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SCATTERING OF THE THREE-DIMENSIONAL CUBIC NONLINEAR SCHRÖDINGER EQUATION WITH PARTIAL HARMONIC POTENTIALS

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We consider the following three-dimensional defocusing cubic nonlinear Schrödinger equation (NLS) with partial harmonic potential:

$$\begin{cases} i\,\partial_t u + (\Delta_{\mathbb{R}^3} - x^2)u = |u|^2 u, \\ u|_{t=0} = u_0. \end{cases} \tag{NLS}$$

Our main result shows that the solution u scatters for any given initial data u_0 with finite mass and energy.

The main new ingredient in our approach is to approximate (NLS) in the large-scale case by a relevant dispersive continuous resonant (DCR) system. The proof of global well-posedness and scattering of the new (DCR) system is greatly inspired by the fundamental works of Dodson (2012, 2016) in his study of scattering for the mass-critical nonlinear Schrödinger equation. The analysis of (DCR) system allows us to utilize the additional regularity of the smooth nonlinear profile so that the celebrated concentration-compactness/rigidity argument of Kenig and Merle (2006, 2008) applies.

1. Introduction

1.1. ***Background and motivation.*** Consider the Cauchy problem for the following family of *nonlinear Schrödinger equations* in $\mathbb{R}^d$, $d \in \mathbb{N}$, with *harmonic oscillators*:

$$\begin{cases} i\,\partial_t u + \Delta_{\mathbb{R}^d} u - (\omega^2|y|^2 + |x|^2)u = \mu|u|^{p-1}u, \\ u|_{t=0} = u_0, \end{cases} \tag{1-1}$$

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where $1 < p < \infty$, $(y, x) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $d = d_1 + d_2$, and $d_1, d_2 \in \mathbb{N}$, $d_1 \geq 1$. The complex-valued function $u = u(t, y, x) \colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is the unknown wave function. The parameter ω is 0 or 1, with $\omega = 1$ corresponding to the quadratic potential case and $\omega = 0$ corresponding to the partial harmonic oscillator on the left-hand side. The parameter $\mu = 1$ corresponds to the *defocusing* case and $\mu = -1$ to the *focusing* case. Equation (1-1) arises as models for diverse physical phenomena, including Bose–Einstein condensates in a laboratory trap [Josserand and Pomeau 2001; Pitaevskii and Stringari 2003] and the envelope dynamics of a general dispersive wave in a weakly nonlinear medium. It can also be derived in the NLS with constant magnetic potential; see, for example, [Fukuizumi and Ohta 2003]. The associated conserved mass and energy of (1-1) are given by

$$\mathcal{M}(u)(t) = \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} |u(t, y, x)|^2 \, \mathrm{d}y \, \mathrm{d}x$$

and

$$\mathcal{E}^{d_1,d_2}_{\omega,\mu,p}(u)(t) = \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \tfrac{1}{2}|\nabla_{y,x} u(t, y, x)|^2 + \tfrac{1}{2}(\omega^2|y|^2 + |x|^2)|u(t, y, x)|^2 + \frac{\mu}{p+1}|u(t, y, x)|^{p+1} \, \mathrm{d}y \, \mathrm{d}x.$$

It is natural to take the initial data from the following weighted Sobolev space:

$$u_0 \in \bigl\{ f = f(y, x) \in L^2_{y,x}(\mathbb{R}^d) : \\ \|\nabla_{y,x} f\|_{L^2_{y,x}(\mathbb{R}^d)} + \||x| f\|_{L^2_{y,x}(\mathbb{R}^d)} + \omega \||y| f\|_{L^2_{y,x}(\mathbb{R}^d)} + \|f\|_{L^2_{y,x}(\mathbb{R}^d)} < \infty \bigr\}.$$

In view of the Sobolev embedding

$$H^1(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d), \qquad \begin{cases} 2 \leq q \leq 2 + \frac{4}{d-2} & \text{if } d \geq 3, \\ 2 \leq q < \infty & \text{if } d = 2, \\ 2 \leq q \leq \infty & \text{if } d = 1, \end{cases}$$

the initial data is of finite energy in the *energy-subcritical* case

$$\begin{cases} 1 < p < 1 + \frac{4}{d-2} & \text{if } d \geq 3, \\ 1 < p < \infty & \text{if } d = 1, 2, \end{cases}$$

and we call the critical case $p = 1 + \frac{4}{d-2}$, $d \geq 3$, the *energy critical* case.

The global well-posedness of (1-1) has been established in the energy-subcritical case by R. Carles [2002b; 2008] in the defocusing case $\mu = 1$, and by J. Zhang [2005] in the focusing case $\mu = -1$ when the initial energy is assumed to be less than the energy of the ground state of the related elliptic equation. The Cauchy problem for (1-1) with quadratic potential (that is, $\omega = 1$) in the energy-critical case was considered by R. Killip, M. Visan, and X. Zhang [Killip et al. 2009b] in the radial case, and in the general case later by C. Jao [2016; 2018]. They proved the global well-posedness for the defocusing case and also for the focusing case when the initial energy (resp. kinetic energy) is less than the energy (resp. kinetic energy) of the ground state. We would also like to mention the work of C. Hao, L. Hsiao and H. Li [Hao et al. 2007; 2008], where the authors proved the global well-posedness for (1-1) (when $\omega = 1$) with an additional angular momentum rotational term.

It is well known that solutions of (1-1) with a quadratic potential (i.e., $\omega = 1$) cannot scatter. However, intuitively, in the defocusing case, if we turn off the confinement in *some instead of all* of the directions, it should suffice for the condensate to evolve asymptotically freely: Indeed, if $\omega = 0$, then the operator $i\,\partial_t + \Delta_y$ should yield large time dispersion and one expects a scattering theory for (1-1). When $\omega = 0$, the scattering phenomena for (1-1) in the defocusing case has already been showed by P. Antonelli, R. Carles and J. D. Silva [Antonelli et al. 2015] (see also [Carles and Gallo 2015]) in the fully weighted space when $\omega = 0$, $\mu = 1$, $d_1 = 1, 2, 3$, $d_2 = 1$ and $1 + \frac{4}{d_1} < p < 1 + \frac{4}{d_1 - 1}$. The focusing case of (1-1) has been investigated by A. H. Ardila and R. Carles [2021] recently when the energy is strictly less than the static energy of the ground state. In this aspect, one expects the global-in-time well-posedness result for the defocusing/focusing (when energy is strictly less than the static energy of the ground state) energy-critical and subcritical cases for (1-1). On the other hand, the potential influences strongly the asymptotic dynamics of the solution. In (1-1), the x-direction is not expected to have a global-in-time dispersive estimate in view of Mehler's formula

$$e^{it(\Delta_x - |x|^2)} f(y,x) = (2\pi i \sin(2t))^{-\frac{d_2}{2}} \int_{\mathbb{R}^{d_2}} e^{\frac{i}{\sin(2t)}\left(\frac{|x|^2 + |\tilde{x}|^2}{2}\cos(2t) - x\cdot\tilde{x}\right)} f(y,\tilde{x})\,d\tilde{x} \quad \text{for all } y \in \mathbb{R}^{d_1},\ x \in \mathbb{R}^{d_2},$$

from which we can only derive the following periodic-in-time dispersive estimate:

$$\|e^{it(\Delta_x - |x|^2)} f(y,x)\|_{L_x^\infty(\mathbb{R}^{d_2})} \lesssim |\sin(2t)|^{-\frac{d_2}{2}} \|f(y,x)\|_{L_x^1(\mathbb{R}^{d_2})} \quad \text{for all } t \notin \tfrac{\pi}{2}\mathbb{Z}, \text{ for all } y \in \mathbb{R}^{d_1}.$$

Nevertheless, we have the following global-in-time dispersive estimate in the y-direction:

$$\|e^{it(\Delta_x + \Delta_y - |x|^2)} f(y,x)\|_{L_y^\infty L_x^2(\mathbb{R}^d)} \lesssim |t|^{-\frac{d_1}{2}} \|f(y,x)\|_{L_y^1 L_x^2(\mathbb{R}^d)},$$

where we used the dispersive estimate for the semigroup $e^{it\Delta_y}$ together with the L^2-norm conservation for the unitary of the operator $e^{it(\Delta_x - |x|^2)}$. Thus, according to the scattering theory for the nonlinear Schrödinger equations without potential, see for instance [Staffilani 2013; Tao 2006], one *expects a scattering result in the weighted Sobolev space* when $\omega = 0$ in the case $1 + \frac{4}{d_1} \le p \le 1 + \frac{4}{d_1 + d_2 - 2}$, with $d_1 + d_2 \ge 2$. Generally, to obtain the scattering in the intercritical case, one relies on the Morawetz estimate; see for instance [Antonelli et al. 2015]. It is difficult to deal with the scattering on the two endpoints $p = 1 + \frac{4}{d_1}$ and $p = 1 + \frac{4}{d_1 + d_2 - 2}$, which correspond to the usual d_1-dimensional mass-critical and $(d_1 + d_2)$-dimensional energy-critical nonlinear Schrödinger equation without potentials respectively. For the endpoint $p = 1 + \frac{4}{d_1 + d_2 - 2}$, the scattering is a byproduct of the proof of the global well-posedness, and we need to use induction on the energy method [Colliander et al. 2008] or the concentration-compactness/rigidity argument [Kenig and Merle 2006; 2008] to prove the global well-posedness. It seems to us one of the main difficulties is to establish a more delicate global-in-time Strichartz estimate which should be a lot combination of the local Strichartz estimate of three-dimensional Schrödinger equations as in [Barron 2021; Hani and Pausader 2014]. We refer to [Killip and Vişan 2013] for more illustration on the proof of the scattering of the nonlinear Schrödinger equations at critical regularity Sobolev space. For the endpoint $p = 1 + \frac{4}{d_1}$, global well-posedness is quite easy to get, and the main obstacle is to show the scattering. We cannot prove the scattering by the Morawetz estimate even when the initial data lies in a better regular Sobolev space $H^1_{y,x}$ because the Morawetz estimate only provides an a priori estimate of the nonendpoint Strichartz norm on the $\dot{H}_y^{1/4} L_x^2$-level but cannot give an a priori estimate of the Strichartz norm on the L^2-level, which is not enough to yield the scattering in this case. Therefore, to show the scattering, we still need to use the concentration-compactness/rigidity argument [Kenig and Merle 2006; Kenig and Merle 2008] and its mass-critical counterpart [Dodson 2012; 2016a; 2016b; Killip et al. 2008; 2009a; Killip and Vişan 2013; Tao et al. 2007a; 2008] to show the finiteness of the L^2-level Strichartz norm. In the L^2-level Strichartz norm, we need to consider not only the space and time translations of (1-1) as in the case $1+\frac{4}{d_1} < p \le 1+\frac{4}{d_1+d_2-2}$, but also the partial Galilean invariance

$$u(t,y,x) \mapsto e^{-it|\xi_0|^2} e^{iy\cdot\xi_0} u(t, y-2\xi_0 t, x),$$

where $\xi_0 \in \mathbb{R}^{d_1}$, of (1-1). In addition, by a limitation operation, it is realized that a new mass-critical nonlinear Schrödinger system can be embedded into (1-1): this new mass-critical nonlinear Schrödinger system inherits the above invariance and also has the scaling invariance in space-time, and its global well-posedness and scattering should be proven by the argument from [Dodson 2012; 2016a; 2016b; Killip et al. 2008; 2009a; Killip and Vişan 2013; Tao et al. 2007a; 2008].

In this paper, we will consider the following Cauchy problem for the defocusing cubic NLS on $\mathbb{R}^3$:

$$\begin{cases} i\partial_t u + (\Delta_{\mathbb{R}^3} - x^2)u = |u|^2 u, \\ u|_{t=0} = u_0, \end{cases} \tag{1-2}$$

where $u = u(t,y,x)\colon \mathbb{R}\times\mathbb{R}^2\times\mathbb{R} \to \mathbb{C}$ is an unknown wave function. The following mass and energy quantities are conserved by the evolution of (1-2):

$$\begin{aligned} \mathcal{M}(u(t)) &= \int_{\mathbb{R}^2\times\mathbb{R}} |u(t,y,x)|^2 \, dy\, dx, \\ \mathcal{E}(u(t)) &= \int_{\mathbb{R}^2\times\mathbb{R}} \tfrac{1}{2}|\nabla_{y,x} u(t,y,x)|^2 + \tfrac{1}{2}x^2|u(t,y,x)|^2 + \tfrac{1}{4}|u(t,y,x)|^4 \, dy\, dx. \end{aligned} \tag{ME}$$

Motivated by the mass and energy formulations, we take the initial data in the following weighted Sobolev space:

$$u_0 \in \Sigma(\mathbb{R}^3) := \{ f \in L^2_{y,x}(\mathbb{R}^3) : \|f\|_{\Sigma(\mathbb{R}^3)} := \|\nabla_y f\|_{L^2_{y,x}(\mathbb{R}^3)} + \|f\|_{L^2_y \mathcal{H}^1_x(\mathbb{R}^2\times\mathbb{R})} < \infty \},$$
$$\text{with } \|f\|_{\mathcal{H}^1_x(\mathbb{R})} = \|f\|_{H^1_x(\mathbb{R})} + \|xf\|_{L^2_x(\mathbb{R})}. \tag{1-3}$$

By the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$, $2 \le q \le 6$, the initial data is of finite mass and energy.

Observe that (1-2) is a special case of (1-1), namely, corresponding to $d_1 = 2$, $d_2 = 1$, $\omega = 0$, $\mu = 1$, $p = 1 + \frac{4}{d_1} = 3$ in (1-1). In this case, the scattering phenomena is not yet clear. As we are in the energy subcritical case $1 < p = 3 < 5$, the equation (1-2) is globally well-posed and the scattering of the solutions follows in the small initial data case $\|u_0\|_\Sigma \ll 1$, which is a byproduct of the small-data well-posedness theorem. We will briefly explore these results in Section 3 and outline the ideas of the proofs, as we did not find them in the literature.

1.2. ***Main results.*** Our main result in this article is the following scattering result for solutions of the defocusing cubic NLS (1-2). Recall that $\Sigma(\mathbb{R}^3)$ is defined in (1-3).

Theorem 1.1. *For any initial data $u_0 \in \Sigma(\mathbb{R}^3)$, there is a unique global solution $u \in C_t^0(\mathbb{R}, \Sigma(\mathbb{R}^3))$ of (1-2). Moreover, the solution scatters; namely there exist $u_\pm \in \Sigma(\mathbb{R}^3)$ such that*

$$\|u(t) - e^{it(\Delta_{\mathbb{R}^3} - x^2)} u_\pm\|_{\Sigma(\mathbb{R}^3)} \to 0 \quad \text{as } t \to \pm\infty.$$

In order to treat the general initial data with finite (but not necessarily small) Σ-norm $\|u_0\|_\Sigma < \infty$, we turn to the celebrated concentration-compactness/rigidity argument developed by C. E. Kenig and F. Merle [2006; 2008], where one key ingredient is the linear and nonlinear profile decompositions for solutions with bounded Σ-norm. The proof of Theorem 1.1 shall rely on (a corollary of) Theorem 1.2 given below. More precisely, we shall use Theorem 1.2 to prove the core result Theorem 4.9 in Section 4.2, which in return gives Theorem 4.10 in Section 4.3. Theorem 4.10 will be used later in the proof of Theorem 1.1 in Section 5.

As for the nonlinear profile decomposition, we will consider a sequence of solutions exhibiting an extreme behavior to study the concentration of the data. More precisely, we need to study the behavior of the nonlinear profile u_λ when $\lambda \to \infty$. The (simplified) nonlinear profile u_λ, $\lambda > 0$, is the solution of (1-2)

$$\begin{cases} i\partial_t u_\lambda + \Delta_y u_\lambda + (\Delta_x - x^2) u_\lambda = |u_\lambda|^2 u_\lambda, \\ u_\lambda(0, y, x) = \frac{1}{\lambda}\phi\left(\frac{y}{\lambda}, x\right), \end{cases} \tag{1-4}$$

taking the initial data by rescaling the function ϕ only in the y-variable. Set

$$w_\lambda(t, y, x) = e^{-it(\Delta_x - x^2)} u_\lambda(t, y, x),$$

and we obtain from (1-4) the following evolutionary equation for w_λ:

$$\begin{cases} (i\partial_t + \Delta_y) w_\lambda = e^{-it(\Delta_x - x^2)}\left(|e^{it(\Delta_x - x^2)} w_\lambda|^2 e^{it(\Delta_x - x^2)} w_\lambda\right), \\ w_\lambda(0, y, x) = \frac{1}{\lambda}\phi\left(\frac{y}{\lambda}, x\right). \end{cases}$$

If we define $w_\lambda(t, y, x) = \frac{\tilde{v}}{\lambda}\left(\frac{t}{\lambda^2}, \frac{y}{\lambda}, x\right)$, then $\tilde{v}$ satisfies

$$\begin{cases} (i\partial_t + \Delta_y)\tilde{v} = e^{-i\lambda^2 t(\Delta_x - x^2)}\left(|e^{i\lambda^2 t(\Delta_x - x^2)}\tilde{v}|^2 e^{i\lambda^2 t(\Delta_x - x^2)}\tilde{v}\right), \\ \tilde{v}(0, y, x) = \phi(y, x). \end{cases}$$

Denote by Π_n the orthogonal projector on the n-th eigenspace of $-\Delta_x + x^2$ (see Section 2 below for more details). Applying Π_n to the equation for $\tilde{v}$, we arrive at the following equation for $\tilde{v}_n = \Pi_n \tilde{v}$:

$$\begin{cases} (i\partial_t + \Delta_y)\tilde{v}_n = e^{i\lambda^2 t(2n+1)}\Pi_n\left(\sum_{n_1, n_2, n_3 \in \mathbb{N}} e^{-i\lambda^2(2n_1 - 2n_2 + 2n_3 + 1)t}\, \tilde{v}_{n_1}\bar{\tilde{v}}_{n_2}\tilde{v}_{n_3}\right), \\ \tilde{v}_n(0, y, x) = \phi_n(y, x) := \Pi_n \phi(y, x). \end{cases}$$

Letting $\lambda \to \infty$, we can formally get a limiting equation

$$\begin{cases} (i\partial_t + \Delta_y) v_n(t, y, x) = \sum_{\substack{n_1, n_2, n_3 \in \mathbb{N} \\ n_1 - n_2 + n_3 = n}} \Pi_n(v_{n_1}\bar{v}_{n_2} v_{n_3})(t, y, x), \\ v_n(0, y, x) = \phi_n(y, x). \end{cases} \tag{1-5}$$

By reversing the above process, we get an approximation solution of u_λ:

$$\tilde{u}_\lambda(t,y,x) = e^{it(\Delta_x - x^2)} \sum_{n\in\mathbb{N}} \left(\frac{1}{\lambda} v_n \left(\frac{t}{\lambda^2}, \frac{y}{\lambda}, x \right) \right), \quad (t,y,x) \in \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}, \tag{1-6}$$

where v_n is the solution of (1-5).

In the above deduction, the following equivalent form dispersive continuous resonant (DCR) system enters naturally:

$$\begin{cases} i\,\partial_t v + \Delta_{\mathbb{R}^2} v = F(v), \\ v(0,y,x) = \phi(y,x), \end{cases} \tag{DCR}$$

where the nonlinear term $F(v)$ is defined by

$$F(v) := \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N} \\ n_1-n_2+n_3=n}} \Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3}).$$

This (DCR) system can be viewed as a dispersive version of the (CR) system derived by E. Faou, P. Germain, and Z. Hani [Faou et al. 2016] in their study of the weak turbulence of the nonlinear Schrödinger equations on compact domains; see also [Buckmaster et al. 2019; Colliander et al. 2010; Germain et al. 2015; 2016; Dartois et al. 2020; Fennell 2019]. This new (DCR) system is very similar to the resonant nonlinear Schrödinger system arising in [Biasi et al. 2018; Cheng et al. 2020a; 2020b; Hani and Pausader 2014; Hani et al. 2015]. The latter has nice local well-posedness theory, and also scatters for small data in $L^2_y\mathcal{H}^1_x$.

In our second main result, we prove the following large-data global well-posedness and scattering theorem for (DCR), which might be of independent interest.

Theorem 1.2. *For any $\phi \in L^2_y\mathcal{H}^1_x(\mathbb{R}^2\times\mathbb{R})$, there exists a unique global solution v of (DCR) in $C^0_t L^2_y\mathcal{H}^1_x(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})$ satisfying*

$$\|v\|_{L^\infty_t L^2_y\mathcal{H}^1_x \cap L^4_{t,y}\mathcal{H}^1_x(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} \le C,$$

where $C = C(\|\phi\|_{L^2_y\mathcal{H}^1_x})$ is a constant. Moreover, the solution scatters; namely there exist $v_\pm \in L^2_y\mathcal{H}^1_x$ such that

$$\|v(t) - e^{it\Delta_y}v_\pm\|_{L^2_y\mathcal{H}^1_x(\mathbb{R}^2\times\mathbb{R})} \to 0 \quad \text{as } t\to\pm\infty.$$

Theorem 1.2 shall be proved in the final two sections and it takes a vast bulk of the paper. We prove it again by the concentration-compactness/rigidity argument from [Kenig and Merle 2006; 2008]. The system (DCR) is essentially a defocusing mass-critical nonlinear Schrödinger system. In the proof, we follow the framework for scattering of mass-critical nonlinear Schrödinger equation [Dodson 2012; 2016a; 2016b; Tao et al. 2008] and our argument is also partly inspired by the scattering of the resonant Schrödinger system derived from the NLS on cylinders [Cheng et al. 2020a; 2020b; Hani and Pausader 2014; Hani et al. 2015; Yang and Zhao 2018; Zhao 2019].

We would like to comment briefly on the relation between (DCR) and weak turbulence.

Remark 1.3 (the (DCR) system and weak turbulence). We can rewrite (1-5) in the Hermite coordinate (see (2-1) below for the definition of the Hermite functions): taking the solution $v_n(t, y, x) = c_n(t, y)h_n(x)$ in (1-5), we get an equivalent but simplified equation

$$(i\partial_t + \Delta_{\mathbb{R}^2})c_n(t, y) = \sum_{\substack{n_1, n_2, n_3 \in \mathbb{N} \\ n_1 - n_2 + n_3 = n}} D_{n_1, n_2, n_3, n} c_{n_1} \bar{c}_{n_2} c_{n_3}, \tag{1-7}$$

where $D_{n_1,n_2,n_3,n}$ is the number such that $\Pi_n(h_{n_1}\bar{h}_{n_2}h_{n_3})(x) = D_{n_1,n_2,n_3,n}h_n(x)$, $x \in \mathbb{R}$. It would be very interesting to understand $D_{n_1,n_2,n_3,n}$.[1] Compared with the success of the proof of the weak turbulence on cylinders given by Z. Hani, B. Pausader, N. Tzvetkov, and N. Visciglia [Hani et al. 2015], the unclear expression of the nonlinear term of the (DCR) system seems to be one of the main obstacles to study the weak turbulence of the nonlinear Schrödinger equations with (partial) harmonic potentials; for more information we refer to [Hani and Thomann 2016]. However, there are some interesting recent attempts toward this direction in [Gérard et al. 2019; Germain and Thomann 2016].

Remark 1.4 (focusing NLS equations with harmonic potentials). In this paper, we only consider the scattering of the defocusing NLS with partial harmonic potentials. It is an interesting problem to study the scattering of the focusing version of (1-2). It seems difficult to find the threshold of the scattering of the focusing NLS, and if we were able to find it, then most likely the scattering can be proven by following the argument in [Dodson 2012; 2015; 2016a; 2016b; Killip et al. 2009a; Tao et al. 2007a; 2008]. We refer to [Ardila and Carles 2021; Bellazzini et al. 2017; Cao et al. 2022; Zhang 2020; Stanislavova and Stefanov 2021] for the study of the instability/stability of soliton which may give some clues on the threshold of the scattering of the focusing NLS.

1.3. ***Brief outline of the proofs.*** The model with partial harmonic potential studied in this paper can be compared to the NLS on wave-guide $\mathbb{R}^2 \times \mathbb{T}$, which was considered previously in [Yang and Zhao 2018; Cheng et al. 2020a]. One key difference is that in our case, the linear operator has more complicated spectral theory; for example the eigenfunctions cannot be written explicitly.

The proof of this paper contains two main ingredients. In the first part, we prove that Theorem 1.2 (or more precisely, the consequence Theorem 4.10 of Theorem 1.2) implies Theorem 1.1. The proof of Theorem 1.1 has a very standard skeleton based on the concentration-compactness/rigidity argument introduced by C. Kenig and F. Merle [2006], and it consists of three main steps: linear profile decomposition, the existence of an almost periodic solution to the defocusing cubic NLS (1-2), and a rigidity theorem.

First of all, we establish the linear profile decomposition of Schrödinger operator with partial harmonic potentials; namely the linear solutions can be divided into several orthogonal bubbles modulo some transforms. This can be viewed as a vector-valued version of linear profile decomposition of the Schrödinger equation in L^2, which was first established by F. Merle and L. Vega [1998] in two dimensions, and then extended to general dimensions; see for instance [Killip and Vişan 2013] for more details. The proof of this part is very similar to the wave-guide case in [Cheng et al. 2020a], and it is essentially related to the

[1] In [Biasi et al. 2019a; 2019b; Evnin 2020], the authors studied a special structure of the constant $D_{n_1,n_2,n_3,n}$, and proved that it satisfies a certain identity.

description of the lack of compactness of the embedding $e^{it(\Delta_{\mathbb{R}^3}-x^2)}:\Sigma(\mathbb{R}^3)\hookrightarrow L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x$ for some fixed $0<\epsilon_0<\frac{1}{2}$.

In the second step, we prove the existence of a critical element by the construction of approximation solutions. Since the nonlinear flow is not commutable with the transform groups derived in the first step, in order to construct the approximation solutions, we need to assume that the limiting equations, which is exactly the (DCR) system, is globally well-posed and scatters, as stated in Theorem 1.2. The idea of using limiting equations was first considered in [Ibrahim et al. 2011], and was widely used in [Cheng et al. 2020a; 2020b; Hani and Pausader 2014; Ionescu and Pausader 2012; Jao 2016]. Then, much as in [Cheng et al. 2020a], we use the normal form method to exploit additional decay to approximate the nonlinear profile. In the wave-guide case [Cheng et al. 2020a], the eigenfunctions, which are the plain waves $e^{iy\cdot j}$, can be easily computed, and thus the Fourier coefficients are summed naturally. The difficulty in this step is that we need to sum up the spectral projections of the solution properly. To some extent, the main innovation of this paper is that we utilize the additional regularity of the smooth nonlinear profile to update the l^1 summation of projections to l^2.

In the third step, we borrow the idea used in [Cheng et al. 2020a] to prove the nonexistence of a nontrivial critical element. The key point is the use of the interaction Morawetz estimate developed by J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao [Colliander et al. 2004], which is very important in the remarkable work [Colliander et al. 2008] on scattering for energy-critical NLS in three dimensions, and was further developed in [Planchon and Vega 2009; Colliander et al. 2009]. Then, we can arrive at the contradiction similar to [Kenig and Merle 2006; 2008] using the compactness property of the critical element.

The second part of this paper is devoted to the proof of Theorem 1.2. The proof is greatly inspired by the fundamental work of B. Dodson [2012; 2016a; 2016b] in his study of mass critical NLS. We also refer to [Yang and Zhao 2018], and the principal difference between that work and this paper is that our system (DCR) involves the spectral projection of Schrödinger operator with harmonic potential. Here, one key observation is that the (DCR) system is scaling invariant, which indicates that the classical method as developed in [Cheng et al. 2020a; 2020b; Tao et al. 2008] could be potentially applied to our situation. Indeed, the linear profile decomposition developed for the Schrödinger propagator in $L^2_y\mathcal{H}^1_x(\mathbb{R}^2\times\mathbb{R})$ (see Theorem 4.1) can be directly applied here. The essential difficulty occurring in the proof of Theorem 1.2 lies in precluding the almost periodic solution to the (DCR) system.

There are two cases of the critical element: high-to-low frequency cascade and the quasisoliton scenarios. We exclude these scenarios based on the rigidity argument of B. Dodson [2012; 2016a; 2016b]. The key tool is to establish a vector-valued version of the two-dimensional long-time Strichartz estimate in [Dodson 2016b]. The long-time Strichartz estimate is developed by B. Dodson to show the scattering of the mass-critical nonlinear Schrödinger equations and has been proved as an important technique in the scattering theory of nonlinear dispersive and wave equation. We refer to [Dodson 2019; Dodson and Lawrie 2015; Dodson et al. 2017; 2020; Killip and Vişan 2012; Vişan 2012; Murphy 2014; Rosenzweig 2018] for more applications of this powerful tool. The proof of the long-time Strichartz estimate in our situation here is rather technical due to the spectral projection and the failure of the two-dimensional endpoint Strichartz estimate. For the high-to-low frequency cascade scenario, it is more delicate and we have to exploit some additional regularity of the critical element through the long-time Strichartz estimate, and then preclude it using energy conservation law. For the quasisoliton scenario, we mainly use the long-time Strichartz estimate to control the error terms of low frequency cut-off of the interaction Morawetz identity. With all these ingredients at hand, the contradiction argument of C. E. Kenig and F. Merle [2006; 2008] allows us to conclude the proof.

The rest of the paper is organized as follows. Section 2 contains some basic notation and preliminaries. In Section 3, we record the local well-posedness, the small-data scattering result and the stability theory for system (1-2). For the convenience of the readers, we present the proofs in the Appendix. In Section 4, we will give the linear profile decomposition for data in $\Sigma(\mathbb{R}^3)$ and also analyze the nonlinear profiles; therefore we reduce the nonscattering in $\Sigma(\mathbb{R}^3)$ to the existence of an almost-periodic solution. In Section 5, we will show the extinction of such an almost-periodic solution. The scattering of the (DCR) system shall be proved in Section 6, where the proofs of two auxiliary theorems are left to the final Section 7.

2. Basic notation and preliminaries

In this section, we introduce some basic notation used in this paper. We will use the notation $X \lesssim Y$ whenever there exists some constant $C > 0$ so that $X \le CY$. Similarly, we will write $X \sim Y$ if $X \lesssim Y \lesssim X$. We use $\mathbb{N}$ to denote the set of all nonnegative integers.

Throughout the paper, we will take ϵ_0 to be some small fixed number in $\left(0, \frac{1}{2}\right)$.

2.1. ***Fourier transform and Sobolev spaces.*** For any $a \in \mathbb{R}^d$, $d \in \mathbb{N}$, the Japanese bracket $\langle a \rangle$ is defined to be $\langle a \rangle = (1 + |a|^2)^{1/2}$. We define the Fourier transform $\hat{f} : \mathbb{R}^d \to \mathbb{C}$ of a function $f : \mathbb{R}^d \to \mathbb{C}$ as

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-iz\cdot\xi} f(z)\,\mathrm{d}z.$$

For each $s \in \mathbb{R}$, the fractional differential operator $|\nabla|^s$ is defined by $\widehat{|\nabla|^s f}(\xi) = |\xi|^s \hat{f}(\xi)$. We also define $\langle\nabla\rangle^s$ as an operator between function spaces by $\widehat{\langle\nabla\rangle^s f}(\xi) = (1 + |\xi|^2)^{s/2} \hat{f}(\xi)$. In the following we will use $\langle\nabla_x\rangle^s$ to emphasize the application of the operator on the x-variable.

We will frequently use the partial Fourier transform $\mathcal{F}_y f$ of a complex-valued function $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}$ defined as

$$\mathcal{F}_y f(\xi, x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-iy\cdot\xi} f(y, x)\,\mathrm{d}y, \quad \xi \in \mathbb{R}^2,$$

where $x \in \mathbb{R}$ is viewed as a parameter.

We shall also use the Littlewood–Paley projections. Take a cut-off function $\chi \in C^\infty((0, \infty))$ such that $\chi(r) = 1$ if $r \leqslant 1$ and $\chi(r) = 0$ if $r > 2$. For $N \in 2^{\mathbb{Z}}$, let $\chi_N(r) = \chi(N^{-1}r)$ and $\phi_N(r) = \chi_N(r) - \chi_{N/2}(r)$. We define the Littlewood–Paley dyadic operator $P_{\leqslant N} f := \mathcal{F}^{-1}(\chi_N(|\xi|)\hat{f}(\xi))$ and $P_N f := \mathcal{F}^{-1}(\phi_N(|\xi|)\hat{f}(\xi))$. We also define the partial Littlewood–Paley projections to be $P^y_{\le N} f(y, x) := \mathcal{F}_y^{-1}(\chi_N(\xi)(\mathcal{F}_y f)(\xi, x))$ and $P^y_N f(y, x) := \mathcal{F}_y^{-1}(\phi_N(|\xi|)(\mathcal{F}_y f)(\xi, x))$.

Next, we denote the usual Lebesgue space as $L^p(\mathbb{R}^d)$, and sometimes we write $\|f\|_p = \|f\|_{L^p(\mathbb{R}^d)}$ for abbreviation. For any $s \in \mathbb{R}$, we define the Sobolev space as

$$W^{s,p}(\mathbb{R}^d) := \{f \in L^p(\mathbb{R}^d) : \|f\|_{W^{s,p}(\mathbb{R}^d)} := \|\langle\nabla\rangle^s f\|_{L^p(\mathbb{R}^d)} < +\infty\}.$$

We also define $H^s(\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d)$.

2.2. *Harmonic oscillator and Hermite-Sobolev spaces.* The harmonic oscillator $-\Delta_x + x^2,\ x \in \mathbb{R}$, has been studied by many authors, and we refer to the lecture notes of B. Helffer [1988] and also the seminal work of H. Koch and D. Tataru [2005b] for a few basic facts that we shall record below. The harmonic oscillator admits a Hilbertian basis of eigenvectors for $L^2(\mathbb{R})$, and, for each $n \in \mathbb{N}$, we will denote the n-th eigenspace by E_n and the corresponding eigenvalue by $\lambda_n = 2n+1$. Each eigenspace E_n is spanned by the Hermite functions h_n, where

$$h_n(x) = \frac{1}{\sqrt{n!}\,2^{\frac{n}{2}}\pi^{\frac{1}{4}}}(-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-x^2}) \tag{2-1}$$

for $n \in \mathbb{N}$. We also let Π_n be the orthogonal projector on the n-th eigenspace E_n of $-\Delta_x + x^2$.

For $s \in \mathbb{R}$ and $p \geq 1$, the Hermite-Sobolev space $\mathcal{W}^{s,p}(\mathbb{R})$ is defined as

$$\mathcal{W}^{s,p}(\mathbb{R}) = \{u \in L^p_x(\mathbb{R}) : \|u\|_{\mathcal{W}^{s,p}} := \|\langle\nabla\rangle^s u\|_{L^p_x} + \||\cdot|^s u\|_{L^p_x} < \infty\}.$$

In particular, if $p = 2$, we denote $\mathcal{W}^{s,2}_x(\mathbb{R})$ by $\mathcal{H}^s_x(\mathbb{R})$, and the $\mathcal{H}^1_x(\mathbb{R})$-norm was given in (1-3). By [Yajima and Zhang 2004], we have

$$\|u\|_{\mathcal{W}^{s,p}_x} \sim \|(-\Delta + x^2)^{\frac{s}{2}} u\|_p + \|u\|_p.$$

The Hermite-Sobolev spaces satisfy the usual Sobolev embedding; see for instance [Cazenave 2003]. These spaces also have other stronger Hermite-Sobolev embedding. For instance, we have

$$L^4_x(\mathbb{R}) \hookrightarrow \mathcal{H}^{-1}(\mathbb{R}). \tag{2-2}$$

By duality, to prove (2-2), we only need to show $\mathcal{H}^1(\mathbb{R}) \hookrightarrow L^{4/3}(\mathbb{R})$. This follows from Hölder's inequality as follows:

$$\|f\|_{L^{4/3}} \lesssim \|\langle x\rangle^{-1}\|_{L^4_x} \|\langle x\rangle f\|_{L^2_x} \lesssim \|(1+|x|^2)^{\frac{1}{2}} f\|_{L^2_x} \lesssim \|f\|_{\mathcal{H}^1_x}.$$

The Hermite-Sobolev space $L^p_y\mathcal{H}^s_x$ with $1 \leq p < \infty$ and $s \in \mathbb{R}$ is defined by

$$L^p_y\mathcal{H}^s_x = \left\{ f \in L^p_y L^2_x(\mathbb{R}^2\times\mathbb{R}) : \|f\|_{L^p_y\mathcal{H}^s_x(\mathbb{R}^2\times\mathbb{R})} := \left(\int_{\mathbb{R}^2} \|f(y,\cdot)\|^p_{\mathcal{H}^s_x(\mathbb{R})}\,\mathrm{d}y\right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^2} \left\| \left(\sum_{n\in\mathbb{N}} (2n+1)^s |f_n(y,x)|^2\right)^{\frac{1}{2}} \right\|^p_{L^2_x(\mathbb{R})} \mathrm{d}y\right)^{\frac{1}{p}} < \infty \right\},$$

where $f_n = \Pi_n f$. Similarly, for any time interval $I \subseteq \mathbb{R}$ and $u : I \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}$, we define the space-time norms $L_t^p W_y^{s,q} L_x^r$ and $L_t^p L_y^q \mathcal{H}_x^s$ of u as

$$\|u\|_{L_t^p W_y^{s,q} L_x^r (I\times\mathbb{R}^2\times\mathbb{R})} := \left(\int_I \left(\int_{\mathbb{R}^2} \|\langle\nabla_y\rangle^s u(t,y,\cdot)\|_{L_x^r(\mathbb{R})}^q \,\mathrm{d}y\right)^{\frac{p}{q}} \mathrm{d}t\right)^{\frac{1}{p}},$$

$$\|u\|_{L_t^p L_y^q \mathcal{H}_x^s (I\times\mathbb{R}^2\times\mathbb{R})} := \left(\int_I \left(\int_{\mathbb{R}^2} \|u(t,y,\cdot)\|_{\mathcal{H}_x^s(\mathbb{R})}^q \,\mathrm{d}y\right)^{\frac{p}{q}} \mathrm{d}t\right)^{\frac{1}{p}},$$

where $1 \le p, q, r \le \infty$, and $s \in \mathbb{R}$. When $s = 0$ and $p = q = r$, we shall write $L_{t,y,x}^p$ for $L_t^p W_y^{s,p} L_x^r$. Similarly, when $p = q$, we shall write $L_{t,y}^p \mathcal{H}_x^s$ for $L_t^p L_y^q \mathcal{H}_x^s$. We also use the following space-time norm. For any $\{u_n(t,y,x)\}_{n\in\mathbb{N}}$, with $(t,y,x) \in I \times \mathbb{R}^2 \times \mathbb{R}$, we set

$$\|u_n\|_{L_t^p L_y^q L_x^r l_n^2 (I\times\mathbb{R}^2\times\mathbb{R}\times\mathbb{N})} = \big\| \|u_n\|_{l_n^2} \big\|_{L_t^p L_y^q L_x^r (I\times\mathbb{R}^2\times\mathbb{R})},$$

where $1 \le p, q, r \le \infty$.

Lemma 2.1. *The Dirac function $\delta_0(x)$ belongs to $\mathcal{H}_x^{-1}(\mathbb{R})$.*

Proof. By definition, we have

$$\|\delta_0(x)\|_{\mathcal{H}_x^{-1}}^2 = \sum_{n=0}^{\infty} (2n+1)^{-1} |c_n|^2, \tag{2-3}$$

where $c_n = \langle \delta_0(x), h_n(x)\rangle = h_n(0)$. Since

$$e^{-x^2} = \sum_{m=0}^{\infty} \frac{(-x^2)^m}{m!} = \sum_{n=0}^{\infty} \frac{d^n}{dx^n}\Big|_{x=0} e^{-x^2} \cdot \frac{x^n}{n!},$$

we have

$$\frac{d^n}{dx^n}\Big|_{x=0} e^{-x^2} = \begin{cases} 0, & n \text{ is odd}, \\ \dfrac{(-1)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} n!\,, & n \text{ is even}. \end{cases}$$

Thus

$$h_n(0) = \begin{cases} 0, & n \text{ is odd}, \\ \dfrac{(-1)^n}{\sqrt{n!}\, 2^{\frac{n}{2}} \pi^{\frac{1}{4}}} \dfrac{(-1)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} n!\,, & n \text{ is even}. \end{cases}$$

Together with (2-3), this implies

$$\|\delta_0(x)\|_{\mathcal{H}_x^{-1}}^2 \le \pi^{-\frac{1}{4}} \sum_{\substack{n=0,\\ n \text{ even}}}^{\infty} \frac{n!}{2^n \left(\left(\frac{n}{2}\right)!\right)^2 (2n+1)} \lesssim \sum_{m=0}^{\infty} \frac{1}{2^m(4m+1)} \lesssim 1. \qquad \square$$

3. Local well-posedness and small-data scattering

In this section, we will review the local well-posedness theorem and the stability theorem for solutions of (1-2), which shall be crucial in proving the existence of the critical element, and then record another important theorem on the scattering norm in Theorem 3.4, which says that a weak space-time norm $L_{t,y}^4 \mathcal{H}_x^{1-\epsilon_0}$

is sufficient to prove the scattering result. We shall only state these results in this section and leave the proofs to the Appendix. In fact, the results in this section can be proved by following the exact arguments as in [Cheng et al. 2020a, Section 2; 2020b], upon noticing the embedding $\mathcal{H}^{(1/2)+}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$.

Different from the Strichartz estimate for the harmonic oscillator, which is a local estimate, we have a global Strichartz estimate for the partial harmonic oscillator similar to the Schrödinger equation on waveguides [Cheng et al. 2020a; 2020b; Tarulli 2017; Tzvetkov and Visciglia 2012]. Before giving the Strichartz estimate, we first introduce the following definition.

Definition 3.1 (Strichartz admissible pair). We call a pair (p,q) Strichartz admissible if $2 < p \le \infty$, $2 \le q < \infty$, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$.

We can now state the Strichartz estimate. The proof is almost identical to [Tzvetkov and Visciglia 2012, proof of Proposition 2.1]; we also refer to Proposition 3.1 in [Antonelli et al. 2015], and we omit the proof here.

Proposition 3.2 (Strichartz estimate for the partial harmonic oscillator). *For any Strichartz admissible pair (p,q), we have*

$$\|e^{it(\Delta_{\mathbb{R}^3}-x^2)} f(y,x)\|_{L_t^p L_y^q L_x^2(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} \lesssim \|f\|_{L^2_{y,x}}.$$

Meanwhile, for $\alpha = 0, 1$, it holds

$$\|e^{it\Delta_y} f(y,x)\|_{L_t^p L_y^q \mathcal{H}_x^\alpha(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} \lesssim \|f\|_{L_y^2 \mathcal{H}_x^\alpha(\mathbb{R}^2\times\mathbb{R})}.$$

The following nonlinear estimate, which follows from the Hölder and Sobolev inequalities, is useful in showing the local well-posedness result.

Proposition 3.3 (nonlinear estimate). *For any $0 < \epsilon_0 < \frac{1}{2}$, we have*

$$\|u_1u_2u_3\|_{L^{4/3}_{t,y}\mathcal{H}_x^{1-\epsilon_0}} \lesssim \|u_1\|_{L^4_{t,y}\mathcal{H}_x^{1-\epsilon_0}} \|u_2\|_{L^4_{t,y}\mathcal{H}_x^{1-\epsilon_0}} \|u_3\|_{L^4_{t,y}\mathcal{H}_x^{1-\epsilon_0}}.$$

Using Propositions 3.2 and 3.3, one can easily prove the following local well-posedness and small-data scattering in $L_y^2\mathcal{H}_x^1(\mathbb{R}^2\times\mathbb{R})$ and $\Sigma(\mathbb{R}^3)$. The local solution can be extended to be global by the conservation of mass and energy; we refer to [Carles 2002b; Tao 2006]. The proof of the local well-posedness is given in the Appendix; see also [Antonelli et al. 2015; Ardila and Carles 2021; Carles 2002a; 2002b; 2003; 2011; Carles and Gallo 2015] for a comparison.

Theorem 3.4 (LWP and scattering in $L_y^2\mathcal{H}_x^1$ and Σ).

(1) (*well-posedness*) *Let $u_0 \in L_y^2\mathcal{H}_x^1$. There exists a unique solution $u \in C_t^0 L_y^2\mathcal{H}_x^1(I\times\mathbb{R}^2\times\mathbb{R})$ of (1-2), where $I \subseteq \mathbb{R}$ is the maximal lifespan. Furthermore, if $u_0 \in \Sigma(\mathbb{R}^3)$, the solution u can be extended to be global in $C_t^0\Sigma_{y,x}(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})$.*

(2) (*scattering norm*) *If the solution $u \in C_t^0\Sigma_{y,x}(\mathbb{R}\times\mathbb{R}^3)$ of (1-2) satisfies $\|u\|_{L^4_{t,y}\mathcal{H}_x^{1-\epsilon_0}(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} \le M$ for some positive constant M. Then u scatters in $\Sigma(\mathbb{R}^3)$; that is, there exist $u_\pm \in \Sigma_{y,x}(\mathbb{R}^2\times\mathbb{R})$ such that*

$$\|u(t,y,x) - e^{it(\Delta_{\mathbb{R}^3}-x^2)}u_\pm(y,x)\|_{\Sigma_{y,x}} \to 0 \quad \textit{as } t \to \pm\infty. \tag{3-1}$$

We next give the existence of wave operators, whose proof can be found in the Appendix.

Theorem 3.5 (existence of the wave operators). *Let $u_- \in \Sigma$. There exists $T_- > 0$ depending on $\|u_-\|_\Sigma$, and a solution $u \in C((-\infty, -T_-], \Sigma)$ to (1-2) such that*

$$\|u(t, y, x) - e^{it(\Delta - x^2)} u_-(y, x)\|_\Sigma \to 0 \quad \text{as } t \to -\infty. \tag{3-2}$$

Similarly, let $u_+ \in \Sigma$. There exists $T_+ > 0$ depending on $\|u_+\|_\Sigma$, and a solution $u \in C([T_+, \infty), \Sigma)$ to (1-2) such that

$$\|u(t, y, x) - e^{it(\Delta - x^2)} u_+(y, x)\|_\Sigma \to 0 \quad \text{as } t \to \infty.$$

We now state the stability theory in $L_y^2 \mathcal{H}_x^1(\mathbb{R}^2 \times \mathbb{R})$. The proof is again given in the Appendix. For a comparison, see [Colliander et al. 2008; Killip and Vişan 2013; Koch et al. 2014]; in particular, [Killip and Vişan 2013, Theorem 3.7]. We also contend that the result in the following theorem can extended to $\Sigma(\mathbb{R}^3)$.

Theorem 3.6 (stability theorem). *Let I be a compact interval and let $\tilde{u}$ be an approximate solution to (1-2) in the sense that $\tilde{u}$ satisfies $i\,\partial_t \tilde{u} + \Delta_{\mathbb{R}^3} \tilde{u} - x^2 \tilde{u} = |\tilde{u}|^2 \tilde{u} + e$ for some function e.*

Suppose

$$\|\tilde{u}\|_{L_t^\infty L_y^2 \mathcal{H}_x^1 \cap L_{t,y}^4 \mathcal{H}_x^1} \le M$$

for some positive constant M.

Let $t_0 \in I$ and let $u(t_0)$ obey

$$\|u(t_0) - \tilde{u}(t_0)\|_{L_y^2 \mathcal{H}_x^1} \le M' \tag{3-3}$$

for some $M' > 0$. Assume in addition that the smallness condition

$$\|e^{i(t-t_0)(\Delta_{\mathbb{R}^3} - x^2)}(u(t_0) - \tilde{u}(t_0))\|_{L_{t,y}^4 \mathcal{H}_x^1} + \|e\|_{L_{t,y}^{4/3} \mathcal{H}_x^1} \le \epsilon \tag{3-4}$$

holds for some $0 < \epsilon \le \epsilon_1$, where $\epsilon_1 = \epsilon_1(M, M') > 0$ is a small constant. Then, there exists a solution u to (1-2) on $I \times \mathbb{R}^2 \times \mathbb{R}$ with an initial data $u(t_0)$ at time $t = t_0$ satisfying

$$\|u - \tilde{u}\|_{L_{t,y}^4 \mathcal{H}_x^1} \le C(M, M')\epsilon, \qquad \|u - \tilde{u}\|_{L_t^\infty L_y^2 \mathcal{H}_x^1} \le C(M, M')M',$$
$$\|u\|_{L_t^\infty L_y^2 \mathcal{H}_x^1 \cap L_{t,y}^4 \mathcal{H}_x^1} \le C(M, M').$$

4. Existence of an almost-periodic solution

In this section, we will show the existence of an almost-periodic solution by the profile decomposition and the nonlinear approximation.

4.1. *Linear profile decomposition.*

In this subsection, we will establish the linear profile decomposition in $\Sigma(\mathbb{R}^3)$, which depends on the corresponding decomposition in $L^2(\mathbb{R}^2)$. The linear profile decomposition in L^2 for the mass-critical nonlinear Schrödinger equation has been established by F. Merle and L. Vega [1998], R. Carles and S. Keraani [2007], and P. Bégout and A. Vargas [2007]. We also refer readers to [Killip and Vişan 2013; Koch et al. 2014] for other versions of the linear profile decomposition.

Theorem 4.1 (linear profile decomposition in $L_y^2 \mathcal{H}_x^1(\mathbb{R}^2 \times \mathbb{R})$ and Σ). *Let $\{u_k\}_{k \ge 1}$ be a bounded sequence in $L_y^2 \mathcal{H}_x^1(\mathbb{R}^2 \times \mathbb{R})$. Then after passing to a subsequence if necessary, there exists $J^* \in \{0, 1, \dots\} \cup \{\infty\}$,* *so that for any* $J \le J^*$ *we have functions* $\phi^j \in L^2_y\mathcal{H}^1_x(\mathbb{R}^2\times\mathbb{R})$, $1 \le j \le J$, $r^J_k \in L^2_y\mathcal{H}^1_x(\mathbb{R}^2\times\mathbb{R})$, *and mutually orthogonal frames* $\{(\lambda^j_k, t^j_k, y^j_k, \xi^j_k)\}_{k\ge1} \subseteq \mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2$ *in the sense that, for any* $j \ne j'$,

$$\frac{\lambda^j_k}{\lambda^{j'}_k} + \frac{\lambda^{j'}_k}{\lambda^j_k} + \lambda^j_k\lambda^{j'}_k|\xi^j_k - \xi^{j'}_k|^2 + \frac{|y^j_k - y^{j'}_k|^2}{\lambda^j_k\lambda^{j'}_k} + \frac{|(\lambda^j_k)^2 t^j_k - (\lambda^{j'}_k)^2 t^{j'}_k|}{\lambda^j_k\lambda^{j'}_k} \to \infty \quad \text{as } k\to\infty \tag{4-1}$$

such that, for every $1 \le j \le J$, *we have a decomposition*

$$u_k(y,x) = \sum_{j=1}^J \frac{1}{\lambda^j_k} e^{iy\cdot\xi^j_k}(e^{it^j_k\Delta_{\mathbb{R}^2}}\phi^j)\left(\frac{y-y^j_k}{\lambda^j_k}, x\right) + r^J_k(y,x).$$

In addition,

$$\lim_{k\to\infty}\left(\|u_k\|^2_{L^2_y\mathcal{H}^1_x} - \sum_{j=1}^J \|\phi^j\|^2_{L^2_y\mathcal{H}^1_x} - \|r^J_k\|^2_{L^2_y\mathcal{H}^1_x}\right) = 0, \tag{4-2}$$

$$\lambda^j_k e^{-it^j_k\Delta_y}\left(e^{-i(\lambda^j_k y + y^j_k)\cdot\xi^j_n} r^J_k(\lambda^j_k y + y^j_k, x)\right) \rightharpoonup 0 \quad \text{in } L^2_y\mathcal{H}^1_x, \text{ as } k\to\infty, \text{ for } j\le J, \tag{4-3}$$

$$\limsup_{k\to\infty}\|e^{it(\Delta_{\mathbb{R}^3}-x^2)} r^J_k\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x} \to 0 \quad \text{as } J\to J^*. \tag{4-4}$$

Furthermore, if $\{u_k\}_{k\ge1}$ *is a bounded sequence in* $\Sigma(\mathbb{R}^3)$, *then in the above conclusion, we can further take* $\lambda^j_k \to 1$ *or* ∞, *as* $k\to\infty$, $|\xi^j_k| \le C_j$, *for every* $1\le j\le J$. *And we have a slight different decomposition*

$$u_k(y,x) = \sum_{j=1}^J \phi^j_k(y,x) + r^J_k(y,x) := \sum_{j=1}^J \frac{1}{\lambda^j_k} e^{iy\cdot\xi^j_k}(e^{it^j_k\Delta_{\mathbb{R}^2}} P^j_k\phi^j)\left(\frac{y-y^j_k}{\lambda^j_k}, x\right) + r^J_k(y,x),$$

where

$$P^j_k\phi^j(y,x) = \begin{cases} \phi^j(y,x) & \text{if } \lim_{k\to\infty}\lambda^j_k = 1,\\ P^y_{\le(\lambda^j_k)^\theta}\phi^j(y,x) & \text{if } \lim_{k\to\infty}\lambda^j_k = \infty,\end{cases}$$

and θ *is some fixed positive sufficiently small number. In addition, we also have a slight different decoupling*

$$\lim_{k\to\infty}\left(\mathcal{E}(u_k) - \sum_{j=1}^J \mathcal{E}(\phi^j_k) - \mathcal{E}(r^J_k)\right) = 0, \tag{4-5}$$

$$\lim_{k\to\infty}\left(\mathcal{M}(u_k) - \sum_{j=1}^J \mathcal{M}(\phi^j_k) - \mathcal{M}(r^J_k)\right) = 0, \tag{4-6}$$

where $\mathcal{E}$ *and* $\mathcal{M}$ *are given in* (ME). *Other conclusions* (4-1)–(4-4) *hold as before.*

To prove the above theorem, we need to establish the inverse Strichartz estimate in Proposition 4.6 below. We first recall the following refined Strichartz estimate which is essentially established in [Cheng et al. 2020a; 2020b].

Proposition 4.2 (refined Strichartz estimate [Cheng et al. 2020a; 2020b]). *For any* $f \in L^2_y\mathcal{H}^{1-\epsilon_0/2}_x$, *we have*

$$\|e^{it\Delta_{\mathbb{R}^2}} f\|_{L^4_{t,y,x}(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} \lesssim \|f\|^{\frac34}_{L^2_y\mathcal{H}^{1-\epsilon_0/2}_x}\left(\sup_{Q\in\mathcal{D}}|Q|^{-\frac{3}{22}}\|e^{it\Delta_{\mathbb{R}^2}} f_Q\|_{L^{11/2}_{t,y,x}}\right)^{\frac14},$$

where

$$\mathcal{D} = \bigcup_{j\in\mathbb{Z}} \{[2^j k_1, 2^j(k_1+1)) \times [2^j k_2, 2^j(k_2+1)) : (k_1,k_2)\in\mathbb{Z}^2\}$$

is the collection of all dyadic cubes, and f_Q *is defined by* $\mathcal{F}_y(f_Q) = \chi_Q \mathcal{F}_y f$.

To prove the inverse Strichartz estimate, we shall need the following two facts:

Proposition 4.3 (local smoothing estimate [Constantin and Saut 1988; Vega 1988]). *For any given* $\epsilon > 0$, *we have*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2\times\mathbb{R}} |(|\nabla_y|^{\frac12} e^{it\Delta_{\mathbb{R}^2}} f)(y,x)|^2 \langle y\rangle^{-1-\epsilon}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \lesssim_\epsilon \|f\|^2_{L^2_{y,x}(\mathbb{R}^2\times\mathbb{R})}.$$

Furthermore, if $\epsilon \ge 1$, *then we have*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2\times\mathbb{R}} |(\langle\nabla_y\rangle^{\frac12} e^{it\Delta_{\mathbb{R}^2}} f)(y,x)|^2 \langle y\rangle^{-1-\epsilon}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \lesssim_\epsilon \|f\|^2_{L^2_{y,x}(\mathbb{R}^2\times\mathbb{R})}.$$

Lemma 4.4. *For each* $f \in \mathcal{H}^1_x(\mathbb{R})$ *and any* $R > 0$, *we have*

$$\|f\|_{L^\infty_x(|x|\ge R)} \lesssim R^{-\frac12}(\|f(x)\|_{L^2_x} + \|xf(x)\|^{\frac12}_{L^2_x}\|f'(x)\|^{\frac12}_{L^2_x}).$$

Proof. For any $f \in \mathcal{H}^1_x(\mathbb{R})$, we have

$$xf^2(x) = \int_0^x (zf^2(z))'\,\mathrm{d}z = \int_0^x f^2(z) + 2zf(z)f'(z)\,\mathrm{d}z.$$

Then by Hölder's inequality, we get for any $R > 0$,

$$\|xf^2(x)\|_{L^\infty_x(|x|\ge R)} \lesssim \|f\|^2_{L^2_x} + \|xf(x)\|_{L^2_x}\|f'(x)\|_{L^2_x}.$$

Therefore,

$$\|f(x)\|_{L^\infty_x(|x|\ge R)} \lesssim R^{-\frac12}(\|f\|_{L^2_x} + \|xf(x)\|^{\frac12}_{L^2_x}\|f'(x)\|^{\frac12}_{L^2_x}).$$ □

We also have the following estimate.

Lemma 4.5. *By interpolation, the Hölder inequality,* (2-2), *and Proposition 3.2, we have*

$$\|e^{it\Delta_y} f\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x} \lesssim \|e^{it\Delta_y} f\|^{\frac{\epsilon_0}{2}}_{L^4_{t,y}\mathcal{H}^{-1}_x}\|e^{it\Delta_y} f\|^{1-\frac{\epsilon_0}{2}}_{L^4_{t,y}\mathcal{H}^1_x} \lesssim \|e^{it\Delta_y} f\|^{\frac{\epsilon_0}{2}}_{L^4_{t,y,x}}\|f\|^{1-\frac{\epsilon_0}{2}}_{L^2_y\mathcal{H}^1_x}. \tag{4-7}$$

We can now prove the inverse Strichartz estimate.

Proposition 4.6 (inverse Strichartz estimate). *For* $\{f_k\}_{k\ge1} \subseteq L^2_y\mathcal{H}^1_x(\mathbb{R}^2\times\mathbb{R})$ *satisfying*

$$\lim_{k\to\infty}\|f_k\|_{L^2_y\mathcal{H}^1_x} = A \quad \textit{and} \quad \lim_{k\to\infty}\|e^{it\Delta_{\mathbb{R}^2}} f_k\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x} = \epsilon, \tag{4-8}$$

there exist $\phi \in L^2_y\mathcal{H}^1_x$ *and* $(\lambda_k, t_k, \xi_k, y_k) \in \mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2$, *so that passing to a further subsequence if necessary, we have*

$$\lambda_k e^{-i\xi_k\cdot(\lambda_k y + y_k)}(e^{it_k\Delta_{\mathbb{R}^2}} f_k)(\lambda_k y + y_k, x) \rightharpoonup \phi(y,x) \quad \textit{in } L^2_y\mathcal{H}^1_x,\ \textit{as } k\to\infty,$$

$$\lim_{k\to\infty}(\|f_k\|^2_{L^2_y\mathcal{H}^1_x}-\|f_k-\phi_k\|^2_{L^2_y\mathcal{H}^1_x})=\|\phi\|^2_{L^2_y\mathcal{H}^1_x}\gtrsim A^2\Big(\frac{\epsilon}{A}\Big)^{\frac{48}{\epsilon_0}},\tag{4-9}$$

$$\limsup_{k\to\infty}\|e^{it\Delta_{\mathbb{R}^2}}(f_k-\phi_k)\|^4_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}\le\epsilon^{\frac{8}{\epsilon_0}}A^{4-\frac{8}{\epsilon_0}}\bigg(1-cA^{2\beta}\Big(\frac{\epsilon}{A}\Big)^{\frac{2\beta}{\epsilon_0}}\bigg),\tag{4-10}$$

where c and β are small positive constants, and

$$\phi_k(y,x)=\frac{1}{\lambda_k}e^{iy\cdot\xi_k}(e^{-i(t_k/\lambda_k^2)\Delta_{\mathbb{R}^2}}\phi)\Big(\frac{y-y_k}{\lambda_k},x\Big).$$

Moreover, if $\{f_k\}_{k\ge1}$ is bounded in $\Sigma(\mathbb{R}^3)$, and also

$$\lim_{k\to\infty}\|f_k\|_\Sigma=A\quad\text{and}\quad\lim_{k\to\infty}\|e^{it\Delta_{\mathbb{R}^2}}f_k\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}=\epsilon,\tag{4-11}$$

then we can take $\lambda_k\ge1$, $|\xi_k|\lesssim1$ and $\phi\in L^2_y\mathcal{H}^1_x(\mathbb{R}^2\times\mathbb{R})$ such that

$$\lambda_ke^{-i\xi_k\cdot(\lambda_ky+y_k)}(e^{it_k\Delta_{\mathbb{R}^2}}f_k)(\lambda_ky+y_k,x)\rightharpoonup\phi(y,x)\quad\text{in } L^2_y\mathcal{H}^1_x,\ \text{as } k\to\infty,\tag{4-12}$$

and

$$\lim_{k\to\infty}(\|f_k\|^2_\Sigma-\|f_k-\phi_k\|^2_\Sigma)=\lim_{k\to\infty}\|\phi_k\|^2_\Sigma\gtrsim A^2\Big(\frac{\epsilon}{A}\Big)^{\frac{48}{\epsilon_0}}.\tag{4-13}$$

Proof. Case 1: $\{f_k\}_{k\ge1}$ is bounded in $L^2_y\mathcal{H}^1_x$. By Proposition 4.2, (4-7) and (4-8), there exists $\{Q_k\}_{k\ge1}\subseteq\mathcal{D}$ so that

$$\epsilon^{\frac{8}{\epsilon_0}}A^{1-\frac{8}{\epsilon_0}}\lesssim\liminf_{k\to\infty}|Q_k|^{-\frac{3}{22}}\|e^{it\Delta_{\mathbb{R}^2}}(f_k)_{Q_k}\|_{L^{11/2}_{t,y,x}}.\tag{4-14}$$

Let λ_k be the inverse of the side length and ξ_k be the center of the cube Q_k. By Hölder's inequality and (4-8), we have

$$\liminf_{k\to\infty}|Q_k|^{-\frac{3}{22}}\|e^{it\Delta_{\mathbb{R}^2}}(f_k)_{Q_k}\|_{L^{11/2}_{t,y,x}}\lesssim\liminf_{k\to\infty}\lambda_k^{\frac{3}{11}}(\epsilon^{\frac{2}{\epsilon_0}}A^{1-\frac{2}{\epsilon_0}})^{\frac{8}{11}}\|e^{it\Delta_{\mathbb{R}^2}}(f_k)_{Q_k}\|^{\frac{3}{11}}_{L^\infty_{t,y,x}}.$$

Together with (4-14), this implies

$$\liminf_{k\to\infty}\lambda_k\|e^{it\Delta_{\mathbb{R}^2}}(f_k)_{Q_k}\|_{L^\infty_{t,y,x}(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})}\gtrsim\epsilon^{\frac{24}{\epsilon_0}}A^{1-\frac{24}{\epsilon_0}}.$$

Then by Lemma 4.4 and Bernstein's inequality, we have

$$\begin{aligned}
&\liminf_{k\to\infty}\lambda_k\|e^{it\Delta_{\mathbb{R}^2}}(f_k)_{Q_k}\|_{L^\infty_{t,y,x}(\mathbb{R}\times\mathbb{R}^2\times\{|x|\ge R\})}\\
&\qquad\lesssim R^{-\frac12}\liminf_{k\to\infty}\lambda_k\big(\|(f_k)_{Q_k}\|_{L^\infty_{t,y}L^2_x}+\||x|(f_k)_{Q_k}(x)\|^{\frac12}_{L^\infty_{t,y}L^2_x}\|\partial_x((f_k)_{Q_k})\|^{\frac12}_{L^\infty_{t,y}L^2_x}\big)\\
&\qquad\lesssim R^{-\frac12}\liminf_{k\to\infty}\lambda_k\big(|Q_k|^{\frac12}\|f_k\|_{L^\infty_tL^2_{y,x}}+|Q_k|^{\frac12}\|xf_k\|^{\frac12}_{L^\infty_tL^2_{y,x}}\|\partial_xf_k\|^{\frac12}_{L^\infty_tL^2_{y,x}}\big)\\
&\qquad\sim R^{-\frac12}\liminf_{k\to\infty}\big(\|f_k\|_{L^\infty_tL^2_{y,x}}+\|xf_k\|^{\frac12}_{L^\infty_tL^2_{y,x}}\|\partial_xf_k\|^{\frac12}_{L^\infty_tL^2_{y,x}}\big)\to0\quad\text{as } R\to\infty.
\end{aligned}$$

Therefore, we can take R large enough such that

$$\liminf_{k\to\infty} \lambda_k \|e^{it\Delta_{\mathbb{R}^2}}(f_k)_{Q_k}\|_{L^\infty_{t,y,x}(|x|\geq R)} \lesssim \tfrac{1}{2}\epsilon^{\frac{24}{\epsilon_0}} A^{1-\frac{24}{\epsilon_0}}.$$

As a consequence, there exists $(t_k, y_k, x_k) \in \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}$ with $|x_k| \leq R$, so that

$$\liminf_{k\to\infty} \lambda_k |(e^{it_k\Delta_{\mathbb{R}^2}}(f_k)_{Q_k})(y_k, x_k)| \gtrsim \epsilon^{\frac{24}{\epsilon_0}} A^{1-\frac{24}{\epsilon_0}}. \tag{4-15}$$

Since $|x_k| \leq R$, we may assume, up to a subsequence, $x_k \to x^*$ as $k\to\infty$, with $|x^*| \lesssim 1$.

By the weak compactness of $L^2_y\mathcal{H}^1_x$, we have

$$\lambda_k e^{-i\xi_k\cdot(\lambda_k y+y_k)}(e^{it_k\Delta_{\mathbb{R}^2}} f_k)(\lambda_k y + y_k, x) \rightharpoonup \phi(y,x) \quad \text{in } L^2_y\mathcal{H}^1_x, \text{ as } k\to\infty.$$

By the very basic fact in Hilbert space H that

$$g_k \rightharpoonup g \quad \text{in } H \quad \Longrightarrow \quad \|g_k\|_H^2 - \|g_k - g\|_H^2 \to \|g\|_H^2,$$

we have

$$\lim_{k\to\infty} (\|f_k\|^2_{L^2_y\mathcal{H}^1_x} - \|f_k - \phi_k\|^2_{L^2_y\mathcal{H}^1_x}) = \|\phi\|^2_{L^2_y\mathcal{H}^1_x}.$$

We now turn to the remaining part (4-9). Define h so that $\mathcal{F}_y h$ is the characteristic function of the cube $\left[-\frac{1}{2},\frac{1}{2}\right]^2$. By Lemma 2.1, $(x,y)\mapsto h(y)\delta_0(x) \in L^2_y\mathcal{H}^{-1}_x(\mathbb{R}^2\times\mathbb{R})$. From (4-15), we obtain

$$\begin{aligned} &|\langle h(y)\delta_0(x), \phi(y, x+x^*)\rangle_{y,x}| \\ &\quad= \lim_{k\to\infty}\left|\left\langle \delta_0(x), \int_{\mathbb{R}^2} \bar{h}(y)\lambda_k e^{-i\xi_k\cdot(\lambda_k y+y_k)}(e^{it_k\Delta_{\mathbb{R}^2}} f_k)(\lambda_k y + y_k, x+x_k)\,dy\right\rangle_x\right| \\ &\quad= \lim_{k\to\infty} \lambda_k |(e^{it_k\Delta_{\mathbb{R}^2}}(f_k)_{Q_k})(y_k, x_k)| \gtrsim \epsilon^{\frac{24}{\epsilon_0}} A^{1-\frac{24}{\epsilon_0}}, \end{aligned} \tag{4-16}$$

from which it follows

$$\|\phi(y, x+x^*)\|_{L^2_y\mathcal{H}^1_x} \gtrsim \epsilon^{\frac{24}{\epsilon_0}} A^{1-\frac{24}{\epsilon_0}}.$$

At the same time, since

$$\begin{aligned} \|\phi(y,x)\|_{L^2_y\mathcal{H}^1_x} &\geq \|\phi(y,x+x^*)\|_{L^2_y H^1_x} + \||x|\phi(y,x+x^*)\|_{L^2_{y,x}} - \||x^*|\phi(y,x+x^*)\|_{L^2_{y,x}} \\ &= \|\phi(y,x+x^*)\|_{L^2_y\mathcal{H}^1_x} - |x^*|\|\phi(y,x)\|_{L^2_{y,x}}, \end{aligned}$$

we get

$$\|\phi(y,x+x^*)\|_{L^2_y\mathcal{H}^1_x} \leq \|\phi\|_{L^2_y\mathcal{H}^1_x} + |x^*|\|\phi\|_{L^2_{y,x}} \lesssim \|\phi\|_{L^2_y\mathcal{H}^1_x}.$$

Therefore $\|\phi\|_{L^2_y\mathcal{H}^1_x} \gtrsim \epsilon^{24/\epsilon_0} A^{1-24/\epsilon_0}$ and (4-9) follows. We turn to (4-10), by Proposition 4.3 and the Rellich–Kondrachov theorem, we have

$$e^{it\Delta_{\mathbb{R}^2}}\big(\lambda_k e^{-i\xi_k\cdot(\lambda_k y+y_k)}(e^{it_k\Delta_{\mathbb{R}^2}} f_k)(\lambda_k y+y_k, x+x_k)\big) \to e^{it\Delta_{\mathbb{R}^2}}\phi(y,x) \quad \text{as } k\to\infty,$$

for almost every $(t,y,x) \in \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}$. By the refined Fatou's lemma [Lieb and Loss 1997], we obtain

$$\|e^{it\Delta_{\mathbb{R}^2}} f_k\|^4_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x} - \|e^{it\Delta_{\mathbb{R}^2}}(f_k - \phi_k)\|^4_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x} - \|e^{it\Delta_{\mathbb{R}^2}}\phi_k\|^4_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x} \to 0 \quad \text{as } k\to\infty.$$

Thus, by the invariance of Galilean transform, we have

$$\begin{aligned}\limsup_{k\to\infty}\|e^{it\Delta_{\mathbb{R}^2}}(f_k-\phi_k)\|^4_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x} &= \limsup_{k\to\infty}\big(\|e^{it\Delta_{\mathbb{R}^2}}f_k\|^4_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}-\|e^{it\Delta_{\mathbb{R}^2}}\phi_k\|^4_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}\big)\\ &= (\epsilon^{\frac{2}{\epsilon_0}}A^{1-\frac{2}{\epsilon_0}})^4-\|e^{it\Delta_{\mathbb{R}^2}}\phi\|^4_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}.\end{aligned}$$

We now take $c(t)\in C^\infty$, which has compact support on $[-1,1]$, such that

$$\|c(t)e^{it\Delta}h\|_{L^{4/3}_{t,y}}=1.$$

Then by (4-16), we have

$$\left|\int_{\mathbb{R}}\langle c(t)h(y)\delta_0(x),\phi(y,x+x^*)\rangle_{y,x}\,\mathrm{d}t\right|\gtrsim\epsilon^{\frac{24}{\epsilon_0}}A^{1-\frac{24}{\epsilon_0}}.$$

On the other hand, by Hölder's inequality, Sobolev's inequality and Lemma 2.1,

$$\begin{aligned}\left|\int_{\mathbb{R}}\langle c(t)h(y)\delta_0(x),\phi(y,x+x^*)\rangle_{y,x}\,\mathrm{d}t\right| &= \left|\int_{\mathbb{R}}\langle e^{it\Delta_y}(c(t)h(y)\delta_0(x)),e^{it\Delta_y}\phi(y,x+x^*)\rangle_{y,x}\,\mathrm{d}t\right|\\ &\lesssim \|e^{it\Delta_y}(c(t)h(y))\|_{L^{4/3}_{t,y}}\|e^{it\Delta_y}\phi(y,x)\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}\\ &\lesssim \|e^{it\Delta_y}\phi(y,x)\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}.\end{aligned}$$

Therefore, by the above two estimates and (4-17), we get (4-10).

Case 2: $\{f_k\}_{k\ge1}$ is bounded in $\Sigma(\mathbb{R}^3)$. In this case, we have

$$\limsup_{k\to\infty}\|P^y_{\ge R}f_k\|_{L^2_y\mathcal{H}^{1-\epsilon_0}_x}\lesssim\langle R\rangle^{-\epsilon_0}\limsup_{k\to\infty}\|f_k\|_{\Sigma(\mathbb{R}^3)}\to0\quad\text{as } R\to\infty.$$

For $R\in2^{\mathbb{Z}}$ large enough depending on A and ϵ, by (4-11), Sobolev embedding, and the Strichartz estimate, $P^y_{\le R}f_k$ satisfies

$$\begin{aligned}\lim_{k\to\infty}\|e^{it\Delta_{\mathbb{R}^2}}P^y_{\le R}f_k\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x} &\ge \lim_{k\to\infty}\|e^{it\Delta_{\mathbb{R}^2}}f_k\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}-\lim_{k\to\infty}\|e^{it\Delta_{\mathbb{R}^2}}P^y_{\ge R}f_k\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}\\ &\ge \lim_{k\to\infty}\|e^{it\Delta_{\mathbb{R}^2}}f_k\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}-C\lim_{k\to\infty}\|P^y_{\ge R}f_k\|_{L^2_y\mathcal{H}^{1-\epsilon_0}_x}\\ &\ge \tfrac{1}{2}\epsilon^{\frac{2}{\epsilon_0}}A^{1-\frac{2}{\epsilon_0}}.\end{aligned}$$

So we can replace f_k by $P^y_{\le R}f_k$ in the above case, and for $R=R(A,\epsilon)>0$ large enough, we may take $\{Q_k\}_{k\ge1}\subseteq\mathcal{D}$ and $|Q_k|\lesssim R^2$ such that $\lambda_k\gtrsim R^{-1}$, and $|\xi_k|\lesssim R$. As in the proof of Case 1, we still have (4-12) and also (4-9), (4-10). Furthermore, if $\limsup_{k\to\infty}\lambda_k<\infty$, then

$$\lambda_k e^{-i\xi_k\cdot(\lambda_k y+y_k)}(e^{it_k\Delta_{\mathbb{R}^2}}f_k)(\lambda_k y+y_k,x)\rightharpoonup\phi(y,x)$$

holds for some $\phi\in\Sigma(\mathbb{R}^2\times\mathbb{R})$. To show (4-13), we just need to consider the case when $\lambda_k\to\infty$ because the situation when $\limsup_{k\to\infty}\lambda_k<\infty$ is as in Case 1. We note

$$\lim_{k\to\infty}\|\phi_k\|^2_\Sigma\ge\lim_{k\to\infty}\|P^y_{\le\lambda_k^\theta}\phi\|^2_{L^2_y\mathcal{H}^1_x}\gtrsim A^2\left(\frac{\epsilon}{A}\right)^{\frac{48}{\epsilon_0}}.$$

Then the decoupling of the Σ-norm comes from $P_{\le\lambda_k^\theta}\to\mathrm{Id}$ in $L^2_y\mathcal{H}^1_x$ and (4-12). □

Proof of Theorem 4.1. The conclusion follows by applying Proposition 4.6 repeatedly until the asymptotically linear evolution of the remainder is trivial in $L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x$. The decoupling (4-5) and (4-6) follow from (4-13) and the orthogonality (4-1). □

Remark 4.7. For a linear profile decomposition for the Schrödinger propagator of the Schrödinger operator $-\Delta+|x|^2$ in L^2, we refer to the work of C. Jao, R. Killip, and M. Vişan [Jao et al. 2019] and C. Jao [2020]; we believe that some part of their argument can be applied in our equation. We also refer to the linear profile decomposition proved by A. Ardila and R. Carles [2021].

4.2. ***Approximation of the nonlinear profile: the case of concentrated initial data.*** In this section, we will show that the nonlinear profile u_λ given in (1-4)

$$\begin{cases} i\,\partial_t u_\lambda+\Delta_{\mathbb{R}^3}u_\lambda-x^2u_\lambda=|u_\lambda|^2u_\lambda,\\ u_\lambda(0,y,x)=\frac{1}{\lambda}\phi(\frac{y}{\lambda},x),\end{cases}$$

can be approximated by $\tilde{u}_\lambda$ given in (1-6)

$$\tilde{u}_\lambda(t,y,x)=e^{it(\Delta_{\mathbb{R}}-x^2)}\sum_{n\in\mathbb{N}}\left(\frac{1}{\lambda}v_n\left(\frac{t}{\lambda^2},\frac{y}{\lambda},x\right)\right),\qquad (t,y,x)\in\mathbb{R}\times\mathbb{R}^2\times\mathbb{R},$$

when λ is sufficiently large. Here v_n is the solution of the (DCR) system (1-5)

$$\begin{cases} (i\,\partial_t+\Delta_y)v_n(t,y,x)=\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})(t,y,x),\\ v_n(0,y,x)=\phi_n(y,x)=\Pi_n\phi(y,x).\end{cases}$$

The following corollary can be proven from Theorem 1.2 by following the argument in [Colliander et al. 2008; Killip and Vişan 2013]. In particular, we refer to [Colliander et al. 2004, Lemma 3.12].

Corollary 4.8 (corollary of Theorem 1.2: preservation of higher regularity). *Suppose $\phi\in L^2_y\mathcal{H}^1_x(\mathbb{R}^2\times\mathbb{R})$ and v is the global solution of (DCR) given as in Theorem 1.2. For any $s_1\ge 0$ and $s_2\ge 1$, if we assume further $v|_{t=0}\in H^{s_1}_y\mathcal{H}^{s_2}_x(\mathbb{R}^2\times\mathbb{R})$, then the solution $v\in C^0_tH^{s_1}_y\mathcal{H}^{s_2}_x(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})$ and satisfies*

$$\|v\|_{L^\infty_tH^{s_1}_y\mathcal{H}^{s_2}_x\cap L^4_tW^{s_1,4}_y\mathcal{H}^{s_2}_x(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})}\le C(\|\phi\|_{H^{s_1}_y\mathcal{H}^{s_2}_x(\mathbb{R}^2\times\mathbb{R})}).$$

Relying on Corollary 4.8, we can now prove the following general result on approximation of the nonlinear profile in the large-scale case. We will prove it with the help of Theorem 3.6.

Theorem 4.9. *For any $\phi\in L^2_y\mathcal{H}^1_x$, $0<\theta\ll 1$, $(\lambda_k,t_k,y_k,\xi_k)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2$, $|\xi_k|\lesssim 1$ and $\lambda_k\to\infty$ when $k\to\infty$. There exists a global solution $u_k\in C^0_tL^2_y\mathcal{H}^1_x$ of*

$$\begin{cases} i\,\partial_t u_k+\Delta_y u_k+\Delta_x u_k-x^2u_k=|u_k|^2u_k,\\ u_k(0,y,x)=\lambda_k^{-1}e^{iy\cdot\xi_k}(e^{it_k\Delta_y}P_{\le\lambda_k^\theta}\phi)(\frac{y-y_k}{\lambda_k}x),\end{cases}$$

for k large enough, satisfying

$$\|u_k\|_{L^\infty_tL^2_y\mathcal{H}^1_x\cap L^4_{t,y}\mathcal{H}^1_x(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})}\lesssim_{\|\phi\|_{L^2_y\mathcal{H}^1_x}}1.$$

Furthermore, assume that $\epsilon_4 = \epsilon_4(\|\phi\|_{L^2_y \mathcal{H}^1_x})$ *is a sufficiently small positive constant and* $\psi \in H^{10}_y \mathcal{H}^{10}_x$ *such that*

$$\|\phi - \psi\|_{L^2_y \mathcal{H}^1_x} \le \epsilon_4.$$

Then there exists a solution $v \in C^0_t H^2_y \mathcal{H}^1_x(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R})$ *of* (DCR), *with*

$$v(0, y, x) = \psi(y, x) \quad \text{if } t_k = 0,$$
$$\lim_{t \to \pm\infty} \|v(t, y, x) - e^{it\Delta_y}\psi\|_{L^2_y \mathcal{H}^1_x} = 0 \quad \text{if } t_k \to \pm\infty,$$

such that for k *large enough we have*

$$\|u_k\|_{L^\infty_t L^2_y \mathcal{H}^1_x \cap L^4_{t,y} \mathcal{H}^1_x(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} \lesssim 1,$$

with

$$\|u_k(t) - w_{\lambda_k}(t)\|_{L^\infty_t L^2_y \mathcal{H}^1_x \cap L^4_{t,y} \mathcal{H}^1_x(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} \to 0 \quad \text{as } k \to \infty,$$

where

$$w_{\lambda_k}(t, y, x) = e^{-i(t-t_k)|\xi_k|^2} e^{iy\cdot\xi_k} \lambda_k^{-1} e^{it(\Delta_x - x^2)} v\left(\frac{t}{\lambda_k^2} + t_k, \frac{y - y_k - 2\xi_k(t - t_k)}{\lambda_k}, x\right).$$

Proof of Theorem 4.9. By translation invariance, we may take $y_k = 0$. By Galilean transformation and $|\xi_k|$ is bounded, we may take $\xi_k = 0$. Then

$$w_{\lambda_k}(t, y, x) = \lambda_k^{-1} e^{it(\Delta_x - x^2)} v\left(\frac{t}{\lambda_k^2} + t_k, \frac{y}{\lambda_k}, x\right).$$

When $t_k = 0$, we will show w_{λ_k} is an approximate solution to (1-2). After a simple computation, we see

$$\begin{aligned} e_{\lambda_k} &:= (i\partial_t + \Delta_y + \Delta_x - x^2) w_{\lambda_k} - |w_{\lambda_k}|^2 w_{\lambda_k} \\ &= -\lambda_k^{-3} \sum_{n\in\mathbb{N}} e^{-it(2n+1)} \sum_{\substack{n_1,n_2,n_3\in\mathbb{N} \\ n_1-n_2+n_3\neq n}} e^{-2it(n_1-n_2+n_3-n)} (\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3}))\left(\frac{t}{\lambda_k^2}, \frac{y}{\lambda_k}, x\right). \end{aligned} \tag{4-17}$$

We will show this error term is small in the dual Strichartz space. Divide the right-hand side of (4-17) into three terms:

$$\begin{aligned} &e_{\lambda_k}(t, y, x) \\ &\quad = -\lambda_k^{-3} \sum_{n\in\mathbb{N}} e^{-it(2n+1)} \sum_{n_1,n_2,n_3\in\mathbb{N}} e^{-2it(n_1-n_2+n_3-n)} P^y_{\ge 2^{-10}}\left(\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\left(\frac{t}{\lambda_k^2}, \frac{y}{\lambda_k}, x\right)\right) \\ &\quad\quad + \lambda_k^{-3} \sum_{n\in\mathbb{N}} e^{-it(2n+1)} \sum_{\substack{n_1,n_2,n_3\in\mathbb{N} \\ n_1-n_2+n_3=n}} e^{-2it(n_1-n_2+n_3-n)} P^y_{\ge 2^{-10}}\left(\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\left(\frac{t}{\lambda_k^2}, \frac{y}{\lambda_k}, x\right)\right) \\ &\quad\quad - \lambda_k^{-3} \sum_{n\in\mathbb{N}} e^{-it(2n+1)} \sum_{\substack{n_1,n_2,n_3\in\mathbb{N} \\ n_1-n_2+n_3\neq n}} e^{-2it(n_1-n_2+n_3-n)} P^y_{\le 2^{-10}}\left(\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\left(\frac{t}{\lambda_k^2}, \frac{y}{\lambda_k}, x\right)\right) \\ &\quad =: e^1_{\lambda_k} + e^2_{\lambda_k} + e^3_{\lambda_k}. \end{aligned}$$

We first consider $e^1_{\lambda_k}$ and shall use Bernstein's inequality, Leibniz's rule, Plancherel's identity and Hölder's inequality to estimate as follows:

$$\begin{aligned}
&\|e^1_{\lambda_k}(t,y,x)\|_{L^{4/3}_{t,y}\mathcal{H}^1_x}\\
&\quad\lesssim \lambda_k^{-1}\Big\|\sum_{n\in\mathbb{N}} e^{-i\lambda_k^2 t(2n+1)}\sum_{n_1,n_2,n_3\in\mathbb{N}} e^{-2i\lambda_k^2 t(n_1-n_2+n_3-n)}\Pi_n(\nabla_y v_{n_1}\cdot\bar v_{n_2}\cdot v_{n_3})(t,y,x)\Big\|_{L^{4/3}_{t,y}\mathcal{H}^1_x}+\cdots\\
&\quad\sim \lambda_k^{-1}\big\|e^{i\lambda_k^2 t(\Delta_x-x^2)}\big(e^{i\lambda_k^2 t(\Delta_x-x^2)}\nabla_y v\cdot\overline{e^{i\lambda_k^2 t(\Delta_x-x^2)}v}\,e^{i\lambda_k^2 t(\Delta_x-x^2)}v\big)(t,y,x)\big\|_{L^{4/3}_{t,y}\mathcal{H}^1_x}\\
&\quad\lesssim \lambda_k^{-1}\|\nabla_y v\|_{L^4_{t,y}\mathcal{H}^1_x}\|v\|^2_{L^4_{t,y}\mathcal{H}^1_x}\to 0\quad\text{as } k\to\infty,
\end{aligned}\tag{4-18}$$

where $\cdots$ are the missing two terms with ∇_y acting on $\bar v_{n_2}$ and v_{n_3}.

We now turn to the estimate of $e^2_{\lambda_k}$. Using Bernstein's inequality and Leibniz's rule as above, we have

$$\begin{aligned}
\|e^2_{\lambda_k}\|_{L^{4/3}_{t,y}\mathcal{H}^1_x}&\lesssim \lambda_k^{-1}\Big\|\sum_{n\in\mathbb{N}} e^{-i\lambda_k^2 t(2n+1)}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(\nabla_y v_{n_1}\cdot\bar v_{n_2}v_{n_3})(t,y,x)\Big\|_{L^{4/3}_{t,y}\mathcal{H}^1_x}+\cdots\\
&\sim \lambda_k^{-1}\Big\|\Big(\sum_{n\in\mathbb{N}}\Big|\langle n\rangle^{\frac12}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(\nabla_y v_{n_1}\cdot\bar v_{n_2}v_{n_3})(t,y,x)\Big|^2\Big)^{\frac12}\Big\|_{L^{4/3}_{t,y}L^2_x}+\cdots,
\end{aligned}\tag{4-19}$$

where $\cdots$ are the missing two terms with ∇_y acting on $\bar v_{n_2}$ and v_{n_3}.

We observe the following elementary inequality: for $n=n_1-n_2+n_3$

$$\langle n\rangle^{\frac12}\leqslant\langle n\rangle^{-1}\langle n\rangle^2\leqslant\langle n\rangle^{-1}\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2.$$

Using the fact $\{\langle n\rangle^{-1}\}_{n\in\mathbb{N}}\in l^2_n$, the Minkowski inequality and boundedness of Π_n, we have

$$\begin{aligned}
&\Big\|\langle n\rangle^{\frac12}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(\nabla_y v_{n_1}\cdot\bar v_{n_2}v_{n_3})(t,y,x)\Big\|_{L^{4/3}_{t,y}L^2_x l^2_n}\\
&\qquad\lesssim\Big\|\langle n\rangle^{-1}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2|\Pi_n(\nabla_y v_{n_1}\cdot\bar v_{n_2}v_{n_3})(t,y,x)|\Big\|_{L^{4/3}_{t,y}L^2_x l^2_n}\\
&\qquad\lesssim\Big\|\sum_{n_1,n_2,n_3\in\mathbb{N}}\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2\|(\nabla_y v_{n_1}\cdot\bar v_{n_2}v_{n_3})(t,y,x)\|_{L^2_x}\Big\|_{L^{4/3}_{t,y}}.
\end{aligned}$$

By Hölder's inequality and the embedding $\mathcal{H}^1(\mathbb{R})\subseteq L^\infty(\mathbb{R})$, we find

$$\|(\nabla_y v_{n_1}\cdot\bar v_{n_2}v_{n_3})(t,y,x)\|_{L^2_x(\mathbb{R})}\lesssim\|\nabla_y v_{n_1}(t,y,x)\|_{L^2_x(\mathbb{R})}\|v_{n_2}(t,y,x)\|_{\mathcal{H}^1_x(\mathbb{R})}\|v_{n_3}(t,y,x)\|_{\mathcal{H}^1_x(\mathbb{R})}.$$

Similar arguments can be applied to the other two terms on the right-hand side of (4-19). All together this leads to the estimate

$$\begin{aligned}
\|e^2_{\lambda_k}\|_{L^{4/3}_{t,y}\mathcal{H}^1_x}\lesssim\lambda_k^{-1}&\|\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2\|(\nabla_y v_{n_1}\cdot\bar v_{n_2}v_{n_3})(t,y,x)\|_{L^2_x}\|_{L^{4/3}_{t,y}l^1_{n_1}l^1_{n_2}l^1_{n_3}}\\
&+\lambda_k^{-1}\|\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2\|(v_{n_1}\cdot\overline{\nabla_y v_{n_2}}v_{n_3})(t,y,x)\|_{L^2_x}\|_{L^{4/3}_{t,y}l^1_{n_1}l^1_{n_2}l^1_{n_3}}\\
&+\lambda_k^{-1}\|\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2\|(v_{n_1}\cdot\bar v_{n_2}\nabla_y v_{n_3})(t,y,x)\|_{L^2_x}\|_{L^{4/3}_{t,y}l^1_{n_1}l^1_{n_2}l^1_{n_3}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \lambda_k^{-1} \big\| \langle n_1\rangle^3 \langle n_2\rangle^3 \langle n_3\rangle^3 \|\nabla_y v_{n_1}(t,y,\cdot)\|_{L^2} \|v_{n_2}(t,y,\cdot)\|_{\mathcal{H}^1} \|v_{n_3}(t,y,\cdot)\|_{\mathcal{H}^1} \big\|_{L^{4/3}_{t,y} l^2_{n_1} l^2_{n_2} l^2_{n_3}} \\
&\lesssim \lambda_k^{-1} \|\nabla_y v\|_{L^4_{t,y}\mathcal{H}^6_x} \|v\|^2_{L^4_{t,y}\mathcal{H}^7_x} \\
&\lesssim \lambda_k^{-1} C(\|\psi\|_{H^1_y \mathcal{H}^6_x}) C(\|\psi\|_{L^2_y \mathcal{H}^7_x}) \to 0 \quad \text{as } k\to\infty.
\end{aligned} \tag{4-20}
$$

Now, we only need to deal with $e^3_{\lambda_k}$. We will use the normal form transform to exploit additional decay of λ_k, since it possesses time nonresonance property. Integrating by parts and direct computation imply

$$
\begin{aligned}
&\int_0^t e^{i(t-\tau)(\Delta_y+\Delta_x-x^2)} e^3_{\lambda_k}(\tau)\,\mathrm{d}\tau \\
&= -\lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} \int_0^t e^{it(\Delta_y-2n-1)} e^{-i\tau\tilde{\Delta}_y} P^y_{\le 2^{-10}} \Big(\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(\frac{\tau}{\lambda_k^2}, \frac{y}{\lambda_k}, x\Big)\Big)\,\mathrm{d}\tau \\
&= -\lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{it(\Delta_y-2n-1)} e^{-it\tilde{\Delta}_y} (-i\tilde{\Delta}_y)^{-1} P^y_{\le 2^{-10}} \Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(\frac{t}{\lambda_k^2}, \frac{y}{\lambda_k}, x\Big) \\
&\quad +\lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{it(\Delta_y-2n-1)} (-i\tilde{\Delta}_y)^{-1} P^y_{\le 2^{-10}} \Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(0, \frac{y}{\lambda_k}, x\Big) \\
&\quad\quad +\lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} \int_0^t e^{it(\Delta_y-2n-1)} e^{-i\tau\tilde{\Delta}_y} (-i\tilde{\Delta}_y)^{-1} \partial_\tau \Big(P^y_{\le 2^{-10}} \Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(\frac{\tau}{\lambda_k^2}, \frac{y}{\lambda_k}, x\Big)\Big)\,\mathrm{d}\tau \\
&= \lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{it(\Delta_y+\Delta_x-x^2)} (-i\tilde{\Delta}_y)^{-1} P^y_{\le 2^{-10}} \Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(0, \frac{y}{\lambda_k}, x\Big) \\
&\quad -\lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{-2it(n_1-n_2+n_3)-it} (-i\tilde{\Delta}_y)^{-1} P^y_{\le 2^{-10}} \Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(\frac{t}{\lambda_k^2}, \frac{y}{\lambda_k}, x\Big) \\
&\quad\quad +\lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{it(\Delta_y-2n-1)} \int_0^t e^{-i\tau\tilde{\Delta}_y} (-i\tilde{\Delta}_y)^{-1} \partial_\tau P^y_{\le 2^{-10}} \Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(\frac{\tau}{\lambda_k^2}, \frac{y}{\lambda_k}, x\Big)\,\mathrm{d}\tau,
\end{aligned}
$$

where the operator $\tilde{\Delta}_y$ is defined to be

$$
\tilde{\Delta}_y := 2(n_1-n_2+n_3-n)+\Delta_y.
$$

This is a perturbation of the Laplacian operator and we suppress the parameters n_1, n_2, n_3, n. The inverse operator $(-\tilde{\Delta}_y)^{-1}$ is defined by the Fourier transform

$$
\mathcal{F}_y((-i\tilde{\Delta}_y)^{-1} f)(\xi, x) = \frac{i(\mathcal{F}_y f)(\xi, x)}{2(n_1-n_2+n_3-n)-|\xi|^2}.
$$

This operator is invertible when $n_1-n_2+n_3-n\neq 0$ and $|\xi|\le 2^{-10}$. We will use this expression in the remaining of the proof.

Define

$$e^{3,1}_{\lambda_k} := \Big\| \lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{it(\Delta_y+\Delta_x-x^2)} P^y_{\leq 2^{-10}}\Big((-i\tilde{\Delta}_y)^{-1}\Big(\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(0,\frac{y}{\lambda_k},x\Big)\Big)\Big)\Big\|_{L^4_{t,y}\mathcal{H}^1_x},$$

$$e^{3,2}_{\lambda_k} := \Big\| \lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{-2it(n_1-n_2+n_3)-it}(-i\tilde{\Delta}_y)^{-1}\Big(P^y_{\leq 2^{-10}}\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\Big\|_{L^4_{t,y}\mathcal{H}^1_x},$$

and

$$e^{3,3}_{\lambda_k} := \Big\| \lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{it(\Delta_y-2n-1)} \cdot \int_0^t e^{-i\tau\tilde{\Delta}_y}(-i\tilde{\Delta}_y)^{-1}\Big(\partial_\tau P^y_{\leq 2^{-10}}\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\,\mathrm{d}\tau\Big\|_{L^4_{t,y}\mathcal{H}^1_x}.$$

Then, we have

$$\Big\|\int_0^t e^{i(t-\tau)(\Delta_y+\Delta_x-x^2)} e^3_{\lambda_k}(\tau,y,x)\,\mathrm{d}\tau\Big\|_{L^4_{t,y}\mathcal{H}^1_x} \sim e^{3,1}_{\lambda_k}+e^{3,2}_{\lambda_k}+e^{3,3}_{\lambda_k}. \tag{4-21}$$

First, we consider the term $e^{3,1}_{\lambda_k}$. By the boundedness of the operator $P^y_{\leq 2^{-10}}(-i\tilde{\Delta}_y)^{-1}$ when $n_1-n_2+n_3\neq n$ and Minkowski's inequality, we may estimate as follows:

$$\begin{aligned}
e^{3,1}_{\lambda_k} &\lesssim \Big\|\Big\|\langle n\rangle^{\frac12}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} \lambda_k^{-3}P^y_{\leq 2^{-10}}(-i\tilde{\Delta}_y)^{-1}\Pi_n\Big((v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(0,\frac{y}{\lambda_k},x\Big)\Big)\Big\|_{l^2_n}\Big\|_{L^2_{y,x}}\\
&\lesssim \lambda_k^{-3}\Big\|\langle n\rangle^{\frac12}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3\neq n}}\Big\|\Pi_n\Big((v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(0,\frac{y}{\lambda_k},x\Big)\Big)\Big\|_{L^2_y}\Big\|_{L^2_x l^2_n}\\
&\lesssim \lambda_k^{-2}\Big\|\langle n\rangle^{\frac12}\sum_{n_1,n_2,n_3\in\mathbb{N}}\|\Pi_n\big((v_{n_1}\bar{v}_{n_2}v_{n_3})(0,y,x)\big)\|_{L^2_y}\Big\|_{L^2_x l^2_n}\\
&\lesssim \lambda_k^{-2}\sum_{n_1,n_2,n_3\in\mathbb{N}}\|\langle n\rangle^{\frac12}\Pi_n\big((v_{n_1}\bar{v}_{n_2}v_{n_3})(0,y,x)\big)\|_{L^2_{y,x}l^2_n} \lesssim \lambda_k^{-2}\sum_{n_1,n_2,n_3\in\mathbb{N}}\|(v_{n_1}\bar{v}_{n_2}v_{n_3})(0,y,x)\|_{L^2_y\mathcal{H}^1_x}\\
&\lesssim \lambda_k^{-2}\sum_{n_1,n_2,n_3\in\mathbb{N}}\|v_{n_1}(0,y,x)\|_{L^6_y\mathcal{H}^1_x}\|v_{n_2}(0,y,x)\|_{L^6_y\mathcal{H}^1_x}\|v_{n_3}(0,y,x)\|_{L^6_y\mathcal{H}^1_x}\\
&\lesssim \lambda_k^{-2}C(\|v(0,y,x)\|_{H^1_y\mathcal{H}^3_x})^3 \to 0 \quad \text{as } k\to\infty.
\end{aligned} \tag{4-22}$$

Next, we consider the term $e^{3,2}_{\lambda_k}$. As in the estimate of $e^{3,1}_{\lambda_k}$, by the boundedness of the operator $P^y_{\leq 2^{-10}}(-i\tilde{\Delta}_y)^{-1}$ when $n_1-n_2+n_3\neq n$, Minkowski's inequality, the fractional Leibniz rule, Sobolev's inequality and Hölder's inequality, we have

$$e^{3,2}_{\lambda_k} \lesssim \lambda_k^{-3}\Big\|\langle n\rangle^{\frac12}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{-2it(n_1-n_2+n_3)-it}(-i\tilde{\Delta}_y)^{-1} \cdot P^y_{\leq 2^{-10}}\Pi_n\Big((v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\Big\|_{L^4_{t,y}L^2_x l^2_n}$$

$$\lesssim \lambda_k^{-3}\Big\| \langle n\rangle^{\frac12} \sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} \|\Pi_n\Big((v_{n_1}\bar v_{n_2}v_{n_3})\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\|_{L^4_y}\Big\|_{L^4_tL^2_xl^2_n}$$

$$\lesssim \lambda_k^{-\frac32}\Big\|\sum_{n_1,n_2,n_3\in\mathbb{N}}\|v_{n_1}\bar v_{n_2}v_{n_3}\|_{H^{1/2}_y\mathcal{H}^1_x}\Big\|_{L^4_t} \lesssim \lambda_k^{-\frac32}\big\|\|v(t,y,x)\|^3_{W^{3/4,4}_y\mathcal{H}^5_x}\big\|_{L^4_t}$$

$$\lesssim \lambda_k^{-\frac32}\|v(t,y,x)\|^3_{L^{12}_tW^{3/4,4}_y\mathcal{H}^5_x} \lesssim \lambda_k^{-\frac32}C(\|v(0,y,x)\|_{H^{13/12}_y\mathcal{H}^5_x})^3\to 0 \quad \text{as } k\to\infty. \tag{4-23}$$

Finally, we are left to consider the term $e^{3,3}_{\lambda_k}$. Applying the Strichartz estimate, we obtain

$$\Big\|\lambda_k^{-3}\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{it(\Delta_y-2n-1)} \cdot\int_0^t e^{-i\tau\tilde\Delta_y}(-i\tilde\Delta_y)^{-1}\Big(\partial_\tau P^y_{\le 2^{-10}}\Pi_n(v_{n_1}\bar v_{n_2}v_{n_3})\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\,\mathrm{d}\tau\Big\|_{L^4_{t,y}\mathcal{H}^1_x}$$

$$\lesssim \Big\|(i\partial_t+\Delta_y+\Delta_x-x^2)\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{it(\Delta_y-2n-1)} \cdot\int_0^t P^y_{\le 2^{-10}}\Big(e^{-i\tau\tilde\Delta_y}(-i\tilde\Delta_y)^{-1} \cdot\partial_\tau\Pi_n\Big(\lambda_k^{-3}v_{n_1}\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\overline{v_{n_2}\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)}v_{n_3}\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\Big)\,\mathrm{d}\tau\Big\|_{L^1_tL^2_y\mathcal{H}^1_x}. \tag{4-24}$$

We observe, after some computation, that

$$(i\partial_t+\Delta_y+\Delta_x-x^2)\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{it(\Delta_y-2n-1)}\int_0^t e^{-i\tau\tilde\Delta_y}(-i\tilde\Delta_y)^{-1}P^y_{\le 2^{-10}} \cdot\partial_\tau\Pi_n\Big(\lambda_k^{-3}v_{n_1}\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\overline{v_{n_2}\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)}v_{n_3}\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\,\mathrm{d}\tau$$
$$=\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{-2it(n_1-n_2+n_3)-it}(-i\tilde\Delta_y)^{-1}P^y_{\le 2^{-10}} \cdot\partial_t\Pi_n\Big(\lambda_k^{-3}v_{n_1}\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\overline{v_{n_2}\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)}v_{n_3}\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big).$$

Therefore, by the above observation, Plancherel's theorem and Leibniz's rule, we have

$$(4\text{-}24)\lesssim\Big\|\Big(\sum_{n\in\mathbb{N}}\Big(\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3\neq n}}\Big|e^{-2it(n_1-n_2+n_3-n)-it}(-i\tilde\Delta_y)^{-1}P^y_{\le 2^{-10}}\partial_t\Pi_n\Big(\lambda_k^{-3}v_{n_1}\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big) \cdot\overline{v_{n_2}\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)}v_{n_3}\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\Big|\Big)^2\langle n\rangle\Big)^{\frac12}\Big\|_{L^1_tL^2_yL^2_x}$$

$$\lesssim\Big\|\Big(\sum_{n\in\mathbb{N}}\Big(\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3\neq n}}\Big\|\partial_t\Pi_n\Big(\lambda_k^{-3}v_{n_1}\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big) \cdot\overline{v_{n_2}\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)}v_{n_3}\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\Big\|_{L^2_{y,x}}\Big)^2\langle n\rangle\Big)^{\frac12}\Big\|_{L^1_t}$$

$$\lesssim \lambda_k^{-2} \sum_{n_1,n_2,n_3\in\mathbb{N}} \|\partial_t v_{n_1}\cdot \bar{v}_{n_2} v_{n_3}\|_{L_t^1 L_y^2 \mathcal{H}_x^1} + \lambda_k^{-2} \sum_{n_1,n_2,n_3\in\mathbb{N}} \|v_{n_1}\cdot \overline{\partial_t v_{n_2}} v_{n_3}\|_{L_t^1 L_y^2 \mathcal{H}_x^1} + \lambda_k^{-2} \sum_{n_1,n_2,n_3\in\mathbb{N}} \|v_{n_1}\cdot \bar{v}_{n_2}\partial_t v_{n_3}\|_{L_t^1 L_y^2 \mathcal{H}_x^1}. \quad (4\text{-}25)$$

We shall only show how to estimate the first term on the right-hand side of (4-25), as the other two terms can be estimated similarly. By Hölder's inequality, and the fact that v satisfies (DCR), we have

$$\begin{aligned}\sum_{n_1,n_2,n_3\in\mathbb{N}} \|\partial_t v_{n_1}\bar{v}_{n_2} v_{n_3}\|_{L_t^1 L_y^2 \mathcal{H}_x^1} &\lesssim \|v\|_{L_t^3 L_y^6 \mathcal{H}_x^3}^2 \|\partial_t v\|_{L_t^3 L_y^6 \mathcal{H}_x^3} \\ &\lesssim \|v\|_{L_t^3 L_y^6 \mathcal{H}_x^3}^2 \|\Delta_y v\|_{L_t^3 L_y^6 \mathcal{H}_x^3} + \|v\|_{L_t^3 L_y^6 \mathcal{H}_x^3}^2 \Bigg\| \sum_{\substack{n_4,n_5,n_6,n_1\in\mathbb{N}\\ n_4-n_5+n_6=n_1}} \Pi_{n_1}(v_{n_4}\bar{v}_{n_5} v_{n_6})\Bigg\|_{L_t^3 L_y^6 \mathcal{H}_x^3}.\end{aligned}$$

Applying Hölder's inequality and the Sobolev embedding, we have

$$\begin{aligned}&\Bigg\| \sum_{\substack{n_4,n_5,n_6,n_1\in\mathbb{N}\\ n_4-n_5+n_6=n_1}} \Pi_{n_1}(v_{n_4}\bar{v}_{n_5} v_{n_6})\Bigg\|_{L_t^3 L_y^6 \mathcal{H}_x^3} \\ &\quad\lesssim \Bigg\| \langle n_1\rangle^{\frac{3}{2}} \sum_{\substack{n_4,n_5,n_6\in\mathbb{N}\\ n_4-n_5+n_6=n_1}} \Pi_{n_1}(v_{n_4}\bar{v}_{n_5} v_{n_6})\Bigg\|_{L_t^3 L_y^6 L_x^2 l_{n_1}^2} \\ &\quad\lesssim \Bigg\| \sum_{\substack{n_4,n_5,n_6\in\mathbb{N}\\ n_4-n_5+n_6=n_1}} \langle n_1\rangle^{-1}\langle n_4\rangle^3\langle n_5\rangle^3\langle n_6\rangle^3 |\Pi_{n_1}(v_{n_4}\bar{v}_{n_5} v_{n_6})|\Bigg\|_{L_t^3 L_y^6 L_x^2 l_{n_1}^2} \\ &\quad\lesssim \Bigg\| \sum_{n_4,n_5,n_6\in\mathbb{N}} \langle n_4\rangle^3\langle n_5\rangle^3\langle n_6\rangle^3 \|v_{n_4}\bar{v}_{n_5} v_{n_6}\|_{L_x^2}\Bigg\|_{L_t^3 L_y^6} \\ &\quad\lesssim \Bigg\| \sum_{n_4,n_5,n_6\in\mathbb{N}} \langle n_4\rangle^3\langle n_5\rangle^3\langle n_6\rangle^3 \|v_{n_4}\|_{\mathcal{H}_x^1}\|v_{n_5}\|_{\mathcal{H}_x^1}\|v_{n_6}\|_{\mathcal{H}_x^1}\Bigg\|_{L_t^3 L_y^6} \lesssim \|v\|_{L_t^9 L_y^{18} \mathcal{H}_x^9}^3 \le C(\|\psi\|_{H_y^{2/3}\mathcal{H}_x^9})^3,\end{aligned}$$

where in the last inequality we use the fact that by Corollary 4.8, we have

$$\|v\|_{L_t^3 L_y^6 \mathcal{H}_x^3} \le C(\|\psi\|_{L_y^2 \mathcal{H}_x^3}),$$
$$\|\Delta_y v\|_{L_t^3 L_y^6 \mathcal{H}_x^3} \le C(\|\Delta_y \psi\|_{L_y^2 \mathcal{H}_x^3}).$$

Combining all these estimates together, we finally obtain

$$e_{\lambda_k}^{3,3} \lesssim \lambda_k^{-2}\big(C(\|\psi\|_{L_y^2 \mathcal{H}_x^3})^2 C(\|\Delta_y\psi\|_{L_y^2 \mathcal{H}_x^3}) + C(\|\psi\|_{L_y^2 \mathcal{H}_x^3})^2 C(\|\psi\|_{H_y^{2/3}\mathcal{H}_x^9})^3\big) \to 0 \quad \text{as } k\to\infty. \quad (4\text{-}26)$$

To apply Theorem 3.6, we see

$$\lim_{k\to\infty} \|w_{\lambda_k}(0,y,x) - u_{\lambda_k}(0,y,x)\|_{L_y^2 \mathcal{H}_x^1} = \|\phi-\psi\|_{L_y^2 \mathcal{H}_x^1} \le \epsilon_4,$$

$$\|w_{\lambda_k}\|_{L_t^\infty L_y^2\mathcal{H}_x^1(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} = \left\|\frac{1}{\lambda_k}v\left(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\right)\right\|_{L_t^\infty L_y^2\mathcal{H}_x^1} = \|v\|_{L_t^\infty L_y^2\mathcal{H}_x^1},$$

$$\|w_{\lambda_k}\|_{L_{t,y}^4\mathcal{H}_x^1(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} = \left\|\frac{1}{\lambda_k}v\left(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\right)\right\|_{L_{t,y}^4\mathcal{H}_x^1} = \|v\|_{L_{t,y}^4\mathcal{H}_x^1}.$$

These together with the estimates (4-18), (4-20), (4-21), (4-22), (4-23), (4-26) and Theorem 1.2 yields Theorem 4.9 when $t_k = 0$.

If $t_k \to \pm\infty$ as $k \to \infty$, then v is the solution of (DCR) with

$$\lim_{t\to\pm\infty} \|v(t,y,x) - e^{it\Delta_y}\psi\|_{L_y^2\mathcal{H}_x^1} = 0.$$

By the argument in the case $t_k = 0$, we can also obtain Theorem 4.9 in this case. □

4.3. *Existence of an almost-periodic solution.* Define

$$\Lambda(L) = \sup \|u\|_{L_{t,y}^4\mathcal{H}_x^{1-\epsilon_0}(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})},$$

where the supremum is taken over all global solutions $u \in C_t^0(\mathbb{R}, \Sigma(\mathbb{R}^3))$ of (1-2) with

$$\mathcal{E}(u(t)) + \mathcal{M}(u(t)) \le L.$$

The proof of Theorem 3.4 implies $\Lambda(L) < \infty$ for sufficiently small L. Let

$$L_{\max} = \sup\{L \ge 0 : \Lambda(L) < \infty\}. \tag{4-27}$$

If $L_{\max} < \infty$, then following the arguments in [Cheng et al. 2020a; 2020b], one can show the existence of an almost periodic solution with the help of Theorems 4.1 and 4.9. The proof is rather standard, we refer to [Cheng et al. 2020a; Cheng et al. 2020b; Killip and Vişan 2013; Kenig and Merle 2006; 2008; Tao et al. 2008] and omit the proof here.

Theorem 4.10 (existence of an almost-periodic solution). *Assume that $L_{\max} < \infty$. Then there exists a solution $u_c \in C_t^0(\mathbb{R}, \Sigma(\mathbb{R}^3))$ of the defocusing cubic NLS with partial harmonic potential (1-2) satisfying*

$$\mathcal{E}(u_c) + \mathcal{M}(u_c) = L_{\max} \quad \textit{and} \quad \|u_c\|_{L_{t,y}^4\mathcal{H}_x^{1-\epsilon_0}(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} = \infty. \tag{4-28}$$

Furthermore, u_c is almost periodic in the sense that for any $\eta > 0$ there is a Lipschitz function $t \mapsto y(t)$ and a sufficiently large positive number $C(\eta)$ such that

$$\int_{|y+y(t)|\ge C(\eta)} \|u_c(t,y,x)\|_{\mathcal{H}_x^1}^2 \,\mathrm{d}y < \eta \quad \textit{for all } t \in \mathbb{R}. \tag{4-29}$$

5. Rigidity theorem

In this section, we will exclude the almost-periodic solution in Theorem 4.10 by the interaction Morawetz estimate with an appropriately chosen weight function. Once the almost-periodic solution is excluded, we can finish the proof of Theorem 1.1.

Proposition 5.1 (nonexistence of the almost-periodic solution). *The almost-periodic solution u_c as in Theorem 4.10 does not exist.*

Proof. For each $r_0 > 0$, we define the interaction Morawetz action

$$M_{r_0}(t) = \int_{\mathbb{R}^2\times\mathbb{R}} \int_{\mathbb{R}^2\times\mathbb{R}} \Im(\overline{u_c(t,y,x)}\nabla_y u_c(t,y,x)) \cdot \nabla_y \psi_{r_0}(|y-\tilde{y}|)|u_c(t,\tilde{y},\tilde{x})|^2 \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x},$$

where $\Im = \operatorname{Im}$ denotes the imaginary part of a complex number and $\psi_{r_0}\colon \mathbb{R}\to\mathbb{R}$ is a radial function defined as in [Colliander et al. 2009; Planchon and Vega 2009], with

$$\Delta\psi_{r_0}(r) = \int_r^\infty s\log\left(\frac{s}{r}\right) w_{r_0}(s)\,\mathrm{d}s,$$

where

$$w_{r_0}(s) = \begin{cases} \frac{1}{s^3} & \text{if } s \geq r_0, \\ 0 & \text{if } s < r_0. \end{cases}$$

It is straightforward to verify that ψ_{r_0} is convex and $|\nabla\psi_{r_0}|$ is uniformly bounded (independent of r_0), with

$$-\Delta^2\psi_{r_0}(r) = \frac{2\pi}{r_0}\delta_0(r) - w_{r_0}(r).$$

Using the above properties of the weight function ψ_{r_0}, one can show (see [Colliander et al. 2009, Section 3.3]) for all $T_0 > 0$

$$\int_{-T_0}^{T_0}\int_{\mathbb{R}^2} \left||\nabla_y|^{\frac{1}{2}}(\|u_c(t,y,x)\|^2_{L^2_x(\mathbb{R})})\right|^2 \mathrm{d}y\,\mathrm{d}t \lesssim \|u_c\|^3_{L^\infty_t L^2_{y,x}} \|\nabla_y u_c\|_{L^\infty_t L^2_{y,x}} \lesssim 1. \tag{5-1}$$

By (4-29) and the conservation of mass, we have

$$\frac{\|u_c\|^2_{L^2_{y,x}}}{2} \leq \int_{|y+y(t)|\leq C(\frac{m_c}{100})} \|u_c(t,y,x)\|^2_{L^2_x}\,\mathrm{d}y, \tag{5-2}$$

where $m_c := \mathcal{M}(u_c) > 0$ by (4-28).

Therefore, for each $T_0 > 0$, by (5-2), Sobolev's inequality, and (5-1), we deduce

$$\begin{aligned} \frac{m_c^2 T_0}{2} &\leq \int_{-T_0}^{T_0}\left(\int_{|y+y(t)|\leq C(\frac{m_c}{100})} \|u_c(t,y,x)\|^2_{L^2_x}\,\mathrm{d}y\right)^2 \mathrm{d}t \\ &\lesssim C\left(\frac{m_c}{100}\right)\int_{-T_0}^{T_0}\left(\left(\int_{\mathbb{R}^2}(\|u_c(t,y,x)\|^2_{L^2_x})^4\,\mathrm{d}y\right)^{\frac{1}{4}}\right)^2 \mathrm{d}t \\ &\lesssim C\left(\frac{m_c}{100}\right)\int_{-T_0}^{T_0}\int_{\mathbb{R}^2}\left||\nabla_y|^{\frac{1}{2}}(\|u_c(t,y,x)\|^2_{L^2_x})\right|^2 \mathrm{d}y\mathrm{d}t \lesssim C\left(\frac{m_c}{100}\right). \end{aligned}$$

Letting $T_0 \to \infty$, we obtain a contradiction, and this concludes the proof. □

Finally, we can now prove Theorem 1.1.

Proof of Theorem 1.1. By Theorem 3.4, to prove the scattering of solutions to (1-2), it suffices to show the finiteness of the $L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x$-norm of the solution of (1-2).

To this end, let $L_{\max}$ be given as in (4-27). Then, equivalently, we need show that $L_{\max} = \infty$. Suppose for a contradiction that $L_{\max} < \infty$. Then Theorem 4.10 would yield an almost-periodic solution of (1-2), which is impossible in view of Proposition 5.1. □

6. Scattering of equation (DCR)

We will now prove Theorem 1.2, that is, the global well-posedness and scattering of the (DCR) system

$$\begin{cases} i\partial_t v + \Delta_{\mathbb{R}^2} v = F(v), \\ v(0, y, x) = \phi(y, x), \end{cases}$$

where

$$F(v) := \sum_{\substack{n_1, n_2, n_3, n \in \mathbb{N} \\ n_1 - n_2 + n_3 = n}} \Pi_n(\Pi_{n_1} v \overline{\Pi_{n_2} v} \Pi_{n_3} v)$$

and Π_n is the orthogonal projector on the n-th eigenspace E_n of $-\Delta_x + x^2$.

We will mainly follow the approach to the global well-posedness and scattering of the two-dimensional mass-critical nonlinear Schrödinger equation as in [Dodson 2016b]. The main ingredient is to establish an infinite-dimensional vector-valued version of the two-dimensional long-time Strichartz estimate, which helps us to preclude certain almost periodic solutions.

The (DCR) system is Hamiltonian with an energy functional

$$\mathcal{E}(v) = \frac{1}{2} \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^2 \times \mathbb{R}} |\nabla_y v_n|^2 \, dy \, dx + \frac{1}{4} \sum_{\substack{n, n_1, n_2, n_3, n_4 \in \mathbb{N} \\ n_1 - n_2 + n_3 - n_4 = n}} \int_{\mathbb{R}^2 \times \mathbb{R}} v_{n_1} \bar{v}_{n_2} v_{n_3} \bar{v}_{n_4} \, dy \, dx,$$

under the symplectic structure on $L^2_{y,x}(\mathbb{R}^2 \times \mathbb{R})$ given by $\omega(f, g) := \Im \int_{\mathbb{R}^2 \times \mathbb{R}} f(y, x) \overline{g(y, x)} \, dy \, dx$. It also conserves the following mass $\mathcal{M}$ and kinetic energy $\mathcal{E}_0$:

$$\mathcal{M}(v) = \int_{\mathbb{R}^2 \times \mathbb{R}} |v(t, y, x)|^2 \, dy \, dx,$$

$$\mathcal{E}_0(v) = \int_{\mathbb{R}^2 \times \mathbb{R}} |(-\Delta_x + x^2)^{\frac{1}{2}} v(t, y, x)|^2 \, dy \, dx = \sum_{n \in \mathbb{N}} (2n+1) \|v_n\|^2_{L^2_{y,x}(\mathbb{R}^2 \times \mathbb{R})} = \|v\|^2_{L^2_y \mathcal{H}^1_x(\mathbb{R}^2 \times \mathbb{R})}.$$

We shall divide this section into three subsections. In Section 6.1, we establish the local well-posedness theory for (DCR) and reduce the scattering to the exclusion of almost periodic solutions. In Section 6.2, we derive the long-time Strichartz estimate and in Section 6.3, we exclude the almost periodic solution.

6.1. ***Local well-posedness and reduction to the almost periodic solution.*** In this subsection, we will present the well-posedness theory of the (DCR). Then following ideas similar to those in [Tao et al. 2008; Cheng et al. 2020a; 2020b; Yang and Zhao 2018], we shall prove that there is an almost periodic solution of (DCR) if the system is not global well-posed and if the solution does not scatter in $L^2_y \mathcal{H}^1_x$. That is, we reduce the global well-posedness and scattering of (DCR) to the exclusion of this almost periodic solution.

6.1.1. *Local well-posedness theory and the existence of an almost periodic solution.* The local well-posedness theory of the (DCR) system follows from a more or less standard argument: the Strichartz estimate in Proposition 3.2 and the nonlinear estimate in Lemma 6.2. The proof of the nonlinear estimate relies on the following Strichartz estimate for the harmonic oscillator.

Lemma 6.1 (Strichartz estimate for the harmonic oscillator [Carles 2002b; Keel and Tao 1998]). *For* $2 \le q, r \le \infty$ *with* $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$, *we have the estimate*

$$\|e^{it(\Delta_x - x^2)} f\|_{L_t^q \mathcal{W}_x^{s,r}([-T_1,T_1]\times\mathbb{R})} \lesssim \|f\|_{\mathcal{H}_x^s(\mathbb{R})}$$

holds for any $T_1 > 0$ *and* $s \ge 0$.

We can now give the nonlinear estimate.

Lemma 6.2. *For functions* F_1, F_2, F_3 *defined on* $\mathbb{R}^2 \times \mathbb{R}$, *we have*

$$\Big\| \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n(\Pi_{n_1}F_1 \overline{\Pi_{n_2}F_2} \Pi_{n_3}F_3) \Big\|_{L_y^{4/3}L_x^2} \lesssim \|F_1\|_{L_y^4 L_x^2} \|F_2\|_{L_y^4 L_x^2} \|F_3\|_{L_y^4 L_x^2}, \tag{6-1}$$

and consequently, *for any* $\beta \ge 0$,

$$\Big\| \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n(\Pi_{n_1}F_1 \overline{\Pi_{n_2}F_2} \Pi_{n_3}F_3) \Big\|_{L_y^{4/3}\mathcal{H}_x^\beta} \lesssim \min_{\tau\in\sigma_3} \|F_{\tau(1)}\|_{L_y^4\mathcal{H}_x^\beta} \|F_{\tau(2)}\|_{L_y^4 L_x^2} \|F_{\tau(3)}\|_{L_y^4 L_x^2}, \tag{6-2}$$

where σ_3 *is a permutation of the set* $\{1,2,3\}$.

Proof. Let $F_0 \in L_y^4 L_x^2$, by Hölder's inequality and Lemma 6.1, we have

$$\begin{aligned}
&\Big\langle \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n(\Pi_{n_1}F_1 \overline{\Pi_{n_2}F_2} \Pi_{n_3}F_3), F_0 \Big\rangle \\
&\quad = \frac{1}{\pi} \sum_{n_1,n_2,n_3,n\in\mathbb{N}} \int_0^\pi e^{2it(n_1-n_2+n_3-n)} \int_{\mathbb{R}^2\times\mathbb{R}} \Pi_{n_1}F_1 \overline{\Pi_{n_2}F_2} \Pi_{n_3}F_3 \overline{\Pi_n F_0} \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \frac{1}{\pi} \int_0^\pi \int_{\mathbb{R}^2\times\mathbb{R}} e^{it(-\Delta_x+x^2)} F_1(y,x) \overline{e^{it(-\Delta_x+x^2)} F_2(y,x)} \\
&\qquad\qquad \cdot e^{it(-\Delta_x+x^2)} F_3(y,x) \overline{e^{it(-\Delta_x+x^2)} F_0(y,x)} \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \lesssim \int_{\mathbb{R}^2} \|e^{it(-\Delta_x+x^2)} F_0(y,x)\|_{L_t^\infty L_x^2([0,\pi]\times\mathbb{R})} \|e^{it(-\Delta_x+x^2)} F_1(y,x)\|_{L_t^2 L_x^4([0,\pi]\times\mathbb{R})} \\
&\qquad\qquad \cdot \|e^{it(-\Delta_x+x^2)} F_2(y,x)\|_{L_t^4 L_x^8([0,\pi]\times\mathbb{R})} \|e^{it(-\Delta_x+x^2)} F_3(y,x)\|_{L_t^4 L_x^8([0,\pi]\times\mathbb{R})} \,\mathrm{d}y \\
&\quad \lesssim \int_{\mathbb{R}^2} \|F_0(y,x)\|_{L_x^2(\mathbb{R})} \|e^{it(-\Delta_x+x^2)} F_1(y,x)\|_{L_t^8 L_x^4([0,\pi]\times\mathbb{R})} \|e^{it(-\Delta_x+x^2)} F_2(y,x)\|_{L_t^{16/3} L_x^8([0,\pi]\times\mathbb{R})} \\
&\qquad\qquad \cdot \|e^{it(-\Delta_x+x^2)} F_3(y,x)\|_{L_t^{16/3} L_x^8([0,\pi]\times\mathbb{R})} \,\mathrm{d}y \\
&\quad \lesssim \int_{\mathbb{R}^2} \|F_0(y,x)\|_{L_x^2} \|F_1(y,x)\|_{L_x^2} \|F_2(y,x)\|_{L_x^2} \|F_3(y,x)\|_{L_x^2} \,\mathrm{d}y.
\end{aligned}$$

Therefore,

$$\Big\| \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n(\Pi_{n_1}F_1 \overline{\Pi_{n_2}F_2} \Pi_{n_3}F_3) \Big\|_{L_y^{4/3}L_x^2} \lesssim \|F_1\|_{L_y^4 L_x^2} \|F_2\|_{L_y^4 L_x^2} \|F_3\|_{L_y^4 L_x^2},$$

which is (6-1). One can similarly prove (6-2) using the fractional Leibniz rule. □

Lemma 6.2 provides the following estimate for the nonlinearity $F(v)$.

Lemma 6.3. *For each solution v of (DCR), we have*

$$\|F(v)\|_{L^{4/3}_{t,y}\mathcal{H}^{\alpha}_x} \lesssim \|v\|^3_{L^4_{t,y}\mathcal{H}^{\alpha}_x}, \quad \text{where } \alpha = 0, 1.$$

Thus by Proposition 3.2, the solution v of (DCR) satisfies the Strichartz estimate

$$\|v\|_{L^p_t L^q_y \mathcal{H}^{\alpha}_x(I\times\mathbb{R}^2\times\mathbb{R})} \lesssim \|v_0\|_{L^2_y\mathcal{H}^{\alpha}_x(\mathbb{R}^2\times\mathbb{R})} + \|v\|^3_{L^4_{t,y}\mathcal{H}^{\alpha}_x(I\times\mathbb{R}^2\times\mathbb{R})} \quad \text{for } \alpha = 0, 1, \tag{6-3}$$

where $I \subseteq \mathbb{R}$, and (p, q) is a Strichartz admissible pair.

As a consequence of Lemma 6.3 and (6-2), we obtain the following well-posedness theory. Since the proof is well known (see for instance [Cheng et al. 2020a; 2020b; Tao 2006; Killip and Vişan 2013]), we omit it.

Theorem 6.4 (well-posedness and scattering of (DCR)).

(1) (*local well-posedness*) *Assume $\|v_0\|_{L^2_y\mathcal{H}^1_x} < \infty$. The (DCR) admits a unique solution*

$$v \in (C^0_t L^2_y \mathcal{H}^1_x \cap L^4_{t,y}\mathcal{H}^1_x)((-T, T)\times\mathbb{R}^2\times\mathbb{R})$$

for some $T > 0$.

(2) (*small-data scattering*) *There is a sufficient small constant $\delta > 0$, such that when $\|v_0\|_{L^2_y\mathcal{H}^1_x} \le \delta$, (DCR) admits a unique global solution v with $v(0) = v_0$, which scatters in $L^2_y\mathcal{H}^1_x$ in the sense that there exist $v^{\pm} \in L^2_y\mathcal{H}^1_x(\mathbb{R}^2\times\mathbb{R})$ such that*

$$\|v(t) - e^{it\Delta_y}v^{\pm}\|_{L^2_y\mathcal{H}^1_x} \to 0 \quad \text{as } t \to \pm\infty.$$

(3) (*scattering norm*) *Suppose v is a maximal lifespan solution on I with $\|v\|_{L^4_{t,y}L^2_x(I\times\mathbb{R}^2\times\mathbb{R})} < \infty$. Then v globally exists and scatters in $L^2_y\mathcal{H}^1_x$.*

We also have the stability theorem by Lemmas 6.2 and 6.3. The argument is similar to the proof of Theorem 3.6, and we also refer to [Colliander et al. 2008; Killip and Vişan 2013].

Theorem 6.5 (stability). *Let $l \in \{0, 1\}$, I be a compact interval and $\tilde{v} \in (C^0_t L^2_y\mathcal{H}^1_x \cap L^4_{t,y}\mathcal{H}^1_x)(I\times\mathbb{R}^2\times\mathbb{R})$ be an approximate solution of (DCR) with the error term $e = i\partial_t\tilde{v} + \Delta_y\tilde{v} - F(\tilde{v})$. Then, for any $\epsilon > 0$, there is $\delta > 0$ such that if*

$$\|e\|_{L^{4/3}_{t,y}\mathcal{H}^l_x(I\times\mathbb{R}^2\times\mathbb{R})} + \|\tilde{v}(t_0) - v_0\|_{L^2_y\mathcal{H}^1_x} \le \delta,$$

then (DCR) admits a solution $v \in (L^\infty_t L^2_y\mathcal{H}^1_x \cap L^4_{t,y}\mathcal{H}^l_x)(I\times\mathbb{R}^2\times\mathbb{R})$ with $v(t_0) = v_0$ and

$$\|\tilde{v} - v\|_{L^4_{t,y}\mathcal{H}^l_x \cap L^\infty_t L^2_y\mathcal{H}^1_x(I\times\mathbb{R}^2\times\mathbb{R})} < \epsilon.$$

To prove (DCR) is globally well-posed and scatters for large data, by Theorem 6.4, we need to prove

$$\|v\|_{L^4_{t,y}L^2_x(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} < \infty,$$

where v is a solution to (DCR) with initial data $v_0 \in L_y^2\mathcal{H}_x^1(\mathbb{R}^2\times\mathbb{R})$. For the solution v of (DCR) with maximal lifespan interval I, let

$$S(m)=\sup\{\|v\|_{L^4_{t,y}L^2_x(I\times\mathbb{R}^2\times\mathbb{R})}:\|v(0)\|_{L^2_y\mathcal{H}^1_x(\mathbb{R}^2\times\mathbb{R})}\le m\},$$

and

$$m_0=\sup\{m:S(\tilde m)<\infty \text{ for all } \tilde m<m\}>0.$$

If we have $m_0=\infty$, then the global well-posedness and scattering in $L_y^2\mathcal{H}_x^1$ of (DCR) hold. Following the argument in [Tao et al. 2008; Killip and Vişan 2013], and using Theorems 6.5 and 4.1 during the proof, we have:

Theorem 6.6 (existence of an almost periodic solution to (DCR)). *Assume $m_0<\infty$. Then there exists an nonzero almost periodic solution $v\in C_t^0L_y^2\mathcal{H}_x^1\cap L^4_{t,y}L^2_x(I\times\mathbb{R}^2\times\mathbb{R})$ to (DCR) with I the maximal lifespan interval such that $\mathcal{M}(v)=m_0$. In addition, for any $\eta>0$, there exists $C(\eta)>0$ and $(y(t),\xi(t),N(t))\in\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}_+$ such that*

$$\int_{|y-y(t)|\ge\frac{C(\eta)}{N(t)}}\|v(t,y,x)\|^2_{\mathcal{H}^1_x}\,\mathrm{d}y+\int_{|\xi-\xi(t)|\ge C(\eta)N(t)}\|(\mathcal{F}_yv)(t,\xi,x)\|^2_{\mathcal{H}^1_x}\,\mathrm{d}\xi<\eta\quad\text{for all } t\in I. \tag{6-4}$$

Furthermore, we can take $[0,\infty)\subseteq I$, and $N(0)=1,\ \xi(0)=y(0)=0$, with

$$N(t)\le 1,\quad |N'(t)|+|\xi'(t)|\lesssim N(t)^3\quad\text{for all } t\in[0,\infty).$$

As in [Dodson 2012; 2016a; 2016b; Killip and Vişan 2013; Rosenzweig 2018], we see the almost periodic solution in Theorem 6.6 has the following property:

Theorem 6.7. (1) *If $J\subseteq I$ is an interval which is partitioned into small intervals J_k in the sense that $\|v\|_{L^4_{t,y}L^2_x(J_k\times\mathbb{R}^2\times\mathbb{R})}=1$, then we have*

$$N(J_k)\sim\int_{J_k}N(t)^3\,\mathrm{d}t\sim\inf_{t\in J_k}N(t)\quad\text{and}\quad\sum_{J_k\subseteq J}N(J_k)\sim\int_J N(t)^3\,\mathrm{d}t, \tag{6-5}$$

where $N(J_k)=\sup_{t\in J_k}N(t)$.

(2) *For any interval $J\subseteq[0,\infty)$, we have*

$$\int_J N(t)^2\,\mathrm{d}t\lesssim\|v\|^4_{L^4_{t,y}L^2_x(J\times\mathbb{R}^2\times\mathbb{R})}\lesssim 1+\int_J N(t)^2\,\mathrm{d}t. \tag{6-6}$$

Proof of Theorems 6.6 and 6.7. With Theorems 4.1 and 6.5 at hand, one can follow the arguments in [Cheng et al. 2020a; 2020b; Dodson 2012; 2016a; 2016b; Rosenzweig 2018; Tao et al. 2008; Killip and Vişan 2013]. □

6.1.2. *Some functional spaces and bilinear Strichartz estimates.* As in [Dodson 2016b], due to the failure of the endpoint Strichartz estimate in two dimensions, we need to utilize the function spaces U^p_Δ and V^p_Δ introduced originally in the seminal work [Koch and Tataru 2005a]; see also [Hadac et al. 2009; Koch and Tataru 2007; Koch et al. 2014] for more detailed study on these spaces. The structure of our (DCR) system motivates us to introduce the Banach spaces $U^p_\Delta(L^2_x)$ and $V^p_\Delta(L^2_x)$ as follows.

Definition 6.8 ($U^p_\Delta(L^2_x)$ space). For $1 \le p < \infty$, let $U^p_\Delta(L^2_x)$ be an atomic space, where an atom v^γ is defined to be

$$v^\gamma(t,y,x)=\sum_{k=0}^{N}\chi_{[t_k,t_{k+1})}(t)e^{it\Delta_y}v_k^\gamma(y,x),\quad \text{with}\ \sum_{k=0}^{N}\|v_k^\gamma(y,x)\|^p_{L^2_{y,x}}=1.$$

In the expansion of v^γ, N may be finite or infinite, $t_0=-\infty$, and $t_{N+1}=\infty$ if N is finite. We impose a norm on $\|\cdot\|_{U^p_\Delta(L^2_x)}$ as

$$\|v\|_{U^p_\Delta(L^2_x)}=\inf\Big\{\sum_\gamma |c_\gamma| : v=\sum_\gamma c_\gamma v^\gamma,\ \text{where}\ v^\gamma\ \text{are}\ U^p_\Delta(L^2_x)\ \text{atoms}\Big\}.$$

For a time interval $I \subseteq \mathbb{R}$, we define

$$\|v\|_{U^p_\Delta(L^2_x,I)}=\|v1_I\|_{U^p_\Delta(L^2_x)}.$$

Let $DU^p_\Delta(L^2_x)$ be the space

$$DU^p_\Delta(L^2_x)=\{(i\partial_t+\Delta_y)v : v\in U^p_\Delta(L^2_x)\},$$

endowed with the norm

$$\|(i\partial_t+\Delta_y)v(t,y,x)\|_{DU^p_\Delta(L^2_x)}=\Big\|\int_0^t e^{i(t-s)\Delta_y}(i\partial_s+\Delta_y)v(s,y,x)\,ds\Big\|_{U^p_\Delta(L^2_x)}.$$

For each time interval $I\subseteq\mathbb{R}$, we can similarly define the restriction space $DU^p_\Delta(L^2_x,I)$.

Definition 6.9 ($V^p_\Delta(L^2_x)$ space). For $1\le p<\infty$, $V^p_\Delta(L^2_x)$ is defined to be the space of right continuous functions $v\in L^\infty_t L^2_{y,x}$ such that

$$\|v\|^p_{V^p_\Delta(L^2_x)}=\|v\|^p_{L^\infty_t L^2_{y,x}}+\sup_{\{t_k\}_k\nearrow}\sum_k\|e^{-it_{k+1}\Delta_y}v(t_{k+1})-e^{-it_k\Delta_y}v(t_k)\|^p_{L^2_{y,x}}<\infty.$$

When the time is restricted to $I\subseteq\mathbb{R}$, We can similarly define the function space $V^p_\Delta(L^2_x,I)$. Then we have

$$(DU^p_\Delta(L^2_x))^*=V^{p'}_\Delta(L^2_x). \tag{6-7}$$

The following basic properties are straightforward to verify. For the proofs, see [Hadac et al. 2009; Koch et al. 2014].

Remark 6.10 (basic properties of $U^p_\Delta(L^2_x)$ and $V^p_\Delta(L^2_x)$). For any $1<p<q<\infty$ and $a\le b\le c$, we have

$$U^p_\Delta(L^2_x)\subseteq V^p_\Delta(L^2_x)\subseteq U^q_\Delta(L^2_x), \tag{6-8}$$

$$\|v\|_{U^p_\Delta(L^2_x,[a,b])}\le\|v\|_{U^p_\Delta(L^2_x,[a,c])}\quad\text{and}\quad \|v\|^p_{U^p_\Delta(L^2_x,[a,c])}\le\|v\|^p_{U^p_\Delta(L^2_x,[a,b])}+\|v\|^p_{U^p_\Delta(L^2_x,[b,c])}, \tag{6-9}$$

$$\|v\|_{U^p_\Delta(L^2_x)}\lesssim\|v|_{t=0}\|_{L^2_{y,x}}+\|(i\partial_t+\Delta_y)v\|_{DU^p_\Delta(L^2_x)}. \tag{6-10}$$

Moreover,

$$L^{p'}_t L^{r'}_y L^2_x\subseteq DU^2_\Delta(L^2_x),\quad\text{and}\quad U^p_\Delta(L^2_x)\subseteq L^p_t L^r_y L^2_x,\quad \text{where } (p,r) \text{ is Strichartz admissible.} \tag{6-11}$$

Following the argument in [Dodson 2016b], we also have:

Lemma 6.11. *Suppose $I=\bigcup_{j=1}^m I^j$, where $I^j=[a_j,b_j]$, $a_{j+1}=b_j$. If $f\in L_t^1L_{y,x}^2(I\times\mathbb{R}^2\times\mathbb{R})$, then, for all $t_0\in I$,*

$$\Big\|\int_{t_0}^t e^{i(t-\tau)\Delta_y}f(\tau,y,x)\,\mathrm{d}\tau\Big\|_{U_\Delta^2(L_x^2,I)}\lesssim\sum_{j=1}^m\Big\|\int_{I^j}e^{-i\tau\Delta_y}f(\tau)\,\mathrm{d}\tau\Big\|_{L_{y,x}^2}+\Big(\sum_{j=1}^m\|f\|_{DU_\Delta^2(L_x^2,I^j)}^2\Big)^{\frac12},$$

where

$$\|f\|_{DU_\Delta^2(L_x^2,I^j)}=\sup_{\|w\|_{V_\Delta^2(L_x^2,I^j)}=1}\int_{I^j}\int_{\mathbb{R}^2\times\mathbb{R}}f(\tau,y,x)\overline{w(\tau,y,x)}\,\mathrm{d}\tau\,\mathrm{d}y\,\mathrm{d}x.$$

By the bilinear Strichartz estimate in [Bourgain 1998], Minkowski's inequality, Hölder's inequality, and interpolation, we have the following two propositions. The proofs are similar to the bilinear Strichartz estimates in [Dodson 2016a; 2016b].

Proposition 6.12 (bilinear Strichartz estimate, I). *Let (p,q) satisfy $1<p,q<\infty$, $\frac1p+\frac1q=1$. For $M\ll N$, assume* $\operatorname{supp}\mathcal{F}_yu_0\subseteq\{\xi:|\xi|\sim N\}$ *and* $\operatorname{supp}\mathcal{F}_yv_0\subseteq\{\xi:|\xi|\sim M\}$. *Then we have*

$$\big\|\|e^{it\Delta_y}u_0\|_{L_x^2}\|e^{it\Delta_y}v_0\|_{L_x^2}\big\|_{L_t^pL_y^q(\mathbb{R}\times\mathbb{R}^2)}\lesssim\Big(\frac MN\Big)^{\frac1p}\|u_0\|_{L_{y,x}^2}\|v_0\|_{L_{y,x}^2}.\tag{6-12}$$

Furthermore, suppose that $g(t,y-\tilde y)$ and $h(t,y-\tilde{\tilde y})$ are convolution kernels with respect to the y-variable and

$$\big\|\sup_{t\in\mathbb{R}}|g(t,y)|\big\|_{L_y^1(\mathbb{R}^2)}+\big\|\sup_{t\in\mathbb{R}}|h(t,y)|\big\|_{L_y^1(\mathbb{R}^2)}\lesssim1.$$

Then we also have

$$\big\|\|g*_ye^{it\Delta_y}u_0\|_{L_x^2}\|h*_ye^{it\Delta_y}v_0\|_{L_x^2}\big\|_{L_t^pL_y^q(\mathbb{R}\times\mathbb{R}^2)}\lesssim\Big(\frac MN\Big)^{\frac1p}\|u_0\|_{L_{y,x}^2}\|v_0\|_{L_{y,x}^2}.$$

Similar to the argument in the proof of Lemma 3.5 in [Dodson 2016b], we can transfer the estimate (6-12) to the U_Δ^p space. Therefore, we have:

Proposition 6.13 (bilinear Strichartz estimate, II). *Let (p,q) satisfy $1<p,q<\infty$, $\frac1p+\frac1q=1$. For $M\ll N$, assume* $\operatorname{supp}\mathcal{F}_yu\subseteq\{\xi:|\xi|\sim N\}$ *and* $\operatorname{supp}\mathcal{F}_yv\subseteq\{\xi:|\xi|\sim M\}$. *Then we have*

$$\big\|\|u\|_{L_x^2}\|v\|_{L_x^2}\big\|_{L_t^pL_y^q}\lesssim\Big(\frac MN\Big)^{\frac1p}\|u\|_{U_\Delta^p(L_x^2)}\|v\|_{U_\Delta^p(L_x^2)}.$$

6.2. ***Long time Strichartz estimate.*** From now on, we shall take our following setting as standard assumptions. Fix

$$0<\epsilon_3\ll\epsilon_2\ll\epsilon_1<1\quad\text{and}\quad\epsilon_3<\epsilon_2^{10}.\tag{6-13}$$

By Theorem 6.6, we can take

$$|N'(t)|+|\xi'(t)|\le2^{-20}\epsilon_1^{-\frac12}N(t)^3\tag{6-14}$$

and

$$\int_{|y-y(t)|\geq 2^{-20}\epsilon_3^{-1/4}/N(t)} \|v(t,y,x)\|_{\mathcal{H}_x^1}^2\,\mathrm{d}y+\int_{|\xi-\xi(t)|\geq 2^{-20}\epsilon_3^{-1/4}N(t)} \|(\mathcal{F}_y v)(t,\xi,x)\|_{\mathcal{H}_x^1}^2\,\mathrm{d}\xi \leq \epsilon_2^2. \quad (6\text{-}15)$$

If $[0,T]$ is an interval with

$$\|v\|_{L_{t,y}^4 L_x^2([0,T]\times\mathbb{R}^2\times\mathbb{R})}^4 = 2^{k_0} \quad \text{and} \quad \int_0^T N(t)^3\,\mathrm{d}t = \epsilon_3 2^{k_0} \quad \text{for some } k_0 \geq 0, \quad (6\text{-}16)$$

then we can partition $[0,T]=\bigcup_{\alpha=0}^{M-1} J^\alpha$, where J^α are intervals that satisfy

$$\int_{J^\alpha} (N(t)^3+\epsilon_3\|v(t)\|_{L_y^4 L_x^2(\mathbb{R}^2\times\mathbb{R})}^4)\,\mathrm{d}t = 2\epsilon_3. \quad (6\text{-}17)$$

We can define the interval G_k^j now.

Definition 6.14. For any nonnegative integer $j<k_0$, and nonnegative integer $k<2^{k_0-j}$, we can define

$$G_k^j = \bigcup_{\alpha=k2^j}^{(k+1)2^j-1} J^\alpha. \quad (6\text{-}18)$$

For $j\geq k_0$, we simply define $G_k^j=[0,T]$. We let $\xi(G_k^j)=\xi(t_k^j)$, where t_k^j is the left endpoint of G_k^j.

On the time interval G_k^j defined above, we have:

Lemma 6.15. (1) *Let J_l be the small intervals contained in G_k^j. By (6-5) and (6-17), the following estimate holds*:

$$\sum_{J_l\subseteq G_k^j} N(J_l) \lesssim \int_{G_k^j} N(t)^3\,\mathrm{d}t \lesssim \sum_{\alpha=k2^j}^{(k+1)2^j-1} \int_{J^\alpha} N(t)^3\,\mathrm{d}t \lesssim 2^j\epsilon_3. \quad (6\text{-}19)$$

(2) *By (6-14) and Definition 6.14, we have, for each $t\in G_k^j$,*

$$|\xi(t)-\xi(G_k^j)| \leq 2^{j-19}\epsilon_3\epsilon_1^{-\frac{1}{2}}. \quad (6\text{-}20)$$

Thus, for any $t\in G_k^j$, and $i\geq j$,

$$\{\xi:2^{i-1}\leq|\xi-\xi(t)|\leq 2^{i+1}\}\subseteq\{\xi:2^{i-2}\leq|\xi-\xi(G_k^j)|\leq 2^{i+2}\}\subseteq\{\xi:2^{i-3}\leq|\xi-\xi(t)|\leq 2^{i+3}\}, \quad (6\text{-}21)$$

and also

$$\{\xi:|\xi-\xi(t)|\leq 2^{i+1}\}\subseteq\{\xi:|\xi-\xi(G_k^j)|\leq 2^{i+2}\}\subseteq\{\xi:|\xi-\xi(t)|\leq 2^{i+3}\}. \quad (6\text{-}22)$$

Lemma 6.16. *For the almost periodic solution $v(t)$ to (DCR), and assume $\|v\|_{L_{t,y}^4 L_x^2(J\times\mathbb{R}^2\times\mathbb{R})}\leq 1$ on $J\subseteq\mathbb{R}$, then we have*

$$\|v\|_{U_\Delta^2(L_x^2,J)} \lesssim 1 \quad \textit{and} \quad \|P^y_{>2^{-4}\epsilon_3^{-1/4}N(J)} v\|_{U_\Delta^2(L_x^2,J)} \lesssim \epsilon_2,$$

where $N(J)=\sup_{t\in J} N(t)$.

Proof. Let $J = [t_0, t_1]$, by (6-10), (6-8), (6-7), and (6-11), we have

$$\|v\|_{U^2_\Delta(L^2_x,J)} \lesssim \|v(t_0)\|_{L^2_{y,x}} + \|v\|^3_{L^4_{t,y}L^2_x(J\times\mathbb{R}^2\times\mathbb{R})} \lesssim 1.$$

By (6-15), we have

$$\|P^y_{>2^{-20}\epsilon_3^{-1/4}N(J)}v\|_{L^\infty_t L^2_{y,x}(J\times\mathbb{R}^2\times\mathbb{R})} \le \|P^y_{>2^{-20}\epsilon_3^{-1/4}N(t)}v\|_{L^\infty_t L^2_{y,x}(J\times\mathbb{R}^2\times\mathbb{R})} \le \epsilon_2.$$

Therefore, by the Strichartz estimate, we have

$$\begin{aligned}
&\|P^y_{>2^{-4}\epsilon_3^{-1/4}N(J)}v\|_{U^2_\Delta(L^2_x,J)} \\
&\quad\lesssim \|P^y_{>2^{-20}\epsilon_3^{-1/4}N(J)}v(t_0)\|_{L^2_{y,x}(J\times\mathbb{R}^2\times\mathbb{R})} + \|P^y_{>2^{-4}\epsilon_3^{-1/4}N(J)}F(v)\|_{L^{3/2}_t L^{6/5}_y L^2_x(J\times\mathbb{R}^2\times\mathbb{R})} \\
&\quad\lesssim \epsilon_2 + \|P^y_{>2^{-20}\epsilon_3^{-1/4}N(J)}v\|_{L^\infty_t L^2_{y,x}}\|v\|^2_{L^3_t L^6_y L^2_x(J\times\mathbb{R}^2\times\mathbb{R})} \\
&\quad\lesssim \epsilon_2 + \epsilon_2\big(\|v(t_0)\|_{L^\infty_t L^2_{y,x}(J\times\mathbb{R}^2\times\mathbb{R})} + \|v\|^3_{L^4_{t,y}L^2_x(J\times\mathbb{R}^2\times\mathbb{R})}\big)^2 \lesssim \epsilon_2.
\end{aligned}$$

□

We also have the following fact as a consequence of the above lemma.

Remark 6.17. If $N(J) < 2^{i-5}\epsilon_3^{1/2}$, we have

$$\begin{aligned}
&\|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}F(v)\|_{L^{3/2}_t L^{6/5}_y L^2_x(J\times\mathbb{R}^2\times\mathbb{R})} \\
&\qquad\lesssim \|P^y_{>2^{-20}\epsilon_3^{-1/4}N(J)}v\|_{L^\infty_t L^2_{y,x}(J\times\mathbb{R}^2\times\mathbb{R})}\|v\|^2_{L^3_t L^6_y L^2_x(J\times\mathbb{R}^2\times\mathbb{R})} \lesssim \epsilon_2,
\end{aligned}$$

where the operator $P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}$ is given in Definition 6.18 below. Thus, for $0 \le i \le 11$, and $N(G^i_\alpha) < 2^{i-5}\epsilon_3^{1/2}$, by the fact that G^i_α is a union of at most 2^{11} small intervals, we have

$$\|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}F(v)\|_{L^{3/2}_t L^{6/5}_y L^2_x(G^i_\alpha\times\mathbb{R}^2\times\mathbb{R})} \lesssim \epsilon_2.$$

We can now define the long-time Strichartz estimate norm as in [Dodson 2012; 2016a; 2016b]; see also [Cheng et al. 2020a; 2020b].

Definition 6.18 (long-time Strichartz estimate norm). For any $G^j_k \subseteq [0,T]$, let

$$\|v\|^2_{X(G^j_k)} = \sum_{0\le i<j} 2^{i-j} \sum_{G^i_\alpha\subseteq G^j_k} \|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}v\|^2_{U^2_\Delta(L^2_x,G^i_\alpha)} + \sum_{i\ge j}\|P^y_{\xi(G^j_k),2^{i-2}\le\cdots\le 2^{i+2}}v\|^2_{U^2_\Delta(L^2_x,G^j_k)},$$

where

$$P^y_{\xi(t),2^{i-2}\le\cdots\le 2^{i+2}}v = e^{iy\cdot\xi(t)}P^y_{2^{i-2}\le\cdots\le 2^{i+2}}(e^{-iy\cdot\xi(t)}v).$$

We define the $\widetilde{X}_{k_0}$ norm to be

$$\|v\|^2_{\widetilde{X}_{k_0}([0,T])} = \sup_{0\le j\le k_0}\ \sup_{G^j_k\subseteq[0,T]} \|v\|^2_{X(G^j_k)}.$$

For any nonnegative integer $k_* \le k_0$, we take

$$\|v\|^2_{\widetilde{X}_{k_*}([0,T])} = \sup_{0\le j\le k_*}\ \sup_{G^j_k\subseteq[0,T]} \|v\|^2_{X(G^j_k)}. \tag{6-23}$$

To close our bootstrap argument in the proof of the long-time Strichartz estimate, we also need to introduce the following norm to measure $\widetilde{X}_{k_0}$ norm of v at scales much higher than $N(t)$.

Definition 6.19. Let

$$\|v\|^2_{Y(G_k^j)} = \sum_{0\le i<j} 2^{i-j} \sum_{\substack{G_\alpha^i\subseteq G_k^j\\ N(G_\alpha^i)\le 2^{i-5}\epsilon_3^{1/2}}} \|P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,G_\alpha^i)} + \sum_{\substack{i\ge j,\, i>0\\ N(G_k^j)\le 2^{i-5}\epsilon_3^{1/2}}} \|P^y_{\xi(G_k^j),2^{i-2}\le\cdot\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,G_k^j)}.$$

We can define the norm $\|v\|_{\widetilde{Y}_{k_*}([0,T])}$ similar to (6-23) in Definition 6.18.

For $i<j$, and the solution v on the time interval G_k^j, we can define the Littlewood–Paley projector around $\xi(t)$ of v as

$$P^y_{\xi(t),2^i} v = e^{iy\cdot\xi(t)} P^y_{2^i}(e^{-iy\cdot\xi(t)}v), \quad P^y_{\xi(t),>2^j} v = e^{iy\cdot\xi(t)} P^y_{>2^j}(e^{-iy\cdot\xi(t)}v).$$

Then, as a consequence of (6-7), (6-8), (6-11), the Littlewood–Paley theorem and Proposition 3.2, we have the following estimates which reveal the relationship between the Strichartz norm $L^p_t L^q_y L^2_x$ of the Littlewood–Paley projector around $\xi(t)$ of v and the long-time Strichartz norm of v. We still refer to [Dodson 2016a; 2016b] for the argument, without presenting the proof here.

Lemma 6.20. *For $i<j$, we have*

$$\|P^y_{\xi(t),2^i} v\|_{L^p_t L^q_y L^2_x(G_k^j\times\mathbb{R}^2\times\mathbb{R})} \lesssim 2^{\frac{j-i}{p}} \|v\|_{\widetilde{X}_j(G_k^j)}, \tag{6-24}$$

$$\|P^y_{\xi(t),\ge 2^j} v\|_{L^p_t L^q_y L^2_x(G_k^j\times\mathbb{R}^2\times\mathbb{R})} \lesssim \|v\|_{X(G_k^j)}, \tag{6-25}$$

where (p,q) is Strichartz admissible pair.

Our aim is to prove the long-time Strichartz estimate.

Theorem 6.21 (long-time Strichartz estimate). *For the almost periodic solution v in Theorem 6.6, which satisfies (6-13), (6-14) and (6-15), there exists a positive constant $C=C(v)$ such that, for any nonnegative integer k_0, v and $N(t)$ satisfy (6-16), we have*

$$\|v\|_{\widetilde{X}_{k_0}([0,T])} \le C.$$

To prove Theorem 6.21, it suffices to show, for any $0\le j\le k_0$ and $G_k^j\subseteq[0,T]$,

$$\sum_{0\le i<j} 2^{i-j} \sum_{G_\alpha^i\subseteq G_k^j} \|P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,G_\alpha^i)} + \sum_{i\ge j} \|P^y_{\xi(G_k^j),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,G_k^j)} \le C.$$

To reach the above estimate, we will perform an induction argument on $0\le k_*\le k_0$ and then a bootstrap argument in Sections 6.2.1 and 6.2.2, respectively.

6.2.1. *Basic inductive estimates.* First we show the basic estimates to start up our induction.

Lemma 6.22 (basic inductive estimate).

$$\|v\|_{\widetilde{X}_0([0,T])} \le C \quad \text{and} \quad \|v\|_{\widetilde{Y}_0([0,T])} \le C\epsilon_2^{\frac{3}{4}}. \tag{6-26}$$

For $0 \le k_* \le k_0$, *we have*

$$\|v\|^2_{\widetilde{X}_{k_*+1}([0,T])} \le 2\|v\|^2_{\widetilde{X}_{k_*}([0,T])} \quad \text{and} \quad \|v\|^2_{\widetilde{Y}_{k_*+1}([0,T])} \le 2\|v\|^2_{\widetilde{Y}_{k_*}([0,T])}. \tag{6-27}$$

Proof. By Lemma 6.16, we have

$$\|v\|_{U^2_\Delta(L^2_x, J^\alpha)} \lesssim 1 \quad \text{for any } J^\alpha \text{ in the decomposition of } G^j_k \text{ in (6-18)}. \tag{6-28}$$

Therefore, by Strichartz estimate, (6-10), (6-11), (6-8), we have, for $t_\alpha \in J^\alpha$,

$$\begin{aligned}\Big(\sum_{i\ge 0} \|P^y_{\xi(J^\alpha),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,J^\alpha)}\Big)^{\frac12} &\lesssim \|v(t_\alpha)\|_{L^2_{y,x}} + \|v\|^3_{L^3_t L^6_y L^2_x(J^\alpha\times\mathbb{R}^2\times\mathbb{R})}\\ &\lesssim \|v(t_\alpha)\|_{L^2_{y,x}} + \|v\|^3_{U^2_\Delta(L^2_x,J^\alpha)} \lesssim 1.\end{aligned}$$

Thus, $\|v\|_{\widetilde{X}_0([0,T])} \le C$.

At the same time, by (6-15), the conservation of mass, and (6-28), we infer that

$$\begin{aligned}&\Big(\sum_{\substack{i\ge 0\\ N(J^\alpha)\le \epsilon_3^{1/2}2^{i-5}}} \|P^y_{\xi(J^\alpha),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,J^\alpha)}\Big)^{\frac12}\\ &\qquad\lesssim \|P^y_{\xi(J^\alpha),\ge 8\epsilon_3^{-1/2}N(J^\alpha)} v(t_\alpha)\|_{L^2_{y,x}} + \|P^y_{\xi(J^\alpha),\ge 8\epsilon_3^{-1/2}N(J^\alpha)} F(v)\|_{L^1_t L^2_{y,x}(J^\alpha\times\mathbb{R}^2\times\mathbb{R})}\\ &\qquad\lesssim \|P^y_{\xi(t),\ge 4\epsilon_3^{-1/2}N(t)} v\|_{L^\infty_t L^2_{y,x}(J^\alpha\times\mathbb{R}^2\times\mathbb{R})} + \|P^y_{\xi(t),\ge 4\epsilon_3^{-1/2}N(t)} F(v)\|_{L^1_t L^2_{y,x}(J^\alpha\times\mathbb{R}^2\times\mathbb{R})}\\ &\qquad\lesssim \|P^y_{\xi(t),\ge \epsilon_3^{-1/2}N(t)} v\|^{\frac34}_{L^\infty_t L^2_{y,x}(J^\alpha\times\mathbb{R}^2\times\mathbb{R})}\big(\|v\|^{\frac14}_{L^\infty_t L^2_{y,x}(J^\alpha\times\mathbb{R}^2\times\mathbb{R})} + \|v\|^{\frac94}_{U^2_\Delta(L^2_x,J^\alpha)}\big) \lesssim \epsilon_2^{\frac34}.\end{aligned}$$

Thus, by Definition 6.19, we have $\|v\|_{\widetilde{Y}_0([0,T])} \le C\epsilon_2^{3/4}$.

By Definition 6.14, we see $G^{j+1}_k = G^j_{2k} \cup G^j_{2k+1}$, with $G^j_{2k} \cap G^j_{2k+1} = \varnothing$. Then for $0 \le i \le j$, if $G^i_\alpha \subseteq G^{j+1}_k$, we have $G^i_\alpha \subseteq G^j_{2k}$ or $G^i_\alpha \subseteq G^j_{2k+1}$. Thus

$$\begin{aligned}&\sum_{0\le i<j+1} 2^{i-(j+1)} \sum_{G^i_\alpha\subseteq G^{j+1}_k} \|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,G^i_\alpha)}\\ &\le 2^{-1}\sum_{0\le i<j} 2^{i-j}\Big(\sum_{G^i_\alpha\subseteq G^j_{2k}} \|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,G^i_\alpha)}\\ &\qquad\qquad + \sum_{G^i_\alpha\subseteq G^j_{2k+1}} \|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,G^i_\alpha)}\Big)\\ &\qquad + 2^{-1}\big(\|P^y_{\xi(G^j_{2k}),2^{j-2}\le\cdots\le 2^{j+2}} v\|^2_{U^2_\Delta(L^2_x,G^j_{2k})} + \|P^y_{\xi(G^j_{2k+1}),2^{j-2}\le\cdots\le 2^{j+2}} v\|^2_{U^2_\Delta(L^2_x,G^j_{2k+1})}\big)\\ &\le \tfrac12\big(\|v\|^2_{X(G^j_{2k})} + \|v\|^2_{X(G^j_{2k+1})}\big).\end{aligned} \tag{6-29}$$

At the same time, by (6-21) and (6-9), we see

$$\begin{aligned}\sum_{i\ge j+1} &\|P^y_{\xi(G_k^{j+1}),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,G_k^{j+1})}\\ &\le \sum_{i\ge j+1}\big(\|P^y_{\xi(G_k^{j+1}),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,G^j_{2k})}+\|P^y_{\xi(G_k^{j+1}),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,G^j_{2k+1})}\big)\\ &\le \sum_{i\ge j+1}\big(\|P^y_{\xi(G^j_{2k}),2^{i-3}\le\cdots\le 2^{i+3}} v\|^2_{U^2_\Delta(L^2_x,G^j_{2k})}+\|P^y_{\xi(G^j_{2k+1}),2^{i-3}\le\cdots\le 2^{i+3}} v\|^2_{U^2_\Delta(L^2_x,G^j_{2k+1})}\big). \quad (6\text{-}30)\end{aligned}$$

Therefore, by (6-29) and (6-30), and Definition 6.18, we get

$$\|v\|^2_{\widetilde{X}_{k_*+1}([0,T])}\le 2\|v\|^2_{\widetilde{X}_{k_*}([0,T])}.$$

By a similar argument, we can deduce

$$\|v\|^2_{\widetilde{Y}_{k_*+1}([0,T])}\le 2\|v\|^2_{\widetilde{Y}_{k_*([0,T])}}.$$

□

6.2.2. *The bootstrap estimate.* In the following, we will establish the bootstrap estimate, which is necessary for the proof of Theorem 6.21. For $0\le j\le k_0$ and $G^j_k\subseteq[0,T]$. By Duhamel's formula, we have, for $0\le i<j$,

$$\begin{aligned}\|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}} v\|_{U^2_\Delta(L^2_x,G^i_\alpha)} &\le \|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}} v(t^i_\alpha)\|_{L^2_{y,x}}\\ &\quad+\Big\|\int_{t^i_\alpha}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}} F(v(\tau))\,d\tau\Big\|_{U^2_\Delta(L^2_x,G^i_\alpha)}. \quad (6\text{-}31)\end{aligned}$$

Here we take t^i_α to satisfy

$$\|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}} v(t^i_\alpha)\|_{L^2_{y,x}}=\inf_{t\in G^i_\alpha}\|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}} v(t)\|_{L^2_{y,x}}.$$

We now consider the first term on the right-hand side of (6-31). By (6-17) and Lemma 6.15, we have

$$\begin{aligned}\sum_{0\le i<j} 2^{i-j}&\sum_{G^i_\alpha\subseteq G^j_k}\|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}} v(t^i_\alpha)\|^2_{L^2_{y,x}}\\ &\lesssim 2^{-j}\epsilon_3^{-1}\int_{G^j_k}(N(t)^3+\epsilon_3\|v(t)\|^4_{L^4_yL^2_x(\mathbb{R}^2\times\mathbb{R})})\sum_{0\le i<j}\|P^y_{\xi(t),2^{i-3}\le\cdots\le 2^{i+3}} v(t)\|^2_{L^2_{y,x}}\,dt\lesssim 1.\end{aligned}$$

For $i\ge j$, we can just take t^j_k to be the left endpoint of G^j_k. Then we have

$$\sum_{i\ge j}\|P^y_{\xi(G^j_k),2^{i-2}\le\cdots\le 2^{i+2}} v(t^j_k)\|^2_{L^2_{y,x}}\lesssim\|v(t^j_k)\|^2_{L^2_{y,x}}\lesssim 1.$$

Thus

$$\sum_{0\le i<j} 2^{i-j}\sum_{G^i_\alpha\subseteq G^j_k}\|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}} v(t^i_\alpha)\|^2_{L^2_{y,x}}+\sum_{i\ge j}\|P^y_{\xi(G^j_k),2^{i-2}\le\cdots\le 2^{i+2}} v(t^j_k)\|^2_{L^2_{y,x}}\lesssim 1. \quad (6\text{-}32)$$

We next consider the second term on the right-hand side of (6-31). Observe that there are at most two small intervals, called for instance J_1 and J_2, which intersect G^j_k but are not contained in G^j_k. Then by

Lemma 6.16 and (6-11), we have

$$\begin{aligned}\sum_{0\le i<j} 2^{i-j} \sum_{G_\alpha^i\subseteq G_k^j} \|F(v)\|^2_{L_t^1 L^2_{y,x}((G_\alpha^i\cap(J_1\cup J_2))\times\mathbb{R}^2\times\mathbb{R})} &\lesssim \sum_{0\le i<j} 2^{i-j}\|F(v)\|^2_{L_t^1L^2_{y,x}((J_1\cup J_2)\times\mathbb{R}^2\times\mathbb{R})}\\ &\lesssim \|v\|^6_{L_t^3L_y^6L_x^2(J_1\times\mathbb{R}^2\times\mathbb{R})} + \|v\|^6_{L_t^3L_y^6L_x^2(J_2\times\mathbb{R}^2\times\mathbb{R})} \lesssim 1. \end{aligned}\tag{6-33}$$

Then by (6-9), (6-11), (6-33), (6-14), (6-19) and Definition 6.14, we obtain

$$\begin{aligned}&\sum_{0\le i<j} 2^{i-j} \sum_{\substack{G_\alpha^i\subseteq G_k^j\\ N(G_\alpha^i)\ge 2^{i-5}\epsilon_3^{1/2}}} \left\|\int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}} F(v(\tau))\,d\tau\right\|^2_{U^2_\Delta(L_x^2,G_\alpha^i)}\\
&\lesssim \sum_{0\le i<j} 2^{i-j} \sum_{\substack{G_\alpha^i\subseteq G_k^j\\ N(G_\alpha^i)\ge 2^{i-5}\epsilon_3^{1/2}}} \sum_{J_l\cap G_k^j\neq\varnothing} \|P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}}F(v)\|^2_{DU^2_\Delta(L_x^2,G_\alpha^i\cap J_l)}\\
&\lesssim \sum_{0\le i<j} 2^{i-j} \sum_{\substack{G_\alpha^i\subseteq G_k^j\\ N(G_\alpha^i)\ge 2^{i-5}\epsilon_3^{1/2}}} \sum_{J_l\cap G_k^j\neq\varnothing} \|F(v)\|^2_{L_t^1L^2_{y,x}((G_\alpha^i\cap J_l)\times\mathbb{R}^2\times\mathbb{R})}\\
&\lesssim 1+\sum_{0\le i<j} 2^{i-j}\Bigg(\sum_{\substack{J_l\subseteq G_k^j\\ N(J_l)\ge 2^{i-6}\epsilon_3^{1/2}}} \|F(v)\|^2_{L_t^1L^2_{y,x}(J_l\times\mathbb{R}^2\times\mathbb{R})}\Bigg) \lesssim 1+\sum_{J_l\subseteq G_k^j}\sum_{\substack{0\le i<j,\\ 2^i\le 2^6\epsilon_3^{-1/2}N(J_l)}} 2^{i-j}\lesssim 1. \end{aligned}\tag{6-34}$$

On the interval G_k^j with $N(G_k^j)\ge 2^{i-5}\epsilon_3^{1/2}$, by (6-14) and (6-17), we have

$$\int_{G_k^j} N(t)^2\,dt \lesssim 1. \tag{6-35}$$

Thus, by Minkowski's inequality, (6-3), (6-19), (6-6), and (6-35), we have

$$\begin{aligned}\sum_{\substack{i\ge j\\ N(G_k^j)\ge 2^{i-5}\epsilon_3^{1/2}}} \|P^y_{\xi(G_k^j),2^{i-2}\le\cdots\le 2^{i+2}}F(v)\|^2_{L_t^1L^2_{y,x}(G_k^j\times\mathbb{R}^2\times\mathbb{R})} &\lesssim \|F(v)\|^2_{L_t^1L^2_{y,x}(G_k^j\times\mathbb{R}^2\times\mathbb{R})}\\ &\lesssim \|v\|^6_{L_t^3L_y^6L_x^2(G_k^j\times\mathbb{R}^2\times\mathbb{R})}\lesssim 1.\end{aligned}\tag{6-36}$$

Thus, by (6-31), (6-32), (6-34), and (6-36), we infer

$$\begin{aligned}\|v\|^2_{X(G_k^j)} \lesssim 1 &+ \sum_{\substack{i\ge j\\ N(G_k^j)\le 2^{i-5}\epsilon_3^{1/2}}} \left\|\int_{t_k^j}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_k^j),2^{i-2}\le\cdots\le 2^{i+2}}F(v(\tau))\,d\tau\right\|^2_{U^2_\Delta(L_x^2,G_k^j)}\\ &+\sum_{0\le i<j}2^{i-j}\sum_{\substack{G_\alpha^i\subseteq G_k^j\\ N(G_\alpha^i)\le 2^{i-5}\epsilon_3^{1/2}}} \left\|\int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}}F(v(\tau))\,d\tau\right\|^2_{U^2_\Delta(L_x^2,G_\alpha^i)}. \end{aligned}\tag{6-37}$$

We can further get

$$\|v\|^2_{X(G_k^j)} \lesssim 1 + \sum_{\substack{i \geq j, \\ N(G_k^j) \leq 2^{i-10}\epsilon_3^{1/2}}} \left\| \int_{t_k^j}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_k^j), 2^{i-2} \leq \cdots \leq 2^{i+2}} F(v(\tau))\, d\tau \right\|^2_{U^2_\Delta(L^2_x, G_k^j)} + \sum_{0 \leq i < j} 2^{i-j} \sum_{\substack{G_\alpha^i \subseteq G_k^j, \\ N(G_\alpha^i) \leq 2^{i-10}\epsilon_3^{1/2}}} \left\| \int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i), 2^{i-2} \leq \cdots \leq 2^{i+2}} F(v(\tau))\, d\tau \right\|^2_{U^2_\Delta(L^2_x, G_\alpha^i)}, \tag{6-38}$$

because the contribution of those terms for i satisfying $2^{i-10}\epsilon_3^{1/2} \leq N(t) \leq 2^{i-5}\epsilon_3^{1/2}$ in the right-hand side of (6-37) is small by similar argument as in the proof of (6-37).

By a similar argument as above for (6-38), we also refer to [Dodson 2016b] for more explanation. Then, we have

$$\|v\|^2_{Y(G_k^j)} \lesssim \epsilon_2^{\frac{3}{2}} + \sum_{\substack{i \geq j \\ N(G_k^j) \leq 2^{i-10}\epsilon_3^{1/2}}} \left\| \int_{t_k^j}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_k^j), 2^{i-2} \leq \cdots \leq 2^{i+2}} F(v(\tau))\, d\tau \right\|^2_{U^2_\Delta(L^2_x, G_k^j)} + \sum_{0 \leq i < j} 2^{i-j} \sum_{\substack{G_\alpha^i \subseteq G_k^j \\ N(G_\alpha^i) \leq 2^{i-10}\epsilon_3^{1/2}}} \left\| \int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i), 2^{i-2} \leq \cdots \leq 2^{i+2}} F(v(\tau))\, d\tau \right\|^2_{U^2_\Delta(L^2_x, G_\alpha^i)}. \tag{6-39}$$

Remark 6.23. By Lemma 6.22, Remark 6.17, and (6-11), we have

$$\sum_{\substack{i \geq j \\ 0 \leq i \leq 11 \\ N(G_k^j) \leq 2^{i-5}\epsilon_3^{1/2}}} \left\| \int_{t_k^j}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_k^j), 2^{i-2} \leq \cdots \leq 2^{i+2}} F(v(\tau))\, d\tau \right\|^2_{U^2_\Delta(L^2_x, G_k^j)} + \sum_{0 \leq i \leq 11} 2^{i-j} \sum_{\substack{G_\alpha^i \subseteq G_k^j, \\ N(G_\alpha^i) \leq 2^{i-5}\epsilon_3^{1/2}}} \left\| \int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i), 2^{i-2} \leq \cdots \leq 2^{i+2}} F(v(\tau))\, d\tau \right\|^2_{U^2_\Delta(L^2_x, G_\alpha^i)} \lesssim 1.$$

So we can further reduce the summation over i on the right-hand side of (6-38) and (6-39) to $i > 11$.

Therefore, we have reduced to the proof of the following estimate.

Theorem 6.24 (reduced estimate).

$$\sum_{\substack{i \geq j \\ N(G_k^j) \leq 2^{i-10}\epsilon_3^{1/2}}} \left\| \int_{t_k^j}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_k^j), 2^{i-2} \leq \cdots \leq 2^{i+2}} F(v(\tau))\, d\tau \right\|^2_{U^2_\Delta(L^2_x, G_k^j)} + \sum_{0 \leq i < j} 2^{i-j} \sum_{\substack{G_\alpha^i \subseteq G_k^j \\ N(G_\alpha^i) \leq 2^{i-10}\epsilon_3^{1/2}}} \left\| \int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i), 2^{i-2} \leq \cdots \leq 2^{i+2}} F(v(\tau))\, d\tau \right\|^2_{U^2_\Delta(L^2_x, G_\alpha^i)}$$
$$\lesssim \epsilon_2^{\frac{1}{3}} \|v\|^{\frac{5}{3}}_{\tilde{X}_j([0,T])} \|v\|^2_{\tilde{Y}_j([0,T])} + \epsilon_2^2 \|v\|^2_{\tilde{Y}_j([0,T])} + \|v\|^4_{\tilde{Y}_j([0,T])} (1 + \|v\|^8_{\tilde{X}_j([0,T])}). \tag{6-40}$$

Once this theorem is proved, we can close the proof of Theorem 6.21 by a bootstrap argument. In the proof given below, we shall assume Theorem 6.24 holds, while leaving its proof to Section 7.1.

Proof of Theorem 6.21. Suppose

$$\|v\|^2_{\widetilde{X}_{k*}([0,T])} \le C_0 \quad \text{and} \quad \|v\|^2_{\widetilde{Y}_{k*}([0,T])} \le C\epsilon_2^{\frac{3}{2}} \le \epsilon_2,$$

and from (6-27), we have

$$\|v\|^2_{\widetilde{X}_{k*+1}([0,T])} \le 2C_0 \quad \text{and} \quad \|v\|^2_{\widetilde{Y}_{k*+1}([0,T])} \le 2\epsilon_2.$$

Then, by (6-38), (6-39), and (6-40), we can further get

$$\|v\|_{\widetilde{X}_{k*+1}([0,T])} \le C(1+\epsilon_2^{\frac{2}{3}}(2C_0)^{\frac{5}{6}}+\epsilon_2^{\frac{3}{2}}+\epsilon_2(1+2C_0)^8),$$

$$\|v\|_{\widetilde{Y}_{k*+1}([0,T])} \le C(\epsilon_2^{\frac{3}{4}}+\epsilon_2^{\frac{2}{3}}(2C_0)^{\frac{5}{6}}+\epsilon_2^{\frac{3}{2}}+\epsilon_2(1+2C_0)^8).$$

If we choose $C_0 = 2^6 C$, and ϵ_2 small enough, then we may deduce

$$\|v\|_{\widetilde{X}_{k*+1}([0,T])} \le C_0^{\frac{1}{2}} \quad \text{and} \quad \|v\|_{\widetilde{Y}_{k*+1}([0,T])} \le \epsilon_2^{\frac{1}{2}}.$$

Theorem 6.21 now follows from this and (6-26) by performing an induction on k_*. □

6.2.3. *The low-frequency localized interaction Morawetz estimate.* As an application of the long-time Strichartz estimate, we can obtain the low-frequency localized interaction Morawetz estimate of the (DCR). The Morawetz estimate is a very important tool to prove the scattering of the nonlinear dispersive equations for the radial case [Lin and Strauss 1978; Morawetz 1968]. In the nonradial case, J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao [Colliander et al. 2004] developed the interaction Morawetz estimate, which is used to prove the scattering of the nonlinear Schrödinger equation [Colliander et al. 2008; Tao et al. 2007a; 2007b; Dodson 2012; 2016a; 2016b] in the nonradial case. The low-frequency localized interaction Morawetz estimate will be used to preclude the soliton-like solution in Theorem 6.26.

Theorem 6.25 (low-frequency localized interaction Morawetz estimate). *Let $v(t,y,x)$ be the almost periodic solution in Theorem 6.6 on $[0,T]$ with $\int_0^T N(t)^3\,\mathrm{d}t = K$. Then we have*

$$\left\| \int_{\mathbb{R}} |\nabla_y|^{\frac{1}{2}}(|P^y_{\le 10\epsilon_1^{-1}K} v(t,y,x)|^2)\,\mathrm{d}x \right\|^2_{L^2_{t,y}([0,T]\times\mathbb{R}^2)} \lesssim o(K). \tag{6-41}$$

The proof of this theorem follows from similar arguments in [Dodson 2012; 2016a; 2016b] and relies on Theorem 6.24 (and also some part of the proof). In our (DCR) system, the interaction Morawetz quantity is

$$M_0(t) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |v(t,\tilde{y},\tilde{x})|^2 \frac{y-\tilde{y}}{|y-\tilde{y}|}\Im(\bar{v}\nabla_y v)(t,y,x)\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x},$$

which is invariant under the Galilean transform in the $\mathbb{R}^2$-component. Following the argument in [Colliander et al. 2009; Planchon and Vega 2009], we can get

$$\left\| \int_{\mathbb{R}} |\nabla_y|^{\frac{1}{2}}(|v(t,y,x)|^2)\,\mathrm{d}x \right\|^2_{L^2_{t,y}([0,T]\times\mathbb{R}^2)} \lesssim |M_0(T)-M_0(0)|.$$

Replacing v by its low-frequency cut-off $P^y_{\leq 10\epsilon_1^{-1}K}v$, we then get the low-frequency localized interaction Morawetz quantity

$$M(t)=\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|P^y_{\leq 10\epsilon_1^{-1}K}v(t,\tilde{y},\tilde{x})|^2\frac{y-\tilde{y}}{|y-\tilde{y}|}\Im(\overline{P^y_{\leq 10\epsilon_1^{-1}K}v}\nabla_y P^y_{\leq 10\epsilon_1^{-1}K}v)(t,y,x)\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}.$$

Because for any $\eta>0$ independent of ϵ_1, by Theorem 6.6 and Bernstein's inequality, we have

$$|M(T)|+|M(0)|\lesssim \eta K,$$

we then obtain

$$\left\|\int_{\mathbb{R}}|\nabla_y|^{\frac{1}{2}}(|v(t,y,x)|^2)\,\mathrm{d}x\right\|^2_{L^2_{t,y}([0,T]\times\mathbb{R}^2)}\lesssim \eta K+\mathscr{E},$$

where $\mathscr{E}$ are the error terms coming from the low-frequency cut-off of the solution of the (DCR). These error terms can be proven to be $o(K)$, using Theorem 6.24 and also some estimates from the proof of it. We shall leave the detailed proof of this theorem to Section 7.2.

6.3. *Exclusion of the almost periodic solution.*

Theorem 6.26. *The almost periodic solution to* (DCR) *in Theorem 6.6 does not exist.*

Proof. We will preclude two scenarios in the following.

Case I: $\int_0^\infty N(t)^3\,\mathrm{d}t<\infty$. By the proof of Theorem 6.21, as in [Dodson 2016a; 2016b], we have

$$\|v(t,y,x)\|_{L^\infty_t\dot{H}^3_y L^2_x([0,\infty)\times\mathbb{R}^2\times\mathbb{R})}\lesssim_{m_0}\left(\int_0^\infty N(t)^3\,\mathrm{d}t\right)^3. \tag{6-42}$$

By (6-42) and (6-4), we have

$$\|e^{iy\cdot\xi(t)}v\|_{\dot{H}^1_y L^2_x}\lesssim N(t)C(\eta(t))+\eta(t)^{\frac{1}{2}}\to 0\quad \text{as } t\to\infty.$$

Thus, for any $\epsilon>0$, we can take a sufficiently large positive constant t_0 such that $\|e^{iy\cdot\xi(t_0)}v(t_0)\|_{\dot{H}^1_y L^2_x}\leq\epsilon$. In the following, we can assume $t_0=0$ because of the Galilean invariance. By Minkowski's inequality, the Gagliardo–Nirenberg inequality, and Hölder's inequality, we have

$$\mathcal{E}(v(t))=\mathcal{E}(v(0))\lesssim\|v(0)\|^2_{\dot{H}^1_y L^2_x}\lesssim\epsilon^2.$$

Because we can take ϵ as small as we wish, this scenario does not exist.

Case II: $\int_0^\infty N(t)^3\,\mathrm{d}t=\infty$. By Hölder's inequality and Sobolev's inequality, we have

$$\int_{|y-y(t)|\leq C\left(\frac{1}{100}\|v(0)\|^2_{L^2_{y,x}}\right)/N(t)}\int_{\mathbb{R}}|P^y_{\leq 10\epsilon_1^{-1}K}v(t,y,x)|^2\,\mathrm{d}y\,\mathrm{d}x$$
$$\lesssim\left(\frac{C\left(\frac{1}{100}\|v(0)\|^2_{L^2_{y,x}}\right)}{N(t)}\right)^{\frac{3}{2}}\left\|\int_{\mathbb{R}}|\nabla_y|^{\frac{1}{2}}(|P^y_{\leq 10\epsilon_1^{-1}K}v(t,y,x)|^2)\,\mathrm{d}x\right\|_{L^2_y}.$$

By Theorem 6.6, we have for $K \geq C\big(\frac{1}{100}\|v\|^2_{L^2_{y,x}}\big)$,

$$\frac{\|v\|^2_{L^2_{y,x}}}{2} \leq \int_{\mathbb{R}} \int_{|y-y(t)|\leq C(\frac{1}{100}\|v\|^2_{L^2_{y,x}})/N(t)} |P^y_{\leq 10\epsilon_1^{-1}K} v(t,y,x)|^2 \, dy \, dx.$$

By the above two estimates, together with Theorem 6.25 and the conservation of mass, we have the following contradiction when K is sufficiently large:

$$\begin{aligned}\|v\|^4_{L^2_{y,x}} K &\lesssim \int_0^T N(t)^3 \bigg(\int_{|y-y(t)|\leq C(\frac{1}{100}\|v\|^2_{L^2_{y,x}})/N(t)} \int_{\mathbb{R}} |P^y_{\leq 10\epsilon_1^{-1}K} v(t,y,x)|^2 \, dx \, dy \bigg)^2 dt \\ &\lesssim \bigg\| \int_{\mathbb{R}} |\nabla_y|^{\frac{1}{2}} (|P^y_{\leq 10\epsilon_1^{-1}K} v(t,y,x)|^2) \, dx \bigg\|^2_{L^2_{t,y}([0,T]\times\mathbb{R}^2)} \lesssim o(K).\end{aligned}$$

This completes the proof of Theorem 6.26. □

Proof of Theorem 1.2. This is an immediate consequence of Theorems 6.6 and 6.26. □

7. Proof of Theorems 6.24 and 6.25

7.1. ***Proof of Theorem 6.24.*** In this section, we complete the proof of Theorem 6.24. To prove this theorem, we decompose the nonlinear term $P^y_{\xi(G^i_\alpha),2^{i-2}\leq\cdots\leq 2^{i+2}} F(v)$ and also use the fact that on the time interval G^i_α, $\xi(t)$ can replace $\xi(G^i_\alpha)$ up to 2^{i-20} by (6-20). Then, we can see it is enough to prove the estimate the left-hand side of (6-40) with $P^y_{\xi(G^i_\alpha),2^{i-2}\leq\cdots\leq 2^{i+2}} F(v)$ being replaced by

$$P^y_{\xi(G^i_\alpha),2^{i-2}\leq\cdots\leq 2^{i+2}} \mathcal{O}\bigg(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n \big(P^y_{\xi(G^i_\alpha),\geq 2^{i-5}} v_{n_1} \overline{P^y_{\xi(t),\geq 2^{i-10}} v_{n_2}} v_{n_3} \big) \bigg) \tag{7-1}$$

$$+ P^y_{\xi(t),2^{i-2}\leq\cdots\leq 2^{i+2}} \mathcal{O}\bigg(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n \big(P^y_{\xi(t),\leq 2^{i-10}} v_{n_1} \overline{P^y_{\xi(G^i_\alpha),\leq 2^{i-10}} v_{n_2}} P^y_{\xi(G^i_\alpha),2^{i-5}\leq\cdots\leq 2^{i+5}} v_{n_3} \big) \bigg), \tag{7-2}$$

we also have a similar fact for the nonlinear term $P^y_{\xi(G^j_k),2^{i-2}\leq\cdots\leq 2^{i+2}} F(v)$, where the symbol $\mathcal{O}$ represents the different frequencies that will be located in different v_{n_l}, $l = 1, 2, 3$. Since their estimates are almost identical, we denote them as a single $\mathcal{O}$. The estimate of the Duhamel propagator of the term (7-1) is very short and easy, and mainly relies on the bilinear Strichartz estimate in Proposition 6.13. The estimate of the Duhamel propagator of the term (7-2) is lengthy. This is because to prove the estimate of the Duhamel propagator of the term (7-2), we need to prove the bilinear Strichartz estimates on the union of the small intervals. It turns out these bilinear Strichartz estimates cannot be proven just by the harmonic analysis but also rely heavily on the structure of the (DCR) system or more precisely the corresponding interaction Morawetz estimate of (DCR). During the proof of this part, some terms can be estimated by the following bilinear Strichartz estimate established recently [Candy 2019] instead of the interaction Morawetz estimate as in [Dodson 2016b]. This new bilinear Strichartz estimate is very useful in [Shen and Wu 2020].

Lemma 7.1 (bilinear Strichartz estimate [Candy 2019]). *Let* $1 \le q, r \le 2$, $\frac{1}{q} + \frac{3}{2r} < \frac{3}{2}$, *and suppose* $M, N \in 2^{\mathbb{Z}}$ *satisfy* $M \ll N$. *Then, for any* $\phi, \psi \in L^2(\mathbb{R}^2)$,

$$\|e^{it\Delta} P_N \phi e^{it\Delta} P_M \psi\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^2)} \lesssim \frac{M^{3-\frac{2}{q}-\frac{3}{r}}}{N^{1-\frac{1}{r}}} \|P_N\phi\|_{L^2(\mathbb{R}^2)} \|P_M\psi\|_{L^2(\mathbb{R}^2)}.$$

7.1.1. *Estimate of* (7-1). We first deal with (7-1).

Theorem 7.2. *For any fixed* $G_k^j \subseteq [0,T]$, $j > 0$, *we have*

$$\begin{aligned}
&\sum_{0\le i<j} 2^{i-j} \sum_{\substack{G_\alpha^i \subseteq G_k^j \\ N(G_\alpha^i)\le 2^{i-10}\epsilon_3^{1/2}}} \left\| \int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}} F_{\alpha,i}^{\text{high}}(v(\tau))\,\mathrm{d}\tau \right\|^2_{U^2_\Delta(L^2_x, G^i_\alpha)} \\
&\qquad + \sum_{\substack{i\ge j,\\ N(G_k^j)\le 2^{i-10}\epsilon_3^{1/2}}} \left\| \int_{t_k^j}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_k^j),2^{i-2}\le\cdots\le 2^{i+2}} F_{k,j}^{\text{high}}(v(\tau))\,\mathrm{d}\tau \right\|^2_{U^2_\Delta(L^2_x, G^j_k)} \\
&\lesssim \epsilon_2^{\frac{1}{3}} \|v\|^{\frac{5}{3}}_{\widetilde{X}_j([0,T])} \|v\|^2_{\widetilde{Y}_j([0,T])},
\end{aligned}$$

where

$$F_{k,j}^{\text{high}}(v(t)) = \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n \big(P^y_{\xi(G_k^j),\ge 2^{i-5}} v_{n_1} \overline{P^y_{\xi(t),\ge 2^{i-10}} v_{n_2}} v_{n_3}\big)(t),$$

$$F_{\alpha,i}^{\text{high}}(v(t)) = \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n \big(P^y_{\xi(G_\alpha^i),\ge 2^{i-5}} v_{n_1} \overline{P^y_{\xi(t),\ge 2^{i-10}} v_{n_2}} v_{n_3}\big)(t).$$

Proof. On the time interval G_α^i with $N(G_\alpha^i) \le 2^{i-10}\epsilon_3^{1/2}$, we take $w \in V^2_\Delta(L^2_x, G^i_\alpha)$ be normalized so that $(\mathcal{F}_y w)(t,\xi,x)$ is supported on

$$\{\xi : 2^{i-2} \le |\xi - \xi(G_\alpha^i)| \le 2^{i+2}\}$$

for any $(t,x) \in \mathbb{R}\times\mathbb{R}$. By the Cauchy–Schwarz inequality, (6-1), Proposition 6.13, the conservation of mass, (6-8), (6-11), Lemma 6.20, and (6-15), we infer

$$\begin{aligned}
&\int_{G_\alpha^i} \langle w(\tau), F_{\alpha,i}^{\text{high}}(v(\tau))\rangle\,\mathrm{d}\tau \\
&\le \sum_{l\ge i-5} \big\| \|w\|_{L^2_x} \|P^y_{\xi(G_\alpha^i),2^l} v\|_{L^2_x} \big\|^{\frac{1}{2}}_{L_t^{5/2} L_y^{5/3}(G_\alpha^i\times\mathbb{R}^2)} \|P^y_{\xi(G_\alpha^i),2^l} v\|^{\frac{1}{2}}_{L_t^{5/2} L_y^{10} L_x^2(G_\alpha^i\times\mathbb{R}^2\times\mathbb{R})} \\
&\qquad \cdot \|w\|^{\frac{1}{2}}_{L_t^{5/2} L_y^{10} L_x^2(G_\alpha^i\times\mathbb{R}^2\times\mathbb{R})} \|P^y_{\xi(t),\ge 2^{i-10}} v\|_{L_t^{5/2} L_y^{10} L_x^2(G_\alpha^i\times\mathbb{R}^2\times\mathbb{R})} \|v\|_{L_t^\infty L^2_{y,x}(G_\alpha^i\times\mathbb{R}^2\times\mathbb{R})} \\
&\lesssim \sum_{l\ge i-5} 2^{\frac{i-l}{5}} \|P^y_{\xi(G_\alpha^i),2^l} v\|_{U^2_\Delta(L^2_x, G^i_\alpha)} \|P^y_{\xi(t),\ge 2^{i-10}} v\|^{\frac{1}{6}}_{L_t^\infty L^2_{y,x}(G_\alpha^i\times\mathbb{R}^2\times\mathbb{R})} \\
&\qquad \cdot \|P^y_{\xi(t),\ge 2^{i-10}} v\|^{\frac{5}{6}}_{L_t^{25/12} L_y^{50} L_x^2(G_\alpha^i\times\mathbb{R}^2\times\mathbb{R})} \\
&\lesssim \epsilon_2^{\frac{1}{6}} \|v\|^{\frac{5}{6}}_{\widetilde{X}_i([0,T])} \sum_{l\ge i-5} 2^{\frac{i-l}{5}} \|P^y_{\xi(G_\alpha^i),2^l} v\|_{U^2_\Delta(L^2_x, G^i_\alpha)} \\
&\lesssim \epsilon_2^{\frac{1}{6}} \|v\|^{\frac{5}{6}}_{\widetilde{X}_j([0,T])} \Big(\sum_{l\ge i-5} 2^{\frac{i-l}{5}} \|P^y_{\xi(G_\alpha^i),2^l} v\|^2_{U^2_\Delta(L^2_x, G^i_\alpha)} \Big)^{\frac{1}{2}}.
\end{aligned} \tag{7-3}$$

As in [Dodson 2016b], we see, for any $0 \le l \le j$, G_k^j overlaps 2^{j-l} intervals G_β^l and, for $0 \le i \le l$, G_β^l overlaps 2^{l-i} intervals G_α^i. In addition, every G_α^i is contained in one G_β^l. Thus, we can divide the summation in the left-hand side of the following (7-4) and (7-5) into different groups according to $l \ge j$ and $0 \le l < j$. Then by some easy calculation and reordering the summation of i and l, we have

$$\sum_{0\le i<j} 2^{i-j} \sum_{\substack{G_\alpha^i \subseteq G_k^j \\ N(G_\alpha^i)\le 2^{i-10}\epsilon_3^{1/2}}} \left(\sum_{l\ge i-5} 2^{\frac{i-l}{5}} \|P^y_{\xi(G_\alpha^i),2^l} v\|^2_{U^2_\Delta(L^2_x,G_\alpha^i)}\right), \tag{7-4}$$

$$\sum_{\substack{i\ge j,\\ N(G_k^j)\le 2^{i-10}\epsilon_3^{1/2}}} \left(\sum_{l\ge i-5} 2^{\frac{i-l}{5}} \|P^y_{\xi(G_k^j),2^l} v\|^2_{U^2_\Delta(L^2_x,G_k^j)}\right) \lesssim \|v\|^2_{\widetilde{Y}_j([0,T])}. \tag{7-5}$$

Theorem 7.2 follows from (6-7), (7-3) and (7-4). □

7.1.2. *Estimate of* (7-2). Now we turn to the estimate of (7-2). Let

$$F^{\text{low}}_{k,j}(v(t)) := \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n\big(P^y_{\xi(t),\le 2^{i-10}} v_{n_1} \overline{P^y_{\xi(t),\le 2^{i-10}} v_{n_2}} P^y_{\xi(G_k^j),2^{i-5}\le\cdots\le 2^{i+5}} v_{n_3}\big),$$

$$F^{\text{low}}_{\alpha,i}(v(t)) := \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n\big(P^y_{\xi(t),\le 2^{i-10}} v_{n_1} \overline{P^y_{\xi(t),\le 2^{i-10}} v_{n_2}} P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}} v_{n_3}\big).$$

Then, we have:

Theorem 7.3. *For any $0 \le i \le j$, on the time interval $G_\alpha^i \subseteq G_k^j$, with $N(G_\alpha^i) \le 2^{i-10}\epsilon_3^{1/2}$, we have*

$$\left\|\int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}} F^{\text{low}}_{\alpha,i}(v(\tau))\,d\tau\right\|_{U^2_\Delta(L^2_x,G_\alpha^i)} \lesssim \|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}} v\|_{U^2_\Delta(L^2_x,G_\alpha^i)}\big(\epsilon_2 + \|v\|_{\widetilde{Y}_i([0,T])}(1+\|v\|_{\widetilde{X}_i([0,T])})^4\big). \tag{7-6}$$

In addition, for $i \ge j$, $N(G_k^j) \le 2^{i-10}\epsilon_3^{1/2}$, we have

$$\left\|\int_{t_k^j}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_k^j),2^{i-2}\le\cdots\le 2^{i+2}} F^{\text{low}}_{k,j}(v(\tau))\,d\tau\right\|_{U^2_\Delta(L^2_x,G_k^j)} \lesssim 2^{\frac{3(j-i)}{4}} \|P^y_{\xi(G_k^j),2^{i-5}\le\cdots\le 2^{i+5}} v\|_{U^2_\Delta(L^2_x,G_k^j)}\big(\epsilon_2 + \|v\|_{\widetilde{Y}_j([0,T])}(1+\|v\|_{\widetilde{X}_j([0,T])})^4\big). \tag{7-7}$$

Proof of Theorem 7.3. We will only prove (7-6), as (7-7) follows by a similar argument. Fix G_α^i with $N(G_\alpha^i) \le 2^{i-10}\epsilon_3^{1/2}$. We can see there are no more than two small intervals J_1 and J_2 which overlap G_α^i but are not contained in G_α^i. Let $\widetilde{G}_\alpha^i = G_\alpha^i \backslash (J_1 \cup J_2)$, by (6-8), (6-7), (6-9), and (6-11), we have

$$\begin{aligned}\left\|\int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}} F^{\text{low}}_{\alpha,i}(v(\tau))\,d\tau\right\|_{U^2_\Delta(L^2_x,G_\alpha^i)} &\lesssim \left\|\int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}} F^{\text{low}}_{\alpha,i}(v(\tau))\,d\tau\right\|_{U^2_\Delta(L^2_x,\widetilde{G}_\alpha^i)} \\ &\quad + \|F^{\text{low}}_{\alpha,i}(v(t))\|_{L^{4/3}_{t,y}L^2_x((J_1\cap G_\alpha^i)\times\mathbb{R}^2\times\mathbb{R})} + \|F^{\text{low}}_{\alpha,i}(v(t))\|_{L^{4/3}_{t,y}L^2_x((J_2\cap G_\alpha^i)\times\mathbb{R}^2\times\mathbb{R})}. \end{aligned} \tag{7-8}$$

Here, we may assume $t^i_\alpha \in \widetilde{G}^i_\alpha$, because if $t^i_\alpha \notin \widetilde{G}^i_\alpha$, we may move t^i_α into $\widetilde{G}^i_\alpha$ with the errors being absorbed by the last two terms on the right-hand side of the above inequality. We can show the last two terms on the right-hand side of (7-8) are small in the following.

On the intervals J_l for $l = 1, 2$, by Propositions 6.12 and 6.13, (6-11), the fact $N(t) \le 2^{i-5}\epsilon_3^{1/2}$ on G^i_α, (6-14), (6-15), Lemma 6.16, and (6-11), we can get

$$
\begin{aligned}
&\|F^{\text{low}}_{\alpha,i}(v(t))\|_{L^{4/3}_{t,y}L^2_x((J_l\cap G^i_\alpha)\times\mathbb{R}^2\times\mathbb{R})}\\
&\lesssim \big\| \|P^y_{\xi(J_l),\le\epsilon_3^{-1/4}N(J_l)}v\|_{L^2_x}\|P^y_{\xi(G^i_\alpha),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{L^2_x}\big\|_{L^2_{t,y}((J_l\cap G^i_\alpha)\times\mathbb{R}^2\times\mathbb{R})}\\
&\qquad\qquad\qquad\qquad\cdot\|P^y_{\xi(t),\le 2^{i-10}}v\|_{L^4_{t,y}L^2_x((J_l\cap G^i_\alpha)\times\mathbb{R}^2\times\mathbb{R})}\\
&\quad+\|P^y_{\xi(J_l),\ge\epsilon_3^{-1/4}N(J_l)}v\|_{L^\infty_t L^2_{y,x}((J_l\cap G^i_\alpha)\times\mathbb{R}^2\times\mathbb{R})}\|P^y_{\xi(G^i_\alpha),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{L^{8/3}_t L^8_y L^2_x((J_l\cap G^i_\alpha)\times\mathbb{R}^2\times\mathbb{R})}\\
&\qquad\qquad\qquad\qquad\cdot\|P^y_{\xi(t),\le 2^{i-10}}v\|_{L^{8/3}_t L^8_y L^2_x((J_l\cap G^i_\alpha)\times\mathbb{R}^2\times\mathbb{R})}\\
&\lesssim \sum_{2^k\le\epsilon_3^{-1/4}N(J_l)} 2^{\frac{k-i}{2}}\|P^y_{\xi(G^i_\alpha),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{U^2_\Delta(L^2_x,G^i_\alpha)}\|P^y_{\xi(J_l),2^k}v\|_{U^2_\Delta(L^2_x,J_l)}\\
&\qquad\qquad\qquad\qquad+\epsilon_2\|P^y_{\xi(G^i_\alpha),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{U^2_\Delta(L^2_x,G^i_\alpha)}\\
&\lesssim \epsilon_2\|P^y_{\xi(G^i_\alpha),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{U^2_\Delta(L^2_x,G^i_\alpha)}.
\end{aligned}
\tag{7-9}
$$

Thus, we can simplify the estimate of (7-8) to the case that G^i_α is the union of finite many small intervals J_l. (If not, we just need to add the right-hand side of (7-9)). Let

$$F^{\text{low},l_2}_{\alpha,i}(v(t)) = \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n\big(P^y_{\xi(t),2^{l_2}}v_{n_1}\overline{P^y_{\xi(t),\le 2^{l_2}}v_{n_2}}P^y_{\xi(G^i_\alpha),2^{i-5}\le\cdots\le 2^{i+5}}v_{n_3}\big).$$

Then by Lemma 6.11, we have

$$\text{LHS of (7-8)} \lesssim A_1 + A_2 + A_3 + A_4,$$

where

$$A_1 = \sum_{0\le l_2\le i-10}\Bigg(\sum_{\substack{J_l\subseteq G^i_\alpha\\ N(J_l)\ge\epsilon_3^{1/2}2^{l_2-5}}}\|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}F^{\text{low},l_2}_{\alpha,i}(v(t))\|^2_{DU^2_\Delta(L^2_x,J_l)}\Bigg)^{\frac12}, \tag{7-10}$$

$$A_2 = \sum_{\substack{0\le l_2\le i-10\\ J_l\subseteq G^i_\alpha\\ N(J_l)\ge\epsilon_3^{1/2}2^{l_2-5}}}\bigg\|\int_{J_l} e^{-it\Delta_y}P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}F^{\text{low},l_2}_{\alpha,i}(v(t))\,\mathrm{d}t\bigg\|_{L^2_{y,x}}, \tag{7-11}$$

$$A_3 = \sum_{0\le l_2\le i-10}\Bigg(\sum_{\substack{G^{l_2}_\beta\subseteq G^i_\alpha\\ N(G^{l_2}_\beta)\le\epsilon_3^{1/2}2^{l_2-5}}}\|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}F^{\text{low},l_2}_{\alpha,i}(v(t))\|^2_{DU^2_\Delta(L^2_x,G^{l_2}_\beta)}\Bigg)^{\frac12}, \tag{7-12}$$

$$A_4 = \sum_{\substack{0\le l_2\le i-10\\ G_\beta^{l_2}\subseteq G_\alpha^i\\ N(G_\beta^{l_2})\le \epsilon_3^{1/2}2^{l_2-5}}} \left\| \int_{G_\beta^{l_2}} e^{-it\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}} F^{\mathrm{low},l_2}_{\alpha,i}(v(t))\,\mathrm{d}t \right\|_{L^2_{y,x}}. \tag{7-13}$$

The proof of the first two terms are easy. We first prove the following auxiliary estimate.

Lemma 7.4. *Let (p_0,q_0) be Strichartz admissible with $q_0\ge 20$. Suppose that*

$$w(t,y,x)\in L_t^{p_0}L_y^{q_0}L_x^2(J_l\times\mathbb{R}^2\times\mathbb{R})$$

satisfies that $\mathcal{F}w(t,\cdot,x)$ is supported on $\{\xi : 2^{i-2}\le|\xi-\xi(G_\alpha^i)|\le 2^{i+2}\}$. If $N(J_l)\ge\epsilon_3^{1/2}2^{l_2-5}$, then we have

$$\left|\int_{J_l}\int_{\mathbb{R}^2}\int_{\mathbb{R}} \overline{w(t,y,x)} F^{\mathrm{low},l_2}_{\alpha,i}(v(t,y,x))\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\right|$$
$$\lesssim 2^{\frac{2l_2}{q_0}}\epsilon_3^{\frac14-\frac1{q_0}}2^{-\frac{i}{2}}\|w\|_{L_t^{p_0}L_y^{q_0}L_x^2(J_l\times\mathbb{R}^2\times\mathbb{R})}\|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}}v(t,y,x)\|_{U^2_\Delta(G_\alpha^i,L_x^2)}. \tag{7-14}$$

Proof. By (6-14), we see $|\xi-\xi(t)|\le 2^{l_2+2}$ implies $|\xi-\xi(J_l)|\le\epsilon_3^{-1/2}N(J_l)$ for $t\in J_l$. By the argument in the proof of Lemma 6.2 and Hölder's inequality, we have

$$\left|\int_{J_l}\int_{\mathbb{R}^2}\int_{\mathbb{R}} \overline{w(t,y,x)} F^{\mathrm{low},l_2}_{\alpha,i}v((t,y,x))\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\right|$$
$$\le \|e^{i\tau(-\Delta_x+x^2)}wP^y_{\xi(t),2^{l_2}}e^{i\tau(-\Delta_x+x^2)}v\|_{L^{p_0}_{\tau,x}L_t^{p_0}L_y^2([0,\pi]\times\mathbb{R}\times J_l\times\mathbb{R}^2)}$$
$$\cdot\|P^y_{\xi(J_l),\lesssim\epsilon_3^{-1/2}N(J_l)}e^{i\tau(-\Delta_x+x^2)}$$
$$\cdot vP^y_{\xi(G_\alpha^i),2^{i-5}\le\cdot 2^{i+5}}e^{i\tau(-\Delta_x+x^2)}v\|_{L^{p_0/(p_0-1)}_{\tau,x}L_t^{p_0/(p_0-1)}L_y^2([0,\pi]\times\mathbb{R}\times J_l\times\mathbb{R}^2)}. \tag{7-15}$$

By Minkowski's inequality, Hölder's inequality, Lemma 6.1, Bernstein's inequality and the conservation of mass, we have

$$\|e^{i\tau(-\Delta_x+x^2)}wP^y_{\xi(t),2^{l_2}}e^{i\tau(-\Delta_x+x^2)}v\|_{L^{p_0}_{\tau,x}L_t^{p_0}L_y^2([0,\pi]\times\mathbb{R}\times J_l\times\mathbb{R}^2)}$$
$$\lesssim \Big\|\|e^{i\tau(-\Delta_x+x^2)}w\|_{L^{2p_0}_{\tau,x}([0,\pi]\times\mathbb{R})}\|P^y_{\xi(t),2^{l_2}}e^{i\tau(-\Delta_x+x^2)}v\|_{L^{2p_0}_{\tau,x}([0,\pi]\times\mathbb{R})}\Big\|_{L_t^{p_0}L_y^2(J_l\times\mathbb{R}^2)}$$
$$\lesssim\|w\|_{L_t^{p_0}L_y^{q_0}L_x^2(J_l\times\mathbb{R}^2\times\mathbb{R})}\|P^y_{\xi(t),2^{l_2}}v\|_{L_t^\infty L_y^{p_0}L_x^2(J_l\times\mathbb{R}^2\times\mathbb{R})}\lesssim 2^{\frac{2l_2}{q_0}}\|w\|_{L_t^{p_0}L_y^{q_0}L_x^2(J_l\times\mathbb{R}^2\times\mathbb{R})}. \tag{7-16}$$

Next, we use the vector-valued version of transference principle to estimate

$$\|P^y_{\xi(J_l),\lesssim\epsilon_3^{-1/2}N(J_l)}e^{i\tau(-\Delta_x+x^2)}vP^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}}e^{i\tau(-\Delta_x+x^2)}v\|_{L^{p_0}_{\tau,x}L_t^{p_0}L_y^2([0,\pi]\times\mathbb{R}\times J_l\times\mathbb{R}^2)}.$$

Then, using the similar argument of Corollary 1.6 in [Candy 2019], we are reduced to considering $P^y_{\xi(J_l),\lesssim\epsilon_3^{-1/2}N(J_l)}v$ and $P^y_{\xi(G^i_\alpha),2^{i-5}\le\cdots\le 2^{i+5}}v$ are $U^2_\Delta(L^2_x)$-atoms. Let

$$P^y_{\xi(J_l),\lesssim\epsilon_3^{-1/2}N(J_l)}v=\sum_{I\in\mathcal{I}}\chi_I e^{it\Delta_y}f_I(y,x),\ \ P^y_{\xi(G^i_\alpha),2^{i-5}\le\cdots\le 2^{i+5}}v=\sum_{J\in\mathcal{J}}\chi_J e^{it\Delta_y}g_J(y,x),$$

where $\mathcal{I}$ and $\mathcal{J}$ are partitions as in the definition of $U^2_\Delta(L^2_x)$. We see by Lemma 7.1, Hölder's inequality and Lemma 6.1,

$$\begin{aligned}
&\|e^{it\Delta_y}e^{i\tau(-\Delta_x+x^2)}f_I e^{it\Delta_y}e^{i\tau(-\Delta_x+x^2)}g_J\|_{L^{p_0/(p_0-1)}_{\tau,x}L^{p_0/(p_0-1)}_t L^2_y([0,\pi]\times\mathbb{R}\times J_l\times\mathbb{R}^2)}\\
&\lesssim 2^{-\frac{i}{2}}(\epsilon_3^{-\frac12}N(J_l))^{\frac12-\frac{2}{q_0}}\|e^{i\tau(-\Delta_x+x^2)}f_I\|_{L^2_y L^{2p_0/(p_0-1)}_{\tau,x}(\mathbb{R}^2\times[0,\pi]\times\mathbb{R})}\\
&\qquad\qquad\cdot\|e^{i\tau(-\Delta_x+x^2)}g_J\|_{L^2_y L^{2p_0/(p_0-1)}_{\tau,x}(\mathbb{R}^2\times[0,\pi]\times\mathbb{R})}\\
&\lesssim 2^{-\frac{i}{2}}(\epsilon_3^{-\frac12}N(J_l))^{\frac12-\frac{2}{q_0}}\|f_I\|_{L^2_{y,x}}\|g_J\|_{L^2_{y,x}}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\|P^y_{\xi(J_l),\lesssim\epsilon_3^{-1/2}N(J_l)}e^{i\tau(-\Delta_x+x^2)}\\
&\qquad\cdot vP^y_{\xi(G^i_\alpha),2^{i-5}\le\cdots\le 2^{i+5}}e^{i\tau(-\Delta_x+x^2)}v\|_{L^{p_0/(p_0-1)}_{\tau,x}L^{p_0/(p_0-1)}_t L^2_y([0,\pi]\times\mathbb{R}\times J_l\times\mathbb{R}^2)}\\
&\lesssim 2^{-\frac{i}{2}}(\epsilon_3^{-1/2}N(J_l))^{\frac12-\frac{2}{q_0}}\|P^y_{\xi(J_l),\lesssim\epsilon_3^{-1/2}N(J_l)}v\|_{U^2_\Delta(J_l,L^2_x)}\|P^y_{\xi(G^i_\alpha),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{U^2_\Delta(J_l,L^2_x)}.
\end{aligned}\tag{7-17}$$

Therefore, (7-14) follows from (7-15), (7-16) and (7-17). □

We first consider (7-10). By duality, we have

$$\text{(7-10)}=\sum_{0\le l_2\le i-10}\Bigg(\sum_{\substack{J_l\subseteq G^i_\alpha,\\ N(J_l)\ge\epsilon_3^{1/2}2^{l_2-5}}}\sup_{\|w\|_{V^2_\Delta(J_l,L^2_x)}=1}\bigg|\int_{J_l}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\overline{w(t,y,x)}P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}\cdot F^{\mathrm{low},l_2}_{\alpha,i}(v(t,y,x))\,dy\,dx\,dt\bigg|^2\Bigg)^{\frac12}.$$

By $\|w\|_{L^{p_0}_t L^{q_0}_y L^2_x(J_l\times\mathbb{R}^2\times\mathbb{R})}\lesssim\|w\|_{V^2_\Delta(J_l,L^2_x)}\lesssim 1$, and (7-14), we get

$$\begin{aligned}
\text{(7-10)}&\lesssim\sum_{0\le l_2\le i-10}\Bigg(\sum_{\substack{J_l\subseteq G^i_\alpha,\\ N(J_l)\ge\epsilon_3^{1/2}2^{l_2-5}}}2^{\frac{4l_2}{q_0}}\epsilon_3^{\frac12-\frac{2}{q_0}}2^{-i}\Bigg)^{\frac12}\|P^y_{\xi(G^i_\alpha),2^{i-5}\le\cdots\le 2^{i+5}}v(t,y,x)\|_{U^2_\Delta(G^i_\alpha,L^2_x)}\\
&\lesssim(\epsilon_3^{-\frac12}N(J_l))\epsilon_3^{\frac14-\frac{1}{q_0}}\|P^y_{\xi(G^i_\alpha),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{U^2_\Delta(G^i_\alpha,L^2_x)}\\
&\lesssim\epsilon_2^2\|P^y_{\xi(G^i_\alpha),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{U^2_\Delta(L^2_x,G^i_\alpha)}.
\end{aligned}$$

We now consider (7-11). By duality and Lemma 7.4, we have

$$\begin{aligned}
&\bigg\|\int_{J_l}e^{-it\Delta_y}P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}F^{\mathrm{low},l_2}_{\alpha,i}(v(t,y,x))\,dt\bigg\|_{L^2_{y,x}}\\
&\lesssim\sup_{\|w_0\|_{L^2_{y,x}}=1}\|e^{it\Delta_y}w_0\|_{L^{p_0}_t L^{q_0}_y L^2_x(J_l\times\mathbb{R}^2\times\mathbb{R})}2^{\frac{2l_2}{q_0}}\epsilon_3^{\frac14-\frac{1}{q_0}}2^{-\frac{i}{2}}\|P^y_{\xi(G^i_\alpha),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{U^2_\Delta(G^i_\alpha,L^2_x)},
\end{aligned}\tag{7-18}$$

where $\mathcal{F}_y w_0$ is supported on $\{\xi : 2^{i-5} \le |\xi - \xi(G_\alpha^i)| \le 2^{i+5}\}$ in the above estimate. For fixed i, we take $q_0 = 20 + 2i$; then $2^{i/q_0} \lesssim 1$. For the right-hand side of (7-18), by Hölder's inequality, Young's inequality, (6-14), (6-5), (6-19), and the conservation of mass, we have

$$
\begin{aligned}
(7\text{-}11) &\lesssim \sum_{\substack{0\le l_2\le i-10\\ J_l\subseteq G_\alpha^i\\ N(J_l)\ge \epsilon_3^{1/2}2^{l_2-5}}} \sup_{\|w_0\|_{L^2_{y,x}}=1} 2^{\frac{2l_2}{q_0}} \epsilon_3^{\frac14-\frac{1}{q_0}} 2^{-\frac{i}{2}} \|e^{it\Delta_y} w_0\|_{L_t^{p_0} L_y^{q_0} L_x^2 (J_l\times\mathbb{R}^2\times\mathbb{R})} \cdot \|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}} v\|_{U^2_\Delta(G_\alpha^i, L_x^2)} \\
&\lesssim \sum_{0\le l_2\le i-10} 2^{\frac{2l_2}{q_0}} \epsilon_3^{\frac14-\frac{1}{q_0}} \Biggl(\sum_{\substack{J_l\subseteq G_\alpha^i\\ N(J_l)\ge \epsilon_3^{1/2}2^{l_2-5}}} \|e^{it\Delta_y} w_0\|^{p_0}_{L_t^{p_0} L_y^{q_0} L_x^2 (J_l\times\mathbb{R}^2\times\mathbb{R})} \Biggr)^{\frac{1}{p_0}} \\
&\qquad\qquad \cdot \Biggl(\sum_{J_l\subseteq G_\alpha^i} (2^{-\frac{i}{2}})^{\frac{p_0}{p_0-1}} \Biggr)^{\frac{p_0-1}{p_0}} \|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}} v\|_{U^2_\Delta(G_\alpha^i, L_x^2)} \\
&\lesssim \sum_{\substack{0\le l_2\le i-10\\ 2^{l_2-5}\le \epsilon_3^{-1/2} N(J_l)}} 2^{\frac{2l_2}{q_0}} \epsilon_3^{\frac14-\frac{1}{q_0}} \|e^{it\Delta_y} w_0\|_{L_t^{p_0} L_y^{q_0} L_x^2 (G_\alpha^i\times\mathbb{R}^2\times\mathbb{R})} (2^{\frac{2i}{q_0+2}})^{\frac12+\frac{1}{q_0}} \cdot \|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}} v\|_{U^2_\Delta(G_\alpha^i, L_x^2)} \\
&\lesssim \sum_{\substack{0\le l_2\le i-10\\ 2^{l_2-5}\le \epsilon_3^{-1/2} N(J_l)}} 2^{\frac{2l_2}{q_0}} \epsilon_3^{\frac14-\frac{1}{q_0}} 2^{\frac{i}{q_0}} \|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}} v\|_{U^2_\Delta(G_\alpha^i, L_x^2)} \\
&\lesssim \epsilon_3^{\frac14} \|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}} v\|_{U^2_\Delta(L_x^2, G_\alpha^i)}.
\end{aligned}
$$

For the estimates of (7-12) and (7-13), we separate the proofs in the next section using two bilinear Strichartz estimates. ☐

7.1.3. *Two bilinear Strichartz estimates.* We have the following:

Theorem 7.5 (first bilinear Strichartz estimate). *Let $w_0 \in L^2_{y,x}(\mathbb{R}^2\times\mathbb{R})$ with $\operatorname{supp} \mathcal{F}_y w_0$ is supported on $\{\xi : 2^{i-5} \le |\xi - \xi(G_\alpha^i)| \le 2^{i+5}\}$. Then, for any $0 \le l_2 \le i-10$, we have on $G_\beta^{l_2} \subseteq G_\alpha^i$*

$$\Bigl\| \|e^{it\Delta_y} w_0\|_{L_x^2} \|P^y_{\xi(t),\le 2^{l_2}} v\|_{L_x^2} \Bigr\|^2_{L^2_{t,y}(G_\beta^{l_2}\times\mathbb{R}^2)} \lesssim \|w_0\|^2_{L^2_{y,x}} (1 + \|v\|^4_{\widetilde{X}_i(G_\alpha^i)}). \tag{7-19}$$

Theorem 7.6 (second bilinear Strichartz estimate). *Let $w_0 \in L^2_{y,x}(\mathbb{R}^2\times\mathbb{R})$ with $\operatorname{supp} \mathcal{F}_y w_0$ supported on $\{\xi : 2^{i-5} \le |\xi - \xi(G_\alpha^i)| \le 2^{i+5}\}$. Then we have*

$$\sum_{0\le l_2\le i-10} \Bigl\| \|e^{it\Delta_y} w_0\|_{L_x^2} \|P^y_{\xi(t),\le 2^{l_2}} v\|_{L_x^2} \Bigr\|^2_{L^2_{t,y}(G_\alpha^i\times\mathbb{R}^2)} \lesssim \|w_0\|^2_{L^2_{y,x}} (1 + \|v\|^6_{\widetilde{X}_i(G_\alpha^i)}). \tag{7-20}$$

With the above two bilinear Strichartz estimates, we can now estimate (7-12) and (7-13).

Estimate of (7-12). For any $0 \le l_2 \le i-10$, by the fact that G_α^i consists of 2^{10} subintervals G_β^{i-10}, Proposition 6.12 and Theorem 7.5 on the subintervals G_β^{i-10}, we get

$$
\begin{aligned}
&\Bigl\| \|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}} v\|_{L_x^2} \|P^y_{\xi(t),\le 2^{l_2}} v\|_{L_x^2} \Bigr\|_{L^2_{t,y}(G_\alpha^i\times\mathbb{R}^2)} \\
&\qquad\qquad \lesssim \|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}} v\|_{U^2_\Delta(L_x^2, G_\alpha^i)} (1 + \|v\|^2_{\widetilde{X}_i(G_\alpha^i)}).
\end{aligned} \tag{7-21}
$$

For any $G_\beta^{l_2}\subseteq G_\alpha^i$, choose $w_\beta^{l_2}\in V_\Delta^2(L_x^2,G_\beta^{l_2})$, with $\operatorname{supp}\mathcal{F}_y w_\beta^{l_2}\subseteq\{\xi:2^{i-2}\le|\xi-\xi(G_\alpha^i)|\le 2^{i+2}\}$ and $\|w_\beta^{l_2}\|_{V_\Delta^2(L_x^2,G_\beta^{l_2})}=1$. Then by Hölder's inequality, (6-1), and (7-21), we have

$$\begin{aligned}&\Bigg(\sum_{\substack{G_\beta^{l_2}\subseteq G_\alpha^i\\ N(G_\beta^{l_2})\le\epsilon_3^{1/2}2^{l_2-5}}}\Big\|\overline{w_\beta^{l_2}}\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n\big(P^y_{\xi(t),2^{l_2}}v_{n_1}\overline{P^y_{\xi(t),\le 2^{l_2}}v_{n_2}}P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}}v_{n_3}\big)\Big\|^2_{L^1_{t,y,x}(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\Bigg)^{\frac12}\\ &\quad\lesssim\Bigg(\sup_{\substack{G_\beta^{l_2}\subseteq G_\alpha^i\\ N(G_\beta^{l_2})\le\epsilon_3^{1/2}2^{l_2-5}}}\big\|\|w_\beta^{l_2}\|_{L_x^2}\|P^y_{\xi(t),2^{l_2}}v\|_{L_x^2}\big\|_{L^2_{t,y}(G_\beta^{l_2}\times\mathbb{R}^2)}\Bigg)\\ &\qquad\qquad\cdot\big\|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}}v\big\|_{U_\Delta^2(L_x^2,G_\alpha^i)}\big(1+\|v\|^2_{\widetilde{X}_i(G_\alpha^i)}\big).\qquad(7\text{-}22)\end{aligned}$$

By Proposition 6.13, (6-8), (6-11), and $N(G_\beta^{l_2})\le\epsilon_3^{1/2}2^{l_2-5}$, we can estimate the term in the first bracket on the right-hand side of (7-22) as follows:

$$\begin{aligned}&\big\|\|w_\beta^{l_2}\|_{L_x^2}\|P^y_{\xi(t),2^{l_2}}v\|_{L_x^2}\big\|_{L^2_{t,y}(G_\beta^{l_2}\times\mathbb{R}^2)}\\ &\quad\lesssim\big\|\|w_\beta^{l_2}\|_{L_x^2}\|P^y_{\xi(t),2^{l_2}}v\|_{L_x^2}\big\|^{\frac12}_{L_t^3L_y^{3/2}(G_\beta^{l_2}\times\mathbb{R}^2)}\|w_\beta^{l_2}\|^{\frac12}_{L_t^3L_y^6L_x^2(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\|P^y_{\xi(t),2^{l_2}}v\|^{\frac12}_{L_t^3L_y^6L_x^2(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\\ &\quad\lesssim 2^{\frac{l_2-i}{6}}\|v\|_{\widetilde{Y}_i(G_\alpha^i)}.\end{aligned}$$

Thus, by the above inequalities, we obtain

$$(7\text{-}12)\lesssim\|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{U_\Delta^2(L_x^2,G_\alpha^i)}\|v\|_{\widetilde{Y}_i(G_\alpha^i)}\big(1+\|v\|^2_{\widetilde{X}_i(G_\alpha^i)}\big).\qquad\square$$

Estimate of (7-13). Let $w_0\in L^2_{y,x}$ have unit norm with $\mathcal{F}_y w_0$ supported on $\{\xi:2^{i-2}\le|\xi-\xi(G_\alpha^i)|\le 2^{i+2}\}$. By the Hölder inequality and Proposition 6.12, we have

$$\begin{aligned}&\Big\|\int_{G_\beta^{l_2}}e^{-it\Delta_y}P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}}F^{\mathrm{low},l_2}_{\alpha,i}(v(t,y,x))\,dt\Big\|_{L^2_{y,x}}\\ &\lesssim\sup_{\|w_0\|_{L^2_{y,x}}=1}\|e^{it\Delta_y}w_0\cdot F^{\mathrm{low},l_2}_{\alpha,i}(v(t,y,x))\|_{L^1_{t,y,x}(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\\ &\lesssim\sup_{\|w_0\|_{L^2_{y,x}}=1}\big\|\|e^{it\Delta_y}w_0\|_{L_x^2}\|P^y_{\xi(t),2^{l_2}}v\|_{L_x^2}\|P^y_{\xi(t),\le 2^{l_2}}v\|_{L_x^2}\|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{L_x^2}\big\|_{L^1_{t,y}(G_\beta^{l_2}\times\mathbb{R}^2)}\\ &\lesssim 2^{\frac{l_2-i}{2}}\|P^y_{\xi(G_\beta^{l_2}),2^{l_2-2}\le\cdots\le 2^{l_2+2}}v\|_{U_\Delta^2(L_x^2,G_\beta^{l_2})}\big\|\|P^y_{\xi(t),\le 2^{l_2}}v\|_{L_x^2}\|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{L_x^2}\big\|_{L^2_{t,y}(G_\beta^{l_2}\times\mathbb{R}^2)}.\end{aligned}$$

Then by the Cauchy–Schwarz inequality and (7-20), we have

$$\begin{aligned}(7\text{-}13)&\lesssim\|v\|_{\widetilde{Y}_i(G_\alpha^i)}\Bigg(\sum_{0\le l_2\le i-10}\big\|\|P^y_{\xi(t),\le 2^{l_2}}v\|_{L_x^2}\|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{L_x^2}\big\|^2_{L^2_{t,y}(G_\alpha^i\times\mathbb{R}^2)}\Bigg)^{\frac12}\\ &\lesssim\|v\|_{\widetilde{Y}_i(G_\alpha^i)}\|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{U_\Delta^2(L_x^2,G_\alpha^i)}\big(1+\|v\|^3_{\widetilde{X}_i(G_\alpha^i)}\big).\qquad\square\end{aligned}$$

Therefore, this completes the proof of Theorem 7.3. Then we can prove Theorem 6.24 by summation with respect to i in the same way as (7-6) and (7-7) in Theorems 7.3 and 7.2.

7.1.4. *Proofs of the bilinear Strichartz estimates.* It remains to prove the two bilinear Strichartz estimates, that is Theorems 7.5 and 7.6. The proofs of these results are basically the same and rely on the interaction Morawetz estimates of the (DCR) system; the argument here follows from that in [Dodson 2016b]. We shall only present the proof of Theorem 7.5 here, because argument of the proof of Theorem 7.6 is similar and also relies on the result of Theorem 7.5 as the proof of the corresponding bilinear Strichartz estimate in [Dodson 2016b].

Proof of Theorem 7.5. Let $w = e^{it\Delta_y} w_0$ and $\tilde{w} = P^y_{\xi(t),\le 2^{l_2}} v$. Then w and $\tilde{w}$ satisfy $i\partial_t w + \Delta_y w = 0$, and

$$i\partial_t \tilde{w} + \Delta_y \tilde{w} = F(\tilde{w}) + N_1 + N_2 = F(\tilde{w}) + N,$$

where

$$N_1 = P^y_{\xi(t),\le 2^{l_2}} F(v) - F(\tilde{w}),$$

and $N_2 = \big(\frac{d}{dt} P^y_{\xi(t),\le 2^{l_2}}\big) v$ with $\frac{d}{dt} P^y_{\xi(t),\le 2^{l_2}}$ being given by the Fourier multiplier

$$-\nabla\phi\left(\frac{\xi - \xi(t)}{2^{l_2}}\right)\frac{\xi'(t)}{2^{l_2}}.$$

We define the interaction Morawetz action

$$\begin{aligned} M(t) = \int_{\mathbb{R}^2\times\mathbb{R}}\int_{\mathbb{R}^2\times\mathbb{R}} |\tilde{w}(t,\tilde{y},\tilde{x})|^2 \frac{y-\tilde{y}}{|y-\tilde{y}|}\Im(\bar{w}\nabla_y w)(t,y,x)\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x} \\ + \int_{\mathbb{R}^2\times\mathbb{R}}\int_{\mathbb{R}^2\times\mathbb{R}} |w(t,\tilde{y},\tilde{x})|^2 \frac{y-\tilde{y}}{|y-\tilde{y}|}\Im(\bar{\tilde{w}}\nabla_y \tilde{w})(t,y,x)\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}. \end{aligned}$$

After some tedious calculation, we get

$$\begin{aligned} &\int_{G^{l_2}_\beta}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}} |\overline{\tilde{w}(t,y,\tilde{x})} w(t,y,x)|^2\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}y\,\mathrm{d}t \\ &\quad\lesssim 2^{l_2-2i}\sup_{t\in G^{l_2}_\beta}|M(t)| \\ &\quad + 2^{l_2-2i}\left|\int_{G^{l_2}_\beta}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}} |w(t,\tilde{y},\tilde{x})|^2\frac{y-\tilde{y}}{|y-\tilde{y}|}\Im(\bar{N}(\nabla_y - i\xi(t))\tilde{w})(t,y,x)\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}y\,\mathrm{d}\tilde{y}\mathrm{d}t\right| && \text{(7-23)} \\ &\quad + 2^{l_2-2i}\left|\int_{G^{l_2}_\beta}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}} |w(t,\tilde{y},\tilde{x})|^2\frac{y-\tilde{y}}{|y-\tilde{y}|}\Im(\bar{\tilde{w}}(\nabla_y - i\xi(t))N)(t,y,x)\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}t\right| && \text{(7-24)} \\ &\quad + 2^{l_2-2i}\left|\int_{G^{l_2}_\beta}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \Im(\bar{w}(\nabla_y - i\xi(t))w)(t,\tilde{y},\tilde{x})\frac{y-\tilde{y}}{|y-\tilde{y}|}\Im(\bar{\tilde{w}}N)(t,y,x)\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}t\right|. && \text{(7-25)} \end{aligned}$$

By the invariance of the Galilean transformation of $M(t)$, Hölder's inequality, and the conservation of mass, we infer that $2^{l_2-2i}\sup_{t\in G^{l_2}_\beta}|M(t)|$ can be bounded by the right-hand side of (7-19).

Estimate of (7-23). By (6-14), (6-17), Bernstein's inequality, the conservation of mass and the Strichartz estimate, we have

$$
\begin{aligned}
|(7\text{-}23)| &\lesssim 2^{l_2-2i}\|N_1\|_{L^{4/3}_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\|(\nabla_y-i\xi(t))\tilde{w}\|_{L^4_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\|w\|^2_{L^\infty_t L^2_{y,x}(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\\
&\quad+2^{-2i}\|w\|^2_{L^\infty_t L^2_{y,x}(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\int_{\mathbb{R}}\int_{G^{l_2}_\beta}|\xi'(t)|\,\|P^y_{\xi(t),2^{l_2-3}\le\cdots\le 2^{l_2+3}}v(t,y,x)\|_{L^2_y}\\
&\qquad\qquad\cdot\|(\nabla_y-i\xi(t))P^y_{\xi(t),\le 2^{l_2}}v(t,\tilde{y},x)\|_{L^2_{\tilde{y}}}\,dx\,dt\\
&\lesssim 2^{l_2-2i}\|N_1\|_{L^{4/3}_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\|(\nabla_y-i\xi(t))\tilde{w}\|_{L^4_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\|w_0\|^2_{L^2_{y,x}}+2^{2l_2-2i}\|w_0\|^2_{L^2_{y,x}}.
\end{aligned}
$$

Let

$$m(t,\xi)=\frac{\xi-\xi(t)}{2^{l_1}}\phi\Bigl(\frac{\xi-\xi(t)}{2^{l_1}}\Bigr).$$

By Minkowski's inequality, Young's inequality, $\sup_t\|(\mathcal{F}^{-1}_\xi m)(t,y)\|_{L^1_y}\lesssim 1$ and (6-24), we get

$$
\begin{aligned}
&\|(\nabla_y-i\xi(t))\tilde{w}\|_{L^4_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\\
&\quad\lesssim\sum_{0\le l_1\le l_2}\|(\nabla_y-i\xi(t))P^y_{\xi(t),2^{l_1}}v\|_{L^4_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\\
&\quad\lesssim\sum_{0\le l_1\le l_2}2^{l_1}\Bigl\|\int|(\mathcal{F}^{-1}_\xi m)(t,y-\tilde{y})|\,\|(P^y_{\xi(t),2^{l_1}}v)(t,\tilde{y},x)\|_{L^2_x}\,d\tilde{y}\Bigr\|_{L^4_{t,y}(G^{l_2}_\beta\times\mathbb{R}^2)}\\
&\quad\lesssim\sum_{0\le l_1\le l_2}2^{l_1}\|P^y_{\xi(t),2^{l_1}}v\|_{L^4_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\\
&\quad\lesssim\sum_{0\le l_1\le l_2}2^{l_1}2^{\frac{l_2-l_1}{4}}\|v\|_{\tilde{X}_{l_2}(G^{l_2}_\beta)}\lesssim 2^{l_2}\|v\|_{\tilde{X}_i(G^i_\alpha)}.
\end{aligned}
$$

Thus, it implies

$$|(7\text{-}23)|\lesssim 2^{2l_2-2i}\|w_0\|^2_{L^2_{y,x}}\|v\|_{\tilde{X}_i(G^i_\alpha)}\|N_1\|_{L^{4/3}_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}+2^{2l_2-2i}\|w_0\|^2_{L^2_{y,x}}.$$

Let

$$v^l=P^y_{\xi(t),\le 2^{l_2-5}}v\quad\text{and}\quad v^h=P^y_{\xi(t),>2^{l_2-5}}v.$$

We can then decompose N_1 as

$$
\begin{aligned}
N_1=\;&P^y_{\xi(t),\le 2^{l_2}}\Bigl(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v^l_{n_1}\overline{v^l_{n_2}}v_{n_3})\Bigr)-\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n\bigl(P^y_{\xi(t),\le 2^{l_2}}v^l_{n_1}\overline{P^y_{\xi(t),\le 2^{l_2}}v^l_{n_2}}P^y_{\xi(t),\le 2^{l_2}}v^l_{n_3}\bigr) && \text{(7-26)}\\
&+2\Bigl(P^y_{\xi(t),\le 2^{l_2}}\Bigl(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v^l_{n_1}\overline{v^l_{n_2}}v^h_{n_3})\Bigr)-\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n\bigl(P^y_{\xi(t),\le 2^{l_2}}v^l_{n_1}\overline{P^y_{\xi(t),\le 2^{l_2}}v^l_{n_2}}P^y_{\xi(t),\le 2^{l_2}}v^h_{n_3}\bigr)\Bigr) && \text{(7-27)}
\end{aligned}
$$

$$+P^y_{\xi(t),\leq l_2}\Big(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v^l_{n_1}\overline{v^h_{n_2}}v^l_{n_3})\Big)-\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n\big(P^y_{\xi(t),\leq 2^{l_2}}v^l_{n_1}\overline{P^y_{\xi(t),\leq 2^{l_2}}v^h_{n_2}}P^y_{\xi(t),\leq 2^{l_2}}v^l_{n_3}\big) \tag{7-28}$$

$$+\mathcal{O}\Big(P^y_{\xi(t),\leq 2^{l_2}}\Big(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v^h_{n_1}\overline{v^h_{n_2}}v_{n_3})\Big)-\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n\big(P^y_{\xi(t),\leq 2^{l_2}}v^h_{n_1}\overline{P^y_{\xi(t),\leq 2^{l_2}}v^h_{n_2}}P^y_{\xi(t),\leq 2^{l_2}}v_{n_3}\big)\Big), \tag{7-29}$$

where the $\mathcal{O}$ in (7-29) means there are two high-frequency factors in it. Observe that

$$(7\text{-}26)=0.$$

We next consider (7-27) and (7-28). Because their estimates are very similar, we only prove (7-27). Since $(\mathcal{F}_y\, v^h_{n_3})(t,\sigma,x)$ is supported on $\{\sigma:|\sigma-\xi(t)|\leq 2^{l_2+10}\}$, we have

$$P^y_{\xi(t),\leq 2^{l_2}}\Big(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v^l_{n_1}\overline{v^l_{n_2}}v^h_{n_3})\Big)-\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N},\\ n_1-n_2+n_3=n}}\Pi_n(P^y_{\xi(t),\leq 2^{l_2}}v^l_{n_1}\overline{P^y_{\xi(t),\leq 2^{l_2}}v^l_{n_2}}P^y_{\xi(t),\leq 2^{l_2}}v^h_{n_3}) \tag{7-30}$$

$$\begin{aligned}=\sum_{l_1\leq l_2}\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\iiint e^{-i\tilde{y}\xi(t)}\Pi_n\big((P^y_{\xi(t),2^{l_1}}v^l_{n_1})(\tilde{y},x)\overline{e^{-iz\xi(t)}(P^y_{\xi(t),\leq 2^{l_1}}v^l_{n_2})(z,x)}v^h_{n_3}(\theta,x)\big)\\ \cdot\iiint e^{i\xi(y-\tilde{y})+i\eta(\tilde{y}-z)+i\sigma(z-\theta)}\Big(\Big(\phi\Big(\frac{\xi-\xi(t)}{2^{l_2}}\Big)-\phi\Big(\frac{\sigma-\xi(t)}{2^{l_2}}\Big)\Big)\\ \cdot\phi\Big(\frac{\sigma-\xi(t)}{2^{l_2+10}}\Big)\psi_{l_1}(\xi-\eta)\phi\Big(\frac{\eta-\sigma}{2^{l_1}}\Big)\Big)\,d\sigma\,d\eta\,d\xi\,dz\,d\tilde{y}\,d\theta\end{aligned} \tag{7-31}$$

$$\begin{aligned}+\sum_{l_1\leq l_2}\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\iiint \Pi_n\big(e^{-i\tilde{y}\xi(t)}(P^y_{\xi(t),\leq 2^{l_1}}v^l_{n_1})(\tilde{y},x)\overline{e^{-iz\xi(t)}(P^y_{\xi(t),2^{l_1}}v^l_{n_2})(z,x)}v^h_{n_3}(\theta,x)\big)\\ \cdot\iiint e^{i\xi(y-\tilde{y})+i\eta(\tilde{y}-z)+i\sigma(z-\theta)}\Big(\Big(\phi\Big(\frac{\xi-\xi(t)}{2^{l_2}}\Big)-\phi\Big(\frac{\sigma-\xi(t)}{2^{l_2}}\Big)\Big)\phi\Big(\frac{\sigma-\xi(t)}{2^{l_2+10}}\Big)\Big)\\ \cdot\psi_{l_1}(\eta-\sigma)\phi\Big(\frac{\xi-\eta}{2^{l_1}}\Big)\,d\sigma\,d\eta\,d\xi\,d\tilde{y}\,dz\,d\theta.\end{aligned} \tag{7-32}$$

We shall only prove estimate (7-31), as the proof of (7-32) is similar.

$$\begin{aligned}(7\text{-}31)=\sum_{l_1\leq l_2}\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\iiint K(t;\tilde{y},z,\theta)\Pi_n\big(e^{-i(y-\tilde{y})\xi(t)}(P^y_{\xi(t),2^{l_1}}v^l_{n_1})(y-\tilde{y},x)\\ \cdot\overline{e^{i(y-\tilde{y}-z)\xi(t)}(P^y_{\xi(t),\leq 2^{l_1}}v^l_{n_2})(y-\tilde{y}-z,x)}v^h_{n_3}(y-\tilde{y}-z-\theta,x)\big)\,d\tilde{y}\,dz\,d\theta,\end{aligned}$$

where

$$\begin{aligned}K(t;\tilde{y},z,\theta)=\iiint e^{i\xi\tilde{y}+i\eta z+i\sigma\theta}\Big(\Big(\phi\Big(\frac{\xi-\xi(t)}{2^{l_2}}\Big)-\phi\Big(\frac{\sigma-\xi(t)}{2^{l_2}}\Big)\Big)\\ \cdot\phi\Big(\frac{\sigma-\xi(t)}{2^{l_2}}\Big)\psi_{l_1}(\xi-\eta)\phi\Big(\frac{\eta-\sigma}{2^{l_1}}\Big)\Big)\,d\xi\,d\eta\,d\sigma.\end{aligned}$$

By the estimates $|\xi-\eta|\sim 2^{l_1}$, $|\eta-\sigma|\lesssim 2^{l_1}$, and the fundamental theorem of calculus, we obtain

$$\left|\phi\left(\frac{\xi-\xi(t)}{2^{l_2}}\right)-\phi\left(\frac{\sigma-\xi(t)}{2^{l_2}}\right)\right|\lesssim 2^{-l_2}|\xi-\sigma|\lesssim 2^{l_1-l_2}. \tag{7-33}$$

This implies

$$\sup_t \int |K(t;\tilde{y},z,\theta)|\,\mathrm{d}\tilde{y}\,\mathrm{d}z\,\mathrm{d}\theta\lesssim 2^{l_1-l_2}. \tag{7-34}$$

Thus, by Minkowski's inequality, Hölder's inequality, (7-34), Lemma 6.20 and the conservation of mass, we infer

$$\begin{aligned}&\Bigg\| P^y_{\xi(t),\le 2^{l_2}}\Pi_n\Bigg(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} v^l_{n_1}\overline{v^l_{n_2}}v^h_{n_3}\Bigg)\\ &\qquad\qquad -\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(P^y_{\xi(t),\le 2^{l_2}}v^l_{n_1}\overline{P^y_{\xi(t),\le 2^{l_2}}v^l_{n_2}}P^y_{\xi(t),\le 2^{l_2}}v^h_{n_3})\Bigg\|_{L^{4/3}_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\\ &\lesssim\sum_{l_1\le l_2}\Bigg\|\iiint |K(t;\tilde{y},z,\theta)|\,\Big\|\|(P^y_{\xi(t),2^{l_1}}v^l)(y-\tilde{y},x)\|_{L^2_x}\|(P^y_{\xi(t),\le 2^{l_1}}v^l)(y-\tilde{y}-z,x)\|_{L^2_x}\\ &\qquad\qquad\qquad\cdot\|v^h(y-\tilde{y}-z-\theta,x)\|_{L^2_x}\Big\|_{L^{4/3}_y}\mathrm{d}\tilde{y}\,\mathrm{d}z\,\mathrm{d}\theta\Bigg\|_{L^{4/3}_t}\\ &\lesssim\sum_{l_1\le l_2}2^{l_1-l_2}\|P^y_{\xi(t),\le 2^{l_1}}v^l\|_{L^\infty_t L^2_{y,x}}\|P^y_{\xi(t),2^{l_1}}u^l\|_{L^{8/3}_t L^8_y L^2_x}\|v^h\|_{L^{8/3}_t L^8_y L^2_x}\lesssim\|v\|^2_{\widetilde{X}_i(G^i_\alpha)}.\end{aligned}$$

We now consider (7-29). Since

$$\begin{aligned}&\Bigg\|\mathcal{O}\Bigg(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n\big(P^y_{\xi(t),\le 2^{l_2}}(v^h_{n_1}\overline{v^h_{n_2}}v_{n_3})\\ &\qquad\qquad -P^y_{\xi(t),\le 2^{l_2}}v^h_{n_1}\overline{P^y_{\xi(t),\le 2^{l_2}}v^h_{n_2}}P^y_{\xi(t),\le 2^{l_2}}v_{n_3}\big)\Bigg)\Bigg\|_{L^{4/3}_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\\ &\quad\lesssim\|v\|_{L^\infty_t L^2_{y,x}}\|v^h\|^2_{L^{8/3}_t L^8_y L^2_x}\lesssim\|v\|^2_{\widetilde{X}_i(G^i_\alpha)},\end{aligned}$$

it follows that

$$\|N_1\|_{L^{4/3}_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\lesssim\|v\|^2_{\widetilde{X}_i(G^i_\alpha)}, \tag{7-35}$$

and therefore, we have

$$\text{(7-23)}\lesssim 2^{2l_2-2i}\|w_0\|^2_{L^2_{y,x}(\mathbb{R}^2\times\mathbb{R})}(1+\|v\|^3_{\widetilde{X}_i(G^i_\alpha)}).$$

<u>Estimate of (7-24)</u>. Applying integration by parts, we have

$$\text{(7-24)}\lesssim\text{(7-23)}+2^{l_2-2i}\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{G^{l_2}_\beta}\iint |w(t,\tilde{y},\tilde{x})|^2\frac{1}{|y-\tilde{y}|}\Re(\bar{\tilde{w}}N)(t,y,x)\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t\right|.$$

By the Strichartz estimate, (7-35), (6-14), (6-17), Bernstein's inequality, and the conservation of mass, we have

$$
\begin{aligned}
&2^{l_2-2i}\left|\int_{G_\beta^{l_2}}\iiiint |w(t,\tilde y,\tilde x)|^2\frac{1}{|y-\tilde y|}\Re(\bar{\tilde w}N)(t,y,x)\,\mathrm{d}y\,\mathrm{d}\tilde y\,\mathrm{d}x\,\mathrm{d}\tilde x\,\mathrm{d}t\right|\\
&\lesssim 2^{l_2-2i}\|w_0\|^2_{L^2_{\tilde y,x}}2^{-\frac{l_2}{2}}\int_{G_\beta^{l_2}}|\xi'(t)|\|P^y_{\xi(t),2^{l_2-3}\le\cdots\le 2^{l_2+3}}v\|_{L^2_{\tilde y,x}}\|(\nabla_y-i\xi(t))^{\frac12}P^y_{\xi(t),\le 2^{l_2}}v\|_{L^2_{\tilde y,x}}\,\mathrm{d}t\\
&\qquad\qquad+2^{l_2-2i}\|w\|^2_{L^4_{t,y}L^2_x(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\|N_1\|_{L^{4/3}_{t,y}L^2_x(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\|\tilde w\|_{L^\infty_{t,y}L^2_x(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\\
&\lesssim 2^{2l_2-2i}\|w_0\|^2_{L^2_{\tilde y,x}}(1+\|v\|^3_{\tilde X_i(G_\alpha^i)}).
\end{aligned}
$$

Thus

$$
\text{(7-24)}\lesssim 2^{2l_2-2i}\|w_0\|^2_{L^2_{\tilde y,x}}(1+\|v\|^3_{\tilde X_i(G_\alpha^i)}).
$$

Estimate of (7-25): By Bernstein's inequality and the conservation of mass, we have

$$
\begin{aligned}
&2^{l_2-2i}\left|\int_{G_\beta^{l_2}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\Im(\bar w(\nabla_y-i\xi(t))w)(t,y,x)\frac{y-\tilde y}{|y-\tilde y|}\Im(\bar{\tilde w}N_2)(t,\tilde y,\tilde x)\,\mathrm{d}y\,\mathrm{d}\tilde y\,\mathrm{d}x\,\mathrm{d}\tilde x\,\mathrm{d}t\right|\\
&\lesssim \|w_0\|^2_{L^2_{\tilde y,x}}\Big(2^{-i-l_2}\int_{G_\beta^{l_2}}|\xi'(t)|\,\|P^y_{\xi(t),2^{l_2-3}\le\cdots\le 2^{l_2+3}}v\|_{L^2_{\tilde y,x}}\|(\nabla_y-i\xi(t))P^y_{\xi(t),\le 2^{l_2}}v\|_{L^2_{\tilde y,x}}\,\mathrm{d}t\Big)\\
&\qquad+\|w_0\|^2_{L^2_{\tilde y,x}}\Big(2^{-i-\frac{l_2}{2}}\int_{G_\beta^{l_2}}|\xi'(t)|\|P^y_{\xi(t),l_2-3\le\cdots\le l_2+3}v\|_{L^2_{\tilde y,x}}\|(\nabla_y-i\xi(t))^{\frac12}P^y_{\xi(t),\le 2^{l_2}}v\|_{L^2_{\tilde y,x}}\,\mathrm{d}t\Big)\\
&\lesssim \|w_0\|^2_{L^2_{\tilde y,x}}.
\end{aligned}
$$

We now turn to the estimate of

$$
2^{l_2-2i}\left|\int_{G_\beta^{l_2}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\Im(\bar w(\nabla_y-i\xi(t))w)(t,y,x)\frac{y-\tilde y}{|y-\tilde y|}\Im(\bar{\tilde w}N_1)(t,\tilde y,\tilde x)\,\mathrm{d}y\,\mathrm{d}\tilde y\,\mathrm{d}x\,\mathrm{d}\tilde x\,\mathrm{d}t\right|.
$$

Since

$$
\int_{\mathbb{R}}\Im\Big(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\bar{\tilde w}\Pi_n(\tilde w_{n_1}\overline{\tilde w_{n_2}}\tilde w_{n_3})\Big)(\tilde x)\,\mathrm{d}\tilde x=0, \tag{7-36}
$$

we see

$$
\int_{\mathbb{R}}\sum_{n\in\mathbb{N}}\Im(\overline{\tilde w_n}N_{1,n})(\tilde x)\,\mathrm{d}\tilde x=\int_{\mathbb{R}}\Im\Big(\sum_{n\in\mathbb{N}}\overline{\tilde w_n}P^y_{\xi(t),\le 2^{l_2}}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v_{n_1}\overline{v_{n_2}}v_{n_3})\Big)(\tilde x)\,\mathrm{d}\tilde x.
$$

Using the decomposition $v=v^h+v^l$, where $v^l=P^y_{\xi(t),\le 2^{l_2-5}}v$, together with the above equality, we have

$$
\int_{\mathbb{R}}\sum_{n\in\mathbb{N}}\Im(\overline{\tilde w_n}N_{1,n})(\tilde x)=\int_{\mathbb{R}}\sum_{n\in\mathbb{N}}(F_{0,n}+F_{1,n}+F_{2,n}+F_{3,n}+F_{4,n})(\tilde x)\,\mathrm{d}\tilde x, \tag{7-37}
$$

where $F_{j,n}$ consists of j v_n^h-terms and $4-j$ v_n^l-terms, for $j=0,1,2,3,4$, in

$$\Im\Big(\overline{w_n}P^y_{\xi(t),\le 2^{l_2}}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v_{n_1}\overline{v_{n_2}}v_{n_3})\Big).$$

We now consider the estimate of the F_j terms, $j=0,1,2,3,4$, as follows.

By (7-36), we have

$$\int_{\mathbb{R}}\sum_{n\in\mathbb{N}}F_{0,n}(t,\tilde{y},\tilde{x})\,\mathrm{d}\tilde{x}=0.$$

By Bernstein's inequality, (6-1) and Lemma 6.20, we have

$$\begin{aligned}2^{l_2-2i}\bigg|\int_{G_\beta^{l_2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Im(\bar{w}(\nabla_y-i\xi(t))w)(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}(F_3+F_4)(t,\tilde{y},\tilde{x})\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t\bigg|\\ \lesssim 2^{l_2-i}\|w_0\|^2_{L^2_{y,x}}\|v^h\|^3_{L^3_tL^6_yL^2_x(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\|v\|_{L^\infty_tL^2_{y,x}(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\\ \lesssim 2^{l_2-i}\|w_0\|^2_{L^2_{y,x}}\|v\|^3_{\widetilde{X}_i(G_\alpha^i)}.\end{aligned}$$

By a direct calculation, we have

$$\begin{aligned}\sum_{n\in\mathbb{N}}F_{1,n}=\sum_{n\in\mathbb{N}}\Im\bigg(&\overline{P^y_{\xi(t),\le 2^{l_2}}P^y_{\xi(t),\ge 2^{l_2-2}}v_n^h}\,P^y_{\xi(t),\le 2^{l_2}}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v^l_{n_1}\overline{v^l_{n_2}}v^l_{n_3})\\ &+\overline{v_n^l}P^y_{\xi(t),\le 2^{l_2}}\bigg(\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v^l_{n_1}\overline{P^y_{\xi(t),\ge 2^{l_2-2}}v^h_{n_2}}v^l_{n_3})\\ &\qquad+2\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v^l_{n_1}\overline{v^l_{n_2}}P^y_{\xi(t),\ge 2^{l_2-2}}v^h_{n_3})\bigg)\bigg).\end{aligned}\tag{7-38}$$

Since the support of the partial Fourier transform with respect to $\tilde{y}$ of $\sum_{n\in\mathbb{N}}F_{1,n}(t,\tilde{y},\tilde{x})$ is contained in $\{\xi:|\xi|\ge 2^{l_2-4}\}$, we can apply the integration by parts with respect to $\tilde{y}$, the Hardy–Littlewood–Sobolev inequality, Bernstein's inequality, the Strichartz estimate, (6-25), and (6-24) to give the following estimate:

$$\begin{aligned}&2^{l_2-2i}\bigg|\sum_{n\in\mathbb{N}}\int_{G_\beta^{l_2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Im(\overline{w_n}(\nabla_y-i\xi(t))w_n)(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}\Big(\sum_{n'\in\mathbb{N}}F_{1,n'}\Big)(t,\tilde{y},\tilde{x})\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t\bigg|\\ &\lesssim 2^{l_2-2i}\int_{G_\beta^{l_2}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\bigg|\sum_{n\in\mathbb{N}}(\overline{w_n}(\nabla_y-i\xi(t))w_n)(t,y,x)\bigg|\\ &\qquad\cdot\frac{1}{|y-\tilde{y}|}\bigg|\partial_{\tilde{y}}(-\Delta_{\tilde{y}})^{-1}\int_{\mathbb{R}}\Big(\sum_{n'\in\mathbb{N}}F_{1,n'}\Big)(t,\tilde{y},\tilde{x})\,\mathrm{d}\tilde{x}\bigg|\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}t\\ &\lesssim 2^{l_2-2i}\|w\|_{L^3_tL^6_yL^2_x(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\|(\nabla_y-i\xi(t))w\|_{L^\infty_tL^2_{y,x}(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\\ &\qquad\cdot\bigg\|\partial_{\tilde{y}}(-\Delta_{\tilde{y}})^{-1}\int_{\mathbb{R}}\Big(\sum_{n'\in\mathbb{N}}F_{1,n'}\Big)\,\mathrm{d}\tilde{x}\bigg\|_{L_t^{3/2}L_{\tilde{y}}^{6/5}(G_\beta^{l_2}\times\mathbb{R}^2)}\\ &\lesssim 2^{-i}\|w_0\|^2_{L^2_{y,x}}\|v^l\|^3_{L^9_tL^{9/2}_yL^2_x(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\|v^h\|_{L^3_tL^6_yL^2_x(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\lesssim 2^{l_2-i}\|w_0\|^2_{L^2_{y,x}}\|v\|^4_{\widetilde{X}_i(G_\alpha^i)}.\end{aligned}$$

We are now left to show

$$2^{l_2-2i}\left|\sum_{n,n'\in\mathbb{N}}\int_{G_\beta^{l_2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Im(\overline{w_n}(\nabla_y-i\xi(t))w_n)(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}F_{2,n'}(t,\tilde{y},\tilde{x})\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t\right| \lesssim \|w_0\|_{L^2_{y,x}}^2(1+\|v\|_{\tilde{X}_i(G_\alpha^i)}^4). \tag{7-39}$$

Similar to the estimate on the term involved F_1 above, from integration by parts, Bernstein's inequality and (6-25), we conclude

$$\begin{aligned}
&2^{l_2-2i}\left|\sum_{n\in\mathbb{N}}\int_{G_\beta^{l_2}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\Im(\overline{w_n}(\nabla_y-i\xi(t))w_n)(t,y,x)\right.\\
&\qquad\qquad\left.\cdot\frac{y-\tilde{y}}{|y-\tilde{y}|}\int_{\mathbb{R}}\Big(\sum_{n'\in\mathbb{N}}P^y_{\geq l_2-10}F_{2,n'}\Big)(t,\tilde{y},\tilde{x})\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}t\right|\\
&\lesssim 2^{l_2-2i}\|w\|_{L_t^3L_y^6L_x^2(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\|(\nabla_y-i(t))w\|_{L_t^\infty L^2_{y,x}(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\\
&\qquad\qquad\cdot\Big\|\partial_{\tilde{y}}(-\Delta_{\tilde{y}})^{-1}\Big(\int_{\mathbb{R}}\sum_{n'\in\mathbb{N}}P^y_{\geq l_2-10}F_{2,n'}\,\mathrm{d}\tilde{x}\Big)\Big\|_{L_t^{3/2}L_y^{6/5}(G_\beta^{l_2}\times\mathbb{R}^2)}\\
&\lesssim 2^{-i}\|w_0\|^2_{L^2_{y,x}}\|v^l\|^2_{L_t^\infty L_y^4L_x^2(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\|v^h\|^2_{L_t^3L_y^6L_x^2(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\lesssim 2^{l_2-i}\|w_0\|^2_{L^2_{y,x}}\|v\|^2_{\tilde{X}_i(G_\alpha^i)}.
\end{aligned}$$

We now turn to the estimate of the low-frequency part of F_2. First of all, we can decompose $F_{2,n'}$ as

$$\begin{aligned}
F_{2,n'}(t,\tilde{y},\tilde{x})=\Im\Bigg(&2\overline{v^l_{n'}}P^y_{\xi(t),\leq 2^{l_2}}\Bigg(\sum_{\substack{n'_1,n'_2,n'_3\in\mathbb{N}\\ n'_1-n'_2+n'_3=n'}}\Pi_{n'}(v^h_{n'_1}\overline{v^h_{n'_2}}v^l_{n'_3})\Bigg)\\
&+\overline{v^l_{n'}}P^y_{\xi(t),\leq 2^{l_2}}\Bigg(\sum_{\substack{n'_1,n'_2,n'_3\in\mathbb{N}\\ n'_1-n'_2+n'_3=n'}}\Pi_{n'}(v^h_{n'_1}\overline{v^l_{n'_2}}v^h_{n'_3})\Bigg)\\
&+2\overline{P^y_{\xi(t),\leq 2^{l_2}}v^h_{n'}}P^y_{\xi(t),\leq 2^{l_2}}\sum_{\substack{n'_1,n'_2,n'_3\in\mathbb{N}\\ n'_1-n'_2+n'_3=n'}}\Pi_{n'}(v^l_{n'_1}\overline{v^l_{n'_2}}v^h_{n'_3})\\
&+\overline{P^y_{\xi(t),\leq 2^{l_2}}v^h_{n'}}P^y_{\xi(t),\leq 2^{l_2}}\sum_{\substack{n'_1,n'_2,n'_3\in\mathbb{N}\\ n'_1-n'_2+n'_3=n'}}\Pi_{n'}(v^l_{n'_1}\overline{v^h_{n'_2}}v^l_{n'_3})\Bigg).
\end{aligned}\tag{7-40}$$

Since

$$\Im\Bigg(\sum_{n'\in\mathbb{N}}\sum_{\substack{n'_1,n'_2,n'_3\in\mathbb{N}\\ n'_1-n'_2+n'_3=n'}}\overline{P^y_{\xi(t),\leq 2^{l_2}}v^h_{n'}}\Pi_{n'}(v^l_{n'_1}\overline{v^l_{n'_2}}P^y_{\xi(t),\leq 2^{l_2}}v^h_{n'_3})\Bigg)=0,$$

$$\Im\Bigg(\sum_{n'\in\mathbb{N}}\sum_{\substack{n'_1,n'_2,n'_3\in\mathbb{N}\\ n'_1-n'_2+n'_3=n'}}\overline{v^l_{n'}}\Pi_{n'}(P^y_{\xi(t),\leq 2^{l_2}}(v^h_{n'_1}\overline{v^h_{n'_2}})v^l_{n'_3})\Bigg)=0,$$

we obtain

$$2\Im\Big(\sum_{n'\in\mathbb{N}}\overline{v^l_{n'}}P^y_{\xi(t),\le 2^{l_2}}\Big(\sum_{\substack{n'_1,n'_2,n'_3\in\mathbb{N}\\ n'_1-n'_2+n'_3=n'}}\Pi_{n'}(v^h_{n'_1}\overline{v^h_{n'_2}}v^l_{n'_3})\Big) + \sum_{n'\in\mathbb{N}}\overline{P^y_{\xi(t),\le 2^{l_2}}v^h_{n'}}P^y_{\xi(t),\le 2^{l_2}}\sum_{\substack{n'_1,n'_2,n'_3\in\mathbb{N}\\ n'_1-n'_2+n'_3=n'}}\Pi_{n'}(v^l_{n'_1}\overline{v^l_{n'_2}}v^h_{n'_3})\Big)$$

$$= 2\Im\Big(\sum_{n'\in\mathbb{N}}\sum_{\substack{n'_1,n'_2,n'_3\in\mathbb{N}\\ n'_1-n'_2+n'_3=n'}}\big(\overline{P^y_{\xi(t),\le 2^{l_2}}v^h_{n'}}P^y_{\xi(t),\le 2^{l_2}}\Pi_{n'}(v^l_{n'_1}\overline{v^l_{n'_2}}v^h_{n'_3}) - \overline{P^y_{\xi(t),\le 2^{l_2}}v^h_{n'}}\Pi_{n'}(v^l_{n'_1}\overline{v^l_{n'_2}}P^y_{\xi(t),\le 2^{l_2}}v^h_{n'_3})\big)\Big) \tag{7-41}$$

$$+ 2\Im\Big(\sum_{n'\in\mathbb{N}}\sum_{\substack{n'_1,n'_2,n'_3\in\mathbb{N}\\ n'_1-n'_2+n'_3=n'}}\big(\overline{v^l_{n'}}P^y_{\xi(t),\le 2^{l_2}}\Pi_{n'}(v^h_{n'_1}\overline{v^h_{n'_2}}v^l_{n'_3}) - \overline{v^l_{n'}}\Pi_{n'}(P^y_{\xi(t),\le 2^{l_2}}(v^h_{n'_1}\overline{v^h_{n'_2}})v^l_{n'_3})\big)\Big). \tag{7-42}$$

For (7-41), by (7-30), (7-34), Lemma 6.20 and the conservation of mass, we have

$$\begin{aligned}
&2^{l_2-i}\Big\|\int_{\mathbb{R}}(7\text{-}41)(x)\,dx\Big\|_{L^1_{t,y}(G^{l_2}_\beta\times\mathbb{R}^2)}\\
&\lesssim 2^{l_2-i}\|v^h\|_{L^4_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\Big\|\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n P^y_{\xi(t),\le 2^{l_2}}(v^l_{n_1}\overline{v^l_{n_2}}v^h_{n_3}) - \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v^l_{n_1}\overline{v^l_{n_2}}P^y_{\xi(t),\le 2^{l_2}}v^h_{n_3})\Big\|_{L^{4/3}_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\\
&\lesssim 2^{l_2-i}\|v\|_{\widetilde{X}_i(G^i_\alpha)}\sum_{l_1\le l_2}2^{l_1-l_2}\|P^y_{\xi(t),\le 2^{l_1}}v\|_{L^\infty_t L^2_{y,x}}\|P^y_{\xi(t),2^{l_1}}v^l\|_{L^{8/3}_t L^8_y L^2_x}\|v^h\|_{L^{8/3}_t L^8_y L^2_x}\\
&\lesssim 2^{l_2-i}\|v\|^3_{\widetilde{X}_i(G^i_\alpha)}.
\end{aligned} \tag{7-43}$$

To estimate (7-42). We note similar to (7-33), we have

$$\Big|\phi\Big(\frac{\xi_1+\xi_2-\xi(t)}{2^{l_2}}\Big)-\phi\Big(\frac{\xi_1}{2^{l_2}}\Big)\Big|\lesssim 2^{-l_2}|\xi_2-\xi(t)|.$$

Then by Lemma 6.20 and the conservation of mass, we have

$$\begin{aligned}
&2^{l_2-i}\Big\|\int_{\mathbb{R}}(7\text{-}42)\,dx\Big\|_{L^1_{t,y}(G^{l_2}_\beta\times\mathbb{R}^2)}\\
&\lesssim 2^{l_2-i}\|v^l\|_{L^\infty_t L^2_{y,x}(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\Big\|\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}P^y_{\xi(t),\le 2^{l_2}}\Pi_n(v^h_{n_1}\overline{v^h_{n_2}}v^l_{n_3}) - \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(P^y_{\xi(t),\le 2^{l_2}}(v^h_{n_1}\overline{v^h_{n_2}})v^l_{n_3})\Big\|_{L^1_t L^2_{y,x}(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\\
&\lesssim 2^{-i}\|v^h\|^2_{L^3_t L^6_y L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\sum_{l_1\le l_2}2^{l_1}\|P^y_{\xi(t),2^{l_1}}v^l\|_{L^3_t L^6_y L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\lesssim 2^{l_2-i}\|v\|^3_{\widetilde{X}_i(G^i_\alpha)}.
\end{aligned}$$

Now we turn to the remaining terms in (7-40). Observe that

$$\Im\Bigg(\sum_{n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}\overline{v_{n'}^l}P^y_{\xi(t),\le 2^{l_2}}\Pi_{n'}(v_{n_1'}^h\overline{v_{n_2'}^l}v_{n_3'}^h)+\sum_{n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}\overline{P^y_{\xi(t),\le 2^{l_2}}v_{n'}^h}P^y_{\xi(t),\le 2^{l_2}}\Pi_{n'}(v_{n_1'}^l\overline{v_{n_2'}^h}v_{n_3'}^l)\Bigg)$$

$$=\sum_{n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}\Im\big(\overline{P^y_{\xi(t),\le 2^{l_2}}v_{n'}^h}P^y_{\xi(t),\le 2^{l_2}}\Pi_{n'}(v_{n_1'}^l\overline{v_{n_2'}^h}v_{n_3'}^l)-\overline{P^y_{\xi(t),\le 2^{l_2}}v_{n'}^h}\Pi_{n'}(v_{n_1'}^l\overline{P^y_{\xi(t),\le 2^{l_2}}v_{n_2'}^h}v_{n_3'}^l)\big) \tag{7-44}$$

$$+\sum_{n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}\Im\big(\overline{v_{n'}^l}P^y_{\xi(t),\le 2^{l_2}}\Pi_{n'}(v_{n_1'}^h\overline{v_{n_2'}^l}v_{n_3'}^l)+\overline{P^y_{\xi(t),\le 2^{l_2}}v_{n'}^h}\Pi_{n'}(v_{n_1'}^l P^y_{\xi(t),\le 2^{l_2}}\overline{v_{n_2'}^h}v_{n_3'}^l)\big). \tag{7-45}$$

Similar to the arguments for (7-43), we have

$$2^{l_2-i}\Big\|\int_{\mathbb{R}}(7\text{-}44)\,\mathrm{d}x\Big\|_{L^1_{t,y}(G_\beta^{l_2}\times\mathbb{R}^2)}\lesssim 2^{l_2-i}\|v\|^3_{\tilde{X}_i(G_\alpha^i)}.$$

Thus, to show (7-39), we just need to consider the term that contains (7-45). By direct calculation, we get

$$\begin{aligned}
&2^{l_2-2i}\int_{G_\beta^{l_2}}\iiint\sum_{n\in\mathbb{N}}\Im(\overline{w_n}(\nabla_y-i\xi(t))w_n)(t,y,x)\\
&\quad\cdot\frac{y-\tilde y}{|y-\tilde y|}\sum_{n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}P^y_{\xi(t),\le 2^{l_2-10}}\Im\big(\overline{v_{n'}^l}P^y_{\xi(t),\le 2^{l_2}}\Pi_{n'}(v_{n_1'}^h\overline{v_{n_2'}^l}v_{n_3'}^h)\\
&\qquad+\overline{P^y_{\xi(t),\le 2^{l_2}}v_{n'}^h}\Pi_{n'}(v_{n_1'}^l\overline{P^y_{\xi(t),\le 2^{l_2}}v_{n_2'}^h}v_{n_3'}^l)\big)(t,\tilde y,\tilde x)\,\mathrm{d}y\,\mathrm{d}\tilde y\,\mathrm{d}x\,\mathrm{d}\tilde x\,\mathrm{d}t\\
&=2^{l_2-2i}\int_{G_\beta^{l_2}}\iint\sum_{n\in\mathbb{N}}\Im(\overline{w_n}(\nabla_y-i\xi(t))w_n)(t,y+2\xi(G_\beta^{l_2})t,x)\frac{y-\tilde y}{|y-\tilde y|}\sum_{n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}P^y_{\xi(t),\le 2^{l_2-10}}\\
&\quad\cdot\Im\big(\overline{v_{n_1'}^l}\,\overline{v_{n_3'}^l}(v_{n'}^h v_{n_2'}^h-P^y_{\xi(t),\le 2^{l_2}}v_{n'}^h P^y_{\xi(t),\le 2^{l_2}}v_{n_2'}^h)\big)(t,\tilde y+2\xi(G_\beta^{l_2})t,\tilde x)\,\mathrm{d}x\,\mathrm{d}\tilde x\,\mathrm{d}y\,\mathrm{d}\tilde y\,\mathrm{d}t.
\end{aligned}\tag{7-46}$$

We may take $\xi(G_\beta^{l_2})=0$ in the right-hand side of the above equality by the invariance of the Galilean transformation. By the inverse Fourier transform, we have

$$\begin{aligned}
&\sum_{n,n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}\int_{G_\beta^{l_2}}\iiiint\Im(\overline{w_n}(\nabla_y-i\xi(t))w_n)(t,y,x)\frac{y-\tilde y}{|y-\tilde y|}\\
&\quad\cdot P^y_{\le 2^{l_2-10}}\Im\big((\overline{v_{n_1'}}v_{n_3'}^l)(v_{n'}^h v_{n_2'}^h-P^y_{\xi(t),\le 2^{l_2}}v_{n'}^h P^y_{\xi(t),\le 2^{l_2}}v_{n_2'}^h)\big)(t,\tilde y,\tilde x)\,\mathrm{d}y\,\mathrm{d}\tilde y\,\mathrm{d}x\,\mathrm{d}\tilde x\,\mathrm{d}t
\end{aligned}$$

$$= \sum_{n,n'\in\mathbb{N}} \sum_{\substack{n_1',n_2',n_3'\in\mathbb{N},\\ n_1'-n_2'+n_3'=n'}} \int_{G_\beta^{l_2}} \iiiint \Im(\overline{w_n}(\nabla_y - i\xi(t))w_n)(t,y,x) \cdot \frac{y-\tilde y}{|y-\tilde y|}\Big(\iiint \phi\Big(\frac{\eta_1+\eta_2+\eta_3+\eta_4}{2^{l_2-10}}\Big) e^{i\tilde y(\eta_1+\eta_2+\eta_3+\eta_4)} \cdot \Pi_{n'}\big((\mathcal{F}_{\tilde y} v^l_{n_1'})(t,\eta_3,\tilde x)(\mathcal{F}_{\tilde y} v^l_{n_3'})(t,\eta_4,\tilde x)(\overline{\mathcal{F}_{\tilde y} v^h_{n'}})(t,\eta_1,\tilde x)(\overline{\mathcal{F}_{\tilde y} v^h_{n_2'}})(t,\eta_2,\tilde x)\big) \cdot \Big(1-\phi\Big(\frac{\eta_1-\xi(t)}{2^{l_2}}\Big)\phi\Big(\frac{\eta_2-\xi(t)}{2^{l_2}}\Big)\Big)\, d\eta_1\, d\eta_2\, d\eta_3\, d\eta_4\Big)\, dy\, d\tilde y\, dx\, d\tilde x\, dt.$$

Let

$$q(\eta) = |\eta_1|^2 + |\eta_2|^2 - |\eta_3|^2 - |\eta_4|^2,$$

as in [Dodson 2016b], we have $1/q(\eta)$ is a convergent sum of terms with operator norm being dominated by $1/(|\eta_1|^2+|\eta_2|^2) \sim 1/(|\eta_1||\eta_2|)$ on the support of

$$\Big(1-\phi\Big(\frac{\eta_1-\xi(t)}{2^{l_2}}\Big)\phi\Big(\frac{\eta_2-\xi(t)}{2^{l_2}}\Big)\Big)\phi\Big(\frac{\eta_3}{2^{l_2-4}}\Big)\phi\Big(\frac{\eta_4}{2^{l_2-4}}\Big).$$

Let $G_\beta^{l_2} = [t_0, t_1]$. Applying integration by parts (with respect to time), we have

$$\int_{G_\beta^{l_2}} \iiiiiiiint \frac{1}{iq(\eta)}\Big(\frac{d}{dt} e^{itq(\eta)}\Big)\Im(\overline{w_n}(\nabla_y - i\xi(t))w_n)(t,y,x)\frac{y-\tilde y}{|y-\tilde y|}\phi\Big(\frac{\eta_0+\eta_1+\eta_2+\eta_3}{2^{l_2-10}}\Big) \cdot e^{i\tilde y(\eta_1+\eta_2+\eta_3+\eta_0)}\Big(1-\phi\Big(\frac{\eta_0-\xi(t)}{2^{l_2}}\Big)\phi\Big(\frac{\eta_2-\xi(t)}{2^{l_2}}\Big)\Big) \cdot \big(e^{-it|\eta_0|^2}(\overline{\mathcal{F}_{\tilde y} v^h_{n'}})(t,\eta_0,\tilde x)e^{-it|\eta_2|^2}(\overline{\mathcal{F}_{\tilde y} v^h_{n_2'}})(t,\eta_2,\tilde x)e^{it|\eta_1|^2}(\mathcal{F}_{\tilde y} v^l_{n_1'})(t,\eta_1,\tilde x)e^{it|\eta_3|^2} \cdot (\mathcal{F}_{\tilde y} v^l_{n_3'})(t,\eta_3,\tilde x)\, d\eta_1\, d\eta_2\, d\eta_3\, d\eta_0\big)\, dy\, d\tilde y\, dx\, d\tilde x\, dt$$

$$:= B_1 + B_2 + B_3 + B_4, \tag{7-47}$$

where

$$B_1 = \iiiiiiiint \frac{1}{iq(\eta)} e^{itq(\eta)}\Im(\overline{w_n}(\nabla_y - i\xi(t))w_n)(t,y,x)\frac{y-\tilde y}{|y-\tilde y|}\phi\Big(\frac{\eta_1+\eta_2+\eta_3+\eta_0}{2^{l_2-10}}\Big)e^{i\tilde y(\eta_1+\eta_2+\eta_3+\eta_0)} \cdot\Big(1-\phi\Big(\frac{\eta_0-\xi(t)}{2^{l_2}}\Big)\phi\Big(\frac{\eta_2-\xi(t)}{2^{l_2}}\Big)\Big)\big(e^{-it|\eta_0|^2}(\overline{\mathcal{F}_{\tilde y} v^h_{n'}})(t,\eta_0,\tilde x)e^{-it|\eta_2|^2}(\overline{\mathcal{F}_{\tilde y} v^h_{n_2'}})(t,\eta_2,\tilde x) \cdot e^{it|\eta_1|^2}(\mathcal{F}_{\tilde y} v^l_{n_1'})(t,\eta_1,\tilde x)e^{it|\eta_3|^2}(\mathcal{F}_{\tilde y} v^l_{n_3'})(t,\eta_3,\tilde x)\, d\eta_0\, d\eta_1\, d\eta_2\, d\eta_3\big)\, dy\, d\tilde y\, dx\, d\tilde x\Big|_{t_0}^{t_1}, \tag{7-48}$$

$$B_2 = -\int_{t_0}^{t_1} \iiiiiiiint \frac{1}{iq(\eta)} e^{itq(\eta)}\frac{\partial}{\partial t}\Im(\overline{w_n}(\nabla_y - i\xi(t))w_n)(t,y,x)\frac{y-\tilde y}{|y-\tilde y|}\phi\Big(\frac{\eta_1+\eta_2+\eta_3+\eta_0}{2^{l_2-10}}\Big) \cdot e^{i\tilde y(\eta_1+\eta_2+\eta_3+\eta_0)}\Big(1-\phi\Big(\frac{\eta_0-\xi(t)}{2^{l_2}}\Big)\phi\Big(\frac{\eta_2-\xi(t)}{2^{l_2}}\Big)\Big) \cdot\big(e^{-it|\eta_0|^2}(\overline{\mathcal{F}_{\tilde y} v^h_{n'}})(t,\eta_0,\tilde x)e^{-it|\eta_2|^2}(\overline{\mathcal{F}_{\tilde y} v^h_{n_2'}})(t,\eta_2,\tilde x)e^{it|\eta_1|^2}(\mathcal{F}_{\tilde y} v^l_{n_1'})(t,\eta_1,\tilde x)e^{it|\eta_3|^2} \cdot(\mathcal{F}_{\tilde y} v^l_{n_3'})(t,\eta_3,\tilde x)\, d\eta_1\, d\eta_2\, d\eta_3\, d\eta_0\big)\, dy\, d\tilde y\, dx\, d\tilde x\, dt, \tag{7-49}$$

$$B_3 = -\int_{t_0}^{t_1} \iiiint\!\!\!\iiiint \frac{1}{iq(\eta)} e^{itq(\eta)} \Im(\overline{w_n}(\nabla_y - i\xi(t))w_n)(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|} \cdot \frac{\partial}{\partial t}\left(\phi\left(\frac{\eta_1+\eta_2+\eta_3+\eta_0}{2^{l_2-10}}\right)e^{i\tilde{y}(\eta_1+\eta_2+\eta_3+\eta_0)}\left(1-\phi\left(\frac{\eta_0-\xi(t)}{2^{l_2}}\right)\phi\left(\frac{\eta_2-\xi(t)}{2^{l_2}}\right)\right)\right) \cdot \left(e^{-it|\eta_0|^2}\overline{(\mathcal{F}_{\tilde{y}}v^h_{n'})}(t,\eta_0,\tilde{x})e^{-it|\eta_2|^2}\overline{(\mathcal{F}_{\tilde{y}}v^h_{n'_2})}(t,\eta_2,\tilde{x})e^{it|\eta_1|^2}(\mathcal{F}_{\tilde{y}}v^l_{n'_1})(t,\eta_1,\tilde{x}) \cdot e^{it|\eta_3|^2}(\mathcal{F}_{\tilde{y}}v^l_{n'_3})(t,\eta_3,\tilde{x})\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3\,\mathrm{d}\eta_0\right)\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t, \tag{7-50}$$

$$B_4 = -\int_{t_0}^{t_1} \iiiint\!\!\!\iiiint \frac{1}{iq(\eta)} e^{itq(\eta)} \Im(\overline{w_n}(\nabla_y - i\xi(t))w_n)(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}\phi\left(\frac{\eta_1+\eta_2+\eta_3+\eta_0}{2^{l_2-10}}\right) \cdot e^{i\tilde{y}(\eta_1+\eta_2+\eta_3+\eta_0)}\left(1-\phi\left(\frac{\eta_0-\xi(t)}{2^{l_2}}\right)\phi\left(\frac{\eta_2-\xi(t)}{2^{l_2}}\right)\right) \cdot \frac{\partial}{\partial t}\left(e^{-it|\eta_0|^2}\overline{(\mathcal{F}_{\tilde{y}}v^h_{n'})}(t,\eta_0,\tilde{x})e^{-it|\eta_2|^2}\overline{(\mathcal{F}_{\tilde{y}}v^h_{n'_2})}(t,\eta_2,\tilde{x}) \cdot e^{it|\eta_1|^2}(\mathcal{F}_{\tilde{y}}v^l_{n'_1})(t,\eta_1,\tilde{x})e^{it|\eta_3|^2}(\mathcal{F}_{\tilde{y}}v^l_{n'_3})(t,\eta_3,\tilde{x})\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3\,\mathrm{d}\eta_0\right)\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t. \tag{7-51}$$

For (7-48), set

$$m(t;\eta_0,\eta_1,\eta_2,\eta_3) = \frac{1}{q(\eta)}\phi\left(\frac{\eta_1+\eta_2+\eta_3+\eta_0}{2^{l_2-10}}\right)\left(1-\phi\left(\frac{\eta_0-\xi(t)}{2^{l_2}}\right)\phi\left(\frac{\eta_2-\xi(t)}{2^{l_2}}\right)\right).$$

Then we have

$$(7\text{-}48) = -i \iiiint\!\!\!\iiiint \Im(\overline{w_n}(\nabla_y - i\xi(t))w_n)(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}K(t;z_0,z_1,z_2,z_3)\overline{v^h_{n'}(t,\tilde{y}-z_0,\tilde{x})} \cdot \overline{v^h_{n'_2}(t,\tilde{y}-z_2,\tilde{x})}v^l_{n'_1}(t,\tilde{y}-z_1,\tilde{x})v^l_{n'_3}(t,\tilde{y}-z_3,\tilde{x})\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_0\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}y\,\mathrm{d}\tilde{y}\Big|_{t_0}^{t_1},$$

where

$$K(t;z_0,z_1,z_2,z_3) = \iiiint m(t;\eta_0,\eta_1,\eta_2,\eta_3)e^{iz_1\eta_1}e^{iz_2\eta_2}e^{iz_3\eta_3}e^{iz_0\eta_0}\,\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3\mathrm{d}\eta_0, \tag{7-52}$$

which satisfies

$$\sup_t \int |K(t;z_0,z_1,z_2,z_3)|\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_0 \lesssim 2^{-2l_2}, \tag{7-53}$$

by the Coifman–Meyer theorem [Germain et al. 2012]. Thus, by Bernstein's inequality, (7-53) and the conservation of mass, we have

$$2^{l_2-2i}\left|\sum_{n,n'\in\mathbb{N}}\ \sum_{\substack{n'_1,n'_2,n'_3\in\mathbb{N}\\ n'_1-n'_2+n'_3=n'}} (7\text{-}48)\right|$$
$$\lesssim 2^{l_2-2i}\|w\|_{L^\infty_t L^2_{y,x}}\|(\nabla_y - i\xi(t))w\|_{L^\infty_t L^2_{y,x}} \cdot \iiiint |K(t;z_0,z_1,z_2,z_3)|\|v^h(t,\tilde{y}-z_0,\tilde{x})\|_{L^2_{\tilde{x}}}\|v^h(t,\tilde{y}-z_2,\tilde{x})\|_{L^2_{\tilde{x}}} \cdot \|v^l(t,\tilde{y}-z_1,\tilde{x})\|_{L^2_{\tilde{x}}}\|v^l(t,\tilde{y}-z_3,\tilde{x})\|_{L^2_{\tilde{x}}}\,\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_0\mathrm{d}\tilde{y}$$
$$\lesssim 2^{l_2-i}\|w_0\|^2_{L^2_{y,x}}2^{-2l_2}\|v^h\|^2_{L^\infty_t L^2_{y,x}}\|v^l\|^2_{L^\infty_{t,y}L^2_x} \lesssim \|w_0\|^2_{L^2_{y,x}}.$$

Next, we turn to the estimate of (7-49). By a direct computation, we have

$$\begin{aligned}\frac{\partial}{\partial t}\Im\int_{\mathbb{R}}(\overline{w_n}(\partial_{y_k}-i\xi_k(t))w_n)(t,y,x)\,\mathrm{d}x\\ =\xi_k'(t)\|w_n(t,y,x)\|_{L_x^2}^2+\sum_{k'=1}^{2}\partial_{y_{k'}}\Re\int_{\mathbb{R}}(\overline{w_n}(\partial_{y_k}-i\xi_k(t))\partial_{y_{k'}}w_n)(t,y,x)\,\mathrm{d}x\\ -\sum_{k'=1}^{2}\partial_{y_{k'}}\Re\int_{\mathbb{R}}(\overline{\partial_{y_{k'}}w_n}(\partial_{y_k}-i\xi_k(t))w_n)(t,y,x)\,\mathrm{d}x.\end{aligned}$$

Thus, we get

$$\begin{aligned}&(7\text{-}49)=\\ &\int_{G_\beta^{l_2}}\iiiint\!\!\!\iiiint\frac{y-\tilde y}{|y-\tilde y|}\xi'(t)\|w_n(t,y,x)\|_{L_x^2}^2K(t;z_0,z_1,z_2,z_3)\overline{v_{n'}^h(t,\tilde y-z_0,\tilde x)}\,\overline{v_{n_2'}^h(t,\tilde y-z_2,\tilde x)}\\ &\qquad\cdot v_{n_1'}^l(t,\tilde y-z_1,\tilde x)v_{n_3'}^l(t,\tilde y-z_3,\tilde x)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_0\,\mathrm{d}y\,\mathrm{d}\tilde y\,\mathrm{d}x\mathrm{d}\tilde x\,\mathrm{d}t \quad (7\text{-}54)\\ &+\sum_{k,k'=1}^{2}\int_{G_\beta^{l_2}}\iiiint\!\!\!\iiiint\frac{(y-\tilde y)_k}{|y-\tilde y|}\partial_{y_{k'}}\Re(\overline{w_n}(\partial_{y_k}-i\xi_k(t))\partial_{y_{k'}}w_n)(t,y,x)\\ &\qquad\cdot K(t;z_0,z_1,z_2,z_3)\overline{v_{n'}^h(t,\tilde y-z_0,\tilde x)}\,\overline{v_{n_2'}^h(t,\tilde y-z_2,\tilde x)}\\ &\qquad\cdot v_{n_1'}^l(t,\tilde y-z_1,\tilde x)v_{n_3'}^l(t,\tilde y-z_3,\tilde x)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_0\,\mathrm{d}y\,\mathrm{d}\tilde y\,\mathrm{d}x\,\mathrm{d}\tilde x\,\mathrm{d}t \quad (7\text{-}55)\\ &-\sum_{k,k'=1}^{2}\int_{G_\beta^{l_2}}\iiiint\!\!\!\iiiint\frac{(y-\tilde y)_k}{|y-\tilde y|}\partial_{y_{k'}}\Re(\partial_{y_{k'}}\bar w_n(\partial_{y_k}-i\xi_k(t))w_n)(t,y,x)\\ &\qquad\cdot K(t;z_0,z_1,z_2,z_3)\overline{v_{n'}^h(t,\tilde y-z_0,\tilde x)}\,\overline{v_{n_2'}^h(t,\tilde y-z_2,\tilde x)}\\ &\qquad\cdot v_{n_1'}^l(t,\tilde y-z_1,\tilde x)v_{n_3'}^l(t,\tilde y-z_3,\tilde x)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_0\,\mathrm{d}y\,\mathrm{d}\tilde y\,\mathrm{d}x\,\mathrm{d}\tilde x\,\mathrm{d}t, \quad (7\text{-}56)\end{aligned}$$

where $K(t;z_0,z_1,z_2,z_3)$ is given in (7-52).

By (7-53), (6-14), (6-19), Bernstein's inequality and the conservation of mass, we have

$$\begin{aligned}2^{l_2-2i}\Bigg|\sum_{n,n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}(7\text{-}54)\Bigg|&\lesssim 2^{l_2-2i}2^{-2l_2}\|w\|_{L_t^\infty L_{y,x}^2}^2\|v^h\|_{L_t^\infty L_{y,x}^2}^2\|v^l\|_{L_{t,y}^\infty L_x^2}^2\bigg(\int_{G_\beta^{l_2}}|\xi'(t)|\,\mathrm{d}t\bigg)\\ &\lesssim\|w_0\|_{L_{y,x}^2}^2.\end{aligned}$$

Integrating (7-55) by parts in space, we derive

$$\begin{aligned}(7\text{-}55)=-\sum_{k,k'=1}^{2}\int_{G_\beta^{l_2}}\iiiint\!\!\!\iiiint\bigg(\frac{\delta_{kk'}}{|y-\tilde y|}+\frac{(y-\tilde y)_{k'}(y-\tilde y)_k}{|y-\tilde y|^3}\bigg)\Re(\overline{w_n}(\partial_{y_k}-i\xi_k(t))\partial_{y_{k'}}w_n)(t,y,x)\\ \cdot K(t;z_0,z_1,z_2,z_3)\overline{v_{n'}^h(t,\tilde y-z_0,\tilde x)}\,\overline{v_{n_2'}^h(t,\tilde y-z_2,\tilde x)}\\ \cdot v_{n_1'}^l(t,\tilde y-z_1,\tilde x)v_{n_3'}^l(t,\tilde y-z_4,\tilde x)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_0\,\mathrm{d}x\,\mathrm{d}\tilde x\,\mathrm{d}y\,\mathrm{d}\tilde y\,\mathrm{d}t.\end{aligned}$$

Therefore, by the Hardy–Littlewood–Sobolev inequality, (7-53), Lemma 6.20, the Sobolev embedding theorem, the fact $G_\beta^{l_2} \subseteq G_\alpha^i$, $|\xi(t)| \ll 2^{l_2}$ and $l_2 \le i$, we have

$$2^{l_2-2i}\Bigg|\sum_{n,n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}(7\text{-}55)\Bigg| \lesssim 2^{l_2-2i}2^{2i}2^{-2l_2}\|w\|^2_{L_t^6L_y^3L_x^2}\|v^h\|^2_{L_t^3L_y^6L_x^2}\|v^l\|^2_{L_t^\infty L_y^4L_x^2} \lesssim \|w_0\|^2_{L^2_{y,x}}\|v\|^2_{\widetilde{X}_i(G_\alpha^i)}.$$

By a similar argument, we infer

$$2^{l_2-2i}\Bigg|\sum_{n,n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}(7\text{-}56)\Bigg| \lesssim \|w_0\|^2_{L^2_{y,x}}\|v\|^2_{\widetilde{X}_i(G_\alpha^i)}.$$

Now we turn to (7-50). As (7-48), we have the corresponding integral kernel

$$\widetilde{K}(t;z_0,z_1,z_2,z_3) = \iiiint \tilde{m}(t;\eta_0,\eta_1,\eta_2,\eta_3)e^{iz_1\eta_1}e^{iz_2\eta_2}e^{i\eta_3z_3}e^{iz_0\eta_0}\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3\,\mathrm{d}\eta_0,$$

where

$$\tilde{m}(t;\eta_0,\eta_1,\eta_2,\eta_3) = -\frac{2^{-l_2}}{q(\eta)}\phi\left(\frac{\eta_1+\eta_2+\eta_3+\eta_0}{2^{l_2-10}}\right) \cdot\left((\nabla\phi)\left(\frac{\eta_0-\xi(t)}{2^{l_2}}\right)\phi\left(\frac{\eta_2-\xi(t)}{2^{l_2}}\right)+\phi\left(\frac{\eta_0-\xi(t)}{2^{l_2}}\right)(\nabla\phi)\left(\frac{\eta_2-\xi(t)}{2^{l_2}}\right)\right).$$

The kernel function $\widetilde{K}(t;z_0,z_1,z_2,z_3)$ satisfies

$$\sup_t\int|\widetilde{K}(t;z_0,z_1,z_2,z_3)|\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_0 \lesssim 2^{-3l_2}.$$

Thus

$$2^{l_2-2i}\Bigg|\sum_{n,n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}(7\text{-}50)\Bigg| \lesssim 2^{-2l_2-i}\|w\|^2_{L_t^\infty L^2_{y,x}}\|v^h\|^2_{L_t^\infty L^2_{y,x}}\|v^l\|^2_{L^\infty_{t,y}L_x^2}\left(\int_{G_\beta^{l_2}}|\xi'(t)|\,\mathrm{d}t\right) \lesssim \|w_0\|^2_{L^2_{y,x}}.$$

Finally, we consider the term (7-51). Following the argument for the estimates (7-48) and (7-50), by the Bernstein inequality, the conservation of mass and Lemma 6.20, we deduce

$$2^{l_2-2i}\Bigg|\sum_{n,n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}(7\text{-}51)\Bigg| \lesssim 2^{l_2-i}\|w_0\|^2_{L^2_{y,x}}(1+\|v\|^4_{\widetilde{X}_i(G_\alpha^i)}).$$

Therefore, we eventually arrive at

$$(7\text{-}25) \lesssim \|w_0\|^2_{L^2_{y,x}}(1+\|v\|^4_{\widetilde{X}_i(G_\alpha^i)}).$$

The proof of Theorem 7.5 is complete. □

7.2. ***Proof of Theorem 6.25.*** By Theorem 6.21, we have

$$\|v_\lambda\|_{\widetilde{X}_{k_0}([0,\lambda^{-2}T])} \lesssim 1, \tag{7-57}$$

where

$$v_\lambda(t,y,x) = \lambda v(\lambda^2 t, \lambda y, x), \quad \text{with } \lambda = \frac{\epsilon_3 2^{k_0}}{K}. \tag{7-58}$$

Let $\tilde{w} = P^y_{\le 2^{k_0}} v_\lambda$. Then $\tilde{w}$ satisfies

$$i\partial_t \tilde{w} + \Delta_y \tilde{w} = F(\tilde{w}) + N,$$

where

$$N = P^y_{\le 2^{k_0}} F(v_\lambda).$$

Let

$$M(t) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\tilde{w}(t,\tilde{y},\tilde{x})|^2 \frac{y-\tilde{y}}{|y-\tilde{y}|} \Im(\bar{\tilde{w}}\nabla_y \tilde{w})(t,y,x)\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}.$$

Then a direct calculation similar to [Dodson 2012; 2016a; 2016b; 2009] gives

$$\left\| \int_{\mathbb{R}} |\nabla_y|^{\frac{1}{2}} (|\tilde{w}(t,y,x)|^2)\,\mathrm{d}x \right\|^2_{L^2_{t,y}([0,\lambda^{-2}T]\times\mathbb{R}^2)} \lesssim \sup_{t\in[0,\lambda^{-2}T]} |M(t)| + \mathscr{E},$$

where

$$\mathscr{E} =$$

$$2\left|\int_0^{\lambda^{-2}T}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Im(\bar{\tilde{w}}(\nabla_y - i\xi(t))\tilde{w})(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}\Im(\bar{\tilde{w}}N)(t,\tilde{y},\tilde{x})\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t\right| \tag{7-59}$$

$$+\left|\int_0^{\lambda^{-2}T}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\tilde{w}(t,\tilde{y},\tilde{x})|^2\frac{y-\tilde{y}}{|y-\tilde{y}|}\Im(\bar{N}(\nabla_y - i\xi(t))\tilde{w})(t,y,x)\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t\right| \tag{7-60}$$

$$+\left|\int_0^{\lambda^{-2}T}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\tilde{w}(t,\tilde{y},\tilde{x})|^2\frac{y-\tilde{y}}{|y-\tilde{y}|}\Im(\bar{\tilde{w}}(\nabla_y - i\xi(t))N)(t,y,x)\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t\right|. \tag{7-61}$$

Since $N(t) \le 1$, we have $N_\lambda(t) \le \epsilon_3 2^{k_0}/K$. By Theorem 6.6 and the Bernstein inequality, for any $\eta > 0$, if $K \ge C(\eta)$, we have

$$\|(\nabla_y - i\xi(t))\tilde{w}\|_{L^\infty_t L^2_{y,x}([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})} \lesssim \eta 2^{k_0}. \tag{7-62}$$

Therefore, by the Galilean transformation and the conservation of mass, we get

$$\sup_{t\in[0,\lambda^{-2}T]} |M(t)| \lesssim \eta 2^{k_0}.$$

We now consider (7-59). As in (7-37), let $v^l_\lambda = P^y_{\le 2^{k_0-3}} v_\lambda$ and $v^h_\lambda = P^y_{>2^{k_0-3}} v_\lambda$. Then we have the decomposition

$$\int_{\mathbb{R}} \Im(\bar{\tilde{w}}N)(t,\tilde{y},\tilde{x})\,\mathrm{d}\tilde{x} = \int_{\mathbb{R}} F_0(t,\tilde{y},\tilde{x}) + F_1(t,\tilde{y},\tilde{x}) + F_2(t,\tilde{y},\tilde{x}) + F_3(t,\tilde{y},\tilde{x}) + F_4(t,\tilde{y},\tilde{x})\,\mathrm{d}\tilde{x}.$$

We can see

$$\int_{\mathbb{R}} F_0(t,\tilde{y},\tilde{x})\,\mathrm{d}\tilde{x} = 0.$$

Following the same argument as the proof of (7-25), we may obtain

$$\int_{\mathbb{R}} \|F_2(t,\tilde{y},\tilde{x})+F_3(t,\tilde{y},\tilde{x})+F_4(t,\tilde{y},\tilde{x})\|_{L^1_{t,\tilde{y}}([0,\lambda^{-2}T]\times\mathbb{R}^2)}\,d\tilde{x} \lesssim 1.$$

Then by (7-62) and the conservation of mass, we have

$$\left|\int_0^{\lambda^{-2}T}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Im(\bar{\tilde{w}}(\nabla_y - i\xi(t))\tilde{w})(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}(F_2+F_3+F_4)(t,\tilde{y},\tilde{x})\,dy\,d\tilde{y}\,dx\,d\tilde{x}\,dt\right| \lesssim \eta 2^{k_0}.$$

To estimate the contribution of the term with F_1 in (7-59), we see the support of the spatial Fourier transform of $\int_{\mathbb{R}} F_1(t,\tilde{y},\tilde{x})\,d\tilde{x}$ is in $\{\xi : |\xi| \geq 2^{k_0-4}\}$ as in (7-38). Therefore, by integration by parts, the Hardy–Littlewood–Sobolev inequality, the Bernstein inequality, Lemma 6.20 and (7-57), we have

$$\begin{aligned}
&\left|\int_0^{\lambda^{-2}T}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Im(\bar{\tilde{w}}(\nabla_y - i\xi(t))\tilde{w})(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}\left(\int_{\mathbb{R}} F_1(t,\tilde{y},\tilde{x})\,d\tilde{x}\right)dy\,d\tilde{y}\,dx\,dt\right| \\
&\lesssim \int_0^{\lambda^{-2}T}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}}(\bar{\tilde{w}}(\nabla_y - i\xi(t))\tilde{w})(t,y,x)\,dx\right|\cdot\frac{1}{|y-\tilde{y}|}\cdot\left|\partial_{\tilde{y}}(-\Delta_{\tilde{y}})^{-1}\int_{\mathbb{R}} F_1(t,\tilde{y},\tilde{x})\,d\tilde{x}\right|dy\,d\tilde{y}\,dt \\
&\lesssim \|\tilde{w}\|_{L^\infty_t L^2_{\tilde{y},x}([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})}\|(\nabla_y - i\xi(t))\tilde{w}\|_{L^4_{t,y}L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})} \\
&\qquad\qquad\cdot\left\|\partial_{\tilde{y}}(-\Delta_{\tilde{y}})^{-1}\int_{\mathbb{R}} F_1(t,\tilde{y},\tilde{x})\,d\tilde{x}\right\|_{L^{4/3}_{t,y}([0,\lambda^{-2}T]\times\mathbb{R}^2)} \\
&\lesssim 2^{-k_0}\|(\nabla_y - i\xi(t))\tilde{w}\|_{L^4_{t,y}L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})}\|v^l_\lambda\|^3_{L^6_{t,y}L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})}.
\end{aligned}$$

By the Bernstein inequality, Lemma 6.20, and (7-57), we have

$$\|v^l_\lambda\|_{L^6_{t,y}L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})} \lesssim \sum_{0\leq l\leq k_0} 2^{\frac{l}{3}}2^{\frac{k_0-l}{6}} \lesssim 2^{\frac{k_0}{3}}.$$

Note that

$$\|(\nabla_y - i\xi(t))\tilde{w}\|_{L^{5/2}_t L^{10}_y L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})} \lesssim \sum_{0\leq l\leq k_0} 2^l 2^{\frac{2}{5}(k_0-l)} \lesssim 2^{k_0}. \tag{7-63}$$

Interpolating (7-63) and (7-62), we obtain

$$\|(\nabla_y - i\xi(t))\tilde{w}\|_{L^4_{t,y}L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})} \lesssim \eta^{\frac{3}{8}}2^{k_0}. \tag{7-64}$$

Thus, by the above estimates, we have

$$\text{(7-59)} \lesssim \eta^{\frac{3}{8}}2^{k_0}.$$

Now, we turn to (7-60). By (7-35) and (7-57), we have

$$\begin{aligned}
\text{(7-60)} &\lesssim \|\tilde{w}\|^2_{L^\infty_t L^2_{\tilde{y},x}([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})}\|N\|_{L^{4/3}_{t,y}L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})}\|(\nabla_y - i\xi(t))\tilde{w}\|_{L^4_{t,y}L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})} \\
&\lesssim \eta^{\frac{3}{8}}2^{k_0}.
\end{aligned}$$

Finally, we consider (7-61). Applying integration by parts, we have

$$\text{(7-61)} \le \text{(7-60)} + \left| \int_0^{\lambda^{-2}T} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\tilde{w}(t,\tilde{y},\tilde{x})|^2 \frac{1}{|y-\tilde{y}|} \Re(\bar{\tilde{w}} N)(t,y,x) \, \mathrm{d}y \, \mathrm{d}\tilde{y} \, \mathrm{d}x \, \mathrm{d}\tilde{x} \, \mathrm{d}t \right|. \tag{7-65}$$

By (6-1) and (7-36), we see

$$\text{(7-65)} \lesssim \int_0^{\lambda^{-2}T} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \|\tilde{w}(t,\tilde{y},\tilde{x})\|_{L^2_{\tilde{x}}}^2 \frac{1}{|y-\tilde{y}|} \|\tilde{w}(t,y,x)\|_{L^2_x} \|v_\lambda^h(t,y,x)\|_{L^2_x}^3 \, \mathrm{d}y \, \mathrm{d}\tilde{y} \, \mathrm{d}t \tag{7-66}$$

$$+ \int_0^{\lambda^{-2}T} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \|\tilde{w}(t,\tilde{y},\tilde{x})\|_{L^2_{\tilde{x}}}^2 \frac{1}{|y-\tilde{y}|} \|\tilde{w}(t,y,x)\|_{L^2_x} \|v_\lambda^l(t,y,x)\|_{L^2_x}^2 \|v_\lambda^h(t,y,x)\|_{L^2_x} \, \mathrm{d}y \, \mathrm{d}\tilde{y} \, \mathrm{d}t. \tag{7-67}$$

By the Hardy–Littlewood–Sobolev inequality, (7-62), (7-64), Lemma 6.20, the Sobolev embedding theorem, the conservation of mass, and interpolation, we have

$$\text{(7-66)} \lesssim \|v_\lambda^h\|_{L^4_{t,y} L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})}^3 \|\tilde{w}\|_{L^4_{t,y} L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})}^3 \lesssim \eta^{\frac{3}{8}} 2^{k_0}$$

and

$$\text{(7-67)} \lesssim \|v_\lambda^h\|_{L^3_t L^6_y L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})} \|\tilde{w}\|_{L^9_t L^{90/29}_y L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})}^3 \cdot \|v_\lambda^l\|_{L^6_t L^{60/11}_y L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})}^2 \lesssim \eta^{\frac{1}{6}} 2^{k_0}.$$

Thus, by the above estimates, we have

$$\left\| \int_{\mathbb{R}} |\nabla_y|^{\frac{1}{2}} (|\tilde{w}(t,y,x)|^2) \, \mathrm{d}x \right\|_{L^2_{t,y}([0,\lambda^{-2}T]\times\mathbb{R}^2)}^2 \lesssim \eta^{\frac{1}{6}} 2^{k_0}.$$

Undoing the scaling in (7-58), we finally reach the desired estimate (6-41). □

Appendix: Well-posedness theory for (1-2)

In this appendix, we present the proofs of the recorded results in Section 3. Let

$$X_1(t) = x\sin(t) - i\cos(t)\,\partial_x \quad \text{and} \quad X_2(t) = x\cos(t) + i\sin(t)\,\partial_x. \tag{A-1}$$

We have the pointwise identity: for any $f \in \mathcal{S}(\mathbb{R}^3)$,

$$|X_1(t)f(y,x)|^2 + |X_2(t)f(y,x)|^2 = |xf(y,x)|^2 + |\partial_x f(y,x)|^2 \quad \text{for all } t \in \mathbb{R}. \tag{A-2}$$

The next result follows by direct calculation. We refer to [Carles 2002b] for more explanation.

Lemma A.1. *The operators $X_1(t)$ and $X_2(t)$ satisfy the following properties*:

(1) *They correspond to the conjugation of gradient and momentum by the free flow*,

$$X_1(t) = e^{it(\Delta_{\mathbb{R}^3}-x^2)}(-i\partial_x)e^{-it(\Delta_{\mathbb{R}^3}-x^2)},$$
$$X_2(t) = e^{it(\Delta_{\mathbb{R}^3}-x^2)} x e^{-it(\Delta_{\mathbb{R}^3}-x^2)}.$$

(2) *They act on the nonlinearity like derivatives, that is, for* $j = 1, 2$, *we have*

$$|X_j(t)(|u|^2 u)| \lesssim |u|^2 |X_j(t)u|.$$

As a consequence, for any $u_\pm \in \Sigma$, we have

$$\|e^{-it(\Delta_{\mathbb{R}^3}-x^2)}u(t)-u_\pm\|_\Sigma = \|u(t)-e^{it(\Delta_{\mathbb{R}^3}-x^2)}u_\pm\|_{L_x^2 H_y^1} + \|X_1(t)(u(t)-e^{it(\Delta_{\mathbb{R}^3}-x^2)}u_\pm)\|_{L^2_{y,x}} + \|X_2(t)(u(t)-e^{it(\Delta_{\mathbb{R}^3}-x^2)}u_\pm)\|_{L^2_{y,x}}.$$

We now show the local well-posedness part of Theorem 3.4 in the following formulation. This is essentially following the argument in [Cazenave 2003; Tao 2006].

Theorem A.2 (local well-posedness). *For any* $E > 0$ *and* u_0 *with* $\|u_0\|_{L_y^2 \mathcal{H}_x^1(\mathbb{R}^2\times\mathbb{R})} \le E$, *there exists* $\delta_0 = \delta_0(E) > 0$ *such that if*

$$\|e^{it(\Delta_{\mathbb{R}^3}-x^2)}u_0\|_{L^4_{t,y}L^2_x(I\times\mathbb{R}^2\times\mathbb{R})} + \|X_1(t)e^{it(\Delta_{\mathbb{R}^3}-x^2)}u_0\|_{L^4_{t,y}L^2_x(I\times\mathbb{R}^2\times\mathbb{R})} + \|X_2(t)e^{it(\Delta_{\mathbb{R}^3}-x^2)}u_0\|_{L^4_{t,y}L^2_x(I\times\mathbb{R}^2\times\mathbb{R})} \le \delta_0,$$

where I *is the time interval, there exists a unique solution* $u \in C_t^0 L_y^2 \mathcal{H}_x^1(I \times \mathbb{R}^2 \times \mathbb{R})$ *of* (1-2) *satisfying*

$$\|u\|_{L^4_{t,y}\mathcal{H}^1_x(I\times\mathbb{R}^2\times\mathbb{R})} \le 2\|e^{it(\Delta_{\mathbb{R}^3}-x^2)}u_0\|_{L^4_{t,y}\mathcal{H}^1_x(I\times\mathbb{R}^2\times\mathbb{R})} \quad \text{and} \quad \|u\|_{L_t^\infty L_y^2\mathcal{H}_x^1(I\times\mathbb{R}^2\times\mathbb{R})} \le C\|u_0\|_{L_y^2\mathcal{H}_x^1}.$$

Proof. Let

$$\Phi(u) = e^{it(\Delta_{\mathbb{R}^3}-x^2)}u_0 - i\int_0^t e^{i(t-s)(\Delta_{\mathbb{R}^3}-x^2)}(|u|^2u)(s)\,\mathrm{d}s,$$

and set the space X to be

$$X = \{u \in C_t^0 L_y^2 \mathcal{H}_x^1 : \|u\|_{L_t^\infty L_y^2 \mathcal{H}_x^1} \le 2E, \|u\|_{L^4_{t,y}\mathcal{H}^1_x} \le 2C\delta_0\}$$

or

$$X = \big\{u \in C_t^0 L_y^2 \mathcal{H}_x^1 : \|u\|_{L_t^\infty L^2_{y,x}} \le 2E,\ \|u\|_{L^4_{t,y}L^2_x} \le 2C\delta_0,\ \|X_j(t)u\|_{L_t^\infty L^2_{y,x}} \le 2E,\ \|X_j(t)u\|_{L^4_{t,y}L^2_x} \le 2C\delta_0,\ j = 1, 2\big\}.$$

For any $u \in X$, by Proposition 3.2, Hölder's inequality, Sobolev's inequality, Lemma A.1, and (A-1), we have

$$\|\Phi(u)\|_{L_t^\infty L^2_{y,x}} \lesssim \|u_0\|_{L^2_{y,x}} + \|u\|_{L^4_{t,y}L^2_x}\|u\|^2_{L^4_{t,y}H^1_x}$$

and

$$\|X_1(t)\Phi(u)\|_{L_t^\infty L^2_{y,x}} + \|X_2(t)\Phi(u)\|_{L_t^\infty L^2_{y,x}} \lesssim \|\nabla_x u_0\|_{L^2_{y,x}} + \|xu_0\|_{L^2_{y,x}} + \|u\|^2_{L^4_{t,y}H^1_x}(\|X_1(t)u\|_{L^4_{t,y}L^2_x} + \|X_2(t)u\|_{L^4_{t,y}L^2_x}).$$

Thus

$$\|\Phi(u)\|_{L_t^\infty L^2_{y,x}} + \|X_1(t)\Phi(u)\|_{L_t^\infty L^2_{y,x}} + \|X_2(t)\Phi(u)\|_{L_t^\infty L^2_{y,x}} \le E + (2C\delta_0)^3 \le 2E. \tag{A-3}$$

Similarly, we can obtain

$$\|\Phi(u)\|_{L^4_{t,y}L^2_x} + \|X_1(t)\Phi(u)\|_{L^4_{t,y}L^2_x} + \|X_2(t)\Phi(u)\|_{L^4_{t,y}L^2_x} \le \delta_0 + (2C\delta_0)^3 \le 2C\delta_0. \tag{A-4}$$

In the same time, for any $u, v \in X$, by the Strichartz estimate, Hölder's inequality, and Sobolev's inequality, we have

$$\begin{aligned}\|\Phi(u)-\Phi(v)\|_{L^4_{t,y}L^2_x} &\lesssim \big\||u|^2u-|v|^2v\big\|_{L^{4/3}_{t,y}L^2_x}\\ &\lesssim \|u-v\|_{L^4_{t,y}L^2_x}(\|u\|^2_{L^4_{t,y}H^1_x}+\|v\|^2_{L^4_{t,y}H^1_x}) \lesssim (2C\delta_0)^2\|u-v\|_{L^4_{t,y}L^2_x}.\end{aligned} \tag{A-5}$$

Combining (A-3), (A-4), and (A-5), we have for δ_0 small enough $\Phi : X \to X$ is a contractive map. Therefore, the theorem follows from the fixed point theorem. □

We now turn to the proof of the scattering norm in Theorem 3.4.

Proof of the scattering norm part of Theorem 3.4. We need to show

$$\|u\|_{L^4_{t,y}\mathcal{H}^1_x \cap L^4_t W^{1,4}_y L^2_x(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} \le C(M). \tag{A-6}$$

Then by the scattering theory of the nonlinear Schrödinger equations [Antonelli et al. 2015; Carles 2011; Tao 2006], we have scattering in (3-1). By the well-posedness part of Theorem 3.4, it suffices to prove (A-6) as an a priori bound.

Divide the time interval $\mathbb{R}$ into $N \sim \big(1+\frac{L}{\delta}\big)^4$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|u\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x(I_j\times\mathbb{R}^2\times\mathbb{R})} \le \delta, \tag{A-7}$$

where $\delta > 0$ will be chosen later.

On each I_j, by (A-1), the Strichartz estimate, the Sobolev embedding and (A-7), we have

$$\begin{aligned}&\|u\|_{L^4_t W^{1,4}_y L^2_x \cap L^4_{t,y}\mathcal{H}^1_x(I_j\times\mathbb{R}^2\times\mathbb{R})}\\ &\le C\big(\|u(t_j)\|_\Sigma + \||u|^2u\|_{L^{4/3}_{t,y}L^2_x} + \|X_1(t)(|u|^2u)\|_{L^{4/3}_{t,y}L^2_x} + \|X_2(t)(|u|^2u)\|_{L^{4/3}_{t,y}L^2_x}\big)\\ &\le C\big(\|u(t_j)\|_\Sigma + \|u\|^2_{L^4_{t,y}H^{1-\epsilon_0}_x}(\|u\|_{L^4_{t,y}L^2_x} + \|X_1(t)u\|_{L^4_{t,y}L^2_x}\\ &\qquad\qquad + \|X_2(t)u\|_{L^4_{t,y}L^2_x}) + \|u\|_{L^4_t W^{1,4}_y L^2_x}\|u\|^2_{L^4_{t,y}H^{1-\epsilon_0}_x}\big)\\ &\le C\big(\|u(t_j)\|_\Sigma + \|u\|^2_{L^4_{t,y}H^{1-\epsilon_0}_x}(\|u\|_{L^4_{t,y}L^2_x} + \|\nabla_x u\|_{L^4_{t,y}L^2_x} + \||x|u\|_{L^4_{t,y}L^2_x} + \|u\|_{L^4_t W^{1,4}_y L^2_x})\big)\\ &\le C\big(\|u(t_j)\|_\Sigma + \delta^2\|u\|_{L^4_t W^{1,4}_y L^2_x \cap L^4_{t,y}\mathcal{H}^1_x}\big).\end{aligned}$$

Choosing $\delta \le \big(\frac{1}{2C}\big)^{1/4}$ leads to the estimate

$$\|u\|_{L^4_t W^{1,4}_y L^2_x \cap L^4_{t,y}\mathcal{H}^1_x(I_j\times\mathbb{R}^2\times\mathbb{R})} \le 2C\|u(t_j)\|_{\Sigma_{y,x}}.$$

The desired bound (A-6) now follows by adding up the bounds on each subintervals I_j. □

Proof of Theorem 3.5. We only give a sketch for the proof of Theorem 3.5, since it follows essentially by the same argument as in the proof of Theorem 3.4.

For $u_- \in \Sigma$, let $\delta > 0$ be a small absolute constant to be taken later. Taking $T_- = T_-(u_-)$ large enough and then applying the monotone convergence theorem, we obtain

$$\|e^{it(\Delta - x^2)}u_-\|_{L_t^4 W_y^{1,4} L_x^2 \cap L_{t,y}^4 \mathcal{H}_x^1((-\infty,-T_-]\times\mathbb{R}^2\times\mathbb{R})} \le \delta.$$

Solving the integral equation

$$u(t) = e^{it(\Delta - x^2)}u_- - i\int_{-\infty}^t e^{i(t-s)(\Delta - x^2)}(|u|^2 u)(s)\,\mathrm{d}s$$

in $L_t^\infty L_y^2 \mathcal{H}_x^1 \cap L_t^\infty H_y^1 L_x^2 \cap L_t^4 W_y^{1,4} L_x^2 \cap L_{t,y}^4 \mathcal{H}_x^1((-\infty,-T_-]\times\mathbb{R}^2\times\mathbb{R})$ and keeping u small in $L_t^4 W_y^{1,4} L_x^2 \cap L_{t,y}^4 \mathcal{H}_x^1((-\infty,-T_-]\times\mathbb{R}^2\times\mathbb{R})$, the argument in the proof of Theorem A.2 implies that there exists a solution of (1-2) on $(-\infty,-T_-]\times\mathbb{R}^2\times\mathbb{R}$, which furthermore satisfies (3-2). This completes the proof for first part of Theorem 3.5. The proof for the second part of Theorem 3.5 is similar and thus we omit it here. □

We now turn to the proof of Theorem 3.6. First, we show the following short-time version.

Lemma A.3 (short-time stability theorem). *Let I be a compact interval and let $\tilde{u}$ be an approximate solution to (1-2) in the sense that $i\partial_t \tilde{u} + \Delta_{\mathbb{R}^3}\tilde{u} - x^2\tilde{u} = |\tilde{u}|^2\tilde{u} + e$ for some function e. Assume that*

$$\|\tilde{u}\|_{L_t^\infty L_y^2 \mathcal{H}_x^1(I\times\mathbb{R}^2\times\mathbb{R})} \le M \tag{A-8}$$

for some positive constant M. Let $t_0 \in I$ and $u(t_0)$ be such that

$$\|u(t_0) - \tilde{u}(t_0)\|_{L_y^2 \mathcal{H}_x^1} \le M' \tag{A-9}$$

for some $M' > 0$.

Assume also the smallness conditions hold:

$$\|\tilde{u}\|_{L_t^4 L_y^4 \mathcal{H}_x^1(I\times\mathbb{R}^2\times\mathbb{R})} \le \epsilon, \tag{A-10}$$

$$\|e^{i(t-t_0)(\Delta_{\mathbb{R}^3} - x^2)}(u(t_0) - \tilde{u}(t_0))\|_{L_t^4 L_y^4 \mathcal{H}_x^1} + \|e\|_{L_t^{4/3} L_y^{4/3} \mathcal{H}_x^1} \le \epsilon \tag{A-11}$$

for some $0 < \epsilon \le \epsilon_1$, where $\epsilon_1 = \epsilon_1(M, M') > 0$ is a small constant. Then, there exists a solution u to (1-2) on $I\times\mathbb{R}^2\times\mathbb{R}$ with initial data $u(t_0)$ at time $t = t_0$ satisfying

$$\|u - \tilde{u}\|_{L_{t,y}^4 \mathcal{H}_x^1} \lesssim \epsilon, \tag{A-12}$$

$$\|u - \tilde{u}\|_{L_t^\infty L_y^2 \mathcal{H}_x^1} \lesssim M', \tag{A-13}$$

$$\|u\|_{L_t^\infty L_y^2 \mathcal{H}_x^1} \lesssim M + M', \tag{A-14}$$

$$\||u|^2 u - |\tilde{u}|^2\tilde{u}\|_{L_t^{4/3} L_y^{4/3} \mathcal{H}_x^1} \lesssim \epsilon. \tag{A-15}$$

Proof. By symmetry, we may assume $t_0 = \inf I$. Let $w = u - \tilde{u}$. Then w satisfies

$$i\,\partial_t w + \Delta_{\mathbb{R}^3} w - x^2 w = |\tilde{u} + w|^2(\tilde{u} + w) - |\tilde{u}|^2\tilde{u} - e.$$

For $t \in I$, we define

$$D(t) = \||\tilde{u} + w|^2(\tilde{u} + w) - |\tilde{u}|^2\tilde{u}\|_{L_{t,y}^{4/3} \mathcal{H}_x^1([t_0,t]\times\mathbb{R}^2\times\mathbb{R})}.$$

By (A-10), we have

$$\begin{aligned} D(t) &\lesssim \|w\|_{L^4_{t,y}\mathcal{H}^1_x}(\|\tilde{u}\|^2_{L^4_{t,y}\mathcal{H}^1_x} + \|w\|^2_{L^4_{t,y}\mathcal{H}^1_x}) \\ &\lesssim \|w\|^3_{L^4_{t,y}\mathcal{H}^1_x([t_0,t]\times\mathbb{R}^2\times\mathbb{R})} + \epsilon_1^2\|w\|_{L^4_{t,y}\mathcal{H}^1_x([t_0,t]\times\mathbb{R}^2\times\mathbb{R})}. \end{aligned} \tag{A-16}$$

On the other hand, by the Strichartz estimate and (A-11), we get

$$\begin{aligned} \|w\|_{L^4_{t,y}\mathcal{H}^1_x([t_0,t]\times\mathbb{R}^2\times\mathbb{R})} &\lesssim \|e^{i(t-t_0)(\Delta_{\mathbb{R}^3}-x^2)}w(t_0)\|_{L^4_{t,y}\mathcal{H}^1_x([t_0,t]\times\mathbb{R}^2\times\mathbb{R})} + D(t) + \|e\|_{L^{4/3}_{t,y}\mathcal{H}^1_x([t_0,t]\times\mathbb{R}^2\times\mathbb{R})} \\ &\lesssim D(t)+\epsilon. \end{aligned} \tag{A-17}$$

Combining (A-16) and (A-17), we obtain

$$D(t) \lesssim (D(t)+\epsilon)^3 + \epsilon_1^2(D(t)+\epsilon).$$

A standard continuity argument then shows that if ϵ_1 is taken sufficiently small, then

$$D(t) \lesssim \epsilon \quad \text{for all } t \in I,$$

which implies (A-15).

Using (A-15) and (A-17), one easily derives (A-12). Moreover, by the Strichartz estimate, (A-9) and (A-15),

$$\|w\|_{L^\infty_t L^2_y \mathcal{H}^1_x(I\times\mathbb{R}^2\times\mathbb{R})} \lesssim \|w(t_0)\|_{L^2_y\mathcal{H}^1_x} + \||\tilde{u}+w|^2(\tilde{u}+w) - |\tilde{u}|^2\tilde{u}\|_{L^{4/3}_{t,y}\mathcal{H}^1_x} + \|e\|_{L^{4/3}_{t,y}\mathcal{H}^1_x} \lesssim M'+\epsilon,$$

which establishes (A-13) for $\epsilon_1 = \epsilon_1(M')$ sufficiently small.

To prove (A-14), we use the Strichartz estimate, (A-8), (A-9), (A-15) and (A-10),

$$\begin{aligned} \|u\|_{L^\infty_t L^2_y\mathcal{H}^1_x(I\times\mathbb{R}^2\times\mathbb{R})} &\lesssim \|\tilde{u}(t_0)\|_{L^2_y\mathcal{H}^1_x} + \|u(t_0)-\tilde{u}(t_0)\|_{L^2_y\mathcal{H}^1_x} + \||u|^2u-|\tilde{u}|^2\tilde{u}\|_{L^{4/3}_{t,y}\mathcal{H}^1_x} + \||\tilde{u}|^2\tilde{u}\|_{L^{4/3}_{t,y}\mathcal{H}^1_x} \\ &\lesssim M+M'+\epsilon+\|\tilde{u}\|^3_{L^4_{t,y}\mathcal{H}^1_x} \lesssim M+M'+\epsilon+\epsilon_1^3. \end{aligned}$$

The proof is complete by choosing $\epsilon_1 = \epsilon_1(M, M')$ sufficiently small. □

We now show the proof of Theorem 3.6.

Proof of Theorem 3.6. We divide the interval I into $N \sim \left(1+\frac{L}{\epsilon_0}\right)^4$ subintervals $I_j = [t_j, t_{j+1}]$, $0 \le j \le N-1$, such that

$$\|\tilde{u}\|_{L^4_{t,y}\mathcal{H}^1_x(I_j\times\mathbb{R}^2\times\mathbb{R})} \le \epsilon_1,$$

where $\epsilon_1 = \epsilon_1(M, 2M')$ is given by Lemma A.3.

By choosing ϵ_1 sufficiently small depending on J, M and M', we can apply Lemma A.3 to obtain, for each j and all $0<\epsilon<\epsilon_1$,

$$\|u-\tilde{u}\|_{L^4_{t,y}\mathcal{H}^1_x(I_j\times\mathbb{R}^2\times\mathbb{R})} \le C(j)\epsilon, \qquad \|u-\tilde{u}\|_{L^\infty_t L^2_y\mathcal{H}^1_x(I_j\times\mathbb{R}^2\times\mathbb{R})} \le C(j)M',$$

$$\|u\|_{L^\infty_t L^2_y\mathcal{H}^1_x(I_j\times\mathbb{R}^2\times\mathbb{R})} \le C(j)(M+M'), \quad \||u|^2u-|\tilde{u}|^2\tilde{u}\|_{L^{4/3}_{t,y}\mathcal{H}^1_x(I_j\times\mathbb{R}^2\times\mathbb{R})} \le C(j)\epsilon,$$

provided we can prove that analogues of (3-3) and (3-4) hold with t_0 replaced by t_j.

In order to verify this, we use an inductive argument. By the Strichartz estimate, (3-3), and the inductive hypothesis,

$$\begin{aligned}\|u(t_j)-\tilde{u}(t_j)\|_{L_y^2\mathcal{H}_x^1} &\lesssim \|u(t_0)-\tilde{u}(t_0)\|_{L_y^2\mathcal{H}_x^1} + \||u|^2u-|\tilde{u}|^2\tilde{u}\|_{L_{t,y}^{4/3}\mathcal{H}_x^1([t_0,t_j]\times\mathbb{R}^2\times\mathbb{R})} + \|e\|_{L_{t,y}^{4/3}\mathcal{H}_x^1([t_0,t_j]\times\mathbb{R}^2\times\mathbb{R})} \\ &\lesssim M' + \sum_{k=0}^{j-1} C(k)\epsilon + \epsilon.\end{aligned}$$

Similarly, by the Strichartz estimate, (3-4), and the inductive hypothesis,

$$\begin{aligned}\|e^{i(t-t_j)(\Delta_{\mathbb{R}^3}-x^2)}(u(t_j)-\tilde{u}(t_j))\|_{L_{t,y}^4\mathcal{H}_x^1(I_j\times\mathbb{R}^2\times\mathbb{R})} &\lesssim \|e^{i(t-t_0)(\Delta_{\mathbb{R}^3}-x^2)}(u(t_0)-\tilde{u}(t_0))\|_{L_{t,y}^4\mathcal{H}_x^1(I_j\times\mathbb{R}^2\times\mathbb{R})} + \|e\|_{L_{t,y}^{4/3}\mathcal{H}_x^1([t_0,t_j]\times\mathbb{R}^2\times\mathbb{R})} \\ &\qquad + \||u|^2u-|\tilde{u}|^2\tilde{u}\|_{L_{t,y}^{4/3}\mathcal{H}_x^1([t_0,t_j]\times\mathbb{R}^2\times\mathbb{R})} \\ &\lesssim \epsilon + \sum_{k=0}^{j-1} C(k)\epsilon.\end{aligned}$$

It is clear now we may choose ϵ_1 sufficiently small, depending on N, M and M', such that the hypotheses of Lemma A.3 continue to hold as j varies. This completes the proof of Theorem 3.6. □

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