt.$$

By Lemma 3.4, for any measure-preserving transformation $\sigma : \mathbb{R}^k \to [0,\infty)$, it follows that

$$\int_0^\infty (v_j)_*(t)K^*(t)\,dt = \int_{\mathbb{R}^k} (v_j)_*(\sigma(y))K^*(\sigma(y))\,dy.$$

By Ryff's theorem (Theorem 3.5), there exists $\sigma_j : A_j \to [0, \mathcal{H}^k(A_j)]$ measure-preserving such that $v_j = (v_j)_* \circ \sigma_j$ in A_j. Therefore,

$$\int_{A_j} \tilde{u}(y,z)K(y,z)\,dy \ge \int_{A_j} \tilde{u}(y,z)K^*(\sigma_j(y))\,dy. \tag{4-3}$$

We claim that $\sigma_{j+1}(y) \le \sigma_j(y)$, for all $y \in A_j$. Indeed, since $A_j \subseteq A_{j+1}$, we have

$$\begin{cases} v_j(y) = v_{j+1}(y) & \text{for all } y \in A_j, \\ v_j(y) \le v_{j+1}(y) & \text{for all } y \in A_{j+1} \setminus A_j. \end{cases}$$

In particular,

$$(v_{j+1})_*(\sigma_{j+1}(y)) = (v_j)_*(\sigma_j(y)) \le (v_{j+1})_*(\sigma_j(y)) \quad \text{for all } \; y \in A_j.$$

Since $(v_{j+1})_*$ is monotone increasing, we must have

$$\sigma_{j+1}(y) \le \sigma_j(y) \quad \text{for all } y \in A_j.$$

Therefore, there exists $\sigma_\infty : A_\infty \to [0,\infty)$ measure-preserving such that

$$\sigma_\infty(y) = \lim_{j\to\infty} \sigma_j(y).$$

Define the kernel K_0 as

$$K_0(y,z) = ((\omega_k^{-1}\sigma_\infty(y))^{k/2} + |z|^2)^{-(n+2s)/2}\chi_{A_\infty}(y).$$

Since $\mathcal{H}^k(A_\infty) = \infty$, we have that $K_0 \in \mathcal{K}_k^s$. Furthermore, we note that $K_0(y,z) = K_0^*(\sigma_\infty(y))$ and $\operatorname{supp} K_0(\,\cdot\,,z) = \overline{A_\infty} = \{y \in \mathbb{R}^k : \tilde{u}(y,z) \le \lambda_0\}$ for all $y \in A_\infty$. Then by Fatou's lemma, Lemma 3.7, and (4-3), we get

$$\begin{aligned} \int_{\mathbb{R}^k} \tilde{u}(y,z)K_0(y,z)\,dy &= \int_{A_\infty} \tilde{u}(y,z)K_0^*(\sigma_\infty(y))\,dy \le \liminf_{j\to\infty} \int_{A_j} \tilde{u}(y,z)K_0^*(\sigma_j(y))\,dy \\ &= \liminf_{j\to\infty} \int_{A_j} \tilde{u}(y,z)K^*(\sigma_j(y))\,dy \le \int_{\mathbb{R}^k} \tilde{u}(y,z)K(y,z)\,dy \end{aligned}$$

for any $K \in \mathcal{K}_k^s$. Integrating over z and taking the infimum over all kernels K, we conclude the result.

Case 2B. Assume that $\mathcal{H}^k(A_\infty) < \infty$. Set $A = \{y \in \mathbb{R}^k : \tilde{u}(y,z) = \lambda_0\}$. Then

$$\mathcal{H}^k(A) = \infty, \tag{4-4}$$

since $\{y \in \mathbb{R}^k : \tilde{u}(y,z) \le \lambda_0\} = A_\infty \cup A$. Fix $\varepsilon > 0$ and define

$$v_\varepsilon(y,z) = \tilde{u}(y,z)\chi_{A_\infty}(y) + \max\{\lambda_0, (\lambda_0+\varepsilon)\phi(y,z)\}\chi_A(y),$$

with $\phi(y,z) = 1 - e^{-|y|^2-|z|^2}$. Note that $0 < \phi \le 1$, $\phi(y,z) \to 1$ as $|(y,z)| \to \infty$, and $\phi(y,z) \approx |y|^2+|z|^2$ as $|(y,z)| \to 0$. Also, $\{v_\varepsilon\}_{\varepsilon>0}$ is a monotone increasing sequence and

$$\begin{aligned}\lim_{\varepsilon\to 0} v_\varepsilon(y,z) &= \tilde{u}(y,z)\chi_{A_\infty}(y) + \max\bigl\{\lambda_0, \lim_{\varepsilon\to 0}(\lambda_0+\varepsilon)\phi(y,z)\bigr\}\chi_A(y) \\ &= \tilde{u}(y,z)\chi_{A_\infty}(y) + \max\{\lambda_0, \lambda_0\phi(y,z)\}\chi_A(y) = \tilde{u}(y,z)\chi_{A_\infty\cup A}(y).\end{aligned} \tag{4-5}$$

For any $j \in \mathbb{N}$ with $j > 1/\varepsilon$, consider the set

$$B_j^\varepsilon(z) = \Bigl\{y \in \mathbb{R}^k : v_\varepsilon(y,z) \le \lambda_0 + \varepsilon - \frac{1}{j}\Bigr\}.$$

Then $B_j^\varepsilon \subseteq B_{j+1}^\varepsilon$ and $B_\infty^\varepsilon = \bigcup_{j>1/\varepsilon} B_j^\varepsilon = \{y \in \mathbb{R}^k : v_\varepsilon(y,z) < \lambda_0 + \varepsilon\}$. Moreover, we have

$$\mathcal{H}^k(B_j^\varepsilon) \le \mathcal{H}^k(A_\infty) + \mathcal{H}^k\Bigl(\Bigl\{y \in A : \max\{\lambda_0, (\lambda_0+\varepsilon)\phi(y,z)\} \le \lambda_0 + \varepsilon - \frac{1}{j}\Bigr\}\Bigr). \tag{4-6}$$

Choose $R > 0$ large enough (depending on ε, j, λ_0, and z) that

$$(\lambda_0+\varepsilon)e^{-R^2-|z|^2} < \frac{1}{j}.$$

Then $(\lambda_0+\varepsilon)\phi(y,z) > \lambda_0 + \varepsilon - 1/j > \lambda_0$ for all $y \in B_R^c$, and thus

$$\mathcal{H}^k\Bigl(\Bigl\{y \in A \cap B_R^c : \max\{\lambda_0, (\lambda_0+\varepsilon)\phi(y,z)\} \le \lambda_0 + \varepsilon - \frac{1}{j}\Bigr\}\Bigr) = 0. \tag{4-7}$$

By (4-6) and (4-7), we see that

$$\mathcal{H}^k(B_j^\varepsilon(z)) \le \mathcal{H}^k(A_\infty) + \mathcal{H}^k(A \cap B_R) < \infty.$$

Furthermore, $A \subseteq B_\infty^\varepsilon$, and thus, by (4-4), we get

$$\mathcal{H}^k(B_\infty^\varepsilon) \ge \mathcal{H}^k(A) = \infty.$$

In particular, v_ε satisfies the assumptions of Case 2A, so there exists $K_\varepsilon \in \mathcal{K}_k^s$ defined by

$$K_\varepsilon(y,z) = ((\omega_k^{-1}\sigma_\varepsilon(y))^{k/2} + |z|^2)^{-(n+2s)/2}\chi_{B_\infty^\varepsilon}(y), \tag{4-8}$$

with $\sigma_\varepsilon : B_\infty^\varepsilon \to [0,\infty)$ measure-preserving, depending on v_ε, such that

$$\inf_{K\in\mathcal{K}_k^s}\int_{\mathbb{R}^{n-k}}\int_{\mathbb{R}^k} v_\varepsilon(y,z)K(y,z)\,dy\,dz = \int_{\mathbb{R}^{n-k}}\int_{\mathbb{R}^k} v_\varepsilon(y,z)K_\varepsilon(y,z)\,dy\,dz. \tag{4-9}$$

Finally, we need to pass to the limit. First, we prove that $\{\sigma_\varepsilon\}_{\varepsilon>0}$ is monotone decreasing. Indeed, let $V_\varepsilon = \{y \in \mathbb{R}^k : v_\varepsilon(y,z) = \tilde{u}(y,z)\}$. In particular, $A_\infty \subseteq V_\varepsilon \subseteq A_\infty \cup A$. Also, $V_{\varepsilon_2} \subseteq V_{\varepsilon_1}$ for any $\varepsilon_1 \le \varepsilon_2$. By Ryff's theorem, recall that

$$v_{\varepsilon_1}(y,z) = (v_{\varepsilon_1})_*(\sigma_{\varepsilon_1}(y)) \quad \text{and} \quad v_{\varepsilon_2}(y,z) = (v_{\varepsilon_2})_*(\sigma_{\varepsilon_2}(y)).$$

Since $v_{\varepsilon_2}(y,z) = v_{\varepsilon_1}(y,z)$ for all $y \in V_{\varepsilon_2}$ and $v_{\varepsilon_1}(y,z) \le v_{\varepsilon_2}(y,z)$ for all $y \in \mathbb{R}^k$, we see that

$$(v_{\varepsilon_2})_*(\sigma_{\varepsilon_2}(y)) = (v_{\varepsilon_1})_*(\sigma_{\varepsilon_1}(y)) \le (v_{\varepsilon_2})_*(\sigma_{\varepsilon_1}(y)) \quad \text{for all } \ y \in V_{\varepsilon_2}.$$

Since $(v_{\varepsilon_2})_*$ is monotone increasing, we must have that $\sigma_{\varepsilon_2}(y) \le \sigma_{\varepsilon_1}(y)$ for all $y \in V_{\varepsilon_2}$. Hence there exists $\sigma_0 : B_\infty \to [0,\infty)$ measure-preserving such that

$$\sigma_0(y) = \lim_{\varepsilon\to 0} \sigma_\varepsilon(y),$$

where $B_\infty = \bigcap_{\varepsilon>0} B_\infty^\varepsilon = \{y \in \mathbb{R}^k : \tilde{u}(y,z) \le \lambda_0\} = A_\infty \cup A$. In particular, the sequence of kernels $\{K_\varepsilon\}_{\varepsilon>0}$ is monotone decreasing. Define

$$K_0(y,z) = \lim_{\varepsilon\to 0} K_\varepsilon(y,z). \tag{4-10}$$

By (4-8) and (4-10), we have

$$K_0(y,z) = ((\omega_k^{-1}\sigma_0(y))^{k/2} + |z|^2)^{-(n+2s)/2} \chi_{B_\infty}(y).$$

Moreover, $K_0 \in \mathcal{K}_k^s$ since $K_\varepsilon \in \mathcal{K}_k^s$, and, for any $r > 0$, it follows that

$$^k(D_0(r)) = \lim_{\varepsilon\to 0} \mathcal{H}^k(D_\varepsilon(r)),$$

where $D_\varepsilon(r) = \{y \in \mathbb{R}^k : K_\varepsilon(y,z) > r^{-(n+2s)}\}$.

Finally, using (4-5), (4-9), (4-10), and the monotone convergence theorem, we get

$$\begin{aligned}
\int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y,z) K_0(y,z)\,dy\,dz &= \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \lim_{\varepsilon\to 0} (v_\varepsilon(y,z) K_\varepsilon(y,z))\,dy\,dz \\
&= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} v_\varepsilon(y,z) K_\varepsilon(y,z)\,dy\,dz \\
&= \lim_{\varepsilon\to 0} \inf_{K\in\mathcal{K}_k^s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} v_\varepsilon(y,z) K(y,z)\,dy\,dz \\
&\le \inf_{K\in\mathcal{K}_k^s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \big(\lim_{\varepsilon\to 0} v_\varepsilon(y,z)\big) K(y,z)\,dy\,dz \\
&= \inf_{K\in\mathcal{K}_k^s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y,z)(K(y,z)\chi_{A_\infty\cup A}(y))\,dy\,dz \\
&= \inf_{K\in\mathcal{K}_k^s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y,z) K(y,z)\,dy\,dz.
\end{aligned}$$

The last equality follows from the observation that, since

$$\widetilde{\mathcal{K}}_k^s = \{K \in \mathcal{K}_k^s : \operatorname{supp} K(\,\cdot\,,z) \subseteq A_\infty \cup A\} \subseteq \mathcal{K}_k^s,$$

the infimum over all kernels in $\mathcal{K}_k^s$ is less than or equal to the infimum over $\widetilde{\mathcal{K}}_k^s$. Moreover, the reverse inequality holds trivially. $\square$

Finally, we deal with Case 3, that is, when all of the level sets of $\tilde{u}$ have infinite measure. In particular, notice that

$$\tilde{u}_{*,k}(x) = 0 \quad \text{for all } x \in \mathbb{R}^n.$$

This is the only case where the infimum is not attained. Indeed, we prove in the following theorem that the infimum is equal to zero.

Theorem 4.8. *Suppose that, for all $\lambda > 0$ and $z \in \mathbb{R}^{n-k}$,*

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda\}) = \infty.$$

Then $\mathcal{F}_k^s u(x_0) = 0$.

Proof. From (P_2), we have that $\mathcal{F}_k^s u(x_0) \geq 0$. To prove the reverse inequality, it is enough to find a sequence of kernels $\{K_\varepsilon\}_{\varepsilon>0} \subset \mathcal{K}_k^s$ such that

$$\liminf_{\varepsilon \to 0} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y, z) K_\varepsilon(y, z)\, dy\, dz = 0. \tag{4-11}$$

Fix $\varepsilon > 0$ and $z \in \mathbb{R}^{n-k}$. For any $j \geq 0$, we define the set

$$U_j(z) = \{y \in \mathbb{R}^k : \tilde{u}(y, z) < \varepsilon 2^{-j(n+2s)} e^{-|z|^2}\}.$$

Note that $U_{j+1} \subseteq U_j$. Also, by assumption, with $\lambda = \varepsilon 2^{-j(1+2s)} e^{-|z|^2}$, we have that

$$\mathcal{H}^k(U_j) = \infty \quad \text{for all } j \geq 0.$$

We will construct $K_\varepsilon \in \mathcal{K}_k^s$ by describing first where to locate each level set of the form

$$\begin{aligned} A_{-1} \equiv A_{-1}(z) &= \{y \in \mathbb{R}^k : 0 < K_\varepsilon(y, z) \leq 1\}, \\ A_j \equiv A_j(z) &= \{y \in \mathbb{R}^k : 2^{j(n+2s)} < K_\varepsilon(y, z) \leq 2^{(j+1)(n+2s)}\} \quad \text{for } j \geq 0. \end{aligned}$$

Recall that $K \in \mathcal{K}_k^s$ if, for all $r > 0$, we have

$$\mathcal{H}^k(\{y \in \mathbb{R}^k : K(y, z) > r^{-(n+2s)}\}) = \mathcal{H}^k(\{y \in \mathbb{R}^k : (|y|^2 + |z|^2)^{-(n+2s)/2} > r^{-(n+2s)}\}).$$

In view of this definition, we define the sets

$$\begin{aligned} B_{-1} \equiv B_{-1}(z) &= \{y \in \mathbb{R}^k : 0 < (|y|^2 + |z|^2)^{-(n+2s)/2} \leq 1\}, \\ B_j \equiv B_j(z) &= \{y \in \mathbb{R}^k : 2^{j(n+2s)} < (|y|^2 + |z|^2)^{-(n+2s)/2} \leq 2^{(j+1)(n+2s)}\} \quad \text{for } j \geq 0. \end{aligned}$$

Note that

$$\begin{cases} \mathcal{H}^k(A_{-1}) = \mathcal{H}^k(B_{-1}) = \infty, \\ \mathcal{H}^k(A_j) = \mathcal{H}^k(B_j) < \infty \quad \text{for all } j \geq 0. \end{cases}$$

More precisely, for $j \geq 0$, if $|z| < 2^{-(j+1)} < 2^{-j}$, then

$$\mathcal{H}^k(A_j) = \mathcal{H}^k(B_{(2^{-2j} - |z|^2)^{1/2}}) - \mathcal{H}^k(B_{(2^{-2(j+1)} - |z|^2)^{1/2}}) = \omega_k (2^{-2j} - |z|^2)^{k/2} - \omega_k (2^{-2(j+1)} - |z|^2)^{k/2} \leq \omega_k 2^{-kj}.$$

If $2^{-(j+1)} \leq |z| < 2^{-j}$, then

$$\mathcal{H}^k(A_j) = \mathcal{H}^k(B_{(2^{-2j} - |z|^2)^{1/2}}) = \omega_k (2^{-2j} - |z|^2)^{k/2} \leq \omega_k \left(\tfrac{3}{4}\right)^{k/2} 2^{-kj}.$$

If $|z| \geq 2^{-j} > 2^{-(j+1)}$, then

$$\mathcal{H}^k(A_j) = 0.$$

Therefore, $\mathcal{H}^k(A_j) \leq c2^{-kj}$, where $c > 0$ only depends on k. It follows that

$$\mathcal{H}^k\left(\bigcup_{j=0}^{\infty} A_j\right) = \sum_{j=0}^{\infty} \mathcal{H}^k(A_j) \leq c \sum_{j=0}^{\infty} 2^{-jk} < \infty. \tag{4-12}$$

For any $i \geq 0$, let $\mathcal{D}_i$ be the collection of all dyadic closed cubes of the form

$$[m2^{-i}, (m+1)2^{-i}]^k = [m2^{-i}, (m+1)2^{-i}] \times \cdots \times [m2^{-i}, (m+1)2^{-i}].$$

Note that if $Q \in \mathcal{D}_i$, then $l(Q) = 2^{-i}$, where $l(Q)$ denotes the side length of the cube Q. For any $j \geq 0$, since U_j is an open set, by a standard covering argument, we have that there exists a family of dyadic cubes $\mathcal{F}_j$ such that

$$U_j = \bigcup_{Q \in \mathcal{F}_j} Q$$

satisfying the following properties:

(1) For any $Q \in \mathcal{F}_j$, there exists some $i \geq 0$ such that $Q \in \mathcal{D}_i$.

(2) $\mathrm{Int}(Q) \cap \mathrm{Int}(\widetilde{Q})\,\mathrm{Int}(\widetilde{Q}) = \varnothing$ for any $Q, \widetilde{Q} \in \mathcal{F}_j$ with $Q \neq \widetilde{Q}$.

(3) If $x \in Q \in \mathcal{F}_j$, then Q is the maximal dyadic cube contained in U_j that contains x.

Analogously, for the sets B_j with $j \geq -1$, there exists a family of dyadic cubes $\widetilde{\mathcal{F}}_j$ satisfying properties (1)–(3) such that

$$\mathrm{Int}(B_j) = \bigcup_{Q \in \widetilde{\mathcal{F}}_j} Q.$$

Note that $\widetilde{\mathcal{F}}_j \cap \widetilde{\mathcal{F}}_{j+1} = \varnothing$ since $B_j \cap B_{j+1} = \varnothing$.

We will construct the sets A_j by properly translating the dyadic cubes partitioning the sets B_j into U_j. In particular, we will prove that

$$\begin{cases} A_0 = T_0(B_0) \subset U_0, \\ A_j = T_j(B_j) \subset U_j \setminus \bigcup_{i=0}^{j-1} A_i \quad \text{for all } j \geq 1, \\ A_{-1} = T_{-1}(B_{-1}) \subset U_0 \setminus \bigcup_{i=0}^{\infty} A_i, \end{cases}$$

for some *translation* mappings $T_j : \widetilde{\mathcal{F}}_j \to \mathcal{F}_j$ to be determined.

We start with the case $j = 0$. For any $i \geq 0$, write

$$m_i = \mathcal{H}^0(\mathcal{F}_0 \cap \mathcal{D}_i) \quad \text{and} \quad n_i = \mathcal{H}^0(\widetilde{\mathcal{F}}_0 \cap \mathcal{D}_i),$$

where $\mathcal{H}^0(E)$ is equal to the cardinal of the set E. Note that $m_i, n_i \in \mathbb{Z}^+ \cup \{\infty\}$.

We will recursively place B_0 into U_0. First, fix $i = 0$. If $m_0 \geq n_0$, then, for any $\widetilde{Q} \in \widetilde{\mathcal{F}}_0 \cap \mathcal{D}_0$, there exists some $\tau \in \mathbb{R}^k$ and some $Q \in \mathcal{F}_0 \cap \mathcal{D}_0$ such that $Q = \widetilde{Q} + \tau$. Then define

$$T_0 : \widetilde{\mathcal{F}}_0 \cap \mathcal{D}_0 \to \mathcal{F}_0 \cap \mathcal{D}_0, \quad T_0(\widetilde{Q}) = Q. \tag{4-13}$$

Moreover, we can define T_0 to be one-to-one since $m_0 \geq n_0$, and we can always choose a different Q for each $\widetilde{Q}$. Note that there are p_0 cubes in $\mathcal{F}_0 \cap D_0$ with $p_0 = m_0 - n_0$ that have not been used. Hence for all of these cubes, divide each side in half, so that each cube gives rise to 2^k cubes with side length 2^{-1}. Call this collection of new cubes $\mathcal{Q} = \{Q_l\}_{l=1}^{2^{kp_0}} \subset \mathcal{D}_1$ and add them to the family $\mathcal{F}_0 \cap \mathcal{D}_1$. Namely, we replace $\mathcal{F}_0 \cap \mathcal{D}_1$ by $(\mathcal{F}_0 \cap \mathcal{D}_1) \cup \mathcal{Q}$.

If $m_0 < n_0$, then take q_0 cubes in $\widetilde{\mathcal{F}}_0 \cap \mathcal{D}_0$ with $q_0 = n_0 - m_0$ and divide each side in half. Call this collection of new cubes $\widetilde{\mathcal{Q}} = \{\widetilde{Q}_l\}_{l=1}^{2^{kq_0}} \subset \mathcal{D}_1$. Then, we replace $\widetilde{\mathcal{F}}_0$ by $\widehat{\mathcal{F}}_0$, where

$$\begin{aligned}
\widehat{\mathcal{F}}_0 \cap \mathcal{D}_0 &= (\widetilde{\mathcal{F}}_0 \setminus \widetilde{\mathcal{Q}}) \cap \mathcal{D}_0, \\
\widehat{\mathcal{F}}_0 \cap \mathcal{D}_1 &= (\widetilde{\mathcal{F}}_0 \cup \widetilde{\mathcal{Q}}) \cap \mathcal{D}_1, \\
\widehat{\mathcal{F}}_0 \cap \mathcal{D}_i &= \widetilde{\mathcal{F}}_0 \cap \mathcal{D}_i \quad \text{for all } i \geq 2.
\end{aligned}$$

If $\hat{n}_0 = \mathcal{H}^0(\widehat{\mathcal{F}} \cap \mathcal{D}_0)$, then $m_0 = \hat{n}_0$. Hence, by the same argument as in the previous case, we find T_0 as in (4-13). For $i \geq 1$, we can repeat the same process until we run out of cubes from $\widetilde{\mathcal{F}}_0$ (or the modified family). We know the process will end since $\mathcal{H}^k(B_0) < \mathcal{H}^k(U_0)$. When this happens, we will have constructed a one-to-one mapping $T_0 : \widetilde{\mathcal{F}}_0 \to \mathcal{F}_0$, since $\widetilde{\mathcal{F}}_0 = \bigcup_{i=0}^{\infty} \widetilde{\mathcal{F}}_0 \cap \mathcal{D}_i$ and $\mathcal{F}_0 = \bigcup_{i=0}^{\infty} \mathcal{F}_0 \cap \mathcal{D}_i$. Then define

$$A_0 = T_0(B_0) \subset U_0.$$

Iterating this process, we find a sequence of translation mappings $\{T_j\}_{j=0}^{\infty}$ with $T_j : \widetilde{F}_j \to \mathcal{F}_j$ and a sequence of disjoint sets $\{A_j\}_{j=0}^{\infty}$ such that

$$A_j = T_j(B_j) \subset U_j \setminus \bigcup_{i=0}^{j-1} A_i.$$

The case $j = -1$ is somewhat special since $\mathcal{H}^k(A_{-1}) = \mathcal{H}^k(B_{-1}) = \infty$. We will see that

$$A_{-1} = T_{-1}(B_{-1}) \subset U_0 \setminus \bigcup_{i=0}^{\infty} A_i.$$

This is possible because $\mathcal{H}^k\big(U_0 \setminus \bigcup_{i=0}^{\infty} A_i\big) = \infty$ using (4-12). Indeed, we can write

$$\{y \in \mathbb{R}^k : 0 < K_\varepsilon(y,z) \leq 1\} = \bigcup_{j=0}^{\infty} \{2^{-(j+1)(n+2s)} < K_\varepsilon(y,z) \leq 2^{-j(n+2s)}\}.$$

Now write

$$C_j = \{2^{-(j+1)(n+2s)} < (|y|^2 + |z|^2)^{-(n+2s)/2} \leq 2^{-j(n+2s)}\} \quad \text{for } j \geq 0.$$

Then $B_{-1} = \bigcup_{j=0}^{\infty} C_j$ with $\mathcal{H}^k(C_j) < \infty$ for all $j \geq 0$. Hence, instead of partitioning all of B_{-1} into dyadic cubes, we partition each of its disjoint components C_j. Arguing as before, we place them into $U_0 \setminus \bigcup_{i=0}^{\infty} A_i$ recursively, according to the following scheme:

$$\begin{cases}
T_{-1}^0(C_0) \subset U_0 \setminus \bigcup_{i=0}^{\infty} A_i, \\
T_{-1}^j(C_j) \subset U_0 \setminus \big(\bigcup_{i=0}^{\infty} A_i \cup \bigcup_{i=0}^{j-1} C_i\big) \quad \text{for } j \geq 1,
\end{cases}$$

where T_{-1}^j is defined as before. At the end of this process, we find a translation map T_{-1} defined by $T_{-1}(Q) = T_{-1}^j(Q)$ for $Q \in C_j$. Therefore, we define

$$A_{-1} = T_{-1}(B_{-1}).$$

Lastly, let $y \in \mathbb{R}^k = A_{-1} \cup \left(\bigcup_{j=0}^{\infty} A_j\right)$. In particular, there exists some $j \geq -1$ such that $y \in A_j$. Furthermore, recall that $A_j = T_j(B_j)$, where T_j is a one-to-one and onto translation map. Hence there exists a unique $w \in B_j$ such that $y = T_j(w) = w + \tau$ for some $\tau \in \mathbb{R}^k$. Let $T_z : \mathbb{R}^k \to \mathbb{R}^k$ be given by $T_z(y) = w$. Note that T_z is measure-preserving. Then we define the kernel

$$K_\varepsilon(y, z) = (|T_z(y)|^2 + |z|^2)^{-(n+2s)/2}.$$

We have

$$\int_{\mathbb{R}^k} \tilde{u}(y, z) K_\varepsilon(y, z)\, dy = \int_{A_{-1}} \tilde{u}(y, z) K_\varepsilon(y, z)\, dy + \sum_{j=0}^{\infty} \int_{A_j} \tilde{u}(y, z) K_\varepsilon(y, z)\, dy \equiv \mathrm{I} + \mathrm{II}.$$

For I, we use that $\tilde{u}(y, z) \leq \varepsilon e^{-|z|^2}$, since $A_{-1} \subset U_0$. Then by Lemmas 3.7 and 3.4,

$$\begin{aligned} \mathrm{I} \leq \varepsilon e^{-|z|^2} \int_{\{0 < K_\varepsilon(y,z) \leq 1\}} K_\varepsilon(y, z)\, dy &= \varepsilon e^{-|z|^2} \int_{\{0 < |\sigma_z(y)|^{-n-2s} \leq 1\}} |\sigma_z(y)|^{-n-2s}\, dy \\ &= \varepsilon e^{-|z|^2} \int_{\{|y| \geq 1\}} |y|^{-n-2s}\, dy = C\varepsilon e^{-|z|^2}, \end{aligned}$$

where $C > 0$ depends only on n and s. For II, we use that $\tilde{u}(y, z) \leq \varepsilon 2^{-j(n+2s)} e^{-|z|^2}$, since $A_j \subset U_j$ and $K_\varepsilon(y, z) \leq 2^{(j+1)(n+2s)}$ in A_j, by definition. Then

$$\mathrm{II} \leq \varepsilon e^{-|z|^2} \sum_{j=0}^{\infty} 2^{-j(n+2s)} 2^{(j+1)(n+2s)} \mathcal{H}^k(A_j) \leq c\varepsilon e^{-|z|^2} 2^{n+2s} \sum_{j=0}^{\infty} 2^{-kj} \leq C\varepsilon e^{-|z|^2},$$

where $C > 0$ depends only on n, s, and k.

Integrating over z, we see that

$$\int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y, z) K_\varepsilon(y, z)\, dy\, dz \leq C\varepsilon \int_{\mathbb{R}^{n-k}} e^{-|z|^2}\, dz \leq \widetilde{C}\varepsilon.$$

Letting $\varepsilon \to 0$, we conclude (4-11). □

Limit as s → 1. Let $u \in C^2(\mathbb{R}^n)$. We define $\mathrm{MA}_k u$ as the Monge–Ampère operator acting on u with respect to the first k variables, that is,

$$\mathrm{MA}_k u(x) = k(\det((u_{ij}(x))_{1 \leq i,j \leq k}))^{1/k},$$

with $D^2 u(x) = (u_{ij}(x))_{1 \leq i,j \leq n}$. We define $\Delta_{n-k} u$ as the Laplacian of u with respect to the last $n - k$ variables, that is,

$$\Delta_{n-k} u(x) = \sum_{i=k+1}^{n} u_{ii}(x).$$

Then under some *special* conditions,

$$\lim_{s\to 1}\mathcal{F}_k^s u(x)=\mathrm{MA}_k u(x)+\Delta_{n-k}u(x). \tag{4-14}$$

In particular, the operators in the family $\{\mathcal{F}_k^s\}_{k=1}^{n-1}$ can be understood as nonlocal analogs of concave second order elliptic operators, which are decomposed into a Monge–Ampère operator restricted to $\mathbb{R}^k$ and a Laplacian restricted to $\mathbb{R}^{n-k}$.

Indeed, by Corollary 4.5, we have $\mathcal{F}_k^s u(x)=\Delta^s\tilde{u}_{*,k}(0)$. Since the k-symmetric rearrangement does not depend on s and $\Delta^s\to\Delta$ as $s\to 1$, passing to the limit we see that

$$\lim_{s\to 1}\mathcal{F}_k^s u(x)=\Delta\tilde{u}_{*,k}(0).$$

Suppose that $\tilde{u}_{*,k}(y,z)=\tilde{u}(\varphi_z^{-1}(y),z)$, where $\varphi_z:\mathbb{R}^k\to\mathbb{R}^k$ is an invertible measure-preserving transformation with $\varphi_z(0)=0$ and

$$\omega_k|\varphi_z(y)|^{1/k}=\sigma_z(y).$$

Recall that σ_z is given in Theorem 4.1 (see also Remark 4.4). In this case,

$$\Delta\tilde{u}_{*,k}(0)=\Delta_y\tilde{u}(\varphi_z^{-1}(y),z)+\Delta_z\tilde{u}(\varphi_z^{-1}(y),z)|_{(y,z)=(0,0)}. \tag{4-15}$$

For the first term, we use

$$\mathrm{MA}_k u(x)=\inf_{\psi\in\Psi}\Delta(\tilde{u}\circ\psi)(0),$$

where $\Psi=\{\psi:\mathbb{R}^k\to\mathbb{R}^k$ measure-preserving such that $\psi(0)=0\}$, and the fact that the infimum is attained when $\tilde{u}\circ\psi$ is a radially symmetric increasing function [Caffarelli and Silvestre 2016]. Hence

$$\Delta_y\tilde{u}(\varphi_z^{-1}(y),z)|_{(y,z)=(0,0)}=\mathrm{MA}_k u(x). \tag{4-16}$$

For the second term, write $\phi(y,z)=(\varphi_z^{-1}(y),z)$ and compute

$$\Delta_z(\tilde{u}\circ\phi)(0)=\mathrm{tr}(D_z\phi(0)^T D_z^2\tilde{u}(\phi(0))D_z\phi(0))+\nabla_z\tilde{u}(\phi(0))^T\cdot\Delta_z\phi(0).$$

Recall that $\phi(0)=0$ and $\tilde{u}(y,z)=u(x+(y,z))-u(x)-\nabla_y u(x)\cdot y-\nabla_z u(x)\cdot z$. Then

$$\nabla_z\tilde{u}(\phi(0))=0,\quad D_z^2\tilde{u}(\phi(0))=D_z^2u(x),\quad\text{and}\quad D_z\phi(0)=(0,I_{n-k}),$$

where I_{n-k} denotes the identity matrix in M_{n-k}. Therefore,

$$\Delta_z\tilde{u}(\varphi_z^{-1}(y),z)|_{(y,z)=(0,0)}=\Delta_z(\tilde{u}\circ\phi)(0)=\mathrm{tr}(D_z^2u(x))=\Delta_{n-k}u(x). \tag{4-17}$$

Combining (4-15)–(4-17) we conclude (4-14).

Connection to optimal transport. In Corollary 4.5 we obtained a representation of the function $\mathcal{F}_k^s u$ in terms of the k-symmetric increasing rearrangement. Using this representation, we find an equivalent expression of $\mathcal{F}_k^s u$ that can be understood from the viewpoint of optimal transport.

Theorem 4.9. *Suppose we are under the assumptions of Theorem 4.1. Then, for any $z \in \mathbb{R}^{n-k}$, $z \neq 0$, there exists an invertible map $\varphi_z : \mathbb{R}^k \to \mathbb{R}^k$ such that*

$$\mathcal{F}_k^s u(x) = c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \frac{\tilde{u}(\varphi_z^{-1}(y), z)}{(|y|^2 + |z|^2)^{(n+2s)/2}} \, dy \, dz. \tag{4-18}$$

Moreover, if $\sigma_z : \mathbb{R}^k \to [0, \infty)$ is the Ryff's map given in Theorem 4.1, then φ_z is measure-preserving if and only if

$$\omega_k |\varphi_z(y)|^k = \sigma_z(y) \quad \textit{for a.e. } y \in \mathbb{R}^k. \tag{4-19}$$

The key tool to prove Theorem 4.9 is Brenier–McCann's theorem, a very well-known result in the theory of optimal transport [Brenier 1991; McCann 1995]. We state it here in the form that we will use it.

Theorem 4.10. *Let $f, g \in L^1(\mathbb{R}^k)$. Assume that*

$$\|f\|_{L^1(\mathbb{R}^k)} = \|g\|_{L^1(\mathbb{R}^k)}.$$

Then there exists a convex function $\psi : \mathbb{R}^k \to \mathbb{R}$ whose gradient $\nabla\psi$ pushes forward $f\,dy$ to $g\,dy$. Namely, for any measurable function h in $\mathbb{R}^k$,

$$\int_{\mathbb{R}^k} h(y) g(y) \, dy = \int_{\mathbb{R}^k} h(\nabla\psi(y)) f(y) \, dy. \tag{4-20}$$

Moreover, $\nabla\psi : \mathbb{R}^k \to \mathbb{R}^k$ is invertible and unique.

In the literature, $\nabla\psi$ is known as the (optimal) transport map.

Proof of Theorem 4.9. Fix $z \in \mathbb{R}^{n-k}$, $z \neq 0$, and consider $f_z, g_z \in L^1(\mathbb{R}^k)$ given by

$$f_z(y) = (|y|^2 + |z|^2)^{-(n+2s)/2} \quad \text{and} \quad g_z(y) = ((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2)^{-(n+2s)/2},$$

where $\sigma_z : \mathbb{R}^k \to [0, \infty)$ is given in Theorem 4.1. Note that

$$\begin{aligned} \|f\|_{L^1(\mathbb{R}^k)} &= \int_{\mathbb{R}^k} ((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2)^{-(n+2s)/2} \, dy \\ &= k\omega_k \int_0^\infty (r^2 + |z|^2)^{-(n+2s)/2} r^{k-1} \, dr \\ &= \int_{\mathbb{R}^k} (|y|^2 + |z|^2)^{-(n+2s)/2} \, dy = \|g\|_{L^1(\mathbb{R}^k)}, \end{aligned}$$

since σ_z is measure-preserving. By Theorem 4.10, there exists a convex function $\psi_z : \mathbb{R}^k \to \mathbb{R}$ (depending on z) whose gradient $\nabla\psi_z$ pushes forward $f_z\,dy$ to $g_z\,dy$. Moreover, $\nabla\psi_z$ is invertible and unique. Write $\varphi_z = (\nabla\psi_z)^{-1}$. Using (4-20) with $h(y) = \tilde{u}(y, z)$, we see that

$$\int_{\mathbb{R}^k} \frac{\tilde{u}(y, z)}{((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2)^{(n+2s)/2}} \, dy = \int_{\mathbb{R}^k} \frac{\tilde{u}(\varphi_z^{-1}(y), z)}{(|y|^2 + |z|^2)^{(n+2s)/2}} \, dy. \tag{4-21}$$

Integrating over $z \in \mathbb{R}^{n-k}$, we obtain (4-18).

It remains to show that φ_z is measure-preserving if and only if (4-19) holds. Indeed, for any measurable set $E \subset \mathbb{R}^k$, we have

$$\begin{aligned}\mathcal{H}^k(\varphi_z^{-1}(E)) = \int_{\varphi_z^{-1}(E)} dy &= \int_{\varphi_z^{-1}(E)} \frac{(|y|^2+|z|^2)^{(n+2s)/2}}{(|y|^2+|z|^2)^{(n+2s)/2}}\, dy \\ &= \int_{\varphi_z^{-1}(E)} \frac{(|\varphi_z(\varphi_z^{-1}(y))|^2+|z|^2)^{(n+2s)/2}}{(|y|^2+|z|^2)^{(n+2s)/2}}\, dy \\ &= \int_E \frac{(|\varphi_z(y)|^2+|z|^2)^{(n+2s)/2}}{((\omega_k^{-1}\sigma_z(y))^{2/k}+|z|^2)^{(n+2s)/2}}\, dy,\end{aligned}$$

where the last equality follows from (4-21) with $h(y) = (|\varphi_z(y)|^2+|z|^2)^{(n+2s)/2}\chi_E(y)$. Therefore,

$$\mathcal{H}^k(\varphi_z^{-1}(E)) = \mathcal{H}^k(E)$$

if and only if $\omega_k|\varphi_z(y)|^k = \sigma_z(y)$ for a.e. $y \in \mathbb{R}^k$. □

5. Regularity of $\mathcal{F}_k^s u$

Given $x_0 \in \mathbb{R}^n$, we define the sections

$$D_{x_0}u(t) = \{x \in \mathbb{R}^n : u(x) - u(x_0) - (x-x_0)\cdot\nabla u(x_0) \le t\} \quad \text{for } t > 0.$$

Our main regularity result is the following.

Theorem 5.1. *Let $s \in \left(\frac{1}{2}, 1\right)$ and $1 \le k < n$. Let $u \in C^{1,1}(\mathbb{R}^n)$ be convex. Fix $x_0 \in \mathbb{R}^n$ and $r_0,\ \varepsilon > 0$. Suppose that $\Lambda = \sup_{x\in B_{r_0}(x_0)} \operatorname{diam}(D_x u(\varepsilon)) < \infty$ and $M = \sup_{x\in B_{r_0}(x_0)} \mathcal{F}_k^s u(x) < \infty$. Then we have $\mathcal{F}_k^s u \in C^{0,1-s}(\overline{B_r(x_0)})$ with $r < \min\{r_0/4, \Lambda, \varepsilon/(8\Lambda)\}$ and*

$$[\mathcal{F}_k^s]_{C^{0,1-s}(\overline{B_r(x_0)})} \le C_0[u]_{C^{1,1}(\mathbb{R}^n)}$$

for some constant $C_0 > 0$ depending only on $n,\ k,\ s,\ \varepsilon,\ \Lambda$, and M.

This theorem will be a consequence of the next proposition.

Proposition 5.2. *Fix $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$. Suppose that $\Lambda = \operatorname{diam}(D_{x_0}u(\varepsilon)) < \infty$ and $[u]_{C^{1,1}(\mathbb{R}^n)} \le 1$. Then, for any $x_1 \in B_r(x_0)$ with $r \le \varepsilon/(4\Lambda)$, we have*

$$\mathcal{F}_k^s u(x_1) - \mathcal{F}_k^s u(x_0) \le C\Lambda^{1-s}|x_1-x_0|^{1-s} + \frac{4\Lambda}{\varepsilon}|x_1-x_0|\mathcal{F}_k^s u(x_0)$$

for some $C > 0$ depending only on $n,\ k$, and s.

First, we prove Theorem 5.1.

Proof of Theorem 5.1. Without loss of generality, we may assume that $[u]_{C^{1,1}(\mathbb{R}^n)} \le 1$. Otherwise, we consider $u/[u]_{C^{1,1}(\mathbb{R}^n)}$. Let $r < \min\{r_0/4, \Lambda, \varepsilon/(8\Lambda)\}$. It is enough to show that

$$[\mathcal{F}_k^s]_{C^{0,1-s}(\overline{B_r(x_0)})} \le C_0 \tag{5-1}$$

for some constant $C_0 > 0$ depending only on $n,\ k,\ s,\ \varepsilon,\ \Lambda$, and M.

Let $x_1,\ x_2 \in \overline{B_r(x_0)}$. Then $x_2 \in \overline{B_{2r}(x_1)} \subset B_{r_0}(x_0)$, since $4r < r_0$. Moreover, $\operatorname{diam}(D_{x_1}u(\varepsilon)) \le \Lambda < \infty$. Hence, applying Proposition 5.2 to u and $B_{2r}(x_1)$ in place of $B_r(x_0)$, we get

$$\mathcal{F}_k^s u(x_2) - \mathcal{F}_k^s u(x_1) \le C\Lambda^{1-s}|x_2 - x_1|^{1-s} + \frac{4\Lambda}{\varepsilon}|x_2 - x_1|\mathcal{F}_k^s u(x_1) \le C_0|x_2 - x_1|^{1-s},$$

where $C_0 = C\Lambda^{1-s} + 4\Lambda^{1+s}M/(\varepsilon 2^s)$. Since x_1 and x_2 are arbitrary, we conclude (5-1). □

Before we prove Proposition 5.2, we need several preliminary results.

Lemma 5.3. *If f is monotone increasing, then*

$$\int_0^\infty f(r)\omega(r)\,dr = \int_0^\infty \int_{\mu_f(t)}^\infty \omega(r)\,dr\,dt,$$

with $\mu_f(t) = |\{r > 0 : f(r) \le t\}|$.

Proof. By Fubini's theorem, we have

$$\int_0^\infty \int_{\mu_f(t)}^\infty \omega(r)\,dr\,dt = \int_0^\infty \omega(r) \int_{\{r>\mu_f(t)\}} dt\,dr.$$

Since f is monotone increasing, $r > \mu_f(t)$ if and only if $t < f(r)$. Therefore,

$$\int_{\{r>\mu_f(t)\}} dt = \int_0^{f(r)} dt = f(r). \qquad \square$$

Proposition 5.4. *Let $x \in \mathbb{R}^n$. Under the assumptions of Corollary 4.5,*

$$\mathcal{F}_k^s u(x) = c_{n,s} \int_0^\infty \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_x u(t,z)^{1/k}}{|z|}\right) dz\,dt,$$

where $\mu_x u(t,z) = \omega_k^{-1}\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}_x(y,z) \le t\})$ and

$$W(\rho) = k\omega_k \int_\rho^\infty \frac{r^{k-1}}{(1+r^2)^{(n+2s)/2}}\,dr. \tag{5-2}$$

Proof. By Corollary 4.5, we have that

$$\begin{aligned}\mathcal{F}_k^s u(x) = \Delta^s \tilde{u}_{*,k}(0) &= c_{n,s} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n+2s}} \left(\int_{\mathbb{R}^k} \frac{\tilde{u}_{*,k}(y,z)}{(||z|^{-1}y|^2+1)^{(n+2s)/2}}\,dy \right) dz \\ &= c_{n,s} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} \left(k\omega_k \int_0^\infty v(|z|r,z) \frac{r^{k-1}}{(r^2+1)^{(n+2s)/2}}\,dr \right) dz,\end{aligned}$$

where $v(r,z) = \tilde{u}_{*,k}(y,z)$ for $|y| = r$.

Next we apply Lemma 5.3 to $f(r) = v(|z|r,z)$ and $\omega(r) = k\omega_k r^{k-1}(r^2+1)^{-(n+2s)/2}$. Note that since v is the k-symmetric increasing rearrangement of $\tilde{u}$, we have

$$\mu_f(t) = \frac{1}{|z|}|\{r > 0 : v(r,z) < t\}| = \frac{\omega_k^{-1/k}}{|z|}\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}(y,z) < t\})^{1/k} = \frac{1}{|z|}\mu_x u(t,z)^{1/k}.$$

Therefore,

$$k\omega_k \int_0^\infty v(|z|r, z)\frac{r^{k-1}}{(r^2+1)^{(n+2s)/2}}\,dr = \int_0^\infty \left(k\omega_k \int_{\mu_x u(t,z)^{1/k}/|z|}^\infty \frac{r^{k-1}}{(r^2+1)^{(n+2s)/2}}\,dr\right)dt$$
$$= \int_0^\infty W\left(\frac{\mu_x u(t,z)^{1/k}}{|z|}\right)dt,$$

where W is given in (5-2). By Fubini's theorem, we conclude that

$$\mathcal{F}_k^s u(x) = c_{n,s}\int_0^\infty\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_x u(t,z)^{1/k}}{|z|}\right)dz\,dt. \qquad \square$$

Lemma 5.5. *Suppose we are under the assumptions of Proposition 5.2. Let $x_1 \in \overline{B_r(x_0)}$ and $d = |x_1 - x_0|$. The following hold*:

(a) *If $t \in (2\Lambda d, \varepsilon]$, then $D_{x_0}u(t-2\Lambda d) \subset D_{x_1}u(t)$.*

(b) *If $t \in (\varepsilon, \infty)$, then $D_{x_0}u(t-2\Lambda dt/\varepsilon) \subset D_{x_1}u(t)$.*

Proof. First we prove (a). Fix $t \in (2\Lambda d, \varepsilon]$, and let $x \in D_{x_0}u(t-2\Lambda d)$. Then

$$u(x) - u(x_0) - (x-x_0)\cdot\nabla u(x_0) \le t - 2\Lambda d. \tag{5-3}$$

Using (5-3), convexity, and $[u]_{C^{1,1}(\mathbb{R}^n)} \le 1$, we see that

$$\begin{aligned} u(x)-u(x_1)-(x-x_1)\cdot\nabla u(x_1) &= u(x)-u(x_0)-(x-x_0)\cdot\nabla u(x_0)\\ &\quad - (u(x_1)-u(x_0)-(x_1-x_0)\cdot\nabla u(x_0))\\ &\quad + (x-x_1)\cdot(\nabla u(x_0)-\nabla u(x_1))\\ &\le t - 2\Lambda d + |x-x_1|d. \end{aligned}$$

Moreover, $x \in D_{x_0}u(\varepsilon)$, since $t \le \varepsilon$, and thus,

$$|x-x_1| \le |x-x_0| + |x_0-x_1| \le \Lambda + d \le 2\Lambda.$$

Therefore, $x \in D_{x_1}u(t)$.

Next we prove (b). Fix $t \in (\varepsilon, \infty)$, and let $x \in D_{x_0}u(t-2\Lambda dt/\varepsilon)$. By the previous computation, we have that

$$u(x)-u(x_1)-(x-x_1)\cdot\nabla u(x_1) \le t - 2\Lambda dt/\varepsilon + (|x-x_0|+\Lambda)d. \tag{5-4}$$

To control $|x-x_0|$, the distance from x to x_0, we need to estimate the diameter of $D_{x_0}u(t)$. We take $y \in D_{x_0}u(t) \setminus D_{x_0}u(\varepsilon)$ and let z be in the intersection of $\partial D_{x_0}u(\varepsilon)$ and the line segment joining x_0 and y. Then there is some $\lambda > 1$ such that $y - x_0 = \lambda(z-x_0)$. By convexity of u,

$$u(z) \le \frac{\lambda-1}{\lambda}u(x_0) + \frac{1}{\lambda}u(y).$$

Therefore,

$$\begin{aligned} \lambda\varepsilon &= \lambda(u(z)-u(x_0)-(z-x_0)\cdot\nabla u(x_0))\\ &\le (\lambda-1)u(x_0)+u(y)-\lambda u(x_0)-(y-x_0)\cdot\nabla u(x_0) = u(y)-u(x_0)-(y-x_0)\cdot\nabla u(x_0) \le t, \end{aligned}$$

so $\lambda \le t/\varepsilon$. By convexity, we have that $D_{x_0}u(t) \subset x_0 + (t/\varepsilon)(D_{x_0}u(\varepsilon) - x_0)$. It follows that

$$\operatorname{diam} D_{x_0}u(t) \le \frac{t}{\varepsilon} \operatorname{diam} D_{x_0}u(\varepsilon) = \frac{\Lambda t}{\varepsilon}.$$

Hence $|x - x_0| \le \Lambda t/\varepsilon$, and, by (5-4), we get

$$u(x) - u(x_1) - (x - x_1)\cdot \nabla u(x_1) \le t - \frac{2\Lambda d t}{\varepsilon} + \left(\frac{\Lambda t}{\varepsilon} + \Lambda\right) d \le t,$$

which means that $x \in D_{x_1}u(t)$. □

We are ready to give the proof of Proposition 5.2.

Proof of Proposition 5.2. Let $x_1 \in B_r(x_0)$ with $r \le \varepsilon/(4\Lambda)$, and write $d = |x_0 - x_1|$. We will estimate $\mathcal{F}_k^s u(x_1)$ using Proposition 5.4:

$$\mathcal{F}_k^s u(x_1) = c_{n,s} \int_0^\infty \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_1}u(t,z)^{1/k}}{|z|}\right) dz\, dt.$$

In view of Lemma 5.5, we separate the above integral into terms I + II + III by dividing the integral with respect to t into three parts as follows:

$$\mathrm{I}:\ t \in (0, 2\Lambda d], \qquad \mathrm{II}:\ t \in (2\Lambda d, \varepsilon], \qquad \mathrm{III}:\ t \in (\varepsilon, \infty).$$

Let us start with I. Since $u \in C^{1,1}(\mathbb{R}^n)$ with $[u]_{C^{1,1}(\mathbb{R}^n)} \le 1$, we have

$$\mu_{x_1}u(t,z) \ge (t - |z|^2)_+^{k/2}.$$

Hence, using that $W(\rho)$ is monotone decreasing, we get

$$W\left(\frac{\mu_{x_1}u(t,z)^{1/k}}{|z|}\right) \le W\left(\left(\frac{t}{|z|^2} - 1\right)_+^{1/2}\right).$$

Therefore,

$$\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_1}u(t,z)^{1/k}}{|z|}\right) dz \le \int_{\{|z|<t^{1/2}\}} \frac{1}{|z|^{n-k+2s}} W\left(\left(\frac{t}{|z|^2} - 1\right)^{1/2}\right) dz \\ + W(0) \int_{\{|z|>t^{1/2}\}} \frac{1}{|z|^{n-k+2s}}\, dz \equiv I_1 + I_2.$$

Note that $W(0) = C(n,k,s) < \infty$. Then

$$I_2 \lesssim \int_{t^{1/2}}^\infty \frac{1}{\rho^{n-k+2s}} \rho^{n-k-1}\, d\rho \approx t^{-s}.$$

For I_1, we make the change of variables $w = z/t^{1/2}$. We see that

$$I_1 = \int_{\{|w|<1\}} \frac{1}{t^{(n-k+2s)/2}|w|^{n-k+2s}} W\left(\left(\frac{1}{|w|^2} - 1\right)^{1/2}\right) t^{(n-k)/2}\, dw \approx \frac{1}{t^s} \int_0^1 \frac{1}{\rho^{1+2s}} W\left(\left(\frac{1}{\rho^2} - 1\right)^{1/2}\right) d\rho.$$

Note that if $0 < \rho \le \frac{1}{2}$, then

$$\left(\frac{1}{\rho^2} - 1\right)^{1/2} \ge \frac{1}{\sqrt{2}\rho}.$$

Hence

$$W\left(\left(\frac{1}{\rho^2}-1\right)^{1/2}\right) \le W\left(\frac{1}{\sqrt{2}\rho}\right) = \int_{1/(\sqrt{2}\rho)}^{\infty} \frac{r^{k-1}}{(1+r^2)^{(n+2s)/2}}\,dr \lesssim \rho^{n-k+2s}.$$

Therefore,

$$I_1 \lesssim t^{-s}\int_0^{1/2} \frac{1}{\rho^{1+2s}}\rho^{n-k+2s}\,d\rho + t^{-s}W(0)\int_{1/2}^{1}\frac{1}{\rho^{1+2s}}\,d\rho \approx t^{-s},$$

since $n-k>0$. We conclude that

$$\begin{aligned}\mathrm{I} = c_{n,s}\int_0^{2\Lambda d}\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_1}u(t,z)^{1/k}}{|z|}\right)dz\,dt &\lesssim \int_0^{2\Lambda d} t^{-s}\,dt \\ &\approx (2\Lambda d)^{1-s} = (2\Lambda)^{1-s}|x_1-x_0|^{1-s}.\end{aligned}$$

Next we estimate the integral for $t\in(2\Lambda d,\varepsilon]$. To this end, we use Lemma 5.5 (a) to get

$$D_{x_0}u(t-2\Lambda d) \subset D_{x_1}u(t).$$

In particular, for any $z\in\mathbb{R}^{n-k}$ fixed, we have

$$\{y\in\mathbb{R}^k : \tilde{u}_{x_0}(y,z)\le t-2\Lambda d\} \subset \{y\in\mathbb{R}^k : \tilde{u}_{x_1}(y,z)\le t\}.$$

Hence $\mu_{x_0}(t-2\Lambda d,z)\le\mu_{x_1}(t,z)$, which yields

$$\begin{aligned}\mathrm{II} &= c_{n,s}\int_{2\Lambda d}^{\varepsilon}\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_1}u(t,z)^{1/k}}{|z|}\right)dz\,dt \\ &\le c_{n,s}\int_0^{\varepsilon-2\Lambda d}\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_0}u(t,z)^{1/k}}{|z|}\right)dz\,dt.\end{aligned}$$

Finally, we estimate the integral for $t\in[\varepsilon,\infty)$. By Lemma 5.5 (b),

$$D_{x_0}u\left(t-\frac{2\Lambda dt}{\varepsilon}\right) \subset D_{x_1}u(t).$$

Hence $\mu_{x_0}u(t-2\Lambda dt/\varepsilon,z)\le\mu_{x_1}u(t,z)$, and

$$\begin{aligned}\mathrm{III} &= c_{n,s}\int_{\varepsilon}^{\infty}\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_1}u(t,z)^{1/k}}{|z|}\right)dz\,dt \\ &\lesssim \int_{\varepsilon}^{\infty}\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_0}u(t-2\Lambda dt/\varepsilon,z)^{1/k}}{|z|}\right)dz\,dt \\ &= \frac{1}{1-2\Lambda d/\varepsilon}\int_{\varepsilon-2\Lambda d}^{\infty}\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_0}u(t,z)^{1/k}}{|z|}\right)dz\,dt.\end{aligned}$$

Note that

$$\mathrm{II}+\mathrm{III} \le \frac{c_{n,s}}{1-2\Lambda d/\varepsilon}\int_0^{\infty}\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_0}u(t,z)^{1/k}}{|z|}\right)dz\,dt = \frac{\varepsilon}{\varepsilon-2\Lambda d}\mathcal{F}_k^s u(x_0).$$

Therefore, we conclude that

$$\begin{aligned}\mathcal{F}_k^s u(x_1) - \mathcal{F}_k^s u(x_0) &\le C\Lambda^{1-s}|x_1 - x_0|^{1-s} + \left(\frac{\varepsilon}{\varepsilon - 2\Lambda d} - 1\right)\mathcal{F}_k^s u(x_0) \\ &\le C\Lambda^{1-s}|x_1 - x_0|^{1-s} + \frac{4\Lambda}{\varepsilon}|x_1 - x_0|\mathcal{F}_k^s u(x_0),\end{aligned}$$

since $d < r \le \varepsilon/(4\Lambda)$, and thus, $\varepsilon - 2\Lambda d \ge \varepsilon/2$. □

6. A global Poisson problem

We consider the following Poisson problem in the full space:

$$\begin{cases} \mathcal{F}_k^s u = u - \varphi & \text{in } \mathbb{R}^n, \\ (u-\varphi)(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{6-1}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is nonnegative, smooth, and strictly convex. Furthermore, we ask that φ behaves asymptotically at infinity as a cone ϕ, that is,

$$\lim_{|x|\to\infty} (\varphi - \phi)(x) = 0. \tag{6-2}$$

Similar problems have been studied for nonlocal Monge–Ampère operators [Caffarelli and Charro 2015; Caffarelli and Silvestre 2016].

We will prove the following theorem.

Theorem 6.1. *There exists a unique solution u to (6-1) such that $u \in C^{1,1}(\mathbb{R}^n)$ with*

$$[u]_{C^{1,1}(\mathbb{R}^n)} \le [\varphi]_{C^{1,1}(\mathbb{R}^n)}.$$

To define the notion of a solution, we introduce a natural pointwise definition of $\mathcal{F}_k^s u$ for functions u that are merely continuous.

Definition 6.2. Let $u \in C^0(\mathbb{R}^n)$.

(a) We say that a linear function $l(y) = y \cdot p + b$, with $p \in \mathbb{R}^n$ and $b \in \mathbb{R}$, is a supporting plane of u at a point x if $l(x) = u(x)$ and $l(y) \le u(y)$ for all $y \in \mathbb{R}^n$.

(b) We define the subdifferential of u at a point x as the set $\partial u(x)$ of all vectors $p \in \mathbb{R}^n$ such that $l(y) = y \cdot p + b$ is a supporting plane of u at x for some $b \in \mathbb{R}$.

Definition 6.3. Let $u \in C^0(\mathbb{R}^n)$ be a convex function. For $x_0 \in \mathbb{R}^n$, we define

$$\mathcal{F}_k^s u(x_0) = c_{n,s} \sup_{p\in\partial u(x_0)} \inf_{K\in\mathcal{K}_k^s} \int_{\mathbb{R}^n} (u(x_0+x) - u(x_0) - x\cdot p)K(x)\,dx.$$

Remark 6.4. Note that if $u \in C^{1,1}(x_0)$, then $\partial u(x_0) = \{\nabla u(x_0)\}$, and the previous definition coincides with Definition 2.4.

The following properties of $\mathcal{F}_k^s u$ will be useful for our purposes. The proof is analogous to the one in [Caffarelli and Silvestre 2016], so we omit it here.

Lemma 6.5. *Let $u,\ v \in C^0(\mathbb{R}^n)$ be convex functions. The following hold*:

(a) (*homogeneity*) *For any* $\lambda > 0$,
$$\mathcal{F}_k^s(\lambda u) = \lambda \mathcal{F}_k^s u.$$

(b) (*monotonicity*) *Assume that* $u(x_0) = v(x_0)$ *and* $u(x) \geq v(x)$ *for all* $x \in \mathbb{R}^n$. *Then*
$$\mathcal{F}_k^s u(x_0) \geq \mathcal{F}_k^s v(x_0).$$

(c) (*concavity*) *For any* $x \in \mathbb{R}^n$,
$$\mathcal{F}_k^s\big(\tfrac{1}{2}(u+v)\big)(x) \geq \tfrac{1}{2}(\mathcal{F}_k^s u(x) + \mathcal{F}_k^s v(x)).$$

(d) (*lower semicontinuity*) *Assume that* $u \in C^{1,1}(\mathbb{R}^n)$. *Then*
$$\mathcal{F}_k^s u(x_0) \leq \liminf_{x \to x_0} \mathcal{F}_k^s u(x).$$

Definition 6.6. Let $u \in C^0(\mathbb{R}^n)$ be a convex function. We say that u is a subsolution to $\mathcal{F}_k^s u = u - \varphi$ in $\mathbb{R}^n$ if
$$\mathcal{F}_k^s u(x_0) \geq u(x_0) - \varphi(x_0) \quad \text{for all } x_0 \in \mathbb{R}^n.$$
Similarly, u is a supersolution if
$$\mathcal{F}_k^s u(x_0) \leq u(x_0) - \varphi(x_0) \quad \text{for all } x_0 \in \mathbb{R}^n.$$
We say that u is a solution if it is both a subsolution and a supersolution.

Lemma 6.7. *If u and v are subsolutions, then* $\max\{u, v\}$ *is a subsolution.*

Proof. Let $w = \max\{u, v\}$. Then w is continuous and convex. Fix $x_0 \in \mathbb{R}^n$. Without loss of generality, we may assume that $u(x_0) \geq v(x_0)$. Then $w(x_0) = u(x_0)$ and $w(x) \geq u(x)$ for any $x \in \mathbb{R}^n$. By monotonicity (see Lemma 6.5), we have
$$\mathcal{F}_k^s w(x_0) \geq \mathcal{F}_k^s u(x_0) \geq u(x_0) - \varphi(x_0) = w(x_0) - \varphi(x_0).$$
Hence w is a subsolution. □

We will show existence and uniqueness of solutions to (6-1) using Perron's method. The key ingredients are the comparison principle and the existence of a subsolution (lower barrier) and a supersolution (upper barrier). We state this in the following proposition. We omit the proof since it is similar to that in [Caffarelli and Silvestre 2016].

Proposition 6.8. *Consider the equation $\mathcal{F}_k^s u = u - \varphi$ in $\mathbb{R}^n$. The following hold*:

(a) (*comparison principle*) *Let u and v be a subsolution and supersolution, respectively. Assume that $u \leq v$ in $\mathbb{R}^n \setminus \Omega$ for some bounded domain $\Omega \subset \mathbb{R}^n$. Then $u \leq v$ in $\mathbb{R}^n$.*

(b) (*lower barrier*) *The function φ is a subsolution.*

(c) (*upper barrier*) *The function $\varphi + w$ is a supersolution, where $w = (I - \Delta^s)^{-1}\Delta^s \varphi$. In particular, $w(x) \leq C(1 + |x|)^{1-2s}$ for some $C > 0$.*

An immediate consequence of the comparison principle is the uniqueness of solutions.

Lemma 6.9 (uniqueness). *There exists at most one solution to (6-1).*

Proof. Suppose by means of contradiction that there exist two functions $u,\ v \in C^0(\mathbb{R}^n)$, with $u \neq v$, satisfying (6-1). Then $|u(x) - v(x)| \to 0$ as $|x| \to \infty$. Hence, for any $\varepsilon > 0$, there exists a compact set $\Omega_\varepsilon \in \mathbb{R}^n$, depending on ε, such that

$$v(x) - \varepsilon \leq u(x) \leq v(x) + \varepsilon \quad \text{for all } x \in \mathbb{R}^n \setminus \Omega_\varepsilon.$$

Moreover, for any $x_0 \in \mathbb{R}^n$, the function $v + \varepsilon$ satisfies

$$\mathcal{F}_k^s(v+\varepsilon)(x_0) = v(x_0) - \varphi(x_0) < (v(x_0) + \varepsilon) - \varphi(x_0).$$

Therefore, v is a supersolution and, by the comparison principle, it follows that $u \leq v + \varepsilon$ in $\mathbb{R}^n$. Similarly, we see that $v - \varepsilon$ is a subsolution and $u \geq v - \varepsilon$ in $\mathbb{R}^n$. Hence

$$\|u - v\|_{L^\infty(\mathbb{R}^n)} \leq \varepsilon,$$

and letting $\varepsilon \to 0$, we get $u = v$ in $\mathbb{R}^n$, which is a contradiction. $\square$

To prove existence of a solution, we define

$$u(x) = \sup_{v \in \mathcal{S}} v(x), \tag{6-3}$$

where $\mathcal{S}$ is the set of admissible subsolutions given by

$$\mathcal{S} = \{v \in C^{0,1}(\mathbb{R}^n) : v \text{ a subsolution},\ \varphi \leq v \leq \varphi + w,\ \text{ and } [v]_{C^{0,1}(\mathbb{R}^n)} \leq [\varphi]_{C^{0,1}(\mathbb{R}^n)}\}.$$

Note that $\mathcal{S} \neq \varnothing$ since $\varphi \in \mathcal{S}$, and the supremum is finite since $v \leq \varphi + w$ for any $v \in \mathcal{S}$. Moreover, u is convex and Lipschitz with

$$[u]_{C^{0,1}(\mathbb{R}^n)} \leq [\varphi]_{C^{0,1}(\mathbb{R}^n)}.$$

From $\varphi \leq u \leq \varphi + w$ and the upper bound for w in Proposition 6.8, it follows that

$$0 \leq (u - \varphi)(x) \leq w(x) \leq C(1 + |x|)^{1-2s} \to 0$$

as $|x| \to \infty$, since $1 - 2s < 0$.

Proposition 6.10. *The function u given in (6-3) is $C^{1,1}(\mathbb{R}^n)$ with*

$$[u]_{C^{1,1}(\mathbb{R}^n)} \leq [\varphi]_{C^{1,1}(\mathbb{R}^n)}.$$

Proof. We will show that, for any $x_0,\ x_1 \in \mathbb{R}^n$,

$$0 \leq u(x_0 + x_1) - u(x_0 - x_1) - 2u(x_0) \leq [\varphi]_{C^{1,1}(\mathbb{R}^n)} |x_1|^2.$$

Indeed, the lower bound follows from convexity of u. Hence we only need to prove the upper bound. Write $M = [\varphi]_{C^{1,1}(\mathbb{R}^n)}$. Then

$$\varphi(x_0 + x_1) - \varphi(x_0 - x_1) - M|x_1|^2 \leq 2\varphi(x_0). \tag{6-4}$$

Take any $v \in \mathcal{S}$ and fix $x_1 \in \mathbb{R}^n$. Define

$$\hat{v}(x_0) = \tfrac{1}{2}(v(x_0+x_1) + v(x_0-x_1) - M|x_1|^2) \quad \text{for } x_0 \in \mathbb{R}^n.$$

We claim that $\hat{v}$ is a subsolution to $\mathcal{F}_k^s u = u - \varphi$ in $\mathbb{R}^n$. Indeed, since $\mathcal{F}_k^s$ is homogeneous of degree 1, concave, and translation-invariant (see Lemma 6.5), we have

$$\begin{aligned}
\mathcal{F}_k^s \hat{v}(x_0) &= \mathcal{F}_k^s\bigl(\tfrac{1}{2}v(x_0+x_1) + \tfrac{1}{2}v(x_0-x_1)\bigr) \\
&\geq \tfrac{1}{2}\mathcal{F}_k^s v(x_0+x_1) + \tfrac{1}{2}\mathcal{F}_k^s v(x_0-x_1) \\
&\geq \tfrac{1}{2}(v(x_0+x_1) - \varphi(x_0+x_1) + v(x_0-x_1) - \varphi(x_0-x_1)) \\
&= \tfrac{1}{2}(v(x_0+x_1) - v(x_0-x_1) - M|x_1|^2) - \tfrac{1}{2}(\varphi(x_0+x_1) + \varphi(x_0-x_1) - M|x_1|^2) \\
&\geq \hat{v}(x_0) - \varphi(x_0).
\end{aligned}$$

Moreover, using that $v \leq \varphi + w$, we get

$$\hat{v}(x_0) \leq \tfrac{1}{2}(\varphi(x_0+x_1) + \varphi(x_0-x_1) - M|x_1|^2) + \tfrac{1}{2}(w(x_0+x_1) + w(x_0-x_1)).$$

By (6-4) and the upper bound of w in Proposition 6.8 (c), we see that

$$\hat{v}(x_0) - \varphi(x_0) \leq \tfrac{1}{2}C(1+|x_0+x_1|^{1-2s}) + \tfrac{1}{2}C(1+|x_0-x_1|^{1-2s}) \to 0$$

as $|x_0| \to \infty$ with x_1 fixed, since $1-2s<0$. Then, for all $\varepsilon>0$, there is some compact set Ω_ε, depending on ε and x_1, such that

$$\hat{v}(x_0) - \varepsilon \leq \varphi(x_0) \quad \text{for all } x_0 \in \mathbb{R}^n \setminus \Omega_\varepsilon.$$

Consider $\hat{v}_\varepsilon = \max\{\hat{v}-\varepsilon,\ \varphi\}$. Then $\hat{v}_\varepsilon$ is a subsolution, since the maximum of subsolutions is a subsolution (see Lemma 6.7). Also, $\hat{v}_\varepsilon = \varphi \leq \varphi + w$ in $\mathbb{R}^n \setminus \Omega_\varepsilon$, and $\varphi + w$ is a supersolution by Proposition 6.8 (c). Applying the comparison principle, we get $\varphi \leq \hat{v}_\varepsilon \leq \varphi + w$. Moreover, $[\hat{v}_\varepsilon]_{C^{0,1}(\mathbb{R}^n)} \leq [\varphi]_{C^{0,1}(\mathbb{R}^n)}$. Therefore, $\hat{v}_\varepsilon \in \mathcal{S}$.

Since $u(x_0) = \sup_{v\in\mathcal{S}} v(x_0)$, it follows that $u(x_0) \geq \hat{v}_\varepsilon(x_0) \geq \hat{v}(x_0) - \varepsilon$. Letting $\varepsilon \to 0$, we conclude that, for any $v \in \mathcal{S}$ and $x_0,\ x_1 \in \mathbb{R}^n$,

$$u(x_0) \geq \tfrac{1}{2}(v(x_0+x_1) + v(x_0-x_1) - M|x_1|^2). \tag{6-5}$$

Finally, by definition of supremum, for any $\delta > 0$ and $x_0,\ x_1 \in \mathbb{R}^n$, there exist $v_1,\ v_2 \in \mathcal{S}$ such that $u(x_0+x_1) - \delta < v_1(x_0+x_1)$ and $u(x_0-x_1) - \delta < v_2(x_0-x_1)$. Let $v = \max\{v_1,\ v_2\}$. Then using (6-5) for this v, we get

$$u(x_0) \geq \tfrac{1}{2}(u(x_0+x_1) - \delta + u(x_0-x_1) - \delta - M|x_1|^2).$$

Letting $\delta \to 0$, we conclude that

$$u(x_0+x_1) - u(x_0-x_1) - 2u(x_0) \leq [\varphi]_{C^{1,1}(\mathbb{R}^n)}|x_1|^2. \qquad \square$$

To complete the proof of Theorem 6.1, it remains to see that u is a solution. Hence, we need to show that u is both a subsolution and a supersolution. We will prove these results in the next two propositions.

Lemma 6.11. *For any $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, the set*

$$D_{x_0}u(\varepsilon) = \{x \in \mathbb{R}^n : u(x) - u(x_0) - (x - x_0) \cdot \nabla u(x_0) \le \varepsilon\}$$

is compact.

Proof. Let $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$. Without loss of generality, we may assume that $x_0 = 0$. Let l be the supporting plane of u at 0, that is, $l(x) = u(0) + x \cdot \nabla u(0)$. Clearly, $D_{x_0}u(\varepsilon)$ is closed. Hence we only need to show that it is bounded. Recall that

$$\phi(x) < \varphi(x) \le u(x) \quad \text{for all } x \in \mathbb{R}^n, \tag{6-6}$$

where ϕ is a cone. Note that the strict inequality in (6-6) follows from the strict convexity of φ. Moreover, by (6-1) and (6-2) we have

$$\lim_{|x|\to\infty} (u - \phi)(x) = 0.$$

Therefore, $D_{x_0}u(\varepsilon) \subset \{\phi < l + \varepsilon\}$. We claim that

$$\lim_{|x|\to\infty} (\phi - l)(x) = \infty. \tag{6-7}$$

If this condition holds, then, for all $M > 0$, there exists $R > 0$ such that

$$\phi(x) - l(x) > M \quad \text{for all } |x| > R.$$

Choosing $M = \varepsilon$, we have $\{\phi < l + \varepsilon\} \subset B_R$ for some R depending on ε. Hence the set $D_{x_0}u(\varepsilon)$ is bounded.

To prove the claim, we distinguish two cases. If $u(0) = 0$, then u attains an absolute minimum at 0, so $\nabla u(0) = 0$. In particular, $l(x) = 0$ for all $x \in \mathbb{R}^n$, and thus (6-7) is clearly satisfied. Hence it remains to show the claim when

$$u(0) > 0.$$

We will prove it by contradiction. If (6-7) is not true, then there exists a sequence of points $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}^n$ such that $|x_j| \to \infty$ as $j \to \infty$ and

$$\lim_{j\to\infty} (\phi - l)(x_j) < \infty.$$

Using that ϕ is continuous and homogeneous of degree 1, and letting $j \to \infty$, we get

$$\frac{\phi(x_j)}{|x_j|} - \frac{l(x_j)}{|x_j|} = \phi\left(\frac{x_j}{|x_j|}\right) - \frac{u(0)}{|x_j|} - \frac{x_j}{|x_j|} \cdot \nabla u(0) \to \phi(e) - D_e u(0) = 0,$$

where $x_j/|x_j| \to e$, up to a subsequence. Therefore, $\phi(e) = D_e u(0)$. For any $\lambda > 0$, we have

$$l(\lambda e) = u(0) + \lambda e \cdot \nabla u(0) = u(0) + \lambda\phi(e) = u(0) + \phi(\lambda e).$$

Since l is a supporting plane of u, we know that $u(x) \ge l(x)$ for all $x \in \mathbb{R}^n$, and thus,

$$u(\lambda e) \ge l(\lambda e) = \phi(\lambda e) + u(0).$$

Letting $\lambda \to \infty$, we see that

$$0 = \lim_{\lambda\to\infty} (u - \phi)(\lambda e) \ge u(0) > 0,$$

which is a contradiction. □

Proposition 6.12 (u is a subsolution). *The function u given in (6-3) satisfies*

$$\mathcal{F}_k^s u(x_0) \geq u(x_0) - \varphi(x_0) \quad \textit{for all } x_0 \in \mathbb{R}^n.$$

Proof. By Proposition 6.10, we know that $u \in C^{1,1}(\mathbb{R}^n)$. Without loss of generality, we may assume that $[u]_{C^{1,1}(\mathbb{R}^n)} = 1$. Otherwise, consider $u/[u]_{C^{1,1}(\mathbb{R}^n)}$.

Let $x_0 \in \mathbb{R}^n$. Then the quadratic polynomial

$$P(x) = u(x_0) + \nabla u(x_0) \cdot (x - x_0) + |x - x_0|^2$$

touches u from above at x_0. Moreover, we may assume that P touches u strictly from above at x_0. If not, we replace P by $P + \varepsilon|x - x_0|^2$ with $\varepsilon > 0$ small.

Fix $\delta > 0$. Then there exists $h > 0$, with $h \to 0$ as $\delta \to 0$, such that

$$P(x) - u(x) \geq h > 0 \quad \text{for all } x \in \mathbb{R}^n \setminus B_\delta(x_0).$$

Since $u(x) = \sup_{v \in \mathcal{S}} v(x)$ and $v \in \mathcal{S}$ is uniformly continuous, there is a monotone sequence $\{v_j\}_{j=1}^\infty \subset \mathcal{S}$ such that $v_j \to u$ uniformly in compact subsets of $\mathbb{R}^n$. In particular, there exists $j_0 \geq 1$, depending on h, such that, for all $j > j_0$,

$$u(x) - h < v_j(x) \quad \text{for all } x \in \overline{B_\delta(x_0)}. \tag{6-8}$$

Write $v = v_j$ for some $j > j_0$. It follows that

$$\begin{cases} P - v \geq h & \text{in } \mathbb{R}^n \setminus B_\delta(x_0), \\ P - v < P - u + h & \text{in } B_\delta(x_0). \end{cases}$$

Let $d = \inf_{\mathbb{R}^n}(P - v)$. Then $d = P(x_1) - v(x_1)$ for some $x_1 \in \overline{B_h(x_0)}$ with $0 \leq d < h$, and

$$\begin{cases} P(x_1) - d = v(x_1), \\ P(x) - d \geq v(x) \quad \text{for all } x \in \mathbb{R}^n. \end{cases}$$

Hence $P - d$ is a quadratic polynomial that touches v from above at x_1. In particular, since v is convex, v has a unique supporting plane l at x_1, so $\partial v(x_1) = \{\nabla l\}$.

Let $\tau \geq 0$ be such that $l + \tau$ is the supporting plane of u at some point x_2. Note that x_2 approaches x_0 as h goes to 0, and thus, there exists some $r = r(h) > 0$ such that $r \to 0$ as $h \to 0$ and $x_2 \in B_r(x_0)$. Furthermore, since $l(x_1) + d = v(x_1) + d = P(x_1) \geq u(x_1)$, then $\tau \leq d < h$ (see Figure 2).

Fix $\varepsilon > 0$. By Lemma 6.11, we have that $D_{x_0}u(\varepsilon)$ is bounded, so $\Lambda = \operatorname{diam} D_{x_0}u(\varepsilon) < \infty$. Choose δ sufficiently small that $r < \varepsilon/(4\Lambda)$. Then by Proposition 5.2,

$$\mathcal{F}_k^s u(x_2) \leq \mathcal{F}_k^s u(x_0) + C\Lambda^{1-s}|x_2 - x_0|^{1-s} + \frac{4\Lambda}{\varepsilon}\mathcal{F}_k^s u(x_0)|x_2 - x_0| \leq \mathcal{F}_k^s u(x_0) + C(r), \tag{6-9}$$

where $C(r) \to 0$ as $r \to 0$. Next we will show that

$$\mathcal{F}_k^s v(x_1) - C\tau^{1-s} \leq \mathcal{F}_k^s u(x_2) \tag{6-10}$$

for some constant $C > 0$ depending only on n, k, and s. Since $\partial v(x_1) = \{\nabla l\}$ we have $v \in C^{1,1}(x_1)$, and using Proposition 5.4 we get

$$\mathcal{F}_k^s v(x_1) = c_{n,s} \int_0^\infty \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_1} v(t, z)^{1/k}}{|z|}\right) dz\, dt,$$

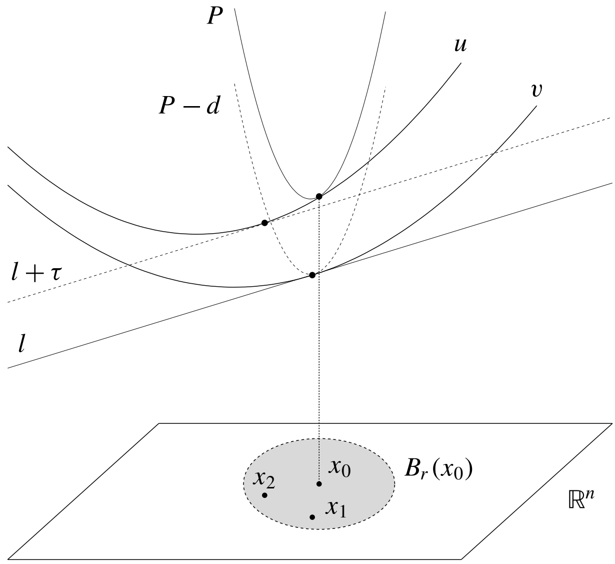

Figure 2. Geometry involved in the proof of Proposition 6.12.

where $\mu_x v(t,z) = \omega_k^{-1}\mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{v}_x(y,z) \le t\})$ and W is the monotone decreasing function given in (5-2). Observe that since $v \le u$, the supporting plane of v at x_1 is l, and the supporting plane of u at x_2 is $l+\tau$. Then, for any $t > 0$, it follows that

$$D_{x_2}u(t) = \{u - (l+\tau) \le t\} \subseteq \{v - l \le t + \tau\} = D_{x_1}v(t+\tau).$$

In particular, $\mu_{x_2}u(t,z) \le \mu_{x_1}v(t+\tau,z)$ for any $z \in \mathbb{R}^{n-k}$. Therefore,

$$W(\mu_{x_2}u(t,z)) \ge W(\mu_{x_1}v(t+\tau,z)),$$

which yields

$$\begin{aligned}
\mathcal{F}_k^s u(x_2) &\ge c_{n,s}\int_\tau^\infty\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_1}v(t,z)^{1/k}}{|z|}\right) dz\,dt \\
&= \mathcal{F}_k^s v(x_1) - c_{n,s}\int_0^\tau\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_1}v(t,z)^{1/k}}{|z|}\right) dz\,dt \\
&\ge \mathcal{F}_k^s v(x_1) - C\tau^{1-s},
\end{aligned}$$

where the last inequality follows from the fact that $\mu_{x_1}v(t,z) \ge C(t-|z|^2)_+^{k/2}$ and W is monotone decreasing.

Combining (6-9) and (6-10), using that v is a subsolution, and using (6-8), we get

$$\mathcal{F}_k^s u(x_0) + C(r) \ge \mathcal{F}_k^s v(x_1) - C\tau^{1-s} \ge v(x_1) - \varphi(x_1) - C\tau^{1-s} > u(x_1) - h - \varphi(x_1) - C\tau^{1-s}.$$

Letting $\delta \to 0$, it follows that $h \to 0$, $C(r) \to 0$, $\tau \to 0$, and $x_1 \to x_0$. By continuity of u and φ, we conclude the result. □

Proposition 6.13 (u is a supersolution). *The function u given in (6-3) satisfies*

$$\mathcal{F}_k^s u(x_0) \le u(x_0) - \varphi(x_0) \quad \textit{for all } x_0 \in \mathbb{R}^n.$$

Proof. Assume the statement is false. Then there exists some $x_0 \in \mathbb{R}^n$ such that

$$\mathcal{F}_k^s u(x_0) > u(x_0) - \varphi(x_0).$$

Without loss of generality, we may assume that $u(x_0) = 0$ and $\nabla u(x_0) = 0$. Otherwise, consider $v(x) = u(x) - u(x_0) - (x - x_0) \cdot \nabla u(x_0)$. Then there exists some $\delta > 0$ such that

$$\mathcal{F}_k^s u(x_0) \ge -\varphi(x_0) + \delta. \tag{6-11}$$

Fix $\varepsilon > 0$ and let $u^\varepsilon(x) = \max\{u(x), \varepsilon\}$. We will show that, for ε sufficiently small, u^ε is an admissible subsolution, and thus reach a contradiction with u being the largest subsolution. Indeed, u^ε is convex and $u^\varepsilon \in C^{0,1}(\mathbb{R}^n)$ with $[u^\varepsilon]_{C^{0,1}(\mathbb{R}^n)} \le [\varphi]_{C^{0,1}(\mathbb{R}^n)}$. Moreover, note that $u^\varepsilon(x) = u(x)$ for x large. Hence, once we show that u^ε is a subsolution, it will follow from the comparison principle that $\varphi \le u^\varepsilon \le \varphi + w$.

If $x \in \{u_\varepsilon = u\}$, then $u_\varepsilon(x) = u(x)$ and $u_\varepsilon \ge u$ in $\mathbb{R}^n$. By monotonicity (Lemma 6.5),

$$\mathcal{F}_k^s u^\varepsilon(x) \ge \mathcal{F}_k^s u(x) \ge u(x) - \varphi(x) = u^\varepsilon(x) - \varphi(x),$$

since u is a subsolution, by Proposition 6.12.

If $x \in \{u^\varepsilon > u\}$, then $u^\varepsilon(x) = \varepsilon$ and $\partial u^\varepsilon(x) = \{0\}$. In particular,

$$\mathcal{F}_k^s u^\varepsilon(x) = \mathcal{F}_k^s u^\varepsilon(x_0). \tag{6-12}$$

Moreover, for any $t > 0$, we have $D_{x_0} u^\varepsilon(t) = \{u^\varepsilon - \varepsilon \le t\} = \{u \le t + \varepsilon\} = D_{x_0} u(t + \varepsilon)$. Therefore, in view of Proposition 5.4, we get

$$\mathcal{F}_k^s u^\varepsilon(x_0) = \mathcal{F}_k^s u(x_0) - \int_0^\varepsilon \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{x_0} u(t, z)^{1/k}}{|z|}\right) dz\, dt \ge \mathcal{F}_k^s u(x_0) - C\varepsilon^{1-s}, \tag{6-13}$$

since $u \in C^{1,1}(\mathbb{R}^n)$ and $\mu_{x_0} u(t, z) \ge (t - |z|^2)_+^{k/2}$.

Combining (6-11)–(6-13), we see that

$$\begin{aligned} \mathcal{F}_k^s u^\varepsilon(x) = \mathcal{F}_k^s u^\varepsilon(x_0) \ge \mathcal{F}_k^s u(x_0) - C\varepsilon^{1-s} &\ge -\varphi(x_0) + \delta - C\varepsilon^{1-s} \\ &= u^\varepsilon(x) - \varphi(x) + (\varphi(x) - \varphi(x_0) + \delta - C\varepsilon^{1-s} - \varepsilon), \end{aligned}$$

since $u^\varepsilon(x) = \varepsilon$. We need the term inside the parenthesis to be nonnegative. Hence it remains to control $\varphi(x) - \varphi(x_0)$. Since φ is smooth,

$$|\varphi(x) - \varphi(x_0)| \le [\varphi]_{C^{0,1}(\mathbb{R}^n)} |x - x_0|.$$

We distinguish two cases. If $\{u = 0\} = \{x_0\}$, then $|x - x_0| \le d_\varepsilon \to 0$ as $\varepsilon \to 0$. Hence, choosing ε sufficiently small, we see that

$$\varphi(x) - \varphi(x_0) + \delta - C\varepsilon^{1-s} - \varepsilon \ge \delta - [\varphi]_{C^{0,1}(\mathbb{R}^n)} d_\varepsilon - C\varepsilon^{1-s} - \varepsilon \ge 0.$$

Therefore, $u^\varepsilon \in \mathcal{S}$, which contradicts $u^\varepsilon(x_0) > u(x_0) = \sup_{v \in \mathcal{S}} v(x_0) \ge u^\varepsilon(x_0)$.

Suppose now that $\{u=0\}$ contains more than one point. By compactness of $\{u=0\}$ and continuity of φ, there exists some $x_1 \in \{u=0\}$ where φ attains its maximum. Then

$$\mathcal{F}_k^s u(x_1) = \mathcal{F}_k^s u(x_0) \geq u(x_0) - \varphi(x_0) + \delta \geq u(x_1) - \varphi(x_1) + \delta.$$

Moreover, by convexity of $\{u=0\}$ (since $u \geq \varphi \geq 0$) and φ, we must have that $x_1 \in \partial\{u=0\}$. Hence there exists $\{x_j\}_{j=2}^{\infty} \subset \{u>0\}$ such that $x_j \to x_1$ and u is strictly convex at x_j. Namely, there is a supporting plane that touches u only at x_j.

By continuity of u, there exists some $j_0 \geq 2$ such that

$$u(x_1) > u(x_j) - \tfrac{1}{4}\delta \quad \text{for all } j > j_0.$$

By continuity of φ, there exists some $j_1 \geq 2$ such that

$$\varphi(x_1) < \varphi(x_j) + \tfrac{1}{4}\delta \quad \text{for all } j > j_1.$$

By lower semicontinuity of $\mathcal{F}_k^s u$, up to a subsequence, there exists some $j_2 \geq 2$ such that

$$\mathcal{F}_k^s u(x_j) > \mathcal{F}_k^s u(x_1) - \tfrac{1}{4}\delta \quad \text{for all } j > j_2.$$

Let $J > \max\{j_0, j_1, j_2\}$. Then

$$\mathcal{F}_k^s u(x_J) > \mathcal{F}_k^s u(x_1) - \tfrac{1}{4}\delta \geq u(x_1) - \varphi(x_1) + \tfrac{3}{4}\delta > u(x_J) - \varphi(x_J) + \tfrac{1}{4}\delta,$$

and we can repeat the previous argument, replacing x_0 by x_J. We conclude that

$$\mathcal{F}_k^s u(x_0) \leq u(x_0) - \varphi(x_0) \quad \text{for all } x_0 \in \mathbb{R}^n. \qquad \square$$

7. Future directions

As mentioned in the introduction, the main idea to define a nonlocal analog to the Monge–Ampère operator is to write it as a concave envelope of linear operators. More precisely,

$$n \det(D^2 u(x))^{1/n} = \inf_{M \in \mathcal{M}} \operatorname{tr}(M D^2 u(x)),$$

where $\mathcal{M} = \{M \in \mathcal{S}^n : M > 0, \det(M) = 1\}$ and $\mathcal{S}^n$ is the set of $n \times n$ symmetric matrices. Note that this identity is equivalent to the one given in (1-2) taking $M = AA^T$ and $B = D^2 u(x)$, since $\operatorname{tr}(A^T B A) = \operatorname{tr}(AA^T B)$. In fact, this extremal property does not only hold for $n \det(B)^{1/n}$ with $B \in \mathcal{S}^n$ and $B > 0$. If $\lambda = (\lambda_1, \dots, \lambda_n)$, where λ_i are the eigenvalues of B, then the function f defined on $\Gamma = \{\lambda \in \mathbb{R}^n : \lambda_i > 0 \text{ for all } i = 1, \dots, n\}$ and given by

$$f(\lambda) = n \left(\prod_{i=1}^{n} \lambda_i \right)^{1/n} = n \det(B)^{1/n}$$

is differentiable, concave, and homogeneous of degree 1. In general, if f satisfies these conditions in an open convex set Γ in $\mathbb{R}^n$, then

$$f(\lambda) = \inf_{\mu \in \Gamma} \{ f(\mu) + \nabla f(\mu) \cdot (\lambda - \mu) \} = \inf_{\mu \in \Gamma} \nabla f(\mu) \cdot \lambda,$$

where the second identity follows by Euler's theorem. Therefore,

$$f(\lambda) = \inf_{M \in \mathcal{M}_f} \operatorname{tr}(MB),$$

where $\mathcal{M}_f = \{M \in \mathcal{S}^n : \lambda(M) \in \nabla f(\Gamma)\}$, $\nabla f(\Gamma) = \{\nabla f(\mu) : \mu \in \Gamma\}$, and $\lambda(M)$ are the eigenvalues of M.

For instance, the k-Hessian functions introduced by Caffarelli, Nirenberg, and Spruck in [Caffarelli et al. 1985] satisfy these conditions and, in fact, fractional analogs have been recently studied by Wu [2019]. It would be interesting to explore fractional analogs to a wider class of fully nonlinear concave operators, like the ones mentioned above.

We remark that the 1-Hessian is equal to the Laplacian, and the n-Hessian is equal to the Monge–Ampère operator. Moreover, for $1 < k < n$, we obtain an intermediate *discrete* family between these operators. In view of this observation, a natural question of finding a *continuous* family connecting the Laplacian with the Monge–Ampère operator arises. Here we suggest possible families that smoothly connect these two operators and pass through the k-Hessians, in some sense. Indeed, let $\alpha \in (0, 1]^n$ and write $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For $\lambda \in \mathbb{R}^n_+$, we consider the functions

$$f_\alpha(\lambda) = \left(\sum_{\sigma \in S} \lambda_{\sigma(1)}^{\alpha_1} \cdots \lambda_{\sigma(n)}^{\alpha_n} \right)^{1/|\alpha|},$$

where S is the set of all cyclic permutations of $\{1, \dots, n\}$. Observe that, for any $1 \le k \le n$, if $\alpha = \sum_{i \in \mathcal{I}} e_i$ with $|\mathcal{I}| = k$, then f_α is precisely the k-Hessian function. Consider any smooth simple curve $\gamma : [0, 1] \to (0, 1]^n$ such that

(1) $\gamma(0) = e_i$ for some $1 \le i \le n$,

(2) $\gamma(t_k) = \sum_{i \in \mathcal{I}_k} e_i$ with $|\mathcal{I}_k| = k$ and $0 < t_k < t_{k+1} < 1$ for all $1 < k < n$, and

(3) $\gamma(1) = (1, \dots, 1)$.

Then the family $\{f_\alpha\}_{\alpha \in \mathrm{Im}(\gamma)}$ is as we described. In particular, fractional analogs of these functions would give a continuous family from the fractional Laplacian to the nonlocal Monge–Ampère. We will study this problem in a forthcoming paper.

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PROPAGATION OF SINGULARITIES FOR GRAVITY-CAPILLARY WATER WAVES

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We obtain two results of propagation for the gravity-capillary water wave system. The first result shows the propagation of oscillations and the spatial decay at infinity; the second result shows a microlocal smoothing effect under the nontrapping condition of the initial free surface. These results extend the works of Craig, Kappeler and Strauss (1995), Wunsch (1999) and Nakamura (2005) to quasilinear dispersive equations. These propagation results are stated for water waves with asymptotically flat free surfaces, of which we also obtain the existence. To prove these results, we generalize the paradifferential calculus of Bony (1979) to weighted Sobolev spaces and develop a semiclassical paradifferential calculus. We also introduce the quasihomogeneous wavefront sets which characterize, in a general manner, the oscillations and the spatial growth/decay of distributions.

1. Introduction

We present two results on the propagation of singularities for the gravity-capillary water wave system, including a microlocal smoothing effect. To the best of our knowledge, these results are the first of this type for quasilinear dispersive equations. Before stating the main results, we shall first revisit classical results of propagation for the linear half-wave equation and the linear Schrödinger equation. They lead us to a more generalized concept of singularities which is adaptive to various dispersive equations.

1A. ***Wavefront set and the linear half-wave equation.*** If $u \in \mathscr{D}'(M)$, where M is a smooth manifold without boundary, then the singular support of u, denoted by $\operatorname{sing\,supp} u$, is the smallest closed subset of M outside of which u is smooth. To study the propagation of singularities when u solves some partial

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differential equations, the information given by $\operatorname{sing\,supp} u$ is usually insufficient. Heuristically, if we consider singularities as accumulations of wavepackets with large wavenumbers, then this is because the propagation direction of a wavepacket is given by its wavenumber rather than its location. It is probably with this mindset that Hörmander [1971] introduced the concept of the wavefront set.

The wavefront set of u, denoted by $\mathrm{WF}(u)$, lifts $\operatorname{sing\,supp} u$ to the cotangent bundle $T^*M\backslash 0$ in the sense that a point $x_0 \in M$ belongs to $\operatorname{sing\,supp} u$ if and only if there exists $\xi_0 \neq 0$ such that $(x_0, \xi_0) \in \mathrm{WF}(u)$. We shall recall an equivalent definition of $\mathrm{WF}(u)$ essentially due to [Guillemin and Sternberg 1977]: in local coordinates, a point $(x_0, \xi_0) \in T^*M\backslash 0$ does not belong to $\mathrm{WF}(u)$ if and only if there exists $a \in C_c^\infty(\mathbb{R}^{2d})$ with $a(x_0, \xi_0) \neq 0$ such that $\|a(x, hD_x)u\|_{L^2} = \mathcal{O}(h^\infty)$ for $h \in (0, 1]$. For the definition of the pseudodifferential operator $a(x, hD_x)$, see (1-7).

In terms of the wavefront set, Hörmander [1971] proved a propagation result for pseudodifferential equations of real principal type, improving previous works [Courant and Lax 1956; Lax 1957] on wave propagation.

Theorem 1.1 [Hörmander 1971]. *Let M be a smooth manifold without boundary. Let $P \in \Psi^1(M)$ admit a real principal symbol $\sigma(P) = \sigma(P)(x, \xi) \in C^\infty(T^*M\backslash 0, \mathbb{R})$, and let $\Phi = \Phi_t(x, \xi) \in C^\infty(\mathbb{R} \times T^*M\backslash 0, T^*M\backslash 0)$ be the Hamiltonian flow of $\sigma(P)$. If u solves the Cauchy problem*

$$\begin{cases} \partial_t u + iPu = 0, \\ u(0) = u_0 \in L^2(M), \end{cases} \tag{1-1}$$

then for all $(x_0, \xi_0) \in \mathrm{WF}(u_0)$ and all $t \in \mathbb{R}$, we have $\Phi_t(x_0, \xi_0) \in \mathrm{WF}(u(t))$.

In particular, if $P = \sqrt{-\Delta_g}$ where g is a Riemannian metric on M, then (1-1) becomes the half-wave equation and Φ is the corresponding cogeodesic flow on T^*M. Therefore, we conclude that, for solutions to the half-wave equation, microlocal singularities travel at speed 1 along cogeodesics. This gives a justification for the Huygens–Fresnel principal of wavefront propagation.

For the propagation of singularities for the semilinear wave equation, we refer to [Bony 1986; Lebeau 1989]. For the propagation and the reflection of singularities for the linear wave equation on manifolds with corners, see [Vasy 2008; Melrose, Vasy and Wunsch 2013].

1B. ***The homogeneous wavefront set and the linear Schrödinger equation.*** Hörmander's theorem (Theorem 1.1) is untrue when the order of P is higher than 1. For example, the Schrödinger propagator $e^{it\Delta/2}$ on $\mathbb{R}^d$ sends $\mathscr{E}'(\mathbb{R}^d)$ to $C^\infty(\mathbb{R}^d)$ whenever $t \neq 0$. We conclude that singularities may appear and disappear along the Schrödinger flow. These phenomena of "microlocal smoothing effect" and "microlocal singularity formation" are due to the infinite speed of propagation of the Schrödinger equation, as wavepackets with large wavenumbers can travel to or back from infinity instantaneously.

The study of the infinite speed of propagation of the Schrödinger equation probably dates back to [Boutet de Monvel 1975; Lascar 1977; 1978]. They proved that space-time singularities, as elements of some space-time wavefront sets, travel along geodesics at an infinite speed. They did not obtain, however, a time-dependent propagation results for wavefront sets with respect to the space variable alone. The study of the smoothing effect for dispersive equations with an infinite speed of propagation was initiated by Kato [1983], who proved a local smoothing effect for generalized KdV equations. Craig, Kappeler and Strauss [1995] proved microlocal smoothing effects for the linear Schrödinger equation under the nontrapping condition of the geometry. Their results were later refined by Wunsch [1999] who obtained a time-dependent propagation after understanding the transformation between singularities and quadratic oscillations at infinity. The simplest example is the identity

$$e^{it\Delta/2}\delta_{x_0}(x)=\frac{1}{(2\pi it)^{d/2}}e^{i|x-x_0|^2/(2t)},$$

where δ_{x_0} is the Dirac measure at $x_0\in\mathbb{R}^d$. Wunsch's results were stated on Riemannian manifolds endowed with a scattering metric. He introduced the quadratic scattering wavefront set to characterize quadratic oscillations.

Similar results were later obtained, independently, by Nakamura [2005] via a simpler calculus but in a less general geometric setting — asymptotically Euclidean geometries, where he introduced the homogeneous wavefront set. By definition, if $u\in\mathscr{S}'(\mathbb{R}^d)$, then the homogeneous wavefront set $\mathrm{HWF}(u)$ is a subset of $\mathbb{R}^{2d}$ whose complement consists of all (x_0,ξ_0) admitting a symbol $a\in C_c^\infty(\mathbb{R}^{2d})$ with $a(x_0,\xi_0)\neq 0$ such that $\|a(hx,hD_x)u\|_{L^2}=\mathcal{O}(h^\infty)$ for $h\in(0,1]$. It was proven by Ito [2006] that the quadratic scattering wavefront set and the homogeneous wavefront set are essentially equivalent in asymptotically Euclidean geometries. In fact, heuristically, if $x_0\neq 0$ and $\xi_0\neq 0$, then the pseudodifferential operator $a(hx,hD_x)$ is a microlocalization in the region of quadratic oscillation:

$$|x|\sim|\xi|\sim h^{-1}.$$

Take for example the free Schrödinger equation in $\mathbb{R}^d$, of which the dispersion relation is $\omega=\frac{1}{2}|\xi|^2$. A wave packet of frequency $\xi\sim h^{-1}$ travels at the group velocity $v=\mathrm{d}\omega/\,\mathrm{d}\xi=\xi\sim h^{-1}$. The homogeneously scaled quantization $a\mapsto a(hx,hD_x)$ thus allows us to keep up with the infinite speed of propagation and obtain an analogue of Hörmander's theorem.

Theorem 1.2 ([Nakamura 2005], similar results in [Wunsch 1999]). *Let g be an asymptotically Euclidean Riemannian metric on $\mathbb{R}^d$, meaning that there exists $\epsilon>0$ such that, for all $\alpha\in\mathbb{N}^d$ and all $i,j\in\{1,\dots,d\}$, we have*

$$|\partial_x^\alpha(g_{ij}(x)-\delta_{ij})|\lesssim\langle x\rangle^{-|\alpha|-\epsilon}. \tag{1-2}$$

Consider the Cauchy problem of the linear Schrödinger equation

$$\begin{cases} i\partial_t u+\frac{1}{2}\Delta_g u=0,\\ u(0)=u_0\in L^2(\mathbb{R}^d).\end{cases}$$

Then the following propagation results hold:

(1) *If $(x_0,\xi_0)\in\mathrm{HWF}(u_0)$ and $t_0\in\mathbb{R}$ such that $\xi_0\neq 0$ and $x_0+t\xi_0\neq 0$ for all t between 0 and t_0, then $(x_0+t_0\xi_0,\xi_0)\in\mathrm{HWF}(u(t_0))$.*

(2) *If $(x_0,\xi_0)\in\mathrm{WF}(u_0)$ is forwardly (resp. backwardly) nontrapping in the sense that the cogeodesic issued from (x_0,ξ_0), denoted by $\{(x_t,\xi_t)\}_{t\in\mathbb{R}}$ (with an abuse of notation), satisfies*

$$\lim_{t\to+\infty}|x_t|=+\infty\quad\bigl(\textit{resp.}\ \lim_{t\to-\infty}|x_t|=+\infty\bigr),$$

then there exists $\xi_+ \in \mathbb{R}^d$ (*resp.* $\xi_- \in \mathbb{R}^d$) *satisfying* $\xi_\pm = \lim_{t\to\pm\infty} \xi_t$, *and moreover, for all* $t_0 > 0$ (*resp.* $t_0 < 0$), *we have*

$$(t_0\xi_+, \xi_+) \in \mathrm{HWF}(u(t_0)) \quad \big(\textit{resp.}\ (t_0\xi_-, \xi_-) \in \mathrm{HWF}(u(t_0))\big).$$

Theorem 1.2(1) studies the propagation of oscillations and spatial growth/decay for Schrödinger waves at infinity and we thus require the condition $x_0 + t\xi_0 \neq 0$. In $\mathbb{R}^d$, this result is a consequence of an Egorov-type argument and the commutation relation

$$\big[i\partial_t + \tfrac{1}{2}\Delta, a(t, hx, hD_x)\big] = (i\partial_t a - \xi\cdot\partial_x a)(t, hx, hD_x) + \mathcal{O}(h^2),$$

where $a \in C_b^\infty(\mathbb{R}\times\mathbb{R}^{2d})$. A similar argument works in asymptotically Euclidean geometries where we replace the role of the semiclassical quantization $x \mapsto hx$ with the spatial decay of the metric g, i.e., the condition (1-2).

Theorem 1.2(2) is a microlocal smoothing effect: if $(t_0\xi_\pm, \xi_\pm)$ does not belong to $\mathrm{HWF}(u(t_0))$, then (x_0, ξ_0) cannot be an element of $\mathrm{WF}(u_0)$. This result is a refinement of the result in [Craig, Kappeler and Strauss 1995] and can be proven via a positive commutator estimate. In $\mathbb{R}^d$, this estimate has the form

$$\big[i\partial_t + \tfrac{1}{2}\Delta, a(t, x, hD_x)\big] \gtrsim \mathcal{O}(h^\infty),$$

where a is some well-chosen symbol. For related results, see [Doi 1996; 2000; Burq 2004] for the necessity of the nontrapping condition; see [Robbiano and Zuily 1999] for a microlocal analytic smoothing effect; see [Kenig, Ponce and Vega 1998; Szeftel 2005] for local and microlocal smoothing effects for the semilinear Schrödinger equation. We should also remark that Hörmander [1991] has also introduced an essentially equivalent counterpart of the homogeneous wavefront set to which a similar definition as that of Nakamura was given. See [Rodino and Wahlberg 2014; Schulz and Wahlberg 2017] for more comments. However, Theorem 1.2(2) is unable, via simply reversing the time, to show how oscillations at infinity form singularities along the Schrödinger flow. Indeed, the information about the locations of singularities is not contained in quadratic oscillations but rather in linear oscillations at infinity. See [Hassell and Wunsch 2005; Nakamura 2009] for more on this subject.

1C. *Quasihomogeneous wavefront set and the gravity-capillary water wave system.* The gravity-capillary water wave system describes the evolution of inviscid, incompressible and irrotational fluid with a free surface, in the presence of a gravitational field and the surface tension.

1C1. *Formulations of the gravity-capillary water wave system.* We shall first recall the Eulerian formulation of the gravity-capillary water wave system. The area occupied by the fluid is a time-dependent simply connected open subset of $\mathbb{R}^{d+1}$ and is denoted by Ω. The boundary of Ω consists of two parts: the free surface Σ and the bottom Γ. The free surface of the fluid is a time-dependent hypersurface which is the graph of a function $\eta = \eta(t, x)$, where $(t, x) \in \mathbb{R}\times\mathbb{R}^d$, whereas the bottom is independent of time and is of depth $b \in (0, \infty)$. Therefore,

$$\Omega = \{-b < y < \eta\}, \quad \Sigma = \{y = \eta\}, \quad \Gamma = \{y = -b\}.$$

The Eulerian formulation describes water waves in the unknowns (η, v, P) where $v : \Omega \to \mathbb{R}^d$ is the Eulerian vector field and $P : \Omega \to \mathbb{R}$ is the pressure of the fluid:

$$\begin{cases} \partial_t v + v \cdot \nabla_{xy} v = -\nabla_{xy}(P + gy) & \text{(Euler equation)}, \\ \nabla_{xy} \cdot v = 0 & \text{(incompressibility)}, \\ \nabla_{xy} \times v = 0 & \text{(irrotationality)}, \\ (v \cdot \boldsymbol{n})_{y=\eta} = \partial_t \eta / \langle \nabla \eta \rangle & \text{(kinetic condition at the free surface)}, \\ (v \cdot \boldsymbol{n})_{y=-b} = 0 & \text{(kinetic condition at the bottom)}, \\ -P|_{y=\eta} = \kappa H(\eta), & \text{(dynamic condition)}. \end{cases} \tag{1-3}$$

Here $g \in \mathbb{R}$ is the gravitational acceleration, $\kappa > 0$ is the surface tension, $\boldsymbol{n} : \partial\Omega \to \mathbb{S}^d$ denotes the exterior unit normal vector field of $\partial\Omega$, while

$$H(\eta) = \nabla \cdot \left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right) \tag{1-4}$$

is the mean curvature of the free surface. In (1-3), the kinetic condition at the free surface implies that fluid particles which are initially on the free surface will stay on the free surface, whereas the kinetic condition at the bottom is a rephrasing of the impenetrability of the bottom. The dynamic condition is the Laplace–Young equation which expresses the balance between the interior pressure P and the surface tension κ.

One of the main difficulties in the study of the Eulerian formulation of the system (1-3) is the time-dependence of the domain Ω. By [Zakharov 1968; Craig and Sulem 1993], we can reformulate (1-3) as a system in $\mathbb{R}^d$. Note that due to the simply connected geometry of Ω and the irrotationality of the fluid, there exists a velocity potential $\phi : \Omega \to \mathbb{R}$ such that $\nabla_{xy}\phi = v$. By the incompressiblity of the fluid, the potential ϕ is harmonic. Therefore ϕ satisfies the Laplace equation with Neumann boundary conditions:

$$\Delta_{xy}\phi = 0, \quad \partial_{\boldsymbol{n}}\phi|_{y=\eta} = \partial_t \eta / \langle \nabla \eta \rangle, \quad \partial_{\boldsymbol{n}}\phi|_{y=-b} = 0.$$

Define $\psi = \phi|_{y=\eta}$ and define

$$G(\eta)\psi = \langle \nabla \eta \rangle \partial_{\boldsymbol{n}}\phi|_{y=\eta}.$$

Here $G(\eta)$ is the Dirichlet–Neumann operator (see Section 5A for a rigorous definition). Then the system (1-3) can be rewritten in terms of the unknowns (η, ψ):

$$\begin{cases} \partial_t \eta - G(\eta)\psi = 0, \\ \partial_t \psi + g\eta - \kappa H(\eta) + \dfrac{1}{2}|\nabla\psi|^2 - \dfrac{1}{2}\dfrac{(\nabla\eta \cdot \nabla\psi + G(\eta)\psi)^2}{1 + |\nabla\eta|^2} = 0. \end{cases} \tag{1-5}$$

We shall assume henceforth that $\kappa = 1$ for simplicity.

1C2. *Quasihomogeneous wavefront set and model equations.* It is known that the linearization of (1-5) about the stationary solution $(\eta, \psi) = (0, 0)$ can be symmetrized, up to a smoothing remainder, to the fraction Schrödinger equation or order $\frac{3}{2}$. Consider the more general model equation

$$\partial_t u + i|D_x|^\gamma u = 0, \quad \gamma \geq 1. \tag{1-6}$$

It is natural to ask ourselves if we can define a new family of wavefront sets and extend the results from Theorems 1.1 and 1.2 to (1-6). Note that a wave packet of (1-6) of frequency $\xi \sim h^{-1}$ travels at the group velocity

$$v = \frac{\mathrm{d}|\xi|^\gamma}{\mathrm{d}\xi} = \gamma|\xi|^{\gamma-2}\xi \sim h^{-(\gamma-1)}.$$

It suggests that we need to use pseudodifferential operators of the form $a(h^{\gamma-1}x, hD_x)$ as test operators. In the following definition, we consider the more general quantization with two parameters.

Definition 1.3. If $u \in \mathscr{S}'(\mathbb{R}^d)$, $\mu \in \mathbb{R}\cup\{\infty\}$, $\delta \geq 0$ and $\rho \geq 0$ with $\delta+\rho > 0$, then the quasihomogeneous wavefront set $\mathrm{WF}^{\mu}_{\delta,\rho}(u)$ is a subset of $\mathbb{R}^{2d}$ defined as follows. A point (x_0, ξ_0) does not belong to $\mathrm{WF}^{\mu}_{\delta,\rho}(u)$ if and only if there exists $a \in C_c^\infty(\mathbb{R}^{2d})$ with $a(x_0, \xi_0) \neq 0$ such that $\|a(h^\delta x, h^\rho D_x)u\|_{L^2} = \mathcal{O}(h^\mu)$ for $h \in (0, 1]$. Here,

$$a(h^\delta x, h^\rho D_x)u(x) = (2\pi)^{-d} \iint_{\mathbb{R}^{2d}} e^{i(x-y)\cdot\xi} a(h^\delta x, h^\rho \xi)u(y)\,\mathrm{d}y\,\mathrm{d}\xi. \tag{1-7}$$

Note that $\mathrm{WF}^{\mu}_{\delta,\rho}(u)$ is invariant under the scaling $(x, \xi) \mapsto (\lambda^\delta x, \lambda^\rho \xi)$ for all $\lambda > 0$. The existence of $(x_0, \xi_0) \in \mathrm{WF}^{\mu}_{\delta,\rho}(u)$ implies an accumulation of mass near the ray $\{(\lambda^\delta x_0, \lambda^\rho \xi_0)\}_{\lambda>0}$. By choosing different parameters, we recover the definitions of various wavefront sets from the quasihomogeneous wavefront set: the wavefront set of Hörmander $(\delta, \rho, \mu) = (0, 1, \infty)$, the homogeneous wavefront set of Nakamura $(\delta, \rho, \mu) = (1, 1, \infty)$ and the scattering wavefront set of [Melrose 1994] $(\delta, \rho, \mu) = (1, 0, \infty)$.

Theorem 1.4. *If u solves (1-6) with initial data $u(0) = u_0 \in L^2(\mathbb{R}^d)$ and $\mu \in \mathbb{R}\cup\{\infty\}$, then the following results of propagation hold*:

(1) *If $\rho\gamma = \delta+\rho$, $(x_0, \xi_0) \in \mathrm{WF}^{\mu}_{\delta,\rho}(u_0)\backslash\{\xi = 0\}$ and $t_0 \in \mathbb{R}$, then*

$$(x_0 + t_0\gamma|\xi_0|^{\gamma-2}\xi_0, \xi_0) \in \mathrm{WF}^{\mu}_{\delta,\rho}(u(t_0)).$$

(2) *If $\gamma > 1$, $\rho\gamma > \delta+\rho$, $(x_0, \xi_0) \in \mathrm{WF}^{\mu}_{\delta,\rho}(u_0)\backslash\{\xi = 0\}$ and $t_0 \neq 0$, then*

$$(t_0\gamma|\xi_0|^{\gamma-2}\xi_0, \xi_0) \in \mathrm{WF}^{\mu}_{\rho(\gamma-1),\rho}(u(t_0)).$$

Note that we do not require $x_0 + t\gamma|\xi_0|^{\gamma-2}\xi_0 \neq 0$ in Theorem 1.4(1), while we require $x_0 + t\xi_0 \neq 0$ in Theorem 1.2(1). This is because in Theorem 1.2 the geometry is only Euclidean at infinity.

1C3. *Asymptotically flat water waves.* Instead of the linearization at $(\eta, \psi) = (0, 0)$, if we paralinearize and symmetrize (1-5) as in [Alazard, Burq and Zuily 2011], then we obtain a quasilinear paradifferential fractional Schrödinger equation of order $\frac{3}{2}$. We require the geometry of the free surface to be Euclidean at infinity and the velocity field to be zero at infinity to avoid problems caused by the infinite speed of propagation and the nonlinearity. We shall fulfill this requirement by proving the existence of gravity-capillary water waves in some weighted Sobolev spaces.

Definition 1.5. If $\mu, k \in \mathbb{R}$, then $H^\mu_k = H^\mu_k(\mathbb{R}^d)$ is the set of all $u \in \mathscr{S}'(\mathbb{R}^d)$ such that

$$\|u\|_{H^\mu_k} = \|\langle x\rangle^k \langle D_x\rangle^\mu u\|_{L^2} < +\infty.$$

If in addition $k \in \mathbb{N}$ and $\delta \geq 0$, then define

$$\mathcal{H}_k^{\mu,\delta} = \bigcap_{j=0}^{k} H_j^{\mu-\delta j}.$$

We are mostly interested in the case where $\delta = \frac{1}{2}$. The weighted Sobolev space $\mathcal{H}_k^{\mu,1/2}$ is a natural space to apply the energy estimate for the fractional Schrödinger equation of order $\frac{3}{2}$ and thus also for the gravity-capillary water wave system.

Theorem 1.6. *If $d \geq 1$, $\mu > 3 + \frac{d}{2}$, $k \leq 2\mu - d - 6$ and $(\eta_0, \psi_0) \in \mathcal{H}_k^{\mu+1/2,1/2} \times \mathcal{H}_k^{\mu,1/2}$, then there exist $T > 0$ and a unique solution*

$$(\eta, \psi) \in C([-T, T], \mathcal{H}_k^{\mu+1/2,1/2} \times \mathcal{H}_k^{\mu,1/2})$$

to the Cauchy problem of (1-5) with initial data (η_0, ψ_0).

The study of the Cauchy problem for the water wave equation dates back to [Nalimov 1974; Kano and Nishida 1979; Yosihara 1982; 1983]. The local well-posedness in Sobolev spaces with general initial data were achieved in [Wu 1997; 1999; Beyer and Günther 1998]. Our analysis of the water wave equation relies on the paradifferential calculus of [Bony 1986], which was introduced to the study of the water wave equation in [Alazard and Métivier 2009] and later allowed Alazard, Burq and Zuily [2011; 2014] to prove the local well-posedness with low Sobolev regularities. For recent progress of the Cauchy problem, see e.g., [Alazard and Delort 2015; de Poyferré and Nguyen 2016; 2017; Deng, Ionescu, Pausader and Pusateri 2017; Hunter, Ifrim and Tataru 2016; Ifrim and Tataru 2017; Ionescu and Pusateri 2018; Ming, Rousset and Tzvetkov 2015; Rousset and Tzvetkov 2011; Wang 2020].

To prove Theorem 1.6, we shall combine the analysis in [Alazard, Burq and Zuily 2011] and a paradifferential calculus in weighted Sobolev spaces. The latter can be achieved by modifying the definition of paradifferential operators via a spatial dyadic decomposition. More precisely, if a is a symbol, then we define

$$\mathcal{P}_a = \sum_{j \in \mathbb{N}} \underline{\psi}_j T_{\psi_j a} \underline{\psi}_j,$$

where $\{\psi_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$ is a dyadic partition of unity of $\mathbb{R}^d$, $\underline{\psi}_j = \sum_{|k-j|\leq N} \psi_k$ for some sufficiently large $N \in \mathbb{N}$, and $T_{\psi_j a}$ is the usual paradifferential operator of Bony. Such dyadic paradifferential calculus inherits the symbolic calculus and the paralinearization of Bony's calculus, while at the same time allows the spatial polynomial growth/decay of symbols to play their roles in estimates.

We do not attempt to lower μ to $> 2 + \frac{d}{2}$ as it was in [Alazard, Burq and Zuily 2011]. The range of k is so chosen such that $\mu - \frac{k}{2} > 3 + \frac{d}{2}$, enabling us to paralinearize (1-5) in $\mathcal{H}_k^{\mu}$. We should mention that the existence of gravity water waves (water waves without surface tension) in uniformly local weighted Sobolev spaces was obtain by [Nguyen 2016] via a periodic spatial decomposition from [Alazard, Burq and Zuily 2016].

1C4. *Propagation at infinity.* Our first main result concerns the propagation of quasihomogeneous wavefront sets with parameters $(\delta, \rho) = \left(\frac{1}{2}, 1\right)$, corresponding to Theorem 1.4(1).

Theorem 1.7. *Suppose that* $d \geq 1$, $\mu > 3 + \frac{d}{2}$, $3 \leq k < 2\mu - K - d$ *for some* $K > 0$, *and*

$$(\eta, \psi) \in C([-T, T], \mathcal{H}_k^{\mu+1/2,1/2} \times \mathcal{H}_k^{\mu,1/2}),$$

where $T > 0$, *solves* (1-5). *If* $t_0 \in [-T, T]$ *and*

$$(x_0, \xi_0) \in \mathrm{WF}_{1/2,1}^{\mu+1/2+\sigma}(\eta(0)) \cup \mathrm{WF}_{1/2,1}^{\mu+\sigma}(\psi(0))$$

such that $\xi_0 \neq 0$, $0 \leq \sigma \leq \frac{k}{2} - \frac{3}{2}$ *and*

$$x_0 + \tfrac{3}{2} t |\xi_0|^{-1/2} \xi_0 \neq 0$$

for all t *between* 0 *and* t_0, *then*

$$\big(x_0 + \tfrac{3}{2} t_0 |\xi_0|^{-1/2} \xi_0, \xi_0\big) \in \mathrm{WF}_{1/2,1}^{\mu+1/2+\sigma}(\eta(t_0)) \cup \mathrm{WF}_{1/2,1}^{\mu+\sigma}(\psi(t_0)).$$

We will see that, by Lemma 2.15, if $(\eta, \psi) \in \mathcal{H}_k^{\mu+1/2,1/2} \times \mathcal{H}_k^{\mu,1/2}$, then

$$\mathrm{WF}_{1/2,1}^{\mu+1/2}(\eta) \cup \mathrm{WF}_{1/2,1}^{\mu}(\psi) \subset \{x = 0\} \cup \{\xi = 0\}.$$

By [Alazard and Métivier 2009], we expect σ to be at most $\mu - \alpha - \frac{d}{2}$ for some $\alpha > 0$, corresponding to the gain of regularity by the remainder in the paralinearization procedure. Theorem 1.7 does not give the optimal upper bound for σ, as it is not our priority, but when $k = 2\mu - K - d$, the parameter σ can still be as large as $\mu - \frac{K}{2} - \frac{d}{2} - \frac{3}{2}$, almost reaching the paradifferential threshold.

1C5. *Microlocal smoothing effect.* Our second main result shows that singularities of the initial data which are nontrapped with respect to the initial geometry instantaneously generate an element in the quasihomogeneous wavefront set with parameters $(\delta, \rho) = \big(\frac{1}{2}, 1\big)$, corresponding to Theorem 1.4(2).

Observe that if η is sufficiently regular, then Σ endowed with the metric inherited from $\mathbb{R}^{d+1}$ is isometric to $(\mathbb{R}^d, \varrho)$, where

$$\varrho = \begin{pmatrix} \mathrm{Id} + (\nabla\eta)\,{}^t(\nabla\eta) & \nabla\eta \\ {}^t(\nabla\eta) & 1 \end{pmatrix}.$$

Define $\Sigma_0 = \Sigma|_{t=0}$ and $\varrho_0 = \varrho|_{t=0}$. We identify the cogeodesic flow $\mathcal{G}$ on $T^*\Sigma_0$ with the Hamiltonian flow on $\mathbb{R}^{2d}$ of the symbol $G(x, \xi) = {}^t\xi \varrho_0(x)^{-1} \xi$. Precisely $\mathcal{G} = \mathcal{G}_s(x, \xi)$ is defined by the equation

$$\partial_s \mathcal{G}_s = (\partial_\xi G, -\partial_x G)(\mathcal{G}_s), \quad \mathcal{G}_0 = \mathrm{Id}_{\mathbb{R}^{2d}}. \tag{1-8}$$

Definition 1.8. A point $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \backslash 0)$ is called forwardly (resp. backwardly) nontrapped with respect to $\mathcal{G}$ if, with an abuse of notation, the cogeodesic $\{(x_s, \xi_s) = \mathcal{G}_s(x_0, \xi_0)\}_{s \in \mathbb{R}}$ satisfies

$$\lim_{s \to +\infty} |x_s| = \infty \quad \big(\text{resp.} \lim_{s \to -\infty} |x_s| = \infty\big).$$

Theorem 1.9. *If* $d \geq 1$, $\mu > 3 + \frac{d}{2}$, $3 \leq k < \frac{2}{3}\big(\mu - 1 - \frac{d}{2}\big)$, *and*

$$(\eta, \psi) \in C([-T, T], \mathcal{H}_k^{\mu+1/2,1/2} \times \mathcal{H}_k^{\mu,1/2}),$$

where $T > 0$, *solves* (1-5). *Let*

$$(x_0, \xi_0) \in \mathrm{WF}_{0,1}^{\mu+1/2+\sigma}(\eta(0)) \cup \mathrm{WF}_{0,1}^{\mu+\sigma}(\psi(0)),$$

where $\xi_0 \neq 0$ *and* $0 \leq \sigma \leq \frac{3}{2}k$. *If* (x_0, ξ_0) *is forwardly (resp. backwardly) nontrapped, and the cogeodesic* $\{(x_s, \xi_s)\}_{s\in\mathbb{R}}$ *is defined as above, then there exists* $\xi_{+\infty}$ *(resp.* $\xi_{-\infty}$*) in* $\mathbb{R}^d\backslash\{0\}$ *such that*

$$\lim_{s\to\infty} \xi_s = \xi_{+\infty} \quad \big(\textit{resp.} \ \lim_{s\to\infty} \xi_{-s} = \xi_{-\infty}\big),$$

and moreover, for all $0 < t_0 \leq T$ *(resp.* $-T \leq t_0 < 0$*), we have*

$$\big(\tfrac{3}{2}t_0|\xi_{+\infty}|^{-1/2}\xi_{+\infty}, \xi_{+\infty}\big) \in \mathrm{WF}^{\mu+1/2+\sigma}_{1/2,1}(\eta(t_0)) \cup \mathrm{WF}^{\mu+\sigma}_{1/2,1}(\psi(t_0)),$$
$$\big(\textit{resp.}\ \big(\tfrac{3}{2}t_0|\xi_{-\infty}|^{-1/2}\xi_{-\infty}, \xi_{-\infty}\big) \in \mathrm{WF}^{\mu+1/2+\sigma}_{1/2,1}(\eta(t_0)) \cup \mathrm{WF}^{\mu+\sigma}_{1/2,1}(\psi(t_0))\big).$$

We remark that the asymptotic directions $\xi_{\pm\infty}$ are determined solely by the geometry of Σ_0. This is due to the infinite speed of propagation. We can also prove that the nontrapping assumption is, at least in the following two cases, unnecessary: if $d = 1$, or if $\nabla\eta(0) \in L^\infty$ and $\|\langle x\rangle\nabla^2\eta(0)\|_{L^\infty}$ is sufficiently small. In both cases we obtain the following local smoothing effect.

Corollary 1.10. *Suppose* d, μ, k, σ *satisfy the hypothesis of the previous theorem,* $T > 0$,

$$(\eta, \psi) \in C([-T, T], \mathcal{H}^{\mu+1/2,1/2}_k \times \mathcal{H}^{\mu,1/2}_k)$$

solves (1-5), and both of the following two conditions are satisfied:

(1) *Either* $d = 1$ *or* $\|\langle x\rangle\nabla^2\eta(0)\|_{L^\infty}$ *is sufficiently small.*

(2) $\mathrm{WF}^{\mu+1/2+\sigma}_{1/2,1}(\eta(0)) \cup \mathrm{WF}^{\mu+\sigma}_{1/2,1}(\psi(0)) \subset \{x = 0\} \cup \{\xi = 0\}$.

Then, for all $t_0 \in [-T, T]\backslash\{0\}$ *and for all* $\epsilon > 0$,

$$(\eta(t_0), \psi(t_0)) \in H^{\mu+1/2+\sigma-\epsilon}_{\mathrm{loc}} \times H^{\mu+\sigma-\epsilon}_{\mathrm{loc}}.$$

The second condition is satisfied if, by Lemma 2.15, there exists $(k, k') \in \mathbb{R}^2$ such that

$$(\eta(0), \psi(0)) \in H^{\mu+1/2+\sigma-k}_{2k} \times H^{\mu+\sigma-k'}_{2k'}.$$

This is particularly the case if $(\eta(0), \psi(0)) \in \mathscr{E}'(\mathbb{R}^d) \times \mathscr{E}'(\mathbb{R}^d)$.

We refer to [Christianson, Hur and Staffilani 2009; Alazard, Burq and Zuily 2011] for local smoothing effects of 2-dimensional capillary-gravity water waves. See also [Alazard, Ifrim and Tataru 2022] for a Morawetz inequality of 2-dimensional gravity water waves.

1D. *Outline of paper.* In Section 2, we present basic properties of weighted Sobolev spaces and the quasihomogeneous wavefront set. In Section 3, we prove Theorem 1.4 by extending the idea of Nakamura. In Section 4, we review the paradifferential calculus of Bony, and extend it to weighted Sobolev spaces by a spatial dyadic decomposition. We also develop a quasihomogeneous semiclassical paradifferential calculus, and study its relations with the quasihomogeneous wavefront set. In Section 5, we study the Dirichlet–Neumann operator in weighted Sobolev spaces and prove the existence of asymptotically flat gravity-capillary water waves, i.e., Theorem 1.6. In Section 6, we prove our main results, i.e., Theorem 1.7, Theorem 1.9 and Corollary 1.10, by extending the proof of Theorem 1.4 to the quasilinear equation using the paradifferential calculus.

2. Quasihomogeneous microlocal analysis

In this section we develop the quasihomogeneous semiclassical calculus and discuss its relation with weighted Sobolev spaces and the quasihomogeneous wavefront set.

2A. *Quasihomogeneous semiclassical calculus.*

Definition 2.1. For $(\mu, k) \in \mathbb{R}^2$, set $m_k^\mu(x,\xi) = \langle x\rangle^k \langle\xi\rangle^\mu$. Let $a_h \in C^\infty(\mathbb{R}^{2d})$. We say that $a_h \in S_k^\mu$ if for all $\alpha, \beta \in \mathbb{N}^d$, there exists $C_{\alpha\beta} > 0$, such that, for all $(x,\xi) \in \mathbb{R}^{2d}$,

$$\sup_{h\in(0,1]} |\partial_x^\alpha \partial_\xi^\beta a_h(x,\xi)| \le C_{\alpha\beta} m_{k-|\alpha|}^{\mu-|\beta|}(x,\xi). \tag{2-1}$$

We say that $a_h \in S_k^\mu$ is (μ,k)-elliptic if there exist $R > 0$, $C > 0$ such that, for $|x| + |\xi| \ge R$,

$$\inf_{h\in(0,1]} |a_h(x,\xi)| \ge C m_k^\mu(x,\xi).$$

Also write $S_\infty^\infty = \bigcup_{(\mu,k)\in\mathbb{R}^2} S_k^\mu$, and $S_{-\infty}^{-\infty} = \bigcap_{(\mu,k)\in\mathbb{R}^2} S_k^\mu$.

We say that $a_h \in S_{-\infty}^{-\infty}$ is elliptic at (x_0,ξ_0) if, for some neighborhood Ω of (x_0,ξ_0),

$$\inf_{h\in(0,1]} \inf_{(x,\xi)\in\Omega} |a_h(x,\xi)| > 0.$$

Definition 2.2. Let $\delta, \rho \in \mathbb{R}$ such that $\delta + \rho > 0$ and, for all $h \in (0,1]$, define the scaling

$$\theta_h^{\delta,\rho} : (x,\xi) \mapsto (h^\delta x, h^\rho \xi), \tag{2-2}$$

which induces a pullback $\theta_{h,*}^{\delta,\rho}$ on S_∞^∞: $\theta_{h,*}^{\delta,\rho} a_h = a_h \circ \theta_h^{\delta,\rho}$. Then define, by (1-7),

$$\mathrm{Op}_h^{\delta,\rho}(a_h) = \mathrm{Op}(\theta_{h,*}^{\delta,\rho} a_h) = a(h^\delta x, h^\rho D_x).$$

The scaling $\vartheta_h^\delta u(x) = h^{\delta d/2} u(h^\delta x)$ defines an isometry on $L^2(\mathbb{R}^d)$. Therefore, by the formula

$$(\vartheta_h^\delta)^{-1} \mathrm{Op}_h^{\delta,\rho}(a) \vartheta_h^\delta = \mathrm{Op}_h^{0,\delta+\rho}(a), \tag{2-3}$$

we deduce the following results from the usual semiclassical calculus, for which we refer to [Zworski 2012].

Proposition 2.3. *There exists $K > 0$ such that, if $a \in C^\infty(\mathbb{R}^{2d})$ with $\|\partial_x^\alpha \partial_\xi^\beta a\|_{L^\infty} \le M$ for all $|\alpha| + |\beta| \le d$, then $\mathrm{Op}_h^{\delta,\rho}(a) : L^2 \to L^2$ and $\|\mathrm{Op}_h^{\delta,\rho}(a)\|_{L^2\to L^2} \le KM$.*

Proposition 2.4. *There exists a bilinear operator $\sharp_h^{\delta,\rho} : S_\infty^\infty \times S_\infty^\infty \to S_\infty^\infty$ such that*

$$\mathrm{Op}_h^{\delta,\rho}(a_h)\, \mathrm{Op}_h^{\delta,\rho}(b_h) = \mathrm{Op}_h^{\delta,\rho}(a_h \sharp_h^{\delta,\rho} b_h).$$

Moreover, if $a_h \in S_k^\mu$ and $b_h \in S_\ell^\nu$, then $a_h \sharp_h^{\delta,\rho} b_h \in S_{k+\ell}^{\mu+\nu}$. For all $r > 0$, define

$$a_h \sharp_{h,r}^{\delta,\rho} b_h = \sum_{|\alpha|<r} \frac{h^{|\alpha|(\delta+\rho)}}{\alpha!} \partial_\xi^\alpha a_h D_x^\alpha b_h. \tag{2-4}$$

Then we have

$$a_h \sharp_h^{\delta,\rho} b_h - a_h \sharp_{h,r}^{\delta,\rho} b_h = \mathcal{O}(h^{r(\delta+\rho)})_{S_{k+\ell-r}^{\mu+\nu-r}}.$$

Proposition 2.5. *There exists a linear operator* $\zeta_h^{\delta,\rho} : S_\infty^\infty \to S_\infty^\infty$ *such that*

$$\mathrm{Op}_h^{\delta,\rho}(a_h)^* = \mathrm{Op}_h^{\delta,\rho}(\zeta_h^{\delta,\rho} a_h).$$

Moreover if $a_h \in S_k^\mu$, *then* $\zeta_h^{\delta,\rho} a_h \in S_k^\mu$. *For* $r > 0$, *define*

$$\zeta_{h,r}^{\delta,\rho} a_h = \sum_{|\alpha|<r} \frac{h^{|\alpha|(\delta+\rho)}}{\alpha!} \partial_\xi^\alpha D_x^\alpha \bar{a}_h. \tag{2-5}$$

Then we have

$$\zeta_h^{\delta,\rho} a_h - \zeta_{h,r}^{\delta,\rho} a_h = \mathcal{O}(h^{r(\delta+\rho)})_{S_{k-r}^{\mu-r}}.$$

Proposition 2.6 (sharp Gårding inequality). *If* $\delta + \rho > 0$ *and* $a_h \in S_0^0$ *such that* $\operatorname{Re} a_h \geq 0$, *then there exists* $C > 0$ *such that, for all* $u \in L^2(\mathbb{R}^d)$ *and* $0 < h < 1$, *we have*

$$\operatorname{Re}(\mathrm{Op}_h^{\delta,\rho}(a_h)u, u)_{L^2} \geq -Ch^{\delta+\rho}\|u\|_{L^2}^2.$$

2B. *Weighted Sobolev spaces.* Recall the weighted Sobolev spaces defined in Definition 1.5.

Proposition 2.7. *We have* $\mathscr{S}(\mathbb{R}^d) = \bigcap_{\mu,k\in\mathbb{R}} H_k^\mu$ *and* $\mathscr{S}'(\mathbb{R}^d) = \bigcup_{\mu,k\in\mathbb{R}} H_k^\mu$.

Proof. Clearly $\mathscr{S}(\mathbb{R}^d) \subset \bigcap_{\mu,k\in\mathbb{R}} H_k^\mu$. The converse follows by the Sobolev embedding theorems. As for the second statement, clearly $\bigcup_{(\mu,k)\in\mathbb{R}^2} H_k^\mu \subset \mathscr{S}'(\mathbb{R}^d)$. Conversely, if $u \in \mathscr{S}'(\mathbb{R}^d)$, then there exists $N > 0$, such that for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$ we have

$$\langle u, \varphi\rangle_{\mathscr{S}',\mathscr{S}} \lesssim \sum_{|\alpha|+|\beta|\leq N} \|x^\alpha \partial_x^\beta \varphi\|_{L^\infty} \lesssim \|\mathrm{Op}(m_N^N)\varphi\|_{L^2}.$$

By duality this implies that $u \in H_{-N}^{-N}$. □

Lemma 2.8. *If* $u \in \mathscr{S}'(\mathbb{R}^d)$, *then there exists* $N > 0$ *such that*

$$u = h^{-N}\,\mathrm{Op}_h^{\delta,\rho}(m_{-N}^{-N})\mathcal{O}(1)_{L^2}.$$

Therefore, if $\delta + \rho > 0$, *and* $a_h \in \mathcal{O}(h^\infty)_{S_{-\infty}^{-\infty}}$, *then* $\mathrm{Op}_h^{\delta,\rho}(a_h)u_h = \mathcal{O}(h^\infty)_{\mathscr{S}}$.

Proof. By the proof of Proposition 2.7, there exists $N > 0$ such that, for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$,

$$\langle u, \varphi\rangle_{\mathscr{S}',\mathscr{S}} \lesssim \sum_{|\alpha|+|\beta|\leq N} \|x^\alpha \partial_x^\beta \varphi\|_{L^\infty} \lesssim h^{-N}\|\mathrm{Op}_h^{\delta,\rho}(m_N^N)\varphi\|_{L^2}.$$

Again we conclude by duality. □

Definition 2.9. We say that a linear operator $\mathcal{A} : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ is of order $(\nu, \ell) \in \mathbb{R}^2$, and write $\mathcal{A} \in \mathscr{O}_\ell^\nu$ if for all $(\mu, k) \in \mathbb{R}^2$ there exists $C > 0$ such that for all $u \in \mathscr{S}(\mathbb{R}^d)$ we have

$$\|\mathcal{A}u\|_{H_{k-\ell}^{\mu-\nu}} \leq C\|u\|_{H_k^\mu}.$$

Therefore $\mathcal{A}$ extends to a bounded linear operator from H_k^μ to $H_{k-\ell}^{\mu-\nu}$. We write $\mathcal{A} \in \mathscr{O}_{-\infty}^{-\infty}$ if $\mathcal{A} \in \mathscr{O}_\ell^\nu$ for all $(\nu, \ell) \in \mathbb{R}^2$.

Let $\mathscr{A}$ be any nonempty set. Let $\mathcal{A}_\alpha : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ and $C_\alpha > 0$ be indexed by $\alpha \in \mathscr{A}$. We say $\mathcal{A}_\alpha = \mathcal{O}(C_\alpha)_{\mathscr{O}^\nu_\ell}$ if for all $(\mu, k) \in \mathbb{R}^2$ there exists $K > 0$ such that, for all $\alpha \in \mathscr{A}$, we have

$$\|\mathcal{A}_\alpha\|_{H^\mu_k \to H^{\mu-\nu}_{k-\ell}} \le K C_\alpha.$$

By Propositions 2.3 and 2.4, we obtain:

Proposition 2.10. *The following mapping properties of pseudodifferential operators hold*:

(1) *If $a_h \in S^\nu_\ell$ with $(\nu, \ell) \in \mathbb{R}^2$, then* $\operatorname{Op}(a_h) \in \mathscr{O}^\nu_\ell$.

(2) *If $u \in \mathscr{S}'(\mathbb{R}^d)$, then $u \in H^\mu_k$ if and only if there exists a (μ, k)-elliptic symbol $a_h \in S^\mu_k$ such that* $\operatorname{Op}(a_h)u = \mathcal{O}(1)_{L^2}$.

Next, we characterize weighted Sobolev spaces by a dyadic decomposition.

Definition 2.11. The set $\mathscr{P}$ consists of all maps of the form

$$\psi : \mathbb{N} \to C^\infty_c(\mathbb{R}^d), \quad j \mapsto \psi_j,$$

such that the following conditions are satisfied:

(1) There exists $C > 1$ such that for all $j \ge 1$ we have

$$\operatorname{supp} \psi_j \subset \{x \in \mathbb{R}^d : C^{-1} 2^j \le |x| \le C 2^j\}.$$

(2) For all $j \ge 0$, the function ψ_j is nonnegative.

(3) There exists $C > 1$ such that $C^{-1} \le \sum_{j\in\mathbb{N}} \psi_j \le C$.

(4) For all $\alpha \in \mathbb{N}$ there exists C_α such that for all $j \in \mathbb{N}$ we have

$$\|\partial^\alpha_x \psi_j\|_{L^\infty} \le C_\alpha 2^{-j|\alpha|}.$$

The set $\mathscr{P}_*$ consists of all $\psi \in \mathscr{P}$ such that

(5) $\sum_{j\in\mathbb{N}} \psi_j = 1$, and

(6) $\operatorname{supp} \psi_j \cap \operatorname{supp} \psi_k = \varnothing$ whenever $|j - k| > 2$.

If $\psi, \tilde{\psi} \in \mathscr{P}$ such that $\psi_j \tilde{\psi}_j = \psi_j$ for all $j \in \mathbb{N}$, then we write $\psi \Subset \tilde{\psi}$.

Proposition 2.12. *If $\mu, k \in \mathbb{R}$, $\psi \in \mathscr{P}$ and $u \in \mathscr{S}'(\mathbb{R}^d)$, then $u \in H^\mu_k$ if and only if*

$$\sum_{j\in\mathbb{N}} 2^{2jk} \|\psi_j u\|^2_{H^\mu} < \infty.$$

Moreover, there exists $C > 1$ such that, for all $u \in H^\mu_k$, we have

$$C^{-1}\|u\|^2_{H^\mu_k} \le \sum_{j\in\mathbb{N}} 2^{2jk} \|\psi_j u\|^2_{H^\mu} \le C\|u\|^2_{H^\mu_k}.$$

Proof. We may assume that $\psi \in \mathscr{P}_*$ because if $\phi^1, \phi^2 \in \mathscr{P}$ then

$$\sum_{j\in\mathbb{N}} 2^{2jk} \|\phi^1_j u\|^2_{H^\mu} \simeq \sum_{j\in\mathbb{N}} 2^{2jk} \|\phi^2_j u\|^2_{H^\mu}.$$

Define $\tilde\psi \in \mathscr{P}$ by setting $\tilde\psi_j = \sum_{|k-j|\le 2} \psi_k$ for all $j \in \mathbb{N}$. Then $\psi \Subset \tilde\psi$. Note that the family of multiplication operators $\{2^{-jk}\langle x\rangle^k \tilde\psi_j\}_{j\in\mathbb{N}}$ is bounded in $\mathscr{O}^0_0$, which implies that, for all $\mu \in \mathbb{N}$, the family of pseudodifferential operators $\{2^{-jk}\langle D_x\rangle^\mu \langle x\rangle^k \tilde\psi_j \langle D_x\rangle^{-\mu}\}_{j\in\mathbb{N}}$ is bounded in $\mathscr{O}^0_0$. Therefore, for all $u \in H^\mu_k$, we have

$$2^{2jk}\|\psi_j u\|^2_{H^\mu} \lesssim \|\langle D_x\rangle^\mu \psi_j \langle x\rangle^k u\|^2_{L^2} \lesssim \|\tilde\psi_j \langle D_x\rangle^\mu \psi_j \langle x\rangle^k u\|^2_{L^2} + \|(1-\tilde\psi_j)\langle D_x\rangle^\mu \psi_j \langle x\rangle^k u\|^2_{L^2}. \tag{2-6}$$

Apply Proposition 2.4 with $(\delta,\rho)=(1,0)$ and $h=2^{-j}$, we obtain that, for all $N>0$, the estimate

$$(1-\tilde\psi_j)\langle D_x\rangle^\mu \psi_j \langle D_x\rangle^{-\mu} = \mathcal{O}(2^{-jN})_{L^2\to L^2} \tag{2-7}$$

holds uniformly for all $j \in \mathbb{N}$. Therefore,

$$\sum_{j\in\mathbb{N}} \|(1-\tilde\psi_j)\langle D_x\rangle^\mu \psi_j \langle x\rangle^k u\|^2_{L^2} \lesssim \sum_{j\in\mathbb{N}} 2^{-2jN}\|u\|^2_{H^\mu_k} \lesssim \|u\|^2_{H^\mu_k}.$$

For $r \in \{0,1,\dots,9\}$, set

$$a_r = \sum_{j\in 10\mathbb{N}+r} \tilde\psi_j \langle\xi\rangle^\mu \sharp(\langle x\rangle^k \psi_j) \in S^\mu_k,$$

where $\sharp = \sharp^{0,0}_1$. Observe that if $0 \ne j-j' \in 10\mathbb{N}$, then $\operatorname{supp}\tilde\psi_j \cap \operatorname{supp}\tilde\psi_{j'} = \varnothing$. Therefore,

$$\sum_{j\in\mathbb{N}} \|\tilde\psi_j \langle D_x\rangle^\mu \psi_j \langle x\rangle^k u\|^2_{L^2} = \sum_{r=0}^{9} \sum_{j\in 10\mathbb{N}+r} \|\tilde\psi_j \langle D_x\rangle^\mu \psi_j \langle x\rangle^k u\|^2_{L^2} = \sum_{r=0}^{9} \|\mathrm{Op}(a_r)u\|^2_{L^2} \lesssim \|u\|^2_{H^\mu_k}. \tag{2-8}$$

Combining (2-6), (2-7) and (2-8), we prove that if $u \in H^\mu_k$ then $\sum_{j\in\mathbb{N}} 2^{2jk}\|\psi_j u\|^2_{H^\mu} \lesssim \|u\|^2_{H^\mu_k}$.

Conversely, assume that $\sum_{j\in\mathbb{N}} 2^{2jk}\|\psi_j u\|^2_{H^\mu} < \infty$. Much as above, we have

$$\begin{aligned}\infty > \sum_{j\in\mathbb{N}} 2^{2jk}\|\langle D_x\rangle^\mu \psi_j u\|^2_{L^2} &\gtrsim \sum_{r=0}^{9} \sum_{j\in 10\mathbb{N}+r} \|\tilde\psi_j \langle D_x\rangle^\mu \psi_j \langle x\rangle^k u\|^2_{L^2} \\ &\gtrsim \sum_{r=0}^{9} \|\mathrm{Op}(a_r)u\|^2_{L^2} \gtrsim \|\mathrm{Op}(a)u\|^2_{L^2},\end{aligned} \tag{2-9}$$

where $a = \sum_{r=0}^{9} a_r$. Observe a is (μ,k)-elliptic, so $u \in H^\mu_k$. By the symbolic calculus, there exists $r \in S^{-\infty}_{-\infty}$ such that

$$\|u\|^2_{H^\mu_k} \lesssim \|\mathrm{Op}(a)u\|^2_{L^2} + \|\mathrm{Op}(r)u\|^2_{L^2}. \tag{2-10}$$

For the remainder term, we have

$$\|\mathrm{Op}(r)u\|^2_{L^2} = (u, \mathrm{Op}(r^*\sharp r)u)_{L^2} = \sum_{j\in\mathbb{N}} (u, \mathrm{Op}(r^*\sharp r)\psi_j u)_{L^2}.$$

For each term in the summation, by the analysis above (2-6), we have, for all $N>0$ and $\varepsilon>0$,

$$\begin{aligned}(u, \mathrm{Op}(r^*\sharp r)\psi_j u)_{L^2} &= (\mathrm{Op}(m^\mu_k)u, \mathrm{Op}(m^{-\mu}_{-k}\sharp r^*\sharp r\sharp m^{-\mu}_{N-k})\langle D_x\rangle^\mu \langle x\rangle^{-N+k}\psi_j u)_{L^2} \\ &\lesssim \|u\|_{H^\mu_k}\|\langle D_x\rangle^\mu \langle x\rangle^{-N+k}\psi_j u\|_{L^2} \\ &\lesssim 2^{-jN}\|u\|_{H^\mu_k} 2^{jk}\|\langle D_x\rangle^\mu \psi_j u\|_{L^2} \\ &\lesssim 2^{-jN}(\varepsilon\|u\|^2_{H^\mu_k} + \varepsilon^{-1}2^{2jk}\|\langle D_x\rangle^\mu \psi_j u\|^2_{L^2}),\end{aligned}$$

where the constants are independent of ε. Summing up in j,

$$\|\mathrm{Op}(r)u\|_{L^2}^2 \lesssim \varepsilon\|u\|_{H_k^\mu}^2 + \varepsilon^{-1}\sum_{j\in\mathbb{N}} 2^{2jk}\|\psi_j u\|_{H^\mu}^2. \tag{2-11}$$

By choosing ε sufficiently small, we conclude by (2-9), (2-10) and (2-11) that

$$\|u\|_{H_k^\mu}^2 \lesssim \sum_{j\in\mathbb{N}} 2^{2jk}\|\psi_j u\|_{H^\mu}^2$$

and finishes the proof. ☐

2C. *The quasihomogeneous wavefront set.* In this section the parameters δ, ρ, μ satisfy the conditions in Definition 1.3 without further specification. By Lemma 2.8, the following characterization of the quasihomogeneous wavefront set is easy to prove by the symbolic calculus.

Proposition 2.13. *If $u \in \mathscr{S}'(\mathbb{R}^d)$, then $(x_0, \xi_0) \notin \mathrm{WF}_{\delta,\rho}^\mu(u)$ if and only if there exists $a_h \in S_{-\infty}^{-\infty}$ which is elliptic at (x_0, ξ_0) such that $\mathrm{Op}_h^{\delta,\rho}(a_h)u = \mathcal{O}(h^\mu)_{L^2}$ for $h \in (0, 1]$.*

Lemma 2.14. *If $u \in \mathscr{S}'(\mathbb{R}^d)$ and $a_h \in S_{-\infty}^{-\infty}$ is such that*

$$\overline{\bigcup_{h\in(0,1]} \operatorname{supp} a_h} \Subset \mathbb{R}^{2d} \setminus \mathrm{WF}_{\delta,\rho}^\mu(u),$$

then $\langle u, \mathrm{Op}_h^{\delta,\rho}(a_h)u\rangle_{\mathscr{S}',\mathscr{S}} = \mathcal{O}(h^{2\mu})$ and consequently $\mathrm{Op}_h^{\delta,\rho}(a_h)u = \mathcal{O}(h^\mu)_{L^2}$ for $h \in (0, 1]$.

Proof. Let $K = \overline{\bigcup_{h\in(0,1]} \operatorname{supp} a_h}$ and let $\{\Omega_i\}_{i\in I}$ be an open cover of K. Let $b_h^i \in S_{-\infty}^{-\infty}$ be elliptic everywhere in Ω_i such that $\mathrm{Op}_h^{\delta,\rho}(b_h^i)u = \mathcal{O}(h^\mu)_{L^2}$. By a partition of unity, we may assume that $K \subset \Omega := \Omega_{i_0}$ for some $i_0 \in I$, and set $b_h = b_h^{i_0}$. By the ellipticity of b_h, we can find $c_h \in S_{-\infty}^{-\infty}$ and $r_h = \mathcal{O}(h^N)_{S_{-\infty}^{-\infty}}$ for some large $N > 0$ such that $a_h = (\zeta_h^{\delta,\rho} b_h)\sharp_h^{\delta,\rho} c_h \sharp_h^{\delta,\rho} b_h + r_h$. Therefore, by Lemma 2.8,

$$\begin{aligned}\langle u, \mathrm{Op}_h^{\delta,\rho}(a_h)u\rangle_{\mathscr{S}',\mathscr{S}} &= (\mathrm{Op}_h^{\delta,\rho}(b_h)u, \mathrm{Op}_h^{\delta,\rho}(c_h)\,\mathrm{Op}_h^{\delta,\rho}(b_h)u)_{L^2} + \langle u, \mathrm{Op}_h^{\delta,\rho}(r_h)u\rangle_{\mathscr{S}',\mathscr{S}}\\ &= \mathcal{O}(h^\mu)^2 + \mathcal{O}(h^\infty) = \mathcal{O}(h^{2\mu}).\end{aligned}$$

Next, observe that there exists $w_h \in S_{-\infty}^{-\infty}$ and $\tilde{r}_h = \mathcal{O}(h^N)_{S_{-\infty}^{-\infty}}$ such that $\operatorname{supp} w_h \subset K$ and

$$\mathrm{Op}_h^{\delta,\rho}(a_h)^*\,\mathrm{Op}_h^{\delta,\rho}(a_h) = \mathrm{Op}_h^{\delta,\rho}(w_h) + \mathrm{Op}_h^{\delta,\rho}(\tilde{r}_h).$$

Therefore,

$$\begin{aligned}\|\mathrm{Op}_h^{\delta,\rho}(a_h)u\|_{L^2}^2 &= \langle u, \mathrm{Op}_h^{\delta,\rho}(a_h)^*\,\mathrm{Op}_h^{\delta,\rho}(a_h)u\rangle_{\mathscr{S}',\mathscr{S}}\\ &= \langle u, \mathrm{Op}_h^{\delta,\rho}(w_h)u\rangle_{\mathscr{S}',\mathscr{S}} + \mathcal{O}(h^{2\mu}) = \mathcal{O}(h^{2\mu}).\end{aligned}$$

☐

Lemma 2.15. *If $u \in \mathscr{S}'(\mathbb{R}^d)$. Then the following statements hold:*

(1) *The quasihomogeneous wavefront set $\mathrm{WF}_{\delta,\rho}^\mu(u)$ is a closed (δ, ρ)-cone. To be precise, this means $\theta_h^{\delta,\rho}\,\mathrm{WF}_{\delta,\rho}^\mu(u) = \mathrm{WF}_{\delta,\rho}^\mu(u)$ for all $h > 0$ where the scaling $\theta_h^{\delta,\rho}$ is defined by (2-2).*

(2) *If $\gamma > 0$ then $\mathrm{WF}_{\delta,\rho}^\mu(u) = \mathrm{WF}_{\delta/\gamma,\rho/\gamma}^{\mu/\gamma}(u)$. Therefore in all situations we can restrict our discussions to the cases where either $\delta = 1$ or $\rho = 1$.*

(3) *For all $(x_0, \xi_0) \in \mathbb{R}^{2d}$, we have $(x_0, \xi_0) \in \mathrm{WF}_{\delta,\rho}^\mu(u)$ if and only if $(\xi_0, -x_0) \in \mathrm{WF}_{\rho,\delta}^\mu(\hat{u})$.*

(4) *For all* $(x_0, \xi_0) \in \mathbb{R}^{2d}$, *we have* $(x_0, \xi_0) \in \mathrm{WF}^{\mu}_{\delta,\rho}(u)$ *if and only if* $(x_0, -\xi_0) \in \mathrm{WF}^{\mu}_{\delta,\rho}(\bar{u})$.

(5) *Define* $\mathrm{WF}^{\mu}_{\delta,\rho}(u)^\circ = \mathrm{WF}^{\mu}_{\delta,\rho}(u) \backslash \mathcal{N}_{\delta,\rho}$, *where*

$$\mathcal{N}_{\delta,\rho} = \begin{cases} \{x=0\} \times \mathbb{R}^d, & \delta > 0,\ \rho = 0, \\ \mathbb{R}^d \times \{\xi = 0\}, & \delta = 0,\ \rho > 0, \\ \{x=0\} \times \mathbb{R}^d \cup \mathbb{R}^d \times \{\xi = 0\}, & \delta > 0,\ \rho > 0. \end{cases} \tag{2-12}$$

If $u \in H^{\mu}_{k}$ *with* $(\mu, k) \in \mathbb{R}^2$ *and* $a_h \in S^{-\infty}_{-\infty}$ *such that*

$$\mathcal{N}_{\delta,\rho} \cap \overline{\bigcup_{0<h<1} \operatorname{supp} a_h} = \varnothing, \tag{2-13}$$

then $\mathrm{Op}^{\delta,\rho}_h(a_h)u = \mathcal{O}(h^{\delta k+\rho\mu})_{L^2}$ *and consequently* $\mathrm{WF}^{\delta k+\rho\mu}_{\delta,\rho}(u)^\circ = \varnothing$.

Proof. The statements (1) and (2) are consequences of the quasihomogeneous scaling (2-2) we used to define the pseudodifferential operators. To prove (3), note that if $a_h \in S^{-\infty}_{-\infty}$ and $\mathcal{F}$ is the Fourier transform operator, then

$$\mathcal{F}^{-1} \mathrm{Op}^{\rho,\delta}_h(a_h)\mathcal{F} = \mathrm{Op}^{\delta,\rho}_h(b_h),$$

where $b_h(x,\xi) = a_h(\xi, -x)$. To prove (4), we use $\overline{\mathrm{Op}(a_h)u_h} = \mathrm{Op}(c_h)\bar{u}_h$, where $c_h(x,\xi) = \overline{a_h(x,-\xi)}$. To prove (5), note that if a_h satisfies the condition (2-13), then

$$(\theta^{\delta,\rho}_{h,*}a_h)\langle\xi\rangle^{-\mu} \sharp^{0,0}_1 \langle x\rangle^{-k} = \mathcal{O}(h^{\delta k+\rho\mu})_{S^0_0}. \qquad \square$$

3. Model equations

We prove Theorem 1.4 by combining the ideas of [Nakamura 2005] and simple scaling arguments.

3A. ***Proof of Theorem 1.4(1).*** If $a \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^{2d})$ and $\mathcal{A} \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}, L^2 \to L^2)$, then define

$$\mathscr{L}_t a = \partial_t a + \{|\xi|^\gamma, a\}, \qquad \mathcal{L}_t \mathcal{A} = \partial_t \mathcal{A} + i[|D_x|^\gamma, \mathcal{A}].$$

Here $\{\cdot,\cdot\}$ denotes the Poisson bracket defined by $\{f, g\} = \partial_\xi f \cdot \partial_x g - \partial_x f \cdot \partial_\xi g$.

Lemma 3.1. *If* $a_h \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}, S^{-\infty}_{-\infty})$ *and satisfies the condition*

$$\overline{\bigcup_{0<h<1} \operatorname{supp} a_h} \cap \{\xi = 0\} = \varnothing,$$

then there exists $b_h \in L^\infty_{\mathrm{loc}}(\mathbb{R}, S^{-\infty}_{-\infty})$, *with* $\operatorname{supp} b_h \subset \operatorname{supp} a_h$, *such that*

$$\mathcal{L}_t \mathrm{Op}^{\delta,\rho}_h(a_h) = \mathrm{Op}^{\delta,\rho}_h(\mathscr{L}_t a_h) + h^{\delta+\rho}\mathrm{Op}^{\delta,\rho}_h(b_h) + \mathcal{O}(h^\infty)_{L^\infty_{\mathrm{loc}}(\mathbb{R}, L^2\to L^2)}.$$

Proof. For all $T > 0$, there exists $\epsilon > 0$ such that

$$\overline{\bigcup_{t\in[-T,T]} \bigcup_{0<h<1} \operatorname{supp} a_h(t,\cdot)} \cap \{|\xi| \le \epsilon\} = \varnothing.$$

Let $\pi \in C^\infty(\mathbb{R}^d)$ such that $0 \le \pi \le 1$, $\pi(\xi) = 0$ for $|\xi| \le \frac{\epsilon}{3}$, and $\pi(\xi) = 1$ for $|\xi| \ge \frac{2\epsilon}{3}$. Then

$$i[|D_x|^\gamma, \mathrm{Op}^{\delta,\rho}_h(a_h)] = ih^{-\rho\gamma}[|h^\rho D_x|^\gamma \pi(h^\rho D_x), \mathrm{Op}^{\delta,\rho}_h(a_h)] + \mathcal{O}(h^\infty)_{L^\infty([-T,T], L^2\to L^2)}.$$

Now that $|\xi|^\gamma \pi(\xi) \in S^\gamma_0$, we conclude by Proposition 2.4 and the hypothesis $\rho\gamma = \delta + \rho$. $\square$

Assume that $\mu = \infty$, as the proof is similar for $\mu < \infty$. Let $(x_0, \xi_0) \notin \mathrm{WF}^{\mu}_{\delta,\rho}(u_0)$ with $\xi_0 \neq 0$. We aim to find $a_h \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}, S^{-\infty}_{-\infty})$ of the asymptotic expansion $a_h \sim \sum_{j\in\mathbb{N}} h^{j(\delta+\rho)} a^j_h$, where $a^j_h \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}, S^{-\infty}_{-\infty})$, such that, for all $t \in \mathbb{R}$, $a_h(t, \cdot)$ is elliptic at $(x_0 + t\gamma|\xi_0|^{\gamma-2}\xi_0, \xi_0)$, and

$$\mathcal{L}_t \operatorname{Op}^{\delta,\rho}_h(a_h) = \mathcal{O}(h^\infty)_{L^\infty_{\mathrm{loc}}(\mathbb{R}, L^2\to L^2)}. \tag{3-1}$$

If such an a_h is found, we let $\mathcal{A}_h(t) = \operatorname{Op}^{\delta,\rho}_h(a_h(t))$ and

$$v_h(t) = e^{it|D_x|^\gamma} \mathcal{A}_h(t) e^{-it|D_x|^\gamma} u_0,$$

then by a direct computation and (3-1), we have

$$\partial_t v_h = e^{it|D_x|^\gamma} \mathcal{L}_t \mathcal{A}_h e^{-it|D_x|^\gamma} u_0 = \mathcal{O}(h^\infty)_{L^\infty_{\mathrm{loc}}(\mathbb{R}, L^2\to L^2)}. \tag{3-2}$$

If we assume that $\operatorname{supp} a_h(0)$ is sufficiently close to (x_0, ξ_0) so that

$$\overline{\bigcup_{h\in(0,1]} \operatorname{supp} a_h(0)} \Subset \mathbb{R}^{2d} \setminus \mathrm{WF}^{\mu}_{\delta,\rho}(u_0),$$

then by Lemma 2.14, we have $v_h(0) = \operatorname{Op}^{\delta,\rho}_h(a_h(0))u_0 = \mathcal{O}(h^\infty)_{L^2}$. Therefore by (3-2), we have $v_h \in \mathcal{O}(h^\infty)_{L^\infty_{\mathrm{loc}}(\mathbb{R}, L^2)}$ and thus $\mathcal{A}_h u \in \mathcal{O}(h^\infty)_{L^\infty_{\mathrm{loc}}(\mathbb{R}, L^2)}$.

To construct a_h, let $\varphi \in C^\infty_c(\mathbb{R}^d \times (\mathbb{R}^d\setminus 0))$ with $\varphi(x_0, \xi_0) \neq 0$, such that $\operatorname{Op}^{\delta,\rho}_h(\varphi)u = \mathcal{O}(h^\infty)_{L^2}$. Then we can construct a_h with $a_h|_{t=0} = \varphi$, with $a^j_h \in W^{\infty,\infty}_{\mathrm{loc}}(\mathbb{R}, S^{-\infty}_{-\infty})$, by solving iteratively the transportation equations

$$\begin{cases} \mathscr{L}_t a^0_h = 0, \\ a^0_h|_{t=0} = \varphi, \end{cases} \qquad \begin{cases} \mathscr{L}_t a^j_h + b^{j-1}_h = 0, \\ a^j_h|_{t=0} = 0, \quad j \geq 1, \end{cases}$$

where $b^j_h \in W^{\infty,\infty}_{\mathrm{loc}}(\mathbb{R}, S^{-\infty}_{-\infty})$ satisfies, by Lemma 3.1, that

$$\mathcal{L}_t \operatorname{Op}^{\delta,\rho}_h(a^j_h) = \operatorname{Op}^{\delta,\rho}_h(\mathscr{L}_t a^j_h) + h^{\delta+\rho} \operatorname{Op}^{\delta,\rho}_h(b^j_h) + \mathcal{O}(h^\infty)_{L^\infty_{\mathrm{loc}}(\mathbb{R}, L^2\to L^2)}.$$

Thus we have proved Theorem 1.4(1).

3B. ***Proof of Theorem 1.4(2).*** Let $\beta = \rho\gamma - (\delta+\rho) > 0$. For all $h > 0$, introduce the semiclassical time variable $s = h^{-\beta}t$, and rewrite (1-6) as

$$\partial_s u + ih^\beta |D_x|^\gamma u = 0. \tag{3-3}$$

If $a = a(s, x, \xi) \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^{2d})$ and $\mathcal{A} = \mathcal{A}(s) \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}, L^2 \to L^2)$, then define

$$\mathscr{L}_s a = \partial_s a + \{|\xi|^\gamma, a\}, \quad \mathcal{L}^h_s \mathcal{A} = \partial_s \mathcal{A} + ih^\beta [|D_x|^\gamma, \mathcal{A}].$$

Lemma 3.2. *If $\phi \in C^\infty_c(\mathbb{R}^d)$ such that $\phi \geq 0$, $\phi(0) > 0$, and $x \cdot \nabla\phi(x) \leq 0$ for all $x \in \mathbb{R}^d$, and we define*

$$\chi(s, x, \xi) = \phi\left(\frac{x - s\gamma|\xi|^{\gamma-2}\xi - x_0}{1+s}\right)\phi\left(\frac{\xi - \xi_0}{\epsilon}\right)$$

for $s \geq 0$, $\epsilon > 0$, $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d\setminus 0)$, then the following statements hold when ϵ is sufficiently small and $|\xi_0|$ is sufficiently large:

(1) $\chi \in W^{\infty,\infty}(\mathbb{R}_{\geq 0}, S_0^{-\infty})$.

(2) $\mathscr{L}_s\chi \in W^{\infty,\infty}(\mathbb{R}_{\geq 0}, S_{-1}^{-\infty})$ *and* $\mathscr{L}_s\chi \geq 0$.

(3) *If* $t_0 > 0$ *and set* $(\tau u)(s, x, \xi) = u\big(s, \frac{s}{t_0}x, \xi\big)$, *then* $\tau\chi \in W^{\infty,\infty}(\mathbb{R}_{\geq 0}, S_{-\infty}^{-\infty})$.

(4) *If* s *is sufficiently large, then* $(\tau\chi)(s, \cdot)$ *is elliptic at* $(t_0\gamma|\xi_0|^{\gamma-2}\xi_0, \xi_0)$.

Proof. Each time we differentiate χ with respect to x, we get a multiplicative factor $(1+s)^{-1}$, which is of size $\langle x\rangle^{-1}$ in $\operatorname{supp}\chi$ as

$$\operatorname{supp}\chi \subset \{C^{-1}s \leq |x| \leq Cs\} \tag{3-4}$$

for some $C > 0$ when $|s|$ and $|\xi_0|$ are sufficiently large and ϵ is sufficiently small. Therefore $\chi \in W^{\infty,\infty}(\mathbb{R}_{\geq 0}, S_0^{-\infty})$. Clearly $\tau\chi(s, \cdot)$ is bounded in $C_c^\infty(\mathbb{R}^{2d})$. We write

$$(\tau\chi)(s, x, \xi) = \phi\left(\frac{x - t_0\gamma|\xi_0|^{\gamma-2}\xi_0}{t_0(1+s)/s} - \frac{\gamma|\xi|^{\gamma-2}\xi - \gamma|\xi_0|^{\gamma-2}\xi_0}{(1+s)/s} - \frac{x_0}{1+s}\right)\phi\left(\frac{\xi - \xi_0}{\epsilon}\right), \tag{3-5}$$

where $|\xi|^{\gamma-2}\xi - |\xi_0|^{\gamma-2}\xi_0 = o(1)$ as $\epsilon \to 0$, whence $\tau\chi(s, \cdot)$ is elliptic at $(t_0\gamma|\xi_0|^{\gamma-2}\xi_0, \xi_0)$ for sufficiently large s. To estimate $\mathscr{L}_s\chi$, we perform an explicit computation:

$$\partial_s\chi(s, x, \xi) = -(\nabla\phi)\left(\frac{x - s\gamma|\xi|^{\gamma-2}\xi - x_0}{1+s}\right)\phi\left(\frac{\xi - \xi_0}{\epsilon}\right)\frac{(x - s\gamma|\xi|^{\gamma-2}\xi - x_0) + (1+s)\gamma|\xi|^{\gamma-2}\xi}{(1+s)^2},$$

$$\{|\xi|^\gamma, \chi\}(s, x, \xi) = \frac{\gamma|\xi|^{\gamma-2}\xi}{1+s}\cdot(\nabla\phi)\left(\frac{x - s\gamma|\xi|^{\gamma-2}\xi - x_0}{1+s}\right)\phi\left(\frac{\xi - \xi_0}{\epsilon}\right).$$

Therefore,

$$\mathscr{L}_s\chi(s, x, \xi) = -(\nabla\phi)\left(\frac{x - s\gamma|\xi|^{\gamma-2}\xi - x_0}{1+s}\right)\phi\left(\frac{\xi - \xi_0}{\epsilon}\right)\cdot\frac{x - s\gamma|\xi|^{\gamma-2}\xi - x_0}{(1+s)^2} \geq 0.$$

Note that on $\operatorname{supp}\mathscr{L}_s\chi$, we have

$$\frac{x - s\gamma|\xi|^{\gamma-2}\xi - x_0}{(1+s)^2} = \mathcal{O}\left(\frac{1+s}{(1+s)^2}\right) = \mathcal{O}\left(\frac{1}{1+s}\right) = \mathcal{O}\left(\frac{1}{\langle x\rangle}\right).$$

So we prove similarly that $\mathscr{L}_s\chi \in W^{\infty,\infty}(\mathbb{R}_{\geq 0}, S_{-1}^{-\infty})$. ☐

Now fix $t_0 > 0$ and let $\mu = \infty$ as the other cases are similar. Let $\epsilon > 0$ be sufficiently small and let $\{\lambda_j\}_{j\in\mathbb{N}} \subset [1, 1+\epsilon)$ be strictly increasing. Choose ϕ as in Lemma 3.2, and set

$$\chi_j(s, x, \xi) = \phi\left(\frac{x - s\gamma|\xi|^{\gamma-2}\xi - x_0}{\lambda_j(1+s)}\right)\phi\left(\frac{\xi - \xi_0}{\lambda_j\epsilon}\right).$$

Then $\operatorname{supp}\chi_j \subset \{\chi_{j+1} > 0\}$ for all $j \in \mathbb{N}$. We aim to construct $a_h \in W^{\infty,\infty}(\mathbb{R}_{\geq 0}, S_0^{-\infty})$ such that the following statements hold:

(1) For all $s \geq 0$ and $h \in (0, 1]$, we have $\operatorname{supp} a_h \subset \bigcup_{j\in\mathbb{N}} \operatorname{supp}\chi_j$.

(2) The symbol $a_h|_{s=0}$ is elliptic at (x_0, ξ_0); more precisely , we have

$$(a_h - (\zeta_h^{\delta,\rho}\chi_0)\sharp_h^{\delta,\rho}\chi_0)|_{s=0} = \mathcal{O}(h^\infty)_{S_{-\infty}^{-\infty}}.$$

(3) For $t_0 > 0$, let τ be defined as in the lemma. Then $\tau a_h \in W^{\infty,\infty}(\mathbb{R}_{\geq 0}, S^{-\infty}_{-\infty})$ and $\tau a_h(s, \cdot)$ is elliptic at $(t_0\gamma|\xi_0|^{\gamma-2}\xi_0, \xi_0)$ when s is sufficiently large.

(4) $\mathcal{L}^h_s \operatorname{Op}^{\delta,\rho}_h(a_h) \geq \mathcal{O}(h^\infty)_{L^\infty(\mathbb{R}_{\geq 0}, L^2 \to L^2)}$.

Assume that such an a_h is found and that

$$(t_0\gamma|\xi_0|^{\gamma-2}\xi_0, \xi_0) \notin \mathrm{WF}^\mu_{\rho(\gamma-1),\rho}(u(t = t_0)).$$

By (1) and (3-5), if we choose ϕ such that $\operatorname{supp}\phi$ is sufficiently close to the origin, then for sufficiently small $h > 0$ we have

$$\operatorname{supp} \theta^{\beta,0}_{1/h,*} a_h|_{s=h^{-\beta}t_0} \Subset \mathbb{R}^{2d} \setminus \mathrm{WF}^\infty_{\rho(\gamma-1),\rho}(u|_{t=t_0}).$$

By (3), the symbol $\theta^{\beta,0}_{1/h,*} a_h|_{s=h^{-\beta}t_0} \in S^{-\infty}_{-\infty}$ is elliptic at $(t_0\gamma|\xi_0|^{\gamma-2}\xi_0, \xi_0)$. Therefore, by Lemma 2.14,

$$(u, \operatorname{Op}^{\delta,\rho}_h(a_h)u)_{L^2}|_{s=h^{-\beta}t_0} = (u, \operatorname{Op}^{\rho(\gamma-1),\rho}_h(\theta^{\beta,0}_{1/h,*} a_h)u)_{L^2}|_{s=h^{-\beta}t_0} = \mathcal{O}(h^\infty).$$

By (3-3), we have

$$\frac{\mathrm{d}}{\mathrm{d}s}(u, \operatorname{Op}^{\delta,\rho}_h(a_h)u)_{L^2} = (u, \mathcal{L}^h_s \operatorname{Op}^{\delta,\rho}_h(a_h)u)_{L^2},$$

which implies, by (4), that

$$\begin{aligned}(u, \operatorname{Op}^{\delta,\rho}_h(a_h)u)_{L^2}|_{s=0} &= (u, \operatorname{Op}^{\delta,\rho}_h(a_h)u)_{L^2}|_{s=h^{-\beta}t_0} - \int_0^{h^{-\beta}t_0} (u, \mathcal{L}^h_s \operatorname{Op}^{\delta,\rho}_h(a_h)u)_{L^2}\,\mathrm{d}s \\ &\leq \mathcal{O}(h^\infty) + \mathcal{O}(h^{-\beta} \times h^\infty) = \mathcal{O}(h^\infty).\end{aligned}$$

Therefore, by (2), we have

$$\|\operatorname{Op}^{\delta,\rho}_h(\chi_0)u|_{s=0}\|^2_{L^2} = (u, \operatorname{Op}^{\delta,\rho}_h(a_h)u)_{L^2}|_{s=0} + \mathcal{O}(h^\infty) = \mathcal{O}(h^\infty).$$

We conclude that $(x_0, \xi_0) \notin \mathrm{WF}^\infty_{\delta,\rho}(u_0)$.

We shall construct a_h in the following form of asymptotic expansion:

$$a_h(s, x, \xi) \sim \sum_{j \in \mathbb{N}} h^{j(\delta+\rho)} \varphi^j(s) a^j_h(s, x, \xi),$$

where $a^j_h \in W^{\infty,\infty}(\mathbb{R}_{\geq 0}, S^{-\infty}_0)$, with $\operatorname{supp} a^j_h \subset \operatorname{supp}\chi_j$, and $\varphi^j \in P_j$, with

$$P_j = \left\{ f(\ln(1+s)) : f(X) = \sum_{k=0}^{j} c_k X^k,\ c_k \geq 0 \text{ for all } k \right\}. \tag{3-6}$$

The above asymptotic expansion is in the weak sense that, for some $\epsilon' > 0$, and all $N \in \mathbb{N}$,

$$a_h - \sum_{j<N} h^{j(\delta+\rho)} \varphi^j a^j_h \in \mathcal{O}(h^{N(\delta+\rho-\epsilon')})_{W^{\infty,\infty}([0,h^{-\beta}T], S^{-\infty}_0)}.$$

The following properties for functions in $\bigcup_{j\in\mathbb{N}} P_j$ will be used in the construction of a_h.

Lemma 3.3. *If $\psi \in P_j$ for some $j \in \mathbb{N}$, then ψ is smooth and nonnegative on* $[0, +\infty)$ *and*

$$((1+s)\partial_s)^{-1}\psi(s) := \int_0^s (1+\sigma)^{-1}\psi(\sigma)\,\mathrm{d}\sigma \in P_{j+1}.$$

Proof. The function ψ is smooth because it is the composition of a polynomial and the smooth function $s \mapsto \ln(1+s)$. The nonnegativity of ψ is the consequence of the nonnegativity of the coefficients c_k in (3-6) and the fact that $\ln(1+s) \geq 0$ when $s \geq 0$. To prove that $((1+s)\partial_s)^{-1}\psi \in P_{j+1}$, note that for all $n \in \mathbb{N}$ we have

$$((1+s)\partial_s)^{-1}(\ln(1+\,\cdot\,))^n = (n+1)^{-1}(\ln(1+\,\cdot\,))^{n+1}.$$

The claim follows by the linearity of the operator $((1+s)\partial_s)^{-1}$. □

To construct a_h, we begin by setting $\varphi^0 \equiv 1$ and choosing a_h^0 satisfying

$$a_h^0 - (\zeta_h^{\delta,\rho}\chi_0)\sharp_h^{\delta,\rho}\chi_0 = \mathcal{O}(h^\infty)_{W^{\infty,\infty}(\mathbb{R}_{\geq 0}, S_0^{-\infty})},$$
$$(a_h^0 - (\zeta_h^{\delta,\rho}\chi_0)\sharp_h^{\delta,\rho}\chi_0)|_{s=0} = \mathcal{O}(h^\infty)_{S_{-\infty}^{-\infty}}.$$

By the definition of β and Propositions 2.4 and 2.5, there exists $r_h^0 \in L^\infty(\mathbb{R}_{\geq 0}, S_{-1}^{-\infty})$ with $\operatorname{supp} r_h^0 \subset \operatorname{supp}\chi_0$ such that

$$\mathcal{L}_s^h \operatorname{Op}_h^{\delta,\rho}(a_h^0) = 2\operatorname{Op}_h^{\delta,\rho}(\chi_0\mathcal{L}_s\chi_0) + h^{\delta+\rho}\operatorname{Op}_h^{\delta,\rho}(r_h^0) + \mathcal{O}(h^\infty)_{L^\infty(\mathbb{R}_{\geq 0}, L^2\to L^2)}. \tag{3-7}$$

By (3-4), we have $\langle s\rangle r_h^0 \in L^\infty(\mathbb{R}_{\geq 0}, S_0^{-\infty})$ and similarly

$$\langle s\rangle \chi_0\mathcal{L}_s\chi_0 \in L^\infty(\mathbb{R}_{\geq 0}, S_0^{-\infty}). \tag{3-8}$$

By Lemma 3.2, we have

$$\chi_0\mathcal{L}_s\chi_0 \geq 0. \tag{3-9}$$

Recall that, by the sharp Gårding inequality (Proposition 2.6), if a symbol $p_h \in S_0^0$ satisfies $p_h \geq 0$, then $\operatorname{Op}_h^{0,1}(p_h) \gtrsim -h$. By (2-3), we deduce that $\operatorname{Op}_h^{0,1}(p_h) \gtrsim -h^{\delta+\rho}$. To be precise, this means there exists $C > 0$ which only depends on a finite number of seminorms defined by (2-1), such that, for all $u \in L^2$,

$$\langle u, \operatorname{Op}_h^{\delta,\rho}(p_h)u\rangle_{L^2} \geq -Ch^{\delta+\rho}\|u\|_{L^2}^2.$$

Take $c_h^0 \in L^\infty(\mathbb{R}_{\geq 0}, S_0^{-\infty})$ such that

$$\operatorname{supp} a_h^0 \Subset \{c_h^0 = 1\} \subset \operatorname{supp} c_h^0 \subset \{\chi_1 > 0\}.$$

By (3-7) and (3-9), for all $u \in L^2$, we have

$$\begin{aligned}\langle u, \mathcal{L}_s^h \operatorname{Op}_h^{\delta,\rho}(a_h^0)u\rangle_{L^2} &= \langle \operatorname{Op}_h^{\delta,\rho}(c_h)u, \mathcal{L}_s^h \operatorname{Op}_h^{\delta,\rho}(a_h^0)\operatorname{Op}_h^{\delta,\rho}(c_h)u\rangle_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}^2\\ &\geq -C\langle s\rangle^{-1}h^{\delta+\rho}\|\operatorname{Op}_h^{\delta,\rho}(c_h)u\|_{L^2}^2 + \mathcal{O}(h^\infty),\end{aligned}$$

where the factor $\langle s\rangle^{-1}$ comes from the estimate (3-8). By the symbolic calculus, there exists $b_h \in L^\infty(\mathbb{R}_{\geq 0}, S_0^{-\infty})$ such that

$$\operatorname{Op}_h^{\delta,\rho}(b_h) - C\operatorname{Op}_h^{\delta,\rho}(c_h)^*\operatorname{Op}_h^{\delta,\rho}(c_h) = \mathcal{O}(h^\infty)_{L^\infty(\mathbb{R}_{\geq 0}, L^2\to L^2)}$$

and $\operatorname{supp} b_h \subset \operatorname{supp} c_h$. Therefore,

$$\begin{aligned}\mathcal{L}_s^h \operatorname{Op}_h^{\delta,\rho}(a_h^0) &\geq -C\langle s\rangle^{-1}h^{\delta+\rho}\operatorname{Op}_h^{\delta,\rho}(c_h)^*\operatorname{Op}_h^{\delta,\rho}(c_h) + \mathcal{O}(h^\infty)_{L^\infty(\mathbb{R}_{\geq 0}, L^2\to L^2)}\\ &\geq -\langle s\rangle^{-1}h^{\delta+\rho}\operatorname{Op}_h^{\delta,\rho}(b_h^0) + \mathcal{O}(h^\infty)_{L^\infty(\mathbb{R}_{\geq 0}, L^2\to L^2)}.\end{aligned} \tag{3-10}$$

Suppose that, for some $\ell \geq 1$, we can find $\varphi^j \in P_j$, a_h^j for $j = 0, \dots, \ell-1$ and $\psi^{\ell-1} \in P_{\ell-1}$, $b_h^{\ell-1} \in L^\infty(\mathbb{R}_{\geq 0}, S_0^{-\infty})$, with $\operatorname{supp} b_h^{\ell-1} \subset \{\chi_\ell > 0\}$, such that

$$\mathcal{L}_s^h \operatorname{Op}_h^{\delta,\rho}\left(\sum_{j=0}^{\ell-1} h^{j(\delta+\rho)} \varphi^j a_h^j\right) \geq -\langle s\rangle^{-1} \psi^{\ell-1} h^{\ell(\delta+\rho)} \operatorname{Op}_h^{\delta,\rho}(b_h^{\ell-1}) + \mathcal{O}(h^\infty)_{L^\infty(\mathbb{R}_{\geq 0}, L^2 \to L^2)}. \tag{3-11}$$

If we choose $B_\ell > 0$ sufficiently large and set $\varphi^\ell = ((1+s)\partial_s)^{-1} \psi^{\ell-1}$ and $a_h^\ell = B_\ell \chi_\ell$, then by a direct calculation, we have

$$\begin{aligned} \mathscr{L}_s(\varphi^\ell a_h^\ell) &= B_\ell (1+s)^{-1} \psi^{\ell-1} \chi_\ell + B_\ell \varphi^\ell \mathscr{L}_s \chi_\ell \\ &\geq B_\ell (1+s)^{-1} \psi^{\ell-1} \chi_\ell \geq \langle s\rangle^{-1} \psi^{\ell-1} b_h^{\ell-1}. \end{aligned}$$

Observe that

$$\mathscr{L}_s(\varphi^\ell a_h^\ell) = \mathcal{O}(\langle s\rangle^{-1}(\psi^{\ell-1} + \varphi^\ell))_{S_0^{-\infty}}, \quad \langle s\rangle^{-1} \psi^{\ell-1} b_h^{\ell-1} = \mathcal{O}(\langle s\rangle^{-1} \psi^{\ell-1})_{S_0^{-\infty}}.$$

Much as above, applying the sharp Gårding inequality to the symbol

$$\mathscr{L}_s(\varphi^\ell a_h^\ell) - \langle s\rangle^{-1} \psi^{\ell-1} b_h^{\ell-1} = \mathcal{O}(\langle s\rangle^{-1}(\varphi^\ell + \psi^{\ell-1}))_{S_0^{-\infty}},$$

we can find $b_h^\ell \in L^\infty(\mathbb{R}_{\geq 0}, S_0^{-\infty})$ with $\operatorname{supp} b_h^\ell \subset \{\chi_{\ell+1} > 0\}$ such that

$$\mathcal{L}_s^h \operatorname{Op}_h^{\delta,\rho}(\varphi^\ell a_h^\ell) - \langle s\rangle^{-1} \psi^{\ell-1} \operatorname{Op}_h^{\delta,\rho}(b_h^{\ell-1}) \geq -\langle s\rangle^{-1} \psi^\ell h^{\delta+\rho} \operatorname{Op}_h^{\delta,\rho}(b_h^\ell) + \mathcal{O}(h^\infty)_{L^2 \to L^2}, \tag{3-12}$$

with $\psi^\ell = \psi^{\ell-1} + \varphi^\ell \in P_\ell$. Summing up (3-11) and $h^{\ell(\delta+\rho)} \times$ (3-12), we close the induction procedure.

Finally we conclude the asymptotic expansion by observing that, for all $\epsilon' > 0$, we have

$$\|\varphi^\ell\|_{L^\infty([0, h^{-\beta} T])} = \mathcal{O}(|\log h|^\ell) = \mathcal{O}(h^{-\epsilon' \ell}).$$

Thus we have proved Theorem 1.4(2).

4. Paradifferential calculus

In this section, we develop a paradifferential calculus on weighted Sobolev spaces and a semiclassical paradifferential calculus.

4A. ***Classical paradifferential calculus.*** We recall some classical results of the paradifferential calculus. We refer to the original work [Bony 1979] and the books [Hörmander 1997; Métivier 2008; Bahouri, Chemin and Danchin 2011]. The results and proofs below are mainly based on [Métivier 2008], so we shall only sketch them. In the meantime, we shall also make some refinements regarding the estimates of the remainder terms, for the sake of the semiclassical paradifferential calculus that will be developed later.

4A1. *Symbol classes and paradifferential operators.*

Definition 4.1. For $m \in \mathbb{R}$, $r \geq 0$, let $\Gamma^{m,r}$ be the space of all $a(x,\xi) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d \times (\mathbb{R}^d \backslash 0))$ such that:

(1) For all $x \to \mathbb{R}^d$, the function $\xi \mapsto a(x,\xi)$ is smooth.

(2) For all $\alpha \in \mathbb{N}^d$, there exists $C_\alpha > 0$ such that for all $\xi \in \mathbb{R}^d$ with $|\xi| \geq \frac{1}{2}$, we have

$$\|\partial_\xi^\alpha a(\cdot,\xi)\|_{W^{r,\infty}} \leq C_\alpha \langle\xi\rangle^{m-|\alpha|}.$$

If $a \in \Gamma^{m,r}$, then we define for all $n \geq 0$ the seminorm

$$M_n^{m,r}(a) = \sup_{|\alpha| \leq n} \sup_{|\xi| \geq 1/2} \langle \xi \rangle^{|\alpha|-m} \|\partial_\xi^\alpha a(\,\cdot\,, \xi)\|_{W^{r,\infty}}.$$

We also define $M^{m,r}(a) = M^{m,r}_{\tilde{d}+r}(a)$, where $\tilde{d} = 1 + \left[\frac{d}{2}\right]$.

Definition 4.2. A pair of nonnegative functions $(\chi, \pi) \in C^\infty(\mathbb{R}^{2d} \backslash 0) \times C^\infty(\mathbb{R}^d)$ is called admissible if the following conditions are satisfied:

(1) The function $1 - \pi$ is a cutoff function of the origin. To be precise, if $|\eta| \geq 1$, then $\pi(\eta) = 1$, and if $|\eta| \leq \frac{1}{2}$, then $\pi(\eta) = 0$.

(2) The function χ is an even and homogeneous of degree 0, and there exist $\epsilon_1, \epsilon_2 \in (0, 1)$ with $\epsilon_1 < \epsilon_2$, such that

$$\begin{cases} \chi(\theta, \eta) = 1, & |\theta| \leq \epsilon_1 |\eta|, \\ \chi(\theta, \eta) = 0, & |\theta| \geq \epsilon_2 |\eta|. \end{cases} \tag{4-1}$$

Definition 4.3. If $m \in \mathbb{R}$ and $a \in \Gamma^{m,0}$, then the paradifferential operator T_a is defined by

$$\widehat{T_a u}(\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} \chi(\xi - \eta, \eta) \pi(\eta) \hat{a}(\xi - \eta, \eta) \hat{u}(\eta)\, d\eta, \tag{4-2}$$

where (χ, π) is admissible and $\hat{a}(\theta, \xi) = \int e^{-ix \cdot \theta} a(x, \xi)\, dx$. In other words $T_a = \mathrm{Op}(\sigma_a)$ where

$$\sigma_a(\,\cdot\,, \xi) = \pi(\xi) \chi(D_x, \xi) a(\,\cdot\,, \xi). \tag{4-3}$$

Proposition 4.4. *If $m \in \mathbb{R}$ and $a \in \Gamma^{m,0}$, then $T_a = \mathcal{O}(M^{m,0}(a))_{\mathscr{O}_0^m}$.*

Remark 4.5. A symbol p satisfies the spectral condition if there exists $\epsilon \in (0, 1)$ such that

$$\operatorname{supp} \hat{p} \subset \{(\eta, \xi) \in \mathbb{R}^{2d} : |\eta| \geq \epsilon \langle \xi \rangle\}.$$

By [Métivier 2008], if $a \in \Gamma^{m,0}$, then $\sigma_a \in \Gamma^{m,0}$ and satisfies the spectral condition. The above Proposition 4.4 is in fact a consequence of the following estimate (4-4) and the mapping property: if $p \in \Gamma^{m,0}$ satisfies the spectral condition, then $\mathrm{Op}(p)$ defines a bounded operator from $H^{\mu+m} \to H^\mu$ for all $\mu \in \mathbb{R}$.

Note the definition (4-2) depends on the choice of admissible pairs of functions. The following lemma and corollary show that if we change the admissible pair, then the error term is regularizing.

Lemma 4.6. *If $m \in \mathbb{R}$, $r \geq 0$ and $a \in \Gamma^{m,r}$, then, for all $n \geq 0$, we have*

$$M_n^{m,r}(\sigma_a) \lesssim M_n^{m,r}(a). \tag{4-4}$$

If in addition $r \in \mathbb{N}$, then, for all $\beta \in \mathbb{N}^d$ with $|\beta| \leq r$, we have

$$M_n^{m-r+|\beta|,0}(\partial_x^\beta(\sigma_a - a\pi)) \lesssim M_n^{m,0}(\nabla^r a). \tag{4-5}$$

Proof. The first statement is proven in [Métivier 2008]. We only prove the second statement. We shall only prove the case where $\beta = 0$ for the rest is similar. By [Métivier 2008], we have

$$(\sigma_a - a\pi)(x, \xi) = \pi(\xi) \int \rho(x, y, \xi) \Phi(y, \xi)\, dy$$

for all $(x,\xi)\in\mathbb{R}^d\times(\mathbb{R}^d\backslash 0)$, where $\Phi(\,\cdot\,,\xi)=\mathcal{F}^{-1}\chi(\,\cdot\,,\xi)$ and

$$\rho(x,y,\xi)=\sum_{|\gamma|=r}\frac{(-y)^\gamma}{\gamma!}\int_0^1 r(1-t)^{r-1}\partial_x^\gamma a(x-ty,\xi)\,\mathrm{d}t.$$

Therefore, if $|\xi|\geq\frac{1}{2}$ and $|\alpha|\leq n$, then

$$\|\partial_\xi^\alpha\rho(\,\cdot\,,y,\xi)\|_{L^\infty}\lesssim|y|^r\|\partial_\xi^\alpha\nabla^r a(\,\cdot\,,\xi)\|_{L^\infty}\lesssim|y|^r|\xi|^{m-|\alpha|}M_n^{m,0}(\nabla^r a). \tag{4-6}$$

Note that the admissibility of (π,χ) implies that, for all $\alpha,\beta\in\mathbb{N}$, there exists $C_{\alpha,\beta}>0$ such that, for all $(x,\xi)\in\mathbb{R}^{2d}$, we have $|x^\beta\partial_\xi^\alpha\Phi(x,\xi)|\leq C_{\alpha,\beta}\langle\xi\rangle^{d-|\alpha|-|\beta|}$. Therefore, for all $\alpha\in\mathbb{N}$ and all $\xi\in\mathbb{R}^d$, there exists $C_\alpha>0$ such that

$$\|\partial_\xi^\alpha\Phi(\,\cdot\,,\xi)\|_{L^1}\leq C_\alpha\langle\xi\rangle^{-|\alpha|}. \tag{4-7}$$

We conclude (4-5) by estimates (4-6) and (4-7). □

Corollary 4.7. *Let $a\in\Gamma^{m,r}$, with $m\in\mathbb{R}$ and $r\in\mathbb{N}$. Let (χ,π) and (χ',π') be admissible. Denote by T_a and T'_a the paradifferential operators respectively defined by these two admissible pairs. Then*

$$T_a-T'_a=\mathcal{O}(M^{m,0}(\nabla^r a))_{\mathscr{O}_0^{m-r}}+\mathcal{O}(M^{m,r}(a))_{\mathscr{O}_0^{-\infty}}.$$

If in addition $a\pi=a\pi'=a$, then

$$T_a-T'_a=\mathcal{O}(M^{m,0}(\nabla^r a))_{\mathscr{O}_0^{m-r}}.$$

Proof. Let T''_a be the paradifferential operator defined with respect to (χ',π); then by Lemma 4.6, $T_a-T''_a=\mathcal{O}(M^{m,0}(\nabla^r a))_{\mathscr{O}_0^{m-r}}$. Note that $T''_a-T'_a$ is a composition with a paradifferential operator with a smoothing operator $\pi(D_x)-\pi'(D_x)$, which implies $T''_a-T'_a=\mathcal{O}(M^{m,r}(a))_{\mathscr{O}_0^{-\infty}}$. This term vanishes if $a\pi=a\pi'=a$. □

Corollary 4.8. *Let $\psi\in C_b^\infty(\mathbb{R}^d)$. Then $T_\psi-\psi\in\mathscr{O}_0^{-\infty}$.*

Proof. This is a consequence of (4-5) and the Calderón–Vaillancourt theorem. □

4A2. *Symbolic calculus and paralinearization.*

Proposition 4.9. *If $a\in\Gamma^{m,r}$ and $b\in\Gamma^{m',r}$, where $r\in\mathbb{N}$, $m\in\mathbb{R}$ and $m'\in\mathbb{R}$, then*

$$T_aT_b-T_{a\sharp b}=\mathcal{O}\big(M^{m,r}(a)M^{m',0}(\nabla^r b)+M^{m,0}(\nabla^r a)M^{m',r}(b)\big)_{\mathscr{O}_0^{m+m'-r}}+\mathcal{O}\big(M^{m,r}(a)M^{m',r}(b)\big)_{\mathscr{O}_0^{-\infty}},$$

where the symbol $a\sharp b=a\sharp_{1,r}^{0,0}b$ is defined by (2-4). If in addition $a\pi=a$ and $b\pi=b$, then

$$T_aT_b-T_{a\sharp b}=\mathcal{O}\big(M^{m,r}(a)M^{m',0}(\nabla^r b)+M^{m,0}(\nabla^r a)M^{m',r}(b)\big)_{\mathscr{O}_0^{m+m'-r}}.$$

Proof. By Corollary 4.7, we may choose an admissible pair (π,χ) to define paradifferential operators, while assuming that $\epsilon_2<\frac{1}{4}$. We shall only prove the case where $a\pi=a$ and $b\pi=b$, as the general case follows easily. The following proof follows [Métivier 2008]. Take the decomposition $T_aT_b-T_{a\sharp b}=$ (I) + (II), where

$$\text{(I)}=\mathrm{Op}(\sigma_a)\,\mathrm{Op}(\sigma_b)-\mathrm{Op}(\sigma_a\sharp\sigma_b),\qquad \text{(II)}=\mathrm{Op}(\sigma_a\sharp\sigma_b)-\mathrm{Op}(\sigma_{a\sharp b}).$$

Write $\mathrm{Op}(\sigma_a)\,\mathrm{Op}(\sigma_b) = \mathrm{Op}(\sigma)$, where

$$\sigma(x,\xi) = \frac{1}{(2\pi)^d}\iint e^{i(x-y)\cdot\eta}\sigma_a(x,\xi+\eta)\theta(\eta,\xi)\sigma_b(y,\xi)\,\mathrm{d}y\,\mathrm{d}\eta.$$

Here $\theta \in C^\infty(\mathbb{R}^{2d}\backslash 0)$ satisfies that (θ,π) is admissible and $\theta\chi = \chi$. By Taylor's formula, we have the decomposition

$$\sigma_a(x,\xi+\eta) = \sum_{|\alpha|<r}\frac{1}{\alpha!}\partial_\xi^\alpha\sigma(x,\xi)\eta^\alpha + \sum_{|\alpha|=r}\rho_\alpha(x,\xi,\eta)\eta^\alpha,$$

where the functions ρ_α depend on $\nabla_\xi^r\sigma_a$. Then write $\sigma = \sigma_a\sharp\sigma_b + \sum_{|\alpha|=r} q_\alpha$, where

$$q_\alpha(x,\xi) = \int R_\alpha(x,x-y,\xi)(D_x^\alpha\sigma_b)(y,\xi)\,\mathrm{d}y,$$

$$R_\alpha(x,y,\xi) = (2\pi)^{-2}\int e^{iy\eta}\rho_\alpha(x,y,\eta)\theta(\eta,\xi)\,\mathrm{d}\eta.$$

By the same estimate in [Métivier 2008],

$$\|\partial_\xi^\beta R_\alpha(x,\cdot,\xi)\|_{L^1} \lesssim M^{m,r}(a)\langle\xi\rangle^{m-r-|\beta|}.$$

Using $D_x^\alpha\sigma_b = \sigma_{D_x^\alpha b}$, we verify that

$$\|\partial_\xi^\beta q_\alpha(\cdot,\xi)\|_{L^\infty} \lesssim M^{m,r}(a)M^{m',0}(\nabla^r b)\langle\xi\rangle^{m+m'-r-|\beta|},$$

and consequently, by Remark 4.5,

$$\|(\mathrm{I})\|_{H^s\to H^{s-m-m'+r}} \lesssim \sum_{|\alpha|=r} M^{m+m'-r,0}(q_\alpha) \lesssim M^{m,r}(a)M^{m',0}(\nabla^r b).$$

To estimate (II), for all $|\alpha|<r$, take the decomposition $\partial_\xi^\alpha\sigma_a D_x^\alpha\sigma_b - \sigma_{\partial_\xi^\alpha a D_x^\alpha b} = \text{(i)}+\text{(ii)}+\text{(iii)}$, where

$$\text{(i)} = \partial_\xi^\alpha(\sigma_a - a)D_x^\alpha\sigma_b,\quad \text{(ii)} = \partial_\xi^\alpha a D_x^\alpha(\sigma_b - b),\quad \text{(iii)} = \partial_\xi^\alpha a D_x^\alpha b - \sigma_{\partial_\xi^\alpha a D_x^\alpha b}.$$

By Lemma 4.6, Leibniz's rule and interpolation,

$$\begin{aligned}
M^{m+m'-r,0}\text{(i)} &\lesssim M^{m-r,0}(\sigma_a - a)M^{m',0}(D_x^\alpha\sigma_b) \lesssim M^{m,0}(\nabla^r a)M^{m',r}(b),\\
M^{m+m'-r,0}\text{(ii)} &\lesssim M^{m,r}(a)M^{m'-r+|\alpha|,0}(D_x^\alpha(\sigma_b - b\pi)) \lesssim M^{m,r}(a)M^{m',0}(\nabla^r b),\\
M^{m+m'-r,0}\text{(iii)} &\lesssim M^{m+m'-|\alpha|,0}(\nabla^{r-|\alpha|}(\partial_\xi^\alpha a D_x^\alpha b))\\
&\lesssim M^{m-|\alpha|,0}(\nabla^r\partial_\xi^\alpha a)M^{m',0}(b) + M^{m-|\alpha|,0}(\partial_\xi^\alpha a)M^{m',0}(\nabla^r b)\\
&\lesssim M^{m,0}(\nabla^r a)M^{m',r}(b) + M^{m,r}(a)M^{m',0}(\nabla^r b).
\end{aligned}$$

By Remark 4.5, these estimates imply that

$$\text{(II)} = \mathcal{O}\big(M^{m,r}(a)M^{m',0}(\nabla^r b) + M^{m,0}(\nabla^r a)M^{m',r}(b)\big)_{\mathscr{O}_0^{m+m'-r}}.$$

The proposition follows. □

Proposition 4.10. *Let $a \in \Gamma^{m,r}$ with $r \in \mathbb{N}$ and $m \in \mathbb{R}$. Then*

$$T_a^* - T_{a^*} = \mathcal{O}(M^{m,0}(\nabla^r a))_{\mathscr{O}_0^{m-r}} + \mathcal{O}(M^{m,r}(a))_{\mathscr{O}_0^{-\infty}},$$

where the symbol $a^* = \zeta^{0,0}_{1,r} a$ *is defined by* (2-5). *If in addition* $a\pi = a$, *then*

$$T_a^* - T_{a^*} = \mathcal{O}(M^{m,0}(\nabla^r a))_{\mathscr{O}_0^{m-r}}.$$

Proof. Much as in the proposition for the composition, we shall only prove the case where $a\pi = a$. Let (θ, π) be admissible such that $\theta\chi = \chi$, then $T_a^* = \mathrm{Op}(\sigma_a^*)$, with

$$\sigma_a^*(x,\xi) = (2\pi)^{-d}\int e^{-iy\cdot\eta}\bar{\sigma}_a(x+y,\xi+\eta)\,\mathrm{d}\eta\,\mathrm{d}y = a^*(x,\xi) + \sum_{|\alpha|=r} r_\alpha(x,\xi),$$

where by Taylor's formula,

$$r_\alpha(x,\xi) = \frac{2\pi}{\alpha!}\iiint_{\mathbb{R}^{2d}\times[0,1]} r(1-t)^{r-1}e^{-iy\cdot\eta}D_x^\alpha\partial_\xi^\alpha\bar{\sigma}_a(x,\xi+t\eta)\theta(\eta,\xi)\,\mathrm{d}t\,\mathrm{d}\eta\,\mathrm{d}y.$$

The term $D_x^\alpha\partial_\xi^\alpha\bar{\sigma}_a(x,\xi+t\eta)$ in the integral and the analysis in [Métivier 2008] imply that

$$M^{m-r,0}(\sigma_a^* - \sigma_{a^*}) \le \sum_{|\alpha|=r} M^{m-r,0}(r_\alpha) + M^{m-r,0}(a^* - \sigma_{a^*}) \lesssim M^{m,0}(\nabla^r a).$$

The proposition follows by Remark 4.5. □

Recall the following results of paralinearization. See, e.g., [Métivier 2008].

Proposition 4.11. *If* $a \in H^\alpha$ *and* $b \in H^\beta$ *with* $\alpha > \frac{d}{2}$ *and* $\beta > \frac{d}{2}$, *then*

$$\|ab - T_a b - T_b a\|_{H^{\alpha+\beta-d/2}} \lesssim \|a\|_{H^\alpha}\|b\|_{H^\beta}.$$

Proposition 4.12. *If* $F \in C^\infty(\mathbb{R})$ *with* $F(0) = 0$, *then for all* $\mu > \frac{d}{2}$, *there exists a monotonically increasing function* $C : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ *such that, for all* $u \in H^\mu$, *we have*

$$\|F(u)\|_{H^\mu} + \|F(u) - T_{F'(u)}u\|_{H^{2\mu-d/2}} \le C(\|u\|_{H^s})\|u\|_{H^\mu}.$$

4B. *Dyadic paradifferential calculus.* Now we develop the theory of paradifferential calculus with weighted symbols on weighted Sobolev spaces via a dyadic decomposition of the space.

4B1. *Weighted symbol classes and dyadic paradifferential operators.* We define a family of symbol classes which take into consideration the spacial decay of symbols.

Definition 4.13. If $r \in \mathbb{N}$, $k \in \mathbb{R}$, and $\delta \in [0,1]$, then $W^{r,\infty}_{k,\delta}$ is the set of all $u \in \mathscr{S}'(\mathbb{R}^d)$ such that

$$\|u\|_{W^{r,\infty}_{k,\delta}} = \sum_{|\alpha|\le r} \|\langle x\rangle^{-k+\delta|\alpha|}\partial_x^\alpha u\|_{L^\infty} < \infty.$$

Definition 4.14. If $m, k \in \mathbb{R}$, $r \in \mathbb{N}$ and $\delta \in [0,1]$, then $\Gamma^{m,r}_{k,\delta}$ is the set of all $a(x,\xi) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d \times (\mathbb{R}^d\backslash 0))$ such that

(1) for all $x \in \mathbb{R}^d$ the function $\xi \mapsto a(x,\xi)$ is smooth, and

(2) for all $\alpha \in \mathbb{N}^d$ there exists $C_\alpha > 0$, such that

$$\|\partial_\xi^\alpha a(\cdot,\xi)\|_{W^{r,\infty}_{k,\delta}} \le C_\alpha\langle\xi\rangle^{m-|\alpha|} \quad \text{for all } |\xi| \ge \tfrac{1}{2}.$$

Moreover, we define

$$M_{k,\delta}^{m,r}(a)=\sup_{|\alpha|\le r+\tilde{d}}\sup_{|\xi|\ge 1/2}\langle\xi\rangle^{|\alpha|-m}\|\partial_\xi^\alpha a(\cdot,\xi)\|_{W_{k,\delta}^{r,\infty}}.$$

Let $\Gamma_{k,\delta}^{-\infty,r}=\bigcap_{m\in\mathbb{R}}\Gamma_{k,\delta}^{m,r}$ and $\Gamma_{-\infty,\delta}^{m,r}=\bigcap_{k\in\mathbb{R}}\Gamma_{k,\delta}^{m,r}$. Then for $(m,k)\in(\mathbb{R}\cup\{-\infty\})^2$, define

$$\Sigma_{k,\delta}^{m,r}=\sum_{0\le j\le r}\Gamma_{k-\delta j,\delta}^{m-j,r-j}.$$

We say that $a_h=\sum_{0\le j\le r}h^j a_h^j\in{}_h\Sigma_{k,\delta}^{m,r}$ if

$$\sup_{0<h<1}\sum_{0\le j\le r}M_{k-\delta j,\delta}^{m-j,r-j}(a_h^j)<\infty.$$

We shall define $\Sigma^{m,r}=\Sigma_{0,0}^{m,r}$ and ${}_h\Sigma^{m,r}={}_h\Sigma_{0,0}^{m,r}$.

We are mostly interested in the cases where $\delta\in\{0,1\}$. Note that $W_{k,0}^{r,\infty}=\langle x\rangle^k W^{r,\infty}$ and thus $\Gamma_{k,0}^{m,r}=\langle x\rangle^k\Gamma^{m,r}$, whereas $\Gamma_{k,1}^{m,r}$ is a natural extension of S_k^m to symbols of finite regularities. We will encounter symbols defined by solutions of the water wave system and thus have coefficients in weighted Sobolev spaces. We need the following lemma.

Lemma 4.15. *If $u\in\mathcal{H}_k^{\mu,\delta}$, where $\mu\ge\tilde{d}$, $k\in\mathbb{N}$ and $\delta\in(0,1]$, then, for all $\alpha\in\mathbb{N}^d$ with $|\alpha|\le\min\{(\mu-\tilde{d})/(1+\delta),k\}$, we have $\langle x\rangle^{|\alpha|}\partial_x^\alpha u\in L^\infty$ and consequently we have the inclusion*

$$\mathcal{H}_k^{\mu,\delta}\subset W_{0,1}^{\min\{[(\mu-\tilde{d})/(1+\delta)],k\},\infty}\cap\langle x\rangle^{-\min\{[\mu-\tilde{d}]/\delta,k\}}L^\infty.$$

In particular $\mathcal{H}_k^{\mu,1/2}\subset W_{0,1}^{\min\{[2(\mu-\tilde{d})/3],k\},\infty}\cap\langle x\rangle^{-\min\{k,2[\mu-\tilde{d}]\}}L^\infty$.

Proof. The lemma follows directly from the Sobolev injection:

$$\|\langle x\rangle^{|\alpha|}\partial_x^\alpha u\|_{L^\infty}\lesssim\|u\|_{W_{-|\alpha|}^{|\alpha|,\infty}}\lesssim\|u\|_{H_{|\alpha|}^{|\alpha|+\tilde{d}}}\lesssim\|u\|_{\mathcal{H}_k^{\mu,\delta}},$$
$$\|\langle x\rangle^n u\|_{L^\infty}\lesssim\|u\|_{W_{-n,0}^{0,\infty}}\lesssim\|u\|_{H_n^{\tilde{d}}}\lesssim\|u\|_{\mathcal{H}_k^{\mu,\delta}},$$

which hold provided $|\alpha|+\tilde{d}\le\mu-\delta|\alpha|$, $|\alpha|\le k$, $\tilde{d}\le\mu-\delta n$ and $n\le k$, that is,

$$|\alpha|\le\min\left\{\frac{\mu-\tilde{d}}{1+\delta},k\right\},\quad n\le\min\left\{\frac{\mu-\tilde{d}}{\delta},k\right\}.\qquad\square$$

Lemma 4.16. *Let $\mathcal{A}$ be a linear operator from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$ and let $m,k\in\mathbb{R}$. If there exists $\{\mathcal{A}_j\}_{j\in\mathbb{N}}\in\ell^\infty(\mathscr{O}_0^m)$ and $\psi,\phi\in\mathscr{P}$ such that $\mathcal{A}=\sum_{j\in\mathbb{N}}2^{jk}\psi_j\mathcal{A}_j\phi_j$, then $\mathcal{A}\in\mathscr{O}_k^m$.*

Proof. The lemma is a consequence of Proposition 2.12. $\square$

Definition 4.17. Let $\psi\in\mathscr{P}_*$ and define $\underline{\psi}\in\mathscr{P}$ by setting $\underline{\psi}_j=\sum_{|j-k|\le 10}\psi_k$. If $a\in\Gamma_{k,\delta}^{m,r}$, where $m,k\in\mathbb{R}$, $r\in\mathbb{N}$ and $\delta\in[0,1]$, then define the dyadic paradifferential operator

$$\mathcal{P}_a=\sum_{j\in\mathbb{N}}\underline{\psi}_j T_{\psi_j a}\underline{\psi}_j.$$

Proposition 4.18. *If $a\in\Gamma_{k,\delta}^{m,r}$, then $\mathcal{P}_a=\mathcal{O}(M_k^{m,r}(a))_{\mathscr{O}_k^m}$.*

Proof. Note that if $a \in \Gamma^{m,r}_{k,\delta}$ then $a \in \langle x\rangle^k \Gamma^{m,0}$. Therefore, by Proposition 4.4, we have

$$\|T_{\psi_j a}\|_{H^\nu \to H^{\nu-m}} \lesssim M^{m,0}(\psi_j a) \lesssim 2^{-jk} M^{m,0}_{k,0}(a).$$

We conclude by Lemma 4.16. □

4B2. *Symbolic calculus.*

Proposition 4.19. *Let $a \in \Gamma^{m,r}_{k,\delta}$, $b \in \Gamma^{m',r}_{k',\delta}$, $r \in \mathbb{N}$, $(m,k), (m',k') \in \mathbb{R}^2$, $0 \le \delta \le 1$, then*

$$\mathcal{P}_a \mathcal{P}_b - \mathcal{P}_{a\sharp b} = \mathcal{O}(M^{m,r}_{k,\delta}(a) M^{m',r}_{k',\delta}(b))_{\mathscr{O}^{m+m'-r}_{k+k'-\delta r} + \mathscr{O}^{-\infty}_{k+k'}},$$

where

$$a\sharp b = \sum_{\alpha \in \mathbb{N}^d}^{|\alpha|<r} \frac{1}{\alpha!} \partial_\xi^\alpha a D_x^\alpha b \in \Sigma^{m+m',r}_{k+k',\delta}.$$

Proof. Let $\tilde{\psi}_j : \mathbb{N} \to C_c^\infty$, $\tilde{\psi}_j = \sum_{|j-j'|\le 50} \psi_{j'}$, so $\underline{\psi}_{j'} \tilde{\psi}_j = \underline{\psi}_{j'}$ if $|j-j'| \le 20$. Then write

$$\mathcal{P}_a \mathcal{P}_b = \sum_{(j,j')\in\mathbb{N}^2}^{|j-j'|\le 20} \underline{\psi}_j T_{\psi_j a} \underline{\psi}_j \underline{\psi}_{j'} T_{\psi_{j'} b} \underline{\psi}_{j'} = \sum_{(j,j')\in\mathbb{N}^2}^{|j-j'|\le 20} \tilde{\psi}_j T_{\psi_j a} T_{\psi_{j'} b} \tilde{\psi}_j + \tilde{\psi}_j R_{j,j'} \tilde{\psi}_j,$$

the remainder being

$$\begin{aligned} R_{j,j'} &= \underline{\psi}_j T_{\psi_j a} \underline{\psi}_j \underline{\psi}_{j'} T_{\psi_{j'} b} \underline{\psi}_{j'} - T_{\psi_j a} T_{\psi_{j'} b} \\ &= \mathcal{O}(2^{j(k+k'-\delta r)} M^{m,r}_{k,\delta}(a) M^{m',r}_{k',\delta}(b))_{\mathscr{O}^{m+m'-r}_0} + \mathcal{O}(2^{j(k+k')} M^{m,r}_{k,\delta}(a) M^{m',r}_{k',\delta}(b))_{\mathscr{O}^{-\infty}_0} \end{aligned}$$

by Propositions 4.4 and 4.9 and Corollary 4.8. More precisely, when composing $T_{\psi_j a}$ and $T_{\underline{\psi}_j}$, we use $\psi_j \underline{\psi}_j = \psi_j$ and have

$$\begin{aligned} T_{\psi_j a} T_{\underline{\psi}_j} = T_{\psi_j a} &+ \mathcal{O}\big(M^{m,r}(\psi_j a) M^{0,0}(\nabla^r \underline{\psi}_j)\big)_{\mathscr{O}^{m-r}_0} \\ &+ \mathcal{O}\big(M^{m,0}(\nabla^r(\psi_j a)) M^{0,r}(\underline{\psi}_j)\big)_{\mathscr{O}^{m-r}_0} + \mathcal{O}\big(M^{m,r}(\psi_j a) M^{0,r}(\underline{\psi}_j)\big)_{\mathscr{O}^{-\infty}_0}, \end{aligned}$$

where $M^{0,0}(\nabla^r \underline{\psi}_j) = \mathcal{O}(2^{-jr})$, $M^{m,r}(\psi_j a) = \mathcal{O}(2^{jk})$, and we use $0 \le \delta \le 1$ to induce that

$$M^{m,0}(\nabla^r(\psi_j a)) = \mathcal{O}\big(\max_{0\le n\le r} \{2^{-j(r-n)+j(k-\delta n)}\}\big) = \mathcal{O}(2^{j(k-\delta r)}).$$

Similar arguments work for the composition $T_{\underline{\psi}_j} T_{\psi_j a}$.

Observe that $\sum_{j':|j-j'|\le 20} (\psi_j a)\sharp \psi_{j'} b = (\psi_j a)\sharp b$, for all $j \in \mathbb{N}$. Hence

$$\sum_{j':|j-j'|\le 20} T_{\psi_j a} T_{\psi_{j'} b} = \underline{\psi}_j T_{\psi_j (a\sharp b)} \underline{\psi}_j + R_j,$$

where the remainder can be estimated much as above:

$$R_j = \mathcal{O}(2^{j(k+k'-\delta r)} M^{m,r}_{k,\delta}(a) M^{m',r}_{k',\delta}(b))_{\mathscr{O}^{m+m'-\delta r}_0} + \mathcal{O}(2^{j(k+k')} M^{m,r}_{k,\delta}(a) M^{m',r}_{k',\delta}(b))_{\mathscr{O}^{-\infty}_0}.$$

We conclude by Lemma 4.16. □

Proposition 4.20. *Let $a \in \Gamma^{m,r}_{k,\delta}$ with $(m,k) \in \mathbb{R}^2$, and $r \in \mathbb{N}$, $0 \le \delta \le 1$, then*

$$\mathcal{P}_a^* - \mathcal{P}_{a^*} = \mathcal{O}(M^{m,r}_{k,\delta}(a))_{\mathscr{O}^{m-r}_{k-\delta r} + \mathscr{O}^{-\infty}_k}, \tag{4-8}$$

where

$$a^* = \sum_{\alpha\in\mathbb{N}^d}^{|\alpha|<r} \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha \bar{a} \in \Sigma_{k,\delta}^{m,r}.$$

Proof. Observe that for any real-valued $\psi \in C_c^\infty(\mathbb{R}^d)$,

$$(\psi a)^* = a^* \sharp \psi. \tag{4-9}$$

More precisely, this means that

$$\begin{aligned}(\psi a)^* &= \sum_{|\gamma|<r} \frac{1}{\gamma!}\partial_\xi^\gamma D_x^\gamma(\psi\bar{a}) = \sum_{|\gamma|<r}\frac{1}{\gamma!}\sum_{\alpha+\beta=\gamma}\frac{\gamma!}{\alpha!\beta!}D_x^\alpha\psi\,\partial_\xi^\gamma D_x^\beta\bar{a}\\ &= \sum_{|\alpha|+|\beta|<r}\frac{1}{\alpha!}\partial_\xi^\alpha\Big(\frac{1}{\beta!}\partial_\xi^\beta D_x^\beta\bar{a}\Big)D_x^\alpha\psi = \sum_{|\beta|<r}\sum_{|\alpha|<r-|\alpha|}\frac{1}{\alpha!}\partial_\xi^\alpha\Big(\frac{1}{\beta!}\partial_\xi^\beta D_x^\beta\bar{a}\Big)D_x^\alpha\psi\\ &= \sum_{|\beta|<r}\Big(\frac{1}{\beta!}\partial_\xi^\beta D_x^\beta\bar{a}\Big)\sharp\psi = a^*\sharp\psi.\end{aligned}$$

Then write $\mathcal{P}_a^* - \mathcal{P}_{a^*} = \sum_{j\in\mathbb{N}} \underline{\psi}_j(R_j^1 + R_j^2)\underline{\psi}_j$, where, by (4-9),

$$R_j^1 = T_{\psi_j a}^* - T_{(\psi_j a)^*}, \qquad R_j^2 = T_{(\psi_j a)^*} - T_{\psi_j a^*} = T_{a^*\sharp\psi_j - \psi_j a^*}.$$

For R_j^1 we use Proposition 4.10,

$$R_j^1 = \mathcal{O}(M^{m,0}(\nabla_x^r(\psi_j a)))_{\mathscr{O}_0^{m-r}} = \mathcal{O}(2^{j(k-\delta r)}M_{k,\delta}^{m,r}(a))_{\mathscr{O}_0^{m-r}}.$$

By Lemma 4.16,

$$\sum_{j\in\mathbb{N}} \underline{\psi}_j R_j^1 \underline{\psi}_j = \mathcal{O}(M_{k,\delta}^{m,r}(a))_{\mathscr{O}_{k-\delta r}^{m-r}+\mathscr{O}_k^{-\infty}}.$$

Using $\sum_{j\in\mathbb{N}}\psi_j \equiv 1$, we induce that

$$\sum_{j\in\mathbb{N}}\partial_x^\alpha\psi_j \equiv 0 \quad \text{for all } \alpha\in\mathbb{N}^d\backslash 0, \qquad \sum_{j\in\mathbb{N}} a^*\sharp\psi_j - \psi_j a^* = 0. \tag{4-10}$$

Then we write

$$a^*\sharp\psi_j - \psi_j a^* = \sum_{\substack{\alpha\neq 0\\ |\alpha|+|\beta|<r}} D_x^\alpha\psi_j\cdot w_{\alpha\beta}, \qquad w_{\alpha\beta}\in\Gamma_{k-|\beta|\delta,\delta}^{m-|\alpha|-|\beta|,r-|\beta|},$$

where the symbols $w_{\alpha\beta}$ are independent of j. Write

$$\sum_{j\in\mathbb{N}}\underline{\psi}_j R_j^2\underline{\psi}_j = \sum_{\alpha,\beta} R_{\alpha\beta}, \qquad R_{\alpha\beta} = \sum_{j\in\mathbb{N}}\underline{\psi}_j T_{D_x^\alpha\psi_j\cdot w_{\alpha\beta}}\underline{\psi}_j.$$

By (4-10), we prove similarly to Proposition 4.19 that

$$\begin{aligned}\psi_j R_{\alpha\beta} &= \psi_j \sum_{|j-j'|\le 20}\underline{\psi}_{j'}T_{D_x^\alpha\psi_{j'}\cdot w_{\alpha\beta}}\underline{\psi}_{j'}\\ &= \mathcal{O}(2^{j(-|\alpha|+k-|\beta|\delta-(r-|\beta|)\delta)}M_{k-|\beta|\delta,\delta}^{m-|\alpha|-|\beta|,r-|\beta|}(w_{\alpha\beta}))_{\mathscr{O}_0^{m-r-|\alpha|}}\\ &\qquad\qquad + \mathcal{O}(2^{j(-|\alpha|+k-|\beta|\delta)}M_{k-|\beta|\delta,\delta}^{m-|\alpha|-|\beta|,r-|\beta|}(w_{\alpha\beta}))_{\mathscr{O}_0^{-\infty}}\\ &= \mathcal{O}(2^{j(k-\delta r)}M_{k,\delta}^{m,r}(a))_{\mathscr{O}_0^{m-r}} + \mathcal{O}(2^{jk}M_{k,\delta}^{m,r}(a))_{\mathscr{O}_0^{-\infty}}.\end{aligned}$$

Setting $\psi'_j = \sum_{|j'-j|\le 100} \psi_{j'}$. We again conclude by Lemma 4.16, and the identity

$$R_{\alpha\beta} = \sum_{j\in\mathbb{N}} \psi_j R_{\alpha\beta} \psi'_j,$$

that $R_{\alpha\beta} = \mathcal{O}(M^{m,r}_{k,\delta}(a))_{\mathscr{O}^{m-r}_{k-\delta r}+\mathscr{O}^{-\infty}_k}$. □

4B3. *Paralinearization.*

Proposition 4.21. *If $a \in H^\alpha_k$, $b \in H^\beta_\ell$ with $\alpha > \frac{d}{2}$, $\beta > \frac{d}{2}$, $k \in \mathbb{R}$, $\ell \in \mathbb{R}$, then, for all $\epsilon > 0$,*

$$\|ab - \mathcal{P}_a b - \mathcal{P}_b a\|_{H^{\alpha+\beta-d/2-\epsilon}_{k+\ell}} \lesssim \|a\|_{H^\alpha_k} \|b\|_{H^\beta_\ell}.$$

Consequently if $a \in \mathcal{H}^{\alpha,\delta}_m$ and $b \in \mathcal{H}^{\beta,\delta}_n$ with $\delta \ge 0$, $\alpha - \delta m > \frac{d}{2}$, $\beta - \delta n > \frac{d}{2}$, then, for all $\epsilon > 0$,

$$\|ab - \mathcal{P}_a b - \mathcal{P}_b a\|_{\mathcal{H}^{\alpha+\beta-d/2-\epsilon,\delta}_{m+n}} \lesssim \|a\|_{\mathcal{H}^{\alpha,\delta}_m} \|b\|_{\mathcal{H}^{\beta,\delta}_n}.$$

Proof. Decompose the product ab as

$$ab = \sum_{j\in\mathbb{N}} \psi_j (\underline{\psi}_j a)(\underline{\psi}_j b) = \mathcal{P}_a b + \mathcal{P}_b a + R^1_j + R^2_j,$$

where the remainders R^1_j and R^2_j are defined by

$$R^1_j = \psi_j \big(\underline{\psi}_j a \cdot \underline{\psi}_j b - T_{\underline{\psi}_j a}(\underline{\psi}_j b) - T_{\underline{\psi}_j b}(\underline{\psi}_j a)\big),$$
$$R^2_j = \underline{\psi}_j (\psi_j T_{\underline{\psi}_j a} - T_{\psi_j a}) \underline{\psi}_j b + \underline{\psi}_j (\psi_j T_{\underline{\psi}_j b} - T_{\psi_j a}) \underline{\psi}_j a.$$

By Proposition 4.11,

$$\|R^1_j\|_{H^{\alpha+\beta-d/2}} \lesssim \|\underline{\psi}_j a\|_{H^\alpha} \|\underline{\psi}_j b\|_{H^\beta} \lesssim 2^{-j(k+\ell)} \|a\|_{H^\alpha_k} \|b\|_{H^\beta_\ell}.$$

By Proposition 4.9 and Corollary 4.8 and the Sobolev embedding theorem, for all $\epsilon > 0$ we have

$$\psi_j T_{\underline{\psi}_j a} \underline{\psi}_j - T_{\psi_j a} = 2^{-jk} \mathcal{O}(\|a\|_{H^\alpha_k})_{\mathscr{O}^{\alpha-d/2-\epsilon}_0},$$
$$\psi_j T_{\underline{\psi}_j b} \underline{\psi}_j - T_{\psi_j b} = 2^{-j\ell} \mathcal{O}(\|b\|_{H^\beta_\ell})_{\mathscr{O}^{\beta-d/2-\epsilon}_0}.$$

We conclude the first statement by Proposition 2.12.

As for the second statement, observe that if $0 \le k \le m$ and $0 \le \ell \le n$, then

$$\|ab - \mathcal{P}_a b - \mathcal{P}_b a\|_{H^{(\alpha-\delta k)+(\beta-\delta\ell)-d/2-\epsilon,\delta}_{k+\ell}} \lesssim \|a\|_{H^{\alpha-\delta k}_k} \|b\|_{H^{\beta-\delta\ell}_\ell} \lesssim \|a\|_{\mathcal{H}^{\alpha,\delta}_m} \|b\|_{\mathcal{H}^{\beta,\delta}_n}.$$

We conclude by noting that for all $p \in \mathbb{N} \cap [0, m+n]$, there exist $k \in \mathbb{N} \cap [0, m]$ and $\ell \in \mathbb{N} \cap [0, n]$ such that $p = k + \ell$. □

Proposition 4.22. *Let $F \in C^\infty(\mathbb{R})$ with $F(0) = 0$. For all $\mu > \frac{d}{2}$, there exists some monotonically increasing function $C : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $k \ge 0$ and all $u \in H^\mu_k$, we have*

$$\|F(u)\|_{H^\mu_k} + \|F(u) - \mathcal{P}_{F'(u)} u\|_{H^{2\mu-d/2}_k} \le C(\|u\|_{H^\mu}) \|u\|_{H^\mu_k}.$$

Consequently, if $u \in \mathcal{H}^{\mu,\delta}_k$ with $\delta \ge 0$ and $\mu - \delta k > \frac{d}{2}$, then

$$\|F(u)\|_{\mathcal{H}^{\mu,\delta}_k} + \|F(u) - \mathcal{P}_{F'(u)} u\|_{\mathcal{H}^{2\mu-d/2,2\delta}_k} \le C(\|u\|_{H^\mu}) \|u\|_{\mathcal{H}^{\mu,\delta}_k}.$$

Proof. Decompose $F(u)$ as $F(u) = \sum_{j\geq 0} \psi_j F(\underline{\psi}_j u)$. By Proposition 4.12,

$$\|F(\underline{\psi}_j u)\|_{H^\mu} \leq C(\|\underline{\psi}_j u\|_{H^\mu})\|\underline{\psi}_j u\|_{H^\mu} \leq C(\|u\|_{H^\mu})\|\underline{\psi}_j u\|_{H^\mu}.$$

Then write $\psi_j F(\underline{\psi}_j u) = \underline{\psi}_j T_{\psi_j F'(u)} \underline{\psi}_j u + \underline{\psi}_j R_j$, where

$$R_j = \psi_j(F(\underline{\psi}_j u) - T_{F'(\underline{\psi}_j u)} \underline{\psi}_j u) + \underline{\psi}_j(\psi_j T_{F'(\underline{\psi}_j u)} - T_{\psi_j F'(\underline{\psi}_j u)})\underline{\psi}_j u.$$

By Propositions 4.12 and 4.11 and Corollary 4.8,

$$\|R_j\|_{H^{2\mu-d/2}} \leq C(\|u\|_{H^\mu})\|\underline{\psi}_j u\|_{H^\mu}.$$

We conclude the first statement with Proposition 2.12. To prove the second statement, note that for all $j \in \mathbb{N} \cap [0, k]$ we have

$$\|F(u)\|_{H_j^{\mu-\delta j}} + \|F(u) - \mathcal{P}_{F'(u)} u\|_{H_j^{2(\mu-\delta j)-d/2}} \leq C(\|u\|_{H^{\mu-\delta j}})\|u\|_{H_j^{\mu-\delta j}}. \qquad \square$$

4C. *Semiclassical paradifferential calculus.* We develop a semiclassical dyadic paradifferential calculus and a quasihomogeneous semiclassical paradifferential calculus, using scaling arguments inspired by [Métivier and Zumbrun 2005].

4C1. *Semiclassical paradifferential operators.*

Definition 4.23. For all $h \in (0, 1]$, define the scaling operator

$$\tau_h : u(\,\cdot\,) \mapsto h^{d/2} u(h\,\cdot\,). \tag{4-11}$$

(1) If $b \in \Gamma^{m,r}$, then define $T_b^h = \tau_h^{-1} T_{\theta_{h,*}^{1,0} b} \tau_h$.

(2) If $a \in \Gamma_{k,\delta}^{m,r}$, then define $\mathcal{P}_a^h = \sum_{j\in\mathbb{N}} \underline{\psi}_j T_{\psi_j a}^h \underline{\psi}_j$.

(3) If $\epsilon \geq 0$, then define $\mathcal{P}_a^{h,\epsilon} = \mathcal{P}_{\theta_{h,*}^{\epsilon,0} a}^h$.

Proposition 4.24. *If $\epsilon \geq 0$ and $a \in \Gamma_{k,0}^{m,0}$, where $m \leq 0$, $k \leq 0$, then $\sup_{h\in(0,1]} \|\mathcal{P}_a^{h,\epsilon}\|_{L^2\to L^2} < \infty$.*

Proof. Observe that $\theta_{h,*}^{1+\epsilon,0} a = \mathcal{O}(1)_{\Gamma^{0,0}}$. We conclude with Lemma 4.16. $\square$

4C2. *Semiclassical symbolic calculus.*

Definition 4.25. If $a_h \in \mathscr{D}'(\mathbb{R}^{2d})$ and $\epsilon \geq 0$, we say that $a_h \in \sigma_\epsilon$ if

$$\overline{\bigcup_{0<h<1} \operatorname{supp} a_h} \cap \mathscr{N}_{\epsilon,1} = \varnothing.$$

Proposition 4.26. *Let $(m, k), (m', k') \in (\mathbb{R} \cup \{-\infty\})^2$, $r \in \mathbb{N}$, with $r \geq m + m'$, $\delta r \geq k + k'$. Let $a_h \in \Gamma_{k,\delta}^{m,r} \cap \sigma_0$ and $b_h \in \Gamma_{k',\delta}^{m',r} \cap \sigma_0$ such that, for some $R_h \geq 0$ depending on h,*

$$\operatorname{supp} a_h \cap \operatorname{supp} b_h \subset \{|x| \geq R_h\} \times \mathbb{R}^d. \tag{4-12}$$

Then, for $h > 0$ sufficiently small,

$$\mathcal{P}_{a_h}^h \mathcal{P}_{b_h}^h - \mathcal{P}_{a_h \sharp_h b_h}^h = \mathcal{O}(h^r (1 + R_h)^{k+k'-\delta r})_{L^2\to L^2},$$

where the symbol $a_h \sharp_h b_h = a_h \sharp_{h,r}^{0,1} b_h \in {}_h\Sigma_{k+k',\delta}^{m+m',r}$ is defined by (2-4).

Proof. By (4-12), if $\psi_j a_h \neq 0$ and $\psi_j b_h \neq 0$, then $j \gtrsim \log_2(1+R_h)$. We claim that

$$\begin{aligned}\mathcal{P}_a^h \mathcal{P}_b^h &= \sum_{\substack{j \gtrsim \log_2(1+R_h)\\ |j'-j|\le 20}} \underline{\psi}_j T^h_{\psi_j a_h} \underline{\psi}_j \underline{\psi}_{j'} T^h_{\psi_{j'} b_h} \underline{\psi}_{j'} \\ &= \sum_{\substack{j \gtrsim \log_2(1+R_h)\\ |j'-j|\le 20}} \underline{\psi}_j T^h_{(\psi_j a_h)\sharp_h(\psi_{j'} b_h)} \underline{\psi}_{j'} + \mathcal{O}(h^r(1+R_h)^{k+k'-\delta r})_{L^2\to L^2}. \end{aligned} \tag{4-13}$$

Then we conclude by

$$\sum_{j':|j'-j|\le 20} (\psi_j a_h)\sharp_h(\psi_{j'} b_h) = \psi_j (a_h \sharp_h b_h).$$

It remains to prove (4-13). We use (4-1) to deduce that $\mathcal{F}(T_{\theta^{1,0}_{h,*}(\psi_{j'} b_h)} u)$ vanishes in a neighborhood of $\xi = 0$. By (4-5), for some $\pi' \in C^\infty(\mathbb{R}^d)$ which vanishes near $\xi = 0$ and equals 1 outside a neighborhood of $\xi = 0$, and, for all $m+m' \le N \in \mathbb{N}$,

$$\begin{aligned}\tau_h T^h_{\psi_j a_h} \underline{\psi}_j \underline{\psi}_{j'} T^h_{\psi_{j'} b_h} \tau_h^{-1} &= T_{\theta^{1,0}_{h,*}(\psi_j a_h)} \theta^{1,0}_{h,*}(\underline{\psi}_j \underline{\psi}_{j'}) \pi'(D_x) T_{\theta^{1,0}_{h,*}(\psi_{j'} b_h)} \\ &= T_{\theta^{1,0}_{h,*}(\psi_j a_h)} T_{\theta^{1,0}_{h,*}(\underline{\psi}_j \underline{\psi}_{j'})\otimes \pi'} T_{\theta^{1,0}_{h,*}(\psi_{j'} b_h)} \\ &\qquad + \mathcal{O}(M^{m,0}(\psi_j a_h))_{\mathscr{O}_0^m} \mathcal{O}(2^{-jN} h^N)_{\mathscr{O}_0^{-N}} \mathcal{O}(M^{m',0}(\psi_j b_h))_{\mathscr{O}_0^{m'}}. \end{aligned} \tag{4-14}$$

Then we use Proposition 4.9 and the fact that $a_h, b_h \in \sigma_0$ to deduce

$$\begin{aligned} & T_{\theta^{1,0}_{h,*}(\psi_j a_h)} T_{\theta^{1,0}_{h,*}(\underline{\psi}_j \underline{\psi}_{j'})\otimes \pi'} T_{\theta^{1,0}_{h,*}(\psi_{j'} b_h)} \\ &\quad = T_{\theta^{1,0}_{h,*}(\psi_j a_h)\sharp \theta^{1,0}_{h,*}(\underline{\psi}_j \underline{\psi}_{j'}\otimes \pi')\sharp \theta^{1,0}_{h,*}(\psi_{j'} b_h)} \\ &\qquad + \mathcal{O}(M^{m,0}(\nabla^r \theta^{1,0}_{h,*}(\psi_j a_h)) M^{0,r}(\theta^{1,0}_{h,*}(\psi_j \psi_{j'})) M^{m',r}(\theta^{1,0}_{h,*}(\psi_{j'} b_h)))_{\mathscr{O}_0^{m+m'-r}} \\ &\qquad + \mathcal{O}(M^{m,r}(\theta^{1,0}_{h,*}(\psi_j a_h)) M^{0,0}(\nabla^r \theta^{1,0}_{h,*}(\psi_j \psi_{j'})) M^{m',r}(\theta^{1,0}_{h,*}(\psi_{j'} b_h)))_{\mathscr{O}_0^{m+m'-r}} \\ &\qquad + \mathcal{O}(M^{m,r}(\theta^{1,0}_{h,*}(\psi_j a_h)) M^{0,r}(\theta^{1,0}_{h,*}(\psi_j \psi_{j'})) M^{m',0}(\nabla^r \theta^{1,0}_{h,*}(\psi_{j'} b_h)))_{\mathscr{O}_0^{m+m'-r}}. \end{aligned}$$

To estimate the remainders, we see that, for each $\alpha \in \mathbb{N}^d$ with $|\alpha| = r$,

$$\begin{aligned}\partial_x^\alpha \theta^{1,0}_{h,*}(\psi_j a_h) &= \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\,\alpha_2!} \partial_x^{\alpha_1} \theta^{1,0}_{h,*} \psi_j \partial_x^{\alpha_2} \theta^{1,0}_{h,*} a_h \\ &= \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\,\alpha_2!} \mathcal{O}(h^{|\alpha_1|} 2^{-j|\alpha_1|} \times h^{|\alpha_2|} 2^{j(k-\delta|\alpha_2|)})_{L^\infty} = \mathcal{O}(h^r 2^{j(k-\delta r)})_{L^\infty}, \end{aligned}$$

where we use $0 \le \delta \le 1$. Therefore, the first term in the remainder is

$$\mathcal{O}(h^r 2^{j(k+k'-\delta r)})_{L^2\to L^2} = \mathcal{O}(h^r(1+R_h)^{k+k'-\delta r})_{L^2\to L^2}.$$

Similar methods apply to the other two terms and we conclude that

$$T_{\theta^{1,0}_{h,*}(\psi_j a_h)} T_{\theta^{1,0}_{h,*}(\underline{\psi}_j \underline{\psi}_{j'})\otimes \pi'} T_{\theta^{1,0}_{h,*}(\psi_{j'} b_h)} = T_{\theta^{1,0}_{h,*}((\psi_j a_h)\sharp_h(\psi_{j'} b_h))} + \mathcal{O}(h^r(1+R_h)^{k+k'-\delta r})_{L^2\to L^2}. \tag{4-15}$$

The estimate (4-13) follows from (4-14) and (4-15). ∎

Combining the analysis of Propositions 4.26 and 4.20, using Proposition 4.9, we obtain a similar result for the adjoint, to the proof of which we shall omit, as it is similar to the above.

Proposition 4.27. *Let $(m,k)\in(\mathbb{R}\cup\{-\infty\})^2$, $r\in\mathbb{N}$, with $r\geq m$, $\delta r\geq k$. Let $a_h\in\Gamma^{m,r}_{k,\delta}\cap\sigma_0$, such that, for some $R_h\geq 0$ depending on h, $\operatorname{supp} a_h\subset\{|x|\geq R_h\}\times\mathbb{R}^d$, then, for $h>0$ sufficiently small,*

$$(\mathcal{P}^h_{a_h})^*-\mathcal{P}^h_{a_h^*}=\mathcal{O}(h^r(1+R_h)^{k-\delta r})_{L^2\to L^2},$$

where $a_h^*=\zeta^{0,1}_{h,r}a_h\in {}_h\Sigma^{m,r}_{k,\delta}$ is defined by (2-5).

Corollary 4.28. *Let $\epsilon\geq 0$, $(m,k),(m',k')\in(\mathbb{R}\cup\{-\infty\})^2$, $r\in\mathbb{N}$, with $r\geq\max\{m+m',k'\}$, $k\leq 0$. If $a_h\in\Gamma^{m,r}_{k,1}\cap\sigma_\epsilon$ and $b_h\in\Gamma^{m',r}_{k',1}\cap\sigma_0$, then*

$$\begin{aligned}\mathcal{P}^{h,\epsilon}_{a_h}\mathcal{P}^h_{b_h}-\mathcal{P}^h_{(\theta^{\epsilon,0}_{h,*}a_h)\sharp_h b_h}&=\mathcal{O}(h^{(1+\epsilon)r-\epsilon(k+k')})_{L^2\to L^2},\\ \mathcal{P}^h_{b_h}\mathcal{P}^{h,\epsilon}_{a_h}-\mathcal{P}^h_{b_h\sharp_h(\theta^{\epsilon,0}_{h,*}a_h)}&=\mathcal{O}(h^{(1+\epsilon)r-\epsilon(k+k')})_{L^2\to L^2}.\end{aligned}$$

Proof. It suffices to observe that, if $\epsilon>0$ then $\operatorname{supp}\theta^{\epsilon,0}_{h,*}a_h\subset\{|x|\gtrsim h^{-\epsilon}\}$ and $\theta^{\epsilon,0}_{h,*}a_h=\mathcal{O}(h^{-\epsilon k})_{\Gamma^{m,r}_{0,1}}$. We conclude by Proposition 4.26. □

Corollary 4.29. *Let $\epsilon\geq 0$, $(m,k),(m',k')\in(\mathbb{R}\cup\{-\infty\})^2$, $r\in\mathbb{N}$, with $r\geq m+m'$, $k\leq 0$, $k'\leq 0$. If $a_h\in\Gamma^{m,r}_{k,1}\cap\sigma_\epsilon$ and $b_h\in\Gamma^{m',r}_{k',1}\cap\sigma_\epsilon$, then, for $h>0$ sufficiently small,*

$$\mathcal{P}^{h,\epsilon}_{a_h}\mathcal{P}^{h,\epsilon}_{b_h}-\mathcal{P}^{h,\epsilon}_{a_h\sharp^\epsilon_h b_h}=\mathcal{O}(h^{(1+\epsilon)r-\epsilon(k+k')})_{L^2\to L^2},$$

where the symbol $a_h\sharp^\epsilon_h b_h=a_h\sharp^{\epsilon,1}_{h,r}\in {}_{h^{1+\epsilon}}\Sigma^{m+m',r}_{k+k',1}$ is defined by (2-4).

Proof. It suffices to use the identity $(\theta^{\epsilon,0}_{h,*}a_h)\sharp_h(\theta^{\epsilon,0}_{h,*}b_h)=\theta^{\epsilon,0}_{h,*}(a_h\sharp^\epsilon_h b_h)$. □

4C3. *Some technical lemmas.* The results above only concerned the high frequency regime as we require the σ_ϵ condition. The next lemma studies the interaction of high frequencies and low frequencies.

Lemma 4.30. *Let $m\in\mathbb{R}$, $a_h\in\Gamma^{0,0}$, $b_h\in\Gamma^{0,0}$ such that, for some $R>0$,*

$$\operatorname{supp} a_h\in\{|\xi|\geq R\},\qquad \operatorname{supp} b_h\in\{|\xi|\leq h^{-1}R/4\}.$$

Then $\mathcal{P}^h_{a_h}\mathcal{P}_{b_h}=\mathcal{O}(h^\infty)_{L^2\to L^2}$.

Remark 4.31. This lemma concerns the estimate of $\mathcal{P}^h_{a_h}\mathcal{P}_{b_h}$, not $\mathcal{P}^h_{a_h}\mathcal{P}^h_{b_h}$. This is not a typo.

Proof. By definition

$$\widehat{T_{\psi_j b_h}u}(\xi)=(2\pi)^{-d}\int\chi(\xi-\eta,\eta)\pi(\eta)\widehat{\psi_j b_h}(\xi-\eta,\eta)\hat{u}(\eta)\,\mathrm{d}\eta.$$

The admissibility of χ implies $\operatorname{supp}\widehat{T_{\psi_j b_h}u}\subset\{|\xi|\leq h^{-1}R/3\}$. Therefore, for any $|j'-j|\leq 20$,

$$\begin{aligned}\underline{\psi}_j T^h_{\psi_j a_h}\underline{\psi}_j\underline{\psi}_{j'}T_{\psi_{j'}b_h}\underline{\psi}_{j'}&=\underline{\psi}_j T^h_{\psi_j a_h}\pi(hD_x/R)\underline{\psi}_j\underline{\psi}_{j'}(1-\pi(2hD_x/R))T_{\psi_{j'}b_h}\underline{\psi}_{j'}\\&=\underline{\psi}_j\mathcal{O}(h^\infty)_{L^2\to L^2}\underline{\psi}_{j'}.\end{aligned}$$

We conclude by Lemma 4.16. □

Corollary 4.32. *If $a \in \Gamma^{m,0}$ is homogeneous of degree m with respect to ξ, then, for $b \in \Gamma^{0,0} \cap \sigma_0$ and $h > 0$ sufficiently small,*

$$\mathcal{P}_b^h(h^m \mathcal{P}_a - \mathcal{P}_a^h) = \mathcal{O}(h^\infty)_{L^2 \to L^2}.$$

Proof. By a direct verification using (4-2), the homogeneity of a and the admissible function χ, and Corollary 4.7, we see that $h^m \mathcal{P}_a - \mathcal{P}_a^h = \mathcal{P}'_{\tilde{a}_h}$, where $\mathcal{P}'$ denotes the paradifferential quantization with any admissible pair (π', χ) such that $\pi\pi' = \pi'$, and the symbol

$$\tilde{a}_h(x,\xi) = (\pi(h\xi) - \pi(\xi))a(x, h\xi) \in \Gamma^{0,0}$$

satisfies the condition

$$\operatorname{supp} \tilde{a}_h \subset \mathbb{R}^d \times \operatorname{supp}(1 - \pi(h\,\cdot)) \subset \mathbb{R}^d \times \{|\xi| \le 2h^{-1}\}.$$

We conclude by Lemma 4.30. □

Lemma 4.33. *If $a_h \in \Gamma^{m,r} \cap \sigma_0$ with $r \ge \max\{m, 0\} + \tilde{d}$, then, for $h > 0$ sufficiently small,*

$$T_{a_h}^h - \operatorname{Op}_h(a_h) = \mathcal{O}(h^r)_{L^2 \to L^2}.$$

Proof. By Calderón–Vaillancourt theorem, we have

$$\begin{aligned} T_{a_h}^h - \operatorname{Op}_h(a_h) &= \tau_h^{-1}(T_{\theta_{h,*}^{1,0} a_h} - \operatorname{Op}(\theta_{h,*}^{1,0} a_h))\tau_h \\ &= \mathcal{O}\Big(\sum_{|\alpha|,|\beta| \le \tilde{d}} \|\partial_\xi^\alpha \partial_x^\beta (\sigma_{\theta_{h,*}^{1,0} a_h} - \theta_{h,*}^{1,0} a_h)\|_{L^\infty}\Big)_{L^2 \to L^2}. \end{aligned}$$

By hypothesis $r \ge \max\{m, 0\} + |\beta| \ge |\beta|$. We use (4-5) to deduce that

$$\begin{aligned} \|\partial_\xi^\alpha \partial_x^\beta (\sigma_{\theta_{h,*}^{1,0} a_h} - \theta_{h,*}^{1,0} a_h)\|_{L^\infty} &\lesssim M^{0,0}(\partial_x^\beta (\sigma_{\theta_{h,*}^{1,0} a_h} - \theta_{h,*}^{1,0} a_h)) \\ &\lesssim M^{\max\{m,0\} - r + |\beta|, 0}(\partial_x^\beta (\sigma_{\theta_{h,*}^{1,0} a_h} - \theta_{h,*}^{1,0} a_h)) \\ &\lesssim M^{\max\{m,0\},0}(\nabla_x^r (\theta_{h,*}^{1,0} a_h)) \\ &\lesssim h^r M^{m,0}(a_h). \end{aligned}$$

□

Lemma 4.34. *If $a_h \in \Gamma^{m,\infty} \cap \sigma_0$ with $m \in \mathbb{R} \cup \{-\infty\}$, then, for $h > 0$ sufficiently small,*

$$\mathcal{P}_{a_h}^h - \operatorname{Op}_h(a_h) = \mathcal{O}(h^\infty)_{L^2 \to L^2}.$$

Proof. By Lemmas 4.33 and 4.16,

$$\mathcal{P}_{a_h}^h - \sum_{j \in \mathbb{N}} \underline{\psi}_j \operatorname{Op}_h(\psi_j a_h) \underline{\psi}_j = \mathcal{O}(h^\infty)_{L^2 \to L^2}.$$

Note that, uniformly in $j \in \mathbb{N}$, we have

$$\underline{\psi}_j \sharp_h (\psi_j a_h) \sharp_h \underline{\psi}_j = \psi_j a_h + \underline{\psi}_j \mathcal{O}(h^\infty)_{\Gamma^{-\infty,\infty}}.$$

Therefore,

$$\sum_{j \in \mathbb{N}} \underline{\psi}_j \sharp_h (\psi_j a_h) \sharp_h \underline{\psi}_j = a_h + \mathcal{O}(h^\infty)_{\Gamma^{-\infty,\infty}}.$$

□

4C4. *Symbols with limited regularities in* (x, ξ). The symbols we have encountered so far have limited regularities in the x-variable but are smooth with respect to the ξ-variable. When studying the propagation of singularities for nonlinear equations, we need to solve Hamiltonian equations which transfer the limited regularity in the x-variable to the ξ-variable. Therefore we need to discuss in this section paradifferential operators with symbols that have limited regularities in both the x- and ξ-variables. As we do not intend to obtain optimal regularities, we shall content ourselves with an approach by approximation.

Definition 4.35. For all $r \in \mathbb{N}$, the symbol class Υ^r is the set of all $a \in L^\infty_{\text{loc}}(\mathbb{R}^d \times (\mathbb{R}^d \backslash 0))$ compactly supported in $\mathbb{R}^d \times (\mathbb{R}^d \backslash 0)$ such that $N^r(a) < +\infty$, where

$$N^r(a) = \sum_{\alpha \in \mathbb{N}^{2d}, |\alpha| \le r} \|\partial^\alpha_{x,\xi} a\|_{L^\infty_x W^{\tilde{d},\infty}_\xi}.$$

If $a \in \Upsilon^{r+1}$ with $r \in \mathbb{N}$, then the paradifferential operator T_a is defined via approximating a by smooth symbols. To be precise, let $\Omega \Subset \mathbb{R}^d \times (\mathbb{R}^d \backslash 0)$ be an open neighborhood of $\operatorname{supp} a$ and let $\{a_n\}_{n \in \mathbb{N}} \subset C^\infty_c(\Omega)$ such that

$$\lim_{n \to \infty} N^r(a_n - a) = 0.$$

Note that such an approximation is always possible because a is compactly supported and we only require the convergence with respect to the N^r-norm (not the N^{r+1}-norm)! By Proposition 4.4 and Lemma 4.6, for all $n, m \in \mathbb{N}$ we have

$$\|T_{a_n} - T_{a_m}\|_{L^2 \to L^2} \lesssim M^{0,0}_{\tilde{d}}(a_n - a_m) \lesssim N^0(a_n - a_m) \le N^r(a_n - a_m).$$

Therefore, for all $u \in L^2$, the sequence $\{T_{a_n} u\}_{n \in \mathbb{N}}$ is Cauchy in L^2 and we define

$$T_a u = \lim_{n \to \infty} T_{a_n} u.$$

Clearly this definition is independent of the choice of the sequence $\{a_n\}_{n \in \mathbb{N}}$ and extends the definition of paradifferential operators with symbols that are smooth with respect to ξ. Then we define the operators T^h_a, $\mathcal{P}_a$, $\mathcal{P}^h_a$ and $\mathcal{P}^h_a$ exactly as before.

Proposition 4.36. *If* $a \in \Upsilon^{r+1}$ *with* $r \ge 0$, *then for all* $h \in (0, 1]$, *we have* $T^h_a : L^2 \to L^2$. *Moreover*,

$$\sup_{h \in (0,1]} \|T^h_a\|_{L^2 \to L^2} \lesssim N^0(a).$$

Consequently, *for all* $\epsilon \ge 0$ *we have*

$$\sup_{h \in (0,1]} \|\mathcal{P}^{h,\epsilon}_a\|_{L^2 \to L^2} \lesssim N^0(a).$$

Proof. The general case $h \in (0, 1]$ follows from the case $h = 1$ and we shall assume $h = 1$. Choose a convergent sequence $\{a_n\}_{n \in \mathbb{N}} \subset C^\infty_c(\Omega)$ as above. For all $u \in L^2$ with $\|u\|_{L^2} = 1$, we have

$$\|T_a u\|_{L^2} \le \|T_a u - T_{a_n} u\|_{L^2} + \|T_{a_n} u\|_{L^2},$$

where $\lim_{n \to \infty} \|T_a u - T_{a_n} u\|_{L^2} = 0$ by the definition of $T_a u$ and

$$\|T_{a_n} u\|_{L^2} \lesssim N^0(a_n) \lesssim N^0(a - a_n) + N^0(a) \to N^0(a).$$

Therefore, passing $n \to \infty$ we conclude that $\|T_a\|_{L^2 \to L^2} \lesssim N^0(a)$. The estimate for $\mathcal{P}_a^{h,\epsilon}$ follows similarly to Proposition 4.18. $\square$

Combining the approximation method above and the analysis in Proposition 4.26, we obtain the following corollaries similarly to Corollaries 4.28 and 4.29.

Corollary 4.37. *Let $\epsilon \geq 0$, $(m, k) \in (\mathbb{R} \cup \{-\infty\})^2$, $r \in \mathbb{N}$, with $r \geq 0$. If $a_h \in \Upsilon^{r+1} \cap \sigma_\epsilon$ and $b_h \in \Gamma_{k,1}^{m,r} \cap \sigma_0$, then, for all $k' \in \mathbb{R}$ such that $r \geq k + k'$, we have*

$$\mathcal{P}_{a_h}^{h,\epsilon} \mathcal{P}_{b_h}^h - \mathcal{P}_{(\theta_{h,*}^{\epsilon,0} a_h) \sharp_h b_h}^h = \mathcal{O}(h^{(1+\epsilon)r - \epsilon(k+k')} M_{k',1}^{-m,r}(a_h) M_{k,1}^{m,r}(b_h))_{L^2 \to L^2},$$
$$\mathcal{P}_{b_h}^h \mathcal{P}_{a_h}^{h,\epsilon} - \mathcal{P}_{b_h \sharp_h (\theta_{h,*}^{\epsilon,0} a_h)}^h = \mathcal{O}(h^{(1+\epsilon)r - \epsilon(k+k')} M_{k',1}^{-m,r}(a_h) M_{k,1}^{m,r}(b_h))_{L^2 \to L^2}.$$

Proof. Let a_h be a sequence of approximating symbols $\{a_h^n\}_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^{2d}) \cap \sigma_\epsilon$ which is bounded, uniformly in $h \in (0, 1]$, with respect to the norm $N^r(\cdot)$. Note that for all $k' \in \mathbb{R}$, we have $M_{k',1}^{-m,r}(\cdot) \lesssim N^r(\cdot)$. And thus, when $n \in \mathbb{N}$ is sufficiently large, we have $M_{k',1}^{-m,r}(a_h^n - a_h) \leq 2N^r(a_h^n - a_h) = o(1)$. By Corollary 4.28, if $r \geq k + k'$, we have

$$\begin{aligned}
\mathcal{P}_{a_h^n}^{h,\epsilon} \mathcal{P}_{b_h}^h - \mathcal{P}_{(\theta_{h,*}^{\epsilon,0} a_h^n) \sharp_h b_h}^h &= \mathcal{O}(h^{(1+\epsilon)r - \epsilon(k+k')} M_{k'1}^{-m,r}(a_h^n) M_{k,1}^{m,r}(b_h))_{L^2 \to L^2} \\
&= \mathcal{O}(h^{(1+\epsilon)r - \epsilon(k+k')} M_{k',1}^{-m,r}(a_h) M_{k,1}^{m,r}(b_h))_{L^2 \to L^2} + o(1)_{L^2 \to L^2}.
\end{aligned}$$

In fact, for all $u \in \mathscr{S}(\mathbb{R}^d)$, as $n \to \infty$, by Proposition 4.36, we have

$$\begin{aligned}
&\|\mathcal{P}_{a_h}^{h,\epsilon} \mathcal{P}_{b_h}^h u - \mathcal{P}_{(\theta_{h,*}^{\epsilon,0} a_h) \sharp_h b_h}^h u\|_{L^2} \\
&\qquad = \|(\mathcal{P}_{a_h}^{h,\epsilon} - \mathcal{P}_{a_h^n}^{h,\epsilon}) \mathcal{P}_{b_h}^h u\|_{L^2} + \|\mathcal{P}_{(\theta_{h,*}^{\epsilon,0}(a_h - a_h^n)) \sharp_h b_h}^h u\|_{L^2} + \|\mathcal{P}_{a_h^n}^{h,\epsilon} \mathcal{P}_{b_h}^h u - \mathcal{P}_{(\theta_{h,*}^{\epsilon,0} a_h^n) \sharp_h b_h}^h u\| \\
&\qquad = o(1)(\|\mathcal{P}_{b_h}^h u\|_{L^2} + \|u\|_{L^2}) + \mathcal{O}(h^{(1+\epsilon)r - \epsilon(k+k')} M_{k',1}^{-m,r}(a_h) M_{k,1}^{m,r}(b_h)) \|u\|_{L^2}.
\end{aligned}$$

Passing $n \to \infty$ and then use the density of $\mathscr{S}(\mathbb{R}^d)$ in L^2, we conclude that for all $u \in L^2$, we have

$$\|\mathcal{P}_{a_h}^{h,\epsilon} \mathcal{P}_{b_h}^h u - \mathcal{P}_{(\theta_{h,*}^{\epsilon,0} a_h) \sharp_h b_h}^h u\|_{L^2} = \mathcal{O}(h^{(1+\epsilon)r - \epsilon(k+k')} M_{k',1}^{-m,r}(a_h) M_{k,1}^{m,r}(b_h)) \|u\|_{L^2}.$$

The estimate for $\mathcal{P}_{b_h}^h \mathcal{P}_{a_h}^{h,\epsilon} - \mathcal{P}_{b_h \sharp_h (\theta_{h,*}^{\epsilon,0} a_h)}^h$ is similar. $\square$

Corollary 4.38. *If $\epsilon \geq 0$ and $a_h, b_h \in \Upsilon^{r+1}$, where $r \in \mathbb{N}$, then*

$$\mathcal{P}_{a_h}^{h,\epsilon} \mathcal{P}_{b_h}^{h,\epsilon} - \mathcal{P}_{a_h \sharp_h^\epsilon b_h}^{h,\epsilon} = \mathcal{O}(h^{(1+\epsilon)r})_{L^2 \to L^2},$$

where the symbol $a_h \sharp_h^\epsilon b_h = a_h \sharp_{h,r}^{0,\epsilon} b_h$ is defined by (2-4).

4C5. *Almost sharp Gårding inequality for paradifferential operators.* We need an almost sharp Gårding inequality for our paradifferential calculus. There are various works on the (almost) sharp Gårding inequality for pseudodifferential operators with limited regularities; see, e.g., [Taylor 1991; Tataru 2002; Hérau 2002].

Lemma 4.39. *If $\epsilon \in (0, 1)$ and $a_h \in M_{n \times n}(\Gamma^{0,r}) \cap \sigma_0$ is compactly supported, where $n \in \mathbb{N}$, $r \geq \max\{\tilde{d}, \epsilon^{-1} - 1\}$ and $\operatorname{Re} a \geq 0$, then, for all $\epsilon \in (0, 1)$, there exists $C > 0$ such that, for all $u \in L^2$,*

$$\operatorname{Re}(T_{a_h}^h u, u)_{L^2} \geq -C h^{1-\epsilon} \|u\|_{L^2}^2.$$

Proof. By Lemma 4.33 and the condition $r \geq \tilde{d}$, we may replace $T^h_{a_h}$ with $\mathrm{Op}_h(a_h)$ in the above inequality. As $a_h \in \sigma_0$ and is compactly supported, we have $\{b_h(x,\xi) = h^{-1+\epsilon} a_h(x, h\xi)\}_{h\in(0,1]}$ is bounded in $\Gamma^{1-\epsilon,r}$. By [Taylor 1991, §2.4 (2.4.6)], as $r \geq \epsilon^{-1} - 1$, we have $1-\epsilon \leq r/(1+r)$ and thus

$$\mathrm{Re}(\mathrm{Op}(b_h)u, u)_{L^2} \gtrsim -\|u\|^2_{L^2}.$$

We conclude by $\mathrm{Op}(b_h) = h^{-1+\epsilon}\,\mathrm{Op}_h(a_h)$. □

We are mostly interested in the case where $\epsilon = \frac{1}{2}$. In this case, the condition for r is simply $r \geq \max\{\tilde{d}, (\frac{1}{2})^{-1} - 1\} = \tilde{d}$. Next we show that the almost sharp Gårding inequality also applies to symbols in Υ^{1+r}.

Lemma 4.40. *If $\epsilon \in (0,1)$ and $a_h \in M_{n\times n}(\Upsilon^{1+r})$, with $n \in \mathbb{N}$, $r \geq \max\{\tilde{d}, \epsilon^{-1} - 1\}$, then there exists $C > 0$ such that, for all $u \in L^2$,*

$$\mathrm{Re}(T^h_{a_h} u, u)_{L^2} \geq -Ch^{1-\epsilon}\|u\|^2_{L^2}, \quad \mathrm{Re}(\mathcal{P}^h_{a_h} u, u)_{L^2} \geq -Ch^{1-\epsilon}\|u\|^2_{L^2}.$$

Proof. Choose a sequence $a^j_h \in M_{n\times n}(\Gamma^{0,r})$ which converges to a_h with respect to the norm $N^r(\,\cdot\,)$ and is uniformly compactly supported in $\mathbb{R}^d \times (\mathbb{R}^d\backslash 0)$. Apply the almost sharp Gårding inequality for a^j_h; there exists a constant $C > 0$ which is independent of j such that, for all $u \in L^2$, we have

$$\mathrm{Re}(T^h_{a_h} u, u)_{L^2} = \mathrm{Re}(T^h_{a_h - a^j_h} u, u)_{L^2} + \mathrm{Re}(T^h_{a^j_h} u, u)_{L^2} \geq o(1) - Ch^{1-\epsilon}\|u\|^2_{L^2}.$$

We conclude the almost sharp Gårding inequality for $T^h_{a_h}$ by passing $j \to \infty$. Therefore,

$$\begin{aligned}\mathrm{Re}(\mathcal{P}^h_{a_h} u, u)_{L^2} &= \sum_{j\in\mathbb{N}} \mathrm{Re}(\underline{\psi}_j T^h_{\psi_j a_h} \underline{\psi}_j u, u)_{L^2} = \sum_{j\in\mathbb{N}} \mathrm{Re}(T^h_{\psi_j a_h} \underline{\psi}_j u, \underline{\psi}_j u)_{L^2} \\ &\gtrsim -h^{1-\epsilon} \sum_{j\in\mathbb{N}} \|\underline{\psi}_j u\|^2_{L^2} \gtrsim -h^{1-\epsilon}\|u\|^2_{L^2}.\end{aligned}$$ □

4C6. *Relation with quasihomogeneous wavefront sets.*

Lemma 4.41. *If $r \geq 0$ and $a_h = \sum_{j=0}^r h^j a^j_h$, where $a^j_h \in \Upsilon^{1+r-j}$ such that a_h is elliptic at $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d\backslash 0)$ in the sense that, for some neighborhood Ω of (x_0,ξ_0), we have*

$$\inf_{0<h<1} \inf_{(x,\xi)\in\Omega} |a_h(x,\xi)| > 0,$$

then for all $u \in L^2$ such that $T^h_{a_h} u = \mathcal{O}(h^\sigma)_{L^2}$, where $0 \leq \sigma \leq r$, we have $(x_0,\xi_0) \notin \mathrm{WF}^\sigma_{0,1}(u)$.

Proof. Assume that $\Omega \subset \mathbb{R}^d \times (\mathbb{R}^d\backslash 0)$. Let $b_h \in S^{-\infty}_{-\infty}$ with $\mathrm{supp}\, b_h \subset \Omega$. Then by the symbolic calculus stated in Corollary 4.38, there exists $c_h = \sum_{j=0}^r h^j c^j_h$, where $c^j_h \in \Upsilon^{1+r-j}$, such that

$$T^h_{b_h} = T^h_{c_h} T^h_{a_h} + \mathcal{O}(h^r)_{L^2\to L^2}.$$

Thus $T^h_{b_h} u = \mathcal{O}(h^\sigma)_{L^2}$. By Lemma 4.33 we have $\mathrm{Op}_h(b_h)u = \mathcal{O}(h^\sigma)_{L^2}$. We conclude by Lemma 2.14. □

Lemma 4.42. *Let $\epsilon \geq 0$, $e \in \Gamma^{m,r}_{0,0}$ if $\epsilon = 0$ and $e \in \Gamma^{m,r}_{0,1}$ if $\epsilon > 0$, and suppose that e is homogeneous of degree m with respect to ξ. Then, for $f \in H^s$ and $0 \leq \sigma \leq (1+\epsilon)r$,*

$$\mathrm{WF}^{s+\sigma-m}_{\epsilon,1}(\mathcal{P}_e f)^\circ \subset \mathrm{WF}^{s+\sigma}_{\epsilon,1}(f)^\circ.$$

If in addition e is elliptic, i.e., for some $C > 0$ and $|\xi|$ sufficiently large, $|e(x,\xi)| \geq C|\xi|^m$, then

$$\mathrm{WF}^{s+\sigma-m}_{\epsilon,1}(\mathcal{P}_e f)^\circ = \mathrm{WF}^{s+\sigma}_{\epsilon,1}(f)^\circ.$$

Proof. For $\mu \in \mathbb{R}$, define $Z^\mu = \mathcal{P}_{|\xi|^\mu}$. Then $Z^{-\mu}Z^\mu - \mathrm{Id} \in \mathscr{O}^{-\infty}_{-\infty}$. Therefore,

$$f - Z^{-s}Z^s f \in H^\infty_\infty, \quad \mathcal{P}_e f - \mathcal{P}_{e\sharp|\xi|^{-s}} Z^s f \in H^{s+r-m}_{\delta r} + H^\infty,$$

where $\delta = 0$ if $\epsilon = 0$, while $\delta = 1$ if $\epsilon > 0$. By Lemma 2.15 and the fact that $Z^{\pm s}$ are pseudodifferential operators with elliptic symbols in $S^{\pm s}_0$, we readily have

$$\mathrm{WF}^{s+\sigma}_{\epsilon,1}(f)^\circ = \mathrm{WF}^{\sigma}_{\epsilon,1}(Z^s f)^\circ, \quad \mathrm{WF}^{s+\sigma-m}_{\epsilon,1}(\mathcal{P}_e f)^\circ = \mathrm{WF}^{\sigma-(m-s)}_{\epsilon,1}(\mathcal{P}_{e|\xi|^{-s}} Z^s f)^\circ.$$

So we may assume that $s = 0$. Let $a, b \in S^{-\infty}_{-\infty} \cap \sigma_\epsilon$ such that

$$\operatorname{supp} b \subset \{a = 1\} \subset \operatorname{supp} a \subset \mathbb{R}^{2d} \setminus \mathrm{WF}^\sigma_{\epsilon,1}(f).$$

Then by Lemma 2.14, $\mathrm{Op}^{\epsilon,1}_h(a) f = \mathcal{O}(h^\sigma)_{L^2}$. By Corollary 4.32, Lemma 4.34, Proposition 4.26, and Corollary 4.28,

$$\begin{aligned}
h^m \mathrm{Op}^{\epsilon,1}_h(b)\mathcal{P}_e f &= \mathrm{Op}^{\epsilon,1}_h(b)\mathcal{P}^h_e f + \mathcal{O}(h^\infty)_{L^2} \\
&= \mathrm{Op}^{\epsilon,1}_h(b)\mathcal{P}^h_e \mathrm{Op}^{\epsilon,1}_h(a) f + \mathrm{Op}^{\epsilon,1}_h(b)\mathcal{P}^h_e \mathrm{Op}^{\epsilon,1}_h(1-a) f + \mathcal{O}(h^\infty)_{L^2} \\
&= \mathcal{O}(1)_{L^2\to L^2} \mathrm{Op}^{\epsilon,1}_h(a) f + \mathcal{O}(h^{r(1+\epsilon)})_{L^2} \\
&= \mathcal{O}(h^\sigma)_{L^2},
\end{aligned}$$

proving the first statement. The second statement follows by a construction of parametrix. □

5. Asymptotically flat water waves

In this section we prove Theorem 1.6. The idea is to combine the analysis in [Alazard, Burq and Zuily 2011] with the dyadic paradifferential calculus in weighted Sobolev spaces. We shall use the following formal notations for simplicity. Let w be a function on $\mathbb{R}^d$ which is nowhere-vanishing. Then for any operator $\mathcal{A}$ between some function spaces on $\mathbb{R}^d$ and, for any function f on $\mathbb{R}^d$, we introduce the following notations whenever they are well-defined:

$$\mathcal{A}^{(w)} = w\mathcal{A}w^{-1}, \quad f^{(w)} = wf.$$

Note that $(\mathcal{A}f)^{(w)} = \mathcal{A}^{(w)} f^{(w)}$. For $k \in \mathbb{R}$, we also define by an abuse of notation

$$\mathcal{A}^{(k)} = \mathcal{A}^{(\langle x\rangle^k)}, \quad f^{(k)} = f^{(\langle x\rangle^k)}.$$

Observe that $L^2_k = H^0_k$ is an Hilbert space with the inner product

$$(f, g)_{L^2_k} = (f^{(k)}, g^{(k)})_{L^2}.$$

5A. ***Dirichlet–Neumann operator.*** We study the Dirichlet–Neumann operator on weighted Sobolev spaces and its paralinearization. The time variable will be temporarily omitted for simplicity.

5A1. *Boundary flattening.* Let $\eta \in W^{1,\infty}(\mathbb{R}^d)$, such that

$$\delta := b + \inf_{x\in\mathbb{R}^d} \eta(x) > 0. \tag{5-1}$$

Define $\tau(x, z) = (x, z + \eta(x))$ and set

$$\begin{aligned}\widetilde{\Omega} &= \tau^{-1}(\Omega) = \{-b-\eta(x) < z < 0\},\\ \widetilde{\Sigma} &= \tau^{-1}(\Sigma) = \{z=0\},\\ \widetilde{\Gamma} &= \tau^{-1}(\Gamma) = \{z=-b-\eta(x)\}.\end{aligned}$$

Let τ_* be the pullback deduced by τ, then

$$\tau_*(\mathrm{d}x^2 + \mathrm{d}y^2) = (\mathrm{d}x\ \mathrm{d}z)\varrho\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}z\end{pmatrix},$$

where

$$\varrho = \begin{pmatrix}\mathrm{Id}+(\nabla\eta)\,{}^t(\nabla\eta) & \nabla\eta\\ {}^t(\nabla\eta) & 1\end{pmatrix}.$$

We verify that

$$\varrho^{-1} = \begin{pmatrix}\mathrm{Id} & -\nabla\eta\\ -{}^t(\nabla\eta) & 1+|\nabla\eta|^2\end{pmatrix}.$$

Let $\nabla_{xz} = (\nabla_x, \partial_z)$. Then the divergence, gradient and Laplacian with respect to the metric ϱ are

$$\begin{aligned}\mathrm{div}_\varrho\, u &= \nabla_{xz}\cdot u,\\ \nabla_\varrho u &= (\nabla u - \nabla\eta\partial_z u, -\nabla\eta\cdot\nabla u + (1+|\nabla\eta|^2)\partial_z u),\\ \Delta_\varrho u &= \partial_z^2 u + (\nabla - \nabla\eta\partial_z)^2 u.\end{aligned}$$

The exterior unit normal to $\partial\widetilde{\Omega} = \widetilde{\Sigma}\cup\widetilde{\Gamma}$ is

$$\boldsymbol{n}_\varrho = \langle (D\tau)^{-1}|_{T\partial\widetilde{\Omega}}, \boldsymbol{n}\rangle = \begin{cases}{}^t(-\nabla\eta, 1+|\nabla\eta|^2)/\sqrt{1+|\nabla\eta|^2}, & \widetilde{\Sigma},\\ {}^t(0,1), & \widetilde{\Gamma}.\end{cases}$$

Let $\psi \in H^{1/2}$, and suppose that ϕ satisfies the equation

$$\Delta_{xy}\phi = 0, \quad \phi|_\Sigma = \psi, \quad \partial_{\boldsymbol{n}}\phi|_\Gamma = 0.$$

Then $v = (\tau|_{\widetilde{\Omega}})_*\phi$ satisfies

$$\Delta_\varrho v = 0, \quad v|_{\widetilde{\Sigma}} = \psi, \quad \partial_{\boldsymbol{n}_\varrho} v|_{\widetilde{\Gamma}} = 0. \tag{5-2}$$

The Dirichlet–Neumann operator can now be written as

$$\sqrt{1+|\nabla\eta|^2}^{-1} G(\eta)\psi = \partial_{\boldsymbol{n}_\varrho} v|_{\widetilde{\Sigma}} = \boldsymbol{n}_\varrho\cdot\nabla_{xz} v|_{z=0}.$$

5A2. *Elliptic estimate.* Let $\chi_0 \in C^\infty(\mathbb{R})$ with $\chi_0(z) = 0$ for $z \le -\frac{\delta}{2}$ and $\chi_0(z) = 1$ for $z \ge 0$. Take the decomposition $v = \tilde v + \underline{\psi}$, where

$$\underline{\psi}(x, z) = \chi_0(z) e^{z\langle D_x\rangle}\psi(x).$$

Lemma 5.1. *Let $n \in \mathbb{N}$, $m \in \mathbb{R}$, $\mu \in \mathbb{R}$, $k \in \mathbb{R}$, $a \in S^m_0$. Then*

$$\|\partial_z^n \operatorname{Op}(a)\underline{\psi}\|_{L^2_z(\mathbb{R}_{\le 0}, H_k^{\mu-n-m+1/2})} \lesssim \|\psi\|_{H_k^\mu}.$$

Proof. We only prove the case where $n=0$. The general case follows with a similar argument and the identity

$$\partial_z^n \underline{\psi}(x,z)=\sum_{j=0}^{n}\binom{n}{j}\chi_0^{(n-j)}(z)\langle D_x\rangle^j e^{z\langle D_x\rangle}\psi(x).$$

Let

$$b(x,\xi)=a(x,\xi)\langle\xi\rangle^{\mu-m}\in S_0^{\mu},$$
$$\lambda(z,\xi)=\chi_0(z)e^{z\langle\xi\rangle}\langle\xi\rangle^{1/2}\in L^\infty_{z\le 0}S_0^{1/2}.$$

Then for all $N\ge 0$,

$$\|\mathrm{Op}(a)\underline{\psi}\|_{L^2_z(\mathbb{R}_{\le 0},H_k^{\mu-m+1/2})}\lesssim\|\mathrm{Op}(\lambda)\,\mathrm{Op}(b)\psi\|_{L^2_z(\mathbb{R}_{\le 0},L^2_k)}+\|\psi\|_{H_k^{-N}}.$$

Observe that

$$\mathrm{Op}(\lambda)^{(k)}-(\mathrm{Op}(\lambda)^{(k)})^*\in L^\infty_{z\le 0}\mathscr{O}_0^{-1/2},$$
$$(\mathrm{Op}(\lambda)^{(k)})^2-\mathrm{Op}(\lambda^2)^{(k)}\in L^\infty_{z\le 0}\mathscr{O}_0^{0}.$$

Also note that

$$\sigma(\xi):=\int_{-\infty}^{0}\lambda^2(z,\xi)\,\mathrm{d}z=\langle\xi\rangle\int_{-\infty}^{0}\chi_0^2(z)e^{2\langle\xi\rangle z}\,\mathrm{d}z\in S_0^0.$$

Therefore,

$$\begin{aligned}\|\mathrm{Op}(\lambda)\,\mathrm{Op}(b)\psi\|^2_{L^2_z(\mathbb{R}_{\le 0},L^2_k)}&=(\mathrm{Op}(\lambda^2)\,\mathrm{Op}(b)\psi,\mathrm{Op}(b)\psi)_{L^2_z(\mathbb{R}_{\le 0},L^2_k)}+\mathcal{O}(\|\psi\|^2_{H_k^\mu})\\&=(\mathrm{Op}(\sigma)\,\mathrm{Op}(b)\psi,\mathrm{Op}(b)\psi)_{L^2_k}+\mathcal{O}(\|\psi\|^2_{H_k^\mu})=\mathcal{O}(\|\psi\|^2_{H_k^\mu}).\end{aligned}$$ □

Lemma 5.2. *For all $k\in\mathbb{R}$, we have $\|\tilde{v}\|_{H^1_k}\le C(\|\eta\|_{W^{1,\infty}})\|\psi\|_{H_k^{1/2}}$.*

Proof. Let $H_\varrho^{1,0}$ be the completion of the space

$$\{f\in C^\infty(\widetilde{\Omega}): f \text{ vanishes in a neighborhood of } \widetilde{\Sigma}\},$$

with respect to the norm

$$\|u\|_{H_\varrho^{1,0}}:=\|\nabla_\varrho u\|_{L^2_\varrho}=(\nabla_\varrho u,\nabla_\varrho u)^{1/2}_{L^2_\varrho},$$

where $(X,Y)_{L^2_\varrho}:=\int_{\widetilde{\Omega}}\varrho(X,Y)\,\mathrm{d}x\,\mathrm{d}z$. As $b<\infty$, by the Poincaré inequality,

$$\|u\|_{L^2}\le C(\|\eta\|_{L^\infty})\|\partial_z u\|_{L^2}\le C(\|\eta\|_{W^{1,\infty}})\|u\|_{H_\varrho^{1,0}}$$

for all $u\in H_\varrho^{1,0}$. Let $0<\zeta\in C^\infty(\mathbb{R})$ be such that $\zeta(z)=1$ for $|z|\le 1$, and $\zeta(z)=z$ for $|z|\ge 2$. For some $R>0$ sufficiently large to be determined later, set $w(x)=R\times\zeta(\langle x\rangle^k/R)$. Then $\langle x\rangle^k\lesssim w(x)\lesssim R\langle x\rangle^k$, $\operatorname{supp}\nabla w\subset\{\langle x\rangle\gtrsim R^{1/k}\}$, and $|\nabla w(x)|\lesssim R^{(k-1)/k}$.

As $\tilde{v}$ satisfies the equation $\Delta_\varrho\tilde{v}=-\Delta_\varrho\underline{\psi}$, we consider $\tilde{v}^{(w)}$ as the variational solution to the equation $B(\tilde{v}^{(w)},\cdot)=-L(\cdot)$, where, for $u,\varphi\in H_\varrho^{1,0}$,

$$B(u,\varphi)=(\nabla_\varrho^{(w)}u,\nabla_\varrho^{(1/w)}\varphi)_{L^2(\widetilde{\Omega})},\qquad L(\varphi)=(\nabla_\varrho^{(w)}\underline{\psi}^{(w)},\nabla_\varrho^{(1/w)}\varphi)_{L^2(\widetilde{\Omega})}.$$

Observe that $\nabla_\varrho^{(w^{\pm 1})} = \nabla_\varrho \mp b_w$, where $b_w = (w^{-1}\nabla w, -\nabla\eta\cdot w^{-1}\nabla w) \in L^\infty$, satisfies $\|b_w\| \le C(\|\eta\|_{W^{1,\infty}})R^{-1/k}$. We verify that L and B are continuous linear and bilinear forms on $H_\varrho^{1,0}$. Moreover B is coercive when R is sufficiently large; indeed,

$$B(\varphi,\varphi) = \|\nabla_\varrho\varphi\|_{L_\varrho^2}^2 - \|b_w\varphi\|_{L_\varrho^2}^2 \ge (1 - C(\|\eta\|_{W^{1,\infty}})R^{-2/k})\|\nabla_\varrho\varphi\|_{L_\varrho^2}^2. \tag{5-3}$$

Therefore, by the Lax–Milgram theorem and Lemma 5.1,

$$\|\tilde v\|_{H_k^1} \lesssim \|\tilde v^{(w)}\|_{H_\varrho^{1,0}} \lesssim \|L\|_{(H_\varrho^{1,0})^*} \lesssim \|\underline{\psi}\|_{H^1} \lesssim \|\psi\|_{H_k^{1/2}}. \qquad \square$$

Proposition 5.3. *Let $(\eta,\psi) \in W^{1,\infty}\times H_k^{1/2}$, $k\in\mathbb{R}$. Then $\|G(\eta)\psi\|_{H_k^{-1/2}} \le C(\|\eta\|_{W^{1,\infty}})\|\psi\|_{H_k^{1/2}}$.*

Proof. By Lemmas 5.1 and 5.2, $v \in L_z^2((-\delta,0), H_k^1)\cap H_z^1((-\delta,0), L_k^2)$. By a classical interpolation result (see, e.g., [Alazard, Burq and Zuily 2014, Lemma 2.19]) and the equation satisfied by v, we deduce that

$$v \in C_z^0([-\delta,0], H_k^{1/2}) \cap C_z^1([-\delta,0], H_k^{-1/2}). \qquad \square$$

5A3. *Higher regularity.*

Proposition 5.4. *Let $(\eta,\psi)\in H^{\mu+1/2}\times H_k^{\sigma+1/2}$, where $k\in\mathbb{R}$, $\mu > \frac12 + \frac d2$, $0\le\sigma\le\left[\mu-\frac12\right]$. Then*

$$\|G(\eta)\psi\|_{H_k^{\sigma-1/2}} \le C(\|\eta\|_{H^{\mu+1/2}})\|\psi\|_{H_k^{\sigma+1/2}}.$$

Consequently, if $(\eta,\psi)\in H^{\mu+1/2}\times \mathcal{H}_k^{\sigma+1/2,\delta}$, with $\delta\ge 0$, $k\in\mathbb{N}$ and $\sigma - k\delta \ge 0$, then

$$\|G(\eta)\psi\|_{\mathcal{H}_k^{\sigma-1/2,\delta}} \le C(\|\eta\|_{H^{\mu+1/2}})\|\psi\|_{\mathcal{H}_k^{\sigma+1/2,\delta}}.$$

Proof. We shall only prove the cases where $\sigma\in\mathbb{N}$. The remaining cases follow by interpolation. By Section 5A2, it suffices to prove that, for all $\sigma\in\left[0,\mu-\frac12\right]\cap\mathbb{N}$, there exists $\delta>0$ such that

$$\tilde v \in L^2((-\delta,0), H_k^{\sigma+1})\cap H^1((-\delta,0), H_k^\sigma).$$

Let N_σ be the corresponding norm of $\tilde v$, we shall prove that $N_\sigma < +\infty$. The case where $\sigma = 0$ has already been proven by Lemma 5.2. It remains to bound $N_{\sigma+1}$ by N_σ via a mathematical induction. Note that if $\chi\in C_c^\infty((-\delta,\delta))$, then $\chi\partial_x^\sigma\tilde v$ satisfies the equation

$$-\Delta_\varrho(\chi\partial_x^\sigma\tilde v) + K\tilde v = \Delta_\varrho(\chi\partial_x^\sigma\underline{\psi}) - K\underline{\psi}. \tag{5-4}$$

where $K = [\Delta_\varrho, \chi\partial_x^\sigma]$. Note that $\Delta_\varrho = P\cdot P$ with $P = (\nabla - \nabla\eta\partial_z, \partial_z)$, so

$$K = P\cdot[P,\chi\partial_x^\sigma] + [P,\chi\partial_x^\sigma]\cdot P.$$

By an explicit calculation

$$[P,\chi\partial_x^\sigma] = (-\chi[\nabla\eta,\partial_x^\sigma]\partial_z - \nabla\eta\chi'\partial_x^\sigma, \chi'\partial_x^\sigma).$$

Integrating the following pairings by parts using $\tilde v|_{z=0}=0$, we have by Lemma 5.1 that

$$|(\chi\partial_x^\sigma\tilde v, K\chi\partial_x^\sigma\tilde v)_{L^2L^2_k}| \lesssim \|P^*(\chi\partial_x^\sigma\tilde v)\|_{L^2L^2_k}\|[P,\chi\partial_x^\sigma]\chi\tilde v\|_{L^2L^2_k}+\|P(\chi\partial_x^\sigma\tilde v)\|_{L^2L^2_k}\|[P,\chi\partial_x^\sigma]^*\chi\tilde v\|_{L^2L^2_k} \lesssim N_\sigma N_{\sigma+1},$$

$$|(\chi\partial_x^\sigma\tilde v, K\chi\partial_x^\sigma\underline{\psi})_{L^2L^2_k}| \lesssim \|P^*(\chi\partial_x^\sigma\tilde v)\|_{L^2L^2_k}\|[P,\chi\partial_x^\sigma]\chi\underline{\psi}\|_{L^2L^2_k}+\|P(\chi\partial_x^\sigma\underline{\psi})\|_{L^2L^2_k}\|[P,\chi\partial_x^\sigma]^*\chi\tilde v\|_{L^2L^2_k} \lesssim \|\psi\|_{H^{\sigma+1/2}}(N_\sigma+N_{\sigma+1}),$$

$$|(\chi\partial_x^\sigma\tilde v, -\Delta_\varrho(\chi\partial_x^\sigma\underline{\psi}))_{L^2L^2_k}| \lesssim \|P(\chi\partial_x^\sigma\tilde v)\|_{L^2L^2_k}\|P(\chi\partial_x^\sigma\underline{\psi})\|^2_{L^2L^2_k} \lesssim \|\psi\|_{H^{\sigma+1/2}}N_{\sigma+1}.$$

In the above inequalities, the adjoint operators are taken with respect to $L^2L^2_k$. Using again the structure of Δ_ϱ, we have by (5-3) that

$$(\chi\partial_x^\sigma\tilde v, -\Delta_\varrho(\chi\partial_x^\sigma\tilde v))_{L^2L^2_k} \gtrsim \|P(\chi\partial_x^\sigma\tilde v)\|^2_{L^2L^2_k} - \|\chi\partial_x^\sigma\tilde v\|^2_{L^2L^2_k} \gtrsim \|\chi\partial_x^\sigma\tilde v\|^2_{H^1H^1_k} - N_\sigma^2.$$

Pairing (5-4) with $\chi\partial_x^\sigma\tilde v$ and using the estimates above, for all $\epsilon>0$,

$$N_{\sigma+1}^2 \lesssim \|\chi\partial_x^\sigma\tilde v\|^2_{H^1H^1_k} \lesssim N_\sigma N_{\sigma+1} + \|\psi\|_{H^{\sigma+1/2}}(N_\sigma+N_{\sigma+1}) \lesssim \epsilon N_{\sigma+1}^2 + \epsilon^{-1}(N_\sigma^2+\|\psi\|^2_{H^{\sigma+1/2}}).$$

All the constants hidden by $\lesssim$ are of the form $C(\|\eta\|_{H^{\mu+1/2}})$. We thus conclude the induction by choosing $\epsilon>0$ sufficiently small. By interpolation as in Proposition 5.3,

$$v\in C_z^0([-\delta,0],H_k^{\sigma+1/2})\cap C_z^1([-\delta,0],H_k^{\sigma-1/2}).$$

When $\psi\in\mathcal{H}_k^{\sigma,\delta}$, we apply the above estimate to $\psi\in H_j^{\sigma-\delta j}$ and conclude. □

5B. *Paralinearization.* Now we paralinearize the system of water waves. The following results are immediate consequences of the analysis in [Alazard, Burq and Zuily 2011] and our dyadic paradifferential calculus on weighted Sobolev spaces.

Proposition 5.5. *Let $(\eta,\psi)\in\mathcal{H}_k^{\mu+1/2,\delta}\times\mathcal{H}_k^{\mu,\delta}$ with $\mu-\frac12\in\mathbb{N}$, $k\in\mathbb{N}$ and $\mu-\delta k>3+\frac d2$. Let*

$$B=\frac{\nabla\eta\cdot\nabla\psi+G(\eta)\psi}{1+|\nabla\eta|^2},\qquad V=\nabla\psi-B\nabla\eta,$$

and $\lambda=\lambda^{(1)}+\lambda^{(0)}\in\Gamma_{0,0}^{3/2,\mu-1/2-\tilde d}+\Gamma_{0,0}^{1/2,\mu-3/2-\tilde d}$, where

$$\lambda^{(1)}(x,\xi)=\sqrt{(1+|\nabla\eta|^2)|\xi|^2-(\nabla\eta\cdot\xi)^2},$$

$$\lambda^{(0)}(x,\xi)=\frac{1+|\nabla\eta|^2}{2\lambda^{(1)}}\{\nabla\cdot(\alpha^{(1)}\nabla\eta)+i\partial_\xi\lambda^{(1)}\cdot\nabla\alpha^{(1)}\},$$

and

$$\alpha^{(1)}(x,\xi)=\frac{\lambda^{(1)}+i\nabla\eta\cdot\xi}{1+|\nabla\eta|^2}.$$

Then

$$G(\eta)\psi=\mathcal{P}_\lambda(\psi-\mathcal{P}_B\eta)-\mathcal{P}_V\cdot\nabla\eta+R(\eta,\psi),$$

where $R(\eta,\psi)\in\mathcal{H}_k^{\mu+1/2,\delta}$.

We shall define $\omega=\psi-\mathcal{P}_B\eta$, which is called the good unknown of Alinhac.

Proof. We only sketch the proof, for the key ingredients are already given in [Alazard, Burq and Zuily 2011]. We simply replace the paradifferential calculus in [loc. cit.] by our dyadic paradifferential calculus. Let v be defined as in Section 5A. Rewrite (5-2) as

$$\alpha\partial_z^2 v+\Delta v+\beta\cdot\nabla\partial_z v-\gamma\partial_z v=0,$$

where $\alpha=1+|\nabla\eta|^2,\ \beta=-2\nabla\eta,\ \gamma=\Delta\eta.$ Applying Proposition 4.21, we obtain as in [loc. cit., Lemma 3.17],

$$\mathcal{P}_\alpha\partial_z^2 u+\Delta u+\mathcal{P}_\beta\cdot\nabla\partial_z u-\mathcal{P}_\gamma\partial_z u\in C([-\delta,0],\mathcal{H}_k^{\mu,\delta}),\tag{5-5}$$

where $u=v-\mathcal{P}_{\partial_z v}\zeta$ with $\zeta(x,z)=z+\eta(x)$. Define $a_\pm=a_\pm^{(1)}+a_\pm^{(0)}\in\Gamma_{0,0}^{1,\mu-1/2-\tilde d}+\Gamma_{0,0}^{0,\mu-3/2-\tilde d}$ by

$$a_\pm^{(1)}(x,\xi)=\frac{1}{2\alpha}\bigl(-\beta\cdot\xi\pm\sqrt{4\alpha|\xi|^2-(\beta\cdot\xi)^2}\bigr),$$
$$a_\pm^{(0)}(x,\xi)=\pm\frac{1}{a_-^{(1)}-a_+^{(1)}}\left(i\partial_\xi a_-^{(1)}\cdot\partial_x a_+^{(1)}-\frac{\gamma}{\alpha}a_\pm^{(1)}\right).$$

Then we factorize (5-5) as

$$\mathcal{P}_\alpha(\partial_z-\mathcal{P}_{a_-})(\partial_z-\mathcal{P}_{a_+})u\in C([-\delta,0],\mathcal{H}_k^{\mu,\delta}).$$

Because $\operatorname{Re}a_-^{(1)}\le 0$, a parabolic estimate (see e.g., [loc. cit., Proposition 3.19]) implies that

$$(\partial_z u-\mathcal{P}_{a_+}u)|_{z=0}\in\mathcal{H}_k^{\mu+1/2,\delta}.$$

We conclude by setting $\lambda=(1+|\nabla\eta|^2)a_+-i\nabla\eta\cdot\xi$. □

The proofs of the following results are in the same spirit and much simpler. Their proofs are exactly the same as in [loc. cit.], simply replacing the usual paradifferential calculus with our dyadic paradifferential calculus, particularly Propositions 4.22 and 4.21. Therefore we shall omit the proofs.

Proposition 5.6. *Let $\eta\in\mathcal{H}_k^{\mu+1/2,\delta}$, with $\mu-\frac12\in\mathbb{N},\ \mu-\delta k>3+\frac{d}{2}$, and define $\ell=\ell^{(2)}+\ell^{(1)}\in\Gamma_{0,0}^{2,\mu-1/2-\tilde d}+\Gamma_{0,0}^{1,\mu-3/2-\tilde d}$, where*

$$\ell^{(2)}=\frac{(1+|\nabla\eta|^2)|\xi|^2-(\nabla\eta\cdot\xi)^2}{(1+|\nabla\eta|^2)^{3/2}},\qquad \ell^{(1)}=\tfrac12\partial_\xi\cdot D_x\ell^{(2)}.$$

Then $H(\eta)=-\mathcal{P}_\ell\eta+f(\eta)$, where $f(\eta)\in\mathcal{H}_k^{2\mu-2-d/2,2\delta}$.

Proposition 5.7. *Let $(\eta,\psi)\in\mathcal{H}_k^{\mu+1/2,\delta}\times\mathcal{H}_k^{\mu,\delta}$, with $\mu-\frac12\in\mathbb{N},\ \mu-\delta k>3+\frac{d}{2}$. Then*

$$\frac12|\nabla\psi|-\frac12\frac{(\nabla\eta\cdot\nabla\psi+G(\eta)\psi)^2}{1+|\nabla\eta|^2}=\mathcal{P}_V\cdot\nabla\psi-\mathcal{P}_B\mathcal{P}_V\cdot\nabla\eta-\mathcal{P}_BG(\eta)\psi+f(\eta,\psi),$$

where $f(\eta,\psi)\in\mathcal{H}_k^{2\mu-2-d/2,2\delta}$.

Note that in the above paralinearization results, we do not use the spatial decay of the symbols, as we only require the symbols to be in the classes $\Gamma_{0,0}^{m,r}$. These results will only be used in the proof of the Cauchy theorem, where the spatial decay of the symbols is not important. Later when we study the propagation of singularities, we will heavily use the spatial decay of the symbols.

Combining Propositions 5.5, 5.6 and 5.7, we obtain the paralinearization of the water wave system.

Proposition 5.8. *Let* $(\eta, \psi) \in \mathcal{H}_k^{\mu+1/2,\delta} \times \mathcal{H}_k^{\mu,\delta}$, *with* $\mu - \frac{1}{2} \in \mathbb{N}$, $\mu - \delta k > 3 + \frac{d}{2}$. *Then* (η, ψ) *solves the water wave equation if and only if*

$$(\partial_t + \mathcal{P}_V \cdot \nabla + \mathcal{L}) \begin{pmatrix} \eta \\ \psi \end{pmatrix} = f(\eta, \psi)$$

where

$$\mathcal{L} = Q^{-1} \begin{pmatrix} 0 & -\mathcal{P}_\lambda \\ \mathcal{P}_\ell & 0 \end{pmatrix} Q, \quad \textit{with } Q = \begin{pmatrix} \mathrm{Id} & 0 \\ -\mathcal{P}_B & \mathcal{P}_\lambda \end{pmatrix},$$

and $f(\eta, \psi) = Q^{-1}\binom{f_1}{f_2} \in \mathcal{H}_k^{\mu+1/2} \times \mathcal{H}_k^{\mu}$ *is defined by*

$$f_1 = G(\eta)\psi - \{\mathcal{P}_\lambda(\psi - \mathcal{P}_B \eta) - \mathcal{P}_V \cdot \nabla \eta\},$$

$$f_2 = -\frac{1}{2}|\nabla \psi|^2 + \frac{1}{2}\frac{(\nabla \eta \cdot \nabla \psi + G(\eta)\psi)^2}{1 + |\nabla \eta|^2} + H(\eta) + \mathcal{P}_V \cdot \nabla \psi - \mathcal{P}_B \mathcal{P}_V \cdot \nabla \eta - \mathcal{P}_B G(\eta)\psi + \mathcal{P}_\ell \eta - g\eta.$$

5C. *Symmetrization.*

Definition 5.9. For $T > 0$, $\gamma \in \mathbb{R}$ and two operators $\mathcal{A}, \mathcal{B} \in L^\infty([0, T], \mathscr{O}_0^\gamma)$, we say that $\mathcal{A} \sim_\gamma \mathcal{B}$, or simply $\mathcal{A} \sim \mathcal{B}$ when there is no ambiguity of the choice of γ, if

$$\mathcal{A} - \mathcal{B} \in L^\infty([0, T], \mathscr{O}_0^{\gamma - 3/2}).$$

By [Alazard, Burq and Zuily 2011], there exist symbols which depend solely on η,

$$\gamma = \gamma^{(3/2)} + \gamma^{(1/2)}, \quad p = p^{(1/2)} + p^{(-1/2)}, \quad q = q^{(0)},$$

whose principal symbols are explicitly

$$\gamma^{(3/2)} = \sqrt{\ell^{(2)}\lambda^{(1)}}, \quad p^{(1/2)} = (1 + |\nabla \eta|^2)^{-1/2}\sqrt{\lambda^{(1)}}, \quad q^{(0)} = (1 + |\nabla \eta|^2)^{1/4}$$

such that

$$\mathcal{P}_p \mathcal{P}_\lambda \sim_{3/2} \mathcal{P}_\gamma \mathcal{P}_q, \quad \mathcal{P}_q \mathcal{P}_\ell \sim_2 \mathcal{P}_\gamma \mathcal{P}_p, \quad \mathcal{P}_\gamma \sim_{3/2} (\mathcal{P}_\gamma)^*. \tag{5-6}$$

Define the symmetrizer

$$S = \begin{pmatrix} \mathcal{P}_p & 0 \\ 0 & \mathcal{P}_q \end{pmatrix} Q.$$

Then the first two relations in (5-6) can be rephrased as

$$S\mathcal{L} \sim \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} S. \tag{5-7}$$

where the equivalence relation $\sim$ is applied separately to each component of the matrices.

5D. ***Approximate system.*** Set the mollifier $J_\varepsilon = \mathcal{P}_{j_\varepsilon}$, where $j_\varepsilon = j_\varepsilon^{(0)} + j_\varepsilon^{(-1)}$:

$$j_\varepsilon^{(0)} = \exp(-\varepsilon \gamma^{(3/2)}), \quad j_\varepsilon^{(-1)} = \tfrac{1}{2}\partial_\xi \cdot D_x j_\varepsilon^{(0)}.$$

Then uniformly for $\varepsilon > 0$, we have

$$J_\varepsilon \mathcal{P}_\gamma \sim_{3/2} \mathcal{P}_\gamma J_\varepsilon, \quad J_\varepsilon^* \sim_0 J_\varepsilon.$$

Let $\tilde{p} = \tilde{p}^{(-1/2)} + \tilde{p}^{(-3/2)}$, with

$$\tilde{p}^{(-1/2)} = \frac{1}{p^{(1/2)}}, \quad \tilde{p}^{(-3/2)} = \frac{-\big(\tilde{p}^{(-1/2)} p^{(-1/2)} + \frac{1}{i}\partial_\xi \tilde{p}^{(-1/2)} \cdot \partial_x p^{(1/2)}\big)}{p^{(1/2)}}.$$

Then we have

$$\mathcal{P}_p \mathcal{P}_{\tilde{p}} \sim_0 \mathrm{Id}, \quad \mathcal{P}_q \mathcal{P}_{1/q} \sim_0 \mathrm{Id}.$$

Let

$$\mathcal{L}_\varepsilon = \mathcal{L} Q^{-1} \begin{pmatrix} \mathcal{P}_{\tilde{p}} J_\varepsilon \mathcal{P}_p & 0 \\ 0 & \mathcal{P}_{1/q} J_\varepsilon \mathcal{P}_q \end{pmatrix} Q.$$

Then as in (5-7) we have

$$S\mathcal{L}_\varepsilon \sim \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} J_\varepsilon S. \tag{5-8}$$

We define the approximate system

$$(\partial_t + \mathcal{P}_V \cdot \nabla J_\varepsilon + \mathcal{L}_\varepsilon)\begin{pmatrix} \eta \\ \psi \end{pmatrix} = f(J_\varepsilon \eta, J_\varepsilon \psi). \tag{5-9}$$

5E. ***A priori estimate.*** From now on we restrict ourselves to the case where $\delta = \frac{1}{2}$. The weighted Sobolev spaces $\mathcal{H}_k^{\mu+1/2,1/2} \times \mathcal{H}_k^{\mu,1/2}$ are the spaces where we do the energy estimates.

Proposition 5.10. *Let* $(\eta, \psi) \in C^1([0, T], \mathcal{H}_k^{\mu+1/2,1/2} \times \mathcal{H}_k^{\mu,1/2})$, *with* $\mu - \frac{1}{2} \in \mathbb{N}$, $\mu - \frac{k}{2} > 3 + \frac{d}{2}$, *solve the approximate system* (5-9). *Define*

$$M_T = \sup_{0 \le t \le T} \|(\eta, \psi)(t)\|_{\mathcal{H}_k^{\mu+1/2} \times \mathcal{H}_k^{\mu}}, \quad M_0 = \|(\eta, \psi)(0)\|_{\mathcal{H}_k^{\mu+1/2} \times \mathcal{H}_k^{\mu}}.$$

Then there exists some nondecaying function $C : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ *such that*

$$M_T \le C(M_0) + TC(M_T).$$

Proof. For $0 \le j \le k$, set

$$M_T^j = \sup_{0 \le t \le T} \|(\eta, \psi)(t)\|_{H_j^{\mu+1/2-j/2} \times H_j^{\mu-j/2}},$$

$$M_0^j = \|(\eta, \psi)(0)\|_{H_j^{\mu+1/2-j/2} \times H_j^{\mu-j/2}}.$$

By [Alazard, Burq and Zuily 2011], we know

$$M_T^0 \le C(M_0^0) + TC(M_T^0).$$

It remains to prove that, for $1 \le j \le k$, we have

$$M_T^j \le C(M_0^j) + TC(M_T).$$

To do this, let $\Lambda_j^\mu = \mathcal{P}_{m_j^{\mu-j/2}}$, and set

$$\Phi = \Lambda_j^\mu S \begin{pmatrix} \eta \\ \psi \end{pmatrix}.$$

Then

$$(\partial_t + \mathcal{P}_V \cdot \nabla J_\varepsilon)\Phi + \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} J_\varepsilon \Phi = F_\varepsilon,$$

where $F_\varepsilon = F_\varepsilon^1 + F_\varepsilon^2 + F_\varepsilon^3$, with

$$F_\varepsilon^1 = \Lambda_k^\mu S f(J_\varepsilon \eta, J_\varepsilon \psi),$$
$$F_\varepsilon^2 = [\partial_t + \mathcal{P}_V \cdot \nabla J_\varepsilon, \Lambda_j^\mu S]\begin{pmatrix}\eta\\ \psi\end{pmatrix},$$
$$F_\varepsilon^3 = \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0\end{pmatrix} J_\varepsilon \Lambda_j^\mu S \begin{pmatrix}\eta\\ \psi\end{pmatrix} - \Lambda_j^\mu S \mathcal{L}_\varepsilon \begin{pmatrix}\eta\\ \psi\end{pmatrix}.$$

By Propositions 5.8, 5.5, 5.6 and 5.7,

$$\|f(J_\varepsilon\eta, J_\varepsilon\psi)\|_{\mathcal{H}_k^{\mu+1/2,1/2}\times\mathcal{H}_k^{\mu,1/2}} \le C(\|(J_\varepsilon\eta, J_\varepsilon\psi)\|_{\mathcal{H}_k^{\mu+1/2,1/2}\times\mathcal{H}_k^{\mu,1/2}}) \le C(\|(\eta,\psi)\|_{\mathcal{H}_k^{\mu+1/2,1/2}\times\mathcal{H}_k^{\mu,1/2}}).$$

Therefore,

$$\|F_\varepsilon^1\|_{L^\infty([0,T],L^2)} \le C(M_T).$$

As $\mathcal{P}_V \cdot \nabla J_\varepsilon$ is a scalar operator, Proposition 4.19 gives

$$\|[\partial_t + \mathcal{P}_V\cdot\nabla J_\varepsilon, \Lambda_j^\mu S]\|_{L^\infty([0,T],H_j^{\mu+1/2-j/2}\times H_j^{\mu-j/2}\to L^2\times L^2)} \le C(M_T),$$

which implies

$$\|F_\varepsilon^2\|_{L^\infty([0,T],L^2)} \le C(M_T).$$

By (5-8), the operator

$$\begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0\end{pmatrix} J_\varepsilon S - S\mathcal{L}_\varepsilon$$

sends $H^{\mu+1/2}\times H^\mu$ to $H^\mu \times H^\mu$. Unfortunately,

$$\begin{aligned} R &:= \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0\end{pmatrix} J_\varepsilon \Lambda_k^\mu S - \Lambda_k^\mu S\mathcal{L}_\varepsilon \\ &= \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0\end{pmatrix} J_\varepsilon[\Lambda_k^\mu, S] + [S\mathcal{L}_\varepsilon, \Lambda_k^\mu] + \left(\begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0\end{pmatrix} J_\varepsilon S - S\mathcal{L}_\varepsilon\right)\Lambda_k^\mu \\ &=: \text{(I)} + \text{(II)} + \text{(III)} \end{aligned}$$

does not send $H_j^{\mu+1/2-j/2}\times H_j^{\mu-j/2}$ to $L^2\times L^2$ because the subprincipal symbol cannot be canceled out in the symbolic calculus, due to the existence of Λ_j^μ. Particularly, we need to use Proposition 4.19 to estimate the commutators $[\Lambda_j^\mu, S]$ and $[S\mathcal{L}_\varepsilon, \Lambda_j^\mu]$, and obtain

$$\left\| R\begin{pmatrix}\eta\\ \psi\end{pmatrix}\right\|_{L^2\times L^2} \lesssim \|(\eta,\psi)\|_{H_{j-1}^{\mu+1-j/2}\times H_{j-1}^{\mu+1/2-j/2}} + \|(\eta,\psi)\|_{H_j^{\mu+1/2-j/2}\times H_j^{\mu-j/2}}.$$

More precisely, the first term on the right-hand side comes from (I) and (II), while the second term comes from (III). When $j \ge 1$,

$$H_{j-1}^{\mu+1-j/2}\times H_{j-1}^{\mu+1/2-j/2} = H_{j-1}^{\mu+1/2-(j-1)/2}\times H_{j-1}^{\mu-(j-1)/2} \supset \mathcal{H}_k^{\mu+1/2,1/2}\times\mathcal{H}_k^{\mu,1/2},$$

and we deduce that

$$\|F_\varepsilon^3\|_{L^\infty([0,T],L^2)} \le C(M_T).$$

Finally by the exact same energy estimate as in [Alazard, Burq and Zuily 2011], we conclude that

$$M_T^j \lesssim \|\Phi\|_{L^\infty([0,T],L^2)} \le C(M_0^j) + TC(M_T). \qquad \square$$

5F. *Existence.*

Lemma 5.11. *For all* $(\eta_0, \psi_0) \in \mathcal{H}_k^{\mu+1/2,1/2} \times \mathcal{H}_k^{\mu,1/2}$, *where* $\mu - \frac{1}{2} \in \mathbb{N}$ *and* $\mu - \frac{k}{2} > 3 + \frac{d}{2}$, *and, for all* $\varepsilon > 0$, *the Cauchy problem of the approximate system* (5-9) *has a unique maximal solution*

$$(\eta_\varepsilon, \psi_\varepsilon) \in C([0, T_\varepsilon), \mathcal{H}_k^{\mu+1/2,1/2} \times \mathcal{H}_k^{\mu,1/2}).$$

Moreover, there exists $T_0 > 0$ *such that*

$$\inf_{\varepsilon\in(0,1]} T_\varepsilon \ge T_0.$$

Proof. Following [Alazard, Burq and Zuily 2011], the existence follows from the existence theory of ODEs by writing (5-9) in the compact form

$$\partial_t X = \mathcal{F}_\varepsilon(X),$$

where $\mathcal{F}_\varepsilon$ is a Lipschitz map on $\mathcal{H}_k^{\mu+1/2,1/2} \times \mathcal{H}_k^{\mu,1/2}$. Indeed, $J_\varepsilon \in \mathscr{O}_0^{-\infty}$ is a smoothing operator.[1] The estimates to proving the Lipschitz regularity can be carried out much as in the proof of Proposition 5.10. The only nontrivial term that remains is the Dirichlet–Neumann operator, whose regularity follows by combining Proposition 5.4 and the shape derivative formula (which goes back to [Zakharov 1998],

$$\langle \mathrm{d}G(\eta)\psi, \varphi\rangle := \lim_{h\to 0} \frac{1}{h}(G(\eta + h\varphi) - G(\eta))\psi = -G(\eta)(B\varphi) - \nabla\cdot(V\varphi).$$

A standard abstract argument then shows that T_ε has a strictly positive lower bound, we refer to [Alazard, Burq and Zuily 2011] for more details. $\square$

Proof of Theorem 1.6. By Lemma 5.11, we obtain a sequence $\{(\eta_\varepsilon, \psi_\varepsilon)\}_{0<\varepsilon\le 1}$ which satisfies (5-9) and is uniformly bounded in $L^\infty([0, T], \mathcal{H}_k^{\mu+1/2,1/2} \times \mathcal{H}_k^{\mu,1/2})$ for some $T > 0$. By (5-9), the time derivatives $\{(\partial_t\eta_\varepsilon, \partial_t\psi_\varepsilon)\}_{0<\varepsilon\le 1}$ are uniformly bounded in $L^\infty([0, T], \mathcal{H}_k^{\mu-1,1/2} \times \mathcal{H}_k^{\mu-3/2,1/2}))$. By [Alazard, Burq and Zuily 2011], there exists

$$(\eta, \psi) \in C([0, T], H^{\mu+1/2} \times H^\mu), \tag{5-10}$$

which solves (1-5), such that as $\varepsilon \to 0$, we have $(\eta_\varepsilon, \psi_\varepsilon) \to (\eta, \psi)$ weakly in $L^2([0, T], \mathcal{H}_k^{\mu+1/2,1/2} \times \mathcal{H}_k^{\mu,1/2})$, and strongly in $C([0, T], \mathcal{H}_k^{\mu-1,1/2} \times \mathcal{H}_k^{\mu-3/2,1/2})$. We then prove that, for $1 \le j \le k$,

$$\Phi = \Phi(\eta, \psi) := \Lambda_j^\mu S(\eta, \psi)\begin{pmatrix}\eta\\ \psi\end{pmatrix}$$

lies in $C([0, T], L^2)$, where Λ_j^μ is defined in Proposition 5.10, and $S = S(\eta, \psi)$ is the symmetrizer. Up to an extraction of a subsequence, we may assume by weak convergence that

$$(\eta, \psi) \in L^\infty([0, T], \mathcal{H}_k^{\mu+1/2,1/2} \times \mathcal{H}_k^{\mu,1/2}),$$
$$(\partial_t\eta, \partial_t\psi) \in L^\infty([0, T], \mathcal{H}_k^{\mu-1,1/2} \times \mathcal{H}_k^{\mu-3/2,1/2}),$$

[1] We do not need $J_\varepsilon \in \mathscr{O}_{-\infty}^{-\infty}$ because the operators such as $\mathcal{P}_V \cdot \nabla$, $\mathcal{L}$, etc., are all of nonpositive orders with respect to the spatial decay.

with

$$\|(\eta,\psi)\|_{L^\infty([0,T],\mathcal{H}_k^{\mu+1/2,1/2}\times\mathcal{H}_k^{\mu})\cap W^{1,\infty}([0,T],\mathcal{H}_k^{\mu-1,1/2}\times\mathcal{H}_k^{\mu-3/2})} \le C(\|(\eta_0,\psi_0)\|_{\mathcal{H}_k^{\mu+1/2,1/2}\times\mathcal{H}_k^{\mu,1/2}}).$$

This already implies that (η,ψ) is weakly continuous in $\mathcal{H}_k^{\mu+1/2,1/2}\times\mathcal{H}_k^{\mu,1/2}$. By the analysis in the previous section,

$$(\partial_t+\mathcal{P}_V\cdot\nabla)\Phi+\begin{pmatrix}0 & -\mathcal{P}_\gamma\\ \mathcal{P}_\gamma & 0\end{pmatrix}\Phi=F,$$

with

$$\|F\|_{L^\infty([0,T],L^2)}\le C(\|(\eta_0,\psi_0)\|_{\mathcal{H}_k^{\mu+1/2,1/2}\times\mathcal{H}_k^{\mu,1/2}}).$$

Let $J_h=\mathrm{Op}_h(e^{-|x|^2-|\xi|^2})$. Now that $e^{-h^2|x|^2-h^2|\xi|^2}\in S_0^0$, we have the commutator estimate

$$[J_h,\mathcal{P}_V\cdot\nabla]=\mathcal{O}(1)_{\mathscr{O}_{-1}^0},\qquad [J_h,\mathcal{P}_\gamma]=\mathcal{O}(1)_{\mathscr{O}_{-1}^{1/2}}.$$

Because $k\ge 1$, by the same spirit of estimating R in Proposition 5.10, we obtain the energy estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|J_h\Phi(t)\|_{L^2}^2\le C(\|(\eta_0,\psi_0)\|_{\mathcal{H}_k^{\mu+1/2,1/2}\times\mathcal{H}_k^{\mu,1/2}}).$$

Therefore, $t\mapsto\|J_h\Phi(t)\|_{L^2}^2$ are uniformly Lipschitzian. Consequently, by the Arzelà–Ascoli theorem, $t\mapsto\|\Phi(t)\|_{L^2}^2$ is continuous, because $J_h\Phi\to\Phi$ as $h\to 0$. Combining the weak continuity, we deduce by functional analysis that $\Phi\in C([0,T],L^2)$. By (5-10), the paradifferential calculus, and the definition of Φ, we deduce that

$$(\eta,\psi)\in C([0,T],\mathcal{H}_k^{\mu+1/2,1/2}\times\mathcal{H}_k^{\mu,1/2}).$$

Thus we finish the proof of Theorem 1.6. □

6. Propagation of singularities for water waves

6A. *Finer paralinearization and symmetrization.* To study the propagation of singularities, we need much finer results of paralinearization and symmetrization than Propositions 5.5 and 5.8 so as to gain regularities in the remainder terms.

Proposition 6.1. *If $(\eta,\psi)\in H^{\mu+1/2}\times H^\mu$, with $\mu-\frac{1}{2}\in\mathbb{N}$ and $\mu>3+\frac{d}{2}$, then there exists $\lambda=\lambda^{(1)}+\lambda^{(0)}+\cdots\in\Sigma^{1,\mu-1/2-\tilde{d}}$ such that*

$$G(\eta)\psi=\mathcal{P}_\lambda(\psi-\mathcal{P}_B\eta)-\mathcal{P}_V\cdot\nabla\eta+R(\eta,\psi),$$

where $R(\eta,\psi)\in H^{2\mu-K-d/2}$ for some $K>0$ independent of the dimension d. Moreover $\lambda^{(1-j)}$, when it is defined, is a function of derivatives $\partial_x^\alpha\eta$, where $|\alpha|\le 1+j$.

Proof. This theorem follows by replacing the usual paradifferential calculus with the dyadic paradifferential calculus in the analysis of [Alazard and Métivier 2009]. In that work, the explicit expression for λ is given. We write it down for the sake of later applications:

$$\lambda=(1+|\nabla\eta|^2)a_+-i\nabla\eta\cdot\xi,$$

where $a_\pm = \sum_{j\le 1} a_\pm^{(j)} \in \Sigma^{1,\mu-1-d/2}$ is defined as follows. Setting $c = 1/(1+|\nabla\eta|^2)$, we have

$$a_-^{(1)} = ic\nabla\eta\cdot\xi - \sqrt{c|\xi|^2-(c\nabla\eta\cdot\xi)^2},\quad a_+^{(1)} = ic\nabla\eta\cdot\xi + \sqrt{c|\xi|^2-(c\nabla\eta\cdot\xi)^2},$$
$$a_-^{(0)} = \frac{i\partial_\xi a_-^{(1)}\cdot\partial_x a_+^{(1)} - c\Delta\eta a_-^{(1)}}{a_+^{(1)}-a_-^{(1)}},\qquad a_+^{(0)} = \frac{i\partial_\xi a_-^{(1)}\cdot\partial_x a_+^{(1)} - c\Delta\eta a_+^{(1)}}{a_-^{(1)}-a_+^{(1)}}.$$

Suppose that $a_\pm^{(j)}$ are defined for $m\le j\le 1$. Then we define

$$a_-^{(m-1)} = \frac{1}{a_-^{(1)}-a_+^{(1)}}\sum_{m\le k\le 1}\sum_{m\le \ell\le 1}\sum_{|\alpha|=k+\ell-m}\frac{1}{\alpha!}\partial_\xi^\alpha a_-^{(k)} D_x^\alpha a_+^{(\ell)},$$
$$a_+^{(m-1)} = -a_-^{(m-1)}.$$

The principal and subprincipal symbols of λ coincide with the ones given by Proposition 5.5. □

Proposition 6.2. *Let $(\eta,\psi)\in H^{\mu+1/2}\times H^\mu$, with $\mu-\frac{1}{2}\in\mathbb{N}$ and $\mu>3+\frac{d}{2}$. Let $\Lambda^\mu = \mathcal{P}_{(\gamma^{(3/2)})^{2\mu/3}}$, and set*

$$w = \Lambda^\mu U S\begin{pmatrix}\eta\\ \psi\end{pmatrix},\quad U = \begin{pmatrix}-i & 1\\ i & 1\end{pmatrix}.$$

Then there exist $Q\in M_{2\times 2}(\Sigma_{0,0}^{0,\mu-1/2-2-\tilde d})$ and $\zeta\in\Sigma_{0,0}^{-1/2,\mu-1/2-2-\tilde d}$ such that, for some $K>0$ which is independent of the dimension d, we have

$$(\partial_t+\mathcal{P}_V\cdot\nabla+\mathcal{P}_Q)w + i\mathcal{P}_\gamma\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}w + \frac{ig}{2}\mathcal{P}_\zeta\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\in H^{\mu-K-d/2}. \tag{6-1}$$

Remark 6.3. Because χ in the definition of paradifferential operators is an even function, we verify that Λ^μ, $\mathcal{P}_p$, $\mathcal{P}_q$, $\mathcal{P}_B$ all map real-valued functions to real-valued functions. Therefore,

$$w = \begin{pmatrix}u\\ \bar u\end{pmatrix},\quad \text{with } u = \Lambda^\mu(-i,1)S\begin{pmatrix}\eta\\ \psi\end{pmatrix} = \Lambda^\mu\mathcal{P}_q\omega - i\Lambda^\mu\mathcal{P}_p\eta, \tag{6-2}$$

recalling that $\omega = \psi - \mathcal{P}_B\eta$ is the good unknown of Alinhac.

Proof. Combining Propositions 6.1 and 5.8, and moving the term $g\eta$ to the left-hand side,

$$(\partial_t+\mathcal{P}_V\cdot\nabla+\mathcal{L})\begin{pmatrix}\eta\\ \psi\end{pmatrix} + g\begin{pmatrix}0\\ \eta\end{pmatrix} = f(\eta,\psi),$$

where

$$f(\eta,\psi) = Q^{-1}\begin{pmatrix}f_1\\ f_2\end{pmatrix}\in H^{2\mu+1/2-K-d/2}\times H^{2\mu-K-d/2}$$

for some $K>0$ and

$$f_1 = G(\eta)\psi - \{\mathcal{P}_\lambda(\psi-\mathcal{P}_B\eta) - \mathcal{P}_V\cdot\nabla\eta\},$$
$$f_2 = -\frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}\frac{(\nabla\eta\cdot\nabla\psi+G(\eta)\psi)^2}{1+|\nabla\eta|^2} + H(\eta) + \mathcal{P}_V\cdot\nabla\psi - \mathcal{P}_B\mathcal{P}_V\cdot\nabla\eta - \mathcal{P}_BG(\eta)\psi + \mathcal{P}_\ell\eta.$$

Given two time-dependent operators $\mathcal{A}, \mathcal{B} : \mathscr{S} \to \mathscr{S}'$, we say that $\mathcal{A} \sim \mathcal{B}$ if

$$\mathcal{A} - \mathcal{B} \in L^\infty([0, T], \mathscr{O}_0^{-\mu+d/2+K}).$$

By the ellipticity of $\gamma^{(3/2)}$, $p^{(1/2)}$ and $q^{(0)}$, we can find paradifferential operators $\widetilde{\Lambda}^\mu$ and $\widetilde{S}$ by a routine construction of a parametrix such that $\widetilde{\Lambda}^\mu \Lambda^\mu \sim \mathrm{Id}$, $\widetilde{S} S \sim \mathrm{Id}$. We can find $\zeta \in \Sigma^{-1/2, \mu-1/2-2-\tilde{d}}$ with principal symbol $\zeta^{(-1/2)} = q^{(0)}/p^{(1/2)}$, which implies (note that the only nonzero entries in the following matrices are in the lower left corners)

$$\begin{pmatrix} 0 & 0 \\ \mathcal{P}_\zeta & 0 \end{pmatrix} \Lambda^\mu S - \Lambda^\mu S \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \sim 0.$$

Then by (5-7) and the fact that the Poisson bracket between the symbol of Λ^μ and γ vanishes, we find by the symbolic calculus two symbols $A, B \in M_{2\times 2}(\Sigma^{0, \mu-1/2-2-\tilde{d}})$ such that

$$\begin{aligned}
\mathcal{A} &:= [\partial_t + \mathcal{P}_V \cdot \nabla, \Lambda^\mu S] \sim [\partial_t + \mathcal{P}_V \cdot \nabla, \Lambda^\mu S] \widetilde{S} \widetilde{\Lambda}^\mu \Lambda^\mu S \sim \mathcal{P}_A \Lambda^\mu S, \\
\mathcal{B} &:= \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} \Lambda^\mu S - \Lambda^\mu S \mathcal{L} \sim \left(\begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} - \Lambda^\mu S \mathcal{L} \widetilde{S} \widetilde{\Lambda}^\mu \right) \Lambda^\mu S \sim \mathcal{P}_B \Lambda^\mu S.
\end{aligned}$$

In fact, by Proposition 4.19, the symbol A is a finite sum of symbols which is given by the symbolic calculus of the operator $[\partial_t + \mathcal{P}_V \cdot \nabla, \Lambda^\mu S] \widetilde{S} \widetilde{\Lambda}^\mu$, whereas the symbol B is given by the symbolic calculus of the operator

$$\begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} - \Lambda^\mu S \mathcal{L} \widetilde{S} \widetilde{\Lambda}^\mu.$$

Clearly A is of zeroth order. The reason why B is of zeroth order is the condition (5-7) according to which we constructed the symbols γ, p, q.

Let $\Phi = \Lambda^\mu S \binom{\eta}{\psi}$, and write

$$g \begin{pmatrix} 0 \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ g & 0 \end{pmatrix} \begin{pmatrix} \eta \\ \psi \end{pmatrix},$$

we obtain by the analysis above that

$$(\partial_t + \mathcal{P}_V \cdot \nabla)\Phi + \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} \Phi + \begin{pmatrix} 0 & 0 \\ g\mathcal{P}_\zeta & 0 \end{pmatrix} \Phi = \mathcal{P}_A \Phi + \mathcal{P}_B \Phi + F,$$

where

$$F = (\mathcal{A} + \mathcal{B}) \begin{pmatrix} \eta \\ \psi \end{pmatrix} - \mathcal{P}_{A+B} \Phi + \begin{pmatrix} 0 & 0 \\ g\mathcal{P}_\zeta & 0 \end{pmatrix} \Phi - g \Lambda^\mu S \begin{pmatrix} 0 \\ \eta \end{pmatrix} + \Lambda^\mu S f(\eta, \psi) \in H^{\mu-K-d/2}.$$

Finally, observe that

$$\begin{aligned}
U \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} U^{-1} &= i \begin{pmatrix} \mathcal{P}_\gamma & 0 \\ 0 & -\mathcal{P}_\gamma \end{pmatrix}, \\
U \begin{pmatrix} 0 & 0 \\ \mathcal{P}_\zeta & 0 \end{pmatrix} U^{-1} &= \frac{i}{2} \begin{pmatrix} \mathcal{P}_\zeta & -\mathcal{P}_\zeta \\ \mathcal{P}_\zeta & -\mathcal{P}_\zeta \end{pmatrix},
\end{aligned}$$

We conclude by setting

$$Q = -\tfrac{1}{2} U (A + B) U^{-1}. \qquad \square$$

Remark 6.4. By Proposition 6.1 and the symbolic calculus, the symbols that we have encountered, such as λ, ζ and Q etc., are of the form $a = a^{(m)} + a^{(m-1)} + \cdots$ such that $a^{(m-j)}$, whenever it is defined, is a function of $(\nabla\eta, \dots, \nabla^{j+1}\eta)$. To be precise $a^{(m-j)} = f_j(\nabla\eta, \dots, \nabla^{j+1}\eta, \xi)$, where f_j is homogeneous of degree $m-j$ in ξ and $f_0(0, \dots, 0, \xi) = |\xi|^m$, $f_j(0, \dots, 0, \xi) = 0$ for $j \geq 1$. Note that if $\eta \in \mathcal{H}_k^{\mu+1/2,1/2}$, then for all $j \leq \mu + \frac{1}{2} - \tilde{d}$, we have $\nabla^j\eta \in \mathcal{H}_k^{\mu+1/2-j,1/2}$. Therefore, by Lemma 4.15,

$$\nabla^j\eta \in W_{0,1}^{\min\{[2(\mu+1/2-j-\tilde{d})/3],k\},\infty} \cap \langle x\rangle^{-\min\{2(\mu+1/2-j-\tilde{d}),k\}} L^\infty, \tag{6-3}$$

and consequently

$$a^{(m)} - |\xi|^m \in \Gamma^{m,0}_{-\min\{2\mu-1-2\tilde{d},k\},0}, \qquad a^{(m-j)} \in \Gamma^{m-j,0}_{-\min\{2\mu-1-2j-2\tilde{d},k\},0}. \tag{6-4}$$

As another consequence of (6-3), we also have

$$\begin{aligned} &a^{(m)} - |\xi|^m \in \Gamma_{0,1}^{m,\min\{[2(\mu-1/2-\tilde{d})/3],k\}}, \\ &\quad a^{(m-j)} \in \Gamma_{-j,1}^{m-j,\min\{[2(\mu-1/2-j-\tilde{d})/3],k\}} \subset \Gamma_{-j,1}^{m-j,\min\{[2(\mu-1/2-\tilde{d})/3],k\}-j}. \end{aligned} \tag{6-5}$$

Lemma 6.5. *Let u be defined as in (6-2). If $(\eta, \psi) \in H^{\mu+1/2} \times H^\mu$, with $\mu - \frac{1}{2} \in \mathbb{N}$, then, for $0 \leq \sigma \leq r \in \mathbb{N}$, with $r < \mu - \frac{1}{2} - 1 - \tilde{d}$,*

$$\mathrm{WF}^\sigma_{0,1}(u)^\circ = \mathrm{WF}^{\mu+1/2+\sigma}_{0,1}(\eta)^\circ \cup \mathrm{WF}^{\mu+\sigma}_{0,1}(\psi)^\circ.$$

If $(\eta, \psi) \in \mathcal{H}_k^{\mu+1/2} \times \mathcal{H}_k^\mu$, with $k \leq \frac{2}{3}(\mu - 1 - \tilde{d})$, then, for $0 \leq \sigma \leq \frac{3}{2}k$,

$$\mathrm{WF}^\sigma_{1/2,1}(u)^\circ = \mathrm{WF}^{\mu+1/2+\sigma}_{1/2,1}(\eta)^\circ \cup \mathrm{WF}^{\mu+\sigma}_{1/2,1}(\psi)^\circ.$$

Proof. Clearly if $\eta \in H^{\mu+1/2}$, then $(\gamma^{(3/2)})^{2\mu/3} \in \Gamma^{\mu,r}$, $p^{(1/2)} \in \Gamma^{1/2,r}$, $q^{(0)} \in \Gamma^{0,r}$, $B \in \Gamma^{0,r}$. By (6-5), if $\eta \in \mathcal{H}_k^{\mu+1/2}$, then $(\gamma^{(3/2)})^{2\mu/3} \in \Gamma^{\mu,k}_{0,1}$, $p^{(1/2)} \in \Gamma^{1/2,k}_{0,1}$, $q^{(0)} \in \Gamma^{0,k}_{0,1}$, $B \in \Gamma^{0,k}_{0,1}$. By Lemma 4.42 and (6-2), for either $\epsilon = 0$ or $\epsilon = \frac{1}{2}$,

$$\begin{aligned} \mathrm{WF}^\sigma_{\epsilon,1}(u)^\circ &= \mathrm{WF}^\sigma_{\epsilon,1}(\Lambda^\mu \mathcal{P}_p \eta)^\circ \cup \mathrm{WF}^\sigma_{\epsilon,1}(\Lambda^\mu \mathcal{P}_q(\psi - \mathcal{P}_B\eta))^\circ \\ &= \mathrm{WF}^{\mu+1/2+\sigma}_{\epsilon,1}(\eta)^\circ \cup \mathrm{WF}^{\mu+\sigma}_{\epsilon,1}(\psi - \mathcal{P}_B\eta)^\circ \\ &\subset \mathrm{WF}^{\mu+1/2+\sigma}_{\epsilon,1}(\eta)^\circ \cup (\mathrm{WF}^{\mu+\sigma}_{\epsilon,1}(\psi)^\circ \cup \mathrm{WF}^{\mu+\sigma}_{\epsilon,1}(\mathcal{P}_B\eta)^\circ) \\ &\subset \mathrm{WF}^{\mu+1/2+\sigma}_{\epsilon,1}(\eta)^\circ \cup (\mathrm{WF}^{\mu+\sigma}_{\epsilon,1}(\psi)^\circ \cup \mathrm{WF}^{\mu+\sigma}_{\epsilon,1}(\eta)^\circ) \\ &= \mathrm{WF}^{\mu+1/2+\sigma}_{\epsilon,1}(\eta)^\circ \cup \mathrm{WF}^{\mu+\sigma}_{\epsilon,1}(\psi)^\circ. \end{aligned}$$

Conversely, as $\mathrm{WF}^{\mu+\sigma}_{\epsilon,1}(\mathcal{P}_B\eta)^\circ \subset \mathrm{WF}^{\mu+1/2+\sigma}_{\epsilon,1}(\eta)^\circ$, we have

$$\begin{aligned} &\mathrm{WF}^{\mu+1/2+\sigma}_{\epsilon,1}(\eta)^\circ \cup \mathrm{WF}^{\mu+\sigma}_{\epsilon,1}(\psi)^\circ \\ &\qquad = \mathrm{WF}^{\mu+1/2+\sigma}_{\epsilon,1}(\eta)^\circ \cup (\mathrm{WF}^{\mu+\sigma}_{\epsilon,1}(\psi)^\circ \setminus \mathrm{WF}^{\mu+1/2+\sigma}_{\epsilon,1}(\eta)^\circ) \\ &\qquad = \mathrm{WF}^{\mu+1/2+\sigma}_{\epsilon,1}(\eta)^\circ \cup (\mathrm{WF}^{\mu+\sigma}_{\epsilon,1}(\psi - \mathcal{P}_B\eta)^\circ \setminus \mathrm{WF}^{\mu+1/2+\sigma}_{\epsilon,1}(\eta)^\circ) \\ &\qquad = \mathrm{WF}^{\mu+1/2+\sigma}_{\epsilon,1}(\eta)^\circ \cup \mathrm{WF}^{\mu+\sigma}_{\epsilon,1}(\psi - \mathcal{P}_B\eta)^\circ \\ &\qquad = \mathrm{WF}^\sigma_{\epsilon,1}(u)^\circ. \end{aligned}$$

The lemma follows. ∎

6B. ***Proof of Theorem 1.7.*** By Lemma 6.5, it is equivalent to prove the following theorem.

Theorem 6.6. *Under the hypothesis of Theorem 1.7, let u be defined by (6-2), and let*

$$(x_0, \xi_0) \in \mathrm{WF}^{\sigma}_{1/2,1}(u_0)^{\circ},$$

with $0 \le \sigma < \frac{k}{2} - \frac{3}{2}$. *Let* $t_0 \in [0, T]$, *and suppose that*

$$x_0 + \tfrac{3}{2} t |\xi_0|^{-1/2} \xi_0 \neq 0$$

for all $t \in [0, t_0]$. *Then*

$$\big(x_0 + \tfrac{3}{2} t_0 |\xi_0|^{-1/2} \xi_0, \xi_0\big) \in \mathrm{WF}^{\sigma}_{1/2,1}(u(t_0))^{\circ}.$$

Proof. For $\nu \in \mathbb{R}$, define

$$X^{\nu} = \sum_{k \in \mathbb{Z}} H_k^{\nu - k/2}.$$

By Lemma 2.15, if $f \in X^{\nu}$, then $\mathrm{WF}^{\nu}_{1/2,1}(f)^{\circ} = \varnothing$. Also note that if $f \in X^{\nu}$ and $a \in \Sigma^{m,r}_{0,1}$, then $\mathcal{P}_a f \in X^{\nu - m}$. As $k < 2\mu - d$, we have $V \in \mathcal{H}^{\mu}_k \subset \langle x \rangle^k H^{\mu - k/2} \subset \langle x \rangle^{-k} L^{\infty}$, which implies

$$\mathcal{P}_V \cdot \nabla w \subset \mathcal{P}_V H^{-1} \subset H_k^{-1} \subset X^{k/2-1}.$$

By Remark 6.4, particularly (6-4),

$$\mathcal{P}_Q w \in \sum_{j < \mu - \tilde{d}} H^{j}_{\min\{2\mu - 1 - 2j - 2\tilde{d}, k\}} \subset \sum_{j < \mu - \tilde{d}} X^{\min\{\mu - 1 - \tilde{d}, j + k/2\}} \subset X^{k/2}.$$

Similarly

$$\mathcal{P}_{\gamma} w - \mathcal{P}_{|\xi|^{3/2}} w \in \sum_{j < \mu - \tilde{d}} H^{j - 3/2}_{\min\{2\mu - 1 - 2j - 2\tilde{d}, k\}} \subset X^{k/2 - 3/2},$$

$$\mathcal{P}_{\zeta} w - \mathcal{P}_{|\xi|^{-1/2}} w \in \sum_{j < \mu - \tilde{d}} H^{j + 1/2}_{\min\{2\mu - 1 - 2j - 2\tilde{d}, k\}} \subset X^{k/2 + 1/2}.$$

By the hypothesis on m, we thus obtain

$$\partial_t w' + i |D_x|^{3/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} w' + \frac{ig}{2} |D_x|^{-1/2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} w' \in X^{k/2 - 3/2}, \tag{6-6}$$

where $w' = \pi(D_x) w$, and $\pi \in C^{\infty}(\mathbb{R}^d)$, which vanishes near the origin, and equals 1 outside a neighborhood of the origin. Moreover, we require that $\operatorname{supp} \pi \subset \{\tilde{\pi} = 1\}$ such that $1 - \tilde{\pi} \in C_c^{\infty}(\mathbb{R}^d)$ and $\tilde{\pi}(\xi) = 0$ if $|\xi|^2 \le |g|$. Observe that the matrix

$$M = |\xi|^{3/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{g}{2} |\xi|^{-1/2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

is symmetrizable when restricted to $\operatorname{supp} \pi$. Indeed, let

$$P = \frac{1}{2} \begin{pmatrix} 1 + \theta & 1 - \theta \\ -(1 - \theta) & -(1 + \theta) \end{pmatrix},$$

where $\theta = \sqrt{\tilde{\pi}(\xi)\cdot(g|\xi|^{-2}+1)}$. Then $P \in \mathscr{O}_0^0$. For $\xi \in \operatorname{supp}\pi$, we have

$$PMP^{-1} = |\xi|^{3/2}\theta(\xi)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Set

$$\tilde{w} = P(D_x)w' = P(D_x)\begin{pmatrix}u'\\ \overline{u'}\end{pmatrix} = \begin{pmatrix}\operatorname{Re}u' + i\theta(D_x)\operatorname{Im}u'\\ -\operatorname{Re}u' + i\theta(D_x)\operatorname{Im}u'\end{pmatrix},$$

where $u' = \pi(D_x)u$, then

$$\partial_t\tilde{w} + |D_x|^{3/2}\theta(D_x)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\tilde{w} \in X^{k/2-3/2}.$$

Finally, let $v = \operatorname{Re}u' + i\theta(D_x)\operatorname{Im}u'$. Then $\mathrm{WF}^{\sigma}_{1/2,1}(u)^\circ = \mathrm{WF}^{\sigma}_{1/2,1}(v)^\circ$, and

$$\partial_t v + |D_x|^{3/2}\theta(D_x)v \in X^{k/2-3/2}.$$

We are left to prove that if $(x_0,\xi_0) \in \mathrm{WF}^{\sigma}_{1/2,1}(v(0))^\circ$, then

$$\big(x_0 + \tfrac{3}{2}t_0|\xi_0|^{-1/2}\xi_0, \xi_0\big) \in \mathrm{WF}^{\sigma}_{1/2,1}(v(t_0)).$$

Because $\theta(\xi) \sim 1$ in the high-frequency regime, a proof similar to that of Theorem 1.4(1) yields the conclusion. □

6C. *Proof of Theorem 1.9.*

6C1. *Hamiltonian flow.* Let $\Phi = \Phi_s : \mathbb{R}^d \times (\mathbb{R}^d\backslash 0) \to \mathbb{R}^d \times (\mathbb{R}^d\backslash 0)$ be the Hamiltonian flow of

$$H(x,\xi) = \gamma^{(3/2)}(0,x,\xi) = \left(|\xi|^2 - \frac{(\nabla\eta_0\cdot\xi)^2}{1+|\nabla\eta_0|^2}\right)^{3/4}.$$

That is

$$\partial_s\Phi_s(x,\xi) = X_H(\Phi_s(x,\xi)), \qquad \Phi|_{s=0} = \mathrm{Id}_{\mathbb{R}^d\times(\mathbb{R}^d\backslash 0)},$$

where $X_H = (\partial_\xi H, -\partial_x H)$. We use s to denote the time variable in accordance to the semiclassical time variable in the following section. Observe that:

Lemma 6.7. *For $(x,\xi) \in \mathbb{R}^d \times (\mathbb{R}^d\backslash 0)$, we have*

$$\Phi_s(x,\xi) = \mathcal{G}_{\varphi_s(x,\xi)}(x,\xi),$$

where $\mathcal{G}$ is the geodesic flow defined in Section 1C5, and

$$\varphi_s(x,\xi) = \frac{3}{4}\int_0^s G(\Phi_\sigma(x,\xi))^{-1/4}\,\mathrm{d}\sigma.$$

Proof. We have $\mathcal{G}_{\varphi_0(x,\xi)}(x,\xi) = \mathcal{G}_0(x,\xi) = (x,\xi) = \Phi_0(x,\xi)$. Then observe that

$$H(x,\xi) = G(x,\xi)^{3/4} = \varrho_x^{-1}(\xi,\xi)^{3/4}.$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{G}_{\varphi_s(x,\xi)}(x,\xi) &= \frac{\mathrm{d}}{\mathrm{d}s}\varphi_s(x,\xi)\Big(\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{G}\Big)_{\varphi_s(x,\xi)}(x,\xi)\\ &= \tfrac{3}{4}G(\mathcal{G}_{\varphi_s(x,\xi)}(x,\xi))^{-1/4}X_G(\mathcal{G}_{\varphi_s(x,\xi)}(x,\xi))\\ &= X_H(\mathcal{G}_{\varphi_s(x,\xi)}(x,\xi)).\end{aligned}$$

We conclude by the uniqueness of solutions to Hamiltonian ODEs. □

Lemma 6.8. *Suppose that, for some $\epsilon > 0$, $\nabla\eta_0 \in W^{0,\infty}_{1/2+\epsilon}$, $\nabla^2\eta_0 \in W^{0,\infty}_{1+\epsilon}$. Let $(x_0,\xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d\backslash 0)$ such that the cogeodesic $\{(x_s,\xi_s) = \Phi_s(x_0,\xi_0)\}_{s\in\mathbb{R}}$ is forwardly nontrapping. Set*

$$z_s = x_s - x_0 - \frac{3}{2}\int_0^s |\xi_\sigma|^{-1/2}\xi_\sigma\,\mathrm{d}\sigma.$$

Then there exists $(z_{+\infty},\xi_{+\infty}) \in \mathbb{R}^d \times (\mathbb{R}^d\backslash 0)$ such that

$$\lim_{s\to+\infty}(z_s,\xi_s) = (z_{+\infty},\xi_{+\infty}).$$

Consequently, by Lemma 6.7, let $(x'_s,\xi'_s) = \mathcal{G}_s(x_0,\xi_0)$, and then

$$\lim_{s\to+\infty}\xi'_s = \xi_{+\infty}.$$

Proof. Because $\{(x_s,\xi_s)\}_{s\in\mathbb{R}}$ is forwardly nontrapping and we only consider the limiting behavior when $s\to+\infty$, we may assume that $\varepsilon_0 := \|\langle x\rangle\nabla^2\eta_0\|_{L^\infty}$ is sufficiently small. As $\nabla\eta_0 \in L^\infty$, we have $H(\,\cdot\,,\xi) \simeq |\xi|^{3/2}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}s}(x_s\cdot\xi_s) = \partial_\xi H(x_s,\xi_s)\cdot\xi_s - x_s\cdot\partial_x H(x_s,\xi_s),$$

where

$$\partial_\xi H(x_s,\xi_s)\cdot\xi_s = \tfrac{3}{2}H(x_s,\xi_s) = \tfrac{3}{2}H(x_0,\xi_0) \simeq |\xi_0|^{3/2}$$

and

$$\begin{aligned}\partial_x H(x_s,\xi_s) &= \tfrac{3}{4}H(x_s,\xi_s)^{-1/3}\partial_x G(x_s,\xi_s)\\ &= \tfrac{3}{4}H(x_s,\xi_s)^{-1/3}\bigg(\frac{2\nabla\eta_0\cdot\xi_s}{1+|\nabla\eta_0|^2}\nabla^2\eta_0\xi_s - \frac{2(\nabla\eta_0\cdot\xi_s)^2}{(1+|\nabla\eta_0|^2)^2}\nabla^2\eta_0\nabla\eta_0\bigg)\bigg|_{x=x_s}.\end{aligned}$$

Therefore

$$x_s\cdot\partial_x H(x_s,\xi_s) = \mathcal{O}(\varepsilon|\xi_s|^{3/2}) = \mathcal{O}(\varepsilon|\xi_0|^2),$$

and consequently,

$$\frac{\mathrm{d}}{\mathrm{d}s}(x_s\cdot\xi_s) \gtrsim |\xi_0|^{3/2}. \tag{6-7}$$

So, for any bounded set $B\subset\mathbb{R}^d$,

$$\lambda(s\geq 0 : x_s\in B) \lesssim \frac{\sup\{|x\cdot\xi| : (x,\xi)\in B\times\mathbb{R}^d,\ H(x,\xi) = H(x_0,\xi_0)\}}{|\xi_0|^{3/2}} \lesssim \sup_{x\in B}|x|\langle\xi_0\rangle^{-1/2}, \tag{6-8}$$

where λ is the Lebesgue measure on $\mathbb{R}$. Let

$$E(x,\xi) = H(x,\xi) - |\xi|^{3/2}.$$

Then by the hypothesis of the decay of η_0, we have $E \in \Gamma^{3/2,1}_{-1-\epsilon,0}$. By the definition of z_s, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}(z_s, \xi_s) = (\partial_\xi E, -\partial_x E)(x_s, \xi_s) = \mathcal{O}(\langle x_s\rangle^{-1-\epsilon}),$$

where we used the conversation of $H(x_s, \xi_s)$ to deduce the boundedness of ξ_s. By (6-8),

$$\int_0^\infty \langle x_s\rangle^{-1-\epsilon}\,\mathrm{d}s = (1+\epsilon)\int_0^\infty t^\epsilon \lambda(s \ge 0 : \langle x_s\rangle^{-1} > t)\,\mathrm{d}t \lesssim \int_0^1 t^\epsilon \sqrt{t^{-2}-1}\,\mathrm{d}t < \infty.$$

Therefore, for any $0 < s^- < s^+$ with $s^- \to \infty$,

$$|(z_{s^+}, \xi_{s^+}) - (z_{s^-}, \xi_{s^-})| \lesssim \int_{s^-}^{s^+} \langle x_\sigma\rangle^{-1-\epsilon}\,\mathrm{d}\sigma \to 0,$$

implying that (x_s, ξ_s) is a Cauchy sequence as $s \to \infty$. □

6C2. *Construction of symbol.* For $h \ge 0$, and $h^{1/2}s \le T$. Set

$$H_h(s, x, \xi) = \gamma^{(3/2)}(h^{1/2}s, x, \xi),$$

so in particular $H(x, \xi) \equiv H_0(s, x, \xi)$. For $h > 0$, the semiclassical time variable $s = h^{-1/2}t$ was inspired by Lebeau [Lebeau 1992]; see also [Zhu 2020] for an application in theory of control for water waves.

For $a \in C^\infty([0, h^{-1/2}T) \times \mathbb{R}^{2d})$, set

$$\mathscr{L}^\pm_{h,s} a = \partial_s a \pm \{H_h, a\}.$$

Lemma 6.9. *Suppose that, for some* $\epsilon > 0$, $\nabla\eta_0 \in W^{0,\infty}_{1/2+\epsilon}$, $\nabla^2\eta_0 \in W^{0,\infty}_{1+\epsilon}$, $\nabla^3\eta_0 \in W^{0,\infty}_{3/2+\epsilon}$. *Let* $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d\backslash 0)$ *such that the cogeodesic* $\{(x_s, \xi_s) = \Phi_s(x_0, \xi_0)\}_{s\in\mathbb{R}}$ *is forwardly nontrapping. Then there exists* $s_0 > 0$, $K > 0$ *and*

$$\chi^\pm \in W^{1,\infty}(\mathbb{R}_{\ge 0}, \Upsilon^{\mu-K-\tilde{d}}) \cap W^{1,\infty}(\mathbb{R}_{\ge s_0}, S^{-\infty}_0) \tag{6-9}$$

in the sense that

$$\|N^{\mu-K-\tilde{d}}(\chi^\pm)\|_{L^\infty(\mathbb{R}_{\ge 0})} + \|N^{\mu-K-\tilde{d}}(\partial_s\chi^\pm)\|_{L^\infty(\mathbb{R}_{\ge 0})} < +\infty,$$

and satisfies the following conditions:

(1) $\chi^\pm(0, x, \xi) \in S^{-\infty}_{-\infty}$ *is elliptic at* $(x_0, \pm\xi_0)$.

(2) *For all* $t_0 > 0$, $\chi^\pm\big(s, \frac{s}{t_0}x, \xi\big) \in S^{-\infty}_{-\infty}$ *is elliptic at* $\big(\frac{3}{2}t_0|\xi_\infty|^{-1/2}\xi_\infty, \pm\xi_\infty\big)$ *for sufficiently large* s.

(3) *If* Ω *is a neighborhood of* $\big(\frac{3}{2}t_0|\xi_\infty|^{-1/2}\xi_\infty, \pm\xi_\infty\big)$, *then* $\chi^\pm$ *can be chosen such that*

$$\operatorname{supp}\chi^\pm\Big(s, \frac{s}{t_0}x, \xi\Big) \subset \Omega$$

for sufficiently large s.

Moreover, if $(\eta, \psi) \in \mathcal{H}^{\mu+1/2}_k \times \mathcal{H}^\mu_k$, *with* $\mu > 3 + \frac{d}{2}$ *and* $m \ge 2$, *then*

$$\mathscr{L}^\pm_{h,s}\chi^\pm \in L^\infty([0, h^{-1/2}T], \langle x\rangle^{-1}\Upsilon^{\mu-K-\tilde{d}-1})$$

and

$$\mathscr{L}^\pm_{h,s}\chi^\pm \ge \mathcal{O}(h^{1/2})_{L^\infty([0,h^{-1/2}T],\langle x\rangle^{-1}\Upsilon^{\mu-K-\tilde{d}-1})}. \tag{6-10}$$

Proof. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ such that

(i) $\phi \geq 0,\ \phi(x) = 1$ for $|x| \leq \frac{1}{2}$ and $\phi(x) = 0$ for $|x| \geq 1,\ \operatorname{supp}\phi = \{|x| \leq 1\}$,

(ii) $x \cdot \nabla\phi(x) \leq 0$ for all $x \in \mathbb{R}^d$,

(iii) $y \cdot \nabla\phi(x) = 0$ for all $x, y \in \mathbb{R}^d$, with $x \cdot y = 0$.

Such ϕ can be constructed by setting $\phi(x) = \varphi(|x|)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies $0 \leq \varphi \leq 1,\ \varphi(z) = 1$ if $z \leq \frac{1}{2},\ \varphi(z) = 0$ if $z \geq 1$. For $\rho > 0,\ \delta > 0,\ \lambda > 0,\ \nu > 0$ and sufficiently large $s > 0$, set

$$\tilde{\chi}^\pm(s, x, \xi) = \phi\left(\frac{x - x_s}{\rho\lambda\delta s}\right)\phi\left(\frac{\xi \mp \xi_s}{\rho(\delta - s^{-\nu})}\right).$$

We verify that $\mathscr{L}_{0,s}^\pm \tilde{\chi}^\pm(s, \cdot) \geq 0$ for $s > 0$ sufficient large. Indeed,

$$\begin{aligned}\mathscr{L}_{0,s}^\pm \tilde{\chi}^\pm(s, x, \xi) = &\left(\pm\frac{\partial_\xi H(x, \xi) - \partial_\xi H(x_s, \mp\xi_s)}{\rho\lambda\delta s} - \frac{x - x_s}{\rho\lambda\delta s^2}\right)\nabla\phi\left(\frac{x - x_s}{\rho\lambda\delta s}\right)\phi\left(\frac{\xi \mp \xi_s}{\rho(\delta - s^{-\nu})}\right)\\ &+\left(\pm\frac{\partial_x H(x_s, \mp\xi_s) - \partial_x H(x, \xi)}{\rho(\delta - s^{-\nu})} - \nu\frac{\xi \mp \xi_s}{\rho(\delta - s^{-\nu})^2 s^{\nu+1}}\right)\phi\left(\frac{x - x_s}{\rho\lambda\delta s}\right)\nabla\phi\left(\frac{\xi \mp \xi_s}{\rho(\delta - s^{-\nu})}\right).\end{aligned}$$

By (i),

$$\begin{aligned}\operatorname{supp}\phi\left(\frac{\cdot - x_s}{\rho\lambda\delta s}\right) &\subset \{x \in \mathbb{R}^d : |x - x_s| \leq \rho\lambda\delta s\},\\ \operatorname{supp}\phi\left(\frac{\cdot \mp \xi_s}{\rho(\delta - s^{-\nu})}\right) &\subset \{\xi \in \mathbb{R}^d : |\xi \mp \xi_s| \leq \rho(\delta - s^{-\nu})\},\\ \operatorname{supp}\nabla\phi\left(\frac{\cdot - x_s}{\rho\lambda\delta s}\right) &\subset \left\{x \in \mathbb{R}^d : \tfrac{1}{2}\rho\lambda\delta s \leq |x - x_s| \leq \rho\lambda\delta s\right\},\\ \operatorname{supp}\nabla\phi\left(\frac{\cdot \mp \xi_s}{\rho(\delta - s^{-\nu})}\right) &\subset \left\{\xi \in \mathbb{R}^d : \tfrac{1}{2}\rho(\delta - s^{-\nu}) \leq |\xi \mp \xi_s| \leq \rho(\delta - s^{-\nu})\right\}.\end{aligned}$$

By Lemma 6.8,

$$x_s = x_0 + \frac{3}{2}\int_0^s |\xi_\sigma|^{-1/2}\xi_\sigma \, \mathrm{d}\sigma + z_s = \frac{3}{2}s|\xi_\infty|^{-1/2}\xi_\infty + o(s).$$

Therefore, by writing

$$\tilde{\chi}^\pm\left(s, \frac{s}{t_0}x, \xi\right) = \phi\left(\frac{x - \frac{3}{2}t_0|\xi_\infty|^{-1/2}\xi_\infty + o(1)}{\rho\lambda\delta t_0}\right)\phi\left(\frac{\xi \mp \xi_\infty + o(1)}{\rho(\delta - s^{-\nu})}\right),$$

we see that $\tilde{\chi}^\pm\big(s, \frac{s}{t_0}x, \xi\big)$ is elliptic at $\big(\frac{3}{2}t_0|\xi_\infty|^{-1/2}\xi_\infty, \pm\xi_\infty\big)$ for sufficiently large s. Moreover, if $\rho\lambda\delta$ is sufficiently small and s is sufficiently large, then

$$\operatorname{supp}\phi\left(\frac{\cdot - x_s}{\rho\lambda\delta s}\right) \subset \{x \in \mathbb{R}^d : |x| \gtrsim s\}.$$

Therefore, by the hypothesis on η_0, we have, for $(x, \xi) \in \operatorname{supp}\tilde{\chi}^\pm(s, \cdot)$,

$$\nabla_{x\xi}^2 H(x, \xi) = \begin{pmatrix} \nabla_x^2 H & \nabla_x\nabla_\xi H \\ \nabla_\xi\nabla_x H & \nabla_\xi^2 H \end{pmatrix}(x, \xi) = \begin{pmatrix} \mathcal{O}(s^{-2-\epsilon}) & \mathcal{O}(s^{-3/2-\epsilon}) \\ \mathcal{O}(s^{-3/2-\epsilon}) & \mathcal{O}(1) \end{pmatrix},$$

and consequently, by the finite increment formula,

$$|\partial_\xi H(x_s, \mp\xi_s) - \partial_\xi H(x,\xi)| \lesssim s^{-3/2-\epsilon}|x-x_s| + |\xi \mp \xi_s| \lesssim s^{-1/2-\epsilon}\rho\lambda\delta + \rho\delta,$$
$$|\partial_x H(x_s, \mp\xi_s) - \partial_x H(x,\xi)| \lesssim s^{-2-\epsilon}|x-x_s| + s^{-3/2-\epsilon}|\xi \mp \xi_s| \lesssim \rho\lambda\delta s^{-1-\epsilon} + \rho\delta s^{-3/2-\epsilon}.$$

By (iii) and the estimates above,

$$\begin{aligned}
&(\partial_\xi H(x,\xi) - \partial_\xi H(x_s,\mp\xi_s))\cdot\nabla\phi\left(\frac{x-x_s}{\rho\lambda\delta s}\right)\\
&\qquad = (\partial_\xi H(x,\xi) - \partial_\xi H(x_s,\mp\xi_s))\cdot\frac{x-x_s}{|x-x_s|^2}(x-x_s)\cdot\nabla\phi\left(\frac{x-x_s}{\rho\lambda\delta s}\right)\\
&\qquad = \mathcal{O}(s^{-3/2-\epsilon} + \lambda^{-1}s^{-1})(x-x_s)\cdot\nabla\phi\left(\frac{x-x_s}{\rho\lambda\delta s}\right),\\
&(\partial_x H(x_s,\mp\xi_s) - \partial_x H(x,\xi))\cdot\nabla\phi\left(\frac{\xi\mp\xi_s}{\rho(\delta - s^{-\nu})}\right)\\
&\qquad = (\partial_x H(x_s,\mp\xi_s) - \partial_x H(x,\xi))\cdot\frac{\xi\mp\xi_s}{|\xi\mp\xi_s|^2}(\xi\mp\xi_s)\cdot\nabla\phi\left(\frac{\xi\mp\xi_s}{\rho(\delta - s^{-\nu})}\right)\\
&\qquad = \mathcal{O}(\lambda s^{-1-\epsilon} + s^{-3/2-\epsilon})(\xi\mp\xi_s)\cdot\nabla\phi\left(\frac{\xi\mp\xi_s}{\rho(\delta - s^{-\nu})}\right).
\end{aligned}$$

Finally, we fix $0 < \nu < \epsilon$, $\delta > 0$. Then, when λ is sufficiently large, and $s \geq s_0 - 1 > 0$, with s_0 being sufficiently large, by (ii),

$$\begin{aligned}
\mathscr{L}^\pm_{0,s}\tilde{\chi}^\pm = &-\frac{1+\mathcal{O}(s^{-1/2-\epsilon}+\lambda^{-1})}{\rho\lambda\delta s^2}(x-x_s)\cdot\nabla\phi\left(\frac{x-x_s}{\rho\lambda\delta s}\right)\phi\left(\frac{\xi\mp\xi_s}{\rho(\delta-s^{-\nu})}\right)\\
&-\frac{\nu-\mathcal{O}(\lambda)s^{\nu-\epsilon}}{\rho(\delta-s^{-\nu})^2 s^{\nu+1}}(\xi\mp\xi_s)\cdot\phi\left(\frac{x-x_s}{\rho\lambda\delta s}\right)\nabla\phi\left(\frac{\xi\mp\xi_s}{\rho(\delta-s^{-\nu})}\right) \geq 0. \qquad (6\text{-}11)
\end{aligned}$$

We verify as in Lemma 3.2 that

$$\tilde{\chi}^\pm \in W^{\infty,\infty}(\mathbb{R}_{\geq s_0}, S^{-\infty}_0), \qquad \mathscr{L}^\pm_{0,s}\tilde{\chi}^\pm \in W^{\infty,\infty}(\mathbb{R}_{\geq s_0}, \Gamma^{-\infty,\mu-K-\tilde{d}}_{-1,0}).$$

We then choose $\rho > 0$ sufficiently small such that $\rho\lambda\delta$ is small and that $\operatorname{supp}\tilde{\chi}^\pm\big(s, \frac{s}{t_0}x, \xi\big) \subset \Omega$ when s is large. Next, we set, for $s \geq s_0$,

$$\chi^\pm(s,x,\xi) = \tilde{\chi}^\pm(s,x,\xi).$$

To define $\chi^\pm$ for $s \leq s_0$, we choose $\rho \in C^\infty(\mathbb{R})$ such that $0 \leq \rho \leq 1$, $\rho(s) = 1$ for $s \geq s_0$, and $\rho(s) = 0$ for $s \leq s_0 - \alpha$ for some small $\alpha > 0$ to be specified later, and solve the transport equation on $[0, s_0]$,

$$\mathscr{L}^\pm_{0,s}\chi^\pm(s,x,\xi) = \rho(s)\mathscr{L}^\pm_{0,s}\tilde{\chi}^\pm(s,x,\xi), \qquad \chi^\pm(s_0,x,\xi) = \tilde{\chi}^\pm(s_0,x,\xi).$$

Because the vector field involved in the definition of $\mathscr{L}^\pm_{0,s}$ is in $W^{\mu-K-\tilde{d},\infty}$ with respect to the x-variable, we deduce that $\chi^\pm \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \Upsilon^{\mu-K-\tilde{d}})$ and thus $\chi^\pm$ satisfies (6-9). Clearly

$$\mathscr{L}^\pm_{0,s}\chi^\pm \geq 0. \qquad (6\text{-}12)$$

Moreover, because

$$\chi^{\pm}(s,x,\xi)=\tilde{\chi}^{\pm}(s_0,\Phi_{\pm(s_0-s)}(x,\xi))-\int_s^{s_0}\rho(\sigma)\mathscr{L}^{\pm}_{0,s}\tilde{\chi}^{\pm}(\sigma,\Phi_{\pm(\sigma-s)}(x,\xi))\,\mathrm{d}\sigma,$$

if we choose $\alpha>0$ sufficiently small, then

$$\begin{aligned}\chi^{\pm}(0,x_0,\pm\xi_0)&=\tilde{\chi}^{\pm}(s_0,x_{s_0},\pm\xi_{s_0})-\int_{s_0-\alpha}^{s_0}\rho(\sigma)\mathscr{L}^{\pm}_{0,s}\tilde{\chi}^{\pm}(\sigma,x_\sigma,\pm\xi_\sigma)\,\mathrm{d}\sigma\\&\geq 1-\|\mathscr{L}^{\pm}_{0,s}\tilde{\chi}^{\pm}(\sigma,x_\sigma,\pm\xi_\sigma)\|_{L^1_\sigma([s_0-\alpha,s_0])}>0.\end{aligned}$$

Therefore, $\chi^{\pm}(0,\cdot)$ is elliptic at $(x_0,\pm\xi_0)$.

To estimate $\mathscr{L}^{\pm}_{h,s}\chi^{\pm}$, we use

$$\begin{aligned}H_h(s,x,\xi)-H_0(s,x,\xi)&=H_h(s,x,\xi)-H_h(0,x,\xi)\\&=\int_0^s(\partial_s H_h)(\sigma,x,\xi)\,\mathrm{d}\sigma=h^{1/2}\int_0^s(\partial_t\gamma^{(3/2)})(h^{1/2}\sigma,x,\xi)\,\mathrm{d}\sigma,\end{aligned}$$

and write

$$\begin{aligned}\mathscr{L}^{\pm}_{h,s}\chi^{\pm}(s,\cdot)-\mathscr{L}^{\pm}_{0,s}\chi^{\pm}(s,\cdot)&=\pm\{H_h-H_0,\chi^{\pm}\}(s,\cdot)\\&=\pm h^{1/2}\int_0^s\{\partial_t\gamma^{(3/2)}(h^{1/2}\sigma,\cdot),\chi^{\pm}(s,\cdot)\}\,\mathrm{d}\sigma.\end{aligned}$$

Observe that

$$\partial_t\gamma^{(3/2)}=-\frac{3}{2}\biggl(|\xi|^2-\frac{(\nabla\eta\cdot\xi)^2}{1+|\nabla\eta|^2}\biggr)^{-1/4}\biggl(\frac{\nabla\eta\cdot\xi}{1+|\nabla\eta|^2}\nabla G(\eta)\psi\cdot\xi-\frac{(\nabla\eta\cdot\xi)^2}{(1+|\nabla\eta|^2)^2}\nabla G(\eta)\psi\cdot\nabla\eta\biggr).$$

By hypothesis and Proposition 5.4, $\nabla G(\eta)\psi\in\mathcal{H}^{\mu-2,1/2}_k\subset H^{\mu-3}_2$ as $k\geq 2$. Therefore,

$$\partial_t\gamma^{(3/2)}(h^{1/2}\cdot,\cdot)\in L^\infty([0,h^{-1/2}T],\Gamma^{3/2,\mu-K-\tilde{d}}_{-2,0}).$$

Using $|x|\sim s$ on $\operatorname{supp}\chi^{\pm}(s,\cdot)$, we have, uniformly for all $s\in[0,h^{-1/2}T]$,

$$\langle s\rangle\{\partial_t\gamma^{(3/2)}(h^{1/2}\sigma,\cdot),\chi^{\pm}(s,\cdot)\}\in L^\infty_\sigma([0,h^{-1/2}T],\langle x\rangle^{-1}\Upsilon^{\mu-K-\tilde{d}-1}).$$

Therefore,

$$\begin{aligned}\mathscr{L}^{\pm}_{h,s}\chi^{\pm}(s,\cdot)-\mathscr{L}^{\pm}_{0,s}\chi^{\pm}(s,\cdot)&=\pm h^{1/2}\langle s\rangle^{-1}\int_0^s\mathcal{O}(1)_{L^\infty([0,h^{-1/2}T],\langle x\rangle^{-1}\Upsilon^{\mu-K-\tilde{d}-1})}\,\mathrm{d}\sigma\\&=\pm h^{1/2}\langle s\rangle^{-1}\mathcal{O}(s)_{\langle x\rangle^{-1}\Upsilon^{\mu-K-\tilde{d}-1}}\\&=\mathcal{O}(h^{1/2})_{\langle x\rangle^{-1}\Upsilon^{\mu-K-\tilde{d}-1}},\end{aligned}$$

which, together with (6-12), proves (6-10). □

6C3. *Propagation.* Now we prove Theorem 1.9. By Lemmas 6.5 and 6.7, it suffices to prove the following propagation theorem for u defined as in (6-2).

Theorem 6.10. *Under the hypothesis of Theorem 1.9, let u be defined as (6-2). Let*

$$(x_0,\xi_0)\in\mathrm{WF}^{\sigma}_{0,1}(u_0)^\circ,$$

with $0 \le \sigma < \min\{(\mu - K - \tilde{d})/2, 3k/2\}$ *for some* $K > 0$, *such that the cogeodesic* $\{(x_s, \xi_s) = \Phi_s(x_0, \xi_0)\}_{s\in\mathbb{R}}$ *is forwardly nontrapping. Set*

$$\xi_\infty = \lim_{s\to+\infty} \xi_s.$$

Then, for all $t_0 \in (0, T]$, *we have*

$$\left(\tfrac{3}{2}t_0|\xi_\infty|^{-1/2}\xi_\infty, \xi_\infty\right) \in \mathrm{WF}^{\sigma}_{1/2,1}(u(t_0)).$$

Under the semiclassical time variable $s = h^{-1/2}t$, (6-1) becomes

$$(\partial_s + h^{1/2}\mathcal{P}_V \cdot \nabla + h^{1/2}\mathcal{P}_Q)w + ih^{1/2}\begin{pmatrix}\mathcal{P}_\gamma & 0\\ 0 & -\mathcal{P}_\gamma\end{pmatrix}w + \frac{ih^{1/2}g}{2}\mathcal{P}_\zeta\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}w = F_h = \mathcal{O}(h^{1/2})_{H^{\mu-K-\tilde{d}}}$$

for some $K > 0$. We define $\mathcal{L}^h_s$, which applies to time-dependent operators $\mathcal{A} : \mathscr{S} \to \mathscr{S}'$,

$$\mathcal{L}^h_s\mathcal{A} = \partial_s\mathcal{A} + h^{1/2}\left[\mathcal{P}_V \cdot \nabla + \mathcal{P}_Q + i\mathcal{P}_\gamma\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{ig}{2}\mathcal{P}_\zeta\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}, \mathcal{A}\right].$$

We also define $\mathscr{L}^h_s$, which applies to symbols of the diagonal form $A = \left(\begin{smallmatrix}A^+ & 0\\ 0 & A^-\end{smallmatrix}\right)$:

$$\mathscr{L}^h_s A = \begin{pmatrix}\mathscr{L}^+_{h,s}A^+ & 0\\ 0 & \mathscr{L}^-_{h,s}A^-\end{pmatrix}.$$

Proof of Theorem 6.10. We shall from now on write $\rho = \mu - K - \tilde{d}$ for some sufficiently large $K > 0$, also define $I_h = [0, h^{-1/2}T]$ and

$$Y^\rho_h = L^\infty\left(I_h, M_{2\times 2}\left(\sum_{j=0}^{\rho} h^j \Upsilon^{\rho-j}\right)\right)$$

for simplicity. More precisely, a symbol $A_h = \sum_{j=0}^{\rho} h^j A^j_h \in Y^\rho_h$ if

$$\sup_{h\in(0,1]}\ \sup_{s\in[0,h^{-1/2}T]} N^{\rho-j}(A^j_h) < +\infty,$$

where the norm $N^{\rho-j}(A^j_h)$ is applied to every component of A^j_h. Choose a strictly increasing sequence $\{\lambda_j\}_{j\ge0} \subset [1, 1+\epsilon)$ with $\epsilon > 0$ being sufficiently small. Define $\chi^\pm_j$ as in Lemma 6.9, where we replace ϕ with $\phi(\cdot/\lambda_j)$. Then

$$\operatorname{supp}\chi^\pm_j \subset \{\chi^\pm_{j+1} > 0\}$$

for all $j \in \mathbb{N}$. Set

$$\chi_j = \begin{pmatrix}\chi^+_j & 0\\ 0 & \chi^-_j\end{pmatrix}.$$

We shall construct an operator $\mathcal{A}_h \in L^\infty(I_h, L^2 \to L^2)$ such that:

(1) $\mathcal{A}_h$ is a paradifferential operator; more precisely, there exists

$$A^\pm_h \in W^{1,\infty}(\mathbb{R}_{\ge0}, \Upsilon^{\rho+1}) \cap W^{1,\infty}(\mathbb{R}_{\ge s_0}, S^{-\infty}_0)$$

for some $s_0 > 0$, such that

$$\mathcal{A}_h - \mathcal{P}^h_{A_h} = \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2\to L^2)}, \quad A_h = \begin{pmatrix}A^+_h & 0\\ 0 & A^-_h\end{pmatrix}.$$

Moreover, we require that

$$\operatorname{supp} A_h^{\pm} \subset \bigcup_{j\geq 0} \operatorname{supp} \chi_j^{\pm}.$$

(2) $A_h^{\pm}(0,x,\xi)$ is elliptic at $(x_0, \pm\xi_0)$.

(3) $A_h^{\pm}\big(s, \frac{s}{t_0}x, \xi\big) \in S^{-\infty}_{-\infty}$ is elliptic at $\big(\frac{3}{2}t_0|\xi_\infty|^{-1/2}\xi_\infty, \xi_\infty\big)$ for $s>0$ sufficiently large.

(4) $\mathcal{L}_s^h \mathcal{A}_h \geq \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2\to L^2)}$.

We shall construct $\mathcal{A}_h$ of the form

$$\mathcal{A}_h = \sum_{j\geq 0}^{2\rho} h^{j/2}\varphi^j \mathcal{A}_h^j,$$

where $\varphi \in P_j$, recalling the definition (3-6), and $\mathcal{A}_h^j \in L^\infty(I_h, L^2 \to L^2)$. We begin by setting

$$\mathcal{A}_h^0 = (\mathcal{P}_{\chi_0}^h)^*\mathcal{P}_{\chi_0}^h, \quad \varphi^0 \equiv 1.$$

Therefore, by the symbolic calculus, Lemma 6.9 and Corollary 4.32 (observe that the symbol of $\mathcal{A}_h^0$ belongs to σ_0, and that γ is a sum of homogeneous symbols),

$$\partial_s \mathcal{A}_h^0 + h^{1/2}\left[i\mathcal{P}_\gamma \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \mathcal{A}_h^0\right] = 2\mathcal{P}^h_{\chi_0 \mathscr{L}_s^h \chi_0} + h\mathcal{P}^h_{b_h^0} + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2\to L^2)}$$

for some symbol b_h^0 such that $\langle x\rangle b_h^0 \in Y_h^\rho$. This $\langle x\rangle$ factor comes from the spatial decay of $\partial_{x,\xi}\gamma$. Moreover, we have $\operatorname{supp} b_h^0 \subset \operatorname{supp} \chi_0$, which implies $\langle s\rangle b_h^0 \in Y_h^\rho$. Similarly,

$$h^{1/2}[\mathcal{P}_V\cdot\nabla, \mathcal{A}_h^0] = h^{1/2}\mathcal{P}^h_{b_h^1} + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2\to L^2)},$$

where $\langle s\rangle b_h^1 \in Y_h^\rho$, with $\operatorname{supp} b_h^1 \subset \operatorname{supp}\chi_0$. Be careful that, because Q and $\mathcal{P}_\zeta\left(\begin{smallmatrix}1 & -1\\ 1 & -1\end{smallmatrix}\right)$ are not diagonal matrices, their commutators with $\mathcal{A}_h^0$ do not gain an extra h, for the principal symbols do not cancel each other. So,

$$h^{1/2}[\mathcal{P}_Q, \mathcal{A}_h^0] = h^{1/2}\mathcal{P}^h_{b_h^2} + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2\to L^2)},$$

$$h^{1/2}\left[\mathcal{P}_\zeta \begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}, \mathcal{A}_h^0\right] = h\mathcal{P}^h_{b_h^3} + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2\to L^2)},$$

where $\langle s\rangle b_h^2, \langle s\rangle b_h^3 \in Y_h^\rho$, with $\operatorname{supp} b_h^2 \cup \operatorname{supp} b_h^3 \subset \operatorname{supp}\chi_0$. By Lemma 6.9,

$$\chi_0 \mathscr{L}_s^h \chi_0 \geq h^{1/2} b_h^4,$$

where $\langle s\rangle b_h^4 \in Y_h^\rho$, with $\operatorname{supp} b_h^4 \subset \operatorname{supp}\chi_0$. Therefore, combining the idea described above (3-10) and the paradifferential Gårding inequality (Lemma 4.40, where we take $\epsilon = \frac{1}{2}$),

$$\mathcal{P}^h_{\chi_0 \mathscr{L}_s^h \chi_0} - h^{1/2}\mathcal{P}^4_{b_h^4} \geq h^{1/2}\mathcal{P}^h_{b_h^5} + \mathcal{O}(h^\rho)_{L^2\to L^2}$$

for some $b_h^5 \in Y_h^\rho$ with $\operatorname{supp} b_h^5 \subset \{\chi_1 > 0\}$. In fact, choose $c_h \in L^\infty(\mathbb{R}_{\geq 0}, S_0^{-\infty})$ such that

$$\operatorname{supp} a_h \subset \{c_h = 1\} \subset \operatorname{supp} c_h \subset \operatorname{supp}\chi_1.$$

Then, for all $v \in L^2$, we have

$$\langle v, (\mathcal{P}^h_{\chi_0 \mathscr{L}^h_s \chi_0} - h^{1/2}\mathcal{P}^4_{b^4_h})v\rangle_{L^2} = \langle \mathcal{P}_{c_h} v, (\mathcal{P}^h_{\chi_0 \mathscr{L}^h_s \chi_0} - h^{1/2}\mathcal{P}^4_{b^4_h})\mathcal{P}_{c_h} v\rangle_{L^2} + \mathcal{O}(h^\rho)$$
$$\gtrsim -Ch^{1/2}\|\mathcal{P}_{c_h} v\|^2_{L^2} + \mathcal{O}(h^\rho).$$

Therefore, it suffices to choose b^5_h such that

$$\mathcal{P}_{b^5_h} - C\mathcal{P}^*_{c_h}\mathcal{P}_{c_h} = \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \to L^2)},$$

which can be achieved by Propositions 4.26 and 4.27. Set

$$\alpha^0_h = \langle s\rangle(b^1_h + b^2_h + 2b^4_h + 2b^5_h) \in Y^\rho_h, \quad \beta^0_h = \langle s\rangle(b^0_h + b^3_h) \in Y^\rho_h.$$

Then

$$\mathcal{L}^h_s \mathcal{A}^0_h \geq h^{1/2}\langle s\rangle^{-1}\mathcal{P}^h_{\alpha^0_h + h^{1/2}\beta^0_h} + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \to L^2)}.$$

Suppose that we have found $\mathcal{A}^j_h \in L^\infty(I_h, L^2 \to L^2)$, $\varphi^j \in P_j$ for $j = 0, \dots, \ell - 1$, and $\psi^{\ell-1} \in P_{\ell-1}$, $\alpha^{\ell-1}_h, \beta^{\ell-1}_h \in Y^\rho_h$, with

$$\operatorname{supp} \alpha^{\ell-1}_h \cup \operatorname{supp} \beta^{\ell-1}_h \subset \{\chi_\ell > 0\},$$

such that

$$\mathcal{L}^h_s\left(\sum_{j=0}^{\ell-1} h^{j/2}\varphi^j \mathcal{A}^j_h\right) \geq h^{\ell/2}\langle s\rangle^{-1}\psi^{\ell-1}\mathcal{P}^h_{\alpha^{\ell-1}_h + h^{1/2}\beta^{\ell-1}_h} + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \to L^2)}. \tag{6-13}$$

Then as in the proof of Theorem 1.4(2), we set

$$\varphi^\ell(s) = \int_0^s (1+\sigma)^{-1}\psi^{\ell-1}(\sigma)\,\mathrm{d}\sigma, \quad \mathcal{A}^\ell_h = C_\ell \varphi^\ell \mathcal{P}^h_{\chi_\ell},$$

where the constant C_ℓ is sufficiently large, such that by Lemma 6.9, in the sense of positivity of matrices,

$$C_\ell \mathscr{L}^s_h(\varphi^\ell \chi_\ell) = C_\ell (1+s)^{-1}\psi^{\ell-1}\chi_\ell + C_\ell \varphi^\ell \mathscr{L}^s_h \chi_\ell$$
$$\geq \langle s\rangle^{-1}\psi^{\ell-1}\alpha^{\ell-1}_h + \varphi^\ell h^{1/2}\langle s\rangle^{-1}\tilde{\beta}^\ell_h$$

for some $\tilde{\beta}^\ell_h \in Y^\rho_h$. By the paradifferential Gårding inequality, and a routine construction of a parametrix, we find $\tilde{\alpha}^\ell_h \in Y^\rho_h$, with $\operatorname{supp}\tilde{\alpha}^\ell_h \subset \{\chi_{\ell+1} > 0\}$, such that

$$\mathcal{P}^h_{C_\ell \mathscr{L}^s_h(\varphi^\ell \chi_\ell)} - \langle s\rangle^{-1}\mathcal{P}^h_{\psi^{\ell-1}\alpha^{\ell-1}_h + h^{1/2}\varphi^\ell \tilde{\beta}^\ell_h} \geq h\langle s\rangle^{-1}\mathcal{P}^h_{(\psi^{\ell-1}+\varphi^\ell)\tilde{\alpha}^\ell_h} + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \to L^2)}.$$

Similar to the estimate of $\mathcal{A}^0_h$, by a symbolic calculus, we find $\underline{\alpha}^\ell_h, \underline{\beta}^\ell_h \in Y^\rho_h$, with

$$\operatorname{supp} \underline{\alpha}^\ell_h \cup \operatorname{supp} \underline{\beta}^\ell_h \subset \operatorname{supp} \chi_\ell$$

such that

$$\mathcal{L}^h_s \mathcal{A}^\ell_h = \mathcal{P}^h_{C_\ell \mathscr{L}^s_h(\varphi^\ell \chi_\ell)} + h^{1/2}\langle s\rangle^{-1}\varphi^\ell \mathcal{P}_{\underline{\alpha}^\ell_h + h^{1/2}\underline{\beta}^\ell_h} + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \to L^2)}.$$

Summing up the two inequalities above,

$$\mathcal{L}^h_s \mathcal{A}^\ell_h - \langle s\rangle^{-1}\psi^{\ell-1}\mathcal{P}^h_{\alpha^{\ell-1}_h} \geq h^{1/2}\langle s\rangle^{-1}\mathcal{P}^h_{\varphi^\ell(\underline{\alpha}^\ell_h + \tilde{\beta}^\ell_h) + h^{1/2}(\psi^{\ell-1}+\varphi^\ell)\tilde{\alpha}^\ell_h + h^{1/2}\varphi^\ell \underline{\beta}^\ell_h} + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \to L^2)}. \tag{6-14}$$

Therefore, combining (6-13) and (6-14),

$$\mathcal{L}_s^h\Bigg(\sum_{j=0}^{\ell} h^{j/2}\varphi^j\mathcal{A}_h^j\Bigg)\geq h^{(\ell+1)/2}\langle s\rangle^{-1}\psi^\ell\mathcal{P}^h_{\alpha_h^\ell+h^{1/2}\beta_h^\ell}+\mathcal{O}(h^\rho)_{L^\infty(I_h,L^2\to L^2)},$$

where

$$\psi^\ell=1+\psi^{\ell-1}+\varphi^\ell,\quad \alpha_h^\ell=\frac{\psi^{\ell-1}}{\psi^\ell}\beta_h^{\ell-1}+\frac{\phi^\ell}{\psi^\ell}(\underline{\alpha}_h^\ell+\tilde{\beta}_h^\ell),\quad \beta_h^\ell=\frac{\psi^{\ell-1}+\varphi^\ell}{\psi^\ell}\tilde{\alpha}_h^\ell+\frac{\varphi^\ell}{\psi^\ell}\underline{\beta}_h^\ell.$$

Thus we close the induction procedure.

To finish the proof, suppose that

$$\big(\tfrac{3}{2}t_0|\xi_\infty|^{-1/2}\xi_\infty,\xi_\infty\big)\notin \mathrm{WF}^\sigma_{1/2,1}(u(t_0)),$$
$$\big(\tfrac{3}{2}t_0|\xi_\infty|^{-1/2}\xi_\infty,-\xi_\infty\big)\notin \mathrm{WF}^\sigma_{1/2,1}(\overline{u(t_0)}).$$

By Lemma 6.9, we can choose ϕ such that, for sufficiently small $h>0$,

$$\operatorname{supp}\theta^{1/2,0}_{1/h,*}\chi_j^+|_{s=h^{-1/2}t_0}\subset\mathbb{R}^{2d}\setminus \mathrm{WF}^\sigma_{1/2,1}(u(t_0)),$$
$$\operatorname{supp}\theta^{1/2,0}_{1/h,*}\chi_j^-|_{s=h^{-1/2}t_0}\subset\mathbb{R}^{2d}\setminus \mathrm{WF}^\sigma_{1/2,1}(\overline{u(t_0)}).$$

So by Lemmas 4.34 and 2.14,

$$(\mathcal{A}_h w,w)_{L^2}|_{s=h^{-1/2}t_0}=\mathcal{O}(h^{2\sigma}).$$

By our construction, $\varphi^\ell(0)=0$ for all $\ell\geq 1$, so

$$\mathcal{A}_h|_{s=0}=\mathcal{A}_h^0|_{s=0}=(\mathcal{P}^h_{\chi_0})^*\mathcal{P}^h_{\chi_0}|_{s=0}.$$

Because $F_h=\mathcal{O}(h^{1/2})_{H^\rho}$, we have, by Lemma 2.15, that $\mathcal{A}_hF_h=\mathcal{O}(h^{\rho+1/2})_{L^2}$. Therefore, by (4),

$$\begin{aligned}\|\mathcal{P}^h_{\chi_0}w|_{s=0}\|^2_{L^2}&=\mathrm{Re}(\mathcal{A}_hw,w)_{L^2}|_{s=h^{-1/2}t_0}-\int_0^{h^{-1/2}t_0}\mathrm{Re}(\mathcal{L}_s^h\mathcal{A}_hw,w)_{L^2}\,\mathrm{d}s-\int_0^{h^{-1/2}t_0}\mathrm{Re}(\mathcal{A}_hF_h,w)_{L^2}\,\mathrm{d}s\\&\leq\mathcal{O}(h^{2\sigma})+\mathcal{O}(h^{\rho-1/2})=\mathcal{O}(h^{2\sigma}).\end{aligned}$$

Observe that $\chi_0|_{s=0}$ is of compact support with respect to x, and we have

$$\mathcal{P}^h_{\chi_0|_{s=0}}=T^h_{\beta_h}+\mathcal{O}(h^\rho)_{L^2\to L^2},$$

where

$$\beta_h=\sum_{j\geq 0}\psi_j\chi_0|_{s=0}\sharp_h\psi_j\in\sum_{j=0}^{\rho}h^j\Upsilon^{\rho-j}$$

is a finite summation. By Lemma 4.41 and (6-5), we conclude that, if $(x_0,\xi_0)\notin\mathrm{WF}^\sigma_{0,1}(u_0)$ provided $\sigma\leq\frac{3}{2}r$, where

$$r=\min\big\{\big[\tfrac{2}{3}(\mu-1-\tilde{d})\big],k\big\},$$

then under the hypothesis of theorem we have $r=k$. □

6D. ***Proof of Corollary 1.10.*** The case when $d=1$ is trivial. For the second case, we shall prove that, on any cogeodesic $\{(x_t,\xi_t)\}_{t\in\mathbb{R}}$,

$$\lim_{t\to+\infty} x_t\cdot\xi_t=\infty, \tag{6-15}$$

so no geodesics can be trapped. The proof of (6-15) is almost finished by the proof of Lemma 6.8. Indeed, similar calculations imply

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_t\cdot\xi_t)\gtrsim|\xi_0|^2.$$

List of notation

$\mathrm{WF}^\mu(u)$	wavefront set
$\mathrm{WF}^\mu_{\delta,\rho}(u)$	quasihomogeneous wavefront set
$\mathrm{Op}(a)$	pseudodifferential operator
$\mathrm{Op}_h(a)$	semiclassical pseudodifferential operator
$\mathrm{Op}_h^{\delta,\rho}(a)$	quasihomogeneous semiclassical pseudodifferential operator
T_a	paradifferential operator
$\mathcal{P}_a$	dyadic paradifferential operator
$\mathcal{P}_a^h$	semiclassical dyadic paradifferential operator
$\mathcal{P}_a^{h,\epsilon}$	quasihomogeneous semiclassical dyadic paradifferential operator
$a\sharp_h^{\delta,\rho}b$	composition of symbols
$\zeta_h^{\delta,\rho}a$	adjoint of symbols
$\mathscr{S}$, $\mathscr{S}'$	Schwartz function space and tempered distribution space
$\mathcal{H}_k^{\mu,\delta}$, $W_{k,\delta}^{r,\infty}$	weighted Sobolev spaces
$\Gamma^{m,r}$	paradifferential symbol class
$\Gamma^{m,r}_{k,\delta}$	weighted paradifferential symbol class
$\Sigma^{m,r}_{k,\delta}$	weighted paradifferential polysymbol class
$M^{m,r}$, $M^{m,r}_{k,\delta}$	symbol norm and weighted symbol norm
$\theta_h^{\delta,\rho}$	phase-space scaling operator

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SHIFT EQUIVALENCES THROUGH THE LENS OF CUNTZ–KRIEGER ALGEBRAS

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Motivated by Williams' problem of measuring novel differences between shift equivalence (SE) and strong shift equivalence (SSE), we introduce three equivalence relations that provide new ways to obstruct SSE while merely assuming SE.

Our shift equivalence relations arise from studying graph C*-algebras, where a variety of intermediary equivalence relations naturally arise. As a consequence we realize a goal sought after by Muhly, Pask and Tomforde, measure a delicate difference between SSE and SE in terms of Pimsner dilations for C*-correspondences of adjacency matrices, and use this distinction to refute a proof from a previous paper.

1. Introduction

Initially recognized in the 40's as the right object to model quantum phenomena, C*-algebras are applied today in a variety of areas including theoretical physics, topology, differential geometry and dynamical systems. Such applications drive the impetus for obtaining structural and classification results for C*-algebras, especially in relation with Elliott's classification programme [14; 45; 49; 50]. One fantastic application of C*-algebras in dynamics, using tools from classification of C*-algebras, is the classification of Cantor minimal systems up to orbit equivalence by their dimension groups [17; 18]. Similar to this, our work here deals with subtle invariants arising from C*-algebras associated to subshifts of finite type (SFTs), with the aim of distinguishing SFTs up to conjugacy.

In a seminal 1973 paper [47], Williams recast conjugacy and eventual conjugacy for SFTs purely in terms of equivalence relations between adjacency matrices of the directed graphs. These are called strong shift equivalence (SSE) and shift equivalence (SE) respectively. Williams expected SSE and SE to be the same [47, Proposition 7.2], but after around 25 years the last hope for a positive answer to Williams' problem, even under the most restrictive conditions, was extinguished by Kim and Roush [28]. Although these counterexamples are concrete, aperiodic and irreducible 7×7 matrices, showing that they are not strong shift equivalent requires an invariant which is very difficult to compute. Thus, finding new

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obstructions to strong shift equivalence when two matrices are only assumed to be shift equivalent is an important endeavor, even just for 2×2 matrices (see [31, Example 7.3.13]).

Definition 1.1 [47]. Let A and B be matrices indexed by sets V and W respectively, with (possibly infinite) cardinal entries. We say that A and B are:

(1) *Shift equivalent* with lag $m \in \mathbb{N} \setminus \{0\}$ if there are a $V \times W$ matrix R and a $W \times V$ matrix S with cardinal entries such that
$$A^m = RS, \quad B^m = SR,$$
$$SA = BS, \quad AR = RB.$$

(2) *Elementary shift related* if they are shift equivalent with lag 1.

(3) *Strong shift equivalent* if they are equivalent in the transitive closure of elementary shift relation.

In tandem with early attacks on Williams' problem, Cuntz and Krieger [8] created a bridgehead between symbolic dynamics and operator algebras, where several natural properties of subshifts of finite type are expressed through associated C*-algebras. In fact, by [8, Proposition 2.17] we know that strong shift equivalence of A and B implies that the Cuntz–Krieger C*-algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are stably isomorphic in a way preserving their gauge actions γ^A and γ^B and their diagonal subalgebras $\mathcal{D}_A$ and $\mathcal{D}_B$. On the other hand, by a theorem of Krieger [29] we know that the dimension group triples of SFTs are isomorphic if and only if the associated matrices are SE. Since these dimension group triples coincide with K-theoretical data of crossed products of Cuntz–Krieger C*-algebras by their gauge action, Krieger's theorem implies as a corollary (see Section 7) that if two Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are stably isomorphic in a way preserving their gauge actions γ^A and γ^B, then their defining adjacency matrices A and B are shift equivalent. Through the lens of Cuntz–Krieger algebras, this provides several natural equivalence relations between SSE and SE, and it then becomes important to orient them and determine whether they coincide with SSE, SE or perhaps a completely new equivalence relation strictly between SSE and SE. Such distinctions may pave the way towards more concrete and computable invariants that distinguish SFTs up to conjugacy.

In this paper we introduce, study and orient three equivalence relations that provide new ways of measuring the difference between SSE and SE. Before we discuss these, let us first mention the state of the art.

A partial converse of the corollary to Krieger's theorem was obtained by Bratteli and Kishimoto [4], using deep machinery from C*-algebra K-theory classification, involving the essential concept of Rokhlin towers. More precisely, using their work it can be shown that if A and B are two *aperiodic and irreducible* adjacency matrices then

$$A \text{ is SE to } B \quad \Longleftrightarrow \quad (\mathcal{O}_A \otimes \mathbb{K}, \gamma^A \otimes \mathrm{id}) \simeq (\mathcal{O}_B \otimes \mathbb{K}, \gamma^B \otimes \mathrm{id}). \tag{1-1}$$

This *converse to Krieger's corollary* for essential matrices remains a subtle and important classification problem in this line of research, and is one of the key motivating problems for our work here.

After a major undertaking pioneered by Matsumoto [33; 34], it is now known that several key concepts in symbolic dynamics may be fully understood in terms of operator algebraic descriptions. Indeed, both

SSE as well as flow equivalence (FE) of subshifts of finite type may be given an inherently operator-algebraic characterization. Suppose A and B are finite adjacency matrices defining two-sided SFTs (X_A, σ_A) and (X_B, σ_B) respectively (see Section 2). When A and B are irreducible, Matsumoto and Matui [35] established that

$$(X_A, \sigma_A) \text{ is FE to } (X_B, \sigma_B) \quad \iff \quad (\mathcal{O}_A \otimes \mathbb{K}, \mathcal{D}_A \otimes c_0) \simeq (\mathcal{O}_B \otimes \mathbb{K}, \mathcal{D}_B \otimes c_0),$$

where $\mathbb{K}$ is the C*-algebra of compact operators on $\ell^2(\mathbb{N})$ and c_0 is its subalgebra of diagonal operators. This was later extended to cover all two-sided SFTs by the first-named and third-named authors with Ortega and Restorff [6].

For conjugacy, the first-named author and Rout [5] proved that (X_A, σ_A) is conjugate to (X_B, σ_B) precisely when the associated Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_A$ are stably isomorphic in a way preserving both the gauge actions and the diagonals (this time with no additional restrictions on A and B). More precisely, and combining with Williams' characterization of conjugacy, we have

$$A \text{ is SSE to } B \quad \iff \quad (\mathcal{O}_A \otimes \mathbb{K}, \gamma^A \otimes \mathrm{id}, \mathcal{D}_A \otimes c_0) \simeq (\mathcal{O}_B \otimes \mathbb{K}, \gamma^B \otimes \mathrm{id}, \mathcal{D}_B \otimes c_0). \tag{1-2}$$

The equivalences in (1-1) and (1-2) provide a fresh perspective for symbolic dynamics via C*-algebras, and reinterpret Williams' problem via the counterexamples provided by Kim and Roush [27; 28]. More precisely, the examples of Kim and Roush show that two adjacency matrices can be shift equivalent and with flow equivalent two-sided SFTs without being strong shift equivalent. Hence, in terms of C*-algebras, this shows it is not always possible to trade in two isomorphisms — one respecting the diagonal, and one respecting the gauge action — for one which respects both.

Another research agenda motivating this work is the *graded isomorphism problem* of Hazrat from the theory of Leavitt path algebras, where one studies graded isomorphisms of pure algebras by means of graded K-theory. In work of Hazrat [19], it was shown that the relevant graded K-theoretical data is in fact the same as Krieger's dimension group triple, and Hazrat conjectured that this invariant is complete when the class of the unit is added to the dimension triple as part of the invariant. Thus, the converse to Krieger's corollary is a topological analogue of Hazrat's conjecture. Although substantial advances have been made [2; 3], Hazrat's conjecture remains elusive.

For a $V \times W$ matrix $F = [F_{ij}]$ with cardinal entries, we define

$$E_F := \{(v, w, \alpha) \mid 0 \leq \alpha < F_{vw},\ v \in V,\ w \in W\},$$

so that $r(v, w, \alpha) = w$ and $s(v, w, \alpha) = v$, α is an ordinal, and F_{vw} is interpreted as the least ordinal with cardinality F_{vw}. When $V = W$, this makes $G_F := (V, E_F)$ into a directed graph in its own right. For two matrices C over $V \times W$ and D over $W \times X$ with cardinal entries, we define the *fibered product*

$$E_C \times E_D := \{cd \mid c \in E_C,\ d \in E_D,\ r(c) = s(d)\}.$$

Note here that we write cd to mean the pair (c, d) for which $r(c) = s(d)$, which should be thought of as concatenation of edges, even if there are no actual graphs. When $V = W$, for any $n \in \mathbb{N}$ we denote by E_C^n the n-fold product of E_C with itself, so that E_C^n is naturally identified with E_{C^n}.

One might also consider $E_C \times E_D$ as a pullback of W, and indicate the role of W in the notation. We refrain from doing so for the sake of readability.

The following definition was dubbed "specified equivalence" by Nasu [38] in his study of shift equivalences between textile systems.

Definition 1.2. Let A and B be matrices with cardinal entries over $V \times W$. A *path isomorphism* is a bijection $\phi : E_A \to E_B$ such that $s(\phi(e)) = s(e)$ and $r(\phi(e)) = r(e)$ for every $e \in E_A$.

Using path isomorphism, we define our first equivalence relation of *compatible shift equivalence* (CSE) (see Definition 4.1). The mere existence of path isomorphisms $\phi_R, \phi_S, \psi_A, \psi_B$ in Definition 4.1 implies shift equivalence, so that CSE is a direct strengthening of shift equivalence. In Theorem 7.3 we show that CSE coincides with SSE, and this allows for a direct comparison between SSE and SE via CSE. Motivated by Williams' problem, another related notion called *adapted shift equivalence* was studied by Parry [39] and was also shown to be equivalent to SSE. More precisely, instead of a requirement on compatibility of path isomorphisms between shift equivalent matrices, adapted shift equivalence is a shift equivalence of lag $m \in \mathbb{N} \setminus \{0\}$ between the adjacency matrices of the m-line graphs of A and B.

It is known that shift equivalence is decidable by Kim and Roush [25; 26], but the problem of decidability of strong shift equivalence remains a fundamental open problem in symbolic dynamics. In fact, this was the original motivation in Williams' paper [47]. The identification between path isomorphisms in the definition of CSE illustrates in what way the algorithm of Kim and Roush would need to improve were SSE turn out to be decidable.

Our second equivalence relation is called *representable shift equivalence* (RSE) (see Definition 5.1). It arises naturally when one attempts to represent shift equivalence as bounded operators on Hilbert space.

Surprisingly, by merely representing the relations of shift equivalence as bounded operator on Hilbert space, we get SSE (see Theorem 7.3). Considering the counterexamples of Kim and Roush, we see that if A and B are shift equivalent but not strong shift equivalent, then it follows that one of the four relations of shift equivalence must fail when representing everything on the same Hilbert space (see Section 5 for more details).

In what follows, we will say that a matrix is *essential* if it has no zero rows and no zero columns. In the work of Pimsner [40], Pimsner dilations were introduced and were subsequently studied by several authors [13; 15; 22; 37]. Pimsner dilations offer a "reversible" perspective for Cuntz–Pimsner C*-algebras, showing that they are always generated by an imprimitivity bimodule that contains the original C*-correspondence. In [37, Remark 5.5] an equivalent formulation of strong shift equivalence in terms of Pimsner dilations was sought. This leads us to our third equivalence relation, which we call *strong Morita shift equivalence* (SMSE) (see Definition 6.1). Let A and B be finite essential matrices. Denote by $X(A)_\infty$ and $X(B)_\infty$ the Pimsner dilations of their graph C*-correspondences (see Section 3) and by $C^*(G_A)$ and $C^*(G_B)$ their Cuntz–Krieger graph C*-algebras. Our main theorem provides an equivalent formulation sought by Muhly, Pask and Tomforde, and orients CSE and RSE in one fell swoop (see Theorem 7.3).

Theorem 1.3. *Suppose A and B are two finite essential matrices with entries in $\mathbb{N}$. Then the following are equivalent*:

(1) *A and B are strong shift equivalent.*
(2) *A and B are compatible shift equivalent.*
(3) *A and B are representable shift equivalent.*
(4) *$X(A)_\infty$ and $X(B)_\infty$ are strong Morita shift equivalent.*
(5) *$C^*(G_A)$ and $C^*(G_B)$ are equivariantly stably isomorphic in a way that respects the diagonals.*

For some time now, several experts have been perplexed about certain Pimsner dilation techniques (see for instance [37, Remark 5.5], the incorrect proof of [22, Theorem 5.8], and the subsequent corrigendum of [22]), and many are still wondering whether they can be used to show that shift equivalence of A and B implies strong Morita equivalence of the Pimsner dilations $X(A)_\infty$ and $X(B)_\infty$ (in the sense of Abadie, Eilers and Exel [1, Section 4] or Muhly and Solel [36]). The importance of this question is further elevated because of Theorem 3.17 and the discussion succeeding it, where we show that a positive answer to it is *equivalent* to the *converse to Krieger's corollary*. Combining our main theorem with the celebrated counterexamples of Kim and Roush [28], as well as the work of Bratteli and Kishimoto [4], we obtain the following cutoff result. This result addresses the ambiguities mentioned above, and refutes the proof of [22, Theorem 5.8] (see Theorem 7.4).

Theorem 1.4. *There exist finite aperiodic 7×7 irreducible matrices A and B with entries in $\mathbb{N}$ such that $X(A)_\infty$ and $X(B)_\infty$ are strong Morita equivalent*, *but not strong Morita shift equivalent.*

Since the validity of [22, Theorem 5.8] is in question, so is the validity of [22, Corollary 5.11]. This latter result states that shift equivalence of A and B implies the (not necessarily equivariant) stable isomorphisms of $C^*(G_A)$ and $C^*(G_B)$. However, thanks to Ara, Hazrat and Li [3] this result can still be recovered. Indeed, in work of the third author with Restorff, Ruiz and Sørensen [12] it was shown that filtered, ordered K-theory classifies unital graph C*-algebras up to stable isomorphism. By showing that filtered ordered K-theory is an invariant of shift equivalence, Ara, Hazrat and Li [3] ipso facto show that shift equivalence implies the stable isomorphisms of unital graph C*-algebras.

This paper contains eight sections, including this introductory section. In Section 2 we discuss some of the basic theory of directed graphs, subshifts of finite type, groupoid C*-algebra description of graph C*-algebras and Cuntz–Krieger C*-algebras. In Section 3 we discuss some of the theory of C*-correspondences, Cuntz–Pimsner algebras, Pimsner dilations and equivariant isomorphisms. We provide there a characterization of the existence of an equivariant stable isomorphism between Cuntz–Pimsner algebras in terms of Pimsner dilations. In Section 4 we introduce CSE, show it is an equivalence relation, and that it is implied by SSE. In Section 5 we introduce RSE, show that it is implied by CSE, and upgrade representations to be faithful on associated graph C*-algebras. In Section 6 we explain how to concretely construct inductive limits related to Pimsner dilations, introduce SMSE, show that RSE implies SMSE and show that SMSE implies the existence of a stable equivariant diagonal-preserving isomorphism between graph C*-algebras. Finally in Section 7 we discuss dimension triples for graph C*-algebras, orient different equivalence relations on adjacency matrices and prove the main results stated above.

2. Preliminaries

In this section we explain some of the basic theory of directed graphs, subshifts of finite type, groupoid C*-algebra descriptions of graph C*-algebras, Cuntz–Krieger algebras and dimension triples. We recommend [31] for the basics of symbolic dynamics.

A directed graph $G = (V, E, r, s)$ is comprised of a vertex set V and an edge set E together with range and source maps $r, s : E \to V$.

We say that a directed graph $G = (V, E)$ has *finite out-degrees* if $s^{-1}(v)$ is finite for every $v \in V$. We say that G is *finite* if both V and E are finite. When G has finite out-degrees and has no sources (i.e., r is surjective) and no sinks (i.e., s is surjective), we may define the *two-sided edge shift* to be the pair (X_E, σ_E) where X_E is the set of bi-infinite paths

$$X_E = \{(e_n)_{n\in\mathbb{Z}} \mid s(e_{i+1}) = r(e_i)\} \subseteq \prod_{i\in\mathbb{Z}} E,$$

with the product topology, and $\sigma_E : X_E \to X_E$ is the left shift homeomorphism given by $\sigma_E((e_n)_{n\in\mathbb{Z}}) = (e_{n+1})_{n\in\mathbb{Z}}$. The *one-sided edge shift* (X_E^+, σ_E^+) is defined as above, by replacing every occurrence of $\mathbb{Z}$ with $\mathbb{N}$.

Given a directed graph $G = (V, E)$, we may always form its $V \times V$ adjacency matrix with cardinal entries given for $v, w \in V$ by

$$A_E(v, w) = |\{e \in E \mid s(e) = v,\ r(e) = w\}|.$$

Conversely, we have seen that given a matrix A indexed by V with cardinal entries, one may form a directed graph $G_A = (V, E_A, r, s)$, where E_A is the set of triples (v, w, α) such that $v, w \in V$, $0 \le \alpha < A_{vw}$ is an ordinal, A_{vw} is interpreted as the least ordinal with cardinality A_{vw}, while $r(v, w, \alpha) = w$ and $s(v, w, \alpha) = v$. It is clear that G_{A_E} and G are isomorphic directed graphs and that $A_{E_A} = A$.

The following shows that under countability/finiteness assumptions, shift equivalence with arbitrary cardinals becomes the standard notion we know from the literature.

Proposition 2.1. *Let A and B be matrices with cardinal entries, indexed by sets V and W. Suppose that V and W are countable/finite, and suppose that A and B are over $\mathbb{N} \cup \{\aleph_0\}$ / over $\mathbb{N}$, respectively. If A and B are shift equivalent, then the matrices R and S realizing shift equivalence can be chosen to be with entries in $\mathbb{N} \cup \{\aleph_0\}$ / in $\mathbb{N}$, respectively.*

Proof. Suppose now that V and W are countable/finite, and that A and B are with entries in $\mathbb{N} \cup \{\aleph_0\}$ / in $\mathbb{N}$ respectively. If R and S implement shift equivalence of lag m between A and B, denote by R' and S' the matrices obtained from R and S by replacing all noncountable/nonfinite entries with zeros (respectively). As the matrices A^m and B^m are with entries in $\mathbb{N} \cup \{\aleph_0\}$ / in $\mathbb{N}$ (respectively), we still have that $R'S' = A^m$ and $S'R' = B^m$, as well as $S'A = BS'$ and $AR' = R'B$. Hence, A and B are shift equivalent via R' and S', and we are done. □

In this paper we will conduct our study through the lens of graph C*-algebras, which include the class of Cuntz–Krieger C*-algebras. We recommend [16; 41] for more on graph C*-algebras. We will sometimes assume that our graphs have finite out-degree, which is often called "row-finiteness" in the literature.

Definition 2.2. Let $G = (V, E, r, s)$ be a directed graph. A family of operators $(S_v, S_e)_{v\in V, e\in E}$ on a Hilbert space $\mathcal{H}$ is called a *Cuntz–Krieger family* if

(1) $(S_v)_{v\in V}$ is a family of pairwise orthogonal projections,

(2) $S_e^* S_e = S_{r(e)}$ for all $e \in E$, and

(3) $\sum_{e\in s^{-1}(v)} S_e S_e^* = S_v$ for all $v \in V$ with $0 < |s^{-1}(v)| < \infty$.

The *graph C*-algebra* $C^*(G)$ of G is the universal C*-algebra generated by a Cuntz–Krieger families.

Universality of $C^*(G)$ gives rise to a point-norm continuous *gauge action* of the unit circle $\gamma : \mathbb{T} \to \mathrm{Aut}(C^*(G))$ given by

$$\gamma_z(S_v) = S_v \quad \text{and} \quad \gamma_z(S_e) = z \cdot S_e \quad \text{for } z \in \mathbb{T},\ v \in V,\ e \in E.$$

With this gauge action, $C^*(G)$ becomes a $\mathbb{Z}$-graded C*-algebra whose graded components are $C^*(G)_n := \{T \in C^*(G) \mid \gamma_z(T) = z^n \cdot T\}$.

We will sometime assume that our graphs have finite out-degree, which is often called “row-finiteness” in the literature. We shall need the groupoid C*-algebra description of $C^*(G)$, as specified in [5]. Indeed, if $G = (V, E)$ is a directed graph with finite out-degree, no sources and no sinks, we may construct the locally compact, Hausdorff etale groupoid

$$\mathcal{G}_E := \{(x, m-n, y) \in X_E^+ \times \mathbb{Z} \times X_E^+ \mid x, y \in X_E,\ m, n \in \mathbb{N},\ \sigma_E^m(x) = \sigma_E^n(y)\},$$

with product $(x, k, y)(w, \ell, z) = (x, k+\ell, z)$ if $y = w$ (otherwise undefined), and inverse $(x, k, y)^{-1} = (y, -k, x)$. The topology on $\mathcal{G}_E$ is generated by subsets of the form

$$Z(U, m, n, V) = \{(x, m-n, y) \in \mathcal{G}_E \mid x \in U,\ y \in V\},$$

where $m, n \in \mathbb{N}$ and $U, V \subseteq X_E^+$ are clopen such that $\sigma_E^m|_U$ is injective, $\sigma_E^n|_V$ is injective, and $\sigma_E^m(U) = \sigma_E^n(V)$. The map $x \mapsto (x, 0, x)$ then provides a homeomorphism from X_E^+ into the unit space $\mathcal{G}_E^0$ of $\mathcal{G}_E$. It is well known that $C^*(G) \cong C^*(\mathcal{G}_E)$ is the groupoid C*-algebra of $\mathcal{G}_E$, and that $\mathcal{D}_E := C_0(\mathcal{G}_E^{(0)}) \cong C_0(X_E^+)$ is the subalgebra of continuous functions on units of $\mathcal{G}_E$. This subalgebra is often called the *diagonal subalgebra* of $C^*(G)$, and is given by

$$\mathcal{D}_E = \overline{\mathrm{span}}\{S_\mu S_\mu^* \mid \mu \in E^n,\ n \in \mathbb{N}\}.$$

In what follows, recall that $\mathbb{K}$ denotes compact operators on $\ell^2(\mathbb{N})$, which contains a natural copy of diagonal compact operators $c_0 \subseteq \mathbb{K}$. We will consider $\gamma \otimes \mathrm{id}_{\mathbb{K}}$ as the standard gauge action on the stabilization $C^*(G) \otimes \mathbb{K}$. It was shown in [5, Theorem 5.1] that for any two finite graphs without sources and sinks $G = (V, E)$ and $G' = (V', E')$ we have that (X_E, σ_E) and $(X_{E'}, \sigma_{E'})$ are conjugate if and only if there is an equivariant isomorphism $\varphi : C^*(G) \otimes \mathbb{K} \to C^*(G') \otimes \mathbb{K}$ such that $\varphi(\mathcal{D}_E \otimes c_0) = \mathcal{D}_{E'} \otimes c_0$.

Suppose that $A = (A_{vw})_{v,w=0}^{n-1}$ is an essential $n \times n$ matrix with nonnegative integer entries. The *Cuntz–Krieger algebra* of A is the universal C*-algebra $\mathcal{O}_A$ generated by a family

$$\{S_{(v,w,m)} \mid v, w, m \in \mathbb{N},\ 0 \le v, w < n,\ 0 \le m < A_{vw}\}$$

of partial isometries satisfying

(1) $S^*_{(v,w,m)}S_{(v,w,m)} = \sum_{u=0}^{n-1}\sum_{\ell=0}^{A_{wu}-1} S_{(w,u,\ell)}S^*_{(w,u,\ell)}$,

(2) $\sum_{v,w=0}^{n-1}\sum_{m=0}^{A_{vw}-1} S_{(v,w,m)}S^*_{(v,w,m)} = 1$.

For each family as above, we get projections $S_w = S^*_{(v,w,m)}S_{(v,w,m)}$ (independent of v and m), so that the family $(S_v, S_{(v',w',m)})$ becomes a Cuntz–Krieger family for the directed graph G_A (see [8, Remark 2.16] and [41, Remark 2.8]). Hence, there is a $*$-isomorphism between $\mathcal{O}_A$ and $C^*(G_A)$ which maps the generators $S_{(v,w,m)}$ of $\mathcal{O}_A$ to edge generators for $C^*(G_A)$. This isomorphism between $\mathcal{O}_A$ and $C^*(G_A)$ induces the usual gauge action on $\mathcal{O}_A$ from [8] (which is also built from the universality of $\mathcal{O}_A$, see [20, Remark 2.2(2)]) and sends the natural diagonal subalgebra of $\mathcal{O}_A$ to the diagonal subalgebra $\mathcal{D}_E$ inside $C^*(G_A)$. Thus, there is no loss of generality arising from considering graph C*-algebras instead of Cuntz–Krieger C*-algebras.

Notation 2.3. Whenever $X, Y \subseteq \mathcal{L}(E)$ are norm-closed subspaces, we denote by XY or by $X \cdot Y$ the closed linear span of products $x \cdot y$ with $x \in X$ and $y \in Y$.

3. Shift equivalence and Cuntz–Pimsner algebras

In this section we discuss C*-correspondences of adjacency matrices, Cuntz–Pimsner C*-algebras and Pimsner dilations. The main result of this section is a characterization of equivariant stable isomorphism of Cuntz–Pimsner algebras in terms of Pimsner dilations.

We will need some of the theory of C*-correspondences. We mention some of the basic definitions, but will assume some familiarity with the theory of Hilbert C*-modules as in [30; 32].

Definition 3.1. A *C*-correspondence* from $\mathcal{A}$ to $\mathcal{B}$ (or $\mathcal{B}$-$\mathcal{A}$ C*-correspondence) is a right Hilbert $\mathcal{A}$-module E with a *-representation $\phi_E : \mathcal{B} \to \mathcal{L}(E)$, where $\mathcal{L}(E)$ is the C*-algebra of adjointable operators on E. We denote by $\mathcal{K}(E)$ the ideal of $\mathcal{L}(E)$ which is closed linear span of rank-1 operators $\theta_{\xi,\eta}$ given by $\theta_{\xi,\eta}(\zeta) = \xi\langle\eta, \zeta\rangle$.

We will say that two $\mathcal{B}$-$\mathcal{A}$ correspondences E and F are *unitarily isomorphic* (and denote this by $E \cong F$) if there is an isometric surjection $U : E \to F$ such that for every $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $\xi \in E$ we have

$$U(\phi_E(b)\xi a) = \phi_F(b)U(\xi)a.$$

We will assume throughout this note that C*-correspondences E are nondegenerate (sometimes called essential) in the sense that $\phi_E(\mathcal{B})E = E$. We say that a $\mathcal{B}$-$\mathcal{A}$ C*-correspondence E is *regular* if its left action ϕ_E is injective and $\phi_E(\mathcal{B}) \subseteq \mathcal{K}(E)$. When the context is clear, we write $b\xi$ to mean $\phi_E(b)\xi$ for $b \in \mathcal{B}$ and $\xi \in E$. Finally, we say that a $\mathcal{B}$-$\mathcal{A}$ C*-correspondence E is *full* if $\mathcal{A}$ is equal to the ideal $\langle E, E\rangle$ defined as the closed linear span of $\langle\xi, \eta\rangle$ for $\xi, \eta \in E$.

The most important examples of C*-correspondences in our study are the ones coming from adjacency matrices. Let V and W be sets, and let C be a $V \times W$ matrix so that C_{vw} is some cardinal. We define

$$E_C := \{(v, w, \alpha) \mid 0 \le \alpha < C_{vw},\ v \in V,\ w \in W\}.$$

When C is a $V \times W$ matrix with cardinal entries, we may construct a $c_0(V)$-$c_0(W)$ correspondence $X(C)$ by taking the Hausdorff completion of all finitely supported functions on E_C with respect to the inner product

$$\langle \xi, \eta \rangle(w) = \sum_{(v,w,\alpha)\in E_C} \overline{\xi(v, w, \alpha)}\eta(v, w, \alpha),$$

where ξ and η are finitely supported on E_C. The left actions of $c_0(V)$ and the right action of $c_0(W)$ on $X(C)$ are then given by

$$(f \cdot \xi \cdot g)(v, w, \alpha) = f(v)\xi(v, w, \alpha)g(w) \quad \text{for } f \in c_0(V) \text{ and } g \in c_0(W).$$

When $V = W$, it is clear that $X(C)$ coincides with the graph C*-correspondence $X(G_C)$ of the directed $G_C = (V, E_C)$, as is explained in the discussion preceding [10, Theorem 6.2], with range and source interchanged in the definitions of inner product and bimodule actions.

We will need to know that in the above way we obtain all $c_0(V)$-$c_0(W)$ correspondences. This type of result was first shown by Kaliszewski, Patani and Quigg [23] when C is a square matrix indexed by a countable set and with countable entries.

Proposition 3.2. *Let V and W be sets, and E a $c_0(V)$-$c_0(W)$ correspondence. Then there exists a unique $V \times W$ matrix C with cardinal entries such that E is unitarily isomorphic to $X(C)$.*

Proof. Let $v \in V$ and $w \in W$, and let $p_v \in c_0(V)$ and $p_w \in c_0(W)$ be the characteristic functions of $\{v\}$ and $\{w\}$ respectively. Since $p_v E p_w$ is a Hilbert space, we let C_{vw} be its dimension, and let $\{e_{(v,w,\alpha)}\}_{0\le\alpha<C_{vw}}$ be an orthonormal basis for it, indexed by ordinals $0 \le \alpha < C_{vw}$. Then clearly C is a $V \times W$ matrix with cardinal values. We define a map $U : X(C) \to E$ on finitely supported functions by setting $U(\xi) = \sum_{(v,w,\alpha)\in E_C} \xi(v, w, \alpha)e_{(v,w,\alpha)}$ for finitely supported $\xi \in X(C)$. Now, for a finitely supported function $\xi \in X(C)$ we have for fixed $w \in W$ that

$$|U(\xi)|^2(w) = \sum_{(v,w,\alpha)\in E_C} |\xi(v, w, \alpha)|^2 = |\xi|^2(w).$$

Hence, U extends to an isometry on $X(C)$, and since the linear span of

$$\{e_{(v,w,\alpha)} \mid v \in V,\ w \in W,\ 0 \le \alpha < C_{vw}\}$$

is dense in E, we get that U is a unitary isomorphism.

As for uniqueness, suppose F is another $c_0(W)$-$c_0(V)$-correspondence which is unitarily isomorphic to E via a unitary U. Hence, if C^E and C^F are the matrices associated to E and F via the first paragraph, we must have $C^F_{vw} = C^E_{vw}$, and we are done. ☐

For a $\mathcal{B}$-$\mathcal{A}$ correspondence E and a $\mathcal{C}$-$\mathcal{B}$ correspondence F, we can form the *interior tensor product* $\mathcal{C}$-$\mathcal{A}$ correspondence $F \otimes_{\mathcal{B}} E$ of E and F as follows. Let $F \otimes_{\text{alg}} E$ denote the quotient of the algebraic tensor product by the subspace generated by elements of the form

$$\eta b \otimes \xi - \eta \otimes b\xi \quad \text{for } \xi \in E,\ \eta \in F,\ b \in \mathcal{B}.$$

Define an $\mathcal{A}$-valued semi-inner product and left $\mathcal{C}$-action by setting

$$\langle \eta_1 \otimes \xi_1, \eta_2 \otimes \xi_2 \rangle = \langle \xi_1, \langle \eta_1, \eta_2 \rangle \cdot \xi_2 \rangle \quad \text{for } \xi_1, \xi_2 \in E,\ \eta_1, \eta_2 \in F,$$
$$c \cdot (\eta \otimes \xi) = (c \cdot \eta) \otimes \xi \quad \text{for } \eta \in E,\ \xi \in F,\ c \in \mathcal{C}.$$

We denote by $F \otimes_{\mathcal{B}} E$ the separated completion of $F \otimes_{\mathrm{alg}} E$ with respect to the $\mathcal{A}$-valued semi-inner product defined above. One then verifies that $F \otimes_{\mathcal{B}} E$ is a $\mathcal{C}$-$\mathcal{A}$ correspondence (see for example [30, Proposition 4.5]). We will often abuse notation and write $F \otimes E$ for $F \otimes_{\mathcal{B}} E$ when the context is clear. If E is an $\mathcal{A}$-$\mathcal{A}$ correspondence (a C*-correspondence over $\mathcal{A}$), then we denote by $E^{\otimes n}$ the n-fold interior tensor product of E with itself.

Proposition 3.3. *Let I, J, K be sets, and let C be an $I \times J$ matrix and D be a $J \times K$ matrix, both with cardinal entries. Then there is a unitary isomorphism $U : X(C) \otimes X(D) \to X(CD)$.*

Proof. For each $i \in I$, $j \in J$ and $k \in K$, let $C_{ij} = |X_{ij}|$, $D_{jk} = |Y_{jk}|$ for some sets X_{ij} and Y_{jk}. Then $(CD)_{ik}$ is the cardinality of the disjoint union of Cartesian products $\bigsqcup_{j \in J} X_{ij} \times Y_{jk}$.

Now, clearly both $X(C) \otimes X(D)$ and $X(CD)$ are $c_0(I)$-$c_0(K)$ correspondences, so that by the uniqueness part of Proposition 3.2 it will suffice to show for every $i \in I$ and $k \in K$ that the dimension of the Hilbert space $p_i X(C) \otimes X(D) p_k$ is equal to $(CD)_{ik}$.

So let $\{e_x\}_{x \in X_{ij}}$ be an orthonormal basis for $p_i X(C) p_j$ for $i \in I$ and $j \in J$ and let $\{e_y\}_{y \in Y_{j'k}}$ be an orthonormal basis for $p_{j'} X(D) p_k$ for $j' \in J$ and $k \in K$. Then, for $x \in X_{ij}$ and $y \in Y_{j'k}$ we have $e_x \otimes e_y \neq 0$ if and only if $j = j'$. Hence, an orthonormal basis for $p_i X(C) \otimes X(D) p_k$ is given by $\{e_x \otimes e_y \mid x \in X_{ij},\ y \in Y_{jk},\ j \in J\}$. The cardinality of this basis is clearly equal to $(CD)_{ik}$, so the proof is concluded. □

The following definition of shift equivalence of C*-correspondences first appeared in [22].

Definition 3.4. Let E and F be C*-correspondences over $\mathcal{A}$ and $\mathcal{B}$ respectively. We say that E and F are:

(1) *Shift equivalent* with lag m if there are $m \in \mathbb{N} \setminus \{0\}$, an $\mathcal{A}$-$\mathcal{B}$ correspondence R and a $\mathcal{B}$-$\mathcal{A}$ correspondence S, together with unitary isomorphisms

$$E^{\otimes m} \cong R \otimes S, \qquad F^{\otimes m} \cong S \otimes R,$$
$$S \otimes E \cong F \otimes S, \quad E \otimes R \cong R \otimes F.$$

(2) *Elementary shift related* if they are shift equivalent with lag 1.

(3) *Strong shift equivalent* if they are equivalent in the transitive closure of the elementary shift relation.

The following shows that shift equivalence between two C*-correspondences generalizes shift equivalence of adjacency matrices.

Proposition 3.5. *Let A and B be matrices with cardinal entries, indexed by sets V and W. Then, A and B are shift equivalent if and only if $X(A)$ and $X(B)$ are shift equivalent.*

Proof. By Proposition 3.3 we see that if A and B are shift equivalent with lag m via matrices C and D, then the C*-correspondences $X(A)$ and $X(B)$ are shift equivalent with lag m via the C*-correspondences $X(C)$ and $X(D)$.

Conversely, suppose R and S are $c_0(V)$-$c_0(W)$ and $c_0(W)$-$c_0(V)$ correspondences implementing shift equivalence of lag m for $X(A)$ and $X(B)$. By Proposition 3.2 there are a $V \times W$ matrix C and $W \times V$ matrix D with unitary isomorphisms $R \cong X(C)$ and $S \cong X(D)$. Hence, the uniqueness portion in Proposition 3.2 guarantees that C and D implement a shift equivalence of A and B with lag m. $\square$

Every path isomorphism $\phi : E_A \to E_B$ induces a unitary isomorphism $\Phi : X(A) \to X(B)$ by setting $\Phi(\xi)(v, w, \alpha) = \xi(\phi^{-1}(v, w, \alpha))$. However, the converse is in general false; it is easy to construct a unitary isomorphism $U : X(A) \to X(B)$ for which there is no path isomorphism $\phi : E_A \to E_B$ such that U is the induced unitary isomorphism from ϕ. The point of Proposition 3.5 is that we only need to find *some* path isomorphisms for each one of the four relations appearing in the definition of shift equivalence, and not necessarily path isomorphisms which induce *the same* four isomorphisms we started with at the level of C*-correspondences.

Remark 3.6. From considerations similar to the above we see that strong shift equivalence of matrices implies the strong shift equivalence of their associated C*-correspondences. The converse, however, is unknown.

Next we discuss Cuntz–Pimsner algebras and Pimsner dilations. More material on Cuntz–Pimsner algebras, with a special emphasis on C*-correspondences of graphs, can be found in [41, Chapter 8]. We note immediately that what we call a *rigged* representation here is often referred to as an *isometric* representation in the literature (see [36]).

Definition 3.7. Let E be a C*-correspondence from $\mathcal{A}$ to $\mathcal{B}$, and $\mathcal{C}$ some C*-algebra. A *rigged representation* of E is a triple $(\pi_{\mathcal{A}}, \pi_{\mathcal{B}}, t)$ such that $\pi_{\mathcal{A}} : \mathcal{A} \to \mathcal{C}$ and $\pi_{\mathcal{B}} : \mathcal{B} \to \mathcal{C}$ are *-homomorphisms and $t : E \to \mathcal{C}$ is a linear map such that

(1) $\pi_{\mathcal{B}}(b)t(\xi)\pi_{\mathcal{A}}(a) = t(b \cdot \xi \cdot a)$ for $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $\xi \in E$,

(2) $t(\xi)^* t(\eta) = \pi_{\mathcal{A}}(\langle \xi, \eta \rangle)$.

We say that $(\pi_{\mathcal{A}}, \pi_{\mathcal{B}}, t)$ is *injective* if both $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ are injective *-homomorphisms. We denote by $C^*(\pi_{\mathcal{A}}, \pi_{\mathcal{B}}, t)$ the C*-algebra generated by the images of $\pi_{\mathcal{A}}$, $\pi_{\mathcal{B}}$ and t.

We will mostly be concerned with the situation where $\mathcal{A} = \mathcal{B}$ and $\pi_{\mathcal{A}} = \pi_{\mathcal{B}}$, in which case we will define $\pi := \pi_{\mathcal{A}} = \pi_{\mathcal{B}}$, and refer to the representation as a pair (π, t), and the generated C*-algebra by $C^*(\pi, t)$.

Now let E be a C*-correspondence over $\mathcal{A}$. The Toeplitz–Pimsner algebra $\mathcal{T}(E)$ is then the universal C*-algebra generated by a rigged representation of E. Universality of $\mathcal{T}(E)$ implies that it comes equipped with a point-norm continuous gauge action $\gamma : \mathbb{T} \to \mathrm{Aut}(\mathcal{T}(E))$ given by

$$\gamma_z(\pi(a)) = \pi(a) \quad \text{and} \quad \gamma_z(t(\xi)) = z \cdot t(\xi) \quad \text{for } z \in \mathbb{T},\ \xi \in E,\ a \in \mathcal{A}.$$

Toeplitz–Pimsner algebras have a canonical quotient, also known as the Cuntz–Pimsner algebra, which was originally defined by Pimsner in [40] and refined by Katsura in [24].

Definition 3.8. For a C*-correspondence E over $\mathcal{A}$, we define *Katsura's ideal* J_E in $\mathcal{A}$ by

$$J_E := \{a \in \mathcal{A} \mid \phi_E(a) \in \mathcal{K}(E) \text{ and } ab = 0 \text{ for all } b \in \ker \phi_E\}.$$

For a rigged representation (π, t) of a C*-correspondence E over $\mathcal{A}$, it is known there is a well-defined *-homomorphism $\psi_t : \mathcal{K}(E) \to C^*(\pi, t)$ given by $\psi_t(\theta_{\xi,\eta}) = t(\xi)t(\eta)^*$ for $\xi, \eta \in E$ (see for instance [21, Lemma 2.2]).

Definition 3.9. A rigged representation (π, t) is said to be *covariant* if $\pi(a) = \psi_t(\phi_E(a))$ for all $a \in J_E$.

The Cuntz–Pimsner algebra $\mathcal{O}(E)$ is then the universal C*-algebra generated by a *covariant* representation of E. Suppose now that (π, t) and $(\bar{\pi}, \bar{t})$ are universal rigged and covariant representations respectively, so that $\mathcal{T}(E) = C^*(\pi, t)$ and $\mathcal{O}(E) = C^*(\bar{\pi}, \bar{t})$. We denote by $\mathcal{J}_E$ the kernel of the natural quotient map from $\mathcal{T}(E)$ onto $\mathcal{O}(E)$, which is the ideal generated by elements of the form $\pi(a) - \psi_t(\phi_E(a))$ for $a \in J_E$. Since the ideal $\mathcal{J}_E$ is gauge-invariant, we see there is an induced point-norm continuous circle action $\gamma : \mathbb{T} \to \mathrm{Aut}(\mathcal{O}(E))$ given by

$$\gamma_z(\pi(a)) = \pi(a) \quad \text{and} \quad \gamma_z(t(\xi)) = z \cdot t(\xi) \quad \text{for } z \in \mathbb{T},\ \xi \in E,\ a \in \mathcal{A}.$$

It follows that $\mathcal{O}(E)$ then becomes a $\mathbb{Z}$-graded C*-algebra, with its n-th graded component given by $\mathcal{O}(E)_n := \{c \in \mathcal{O}(E) \mid \gamma_z(c) = z^n \cdot c\}$. By [42, Theorem 3] we see that there is a bijective correspondence between topologically $\mathbb{Z}$-graded C^*-algebras, with graded *-homomorphisms and C^*-algebras equipped with a circle action, together with equivariant *-homomorphisms. In particular, an isomorphism $\varphi : \mathcal{O}(E) \to \mathcal{O}(F)$ between two Cuntz–Pimsner algebras is equivariant if and only if it is graded.

Example 3.10. When $G = (V, E)$ is a directed graph we know that the graph C*-algebra $C^*(G)$ coincides with the Cuntz–Pimsner algebra of the correspondence $X(A_E)$ by an isomorphism that intertwines the gauge action of $C^*(G)$ and the gauge action of $\mathcal{O}(X(A_E))$. See [41, Section 8] for more details.

Definition 3.11. Let E, F be C*-correspondences over $\mathcal{A}$ and $\mathcal{B}$, respectively. Denote by γ^E and γ^F the gauge actions on $\mathcal{O}(E)$ and $\mathcal{O}(F)$ respectively. We say that $\mathcal{O}(E)$ and $\mathcal{O}(F)$ are *equivariantly stably isomorphic* if there is a $*$-isomorphism $\varphi : \mathcal{O}(E) \otimes \mathbb{K} \to \mathcal{O}(F) \otimes \mathbb{K}$ such that $\varphi \circ (\gamma_z^E \otimes \mathrm{id}) = (\gamma_z^F \otimes \mathrm{id}) \circ \varphi$ for every $z \in \mathbb{T}$.

Equivariant stable isomorphisms arise naturally in the classification of groupoid C*-algebras, but always in a way which respects diagonal subalgebras. For instance in [7], equivariant stable isomorphisms which respect diagonal subalgebras are characterized in terms of isomorphisms of graded groupoids. We will get back to such isomorphisms in Section 6.

The following definition is similar to Definition 3.1, and we will provide a precise distinction between the two in the remark that follows.

Definition 3.12. Suppose E and F are C*-correspondences over C*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively. We say that E and F are *unitarily isomorphic* (denoted by $E \cong F$) if there exist a surjective, isometric map $U : E \to F$ and a *-isomorphism $\rho : \mathcal{A} \to \mathcal{B}$ such that $U(b \cdot \xi \cdot a) = \rho(b) \cdot U(\xi) \cdot \rho(a)$ for all $a, b \in \mathcal{A}$, $\xi \in E$.

Remark 3.13. Suppose now that E and F are C*-correspondences over C*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, and that $U : E \to F$ is a unitary isomorphism implemented by a *-isomorphism $\rho : \mathcal{A} \to \mathcal{B}$. By the discussion in [9, Subsection 2.1] we may "twist" the C*-correspondence F to a C*-correspondence F_ρ

over $\mathcal{A}$ so that $U : E \to F_\rho$ becomes a unitary isomorphism as in Definition 3.1. More precisely, the new operations on F_ρ are given by

$$\langle \xi, \eta \rangle_\rho := \rho^{-1}(\langle \xi, \eta \rangle_{\mathcal{B}}) \quad \text{for } \xi, \eta \in F,$$
$$a \cdot \xi = \rho(a) \cdot \xi \qquad \text{and} \qquad \xi \cdot a := \xi \cdot \rho(a) \quad \text{for all } \xi \in F \text{ and } a \in \mathcal{A},$$

and the identity map $\mathrm{id}_F^\rho : F \to F_\rho$becomes a unitary isomorphism as in Definition 3.12. Then, the isometric surjection $\mathrm{id}_F^\rho \circ U$ is a unitary isomorphism as in Definition 3.1. Hence, we can go back and forth between the two definitions of unitary isomorphism.

We warn the reader that unitary isomorphism as in Definition 3.12 is not the same as having an isometric surjection U implemented via two potentially *different* *-isomorphisms $\rho_1 : \mathcal{A} \to \mathcal{B}$ and $\rho_2 : \mathcal{A} \to \mathcal{B}$ in the sense that $U(b \cdot \xi \cdot a) = \rho_1(a)U(\xi)\rho_2(b)$ for $a, b \in \mathcal{A}$. We also note that whenever one of our C*-correspondences has possibly different left and right coefficient C*-algebras, we only consider one notion of unitary isomorphism, which is the one in Definition 3.1.

In what follows, we say that M is an *imprimitivity* $\mathcal{A}$-$\mathcal{B}$ correspondence (from $\mathcal{B}$ to $\mathcal{A}$) if it is full and its left action ϕ_M is a $*$-isomorphism onto $\mathcal{K}(E)$. The following was introduced by Muhly and Solel [36] in their study of tensor algebras of C*-correspondences.

Definition 3.14. Let E and F be C*-correspondences over C*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively. We say that E and F are *strongly Morita equivalent* if there are an imprimitivity $\mathcal{A}$-$\mathcal{B}$ bimodule M and an isometric surjective linear map $U : E \otimes M \to M \otimes F$ such that for every $\xi \in E \otimes M$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$ we have $U(a\xi b) = aU(\xi)b$.

When M is an imprimitivity $\mathcal{A}$-$\mathcal{B}$-bimodule, there is an "inverse" imprimitivity $\mathcal{B}$-$\mathcal{A}$-bimodule M^* satisfying $M \otimes M^* \cong \mathcal{A}$ and $M^* \otimes M \cong \mathcal{B}$. If moreover $\mathcal{A} = \mathcal{B}$, it follows from [1, Theorem 2.9] that the $\mathbb{Z}$-graded components of $\mathcal{O}(M)$ are given for $n \in \mathbb{N}$ by

- $\mathcal{O}(M)_n \cong M^{\otimes n}$ for $n > 0$,
- $\mathcal{O}(M)_0 \cong \mathcal{A}$, and
- $\mathcal{O}(M)_n \cong (M^{\otimes n})^*$ for $n < 0$.

Proposition 3.15. *Let E and F be C*-correspondences over C*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively. If E and F are unitarily equivalent, then $\mathcal{O}(E)$ and $\mathcal{O}(F)$ are equivariantly isomorphic. If moreover E and F are imprimitivity bimodules, then E and F are unitarily equivalent if and only if $\mathcal{O}(E)$ and $\mathcal{O}(F)$ are equivariantly isomorphic.*

Proof. Let $(\bar{\pi}, \bar{t})$ be a universal covariant representation for $\mathcal{O}(E)$. Assume that E and F are unitarily equivalent. By the implication (1) $\Rightarrow$ (4) of [10, Corollary 3.5] we get that $\mathcal{T}(E)$ and $\mathcal{T}(F)$ are graded isomorphic, and hence equivariantly isomorphic. Then, [10, Theorem 3.1] gives rise to an induced isomorphism between $\mathcal{O}(E)$ and $\mathcal{O}(F)$, which is actually equivariant since the ideals $\mathcal{J}_E$ and $\mathcal{J}_F$ are gauge-invariant.

Conversely, if E and F are imprimitivity bimodules, and $\varphi : \mathcal{O}(E) \to \mathcal{O}(F)$ is an equivariant isomorphism, then φ is $\mathbb{Z}$-graded, and we get that the restriction $U := \varphi|_{\mathcal{O}(E)_1} : \mathcal{O}(E)_1 \to \mathcal{O}(F)_1$ is an isometric

$\mathcal{O}(E)_0$-$\mathcal{O}(F)_0$ bimodule isomorphism, implemented by the $*$-isomorphism $\rho := \varphi|_{\mathcal{O}(E)_0} : \mathcal{O}(E)_0 \to \mathcal{O}(F)_0$. From the identifications in the discussion preceding the theorem, we get that E and F are unitarily isomorphic. □

Given a C*-correspondence E over $\mathcal{A}$, we may form the external minimal tensor product $E \otimes \mathbb{K}$, which is a C*-correspondence over $\mathcal{A} \otimes \mathbb{K}$ as defined in [30, p. 34]. A consequence of [10, Proposition 2.10] is that $\mathcal{O}(E \otimes \mathbb{K})$ is canonically isomorphic to $\mathcal{O}(E) \otimes \mathbb{K}$ via a map induced by the representation $(\bar{\pi} \otimes \mathrm{id}, \bar{t} \otimes \mathrm{id})$.

Corollary 3.16. *Let E and F be C*-correspondences over C*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively. If $E \otimes \mathbb{K}$ and $F \otimes \mathbb{K}$ are unitarily equivalent, then $\mathcal{O}(E)$ and $\mathcal{O}(F)$ are equivariantly stably isomorphic. If moreover E and F are* ***imprimitivity bimodules****, then $E \otimes \mathbb{K}$ and $F \otimes \mathbb{K}$ are unitarily equivalent if and only if $\mathcal{O}(E)$ and $\mathcal{O}(F)$ are equivariantly stably isomorphic.*

Proof. Since $E \otimes \mathbb{K}$ and $F \otimes \mathbb{K}$ are unitarily equivalent, by Proposition 3.15 we get that $\mathcal{O}(E \otimes \mathbb{K}) \cong \mathcal{O}(F \otimes \mathbb{K})$ equivariantly.

Suppose now that $(\bar{\pi}, \bar{t})$ is a universal covariant representation of E. By [10, Proposition 2.10], we get that the representation $(\bar{\pi} \otimes \mathrm{id}, \bar{t} \otimes \mathrm{id})$ induces an isomorphism $\bar{\rho} : \mathcal{O}(E \otimes \mathbb{K}) \to \mathcal{O}(E) \otimes \mathbb{K}$ satisfying $\bar{\rho}(\xi \otimes K) = \bar{t}(\xi) \otimes K$ for every $\xi \in E$ and $K \in \mathbb{K}$. Hence $\bar{\rho}$ must be equivariant. It is similarly shown that $\mathcal{O}(F \otimes \mathbb{K}) \cong \mathcal{O}(F) \otimes \mathbb{K}$ equivariantly. Hence, we get equivariantly that

$$\mathcal{O}(E) \otimes \mathbb{K} \cong \mathcal{O}(E \otimes \mathbb{K}) \cong \mathcal{O}(F \otimes \mathbb{K}) \cong \mathcal{O}(F) \otimes \mathbb{K}.$$

With this, we obtain the first part of our result.

Conversely, if moreover E and F are imprimitivity bimodules, then so are $E \otimes \mathbb{K}$ and $F \otimes \mathbb{K}$. Thus, we are done by Proposition 3.15 and the above identifications. □

Let E be a C*-correspondence over a C*-algebra $\mathcal{A}$, and $(\bar{\pi}, \bar{t})$ a universal covariant representation of E. We denote by

$$\mathcal{A}_\infty = \mathcal{O}(E)_0 = \overline{\operatorname{span}}\{\bar{\pi}(\mathcal{A}),\ \psi_{\bar{t}^m}(\mathcal{K}(E^{\otimes m})) \mid m \geq 1\}$$

the fixed point algebra of $\mathcal{O}(E)$ under the gauge action γ. It is well known that $\mathcal{A}_\infty$ is the direct limit of C*-subalgebras $\mathcal{A}_n$ given by

$$\mathcal{A}_n = \overline{\operatorname{span}}\{\bar{\pi}(\mathcal{A}),\ \psi_{\bar{t}^m}(\mathcal{K}(E^{\otimes m})) \mid 1 \leq m \leq n\}.$$

When E has an injective left action, we get that $\mathcal{A}_n = \psi_{\bar{t}^n}(\mathcal{K}(E^{\otimes m}))$ for each $n \in \mathbb{N}$, and we define $E_n := \bar{t}(E)\mathcal{A}_n$ inside $\mathcal{O}(E)$. Then, E_n becomes a C*-correspondence from $\mathcal{A}_n$ to $\mathcal{A}_{n+1}$, where the left action of $\mathcal{A}_{n+1}$ is defined by left multiplication in $\mathcal{O}(E)$. By taking the direct limit $E_\infty := \bar{t}(E) \cdot \mathcal{A}_\infty = \overline{\bigcup_{n=1}^\infty E_n}$, we obtain a C*-correspondence over $\mathcal{A}_\infty$ called the *Pimsner dilation* of E.

It was shown by Pimsner in [40, Theorem 2.5(2)] that the identification $E \otimes \mathcal{A}_\infty \cong E_\infty$ gives rise to an equivariant $*$-injection $\mathcal{T}(E) \to \mathcal{T}(E_\infty)$, which then induces an equivariant isomorphism $\mathcal{O}(E) \cong \mathcal{O}(E_\infty)$ between the quotients. When E is also *regular* and *full*, the C*-correspondence E_∞ becomes an imprimitivity bimodule.

Theorem 3.17. *Let E and F be* **regular** *and* **full** *C*-correspondences over σ-unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively. Then, $\mathcal{O}(E)$ and $\mathcal{O}(F)$ are equivariantly stably isomorphic if and only if E_∞ and F_∞ are strong Morita equivalent.*

Proof. Since E and F are regular and full, we get that E_∞ and F_∞ are imprimitivity bimodules. Hence, by Corollary 3.16 we get that $\mathcal{O}(E)$ and $\mathcal{O}(F)$ are equivariantly stably isomorphic if and only if $E_\infty \otimes \mathbb{K}$ and $F_\infty \otimes \mathbb{K}$ are unitarily isomorphic.

Since $\mathcal{A}$ and $\mathcal{B}$ are σ-unital C*-algebras, so are $\mathcal{A}_\infty$ and $\mathcal{B}_\infty$. Hence, every strong Morita equivalence M for E_∞ and F_∞ must be a σ-TRO in the sense of [13, p. 6], so that a combination of [13, Theorem 5.2] and [13, Proposition 3.1] shows that $E_\infty \otimes \mathbb{K}$ and $F_\infty \otimes \mathbb{K}$ are unitarily equivalent if and only if E_∞ and F_∞ are strongly Morita equivalent. □

It is important to say a few words about the assumptions in Theorem 3.17 and what they mean for C*-correspondences of adjacency matrices with entries in $\mathbb{N}$. Suppose A is a square adjacency matrix with entries in $\mathbb{N}$, and indexed by a set V. First note that V is countable if and only if $c_0(V)$ is σ-unital if and only if $C^*(G_A)_0$ is σ-unital. Moreover, by [41, Proposition 8.8] the following hold:

(1) A has finitely supported rows if and only if G_A has finite out-degrees, if and only if the left action of $X(A)$ has image in $\mathcal{K}(X(A))$.

(2) A has no zero rows if and only if G_A has no sinks, if and only if $X(A)$ has an injective left action.

(3) A has no zero columns if and only if G_A has no sources, if and only if $X(A)$ is full.

So we see that in order to apply Theorem 3.17 to $X(A)$, we must verify that A is over a countable set V, has finitely supported rows and is essential. In this case, Theorem 3.17 shows that the existence of an equivariant stable isomorphism between graph C*-algebras $C^*(G_A)$ and $C^*(G_B)$ coincides with the existence of a strong Morita equivalence of the Pimsner dilations $X(A)_\infty$ and $X(B)_\infty$.

4. Compatible shift equivalence

In this section we introduce and study *compatible shift equivalence*, which is formulated in terms of adjacency matrices and path isomorphisms. We show that it is indeed an equivalence relation, and that strong shift equivalence implies compatible shift equivalence. In what follows, for a matrix A we write id_A to mean id_{E_A}.

Definition 4.1. Let A and B be matrices indexed by V and W respectively, with entries in $\mathbb{N}$. Suppose there are a *lag* $m \in \mathbb{N} \setminus \{0\}$ and matrices R over $V \times W$ and S over $W \times V$ with entries in $\mathbb{N}$ together with path isomorphisms

$$\begin{aligned} \phi_R &: E_A \times E_R \to E_R \times E_B, & \phi_S &: E_B \times E_S \to E_S \times E_A, \\ \psi_A &: E_R \times E_S \to E_A^m, & \psi_B &: E_S \times E_R \to E_B^m. \end{aligned}$$

We say that R and S are *compatible* if

$$\phi_R^{(m)} = (\mathrm{id}_R \times \psi_B)(\psi_A^{-1} \times \mathrm{id}_R), \quad \phi_S^{(m)} = (\mathrm{id}_S \times \psi_A)(\psi_B^{-1} \times \mathrm{id}_S), \tag{4-1}$$

where

$$\phi_R^{(m)} := (\phi_R \times \mathrm{id}_{B^{m-1}})(\mathrm{id}_A \times \phi_R \times \mathrm{id}_{B^{m-2}}) \cdots (\mathrm{id}_{A^{m-1}} \times \phi_R),$$
$$\phi_S^{(m)} := (\phi_S \times \mathrm{id}_{A^{m-1}})(\mathrm{id}_B \times \phi_S \times \mathrm{id}_{A^{m-2}}) \cdots (\mathrm{id}_{B^{m-1}} \times \phi_S).$$

Finally, we say that A and B are *compatibly shift equivalent* if they are shift equivalent via a compatible pair of matrices R and S.

The following lemma shows that certain relations between the maps $\phi_R, \phi_S, \psi_A, \psi_B$ are automatic when R and S are compatible.

Lemma 4.2. *Suppose A and B are essential matrices over V and W respectively, with entries in $\mathbb{N}$. Suppose A and B are compatibly shift equivalent with lag m, matrices R and S, and path isomorphisms $\phi_R, \phi_S, \psi_A, \psi_B$. Then*

$$(\psi_A \times \mathrm{id}_A)(\mathrm{id}_R \times \phi_S) = (\mathrm{id}_A \times \psi_A)(\phi_R^{-1} \times \mathrm{id}_S),$$
$$(\psi_B \times \mathrm{id}_B)(\mathrm{id}_S \times \phi_R) = (\mathrm{id}_B \times \psi_B)(\phi_S^{-1} \times \mathrm{id}_R).$$

Proof. First note that by compatible shift equivalence we have that

$$\begin{aligned}(\psi_A \times \mathrm{id}_R \times \mathrm{id}_S) &= (\phi_R^{(m)} \times \mathrm{id}_S)^{-1}(\mathrm{id}_R \times \psi_B \times \mathrm{id}_S)\\ &= (\phi_R^{(m)} \times \mathrm{id}_S)^{-1}(\mathrm{id}_R \times \phi_S^{(m)})^{-1}(\mathrm{id}_R \times \mathrm{id}_S \times \psi_A).\end{aligned}$$

However, since

$$(\psi_A \times \mathrm{id}_R \times \mathrm{id}_S)(\mathrm{id}_R \times \mathrm{id}_S \times \psi_A^{-1}) = (\mathrm{id}_{A^m} \times \psi_A^{-1})(\psi_A \times \mathrm{id}_{A^m}),$$

we actually get that

$$(\psi_A \times \mathrm{id}_{A^m}) = (\mathrm{id}_{A^m} \times \psi_A)(\phi_R^{(m)} \times \mathrm{id}_S)^{-1}(\mathrm{id}_R \times \phi_S^{(m)})^{-1}. \tag{4-2}$$

Now let $(r, s, a) \in E_R \times E_S \times E_A$. Define $r_0 = r$, $s_0 = s$ and $a_0 = a$. Since A is essential, we can find $a_1, \dots, a_m, a_{m+1}, \dots, a_{2m-1} \in E_A$ so that $a_0 a_1 \cdots a_m a_{m+1} \cdot \dots \cdot a_{2m-1} \in E_A^{m+1}$. We may then define inductively $r_k \in E_R$, $s_k \in E_S$ and $a'_k \in E_A$ for $0 \le k \le 2m-1$ so that

$$(\phi_R^{-1} \times \mathrm{id}_S)(\mathrm{id}_R \times \phi_S^{-1})(r_k, s_k, a_k) = (a'_k, r_{k+1}, s_{k+1}).$$

From (4-2) we get that $\psi_A(r_0, s_0) = a'_0 \cdots a'_{m-1}$, that $\psi_A(r_1, s_1) = a'_1 \cdots a'_m$ and that $a_0 \cdots a_{m-1} = \psi_A(r_m, s_m) = a'_m \cdots a'_{2m-1}$. In particular, we see that $a = a_0 = a'_m$.

To prove the first equality in the statement, we compute

$$\begin{aligned}(\mathrm{id}_A \times \psi_A)(\phi_R^{-1} \times \mathrm{id}_S)(\mathrm{id}_R \times \phi_S^{-1})(r, s, a) &= (\mathrm{id}_A \times \psi)(a'_0, r_1, s_1)\\ &= a'_0 a'_1 \cdots a'_m = \psi_A(r, s) a'_m = \psi_A(r, s) a = (\psi_A \times \mathrm{id})(r, s, a).\end{aligned}$$

This shows that $(\mathrm{id}_A \times \psi_A)(\phi_R^{-1} \times \mathrm{id}_S)(\mathrm{id}_R \times \phi_S^{-1}) = (\psi_A \times \mathrm{id})$, which is then equivalent to the first equality $(\psi_A \times \mathrm{id}_A)(\mathrm{id}_R \circ \phi_S) = (\mathrm{id}_A \times \psi_A)(\phi_R^{-1} \times \mathrm{id}_S)$. A symmetric argument works to show the second equality as well. $\square$

Remark 4.3. Notice from the proof that we also get that ψ_A and ψ_B are uniquely determined by ϕ_R and ϕ_S. Indeed, from (4-2) we get that

$$(\psi_A \times \psi_A^{-1}) = (\phi_R^{(m)} \times \mathrm{id}_S)^{-1}(\mathrm{id}_R \times \phi_S^{(m)})^{-1},$$

so that compressing to the first part yields back ψ_A. A similar argument then works for ψ_B as well.

Proposition 4.4. *Compatible shift equivalence is an equivalence relation on the collection of essential matrices with entries in* $\mathbb{N}$.

Proof. That compatible shift equivalence is reflexive and symmetric is clear. Thus, we need to show transitivity.

Let A, B and C be essential matrices over $\mathbb{N}$. Suppose that A and B are compatibly shift equivalent with lag m, matrices R, S and path isomorphisms ψ_A, ψ_B, ϕ_R, ϕ_S, while B and C are compatibly shift equivalent with lag m', matrices R', S' and path isomorphisms ψ'_B, ψ'_C, $\phi_{R'}$, $\phi_{S'}$. We claim that A and C are compatibly shift equivalent with lag $m+m'$, matrices RR', $S'S$ and path isomorphisms ψ'_A, ψ_C, $\phi_{RR'}$, $\phi_{S'S}$ given by

$$\begin{aligned}
\psi'_A &:= (\mathrm{id}_{A^{m'}} \times \psi_A)((\phi_R^{(m')})^{-1} \times \mathrm{id}_S)(\mathrm{id}_R \times \psi'_B \times \mathrm{id}_S),\\
\psi_C &:= (\mathrm{id}_{C^m} \times \psi'_C)((\phi_{S'}^{(m)})^{-1} \times \mathrm{id}_{R'})(\mathrm{id}_{S'} \times \psi_B \times \mathrm{id}_{R'}),\\
\phi_{RR'} &:= (\mathrm{id}_R \times \phi_{R'})(\phi_R \times \mathrm{id}_{R'}),\\
\phi_{S'S} &:= (\mathrm{id}_S \times \phi_{S'})(\phi_S \times \mathrm{id}_{S'}).
\end{aligned}$$

It is easy to see that $RR'S'S = A^{m+m'}$, $S'SRR' = C^{m+m'}$, $RR'C = ARR'$ and $CS'S = S'SA$. Moreover, it is clear that ψ'_A, ψ_C, $\phi_{RR'}$, $\phi_{S'S}$ are path isomorphisms. Thus, in order to show that the above data yields a compatible shift equivalence, we need only show

$$\begin{aligned}
\phi_{RR'}^{(m+m')}(\psi'_A \times \mathrm{id}_{RR'}) &= \mathrm{id}_{RR'} \times \psi_C,\\
\phi_{S'S}^{(m+m')}(\psi_C \times \mathrm{id}_{S'S}) &= \mathrm{id}_{S'S} \times \psi'_A.
\end{aligned}$$

We will show the first of these equalities, and the second will follow from a symmetric argument.

First, let $r_1r'_1s'sr_2r'_2 \in E_{RR'} \times E_{S'S} \times E_{RR'}$, and define

$$\mu_A r_3 := (\phi_R^{(m')})^{-1}(r_1\psi'_B(r'_1s')) \quad \text{and} \quad r'_3\mu_C := \phi_{R'}^{(m)}(\psi_B(sr_2)r'_2). \tag{4-3}$$

Then we get that

$$\begin{aligned}
\psi'_A(r_1r'_1s's) &= (\mathrm{id}_{A^m} \times \psi_A)((\phi_R^{(m)})^{-1} \times \mathrm{id}_S)(r_1\psi'_B(r's')s)\\
&= (\mathrm{id}_{A^m} \times \psi_A)(\mu_A r_3 s) = \mu_A\psi_A(r_3s),
\end{aligned}$$

and from Lemma 4.2 we also get that

$$\begin{aligned}
\psi_C(s'sr_2r'_2) &= (\mathrm{id}_{C^m} \times \psi'_C)((\phi_{S'}^{(m)})^{-1} \times \mathrm{id}_{R'})(s'\psi_B(sr_2)r'_2)\\
&= (\psi'_C \times \mathrm{id}_{C^m})(\mathrm{id}_{S'} \times \phi_{R'}^{(m)})(s'\psi_B(sr_2)r'_2) = \psi'_C(s'r'_3)\mu_C.
\end{aligned}$$

Thus, together we obtain

$$\psi'_A(r_1r'_1s's) = \mu_A\psi_A(r_3s) \quad \text{and} \quad \psi_C(s'sr_2r'_2) = \psi'_C(s'r'_3)\mu_C. \tag{4-4}$$

Next, from compatibility we also get

$$\phi_R^{(m)}(\psi_A(r_3 s)r_2) = r_3\psi_B(sr_2) \quad \text{and} \quad \phi_{R'}^{(m)}(\psi'_B(r'_1 s')r'_3) = r'_1\psi'_C(s'r'_3). \tag{4-5}$$

Combining (4-3), (4-4), (4-5), we compute

$$\begin{aligned}
\phi_{RR'}^{(m+m')}(\psi'_A \times \mathrm{id}_{RR'})(r_1 r'_1 s' s r_2 r'_2) &= \phi_{RR'}^{(m+m')}(\mu_A \psi_A(r_3 s) r_2 r'_2) \\
&= (\mathrm{id}_R \times \phi_{R'}^{(m+m')})(\phi_R^{(m+m')} \times \mathrm{id}_{R'})(\mu_A \psi_A(r_3 s) r_2 r'_2) \\
&= (\mathrm{id}_R \times \phi_{R'}^{(m+m')})(\phi_R^{(m')} \times \mathrm{id}_{B^m} \times \mathrm{id}_{R'})(\mu_A r_3 \psi_B(s r_2) r'_2) \\
&= (\mathrm{id}_R \times \phi_{R'}^{(m+m')})(r_1 \psi'_B(r'_1 s')\psi_B(s r_2) r'_2) \\
&= (\mathrm{id}_R \times \phi_{R'}^{(m')} \times \mathrm{id}_{C^m})(r_1 \psi'_B(r'_1 s') r'_3 \mu_C) \\
&= r_1 r'_1 \psi'_C(s' r'_3)\mu_C \\
&= r_1 r'_1 \psi_C(s' s r_2 r'_2) \\
&= (\mathrm{id}_{RR'} \times \psi_C)(r_1 r'_1 s' s r_2 r'_2).
\end{aligned}$$

Thus, we have shown $\phi_{RR'}^{(m+m')}(\psi'_A \times \mathrm{id}_{RR'}) = \mathrm{id}_{RR'} \times \psi_C$ as desired. □

Corollary 4.5. *Let A and B be essential matrices over V and W respectively, with entries in $\mathbb{N}$. If A and B are strong shift equivalent, then they are compatibly shift equivalent.*

Proof. By Proposition 4.4 we know that compatible shift equivalence is an equivalence relation. Hence, it will suffice to show that if A and B are elementary shift related via R and S, then R and S are compatible.

Since A and B are elementary shift related via R and S, we have that $A = RS$ and $B = SR$. So choose some path isomorphisms $\psi_A : E_R \times E_S \to E_A$ and $\psi_B : E_S \times E_S \to E_B$ and define

$$\phi_R : E_A \times E_R \to E_R \times E_A \quad \text{and} \quad \phi_S : E_B \times E_S \to E_S \times E_A$$

by setting $\phi_R := (\mathrm{id}_R \times \psi_B)(\psi_A^{-1} \times \mathrm{id}_R)$ and $\phi_S := (\mathrm{id}_S \times \psi_A)(\psi_B^{-1} \times \mathrm{id}_S)$. Since the lag is $m = 1$, compatibility follows by definition of ϕ_R and ϕ_S. □

Remark 4.6. When A and B are essential matrices with entries in $\mathbb{N}$ over V and W respectively, it can be shown directly that compatible shift equivalence implies conjugacy of (X_{E_A}, σ_{E_A}) and (X_{E_B}, σ_{E_B}) with a formula for the homeomorphism $h : X_{E_A} \to X_{E_B}$ which implements the conjugacy. Thus, by Williams' theorem, it follows that A and B are strong shift equivalent. We skip the argument here, because it will follow from Theorem 7.3 that the converse of Corollary 4.5 holds.

5. Representable shift equivalence

Our goal in this section is to determine when a shift equivalence between two matrices can be represented as operators on Hilbert space. This leads to the notion of representable shift equivalence. We show that compatible shift equivalence implies representable shift equivalence, and that a representation of shift equivalence can be chosen so that the graph C*-algebras act *faithfully* on the Hilbert space.

Let A and B be matrices over $\mathbb{N}$ indexed by V and W respectively, and suppose that R and S are matrices over $\mathbb{N}$ indexed by $V \times W$ and $W \times V$ respectively. To ease some of our notation, we define two matrices indexed by $V \sqcup W$,

$$C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix},$$

so that A and B are shift equivalent with lag m via R and S if and only if

$$CD = DC \quad \text{and} \quad D^2 = C^m.$$

From shift equivalence, there exist path isomorphisms

$$\begin{aligned} \phi_R &: E_A \times E_R \to E_R \times E_B, & \phi_S &: E_B \times E_S \to E_S \times E_A, \\ \psi_A &: E_R \times E_S \to E_A^m, & \psi_B &: E_S \times E_R \to E_B^m. \end{aligned}$$

We may define path isomorphisms $\phi : E_C \times E_D \to E_D \times E_C$ and $\psi : E_D^2 \to E_C^m$ given by

$$\psi = \begin{bmatrix} \psi_A & 0 \\ 0 & \psi_B \end{bmatrix} \quad \text{and} \quad \phi = \begin{bmatrix} 0 & \phi_R \\ \phi_S & 0 \end{bmatrix}.$$

Conversely, all path isomorphisms $\phi : E_C \times E_D \to E_D \times E_C$ and $\psi : E_D^2 \to E_C^m$ must be of the above form for some path isomorphisms $\phi_R, \phi_S, \psi_A, \psi_B$ as above. Hence, we see that A and B are compatibly shift equivalent with lag m via R and S if and only if there exist path isomorphisms $\phi : E_C \times E_D \to E_D \times E_C$ and $\psi : E_D^2 \to E_C^m$ such that

$$\phi^{(m)} = (\mathrm{id}_D \times \psi)(\psi^{-1} \times \mathrm{id}_D), \tag{5-1}$$

where $\phi^{(m)} : E_C^m \times E_D \to E_D \times E_C^m$ is the path isomorphism given by

$$\phi^{(m)} := (\phi \times \mathrm{id}_{C^{m-1}})(\mathrm{id}_C \times \phi \times \mathrm{id}_{C^{m-2}}) \cdots (\mathrm{id}_{C^{m-1}} \times \phi).$$

The following is the natural way to represent shift equivalence as bounded operators on Hilbert space, via some choice of path isomorphisms as above.

Definition 5.1. Let A and B be square matrices indexed by V and W respectively, with entries in $\mathbb{N}$. Let $m \in \mathbb{N} \setminus \{0\}$ and suppose there are matrices R over $V \times W$ and S over $W \times V$ with entries in $\mathbb{N}$ for which there exist path isomorphisms

$$\phi : E_C \times E_D \to E_D \times E_C \quad \text{and} \quad \psi : E_D^2 \to E_C^m.$$

We say that R and S are *representable* via ϕ and ψ if there are Cuntz–Krieger families (P_v, S_c) for G_C and (P_v, T_d) for G_D on the same Hilbert space $\mathcal{H}$ with $P_v \neq 0$ for all $v \in V \sqcup W$ such that

$$T_{d_1 d_2} = S_{c_1 \cdots c_m}, \quad \text{when } \psi(d_1 d_2) = c_1 \cdots c_m \text{ for } d_1 d_2 \in E_D^2, \tag{5-2}$$

$$S_c T_d = T_{d'} S_{c'}, \quad \text{when } \phi(cd) = d'c' \text{ for } cd \in E_C \times E_D. \tag{5-3}$$

We say that A and B are *representable shift equivalent* if there are R and S and ϕ and ψ as above so that R and S are representable via ϕ and ψ.

Remark 5.2. When A and B both have no zero columns we may define R and S to be representable via ϕ and ψ by specifying a priori separate Cuntz–Krieger families (S_v, S_c) for G_C and (T_w, T_d) for G_D. It then follows from (5-2) that $S_v = T_v$ for every $v \in V \sqcup W$. Indeed, since no vertex is a source, for every $v \in V$ there exists $c_1 \cdots c_m \in E_C^m$ with $r(c_m) = v$, so we may take $\psi^{-1}(c_1 \cdots c_m) = d_1 d_2$ with $d_1 d_2 \in E_D^2$ and $r(d_2) = v$. So that

$$S_v = S^*_{c_1 \cdots c_m} S_{c_1 \cdots c_m} = T^*_{d_1 d_2} T_{d_1 d_2} = T_v.$$

Proposition 5.3. *Let A and B be essential matrices indexed by V and W respectively, with entries in $\mathbb{N}$ and finitely supported rows. Suppose there are a lag $m \in \mathbb{N} \setminus \{0\}$ and matrices R over $V \times W$ and S over $W \times V$ with entries in $\mathbb{N}$, and path isomorphisms*

$$\phi_R : E_A \times E_R \to E_R \times E_B, \quad \phi_S : E_B \times E_S \to E_S \times E_A,$$
$$\psi_A : E_R \times E_S \to E_A^m, \quad \psi_B : E_S \times E_R \to E_B^m$$

such that

$$(\psi_A \times \mathrm{id}_A)(\mathrm{id}_R \times \phi_S) = (\mathrm{id}_A \times \psi_A)(\phi_R^{-1} \times \mathrm{id}_S) \tag{5-4}$$

and

$$\phi_S^{(m)} = (\mathrm{id}_S \times \psi_A)(\psi_B^{-1} \times \mathrm{id}_S). \tag{5-5}$$

Then R and S are representable via ϕ and ψ.

Proof. Let $X^+_{E_D}$ be the one-sided subshift for $G_D = (V \sqcup W, E_D)$, and let $\{e_x\}_{x \in X^+_{E_D}}$ be an orthonormal basis for $\ell^2(X^+_{E_D})$. We denote by $VX^+_{E_D}$ and $WX^+_{E_D}$ the clopen subsets of those $x \in X^+_{E_D}$ such that $s(x) \in V$ and $s(x) \in W$ respectively. We then have the homeomorphisms $\psi_A^\infty : VX^+_{E_D} \to X^+_{E_A}$ and $\psi_B^\infty : WX^+_{E_D} \to X^+_{E_B}$ given by

$$\psi_A^\infty(r_0 s_0 r_1 s_1 \cdots) = \psi_A(r_0 s_0)\psi_A(r_1 s_1) \cdots,$$
$$\psi_B^\infty(s_0 r_0 s_1 r_1 \cdots) = \psi_B(s_0 r_0)\psi_B(s_1 r_1) \cdots$$

for $r_i \in E_R$ and $s_i \in E_S$.

We define P_v for $v \in V \sqcup W$ by

$$P_v(e_x) = \begin{cases} e_x & \text{if } v = s(x), \\ 0 & \text{otherwise.} \end{cases}$$

For $a \in E_A$ and $x \in X^+_{E_D}$, define S_a via

$$S_a(e_x) = \begin{cases} e_y & \text{if } r(a) = s(x), \\ 0 & \text{otherwise,} \end{cases}$$

where $y := (\psi_A^\infty)^{-1}(a \psi_A^\infty(r_0 s_0 r_1 s_1 \cdots))$ when we write $x = r_0 s_0 \cdots$ for elements $r_i \in E_R$ and $s_i \in E_S$ in case that $r(a) = s(x)$.

For $b \in E_B$ and $x \in X^+_{E_D}$, define S_b via

$$S_b(e_x) = \begin{cases} e_y & \text{if } r(b) = s(x), \\ 0 & \text{otherwise,} \end{cases}$$

where $y := (\mathrm{id}_S \times \psi_A^\infty)^{-1}(\phi_S(b s_0) \psi_A^\infty(r_0 s_1 r_1 s_2 \cdots))$ when we write $x = s_0 r_0 \cdots$ for elements $s_i \in E_S$, $r_i \in E_R$ in the case that $r(b) = s(x)$.

Finally, we define for $d \in E_D$ the operator T_d via

$$T_d(e_x) = \begin{cases} e_{dx} & \text{if } r(d) = s(x), \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that (P_v, T_d) is a Cuntz–Krieger family for G_D. So we first verify that (P_v, S_c) is a Cuntz–Krieger family for G_C. Since concatenation for S_a is done simply through the homeomorphism ψ_A^∞, it is easy to show that, for $a \in E_A$ or $v \in V$, we have $S_a^* S_a = P_{r(a)}$ and $\sum_{s(e)=v} S_e S_e^* = P_v$. Now, to show the same for $b \in E_B$, for $y = s_0' r_0' s_1' r_1' \cdots$, when $s_i' \in E_S$ and $r_i' \in E_R$ we write $(\mathrm{id}_B \times \mathrm{id}_S \times \psi_A^\infty)^{-1}(\phi_S^{-1} \times \mathrm{id})(s_0' \psi_A^\infty(r_0' s_1' r_1' \cdots)) = b' s_0 r_0 \cdots$ for $b' \in E_B$, $s_i \in E_S$, $r_i \in E_R$, so that

$$S_b^*(e_y) = \begin{cases} e_{s_0 r_0 \cdots} & \text{if } r(b) = s(x),\, b = b', \\ 0 & \text{otherwise.} \end{cases}$$

From this formula it follows that $S_b^* S_b = P_{r(b)}$, and that for $w \in W$ we have $\sum_{s(e)=w} S_e S_e^* = P_w$. Thus, we see that (P_v, S_c) is a Cuntz–Krieger family for G_C. Since clearly $P_v \neq 0$ for every $v \in V \sqcup W$, we are left with verifying (5-2) and (5-3).

It is clear from the definition of S_a for $a \in E_A$ that for $r \in E_R$ and $s \in E_S$ we have $T_{rs} = S_{a_1 \cdots a_m}$ when $\psi_A(rs) = a_1 \cdots a_m$. We next show that $S_b T_s = T_{s'} S_a$ when $\phi_S(bs) = s'a$ for $b \in E_B$ and $s \in E_S$. Indeed, let $x \in X_{E_D}^+$ with $s(x) \in V$, and write $x = r_0 s_0 \cdots$ so that $S_b T_s(e_x) = e_z$ and $T_{s'} S_a(e_x) = e_{z'}$, where

$$\begin{aligned} z &= (\mathrm{id}_S \times \psi_A^\infty)^{-1} \phi_S(bs) \psi_A^\infty(r_0 s_0 \cdots), \\ z' &= s'(\psi_A^\infty)^{-1}(a \psi_A^\infty(r_0 s_0 \cdots)). \end{aligned}$$

Since $\phi(bs) = s'a$, it follows that $z = z'$, so that $S_b T_s = T_{s'} S_a$.

Next, we show that $T_{sr} = S_{b_1 \cdots b_m}$ when $\psi_B(sr) = b_1 \cdots b_m$. Indeed, let $x \in X_{E_D}^+$ with $s(x) \in W$, and write $x = s_0 r_0 \cdots$. Then we have that $S_{b_1 \cdots b_m}(e_x) = e_z$, where

$$z = (\mathrm{id}_S \times \psi_A^\infty)^{-1}(\phi_S^{(m)}(b_1 \cdots b_m s_0) \psi_A^\infty(r_0 s_1 \cdots)).$$

From (5-5) it follows that

$$z = (\psi_B)^{-1}(b_1 \cdots b_m) s_0 r_0 s_1 \cdots .$$

Thus, we see that if $\psi_B(sr) = b_1 \cdots b_m$, then $T_{sr} = S_{b_1 \cdots b_m}$.

Finally, we show that when $\phi_R(ar) = r'b$ we have $S_a T_r = T_{r'} S_b$. Indeed, let $x \in X_{E_D}^+$ with $s(x) \in W$, and write $x = s_0 r_0 \cdots$ so that $S_a T_r(e_x) = e_z$ and $T_{r'} S_b(e_x) = e_{z'}$, where

$$\begin{aligned} z &= (\psi_A^\infty)^{-1}(a \psi_A^\infty(r s_0 r_0 \cdots)), \\ z' &= r'(\mathrm{id}_S \times \psi_A^\infty)^{-1}(\phi_S(b s_0) \psi_A^\infty(r_0 s_1 r_1 \cdots)). \end{aligned}$$

But by (5-4) and the fact that $\phi_R(ar) = r'b$ we get

$$\begin{aligned} z' &= (\psi_A^\infty)^{-1}(\mathrm{id}_A \times \psi_A)(\phi_R^{-1} \times \mathrm{id}_S)(r' b s_0) \psi_A^\infty(r_0 s_1 r_1 \cdots) \\ &= (\psi_A^\infty)^{-1}(\mathrm{id}_A \times \psi_A)(a r s_0 \cdot \psi_A^\infty(r_0 s_1 r_1 \cdots)) = (\psi_A^\infty)^{-1}(a \psi_A^\infty(r s_0 r_0 \cdots)) = z. \end{aligned}$$

Hence, we get that $z' = z$, so that $S_a T_r = T_{r'} S_b$. Thus, we have shown that R and S are representable via ϕ and ψ. ∎

Remark 5.4. We note that for two essential matrices A and B with entries in $\mathbb{N}$ to be representable shift equivalent, we need only know the validity of the two asymmetric equations (5-4) and (5-5), as opposed to the symmetric equations in (4-1).

Using Szymański's uniqueness theorem [44, Theorem 1.2], we can upgrade a representation of a shift equivalence to be injective in the following sense.

Corollary 5.5. *Let A and B be essential matrices indexed by V and W respectively, with entries in $\mathbb{N}$ and finitely supported rows. Suppose that A and B are compatibly shift equivalent via R and S and path isomorphisms ϕ and ψ. Then R and S are representable via ϕ and ψ. In fact, there are Cuntz–Krieger families (P_v, S_c) for G_C and (P_v, T_d) for G_D satisfying (5-2) and (5-3) so that both of the canonical surjections $q_C : C^*(G_C) \to C^*(P_v, S_c)$ and $q_D : C^*(G_D) \to C^*(P_v, T_d)$ are injective.*

Proof. From Lemma 4.2 we know (5-4) holds. Since (5-5) holds by definition, by Proposition 5.3 there are Cuntz–Krieger families $S := (P_v, S_c)$ for G_C and $T := (P_v, T_d)$ for G_D satisfying (5-2) and (5-3).

Let $z \in \mathbb{T}$ be some unimodular scalar. Then we may define two operator families $S^{(z)} := (P_v, z \cdot S_c)$ and $T^{(z)} := (P_v, T_d^{(z)})$, where
$$T_d^{(z)} := \begin{cases} T_d & \text{if } d \in E_R, \\ z^m \cdot T_d & \text{if } d \in E_S. \end{cases}$$
Then clearly $S^{(z)}$ and $T^{(z)}$ are Cuntz–Krieger families satisfying (5-2) and (5-3).

Let $(z_n)_{n\in\mathbb{N}}$ be a countable dense subset of $\mathbb{T}$. We take $S' := \bigoplus_{n=1}^{\infty} S^{(z_n)}$ and $T' := \bigoplus_{n=1}^{\infty} T^{(z_n)}$, which are still Cuntz–Krieger families satisfying (5-2) and (5-3). By Szymański's uniqueness theorem [44, Theorem 1.2], it suffices to show that for every cycle $c_1 \cdots c_\ell$ without exits in G_C, and every cycle $d_1 \cdots d_{2t}$ without exits in G_D (which is necessarily of even length since G_D is bipartite), the spectrum of the operators $S_{c_1\cdots c_\ell}$ and $T_{d_1\cdots d_{2t}}$ contains the entire unit circle.

Since $S_{c_1\cdots c_\ell}$ and $T_{d_1\cdots d_{2t}}$ are unitaries on the ranges of $P_{s(c_1)}$ and $P_{s(d_1)}$ respectively, each of their spectra must contain some element in the unit circle. But since for every $n \in \mathbb{N}$ we have that $z_n^\ell \cdot S_{c_1\cdots c_\ell}$ is a direct summand of $S'_{c_1\cdots c_\ell}$, and $z_n^{m\cdot t} \cdot T_{d_1\cdots d_{2t}}$ is a direct summand of $T'_{d_1\cdots d_{2t}}$, we see that the spectra of $S'_{c_1\cdots c_\ell}$ and $T'_{d_1\cdots d_{2t}}$ both contain a dense subset of $\mathbb{T}$, and hence $\mathbb{T}$ itself. Thus, by Szymański's uniqueness theorem we get that the canonical surjections $C^*(G_C) \to C^*(P'_v, S'_c)$ and $C^*(G_D) \to C^*(P'_v, T'_d)$ are injective. □

Remark 5.6. Suppose we have two essential matrices A and B with entries in $\mathbb{N}$ indexed by V and W respectively, and that R, S are matrices that comprise a representable shift equivalence of lag m via path isomorphisms ϕ and ψ. It can be shown directly that A and B are compatible shift equivalent with lag m via R and S, together with the 1-1 and 2-2 corners of ψ and the 1-2 and 2-1 corners of ϕ. We skip the proof since representable shift equivalence implies compatible shift equivalence by Theorem 7.3.

6. Strong Morita shift equivalence

In this section we introduce and study strong Morita shift equivalence. This equivalence relation is expressed in terms of a specific strong Morita equivalence between Pimsner dilations and is implied by representable shift equivalence. Strong Morita shift equivalence turns out to imply the existence of a stable equivariant isomorphism of graph C*-algebras that also preserves the diagonal subalgebras.

Suppose A and B are essential matrices over $\mathbb{N}$ indexed by V and W respectively and have finitely supported rows. Suppose there are matrices R and S with entries in $\mathbb{N}$, and let C and D be as described in the previous section. Suppose further that we (only) have a path isomorphism $\psi : E_D^2 \to E_C^m$, which is then the direct sum of path isomorphisms

$$\psi_A : E_R \times E_S \to E_A^m \quad \text{and} \quad \psi_B : E_S \times E_R \to E_B^m.$$

Now let (S_v, S_c) be a CK family that generates the graph C*-algebra $C^*(G_C) = C^*(G_A) \oplus C^*(G_B)$ of G_C, and let (T_v, T_d) be a CK family that generates the graph C*-algebra $C^*(G_D)$. We define

$$\mathcal{A}_n^C := \overline{\operatorname{span}}\{S_\lambda S_{\lambda'}^* \mid \lambda, \lambda' \in E_C^n,\ r(\lambda) \in V\}, \qquad \mathcal{B}_n^C := \overline{\operatorname{span}}\{S_\lambda S_{\lambda'}^* \mid \lambda, \lambda' \in E_C^n,\ r(\lambda) \in W\},$$
$$\mathcal{A}_n^D := \overline{\operatorname{span}}\{T_\mu T_{\mu'}^* \mid \mu, \mu' \in E_D^n,\ r(\mu) \in V\}, \quad \mathcal{B}_n^D := \overline{\operatorname{span}}\{T_\mu T_{\mu'}^* \mid \mu, \mu' \in E_D^n,\ r(\mu) \in W\},$$

with direct limits $\mathcal{A}_\infty^C$, $\mathcal{B}_\infty^C$, $\mathcal{A}_\infty^D$ and $\mathcal{B}_\infty^D$. Then it follows that $\mathcal{A}_\infty^C \oplus \mathcal{B}_\infty^C$ is canonically isomorphic to the fixed-point algebra $C^*(G_C)_0$ of $C^*(G_C)$ with its canonical gauge action, and that $\mathcal{A}_\infty^D \oplus \mathcal{B}_\infty^D$ is canonically isomorphic to the fixed-point algebra $C^*(G_D)_0$ of $C^*(G_D)$ with its canonical gauge action.

Thinking of $X(A)$ and $X(B)$ as block diagonal C*-subcorrespondences of $X(C)$ in the natural way, we get that the Pimsner dilations $X(A)_\infty$ and $X(B)_\infty$ are naturally identified with the direct limits of C*-correspondences

$$X(A)_n := \overline{\operatorname{span}}\{S_\lambda S_{\lambda'}^* \mid \lambda \in E_C^{n+1},\ \lambda' \in E_C^n,\ s(\lambda) \in V\},$$

which is a C*-correspondence from $\mathcal{A}_n^C$ to $\mathcal{A}_{n+1}^C$, and

$$X(B)_n := \overline{\operatorname{span}}\{S_\lambda S_{\lambda'}^* \mid \lambda \in E_C^{n+1},\ \lambda' \in E_C^n,\ s(\lambda) \in W\},$$

which is a C*-correspondence from $\mathcal{B}_n^C$ to $\mathcal{B}_{n+1}^C$.

Similarly, thinking of $X(R)$ and $X(S)$ as the off-diagonal C*-subcorrespondences of $X(D)$ in the natural way, we get the C*-correspondences (which are only denoted as) $X(R)_\infty$ and $X(S)_\infty$ as the direct limits of C*-correspondences

$$X(R)_n := \overline{\operatorname{span}}\{T_\mu T_{\mu'}^* \mid \mu \in E_D^{n+1},\ \mu' \in E_D^n,\ s(\mu) \in V\},$$

which is a C*-correspondence from $\mathcal{B}_n^D$ to $\mathcal{A}_{n+1}^D$, and

$$X(S)_n := \overline{\operatorname{span}}\{T_\mu T_{\mu'}^* \mid \mu \in E_D^{n+1},\ \mu' \in E_D^n,\ s(\mu) \in W\},$$

which is a C*-correspondence from $\mathcal{A}_n^D$ to $\mathcal{B}_{n+1}^D$. Note, however, that $X(R)_\infty$ and $X(S)_\infty$ are not Pimsner dilations in the sense described in Section 3, because they are C*-correspondences over possibly different left and right coefficient C*-algebras.

Recall that $X(D)_\infty$ is the C*-correspondence over $C^*(G_D)_0 = \mathcal{A}_\infty^D \oplus \mathcal{B}_\infty^D$ given by $X(D)_\infty = X(D) \cdot (\mathcal{A}_\infty^D \oplus \mathcal{B}_\infty^D)$, which is identified with the direct limit

$$\overline{\operatorname{span}}\{T_\mu T_{\mu'}^* \mid \mu \in E_D^{n+1},\ \mu' \in E_D^n,\ n \in \mathbb{N}\}.$$

The $\mathcal{B}^D_\infty$-$\mathcal{A}^D_\infty$ correspondence $X(R)_\infty$ and the $\mathcal{A}^D_\infty$-$\mathcal{B}^D_\infty$ correspondence $X(S)_\infty$ coincide with the 1-2 corner and 2-1 corner (respectively) of the Pimsner dilation $X(D)_\infty$ and satisfy the equalities

$$X(R)_\infty = X(R)\cdot\mathcal{B}^D_\infty, \quad X(S)_\infty = X(S)\cdot\mathcal{A}^D_\infty.$$

Now, using the map $\psi : E^2_D \to E^m_C$, for each $k\in\mathbb{N}$ and $\mu_1,\dots,\mu_k\in E^2_D$ such that $\mu_1\cdots\mu_k\in E^{2k}_D$ we may define $\psi_k : E^{2k}_D \to E^{mk}_C$ by setting $\psi_k(\mu_1\cdots\mu_k)=\psi(\mu_1)\cdots\psi(\mu_k)$. This then gives rise to $*$-isomorphisms $\Psi^A_k : \mathcal{A}^D_{2k}\to\mathcal{A}^C_{mk}$ and $\Psi^B_k : \mathcal{B}^D_{2k}\to\mathcal{B}^C_{mk}$ by setting $\Psi^A_k(T_\mu T^*_{\mu'_1}) = S_{\psi_k(\mu)}S^*_{\psi_k(\mu')}$ and $\Psi^B_k(T_\mu T^*_{\mu'}) = S_{\psi_k(\mu)}S^*_{\psi_k(\mu')}$. Since these maps are compatible with direct limits, we obtain two $*$-isomorphisms $\Psi^A_\infty : \mathcal{A}^D_\infty\to\mathcal{A}^C_\infty$ and $\Psi^B_\infty : \mathcal{B}^D_\infty\to\mathcal{B}^C_\infty$ that we will use as identifications between the coefficient C*-algebras. For instance, this allows us to turn $X(R)_\infty$ into a $\mathcal{B}^C_\infty$-$\mathcal{A}^C_\infty$-bimodule, where the left and right actions are implemented via Ψ^A_∞ and Ψ^B_∞ respectively, and the inner product via Ψ^B_∞.

Definition 6.1. Let A and B be essential matrices over $\mathbb{N}$ indexed by V and W respectively, with finitely supported rows. We say that $X(A)_\infty$ and $X(B)_\infty$ are *strong Morita shift equivalent* if there are a lag $m\in\mathbb{N}\setminus\{0\}$ and matrices R over $V\times W$ and S over $W\times V$ over $\mathbb{N}$ together with path isomorphisms

$$\psi_A : E_R\times E_S\to E^m_A, \quad \psi_B : E_S\times E_R\to E^m_B$$

such that $X(R)_\infty$ is a strong Morita equivalence between $X(A)_\infty$ and $X(B)_\infty$, up to the identifications Ψ^A_∞ and Ψ^B_∞. More precisely, when $X(R)_\infty$ is considered as a $\mathcal{B}^C_\infty$-$\mathcal{A}^C_\infty$-bimodule via Ψ^A_∞ and Ψ^B_∞, there exists a unitary bimodule isomorphism $U : X(A)_\infty\otimes X(R)_\infty\to X(R)_\infty\otimes X(B)_\infty$ such that for every $a\in\mathcal{A}^C_\infty$, $b\in\mathcal{B}^C_\infty$ and $\xi\in X(A)_\infty\otimes X(R)_\infty$ we have

$$U(a\cdot\xi\cdot b) = a\cdot U(\xi)\cdot b.$$

Henceforth, we will no longer belabor the point of distinguishing between $\mathcal{A}^C_\infty$ and $\mathcal{A}^D_\infty$ and between $\mathcal{B}^C_\infty$ and $\mathcal{B}^D_\infty$. However, we emphasize that this identification is important in the above definition, and depends on the choice of the maps ψ_A and ψ_B. We have already seen that

$$X(A)_\infty = X(A)\cdot\mathcal{A}_\infty, \quad X(B)_\infty = X(B)\cdot\mathcal{B}_\infty,$$
$$X(R)_\infty = X(R)\cdot\mathcal{B}_\infty, \quad X(S)_\infty = X(S)\cdot\mathcal{A}_\infty,$$

so that by the above discussion and identifications using Ψ^A_∞ and Ψ^B_∞, as $\mathcal{B}_\infty$-$\mathcal{A}_\infty$ correspondences we may canonically identify

$$X(A)_\infty\otimes_{\mathcal{A}_\infty} X(R)_\infty \cong X(A)\otimes_{\mathcal{A}} X(R)\cdot\mathcal{B}_\infty,$$
$$X(R)_\infty\otimes_{\mathcal{B}_\infty} X(B)_\infty \cong X(R)\otimes_{\mathcal{B}} X(B)\cdot\mathcal{B}_\infty.$$

Proposition 6.2. *Let A and B be essential matrices over $\mathbb{N}$ indexed by V and W respectively, with finitely supported rows. Suppose A and B are representable shift equivalent with lag $m\in\mathbb{N}$ via R and S, together with path isomorphisms ϕ and ψ. Then $X(A)_\infty$ and $X(B)_\infty$ are strong Morita shift equivalent with lag m via R, S and ψ.*

Proof. By Corollary 5.5 we have a Cuntz–Krieger family (P_v, S_c) for G_C generating $C^*(G_C)$ and a Cuntz–Krieger family (P_v, T_d) for G_D generating $C^*(G_D)$ on the same Hilbert space $\mathcal{H}$, satisfying (5-2) and (5-3). Thus, we are in the context of the discussion above.

By the identifications preceding the theorem, we may define maps

$$U_{AR} : X(A)_\infty \otimes_{\mathcal{A}_\infty} X(R)_\infty \to X(A) \cdot X(R) \cdot \mathcal{B}_\infty,$$
$$U_{RB} : X(R)_\infty \otimes_{\mathcal{B}_\infty} X(B)_\infty \to X(R) \cdot X(B) \cdot \mathcal{B}_\infty$$

(where the notation $X(A) \cdot X(R) \cdot \mathcal{B}_\infty$ and $X(R) \cdot X(B) \cdot \mathcal{B}_\infty$ are understood as the closed linear span of products), by setting

$$U_{AR}(S_a \otimes T_r \cdot w) = S_a T_r \cdot w \quad \text{and} \quad U_{RB}(T_r \otimes S_b \cdot w) = T_r S_b \cdot w$$

for $a \in E_A,\ r \in E_r,\ b \in E_B$ and $w \in \mathcal{B}_\infty$. It is straightforward to show that U_{AR} and U_{RB} are well-defined unitary right $\mathcal{B}_\infty$-module maps. Thus, we are left with showing that U_{AR} and U_{RB} are left $\mathcal{A}_\infty$-module maps.

We first show that U_{AR} is a left $\mathcal{A}_\infty$-module map. We let $S_\lambda S_{\lambda'}^* \in \mathcal{A}_\infty$ for $\lambda, \lambda' \in E_A^{mk}$, and note that it will suffice to show that for $a \in E_A$ and $r \in E_R$ we have

$$U_{AR}(S_\lambda S_{\lambda'}^* (S_a \otimes T_r)) = S_\lambda S_{\lambda'}^* S_a T_r.$$

To prove this, let $e \in E_A$ be some edge so that $s(e) = r(\lambda) = r(\lambda')$, and suppose that $\lambda = a_1 \nu$ and $\lambda' = a_1' \nu'$ for $a_1, a_1' \in E_A$. Write

$$\nu e = \psi(r_1 s_1) \cdots \psi(r_k s_k) \quad \text{and} \quad \nu' e = \psi(r_1' s_1') \cdots \psi(r_k' s_k')$$

for $r_i, r_i' \in E_R$ and $s_i, s_i' \in E_S$. Then, we have

$$\begin{aligned}
S_\lambda S_{\lambda'}^* S_a \otimes T_r &= \delta_{a,a_1'} \cdot S_{a_1} S_\nu S_{\nu'}^* \otimes T_r = \delta_{a,a_1'} \cdot S_{a_1} \sum_{e \in s^{-1}(r(\lambda))} S_{\nu e} S_{\nu' e}^* \otimes T_r \\
&= \delta_{a,a_1'} \cdot S_{a_1} \otimes \sum_{e \in s^{-1}(r(\lambda))} T_{r_1 s_1 \cdots r_k s_k} T^*_{r_1' s_1' \cdots r_k' s_k'} T_r \\
&= \delta_{a,a_1'} \cdot S_{a_1} \otimes T_{r_1} \sum_{e \in s^{-1}(r(\lambda))} \delta_{r_1',r} \cdot T_{s_1 \cdots r_k s_k} T^*_{s_1' \cdots r_k' s_k'} \\
&= \delta_{a,a_1'} \cdot S_{a_1} \otimes T_{r_1} \sum_{e \in s^{-1}(r(\lambda))} \delta_{r_1',r} \cdot \sum_{f \in s^{-1}(r(\lambda e))} S_{\psi_k^{-1}(s_1 \cdots r_k s_k f)} S_{\psi_k^{-1}(s_1' \cdots r_k' s_k' f)},
\end{aligned}$$

where in the above calculations we used the identifications via Ψ_∞^A and Ψ_∞^B. Essentially the same calculation, using (5-2) instead, will show that

$$S_\lambda S_{\lambda'}^* S_a T_r = \delta_{a,a_1'} \cdot S_{a_1} T_{r_1} \sum_{e \in s^{-1}(r(\lambda))} \delta_{r_1,r} \cdot \sum_{f \in s^{-1}(r(\lambda e))} S_{\psi_k^{-1}(s_1 \cdots r_k s_k f)} S_{\psi_k^{-1}(s_1' \cdots r_k' s_k' f)}.$$

Thus, we see that U_{AR} is a left $\mathcal{A}_\infty$-module map.

Next, we show that U_{RB} is a left $\mathcal{A}_\infty$-module map. We let $S_\lambda S_{\lambda'}^* \in \mathcal{A}_\infty$ with $\lambda, \lambda' \in E_A^{mk}$, and note that it will suffice to show that for $r \in E_R$ and $b \in E_B$ we have

$$U_{RB}(S_\lambda S_{\lambda'}^* T_r \otimes S_b) = S_\lambda S_{\lambda'}^* T_r S_b.$$

Write $\lambda = \psi(r_1s_1)\cdots\psi(r_ks_k)$ and $\lambda' = \psi(r_1's_1')\cdots\psi(r_k's_k')$, and then further write $\psi(s_1r_2) = b_1\sigma$ and $\psi(s_1'r_2') = b_1'\sigma'$ for $b_1, b_1' \in E_B$. Then, we have

$$\begin{aligned} S_\lambda S_{\lambda'}^* T_r \otimes S_b &= T_{r_1s_1\cdots r_ks_k} T^*_{r_1's_1'\cdot r_k's_k'} T_r \otimes S_b \\ &= \delta_{r_1',r} T_{r_1} \sum_{r\in r^{-1}(\lambda)} T_{s_1r_2\cdots s_kr} T^*_{s_1'r_2'\cdots s_k'r} \otimes S_b \\ &= \delta_{r_1',r} T_{r_1} \otimes \sum_{r\in r^{-1}(\lambda)} S_{\psi(s_1r_2)\cdots\psi(s_kr)} S^*_{\psi(s_1'r_2')\cdots\psi(s_k'r)} S_b \\ &= \delta_{r_1',r}\delta_{b_1',b}\cdot T_{r_1}\otimes S_{b_1} \sum_{r\in r^{-1}(\lambda)} S_{\sigma\psi(s_2r_3)\cdots\psi(s_kr)} S^*_{\sigma'\psi(s_2'r_3')\cdots\psi(s_k'r)}, \end{aligned}$$

where in the above calculation we used the identifications via Ψ_∞^A and Ψ_∞^B. Essentially the same calculation, using (5-2) instead, will show that

$$S_\lambda S_{\lambda'}^* T_r S_b = \delta_{r_1',r}\delta_{b_1',b}\cdot T_{r_1} S_{b_1} \sum_{r\in r^{-1}(\lambda)} S_{\sigma\psi(s_2r_3)\cdots\psi(s_kr)} S^*_{\sigma'\psi(s_2'r_3')\cdots\psi(s_k'r)}.$$

Thus, we see that U_{BR} is a left $\mathcal{A}_\infty$-module map.

Thus, to conclude the proof we need only show the equality

$$X(A)\cdot X(R)\cdot \mathcal{B}_\infty = X(R)\cdot X(B)\cdot \mathcal{B}_\infty.$$

However, it is clear from (5-3) that $X(A)\cdot X(R) = X(R)\cdot X(B)$, so we are done. Thus, the map $U_{RB}^{-1}\circ U_{AR}$ is a unitary isomorphism, showing that $X(R)_\infty$ is a strong Morita equivalence between $X(A)_\infty$ and $X(B)_\infty$. □

Suppose A and B are matrices indexed by V and W respectively, and suppose there are matrices R and S, so that C and D are as described in the previous section. Suppose we have a path isomorphism $\psi : E_D^2 \to E_C^m$.

Assume we have a unitary $U : X(A)_\infty \otimes X(R)_\infty \to X(R)_\infty \otimes X(B)_\infty$ such that for every $\xi \in X(A)_\infty \otimes X(R)_\infty$ and every $a \in \mathcal{A}_\infty$ and $b \in \mathcal{B}_\infty$ we have $U(b\xi a) = bU(\xi)a$. We refer the reader to [1] and [36, Section 2] for the basic theory of Morita equivalence, linking algebras and crossed products by Hilbert bimodules that we shall need in what follows. We denote by $\mathcal{L}$ the linking algebra of $X(R)_\infty$ given by

$$\mathcal{L} := \begin{bmatrix} \mathcal{A}_\infty & X(R)_\infty \\ X(R)_\infty^* & \mathcal{B}_\infty \end{bmatrix},$$

and by W the $\mathcal{L}$-imprimitivity bimodule

$$W := \begin{bmatrix} X(A)_\infty & X(R)_\infty \otimes_{\mathcal{B}_\infty} X(B)_\infty \\ X(R)_\infty^* \otimes_{\mathcal{A}_\infty} X(A)_\infty & X(B)_\infty \end{bmatrix}.$$

It is shown in the proof of [1, Theorem 4.2] that there are two complementary and full projections $p_A, p_B \in \mathcal{M}(\mathcal{O}(W))$ (corresponding to the 2×2 block form of W) together with two $*$-isomorphisms $\varphi_E : \mathcal{O}(X(A)_\infty) \to p_A\mathcal{O}(W)p_A$ and $\varphi_F : \mathcal{O}(X(B)_\infty) \to p_B\mathcal{O}(W)p_B$ given as follows. For $\xi \in X(A)_\infty$

and $\eta \in X(B)_\infty$ we let

$$\tilde{\xi} := \begin{bmatrix} \xi & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \tilde{\eta} := \begin{bmatrix} 0 & 0 \\ 0 & \eta \end{bmatrix},$$

so that the maps φ_E and φ_F are given by

$$\varphi_E(S_\xi) = S_{\tilde{\xi}} \quad \text{and} \quad \varphi_F(S_\eta) = S_{\tilde{\eta}}.$$

In particular, we see that φ_E and φ_F are gauge equivariant.

Since W is an imprimitivity bimodule, again by [1, Theorem 2.9] we get that $\mathcal{O}(W)_0 \cong \mathcal{L}$. In particular, we get that up to the same identification, the direct sum of diagonal subalgebras $\mathcal{D}_{E_A} \oplus \mathcal{D}_{E_B}$, according to $p_A, p_B \in \mathcal{M}(\mathcal{O}(W))$, coincides with

$$\mathcal{D}_W := \overline{\operatorname{span}}\{S_\zeta S_{\zeta'}^* \mid \zeta,\ \zeta' \in W^{\otimes n},\ n \in \mathbb{N}\}.$$

Hence, we get that $\varphi_E(\mathcal{D}_{E_A}) = p_A \mathcal{D}_W p_A$ and $\varphi_F(\mathcal{D}_{E_B}) = p_B \mathcal{D}_W p_B$. Let

$$N(\mathcal{D}_W) = \{n \in \mathcal{O}(W) \mid n^* \mathcal{D}_W n,\ n \mathcal{D}_W n^* \subseteq \mathcal{D}_W\}$$

be the set of normalizers of $\mathcal{D}_W$, and denote by

$$N_\star(\mathcal{D}_W) = \{n \in N(\mathcal{D}_W) \mid \text{there exists } k \in \mathbb{Z} \text{ such that } \gamma_z(n) = z^k n\}$$

the *homogeneous* normalizers of $\mathcal{D}_W$.

Proposition 6.3. *Let A and B be essential matrices over $\mathbb{N}$, indexed by V and W respectively, with finitely supported rows. Suppose that $X(A)_\infty$ and $X(B)_\infty$ are strong Morita shift equivalent. Then there is an equivariant $*$-isomorphism $\varphi : C^*(G_A) \otimes \mathbb{K} \to C^*(G_B) \otimes \mathbb{K}$, with $\varphi(\mathcal{D}_{E_A} \otimes c_0) = \mathcal{D}_{E_B} \otimes c_0$.*

Proof. We are in the situation where we can apply the implication (7) $\Rightarrow$ (8) in [7, Corollary 11.3]. By the description of $C^*(G_A) \cong C^*(\mathcal{G}_{E_A})$ and $C^*(G_B) \cong C^*(\mathcal{G}_{E_B})$ as groupoid C*-algebras, we get that item (8) in [7, Corollary 11.3] is equivalent to $C^*(G_A)$ and $C^*(G_B)$ being stably equivariant diagonal-isomorphic. Indeed, this is because $\mathbb{Z}$-coactions correspond to topological $\mathbb{Z}$-gradings by [42, Remark 6], which in turn correspond to $\mathbb{T}$-actions by [42, Theorem 3]. This means that the second equality in item (8) of [7, Corollary 11.3] is equivalent to equivariance of the isomorphism.

Thus, to prove our result it will suffice to prove item (7) in [7, Corollary 11.3]. Everything is set up in the discussion preceding the proposition, except for one thing. We are left with showing that $p_A \mathcal{O}(W) p_B$ is the closed linear span of $p_A N_*(\mathcal{D}_W) p_B$. However, we know that $\mathcal{O}(W)$ is the closed linear span of its graded subspaces $\mathcal{O}(W)_n$ for $n \in \mathbb{Z}$. Hence, $p_A \mathcal{O}(W) p_B$ is the closed linear span of $p_A \mathcal{O}(W)_n p_B$. By [1, Theorem 2.9] we have for all $n \in \mathbb{Z}$ that

(1) $\mathcal{O}(W)_n \cong W^{\otimes n}$ for $n > 0$,

(2) $\mathcal{O}(W)_0 \cong \mathcal{L}$, and

(3) $\mathcal{O}(W)_n \cong (W^{\otimes n})^*$ for $n < 0$.

Thus, we are left with showing that each $p_A \mathcal{O}(W)_n p_B$ is the closed linear span of its normalizers (which are automatically n-homogeneous).

For $n=0$ we get that $p_A\mathcal{O}(W)_0 p_B \cong X(R)_\infty$, for $n>0$ we get that $p_A\mathcal{O}(W)_n p_B \cong X(A)_\infty^n \otimes X(R)_\infty$, and for $n<0$ we get that $p_A\mathcal{O}(W)_n p_B \cong (X(A)_\infty^*)^n \otimes X(R)_\infty$.

Thus, $p_A\mathcal{O}(W)_n p_B$ is the closed linear span of elements of the form $S_\lambda S_{\lambda'}^* S_\alpha T_r$ (for $n \geq 0$) or $S_\lambda S_{\lambda'}^* S_\alpha^* T_r$ (for $n<0$) for $\lambda, \lambda' \in E_A^\ell$, $\alpha \in E_A^{|n|}$ and $r \in E_R$ for some $\ell \in \mathbb{N}$. In order to show that these elements are normalizers for $\mathcal{D}_W$, and since $S_\lambda S_{\lambda'}^* S_\alpha, S_\lambda S_{\lambda'}^* S_\alpha^* \in N(\mathcal{D}_{E_A})$, it will suffice to show that $T_r \mathcal{D}_{E_B} T_r^* \subseteq \mathcal{D}_{E_A}$ and that $T_r^* \mathcal{D}_{E_A} T_r \subseteq \mathcal{D}_{E_B}$ for $r \in E_R$.

We know already from the inclusions $\mathcal{D}_{E_A} \subseteq \mathcal{A}_\infty$ and $\mathcal{D}_{E_B} \subseteq \mathcal{B}_\infty$ that $\mathcal{D}_{E_A}$ is the closed linear span of elements of the form

$$S_{\psi_k(r_1 s_1 \cdots r_k s_k)} S^*_{\psi_k(r_1 s_1 \cdots r_k s_k)}$$

for paths $r_1 s_1 \cdots r_k s_k \in E_D^{2k}$ with $r_i \in E_R$ and $s_i \in E_S$, and that $\mathcal{D}_{E_B}$ is the closed linear span of elements of the form

$$S_{\psi_k(s_1 r_1 \cdots s_k r_k)} S^*_{\psi_k(s_1 r_1 \cdots s_k r_k)}$$

for paths $s_1 r_1 \cdots s_k r_k \in E_D^{2k}$, with $r_i \in E_R$ and $s_i \in E_S$. So for such paths we compute

$$T_r S_{\psi_k(s_1 r_1 \cdots s_k r_k)} S^*_{\psi_k(s_1 r_1 \cdots s_k r_k)} T_r^* = \sum_{s \in E_S} T_r T_{s_1 r_1 \cdots s_k r_k s} T^*_{s_1 r_1 \cdots s_k r_k s} T_r^* = \sum_{s \in E_S} S_{\psi_{k+1}(r s_1 r_1 \cdots s_k r_k s)} S^*_{\psi_{k+1}(r s_1 r_1 \cdots s_k r_k s)}$$

is in $\mathcal{D}_{E_B}$. On the other hand,

$$T_r^* S_{\psi_k(r_1 s_1 \cdots r_k s_k)} S^*_{\psi_k(r_1 s_1 \cdots r_k s_k)} T_r = \delta_{r,r_1} \cdot \sum_{r \in E_R} S_{\psi_k(s_1 \cdots r_k s_k r)} S^*_{\psi_k(s_1 \cdots r_k s_k r)}$$

is in $\mathcal{D}_{E_A}$. Thus, $T_r \mathcal{D}_{E_B} T_r^* \subseteq \mathcal{D}_{E_A}$ and $T_r^* \mathcal{D}_{E_A} T_r \subseteq \mathcal{D}_{E_B}$ as required. $\square$

7. Shift equivalences through the lens

In this section we orient various equivalence relations between strong shift equivalence and shift equivalence. We will assume some familiarity with crossed product C*-algebras and K-theory of C*-algebras. We recommend [48] for the basic theory of crossed product C*-algebras, [43] for the basic K-theory for C*-algebras, and especially [41, Chapter 7] for K-theory and crossed products of graph algebras by their gauge actions.

Suppose now that $G=(V,E)$ is a directed graph, and let γ be the gauge unit circle action on $C^*(G)$. We denote by $\hat{\gamma}$ the dual $\mathbb{Z}$ action on $C^*(G) \rtimes_\gamma \mathbb{T}$. By [48, Lemma 2.75] we have that $[C^*(G) \rtimes \mathbb{T}] \otimes \mathbb{K}$ and $[C^*(G) \otimes \mathbb{K}] \rtimes \mathbb{T}$ are $*$-isomorphic via the map $f \otimes K \mapsto f \cdot K$ defined for $f \in C(\mathbb{T}; C^*(G))$ and $K \in \mathbb{K}$, and that this map intertwines the action $\hat{\gamma}^G \otimes \mathrm{id}$ with the dual action $\widehat{\gamma^G \otimes \mathrm{id}}$. Using this equivariant identification together with [48, Corollary 2.48] we obtain the following standard fact which we leave for the reader to verify.

Proposition 7.1. *Let G and G' be directed graphs, and suppose there exists a $*$-isomorphism $\varphi : C^*(G) \otimes \mathbb{K} \to C^*(G') \otimes \mathbb{K}$ such that $\varphi \circ (\gamma^G \otimes \mathrm{id})_z = (\gamma^{G'} \otimes \mathrm{id})_z \circ \varphi$ for all $z \in \mathbb{T}$. Then there exists a $*$-isomorphism $\varphi \rtimes \mathrm{id} : [C^*(G) \rtimes_{\gamma^G} \mathbb{T}] \otimes \mathbb{K} \to [C^*(G') \rtimes_{\gamma^G} \mathbb{T}] \otimes \mathbb{K}$ such that $[\varphi \rtimes \mathrm{id}] \circ [\hat{\gamma}^G \otimes \mathrm{id}] = [\hat{\gamma}^{G'} \otimes \mathrm{id}] \circ [\varphi \rtimes \mathrm{id}]$.*

Let A be a finite essential matrix with entries in $\mathbb{N}$. We denote by (D_A, D_A^+) the inductive limit of the inductive system of ordered abelian groups acting on columns

$$(\mathbb{Z}^V, \mathbb{Z}_+^V) \xrightarrow{A^T} (\mathbb{Z}^V, \mathbb{Z}_+^V) \xrightarrow{A^T} (\mathbb{Z}^V, \mathbb{Z}_+^V) \xrightarrow{A^T} \cdots$$

and let $d_A : D_A \to D_A$ be the homomorphism induced by the following diagram:

$$\begin{tikzcd}
\mathbb{Z}^V \arrow[r, "A^T"] \arrow[d, "A^T"] & \mathbb{Z}^V \arrow[r, "A^T"] \arrow[d, "A^T"] & \mathbb{Z}^V \arrow[r, "A^T"] \arrow[d, "A^T"] & \cdots D_A \arrow[d, "d_A"] \\
\mathbb{Z}^V \arrow[r, "A^T"] & \mathbb{Z}^V \arrow[r, "A^T"] & \mathbb{Z}^V \arrow[r, "A^T"] & \cdots D_A
\end{tikzcd}$$

The triple (D_A, D_A^+, d_A) is called the *dimension group triple* of A.

Let H_i be an abelian group, H_i^+ a submonoid of H_i, and α_i an automorphism of H_i for $i = 1, 2$. We say two triples (H_i, H_i^+, α_i) are isomorphic if there is a group isomorphism $\phi : H_1 \to H_2$ such that $\phi(H_1^+) = H_2^+$ and $\phi \circ \alpha_1 = \alpha_2 \circ \phi$.

From work of Wagoner [46, Corollary 2.9] we have that (D_A, D_A^+, d_A) is isomorphic to a dimension group triple constructed from the edge shift (X_{E_A}, σ_{E_A}) associated to A. On the other hand, the dimension group triple (D_A, D_A^+, d_A) also coincides with a K-theory triple arising from the crossed product C*-algebra $C^*(G_A) \rtimes_{\gamma^A} \mathbb{T}$, where γ^A is the gauge unit circle action. More precisely, noting that arrow directions in [41] are reversed to ours, it follows from [41, Corollary 7.14] together with the discussion preceding [41, Lemma 7.15] that the triple (D_A, D_A^+, d_A) is isomorphic to the triple

$$\big(K_0(C^*(G_A) \rtimes_{\gamma^A} \mathbb{T}), K_0(C^*(G_A) \rtimes_{\gamma^A} \mathbb{T})^+, K_0(\hat{\gamma}_1^A)^{-1}\big),$$

where $\hat{\gamma}^A$ is the dual action of $\mathbb{Z}$ on $C^*(G_A) \rtimes_{\gamma^A} \mathbb{T}$. The implication (3) $\Rightarrow$ (4) below is what we referred to as *Krieger's corollary* in the Introduction.

Corollary 7.2. *Suppose now that A and B are two finite essential matrices with entries in $\mathbb{N}$. The former conditions imply the latter*:

(1) *A and B are strong shift equivalent.*

(2) *$C^*(G_A)$ and $C^*(G_B)$ are equivariantly stably isomorphic in a way that respects the diagonals.*

(3) *$C^*(G_A)$ and $C^*(G_B)$ are equivariantly stably isomorphic.*

(4) *A and B are shift equivalent.*

Proof. By Williams' theorem [47, Theorem 7.5.8] (see also [31]), strong shift equivalence of A and B coincides with conjugacy of two-sided edge shifts (X_{E_A}, σ_{E_A}) and (X_{E_B}, σ_{E_B}) (see [31, Theorem 7.2.7]), so we get that (1) implies (2) by the implication $(III) \Rightarrow (II)$ of [5, Theorem 5.1] (see also [8, Proposition 2.17]).

Clearly (2) $\Rightarrow$ (3), so we are left with showing (3) $\Rightarrow$ (4). To do this, we use Proposition 7.1 to get that $[C^*(G) \rtimes_{\gamma^A} \mathbb{T}] \otimes \mathbb{K}$ and $[C^*(G') \rtimes_{\gamma^B} \mathbb{T}] \otimes \mathbb{K}$ are equivariantly isomorphic with actions $\hat{\gamma}^A \otimes \mathrm{id}$ and

$\hat{\gamma}^B \otimes \mathrm{id}$ respectively. Hence, by applying K-theory we get an isomorphism of the triples

$$\big(K_0(C^*(G) \rtimes_{\gamma^A} \mathbb{T}),\, K_0(C^*(G) \rtimes_{\gamma^A} \mathbb{T})^+,\, K_0(\hat{\gamma}_1^A)\big),$$
$$\big(K_0(C^*(G') \rtimes_{\gamma^B} \mathbb{T}),\, K_0(C^*(G') \rtimes_{\gamma^B} \mathbb{T})^+,\, K_0(\hat{\gamma}_1^B)\big).$$

By Krieger's theorem [11, Theorem 6.4] (see also [29]), and up to the identification with dimension triples, we get that the triples above are isomorphic if and only if A and B are shift equivalent. Hence, (3) $\Rightarrow$ (4). □

From the perspective of C*-algebras, we get that compatible, representable, and strong Morita shift equivalences are between strong shift equivalence and shift equivalence. The work done in the previous sections allows us to orient them, and show that each one provides a new way of obstructing SSE when merely assuming SE. The equivalence between (1) and (4) in the theorem below realizes a goal sought after by Muhly, Pask and Tomforde in [37, Remark 5.5], and provides a characterization of strong shift equivalence of matrices in terms of strong Morita shift equivalence.

Theorem 7.3. *Suppose A and B are two finite essential matrices with entries in $\mathbb{N}$. Then the following are equivalent*:

(1) *A and B are strong shift equivalent.*

(2) *A and B are compatibly shift equivalent.*

(3) *A and B are representable shift equivalent.*

(4) *$X(A)_\infty$ and $X(B)_\infty$ are strong Morita shift equivalent.*

(5) *$C^*(G_A)$ and $C^*(G_B)$ are equivariantly stably isomorphic in a way that respects the diagonals.*

Proof. It follows from Corollary 4.5 that (1) $\Rightarrow$ (2), from Corollary 5.5 that (2) $\Rightarrow$ (3), and from Proposition 6.2 that (3) $\Rightarrow$ (4). The implication (4) $\Rightarrow$ (5) is provided by Proposition 6.3. Finally, the implication (5) $\Rightarrow$ (1) is granted to us by combining the implication $(II) \Rightarrow (III)$ of [5, Theorem 5.1] and the fact that conjugacy of two-sided subshifts coincides with strong shift equivalence (Williams' theorem [47]). □

We note here that if the construction in [22, Section 5] is applied to $E := X(A)$, $F := X(B)$ and $X := X(D)$ (as in the notation of [22]), we obtain again the correspondences E_∞, F_∞ and X_∞ as in [22, Section 5]. This follows from uniqueness of Pimsner dilations [22, Theorem 3.9]. In particular, the correspondence $X(R)_\infty$ coincides with "R_∞" in [22, Section 5] as the 1-2 corner of $X_\infty = X(D)_\infty$.

Hence, when we specify to $E := X(A)$ and $F := X(B)$, in the proof of [22, Theorem 5.8] it is erroneously claimed that $X(R)_\infty$ is the imprimitivity bimodule that implements a strong Morita equivalence between $X(A)_\infty$ and $X(B)_\infty$, through the identifications coming from ψ_A and ψ_B. However, by our result this would show that shift equivalence implies strong shift equivalence. Hence, the strategy of proof in [22, Theorem 5.8] cannot be made to work, as the following cutoff result demonstrates.

Theorem 7.4. *There exist finite aperiodic irreducible matrices A and B with entries in $\mathbb{N}$ that such that $X(A)_\infty$ and $X(B)_\infty$ are strong Morita equivalent, but not strong Morita shift equivalent.*

Proof. Let A and B be the aperiodic and irreducible counterexamples of Kim and Roush from [28], so that they are shift equivalent but not strong shift equivalent. By the result of Bratteli and Kishimoto [4] we know that $C^*(G_A)$ and $C^*(G_B)$ are equivariantly stably isomorphic, so that by Theorem 3.17 we get that $X(A)_\infty$ and $X(B)_\infty$ are strong Morita equivalent.

On the other hand, since A and B are not strong shift equivalent, by Theorem 7.3 we get that $X(A)_\infty$ and $X(B)_\infty$ cannot be strong Morita shift equivalent. □

Remark 7.5. Suppose A and B are aperiodic and irreducible matrices over $\mathbb{N}$, and not 1×1. Then the following conditions are equivalent:

(1) A and B are shift equivalent.

(2) $X(A)$ and $X(B)$ are shift equivalent.

(3) $X(A)_\infty$ and $X(B)_\infty$ are strong Morita equivalent.

(4) $C^*(G_A)$ and $C^*(G_B)$ are equivariantly stably isomorphic.

(5) The triples (D_A, D_A^+, d_A) and (D_B, D_B^+, d_B) are isomorphic.

Indeed, Krieger's theorem shows that (1) $\Leftrightarrow$ (5), Proposition 3.5 shows that (1) $\Leftrightarrow$ (2), Theorem 3.17 shows that (3) $\Leftrightarrow$ (4), and Corollary 7.2 shows that (1) $\Rightarrow$ (4). Finally, from Bratteli and Kishimoto [4, Corollary 4.3] we get (5) $\Rightarrow$ (4), which finishes the proof.

Note that the above proof avoids the implication (2) $\Rightarrow$ (3). Since now the validity of [22, Theorem 5.8] is in question, it is unknown whether one can prove (2) $\Rightarrow$ (3) directly, without classification techniques as in [4].

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