Analysis & PDE

Volume 17 No. 10 2024

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ANALYSIS AND PDE
Vol. 17 (2024), No. 10, pp. 3589–3621
DOI: 10.2140/apde.2024.17.3589

THE QUERMASSINTEGRAL-PRESERVING MEAN CURVATURE FLOW IN THE SPHERE

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We introduce a mean curvature flow with global term of convex hypersurfaces in the sphere, for which the global term can be chosen to keep any quermassintegral fixed. Then, starting from a strictly convex initial hypersurface, we prove that the flow exists for all times and converges smoothly to a geodesic sphere. This provides a workaround to an issue present in the volume-preserving mean curvature flow in the sphere introduced by Huisken (1987). We also classify solutions for some constant curvature-type equations in space forms, as well as solitons in the sphere and in the upper branch of the De Sitter space.

1. Introduction and statement of main results

Let $n \geq 2$, and let $M^n \subset \mathbb{M}_K^{n+1}$ be a smooth, closed, embedded hypersurface in a simply connected space form $\mathbb{M}_K^{n+1}$ of constant curvature $K \in \mathbb{R}$, given by the embedding x_0. We consider a family of embeddings $x = x(t, \cdot\,)$ satisfying the mean curvature-type flow with a global forcing term

$$\partial_t x = (\mu(t)\mathrm{c}_K(r) - H)\nu, \tag{1-1}$$

which has initial condition $x(0, \cdot\,) = x_0$. Here H is the mean curvature and ν the outward unit normal of the evolving hypersurfaces M_t. For convex hypersurfaces (i.e., with $\kappa_1 \geq 0$, where $\kappa_1 \leq \cdots \leq \kappa_n$ denote the principal curvatures), the sign conventions are taken so that $-H\nu$ points inwards. Moreover, let r denote the radial distance to a given point $\mathcal{O} \in \mathbb{M}_K^{n+1}$, which we call the *origin* in the sequel. This means that the flow (1-1) depends on the choice of the origin and, in fact, along the flow we will change the origin in a controlled way. We use the notation

$$\mathrm{c}_K(r) = \mathrm{s}_K'(r), \quad \text{where } \mathrm{s}_K(r) = \begin{cases} K^{-1/2}\sin(\sqrt{K}r) & \text{if } K > 0, \\ r & \text{if } K = 0, \\ |K|^{-1/2}\sinh(\sqrt{|K|}r) & \text{if } K < 0. \end{cases} \tag{1-2}$$

If σ_ℓ represents the ℓ-th elementary symmetric function, we define the time-dependent term by

$$\mu(t) = \frac{\int_M H\sigma_\ell \, dV_t}{\int_M \mathrm{c}_K \sigma_\ell \, dV_t} \tag{1-3}$$

The work of Cabezas-Rivas is partially supported by the projects PID2019-105019GB-C21 and PID2022-136589NB-I00, both funded by MCIN/AEI/10.13039/501100011033 and by ERDF A way of making Europe; by the project AICO/2021/252 funded by Conselleria d'Educació, Universitats i Ocupació; and by the network RED2022-134077 funded by MCIN/AEI/10.13039/501100011033 and by ERDF A way of making Europe.
MSC2020: 53C21, 53E10.
Keywords: volume-preserving mean curvature flow, spherical geometry.

for each $\ell = 0, 1, \dots, n$, where dV_t denotes the volume element of M_t. This choice guarantees that (1-1) gives a family of globally constrained mean curvature flows, where μ can be chosen to preserve any of the $n+1$ quermassintegrals $W_\ell(\Omega_t)$ of the evolving hypersurfaces M_t (see Section 2.1 for a review of the quermassintegrals). Here Ω_t denotes the convex region enclosed by $M_t = x(t, \mathbb{S}^n)$ — note that we may assume that the common domain of the embeddings is $\mathbb{S}^n$ due to convexity.

Let us stress that, for $\ell = 0$ and $K = 0$, the flow (1-1) with nonlocal term as in (1-3) coincides with the *volume-preserving mean curvature flow* (VPMCF) introduced by Huisken [1987]. He proved that strictly convex hypersurfaces in $\mathbb{R}^{n+1}$ remain convex and embedded under the flow, and the solution exists for all times and converges to a round sphere smoothly as $t \to \infty$. Since then it was still an open question of extending the result to an $(n+1)$-dimensional sphere $\mathbb{S}^{n+1}_K$, $K > 0$, where convexity can be lost under VPMCF, as pointed out in [Huisken 1987, p. 38].

Our main result settles this question by proposing the flow (1-1) as the most natural generalization of the VPMCF to a space form with positive curvature. Indeed, such a definition preserves convexity under the flow, and allows us to prove the following version of Huisken's original result within the half sphere, where, for a point $p \in \mathbb{S}^{n+1}_K$, we denote by $\mathcal{H}(p)$ the open hemisphere around p.

Theorem 1.1. *Let $n \geq 2$, and let $M_0 \subset \mathbb{S}^{n+1}_K$ be a strictly convex hypersurface enclosing a domain Ω_0. Then there exists a finite system of origins $(\mathcal{O}_i)_{0 \leq i \leq m}$ and numbers $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = \infty$ such that the problem*

$$\begin{aligned} \partial_t x &= (\mu_i(t)\mathrm{c}_K(r_i) - H)\nu, \quad t \in [t_i, t_{i+1}), \quad 0 \leq i \leq m, \\ x(0, M) &= M_0, \\ x(t_i, M) &= \lim_{t \nearrow t_i} M_t, \quad 1 \leq i \leq m, \end{aligned}$$

where r_i is the distance to $\mathcal{O}_i$ and μ_i is given as in (1-3) to keep the quermassintegral $W_\ell(\Omega_t)$ fixed for any $\ell = 0, 1, \dots, n$, has a solution

$$x : [0, \infty) \times \mathbb{S}^n \to \mathbb{S}^{n+1}_K.$$

For every $t \geq 0$, the embeddings $x(t, \cdot)$ smoothly map $\mathbb{S}^n$ to strictly convex hypersurfaces, with

$$\mathcal{O}_i \in \Omega_t \quad \text{and} \quad M_t \subset \mathcal{H}(\mathcal{O}_i) \qquad \text{for all } t \in [t_i, t_{i+1}),$$

and satisfy spatial C^∞-estimates which are uniform in time. The restriction

$$x : [t_m, \infty) \times \mathbb{S}^n \to \mathbb{S}^{n+1}_K$$

is smooth and converges for $t \to \infty$ in C^∞ to a geodesic sphere around $\mathcal{O}_m$ with radius determined by $W_\ell(B_r) = W_\ell(\Omega_0)$.

At this stage we should mention that Guan and Li [2015] invented a *purely local* mean curvature-type flow in the sphere, which is volume-preserving and drives star-shaped hypersurfaces to geodesic spheres. There is also a flow of Guan–Li-type that preserves $W_{\ell-1}(\Omega_t)$ and decreases $W_\ell(\Omega_t)$, which has so far refused to allow curvature estimates (see [Chen et al. 2022] for an overview of known results).

However, notice that nonlocal flows are more challenging than their corresponding local counterparts, as the evolution depends heavily on the global shape of the hypersurface M_t and the presence of the term $\mu(t)$ in all the relevant evolution equations causes a plethora of extra complications; e.g., comparison principles and preservation of key properties fail (see [Cabezas-Rivas and Miquel 2016]), and embedded hypersurfaces may develop self-intersections (see [Mayer and Simonett 2000]).

In this framework, our flow (1-1) is, to the best of our knowledge, the first known curvature flow in the sphere which preserves any desired quermassintegral by a suitable choice of the global term μ and which enjoys smooth convergence to a geodesic sphere.

***Elliptic counterpart*: *rigidity results*.** As a byproduct of the pinching estimates (see Proposition 4.3) that we require to prove Theorem 1.1, we can also classify hypersurfaces in a space form $\mathbb{M}^{n+1}_K$ which have a rotationally symmetric curvature function under suitable assumptions on the sectional curvature.

With this goal, we work with curvature functions more general than H. Let $\Gamma \subset \mathbb{R}^n$ be a symmetric, open cone containing the positive cone

$$\Gamma_+ = \{\kappa \in \mathbb{R}^n : \kappa_i > 0, \forall i = 1, \ldots, n\},$$

and consider a symmetric function $f \in C^2(\Gamma)$. Let

$$F(A) = f(\kappa_1, \ldots, \kappa_n)$$

be the corresponding operator-dependent function, where A denotes the Weingarten or shape operator. We assume the following:

Conditions 1.2. Let $f(\kappa) = F(A)$ be a C^2 symmetric function defined on an open, symmetric cone $\Gamma \supset \Gamma_+$. We ask further that

(a) f is strictly increasing in each argument,

(b) f is homogeneous of degree 1,

(c) f is normalized so that $f(1, \ldots, 1) = n$.

Notice that (a) implies that F defines a strictly elliptic operator on M, as proved in [Huisken and Polden 1999]. We say that f is *inverse concave/convex* if the dual function

$$\tilde{f}(\kappa_1, \ldots, \kappa_n) = f(\kappa_1^{-1}, \ldots, \kappa_n^{-1})$$

is concave/convex (see Section 8.1 for a more detailed introduction).

A classical result by Alexandrov [1962] says that if a compact hypersurface embedded in $\mathbb{R}^{n+1}$ has H equal to a constant, it must be a round sphere. Later on, Ros [1987] extended this result to the constancy of higher-order symmetric functions σ_ℓ. Hypersurfaces in a model space $\mathbb{M}^{n+1}_K$ which have a constant curvature function F were often called *Weingarten hypersurfaces* in the previous literature. It was shown in [Espinar et al. 2009, Theorem 28] that round spheres are the only examples of compact Weingarten hypersurfaces in the hyperbolic space $\mathbb{H}^{n+1}$. In this spirit, we obtain similar rigidity results for F radially symmetric instead of constant.

Theorem 1.3. *Let $n \geq 2$, $K \in \mathbb{R}$ and $|\alpha| \geq 1$, and let $M^n \subset \mathbb{M}^{n+1}_K$ be a convex hypersurface, which is located in the northern hemisphere for $K > 0$, such that*

$$\mathrm{Sec}_M \geq -\alpha K.$$

Suppose that F is a convex function satisfying Conditions 1.2, which is a solution to

$$F = \gamma \mathrm{c}_K^\alpha$$

for some constant γ. Then M is a geodesic sphere which is centered at the origin provided $K \neq 0$. If $\alpha = 1$ the convexity assumption on M can be dropped, while if $\alpha = -1$ the convexity of F may be replaced by inverse concavity.

Such results have been obtained for σ_ℓ by integral methods: for instance, [Wu and Xia 2014] studies constant linear combinations of higher-order mean curvatures, [Wu 2016] analyses the constancy of $\mathrm{c}_K \sigma_\ell$ in $\mathbb{H}^{n+1}$, and [Kwong et al. 2018] deals with hypersurfaces having radially symmetric higher-order mean curvatures in general $\mathbb{M}^{n+1}_K$ under mild convexity assumptions. But those integral techniques are restricted to the σ_ℓ because they are divergence-free in space forms. Our maximum principle approach enables us to relax the assumptions on the curvature functions at the cost of having to impose a condition on the sectional curvature of the hypersurface. However, notice that if $(1+\alpha)K > 0$ this assumption is weaker than convexity, while if this product is ≤ 0 the condition already implies convexity.

Classification of solitons. In the study of singularity formation along curvature flows, especially the mean curvature flow, the class of *self-shrinking solutions*, simply called *solitons* subsequently, plays an important role. For the mean curvature flow in Euclidean space, they arise as blow-up limits of type-I singularities (see [Huisken 1990]), and they satisfy the elliptic equation

$$H = \langle x, \nu \rangle.$$

Huisken [1990] showed that the only compact mean-convex solitons are spheres.

A similar recent result with H replaced by the Gauss curvature K (see [Brendle et al. 2017]) settled the long-standing open problem of whether the flow by certain powers of the Gauss curvature of n-dimensional hypersurfaces, $n \geq 3$, converges to a round sphere; the convergence to a soliton had already been proved in [Andrews et al. 2016].

The study of solitons for more general curvature functions has received plenty of attention, as well as in space forms; see, e.g., [Gao and Ma 2019; Gao et al. 2018; 2022; McCoy 2011]. Here one considers the general equation

$$F^\beta = u, \tag{1-4}$$

where $\beta \in \mathbb{R}$, F is a function of the principal curvatures with suitable assumptions, and

$$u = \mathrm{s}_K(r)\langle \partial_r, \nu \rangle \tag{1-5}$$

is the generalized support function. From a well-known duality relation by means of the Gauss map for hypersurfaces of the sphere and itself, from Theorem 1.3 we can deduce a new classification result for convex solitons in the sphere $\mathbb{S}^{n+1}_1$, and similarly, from a duality relation between hypersurfaces of the hyperbolic and De Sitter space, from Theorem 1.3 we can deduce a new classification result for convex solitons in the upper branch of the $(n+1)$-dimensional De Sitter space $\mathbb{S}^{n,1}$ with sectional curvature $K = 1$, i.e.,

$$\mathbb{S}^{n,1} = \left\{ y \in \mathbb{R}^{n+2} : -(y^0)^2 + \sum_{i=1}^{n+1} (y^i)^2 = 1,\, y^0 > 0 \right\}.$$

More precisely, with the notation

$$\operatorname{sgn}(\mathbb{M}) = \begin{cases} 1, & \mathbb{M} = \mathbb{S}^{n+1}_1, \\ -1, & \mathbb{M} = \mathbb{S}^{n,1}, \end{cases} \tag{1-6}$$

we prove the following result.

Corollary 1.4. *Let $n \ge 2$, $|\beta| \le 1$, $\beta \ne 0$, and let $\mathbb{M}$ be either $\mathbb{S}^{n+1}_1$ or $\mathbb{S}^{n,1}$. Consider $M^n \subset \mathbb{M}$ a closed strictly convex hypersurface, and, in the case $((1-\beta)/\beta)\operatorname{sgn}(\mathbb{M}) > 0$, we assume further that*

$$\operatorname{Sec}_M \le \frac{\operatorname{sgn}(\mathbb{M})}{1-\beta}.$$

If F is an inverse convex function satisfying Conditions 1.2, which is a solution to the soliton equation (1-4), then M is a geodesic sphere centered at the origin. In the case $\beta = 1$, the inverse convexity may be replaced by concavity.

Remark 1.5. Corollary 1.4 is remarkable in several ways. Firstly, to our knowledge this is the first such result, where the β-regime ranges down to zero. This is surprising, as in the Euclidean space, for $F = K^{1/n}$ and $\beta \le n/(n+2)$ the result is false; see [Andrews 2000; Brendle et al. 2017]. Note however that $K^{1/n}$ is not inverse convex. Secondly, in all of the previous results of this type, the inverse concavity of F was exploited crucially. The duality approach allows us to deal with a further class of curvature functions, which could not be treated by earlier methods.

Notice that, while Weingarten hypersurfaces are known to be geodesic spheres in $\mathbb{S}^{n,1}$ (see [Roldán 2022]), we are not aware of rigidity results for solitons in this setting. On the other hand, to have some model examples in mind, $F = |A|$ satisfies the assumptions of Theorem 1.3, and the harmonic mean curvature is suitable for Corollary 1.4.

The problem of extending a nonlocal flow to curved spaces. As said before, Huisken [1987] introduced the VPMCF of convex hypersurfaces in the Euclidean space:

$$\partial_t x = (\mu(t) - H)\nu, \tag{1-7}$$

where the global term is the average mean curvature $\mu = \overline{H} = ⨍_{M_t} H$. Taking μ as in (1-3) for $K = 0$, McCoy [2004] obtained convergence of convex hypersurfaces in $\mathbb{R}^{n+1}$ to round spheres under a flow that preserves any quermassintegral (which in the Euclidean case coincide with the mixed volumes; see Section 2.1).

Huisken already pointed out that an interesting problem is to extend his result to non-Euclidean ambient spaces, with the warning that the generalization will not be straightforward because (1-7) does not preserve convexity in general Riemannian manifolds, due to terms with an unfavorable sign in the evolution equation of the second fundamental form. In particular, for hypersurfaces in $\mathbb{M}^{n+1}_K$, the Weingarten matrix h^i_j evolves according to

$$(\partial_t - \Delta)h^i_j = (|A|^2 - nK)h^i_j + 2KH\delta^i_j - \mu(h^i_\ell h^\ell_j + K\delta^i_j).$$

Notice that for $K < 0$ the bad term is $2KH\delta^i_j$, which comes from the background geometry and causes that convexity is not preserved in general. This failure is independent of the nonlocal nature of the flow; indeed, if we replace convexity by h-convexity ($\kappa_1 > |K|$), Miquel and the first author [Cabezas-Rivas and Miquel 2007] proved that h-convex hypersurfaces can be deformed under (1-7) to a geodesic sphere; this was extended by Andrews and Wei [2018] for a class of quermassintegral-preserving flows. The curvature condition was relaxed to positive sectional curvature ($\kappa_1\kappa_2 > |K|$) by Andrews, Chen and Wei [Andrews et al. 2021] in the volume-preserving case.

Notice that the complication for $K > 0$ is of a completely different nature, since the fatal term is now $-\mu K\delta^i_j$, and thus comes directly from the global term. Indeed, Huisken [1987] illustrated this with an intuitive example: if the flow starts with a convex hypersurface of $\mathbb{S}^{n+1}$ with a portion M^* C^2-close to the equator, then in this region $\overline{H} \gg H$ and hence M^* moves in the outward direction crossing the equator, and thus the evolving hypersurface becomes nonconvex.

This obstruction to the preservation of convexity in an ambient sphere supports the claim that the flow (1-7) is, geometrically, not the most natural generalization of the same flow in the Euclidean case to the spherical ambient space. Indeed, our alternative flow (1-1) does preserve pinching of the principal curvatures, and hence, it succeeds in driving any convex initial hypersurface to a geodesic sphere. Notice that Huisken's example is actually the motivation for the definition of (1-1), as the effect of multiplying the global term by $\mathrm{c}_K(r)$ is to slow down the motion as the hypersurfaces approach the equator.

In short, to extend Huisken's results to the hyperbolic space one needs to strengthen the notion of convexity, whereas for the ambient sphere we propose a different generalization of the flow (notice that (1-1) and (1-7) coincide for the Euclidean space), which works for convex hypersurfaces.

The isoperimetric nature of the flow. In addition, under (1-7) the surface area is nonincreasing, and hence Huisken's theorem provides an alternative proof of the isoperimetric inequality for convex hypersurfaces of $\mathbb{R}^{n+1}$. An interesting side effect of the extra term in (1-1) is that this flow is no longer of isoperimetric nature in the classical sense, because if we choose μ to preserve enclosed volume, the surface area is no longer decreasing necessarily.

However, the flow (1-1), with global term chosen to preserve the weighted volume $\int_{\Omega_t} \mathrm{c}_K$, has decreasing surface area, which suggests that in principle it is the right flow to prove the isoperimetric type inequality

$$\int_{\Omega_0} \mathrm{c}_K \le \phi(|M_0|),$$

with equality if and only if Ω_0 is a ball centered at the origin. Here ϕ is a function that gives equality on the slices. This was originally shown in [Girão and Pinheiro 2017, Proposition 4] by other means, and hence we do not pursue any further investigation in this matter here.

This reinforces the idea that our new flow has a geometric meaning beyond the generalization of Huisken's result, and we hope that in the future some interesting new applications will follow.

Structure of the paper. The contents of this paper are organized as follows. We first introduce in Section 2 the basic notation and evolution equations that ensure that our flow preserves the quermassintegrals, while Section 3 gathers new estimates for strictly convex hypersurfaces in the sphere, which may be of independent interest, like a refined outradius bound (Theorem 3.3) or inradius control in terms of pinching (Corollary 3.2). Then in Section 4 we prove that the *pinching deficit* decreases exponentially under the flow as time evolves, which is the key to get convergence of the evolving hypersurfaces. To achieve upper curvature bounds, we perform a technically intricate process in Section 5, which includes a delicate iterative changing of origin to ensure an optimal configuration that enables us to gain some uniform bound on the global term for some controlled time interval. This is a novel method, providing an alternative to proving initial value-independent curvature bounds after a waiting time. To finish the proof of Theorem 1.1, in Section 6 we establish long-time existence, and convergence to a geodesic sphere is done in Section 7. Finally, the elliptic results are proved in Section 8.

2. Notation, conventions and preliminary results

Hypersurfaces in space forms. Let $x : M \hookrightarrow \mathbb{M}_K^{n+1}$ be the embedding of a smooth hypersurface in a simply connected space form $\mathbb{M}_K^{n+1}$ enclosing a bounded domain Ω. Then the metric in polar coordinates is given by

$$\bar{g} = dr^2 + \mathrm{s}_K^2(r)\sigma,$$

where r is the radial distance to a fixed point $\mathcal{O} \in \mathbb{M}_K^{n+1}$ and σ is the round metric on $\mathbb{S}^n$.

The trigonometric functions in (1-2) satisfy the computational rules

$$\mathrm{c}_K' = -K\mathrm{s}_K, \quad \mathrm{c}_K^2 + K\mathrm{s}_K^2 = 1.$$

We will also use the related notation $\mathrm{co}_K(r) = \mathrm{c}_K(r)/\mathrm{s}_K(r)$.

For the outward pointing unit normal ν, we define the second fundamental form $h = (h_{ij})$ by

$$\overline{\nabla}_X Y = \nabla_X Y - h(X, Y)\nu,$$

where $\overline{\nabla}$ is the Levi-Civita connection of the metric $\bar{g} = \langle\cdot,\cdot\rangle$ on $\mathbb{M}_K^{n+1}$, and X and Y are vector fields on M. We adopt the summation convention throughout, and latin indices indicate components with respect to a coordinate frame $(\partial_i)_{1\le i\le n}$ on the domain of the embedding x.

If the induced metric on M is denoted by g, then we write Δ for its Laplace–Beltrami operator and define the Weingarten operator $A = (h^i_j)$ via

$$h_{ij} = g(A(\partial_i), \partial_j) = g_{ik}h^k_j.$$

Recall that the symmetry of h and the *Codazzi equations*

$$\nabla_i h_{jk} = \nabla_j h_{ik}$$

imply that the tensor ∇A is totally symmetric. Moreover, one can relate the geometry of a hypersurface M with the ambient manifold $\mathbb{M}_K^{n+1}$ by means of the *Gauss equation*

$$R_{ijk\ell} = h_{ik}h_{j\ell} - h_{i\ell}h_{jk} + K(g_{ik}g_{j\ell} - g_{i\ell}g_{jk}). \tag{2-1}$$

On the other hand, if

$$\kappa_1 \le \cdots \le \kappa_n$$

denote the eigenvalues of the operator A, that is, the principal curvatures of M, we consider the normalized mean curvatures H_ℓ defined as

$$H_\ell = \binom{n}{\ell}^{-1} \sigma_\ell, \quad \text{with } \sigma_\ell = \sum_{1 \le i_1 < \cdots < i_\ell \le n} \kappa_{i_1} \cdots \kappa_{i_\ell}.$$

In particular, $H_1 = H/n$ and H_n equals the Gauss curvature. We use the convention that $H_0 = 1$. For convex hypersurfaces, these symmetric functions satisfy the *Newton–MacLaurin* inequalities [Wang and Xia 2014]

$$H_{\ell-1}H_k \ge H_\ell H_{k-1} \quad \text{for } 1 \le k < \ell \le n. \tag{2-2}$$

We will also use the *Hsiung–Minkowski* identities [Guan and Li 2015]

$$(\ell+1)\int_M u\sigma_{\ell+1} = (n-\ell)\int_M \mathrm{c}_K \sigma_\ell \tag{2-3}$$

for $\ell = 0, \dots, n-1$.

Later on, we need to control the support function from below, given that there is a uniform ball enclosed by the evolving domain. Fortunately, for strictly convex domains, such control is easy to obtain. We furnish quantities like r and u with a hat if their reference point is not the origin. The right reference point will then be displayed as a subscript, and in cases where the reference point is clear by context, it is suppressed.

Lemma 2.1. *Let $\Omega \subset \mathbb{M}_K^{n+1}$ be a strictly convex domain with $p \in \Omega$ and $M = \partial\Omega$. Then the support function*

$$\hat{u}_p = \mathrm{s}_K(\hat{r}_p)\langle \partial_{\hat{r}_p}, \nu \rangle,$$

where $\hat{r}_p$ is the distance to the point p, satisfies

$$\hat{u}_p \ge \min_M \hat{u}_p = \min_M \mathrm{s}_K(\hat{r}_p) = \mathrm{s}_K(\operatorname{dist}(p, M)).$$

Proof. At a global minimum of the support function, we have $\nabla \hat{u}_p = 0$. It is well known (see [Guan and Li 2015]) that

$$\nabla_i \hat{u}_p = h_i^j \nabla_j \left(\frac{1 - \mathrm{c}_K}{K} \right) = -h_i^j \frac{\mathrm{c}_K'}{K} \nabla_j \hat{r}_p = \mathrm{s}_K(\hat{r}_p) h_i^j \nabla_j \hat{r}_p. \tag{2-4}$$

Accordingly, due to the invertibility of A, we also have $\nabla \hat{r}_p = 0$. Hence, at such a point and for $K > 0$,

$$\hat{u}_p = \mathrm{s}_K(\hat{r}_p) \ge \min\big(\mathrm{s}_K\big(\min_M \hat{r}_p\big), \mathrm{s}_K\big(\max_M \hat{r}_p\big)\big)$$

due to the concavity of s_K within the interval $[0, \pi/\sqrt{K}]$. In the case $K \leq 0$, we have that s_K is increasing, so we are done. Now suppose that

$$\mathrm{s}_K\big(\max_M \hat{r}_p\big) < \mathrm{s}_K\big(\min_M \hat{r}_p\big). \tag{2-5}$$

Due to the symmetries of the sine function we must then have

$$\min_M \hat{r}_p > \frac{\pi}{\sqrt{K}} - \max_M \hat{r}_p.$$

At a point $\xi \in M$, where $\hat{r}_p$ is maximized, the geodesic which connects p and ξ intersects M in another point, say $\zeta \in M$. Due to the convexity of M, we have

$$\frac{\pi}{\sqrt{K}} > \operatorname{dist}(\zeta, \xi) = \hat{r}_p(\zeta) + \hat{r}_p(\xi) \geq \hat{r}_p(\zeta) + \frac{\pi}{\sqrt{K}} - \min_M \hat{r}_p \geq \frac{\pi}{\sqrt{K}},$$

a contradiction. Hence (2-5) cannot be valid and the proof is complete. □

2.1. *Mixed volumes and quermassintegrals.* We define the *curvature integrals* or *mixed volumes* as

$$V_{n-\ell}(\Omega) = \int_M H_\ell \, dV \quad \text{for } \ell = 0, \dots, n.$$

On the other hand, for any connected domain $\Omega \subset \mathbb{M}_K^{n+1}$ bounded by a compact hypersurface of class C^3, the *quermassintegrals* are given by (see [Solanes 2006] or [Santaló 1976, Chapter 17])

$$W_\ell(\Omega) = \frac{(n+1-\ell)\omega_{\ell-1}\cdots\omega_0}{(n+1)\omega_{n-1}\cdots\omega_{n-\ell}} \int_{\mathcal{L}_\ell} \chi(L_\ell \cap \Omega) \, dL_\ell, \quad \ell = 1, \dots, n. \tag{2-6}$$

Here $\mathcal{L}_\ell$ represents the space of ℓ-dimensional totally geodesic subspaces L_ℓ in $\mathbb{M}_K^{n+1}$, where one can define a natural invariant measure dL_ℓ, and $\omega_n = |\mathbb{S}^n|$ is the area of the n-dimensional unit sphere in $\mathbb{R}^{n+1}$. If Ω is a convex set, then the function χ is equal to 1 if $L_\ell \cap \Omega \neq \varnothing$ and 0 otherwise.

One typically sets

$$W_0(\Omega) = |\Omega| \quad \text{and} \quad W_{n+1}(\Omega) = \frac{\omega_n}{n+1}.$$

Moreover, using the Cauchy–Crofton formula (see [Santaló 1976]), we recover the area of the hypersurface

$$|\partial\Omega| = (n+1) W_1(\Omega).$$

Accordingly, volume- and area-preserving flows can be regarded as particular cases of quermassintegral-preserving flows.

Mixed volumes and quermassintegrals are related (see [Solanes 2006, Proposition 7]) in a space of constant curvature $\mathbb{M}_K^{n+1}$ by means of

$$\begin{aligned} \frac{1}{n+1} V_{n-\ell}(\Omega) &= W_{\ell+1}(\Omega) - K \frac{\ell}{n+2-\ell} W_{\ell-1}(\Omega), \quad \ell = 1, \dots, n, \\ V_n(\Omega) &= (n+1) W_1(\Omega) = |\partial\Omega|. \end{aligned} \tag{2-7}$$

Notice that in $\mathbb{R}^{n+1}$ the mixed volumes coincide with the quermassintegrals, up to a constant factor.

The next result gathers the evolution equations of the quantities defined above under a normal variation.

Lemma 2.2. *If M_t is a hypersurface of $\mathbb{M}_K^{n+1}$ evolving along a flow given by $\partial_t x = \varphi \nu$, then*

(a) $\partial_t \operatorname{Vol}(\Omega_t) = \int_M \varphi \, dV_t$ *and* $\partial_t |M_t| = \int_M \varphi H \, dV_t$,

(b) $\partial_t \int_M H_\ell \, dV_t = \int_M \varphi((n-\ell)H_{\ell+1} - K\ell H_{\ell-1}) \, dV_t, \quad \ell = 0, \dots, n,$

(c) $\partial_t W_\ell(\Omega_t) = \frac{n+1-\ell}{n+1} \int_M \varphi H_\ell \, dV_t, \quad \ell = 0, \dots, n,$

(d) $\partial_t g_{ij} = 2\varphi h_{ij}$,

(e) $\partial_t h^i_j = -g^{ik} \nabla^2_{kj} \varphi - \varphi h^i_k h^k_j - K \varphi \delta^i_j$.

Proof. The formulas in (a) and (b) were deduced in [Reilly 1973]. The evolution in (c) follows by induction on ℓ and using the relation (2-7); see [Wang and Xia 2014, Proposition 3.1] for $K = -1$. The evolution for the metric and the Weingarten operator are standard, e.g., [Gerhardt 2006, Chapter 2]. □

Corollary 2.3. *If the global term in (1-1) is chosen as in (1-3), then the quermassintegral $W_\ell(\Omega_t)$ is constant along the flow (1-1).*

The fact that $\mu(t) > 0$ for strictly convex hypersurfaces is heavily used within the proof of Theorem 1.1.

Remark 2.4. Notice that a global term given by

$$\mu(t) = \frac{\int_M H((n-\ell)H_{\ell+1} - K\ell H_{\ell-1}) \, dV_t}{\int_M \mathrm{c}_K((n-\ell)H_{\ell+1} - K\ell H_{\ell-1}) \, dV_t}$$

leads to a flow that preserves the mixed volume $V_{n-\ell}(\Omega_t)$. Unlike the quermassintegral-preserving case, this term does not have a sign for convex hypersurfaces if $K > 0$. For $K < 0$ this difficulty disappears, but another type of mixed volume-preserving curvature flows for h-convex hypersurfaces in the hyperbolic space was already studied in [Makowski 2012].

We use the following conventions for constants. Indexed letters C, i.e., C_0, C_1, etc. will retain a specific meaning throughout the whole paper, while the letter C denotes a generic constant, which is always allowed to change from line to line and depends on the quantities listed in the formulation of the lemma or theorem. Capital letters also stand for "large" constants. A similar convention holds for lower case letters, which stand for "small" constants. The only exception from this convention concerns the use of various versions of the letter t, like T, τ, $\hat{\tau}$, etc. Those always refer in some way to time and t denotes the time variable, while T, τ, $\hat{\tau}$, etc. will, once defined, not change value.

3. Geometry and location of strictly convex hypersurfaces in the sphere

This section presents some geometric results for strictly convex hypersurfaces of the sphere, which are required to prove Theorem 1.1. In particular, we obtain inradius estimates in terms of pinching, as well as a suitable outball configuration in terms of pinching and the value of any given $W_\ell(\Omega)$. *Throughout Sections 3–5, we make the standing assumption that $M \subset \mathbb{S}_K^{n+1}$ is a strictly convex hypersurface enclosing a domain Ω.*

Let B_r denote a geodesic ball of radius r in $\mathbb{S}^{n+1}_K$. The outer radius of Ω is given by

$$\rho_+(\Omega) = \inf\{R > 0 : \Omega \subset B_R(q) \text{ for some } q \in \mathbb{S}^{n+1}_K\},$$

and the inner radius is given by

$$\rho_-(\Omega) = \sup\{\rho > 0 : B_\rho(p) \subset \Omega \text{ for some } p \in \mathbb{S}^{n+1}_K\}.$$

In [Andrews 1994a], it was shown that pinched hypersurfaces of the Euclidean space satisfy a uniform control of the outer radius by inner radius, and a version for a positive ambient space can be found in [Gerhardt 2015, Section 6]. The proof of this version relied on uniform positivity of the smallest principal curvature, which is insufficient for our purposes. Hence we provide a more general version in the following proposition.

Proposition 3.1. *If, for some number $C_0 > 0$, we have the pinching estimate $\kappa_n \le C_0\kappa_1$ in M, then the outer radius is estimated from above according to*

$$\rho_+(\Omega) \le C_1\rho_-(\Omega)$$

for some positive constant $C_1 = C_1(n, K, C_0)$.

Proof. For simplicity but without loss of generality, we assume $K = 1$. Due to a classical result [do Carmo and Warner 1970],

$$0 < \rho_+(\Omega) < \tfrac{1}{2}\pi$$

because M lies in some open hemisphere. Hence, there is a center q such that

$$\Omega \subset B_{\rho_+}(q),$$

and it is true that $q \in \bar{\Omega}$ (see [Santaló 1946, p. 455]). By moving q slightly inwards, we can achieve $M \subset B_{\pi/2}(q)$, and that M is star-shaped around q.

Now consider the stereographic projection from the antipodal point $-q$, where q is mapped to the origin $0 \in \mathbb{R}^{n+1}$. It follows (see [Gerhardt 2015, (6.15)]) that the metric $\bar{g}$ of $\mathbb{S}^{n+1}$ is conformal to the Euclidean metric; more precisely,

$$\bar{g} = e^{2\psi}(dr^2 + r^2\sigma), \quad \text{with } \psi(r) = -\ln\bigl(1 + \tfrac{1}{4}r^2\bigr).$$

Hereafter, we denote by tilde the Euclidean geometric quantities. On $\bar{B}_{\pi/2}(q)$ the metric $\bar{g}$ is uniformly equivalent to the Euclidean metric.

Next, from [Gerhardt 2006, (1.1.51)], we get

$$e^{\psi}\kappa_i = \tilde{\kappa}_i + d\psi(\tilde{\nu})$$

and hence, from our pinching assumption,

$$0 < C_0^{-1} \le \frac{\kappa_1}{\kappa_n} = \frac{\tilde{\kappa}_1 + d\psi(\tilde{\nu})}{\tilde{\kappa}_n + d\psi(\tilde{\nu})} \le 1.$$

Thus

$$\tilde{\kappa}_1 + d\psi(\tilde{\nu}) \ge C_0^{-1}(\tilde{\kappa}_n + d\psi(\tilde{\nu}))$$

and

$$\tilde{\kappa}_1 \ge C_0^{-1}\tilde{\kappa}_n + (C_0^{-1} - 1)\psi'(r)\langle\partial_r, \tilde{\nu}\rangle \ge C_0^{-1}\,\tilde{\kappa}_n$$

because $\psi' < 0$ and $\widetilde{M}$ is star-shaped, i.e., $\langle \partial_r, \tilde{\nu} \rangle > 0$. Therefore, the Euclidean hypersurface $\widetilde{M} \subset \widetilde{B}_2(0)$ is pinched, which from [Andrews 1994a, Lemma 5.4] leads to

$$\rho_+(\Omega) \le C\tilde{\rho}_+(\Omega) \le C\tilde{\rho}_-(\Omega) \le C_1\rho_-(\Omega),$$

where we have used the uniform equivalence of the ambient metrics. □

Corollary 3.2. *If, for some number $C_0 > 0$, we have the pinching estimate $\kappa_n \le C_0\kappa_1$ in M, then one can find positive constants d_1 and C_2, depending on n, K, C_0 and $W_\ell(\Omega)$, such that*

$$d_1 \le \rho_-(\Omega) \le C_2 < \frac{\pi}{2\sqrt{K}}.$$

Proof. By the definition of inner and outer radius, we can find points $p, q \in \mathbb{S}^{n+1}_K$ such that

$$B_{\rho_-(\Omega)}(p) \subset \Omega \subset B_{\rho_+(\Omega)}(q).$$

From (2-6), the quermassintegrals W_ℓ are clearly monotone under the inclusion of convex domains, and hence

$$W_\ell(B_{\rho_-(\Omega)}(p)) \le W_\ell(\Omega) \le W_\ell(B_{\rho_+(\Omega)}(q)).$$

We obtain, with Proposition 3.1,

$$C_2 := f_\ell^{-1}(W_\ell(\Omega)) \ge \rho_-(\Omega) \ge C_1^{-1}\rho_+(\Omega) \ge C_1^{-1} f_\ell^{-1}(W_\ell(\Omega)) =: d_1,$$

where f_ℓ denotes the increasing function given by $f_\ell(r) = W_\ell(B_r)$. □

The trivial outer radius estimate

$$\rho_+(\Omega) < \frac{\pi}{2\sqrt{K}}$$

is not good enough for our purposes. Now we present a refined estimate which should be of independent interest in the future.

Theorem 3.3. *The outer radius satisfies*

$$\rho_+(\Omega) \le \frac{\pi}{2\sqrt{K}} - \frac{\log W_\ell(\mathcal{H}) - \log W_\ell(\Omega)}{(n+1-\ell)\max_M H} =: \frac{\pi}{2\sqrt{K}} - \frac{d_2}{\max_M H},$$

where $\mathcal{H}$ is an open hemisphere.

Proof. From the initial hypersurface $M_0 = M$, we start the curvature flow

$$x : [0, T^*) \times \mathbb{S}^n \to \mathbb{S}^{n+1}_K, \quad \partial_t x = \frac{H_{\ell-1}}{H_\ell}\nu. \tag{3-1}$$

Then Lemma 2.2 (c) ensures that W_ℓ evolves in time according to

$$\partial_t W_\ell(\Omega_t) = \frac{n+1-\ell}{n+1} V_{n-\ell+1}(\Omega_t) \le (n+1-\ell)W_\ell(\Omega_t),$$

where the inequality follows from (2-7).

From [Gerhardt 2015; Makowski and Scheuer 2016], we know that (3-1) preserves the strict convexity and the solution converges smoothly to an equator, while we also have the estimate

$$W_\ell(\Omega_t) \le W_\ell(\Omega) e^{(n+1-\ell)t}.$$

Hence the maximal existence time of (3-1) is at least

$$T^* \ge \frac{1}{n+1-\ell}(\log W_\ell(\mathcal{H}) - \log W_\ell(\Omega))$$

because we know that $W_\ell(\Omega)$ must converge to $W_\ell(\mathcal{H})$ at T^*.

Now [Makowski and Scheuer 2016, Lemma 4.7] leads to the curvature bound

$$\max_{M_t} H \le \max_{M_0} H \quad \text{for all } 0 \le t < T^*.$$

Due to the convexity this implies a full second fundamental form bound, as well as a bound

$$\frac{H_\ell}{H_{\ell-1}} \le H_1 \le \max_{M_0} H, \tag{3-2}$$

which follows by application of (2-2) for $k = 1$.

Next, let $E = \partial\mathcal{H}$ be the limiting equator of the flow, and let r be the radial distance from the center of $\mathcal{H}$, which contains all M_t. Define

$$\tilde{r}(t) = \max_{\mathbb{S}^n} r(t, \cdot\,) = r(t, \xi_t),$$

where ξ_t is chosen to be any point where the maximum is realized. The function $\tilde{r}$ is Lipschitz and hence differentiable almost everywhere. At each time t, where $\tilde{r}$ is differentiable, we have

$$\frac{d}{dt}\tilde{r}(t) = \frac{H_{\ell-1}}{H_\ell}(t, \xi_t),$$

where we used that $\nu(t, \xi_t) = \partial_r$. Integration and (3-2) yield

$$\frac{\pi}{2\sqrt{K}} - \tilde{r}(0) = \tilde{r}(T^*) - \tilde{r}(0) = \int_0^{T^*} \frac{d}{dt}\tilde{r} \ge \frac{\log W_\ell(\mathcal{H}) - \log W_\ell(\Omega)}{(n+1-\ell)\max_{M_0} H}.$$

Hence

$$\max_{M_0} r = \tilde{r}(0) \le \frac{\pi}{2\sqrt{K}} - \frac{\log W_\ell(\mathcal{H}) - \log W_\ell(\Omega)}{(n+1-\ell)\max_{M_0} H}.$$

Accordingly, M_0 fits into a neighborhood of the origin of size given by the right-hand side of the latter inequality, and therefore the outer radius is controlled by the very same quantity. □

Remark 3.4. A lune, i.e., the intersection of two hemispheres, shows that an outer radius bound in terms of $W_\ell(\Omega)$ cannot be independent of $\max H$.

Theorem 3.3 enables us to find, for a given strictly convex hypersurface of the sphere, a suitable origin, which allows a ball of controlled size within Ω and at the same time ensures a controlled positive distance of M to the equator.

Lemma 3.5. *There exists an origin $\mathcal{O} \in \Omega$ such that, with the constant d_2 from Theorem 3.3, we have*

$$B_{4\epsilon}(\mathcal{O}) \subset \Omega \quad \textit{and} \quad \max_M r \le \frac{\pi}{2\sqrt{K}} - 4\epsilon \tag{3-3}$$

for all

$$\epsilon \le \frac{1}{4} \min\left(\frac{d_2}{2 \max_M H}, \frac{\frac{1}{2}\pi - \tan^{-1}(\max_M H/\sqrt{K})}{2\sqrt{K}} \right).$$

Proof. If $B_{\rho_+(\Omega)}$ denotes an outball for Ω with center $\mathcal{O}$, then we know that $\mathcal{O} \in \bar{\Omega}$. The distance of focal points from M can be calculated from the evolution of the Weingarten operator in Lemma 2.2 along the normal variation with speed $\varphi = -1$. Then the largest principal curvature is controlled by the solution to the ODE

$$y' = y^2 + K,$$
$$y(0) = \max_M H,$$

which exists for all $t < t_0$, with

$$t_0 := \frac{\pi}{2\sqrt{K}} - \frac{\tan^{-1}(\max_M H/\sqrt{K})}{\sqrt{K}}.$$

Hence, around all points belonging to the set

$$\left\{x \in \Omega : \operatorname{dist}(x, M) = \tfrac{1}{2} t_0\right\},$$

there exists an interior ball of radius $\frac{1}{2}t_0$. In addition, if we shift $\mathcal{O}$ by a distance of $d_2/(2 \max_M H)$ in any direction, there still is the same amount of space between M and the new equator. Therefore, if we shift $\mathcal{O}$ into Ω along a perpendicular geodesic only by the amount

$$\epsilon \le \epsilon_0 = \frac{1}{4} \min\left(\frac{d_2}{2 \max_M H}, \frac{t_0}{2} \right),$$

then (3-3) holds. □

4. Monotonicity of the pinching deficit

The geometric results from Section 3 depend on the quality of the pinching and the size of the quermassintegral. In the following we investigate how these quantities behave under the flow (1-1). As this flow is defined to be quermassintegral-preserving, the key ingredient for proving Theorem 1.1 is the pinching estimate to be proven in this section.

Again, we assume $K > 0$, unless stated otherwise. As a first step, we need the following evolution equations.

Lemma 4.1. *For every choice of origin $\mathcal{O}$, along (1-1), the induced metric g and second fundamental form h satisfy the evolution equations*

$$\partial_t g_{ij} = 2(\mu c_K - H) h_{ij},$$
$$\partial_t h^i_j = \Delta h^i_j + (|A|^2 - nK) h^i_j + 2KH\delta^i_j - \mu(c_K h^i_k h^k_j + K u h^i_j),$$

where u is the generalized support function in (1-5) of M_t with respect to the origin $\mathcal{O}$.

Proof. The evolution of the metric comes from Lemma 2.2 (d). For the evolution of A, we depart from the standard evolution equation

$$\partial_t h_{ij} = \nabla^2_{ij}(H - \mu c_K) + (\mu c_K - H)(h_{ik}h^k_j - K g_{ij})$$

(see [Andrews 1994b, Theorem 3-15]) and use the Simons-type identity

$$\nabla^2_{ij} H = \Delta h_{ij} + (|A|^2 - nK)h_{ij} + H(K g_{ij} - h^k_i h_{jk}).$$

Now we expand the second derivatives of c_K:

$$-\nabla^2_{ij} c_K = d c_K(\nu) h_{ij} - \overline{\nabla}^2 c_K(x_i, x_j) = c'_K \langle \partial_r, \nu\rangle h_{ij} + K c_K g_{ij}, \tag{4-1}$$

where we have used

$$\overline{\nabla}^2 c_K = -K c_K \bar{g}.$$

The proof is complete, using the evolution of the metric to revert to h^i_j. □

Remark 4.2. Notice that the strong maximum principle for tensors applied to the evolution of h^i_j already implies that the property of strict convexity is preserved for all times. Accordingly, $H > 0$, and the quotient κ_1/H is well defined as long as the flow exists.

Next we deduce an evolution equation that is the key to convergence of the flow.

Proposition 4.3. *Let κ_1 be the smallest eigenvalue of A. Then, for every choice of origin $\mathcal{O}$, under the flow (1-1) with initial data M, the function*

$$p = \frac{\kappa_1}{H}$$

is a supersolution to the following evolution equation in the viscosity sense, as long as the flow exists:

$$\partial_t p - \Delta p = \frac{2}{H}\sum_{k=1}^{n}\sum_{j>D}\frac{(\nabla_k h^1_j)^2}{\kappa_j - \kappa_1} + 2\, dp(\nabla \log H) + \frac{\mu}{H} c_K |\mathring{A}|^2 p + \mu c_K \kappa_1\Big(\frac{1}{n} - p\Big) + 2nK\Big(\frac{1}{n} - p\Big),$$

where D is the multiplicity of κ_1.

Proof. Assume that the flow is defined on a maximal time interval $[0, T)$. Let $(t_0, \xi_0) \in (0, T) \times M$, and let η be a smooth lower support of p at (t_0, ξ_0), i.e., η is defined on a spacetime neighborhood $\mathcal{U}$ of (t_0, ξ_0), and we have

$$\eta(t_0, \xi_0) = p(t_0, \xi_0), \qquad \eta \le p_{|\mathcal{U}}.$$

Hence $\varphi = H\eta$ is a smooth lower support for κ_1.

Now we take coordinates with the properties

$$g_{ij} = \delta_{ij}, \qquad h^i_j = \kappa_j \delta^i_j \quad \text{at } (t_0, \xi_0).$$

If we denote by D the multiplicity of $\kappa_1(t_0, \xi_0)$, then at the point (t_0, ξ_0) and for all $1 \le i,\ j \le D$, we have (see [Brendle et al. 2017, Lemma 5])

$$\partial_t h^i_j = \delta^i_j \partial_t \varphi, \qquad \nabla_k h^i_j = \delta^i_j \nabla_k \varphi$$

and

$$\nabla^2_{kk}\varphi \le \nabla^2_{kk}h^1_1 - 2\sum_{j>D}\frac{(\nabla_k h^1_j)^2}{\kappa_j-\kappa_1}. \tag{4-2}$$

Next, write

$$G^i_j = \partial_t h^i_j - \Delta h^i_j,$$

and compute at (t_0, ξ_0)

$$\begin{aligned}\partial_t\eta = \frac{\partial_t h^1_1}{H} - \frac{\varphi}{H^2}\partial_t H &= \frac{G^1_1 + g^{kl}\nabla^2_{kl}h^1_1}{H} - \frac{\varphi}{H^2}\partial_t H\\ &\ge \frac{1}{H}\sum_{k=1}^n\biggl(\nabla^2_{kk}\varphi + 2\sum_{j>D}\frac{(\nabla_k h^1_j)^2}{\kappa_j-\kappa_1}\biggr) + \frac{G^1_1}{H} - \frac{\varphi}{H^2}\partial_t H\\ &= \frac{2}{H}\sum_{k=1}^n\sum_{j>D}\frac{(\nabla_k h^1_j)^2}{\kappa_j-\kappa_1} + \frac{G^1_1}{H} + \Delta\eta + 2\,d\eta(\nabla\log H) - \frac{\eta}{H}G^k_k.\end{aligned}$$

On the other hand, we know by Lemma 4.1 that

$$G^i_j = (|A|^2 - nK)h^i_j + 2KH\delta^i_j - \mu(\mathrm{c}_K h^i_k h^k_j + Kuh^i_j).$$

Accordingly, we get

$$G^1_1 - \eta G^k_k = -\mu\mathrm{c}_K\kappa_1^2 + \eta\mu\mathrm{c}_K|A|^2 + 2KH(1-n\eta).$$

Finally, by means of

$$|A|^2 = |\mathring{A}|^2 + \frac{1}{n}H^2,$$

we obtain

$$G^1_1 - \eta G^k_k = \eta\mu\mathrm{c}_K|\mathring{A}|^2 + \mu\mathrm{c}_K\kappa_1 H\Bigl(\frac{1}{n}-\eta\Bigr) + 2KH(1-n\eta). \qquad \square$$

Corollary 4.4. *For every choice of origin $\mathcal{O}$ for which $M \subset \mathcal{H}(\mathcal{O})$, the flow (1-1) with initial data M stays in $\mathcal{H}(\mathcal{O})$ and improves every pinching, i.e., the* ***pinching deficit***

$$\omega(t) = \frac{1}{n} - \min_{M_t}\frac{\kappa_1}{H}$$

is exponentially decreasing:

$$\omega(t) \le \omega(s)e^{-2nK(t-s)} \quad \text{for all } 0 \le s \le t < T,$$

where T is the maximal time of existence of the flow with initial data M.

Proof. We need the evolution equation of c_K. From (1-1) and (4-1), we obtain

$$\partial_t\mathrm{c}_K = \mathrm{c}'_K\partial_t r = -K\mathrm{s}_K\,dr(\partial_t x) = Ku(H-\mu\mathrm{c}_K) = \Delta\mathrm{c}_K + K\mathrm{c}_K(n-\mu u).$$

The preservation of $\mathrm{c}_K > 0$ follows immediately by the strong maximum principle, as long as the flow exists.

The strict convexity is also preserved from Remark 4.2. The statement about the pinching deficit follows from the strong maximum principle for viscosity solutions, e.g., see [Da Lio 2004], and from the fact that

$$\partial_t \omega \leq -2nK\omega,$$

where we can discard the terms including μ because $c_K > 0$ and $\mu(t) > 0$ by convexity. □

In particular, for $s = 0$, we reach a pinching relation between the biggest and smallest principal curvatures of M_t:

$$\kappa_1 \geq \left(\min_{M_0} \frac{\kappa_1}{H}\right) H \geq C_0^{-1} \kappa_n, \tag{4-3}$$

provided that an origin is chosen such that the strictly convex initial hypersurface is contained in the open hemisphere centered at that origin.

5. Upper curvature bounds

Notice that, unlike in previous treatments of quermassintegral-preserving curvature flows, an upper bound for the global term does not come automatically from an upper bound for H, since the c_K in the denominator of μ is not uniformly bounded away from zero, at least not without further work. Moreover, we need some uniform control of μ to get bounds for H.

To overcome these difficulties, the idea is to choose the origin such that a configuration as in Lemma 3.5 is achieved, which will allow us to deduce uniform bounds on the curvature and the global term in a short but controlled interval $[0, \tau(\epsilon)]$ (Lemma 5.4). Then, since the pinching is at least as good as at the beginning, we can repeat this process as often as needed, in order to keep the flow going as long as we like (Lemma 6.1). During this evolution, the pinching improves exponentially and at some point will be strong enough that M_t is very close to a sphere. From here the flow is very easy to estimate and no further shifting of the origin is necessary. Now we will implement all the required steps to make this argument rigorous.

A key idea to obtain a curvature bound is to adapt a well-known trick from [Tso 1985], which consists in a suitable combination of the generalized support function with the mean curvature. For this we need control on the size of inballs during the flow.

A lower bound for the support function of an arbitrary inball. For a domain Ω and a point $p \in \Omega$, we say that B is an *inball at* p if p is the center of B and B has maximal radius with the property that $B \subset \Omega$. In the sequel we are going to prove that, along the flow, the radii of inballs at p don't decrease too quickly. We give a quantitative estimate.

We need to be careful because now we are dealing with two different support functions: we denote by u the support function with respect to the origin $\mathcal{O}$ that is implicit in the flow equation, while for a given point $p \in \Omega$,

$$\hat{u} \equiv \hat{u}_p = s_K(\hat{r}_p)\langle \partial_{\hat{r}_p}, \nu \rangle$$

takes another interior point p as the origin of distances. Accordingly, r and $\hat{r}$ mean distance from $\mathcal{O}$ and p, respectively. Similar notations will apply to the corresponding trigonometric functions, i.e.,

$$\hat{\mathrm{s}}_K := \mathrm{s}_K(\hat{r}_p) \quad \text{and} \quad \hat{\mathrm{c}}_K := \mathrm{c}_K(\hat{r}_p).$$

Note that for brevity we suppress the dependence on the point p within the notation $\hat{u}$ and $\hat{r}$.

Lemma 5.1. *For every choice of origin, along (1-1), the evolution equations of the mean curvature H and the support function $\hat{u} = \hat{u}_p$ are given by*

$$\partial_t H = \Delta H + H(|A|^2 + Kn) - \mu(\mathrm{c}_K|A|^2 + uKH),$$
$$\partial_t \hat{u} = \Delta\hat{u} + \hat{u}|A|^2 + (\mu\mathrm{c}_K - 2H)\hat{\mathrm{c}}_K + \mu K\mathrm{s}_K\hat{\mathrm{s}}_K\langle\nabla r, \nabla\hat{r}\rangle.$$

Proof. The formula for H follows directly by taking the trace in the evolution equation for h^i_j from Lemma 4.1. On the other hand, a standard calculation leads to

$$\begin{aligned}\partial_t\hat{u} &= \langle\overline{\nabla}_t(\hat{\mathrm{s}}_K\partial_{\hat{r}}), \nu\rangle + \langle\hat{\mathrm{s}}_K\partial_{\hat{r}}, \overline{\nabla}_t\nu\rangle\\ &= (\mu\mathrm{c}_K - H)\hat{\mathrm{c}}_K + \hat{\mathrm{s}}_K\langle\partial_{\hat{r}}, \nabla(H - \mu\mathrm{c}_K)\rangle\\ &= \Delta\hat{u} + (\mu\mathrm{c}_K - 2H)\hat{\mathrm{c}}_K + \hat{u}|A|^2 - \mu\,\hat{\mathrm{s}}_K\langle\partial_{\hat{r}}, \nabla\mathrm{c}_K\rangle,\end{aligned}$$

where we applied well-known formulas for $\Delta\hat{u}$ and $\overline{\nabla}_t(\hat{\mathrm{s}}_K\partial_{\hat{r}})$ on $\mathbb{M}^{n+1}_K$ (see [Cabezas-Rivas and Miquel 2007, (4.6) and (4.11)]). The stated formula follows by realizing that

$$\langle\partial_{\hat{r}}, \nabla\mathrm{c}_K\rangle = -K\mathrm{s}_K\langle\partial_{\hat{r}}, \nabla r\rangle. \qquad \square$$

Proposition 5.2. *For every choice of origin $\mathcal{O}$ for which $M \subset \mathcal{H}(\mathcal{O})$, and for every $p \in \Omega$ and radius ρ with the property $B_\rho(p) \subset \Omega$, the solution $M_t = \partial\Omega_t$ of (1-1), with initial data M and maximal existence time $T > 0$, satisfies the following:*

(i) *There is a positive constant $\tilde{\tau} = \tilde{\tau}(n, K, \rho)$ with the property*

$$B_{\rho/4}(p) \subset \Omega_t \quad \textit{for all } t \in [0, \min(\tilde{\tau}, T)).$$

(ii) *One can find positive constants $d_3 \le 1/(2\sqrt{K})$ and τ, depending on $n,\ K,\ C_0$ and $W_\ell(\Omega)$, with the property*

$$\hat{u}_{p_\Omega} - 2d_3 \ge 2d_3 > 0 \quad \textit{for all } t \in [0, \min(\tau, T)),$$

where p_Ω is the center of an inball corresponding to the inradius $\rho_-(\Omega)$.

Proof. (i) Let us first obtain the evolution of the distance $\hat{r} = \hat{r}_p$ from the fixed point p to the points on M_t under the flow (1-1):

$$\partial_t\hat{r} = d\hat{r}(\partial_t x) = (\mu\mathrm{c}_K - H)\langle\nu, \partial_{\hat{r}}\rangle. \tag{5-1}$$

On the other hand, $\mathtt{r}(t)$ denotes the radius of a geodesic sphere centered at p that moves under the ordinary mean curvature flow starting at $\mathtt{r}(0) = \frac{1}{2}\rho$, that is,

$$\mathtt{r}'(t) = -n\mathrm{co}_K(\mathtt{r}(t)),$$

whose solution is given by

$$\mathrm{c}_K(\mathtt{r}(t)) = e^{Knt}\mathrm{c}_K\left(\tfrac{1}{2}\rho\right) \quad \text{for } t \geq 0.$$

As c_K is a decreasing function,

$$\mathtt{r}(t) \geq \tfrac{1}{4}\rho \quad \Longleftrightarrow \quad e^{Knt}\mathrm{c}_K\left(\tfrac{1}{2}\rho\right) \leq \mathrm{c}_K\left(\tfrac{1}{4}\rho\right),$$

meaning that

$$\mathtt{r}(t) \geq \tfrac{1}{4}\rho \quad \Longleftrightarrow \quad t \leq \frac{1}{Kn}\log\frac{\mathrm{c}_K\left(\frac{1}{4}\rho\right)}{\mathrm{c}_K\left(\frac{1}{2}\rho\right)} =: \tilde{\tau}.$$

Set $f(t,\cdot) = \hat{r}(t,\cdot) - \mathtt{r}(t)$ for $t \in [0, \min(\tau, T))$. Then $f(0,\cdot) > 0$, and f evolves according to

$$\partial_t f = (\mu \mathrm{c}_K - H)\langle \nu, \partial_{\hat{r}}\rangle + n\mathrm{co}_K(\mathtt{r}(t)).$$

If there exists a first time t_1 such that the geodesic sphere $B_{\mathtt{r}(t_1)}$ touches the hypersurface M_{t_1} at some point x_1, then at this first minimum for f we have $H(x_1,t_1) \leq n\mathrm{co}_K(\mathtt{r}(t_1))$, $\langle \partial_{\hat{r}}, \nu\rangle = 1$ and $\partial_t f(x_1,t_1) \leq 0$. Consequently, taking into account that $\mathrm{c}_K > 0$ and strict convexity is preserved, we have

$$\partial_t f(x_1,t_1) \geq \mu(t_1)\mathrm{c}_K(r(t_1)) > 0,$$

which is a contradiction, and hence the statement follows.

(ii) Apply (i) with $p = p_\Omega$ and $\rho = \rho_-(\Omega)$, and obtain the desired τ due to Corollary 3.2. Using Lemma 2.1, we get

$$\hat{u}_{p_\Omega} \geq \mathrm{s}_K\left(\tfrac{1}{4}\rho_-(\Omega)\right) \geq \mathrm{s}_K\left(\tfrac{1}{4}d_1\right) =: 4d_3, \tag{5-2}$$

where we applied the lower bound in Corollary 3.2. □

An upper bound for the mean curvature. To estimate the mean curvature along a solution with suitably located initial data, we use the well-known auxiliary function

$$\Phi_p = \frac{H}{\hat{u}_p - d_3},$$

which, after choosing the origin $\mathcal{O}$ as in Proposition 5.2, is well defined for a while for some suitable $p \in \Omega$. Routine computations lead to the evolution of Φ, where we suppress the dependence on p within the notation.

Lemma 5.3. *Under the assumptions of Proposition 5.2, along the flow (1-1), the function Φ evolves according to*

$$\partial_t\Phi = \Delta\Phi + \frac{2}{\hat{u}-d_3}\langle\nabla\Phi,\nabla\hat{u}\rangle + \Phi\left(nK - \frac{d_3}{\hat{u}-d_3}|A|^2\right) + 2\Phi^2\hat{\mathrm{c}}_K$$
$$- \mu K\Phi u - \frac{\mu}{\hat{u}-d_3}(\mathrm{c}_K|A|^2 + \Phi(K\mathrm{s}_K\hat{\mathrm{s}}_K\langle\nabla r,\nabla\hat{r}\rangle + \mathrm{c}_K\hat{\mathrm{c}}_K)).$$

Unlike in previous literature, we cannot neglect all the terms including μ, as some do not have a sign. Nor is it known at this point that μ is bounded. The novelty of our approach is to make use of Lemma 3.5, a configuration which enables us to gain some control on μ and then get an estimate on Φ and H.

Another complication arises from the necessity of using an iterative change of origin. The configuration of Lemma 3.5 depends on curvature. Hence we need a very precise estimate of curvature as the flow progresses, and it is insufficient to estimate the curvature by a multiple of its initial value, as then our time interval, along which $B_\epsilon(\mathcal{O}) \subset \Omega_t$ is valid, would decrease and we would not be able to prove long-time existence.

For this reason, we introduce a novel method, which also gives an interesting alternative to proving initial value-independent curvature bounds after a waiting time, as for example in [McCoy 2004, Equation (17)]. It provides a bound on Φ, which is uniform in p lying within a certain region.

With this purpose, we define a modified auxiliary function

$$\Psi : [0,T) \times \mathbb{S}^n \times \mathbb{S}^{n+1}_K \to \mathbb{R}, \quad (t,\xi,p) \mapsto \begin{cases} (\operatorname{dist}(p,M_t) - 2d_3)\Phi_p, & p \in V_t, \\ 0, & \text{otherwise}, \end{cases}$$

with

$$V_t := \{p \in \Omega_t : \operatorname{dist}(p, M_t) > 2d_3\},$$

and where for $p \notin \bar{\Omega}$ the distance to M_t is defined to be negative. In short,

$$\Psi(t,\xi,p) = \max\bigl(0, \min_{M_t} \hat{r}_p - 2d_3\bigr)\Phi_p.$$

Note that Ψ is Lipschitz, because when $\min_{M_t} \hat{r}_p = \operatorname{dist}(p, M_t) \geq 2d_3$, then, by Lemma 2.1,

$$\hat{u}_p \geq \mathrm{s}_K\bigl(\min_{M_t} \hat{r}_p\bigr) \geq \mathrm{s}_K(2d_3)$$

due to $\min_{M_t} \hat{r}_p < \pi/(2\sqrt{K})$. Furthermore,

$$\hat{u}_p \geq \mathrm{s}_K(2d_3) = \frac{1}{\sqrt{K}} \sin(2d_3\sqrt{K}) \geq \tfrac{6}{5} d_3,$$

where we used $\sin x \geq \frac{3}{5}x$ for $x \in \bigl[0, \frac{1}{2}\pi\bigr]$. Hence Φ_p is well defined for $p \in \bar{V}_t$. In the sequel we write, for brevity,

$$\max_{M_t} \Psi = \max_{\mathbb{S}^n \times \mathbb{S}^{n+1}_K} \Psi(t, \cdot).$$

Lemma 5.4. *There exists an origin $\mathcal{O} \in \Omega$, a constant $C_3(n, K, C_0, W_\ell(\Omega))$, and constants τ and ϵ_1, depending on n, K, C_0, $W_\ell(\Omega)$, $\max(C_3, \max_{M_0} \Psi)$, such that, for the solution $M_t = \partial\Omega_t$ of (1-1) with initial data $M_0 = M$ and maximal existence time $T > 0$, we have:*

(i) *$M_t \subset \mathcal{H}(\mathcal{O})$ for all $t \in [0, \min(\tau, T)]$.*

(ii) *$\hat{u}_{p_\Omega} \geq 4d_3$ and $B_{\epsilon_1}(\mathcal{O}) \subset \Omega_t$ for all $t \in [0, \min(\tau, T))$.*

(iii) *The function Ψ satisfies*

$$\max\bigl(C_3, \max_{M_t} \Psi\bigr) \leq \max\bigl(C_3, \max_{M} \Psi\bigr) \quad \textit{for all } t \in [0, \min(\tau, T)).$$

Proof. Define the functions

$$\epsilon_0(y) = \frac{1}{4}\min\left(\frac{d_2 d_3\sqrt{K}}{2y}, \frac{\frac{1}{2}\pi - \tan^{-1}(y/d_3K)}{2\sqrt{K}}\right) \quad \text{for } y > 0$$

and

$$q(y) = \frac{n\pi C_0^{\ell+1}}{\sqrt{K}}\left(1 + \frac{5\pi}{\sqrt{K}d_3}\right)\frac{y}{\epsilon_0(y)} + \frac{n\pi^2}{4}y + \frac{\pi}{\sqrt{K}}y^2 - \frac{d_3^2}{n}y^3,$$

where d_3 and C_0 are the constants from Proposition 5.2 and (4-3), respectively. It is clear that $q(y)$ converges to $-\infty$ as $y \to \infty$, and hence it has a largest zero $\bar{y}$, which only depends on n, K, C_0 and $W_\ell(\Omega)$. Let $\psi_0 = \max_{t=0}\Psi$, and define

$$\epsilon_1 = \epsilon_0(\max(\bar{y}, \psi_0)).$$

(i) & (ii) With p_Ω from Proposition 5.2 (ii) and by definition of d_3 in (5-2), along $M_0 = M$ we have

$$H = (\hat{u}_{p_\Omega} - d_3)\Phi_{p_\Omega} \le \frac{\Phi_{p_\Omega}}{\sqrt{K}} \le \frac{\mathrm{s}_K(\min_M \hat{r}_{p_\Omega}) - 2d_3}{2d_3}\frac{\Phi_{p_\Omega}}{\sqrt{K}} \le \frac{1}{d_3\sqrt{K}}\psi_0,$$

and hence

$$\epsilon_1 \le \epsilon_0\big(d_3\sqrt{K}\max_M H\big) < \frac{\pi}{2\sqrt{K}},$$

where the latter estimate is due to the definition of ϵ_0. This is exactly the threshold required to apply Lemma 3.5.

Then we can apply Lemma 3.5 with $\epsilon = \epsilon_1$ in order to obtain a suitable origin $\mathcal{O} \in \Omega$ with the property (3-3). From the first part of Proposition 5.2 applied to $p = \mathcal{O}$ and $\rho = 4\epsilon_1$, as well as from the second part of Proposition 5.2, we obtain $\tau = \tau(n, K, C_0, W_\ell(\Omega), \epsilon_1)$, up to which the claimed properties of (ii) are satisfied. Property (i) is then clear from the fact that, at the equator, $\mathrm{c}_K = 0$, and hence $(d/dt)\max r < 0$.

(iii) Next, we bound the function Ψ. Suppose Ψ attains a positive maximum over the set $[0, \bar{t}] \times \mathbb{S}^n \times \mathbb{S}^{n+1}_K$ at some $(\bar{t}, \bar{\xi}, \bar{p})$. Define the Lipschitz function

$$\overline{\Psi}_{\bar{p}}(t) = \max_{\mathbb{S}^n}\Psi(t, \cdot, \bar{p}),$$

which is positive in some small interval $J := [\bar{t} - \delta, \bar{t}]$. Thus we have

$$\mathrm{dist}(\bar{p}, M_t) > 2d_3 \quad \text{for all } t \in J.$$

Hence in J the function $\Phi_{\bar{p}}$ is smooth, and

$$\overline{\Psi}_{\bar{p}}(t) = \big(\min_{M_t}\hat{r}_{\bar{p}} - 2d_3\big)\max_{M_t}\Phi_{\bar{p}}$$

is differentiable almost everywhere in J.

For almost every $t \in J$, using (5-1) and $\langle \partial_{\hat{r}_{\bar{p}}}, \nu \rangle > 0$ because $\bar{p} \in \Omega_t$ for all $t \in J$, we have

$$\begin{aligned}\frac{d}{dt}\overline{\Psi}_{\bar{p}} &= \frac{d}{dt}\min_{M_t}\hat{r}_{\bar{p}}\max_{M_t}\Phi_{\bar{p}} + \bigl(\min_{M_t}\hat{r}_{\bar{p}} - 2d_3\bigr)\frac{d}{dt}\max_{M_t}\Phi_{\bar{p}} \\ &= \bigl(\mathrm{c}_K(r)\mu - H\bigl(\bar{t}, x\bigl(\bar{t}, \operatorname{argmin}_{M_{\bar{t}}}\hat{r}_{\bar{p}}\bigr)\bigr)\bigr)\langle \partial_{\hat{r}_{\bar{p}}}, \nu \rangle \max_{M_t}\Phi_{\bar{p}} + \bigl(\min_{M_{\bar{t}}}\hat{r}_{\bar{p}} - 2d_3\bigr)\frac{d}{dt}\max_{M_t}\Phi_{\bar{p}} \\ &\le \mu\max_{M_t}\Phi_{\bar{p}} + \bigl(\min_{M_{\bar{t}}}\hat{r}_{\bar{p}} - 2d_3\bigr)\frac{d}{dt}\max_{M_t}\Phi_{\bar{p}}.\end{aligned}$$

Inserting the evolution equation from Lemma 5.3, discarding two good terms, and using $\nabla\Phi_{\bar{p}}(t, \xi_t) = 0$ at all maximizers ξ_t, we obtain

$$\begin{aligned}\frac{d}{dt}\overline{\Psi}_{\bar{p}} \le \mu\max_{M_t}\Phi_{\bar{p}} + \bigl(\min_{M_{\bar{t}}}\hat{r}_{\bar{p}} - 2d_3\bigr)\biggl(\max_{M_t}\Phi_{\bar{p}}\biggl(nK - \frac{d_3}{\hat{u}_{\bar{p}} - d_3}|A|^2\biggr) + 2\max_{M_t}\Phi_{\bar{p}}^2 \\ - \frac{\mu}{\hat{u}_{\bar{p}} - d_3}\Bigl(\max_{M_t}\Phi_{\bar{p}}(K\mathrm{s}_K\hat{\mathrm{s}}_K\langle\nabla r, \nabla\hat{r}_{\bar{p}}\rangle + \mathrm{c}_K\hat{\mathrm{c}}_K)\Bigr)\biggr),\end{aligned}$$

where we used that u and c_K are positive due to $M_t \subset \mathcal{H}(\mathcal{O})$ and $\mathcal{O} \in \Omega_t$. Using $\mathrm{s}_K \le K^{-1/2}$ and

$$\frac{\min_{M_{\bar{t}}}\hat{r}_{\bar{p}} - 2d_3}{\hat{u}_{\bar{p}} - d_3}(K\mathrm{s}_K\hat{\mathrm{s}}_K\langle\nabla r, \nabla\hat{r}_{\bar{p}}\rangle + \mathrm{c}_K\hat{\mathrm{c}}_K) \le 2\cdot\frac{\pi/(2\sqrt{K})}{\frac{1}{5}d_3} = \frac{5\pi}{d_3\sqrt{K}},$$

we get

$$\frac{d}{dt}\overline{\Psi}_{\bar{p}} \le \biggl(1 + \frac{5\pi}{\sqrt{K}d_3}\biggr)\mu\max_{M_t}\Phi_{\bar{p}} + nK\overline{\Psi}_{\bar{p}} + 2\overline{\Psi}_{\bar{p}}\max_{M_t}\Phi_{\bar{p}} - \frac{d_3^2}{n}\overline{\Psi}_{\bar{p}}\max_{M_t}\Phi_{\bar{p}}^2.$$

By means of the Hsiung–Minkowski identity (2-3) and Lemma 2.1, we estimate, for all $0 \le t \le \tau$,

$$\mu(t) \le n\binom{n}{\ell+1}\frac{\int_{M_t}\kappa_n^{\ell+1}}{\int_{M_t}u\sigma_{\ell+1}} \le \frac{n}{\mathrm{s}_K(\epsilon_1)}\frac{\int_{M_t}\kappa_n^{\ell+1}}{\int_{M_t}\kappa_1^{\ell+1}} \le \frac{2nC_0^{\ell+1}}{\epsilon_1}, \tag{5-3}$$

where the constant C_0 comes from the pinching (4-3). Hence

$$\begin{aligned}\frac{d}{dt}\overline{\Psi}_{\bar{p}} &\le 2nC_0^{\ell+1}\biggl(1 + \frac{5\pi}{\sqrt{K}d_3}\biggr)\frac{\max_{M_t}\Phi_{\bar{p}}}{\epsilon_0(\max(\bar{y}, \psi_0))} + nK\overline{\Psi}_{\bar{p}} + 2\overline{\Psi}_{\bar{p}}\max_{M_t}\Phi_{\bar{p}} - \frac{d_3^2}{n}\overline{\Psi}_{\bar{p}}\max_{M_t}\Phi_{\bar{p}}^2 \\ &= \frac{2nC_0^{\ell+1}}{\min_{M_{\bar{t}}}\hat{r}_{\bar{p}} - 2d_3}\biggl(1 + \frac{5\pi}{\sqrt{K}d_3}\biggr)\frac{\overline{\Psi}_{\bar{p}}}{\epsilon_0(\max(\bar{y}, \psi_0))} + nK\overline{\Psi}_{\bar{p}} \\ &\qquad + \frac{2}{\min_{M_{\bar{t}}}\hat{r}_{\bar{p}} - 2d_3}\overline{\Psi}_{\bar{p}}^2 - \frac{d_3^2}{n(\min_{M_{\bar{t}}}\hat{r}_{\bar{p}} - 2d_3)^2}\overline{\Psi}_{\bar{p}}^3.\end{aligned}$$

Multiplication with $(\min_{M_{\bar{t}}}\hat{r}_{\bar{p}} - 2d_3)^2$ gives, for almost every $t \in J$,

$$\bigl(\min_{M_{\bar{t}}}\hat{r}_{\bar{p}} - 2d_3\bigr)^2\frac{d}{dt}\overline{\Psi}_{\bar{p}} \le \frac{n\pi C_0^{\ell+1}}{\sqrt{K}}\biggl(1 + \frac{5\pi}{\sqrt{K}d_3}\biggr)\frac{\overline{\Psi}_{\bar{p}}}{\epsilon_0(\max(\bar{y}, \psi_0))} + \frac{n\pi^2}{4}\overline{\Psi}_{\bar{p}} + \frac{\pi}{\sqrt{K}}\overline{\Psi}_{\bar{p}}^2 - \frac{d_3^2}{n}\overline{\Psi}_{\bar{p}}^3,$$

which is strictly negative whenever $\max(\bar{y}, \psi_0) < \overline{\Psi}_{\bar{p}}(t)$. To conclude the argument, suppose that

$$\Psi(\bar{t}, \bar{\xi}, \bar{p}) = \max_{[0,\bar{t}]\times\mathbb{S}^n\times\mathbb{S}^{n+1}_K} \Psi > \max(\bar{y}, \psi_0).$$

We know that

$$\overline{\Psi}_{\bar{p}}(t) \le \Psi(\bar{t}, \bar{\xi}, \bar{p}),$$

with equality at $t = \bar{t}$. But previously we showed that, for t close to $\bar{t}$, we have $(d/dt)\overline{\Psi}_{\bar{p}} < 0$ almost everywhere, which is impossible. Hence we obtain the desired estimate with $C_3 = \bar{y}$. □

Higher-order curvature bounds. We use the estimate from Lemma 5.4 to control the global term and estimate the derivatives of curvature.

Lemma 5.5. *For the origin $\mathcal{O} \in \Omega$ from Lemma 5.4 and the solution $M_t = \partial\Omega_t$ of (1-1) with initial data M and maximal existence time $T > 0$ and for all $m \in \mathbb{N}$, there exists*

$$C_4 = C_4\bigl(n, m, K, C_0, W_\ell(\Omega), \max\bigl(C_3, \max_M \Psi\bigr)\bigr)$$

with the property

$$\mu(t) + |\nabla^m A| \le C_4 \quad \textit{for all } t \in [0, \min(\tau, T)),$$

where τ is the number from Lemma 5.4. In particular, we have $T \ge \tau$, and the flow exists smoothly on $[0, \tau]$.

Proof. Up to the time $\min(\tau, T)$, (5-3) holds, i.e., with the notation from the proof of Lemma 5.4, we have

$$\mu(t) \le \frac{2nC_0^{\ell+1}}{\epsilon_1} = \frac{2nC_0^{\ell+1}}{\epsilon_0(\max(C_3, \psi_0))}.$$

The curvature derivative bound can be proved by a well-known induction argument, as for example in [Huisken 1984]. First, due to convexity and Lemma 5.4 (ii) and (iii),

$$|A|^2 \le H^2 \le \frac{\Phi^2_{p\Omega}}{K} \le \frac{1}{d_3^2 K}\bigl(\max\bigl(C_3, \max_M \Psi\bigr)\bigr)^2.$$

Assuming that all derivatives up to order $m-1$ are bounded by a constant of the form C_4, we obtain the evolution equation of $|\nabla^m A|$:

$$\partial_t|\nabla^m A|^2 \le \Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 + C(\mu u + K)|\nabla^m A|^2 + C\sum_{i+j+k=m}(\mu|\nabla^i \mathrm{c}_K| + |\nabla^i A|)|\nabla^j A||\nabla^k A||\nabla^m A|,$$

where we used that $\nabla u = A * \nabla \mathrm{c}_K$; see (2-4). Here $S * T$ denotes any linear combination of tensors formed by contracting S and T by means of g.

Then we claim

$$|\nabla^m A|^2 \le C \quad \text{for all } t \in [0, \min(\tau, T)). \tag{5-4}$$

As μ and u are bounded, we get

$$(\partial_t - \Delta)|\nabla^m A|^2 \le -2|\nabla^{m+1} A|^2 + C|\nabla^m A|^2 + C\left(\sum_{i=0}^{m} |\nabla^i \mathrm{c}_K| + 1\right)|\nabla^m A|.$$

Notice that $\mathrm{c}_K(r)$ and $|\nabla \mathrm{c}_K|$ are bounded as well. Moreover, from (4-1), one has, for $\ell \ge 0$, the covariant derivatives

$$\nabla^{\ell+2}\mathrm{c}_K = \nabla^\ell \mathrm{c}_K * K + u * \nabla^\ell A + \sum_{i+j+k=\ell} \nabla^i \mathrm{c}_K * \nabla^j A * \nabla^k A,$$

which are controlled by uniform constants arguing by induction. In short, we reach

$$(\partial_t - \Delta)|\nabla^m A|^2 \le -2|\nabla^{m+1} A|^2 + C|\nabla^m A|(1 + |\nabla^m A|),$$

which leads to (5-4) by standard maximum principle arguments, as for example in the proof of [Huisken 1987, Theorem 4.1].

As the right-hand side of the flow equation and all higher derivatives of the curvature remain uniformly bounded, we conclude (as in [Huisken 1984, pp. 257 ff.]) that, if $T < \tau$, then M_t converges (as $t \to T$ in the C^∞-topology) to a unique, smooth and strictly convex hypersurface.[1] Now we can apply short-time existence to continue the solution after T, contradicting the maximality of T. Hence the solution of (1-1) starting at a strictly convex hypersurface exists on $[0, \tau)$. On this interval we have uniform smooth estimates, and hence the flow also exists on $[0, \tau]$. □

6. Construction of a global solution

In the previous section we achieved existence and uniform estimates of any solution to (1-1) with strictly convex initial data M on a time interval $[0, \tau]$ from Lemma 5.4, the length of which only depends on preserved data of the problem. Those are, in particular, the hemisphere $\mathcal{H}(\mathcal{O})$, the pinching constant C_0, the quermassintegral $W_\ell(\Omega)$ and the number $\max(C_3, \max_M \Psi)$. The full curvature derivative bounds also only depend on those quantities.

Hence we can start an iteration process and shift, at time $i\tau$ with $i \in \mathbb{N}$, the origin according to Lemma 5.4 applied to the new strictly convex initial hypersurface $M_{i\tau}$. The constant C_4 from Lemma 5.5 is then uniform among the integers i, because it only depends on quantities which are always preserved. The following lemma makes this precise.

Lemma 6.1. *Let $M_0 \subset \mathbb{S}^{n+1}_K$ be a strictly convex hypersurface enclosing a domain Ω. Then there exists a sequence of origins $(\mathcal{O}_i)_{i \in \mathbb{N} \cup \{0\}}$ and positive numbers τ_0, $\epsilon_1(0)$ depending only on n, K, C_0, $W_\ell(\Omega_0)$, $\max(C_3, \max_{M_0} \Psi)$, such that the problem*

$$\begin{aligned} \partial_t x &= (\mu_i(t)\mathrm{c}_K(r_i) - H)\nu \quad \textit{for all } t \in [i\tau_0, (i+1)\tau_0), \\ x(0, \mathbb{S}^n) &= M_0, \\ x((i+1)\tau_0, \mathbb{S}^n) &= \lim_{t \nearrow (i+1)\tau_0} M_t, \end{aligned}$$

[1] Note that, due to the bound on μ, it can be seen from Lemma 5.1 that H is bounded from below on every finite time interval.

where r_i is the distance to $\mathcal{O}_i$ and μ_i is given as in (1-3) to keep the quermassintegral $W_\ell(\Omega_t)$ fixed for any $\ell = 0, 1, \dots, n$, has a solution

$$x : [0, \infty) \times \mathbb{S}^n \to \mathbb{S}^{n+1}_K.$$

For every $t \ge 0$, M_t is strictly convex and we have

$$B_{\epsilon_1(0)}(\mathcal{O}_i) \subset \Omega_t \quad \text{and} \quad M_t \subset \mathcal{H}(\mathcal{O}_i) \quad \text{for all } t \in [i\tau_0, (i+1)\tau_0). \tag{6-1}$$

The mappings $x(t, \cdot)$ satisfy spatial C^∞-estimates which are uniform in time.

Proof. For M_0, pick $\mathcal{O}_0$ according to Lemma 5.4. From Lemma 5.5 we conclude that the solution M_t of (1-1) with initial data M_0 exists on $[0, \tau_0]$, where τ_0 and $\epsilon_1(0)$ depend on n, K, C_0, $W_\ell(\Omega_0)$, and $\max(C_3, \max_{M_0} \Psi)$. The derivatives of A are bounded by

$$C_4 = C_4\big(n, m, K, C_0, W_\ell(\Omega_0), \max\big(C_3, \max_{M_0} \Psi\big)\big).$$

Now suppose that, for $i \ge 0$, the hypersurface $M_{i\tau_0}$, the origin $\mathcal{O}_i$ and the solution $(M_t)_{t\in[i\tau_0,(i+1)\tau_0)}$ were already constructed such that (6-1),

$$\max\big(C_3, \max_{M_t} \Psi\big) \le \max\big(C_3, \max_{M_0} \Psi\big),$$

as well as

$$\mu_i(t) + |\nabla^m A| \le C_4\big(n, m, K, C_0, W_\ell(\Omega_0), \max\big(C_3, \max_{M_0} \Psi\big)\big) \tag{6-2}$$

all hold for all $t \in [i\tau_0, (i+1)\tau_0]$. Then apply Lemma 5.4 to the initial hypersurface $M_{(i+1)\tau_0}$, and obtain an origin $\mathcal{O}_{i+1}$ such that the solution M_t of (1-1) with initial data $M_{(i+1)\tau_0}$ satisfies (6-1),

$$\max\big(C_3, \max_{M_t} \Psi\big) \le \max\big(C_3, \max_{M_{(i+1)\tau_0}} \Psi\big) \le \max\big(C_3, \max_{M_0} \Psi\big)$$

and

$$\mu_{i+1}(t) + |\nabla^m A| \le C_4\big(n, m, K, C_0, W_\ell(\Omega_0), \max\big(C_3, \max_{M_0} \Psi\big)\big)$$

during the interval $[(i+1)\tau_0, (i+2)\tau_0]$ and with the same $\epsilon_1(0)$. Here we also used that C_0 and W_ℓ are preserved. This means that the construction can be carried out infinitely often to obtain the desired long-time solution. □

7. Asymptotic estimates and convergence to a spherical cap

In the previous sections we have put ourselves into a position where we have a strictly convex flow $(M_t)_{0\le t<\infty}$ in the sphere. This flow is not necessarily smooth in time, but it satisfies spatial C^k-estimates which are uniform with respect to time and has a uniformly bounded global term, due to the proof of Lemma 6.1.

Additionally, by means of Corollary 4.4 and the curvature bounds, we get

$$\sum_{i=1}^{n} (\kappa_i - \kappa_1) = H - n\kappa_1 \le nH\omega(t) \le nCe^{-2nKt},$$

which implies exponential decay of the traceless second fundamental form

$$|\mathring{A}|^2 = \frac{1}{n}\sum_{i<j}(\kappa_j - \kappa_i)^2 \leq Ce^{-4nKt}. \tag{7-1}$$

Using this property, in the following we are going to apply some recent estimates of almost-umbilical type due to De Rosa and Gioffré [2021], to show that the process of picking new origins actually terminates after finitely many steps and that the flow will then converge to a geodesic sphere of a uniquely determined radius. The crucial ingredient is the following result.

Theorem 7.1 [De Rosa and Gioffrè 2021, Theorem 1.3]. *Let $n \geq 2$, let Σ be a closed hypersurface in $\mathbb{R}^{n+1}$ and let $p > n$ be given. We assume that there exists $c_0 > 0$ such that Σ satisfies the conditions*

$$|\Sigma| = |\mathbb{S}^n|, \quad \|A\|_{L^p(\Sigma)} \leq c_0.$$

There exist positive numbers $\delta, C > 0$, depending only on $n,\ p,\ c_0$, with the following property: if

$$\|\mathring{A}\|_{L^p(\Sigma)} \leq \delta,$$

then there exists a vector $c = c(\Sigma)$, such that $\Sigma - c$ is a graph over the sphere; namely, there exists a parametrization

$$\psi : \mathbb{S}^n \to \Sigma, \quad \psi(x) = e^{f(x)}x + c,$$

and f satisfies the estimate

$$\|f\|_{W^{2,p}(\mathbb{S}^n)} \leq C\|\mathring{A}\|_{L^p(\Sigma)}.$$

In the following we will use this result to prove that the surfaces become exponentially C^2-close to geodesic spheres and that the necessity to pick new origins vanishes.

Lemma 7.2. *In the situation of Lemma 6.1, there exists an integer $m > 0$ depending on $n,\ K$ and M, such that, in Lemma 6.1, the origins $\mathcal{O}_i,\ i > m$, may be chosen constantly equal to $\mathcal{O}_m$.*

Proof. Let m be a positive integer to be specified during the proof. Let $\mathcal{O}_m$ be the flow origin associated to the interval $I_m := [m\tau_0, (m+1)\tau_0)$. By stereographic projection from the antipodal point of $\mathcal{O}_m$, the family $(M_t)_{t\in I_m}$ can be viewed as a flow in the Euclidean space, which we denote by $(\widetilde{M}_t)_{t\in I_m}$. Geometric quantities of this flow, denoted by a tilde, are related to the original ones as follows, see [Gerhardt 2006, Proposition 1.1.11], where $e^{2\varphi}$ is the conformal factor:

$$g = e^{2\varphi}\tilde{g}, \qquad \nu = e^{-\varphi}\tilde{\nu},$$
$$e^{\varphi}A = \tilde{A} + d\varphi(\tilde{\nu})\,\mathrm{id}, \quad e^{\varphi}H = \widetilde{H} + n\,d\varphi(\tilde{\nu}).$$

In particular, we obtain

$$\mathring{\tilde{A}} = e^{\varphi}\mathring{A}.$$

The surface areas of M_t and $\widetilde{M}_t$ are related by

$$|M_t| = \int_{M_t} 1 = \int_{\widetilde{M}_t} e^{n\varphi},$$

and hence

$$C^{-1}|M_t| \le |\widetilde{M}_t| \le |M_t|,$$

where C depends on $|\varphi|_{C^0(M_t)}$. Now define the scaled hypersurface

$$\widehat{M}_t = \lambda \widetilde{M}_t, \quad \text{with } \lambda^n = \frac{|\mathbb{S}^n|}{|\widetilde{M}_t|},$$

so that $|\widehat{M}_t| = |\mathbb{S}^n|$. Now the associated Weingarten operator is

$$\hat{A} = \lambda^{-1}\tilde{A} = \frac{e^{\varphi}}{\lambda} A - \lambda^{-1}\, d\varphi(\tilde{\nu})\,\mathrm{id},$$

and similarly for the traceless Weingarten operator. Hence

$$\|\hat{A}\|_{L^\infty(\widehat{M}_t)} \le C(\|A\|_{L^\infty(M_t)} + 1)$$

and

$$\|\mathring{\hat{A}}\|_{L^\infty(\widehat{M}_t)} \le C\|\mathring{A}\|_{L^\infty(M_t)},$$

where C depends on n, $|\varphi|_{C^1(M_t)}$ and $|M_t|$.

Then $\|\hat{A}\|_{L^\infty(\widehat{M}_t)}$ is bounded by Lemma 5.5 and (7-1) ensures that $\|\mathring{\hat{A}}\|_{L^\infty}$ is as small as needed for m big enough. Therefore, we can apply Theorem 7.1 for sufficiently large m to get a function $\hat{f}$ which, from the embedding theorems of Sobolev spaces into Hölder spaces, satisfies

$$\|\hat{f}\|_{C^1(\mathbb{S}^n)} \le C\|\mathring{\hat{A}}\|_{L^\infty(\widehat{M}_t)} \le C\|\mathring{A}\|_{L^\infty(M_t)} \le Ce^{-2nKt} \quad \text{for all } t \in I_m.$$

Then, due to our curvature bounds (5-4), we have full C^k-bounds on $\hat{f}$ for all k. By iteration of interpolation arguments for C^k bounds (see [Gerhardt 2011, Corollary 6.2]), this implies that

$$\|\hat{f}\|_{C^k(\mathbb{S}^n)} \le Ce^{-2nKt} \quad \text{for all } t \in I_m.$$

In other words, $\widehat{M}_t$ is exponentially C^k-close to a sphere $\hat{\mathcal{S}}_t$ for all $k \in \mathbb{N}$ and for all $t \in I_m$. As the area along the M_t is uniformly bounded above and below by Corollary 3.2, we get C^k bounds for the conformal factor as well, and this property of closeness to a sphere $\mathcal{S}_t$ carries over to the original flow in $\mathbb{S}^{n+1}_K$. Note that the radii of the spheres $\mathcal{S}_t$ converge to a well-defined limit, which is strictly less than $\pi/(2\sqrt{K})$, determined by the initial value of $W_\ell(\Omega_0)$. Hence the curvature of M_t is uniformly bounded from below.

On the other hand, the radial distance to the origin $\mathcal{O}_m$ satisfies

$$\partial_t r = (\mu \mathrm{c}_K - H)v^{-1}, \quad \text{with } v^2 = 1 + \mathrm{s}_K^{-2}|dr|^2_\sigma. \tag{7-2}$$

Hence, for an error δ_m that converges to zero when $m \to \infty$,

$$\partial_t r = \left(\frac{\int_{M_t}\sigma_\ell H}{\int_{M_t}\sigma_\ell \mathrm{c}_K}\mathrm{c}_K - H\right)v^{-1} \le \left(\frac{\mathrm{c}_K}{⨍_{\mathcal{S}_t}\mathrm{c}_K} - 1\right)\frac{H_{\mathcal{S}_t}}{v} + \delta_m, \tag{7-3}$$

where $H_{\mathcal{S}_t}$ is the mean curvature of the sphere $\mathcal{S}_t$. At points which maximize r, we have that c_K is minimized. At such points the first term on the right-hand side of (7-3) is strictly negative if c_K is not constant. Hence, for large m, if $\mathcal{S}_t$ is uniformly off-center, $\max_{M_t} r$ is decreasing and a similar estimate shows that $\min_{M_t} r$ is increasing. Hence, from then on, there is no longer a need to adjust the origin. □

Convergence to a spherical cap. To complete the proof of Theorem 1.1, it only remains to show that the immortal solution coming from Lemma 6.1 actually converges to a limit geodesic sphere. After m-fold picking of a new origin, we now may, without loss of generality, assume that origins have not been changed at all. We will exploit the C^∞-estimates for the flow hypersurfaces M_t coming from (5-4). We already know from Lemma 7.2 that every limit point of the flow must be a round sphere.

Now we prove that only the sphere centered at the origin can arise as a limit. Notice that the radius R of any limit sphere is determined by the initial hypersurfaces by means of the equality $W_\ell(B_R) = W_\ell(\Omega_0)$. Denote by H_R the mean curvature of such a sphere S_R. Hence, from the evolution (7-2) of the radial distance and for an error δ that converges to zero when $M_{t_k} \to S_R$, we get

$$\partial_t r = \left(\frac{\mathrm{c}_K}{\fint_M \mathrm{c}_K} - 1\right)\frac{H_R}{v} + \delta. \tag{7-4}$$

As above, at points which maximize r, we have that c_K is minimized, thus at such points the first term on the right-hand side of (7-4) is strictly negative if c_K is not constant. Therefore the function $\max_{M_t} r$ is strictly decreasing in sufficiently small C^2-neighborhoods of any noncentered sphere, which excludes those as limits. Thus subsequential limits are unique and the whole flow must converge.

8. The elliptic case: rigidity results

In order to prove Theorem 1.3, we first have to get the elliptic viscosity equation of the pinching deficit for general curvature function F. Let us first gather some prerequisites about these functions.

8.1. ***Symmetric curvature functions.*** If M is a hypersurface of $\mathbb{M}^{n+1}_K$, then we set

$$F(x) = f(\kappa_1(x), \dots, \kappa_n(x)),$$

which can be alternatively seen as a function defined on the diagonalizable endomorphisms, $F = F(A)$, or as a function of a symmetric and a positive definite bilinear form, $F = F(g, h)$. In the latter case, we write

$$F^{ij} = \frac{\partial F}{\partial h_{ij}}, \qquad F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij}\partial h_{kl}}.$$

With these conventions, the covariant derivatives are given by

$$\nabla_i F = F^{jk}\nabla_i h_{jk}. \tag{8-1}$$

On strictly convex hypersurfaces M, we can define the so-called *inverse curvature function* by

$$\widetilde{F}(A) = \frac{1}{F(A^{-1})}.$$

A curvature function F is called *inverse concave* if $\widetilde{F}$ is concave. Notice that concavity/convexity with respect to the matrix variables is equivalent to the same property with respect to the eigenvalues (see [Andrews 2007; Gerhardt 2006] for further information). Next we gather several useful properties.

Lemma 8.1. (a) *If F is inverse concave and M is strictly convex, then*

$$F^{ij,kl}\nabla_1 h_{ij}\nabla_1 h_{kl} + 2\sum_j \frac{F^{ii}}{\kappa_j}(\nabla_1 h_{ij})^2 \geq \frac{2}{F}(\nabla_1 F)^2. \tag{8-2}$$

(b) *Under Conditions 1.2, we have:*

(i) *If F is convex, then it is inverse concave* (*see* [Gerhardt 2006, Lemmas 2.2.12 and 2.2.14]).

(ii) *Euler's formula $F^{ii}\kappa_i = F$ implies that F is strictly positive.*

Rigidity for radial curvature functions. We start with a result that contains an elliptic version of Proposition 4.3.

Lemma 8.2. *Suppose F satisfies Conditions 1.2. Then:*

(i) *The Weingarten operator satisfies the elliptic equation*

$$-F^{rs}\nabla^2_{rs}h_{ij} = F^{pq,rs}\nabla_i h_{pq}\nabla_j h_{rs} - \nabla^2_{ij}F + F^{rs}h_{ms}h^m_r h_{ij} + KFg_{ij} - Fh_{mj}h^m_i - KF^{rs}g_{rs}h_{ij}.$$

(ii) *For the function $p = \kappa_1/F$, we have*

$$-FF^{kl}\nabla^2_{kl}p \geq 2\sum_{j>D}\frac{F^{kk}(\nabla_k h^1_j)^2}{\kappa_j - \kappa_1} + F^{pq,rs}\nabla_1 h_{pq}\nabla_1 h_{rs} + 2F^{kl}\nabla_k p\nabla_l F + pF^{kl}\nabla_{kl}F - \nabla_{11}F + F^{ii}\,\mathrm{Sec}_{i1}(\kappa_i - \kappa_1)$$

in the viscosity sense, where D is the multiplicity of κ_1.

Proof. (i) We differentiate (8-1) to get

$$\nabla^2_{ij}F = F^{pq,rs}\nabla_i h_{pq}\nabla_j h_{rs} + F^{rs}\nabla^2_{ij}h_{rs}.$$

Now combine the commutator formula for second covariant derivatives with the Gauss equation (2-1) to deduce

$$\begin{aligned} F^{rs}\nabla^2_{ij}h_{rs} &= F^{rs}(\nabla^2_{rs}h_{ij} + R^m_{isj}h_{mr} + R^m_{rsj}h_{mi}) \\ &= F^{rs}\nabla^2_{rs}h_{ij} + F^{rs}(h_{ms}h_{ij} - h_{mj}h_{is} + Kg_{ms}g_{ij} - Kg_{mj}g_{is})h^m_r \\ &\qquad + F^{rs}(h_{ms}h_{rj} - h_{mj}h_{rs} + Kg_{ms}g_{rj} - Kg_{mj}g_{rs})h^m_i \\ &= F^{rs}(\nabla^2_{rs}h_{ij} + h_{ms}h^m_r h_{ij}) + KFg_{ij} - Fh_{mj}h^m_i - KF^{rs}g_{rs}h_{ij}. \end{aligned}$$

(ii) As in the proof of Proposition 4.3, let η be a smooth lower support of p at $\xi_0 \in M$, and define $\varphi = \eta F$. Then, in coordinates with

$$g_{ij} = \delta_{ij}, \quad h_{ij} = \kappa_i\delta_{ij}, \tag{8-3}$$

we have inequality (4-2), which combined with (i) leads to

$$
\begin{aligned}
-FF^{kl}\nabla^2_{kl}\eta &= -F^{kl}\nabla^2_{kl}\varphi + \frac{\varphi}{F}F^{kl}\nabla^2_{kl}F + 2F^{kl}\nabla_k\eta\nabla_l F \\
&\geq 2F^{kl}\nabla_k\eta\nabla_l F + 2\sum_{j>D}\frac{F^{kk}(\nabla_k h^1_j)^2}{\kappa_j-\kappa_1} + F^{pq,rs}\nabla_1 h_{pq}\nabla_1 h_{rs} \\
&\qquad + \eta F^{kl}\nabla^2_{kl}F - \nabla^2_{11}F + F^{ii}(\kappa_i\kappa_1 + K)(\kappa_i-\kappa_1). \qquad \square
\end{aligned}
$$

In order to prove Theorem 1.3, let $F = \gamma \mathrm{c}_K^{\alpha}$. Then taking derivatives and using (4-1), we get

$$
\nabla^2_{kl}F = \alpha\gamma \mathrm{c}_K^{\alpha-1}\nabla^2_{kl}\mathrm{c}_K + \alpha(\alpha-1)\gamma \mathrm{c}_K^{\alpha-2}\nabla_k \mathrm{c}_K\nabla_l \mathrm{c}_K = K\Big(\frac{u}{\mathrm{c}_K}h_{kl} - g_{kl}\Big)\alpha F + \frac{2\epsilon}{F}\nabla_k F\nabla_l F,
$$

where

$$
\epsilon = \frac{\alpha-1}{2\alpha} = \frac{1}{2} - \frac{1}{2\alpha}. \tag{8-4}
$$

This implies

$$
\begin{aligned}
pF^{kl}\nabla^2_{kl}F - \nabla^2_{11}F &= p\alpha KF\Big(\frac{u}{\mathrm{c}_K}F - F^{kl}g_{kl}\Big) - \alpha KF\Big(\frac{u}{\mathrm{c}_K}\kappa_1 - 1\Big) + \frac{2\epsilon p}{F}F^{kl}\nabla_k F\nabla_l F - \frac{2\epsilon}{F}(\nabla_1 F)^2 \\
&= K\alpha F^{ii}(\kappa_i-\kappa_1) + \frac{2\epsilon}{F}(pF^{kl}\nabla_k F\nabla_l F - (\nabla_1 F)^2).
\end{aligned} \tag{8-5}
$$

Here we have used Euler's relation, and computations are done in the coordinates (8-3).

Proof of Theorem 1.3. As $|\alpha| \geq 1$, we have that $\epsilon \in [0, 1]$ for ϵ defined as in (8-4). Then using that M is convex, we can estimate

$$
2\sum_{j>D}\frac{F^{kk}(\nabla_k h^1_j)^2}{\kappa_j-\kappa_1} \geq 2\epsilon\sum_{j=1}^n \frac{F^{kk}(\nabla_k h^1_j)^2}{\kappa_j} - 2\epsilon\frac{p}{F}F^{kl}\nabla_k F\nabla_l F + T * \nabla p,
$$

where we also used the following (see [Brendle et al. 2017, Lemma 5]):

$$
\nabla_k\varphi\delta_{ij} = \nabla_k h_{ij} \quad \text{for all } 1 \leq i,\ j \leq D.
$$

Plugging this into Lemma 8.2 (ii) and using the convexity of F, we get

$$
\begin{aligned}
-FF^{kl}\nabla^2_{kl}p \ &\geq\ 2\epsilon\sum_{j=1}^n \frac{F^{kk}(\nabla_k h^1_j)^2}{\kappa_j} - 2\epsilon\frac{p}{F}F^{kl}\nabla_k F\nabla_l F + \epsilon F^{pq,rs}\nabla_1 h_{pq}\nabla_1 h_{rs} \\
&\qquad + pF^{kl}\nabla_{kl}F - \nabla_{11}F + F^{ii}\,\mathrm{Sec}_{i1}(\kappa_i-\kappa_1) + T*\nabla p \\
&\underset{(8\text{-}5)}{\geq}\ 2\epsilon\sum_{j=1}^n \frac{F^{kk}(\nabla_k h^1_j)^2}{\kappa_j} + \epsilon F^{pq,rs}\nabla_1 h_{pq}\nabla_1 h_{rs} - \frac{2\epsilon}{F}(\nabla_1 F)^2 \\
&\qquad + F^{ii}(\mathrm{Sec}_{i1} + \alpha K)(\kappa_i-\kappa_1) + T*\nabla p.
\end{aligned}
$$

Notice that in the case $\alpha = -1$, we have $\epsilon = 1$ and therefore we do not need the convexity of F in the first inequality above.

By Lemma 8.1 (b) our F is inverse concave, and hence (8-2) leads to

$$-FF^{kl}\nabla^2_{kl}p + T * \nabla p \geq F^{ii}(\mathrm{Sec}_{i1} + \alpha K)(\kappa_i - \kappa_1).$$

This completes the proof using the strong maximum principle for viscosity solutions. We conclude that M has to be centered at the origin provided $K \neq 0$, since only in this case can c_K be constant. □

Remark 8.3. Note that this approach also provides a direct maximum principle proof of Liebmann's soap bubble theorem (the convex case of Alexandrov's theorem); see [Liebmann 1900].

Solitons. We complete this paper by proving Corollary 1.4.

Proof of Corollary 1.4. For the given hypersurface M, there is a dual hypersurface $\widetilde{M} \subset \mathbb{M}^{n+1}_K$, where $K = \mathrm{sgn}(\mathbb{M})$ as in (1-6) and with the properties

$$\tilde{\kappa}_i = \frac{1}{\kappa_i}, \quad \tilde{\mathrm{c}}_K = u;$$

see [Gerhardt 2006, Theorems 10.4.4 and 10.4.9] and [Scheuer 2021]. Hence $\widetilde{M}$ satisfies the equation

$$\tilde{c}_K^{1/\beta} = F(\kappa_i) = F(\tilde{\kappa}_i^{-1}) = \frac{1}{\widetilde{F}(\tilde{\kappa}_i)},$$

i.e., with $\alpha = -1/\beta$, we have

$$\widetilde{F}_{|\widetilde{M}} = \tilde{\mathrm{c}}_K^{\alpha},$$

where $\widetilde{F}$ is the inverse curvature function of F. Therefore to complete the proof it only remains to check the conditions of Theorem 1.3 for $\widetilde{M}$, which hold provided that, for any $\tilde{g}$-orthonormal frame, we have

$$\widetilde{\mathrm{Sec}}_{ij} \geq -\alpha K.$$

In coordinates that diagonalize $\tilde{A}$, the Gauss equation (2-1) for $\widetilde{M}$ gives

$$\widetilde{R}_{ijij} + \alpha K = \tilde{h}_{ii}\tilde{h}_{jj} - \tilde{h}_{ij}\tilde{h}_{ij} + (1+\alpha)K = \frac{1}{\kappa_i\kappa_j} + (1+\alpha)K \geq 0,$$

provided

$$\frac{1}{\kappa_i\kappa_j} \geq \frac{1-\beta}{\beta}K.$$

Notice that if $((1-\beta)/\beta)K \leq 0$, this is guaranteed by convexity of M; otherwise, the inequality follows by the assumption on Sec_M. Hence the statement follows by direct application of Theorem 1.3. □

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Received 5 Dec 2022. Accepted 31 Aug 2023.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

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ANALYSIS & PDE

Volume 17 No. 10 2024