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**UPPER BOUND ON THE NUMBER OF RESONANCES
FOR EVEN ASYMPTOTICALLY HYPERBOLIC MANIFOLDS
WITH REAL-ANALYTIC ENDS**



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We prove a polynomial upper bound on the number of resonances in a disk whose radius tends to $+\infty$ for even asymptotically hyperbolic manifolds with real-analytic ends. Our analysis also gives a similar upper bound on the number of quasinormal frequencies for Schwarzschild–de Sitter spacetimes.

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1. Introduction

The purpose of this work is to prove an upper bound for the number of resonances for even asymptotically hyperbolic manifolds with real-analytic (but a priori not exactly hyperbolic) ends. Let us recall that an asymptotically hyperbolic manifold is a Riemannian manifold (M, g) such that M is the interior of a compact manifold with boundary \bar{M} and there is an identification of a neighborhood of $\partial\bar{M}$ with $[0, \epsilon]_{y_1} \times \partial\bar{M}_{y'}$ that puts the metric g into the form

$$g = \frac{dy_1^2 + g_1(y_1, y', dy')}{y_1^2}, \quad (1)$$

where $g_1(y_1, y', dy')$ is a family of metrics on $\partial\bar{M}$ depending on y_1 . We say that (M, g) is even if g_1 is a smooth function of y_1^2 . We refer to [Dyatlov and Zworski 2019, §5.1] for a detailed discussion of this notion.

Letting Δ denote the (nonpositive) Laplace operator on an even asymptotically hyperbolic manifold (M, g) of dimension n , one commonly introduces the family of operators, depending on the complex parameter λ ,

$$\left(-\Delta - \frac{1}{4}(n-1)^2 - \lambda^2\right)^{-1} : L^2(M) \rightarrow L^2(M), \quad \text{Im } \lambda > 0. \quad (2)$$

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Since the essential spectrum of $-\Delta$ is $[(n - 1)^2/4, +\infty[$, this family of operators is well-defined and meromorphic for $\text{Im } \lambda > 0$, with maybe a finite number of poles between 0 and $i(n - 1)/2$ on the imaginary axis, corresponding to the eigenvalues of $-\Delta$ in $]0, (n - 1)^2/4[$. Notice that the residues of these poles have finite ranks.

The *scattering resolvent* of (M, g) is then defined as the meromorphic continuation of (2), as provided by the following result.

Theorem 1 [Mazzeo and Melrose 1987; Guillarmou 2005]. *Let (M, g) be an even asymptotically hyperbolic manifold of dimension n . Then the resolvent (2) admits a meromorphic extension $R_{\text{scat}}(\lambda)$ to \mathbb{C} as an operator from $C_c^\infty(M)$ to $\mathcal{D}'(M)$, with residues of finite rank.*

In the case of manifolds that are exactly hyperbolic near infinity, one may also refer to [Guillopé and Zworski 1995a]. Notice that we do not use here the same spectral parameter as in [Mazzeo and Melrose 1987; Guillarmou 2005; Guillopé and Zworski 1995a]. The spectral parameter from these references is given in terms of our λ as $\zeta = (n - 1)/2 - i\lambda$. Another proof of Theorem 1 has been given by Vasy [2013a] (see also [Vasy 2013b; Zworski 2016; Dyatlov and Zworski 2019, Chapter 5]).

The poles of the scattering resolvent (the meromorphic continuation of (2)) are called the resonances of (M, g) . If $\mu \neq 0$ is a scattering resonance for (M, g) then we define the multiplicity of μ as the rank of the operator

$$\frac{i}{\pi} \int_\gamma \lambda R_{\text{scat}}(\lambda) \, d\lambda, \tag{3}$$

where γ is a small positively oriented circle around μ (so that the index of μ with respect to γ is 1, and the index of any other resonance is zero). That this operator has finite rank follows from the fact that the residues of $R_{\text{scat}}(\lambda)$ have finite ranks. Another definition for the multiplicity of resonances may be found for instance in [Guillopé and Zworski 1997, Definition 1.2], but it coincides with the one we gave when μ is nonzero (see [Guillopé and Zworski 1997, Proposition 2.11]). The definition of the multiplicity of 0 as a resonance is more subtle (and will not matter in our case), see the discussion after Theorem 1 in [Zworski 1997]. Notice that in [Mazzeo and Melrose 1987; Vasy 2013a], the scattering resolvent $R_{\text{scat}}(\lambda)$ is constructed as an operator from the space $\dot{C}^\infty(M)$ of smooth functions on \bar{M} that vanish at infinite orders on $\partial\bar{M}$ to its dual. Since $C_c^\infty(M)$ is contained in $\dot{C}^\infty(M)$, we stated in Theorem 1 a weaker result. Notice however that, since $C_c^\infty(M)$ is dense in $\dot{C}^\infty(M)$, this simplification does not modify the notion of multiplicity of a resonance.

Our main result is an upper bound on the number of resonances for even asymptotically hyperbolic manifolds with real-analytic ends (as defined in Section 4.1).

Theorem 2. *Let (M, g) be an even asymptotically hyperbolic manifold real-analytic near infinity (as defined in Section 4.1) of dimension n . For $r > 0$, let $N(r)$ denote the number of resonances of (M, g) of modulus less than r , counted with multiplicities. Then*

$$N(r) \underset{r \rightarrow +\infty}{=} \mathcal{O}(r^n). \tag{4}$$

This upper bound is natural, since it is coherent with the asymptotic for the number of eigenvalues for the Laplacian on a closed Riemannian manifold given by Weyl law. There are also noncompact examples for which the bound (4) is optimal; see the lower bounds from [Guillopé and Zworski 1997; Borthwick 2008] discussed below.

There is a long tradition of studies of such counting problems in scattering theory, going back to the work of Tullio Regge [1958]. Results similar to Theorem 2 have been established in the context of scattering (e.g., by a compactly supported potential or by certain black boxes) on odd-dimensional Euclidean spaces [Melrose 1984; Zworski 1989; Sjöstrand and Zworski 1991; Vodev 1992]. In the context of asymptotically hyperbolic manifolds, the bound (4) is known for manifolds with *exactly* hyperbolic ends [Guillopé and Zworski 1995b; Cuevas and Vodev 2003; Borthwick 2008]. Still in the case of manifolds with exactly hyperbolic ends, we also have some lower bounds available: in the case of surfaces Guillopé and Zworski [1997] proved that $r^2 = \mathcal{O}(N(r))$, which implies that (4) is optimal in that case. In higher dimension n , Borthwick [2008] proved a similar lower bound $r^n = \mathcal{O}(N^{\text{sc}}(r))$ for compact perturbations of conformally compact hyperbolic manifolds (a stronger assumption than just having exactly hyperbolic ends). This lower bound is obtained for the counting function $N^{\text{sc}}(r)$ associated to a larger set of resonances than $N(r)$, and that also satisfies (4). However, a few cases in which the same lower bound for $N(r)$ follows are given in [Borthwick 2008]. Finally, a lower bound for $N(r)$ of the form

$$\limsup_{r \rightarrow +\infty} \frac{\log N(r)}{\log r} = n$$

is proven for generic compact perturbations of a manifold with exactly hyperbolic ends in [Borthwick et al. 2011].

Leaving the context of manifolds with exactly hyperbolic ends, much less is known on the asymptotic of the counting function $N(r)$. The bound (4) was established by Borthwick and Philipp [2014] in the case of asymptotically hyperbolic manifolds with warped-product ends, that is, for which the coordinates (y_1, y') in (1) may be chosen so that $g_1(y_1, y', dy') = g_1(y', dy')$ does not depend on y_1 . The proof of a similar bound is sketched in [Froese and Hislop 2000] for a class of asymptotically hyperbolic manifolds with ends that are asymptotically warped. Wang [2019] established, for certain real-analytic asymptotically hyperbolic metrics on \mathbb{R}^3 , a polynomial bound $\mathcal{O}(r^6)$ for the number of resonances in a sector of the form

$$\{z \in \mathbb{C} : \epsilon < |z| < r, -\frac{1}{2}\pi + \epsilon < \arg z < \frac{3}{2}\pi - \epsilon\} \quad (5)$$

when r tends to $+\infty$ while $\epsilon > 0$ is fixed. The evenness assumption is not made in [Wang 2019], hence the necessity to count resonances in sectors of the form (5) rather than in disks (one has to avoid the essential singularities that can appear in the noneven case according to [Guillarmou 2005]). In the even case, our result, Theorem 2, improves the bound from [Wang 2019], not only because we can count resonances in a disk, but also because our result, valid in any dimension, gives a better exponent in the 3-dimensional case.

Let us point out that the upper bound (4) is also satisfied by the counting functions for the *Ruelle resonances* of a real-analytic Anosov flow, as follows from a result of Fried [1995] based on techniques introduced by Rugh [1992; 1996]. We gave a new proof of this result in [Bonthonneau and Jézéquel 2020],

adapting techniques originally developed in [Helffer and Sjöstrand 1986; Sjöstrand 1996]. The tools of real-analytic microlocal analysis that we use in the present paper rely heavily on [Bonthonneau and Jézéquel 2020].

The main idea behind the proof of [Theorem 2](#) is to adapt the method of Vasy [2013a] to construct the scattering resolvent, by introducing tools of real-analytic microlocal analysis. The method of Vasy does not only apply to even asymptotically hyperbolic manifolds, it may also be used to study resonances associated to the wave equation on *Schwarzschild–de Sitter spacetimes* (in this context, resonances are also called *quasinormal frequencies*). The interested reader may for instance refer to [Dafermos and Rodnianski 2013, §6] for a description of the geometry of Schwarzschild–de Sitter spacetimes. Consequently, our method also gives an upper bound on the number of resonances (or quasinormal frequencies) for Schwarzschild–de Sitter spacetimes.

Theorem 3. *The number of quasinormal frequencies of modulus less than r for a Schwarzschild–de Sitter spacetime is $\mathcal{O}(r^3)$ when r tends to $+\infty$.*

It is proven in [Sá Barreto and Zworski 1997] that the quasinormal frequencies for a Schwarzschild–de Sitter spacetime are well approximated by the pseudopoles

$$c(\pm\ell \pm \frac{1}{2} - i(k + \frac{1}{2})),$$

for $k \in \mathbb{N}$ and $\ell \in \mathbb{N}^*$, the corresponding pole having multiplicity $2\ell + 1$. Here, c is a constant depending on the mass of the black hole and the cosmological constant. However, the approximation given in [loc. cit.] is only effective for a pseudopole μ such that $|\mu|$ tends to $+\infty$ while the imaginary part of μ remains bounded from below. Consequently, while [Theorem 3](#) seems reasonable in view of the approximation result from [loc. cit.], these two results discuss two different asymptotics. The result from [loc. cit.] cannot be used to prove [Theorem 3](#), nor to prove that [Theorem 3](#) is sharp (even though it suggest that it should be the case).

It may be possible that the method of the proof of [Theorems 2](#) and [3](#) generalizes to the case of slowly rotating Kerr–de Sitter black holes (as the method of Vasy [2013a, §6] also applies in this context). However, there are some additional technical difficulties that would probably arise in that case, due to the microlocal geometry being more complicated than in the Schwarzschild–de Sitter case. In particular, there are bicharacteristics that originate at the source above the event horizon, then enter the domain of outer communication and eventually leave it. Our strategy of proof would require the propagation of singularities along these bicharacteristics using real-analytic microlocal analysis. Consequently, in order to deal with Kerr–de Sitter spacetimes, one cannot use real-analytic tools only near the event and cosmological horizon, as it is the case in the proof of [Theorem 3](#); see [Remark 4](#). Since the coefficients of Kerr–de Sitter spacetimes are real-analytic on the whole domain of outer communication, it is not unlikely that this problem may be solved. In any case, we expect that one would need to use an escape function more carefully designed than in our analysis below.

Idea of the proof. As mentioned above, the proof of [Theorems 2](#) and [3](#) is based on an adaptation of the method of Vasy [2013a] to construct the scattering resolvent, by introducing tools of real-analytic

microlocal analysis. Our approach of the method of Vasy is mostly based on the exposition from [Dyatlov and Zworski 2019, Chapter 5].

The starting point of the proof of Theorems 2 and 3 is the following observation. When using the method of Vasy to construct the scattering resolvent, one will construct a meromorphic extension to (2) on a half plane of the form

$$\{\lambda \in \mathbb{C} : \text{Im } \lambda > -C\} \quad (6)$$

for a given $C > 0$, by studying the action of a modified Laplacian on a functional space H_C that depends on C . The constant C may be chosen arbitrarily large, so that we get indeed a meromorphic continuation to \mathbb{C} , but this requires a change in the space on which the modified Laplacian is acting.

In the context of even asymptotically hyperbolic manifolds, the space H_C is constructed in the following manner: one embeds M as a relatively compact subset of a manifold X , and replaces the operator $-\Delta - (n-1)^2/4 - \lambda^2$ by a family of modified Laplacians. These modified Laplacians are elliptic on M but have a source/sink structure above the boundary of M in X . One can then set up a Fredholm theory for the modified Laplacians by using microlocal radial estimates (see for instance [Dyatlov and Zworski 2019, §E.4]). However, radial estimates in the C^∞ category are limited by a threshold condition. In our setting, it imposes choosing space H_C as a space of functions with a number of derivatives proportional to C in order to get a meromorphic continuation of (2) on the half-plane (6).

Consequently, working only with C^∞ tools, one will a priori only have access to bound on the number of resonances when restricting to a half-plane of the form (6). A natural idea to tackle this difficulty is to work with a space “ H_∞ ” of functions that are smooth near the boundary of M in X (in our case, this would be real-analytic objects). If one is able to prove a real-analytic version of the radial estimates, it should be possible to bypass the threshold condition and construct directly the meromorphic continuation of (2) to the whole \mathbb{C} , working on a single space H_∞ . One can then hope that this functional analytic setting can be used to prove a global bound on the number of resonances, without the need to restrict to a half-plane of the form (6). We will use the tools from [Bonthonneau and Jézéquel 2020], based on [Helffer and Sjöstrand 1986; Sjöstrand 1996], to prove an estimate that is in some sense a real-analytic version of a radial estimate (see also [Galkowski and Zworski 2022]). Notice that similar estimates are proved in [Galkowski and Zworski 2021; Guedes-Bonthonneau et al. 2024] in different geometric contexts, and with a focus more on the hypoellipticity statement that may be deduced from the radial estimates rather than on the functional analytic consequences. In some sense, the results on resonances for zeroth order pseudodifferential operators in [Galkowski and Zworski 2022] and the results on real-analytic and Gevrey Anosov flows from [Bonthonneau and Jézéquel 2020] are already implicitly based on real-analytic radial estimates.

There is an important technical difference between the idea of the proof of Theorems 2 and 3 as depicted above and the way the proof is actually written. Indeed, we cannot work with a space H_∞ of functions that are analytic everywhere on X (in particular because we do not want to assume that g is analytic everywhere in M). Due to the lack of real-analytic bump functions, it is not easy to construct a space of functions that are real-analytic somewhere but have (at most) finite differentiability somewhere else, and that can be used to construct the scattering resolvent. We solve this issue using a strategy that

was already present in [Guedes-Bonthonneau et al. 2024]: we introduce a semiclassical parameter $h > 0$ and work with a space of distributions \mathcal{H} on X that depends on h . Let us point out that the *space* \mathcal{H} really depends on h , not only its norm. As h tends to 0, the elements of \mathcal{H} are more and more regular near the boundary of M in X . We can then invert a rescaled modified Laplacian acting on \mathcal{H} after the addition of a trace class operator whose trace class norm is controlled as h tends to 0, and the upper bound from Theorems 2 and 3 will follow.

Structure of the paper. In Section 2, we introduce a set of general assumptions that will allow us to deal simultaneously with the analysis in the context of Theorems 2 and 3. The point of these assumptions is not to cover a wide generality, but to avoid to write the same proof twice with only notational changes. We state in Section 2 a general result, Proposition 5, from which Theorems 2 and 3 will be deduced.

In Sections 3 and 4, we prove respectively Theorems 3 and 2.

In Section 5, we recall and extend some results from [Bonthonneau and Jézéquel 2020] that will be needed for the proof of Proposition 5.

Finally, Section 6 is the main technical part of the paper, as it contains the proof of Proposition 5.

2. A general statement

In order to deal with the cases of asymptotically hyperbolic manifolds and of Schwarzschild–de Sitter spacetimes simultaneously, we introduce here an abstract set of assumptions that are enough to make our analysis work.

2.1. General assumption. We will use the notion of semiclassical differential operator, so let us recall very briefly what it means (see [Zworski 2012] or [Dyatlov and Zworski 2019, Appendix E] for more details on semiclassical analysis). A semiclassical differential operator Q of order $m \in \mathbb{N}$ on a smooth manifold X is a differential operator on X , depending on a small, so-called semiclassical, implicit parameter $h > 0$, of the form

$$Q = \sum_{k=0}^m h^k Q_k,$$

where Q_k is a differential operator of order k on X that does not depend on h , for $k = 0, \dots, m$. With Q one may associate its (semiclassical) principal symbol $q : T^*X \rightarrow \mathbb{C}$, which is a polynomial of degree m in each fiber of T^*X . We may define q as the unique h -independent function on T^*X such that, for every smooth function $\varphi : M \rightarrow \mathbb{C}$ and $x \in X$, we have

$$e^{-i\frac{\varphi(x)}{h}} Q(e^{i\frac{\varphi}{h}})(x) \underset{h \rightarrow 0}{=} q(x, d_x \varphi) + \mathcal{O}(h).$$

Notice that $q = \sum_{k=0}^m q_k$, where q_k denotes the (classical) homogeneous principal symbol of the differential operator Q_k for $k = 0, \dots, m$. In the applications from Sections 3 and 4, the introduction of the semiclassical parameter h will be somehow artificial, this is just a technical trick.

Now that this reminder is done, we are ready to state our set of general assumptions.

Let X be a closed real-analytic manifold of dimension n . We endow X with a real-analytic Riemannian metric (this is always possible; see [Morrey 1958]). Let Y be an open subset of X with real-analytic boundary ∂Y . Consider a family of differential operators

$$\mathcal{P}_h(\omega) = P_2 + \omega P_1 + \omega^2 P_0, \quad (7)$$

where $\omega \in \mathbb{C}$ and the operator P_j for $j \in \{0, 1, 2\}$ is a semiclassical differential operator (that does not depend on ω) on X of order j with principal symbol p_j . We assume that there is $\epsilon > 0$ and a neighborhood U of ∂Y with real-analytic coordinates $(x_1, x') : U \rightarrow]-\epsilon, \epsilon[\times \partial Y$ such that $\{x_1 = 0\} = \partial Y$ and $\{x_1 > 0\} = Y \cap U$. We require in addition that P_0, P_1 and P_2 have real-analytic coefficients in U and that the following properties hold:

- (a) For $(x_1, x', \xi_1, \xi') \in T^*U \simeq T^*(]-\epsilon, \epsilon[\times \partial Y)$, we have $p_2(x_1, x', \xi_1, \xi') = w(x_1)\xi_1^2 + q_1(x_1, x', \xi')$ where q_1 is a homogeneous real-valued symbol of order 2 on $]-\epsilon, \epsilon[\times T^*\partial Y$ and $w :]-\epsilon, \epsilon[\rightarrow \mathbb{R}$ is a real-analytic function such that $w(0) = 0$ and $w'(0) > 0$.
- (b) There is a constant $C > 0$ such that for $(x_1, x', \xi_1, \xi') \in T^*U$ we have $q_1(x_1, x', \xi') \geq C^{-1}|\xi'|^2$.
- (c) The symbol p_1 is real-valued, $p_1(x_1, x', \xi_1, \xi') = p_1(x_1, \xi_1)$ does not depend on (x', ξ') for $(x_1, x', \xi_1, \xi') \in T^*U$, and there is $C > 0$ such that

$$\frac{p_1(x_1, \xi_1)}{\xi_1} \leq -C^{-1},$$

in particular the sign of $p_1(x_1, \xi_1)$ is the same as the sign of $-\xi_1$.

- (d) The symbol p_2 is real-valued and positive on $T^*Y \setminus \{0\}$.
- (e) The symbol p_0 is real-valued and negative on a neighborhood of \bar{Y} .

Remark 4. Let us explain the significance of these assumptions. In the context of the proof of [Theorem 2](#), the manifold X will be an even extension for M , and Y will be M seen as a subset of the even extension X . In the context of [Theorem 3](#), Y will be the domain of outer communication and ∂Y corresponds to the event and cosmological horizons. In both cases, $\mathcal{P}_h(\omega)$ will be a (semiclassically rescaled) family of modified operators. For instance, in the context of [Theorem 2](#), we replace the operator $-h^2\Delta - h^2(n-1)^2/4 - \omega^2$ by a modified Laplacian $\mathcal{P}_h(\omega)$ (see for instance [Dyatlov and Zworski 2019, §5.3]). The new operator $\mathcal{P}_h(\omega)$ is defined on the whole X , and, for f smooth and compactly supported in Y , solving for u the equation $\mathcal{P}_h(\omega)u = f$ with u satisfying a regularity condition near ∂Y amounts to solving for \tilde{u} the equation $(-h^2\Delta - h^2(n-1)^2/4 - \omega^2)\tilde{u} = \tilde{f}$ while imposing a certain behavior at infinity for \tilde{u} (here \tilde{f} depends on f and is smooth and compactly supported in M).

A method to construct the scattering resolvent is then to construct a meromorphic inverse $\mathcal{P}_h(\omega)^{-1}$ for $\mathcal{P}_h(\omega)$. In [Proposition 5](#) below, we give a new construction of this meromorphic inverse (maybe after modifying $\mathcal{P}_h(\omega)$ away from \bar{Y} , which is harmless since we only care about what happens on Y). This new construction is inspired by the method of Vasy [2013a] (see also [Dyatlov and Zworski 2019, Chapter 5]) with the addition of tools of real-analytic microlocal analysis near ∂Y .

Let us explain very briefly how it works. The idea is to set up a Fredholm theory for $\mathcal{P}_h(\omega)$. Inside Y , the operator $\mathcal{P}_h(\omega)$ is elliptic (due to (d)), so there is no problem here. Outside of \bar{Y} , we are allowed to modify $\mathcal{P}_h(\omega)$, and we can consequently deal with this part of X by adding to $\mathcal{P}_h(\omega)$ a well-chosen elliptic operator. This is similar to the addition of a complex absorbing potential in [Vasy 2013a], and possible because of the hyperbolic structure of $\mathcal{P}_h(\omega)$ near ∂Y in $X \setminus \bar{Y}$. Hence, the most important point is to understand what happens at ∂Y , where $\mathcal{P}_h(\omega)$ stops being elliptic. At that place, the operator $\mathcal{P}_h(\omega)$ has a source/sink structure on its characteristic set (this is a consequence of the assumptions (a) and (b)), so that one can use radial estimates (see for instance to [Dyatlov and Zworski 2019, §E.4]) to set up a Fredholm theory for $\mathcal{P}_h(\omega)$. However, the C^∞ versions of the radial estimates are restricted by a threshold condition: they can be used to construct the scattering resolvent, but they do not give a bound on the number of resonances in disks as in Theorem 2. This is where real-analytic microlocal analysis becomes useful: using methods as in [Bonthonneau and Jézéquel 2020; Galkowski and Zworski 2022] (see also [Galkowski and Zworski 2021; Guedes-Bonthonneau et al. 2024]), we are able to get an estimate which is in some sense a C^ω version of a radial estimate and allows us to prove Theorem 2. This estimate corresponds to the fourth and fifth case in the proof of Lemma 26.

There are some technical reasons that make our set of assumptions very specific. The (e) is rather artificial, this is just a way to ensure that our family of Fredholm operators will be invertible at a point. The assumptions (a), (b) and (c) impose that the source/sink structure of $\mathcal{P}_h(\omega)$ on its characteristic set is very particular. This specific structure will allow us to work in the real-analytic category only near ∂Y , which is essential because we are not able to ensure that $\mathcal{P}_h(\omega)$ is analytic away from ∂Y . Concretely, this ensures that near ∂Y in $X \setminus \bar{Y}$, the projection on X of the bicharacteristics curve of $\mathcal{P}_h(\omega)$ that are contained in its characteristic set go either toward or away from ∂Y . This allows us to set up a propagation estimate by working on spaces weighted by $e^{\psi/h}$, where ψ is a function on X monotone along the projection to X of the bicharacteristics of $\mathcal{P}_h(\omega)$. This estimate does not require real-analytic coefficients, so it can be used to make the link between ∂Y (where we really need real-analytic machinery) and the place in $X \setminus \bar{Y}$ where $\mathcal{P}_h(\omega)$ is artificially made elliptic by the addition of a differential operator with C^∞ coefficients.

2.2. General result. The assumptions from Section 2.1 allow us to state an abstract result from which Theorems 2 and 3 follow.

Proposition 5. *Under the assumptions from Section 2.1, we may modify the operator $\mathcal{P}_h(\omega)$ away from \bar{Y} into a new operator $P_h(\omega)$ so that the following holds. There are two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ (depending on h) and a constant $\kappa > 0$ (that does not depend on h) such that the following properties hold when h is small enough:*

- (i) For $j = 1, 2$, there are continuous inclusions $C^\infty(X) \subseteq \mathcal{H}_j \subseteq \mathcal{D}'(X)$.
- (ii) For $j = 1, 2$, the elements of \mathcal{H}_j are continuous on a neighborhood of ∂Y .
- (iii) $P_h(\omega) : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a holomorphic family of bounded operators.
- (iv) There is $\nu > 0$ such that $P_h(i\nu) : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is invertible.

- (v) For every open and relatively compact subset V of $\{z \in \mathbb{C} : \text{Im } z > -\kappa\}$, if h is small enough then, for every $\omega \in V$, the operator $P_h(\omega) : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is Fredholm of index 0. Moreover, this operator has a meromorphic inverse $\omega \mapsto P_h(\omega)^{-1}$ on V with poles of finite rank.
- (vi) If $\delta \in]0, \kappa[$, there is $C > 0$ such that for every h small enough, the number of ω in the disk of center 0 and radius δ such that $P_h(\omega) : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is not invertible (counted with null multiplicity) is less than Ch^{-n} .

Remark 6. The notion of null multiplicity used in the statement of Proposition 5 is defined using Gohberg–Sigal theory (see for instance [Dyatlov and Zworski 2019, §C.4]). In our context, we can use the following definition: if ω_0 is such that the meromorphic inverse $\omega \mapsto P_h(\omega)$ is defined near ω_0 , then the null multiplicity of $P_h(\omega)$ at ω_0 is the trace of the residue of $\omega \mapsto P_h(\omega)^{-1} \partial_\omega P_h(\omega)$ at ω_0 (which is a finite rank operator).

Remark 7. The modification of $\mathcal{P}_h(\omega)$ needed to get Proposition 5 will be obtained by modifying the coefficients of P_0, P_1 and P_2 away from \bar{Y} , so that the general assumption are still satisfied by $P_h(\omega)$ after this modification.

3. Schwarzschild–de Sitter spacetimes (proof of Theorem 3)

In this section, we explain how the general framework from Section 2 can be used to prove Theorem 3. We start with this case because the setting is slightly simpler than in Theorem 2 that we prove in Section 4 below. We recall a few basic facts about Schwarzschild–de Sitter spacetimes in Section 3.1 and then apply Proposition 5 in Section 3.2. Finally, in Section 3.3, we discuss the number of resonances for the operators obtained by decomposing functions on Schwarzschild–de Sitter spacetimes on spherical harmonics.

3.1. The model. We start by recalling the definition of Schwarzschild–de Sitter spacetimes and of the associated quasinormal frequencies. The interested reader may refer to [Dafermos and Rodnianski 2013] for the geometry of Schwarzschild–de Sitter spacetimes (and other notions from general relativity). For the definition of the resonances, one may refer to [Sá Barreto and Zworski 1997] or [Vasy 2013a]. Fix two constants

$$M_0 > 0 \quad \text{and} \quad 0 < \Lambda < \frac{1}{9M_0^2}.$$

The constant M_0 is called the *mass of the black hole* and Λ the *cosmological constant*. We define the function

$$G(r) = 1 - \frac{\Lambda r^2}{3} - \frac{2M_0}{r} \quad \text{for } r > 0.$$

Let then $r_- < r_+$ be the positive roots of the polynomial $rG(r)$. Define $M =]r_-, r_+[\times \mathbb{S}_y^2$ and $\widehat{M} = \mathbb{R}_t \times M$. Let g be the Lorentzian metric

$$g = -Gdt^2 + G^{-1}dr^2 + r^2g_S(y, dy),$$

where g_S denotes the standard metric on \mathbb{S}^2 . The Lorentzian manifold (\widehat{M}, g) is called a Schwarzschild–de Sitter spacetime. The hypersurfaces $\{r_-\} \times \mathbb{S}^2$ and $\{r_+\} \times \mathbb{S}^2$ are called respectively the event and the cosmological horizons.

In order to understand the asymptotic of the solution to the wave equation on (\widehat{M}, g) , one studies the meromorphic continuation of the resolvent $(P_{\text{SdS}} - \lambda^2)^{-1}$, where

$$P_{\text{SdS}} = Gr^{-2}D_r(r^2G)D_r - Gr^{-2}\Delta_{\mathbb{S}^2}.$$

Here, $D_r = -i\partial_r$ and $\Delta_{\mathbb{S}^2}$ is the (nonpositive) Laplace operator on the sphere \mathbb{S}^2 . The operator P_{SdS} is self-adjoint and nonnegative on the Hilbert space $L^2(]r_-, r_+[\times \mathbb{S}^2; G^{-1}r^2 dr dy)$, where dy denotes the standard volume form on \mathbb{S}^2 . Consequently, the operator $(P_{\text{SdS}} - \lambda^2)^{-1}$ is well-defined on this space when $\text{Im } \lambda > 0$. It is proven for instance in [Sá Barreto and Zworski 1997, §2] that $(P_{\text{SdS}} - \lambda^2)^{-1}$ has a meromorphic continuation $R_{\text{SdS}}(\lambda)$ to \mathbb{C} , with poles of finite rank, as an operator from $C_c^\infty(]r_-, r_+[\times \mathbb{S}^2)$ to $\mathcal{D}'(]r_-, r_+[\times \mathbb{S}^2)$. The poles of this meromorphic continuation are called the *quasinormal frequencies* for the Schwarzschild–de Sitter spacetime. If $\lambda_0 \neq 0$ is a quasinormal frequency, we define its multiplicity as the rank of the operator

$$\frac{i}{\pi} \int_\gamma \lambda R_{\text{SdS}}(\lambda) d\lambda,$$

where γ is a positively oriented circle around λ_0 , small enough so that the index of any other quasinormal frequency with respect to γ is zero.

3.2. Upper bound on the number of quasinormal frequencies. Our proof of Theorem 3 is based on the method of Vasy [2013a] to construct the resolvent $R_{\text{SdS}}(\lambda)$, following mostly the exposition from [Dyatlov and Zworski 2019, Exercise 16, p. 376]. We start with a standard modification of the operator $P_{\text{SdS}} - \lambda^2$, with some minor changes that will be convenient to check the assumptions from Section 2.1.

Let us embed a neighborhood of $[r_-, r_+]$ in the circle \mathbb{S}^1 and set $X = \mathbb{S}^1 \times \mathbb{S}^2$ and $Y =]r_-, r_+[\times \mathbb{S}^2$. Let $\rho :]r_-, r_+[\rightarrow [-1, 1]$ be a C^∞ function, identically equal to ± 1 near r_\pm . Let then $F :]r_-, r_+[\rightarrow \mathbb{R}$ be a primitive of

$$F'(r) = \rho(r) \left(\frac{1}{G(r)} - \frac{1}{2(1 - (9M_0^2\Lambda)^{1/3})} \right) \tag{8}$$

and introduce, for $\lambda \in \mathbb{C}$, the operator

$$G^{-1}e^{-i\lambda F(r)}(P_{\text{SdS}} - \lambda^2)e^{i\lambda F(r)},$$

which is explicitly given by the formula

$$GD_r^2 - r^{-2}\Delta_{\mathbb{S}^2} + \left(2\lambda F'G - i \left(\frac{2G}{r} + G' \right) \right) D_r - i\lambda \left(\frac{2GF'}{r} + G'F' + GF'' \right) - \lambda^2 \frac{(1 - G^2(F')^2)}{G}. \tag{9}$$

The coefficients of this differential operator extend as real-analytic functions near r_- and r_+ . Indeed, the definition of F ensures that $F'G$ continues analytically passed r_- and r_+ . Moreover, near r_\pm a direct computation yields

$$G'F' + GF'' = \mp \frac{G'}{2(1 - (9M_0^2\Lambda)^{1/3})}$$

and

$$\frac{1 - G^2(F')^2}{G} = \frac{1}{1 - (9M_0^2\Lambda)^{1/3}} - \frac{G}{4(1 - (9M_0^2\Lambda)^{1/3})^2}.$$

Letting χ be a C^∞ function supported in a small neighborhood of $[r_-, r_+]$ and identically equal to 1 on a smaller neighborhood of $[r_-, r_+]$, we can define a family of operators on X by

$$\Theta(\lambda) = \chi(r) \times (9).$$

Finally, for $\omega \in \mathbb{C}$, we define the semiclassical differential operator

$$\mathcal{P}_h(\omega) := h^2\Theta(h^{-1}\omega).$$

Notice that this operator depends on the implicit semiclassical parameter h as in Section 2. It is of the form (7) with

$$\begin{aligned} P_0 &= \chi(r) \left(Gh^2D_r^2 - r^{-2}h^2\Delta_{\mathbb{S}^2} - i \left(\frac{2G}{r} + G' \right) h^2D_r \right), \\ P_1 &= \chi(r) \left(2F'GhD_r - ih \left(\frac{2GF'}{r} + G'F' + GF'' \right) \right), \\ P_2 &= -\chi(r) \frac{1 - G^2(F')^2}{G}, \end{aligned}$$

where it is understood that the factor in parentheses continues analytically in r passed r_- and r_+ . Let us check that the general assumptions from Section 2.1 are satisfied by this family of operator.

We already mentioned that $\mathcal{P}_h(\omega)$ is of the form (7), and it follows from the expression for the P_j 's given above that they are semiclassical differential operators of order j with analytic coefficients on a neighborhood of ∂Y . Moreover, the principal symbols of the P_j 's are given on Y by

$$p_2(r, y, \rho, \eta) = G(r)\rho^2 + r^{-2}\eta^2, \quad p_1(r, y, \rho, \eta) = 2F'(r)G(r)\rho, \quad p_0(r, y) = -\frac{1 - G(r)^2F'(r)^2}{G(r)}.$$

We get the values of these symbols on a neighborhood of \bar{Y} by continuing these formulas analytically in r .

We can define the coordinates (x_1, x') near ∂Y by taking $x_1 = r - r_-$ (when r is near r_-) or $x_1 = r_+ - r$ (when r is near r_+) and $x' = y$. Beware here that this change of coordinates reverses the orientation of the real line near r_+ . Then, we see that the (a) holds with $w(x_1) = G(r_\pm \mp x_1)$ and $q_1(x_1, y, \eta) = (r_\pm \mp x_1)^{-2}\eta^2$. In particular, we have $w'(0) = \mp G'(r_\pm) > 0$. The point (b) follows from the definition of q_1 . To get (c), one only needs to notice that the value at r_\pm of the real-analytic extension of $F'(r)G(r)$ is ± 1 (and that our change of variable reverses orientation near r_+). Since G is positive on $]r_-, r_+[$, we get (d). In order to check (e), write

$$p_0(r, y) = \frac{\rho(r)^2 \left(1 - \frac{G(r)}{2(1 - (9M_0^2\Lambda)^{1/3})} \right)^2 - 1}{G(r)}.$$

Since $1 - (9M_0^2\Lambda)^{1/3}$ is an upper bound for G on $]r_-, r_+[$, we find that $p_0(r, y) < 0$ for $r \in]r_-, r_+[$. Using that $\rho(r)^2$ is equal to 1 when r is near r_\pm , we find that

$$p_0(r_\pm, y) = -\frac{1}{1 - (9M_0^2\Lambda)^{1/3}} < 0,$$

and thus (e) holds.

Consequently, we can modify $\mathcal{P}_h(\omega)$ away from \bar{Y} in order to get a family of operator $P_h(\omega)$ that satisfies [Proposition 5](#). With κ as in [Proposition 5](#), we let V be a connected, relatively compact and open subset of $\{z \in \mathbb{C} : \text{Im } z > -\kappa\}$ that contains the closed disk of center 0 and radius $3\kappa/4$. Let ι_2 denote the injection of $C_c^\infty(Y)$ in \mathcal{H}_2 and ι_1 denote the map from \mathcal{H}_1 to $\mathcal{D}'(Y)$ obtained by composing the injection $\mathcal{H}_1 \rightarrow \mathcal{D}'(X)$ with the restriction map $\mathcal{D}'(X) \rightarrow \mathcal{D}'(Y)$.

If $\lambda \in h^{-1}V$, we define the resolvent

$$R_h(\lambda) = e^{i\lambda F(r)} h^2 \iota_1 P_h(\lambda h)^{-1} \iota_2 e^{-i\lambda F(r)} G^{-1} : C_c^\infty(Y) \rightarrow \mathcal{D}'(Y). \tag{10}$$

This is a meromorphic family of operators. We just got a new construction of the meromorphic continuation $R_{\text{Sds}}(\lambda)$ of the L^2 resolvent $(P_{\text{Sds}} - \lambda^2)^{-1}$, as we will now demonstrate.

Lemma 8. *If $\lambda \in h^{-1}V$ is such that $\text{Im } \lambda > 0$, then $R_h(\lambda)$ is the restriction to $C_c^\infty(Y)$ of the L^2 resolvent $(P_{\text{Sds}} - \lambda^2)^{-1}$. In particular $R_h(\lambda)$ does not depend on h .*

Proof. Let $\lambda \in h^{-1}V$ be such that $\text{Im } \lambda > 0$. Let $u \in C_c^\infty(Y)$. Notice that

$$\begin{aligned} (P_{\text{Sds}} - \lambda^2)R_h(\lambda)u &= G e^{i\lambda F(r)} G^{-1} e^{-i\lambda F(r)} (P_{\text{Sds}} - \lambda^2) e^{i\lambda F(r)} h^2 \iota_1 P_h(\lambda h)^{-1} \iota_2 e^{-i\lambda F(r)} G^{-1} u \\ &= G e^{i\lambda F(r)} P_h(\lambda h) \iota_1 P_h(\lambda h)^{-1} \iota_2 e^{-i\lambda F(r)} G^{-1} u = u, \end{aligned}$$

where we used that $h^2 G^{-1} e^{-i\lambda F(r)} (P_{\text{Sds}} - \lambda^2) e^{i\lambda F(r)} \iota_1 = P_h(\lambda h) \iota_1 = \iota_3 P_h(\lambda h)$, where ι_3 is the map obtained by composing the injection $\mathcal{H}_2 \rightarrow \mathcal{D}'(X)$ with the restriction map $\mathcal{D}'(X) \rightarrow \mathcal{D}'(Y)$. Consequently, we only need to prove that the distribution $R_h(\lambda)u$ belongs to the space $L^2(\text{]}r_-, r_+[\times \mathbb{S}^2; G^{-1}r^2 \text{ dr dy})$. Since P_{Sds} is elliptic, we know that u is smooth, and thus bounded on all compact subsets of Y . It remains to understand the behavior of u near ∂Y .

Notice that $R_h(\lambda)u = e^{i\lambda F(r)}v$, where v is the restriction to Y of an element of \mathcal{H}_1 . In particular, since the elements of \mathcal{H}_1 are continuous near ∂Y , there is a compact subset K of Y such that v is continuous and bounded outside of K . Let us study for instance the behavior of u near $r = r_-$ (the behavior near r_+ is similar). From [\(8\)](#), we see that

$$F(r) \underset{r \rightarrow r_-}{=} -\frac{\ln|r - r_-|}{G'(r_-)} + \mathcal{O}(1).$$

Consequently, we have that $e^{i\lambda F(r)}$ is $\mathcal{O}(|r - r_-|^{\text{Im } \lambda / G'(r_-)})$ when r tends to r_- . Working similarly near r_+ , we find that u belongs to the Hilbert space $L^2(\text{]}r_-, r_+[\times \mathbb{S}^2; G^{-1}r^2 \text{ dr dy})$. □

Remark 9. It follows from [Lemma 8](#) that $R_h(\lambda) = R_{\text{Sds}}(\lambda)$ on $h^{-1}V$. In particular, $\lambda \in h^{-1}V$ is a quasinormal frequency if and only if it is a pole of $R_h(\lambda)$ and, if in addition $\lambda \neq 0$, its multiplicity is the rank of the operator

$$\frac{i}{\pi} \int_\gamma \mu R_h(\mu) \text{ d}\mu,$$

where γ is a small circle around λ .

With this new construction of the resolvent $R_{\text{Sds}}(\lambda)$ at our disposal, we are ready to prove [Theorem 3](#).

Proof of Theorem 3. Considering the bound on the number of points where $P_h(\omega)$ is not invertible given in Proposition 5, we only need to prove that if λ is a nonzero complex number of modulus less than $\kappa/(4h)$ then its multiplicity as a quasinormal frequency is less than the null multiplicity of $\omega \mapsto P_h(\omega)$ at λh .

Let us consider a quasinormal frequency λ of modulus less than $\kappa/(4h)$. Since $P_h(\omega)$ is a holomorphic family of operators with a meromorphic inverse near λh (because λh belongs to V), it follows from the Gohberg–Sigal theory [Dyatlov and Zworski 2019, Theorem C.10], that there are holomorphic families of invertible operators $U_1(\omega)$ and $U_2(\omega)$ for ω near λh , respectively on \mathcal{H}_2 and from \mathcal{H}_1 to \mathcal{H}_2 , an integer $M \geq 0$, operators P_0, \dots, P_M on \mathcal{H}_2 and nonzero integers k_1, \dots, k_M such that

$$P_h(\omega) = U_1(\omega) \left(P_0 + \sum_{m=1}^M (\omega - \lambda h)^{k_m} P_m \right) U_2(\omega) \tag{11}$$

for ω near λh . Moreover, P_1, \dots, P_M are rank 1 and $P_\ell P_m = \delta_{\ell,m} P_m$ for $0 \leq \ell, m \leq M$. We also have that $I = \sum_{m=0}^M P_m$, since $P_h(\omega)$ is invertible for $\omega \neq \lambda h$ near λh . Notice that the k_m 's must be positive, since $P_h(\omega)$ is holomorphic in ω , and that the null multiplicity of $P_h(\omega)$ at λh is $\sum_{m=1}^M k_m$.

It follows from (11) that

$$P_h(\omega)^{-1} = U_2(\omega)^{-1} \left(P_0 + \sum_{m=1}^M (\omega - \lambda h)^{-k_m} P_m \right) U_1(\omega)^{-1}. \tag{12}$$

From (10) we get

$$R_h(\mu) = A_1(\mu) + A_2(\mu),$$

where A_1 and A_2 are obtained by replacing the inverse $P_h(\omega)^{-1}$ respectively by $U_2(\omega)^{-1} P_0 U_1(\omega)^{-1}$ and by $U_2(\omega)^{-1} \sum_{m=1}^M (\omega - \lambda h)^{-k_m} P_m U_1(\omega)^{-1}$ in (10), with $\omega = \mu h$. Notice that $A_1(\mu)$ is holomorphic in μ , so that

$$\int_\gamma \mu R_h(\mu) \, d\mu = \int_\gamma \mu A_2(\mu) \, d\mu. \tag{13}$$

The operator $\mu A_2(\mu)$ is of the form $B_1(\mu) (\sum_{k=1}^M (\mu - \lambda)^{-k_m} P_m) B_2(\mu)$, where $B_1(\mu)$ and $B_2(\mu)$ are holomorphic near λ . Writing the Taylor expansions for $B_1(\mu)$ and $B_2(\mu)$,

$$B_j(\mu) = \sum_{\ell \geq 0} (\mu - \lambda)^\ell C_{j,\ell},$$

we find that the residue of $\mu A_2(\mu)$ at λ is

$$\sum_{\substack{m,k,\ell \\ k+\ell=k_m-1}} C_{1,k} P_m C_{2,\ell}.$$

This operator is the sum of $\sum_{m=1}^M k_m$ operators of rank at most 1, and thus is of rank at most $\sum_{m=1}^M k_m$. It follows then from Remark 9 and (13) that the multiplicity of λ as a scattering resonance is at most $\sum_{m=1}^M k_m$, which is the null multiplicity of $\omega \mapsto P_h(\omega)$ at λh . \square

3.3. Decomposition on spherical harmonics. Notice that the Schwarzschild–de Sitter spacetime is radially symmetric. It is standard to use this kind of symmetry to study quasinormal frequencies by decomposing the operator P_{SdS} on spherical harmonics (see for instance [Sá Barreto and Zworski 1997] or [Hintz and Xie 2022]). Let $\ell \in \mathbb{N}$ and Y be a spherical harmonics satisfying $-\Delta_{\mathbb{S}^2} Y = \ell(\ell + 1)Y$. The action of P_{SdS} on functions of the form $u(r)Y(y)$ is then equivalent to the action of the operator

$$P_{\text{SdS}}^\ell = Gr^{-2}D_r(r^2G)D_r + Gr^{-2}\ell(\ell + 1).$$

The operator $(P_{\text{SdS}}^\ell - \lambda^2)^{-1}$ defined for $\text{Im } \lambda > 0$ by the spectral theory on $L^2([r_-, r_+]; G^{-1}r^2 dr)$ admits a meromorphic continuation to \mathbb{C} . The poles of this extension are quasinormal frequencies corresponding to angular momentum ℓ .

We can then apply Proposition 5 as in Section 3.2 to get:

Theorem 10. *The number of quasinormal frequencies corresponding to the angular momentum ℓ of modulus less than r is $\mathcal{O}(r)$ when r tends to $+\infty$.*

4. Scattering on asymptotically hyperbolic manifolds (proof of Theorem 2)

In this section, we specify the geometric assumptions from Theorem 2 and explain how one can use Proposition 5 to prove Theorem 2. In Section 4.1 we describe the class of asymptotically hyperbolic manifolds with real-analytic ends that we are going to study. In Sections 4.2 and 4.3, we check the assumptions from Section 2.1 in order to use Proposition 5 and prove Theorem 2 in Section 4.4.

Sections 4.1, 4.2 and 4.3 are based on the exposition in [Dyatlov and Zworski 2019, Chapter 5] of the method of Vasy [2013a] to construct the scattering resolvent, with a few additional technicalities required to deal with real-analytic ends and apply Proposition 5.

4.1. Geometric assumptions. We explain here how the definition of asymptotically hyperbolic manifold may be modified to obtain the definition of asymptotically hyperbolic manifolds with real-analytic ends that appears in Theorem 2. Let us consider a Riemannian manifold (M, g) where M is a real-analytic manifold but the metric g is a priori only C^∞ . One could just say that (M, g) is asymptotically hyperbolic with real-analytic ends if M is the interior of a compact real-analytic manifold with boundary \bar{M} such that g may be put into the form (1), with g_1 real-analytic, near $\partial\bar{M}$, using a real-analytic diffeomorphism between $[0, \epsilon[\times \partial\bar{M}$ and a neighborhood of $\partial\bar{M}$. This is for instance the assumption that is made in [Zuily 2017]. However, it may seem a priori too restrictive to assume the existence of such coordinates defined on a neighborhood of the whole $\partial\bar{M}$. Consequently, we will rather make a local assumption on g and then see that it implies that g takes the form (1) in real-analytic coordinates.

Definition 11. Let M be a real-analytic manifold and g be a smooth (C^∞) Riemannian metric on M . We assume that M is the interior of a compact real-analytic manifold with boundary \bar{M} . Assume that, for every $x_0 \in \partial\bar{M}$, there is a neighborhood U of x_0 in \bar{M} and a real-analytic function y_1 from U to \mathbb{R} such that

- (i) $y_1 \geq 0$ on U and $\partial\bar{M} \cap U = \{y_1 = 0\}$;
- (ii) $dy_1(x) \neq 0$ for every $x \in \partial\bar{M} \cap U$;

- (iii) $y_1^2 g$ extends to a real-analytic metric \tilde{g} on U ;
- (iv) $|dy_1|_{\tilde{g}} = 1$ on $\partial\bar{M} \cap U$.

Then we say that (M, g) is an asymptotically hyperbolic manifold real-analytic near infinity.

A function that satisfies (i) and (ii) is called a boundary defining function for \bar{M} . Notice that if y_1 and \tilde{y}_1 are two real-analytic boundary defining functions, then there is a real-analytic real-valued function f , defined wherever y_1 and \tilde{y}_1 are both defined, and such that $\tilde{y}_1 = e^f y_1$. In particular, the validity of (iii) and (iv) does not depend on the choice of the boundary defining function y_1 . One can check that if (M, g) is an asymptotically hyperbolic manifold real-analytic near infinity, then it is also an asymptotically hyperbolic manifold in the standard (C^∞) sense (see for instance [Dyatlov and Zworski 2019, Definition 5.2]).

Let us fix an asymptotically hyperbolic manifold real-analytic near infinity (M, g) , and let \bar{M} be as in Definition 11. The existence of a real-analytic boundary defining function defined on a neighborhood of $\partial\bar{M}$ does not seem obvious, and will be established in Lemma 13 below. However, notice that one easily shows that there are C^∞ boundary defining functions defined on the whole \bar{M} and let us define the conformal class of Riemannian metrics on ∂M :

$$[g]_{\partial\bar{M}} = \{(y_1^2 g)|_{\partial\bar{M}} : y_1 \in C^\infty(\bar{M}) \text{ is a boundary defining function}\}.$$

It will be convenient to know that:

Lemma 12. *The conformal class $[g]_{\partial\bar{M}}$ admits a real-analytic representative.*

Proof. Let g_0 be any C^∞ representative of $[g]_{\partial\bar{M}}$. Let \hat{g} be a real-analytic Riemannian metric on $\partial\bar{M}$ (whose existence is guaranteed by [Morrey 1958]). For every $x \in \partial\bar{M}$, let $B(x)$ be the self-adjoint (for $\hat{g}(x)$) endomorphism of $T_x \partial M$ such that $g_0(x) = \hat{g}(x)(B(x) \cdot, \cdot)$. Let g_1 be the metric defined by $g_1(x) = g_0(x) / \|B(x)\|$, where the operator norm of $B(x)$ is defined using the metric $\hat{g}(x)$. From its very definition, g_1 is a representative of $[g]_{\partial M}$. Let us prove that g_1 is real-analytic.

Let $x_0 \in \partial\bar{M}$. From our assumption above (Definition 11), there is a neighborhood V of x_0 in $\partial\bar{M}$ and a real-analytic metric g_2 on V such that g_2 is conformal to g_0 on V . We have $g_0 = e^{2f} g_2$ for some C^∞ function f on V . For $x \in V$, we have

$$\begin{aligned} g_1(x) &= \frac{g_0(x)}{\|B(x)\|} = \hat{g}(x) \left(\frac{B(x)}{\|B(x)\|} \cdot, \cdot \right) \\ &= \hat{g}(x) \left(\frac{e^{-2f(x)} B(x)}{\|e^{-2f(x)} B(x)\|} \cdot, \cdot \right). \end{aligned}$$

On the other hand, for $x \in V$, we have

$$g_2(x) = \hat{g}(x)(e^{-2f(x)} B(x) \cdot, \cdot).$$

Since g_2 and \hat{g} are real-analytic, it follows that $x \mapsto e^{-2f(x)} B(x)$ is real-analytic on V , and thus so is g_1 . \square

We can then establish the existence of a real-analytic diffeomorphism on a neighborhood of $\partial\bar{M}$ that puts the metric g into the form (1) (this is also known as a canonical product structure). The C^∞ version of this result is standard; see for instance [Dyatlov and Zworski 2019, Theorem 5.4].

Lemma 13. *Let g_0 be a real-analytic representative of $[g]_{\partial\bar{M}}$. Then there is a real-analytic boundary function y_1 defined on a neighborhood U of $\partial\bar{M}$ such that*

$$|dy_1|_{y_1^2g} = 1 \text{ on a neighborhood of } \partial\bar{M} \text{ and } g_0 = (y_1^2g)|_{\partial\bar{M}}. \tag{14}$$

Moreover, there is a real-analytic map y' from U to $\partial\bar{M}$ such that y' is the identity on $\partial\bar{M}$, the map $\Psi = (y_1, y')$ is a diffeomorphism from U to $[0, \epsilon[\times \partial\bar{M}$ for some $\epsilon > 0$, and the pushforward of g under this map has the form

$$(\Psi^{-1})^*g = \frac{dy_1^2 + g_1(y_1, y', dy')}{y_1^2},$$

where $g_1(y_1, y', dy')$ is a real-analytic family of Riemannian metrics on $\partial\bar{M}$.

Proof. We start by constructing y_1 locally. Let $x_0 \in \partial\bar{M}$. Let \tilde{y}_1 be a real-analytic boundary function defined on a neighborhood U of x_0 as in Definition 11. Up to multiplying \tilde{y}_1 by a real-analytic function, we may assume that $(\tilde{y}_1^2g)|_{\partial\bar{M} \cap U} = g_0$. We want to construct y_1 on a neighborhood of x_0 of the form $y_1 = e^f \tilde{y}_1$ with f real-analytic that vanishes on $\partial\bar{M}$. The condition $|dy_1|_{y_1^2g} = 1$ may be rewritten as an eikonal equation, $F(x, df(x)) = 0$, noncharacteristic with respect to $\partial\bar{M}$, like in [Dyatlov and Zworski 2019, (5.1.11)–(5.1.12)], which in our case has real-analytic coefficients. We can then use [Taylor 2011, Theorem 1.15.3] to find a (unique) solution f to this equation near x_0 , which happens to be real-analytic. Thus, we constructed a boundary defining function y_1 that satisfies (14) near x_0 .

Notice that if y_1 and y_2 are boundary defining functions that satisfy (14) on open sets U_1 and U_2 of \bar{M} , then y_1 and y_2 coincide on all the connected components of $U_1 \cap U_2$ that intersect $\partial\bar{M}$. Indeed, we can write $y_1 = e^f y_2$ with f that satisfies an eikonal equation as above and vanishes on $\partial\bar{M}$, and there is only one solution to this equation near $\partial\bar{M}$. We get the coincidence of y_1 and y_2 on the whole connected component of $U_1 \cap U_2$ by analytic continuation.

We can consequently glue the local solutions to (14) to get a solution defined on a neighborhood of the whole $\partial\bar{M}$.

Finally, we construct the normal coordinates (y_1, y') by integrating the gradient vector field $\nabla^{y_1^2g} y_1$ starting on $\partial\bar{M}$ as in the proof of [Dyatlov and Zworski 2019, Theorem 5.4]. □

Definition 14. Using the notation from Lemma 13, we say that (M, g) is even if for every integer k , we have

$$\partial_{y_1}^{2k+1} g_1(0, y', dy') = 0. \tag{15}$$

From now on, we will always assume that (M, g) satisfies the evenness assumption Definition 14. Notice that Definitions 11 and 14 together are the hypotheses from Theorem 2. It is also worth noticing that the evenness assumption (15) does not depend on the choice of the canonical product structure; see [Dyatlov and Zworski 2019, Theorem 5.6].

4.2. Even extension. We define an even extension X for M in the following way. We fix a canonical

product structure (y_1, y') on a neighborhood $U \simeq]0, \epsilon[\times \partial\bar{M}$ of $\partial\bar{M}$, as in [Lemma 13](#). Let us define the real-analytic diffeomorphisms

$$\begin{aligned} \psi_+ : U \cap M &\rightarrow]0, \epsilon^2[\times \partial\bar{M}, & x &\mapsto (y_1(x)^2, y'(x)), \\ \psi_- : U \cap M &\rightarrow]-1 - \epsilon^2, -1[\times \partial\bar{M}, & x &\mapsto (-1 - y_1(x)^2, y'(x)). \end{aligned}$$

We let X be the closed real-analytic manifold obtained by gluing $]-1 - \epsilon^2, \epsilon^2[\times \partial\bar{M}$ with two distinct copies of M using the maps ψ_- and ψ_+ . We let x_1 be the function on X given by the first coordinate in $]-1 - \epsilon^2, \epsilon^2[\times \partial\bar{M}$. Up to making ϵ smaller, we extend x_1 to a smooth function on X , real-analytic on $]-1 - \epsilon^2, \epsilon^2[\times \partial\bar{M}$, and such that $]-1 - \epsilon^2, \epsilon^2[\times \partial\bar{M} = \{-1 - \epsilon^2 < x_1 < \epsilon^2\}$.

The features of the even extension X of M in $\{x_1 < 0\}$ are somehow irrelevant: we are only concerned by the analysis in $\{x_1 \geq 0\}$ (but it is more convenient to work on a closed real-analytic manifold). In particular, we will identify $Y := \{x_1 > 0\}$ with M . We will never do that with $\{x_1 < -1\}$. Notice however that $\bar{Y} \subseteq X$ does not have the same smooth structure as \bar{M} as defined above (the manifold \bar{Y} is the even compactification of M).

Notice that the diffeomorphism $\psi_+ : U \cap M \rightarrow]0, \epsilon^2[_{x_1} \times \partial\bar{M}_{x'}$ puts the metric g into the form

$$(\psi_+^{-1})^* g = \frac{dx_1^2}{4x_1^2} + \frac{g_1(\sqrt{x_1}, x', dx')}{x_1}.$$

It follows from our evenness assumption, [Definition 14](#), that the family $x_1 \mapsto g_1(\sqrt{x_1}, x', dx')$ of real-analytic metrics on $\partial\bar{M}$ has a real-analytic extension to $\{-\zeta < x_1 < \zeta\}$ for some $\zeta > 0$.

4.3. The modified Laplacian. Let $\eta > 0$ be smaller than $\zeta/2, \epsilon^2/2$ and 1 (where ζ and ϵ are defined in the previous section), and choose a function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ such that $\rho(x) = x$ for $|x| \leq \eta$ and $\rho(x) = \pm 3\eta/2$ for $|x| \geq 2\eta$ (where \pm is the sign of x). Notice that we can choose ρ such that $\rho'(x)x/\rho(x) \leq 1$ for positive x . Define then the function

$$\tilde{x}_1 = \rho\left(\frac{4x_1}{(1+x_1)^2}\right)$$

on X . For $\lambda \in \mathbb{C}$, let us consider the operator on $M \simeq Y$

$$\tilde{x}_1^{\frac{i\lambda}{2} - \frac{n+3}{4}} \left(-\Delta_g - \frac{(n-1)^2}{4} - \lambda^2\right) \tilde{x}_1^{\frac{n-1}{4} - \frac{i\lambda}{2}}, \tag{16}$$

where Δ_g is the (nonpositive) Laplacian on M . Using ψ_+ to identify the set $\{0 < x_1 < \eta\}$ with $]0, \eta[_{x_1} \times \partial\bar{M}_{x'}$, we see that the operator [\(16\)](#) takes the form

$$\begin{aligned} -x_1(1+x_1)^2 \partial_{x_1}^2 - \frac{(1+x_1)^2}{4} \Delta_{g_1} + (1+x_1)((n-2-i\lambda)x_1 + i\lambda - 1 - \gamma x_1(1+x_1)) \partial_{x_1} \\ - \left(\frac{n-1}{2} - i\lambda\right) \left(x_1 \frac{n-1}{2} + i\lambda - 1 - \gamma \frac{(1+x_1)(1-x_1)}{2}\right) \end{aligned} \tag{17}$$

there. Here Δ_{g_1} is the Laplacian for the metric $g_1(\sqrt{x_1}, x', dx')$ on $\partial\bar{M}$, the function γ is the logarithmic derivative $J^{-1} \frac{\partial J}{\partial x_1}$ with respect to x_1 of the Jacobian J of the metric $g_1(\sqrt{x_1}, x', dx')$ on $\partial\bar{M}$. The

Jacobian J may be defined by taking local coordinates on $\partial\bar{M}$. While J depends on the choice of coordinates, the logarithmic derivative γ does not. It follows from our evenness assumption that γ extends to a real-analytic function on $\{-\eta < x_1 < \eta\} \subseteq X$. Notice that the expression (17) extends real-analytically to $\{-\eta < x_1 < \eta\} \subseteq X$.

Remark 15. Here, we differ from the exposition in [Dyatlov and Zworski 2019, Chapter 5] where, instead of (16), the operator

$$x_1^{\frac{i\lambda}{2} - \frac{n+3}{4}} \left(-\Delta_g - \frac{(n-1)^2}{4} - \lambda^2 \right) x_1^{\frac{n-1}{4} - \frac{i\lambda}{2}} \tag{18}$$

is considered. This is an artificial modification that we introduce in order to be able to check (e) from Section 2.1. The formula (17) for (16) can be deduced from the formula for (18) given in [Dyatlov and Zworski 2019, Lemma 5.10].

Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that $\chi(t) = 0$ for $t \leq -2\eta/3$ and $\chi(t) = 1$ for $t \geq -\eta/3$. Define then for $\lambda \in \mathbb{C}$ the differential operator $P(\lambda)$ on X by

$$P(\lambda) = \begin{cases} \tilde{x}_1^{\frac{i\lambda}{2} - \frac{n+3}{4}} \left(-\Delta_g - \frac{(n-1)^2}{4} - \lambda^2 \right) \tilde{x}_1^{\frac{n-1}{4} - \frac{i\lambda}{2}} & \text{on } Y \simeq M, \\ \chi(x_1) \times (17) & \text{on } \{-\eta < x_1 < \eta\}, \\ 0 & \text{on } \{x_1 < -2\eta/3\}. \end{cases}$$

Notice that the differential operator $P(\lambda)$ has real-analytic coefficients on the set $\{-\eta/3 < x_1 < \eta\}$. Let us define for $\omega \in \mathbb{C}$ and $h > 0$ the semiclassical operator

$$\mathcal{P}_h(\omega) = h^2 P(\omega/h).$$

Let us check that this family of operators satisfy the general assumptions from Section 2.1. We recall that the manifold X and its open subset Y have been defined at the end of Section 4.2. It follows from (17) that $\mathcal{P}_h(\omega)$ is of the form (7) with P_0, P_1 and P_2 that have real-analytic coefficients in the neighborhood $\{-\eta/3 < x_1 < \eta\}$ of ∂Y .

Let p_j denote the principal symbol of P_j for $j = 0, 1, 2$. For x in the interior of Y , we have

$$\begin{aligned} p_2(x, \xi) &= \frac{(1+x_1)^2}{4x_1} |\xi|_{g(x)}^2, \\ p_1(x, \xi) &= -\frac{(1+x_1)^2}{4x_1} \left\langle \xi, \frac{d\tilde{x}_1}{\tilde{x}_1} \right\rangle_{g(x)}, \\ p_0(x) &= \frac{(1+x_1)^2}{4x_1} \left(\left| \frac{d\tilde{x}_1}{2\tilde{x}_1} \right|_{g(x)}^2 - 1 \right). \end{aligned}$$

Near ∂Y , we can express these symbols in the (x_1, x') coordinates to find

$$\begin{aligned} p_2(x_1, x, \xi_1, \xi') &= x_1(1+x_1)^2 \xi_1^2 + \frac{(1+x_1)^2}{4} |\xi'|_{g_1(\sqrt{x_1}, x')}, \\ p_1(x_1, x, \xi_1, \xi') &= -(1+x_1)(1-x_1)\xi_1, \\ p_0(x_1, x') &= -1. \end{aligned}$$

We are now in position to check that the assumptions from Section 2.1 are satisfied. We see that (a) holds with $w(x_1) = x_1(1 + x_1)$ and $q_1(x_1, x', \xi') = \frac{1}{4}(1 + x_1)|\xi'|^2_{g_1(\sqrt{x_1}, x')}$. It is clear from the definition of q_1 that (b) also holds. The validity of (c) and (d) follows immediately from the formulae for $p_1(x_1, x, \xi_1, \xi')$ and $p_2(x, \xi)$ above.

It remains to prove (e), that is, that p_0 is negative on a neighborhood of \bar{Y} . It is clear that p_0 is negative on a neighborhood of ∂Y from the formula above, so that we only need to check that

$$\left| \frac{d\tilde{x}_1}{2\tilde{x}_1} \right|_{g(x)} < 1$$

on the interior of Y .

Notice that we have

$$\frac{d\tilde{x}_1}{2\tilde{x}_1} = \frac{\rho'\left(\frac{4x_1}{(1+x_1)^2}\right)}{\rho\left(\frac{4x_1}{(1+x_1)^2}\right)} \frac{4x_1}{(1+x_1)^2} \frac{1-x_1}{1+x_1} \frac{dx_1}{2x_1}.$$

Since $\left| \frac{dx_1}{2x_1} \right|_{g(x)} = 1$ when $0 < x_1 < 2\eta$, we get

$$\left| \frac{d\tilde{x}_1}{2\tilde{x}_1} \right|_{g(x)} = \left| \frac{\rho'\left(\frac{4x_1}{(1+x_1)^2}\right)}{\rho\left(\frac{4x_1}{(1+x_1)^2}\right)} \frac{4x_1}{(1+x_1)^2} \right| \frac{1-x_1}{1+x_1} \leq \frac{1-x_1}{1+x_1},$$

and the validity of the (e) follows.

4.4. Upper bound on the number of resonances. Since the assumptions from Section 2.1 are satisfied by the operator $\mathcal{P}_h(\omega)$ introduced in Section 4.3, we may modify $\mathcal{P}_h(\omega)$ to get an operator $P_h(\omega)$ that satisfies Proposition 5.

From here, the strategy to prove Theorem 2 is the same as in Section 3.2. We let κ be as in Proposition 5 and choose a connected, relatively compact and open subset V of $\{z \in \mathbb{C} : \text{Im } z > -\kappa\}$ that contains the closed disk of center 0 and radius $3\kappa/4$. We write ι_2 for the inclusion of $C_c^\infty(M)$ in \mathcal{H}_2 and ι_1 for the map obtained by composition of the inclusion of \mathcal{H}_1 in $\mathcal{D}'(X)$ and the restriction map $\mathcal{D}'(X) \rightarrow \mathcal{D}'(M)$.

For $\lambda \in h^{-1}V$, define the resolvent

$$R_h(\lambda) = \tilde{x}_1^{\frac{n-1}{4} - \frac{i\lambda}{2}} \iota_1 h^2 P_h(h\lambda)^{-1} \iota_2 \tilde{x}_1^{\frac{i\lambda}{2} - \frac{n+3}{4}} : C_c^\infty(M) \rightarrow \mathcal{D}'(M).$$

As in Section 3.2, we get:

Lemma 16. *If h is small enough, λ is in $h^{-1}V$ and $\text{Im } \lambda > 0$, then $R_h(\lambda)$ coincides with the inverse of $-\Delta_g - \frac{1}{4}(n-1)^2 - \lambda^2$ on $L^2(M)$. In particular, $R_h(\lambda)$ does not depend on h for $\lambda \in h^{-1}K$.*

Proof. The proof is the same as for Lemma 8. One just needs to notice that if $\text{Im } \lambda > 0$ then the function $\tilde{x}_1^{(n-1)/4 - i\lambda/2}$ belongs to $L^2(M)$. □

Notice that Lemma 16 implies that for $\lambda \in h^{-1}V$ the scattering resolvent $R_{\text{scat}}(\lambda)$ coincides with $R_h(\lambda)$. With Proposition 5 and Lemma 16 at our disposal, the proof of Theorem 2 follows exactly the same lines as the proof of Theorem 3 given in Section 3.2. Consequently, we do not repeat it.

5. Real-analytic Fourier–Bros–Iagolnitzer transform

In this section, we detail the tools of real-analytic microlocal analysis that will be used in the proof of Proposition 5 in Section 6. The main ingredient that we need is a real-analytic Fourier–Bros–Iagolnitzer transform as we studied in [Bonthonneau and Jézéquel 2020].

In Section 5.1, we recall the main feature of such an FBI transform, and prove a slight generalization, Proposition 18, of [Bonthonneau and Jézéquel 2020, Proposition 2.10]. In Section 5.2, we give a description, Proposition 20, of the dual of a Hilbert space defined in Section 5.1. This result will be useful to construct the injection of the spaces \mathcal{H}_1 and \mathcal{H}_2 in $\mathcal{D}'(X)$ in the proof of Proposition 5 (see Proposition 21) and to reuse results from [Guedes-Bonthonneau et al. 2024] in Section 5.3, where we study the specificities of certain spaces defined using FBI transform and logarithmic weights (rather than weight of order 1 as in [Bonthonneau and Jézéquel 2020]).

5.1. Generality. Let us recall the tools from [Bonthonneau and Jézéquel 2020] that we need for the proof of Proposition 5. As in Section 2, we let X be a closed real-analytic manifold, and we endow it with a real-analytic metric g_X (which is possible due to [Morrey 1958]). We endow T^*X with an associated metric g_{KN} which is given, using the decomposition into horizontal and vertical direction

$$T_\alpha(T^*X) \simeq T_{\alpha_x}X \oplus T_{\alpha_x}^*X \simeq T_{\alpha_x}X \oplus T_{\alpha_x}X$$

for $\alpha = (\alpha_x, \alpha_\xi) \in T^*M$, by the formula

$$g_{\text{KN},\alpha}((u, v), (u, v)) = g_{X,\alpha_x}(u, u) + \frac{g_{X,\alpha_x}(v, v)}{1 + g_{X,\alpha_x}(\alpha_\xi, \alpha_\xi)}$$

for $(u, v) \in T_{\alpha_x}X \oplus T_{\alpha_x}X$. This metric can be used to give a characterization of Kohn–Nirenberg symbols (see for instance [Bonthonneau and Jézéquel 2020, Remark 2.5]), and we will consequently call it a Kohn–Nirenberg metric. Let \tilde{X} be a complexification of X (endowed with any smooth distance) and $T^*\tilde{X}$ its cotangent bundle. If $r > 0$ is small, we let $(X)_r$ denote the Grauert tube (see for instance [Guillemin and Stenzel 1991; 1992]) of size r for X , that is, the image of

$$\{(x, v) \in TX : g_{X,x}(v, v) \leq r^2\} \tag{19}$$

by the map

$$(x, v) \mapsto \exp_x(iv),$$

which is well-defined on (19) if r is small enough (here we use the holomorphic extension of the exponential map for g_X). We define similarly the Grauert tube $(T^*X)_r \subseteq T^*\tilde{X}$ by using the Kohn–Nirenberg metric on T^*X . Because of the noncompactness of T^*X , it is not clear a priori that $(T^*X)_r$ is well-defined. However, one can reduce the study of the Kohn–Nirenberg metric on T^*X to its study near the zero section and the study of its pullbacks by the dilations $(\alpha_x, \alpha_\xi) \mapsto (\alpha_x, \lambda\alpha_\xi)$ for $\lambda \geq 1$ on a bounded subset of T^*X (for instance the space between the spheres of radii 1 and 2 in each fiber). Since these pullbacks are uniformly analytic and positive definite, we see in particular that $(T^*X)_r$ is well-defined when r is small enough.

Working in the holomorphic extension of real-analytic coordinates on X , we get a holomorphic trivialization $(\tilde{x}, \tilde{\xi}) = (x + iy, \xi + i\eta)$ of $T^*\tilde{X}$ in which T^*X is described by $\{y = \eta = 0\}$. Using the same dilation trick as above, one may then check that, for every compact subset K of the domain of the coordinate patch \tilde{x} , there is $C > 0$ such that, for every $r > 0$ small enough, the image of $(T^*X)_r$ above K in this trivialization is intermediate between

$$T_K^*\tilde{X} \cap \{|y| \leq C^{-1}r, |\eta| \leq C^{-1}(1 + |\xi|)r\}$$

and

$$T_K^*\tilde{X} \cap \{|y| \leq Cr, |\eta| \leq C(1 + |\xi|)r\}.$$

Here, we write $T_K^*\tilde{X}$ for the reciprocal image of K by the canonical projection $T^*\tilde{X} \rightarrow \tilde{X}$.

If m is a real number, $r > 0$ is small and a is a smooth function on $(T^*X)_r$, we say that $a \in S_{\text{KN}}^m((T^*X)_r)$ is a Kohn–Nirenberg symbol of order m on $(T^*X)_r$ if, for every compact subset of the domain of a coordinate patch as above and every $k, k', \ell, \ell' \in \mathbb{N}^n$ there is a constant $C > 0$ such that on the image of $T_K^*\tilde{X} \cap (T^*X)_r$ by the trivialization of $T^*\tilde{X}$ associated to the coordinate patch, we have

$$|\partial_x^k \partial_y^{k'} \partial_\xi^\ell \partial_\eta^{\ell'} a(\tilde{x}, \tilde{\xi})| \leq C(1 + |\xi|)^m (1 + |\xi|)^{-|\ell| - |\ell'|}.$$

We define similarly symbols of logarithmic order by replacing $(1 + |\xi|)^m$ by $\log(2 + |\xi|)$.

Let us fix a real C^∞ metric \tilde{g} on the vector bundle $T^*\tilde{X} \rightarrow X$ (seen as a real vector bundle) and define for $\alpha = (x, \xi) \in T^*\tilde{X}$ the Japanese bracket

$$\langle |\alpha| \rangle = \sqrt{2 + \tilde{g}_x(\xi)}.$$

This is just a more convenient way to denote the size of α than taking the norm of ξ directly, notice in particular that $\langle |\alpha| \rangle$ and $\log \langle |\alpha| \rangle$ are bounded from below. Notice that if $r > 0$ is small enough, then the function $\alpha \mapsto \langle |\alpha| \rangle$ is a Kohn–Nirenberg symbol of order 1 on $(T^*X)_r$, as defined above.

It will also be useful to endow $T^*\tilde{X}$ with a distance adapted to Kohn–Nirenberg symbols. One way to do that is to endow $T\tilde{X}$ with a smooth Hermitian metric, which gives an identification of $T^*\tilde{X}$ with $T\tilde{X}$. Then, one may define a Kohn–Nirenberg metric on $T\tilde{X}$ as above when \tilde{X} , seen as a real manifold, is endowed with a smooth Riemannian metric (e.g., the real part of the Hermitian metric). We let d_{KN} denote the associated distance. Restricting to a compact subset K of \tilde{X} , one may check that $\alpha, \beta \in T_K^*\tilde{X}$ are close for d_{KN} if their position variables are close to each other and, in local coordinates, their momentum variables have the same order of magnitude and the Euclidean distance between them is small with respect to this order of magnitude. This can be proved using a rescaling argument as described above.

For $R \gg 1$, so that $(X)_{1/R}$ is defined, we let $\tilde{E}_R(X)$ denote the space of bounded holomorphic functions on the interior of $(X)_{1/R}$, endowed with the supremum norm. Then, we let $E_R(X)$ denote the closure of $\tilde{E}_{R'}(X)$ in $E_R(X)$ for any $R' < R$ large enough so that $(X)_{1/R'}$ is well-defined. It follows from the Oka–Weil theorem [Forstnerič 2017, Theorems 2.3.1 and 2.5.2] that the space $E_R(X)$ does not depend on the choice of R' . Let $E'_R(X)$ denote the dual of $E_R(X)$, and notice that if $R > R'$ are such that $(X)_{1/R}$ and $(X)_{1/R'}$ are well-defined, then the injection of $E_{R'}(X)$ in $E_R(X)$ has dense image (because it contains $\tilde{E}_{R''}(X)$ for some $R'' < R'$), so that the adjoint of this map defines an injection of $E'_R(X)$ into $E'_{R'}(X)$.

We choose a real-analytic FBI transform $T : \mathcal{D}'(X) \rightarrow C^\infty(T^*X)$ on X , as defined in [Bonthonneau and Jézéquel 2020, Definition 2.1]. This is a transform defined by a real-analytic kernel K_T :

$$Tu(\alpha) = \int_X K_T(\alpha, x)u(x) dx$$

for $u \in \mathcal{D}'(X)$ and $\alpha \in T^*X$. Here, dx denotes the Lebesgue density associated to the Riemannian metric g_X on X . The kernel K_T , and thus T , depends on the implicit semiclassical parameter $h > 0$ introduced in the beginning of Section 2.1. Unless the opposite is explicitly stated, all the estimates below will be uniform in h . The fact that T is a real-analytic FBI transform [loc. cit., Definition 2.1] means that the kernel K_T has a holomorphic extension to $(T^*X)_{r'} \times (X)_{r'}$ for some small $r > 0$, which satisfies the following properties:

- For every $\delta > 0$, there is $r' > 0$ such that if $(\alpha, x) \in (T^*X)_{r'} \times (X)_{r'}$ are such that $d(\alpha_x, x) \geq \delta$ then

$$|K_T(\alpha, x)| \leq (r')^{-1} \exp\left(-r' \frac{\langle |\alpha| \rangle}{h}\right). \tag{20}$$

- There is $\delta > 0$ and $r' > 0$ such that if $(\alpha, x) \in (T^*X)_{r'} \times (X)_{r'}$ are such that $d(\alpha_x, x) \leq \delta$ then

$$|K_T(\alpha, x) - e^{i \frac{\Phi_T(\alpha, x)}{h}} a(\alpha, x)| \leq (r')^{-1} \exp\left(-r' \frac{\langle |\alpha| \rangle}{h}\right). \tag{21}$$

Here, $a(\alpha, x)$ is an analytic symbol defined near the diagonal, elliptic in the class of $h^{-3n/4} \langle |\alpha| \rangle^{n/4}$, meaning that for $r', \delta > 0$ small enough, there is a constant $C > 0$ such that $a(\alpha, x)$ is holomorphic in $\{(\alpha, x) \in (T^*X)_{r'} \times (X)_{r'} : d(\alpha_x, x) < \delta\}$ and satisfies on that set the estimate

$$C^{-1}h^{-\frac{3n}{4}} \langle |\alpha| \rangle^{\frac{n}{4}} \leq |a(\alpha, x)| \leq Ch^{-\frac{3n}{4}} \langle |\alpha| \rangle^{\frac{n}{4}}.$$

The phase $\Phi_T(\alpha, x)$ from (21) is an analytic symbol of order 1 on the set

$$\{(\alpha, x) \in (T^*X)_{r'} \times (X)_{r'} : d(\alpha_x, x) < \delta\}$$

(it is holomorphic and bounded by $C\langle \alpha \rangle$ for some $C > 0$), which satisfies in addition the following properties:

- For $\alpha \in T^*X$, we have $\Phi_T(\alpha, \alpha_x) = 0$.
- For $\alpha \in T^*X$, we have $d_x \Phi_T(\alpha, \alpha_x) = -\alpha_\xi$.
- There is $C > 0$ such that, if $(\alpha, x) \in T^*X \times X$ and $d(\alpha_x, x) < \delta$, then

$$\text{Im}(\Phi_T(\alpha, x)) \geq C^{-1} \langle |\alpha| \rangle d(\alpha_x, x)^2. \tag{22}$$

According to [loc. cit., Theorem 6], such a FBI transform exists. Moreover, if we endow T^*X with the volume associated to the canonical symplectic form, then we may assume that the formal adjoint $S := T^*$ of T is a left inverse for T , i.e., that T is an isometry on its image. Notice that S has a real-analytic kernel K_S that satisfies for α and x real

$$K_S(x, \alpha) = \overline{K_T(\alpha, x)}.$$

In particular, K_S is negligible away from the diagonal, and may be described near the diagonal in a similar fashion as K_T .

Let us fix some small $r > 0$, and let G_0 be a Kohn–Nirenberg symbol of order 1 on $(T^*X)_r$ and set $G = \tau G_0$ for some small $\tau > 0$ (the function G is sometimes called an escape function). We let $\Lambda = \Lambda_G$ be the submanifold of $(T^*X)_r$ defined by

$$\Lambda = e^{H_G^{\omega_I}} T^*X, \tag{23}$$

where $H_G^{\omega_I}$ is the Hamiltonian vector field of G for the symplectic form $\omega_I = \text{Im } \omega$, where ω denotes the canonical complex symplectic form on $T^*\tilde{X}$. By taking τ small, we ensure that Λ is C^∞ close to T^*X (this statement can be made uniform by pulling back Λ to a bounded subset of $T^*\tilde{X}$ using dilation in the fibers as above). Notice that in [Bonthonneau and Jézéquel 2020, Definition 2.2], the symbol G_0 was assumed to be supported in $(T^*X)_{r'}$ for some $r' < r$. The only reason for that was to ensure that the flow of $H_G^{\omega_I}$ is complete, which implies that (23) makes sense. However, taking τ small (which we will always do) is enough to ensure that (23) is well-defined. Moreover, we see that Λ only depends on the values of G on $(T^*X)_{r'}$ for some $r' < r$, so that the assumption on the support of G_0 from [loc. cit., Definition 2.2] may be lifted without harm.

We will say that a smooth function a on Λ is a symbol of order $m \in \mathbb{R}$, and write $a \in S_{\text{KN}}^m(\Lambda)$, if the function $a \circ e^{H_G^{\omega_I}}$ is a symbol of order m , in the standard Kohn–Nirenberg class on T^*X . We define similarly symbols on $\Lambda \times \Lambda$.

On Λ , we can construct a real-valued symbol H of order 1 such that $dH = -\text{Im } \theta$ where θ denotes the canonical complex 1-form on $T^*\tilde{X}$ (see [loc. cit., §2.1.1], in particular equation (2.9) there). Notice also that $\omega_R = \text{Re } \omega$ is a symplectic form on Λ if τ is small enough. We let $d\alpha = \omega_R^n/n!$ denote the associated volume form.

Notice that if $u \in E'_R(X)$ with R large enough, then Tu is well-defined and holomorphic on $(T^*X)_r$ for some small $r > 0$, so that if τ is small enough, Tu is defined on Λ . We can consequently define the FBI transform T_Λ associated to Λ by restriction $T_\Lambda u = (Tu)|_\Lambda$. Notice that since the kernel of S is holomorphic, we also have an operator S_Λ that is a left inverse for T_Λ (see [loc. cit., Lemma 2.7]). We will work with the spaces

$$L_k^2(\Lambda) := L^2(\Lambda, \langle |\alpha| \rangle^{2k} e^{-\frac{2H}{h}} d\alpha) \quad \text{for } k \in \mathbb{R},$$

$$\mathcal{H}_\Lambda^k := \{u \in E'_R(X) : T_\Lambda u \in L_k^2(\Lambda)\}.$$

Here, R needs to be large enough so that $E_R(X)$ is well-defined, and τ small enough depending on R (but the particular choice of R is irrelevant when τ is small). According to [loc. cit., Corollary 2.2], we know that \mathcal{H}_Λ^k is a Hilbert space. We let also $\mathcal{H}_{\Lambda, \text{FBI}}^k \subseteq L_k^2(\Lambda)$ denote the (closed) image of \mathcal{H}_Λ^k by T_Λ . The structure of the projector $\Pi_\Lambda := T_\Lambda S_\Lambda$ on the image of T_Λ has been studied in [loc. cit., §2.2]. The orthogonal projector B_Λ on $\mathcal{H}_{\Lambda, \text{FBI}}^0$ in $L_0^2(\Lambda)$ is studied in [loc. cit., §2.3]. Notice that in order to prove Proposition 5, we will work with a symbol G_0 which is of logarithmic order. As explained in Section 5.3 (see also [Guedes-Bonthonneau et al. 2024]), it implies that \mathcal{H}_Λ^k is in fact a space of distributions. Consequently, we could have worked from the beginning only with distributions (and avoid

the introduction of the space $E'_R(X)$). However, we decided to start from the context of [Bonthonneau and Jézéquel 2020] and then specify to the case of logarithmic weights in Section 5.3. This is because we will need some extensions of the results from [loc. cit.] that are not made easier by assuming that G_0 is of logarithmic order. It is also useful to see the case of logarithmic weights as a particular case of [loc. cit.], as it allows us to use the results from this reference.

Assume that $A(\alpha, \beta)$ is a smooth function on $\Lambda \times \Lambda$ and let A be the associated operator

$$Au(\alpha) = \int_{\Lambda} A(\alpha, \beta)u(\beta) \, d\beta \quad \text{for } \alpha \in \Lambda.$$

The operator A may be defined for instance as an operator from the space of smooth compactly supported functions u on Λ to the space of smooth functions on Λ . In order to understand the action of A on $L^2_0(\Lambda)$, one has to study the reduced kernel of A :

$$A_{\text{red}}(\alpha, \beta) = A(\alpha, \beta)e^{\frac{H(\beta)-H(\alpha)}{h}}.$$

To study the action of A from $L^2_k(\Lambda)$ to $L^2_\ell(\Lambda)$, one can study the kernel $A_{\text{red}}(\alpha, \beta)\langle|\beta|\rangle^{-2k}\langle|\alpha|\rangle^{2\ell}$. We will say that the kernel A is negligible if

$$A_{\text{red}}(\alpha, \beta) = \mathcal{O}_{C^\infty}(h^\infty(\langle|\alpha|\rangle + \langle|\beta|\rangle)^{-\infty}). \tag{24}$$

Here, the C^∞ estimates may be understood by identifying Λ with T^*X using $e^{H_G^{\omega_I}}$, taking a trivialization for T^*X and then asking for all partial derivatives of A_{red} to be $\mathcal{O}(h^\infty(\langle|\alpha|\rangle + \langle|\beta|\rangle)^{-\infty})$. We do not need to ask for symbolic estimates in that case, as it is automatic for something that decays that fast. Notice that an operator whose reduced kernel satisfies (24) is bounded from $L^2_k(\Lambda)$ to $L^2_\ell(\Lambda)$ for every $k, \ell \in \mathbb{R}$, with norm $\mathcal{O}(h^\infty)$. An operator whose reduced kernel satisfy (24) will be called a negligible operator.

Recall the phase $\Phi_{TS}(\alpha, \beta)$ from [Bonthonneau and Jézéquel 2020, §2.2], which is the critical value of $y \mapsto \Phi_T(\alpha, y) + \Phi_S(y, \beta)$. Here, Φ_S is the phase that appear when describing the kernel $K_S(y, \beta)$ of S locally as we do for K_T in (21). That is, $\Phi_S(y, \beta) = -\overline{\Phi_T(\beta, \bar{y})}$. The following fact follows from the analysis in [loc. cit.].

Lemma 17. *Let $\delta > 0$ be small enough. Assume that $\tau > 0$ and $h > 0$ are small enough. Assume that $A(\alpha, \beta)$ is a smooth function on $\Lambda \times \Lambda$ and let A be the associated operator. Let $m \in \mathbb{R}$. Assume that there is a symbol $a \in S^m_{\text{KN}}(\Lambda \times \Lambda)$ supported in $\{(\alpha, \beta) \in \Lambda \times \Lambda, d_{\text{KN}}(\alpha, \beta) < \delta\}$ such that*

$$A_{\text{red}}(\alpha, \beta) = \frac{1}{(2\pi h)^n} e^{\frac{H(\beta)+i\Phi_{TS}(\alpha,\beta)-H(\alpha)}{h}} a(\alpha, \beta) + \mathcal{O}_{C^\infty}(h^\infty(\langle|\alpha|\rangle + \langle|\beta|\rangle)^{-\infty}). \tag{25}$$

Then, A is bounded from $L^2_k(\Lambda)$ to $L^2_{k-m}(\Lambda)$ for every $k \in \mathbb{R}$, and there is a symbol $\sigma \in S^m_{\text{KN}}(\Lambda)$ such that the operators $B_\Lambda A B_\Lambda$ and $B_\Lambda \sigma B_\Lambda$ differ by a negligible operator.

Moreover, σ coincides with $\alpha \mapsto a(\alpha, \alpha)$ up to $\mathcal{O}(h)$ in $S^{m-1}_{\text{KN}}(\Lambda)$.

Indeed, the boundedness statement follows from the proof of [loc. cit., Proposition 2.4]. Our assumption on the kernel of A implies that A belongs to the class of FIO from [loc. cit., Definition 2.5], and thus the proof of [loc. cit., Proposition 2.10] may be rewritten replacing the operator “ $fT_\Lambda P S_\Lambda$ ” by the operator A .

This gives the symbol σ such that $B_\Lambda A B_\Lambda - B_\Lambda \sigma B_\Lambda$ is a negligible operator. The proof gives that σ coincides with $\alpha \mapsto g_0(\alpha) a(\alpha, \alpha)$ for a symbol g_0 of order 0 that does not depend on A . To see that one can take $g_0 = 1$, just notice that the operator $\Pi_\Lambda = T_\Lambda S_\Lambda$ satisfies the hypotheses from [Lemma 17](#) with $\alpha \mapsto a(\alpha, \alpha)$ identically equal to 1 up to $\mathcal{O}(h)$ in $S_{\text{KN}}^{-1}(\Lambda)$, according to [\[Bonthonneau and Jézéquel 2020, Lemma 2.10\]](#), and that $B_\Lambda \Pi_\Lambda B_\Lambda = B_\Lambda B_\Lambda$. Moreover, one may retrieve the leading part of a symbol σ from restriction to the diagonal of the kernel of the operator $B_\Lambda \sigma B_\Lambda$ (the kernel may be computed by the stationary phase method as in [\[loc. cit., Lemma 2.16\]](#)).

We need to extend certain results from [\[loc. cit.\]](#) to a slightly more general context in order to prove [Proposition 5](#). Let P be a semiclassical differential operator of order m with C^∞ coefficients and let p be the principal symbol of P . We make the following assumption:

$$\begin{aligned} & \text{for every } x \in X \text{ either } G_0(y, \xi) = 0 \text{ for every } y \text{ near } x \\ & \text{and } \xi \in T_y^* X, \text{ or } P \text{ has real-analytic coefficients near } x. \end{aligned} \quad (26)$$

Notice that under the assumption [\(26\)](#) the principal symbol p of P may be restricted to Λ provided τ is small enough. Indeed, for every $x \in X$, either p has a holomorphic extension near $T_x^* X$ or Λ coincides with $T^* X$ near $T_x^* X$. We let p_Λ denote this restriction. If P is an operator that satisfies [\(26\)](#), we may define $T_\Lambda P S_\Lambda$ as the operator with kernel

$$T_\Lambda P S_\Lambda(\alpha, \beta) = \int_M K_T(\alpha, y) P_y(K_S(y, \beta)) dy. \quad (27)$$

The reason for which we use this definition is because since P is a priori not an operator with real-analytic coefficients, it is not straightforward to define the action of P on elements of $E'_R(X)$. Notice that the following result allows to define P as an operator from \mathcal{H}_Λ^k to $\mathcal{H}_\Lambda^{k-m}$. When we will specify to the case of logarithmic weights in [Section 5.3](#), the spaces \mathcal{H}_Λ^k 's will be included in $\mathcal{D}'(M)$, and the natural relation $T_\Lambda P u = T_\Lambda P S_\Lambda T_\Lambda u$ will be satisfied; see [Lemma 23](#).

Proposition 18. *Under the assumption [\(26\)](#), if τ is small enough, then the operator $T_\Lambda P S_\Lambda$ is bounded from $L_k^2(\Lambda)$ to $L_{k-m}^2(\Lambda)$. Moreover, if $\ell \in \mathbb{R}$ and $f \in S_{\text{KN}}^\ell(\Lambda)$, there is a symbol $\sigma \in S_{\text{KN}}^{m+\ell}(\Lambda)$ and an operator L with negligible kernel such that*

$$B_\Lambda f T_\Lambda P S_\Lambda B_\Lambda = B_\Lambda \sigma B_\Lambda + L.$$

In addition, σ coincides with $f p_\Lambda$ up to $\mathcal{O}(h \langle |\alpha| \rangle^{m+\ell-1})$.

The proof of [Proposition 18](#) is based on applications of the stationary and nonstationary phase methods with complex phase. We will apply both the C^∞ and the holomorphic versions of these methods. We are not aware of a reference stating the C^∞ version of the nonstationary phase method with complex phase that would cover all the cases we are going to consider (for the stationary phase method, see [\[Melin and Sjöstrand 1974\]](#), and for a standard version of the non stationary phase method with complex phase, see [\[Hörmander 1983, Theorem 7.7.1\]](#)), so that we prove here a statement adapted to our needs. This result and its proof should be no surprise for specialists.

Lemma 19. *Let m, n be integer. Let U, V be open subsets respectively of \mathbb{R}^m and \mathbb{R}^n . Let $\Phi : U \times V \rightarrow \mathbb{C}$ be a C^∞ function. Let K_1 and K_2 be compact subsets respectively of U and V . Assume that for every $(x, y) \in K_1 \times K_2$ we have $\text{Im } \Phi(x, y) \geq 0$ and $d_y \Phi(x, y) \neq 0$. Then, for every $L, N > 0$, there are constants $k \in \mathbb{N}$ and $\lambda_0 > 0$ such that, for every $\lambda \geq \lambda_0$, every C^k function u supported in K_2 and every $x \in U$ such that $d(x, K_1) \leq L \log \lambda / \lambda$, we have*

$$\left| \int_V e^{i\lambda\Phi(x,y)} u(y) dy \right| \leq \lambda^{-N} \|u\|_{C^k}.$$

Proof. Let $x \in U$ be such that $d(x, K_1) \leq L \log \lambda / \lambda$. From our nonstationary assumption, we see that if λ is large enough then $d_y \Phi(x, y) \neq 0$ for every $y \in K_2$. We can consequently introduce the differential operator

$$L_x = -i \sum_{j=1}^n \frac{\partial_{y_j} \overline{\Phi(x, y)}}{|\nabla_y \Phi(x, y)|^2} \partial_{y_j},$$

and notice that $L_x(e^{i\lambda\Phi(x,y)}) = \lambda e^{i\lambda\Phi(x,y)}$. Letting k be a large integer and tL_x denote the formal adjoint of L_x , we find that

$$\int_V e^{i\lambda\Phi(x,y)} u(y) dy = \lambda^{-k} \int_V e^{i\lambda\Phi(x,y)t} L_x^k u(y) dy.$$

Then, we notice that the L^∞ norm of ${}^tL_x^k u$ is controlled by the C^k norm of u . Moreover, since $d(x, K_1) \leq L \log \lambda / \lambda$, we find that for every $y \in K_2$, if λ is large enough, we have $\text{Im } \Phi(x, y) \geq -C_\Phi L \log \lambda / \lambda$ for some constant C_Φ that does not depend on k nor u . Consequently, for λ large, we have

$$\left| \int_V e^{i\lambda\Phi(x,y)} u(y) dy \right| \leq C \lambda^{-k+C_\Phi L} \|u\|_{C^k}.$$

Here the constant C may depend on k and Φ , but not on λ nor u . Taking k large enough, we ensure that $k - C_\Phi L > N$ and the result follows. □

We have now at our disposal all the tools to prove [Proposition 18](#).

Proof of Proposition 18. We want to apply [Lemma 17](#) to the operators $T_\Lambda P S_\Lambda$ and $f T_\Lambda P S_\Lambda$. Let us introduce the open sets

$$U_1 = \{x \in X : G_0(y, \xi) = 0 \text{ for every } y \text{ near } x \text{ and } \xi \in T_y^* X\},$$

$$U_2 = \{x \in X : P \text{ has real-analytic coefficients near } x\}.$$

By assumption $X = U_1 \cup U_2$. We start by proving that for every $\delta > 0$, provided τ is small enough, we have

$$T_\Lambda P S_\Lambda(\alpha, \beta) e^{\frac{H(\beta) - H(\alpha)}{h}} = \mathcal{O}(h^\infty (\|\alpha\| + \|\beta\|)^{-\infty}) \tag{28}$$

whenever $\alpha, \beta \in \Lambda$ are such that $d_{\text{KN}}(\alpha, \beta) \geq \delta$. Let us write $\alpha = e^{H_G^{\omega_1}}(x, \xi)$ and $\beta = e^{H_G^{\omega_1}}(y, \eta)$ where (x, ξ) and (y, η) are in T^*X . Assume first that x and y are at distance larger than δ/L for some large constant $L \gg 1$. We can then write

$$T_\Lambda P S_\Lambda(\alpha, \beta) = \left(\int_{D(x, \delta/10L)} + \int_{D(y, \delta/10L)} + \int_{X \setminus (D(x, \delta/10L) \cup D(y, \delta/10L))} \right) K_T(\alpha, z) P_z(K_S(z, \beta)) dz. \tag{29}$$

We write $D(w, r)$ for the ball of center w and radius r in X . Notice that, provided τ is small enough, the third integral in (29) is $\mathcal{O}(\exp(-(\langle|\alpha\rangle + \langle|\beta\rangle)/Ch))$ since the kernel K_T and K_S are negligible away from the diagonal (20). Since $e^{(H(\beta)-H(\alpha))/h}$ is $\mathcal{O}(\exp(C\tau(\langle|\alpha\rangle + \langle|\beta\rangle)))$, we see that for τ small enough we have

$$e^{\frac{H(\beta)-H(\alpha)}{h}} \int_{X \setminus (D(x, \delta/10L) \cup D(y, \delta/10L))} K_T(\alpha, z) P_z(K_S(z, \beta)) dz = \mathcal{O}\left(\exp\left(-\frac{\langle|\alpha\rangle + \langle|\beta\rangle}{Ch}\right)\right) = \mathcal{O}(h^\infty(\langle|\alpha\rangle + \langle|\beta\rangle)^{-\infty}),$$

and we only need to care about the two other terms.

Let us deal with the first term in (29). Up to a negligible term, it is given by

$$\int_{D(x, \delta/10L)} e^{i\frac{\Phi_T(\alpha, z)}{h}} a(\alpha, z) P_z(K_S(z, \beta)) dz. \tag{30}$$

By taking L large enough, we have either $D(x, \delta/10L) \subseteq U_1$ or $D(x, \delta/10L) \subseteq U_2$.

Let us begin with the case of $D(x, \delta/10L) \subseteq U_1$. In that case, the differential operator P has a priori only C^∞ coefficients on $D(x, \delta/10L)$ so that we find that $P_z(K_S(z, \beta))$ is $\mathcal{O}(\exp(-\langle|\beta\rangle)/Ch))$ in C^∞ . Notice also that $d_y \Phi_T(\alpha, \alpha_x) = -\alpha_\xi$ and that the imaginary part of $\Phi_T(\alpha, z)$ is nonnegative when $z \in D(x, \delta/10L)$. Hence, provided L is large enough, we can use the C^∞ nonstationary phase method (apply Lemma 19 with a rescaling argument) to find that (30) is

$$\mathcal{O}(h^\infty \langle|\alpha\rangle^{-\infty} \exp(-\langle|\beta\rangle)/Ch).$$

Here, the integrand is not supported away from the boundary of the domain of integration, but since the imaginary part of the phase is larger than $C^{-1}\langle|\alpha\rangle/h$ near the boundary of the domain of integration, we may just introduce a bump function to fix that. The same trick allows to remove the dependence on x of the domain of integration. Using that $x \in U_1$, we find that $\alpha = (x, \xi)$ and that $H(\alpha) = 0$ (see [Bonthonneau and Jézéquel 2020, (2.9)]), so that

$$e^{\frac{H(\beta)-H(\alpha)}{h}} = e^{\frac{H(\beta)}{h}} = \mathcal{O}\left(\exp\left(C\tau \frac{\langle|\beta\rangle}{h}\right)\right).$$

Hence, for τ small enough, we find that

$$e^{\frac{H(\beta)-H(\alpha)}{h}} \int_{D(x, \delta/10L)} e^{i\frac{\Phi_T(\alpha, z)}{h}} a(\alpha, z) P_z(K_S(z, \beta)) dz = \mathcal{O}(h^\infty(\langle|\alpha\rangle + \langle|\beta\rangle)^{-\infty}). \tag{31}$$

When $D(x, \delta/10L) \subseteq U_2$, the coefficients of P are analytic, and $P_z(K_S(z, \beta))$ is $\mathcal{O}(\exp(-\langle|\beta\rangle)/Ch))$ as a real-analytic function. Hence, provided L is large enough, we can use the holomorphic nonstationary phase method (see for instance [loc. cit., Proposition 1.1], and use a rescaling argument) as in the proof of [loc. cit., Lemma 2.9] to see that (30) is $\mathcal{O}(\exp(-(\langle|\alpha\rangle + \langle|\beta\rangle)/Ch))$, provided τ is small enough. Hence, if τ is small enough, this is enough to beat the potential growth of the factor $e^{(H(\beta)-H(\alpha))/h}$, so that we also have (31) in that case.

We deal similarly with the second term in (29), distinguishing the cases $D(y, \delta/10L) \subseteq U_1$ and $D(y, \delta/10L) \subseteq U_2$.

Let us now prove (28) when the distance between x and y is less than δ/L (and consequently ξ and η are away from each other in a trivialization of T^*X). As above, we can discard the z 's that are away from x (and thus from y) and write up to a negligible term the kernel of $T_\Lambda P S_\Lambda$ as (the error term coming from the approximation (21) is dealt with by an application of the nonstationary phase method as in the previous case)

$$\int_{D(x, 10\delta/L)} e^{i \frac{\Phi_T(\alpha, z) + \Phi_S(z, \beta)}{h}} a(\alpha, z) \tilde{b}(z, \beta) dz, \tag{32}$$

where the symbol \tilde{b} is defined by

$$\tilde{b}(z, \beta) = e^{-i \frac{\Phi_S(z, \beta)}{h}} P_z(e^{i \frac{\Phi_S(z, \beta)}{h}} b(z, \beta)).$$

Notice that the phase in (32) is holomorphic and nonstationary. Indeed, working in coordinates and assuming that L is large enough, we find that, for some $C > 0$ and every $z \in D(x, 10\delta/L)$,

$$\begin{aligned} |\nabla_z(\Phi_T(\alpha, z) + \Phi_S(z, \beta))| &= |\beta_\xi - \alpha_\xi| + \mathcal{O}\left(\frac{\max(\langle|\alpha|\rangle, \langle|\beta|\rangle)}{L}\right) \\ &\geq C^{-1} \max(\langle|\alpha|\rangle, \langle|\beta|\rangle). \end{aligned}$$

Moreover, provided τ is small enough, the imaginary part of the phase is larger than $C^{-1} \max(\langle|\alpha|\rangle, \langle|\beta|\rangle)$ when z is on the boundary of $D(x, 10\delta/L)$ (because z is away from α_x and β_x), and is always nonnegative when $D(x, 10\delta/L) \subseteq U_1$. We can apply the C^∞ nonstationary phase method when $D(x, 10\delta/L) \subseteq U_1$ and the holomorphic nonstationary phase method when $D(x, 10\delta/L) \subseteq U_2$ (for this second case, see the similar computation in the proof of [Bonthonneau and Jézéquel 2020, Lemma 2.9]). Indeed, in the latter case \tilde{b} is holomorphic in z , while in the first case it is only C^∞ . In the first case, we get that (32) is $\mathcal{O}(h^\infty(\langle|\alpha|\rangle + \langle|\beta|\rangle)^{-\infty})$ and in the second case that it is $\mathcal{O}(\exp(-(\langle|\alpha|\rangle + \langle|\beta|\rangle)/Ch))$. Noticing that in the first case $H(\alpha) = H(\beta) = 0$, we find that (28) holds.

Notice that differentiating the kernel of K_T or of K_S (in a local trivialization of T^*X) amount to replace the symbols a and b by symbols of higher orders (in terms of α, β and h). Thus, all the estimates that we established when α and β are away from each other actually hold in C^∞ .

We must now understand what happens when α and β are close to each other. We write as above $\alpha = e^{H_G^{\omega l}}(x, \xi)$ and $\beta = e^{H_G^{\omega l}}(y, \eta)$ where (x, ξ) and (y, η) are in T^*X . Then, up to negligible terms, the kernel of $T_\Lambda P S_\Lambda$ at (α, β) is given as above, for some small $\delta > 0$, by

$$\int_{D(x, \delta)} e^{i \frac{\Phi_T(\alpha, z) + \Phi_S(z, \beta)}{h}} a(\alpha, z) \tilde{b}(z, \beta) dz.$$

As above, the error coming from the approximation (21) is dealt with by an application of the nonstationary phase method. The asymptotic of this integral when $\langle|\alpha|\rangle/h$ tends to $+\infty$ is given by the stationary phase method. Indeed, when $\alpha = \beta$, the rescaled phase $y \mapsto (\Phi_T(\alpha, y) + \Phi_S(y, \beta))/\langle|\alpha|\rangle$ has a uniformly nondegenerate critical point at $y = \alpha_x = \beta_x$, as a consequence of (22). Moreover, when $D(x, \delta) \subseteq U_1$, the imaginary part of this phase is nonnegative on $D(x, \delta)$, provided the distance between α_x and β_x is way smaller than δ . When $D(x, \delta) \subseteq U_2$, we may ensure that the imaginary part of the (rescaled) phase is uniformly positive on the boundary of $D(x, \delta)$ by taking τ small enough. As above, we apply the

stationary phase method in the C^∞ category (see [Melin and Sjöstrand 1974, §2]) when $D(x, \delta) \subseteq U_1$ and in the C^ω category when $D(x, \delta) \subseteq U_2$ (see [Sjöstrand 1982, §2] for the general method and the proof of [Bonthonneau and Jézéquel 2020, Lemma 2.10] in the case $s = 1$, page 111, for the details of the computation in our particular setting). In both cases, we can use the fact that the imaginary part of the phase is positive on the boundary of the domain of integration to remove the dependence of this domain on x . In the first case we get an expansion with an error term of the form $\mathcal{O}(h^\infty \langle |\alpha| \rangle^{-\infty})$ and in the second case of the form $\mathcal{O}(\exp(-\langle |\alpha| \rangle / Ch))$. Since in the first case we have $H(\alpha) = H(\beta) = 0$, we see that in both cases we get the desired expansion (25) for the reduced kernel of $T_\Lambda P S_\Lambda$, with an error term of the required size.

We can then apply Lemma 17 to end the proof. Indeed, we just saw that the kernel of $fT_\Lambda P S_\Lambda$ is of the form (25). Moreover, it follows from the application of the stationary phase method that, up to $\mathcal{O}(h)$ in $S_{\text{KN}}^{m-1}(\Lambda)$, the symbol $\alpha \mapsto a(\alpha, \alpha)$ coincides with $fp_\Lambda g_0$, where g_0 is a symbol of order 0 that does not depend on P . Thus, the operator $fT_\Lambda P S_\Lambda - fp_\Lambda \Pi_\Lambda$ is also of the form (25) but with an a such that $a \mapsto a(\alpha, \alpha)$ is $\mathcal{O}(h)$ in $S_{\text{KN}}^{m+\ell-1}(\Lambda)$. Consequently, there is a symbol $\tilde{\sigma} \in hS_{\text{KN}}^{m+\ell-1}(\Lambda)$ such that $B_\Lambda(fT_\Lambda P S_\Lambda - fp_\Lambda \Pi_\Lambda)B_\Lambda - B_\Lambda \tilde{\sigma} B_\Lambda = B_\Lambda fT_\Lambda P S_\Lambda B_\Lambda - B_\Lambda(fp_\Lambda + \tilde{\sigma})B_\Lambda$ is a negligible operator. We get the announced result with $\sigma = fp_\Lambda + \tilde{\sigma}$. \square

5.2. Duality statement. In [Bonthonneau and Jézéquel 2020, Lemma 2.24], an identification between \mathcal{H}_Λ^{-k} and the dual of \mathcal{H}_Λ^k is given. However, the pairing used to define this identification is not the L^2 pairing. We explain here how to describe the dual of \mathcal{H}_Λ^k using the L^2 pairing. This will allow us in particular to reuse results from [Guedes-Bonthonneau et al. 2024] in Section 5.3.

Let us first recall that there is an antiholomorphic involution $\alpha \mapsto \bar{\alpha}$ on $(T^*X)_r$ such that

$$\{\alpha \in (T^*X)_r : \alpha = \bar{\alpha}\} = T^*X;$$

see [Guillemin and Stenzel 1991]. Let G be a symbol of order 1 on $(T^*X)_r$ as above (of the form $G = \tau G_0$ with τ small) and Λ be defined by (23). Let us introduce a new symbol $G^*(\alpha) = -G(\bar{\alpha})$, and notice that the Lagrangian associated to G^* by (23) is $\bar{\Lambda}$, that is, the image of Λ by the involution $\alpha \mapsto \bar{\alpha}$. Notice also that changing G to G^* , we have to replace H by the function H^* on $\bar{\Lambda}$ given by $H^*(\alpha) = -H(\bar{\alpha})$.

Consequently, if $u \in \mathcal{H}_\Lambda^k$ and $v \in \mathcal{H}_{\bar{\Lambda}}^{-k}$, we may define the pairing

$$\langle u, v \rangle = \int_\Lambda T_\Lambda u(\alpha) \overline{T_\Lambda v(\bar{\alpha})} \, d\alpha, \tag{33}$$

for which we can prove:

Proposition 20. *Let $R \gg 1$. Assume that τ is small enough. The pairing (33) induces an identification between $\mathcal{H}_{\bar{\Lambda}}^{-k}$ and the dual of \mathcal{H}_Λ^k . Moreover, if u or v belongs to $E_R(X)$ then (33) is just the natural (sesquilinear) pairing between elements of $E_R(X)$ and $E'_R(X)$.*

Proof. Assume that u is in $E_R(X)$ and that $v \in \mathcal{H}_{\bar{\Lambda}}^{-k}$. Since T is an isometry on its image, we know that

$$\int_X u \bar{v} \, dx = \int_{T^*X} Tu \overline{Tv} \, d\alpha. \tag{34}$$

Notice that the function $\alpha \mapsto Tu(\alpha)\overline{Tv(\bar{\alpha})}$ is holomorphic on $(T^*X)_r$. Moreover, from [Bonthonneau and Jézéquel 2020, Lemmas 2.4 and 2.5, Corollary 2.2], we see that, provided τ is small enough, there is $r > 0$ such that $Tu(\alpha)\overline{Tv(\bar{\alpha})}$ decays exponentially fast in $(T^*X)_r$. This allows us to shift contour in (34) to find that $\int_X u\bar{v} dx$ coincides with (33), provided τ is small enough. By symmetry, we have the same equality when v is assumed to belong to $E_R(X)$.

Consequently, the (antilinear) map from \mathcal{H}_Λ^{-k} to the dual of \mathcal{H}_Λ^k induced by the pairing (33) is injective. Let us prove that it is surjective. Let l be a continuous linear form on \mathcal{H}_Λ^k . It follows from [loc. cit., Proposition 2.4] that S_Λ is bounded from $L_k^2(\Lambda)$ to \mathcal{H}_Λ^k , and we can thus define a linear form \tilde{l} on $L_k^2(\Lambda)$ by the formula $\tilde{l}(w) = l(S_\Lambda w)$. Notice that if $u \in \mathcal{H}_\Lambda^k$ then $l(u) = \tilde{l}(T_\Lambda u)$. Let then h_1 be the element of $L_k^2(\Lambda)$ such that

$$\tilde{l}(w) = \int_\Lambda w(\alpha)\overline{h_1(\alpha)}\langle|\alpha|\rangle^{2k} e^{-\frac{2H(\alpha)}{h}} d\alpha$$

for every $w \in L_k^2(\Lambda)$. Let us define the function h_2 on $\bar{\Lambda}$ by

$$h_2(\alpha) = h_1(\bar{\alpha})\langle|\bar{\alpha}|\rangle^{2k} e^{-\frac{2H(\bar{\alpha})}{h}},$$

and notice that h_2 belongs to $L_{-k}^2(\bar{\Lambda})$, so that $v := S_{\bar{\Lambda}}h_2$ belongs to $\mathcal{H}_{\bar{\Lambda}}^{-k}$. Let $u \in E_R(X)$, then with the pairing above, we have

$$\langle u, v \rangle = \int_\Lambda T_\Lambda u(\alpha)\overline{\Pi_{\bar{\Lambda}}h_2(\bar{\alpha})} d\alpha.$$

Notice that the kernel of the operators Π_Λ and $\Pi_{\bar{\Lambda}}$ are obtained by restricting respectively to $\Lambda \times \Lambda$ and $\bar{\Lambda} \times \bar{\Lambda}$ the holomorphic kernel of the operator $\Pi = TS$. We write $\Pi(\alpha, \beta)$ for this kernel. Since S is the adjoint of T , we find by analytic continuation that $\Pi(\bar{\alpha}, \bar{\beta}) = \Pi(\beta, \alpha)$. It follows then from Fubini’s theorem that

$$\begin{aligned} \langle u, v \rangle &= \int_\Lambda \Pi_\Lambda T_\Lambda u(\alpha)\overline{h_2(\bar{\alpha})} d\alpha \\ &= \int_\Lambda T_\Lambda u(\alpha)\overline{h_1(\alpha)}\langle|\alpha|\rangle^{2k} e^{-\frac{2H(\alpha)}{h}} d\alpha = l(u). \end{aligned}$$

The equality on the first line can be proved first by replacing h_2 by a rapidly decaying function and then using an approximation argument. It follows from [loc. cit., Corollary 2.3] and the Oka–Weil theorem that $E_R(X)$ is dense in \mathcal{H}_Λ^k and the result follows. □

5.3. Particularity of logarithmic weights. When applying the FBI transform techniques that we describe here in Section 6, the weight G_0 will be of logarithmic order. This is a strategy that we already applied in [Guedes-Bonthonneau et al. 2024]. It amounts to doing C^∞ microlocal analysis with respect to the large parameter $\langle|\alpha|\rangle$ but real-analytic microlocal analysis with respect to the small parameter h .

Using a logarithmic weight allows us to construct spaces that are intermediate between $C^\infty(X)$ and $\mathcal{D}'(X)$.

Proposition 21. *Assume that G_0 has logarithmic order. Assume that τ and h are small enough. Then, for every $k \in \mathbb{R}$, there are continuous injections $C^\infty(X) \subseteq \mathcal{H}_\Lambda^k \subseteq \mathcal{D}'(X)$. Moreover, these injections are natural in the following sense: the diagram*

$$\begin{array}{ccccc}
 C^\infty(X) & \longrightarrow & \mathcal{H}_\Lambda^k(X) & \longrightarrow & E'_R(X) \\
 & \searrow & \downarrow & \nearrow & \\
 & & \mathcal{D}'(X) & &
 \end{array}$$

is commutative, with R as in the definition of \mathcal{H}_Λ^k . The arrows that are not given by the proposition are the standard injections.

Proof. It follows from Lemma 19, using for instance [Bonthonneau and Jézéquel 2020, (2.9)] to bound H , that $C^\infty(X)$ is contained in \mathcal{H}_Λ^k , where we identify an element of $C^\infty(X)$ with an element of $E'_R(X)$ using the L^2 pairing (see also [Guedes-Bonthonneau et al. 2024, Lemma 4.10]). The proof of this result actually proves that the injection is continuous (even if the estimates are not uniform in h). Notice that $C^\infty(X)$ is dense in \mathcal{H}_Λ^k as a consequence of [Bonthonneau and Jézéquel 2020, Corollary 2.3]. Replacing G by G^* , we find that $C^\infty(X)$ is also a dense subset of \mathcal{H}_Λ^{-k} , with continuous injection. Consequently, the pairing (33) induces a continuous injection of \mathcal{H}_Λ^k into $\mathcal{D}'(X)$ according to Proposition 20. Since the pairing (33) coincides with the L^2 pairing when u or v is in $E_R(X)$, we see that the diagram above is indeed commutative. □

Remark 22. It follows from Propositions 20 and 21 that if $u \in \mathcal{H}_\Lambda^k$ and $v \in \mathcal{H}_\Lambda^{-k}$ are such that u or v is in $C^\infty(X)$, then the pairing (33) coincides with the natural pairing between a smooth function and a distribution.

When G_0 is of logarithmic order, we may identify the \mathcal{H}_Λ^k 's with spaces of distributions, and consequently it makes sense to let a differential operator P with C^∞ coefficients act on the elements of the \mathcal{H}_Λ^k 's. In the following lemma, we see that under the assumption (26) we can relate the action of P on these spaces with the action of the operator $T_\Lambda P S_\Lambda$ that we studied in Proposition 18.

Lemma 23. *Assume that G_0 has logarithmic order. Let P be a semiclassical operator of order $m \in \mathbb{N}$ that satisfy (26). Assume that τ is small enough. Then, for every $k \in \mathbb{R}$, the operator P is bounded from \mathcal{H}_Λ^k to $\mathcal{H}_\Lambda^{k-m}$ and for every $u \in \mathcal{H}_\Lambda^k$ we have*

$$T_\Lambda P u = (T_\Lambda P S_\Lambda) T_\Lambda u,$$

where we recall that $T_\Lambda P S_\Lambda$ is the operator with kernel (27).

Proof. For $\alpha \in \Lambda$, we have by definition

$$T_\Lambda P u(\alpha) = \int_M P u(y) K_T(\alpha, y) dy = \int_M u(y) {}^t P_y(K_T(\alpha, y)) dy, \tag{35}$$

where ${}^t P$ denotes the adjoint of P for the bilinear (rather than sesquilinear) L^2 pairing on M . Notice that for $\alpha \in \Lambda$, the function $h_\alpha : y \mapsto {}^t P_y(K_T(\alpha, y))$ is C^∞ . Consequently, one may use the C^∞ nonstationary

phase method, [Lemma 19](#), to find that ${}^tSh_\alpha(\beta)$ decays faster than the inverse of any polynomial when β becomes large while its imaginary part remains bounded (from the Kohn–Nirenberg point of view) by $L \log(|\beta|)/|\beta|$ (for any large constant L). Notice however that this estimate is not uniform in h (we apply [Lemma 19](#) with h fixed and $\langle|\beta|\rangle$, rather than $\langle|\beta|\rangle/h$, as a large parameter). Consequently, we can shift contour in the integral equality ${}^tT{}^tSh_\alpha = {}^t(ST)h_\alpha = h_\alpha$ to find

$$\begin{aligned} h_\alpha(x) &= \int_\Lambda K_T(\beta, x) \left(\int_M K_S(y, \beta) h_\alpha(y) dy \right) d\beta \\ &= \int_\Lambda K_T(\beta, x) T_\Lambda P S_\Lambda(\alpha, \beta) d\beta. \end{aligned}$$

Using the fast decay of ${}^tSh_\alpha$, we see that this integral actually converges in $C^\infty(X)$, and plugging this equality into [\(35\)](#), we get $T_\Lambda Pu = (T_\Lambda P S_\Lambda)T_\Lambda u$. It follows from [Proposition 18](#) that $T_\Lambda Pu \in L^2_{k-m}(\Lambda)$, that is, $Pu \in \mathcal{H}^{k-m}_\Lambda(\Lambda)$. □

The following result will be used in the demonstration of [Proposition 5](#) to prove that the elements of the spaces \mathcal{H}_1 and \mathcal{H}_2 are bounded near ∂Y .

Proposition 24. *Let K be a compact subset of X . Assume that G_0 has logarithmic order and that there is $C > 0$ such that if $\alpha \in T^*_K X$ is large enough then*

$$G_0(\alpha) \leq -C^{-1} \log\langle|\alpha|\rangle.$$

Assume that τ is small enough. Then, for every $k \in \mathbb{R}$, if h is small enough then the elements of \mathcal{H}^k_Λ are continuous on a neighborhood of K .

Proof. Let $N > n$. It follows from [[Guedes-Bonthonneau et al. 2024](#), Lemmas 4.2 and 4.9] that, for h small enough, there is a neighborhood U of K such that if v is in $H^{-N}(X)$ and supported in U then v belongs to \mathcal{H}^{-k}_Λ and its norm in this space is less than $C\|v\|_{H^{-N}}$, where the constant C may depend on h but not on v .

Let $u \in \mathcal{H}^k_\Lambda$. If χ is a C^∞ function supported in the intersection of U with a coordinates patch, then we see that in these coordinates the Fourier transform of χu decays faster than $\langle\xi\rangle^{-N}$. Indeed, the H^{-N} norm of the functions given in coordinates by $\chi(x)e^{ix\xi}$ decays like $\langle\xi\rangle^{-N}$ when ξ tends to $+\infty$. Thus, the same is true for the norm of these functions in \mathcal{H}^{-k}_Λ . It follows then from [Remark 22](#) that $\widehat{\chi u}(\xi)$, which is the L^2 pairing of u with one of these functions, decays like $\langle\xi\rangle^{-N}$ when ξ tends to $+\infty$. Consequently, the distribution χu is a continuous function, and the result follows by a partition of unity argument. □

6. General construction (proof of [Proposition 5](#))

The aim of this section is to prove [Proposition 5](#). We will use the notation that we introduced in [Section 2.1](#).

In [Section 6.1](#), we fix the value of certain parameters that play an important role in the proof of [Proposition 5](#) and define the modification $P_h(\omega)$ of $\mathcal{P}_h(\omega)$. In [Section 6.2](#), we define the spaces that will be \mathcal{H}_1 and \mathcal{H}_2 in [Proposition 5](#), and explain how the action of $P_h(\omega)$ on these spaces is related to the values of a certain symbol ([Proposition 25](#)). In [Section 6.3](#), we prove ellipticity estimates on this symbol (Lemmas [26](#) and [28](#)). In [Section 6.4](#), we use these estimates to study the functional analytic properties

of $P_h(\omega)$ acting on the spaces defined in Section 6.3: we prove that $P_h(\omega)$ is Fredholm by proving that it is invertible after perturbation by a compact operator (Lemmas 30 and 32 and Proposition 36), and that $P_h(i\nu)$ is invertible for some $\nu > 0$ (Lemmas 31 and 33 and Proposition 36). In Section 6.5, we prove the crucial point (vi) from Proposition 5 (from which our upper bounds on resonances, Theorems 2 and 3, follow). This is done by evaluating the trace class norm of the compact perturbation that we use to make $P_h(\omega)$ invertible (Lemmas 37 and 38). Finally, in Section 6.6, we put all these information together in order to get a full proof of Proposition 5.

Notice that, in most of this section, we are not working directly with the operator $P_h(\omega)$, but rather with an operator $\tilde{P}_h(\omega)$, defined in Section 6.1, which is conjugated to $P_h(\omega)$, but simpler to apprehend.

6.1. Choice of parameters and modification of the operator. We use the notation from Section 2.1. Up to making ϵ smaller, we may assume that $w'(x_1) > \epsilon$ for every $x_1 \in]-\epsilon, \epsilon[$ and $p_0(x) < -\epsilon$ for every $x \in U \cup Y$ (this second point is a consequence of assumption (e)). We may also assume that x_1 extends to a smooth function on the whole X (analytic on U) such that $U = \{-\epsilon < x_1 < \epsilon\}$, $Y = \{x_1 > 0\}$ and $X \setminus \bar{Y} = \{x_1 < 0\}$.

Let us introduce on $T^*U \simeq T^*(]-\epsilon, \epsilon[)_{(x_1, \xi_1)} \times T^*\partial Y_{(x', \xi')}$ the symbol of logarithmic order

$$G_1(x_1, x', \xi_1, \xi') = \log(2 + \xi_1^2 + |\xi'|^2)$$

and denote by H_{G_1} the Hamiltonian flow of G_1 for the canonical symplectic form on T^*U . Here, the quantity $|\xi'|^2$ is computed using any smooth Riemannian metric on ∂Y , e.g., the restriction of g_X . Let us compute $H_{G_1}p_2$ where we recall that p_2 is the principal symbol of the order 2 differential operator P_2 from (7). Using local coordinates on ∂Y , we find that

$$\begin{aligned} H_{G_1}p_2(x_1, x', \xi_1, \xi') &= \frac{2\xi_1}{2 + \xi_1^2 + |\xi'|^2} w'(x_1)\xi_1^2 + \frac{2\xi_1}{2 + \xi_1^2 + |\xi'|^2} \frac{\partial q_1}{\partial x_1}(x_1, x', \xi') \\ &\quad + \frac{\nabla_{\xi'}(|\xi'|^2)}{2 + \xi_1^2 + |\xi'|^2} \cdot \nabla_{x'}q_1(x_1, x', \xi') - \frac{\nabla_{x'}(|\xi'|^2)}{2 + \xi_1^2 + |\xi'|^2} \cdot \nabla_{\xi'}q_1(x_1, x', \xi'). \end{aligned}$$

Since $w'(x_1) > \epsilon$, the first term on the right-hand side is elliptic of order 1 whenever ξ_1 is larger than a fixed proportion of $|\xi'|$. Moreover, this term has the same sign as ξ_1 . The other terms are also of order 1, and they can be made arbitrarily small by assuming that ξ_1 is much larger than ξ' . Hence, there is some small $\epsilon_1 \in]0, \epsilon[$ such that if $(x_1, x', \xi_1, \xi') \in T^*U$ and $|\xi_1| \geq \epsilon_1^{-1}(1 + |\xi'|)$ we have

$$\frac{H_{G_1}p_2(x_1, x', \xi_1, \xi')}{\xi_1} \geq C^{-1} \tag{36}$$

for some constant $C > 0$.

Let then C_0 be a bound for the derivative of w on $]-\epsilon, \epsilon[$. We choose $\epsilon_0 \in]0, \epsilon[$ small enough so that if $(x_1, x', \xi_1, \xi') \in T^*U$ and $|\xi_1| \leq 2\epsilon_0^{-1}(1 + |\xi'|)$ we have, with $\xi = (\xi_1, \xi')$,

$$-C_0\epsilon_0\xi_1^2 + |\xi|^2 + 1 \geq C^{-1}(1 + |\xi|^2), \tag{37}$$

$$-C_0\epsilon_0\xi_1^2 + q_1(x_1, x', \xi') + 1 \geq C^{-1}(1 + |\xi|^2). \tag{38}$$

Here, $|\xi|^2$ is defined using the metric g_X on X , and we used the ellipticity condition on q_1 (assumption (b)).

Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that $\chi(t) = 0$ for $t \leq -\epsilon_0$ and $\chi(t) = 1$ for $t \geq -5\epsilon_0/6$. Let ψ be a real-analytic function from \mathbb{R} to \mathbb{R} such that $t\psi'(t) \leq 0$ for $t \in \mathbb{R}$ and $t\psi'(t) < 0$ for $t \neq 0$. One can take for instance $\psi(t) = -t^2/2$.

Let Q be a semiclassical differential operator of order 2 with principal symbol q . We assume that Q has the following properties:

- The coefficients of Q are supported in $\{x_1 < -\epsilon_0/2\}$.
- The principal symbol of Q is

$$q(x, \xi) = \chi_1(x_1)(1 + |\xi|^2),$$

where $\chi_1 : \mathbb{R} \rightarrow [0, 1]$ is a smooth function supported in $]-\infty, -\epsilon_0/2]$ and that takes value 1 on $]-\infty, -2\epsilon_0/3]$. For instance, one can take $Q = \chi_1(I - h^2 \Delta_{g_X})$.

The modification $P_h(\omega)$ of the operator $\mathcal{P}_h(\omega)$ for which [Proposition 5](#) will be established is

$$P_h(\omega) = \chi(x_1)\mathcal{P}_h(\omega) + e^{-\frac{\psi(x_1)}{h}} Q e^{\frac{\psi(x_1)}{h}}, \tag{39}$$

but we will rather study the conjugated operator

$$\tilde{P}_h(\omega) = e^{\frac{\psi(x_1)}{h}} \chi(x_1)\mathcal{P}_h(\omega)e^{-\frac{\psi(x_1)}{h}} + Q.$$

For $(x, \xi) = (x_1, x', \xi_1, \xi') \in T^*U \simeq T^*(]-\epsilon, \epsilon[\times \partial Y)$, the principal symbol of $\tilde{p}(x, \xi; \omega)$ of $\tilde{P}_h(\omega)$ is given by

$$\begin{aligned} \tilde{p}(x, \xi; \omega) = & \chi(x_1)(w(x_1)\xi_1^2 + q_1(x_1, x', \xi') + 2iw(x_1)\psi'(x_1)\xi_1 + \omega p_1(x_1, \xi_1) \\ & + i\omega p_1(x_1, \psi'(x_1)) + \omega^2 p_0(x) - \psi'(x_1)^2 w(x_1)) + \chi_1(x_1)(1 + |\xi|^2). \end{aligned} \tag{40}$$

Finally, let ϕ be a C^∞ function from \mathbb{R} to $[0, 1]$, supported in $]-\epsilon_0/3, \epsilon_0[$, such that $\phi(t) = 1$ for $t \in [-\epsilon_0/6, 2\epsilon_0/3]$ and $t\phi'(t) \leq 0$ for every $t \in \mathbb{R}$.

Our choices of parameters are summed up in [Figure 1](#), where the black line represent the x_1 -axis. The colored zones in this drawing correspond to places where we will use different mechanisms to prove the Fredholm property for $\tilde{P}_h(\omega)$. In the purple zone (which is compactly contained in Y), we will use the ellipticity of $\tilde{P}_h(\omega)$, which follows from our assumption [\(d\)](#) in [Section 2.1](#). In the green zone (which is away from \bar{Y}), the operator $\tilde{P}_h(\omega)$ is also elliptic, but this is just because Q is. Finally, the most interesting part is the blue zone, where two phenomena occur: in some places $\tilde{P}_h(\omega)$ is elliptic and in other places we need to use propagation and radial estimates to get the Fredholm property. See [Section 6.3](#) for the details on how [Figure 1](#) can be turned into actual estimates.

6.2. Definition of the spaces. We define the symbol G_0 on T^*X by

$$G_0(x, \xi) = -\phi(x_1)G_1(x, \xi) \quad \text{for } (x, \xi) \in T^*X.$$

Then, for some small $r > 0$, we extend G_0 to $(T^*X)_r$ as a symbol of logarithmic order. The particular features of the extension are irrelevant as soon as we have symbolic estimates, and that G_0 is identically equal to 0 away from a small neighborhood of the support of ϕ , so that all derivatives of G_0 vanish at any

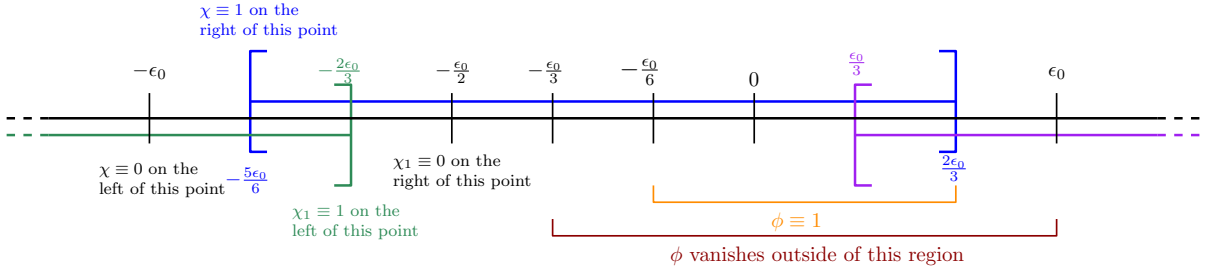


Figure 1. Some relevant places near ∂Y .

point of T^*X such that $x_1 \leq -\epsilon_0/3$ or $x_1 \geq \epsilon_0$ (even derivatives in directions that are not tangent to T^*X). As above we define the escape function $G = \tau G_0$ for some small $\tau > 0$. We let $\Lambda = e^{H_G^{(0)}} T^*X$ be the associated Lagrangian deformation and $(\mathcal{H}_\Lambda^k)_{k \in \mathbb{R}}$ the associated family of Hilbert spaces (see Section 5). Notice that these are spaces of distributions according to Proposition 21. For $k \in \mathbb{R}$, define the Hilbert space

$$\mathcal{F}_k = \{u \in \mathcal{D}'(X) : e^{\frac{\psi}{h}} u \in \mathcal{H}_\Lambda^k\}, \tag{41}$$

where we recall that ψ is defined in Section 6.1. The spaces \mathcal{H}_1 and \mathcal{H}_2 in Proposition 5 will be respectively $\{u \in \mathcal{F}_k : P_h(0)u \in \mathcal{F}_{k-1}\}$ and \mathcal{F}_{k-1} .

Notice that it is equivalent to study $P_h(\omega)$ acting on the \mathcal{F}_k 's or $\tilde{P}_h(\omega)$ acting on the \mathcal{H}_Λ^k 's. Also, we can write $\tilde{P}_h(\omega) = \tilde{P}_2 + \omega \tilde{P}_1 + \omega^2 \tilde{P}_0$ where the \tilde{P}_j 's are semiclassical differential operator with analytic coefficients near the support of ϕ . Consequently, these operators satisfy the assumption (26) and it makes sense to restrict their principal symbols (and thus the principal symbol $\tilde{p}(\cdot; \omega)$ of $\tilde{P}_h(\omega)$) to Λ , see the remark below (26). Applying Proposition 18 and Lemma 23 to the operators \tilde{P}_0 , \tilde{P}_1 and \tilde{P}_2 , we find:

Proposition 25. *Assume that τ and h are small enough. Let $m \in \mathbb{R}$ and f be a symbol of order m on Λ . Let $\omega \in \mathbb{C}$. Let k_1 and k_2 be such that $k_1 + k_2 = m + 1$. Then, there is a constant C such that for every $u, v \in \mathcal{H}_\Lambda^\infty$, we have*

$$\left| \int_\Lambda f T_\Lambda \tilde{P}_h(\omega) u \overline{T_\Lambda v} e^{-\frac{2H}{h}} d\alpha - \int_\Lambda f(\alpha) \tilde{p}(\alpha; \omega) T_\Lambda u \overline{T_\Lambda v} e^{-\frac{2H}{h}} d\alpha \right| \leq Ch \|u\|_{\mathcal{H}_\Lambda^{k_1}} \|v\|_{\mathcal{H}_\Lambda^{k_2}}.$$

Here, the constant C depends continuously on ω and $\tilde{p}(\cdot; \omega)$ denotes the principal symbol of $\tilde{P}_h(\omega)$. We also wrote $\mathcal{H}_\Lambda^\infty$ for $\bigcap_{k \in \mathbb{R}} \mathcal{H}_\Lambda^k$.

Another consequence of Proposition 18 and Lemma 23 that it will be useful to remember is that, under the assumptions of Proposition 25, for every $k \in \mathbb{R}$ the family $\omega \mapsto \tilde{P}_h(\omega) - \tilde{P}_h(0)$ is a holomorphic family of bounded operators from \mathcal{H}_Λ^k to $\mathcal{H}_\Lambda^{k-1}$.

6.3. Ellipticity estimates. In order to use Proposition 25, let us introduce the following subsets of T^*X :

$$\begin{aligned} V_R &= \{x_1 \leq -2\epsilon_0/3\} \cup \{x_1 \geq \epsilon_0/3\} \cup (\{-5\epsilon_0/6 \leq x_1 \leq 2\epsilon_0/3\} \cap \{|\xi_1| \leq 2\epsilon_1^{-1}(1 + |\xi'|)\}), \\ V_+ &= \{-5\epsilon_0/6 \leq x_1 \leq 2\epsilon_0/3\} \cap \{\xi_1 \geq \epsilon_1^{-1}(1 + |\xi'|)\}, \\ V_- &= \{-5\epsilon_0/6 \leq x_1 \leq 2\epsilon_0/3\} \cap \{\xi_1 \leq -\epsilon_1^{-1}(1 + |\xi'|)\}. \end{aligned}$$

Here, we see the function $x_1 : X \rightarrow \mathbb{R}$ as a function on T^*X by composition by the canonical projection $T^*X \rightarrow X$, and the constant ϵ_1 has been defined in Section 6.1. Notice that a point of T^*X for which x_1 is between $-\epsilon$ and ϵ is in $T^*U \simeq T^*(]-\epsilon, \epsilon[) \times T^*(\partial Y)$, and may consequently be written as (x_1, x', ξ_1, ξ') with $(x_1, \xi_1) \in T^*(]-\epsilon, \epsilon[)$ and $(x', \xi') \in T^*\partial Y$. This is how we make sense of ξ' in the equation above. We use the same metric on $T^*\partial Y$ as in Section 6.1.

We let also W_R, W_+ and W_- denote the images respectively of V_R, V_+ and V_- by $e^{H_G^{\omega_l}}$. Notice that $T^*X = V_R \cup V_+ \cup V_-$ so that $\Lambda = W_R \cup W_+ \cup W_-$. We are going to prove two ellipticity estimates, Lemmas 26 and 28, that will be used in Section 6.4 below to prove that $\tilde{P}_h(\omega)$ is Fredholm for a certain range of ω 's and invertible for at least one of these ω 's.

Lemma 26. *Let $\tau > 0$ be small and fixed. There is $\kappa > 0$ (depending on τ) such that the following holds. For every compact subset K of $\{z \in \mathbb{C} : \text{Im } z \geq -\kappa\}$, there is a constant C_K such that for $\omega \in K$ and $\alpha \in \Lambda$ such that $\langle |\alpha| \rangle \geq C_K$ we have*

$$\begin{aligned} \text{Re } \tilde{p}(\alpha; \omega) &\geq C_K^{-1} \langle |\alpha| \rangle && \text{if } \alpha \in W_R, \\ \text{Im } \tilde{p}(\alpha; \omega) &\leq -C_K^{-1} \langle |\alpha| \rangle && \text{if } \alpha \in W_+, \\ \text{Im } \tilde{p}(\alpha; \omega) &\geq C_K^{-1} \langle |\alpha| \rangle && \text{if } \alpha \in W_-. \end{aligned}$$

Proof. Let us write $\alpha = e^{H_G^{\omega_l}}(x, \xi)$ for $(x, \xi) \in T^*X$. We will distinguish the different cases that appear in the definitions of V_R, V_+ and V_- .

First case: $(x, \xi) \in \{x_1 \leq -2\epsilon_0/3\}$. In that case, we see that G_0 is null on a neighborhood of (x, ξ) so that $\alpha = (x, \xi)$. Moreover $\chi_1(x_1) = 1$, so that $q(x, \xi) = 1 + |\xi|^2$. Using that $\chi(x_1) = 0$ for $x_1 \leq -\epsilon_0$, we find from (40) that the real part of $\tilde{p}(x, \xi; \omega)$ is greater than

$$-C_0\epsilon_0\xi_1^2 - C(1 + |\xi_1|) + |\xi|^2. \tag{42}$$

Here, $C > 0$ is some constant that depends continuously on ω (and does not depend on α). Thanks to our assumption (37) on ϵ_0 , we see that (42) is larger than $C^{-1} \langle |\alpha| \rangle^2$ and hence that $C^{-1} \langle |\alpha| \rangle$ when α is large enough.

Second case: $(x, \xi) \in \{x_1 \geq \epsilon_0/3\}$. Notice that $H_G^{\omega_l} \tilde{p}(\alpha; h)$ is $\mathcal{O}(\langle |\alpha| \rangle \log \langle |\alpha| \rangle)$, with symbolic estimates (it follows from the fact that G has logarithmic order). Consequently, we have

$$\begin{aligned} \tilde{p}(\alpha; h) &= \tilde{p}(x, \xi; \omega) + \mathcal{O}(\langle |\alpha| \rangle \log \langle |\alpha| \rangle) \\ &= p(x, \xi + i\psi'(x_1)dx_1; \omega) + \mathcal{O}(\langle |\alpha| \rangle \log \langle |\alpha| \rangle) \\ &= p_2(x, \xi) + \mathcal{O}(\langle |\alpha| \rangle \log \langle |\alpha| \rangle). \end{aligned}$$

Thanks to our assumption (d) of ellipticity on p_2 in Section 2.1, we see that this quantity is larger than $C^{-1} \langle |\alpha| \rangle^2$ and hence that $C^{-1} \langle |\alpha| \rangle$ when α is large enough.

Third case: $(x, \xi) \in \{-5\epsilon_0/6 \leq x_1 \leq 2\epsilon_0/3\} \cap \{|\xi_1| \leq 2\epsilon_1^{-1}(1 + |\xi'|)\}$. In that case, we notice that $\chi(x_1) = 1$ and that $\tilde{p}(\alpha; \omega) = \tilde{p}(x, \xi; \omega) + \mathcal{O}(\langle |\alpha| \rangle \log \langle |\alpha| \rangle)$ as above. Using (40), we find that

$$\text{Re } \tilde{p}(\alpha; \omega) \geq q_1(x_1, x', \xi') - C_0\epsilon_0\xi_1^2 - C \langle |\alpha| \rangle \log \langle |\alpha| \rangle.$$

Using that $|\xi_1| \leq 2\epsilon_1^{-1}(1 + |\xi'|)$ and our assumption (38) on ϵ_0 , we see that for α large enough, this real part is larger than $C^{-1}\langle|\alpha|\rangle^2$ and hence that $C^{-1}\langle|\alpha|\rangle$.

Fourth case: $(x, \xi) \in V_+$. Notice that we have

$$\text{Im } \tilde{p}(\alpha; \omega) = \text{Im } \tilde{p}(x, \xi; \omega) + \tau H_{G_0}^{\omega_I} \text{Im } \tilde{p}(x, \xi; \omega) + \mathcal{O}(\log\langle|\alpha|\rangle)^2. \tag{43}$$

We want to estimate $H_{G_0}^{\omega_I} \text{Im } \tilde{p}(x, \xi; \omega)$. First, notice that $\tilde{p}(\cdot; \omega) - p_2$ is a symbol of order 1 on a neighborhood of the support of G_0 , so that

$$\begin{aligned} H_{G_0}^{\omega_I} \text{Im } \tilde{p}(x, \xi; \omega) &= H_{G_0}^{\omega_I} \text{Im } p_2(x, \xi) + \mathcal{O}(\log\langle|\alpha|\rangle) \\ &= -H_{\text{Im } p_2}^{\omega_I} G_0(x, \xi) + \mathcal{O}(\log\langle|\alpha|\rangle). \end{aligned}$$

Notice that the symbol $\text{Im } p_2$ vanishes on the real cotangent bundle T^*X , which is a Lagrangian submanifold for the symplectic form ω_I . Consequently, the Hamiltonian vector field $H_{\text{Im } p_2}^{\omega_I}$ is tangent to T^*X (this is why we only care about the value of G_0 on T^*X). Recall that ω_R denotes the real part of the canonical symplectic form ω on $(T^*X)_r$. For u tangent to T^*X , we have

$$\begin{aligned} \omega_R(u, H_{\text{Im } p_2}^{\omega_I}) &= \text{Im}(i\omega(u, H_{\text{Im } p_2}^{\omega_I})) = \text{Im}(\omega(iu, H_{\text{Im } p_2}^{\omega_I})) \\ &= d(\text{Im } p_2) \cdot (iu) = d(\text{Re } p_2) \cdot u \\ &= \omega_R(u, H_{p_2}), \end{aligned} \tag{44}$$

where H_{p_2} is the Hamiltonian vector field of p_2 for the (real) canonical symplectic form on the real cotangent bundle T^*X . We used the Cauchy–Riemann equation on the second line of (44). On the last line, we used the fact that p_2 is real-valued on T^*X and that the pullback of ω_R on T^*X is the canonical symplectic form on T^*X . Since ω_R is symplectic on T^*X and the vector fields H_{p_2} and $H_{\text{Im } p_2}^{\omega_I}$ are parallel to T^*X , we find that $H_{\text{Im } p_2}^{\omega_I}$ coincides with H_{p_2} on T^*X . It follows that

$$\begin{aligned} H_{G_0}^{\omega_I} \text{Im } \tilde{p}(x, \xi; \omega) &= H_{G_0} p_2(x, \xi) + \mathcal{O}(\log\langle|\alpha|\rangle) \\ &= -\phi(x_1)H_{G_1} p_2(x, \xi) + 2w(x_1)\phi'(x_1)G_1(x, \xi)\xi_1 + \mathcal{O}(\log\langle|\alpha|\rangle) \\ &\leq -C^{-1}\phi(x_1)\xi_1 + C_\omega \log\langle|\alpha|\rangle \end{aligned}$$

for some constant $C_\omega > 0$ that depends continuously on ω and some constant C that does not depend on ω . Here, we used (36), which is valid thanks to the assumption $(x, \xi) \in V_+$, and the fact that $w(x_1)\phi'(x_1) \leq 0$. Then, we plug this estimate into (43) to find that

$$\begin{aligned} \text{Im } \tilde{p}(\alpha; \omega) &\leq 2w(x_1)\psi'(x_1)\xi_1 + \text{Im } \omega p_1(x_1, \xi_1) - C^{-1}\tau\phi(x_1)\xi_1 + C_\omega(\log\langle|\alpha|\rangle)^2 \\ &\leq -C^{-1}(-w(x_1)\psi'(x_1) + \tau\phi(x_1) + \text{Im } \omega)\xi_1 + C_\omega(\log\langle|\alpha|\rangle)^2, \end{aligned}$$

where the constants C may change from one line to another but still does not depend on ω . We used here that $p_1(x_1, \xi_1)$ is elliptic of order 1 with the same sign as $-\xi_1$, that is, our (c) from Section 2.1. Notice that $w(x_1)\psi'(x_1)$ has the same sign as $x_1\psi'(x_1)$, and consequently there is a constant $\kappa > 0$ such that $-w(x_1)\psi'(x_1) + \tau\phi(x_1) > \kappa$ for $-5\epsilon_0/6 < x_1 < 2\epsilon_0/3$. Hence, if $\text{Im } \omega > -\kappa$, we see that $\text{Im } \tilde{p}(\alpha; \omega)$ is less than $-C^{-1}\langle|\alpha|\rangle$ when α is large.

Fifth case: $(x, \xi) \in V_-$. This is the same as the fourth case up to a few sign flips. □

Remark 27. Let us point out how the five cases in the proof of Lemma 26 correspond to different places in Figure 1. The first and the second cases correspond respectively to the green and the purple zone. The last three cases correspond to the blue zone (to distinguish these cases one need to consider the momentum variable which is not represented on Figure 1).

Lemma 28. Assume that ν is large enough. Assume that τ is small enough (depending on ν). Then there is a constant $C > 0$ such that for every $\alpha \in \Lambda$ we have

$$\begin{aligned} \operatorname{Re} \tilde{p}(\alpha; i\nu) &\geq C^{-1}\langle|\alpha|\rangle && \text{if } \alpha \in W_R, \\ \operatorname{Im} \tilde{p}(\alpha; i\nu) &\leq -C^{-1}\langle|\alpha|\rangle && \text{if } \alpha \in W_+, \\ \operatorname{Im} \tilde{p}(\alpha; i\nu) &\geq C^{-1}\langle|\alpha|\rangle && \text{if } \alpha \in W_-. \end{aligned}$$

Proof. We write as above $\alpha = e^{H_G^{\omega_l}}(x, \xi)$ for $(x, \xi) \in T^*X$. We review the same five cases as in the proof of Lemma 26, with the additional assumption that $\omega = i\nu$ with $\nu > 0$ large.

First case: $(x, \xi) \in \{x_1 \leq -2\epsilon_0/3\}$. The symbol $q(x, \xi)$ is still $1 + |\xi|^2$ in that case. Notice that we have here $p_0(x) < -\epsilon$ (see the beginning of Section 6.1). Looking at (40), we find that, for some constant $C > 0$, the real part of $\tilde{p}(x, \xi; i\nu)$ is larger than

$$\chi(x_1)(-C_0\epsilon_0\xi_1^2 + \epsilon\nu^2 - C(1 + \nu)) + (1 + |\xi|^2) \geq -C_0\epsilon_0\xi_1^2 + (1 + |\xi|^2),$$

provided ν is large enough so that $\epsilon\nu^2 - C(1 + \nu) \geq 0$. Using our assumption (37), we see that the real part of $\tilde{p}(\alpha; i\nu)$ is indeed larger than $C^{-1}\langle|\alpha|\rangle$.

Second case: $(x, \xi) \in \{x_1 \geq \epsilon_0/3\}$. Notice that $\tilde{p}(\cdot; i\nu) = \tilde{p}_2 + i\nu\tilde{p}_1 - \nu^2\tilde{p}_0$, where for $j = 0, 1, 2$, the principal symbol \tilde{p}_j of \tilde{P}_j is a symbol of order j that does not depend on ν . It follows that $H_G^{\omega_l} \tilde{p}(\cdot; i\nu)$ is $\mathcal{O}(\tau(\langle|\alpha|\rangle \log\langle|\alpha|\rangle + \nu^2 \log\langle|\alpha|\rangle/\langle|\alpha|\rangle))$, uniformly in ν and τ and with symbolic estimates.

Consequently, we have in this second case

$$\begin{aligned} \operatorname{Re} \tilde{p}(\alpha; i\nu) &= \operatorname{Re} \tilde{p}(x, \xi; i\nu) + \mathcal{O}\left(\tau\left(\langle|\alpha|\rangle \log\langle|\alpha|\rangle + \nu^2 \frac{\log\langle|\alpha|\rangle}{\langle|\alpha|\rangle}\right)\right) \\ &= \operatorname{Re} p(x, \xi + i\psi'(x_1)dx_1; i\nu) + \mathcal{O}(\tau\nu^2\langle|\alpha|\rangle \log\langle|\alpha|\rangle) \\ &= p_2(x, \xi) - \nu^2 p_0(x) + \mathcal{O}(\nu + \tau\nu^2\langle|\alpha|\rangle \log\langle|\alpha|\rangle), \end{aligned}$$

uniformly in τ and ν . We start by taking ν large enough so that $p_2(x, \xi) - \nu^2 p_0(x) + \mathcal{O}(\nu)$ is larger than $C^{-1}\langle|\alpha|\rangle^2$ (which is possible by our ellipticity assumptions on p_2 and p_0). Then, by taking τ small enough, we ensure that $\mathcal{O}(\tau\nu^2\langle|\alpha|\rangle \log\langle|\alpha|\rangle)$ is smaller than $C^{-1}\langle|\alpha|\rangle^2$, which gives the required estimate. Let us point out here that how small τ needs to be depend on ν , but how large ν has to be does not depend on τ .

Third case: $(x, \xi) \in \{-5\epsilon_0/6 \leq x_1 \leq 2\epsilon_0/3\} \cap \{|\xi_1| \leq 2\epsilon_1^{-1}(1 + |\xi'|)\}$. As in the previous case, we notice that

$$\tilde{p}(\alpha; i\nu) = \tilde{p}(x, \xi; i\nu) + \mathcal{O}(\tau\nu^2\langle|\alpha|\rangle \log\langle|\alpha|\rangle).$$

Then, we use (40) to find that

$$\operatorname{Re} \tilde{p}(x, \xi; i\nu) \geq -C_0 \epsilon_0 \xi_1^2 + q_1(x_1, x', \xi') + \epsilon \nu^2 - C(1 + \nu),$$

for some $C > 0$ that does not depend on ν , nor τ . Using (38), we find that if ν is large enough we have

$$\operatorname{Re} \tilde{p}(x, \xi; i\nu) \geq C^{-1}(1 + |\xi|^2).$$

Consequently, we have

$$\operatorname{Re} \tilde{p}(\alpha; i\nu) \geq C^{-1}(1 + |\xi|^2) - C\tau\nu^2 \langle |\alpha| \rangle \log \langle |\alpha| \rangle,$$

where the constant $C > 0$ may have changed, but still does not depend on ν , nor τ . Taking τ small enough (depending on ν), we get rid of the term $-C\tau\nu^2 \langle |\alpha| \rangle \log \langle |\alpha| \rangle$. Thus, we get the required estimate. As above, it is crucial here that how small τ needs to be depend on ν , but how large ν has to be does not depend on τ .

Fourth case: $(x, \xi) \in V_+$. Writing $\tilde{p}(\cdot; i\nu) = \tilde{p}_2 + i\nu\tilde{p}_1 - \nu^2\tilde{p}_0$, we find that

$$\operatorname{Im} \tilde{p}(\alpha; i\nu) = \operatorname{Im} \tilde{p}(x, \xi; i\nu) + \tau H_{G_0}^{\omega_l} \operatorname{Im} \tilde{p}(x, \xi; i\nu) + \mathcal{O}(\tau^2 \nu^2 (\log \langle |\alpha| \rangle)^2).$$

Then, we notice that on a neighborhood of the support of G_0 , the symbol $\tilde{p}(\cdot; i\nu) - p_2$ is the sum of a symbol of order 1, a symbol of order 1 multiplied by ν and a symbol of order 0 multiplied by ν^2 . Consequently, we have

$$H_{G_0}^{\omega_l} \operatorname{Im} \tilde{p}(x, \xi; i\nu) = H_{G_0}^{\omega_l} \operatorname{Im} p_2(x, \xi) + \mathcal{O}(\nu^2 \log \langle |\alpha| \rangle).$$

Using (36) as in the proof of Lemma 26, we see that $H_{G_0}^{\omega_l} \operatorname{Im} p_2(x, \xi)$ is nonpositive. We recall (40) and the fact that $w(x_1)\psi'(x_1)$ is nonpositive, to find, for some $C > 0$ that does not depend on τ nor α , that

$$\operatorname{Im} \tilde{p}(\alpha; i\nu) \leq \nu p_1(x_1, \xi_1) + C\tau\nu^2 (\log \langle |\alpha| \rangle)^2. \quad (45)$$

Since $(x, \xi) \in V_+$, we know that ξ_1 is nonzero. Moreover, $p_1(x_1, \xi_1)$ is negative and elliptic. Thus, we only need to take τ small enough to get rid of the last term in (45) and the required estimate follows. Once again here, see that τ depends on ν , but ν does not depend on τ .

Fifth case: $(x, \xi) \in V_-$. This is the same as the fourth case up to a few sign flips. \square

6.4. Invertibility and Fredholm properties. With the estimates from Section 6.3, we are now ready to study the functional analytic properties of $\tilde{P}_h(\omega)$ acting on suitable spaces.

Let ν be large enough and τ be small enough so that Lemma 28 and Proposition 25 hold. Let then κ be as in Lemma 26. Let $\delta \in]0, \kappa[$ and V be a relatively compact open subset of $\{z \in \mathbb{C} : \operatorname{Im} z > -\kappa\}$. Without loss of generality, we may assume that V is connected and contains the compact set

$$\{z \in \mathbb{C} : |\operatorname{Re} z| \leq \nu + \kappa, -\delta \leq \operatorname{Im} z \leq 2\nu + \kappa\}.$$

Let then C_K be the constant from Lemma 26 applied with $K = \bar{V}$. We shall always assume that h is small enough so that Proposition 25 holds. Let k be any real number.

Let a be a compactly supported smooth function from Λ to \mathbb{R}_+ such that

$$\inf_{\omega \in K} \inf_{\substack{\alpha \in \Lambda \\ \langle |\alpha| \rangle \leq 2C_K}} a(\alpha) + \operatorname{Re} \tilde{p}(\alpha; \omega) > 0. \tag{46}$$

We let then A be the operator

$$A := S_\Lambda B_\Lambda a B_\Lambda T_\Lambda, \tag{47}$$

where we recall that S_Λ is a left inverse for T_Λ , and B_Λ is the orthogonal projector on $\mathcal{H}_{\Lambda, \text{FBI}}^0$ in $L^2_0(\Lambda)$ (see page 3645). The operator $A : C^\infty(X) \rightarrow \mathcal{D}'(X)$ extends to a bounded operator from \mathcal{H}_Λ^m to \mathcal{H}_Λ^ℓ for every $m, \ell \in \mathbb{R}$, see for instance [Bonthonneau and Jézéquel 2020, Proposition 2.4 and Remark 2.20].

Let us define the domain of $\tilde{P}_h(\omega)$ on \mathcal{H}_Λ^k as

$$D_k = \{u \in \mathcal{H}_\Lambda^k : \tilde{P}_h(0)u \in \mathcal{H}_\Lambda^{k-1}\}.$$

We put a Hilbert space structure on D_k by endowing it with the norm

$$\|u\|_{D_k}^2 = \|u\|_{\mathcal{H}_\Lambda^k}^2 + \|\tilde{P}_h(0)u\|_{\mathcal{H}_\Lambda^{k-1}}^2.$$

We will need the following approximation result.

Lemma 29. *Let $u \in D_k$. Then $\tilde{P}_h(\omega)u$ belongs to $\mathcal{H}_\Lambda^{k-1}$ and there is a sequence $(u_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{H}_\Lambda^\infty$ such that $(u_n)_{n \in \mathbb{N}}$ tends to u in \mathcal{H}_Λ^k and $(\tilde{P}_h(\omega)u_n)_{n \in \mathbb{N}}$ converges to $\tilde{P}_h(\omega)u$ in $\mathcal{H}_\Lambda^{k-1}$.*

Proof. Start by noticing that

$$\tilde{P}_h(\omega)u = (\tilde{P}_h(\omega) - \tilde{P}_h(0))u + \tilde{P}_h(0)u.$$

Since $\tilde{P}_h(\omega) - \tilde{P}_h(0)$ is bounded from \mathcal{H}_Λ^k to $\mathcal{H}_\Lambda^{k-1}$, we see that $\tilde{P}_h(\omega)u$ belongs to $\mathcal{H}_\Lambda^{k-1}$ when u belongs to D_k .

Let then I_ϵ be the operator

$$I_\epsilon = S_\Lambda B_\Lambda s_\epsilon B_\Lambda T_\Lambda,$$

where s_ϵ is a symbol on Λ defined by $s_\epsilon(\alpha) = \theta(\langle |\alpha| \rangle)$, where θ is a compactly supported function in \mathbb{R} , identically equal to 1 near 0. It follows for instance from [loc. cit., Proposition 2.4 and Remark 2.20] that if $u \in \mathcal{H}_\Lambda^k$ then $I_\epsilon u \in \mathcal{H}_\Lambda^\infty$. We see that for $u \in \mathcal{H}_\Lambda^k$, we have

$$\begin{aligned} \|I_\epsilon u - u\|_{\mathcal{H}_\Lambda^k} &= \|\Pi_\Lambda B_\Lambda s_\epsilon T_\Lambda u - T_\Lambda u\|_{L^2_k(\Lambda)} \\ &= \|B_\Lambda (s_\epsilon - 1)T_\Lambda u\|_{L^2_k(\Lambda)} \leq C \|(s_\epsilon - 1)T_\Lambda u\|_{L^2_k(\Lambda)}. \end{aligned}$$

It follows that $I_\epsilon u$ converges to u in \mathcal{H}_Λ^k when ϵ tends to 0.

If u belongs to D_k , we see that

$$\tilde{P}_h(\omega)I_\epsilon u = I_\epsilon \tilde{P}_h(\omega)u + [\tilde{P}_h(\omega), I_\epsilon]u.$$

From the analysis above, we have that $I_\epsilon \tilde{P}_h(\omega)u$ converges to $\tilde{P}_h(\omega)u$ in $\mathcal{H}_\Lambda^{k-1}$ when ϵ tends to 0. Notice that the symbol s_ϵ is uniformly bounded as a symbol of order 0 on Λ . Hence, it follows from [loc. cit., Proposition 2.12], as in the proof of [loc. cit., Lemma 3.4], that the operator $[\tilde{P}_h(\omega), I_\epsilon]$ is uniformly

bounded from \mathcal{H}_Λ^k to $\mathcal{H}_\Lambda^{k-1}$ when ϵ tends to 0. If u is in $\mathcal{H}_\Lambda^\infty$, the analysis above implies that $[\tilde{P}_h(\omega), I_\epsilon]u$ tends to 0 in $\mathcal{H}_\Lambda^{k-1}$. Thanks to the uniform boundedness of $[\tilde{P}_h(\omega), I_\epsilon]$ when ϵ tends to 0, we see that the same holds when u is only in $\mathcal{H}_\Lambda^{k-1}$. \square

We first use [Lemma 26](#) to find:

Lemma 30. *There is $C > 0$ such that for h small enough and every $\omega \in V$ and $u \in D_k$ we have*

$$\|u\|_{\mathcal{H}_\Lambda^k} \leq C \|(\tilde{P}_h(\omega) + A)u\|_{\mathcal{H}_\Lambda^{k-1}}.$$

Proof. Thanks to [Lemma 29](#), we only need to prove this estimate for $u \in \mathcal{H}_\Lambda^\infty$. Let f_+, f_-, f_R and f_a be symbols of order 0 on Λ such that $f_+ + f_- + f_R + f_a = 1$. Moreover, we assume that f_+, f_- and f_R are supported in the intersection of $\{(|\alpha|) \geq C_K\}$ respectively with W_+, W_- and W_R and that f_a is supported in $\{(|\alpha|) \leq 2C_K\}$.

For $u \in \mathcal{H}_\Lambda^\infty$ and $\omega \in K$, we have, from [Proposition 25](#),

$$\begin{aligned} \operatorname{Re} \left(\int_\Lambda f_R(\alpha) \langle |\alpha| \rangle^{2k-1} T_\Lambda(\tilde{P}_h(\omega) + A)u \overline{T_\Lambda u} e^{-\frac{2H(\alpha)}{h}} \, d\alpha \right) \\ \geq \int_\Lambda f_R(\alpha) \langle |\alpha| \rangle^{2k-1} \operatorname{Re}(\tilde{p}(\alpha; \omega) + a(\alpha)) |T_\Lambda u(\alpha)|^2 e^{-\frac{2H(\alpha)}{h}} \, d\alpha - Ch \|u\|_{\mathcal{H}_\Lambda^k}^2 \\ \geq C^{-1} \int_\Lambda f_R(\alpha) \langle |\alpha| \rangle^{2k} |T_\Lambda u(\alpha)|^2 e^{-\frac{2H(\alpha)}{h}} \, d\alpha - Ch \|u\|_{\mathcal{H}_\Lambda^k}^2, \end{aligned}$$

where we used [Lemma 26](#) in the last line (since a takes positive values it does not harm the positivity of the real part of \tilde{p}). From Cauchy–Schwarz inequality, we find that

$$\int_\Lambda f_R(\alpha) \langle |\alpha| \rangle^{2k} |T_\Lambda u(\alpha)|^2 e^{-\frac{2H(\alpha)}{h}} \, d\alpha \leq C \|(\tilde{P}_h(\omega) + A)u\|_{\mathcal{H}_\Lambda^{k-1}} \|u\|_{\mathcal{H}_\Lambda^k} + Ch \|u\|_{\mathcal{H}_\Lambda^k}^2.$$

Replacing the real part by an imaginary part, and varying the sign, we get the same estimates with f_R replaced by f_+ and f_- . Using (46), we get the same estimates with f_R replaced by f_a . Summing these four estimates, we find that

$$\|u\|_{\mathcal{H}_\Lambda^k}^2 \leq C \|(\tilde{P}_h(\omega) + A)u\|_{\mathcal{H}_\Lambda^{k-1}} \|u\|_{\mathcal{H}_\Lambda^k} + Ch \|u\|_{\mathcal{H}_\Lambda^k}^2.$$

When h is small enough, we can get rid of the second term in the right-hand side. Dividing by $\|u\|_{\mathcal{H}_\Lambda^k}$ the result follows (the result is trivial when $u = 0$). \square

The same proof using [Lemma 28](#) instead of [Lemma 26](#) gives:

Lemma 31. *There is $C > 0$ such that for h small enough and every $u \in D_k$ we have*

$$\|u\|_{\mathcal{H}_\Lambda^k} \leq C \|\tilde{P}_h(i\nu)u\|_{\mathcal{H}_\Lambda^{k-1}}.$$

Applying [Proposition 18](#) as in the justification of [Proposition 25](#), we find that, for every $\omega \in V$, there is a symbol σ_ω of order 2 on Λ and an operator Z with negligible kernel on $\Lambda \times \Lambda$ such that

$$B_\Lambda T_\Lambda \tilde{P}_h(\omega) S_\Lambda B_\Lambda = B_\Lambda \sigma_\omega B_\Lambda + Z = B_\Lambda \sigma_\omega B_\Lambda + B_\Lambda Z B_\Lambda.$$

Let us identify the dual of \mathcal{H}_Λ^k with \mathcal{H}_Λ^{-k} as in [Bonthonneau and Jézéquel 2020, Lemma 2.24], that is using the pairing

$$\langle u, v \rangle_\Lambda := \int_\Lambda T_\Lambda u \overline{T_\Lambda v} e^{-\frac{2H}{h}} d\alpha. \tag{48}$$

Notice that it is a priori not the L^2 pairing (recall that the L^2 pairing identifies the dual of \mathcal{H}_Λ^k with \mathcal{H}_Λ^{-k} , see Section 5.2). Under this identification, the formal adjoint of $\tilde{P}_h(\omega)$ may be defined as

$$\tilde{P}_h(\omega)^* = S_\Lambda (B_\Lambda \bar{\sigma}_\omega B_\Lambda + B_\Lambda Z^* B_\Lambda) T_\Lambda$$

By this, we just mean that if $u, v \in \mathcal{H}_\Lambda^\infty$ then

$$\langle \tilde{P}_h(\omega)u, v \rangle_\Lambda = \langle u, \tilde{P}_h(\omega)^*v \rangle_\Lambda.$$

Notice that we do not claim that $\tilde{P}_h(\omega)^*$ is the adjoint of $\tilde{P}_h(\omega)$ for a Hilbert space structure. We define the domain of $\tilde{P}_h(\omega)^*$ as

$$D_{-k}^* = \{u \in \mathcal{H}_\Lambda^{-k+1} : \tilde{P}_h(\omega)^*u \in \mathcal{H}_\Lambda^{-k}\}.$$

Notice that we have $\bar{\sigma}_\omega(\alpha) = \overline{\tilde{p}(\alpha; \omega)} + \mathcal{O}(h\langle|\alpha| \rangle)$. Hence, the operator $\tilde{P}_h(\omega)^*$ satisfies Proposition 25 with \tilde{p} replaced by $\overline{\tilde{p}}$. In order to introduce the symbol f , one may use [loc. cit., Proposition 2.12]. Consequently, we can use Lemmas 26 and 28, as in the proofs of Lemmas 30 and 31, to get:

Lemma 32. *There is $C > 0$ such that for h small enough and every $\omega \in V$ and $u \in D_{-k}^*$ we have*

$$\|u\|_{\mathcal{H}_\Lambda^{-k+1}} \leq C \|(\tilde{P}_h(\omega) + A)^*u\|_{\mathcal{H}_\Lambda^{-k}}.$$

In this statement, $(\tilde{P}_h(\omega) + A)^* = \tilde{P}_h(\omega)^* + A$ is the formal adjoint of $\tilde{P}_h(\omega) + A$ for the pairing (48).

Lemma 33. *There is $C > 0$ such that for h small enough and every $u \in D_{-k}^*$ we have*

$$\|u\|_{\mathcal{H}_\Lambda^{-k+1}} \leq C \|\tilde{P}_h(iv)^*u\|_{\mathcal{H}_\Lambda^{-k}}.$$

Remark 34. Here, we used (48) rather than the L^2 pairing to describe the dual of \mathcal{H}_Λ^k because this identification makes A self-adjoint, so that we can reuse directly the estimates from Lemmas 26 and 28. We expect however that the L^2 pairing studied in Section 5.2 would allow to get similar estimates that we could also use in the proofs below.

From Lemmas 30, 31, 32 and 33, we deduce:

Proposition 35. *There is $C > 0$ such that for h small enough and $\omega \in V$ the operators $\tilde{P}_h(\omega) + A$ and $\tilde{P}_h(iv)$ are invertible as operators from D_k to $\mathcal{H}_\Lambda^{k-1}$. Moreover, the operator norms of their inverses is bounded by C .*

Proof. From Lemma 30, we find that $\tilde{P}_h(\omega) + A$ is injective on D_k and that its image is closed in $\mathcal{H}_\Lambda^{k-1}$.

Let us consider an element $v \in \mathcal{H}_\Lambda^{-k+1}$ such that $\langle u, v \rangle_\Lambda = 0$ for every $u \in \mathcal{H}_\Lambda^{k-1}$ in the image of $\tilde{P}_h(\omega) + A$. In particular, if $u \in \mathcal{H}_\Lambda^\infty$, we have $\langle (\tilde{P}_h(\omega) + A)u, v \rangle = 0$. Notice that $\mathcal{H}_\Lambda^\infty$ is dense in $\mathcal{H}_\Lambda^{-k+1}$ (for instance because it contains all real-analytic functions due to Proposition 20, and they form a dense

subset of $\mathcal{H}_\Lambda^{-k+1}$ according to [Bonthonneau and Jézéquel 2020, Corollary 2.3], one can also work as in Lemma 29). Consequently, we have

$$\langle (\tilde{P}_h(\omega) + A)u, v \rangle_\Lambda = \langle u, (\tilde{P}_h(\omega) + A)^*v \rangle_\Lambda$$

for every $u \in \mathcal{H}_\Lambda^\infty$, since this equality holds when $v \in \mathcal{H}_\Lambda^\infty$. Hence, we have $\langle u, (\tilde{P}_h(\omega) + A)^*v \rangle = 0$ for every $u \in \mathcal{H}_\Lambda^\infty$, and thus $(\tilde{P}_h(\omega) + A)^*v = 0$. It follows from Lemma 32 that $v = 0$.

We just proved that the image of $\tilde{P}_h(\omega) + A$ is dense in $\mathcal{H}_\Lambda^{k-1}$, and thus $\tilde{P}_h(\omega) + A$ is invertible. The estimate on the operator norm of the inverse immediately follows from Lemma 30.

The argument to invert $\tilde{P}_h(i\nu)$ is the same using Lemmas 31 and 33 instead of Lemmas 30 and 32. \square

The analytic Fredholm theory then implies that:

Proposition 36. *Assume that h is small enough. For every $\omega \in V$, the operator $\tilde{P}_h(\omega) : D_k \rightarrow \mathcal{H}_\Lambda^{k-1}$ is Fredholm of index 0. Moreover, the operator $\tilde{P}_h(\omega) : D_k \rightarrow \mathcal{H}_\Lambda^{k-1}$ has a meromorphic inverse $\omega \mapsto \tilde{P}_h(\omega)^{-1}$ with poles of finite rank on V .*

Proof. From [loc. cit., Proposition 21.3] or Lemma 38 below, we find that A is a compact operator from D_k to $\mathcal{H}_\Lambda^{k-1}$. Hence, it follows from Proposition 35 that $\tilde{P}_h(\omega) : D_k \rightarrow \mathcal{H}_\Lambda^{k-1}$ is Fredholm for $\omega \in V$.

Since $\tilde{P}_h(\omega) - \tilde{P}_h(0)$ is a holomorphic family of bounded operators from \mathcal{H}_Λ^k to $\mathcal{H}_\Lambda^{k-1}$, we see that $\tilde{P}_h(\omega)$ is a holomorphic family of operators from D_k to $\mathcal{H}_\Lambda^{k-1}$, for ω in V . Since this operator is invertible for $\omega = i\nu$ and V is connected, we find that the index of $\tilde{P}_h(\omega)$ is 0. Finally, the analytic Fredholm theorem [Dyatlov and Zworski 2019, Theorem C.8] implies the existence of the meromorphic inverse $\omega \mapsto \tilde{P}_h(\omega)^{-1}$, with poles of finite rank. \square

6.5. Counting resonances. We will now use the functional analytic framework from Section 6.4 to prove the point (vi) in Proposition 5. The bounds on the number of resonances from Theorems 2 and 3 ultimately come from the following lemma.

Lemma 37. *Recall that $\delta \in]0, \kappa[$. There is $C > 0$ such that, for every h small enough, the number of ω 's in the disk of center 0 and radius δ such that $\tilde{P}_h(\omega) : D_k \rightarrow \mathcal{H}_\Lambda^{k-1}$ is not invertible (counted with null multiplicity) is less than Ch^{-n} .*

Before being able to prove Lemma 37, we need to establish a bound on the trace class operator norm of A , which is defined by (47).

Lemma 38. *The operator $A : D_k \rightarrow \mathcal{H}_\Lambda^{k-1}$ is trace class, with trace class norm $\mathcal{O}(h^{-n})$.*

Proof. We only need to prove that the operator $\tilde{A} = B_\Lambda a B_\Lambda$ is trace class from $L^2_k(\Lambda)$ to $L^2_{k-1}(\Lambda)$, with trace class norm $\mathcal{O}(h^{-n})$.

For every $N > 0$, introduce the operator $\square_N := B_\Lambda \langle |\alpha| \rangle^N B_\Lambda$. Using [Bonthonneau and Jézéquel 2020, Proposition 2.12] to make a parametrix construction, we see that there is a symbol σ_N of order $-N$ on Λ such that $\square_N B_\Lambda \sigma_N B_\Lambda - B_\Lambda$ and $B_\Lambda \sigma_N B_\Lambda \square_N - B_\Lambda$ are negligible operators, in particular they are $\mathcal{O}(h^\infty)$ as operators from $L^2_{s_1}(\Lambda) \rightarrow L^2_{s_2}(\Lambda)$ for any $s_1, s_2 \in \mathbb{R}$. Hence, for h small enough, we get an inverse $\square_N^{-1} : \mathcal{H}_{\Lambda, \text{FBI}}^0 \rightarrow \mathcal{H}_{\Lambda, \text{FBI}}^N$ for \square_N , which is bounded uniformly in h and satisfies the equation

$$\square_N^{-1} B_\Lambda = B_\Lambda \sigma_N B_\Lambda + B_\Lambda \sigma_N B_\Lambda (B_\Lambda - \square_N B_\Lambda \sigma_N B_\Lambda) + (B_\Lambda - B_\Lambda \sigma_N B_\Lambda \square_N) \square_N^{-1} (B_\Lambda - \square_N B_\Lambda \sigma_N B_\Lambda).$$

Thus, we see that $\square_N^{-1}B_\Lambda$ is equal to $B_\Lambda\sigma_N B_\Lambda$ up to a negligible operator. Let us recall that $\mathcal{H}_{\Lambda,\text{FBI}}^k$ is the image of \mathcal{H}_Λ^k by T_Λ (which is also the image of $L_k^2(\Lambda)$ by B_Λ).

Fix $N > n$. Notice that \tilde{A} is bounded, uniformly in h , as an operator from $L_k^2(\Lambda)$ to $L_{k+N}^2(\Lambda)$ (since B_Λ is bounded on $L_k^2(\Lambda)$ and on $L_{k+N}^2(\Lambda)$). We can then write

$$\tilde{A} = \iota \square_k^{-1} \square_N^{-1} B_\Lambda \square_N \square_k \tilde{A}. \tag{49}$$

On the left-hand side, \tilde{A} is seen as an operator from $L_k^2(\Lambda)$ to $L_{k-1}^2(\Lambda)$. On the right-hand side, \tilde{A} sends $L_k^2(\Lambda)$ into $\mathcal{H}_{\Lambda,\text{FBI}}^{k+N}$, the operator \square_k sends $\mathcal{H}_{\Lambda,\text{FBI}}^{k+N}$ into $\mathcal{H}_{\Lambda,\text{FBI}}^N$, the operator \square_N sends $\mathcal{H}_{\Lambda,\text{FBI}}^N$ into $\mathcal{H}_{\Lambda,\text{FBI}}^0$, the operator $\square_N^{-1}B_\Lambda$ sends $\mathcal{H}_{\Lambda,\text{FBI}}^0$ into $\mathcal{H}_{\Lambda,\text{FBI}}^0$, the operator \square_k^{-1} sends $\mathcal{H}_{\Lambda,\text{FBI}}^0$ into $\mathcal{H}_{\Lambda,\text{FBI}}^k$ and ι is the inclusion of $\mathcal{H}_{\Lambda,\text{FBI}}^k$ into $L_{k-1}^2(\Lambda)$. With these mapping properties, the operators $\tilde{A}, \square_k, \square_N, \square_k^{-1}$ and ι on the right-hand side of (49) are bounded uniformly in h . From [Bonthonneau and Jézéquel 2020, Lemma 2.25], we see that $\square_N^{-1}B_\Lambda$ is trace class on $L_0^2(\Lambda)$ (since $B_\Lambda\sigma_N B_\Lambda$ is). Moreover, its trace is given by the integral of its kernel on the diagonal, which is $\mathcal{O}(h^{-n})$. Indeed, $\square_N^{-1}B_\Lambda$ is a “complex FIO associated to Δ_Λ of order $-N$ ” in the sense of [loc. cit., Definition 2.5] as a consequence of [loc. cit., Lemmas 2.16 and 2.23]. Since $\mathcal{H}_{\Lambda,\text{FBI}}^0$ is a closed subset of $L_0^2(\Lambda)$, we see that $\square_N^{-1}B_\Lambda$ is also a trace class operator from $\mathcal{H}_{\Lambda,\text{FBI}}^0$ to itself, with the same trace. Moreover, $\square_N^{-1}B_\Lambda$ is a positive self adjoint operator on $\mathcal{H}_{\Lambda,\text{FBI}}^0$ with h small enough (because $\langle |\alpha| \rangle^N$ is positive), so that its trace class norm coincides with its trace. This ends the proof of the lemma. \square

Proof of Lemma 37. For $\omega \in V$, let us introduce the spectral determinant

$$f_h(\omega) = \det(I - (\tilde{P}_h(\omega) + A)^{-1}A).$$

Since $\tilde{P}_h(\omega) - \tilde{P}_h(0)$ is a holomorphic family of bounded operators from \mathcal{H}_Λ^k to $\mathcal{H}_\Lambda^{k-1}$, we see that $\tilde{P}_h(\omega) + A$ is a holomorphic family of operators from D_k to $\mathcal{H}_\Lambda^{k-1}$. From Proposition 35, the operators $(\tilde{P}_h(\omega) + A)^{-1} : \mathcal{H}_\Lambda^{k-1} \rightarrow D_k$ are bounded uniformly in $\omega \in V$, and thus it is a holomorphic family of operators in V . Consequently, the spectral determinant $f_h(\omega)$ is holomorphic in V .

The logarithmic derivative of f_h is given by

$$\begin{aligned} \frac{f'_h(\omega)}{f_h(\omega)} &= \text{tr}((I - (\tilde{P}_h(\omega) + A)^{-1}A)^{-1}(\tilde{P}_h(\omega) + A)^{-1}\partial_\omega \tilde{P}_h(\omega)(\tilde{P}_h(\omega) + A)^{-1}A) \\ &= \text{tr}((\tilde{P}_h(\omega) + A)^{-1}A(I - (\tilde{P}_h(\omega) + A)^{-1}A)^{-1}(\tilde{P}_h(\omega) + A)^{-1}\partial_\omega \tilde{P}_h(\omega)). \end{aligned}$$

Let us then write

$$\begin{aligned} (\tilde{P}_h(\omega) + A)^{-1}A(I - (\tilde{P}_h(\omega) + A)^{-1}A)^{-1}(\tilde{P}_h(\omega) + A)^{-1}\partial_\omega \tilde{P}_h(\omega) &= ((I - (\tilde{P}_h(\omega) + A)^{-1}A)^{-1} - I)(\tilde{P}_h(\omega) + A)^{-1}\partial_\omega \tilde{P}_h(\omega) \\ &= \tilde{P}_h(\omega)^{-1}\partial_\omega \tilde{P}_h(\omega) - (\tilde{P}_h(\omega) + A)^{-1}\partial_\omega \tilde{P}_h(\omega) \end{aligned} \tag{50}$$

Hence, if ω_0 is in V , the residue of the family of operators (50) at ω_0 is the same as the residue of the family of operators $\omega \mapsto \tilde{P}_h(\omega)^{-1}\partial_\omega \tilde{P}_h(\omega)$. Consequently, the order of annulation of f_h at ω_0 coincides with the null multiplicity of $\omega \mapsto \tilde{P}_h(\omega)$ at ω_0 .

Since V is open, there is $\eta > 0$ such that the closed disk of center $i\nu$ and radius $\nu + \delta + 2\eta$ is contained in V . Since the poles of $\tilde{P}_h(\omega)^{-1}$ are isolated, we may choose $0 \leq \eta' \leq \eta$ such that there is no poles of $\tilde{P}_h(\omega)^{-1}$ on the circle of center $i\nu$ and radius $\nu + \delta + \eta + \eta'$. For $r \geq 0$, let $n_h(r)$ denote the number of zeros of f_h in the disk of center $i\nu$ and radius r . Notice that

$$n_h(\nu + \delta) \leq \frac{\nu + \delta + \eta}{\eta} \int_{\nu+\delta}^{\nu+\delta+\eta} \frac{n_h(r)}{r} dr \leq \frac{\nu + \delta + \eta}{\eta} \int_0^{\nu+\delta+\eta+\eta'} \frac{n_h(r)}{r} dr. \tag{51}$$

From Jensen’s formula, we know that

$$\int_0^{\nu+\delta+\eta+\eta'} \frac{n_h(r)}{r} dr \leq -\log |f_h(i\nu)| + \sup_{|\omega-i\nu|=\nu+\delta+\eta+\eta'} \log |f_h(\omega)|. \tag{52}$$

From Proposition 35 and Lemma 38, we know that the trace class norm of the operator $(\tilde{P}_h(\omega) + A)^{-1}A$ is $\mathcal{O}(h^{-n})$ uniformly in h and in ω on the circle of center $i\nu$ and radius $\nu + \delta + \eta + \eta'$. Then, from [Gohberg et al. 2000, Theorem IV.5.2], we find that

$$\sup_{|\omega-i\nu|=\nu+\delta+\eta+\eta'} \log |f_h(\omega)| \leq Ch^{-n}, \tag{53}$$

for some $C > 0$ and h small enough. In order to estimate $|f_h(i\nu)|$ from below, let us write

$$(I - (\tilde{P}_h(i\nu) + A)^{-1}A)^{-1} = I + \tilde{P}_h(i\nu)^{-1}A.$$

From Proposition 35 and Lemma 38, we see that the trace class operator norm of $(I - (\tilde{P}_h(i\nu) + A)^{-1})^{-1} - I$ is $\mathcal{O}(h^{-n})$. Since

$$f_h(i\nu)^{-1} = \det((I - (\tilde{P}_h(i\nu) + A)^{-1})^{-1}),$$

we find using [Gohberg et al. 2000, Theorem IV.5.2] again that

$$-\log |f_h(i\nu)| \leq Ch^{-n} \tag{54}$$

for some $C > 0$ and h small enough. From (51), (52), (53) and (54), we find that $n_h(\nu + \delta)$ is $\mathcal{O}(h^{-n})$. The result follows since the disk of center 0 and radius δ is contained in the disk of center $i\nu$ and radius $\nu + \delta$. \square

6.6. Summary. Let us put together the definitions from Sections 6.1 and 6.2 and the results from Sections 6.4 and 6.5 to check that Proposition 5 holds.

Proof of Proposition 5. We just need to collect facts that we already proved. We recall that the modification $P_h(\omega)$ of $\mathcal{P}_h(\omega)$ is given by (39). Recalling (41), we let $\mathcal{H}_1 = \{u \in \mathcal{D}'(X) : e^{\psi/h}u \in D_k\} = \{u \in \mathcal{F}_k : P_h(0)u \in \mathcal{F}_{k-1}\}$ and $\mathcal{H}_2 = \mathcal{F}_{k-1}$ (for any value of $k \in \mathbb{R}$).

The inclusions $C^\infty(X) \subseteq \mathcal{H}_j \subseteq \mathcal{D}'(X)$ for $j = 1, 2$ are given by Proposition 21. The fact that the elements of \mathcal{H}_j are continuous near ∂Y follows from Proposition 24.

All the properties needed for $P_h(\omega) : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ follow from the same properties for $\tilde{P}_h(\omega) : D_k \rightarrow \mathcal{H}_\Lambda^{k-1}$. The holomorphic dependence on ω follows from the remark after Proposition 25. The invertibility for $\omega = i\nu$ with a $\nu > 0$ is given by Proposition 35. Point (v) follows from Proposition 36.

Finally, the counting bound (vi) is given by Lemma 37. \square

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