ANALYSIS & PDE

Volume 17 No. 10 2024

msp

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2024 is US $440/year for the electronic version, and $690/year (+$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS AND PDE
Vol. 17 (2024), No. 10, pp. 3371–3446
DOI: 10.2140/apde.2024.17.3371

SCATTERING OF THE THREE-DIMENSIONAL CUBIC NONLINEAR SCHRÖDINGER EQUATION WITH PARTIAL HARMONIC POTENTIALS

XING CHENG, CHANG-YU GUO, ZIHUA GUO, XIAN LIAO AND JIA SHEN

We consider the following three-dimensional defocusing cubic nonlinear Schrödinger equation (NLS) with partial harmonic potential:

$$\begin{cases} i\,\partial_t u + (\Delta_{\mathbb{R}^3} - x^2)u = |u|^2 u, \\ u|_{t=0} = u_0. \end{cases} \tag{NLS}$$

Our main result shows that the solution u scatters for any given initial data u_0 with finite mass and energy.

The main new ingredient in our approach is to approximate (NLS) in the large-scale case by a relevant dispersive continuous resonant (DCR) system. The proof of global well-posedness and scattering of the new (DCR) system is greatly inspired by the fundamental works of Dodson (2012, 2016) in his study of scattering for the mass-critical nonlinear Schrödinger equation. The analysis of (DCR) system allows us to utilize the additional regularity of the smooth nonlinear profile so that the celebrated concentration-compactness/rigidity argument of Kenig and Merle (2006, 2008) applies.

1. Introduction

1.1. *Background and motivation.* Consider the Cauchy problem for the following family of *nonlinear Schrödinger equations* in $\mathbb{R}^d$, $d \in \mathbb{N}$, with *harmonic oscillators*:

$$\begin{cases} i\,\partial_t u + \Delta_{\mathbb{R}^d} u - (\omega^2|y|^2 + |x|^2)u = \mu|u|^{p-1}u, \\ u|_{t=0} = u_0, \end{cases} \tag{1-1}$$

MSC2020: 35Q55, 35P25, 35B40.
Keywords: Schrödinger equation, scattering, partial harmonic potentials, dispersive continuous resonant system, profile decomposition.

where $1 < p < \infty$, $(y, x) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $d = d_1 + d_2$, and $d_1, d_2 \in \mathbb{N}$, $d_1 \geq 1$. The complex-valued function $u = u(t, y, x): \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is the unknown wave function. The parameter ω is 0 or 1, with $\omega = 1$ corresponding to the quadratic potential case and $\omega = 0$ corresponding to the partial harmonic oscillator on the left-hand side. The parameter $\mu = 1$ corresponds to the *defocusing* case and $\mu = -1$ to the *focusing* case. Equation (1-1) arises as models for diverse physical phenomena, including Bose–Einstein condensates in a laboratory trap [Josserand and Pomeau 2001; Pitaevskii and Stringari 2003] and the envelope dynamics of a general dispersive wave in a weakly nonlinear medium. It can also be derived in the NLS with constant magnetic potential; see, for example, [Fukuizumi and Ohta 2003]. The associated conserved mass and energy of (1-1) are given by

$$\mathcal{M}(u)(t) = \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} |u(t, y, x)|^2 \, \mathrm{d}y \, \mathrm{d}x$$

and

$$\mathcal{E}^{d_1,d_2}_{\omega,\mu,p}(u)(t) = \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \tfrac{1}{2}|\nabla_{y,x} u(t, y, x)|^2 + \tfrac{1}{2}(\omega^2|y|^2 + |x|^2)|u(t, y, x)|^2 + \frac{\mu}{p+1}|u(t, y, x)|^{p+1} \, \mathrm{d}y \, \mathrm{d}x.$$

It is natural to take the initial data from the following weighted Sobolev space:

$$u_0 \in \bigl\{ f = f(y, x) \in L^2_{y,x}(\mathbb{R}^d) : \\ \|\nabla_{y,x} f\|_{L^2_{y,x}(\mathbb{R}^d)} + \||x| f\|_{L^2_{y,x}(\mathbb{R}^d)} + \omega \||y| f\|_{L^2_{y,x}(\mathbb{R}^d)} + \|f\|_{L^2_{y,x}(\mathbb{R}^d)} < \infty \bigr\}.$$

In view of the Sobolev embedding

$$H^1(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d), \qquad \begin{cases} 2 \leq q \leq 2 + \frac{4}{d-2} & \text{if } d \geq 3, \\ 2 \leq q < \infty & \text{if } d = 2, \\ 2 \leq q \leq \infty & \text{if } d = 1, \end{cases}$$

the initial data is of finite energy in the *energy-subcritical* case

$$\begin{cases} 1 < p < 1 + \frac{4}{d-2} & \text{if } d \geq 3, \\ 1 < p < \infty & \text{if } d = 1, 2, \end{cases}$$

and we call the critical case $p = 1 + \frac{4}{d-2}$, $d \geq 3$, the *energy critical* case.

The global well-posedness of (1-1) has been established in the energy-subcritical case by R. Carles [2002b; 2008] in the defocusing case $\mu = 1$, and by J. Zhang [2005] in the focusing case $\mu = -1$ when the initial energy is assumed to be less than the energy of the ground state of the related elliptic equation. The Cauchy problem for (1-1) with quadratic potential (that is, $\omega = 1$) in the energy-critical case was considered by R. Killip, M. Visan, and X. Zhang [Killip et al. 2009b] in the radial case, and in the general case later by C. Jao [2016; 2018]. They proved the global well-posedness for the defocusing case and also for the focusing case when the initial energy (resp. kinetic energy) is less than the energy (resp. kinetic energy) of the ground state. We would also like to mention the work of C. Hao, L. Hsiao and H. Li [Hao et al. 2007; 2008], where the authors proved the global well-posedness for (1-1) (when $\omega = 1$) with an additional angular momentum rotational term.

It is well known that solutions of (1-1) with a quadratic potential (i.e., $\omega = 1$) cannot scatter. However, intuitively, in the defocusing case, if we turn off the confinement in *some instead of all* of the directions, it should suffice for the condensate to evolve asymptotically freely: Indeed, if $\omega = 0$, then the operator $i\,\partial_t + \Delta_y$ should yield large time dispersion and one expects a scattering theory for (1-1). When $\omega = 0$, the scattering phenomena for (1-1) in the defocusing case has already been showed by P. Antonelli, R. Carles and J. D. Silva [Antonelli et al. 2015] (see also [Carles and Gallo 2015]) in the fully weighted space when $\omega = 0,\ \mu = 1,\ d_1 = 1, 2, 3,\ d_2 = 1$ and $1 + \frac{4}{d_1} < p < 1 + \frac{4}{d_1 - 1}$. The focusing case of (1-1) has been investigated by A. H. Ardila and R. Carles [2021] recently when the energy is strictly less than the static energy of the ground state. In this aspect, one expects the global-in-time well-posedness result for the defocusing/focusing (when energy is strictly less than the static energy of the ground state) energy-critical and subcritical cases for (1-1). On the other hand, the potential influences strongly the asymptotic dynamics of the solution. In (1-1), the x-direction is not expected to have a global-in-time dispersive estimate in view of Mehler's formula

$$e^{it(\Delta_x - |x|^2)} f(y, x) = (2\pi i \sin(2t))^{-\frac{d_2}{2}} \int_{\mathbb{R}^{d_2}} e^{\frac{i}{\sin(2t)}\left(\frac{|x|^2 + |\tilde{x}|^2}{2}\cos(2t) - x\cdot\tilde{x}\right)} f(y, \tilde{x})\,\mathrm{d}\tilde{x} \quad \text{for all } y \in \mathbb{R}^{d_1},\ x \in \mathbb{R}^{d_2},$$

from which we can only derive the following periodic-in-time dispersive estimate:

$$\|e^{it(\Delta_x - |x|^2)} f(y, x)\|_{L_x^\infty(\mathbb{R}^{d_2})} \lesssim |\sin(2t)|^{-\frac{d_2}{2}} \|f(y, x)\|_{L_x^1(\mathbb{R}^{d_2})} \quad \text{for all } t \notin \tfrac{\pi}{2}\mathbb{Z}, \text{ for all } y \in \mathbb{R}^{d_1}.$$

Nevertheless, we have the following global-in-time dispersive estimate in the y-direction:

$$\|e^{it(\Delta_x + \Delta_y - |x|^2)} f(y, x)\|_{L_y^\infty L_x^2(\mathbb{R}^d)} \lesssim |t|^{-\frac{d_1}{2}} \|f(y, x)\|_{L_y^1 L_x^2(\mathbb{R}^d)},$$

where we used the dispersive estimate for the semigroup $e^{it\Delta_y}$ together with the L^2-norm conservation for the unitary of the operator $e^{it(\Delta_x - |x|^2)}$. Thus, according to the scattering theory for the nonlinear Schrödinger equations without potential, see for instance [Staffilani 2013; Tao 2006], one *expects a scattering result in the weighted Sobolev space* when $\omega = 0$ in the case $1 + \frac{4}{d_1} \le p \le 1 + \frac{4}{d_1 + d_2 - 2}$, with $d_1 + d_2 \ge 2$. Generally, to obtain the scattering in the intercritical case, one relies on the Morawetz estimate; see for instance [Antonelli et al. 2015]. It is difficult to deal with the scattering on the two endpoints $p = 1 + \frac{4}{d_1}$ and $p = 1 + \frac{4}{d_1 + d_2 - 2}$, which correspond to the usual d_1-dimensional mass-critical and $(d_1 + d_2)$-dimensional energy-critical nonlinear Schrödinger equation without potentials respectively. For the endpoint $p = 1 + \frac{4}{d_1 + d_2 - 2}$, the scattering is a byproduct of the proof of the global well-posedness, and we need to use induction on the energy method [Colliander et al. 2008] or the concentration-compactness/rigidity argument [Kenig and Merle 2006; 2008] to prove the global well-posedness. It seems to us one of the main difficulties is to establish a more delicate global-in-time Strichartz estimate which should be a lot combination of the local Strichartz estimate of three-dimensional Schrödinger equations as in [Barron 2021; Hani and Pausader 2014]. We refer to [Killip and Vişan 2013] for more illustration on the proof of the scattering of the nonlinear Schrödinger equations at critical regularity Sobolev space. For the endpoint $p = 1 + \frac{4}{d_1}$, global well-posedness is quite easy to get, and the main obstacle is to show the scattering. We cannot prove the scattering by the Morawetz estimate even when the initial data lies in a better regular Sobolev space $H^1_{y,x}$ because the Morawetz estimate only provides an a priori estimate of the nonendpoint Strichartz norm on the $\dot{H}_y^{1/4}L_x^2$-level but cannot give an a priori estimate of the Strichartz norm on the L^2-level, which is not enough to yield the scattering in this case. Therefore, to show the scattering, we still need to use the concentration-compactness/rigidity argument [Kenig and Merle 2006; Kenig and Merle 2008] and its mass-critical counterpart [Dodson 2012; 2016a; 2016b; Killip et al. 2008; 2009a; Killip and Vişan 2013; Tao et al. 2007a; 2008] to show the finiteness of the L^2-level Strichartz norm. In the L^2-level Strichartz norm, we need to consider not only the space and time translations of (1-1) as in the case $1+\frac{4}{d_1} < p \le 1+\frac{4}{d_1+d_2-2}$, but also the partial Galilean invariance

$$u(t,y,x) \mapsto e^{-it|\xi_0|^2} e^{iy\cdot\xi_0} u(t, y-2\xi_0 t, x),$$

where $\xi_0 \in \mathbb{R}^{d_1}$, of (1-1). In addition, by a limitation operation, it is realized that a new mass-critical nonlinear Schrödinger system can be embedded into (1-1): this new mass-critical nonlinear Schrödinger system inherits the above invariance and also has the scaling invariance in space-time, and its global well-posedness and scattering should be proven by the argument from [Dodson 2012; 2016a; 2016b; Killip et al. 2008; 2009a; Killip and Vişan 2013; Tao et al. 2007a; 2008].

In this paper, we will consider the following Cauchy problem for the defocusing cubic NLS on $\mathbb{R}^3$:

$$\begin{cases} i\partial_t u + (\Delta_{\mathbb{R}^3} - x^2)u = |u|^2 u, \\ u|_{t=0} = u_0, \end{cases} \tag{1-2}$$

where $u = u(t,y,x)\colon \mathbb{R}\times\mathbb{R}^2\times\mathbb{R} \to \mathbb{C}$ is an unknown wave function. The following mass and energy quantities are conserved by the evolution of (1-2):

$$\begin{aligned} \mathcal{M}(u(t)) &= \int_{\mathbb{R}^2\times\mathbb{R}} |u(t,y,x)|^2 \,dy\,dx, \\ \mathcal{E}(u(t)) &= \int_{\mathbb{R}^2\times\mathbb{R}} \tfrac{1}{2}|\nabla_{y,x} u(t,y,x)|^2 + \tfrac{1}{2}x^2|u(t,y,x)|^2 + \tfrac{1}{4}|u(t,y,x)|^4 \,dy\,dx. \end{aligned} \tag{ME}$$

Motivated by the mass and energy formulations, we take the initial data in the following weighted Sobolev space:

$$u_0 \in \Sigma(\mathbb{R}^3) := \{ f \in L^2_{y,x}(\mathbb{R}^3) : \|f\|_{\Sigma(\mathbb{R}^3)} := \|\nabla_y f\|_{L^2_{y,x}(\mathbb{R}^3)} + \|f\|_{L^2_y \mathcal{H}^1_x(\mathbb{R}^2\times\mathbb{R})} < \infty \},$$
$$\text{with } \|f\|_{\mathcal{H}^1_x(\mathbb{R})} = \|f\|_{H^1_x(\mathbb{R})} + \|xf\|_{L^2_x(\mathbb{R})}. \tag{1-3}$$

By the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$, $2 \le q \le 6$, the initial data is of finite mass and energy.

Observe that (1-2) is a special case of (1-1), namely, corresponding to $d_1 = 2$, $d_2 = 1$, $\omega = 0$, $\mu = 1$, $p = 1 + \frac{4}{d_1} = 3$ in (1-1). In this case, the scattering phenomena is not yet clear. As we are in the energy subcritical case $1 < p = 3 < 5$, the equation (1-2) is globally well-posed and the scattering of the solutions follows in the small initial data case $\|u_0\|_\Sigma \ll 1$, which is a byproduct of the small-data well-posedness theorem. We will briefly explore these results in Section 3 and outline the ideas of the proofs, as we did not find them in the literature.

1.2. *Main results.* Our main result in this article is the following scattering result for solutions of the defocusing cubic NLS (1-2). Recall that $\Sigma(\mathbb{R}^3)$ is defined in (1-3).

Theorem 1.1. *For any initial data $u_0 \in \Sigma(\mathbb{R}^3)$, there is a unique global solution $u \in C_t^0(\mathbb{R}, \Sigma(\mathbb{R}^3))$ of (1-2). Moreover, the solution scatters; namely there exist $u_\pm \in \Sigma(\mathbb{R}^3)$ such that*

$$\|u(t) - e^{it(\Delta_{\mathbb{R}^3} - x^2)} u_\pm\|_{\Sigma(\mathbb{R}^3)} \to 0 \quad \text{as } t \to \pm\infty.$$

In order to treat the general initial data with finite (but not necessarily small) Σ-norm $\|u_0\|_\Sigma < \infty$, we turn to the celebrated concentration-compactness/rigidity argument developed by C. E. Kenig and F. Merle [2006; 2008], where one key ingredient is the linear and nonlinear profile decompositions for solutions with bounded Σ-norm. The proof of Theorem 1.1 shall rely on (a corollary of) Theorem 1.2 given below. More precisely, we shall use Theorem 1.2 to prove the core result Theorem 4.9 in Section 4.2, which in return gives Theorem 4.10 in Section 4.3. Theorem 4.10 will be used later in the proof of Theorem 1.1 in Section 5.

As for the nonlinear profile decomposition, we will consider a sequence of solutions exhibiting an extreme behavior to study the concentration of the data. More precisely, we need to study the behavior of the nonlinear profile u_λ when $\lambda \to \infty$. The (simplified) nonlinear profile u_λ, $\lambda > 0$, is the solution of (1-2)

$$\begin{cases} i\partial_t u_\lambda + \Delta_y u_\lambda + (\Delta_x - x^2) u_\lambda = |u_\lambda|^2 u_\lambda, \\ u_\lambda(0, y, x) = \frac{1}{\lambda}\phi\big(\frac{y}{\lambda}, x\big), \end{cases} \tag{1-4}$$

taking the initial data by rescaling the function ϕ only in the y-variable. Set

$$w_\lambda(t, y, x) = e^{-it(\Delta_x - x^2)} u_\lambda(t, y, x),$$

and we obtain from (1-4) the following evolutionary equation for w_λ:

$$\begin{cases} (i\partial_t + \Delta_y) w_\lambda = e^{-it(\Delta_x - x^2)}(|e^{it(\Delta_x - x^2)} w_\lambda|^2 e^{it(\Delta_x - x^2)} w_\lambda), \\ w_\lambda(0, y, x) = \frac{1}{\lambda}\phi\big(\frac{y}{\lambda}, x\big). \end{cases}$$

If we define $w_\lambda(t, y, x) = \frac{\tilde{v}}{\lambda}\big(\frac{t}{\lambda^2}, \frac{y}{\lambda}, x\big)$, then $\tilde{v}$ satisfies

$$\begin{cases} (i\partial_t + \Delta_y)\tilde{v} = e^{-i\lambda^2 t(\Delta_x - x^2)}(|e^{i\lambda^2 t(\Delta_x - x^2)}\tilde{v}|^2 e^{i\lambda^2 t(\Delta_x - x^2)}\tilde{v}), \\ \tilde{v}(0, y, x) = \phi(y, x). \end{cases}$$

Denote by Π_n the orthogonal projector on the n-th eigenspace of $-\Delta_x + x^2$ (see Section 2 below for more details). Applying Π_n to the equation for $\tilde{v}$, we arrive at the following equation for $\tilde{v}_n = \Pi_n \tilde{v}$:

$$\begin{cases} (i\partial_t + \Delta_y)\tilde{v}_n = e^{i\lambda^2 t(2n+1)}\Pi_n\big(\sum_{n_1, n_2, n_3 \in \mathbb{N}} e^{-i\lambda^2(2n_1 - 2n_2 + 2n_3 + 1)t}\, \tilde{v}_{n_1}\bar{\tilde{v}}_{n_2}\tilde{v}_{n_3}\big), \\ \tilde{v}_n(0, y, x) = \phi_n(y, x) := \Pi_n \phi(y, x). \end{cases}$$

Letting $\lambda \to \infty$, we can formally get a limiting equation

$$\begin{cases} (i\partial_t + \Delta_y) v_n(t, y, x) = \sum_{\substack{n_1, n_2, n_3 \in \mathbb{N} \\ n_1 - n_2 + n_3 = n}} \Pi_n(v_{n_1}\bar{v}_{n_2} v_{n_3})(t, y, x), \\ v_n(0, y, x) = \phi_n(y, x). \end{cases} \tag{1-5}$$

By reversing the above process, we get an approximation solution of u_λ:

$$\tilde{u}_\lambda(t,y,x) = e^{it(\Delta_x - x^2)} \sum_{n\in\mathbb{N}} \left(\frac{1}{\lambda} v_n \left(\frac{t}{\lambda^2}, \frac{y}{\lambda}, x \right) \right), \quad (t,y,x) \in \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}, \tag{1-6}$$

where v_n is the solution of (1-5).

In the above deduction, the following equivalent form dispersive continuous resonant (DCR) system enters naturally:

$$\begin{cases} i\,\partial_t v + \Delta_{\mathbb{R}^2} v = F(v), \\ v(0,y,x) = \phi(y,x), \end{cases} \tag{DCR}$$

where the nonlinear term $F(v)$ is defined by

$$F(v) := \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N} \\ n_1-n_2+n_3=n}} \Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3}).$$

This (DCR) system can be viewed as a dispersive version of the (CR) system derived by E. Faou, P. Germain, and Z. Hani [Faou et al. 2016] in their study of the weak turbulence of the nonlinear Schrödinger equations on compact domains; see also [Buckmaster et al. 2019; Colliander et al. 2010; Germain et al. 2015; 2016; Dartois et al. 2020; Fennell 2019]. This new (DCR) system is very similar to the resonant nonlinear Schrödinger system arising in [Biasi et al. 2018; Cheng et al. 2020a; 2020b; Hani and Pausader 2014; Hani et al. 2015]. The latter has nice local well-posedness theory, and also scatters for small data in $L_y^2\mathcal{H}_x^1$.

In our second main result, we prove the following large-data global well-posedness and scattering theorem for (DCR), which might be of independent interest.

Theorem 1.2. *For any $\phi \in L_y^2\mathcal{H}_x^1(\mathbb{R}^2\times\mathbb{R})$, there exists a unique global solution v of* (DCR) *in $C_t^0L_y^2\mathcal{H}_x^1(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})$ satisfying*

$$\|v\|_{L_t^\infty L_y^2\mathcal{H}_x^1 \cap L_{t,y}^4\mathcal{H}_x^1(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} \le C,$$

where $C = C(\|\phi\|_{L_y^2\mathcal{H}_x^1})$ is a constant. Moreover, the solution scatters; namely there exist $v_\pm \in L_y^2\mathcal{H}_x^1$ such that

$$\|v(t) - e^{it\Delta_y}v_\pm\|_{L_y^2\mathcal{H}_x^1(\mathbb{R}^2\times\mathbb{R})} \to 0 \quad \textit{as } t\to\pm\infty.$$

Theorem 1.2 shall be proved in the final two sections and it takes a vast bulk of the paper. We prove it again by the concentration-compactness/rigidity argument from [Kenig and Merle 2006; 2008]. The system (DCR) is essentially a defocusing mass-critical nonlinear Schrödinger system. In the proof, we follow the framework for scattering of mass-critical nonlinear Schrödinger equation [Dodson 2012; 2016a; 2016b; Tao et al. 2008] and our argument is also partly inspired by the scattering of the resonant Schrödinger system derived from the NLS on cylinders [Cheng et al. 2020a; 2020b; Hani and Pausader 2014; Hani et al. 2015; Yang and Zhao 2018; Zhao 2019].

We would like to comment briefly on the relation between (DCR) and weak turbulence.

Remark 1.3 (the (DCR) system and weak turbulence). We can rewrite (1-5) in the Hermite coordinate (see (2-1) below for the definition of the Hermite functions): taking the solution $v_n(t,y,x) = c_n(t,y)h_n(x)$ in (1-5), we get an equivalent but simplified equation

$$(i\partial_t + \Delta_{\mathbb{R}^2})c_n(t,y) = \sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}} D_{n_1,n_2,n_3,n}\, c_{n_1}\bar{c}_{n_2}c_{n_3}, \tag{1-7}$$

where $D_{n_1,n_2,n_3,n}$ is the number such that $\Pi_n(h_{n_1}\bar{h}_{n_2}h_{n_3})(x) = D_{n_1,n_2,n_3,n}h_n(x)$, $x\in\mathbb{R}$. It would be very interesting to understand $D_{n_1,n_2,n_3,n}$.[1] Compared with the success of the proof of the weak turbulence on cylinders given by Z. Hani, B. Pausader, N. Tzvetkov, and N. Visciglia [Hani et al. 2015], the unclear expression of the nonlinear term of the (DCR) system seems to be one of the main obstacles to study the weak turbulence of the nonlinear Schrödinger equations with (partial) harmonic potentials; for more information we refer to [Hani and Thomann 2016]. However, there are some interesting recent attempts toward this direction in [Gérard et al. 2019; Germain and Thomann 2016].

Remark 1.4 (focusing NLS equations with harmonic potentials). In this paper, we only consider the scattering of the defocusing NLS with partial harmonic potentials. It is an interesting problem to study the scattering of the focusing version of (1-2). It seems difficult to find the threshold of the scattering of the focusing NLS, and if we were able to find it, then most likely the scattering can be proven by following the argument in [Dodson 2012; 2015; 2016a; 2016b; Killip et al. 2009a; Tao et al. 2007a; 2008]. We refer to [Ardila and Carles 2021; Bellazzini et al. 2017; Cao et al. 2022; Zhang 2020; Stanislavova and Stefanov 2021] for the study of the instability/stability of soliton which may give some clues on the threshold of the scattering of the focusing NLS.

1.3. ***Brief outline of the proofs.*** The model with partial harmonic potential studied in this paper can be compared to the NLS on wave-guide $\mathbb{R}^2\times\mathbb{T}$, which was considered previously in [Yang and Zhao 2018; Cheng et al. 2020a]. One key difference is that in our case, the linear operator has more complicated spectral theory; for example the eigenfunctions cannot be written explicitly.

The proof of this paper contains two main ingredients. In the first part, we prove that Theorem 1.2 (or more precisely, the consequence Theorem 4.10 of Theorem 1.2) implies Theorem 1.1. The proof of Theorem 1.1 has a very standard skeleton based on the concentration-compactness/rigidity argument introduced by C. Kenig and F. Merle [2006], and it consists of three main steps: linear profile decomposition, the existence of an almost periodic solution to the defocusing cubic NLS (1-2), and a rigidity theorem.

First of all, we establish the linear profile decomposition of Schrödinger operator with partial harmonic potentials; namely the linear solutions can be divided into several orthogonal bubbles modulo some transforms. This can be viewed as a vector-valued version of linear profile decomposition of the Schrödinger equation in L^2, which was first established by F. Merle and L. Vega [1998] in two dimensions, and then extended to general dimensions; see for instance [Killip and Vişan 2013] for more details. The proof of this part is very similar to the wave-guide case in [Cheng et al. 2020a], and it is essentially related to the

[1] In [Biasi et al. 2019a; 2019b; Evnin 2020], the authors studied a special structure of the constant $D_{n_1,n_2,n_3,n}$, and proved that it satisfies a certain identity.

description of the lack of compactness of the embedding $e^{it(\Delta_{\mathbb{R}^3}-x^2)}:\Sigma(\mathbb{R}^3)\hookrightarrow L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x$ for some fixed $0<\epsilon_0<\frac{1}{2}$.

In the second step, we prove the existence of a critical element by the construction of approximation solutions. Since the nonlinear flow is not commutable with the transform groups derived in the first step, in order to construct the approximation solutions, we need to assume that the limiting equations, which is exactly the (DCR) system, is globally well-posed and scatters, as stated in Theorem 1.2. The idea of using limiting equations was first considered in [Ibrahim et al. 2011], and was widely used in [Cheng et al. 2020a; 2020b; Hani and Pausader 2014; Ionescu and Pausader 2012; Jao 2016]. Then, much as in [Cheng et al. 2020a], we use the normal form method to exploit additional decay to approximate the nonlinear profile. In the wave-guide case [Cheng et al. 2020a], the eigenfunctions, which are the plain waves $e^{iy\cdot j}$, can be easily computed, and thus the Fourier coefficients are summed naturally. The difficulty in this step is that we need to sum up the spectral projections of the solution properly. To some extent, the main innovation of this paper is that we utilize the additional regularity of the smooth nonlinear profile to update the l^1 summation of projections to l^2.

In the third step, we borrow the idea used in [Cheng et al. 2020a] to prove the nonexistence of a nontrivial critical element. The key point is the use of the interaction Morawetz estimate developed by J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao [Colliander et al. 2004], which is very important in the remarkable work [Colliander et al. 2008] on scattering for energy-critical NLS in three dimensions, and was further developed in [Planchon and Vega 2009; Colliander et al. 2009]. Then, we can arrive at the contradiction similar to [Kenig and Merle 2006; 2008] using the compactness property of the critical element.

The second part of this paper is devoted to the proof of Theorem 1.2. The proof is greatly inspired by the fundamental work of B. Dodson [2012; 2016a; 2016b] in his study of mass critical NLS. We also refer to [Yang and Zhao 2018], and the principal difference between that work and this paper is that our system (DCR) involves the spectral projection of Schrödinger operator with harmonic potential. Here, one key observation is that the (DCR) system is scaling invariant, which indicates that the classical method as developed in [Cheng et al. 2020a; 2020b; Tao et al. 2008] could be potentially applied to our situation. Indeed, the linear profile decomposition developed for the Schrödinger propagator in $L^2_y\mathcal{H}^1_x(\mathbb{R}^2\times\mathbb{R})$ (see Theorem 4.1) can be directly applied here. The essential difficulty occurring in the proof of Theorem 1.2 lies in precluding the almost periodic solution to the (DCR) system.

There are two cases of the critical element: high-to-low frequency cascade and the quasisoliton scenarios. We exclude these scenarios based on the rigidity argument of B. Dodson [2012; 2016a; 2016b]. The key tool is to establish a vector-valued version of the two-dimensional long-time Strichartz estimate in [Dodson 2016b]. The long-time Strichartz estimate is developed by B. Dodson to show the scattering of the mass-critical nonlinear Schrödinger equations and has been proved as an important technique in the scattering theory of nonlinear dispersive and wave equation. We refer to [Dodson 2019; Dodson and Lawrie 2015; Dodson et al. 2017; 2020; Killip and Vişan 2012; Vişan 2012; Murphy 2014; Rosenzweig 2018] for more applications of this powerful tool. The proof of the long-time Strichartz estimate in our situation here is rather technical due to the spectral projection and the failure of the two-dimensional endpoint Strichartz estimate. For the high-to-low frequency cascade scenario, it is more delicate and we have to exploit some additional regularity of the critical element through the long-time Strichartz estimate, and then preclude it using energy conservation law. For the quasisoliton scenario, we mainly use the long-time Strichartz estimate to control the error terms of low frequency cut-off of the interaction Morawetz identity. With all these ingredients at hand, the contradiction argument of C. E. Kenig and F. Merle [2006; 2008] allows us to conclude the proof.

The rest of the paper is organized as follows. Section 2 contains some basic notation and preliminaries. In Section 3, we record the local well-posedness, the small-data scattering result and the stability theory for system (1-2). For the convenience of the readers, we present the proofs in the Appendix. In Section 4, we will give the linear profile decomposition for data in $\Sigma(\mathbb{R}^3)$ and also analyze the nonlinear profiles; therefore we reduce the nonscattering in $\Sigma(\mathbb{R}^3)$ to the existence of an almost-periodic solution. In Section 5, we will show the extinction of such an almost-periodic solution. The scattering of the (DCR) system shall be proved in Section 6, where the proofs of two auxiliary theorems are left to the final Section 7.

2. Basic notation and preliminaries

In this section, we introduce some basic notation used in this paper. We will use the notation $X \lesssim Y$ whenever there exists some constant $C > 0$ so that $X \le CY$. Similarly, we will write $X \sim Y$ if $X \lesssim Y \lesssim X$. We use $\mathbb{N}$ to denote the set of all nonnegative integers.

Throughout the paper, we will take ϵ_0 to be some small fixed number in $\left(0, \frac{1}{2}\right)$.

2.1. ***Fourier transform and Sobolev spaces.*** For any $a \in \mathbb{R}^d$, $d \in \mathbb{N}$, the Japanese bracket $\langle a\rangle$ is defined to be $\langle a\rangle = (1+|a|^2)^{1/2}$. We define the Fourier transform $\hat{f}\colon \mathbb{R}^d \to \mathbb{C}$ of a function $f\colon \mathbb{R}^d \to \mathbb{C}$ as

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-iz\cdot\xi} f(z)\,\mathrm{d}z.$$

For each $s \in \mathbb{R}$, the fractional differential operator $|\nabla|^s$ is defined by $\widehat{|\nabla|^s f}(\xi) = |\xi|^s \hat{f}(\xi)$. We also define $\langle\nabla\rangle^s$ as an operator between function spaces by $\widehat{\langle\nabla\rangle^s f}(\xi) = (1+|\xi|^2)^{s/2}\hat{f}(\xi)$. In the following we will use $\langle\nabla_x\rangle^s$ to emphasize the application of the operator on the x-variable.

We will frequently use the partial Fourier transform $\mathcal{F}_y f$ of a complex-valued function $f\colon \mathbb{R}^2\times\mathbb{R}\to\mathbb{C}$ defined as

$$\mathcal{F}_y f(\xi, x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-iy\cdot\xi} f(y,x)\,\mathrm{d}y, \quad \xi \in \mathbb{R}^2,$$

where $x \in \mathbb{R}$ is viewed as a parameter.

We shall also use the Littlewood–Paley projections. Take a cut-off function $\chi \in C^\infty((0,\infty))$ such that $\chi(r) = 1$ if $r \leqslant 1$ and $\chi(r) = 0$ if $r > 2$. For $N \in 2^{\mathbb{Z}}$, let $\chi_N(r) = \chi(N^{-1}r)$ and $\phi_N(r) = \chi_N(r) - \chi_{N/2}(r)$. We define the Littlewood–Paley dyadic operator $P_{\leqslant N} f := \mathcal{F}^{-1}(\chi_N(|\xi|)\hat{f}(\xi))$ and $P_N f := \mathcal{F}^{-1}(\phi_N(|\xi|)\hat{f}(\xi))$. We also define the partial Littlewood–Paley projections to be $P^y_{\le N} f(y,x) := \mathcal{F}_y^{-1}(\chi_N(\xi)(\mathcal{F}_y f)(\xi,x))$ and $P^y_N f(y,x) := \mathcal{F}_y^{-1}(\phi_N(|\xi|)(\mathcal{F}_y f)(\xi,x))$.

Next, we denote the usual Lebesgue space as $L^p(\mathbb{R}^d)$, and sometimes we write $\|f\|_p = \|f\|_{L^p(\mathbb{R}^d)}$ for abbreviation. For any $s \in \mathbb{R}$, we define the Sobolev space as

$$W^{s,p}(\mathbb{R}^d) := \{f \in L^p(\mathbb{R}^d) : \|f\|_{W^{s,p}(\mathbb{R}^d)} := \|\langle\nabla\rangle^s f\|_{L^p(\mathbb{R}^d)} < +\infty\}.$$

We also define $H^s(\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d)$.

2.2. ***Harmonic oscillator and Hermite-Sobolev spaces.*** The harmonic oscillator $-\Delta_x + x^2,\ x \in \mathbb{R}$, has been studied by many authors, and we refer to the lecture notes of B. Helffer [1988] and also the seminal work of H. Koch and D. Tataru [2005b] for a few basic facts that we shall record below. The harmonic oscillator admits a Hilbertian basis of eigenvectors for $L^2(\mathbb{R})$, and, for each $n \in \mathbb{N}$, we will denote the n-th eigenspace by E_n and the corresponding eigenvalue by $\lambda_n = 2n+1$. Each eigenspace E_n is spanned by the Hermite functions h_n, where

$$h_n(x) = \frac{1}{\sqrt{n!}\,2^{\frac{n}{2}}\pi^{\frac{1}{4}}}(-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-x^2}) \tag{2-1}$$

for $n \in \mathbb{N}$. We also let Π_n be the orthogonal projector on the n-th eigenspace E_n of $-\Delta_x + x^2$.

For $s \in \mathbb{R}$ and $p \geq 1$, the Hermite-Sobolev space $\mathcal{W}^{s,p}(\mathbb{R})$ is defined as

$$\mathcal{W}^{s,p}(\mathbb{R}) = \{u \in L^p_x(\mathbb{R}) : \|u\|_{\mathcal{W}^{s,p}} := \|\langle\nabla\rangle^s u\|_{L^p_x} + \||\cdot|^s u\|_{L^p_x} < \infty\}.$$

In particular, if $p = 2$, we denote $\mathcal{W}^{s,2}_x(\mathbb{R})$ by $\mathcal{H}^s_x(\mathbb{R})$, and the $\mathcal{H}^1_x(\mathbb{R})$-norm was given in (1-3). By [Yajima and Zhang 2004], we have

$$\|u\|_{\mathcal{W}^{s,p}_x} \sim \|(-\Delta + x^2)^{\frac{s}{2}} u\|_p + \|u\|_p.$$

The Hermite-Sobolev spaces satisfy the usual Sobolev embedding; see for instance [Cazenave 2003]. These spaces also have other stronger Hermite-Sobolev embedding. For instance, we have

$$L^4_x(\mathbb{R}) \hookrightarrow \mathcal{H}^{-1}(\mathbb{R}). \tag{2-2}$$

By duality, to prove (2-2), we only need to show $\mathcal{H}^1(\mathbb{R}) \hookrightarrow L^{4/3}(\mathbb{R})$. This follows from Hölder's inequality as follows:

$$\|f\|_{L^{4/3}} \lesssim \|\langle x\rangle^{-1}\|_{L^4_x} \|\langle x\rangle f\|_{L^2_x} \lesssim \|(1+|x|^2)^{\frac{1}{2}} f\|_{L^2_x} \lesssim \|f\|_{\mathcal{H}^1_x}.$$

The Hermite-Sobolev space $L^p_y\mathcal{H}^s_x$ with $1 \leq p < \infty$ and $s \in \mathbb{R}$ is defined by

$$L^p_y\mathcal{H}^s_x = \Big\{f \in L^p_y L^2_x(\mathbb{R}^2\times\mathbb{R}) : \|f\|_{L^p_y\mathcal{H}^s_x(\mathbb{R}^2\times\mathbb{R})} := \Big(\int_{\mathbb{R}^2} \|f(y,\cdot)\|^p_{\mathcal{H}^s_x(\mathbb{R})}\,\mathrm{d}y\Big)^{\frac{1}{p}} \\ = \Big(\int_{\mathbb{R}^2} \Big\|\Big(\sum_{n\in\mathbb{N}} (2n+1)^s |f_n(y,x)|^2\Big)^{\frac{1}{2}}\Big\|^p_{L^2_x(\mathbb{R})}\,\mathrm{d}y\Big)^{\frac{1}{p}} < \infty\Big\},$$

where $f_n = \Pi_n f$. Similarly, for any time interval $I \subseteq \mathbb{R}$ and $u : I \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}$, we define the space-time norms $L_t^p W_y^{s,q} L_x^r$ and $L_t^p L_y^q \mathcal{H}_x^s$ of u as

$$\|u\|_{L_t^p W_y^{s,q} L_x^r(I\times\mathbb{R}^2\times\mathbb{R})} := \left(\int_I \left(\int_{\mathbb{R}^2} \|\langle\nabla_y\rangle^s u(t,y,\cdot)\|_{L_x^r(\mathbb{R})}^q \, \mathrm{d}y\right)^{\frac{p}{q}} \mathrm{d}t\right)^{\frac{1}{p}},$$

$$\|u\|_{L_t^p L_y^q \mathcal{H}_x^s(I\times\mathbb{R}^2\times\mathbb{R})} := \left(\int_I \left(\int_{\mathbb{R}^2} \|u(t,y,\cdot)\|_{\mathcal{H}_x^s(\mathbb{R})}^q \, \mathrm{d}y\right)^{\frac{p}{q}} \mathrm{d}t\right)^{\frac{1}{p}},$$

where $1 \le p,q,r \le \infty$, and $s \in \mathbb{R}$. When $s = 0$ and $p = q = r$, we shall write $L_{t,y,x}^p$ for $L_t^p W_y^{s,p} L_x^r$. Similarly, when $p = q$, we shall write $L_{t,y}^p \mathcal{H}_x^s$ for $L_t^p L_y^q \mathcal{H}_x^s$. We also use the following space-time norm. For any $\{u_n(t,y,x)\}_{n\in\mathbb{N}}$, with $(t,y,x) \in I \times \mathbb{R}^2 \times \mathbb{R}$, we set

$$\|u_n\|_{L_t^p L_y^q L_x^r l_n^2(I\times\mathbb{R}^2\times\mathbb{R}\times\mathbb{N})} = \big\| \|u_n\|_{l_n^2} \big\|_{L_t^p L_y^q L_x^r(I\times\mathbb{R}^2\times\mathbb{R})},$$

where $1 \le p,q,r \le \infty$.

Lemma 2.1. *The Dirac function $\delta_0(x)$ belongs to $\mathcal{H}_x^{-1}(\mathbb{R})$.*

Proof. By definition, we have

$$\|\delta_0(x)\|_{\mathcal{H}_x^{-1}}^2 = \sum_{n=0}^{\infty} (2n+1)^{-1} |c_n|^2, \tag{2-3}$$

where $c_n = \langle \delta_0(x), h_n(x)\rangle = h_n(0)$. Since

$$e^{-x^2} = \sum_{m=0}^{\infty} \frac{(-x^2)^m}{m!} = \sum_{n=0}^{\infty} \frac{d^n}{dx^n}\Big|_{x=0} e^{-x^2} \cdot \frac{x^n}{n!},$$

we have

$$\frac{d^n}{dx^n}\Big|_{x=0} e^{-x^2} = \begin{cases} 0, & n \text{ is odd}, \\ \dfrac{(-1)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} n!, & n \text{ is even}. \end{cases}$$

Thus

$$h_n(0) = \begin{cases} 0, & n \text{ is odd}, \\ \dfrac{(-1)^n}{\sqrt{n!}\, 2^{\frac{n}{2}} \pi^{\frac{1}{4}}} \dfrac{(-1)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} n!, & n \text{ is even}. \end{cases}$$

Together with (2-3), this implies

$$\|\delta_0(x)\|_{\mathcal{H}_x^{-1}}^2 \le \pi^{-\frac{1}{4}} \sum_{\substack{n=0,\\ n \text{ even}}}^{\infty} \frac{n!}{2^n \left(\left(\frac{n}{2}\right)!\right)^2 (2n+1)} \lesssim \sum_{m=0}^{\infty} \frac{1}{2^m(4m+1)} \lesssim 1. \qquad \square$$

3. Local well-posedness and small-data scattering

In this section, we will review the local well-posedness theorem and the stability theorem for solutions of (1-2), which shall be crucial in proving the existence of the critical element, and then record another important theorem on the scattering norm in Theorem 3.4, which says that a weak space-time norm $L_{t,y}^4 \mathcal{H}_x^{1-\epsilon_0}$

is sufficient to prove the scattering result. We shall only state these results in this section and leave the proofs to the Appendix. In fact, the results in this section can be proved by following the exact arguments as in [Cheng et al. 2020a, Section 2; 2020b], upon noticing the embedding $\mathcal{H}^{(1/2)+}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$.

Different from the Strichartz estimate for the harmonic oscillator, which is a local estimate, we have a global Strichartz estimate for the partial harmonic oscillator similar to the Schrödinger equation on waveguides [Cheng et al. 2020a; 2020b; Tarulli 2017; Tzvetkov and Visciglia 2012]. Before giving the Strichartz estimate, we first introduce the following definition.

Definition 3.1 (Strichartz admissible pair). We call a pair (p,q) Strichartz admissible if $2 < p \le \infty$, $2 \le q < \infty$, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$.

We can now state the Strichartz estimate. The proof is almost identical to [Tzvetkov and Visciglia 2012, proof of Proposition 2.1]; we also refer to Proposition 3.1 in [Antonelli et al. 2015], and we omit the proof here.

Proposition 3.2 (Strichartz estimate for the partial harmonic oscillator). *For any Strichartz admissible pair (p,q), we have*

$$\|e^{it(\Delta_{\mathbb{R}^3}-x^2)} f(y,x)\|_{L_t^p L_y^q L_x^2(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} \lesssim \|f\|_{L^2_{y,x}}.$$

Meanwhile, for $\alpha = 0, 1$, it holds

$$\|e^{it\Delta_y} f(y,x)\|_{L_t^p L_y^q \mathcal{H}_x^\alpha(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} \lesssim \|f\|_{L_y^2 \mathcal{H}_x^\alpha(\mathbb{R}^2\times\mathbb{R})}.$$

The following nonlinear estimate, which follows from the Hölder and Sobolev inequalities, is useful in showing the local well-posedness result.

Proposition 3.3 (nonlinear estimate). *For any $0 < \epsilon_0 < \frac{1}{2}$, we have*

$$\|u_1 u_2 u_3\|_{L_{t,y}^{4/3}\mathcal{H}_x^{1-\epsilon_0}} \lesssim \|u_1\|_{L_{t,y}^4\mathcal{H}_x^{1-\epsilon_0}} \|u_2\|_{L_{t,y}^4\mathcal{H}_x^{1-\epsilon_0}} \|u_3\|_{L_{t,y}^4\mathcal{H}_x^{1-\epsilon_0}}.$$

Using Propositions 3.2 and 3.3, one can easily prove the following local well-posedness and small-data scattering in $L_y^2\mathcal{H}_x^1(\mathbb{R}^2\times\mathbb{R})$ and $\Sigma(\mathbb{R}^3)$. The local solution can be extended to be global by the conservation of mass and energy; we refer to [Carles 2002b; Tao 2006]. The proof of the local well-posedness is given in the Appendix; see also [Antonelli et al. 2015; Ardila and Carles 2021; Carles 2002a; 2002b; 2003; 2011; Carles and Gallo 2015] for a comparison.

Theorem 3.4 (LWP and scattering in $L_y^2\mathcal{H}_x^1$ and Σ).

(1) (*well-posedness*) *Let $u_0 \in L_y^2\mathcal{H}_x^1$. There exists a unique solution $u \in C_t^0 L_y^2\mathcal{H}_x^1(I\times\mathbb{R}^2\times\mathbb{R})$ of (1-2), where $I \subseteq \mathbb{R}$ is the maximal lifespan. Furthermore, if $u_0 \in \Sigma(\mathbb{R}^3)$, the solution u can be extended to be global in $C_t^0\Sigma_{y,x}(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})$.*

(2) (*scattering norm*) *If the solution $u \in C_t^0\Sigma_{y,x}(\mathbb{R}\times\mathbb{R}^3)$ of (1-2) satisfies $\|u\|_{L_{t,y}^4\mathcal{H}_x^{1-\epsilon_0}(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} \le M$ for some positive constant M. Then u scatters in $\Sigma(\mathbb{R}^3)$; that is, there exist $u_\pm \in \Sigma_{y,x}(\mathbb{R}^2\times\mathbb{R})$ such that*

$$\|u(t,y,x) - e^{it(\Delta_{\mathbb{R}^3}-x^2)}u_\pm(y,x)\|_{\Sigma_{y,x}} \to 0 \quad \text{as } t \to \pm\infty. \tag{3-1}$$

We next give the existence of wave operators, whose proof can be found in the Appendix.

Theorem 3.5 (existence of the wave operators). *Let $u_- \in \Sigma$. There exists $T_- > 0$ depending on $\|u_-\|_\Sigma$, and a solution $u \in C((-\infty, -T_-], \Sigma)$ to (1-2) such that*

$$\|u(t, y, x) - e^{it(\Delta - x^2)} u_-(y, x)\|_\Sigma \to 0 \quad \text{as } t \to -\infty. \tag{3-2}$$

Similarly, let $u_+ \in \Sigma$. There exists $T_+ > 0$ depending on $\|u_+\|_\Sigma$, and a solution $u \in C([T_+, \infty), \Sigma)$ to (1-2) such that

$$\|u(t, y, x) - e^{it(\Delta - x^2)} u_+(y, x)\|_\Sigma \to 0 \quad \text{as } t \to \infty.$$

We now state the stability theory in $L^2_y \mathcal{H}^1_x(\mathbb{R}^2 \times \mathbb{R})$. The proof is again given in the Appendix. For a comparison, see [Colliander et al. 2008; Killip and Vişan 2013; Koch et al. 2014]; in particular, [Killip and Vişan 2013, Theorem 3.7]. We also contend that the result in the following theorem can extended to $\Sigma(\mathbb{R}^3)$.

Theorem 3.6 (stability theorem). *Let I be a compact interval and let $\tilde{u}$ be an approximate solution to (1-2) in the sense that $\tilde{u}$ satisfies $i\,\partial_t \tilde{u} + \Delta_{\mathbb{R}^3} \tilde{u} - x^2 \tilde{u} = |\tilde{u}|^2 \tilde{u} + e$ for some function e.*

Suppose

$$\|\tilde{u}\|_{L^\infty_t L^2_y \mathcal{H}^1_x \cap L^4_{t,y} \mathcal{H}^1_x} \le M$$

for some positive constant M.

Let $t_0 \in I$ and let $u(t_0)$ obey

$$\|u(t_0) - \tilde{u}(t_0)\|_{L^2_y \mathcal{H}^1_x} \le M' \tag{3-3}$$

for some $M' > 0$. Assume in addition that the smallness condition

$$\|e^{i(t-t_0)(\Delta_{\mathbb{R}^3} - x^2)}(u(t_0) - \tilde{u}(t_0))\|_{L^4_{t,y} \mathcal{H}^1_x} + \|e\|_{L^{4/3}_{t,y} \mathcal{H}^1_x} \le \epsilon \tag{3-4}$$

holds for some $0 < \epsilon \le \epsilon_1$, where $\epsilon_1 = \epsilon_1(M, M') > 0$ is a small constant. Then, there exists a solution u to (1-2) on $I \times \mathbb{R}^2 \times \mathbb{R}$ with an initial data $u(t_0)$ at time $t = t_0$ satisfying

$$\|u - \tilde{u}\|_{L^4_{t,y} \mathcal{H}^1_x} \le C(M, M')\epsilon, \qquad \|u - \tilde{u}\|_{L^\infty_t L^2_y \mathcal{H}^1_x} \le C(M, M')M',$$
$$\|u\|_{L^\infty_t L^2_y \mathcal{H}^1_x \cap L^4_{t,y} \mathcal{H}^1_x} \le C(M, M').$$

4. Existence of an almost-periodic solution

In this section, we will show the existence of an almost-periodic solution by the profile decomposition and the nonlinear approximation.

4.1. ***Linear profile decomposition.*** In this subsection, we will establish the linear profile decomposition in $\Sigma(\mathbb{R}^3)$, which depends on the corresponding decomposition in $L^2(\mathbb{R}^2)$. The linear profile decomposition in L^2 for the mass-critical nonlinear Schrödinger equation has been established by F. Merle and L. Vega [1998], R. Carles and S. Keraani [2007], and P. Bégout and A. Vargas [2007]. We also refer readers to [Killip and Vişan 2013; Koch et al. 2014] for other versions of the linear profile decomposition.

Theorem 4.1 (linear profile decomposition in $L^2_y \mathcal{H}^1_x(\mathbb{R}^2 \times \mathbb{R})$ and Σ). *Let $\{u_k\}_{k \ge 1}$ be a bounded sequence in $L^2_y \mathcal{H}^1_x(\mathbb{R}^2 \times \mathbb{R})$. Then after passing to a subsequence if necessary, there exists $J^* \in \{0, 1, \ldots\} \cup \{\infty\}$,* *so that for any $J \le J^*$ we have functions $\phi^j \in L_y^2\mathcal{H}_x^1(\mathbb{R}^2\times\mathbb{R})$, $1 \le j \le J$, $r_k^J \in L_y^2\mathcal{H}_x^1(\mathbb{R}^2\times\mathbb{R})$, and mutually orthogonal frames $\{(\lambda_k^j, t_k^j, y_k^j, \xi_k^j)\}_{k\ge1} \subseteq \mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2$ in the sense that, for any $j \ne j'$,*

$$\frac{\lambda_k^j}{\lambda_k^{j'}} + \frac{\lambda_k^{j'}}{\lambda_k^j} + \lambda_k^j\lambda_k^{j'}|\xi_k^j - \xi_k^{j'}|^2 + \frac{|y_k^j - y_k^{j'}|^2}{\lambda_k^j\lambda_k^{j'}} + \frac{|(\lambda_k^j)^2 t_k^j - (\lambda_k^{j'})^2 t_k^{j'}|}{\lambda_k^j\lambda_k^{j'}} \to \infty \quad \text{as } k\to\infty \tag{4-1}$$

such that, for every $1 \le j \le J$, we have a decomposition

$$u_k(y,x) = \sum_{j=1}^{J} \frac{1}{\lambda_k^j} e^{iy\cdot\xi_k^j}(e^{it_k^j\Delta_{\mathbb{R}^2}}\phi^j)\left(\frac{y-y_k^j}{\lambda_k^j}, x\right) + r_k^J(y,x).$$

In addition,

$$\lim_{k\to\infty}\left(\|u_k\|^2_{L_y^2\mathcal{H}_x^1} - \sum_{j=1}^{J}\|\phi^j\|^2_{L_y^2\mathcal{H}_x^1} - \|r_k^J\|^2_{L_y^2\mathcal{H}_x^1}\right) = 0, \tag{4-2}$$

$$\lambda_k^j e^{-it_k^j\Delta_y}\left(e^{-i(\lambda_k^j y + y_k^j)\cdot\xi_n^j} r_k^J(\lambda_k^j y + y_k^j, x)\right) \rightharpoonup 0 \quad \text{in } L_y^2\mathcal{H}_x^1, \text{ as } k\to\infty, \text{ for } j \le J, \tag{4-3}$$

$$\limsup_{k\to\infty}\|e^{it(\Delta_{\mathbb{R}^3}-x^2)} r_k^J\|_{L^4_{t,y}\mathcal{H}_x^{1-\epsilon_0}} \to 0 \quad \text{as } J \to J^*. \tag{4-4}$$

Furthermore, if $\{u_k\}_{k\ge1}$ is a bounded sequence in $\Sigma(\mathbb{R}^3)$, then in the above conclusion, we can further take $\lambda_k^j \to 1$ or ∞, as $k\to\infty$, $|\xi_k^j| \le C_j$, for every $1 \le j \le J$. And we have a slight different decomposition

$$u_k(y,x) = \sum_{j=1}^{J}\phi_k^j(y,x) + r_k^J(y,x) := \sum_{j=1}^{J}\frac{1}{\lambda_k^j}e^{iy\cdot\xi_k^j}(e^{it_k^j\Delta_{\mathbb{R}^2}}P_k^j\phi^j)\left(\frac{y-y_k^j}{\lambda_k^j}, x\right) + r_k^J(y,x),$$

where

$$P_k^j\phi^j(y,x) = \begin{cases} \phi^j(y,x) & \text{if } \lim_{k\to\infty}\lambda_k^j = 1, \\ P^y_{\le(\lambda_k^j)^\theta}\phi^j(y,x) & \text{if } \lim_{k\to\infty}\lambda_k^j = \infty, \end{cases}$$

and θ is some fixed positive sufficiently small number. In addition, we also have a slight different decoupling

$$\lim_{k\to\infty}\left(\mathcal{E}(u_k) - \sum_{j=1}^{J}\mathcal{E}(\phi_k^j) - \mathcal{E}(r_k^J)\right) = 0, \tag{4-5}$$

$$\lim_{k\to\infty}\left(\mathcal{M}(u_k) - \sum_{j=1}^{J}\mathcal{M}(\phi_k^j) - \mathcal{M}(r_k^J)\right) = 0, \tag{4-6}$$

where $\mathcal{E}$ and $\mathcal{M}$ are given in (ME). Other conclusions (4-1)–(4-4) hold as before.

To prove the above theorem, we need to establish the inverse Strichartz estimate in Proposition 4.6 below. We first recall the following refined Strichartz estimate which is essentially established in [Cheng et al. 2020a; 2020b].

Proposition 4.2 (refined Strichartz estimate [Cheng et al. 2020a; 2020b]). *For any $f \in L_y^2\mathcal{H}_x^{1-\epsilon_0/2}$, we have*

$$\|e^{it\Delta_{\mathbb{R}^2}} f\|_{L^4_{t,y,x}(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} \lesssim \|f\|^{\frac{3}{4}}_{L_y^2\mathcal{H}_x^{1-\epsilon_0/2}}\left(\sup_{Q\in\mathcal{D}}|Q|^{-\frac{3}{22}}\|e^{it\Delta_{\mathbb{R}^2}} f_Q\|_{L^{11/2}_{t,y,x}}\right)^{\frac{1}{4}},$$

where

$$\mathcal{D} = \bigcup_{j\in\mathbb{Z}} \{[2^j k_1, 2^j(k_1+1)) \times [2^j k_2, 2^j(k_2+1)) : (k_1,k_2) \in \mathbb{Z}^2\}$$

is the collection of all dyadic cubes, and f_Q *is defined by* $\mathcal{F}_y(f_Q) = \chi_Q \mathcal{F}_y f$.

To prove the inverse Strichartz estimate, we shall need the following two facts:

Proposition 4.3 (local smoothing estimate [Constantin and Saut 1988; Vega 1988]). *For any given* $\epsilon > 0$, *we have*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2\times\mathbb{R}} |(|\nabla_y|^{\frac{1}{2}} e^{it\Delta_{\mathbb{R}^2}} f)(y,x)|^2 \langle y\rangle^{-1-\epsilon} \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \lesssim_\epsilon \|f\|^2_{L^2_{y,x}(\mathbb{R}^2\times\mathbb{R})}.$$

Furthermore, if $\epsilon \geq 1$, *then we have*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2\times\mathbb{R}} |(\langle\nabla_y\rangle^{\frac{1}{2}} e^{it\Delta_{\mathbb{R}^2}} f)(y,x)|^2 \langle y\rangle^{-1-\epsilon} \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \lesssim_\epsilon \|f\|^2_{L^2_{y,x}(\mathbb{R}^2\times\mathbb{R})}.$$

Lemma 4.4. *For each* $f \in \mathcal{H}^1_x(\mathbb{R})$ *and any* $R > 0$, *we have*

$$\|f\|_{L^\infty_x(|x|\geq R)} \lesssim R^{-\frac{1}{2}}(\|f(x)\|_{L^2_x} + \|xf(x)\|^{\frac{1}{2}}_{L^2_x} \|f'(x)\|^{\frac{1}{2}}_{L^2_x}).$$

Proof. For any $f \in \mathcal{H}^1_x(\mathbb{R})$, we have

$$xf^2(x) = \int_0^x (zf^2(z))'\,\mathrm{d}z = \int_0^x f^2(z) + 2zf(z)f'(z)\,\mathrm{d}z.$$

Then by Hölder's inequality, we get for any $R > 0$,

$$\|xf^2(x)\|_{L^\infty_x(|x|\geq R)} \lesssim \|f\|^2_{L^2_x} + \|xf(x)\|_{L^2_x}\|f'(x)\|_{L^2_x}.$$

Therefore,

$$\|f(x)\|_{L^\infty_x(|x|\geq R)} \lesssim R^{-\frac{1}{2}}(\|f\|_{L^2_x} + \|xf(x)\|^{\frac{1}{2}}_{L^2_x} \|f'(x)\|^{\frac{1}{2}}_{L^2_x}). \qquad \square$$

We also have the following estimate.

Lemma 4.5. *By interpolation, the Hölder inequality,* (2-2), *and* *Proposition 3.2*, *we have*

$$\|e^{it\Delta_y} f\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x} \lesssim \|e^{it\Delta_y} f\|^{\frac{\epsilon_0}{2}}_{L^4_{t,y}\mathcal{H}^{-1}_x} \|e^{it\Delta_y} f\|^{1-\frac{\epsilon_0}{2}}_{L^4_{t,y}\mathcal{H}^1_x} \lesssim \|e^{it\Delta_y} f\|^{\frac{\epsilon_0}{2}}_{L^4_{t,y,x}} \|f\|^{1-\frac{\epsilon_0}{2}}_{L^2_y\mathcal{H}^1_x}. \tag{4-7}$$

We can now prove the inverse Strichartz estimate.

Proposition 4.6 (inverse Strichartz estimate). *For* $\{f_k\}_{k\geq 1} \subseteq L^2_y\mathcal{H}^1_x(\mathbb{R}^2\times\mathbb{R})$ *satisfying*

$$\lim_{k\to\infty} \|f_k\|_{L^2_y\mathcal{H}^1_x} = A \quad \textit{and} \quad \lim_{k\to\infty} \|e^{it\Delta_{\mathbb{R}^2}} f_k\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x} = \epsilon, \tag{4-8}$$

there exist $\phi \in L^2_y\mathcal{H}^1_x$ *and* $(\lambda_k, t_k, \xi_k, y_k) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$, *so that passing to a further subsequence if necessary, we have*

$$\lambda_k e^{-i\xi_k\cdot(\lambda_k y + y_k)}(e^{it_k\Delta_{\mathbb{R}^2}} f_k)(\lambda_k y + y_k, x) \rightharpoonup \phi(y,x) \quad \textit{in } L^2_y\mathcal{H}^1_x, \textit{ as } k\to\infty,$$

$$\lim_{k\to\infty}(\|f_k\|^2_{L^2_y\mathcal{H}^1_x}-\|f_k-\phi_k\|^2_{L^2_y\mathcal{H}^1_x})=\|\phi\|^2_{L^2_y\mathcal{H}^1_x}\gtrsim A^2\Big(\frac{\epsilon}{A}\Big)^{\frac{48}{\epsilon_0}},\tag{4-9}$$

$$\limsup_{k\to\infty}\|e^{it\Delta_{\mathbb{R}^2}}(f_k-\phi_k)\|^4_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}\le\epsilon^{\frac{8}{\epsilon_0}}A^{4-\frac{8}{\epsilon_0}}\bigg(1-cA^{2\beta}\Big(\frac{\epsilon}{A}\Big)^{\frac{2\beta}{\epsilon_0}}\bigg),\tag{4-10}$$

where c and β are small positive constants, and

$$\phi_k(y,x)=\frac{1}{\lambda_k}e^{iy\cdot\xi_k}(e^{-i(t_k/\lambda_k^2)\Delta_{\mathbb{R}^2}}\phi)\Big(\frac{y-y_k}{\lambda_k},x\Big).$$

Moreover, if $\{f_k\}_{k\ge1}$ is bounded in $\Sigma(\mathbb{R}^3)$, and also

$$\lim_{k\to\infty}\|f_k\|_\Sigma=A\quad\text{and}\quad\lim_{k\to\infty}\|e^{it\Delta_{\mathbb{R}^2}}f_k\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}=\epsilon,\tag{4-11}$$

then we can take $\lambda_k\ge1$, $|\xi_k|\lesssim1$ and $\phi\in L^2_y\mathcal{H}^1_x(\mathbb{R}^2\times\mathbb{R})$ such that

$$\lambda_ke^{-i\xi_k\cdot(\lambda_ky+y_k)}(e^{it_k\Delta_{\mathbb{R}^2}}f_k)(\lambda_ky+y_k,x)\rightharpoonup\phi(y,x)\quad\text{in } L^2_y\mathcal{H}^1_x,\ \text{as } k\to\infty,\tag{4-12}$$

and

$$\lim_{k\to\infty}(\|f_k\|^2_\Sigma-\|f_k-\phi_k\|^2_\Sigma)=\lim_{k\to\infty}\|\phi_k\|^2_\Sigma\gtrsim A^2\Big(\frac{\epsilon}{A}\Big)^{\frac{48}{\epsilon_0}}.\tag{4-13}$$

Proof. Case 1: $\{f_k\}_{k\ge1}$ is bounded in $L^2_y\mathcal{H}^1_x$. By Proposition 4.2, (4-7) and (4-8), there exists $\{Q_k\}_{k\ge1}\subseteq\mathcal{D}$ so that

$$\epsilon^{\frac{8}{\epsilon_0}}A^{1-\frac{8}{\epsilon_0}}\lesssim\liminf_{k\to\infty}|Q_k|^{-\frac{3}{22}}\|e^{it\Delta_{\mathbb{R}^2}}(f_k)_{Q_k}\|_{L^{11/2}_{t,y,x}}.\tag{4-14}$$

Let λ_k be the inverse of the side length and ξ_k be the center of the cube Q_k. By Hölder's inequality and (4-8), we have

$$\liminf_{k\to\infty}|Q_k|^{-\frac{3}{22}}\|e^{it\Delta_{\mathbb{R}^2}}(f_k)_{Q_k}\|_{L^{11/2}_{t,y,x}}\lesssim\liminf_{k\to\infty}\lambda_k^{\frac{3}{11}}(\epsilon^{\frac{2}{\epsilon_0}}A^{1-\frac{2}{\epsilon_0}})^{\frac{8}{11}}\|e^{it\Delta_{\mathbb{R}^2}}(f_k)_{Q_k}\|^{\frac{3}{11}}_{L^\infty_{t,y,x}}.$$

Together with (4-14), this implies

$$\liminf_{k\to\infty}\lambda_k\|e^{it\Delta_{\mathbb{R}^2}}(f_k)_{Q_k}\|_{L^\infty_{t,y,x}(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})}\gtrsim\epsilon^{\frac{24}{\epsilon_0}}A^{1-\frac{24}{\epsilon_0}}.$$

Then by Lemma 4.4 and Bernstein's inequality, we have

$$\begin{aligned}&\liminf_{k\to\infty}\lambda_k\|e^{it\Delta_{\mathbb{R}^2}}(f_k)_{Q_k}\|_{L^\infty_{t,y,x}(\mathbb{R}\times\mathbb{R}^2\times\{|x|\ge R\})}\\&\quad\lesssim R^{-\frac12}\liminf_{k\to\infty}\lambda_k\big(\|(f_k)_{Q_k}\|_{L^\infty_{t,y}L^2_x}+\||x|(f_k)_{Q_k}(x)\|^{\frac12}_{L^\infty_{t,y}L^2_x}\|\partial_x((f_k)_{Q_k})\|^{\frac12}_{L^\infty_{t,y}L^2_x}\big)\\&\quad\lesssim R^{-\frac12}\liminf_{k\to\infty}\lambda_k\big(|Q_k|^{\frac12}\|f_k\|_{L^\infty_tL^2_{y,x}}+|Q_k|^{\frac12}\|xf_k\|^{\frac12}_{L^\infty_tL^2_{y,x}}\|\partial_xf_k\|^{\frac12}_{L^\infty_tL^2_{y,x}}\big)\\&\quad\sim R^{-\frac12}\liminf_{k\to\infty}\big(\|f_k\|_{L^\infty_tL^2_{y,x}}+\|xf_k\|^{\frac12}_{L^\infty_tL^2_{y,x}}\|\partial_xf_k\|^{\frac12}_{L^\infty_tL^2_{y,x}}\big)\to0\quad\text{as } R\to\infty.\end{aligned}$$

Therefore, we can take R large enough such that

$$\liminf_{k\to\infty} \lambda_k \|e^{it\Delta_{\mathbb{R}^2}}(f_k)_{Q_k}\|_{L^\infty_{t,y,x}(|x|\ge R)} \lesssim \tfrac{1}{2}\epsilon^{\frac{24}{\epsilon_0}} A^{1-\frac{24}{\epsilon_0}}.$$

As a consequence, there exists $(t_k, y_k, x_k) \in \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}$ with $|x_k| \le R$, so that

$$\liminf_{k\to\infty} \lambda_k |(e^{it_k\Delta_{\mathbb{R}^2}}(f_k)_{Q_k})(y_k, x_k)| \gtrsim \epsilon^{\frac{24}{\epsilon_0}} A^{1-\frac{24}{\epsilon_0}}. \tag{4-15}$$

Since $|x_k| \le R$, we may assume, up to a subsequence, $x_k \to x^*$ as $k\to\infty$, with $|x^*| \lesssim 1$.

By the weak compactness of $L^2_y\mathcal{H}^1_x$, we have

$$\lambda_k e^{-i\xi_k\cdot(\lambda_k y+y_k)}(e^{it_k\Delta_{\mathbb{R}^2}} f_k)(\lambda_k y + y_k, x) \rightharpoonup \phi(y,x) \quad \text{in } L^2_y\mathcal{H}^1_x, \text{ as } k\to\infty.$$

By the very basic fact in Hilbert space H that

$$g_k \rightharpoonup g \quad \text{in } H \quad \Longrightarrow \quad \|g_k\|_H^2 - \|g_k - g\|_H^2 \to \|g\|_H^2,$$

we have

$$\lim_{k\to\infty} (\|f_k\|^2_{L^2_y\mathcal{H}^1_x} - \|f_k - \phi_k\|^2_{L^2_y\mathcal{H}^1_x}) = \|\phi\|^2_{L^2_y\mathcal{H}^1_x}.$$

We now turn to the remaining part (4-9). Define h so that $\mathcal{F}_y h$ is the characteristic function of the cube $\left[-\frac{1}{2},\frac{1}{2}\right]^2$. By Lemma 2.1, $(x,y)\mapsto h(y)\delta_0(x) \in L^2_y\mathcal{H}^{-1}_x(\mathbb{R}^2\times\mathbb{R})$. From (4-15), we obtain

$$\begin{aligned}
&|\langle h(y)\delta_0(x), \phi(y, x+x^*)\rangle_{y,x}| \\
&\quad = \lim_{k\to\infty}\left|\left\langle \delta_0(x), \int_{\mathbb{R}^2} \bar{h}(y)\lambda_k e^{-i\xi_k\cdot(\lambda_k y+y_k)}(e^{it_k\Delta_{\mathbb{R}^2}} f_k)(\lambda_k y + y_k, x + x_k)\,dy\right\rangle_x\right| \\
&\quad = \lim_{k\to\infty} \lambda_k |(e^{it_k\Delta_{\mathbb{R}^2}}(f_k)_{Q_k})(y_k, x_k)| \gtrsim \epsilon^{\frac{24}{\epsilon_0}} A^{1-\frac{24}{\epsilon_0}},
\end{aligned} \tag{4-16}$$

from which it follows

$$\|\phi(y, x+x^*)\|_{L^2_y\mathcal{H}^1_x} \gtrsim \epsilon^{\frac{24}{\epsilon_0}} A^{1-\frac{24}{\epsilon_0}}.$$

At the same time, since

$$\begin{aligned}
\|\phi(y,x)\|_{L^2_y\mathcal{H}^1_x} &\ge \|\phi(y,x+x^*)\|_{L^2_y H^1_x} + \||x|\phi(y,x+x^*)\|_{L^2_{y,x}} - \||x^*|\phi(y,x+x^*)\|_{L^2_{y,x}} \\
&= \|\phi(y,x+x^*)\|_{L^2_y\mathcal{H}^1_x} - |x^*|\|\phi(y,x)\|_{L^2_{y,x}},
\end{aligned}$$

we get

$$\|\phi(y,x+x^*)\|_{L^2_y\mathcal{H}^1_x} \le \|\phi\|_{L^2_y\mathcal{H}^1_x} + |x^*|\|\phi\|_{L^2_{y,x}} \lesssim \|\phi\|_{L^2_y\mathcal{H}^1_x}.$$

Therefore $\|\phi\|_{L^2_y\mathcal{H}^1_x} \gtrsim \epsilon^{24/\epsilon_0} A^{1-24/\epsilon_0}$ and (4-9) follows. We turn to (4-10), by Proposition 4.3 and the Rellich–Kondrachov theorem, we have

$$e^{it\Delta_{\mathbb{R}^2}}\big(\lambda_k e^{-i\xi_k\cdot(\lambda_k y+y_k)}(e^{it_k\Delta_{\mathbb{R}^2}} f_k)(\lambda_k y + y_k, x + x_k)\big) \to e^{it\Delta_{\mathbb{R}^2}}\phi(y,x) \quad \text{as } k\to\infty,$$

for almost every $(t,y,x) \in \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}$. By the refined Fatou's lemma [Lieb and Loss 1997], we obtain

$$\|e^{it\Delta_{\mathbb{R}^2}} f_k\|^4_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x} - \|e^{it\Delta_{\mathbb{R}^2}}(f_k - \phi_k)\|^4_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x} - \|e^{it\Delta_{\mathbb{R}^2}}\phi_k\|^4_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x} \to 0 \quad \text{as } k\to\infty.$$

Thus, by the invariance of Galilean transform, we have

$$\begin{aligned}\limsup_{k\to\infty}\|e^{it\Delta_{\mathbb{R}^2}}(f_k-\phi_k)\|^4_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x} &= \limsup_{k\to\infty}\big(\|e^{it\Delta_{\mathbb{R}^2}}f_k\|^4_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}-\|e^{it\Delta_{\mathbb{R}^2}}\phi_k\|^4_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}\big)\\ &= (\epsilon^{\frac{2}{\epsilon_0}}A^{1-\frac{2}{\epsilon_0}})^4-\|e^{it\Delta_{\mathbb{R}^2}}\phi\|^4_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}.\end{aligned}$$

We now take $c(t)\in C^\infty$, which has compact support on $[-1,1]$, such that

$$\|c(t)e^{it\Delta}h\|_{L^{4/3}_{t,y}}=1.$$

Then by (4-16), we have

$$\left|\int_{\mathbb{R}}\langle c(t)h(y)\delta_0(x),\phi(y,x+x^*)\rangle_{y,x}\,\mathrm{d}t\right|\gtrsim\epsilon^{\frac{24}{\epsilon_0}}A^{1-\frac{24}{\epsilon_0}}.$$

On the other hand, by Hölder's inequality, Sobolev's inequality and Lemma 2.1,

$$\begin{aligned}\left|\int_{\mathbb{R}}\langle c(t)h(y)\delta_0(x),\phi(y,x+x^*)\rangle_{y,x}\,\mathrm{d}t\right| &= \left|\int_{\mathbb{R}}\langle e^{it\Delta_y}(c(t)h(y)\delta_0(x)),e^{it\Delta_y}\phi(y,x+x^*)\rangle_{y,x}\,\mathrm{d}t\right|\\ &\lesssim \|e^{it\Delta_y}(c(t)h(y))\|_{L^{4/3}_{t,y}}\|e^{it\Delta_y}\phi(y,x)\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}\\ &\lesssim \|e^{it\Delta_y}\phi(y,x)\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}.\end{aligned}$$

Therefore, by the above two estimates and (4-17), we get (4-10).

Case 2: $\{f_k\}_{k\geq1}$ is bounded in $\Sigma(\mathbb{R}^3)$. In this case, we have

$$\limsup_{k\to\infty}\|P^y_{\geq R}f_k\|_{L^2_y\mathcal{H}^{1-\epsilon_0}_x}\lesssim\langle R\rangle^{-\epsilon_0}\limsup_{k\to\infty}\|f_k\|_{\Sigma(\mathbb{R}^3)}\to0\quad\text{as } R\to\infty.$$

For $R\in2^{\mathbb{Z}}$ large enough depending on A and ϵ, by (4-11), Sobolev embedding, and the Strichartz estimate, $P^y_{\leq R}f_k$ satisfies

$$\begin{aligned}\lim_{k\to\infty}\|e^{it\Delta_{\mathbb{R}^2}}P^y_{\leq R}f_k\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x} &\geq \lim_{k\to\infty}\|e^{it\Delta_{\mathbb{R}^2}}f_k\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}-\lim_{k\to\infty}\|e^{it\Delta_{\mathbb{R}^2}}P^y_{\geq R}f_k\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}\\ &\geq \lim_{k\to\infty}\|e^{it\Delta_{\mathbb{R}^2}}f_k\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x}-C\lim_{k\to\infty}\|P^y_{\geq R}f_k\|_{L^2_y\mathcal{H}^{1-\epsilon_0}_x}\\ &\geq \tfrac{1}{2}\epsilon^{\frac{2}{\epsilon_0}}A^{1-\frac{2}{\epsilon_0}}.\end{aligned}$$

So we can replace f_k by $P^y_{\leq R}f_k$ in the above case, and for $R=R(A,\epsilon)>0$ large enough, we may take $\{Q_k\}_{k\geq1}\subseteq\mathcal{D}$ and $|Q_k|\lesssim R^2$ such that $\lambda_k\gtrsim R^{-1}$, and $|\xi_k|\lesssim R$. As in the proof of Case 1, we still have (4-12) and also (4-9), (4-10). Furthermore, if $\limsup_{k\to\infty}\lambda_k<\infty$, then

$$\lambda_k e^{-i\xi_k\cdot(\lambda_k y+y_k)}(e^{it_k\Delta_{\mathbb{R}^2}}f_k)(\lambda_k y+y_k,x)\rightharpoonup\phi(y,x)$$

holds for some $\phi\in\Sigma(\mathbb{R}^2\times\mathbb{R})$. To show (4-13), we just need to consider the case when $\lambda_k\to\infty$ because the situation when $\limsup_{k\to\infty}\lambda_k<\infty$ is as in Case 1. We note

$$\lim_{k\to\infty}\|\phi_k\|^2_\Sigma\geq\lim_{k\to\infty}\|P^y_{\leq\lambda_k^\theta}\phi\|^2_{L^2_y\mathcal{H}^1_x}\gtrsim A^2\left(\frac{\epsilon}{A}\right)^{\frac{48}{\epsilon_0}}.$$

Then the decoupling of the Σ-norm comes from $P_{\leq\lambda_k^\theta}\to\mathrm{Id}$ in $L^2_y\mathcal{H}^1_x$ and (4-12). □

Proof of Theorem 4.1. The conclusion follows by applying Proposition 4.6 repeatedly until the asymptotically linear evolution of the remainder is trivial in $L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x$. The decoupling (4-5) and (4-6) follow from (4-13) and the orthogonality (4-1). □

Remark 4.7. For a linear profile decomposition for the Schrödinger propagator of the Schrödinger operator $-\Delta+|x|^2$ in L^2, we refer to the work of C. Jao, R. Killip, and M. Vişan [Jao et al. 2019] and C. Jao [2020]; we believe that some part of their argument can be applied in our equation. We also refer to the linear profile decomposition proved by A. Ardila and R. Carles [2021].

4.2. ***Approximation of the nonlinear profile: the case of concentrated initial data.*** In this section, we will show that the nonlinear profile u_λ given in (1-4)

$$\begin{cases} i\,\partial_t u_\lambda+\Delta_{\mathbb{R}^3}u_\lambda-x^2u_\lambda=|u_\lambda|^2u_\lambda,\\ u_\lambda(0,y,x)=\frac{1}{\lambda}\phi\left(\frac{y}{\lambda},x\right),\end{cases}$$

can be approximated by $\tilde{u}_\lambda$ given in (1-6)

$$\tilde{u}_\lambda(t,y,x)=e^{it(\Delta_{\mathbb{R}}-x^2)}\sum_{n\in\mathbb{N}}\left(\frac{1}{\lambda}v_n\left(\frac{t}{\lambda^2},\frac{y}{\lambda},x\right)\right),\qquad (t,y,x)\in\mathbb{R}\times\mathbb{R}^2\times\mathbb{R},$$

when λ is sufficiently large. Here v_n is the solution of the (DCR) system (1-5)

$$\begin{cases} (i\,\partial_t+\Delta_y)v_n(t,y,x)=\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})(t,y,x),\\ v_n(0,y,x)=\phi_n(y,x)=\Pi_n\phi(y,x).\end{cases}$$

The following corollary can be proven from Theorem 1.2 by following the argument in [Colliander et al. 2008; Killip and Vişan 2013]. In particular, we refer to [Colliander et al. 2004, Lemma 3.12].

Corollary 4.8 (corollary of Theorem 1.2: preservation of higher regularity). *Suppose $\phi\in L^2_y\mathcal{H}^1_x(\mathbb{R}^2\times\mathbb{R})$ and v is the global solution of (DCR) given as in Theorem 1.2. For any $s_1\ge 0$ and $s_2\ge 1$, if we assume further $v|_{t=0}\in H^{s_1}_y\mathcal{H}^{s_2}_x(\mathbb{R}^2\times\mathbb{R})$, then the solution $v\in C^0_tH^{s_1}_y\mathcal{H}^{s_2}_x(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})$ and satisfies*

$$\|v\|_{L^\infty_tH^{s_1}_y\mathcal{H}^{s_2}_x\cap L^4_tW^{s_1,4}_y\mathcal{H}^{s_2}_x(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})}\le C(\|\phi\|_{H^{s_1}_y\mathcal{H}^{s_2}_x(\mathbb{R}^2\times\mathbb{R})}).$$

Relying on Corollary 4.8, we can now prove the following general result on approximation of the nonlinear profile in the large-scale case. We will prove it with the help of Theorem 3.6.

Theorem 4.9. *For any $\phi\in L^2_y\mathcal{H}^1_x$, $0<\theta\ll 1$, $(\lambda_k,t_k,y_k,\xi_k)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2$, $|\xi_k|\lesssim 1$ and $\lambda_k\to\infty$ when $k\to\infty$. There exists a global solution $u_k\in C^0_tL^2_y\mathcal{H}^1_x$ of*

$$\begin{cases} i\,\partial_tu_k+\Delta_yu_k+\Delta_xu_k-x^2u_k=|u_k|^2u_k,\\ u_k(0,y,x)=\lambda_k^{-1}e^{iy\cdot\xi_k}(e^{it_k\Delta_y}P_{\le\lambda_k^\theta}\phi)\left(\frac{y-y_k}{\lambda_k}x\right),\end{cases}$$

for k large enough, satisfying

$$\|u_k\|_{L^\infty_tL^2_y\mathcal{H}^1_x\cap L^4_{t,y}\mathcal{H}^1_x(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})}\lesssim_{\|\phi\|_{L^2_y\mathcal{H}^1_x}}1.$$

Furthermore, assume that $\epsilon_4 = \epsilon_4(\|\phi\|_{L_y^2\mathcal{H}_x^1})$ *is a sufficiently small positive constant and* $\psi \in H_y^{10}\mathcal{H}_x^{10}$ *such that*

$$\|\phi - \psi\|_{L_y^2\mathcal{H}_x^1} \le \epsilon_4.$$

Then there exists a solution $v \in C_t^0 H_y^2 \mathcal{H}_x^1(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})$ *of* (DCR), *with*

$$v(0,y,x) = \psi(y,x) \quad \text{if } t_k = 0,$$
$$\lim_{t\to\pm\infty} \|v(t,y,x) - e^{it\Delta_y}\psi\|_{L_y^2\mathcal{H}_x^1} = 0 \quad \text{if } t_k \to \pm\infty,$$

such that for k *large enough we have*

$$\|u_k\|_{L_t^\infty L_y^2\mathcal{H}_x^1 \cap L_{t,y}^4\mathcal{H}_x^1(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} \lesssim 1,$$

with

$$\|u_k(t) - w_{\lambda_k}(t)\|_{L_t^\infty L_y^2\mathcal{H}_x^1 \cap L_{t,y}^4\mathcal{H}_x^1(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} \to 0 \quad \text{as } k\to\infty,$$

where

$$w_{\lambda_k}(t,y,x) = e^{-i(t-t_k)|\xi_k|^2} e^{iy\cdot\xi_k} \lambda_k^{-1} e^{it(\Delta_x - x^2)} v\left(\frac{t}{\lambda_k^2} + t_k, \frac{y - y_k - 2\xi_k(t-t_k)}{\lambda_k}, x\right).$$

Proof of Theorem 4.9. By translation invariance, we may take $y_k = 0$. By Galilean transformation and $|\xi_k|$ is bounded, we may take $\xi_k = 0$. Then

$$w_{\lambda_k}(t,y,x) = \lambda_k^{-1} e^{it(\Delta_x - x^2)} v\left(\frac{t}{\lambda_k^2} + t_k, \frac{y}{\lambda_k}, x\right).$$

When $t_k = 0$, we will show w_{λ_k} is an approximate solution to (1-2). After a simple computation, we see

$$\begin{aligned} e_{\lambda_k} &:= (i\partial_t + \Delta_y + \Delta_x - x^2) w_{\lambda_k} - |w_{\lambda_k}|^2 w_{\lambda_k} \\ &= -\lambda_k^{-3} \sum_{n\in\mathbb{N}} e^{-it(2n+1)} \sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3\ne n}} e^{-2it(n_1-n_2+n_3-n)} (\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3}))\left(\frac{t}{\lambda_k^2}, \frac{y}{\lambda_k}, x\right). \end{aligned} \tag{4-17}$$

We will show this error term is small in the dual Strichartz space. Divide the right-hand side of (4-17) into three terms:

$$\begin{aligned} &e_{\lambda_k}(t,y,x) \\ &= -\lambda_k^{-3} \sum_{n\in\mathbb{N}} e^{-it(2n+1)} \sum_{n_1,n_2,n_3\in\mathbb{N}} e^{-2it(n_1-n_2+n_3-n)} P^y_{\ge 2^{-10}}\left(\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\left(\frac{t}{\lambda_k^2}, \frac{y}{\lambda_k}, x\right)\right) \\ &\quad + \lambda_k^{-3} \sum_{n\in\mathbb{N}} e^{-it(2n+1)} \sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}} e^{-2it(n_1-n_2+n_3-n)} P^y_{\ge 2^{-10}}\left(\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\left(\frac{t}{\lambda_k^2}, \frac{y}{\lambda_k}, x\right)\right) \\ &\quad - \lambda_k^{-3} \sum_{n\in\mathbb{N}} e^{-it(2n+1)} \sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3\ne n}} e^{-2it(n_1-n_2+n_3-n)} P^y_{\le 2^{-10}}\left(\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\left(\frac{t}{\lambda_k^2}, \frac{y}{\lambda_k}, x\right)\right) \\ &=: e^1_{\lambda_k} + e^2_{\lambda_k} + e^3_{\lambda_k}. \end{aligned}$$

We first consider $e^1_{\lambda_k}$ and shall use Bernstein's inequality, Leibniz's rule, Plancherel's identity and Hölder's inequality to estimate as follows:

$$\begin{aligned}
&\|e^1_{\lambda_k}(t,y,x)\|_{L^{4/3}_{t,y}\mathcal{H}^1_x}\\
&\quad\lesssim \lambda_k^{-1}\Big\|\sum_{n\in\mathbb{N}} e^{-i\lambda_k^2 t(2n+1)}\sum_{n_1,n_2,n_3\in\mathbb{N}} e^{-2i\lambda_k^2 t(n_1-n_2+n_3-n)}\Pi_n(\nabla_y v_{n_1}\cdot\bar v_{n_2}\cdot v_{n_3})(t,y,x)\Big\|_{L^{4/3}_{t,y}\mathcal{H}^1_x}+\cdots\\
&\quad\sim \lambda_k^{-1}\big\|e^{i\lambda_k^2 t(\Delta_x-x^2)}\big(e^{i\lambda_k^2 t(\Delta_x-x^2)}\nabla_y v\cdot\overline{e^{i\lambda_k^2 t(\Delta_x-x^2)}v}\,e^{i\lambda_k^2 t(\Delta_x-x^2)}v\big)(t,y,x)\big\|_{L^{4/3}_{t,y}\mathcal{H}^1_x}\\
&\quad\lesssim \lambda_k^{-1}\|\nabla_y v\|_{L^4_{t,y}\mathcal{H}^1_x}\|v\|^2_{L^4_{t,y}\mathcal{H}^1_x}\to 0 \quad\text{as } k\to\infty,
\end{aligned}\tag{4-18}$$

where $\cdots$ are the missing two terms with ∇_y acting on $\bar v_{n_2}$ and v_{n_3}.

We now turn to the estimate of $e^2_{\lambda_k}$. Using Bernstein's inequality and Leibniz's rule as above, we have

$$\begin{aligned}
\|e^2_{\lambda_k}\|_{L^{4/3}_{t,y}\mathcal{H}^1_x} &\lesssim \lambda_k^{-1}\Big\|\sum_{n\in\mathbb{N}} e^{-i\lambda_k^2 t(2n+1)}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(\nabla_y v_{n_1}\cdot\bar v_{n_2}v_{n_3})(t,y,x)\Big\|_{L^{4/3}_{t,y}\mathcal{H}^1_x}+\cdots\\
&\sim \lambda_k^{-1}\Big\|\Big(\sum_{n\in\mathbb{N}}\Big|\langle n\rangle^{\frac12}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(\nabla_y v_{n_1}\cdot\bar v_{n_2}v_{n_3})(t,y,x)\Big|^2\Big)^{\frac12}\Big\|_{L^{4/3}_{t,y}L^2_x}+\cdots,
\end{aligned}\tag{4-19}$$

where $\cdots$ are the missing two terms with ∇_y acting on $\bar v_{n_2}$ and v_{n_3}.

We observe the following elementary inequality: for $n=n_1-n_2+n_3$

$$\langle n\rangle^{\frac12}\leqslant\langle n\rangle^{-1}\langle n\rangle^2\leqslant\langle n\rangle^{-1}\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2.$$

Using the fact $\{\langle n\rangle^{-1}\}_{n\in\mathbb{N}}\in l^2_n$, the Minkowski inequality and boundedness of Π_n, we have

$$\begin{aligned}
&\Big\|\langle n\rangle^{\frac12}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(\nabla_y v_{n_1}\cdot\bar v_{n_2}v_{n_3})(t,y,x)\Big\|_{L^{4/3}_{t,y}L^2_x l^2_n}\\
&\qquad\lesssim\Big\|\langle n\rangle^{-1}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2|\Pi_n(\nabla_y v_{n_1}\cdot\bar v_{n_2}v_{n_3})(t,y,x)|\Big\|_{L^{4/3}_{t,y}L^2_x l^2_n}\\
&\qquad\lesssim\Big\|\sum_{n_1,n_2,n_3\in\mathbb{N}}\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2\|(\nabla_y v_{n_1}\cdot\bar v_{n_2}v_{n_3})(t,y,x)\|_{L^2_x}\Big\|_{L^{4/3}_{t,y}}.
\end{aligned}$$

By Hölder's inequality and the embedding $\mathcal{H}^1(\mathbb{R})\subseteq L^\infty(\mathbb{R})$, we find

$$\|(\nabla_y v_{n_1}\cdot\bar v_{n_2}v_{n_3})(t,y,x)\|_{L^2_x(\mathbb{R})}\lesssim\|\nabla_y v_{n_1}(t,y,x)\|_{L^2_x(\mathbb{R})}\|v_{n_2}(t,y,x)\|_{\mathcal{H}^1_x(\mathbb{R})}\|v_{n_3}(t,y,x)\|_{\mathcal{H}^1_x(\mathbb{R})}.$$

Similar arguments can be applied to the other two terms on the right-hand side of (4-19). All together this leads to the estimate

$$\begin{aligned}
\|e^2_{\lambda_k}\|_{L^{4/3}_{t,y}\mathcal{H}^1_x}\lesssim{}&\lambda_k^{-1}\big\|\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2\|(\nabla_y v_{n_1}\cdot\bar v_{n_2}v_{n_3})(t,y,x)\|_{L^2_x}\big\|_{L^{4/3}_{t,y}l^1_{n_1}l^1_{n_2}l^1_{n_3}}\\
&+\lambda_k^{-1}\big\|\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2\|(v_{n_1}\cdot\overline{\nabla_y v_{n_2}}v_{n_3})(t,y,x)\|_{L^2_x}\big\|_{L^{4/3}_{t,y}l^1_{n_1}l^1_{n_2}l^1_{n_3}}\\
&+\lambda_k^{-1}\big\|\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2\|(v_{n_1}\cdot\bar v_{n_2}\nabla_y v_{n_3})(t,y,x)\|_{L^2_x}\big\|_{L^{4/3}_{t,y}l^1_{n_1}l^1_{n_2}l^1_{n_3}}
\end{aligned}$$

$$\lesssim \lambda_k^{-1} \big\| \langle n_1\rangle^3 \langle n_2\rangle^3 \langle n_3\rangle^3 \|\nabla_y v_{n_1}(t,y,\cdot)\|_{L^2} \|v_{n_2}(t,y,\cdot)\|_{\mathcal{H}^1} \|v_{n_3}(t,y,\cdot)\|_{\mathcal{H}^1} \big\|_{L^{4/3}_{t,y} l^2_{n_1} l^2_{n_2} l^2_{n_3}}$$
$$\lesssim \lambda_k^{-1} \|\nabla_y v\|_{L^4_{t,y}\mathcal{H}^6_x} \|v\|^2_{L^4_{t,y}\mathcal{H}^7_x}$$
$$\lesssim \lambda_k^{-1} C(\|\psi\|_{H^1_y\mathcal{H}^6_x}) C(\|\psi\|_{L^2_y\mathcal{H}^7_x}) \to 0 \quad \text{as } k\to\infty. \tag{4-20}$$

Now, we only need to deal with $e^3_{\lambda_k}$. We will use the normal form transform to exploit additional decay of λ_k, since it possesses time nonresonance property. Integrating by parts and direct computation imply

$$\begin{aligned}
&\int_0^t e^{i(t-\tau)(\Delta_y+\Delta_x-x^2)} e^3_{\lambda_k}(\tau)\,\mathrm{d}\tau \\
&= -\lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} \int_0^t e^{it(\Delta_y-2n-1)} e^{-i\tau\tilde{\Delta}_y} P^y_{\le 2^{-10}}\Big(\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\,\mathrm{d}\tau \\
&= -\lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{it(\Delta_y-2n-1)} e^{-it\tilde{\Delta}_y} (-i\tilde{\Delta}_y)^{-1} P^y_{\le 2^{-10}}\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big) \\
&\quad +\lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{it(\Delta_y-2n-1)} (-i\tilde{\Delta}_y)^{-1} P^y_{\le 2^{-10}}\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(0,\frac{y}{\lambda_k},x\Big) \\
&\quad +\lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} \int_0^t e^{it(\Delta_y-2n-1)} e^{-i\tau\tilde{\Delta}_y} (-i\tilde{\Delta}_y)^{-1} \partial_\tau\Big(P^y_{\le 2^{-10}}\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\,\mathrm{d}\tau \\
&= \lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{it(\Delta_y+\Delta_x-x^2)} (-i\tilde{\Delta}_y)^{-1} P^y_{\le 2^{-10}}\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(0,\frac{y}{\lambda_k},x\Big) \\
&\quad -\lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{-2it(n_1-n_2+n_3)-it} (-i\tilde{\Delta}_y)^{-1} P^y_{\le 2^{-10}}\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big) \\
&\quad +\lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{it(\Delta_y-2n-1)} \int_0^t e^{-i\tau\tilde{\Delta}_y} (-i\tilde{\Delta}_y)^{-1} \partial_\tau P^y_{\le 2^{-10}}\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\,\mathrm{d}\tau,
\end{aligned}$$

where the operator $\tilde{\Delta}_y$ is defined to be

$$\tilde{\Delta}_y := 2(n_1-n_2+n_3-n)+\Delta_y.$$

This is a perturbation of the Laplacian operator and we suppress the parameters n_1, n_2, n_3, n. The inverse operator $(-\tilde{\Delta}_y)^{-1}$ is defined by the Fourier transform

$$\mathcal{F}_y((-i\tilde{\Delta}_y)^{-1}f)(\xi,x) = \frac{i(\mathcal{F}_y f)(\xi,x)}{2(n_1-n_2+n_3-n)-|\xi|^2}.$$

This operator is invertible when $n_1-n_2+n_3-n\neq 0$ and $|\xi|\le 2^{-10}$. We will use this expression in the remaining of the proof.

Define

$$e^{3,1}_{\lambda_k} := \Big\| \lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{it(\Delta_y+\Delta_x-x^2)} P^y_{\le 2^{-10}}\Big((-i\tilde{\Delta}_y)^{-1}\Big(\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(0,\frac{y}{\lambda_k},x\Big)\Big)\Big)\Big\|_{L^4_{t,y}\mathcal{H}^1_x},$$

$$e^{3,2}_{\lambda_k} := \Big\| \lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{-2it(n_1-n_2+n_3)-it}(-i\tilde{\Delta}_y)^{-1}\Big(P^y_{\le 2^{-10}}\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\Big\|_{L^4_{t,y}\mathcal{H}^1_x},$$

and

$$e^{3,3}_{\lambda_k} := \Big\| \lambda_k^{-3} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{it(\Delta_y-2n-1)} \cdot\int_0^t e^{-i\tau\tilde{\Delta}_y}(-i\tilde{\Delta}_y)^{-1}\Big(\partial_\tau P^y_{\le 2^{-10}}\Pi_n(v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\,\mathrm{d}\tau\Big\|_{L^4_{t,y}\mathcal{H}^1_x}.$$

Then, we have

$$\Big\|\int_0^t e^{i(t-\tau)(\Delta_y+\Delta_x-x^2)} e^3_{\lambda_k}(\tau,y,x)\,\mathrm{d}\tau\Big\|_{L^4_{t,y}\mathcal{H}^1_x} \sim e^{3,1}_{\lambda_k}+e^{3,2}_{\lambda_k}+e^{3,3}_{\lambda_k}. \tag{4-21}$$

First, we consider the term $e^{3,1}_{\lambda_k}$. By the boundedness of the operator $P^y_{\le 2^{-10}}(-i\tilde{\Delta}_y)^{-1}$ when $n_1-n_2+n_3\neq n$ and Minkowski's inequality, we may estimate as follows:

$$\begin{aligned}
e^{3,1}_{\lambda_k} &\lesssim \Big\|\Big\|\langle n\rangle^{\frac12} \sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} \lambda_k^{-3} P^y_{\le 2^{-10}}(-i\tilde{\Delta}_y)^{-1}\Pi_n\Big((v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(0,\frac{y}{\lambda_k},x\Big)\Big)\Big\|_{l^2_n}\Big\|_{L^2_{y,x}}\\
&\lesssim \lambda_k^{-3}\Big\|\langle n\rangle^{\frac12} \sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} \Big\|\Pi_n\Big((v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(0,\frac{y}{\lambda_k},x\Big)\Big)\Big\|_{L^2_y}\Big\|_{L^2_x l^2_n}\\
&\lesssim \lambda_k^{-2}\Big\|\langle n\rangle^{\frac12} \sum_{n_1,n_2,n_3\in\mathbb{N}} \|\Pi_n\big((v_{n_1}\bar{v}_{n_2}v_{n_3})(0,y,x)\big)\|_{L^2_y}\Big\|_{L^2_x l^2_n}\\
&\lesssim \lambda_k^{-2}\sum_{n_1,n_2,n_3\in\mathbb{N}} \|\langle n\rangle^{\frac12}\Pi_n\big((v_{n_1}\bar{v}_{n_2}v_{n_3})(0,y,x)\big)\|_{L^2_{y,x}l^2_n} \lesssim \lambda_k^{-2}\sum_{n_1,n_2,n_3\in\mathbb{N}} \|(v_{n_1}\bar{v}_{n_2}v_{n_3})(0,y,x)\|_{L^2_y\mathcal{H}^1_x}\\
&\lesssim \lambda_k^{-2}\sum_{n_1,n_2,n_3\in\mathbb{N}} \|v_{n_1}(0,y,x)\|_{L^6_y\mathcal{H}^1_x}\|v_{n_2}(0,y,x)\|_{L^6_y\mathcal{H}^1_x}\|v_{n_3}(0,y,x)\|_{L^6_y\mathcal{H}^1_x}\\
&\lesssim \lambda_k^{-2}C(\|v(0,y,x)\|_{H^1_y\mathcal{H}^3_x})^3 \to 0 \quad \text{as } k\to\infty.
\end{aligned} \tag{4-22}$$

Next, we consider the term $e^{3,2}_{\lambda_k}$. As in the estimate of $e^{3,1}_{\lambda_k}$, by the boundedness of the operator $P^y_{\le 2^{-10}}(-i\tilde{\Delta}_y)^{-1}$ when $n_1-n_2+n_3\neq n$, Minkowski's inequality, the fractional Leibniz rule, Sobolev's inequality and Hölder's inequality, we have

$$e^{3,2}_{\lambda_k} \lesssim \lambda_k^{-3}\Big\|\langle n\rangle^{\frac12} \sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3\neq n}} e^{-2it(n_1-n_2+n_3)-it}(-i\tilde{\Delta}_y)^{-1} \cdot P^y_{\le 2^{-10}}\Pi_n\Big((v_{n_1}\bar{v}_{n_2}v_{n_3})\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\Big\|_{L^4_{t,y}L^2_x l^2_n}$$

$$
\begin{aligned}
&\lesssim \lambda_k^{-3}\Big\|\langle n\rangle^{\frac12}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3\neq n}}\|\Pi_n\Big((v_{n_1}\bar v_{n_2}v_{n_3})\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\|_{L_y^4}\Big\|_{L_t^4L_x^2l_n^2}\\
&\lesssim \lambda_k^{-\frac32}\Big\|\sum_{n_1,n_2,n_3\in\mathbb{N}}\|v_{n_1}\bar v_{n_2}v_{n_3}\|_{H_y^{1/2}\mathcal{H}_x^1}\Big\|_{L_t^4}\lesssim \lambda_k^{-\frac32}\big\|\|v(t,y,x)\|^3_{W_y^{3/4,4}\mathcal{H}_x^5}\big\|_{L_t^4}\\
&\lesssim \lambda_k^{-\frac32}\|v(t,y,x)\|^3_{L_t^{12}W_y^{3/4,4}\mathcal{H}_x^5}\lesssim \lambda_k^{-\frac32}C(\|v(0,y,x)\|_{H_y^{13/12}\mathcal{H}_x^5})^3\to 0\quad\text{as } k\to\infty. \qquad (4\text{-}23)
\end{aligned}
$$

Finally, we are left to consider the term $e^{3,3}_{\lambda_k}$. Applying the Strichartz estimate, we obtain

$$
\begin{aligned}
&\Big\|\lambda_k^{-3}\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}}e^{it(\Delta_y-2n-1)}\cdot\int_0^t e^{-i\tau\tilde\Delta_y}(-i\tilde\Delta_y)^{-1}\Big(\partial_\tau P^y_{\le 2^{-10}}\Pi_n(v_{n_1}\bar v_{n_2}v_{n_3})\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\,d\tau\Big\|_{L^4_{t,y}\mathcal{H}^1_x}\\
&\lesssim\Big\|(i\partial_t+\Delta_y+\Delta_x-x^2)\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}}e^{it(\Delta_y-2n-1)}\\
&\quad\cdot\int_0^t P^y_{\le 2^{-10}}\Big(e^{-i\tau\tilde\Delta_y}(-i\tilde\Delta_y)^{-1}\\
&\quad\cdot\partial_\tau\Pi_n\Big(\lambda_k^{-3}v_{n_1}\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\overline{v_{n_2}\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)}v_{n_3}\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\Big)\,d\tau\Big\|_{L_t^1L_y^2\mathcal{H}_x^1}. \qquad (4\text{-}24)
\end{aligned}
$$

We observe, after some computation, that

$$
\begin{aligned}
&(i\partial_t+\Delta_y+\Delta_x-x^2)\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}}e^{it(\Delta_y-2n-1)}\int_0^t e^{-i\tau\tilde\Delta_y}(-i\tilde\Delta_y)^{-1}P^y_{\le 2^{-10}}\\
&\qquad\cdot\partial_\tau\Pi_n\Big(\lambda_k^{-3}v_{n_1}\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\overline{v_{n_2}\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)}v_{n_3}\Big(\frac{\tau}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\,d\tau\\
&=\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3\neq n}}e^{-2it(n_1-n_2+n_3)-it}(-i\tilde\Delta_y)^{-1}P^y_{\le 2^{-10}}\\
&\qquad\cdot\partial_t\Pi_n\Big(\lambda_k^{-3}v_{n_1}\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\overline{v_{n_2}\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)}v_{n_3}\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big).
\end{aligned}
$$

Therefore, by the above observation, Plancherel's theorem and Leibniz's rule, we have

$$
\begin{aligned}
(4\text{-}24)&\lesssim\Big\|\Big(\sum_{n\in\mathbb{N}}\Big(\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3\neq n}}\Big|e^{-2it(n_1-n_2+n_3-n)-it}(-i\tilde\Delta_y)^{-1}P^y_{\le 2^{-10}}\partial_t\Pi_n\Big(\lambda_k^{-3}v_{n_1}\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\\
&\qquad\cdot\overline{v_{n_2}\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)}v_{n_3}\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\Big|\Big)^2\langle n\rangle\Big)^{\frac12}\Big\|_{L_t^1L_y^2L_x^2}\\
&\lesssim\Big\|\Big(\sum_{n\in\mathbb{N}}\Big(\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3\neq n}}\Big\|\partial_t\Pi_n\Big(\lambda_k^{-3}v_{n_1}\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\\
&\qquad\cdot\overline{v_{n_2}\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)}v_{n_3}\Big(\frac{t}{\lambda_k^2},\frac{y}{\lambda_k},x\Big)\Big)\Big\|_{L^2_{y,x}}\Big)^2\langle n\rangle\Big)^{\frac12}\Big\|_{L_t^1}
\end{aligned}
$$

$$\lesssim \lambda_k^{-2} \sum_{n_1,n_2,n_3\in\mathbb{N}} \|\partial_t v_{n_1} \cdot \bar{v}_{n_2} v_{n_3}\|_{L_t^1 L_y^2 \mathcal{H}_x^1} + \lambda_k^{-2} \sum_{n_1,n_2,n_3\in\mathbb{N}} \|v_{n_1} \cdot \overline{\partial_t v_{n_2}} v_{n_3}\|_{L_t^1 L_y^2 \mathcal{H}_x^1} + \lambda_k^{-2} \sum_{n_1,n_2,n_3\in\mathbb{N}} \|v_{n_1} \cdot \bar{v}_{n_2} \partial_t v_{n_3}\|_{L_t^1 L_y^2 \mathcal{H}_x^1}. \tag{4-25}$$

We shall only show how to estimate the first term on the right-hand side of (4-25), as the other two terms can be estimated similarly. By Hölder's inequality, and the fact that v satisfies (DCR), we have

$$\begin{aligned}\sum_{n_1,n_2,n_3\in\mathbb{N}} \|\partial_t v_{n_1} \bar{v}_{n_2} v_{n_3}\|_{L_t^1 L_y^2 \mathcal{H}_x^1} &\lesssim \|v\|^2_{L_t^3 L_y^6 \mathcal{H}_x^3} \|\partial_t v\|_{L_t^3 L_y^6 \mathcal{H}_x^3} \\ &\lesssim \|v\|^2_{L_t^3 L_y^6 \mathcal{H}_x^3} \|\Delta_y v\|_{L_t^3 L_y^6 \mathcal{H}_x^3} + \|v\|^2_{L_t^3 L_y^6 \mathcal{H}_x^3} \Bigg\| \sum_{\substack{n_4,n_5,n_6,n_1\in\mathbb{N} \\ n_4-n_5+n_6=n_1}} \Pi_{n_1}(v_{n_4} \bar{v}_{n_5} v_{n_6}) \Bigg\|_{L_t^3 L_y^6 \mathcal{H}_x^3}.\end{aligned}$$

Applying Hölder's inequality and the Sobolev embedding, we have

$$\begin{aligned}&\Bigg\| \sum_{\substack{n_4,n_5,n_6,n_1\in\mathbb{N} \\ n_4-n_5+n_6=n_1}} \Pi_{n_1}(v_{n_4} \bar{v}_{n_5} v_{n_6}) \Bigg\|_{L_t^3 L_y^6 \mathcal{H}_x^3} \\ &\quad\lesssim \Bigg\| \langle n_1\rangle^{\frac{3}{2}} \sum_{\substack{n_4,n_5,n_6\in\mathbb{N} \\ n_4-n_5+n_6=n_1}} \Pi_{n_1}(v_{n_4} \bar{v}_{n_5} v_{n_6}) \Bigg\|_{L_t^3 L_y^6 L_x^2 l_{n_1}^2} \\ &\quad\lesssim \Bigg\| \sum_{\substack{n_4,n_5,n_6\in\mathbb{N} \\ n_4-n_5+n_6=n_1}} \langle n_1\rangle^{-1} \langle n_4\rangle^3 \langle n_5\rangle^3 \langle n_6\rangle^3 |\Pi_{n_1}(v_{n_4} \bar{v}_{n_5} v_{n_6})| \Bigg\|_{L_t^3 L_y^6 L_x^2 l_{n_1}^2} \\ &\quad\lesssim \Bigg\| \sum_{n_4,n_5,n_6\in\mathbb{N}} \langle n_4\rangle^3 \langle n_5\rangle^3 \langle n_6\rangle^3 \|v_{n_4} \bar{v}_{n_5} v_{n_6}\|_{L_x^2} \Bigg\|_{L_t^3 L_y^6} \\ &\quad\lesssim \Bigg\| \sum_{n_4,n_5,n_6\in\mathbb{N}} \langle n_4\rangle^3 \langle n_5\rangle^3 \langle n_6\rangle^3 \|v_{n_4}\|_{\mathcal{H}_x^1} \|v_{n_5}\|_{\mathcal{H}_x^1} \|v_{n_6}\|_{\mathcal{H}_x^1} \Bigg\|_{L_t^3 L_y^6} \lesssim \|v\|^3_{L_t^9 L_y^{18} \mathcal{H}_x^9} \le C(\|\psi\|_{H_y^{2/3} \mathcal{H}_x^9})^3,\end{aligned}$$

where in the last inequality we use the fact that by Corollary 4.8, we have

$$\|v\|_{L_t^3 L_y^6 \mathcal{H}_x^3} \le C(\|\psi\|_{L_y^2 \mathcal{H}_x^3}),$$
$$\|\Delta_y v\|_{L_t^3 L_y^6 \mathcal{H}_x^3} \le C(\|\Delta_y \psi\|_{L_y^2 \mathcal{H}_x^3}).$$

Combining all these estimates together, we finally obtain

$$e^{3,3}_{\lambda_k} \lesssim \lambda_k^{-2} \big(C(\|\psi\|_{L_y^2 \mathcal{H}_x^3})^2 C(\|\Delta_y \psi\|_{L_y^2 \mathcal{H}_x^3}) + C(\|\psi\|_{L_y^2 \mathcal{H}_x^3})^2 C(\|\psi\|_{H_y^{2/3} \mathcal{H}_x^9})^3 \big) \to 0 \quad \text{as } k \to \infty. \tag{4-26}$$

To apply Theorem 3.6, we see

$$\lim_{k\to\infty} \|w_{\lambda_k}(0,y,x) - u_{\lambda_k}(0,y,x)\|_{L_y^2 \mathcal{H}_x^1} = \|\phi - \psi\|_{L_y^2 \mathcal{H}_x^1} \le \epsilon_4,$$

$$\|w_{\lambda_k}\|_{L_t^\infty L_y^2 \mathcal{H}_x^1(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} = \left\| \frac{1}{\lambda_k} v\left(\frac{t}{\lambda_k^2}, \frac{y}{\lambda_k}, x\right)\right\|_{L_t^\infty L_y^2 \mathcal{H}_x^1} = \|v\|_{L_t^\infty L_y^2 \mathcal{H}_x^1},$$

$$\|w_{\lambda_k}\|_{L_{t,y}^4 \mathcal{H}_x^1(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} = \left\| \frac{1}{\lambda_k} v\left(\frac{t}{\lambda_k^2}, \frac{y}{\lambda_k}, x\right)\right\|_{L_{t,y}^4 \mathcal{H}_x^1} = \|v\|_{L_{t,y}^4 \mathcal{H}_x^1}.$$

These together with the estimates (4-18), (4-20), (4-21), (4-22), (4-23), (4-26) and Theorem 1.2 yields Theorem 4.9 when $t_k = 0$.

If $t_k \to \pm\infty$ as $k \to \infty$, then v is the solution of (DCR) with

$$\lim_{t\to\pm\infty} \|v(t, y, x) - e^{it\Delta_y}\psi\|_{L_y^2 \mathcal{H}_x^1} = 0.$$

By the argument in the case $t_k = 0$, we can also obtain Theorem 4.9 in this case. □

4.3. ***Existence of an almost-periodic solution.*** Define

$$\Lambda(L) = \sup \|u\|_{L_{t,y}^4 \mathcal{H}_x^{1-\epsilon_0}(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})},$$

where the supremum is taken over all global solutions $u \in C_t^0(\mathbb{R}, \Sigma(\mathbb{R}^3))$ of (1-2) with

$$\mathcal{E}(u(t)) + \mathcal{M}(u(t)) \le L.$$

The proof of Theorem 3.4 implies $\Lambda(L) < \infty$ for sufficiently small L. Let

$$L_{\max} = \sup\{L \ge 0 : \Lambda(L) < \infty\}. \tag{4-27}$$

If $L_{\max} < \infty$, then following the arguments in [Cheng et al. 2020a; 2020b], one can show the existence of an almost periodic solution with the help of Theorems 4.1 and 4.9. The proof is rather standard, we refer to [Cheng et al. 2020a; Cheng et al. 2020b; Killip and Vişan 2013; Kenig and Merle 2006; 2008; Tao et al. 2008] and omit the proof here.

Theorem 4.10 (existence of an almost-periodic solution). *Assume that $L_{\max} < \infty$. Then there exists a solution $u_c \in C_t^0(\mathbb{R}, \Sigma(\mathbb{R}^3))$ of the defocusing cubic NLS with partial harmonic potential (1-2) satisfying*

$$\mathcal{E}(u_c) + \mathcal{M}(u_c) = L_{\max} \quad \textit{and} \quad \|u_c\|_{L_{t,y}^4 \mathcal{H}_x^{1-\epsilon_0}(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} = \infty. \tag{4-28}$$

Furthermore, u_c is almost periodic in the sense that for any $\eta > 0$ there is a Lipschitz function $t \mapsto y(t)$ and a sufficiently large positive number $C(\eta)$ such that

$$\int_{|y+y(t)|\ge C(\eta)} \|u_c(t, y, x)\|_{\mathcal{H}_x^1}^2 \,\mathrm{d}y < \eta \quad \textit{for all } t \in \mathbb{R}. \tag{4-29}$$

5. Rigidity theorem

In this section, we will exclude the almost-periodic solution in Theorem 4.10 by the interaction Morawetz estimate with an appropriately chosen weight function. Once the almost-periodic solution is excluded, we can finish the proof of Theorem 1.1.

Proposition 5.1 (nonexistence of the almost-periodic solution). *The almost-periodic solution u_c as in Theorem 4.10 does not exist.*

Proof. For each $r_0 > 0$, we define the interaction Morawetz action

$$M_{r_0}(t) = \int_{\mathbb{R}^2\times\mathbb{R}} \int_{\mathbb{R}^2\times\mathbb{R}} \Im(\overline{u_c(t,y,x)}\nabla_y u_c(t,y,x)) \cdot \nabla_y \psi_{r_0}(|y-\tilde{y}|)|u_c(t,\tilde{y},\tilde{x})|^2 \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{x},$$

where $\Im = \mathrm{Im}$ denotes the imaginary part of a complex number and $\psi_{r_0}\colon \mathbb{R}\to\mathbb{R}$ is a radial function defined as in [Colliander et al. 2009; Planchon and Vega 2009], with

$$\Delta\psi_{r_0}(r) = \int_r^\infty s\log\left(\frac{s}{r}\right) w_{r_0}(s)\,\mathrm{d}s,$$

where

$$w_{r_0}(s) = \begin{cases} \frac{1}{s^3} & \text{if } s \ge r_0, \\ 0 & \text{if } s < r_0. \end{cases}$$

It is straightforward to verify that ψ_{r_0} is convex and $|\nabla\psi_{r_0}|$ is uniformly bounded (independent of r_0), with

$$-\Delta^2\psi_{r_0}(r) = \frac{2\pi}{r_0}\delta_0(r) - w_{r_0}(r).$$

Using the above properties of the weight function ψ_{r_0}, one can show (see [Colliander et al. 2009, Section 3.3]) for all $T_0 > 0$

$$\int_{-T_0}^{T_0}\int_{\mathbb{R}^2} \left||\nabla_y|^{\frac{1}{2}}(\|u_c(t,y,x)\|^2_{L_x^2(\mathbb{R})})\right|^2 \mathrm{d}y\,\mathrm{d}t \lesssim \|u_c\|^3_{L_t^\infty L^2_{y,x}} \|\nabla_y u_c\|_{L_t^\infty L^2_{y,x}} \lesssim 1. \tag{5-1}$$

By (4-29) and the conservation of mass, we have

$$\frac{\|u_c\|^2_{L^2_{y,x}}}{2} \le \int_{|y+y(t)|\le C(\frac{m_c}{100})} \|u_c(t,y,x)\|^2_{L_x^2}\,\mathrm{d}y, \tag{5-2}$$

where $m_c := \mathcal{M}(u_c) > 0$ by (4-28).

Therefore, for each $T_0 > 0$, by (5-2), Sobolev's inequality, and (5-1), we deduce

$$\begin{aligned}
\frac{m_c^2 T_0}{2} &\le \int_{-T_0}^{T_0}\left(\int_{|y+y(t)|\le C(\frac{m_c}{100})} \|u_c(t,y,x)\|^2_{L_x^2}\,\mathrm{d}y\right)^2 \mathrm{d}t \\
&\lesssim C\left(\frac{m_c}{100}\right)\int_{-T_0}^{T_0}\left(\left(\int_{\mathbb{R}^2}(\|u_c(t,y,x)\|^2_{L_x^2})^4\,\mathrm{d}y\right)^{\frac{1}{4}}\right)^2 \mathrm{d}t \\
&\lesssim C\left(\frac{m_c}{100}\right)\int_{-T_0}^{T_0}\int_{\mathbb{R}^2}\left||\nabla_y|^{\frac{1}{2}}(\|u_c(t,y,x)\|^2_{L_x^2})\right|^2 \mathrm{d}y\mathrm{d}t \lesssim C\left(\frac{m_c}{100}\right).
\end{aligned}$$

Letting $T_0\to\infty$, we obtain a contradiction, and this concludes the proof. □

Finally, we can now prove Theorem 1.1.

Proof of Theorem 1.1. By Theorem 3.4, to prove the scattering of solutions to (1-2), it suffices to show the finiteness of the $L^4_{t,y}\mathcal{H}_x^{1-\epsilon_0}$-norm of the solution of (1-2).

To this end, let $L_{\max}$ be given as in (4-27). Then, equivalently, we need show that $L_{\max} = \infty$. Suppose for a contradiction that $L_{\max} < \infty$. Then Theorem 4.10 would yield an almost-periodic solution of (1-2), which is impossible in view of Proposition 5.1. □

6. Scattering of equation (DCR)

We will now prove Theorem 1.2, that is, the global well-posedness and scattering of the (DCR) system

$$\begin{cases} i\partial_t v + \Delta_{\mathbb{R}^2} v = F(v), \\ v(0, y, x) = \phi(y, x), \end{cases}$$

where

$$F(v) := \sum_{\substack{n_1, n_2, n_3, n \in \mathbb{N} \\ n_1 - n_2 + n_3 = n}} \Pi_n(\Pi_{n_1} v \overline{\Pi_{n_2} v} \Pi_{n_3} v)$$

and Π_n is the orthogonal projector on the n-th eigenspace E_n of $-\Delta_x + x^2$.

We will mainly follow the approach to the global well-posedness and scattering of the two-dimensional mass-critical nonlinear Schrödinger equation as in [Dodson 2016b]. The main ingredient is to establish an infinite-dimensional vector-valued version of the two-dimensional long-time Strichartz estimate, which helps us to preclude certain almost periodic solutions.

The (DCR) system is Hamiltonian with an energy functional

$$\mathcal{E}(v) = \frac{1}{2} \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^2 \times \mathbb{R}} |\nabla_y v_n|^2 \, dy \, dx + \frac{1}{4} \sum_{\substack{n, n_1, n_2, n_3, n_4 \in \mathbb{N} \\ n_1 - n_2 + n_3 - n_4 = n}} \int_{\mathbb{R}^2 \times \mathbb{R}} v_{n_1} \bar{v}_{n_2} v_{n_3} \bar{v}_{n_4} \, dy \, dx,$$

under the symplectic structure on $L^2_{y,x}(\mathbb{R}^2 \times \mathbb{R})$ given by $\omega(f, g) := \Im \int_{\mathbb{R}^2 \times \mathbb{R}} f(y, x) \overline{g(y, x)} \, dy \, dx$. It also conserves the following mass $\mathcal{M}$ and kinetic energy $\mathcal{E}_0$:

$$\mathcal{M}(v) = \int_{\mathbb{R}^2 \times \mathbb{R}} |v(t, y, x)|^2 \, dy \, dx,$$

$$\mathcal{E}_0(v) = \int_{\mathbb{R}^2 \times \mathbb{R}} |(-\Delta_x + x^2)^{\frac{1}{2}} v(t, y, x)|^2 \, dy \, dx = \sum_{n \in \mathbb{N}} (2n+1) \|v_n\|^2_{L^2_{y,x}(\mathbb{R}^2 \times \mathbb{R})} = \|v\|^2_{L^2_y \mathcal{H}^1_x(\mathbb{R}^2 \times \mathbb{R})}.$$

We shall divide this section into three subsections. In Section 6.1, we establish the local well-posedness theory for (DCR) and reduce the scattering to the exclusion of almost periodic solutions. In Section 6.2, we derive the long-time Strichartz estimate and in Section 6.3, we exclude the almost periodic solution.

6.1. ***Local well-posedness and reduction to the almost periodic solution.*** In this subsection, we will present the well-posedness theory of the (DCR). Then following ideas similar to those in [Tao et al. 2008; Cheng et al. 2020a; 2020b; Yang and Zhao 2018], we shall prove that there is an almost periodic solution of (DCR) if the system is not global well-posed and if the solution does not scatter in $L^2_y \mathcal{H}^1_x$. That is, we reduce the global well-posedness and scattering of (DCR) to the exclusion of this almost periodic solution.

6.1.1. *Local well-posedness theory and the existence of an almost periodic solution.* The local well-posedness theory of the (DCR) system follows from a more or less standard argument: the Strichartz estimate in Proposition 3.2 and the nonlinear estimate in Lemma 6.2. The proof of the nonlinear estimate relies on the following Strichartz estimate for the harmonic oscillator.

Lemma 6.1 (Strichartz estimate for the harmonic oscillator [Carles 2002b; Keel and Tao 1998]). *For* $2 \le q, r \le \infty$ *with* $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$, *we have the estimate*

$$\|e^{it(\Delta_x - x^2)} f\|_{L_t^q \mathcal{W}_x^{s,r}([-T_1,T_1]\times\mathbb{R})} \lesssim \|f\|_{\mathcal{H}_x^s(\mathbb{R})}$$

holds for any $T_1 > 0$ *and* $s \ge 0$.

We can now give the nonlinear estimate.

Lemma 6.2. *For functions* F_1, F_2, F_3 *defined on* $\mathbb{R}^2 \times \mathbb{R}$, *we have*

$$\Big\| \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n(\Pi_{n_1}F_1 \overline{\Pi_{n_2}F_2}\Pi_{n_3}F_3)\Big\|_{L_y^{4/3}L_x^2} \lesssim \|F_1\|_{L_y^4L_x^2}\|F_2\|_{L_y^4L_x^2}\|F_3\|_{L_y^4L_x^2}, \tag{6-1}$$

and consequently, for any $\beta \ge 0$,

$$\Big\| \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n(\Pi_{n_1}F_1 \overline{\Pi_{n_2}F_2}\Pi_{n_3}F_3)\Big\|_{L_y^{4/3}\mathcal{H}_x^\beta} \lesssim \min_{\tau\in\sigma_3}\|F_{\tau(1)}\|_{L_y^4\mathcal{H}_x^\beta}\|F_{\tau(2)}\|_{L_y^4L_x^2}\|F_{\tau(3)}\|_{L_y^4L_x^2}, \tag{6-2}$$

where σ_3 *is a permutation of the set* $\{1,2,3\}$.

Proof. Let $F_0 \in L_y^4L_x^2$, by Hölder's inequality and Lemma 6.1, we have

$$\begin{aligned}
&\Big\langle \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n(\Pi_{n_1}F_1 \overline{\Pi_{n_2}F_2}\Pi_{n_3}F_3),\, F_0\Big\rangle \\
&\quad= \frac{1}{\pi}\sum_{n_1,n_2,n_3,n\in\mathbb{N}}\int_0^\pi e^{2it(n_1-n_2+n_3-n)}\int_{\mathbb{R}^2\times\mathbb{R}} \Pi_{n_1}F_1\overline{\Pi_{n_2}F_2}\Pi_{n_3}F_3\overline{\Pi_nF_0}\,dy\,dx\,dt \\
&\quad= \frac{1}{\pi}\int_0^\pi\int_{\mathbb{R}^2\times\mathbb{R}} e^{it(-\Delta_x+x^2)}F_1(y,x)\overline{e^{it(-\Delta_x+x^2)}F_2(y,x)} \\
&\qquad\qquad\cdot e^{it(-\Delta_x+x^2)}F_3(y,x)\overline{e^{it(-\Delta_x+x^2)}F_0(y,x)}\,dy\,dx\,dt \\
&\quad\lesssim \int_{\mathbb{R}^2} \|e^{it(-\Delta_x+x^2)}F_0(y,x)\|_{L_t^\infty L_x^2([0,\pi]\times\mathbb{R})}\|e^{it(-\Delta_x+x^2)}F_1(y,x)\|_{L_t^2L_x^4([0,\pi]\times\mathbb{R})} \\
&\qquad\qquad\cdot\|e^{it(-\Delta_x+x^2)}F_2(y,x)\|_{L_t^4L_x^8([0,\pi]\times\mathbb{R})}\|e^{it(-\Delta_x+x^2)}F_3(y,x)\|_{L_t^4L_x^8([0,\pi]\times\mathbb{R})}\,dy \\
&\quad\lesssim \int_{\mathbb{R}^2} \|F_0(y,x)\|_{L_x^2(\mathbb{R})}\|e^{it(-\Delta_x+x^2)}F_1(y,x)\|_{L_t^8L_x^4([0,\pi]\times\mathbb{R})}\|e^{it(-\Delta_x+x^2)}F_2(y,x)\|_{L_t^{16/3}L_x^8([0,\pi]\times\mathbb{R})} \\
&\qquad\qquad\cdot\|e^{it(-\Delta_x+x^2)}F_3(y,x)\|_{L_t^{16/3}L_x^8([0,\pi]\times\mathbb{R})}\,dy \\
&\quad\lesssim \int_{\mathbb{R}^2}\|F_0(y,x)\|_{L_x^2}\|F_1(y,x)\|_{L_x^2}\|F_2(y,x)\|_{L_x^2}\|F_3(y,x)\|_{L_x^2}\,dy.
\end{aligned}$$

Therefore,

$$\Big\| \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n(\Pi_{n_1}F_1 \overline{\Pi_{n_2}F_2}\Pi_{n_3}F_3)\Big\|_{L_y^{4/3}L_x^2} \lesssim \|F_1\|_{L_y^4L_x^2}\|F_2\|_{L_y^4L_x^2}\|F_3\|_{L_y^4L_x^2},$$

which is (6-1). One can similarly prove (6-2) using the fractional Leibniz rule. □

Lemma 6.2 provides the following estimate for the nonlinearity $F(v)$.

Lemma 6.3. *For each solution v of (DCR), we have*

$$\|F(v)\|_{L^{4/3}_{t,y}\mathcal{H}^\alpha_x} \lesssim \|v\|^3_{L^4_{t,y}\mathcal{H}^\alpha_x}, \quad \text{where } \alpha = 0, 1.$$

Thus by Proposition 3.2, the solution v of (DCR) satisfies the Strichartz estimate

$$\|v\|_{L^p_t L^q_y \mathcal{H}^\alpha_x(I\times\mathbb{R}^2\times\mathbb{R})} \lesssim \|v_0\|_{L^2_y\mathcal{H}^\alpha_x(\mathbb{R}^2\times\mathbb{R})} + \|v\|^3_{L^4_{t,y}\mathcal{H}^\alpha_x(I\times\mathbb{R}^2\times\mathbb{R})} \quad \text{for } \alpha = 0, 1, \tag{6-3}$$

where $I \subseteq \mathbb{R}$, and (p, q) is a Strichartz admissible pair.

As a consequence of Lemma 6.3 and (6-2), we obtain the following well-posedness theory. Since the proof is well known (see for instance [Cheng et al. 2020a; 2020b; Tao 2006; Killip and Vişan 2013]), we omit it.

Theorem 6.4 (well-posedness and scattering of (DCR)).

(1) (*local well-posedness*) *Assume $\|v_0\|_{L^2_y\mathcal{H}^1_x} < \infty$. The (DCR) admits a unique solution*

$$v \in (C^0_t L^2_y \mathcal{H}^1_x \cap L^4_{t,y}\mathcal{H}^1_x)((-T, T)\times\mathbb{R}^2\times\mathbb{R})$$

for some $T > 0$.

(2) (*small-data scattering*) *There is a sufficient small constant $\delta > 0$, such that when $\|v_0\|_{L^2_y\mathcal{H}^1_x} \le \delta$, (DCR) admits a unique global solution v with $v(0) = v_0$, which scatters in $L^2_y\mathcal{H}^1_x$ in the sense that there exist $v^\pm \in L^2_y\mathcal{H}^1_x(\mathbb{R}^2\times\mathbb{R})$ such that*

$$\|v(t) - e^{it\Delta_y}v^\pm\|_{L^2_y\mathcal{H}^1_x} \to 0 \quad \text{as } t \to \pm\infty.$$

(3) (*scattering norm*) *Suppose v is a maximal lifespan solution on I with $\|v\|_{L^4_{t,y}L^2_x(I\times\mathbb{R}^2\times\mathbb{R})} < \infty$. Then v globally exists and scatters in $L^2_y\mathcal{H}^1_x$.*

We also have the stability theorem by Lemmas 6.2 and 6.3. The argument is similar to the proof of Theorem 3.6, and we also refer to [Colliander et al. 2008; Killip and Vişan 2013].

Theorem 6.5 (stability). *Let $l \in \{0, 1\}$, I be a compact interval and $\tilde v \in (C^0_t L^2_y\mathcal{H}^1_x \cap L^4_{t,y}\mathcal{H}^1_x)(I\times\mathbb{R}^2\times\mathbb{R})$ be an approximate solution of (DCR) with the error term $e = i\partial_t\tilde v + \Delta_y\tilde v - F(\tilde v)$. Then, for any $\epsilon > 0$, there is $\delta > 0$ such that if*

$$\|e\|_{L^{4/3}_{t,y}\mathcal{H}^l_x(I\times\mathbb{R}^2\times\mathbb{R})} + \|\tilde v(t_0) - v_0\|_{L^2_y\mathcal{H}^1_x} \le \delta,$$

then (DCR) admits a solution $v \in (L^\infty_t L^2_y\mathcal{H}^1_x \cap L^4_{t,y}\mathcal{H}^l_x)(I\times\mathbb{R}^2\times\mathbb{R})$ with $v(t_0) = v_0$ and

$$\|\tilde v - v\|_{L^4_{t,y}\mathcal{H}^l_x \cap L^\infty_t L^2_y\mathcal{H}^1_x(I\times\mathbb{R}^2\times\mathbb{R})} < \epsilon.$$

To prove (DCR) is globally well-posed and scatters for large data, by Theorem 6.4, we need to prove

$$\|v\|_{L^4_{t,y}L^2_x(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} < \infty,$$

where v is a solution to (DCR) with initial data $v_0 \in L_y^2\mathcal{H}_x^1(\mathbb{R}^2 \times \mathbb{R})$. For the solution v of (DCR) with maximal lifespan interval I, let

$$S(m) = \sup\{\|v\|_{L_{t,y}^4 L_x^2(I\times\mathbb{R}^2\times\mathbb{R})} : \|v(0)\|_{L_y^2\mathcal{H}_x^1(\mathbb{R}^2\times\mathbb{R})} \leq m\},$$

and

$$m_0 = \sup\{m : S(\tilde{m}) < \infty \text{ for all } \tilde{m} < m\} > 0.$$

If we have $m_0 = \infty$, then the global well-posedness and scattering in $L_y^2\mathcal{H}_x^1$ of (DCR) hold. Following the argument in [Tao et al. 2008; Killip and Vişan 2013], and using Theorems 6.5 and 4.1 during the proof, we have:

Theorem 6.6 (existence of an almost periodic solution to (DCR)). *Assume $m_0 < \infty$. Then there exists an nonzero almost periodic solution $v \in C_t^0 L_y^2\mathcal{H}_x^1 \cap L_{t,y}^4 L_x^2(I \times \mathbb{R}^2 \times \mathbb{R})$ to (DCR) with I the maximal lifespan interval such that $\mathcal{M}(v) = m_0$. In addition, for any $\eta > 0$, there exists $C(\eta) > 0$ and $(y(t), \xi(t), N(t)) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}_+$ such that*

$$\int_{|y-y(t)|\geq \frac{C(\eta)}{N(t)}} \|v(t,y,x)\|_{\mathcal{H}_x^1}^2 \,\mathrm{d}y + \int_{|\xi-\xi(t)|\geq C(\eta)N(t)} \|(\mathcal{F}_y v)(t,\xi,x)\|_{\mathcal{H}_x^1}^2 \,\mathrm{d}\xi < \eta \quad \text{for all } t \in I. \tag{6-4}$$

Furthermore, we can take $[0,\infty) \subseteq I$, and $N(0) = 1$, $\xi(0) = y(0) = 0$, with

$$N(t) \leq 1, \quad |N'(t)| + |\xi'(t)| \lesssim N(t)^3 \quad \text{for all } t \in [0,\infty).$$

As in [Dodson 2012; 2016a; 2016b; Killip and Vişan 2013; Rosenzweig 2018], we see the almost periodic solution in Theorem 6.6 has the following property:

Theorem 6.7. (1) *If $J \subseteq I$ is an interval which is partitioned into small intervals J_k in the sense that $\|v\|_{L_{t,y}^4 L_x^2(J_k\times\mathbb{R}^2\times\mathbb{R})} = 1$, then we have*

$$N(J_k) \sim \int_{J_k} N(t)^3 \,\mathrm{d}t \sim \inf_{t\in J_k} N(t) \quad \text{and} \quad \sum_{J_k\subseteq J} N(J_k) \sim \int_J N(t)^3 \,\mathrm{d}t, \tag{6-5}$$

where $N(J_k) = \sup_{t\in J_k} N(t)$.

(2) *For any interval $J \subseteq [0,\infty)$, we have*

$$\int_J N(t)^2 \,\mathrm{d}t \lesssim \|v\|_{L_{t,y}^4 L_x^2(J\times\mathbb{R}^2\times\mathbb{R})}^4 \lesssim 1 + \int_J N(t)^2 \,\mathrm{d}t. \tag{6-6}$$

Proof of Theorems 6.6 and 6.7. With Theorems 4.1 and 6.5 at hand, one can follow the arguments in [Cheng et al. 2020a; 2020b; Dodson 2012; 2016a; 2016b; Rosenzweig 2018; Tao et al. 2008; Killip and Vişan 2013]. □

6.1.2. *Some functional spaces and bilinear Strichartz estimates.* As in [Dodson 2016b], due to the failure of the endpoint Strichartz estimate in two dimensions, we need to utilize the function spaces U_Δ^p and V_Δ^p introduced originally in the seminal work [Koch and Tataru 2005a]; see also [Hadac et al. 2009; Koch and Tataru 2007; Koch et al. 2014] for more detailed study on these spaces. The structure of our (DCR) system motivates us to introduce the Banach spaces $U_\Delta^p(L_x^2)$ and $V_\Delta^p(L_x^2)$ as follows.

Definition 6.8 ($U^p_\Delta(L^2_x)$ space). For $1 \le p < \infty$, let $U^p_\Delta(L^2_x)$ be an atomic space, where an atom v^γ is defined to be

$$v^\gamma(t,y,x) = \sum_{k=0}^{N} \chi_{[t_k,t_{k+1})}(t) e^{it\Delta_y} v^\gamma_k(y,x), \quad \text{with } \sum_{k=0}^{N} \|v^\gamma_k(y,x)\|^p_{L^2_{y,x}} = 1.$$

In the expansion of v^γ, N may be finite or infinite, $t_0 = -\infty$, and $t_{N+1} = \infty$ if N is finite. We impose a norm on $\|\cdot\|_{U^p_\Delta(L^2_x)}$ as

$$\|v\|_{U^p_\Delta(L^2_x)} = \inf\left\{ \sum_\gamma |c_\gamma| : v = \sum_\gamma c_\gamma v^\gamma, \text{ where } v^\gamma \text{ are } U^p_\Delta(L^2_x) \text{ atoms} \right\}.$$

For a time interval $I \subseteq \mathbb{R}$, we define

$$\|v\|_{U^p_\Delta(L^2_x, I)} = \|v 1_I\|_{U^p_\Delta(L^2_x)}.$$

Let $DU^p_\Delta(L^2_x)$ be the space

$$DU^p_\Delta(L^2_x) = \{(i\partial_t + \Delta_y)v : v \in U^p_\Delta(L^2_x)\},$$

endowed with the norm

$$\|(i\partial_t + \Delta_y)v(t,y,x)\|_{DU^p_\Delta(L^2_x)} = \left\| \int_0^t e^{i(t-s)\Delta_y}(i\partial_s + \Delta_y)v(s,y,x)\,ds \right\|_{U^p_\Delta(L^2_x)}.$$

For each time interval $I \subseteq \mathbb{R}$, we can similarly define the restriction space $DU^p_\Delta(L^2_x, I)$.

Definition 6.9 ($V^p_\Delta(L^2_x)$ space). For $1 \le p < \infty$, $V^p_\Delta(L^2_x)$ is defined to be the space of right continuous functions $v \in L^\infty_t L^2_{y,x}$ such that

$$\|v\|^p_{V^p_\Delta(L^2_x)} = \|v\|^p_{L^\infty_t L^2_{y,x}} + \sup_{\{t_k\}_k \nearrow} \sum_k \|e^{-it_{k+1}\Delta_y} v(t_{k+1}) - e^{-it_k\Delta_y} v(t_k)\|^p_{L^2_{y,x}} < \infty.$$

When the time is restricted to $I \subseteq \mathbb{R}$, We can similarly define the function space $V^p_\Delta(L^2_x, I)$. Then we have

$$(DU^p_\Delta(L^2_x))^* = V^{p'}_\Delta(L^2_x). \tag{6-7}$$

The following basic properties are straightforward to verify. For the proofs, see [Hadac et al. 2009; Koch et al. 2014].

Remark 6.10 (basic properties of $U^p_\Delta(L^2_x)$ and $V^p_\Delta(L^2_x)$). For any $1 < p < q < \infty$ and $a \le b \le c$, we have

$$U^p_\Delta(L^2_x) \subseteq V^p_\Delta(L^2_x) \subseteq U^q_\Delta(L^2_x), \tag{6-8}$$

$$\|v\|_{U^p_\Delta(L^2_x,[a,b])} \le \|v\|_{U^p_\Delta(L^2_x,[a,c])} \quad \text{and} \quad \|v\|^p_{U^p_\Delta(L^2_x,[a,c])} \le \|v\|^p_{U^p_\Delta(L^2_x,[a,b])} + \|v\|^p_{U^p_\Delta(L^2_x,[b,c])}, \tag{6-9}$$

$$\|v\|_{U^p_\Delta(L^2_x)} \lesssim \|v|_{t=0}\|_{L^2_{y,x}} + \|(i\partial_t + \Delta_y)v\|_{DU^p_\Delta(L^2_x)}. \tag{6-10}$$

Moreover,

$$L^{p'}_t L^{r'}_y L^2_x \subseteq DU^2_\Delta(L^2_x), \quad \text{and} \quad U^p_\Delta(L^2_x) \subseteq L^p_t L^r_y L^2_x, \quad \text{where } (p,r) \text{ is Strichartz admissible.} \tag{6-11}$$

Following the argument in [Dodson 2016b], we also have:

Lemma 6.11. *Suppose* $I = \bigcup_{j=1}^{m} I^j$, *where* $I^j = [a_j, b_j]$, $a_{j+1} = b_j$. *If* $f \in L_t^1 L_{y,x}^2(I \times \mathbb{R}^2 \times \mathbb{R})$, *then, for all* $t_0 \in I$,

$$\left\| \int_{t_0}^{t} e^{i(t-\tau)\Delta_y} f(\tau, y, x)\, \mathrm{d}\tau \right\|_{U_\Delta^2(L_x^2, I)} \lesssim \sum_{j=1}^{m} \left\| \int_{I^j} e^{-i\tau\Delta_y} f(\tau)\, \mathrm{d}\tau \right\|_{L_{y,x}^2} + \left(\sum_{j=1}^{m} \|f\|_{DU_\Delta^2(L_x^2, I^j)}^2 \right)^{\frac{1}{2}},$$

where

$$\|f\|_{DU_\Delta^2(L_x^2, I^j)} = \sup_{\|w\|_{V_\Delta^2(L_x^2, I^j)} = 1} \int_{I^j} \int_{\mathbb{R}^2 \times \mathbb{R}} f(\tau, y, x) \overline{w(\tau, y, x)}\, \mathrm{d}\tau\, \mathrm{d}y\, \mathrm{d}x.$$

By the bilinear Strichartz estimate in [Bourgain 1998], Minkowski's inequality, Hölder's inequality, and interpolation, we have the following two propositions. The proofs are similar to the bilinear Strichartz estimates in [Dodson 2016a; 2016b].

Proposition 6.12 (bilinear Strichartz estimate, I). *Let* (p, q) *satisfy* $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *For* $M \ll N$, *assume* $\operatorname{supp} \mathcal{F}_y u_0 \subseteq \{\xi : |\xi| \sim N\}$ *and* $\operatorname{supp} \mathcal{F}_y v_0 \subseteq \{\xi : |\xi| \sim M\}$. *Then we have*

$$\big\| \|e^{it\Delta_y} u_0\|_{L_x^2} \|e^{it\Delta_y} v_0\|_{L_x^2} \big\|_{L_t^p L_y^q(\mathbb{R} \times \mathbb{R}^2)} \lesssim \left(\frac{M}{N}\right)^{\frac{1}{p}} \|u_0\|_{L_{y,x}^2} \|v_0\|_{L_{y,x}^2}. \tag{6-12}$$

Furthermore, suppose that $g(t, y - \tilde{y})$ *and* $h(t, y - \tilde{\tilde{y}})$ *are convolution kernels with respect to the* y*-variable and*

$$\big\| \sup_{t \in \mathbb{R}} |g(t, y)| \big\|_{L_y^1(\mathbb{R}^2)} + \big\| \sup_{t \in \mathbb{R}} |h(t, y)| \big\|_{L_y^1(\mathbb{R}^2)} \lesssim 1.$$

Then we also have

$$\big\| \|g *_y e^{it\Delta_y} u_0\|_{L_x^2} \|h *_y e^{it\Delta_y} v_0\|_{L_x^2} \big\|_{L_t^p L_y^q(\mathbb{R} \times \mathbb{R}^2)} \lesssim \left(\frac{M}{N}\right)^{\frac{1}{p}} \|u_0\|_{L_{y,x}^2} \|v_0\|_{L_{y,x}^2}.$$

Similar to the argument in the proof of Lemma 3.5 in [Dodson 2016b], we can transfer the estimate (6-12) to the U_Δ^p space. Therefore, we have:

Proposition 6.13 (bilinear Strichartz estimate, II). *Let* (p, q) *satisfy* $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *For* $M \ll N$, *assume* $\operatorname{supp} \mathcal{F}_y u \subseteq \{\xi : |\xi| \sim N\}$ *and* $\operatorname{supp} \mathcal{F}_y v \subseteq \{\xi : |\xi| \sim M\}$. *Then we have*

$$\big\| \|u\|_{L_x^2} \|v\|_{L_x^2} \big\|_{L_t^p L_y^q} \lesssim \left(\frac{M}{N}\right)^{\frac{1}{p}} \|u\|_{U_\Delta^p(L_x^2)} \|v\|_{U_\Delta^p(L_x^2)}.$$

6.2. ***Long time Strichartz estimate.*** From now on, we shall take our following setting as standard assumptions. Fix

$$0 < \epsilon_3 \ll \epsilon_2 \ll \epsilon_1 < 1 \quad \text{and} \quad \epsilon_3 < \epsilon_2^{10}. \tag{6-13}$$

By Theorem 6.6, we can take

$$|N'(t)| + |\xi'(t)| \le 2^{-20} \epsilon_1^{-\frac{1}{2}} N(t)^3 \tag{6-14}$$

and

$$\int_{|y-y(t)|\geq 2^{-20}\epsilon_3^{-1/4}/N(t)} \|v(t,y,x)\|_{\mathcal{H}_x^1}^2 \,\mathrm{d}y + \int_{|\xi-\xi(t)|\geq 2^{-20}\epsilon_3^{-1/4}N(t)} \|(\mathcal{F}_y v)(t,\xi,x)\|_{\mathcal{H}_x^1}^2 \,\mathrm{d}\xi \leq \epsilon_2^2. \tag{6-15}$$

If $[0,T]$ is an interval with

$$\|v\|_{L_{t,y}^4 L_x^2([0,T]\times\mathbb{R}^2\times\mathbb{R})}^4 = 2^{k_0} \quad \text{and} \quad \int_0^T N(t)^3 \,\mathrm{d}t = \epsilon_3 2^{k_0} \quad \text{for some } k_0 \geq 0, \tag{6-16}$$

then we can partition $[0,T] = \bigcup_{\alpha=0}^{M-1} J^\alpha$, where J^α are intervals that satisfy

$$\int_{J^\alpha} (N(t)^3 + \epsilon_3 \|v(t)\|_{L_y^4 L_x^2(\mathbb{R}^2\times\mathbb{R})}^4) \,\mathrm{d}t = 2\epsilon_3. \tag{6-17}$$

We can define the interval G_k^j now.

Definition 6.14. For any nonnegative integer $j < k_0$, and nonnegative integer $k < 2^{k_0-j}$, we can define

$$G_k^j = \bigcup_{\alpha=k2^j}^{(k+1)2^j-1} J^\alpha. \tag{6-18}$$

For $j \geq k_0$, we simply define $G_k^j = [0,T]$. We let $\xi(G_k^j) = \xi(t_k^j)$, where t_k^j is the left endpoint of G_k^j.

On the time interval G_k^j defined above, we have:

Lemma 6.15. (1) *Let J_l be the small intervals contained in G_k^j. By (6-5) and (6-17), the following estimate holds*:

$$\sum_{J_l\subseteq G_k^j} N(J_l) \lesssim \int_{G_k^j} N(t)^3 \,\mathrm{d}t \lesssim \sum_{\alpha=k2^j}^{(k+1)2^j-1} \int_{J^\alpha} N(t)^3 \,\mathrm{d}t \lesssim 2^j \epsilon_3. \tag{6-19}$$

(2) *By (6-14) and Definition 6.14, we have, for each $t \in G_k^j$,*

$$|\xi(t) - \xi(G_k^j)| \leq 2^{j-19} \epsilon_3 \epsilon_1^{-\frac{1}{2}}. \tag{6-20}$$

Thus, for any $t \in G_k^j$, and $i \geq j$,

$$\{\xi : 2^{i-1} \leq |\xi-\xi(t)| \leq 2^{i+1}\} \subseteq \{\xi : 2^{i-2} \leq |\xi-\xi(G_k^j)| \leq 2^{i+2}\} \subseteq \{\xi : 2^{i-3} \leq |\xi-\xi(t)| \leq 2^{i+3}\}, \tag{6-21}$$

and also

$$\{\xi : |\xi-\xi(t)| \leq 2^{i+1}\} \subseteq \{\xi : |\xi-\xi(G_k^j)| \leq 2^{i+2}\} \subseteq \{\xi : |\xi-\xi(t)| \leq 2^{i+3}\}. \tag{6-22}$$

Lemma 6.16. *For the almost periodic solution $v(t)$ to (DCR), and assume $\|v\|_{L_{t,y}^4 L_x^2(J\times\mathbb{R}^2\times\mathbb{R})} \leq 1$ on $J \subseteq \mathbb{R}$, then we have*

$$\|v\|_{U_\Delta^2(L_x^2,J)} \lesssim 1 \quad \text{and} \quad \|P^y_{>2^{-4}\epsilon_3^{-1/4}N(J)} v\|_{U_\Delta^2(L_x^2,J)} \lesssim \epsilon_2,$$

where $N(J) = \sup_{t\in J} N(t)$.

Proof. Let $J=[t_0,t_1]$, by (6-10), (6-8), (6-7), and (6-11), we have

$$\|v\|_{U^2_\Delta(L^2_x,J)} \lesssim \|v(t_0)\|_{L^2_{y,x}} + \|v\|^3_{L^4_{t,y}L^2_x(J\times\mathbb{R}^2\times\mathbb{R})} \lesssim 1.$$

By (6-15), we have

$$\|P^y_{>2^{-20}\epsilon_3^{-1/4}N(J)}v\|_{L^\infty_t L^2_{y,x}(J\times\mathbb{R}^2\times\mathbb{R})} \le \|P^y_{>2^{-20}\epsilon_3^{-1/4}N(t)}v\|_{L^\infty_t L^2_{y,x}(J\times\mathbb{R}^2\times\mathbb{R})} \le \epsilon_2.$$

Therefore, by the Strichartz estimate, we have

$$\begin{aligned}
&\|P^y_{>2^{-4}\epsilon_3^{-1/4}N(J)}v\|_{U^2_\Delta(L^2_x,J)}\\
&\qquad\lesssim \|P^y_{>2^{-20}\epsilon_3^{-1/4}N(J)}v(t_0)\|_{L^2_{y,x}(J\times\mathbb{R}^2\times\mathbb{R})} + \|P^y_{>2^{-4}\epsilon_3^{-1/4}N(J)}F(v)\|_{L^{3/2}_t L^{6/5}_y L^2_x(J\times\mathbb{R}^2\times\mathbb{R})}\\
&\qquad\lesssim \epsilon_2 + \|P^y_{>2^{-20}\epsilon_3^{-1/4}N(J)}v\|_{L^\infty_t L^2_{y,x}}\|v\|^2_{L^3_t L^6_y L^2_x(J\times\mathbb{R}^2\times\mathbb{R})}\\
&\qquad\lesssim \epsilon_2 + \epsilon_2\big(\|v(t_0)\|_{L^\infty_t L^2_{y,x}(J\times\mathbb{R}^2\times\mathbb{R})} + \|v\|^3_{L^4_{t,y}L^2_x(J\times\mathbb{R}^2\times\mathbb{R})}\big)^2 \lesssim \epsilon_2. \qquad\square
\end{aligned}$$

We also have the following fact as a consequence of the above lemma.

Remark 6.17. If $N(J) < 2^{i-5}\epsilon_3^{1/2}$, we have

$$\begin{aligned}
&\|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}F(v)\|_{L^{3/2}_t L^{6/5}_y L^2_x(J\times\mathbb{R}^2\times\mathbb{R})}\\
&\qquad\qquad\lesssim \|P^y_{>2^{-20}\epsilon_3^{-1/4}N(J)}v\|_{L^\infty_t L^2_{y,x}(J\times\mathbb{R}^2\times\mathbb{R})}\|v\|^2_{L^3_t L^6_y L^2_x(J\times\mathbb{R}^2\times\mathbb{R})} \lesssim \epsilon_2,
\end{aligned}$$

where the operator $P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}$ is given in Definition 6.18 below. Thus, for $0\le i\le 11$, and $N(G^i_\alpha) < 2^{i-5}\epsilon_3^{1/2}$, by the fact that G^i_α is a union of at most 2^{11} small intervals, we have

$$\|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}F(v)\|_{L^{3/2}_t L^{6/5}_y L^2_x(G^i_\alpha\times\mathbb{R}^2\times\mathbb{R})} \lesssim \epsilon_2.$$

We can now define the long-time Strichartz estimate norm as in [Dodson 2012; 2016a; 2016b]; see also [Cheng et al. 2020a; 2020b].

Definition 6.18 (long-time Strichartz estimate norm). For any $G^j_k \subseteq [0,T]$, let

$$\|v\|^2_{X(G^j_k)} = \sum_{0\le i<j} 2^{i-j} \sum_{G^i_\alpha\subseteq G^j_k} \|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}v\|^2_{U^2_\Delta(L^2_x,G^i_\alpha)} + \sum_{i\ge j}\|P^y_{\xi(G^j_k),2^{i-2}\le\cdots\le 2^{i+2}}v\|^2_{U^2_\Delta(L^2_x,G^j_k)},$$

where

$$P^y_{\xi(t),2^{i-2}\le\cdots\le 2^{i+2}}v = e^{iy\cdot\xi(t)}P^y_{2^{i-2}\le\cdots\le 2^{i+2}}(e^{-iy\cdot\xi(t)}v).$$

We define the $\widetilde{X}_{k_0}$ norm to be

$$\|v\|^2_{\widetilde{X}_{k_0}([0,T])} = \sup_{0\le j\le k_0}\ \sup_{G^j_k\subseteq[0,T]} \|v\|^2_{X(G^j_k)}.$$

For any nonnegative integer $k_*\le k_0$, we take

$$\|v\|^2_{\widetilde{X}_{k_*}([0,T])} = \sup_{0\le j\le k_*}\ \sup_{G^j_k\subseteq[0,T]} \|v\|^2_{X(G^j_k)}. \tag{6-23}$$

To close our bootstrap argument in the proof of the long-time Strichartz estimate, we also need to introduce the following norm to measure $\widetilde{X}_{k_0}$ norm of v at scales much higher than $N(t)$.

Definition 6.19. Let

$$\|v\|^2_{Y(G_k^j)} = \sum_{0\le i<j} 2^{i-j} \sum_{\substack{G_\alpha^i\subseteq G_k^j \\ N(G_\alpha^i)\le 2^{i-5}\epsilon_3^{1/2}}} \|P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,G_\alpha^i)} + \sum_{\substack{i\ge j,\, i>0 \\ N(G_k^j)\le 2^{i-5}\epsilon_3^{1/2}}} \|P^y_{\xi(G_k^j),2^{i-2}\le\cdot\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,G_k^j)}.$$

We can define the norm $\|v\|_{\widetilde{Y}_{k_*}([0,T])}$ similar to (6-23) in Definition 6.18.

For $i<j$, and the solution v on the time interval G_k^j, we can define the Littlewood–Paley projector around $\xi(t)$ of v as

$$P^y_{\xi(t),2^i} v = e^{iy\cdot\xi(t)} P^y_{2^i}(e^{-iy\cdot\xi(t)}v), \quad P^y_{\xi(t),>2^j} v = e^{iy\cdot\xi(t)} P^y_{>2^j}(e^{-iy\cdot\xi(t)}v).$$

Then, as a consequence of (6-7), (6-8), (6-11), the Littlewood–Paley theorem and Proposition 3.2, we have the following estimates which reveal the relationship between the Strichartz norm $L^p_t L^q_y L^2_x$ of the Littlewood–Paley projector around $\xi(t)$ of v and the long-time Strichartz norm of v. We still refer to [Dodson 2016a; 2016b] for the argument, without presenting the proof here.

Lemma 6.20. *For $i<j$, we have*

$$\|P^y_{\xi(t),2^i} v\|_{L^p_t L^q_y L^2_x(G_k^j\times\mathbb{R}^2\times\mathbb{R})} \lesssim 2^{\frac{j-i}{p}} \|v\|_{\widetilde{X}_j(G_k^j)}, \tag{6-24}$$

$$\|P^y_{\xi(t),\ge 2^j} v\|_{L^p_t L^q_y L^2_x(G_k^j\times\mathbb{R}^2\times\mathbb{R})} \lesssim \|v\|_{X(G_k^j)}, \tag{6-25}$$

where (p,q) is Strichartz admissible pair.

Our aim is to prove the long-time Strichartz estimate.

Theorem 6.21 (long-time Strichartz estimate). *For the almost periodic solution v in Theorem 6.6, which satisfies (6-13), (6-14) and (6-15), there exists a positive constant $C=C(v)$ such that, for any nonnegative integer k_0, v and $N(t)$ satisfy (6-16), we have*

$$\|v\|_{\widetilde{X}_{k_0}([0,T])} \le C.$$

To prove Theorem 6.21, it suffices to show, for any $0\le j\le k_0$ and $G_k^j\subseteq[0,T]$,

$$\sum_{0\le i<j} 2^{i-j} \sum_{G_\alpha^i\subseteq G_k^j} \|P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,G_\alpha^i)} + \sum_{i\ge j} \|P^y_{\xi(G_k^j),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,G_k^j)} \le C.$$

To reach the above estimate, we will perform an induction argument on $0\le k_*\le k_0$ and then a bootstrap argument in Sections 6.2.1 and 6.2.2, respectively.

6.2.1. *Basic inductive estimates.* First we show the basic estimates to start up our induction.

Lemma 6.22 (basic inductive estimate).

$$\|v\|_{\widetilde{X}_0([0,T])} \le C \quad \textit{and} \quad \|v\|_{\widetilde{Y}_0([0,T])} \le C\epsilon_2^{\frac{3}{4}}. \tag{6-26}$$

For $0 \le k_* \le k_0$, *we have*

$$\|v\|^2_{\widetilde{X}_{k_*+1}([0,T])} \le 2\|v\|^2_{\widetilde{X}_{k_*}([0,T])} \quad \textit{and} \quad \|v\|^2_{\widetilde{Y}_{k_*+1}([0,T])} \le 2\|v\|^2_{\widetilde{Y}_{k_*}([0,T])}. \tag{6-27}$$

Proof. By Lemma 6.16, we have

$$\|v\|_{U^2_\Delta(L^2_x, J^\alpha)} \lesssim 1 \quad \text{for any } J^\alpha \text{ in the decomposition of } G^j_k \text{ in (6-18).} \tag{6-28}$$

Therefore, by Strichartz estimate, (6-10), (6-11), (6-8), we have, for $t_\alpha \in J^\alpha$,

$$\begin{aligned}\Big(\sum_{i\ge 0} \|P^y_{\xi(J^\alpha),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,J^\alpha)}\Big)^{\frac{1}{2}} &\lesssim \|v(t_\alpha)\|_{L^2_{y,x}} + \|v\|^3_{L^3_t L^6_y L^2_x(J^\alpha\times\mathbb{R}^2\times\mathbb{R})} \\ &\lesssim \|v(t_\alpha)\|_{L^2_{y,x}} + \|v\|^3_{U^2_\Delta(L^2_x,J^\alpha)} \lesssim 1.\end{aligned}$$

Thus, $\|v\|_{\widetilde{X}_0([0,T])} \le C$.

At the same time, by (6-15), the conservation of mass, and (6-28), we infer that

$$\begin{aligned}&\Big(\sum_{\substack{i\ge 0\\ N(J^\alpha)\le\epsilon_3^{1/2}2^{i-5}}} \|P^y_{\xi(J^\alpha),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,J^\alpha)}\Big)^{\frac{1}{2}} \\ &\qquad\lesssim \|P^y_{\xi(J^\alpha),\ge 8\epsilon_3^{-1/2}N(J^\alpha)} v(t_\alpha)\|_{L^2_{y,x}} + \|P^y_{\xi(J^\alpha),\ge 8\epsilon_3^{-1/2}N(J^\alpha)} F(v)\|_{L^1_t L^2_{y,x}(J^\alpha\times\mathbb{R}^2\times\mathbb{R})} \\ &\qquad\lesssim \|P^y_{\xi(t),\ge 4\epsilon_3^{-1/2}N(t)} v\|_{L^\infty_t L^2_{y,x}(J^\alpha\times\mathbb{R}^2\times\mathbb{R})} + \|P^y_{\xi(t),\ge 4\epsilon_3^{-1/2}N(t)} F(v)\|_{L^1_t L^2_{y,x}(J^\alpha\times\mathbb{R}^2\times\mathbb{R})} \\ &\qquad\lesssim \|P^y_{\xi(t),\ge\epsilon_3^{-1/2}N(t)} v\|^{\frac{3}{4}}_{L^\infty_t L^2_{y,x}(J^\alpha\times\mathbb{R}^2\times\mathbb{R})} \big(\|v\|^{\frac{1}{4}}_{L^\infty_t L^2_{y,x}(J^\alpha\times\mathbb{R}^2\times\mathbb{R})} + \|v\|^{\frac{9}{4}}_{U^2_\Delta(L^2_x,J^\alpha)}\big) \lesssim \epsilon_2^{\frac{3}{4}}.\end{aligned}$$

Thus, by Definition 6.19, we have $\|v\|_{\widetilde{Y}_0([0,T])} \le C\epsilon_2^{3/4}$.

By Definition 6.14, we see $G^{j+1}_k = G^j_{2k} \cup G^j_{2k+1}$, with $G^j_{2k} \cap G^j_{2k+1} = \varnothing$. Then for $0 \le i \le j$, if $G^i_\alpha \subseteq G^{j+1}_k$, we have $G^i_\alpha \subseteq G^j_{2k}$ or $G^i_\alpha \subseteq G^j_{2k+1}$. Thus

$$\begin{aligned}&\sum_{0\le i<j+1} 2^{i-(j+1)} \sum_{G^i_\alpha\subseteq G^{j+1}_k} \|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,G^i_\alpha)} \\ &\le 2^{-1} \sum_{0\le i<j} 2^{i-j} \Big(\sum_{G^i_\alpha\subseteq G^j_{2k}} \|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,G^i_\alpha)} \\ &\qquad\qquad + \sum_{G^i_\alpha\subseteq G^j_{2k+1}} \|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}} v\|^2_{U^2_\Delta(L^2_x,G^i_\alpha)}\Big) \\ &\qquad + 2^{-1}\big(\|P^y_{\xi(G^j_{2k}),2^{j-2}\le\cdots\le 2^{j+2}} v\|^2_{U^2_\Delta(L^2_x,G^j_{2k})} + \|P^y_{\xi(G^j_{2k+1}),2^{j-2}\le\cdots\le 2^{j+2}} v\|^2_{U^2_\Delta(L^2_x,G^j_{2k+1})}\big) \\ &\le \tfrac{1}{2}\big(\|v\|^2_{X(G^j_{2k})} + \|v\|^2_{X(G^j_{2k+1})}\big).\end{aligned} \tag{6-29}$$

At the same time, by (6-21) and (6-9), we see

$$\begin{aligned}\sum_{i\ge j+1} &\|P^y_{\xi(G_k^{j+1}),2^{i-2}\le\cdots\le 2^{i+2}}v\|^2_{U^2_\Delta(L^2_x,G_k^{j+1})}\\ &\le \sum_{i\ge j+1}\big(\|P^y_{\xi(G_k^{j+1}),2^{i-2}\le\cdots\le 2^{i+2}}v\|^2_{U^2_\Delta(L^2_x,G^j_{2k})}+\|P^y_{\xi(G_k^{j+1}),2^{i-2}\le\cdots\le 2^{i+2}}v\|^2_{U^2_\Delta(L^2_x,G^j_{2k+1})}\big)\\ &\le \sum_{i\ge j+1}\big(\|P^y_{\xi(G^j_{2k}),2^{i-3}\le\cdots\le 2^{i+3}}v\|^2_{U^2_\Delta(L^2_x,G^j_{2k})}+\|P^y_{\xi(G^j_{2k+1}),2^{i-3}\le\cdots\le 2^{i+3}}v\|^2_{U^2_\Delta(L^2_x,G^j_{2k+1})}\big).\end{aligned}\tag{6-30}$$

Therefore, by (6-29) and (6-30), and Definition 6.18, we get

$$\|v\|^2_{\widetilde{X}_{k_*+1}([0,T])}\le 2\|v\|^2_{\widetilde{X}_{k_*}([0,T])}.$$

By a similar argument, we can deduce

$$\|v\|^2_{\widetilde{Y}_{k_*+1}([0,T])}\le 2\|v\|^2_{\widetilde{Y}_{k_*}([0,T])}.$$ □

6.2.2. *The bootstrap estimate.* In the following, we will establish the bootstrap estimate, which is necessary for the proof of Theorem 6.21. For $0\le j\le k_0$ and $G_k^j\subseteq[0,T]$. By Duhamel's formula, we have, for $0\le i<j$,

$$\begin{aligned}\|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}v\|_{U^2_\Delta(L^2_x,G^i_\alpha)}\le{}&\|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}v(t^i_\alpha)\|_{L^2_{y,x}}\\ &+\Big\|\int_{t^i_\alpha}^t e^{i(t-\tau)\Delta_y}P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}F(v(\tau))\,d\tau\Big\|_{U^2_\Delta(L^2_x,G^i_\alpha)}.\end{aligned}\tag{6-31}$$

Here we take t^i_α to satisfy

$$\|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}v(t^i_\alpha)\|_{L^2_{y,x}}=\inf_{t\in G^i_\alpha}\|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}v(t)\|_{L^2_{y,x}}.$$

We now consider the first term on the right-hand side of (6-31). By (6-17) and Lemma 6.15, we have

$$\begin{aligned}\sum_{0\le i<j}2^{i-j}\sum_{G^i_\alpha\subseteq G^j_k}&\|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}v(t^i_\alpha)\|^2_{L^2_{y,x}}\\ &\lesssim 2^{-j}\epsilon_3^{-1}\int_{G^j_k}(N(t)^3+\epsilon_3\|v(t)\|^4_{L^4_yL^2_x(\mathbb{R}^2\times\mathbb{R})})\sum_{0\le i<j}\|P^y_{\xi(t),2^{i-3}\le\cdots\le 2^{i+3}}v(t)\|^2_{L^2_{y,x}}\,dt\lesssim 1.\end{aligned}$$

For $i\ge j$, we can just take t^j_k to be the left endpoint of G^j_k. Then we have

$$\sum_{i\ge j}\|P^y_{\xi(G^j_k),2^{i-2}\le\cdots\le 2^{i+2}}v(t^j_k)\|^2_{L^2_{y,x}}\lesssim\|v(t^j_k)\|^2_{L^2_{y,x}}\lesssim 1.$$

Thus

$$\sum_{0\le i<j}2^{i-j}\sum_{G^i_\alpha\subseteq G^j_k}\|P^y_{\xi(G^i_\alpha),2^{i-2}\le\cdots\le 2^{i+2}}v(t^i_\alpha)\|^2_{L^2_{y,x}}+\sum_{i\ge j}\|P^y_{\xi(G^j_k),2^{i-2}\le\cdots\le 2^{i+2}}v(t^j_k)\|^2_{L^2_{y,x}}\lesssim 1.\tag{6-32}$$

We next consider the second term on the right-hand side of (6-31). Observe that there are at most two small intervals, called for instance J_1 and J_2, which intersect G^j_k but are not contained in G^j_k. Then by

Lemma 6.16 and (6-11), we have

$$\sum_{0\le i<j} 2^{i-j} \sum_{G_\alpha^i\subseteq G_k^j} \|F(v)\|^2_{L_t^1L^2_{y,x}((G_\alpha^i\cap(J_1\cup J_2))\times\mathbb{R}^2\times\mathbb{R})} \lesssim \sum_{0\le i<j} 2^{i-j}\|F(v)\|^2_{L_t^1L^2_{y,x}((J_1\cup J_2)\times\mathbb{R}^2\times\mathbb{R})} \lesssim \|v\|^6_{L_t^3L_y^6L_x^2(J_1\times\mathbb{R}^2\times\mathbb{R})}+\|v\|^6_{L_t^3L_y^6L_x^2(J_2\times\mathbb{R}^2\times\mathbb{R})}\lesssim 1. \tag{6-33}$$

Then by (6-9), (6-11), (6-33), (6-14), (6-19) and Definition 6.14, we obtain

$$\begin{aligned}
&\sum_{0\le i<j} 2^{i-j} \sum_{\substack{G_\alpha^i\subseteq G_k^j\\ N(G_\alpha^i)\ge 2^{i-5}\epsilon_3^{1/2}}} \left\| \int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}} F(v(\tau))\,d\tau\right\|^2_{U^2_\Delta(L_x^2,G_\alpha^i)}\\
&\lesssim \sum_{0\le i<j} 2^{i-j} \sum_{\substack{G_\alpha^i\subseteq G_k^j\\ N(G_\alpha^i)\ge 2^{i-5}\epsilon_3^{1/2}}} \sum_{J_l\cap G_k^j\neq\varnothing} \|P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}}F(v)\|^2_{DU^2_\Delta(L_x^2,G_\alpha^i\cap J_l)}\\
&\lesssim \sum_{0\le i<j} 2^{i-j} \sum_{\substack{G_\alpha^i\subseteq G_k^j\\ N(G_\alpha^i)\ge 2^{i-5}\epsilon_3^{1/2}}} \sum_{J_l\cap G_k^j\neq\varnothing} \|F(v)\|^2_{L_t^1L^2_{y,x}((G_\alpha^i\cap J_l)\times\mathbb{R}^2\times\mathbb{R})}\\
&\lesssim 1+\sum_{0\le i<j}2^{i-j}\Bigg(\sum_{\substack{J_l\subseteq G_k^j\\ N(J_l)\ge 2^{i-6}\epsilon_3^{1/2}}} \|F(v)\|^2_{L_t^1L^2_{y,x}(J_l\times\mathbb{R}^2\times\mathbb{R})}\Bigg) \lesssim 1+\sum_{J_l\subseteq G_k^j}\sum_{\substack{0\le i<j,\\ 2^i\le 2^6\epsilon_3^{-1/2}N(J_l)}} 2^{i-j}\lesssim 1.
\end{aligned} \tag{6-34}$$

On the interval G_k^j with $N(G_k^j)\ge 2^{i-5}\epsilon_3^{1/2}$, by (6-14) and (6-17), we have

$$\int_{G_k^j} N(t)^2\,dt\lesssim 1. \tag{6-35}$$

Thus, by Minkowski's inequality, (6-3), (6-19), (6-6), and (6-35), we have

$$\sum_{\substack{i\ge j\\ N(G_k^j)\ge 2^{i-5}\epsilon_3^{1/2}}} \|P^y_{\xi(G_k^j),2^{i-2}\le\cdots\le 2^{i+2}}F(v)\|^2_{L_t^1L^2_{y,x}(G_k^j\times\mathbb{R}^2\times\mathbb{R})} \lesssim \|F(v)\|^2_{L_t^1L^2_{y,x}(G_k^j\times\mathbb{R}^2\times\mathbb{R})} \lesssim \|v\|^6_{L_t^3L_y^6L_x^2(G_k^j\times\mathbb{R}^2\times\mathbb{R})}\lesssim 1. \tag{6-36}$$

Thus, by (6-31), (6-32), (6-34), and (6-36), we infer

$$\begin{aligned}
\|v\|^2_{X(G_k^j)} \lesssim 1+ &\sum_{\substack{i\ge j\\ N(G_k^j)\le 2^{i-5}\epsilon_3^{1/2}}} \left\|\int_{t_k^j}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_k^j),2^{i-2}\le\cdots\le 2^{i+2}}F(v(\tau))\,d\tau\right\|^2_{U^2_\Delta(L_x^2,G_k^j)}\\
&+\sum_{0\le i<j}2^{i-j}\sum_{\substack{G_\alpha^i\subseteq G_k^j\\ N(G_\alpha^i)\le 2^{i-5}\epsilon_3^{1/2}}} \left\|\int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}}F(v(\tau))\,d\tau\right\|^2_{U^2_\Delta(L_x^2,G_\alpha^i)}.
\end{aligned} \tag{6-37}$$

We can further get

$$\|v\|^2_{X(G_k^j)} \lesssim 1 + \sum_{\substack{i\geq j,\\ N(G_k^j)\leq 2^{i-10}\epsilon_3^{1/2}}} \left\| \int_{t_k^j}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_k^j),2^{i-2}\leq\cdots\leq 2^{i+2}} F(v(\tau))\,d\tau \right\|^2_{U^2_\Delta(L^2_x,G_k^j)} + \sum_{0\leq i<j} 2^{i-j} \sum_{\substack{G_\alpha^i\subseteq G_k^j,\\ N(G_\alpha^i)\leq 2^{i-10}\epsilon_3^{1/2}}} \left\| \int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2}\leq\cdots\leq 2^{i+2}} F(v(\tau))\,d\tau \right\|^2_{U^2_\Delta(L^2_x,G_\alpha^i)}, \tag{6-38}$$

because the contribution of those terms for i satisfying $2^{i-10}\epsilon_3^{1/2} \leq N(t) \leq 2^{i-5}\epsilon_3^{1/2}$ in the right-hand side of (6-37) is small by similar argument as in the proof of (6-37).

By a similar argument as above for (6-38), we also refer to [Dodson 2016b] for more explanation. Then, we have

$$\|v\|^2_{Y(G_k^j)} \lesssim \epsilon_2^{\frac{3}{2}} + \sum_{\substack{i\geq j\\ N(G_k^j)\leq 2^{i-10}\epsilon_3^{1/2}}} \left\| \int_{t_k^j}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_k^j),2^{i-2}\leq\cdots\leq 2^{i+2}} F(v(\tau))\,d\tau \right\|^2_{U^2_\Delta(L^2_x,G_k^j)} + \sum_{0\leq i<j} 2^{i-j} \sum_{\substack{G_\alpha^i\subseteq G_k^j\\ N(G_\alpha^i)\leq 2^{i-10}\epsilon_3^{1/2}}} \left\| \int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2}\leq\cdots\leq 2^{i+2}} F(v(\tau))\,d\tau \right\|^2_{U^2_\Delta(L^2_x,G_\alpha^i)}. \tag{6-39}$$

Remark 6.23. By Lemma 6.22, Remark 6.17, and (6-11), we have

$$\sum_{\substack{i\geq j\\ 0\leq i\leq 11\\ N(G_k^j)\leq 2^{i-5}\epsilon_3^{1/2}}} \left\| \int_{t_k^j}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_k^j),2^{i-2}\leq\cdots\leq 2^{i+2}} F(v(\tau))\,d\tau \right\|^2_{U^2_\Delta(L^2_x,G_k^j)} + \sum_{0\leq i\leq 11} 2^{i-j} \sum_{\substack{G_\alpha^i\subseteq G_k^j,\\ N(G_\alpha^i)\leq 2^{i-5}\epsilon_3^{1/2}}} \left\| \int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2}\leq\cdots\leq 2^{i+2}} F(v(\tau))\,d\tau \right\|^2_{U^2_\Delta(L^2_x,G_\alpha^i)} \lesssim 1.$$

So we can further reduce the summation over i on the right-hand side of (6-38) and (6-39) to $i > 11$.

Therefore, we have reduced to the proof of the following estimate.

Theorem 6.24 (reduced estimate).

$$\sum_{\substack{i\geq j\\ N(G_k^j)\leq 2^{i-10}\epsilon_3^{1/2}}} \left\| \int_{t_k^j}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_k^j),2^{i-2}\leq\cdots\leq 2^{i+2}} F(v(\tau))\,d\tau \right\|^2_{U^2_\Delta(L^2_x,G_k^j)} + \sum_{0\leq i<j} 2^{i-j} \sum_{\substack{G_\alpha^i\subseteq G_k^j\\ N(G_\alpha^i)\leq 2^{i-10}\epsilon_3^{1/2}}} \left\| \int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2}\leq\cdots\leq 2^{i+2}} F(v(\tau))\,d\tau \right\|^2_{U^2_\Delta(L^2_x,G_\alpha^i)}$$
$$\lesssim \epsilon_2^{\frac{1}{3}} \|v\|^{\frac{5}{3}}_{\tilde{X}_j([0,T])} \|v\|^2_{\tilde{Y}_j([0,T])} + \epsilon_2^2 \|v\|^2_{\tilde{Y}_j([0,T])} + \|v\|^4_{\tilde{Y}_j([0,T])}(1+\|v\|^8_{\tilde{X}_j([0,T])}). \tag{6-40}$$

Once this theorem is proved, we can close the proof of Theorem 6.21 by a bootstrap argument. In the proof given below, we shall assume Theorem 6.24 holds, while leaving its proof to Section 7.1.

Proof of Theorem 6.21. Suppose

$$\|v\|^2_{\widetilde{X}_{k*}([0,T])} \le C_0 \quad \text{and} \quad \|v\|^2_{\widetilde{Y}_{k*}([0,T])} \le C\epsilon_2^{\frac{3}{2}} \le \epsilon_2,$$

and from (6-27), we have

$$\|v\|^2_{\widetilde{X}_{k*+1}([0,T])} \le 2C_0 \quad \text{and} \quad \|v\|^2_{\widetilde{Y}_{k*+1}([0,T])} \le 2\epsilon_2.$$

Then, by (6-38), (6-39), and (6-40), we can further get

$$\|v\|_{\widetilde{X}_{k*+1}([0,T])} \le C(1+\epsilon_2^{\frac{2}{3}}(2C_0)^{\frac{5}{6}} + \epsilon_2^{\frac{3}{2}} + \epsilon_2(1+2C_0)^8),$$

$$\|v\|_{\widetilde{Y}_{k*+1}([0,T])} \le C(\epsilon_2^{\frac{3}{4}} + \epsilon_2^{\frac{2}{3}}(2C_0)^{\frac{5}{6}} + \epsilon_2^{\frac{3}{2}} + \epsilon_2(1+2C_0)^8).$$

If we choose $C_0 = 2^6 C$, and ϵ_2 small enough, then we may deduce

$$\|v\|_{\widetilde{X}_{k*+1}([0,T])} \le C_0^{\frac{1}{2}} \quad \text{and} \quad \|v\|_{\widetilde{Y}_{k*+1}([0,T])} \le \epsilon_2^{\frac{1}{2}}.$$

Theorem 6.21 now follows from this and (6-26) by performing an induction on k_*. □

6.2.3. *The low-frequency localized interaction Morawetz estimate.* As an application of the long-time Strichartz estimate, we can obtain the low-frequency localized interaction Morawetz estimate of the (DCR). The Morawetz estimate is a very important tool to prove the scattering of the nonlinear dispersive equations for the radial case [Lin and Strauss 1978; Morawetz 1968]. In the nonradial case, J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao [Colliander et al. 2004] developed the interaction Morawetz estimate, which is used to prove the scattering of the nonlinear Schrödinger equation [Colliander et al. 2008; Tao et al. 2007a; 2007b; Dodson 2012; 2016a; 2016b] in the nonradial case. The low-frequency localized interaction Morawetz estimate will be used to preclude the soliton-like solution in Theorem 6.26.

Theorem 6.25 (low-frequency localized interaction Morawetz estimate). *Let $v(t,y,x)$ be the almost periodic solution in Theorem 6.6 on $[0,T]$ with $\int_0^T N(t)^3\,\mathrm{d}t = K$. Then we have*

$$\left\| \int_{\mathbb{R}} |\nabla_y|^{\frac{1}{2}} (|P^y_{\le 10\epsilon_1^{-1}K} v(t,y,x)|^2)\,\mathrm{d}x \right\|^2_{L^2_{t,y}([0,T]\times\mathbb{R}^2)} \lesssim o(K). \tag{6-41}$$

The proof of this theorem follows from similar arguments in [Dodson 2012; 2016a; 2016b] and relies on Theorem 6.24 (and also some part of the proof). In our (DCR) system, the interaction Morawetz quantity is

$$M_0(t) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |v(t,\tilde{y},\tilde{x})|^2 \frac{y-\tilde{y}}{|y-\tilde{y}|} \Im(\bar{v}\nabla_y v)(t,y,x)\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x},$$

which is invariant under the Galilean transform in the $\mathbb{R}^2$-component. Following the argument in [Colliander et al. 2009; Planchon and Vega 2009], we can get

$$\left\| \int_{\mathbb{R}} |\nabla_y|^{\frac{1}{2}} (|v(t,y,x)|^2)\,\mathrm{d}x \right\|^2_{L^2_{t,y}([0,T]\times\mathbb{R}^2)} \lesssim |M_0(T) - M_0(0)|.$$

Replacing v by its low-frequency cut-off $P^y_{\le 10\epsilon_1^{-1}K}v$, we then get the low-frequency localized interaction Morawetz quantity

$$M(t)=\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|P^y_{\le 10\epsilon_1^{-1}K}v(t,\tilde{y},\tilde{x})|^2\frac{y-\tilde{y}}{|y-\tilde{y}|}\Im(\overline{P^y_{\le 10\epsilon_1^{-1}K}v}\nabla_y P^y_{\le 10\epsilon_1^{-1}K}v)(t,y,x)\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}.$$

Because for any $\eta>0$ independent of ϵ_1, by Theorem 6.6 and Bernstein's inequality, we have

$$|M(T)|+|M(0)|\lesssim \eta K,$$

we then obtain

$$\left\|\int_{\mathbb{R}}|\nabla_y|^{\frac{1}{2}}(|v(t,y,x)|^2)\,\mathrm{d}x\right\|^2_{L^2_{t,y}([0,T]\times\mathbb{R}^2)}\lesssim \eta K+\mathscr{E},$$

where $\mathscr{E}$ are the error terms coming from the low-frequency cut-off of the solution of the (DCR). These error terms can be proven to be $o(K)$, using Theorem 6.24 and also some estimates from the proof of it. We shall leave the detailed proof of this theorem to Section 7.2.

6.3. *Exclusion of the almost periodic solution.*

Theorem 6.26. *The almost periodic solution to (DCR) in Theorem 6.6 does not exist.*

Proof. We will preclude two scenarios in the following.

Case I: $\int_0^\infty N(t)^3\,\mathrm{d}t<\infty$. By the proof of Theorem 6.21, as in [Dodson 2016a; 2016b], we have

$$\|v(t,y,x)\|_{L^\infty_t\dot{H}^3_yL^2_x([0,\infty)\times\mathbb{R}^2\times\mathbb{R})}\lesssim_{m_0}\left(\int_0^\infty N(t)^3\,\mathrm{d}t\right)^3. \tag{6-42}$$

By (6-42) and (6-4), we have

$$\|e^{iy\cdot\xi(t)}v\|_{\dot{H}^1_yL^2_x}\lesssim N(t)C(\eta(t))+\eta(t)^{\frac{1}{2}}\to 0\quad \text{as } t\to\infty.$$

Thus, for any $\epsilon>0$, we can take a sufficiently large positive constant t_0 such that $\|e^{iy\cdot\xi(t_0)}v(t_0)\|_{\dot{H}^1_yL^2_x}\le\epsilon$. In the following, we can assume $t_0=0$ because of the Galilean invariance. By Minkowski's inequality, the Gagliardo–Nirenberg inequality, and Hölder's inequality, we have

$$\mathcal{E}(v(t))=\mathcal{E}(v(0))\lesssim\|v(0)\|^2_{\dot{H}^1_yL^2_x}\lesssim\epsilon^2.$$

Because we can take ϵ as small as we wish, this scenario does not exist.

Case II: $\int_0^\infty N(t)^3\,\mathrm{d}t=\infty$. By Hölder's inequality and Sobolev's inequality, we have

$$\begin{aligned}\int_{|y-y(t)|\le C\left(\frac{1}{100}\|v(0)\|^2_{L^2_{y,x}}\right)/N(t)}\int_{\mathbb{R}}|P^y_{\le 10\epsilon_1^{-1}K}v(t,y,x)|^2\,\mathrm{d}y\,\mathrm{d}x&\\ \lesssim\left(\frac{C\left(\frac{1}{100}\|v(0)\|^2_{L^2_{y,x}}\right)}{N(t)}\right)^{\frac{3}{2}}\left\|\int_{\mathbb{R}}|\nabla_y|^{\frac{1}{2}}(|P^y_{\le 10\epsilon_1^{-1}K}v(t,y,x)|^2)\,\mathrm{d}x\right\|_{L^2_y}&.\end{aligned}$$

By Theorem 6.6, we have for $K \geq C\big(\frac{1}{100}\|v\|^2_{L^2_{y,x}}\big)$,

$$\frac{\|v\|^2_{L^2_{y,x}}}{2} \leq \int_{\mathbb{R}} \int_{|y-y(t)| \leq C(\frac{1}{100}\|v\|^2_{L^2_{y,x}})/N(t)} |P^y_{\leq 10\epsilon_1^{-1}K} v(t,y,x)|^2 \, dy \, dx.$$

By the above two estimates, together with Theorem 6.25 and the conservation of mass, we have the following contradiction when K is sufficiently large:

$$\begin{aligned} \|v\|^4_{L^2_{y,x}} K &\lesssim \int_0^T N(t)^3 \bigg(\int_{|y-y(t)| \leq C(\frac{1}{100}\|v\|^2_{L^2_{y,x}})/N(t)} \int_{\mathbb{R}} |P^y_{\leq 10\epsilon_1^{-1}K} v(t,y,x)|^2 \, dx \, dy \bigg)^2 dt \\ &\lesssim \bigg\| \int_{\mathbb{R}} |\nabla_y|^{\frac{1}{2}} (|P^y_{\leq 10\epsilon_1^{-1}K} v(t,y,x)|^2) \, dx \bigg\|^2_{L^2_{t,y}([0,T]\times\mathbb{R}^2)} \lesssim o(K). \end{aligned}$$

This completes the proof of Theorem 6.26. □

Proof of Theorem 1.2. This is an immediate consequence of Theorems 6.6 and 6.26. □

7. Proof of Theorems 6.24 and 6.25

7.1. ***Proof of Theorem 6.24.*** In this section, we complete the proof of Theorem 6.24. To prove this theorem, we decompose the nonlinear term $P^y_{\xi(G^i_\alpha),2^{i-2}\leq\cdots\leq 2^{i+2}} F(v)$ and also use the fact that on the time interval G^i_α, $\xi(t)$ can replace $\xi(G^i_\alpha)$ up to 2^{i-20} by (6-20). Then, we can see it is enough to prove the estimate the left-hand side of (6-40) with $P^y_{\xi(G^i_\alpha),2^{i-2}\leq\cdots\leq 2^{i+2}} F(v)$ being replaced by

$$P^y_{\xi(G^i_\alpha),2^{i-2}\leq\cdots\leq 2^{i+2}} \mathcal{O}\bigg(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n \big(P^y_{\xi(G^i_\alpha),\geq 2^{i-5}} v_{n_1} \overline{P^y_{\xi(t),\geq 2^{i-10}} v_{n_2}} v_{n_3} \big) \bigg) \tag{7-1}$$

$$+ P^y_{\xi(t),2^{i-2}\leq\cdots\leq 2^{i+2}} \mathcal{O}\bigg(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n \big(P^y_{\xi(t),\leq 2^{i-10}} v_{n_1} \overline{P^y_{\xi(G^i_\alpha),\leq 2^{i-10}} v_{n_2}} P^y_{\xi(G^i_\alpha),2^{i-5}\leq\cdots\leq 2^{i+5}} v_{n_3} \big) \bigg), \tag{7-2}$$

we also have a similar fact for the nonlinear term $P^y_{\xi(G^j_k),2^{i-2}\leq\cdots\leq 2^{i+2}} F(v)$, where the symbol $\mathcal{O}$ represents the different frequencies that will be located in different v_{n_l}, $l = 1,2,3$. Since their estimates are almost identical, we denote them as a single $\mathcal{O}$. The estimate of the Duhamel propagator of the term (7-1) is very short and easy, and mainly relies on the bilinear Strichartz estimate in Proposition 6.13. The estimate of the Duhamel propagator of the term (7-2) is lengthy. This is because to prove the estimate of the Duhamel propagator of the term (7-2), we need to prove the bilinear Strichartz estimates on the union of the small intervals. It turns out these bilinear Strichartz estimates cannot be proven just by the harmonic analysis but also rely heavily on the structure of the (DCR) system or more precisely the corresponding interaction Morawetz estimate of (DCR). During the proof of this part, some terms can be estimated by the following bilinear Strichartz estimate established recently [Candy 2019] instead of the interaction Morawetz estimate as in [Dodson 2016b]. This new bilinear Strichartz estimate is very useful in [Shen and Wu 2020].

Lemma 7.1 (bilinear Strichartz estimate [Candy 2019]). *Let* $1 \le q, r \le 2$, $\frac{1}{q} + \frac{3}{2r} < \frac{3}{2}$, *and suppose* $M, N \in 2^{\mathbb{Z}}$ *satisfy* $M \ll N$. *Then, for any* $\phi, \psi \in L^2(\mathbb{R}^2)$,

$$\|e^{it\Delta} P_N \phi e^{it\Delta} P_M \psi\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^2)} \lesssim \frac{M^{3-\frac{2}{q}-\frac{3}{r}}}{N^{1-\frac{1}{r}}} \|P_N\phi\|_{L^2(\mathbb{R}^2)} \|P_M\psi\|_{L^2(\mathbb{R}^2)}.$$

7.1.1. *Estimate of* (7-1). We first deal with (7-1).

Theorem 7.2. *For any fixed* $G_k^j \subseteq [0,T]$, $j > 0$, *we have*

$$\begin{aligned}\sum_{0\le i<j} 2^{i-j} \sum_{\substack{G_\alpha^i \subseteq G_k^j \\ N(G_\alpha^i)\le 2^{i-10}\epsilon_3^{1/2}}} \Big\| \int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}} F^{\text{high}}_{\alpha,i}(v(\tau))\, d\tau \Big\|^2_{U^2_\Delta(L^2_x, G_\alpha^i)} \\ + \sum_{\substack{i\ge j,\\ N(G_k^j)\le 2^{i-10}\epsilon_3^{1/2}}} \Big\| \int_{t_k^j}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_k^j),2^{i-2}\le\cdots\le 2^{i+2}} F^{\text{high}}_{k,j}(v(\tau))\, d\tau \Big\|^2_{U^2_\Delta(L^2_x, G_k^j)} \\ \lesssim \epsilon_2^{\frac{1}{3}} \|v\|^{\frac{5}{3}}_{\widetilde{X}_j([0,T])} \|v\|^2_{\widetilde{Y}_j([0,T])},\end{aligned}$$

where

$$F^{\text{high}}_{k,j}(v(t)) = \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n \big(P^y_{\xi(G_k^j),\ge 2^{i-5}} v_{n_1} \overline{P^y_{\xi(t),\ge 2^{i-10}} v_{n_2}} v_{n_3}\big)(t),$$

$$F^{\text{high}}_{\alpha,i}(v(t)) = \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n \big(P^y_{\xi(G_\alpha^i),\ge 2^{i-5}} v_{n_1} \overline{P^y_{\xi(t),\ge 2^{i-10}} v_{n_2}} v_{n_3}\big)(t).$$

Proof. On the time interval G_α^i with $N(G_\alpha^i) \le 2^{i-10}\epsilon_3^{1/2}$, we take $w \in V^2_\Delta(L^2_x, G_\alpha^i)$ be normalized so that $(\mathcal{F}_y w)(t,\xi,x)$ is supported on

$$\{\xi : 2^{i-2} \le |\xi - \xi(G_\alpha^i)| \le 2^{i+2}\}$$

for any $(t,x) \in \mathbb{R}\times\mathbb{R}$. By the Cauchy–Schwarz inequality, (6-1), Proposition 6.13, the conservation of mass, (6-8), (6-11), Lemma 6.20, and (6-15), we infer

$$\begin{aligned}&\int_{G_\alpha^i} \langle w(\tau), F^{\text{high}}_{\alpha,i}(v(\tau))\rangle\, d\tau \\ &\le \sum_{l\ge i-5} \Big\| \|w\|_{L^2_x} \|P^y_{\xi(G_\alpha^i),2^l} v\|_{L^2_x} \Big\|^{\frac12}_{L_t^{5/2} L_y^{5/3}(G_\alpha^i\times\mathbb{R}^2)} \|P^y_{\xi(G_\alpha^i),2^l} v\|^{\frac12}_{L_t^{5/2}L_y^{10}L_x^2(G_\alpha^i\times\mathbb{R}^2\times\mathbb{R})} \\ &\qquad \cdot \|w\|^{\frac12}_{L_t^{5/2}L_y^{10}L_x^2(G_\alpha^i\times\mathbb{R}^2\times\mathbb{R})} \|P^y_{\xi(t),\ge 2^{i-10}} v\|_{L_t^{5/2}L_y^{10}L_x^2(G_\alpha^i\times\mathbb{R}^2\times\mathbb{R})} \|v\|_{L_t^\infty L^2_{y,x}(G_\alpha^i\times\mathbb{R}^2\times\mathbb{R})} \\ &\lesssim \sum_{l\ge i-5} 2^{\frac{i-l}{5}} \|P^y_{\xi(G_\alpha^i),2^l} v\|_{U^2_\Delta(L^2_x,G_\alpha^i)} \|P^y_{\xi(t),\ge 2^{i-10}} v\|^{\frac16}_{L_t^\infty L^2_{y,x}(G_\alpha^i\times\mathbb{R}^2\times\mathbb{R})} \\ &\qquad \cdot \|P^y_{\xi(t),\ge 2^{i-10}} v\|^{\frac56}_{L_t^{25/12}L_y^{50}L_x^2(G_\alpha^i\times\mathbb{R}^2\times\mathbb{R})} \\ &\lesssim \epsilon_2^{\frac16} \|v\|^{\frac56}_{\widetilde{X}_i([0,T])} \sum_{l\ge i-5} 2^{\frac{i-l}{5}} \|P^y_{\xi(G_\alpha^i),2^l} v\|_{U^2_\Delta(L^2_x,G_\alpha^i)} \\ &\lesssim \epsilon_2^{\frac16} \|v\|^{\frac56}_{\widetilde{X}_j([0,T])} \Big(\sum_{l\ge i-5} 2^{\frac{i-l}{5}} \|P^y_{\xi(G_\alpha^i),2^l} v\|^2_{U^2_\Delta(L^2_x,G_\alpha^i)} \Big)^{\frac12}. \qquad (7\text{-}3)\end{aligned}$$

As in [Dodson 2016b], we see, for any $0 \le l \le j$, G_k^j overlaps 2^{j-l} intervals G_β^l and, for $0 \le i \le l$, G_β^l overlaps 2^{l-i} intervals G_α^i. In addition, every G_α^i is contained in one G_β^l. Thus, we can divide the summation in the left-hand side of the following (7-4) and (7-5) into different groups according to $l \ge j$ and $0 \le l < j$. Then by some easy calculation and reordering the summation of i and l, we have

$$\sum_{0 \le i < j} 2^{i-j} \sum_{\substack{G_\alpha^i \subseteq G_k^j \\ N(G_\alpha^i) \le 2^{i-10}\epsilon_3^{1/2}}} \left(\sum_{l \ge i-5} 2^{\frac{i-l}{5}} \| P^y_{\xi(G_\alpha^i),2^l} v \|^2_{U^2_\Delta(L^2_x, G_\alpha^i)} \right), \tag{7-4}$$

$$\sum_{\substack{i \ge j, \\ N(G_k^j) \le 2^{i-10}\epsilon_3^{1/2}}} \left(\sum_{l \ge i-5} 2^{\frac{i-l}{5}} \| P^y_{\xi(G_k^j),2^l} v \|^2_{U^2_\Delta(L^2_x, G_k^j)} \right) \lesssim \|v\|^2_{\widetilde{Y}_j([0,T])}. \tag{7-5}$$

Theorem 7.2 follows from (6-7), (7-3) and (7-4). □

7.1.2. *Estimate of* (7-2). Now we turn to the estimate of (7-2). Let

$$F^{\text{low}}_{k,j}(v(t)) := \sum_{\substack{n_1,n_2,n_3,n \in \mathbb{N} \\ n_1 - n_2 + n_3 = n}} \Pi_n \big(P^y_{\xi(t),\le 2^{i-10}} v_{n_1} \overline{P^y_{\xi(t),\le 2^{i-10}} v_{n_2}} P^y_{\xi(G_k^j),2^{i-5} \le \cdots \le 2^{i+5}} v_{n_3} \big),$$

$$F^{\text{low}}_{\alpha,i}(v(t)) := \sum_{\substack{n_1,n_2,n_3,n \in \mathbb{N} \\ n_1 - n_2 + n_3 = n}} \Pi_n \big(P^y_{\xi(t),\le 2^{i-10}} v_{n_1} \overline{P^y_{\xi(t),\le 2^{i-10}} v_{n_2}} P^y_{\xi(G_\alpha^i),2^{i-5} \le \cdots \le 2^{i+5}} v_{n_3} \big).$$

Then, we have:

Theorem 7.3. *For any* $0 \le i \le j$, *on the time interval* $G_\alpha^i \subseteq G_k^j$, *with* $N(G_\alpha^i) \le 2^{i-10}\epsilon_3^{1/2}$, *we have*

$$\left\| \int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2} \le \cdots \le 2^{i+2}} F^{\text{low}}_{\alpha,i}(v(\tau))\, d\tau \right\|_{U^2_\Delta(L^2_x, G_\alpha^i)} \lesssim \| P^y_{\xi(G_\alpha^i),2^{i-5} \le \cdots \le 2^{i+5}} v \|_{U^2_\Delta(L^2_x, G_\alpha^i)} (\epsilon_2 + \|v\|_{\widetilde{Y}_i([0,T])} (1 + \|v\|_{\widetilde{X}_i([0,T])})^4). \tag{7-6}$$

In addition, for $i \ge j$, $N(G_k^j) \le 2^{i-10}\epsilon_3^{1/2}$, *we have*

$$\left\| \int_{t_k^j}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_k^j),2^{i-2} \le \cdots \le 2^{i+2}} F^{\text{low}}_{k,j}(v(\tau))\, d\tau \right\|_{U^2_\Delta(L^2_x, G_k^j)} \lesssim 2^{\frac{3(j-i)}{4}} \| P^y_{\xi(G_k^j),2^{i-5} \le \cdots \le 2^{i+5}} v \|_{U^2_\Delta(L^2_x, G_k^j)} (\epsilon_2 + \|v\|_{\widetilde{Y}_j([0,T])} (1 + \|v\|_{\widetilde{X}_j([0,T])})^4). \tag{7-7}$$

Proof of Theorem 7.3. We will only prove (7-6), as (7-7) follows by a similar argument. Fix G_α^i with $N(G_\alpha^i) \le 2^{i-10}\epsilon_3^{1/2}$. We can see there are no more than two small intervals J_1 and J_2 which overlap G_α^i but are not contained in G_α^i. Let $\widetilde{G}_\alpha^i = G_\alpha^i \setminus (J_1 \cup J_2)$, by (6-8), (6-7), (6-9), and (6-11), we have

$$\begin{aligned} &\left\| \int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2} \le \cdots \le 2^{i+2}} F^{\text{low}}_{\alpha,i}(v(\tau))\, d\tau \right\|_{U^2_\Delta(L^2_x, G_\alpha^i)} \\ &\quad \lesssim \left\| \int_{t_\alpha^i}^t e^{i(t-\tau)\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2} \le \cdots \le 2^{i+2}} F^{\text{low}}_{\alpha,i}(v(\tau))\, d\tau \right\|_{U^2_\Delta(L^2_x, \widetilde{G}_\alpha^i)} \\ &\qquad + \|F^{\text{low}}_{\alpha,i}(v(t))\|_{L^{4/3}_{t,y} L^2_x((J_1 \cap G_\alpha^i) \times \mathbb{R}^2 \times \mathbb{R})} + \|F^{\text{low}}_{\alpha,i}(v(t))\|_{L^{4/3}_{t,y} L^2_x((J_2 \cap G_\alpha^i) \times \mathbb{R}^2 \times \mathbb{R})}. \end{aligned} \tag{7-8}$$

Here, we may assume $t_\alpha^i \in \widetilde{G}_\alpha^i$, because if $t_\alpha^i \notin \widetilde{G}_\alpha^i$, we may move t_α^i into $\widetilde{G}_\alpha^i$ with the errors being absorbed by the last two terms on the right-hand side of the above inequality. We can show the last two terms on the right-hand side of (7-8) are small in the following.

On the intervals J_l for $l = 1, 2$, by Propositions 6.12 and 6.13, (6-11), the fact $N(t) \le 2^{i-5}\epsilon_3^{1/2}$ on G_α^i, (6-14), (6-15), Lemma 6.16, and (6-11), we can get

$$\begin{aligned}
&\|F_{\alpha,i}^{\mathrm{low}}(v(t))\|_{L_{t,y}^{4/3}L_x^2((J_l\cap G_\alpha^i)\times\mathbb{R}^2\times\mathbb{R})}\\
&\lesssim \big\|\|P^y_{\xi(J_l),\le\epsilon_3^{-1/4}N(J_l)}v\|_{L_x^2}\|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{L_x^2}\big\|_{L_{t,y}^2((J_l\cap G_\alpha^i)\times\mathbb{R}^2\times\mathbb{R})}\\
&\qquad\qquad\qquad\qquad\cdot\|P^y_{\xi(t),\le 2^{i-10}}v\|_{L_{t,y}^4L_x^2((J_l\cap G_\alpha^i)\times\mathbb{R}^2\times\mathbb{R})}\\
&\quad+\|P^y_{\xi(J_l),\ge\epsilon_3^{-1/4}N(J_l)}v\|_{L_t^\infty L_{y,x}^2((J_l\cap G_\alpha^i)\times\mathbb{R}^2\times\mathbb{R})}\|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{L_t^{8/3}L_y^8L_x^2((J_l\cap G_\alpha^i)\times\mathbb{R}^2\times\mathbb{R})}\\
&\qquad\qquad\qquad\qquad\cdot\|P^y_{\xi(t),\le 2^{i-10}}v\|_{L_t^{8/3}L_y^8L_x^2((J_l\cap G_\alpha^i)\times\mathbb{R}^2\times\mathbb{R})}\\
&\lesssim \sum_{2^k\le\epsilon_3^{-1/4}N(J_l)} 2^{\frac{k-i}{2}}\|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{U^2_\Delta(L_x^2,G_\alpha^i)}\|P^y_{\xi(J_l),2^k}v\|_{U^2_\Delta(L_x^2,J_l)}\\
&\qquad\qquad\qquad\qquad+\epsilon_2\|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{U^2_\Delta(L_x^2,G_\alpha^i)}\\
&\lesssim \epsilon_2\|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}}v\|_{U^2_\Delta(L_x^2,G_\alpha^i)}.
\end{aligned}\tag{7-9}$$

Thus, we can simplify the estimate of (7-8) to the case that G_α^i is the union of finite many small intervals J_l. (If not, we just need to add the right-hand side of (7-9)). Let

$$F_{\alpha,i}^{\mathrm{low},l_2}(v(t)) = \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} \Pi_n\big(P^y_{\xi(t),2^{l_2}}v_{n_1}\overline{P^y_{\xi(t),\le 2^{l_2}}v_{n_2}}P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}}v_{n_3}\big).$$

Then by Lemma 6.11, we have

$$\text{LHS of (7-8)} \lesssim A_1 + A_2 + A_3 + A_4,$$

where

$$A_1 = \sum_{0\le l_2\le i-10}\Bigg(\sum_{\substack{J_l\subseteq G_\alpha^i\\ N(J_l)\ge\epsilon_3^{1/2}2^{l_2-5}}} \|P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}}F_{\alpha,i}^{\mathrm{low},l_2}(v(t))\|^2_{DU^2_\Delta(L_x^2,J_l)}\Bigg)^{\frac12},\tag{7-10}$$

$$A_2 = \sum_{\substack{0\le l_2\le i-10\\ J_l\subseteq G_\alpha^i\\ N(J_l)\ge\epsilon_3^{1/2}2^{l_2-5}}} \Bigg\|\int_{J_l} e^{-it\Delta_y}P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}}F_{\alpha,i}^{\mathrm{low},l_2}(v(t))\,dt\Bigg\|_{L^2_{y,x}},\tag{7-11}$$

$$A_3 = \sum_{0\le l_2\le i-10}\Bigg(\sum_{\substack{G_\beta^{l_2}\subseteq G_\alpha^i\\ N(G_\beta^{l_2})\le\epsilon_3^{1/2}2^{l_2-5}}} \|P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}}F_{\alpha,i}^{\mathrm{low},l_2}(v(t))\|^2_{DU^2_\Delta(L_x^2,G_\beta^{l_2})}\Bigg)^{\frac12},\tag{7-12}$$

$$A_4 = \sum_{\substack{0\le l_2\le i-10\\ G_\beta^{l_2}\subseteq G_\alpha^i\\ N(G_\beta^{l_2})\le \epsilon_3^{1/2}2^{l_2-5}}} \left\| \int_{G_\beta^{l_2}} e^{-it\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}} F^{\mathrm{low},l_2}_{\alpha,i}(v(t))\,\mathrm{d}t \right\|_{L^2_{y,x}}. \tag{7-13}$$

The proof of the first two terms are easy. We first prove the following auxiliary estimate.

Lemma 7.4. *Let (p_0,q_0) be Strichartz admissible with $q_0\ge 20$. Suppose that*

$$w(t,y,x)\in L_t^{p_0}L_y^{q_0}L_x^2(J_l\times\mathbb{R}^2\times\mathbb{R})$$

satisfies that $\mathcal{F}w(t,\cdot,x)$ is supported on $\{\xi : 2^{i-2}\le|\xi-\xi(G_\alpha^i)|\le 2^{i+2}\}$. If $N(J_l)\ge\epsilon_3^{1/2}2^{l_2-5}$, then we have

$$\left|\int_{J_l}\int_{\mathbb{R}^2}\int_{\mathbb{R}} \overline{w(t,y,x)}F^{\mathrm{low},l_2}_{\alpha,i}(v(t,y,x))\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\right| \lesssim 2^{\frac{2l_2}{q_0}}\epsilon_3^{\frac14-\frac1{q_0}}2^{-\frac{i}{2}}\|w\|_{L_t^{p_0}L_y^{q_0}L_x^2(J_l\times\mathbb{R}^2\times\mathbb{R})}\|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}}v(t,y,x)\|_{U^2_\Delta(G_\alpha^i,L_x^2)}. \tag{7-14}$$

Proof. By (6-14), we see $|\xi-\xi(t)|\le 2^{l_2+2}$ implies $|\xi-\xi(J_l)|\le\epsilon_3^{-1/2}N(J_l)$ for $t\in J_l$. By the argument in the proof of Lemma 6.2 and Hölder's inequality, we have

$$\begin{aligned}
&\left|\int_{J_l}\int_{\mathbb{R}^2}\int_{\mathbb{R}} \overline{w(t,y,x)}F^{\mathrm{low},l_2}_{\alpha,i}v((t,y,x))\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\quad\le \|e^{i\tau(-\Delta_x+x^2)}wP^y_{\xi(t),2^{l_2}}e^{i\tau(-\Delta_x+x^2)}v\|_{L^{p_0}_{\tau,x}L^{p_0}_tL^2_y([0,\pi]\times\mathbb{R}\times J_l\times\mathbb{R}^2)}\\
&\qquad\cdot\|P^y_{\xi(J_l),\lesssim\epsilon_3^{-1/2}N(J_l)}e^{i\tau(-\Delta_x+x^2)}\\
&\qquad\qquad\cdot vP^y_{\xi(G_\alpha^i),2^{i-5}\le\cdot 2^{i+5}}e^{i\tau(-\Delta_x+x^2)}v\|_{L^{p_0/(p_0-1)}_{\tau,x}L^{p_0/(p_0-1)}_tL^2_y([0,\pi]\times\mathbb{R}\times J_l\times\mathbb{R}^2)}.
\end{aligned} \tag{7-15}$$

By Minkowski's inequality, Hölder's inequality, Lemma 6.1, Bernstein's inequality and the conservation of mass, we have

$$\begin{aligned}
&\|e^{i\tau(-\Delta_x+x^2)}wP^y_{\xi(t),2^{l_2}}e^{i\tau(-\Delta_x+x^2)}v\|_{L^{p_0}_{\tau,x}L^{p_0}_tL^2_y([0,\pi]\times\mathbb{R}\times J_l\times\mathbb{R}^2)}\\
&\quad\lesssim \Big\|\|e^{i\tau(-\Delta_x+x^2)}w\|_{L^{2p_0}_{\tau,x}([0,\pi]\times\mathbb{R})}\|P^y_{\xi(t),2^{l_2}}e^{i\tau(-\Delta_x+x^2)}v\|_{L^{2p_0}_{\tau,x}([0,\pi]\times\mathbb{R})}\Big\|_{L_t^{p_0}L_y^2(J_l\times\mathbb{R}^2)}\\
&\quad\lesssim \|w\|_{L_t^{p_0}L_y^{q_0}L_x^2(J_l\times\mathbb{R}^2\times\mathbb{R})}\|P^y_{\xi(t),2^{l_2}}v\|_{L_t^\infty L_y^{p_0}L_x^2(J_l\times\mathbb{R}^2\times\mathbb{R})}\lesssim 2^{\frac{2l_2}{q_0}}\|w\|_{L_t^{p_0}L_y^{q_0}L_x^2(J_l\times\mathbb{R}^2\times\mathbb{R})}.
\end{aligned} \tag{7-16}$$

Next, we use the vector-valued version of transference principle to estimate

$$\|P^y_{\xi(J_l),\lesssim\epsilon_3^{-1/2}N(J_l)}e^{i\tau(-\Delta_x+x^2)}vP^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}}e^{i\tau(-\Delta_x+x^2)}v\|_{L^{p_0}_{\tau,x}L^{p_0}_tL^2_y([0,\pi]\times\mathbb{R}\times J_l\times\mathbb{R}^2)}.$$

Then, using the similar argument of Corollary 1.6 in [Candy 2019], we are reduced to considering $P^y_{\xi(J_l),\lesssim\epsilon_3^{-1/2}N(J_l)}v$ and $P^y_{\xi(G^i_\alpha),2^{i-5}\leq\cdots\leq 2^{i+5}}v$ are $U^2_\Delta(L^2_x)$-atoms. Let

$$P^y_{\xi(J_l),\lesssim\epsilon_3^{-1/2}N(J_l)}v=\sum_{I\in\mathcal{I}}\chi_I e^{it\Delta_y}f_I(y,x),\ \ P^y_{\xi(G^i_\alpha),2^{i-5}\leq\cdots\leq 2^{i+5}}v=\sum_{J\in\mathcal{J}}\chi_J e^{it\Delta_y}g_J(y,x),$$

where $\mathcal{I}$ and $\mathcal{J}$ are partitions as in the definition of $U^2_\Delta(L^2_x)$. We see by Lemma 7.1, Hölder's inequality and Lemma 6.1,

$$\begin{aligned}
&\|e^{it\Delta_y}e^{i\tau(-\Delta_x+x^2)}f_I e^{it\Delta_y}e^{i\tau(-\Delta_x+x^2)}g_J\|_{L^{p_0/(p_0-1)}_{\tau,x}L^{p_0/(p_0-1)}_t L^2_y([0,\pi]\times\mathbb{R}\times J_l\times\mathbb{R}^2)}\\
&\lesssim 2^{-\frac{i}{2}}(\epsilon_3^{-\frac12}N(J_l))^{\frac12-\frac{2}{q_0}}\|e^{i\tau(-\Delta_x+x^2)}f_I\|_{L^2_y L^{2p_0/(p_0-1)}_{\tau,x}(\mathbb{R}^2\times[0,\pi]\times\mathbb{R})}\\
&\qquad\qquad\cdot\|e^{i\tau(-\Delta_x+x^2)}g_J\|_{L^2_y L^{2p_0/(p_0-1)}_{\tau,x}(\mathbb{R}^2\times[0,\pi]\times\mathbb{R})}\\
&\lesssim 2^{-\frac{i}{2}}(\epsilon_3^{-\frac12}N(J_l))^{\frac12-\frac{2}{q_0}}\|f_I\|_{L^2_{y,x}}\|g_J\|_{L^2_{y,x}}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\|P^y_{\xi(J_l),\lesssim\epsilon_3^{-1/2}N(J_l)}e^{i\tau(-\Delta_x+x^2)}\\
&\qquad\cdot vP^y_{\xi(G^i_\alpha),2^{i-5}\leq\cdots\leq 2^{i+5}}e^{i\tau(-\Delta_x+x^2)}v\|_{L^{p_0/(p_0-1)}_{\tau,x}L^{p_0/(p_0-1)}_t L^2_y([0,\pi]\times\mathbb{R}\times J_l\times\mathbb{R}^2)}\\
&\lesssim 2^{-\frac{i}{2}}(\epsilon_3^{-1/2}N(J_l))^{\frac12-\frac{2}{q_0}}\|P^y_{\xi(J_l),\lesssim\epsilon_3^{-1/2}N(J_l)}v\|_{U^2_\Delta(J_l,L^2_x)}\|P^y_{\xi(G^i_\alpha),2^{i-5}\leq\cdots\leq 2^{i+5}}v\|_{U^2_\Delta(J_l,L^2_x)}. \qquad (7\text{-}17)
\end{aligned}$$

Therefore, (7-14) follows from (7-15), (7-16) and (7-17). □

We first consider (7-10). By duality, we have

$$(7\text{-}10)=\sum_{0\leq l_2\leq i-10}\Bigg(\sum_{\substack{J_l\subseteq G^i_\alpha,\\ N(J_l)\geq\epsilon_3^{1/2}2^{l_2-5}}}\sup_{\|w\|_{V^2_\Delta(J_l,L^2_x)}=1}\bigg|\int_{J_l}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\overline{w(t,y,x)}P^y_{\xi(G^i_\alpha),2^{i-2}\leq\cdots\leq 2^{i+2}}\cdot F^{\mathrm{low},l_2}_{\alpha,i}(v(t,y,x))\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\bigg|^2\Bigg)^{\frac12}.$$

By $\|w\|_{L^{p_0}_t L^{q_0}_y L^2_x(J_l\times\mathbb{R}^2\times\mathbb{R})}\lesssim\|w\|_{V^2_\Delta(J_l,L^2_x)}\lesssim 1$, and (7-14), we get

$$\begin{aligned}
(7\text{-}10)&\lesssim\sum_{0\leq l_2\leq i-10}\Bigg(\sum_{\substack{J_l\subseteq G^i_\alpha,\\ N(J_l)\geq\epsilon_3^{1/2}2^{l_2-5}}}2^{\frac{4l_2}{q_0}}\epsilon_3^{\frac12-\frac{2}{q_0}}2^{-i}\Bigg)^{\frac12}\|P^y_{\xi(G^i_\alpha),2^{i-5}\leq\cdots\leq 2^{i+5}}v(t,y,x)\|_{U^2_\Delta(G^i_\alpha,L^2_x)}\\
&\lesssim(\epsilon_3^{-\frac12}N(J_l))\epsilon_3^{\frac14-\frac{1}{q_0}}\|P^y_{\xi(G^i_\alpha),2^{i-5}\leq\cdots\leq 2^{i+5}}v\|_{U^2_\Delta(G^i_\alpha,L^2_x)}\\
&\lesssim\epsilon_2^2\|P^y_{\xi(G^i_\alpha),2^{i-5}\leq\cdots\leq 2^{i+5}}v\|_{U^2_\Delta(L^2_x,G^i_\alpha)}.
\end{aligned}$$

We now consider (7-11). By duality and Lemma 7.4, we have

$$\begin{aligned}
&\bigg\|\int_{J_l}e^{-it\Delta_y}P^y_{\xi(G^i_\alpha),2^{i-2}\leq\cdots\leq 2^{i+2}}F^{\mathrm{low},l_2}_{\alpha,i}(v(t,y,x))\,\mathrm{d}t\bigg\|_{L^2_{y,x}}\\
&\lesssim\sup_{\|w_0\|_{L^2_{y,x}}=1}\|e^{it\Delta_y}w_0\|_{L^{p_0}_t L^{q_0}_y L^2_x(J_l\times\mathbb{R}^2\times\mathbb{R})}2^{\frac{2l_2}{q_0}}\epsilon_3^{\frac14-\frac{1}{q_0}}2^{-\frac{i}{2}}\|P^y_{\xi(G^i_\alpha),2^{i-5}\leq\cdots\leq 2^{i+5}}v\|_{U^2_\Delta(G^i_\alpha,L^2_x)}, \qquad (7\text{-}18)
\end{aligned}$$

where $\mathcal{F}_y w_0$ is supported on $\{\xi : 2^{i-5} \le |\xi - \xi(G_\alpha^i)| \le 2^{i+5}\}$ in the above estimate. For fixed i, we take $q_0 = 20 + 2i$; then $2^{i/q_0} \lesssim 1$. For the right-hand side of (7-18), by Hölder's inequality, Young's inequality, (6-14), (6-5), (6-19), and the conservation of mass, we have

$$
\begin{aligned}
(7\text{-}11) &\lesssim \sum_{\substack{0 \le l_2 \le i-10 \\ J_l \subseteq G_\alpha^i \\ N(J_l) \ge \epsilon_3^{1/2} 2^{l_2-5}}} \sup_{\|w_0\|_{L^2_{y,x}}=1} 2^{\frac{2l_2}{q_0}} \epsilon_3^{\frac14 - \frac{1}{q_0}} 2^{-\frac{i}{2}} \|e^{it\Delta_y} w_0\|_{L_t^{p_0} L_y^{q_0} L_x^2(J_l \times \mathbb{R}^2 \times \mathbb{R})} \\
&\qquad\qquad \cdot \|P^y_{\xi(G_\alpha^i), 2^{i-5} \le \cdots \le 2^{i+5}} v\|_{U^2_\Delta(G_\alpha^i, L_x^2)} \\
&\lesssim \sum_{0 \le l_2 \le i-10} 2^{\frac{2l_2}{q_0}} \epsilon_3^{\frac14 - \frac{1}{q_0}} \Bigg(\sum_{\substack{J_l \subseteq G_\alpha^i \\ N(J_l) \ge \epsilon_3^{1/2} 2^{l_2-5}}} \|e^{it\Delta_y} w_0\|^{p_0}_{L_t^{p_0} L_y^{q_0} L_x^2(J_l \times \mathbb{R}^2 \times \mathbb{R})} \Bigg)^{\frac{1}{p_0}} \\
&\qquad\qquad \cdot \Bigg(\sum_{J_l \subseteq G_\alpha^i} (2^{-\frac{i}{2}})^{\frac{p_0}{p_0-1}} \Bigg)^{\frac{p_0-1}{p_0}} \|P^y_{\xi(G_\alpha^i), 2^{i-5} \le \cdots \le 2^{i+5}} v\|_{U^2_\Delta(G_\alpha^i, L_x^2)} \\
&\lesssim \sum_{\substack{0 \le l_2 \le i-10 \\ 2^{l_2-5} \le \epsilon_3^{-1/2} N(J_l)}} 2^{\frac{2l_2}{q_0}} \epsilon_3^{\frac14 - \frac{1}{q_0}} \|e^{it\Delta_y} w_0\|_{L_t^{p_0} L_y^{q_0} L_x^2(G_\alpha^i \times \mathbb{R}^2 \times \mathbb{R})} (2^{\frac{2i}{q_0+2}})^{\frac12 + \frac{1}{q_0}} \\
&\qquad\qquad \cdot \|P^y_{\xi(G_\alpha^i), 2^{i-5} \le \cdots \le 2^{i+5}} v\|_{U^2_\Delta(G_\alpha^i, L_x^2)} \\
&\lesssim \sum_{\substack{0 \le l_2 \le i-10 \\ 2^{l_2-5} \le \epsilon_3^{-1/2} N(J_l)}} 2^{\frac{2l_2}{q_0}} \epsilon_3^{\frac14 - \frac{1}{q_0}} 2^{\frac{i}{q_0}} \|P^y_{\xi(G_\alpha^i), 2^{i-5} \le \cdots \le 2^{i+5}} v\|_{U^2_\Delta(G_\alpha^i, L_x^2)} \\
&\lesssim \epsilon_3^{\frac14} \|P^y_{\xi(G_\alpha^i), 2^{i-5} \le \cdots \le 2^{i+5}} v\|_{U^2_\Delta(L_x^2, G_\alpha^i)}.
\end{aligned}
$$

For the estimates of (7-12) and (7-13), we separate the proofs in the next section using two bilinear Strichartz estimates. ☐

7.1.3. *Two bilinear Strichartz estimates.* We have the following:

Theorem 7.5 (first bilinear Strichartz estimate). *Let $w_0 \in L^2_{y,x}(\mathbb{R}^2 \times \mathbb{R})$ with $\operatorname{supp} \mathcal{F}_y w_0$ is supported on $\{\xi : 2^{i-5} \le |\xi - \xi(G_\alpha^i)| \le 2^{i+5}\}$. Then, for any $0 \le l_2 \le i - 10$, we have on $G_\beta^{l_2} \subseteq G_\alpha^i$*

$$
\Big\| \|e^{it\Delta_y} w_0\|_{L_x^2} \|P^y_{\xi(t), \le 2^{l_2}} v\|_{L_x^2} \Big\|^2_{L^2_{t,y}(G_\beta^{l_2} \times \mathbb{R}^2)} \lesssim \|w_0\|^2_{L^2_{y,x}} (1 + \|v\|^4_{\tilde{X}_i(G_\alpha^i)}). \tag{7-19}
$$

Theorem 7.6 (second bilinear Strichartz estimate). *Let $w_0 \in L^2_{y,x}(\mathbb{R}^2 \times \mathbb{R})$ with $\operatorname{supp} \mathcal{F}_y w_0$ supported on $\{\xi : 2^{i-5} \le |\xi - \xi(G_\alpha^i)| \le 2^{i+5}\}$. Then we have*

$$
\sum_{0 \le l_2 \le i-10} \Big\| \|e^{it\Delta_y} w_0\|_{L_x^2} \|P^y_{\xi(t), \le 2^{l_2}} v\|_{L_x^2} \Big\|^2_{L^2_{t,y}(G_\alpha^i \times \mathbb{R}^2)} \lesssim \|w_0\|^2_{L^2_{y,x}} (1 + \|v\|^6_{\tilde{X}_i(G_\alpha^i)}). \tag{7-20}
$$

With the above two bilinear Strichartz estimates, we can now estimate (7-12) and (7-13).

Estimate of (7-12). For any $0 \le l_2 \le i - 10$, by the fact that G_α^i consists of 2^{10} subintervals G_β^{i-10}, Proposition 6.12 and Theorem 7.5 on the subintervals G_β^{i-10}, we get

$$
\begin{aligned}
&\Big\| \|P^y_{\xi(G_\alpha^i), 2^{i-5} \le \cdots \le 2^{i+5}} v\|_{L_x^2} \|P^y_{\xi(t), \le 2^{l_2}} v\|_{L_x^2} \Big\|_{L^2_{t,y}(G_\alpha^i \times \mathbb{R}^2)} \\
&\qquad\qquad \lesssim \|P^y_{\xi(G_\alpha^i), 2^{i-5} \le \cdots \le 2^{i+5}} v\|_{U^2_\Delta(L_x^2, G_\alpha^i)} (1 + \|v\|^2_{\tilde{X}_i(G_\alpha^i)}).
\end{aligned} \tag{7-21}
$$

For any $G_\beta^{l_2} \subseteq G_\alpha^i$, choose $w_\beta^{l_2} \in V_\Delta^2(L_x^2, G_\beta^{l_2})$, with $\operatorname{supp} \mathcal{F}_y w_\beta^{l_2} \subseteq \{\xi : 2^{i-2} \le |\xi - \xi(G_\alpha^i)| \le 2^{i+2}\}$ and $\|w_\beta^{l_2}\|_{V_\Delta^2(L_x^2, G_\beta^{l_2})} = 1$. Then by Hölder's inequality, (6-1), and (7-21), we have

$$\begin{aligned}
&\Bigg(\sum_{\substack{G_\beta^{l_2} \subseteq G_\alpha^i \\ N(G_\beta^{l_2}) \le \epsilon_3^{1/2} 2^{l_2-5}}} \Big\| \overline{w_\beta^{l_2}} \sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N} \\ n_1-n_2+n_3=n}} \Pi_n \big(P^y_{\xi(t),2^{l_2}} v_{n_1} \overline{P^y_{\xi(t),\le 2^{l_2}} v_{n_2}} P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}} v_{n_3} \big) \Big\|^2_{L^1_{t,y,x}(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})} \Bigg)^{\frac12} \\
&\quad \lesssim \Bigg(\sup_{\substack{G_\beta^{l_2} \subseteq G_\alpha^i \\ N(G_\beta^{l_2}) \le \epsilon_3^{1/2} 2^{l_2-5}}} \big\| \|w_\beta^{l_2}\|_{L_x^2} \|P^y_{\xi(t),2^{l_2}} v\|_{L_x^2} \big\|_{L^2_{t,y}(G_\beta^{l_2}\times\mathbb{R}^2)} \Bigg) \\
&\qquad\qquad \cdot \big\| P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}} v \big\|_{U_\Delta^2(L_x^2, G_\alpha^i)} \big(1 + \|v\|^2_{\widetilde{X}_i(G_\alpha^i)}\big). \qquad (7\text{-}22)
\end{aligned}$$

By Proposition 6.13, (6-8), (6-11), and $N(G_\beta^{l_2}) \le \epsilon_3^{1/2} 2^{l_2-5}$, we can estimate the term in the first bracket on the right-hand side of (7-22) as follows:

$$\begin{aligned}
&\big\| \|w_\beta^{l_2}\|_{L_x^2} \|P^y_{\xi(t),2^{l_2}} v\|_{L_x^2} \big\|_{L^2_{t,y}(G_\beta^{l_2}\times\mathbb{R}^2)} \\
&\quad \lesssim \big\| \|w_\beta^{l_2}\|_{L_x^2} \|P^y_{\xi(t),2^{l_2}} v\|_{L_x^2} \big\|^{\frac12}_{L_t^3 L_y^{3/2}(G_\beta^{l_2}\times\mathbb{R}^2)} \|w_\beta^{l_2}\|^{\frac12}_{L_t^3 L_y^6 L_x^2(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})} \|P^y_{\xi(t),2^{l_2}} v\|^{\frac12}_{L_t^3 L_y^6 L_x^2(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})} \\
&\quad \lesssim 2^{\frac{l_2-i}{6}} \|v\|_{\widetilde{Y}_i(G_\alpha^i)}.
\end{aligned}$$

Thus, by the above inequalities, we obtain

$$(7\text{-}12) \lesssim \|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}} v\|_{U_\Delta^2(L_x^2, G_\alpha^i)} \|v\|_{\widetilde{Y}_i(G_\alpha^i)} \big(1 + \|v\|^2_{\widetilde{X}_i(G_\alpha^i)}\big). \qquad \square$$

Estimate of (7-13). Let $w_0 \in L^2_{y,x}$ have unit norm with $\mathcal{F}_y w_0$ supported on $\{\xi : 2^{i-2} \le |\xi - \xi(G_\alpha^i)| \le 2^{i+2}\}$. By the Hölder inequality and Proposition 6.12, we have

$$\begin{aligned}
&\Big\| \int_{G_\beta^{l_2}} e^{-it\Delta_y} P^y_{\xi(G_\alpha^i),2^{i-2}\le\cdots\le 2^{i+2}} F^{\mathrm{low},l_2}_{\alpha,i}(v(t,y,x))\,dt \Big\|_{L^2_{y,x}} \\
&\lesssim \sup_{\|w_0\|_{L^2_{y,x}}=1} \|e^{it\Delta_y} w_0 \cdot F^{\mathrm{low},l_2}_{\alpha,i}(v(t,y,x))\|_{L^1_{t,y,x}(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})} \\
&\lesssim \sup_{\|w_0\|_{L^2_{y,x}}=1} \big\| \|e^{it\Delta_y} w_0\|_{L_x^2} \|P^y_{\xi(t),2^{l_2}} v\|_{L_x^2} \|P^y_{\xi(t),\le 2^{l_2}} v\|_{L_x^2} \|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}} v\|_{L_x^2} \big\|_{L^1_{t,y}(G_\beta^{l_2}\times\mathbb{R}^2)} \\
&\lesssim 2^{\frac{l_2-i}{2}} \|P^y_{\xi(G_\beta^{l_2}),2^{l_2-2}\le\cdots\le 2^{l_2+2}} v\|_{U_\Delta^2(L_x^2, G_\beta^{l_2})} \big\| \|P^y_{\xi(t),\le 2^{l_2}} v\|_{L_x^2} \|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}} v\|_{L_x^2} \big\|_{L^2_{t,y}(G_\beta^{l_2}\times\mathbb{R}^2)}.
\end{aligned}$$

Then by the Cauchy–Schwarz inequality and (7-20), we have

$$\begin{aligned}
(7\text{-}13) &\lesssim \|v\|_{\widetilde{Y}_i(G_\alpha^i)} \Bigg(\sum_{0\le l_2\le i-10} \big\| \|P^y_{\xi(t),\le 2^{l_2}} v\|_{L_x^2} \|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}} v\|_{L_x^2} \big\|^2_{L^2_{t,y}(G_\alpha^i\times\mathbb{R}^2)} \Bigg)^{\frac12} \\
&\lesssim \|v\|_{\widetilde{Y}_i(G_\alpha^i)} \|P^y_{\xi(G_\alpha^i),2^{i-5}\le\cdots\le 2^{i+5}} v\|_{U_\Delta^2(L_x^2, G_\alpha^i)} \big(1 + \|v\|^3_{\widetilde{X}_i(G_\alpha^i)}\big). \qquad \square
\end{aligned}$$

Therefore, this completes the proof of Theorem 7.3. Then we can prove Theorem 6.24 by summation with respect to i in the same way as (7-6) and (7-7) in Theorems 7.3 and 7.2.

7.1.4. *Proofs of the bilinear Strichartz estimates.* It remains to prove the two bilinear Strichartz estimates, that is Theorems 7.5 and 7.6. The proofs of these results are basically the same and rely on the interaction Morawetz estimates of the (DCR) system; the argument here follows from that in [Dodson 2016b]. We shall only present the proof of Theorem 7.5 here, because argument of the proof of Theorem 7.6 is similar and also relies on the result of Theorem 7.5 as the proof of the corresponding bilinear Strichartz estimate in [Dodson 2016b].

Proof of Theorem 7.5. Let $w = e^{it\Delta_y} w_0$ and $\tilde{w} = P^y_{\xi(t),\le 2^{l_2}} v$. Then w and $\tilde{w}$ satisfy $i\partial_t w + \Delta_y w = 0$, and

$$i\partial_t \tilde{w} + \Delta_y \tilde{w} = F(\tilde{w}) + N_1 + N_2 = F(\tilde{w}) + N,$$

where

$$N_1 = P^y_{\xi(t),\le 2^{l_2}} F(v) - F(\tilde{w}),$$

and $N_2 = \left(\frac{d}{dt} P^y_{\xi(t),\le 2^{l_2}}\right) v$ with $\frac{d}{dt} P^y_{\xi(t),\le 2^{l_2}}$ being given by the Fourier multiplier

$$-\nabla\phi\left(\frac{\xi - \xi(t)}{2^{l_2}}\right)\frac{\xi'(t)}{2^{l_2}}.$$

We define the interaction Morawetz action

$$M(t) = \int_{\mathbb{R}^2\times\mathbb{R}} \int_{\mathbb{R}^2\times\mathbb{R}} |\tilde{w}(t,\tilde{y},\tilde{x})|^2 \frac{y-\tilde{y}}{|y-\tilde{y}|} \Im(\bar{w}\nabla_y w)(t,y,x)\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x} \\ + \int_{\mathbb{R}^2\times\mathbb{R}} \int_{\mathbb{R}^2\times\mathbb{R}} |w(t,\tilde{y},\tilde{x})|^2 \frac{y-\tilde{y}}{|y-\tilde{y}|} \Im(\bar{\tilde{w}}\nabla_y \tilde{w})(t,y,x)\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}.$$

After some tedious calculation, we get

$$\begin{aligned}
&\int_{G^{l_2}_\beta}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}} |\overline{\tilde{w}(t,y,\tilde{x})} w(t,y,x)|^2\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}y\,\mathrm{d}t \\
&\lesssim 2^{l_2-2i} \sup_{t\in G^{l_2}_\beta} |M(t)| \\
&\quad + 2^{l_2-2i}\left|\int_{G^{l_2}_\beta}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}} |w(t,\tilde{y},\tilde{x})|^2 \frac{y-\tilde{y}}{|y-\tilde{y}|}\Im(\bar{N}(\nabla_y - i\xi(t))\tilde{w})(t,y,x)\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}y\,\mathrm{d}\tilde{y}\mathrm{d}t\right| && \text{(7-23)} \\
&\quad + 2^{l_2-2i}\left|\int_{G^{l_2}_\beta}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}} |w(t,\tilde{y},\tilde{x})|^2 \frac{y-\tilde{y}}{|y-\tilde{y}|}\Im(\bar{\tilde{w}}(\nabla_y - i\xi(t))N)(t,y,x)\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}t\right| && \text{(7-24)} \\
&\quad + 2^{l_2-2i}\left|\int_{G^{l_2}_\beta}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \Im(\bar{w}(\nabla_y - i\xi(t))w)(t,\tilde{y},\tilde{x}) \frac{y-\tilde{y}}{|y-\tilde{y}|}\Im(\bar{\tilde{w}}N)(t,y,x)\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}t\right|. && \text{(7-25)}
\end{aligned}$$

By the invariance of the Galilean transformation of $M(t)$, Hölder's inequality, and the conservation of mass, we infer that $2^{l_2-2i}\sup_{t\in G^{l_2}_\beta}|M(t)|$ can be bounded by the right-hand side of (7-19).

Estimate of (7-23). By (6-14), (6-17), Bernstein's inequality, the conservation of mass and the Strichartz estimate, we have

$$
\begin{aligned}
|(7\text{-}23)| &\lesssim 2^{l_2-2i}\|N_1\|_{L^{4/3}_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\|(\nabla_y-i\xi(t))\tilde{w}\|_{L^4_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\|w\|^2_{L^\infty_t L^2_{y,x}(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})} \\
&\quad + 2^{-2i}\|w\|^2_{L^\infty_t L^2_{y,x}(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\int_{\mathbb{R}}\int_{G^{l_2}_\beta}|\xi'(t)|\,\|P^y_{\xi(t),2^{l_2-3}\le\cdots\le 2^{l_2+3}}v(t,y,x)\|_{L^2_y} \\
&\qquad\qquad \cdot\|(\nabla_y-i\xi(t))P^y_{\xi(t),\le 2^{l_2}}v(t,\tilde{y},x)\|_{L^2_{\tilde{y}}}\,dx\,dt \\
&\lesssim 2^{l_2-2i}\|N_1\|_{L^{4/3}_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\|(\nabla_y-i\xi(t))\tilde{w}\|_{L^4_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\|w_0\|^2_{L^2_{y,x}}+2^{2l_2-2i}\|w_0\|^2_{L^2_{y,x}}.
\end{aligned}
$$

Let

$$
m(t,\xi)=\frac{\xi-\xi(t)}{2^{l_1}}\phi\Big(\frac{\xi-\xi(t)}{2^{l_1}}\Big).
$$

By Minkowski's inequality, Young's inequality, $\sup_t\|(\mathcal{F}^{-1}_\xi m)(t,y)\|_{L^1_y}\lesssim 1$ and (6-24), we get

$$
\begin{aligned}
&\|(\nabla_y-i\xi(t))\tilde{w}\|_{L^4_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})} \\
&\quad\lesssim \sum_{0\le l_1\le l_2}\|(\nabla_y-i\xi(t))P^y_{\xi(t),2^{l_1}}v\|_{L^4_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})} \\
&\quad\lesssim \sum_{0\le l_1\le l_2}2^{l_1}\Big\|\int|(\mathcal{F}^{-1}_\xi m)(t,y-\tilde{y})|\,\|(P^y_{\xi(t),2^{l_1}}v)(t,\tilde{y},x)\|_{L^2_x}\,d\tilde{y}\Big\|_{L^4_{t,y}(G^{l_2}_\beta\times\mathbb{R}^2)} \\
&\quad\lesssim \sum_{0\le l_1\le l_2}2^{l_1}\|P^y_{\xi(t),2^{l_1}}v\|_{L^4_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})} \\
&\quad\lesssim \sum_{0\le l_1\le l_2}2^{l_1}2^{\frac{l_2-l_1}{4}}\|v\|_{\tilde{X}_{l_2}(G^{l_2}_\beta)}\lesssim 2^{l_2}\|v\|_{\tilde{X}_i(G^i_\alpha)}.
\end{aligned}
$$

Thus, it implies

$$
|(7\text{-}23)|\lesssim 2^{2l_2-2i}\|w_0\|^2_{L^2_{y,x}}\|v\|_{\tilde{X}_i(G^i_\alpha)}\|N_1\|_{L^{4/3}_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}+2^{2l_2-2i}\|w_0\|^2_{L^2_{y,x}}.
$$

Let

$$
v^l=P^y_{\xi(t),\le 2^{l_2-5}}v \quad\text{and}\quad v^h=P^y_{\xi(t),>2^{l_2-5}}v.
$$

We can then decompose N_1 as

$$
\begin{aligned}
N_1 = {} & P^y_{\xi(t),\le 2^{l_2}}\Big(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v^l_{n_1}\overline{v^l_{n_2}}v_{n_3})\Big)-\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n\big(P^y_{\xi(t),\le 2^{l_2}}v^l_{n_1}\overline{P^y_{\xi(t),\le 2^{l_2}}v^l_{n_2}}P^y_{\xi(t),\le 2^{l_2}}v^l_{n_3}\big) \qquad (7\text{-}26) \\
& +2\Big(P^y_{\xi(t),\le 2^{l_2}}\Big(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v^l_{n_1}\overline{v^l_{n_2}}v^h_{n_3})\Big)-\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n\big(P^y_{\xi(t),\le 2^{l_2}}v^l_{n_1}\overline{P^y_{\xi(t),\le 2^{l_2}}v^l_{n_2}}P^y_{\xi(t),\le 2^{l_2}}v^h_{n_3}\big)\Big) \qquad (7\text{-}27)
\end{aligned}
$$

$$+P^y_{\xi(t),\le l_2}\Big(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v^l_{n_1}\overline{v^h_{n_2}}v^l_{n_3})\Big)-\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n\big(P^y_{\xi(t),\le 2^{l_2}}v^l_{n_1}\overline{P^y_{\xi(t),\le 2^{l_2}}v^h_{n_2}}P^y_{\xi(t),\le 2^{l_2}}v^l_{n_3}\big) \tag{7-28}$$

$$+\mathcal{O}\Big(P^y_{\xi(t),\le 2^{l_2}}\Big(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v^h_{n_1}\overline{v^h_{n_2}}v_{n_3})\Big)-\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n\big(P^y_{\xi(t),\le 2^{l_2}}v^h_{n_1}\overline{P^y_{\xi(t),\le 2^{l_2}}v^h_{n_2}}P^y_{\xi(t),\le 2^{l_2}}v_{n_3}\big)\Big), \tag{7-29}$$

where the $\mathcal{O}$ in (7-29) means there are two high-frequency factors in it. Observe that

$$(7\text{-}26)=0.$$

We next consider (7-27) and (7-28). Because their estimates are very similar, we only prove (7-27). Since $(\mathcal{F}_y v^h_{n_3})(t,\sigma,x)$ is supported on $\{\sigma:|\sigma-\xi(t)|\le 2^{l_2+10}\}$, we have

$$P^y_{\xi(t),\le 2^{l_2}}\Big(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v^l_{n_1}\overline{v^l_{n_2}}v^h_{n_3})\Big)-\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N},\\ n_1-n_2+n_3=n}}\Pi_n(P^y_{\xi(t),\le 2^{l_2}}v^l_{n_1}\overline{P^y_{\xi(t),\le 2^{l_2}}v^l_{n_2}}P^y_{\xi(t),\le 2^{l_2}}v^h_{n_3}) \tag{7-30}$$

$$\begin{aligned}=\sum_{l_1\le l_2}\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\iiint e^{-i\tilde{y}\xi(t)}\Pi_n\big((P^y_{\xi(t),2^{l_1}}v^l_{n_1})(\tilde{y},x)\overline{e^{-iz\xi(t)}(P^y_{\xi(t),\le 2^{l_1}}v^l_{n_2})(z,x)}v^h_{n_3}(\theta,x)\big)\\
\cdot\iiint e^{i\xi(y-\tilde{y})+i\eta(\tilde{y}-z)+i\sigma(z-\theta)}\Big(\Big(\phi\Big(\frac{\xi-\xi(t)}{2^{l_2}}\Big)-\phi\Big(\frac{\sigma-\xi(t)}{2^{l_2}}\Big)\Big)\\
\cdot\phi\Big(\frac{\sigma-\xi(t)}{2^{l_2+10}}\Big)\psi_{l_1}(\xi-\eta)\phi\Big(\frac{\eta-\sigma}{2^{l_1}}\Big)\Big)\,d\sigma\,d\eta\,d\xi\,dz\,d\tilde{y}\,d\theta\end{aligned} \tag{7-31}$$

$$\begin{aligned}+\sum_{l_1\le l_2}\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\iiint \Pi_n\big(e^{-i\tilde{y}\xi(t)}(P^y_{\xi(t),\le 2^{l_1}}v^l_{n_1})(\tilde{y},x)\overline{e^{-iz\xi(t)}(P^y_{\xi(t),2^{l_1}}v^l_{n_2})(z,x)}v^h_{n_3}(\theta,x)\big)\\
\cdot\iiint e^{i\xi(y-\tilde{y})+i\eta(\tilde{y}-z)+i\sigma(z-\theta)}\Big(\Big(\phi\Big(\frac{\xi-\xi(t)}{2^{l_2}}\Big)-\phi\Big(\frac{\sigma-\xi(t)}{2^{l_2}}\Big)\Big)\phi\Big(\frac{\sigma-\xi(t)}{2^{l_2+10}}\Big)\\
\cdot\psi_{l_1}(\eta-\sigma)\phi\Big(\frac{\xi-\eta}{2^{l_1}}\Big)\,d\sigma\,d\eta\,d\xi\,d\tilde{y}\,dz\,d\theta.\end{aligned} \tag{7-32}$$

We shall only prove estimate (7-31), as the proof of (7-32) is similar.

$$\begin{aligned}(7\text{-}31)=\sum_{l_1\le l_2}\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\iiint K(t;\tilde{y},z,\theta)\Pi_n\big(e^{-i(y-\tilde{y})\xi(t)}(P^y_{\xi(t),2^{l_1}}v^l_{n_1})(y-\tilde{y},x)\\
\cdot\overline{e^{i(y-\tilde{y}-z)\xi(t)}(P^y_{\xi(t),\le 2^{l_1}}v^l_{n_2})(y-\tilde{y}-z,x)}v^h_{n_3}(y-\tilde{y}-z-\theta,x)\big)\,d\tilde{y}\,dz\,d\theta,\end{aligned}$$

where

$$\begin{aligned}K(t;\tilde{y},z,\theta)=\iiint e^{i\xi\tilde{y}+i\eta z+i\sigma\theta}\Big(\Big(\phi\Big(\frac{\xi-\xi(t)}{2^{l_2}}\Big)-\phi\Big(\frac{\sigma-\xi(t)}{2^{l_2}}\Big)\Big)\\
\cdot\phi\Big(\frac{\sigma-\xi(t)}{2^{l_2}}\Big)\psi_{l_1}(\xi-\eta)\phi\Big(\frac{\eta-\sigma}{2^{l_1}}\Big)\Big)\,d\xi\,d\eta\,d\sigma.\end{aligned}$$

By the estimates $|\xi-\eta|\sim 2^{l_1}$, $|\eta-\sigma|\lesssim 2^{l_1}$, and the fundamental theorem of calculus, we obtain

$$\left|\phi\left(\frac{\xi-\xi(t)}{2^{l_2}}\right)-\phi\left(\frac{\sigma-\xi(t)}{2^{l_2}}\right)\right|\lesssim 2^{-l_2}|\xi-\sigma|\lesssim 2^{l_1-l_2}. \tag{7-33}$$

This implies

$$\sup_t \int |K(t;\tilde{y},z,\theta)|\,\mathrm{d}\tilde{y}\,\mathrm{d}z\,\mathrm{d}\theta\lesssim 2^{l_1-l_2}. \tag{7-34}$$

Thus, by Minkowski's inequality, Hölder's inequality, (7-34), Lemma 6.20 and the conservation of mass, we infer

$$\begin{aligned}
&\Bigg\| P^y_{\xi(t),\le 2^{l_2}}\Pi_n\Bigg(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}} v^l_{n_1}\overline{v^l_{n_2}}v^h_{n_3}\Bigg)\\
&\qquad\qquad -\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n\big(P^y_{\xi(t),\le 2^{l_2}}v^l_{n_1}\,\overline{P^y_{\xi(t),\le 2^{l_2}}v^l_{n_2}}\,P^y_{\xi(t),\le 2^{l_2}}v^h_{n_3}\big)\Bigg\|_{L^{4/3}_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\\
&\lesssim\sum_{l_1\le l_2}\Bigg\|\iiint |K(t;\tilde{y},z,\theta)|\,\Big\|\|(P^y_{\xi(t),2^{l_1}}v^l)(y-\tilde{y},x)\|_{L^2_x}\|(P^y_{\xi(t),\le 2^{l_1}}v^l)(y-\tilde{y}-z,x)\|_{L^2_x}\\
&\qquad\qquad\qquad\qquad\cdot\|v^h(y-\tilde{y}-z-\theta,x)\|_{L^2_x}\Big\|_{L^{4/3}_y}\mathrm{d}\tilde{y}\,\mathrm{d}z\,\mathrm{d}\theta\Bigg\|_{L^{4/3}_t}\\
&\lesssim\sum_{l_1\le l_2}2^{l_1-l_2}\|P^y_{\xi(t),\le 2^{l_1}}v^l\|_{L^\infty_t L^2_{y,x}}\|P^y_{\xi(t),2^{l_1}}u^l\|_{L^{8/3}_t L^8_y L^2_x}\|v^h\|_{L^{8/3}_t L^8_y L^2_x}\lesssim\|v\|^2_{\tilde{X}_i(G^i_\alpha)}.
\end{aligned}$$

We now consider (7-29). Since

$$\begin{aligned}
&\Bigg\|\mathcal{O}\Bigg(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n\big(P^y_{\xi(t),\le 2^{l_2}}(v^h_{n_1}\overline{v^h_{n_2}}v_{n_3})\\
&\qquad\qquad -P^y_{\xi(t),\le 2^{l_2}}v^h_{n_1}\,\overline{P^y_{\xi(t),\le 2^{l_2}}v^h_{n_2}}\,P^y_{\xi(t),\le 2^{l_2}}v_{n_3}\big)\Bigg)\Bigg\|_{L^{4/3}_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\\
&\qquad\lesssim\|v\|_{L^\infty_t L^2_{y,x}}\|v^h\|^2_{L^{8/3}_t L^8_y L^2_x}\lesssim\|v\|^2_{\tilde{X}_i(G^i_\alpha)},
\end{aligned}$$

it follows that

$$\|N_1\|_{L^{4/3}_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\lesssim\|v\|^2_{\tilde{X}_i(G^i_\alpha)}, \tag{7-35}$$

and therefore, we have

$$\text{(7-23)}\lesssim 2^{2l_2-2i}\|w_0\|^2_{L^2_{y,x}(\mathbb{R}^2\times\mathbb{R})}\big(1+\|v\|^3_{\tilde{X}_i(G^i_\alpha)}\big).$$

<u>Estimate of (7-24)</u>. Applying integration by parts, we have

$$\text{(7-24)}\lesssim\text{(7-23)}+2^{l_2-2i}\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{G^{l_2}_\beta}\iint |w(t,\tilde{y},\tilde{x})|^2\frac{1}{|y-\tilde{y}|}\Re(\bar{\tilde{w}}N)(t,y,x)\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t\right|.$$

By the Strichartz estimate, (7-35), (6-14), (6-17), Bernstein's inequality, and the conservation of mass, we have

$$
\begin{aligned}
&2^{l_2-2i}\left|\int_{G_\beta^{l_2}}\iiiint |w(t,\tilde{y},\tilde{x})|^2\frac{1}{|y-\tilde{y}|}\Re(\bar{\tilde{w}}N)(t,y,x)\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t\right|\\
&\lesssim 2^{l_2-2i}\|w_0\|_{L^2_{\tilde{y},x}}^2 2^{-\frac{l_2}{2}}\int_{G_\beta^{l_2}}|\xi'(t)|\|P^y_{\xi(t),2^{l_2-3}\le\cdots\le 2^{l_2+3}}v\|_{L^2_{\tilde{y},x}}\|(\nabla_y-i\xi(t))^{\frac{1}{2}}P^y_{\xi(t),\le 2^{l_2}}v\|_{L^2_{\tilde{y},x}}\,\mathrm{d}t\\
&\qquad\qquad+2^{l_2-2i}\|w\|^2_{L^4_{t,y}L^2_x(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\|N_1\|_{L^{4/3}_{t,y}L^2_x(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\|\tilde{w}\|_{L^\infty_{t,y}L^2_x(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\\
&\lesssim 2^{2l_2-2i}\|w_0\|^2_{L^2_{\tilde{y},x}}(1+\|v\|^3_{\tilde{X}_i(G_\alpha^i)}).
\end{aligned}
$$

Thus

$$
\text{(7-24)}\lesssim 2^{2l_2-2i}\|w_0\|^2_{L^2_{\tilde{y},x}}(1+\|v\|^3_{\tilde{X}_i(G_\alpha^i)}).
$$

Estimate of (7-25): By Bernstein's inequality and the conservation of mass, we have

$$
\begin{aligned}
&2^{l_2-2i}\left|\int_{G_\beta^{l_2}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\Im(\bar{w}(\nabla_y-i\xi(t))w)(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}\Im(\bar{\tilde{w}}N_2)(t,\tilde{y},\tilde{x})\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t\right|\\
&\lesssim\|w_0\|^2_{L^2_{\tilde{y},x}}\Big(2^{-i-l_2}\int_{G_\beta^{l_2}}|\xi'(t)|\,\|P^y_{\xi(t),2^{l_2-3}\le\cdots\le 2^{l_2+3}}v\|_{L^2_{\tilde{y},x}}\|(\nabla_y-i\xi(t))P^y_{\xi(t),\le 2^{l_2}}v\|_{L^2_{\tilde{y},x}}\,\mathrm{d}t\Big)\\
&\qquad+\|w_0\|^2_{L^2_{\tilde{y},x}}\Big(2^{-i-\frac{l_2}{2}}\int_{G_\beta^{l_2}}|\xi'(t)|\|P^y_{\xi(t),l_2-3\le\cdots\le l_2+3}v\|_{L^2_{\tilde{y},x}}\|(\nabla_y-i\xi(t))^{\frac{1}{2}}P^y_{\xi(t),\le 2^{l_2}}v\|_{L^2_{\tilde{y},x}}\,\mathrm{d}t\Big)\\
&\lesssim\|w_0\|^2_{L^2_{\tilde{y},x}}.
\end{aligned}
$$

We now turn to the estimate of

$$
2^{l_2-2i}\left|\int_{G_\beta^{l_2}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\Im(\bar{w}(\nabla_y-i\xi(t))w)(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}\Im(\bar{\tilde{w}}N_1)(t,\tilde{y},\tilde{x})\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t\right|.
$$

Since

$$
\int_{\mathbb{R}}\Im\Bigg(\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\bar{\tilde{w}}\Pi_n(\tilde{w}_{n_1}\overline{\tilde{w}_{n_2}}\tilde{w}_{n_3})\Bigg)(\tilde{x})\,\mathrm{d}\tilde{x}=0, \tag{7-36}
$$

we see

$$
\int_{\mathbb{R}}\sum_{n\in\mathbb{N}}\Im(\overline{\tilde{w}_n}N_{1,n})(\tilde{x})\,\mathrm{d}\tilde{x}=\int_{\mathbb{R}}\Im\Bigg(\sum_{n\in\mathbb{N}}\overline{\tilde{w}_n}P^y_{\xi(t),\le 2^{l_2}}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v_{n_1}\overline{v_{n_2}}v_{n_3})\Bigg)(\tilde{x})\,\mathrm{d}\tilde{x}.
$$

Using the decomposition $v=v^h+v^l$, where $v^l=P^y_{\xi(t),\le 2^{l_2-5}}v$, together with the above equality, we have

$$
\int_{\mathbb{R}}\sum_{n\in\mathbb{N}}\Im(\overline{\tilde{w}_n}N_{1,n})(\tilde{x})=\int_{\mathbb{R}}\sum_{n\in\mathbb{N}}(F_{0,n}+F_{1,n}+F_{2,n}+F_{3,n}+F_{4,n})(\tilde{x})\,\mathrm{d}\tilde{x}, \tag{7-37}
$$

where $F_{j,n}$ consists of j v_n^h-terms and $4-j$ v_n^l-terms, for $j=0,1,2,3,4$, in

$$\Im\Big(\overline{w_n}\,P^y_{\xi(t),\le 2^{l_2}}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v_{n_1}\overline{v_{n_2}}v_{n_3})\Big).$$

We now consider the estimate of the F_j terms, $j=0,1,2,3,4$, as follows.

By (7-36), we have

$$\int_{\mathbb{R}}\sum_{n\in\mathbb{N}}F_{0,n}(t,\tilde{y},\tilde{x})\,\mathrm{d}\tilde{x}=0.$$

By Bernstein's inequality, (6-1) and Lemma 6.20, we have

$$\begin{aligned}
2^{l_2-2i}\Big|\int_{G_\beta^{l_2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Im(\bar{w}(\nabla_y-i\xi(t))w)(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}(F_3+F_4)(t,\tilde{y},\tilde{x})\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t\Big|&\\
\lesssim 2^{l_2-i}\|w_0\|^2_{L^2_{y,x}}\|v^h\|^3_{L^3_tL^6_yL^2_x(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\|v\|_{L^\infty_tL^2_{y,x}(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}&\\
\lesssim 2^{l_2-i}\|w_0\|^2_{L^2_{y,x}}\|v\|^3_{\widetilde{X}_i(G_\alpha^i)}.&
\end{aligned}$$

By a direct calculation, we have

$$\begin{aligned}
\sum_{n\in\mathbb{N}}F_{1,n}=\sum_{n\in\mathbb{N}}\Im\Big(&\overline{P^y_{\xi(t),\le 2^{l_2}}P^y_{\xi(t),\ge 2^{l_2-2}}v_n^h}\,P^y_{\xi(t),\le 2^{l_2}}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v^l_{n_1}\overline{v^l_{n_2}}v^l_{n_3})\\
&+\overline{v_n^l}\,P^y_{\xi(t),\le 2^{l_2}}\Big(\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v^l_{n_1}\overline{P^y_{\xi(t),\ge 2^{l_2-2}}v^h_{n_2}}v^l_{n_3})\\
&\qquad+2\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v^l_{n_1}\overline{v^l_{n_2}}P^y_{\xi(t),\ge 2^{l_2-2}}v^h_{n_3})\Big)\Big).
\end{aligned}\tag{7-38}$$

Since the support of the partial Fourier transform with respect to $\tilde{y}$ of $\sum_{n\in\mathbb{N}}F_{1,n}(t,\tilde{y},\tilde{x})$ is contained in $\{\xi:|\xi|\ge 2^{l_2-4}\}$, we can apply the integration by parts with respect to $\tilde{y}$, the Hardy–Littlewood–Sobolev inequality, Bernstein's inequality, the Strichartz estimate, (6-25), and (6-24) to give the following estimate:

$$\begin{aligned}
&2^{l_2-2i}\Big|\sum_{n\in\mathbb{N}}\int_{G_\beta^{l_2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Im(\overline{w_n}(\nabla_y-i\xi(t))w_n)(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}\Big(\sum_{n'\in\mathbb{N}}F_{1,n'}\Big)(t,\tilde{y},\tilde{x})\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t\Big|\\
&\lesssim 2^{l_2-2i}\int_{G_\beta^{l_2}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\Big|\sum_{n\in\mathbb{N}}(\overline{w_n}(\nabla_y-i\xi(t))w_n)(t,y,x)\Big|\\
&\qquad\qquad\cdot\frac{1}{|y-\tilde{y}|}\Big|\partial_{\tilde{y}}(-\Delta_{\tilde{y}})^{-1}\int_{\mathbb{R}}\Big(\sum_{n'\in\mathbb{N}}F_{1,n'}\Big)(t,\tilde{y},\tilde{x})\,\mathrm{d}\tilde{x}\Big|\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim 2^{l_2-2i}\|w\|_{L^3_tL^6_yL^2_x(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\|(\nabla_y-i\xi(t))w\|_{L^\infty_tL^2_{y,x}(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\\
&\qquad\qquad\cdot\Big\|\partial_{\tilde{y}}(-\Delta_{\tilde{y}})^{-1}\int_{\mathbb{R}}\Big(\sum_{n'\in\mathbb{N}}F_{1,n'}\Big)\,\mathrm{d}\tilde{x}\Big\|_{L_t^{3/2}L^{6/5}_{\tilde{y}}(G_\beta^{l_2}\times\mathbb{R}^2)}\\
&\lesssim 2^{-i}\|w_0\|^2_{L^2_{y,x}}\|v^l\|^3_{L^9_tL^{9/2}_yL^2_x(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\|v^h\|_{L^3_tL^6_yL^2_x(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\lesssim 2^{l_2-i}\|w_0\|^2_{L^2_{y,x}}\|v\|^4_{\widetilde{X}_i(G_\alpha^i)}.
\end{aligned}$$

We are now left to show

$$2^{l_2-2i}\left|\sum_{n,n'\in\mathbb{N}}\int_{G_\beta^{l_2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Im(\overline{w_n}(\nabla_y-i\xi(t))w_n)(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}F_{2,n'}(t,\tilde{y},\tilde{x})\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t\right| \lesssim \|w_0\|_{L^2_{y,x}}^2(1+\|v\|_{\widetilde{X}_i(G_\alpha^i)}^4). \quad (7\text{-}39)$$

Similar to the estimate on the term involved F_1 above, from integration by parts, Bernstein's inequality and (6-25), we conclude

$$\begin{aligned}
&2^{l_2-2i}\left|\sum_{n\in\mathbb{N}}\int_{G_\beta^{l_2}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\Im(\overline{w_n}(\nabla_y-i\xi(t))w_n)(t,y,x)\right.\\
&\qquad\left.\cdot\frac{y-\tilde{y}}{|y-\tilde{y}|}\int_{\mathbb{R}}\Big(\sum_{n'\in\mathbb{N}}P^y_{\geq l_2-10}F_{2,n'}\Big)(t,\tilde{y},\tilde{x})\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}t\right|\\
&\lesssim 2^{l_2-2i}\|w\|_{L_t^3L_y^6L_x^2(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\|(\nabla_y-i(t))w\|_{L_t^\infty L^2_{y,x}(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\\
&\qquad\cdot\left\|\partial_{\tilde{y}}(-\Delta_{\tilde{y}})^{-1}\Big(\int_{\mathbb{R}}\sum_{n'\in\mathbb{N}}P^y_{\geq l_2-10}F_{2,n'}\,\mathrm{d}\tilde{x}\Big)\right\|_{L_t^{3/2}L_y^{6/5}(G_\beta^{l_2}\times\mathbb{R}^2)}\\
&\lesssim 2^{-i}\|w_0\|^2_{L^2_{y,x}}\|v^l\|^2_{L_t^\infty L_y^4L_x^2(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\|v^h\|^2_{L_t^3L_y^6L_x^2(G_\beta^{l_2}\times\mathbb{R}^2\times\mathbb{R})}\lesssim 2^{l_2-i}\|w_0\|^2_{L^2_{y,x}}\|v\|^2_{\widetilde{X}_i(G_\alpha^i)}.
\end{aligned}$$

We now turn to the estimate of the low-frequency part of F_2. First of all, we can decompose $F_{2,n'}$ as

$$\begin{aligned}
F_{2,n'}(t,\tilde{y},\tilde{x})=\Im\bigg(&2\overline{v^l_{n'}}P^y_{\xi(t),\leq 2^{l_2}}\Big(\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}\Pi_{n'}(v^h_{n_1'}\overline{v^h_{n_2'}}v^l_{n_3'})\Big)\\
&+\overline{v^l_{n'}}P^y_{\xi(t),\leq 2^{l_2}}\Big(\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}\Pi_{n'}(v^h_{n_1'}\overline{v^l_{n_2'}}v^h_{n_3'})\Big)\\
&+2\overline{P^y_{\xi(t),\leq 2^{l_2}}v^h_{n'}}P^y_{\xi(t),\leq 2^{l_2}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}\Pi_{n'}(v^l_{n_1'}\overline{v^l_{n_2'}}v^h_{n_3'})\\
&+\overline{P^y_{\xi(t),\leq 2^{l_2}}v^h_{n'}}P^y_{\xi(t),\leq 2^{l_2}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}\Pi_{n'}(v^l_{n_1'}\overline{v^h_{n_2'}}v^l_{n_3'})\bigg).
\end{aligned}\quad (7\text{-}40)$$

Since

$$\Im\Big(\sum_{n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}\overline{P^y_{\xi(t),\leq 2^{l_2}}v^h_{n'}}\Pi_{n'}(v^l_{n_1'}\overline{v^l_{n_2'}}P^y_{\xi(t),\leq 2^{l_2}}v^h_{n_3'})\Big)=0,$$

$$\Im\Big(\sum_{n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}\overline{v^l_{n'}}\Pi_{n'}(P^y_{\xi(t),\leq 2^{l_2}}(v^h_{n_1'}\overline{v^h_{n_2'}})v^l_{n_3'})\Big)=0,$$

we obtain

$$
\begin{aligned}
&2\Im\Biggl(\sum_{n'\in\mathbb{N}}\overline{v^l_{n'}}P^y_{\xi(t),\le 2^{l_2}}\Biggl(\sum_{\substack{n'_1,n'_2,n'_3\in\mathbb{N}\\ n'_1-n'_2+n'_3=n'}}\Pi_{n'}(v^h_{n'_1}\overline{v^h_{n'_2}}v^l_{n'_3})\Biggr)\\
&\qquad\qquad+\sum_{n'\in\mathbb{N}}\overline{P^y_{\xi(t),\le 2^{l_2}}v^h_{n'}}P^y_{\xi(t),\le 2^{l_2}}\sum_{\substack{n'_1,n'_2,n'_3\in\mathbb{N}\\ n'_1-n'_2+n'_3=n'}}\Pi_{n'}(v^l_{n'_1}\overline{v^l_{n'_2}}v^h_{n'_3})\Biggr)\\
&=2\Im\Biggl(\sum_{n'\in\mathbb{N}}\sum_{\substack{n'_1,n'_2,n'_3\in\mathbb{N}\\ n'_1-n'_2+n'_3=n'}}\bigl(\overline{P^y_{\xi(t),\le 2^{l_2}}v^h_{n'}}P^y_{\xi(t),\le 2^{l_2}}\Pi_{n'}(v^l_{n'_1}\overline{v^l_{n'_2}}v^h_{n'_3})\\
&\qquad\qquad-\overline{P^y_{\xi(t),\le 2^{l_2}}v^h_{n'}}\Pi_{n'}(v^l_{n'_1}\overline{v^l_{n'_2}}P^y_{\xi(t),\le 2^{l_2}}v^h_{n'_3})\bigr)\Biggr) \qquad (7\text{-}41)\\
&\quad+2\Im\Biggl(\sum_{n'\in\mathbb{N}}\sum_{\substack{n'_1,n'_2,n'_3\in\mathbb{N}\\ n'_1-n'_2+n'_3=n'}}\bigl(\overline{v^l_{n'}}P^y_{\xi(t),\le 2^{l_2}}\Pi_{n'}(v^h_{n'_1}\overline{v^h_{n'_2}}v^l_{n'_3})-\overline{v^l_{n'}}\Pi_{n'}(P^y_{\xi(t),\le 2^{l_2}}(v^h_{n'_1}\overline{v^h_{n'_2}})v^l_{n'_3})\bigr)\Biggr). \qquad (7\text{-}42)
\end{aligned}
$$

For (7-41), by (7-30), (7-34), Lemma 6.20 and the conservation of mass, we have

$$
\begin{aligned}
&2^{l_2-i}\Bigl\|\int_{\mathbb{R}}(7\text{-}41)(x)\,dx\Bigr\|_{L^1_{t,y}(G^{l_2}_\beta\times\mathbb{R}^2)}\\
&\lesssim 2^{l_2-i}\|v^h\|_{L^4_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\Bigl\|\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n P^y_{\xi(t),\le 2^{l_2}}(v^l_{n_1}\overline{v^l_{n_2}}v^h_{n_3})\\
&\qquad\qquad-\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(v^l_{n_1}\overline{v^l_{n_2}}P^y_{\xi(t),\le 2^{l_2}}v^h_{n_3})\Bigr\|_{L^{4/3}_{t,y}L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\\
&\lesssim 2^{l_2-i}\|v\|_{\widetilde{X}_i(G^i_\alpha)}\sum_{l_1\le l_2}2^{l_1-l_2}\|P^y_{\xi(t),\le 2^{l_1}}v\|_{L^\infty_t L^2_{y,x}}\|P^y_{\xi(t),2^{l_1}}v^l\|_{L^{8/3}_t L^8_y L^2_x}\|v^h\|_{L^{8/3}_t L^8_y L^2_x}\\
&\lesssim 2^{l_2-i}\|v\|^3_{\widetilde{X}_i(G^i_\alpha)}. \qquad (7\text{-}43)
\end{aligned}
$$

To estimate (7-42). We note similar to (7-33), we have

$$
\Bigl|\phi\Bigl(\frac{\xi_1+\xi_2-\xi(t)}{2^{l_2}}\Bigr)-\phi\Bigl(\frac{\xi_1}{2^{l_2}}\Bigr)\Bigr|\lesssim 2^{-l_2}|\xi_2-\xi(t)|.
$$

Then by Lemma 6.20 and the conservation of mass, we have

$$
\begin{aligned}
&2^{l_2-i}\Bigl\|\int_{\mathbb{R}}(7\text{-}42)\,dx\Bigr\|_{L^1_{t,y}(G^{l_2}_\beta\times\mathbb{R}^2)}\\
&\lesssim 2^{l_2-i}\|v^l\|_{L^\infty_t L^2_{y,x}(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\Bigl\|\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}P^y_{\xi(t),\le 2^{l_2}}\Pi_n(v^h_{n_1}\overline{v^h_{n_2}}v^l_{n_3})\\
&\qquad\qquad-\sum_{\substack{n_1,n_2,n_3,n\in\mathbb{N}\\ n_1-n_2+n_3=n}}\Pi_n(P^y_{\xi(t),\le 2^{l_2}}(v^h_{n_1}\overline{v^h_{n_2}})v^l_{n_3})\Bigr\|_{L^1_t L^2_{y,x}(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\\
&\qquad\lesssim 2^{-i}\|v^h\|^2_{L^3_t L^6_y L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\sum_{l_1\le l_2}2^{l_1}\|P^y_{\xi(t),2^{l_1}}v^l\|_{L^3_t L^6_y L^2_x(G^{l_2}_\beta\times\mathbb{R}^2\times\mathbb{R})}\lesssim 2^{l_2-i}\|v\|^3_{\widetilde{X}_i(G^i_\alpha)}.
\end{aligned}
$$

Now we turn to the remaining terms in (7-40). Observe that

$$\Im\Big(\sum_{n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}\overline{v_{n'}^l}P^y_{\xi(t),\le 2^{l_2}}\Pi_{n'}(v^h_{n_1'}\overline{v^l_{n_2'}}v^h_{n_3'}) + \sum_{n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}\overline{P^y_{\xi(t),\le 2^{l_2}}v^h_{n'}}P^y_{\xi(t),\le 2^{l_2}}\Pi_{n'}(v^l_{n_1'}\overline{v^h_{n_2'}}v^l_{n_3'})\Big)$$

$$= \sum_{n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}\Im\big(\overline{P^y_{\xi(t),\le 2^{l_2}}v^h_{n'}}P^y_{\xi(t),\le 2^{l_2}}\Pi_{n'}(v^l_{n_1'}\overline{v^h_{n_2'}}v^l_{n_3'}) - \overline{P^y_{\xi(t),\le 2^{l_2}}v^h_{n'}}\Pi_{n'}(v^l_{n_1'}\overline{P^y_{\xi(t),\le 2^{l_2}}v^h_{n_2'}}v^l_{n_3'})\big) \tag{7-44}$$

$$+ \sum_{n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}\Im\big(\overline{v^l_{n'}}P^y_{\xi(t),\le 2^{l_2}}\Pi_{n'}(v^h_{n_1'}\overline{v^l_{n_2'}}v^l_{n_3'}) + \overline{P^y_{\xi(t),\le 2^{l_2}}v^h_{n'}}\Pi_{n'}(v^l_{n_1'}P^y_{\xi(t),\le 2^{l_2}}\overline{v^h_{n_2'}}v^l_{n_3'})\big). \tag{7-45}$$

Similar to the arguments for (7-43), we have

$$2^{l_2-i}\Big\|\int_{\mathbb{R}}(7\text{-}44)\,\mathrm{d}x\Big\|_{L^1_{t,y}(G^{l_2}_\beta\times\mathbb{R}^2)} \lesssim 2^{l_2-i}\|v\|^3_{\tilde{X}_i(G^i_\alpha)}.$$

Thus, to show (7-39), we just need to consider the term that contains (7-45). By direct calculation, we get

$$2^{l_2-2i}\int_{G^{l_2}_\beta}\iiint\sum_{n\in\mathbb{N}}\Im(\overline{w_n}(\nabla_y - i\xi(t))w_n)(t,y,x)\cdot\frac{y-\tilde{y}}{|y-\tilde{y}|}\sum_{n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}P^y_{\xi(t),\le 2^{l_2-10}}\Im\big(\overline{v^l_{n'}}P^y_{\xi(t),\le 2^{l_2}}\Pi_{n'}(v^h_{n_1'}\overline{v^l_{n_2'}}v^h_{n_3'}) + \overline{P^y_{\xi(t),\le 2^{l_2}}v^h_{n'}}\Pi_{n'}(v^l_{n_1'}\overline{P^y_{\xi(t),\le 2^{l_2}}v^h_{n_2'}}v^l_{n_3'})\big)(t,\tilde{y},\tilde{x})\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t$$

$$= 2^{l_2-2i}\int_{G^{l_2}_\beta}\iint\sum_{n\in\mathbb{N}}\Im(\overline{w_n}(\nabla_y - i\xi(t))w_n)(t,y+2\xi(G^{l_2}_\beta)t,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}\sum_{n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}P^y_{\xi(t),\le 2^{l_2-10}}\cdot\Im\big(\overline{v^l_{n_1'}}\,\overline{v^l_{n_3'}}(v^h_{n'}v^h_{n_2'} - P^y_{\xi(t),\le 2^{l_2}}v^h_{n'}P^y_{\xi(t),\le 2^{l_2}}v^h_{n_2'})\big)(t,\tilde{y}+2\xi(G^{l_2}_\beta)t,\tilde{x})\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}t. \tag{7-46}$$

We may take $\xi(G^{l_2}_\beta)=0$ in the right-hand side of the above equality by the invariance of the Galilean transformation. By the inverse Fourier transform, we have

$$\sum_{n,n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}\int_{G^{l_2}_\beta}\iiiint\Im(\overline{w_n}(\nabla_y - i\xi(t))w_n)(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}\cdot P^y_{\le 2^{l_2-10}}\Im\big((\overline{v_{n_1'}}v^l_{n_3'})(v^h_{n'}v^h_{n_2'} - P^y_{\xi(t),\le 2^{l_2}}v^h_{n'}P^y_{\xi(t),\le 2^{l_2}}v^h_{n_2'})\big)(t,\tilde{y},\tilde{x})\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t$$

$$= \sum_{n,n'\in\mathbb{N}} \sum_{\substack{n_1',n_2',n_3'\in\mathbb{N},\\ n_1'-n_2'+n_3'=n'}} \int_{G_\beta^{l_2}} \iiiint \Im(\overline{w_n}(\nabla_y - i\xi(t))w_n)(t,y,x) \cdot \frac{y-\tilde{y}}{|y-\tilde{y}|}\biggl(\iiint \phi\Bigl(\frac{\eta_1+\eta_2+\eta_3+\eta_4}{2^{l_2-10}}\Bigr) e^{i\tilde{y}(\eta_1+\eta_2+\eta_3+\eta_4)} \cdot \Pi_{n'}\bigl((\mathcal{F}_{\tilde{y}} v^l_{n_1'})(t,\eta_3,\tilde{x})(\mathcal{F}_{\tilde{y}} v^l_{n_3'})(t,\eta_4,\tilde{x})(\overline{\mathcal{F}_{\tilde{y}} v^h_{n'}})(t,\eta_1,\tilde{x})(\overline{\mathcal{F}_{\tilde{y}} v^h_{n_2'}})(t,\eta_2,\tilde{x})\bigr) \cdot \Bigl(1-\phi\Bigl(\frac{\eta_1-\xi(t)}{2^{l_2}}\Bigr)\phi\Bigl(\frac{\eta_2-\xi(t)}{2^{l_2}}\Bigr)\Bigr)\,d\eta_1\,d\eta_2\,d\eta_3\,d\eta_4\biggr)\,dy\,d\tilde{y}\,dx\,d\tilde{x}\,dt.$$

Let

$$q(\eta) = |\eta_1|^2 + |\eta_2|^2 - |\eta_3|^2 - |\eta_4|^2,$$

as in [Dodson 2016b], we have $1/q(\eta)$ is a convergent sum of terms with operator norm being dominated by $1/(|\eta_1|^2+|\eta_2|^2) \sim 1/(|\eta_1||\eta_2|)$ on the support of

$$\Bigl(1-\phi\Bigl(\frac{\eta_1-\xi(t)}{2^{l_2}}\Bigr)\phi\Bigl(\frac{\eta_2-\xi(t)}{2^{l_2}}\Bigr)\Bigr)\phi\Bigl(\frac{\eta_3}{2^{l_2-4}}\Bigr)\phi\Bigl(\frac{\eta_4}{2^{l_2-4}}\Bigr).$$

Let $G_\beta^{l_2} = [t_0,t_1]$. Applying integration by parts (with respect to time), we have

$$\int_{G_\beta^{l_2}} \iiiiiiiint \frac{1}{iq(\eta)}\Bigl(\frac{d}{dt}e^{itq(\eta)}\Bigr)\Im(\overline{w_n}(\nabla_y - i\xi(t))w_n)(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}\phi\Bigl(\frac{\eta_0+\eta_1+\eta_2+\eta_3}{2^{l_2-10}}\Bigr) \cdot e^{i\tilde{y}(\eta_1+\eta_2+\eta_3+\eta_0)}\Bigl(1-\phi\Bigl(\frac{\eta_0-\xi(t)}{2^{l_2}}\Bigr)\phi\Bigl(\frac{\eta_2-\xi(t)}{2^{l_2}}\Bigr)\Bigr) \cdot \bigl(e^{-it|\eta_0|^2}(\overline{\mathcal{F}_{\tilde{y}} v^h_{n'}})(t,\eta_0,\tilde{x})e^{-it|\eta_2|^2}(\overline{\mathcal{F}_{\tilde{y}} v^h_{n_2'}})(t,\eta_2,\tilde{x})e^{it|\eta_1|^2}(\mathcal{F}_{\tilde{y}} v^l_{n_1'})(t,\eta_1,\tilde{x})e^{it|\eta_3|^2} \cdot (\mathcal{F}_{\tilde{y}} v^l_{n_3'})(t,\eta_3,\tilde{x})\,d\eta_1\,d\eta_2\,d\eta_3\,d\eta_0\bigr)\,dy\,d\tilde{y}\,dx\,d\tilde{x}\,dt$$

$$:= B_1 + B_2 + B_3 + B_4, \tag{7-47}$$

where

$$B_1 = \iiiiiiiint \frac{1}{iq(\eta)}e^{itq(\eta)}\Im(\overline{w_n}(\nabla_y - i\xi(t))w_n)(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}\phi\Bigl(\frac{\eta_1+\eta_2+\eta_3+\eta_0}{2^{l_2-10}}\Bigr)e^{i\tilde{y}(\eta_1+\eta_2+\eta_3+\eta_0)} \cdot \Bigl(1-\phi\Bigl(\frac{\eta_0-\xi(t)}{2^{l_2}}\Bigr)\phi\Bigl(\frac{\eta_2-\xi(t)}{2^{l_2}}\Bigr)\Bigr)\bigl(e^{-it|\eta_0|^2}(\overline{\mathcal{F}_{\tilde{y}} v^h_{n'}})(t,\eta_0,\tilde{x})e^{-it|\eta_2|^2}(\overline{\mathcal{F}_{\tilde{y}} v^h_{n_2'}})(t,\eta_2,\tilde{x}) \cdot e^{it|\eta_1|^2}(\mathcal{F}_{\tilde{y}} v^l_{n_1'})(t,\eta_1,\tilde{x})e^{it|\eta_3|^2}(\mathcal{F}_{\tilde{y}} v^l_{n_3'})(t,\eta_3,\tilde{x})\,d\eta_0\,d\eta_1\,d\eta_2\,d\eta_3\bigr)\,dy\,d\tilde{y}\,dx\,d\tilde{x}\Big|_{t_0}^{t_1}, \tag{7-48}$$

$$B_2 = -\int_{t_0}^{t_1} \iiiiiiiint \frac{1}{iq(\eta)}e^{itq(\eta)}\frac{\partial}{\partial t}\Im(\overline{w_n}(\nabla_y - i\xi(t))w_n)(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}\phi\Bigl(\frac{\eta_1+\eta_2+\eta_3+\eta_0}{2^{l_2-10}}\Bigr) \cdot e^{i\tilde{y}(\eta_1+\eta_2+\eta_3+\eta_0)}\Bigl(1-\phi\Bigl(\frac{\eta_0-\xi(t)}{2^{l_2}}\Bigr)\phi\Bigl(\frac{\eta_2-\xi(t)}{2^{l_2}}\Bigr)\Bigr) \cdot \bigl(e^{-it|\eta_0|^2}(\overline{\mathcal{F}_{\tilde{y}} v^h_{n'}})(t,\eta_0,\tilde{x})e^{-it|\eta_2|^2}(\overline{\mathcal{F}_{\tilde{y}} v^h_{n_2'}})(t,\eta_2,\tilde{x})e^{it|\eta_1|^2}(\mathcal{F}_{\tilde{y}} v^l_{n_1'})(t,\eta_1,\tilde{x})e^{it|\eta_3|^2} \cdot (\mathcal{F}_{\tilde{y}} v^l_{n_3'})(t,\eta_3,\tilde{x})\,d\eta_1\,d\eta_2\,d\eta_3\,d\eta_0\bigr)\,dy\,d\tilde{y}\,dx\,d\tilde{x}\,dt, \tag{7-49}$$

$$B_3=-\int_{t_0}^{t_1}\iiiint\!\!\!\!\iiiint \frac{1}{iq(\eta)}e^{itq(\eta)}\Im(\overline{w_n}(\nabla_y-i\xi(t))w_n)(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}$$
$$\cdot\frac{\partial}{\partial t}\left(\phi\left(\frac{\eta_1+\eta_2+\eta_3+\eta_0}{2^{l_2-10}}\right)e^{i\tilde{y}(\eta_1+\eta_2+\eta_3+\eta_0)}\left(1-\phi\left(\frac{\eta_0-\xi(t)}{2^{l_2}}\right)\phi\left(\frac{\eta_2-\xi(t)}{2^{l_2}}\right)\right)\right)$$
$$\cdot\big(e^{-it|\eta_0|^2}\overline{(\mathcal{F}_{\tilde{y}}v^h_{n'})}(t,\eta_0,\tilde{x})e^{-it|\eta_2|^2}\overline{(\mathcal{F}_{\tilde{y}}v^h_{n'_2})}(t,\eta_2,\tilde{x})e^{it|\eta_1|^2}(\mathcal{F}_{\tilde{y}}v^l_{n'_1})(t,\eta_1,\tilde{x})$$
$$\cdot e^{it|\eta_3|^2}(\mathcal{F}_{\tilde{y}}v^l_{n'_3})(t,\eta_3,\tilde{x})\,d\eta_1\,d\eta_2\,d\eta_3\,d\eta_0\big)\,dy\,d\tilde{y}\,dx\,d\tilde{x}\,dt, \tag{7-50}$$

$$B_4=-\int_{t_0}^{t_1}\iiiint\!\!\!\!\iiiint \frac{1}{iq(\eta)}e^{itq(\eta)}\Im(\overline{w_n}(\nabla_y-i\xi(t))w_n)(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}\phi\left(\frac{\eta_1+\eta_2+\eta_3+\eta_0}{2^{l_2-10}}\right)$$
$$\cdot e^{i\tilde{y}(\eta_1+\eta_2+\eta_3+\eta_0)}\left(1-\phi\left(\frac{\eta_0-\xi(t)}{2^{l_2}}\right)\phi\left(\frac{\eta_2-\xi(t)}{2^{l_2}}\right)\right)$$
$$\cdot\frac{\partial}{\partial t}\big(e^{-it|\eta_0|^2}\overline{(\mathcal{F}_{\tilde{y}}v^h_{n'})}(t,\eta_0,\tilde{x})e^{-it|\eta_2|^2}\overline{(\mathcal{F}_{\tilde{y}}v^h_{n'_2})}(t,\eta_2,\tilde{x})$$
$$\cdot e^{it|\eta_1|^2}(\mathcal{F}_{\tilde{y}}v^l_{n'_1})(t,\eta_1,\tilde{x})e^{it|\eta_3|^2}(\mathcal{F}_{\tilde{y}}v^l_{n'_3})(t,\eta_3,\tilde{x})\,d\eta_1\,d\eta_2\,d\eta_3\,d\eta_0\big)\,dy\,d\tilde{y}\,dx\,d\tilde{x}\,dt. \tag{7-51}$$

For (7-48), set

$$m(t;\eta_0,\eta_1,\eta_2,\eta_3)=\frac{1}{q(\eta)}\phi\left(\frac{\eta_1+\eta_2+\eta_3+\eta_0}{2^{l_2-10}}\right)\left(1-\phi\left(\frac{\eta_0-\xi(t)}{2^{l_2}}\right)\phi\left(\frac{\eta_2-\xi(t)}{2^{l_2}}\right)\right).$$

Then we have

$$(7\text{-}48)=-i\iiiint\!\!\!\!\iiiint \Im(\overline{w_n}(\nabla_y-i\xi(t))w_n)(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}K(t;z_0,z_1,z_2,z_3)\overline{v^h_{n'}(t,\tilde{y}-z_0,\tilde{x})}$$
$$\cdot\overline{v^h_{n'_2}(t,\tilde{y}-z_2,\tilde{x})}v^l_{n'_1}(t,\tilde{y}-z_1,\tilde{x})v^l_{n'_3}(t,\tilde{y}-z_3,\tilde{x})\,dz_1\,dz_2\,dz_3\,dz_0\,dx\,d\tilde{x}\,dy\,d\tilde{y}\Big|_{t_0}^{t_1},$$

where

$$K(t;z_0,z_1,z_2,z_3)=\iiiint m(t;\eta_0,\eta_1,\eta_2,\eta_3)e^{iz_1\eta_1}e^{iz_2\eta_2}e^{iz_3\eta_3}e^{iz_0\eta_0}\,d\eta_1d\eta_2d\eta_3d\eta_0, \tag{7-52}$$

which satisfies

$$\sup_t\int|K(t;z_0,z_1,z_2,z_3)|\,dz_1\,dz_2\,dz_3\,dz_0\lesssim 2^{-2l_2}, \tag{7-53}$$

by the Coifman–Meyer theorem [Germain et al. 2012]. Thus, by Bernstein's inequality, (7-53) and the conservation of mass, we have

$$2^{l_2-2i}\Bigg|\sum_{n,n'\in\mathbb{N}}\sum_{\substack{n'_1,n'_2,n'_3\in\mathbb{N}\\ n'_1-n'_2+n'_3=n'}}(7\text{-}48)\Bigg|$$
$$\lesssim 2^{l_2-2i}\|w\|_{L^\infty_t L^2_{y,x}}\|(\nabla_y-i\xi(t))w\|_{L^\infty_t L^2_{y,x}}$$
$$\cdot\iiiint|K(t;z_0,z_1,z_2,z_3)|\|v^h(t,\tilde{y}-z_0,\tilde{x})\|_{L^2_{\tilde{x}}}\|v^h(t,\tilde{y}-z_2,\tilde{x})\|_{L^2_{\tilde{x}}}$$
$$\cdot\|v^l(t,\tilde{y}-z_1,\tilde{x})\|_{L^2_{\tilde{x}}}\|v^l(t,\tilde{y}-z_3,\tilde{x})\|_{L^2_{\tilde{x}}}\,dz_1dz_2dz_3dz_0d\tilde{y}$$
$$\lesssim 2^{l_2-i}\|w_0\|^2_{L^2_{y,x}}2^{-2l_2}\|v^h\|^2_{L^\infty_t L^2_{y,x}}\|v^l\|^2_{L^\infty_{t,y}L^2_x}\lesssim\|w_0\|^2_{L^2_{y,x}}.$$

Next, we turn to the estimate of (7-49). By a direct computation, we have

$$\begin{aligned}\frac{\partial}{\partial t}\Im\int_{\mathbb{R}}(\overline{w_n}(\partial_{y_k}-i\xi_k(t))w_n)(t,y,x)\,\mathrm{d}x\\ =\xi_k'(t)\|w_n(t,y,x)\|_{L_x^2}^2+\sum_{k'=1}^{2}\partial_{y_{k'}}\Re\int_{\mathbb{R}}(\overline{w_n}(\partial_{y_k}-i\xi_k(t))\partial_{y_{k'}}w_n)(t,y,x)\,\mathrm{d}x\\ -\sum_{k'=1}^{2}\partial_{y_{k'}}\Re\int_{\mathbb{R}}(\overline{\partial_{y_{k'}}w_n}(\partial_{y_k}-i\xi_k(t))w_n)(t,y,x)\,\mathrm{d}x.\end{aligned}$$

Thus, we get

$$\begin{aligned}&(7\text{-}49)=\\ &\int_{G_\beta^{l_2}}\iiiint\!\!\!\iiiint\frac{y-\tilde y}{|y-\tilde y|}\xi'(t)\|w_n(t,y,x)\|_{L_x^2}^2K(t;z_0,z_1,z_2,z_3)\overline{v_{n'}^h(t,\tilde y-z_0,\tilde x)}\,\overline{v_{n_2'}^h(t,\tilde y-z_2,\tilde x)}\\ &\qquad\cdot v_{n_1'}^l(t,\tilde y-z_1,\tilde x)v_{n_3'}^l(t,\tilde y-z_3,\tilde x)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_0\,\mathrm{d}y\,\mathrm{d}\tilde y\,\mathrm{d}x\mathrm{d}\tilde x\,\mathrm{d}t \quad (7\text{-}54)\\ &+\sum_{k,k'=1}^{2}\int_{G_\beta^{l_2}}\iiiint\!\!\!\iiiint\frac{(y-\tilde y)_k}{|y-\tilde y|}\partial_{y_{k'}}\Re(\overline{w_n}(\partial_{y_k}-i\xi_k(t))\partial_{y_{k'}}w_n)(t,y,x)\\ &\qquad\cdot K(t;z_0,z_1,z_2,z_3)\overline{v_{n'}^h(t,\tilde y-z_0,\tilde x)}\,\overline{v_{n_2'}^h(t,\tilde y-z_2,\tilde x)}\\ &\qquad\cdot v_{n_1'}^l(t,\tilde y-z_1,\tilde x)v_{n_3'}^l(t,\tilde y-z_3,\tilde x)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_0\,\mathrm{d}y\,\mathrm{d}\tilde y\,\mathrm{d}x\,\mathrm{d}\tilde x\,\mathrm{d}t \quad (7\text{-}55)\\ &-\sum_{k,k'=1}^{2}\int_{G_\beta^{l_2}}\iiiint\!\!\!\iiiint\frac{(y-\tilde y)_k}{|y-\tilde y|}\partial_{y_{k'}}\Re(\partial_{y_{k'}}\bar w_n(\partial_{y_k}-i\xi_k(t))w_n)(t,y,x)\\ &\qquad\cdot K(t;z_0,z_1,z_2,z_3)\overline{v_{n'}^h(t,\tilde y-z_0,\tilde x)}\,\overline{v_{n_2'}^h(t,\tilde y-z_2,\tilde x)}\\ &\qquad\cdot v_{n_1'}^l(t,\tilde y-z_1,\tilde x)v_{n_3'}^l(t,\tilde y-z_3,\tilde x)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_0\,\mathrm{d}y\,\mathrm{d}\tilde y\,\mathrm{d}x\,\mathrm{d}\tilde x\,\mathrm{d}t, \quad (7\text{-}56)\end{aligned}$$

where $K(t;z_0,z_1,z_2,z_3)$ is given in (7-52).

By (7-53), (6-14), (6-19), Bernstein's inequality and the conservation of mass, we have

$$\begin{aligned}2^{l_2-2i}\Bigg|\sum_{n,n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}(7\text{-}54)\Bigg|&\lesssim 2^{l_2-2i}2^{-2l_2}\|w\|_{L_t^\infty L_{y,x}^2}^2\|v^h\|_{L_t^\infty L_{y,x}^2}^2\|v^l\|_{L_{t,y}^\infty L_x^2}^2\bigg(\int_{G_\beta^{l_2}}|\xi'(t)|\,\mathrm{d}t\bigg)\\ &\lesssim\|w_0\|_{L_{y,x}^2}^2.\end{aligned}$$

Integrating (7-55) by parts in space, we derive

$$\begin{aligned}(7\text{-}55)=-\sum_{k,k'=1}^{2}\int_{G_\beta^{l_2}}\iiiint\!\!\!\iiiint\bigg(\frac{\delta_{kk'}}{|y-\tilde y|}+\frac{(y-\tilde y)_{k'}(y-\tilde y)_k}{|y-\tilde y|^3}\bigg)\Re(\overline{w_n}(\partial_{y_k}-i\xi_k(t))\partial_{y_{k'}}w_n)(t,y,x)\\ \cdot K(t;z_0,z_1,z_2,z_3)\overline{v_{n'}^h(t,\tilde y-z_0,\tilde x)}\,\overline{v_{n_2'}^h(t,\tilde y-z_2,\tilde x)}\\ \cdot v_{n_1'}^l(t,\tilde y-z_1,\tilde x)v_{n_3'}^l(t,\tilde y-z_4,\tilde x)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_0\,\mathrm{d}x\,\mathrm{d}\tilde x\,\mathrm{d}y\,\mathrm{d}\tilde y\,\mathrm{d}t.\end{aligned}$$

Therefore, by the Hardy–Littlewood–Sobolev inequality, (7-53), Lemma 6.20, the Sobolev embedding theorem, the fact $G_\beta^{l_2} \subseteq G_\alpha^i$, $|\xi(t)| \ll 2^{l_2}$ and $l_2 \le i$, we have

$$\begin{aligned} 2^{l_2-2i}\Bigg|\sum_{n,n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}(7\text{-}55)\Bigg| &\lesssim 2^{l_2-2i}2^{2i}2^{-2l_2}\|w\|^2_{L_t^6L_y^3L_x^2}\|v^h\|^2_{L_t^3L_y^6L_x^2}\|v^l\|^2_{L_t^\infty L_y^4L_x^2} \\ &\lesssim \|w_0\|^2_{L^2_{y,x}}\|v\|^2_{\widetilde{X}_i(G_\alpha^i)}. \end{aligned}$$

By a similar argument, we infer

$$2^{l_2-2i}\Bigg|\sum_{n,n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}(7\text{-}56)\Bigg| \lesssim \|w_0\|^2_{L^2_{y,x}}\|v\|^2_{\widetilde{X}_i(G_\alpha^i)}.$$

Now we turn to (7-50). As (7-48), we have the corresponding integral kernel

$$\widetilde{K}(t;z_0,z_1,z_2,z_3) = \iiiint \tilde{m}(t;\eta_0,\eta_1,\eta_2,\eta_3)e^{iz_1\eta_1}e^{iz_2\eta_2}e^{i\eta_3z_3}e^{iz_0\eta_0}\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3\,\mathrm{d}\eta_0,$$

where

$$\begin{aligned} \tilde{m}(t;\eta_0,\eta_1,\eta_2,\eta_3) = -\frac{2^{-l_2}}{q(\eta)}\phi\left(\frac{\eta_1+\eta_2+\eta_3+\eta_0}{2^{l_2-10}}\right) \\ \cdot\left((\nabla\phi)\left(\frac{\eta_0-\xi(t)}{2^{l_2}}\right)\phi\left(\frac{\eta_2-\xi(t)}{2^{l_2}}\right)+\phi\left(\frac{\eta_0-\xi(t)}{2^{l_2}}\right)(\nabla\phi)\left(\frac{\eta_2-\xi(t)}{2^{l_2}}\right)\right). \end{aligned}$$

The kernel function $\widetilde{K}(t;z_0,z_1,z_2,z_3)$ satisfies

$$\sup_t\int|\widetilde{K}(t;z_0,z_1,z_2,z_3)|\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_0 \lesssim 2^{-3l_2}.$$

Thus

$$\begin{aligned} 2^{l_2-2i}\Bigg|\sum_{n,n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}(7\text{-}50)\Bigg| &\lesssim 2^{-2l_2-i}\|w\|^2_{L_t^\infty L^2_{y,x}}\|v^h\|^2_{L_t^\infty L^2_{y,x}}\|v^l\|^2_{L^\infty_{t,y}L^2_x}\left(\int_{G_\beta^{l_2}}|\xi'(t)|\,\mathrm{d}t\right) \\ &\lesssim \|w_0\|^2_{L^2_{y,x}}. \end{aligned}$$

Finally, we consider the term (7-51). Following the argument for the estimates (7-48) and (7-50), by the Bernstein inequality, the conservation of mass and Lemma 6.20, we deduce

$$2^{l_2-2i}\Bigg|\sum_{n,n'\in\mathbb{N}}\sum_{\substack{n_1',n_2',n_3'\in\mathbb{N}\\ n_1'-n_2'+n_3'=n'}}(7\text{-}51)\Bigg| \lesssim 2^{l_2-i}\|w_0\|^2_{L^2_{y,x}}(1+\|v\|^4_{\widetilde{X}_i(G_\alpha^i)}).$$

Therefore, we eventually arrive at

$$(7\text{-}25) \lesssim \|w_0\|^2_{L^2_{y,x}}(1+\|v\|^4_{\widetilde{X}_i(G_\alpha^i)}).$$

The proof of Theorem 7.5 is complete. □

7.2. ***Proof of Theorem 6.25.*** By Theorem 6.21, we have

$$\|v_\lambda\|_{\tilde{X}_{k_0}([0,\lambda^{-2}T])} \lesssim 1, \tag{7-57}$$

where

$$v_\lambda(t,y,x) = \lambda v(\lambda^2 t, \lambda y, x), \quad \text{with } \lambda = \frac{\epsilon_3 2^{k_0}}{K}. \tag{7-58}$$

Let $\tilde{w} = P^y_{\le 2^{k_0}} v_\lambda$. Then $\tilde{w}$ satisfies

$$i\partial_t \tilde{w} + \Delta_y \tilde{w} = F(\tilde{w}) + N,$$

where

$$N = P^y_{\le 2^{k_0}} F(v_\lambda).$$

Let

$$M(t) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\tilde{w}(t,\tilde{y},\tilde{x})|^2 \frac{y-\tilde{y}}{|y-\tilde{y}|} \Im(\bar{\tilde{w}} \nabla_y \tilde{w})(t,y,x)\, \mathrm{d}y\, \mathrm{d}\tilde{y}\, \mathrm{d}x\, \mathrm{d}\tilde{x}.$$

Then a direct calculation similar to [Dodson 2012; 2016a; 2016b; 2009] gives

$$\left\| \int_{\mathbb{R}} |\nabla_y|^{\frac{1}{2}} (|\tilde{w}(t,y,x)|^2)\, \mathrm{d}x \right\|^2_{L^2_{t,y}([0,\lambda^{-2}T]\times\mathbb{R}^2)} \lesssim \sup_{t\in[0,\lambda^{-2}T]} |M(t)| + \mathscr{E},$$

where

$$\mathscr{E} =$$

$$2\left|\int_0^{\lambda^{-2}T}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Im(\bar{\tilde{w}}(\nabla_y - i\xi(t))\tilde{w})(t,y,x) \frac{y-\tilde{y}}{|y-\tilde{y}|} \Im(\bar{\tilde{w}} N)(t,\tilde{y},\tilde{x})\, \mathrm{d}y\, \mathrm{d}\tilde{y}\, \mathrm{d}x\, \mathrm{d}\tilde{x}\, \mathrm{d}t\right| \tag{7-59}$$

$$+\left|\int_0^{\lambda^{-2}T}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\tilde{w}(t,\tilde{y},\tilde{x})|^2 \frac{y-\tilde{y}}{|y-\tilde{y}|} \Im(\bar{N}(\nabla_y - i\xi(t))\tilde{w})(t,y,x)\, \mathrm{d}y\, \mathrm{d}\tilde{y}\, \mathrm{d}x\, \mathrm{d}\tilde{x}\, \mathrm{d}t\right| \tag{7-60}$$

$$+\left|\int_0^{\lambda^{-2}T}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\tilde{w}(t,\tilde{y},\tilde{x})|^2 \frac{y-\tilde{y}}{|y-\tilde{y}|} \Im(\bar{\tilde{w}}(\nabla_y - i\xi(t))N)(t,y,x)\, \mathrm{d}y\, \mathrm{d}\tilde{y}\, \mathrm{d}x\, \mathrm{d}\tilde{x}\, \mathrm{d}t\right|. \tag{7-61}$$

Since $N(t) \le 1$, we have $N_\lambda(t) \le \epsilon_3 2^{k_0}/K$. By Theorem 6.6 and the Bernstein inequality, for any $\eta > 0$, if $K \ge C(\eta)$, we have

$$\|(\nabla_y - i\xi(t))\tilde{w}\|_{L^\infty_t L^2_{y,x}([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})} \lesssim \eta 2^{k_0}. \tag{7-62}$$

Therefore, by the Galilean transformation and the conservation of mass, we get

$$\sup_{t\in[0,\lambda^{-2}T]} |M(t)| \lesssim \eta 2^{k_0}.$$

We now consider (7-59). As in (7-37), let $v^l_\lambda = P^y_{\le 2^{k_0-3}} v_\lambda$ and $v^h_\lambda = P^y_{>2^{k_0-3}} v_\lambda$. Then we have the decomposition

$$\int_{\mathbb{R}} \Im(\bar{\tilde{w}} N)(t,\tilde{y},\tilde{x})\, \mathrm{d}\tilde{x} = \int_{\mathbb{R}} F_0(t,\tilde{y},\tilde{x}) + F_1(t,\tilde{y},\tilde{x}) + F_2(t,\tilde{y},\tilde{x}) + F_3(t,\tilde{y},\tilde{x}) + F_4(t,\tilde{y},\tilde{x})\, \mathrm{d}\tilde{x}.$$

We can see

$$\int_{\mathbb{R}} F_0(t,\tilde{y},\tilde{x})\, \mathrm{d}\tilde{x} = 0.$$

Following the same argument as the proof of (7-25), we may obtain

$$\int_{\mathbb{R}} \|F_2(t,\tilde{y},\tilde{x}) + F_3(t,\tilde{y},\tilde{x}) + F_4(t,\tilde{y},\tilde{x})\|_{L^1_{t,\tilde{y}}([0,\lambda^{-2}T]\times\mathbb{R}^2)}\,\mathrm{d}\tilde{x} \lesssim 1.$$

Then by (7-62) and the conservation of mass, we have

$$\left|\int_0^{\lambda^{-2}T}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Im(\bar{\tilde{w}}(\nabla_y - i\xi(t))\tilde{w})(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}(F_2+F_3+F_4)(t,\tilde{y},\tilde{x})\,\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,\mathrm{d}t\right| \lesssim \eta 2^{k_0}.$$

To estimate the contribution of the term with F_1 in (7-59), we see the support of the spatial Fourier transform of $\int_{\mathbb{R}} F_1(t,\tilde{y},\tilde{x})\,\mathrm{d}\tilde{x}$ is in $\{\xi : |\xi| \geq 2^{k_0-4}\}$ as in (7-38). Therefore, by integration by parts, the Hardy–Littlewood–Sobolev inequality, the Bernstein inequality, Lemma 6.20 and (7-57), we have

$$\begin{aligned}
&\left|\int_0^{\lambda^{-2}T}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Im(\bar{\tilde{w}}(\nabla_y - i\xi(t))\tilde{w})(t,y,x)\frac{y-\tilde{y}}{|y-\tilde{y}|}\left(\int_{\mathbb{R}} F_1(t,\tilde{y},\tilde{x})\,\mathrm{d}\tilde{x}\right)\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\lesssim \int_0^{\lambda^{-2}T}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}}(\bar{\tilde{w}}(\nabla_y - i\xi(t))\tilde{w})(t,y,x)\,\mathrm{d}x\right|\cdot\frac{1}{|y-\tilde{y}|}\cdot\left|\partial_{\tilde{y}}(-\Delta_{\tilde{y}})^{-1}\int_{\mathbb{R}} F_1(t,\tilde{y},\tilde{x})\,\mathrm{d}\tilde{x}\right|\mathrm{d}y\,\mathrm{d}\tilde{y}\,\mathrm{d}t \\
&\lesssim \|\tilde{w}\|_{L^\infty_t L^2_{y,x}([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})}\|(\nabla_y - i\xi(t))\tilde{w}\|_{L^4_{t,y}L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})} \\
&\qquad\qquad\cdot\left\|\partial_{\tilde{y}}(-\Delta_{\tilde{y}})^{-1}\int_{\mathbb{R}} F_1(t,\tilde{y},\tilde{x})\,\mathrm{d}\tilde{x}\right\|_{L^{4/3}_{t,y}([0,\lambda^{-2}T]\times\mathbb{R}^2)} \\
&\lesssim 2^{-k_0}\|(\nabla_y - i\xi(t))\tilde{w}\|_{L^4_{t,y}L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})}\|v^l_\lambda\|^3_{L^6_{t,y}L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})}.
\end{aligned}$$

By the Bernstein inequality, Lemma 6.20, and (7-57), we have

$$\|v^l_\lambda\|_{L^6_{t,y}L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})} \lesssim \sum_{0\leq l\leq k_0} 2^{\frac{l}{3}}2^{\frac{k_0-l}{6}} \lesssim 2^{\frac{k_0}{3}}.$$

Note that

$$\|(\nabla_y - i\xi(t))\tilde{w}\|_{L^{5/2}_t L^{10}_y L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})} \lesssim \sum_{0\leq l\leq k_0} 2^l 2^{\frac{2}{5}(k_0-l)} \lesssim 2^{k_0}. \tag{7-63}$$

Interpolating (7-63) and (7-62), we obtain

$$\|(\nabla_y - i\xi(t))\tilde{w}\|_{L^4_{t,y}L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})} \lesssim \eta^{\frac{3}{8}}2^{k_0}. \tag{7-64}$$

Thus, by the above estimates, we have

$$\text{(7-59)} \lesssim \eta^{\frac{3}{8}}2^{k_0}.$$

Now, we turn to (7-60). By (7-35) and (7-57), we have

$$\begin{aligned}
\text{(7-60)} &\lesssim \|\tilde{w}\|^2_{L^\infty_t L^2_{y,x}([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})}\|N\|_{L^{4/3}_{t,y}L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})}\|(\nabla_y - i\xi(t))\tilde{w}\|_{L^4_{t,y}L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})} \\
&\lesssim \eta^{\frac{3}{8}}2^{k_0}.
\end{aligned}$$

Finally, we consider (7-61). Applying integration by parts, we have

$$(7\text{-}61) \le (7\text{-}60) + \left| \int_0^{\lambda^{-2}T} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\tilde{w}(t,\tilde{y},\tilde{x})|^2 \frac{1}{|y-\tilde{y}|} \Re(\bar{\tilde{w}} N)(t,y,x)\, \mathrm{d}y\, \mathrm{d}\tilde{y}\, \mathrm{d}x\, \mathrm{d}\tilde{x}\, \mathrm{d}t \right|. \tag{7-65}$$

By (6-1) and (7-36), we see

$$(7\text{-}65) \lesssim \int_0^{\lambda^{-2}T} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \|\tilde{w}(t,\tilde{y},\tilde{x})\|^2_{L^2_{\tilde{x}}} \frac{1}{|y-\tilde{y}|} \|\tilde{w}(t,y,x)\|_{L^2_x} \|v^h_\lambda(t,y,x)\|^3_{L^2_x}\, \mathrm{d}y\, \mathrm{d}\tilde{y}\, \mathrm{d}t \tag{7-66}$$

$$+ \int_0^{\lambda^{-2}T} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \|\tilde{w}(t,\tilde{y},\tilde{x})\|^2_{L^2_{\tilde{x}}} \frac{1}{|y-\tilde{y}|} \|\tilde{w}(t,y,x)\|_{L^2_x} \|v^l_\lambda(t,y,x)\|^2_{L^2_x} \|v^h_\lambda(t,y,x)\|_{L^2_x}\, \mathrm{d}y\, \mathrm{d}\tilde{y}\, \mathrm{d}t. \tag{7-67}$$

By the Hardy–Littlewood–Sobolev inequality, (7-62), (7-64), Lemma 6.20, the Sobolev embedding theorem, the conservation of mass, and interpolation, we have

$$(7\text{-}66) \lesssim \|v^h_\lambda\|^3_{L^4_{t,y} L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})} \|\tilde{w}\|^3_{L^4_{t,y} L^2_{\tilde{x}}([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})} \lesssim \eta^{\frac{3}{8}} 2^{k_0}$$

and

$$(7\text{-}67) \lesssim \|v^h_\lambda\|_{L^3_t L^6_y L^2_x([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})} \|\tilde{w}\|^3_{L^9_t L^{90/29}_y L^2_{\tilde{x}}([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})} \cdot \|v^l_\lambda\|^2_{L^6_t L^{60/11}_y L^2_{\tilde{x}}([0,\lambda^{-2}T]\times\mathbb{R}^2\times\mathbb{R})} \lesssim \eta^{\frac{1}{6}} 2^{k_0}.$$

Thus, by the above estimates, we have

$$\left\| \int_{\mathbb{R}} |\nabla_y|^{\frac{1}{2}} (|\tilde{w}(t,y,x)|^2)\, \mathrm{d}x \right\|^2_{L^2_{t,y}([0,\lambda^{-2}T]\times\mathbb{R}^2)} \lesssim \eta^{\frac{1}{6}} 2^{k_0}.$$

Undoing the scaling in (7-58), we finally reach the desired estimate (6-41). □

Appendix: Well-posedness theory for (1-2)

In this appendix, we present the proofs of the recorded results in Section 3. Let

$$X_1(t) = x\sin(t) - i\cos(t)\,\partial_x \quad\text{and}\quad X_2(t) = x\cos(t) + i\sin(t)\,\partial_x. \tag{A-1}$$

We have the pointwise identity: for any $f \in \mathcal{S}(\mathbb{R}^3)$,

$$|X_1(t)f(y,x)|^2 + |X_2(t)f(y,x)|^2 = |xf(y,x)|^2 + |\partial_x f(y,x)|^2 \quad \text{for all } t \in \mathbb{R}. \tag{A-2}$$

The next result follows by direct calculation. We refer to [Carles 2002b] for more explanation.

Lemma A.1. *The operators $X_1(t)$ and $X_2(t)$ satisfy the following properties*:

(1) *They correspond to the conjugation of gradient and momentum by the free flow*,

$$X_1(t) = e^{it(\Delta_{\mathbb{R}^3}-x^2)}(-i\partial_x)e^{-it(\Delta_{\mathbb{R}^3}-x^2)},$$
$$X_2(t) = e^{it(\Delta_{\mathbb{R}^3}-x^2)}x e^{-it(\Delta_{\mathbb{R}^3}-x^2)}.$$

(2) *They act on the nonlinearity like derivatives, that is, for $j=1,2$, we have*

$$|X_j(t)(|u|^2u)| \lesssim |u|^2|X_j(t)u|.$$

As a consequence, for any $u_\pm \in \Sigma$, we have

$$\|e^{-it(\Delta_{\mathbb{R}^3}-x^2)}u(t)-u_\pm\|_\Sigma = \|u(t)-e^{it(\Delta_{\mathbb{R}^3}-x^2)}u_\pm\|_{L_x^2H_y^1} + \|X_1(t)(u(t)-e^{it(\Delta_{\mathbb{R}^3}-x^2)}u_\pm)\|_{L^2_{y,x}} + \|X_2(t)(u(t)-e^{it(\Delta_{\mathbb{R}^3}-x^2)}u_\pm)\|_{L^2_{y,x}}.$$

We now show the local well-posedness part of Theorem 3.4 in the following formulation. This is essentially following the argument in [Cazenave 2003; Tao 2006].

Theorem A.2 (local well-posedness). *For any $E>0$ and u_0 with $\|u_0\|_{L_y^2\mathcal{H}_x^1(\mathbb{R}^2\times\mathbb{R})} \le E$, there exists $\delta_0=\delta_0(E)>0$ such that if*

$$\|e^{it(\Delta_{\mathbb{R}^3}-x^2)}u_0\|_{L^4_{t,y}L^2_x(I\times\mathbb{R}^2\times\mathbb{R})} + \|X_1(t)e^{it(\Delta_{\mathbb{R}^3}-x^2)}u_0\|_{L^4_{t,y}L^2_x(I\times\mathbb{R}^2\times\mathbb{R})} + \|X_2(t)e^{it(\Delta_{\mathbb{R}^3}-x^2)}u_0\|_{L^4_{t,y}L^2_x(I\times\mathbb{R}^2\times\mathbb{R})} \le \delta_0,$$

where I is the time interval, there exists a unique solution $u\in C_t^0L_y^2\mathcal{H}_x^1(I\times\mathbb{R}^2\times\mathbb{R})$ of (1-2) satisfying

$$\|u\|_{L^4_{t,y}\mathcal{H}^1_x(I\times\mathbb{R}^2\times\mathbb{R})} \le 2\|e^{it(\Delta_{\mathbb{R}^3}-x^2)}u_0\|_{L^4_{t,y}\mathcal{H}^1_x(I\times\mathbb{R}^2\times\mathbb{R})} \quad \textit{and} \quad \|u\|_{L_t^\infty L_y^2\mathcal{H}_x^1(I\times\mathbb{R}^2\times\mathbb{R})} \le C\|u_0\|_{L_y^2\mathcal{H}_x^1}.$$

Proof. Let

$$\Phi(u) = e^{it(\Delta_{\mathbb{R}^3}-x^2)}u_0 - i\int_0^t e^{i(t-s)(\Delta_{\mathbb{R}^3}-x^2)}(|u|^2u)(s)\,\mathrm{d}s,$$

and set the space X to be

$$X=\{u\in C_t^0L_y^2\mathcal{H}_x^1 : \|u\|_{L_t^\infty L_y^2\mathcal{H}_x^1} \le 2E,\ \|u\|_{L^4_{t,y}\mathcal{H}^1_x} \le 2C\delta_0\}$$

or

$$X=\Big\{u\in C_t^0L_y^2\mathcal{H}_x^1 : \|u\|_{L_t^\infty L^2_{y,x}} \le 2E,\ \|u\|_{L^4_{t,y}L^2_x} \le 2C\delta_0,\ \|X_j(t)u\|_{L_t^\infty L^2_{y,x}} \le 2E,\ \|X_j(t)u\|_{L^4_{t,y}L^2_x} \le 2C\delta_0,\ j=1,2\Big\}.$$

For any $u\in X$, by Proposition 3.2, Hölder's inequality, Sobolev's inequality, Lemma A.1, and (A-1), we have

$$\|\Phi(u)\|_{L_t^\infty L^2_{y,x}} \lesssim \|u_0\|_{L^2_{y,x}} + \|u\|_{L^4_{t,y}L^2_x}\|u\|^2_{L^4_{t,y}H^1_x}$$

and

$$\|X_1(t)\Phi(u)\|_{L_t^\infty L^2_{y,x}} + \|X_2(t)\Phi(u)\|_{L_t^\infty L^2_{y,x}} \lesssim \|\nabla_x u_0\|_{L^2_{y,x}} + \|xu_0\|_{L^2_{y,x}} + \|u\|^2_{L^4_{t,y}H^1_x}(\|X_1(t)u\|_{L^4_{t,y}L^2_x} + \|X_2(t)u\|_{L^4_{t,y}L^2_x}).$$

Thus

$$\|\Phi(u)\|_{L_t^\infty L^2_{y,x}} + \|X_1(t)\Phi(u)\|_{L_t^\infty L^2_{y,x}} + \|X_2(t)\Phi(u)\|_{L_t^\infty L^2_{y,x}} \le E + (2C\delta_0)^3 \le 2E. \tag{A-3}$$

Similarly, we can obtain

$$\|\Phi(u)\|_{L^4_{t,y}L^2_x} + \|X_1(t)\Phi(u)\|_{L^4_{t,y}L^2_x} + \|X_2(t)\Phi(u)\|_{L^4_{t,y}L^2_x} \le \delta_0 + (2C\delta_0)^3 \le 2C\delta_0. \tag{A-4}$$

In the same time, for any $u, v \in X$, by the Strichartz estimate, Hölder's inequality, and Sobolev's inequality, we have

$$\begin{aligned}\|\Phi(u)-\Phi(v)\|_{L^4_{t,y}L^2_x} &\lesssim \big\||u|^2u - |v|^2v\big\|_{L^{4/3}_{t,y}L^2_x} \\ &\lesssim \|u-v\|_{L^4_{t,y}L^2_x}(\|u\|^2_{L^4_{t,y}H^1_x} + \|v\|^2_{L^4_{t,y}H^1_x}) \lesssim (2C\delta_0)^2\|u-v\|_{L^4_{t,y}L^2_x}.\end{aligned} \tag{A-5}$$

Combining (A-3), (A-4), and (A-5), we have for δ_0 small enough $\Phi : X \to X$ is a contractive map. Therefore, the theorem follows from the fixed point theorem. □

We now turn to the proof of the scattering norm in Theorem 3.4.

Proof of the scattering norm part of Theorem 3.4. We need to show

$$\|u\|_{L^4_{t,y}\mathcal{H}^1_x \cap L^4_t W^{1,4}_y L^2_x(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})} \le C(M). \tag{A-6}$$

Then by the scattering theory of the nonlinear Schrödinger equations [Antonelli et al. 2015; Carles 2011; Tao 2006], we have scattering in (3-1). By the well-posedness part of Theorem 3.4, it suffices to prove (A-6) as an a priori bound.

Divide the time interval $\mathbb{R}$ into $N \sim \big(1+\frac{L}{\delta}\big)^4$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|u\|_{L^4_{t,y}\mathcal{H}^{1-\epsilon_0}_x(I_j\times\mathbb{R}^2\times\mathbb{R})} \le \delta, \tag{A-7}$$

where $\delta > 0$ will be chosen later.

On each I_j, by (A-1), the Strichartz estimate, the Sobolev embedding and (A-7), we have

$$\begin{aligned}&\|u\|_{L^4_t W^{1,4}_y L^2_x \cap L^4_{t,y}\mathcal{H}^1_x(I_j\times\mathbb{R}^2\times\mathbb{R})} \\ &\le C\big(\|u(t_j)\|_\Sigma + \||u|^2u\|_{L^{4/3}_{t,y}L^2_x} + \|X_1(t)(|u|^2u)\|_{L^{4/3}_{t,y}L^2_x} + \|X_2(t)(|u|^2u)\|_{L^{4/3}_{t,y}L^2_x}\big) \\ &\le C\big(\|u(t_j)\|_\Sigma + \|u\|^2_{L^4_{t,y}H^{1-\epsilon_0}_x}(\|u\|_{L^4_{t,y}L^2_x} + \|X_1(t)u\|_{L^4_{t,y}L^2_x} \\ &\qquad\qquad + \|X_2(t)u\|_{L^4_{t,y}L^2_x}) + \|u\|_{L^4_tW^{1,4}_yL^2_x}\|u\|^2_{L^4_{t,y}H^{1-\epsilon_0}_x}\big) \\ &\le C\big(\|u(t_j)\|_\Sigma + \|u\|^2_{L^4_{t,y}H^{1-\epsilon_0}_x}(\|u\|_{L^4_{t,y}L^2_x} + \|\nabla_x u\|_{L^4_{t,y}L^2_x} + \||x|u\|_{L^4_{t,y}L^2_x} + \|u\|_{L^4_tW^{1,4}_yL^2_x})\big) \\ &\le C\big(\|u(t_j)\|_\Sigma + \delta^2\|u\|_{L^4_tW^{1,4}_yL^2_x\cap L^4_{t,y}\mathcal{H}^1_x}\big).\end{aligned}$$

Choosing $\delta \le \big(\frac{1}{2C}\big)^{1/4}$ leads to the estimate

$$\|u\|_{L^4_tW^{1,4}_yL^2_x\cap L^4_{t,y}\mathcal{H}^1_x(I_j\times\mathbb{R}^2\times\mathbb{R})} \le 2C\|u(t_j)\|_{\Sigma_{y,x}}.$$

The desired bound (A-6) now follows by adding up the bounds on each subintervals I_j. □

Proof of Theorem 3.5. We only give a sketch for the proof of Theorem 3.5, since it follows essentially by the same argument as in the proof of Theorem 3.4.

For $u_- \in \Sigma$, let $\delta > 0$ be a small absolute constant to be taken later. Taking $T_- = T_-(u_-)$ large enough and then applying the monotone convergence theorem, we obtain

$$\|e^{it(\Delta - x^2)} u_-\|_{L_t^4 W_y^{1,4} L_x^2 \cap L_{t,y}^4 \mathcal{H}_x^1((-\infty,-T_-]\times\mathbb{R}^2\times\mathbb{R})} \le \delta.$$

Solving the integral equation

$$u(t) = e^{it(\Delta - x^2)} u_- - i \int_{-\infty}^t e^{i(t-s)(\Delta - x^2)} (|u|^2 u)(s)\,\mathrm{d}s$$

in $L_t^\infty L_y^2 \mathcal{H}_x^1 \cap L_t^\infty H_y^1 L_x^2 \cap L_t^4 W_y^{1,4} L_x^2 \cap L_{t,y}^4 \mathcal{H}_x^1((-\infty, -T_-] \times \mathbb{R}^2 \times \mathbb{R})$ and keeping u small in $L_t^4 W_y^{1,4} L_x^2 \cap L_{t,y}^4 \mathcal{H}_x^1((-\infty, -T_-] \times \mathbb{R}^2 \times \mathbb{R})$, the argument in the proof of Theorem A.2 implies that there exists a solution of (1-2) on $(-\infty, -T_-] \times \mathbb{R}^2 \times \mathbb{R}$, which furthermore satisfies (3-2). This completes the proof for first part of Theorem 3.5. The proof for the second part of Theorem 3.5 is similar and thus we omit it here. □

We now turn to the proof of Theorem 3.6. First, we show the following short-time version.

Lemma A.3 (short-time stability theorem). *Let I be a compact interval and let $\tilde{u}$ be an approximate solution to (1-2) in the sense that $i\partial_t \tilde{u} + \Delta_{\mathbb{R}^3} \tilde{u} - x^2 \tilde{u} = |\tilde{u}|^2 \tilde{u} + e$ for some function e. Assume that*

$$\|\tilde{u}\|_{L_t^\infty L_y^2 \mathcal{H}_x^1(I\times\mathbb{R}^2\times\mathbb{R})} \le M \tag{A-8}$$

for some positive constant M. Let $t_0 \in I$ and $u(t_0)$ be such that

$$\|u(t_0) - \tilde{u}(t_0)\|_{L_y^2 \mathcal{H}_x^1} \le M' \tag{A-9}$$

for some $M' > 0$.

Assume also the smallness conditions hold:

$$\|\tilde{u}\|_{L_t^4 L_y^4 \mathcal{H}_x^1(I\times\mathbb{R}^2\times\mathbb{R})} \le \epsilon, \tag{A-10}$$

$$\|e^{i(t-t_0)(\Delta_{\mathbb{R}^3} - x^2)}(u(t_0) - \tilde{u}(t_0))\|_{L_t^4 L_y^4 \mathcal{H}_x^1} + \|e\|_{L_t^{4/3} L_y^{4/3} \mathcal{H}_x^1} \le \epsilon \tag{A-11}$$

for some $0 < \epsilon \le \epsilon_1$, where $\epsilon_1 = \epsilon_1(M, M') > 0$ is a small constant. Then, there exists a solution u to (1-2) on $I \times \mathbb{R}^2 \times \mathbb{R}$ with initial data $u(t_0)$ at time $t = t_0$ satisfying

$$\|u - \tilde{u}\|_{L_{t,y}^4 \mathcal{H}_x^1} \lesssim \epsilon, \tag{A-12}$$

$$\|u - \tilde{u}\|_{L_t^\infty L_y^2 \mathcal{H}_x^1} \lesssim M', \tag{A-13}$$

$$\|u\|_{L_t^\infty L_y^2 \mathcal{H}_x^1} \lesssim M + M', \tag{A-14}$$

$$\||u|^2 u - |\tilde{u}|^2 \tilde{u}\|_{L_t^{4/3} L_y^{4/3} \mathcal{H}_x^1} \lesssim \epsilon. \tag{A-15}$$

Proof. By symmetry, we may assume $t_0 = \inf I$. Let $w = u - \tilde{u}$. Then w satisfies

$$i\,\partial_t w + \Delta_{\mathbb{R}^3} w - x^2 w = |\tilde{u} + w|^2 (\tilde{u} + w) - |\tilde{u}|^2 \tilde{u} - e.$$

For $t \in I$, we define

$$D(t) = \||\tilde{u} + w|^2 (\tilde{u} + w) - |\tilde{u}|^2 \tilde{u}\|_{L_{t,y}^{4/3} \mathcal{H}_x^1([t_0,t]\times\mathbb{R}^2\times\mathbb{R})}.$$

By (A-10), we have

$$D(t) \lesssim \|w\|_{L^4_{t,y}\mathcal{H}^1_x}(\|\tilde{u}\|^2_{L^4_{t,y}\mathcal{H}^1_x} + \|w\|^2_{L^4_{t,y}\mathcal{H}^1_x}) \lesssim \|w\|^3_{L^4_{t,y}\mathcal{H}^1_x([t_0,t]\times\mathbb{R}^2\times\mathbb{R})} + \epsilon_1^2 \|w\|_{L^4_{t,y}\mathcal{H}^1_x([t_0,t]\times\mathbb{R}^2\times\mathbb{R})}. \tag{A-16}$$

On the other hand, by the Strichartz estimate and (A-11), we get

$$\|w\|_{L^4_{t,y}\mathcal{H}^1_x([t_0,t]\times\mathbb{R}^2\times\mathbb{R})} \lesssim \|e^{i(t-t_0)(\Delta_{\mathbb{R}^3}-x^2)}w(t_0)\|_{L^4_{t,y}\mathcal{H}^1_x([t_0,t]\times\mathbb{R}^2\times\mathbb{R})} + D(t) + \|e\|_{L^{4/3}_{t,y}\mathcal{H}^1_x([t_0,t]\times\mathbb{R}^2\times\mathbb{R})} \lesssim D(t)+\epsilon. \tag{A-17}$$

Combining (A-16) and (A-17), we obtain

$$D(t) \lesssim (D(t)+\epsilon)^3 + \epsilon_1^2(D(t)+\epsilon).$$

A standard continuity argument then shows that if ϵ_1 is taken sufficiently small, then

$$D(t) \lesssim \epsilon \quad \text{for all } t \in I,$$

which implies (A-15).

Using (A-15) and (A-17), one easily derives (A-12). Moreover, by the Strichartz estimate, (A-9) and (A-15),

$$\|w\|_{L^\infty_t L^2_y \mathcal{H}^1_x(I\times\mathbb{R}^2\times\mathbb{R})} \lesssim \|w(t_0)\|_{L^2_y\mathcal{H}^1_x} + \||\tilde{u}+w|^2(\tilde{u}+w) - |\tilde{u}|^2\tilde{u}\|_{L^{4/3}_{t,y}\mathcal{H}^1_x} + \|e\|_{L^{4/3}_{t,y}\mathcal{H}^1_x} \lesssim M' + \epsilon,$$

which establishes (A-13) for $\epsilon_1 = \epsilon_1(M')$ sufficiently small.

To prove (A-14), we use the Strichartz estimate, (A-8), (A-9), (A-15) and (A-10),

$$\|u\|_{L^\infty_t L^2_y \mathcal{H}^1_x(I\times\mathbb{R}^2\times\mathbb{R})} \lesssim \|\tilde{u}(t_0)\|_{L^2_y\mathcal{H}^1_x} + \|u(t_0)-\tilde{u}(t_0)\|_{L^2_y\mathcal{H}^1_x} + \||u|^2u - |\tilde{u}|^2\tilde{u}\|_{L^{4/3}_{t,y}\mathcal{H}^1_x} + \||\tilde{u}|^2\tilde{u}\|_{L^{4/3}_{t,y}\mathcal{H}^1_x} \lesssim M+M'+\epsilon+\|\tilde{u}\|^3_{L^4_{t,y}\mathcal{H}^1_x} \lesssim M+M'+\epsilon+\epsilon_1^3.$$

The proof is complete by choosing $\epsilon_1 = \epsilon_1(M, M')$ sufficiently small. □

We now show the proof of Theorem 3.6.

Proof of Theorem 3.6. We divide the interval I into $N \sim \left(1+\frac{L}{\epsilon_0}\right)^4$ subintervals $I_j = [t_j, t_{j+1}]$, $0 \le j \le N-1$, such that

$$\|\tilde{u}\|_{L^4_{t,y}\mathcal{H}^1_x(I_j\times\mathbb{R}^2\times\mathbb{R})} \le \epsilon_1,$$

where $\epsilon_1 = \epsilon_1(M, 2M')$ is given by Lemma A.3.

By choosing ϵ_1 sufficiently small depending on J, M and M', we can apply Lemma A.3 to obtain, for each j and all $0<\epsilon<\epsilon_1$,

$$\|u-\tilde{u}\|_{L^4_{t,y}\mathcal{H}^1_x(I_j\times\mathbb{R}^2\times\mathbb{R})} \le C(j)\epsilon, \qquad \|u-\tilde{u}\|_{L^\infty_t L^2_y\mathcal{H}^1_x(I_j\times\mathbb{R}^2\times\mathbb{R})} \le C(j)M',$$

$$\|u\|_{L^\infty_t L^2_y\mathcal{H}^1_x(I_j\times\mathbb{R}^2\times\mathbb{R})} \le C(j)(M+M'), \quad \||u|^2u - |\tilde{u}|^2\tilde{u}\|_{L^{4/3}_{t,y}\mathcal{H}^1_x(I_j\times\mathbb{R}^2\times\mathbb{R})} \le C(j)\epsilon,$$

provided we can prove that analogues of (3-3) and (3-4) hold with t_0 replaced by t_j.

In order to verify this, we use an inductive argument. By the Strichartz estimate, (3-3), and the inductive hypothesis,

$$\begin{aligned}\|u(t_j)-\tilde{u}(t_j)\|_{L_y^2\mathcal{H}_x^1} &\lesssim \|u(t_0)-\tilde{u}(t_0)\|_{L_y^2\mathcal{H}_x^1} + \||u|^2u-|\tilde{u}|^2\tilde{u}\|_{L_{t,y}^{4/3}\mathcal{H}_x^1([t_0,t_j]\times\mathbb{R}^2\times\mathbb{R})} + \|e\|_{L_{t,y}^{4/3}\mathcal{H}_x^1([t_0,t_j]\times\mathbb{R}^2\times\mathbb{R})}\\ &\lesssim M' + \sum_{k=0}^{j-1} C(k)\epsilon + \epsilon.\end{aligned}$$

Similarly, by the Strichartz estimate, (3-4), and the inductive hypothesis,

$$\begin{aligned}\|e^{i(t-t_j)(\Delta_{\mathbb{R}^3}-x^2)}(u(t_j)-\tilde{u}(t_j))\|_{L_{t,y}^4\mathcal{H}_x^1(I_j\times\mathbb{R}^2\times\mathbb{R})} &\lesssim \|e^{i(t-t_0)(\Delta_{\mathbb{R}^3}-x^2)}(u(t_0)-\tilde{u}(t_0))\|_{L_{t,y}^4\mathcal{H}_x^1(I_j\times\mathbb{R}^2\times\mathbb{R})} + \|e\|_{L_{t,y}^{4/3}\mathcal{H}_x^1([t_0,t_j]\times\mathbb{R}^2\times\mathbb{R})}\\ &\qquad + \||u|^2u-|\tilde{u}|^2\tilde{u}\|_{L_{t,y}^{4/3}\mathcal{H}_x^1([t_0,t_j]\times\mathbb{R}^2\times\mathbb{R})}\\ &\lesssim \epsilon + \sum_{k=0}^{j-1} C(k)\epsilon.\end{aligned}$$

It is clear now we may choose ϵ_1 sufficiently small, depending on N, M and M', such that the hypotheses of Lemma A.3 continue to hold as j varies. This completes the proof of Theorem 3.6. □

Acknowledgments

Xing Cheng is grateful to Professor Rémi Carles for explaining his work on the nonlinear Schrödinger equation with quadratic potentials and for pointing out some typos and misunderstanding in an initial version of this manuscript. Xing Cheng also thanks Binghua Feng for calling his attention to [Fukuizumi and Ohta 2003; Cao et al. 2022]. The authors thank Anxo Biasi for helpful discussion on the (DCR) system and for calling their attention to [Biasi et al. 2019a; 2019b; Evnin 2020]. The authors are grateful for the anonymous referees, for their valuable comments that greatly improved the exposition.

Xing Cheng has been supported by the NSF of Jiangsu Province (grant no. BK20221497), the NSF grant of China (grant U2340221), and NSF of Jiangsu Province (grant no. BK20230036). The corresponding author Chang-Yu Guo is supported by the Young Scientist Program of the Ministry of Science and Technology of China (no. 2021YFA1002200), the NSF of China (no. 12101362 and 12311530037), the Taishan Scholar Project and the NSF of Shandong Province (no. ZR2022YQ01). Zihua Guo was supported by the ARC project (no. DP200101065). Xian Liao is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) Project-ID 258734477 -SFB 1173. Jia Shen is partially supported by NSFC 12171356.

References

[Antonelli et al. 2015] P. Antonelli, R. Carles, and J. D. Silva, "Scattering for nonlinear Schrödinger equation under partial harmonic confinement", *Comm. Math. Phys.* **334**:1 (2015), 367–396. MR Zbl

[Ardila and Carles 2021] A. H. Ardila and R. Carles, "Global dynamics below the ground states for NLS under partial harmonic confinement", *Commun. Math. Sci.* **19**:4 (2021), 993–1032. MR Zbl

[Barron 2021] A. Barron, "On global-in-time Strichartz estimates for the semiperiodic Schrödinger equation", *Anal. PDE* **14**:4 (2021), 1125–1152. MR Zbl

[Bégout and Vargas 2007] P. Bégout and A. Vargas, "Mass concentration phenomena for the L^2-critical nonlinear Schrödinger equation", *Trans. Amer. Math. Soc.* **359**:11 (2007), 5257–5282. MR Zbl

[Bellazzini et al. 2017] J. Bellazzini, N. Boussaïd, L. Jeanjean, and N. Visciglia, "Existence and stability of standing waves for supercritical NLS with a partial confinement", *Comm. Math. Phys.* **353**:1 (2017), 229–251. MR Zbl

[Biasi et al. 2018] A. Biasi, P. Bizoń, B. Craps, and O. Evnin, "Two infinite families of resonant solutions for the Gross–Pitaevskii equation", *Phys. Rev. E* **98**:3 (2018), art. id. 032222. MR

[Biasi et al. 2019a] A. Biasi, P. Bizoń, and O. Evnin, "Complex plane representations and stationary states in cubic and quintic resonant systems", *J. Phys. A* **52**:43 (2019), art. id. 435201. MR Zbl

[Biasi et al. 2019b] A. Biasi, P. Bizoń, and O. Evnin, "Solvable cubic resonant systems", *Comm. Math. Phys.* **369**:2 (2019), 433–456. MR Zbl

[Bourgain 1998] J. Bourgain, "Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity", *Int. Math. Res. Not.* **1998**:5 (1998), 253–283. MR Zbl

[Buckmaster et al. 2019] T. Buckmaster, P. Germain, Z. Hani, and J. Shatah, "Analysis of (CR) in higher dimension", *Int. Math. Res. Not.* **2019**:4 (2019), 1265–1280. MR Zbl

[Candy 2019] T. Candy, "Multi-scale bilinear restriction estimates for general phases", *Math. Ann.* **375**:1-2 (2019), 777–843. MR Zbl

[Cao et al. 2022] D. Cao, B. Feng, and T. Luo, "On the standing waves for the X-ray free electron laser Schrödinger equation", *Discrete Contin. Dyn. Syst.* **42**:12 (2022), 6097–6137. MR Zbl

[Carles 2002a] R. Carles, "Critical nonlinear Schrödinger equations with and without harmonic potential", *Math. Models Methods Appl. Sci.* **12**:10 (2002), 1513–1523. MR Zbl

[Carles 2002b] R. Carles, "Remarks on nonlinear Schrödinger equations with harmonic potential", *Ann. Henri Poincaré* **3**:4 (2002), 757–772. MR Zbl

[Carles 2003] R. Carles, "Nonlinear Schrödinger equations with repulsive harmonic potential and applications", *SIAM J. Math. Anal.* **35**:4 (2003), 823–843. MR Zbl

[Carles 2008] R. Carles, *Semi-classical analysis for nonlinear Schrödinger equations*, World Sci., Hackensack, NJ, 2008. MR Zbl

[Carles 2011] R. Carles, "Nonlinear Schrödinger equation with time dependent potential", *Commun. Math. Sci.* **9**:4 (2011), 937–964. MR Zbl

[Carles and Gallo 2015] R. Carles and C. Gallo, "Scattering for the nonlinear Schrödinger equation with a general one-dimensional confinement", *J. Math. Phys.* **56**:10 (2015), art. id. 101503. MR Zbl

[Carles and Keraani 2007] R. Carles and S. Keraani, "On the role of quadratic oscillations in nonlinear Schrödinger equations, II: The L^2-critical case", *Trans. Amer. Math. Soc.* **359**:1 (2007), 33–62. MR Zbl

[Cazenave 2003] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Math. **10**, Amer. Math. Soc., Providence, RI, 2003. MR Zbl

[Cheng et al. 2020a] X. Cheng, Z. Guo, K. Yang, and L. Zhao, "On scattering for the cubic defocusing nonlinear Schrödinger equation on the waveguide $\mathbb{R}^2 \times \mathbb{T}$", *Rev. Mat. Iberoam.* **36**:4 (2020), 985–1011. MR Zbl

[Cheng et al. 2020b] X. Cheng, Z. Guo, and Z. Zhao, "On scattering for the defocusing quintic nonlinear Schrödinger equation on the two-dimensional cylinder", *SIAM J. Math. Anal.* **52**:5 (2020), 4185–4237. MR Zbl

[Colliander et al. 2004] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, "Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on $\mathbb{R}^3$", *Comm. Pure Appl. Math.* **57**:8 (2004), 987–1014. MR Zbl

[Colliander et al. 2008] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, "Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$", *Ann. of Math.* (2) **167**:3 (2008), 767–865. MR Zbl

[Colliander et al. 2009] J. Colliander, M. Grillakis, and N. Tzirakis, "Tensor products and correlation estimates with applications to nonlinear Schrödinger equations", *Comm. Pure Appl. Math.* **62**:7 (2009), 920–968. MR Zbl

[Colliander et al. 2010] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, "Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation", *Invent. Math.* **181**:1 (2010), 39–113. MR Zbl

[Constantin and Saut 1988] P. Constantin and J.-C. Saut, "Local smoothing properties of dispersive equations", *J. Amer. Math. Soc.* **1**:2 (1988), 413–439. MR Zbl

[Dartois et al. 2020] S. Dartois, O. Evnin, L. Lionni, V. Rivasseau, and G. Valette, "Melonic turbulence", *Comm. Math. Phys.* **374**:2 (2020), 1179–1228. MR Zbl

[Dodson 2012] B. Dodson, "Global well-posedness and scattering for the defocusing, L^2-critical nonlinear Schrödinger equation when $d \geq 3$", *J. Amer. Math. Soc.* **25**:2 (2012), 429–463. MR Zbl

[Dodson 2015] B. Dodson, "Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state", *Adv. Math.* **285** (2015), 1589–1618. MR Zbl

[Dodson 2016a] B. Dodson, "Global well-posedness and scattering for the defocusing, L^2 critical, nonlinear Schrödinger equation when $d = 1$", *Amer. J. Math.* **138**:2 (2016), 531–569. MR Zbl

[Dodson 2016b] B. Dodson, "Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when $d = 2$", *Duke Math. J.* **165**:18 (2016), 3435–3516. MR Zbl

[Dodson 2019] B. Dodson, "Global well-posedness and scattering for the focusing, cubic Schrödinger equation in dimension $d = 4$", *Ann. Sci. École Norm. Sup.* (4) **52**:1 (2019), 139–180. MR Zbl

[Dodson and Lawrie 2015] B. Dodson and A. Lawrie, "Scattering for the radial 3D cubic wave equation", *Anal. PDE* **8**:2 (2015), 467–497. MR Zbl

[Dodson et al. 2017] B. Dodson, C. Miao, J. Murphy, and J. Zheng, "The defocusing quintic NLS in four space dimensions", *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **34**:3 (2017), 759–787. MR Zbl

[Dodson et al. 2020] B. Dodson, A. Lawrie, D. Mendelson, and J. Murphy, "Scattering for defocusing energy subcritical nonlinear wave equations", *Anal. PDE* **13**:7 (2020), 1995–2090. MR Zbl

[Evnin 2020] O. Evnin, "Breathing modes, quartic nonlinearities and effective resonant systems", *SIGMA Symmetry Integrability Geom. Methods Appl.* **16** (2020), art. id. 034. MR Zbl

[Faou et al. 2016] E. Faou, P. Germain, and Z. Hani, "The weakly nonlinear large-box limit of the 2D cubic nonlinear Schrödinger equation", *J. Amer. Math. Soc.* **29**:4 (2016), 915–982. MR Zbl

[Fennell 2019] J. Fennell, "Resonant Hamiltonian systems associated to the one-dimensional nonlinear Schrödinger equation with harmonic trapping", *Comm. Partial Differential Equations* **44**:12 (2019), 1299–1344. MR Zbl

[Fukuizumi and Ohta 2003] R. Fukuizumi and M. Ohta, "Instability of standing waves for nonlinear Schrödinger equations with potentials", *Differential Integral Equations* **16**:6 (2003), 691–706. MR Zbl

[Gérard et al. 2019] P. Gérard, P. Germain, and L. Thomann, "On the cubic lowest Landau level equation", *Arch. Ration. Mech. Anal.* **231**:2 (2019), 1073–1128. MR Zbl

[Germain and Thomann 2016] P. Germain and L. Thomann, "On the high frequency limit of the LLL equation", *Quart. Appl. Math.* **74**:4 (2016), 633–641. MR Zbl

[Germain et al. 2012] P. Germain, N. Masmoudi, and J. Shatah, "Global solutions for 2D quadratic Schrödinger equations", *J. Math. Pures Appl.* (9) **97**:5 (2012), 505–543. MR Zbl

[Germain et al. 2015] P. Germain, Z. Hani, and L. Thomann, "On the continuous resonant equation for NLS, II: Statistical study", *Anal. PDE* **8**:7 (2015), 1733–1756. MR Zbl

[Germain et al. 2016] P. Germain, Z. Hani, and L. Thomann, "On the continuous resonant equation for NLS, I: Deterministic analysis", *J. Math. Pures Appl.* (9) **105**:1 (2016), 131–163. MR Zbl

[Hadac et al. 2009] M. Hadac, S. Herr, and H. Koch, "Well-posedness and scattering for the KP-II equation in a critical space", *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **26**:3 (2009), 917–941. MR Zbl

[Hani and Pausader 2014] Z. Hani and B. Pausader, "On scattering for the quintic defocusing nonlinear Schrödinger equation on $\mathbb{R} \times \mathbb{T}^2$", *Comm. Pure Appl. Math.* **67**:9 (2014), 1466–1542. MR Zbl

[Hani and Thomann 2016] Z. Hani and L. Thomann, "Asymptotic behavior of the nonlinear Schrödinger equation with harmonic trapping", *Comm. Pure Appl. Math.* **69**:9 (2016), 1727–1776. MR Zbl

[Hani et al. 2015] Z. Hani, B. Pausader, N. Tzvetkov, and N. Visciglia, "Modified scattering for the cubic Schrödinger equation on product spaces and applications", *Forum Math. Pi* **3** (2015), art. id. e4. MR Zbl

[Hao et al. 2007] C. Hao, L. Hsiao, and H.-L. Li, "Global well posedness for the Gross–Pitaevskii equation with an angular momentum rotational term in three dimensions", *J. Math. Phys.* **48**:10 (2007), art. id. 102105. MR Zbl

[Hao et al. 2008] C. Hao, L. Hsiao, and H.-L. Li, "Global well posedness for the Gross–Pitaevskii equation with an angular momentum rotational term", *Math. Methods Appl. Sci.* **31**:6 (2008), 655–664. MR Zbl

[Helffer 1988] B. Helffer, *Semi-classical analysis for the Schrödinger operator and applications*, Lecture Notes in Math. **1336**, Springer, 1988. MR Zbl

[Ibrahim et al. 2011] S. Ibrahim, N. Masmoudi, and K. Nakanishi, "Scattering threshold for the focusing nonlinear Klein–Gordon equation", *Anal. PDE* **4**:3 (2011), 405–460. MR Zbl

[Ionescu and Pausader 2012] A. D. Ionescu and B. Pausader, "Global well-posedness of the energy-critical defocusing NLS on $\mathbb{R}\times\mathbb{T}^3$", *Comm. Math. Phys.* **312**:3 (2012), 781–831. MR Zbl

[Jao 2016] C. Jao, "The energy-critical quantum harmonic oscillator", *Comm. Partial Differential Equations* **41**:1 (2016), 79–133. MR Zbl

[Jao 2018] C. Jao, "Energy-critical NLS with potentials of quadratic growth", *Discrete Contin. Dyn. Syst.* **38**:2 (2018), 563–587. MR Zbl

[Jao 2020] C. Jao, "Refined mass-critical Strichartz estimates for Schrödinger operators", *Anal. PDE* **13**:7 (2020), 1955–1994. MR Zbl

[Jao et al. 2019] C. Jao, R. Killip, and M. Vişan, "Mass-critical inverse Strichartz theorems for 1d Schrödinger operators", *Rev. Mat. Iberoam.* **35**:3 (2019), 703–730. MR Zbl

[Josserand and Pomeau 2001] C. Josserand and Y. Pomeau, "Nonlinear aspects of the theory of Bose–Einstein condensates", *Nonlinearity* **14**:5 (2001), R25–R62. MR Zbl

[Keel and Tao 1998] M. Keel and T. Tao, "Endpoint Strichartz estimates", *Amer. J. Math.* **120**:5 (1998), 955–980. MR Zbl

[Kenig and Merle 2006] C. E. Kenig and F. Merle, "Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case", *Invent. Math.* **166**:3 (2006), 645–675. MR Zbl

[Kenig and Merle 2008] C. E. Kenig and F. Merle, "Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation", *Acta Math.* **201**:2 (2008), 147–212. MR Zbl

[Killip and Vişan 2012] R. Killip and M. Vişan, "Global well-posedness and scattering for the defocusing quintic NLS in three dimensions", *Anal. PDE* **5**:4 (2012), 855–885. MR Zbl

[Killip and Vişan 2013] R. Killip and M. Vişan, "Nonlinear Schrödinger equations at critical regularity", pp. 325–437 in *Evolution equations* (Zürich, 2008), edited by D. Ellwood et al., Clay Math. Proc. **17**, Amer. Math. Soc., Providence, RI, 2013. MR Zbl

[Killip et al. 2008] R. Killip, M. Visan, and X. Zhang, "The mass-critical nonlinear Schrödinger equation with radial data in dimensions three and higher", *Anal. PDE* **1**:2 (2008), 229–266. MR Zbl

[Killip et al. 2009a] R. Killip, T. Tao, and M. Vişan, "The cubic nonlinear Schrödinger equation in two dimensions with radial data", *J. Eur. Math. Soc.* **11**:6 (2009), 1203–1258. MR Zbl

[Killip et al. 2009b] R. Killip, M. Visan, and X. Zhang, "Energy-critical NLS with quadratic potentials", *Comm. Partial Differential Equations* **34**:10-12 (2009), 1531–1565. MR Zbl

[Koch and Tataru 2005a] H. Koch and D. Tataru, "Dispersive estimates for principally normal pseudodifferential operators", *Comm. Pure Appl. Math.* **58**:2 (2005), 217–284. MR Zbl

[Koch and Tataru 2005b] H. Koch and D. Tataru, "L^p eigenfunction bounds for the Hermite operator", *Duke Math. J.* **128**:2 (2005), 369–392. MR Zbl

[Koch and Tataru 2007] H. Koch and D. Tataru, "A priori bounds for the 1D cubic NLS in negative Sobolev spaces", *Int. Math. Res. Not.* **2007**:16 (2007), art. id. rnm053. MR Zbl

[Koch et al. 2014] H. Koch, D. Tataru, and M. Vişan, *Dispersive equations and nonlinear waves: generalized Korteweg–de Vries, nonlinear Schrödinger, wave and Schrödinger maps*, Oberwolfach Sem. **45**, Birkhäuser, Basel, 2014. MR Zbl

[Lieb and Loss 1997] E. H. Lieb and M. Loss, *Analysis*, Grad. Stud. in Math. **14**, Amer. Math. Soc., Providence, RI, 1997. MR Zbl

[Lin and Strauss 1978] J. E. Lin and W. A. Strauss, "Decay and scattering of solutions of a nonlinear Schrödinger equation", *J. Funct. Anal.* **30**:2 (1978), 245–263. MR Zbl

[Merle and Vega 1998] F. Merle and L. Vega, "Compactness at blow-up time for L^2 solutions of the critical nonlinear Schrödinger equation in 2D", *Int. Math. Res. Not.* **1998**:8 (1998), 399–425. MR Zbl

[Morawetz 1968] C. S. Morawetz, "Time decay for the nonlinear Klein–Gordon equation", *Proc. A* **306** (1968), 291–296. MR Zbl

[Murphy 2014] J. Murphy, "Intercritical NLS: critical $\dot{H}^s$-bounds imply scattering", *SIAM J. Math. Anal.* **46**:1 (2014), 939–997. MR Zbl

[Pitaevskii and Stringari 2003] L. Pitaevskii and S. Stringari, *Bose–Einstein condensation*, Int. Ser. Monogr. Phys. **116**, Oxford Univ. Press, 2003. MR Zbl

[Planchon and Vega 2009] F. Planchon and L. Vega, "Bilinear virial identities and applications", *Ann. Sci. École Norm. Sup.* (4) **42**:2 (2009), 261–290. MR Zbl

[Rosenzweig 2018] M. Rosenzweig, "Global well-posedness and scattering for the elliptic-elliptic Davey–Stewartson system at L^2-critical regularity", preprint, 2018. arXiv 1808.01955

[Shen and Wu 2020] J. Shen and Y. Wu, "Global well-posedness and scattering of 3D defocusing, cubic Schrödinger equation", preprint, 2020. arXiv 2008.10019

[Staffilani 2013] G. Staffilani, "The theory of nonlinear Schrödinger equations", pp. 207–267 in *Evolution equations* (Zürich, 2008), edited by D. Ellwood et al., Clay Math. Proc. **17**, Amer. Math. Soc., Providence, RI, 2013. MR Zbl

[Stanislavova and Stefanov 2021] M. Stanislavova and A. G. Stefanov, "Ground states for the nonlinear Schrödinger equation under a general trapping potential", *J. Evol. Equ.* **21**:1 (2021), 671–697. MR Zbl

[Tao 2006] T. Tao, *Nonlinear dispersive equations: local and global analysis*, CBMS Reg. Conf. Ser. Math. **106**, Amer. Math. Soc., Providence, RI, 2006. MR Zbl

[Tao et al. 2007a] T. Tao, M. Visan, and X. Zhang, "Global well-posedness and scattering for the defocusing mass-critical nonlinear Schrödinger equation for radial data in high dimensions", *Duke Math. J.* **140**:1 (2007), 165–202. MR Zbl

[Tao et al. 2007b] T. Tao, M. Visan, and X. Zhang, "The nonlinear Schrödinger equation with combined power-type nonlinearities", *Comm. Partial Differential Equations* **32**:7-9 (2007), 1281–1343. MR Zbl

[Tao et al. 2008] T. Tao, M. Visan, and X. Zhang, "Minimal-mass blowup solutions of the mass-critical NLS", *Forum Math.* **20**:5 (2008), 881–919. MR Zbl

[Tarulli 2017] M. Tarulli, "Well-posedness and scattering for the mass-energy NLS on $\mathbb{R}^n \times \mathscr{M}^k$", *Analysis (Berlin)* **37**:3 (2017), 117–131. MR Zbl

[Tzvetkov and Visciglia 2012] N. Tzvetkov and N. Visciglia, "Small data scattering for the nonlinear Schrödinger equation on product spaces", *Comm. Partial Differential Equations* **37**:1 (2012), 125–135. MR Zbl

[Vega 1988] L. Vega, "Schrödinger equations: pointwise convergence to the initial data", *Proc. Amer. Math. Soc.* **102**:4 (1988), 874–878. MR Zbl

[Vişan 2012] M. Vişan, "Global well-posedness and scattering for the defocusing cubic nonlinear Schrödinger equation in four dimensions", *Int. Math. Res. Not.* **2012**:5 (2012), 1037–1067. MR Zbl

[Yajima and Zhang 2004] K. Yajima and G. Zhang, "Local smoothing property and Strichartz inequality for Schrödinger equations with potentials superquadratic at infinity", *J. Differential Equations* **202**:1 (2004), 81–110. MR Zbl

[Yang and Zhao 2018] K. Yang and L. Zhao, "Global well-posedness and scattering for mass-critical, defocusing, infinite dimensional vector-valued resonant nonlinear Schrödinger system", *SIAM J. Math. Anal.* **50**:2 (2018), 1593–1655. MR Zbl

[Zhang 2005] J. Zhang, "Sharp threshold for blowup and global existence in nonlinear Schrödinger equations under a harmonic potential", *Comm. Partial Differential Equations* **30**:10-12 (2005), 1429–1443. MR Zbl

[Zhang 2020] J. Zhang, "Sharp threshold of global existence for nonlinear Schrödinger equation with partial confinement", *Nonlinear Anal.* **196** (2020), art. id. 111832. MR Zbl

[Zhao 2019] Z. Zhao, "Global well-posedness and scattering for the defocusing cubic Schrödinger equation on waveguide $\mathbb{R}^2 \times \mathbb{T}^2$", *J. Hyperbolic Differ. Equ.* **16**:1 (2019), 73–129. MR Zbl

Received 12 Apr 2021. Revised 20 Jul 2022. Accepted 31 Aug 2023.

XING CHENG: chengx@hhu.edu.cn
School of Mathematics, Hohai University, Nanjing, China

CHANG-YU GUO: changyu.guo@sdu.edu.cn
Research Center for Mathematics and Interdisciplinary Sciences, Shandong University, Qingdao, China

and

Department of Physics and Mathematics, University of Eastern Finland, Joensuu, Finland

ZIHUA GUO: zihua.guo@monash.edu
School of Mathematics, Monash University, Clayton, VIC, Australia

XIAN LIAO: xian.liao@kit.edu
Institute for Analysis, Karlsruhe Institute of Technology, Karlsruhe, Germany

JIA SHEN: shenjia@nankai.edu.cn
School of Mathematical Sciences and LPMC, Nankai University, Tianjin, China

ANALYSIS AND PDE
Vol. 17 (2024), No. 10, pp. 3447–3476
DOI: 10.2140/apde.2024.17.3447

ON GAGLIARDO–NIRENBERG INEQUALITIES WITH VANISHING SYMBOLS

RAINER MANDEL

We prove interpolation inequalities of Gagliardo–Nirenberg type involving Fourier symbols that vanish on hypersurfaces in $\mathbb{R}^d$.

1. Introduction

In a recent paper by Fernández, Jeanjean, Mariş and the author the following inequality of Gagliardo–Nirenberg-type was proved:

$$\|u\|_q \lesssim \|(|D|^s - 1)u\|_2^{1-\kappa} \|u\|_2^{\kappa}, \quad u \in \mathcal{S}(\mathbb{R}^d). \tag{1}$$

Here, $(|D|^s - 1)u = \mathcal{F}^{-1}((|\cdot|^s - 1)\hat{u})$, the symbol $\lesssim$ stands for $\leq C$ for some positive number C independent of u and the parameters are supposed to satisfy

$$s > 0, \quad \kappa \geq \frac{1}{2}, \quad 2 \leq q < \infty, \quad d \in \mathbb{N}, \quad d \geq 2 \quad \text{and} \quad \frac{2(1-\kappa)}{d+1} \leq \frac{1}{2} - \frac{1}{q} \leq \frac{(1-\kappa)s}{d}; \tag{2}$$

see [Fernández et al. 2022, Theorem 2.6]. In this paper we investigate such inequalities in greater generality both by extending the analysis to a larger class of exponents, but also by allowing for more general Fourier symbols. We expect applications in the context of normalized solutions of elliptic PDEs and orbital stability [Cazenave and Lions 1982; Bartsch et al. 2016; Noris et al. 2014] or long-time behaviour [Weinstein 1982/1983] of time-dependent PDEs just as in the case of the classical Gagliardo–Nirenberg inequality [Nirenberg 1959]. In [Fernández et al. 2022; Lenzmann and Weth 2024] applications of (1) to variational existence results and symmetry-breaking phenomena for biharmonic nonlinear Schrödinger equations are given. For the existence and qualitative properties of maximizers in classical Gagliardo–Nirenberg inequalities we refer to [Weinstein 1982/1983; Del Pino and Dolbeault 2002; Bellazzini et al. 2014; Lenzmann and Sok 2021; Zhang 2021]. Interpolation inequalities in different spaces like Lorentz spaces, Besov spaces, BMO or weighted Lebesgue spaces can be found in [Brezis et al. 2021; Hajaiej et al. 2011; Brezis and Mironescu 2019; Dao et al. 2022; Caffarelli et al. 1984; McCormick et al. 2013].

We shall be concerned with inequalities of the form

$$\|u\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa} \|P_2(D)u\|_{r_2}^{\kappa}, \tag{3}$$

where $q, r_1, r_2 \in [1, \infty]$, $\kappa \in [0, 1]$ and $P_1, P_2 : \mathbb{R}^d \to \mathbb{R}$ are Fourier symbols that may vanish on a given smooth compact hypersurface $S \subset \mathbb{R}^d$, $d \geq 2$, with at least $k \in \{1, \dots, d-1\}$ nonvanishing principal

MSC2020: 35A23.
Keywords: Gagliardo–Nirenberg inequality, interpolation.

curvatures in each point. In the case $d = 1$ the symbols are allowed to have a finite set of zeros $S \subset \mathbb{R}$. We will assume that P_i vanishes of order α_i on S and behaves like $|\cdot|^{s_i}$ at infinity; see assumptions (A1), (A2) below for a precise statement. This covers (1) as a special case, where $d \geq 2$, $(\alpha_1, \alpha_2, s_1, s_2) = (1, 0, s, 0)$ and S is the unit sphere in $\mathbb{R}^d$, so $k = d - 1$. As an application of our results for (3) we obtain the following generalization of [Fernández et al. 2022, Theorem 2.6].

Theorem 1. *Assume $d \in \mathbb{N}$, $d \geq 2$, $\kappa \in [0, 1]$, $s > 0$. Then*

$$\|u\|_q \lesssim \|(|D|^s - 1)u\|_r^{1-\kappa} \|u\|_r^\kappa, \quad u \in \mathcal{S}(\mathbb{R}^d),$$

holds provided that the exponents $r \in [1, 2]$, $q \in [2, \infty]$ satisfy

$$\frac{2(1-\kappa)}{d+1} \leq \frac{1}{r} - \frac{1}{q} \leq \frac{(1-\kappa)s}{d} \qquad \text{and} \qquad \min\left\{\frac{1}{r}, \frac{1}{q'}\right\} \begin{cases} \geq \frac{d+1-2\kappa}{2d} & \text{if } \kappa > 0, \\ > \frac{d+1}{2d} & \text{if } \kappa = 0. \end{cases}$$

So our result from [Fernández et al. 2022] is recovered, as (2) is nothing but the special case $r = 2$ in the above theorem. We even obtain sufficient conditions for general $q, r_1, r_2 \in [1, \infty]$. In the one-dimensional case we obtain the following generalization of [Fernández et al. 2022, Theorem 2.3].

Theorem 2. *Assume $\kappa \in [0, 1]$, $s > 0$. Then*

$$\|u\|_q \lesssim \|(|D|^s - 1)u\|_{r_1}^{1-\kappa} \|u\|_{r_2}^\kappa, \quad u \in \mathcal{S}(\mathbb{R}),$$

holds provided that $q, r_1, r_2 \in [1, \infty]$ satisfy

$$1 - \kappa \leq \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} \leq (1-\kappa)s.$$

Both our main results arise as special cases of Theorems 18 and 19 where interpolation inequalities of the form (3) are proved for symbols $P_1, P_2 : \mathbb{R}^d \to \mathbb{R}$ that satisfy the following abstract conditions:

(A1) There is a compact hypersurface $S = \{\xi \in \mathbb{R}^d : F(\xi) = 0\}$, with $F \in C^\infty(\mathbb{R}^d)$, $|\nabla F| \neq 0$ on S and at least $k \in \{1, \dots, d-1\}$ nonvanishing principal curvatures at each point such that $\{\xi \in \mathbb{R}^d : P_i(\xi) = 0\} \subset S$. For ξ near S we have $P_i(\xi) = a_{i+}(\xi)F(\xi)_+^{\alpha_i} + a_{i-}(\xi)F(\xi)_-^{\alpha_i}$ for smooth nonvanishing functions a_{i+}, a_{i-} and $\alpha_i > -1$. In the case $\alpha_i = 1$, additionally assume $a_{i-} = -a_{i-}$, and in the case $\alpha_i = 0$, additionally assume $a_{i-} = a_{i+}$.

(A2) There are $s_1, s_2 \in \mathbb{R}$, $\delta > 0$ such that for $\operatorname{dist}(\xi, S) \geq \delta > 0$ the functions $Q_i(\xi) := \langle\xi\rangle^{s_i}/P_i(\xi)$ satisfy for some $\varepsilon > 0$

$$\begin{aligned} |\partial^\gamma Q_i(\xi)| &\lesssim \langle\xi\rangle^{-|\gamma|} && \text{if } \gamma \in \mathbb{N}_0^d,\ 0 \leq |\gamma| \leq \lfloor d/2 \rfloor, \\ |\partial^\gamma Q_i(\xi)| &\lesssim \langle\xi\rangle^{-\varepsilon-|\gamma|} && \text{if } \gamma \in \mathbb{N}_0^d,\ |\gamma| = \lfloor d/2 \rfloor + 1. \end{aligned}$$

Here and in the following we set $\langle\xi\rangle := (1 + |\xi|^2)^{1/2}$ and $|\gamma| := |(\gamma_1, \dots, \gamma_d)| := \gamma_1 + \dots + \gamma_d$ for multi-indices $\gamma \in \mathbb{N}_0^d$, $F(\xi)_+ := \max\{F(\xi), 0\}$ and $F(\xi)_- =: -\min\{F(\xi), 0\}$. In the case $d = 1$ assumption (A1) is supposed to mean $S = \{\xi \in \mathbb{R} : F(\xi) = 0\} = \{\xi_1^*, \dots, \xi_L^*\}$, with F, P_i, a_{i+}, a_{i-}

as above. Given the importance of the fractional Laplacian $(-\Delta)^{s/2} = |D|^s$ we mention that one may generalize this further by allowing the symbols P_1, P_2 to vanish at some finite set of points in $\mathbb{R}^d \setminus S$; see Remark 10. The choice $P_1 = P_2$ or $\kappa \in \{0, 1\}$ leads to Sobolev inequalities. In the elliptic case $-\Delta - 1 = |D|^2 - 1$ such results are due to Kenig, Ruiz and Sogge [Kenig et al. 1987, Theorem 2.3], Gutiérrez [2004, Theorem 6] and Evequoz [2017]. Our most general result from Theorem 19 contains these results as a special case $(k, s_1, \alpha_1, \kappa) = (d-1, 2, 1, 0)$. Sharp results for special nonelliptic symbols with unbounded characteristic set S are due to Kenig, Ruiz and Sogge [Kenig et al. 1987, Theorem 2.1], Koch and Tataru [2005] and Jeong, Kwon and Lee [Jeong et al. 2016, Theorem 1.1].

Remark 3. (a) In the case $S = \varnothing$ the main results of this paper hold without any assumptions on α_1, α_2. Similarly, if the Fourier support of the given functions is contained in a fixed compact subset of $\mathbb{R}^d$, then all conditions involving s_1, s_2 can be ignored.

(b) Theorems 1 and 2 equally hold for symbols $P_i(|D|)$, where P_i are polynomials of degree s with simple zeros only or no zeros at all.

(c) Our analysis may be extended to vectorial differential operators with constant coefficients $P_1(D)$, $P_2(D)$, where, according to Cramer's rule, the characteristic set S is then supposed to satisfy $\{\det(P_i(\xi)) = 0\} \subset S$ for $i = 1, 2$. Such a situation occurs in the context of Maxwell's equations, Dirac equations or Lamé equations with constant coefficients.

(d) The Gagliardo–Nirenberg inequalities from this paper hold for functions with Fourier support in bounded smooth pieces of more general sets $S \subset \mathbb{R}^d$. In this way, unbounded characteristic sets S or characteristic sets with singularities as in [Mandel and Schippa 2022, Section 3] may be partially analyzed, but a full analysis remains to be done. In the special case of the wave and Schrödinger operator one may nevertheless implement the strategy from [Fernández et al. 2022] to get such inequalities at least for $r = 2$; see Section 7.

(e) The admissible set of exponents for Gagliardo–Nirenberg inequalities may become larger by imposing symmetries. For instance, the Stein–Tomas theorem for $O(d-k) \times O(k)$-symmetric functions from [Mandel and Oliveira e Silva 2023] may substitute the classical Stein–Tomas theorem in Lemma 13 to prove better dyadic estimates. The latter yield larger values for $A_\varepsilon(p, q)$ in (17), which allows one to deduce Gagliardo–Nirenberg inequalities for a wider range of exponents.

Our strategy is as follows. We decompose the pseudodifferential operators $P_1(D)$, $P_2(D)$ dyadically, both for frequencies close to the critical surface S and at infinity. Assumption (A1) allows us to analyze the first-mentioned part with the aid of Bochner–Riesz estimates from [Mandel and Schippa 2022; Cho et al. 2005]. Here, only the parameters α_1, α_2 will play a role. Assumption (A2) will be used to estimate the second-mentioned part that only involves s_1, s_2. Interpolating the bounds for the dyadic operators in both frequency regimes then allows us to conclude. We stress that the proof from [Fernández et al. 2022] does not carry over from the $L^2(\mathbb{R}^d)$-setting since Plancherel's theorem does not have a counterpart in $L^r(\mathbb{R}^d)$ with $r \neq 2$.

2. Preliminaries

In the following we decompose a given Schwartz function $u \in \mathcal{S}(\mathbb{R}^d)$ in frequency space. We start by separating the frequencies close to the critical surface from the others by defining

$$u_1 := \mathcal{F}^{-1}(\tau\hat{u}), \quad u_2 := \mathcal{F}^{-1}((1-\tau)\hat{u}), \quad \text{where } \tau \in C_0^\infty(\mathbb{R}^d),\ \tau = 1 \text{ near } S. \tag{4}$$

More precisely, τ is chosen in such a way that S admits local parametrizations in Euclidean coordinates within $\operatorname{supp}(\tau)$, that a_{i+}, a_{i-} from (A1) are uniformly positive near S and that the functions Q_i from (A2) behave as required for $\xi \in \mathbb{R}^d \setminus \operatorname{supp}(\tau)$. The function τ is considered as fixed from now on. For both u_1 and u_2 we will introduce a dyadic decomposition into infinitely many annular regions in order to prove our estimates mostly via Bourgain's summation argument [1985]. We will need the following abstract version of this result from [Carbery et al. 1999, p. 604].

Lemma 4. *Let $\beta_1, \beta_2 \in \mathbb{R}$, $\theta \in (0,1)$, and let (X_1, X_2) and (Y_1, Y_2) be real interpolation pairs of Banach spaces. For $j \in \mathbb{N}$ let $\mathcal{T}_j$ be linear operators satisfying*

$$\|\mathcal{T}_j f\|_{Y_1} \le M_1 2^{\beta_1 j} \|f\|_{X_1}, \quad \|\mathcal{T}_j f\|_{Y_2} \le M_2 2^{\beta_2 j} \|f\|_{X_2}.$$

Then we have

$$\Big\|\sum_{j\in\mathbb{N}} \mathcal{T}_j f\Big\|_{(Y_1,Y_2)_{\theta,\infty}} \le C(\beta_1,\beta_2) M_1^{1-\theta} M_2^\theta \|f\|_{(X_1,X_2)_{\theta,1}} \tag{5}$$

provided that $(1-\theta)\beta_1 + \theta\beta_2 = 0$, with $\beta_1, \beta_2 \neq 0$. In the case $(1-\theta)\beta_1 + \theta\beta_2 < 0$ we have for all $r \in [1,\infty]$

$$\Big\|\sum_{j\in\mathbb{N}} \mathcal{T}_j f\Big\|_{(Y_1,Y_2)_{\theta,r}} \le C M_1^{1-\theta} M_2^\theta \|f\|_{(X_1,X_2)_{\theta,r}}. \tag{6}$$

The whole point of this result is (5); the estimate (6) is a rather trivial consequence of the summability of the interpolated bounds

$$\|\mathcal{T}_j f\|_{(Y_1,Y_2)_{\theta,r}} \lesssim 2^{j((1-\theta)\beta_1+\theta\beta_2)} \|f\|_{(X_1,X_2)_{\theta,r}} \quad \text{for all } r \in [1,\infty].$$

Here, $(Y_1,Y_2)_{\theta,r}, (X_1,X_2)_{\theta,r}$ denote real interpolation spaces [Bergh and Löfström 1976]. The choice $Y_1 = L^{q_1}(\mathbb{R}^d)$, $Y_2 = L^{q_2}(\mathbb{R}^d)$, with

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad q_1 \neq q_2,$$

yields the Lorentz space $(Y_1,Y_2)_{\theta,r} = L^{q,r}(\mathbb{R}^d)$, whereas $q_1 = q_2 = q$ leads to $(Y_1,Y_2)_{\theta,r} = L^q(\mathbb{R}^d)$. In our context, the spaces X_i are defined as the completion of $\{u \in \mathcal{S}(\mathbb{R}^d) : P_i(D)u \in L^r(\mathbb{R}^d)\}$ with respect to the norm $\|u\|_{X_i} := \|P_i(D)u\|_r$. Exploiting assumptions (A1), (A2) we find that for any given $u \in \mathcal{S}(\mathbb{R}^d)$ the function $P_i(D)u$ is a priori well-defined as a function in $L^\infty(\mathbb{R}^d)$ because $\xi \mapsto P_i(\xi)\hat{u}(\xi)$ is integrable due to $\alpha_i > -1$. (Choosing the completion of a smaller set one may extend the analysis to $\alpha_i \le -1$.) The link to Gagliardo–Nirenberg-type inequalities is provided by the general interpolation property [Bergh and Löfström 1976, Theorem 3.1.2], namely

$$\|f\|_{(X_1,X_2)_{\kappa,r}} \le \|f\|_{X_1}^{1-\kappa}\|f\|_{X_2}^{\kappa}, \quad 0<\kappa<1,\ 1\le r\le\infty.$$

In fact, choosing X_1, X_2 as above we obtain for $u \in \mathcal{S}(\mathbb{R}^d)$

$$\|u\|_{(X_1,X_2)_{\kappa,r}} \le \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}, \quad 0<\kappa<1, 1\le r\le\infty. \tag{7}$$

The same estimate holds for $(X_1, X_2)_{\kappa,r}$ replaced by the complex interpolation space $[X_1, X_2]_\kappa$. This can be deduced from (7) and $[X_1, X_2]_\kappa \subset (X_1, X_2)_{\kappa,\infty}$; see [Bergh and Löfström 1976, Theorem 4.7.1].

3. Large frequency analysis

We start with our analysis for large frequencies or, more precisely, for those frequencies with uniformly positive distance to the critical surface S given by our assumption (A1). To this end we first choose a function η such that

$$\eta \in C_0^\infty(\mathbb{R}), \quad \operatorname{supp}(\eta) \subset \left[-2,-\tfrac{1}{2}\right] \cup \left[\tfrac{1}{2},2\right], \quad \sum_{j\in\mathbb{Z}} \eta(2^j\,\cdot) = 1 \text{ almost everywhere on } \mathbb{R};$$

see [Bergh and Löfström 1976, Lemma 6.1.7]. For $\xi_0 \in \mathbb{R}^d$ define

$$\begin{aligned} T_j f &:= \mathcal{F}^{-1}(\eta(2^j|\xi-\xi_0|)\hat{f}) = K_j * f, \quad \text{where} \\ K_j(x) &:= \mathcal{F}^{-1}(\eta(2^j|\xi-\xi_0|))(x) = 2^{-jd}\mathcal{F}^{-1}(\eta(|\cdot|))(2^{-j}x)e^{ix\cdot\xi_0}. \end{aligned} \tag{8}$$

Later on, we will choose $\xi_0 \in S$ in order to have $T_j u_2 = 0$ for $j \ge j_0$, where $j_0 \in \mathbb{Z}$ only depends on ξ_0 and τ. Indeed, (4) implies that $\hat{u}_2(\xi) = (1-\tau(\xi))\hat{u}(\xi)$ vanishes for frequencies ξ close to S. As a consequence, only the bounds for $j \searrow -\infty$ will be of importance.

Lemma 5. *Assume $d \in \mathbb{N}$ and let $\eta \in C_0^\infty(\mathbb{R})$, $\xi_0 \in \mathbb{R}^d$. Then we have for $j \in \mathbb{Z}$*

$$\|T_j\|_{p\to q} \lesssim 2^{-jd\left(\frac{1}{p}-\frac{1}{q}\right)} \quad \textit{for } 1\le p\le q\le\infty.$$

Proof. For all $r \in [1,\infty]$ we have $\|K_j\|_r = 2^{-jd}\|\mathcal{F}^{-1}(\eta(|\cdot|))(2^{-j}\cdot)\|_r \lesssim 2^{-jd/r'}$. Hence, for $1 \le p \le q \le \infty$ and $\frac{1}{r} := 1 + \frac{1}{q} - \frac{1}{p}$ we get from Young's convolution inequality

$$\|T_j f\|_q \lesssim \|K_j\|_r\|f\|_p \lesssim 2^{-j\frac{d}{r'}}\|f\|_p \lesssim 2^{-jd\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_p. \qquad \square$$

In the following, we will need a multiplier theorem in $L^\mu(\mathbb{R}^d)$ for arbitrary $\mu \in [1,\infty]$. The natural candidate — Mikhlin's multiplier theorem [Bergh and Löfström 1976, Theorem 6.1.6] — is only available for $\mu \in (1,\infty)$. In order to avoid tiresome separate discussions we first provide a simple sufficient condition for a given function $m : \mathbb{R}^d \to \mathbb{R}$ to be an L^μ-multiplier for all $\mu \in [1,\infty]$. The following result essentially says that a function m serves our purpose provided that its derivatives grow a bit slower near zero and decay a bit faster near infinity compared to the requirements of Mikhlin's multiplier theorem.

Proposition 6. *Let $d \in \mathbb{N}$, $k := \lfloor d/2 \rfloor + 1$ and $m \in C^k(\mathbb{R}^d \setminus \{0\})$. Then m is an L^μ multiplier for all $\mu \in [1,\infty]$ provided that there is $\varepsilon > 0$ such that*

$$|\partial^\alpha m(\xi)| \lesssim \langle \xi \rangle^{-2\varepsilon} |\xi|^{-k+\varepsilon} \quad \textit{for all } \alpha \in \mathbb{N}_0^d \textit{ such that } |\alpha| = k.$$

Proof. We show that the assumptions imply that $\rho := \mathcal{F}^{-1} m$ is integrable. Once this is shown, the result follows from Young's convolution inequality because of

$$\|\mathcal{F}^{-1}(m\hat{f})\|_\mu = \|\rho * f\|_\mu \le \|\rho\|_1 \|f\|_\mu.$$

We may without loss of generality assume $0 < \varepsilon \le 2k - d$. For all $\alpha \in \mathbb{N}_0^d$, $|\alpha| = k$ we have

$$|\mathcal{F}((-ix)^\alpha \rho)(\xi)| = |\partial^\alpha \hat{\rho}(\xi)| = |\partial^\alpha m(\xi)| \lesssim \langle \xi \rangle^{-2\varepsilon} |\xi|^{-k+\varepsilon}.$$

Hence, $\mathcal{F}(x^\alpha \rho)$ belongs to the space $L^{\sigma_1}(\mathbb{R}^d) \cap L^{\sigma_2}(\mathbb{R}^d)$, where

$$\sigma_1 := \frac{d}{k + \varepsilon/2}, \quad \sigma_2 := \frac{d}{k - \varepsilon/2}.$$

Our choice for ε implies $1 \le \sigma_1 \le \sigma_2 \le 2$, so the Hausdorff–Young inequality gives

$$|x|^k \rho \in L^{\sigma_1'}(\mathbb{R}^d) \cap L^{\sigma_2'}(\mathbb{R}^d).$$

To conclude $\rho \in L^1(\mathbb{R}^d)$ with Hölder's inequality it remains to check

$$|x|^{-k} \in L^{\sigma_1}(\mathbb{R}^d) + L^{\sigma_2}(\mathbb{R}^d).$$

But this follows from $|x|^{-k} \mathbb{1}_{|x| \le 1} \in L^{\sigma_1}(\mathbb{R}^d)$ and $|x|^{-k} \mathbb{1}_{|x|>1} \in L^{\sigma_2}(\mathbb{R}^d)$ due to $k\sigma_1 < d < k\sigma_2$, which finishes the proof. □

Next we provide our estimates in the large-frequency regime. To this end we analyze the mapping properties of $\mathcal{T}_j u := T_j(u_2)$, where T_j and $u_2 = \mathcal{F}^{-1}((1-\tau)\hat{u})$ were defined in (8), (4), respectively.

Proposition 7. *Assume $d \in \mathbb{N}$ and (A2) with $s_1, s_2 \in \mathbb{R}$. Then, for $i = 1, 2$,*

$$\|\mathcal{T}_j u\|_q \lesssim 2^{j(s_i - d(\frac{1}{p} - \frac{1}{q}))} \|P_i(D) u\|_p \quad \textit{for } 1 \le p \le q \le \infty,\ j \in \mathbb{Z}.$$

Proof. In order to use Lemma 5 for $\xi_0 \in S$ we set $\eta_i(z) := \eta(z)|z|^{-s_i}$ for $z \in \mathbb{R}$. Then $\eta \in C_0^\infty(\mathbb{R})$, $0 \notin \operatorname{supp}(\eta)$ implies $\eta_i \in C_0^\infty(\mathbb{R})$ for $i = 1, 2$. Moreover, we have for $i = 1, 2$ and $j \in \mathbb{Z}$

$$\begin{aligned}
\mathcal{T}_j u &= \mathcal{F}^{-1}\big(\eta(2^j |\xi - \xi_0|) \hat{u}_2(\xi)\big) \\
&= \mathcal{F}^{-1}\big(\eta_i(2^j |\xi - \xi_0|)\, (2^j |\xi - \xi_0|)^{s_i}\, \hat{u}_2(\xi)\big) \\
&= 2^{js_i} \mathcal{F}^{-1}\big(\eta_i(2^j |\xi - \xi_0|) m_i(\xi) P_i(\xi) \hat{u}(\xi)\big),
\end{aligned}$$

where $m_i(\xi) := (1 - \tau(\xi))|\xi - \xi_0|^{s_i} / P_i(\xi)$. Since τ is smooth and identically 1 near $\xi_0 \in S$, a calculation shows that our assumptions on P_i from (A2) imply that m_i satisfies the assumptions of Proposition 6. In fact, for $|\alpha| = k := \lfloor d/2 \rfloor + 1$ and $Q_i, \varepsilon > 0$ as in assumption (A2),

$$\begin{aligned}|\partial^\alpha m_i(\xi)| &\lesssim \sum_{0\le\gamma\le\alpha} \binom{\alpha}{\gamma} \big|\partial^{\alpha-\gamma}\big((1-\tau(\xi))|\xi-\xi_0|^{s_i}\langle\xi\rangle^{-s_i}\big)\big|\,|\partial^\gamma Q_i(\xi)| \\ &\lesssim 1\cdot|\partial^\alpha Q_i(\xi)| + \sum_{0\le\gamma<\alpha} \langle\xi\rangle^{-|\alpha-\gamma|-1}|\partial^\gamma Q_i(\xi)| \\ &\lesssim \langle\xi\rangle^{-\varepsilon-|\gamma|} + \langle\xi\rangle^{-|\alpha-\gamma|-1}\langle\xi\rangle^{-|\gamma|} \lesssim \langle\xi\rangle^{-\min\{1,\varepsilon\}-|\alpha|}.\end{aligned}$$

Here we used the Leibniz rule. So, by Proposition 6, m_i is an L^μ-multiplier for all $\mu \in [1,\infty]$. Hence, Lemma 5 yields for all $q \in [p,\infty]$

$$\begin{aligned}\|\mathcal{T}_j u\|_q &\lesssim 2^{js_i}\|\mathcal{F}^{-1}(\eta_i(2^j|\xi-\xi_0|)m_i(\xi)\widehat{P_i(D)u}(\xi))\|_q \\ &\lesssim 2^{j(s_i-d(\frac{1}{p}-\frac{1}{q}))}\|\mathcal{F}^{-1}(m_i(\xi)\widehat{P_i(D)u}(\xi))\|_p \\ &\lesssim 2^{j(s_i-d(\frac{1}{p}-\frac{1}{q}))}\|P_i(D)u\|_p. \qquad\square\end{aligned}$$

Next we use these dyadic estimates to prove estimates of Gagliardo–Nirenberg type. We deduce our results from a detailed analysis of the special case $P_i(D) = \langle D\rangle^{s_i}$ for $s_1, s_2 \in \mathbb{R}$. This is possible due to

$$\|\langle D\rangle^{s_i}u_2\|_p \lesssim \|P_i(D)u\|_p, \quad 1 \le p \le \infty, \tag{9}$$

for symbols P_1, P_2 as in (A2) thanks to Proposition 6. So we collect some mapping properties of the Bessel potential operators $\langle D\rangle^{-s}$, where $s > 0$.

Proposition 8. *Assume $d \in \mathbb{N}$, $s > 0$ and $p, q, r \in [1,\infty]$, $u \in \mathcal{S}(\mathbb{R}^d)$.*

(i) *If $0 \le \frac{1}{p} - \frac{1}{q} < \frac{s}{d}$ then $\|u\|_q \lesssim \|\langle D\rangle^s u\|_p$.*
(ii) *If $0 \le \frac{1}{p} - \frac{1}{q} = \frac{s}{d}$ and $1 < p, q < \infty$ then $\|u\|_{q,r} \lesssim \|\langle D\rangle^s u\|_{p,r}$ and $\|u\|_q \lesssim \|\langle D\rangle^s u\|_p$.*
(iii) *If $0 \le \frac{1}{p} - \frac{1}{q} = \frac{s}{d}$ and $s = d = 1$ then $\|u\|_\infty \lesssim \|\langle D\rangle u\|_1$.*
(iv) *If $0 \le \frac{1}{p} - \frac{1}{q} = \frac{s}{d}$ and $1 = p < q < \infty$ then $\|u\|_{q,\infty} \lesssim \|\langle D\rangle^s u\|_1$.*

Proof. The parts (i), (iv) and the second part of (ii) are given in [Grafakos 2014, Corollary 1.2.6]; the Lorentz space mapping properties from (ii) follow from real interpolation. The estimate (iii) follows from

$$\|u\|_\infty \lesssim \|u'\|_1 = \|m(D)(\langle D\rangle u)\|_1 \lesssim \|\langle D\rangle u\|_1, \quad u \in \mathcal{S}(\mathbb{R}).$$

Here we used that $m(\xi) := \xi(1+|\xi|^2)^{-1/2}$ satisfies the assumptions of Proposition 6. $\square$

We finally use these estimates to prove Gagliardo–Nirenberg inequalities for large frequencies.

Proposition 9. *Assume $d \in \mathbb{N}$, $\kappa \in [0,1]$ and (A2) for $s_1, s_2 \in \mathbb{R}$. Then*

$$\|u_2\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}, \quad u \in \mathcal{S}(\mathbb{R}^d), \tag{10}$$

holds provided that the exponents $q, r_1, r_2 \in [1,\infty]$ satisfy $0 \le \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} \le \frac{\bar{s}}{d}$, as well as the following conditions in the endpoint case $\frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \frac{\bar{s}}{d}$:

(i) *If* $q = \infty$ *then* $\frac{1}{r_1} - \frac{s_1}{d} \neq 0 \neq \frac{1}{r_2} - \frac{s_2}{d}$ *or* $r_1 = r_2 = \infty,\ s_1 = s_2 = 0$ *or* $d = 1, (r_1, r_2) = (\frac{1}{s_1}, \frac{1}{s_2})$, $s_1, s_2 \in \{0, 1\}$.

(ii) *If* $1 < q < \infty$ *and* $\frac{1}{r_1} - \frac{s_1}{d} = \frac{1}{q} = \frac{1}{r_2} - \frac{s_2}{d}$ *and additionally, if* $r_1 = 1, \kappa < 1$, *then* $1 < r_2 < q,\ \kappa \geq \frac{r_2}{q}$ *or* $r_2 = \infty,\ \frac{1}{q} \leq \kappa \leq \frac{1}{q'}$.

(iii) *If* $1 < q < \infty$ *and* $\frac{1}{r_1} - \frac{s_1}{d} = \frac{1}{q} = \frac{1}{r_2} - \frac{s_2}{d}$ *and additionally, if* $r_2 = 1,\ \kappa > 0$, *then* $1 < r_1 < q$, $1 - \kappa \geq \frac{r_1}{q}$ *or* $r_1 = \infty,\ \frac{1}{q} \leq 1 - \kappa \leq \frac{1}{q'}$.

Proof. As mentioned before, it is sufficient to prove the estimates in the prototypical case $P_i(D) = \langle D \rangle^{s_i}$. The case $\kappa \in \{0, 1\}$ is covered by Proposition 8(i), (ii), (iii). So we may concentrate on $\kappa \in (0, 1)$ in the following. We combine Proposition 7 and Lemma 4 for the Bessel potential spaces $X_i := P_i(D)^{-1} L^{r_i}(\mathbb{R}^d) = \langle D \rangle^{-s_i} L^{r_i}(\mathbb{R}^d)$ and $i = 1, 2$. Here we use the identity

$$u_2 = \sum_{j=-\infty}^{j_0} \mathcal{T}_j u, \quad \text{where } \|\mathcal{T}_j u\|_{q_i} \lesssim 2^{j(s_i - d(\frac{1}{r_i} - \frac{1}{q_i}))} \|u\|_{X_i} \quad (j \in \mathbb{Z},\ 1 \leq r_i \leq q_i \leq \infty);$$

see Proposition 7. Our strategy is as follows. We first prove apply Lemma 4 to get strong bounds. This will cover all nonendpoint cases $0 \leq \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} < \frac{\bar{s}}{d}$, as well as the endpoint cases involving $q \in \{1, \infty\}$. The remaining discussion for $1 < q < \infty$ and $1 < r_1, r_2 < \infty$ can be taken from the literature, but the analysis for $\{r_1, r_2\} \cap \{1, \infty\} \neq \emptyset$ is more delicate. We will first address the case $\frac{1}{r_1} - \frac{1}{r_2} = \frac{s_1 - s_2}{d}$, where we prove our claim using complex and real interpolation theory. Finally, in the case $\frac{1}{r_1} - \frac{1}{r_2} \neq \frac{s_1 - s_2}{d}$ we will first deduce restricted weak-type bounds from Lemma 4 and upgrade them to strong bounds by interpolating the restricted weak-type bounds with each other. We will need in the following that our assumptions imply $\bar{s} \geq 0$.

<u>Step 1</u>: We start the interpolation procedure with (nonendpoint) exponents satisfying

$$0 \leq \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} < \frac{\bar{s}}{d}. \tag{11}$$

In that case the interpolation estimate (6) with $(Y_1, Y_2, \theta, r) := (L^{q_1}(\mathbb{R}^d), L^{q_2}(\mathbb{R}^d), \kappa, q)$ gives the bound

$$\|u_2\|_q = \left\| \sum_{j=-\infty}^{j_0} \mathcal{T}_j u \right\|_q \overset{(6)}{\lesssim} \|u\|_{(X_1, X_2)_{\kappa, q}} \overset{(7)}{\lesssim} \|\langle D \rangle^{s_1} u\|_{r_1}^{1-\kappa} \|\langle D \rangle^{s_2} u\|_{r_2}^{\kappa}.$$

Here, (6) applies because (11) allows us to find $q_i \in [r_i, \infty]$ such that

$$(1-\kappa)\left(s_1 - d\left(\frac{1}{r_1} - \frac{1}{q_1}\right)\right) + \kappa\left(s_2 - d\left(\frac{1}{r_2} - \frac{1}{q_2}\right)\right) > 0, \quad \frac{1}{q} = \frac{1-\kappa}{q_1} + \frac{\kappa}{q_2}.$$

So the claim is proved for all nonendpoint exponents given by (11).

It remains to discuss the endpoint case $0 \leq \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \frac{\bar{s}}{d}$. Using (5) for $Y_1 = Y_2 = L^q(\mathbb{R}^d)$ we get the claim for all exponents satisfying

$$0 \leq \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \frac{\bar{s}}{d} \quad \text{and} \quad q \geq \max\{r_1, r_2\},\ \frac{1}{r_1} - \frac{s_1}{d} \neq \frac{1}{q} \neq \frac{1}{r_2} - \frac{s_2}{d}. \tag{12}$$

Here the latter two inequalities correspond to $\beta_1, \beta_2 \neq 0$ in Lemma 4. From this we infer that the claimed endpoint estimates hold for $q \in \{1, \infty\}$ via the following cases:

- Case $q = 1$: $r_1 = r_2 = 1,\ s_1 = s_2 = 0$ is trivial.
- Case $q = 1$: $r_1 = r_2 = 1,\ \bar{s} = 0,\ s_1 \neq 0 \neq s_2$ is covered by (12).
- Case $q = \infty$: $r_1 = r_2 = \infty,\ s_1 = s_2 = 0$ is trivial.
- Case $q = \infty$: $\frac{1}{r_1} - \frac{s_1}{d} \neq 0 \neq \frac{1}{r_2} - \frac{s_2}{d}$ is covered by (12).
- Case $q = \infty$: $(d, r_1, r_2) = \left(1, \frac{1}{s_1}, \frac{1}{s_2}\right),\ s_1, s_2 \in \{0, 1\}$ is covered by Proposition 8(iii).

These are all cases involving $q \in \{1, \infty\}$ and in particular claim (i) is proved. So we are left with those endpoint estimates for $1 < q < \infty$ that are not covered by (12).

Step 2: The claim holds for $1 < r_1, r_2 < \infty$ due to

$$\|u\|_q \lesssim \|\langle D\rangle^{\bar{s}} u\|_{\bar{r}} \lesssim \|\langle D\rangle^{s_1} u\|_{r_1}^{1-\kappa} \|\langle D\rangle^{s_2} u\|_{r_2}^{\kappa},$$

where $\frac{1}{\bar{r}} := \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2}$. This is a consequence of Sobolev's embedding theorem [Bergh and Löfström 1976, Theorem 6.5.1] and the complex interpolation result from [loc. cit., Theorem 6.4.5(7)]. So we may in the following assume $\{r_1, r_2\} \cap \{1, \infty\} \neq \varnothing$. As announced earlier, we first deal with $\frac{1}{r_1} - \frac{1}{r_2} = \frac{s_1 - s_2}{d}$.

Step 3: Assume we are in the endpoint case with $1 < q < \infty$, $\frac{1}{r_1} - \frac{1}{r_2} = \frac{s_1 - s_2}{d}$, $r_1 \leq r_2$ (without loss of generality) and $\{r_1, r_2\} \cap \{1, \infty\} \neq \varnothing$. Then $\frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \frac{\bar{s}}{d}$ implies $\frac{1}{r_1} - \frac{s_1}{d} = \frac{1}{q} = \frac{1}{r_2} - \frac{s_2}{d}$. We distinguish the following cases:

- Case $r_1 = 1,\ r_2 = 1$: This case is excluded, so there is nothing to prove.
- Case $r_1 = 1,\ 1 < r_2 < q$: By Proposition 8(ii), (iv), we have $\|u\|_{q,\infty} \lesssim \|\langle D\rangle^{s_1} u\|_1$, as well as $\|u\|_{q,r_2} \lesssim \|\langle D\rangle^{s_2} u\|_{r_2}$. Applying the interpolation identity [loc. cit., Theorem 5.3.1]

$$L^q(\mathbb{R}^d) = (L^{q,\infty}(\mathbb{R}^d), L^{q,\kappa q}(\mathbb{R}^d))_{\kappa,q}, \quad \kappa \in (0, 1], \tag{13}$$

we infer for all $\kappa \in \left[\frac{r_2}{q}, 1\right]$

$$\|u\|_q \lesssim \|u\|_{q,\infty}^{1-\kappa} \|u\|_{q,\kappa q}^{\kappa} \lesssim \|u\|_{q,\infty}^{1-\kappa} \|u\|_{q,r_2}^{\kappa} \lesssim \|\langle D\rangle^{s_1} u\|_1^{1-\kappa} \|\langle D\rangle^{s_2} u\|_{r_2}^{\kappa}.$$

- Case $r_1 = 1, r_2 = \infty$: We have to prove (10) for $\frac{1}{q} \leq \kappa \leq \frac{1}{q'}$. It is sufficient to prove the claim first for $\kappa = \frac{1}{q}$ and then for $\kappa = \frac{1}{q'}$. We use $\|u\|_{q,\infty} \lesssim \|\langle D\rangle^{s_1} u\|_1$ and

$$\|u\|_{q,2}^2 \lesssim \|\langle D\rangle^{\frac{d}{2} - \frac{d}{q}} u\|_2^2 = \int_{\mathbb{R}^d} \langle D\rangle^{\frac{d}{q'}} u \cdot \langle D\rangle^{-\frac{d}{q}} u \, dx \leq \|\langle D\rangle^{s_1} u\|_1 \|\langle D\rangle^{s_2} u\|_\infty. \tag{14}$$

In (14) we subsequently used Proposition 8(ii) and the L^2-isometry property of the Fourier transform, as well as $s_1 = \frac{d}{q'}$, $s_2 = -\frac{d}{q}$. Real interpolation of these two estimates and $L^q(\mathbb{R}^d) = (L^{q,\infty}(\mathbb{R}^d), L^{q,2}(\mathbb{R}^d))_{2/q,q}$, which is (13) for $\kappa = \frac{2}{q}$, gives

$$\|u\|_q \lesssim \|u\|_{q,\infty}^{1-\frac{2}{q}} \|u\|_{q,2}^{\frac{2}{q}} \lesssim \|\langle D\rangle^{s_1} u\|_1^{\frac{1}{q'}} \|\langle D\rangle^{s_2} u\|_\infty^{\frac{1}{q}}. \tag{15}$$

So the claim holds for $\kappa = \frac{1}{q}$ and we now consider $\kappa = \frac{1}{q'}$. Here we use Stein's interpolation theorem [1956] in a more general setting [Voigt 1992, Theorem 2.1] for the family of linear operators $\mathcal{T}^s u := e^{s^2}\langle D\rangle^{s/2-d/q}u$, with $s \in \mathbb{C},\ 0 \le \mathrm{Re}(s) \le 1$. We have

$$\|\mathcal{T}^{it}u\|_{\mathrm{BMO}(\mathbb{R}^d)} = e^{-t^2}\|\langle D\rangle^{it}(\langle D\rangle^{-\frac{d}{q}}u)\|_{\mathrm{BMO}(\mathbb{R}^d)} \lesssim \|\langle D\rangle^{-\frac{d}{q}}u\|_\infty,$$
$$\|\mathcal{T}^{1+it}u\|_2 = e^{1-t^2}\|\langle D\rangle^{\frac{d}{2}-\frac{d}{q}}u\|_2 \overset{(14)}{\lesssim} \|\langle D\rangle^{\frac{d}{q'}}u\|_1^{\frac{1}{2}}\|\langle D\rangle^{-\frac{d}{q}}u\|_\infty^{\frac{1}{2}}.$$

Here we used the validity of Mikhlin's multiplier theorem in $\mathrm{BMO}(\mathbb{R}^d)$ to deduce that the operator norm $\langle D\rangle^{it} : L^\infty(\mathbb{R}^d) \to \mathrm{BMO}(\mathbb{R}^d)$ is polynomially bounded with respect to t and thus compensated for by the mitigating factor e^{-t^2} as $|t| \to \infty$. We refer to Proposition 3.4, Theorem 4.4 and the comments on pages 20-21 in Tao's lecture notes [2018], where such an application in the context of Stein's interpolation theorem is explicitly mentioned. In view of $[\mathrm{BMO}(\mathbb{R}^d), L^2(\mathbb{R}^d)]_\theta = L^{2/\theta}(\mathbb{R}^d)$ for $0 < \theta \le 1$ we may plug in $\theta = \frac{2}{q}$ and get in view of $s_1 = \frac{d}{q'}, s_2 = -\frac{d}{q}$

$$\|u\|_q = \|\mathcal{T}^{\frac{2}{q}}u\|_q \lesssim \|\langle D\rangle^{-\frac{d}{q}}u\|_\infty^{1-\theta}(\|\langle D\rangle^{\frac{d}{q'}}u\|_1^{\frac{1}{2}}\|\langle D\rangle^{-\frac{d}{q}}u\|_\infty^{\frac{1}{2}})^\theta = \|\langle D\rangle^{s_1}u\|_1^{\frac{1}{q}}\|\langle D\rangle^{s_2}u\|_\infty^{\frac{1}{q'}}.$$

- Case $1 < r_1 < r_2 = \infty$: We have to prove (10) for $1 < q < r_1,\ \kappa \ge \frac{r_1}{q}$. We consider $\mathcal{T}^s u := e^{s^2}\langle D\rangle^{s_2+s(s_1-s_2)}u$ and obtain as before

$$\|\mathcal{T}^{it}u\|_{\mathrm{BMO}(\mathbb{R}^d)} \lesssim \|\langle D\rangle^{s_2}u\|_\infty, \quad \|\mathcal{T}^{1+it}u\|_{r_1} \lesssim \|\langle D\rangle^{s_1}u\|_{r_1}.$$

So we conclude for $\kappa := \frac{r_1}{q} = \frac{s_2}{s_2-s_1}$

$$\|u\|_q = \|\mathcal{T}^\kappa u\|_{\frac{r_1}{\kappa}} \lesssim \|\langle D\rangle^{s_2}u\|_\infty^{1-\kappa}\|\langle D\rangle^{s_1}u\|_{r_1}^\kappa.$$

This proves the claim for $\kappa = \frac{r_1}{q}$. Since the desired bound for $\kappa = 1$ follows from Proposition 8(ii), we get the claim for $\kappa \in \left[\frac{r_1}{q}, 1\right]$.

- Case $1 < r_1 = r_2 = \infty$: This case does not occur because $\frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = -\frac{1}{q} < 0$.

Step 4: To prove the remaining estimates we first prove restricted weak-type estimates $\|u_2\|_{q,\infty} \lesssim \|u\|_{(X_1,X_2)_{\kappa,1}}$ for all exponents satisfying

$$0 \le \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \frac{\bar{s}}{d} \quad \text{and} \quad 1 < q < \infty \quad \text{and} \quad \frac{1}{r_1} - \frac{1}{r_2} \ne \frac{s_1 - s_2}{d}. \tag{16}$$

For $s_1 = s_2 = 0$ this is implied by Hölder's inequality, so we may assume $\bar{s} > 0$ or $\bar{s} = 0,\ (s_1, s_2) \ne (0,0)$. For $\bar{s} = 0,\ (s_1, s_2) \ne (0,0),\ q = r_1 = r_2$ this is implied by the strong estimates in the case (12), so we may even assume $\bar{s} > 0$ or $\bar{s} = 0,\ (s_1, s_2) \ne (0,0),\ (r_1, r_2) \ne (q,q)$. For the remaining exponents the weak estimate is a consequence of (6) because one can find $q_i \in [r_i, \infty]$ such that

$$(1-\kappa)\left(s_1 - d\left(\frac{1}{r_1} - \frac{1}{q_1}\right)\right) + \kappa\left(s_2 - d\left(\frac{1}{r_2} - \frac{1}{q_2}\right)\right) = 0,$$
$$\frac{1}{q} = \frac{1-\kappa}{q_1} + \frac{\kappa}{q_2}, \quad s_i - d\left(\frac{1}{r_i} - \frac{1}{q_i}\right) \ne 0, \quad q_1 \ne q_2.$$

Indeed, this condition is equivalent to $\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}=\frac{\bar{s}}{d}$ and finding q_2 such that

$$\frac{1}{q}-\frac{1-\kappa}{r_1}\le\frac{\kappa}{q_2}\le\frac{\kappa}{r_2},\quad q_2\neq q,\quad \frac{1}{q}-(1-\kappa)\Big(\frac{1}{r_1}-\frac{s_1}{d}\Big)\neq\frac{\kappa}{q_2}\neq\kappa\Big(\frac{1}{r_2}-\frac{s_2}{d}\Big),$$

and such a choice is possible due to our assumptions. (In the case $\bar{s}=0$, $(s_1,s_2)\neq(0,0)$, $(r_1,r_2)\neq(q,q)$ choose $q_2=r_2$, $q_1=r_1$.) In this way we obtain $\|u_2\|_{q,\infty}\lesssim\|u\|_{(X_1,X_2)_{\kappa,1}}$ for all exponents satisfying (16). We finally interpolate these restricted weak-type estimates with each other to prove strong estimates for exponents as in (16). To this end let $\delta>0$ be sufficiently small (but fixed) and $\varepsilon:=\delta\big(\frac{s_1-s_2}{d}-\frac{1}{r_1}+\frac{1}{r_2}\big)\neq 0$ and define $\tilde{q},q^*,\tilde{\kappa},\kappa^*$ via $\frac{1}{\tilde{q}}-\varepsilon=\frac{1}{q}=\frac{1}{q^*}+\varepsilon$ and $\tilde{\kappa}-\delta=\kappa=\kappa^*+\delta$. Then $(\tilde{q},r_1,r_2,\tilde{\kappa})$, (q^*,r_1,r_2,κ^*) satisfies (16) and the reiteration property of real interpolation [Bergh and Löfström 1976, Theorem 3.5.3] gives

$$\begin{aligned}\|u_1\|_q &\lesssim \|u_1\|_{(L^{q^*}(\mathbb{R}^d),L^{\tilde{q}}(\mathbb{R}^d))_{\frac12,q}}\\ &\lesssim \|u\|_{((X_1,X_2)_{\kappa^*,1},(X_1,X_2)_{\tilde{\kappa},1})_{\frac12,q}}\\ &\lesssim \|u\|_{(X_1,X_2)_{\kappa,q}} \overset{(7)}{\lesssim} \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}.\end{aligned}$$

Here the first bound uses $\frac{1}{q}=\frac12\big(\frac{1}{q^*}+\frac{1}{\tilde{q}}\big)$ and the third uses $\kappa=\frac12(\tilde{\kappa}+\kappa^*)$. □

We have thus proved that the Gagliardo–Nirenberg inequality (3) holds for noncritical frequencies whenever the exponents belong to the set

$$\mathcal{B}(\kappa):=\{(q,r_1,r_2)\in[1,\infty]^3:\ (q,r_1,r_2)\text{ as in Proposition 9}\}.$$

Remark 10. (a) The original Gagliardo–Nirenberg inequality $\|\nabla^j v\|_q\lesssim\|\nabla^m v\|_{r_1}^{1-\kappa}\|v\|_{r_2}^{\kappa}$ from [Nirenberg 1959, p. 125] holds for $j,m\in\mathbb{N}$ provided that $\frac{1}{q}-\frac{j}{d}=(1-\kappa)\big(\frac{1}{r_1}-\frac{m}{d}\big)+\frac{\kappa}{r_2}$ and $\frac{j}{m}\le 1-\kappa<1$. Our result shows that "in most cases" the large-frequency part of this estimate holds provided that $\frac{j}{m}\le 1-\kappa<1$ holds and $\frac{1}{q}-\frac{j}{d}\ge(1-\kappa)\big(\frac{1}{r_1}-\frac{m}{d}\big)+\frac{\kappa}{r_2}$. The exceptions are due to the fact that, in $L^1(\mathbb{R}^d)$ or $L^\infty(\mathbb{R}^d)$, the term $\langle D\rangle^j u$ does not control $D^j u$, i.e., not every single partial derivative of order j. This is a consequence of the unboundedness of the Riesz transform on these spaces.

(b) Our proof indicates which function spaces to choose in order to get some endpoint estimates in the exceptional cases as well. Roughly speaking, one may replace $L^q(\mathbb{R}^d)$ by $L^{q,r}(\mathbb{R}^d)$ for suitable $r>q$ and $L^\infty(\mathbb{R}^d)$ by $\mathrm{BMO}(\mathbb{R}^d)$ on the left-hand side. On the right-hand side the Hardy space $\mathcal{H}^1(\mathbb{R}^d)$ may replace $L^1(\mathbb{R}^d)$.

(c) One may as well consider symbols $P_i(D)$ that vanish at some finite set of points in $\mathbb{R}^d\setminus S$. If for instance one has $P_i(\xi)=b_i(\xi)|\xi-\xi^*|^{t_i}$ near $\xi^*\in\mathbb{R}^d\setminus S$ for $t_1,t_2>-d$ and nonvanishing $b_i\in C^\infty(\mathbb{R}^d)$, then one finds as in Proposition 9 that the interpolation estimate holds in this frequency regime whenever $\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}>\frac{\bar{t}}{d}$, where $\bar{t}:=(1-\kappa)t_1+\kappa t_2$. Under suitable extra conditions similar to the ones above, this may be extended to the endpoint case $\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}=\frac{\bar{t}}{d}$.

(d) The proof in the important special case $1<r_1,r_2,q<\infty$ is much shorter than the complete analysis; see the beginning of Step 2.

4. Critical frequency analysis

We introduce a real number $A_\varepsilon(p,q)$ such that $\|\widetilde{T}_j\|_{p\to q} \lesssim 2^{-jA_\varepsilon(p,q)}$ holds for suitably defined dyadic operators $\widetilde{T}_j$ that play the role of the T_j in the previous section. Unfortunately, the definition of $A_\varepsilon(p,q)$ is rather complicated for $d \geq 2$. It involves the number

$$A(p,q) := \min\{A_0, A_1, A_2, A_2', A_3, A_3', A_4, A_4'\},$$

where $A_i = A_i(p,q)$ and $A_i' = A_i(q',p')$ are given by

$$A_0 = 1, \quad A_1 = \frac{k+2}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big), \quad A_2 = \frac{k+2}{2} - \frac{k+1}{q},$$

as well as

$$A_3 = \frac{2d-k}{2} - \frac{2d-k-1}{q}, \quad A_4 = \frac{k+2}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big) + \frac{2d-k-2}{2} - \frac{2d-k-2}{q}.$$

The values $A_0, A_1, A_1', A_2, A_2'$ will be important for $1 \leq p \leq 2 \leq q \leq \infty$, whereas all other exponents satisfying $1 \leq p \leq q \leq \infty$ come with A_3, A_3', A_4, A_4'. Then we define for $\varepsilon > 0$

$$A_\varepsilon(p,q) := \frac{1}{p} - \frac{1}{q} \quad \text{if } d = 1, \qquad A_\varepsilon(p,q) := A(p,q) - \varepsilon \cdot \mathbb{1}_{(p,q)\in\mathcal{E}} \quad \text{if } d \geq 2. \tag{17}$$

Here, $\mathcal{E}$ denotes a set of exceptional points where we do not have strong bounds, but only weak bounds or restricted weak-type bounds. It is given by

$$\mathcal{E} := \Big\{(p,q)\in[1,\infty]^2 : \frac{1}{p} = \frac{k+2}{2(k+1)}, \ \frac{1}{q} \leq \frac{k^2}{2(k+1)(k+2)} \quad \text{or} \quad \frac{1}{q} = \frac{k}{2(k+1)}, \ \frac{1}{p} \geq \frac{k^2+6k+4}{2(k+1)(k+2)}\Big\}$$

and coincides with the red points in Figure 1.

We first prove dyadic estimates in the frequency regime close to the critical surface S. The latter can be locally parametrized as a graph $\xi_d = \psi(\xi')$ after some permutation of coordinates, where $\xi = (\xi', \xi_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \simeq \mathbb{R}^d$. In view of (A1) we study operators of the form

$$\begin{aligned} \widetilde{T}_j f &:= \mathcal{F}^{-1}\big(\eta(2^j(\xi_d - \psi(\xi')))\chi(\xi')\hat{f}(\xi)\big) = \widetilde{K}_j * f, \quad \text{where} \\ \widetilde{K}_j &:= \mathcal{F}^{-1}\big(\eta(2^j(\xi_d - \psi(\xi')))\chi(\xi')\big) \end{aligned} \tag{18}$$

and

$$\psi \in C^\infty(\mathbb{R}^{d-1}),\ \chi \in C_0^\infty(\mathbb{R}^{d-1}) \text{ and at least } k \in \{1,\dots,d-1\} \text{ eigenvalues of the Hessian } D^2\psi \text{ are nonzero on } \operatorname{supp}(\chi). \tag{19}$$

In the degenerate case $d = 1$ we interpret $\eta(2^j(\xi_d - \psi(\xi')))\chi(\xi')$ as $\eta(2^j(\xi - c))$ for some constant $c \in \mathbb{R}$. Our analysis of the mapping properties of $\widetilde{T}_j$ follows [Mandel and Schippa 2022, Section 4]. Contrary to the situation for T_j, only the bounds for $j \nearrow +\infty$ will be of importance. Repeating the proof of Lemma 5 gives the following result in the one-dimensional case.

Lemma 11. *Assume $d = 1$ and $\eta \in C_0^\infty(\mathbb{R})$. Then we have*

$$\|\widetilde{T}_j\|_{p\to q} \lesssim 2^{-j(\frac{1}{p}-\frac{1}{q})} \quad \text{for } 1 \leq p \leq q \leq \infty,\ j \in \mathbb{Z}.$$

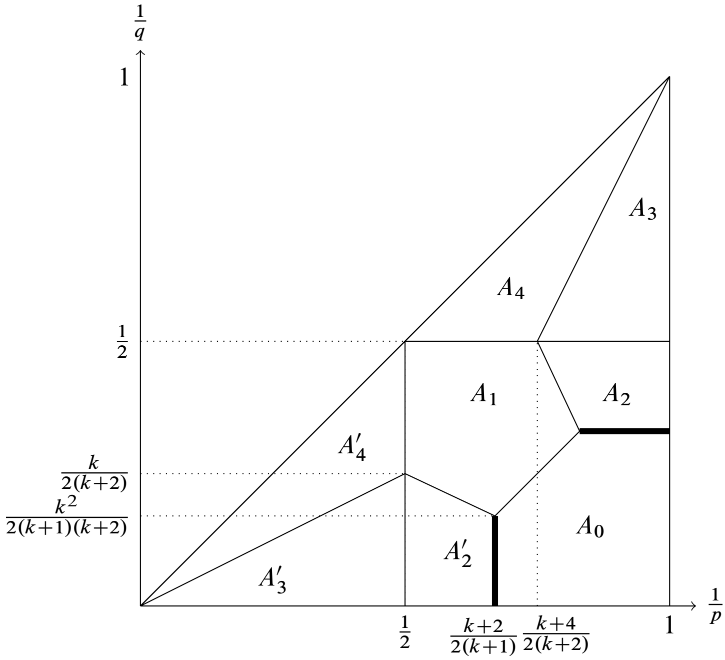

Figure 1. Riesz diagram with the bounds for the mapping constant of $\widetilde{T}_j$ from Lemma 13. The exceptional points from $\mathcal{E}$ are in bold.

The bounds in higher dimensions are more complicated and depend on the number $k \in \{1, \dots, d-1\}$ of nonvanishing principal curvatures of S. We first analyze the kernel function $\widetilde{K}_j$ following [Mandel and Schippa 2022, Lemma 4.4].

Proposition 12. *Assume $d \in \mathbb{N}$, $d \geq 2$, let χ, ψ, k be as in (19) and $\eta \in C_0^\infty(\mathbb{R})$. Then the kernel function $\widetilde{K}_j$ satisfies for $j \in \mathbb{Z},\ j \geq j_0$*

$$\|\widetilde{K}_j\|_r \lesssim 2^{-j\left(\frac{2d-k}{2} - \frac{2d-k-1}{r}\right)} \quad \text{if } 1 \leq r \leq 2, \qquad \|\widetilde{K}_j\|_\infty \lesssim 2^{-j}. \tag{20}$$

Proof. The bound $\|\widetilde{K}_j\|_2 \lesssim 2^{-j/2}$ follows from Plancherel's identity and (18). Indeed,

$$\begin{aligned}\|\widetilde{K}_j\|_2^2 &= \int_{\mathbb{R}^d} \eta(2^j(\xi_d - \psi(\xi')))^2 \chi(\xi')^2 \, d(\xi', \xi_d) \\ &= \int_{\mathbb{R}^{d-1}} \chi(\xi')^2 \left(\int_{\mathbb{R}} \eta(2^j t)^2 \, dt \right) d\xi' \\ &= 2^{-j} \|\chi\|_2^2 \|\eta\|_2^2.\end{aligned}$$

To prove (20) it thus suffices to show $\|\widetilde{K}_j\|_1 \lesssim 2^{-j((k+2)/2-d)}$, as well as $\|\widetilde{K}_j\|_\infty \lesssim 2^{-j}$, and to apply the Riesz–Thorin interpolation theorem. These two norm bounds for the kernel function are consequences of the pointwise bounds for arbitrary $N, M \in \mathbb{N}_0$

$$\begin{aligned}|\widetilde{K}_j(x)| &\lesssim_{N,M} 2^{-j}(1 + 2^{-j}|x_d|)^{-M}(1 + |x'|)^{-N} \quad && \text{if } |x'| \geq c|x_d|, \\ |\widetilde{K}_j(x)| &\lesssim_M 2^{-j}(1 + 2^{-j}|x_d|)^{-M}(1 + |x_d|)^{-\frac{k}{2}} \quad && \text{if } |x'| \leq c|x_d|,\end{aligned} \tag{21}$$

where $c > 0$ is suitably chosen. Indeed, choosing M, N sufficiently large we get

$$\begin{aligned}\|\widetilde{K}_j\|_1 &\lesssim_{N,M} \int_{\mathbb{R}}\biggl(\int_{|x'|\le cx_d} 2^{-j}(1+2^{-j}|x_d|)^{-M}(1+|x_d|)^{-\frac{k}{2}}\,dx'\biggr)\,dx_d \\ &\qquad + \int_{\mathbb{R}}\biggl(\int_{|x'|\ge cx_d} 2^{-j}(1+2^{-j}|x_d|)^{-M}(1+|x'|)^{-N}\,dx'\biggr)\,dx_d \\ &\lesssim_{M,N} 2^{-j}\int_{\mathbb{R}}(1+2^{-j}|x_d|)^{-M}|x_d|^{d-1}(1+|x_d|)^{-\frac{k}{2}}\,dx_d \\ &\qquad + 2^{-j}\int_{\mathbb{R}}(1+2^{-j}|x_d|)^{-M}(1+|x_d|)^{d-N}\,dx_d \\ &\lesssim_{M,N} 2^{-j}\int_0^{2^j}|x_d|^{d-1}(1+|x_d|)^{-\frac{k}{2}}\,dx_d + 2^{(M-1)j}\int_{2^j}^{\infty}|x_d|^{d-\frac{k}{2}-1-M}\,dx_d \\ &\lesssim_{M,N} 2^{-j(\frac{k+2}{2}-d)}.\end{aligned}$$

Here we used $2^j \ge 2^{j_0} > 0$. So it remains to prove the pointwise bounds by adapting the arguments from [Mandel and Schippa 2022]. We have

$$\widetilde{K}_j(x) = c_d\, 2^{-j}(\mathcal{F}^{-1}\eta)(2^{-j}x_d)\int_{\mathbb{R}^{d-1}} e^{i(x'\cdot\xi'+x_d\psi(\xi'))}\chi(\xi')\,d\xi'$$

for some dimensional constant $c_d > 0$. We choose $c > 0$ so large that the smooth phase function $\Phi(\xi') = x'\cdot\xi' + x_d\psi(\xi')$ satisfies $|\nabla\Phi(\xi')| \ge c^{-1}|x'|$ for all $\xi' \in \mathbb{R}^{d-1}$ whenever $|x'| \ge c|x_d|$. In view of $\chi \in C_0^\infty(\mathbb{R}^{d-1})$ the method of nonstationary phase gives

$$\begin{aligned}|\widetilde{K}_j(x)| &\lesssim_N 2^{-j}|(\mathcal{F}^{-1}\eta)(2^{-j}x_d)|(1+|x'|)^{-N} \\ &\lesssim_{N,M} 2^{-j}(1+2^{-j}|x_d|)^{-M}(1+|x'|)^{-N} \quad \text{for } |x'| \ge c|x_d|.\end{aligned}$$

In the second estimate we used that $\mathcal{F}^{-1}\eta$ is a Schwartz function. On the other hand, the theory of oscillatory integrals gives (see [Stein 1993, p. 361])

$$|\widetilde{K}_j(x)| \lesssim_M 2^{-j}(1+2^{-j}|x_d|)^{-M}(1+|x_d|)^{-\frac{k}{2}} \quad \text{for } |x'| \le c|x_d|. \qquad \square$$

Next we use Proposition 12 to find upper bounds for the operator norms of $\widetilde{T}_j$ as maps from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$, where $1 \le p \le q \le \infty$. The latter condition is mandatory since $\widetilde{T}_j$ is a translation-invariant operator covered by Hörmander's result from [Hörmander 1960, Theorem 1.1].

Lemma 13. *Assume $d \in \mathbb{N}$, $d \ge 2$ and let χ, ψ, k be as in (19) and $\eta \in C_0^\infty(\mathbb{R})$. Then, for any fixed $\varepsilon > 0$,*

$$\|\widetilde{T}_j\|_{p\to q} \lesssim 2^{-jA_\varepsilon(p,q)} \quad \textit{for } 1 \le p \le q \le \infty,\ j \in \mathbb{Z},\ j \ge j_0.$$

Proof. We first analyze the range $1 \le p \le 2 \le q \le \infty$. Plancherel's theorem gives

$$\|\widetilde{T}_j f\|_2 = \|\eta(2^j(\xi_d - \psi(\xi')))\chi(\xi')\hat{f}\|_2 \lesssim \|\hat{f}\|_2 = \|f\|_2$$

due to $\eta, \chi \in L^\infty(\mathbb{R}^d)$. The Stein–Tomas theorem for surfaces with k nonvanishing principal curvatures [Stein 1993, p. 365] yields as in [Mandel and Schippa 2022, Lemma 4.3]

$$\|\widetilde{T}_j f\|_q \lesssim 2^{-\frac{j}{2}}\|f\|_2, \quad \|\widetilde{T}_j f\|_2 \lesssim 2^{-\frac{j}{2}}\|f\|_{q'} \quad \text{if } \frac{1}{q} \le \frac{k}{2(k+2)}.$$

The Restriction-Extension operator $f \mapsto \mathcal{F}^{-1}(\hat{f}\, d\sigma_M)$ for compact pieces M of hypersurfaces with k nonvanishing principal curvatures has the mapping properties from [Mandel and Schippa 2022, Corollary 5.1], so it is bounded for (p,q) belonging to the pentagonal region

$$\frac{1}{p} > \frac{k+2}{2(k+1)}, \quad \frac{1}{q} < \frac{k}{2(k+1)}, \quad \frac{1}{p} - \frac{1}{q} \ge \frac{2}{k+2}. \tag{22}$$

So for these exponents and $M_t := \{\xi = (\xi', \xi_d) \in \operatorname{supp}(\chi) \times \mathbb{R} : \xi_d - \psi(\xi') = t\}$ with induced surface measure $d\sigma_{M_t} = (1 + |\nabla\psi(\xi')|^2)^{1/2}\, d\xi'$ we have for $\hat{g}(\xi) := \chi(\xi')\hat{f}(\xi)(1 + |\nabla\psi(\xi')|^2)^{-1/2}$

$$\|\widetilde{T}_j f\|_q \lesssim \int_{\mathbb{R}} |\eta(2^j t)|\, \|\mathcal{F}^{-1}(\hat{g}\, d\sigma_{M_t})\|_q\, dt \lesssim \int_{\mathbb{R}} |\eta(2^j t)|\, \|g\|_p\, dt \lesssim 2^{-j}\|f\|_p.$$

Moreover, [Mandel and Schippa 2022, Corollary 5.1] yields restricted weak-type bounds from $L^{p,1}(\mathbb{R}^d)$ to $L^{q,\infty}(\mathbb{R}^d)$ for all (p,q) belonging to the closure of the above-mentioned pentagon, which implies $\|\widetilde{T}_j f\|_{q,\infty} \lesssim 2^{-j}\|f\|_{p,1}$ in the same manner. Interpolating all these bounds gives

$$\|\widetilde{T}_j\|_{p\to q} \lesssim 2^{-j(\min\{A_0, A_1, A_2, A_2'\} - \varepsilon\cdot \mathbb{1}_{(p,q)\in\mathcal{E}})} = 2^{-jA_\varepsilon(p,q)} \quad \text{for } 1 \le p \le 2 \le q \le \infty,\ \varepsilon > 0.$$

This finishes the analysis in the case $1 \le p \le 2 \le q \le \infty$. For $2 \le p \le q \le \infty$ or $1 \le p \le q \le 2$ we get from Proposition 12

$$\|\widetilde{T}_j\|_{1\to 1} + \|\widetilde{T}_j\|_{\infty\to\infty} \lesssim \|\widetilde{K}_j\|_1 \lesssim 2^{-j(\frac{k+2}{2} - d)}.$$

Interpolating the estimates for $(p,q) = (\infty,\infty)$ with the ones for $p = 2,\ q \ge 2$ from above yields the estimates in the region A_3', A_4'; the dual ones follow analogously. So we get

$$\|\widetilde{T}_j\|_{p\to q} \lesssim 2^{-j\min\{A_3, A_3', A_4, A_4'\}} = 2^{-jA_\varepsilon(p,q)},$$

which proves the claim. □

The optimality of our constants is open. It would be interesting to see whether recent results and techniques for oscillatory integral operators by Guth, Hickman and Iliopolou [Guth et al. 2019] or Kwon and Lee [2020] (Proposition 2.4, Proposition 2.5) can be adapted to prove better bounds, especially in the range $1 \le p \le q < 2$ or $2 < p \le q \le \infty$. Any theorem leading to a larger value of $A_\varepsilon(p,q)$ will automatically provide a larger range of exponents q, r_1, r_2 for which our Gagliardo–Nirenberg inequalities hold. Candidates for such values $\ge A_\varepsilon(p,q)$ are given in [Cho et al. 2005, Lemma 2.2] and [Mandel and Schippa 2022, Lemma 4.4], but it seems nontrivial to make use of those in our setting. Next we use the estimates for $\widetilde{T}_j$ to discuss the relevant operators at distance 2^{-j} from the critical surface where $j \nearrow +\infty$.

Proposition 14. *Assume $d \in \mathbb{N}$ and (A1) with $\alpha_1, \alpha_2 > -1$. Then there are bounded linear operators $\mathcal{T}_j : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ and $j_0 \in \mathbb{Z}$ with $\sum_{j=j_0}^{\infty} \mathcal{T}_j u = u_1$ such that, for $i = 1, 2$ and any given $\varepsilon > 0$, we have, for all $u \in \mathcal{S}(\mathbb{R}^d)$,*

$$\|\mathcal{T}_j u\|_q \lesssim 2^{j(\alpha_i - A_\varepsilon(p,q))} \|P_i(D)u\|_p \quad \text{for } 1 \le p \le q \le \infty,\ j \in \mathbb{Z},\ j \ge j_0.$$

Proof. Recall $u_1 = \mathcal{F}^{-1}(\tau \hat{u})$, where τ was chosen in (4); we first consider the case $d \ge 2$. According to assumption (A1) there are $\tau_1, \dots, \tau_L \in C_0^\infty(\mathbb{R}^d)$ such that $\tau_1 + \dots + \tau_L = \tau$ holds and $S \cap \operatorname{supp}(\tau_l) = \{\xi \in \operatorname{supp}(\tau_l) : \tilde{\xi}_d = \psi_l(\tilde{\xi}'), \text{where } \tilde{\xi} = \Pi_l \xi\}$. Here, Π_l denotes some permutation of coordinates in $\mathbb{R}^d$. Since P_i vanishes of order α_i near the surface in the sense of assumption (A1), we may write

$$P_i(\xi)^{-1}\tau_l(\xi) = [\tau_{l+}(\xi)(\tilde{\xi}_d - \psi_l(\tilde{\xi}'))_+^{-\alpha_i} + \tau_{l-}(\xi)(\tilde{\xi}_d - \psi_l(\tilde{\xi}'))_-^{-\alpha_i}]\chi_l(\tilde{\xi}'),$$
$$\text{with } \tau_{l+}, \tau_{l-} \in C_0^\infty(\mathbb{R}^d),\ \chi_l \in C_0^\infty(\mathbb{R}^{d-1}),\ \tilde{\xi} := \Pi_l \xi, \qquad (23)$$

for suitable functions χ_l, ψ_l that satisfy (19). In view of this we define

$$\mathcal{T}_j := \sum_{l=1}^{L} \mathcal{T}_j^l, \quad \text{where } \mathcal{T}_j^l u := \mathcal{F}^{-1}\big(\tau_l(\xi)\hat{u}(\xi)\,\eta(2^j(\tilde{\xi}_d - \psi_l(\tilde{\xi}')))\chi_l(\tilde{\xi}')\big) \quad (\tilde{\xi} = \Pi_l \xi).$$

Since 0 does not belong to the support of η, there is $j_0 \in \mathbb{Z}$ such that $u_1 = \sum_{j=j_0}^{\infty} \mathcal{T}_j u$ in the sense of distributions. We introduce the smooth function $\eta_i(z) := \eta(z)|z|^{-\alpha_i}$. Then Lemma 13 yields

$$\begin{aligned}
\|\mathcal{T}_j u\|_q &\lesssim \sum_{l=1}^{L} \|\mathcal{T}_j^l u\|_q \\
&= \sum_{l=1}^{L} \big\|\mathcal{F}^{-1}\big(\eta(2^j(\tilde{\xi}_d - \psi_l(\tilde{\xi}')))\chi_l(\tilde{\xi}')\,\tau_l(\xi)\hat{u}(\xi)\big)\big\|_q \\
&= \sum_{l=1}^{L} \big\|\mathcal{F}^{-1}\big(\eta(2^j(\tilde{\xi}_d - \psi_l(\tilde{\xi}')))\chi_l(\tilde{\xi}')\,P_i(\xi)^{-1}\tau_l(\xi)\widehat{P_i(D)u}(\xi)\big)\big\|_q \\
&\overset{(23)}{=} \sum_{l=1}^{L} 2^{j\alpha_i} \big\|\mathcal{F}^{-1}\big(\eta_i(2^j(\tilde{\xi}_d - \psi_l(\tilde{\xi}')))\chi_l(\tilde{\xi}')(\tau_{li+}(\xi) + \tau_{li-}(\xi))\widehat{P_i(D)u}(\xi)\big)\big\|_q \\
&\lesssim \sum_{l=1}^{L} 2^{j(\alpha_i - A_\varepsilon(p,q))} \big\|\mathcal{F}^{-1}\big((\tau_{li+}(\xi) + \tau_{li-}(\xi))\widehat{P_i(D)u}(\xi)\big)\big\|_p \\
&\lesssim 2^{j(\alpha_i - A_\varepsilon(p,q))} \|P_i(D)u\|_p.
\end{aligned}$$

In the last inequality we used that τ_{li+}, τ_{li-} are L^p-multipliers since their Fourier transforms are integrable. □

In the forthcoming analysis we shall need the following auxiliary result. The proof mainly follows Stein's analysis of oscillatory integrals [1993, p. 380–386].

Proposition 15. *Assume $0 \le \alpha < \frac{1}{2}$ and that χ, ψ are as in (19), $\tau \in C_0^\infty(\mathbb{R}^d)$; set*

$$L_\alpha u := \mathcal{F}^{-1}((\xi_d - \psi(\xi'))_+^{-\alpha}\chi(\xi')\tau(\xi)u).$$

Then $L_\alpha : L^2(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is a bounded linear operator for $q := \frac{2(k+2)}{k+2-4\alpha}$.

Proof. Define the family of distributions γ_s as in [Stein 1993, p. 381] (called α_s in this book) via

$$\gamma_s(y) = \frac{e^{s^2}}{\Gamma(s)} y^{s-1}\zeta(y)1_{y>0} \quad \text{if } \Re(s) > 0,$$

where ζ is smooth with compact support and $\zeta(y) = 1$ for $|y| \le y_0$, where y_0 is chosen so large that $\zeta(\xi_d - \psi(\xi')) = 1$ holds whenever $\chi(\xi')\tau(\xi) \neq 0$. The family (γ_s) is extended to all $s \in \mathbb{C}$ via analytic continuation. Then introduce the family of linear operators

$$M_s f := \mathcal{F}^{-1}(\chi(\xi')^2\gamma_s(\xi_d - \psi(\xi'))\hat{f}).$$

Plancherel's identity gives

$$\|M_s f\|_2 \lesssim \|f\|_2 \quad \text{if } \Re(s) = 1.$$

On the other hand

$$M_s f = \Phi * f, \quad \Phi(z) := \hat{\gamma}_s(-z_d) \cdot \int_{\mathbb{R}^{d-1}} \chi(\xi')^2 e^{iz\cdot(\xi',\psi(\xi'))}\, d\xi'.$$

From equation (15) in [Stein 1993] and equation (32) in [Mandel and Schippa 2022] we infer

$$|\Phi(z)| \lesssim (1+|z_d|)^{-\Re(s)}(1+|z_d|)^{-\frac{k}{2}} \lesssim 1 \quad \text{if } \Re(s) = -\frac{k}{2}.$$

We conclude

$$\|M_s f\|_\infty \lesssim \|f\|_1 \quad \text{if } \Re(s) = -\frac{k}{2}.$$

Furthermore, for any given Schwartz functions f, g the function $s \mapsto \int_{\mathbb{R}^d}(M_s f)g$ is holomorphic in the open strip $-\frac{k}{2} < \Re(s) < 1$ with continuous extension to the boundary. So the family M_s is admissible for Stein's interpolation theorem [1956, Theorem 1] and we obtain

$$\|M_{1-2\alpha} f\|_q \lesssim \|f\|_{q'} \quad \text{if } \theta \in [0,1],\ 1-2\alpha = (1-\theta)\cdot\left(-\frac{k}{2}\right) + \theta\cdot 1,\ \frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2}.$$

This leads to $\theta = \frac{2(k+2-4\alpha)}{2(k+2)}$ and $q = \frac{2(k+2)}{k+2-4\alpha}$. In view of $0 < 2\alpha < 1$ this implies

$$\left\|\mathcal{F}^{-1}\big(\chi(\xi')^2(\xi_d - \psi(\xi'))_+^{-2\alpha}\zeta(\xi_d - \psi(\xi'))\hat{f}\big)\right\|_q \lesssim \|f\|_{q'}.$$

Now we consider functions $\hat{f} = \tau^2\hat{g}$. By choice of ζ and of y_0 we then have

$$\left\|\mathcal{F}^{-1}\big(\chi(\xi')^2(\xi_d - \psi(\xi'))_+^{-2\alpha}\tau(\xi)^2\hat{g}\big)\right\|_q \lesssim \|\mathcal{F}^{-1}(\tau^2\hat{g})\|_{q'} \lesssim \|g\|_{q'}.$$

This implies the claim given that this operator coincides with $L_\alpha L_\alpha^*$. ☐

We now use the dyadic estimates from Proposition 14 to prove Gagliardo–Nirenberg inequalities in the special case $P_1(D) = P_2(D)$ where the exponents satisfy $A_\varepsilon(p,q) = \alpha \in [0,1]$. This result plays the same role in the critical frequency regime as Proposition 8 does in the noncritical regime. For $d \ge 2$ we concentrate on exponents with $1 \le p \le 2 \le q \le \infty$.

Lemma 16. *Assume $d \in \mathbb{N}$ and let $P := P_1 = P_2$ satisfy (A1) for $\alpha := \alpha_1 = \alpha_2 \in [0,1]$. Then $\|u_1\|_q \lesssim \|P(D)u\|_p$ holds for all $u \in \mathcal{S}(\mathbb{R}^d)$ provided that*

(i) $d = 1$ *and* $1 \le p, q \le \infty$ *satisfy* $\frac{1}{p} - \frac{1}{q} = \alpha$ *and, if* $0 < \alpha < 1$, $(p,q) \notin \{(1, \frac{1}{1-\alpha}), (\frac{1}{\alpha}, \infty)\}$,

(ii) $d \ge 2$ *and* $1 \le p \le 2 \le q \le \infty$ *satisfy* $\frac{1}{p} - \frac{1}{q} = \frac{2\alpha}{k+2}$ *and* $\min\{\frac{1}{p}, \frac{1}{q'}\} > \frac{k+2\alpha}{2(k+1)}$.

The estimate $\|u_1\|_{q,\infty} \lesssim \|P(D)u\|_p$ *holds for exponents as in* (i), (ii) *or*

(iii) $d = 1$, $p = 1$, $q = \frac{1}{1-\alpha}$ *if* $\alpha \in (0,1)$,

(iv) $d \ge 2$, $1 \le p < \frac{2(k+1)}{k+2\alpha}$, $q = \frac{2(k+1)}{k+2-2\alpha}$ *if* $\alpha \in (\frac{1}{2}, 1]$.

Proof. With the same notation as before we have

$$P(\xi)^{-1}\tau_l(\xi) = [\tau_{l+}(\xi)(\tilde{\xi}_d - \psi_l(\tilde{\xi}'))_+^{-\alpha} + \tau_{l-}(\xi)(\tilde{\xi}_d - \psi_l(\tilde{\xi}'))_-^{-\alpha}]\chi_l(\tilde{\xi}'),$$
$$\text{with } \tau_{l+}, \tau_{l-} \in C_0^\infty(\mathbb{R}^d),\ \chi_l \in C_0^\infty(\mathbb{R}^{d-1}),\ \tilde{\xi} := \Pi_l \xi,$$

for functions χ_l, ψ_l that satisfy (19). So $u_1 = \sum_{j=j_0}^\infty \mathcal{T}_j u$. Assuming $1 \le p \le 2 \le q \le \infty$ are chosen as above we obtain (ii), (iv) as follows:

• Case $d \ge 2$, $\alpha = 0$. Our assumptions give that $A_\varepsilon(p,q) = \alpha = 0$ only occurs for $p = q = 2$. Here the estimate $\|u_1\|_2 \lesssim \|P(D)u\|_2$ follows from Plancherel's theorem.

• Case $d \ge 2$, $\alpha \in (0,1)$. We first consider the case $\alpha < \frac{1}{2}$. By assumption, $(\frac{1}{p}, \frac{1}{q})$ lies on the green diagonal line in Figure 2. By Proposition 15, the claimed inequality holds for the endpoints of that line given by $p = 2$, $q = \frac{2(k+2)}{k+2-4\alpha}$ and its dual $p = \frac{2(k+2)}{k+2+4\alpha}$, $q = 2$. Interpolating these two estimates with each other provides the desired inequality for all tuples on the green line in Figure 2 and thus proves the claim for $\alpha < \frac{1}{2}$.

Now consider the case $\alpha \ge \frac{1}{2}$. Our assumptions imply that $(\frac{1}{p}, \frac{1}{q})$ lies on the blue line in Figure 2 with endpoints excluded. In particular, $(\frac{1}{p}, \frac{1}{q})$ is in the interior of the A_1-region, so $A(\tilde{p}, \tilde{q}) = \frac{k+2}{2}(\frac{1}{\tilde{p}} - \frac{1}{\tilde{q}})$ for all $(\tilde{p}, \tilde{q})$ close to (p,q). For small $\delta > 0$ we choose $\frac{1}{q_1} = \frac{1}{q} + \delta$, $\frac{1}{q_2} = \frac{1}{q} - \delta$. Interpolating the estimates for (p, q_1) and (p, q_2) with interpolation parameter $\theta = \frac{1}{2}$ gives, due to $(1-\theta)A_\varepsilon(p,q_1) + \theta A_\varepsilon(p,q_2) = \alpha$, the weak estimate $\|u\|_{q,\infty} \lesssim \|P(D)u\|_p$. Here we used $u_1 = \sum_{j=j_0}^\infty \mathcal{T}_j u$, the dyadic estimates from Proposition 14 and the interpolation lemma, Lemma 4. These weak estimates hold for all $(\frac{1}{p}, \frac{1}{q})$ on the blue line with endpoints excluded. Interpolating these inequalities with each other gives $\|u\|_q \lesssim \|P(D)u\|_p$ for the same set of exponents, which proves (ii) for $\alpha \in (0,1)$.

To prove the weak estimate from (iv) assume $\alpha \in (\frac{1}{2}, 1)$. For any given $(\frac{1}{p}, \frac{1}{q})$ on the dashed horizontal blue line in Figure 2 with left endpoint excluded we can choose q_1, q_2 as above and the same argument gives $\|u\|_{q,\infty} \lesssim \|P(D)u\|_p$. Since these exponents are given by $1 \le p < \frac{2(k+1)}{k+2\alpha}$ and $q = \frac{2(k+1)}{k+2-2\alpha}$, we are done.

• Case $d \ge 2$, $\alpha = 1$. It was shown in [Mandel and Schippa 2022, Section 5] that the linear operators $(P(D) + i\delta)^{-1} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ are uniformly bounded with respect to small $|\delta| > 0$ given that our additional regularity assumptions on P from (A1) imply that $S = \{\xi \in \mathbb{R}^d : P(\xi) = 0\}$ is a smooth compact manifold with $|\nabla P| \ne 0$ on S. This implies $\|u_1\|_q \lesssim \|P(D)u\|_p$ and analogous arguments yield the weak bounds claimed in (iv).

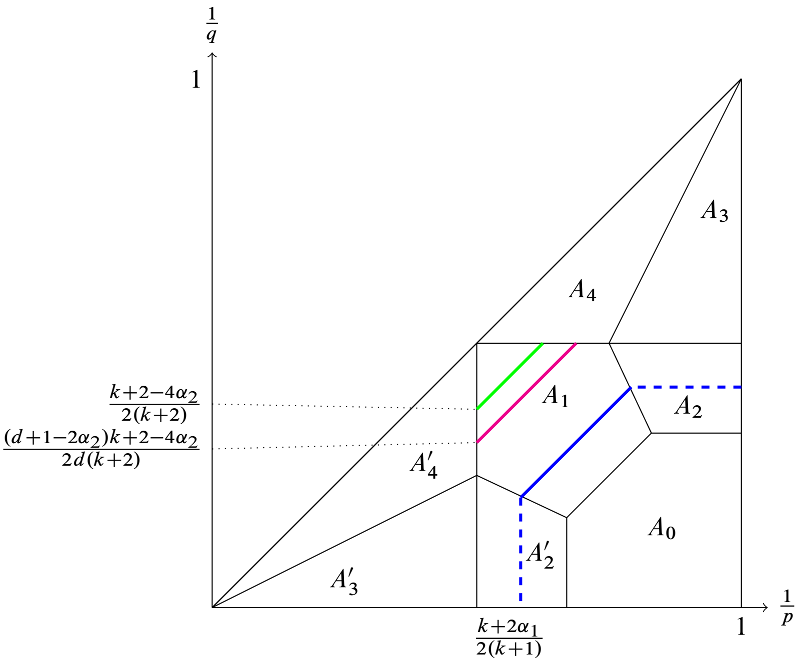

Figure 2. Riesz diagram showing the exponents $1 \le p \le 2 \le q \le \infty$ satisfying $A_\varepsilon(p,q) = \alpha$ in the case $\alpha = \alpha_1 \in \left(\frac{1}{2}, 1\right)$ (blue) and for $\alpha = \alpha_2 \in \left(0, \frac{1}{2}\right)$ (green). For the green resp. nondashed blue, exponent pairs Lemma 16 (i), (ii) give $\|u\|_q \le \|P(D)u\|_p$. In the case $\alpha = \alpha_2$ the corresponding estimates from [Mandel and Schippa 2022, Theorem 1.4(ii)] only hold for exponents on the magenta line. The picture was produced with parameter values $(d, k, \alpha_1, \alpha_2) = \left(4, 2, \frac{3}{4}, \frac{1}{4}\right)$.

Next we turn to the one-dimensional case $d = 1$. The representation formula then reads

$$u_1 = \sum_{l=1}^{L} \mathcal{F}^{-1}\left([\tau_{l+}(\xi)(\xi - \xi_l^*)_+^{-\alpha} + \tau_{l-}(\xi)(\xi - \xi_l^*)_-^{-\alpha}]\widehat{P(D)u}\right), \tag{24}$$

where $\{P(\xi) = 0\} = \{\xi_1^*, \ldots, \xi_L^*\}$. Using our assumption $\frac{1}{p} - \frac{1}{q} = \alpha$ we obtain the claims (i), (iii) from the following arguments:

- Case $d = 1$, $\alpha = 0$. We then have $p = q$ and we first analyze $1 < p = q < \infty$. In this case the Hilbert transform $f \mapsto \mathcal{F}^{-1}(\operatorname{sign}(\xi)\hat{f})$ is bounded on $L^p(\mathbb{R})$, and so is $f \mapsto \mathcal{F}^{-1}(\operatorname{sign}(\xi - \xi_l^*)\hat{f})$ for $l = 1, \ldots, L$. So the representation formula (24) implies $\|u_1\|_p \lesssim \|P(D)u\|_p$. In the case $p = q \in \{1, \infty\}$ we make use of our additional regularity assumption $\tau_l := \tau_{l+} = \tau_{l-}$ from (A1), so

$$\|u_1\|_p \le \sum_{l=1}^{L} \|\mathcal{F}^{-1}(\tau_l \widehat{P(D)u})\|_p \lesssim \sum_{l=1}^{L} \|\mathcal{F}^{-1}(\tau_l) * (P(D)u)\|_p \lesssim \|P(D)u\|_p.$$

Here we used that $\mathcal{F}^{-1}(\tau_l)$ is a Schwartz function for $l = 1, \ldots, L$.

- Case $d = 1$, $\alpha \in (0, 1)$. If $1 < p < q < \infty$ we deduce the claimed estimate from the boundedness of the Hilbert transform on $L^q(\mathbb{R})$ and the Riesz potential estimate $\|\mathcal{F}^{-1}(|\cdot|^{-\alpha}\hat{f})\|_q \lesssim \|f\|_p$. For $p = 1$,

$0 < \alpha < 1$ we have a weak estimate $\|\mathcal{F}^{-1}(|\cdot|^{-\alpha}\hat{f})\|_{q,\infty} \lesssim \|f\|_1$; see [Grafakos 2014, Theorem 1.2.3]. Note that the Hilbert transform is bounded on $L^{q,\infty}(\mathbb{R})$ as well by real interpolation.

• Case $d = 1$, $\alpha = 1$. We now have $\frac{1}{p} - \frac{1}{q} = 1$, so $p = 1, q = \infty$. We exploit the additional smoothness assumption $\tau_{l+} = -\tau_{l-}$ from (A1). Then $P \in C^\infty(\mathbb{R})$ is a smooth function with simple zeros $\xi_1^*, \dots, \xi_L^*$. To prove the claimed inequality we start with the trivial estimate $\|v\|_\infty \lesssim \|v'\|_1 = \|\mathcal{F}^{-1}(i\xi\hat{v})\|_1$ for all $v \in \mathcal{S}(\mathbb{R})$. Translation in Fourier space gives $\|v\|_\infty \lesssim \|\mathcal{F}^{-1}(i(\xi-\xi_l^*)\hat{v})\|_1$ for all $u \in \mathcal{S}(\mathbb{R}), l = 1, \dots, L$. So (24) implies as above

$$\|u_1\|_\infty \lesssim \sum_{l=1}^{L} \|\mathcal{F}^{-1}((\xi-\xi_l^*)^{-1}\tau_l \widehat{P(D)u})\|_\infty \lesssim \sum_{l=1}^{L} \|\mathcal{F}^{-1}(\tau_l \widehat{P(D)u})\|_1 \lesssim \|P(D)u\|_1. \qquad \square$$

As remarked in Figure 2, claim (ii) of the previous lemma improves upon the corresponding bounds from [Mandel and Schippa 2022, Theorem 1.4] in the case $0 < \alpha < \frac{1}{2}$. We finally combine all these estimates to prove Gagliardo–Nirenberg inequalities in the critical frequency regime. Given the rather complicated definition of $A_\varepsilon(p,q)$, an explicit characterization of the admissible exponents is possible in principle, but extremely laborious. We prefer to avoid most of the computations. Instead, we describe the set of admissible exponents in an abstract way and provide the required computations in the reasonably simple special case $1 \le p \le 2 \le q \le \infty$ that allows us to prove our main results. Proceeding in this way it becomes clear how eventual improvements of Lemma 13 affect the final range of exponents. Once more we exploit Bourgain's summation argument, which allows us to argue almost as in the large-frequency regime. On a formal level, comparing Lemma 5 (large frequencies) with Lemma 13 (critical frequencies), we essentially have to replace $s_i - d\left(\frac{1}{r_i} - \frac{1}{q_i}\right)$ by $A_\varepsilon(r_i, q_i) - \alpha_i$ because the summation index now ranges from some $j = j_0$ to $+\infty$ and not from $j = j_0$ to $-\infty$. It will be convenient to formulate our sufficient conditions in terms of $\bar{\alpha} := (1-\kappa)\alpha_1 + \kappa\alpha_2$.

We provide a definition of the set $\mathcal{A}(\kappa)$ of exponents (q, r_1, r_2) that are admissible for

$$\|u_1\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa} \|P_2(D)u\|_{r_2}^{\kappa}, \quad u \in \mathcal{S}(\mathbb{R}^d). \tag{25}$$

Lemma 16 provides the definition for $\kappa \in \{0, 1\}$, namely

$$\begin{aligned} \mathcal{A}(0) &:= \{(q, r_1, r_2) \in [1,\infty]^3 : (q, r_1, \alpha_1) \text{ as in Lemma 16(i),(ii)}\}, \\ \mathcal{A}(1) &:= \{(q, r_1, r_2) \in [1,\infty]^3 : (q, r_2, \alpha_2) \text{ as in Lemma 16(i),(ii)}\}. \end{aligned} \tag{26}$$

In the case $0 < \kappa < 1$ the definition is more involved and relies on the interpolation lemma (Lemma 4) and the dyadic estimates for critical frequencies from Proposition 14. Combining the latter with (6) we obtain $\|u_1\|_q \lesssim \|u\|_{(X_1,X_2)_{\kappa,q}}$ and deduce (25) for exponents (q, r_1, r_2) belonging to the set

$$\mathcal{A}_1(\kappa) := \Big\{(q, r_1, r_2) \in [1,\infty]^3 : \text{ there are } \varepsilon > 0,\ q_1 \in [r_1,\infty],\ q_2 \in [r_2,\infty] \text{ such that} \\ \frac{1}{q} = \frac{1-\kappa}{q_1} + \frac{\kappa}{q_2} \text{ and } (1-\kappa)A_\varepsilon(r_1,q_1) + \kappa A_\varepsilon(r_2,q_2) > \bar{\alpha}\Big\}.$$

This result covers all nonendpoint cases in our considerations further below. Using (5) with $Y_1 = Y_2 = L^q(\mathbb{R}^d)$ we obtain $\|u_1\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}$ for exponents in

$$\mathcal{A}_2(\kappa) := \big\{(q,r_1,r_2) \in [1,\infty]^3 : q \geq \max\{r_1,r_2\} \text{ and there is } \varepsilon > 0 \text{ such that} \\ (1-\kappa)A_\varepsilon(r_1,q) + \kappa A_\varepsilon(r_2,q) = \bar{\alpha},\ A_\varepsilon(r_i,q) \neq \alpha_i,\ i = 1,2\big\}.$$

Next we use $\|u\|_q = \|u\|_q^{1-\kappa}\|u\|_q^{\kappa}$ to deduce further estimates from Lemma 16 for exponents in

$$\mathcal{A}_3(\kappa) := \{(q,r_1,r_2) \in [1,\infty]^3 : (q,r_1,\alpha_1),\ (q,r_2,\alpha_2) \text{ as in Lemma 16(i), (ii)}\}.$$

Using (5) with $Y_1 = L^{q_1}(\mathbb{R}^d),\ Y_2 = L^{q_2}(\mathbb{R}^d)$, we get the weak bound $\|u_1\|_{q,\infty} \lesssim \|u\|_{(X_1,X_2)_{\kappa,1}}$ for exponents belonging to

$$\mathcal{A}_4^w(\kappa) := \Big\{(q,r_1,r_2) \in [1,\infty]^3 : \text{there are } \varepsilon > 0,\ q_1 \in [r_1,\infty],\ q_2 \in [r_2,\infty] \text{ such that} \\ (1-\kappa)A_\varepsilon(r_1,q_1) + \kappa A_\varepsilon(r_2,q_2) = \bar{\alpha},\ \frac{1}{q} = \frac{1-\kappa}{q_1} + \frac{\kappa}{q_2},\ \alpha_i \neq A_\varepsilon(r_i,q_i),\ q_1 \neq q_2\Big\}.$$

Interpolating the (weak or strong) endpoint estimates for $\mathcal{A}_2(\kappa)\cup\mathcal{A}_3(\kappa)\cup\mathcal{A}_4^w(\kappa)$ with each other exactly as in the final step of the proof of Proposition 9 we deduce $\|u_1\|_q \lesssim \|u\|_{(X_1,X_2)_{\kappa,q}} \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}$ for exponents from

$$\mathcal{A}_4(\kappa) := \Big\{(q,r_1,r_2) \in [1,\infty]^3 : \text{there are } \varepsilon \neq 0, \delta > 0,\ \tilde{q}, q^* \in [1,\infty],\ \tilde{\kappa},\kappa^* \in (0,1) \text{ with} \\ \frac{1}{\tilde{q}} - \varepsilon = \frac{1}{q} = \frac{1}{q^*} + \varepsilon,\ \tilde{\kappa} - \delta = \kappa = \kappa^* + \delta \text{ and} \\ (\tilde{q},r_1,r_2) \in \mathcal{A}_4^w(\tilde{\kappa})\cup\mathcal{A}_3(\tilde{\kappa})\cup\mathcal{A}_2(\tilde{\kappa}),\ (q^*,r_1,r_2) \in \mathcal{A}_4^w(\kappa^*)\cup\mathcal{A}_3(\kappa^*)\cup\mathcal{A}_2(\kappa^*)\Big\}.$$

Summarizing these interpolation results we obtain the following interpolation inequality in the critical frequency regime.

Proposition 17. *Assume $d \in \mathbb{N}$, $\kappa \in [0,1]$ and (A1) for $\alpha_1, \alpha_2 > -1$. Then*

$$\|u_1\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}, \quad u \in \mathcal{S}(\mathbb{R}^d),$$

holds provided that $(q,r_1,r_2) \in \mathcal{A}(\kappa) := \mathcal{A}_1(\kappa)\cup\mathcal{A}_2(\kappa)\cup\mathcal{A}_3(\kappa)\cup\mathcal{A}_4(\kappa)$.

5. Gagliardo–Nirenberg inequalities and proofs of Theorems 1 and 2

We first discuss the one-dimensional case. As before, we use the notation

$$\bar{\alpha} := (1-\kappa)\alpha_1 + \kappa\alpha_2 \quad \text{and} \quad \bar{s} := (1-\kappa)s_1 + \kappa s_2.$$

Theorem 18. *Assume $d = 1$, $\kappa \in [0,1]$ and that (A1), (A2) hold for $s_1, s_2 \in \mathbb{R}$ and $\alpha_1, \alpha_2 > -1$ such that $0 < \bar{\alpha} \leq \bar{s}$. Then*

$$\|u\|_q \lesssim \|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}, \quad u \in \mathcal{S}(\mathbb{R}),$$

holds provided that $q, r_1, r_2 \in [1,\infty]$ satisfy $\bar{\alpha} \leq \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} \leq \bar{s}$, as well as the conditions (i), (ii), (iii) *and* (iv), (v), (vi) *in the endpoint cases $\frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \bar{s}$ and $\bar{\alpha} = \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q}$, respectively:*

(i) *If* $q=\infty$ *then* $\frac{1}{r_1}-s_1\neq 0\neq \frac{1}{r_2}-s_2$ *or* $(r_1,r_2)=\left(\frac{1}{s_1},\frac{1}{s_2}\right)$, $s_1,s_2\in\{0,1\}$.

(ii) *If* $1<q<\infty$, $\frac{1}{r_1}-\frac{s_1}{d}=\frac{1}{q}=\frac{1}{r_2}-\frac{s_2}{d}$ *and* $r_1=1$ *then* $1<r_2<q$, $\kappa\geq\frac{r_2}{q}$ *or* $r_2=\infty$, $\frac{1}{q}\leq\kappa\leq\frac{1}{q'}$.

(iii) *If* $1<q<\infty$ *and* $\frac{1}{r_1}-\frac{s_1}{d}=\frac{1}{q}=\frac{1}{r_2}-\frac{s_2}{d}$ *and* $r_2=1$ *then* $1<r_1<q$, $1-\kappa\geq\frac{r_1}{q}$ *or* $r_1=\infty$, $\frac{1}{q}\leq 1-\kappa\leq\frac{1}{q'}$.

(iv) *If* $q=\infty$ *then* $\frac{1}{r_1}-\alpha_1\neq 0\neq\frac{1}{r_2}-\alpha_2$ *or* $(r_1,r_2)=\left(\frac{1}{\alpha_1},\frac{1}{\alpha_2}\right)$, $\alpha_1,\alpha_2\in\{0,1\}$.

(v) *If* $1<q<\infty$, $\frac{1}{r_1}-\alpha_1=\frac{1}{q}=\frac{1}{r_2}-\alpha_2$ *then* $\alpha_1,\alpha_2\in[0,1]$ *and* $r_1=1,\kappa<1$ *only if* $1<r_2<q$, $\kappa\geq\frac{r_2}{q}$.

(vi) *If* $1<q<\infty$, $\frac{1}{r_1}-\alpha_1=\frac{1}{q}=\frac{1}{r_2}-\alpha_2$ *then* $\alpha_1,\alpha_2\in[0,1]$ *and* $r_2=1,\kappa>0$ *only if* $1<r_1<q$, $1-\kappa\geq\frac{r_1}{q}$.

Proof. Proposition 9 shows that the large-frequency part of the inequality (involving s_1,s_2 and thus (i), (ii) and (iii)) holds. In view of Proposition 17 it remains to show that all exponents satisfying $\bar\alpha\leq\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}$ with (iv), (v) and (vi) in the endpoint case $\bar\alpha=\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}$ are covered by $\mathcal{A}(\kappa)$. In the case $\kappa=0$ this holds by definition of $\mathcal{A}(0)$ from (26) because the requirement $(r_1,q)\notin\left\{1,\frac{1}{1-\alpha},\frac{1}{\alpha},\infty\right\}$ if $0<\alpha<1$ from Lemma 16 (i) is met by (iv), (v) and (vi). The discussion for $\kappa=1$ is analogous. So from now on consider the case $0<\kappa<1$.

We now retrieve some information about $\mathcal{A}(\kappa)$ by exploiting the formula $A_\varepsilon(p,q)=\frac{1}{p}-\frac{1}{q}$ for $1\leq p\leq q\leq\infty$; see (17). Going back to the definition of the sets $\mathcal{A}_i(\kappa)$ we find

$$\mathcal{A}_1(\kappa)=\left\{(q,r_1,r_2)\in[1,\infty]^3:\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}>\bar\alpha\right\},$$
$$\mathcal{A}_2(\kappa)\supset\left\{(q,r_1,r_2)\in[1,\infty]^3:\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}=\bar\alpha,\ 0\leq\frac{1}{r_i}-\frac{1}{q}\neq\alpha_i\text{ for }i=1,2\right\},$$
$$\mathcal{A}_3(\kappa)\supset\left\{(q,r_1,r_2)\in[1,\infty]^3:\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}=\bar\alpha,\ \frac{1}{r_i}-\frac{1}{q}=\alpha_i\in[0,1]\text{ and }(r_i,q)\notin\left\{\left(1,\frac{1}{1-\alpha_i}\right),\left(\frac{1}{\alpha_i},\infty\right)\right\}\text{ if }\alpha_i\in(0,1)\text{ for }i=1,2\right\}.$$

Since the interpolation inequality holds for these exponents, our claim is proved in the following cases:

- $\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}>\bar\alpha$: see $\mathcal{A}_1(\kappa)$.
- $\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}=\bar\alpha$ and $q=1$: we necessarily have $\bar\alpha=0$, $r_1=r_2=1$, which is covered by $\mathcal{A}_2(\kappa)$ for $\alpha_1,\alpha_2\neq 0$ or $\mathcal{A}_3(\kappa)$ for $\alpha_1=\alpha_2=0$, respectively.
- $\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}=\bar\alpha$ and $q=\infty$: $\frac{1}{r_1}-\alpha_1\neq 0\neq\frac{1}{r_2}-\alpha_2$ is covered by $\mathcal{A}_2(\kappa)$ and $\frac{1}{r_1}-\alpha_1=0=\frac{1}{r_2}-\alpha_2$ with $\alpha_1,\alpha_2\in\{0,1\}$ is covered by $\mathcal{A}_3(\kappa)$.

So it remains to show the remaining endpoint estimates dealing with $1<q<\infty$. By the definition of $\mathcal{A}_4^w(\kappa)$ we have restricted weak-type estimates for exponents from

$$\begin{aligned}\mathcal{A}_4^w(\kappa)&=\left\{(q,r_1,r_2)\in[1,\infty]^3:\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}=\bar\alpha\text{ and there are }q_1\in[r_1,\infty],\ q_2\in[r_2,\infty]\right.\\&\qquad\left.\text{such that }q_1\neq q_2,\ \frac{1}{r_i}-\frac{1}{q_i}\neq\alpha_i\ (i=1,2),\ \frac{1-\kappa}{q_1}+\frac{\kappa}{q_2}=\frac{1}{q}\right\}\\&=\left\{(q,r_1,r_2)\in[1,\infty]^3:\frac{1-\kappa}{r_1}+\frac{\kappa}{r_2}-\frac{1}{q}=\bar\alpha,\ 1<q<\infty\right\}.\end{aligned}$$

(Indeed, thanks to $\bar{\alpha} > 0$ we may choose $\frac{1}{q_1} := \frac{1}{r_1} - \varepsilon$ and $\frac{\kappa}{q_2} := \frac{1}{q} - \frac{1-\kappa}{q_1}$ for small $\varepsilon > 0$ provided that $1 \le r_1 < \infty$, analogously for $r_2 < \infty$.) This implies

$$\mathcal{A}_4(\kappa) \supset \left\{(q, r_1, r_2) \in [1,\infty]^3 : \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \bar{\alpha},\ 1 < q < \infty,\ \frac{1}{r_1} - \frac{1}{r_2} \ne \alpha_1 - \alpha_2\right\}.$$

This yields the claim for the following exponents:

- $\frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \bar{\alpha},\ 1 < q < \infty$ and $\frac{1}{r_1} - \frac{1}{r_2} \ne \alpha_1 - \alpha_2$, which is covered by $\mathcal{A}_4(\kappa)$,
- $\frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} = \bar{\alpha},\ 1 < q < \infty$ and $\frac{1}{r_i} - \frac{1}{q} = \alpha_i \in [0,1]$ with $(r_i, q) \ne \left(1, \frac{1}{1-\alpha_i}\right)$ if $\alpha_i \in (0,1)$, which is covered by $\mathcal{A}_3(\kappa)$.

So it remains to prove the claim for

$$1 < q < \infty, \quad \frac{1}{r_1} - \alpha_1 = \frac{1}{q} = \frac{1}{r_2} - \alpha_2 \quad \text{and}$$
$$\left[r_1 = 1 < r_2 < q,\quad 1 > \kappa \ge \frac{r_2}{q} \qquad \text{or} \qquad r_2 = 1 < r_1 < q,\quad 1 > 1-\kappa \ge \frac{r_1}{q}\right].$$

By symmetry we may concentrate on $r_1 = 1 < r_2 < q,\ 1 > \kappa \ge \frac{r_2}{q}$, where the estimate follows from

$$\|u\|_q \overset{(13)}{\lesssim} \|u\|_{q,\infty}^{1-\kappa} \|u\|_{q,\kappa q}^{\kappa} \lesssim \|u\|_{q,\infty}^{1-\kappa} \|u\|_{q,r_2}^{\kappa} \lesssim \|P_1(D)u\|_1^{1-\kappa} \|P_2(D)u\|_{r_2}^{\kappa}.$$

Here we used Proposition 8(iv) and (ii) (for $r = r_2$). ☐

Proof of Theorem 2. We apply Theorem 18 to the symbols $P_1(D) = |D|^s - 1, s > 0$ and $P_2(D) = I$ that satisfy the hypotheses of the theorem for $(\alpha_1, \alpha_2, s_1, s_2) = (1, 0, s, 0)$. Then $\bar{\alpha} = 1-\kappa$, $\bar{s} = (1-\kappa)s$, so Theorem 18 implies that the Gagliardo–Nirenberg inequality holds provided that $1-\kappa \le \frac{1-\kappa}{r_1} + \frac{\kappa}{r_2} - \frac{1}{q} \le (1-\kappa)s$. The latter restriction comes from Theorem 18(i) and one checks that (ii)–(vi) are not restrictive for our choice of parameters $(\alpha_1, \alpha_2, s_1, s_2) = (1, 0, s, 0),\ s > 0$. ☐

We continue with the higher-dimensional case where a computation of $\mathcal{A}(\kappa) \cap \mathcal{B}(\kappa)$ is rather cumbersome. To simplify the discussion we concentrate on the special case $r_1 = r_2 = r \in [1,2]$ and $q \in [2,\infty]$ and only consider the special ansatz $q_1 = q_2 = q$ in the definition of the sets $\mathcal{A}_i(\kappa)$.

Theorem 19. *Assume $d \in \mathbb{N},\ d \ge 2, \kappa \in [0,1]$ and that (A1), (A2) hold for $s_1, s_2 \in \mathbb{R}$ and $\alpha_1, \alpha_2 > -1$ such that $0 \le \bar{\alpha} \le 1$. Then*

$$\|u\|_q \lesssim \|P_1(D)u\|_r^{1-\kappa} \|P_2(D)u\|_r^{\kappa}, \quad u \in \mathcal{S}(\mathbb{R}^d),$$

holds provided that $\bar{\alpha} < 1,\ \alpha_1 \ne \alpha_2,\ 0 < \kappa < 1$ and the exponents $r \in [1,2],\ q \in [2,\infty]$ satisfy

$$\frac{2\bar{\alpha}}{k+2} \le \frac{1}{r} - \frac{1}{q} \le \frac{\bar{s}}{d} \quad \text{and} \quad \min\left\{\frac{1}{r}, \frac{1}{q'}\right\} \ge \frac{k+2\bar{\alpha}}{2(k+1)}, \tag{27}$$

as well as $(q,r) \ne \left(\infty, \frac{d}{\bar{s}}\right)$ if $s_1 = s_2 = \bar{s} \in (0,d]$. In the case $\bar{\alpha} = 1$ or $\alpha_1 = \alpha_2$ or $\kappa \in \{0,1\}$ the same is true provided that the last condition in (27) is replaced by $\min\left\{\frac{1}{r}, \frac{1}{q'}\right\} > \frac{k+2\bar{\alpha}}{2(k+1)}$.

Proof. The conditions for large frequencies (involving s_1, s_2) were shown to be sufficient in Proposition 9. So we concentrate on the critical frequency part involving α_1, α_2. The following computations are based on the formula $A_\varepsilon(r,q) = A(r,q) - \varepsilon \cdot \mathbb{1}_{(p,q)\in\mathcal{E}}$, where

$$A(r,q) = \min\left\{1, \frac{k+2}{2}\left(\frac{1}{r}-\frac{1}{q}\right), \frac{k+2}{2} - \frac{k+1}{q}, -\frac{k}{2} + \frac{k+1}{r}\right\}$$

for $1 \le r \le 2 \le q \le \infty$; see (17) and Figure 1. Our definitions of $\mathcal{A}_1(\kappa), \mathcal{A}_2(\kappa), \mathcal{A}_3(\kappa)$ yield in the case $0 < \kappa < 1$

$$\mathcal{A}_1(\kappa) \supset \{(q,r,r) \in [2,\infty] \times [1,2]^2 : A_\varepsilon(r,q) > \bar{\alpha} \text{ for some } \varepsilon > 0\},$$
$$\mathcal{A}_2(\kappa) \supset \{(q,r,r) \in [2,\infty] \times [1,2]^2 : A_\varepsilon(r,q) = \bar{\alpha} \text{ for some } \varepsilon > 0,\ \alpha_1 \ne \bar{\alpha} \ne \alpha_2\},$$
$$\mathcal{A}_3(\kappa) \supset \left\{(q,r,r) \in [2,\infty] \times [1,2]^2 : A_\varepsilon(r,q) = \bar{\alpha} \text{ for some } \varepsilon > 0,\ \alpha_1 = \bar{\alpha} = \alpha_2 \in [0,1] \text{ and } \min\left\{\frac{1}{r}, \frac{1}{q'}\right\} > \frac{k+2\bar{\alpha}}{2(k+1)}\right\}.$$

From $\mathcal{A}(\kappa) \supset \mathcal{A}_1(\kappa) \cup \mathcal{A}_2(\kappa) \cup \mathcal{A}_3(\kappa)$ we thus get

$$\mathcal{A}(\kappa) \supset \left\{(q,r,r) \in [2,\infty] \times [1,2]^2 : A_\varepsilon(r,q) \ge \bar{\alpha} \text{ for some } \varepsilon > 0 \text{ and if } A_\varepsilon(r,q) = \bar{\alpha} = \alpha_1 = \alpha_2 \in [0,1] \text{ then } \min\left\{\frac{1}{r}, \frac{1}{q'}\right\} > \frac{k+2\bar{\alpha}}{2(k+1)}\right\}.$$

Since $A_\varepsilon(r,q) \ge \bar{\alpha}$ for some $\varepsilon > 0$ is equivalent to

$$\frac{1}{r} - \frac{1}{q} \ge \frac{2\bar{\alpha}}{k+2} \quad \text{and} \quad \min\left\{\frac{1}{r}, \frac{1}{q'}\right\} \begin{cases} \ge \frac{k+2\bar{\alpha}}{2(k+1)} & \text{if } \bar{\alpha} < 1 \text{ and } \alpha_1 \ne \alpha_2, \\ > \frac{k+2\bar{\alpha}}{2(k+1)} & \text{if } \bar{\alpha} = 1 \text{ or } \alpha_1 = \alpha_2. \end{cases}$$

This proves the claim for $0 < \kappa < 1$. When $\kappa \in \{0,1\}$ the claim follows from (26) and Lemma 16(i), (ii). □

Proof of Theorem 1. We apply Theorem 19 to $P_1(D) = |D|^s - 1$, $P_2(D) = I$. Again, the hypotheses of the theorem hold for $(\alpha_1, \alpha_2, s_1, s_2, k) = (1, 0, s, 0, d-1)$ because S is the unit sphere with $d-1$ nonvanishing principal curvatures. □

6. Local Gagliardo–Nirenberg inequalities

In [Fernández et al. 2022] it was shown that a "local" version of Gagliardo–Nirenberg inequalities is of interest, too. Here one looks for a larger set of exponents where (3) holds under the additional hypothesis $\|P_1(D)u\|_{r_1} \le R\|P_2(D)u\|_{r_2}$, where $R > 0$ is fixed; see Corollary 2.10 in that paper. A simple consequence of our estimates above is the following.

Corollary 20. *Assume $d \in \mathbb{N}$, $\kappa \in [0,1]$ and (A1), (A2) for $s_1, s_2 \in \mathbb{R}$ and $\alpha_1, \alpha_2 > -1$. Then the inequality*

$$\|u\|_q \lesssim (R^{\kappa-\kappa_1} + R^{\kappa-\kappa_2})\|P_1(D)u\|_{r_1}^{1-\kappa}\,\|P_2(D)u\|_{r_2}^{\kappa}$$

holds for all $u \in \mathcal{S}(\mathbb{R}^d)$ and satisfying $\|P_1(D)u\|_{r_1} \le R\|P_2(D)u\|_{r_2}$ provided that for some $\kappa_1, \kappa_2 \in [0,\kappa]$ we have $(q, r_1, r_2) \in \mathcal{A}(\kappa_1) \cap \mathcal{B}(\kappa_2)$.

Proof. Choose κ_1, κ_2 as required. Then Proposition 17 gives

$$\begin{aligned}\|u_1\|_q &\lesssim \|P_1(D)u\|_{r_1}^{1-\kappa_1}\|P_2(D)u\|_{r_2}^{\kappa_1} \\ &= (\|P_1(D)u\|_{r_1}\|P_2(D)u\|_{r_2}^{-1})^{\kappa-\kappa_1}\cdot\|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa} \\ &\lesssim R^{\kappa-\kappa_1}\|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}.\end{aligned}$$

Similarly, Proposition 9 implies

$$\|u_2\|_q \lesssim R^{\kappa-\kappa_2}\|P_1(D)u\|_{r_1}^{1-\kappa}\|P_2(D)u\|_{r_2}^{\kappa}.$$

Summing up these inequalities gives the claim. □

In the context of our particular example $P_1(D) = |D|^s - 1$, $s > 0$, and $P_2(D) = I$ this gives the following generalization of [Fernández et al. 2022, Corollary 2.10].

Corollary 21. *Assume* $d \in \mathbb{N}$, $d \geq 2$, $\kappa \in (0,1)$, $s > 0$. *Then*

$$\|u\|_q \lesssim (R^\kappa + 1)\|(|D|^s - 1)u\|_r^{1-\kappa}\|u\|_r^\kappa$$

holds for all $u \in \mathcal{S}(\mathbb{R}^d)$ *satisfying* $\|(|D|^s - 1)u\|_r \leq R\|u\|_r$ *provided that* $(q,r) \neq \left(\infty, \frac{d}{s}\right)$ *if* $0 < s \leq d$ *and*

(i) $d = 1$, $1 \leq r, q \leq \infty$ *and* $1 - \kappa \leq \frac{1}{r} - \frac{1}{q} \leq s$ *or*

(ii) $d \geq 2$, $1 \leq r \leq 2 \leq q \leq \infty$ *and* $\frac{2(1-\kappa)}{k+2} \leq \frac{1}{r} - \frac{1}{q} \leq \frac{s}{d}$, $\min\{\frac{1}{r}, \frac{1}{q'}\} \geq \frac{k+2-2\kappa}{2(k+1)}$.

Proof. This corresponds to the special case

$$(\kappa_1, \kappa_2) = (\kappa, 0) \quad \text{and} \quad (\alpha_1, \alpha_2, s_1, s_2, k, r_1, r_2) = (1, 0, s, 0, d-1, r, r)$$

in Corollary 20. The computation of $\mathcal{A}(\kappa)$ and $\mathcal{B}(0)$ can be done as in the proof of Theorem 19. Note that the assumptions imply $\bar{\alpha} = 1 - \kappa \in (0,1)$, $\alpha_1 \neq \alpha_2$ and $0 < \kappa < 1$. □

7. Gagliardo–Nirenberg inequalities with unbounded characteristic sets

In the previous sections we provided a systematic study of Gagliardo–Nirenberg inequalities, where the characteristic set S of the symbols is smooth and compact. In the case of unbounded characteristic sets our analysis works for Schwartz functions whose Fourier transform is supported in a given bounded set, but an argument for general Schwartz functions is lacking so far, even in the case of simple differentiable operators with suitable scaling behaviour like the wave operator or the Schrödinger operator. In the L^2-setting, a less technical approach based on Plancherel's identity can be used. We follow the ideas presented in [Fernández et al. 2022] to prove Gagliardo–Nirenberg inequalities of the form

$$\|u\|_q \lesssim \|\partial_{tt}u - \Delta u\|_r^{1-\kappa}\|u\|_r^\kappa, \quad u \in \mathcal{S}(\mathbb{R}^d), \tag{28}$$

$$\|v\|_q \lesssim \|i\partial_t v - \Delta v\|_r^{1-\kappa}\|v\|_r^\kappa, \quad v \in \mathcal{S}(\mathbb{R}^d), \tag{29}$$

where $r = 2$. We denote the space-time variable by $z = (x,t) \in \mathbb{R}^{d-1} \times \mathbb{R} = \mathbb{R}^d$.

Theorem 22. *Let $d \in \mathbb{N}$. Then (28) holds provided that $r = 2$, $q = \frac{2d}{d-4+4\kappa}$, where $\frac{1}{2} \le \kappa \le 1$ if $d \ge 3$ and $\frac{1}{2} < \kappa \le 1$ if $d = 2$.*

Proof. We first consider the case $d \ge 3$, and define $\mathcal{C}_t := \{\xi = (\xi', \xi_d) \in \mathbb{R}^d : \xi_d^2 - |\xi'|^2 = t\}$ and the induced surface measure σ_t. Then we have the representation formula

$$u(z) = c_d \int_{\mathbb{R}^d} \hat{u}(\xi) e^{iz\cdot\xi}\, d\xi = \frac{c_d}{2} \int_{\mathbb{R}} \int_{\mathcal{C}_t} \hat{u}(\xi) |\xi|^{-1} e^{iz\cdot\xi}\, d\sigma_t(\xi)\, dt,$$

where $c_d = (2\pi)^{-d/2}$. Strichartz' inequality [1977, Theorem I, case III(b)] implies that we have for $\frac{2(d+1)}{d-1} \le q \le \frac{2d}{d-2}$

$$\begin{aligned} \|u\|_q &\lesssim \int_{\mathbb{R}} \|\mathcal{F}^{-1}(\hat{u}|\cdot|^{-1}\, d\sigma_t)\|_q\, dt \\ &\lesssim \int_{\mathbb{R}} |t|^{\frac{d-1}{4} - \frac{d}{2q}} \|\hat{u}|\cdot|^{-1}\|_{L^2(\mathcal{C}_t, d\sigma_t)}\, dt \\ &\lesssim \int_{\mathbb{R}} |t|^{\frac{d-2}{4} - \frac{d}{2q}} \|\hat{u}|\cdot|^{-\frac{1}{2}}\|_{L^2(\mathcal{C}_t, d\sigma_t)}\, dt. \end{aligned}$$

Here, the factor $|t|^{(d-1)/4 - d/(2q)}$ is obtained via scaling and in the last estimate we used $|\xi| \ge \sqrt{|t|}$ for $\xi \in \mathcal{C}_t$. On the other hand, Plancherel's theorem gives

$$\begin{aligned} \|\partial_{tt} u - \Delta u\|_2^2 &= \int_{\mathbb{R}^d} |\xi_d^2 - |\xi'|^2|^2 |\hat{u}(\xi)|^2\, d\xi \\ &= \frac{1}{2} \int_{\mathbb{R}} \int_{\mathcal{C}_t} |t|^2 |\hat{u}(\xi)|^2 |\xi|^{-1}\, d\sigma_t(\xi)\, dt \\ &= \frac{1}{2} \int_{\mathbb{R}} t^2 \|\hat{u}|\cdot|^{-\frac{1}{2}}\|_{L^2(\mathcal{C}_t, d\sigma_t)}^2\, dt \end{aligned}$$

and

$$\|u\|_2^2 = \frac{1}{2} \int_{\mathbb{R}} \|\hat{u}|\cdot|^{-\frac{1}{2}}\|_{L^2(\mathcal{C}_t, d\sigma_t)}^2\, dt.$$

Writing $\varphi(t) := \|\hat{u}|\cdot|^{-1/2}\|_{L^2(\mathcal{C}_t, d\sigma_t)}$ it remains to prove that the quotient

$$\frac{\int_{\mathbb{R}} |t|^{\frac{d-2}{4} - \frac{d}{2q}} \varphi(t)\, dt}{\left(\int_{\mathbb{R}} t^2 \varphi(t)^2\, dt\right)^{\frac{1-\kappa}{2}} \left(\int_{\mathbb{R}} \varphi(t)^2\, dt\right)^{\frac{\kappa}{2}}}$$

is bounded independently of φ. According to [Fernández et al. 2022, Lemma 2.1], with $w(t) = |t|^{(d-2)/4 - d/(2q)}$, $w_1(t) = 1$ and $w_2(t) = t$, this is the case if and only if the following quantity is finite:

$$\begin{aligned} \sup_{s>0} s^{\frac{1-\kappa}{2}} \left\| \frac{w}{(w_1^2 + s w_2^2)^{\frac{1}{2}}} \right\|_{L^2(\mathbb{R})} &= \sup_{s>0} s^{\frac{1-\kappa}{2}} \left(\int_{\mathbb{R}} \frac{|t|^{\frac{d-2}{2} - \frac{d}{q}}}{1 + st^2}\, dt \right)^{\frac{1}{2}} \\ &= \sup_{s>0} s^{\frac{1-\kappa}{2} - \frac{1}{4}\left(\frac{d}{2} - \frac{d}{q}\right)} \left(\int_{\mathbb{R}} \frac{|\rho|^{\frac{d-2}{2} - \frac{d}{q}}}{1 + \rho^2}\, d\rho \right)^{\frac{1}{2}}. \end{aligned}$$

This leads to $q = \frac{2d}{d-4+4\kappa}$. In view of $\frac{2(d+1)}{d-1} \le q \le \frac{2d}{d-2}$ this requires $\frac{1}{2} \le \kappa \le \frac{d+2}{2(d+1)}$, but the upper bound for κ may be removed just as in [Fernández et al. 2022, p. 20–21] by combining the already established inequality for $\frac{2(d+1)}{d-1}$ with

$$\|u\|_q \le \|u\|_2^{1-\theta}\|u\|_{\frac{2(d+1)}{d-1}}^{\theta}, \quad 2\le q\le \frac{2(d+1)}{d-1}, \quad \frac{1}{q}=\frac{1-\theta}{2}+\frac{\theta}{\frac{2(d+1)}{d-1}}.$$

In the case $d=2$ the analogous reasoning based on [Strichartz 1977, Theorem I, case III(c)]. It is also shown in that work that the above estimates are valid for $6=\frac{2(d+1)}{d-1}\le q<\frac{2d}{d-2}=\infty$ and thus $\frac{1}{2}<\kappa\le\frac{d+2}{2(d+1)}$. The same interpolation trick then allows to extend this to the whole range $\kappa>\frac{1}{2}$. □

We now apply this method to the Schrödinger operator.

Theorem 23. *Let $d\in\mathbb{N}, d\ge 2$. Then (29) holds provided that $r=2, q=\frac{2(d+1)}{d-3+4\kappa}$ and $\frac{1}{2}\le\kappa\le 1$.*

Proof. Define $\mathcal{P}_t:=\{\xi=(\xi',\xi_d)\in\mathbb{R}^d:\xi_d-|\xi'|^2=t\}$ and the induced surface measure σ_t. Plancherel's identity gives

$$\begin{aligned}\|v\|_2^2 &= \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}} |\hat v(\xi',t+|\xi'|^2)|^2\,d\xi'\,dt\\ &= \int_{\mathbb{R}}|t|^{\frac{d-1}{2}}\int_{\mathbb{R}^{d-1}} |\hat v(\sqrt{t}\xi',t(1+|\xi'|^2))|^2\,d\xi'\,dt\\ &= \int_{\mathbb{R}}|t|^{\frac{d-1}{2}}\int_{\mathbb{R}^{d-1}} |\hat v_t|^2\sqrt{1+4|\xi'|^2}\,d\xi'\,dt\\ &= \int_{\mathbb{R}}|t|^{\frac{d-1}{2}}\|\hat v_t\|^2_{L^2(\mathcal{P}_1,d\sigma_1)}\,dt,\end{aligned}$$

where $\hat v_t(\xi):=\hat v(\sqrt{t}\xi',t\xi_d)(1+4|\xi'|^2)^{-1/4}$. Similarly,

$$\|i\partial_t v-\Delta v\|_2^2=\int_{\mathbb{R}} t^{2+\frac{d-1}{2}}\|\hat v_t\|^2_{L^2(\mathcal{P}_1,d\sigma_1)}\,dt.$$

Strichartz' inequality from [Strichartz 1977, Theorem I, case I] implies for $q=\frac{2(d+1)}{d-1}$

$$\begin{aligned}\|v\|_q &= \left\|c_d\int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}}\hat v(\xi',t+|\xi'|^2)e^{iz\cdot(\xi',t+|\xi'|^2)}\,d\xi'\,dt\right\|_q\\ &\lesssim \int_{\mathbb{R}}\left\|\int_{\mathbb{R}^{d-1}}\hat v(\xi',t+|\xi'|^2)e^{iz\cdot(\xi',t+|\xi'|^2)}\,d\xi'\right\|_q dt\\ &\lesssim \int_{\mathbb{R}}|t|^{\frac{d-1}{2}}\left\|\int_{\mathbb{R}^{d-1}}\hat v(\sqrt{t}\xi',t(1+|\xi'|^2))e^{iz\cdot(\sqrt{t}\xi',t(1+|\xi'|^2))}\,d\xi'\right\|_q dt\\ &\lesssim \int_{\mathbb{R}}|t|^{\frac{d-1}{2}}\left\|\mathcal{F}^{-1}\big(\hat v(\sqrt{t}\xi',t\xi_d)(1+4|\xi'|^2)^{-\frac12}\,d\sigma_1\big)(\sqrt{t}z',tz_1)\right\|_q dt\\ &= \int_{\mathbb{R}}|t|^{\frac{d-1}{2}-\frac{d+1}{2q}}\left\|\mathcal{F}^{-1}\big(\hat v(\sqrt{t}\xi',t\xi_d)(1+4|\xi'|^2)^{-\frac12}\,d\sigma_1\big)\right\|_q dt\\ &\lesssim \int_{\mathbb{R}}|t|^{\frac{d-1}{2}-\frac{d+1}{2q}}\|\hat v(\sqrt{t}\xi',t\xi_d)(1+4|\xi'|^2)^{-\frac12}\|_{L^2(\mathcal{P}_1,d\sigma_1)}\,dt\\ &\lesssim \int_{\mathbb{R}}|t|^{\frac{d-1}{2}-\frac{d+1}{2q}}\|\hat v_t\|_{L^2(\mathcal{P}_1,d\sigma_1)}\,dt.\end{aligned}$$

We set $\varphi(t) := |t|^{(d-1)/4}\|\hat{v}_t\|_{L^2(\mathcal{P}_1, d\sigma_1)}$ and it remains to show that the quotient

$$\frac{\int_{\mathbb{R}} |t|^{\frac{d-1}{4}-\frac{d+1}{2q}}\varphi(t)\,dt}{\left(\int_{\mathbb{R}} t^2\varphi(t)^2\,dt\right)^{\frac{1-\kappa}{2}}\left(\int_{\mathbb{R}}\varphi(t)^2\,dt\right)^{\frac{\kappa}{2}}}$$

is bounded independently of φ. We apply [Fernández et al. 2022, Lemma 2.1] once more:

$$\sup_{s>0} s^{\frac{1-\kappa}{2}}\left(\int_{\mathbb{R}} \frac{|t|^{\frac{d-1}{2}-\frac{d+1}{q}}}{1+st^2}\,dt\right)^{\frac{1}{2}} = \sup_{s>0} s^{\frac{1-\kappa}{2}}\left(\left(\frac{1}{\sqrt{s}}\right)^{\frac{d+1}{2}-\frac{d+1}{q}}\int_{\mathbb{R}} \frac{|\rho|^{\frac{d-1}{2}-\frac{d+1}{q}}}{1+\rho^2}\,d\rho\right)^{\frac{1}{2}}$$
$$= \sup_{s>0} s^{\frac{1-\kappa}{2}-\frac{d+1}{8}+\frac{d+1}{4q}}\left(\int_{\mathbb{R}} \frac{|\rho|^{\frac{d-1}{2}-\frac{d+1}{q}}}{1+\rho^2}\,d\rho\right)^{\frac{1}{2}}.$$

This term is indeed finite for $q = \frac{2(d+1)}{d-1}$ and $\kappa = \frac{1}{2}$, which proves the claim in this special case. The claim for general $\kappa \geq \frac{1}{2}$ follows as above by interpolation. □

We conjecture that at least for $1 < r \leq 2 \leq q < \infty$ and $0 < \kappa < 1$ the inequality (28) actually holds for exponents

$$\frac{1}{r} - \frac{1}{q} = \frac{2(1-\kappa)}{d}, \quad \min\left\{\frac{1}{r}, \frac{1}{q'}\right\} \geq \frac{d-2\kappa}{2(d-1)}, \tag{30}$$

whereas the corresponding inequality involving the Schrödinger operator holds whenever

$$\frac{1}{r} - \frac{1}{q} = \frac{2(1-\kappa)}{d+1}, \quad \min\left\{\frac{1}{r}, \frac{1}{q'}\right\} \geq \frac{d+1-2\kappa}{2d}.$$

Note that the Sobolev inequalities [Jeong et al. 2016, Theorem 1.1] then take the form of the endpoint estimate $\kappa = 0$ in (30).

Acknowledgments

The author thanks Robert Schippa and Louis Jeanjean for helpful comments related to this work. This work was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)–Project-ID 258734477–SFB 1173.

References

[Bartsch et al. 2016] T. Bartsch, L. Jeanjean, and N. Soave, "Normalized solutions for a system of coupled cubic Schrödinger equations on $\mathbb{R}^3$", *J. Math. Pures Appl.* (9) **106**:4 (2016), 583–614. MR Zbl

[Bellazzini et al. 2014] J. Bellazzini, R. L. Frank, and N. Visciglia, "Maximizers for Gagliardo–Nirenberg inequalities and related non-local problems", *Math. Ann.* **360**:3-4 (2014), 653–673. MR Zbl

[Bergh and Löfström 1976] J. Bergh and J. Löfström, *Interpolation spaces: an introduction*, Grundl. Math. Wissen. **223**, Springer, 1976. MR Zbl

[Bourgain 1985] J. Bourgain, "Estimations de certaines fonctions maximales", *C. R. Acad. Sci. Paris Sér. I Math.* **301**:10 (1985), 499–502. MR

[Brezis and Mironescu 2019] H. Brezis and P. Mironescu, "Where Sobolev interacts with Gagliardo–Nirenberg", *J. Funct. Anal.* **277**:8 (2019), 2839–2864. MR Zbl

[Brezis et al. 2021] H. Brezis, J. Van Schaftingen, and P.-L. Yung, "Going to Lorentz when fractional Sobolev, Gagliardo and Nirenberg estimates fail", *Calc. Var. Partial Differential Equations* **60**:4 (2021), art. id. 129. MR Zbl

[Caffarelli et al. 1984] L. Caffarelli, R. Kohn, and L. Nirenberg, "First order interpolation inequalities with weights", *Compos. Math.* **53**:3 (1984), 259–275. MR Zbl

[Carbery et al. 1999] A. Carbery, A. Seeger, S. Wainger, and J. Wright, "Classes of singular integral operators along variable lines", *J. Geom. Anal.* **9**:4 (1999), 583–605. MR Zbl

[Cazenave and Lions 1982] T. Cazenave and P.-L. Lions, "Orbital stability of standing waves for some nonlinear Schrödinger equations", *Comm. Math. Phys.* **85**:4 (1982), 549–561. MR Zbl

[Cho et al. 2005] Y. Cho, Y. Kim, S. Lee, and Y. Shim, "Sharp L^p-L^q estimates for Bochner–Riesz operators of negative index in $\mathbb{R}^n$, $n \geq 3$", *J. Funct. Anal.* **218**:1 (2005), 150–167. MR Zbl

[Dao et al. 2022] N. A. Dao, N. Lam, and G. Lu, "Gagliardo–Nirenberg and Sobolev interpolation inequalities on Besov spaces", *Proc. Amer. Math. Soc.* **150**:2 (2022), 605–616. MR Zbl

[Del Pino and Dolbeault 2002] M. Del Pino and J. Dolbeault, "Best constants for Gagliardo–Nirenberg inequalities and applications to nonlinear diffusions", *J. Math. Pures Appl.* (9) **81**:9 (2002), 847–875. MR Zbl

[Evéquoz 2017] G. Evéquoz, "Existence and asymptotic behavior of standing waves of the nonlinear Helmholtz equation in the plane", *Analysis* (*Berlin*) **37**:2 (2017), 55–68. MR Zbl

[Fernández et al. 2022] A. J. Fernández, L. Jeanjean, R. Mandel, and M. Mariş, "Non-homogeneous Gagliardo–Nirenberg inequalities in $\mathbb{R}^N$ and application to a biharmonic non-linear Schrödinger equation", *J. Differential Equations* **330** (2022), 1–65. MR Zbl

[Grafakos 2014] L. Grafakos, *Modern Fourier analysis*, 3rd ed., Grad. Texts in Math. **250**, Springer, 2014. MR Zbl

[Guth et al. 2019] L. Guth, J. Hickman, and M. Iliopoulou, "Sharp estimates for oscillatory integral operators via polynomial partitioning", *Acta Math.* **223**:2 (2019), 251–376. MR Zbl

[Gutiérrez 2004] S. Gutiérrez, "Non trivial L^q solutions to the Ginzburg–Landau equation", *Math. Ann.* **328**:1-2 (2004), 1–25. MR Zbl

[Hajaiej et al. 2011] H. Hajaiej, L. Molinet, T. Ozawa, and B. Wang, "Necessary and sufficient conditions for the fractional Gagliardo–Nirenberg inequalities and applications to Navier–Stokes and generalized boson equations", pp. 159–175 in *Harmonic analysis and nonlinear partial differential equations* (Kyoto, 2010), edited by T. Ozawa and M. Sugimoto, RIMS Kôkyûroku Bessatsu **B26**, Res. Inst. Math. Sci., Kyoto, 2011. MR Zbl

[Hörmander 1960] L. Hörmander, "Estimates for translation invariant operators in L^p spaces", *Acta Math.* **104** (1960), 93–140. MR Zbl

[Jeong et al. 2016] E. Jeong, Y. Kwon, and S. Lee, "Uniform Sobolev inequalities for second order non-elliptic differential operators", *Adv. Math.* **302** (2016), 323–350. MR Zbl

[Kenig et al. 1987] C. E. Kenig, A. Ruiz, and C. D. Sogge, "Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators", *Duke Math. J.* **55**:2 (1987), 329–347. MR Zbl

[Koch and Tataru 2005] H. Koch and D. Tataru, "Dispersive estimates for principally normal pseudodifferential operators", *Comm. Pure Appl. Math.* **58**:2 (2005), 217–284. MR Zbl

[Kwon and Lee 2020] Y. Kwon and S. Lee, "Sharp resolvent estimates outside of the uniform boundedness range", *Comm. Math. Phys.* **374**:3 (2020), 1417–1467. MR Zbl

[Lenzmann and Sok 2021] E. Lenzmann and J. Sok, "A sharp rearrangement principle in Fourier space and symmetry results for PDEs with arbitrary order", *Int. Math. Res. Not.* **2021**:19 (2021), 15040–15081. MR Zbl

[Lenzmann and Weth 2024] E. Lenzmann and T. Weth, "Symmetry breaking for ground states of biharmonic NLS via Fourier extension estimates", *J. Anal. Math.* **152**:2 (2024), 777–800. MR Zbl

[Mandel and Oliveira e Silva 2023] R. Mandel and D. Oliveira e Silva, "The Stein–Tomas inequality under the effect of symmetries", *J. Anal. Math.* **150**:2 (2023), 547–582. MR Zbl

[Mandel and Schippa 2022] R. Mandel and R. Schippa, "Time-harmonic solutions for Maxwell's equations in anisotropic media and Bochner–Riesz estimates with negative index for non-elliptic surfaces", *Ann. Henri Poincaré* **23**:5 (2022), 1831–1882. MR Zbl

[McCormick et al. 2013] D. S. McCormick, J. C. Robinson, and J. L. Rodrigo, "Generalised Gagliardo–Nirenberg inequalities using weak Lebesgue spaces and BMO", *Milan J. Math.* **81**:2 (2013), 265–289. MR Zbl

[Nirenberg 1959] L. Nirenberg, "On elliptic partial differential equations", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (3) **13** (1959), 115–162. MR Zbl

[Noris et al. 2014] B. Noris, H. Tavares, and G. Verzini, "Existence and orbital stability of the ground states with prescribed mass for the L^2-critical and supercritical NLS on bounded domains", *Anal. PDE* **7**:8 (2014), 1807–1838. MR Zbl

[Stein 1956] E. M. Stein, "Interpolation of linear operators", *Trans. Amer. Math. Soc.* **83** (1956), 482–492. MR Zbl

[Stein 1993] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Math. Ser. **43**, Princeton Univ. Press, 1993. MR Zbl

[Strichartz 1977] R. S. Strichartz, "Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations", *Duke Math. J.* **44**:3 (1977), 705–714. MR Zbl

[Tao 2018] T. Tao, "Fourier analysis", course notes, UCLA, 2018, available at https://www.math.ucla.edu/~tao/247a.1.06f/.

[Voigt 1992] J. Voigt, "Abstract Stein interpolation", *Math. Nachr.* **157** (1992), 197–199. MR Zbl

[Weinstein 1982/1983] M. I. Weinstein, "Nonlinear Schrödinger equations and sharp interpolation estimates", *Comm. Math. Phys.* **87**:4 (1982/1983), 567–576. MR Zbl

[Zhang 2021] Y. Zhang, "Optimizers of the Sobolev and Gagliardo–Nirenberg inequalities in $\dot{W}^{s,p}$", *Calc. Var. Partial Differential Equations* **60**:1 (2021), art. id. 10. MR Zbl

Received 28 Mar 2022. Revised 23 Dec 2022. Accepted 31 Aug 2023.

RAINER MANDEL: rainer.mandel@kit.edu
Institute for Analysis, Karlsruhe Institute of Technology, Karlsruhe, Germany

ANALYSIS AND PDE
Vol. 17 (2024), No. 10, pp. 3477–3550
DOI: 10.2140/apde.2024.17.3477

SEMICLASSICAL PROPAGATION THROUGH CONE POINTS

PETER HINTZ

We introduce a general framework for the study of the diffraction of waves by cone points at high frequencies. We prove that semiclassical regularity propagates through cone points with an almost sharp loss even when the underlying operator has leading-order terms at the conic singularity which fail to be symmetric. We moreover show improved regularity along strictly diffractive geodesics. Applications include high-energy resolvent estimates for complex- or matrix-valued inverse square potentials and for the Dirac–Coulomb equation. We also prove a sharp propagation estimate for the semiclassical conic Laplacian.

The proofs use the semiclassical cone calculus, introduced recently by the author, and combine radial point estimates with estimates for a scattering problem on an exact cone. A second microlocal refinement of the calculus captures semiclassical conormal regularity at the cone point and thus facilitates a unified treatment of semiclassical cone and b-regularity.

1. Introduction

We present a systematic analysis of the propagation of semiclassical regularity through points which are geometrically singular (cone points), analytically singular (e.g., including inverse square potentials), or both. The novel aspect of our approach is that it handles leading-order singular terms with ease, *regardless of symmetry or sign conditions*.

As a simple application of our main microlocal propagation result, we consider high-energy scattering by complex-valued potentials on $\mathbb{R}^n$ with an inverse square singularity. Denote by $H_0^2(\mathbb{R}^n \setminus \{0\})$ the closure of $\mathcal{C}_c^\infty(\mathbb{R}^n \setminus \{0\})$ in the topology of $H^2(\mathbb{R}^n)$; denote further by $\Delta = \sum_{j=1}^n D_{x^j}^2$ (where $D = (1/i)\partial$) the nonnegative Laplacian, and denote polar coordinates on $\mathbb{R}^n$ by $(r, \omega) \in (0, \infty) \times \mathbb{S}^{n-1}$.

Theorem 1.1 (high-energy estimates for potential scattering). *Let $V(x) = V_0(x)/|x|^2$, where $V_0 = V_0(r, \omega) \in \mathcal{C}_c^\infty([0, \infty)_r \times \mathbb{S}^{n-1}; \mathbb{C})$ and $V_0(0, \omega) \equiv \mathsf{Z} \in \mathbb{C}$. (Thus $V(x) = \mathsf{Z}/|x|^2 + \mathcal{O}(|x|^{-1})$ near the origin.) Suppose that $n \geq 5$ and $\operatorname{Re}\sqrt{((n-2)/2)^2 + \mathsf{Z}} > 1$. Then there exists $\lambda_0 > 0$ so that for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \lambda_0$ and $0 < \operatorname{Im}\lambda < 1$, the operator*

$$\Delta + V - \lambda : H_0^2(\mathbb{R}^n \setminus \{0\}) \to L^2(\mathbb{R}^n) \tag{1-1}$$

is invertible, and its inverse obeys the operator norm bound

$$\|\chi(\Delta + V - \lambda)^{-1}\chi\|_{L^2 \to L^2} \leq C_{\chi,\epsilon}|\lambda|^{-\frac{1}{2}+\epsilon} \tag{1-2}$$

MSC2020: primary 58J47; secondary 35L81, 35P25, 35S05.
Keywords: propagation of singularities, conic singularities, inverse square potentials, semiclassical cone calculus.

for all $\chi \in \mathcal{C}_c^\infty(X)$ *and* $\epsilon > 0$. *More generally*, $\Delta + V - \lambda : \mathcal{D} \to L^2(\mathbb{R}^n)$ *is invertible for* $n \geq 2$ *and* $\mathsf{Z} \in \mathbb{C} \setminus (-\infty, -((n-2)/2)^2]$ *for a suitable domain* $\mathcal{D}$ (*see* (5-29) *with* $l = 1$), *and the estimate* (1-2) *holds in this generality as well.*

The point is that we can allow for Z to be nonreal, in which case $\Delta + V$ is not a symmetric operator on $\mathcal{C}_c^\infty(\mathbb{R}^n \setminus \{0\})$. (The compact support assumption on V is made only to focus the attention of the reader on a neighborhood of the singularity at $x = 0$. The result holds also for V with sufficient decay at infinity, such as $|\partial_x^\alpha V| \lesssim \langle x \rangle^{-1-\delta}$ for all α where $\delta > 0$.) For a general result for matrix-valued inverse square potentials without symmetry conditions, see Theorem 5.7; Lemma 5.10 verifies the assumptions of Theorem 5.7 for the case considered in Theorem 1.1. Typical applications of high-energy resolvent estimates include decay and local smoothing estimates for solutions to wave and Schrödinger equations; since such applications are orthogonal to the focus of the present paper, we shall not discuss them here.

Burq and Planchon, Stalker, and Tahvildar-Zadeh [Planchon et al. 2003; Burq et al. 2003] proved Strichartz estimates for exact inverse square potentials in the case of real $\mathsf{Z} > -((n-2)/2)^2$. Duyckaerts [2006] obtained, by means of estimates for semiclassical defect measures, high-energy resolvent estimates (without the ϵ-loss) in the more general setting of inverse square potentials at a finite collection of points p_j in $\mathbb{R}^n$, at each of which the coefficient Z_j satisfies $\mathsf{Z}_j > -((n-2)/2)^2$. We also mention the work [Baskin and Wunsch 2013] on lossless resolvent estimates in a *geometric* setting, namely in the presence of finitely many conic singularities, and the work by [Hillairet and Wunsch 2020] on resonances in this setting (see also [Galkowski 2017]).

Remark 1.2 (more natural settings). The setting of Theorem 1.1 is chosen here for its simplicity. More natural examples in which leading-order terms without signs or symmetry properties are present arise in particular in the study of PDEs on vector bundles. As an example, motivated by the recent work [Baskin and Wunsch 2023], we prove high-energy resolvent estimates for the Dirac–Coulomb equation in Section 5C; see Theorem 5.14.

The heart of the proof of Theorem 1.1 is the propagation of semiclassical regularity through $r = 0$,[1] which we prove in this paper for a general class of *admissible operators*; see Definition 4.1 and Theorem 4.10. Thus, in addition to inverse square singularities (which may be anisotropic), we allow for the underlying metric g to have a conic singularity at $r = 0$, so $g = \mathrm{d}r^2 + r^2 k(r, y, \mathrm{d}r, \mathrm{d}y)$ for some smooth r-dependent tensor k, with $k|_{r=0}$ a Riemannian metric on a closed manifold Y. We moreover allow for further first-order differential operators of the schematic form $r^{-1} D_r$, $r^{-2} D_y$ to be present. All these singular terms are allowed to be of the same strength at $r = 0$: they are, to leading order at $r = 0$, homogeneous of degree -2 with respect to dilations.

In order to explain the main features of Theorem 1.1, note that the degree -2 homogeneity of the Laplacian and of the potential r^{-2} is reflected also in the Hardy inequality, which demonstrates that any factor of r^{-1} should be regarded as a derivative as far as analysis near the cone point $r = 0$ is concerned. Therefore, when Z in Theorem 1.1 is nonreal, the operator $\Delta + V - \lambda$ is, *even to leading order at the cone*

[1] In particular, the choice of the large end of the space, here $\mathbb{R}^n$, is only made for convenience and allows for simple control of the global structure of the geodesic flow. Thus, we do not discuss the large literature on limiting absorption principles here.

point, not symmetric. Therefore, techniques rooted in the spectral theory of self-adjoint operators do not apply. Furthermore, recall that even for solutions of smooth-coefficient PDEs $Pu = f$ where the principal symbol of P is complex-valued, microlocal regularity of u propagates along the null-bicharacteristics of $\mathrm{Re}\, P$ *only under a sign condition on* $\mathrm{Im}\, P$ near the boundary of the support of $\mathrm{Im}\, P$ [Vasy 2018, §4.5]; on a technical level, the term $\mathrm{Im}\, P$ contributes the leading term in a positive commutator argument for proving the propagation of regularity along null-bicharacteristics of $\mathrm{Re}\, P$. The absence of sign conditions on $\mathrm{Im}\, V$ in Theorem 1.1 is thus a significant obstacle for the applicability of existing methods.

In general geometric or analytic settings where one cannot separate variables, propagation estimates through cone points and other types of singularities have so far largely been restricted to self-adjoint settings. Melrose and Wunsch [2004] studied the diffraction of waves by conic singularities by combining microlocal propagation estimates in the edge calculus developed by Mazzeo [1991] with the inversion of a suitable model operator on an exact cone. This point of view is closely related to that adopted in the present paper, see Remark 1.4, though by contrast to the present work, [Melrose and Wunsch 2004] takes full advantage of the self-adjointness of the underlying Laplace operator.

Later works on wave propagation in singular geometries have been based on positive commutator arguments relative to a quadratic form domain (thus still in self-adjoint settings), following the blueprint of [Vasy 2008] on the propagation of singularities on smooth manifolds with corners (see [Lebeau 1997] for the analytic setting). Vasy's work was extended to the setting of manifolds with edge singularities by Melrose, Vasy and Wunsch [Melrose et al. 2008], and the same authors established improved regularity of the strictly diffracted front on manifolds with corners [Melrose et al. 2013]. See [Qian 2009] for the case of inverse square potentials. We remark that in these works, the underlying geometry near the singularity is *not* reflected in the type of singularities which propagate or diffract—for instance, in the case of [Vasy 2008], the geometry is that of a manifold with corners equipped with a smooth (incomplete!) Riemannian metric, but the correct notion of regularity is conormality at the boundary; thus, these works introduce mixed differential-pseudodifferential calculi which are compatible with both structures.

Baskin and Marzuola [2022] combined the techniques of [Vasy 2008] with those developed in [Baskin et al. 2015] to study the long-time behavior of waves on manifolds with conic singularities. An important ingredient in their work is a high-energy estimate for propagation through the conic singularity. In the present paper we give an alternative proof which in particular avoids the use of a mixed calculus; see also Remark 1.5. We also mention that Gannot and Wunsch [2023] analyzed the diffraction by conormal potentials in the semiclassical setting using direct commutator methods involving paired Lagrangian distributions, inspired by [de Hoop et al. 2015].

The recent work [Baskin and Wunsch 2023] on diffraction for the Dirac–Coulomb equation is also rooted in [Melrose and Wunsch 2004; Melrose et al. 2008]. While the (first-order) Dirac–Coulomb operator is self-adjoint for the range of Coulomb charges considered in [Baskin and Wunsch 2023], the wave-type operator obtained by taking an appropriate square has nonsymmetric leading-order terms at the central singularity; thus, the authors work directly with the first-order operator in their proofs of propagation results. We are able to give a direct proof of high-energy estimates for the resolvent associated with the wave-type operator arising in [Baskin and Wunsch 2023]; see Section 5C.

In the high-energy regime under study in the present paper, the strategy for overcoming the issues caused by the absence of symmetry or self-adjointness properties is the following. We distill the contribution of V to the high-frequency propagation of regularity (i.e., in Theorem 1.1, the inverse powers of $|\lambda|$ appearing in uniform estimates of L^2 norms) into a model problem right at the cone point, thus decoupling it from the real-principal-type propagation away from the cone point (where V plays no role due to its subprincipal nature). More precisely, in the setting of Theorem 1.1, set $h := |\lambda|^{-1/2}$ and $z = h^2\lambda = 1 + \mathcal{O}(h)$, and define the semiclassical rescaling

$$P_{h,z} = h^2(\Delta + V - \lambda) = h^2\Delta - z + \frac{h^2}{|x|^2}V_0 = (hD_r)^2 - i(n-1)\frac{h}{r}hD_r + h^2r^{-2}\Delta_{\mathbb{S}^{n-1}} - z + \frac{h^2}{r^2}V_0. \tag{1-3}$$

Recall here that $V_0 \in \mathcal{C}^\infty_{\mathrm{c}}([0,\infty)_r \times \mathbb{S}^{n-1}; \mathbb{C})$ is equal to a constant $\mathsf{Z} \in \mathbb{C}$ at $r = 0$. The operator $P_{h,z}$ is a semiclassical differential operator in $r > 0$. Its uniform analysis as $h \to 0$, as far as the novel bit near $r = 0$ is concerned, is based on two ingredients, discussed in more detail in Section 1A:

(1) *Symbolic propagation estimates*: real-principal-type propagation in $r > 0$ in the spirit of [Duistermaat and Hörmander 1972], and radial point estimates down to $r = 0$ in the spirit of [Melrose 1994; Vasy 2013] but taking place in the semiclassical cone algebra introduced by the author in [Hintz 2022]. The advantage of this algebra in the present setting is that $P_{h,z}$ has a smooth and nondegenerate principal symbol in this algebra down to $r = 0$; in this algebra, the proofs of the relevant symbolic estimates are then essentially standard.

(2) *Inversion of a model problem*: Passing to the rescaled variable $\hat{r} = r/h$ and letting $h \to 0$ for fixed $\hat{r}$ in the resulting expression of $P_{h,z}$ gives

$$N(P) = D_{\hat{r}}^2 - i(n-1)\hat{r}^{-1}D_{\hat{r}} + \hat{r}^{-2}\Delta_{\mathbb{S}^{n-1}} - 1 + \frac{\mathsf{Z}}{\hat{r}^2}. \tag{1-4}$$

The inversion of $N(P)$ is a scattering problem on an exact cone at unit frequency and requires the existence of the limiting (outgoing) resolvent. Its analysis is based on b-analysis near the small end of the cone [Melrose 1993] and on the microlocal approach to scattering theory on spaces with conic infinite ends pioneered in [Melrose 1994].

The ϵ-loss in the estimate (1-2) is then due to the analogous loss in the limiting absorption principle for the scattering problem, as one needs to exclude incoming but allow outgoing spherical waves; see Remark 5.3. (We shall in fact deduce the lossy estimates stated in Theorem 1.1 from sharp results — as far as the relationship of domain and codomain of $P_{h,z}$ is concerned — on spaces with variable semiclassical orders.) For general admissible operators, the decay rates of incoming and outgoing solutions of the model problem are typically different, and the semiclassical loss upon propagation through the cone point is equal to their difference (up to an additional ϵ-loss); we give explicit examples in which this loss indeed occurs in Appendix A, demonstrating that our analysis is sharp up to an ϵ-loss. It seems impossible to avoid this ϵ-loss if one proves the propagation estimates in the above step-by-step manner: the microlocal radial point estimates force inequalities on the semiclassical orders (see, however, [Wang 2020] in a Besov space setting), and also on the incoming and outgoing decay orders of the function spaces on which the model problem is invertible. Thus, avoiding this ϵ-loss requires the development of propagation arguments which provide control near $r = 0$ in one fell swoop. We demonstrate this for operators $P_{h,z} = h^2\Delta_g - z$, $z = 1 + \mathcal{O}(h)$, on (nonexact) conic manifolds, for which we are able to obtain a *lossless* propagation estimate by means of a positive commutator argument which is global on the level of the normal operator $N(P)$, i.e., which involves the construction of a commutator which is positive *as an operator* on an exact cone, in the spirit of Mourre's construction [1980/81] and Vasy's approach [2000; 2001] to many-body scattering; see Theorem 5.4, in particular the estimate (5-6). It is not clear at present however how to generalize such an argument to more general (in particular nonsymmetric) settings.

The close connection between diffraction by conic singularities and scattering on large ends of cones was recently studied for exact (or "product") cones (i.e., the metric is $g = \mathrm{d}r^2 + r^2 k(y, \mathrm{d}y)$) by Yang [2022], resulting in a partial improvement of the classical analysis by Cheeger and Taylor [1982a; 1982b] which was based on separation of variables and Bessel function analysis. Recently, Chen Xi [2022] constructed a detailed parametrix for high-frequency diffraction by (nonexact) conic singularities, i.e., for the operator $(h^2\Delta_g - (1 \pm i0))^{-1}$, with applications to short-time Strichartz estimates for the Schrödinger equation; an important ingredient in his work is the precise resolvent construction by Guillarmou, Hassell and Sikora [Guillarmou et al. 2013], applied on an exact cone which arises similarly to (1-4). (The history of the study of propagation and diffraction phenomena for solutions of wave-type equations on manifolds with singularities is long, starting with Sommerfeld's example [1896] and early developments by Friedlander [1958] and Keller [1985]. The use of geometric and microlocal techniques for the analysis of singularities goes back to [Melrose and Sjöstrand 1978; 1982] on manifolds with boundary using commutator techniques, and [Melrose 1975; Taylor 1976; Melrose and Taylor 2018] using parametrix constructions.) We also mention the recent work by Keeler and Marzuola [2022] who use estimates for the resolvent on exact cones perturbed by a radial (but not homogeneous) potential in order to obtain dispersive estimates for the Schrödinger equation.

Finally, we prove a *diffractive improvement* which gives finer control on the strength of singularities as they propagate through the cone point. Combining our framework with the arguments in [Melrose and Wunsch 2004; Melrose et al. 2008] for the propagation of coisotropic regularity, we show that, under a nonfocusing condition, the strongest singularities propagating towards the cone point only continue along geometric geodesics (limits of geodesics barely missing the cone point), whereas away from those, the diffracted front is smoother; see Section 4F. We do not address here the interesting question of whether one can prove estimates in the presence of multiple scatterers using such diffractive improvements, as done in [Baskin and Wunsch 2013]; in particular, we do not recover Duyckaerts' results [2006] here. See Remark 5.11.

Regarding applications of our high-frequency estimates, we content ourselves with a few conjectural remarks. First, in the context of [Baskin et al. 2015; 2018; Baskin and Marzuola 2022] and Remark 5.12, it should be possible to use our results to justify contour-shifting arguments for obtaining asymptotic expansions (including radiation fields) of solutions to wave-type equations on static conic manifolds with rather general inverse square potentials in the forward cone. Due to the relationship between edge Sobolev spaces and semiclassical cone Sobolev spaces discussed in Remark 3.4, it is likely not necessary to prove b-regularity at the spatial cone point $r = 0$ (unlike in [Baskin and Marzuola 2022, §8.2.2–8.2.3]); instead,

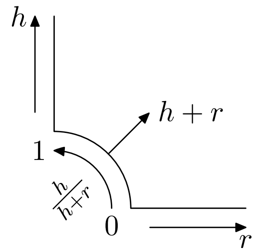

Figure 1. The semiclassical cone single space $X_{c\hbar}$.

edge propagation results at $r = 0$ (with uniformity as $t \to \infty$ encoded by b-spaces), which directly apply also to nonsymmetric operators, likely suffice. One does need to show, however, the existence of global forward solutions by means distinct from energy methods since spectral methods are no longer available for defining domains of the spatial operator.[2] Second, it is not immediately clear how to generalize local smoothing estimates for Schrödinger equations (e.g., as in [Duyckaerts 2006, Corollaire (2)]) if the underlying Hamiltonian is not self-adjoint; it is an interesting question however whether, say in the context of Theorem 1.1, some version of local smoothing (with an ϵ-loss) holds for the evolution defined in terms of the inverse Fourier transform of the resolvent $(\Delta + V - \lambda + i0)^{-1}$, $\lambda \in \mathbb{R}$, assuming it exists. (Estimates in the elliptic regime $\lambda \ll -1$ are discussed in [Hintz 2022].) A similar remark applies to Strichartz estimates.

1A. ***Sketch of the proof.*** Consider again the operator $P_{h,z}$ from (1-3); we work locally near $r = 0$, thus on $X = [0, 1)_r \times \mathbb{S}^{n-1}$. In order to achieve a clean separation of the regimes $h \to 0$, $r > 0$ (corresponding to semiclassical analysis away from the cone point) and $h \sim r \to 0$ (where the normal operator $N(P)$ in (1-4) enters and semiclassical tools cease to be applicable), we work on a resolution of the total space $[0, 1)_h \times X$ obtained by a real blow-up of $h = r = 0$,[3]

$$X_{c\hbar} := \big[[0, 1)_h \times X; \{0\} \times \partial X\big].$$

See Figure 1. We wish to regard $h/(h + r)$ as the "true" semiclassical parameter; we proceed to make this more precise.

Note first that for $h = 1$, the rescaling $r^2 P_{1,z}$ is a Fuchs-type operator, or b-differential operator in the terminology of [Melrose 1993], namely a differential operator built out of the vector fields $r\partial_r$ and ∂_y (which span the space of *b-vector fields*), where $y \in \mathbb{R}^{n-1}$ denotes local coordinates on ∂X. In this sense, the rescaled operator $r^2 P_{1,z}$ has elliptic principal part given by $(r D_r)^2 + k^{ij} D_{y^i} D_{y^j}$, where k^{ij} is the inverse metric on $\mathbb{S}^{n-1}$. As h tends to 0, the operator $r^2 P_{h,z}$ is built out of the semiclassical vector fields $hr\partial_r$ and $h\partial_y$ (which span the space of *semiclassical b-vector fields*). In this semiclassical sense (i.e.,

[2]A closely related setting in which many of these points are addressed, with the exception of the analysis at a spatial inverse square singularity, is described in [Hintz 2023]; see in particular §5.3 of that work.

[3]Recall here that the real blow-up gives an invariant way of introducing polar coordinates around $\{0\} \times \partial X$. Thus, a neighborhood of $h = r = 0$ in $X_{c\hbar}$ is diffeomorphic to $[0, 1)_\rho \times \big[0, \frac{\pi}{2}\big]_\theta \times \partial X$ and equipped with a smooth map (the blow-down map) to $[0, 1) \times X = [0, 1) \times ([0, 1) \times \mathbb{S}^{n-1})$ given by $(\rho, \theta, y) \mapsto (\rho \sin\theta, (\rho\cos\theta, y))$, which is a diffeomorphism away from the front face $\rho^{-1}(0)$. In practice, it is more convenient to work with the smooth functions $h+r$, $h/(h+r)$ on $X_{c\hbar}$ instead of ρ, θ.

ignoring terms with extra powers of h), its principal part is

$$r^2 P_{h,z} \sim (hr D_r)^2 + k^{ij} h D_{y^i} h D_{y^j} - r^2 z.$$

The characteristic set, i.e., the zero set of its principal symbol $\xi_{\mathrm{b}\hbar}^2 + |\eta_{\mathrm{b}\hbar}|^2 - r^2$, becomes singular at $r = 0$, which is indicative of the inadequacy of the semiclassical b-setting to capture the behavior of $P_{h,z}$ microlocally near $h = r = 0$ (cf. the above discussion regarding the tension between the geometry and the notion of regularity in [Melrose and Wunsch 2004; Vasy 2008] and subsequent works). The way out is to divide by $(h+r)^2$ and thus consider

$$\left(\frac{r}{h+r}\right)^2 P_{h,z} =: p_{h,z}\left(r, y, \frac{h}{h+r} r D_r, \frac{h}{h+r} D_y\right)$$

as a differential operator built out of $(h/(h+r))\, r\partial_r$ and $h/((h+r))\, \partial_{y^i}$, which are the prototypical *semiclassical cone vector fields* introduced in [Hintz 2022]; see Section 3A. In this sense, the principal part of $(r/(h+r))^2 P_{h,z}$ (i.e., ignoring terms of size $\mathcal{O}(h/(h+r))$) is

$$\left(\frac{r}{h+r}\right)^2 P_{h,z} \sim \left(\frac{h}{h+r} r D_r\right)^2 + k^{ij}\left(\frac{h}{h+r} D_{y^i}\right)\left(\frac{h}{h+r} D_{y^j}\right) - \left(\frac{r}{h+r}\right)^2 z. \tag{1-5}$$

Put differently, we may note that $P_{h,z} \sim (hD_r)^2 + k^{ij}(hr^{-1}D_{y^i})(hr^{-1}D_{y^j}) - z$ is homogeneous of degree 0 with respect to scaling in (h, r), and approximately homogeneous of degree -2 with respect to scaling in r; this suggests expressing $P_{h,z}$ in terms of $r/(h+r) = 1 - h/(h+r)$, leading again to (1-5).

Note that in the regime $h/(h+r) \ll 1$, where we are aiming to use semiclassical methods, the operator (1-5) is now nondegenerate in the sense that its principal symbol

$$p_{0,1}(r, y, \xi, \eta) = \xi^2 + |\eta|^2 - 1$$

(recall $z = 1 + \mathcal{O}(h)$) has a smooth zero set on which $p_{0,1}$ vanishes simply. (The microlocal analysis of *semiclassical cone operators* in the semiclassical regime is thus concerned with tracking amplitudes of oscillations

$$r^{\frac{i}{h/(h+r)}\xi} e^{\frac{i}{h/(h+r)}\eta\cdot y}$$

through the phase space over $X_{\mathrm{c}\hbar}$ — more precisely, over the "semiclassical face" $h/(h+r) = 0$ — whose fiber variables are (ξ, η).)

The semiclassical cone calculus $\Psi_{\mathrm{c}\hbar}(X)$, introduced in [Hintz 2022] and developed further in Section 3, makes this rigorous. It allows for the symbolic analysis of pseudodifferential operators of the form

$$\mathrm{Op}_{\mathrm{c},h}(p) = \text{“}p\left(\frac{h}{h+r}, h+r, y, \frac{h}{h+r} r D_r, \frac{h}{h+r} D_y\right)\text{”}$$

using standard methods from microlocal analysis: there is a semiclassical principal symbol $p(0, r, y, \xi, \eta)$, which is a symbol on the aforementioned phase space (defined rigorously after Lemma 3.2). Moreover, as usual, the commutator $i[\mathrm{Op}_{\mathrm{c},h}(p), \mathrm{Op}_{\mathrm{c},h}(q)]$ is given by the quantization of the Poisson bracket of p and q up to operators with an extra factor of $h/(h+r)$. For the operator $P_{h,z}$ in (1-3), the Hamilton vector field of its principal symbol is nondegenerate except at two submanifolds of critical points over $r = 0$;

these critical sets are saddle points for the Hamilton flow, and are end or starting points of geodesics hitting the cone point or emanating from it. (See Figure 10.) One can thus prove quantitative microlocal propagation and radial point estimates on the associated scale of *weighted semiclassical cone Sobolev spaces*, which measure L^2 norms of derivatives along $(h/(h+r))\,r D_r$, $(h/(h+r))\,D_y$, and which feature weights which are real powers of $r/(h+r)$ and $h+r$.

Remark 1.3 (semiclassical cone ps.d.o.s as tools). The (large) pseudodifferential calculus $\Psi_{c\hbar}(X)$ was introduced in [Hintz 2022] as the space in which inverses and complex powers of elliptic semiclassical cone operators, such as $h^2\Delta+1$, live; the goal there was a precise description of their Schwartz kernels. Here, by contrast, we use semiclassical cone pseudodifferential operators (ps.d.o.s) as *tools* to understand propagation phenomena. Correspondingly, we only need to consider the *small* semiclassical cone calculus, as our analysis will be based on proving estimates, rather than on the construction and usage of parametrices. (Parametrices are typically significantly more challenging to construct [Chin 2022] and are very precise tools; on the flipside, they tend to be less convenient when the need for generalizations or for proofs of sharp mapping properties on various function spaces arises.) Thus, in Section 3, we provide a perspective on $\Psi_{c\hbar}(X)$ which makes it easy to work with in nonelliptic settings.

At this point, we control the semiclassical regularity of solutions of $P_{h,z}u=f$ at $h/(h+r)=0$. This means that we have an estimate of the schematic form

$$\|u\| \lesssim \|P_{h,z}u\| + \|Eu\| + \left\|\frac{h}{h+r}u\right\|, \tag{1-6}$$

where Eu controls u on a transversal to the collection of forward geodesics which encounter $r=0$; the function spaces here are semiclassical cone Sobolev spaces. That is, control of Eu together with weak control of u at $h/(h+r)=0$ (finiteness of the final term) gives stronger control of u (finiteness of the left-hand side), provided the forcing term $P_{h,z}u$ has suitable bounds (e.g., equals 0). Notice that the weights 1 and $h/(h+r)$ in the norms $\|u\|$ and $\|hu/(h+r)\|$ are comparable for $h\sim r$, i.e., at the front face in Figure 1; thus, the estimate (1-6) does not provide control of u in this regime.

In order to control u globally, including at $h=r=0$, one needs to invert the *normal operator* $N(P)$ of $P_{h,z}$, which is the restriction of $P_{h,z}$ to the front face of $X_{c\hbar}$; see (1-3)–(1-4) for a concrete example. The function spaces on which one inverts $N(P)$ need to match the function spaces in which the symbolic propagation estimates are obtained. As already observed in [Hintz 2022] (see also the earlier paper [Loya 2002]) and demonstrated in detail on the level of function spaces in Section 3C, the correct function spaces for $N(P)$ are standard Sobolev spaces when $\hat r=r/h\gtrsim 1$ (i.e., measuring regularity with respect to $D_{\hat r}$ and $\hat r^{-1}D_{y^i}$) and b-Sobolev spaces in $\hat r\lesssim 1$ (i.e., measuring regularity with respect to $\hat r D_{\hat r}$ and D_{y^i}). Following [Melrose 1994], we show in Section 4D that the analysis of $N(P)$ on spaces with variable orders of decay as $\hat r\to\infty$ *precisely* matches the above symbolic analysis which involves variable semiclassical orders (powers of $h/(h+r)$) to accommodate the threshold requirements for propagation into/out of the radial sets; see [Dyatlov and Zworski 2019, Appendix E.4].

We stress the *global* (rather than microlocal or symbolic) nature of the requirement that the normal operator $N(P)$ be invertible; while verifying this in concrete situations is nontrivial, one has many standard techniques at one's disposal (such as boundary-pairing arguments, unique continuation, separation of variables, etc.). We also remark that the necessity to invert model (or "normal") operators for the purpose of controlling solutions of PDE in a singular regime is a typical feature of singular PDE; see for example the role of the invertibility of the Mellin-transformed normal operator family in the asymptotic behavior of waves in [Baskin et al. 2015; 2018; Baskin and Marzuola 2019; 2022].

In combination, the symbolic estimates and the normal operator invertibility provide control of u at both hypersurfaces $h/(h+r)=0$ and $h+r=0$ of $X_{c\hbar}$: schematically, one estimates the final term in (1-6) by $\|(h/(h+r))\,N(P)u\|$ and then replaces $N(P)$ by $P_{h,z}$, thereby committing an error term which vanishes to leading order at the front face $h+r=0$; one obtains

$$\|u\| \lesssim \|P_{h,z}u\| + \|Eu\| + \left\|(h+r)\frac{h}{h+r}u\right\| \lesssim \|P_{h,z}u\| + \|Eu\| + h\|u\|.$$

The final term can be absorbed into the left-hand side when h is sufficiently small. Thus, we have uniform control of u as $h\to 0$. (One can package this into an invertibility statement for a modification of $P_{h,z}$ by placing complex-absorbing potentials away from $r=0$ in the spirit of [Nonnenmacher and Zworski 2009; Wunsch and Zworski 2011; Datchev and Vasy 2012; Vasy 2013]; see Section 4E.)

Remark 1.4 (relation to edge propagation). The proof of symbolic propagation estimates for wave equations on conic or edge manifolds using the edge calculus [Mazzeo 1991], as done in [Melrose and Wunsch 2004, §8] and [Melrose et al. 2008, §11], is closely related, via the Fourier transform in time, to the semiclassical cone Sobolev spaces associated with $\Psi_{c\hbar}(X)$; see Remark 3.4. Going one step further in the comparison, we note that the fine analysis of diffraction of [Melrose and Wunsch 2004] for waves on a conic manifolds uses a normal operator at the cone point which is defined via a rescaled FBI (Fourier–Bros–Iagolnitzer) transform in time; this normal operator is thus equivalent to the operator $N(P)$ considered here, but used in a different manner.

Remark 1.5 (second microlocalization). Writing

$$hrD_r = (h+r)\frac{h}{h+r}rD_r \quad \text{and} \quad hD_y = (h+r)\frac{h}{h+r}D_y$$

suggests that semiclassical conormal regularity at the cone point (regularity under application of hrD_r and hD_y) can be captured on the scale of semiclassical cone Sobolev spaces as well. We present a systematic second microlocal perspective on this in Section 3D, inspired by recent work of Vasy [2021a; 2021b] on the limiting absorption principle on asymptotically conic manifolds (with the conic nature referring to the *large* end of the manifold). In view of the characterization of the quadratic form domain of $h^2\Delta_g+1$ as a semiclassical cone Sobolev space in [Hintz 2022, Theorem 6.1], we can thus eliminate the need of working with a mixed differential-pseudodifferential calculus as in [Baskin and Marzuola 2022], and instead work in a single microlocal framework.

1B. ***Outline of the paper.*** In Section 2, we review basic notions from b- and scattering analysis, with an eye towards the relationship with semiclassical cone analysis. In Section 3, we describe a hands-on perspective on the semiclassical cone algebra $\Psi_{c\hbar}(X)$ with a focus on its use for symbolic computations.

The heart of the paper is Section 4: we define the general class of operators to which the analysis sketched in Section 1A applies (Section 4A) and analyze in detail their symbolic properties (Section 4B), followed by a general analysis of $N(P)$ (Section 4C). We state and prove the main microlocal result, Theorem 4.10, in Section 4D. We prove the diffractive improvement in Section 4F. Finally, Section 5 contains applications of the general theory: a sharp version of propagation estimates for $h^2\Delta_g - 1$ on conic manifolds in Section 5A, and high-energy resolvent estimates for scattering by inverse square potentials and the Dirac–Coulomb equation in Sections 5B and 5C.

In Appendix B, we provide a brief summary of the Sobolev spaces and pseudodifferential calculi used, to aid the reader in keeping track of the (meanings of the) various orders involved.

2. Review of b- and scattering calculi

We denote by X a smooth n-dimensional compact manifold with nonempty, connected, and embedded boundary ∂X. The Lie algebra $\mathcal{V}_{\mathrm{b}}(X) \subset \mathcal{V}(X) = \mathcal{C}^\infty(X; TX)$ of *b-vector fields* consists of all smooth vector fields on X which are tangent to ∂X. The Lie subalgebra $\mathcal{V}_{\mathrm{sc}}(X) \subset \mathcal{V}_{\mathrm{b}}(X)$ of *scattering vector fields* consists of all b-vector fields which vanish, *as b-vector fields*, at ∂X. Thus, if $x \in \mathcal{C}^\infty(X)$ denotes a boundary-defining function (meaning: $\partial X = x^{-1}(0)$, and $\mathrm{d}x$ does not vanish on ∂X), then $\mathcal{V}_{\mathrm{sc}}(X) = x\mathcal{V}_{\mathrm{b}}(X)$. In local coordinates $(x, y) \in [0, \infty) \times \mathbb{R}^{n-1}$ near a point on ∂X, b-vector fields are of the form

$$a(x, y)x\partial_x + \sum_{j=1}^{n-1} b^j(x, y)\partial_{y^j}, \quad a, b^1, \dots, b^{n-1} \in \mathcal{C}^\infty, \tag{2-1}$$

while scattering vector fields are of the form

$$a(x, y)x^2\partial_x + \sum_{j=1}^{n-1} b^j(x, y)x\partial_{y^j}, \quad a, b^1, \dots, b^{n-1} \in \mathcal{C}^\infty.$$

Correspondingly, there are natural vector bundles

$${}^{\mathrm{b}}TX \to X, \quad {}^{\mathrm{sc}}TX \to X, \tag{2-2}$$

isomorphic to TX° over X°, but with local frames (in local coordinates as above) given by $x\partial_x, \partial_{y^1}, \dots, \partial_{y^{n-1}}$ and $x^2\partial_x, x\partial_{y^1}, \dots, x\partial_{y^{n-1}}$ respectively, so that $\mathcal{V}_{\mathrm{b}}(X) = \mathcal{C}^\infty(X; {}^{\mathrm{b}}TX)$ and $\mathcal{V}_{\mathrm{sc}}(X) = \mathcal{C}^\infty(X; {}^{\mathrm{sc}}TX)$. Here, we implicitly use the bundle maps ${}^{\mathrm{b}}TX \to TX$ and ${}^{\mathrm{sc}}TX \to TX$ (which are isomorphisms over X° but not over ∂X) to identify $\mathcal{C}^\infty(X; {}^{\mathrm{b}}TX)$ and $\mathcal{C}^\infty(X; {}^{\mathrm{sc}}TX)$ with subspaces of $\mathcal{C}^\infty(X; TX) = \mathcal{V}(X)$. The dual bundles of (2-2) are the b-cotangent bundle and scattering cotangent bundle, ${}^{\mathrm{b}}T^*X \to X$ and ${}^{\mathrm{sc}}T^*X \to X$, with local frames $\mathrm{d}x/x, \mathrm{d}y^1, \dots, \mathrm{d}y^{n-1}$ and $\mathrm{d}x/x^2, \mathrm{d}y^1/x, \dots, \mathrm{d}y^{n-1}/x$, respectively. (These 1-forms are thus *smooth*, *nonzero* sections of ${}^{\mathrm{b}}T^*X$, resp. ${}^{\mathrm{sc}}T^*X$, down to ∂X.) Writing the canonical 1-form on T^*X° as

$$\xi_{\mathrm{b}}\frac{\mathrm{d}x}{x} + \sum_{j=1}^{n-1}(\eta_{\mathrm{b}})_j\,\mathrm{d}y^j, \quad \text{resp.} \quad \xi_{\mathrm{sc}}\frac{\mathrm{d}x}{x^2} + \sum_{j=1}^{n-1}(\eta_{\mathrm{sc}})_j\frac{\mathrm{d}y^j}{x}, \tag{2-3}$$

thus defines fiber-linear coordinates $(\xi_{\mathrm{b}}, \eta_{\mathrm{b}})$, resp. $(\xi_{\mathrm{sc}}, \eta_{\mathrm{sc}}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, on ${}^{\mathrm{b}}T^*X$, resp. ${}^{\mathrm{sc}}T^*X$. The b-density bundle is denoted by ${}^{\mathrm{b}}\Omega^1X = |\Lambda^n\,{}^{\mathrm{b}}T^*X|$; in local coordinates, its smooth sections are of the form $a|(\mathrm{d}x/x)\mathrm{d}y^1\cdots\mathrm{d}y^{n-1}|$, $a \in \mathcal{C}^\infty$.

The space of finite linear combinations of up to k-fold compositions of elements of $\mathcal{V}_\bullet(X)$, where $\bullet = \mathrm{b}, \mathrm{sc}$, is denoted by $\mathrm{Diff}_\bullet^k(X)$, and we put $\mathrm{Diff}_\bullet(X) = \bigoplus_{k\in\mathbb{N}_0} \mathrm{Diff}_\bullet^k(X)$. The space $\mathrm{Diff}_\mathrm{b}(X)$ gives rise to the notion of *conormality* (relative to a fixed function space) of distributions on X°: concretely, the space

$$\mathcal{A}^\alpha(X) \subset x^\alpha L^\infty(X^\circ)$$

consists of all u so that $Au \in x^\alpha L^\infty(X^\circ)$ for all $A \in \mathrm{Diff}_\mathrm{b}(X)$. More generally, for $\delta \le 1$, one can consider the space

$$\mathcal{A}^\alpha_{1-\delta}(X) \subset x^\alpha L^\infty(X^\circ)$$

of conormal distributions u of type $1-\delta$, defined by the condition that for any $k \in \mathbb{N}_0$ and $A \in \mathrm{Diff}_\mathrm{b}^k(X)$, one has $Au \in x^{\alpha-k\delta} L^\infty(X^\circ)$. (Thus, $\mathcal{A}^\alpha(X) = \mathcal{A}^\alpha_1(X)$.) A more restrictive class than $\mathcal{A}^\alpha(X)$ is the class of *classical conormal distributions*, $\mathcal{A}^\alpha_{\mathrm{cl}}(X)$, which is defined simply as

$$\mathcal{A}^\alpha_{\mathrm{cl}}(X) = x^\alpha \mathcal{C}^\infty(X) \subset \mathcal{A}^\alpha(X).$$

Given an element $u = x^\alpha u_0 \in \mathcal{A}^\alpha_{\mathrm{cl}}(X)$, the function u_0 is thus not merely conormal (regularity under $x\partial_x, \partial_y$), but smooth (regularity under ∂_x, ∂_y).

As an important example, let $E \to X$ denote a smooth real vector bundle of rank N, and consider the radial compactification $\bar{E} \to X$, i.e., the fiber bundle whose fiber $\bar{E}_x$ over $x \in X$ is equal to the radial compactification of $E_x \cong \mathbb{R}^N$ defined by

$$\overline{\mathbb{R}^N} := \big(\mathbb{R}^N \sqcup ([0,\infty)_\rho \times \mathbb{S}^{n-1})\big)/\sim,$$
$$\mathbb{R}^N \setminus \{0\} \ni \rho^{-1}\omega \sim (\rho,\omega) \in [0,\infty) \times \mathbb{S}^{n-1}.$$

Then the total space $\bar{E}$ is a manifold with corners which has two boundary hypersurfaces, $\bar{E}_{\partial X}$ (the radial compactification of $E_{\partial X}$) and $S\bar{E}$ (fiber infinity, locally defined by $\rho = 0$). On $\bar{E}$, we regard only $S\bar{E}$ as a boundary, in the sense that we declare $\mathcal{V}_\mathrm{b}(\bar{E})$ to consist of all smooth vector fields on $\bar{E}$ which are tangent to $S\bar{E}$ (but not necessarily to $\bar{E}_{\partial X}$). For $s \in \mathbb{R}$, we then put

$$S^s(\bar{E}) := \mathcal{A}^{-s}(\bar{E}).$$

One can of course consider variants of this, e.g., requiring elements of $\mathcal{V}_\mathrm{b}(\bar{E})$ to be tangent to *both* boundary hypersurfaces and defining spaces $S^{s,r}(\bar{E})$ which are conormal of weight $-s, -r$ at $S\bar{E}$, $\bar{E}_{\partial X}$, respectively; or one may require classicality at one or both of the boundary hypersurfaces.

2A. *b-pseudodifferential operators.* We denote fiber infinity of the radial compactification $\overline{{}^\mathrm{b}T^*}X$ of ${}^\mathrm{b}T^*X$ by ${}^\mathrm{b}S^*X$. Elements of $S^s(\overline{{}^\mathrm{b}T^*}X)$ will be symbols of b-pseudodifferential operators (of type $(1,0)$, in Hörmander's (ρ,δ)-terminology [1971, §1.1]). Concretely, consider $a \in S^s(\overline{{}^\mathrm{b}T^*}X)$ with support contained in a local coordinate patch near a point on ∂X; thus, for all $i, j \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{N}_0^{n-1}$, there exists a constant $C_{ij\alpha\beta}$ so that

$$|\partial_x^i \partial_y^\alpha \partial_{\xi_\mathrm{b}}^j \partial_{\eta_\mathrm{b}}^\beta a(x,y,\xi_\mathrm{b},\eta_\mathrm{b})| \le C_{ij\alpha\beta}(1+|\xi_\mathrm{b}|+|\eta_\mathrm{b}|)^{s-(j+|\beta|)}.$$

The (left) quantization of a is then defined by

$$(\mathrm{Op_b}(a)u)(x,y) := (2\pi)^{-n}\iiiint \exp\Bigl(i\Bigl(\frac{x-x'}{x'}\xi_{\mathrm b}+(y-y')\cdot\eta_{\mathrm b}\Bigr)\Bigr)\phi\Bigl(\Bigl|\log\frac{x}{x'}\Bigr|\Bigr)\phi(|y-y'|) \times a(x,y,\xi_{\mathrm b},\eta_{\mathrm b})u(x',y')\,\frac{dx'}{x'}\,dy'\,d\xi_{\mathrm b}\,d\eta_{\mathrm b},$$

where $\phi\in\mathcal{C}^\infty_c((-1,1))$ is identically 1 near 0. The cutoff ϕ serves to make $C^{-1}x'\le x\le Cx'$ and $|y-y'|<c$ on the support of the Schwartz kernel of $\mathrm{Op_b}(a)$ for some $C>1$, $c>0$, i.e., it localizes near the diagonal. We define

$$\Psi^s_{\mathrm b}(X) := \mathrm{Op_b}(S^s(\overline{{}^{\mathrm b}T^*}X)) + \Psi^{-\infty}_{\mathrm b}(X).$$

Here, if we write $\pi_{L/R}: X^2\to X$ for the left/right projection, the space $\Psi^{-\infty}_{\mathrm b}(X)$ of residual operators consists of all operators $\dot{\mathcal{C}}^\infty(X)\to\dot{\mathcal{C}}^\infty(X)$ (with $\dot{\mathcal{C}}^\infty(X)$ denoting the space of smooth functions on X vanishing to infinite order at ∂X) whose Schwartz kernels $\kappa\in\mathcal{C}^{-\infty}(X^2;\pi_R^{*}{}^{\mathrm b}\Omega^1X)$ (the dual space of $\dot{\mathcal{C}}^\infty(X^2;\pi_L^{*}{}^{\mathrm b}\Omega^1X)$) pull back to smooth right b-densities on the b-double space[4]

$$X^2_{\mathrm b} := [X^2;(\partial X)^2] \tag{2-4}$$

which vanish to infinite order at the left boundary $\mathrm{lb_b}$ (the lift of $\partial X\times X$) and the right boundary $\mathrm{rb_b}$ (the lift of $X\times\partial X$) but are smooth down to the front face $\mathrm{ff_b}$. (See [Vasy 2018, §6] for more details, and also [Melrose 1993; Grieser 2001].) One often encounters weighted operators as well,

$$\mathrm{Diff}^{k,l}_{\mathrm b}(X) := x^l\,\mathrm{Diff}^k_{\mathrm b}(X),\qquad \Psi^{s,l}_{\mathrm b}(X) := x^l\Psi^s_{\mathrm b}(X).$$

More generally still, one can consider quantizations of symbols which are conormal of order s at ${}^{\mathrm b}S^*X$ and of order l at $\overline{{}^{\mathrm b}T^*_{\partial X}}X$; this level of generality is occasionally useful; see, e.g., [Vasy 2021b, §5] and Section 3D. Given an operator $A\in\Psi^{s,l}_{\mathrm b}(X)$, we denote its Schwartz kernel by K_A.

Elements of $\Psi^{s,l}_{\mathrm b}(X)$ define continuous linear operators on $\dot{\mathcal{C}}^\infty(X)$, and the composition of two b-ps.d.o.s is again a b-ps.d.o., with orders equal to the sum of the orders of the two factors. The principal symbol ${}^{\mathrm b}\sigma_s:\Psi^{s,l}_{\mathrm b}(X)\to(x^lS^s/x^lS^{s-1})(\overline{{}^{\mathrm b}T^*}X)$ is a $*$-homomorphism, and maps commutators into Poisson brackets. In local coordinates (and omitting orders for brevity), this means that for two operators $A,B\in\Psi_{\mathrm b}(X)$ with principal symbols a,b, we have

$$\begin{gathered}{}^{\mathrm b}\sigma(i[A,B]) = \{a,b\} = H_ab,\\ H_a = (\partial_{\xi_{\mathrm b}}a)x\partial_x + (\partial_{\eta_{\mathrm b}}a)\partial_y - (x\partial_xa)\partial_{\xi_{\mathrm b}} - (\partial_ya)\partial_{\eta_{\mathrm b}}.\end{gathered} \tag{2-5}$$

2B. *Scattering pseudodifferential operators.* It is important to consider more general symbol classes than merely $S^s(\overline{{}^{\mathrm{sc}}T^*}X)$ or $x^{-r}S^s(\overline{{}^{\mathrm{sc}}T^*}X)$. Namely, for $\delta\in[0,\frac12)$, we shall consider the class

$$S^{s,r}_{1-\delta,\delta}(\overline{{}^{\mathrm{sc}}T^*}X)$$

of symbols which are conormal at ${}^{\mathrm{sc}}S^*X$ with weight $-s$, and conormal of type $1-\delta$ with weight $-r$ at $\overline{{}^{\mathrm{sc}}T^*_{\partial X}}X$. (The presence of both $1-\delta$ and δ as subscripts follows the classical literature on symbol

[4]For a detailed discussion of real blow-ups such as (2-4), we refer the reader to [Melrose 1996]. See [Hintz 2022, Appendix A] for a brief summary which is sufficient for our purposes.

classes; see, e.g., [Hörmander 1971].) This means that $S^{s,r}_{1-\delta,\delta}(\overline{{}^{\mathrm{sc}}T^*}X)$ consists of all smooth functions a on ${}^{\mathrm{sc}}T^*X$ which over X° are symbols of type $(1,0)$ and order s, i.e., $a|_{T^*X^\circ} \in S^s_{1,0}(T^*X^\circ) = S^s(\overline{T^*}X^\circ)$, and which near ∂X satisfy for all $i, j \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{N}_0^{n-1}$ an estimate

$$|(x\partial_x)^i \partial_y^\alpha \partial_{\xi_{\mathrm{sc}}}^j \partial_{\eta_{\mathrm{sc}}}^\beta a(x, y, \xi_{\mathrm{sc}}, \eta_{\mathrm{sc}})| \le C_{ij\alpha\beta} x^{-r-(i+j+|\alpha|+|\beta|)\delta}(1 + |\xi_{\mathrm{sc}}| + |\eta_{\mathrm{sc}}|)^{s-(j+|\beta|)}.$$

In the case $\delta = 0$, we omit the subscript "$1-\delta, \delta$". We then define the (left) scattering quantization of a by

$$(\mathrm{Op}_{\mathrm{sc}}(a)u)(x, y) := (2\pi)^{-n} \iiiint \exp\biggl(i\biggl[\frac{x-x'}{x^2}\xi_{\mathrm{sc}} + \frac{y-y'}{x}\cdot\eta_{\mathrm{sc}}\biggr]\biggr)\phi\biggl(\biggl|\log\frac{x}{x'}\biggr|\biggr)\phi(|y-y'|) \\ \times a(x, y, \xi_{\mathrm{sc}}, \eta_{\mathrm{sc}})u(x', y')\,\frac{\mathrm{d}x'}{x'^2}\,\frac{\mathrm{d}y'}{x'^{n-1}}\,\mathrm{d}\xi_{\mathrm{sc}}\,\mathrm{d}\eta_{\mathrm{sc}}.$$

(In this generality, scattering ps.d.o.s were introduced by Melrose [1994].) If one were working with global coordinates, one could remove the cutoffs here due to the rapid decay of the partial (in the fiber variables) inverse Fourier transform of a as $|1/x - 1/x'| + |y/x - y'/x'| \to \infty$.[5] We then set

$$\Psi^{s,r}_{\mathrm{sc},1-\delta,\delta}(X) := \mathrm{Op}_{\mathrm{sc}}(S^{s,r}_{1-\delta,\delta}(\overline{{}^{\mathrm{sc}}T^*}X)) + \Psi^{-\infty,-\infty}_{\mathrm{sc}}(X),$$

where $\Psi^{-\infty,-\infty}_{\mathrm{sc}}(X)$ consists of all operators whose Schwartz kernels lie in $\dot{\mathcal{C}}^\infty(X^2; \pi_R^*\Omega^1 X)$. We shall refer to s as the (scattering) differential order, and to r as the (scattering) decay order.

The principal symbol of scattering operators captures their leading-order behavior for large frequencies as well as at ∂X:

$${}^{\mathrm{sc}}\sigma_{s,r} : \Psi^{s,r}_{\mathrm{sc},1-\delta,\delta}(X) \to (S^{s,r}_{1-\delta,\delta}/S^{s-1,r-1+2\delta}_{1-\delta,\delta})(\overline{{}^{\mathrm{sc}}T^*}X).$$

This is a *-homomorphism. Thus, for $A_j \in \Psi^{s_j,r_j}_{\mathrm{sc},1-\delta,\delta}(X)$, $j = 1, 2$, we have

$$[A_1, A_2] \in \Psi^{s_1+s_2-1,r_1+r_2-1+2\delta}_{\mathrm{sc},1-\delta,\delta}(X); \tag{2-6}$$

the principal symbol (which captures the commutator modulo $\Psi^{s_1+s_2-2,r_1+r_2-2+4\delta}_{\mathrm{sc},1-\delta,\delta}(X)$) is given in terms of the principal symbols a_1, a_2 of A_1, A_2 by

$$\begin{gathered}{}^{\mathrm{sc}}\sigma_{s_1+s_2-1,r_1+r_2-1+2\delta}(i[A_1, A_2]) = H_{a_1}a_2, \\ x^{-1}H_{a_1} = (\partial_{\xi_{\mathrm{sc}}}a_1)(x\partial_x + \eta_{\mathrm{sc}}\partial_{\eta_{\mathrm{sc}}}) + (\partial_{\eta_{\mathrm{sc}}}a_1)\partial_y - ((x\partial_x + \eta_{\mathrm{sc}}\partial_{\eta_{\mathrm{sc}}})a_1)\partial_{\xi_{\mathrm{sc}}} - (\partial_y a_1)\partial_{\eta_{\mathrm{sc}}}.\end{gathered} \tag{2-7}$$

We refer the reader to [Vasy 2018, §3] for more details in the special case $X = \overline{\mathbb{R}^n}$, in which case the scattering calculus is the same as the standard ps.d.o. calculus on $\mathbb{R}^n$ for amplitudes which are product-type symbols in the base and fiber variables.

A natural setting where one must work with $\delta > 0$ arises when working with operators which have a variable scattering decay order

$$\mathsf{r} \in \mathcal{C}^\infty(\overline{{}^{\mathrm{sc}}T^*}X).$$

To wit, for $s \in \mathbb{R}$, we define

$$S^{s,\mathsf{r}}(\overline{{}^{\mathrm{sc}}T^*}X)$$

[5]Importantly, one typically does *not* want to localize more sharply to $|1/x - 1/x'| + |y/x - y'/x'| \lesssim 1$ (which is a small neighborhood of the lifted diagonal in the scattering double space; see [Melrose 1994, §21]), as this would thus destroy the leading-order commutativity of the scattering calculus at ∂X.

to consist of all a of the form $a = x^{-\mathsf{r}} a_0$, where $a_0 \in \bigcap_{\delta\in(0,1/2)} S^{s,0}_{1-\delta,\delta}(\overline{{}^{\mathrm{sc}}T^*}X)$. It is easy to check that $S^{s,\mathsf{r}}(\overline{{}^{\mathrm{sc}}T^*}X) \subset \bigcap_{\delta\in(0,1/2)} S^{s,r_0}_{1-\delta,\delta}(\overline{{}^{\mathrm{sc}}T^*}X)$ for any $r_0 > \sup \mathsf{r}$; in fact, differentiating variable-order symbols produces only logarithmic factors in the boundary-defining function x. Thus, we can quantize such symbols, giving rise to the space

$$\Psi^{s,\mathsf{r}}_{\mathrm{sc}}(X) := \mathrm{Op}_{\mathrm{sc}}(S^{s,\mathsf{r}}(\overline{{}^{\mathrm{sc}}T^*}X)) + \Psi^{-\infty,-\infty}_{\mathrm{sc}}(X) \subset \bigcap_{\delta\in(0,\frac{1}{2})} \Psi^{s,r_0}_{\mathrm{sc},1-\delta,\delta}(X).$$

Principal symbols of elements of $\Psi^{s,\mathsf{r}}_{\mathrm{sc}}(X)$ lie in $\big(S^{s,\mathsf{r}}/\bigcap_{\delta>0} S^{s-1,\mathsf{r}-1+2\delta}\big)(\overline{{}^{\mathrm{sc}}T^*}X)$. The (variable) orders are additive under operator composition; this is a consequence of the formula for the full symbol (in local coordinates) of the composition of two ps.d.o.s.

We point out that for fixed $s \in \mathbb{R}$ the space $S^{s,\mathsf{r}}(\overline{{}^{\mathrm{sc}}T^*}X)$ (and thus $\Psi^{s,\mathsf{r}}_{\mathrm{sc}}(X)$) only depends on the restriction of r to $\overline{{}^{\mathrm{sc}}T^*_{\partial X}}X$. Indeed, given $\mathsf{r}' \in \mathcal{C}^\infty(\overline{{}^{\mathrm{sc}}T^*}X)$ with $\mathsf{r}' - \mathsf{r} = 0$ at $\overline{{}^{\mathrm{sc}}T^*_{\partial X}}X$, we can write $\mathsf{r}' - \mathsf{r} = x\mathsf{w}$, $\mathsf{w} \in \mathcal{C}^\infty(\overline{{}^{\mathrm{sc}}T^*}X)$, and therefore $x^{-\mathsf{r}'} = x^{-\mathsf{r}} \exp(-\mathsf{w}x \log x)$; by direct differentiation, one then finds that $\exp(-\mathsf{w}x \log x) \in S^{0,0}_{1-\delta,\delta}(\overline{{}^{\mathrm{sc}}T^*}X)$ for any $\delta > 0$. In view of this, we can define $S^{s,\mathsf{r}}(\overline{{}^{\mathrm{sc}}T^*}X)$ and $\Psi^{s,\mathsf{r}}_{\mathrm{sc}}(X)$, given a variable order

$$\mathsf{r} \in \mathcal{C}^\infty(\overline{{}^{\mathrm{sc}}T^*_{\partial X}}X),$$

to be equal to $S^{s,\tilde{\mathsf{r}}}(\overline{{}^{\mathrm{sc}}T^*}X)$ and $\Psi^{s,\tilde{\mathsf{r}}}_{\mathrm{sc}}(X)$, respectively, where $\tilde{\mathsf{r}} \in \mathcal{C}^\infty(\overline{{}^{\mathrm{sc}}T^*}X)$ is any smooth extension of r.

2C. ***Sobolev spaces.*** We next recall the corresponding scales of weighted Sobolev spaces. We have some flexibility in the choice of the underlying L^2-space. Thus, fix any smooth positive b-density $\mu_0 \in \mathcal{C}^\infty(X; {}^{\mathrm{b}}\Omega^1 X)$, and fix $a_\mu \in \mathbb{R}$. We then set $\mu := x^{\alpha_\mu}\mu_0$ and

$$H^0_{\mathrm{b}}(X;\mu) \equiv L^2_{\mathrm{b}}(X;\mu) \equiv H^0_{\mathrm{sc}}(X;\mu) \equiv L^2_{\mathrm{sc}}(X;\mu) := L^2(X;\mu). \tag{2-8}$$

These spaces are independent of the choice of μ_0 (but not a_μ), up to equivalence of norms; the same will be true for the spaces defined in the sequel. When the density μ is fixed and clear from the context, we drop it from the notation. Let $\bullet = \mathrm{b}, \mathrm{sc}$. For $s \geq 0$, we then let

$$H^s_\bullet(X) := \{u \in H^0_\bullet(X) : Au \in H^0_\bullet(X)\},$$

where $A \in \Psi^s_\bullet(X)$ denotes any fixed elliptic operator. For $s < 0$, we define $H^s_\bullet(X) = (H^{-s}_\bullet(X))^*$ with respect to the $L^2_\bullet(X)$ inner product; an equivalent definition is given by $H^s_\bullet(X) = \{u_1 + Au_2 : u_1, u_2 \in H^0_\bullet(X)\}$, where $A \in \Psi^{-s}_\bullet(X)$ is elliptic. Weighted spaces are defined by

$$H^{s,l}_{\mathrm{b}}(X) = x^l H^s_{\mathrm{b}}(X), \quad H^{s,r}_{\mathrm{sc}}(X) = x^r H^s_{\mathrm{sc}}(X).$$

Finally, we define scattering Sobolev spaces with variable decay orders $\mathsf{r} \in \mathcal{C}^\infty(\overline{{}^{\mathrm{sc}}T^*_{\partial X}}X)$ by taking $r_0 < \inf \mathsf{r}$ and putting

$$H^{s,\mathsf{r}}_{\mathrm{sc}}(X) := \{u \in H^{s,r_0}_{\mathrm{sc}}(X) : Au \in H^0_{\mathrm{sc}}(X)\},$$

where $A \in \Psi^{s,\mathsf{r}}_{\mathrm{sc}}(X)$ is any fixed elliptic operator.

2D. ***b-scattering operators and Sobolev spaces.*** In our application, we shall encounter a compact manifold X whose boundary ∂X has *two* connected components, say H_1, H_2, both of which are embedded.

We can then consider the space $\mathcal{V}_{\mathrm{b,sc}}(X)$ of b-scattering vector fields (which localized to a neighborhood of H_1, resp. H_2, lie in $\mathcal{V}_{\mathrm{b}}$, resp. $\mathcal{V}_{\mathrm{sc}}$), the corresponding b-scattering tangent bundle ${}^{\mathrm{b,sc}}TX$ and its dual ${}^{\mathrm{b,sc}}T^*X$, as well as weighted b-scattering Sobolev spaces,

$$H_{\mathrm{b,sc}}^{s,l,\mathsf{r}}(X), \quad s, l \in \mathbb{R},\ \mathsf{r} \in \mathcal{C}^\infty(\overline{{}^{\mathrm{sc}}T^*_{H_2}}X).$$

Localized to a neighborhood of H_1, its elements lie in $H_{\mathrm{b}}^{s,l}$, and localized to a neighborhood of H_2, they lie in $H_{\mathrm{sc}}^{s,\mathsf{r}}$.

Let us make this even more concrete in the setting which will arise below,

$$X = [0,\infty]_{\hat{x}} \times Y, \quad H_1 = \hat{x}^{-1}(0), \quad H_2 = \hat{x}^{-1}(\infty), \tag{2-9}$$

where we write $[0,\infty]$ for the closure of $[0,\infty)$ inside of $\overline{\mathbb{R}}$; here Y is a compact $(n-1)$-dimensional manifold without boundary. Then $\hat{x}/(\hat{x}+1)$ and $(1+\hat{x})^{-1}$ are defining functions of H_1 and H_2, respectively; hence $\mathcal{V}_{\mathrm{b,sc}}(X) = (1+\hat{x})^{-1}\mathcal{V}_{\mathrm{b}}(X)$. Using local coordinates $y^1, \dots, y^{n-1}$ on an open subset $U \subset Y$, the collection of 1-forms

$$(1+\hat{x})\frac{\mathrm{d}\hat{x}}{\hat{x}}, \quad (1+\hat{x})\mathrm{d}y^1, \quad \dots, \quad (1+\hat{x})\mathrm{d}y^{n-1}$$

is a smooth frame of ${}^{\mathrm{b,sc}}T^*X$ over $[0,\infty] \times U$. Denoting the corresponding fiber-linear coordinates on ${}^{\mathrm{b,sc}}T^*X$ by $(\xi_{\mathrm{b,sc}}, \eta_{\mathrm{b,sc}}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we can then quantize a symbol[6]

$$a \in S^{s,l,r}(\overline{{}^{\mathrm{b,sc}}T^*}X) = \left(\frac{\hat{x}}{\hat{x}+1}\right)^{-l}(1+\hat{x})^r S^s(\overline{{}^{\mathrm{b,sc}}T^*}X)$$

by

$$\begin{aligned}(\mathrm{Op}_{\mathrm{b,sc}}(a)u)(\hat{x}, y) := (2\pi)^{-n} \iiint \exp\biggl(i\biggl(\frac{\hat{x}-\hat{x}'}{\hat{x}\frac{1}{1+\hat{x}}}\xi_{\mathrm{b,sc}} + \frac{y-y'}{\frac{1}{1+\hat{x}}} \cdot \eta_{\mathrm{b,sc}}\biggr)\biggr) \\ \times \phi\biggl(\biggl|\log\frac{\hat{x}}{\hat{x}'}\biggr|\biggr)\phi(|y-y'|)a(\hat{x}, y, \xi_{\mathrm{b,sc}}, \eta_{\mathrm{b,sc}})u(\hat{x}', y') \\ \times \frac{\mathrm{d}\hat{x}'}{\hat{x}'\frac{1}{1+\hat{x}'}}\frac{\mathrm{d}y'}{\bigl(\frac{1}{1+\hat{x}'}\bigr)^{n-1}}\,\mathrm{d}\xi_{\mathrm{b,sc}}\,\mathrm{d}\eta_{\mathrm{b,sc}}.\end{aligned} \tag{2-10}$$

The space $\Psi_{\mathrm{b,sc}}^{s,l,r}(X)$ of b-scattering ps.d.o.s is then the sum

$$\Psi_{\mathrm{b,sc}}^{s,l,r}(X) = \mathrm{Op}_{\mathrm{b,sc}}(S^{s,l,r}(\overline{{}^{\mathrm{b,sc}}T^*}X)) + \Psi_{\mathrm{b,sc}}^{-\infty,l,-\infty}(X).$$

Here,

$$\Psi_{\mathrm{b,sc}}^{-\infty,l,-\infty}(X) = \left(\frac{\hat{x}}{\hat{x}+1}\right)^{-l}\Psi_{\mathrm{b,sc}}^{-\infty,0,-\infty}(X)$$

is defined momentarily. First define the double space

$$X_{\mathrm{b,sc}}^2 := \bigl[\overline{[0,\infty)^2} \times Y^2; (\{0\} \times Y)^2; \Delta \cap (\{\infty\} \times Y)^2\bigr], \tag{2-11}$$

where $\overline{[0,\infty)^2}$ is the radial compactification (equivalently, the closure of $[0,\infty)^2$ inside of $\overline{\mathbb{R}^2}$), and $\Delta \subset \overline{[0,\infty)^2} \times Y^2$ is the diagonal. Then the space $\Psi_{\mathrm{b,sc}}^{-\infty,0,-\infty}(X)$ consists of all operators whose Schwartz

[6]We leave the minor, largely notational, changes to accommodate symbols with variable scattering decay orders r to the reader.

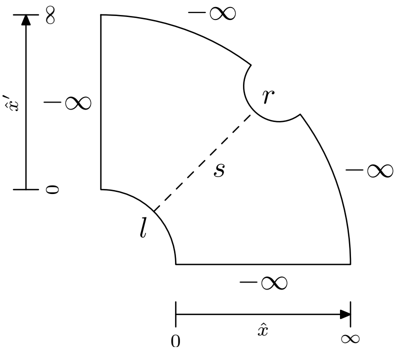

Figure 2. The double space $X^2_{\mathrm{b,sc}}$ without the factor Y^2. The dashed line is the lifted diagonal. Indicated are the symbolic orders of Schwartz kernels of elements of $\Psi^{s,l,r}_{\mathrm{b,sc}}(X)$.

kernels are smooth right b-densities on $X^2_{\mathrm{b,sc}}$ which vanish to infinite order at all boundary hypersurfaces except for the lift of $(\{0\}\times Y)^2$. See Figure 2. Moreover, Schwartz kernels of elements of $\Psi^{s,0,r}_{\mathrm{b,sc}}(X)$ are conormal of order s to the lifted diagonal in $X^2_{\mathrm{b,sc}}$ smoothly down to the lift of $(\{0\}\times Y)^2$, conormal with weight $-r$ down to the lift of $\Delta\cap(\{\infty\}\times Y)^2$, and vanish to infinite order at all other boundary hypersurfaces.

3. Semiclassical cone calculus

We revisit and generalize the algebra $\Psi_{\mathrm{c}\hbar}(X)$ and the associated scale of weighted Sobolev spaces from [Hintz 2022], give a user-friendly treatment of the symbol calculus (including Poisson brackets), and study operators and function spaces with variable (semiclassical) orders and their behavior upon restriction to the transition faces of the semiclassical cone single and double spaces (recalled later in this section). Throughout this section, we denote by X a compact n-dimensional manifold with nonempty, connected, and embedded boundary ∂X. We denote by $x\in\mathcal{C}^\infty(X)$ a boundary-defining function.

3A. ***Vector fields, bundles, Poisson brackets.*** We recall from Section 1A the *semiclassical cone single space*

$$X_{\mathrm{c}\hbar} := \big[[0,1)_h\times X;\{0\}\times\partial X\big],$$

the boundary hypersurfaces of which we denote by cf (conic face, lift of $[0,1)\times\partial X$), tf (transition face, the front face@), and sf (semiclassical face, lift of $\{0\}\times X$). See Figure 3. Defining functions of these boundary hypersurfaces are $x/(x+h)$, $x+h$, and $h/(h+x)$, respectively. On $X_{\mathrm{c}\hbar}\setminus\mathrm{cf}$, it is convenient to use the local defining functions x of tf and h/x of sf.

Definition 3.1 (vector fields). We define the space

$$\mathcal{V}_{\mathrm{c}\hbar}(X_{\mathrm{c}\hbar})$$

of *semiclassical cone vector fields* to consist of all b-vector fields $V\in\mathcal{V}_{\mathrm{b}}(X_{\mathrm{c}\hbar})$ which are horizontal, i.e., tangent to the fibers of $X_{\mathrm{c}\hbar}\to[0,1)_h$, and whose restriction to sf vanishes.

Lemma 3.2 (spanning set). *Identifying a vector field $V\in\mathcal{V}_{\mathrm{b}}(X)$ with its horizontal lift to $X_{\mathrm{c}\hbar}$ along $X_{\mathrm{c}\hbar}\to X$, the space $\mathcal{V}_{\mathrm{c}\hbar}(X_{\mathrm{c}\hbar})$ is spanned over $\mathcal{C}^\infty(X_{\mathrm{c}\hbar})$ by $(h/(h+x))\mathcal{V}_{\mathrm{b}}(X)$. Moreover, given $V,W\in\mathcal{V}_{\mathrm{c}\hbar}(X_{\mathrm{c}\hbar})$, we have $[V,W]\in(h/(h+x))\mathcal{V}_{\mathrm{c}\hbar}(X_{\mathrm{c}\hbar})$.*

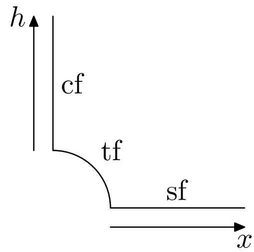

Figure 3. The semiclassical cone single space $X_{c\hbar}$.

This allows us to define the graded ring

$$\mathrm{Diff}_{c\hbar}(X) = \bigoplus_{k\in\mathbb{N}_0} \mathrm{Diff}^k_{c\hbar}(X)$$

of differential operators in the usual manner.

Proof. Directly from the definition, we have $(h/(h+x))\mathcal{V}_b(X) \subset \mathcal{V}_{c\hbar}(X_{c\hbar})$. Conversely, suppose $V \in \mathcal{V}_{c\hbar}(X_{c\hbar})$. Let us work in local coordinates $(x, y) \in [0,\infty)\times\mathbb{R}^{n-1}$ near a point in ∂X. Near cf, we use the local coordinates $(h, \hat{x}, y)$ with $\hat{x} := x/h$. From the definition, we have

$$V = a(h,\hat{x},y)\hat{x}\partial_{\hat{x}} + \sum_{j=1}^{n-1} b^j(h,\hat{x},y)\partial_{y^j}, \tag{3-1}$$

with $a, b^j \in \mathcal{C}^\infty$. Since $\hat{x}\partial_{\hat{x}} = x\partial_x$, this expresses V in the desired form.

Near sf on the other hand, we use $(\hat{h}, x, y)$ with $\hat{h} := h/x$. Since $V \in \mathcal{V}_b(X_{c\hbar})$, we can write

$$V = a(\hat{h},x,y)(x\partial_x - \hat{h}\partial_{\hat{h}}) + \tilde{a}(\hat{h},x,y)\hat{h}\partial_{\hat{h}} + \sum_{j=1}^{n-1} b^j(\hat{h},x,y)\partial_{y^j}.$$

The horizontal nature of V means $0 = Vh = V(x\hat{h}) = \tilde{a}x\hat{h}$, which implies $\tilde{a}\equiv 0$ by continuity from $(X_{c\hbar})^\circ = \{x>0, \hat{h}>0\}$. The vanishing of V at $\hat{h}=0$ as a b-vector field implies, in addition, that $a = \hat{h}a'$ and $b^j = \hat{h}b'_j$ with $a', b'_j \in \mathcal{C}^\infty$. Since the horizontal lifts of $x\partial_x, \partial_{y^j} \in \mathcal{V}_b(X)$ to $X_{c\hbar}$ are equal to $x\partial_x - \hat{h}\partial_{\hat{h}}, \partial_{y^j}$, the claim follows.

Regarding the Lie algebra structure, we compute, for $V, W \in \mathcal{V}_b(X)$,

$$\left[\frac{h}{h+x}V, \frac{h}{h+x}W\right] = \frac{h}{h+x}\left(\frac{h}{h+x}[V,W] + V\left(\frac{h}{h+x}\right)W - W\left(\frac{h}{h+x}\right)V\right).$$

Since $V, W \in \mathcal{V}_b(X_{c\hbar})$, we have $V(h/(h+x)), W(h/(h+x)) \in (h/(h+x))\mathcal{C}^\infty(X_{c\hbar})$. □

There exists a vector bundle

$$^{c\hbar}TX_{c\hbar} \to X_{c\hbar}$$

together with a smooth bundle map $^{c\hbar}TX_{c\hbar} \to {}^{b}TX_{c\hbar}$ so that the space $\mathcal{V}_{c\hbar}(X_{c\hbar})$ is equal to the space of smooth sections of $^{c\hbar}TX_{c\hbar}$. In local coordinates on X, a local frame of $^{c\hbar}TX_{c\hbar}$ is given by (the horizontal lifts to $X_{c\hbar}$ of)

$$\frac{h}{h+x}x\partial_x, \quad \frac{h}{h+x}\partial_{y^1}, \quad \ldots, \quad \frac{h}{h+x}\partial_{y^{n-1}}.$$

We call ${}^{\mathrm{c}\hbar}TX_{\mathrm{c}\hbar}$ the c$\hbar$-*tangent bundle* and its dual ${}^{\mathrm{c}\hbar}T^*X_{\mathrm{c}\hbar}$ the c$\hbar$-*cotangent bundle*, with local frame

$$\frac{x+h}{h}\frac{\mathrm{d}x}{x}, \quad \frac{x+h}{h}\mathrm{d}y^1, \quad \ldots, \quad \frac{x+h}{h}\mathrm{d}y^{n-1}.$$

A choice of local coordinates $(x, y) \in [0, \infty) \times \mathbb{R}^{n-1}$ on an open set $U \subset X$ induces a trivialization of ${}^{\mathrm{c}\hbar}T^*X_{\mathrm{c}\hbar}$ over the preimage of $[0, 1) \times U$ under $X_{\mathrm{c}\hbar} \to X$, with fiber-linear coordinates $(\xi_{\mathrm{c}\hbar}, \eta_{\mathrm{c}\hbar}) \in \mathbb{R} \times \mathbb{R}^{n-1}$ defined by the requirement that the canonical 1-form on T^*X° be equal to

$$\xi_{\mathrm{c}\hbar}\frac{x+h}{h}\frac{\mathrm{d}x}{x} + \sum_{j=1}^{n-1}(\eta_{\mathrm{c}\hbar})_j\frac{x+h}{h}\mathrm{d}y^j. \tag{3-2}$$

In $X_{\mathrm{c}\hbar} \setminus \mathrm{cf}$, where a smooth frame of ${}^{\mathrm{c}\hbar}TX_{\mathrm{c}\hbar}$ is given by $(h/x)x\partial_x$, $(h/x)\partial_{y^1}$, ..., $(h/x)\partial_{y^{n-1}}$, it is computationally simpler to use the fiber-linear coordinates (ξ, η) in which the canonical 1-form takes the form

$$\xi\frac{x}{h}\frac{\mathrm{d}x}{x} + \sum_{j=1}^{n-1}\eta_j\frac{x}{h}\mathrm{d}y^j. \tag{3-3}$$

We compute the form of the Hamilton vector field H_a of a smooth function $a \in \mathcal{C}^\infty({}^{\mathrm{c}\hbar}T^*X_{\mathrm{c}\hbar})$ in these fiber coordinates, and using $(\hat h, x, y)$ with $\hat h = h/x$ as coordinates on the base. In terms of the coordinates on ${}^{\mathrm{b}}T^*X$ used in (2-3), we have $(\xi, \eta) = (h/x)(\xi_{\mathrm{b}}, \eta_{\mathrm{b}})$ and thus, by changing coordinates in the expression (2-5),

$$H_a = \hat h\big((\partial_\xi a)(x\partial_x - \hat h\partial_{\hat h} - \eta\partial_\eta) + (\partial_\eta a)\partial_y - ((x\partial_x - \hat h\partial_{\hat h} - \eta\partial_\eta)a)\partial_\xi a - (\partial_y a)\partial_\eta\big). \tag{3-4}$$

3B. ***Symbols, pseudodifferential operators, Sobolev spaces.*** A simple symbol class for c$\hbar$-operators is $S^s({}^{\mathrm{c}\hbar}T^*X_{\mathrm{c}\hbar}) = \mathcal{A}^{-s}(\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar})$, where we only regard fiber infinity ${}^{\mathrm{c}\hbar}S^*X_{\mathrm{c}\hbar}$ as a boundary, i.e., we require symbols to be smooth down to $\overline{{}^{\mathrm{c}\hbar}T^*_\bullet}X_{\mathrm{c}\hbar}$ for $\bullet = \mathrm{cf}, \mathrm{tf}, \mathrm{sf}$. In practice, we need more general symbols: for $\delta \in \big[0, \tfrac{1}{2}\big)$ and for $s, l, \alpha, b \in \mathbb{R}$, we define

$$S^{s,l,\alpha,b}_{1-\delta,\delta}(\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar}) = \left(\frac{x}{x+h}\right)^{-l}(x+h)^{-\alpha}\left(\frac{h}{h+x}\right)^{-b}S^{s,0,0,0}_{1-\delta,\delta}(\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar})$$

to consist of all symbols which are conormal with weight $-s$ at ${}^{\mathrm{c}\hbar}S^*X_{\mathrm{c}\hbar}$, conormal with weight $-l$ at $\overline{{}^{\mathrm{c}\hbar}T^*_{\mathrm{cf}}}X_{\mathrm{c}\hbar}$ and with weight $-\alpha$ at $\overline{{}^{\mathrm{c}\hbar}T^*_{\mathrm{tf}}}X_{\mathrm{c}\hbar}$, and conormal of type $1-\delta$ at $\overline{{}^{\mathrm{c}\hbar}T^*_{\mathrm{sf}}}X_{\mathrm{c}\hbar}$ with weight $-b$. In the coordinates (3-3), the membership $a \in S^{s,0,0,0}_{1-\delta,\delta}(\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar})$ is equivalent to $a = a(\hat h, x, y, \xi, \eta)$ (with $\hat h = h/x$) satisfying estimates

$$|(x\partial_x)^i\partial_y^\alpha(\hat h\partial_{\hat h})^j\partial_\xi^k\partial_\eta^\beta a(\hat h, x, y, \xi, \eta)| \le C_{ijk\alpha\beta}(1+|\xi|+|\eta|)^{s-(k+|\beta|)}\hat h^{-(i+j+k+|\alpha|+|\beta|)\delta}$$

for all $i, j, k \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{N}_0^{n-1}$; in coordinates $(h, \hat x, y, \xi_{\mathrm{b}}, \eta_{\mathrm{b}})$ on the c$\hbar$-cotangent bundle over $X_{\mathrm{c}\hbar} \setminus \mathrm{sf}$, with $\hat x = x/h$ and with the canonical 1-form given by (2-3), a must satisfy

$$\big|(\hat x\partial_{\hat x})^i\partial_y^\alpha(h\partial_h)^j\partial_{\xi_{\mathrm{b}}}^k\partial_{\eta_{\mathrm{b}}}^\beta a(h, \hat x, y, \xi_{\mathrm{b}}, \eta_{\mathrm{b}})\big| \le C_{ijk\alpha\beta}(1+|\xi_{\mathrm{b}}|+|\eta_{\mathrm{b}}|)^{s-(k+|\beta|)}.$$

See Figure 4. As usual, we omit the subscript "$1-\delta, \delta$" when $\delta = 0$.

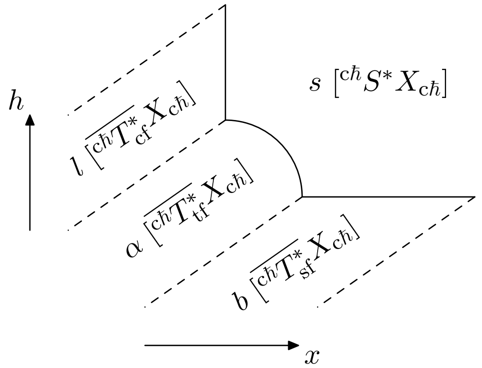

Figure 4. Illustration of $\overline{{}^{c\hbar}T^*}X_{c\hbar}$ (showing only part of the compactified fibers) and the symbol class $S^{s,l,\alpha,b}(\overline{{}^{c\hbar}T^*}X_{c\hbar})$, indicating the orders at the various boundary hypersurfaces of $\overline{{}^{c\hbar}T^*}X_{c\hbar}$.

It is occasionally useful to restrict attention to symbols which are *classical* conormal down to tf, which amounts to replacing $x\partial_x$, $h\partial_h$ in the above symbol estimates (which are for symbols of order 0 at tf) by ∂_x, ∂_h. We denote the corresponding symbol classes with a subscript "cl" as in $S^{s,l,\alpha,b}_{\mathrm{cl}}(\overline{{}^{c\hbar}T^*}X_{c\hbar})$.

As in Section 2, the main use of $\delta > 0$ is to accommodate symbols with variable orders. Here, we only discuss the case of variable semiclassical orders. Thus, consider $\mathsf{b} \in \mathcal{C}^\infty(\overline{{}^{c\hbar}T^*_{\mathrm{sf}}}X_{c\hbar})$, an arbitrary extension of which to an element of $\mathcal{C}^\infty(\overline{{}^{c\hbar}T^*}X_{c\hbar})$ we denote by the same letter; we then put

$$S^{s,l,\alpha,\mathsf{b}}(\overline{{}^{c\hbar}T^*}X_{c\hbar}) := \left\{ \left(\frac{h}{h+x}\right)^{\mathsf{b}} a_0 : a_0 \in \bigcap_{\delta\in(0,\frac{1}{2})} S^{s,l,\alpha,0}_{1-\delta,\delta}(\overline{{}^{c\hbar}T^*}X_{c\hbar}) \right\},$$

which is a subset of $\bigcap_{\delta\in(0,1/2)} S^{s,l,\alpha,b_0}_{1-\delta,\delta}(\overline{{}^{c\hbar}T^*}X_{c\hbar})$ for any $b_0 > \sup \mathsf{b}$.

We now proceed to quantize symbols $a = a(h, x, y, \xi_{c\hbar}, \eta_{c\hbar})$, thereby giving meaning to the formal expression

$$\text{“}\mathrm{Op}_{c,h}(a) = a\left(h, x, y, \frac{h}{h+x} x D_x, \frac{h}{h+x} D_y\right)\text{”}.$$

Thus, fixing $\phi \in \mathcal{C}^\infty_c((-1, 1))$, identically 1 near 0, we define, in local coordinates (x, y) on X,

$$(\mathrm{Op}_{c,h}(a)u)(h, x, y) := (2\pi)^{-n} \iiiint \exp\left(i\left[\frac{x-x'}{x\frac{h}{h+x}}\xi_{c\hbar} + \frac{y-y'}{\frac{h}{h+x}}\cdot\eta_{c\hbar}\right]\right) \times \phi\left(\left|\log\frac{x}{x'}\right|\right)\phi(|y-y'|)a(h, x, y, \xi_{c\hbar}, \eta_{c\hbar})u(h, x', y') \times \frac{dx'}{x'\frac{h}{h+x'}} \frac{dy'}{\left(\frac{h}{h+x'}\right)^{n-1}} \, d\xi_{c\hbar}\, d\eta_{c\hbar} \tag{3-5}$$

for a and u supported in the coordinate chart; for general a, u, one defines $\mathrm{Op}_{c,h}(a)u$ using a partition of unity.

We interpret this in terms of the c$\hbar$-double space

$$X^2_{c\hbar} := \left[[0, 1)_h \times X^2_{\mathrm{b}}; \{0\} \times \mathrm{ff}_{\mathrm{b}}; \{0\} \times \Delta_{\mathrm{b}}\right],$$

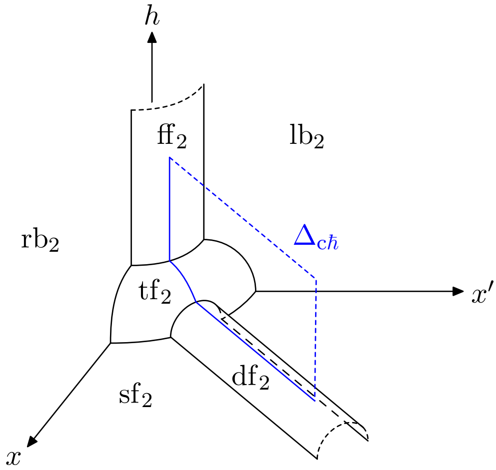

Figure 5. The c$\hbar$-double space $X^2_{\mathrm{c}\hbar}$.

where we denote by $\Delta_{\mathrm{b}} \subset X^2_{\mathrm{b}}$ the lift of the diagonal in X^2 to X^2_{b}; see (2-4) and the subsequent paragraph for the definition of X^2_{b} and its boundary hypersurfaces $\mathrm{lb}_{\mathrm{b}}, \mathrm{ff}_{\mathrm{b}}, \mathrm{rb}_{\mathrm{b}}$. First, recall from[7] [Hintz 2022, Definition 3.1] that $\mathrm{lb}_2, \mathrm{rb}_2, \mathrm{ff}_2, \mathrm{tf}_2, \mathrm{sf}_2$, and df_2 are the lifts of $[0,1) \times \mathrm{lb}_{\mathrm{b}}$, $[0,1) \times \mathrm{rb}_{\mathrm{b}}$, $[0,1) \times \mathrm{ff}_{\mathrm{b}}$, $\{0\} \times \mathrm{ff}_{\mathrm{b}}$, $\{0\} \times X^2_{\mathrm{b}}$, and $\{0\} \times \Delta_{\mathrm{b}}$, respectively; moreover, $\Delta_{\mathrm{c}\hbar}$ denotes the lift of $[0,1) \times \Delta_{\mathrm{b}}$. See Figure 5.

Then the Schwartz kernel of $\mathrm{Op}_{\mathrm{c},h}(a)$ is a conormal distribution of order $s - \frac{1}{4}$ at $\Delta_{\mathrm{c}\hbar}$, conormal down to $\mathrm{ff}_2, \mathrm{tf}_2, \mathrm{df}_2$ with weights $-l, -\alpha, -b$, and vanishes identically in a neighborhood of $\mathrm{lb}_2, \mathrm{rb}_2, \mathrm{sf}_2$.

The composition of two c$\hbar$-quantizations is almost a c$\hbar$-quantization itself; one merely has to allow for additional residual terms: Define the space $\Psi^{-\infty}_{\mathrm{c}\hbar}(X)$ of residual operators to consist of all operators whose Schwartz kernels are conormal sections of the right b-density bundle on $X^2_{\mathrm{c}\hbar}$, with weight 0 at ff_2 and tf_2, and with infinite order vanishing at $\mathrm{lb}_2, \mathrm{rb}_2, \mathrm{df}_2, \mathrm{sf}_2$. We then put

$$\Psi^s_{\mathrm{c}\hbar}(X) := \mathrm{Op}_{\mathrm{c}\hbar}(S^s(\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar})) + \Psi^{-\infty}_{\mathrm{c}\hbar}(X),$$

where $\mathrm{Op}_{\mathrm{c}\hbar} = (\mathrm{Op}_{\mathrm{c},h})_{h\in(0,1)}$; this gives the same space as [Hintz 2022, Definition 3.2]. More generally, we define the quantization of symbols $a \in S^{s,l,\alpha,b}_{1-\delta,\delta}(\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar})$ by the same formula (3-5); the space of residual operators is now

$$\Psi^{-\infty,l,\alpha,-\infty}_{\mathrm{c}\hbar}(X) := \left(\frac{x}{x+h}\right)^{-l} (x+h)^{-\alpha} \Psi^{-\infty}_{\mathrm{c}\hbar}(X).$$

Thus, we can now define the spaces

$$\Psi^{s,l,\alpha,b}_{\mathrm{c}\hbar,1-\delta,\delta}(X) := \mathrm{Op}_{\mathrm{c}\hbar}(S^{s,l,\alpha,b}_{1-\delta,\delta}(\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar})) + \Psi^{-\infty,l,\alpha,-\infty}_{\mathrm{c}\hbar}(X),$$
$$\Psi^{s,l,\alpha,\mathsf{b}}_{\mathrm{c}\hbar}(X) := \mathrm{Op}_{\mathrm{c}\hbar}(S^{s,l,\alpha,\mathsf{b}}(\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar})) + \Psi^{-\infty,l,\alpha,-\infty}_{\mathrm{c}\hbar}(X),$$

where in the second line $\mathsf{b} \in \mathcal{C}^\infty(\overline{{}^{\mathrm{c}\hbar}T^*_{\mathrm{sf}}}X_{\mathrm{c}\hbar})$ is a variable-order function. Their Schwartz kernels can be characterized as being conormal distributions (of order $s - \frac{1}{4}$ and type $(1,0)$) at $\Delta_{\mathrm{c}\hbar}$ which are conormal at ff_2 (with weight $-l$), tf_2 (with weight $-\alpha$), and conormal of type $1-\delta$ at df_2 (with weight $-b$), and

[7] We add subscripts "2" here in order to avoid confusion during the frequent changes between $X_{\mathrm{c}\hbar}$ and $X^2_{\mathrm{c}\hbar}$ later on.

which vanish to infinite order at lb_2, rb_2, sf_2. One can also consider subalgebras which are classical at tf, i.e., the symbols are required to be classical conormal at tf, and the residual operators are required to have classical conormal Schwartz kernels at tf_2; we denote these algebras by a subscript "cl", such as

$$\Psi_{\mathrm{c}\hbar,\mathrm{cl}}^{s,l,\alpha,\mathsf{b}}(X).$$

All such ps.d.o.s define h-dependent families of bounded[8] linear maps on $\dot{\mathcal{C}}^\infty(X)$; compositions of two such ps.d.o.s give a ps.d.o. in the same class, with orders given by the sum of the orders of the two factors. The principal symbol map is

$$^{\mathrm{c}\hbar}\sigma_{s,l,\alpha,b} : \Psi_{\mathrm{c}\hbar,1-\delta,\delta}^{s,l,\alpha,b}(X) \to (S^{s,l,\alpha,b}/S^{s-1,l,\alpha,b-1+2\delta})(\overline{^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar}),$$

similarly for the variable-order spaces (with $\delta > 0$ then arbitrary), and it is a $*$-homomorphism. These facts follow from a minor variation of [Hintz 2022, Proposition 3.9] (using weights instead of index sets), with the statements about principal symbols following by continuity from the corresponding statements for standard semiclassical operators (of type $(1-\delta,\delta)$) in $x > 0$ and b-ps.d.o.s in $h > 0$; we leave the details to the reader. We moreover have, for $A_j \in \Psi_{\mathrm{c}\hbar,1-\delta,\delta}^{s_j,l_j,\alpha_j,b_j}(X)$, $j = 1, 2$, with principal symbols a_j,

$$\mathrm{Op}_{\mathrm{c},h}(i[A_1, A_2]) - \mathrm{Op}_{\mathrm{c},h}(H_{a_1}a_2) \in \Psi_{\mathrm{c}\hbar,1-\delta,\delta}^{s-2,l,\alpha,b-2+4\delta}(X),$$

analogously for variable-order operators. One can evaluate $H_{a_1}a_2$ using the formula (3-4).

Since the principal symbol captures operators to leading order at the union of boundary hypersurfaces $^{\mathrm{c}\hbar}S^*X_{\mathrm{c}\hbar} \cup \overline{^{\mathrm{c}\hbar}T^*_{\mathrm{sf}}}X_{\mathrm{c}\hbar}$, the latter set is also the locus of the elliptic and wave front sets of an operator. Thus, for $A \in \Psi_{\mathrm{c}\hbar}^{s,l,\alpha,b}(X)$, we define

$$\mathrm{Ell}_{\mathrm{c}\hbar}^{s,l,\alpha,b}(A),\ \ \mathrm{WF}'^{l,\alpha}_{\mathrm{c}\hbar}(A) \subset {}^{\mathrm{c}\hbar}S^*X_{\mathrm{c}\hbar} \cup \overline{^{\mathrm{c}\hbar}T^*_{\mathrm{sf}}}X_{\mathrm{c}\hbar}$$

as follows: $\mathrm{Ell}_{\mathrm{c}\hbar}^{s,l,\alpha,b}(A)$ is the set of all ζ so that $^{\mathrm{c}\hbar}\sigma_{s,l,\alpha,b}(A)$ is elliptic in a neighborhood of ζ, and $\mathrm{WF}'^{l,\alpha}_{\mathrm{c}\hbar}(A)$ is the complement of the set of ζ so that the full symbol of A lies in $S^{-\infty,l,\alpha,-\infty}(\overline{^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar})$ when localized to a sufficiently small neighborhood of ζ. In particular, we have $\mathrm{WF}'^{l,\alpha}_{\mathrm{c}\hbar}(A) = \varnothing$ if and only if $A \in \Psi_{\mathrm{c}\hbar}^{-\infty,l,\alpha,-\infty}(X)$. We omit the orders s, l, α, b and l, α when they are clear from the context. The definitions for type-$(1-\delta,\delta)$ and variable-order operators are analogous. See Figure 6.

Finally, we define the corresponding weighted Sobolev spaces. As in (2-8), we first fix a weighted b-density $\mu = x^{\alpha_\mu}\mu_0$, where $0 < \mu_0 \in \mathcal{C}^\infty(X; {}^{\mathrm{b}}\Omega^1 X)$ and $\alpha_\mu \in \mathbb{R}$, and define

$$H^0_{\mathrm{c},h}(X;\mu) := L^2(X;\mu), \quad H^{0,l,\alpha,b}_{\mathrm{c},h}(X;\mu) := \left(\frac{x}{x+h}\right)^l (x+h)^\alpha \left(\frac{h}{h+x}\right)^b H^0_{\mathrm{c},h}(X;\mu).$$

These spaces depend on α_μ, but are independent of μ_0 (up to equivalence of norms). When the choice of μ is clear from the context, we will omit it from the notation. For $s \geq 0$, we then define $H^{s,l,\alpha,b}_{\mathrm{c},h}(X)$ to consist of all $u \in H^{0,l,\alpha,b}_{\mathrm{c},h}(X)$ so that $Au \in H^0_{\mathrm{c},h}(X)$ for any (thus all) elliptic $A \in \Psi^{s,l,\alpha,b}_{\mathrm{c}\hbar}(X)$. We note for $s \in \mathbb{N}_0$ the equivalent characterization

$$H^{s,l,\alpha,b}_{\mathrm{c},h}(X) = \{u \in H^{0,l,\alpha,b}_{\mathrm{c},h}(X) : V_1 \cdots V_j u \in H^{0,l,\alpha,b}_{\mathrm{c},h}(X) \text{ for all } V_i \in \mathcal{V}_{\mathrm{c}\hbar}(X_{\mathrm{c}\hbar}),\ 0 \leq i \leq j \leq s\}.$$

[8] Though not uniformly in h unless $\mathsf{b} \geq 0$.

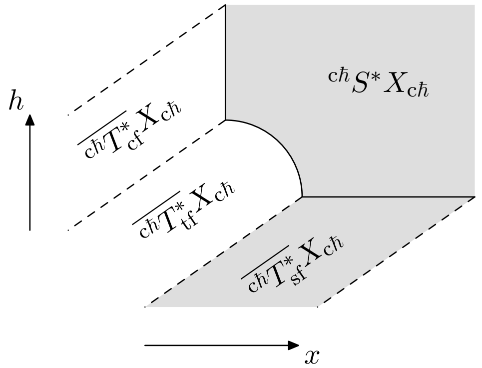

Figure 6. The shaded boundary hypersurfaces are the locus of the elliptic set as well as of operator and distributional wave front sets; see also Figure 4.

For $s < 0$, the space $H^{s,l,\alpha,b}_{c,h}(X)$ can be defined either as the dual space $(H^{-s,-l,-\alpha,-b}_{c,h}(X))^*$, or as the space of all $u_1 + Au_2$ where $u_1, u_2 \in H^{0,l,\alpha,b}_{c,h}(X)$ and $A \in \Psi^{-s}_{c\hbar}(X)$. Lastly, for a variable order $\mathsf{b} \in \mathcal{C}^\infty(\overline{{}^{c\hbar}T^*_{sf}}X_{c\hbar})$, we pick $b_0 < \inf \mathsf{b}$ and put

$$H^{s,l,\alpha,\mathsf{b}}_{c,h}(X) := \{u \in H^{s,l,\alpha,b_0}_{c,h}(X) : Au \in H^0_{c,h}(X)\},$$

where $A \in \Psi^{s,l,\alpha,\mathsf{b}}_{c\hbar}(X)$ is any elliptic operator; the space $H^{s,l,\alpha,\mathsf{b}}_{c,h}(X)$ is independent of the choices of b_0 and A, up to equivalence of norms.

We can define Sobolev wave front sets in the usual manner. Let $l, \alpha \in \mathbb{R}$, and suppose that we are given a distribution $u \in H^{-\infty,l,\alpha,-\infty}_{c,h}(X)$, meaning $u \in H^{-N,l,\alpha,-N}_{c,h}(X)$ for some $N \in \mathbb{R}$. Let $s, b \in \mathbb{R}$. Then

$$\mathrm{WF}^{s,l,\alpha,b}_{c\hbar}(u) \subset {}^{c\hbar}S^*X_{c\hbar} \cup \overline{{}^{c\hbar}T^*_{sf}}X_{c\hbar}$$

is the complement of all α so that there exists an operator $A \in \Psi^{s,l,\alpha,b}_{c\hbar}(X)$, elliptic at α, so that $Au \in H^0_{c,h}(X)$. (The a priori assumption on u is familiar from the definition of the b-wave front set, see, e.g., [Vasy 2018, Definition 6.2], and ensures that one then also has $Bu \in H^0_{c,h}(X)$ for any $B \in \Psi^{s,l,\alpha,b}_{c\hbar}(X)$ with $\mathrm{WF}'_{c\hbar}(B) \subset \mathrm{Ell}'_{c\hbar}(A)$.)

Remark 3.3 (operators on vector bundles). If $E, F \to X$ are smooth vector bundles, one can consider semiclassical cone ps.d.o.s acting between sections of E, F, giving rise to classes $\Psi^s_{c\hbar}(X; E, F)$ and function spaces $H^s_{c,h}(X; E)$ etc. More generally, one can allow E, F to be vector bundles $E, F \to X_{c\hbar}$ over the semiclassical single space, with Schwartz kernels of elements of $\Psi^s_{c\hbar}(X; E, F)$ defined by taking the tensor product of $\Psi^s_{c\hbar}(X)$ over $\mathcal{C}^\infty(X^2_{c\hbar})$ with $\mathcal{C}^\infty(X^2_{c\hbar}; \pi_L^* E \boxtimes \pi_R^* F^*)$, where $\pi_L, \pi_R : X^2_{c\hbar} \to X_{c\hbar}$ are the stretched left and right projections. Using such ps.d.o.s, one can define Sobolev spaces $H^s_{c,h}(X; E)$ etc. in this generality.

Remark 3.4 (relationship with edge Sobolev spaces). For the propagation through cone points in the spacetime setting, many authors [Melrose and Wunsch 2004; Melrose et al. 2008] have utilized Mazzeo's edge algebra [1991]. A typical example is the operator $-D_t^2 + \Delta_g$, where $g = g(x, y, dy)$ is a conic metric on a manifold X with boundary (see (4-2)); upon multiplication by x^2, this is a second-order differential operator, the principal part of which is a Lorentzian signature quadratic form in the collection

(xD_t, xD_x, D_y) of edge vector fields. The membership $u \in H^1_{\mathrm{e}}(\mathbb{R}_t \times X, |dt\, dg|)$ — meaning that u, $xD_t u$, $xD_x u$, $D_y u \in L^2$ — can then be characterized by taking the Fourier transform in t as

$$\hat{u}(\sigma),\ x|\sigma|\hat{u}(\sigma),\ xD_x\hat{u}(\sigma),\ D_y\hat{u}(\sigma) \in L^2(\mathbb{R}_\sigma; L^2(X; |dg|)).$$

Introducing $h = \langle\sigma\rangle^{-1}$, this is equivalent to the $L^2(\mathbb{R}_\sigma; L^2(X))$ membership of $((h+x)/h)\hat{u}$, $xD_x\hat{u}$, $D_y\hat{u}$. Upon multiplication by $h/(h+x)$, we thus find

$$u \in H^1_{\mathrm{e}}(\mathbb{R}_t \times X, |dt\, dg|) \iff \hat{u} \in L^2(\mathbb{R}_\sigma; H^{1,0,0,1}_{\mathrm{c},\langle\sigma\rangle^{-1}}(X; |dg|)),$$

and the respective norms of u and $\hat{u}$ are equivalent. (One can show that similar spectral characterizations of edge Sobolev spaces remain valid also for spaces with weights and with variable differential orders; the details will be given elsewhere.)

3C. ***Restriction to* tf.** Symbolic arguments for the analysis of semiclassical cone PDEs $Pu = f$ can at best control u microlocally at ${}^{c\hbar}S^*X_{c\hbar} \cup \overline{{}^{c\hbar}T^*_{\mathrm{sf}}}X_{c\hbar}$, i.e., modulo errors which are trivial at infinite frequencies and at sf. Crucially however, such errors may well be nontrivial at tf, and thus nontrivial (meaning in particular: not small) as $h \to 0$. To obtain control at tf, one needs to invert the normal operator $N(P)$, defined in [Hintz 2022, §3.1.2] (denoted by $N_{\mathrm{tf}}(P)$ there) and recalled below. The following result, already implicit in the definition of the normal operator in [Hintz 2022, §3.1.2], lays the groundwork for the analysis of $N(P)$.

Lemma 3.5 (restriction to tf: vector fields). *The restriction map*

$$\mathcal{V}_{\mathrm{b}}(X_{c\hbar}) \ni V \mapsto V|_{\mathrm{tf}} \in \mathcal{V}_{\mathrm{b}}(\mathrm{tf})$$

restricts to a surjective map

$$N : \mathcal{V}_{c\hbar}(X_{c\hbar}) \to \mathcal{V}_{\mathrm{b,sc}}(\mathrm{tf}) \tag{3-6}$$

onto the space $\mathcal{V}_{\mathrm{b,sc}}(\mathrm{tf}) = (h/(h+x))\mathcal{V}_{\mathrm{b}}(\mathrm{tf})$ *of vector fields which are b-vector fields near* $\mathrm{tf} \cap \mathrm{cf}$ *and scattering vector fields near* $\mathrm{tf} \cap \mathrm{sf}$. *The map* (3-6) *induces bundle isomorphisms*

$${}^{c\hbar}T_{\mathrm{tf}}X_{c\hbar} \cong {}^{\mathrm{b,sc}}T\mathrm{tf}, \qquad {}^{c\hbar}T^*_{\mathrm{tf}}X_{c\hbar} \cong {}^{\mathrm{b,sc}}T^*\mathrm{tf}. \tag{3-7}$$

Proof. Near $\mathrm{tf} \setminus \mathrm{sf}$, we write $V \in \mathcal{V}_{c\hbar}(X_{c\hbar})$ in the coordinates $(h, \hat{x}, y)$, with $\hat{x} = x/h$, as

$$V = a(h, \hat{x}, y)\hat{x}\partial_{\hat{x}} + \sum_{j=1}^{n-1} b^j(h, \hat{x}, y)\partial_{y^j}; \tag{3-8}$$

see (3-1). The restriction to tf, in local coordinates given by $h^{-1}(0) = [0, \infty)_{\hat{x}} \times \mathbb{R}^{n-1}_y$, is the b-vector field

$$V|_{\mathrm{tf}} = a(0, \hat{x}, y)\hat{x}\partial_{\hat{x}} + \sum_{j=1}^{n-1} b^j(0, \hat{x}, y)\partial_{y^j}. \tag{3-9}$$

Conversely, every b-vector field W on $[0, \infty) \times \mathbb{R}^{n-1}$ can be written in the form $W = a(\hat{x}, y)\hat{x}\partial_{\hat{x}} + \sum_{j=1}^{n-1} b^j(\hat{x}, y)\partial_{y^j}$, and upon taking $a(h, \hat{x}, y)$ and $b^j(h, \hat{x}, y)$ to be smooth functions which restrict at $h = 0$ to the coefficients $a(\hat{x}, y)$ and $b^j(\hat{x}, y)$ of W defines a $c\hbar$-vector field V through (3-8) whose restriction (3-9) to tf is precisely W. We remark moreover that (3-9) vanishes if and only if $a=h\tilde a$ and $b^j=h\tilde b^j$, where $\tilde a,\tilde b^j$ are smooth functions of $(h,\hat x,y)$, i.e., if and only if V vanishes at tf as a c$\hbar$-vector field.

On tf$\setminus$cf on the other hand, and using coordinates $(\hat h,x,y)$ with $\hat h=h/x$, we can write $V\in\mathcal{V}_{c\hbar}(X_{c\hbar})$ as

$$V=\hat h a(\hat h,x,y)(x\partial_x-\hat h\partial_{\hat h})+\sum_{j=1}^{n-1}\hat h b^j(\hat h,x,y)\partial_{y^j}, \tag{3-10}$$

with smooth coefficients $a,b^1,\dots,b^{n-1}$. Restriction to tf, which in these coordinates is given by $x=0$, produces

$$V|_{\mathrm{tf}}=-a(\hat h,0,y)\hat h^2\partial_{\hat h}+\sum_{j=1}^{n-1}b^j(\hat h,0,y)\hat h\partial_{y^j}, \tag{3-11}$$

which is a scattering vector field on $[0,\infty)_{\hat h}\times\mathbb{R}^{n-1}_y$, as claimed. Conversely, every scattering vector field W on $[0,\infty)\times\mathbb{R}^{n-1}$ can be written in the form $W=a(\hat h,y)\hat h^2\partial_{\hat h}+\sum_{j=1}^{n-1}b^j(\hat h,y)\hat h\partial_{y^j}$, and upon taking $a(\hat h,x,y)$ and $b^j(\hat h,x,y)$ to be smooth functions which restrict at $x=0$ to the coefficients $-a(\hat h,y)$ and $b^j(\hat h,y)$ of W defines a c$\hbar$-vector field V through (3-10) whose restriction (3-11) to tf is W. Note also that (3-11) vanishes if and only if $a=x\tilde a$ and $b^j=x\tilde b^j$ for smooth $\tilde a,\tilde b^j$, i.e., if and only if V vanishes at tf as a c$\hbar$-vector field.

The surjectivity of (3-6) follows from these two calculations via a partition of unity subordinate to a cover $X_{c\hbar}=U\cup V$, where $U\cap\mathrm{sf}=\varnothing$ and $V\cap\mathrm{cf}=\varnothing$. Our arguments above also prove that $\ker N=(x+h)\mathcal{V}_{c\hbar}(X_{c\hbar})$. Thus, we have an isomorphism

$$\mathcal{V}_{c\hbar}(X_{c\hbar})/(x+h)\mathcal{V}_{c\hbar}(X_{c\hbar})\to\mathcal{V}_{\mathrm{b,sc}}(\mathrm{tf}).$$

This induces the first isomorphism in (3-7) abstractly as follows: if $p\in\mathrm{tf}$, then

$${}^{\mathrm{b,sc}}T_p\mathrm{tf}=\mathcal{V}_{\mathrm{b,sc}}(\mathrm{tf})/\mathcal{I}_p\mathcal{V}_{\mathrm{b,sc}}(\mathrm{tf}),$$

where $\mathcal{I}_p\subset\mathcal{C}^\infty(\mathrm{tf})$ is the ideal of functions vanishing at p. The ideal in $\mathcal{C}^\infty(X_{c\hbar})/(x+h)\mathcal{C}^\infty(X_{c\hbar})$ of elements that restrict to an element of $\mathcal{I}_p$ at tf is $\mathcal{J}_p/(x+h)\mathcal{C}^\infty(X_{c\hbar})$, where $\mathcal{J}_p\subset\mathcal{C}^\infty(X_{c\hbar})$ is the ideal of functions vanishing at p. Since $\mathcal{V}_{c\hbar}(X_{c\hbar})/\mathcal{J}_p\mathcal{V}_{c\hbar}(X_{c\hbar})={}^{c\hbar}T_pX_{c\hbar}$, we obtain (3-7). Concretely, the first isomorphism in (3-7) maps $\hat x\partial_{\hat x}$, ∂_{y^j} in the coordinates used in (3-8) to $\hat x\partial_{\hat x}$, ∂_{y^j} (cf. (3-9)), and $\hat h(x\partial_x-\hat h\partial_{\hat h})$, $\hat h\partial_{y^j}$ in the coordinates used in (3-10) to $-\hat h^2\partial_{\hat h}$, $\hat h\partial_{y^j}$ (cf. (3-11)). □

The map (3-6) induces a surjective map

$$N:\Big(\frac{x}{x+h}\Big)^{-l}\Big(\frac{h}{h+x}\Big)^{-b}\mathrm{Diff}^k_{c\hbar}(X)\to\Big(\frac{\hat x}{\hat x+1}\Big)^{-l}(\hat x+1)^b\,\mathrm{Diff}^k_{\mathrm{b,sc}}(\mathrm{tf}),\quad \hat x:=\frac{x}{h}, \tag{3-12}$$

into weighted b-scattering differential operators on tf. More generally:

Lemma 3.6 (restriction to the transition face: ps.d.o.s). *Let $s,l,b\in\mathbb{R}$. Restriction to* $\mathrm{tf}_2\subset X^2_{c\hbar}$ *induces a surjective map* $N:\Psi^{s,l,0,b}_{c\hbar,\mathrm{cl}}(X)\to\Psi^{s,l,b}_{\mathrm{b,sc}}(\mathrm{tf})$.[9] *More generally, if* $\mathsf{b}\in\mathcal{C}^\infty(\overline{{}^{c\hbar}T^*_{\mathrm{sf}}}X_{c\hbar})$, *then*

[9] Recall that the subscript "cl" refers to classicality at tf_2, i.e., smoothness of the Schwartz kernels down to tf_2.

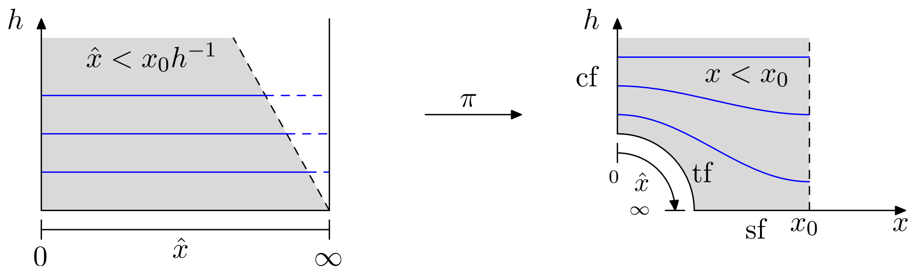

Figure 7. Illustration of the map π defined in (3-14).

$\mathsf{b}' := \mathsf{b}|_{\mathrm{tf}\cap\mathrm{sf}} \in \mathcal{C}^\infty(\overline{{}^{\mathrm{b,sc}}T^*_{\mathrm{tf}\cap\mathrm{sf}}}\,\mathrm{tf})$, *and restriction to* $\mathrm{tf}_2 \subset X^2_{\mathrm{c}\hbar}$ *induces a surjective map*

$$N : \Psi^{s,l,0,\mathsf{b}}_{\mathrm{c}\hbar,\mathrm{cl}}(X) \to \Psi^{s,l,\mathsf{b}'}_{\mathrm{b,sc}}(\mathrm{tf}). \tag{3-13}$$

Proof. This can be proved entirely on the level of Schwartz kernels, since memberships in $\Psi_{\mathrm{c}\hbar}$ or $\Psi_{\mathrm{b,sc}}$ are characterized as conormal distributions with conormal regularity at various boundary hypersurfaces. The point then is that tf_2 is naturally diffeomorphic to the double space $\mathrm{tf}^2_{\mathrm{b,sc}}$ in the notation of (2-11), where we note that $\mathrm{tf} \cong [0,\infty]_{\hat{x}} \times \partial X$ is indeed of the form (2-9). This is the route taken in [Hintz 2022, §3.1.2].

Alternatively, we can proceed explicitly for the symbolically nontrivial part using the quantization map (3-5), and use the Schwartz kernel perspective only to deduce the surjectivity of the restriction map for residual operators, $\Psi^{-\infty,l,0,-\infty}_{\mathrm{c}\hbar,\mathrm{cl}}(X) \to \Psi^{-\infty,l,-\infty}_{\mathrm{b,sc}}(\mathrm{tf})$. Indeed, on the level of symbols, note that with $\hat{x} = x/h$, we have

$$\begin{aligned} S^{s,l,0,b}(\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar}) &= \left(\frac{x}{x+h}\right)^{-l}\left(\frac{h}{h+x}\right)^{-b} S^{s,0,0,0}(\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar}) \\ &= \left(\frac{\hat{x}}{\hat{x}+1}\right)^{-l}(\hat{x}+1)^b S^{s,0,0,0}(\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar}), \end{aligned}$$

and hence Lemma 3.5 implies that restriction to $\overline{{}^{\mathrm{c}\hbar}T^*_{\mathrm{tf}}}X_{\mathrm{c}\hbar}$ induces a surjective map

$$S^{s,l,0,b}_{\mathrm{cl}}(\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar}) \to S^{s,l,b}(\overline{{}^{\mathrm{b,sc}}T^*}\mathrm{tf}).$$

But changing variables in the c$\hbar$-quantization (3-5) to $\hat{x} = x/h$, $\hat{x}' = x'/h$ produces precisely the b-scattering quantization (2-10). This proves the lemma for constant orders; the proof in the variable-order case is the same. □

As a consequence, we can relate semiclassical cone Sobolev spaces to b-scattering Sobolev spaces. In order to state this, we fix a collar neighborhood $\mathcal{U} = [0, x_0)_x \times \partial X$ of ∂X, and define the map

$$\pi : [0,1)_h \times [0,\infty)_{\hat{x}} \times \partial X \to [0,1)_h \times [0,\infty)_x \times \partial X, \quad \pi(h,\hat{x},y) = (h, h\hat{x}, y). \tag{3-14}$$

Note that $(h\hat{x}, y) \in \mathcal{U}$ if and only if $\hat{x} < x_0 h^{-1}$. Now, since $\mathrm{tf} = [0,\infty]_{\hat{x}} \times \partial X$, the domain of π is $[0,1)_h \times (\mathrm{tf} \setminus \mathrm{sf})$; moreover, for any fixed $\hat{x} \in [0,\infty)$ and $y \in \partial X$, the point $\pi(h,\hat{x},y)$ converges, as $h \searrow 0$, to the point $(\hat{x}, y)$ on the transition face of $X_{\mathrm{c}\hbar}$. See Figure 7.

With this setup, we have:

Corollary 3.7 (restriction to tf: Sobolev spaces). *Suppose* $\chi \in \mathcal{A}^0([0,1)_h \times X)$ *has compact support in* $[0,1) \times \mathcal{U}$. *Let* $0 < \mu_0 \in \mathcal{C}^\infty(X; {}^{\mathrm{b}}\Omega^1 X)$ *and* $0 < \hat\mu_0 \in \mathcal{C}^\infty(\mathrm{tf}; {}^{\mathrm{b}}\Omega^1 X)$, *let* $\alpha_\mu \in \mathbb{R}$, *and put*

$$\mu := x^{\alpha_\mu} \mu_0, \quad \hat\mu := \hat x^{\alpha_\mu} \hat\mu_0.$$

(1) (constant orders) *Let* $s, l, \alpha, b \in \mathbb{R}$. *Then*

$$\|\chi u\|_{H^{s,l,\alpha,b}_{\mathrm{c},h}(X;\mu)} \sim h^{\frac{\alpha_\mu}{2}-\alpha} \|\pi^*(\chi u)\|_{H^{s,l,b-\alpha}_{\mathrm{b,sc}}(\mathrm{tf};\hat\mu)}, \quad u \in H^{s,l,\alpha,b}_{\mathrm{c},h}(X;\mu), \tag{3-15}$$

in the sense that the left-hand side is bounded by a uniform constant (*independent of* h *and* u) *times the right-hand side and vice versa.*

(2) (variable orders) *Let* $\mathsf{b} \in \mathcal{C}^\infty(\overline{{}^{\mathrm{c}\hbar}T^*_{\mathrm{sf}}}X_{\mathrm{c}\hbar})$ *denote a variable order, and let* $\mathsf{b}' := \mathsf{b}|_{\mathrm{tf}\cap\mathrm{sf}}$. *If* b *is invariant under the lift of the dilation action* $(x,y) \mapsto (\lambda x, y)$ *in* $\mathcal{U}$, *then*

$$\|\chi u\|_{H^{s,l,\alpha,\mathsf{b}}_{\mathrm{c},h}(X;\mu)} \sim h^{\frac{\alpha_\mu}{2}-\alpha} \|\pi^*(\chi u)\|_{H^{s,l,\mathsf{b}'-\alpha}_{\mathrm{b,sc}}(\mathrm{tf};\hat\mu)}.$$

For general b, *and given* $\delta > 0$, *there exists* $x_0(\delta) \in (0, x_0]$ *so that for* $\chi \in \mathcal{C}^\infty_{\mathrm{c}}([0, x_0(\delta)) \times \partial X)$, *we have*

$$C^{-1} h^{\frac{\alpha_\mu}{2}-\alpha} \|\pi^*(\chi u)\|_{H^{s,l,\mathsf{b}'-\alpha-\delta}_{\mathrm{b,sc}}(\mathrm{tf};\hat\mu)} \le \|\chi u\|_{H^{s,l,\alpha,\mathsf{b}}_{\mathrm{c},h}(X;\mu)} \le C h^{\frac{\alpha_\mu}{2}-\alpha} \|\pi^*(\chi u)\|_{H^{s,l,\mathsf{b}'-\alpha+\delta}_{\mathrm{b,sc}}(\mathrm{tf};\hat\mu)}, \tag{3-16}$$

where C *does not depend on* h, u.

Proof. By factoring out $h^{-\alpha}$, it suffices to consider the case $\alpha = 0$. Consider first the case of constant orders. Factoring out the appropriate powers of $x/(x+h) = \hat x/(\hat x + 1)$ and $h/(h+x) = (\hat x + 1)^{-1}$, we reduce to the case $l = b = 0$. For $s = 0$, the equivalence of norms (3-15) then follows from

$$\iint |\chi u|^2 x^{\alpha_\mu} \frac{\mathrm{d}x}{x} \,\mathrm{d}y = \iint |\pi^*(\chi u)|^2 h^{\alpha_\mu} \hat x^{\alpha_\mu} \frac{\mathrm{d}\hat x}{\hat x} \,\mathrm{d}y.$$

For $s \in \mathbb{Z}$, the conclusion follows from (3-12); for general $s \in \mathbb{R}$, use duality and interpolation.

For variable semiclassical orders b (and still with $\alpha = 0$), and under the assumption of dilation-invariance near tf_2, we first pick an elliptic operator $\hat A \in \Psi^{s,l,\mathsf{b}'}_{\mathrm{b,sc}}(\mathrm{tf})$; we can then extend its Schwartz kernel to a neighborhood of tf_2 to be constant along the orbits of $(h,x) \mapsto (\lambda h, \lambda x)$, and then extend it further to an elliptic operator $A \in \Psi^{s,l,0,\mathsf{b}}_{\mathrm{c}\hbar,\mathrm{cl}}(X)$. In this manner, we obtain a right inverse (with special properties) of the restriction map (3-13). For any fixed $b_0 < \inf \mathsf{b}$, we thus have

$$\begin{aligned} \|\chi u\|^2_{H^{s,l,0,\mathsf{b}}_{\mathrm{c},h}(X;\mu)} &\sim \|\chi u\|^2_{H^{s,l,0,b_0}_{\mathrm{c},h}(X;\mu)} + \|A(\chi u)\|^2_{H^0_{\mathrm{c},h}(X;\mu)} \\ &\sim h^{\frac{\alpha_\mu}{2}} (\|\pi^*(\chi u)\|^2_{H^{s,l,b_0}_{\mathrm{b,sc}}(\mathrm{tf};\hat\mu)} + \|\hat A(\pi^*(\chi u))\|^2_{H^0_{\mathrm{b,sc}}(\mathrm{tf};\hat\mu)}) \\ &\sim h^{\frac{\alpha_\mu}{2}} \|\chi u\|^2_{H^{s,l,\mathsf{b}'}_{\mathrm{b,sc}}(\mathrm{tf};\hat\mu)}. \end{aligned}$$

The lossy estimate (3-16) is a consequence of this, as the dilation-invariant extension of $\mathsf{b}' - \delta$, resp. $\mathsf{b}' + \delta$ is less, resp. greater than b in a sufficiently small (depending on b and δ) neighborhood of ∂X. ☐

3D. ***Relative semiclassical b-regularity.*** We now make Remark 1.5 precise and demonstrate how to combine the notions of semiclassical cone regularity and semiclassical b- (i.e., conormal) regularity. Recall here that a semiclassical b-vector field is a particular type of h-dependent b-vector field on X; namely, it is a vector field on $[0,1)_h \times X$ which is horizontal and which vanishes at $h=0$. In local coordinates as in (2-1), such a vector fields can be written as

$$a(h,x,y)hx\partial_x + \sum_{j=1}^{n-1} b^j(h,x,y)h\partial_{y^j}. \tag{3-17}$$

The main insight is that the semiclassical b-algebra can be embedded into the semiclassical cone algebra via a phase space resolution; see Lemma 3.8 below. This can alternatively be phrased as a second microlocalization of the semiclassical b-algebra at the zero section over ∂X at $h=0$; see Remark 3.10.

First, we explain a slightly nonstandard perspective on semiclassical (b-)phase spaces. Let X be an n-dimensional manifold with nonempty embedded boundary ∂X. Thus, paralleling Definition 3.1, we define

$$\begin{aligned} X_{\mathrm{b}\hbar} &:= [0,1) \times X, \\ \mathcal{V}_{\mathrm{b}\hbar}(X_{\mathrm{b}\hbar}) &:= \{V \in \mathcal{V}_{\mathrm{b}}(X_{\mathrm{b}\hbar}) : V \text{ is horizontal, } V|_{h=0} = 0\}. \end{aligned}$$

It is then easy to see that $\mathcal{V}_{\mathrm{b}\hbar}(X_{\mathrm{b}\hbar})$ is spanned over $\mathcal{C}^\infty(X_{\mathrm{b}\hbar})$ by hV for $V \in \mathcal{V}_{\mathrm{b}}(X)$ (see (3-17)), where we identify V with an h-independent horizontal vector field on $X_{\mathrm{b}\hbar}$. We then have $\mathcal{V}_{\mathrm{b}\hbar}(X_{\mathrm{b}\hbar}) = \mathcal{C}^\infty(X_{\mathrm{b}\hbar}; {}^{\mathrm{b}\hbar}TX_{\mathrm{b}\hbar})$ for a rank-n vector bundle

$${}^{\mathrm{b}\hbar}TX_{\mathrm{b}\hbar} \to X_{\mathrm{b}\hbar}.$$

In local coordinates $[0,\infty)_x \times \mathbb{R}^{n-1}_y$, a smooth frame of this bundle is $hx\partial_x, h\partial_{y^1}, \dots, h\partial_{y^{n-1}}$. We can introduce fiber-linear coordinates on the dual bundle ${}^{\mathrm{b}\hbar}T^*X_{\mathrm{b}\hbar}$ by writing the canonical 1-form as

$$\xi_{\mathrm{b}\hbar} h^{-1}\frac{\mathrm{d}x}{x} + \sum_{j=1}^{n-1} (\eta_{\mathrm{b}\hbar})_j h^{-1}\mathrm{d}y^j.$$

Thus, for example, the symbol of the semiclassical b-differential operator hxD_x is $\xi_{\mathrm{b}\hbar}$.[10] Denote fiber infinity of the radial compactification $\overline{{}^{\mathrm{b}\hbar}T^*}X_{\mathrm{b}\hbar}$ by ${}^{\mathrm{b}\hbar}S^*X_{\mathrm{b}\hbar}$. Given a symbol

$$a \in S^{s,l,b}(\overline{{}^{\mathrm{b}\hbar}T^*}X_{\mathrm{b}\hbar}) = x^{-l}h^{-b}S^{s,0,0}(\overline{{}^{\mathrm{b}\hbar}T^*}X_{\mathrm{b}\hbar})$$

(i.e., a is conormal with weight $-s$ at ${}^{\mathrm{b}\hbar}S^*X_{\mathrm{b}\hbar}$, conormal with weight $-l$ at $\overline{{}^{\mathrm{b}\hbar}T^*_{[0,1)\times X}}X_{\mathrm{b}\hbar}$, and conormal with weight $-b$ at $h=0$), we can then define the semiclassical quantization

$$(\mathrm{Op}_{\mathrm{b},h}(a)u)(x,y) := (2\pi)^{-n} \iiiint \exp\Bigl(i\Bigl[\frac{x-x'}{hx}\xi_{\mathrm{b}\hbar} + \frac{y-y'}{h}\cdot\eta_{\mathrm{b}\hbar}\Bigr]\Bigr)\phi\Bigl(\Bigl|\log\frac{x}{x'}\Bigr|\Bigr)\phi(|y-y'|) \\ a(x,y,\xi_{\mathrm{b}\hbar},\eta_{\mathrm{b}\hbar})u(x',y')\,\frac{\mathrm{d}x'}{hx'}\,\frac{\mathrm{d}y'}{h^{n-1}}\,\mathrm{d}\xi_{\mathrm{b}\hbar}\,\mathrm{d}\eta_{\mathrm{b}\hbar}.$$

---

[10] By contrast, the standard convention is to introduce fiber-linear coordinates $(\xi_{\mathrm{b}},\eta_{\mathrm{b}})$ on ${}^{\mathrm{b}}T^*X$ as in (2-3) and *declare* the principal symbol of hxD_x to be ξ_{b}; the translation to the present convention is accomplished by using (the adjoint of) the bundle isomorphism ${}^{\mathrm{b}\hbar}TX_{\mathrm{b}\hbar} \cong [0,1)_h \times {}^{\mathrm{b}}TX$ induced by division by h (i.e., induced by the map $\mathcal{V}_{\mathrm{b}\hbar}(X_{\mathrm{b}\hbar}) \ni V \mapsto (h^{-1}V)_{h\in[0,1)}$).

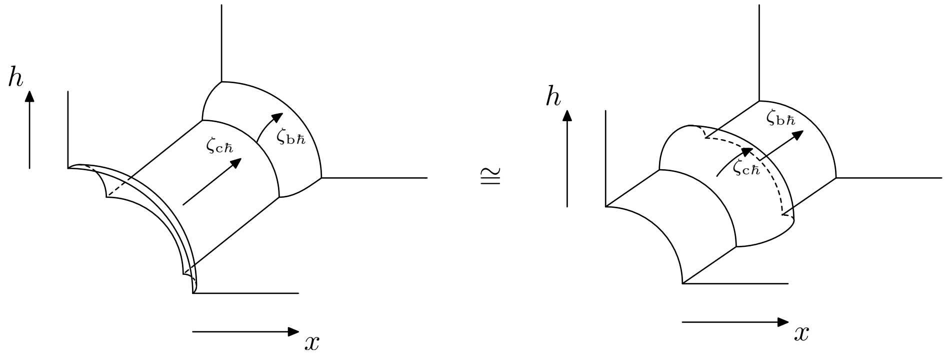

Figure 8. Left: The resolution ${}^{\mathrm{cb}\hbar}\overline{T^*}X_{\mathrm{c}\hbar}$ of the fiber-compactified semiclassical cone phase space at fiber infinity over tf; see (3-19). (Unlike in Figure 4, we show the full compactified fibers here.) Right: The resolution of the fiber-compactified semiclassical b-phase space at $x = h = 0$ and at the zero section over $x = h = 0$; see (3-20).

If we make the change of variables

$$(\xi_{\mathrm{b}\hbar}, \eta_{\mathrm{b}\hbar}) = (x+h)(\xi_{\mathrm{c}\hbar}, \eta_{\mathrm{c}\hbar}), \tag{3-18}$$

see (3-2), this exactly matches the c$\hbar$-quantization (3-5). The key point is now that this match has a clean interpretation on the level of symbol classes on a joint resolution of the semiclassical cone and b-phase spaces:

Lemma 3.8 (relationship between semiclassical cone and b-phase spaces). *Define the* cb$\hbar$*-phase space*

$$\overline{{}^{\mathrm{cb}\hbar}T^*}X_{\mathrm{c}\hbar} := [\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar}; {}^{\mathrm{c}\hbar}S^*_{\mathrm{tf}}X_{\mathrm{c}\hbar}]. \tag{3-19}$$

Denote by $\mathcal{C} := \overline{{}^{\mathrm{b}\hbar}T^*_{\{0\}\times\partial X}}X_{\mathrm{b}\hbar}$ *the semiclassical b-phase space over the corner* $h = x = 0$, *and denote by* $o_{\mathcal{C}} \subset \mathcal{C}$ *the zero section. Then the identity map on* $(0,1)_h \times T^*X^\circ$ *extends by continuity to a diffeomorphism*

$$\overline{{}^{\mathrm{cb}\hbar}T^*}X_{\mathrm{c}\hbar} \xrightarrow{\cong} [\overline{{}^{\mathrm{b}\hbar}T^*}X_{\mathrm{b}\hbar}; \mathcal{C}; o_{\mathcal{C}}]. \tag{3-20}$$

We refer to the front face of (3-19) as fbf ("finite b-frequencies"). See Figure 8.

Proof of Lemma 3.8. We work in polar coordinates $\rho = x + h$, $\theta = (x,h)/|(x,h)|$ in the (x,h)-variables. Thus, local coordinates near ${}^{\mathrm{c}\hbar}T^*_{\mathrm{tf}}X_{\mathrm{c}\hbar}$ are $(\rho, y, \theta, \zeta_{\mathrm{c}\hbar})$, $\zeta_{\mathrm{c}\hbar} := (\xi_{\mathrm{c}\hbar}, \eta_{\mathrm{c}\hbar})$, while local coordinates near the front face of $[\overline{{}^{\mathrm{b}\hbar}T^*}X_{\mathrm{b}\hbar}; \mathcal{C}]$, away from fiber infinity, are $(\rho, y, \theta, \zeta_{\mathrm{b}\hbar})$, $\zeta_{\mathrm{b}\hbar} = (\xi_{\mathrm{b}\hbar}, \eta_{\mathrm{b}\hbar})$. Coordinates near the interior of the front face of the final blow-up in (3-20) are then $(\rho, y, \theta, \zeta_{\mathrm{b}\hbar}/\rho) = (\rho, y, \theta, \zeta_{\mathrm{c}\hbar})$; see (3-18). Near the intersection of the lift of $o_{\mathcal{C}}$ with that of $\mathcal{C}$, smooth coordinates can be constructed by introducing polar coordinates in the fiber variables, giving $(\rho/|\zeta_{\mathrm{b}\hbar}|, y, \theta, |\zeta_{\mathrm{b}\hbar}|, \zeta_{\mathrm{b}\hbar}/|\zeta_{\mathrm{b}\hbar}|)$; this matches, up to a permutation, the local coordinates on $\overline{{}^{\mathrm{cb}\hbar}T^*}X_{\mathrm{c}\hbar}$ near the lift of $\overline{{}^{\mathrm{c}\hbar}T^*_{\mathrm{tf}}}X_{\mathrm{c}\hbar}$ given by $(\rho/|\zeta_{\mathrm{c}\hbar}|^{-1}, y, \theta, |\zeta_{\mathrm{c}\hbar}|^{-1}, \zeta_{\mathrm{c}\hbar}/|\zeta_{\mathrm{c}\hbar}|)$. Lastly, near the lift of fiber infinity on the resolved b-phase space, we can use coordinates $(\rho, y, \theta, |\zeta_{\mathrm{b}\hbar}|^{-1}, \zeta_{\mathrm{b}\hbar}/|\zeta_{\mathrm{b}\hbar}|)$, which matches the local coordinates near the lift of ${}^{\mathrm{c}\hbar}S^*X_{\mathrm{c}\hbar}$ given by $(\rho, y, \theta, |\zeta_{\mathrm{c}\hbar}|^{-1}/\rho, \zeta_{\mathrm{c}\hbar}/|\zeta_{\mathrm{c}\hbar}|)$. □

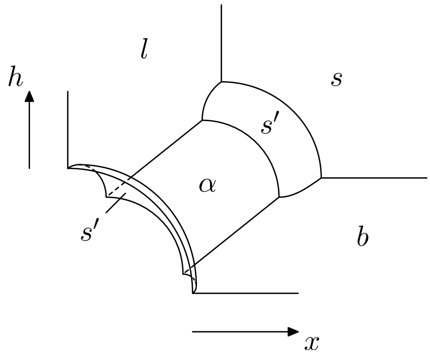

Figure 9. Illustration of the orders of (symbols of) cb$\hbar$-pseudodifferential operators in (3-21) and Definition 3.9.

The blow-up of a boundary face does not enlarge the space of conormal distributions, but allows for more precise accounting of weights. Concretely, define for $s, s', l, \alpha, b \in \mathbb{R}$ the symbol space

$$S^{s,s',l,\alpha,b}(\overline{{}^{\mathrm{cb}\hbar}T^*}X_{\mathrm{c}\hbar}), \tag{3-21}$$

where the orders refer, in this order, to fiber infinity, the front face fbf of (3-19), and the phase space over the lifts of cf, tf and sf, see Figure 9. Then we have

$$\begin{aligned} S^{s,l,\alpha,b}(\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar}) &= S^{s,s+\alpha,l,\alpha,b}(\overline{{}^{\mathrm{cb}\hbar}T^*}X_{\mathrm{c}\hbar}), \\ S^{s,s',l,\alpha,b}(\overline{{}^{\mathrm{cb}\hbar}T^*}X_{\mathrm{c}\hbar}) &\subset S^{\max(s,s'-\alpha),l,\alpha,b}(\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar}). \end{aligned} \tag{3-22}$$

Note that the second inclusion is *false* if we use spaces of *classical* symbols on both sides; after all, blow-ups do enlarge the space of smooth functions (but preserve the space of conormal functions). Since we worked with general conormal symbols and ps.d.o.s in Section 3B, we can immediately quantize symbols on the cb$\hbar$-phase space:

Definition 3.9 (cb$\hbar$-pseudodifferential operators). Let $s, s', l, \alpha, b \in \mathbb{R}$. Then we define

$$\Psi_{\mathrm{cb}\hbar}^{s,s',l,\alpha,b}(X) := \mathrm{Op}_{\mathrm{c}\hbar}(S^{s,s',l,\alpha,b}(\overline{{}^{\mathrm{cb}\hbar}T^*}X_{\mathrm{c}\hbar})) + \Psi_{\mathrm{c}\hbar}^{-\infty,l,\alpha,-\infty}(X).$$

Operators with variable semiclassical orders $\mathsf{b} \in \overline{{}^{\mathrm{c}\hbar}T^*_{\mathrm{sf}}}X_{\mathrm{c}\hbar}$ are defined similarly.

Remark 3.10 (second microlocalization). In view of Lemma 3.8, one can view $\Psi_{\mathrm{cb}\hbar}(X)$ as a second microlocalization of the (conormal) semiclassical b-algebra $\Psi_{\mathrm{b}\hbar}(X)$ at the zero section over $h = x = 0$. In terms of symbol classes, we have

$$\begin{aligned} S^{s,l,b}(\overline{{}^{\mathrm{b}\hbar}T^*}X_{\mathrm{b}\hbar}) &= S^{s,l+b,l,l+b,b}(\overline{{}^{\mathrm{cb}\hbar}T^*}X_{\mathrm{c}\hbar}), \\ S^{s,s',l,\alpha,b}(\overline{{}^{\mathrm{cb}\hbar}T^*}X_{\mathrm{c}\hbar}) &\subset S^{s,l,\max(b,s'-l,\alpha-l)}(\overline{{}^{\mathrm{b}\hbar}T^*}X_{\mathrm{b}\hbar}), \end{aligned} \tag{3-23}$$

and analogous statements hold for ps.d.o.s. However, similarly to [Vasy 2021b, §5] in the context of b- and scattering algebras, it is analytically advantageous to resolve $\Psi_{\mathrm{c}\hbar}(X)$ as in Definition 3.9, as the second microlocal/resolved algebra involves *global* (noncommutative) phenomena at $h = x = 0$ (i.e., the lift of tf, associated to which is the normal operator homomorphism into a noncommutative algebra) which are directly inherited from $\Psi_{\mathrm{c}\hbar}(X)$, but which are not visible on the level of $\Psi_{\mathrm{b}\hbar}(X)$.

For two ps.d.o.s $A_j \in \Psi_{\mathrm{cb}\hbar}^{s_j,s_j',l_j,\alpha_j,b_j}(X)$, one can compute the full symbol, i.e., the symbol modulo

$$S^{-\infty,l_1+l_2,\alpha_1+\alpha_2,-\infty}(\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar}) = S^{-\infty,-\infty,l_1+l_2,\alpha_1+\alpha_2,-\infty}(\overline{{}^{\mathrm{cb}\hbar}T^*}X_{\mathrm{c}\hbar}),$$

of the composition $A_1 \circ A_2 \in \Psi_{\mathrm{cb}\hbar}^{\max(s_1,s_1'-\alpha_1)+\max(s_2,s_2'-\alpha_2),l_1+l_2,\alpha_1+\alpha_2,b_1+b_2}(X)$ in local coordinates using the usual symbol expansion to be the sum of products of derivatives of the full symbols of the two factors along b-vector fields on $\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar}$ which vanish, as b-vector fields, at ${}^{\mathrm{c}\hbar}S^*X_{\mathrm{c}\hbar}$ (thus vanishing as b-vector fields at the lift of ${}^{\mathrm{c}\hbar}S^*X_{\mathrm{c}\hbar}$ as well as at the front face of (3-19)) and at the lift of $\overline{{}^{\mathrm{c}\hbar}T^*_{\mathrm{sf}}}X_{\mathrm{c}\hbar}$. Plugging the cb$\hbar$-symbols of A_1, A_2 into such an expansion thus shows that, in fact,

$$A_1 \circ A_2 \in \Psi_{\mathrm{cb}\hbar}^{s_1+s_2,s_1'+s_2',l_1+l_2,\alpha_1+\alpha_2,b_1+b_2}(X).$$

Similar arguments show that the principal symbol map

$${}^{\mathrm{cb}\hbar}\sigma : \Psi_{\mathrm{cb}\hbar}^{s,s',l,\alpha,b}(X) \to (S^{s,s',l,\alpha,b}/S^{s-1,s'-1,l,\alpha,b-1})(\overline{{}^{\mathrm{cb}\hbar}T^*}X_{\mathrm{c}\hbar})$$

is well-defined (and a *-homomorphism as usual). One can moreover define an associated scale of Sobolev spaces

$$H_{\mathrm{cb},h}^{s,s',l,\alpha,b}(X) = \{u \in H_{\mathrm{c},h}^{\min(s,s'-\alpha),l,\alpha,b}(X) : Au \in L^2(X) \text{ for all } A \in \Psi_{\mathrm{cb}\hbar}^{s,s',l,\alpha,b}(X)\}. \tag{3-24}$$

The relationships (3-22) and (3-23) imply:

Proposition 3.11 (relationships between Sobolev spaces). *Let $s, s', l, \alpha, b \in \mathbb{R}$. Define L^2 using the volume density $\mu = x^{\alpha_\mu}\mu_0$, $0 < \mu_0 \in \mathcal{C}^\infty(X; {}^{\mathrm{b}}\Omega^1 X)$, with $\alpha_\mu \in \mathbb{R}$. Then*

$$H_{\mathrm{c},h}^{s,l,\alpha,b}(X) = H_{\mathrm{cb},h}^{s,s+\alpha,l,\alpha,b}(X),$$
$$H_{\mathrm{b},h}^{s,l,b}(X) = H_{\mathrm{cb},h}^{s,l+b,l,l+b,b}(X).$$

One can conversely embed $H_{\mathrm{cb},h}^{s,s',l,\alpha,b}(X)$ into $H_{\mathrm{c},h}^{\tilde s,\tilde l,\tilde\alpha,\tilde b}(X)$ and $H_{\mathrm{b},h}^{\tilde s,\tilde l,\tilde b}(X)$ under suitable inequalities (which can be read off from Proposition 3.11) between the orders. In particular, this allows us to give a direct proof of [Hintz 2022, Proposition 3.18] on the relationship between $H_{\mathrm{c},h}(X)$ and $H_{\mathrm{b},h}(X)$; for instance, for $s, l, \alpha \in \mathbb{R}$ (denoted by s, α, τ in the reference), we have

$$H_{\mathrm{c},h}^{s,l,\alpha,0}(X) = H_{\mathrm{cb},h}^{s,s+\alpha,l,\alpha,0}(X) \subset H_{\mathrm{b},h}^{s,l,\min(0,\alpha-l,\alpha-l+s)}(X), \tag{3-25}$$

which implies (and is slightly sharper than) the first part of [Hintz 2022, Proposition 3.18]. If one wishes to translate estimates on cone spaces to b-spaces, the advantage of the resolved cb$\hbar$-Sobolev spaces, compared with c$\hbar$-Sobolev spaces, is that one can reduce losses in powers of h (or in regularity) in the conversion; as a simple concrete example, we have

$$H_{\mathrm{cb},h}^{s,s'+\alpha,l,\alpha,0}(X) \subset H_{\mathrm{b},h}^{s,l,\min(0,\alpha-l,\alpha-l+s')}(X),$$

which for $s' \geq -s_-$ gives an improved bound at $h = 0$, and for $s' \geq 0$ a bound which is independent of the differential orders s, s', unlike (3-25), which gets lossier as s decreases.

Remark 3.12 (variable semiclassical orders). The above discussion applies, mutatis mutandis, to symbols and operators with variable semiclassical orders b as well; here b is a smooth function on the lift of $\overline{{}^{\mathrm{c}\hbar}T^*_{\mathrm{sf}}}X_{\mathrm{c}\hbar}$ to $\overline{{}^{\mathrm{cb}\hbar}T^*}X_{\mathrm{c}\hbar}$.

4. Microlocal propagation estimates at cone points and generalizations

Let $n \geq 1$. We work locally near a cone point; thus on an n-dimensional manifold

$$X = [0, 2x_0)_x \times Y, \quad x_0 > 0, \tag{4-1}$$

where Y is a closed connected $(n-1)$-dimensional manifold, and where $X^\circ = (0, 2x_0) \times Y$ is equipped with a smooth Riemannian metric g of the form

$$g = \mathrm{d}x^2 + x^2 k(x, y, \mathrm{d}y), \tag{4-2}$$

where $k \in \mathcal{C}^\infty([0, x_0); \mathcal{C}^\infty(Y, S^2T^*Y))$ is a smooth family of smooth Riemannian metrics on the cross section Y. Any metric which locally near ∂X is of the form $\mathrm{d}\tilde{x}^2 + \tilde{x}^2 k(\tilde{x}, y, \mathrm{d}\tilde{x}, \mathrm{d}y)$, with $k|_{\partial X}$ a Riemannian metric on ∂X, is of the form (4-2) in a suitable smooth collar neighborhood of ∂X, as shown in [Melrose and Wunsch 2004, §1].

While the above X is not compact, all calculations and estimates will take place in the compact subset $[0, x_0] \times Y$ of X; thus, we shall commit a slight abuse of notation and write $\|u\|_{H^s_{\mathrm{c},h}(X)}$ etc. for norms of functions u on X which will always have support in $x^{-1}([0, x_0])$. We fix the volume density

$$\mu = |\mathrm{d}g| = x^{n-1}|\mathrm{d}x\, \mathrm{d}k| \in x^n \mathcal{C}^\infty(X; {}^{\mathrm{b}}\Omega^1 X) \tag{4-3}$$

on X, and define Sobolev spaces relative to $L^2(X) := L^2(X; \mu)$. We moreover define

$$\begin{aligned} \hat{x} &:= \frac{x}{h}, & \hat{g} &:= \mathrm{d}\hat{x}^2 + \hat{x}^2 k(0, y, \mathrm{d}y), \\ \hat{h} &:= \hat{x}^{-1} = \frac{h}{x}, & \hat{\mu} &:= |\mathrm{d}\hat{g}| = \hat{x}^{n-1}|\mathrm{d}\hat{x}\, \mathrm{d}k(0)|. \end{aligned} \tag{4-4}$$

4A. *Admissible operators.*

The class of operators of interest to us is the following.

Definition 4.1 (admissible operators). We call an h-dependent differential operator $P_{h,z}$ on X° *admissible* if it is of the form

$$P_{h,z} = h^2 \Delta_g - z + h^2 x^{-2} Q_{1,z} + h x^{-1} q_{0,z}, \tag{4-5}$$

where $Q_{1,z} \in \mathrm{Diff}^1_{\mathrm{b}}(X)$ and $q_{0,z} \in \mathcal{C}^\infty(X)$ depend smoothly on $z \in \mathbb{C}$, $|z - 1| < Ch$.

We shall henceforth take $z = z(h)$ to be a smooth function of $h \in [0, 1)$ with $z(0) = 1$.

Remark 4.2 (vector bundles). Our analysis applies also to operators acting on sections of a vector bundle $E \to X$; we explain the necessary (largely notational) changes in Remark 4.11.

Using local coordinates $y \in \mathbb{R}^{n-1}$ on ∂X, let us write

$$Q_{1,z} = q_{1,z}(x, y, xD_x, D_y), \quad q_{0,z} = q_{0,z}(x, y).$$

The normal operator of $P_{h,z}$ is

$$\begin{aligned} N(P) &:= \Delta_{\hat g} - 1 + \hat x^{-2} q_{1,1}(0, y, \hat x D_{\hat x}, D_y) + \hat x^{-1} q_{0,1}(0, y) \\ &= D_{\hat x}^2 - i(n-1) D_{\hat x} + \hat x^{-2}\Delta_{k(0)} - 1 + \hat x^{-2} q_{1,1}(0, y, \hat x D_{\hat x}, D_y) + \hat x^{-1} q_{0,1}(0, y) \end{aligned} \tag{4-6}$$

on $\mathrm{tf} = [0, \infty]_{\hat x} \times \partial X$.[11]

Lemma 4.3 (structural properties). *We have*

$$P_{h,z} \in \left(\frac{x}{x+h}\right)^{-2} \mathrm{Diff}^2_{\mathrm{c}\hbar}(X) \quad \textit{and} \quad N(P) \in \left(\frac{\hat x}{\hat x + 1}\right)^{-2} \mathrm{Diff}^2_{\mathrm{b,sc}}(\mathrm{tf}).$$

Furthermore, *we have*

$$P_{h,z} - N(P) \in (x+h)\left(\frac{x}{x+h}\right)^{-2} \mathrm{Diff}^2_{\mathrm{c}\hbar}(X),$$

where we abuse notation and write $N(P) \in (x/(x+h))^{-2}\,\mathrm{Diff}^2_{\mathrm{c}\hbar}(X)$ for any operator whose normal operator is equal to $N(P)$.

Proof. In local coordinates $y^1, \dots, y^{n-1}$ on Y, the metric $k(x, y, \mathrm{d}y)$ is given by an $(n-1) \times (n-1)$ matrix (k_{ij}) with determinant $|k| > 0$ and inverse (k^{ij}), and we have

$$\begin{aligned} \Delta_g &= |k|^{-\frac12} x^{-n+1} D_x(|k|^{\frac12} x^{n-1} D_x) + x^{-2}\Delta_{k(x)} \\ &= D_x^2 - i(n-1+x\gamma)x^{-1} D_x + \sum_{i,j=1}^{n-1} x^{-2} |k|^{-\frac12} D_{y^i}(|k|^{\frac12} k^{ij} D_{y^j}), \end{aligned}$$

where $\gamma = \frac12 \partial_x \log|k| \in \mathcal{C}^\infty$. Since

$$\begin{aligned} hD_x &= \frac{x+h}{x} \cdot \frac{h}{h+x} x D_x \in \frac{x+h}{x} \mathcal{V}_{\mathrm{c}\hbar}(X_{\mathrm{c}\hbar}), \\ hx^{-1} D_{y^i} &= \frac{x+h}{x} \cdot \frac{h}{h+x} D_{y^i} \in \frac{x+h}{x} \mathcal{V}_{\mathrm{c}\hbar}(X_{\mathrm{c}\hbar}), \\ hx^{-1} &= \frac{x+h}{x} \cdot \frac{h}{h+x} \in \frac{x+h}{x} \mathrm{Diff}^1_{\mathrm{c}\hbar}(X), \end{aligned} \tag{4-7}$$

we find $h^2 \Delta_g \in (x/(x+h))^{-2}\,\mathrm{Diff}^2_{\mathrm{c}\hbar}(X)$, and its normal operator is $D_{\hat x}^2 - i(n-1)\hat x^{-1} D_{\hat x} + \hat x^{-2}\Delta_{k(0)} = \Delta_{\hat g}$. The remaining terms in (4-5) are analyzed similarly. □

4B. *Characteristic set, Hamilton flow.* Using the fiber-linear coordinates $(\xi_{\mathrm{c}\hbar}, \eta_{\mathrm{c}\hbar})$ on ${}^{\mathrm{c}\hbar}T^* X_{\mathrm{c}\hbar}$ from (3-2), we can read off the principal symbol from (4-7) to be

$$p := \left(\frac{x}{x+h}\right)^2 \cdot {}^{\mathrm{c}\hbar}\sigma(P_{h,z}) = \xi_{\mathrm{c}\hbar}^2 + |\eta_{\mathrm{c}\hbar}|^2_{k^{-1}} - 1.$$

(Here, we use that $z = 1 + \mathcal{O}(h)$, hence the principal symbol of z is 1.) This is elliptic at fiber infinity ${}^{\mathrm{c}\hbar}S^* X_{\mathrm{c}\hbar}$, but has a nonempty characteristic set at finite frequencies. Near sf, it is more convenient to use

[11] This can be defined more invariantly as an operator on the inward pointing normal bundle ${}^+N\partial X$, which is the natural place for the b-normal operators $q_{1,1}(0, y, xD_x, D_y)$ and $q_{0,1}(0, y)$ to live; see [Melrose 1993, §4.15; Hintz 2022, §3] for details.

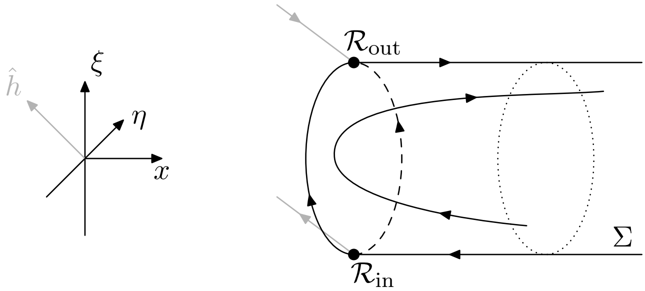

Figure 10. Illustration of the flow along the rescaled Hamilton vector field H, see (4-9), through the radial sets $\mathcal{R}_{\rm in}$ and $\mathcal{R}_{\rm out}$. Shown is the characteristic set, the fibers of which over sf are spheres (here 1-spheres); one fiber is drawn as a dotted circle. Also indicated is (in gray) the linearization of H at $\mathcal{R}_{\rm in/out}$ over tf.

the fiber coordinates (ξ, η) from (3-3), and $\hat h = h/x$, x, y as coordinates on the base, so that

$$\begin{aligned} p &= \xi^2 + k^{ij}(x, y)\eta_i\eta_j - 1, \\ \Sigma &= p^{-1}(0) \cap {}^{\rm c\hbar}T^*_{\rm sf}X_{\rm c\hbar} = \{\hat h = 0,\ \xi^2 + |\eta|^2_{k^{-1}} = 1\}. \end{aligned} \tag{4-8}$$

Using (3-4) and writing $|\eta|^2 = k^{ij}\eta_i\eta_j$, we then compute

$$\mathsf{H} := \hat h^{-1} H_p = 2\xi(x\partial_x - \hat h\partial_{\hat h} - \eta\partial_\eta) + (2|\eta|^2 - x\partial_x k^{ij}\eta_i\eta_j)\partial_\xi + 2k^{ij}\eta_i\partial_{y^j} - (\partial_{y^k}k^{ij})\eta_i\eta_j\partial_{\eta_k}. \tag{4-9}$$

Restricted to $x = 0$ as a b-vector field on $\overline{{}^{\rm c\hbar}T^*}X_{\rm c\hbar}$, this is

$$\mathsf{H}|_{x=0} = 2\xi(x\partial_x - \hat h\partial_{\hat h} - \eta\partial_\eta) + 2|\eta|^2\partial_\xi + (2k^{ij}\eta_i\partial_{y^j} - (\partial_{y^k}k^{ij})\eta_i\eta_j\partial_{\eta_k}). \tag{4-10}$$

This vanishes as a standard vector field on $\hat h = x = 0$ if and only if $\eta = 0$. The intersection of $\eta^{-1}(0)$ with $\Sigma \cap x^{-1}(0)$ has two components: the *incoming* and *outgoing radial sets* $\mathcal{R}_{\rm in/out} \subset {}^{\rm c\hbar}T^*_{\rm sf}X_{\rm c\hbar}$,

$$\begin{aligned} \mathcal{R}_{\rm in} &:= \{(\hat h, x, y, \xi, \eta) : \hat h = 0,\ x = 0,\ y \in \partial X,\ \xi = -1,\ \eta = 0\}, \\ \mathcal{R}_{\rm out} &:= \{(\hat h, x, y, \xi, \eta) : \hat h = 0,\ x = 0,\ y \in \partial X,\ \xi = +1,\ \eta = 0\}. \end{aligned} \tag{4-11}$$

These are saddle points for the rescaled Hamilton vector field H since

$$x^{-1}\mathsf{H}x = \mp 2, \quad \hat h^{-1}\mathsf{H}\hat h = \pm 2, \quad |\eta|^{-2}\mathsf{H}|\eta|^2 = \pm 4 \quad \text{at } \mathcal{R}_{\rm in/out}. \tag{4-12}$$

(The top sign is for “in”, the bottom sign for “out”.) See Figure 10.

Over ${}^{\rm c\hbar}T^*_{\rm tf}X_{\rm c\hbar}$, the set $\mathcal{R}_{\rm in}$ is a radial *source* (though this really only makes sense infinitesimally at $\mathrm{tf} \cap \mathrm{sf}$ since the c$\hbar$-calculus is not symbolic over $\mathrm{tf} \setminus \mathrm{sf}$), and $\mathcal{R}_{\rm out}$ is a radial *sink*. This matches precisely the familiar situation of scattering theory on the asymptotically conic space $(\mathrm{tf}, \hat g)$, see [Melrose 1994], which we discuss in detail in Section 4C.

In $x > 0$, the flow of H is a reparametrization of the flow of $h^{-1}H_p = x^{-1}\mathsf{H}$. Integral curves of H starting over a point in X° never reach ∂X in finite time. Instead, we consider

$$\begin{aligned} \mathsf{H}_{\rm sf} &:= h^{-1}H_p|_{\hat h=0} \\ &= 2\xi(\partial_x - x^{-1}\eta\partial_\eta) + (2x^{-1}|\eta|^2 - \partial_x k^{ij}\eta_i\eta_j)\partial_\xi + x^{-1}(2k^{ij}\eta_i\partial_{y^j} - (\partial_{y^k}k^{ij})\eta_i\eta_j\partial_{\eta_k}). \end{aligned} \tag{4-13}$$

Given $y_0 \in \partial X$, the curves

$$\begin{aligned} \gamma_{I,y_0}(s) &:= (-2s, y_0, -1, 0), \quad && s \in (-x_0, 0), \\ \gamma_{O,y_0}(s) &:= (2s, y_0, 1, 0), && s \in (0, x_0), \end{aligned} \tag{4-14}$$

are integral curves of H_{sf}. Here, γ_{I,y_0} strikes ∂X at $s = 0$ at the incoming radial set over point y_0, whereas γ_{O,y_0} emanates from the outgoing radial set over y_0 at $s = 0$.

Lemma 4.4 (incoming/outgoing null-bicharacteristics). *Let $0 < s_0 < x_0$. Suppose $\gamma : (0, s_0) \to \Sigma \cap {}^{c\hbar}T^*_{\mathrm{sf}\setminus\mathrm{tf}} X_{c\hbar}$ is an integral curve of H_{sf} tending to ∂X as $s \searrow 0$ in the weak sense that $\liminf_{s\searrow 0} x(\gamma(s)) = 0$. Then in the coordinates (x, y, ξ, η), γ is necessarily of the form $\gamma(s) = \gamma_{O,y_0}(s)$ for some $y_0 \in \partial X$. Similarly, if $\gamma : (-s_0, 0) \to \Sigma \cap {}^{c\hbar}T^*_{\mathrm{sf}\setminus\mathrm{tf}} X_{c\hbar}$ is an integral curve of H_{sf} with $\liminf_{s\nearrow 0} x(\gamma(s)) = 0$, then $\gamma(s) = \gamma_{I,y_0}(s)$ for some $y_0 \in \partial X$.*

Proof. The vector field

$$x\mathsf{H}_{\mathrm{sf}} = \mathsf{H}|_{x=0} = 2\xi(x\partial_x - \eta\partial_\eta) + (2|\eta|^2 - x\partial_x k^{ij}\eta_i\eta_j)\partial_\xi + (2k^{ij}\eta_i\partial_{y^j} - (\partial_{y^k}k^{ij})\eta_i\eta_j\partial_{\eta_k})$$

vanishes identically at $\mathcal{R}_{\mathrm{out}}$. We study the behavior of $x\mathsf{H}_{\mathrm{sf}}$ as a vector field on Σ near $\mathcal{R}_{\mathrm{out}}$; we may use the coordinates $x \geq 0$, $y \in \mathbb{R}^{n-1}$, and $\eta \in \mathbb{R}^{n-1}$, in which ξ is determined by $p = 0$ as the positive square root of $1 - k^{ij}(x, y)\eta_i\eta_j$. The linearization of $(x\mathsf{H}_{\mathrm{sf}})|_\Sigma$ in the normal directions at $\mathcal{R}_{\mathrm{out}}$, defined by mapping $\mathrm{d}f$ to $\mathrm{d}(x\mathsf{H}_{\mathrm{sf}}f)$ where $f \in \mathcal{C}^\infty(\Sigma)$, $f|_{\mathcal{R}_{\mathrm{out}}} = 0$, maps

$$\mathrm{d}x \mapsto 2\,\mathrm{d}x, \quad \mathrm{d}\eta \mapsto -2\,\mathrm{d}\eta,$$

and is thus hyperbolic; the unstable and stable subbundles of $T_{\mathcal{R}_{\mathrm{out}}}\Sigma$ for the $(x\mathsf{H}_{\mathrm{sf}})|_\Sigma$-flow are correspondingly the span of ∂_x and ∂_η. The unstable manifold theorem, in the form given in [Hirsch et al. 1977, Theorem 4.1], thus applies inside Σ at $\mathcal{R}_{\mathrm{out}}$ and produces an unstable manifold whose tangent space at a point $\zeta \in \mathcal{R}_{\mathrm{out}}$ is the sum of $T_\zeta(\mathcal{R}_{\mathrm{out}})$ and $\mathbb{R}\partial_x$. (See the proof of [Melrose and Wunsch 2004, Theorem 1.2] for similar, albeit more general, considerations.) Since the manifold $\mathcal{R}_{\mathrm{out}} \cup \{\gamma_{O,y_0}(s) : y_0 \in \partial X,\ s \in (0, s_0)\}$ is H-invariant with the same tangent space, it must be equal to this unstable manifold. The first part of the lemma follows from this observation; the second part is completely analogous. □

Definition 4.5 (generalized broken bicharacteristics). Denote by $\dot{\Sigma}$ the topological space defined as the quotient $\Sigma/\partial\Sigma$. Let $I \subset \mathbb{R}$ denote an open interval. We then say that a continuous curve $\gamma : I \to \dot{\Sigma}$ is a *generalized broken bicharacteristic* (GBB) if either $\gamma(I) \subset \Sigma \setminus \partial\Sigma$ and γ is an integral curve of H_{sf}, or there exist $s_0 \in I$ and $y_I, y_O \in \partial X$ so that $\gamma(s_0+t) = \gamma_{O,y_O}(t)$ for $t > 0$, $s_0+t \in I$ and $\gamma(s_0+t) = \gamma_{I,y_I}(t)$ for $t < 0$, $s_0 + t \in I$.[12] If y_O is at distance π from y_I with respect to the metric $k(0)$ on ∂X, we say that γ is a *geometric GBB*; otherwise γ is a *strictly diffractive GBB*.

See Figure 11. We remark without proof that geometric GBB are uniform limits of H_{sf}-integral curves just barely missing ∂X (see also [Melrose and Wunsch 2004, Lemma 1.5]).

[12] In light of Lemma 4.4, this is equivalent to the condition that γ is an H_{sf}-integral curve outside of $\partial\Sigma$, but may enter and exit the characteristic set over ∂X at different points.

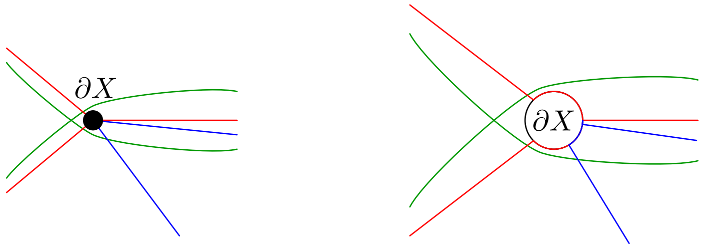

Figure 11. The projection of strictly diffractive (blue) and geometric (red) GBBs to the base X, as well as geodesics (green) just barely missing the cone tip ∂X. Left: the geometric picture, where ∂X is collapsed to a point. Right: the resolved picture.

4C. ***Scattering theory for the normal operator.*** Propagation through the "cone point" ∂X will require *global* control of the normal operator, namely the absence of purely outgoing or purely incoming solutions (depending on the direction in which one wants to propagate estimates). Let us define fiber-linear coordinates on the scattering cotangent bundle ${}^{\mathrm{sc}}T^*(\mathrm{tf}\setminus\mathrm{cf})$ via

$$\xi_{\mathrm{sc}}\frac{\mathrm{d}\hat{h}}{\hat{h}^2}+\sum_{j=1}^{n-1}(\eta_{\mathrm{sc}})_j\frac{\mathrm{d}y^j}{\hat{h}}.$$

Via the identification (3-7), the radial sets $\mathcal{R}_{\mathrm{in/out}}$ defined in (4-11) are then equal to the sets ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in/out}}\subset {}^{\mathrm{sc}}T^*(\mathrm{tf}\setminus\mathrm{cf})$, where

$$\begin{aligned}{}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in}}&:=\{(\hat{h},y,\xi_{\mathrm{sc}},\eta_{\mathrm{sc}}):\hat{h}=0,\ y\in\partial X,\ \xi_{\mathrm{sc}}=+1,\ \eta_{\mathrm{sc}}=0\},\\ {}^{\mathrm{sc}}\mathcal{R}_{\mathrm{out}}&:=\{(\hat{h},y,\xi_{\mathrm{sc}},\eta_{\mathrm{sc}}):\hat{h}=0,\ y\in\partial X,\ \xi_{\mathrm{sc}}=-1,\ \eta_{\mathrm{sc}}=0\}.\end{aligned}$$

Invariantly, $\mathcal{R}_{\mathrm{in}}={}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in}}$ is the graph of $-(x/h)(\mathrm{d}x/x)=-\mathrm{d}(\hat{h}^{-1})=\mathrm{d}\hat{h}/\hat{h}^2$, and likewise for $\mathcal{R}_{\mathrm{out}}={}^{\mathrm{sc}}\mathcal{R}_{\mathrm{out}}$ but with an overall sign switch.

Definition 4.6 (conditions on the normal operator). Let $l,l'\in\mathbb{R}$, and recall (4-4).

(1) We say that $N(P)$ is *injective at weight l on outgoing functions* if the only solution u to the equation $N(P)u=0$ satisfying $u\in\bigcup_{N\in\mathbb{R}}H^{\infty,l,-N}_{\mathrm{b,sc}}(\mathrm{tf};\hat{\mu})$ and $\mathrm{WF}_{\mathrm{sc}}(u)\subset{}^{\mathrm{sc}}\mathcal{R}_{\mathrm{out}}$ is trivial: $u\equiv 0$.

(2) We say that $N(P)^*$ (the formal adjoint with respect to $L^2(\mathrm{tf};\hat{\mu})$) is *injective at weight l' on incoming functions* if the only solution v to the equation $N(P)^*v=0$ satisfying $v\in\bigcup_{N\in\mathbb{R}}H^{\infty,l',-N}_{\mathrm{b,sc}}(\mathrm{tf};\hat{\mu})$ and $\mathrm{WF}_{\mathrm{sc}}(v)\subset{}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in}}$ is trivial: $v\equiv 0$.

(3) If condition (1) and condition (2) with $l'=-l+2$ hold, we say that $N(P)$ is *invertible at weight l*.

The wave front set assumptions here are the microlocal formulations of outgoing/incoming radiation conditions. In the special case that $N(P)=\Delta_{\hat{g}}-1$, these assumptions are indeed equivalent to the standard Sommerfeld radiation condition. Our goal is to elevate the qualitative conditions of Definition 4.6 to quantitative estimates; see Lemma 4.8.

Changing variables in the expression (4-6) for $N(P)$ to $(\hat{h},y)$ gives

$$N(P)=(\hat{h}^2D_{\hat{h}})^2+i(n-1)\hat{h}^2D_{\hat{h}}+\hat{h}^2\Delta_{k(0)}-1+\hat{h}^2q_{1,1}(0,y,-\hat{h}D_{\hat{h}},D_y)+\hat{h}q_{0,1}(0,y),$$

with scattering principal symbol at ${}^{\mathrm{sc}}T^*_{\mathrm{tf}\cap\mathrm{sf}}\mathrm{tf}$ given by

$$p_{\mathrm{tf}} = \xi_{\mathrm{sc}}^2 + |\eta_{\mathrm{sc}}|^2 - 1. \tag{4-15a}$$

Its Hamilton vector field is

$$\mathsf{H}_{\mathrm{tf}} := \hat h^{-1} H_{p_{\mathrm{tf}}} = 2\xi_{\mathrm{sc}}(\hat h\partial_{\hat h} + \eta_{\mathrm{sc}}\partial_{\eta_{\mathrm{sc}}}) - 2|\eta_{\mathrm{sc}}|^2\partial_{\xi_{\mathrm{sc}}} + \hat h^{-1} H_{|\eta_{\mathrm{sc}}|^2} \tag{4-15b}$$

by (2-7), which has a source, resp. sink structure at ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in}}$, resp. ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{out}}$ within the characteristic set $p_{\mathrm{tf}}^{-1}(0)$. Recall then that microlocal propagation estimates near the radial sets ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in/out}}$ require suitable orders — here the decay order — of weighted Sobolev spaces to be above or below certain threshold values; see [Melrose 1994, §9], [Vasy 2018, §4.7], and [Dyatlov and Zworski 2019, Appendix E.4].

Definition 4.7 (threshold quantities). Define the functions

$$r_1 := \mathrm{Im}\Big({}^{\mathrm{b}}\sigma_1(Q_{1,1})\Big(-\frac{\mathrm{d}x}{x}\Big)\Big|_{x=0}\Big) \in \mathcal{C}^\infty(\partial X),$$
$$r_0 := \mathrm{Im}(q_{0,1}|_{x=0}) \in \mathcal{C}^\infty(\partial X).$$

Then the threshold quantities $r_{\mathrm{in/out}} \in \mathbb{R}$ are defined as

$$r_{\mathrm{in}} := -\tfrac12 + \tfrac12 \sup_{\partial X}(r_1 + r_0), \qquad r_{\mathrm{out}} := -\tfrac12 + \tfrac12 \inf_{\partial X}(r_1 - r_0).$$

We next recall that at the other end of tf, i.e., the "b-end" $\mathrm{tf}\cap\mathrm{cf}$, the weights l, l' in Definition 4.6 are related to the *boundary spectrum* of $N(P)$. Concretely, from the expression (4-6), we read off

$$\hat x^2 N(P) \in (\hat x D_{\hat x})^2 - i(n-2)\hat x D_{\hat x} + \Delta_{k(0)} + q_{1,1}(0, y, \hat x D_{\hat x}, D_y) + \hat x\,\mathrm{Diff}_{\mathrm{b}}(\mathrm{tf}\setminus\mathrm{sf}). \tag{4-16}$$

Its (dilation-invariant in $\hat x$) normal operator at $\hat x = 0$ is given by the sum of the first four terms, and the Mellin transformed normal operator family is defined by formally replacing $\hat x D_{\hat x}$ by multiplication with $\lambda \in \mathbb{C}$, giving

$$\widehat{N}(P)(\lambda, y, D_y) := \lambda^2 - i(n-2)\lambda + \Delta_{k(0)} + q_{1,1}(0, y, \lambda, D_y). \tag{4-17}$$

This is a holomorphic family in λ taking values in elliptic elements of $\mathrm{Diff}^2(\partial X)$. The *boundary spectrum* of $N(P)$ is then

$$\mathrm{spec}_{\mathrm{b}}(N(P)) := \{\lambda \in \mathbb{C} : \widehat{N}(P)(\lambda) : \mathcal{C}^\infty(\partial X) \to \mathcal{C}^\infty(\partial X) \text{ is not invertible}\};$$

it is a discrete subset of $\mathbb{C}$, and its intersection with $|\mathrm{Im}\,\lambda| < C$ is finite for any fixed value of C [Melrose 1993, §5.3]. Let us now put

$$\Lambda := \{-\mathrm{Im}\,\lambda : \lambda \in \mathrm{spec}_{\mathrm{b}}(N(P))\}; \tag{4-18}$$

this is a discrete subset of $\mathbb{R}$.

Lemma 4.8 (estimates for $N(P)$). *Let $s, l \in \mathbb{R}$ and $\mathsf{r} \in \mathcal{C}^\infty(\overline{{}^{\mathrm{sc}}T^*_{\mathrm{tf}\cap\mathrm{sf}}}\mathrm{tf})$. Suppose that r is constant near ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in/out}}$ and satisfies $\mathsf{r} > r_{\mathrm{in}}$ at ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in}}$, $\mathsf{r} < r_{\mathrm{out}}$ at ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{out}}$. Suppose moreover that $\mathsf{H}_{\mathrm{tf}}\mathsf{r} \le 0$ and $\sqrt{-\mathsf{H}_{\mathrm{tf}}\mathsf{r}} \in \mathcal{C}^\infty$ on ${}^{\mathrm{sc}}T^*_{\mathrm{tf}\cap\mathrm{sf}}\mathrm{tf} \cap \{p_{\mathrm{tf}} = 0\}$ in the notation of (4-15a)–(4-15b).*

(1) *If $N(P)$ is injective at weight l on outgoing functions and $l - n/2 \notin \Lambda$, then*

$$\|u\|_{H_{\mathrm{b,sc}}^{s,l,\mathsf{r}}(\mathrm{tf};\hat\mu)} \le C\|N(P)u\|_{H_{\mathrm{b,sc}}^{s-2,l-2,\mathsf{r}+1}(\mathrm{tf};\hat\mu)} \tag{4-19}$$

for all u for which both sides are finite.

(2) *If $N(P)^*$ is injective at weight $-l+2$ on incoming functions and $-l+2-n/2 \notin \Lambda$, then*

$$\|v\|_{H_{\mathrm{b,sc}}^{-s+2,-l+2,-\mathsf{r}-1}(\mathrm{tf};\hat\mu)} \le C\|N(P)^*v\|_{H_{\mathrm{b,sc}}^{-s,-l,-\mathsf{r}}(\mathrm{tf};\hat\mu)} \tag{4-20}$$

for all v for which both sides are finite.

(3) *If $N(P)$ is invertible at weight l and $l - n/2 \notin \Lambda$,*[13] *then the operator $N(P)$ is invertible as a map*

$$\{u \in H_{\mathrm{b,sc}}^{s,l,\mathsf{r}}(\mathrm{tf};\hat\mu) : N(P)u \in H_{\mathrm{b,sc}}^{s-2,l-2,\mathsf{r}+1}(\mathrm{tf};\hat\mu)\} \to H_{\mathrm{b,sc}}^{s-2,l-2,\mathsf{r}+1}(\mathrm{tf};\hat\mu).$$

Proof. This is a standard application of elliptic b-theory at $\mathrm{tf}\cap\mathrm{cf}$ and radial point estimates at $\mathrm{tf}\cap\mathrm{sf}$ in the scattering calculus as in [Melrose 1994] and [Vasy 2018, §4.8].

We first prove symbolic estimates for $N(P)$ and $N(P)^*$ which do not use the injectivity assumptions. In $\mathrm{tf}\setminus\mathrm{sf}$, the operator $N(P)$ is an elliptic weighted b-differential operator. Let $\phi_j \in \mathcal{C}_c^\infty(\mathrm{tf}\setminus\mathrm{sf})$, $j = 0, 1, 2, 3$, be identically 1 near $\mathrm{tf}\cap\mathrm{cf}$, with $\phi_{j+1} \equiv 1$ on $\operatorname{supp}\phi_j$. Then, only recording the b-regularity and the weight at cf, we have

$$\|\phi_1 u\|_{H_{\mathrm{b}}^{s,l}} \le C(\|\phi_2 N(P)u\|_{H_{\mathrm{b}}^{s-2,l-2}} + \|\phi_2 u\|_{H_{\mathrm{b}}^{-N,l}}) \tag{4-21}$$

for any fixed N. Now, recalling (4-4), we have

$$H_{\mathrm{b}}^{s,l}([0,\infty)_{\hat x}\times\partial X;\hat\mu) = H_{\mathrm{b}}^{s,l-\frac{n}{2}}\Big([0,\infty)\times\partial X;\Big|\frac{d\hat x}{\hat x}dk(0)\Big|\Big).$$

Using now that $l - n/2 \notin \Lambda$, we can estimate

$$\|\phi_2 u\|_{H_{\mathrm{b}}^{-N,l}} \le C\|\hat x^{-2}\widehat{N}(P)(\hat x D_{\hat x}, y, D_y)(\phi_2 u)\|_{H_{\mathrm{b}}^{-N-2,l-2}}$$

by passing to the Mellin transform. Since $N(P) - \hat x^{-2}\widehat{N}(P)(\hat x D_x, y, D_y) \in \hat x^{-1}\,\mathrm{Diff_b}$ by (4-16), this can be plugged into (4-21) and yields (putting back the scattering decay orders, which at this point are still arbitrary due to the localizers)

$$\|\phi_1 u\|_{H_{\mathrm{b,sc}}^{s,l,\mathsf{r}}} \le C(\|\phi_3 N(P)u\|_{H_{\mathrm{b,sc}}^{s-2,l-2,\mathsf{r}+1}} + \|\phi_3 u\|_{H_{\mathrm{b,sc}}^{-N,l-1,-N}}). \tag{4-22}$$

Turning to the scattering end, and with $\psi_j = 1 - \phi_j$, we claim that (now with the b-decay orders being arbitrary)

$$\|\psi_1 u\|_{H_{\mathrm{b,sc}}^{s,l,\mathsf{r}}} \le C(\|\psi_0 N(P)u\|_{H_{\mathrm{b,sc}}^{s-2,l-2,\mathsf{r}+1}} + \|\psi_0 u\|_{H_{\mathrm{b,sc}}^{-N,-N,-N}}). \tag{4-23}$$

This is proved by means of the scattering calculus by a combination of elliptic estimates (controlling $\psi_1 u$ away from $\Sigma_{\mathrm{tf}} := p_{\mathrm{tf}}^{-1}(0)$), radial point estimates at ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in/out}}$, and microlocal real-principal-type estimates on $\Sigma_{\mathrm{tf}}\setminus({}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in}}\cup{}^{\mathrm{sc}}\mathcal{R}_{\mathrm{out}})$. We only sketch the argument for the radial points in order to explain the emergence of the threshold condition on r; details can be found, e.g., in [Vasy 2018, §4.7].

[13] This condition is automatically satisfied since, for $l - n/2 \notin \Lambda$, the operator $N(P)$ is not even Fredholm; see [Melrose 1993, §6.2].

We work in $[0, 1)_{\hat h}\times\partial X\subset\mathrm{tf}$, and consider estimates near ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in}}$. Fixing a cutoff function $\chi\in\mathcal{C}^\infty_{\mathrm{c}}\bigl(\bigl[0,\tfrac12\bigr)\bigr)$, identically 1 near 0 and with $\chi'\le 0$ on $\bigl[0,\tfrac12\bigr)$, we consider a commutant

$$
\begin{aligned}
a &:= \hat h^{-2\mathsf{r}-1}\chi(\hat h/\delta)\chi(|\eta_{\mathrm{sc}}|^2)\chi((\xi_{\mathrm{sc}}-1)^2),\\
A &:= \tfrac12(\mathrm{Op}_{\mathrm{sc}}(a)+\mathrm{Op}_{\mathrm{sc}}(a)^*)\in\Psi_{\mathrm{sc}}^{-\infty,2\mathsf{r}+1},
\end{aligned}
$$

where $\delta>0$ controls the localization near $\hat h=0$. We compute the commutator

$$
2\,\mathrm{Im}\langle N(P)u, Au\rangle=\Bigl\langle\Bigl(i[N(P),A]+2\frac{N(P)-N(P)*}{2i}A\Bigr)u,u\Bigr\rangle.
$$

(This holds directly for sufficiently decaying u, and for u as in the statement of the lemma can be justified using a regularization argument.) The principal symbol of $i[N(P),A]$ is equal to $\hat h\mathsf{H}_{\mathrm{tf}}a$. When H_{tf} falls on the cutoff in $\hat h$, the result is supported in the elliptic set of $N(P)$, and hence easily controlled. When H_{tf} falls on either of the second or third cutoff functions, the result is ≤ 0 on $\operatorname{supp} a$ in view of the source character of ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in}}$ (or directly using (4-15b)), provided $\delta>0$ is sufficiently small; at ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in}}$ then, the principal symbol of

$$
i[N(P),A]+2\frac{N(P)-N(P)^*}{2i}A
$$

has a matching definite sign, i.e., is a *negative* multiple of $\hat h^{-2\mathsf{r}}$, provided that

$$
\begin{aligned}
&2\cdot(+1)\cdot(-2\mathsf{r}-1)+2\cdot{}^{\mathrm{sc}}\sigma\Bigl(\frac{Q-Q^*}{2i}\Bigr)<0,\\
&Q:=\hat h q_{1,1}(0,y,-\hat h D_{\hat h},D_y)+q_{0,1}(0,y),
\end{aligned}\tag{4-24}
$$

at ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in}}$. But ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in}}$ is the graph of the 1-form $\mathrm{d}\hat h/\hat h^2$; hence

$$
{}^{\mathrm{sc}}\sigma\Bigl(\frac{Q-Q^*}{2i}\Bigr)=\mathrm{Im}\,{}^{\mathrm{sc}}\sigma(Q)
$$

at ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in}}$ is equal to

$$
\mathrm{Im}\,{}^{\mathrm{b}}\sigma(q_{1,1}(0,y,-\hat h D_{\hat h},D_y))\Bigl(\frac{\mathrm{d}\hat h}{\hat h}\Bigr)+\mathrm{Im}\,q_{0,1}=r_1+r_0
$$

in the notation of Definition 4.7. The condition (4-24) thus becomes $-2\mathsf{r}-1+(r_1+r_0)<0$, which is satisfied on all of ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in}}$ provided that $\mathsf{r}>-\frac12+\frac12\sup(r_1+r_0)=r_{\mathrm{in}}$ there. Under this assumption, one thus obtains control on u microlocally near ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in}}$ in the space $H_{\mathrm{sc}}^{s,\mathsf{r}}$ by $N(P)u$ measured in $H_{\mathrm{sc}}^{s-2,\mathsf{r}+1}$.

The analysis at ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{out}}$ is similar, now using the commutant

$$
\hat h^{-2\mathsf{r}-1}\chi(\hat h/\delta)\chi(|\eta_{\mathrm{sc}}|^2)\chi((\xi_{\mathrm{sc}}+1)^2).
$$

The derivatives of the latter two cutoffs along H_{tf} are now positive due to the sink character of ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{out}}$, and the principal symbol of the commutator at the radial set is a negative multiple of $\hat h^{-2\mathsf{r}}$ (thus allowing us to propagate control from a punctured neighborhood of the radial set into the radial set itself) provided that

$$
2\cdot(-1)\cdot(-2\mathsf{r}-1)+2\cdot{}^{\mathrm{sc}}\sigma\Bigl(\frac{Q-Q^*}{2i}\Bigr)<0\quad\text{at }{}^{\mathrm{sc}}\mathcal{R}_{\mathrm{out}}.\tag{4-25}
$$

In view of ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{out}}$ being the graph of $-\mathrm{d}\hat{h}/\hat{h}^2$ and the calculation

$$\operatorname{Im}{}^{\mathrm{sc}}\sigma(Q)|_{{}^{\mathrm{sc}}\mathcal{R}_{\mathrm{out}}} = \operatorname{Im}{}^{\mathrm{b}}\sigma(q_{1,1}(0, y, -\hat{h}D_{\hat{h}}, D_y))\left(-\frac{\mathrm{d}\hat{h}}{\hat{h}}\right) + \operatorname{Im} q_{0,1} = -r_1 + r_0,$$

the condition (4-25) reads $2\mathsf{r} + 1 - r_1 + r_0 < 0$, so $\mathsf{r} < -\frac{1}{2} + \frac{1}{2}\inf(r_1 - r_0) = r_{\mathrm{out}}$.

Putting (4-22) and (4-23) together, we obtain the estimate

$$\|u\|_{H_{\mathrm{b,sc}}^{s,l,\mathsf{r}}} \leq C(\|N(P)u\|_{H_{\mathrm{b,sc}}^{s-2,l-2,\mathsf{r}+1}} + \|u\|_{H_{\mathrm{b,sc}}^{-N,l-1,-N}}) \tag{4-26}$$

for any N; we choose N to satisfy $-N < s$ and $-N < \min \mathsf{r}$.

The estimate (4-26) implies that $N(P)$, acting on $H_{\mathrm{b,sc}}^{s,l,\mathsf{r}}$, has finite-dimensional kernel; any element u in the kernel automatically lies in $H_{\mathrm{b,sc}}^{\infty,l,\mathsf{r}'}$ for any variable-order function r' satisfying $\mathsf{r}' < r_{\mathrm{out}}$ at ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{out}}$. Thus, $\mathrm{WF}_{\mathrm{sc}}(u) \subset {}^{\mathrm{sc}}\mathcal{R}_{\mathrm{out}}$. Under the injectivity assumption on $N(P)$, we thus conclude that $u = 0$. A standard functional analytic argument then allows one to drop the error term in (4-26), which gives the estimate (4-19).

The proof of part (2) is analogous; the direction of propagation in the characteristic set is now reversed, which is precisely matched by the sign switches in the orders in the estimate (4-20). Part (3) is an immediate consequence of the first two parts. □

Remark 4.9 (flexibility in the choice of l). If the assumptions of part (1) of the lemma are satisfied for some value of l, then they continue to hold for all values $\tilde{l}$ with $\tilde{l} - n/2 \notin \Lambda$ for which either $\tilde{l} > l$, or $\tilde{l} \leq l$ but $\tilde{l} - n/2$ and $l - n/2$ lie in the same connected component (a, b) of $\mathbb{R} \setminus \Lambda$. (Indeed, the claim for $\tilde{l} \leq l$ follows from the fact — proved using the Mellin transform upon localizing near $\mathrm{tf} \cap \mathrm{cf}$ — that any element in the kernel of $N(P)$ on $H_{\mathrm{b,sc}}^{s,\tilde{l},\mathsf{r}}$ automatically lies in $H_{\mathrm{b,sc}}^{s,b+n/2-\epsilon,\mathsf{r}}$ for any $\epsilon > 0$.) A similar statement holds for part (2): we may increase $-l + 2$ (or stay in the same connected component of $(\mathbb{R} \setminus \Lambda) + n/2$), i.e., decrease l. Altogether then, there typically only exists an interval of finite length (possibly empty) of weights l so that the invertibility condition of part (3) is satisfied.

4D. *Statement and proof of the microlocal propagation estimate.*

We are now ready to state the main result of the paper:

Theorem 4.10 (microlocal propagation through the cone point). *Let $P_{h,z}$ denote an admissible operator in the sense of Definition 4.1, and define the threshold quantities $r_{\mathrm{in}}, r_{\mathrm{out}}$ as in Definition 4.7. Let $\Sigma \subset {}^{\mathrm{c}\hbar}T^*_{\mathrm{sf}}X_{\mathrm{c}\hbar}$ denote the characteristic set of $P_{h,z}$* (see (4-8)). *Denote by $\mathsf{H} = (x/h)H_p \in \mathcal{V}({}^{\mathrm{c}\hbar}T^*_{\mathrm{sf}}X_{\mathrm{c}\hbar})$ the rescaled Hamilton vector field* (see (4-9)). *Let $s, l, \alpha \in \mathbb{R}$, $\mathsf{b} \in \mathcal{C}^\infty(\overline{{}^{\mathrm{c}\hbar}T^*_{\mathrm{sf}}X_{\mathrm{c}\hbar}})$. Assume that b is constant near the radial sets $\mathcal{R}_{\mathrm{in/out}}$* (see (4-11)) *and satisfies $\mathsf{b} - \alpha > r_{\mathrm{in}}$ at $\mathcal{R}_{\mathrm{in}}$ and $\mathsf{b} - \alpha < r_{\mathrm{out}}$ at $\mathcal{R}_{\mathrm{out}}$; assume moreover that $\mathsf{H}\mathsf{b} \leq 0$ and $\sqrt{-\mathsf{H}\mathsf{b}} \in \mathcal{C}^\infty$ on Σ. Let $\chi, \tilde{\chi} \in \mathcal{C}_{\mathrm{c}}^\infty(X)$ be cutoffs, identically 1 near ∂X, and with $\tilde{\chi} \equiv 1$ on $\operatorname{supp}\chi$. Let $E \in \Psi^0_{\mathrm{c}\hbar}(X)$, with Schwartz kernel supported in $[0, 1)_h \times (\tilde{\chi}^{-1}(1) \times \tilde{\chi}^{-1}(1))$.*

(1) (forward propagation) *Suppose $N(P)$ is injective at weight l on outgoing functions* (see Definition 4.6(1)). *Suppose that* (*the preimage in Σ of*) *all backward GBBs* (see Definition 4.5) *starting in $\Sigma \cap \operatorname{supp}\chi$ reach $\mathrm{Ell}_{\mathrm{c}\hbar}(E)$ in finite time while remaining inside $\tilde{\chi}^{-1}(1)$. Then for some small $\delta > 0$, we have*

$$\|\chi u\|_{H_{\mathrm{c},h}^{s,l,\alpha,\mathsf{b}}(X)} \leq C(\|\tilde{\chi}P_{h,z}u\|_{H_{\mathrm{c},h}^{s-2,l-2,\alpha,\mathsf{b}+1}(X)} + \|Eu\|_{H_{\mathrm{c},h}^{s,l,\alpha,\mathsf{b}}(X)} + h^\delta\|\tilde{\chi}u\|_{H_{\mathrm{c},h}^{-N,l,\alpha,\mathsf{b}}(X)}). \tag{4-27}$$

(2) (backward propagation) *Suppose $N(P)^*$ is injective at weight $-l+2$ on incoming functions* (see Definition 4.6(2)). *Suppose that* (*the preimage in Σ of*) *all forward GBBs starting in $\Sigma\cap\operatorname{supp}\chi$ reach $\mathrm{Ell}_{c\hbar}(E)$ in finite time while remaining inside $\tilde\chi^{-1}(1)$. Then for some small $\delta>0$, we have*

$$\|\chi u\|_{H_{c,h}^{-s+2,-l+2,-\alpha,-b-1}(X)} \leq C\big(\|\tilde\chi P_{h,z}^* u\|_{H_{c,h}^{-s,-l,-\alpha,-b}(X)} + \|Eu\|_{H_{c,h}^{-s+2,-l+2,-\alpha,-b-1}(X)} + h^\delta\|\tilde\chi u\|_{H_{c,h}^{-N,-l+2,-\alpha,-b-1}(X)}\big). \tag{4-28}$$

Since by Lemma 4.3 and the calculations in Section 4B, the operator $P_{h,z}\in\Psi_{c\hbar}^{2,2,0,0}(X)$ is elliptic at fiber infinity, and is of real-principal-type (except at the radial points) at sf, the estimates (4-27) and (4-28) are sharp as far as the relative orders in the norms on u on the left and $P_{h,z}^{(*)}u$ on the right are concerned. Indeed, it has the well-known real-principal-type loss of one order at sf and is an elliptic estimate in the $c\hbar$-differentiability sense.

The improvement of the final (error) terms on the right-hand sides in (4-27) and (4-28) relative to the space on the left-hand sides is accomplished at sf by microlocal symbolic means, and at tf using global normal operator estimates. The overall improvement by a positive power of h between error term and left-hand side allows for the inversion of $P_{h,z}$ for small $h>0$ under suitable assumptions on the *global* behavior of the null-bicharacteristic flow; see Sections 4E and 5 for examples.

Remark 4.11 (operators on vector bundles). Let $E\to X$ denote a smooth vector bundle. Theorem 4.10 then holds (with the same proof) also for operators $P_{h,z}$ acting on sections of E, provided $P_{h,z}$ is *admissible* in the sense that

$$P_{h,z} = h^2x^{-2}Q_{2,z} + hx^{-1}q_{0,z} - z,$$
$$Q_{2,z}\in\mathrm{Diff}_b^2(X;E),\qquad q_{0,z}\in\mathcal{C}^\infty(X;\mathrm{End}(E)),$$

where $x^{-2}Q_{2,z}$ (replacing the combination $h^2\Delta_g + h^2x^{-2}Q_{1,z}$ in Definition 4.1) has scalar principal symbol ${}^b\sigma(x^{-2}Q_{2,z}) = {}^b\sigma(x^{-2}\Delta_g)$. That is, ${}^b\sigma(x^{-2}Q_{2,z})(\zeta) = |\zeta|_{g^{-1}}^2$ for $\zeta\in{}^bT^*X$, with g the conic metric (4-2). The normal operator is of class

$$N(P)\in\left(\frac{\hat x}{\hat x+1}\right)^{-2}\mathrm{Diff}_{b,sc}^2(\mathrm{tf};\pi^*E_{\partial X}),$$

where $\pi:\mathrm{tf}=[0,\infty]_{\hat x}\times\partial X\to\partial X$ denotes the projection map. The injectivity conditions of Definition 4.6 are unchanged. The definition of the threshold quantities $r_{\mathrm{in/out}}$ in Definition 4.7 requires a minor change; to wit, with respect to a choice of a positive definite fiber inner product on $E_{\partial X}$, we set (top sign for “in”, bottom sign for “out”)

$$r_{\mathrm{in/out}} := -\frac12 \pm \frac12 \sup_{\partial X} {}^{sc}\sigma\left(\frac{x^{-2}Q_{2,1} - (x^{-2}Q_{2,1})^* + q_{0,1} - q_{0,1}^*}{2i}\right)\bigg|_{\mp\frac{dx}{x^2}}, \tag{4-29}$$

where the sup is defined to be the supremum of the largest eigenvalue of the scattering symbol (which takes values in self-adjoint endomorphisms of E). One may choose different fiber inner products in the calculation of r_{in} and r_{out}, respectively. A (near-)optimal choice of fiber inner products, resulting in (almost) the smallest possible r_{in} and largest possible r_{out}, is typically easy to read off in concrete situations. For example, if $Q_{2,1}=0$ and $q_{0,1}|_{\partial X}$ is block-diagonal (or more generally lower triangular) with respect to some bundle splitting of $E|_{\partial X}$, then the supremum in (4-29) can be made to be arbitrarily close to the supremum of $\operatorname{Im}\lambda$, where λ ranges over all eigenvalues of the diagonal entries of $q_{0,1}(y)$, $y \in \partial X$, if one chooses the fiber inner product appropriately.

Remark 4.12 (technical assumptions on the variable order). One can replace the assumptions that b be locally constant near $\mathcal{R}_{\mathrm{in/out}}$ and satisfy $\sqrt{-\mathsf{H}\mathsf{b}} \in \mathcal{C}^\infty$ on Σ by the simpler assumption that $\mathsf{H}\mathsf{b} \le 0$ on Σ. This would require the use of the sharp Gårding inequality for the c$\hbar$-calculus, which however we do not prove here.

Proof of Theorem 4.10. We give details for the proof of part (1); the proof of part (2) is completely analogous. If backward GBBs starting in $\mathrm{WF}'_{\mathrm{c}\hbar}(B)$ never pass through $\partial\Sigma \subset {}^{\mathrm{c}\hbar}T^*_{\mathrm{sf}\cap\mathrm{tf}}X_{\mathrm{c}\hbar}$, the orders l and a are irrelevant, and the estimate (4-27) follows from standard elliptic regularity and real-principal-type propagation in the (variable-order) semiclassical calculus on X°. We shall thus work in a small neighborhood of $x=0$.

Step 1: symbolic positive commutator estimate. We first work near the incoming radial set $\mathcal{R}_{\mathrm{in}}$ defined in (4-11); we shall use the coordinates $(\hat h, x, y, \xi, \eta)$ near ${}^{\mathrm{c}\hbar}T^*_{\mathrm{sf}\cap\mathrm{tf}}X_{\mathrm{c}\hbar}$ defined by (3-3) and (4-4). Fix cutoffs $\chi_\partial, \chi_{\mathrm{sf}}, \chi_{\mathcal{R}} \in \mathcal{C}^\infty_{\mathrm{c}}([0,1))$, identically 1 near 0 and satisfying $\chi'_\bullet \le 0$ and $\sqrt{-\chi_\bullet\chi'_\bullet} \in \mathcal{C}^\infty([0,1))$. Denote a smooth extension of b to $\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar}$ by the same symbol. For small $\delta > 0$, fixed momentarily, we then consider a commutant

$$\check a = \hat h^{-\mathsf{b}-\frac12} x^{-\alpha} \chi_\partial\Big(\frac{x}{\delta}\Big)\chi_{\mathrm{sf}}\Big(\frac{\hat h}{\delta}\Big)\chi_{\mathcal{R}}\Big(\frac{\omega}{\delta}\Big), \qquad \omega := \sqrt{|\eta|^2 + |\xi+1|^2}. \tag{4-30}$$

Thus, $\operatorname{supp} a$ is contained in any fixed open neighborhood of $\mathcal{R}_{\mathrm{in}}$ when $\delta > 0$ is sufficiently small. We have $\check a \in S^{-\infty,-\infty,\alpha,\mathsf{b}+1/2}(\overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar})$. Let

$$\check A \in \Psi_{\mathrm{c}\hbar}^{-\infty,-\infty,\alpha,\mathsf{b}+\frac12}(X), \quad {}^{\mathrm{c}\hbar}\sigma(\check A) = \check a, \;\; A := \check A^* \check A.$$

Using the $L^2(X;\mu)$ inner product, we then evaluate the commutator

$$\begin{gathered} 2\operatorname{Im}\langle \check A P_{h,z} u, \check A u\rangle = \langle \mathscr{C} u, u\rangle, \\ \mathscr{C} = i[P_{h,z}, A] + 2\frac{P_{h,z} - P^*_{h,z}}{2i} A \in \Psi_{\mathrm{c}\hbar}^{-\infty,-\infty,2a,2\mathsf{b}}(X). \end{gathered} \tag{4-31}$$

The principal symbol of $\mathscr{C}$ is

$$2\hat h \check a \mathsf{H} \check a + 2 \cdot {}^{\mathrm{c}\hbar}\sigma\Big(\frac{P_{h,z} - P^*_{h,z}}{2i\hat h}\Big)\hat h \check a^2. \tag{4-32}$$

When H hits χ_∂, we obtain a nonnegative contribution (in fact, the square e^2 of a smooth function e), while differentiation of $\chi_{\mathcal{R}}$ gives a nonpositive contribution (in fact, a negative square $-b^2_{\mathcal{R}}$), consistently with the saddle point structure of H at $\mathcal{R}_{\mathrm{in}}$. Differentiation of χ_{sf} produces a symbol with semiclassical order $-\infty$.

The main term of $\hat h \check a \mathsf{H} \check a$ near $\mathcal{R}_{\mathrm{in}}$ arises from differentiation of the weight $\hat h^{-\mathsf{b}-1/2} x^{-\alpha}$; since $H_p = \hat h \mathsf{H}$ is, modulo $\hat h x \mathcal{V}_{\mathrm{b}}$, given by the expression (4-10), we can compute this modulo $x S^{-\infty,-\infty,2a,2\mathsf{b}}$ by substituting the expression (4-10) of $\mathsf{H}|_{x=0}$ for H. Thus, the main term is

$$\hat{h}^{-2\mathsf{b}}x^{-2\alpha}(2\xi(-2\alpha+2\mathsf{b}+1)+\mathcal{O}(x))\chi_\partial^2\chi_{\mathrm{sf}}^2\chi_{\mathcal{R}}^2.$$

A further contribution arises from the skew-adjoint part of $P_{h,z}$ at $\mathcal{R}_{\mathrm{in}}$, which is the same as the skew-adjoint part of $N(P)$ at ${}^{\mathrm{sc}}\mathcal{R}_{\mathrm{in}}$ upon making the identification (3-7); this was already computed in the proof of Lemma 4.8. Overall then, we can write

$$ {}^{\mathrm{c}\hbar}\sigma(\mathscr{C}) = e^2 - b_{\mathcal{R}}^2 - \epsilon\hat{h}\check{a}^2 - f^2\hat{h}\check{a}^2, \tag{4-33}$$

where

$$f = \sqrt{-[2(-2(\mathsf{b}-\alpha)-1)+2(r_1+r_0)]-\epsilon}$$

is positive (and smooth) at $\mathcal{R}_{\mathrm{in}}$ for small $\epsilon>0$. Denoting c$\hbar$-quantizations of the lower case symbols by the corresponding upper case letters, we thus have

$$\mathscr{C} = E^*E - B_{\mathcal{R}}^*B_{\mathcal{R}} - \epsilon\|\hat{h}^{\frac{1}{2}}\check{A}u\|^2 - \|\hat{h}^{\frac{1}{2}}F\check{A}u\|^2 + R,$$

where $R\in\Psi_{\mathrm{c}\hbar}^{-\infty,-\infty,2\alpha,2\mathsf{b}-1}(X)$ has $\mathrm{WF}'_{\mathrm{c}\hbar}(R)\subset\operatorname{supp}\check{a}$ and arises as the remainder term not controlled by the previous symbolic considerations. We will plug this into the right-hand side of (4-31); the left-hand side is bounded from below by

$$-\epsilon\|\hat{h}^{\frac{1}{2}}\check{A}u\|^2 - \epsilon^{-1}\|\hat{h}^{-\frac{1}{2}}\check{A}P_{h,z}u\|^2 \geq -\epsilon\|\hat{h}^{\frac{1}{2}}\check{A}u\|^2 - C\epsilon^{-1}\|GP_{h,z}u\|_{H_{\mathrm{c},h}^{-N,-N,\alpha,\mathsf{b}+1}} - C\epsilon^{-1}\|\tilde{\chi}u\|^2_{H_{\mathrm{c},h}^{-N,-N,\alpha,-N}},$$

where $G\in\Psi_{\mathrm{c}\hbar}^0(X)$ is elliptic on $\operatorname{supp}\check{a}$; here $N\in\mathbb{R}$ is arbitrary. Putting $B_0:=\hat{h}^{1/2}F\check{A}\in\Psi_{\mathrm{c}\hbar}^{-\infty,-\infty,\alpha,\mathsf{b}}(X)$ and dropping the contribution of $B_{\mathcal{R}}$, we thus obtain the estimate

$$\|B_0u\|^2 \lesssim \|GP_{h,z}u\|^2_{H_{\mathrm{c},h}^{-N,-N,\alpha,\mathsf{b}+1}} + \|Eu\|^2 + |\langle Ru,u\rangle| + \|\tilde{\chi}u\|^2_{H_{\mathrm{c},h}^{-N,-N,\alpha,-N}}, \tag{4-34}$$

which provides $H_{\mathrm{c},h}^{-N,-N,\alpha,\mathsf{b}}$-control of u microlocally near $\mathcal{R}_{\mathrm{in}}$ provided one has microlocal $H_{\mathrm{c},h}^{-N,-N,-N,\mathsf{b}}$-control of u on $\mathrm{WF}'_{\mathrm{c}\hbar}(E)\subset\{0<x<\delta,\ |\xi+1|<\delta\}$, and provided $|\langle Ru,u\rangle|$ is finite; since G is elliptic near $\mathrm{WF}'_{\mathrm{c}\hbar}(R)$, we can insert the estimate

$$|\langle Ru,u\rangle| \lesssim \|Gu\|^2_{H_{\mathrm{c},h}^{-N,-N,\alpha,\mathsf{b}-1/2}} + \|\tilde{\chi}u\|^2_{H_{\mathrm{c},h}^{-N,-N,\alpha,-N}}$$

into the right-hand side of (4-34).

Concatenating this radial point estimate with the propagation of regularity from a punctured neighborhood of $\mathcal{R}_{\mathrm{in}}$ to a punctured neighborhood of $\mathcal{R}_{\mathrm{out}}$ and then a radial point estimate at $\mathcal{R}_{\mathrm{out}}$ — proved by the same method, with the commutant $\check{a}$ again given by (4-30) but now with $\omega=(|\eta|^2+|\xi-1|^2)^{1/2}$ and using that $2(\mathsf{b}-\alpha)+1-r_1+r_0<0$ — and using elliptic estimates away from Σ, we obtain the propagation estimate

$$\|B_1u\|_{H_{\mathrm{c},h}^{s,l,\alpha,\mathsf{b}}} \lesssim \|GP_{h,z}u\|_{H_{\mathrm{c},h}^{s-2,l-2,\alpha,\mathsf{b}+1}} + \|Eu\|_{H_{\mathrm{c},h}^{s,-N,-N,\mathsf{b}}} + \|\tilde{\chi}u\|_{H_{\mathrm{c},h}^{-N,l,\alpha,\mathsf{b}-1/2}}; \tag{4-35}$$

the operators $B_1,G,E\in\Psi_{\mathrm{c}\hbar}^{0,0,0,0}(X)$ appearing here are subject to the following conditions: $\mathrm{WF}'_{\mathrm{c}\hbar}(B_1)\subset\mathrm{Ell}_{\mathrm{c}\hbar}(G)$, furthermore G is elliptic on $\Sigma\cap x^{-1}(0)$, and all backward null-bicharacteristics from $\mathrm{WF}'_{\mathrm{c}\hbar}(B_1)$

either tend to $\mathcal{R}_{\text{out}}$ or enter $\text{Ell}_{c\hbar}(E)$. The orders at cf are unconstrained at this point, but chosen for compatibility with the normal operator argument below.

Fixing a variable-order function $\mathsf{b}^\flat$ so that

$$\mathsf{b}^\flat - \alpha > r_{\text{in}} \quad \text{at } \mathcal{R}_{\text{in}}, \tag{4-36}$$

we then have the estimate

$$\|\chi u\|_{H^{s,l,\alpha,\mathsf{b}}_{c,h}} \lesssim \|\tilde{\chi} P_{h,z} u\|_{H^{s-2,l-2,\alpha,\mathsf{b}+1}_{c,h}} + \|Eu\|_{H^{s,-N,-N,\mathsf{b}}_{c,h}} + \|\tilde{\chi} u\|_{H^{-N,l,\alpha,\mathsf{b}^\flat}_{c,h}} \tag{4-37}$$

under the assumptions on E, χ, $\tilde{\chi}$ stated in the theorem. For $\mathsf{b}^\flat \geq \mathsf{b} - \frac{1}{2}$, this follows directly from (4-35) (upon replacing the microlocal cutoff G by the less precise cutoff $\tilde{\chi}$). For general $\mathsf{b}^\flat$, note that as long as (4-36) is satisfied, one can apply this estimate inductively to the error term $\tilde{\chi} u$ provided $\operatorname{supp} \tilde{\chi}$ is sufficiently close to $\operatorname{supp} \chi$ (so that the same operator Eu satisfies the geometric control assumption for $\tilde{\chi}$ in place of χ), increasing supports of the involved cutoff functions by an arbitrarily small but positive amount and gaining half a semiclassical order at each step. Thus, away from $\mathcal{R}_{\text{in}}$, one can ultimately take $\mathsf{b}^\flat$ to be arbitrarily negative, while at $\mathcal{R}_{\text{in}}$, one always needs to have (4-36).

Step 2: normal operator estimate. We now work on the error term $\tilde{\chi} u$ in (4-37). We first prove the desired estimate (4-27) under the stronger condition that $\mathsf{b} - \alpha > r_{\text{in}} + 1$ at $\mathcal{R}_{\text{in}}$. We split $\tilde{\chi} u = \chi^\flat u + (1 - \chi^\flat)\chi \tilde{u}$, where $\chi^\flat \in \mathcal{C}^\infty(X)$ is identically 1 near ∂X and supported in a very small neighborhood of ∂X; the part $(1 - \chi^\flat)\tilde{\chi} u$ is supported away from $\text{cf} \cup \text{tf}$; hence

$$\|(1 - \chi^\flat)\tilde{\chi} u\|_{H^{-N,l,\alpha,\mathsf{b}^\flat}_{c,h}} \lesssim \|(1 - \chi^\flat)\tilde{\chi} u\|_{H^{-N,-N,-N,\mathsf{b}^\flat}_{c,h}}$$

for any N. To estimate $\chi^\flat u$, we use the injectivity assumption on $N(P)$ and the resulting estimate (4-19) together with Corollary 3.7(2) (with $\alpha_\mu = n$). For $0 < \delta < 1$ with $\mathsf{b}^\flat + 2\delta < \mathsf{b}$, and choosing $\operatorname{supp} \chi^\flat$ sufficiently small, we obtain

$$\begin{aligned} \|\chi^\flat u\|_{H^{-N,l,\alpha,\mathsf{b}^\flat}_{c,h}} &\lesssim h^{\frac{n}{2}-\alpha} \|\pi^*(\chi^\flat u)\|_{H^{-N,l,\mathsf{b}^\flat|_{\text{sf}\cap\text{tf}}-\alpha+\delta}_{\text{b,sc}}} \\ &\lesssim h^{\frac{n}{2}-\alpha} \|N(P)(\pi^*(\chi^\flat u))\|_{H^{-N-2,l-2,\mathsf{b}^\flat|_{\text{sf}\cap\text{tf}}-\alpha+\delta+1}_{\text{b,sc}}} \\ &\lesssim \|N(P)(\chi^\flat u)\|_{H^{-N-2,l-2,\alpha,\mathsf{b}^\flat+1+2\delta}_{c,h}}. \end{aligned} \tag{4-38}$$

In the final line, we abuse notation and denote by $N(P) \in \Psi^{2,2,0,0}_{c\hbar}(X)$ any operator whose normal operator is equal to $N(P)$. Put $\mathsf{b}^\sharp := \mathsf{b}^\flat + 1 + 2\delta$. Using Lemma 4.3, which gives $N(P) - P_{h,z} \in \Psi^{2,2,-1,0}_{c\hbar}(X)$, we further estimate

$$\begin{aligned} &\|N(P)(\chi^\flat u)\|_{H^{-N-2,l-2,\alpha,\mathsf{b}^\sharp}_{c,h}} \\ &\quad \leq \|\chi^\flat P_{h,z} u\|_{H^{-N-2,l-2,\alpha,\mathsf{b}^\sharp}_{c,h}} + \|\chi^\flat (P_{h,z} - N(P))u\|_{H^{-N-2,l-2,\alpha,\mathsf{b}^\sharp}_{c,h}} + \|[N(P), \chi^\flat]u\|_{H^{-N-2,l-2,\alpha,\mathsf{b}^\sharp}_{c,h}} && \text{(4-39)} \\ &\quad \lesssim \|\chi^\flat P_{h,z} u\|_{H^{-N-2,l-2,\alpha,\mathsf{b}^\sharp}_{c,h}} + \|\chi^\flat u\|_{H^{-N,l,\alpha-1,\mathsf{b}^\sharp}_{c,h}} + \|\chi^\sharp u\|_{H^{-N-1,-N,-N,\mathsf{b}^\sharp-1}_{c,h}}, && \text{(4-40)} \end{aligned}$$

where $\chi^\sharp \equiv 1$ on $\operatorname{supp} \chi^\flat$. Under the present condition that $\mathsf{b} - \alpha > r_{\text{in}} + 1$ at $\mathcal{R}_{\text{in}}$, we can choose $\mathsf{b}^\flat$ as in (4-36) so that $\mathsf{b}^\sharp < \mathsf{b}$ still. Plugging this into (4-37) finishes the proof of part (1) under this condition.

In order to prove the theorem as stated, thus only assuming $\mathsf{b}-\alpha > r_{\mathrm{in}}$ at $\mathcal{R}_{\mathrm{in}}$, we note that the norm on second term on the right in (4-40) is one order weaker at tf than the left-hand side of the desired estimate (4-27), but only $\mathsf{b}^\sharp - \mathsf{b} = \mathsf{b}^\flat + 1 + 2\delta - \mathsf{b} < 1$ orders stronger at sf. This suggests revisiting the estimates (4-38)–(4-40) using a more precise cutoff which distinguishes between the regimes $\hat h \lesssim x$ and $\hat h \gtrsim x$. To wit, consider $\psi^\flat = \tilde\psi^\flat(\hat h/x)$, where $\tilde\psi^\flat \equiv 0$ on $[0,1]$ and $\tilde\psi^\flat \equiv 1$ on $[2,\infty)$. This is a smooth function on $[X_{\mathrm{c}\hbar}; \mathrm{sf}\cap\mathrm{tf}]$, and thus conormal on $X_{\mathrm{c}\hbar}$; in fact, we have

$$1-\psi^\flat \in \mathcal{A}^{0,\zeta,-\zeta}(X_{\mathrm{c}\hbar}) = (x+h)^\zeta \left(\frac{h}{h+x}\right)^{-\zeta} \mathcal{A}^0(X_{\mathrm{c}\hbar}) \subset \Psi_{\mathrm{c}\hbar}^{0,0,-\zeta,\zeta}(X) \tag{4-41}$$

for any $\zeta \ge 0$, since on $\operatorname{supp}(1-\psi^\flat)$ we have $x \lesssim \hat h$; thus $x+h \lesssim h/(h+x)$. Taking $\zeta=\delta$, we can therefore estimate

$$\|(1-\psi^\flat)\tilde\chi u\|_{H_{\mathrm{c},h}^{-N,l,\alpha,\mathsf{b}^\flat}} \lesssim \|\tilde\chi u\|_{H_{\mathrm{c},h}^{-N,l,\alpha-\delta,\mathsf{b}^\flat+\delta}} \le h^\delta \|\tilde\chi u\|_{H_{\mathrm{c},h}^{-N,l,\alpha,\mathsf{b}}}.$$

Next, the estimate (4-38) holds without change. (Note that Corollary 3.7 applies for merely conormal cutoffs.) Finally, we need to estimate (4-39) more carefully. Note that

$$\psi^\flat \in \mathcal{A}^{0,-\zeta,\zeta}(X_{\mathrm{c}\hbar}) \subset \Psi_{\mathrm{c}\hbar}^{0,0,\zeta,-\zeta}(X)$$

for any $\zeta \ge 0$. Taking $\zeta = 1-\delta$, this gives $\psi^\flat(P_{h,z}-N(P)) \in \Psi_{\mathrm{c}\hbar}^{2,2,-\delta,-1+\delta}(X)$; hence

$$\|\psi^\flat(P_{h,z}-N(P))u\|_{H_{\mathrm{c},h}^{-N-2,l-2,\alpha,\mathsf{b}^\sharp}} \lesssim \|\tilde\chi u\|_{H_{\mathrm{c},h}^{-N,l,\alpha-\delta,\mathsf{b}^\sharp-1+\delta}} \le h^\delta \|\tilde\chi u\|_{H_{\mathrm{c},h}^{-N,l,\alpha,\mathsf{b}}}.$$

For the final, commutator, term in (4-39), we note that we can replace $\psi^\flat$ by $1-\psi^\flat$ and use (4-41) with $\zeta=\delta$, so $[N(P),\psi^\flat] \in \Psi_{\mathrm{c}\hbar}^{1,-\infty,-\delta,-1+\delta}(X)$, which gives

$$\|[N(P),\psi^\flat]u\|_{H_{\mathrm{c},h}^{-N-2,l-2,\alpha,\mathsf{b}^\sharp}} \lesssim \|\tilde\chi u\|_{H_{\mathrm{c},h}^{-N-1,-N,\alpha-\delta,\mathsf{b}^\sharp-1+\delta}} \le h^\delta \|\tilde\chi u\|_{H_{\mathrm{c},h}^{-N-1,-N,\alpha,\mathsf{b}}}.$$

Altogether, we have shown

$$\|\tilde\chi u\|_{H_{\mathrm{c},h}^{-N,l,\alpha,\mathsf{b}^\flat}} \lesssim \|\tilde\chi P_{h,z} u\|_{H_{\mathrm{c},h}^{-N-2,l-2,\alpha,\mathsf{b}^\sharp}} + h^\delta \|\tilde\chi u\|_{H_{\mathrm{c},h}^{-N,l,\alpha,\mathsf{b}}}.$$

Plugged into (4-37), we have now established the desired estimate (4-27). □

We can sharpen Theorem 4.10 by working with the resolved Sobolev spaces defined in (3-24). This is straightforward since admissible operators

$$P_{h,z} \in \Psi_{\mathrm{c}\hbar}^{2,2,0,0}(X) \subset \Psi_{\mathrm{cb}\hbar}^{2,2,2,0,0}(X)$$

are elliptic at the front face fbf of $\overline{{}^{\mathrm{cb}\hbar}T^*}X_{\mathrm{c}\hbar}$; indeed, this follows from the ellipticity at fiber infinity ${}^{\mathrm{c}\hbar}S^*_{\mathrm{tf}}X_{\mathrm{c}\hbar} \subset \overline{{}^{\mathrm{c}\hbar}T^*}X_{\mathrm{c}\hbar}$ and the classical nature of the principal symbol of $P_{h,z}$. Therefore:

Theorem 4.13 (propagation estimates with relative b-regularity). *In the notation of Theorem 4.10, and for any $s' \in \mathbb{R}$, the forward propagation estimate (4-27) generalizes to an estimate on* cbℏ*-Sobolev spaces,*

$$\|\chi u\|_{H_{\mathrm{cb},h}^{s,s',l,\alpha,\mathsf{b}}(X)} \le C\big(\|\tilde\chi P_{h,z}u\|_{H_{\mathrm{cb},h}^{s-2,s'-2,l-2,\alpha,\mathsf{b}+1}(X)} + \|Eu\|_{H_{\mathrm{cb},h}^{s,s',l,\alpha,\mathsf{b}}(X)} + h^\delta \|\tilde\chi u\|_{H_{\mathrm{cb},h}^{-N,-N,l,\alpha,\mathsf{b}}(X)}\big).$$

The backward propagation estimate (4-28) generalizes similarly.

4E. ***Global estimates with complex absorption.*** We upgrade the microlocal estimate proved above into a quantitative invertibility statement for an operator which effectively localizes the interesting nonelliptic phenomena near the cone point into a small neighborhood of ∂X via complex absorption.

Let us assume first for simplicity that Y is null-cobordant; see Remark 4.16 below on how to remove this assumption. With $X = [0, 2x_0) \times Y$ and g as in (4-1)–(4-2), consider then a compact n-dimensional manifold $X' \supset X$ with boundary $\partial X' = \partial X$, equipped with a smooth metric g' which is equal to g on $X^\flat := [0, x_0] \times Y$. Given an admissible operator $P_{h,z}$ on X, let $P'_{h,z} \in (x/(x+h))^{-2}\,\mathrm{Diff}^2_{\mathrm{c}\hbar}(X')$ denote an extension of $P_{h,z}$ from $[0, 1) \times X^\flat$ to $[0, 1) \times X'$ with principal part equal to $h^2\Delta_{g'}$. For $c \in (0, 1)$, let

$$K_c := X' \setminus ([0, cx_0] \times Y).$$

In order to implement complex absorption, let us take $c \in \left(0, \frac{1}{2}\right)$ small and fix an operator

$$Q \in \Psi_h^{-\infty}((X')^\circ)$$

whose Schwartz kernel is supported in $K_c \times K_c$, and so that Q is elliptic on T^*K_{2c} with nonnegative principal symbol. We then consider

$$\mathcal{P}_{h,z} := P'_{h,z} - iQ, \tag{4-42}$$

and assume that

$$\text{all backward GBBs of } P'_{h,z} \text{ enter } \mathrm{Ell}_\hbar(Q) \text{ in finite time.} \tag{4-43}$$

By construction, $\mathcal{P}_{h,z}$ is a semiclassically elliptic second-order semiclassical ps.d.o. on $(X')^\circ$ which is elliptic over K_{2c}. Moreover, due to the sign condition on the principal symbol of Q, one can propagate semiclassical regularity for solutions of $\mathcal{P}_{h,z}u = f$ along *forward* null-bicharacteristics of $P'_{h,z}$; see [Vasy 2013, §2.5] and [Dyatlov and Zworski 2019, §5.6.3]. For our fixed metric g on $[0, 2x_0) \times Y$, the control condition (4-43) is satisfied if we choose $c > 0$ sufficiently small. Indeed, from the expression (4-13), one finds that if $\mathsf{H}_{\mathrm{sf}}x = 2\xi = 0$ on the characteristic set, then $|\eta|^2 = 1$ and thus

$$\mathsf{H}^2_{\mathrm{sf}}x = 2\mathsf{H}\xi \geq 2x^{-1}|\eta|^2 - C|\eta|^2 = 2x^{-1} - C > c^{-1} - C > 0$$

in $x < 2c$ when c is sufficiently small; hence the level sets of x are geodesically convex in $x < 2c$, which implies the claim.

Remark 4.14 (relaxed conditions on Q). One can more generally allow Q to be a second-order operator with real-principal symbol; a concrete choice is then $Q = \psi \cdot (h^2\Delta_{g'} + 1)$, where $\psi \in \mathcal{C}^\infty_{\mathrm{c}}(K_c^\circ)$ is identically 1 on K_{2c}.

We then have:

Proposition 4.15 (global estimates with complex absorption). *Let s, l, α, b be as in the statement of Theorem 4.10 (for the operator $P_{h,z}$). Fix the volume density on X' to be the metric density $|\mathrm{d}g'|$. Then for small $h > 0$, the operator $\mathcal{P}_{h,z}$ defined by (4-42) is invertible as a map $H^{s,l}_{\mathrm{b}}(X') \to H^{s-2,l-2}_{\mathrm{b}}(X')$, and it satisfies the uniform estimate*

$$\|u\|_{H^{s,l,\alpha,\mathsf{b}}_{\mathrm{c},h}(X')} \leq C\|\mathcal{P}_{h,z}u\|_{H^{s-2,l-2,\alpha,\mathsf{b}+1}_{\mathrm{c},h}(X')} = Ch^{-1}\|(x+h)\mathcal{P}_{h,z}u\|_{H^{s-2,l-2,\alpha,\mathsf{b}}_{\mathrm{c},h}}. \tag{4-44}$$

More generally, for any $s' \in \mathbb{R}$, *we have*

$$\|u\|_{H^{s,s',l,\alpha,\mathsf{b}}_{\mathrm{cb},h}(X')} \le C\|\mathcal{P}_{h,z}u\|_{H^{s-2,s'-2,l-2,\alpha,\mathsf{b}+1}_{\mathrm{cb},h}}.$$

Proof. By our assumptions on the complex-absorbing potential Q, we can apply Theorem 4.10(1) with E and χ supported in $X' \setminus K_c$. We thus have

$$\|\chi u\|_{H^{s,l,\alpha,\mathsf{b}}_{\mathrm{c},h}} \le C(\|\mathcal{P}_{h,z}u\|_{H^{s-2,l-2,\alpha,\mathsf{b}+1}_{\mathrm{c},h}} + \|Eu\|_{H^{s,-N,-N,\mathsf{b}}_{\mathrm{c},h}} + h^\delta\|\tilde{\chi}u\|_{H^{-N,l,\alpha-1,\mathsf{b}^\flat}_{\mathrm{c},h}}).$$

On the other hand, we can control Eu and $(1-\chi)u$ in $H^{s,l,\alpha,\mathsf{b}}_{\mathrm{c},h}$ (or simply $h^{\mathsf{b}}H^s_h$ if we take E to be localized away from $x=0$, as we may arrange) by $\mathcal{P}_{h,z}u$ in $H^{s-2,l-2,\alpha,\mathsf{b}+1}_{\mathrm{c},h}$ using a combination of elliptic estimates and real-principal-type propagation estimates (with complex absorption), starting either from $\mathrm{Ell}_\hbar(Q)$ or $\{\chi = 1\}$. Altogether, we obtain

$$\|u\|_{H^{s,l,\alpha,\mathsf{b}}_{\mathrm{c},h}} \le C(\|\mathcal{P}_{h,z}u\|_{H^{s-2,l-2,\alpha,\mathsf{b}+1}_{\mathrm{c},h}} + h^\delta\|u\|_{H^{-N,l,\alpha,\mathsf{b}}_{\mathrm{c},h}}). \tag{4-45}$$

For $h_0 > 0$ with $Ch_0^\delta < \frac{1}{2}$, we can now drop the error term in (4-45) for $0 < h < h_0$. This proves the injectivity of $\mathcal{P}_{h,z}$ (with a quantitative estimate). Analogous arguments prove the dual estimate

$$\|v\|_{H^{-s+2,-l+2,-\alpha,-\mathsf{b}-1}_{\mathrm{c},h}} \le C\|\mathcal{P}^*_{h,z}v\|_{H^{-s,-l,-\alpha,-\mathsf{b}}_{\mathrm{c},h}},$$

which implies the surjectivity of $\mathcal{P}_{h,z}$. □

Remark 4.16 (links Y that are not null-cobordant). When Y is not null-cobordant, we cannot choose X' as above. This is a technical issue, independent of the analysis near the cone point $x^{-1}(0)$, which we circumvent here with the following artificial device: we set $X' := [0, 4x_0] \times Y$, and consider h-dependent families of operators on X' which are semiclassical cone operators near $x^{-1}(0)$ and semiclassical scattering operators [Vasy and Zworski 2000] near $x^{-1}(4x_0)$. We then take $P'_{h,z} \in (x/(x+h))^{-2}\,\mathrm{Diff}^2_{\mathrm{c}\hbar,\mathrm{sc}\hbar}(X')$ — the second subscript referring to the semiclassical scattering behavior near $x^{-1}(4x_0)$ — to be equal to $P_{h,z}$ on $[0,1) \times X^\flat$. We can arrange for $P'_{h,z}$ to be elliptic near $x^{-1}(4x_0)$ in the semiclassical scattering algebra, e.g., by taking it to be equal to $h^2\Delta_{g'} + 1$ near $x = 4x_0$, where g' is a scattering metric on $(x_0, 4x_0] \times Y$. We choose the complex absorbing operator Q as before, and so that $\mathcal{P}_{h,z} = P'_{h,z} - iQ$ is elliptic in $x > \frac{1}{2}x_0$. Proposition 4.15 then remains valid upon using function spaces for u which near $x^{-1}(4x_0)$ are semiclassical scattering Sobolev spaces with differential order s, semiclassical order b, and arbitrary decay order r, and similarly for $\mathcal{P}_{h,z}u$ with orders $s-2$, $\mathsf{b}+1$, r; this uses elliptic estimates in the semiclassical scattering algebra near $x^{-1}(4x_0)$. (This usage of the semiclassical scattering algebra is only one of several possibilities in which the invertibility of $\mathcal{P}_{h,z}$ for small h is easy to obtain despite the presence of a boundary.)

4F. ***Propagation of Lagrangian regularity: diffractive improvement.*** By adapting arguments from [Melrose and Wunsch 2004; Melrose et al. 2008], we improve upon Theorem 4.10 by demonstrating that, under a nonfocusing condition, strong singularities can only propagate along geometric GBBs. The key technical result concerns the propagation of Lagrangian regularity with respect to the incoming and outgoing Lagrangian submanifolds, localized near geometric continuations of a GBB striking the cone point. Using the coordinates $(\hat{h}, x, y, \xi, \eta)$ and the notation of (4-14), the incoming and outgoing Lagrangians are given by

$$\mathcal{F}_\bullet := \bigcup_{y_0 \in \partial X} \mathcal{F}_{\bullet, y_0}, \quad \bullet = I, O,$$

where

$$\begin{aligned} \mathcal{F}_{I,y_0} &:= \gamma_{I,y_0}((-x_0, 0)) = \{(0, x, y_0, -1, 0) : x < 2x_0\}, \\ \mathcal{F}_{O,y_0} &:= \gamma_{O,y_0}((-x_0, 0)) = \{(0, x, y_0, 1, 0) : x < 2x_0\}. \end{aligned} \tag{4-46}$$

(We are making the $\hat{h}$-coordinate, which was set to 0 in (4-14), explicit here.)

We shall first show that one can control the Lagrangian regularity of a solution u of $P_{h,z}u = f$, with sufficiently regular forcing f, near $\mathcal{F}_{O,y_0}$ by propagating Lagrangian regularity from the union of all $\mathcal{F}^\circ_{I,y'}$, with y' at distance π from y_0, into ∂X and then within ∂X to $\mathcal{F}_{O,y_0} \cap x^{-1}(0)$. Localization within the radial sets $\mathcal{R}_{\text{in/out}}$ requires a more careful choice of commutants compared to the symbolic part of the proof of Theorem 4.10, and the extra Lagrangian regularity is captured using test modules, as introduced in [Hassell et al. 2008] and used for this purpose in [Melrose et al. 2008; 2013]; see also [Haber and Vasy 2013]. (Test modules also feature prominently in [Baskin et al. 2015; 2018; Gell-Redman et al. 2020].) Fix $x_0 < x_1 < x_2 < x_3 < 2x_0$ and cutoffs

$$\chi_j \in \mathcal{C}^\infty_c([0, x_j)), \qquad \chi_j \equiv 1 \quad \text{on } [0, x_{j-1}], \quad j = 1, 2, 3.$$

Mirroring [Melrose et al. 2008, Definition 4.2], we then introduce:

Definition 4.17 (test module). Let $\mathcal{F} = \mathcal{F}_I \cup \mathcal{F}_O$. Define the $\Psi^0_{c\hbar}(X)$-module[14]

$$\mathcal{M} := \left\{ A \in \Psi^{0,0,0,1}_{c\hbar}(X) : \operatorname{supp} K_A \subset [0,1)_h \times (\operatorname{supp} \chi_2)^2,\ {}^{c\hbar}\sigma_0\left(\frac{h}{h+x} A\right)\bigg|_{\mathcal{F}} = 0 \right\}.$$

Denote by $\mathcal{M}^k \subset \Psi^{0,0,0,k}_{c\hbar}(X)$ the set of finite linear combinations of up to k-fold products of elements of $\mathcal{M}$. If $\mathcal{X}$ is a function space on which $\Psi^0_{c\hbar}(X)$ acts continuously, we say that u has *Lagrangian regularity of order k relative to* $\mathcal{X}$ if $\mathcal{M}^k u \subset \mathcal{X}$. We say that elements of the space $\mathcal{M}^k \mathcal{X}$ satisfy the *nonfocusing condition of degree k relative to* $\mathcal{X}$.

Since

$$\Psi^{0,0,0,1}_{c\hbar}(X) = \left(\frac{h}{h+x}\right)^{-1} \Psi^0_{c\hbar}(X),$$

regularity with respect to elements of $\mathcal{M}$ means that the semiclassical order improves upon differentiation along suitable elements of $\Psi^0_{c\hbar}(X)$. A concrete example of an element of $\mathcal{M}$ in local coordinates is

$$\frac{h+x}{h}\left(\frac{h}{h+x} D_{y^j}\right) = D_{y^j}.$$

Lemma 4.18 (properties of $\mathcal{M}$; see [Melrose et al. 2008, Lemma 4.4]). *The set $\mathcal{M}$ is closed under commutators. Moreover, $\mathcal{M}$ is finitely generated in the sense that there exist $A_1, \dots, A_N \in \Psi^{0,0,0,1}_{c\hbar}(X)$*

[14] Recall that K_A denotes the Schwartz kernel of A.

with $\operatorname{supp} K_{A_j} \in [0,1)_h \times (\operatorname{supp} \chi_3)^2$ *so that, with* $A_0 := I$, *we have*

$$\mathcal{M} = \left\{ A \in \Psi_{c\hbar}^{0,0,0,1}(X) : \text{there exists } Q_j \in \Psi_{c\hbar}^0(X) \text{ such that } A = \sum_{j=0}^{N} Q_j A_j \right\}.$$

Concretely, one can take A_N *to have principal symbol*

$$\left(\frac{h+x}{h}\right) \cdot {}^{c\hbar}\sigma\left(\left(\frac{x}{x+h}\right)^2 P_{h,z}\right),$$

and one may take A_j, $1 \le j \le N-1$, *to have principal symbol* $((h+x)/h)a_j$, *where* $a_j \in \mathcal{C}^\infty(\overline{{}^{c\hbar}T^*}X_{c\hbar})$ *vanishes on* $\mathcal{F}$ *and has differential* $\mathrm{d}a_j$ *which at a point* $\zeta \in \mathcal{R}_{\mathrm{in}}$, *resp.* $\zeta \in \mathcal{R}_{\mathrm{out}}$ *lies in the unstable, resp. stable eigenspace of the linearization of* H (*as a vector field on* ${}^{c\hbar}T^*_{\mathrm{sf}}X_{c\hbar}$) *at* ζ.

Proof. Let $B = ((h+x)/h)B_0$, $C = ((h+x)/h)C_0 \in \mathcal{M}$. Denote the principal symbols of $B_0, C_0 \in \Psi_{c\hbar}^0(X)$ by b_0, c_0. We then have $[B, C] \in \Psi_{c\hbar}^{-1,0,0,1}(X)$, and

$$d := {}^{c\hbar}\sigma_0\left(\frac{h}{h+x} i[B,C]\right) = \frac{h}{h+x} H_b\left(\frac{h+x}{h} c_0\right) = \frac{h}{h+x} H_b\left(\frac{h+x}{h}\right) c_0 + H_b c_0.$$

But by (3-4), $H_b|_{\hat{h}=0}$ is a smooth b-vector field for $b \in S^{0,0,0,1}$; thus $d \in S^0(\overline{{}^{c\hbar}T^*}X_{c\hbar})$. Moreover, since $\mathcal{F}$ is a Lagrangian submanifold, H_b is tangent to $\mathcal{F}$; therefore, $H_b c_0 = 0$ on $\mathcal{F}$ since $c_0|_{\mathcal{F}} = 0$, and thus $d|_{\mathcal{F}} = 0$ as well. This proves $[B, C] \in \mathcal{M}$.

Let us now work in local coordinates $(\hat{h}, x, y, \xi, \eta)$ in which the rescaled Hamilton vector field $\mathsf{H} = \hat{h}^{-1} H_p$ of $P_{h,z}$ takes the form (4-9). The linearization of H at $\mathcal{R}_{\mathrm{out/in}}$ as a vector field on ${}^{c\hbar}T^*X_{c\hbar}$ is (top sign for "in", bottom sign for "out")

$$\mp 2(x\partial_x - \hat{h}\partial_{\hat{h}} - \eta\partial_\eta) + 2k^{ij}\eta_i\partial_{y^j}, \tag{4-47}$$

which thus has eigenvalue ∓ 2 (with eigenvector $\mathrm{d}x$), ± 2 (with eigenspace spanned by $\mathrm{d}\hat{h}$ and $\mathrm{d}\eta_j$), and 0 (with eigenspace spanned by $\mathrm{d}\xi$ and $\mathrm{d}y^j \pm k^{ij}\mathrm{d}\eta_i$). Upon restriction to $\hat{h} = 0$, the same statements remain true except there is no contribution from $\mathrm{d}\hat{h}$ anymore. Since $\mathcal{F}$ is locally the joint zero set of $\eta^1, \dots, \eta^{n-1}$, and p, which have linearly independent differentials, every smooth function vanishing on $\mathcal{F}$ can be written as a linear combination (with smooth coefficients) of p and η_j. Thus, we may take quantizations of $\hat{h}^{-1}\eta_j$ for the operators A_j in local coordinates. The full collection of A_j can be defined using a partition of unity. □

The fact that $\mathcal{M}$ is a $\Psi_{c\hbar}^0(X)$-module and a Lie algebra implies that

$$\mathcal{M}^k = \left\{ \sum_{|\alpha| \le k} Q_\alpha A^\alpha : Q_\alpha \in \Psi_{c\hbar}^0(X) \right\}, \quad A^\alpha := \prod_{i=1}^{N} A_i^{\alpha_i}, \tag{4-48}$$

where $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$. Modulo $\Psi_{c\hbar}^0(X)$, the operator A_N is a multiple of $P_{h,z}$; therefore, regularity of solutions u of $P_{h,z}u = f$, with f having Lagrangian regularity of order k, under application of an element $Q_\alpha \prod_{i=1}^N A_i^{\alpha_i} \in \mathcal{M}^k$ with $\alpha_N > 0$ is automatic once Lagrangian regularity of order $k-1$ has been established. In order to prove regularity of solutions of $P_{h,z}u = f$ under application of A_j, $1 \le j \le N-1$, we need to control the commutators of $P_{h,z}$ with the A_j chosen in Lemma 4.18:

Lemma 4.19 (commutators; see [Melrose et al. 2008, Lemma 4.5]). *With the A_j chosen as in Lemma 4.18, we have, for $j = 1, \dots, N-1$,*

$$i[P_{h,z}, A_j] = \sum_{k=0}^{N} C_{jk} A_k, \quad C_{jk} \in \Psi_{c\hbar}^{1,2,0,-1}(X),$$

and ${}^{c\hbar}\sigma_{1,2,0,-1}(C_{jk})|_{\mathcal{F}\cap x^{-1}(0)} = 0$ for $k \neq 0$.

Proof. Denote by a_j the principal symbol of $\hat{h}A_j$ for $j = 1, \dots, N-1$, so ${}^{c\hbar}\sigma(A_j) = \hat{h}^{-1}a_j$. Since $P_{h,z} \in \Psi_{c\hbar}^{2,2,0,0}(X)$, we have $i[P_{h,z}, A_j] \in \Psi_{c\hbar}^{1,2,0,0}(X)$, with principal symbol at sf given by $\hat{h}\mathsf{H}(\hat{h}^{-1}a_j)$ in the notation used in (4-9). It thus suffices to prove the existence of $c_{jk} \in S^{1,2,0,0}(\overline{{}^{c\hbar}T^*}X_{c\hbar})$ such that near $\hat{h} = 0$,

$$\hat{h}\mathsf{H}(\hat{h}^{-1}a_j) = \sum_{k=1}^{N} \hat{h} c_{jk} \hat{h}^{-1} a_k, \quad c_{jk}|_{\mathcal{F}\cap x^{-1}(0)} = 0 \ \ (k \neq 0); \tag{4-49}$$

indeed, if $C_{jk} \in \Psi_{c\hbar}^{1,2,0,-1}(X)$ is a quantization of $\hat{h}c_{jk}$ times a cutoff to a neighborhood of sf, then (4-49) implies that

$$C_{j0} := i[P_{h,z}, A_j] - \sum_{k=1}^{N} C_{jk} A_k \in \Psi_{c\hbar}^{0,2,0,-1}(X).$$

In order to verify (4-49), we note that the left-hand side equals $\hat{h}^{-1}(\hat{h}\mathsf{H}a_j - a_j\mathsf{H}\hat{h})$; but since at $\mathcal{F}\cap x^{-1}(0)$, the differentials $\mathrm{d}a_j$ and $\mathrm{d}\hat{h}$ are eigenvectors of the linearization of H with the same eigenvalue, as discussed after (4-47), this vanishes quadratically at $\mathcal{F}\cap x^{-1}(0)$, completing the proof. □

We are now ready to propagate Lagrangian regularity through the radial sets. For $s, l, \alpha, b \in \mathbb{R}$ and $k \in \mathbb{N}_0$, and using the notation (4-48), let

$$H_{c,h}^{s,l,\alpha,b;k}(X) := \{u \in H_{c,h}^{s,l,\alpha,b}(X) : A^\alpha u \in H_{c,h}^{s,l,\alpha,b}(X) \text{ for all } |\alpha| \leq k\}.$$

We recall that we will only encounter distributions on X with compact support, justifying the convenient, albeit slightly imprecise, notation here.

Proposition 4.20 (microlocalized propagation near the radial sets). *Let $s, l, \alpha, b \in \mathbb{R}$. Let $B, E, G \in \Psi_{c\hbar}^{0}(X)$ denote operators with Schwartz kernels supported in $[0,1)_h \times (\operatorname{supp}\chi_1)^2$. Recall the quantities $r_{\mathrm{in/out}}$ from Definition 4.7.*

(1) (propagation into $\mathcal{R}_{\mathrm{in}}$) *Suppose that all backward integral curves of H starting in $\Sigma \cap \mathrm{WF}'_{c\hbar}(B)$ either tend to a subset $\mathcal{S} \subset \mathcal{R}_{\mathrm{in}}$ or enter $\mathrm{Ell}'_{c\hbar}(E)$ in finite time while remaining inside $\mathrm{Ell}_{c\hbar}(G)$; and suppose that, for all incoming null-bicharacteristics $\gamma_{I,y_0} : (-x_0, 0) \to \Sigma$ with $\gamma_{I,y_0}(0) \in \mathcal{S}$, there exists $s \in (-x_0, 0)$ (depending on y_0) such that $\gamma_{I,y_0}((s,0]) \subset \mathrm{Ell}_{c\hbar}(G)$ and $\gamma_{I,y_0}(s) \in \mathrm{Ell}_{c\hbar}(E)$. Under the condition $b - \alpha > r_{\mathrm{in}}$, we then have*

$$\|Bu\|_{H_{c,h}^{s,l,\alpha,b;k}(X)} \leq C(\|GP_{h,z}u\|_{H_{c,h}^{s-2,l-2,\alpha,b+1;k}(X)} + \|Eu\|_{H_{c,h}^{-N,l,\alpha,b;k}(X)} + \|\chi_2 u\|_{H_{c,h}^{-N,l,\alpha,b-1/2}(X)}). \tag{4-50}$$

(2) (propagation out of $\mathcal{R}_{\mathrm{out}}$) *Suppose that all backward integral curves of H starting in $\Sigma \cap \mathrm{WF}'_{c\hbar}(B)$ either tend to a subset $\mathcal{S} \subset \mathcal{R}_{\mathrm{out}}$ or enter $\mathrm{WF}'_{c\hbar}(E)$ in finite time while remaining inside $\mathrm{Ell}_{c\hbar}(G)$. Suppose moreover that $\mathcal{S} \subset \mathrm{Ell}_{c\hbar}(G)$, and that, for every integral curve $\gamma \subset \Sigma \cap x^{-1}(0) \setminus \mathcal{R}_{\mathrm{out}}$ of H*

with $\lim_{s\to\infty}\gamma(s)\in\mathcal{S}$, *there exists* s *so that* $\gamma((s,\infty))\subset \mathrm{Ell}_{c\hbar}(G)$ *and* $\gamma(s)\in\mathrm{Ell}_{c\hbar}(E)$. *Then the estimate* (4-50) *holds under the condition* $b-\alpha<r_{\mathrm{out}}$.

Proof. We begin with the proof of part (1). By compactness of $\mathcal{R}_{\mathrm{in/out}}$ and since $\mathrm{Ell}_{c\hbar}$ is open, it suffices to prove microlocal estimates near a single point $\zeta_0\in\mathcal{R}_{\mathrm{in}}$, which in the coordinate system $(\hat h, x, y, \xi, \eta)$ used in (4-46) has coordinates $\zeta_0=(0,0,y_0,-1,0)$.

Now, restricted to $x=\hat h=0$ and writing $k=k(y,\eta)$ for the dual metric function of the metric $k(0)$ on ∂X in local coordinates, we have

$$\mathsf{H}=-2\xi\eta\partial_\eta+2|\eta|^2\partial_\xi+(\partial_\eta k)\partial_y-(\partial_y k)\partial_\eta.$$

Following [Melrose and Zworski 1996, Lemma 2], introducing $|\eta|$, $\hat\eta=\eta/|\eta|$, one has

$$\partial_s y=(\partial_\eta k)(y,\hat\eta)|\eta|,\qquad \partial_s\hat\eta=-(\partial_y k)(y,\hat\eta)|\eta|$$

along H-integral curves; reparametrizing to $t=t(s)$ satisfying $t'=2|\eta|$, one thus obtains

$$\partial_t y=\tfrac12(\partial_\eta k)(y,\hat\eta),\qquad \partial_t\hat\eta=-\tfrac12(\partial_y k)(y,\hat\eta),\qquad \partial_t|\eta|=-\xi,\qquad \partial_t\xi=|\eta|.$$

Thus, $\xi(t)=a\cos(t+\varphi_0)$ and $|\eta(t)|=a\sin(t+\varphi_0)$, where $a=\sqrt{\xi^2+|\eta|^2}$ is constant, and $\varphi_0\in[0,\pi]$. Therefore, the function Υ assigning to (y,ξ,η) near $(y_0,-1,0)$ the limiting point along the backward H-integral curve is given by evaluation at $t=-\varphi_0$, so

$$\Upsilon(y,\xi,\eta)=\biggl(\exp_y\biggl(\biggl(-\arccos\frac{\xi}{\sqrt{\xi^2+|\eta|^2}}\biggr)\frac{\eta}{|\eta|}\biggr),-1,0\biggr).$$

In particular, Υ is smooth, and $\mathsf{H}\Upsilon=0$ at $\hat h=x=0$. Extending Υ to a smooth function in a neighborhood of $x=\hat h=0$, with values in $\mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R}^{n-1}$, we thus have $\mathsf{H}\Upsilon=\mathcal{O}(x)$ at $\hat h=0$. Since $x^{-1}\mathsf{H}x=-2$ at ζ_0, we can choose C so that in any sufficiently small neighborhood $\mathcal{V}$ of ζ_0,

$$\mathsf{H}(|\Upsilon-\zeta_0|^2-Cx)\geq x>0\quad\text{in }\mathcal{V}.\tag{4-51}$$

Fix now cutoffs $\chi_{\mathcal{S}},\chi_\partial,\chi_{\mathrm{sf}},\chi_{\mathcal{F}},\chi_\Sigma\in\mathcal{C}^\infty_{\mathrm{c}}([0,1))$, identically 1 near 0, with nonpositive derivative and with $\sqrt{-\chi_\bullet\chi_\bullet'}\in\mathcal{C}^\infty$, and consider the commutant

$$\check a=\hat h^{-b-\frac12}x^{-\alpha}\chi_\partial\Bigl(\frac{x}{\delta}\Bigr)\chi_{\mathrm{sf}}\Bigl(\frac{\hat h}{\delta}\Bigr)\chi_{\mathcal{F}}\Bigl(\delta^{-1}\sum_{j=1}^{N-1}a_j^2\Bigr)\chi_\Sigma\Bigl(\frac{p^2}{\delta}\Bigr)\chi_{\mathcal{S}}(\delta^{-1}(|\Upsilon-\zeta_0|^2-Cx)),$$

where $\delta>0$ controls the size of $\operatorname{supp}\check a$. We now proceed as in the first step of the proof of Theorem 4.10. Thus, in the symbol (4-32) of the commutator appearing in (4-31), and specifically in the term $2\hat h\check a\mathsf{H}\check a$, the main contribution near ζ_0 arises from differentiation of the weights (and then the subprincipal symbol of $P_{h,z}$ enters in the threshold condition on $b-\alpha$ as there), giving a negative multiple of $\hat h^{-2b}x^{-2\alpha}$. Differentiation of $\chi_{\mathcal{F}}$ gives a term of the same sign, namely a negative square, since $\sum a_j^2$ is a local quadratic defining function of $\mathcal{R}_{\mathrm{in}}$ inside of $\Sigma\cap x^{-1}(0)$. In view of (4-51), differentiation of $\chi_{\mathcal{S}}$ produces x times the negative of a square, thus another term with sign matching that of the main term. Derivatives falling on χ_∂ produce a nonnegative square, corresponding to the a priori control required along $\gamma_{I,y}$ for y near y_0, at $x\sim\delta$. Finally, differentiation of χ_Σ produces a term vanishing near Σ which thus can be controlled by elliptic regularity, and differentiation of χ_{sf} produces a semiclassically trivial (namely, vanishing near $\hat{h}=0$) term. We can then proceed as in (4-33), obtaining the desired propagation estimate.

For $k \geq 1$, we argue as in the proof of [Baskin et al. 2015, Proposition 4.4]: rather than using $\check{A} = \mathrm{Op}_{\mathrm{c},h}(\check{a})$ as the commutant, we use (in the notation (4-48)) the vector of ps.d.o.s $(\check{A}A^\alpha)_{\alpha\in\mathcal{I}}$, where $\mathcal{I} \subset \mathbb{N}_0^N$ consists of all $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = k$ and $\alpha_0 = \alpha_N = 0$. The main term of the commutator arises from $\check{A}$ as before; the new contributions, from commutators of $P_{h,z}$ with a factor A_j, can be expanded as in Lemma 4.18, and those which have the maximal number of module factors A_l, $1 \leq l \leq N-1$, can be absorbed into this main term due to the vanishing property of the C_{jk} in Lemma 4.18. Thus, one can control k module derivatives of u in a neighborhood of ζ_0 provided one has control of $k-1$ module derivatives in a slightly bigger neighborhood. Thus, one obtains the estimate (4-50) inductively.

The proof of part (2) is completely analogous; one now takes Υ at $x = \hat{h} = 0$ to be the limiting point along *forward* H-integral curves. □

Note that for any $\zeta \in \overline{{}^{\mathrm{c}\hbar}T^*_{\mathrm{sf}}}X_{\mathrm{c}\hbar} \setminus \mathcal{F}$, there exists an element $A \in \mathcal{M}$ which is elliptic at ζ; hence microlocally near such ζ, membership in $H^{s,l,\alpha,b;k}_{\mathrm{c},h}$ is equivalent to membership in $H^{s,l,\alpha,b+k}_{\mathrm{c},h}$.[15] In particular, in $\Sigma \cap x^{-1}(0)$ but away from the radial sets, the propagation of $H^{s,l,\alpha,b;k}_{\mathrm{c},h}$ regularity is equivalent to the standard (real-principal-type) propagation of $H^{s,l,\alpha,b+k}_{\mathrm{c},h}$ regularity. One can thus concatenate the radial point estimates of Proposition 4.20 with such real-principal-type estimates. To state this succinctly, we introduce:

Definition 4.21 (integral curves connecting the radial sets). (1) For a point $y \in \partial X$, denote by $\Gamma^{\rightarrow}(y) \subset \mathcal{C}^0([0,\pi];\Sigma)$ the set of integral curves of H inside $\Sigma \cap x^{-1}(0)$, smoothly reparametrized to uniformly continuous curves $\gamma : (0,\pi) \to \Sigma \cap x^{-1}(0)$, which satisfy $\gamma(\pi) = (0,0,y,1,0) \in \mathcal{R}_{\mathrm{out}}$ and $\gamma(0) \in \mathcal{R}_{\mathrm{in}}$. Denoting by $\Pi : \Sigma \cap x^{-1}(0) \to \partial X$ the projection to the base, define the set of starting points of such curves by

$$\mathcal{Y}^{\rightarrow}(y) = \{\Pi(\gamma(0)) : \gamma \in \Gamma^{\rightarrow}(y)\}.$$

(2) We call a continuous curve $\gamma : I \to \Sigma$ a *resolved GBB* if it is either an integral curve of $h^{-1}H_p$ disjoint from $x^{-1}(0)$, or otherwise if, for some $y \in \partial X$ and $y_0 \in \mathcal{Y}^{\rightarrow}(y)$, the curve γ is the concatenation of γ_{I,y_0}, an element γ of $\Gamma^{\rightarrow}(y)$ with $\Pi(\gamma(0)) = y_0$, $\Pi(\gamma(\pi)) = y$, and the curve $\gamma_{O,y}$.

See Figure 12.

Corollary 4.22 (microlocalized propagation of Lagrangian regularity). *Let $s, l, \alpha \in \mathbb{R}$, $\mathsf{b} \in \mathcal{C}^\infty(\overline{{}^{\mathrm{c}\hbar}T^*_{\mathrm{sf}}}X_{\mathrm{c}\hbar})$ and $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(X)$ with $\operatorname{supp}\chi \subset \operatorname{supp}\chi_1$ be as in Theorem 4.10, with $\mathsf{b}-\alpha$ satisfying the monotonicity and threshold conditions stated there. Let $k \in \mathbb{N}_0$. Let $B, E, G \in \Psi^0_{\mathrm{c}\hbar}(X)$, with Schwartz kernels supported in $[0,1)_h \times (\chi^{-1}(1) \times \chi^{-1}(1))$. Suppose that all backward resolved GBBs starting at a point in $\mathrm{WF}'_{\mathrm{c}\hbar}(B)$ reach $\mathrm{Ell}_{\mathrm{c}\hbar}(E)$ in finite time while remaining in $\mathrm{Ell}_{\mathrm{c}\hbar}(G)$. Then the estimate (4-50) holds.*

A dualization argument gives the propagation of the nonfocusing condition through ∂X. The simplest setting uses the modification of $P_{h,z}$ via extension to an operator $P'_{h,z}$ on compact manifold $X' \supset X$ and

[15] That is, for $B, \widetilde{B} \in \Psi^0_{\mathrm{c}\hbar}(X)$ with $\mathrm{WF}'_{\mathrm{c}\hbar}(B) \subset \mathrm{Ell}_{\mathrm{c}\hbar}(\widetilde{B}) \setminus \mathcal{F}$, one has $\|Bu\|_{H^{s,l,\alpha,b+k}_{\mathrm{c},h}} \lesssim \|\widetilde{B}u\|_{H^{s,l,\alpha,b;k}_{\mathrm{c},h}} + \|u\|_{H^{-N,l,\alpha,-N}_{\mathrm{c},h}}$.

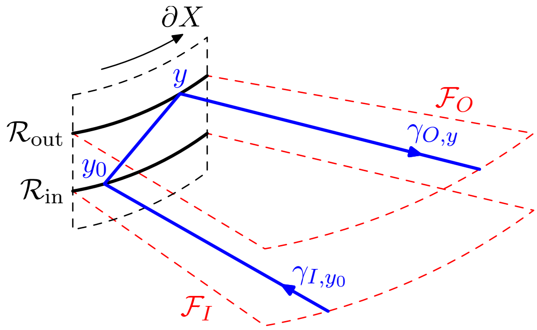

Figure 12. Illustration of a part of the characteristic set, with the Lagrangians $\mathcal{F}_I$ and $\mathcal{F}_O$ in red, and a resolved GBB in blue.

the inclusion of a complex absorbing term $Q \in \Psi_\hbar^{-\infty}((X')^\circ)$ as in Section 4E, resulting in the operator

$$\mathcal{P}_{h,z} = P'_{h,z} - iQ$$

in (4-42). (This requires ∂X to be null-cobordant; if this is not true, one can use the modification described in Remark 4.16.) Recall that the Schwartz kernel of Q has empty intersection with $x^{-1}([0, cx_0]) \times x^{-1}([0, cx_0])$, where $0 < c \ll \frac{1}{2}$. We shall use the notation of Proposition 4.15.

Theorem 4.23 (diffractive improvement). *Let s, l, α, b be as in the statement of Theorem 4.10 (for the operator $P_{h,z}$). Let $E, G \in \Psi^0_{c\hbar}(X')$ be such that all forward resolved GBBs starting at a point in $\mathrm{WF}'_{c\hbar}(E) \subset \mathrm{Ell}_{c\hbar}(G)$ remain in $\mathrm{Ell}_{c\hbar}(G)$ until they enter $\mathrm{Ell}_h(Q)$. Let $f_+ \in H^{s-2,l-2,\alpha,\mathsf{b}+1}_{\mathrm{c},h}(X')$, $f_- \in \mathcal{M}^k H^{s-2,l-2,\alpha,\mathsf{b}+1}_{\mathrm{c},h}(X')$ be such that $\operatorname{supp} f_\pm \subset x^{-1}([0, cx_0])$. Then the solution u of*

$$\mathcal{P}_{h,z} u = f := f_+ + Ef_-$$

can be written in the form

$$u = u_+ + Gu_-, \quad u_+ \in H^{s,l,\alpha,\mathsf{b}}_{\mathrm{c},h}(X'), \quad u_- \in \mathcal{M}^k H^{s,l,\alpha,\mathsf{b}}_{\mathrm{c},h}(X').$$

Note that on the scale of semiclassical cone Sobolev spaces, we have $Ef_- \in H^{s-2,l-2,\alpha,\mathsf{b}'+1}_{\mathrm{c},h}$ with $\mathsf{b}' = \mathsf{b} - k$, but typically Ef_- is no better than this. Thus, Theorem 4.23 (for $f_+ = 0$ for concreteness) implies that the strong semiclassical singularities of u resulting from the forcing term Ef_- only propagate along geometric GBBs (resulting in the term Gu_-), whereas microlocally away from these, u has $H^{s,l,\alpha,\mathsf{b}}_{\mathrm{c},h}$-regularity.

In a simple case, a formulation of Theorem 4.23 which highlights regularity rather than singularities reads as follows: Fix $y_0 \in \partial X$, and define the set

$$K := \gamma_{O,y_0} \cup \bigcup_{\gamma \in \Gamma^{\to}(y_0)} \gamma([0,\pi]) \cup \bigcup_{y \in \mathcal{Y}^{\to}(y_0)} \gamma_{I,y}.$$

Thus, the quotient $K/(K \cap x^{-1}(0))$ contains the image of all backward geometric GBB continuing γ_{O,y_0}, and K in addition contains all curves inside of $\Sigma \cap x^{-1}(0)$ which connect an incoming base point y (at distance π from y_0) with the outgoing base point y_0 of geometric GBBs. Fixing any $E \in \Psi^0_{c\hbar}(X')$ with $\mathrm{WF}'_{c\hbar}(E) \cap K = \varnothing$, there then exists $G \in \Psi^0_{c\hbar}(X')$ with $\mathrm{WF}'_{c\hbar}(G) \cap K = \varnothing$ which satisfies the conditions of Theorem 4.23. Thus, if f satisfies the nonfocusing condition (of some degree k) relative to $H_{c,h}^{s-2,l-2,\alpha,b+1}$, and with f microlocally near K lying in $H_{c,h}^{s-2,l-2,\alpha,b+1}$ (thus f in particular does not have strong singularities along the incoming directions $\gamma_{I,y}$), then the semiclassical wave u forced by f lies in $H_{c,h}^{s,l,\alpha,b}$ microlocally near K (thus u in particular does not have a strong singularity along γ_{O,y_0}).

Proof of Theorem 4.23. As follows from Proposition 4.15 by taking adjoints (or directly from the proof of Proposition 4.15), the adjoint $\mathcal{P}_{h,z}^*$ is invertible, and

$$(\mathcal{P}_{h,z}^*)^{-1} : H_{c,h}^{-s,-l,-\alpha,-b} \to H_{c,h}^{-s+2,-l+2,-\alpha,-b-1}$$

is uniformly bounded. We now apply a backward propagation version of Corollary 4.22 to $P_{h,z}^*$: For E^*, G^* the adjoints of the operators E, G in the statement of the theorem, and for $B^* \in \Psi_{c\hbar}^0(X')$ so that all forward resolved GBBs starting at a point in $\mathrm{WF}'_{c\hbar}(E^*)$ remain in $\mathrm{Ell}_{c\hbar}(G^*)$ until they enter $\mathrm{Ell}_{c\hbar}(B^*)$, we have

$$\|E^*v\|_{H_{c,h}^{-s+2,-l+2,-\alpha,-b-1;k}} \le C(\|G^*P_{h,z}^*v\|_{H_{c,h}^{-s,-l,-\alpha,-b;k}} + \|B^*v\|_{H_{c,h}^{-s+2,-l+2,-\alpha,-b-1;k}} + \|\chi u\|_{H_{c,h}^{-N,-l+2,-\alpha,-b-3/2}})$$

for any $k \in \mathbb{N}_0$. In particular, we may take B^* so that all forward null-bicharacteristics of $P_{h,z}$ starting in $\mathrm{WF}'_{c\hbar}(B^*)$ miss the cone point and enter $\mathrm{Ell}_{c\hbar}(Q^*)$ in finite time. The term B^*u is then automatically controlled for solutions of $\mathcal{P}_{h,z}^*v = w$ when $G^*w \in H_{c,h}^{-s,-l,-\alpha,-b;k}$ by elliptic regularity (on $\mathrm{Ell}_{c\hbar}(Q)$) and real-principal-type propagation (along the backward null-bicharacteristic flow) with complex absorption. We conclude that

$$(\mathcal{P}_{h,z}^*)^{-1} : \{w \in H_{c,h}^{-s,-l,-\alpha,-b} : G^*w \in H_{c,h}^{-s,-l,-\alpha,-b;k}\} \\ \to \{v \in H_{c,h}^{-s+2,-l+2,-\alpha,-b-1} : E^*v \in H_{c,h}^{-s+2,-l+2,-\alpha,-b-1;k}\}.$$

Upon taking adjoints (see also [Melrose et al. 2013, Appendix A]), this implies that

$$\mathcal{P}_{h,z}^{-1} : H_{c,h}^{s-2,l-2,\alpha,b+1} + E(\mathcal{M}^k H_{c,h}^{s-2,l-2,\alpha,b+1}) \to H_{c,h}^{s,l,\alpha,b} + G(\mathcal{M}^k H_{c,h}^{s,l,\alpha,b})$$

is a bounded map. □

Remark 4.24 (second microlocalization at $\mathcal{F}$). A sharper approach would be to second microlocalize at $\mathcal{F}_I$ and $\mathcal{F}_O$, thus cleanly decoupling the semiclassical orders at $\mathcal{F}_I$ and $\mathcal{F}_O$ (subject to threshold conditions at the radial sets) and the semiclassical order away from $\mathcal{F}$; this would allow for a unified treatment of Lagrangian and nonfocusing spaces and thus for a direct proof of Theorem 4.23. We leave such refinements for future work. We note that second microlocalization in the semiclassical setting was studied in [Sjöstrand and Zworski 2007; Vasy and Wunsch 2009] following [Bony 1986]; a second microlocal refinement (at the outgoing radial set) for the scattering theory of the corresponding normal operator was recently obtained in [Vasy 2021c].

5. Applications

We now present applications of the propagation estimates proved in Section 4. First, we discuss the familiar geometric case of $h^2\Delta_g - 1$ in Section 5A, where we can moreover prove a result sharpening both Theorem 4.10 and the propagation results of [Baskin and Marzuola 2022]. We discuss high-frequency scattering by inverse square potentials on Euclidean space in Section 5B, and high-frequency scattering for the Dirac–Coulomb equation in Section 5C.

5A. ***Propagation estimates for conic Laplacians.*** For a conic metric g as in (4-2) on the manifold $X = [0, 2x_0)_x \times Y$ of dimension $n = \dim X \geq 3$, we consider

$$P_{h,z} = h^2\Delta_g - z, \quad |z-1| < Ch.$$

We fix the volume density $\mu = |dg|$ on X.

Lemma 5.1 (admissibility, thresholds, invertibility). *The operator $P_{h,z}$ is admissible in the sense of Definition 4.1, with threshold quantities $r_{\mathrm{in}} = -\frac{1}{2}$ and $r_{\mathrm{out}} = -\frac{1}{2}$* (see Definition 4.7). *Moreover, the normal operator $N(P) = \Delta_{\hat g} - 1$, with $\hat g$ given in (4-4), is invertible at weight l* (in the sense of Definition 4.6(3)) *for all*

$$l \in \left(1 - \frac{n-2}{2}, 1 + \frac{n-2}{2}\right). \tag{5-1}$$

Proof. Only the final statement is nontrivial. In the notation (4-17), and passing to a spectral decomposition of $\Delta_{k(0)}$ whose eigenvalues we denote by $0 \leq \lambda_j^2$, $j = 0, 1, 2, \ldots$, one finds that $\lambda \in \mathrm{spec}_{\mathrm{b}}(N(P))$ if and only if there exists j with $\lambda^2 - i(n-2)\lambda + \lambda_j^2 = 0$, so

$$\mathrm{spec}_{\mathrm{b}}(N(P)) = \left\{ i\left(\frac{n-2}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_j^2}\right) : j = 0, 1, 2, \ldots \right\}.$$

Therefore, the complement of the set Λ defined in (4-18) contains $(-n+2, 0)$. As noted in Remark 4.9, the invertibility of $N(P)$ at weight l is independent of the choice of l inside the shifted interval

$$\frac{n}{2} + (-n+2, 0) = \left(1 - \frac{n-2}{2}, 1 + \frac{n-2}{2}\right).$$

The choice $l = 1$ is particularly natural, as the space $H^{1,1,0}_{\mathrm{b,sc}}(\mathrm{tf}; \hat\mu)$ is the quadratic form domain of $\Delta_{\hat g}$ (as follows from Hardy's inequality). The invertibility of $N(P)$ at weight $l = 1$ is then equivalent to the limiting absorption principle for the exact conic metric $\hat g$, the proof of which is a standard application of a boundary-pairing argument [Melrose 1995, §2.3] and unique continuation at infinity. See Lemma 5.10 below for a proof is a more general setting. □

As a consequence, we may apply Theorem 4.10 for l in the range (5-1), any value of $s \in \mathbb{R}$, $\alpha = 0$, and variable orders b satisfying in particular $\mathsf{b} > -\frac{1}{2}$ at $\mathcal{R}_{\mathrm{in}}$, $\mathsf{b} < -\frac{1}{2}$ at $\mathcal{R}_{\mathrm{out}}$, and we may arrange that $\left|\mathsf{b} - \left(-\frac{1}{2}\right)\right| < \epsilon$ for any fixed $\epsilon > 0$. Packaged in the form of Proposition 4.15 using complex absorption, we thus have, using the volume density $|dg|$ near ∂X,

$$\|u\|_{H^{s,l,0,\mathsf{b}}_{\mathrm{c},h}} \leq C\|\mathcal{P}_{h,z}u\|_{H^{s-2,l-2,0,\mathsf{b}+1}_{\mathrm{c},h}}; \tag{5-2}$$

this estimate is sharp in the sense explained after the statement of Theorem 4.10. Lossy estimates on constant order spaces are given by

$$\|(x+h)^{-\frac{1}{2}-\epsilon}u\|_{H^{s,l,0,0}_{\mathrm{c},h}} \sim h^{-\frac{1}{2}-\epsilon}\|u\|_{H^{s,l,0,-1/2-\epsilon}_{\mathrm{c},h}}$$
$$\lesssim h^{-\frac{1}{2}-\epsilon}\|\mathcal{P}_{h,z}u\|_{H^{s-2,l-2,0,1/2+\epsilon}_{\mathrm{c},h}} \sim h^{-1-2\epsilon}\|(x+h)^{\frac{1}{2}+\epsilon}\mathcal{P}_{h,z}u\|_{H^{s-2,l-2,0,0}_{\mathrm{c},h}}.$$

In the special case $s = l$, and recalling from [Hintz 2022, Theorem 6.3] that the domain

$$\mathcal{D}_h^l = \mathcal{D}((h^2\Delta_{g'}+1)^{l/2})$$

of the $(l/2)$-th power of the Friedrichs extension of the conic Laplacian $h^2\Delta_{g'}+1$ is equal to $H_{c,h}^{l,l,0,0}$ in present notation, this gives:

Proposition 5.2 (constant-order estimates). *In the above setting and with l as in (5-1), we have for all $\epsilon > 0$ the estimates*

$$\|(x+h)^{-\frac{1}{2}-\epsilon}u\|_{\mathcal{D}_h^l} \le C_\epsilon h^{-1-2\epsilon}\|(x+h)^{\frac{1}{2}+\epsilon}\mathcal{P}_{h,z}u\|_{\mathcal{D}_h^{l-2}}, \tag{5-3a}$$

$$\|u\|_{\mathcal{D}_h^l} \le C_\epsilon h^{-1-2\epsilon}\|\mathcal{P}_{h,z}u\|_{\mathcal{D}_h^{l-2}}, \tag{5-3b}$$

as well as more general estimates with $\mathcal{D}_h^l$ and $\mathcal{D}_h^{l-2}$ replaced by $H_{c,h}^{s,l,0,0}$ and $H_{c,h}^{s-2,l-2,0,0}$.

The estimate (5-3b) is an immediate consequence of (5-3a). We recall that in the case $l = 1$, the (arbitrarily small) 2ϵ-loss in (5-3b) can be removed, as shown in the semiclassical cone setting by Baskin and Marzuola [2022] following arguments by Melrose, Wunsch, and Vasy [Melrose and Wunsch 2004; Melrose et al. 2008]; in the full range of weights l considered here, a lossless estimate was obtained by the author in [Hintz 2022, §6.2] via reduction to the case $l = 1$ via conjugation by $(1+h^2\Delta_{g'})^{(l-1)/2}$ and reduction to the case $l = 1$. On the other hand, the estimate (5-3b), even for $\epsilon = 0$, loses a full order at tf compared to the sharper estimate (5-3a).

Remark 5.3 (limiting absorption principle). The $h^{-1-2\epsilon}$ loss in Proposition 5.2 is familiar from (and essentially arises from) the loss of slightly more than one power of $\langle z\rangle$ in the limiting absorption principle

$$(\Delta - 1 \pm i0)^{-1} : \langle z\rangle^{-\frac{1}{2}-\epsilon}L^2(\mathbb{R}^n) \to \langle z\rangle^{\frac{1}{2}+\epsilon}H^2(\mathbb{R}^n)$$

on Euclidean space, which is a consequence of a sharp variable-order estimate akin to (5-2); see Lemma 4.8.

A natural question is whether one can prove an estimate which removes both the ϵ-loss of (5-3b) while retaining the lossless character of (5-3a) (or (5-2)) at tf. We answer this in the affirmative:

Theorem 5.4 (sharp propagation estimate). *Consider a conic manifold (X, g) as in (4-1)–(4-2) and with* $\dim X \ge 3$. *Let $P_{h,z} = h^2\Delta_g - z$, $|z-1| < Ch$. Denote the characteristic set of $P_{h,z}$ by $\Sigma \subset {}^{c\hbar}T^*_{sf}X_{c\hbar}$; see (4-8). Let $\chi, \tilde\chi \in \mathcal{C}_c^\infty(X)$, with $\tilde\chi \equiv 1$ near* $\operatorname{supp}\chi$, *and $E \in \Psi_{c\hbar}^{-\infty}(X)$. Suppose that all backward GBB from $\Sigma \cap \operatorname{supp}\chi$ enter $\mathrm{Ell}_{c\hbar}(E)$ in finite time while remaining inside $\operatorname{supp}\tilde\chi$. Then, for any $s, N \in \mathbb{R}$, we have an estimate*

$$\|\chi u\|_{H_{c,h}^{s,1,0,0}(X)} \le C(\|\tilde\chi P_{h,z}u\|_{H_{c,h}^{s-2,-1,0,1}(X)} + \|Eu\|_{H_{c,h}^{-N,-N,-N,0}(X)} + h^{\frac{1}{2}}\|\tilde\chi u\|_{H_{c,h}^{-N,-N,0,0}(X)}). \tag{5-4}$$

This holds more generally if the first two norms above are replaced by $\|\chi u\|_{H_{cb,h}^{s,s',1,0,0}}$ and $\|\tilde\chi P_{h,z}u\|_{H_{cb,h}^{s-2,s'-2,-1,0,1}}$, with $s' \in \mathbb{R}$ arbitrary. Taking $s = 1$, $N = 0$ in (5-4) gives

$$\|\chi u\|_{\mathcal{D}_h^1} \lesssim h^{-1}\|(x+h)\tilde\chi P_{h,z}u\|_{\mathcal{D}_h^{-1}} + \|Eu\|_{L^2} + h^{\frac{1}{2}}\|\tilde\chi u\|_{L^2}, \tag{5-5}$$

and upon adding complex absorption as in Section 4E and (4-42), we have

$$\|u\|_{\mathcal{D}_h^1} = \|u\|_{H_{c,h}^{1,1,0,0}} \lesssim h^{-1}\|(x+h)\mathcal{P}_{h,z}u\|_{\mathcal{D}_h^{-1}} = \|\mathcal{P}_{h,z}u\|_{H_{c,h}^{-1,-1,0,1}}. \tag{5-6}$$

For comparison, the h-lossless version of (5-3b) for $l = 1$ reads

$$\|u\|_{H_{c,h}^{1,1,0,0}} \lesssim h^{-1}\|\mathcal{P}_{h,z}u\|_{H_{c,h}^{-1,-1,0,0}} = \|\mathcal{P}_{h,z}u\|_{H_{c,h}^{-1,-1,1,1}}, \tag{5-7}$$

which is weaker than Theorem 5.4 in that the required control on $\mathcal{P}_{h,z}$ at tf is one order stronger than in the theorem.

As discussed after Theorem 4.10, the estimate (5-4) is sharp in the sense that the relative orders on u on the left and $P_{h,z}u$ on the right cannot be improved, but here the semiclassical order remains fixed upon propagation through the cone point.

The proof of Theorem 5.4 uses the global positivity (as an operator) of a commutator on tf, reminiscent of proofs of similar lossless results in N-body scattering [Vasy 2000; 2001], as well as a splitting of u, using the functional calculus for $h^2\Delta_g$, into a part localized near the characteristic set and a part where $h^2\Delta_g - 1$ is elliptic and can be inverted by spectral theory. The following technical result is proved at the end of this section.

Lemma 5.5 (functional calculus). *Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$. Then, for all $\epsilon > 0$,*

$$\phi(h^2\Delta_g) \in \mathcal{A}^{-\epsilon,n-\epsilon,n-\epsilon,0,0,\infty}(X_{c\hbar}^2; {}^{b}\Omega^{\frac{1}{2}}), \tag{5-8}$$

where the orders of the conormal space refer to $\mathrm{lb}_2, \mathrm{ff}_2, \mathrm{rb}_2, \mathrm{tf}_2, \mathrm{df}_2, \mathrm{sf}_2 \subset X_{c\hbar}^2$ *in this order.*

Remark 5.6 (dimension). We only study the case $\dim X \geq 3$ here. The methods used in [Melrose et al. 2008; Baskin and Marzuola 2022], which are based on quadratic forms, and also the methods in [Melrose and Wunsch 2004], work in the case $\dim X = 2$ as well. However, the identification of the quadratic form domain with a semiclassical cone Sobolev space fails in this case (see [Melrose and Wunsch 2004, equation (3.11)] for $h = 1$), which is why we do not consider it here.

Proof of Theorem 5.4. We present the proof in the case that $P_{h,z}$ agrees with its normal operator, equivalently $P_{h,z} = h^2\Delta_g - 1$ with $g = dx^2 + x^2k(y, dy)$ an exact conic metric. In the general case, the error terms arising from $P_{h,z} - N(P) \in \Psi_{c\hbar}^{2,-2,-1,0}(X)$ are handled easily; we leave the details to the reader. (In particular, since we shall use a global commutator argument which controls u at sf and tf in one fell swoop, there is no need for a delicate argument for the combination of the symbolic estimate at sf and a normal operator estimate at tf as in the end of the proof of Theorem 4.10.) We write $P \equiv P_{h,z}$ for brevity.

Positive commutator argument: Define the operator

$$A := \frac{h}{2}(xD_x + (xD_x)^*) - 1 = hxD_x - \frac{inh}{2} - 1, \quad a := {}^{c\hbar}\sigma(A) = x\xi - 1, \tag{5-9}$$

where we use the coordinates (3-3). This will be the main piece of the commutant in a positive commutator calculation, and it is in essence the key term both in the commutator argument of [Vasy 2008], as well as in the Mourre commutant [1980/81] in classical scattering theory. Let $\chi = \chi(x)$ be identically 1 near $\partial X = x^{-1}(0)$, with support in any prespecified neighborhood of ∂X, and so that $\chi' \leq 0$, $\sqrt{-\chi\chi'} \in \mathcal{C}^\infty$, and so that $a < 0$ has a constant (negative) sign on $\Sigma \cap \operatorname{supp}\chi$; arranging the latter property is what the constant term in (5-9) is for. We then consider the operator

$$\tilde{A} := \chi A \chi,$$

and estimate in two different ways the expression

$$2h^{-1}\operatorname{Im}\langle Pu,\tilde{A}u\rangle=\Bigl\langle\frac{i}{h}[P,\tilde{A}]u,u\Bigr\rangle. \tag{5-10}$$

Consider first the commutator term. Since $h^2\Delta_g$ is homogeneous of degree -2, we have $(i/h)[P,A]=-[x\partial_x,P]=2h^2\Delta_g$, which is the crucial global positive commutator. Therefore,

$$\frac{i}{h}[P,\chi A\chi]=2\chi h^2\Delta_g\chi+\frac{i}{h}(\chi A[P,\chi]+[P,\chi]A\chi).$$

The contribution of the first term to the right-hand side of (5-10) is

$$2\|h\nabla_g(\chi u)\|^2,$$

where we write $\|\cdot\|\equiv\|\cdot\|_{L^2}$. The second term on the other hand consists of operators with coefficients supported strictly away from $x=0$. It suffices to merely capture its principal symbol, which by (4-9) is $2\chi a h^{-1}H_p\chi=ax^{-1}\mathsf{H}(\chi^2)=4a\xi\chi\chi'$; near incoming directions, where $\xi<0$, this is negative, whereas near outgoing directions, where $\xi>0$, this is positive and thus has a sign matching that of the above main commutator term. For a suitable microlocal cutoff $E\in\Psi^0_{c\hbar}(X)$ which is elliptic on Σ in the region $\xi<\epsilon$ for some fixed small $\epsilon\in(0,1)$, we thus conclude from (5-10) that

$$2\|h\nabla_g(\chi u)\|^2\le 2\operatorname{Im}\Bigl\langle\Bigl(hD_x-\frac{in}{2}\frac{h}{x}\Bigr)(\chi u),h^{-1}x\chi Pu\Bigr\rangle-2h^{-1}\operatorname{Im}\langle\chi u,\chi Pu\rangle+\|Eu\|^2. \tag{5-11}$$

Hardy's inequality gives $\bigl\|(h/x)\chi u\bigr\|\le C_n\|hD_x(\chi u)\|$; hence the first term on the right is bounded from above by

$$\epsilon\|hD_x(\chi u)\|^2+C_\epsilon\|h^{-1}x\chi Pu\|^2.$$

For the second term in (5-11), we rewrite

$$\begin{aligned}\langle\chi u,\chi Pu\rangle&=-\|\chi u\|^2+\langle h\nabla_g(\chi^2u),h\nabla_g u\rangle\\&=-\|\chi u\|^2+\|\chi h\nabla_g u\|^2+\langle h(\nabla_g\chi^2)u,h\nabla_g u\rangle;\end{aligned}$$

taking the imaginary part annihilates the first two terms, while for the final term we have

$$\begin{aligned}\langle h(\nabla_g\chi^2)u,h\nabla_g u\rangle-\langle h\nabla_g u,h(\nabla_g\chi^2)u\rangle&=ah^2\nabla_g((\nabla_g\chi^2)u)-h^2(\nabla_g\chi^2)\cdot\nabla_g u,u\rangle\\&=\langle h^2\Delta_g(\chi^2)u,u\rangle,\end{aligned} \tag{5-12}$$

with no derivatives falling on u anymore. Altogether, we obtain from (5-11) the estimate

$$(2-C\epsilon)\|h\nabla_g(\chi u)\|^2\le C_\epsilon\|h^{-1}x\chi Pu\|^2+Ch\|\tilde{\chi}u\|^2+\|Eu\|^2, \tag{5-13}$$

where $\tilde{\chi}\equiv 1$ on $\operatorname{supp}\chi$, used to bound the contribution of (5-12).

Control of χu in $H^{1,1,0,0}_{c,h}$: Since the principal symbol of

$$h\nabla_g\in\Psi^{1,1,0,0}_{c\hbar}(X;\mathbb{C},{}^{c\hbar}T^*X_{c\hbar})$$

(mapping complex-valued functions into sections of ${}^{c\hbar}T^*X_{c\hbar}$, see Remark 3.3) is not injective at the zero section over sf, the estimate (5-13) does not yet give full control of χu in $H^{1,1,0,0}_{c,h}$: an estimate of $\|\chi u\|^2_{L^2}$

is lacking at this point. (Note that the control of $(h/x)\chi u$ via Hardy's inequality degenerates precisely at sf, i.e., the lift of $h=0$.) The key observation is that the characteristic set of P and the set where the principal symbol of $h\nabla_g$ fails to be injective are disjoint. Thus, for some $A_1\in\Psi_{c\hbar}^{-1,-1,0,0}(X;{}^{c\hbar}T^*X_{c\hbar},\mathbb{C})$ and $A_2\in\Psi_{c\hbar}^{-2,-2,0,0}(X)$, we have

$$I=A_1\circ h\nabla_g+A_2P+R,\qquad R\in\Psi_{c\hbar}^{-\infty,0,0,-\infty}(X);$$

this implies

$$\|\chi u\|\le C(\|h\nabla_g(\chi u)\|_{H_{c,h}^{-1,-1,0,0}(X)}+\|P(\chi u)\|_{H_{c,h}^{-2,-2,0,0}(X)})+\|R(\chi u)\|_{L^2}. \tag{5-14}$$

Using $[P,\chi]\in\Psi_{c\hbar}^{1,-\infty,-\infty,-1}(X)$, we can estimate the second term by

$$\begin{aligned}\|P(\chi u)\|_{H_{c,h}^{-2,-2,0,0}}&\le\|\chi Pu\|_{H_{c,h}^{-2,-2,0,0}}+\|[P,\chi]u\|_{H_{c,h}^{-2,-2,0,0}}\\&\lesssim\|h^{-1}x\chi Pu\|_{H_{c,h}^{-2,-1,0,-1}}+\|\tilde\chi u\|_{H_{c,h}^{-1,-N,-N,-1}}\end{aligned}$$

for any $N\in\mathbb{R}$. The remainder term in (5-14) is simply estimated by

$$\|R(\chi u)\|_{L^2}\le C\left\|\frac{h}{h+x}\chi u\right\|_{L^2}\le C\left\|\frac{h}{x}\chi u\right\|_{L^2}.$$

Applying Hardy's inequality to this term, the estimate (5-14) then implies, a fortiori,

$$\|\chi u\|_{L^2}\le C'(\|h\nabla_g(\chi u)\|_{L^2}+\|h^{-1}x\chi Pu\|_{L^2}+h\|\tilde\chi u\|_{L^2}).$$

We can now add η times this, with $\eta C'<\frac12$, to the estimate (5-13) (in which we fix $\epsilon<C^{-1}$), in order to obtain

$$\|\chi u\|^2_{\mathcal{D}_h^1}=\|\chi u\|^2+\|h\nabla_g(\chi u)\|^2\lesssim\|h^{-1}x\chi Pu\|^2+\|Eu\|^2+h\|\tilde\chi u\|^2. \tag{5-15}$$

As far as weights in h and x are concerned, this is already the desired estimate. However, the differential order is forced to be 1 here, and in addition the order of differentiability required on Pu in (5-15) is too strong (0 instead of -1) even in this special case.

Sharp improvement at tf: The basic idea is to apply the estimate (5-15) to $\phi(h^2\Delta_g)u$, where $\phi\in\mathcal{C}_c^\infty(\mathbb{R})$ is equal to 1 on $[-4,4]$; on the remaining piece $(1-\phi(h^2\Delta_g))u$, the operator $h^2\Delta_g-1$ can be inverted directly using the functional calculus. In order to define $h^2\Delta_g$ as a self-adjoint operator, we need to pass from X to a compact manifold X'. If $Y=\partial X$ is null-cobordant, we may choose X' so that $\partial X'=\partial X$, and we then extend g to a Riemannian metric on X' which we continue to denote by g. The operator $\phi(h^2\Delta_g)$ does depend on the choice of extension, but its structural properties, as used in the following argument, do not. If Y is not null-cobordant, we may set $X'=[0,6x_0]_x\times Y$ and define a smooth metric on X' which is equal to g on $[0,2x_0)\times Y$ and equal to the pullback of g along the map $(x,y)\mapsto(6x_0-x,y)$ on $(4x_0,6x_0]\times Y$; we denote this metric g again. Thus, we have two identical cone points at $x=0$ and at $6x_0-x=0$.

Concretely then, $\Phi:=\phi(h^2\Delta_g)$ is given by Lemma 5.5, the notation of which we shall use here. Now, in order to remain localized near ∂X, we apply the estimate (5-15) to

$$u_1:=\tilde\chi\Phi\tilde\chi u.$$

Using $[P, \Phi] \equiv 0$ and $\chi[P, \tilde{\chi}] \equiv 0$, we estimate the first term on the right in (5-15) by

$$\|h^{-1} x \chi P \tilde{\chi} \Phi \tilde{\chi} u\| \leq \left\| \frac{x}{h} \chi \Phi \frac{h}{h+x} (h^{-1}(x+h) \tilde{\chi} P u) \right\| + \|h^{-1} x \chi \Phi [P, \tilde{\chi}] u\|. \tag{5-16}$$

Denoting the lift of x to the left, resp. right factor of $X^2_{c\hbar}$ by x, resp. x', we note that

$$\frac{x}{h} \frac{h}{x'+h} \in \mathcal{A}^{1,1,0,0,0,0}(X^2_{c\hbar}) \quad \Longrightarrow \quad \Psi := \frac{x}{h} \chi \Phi \frac{h}{x'+h} \in \mathcal{A}^{1-\epsilon, n+1-\epsilon, n-\epsilon, 0,0,\infty}(X^2_{c\hbar}).$$

Passing to a b-density $0 < \mu_0 \in \mathcal{C}^\infty(X; {}^{\mathrm{b}}\Omega^1 X)$, we claim that Ψ is continuous as a map

$$H^{-\infty,-1,0,0}_{\mathrm{c},h}(X; |\mathrm{d}g|) = H^{-\infty,-1-\frac{n}{2},-\frac{n}{2},0}_{\mathrm{c},h}(X; \mu_0) \to H^{\infty,-\frac{n}{2},-\frac{n}{2},0}_{\mathrm{c},h}(X; \mu_0) = H^{\infty,0,0,0}_{\mathrm{c},h}(X; |\mathrm{d}g|),$$

but since Ψ is smoothing in the sense of c$\hbar$-differentiability, it suffices to show the boundedness on $L^2(X; \mu_0)$ of

$$x^{\frac{n}{2}} \Psi (x')^{-\frac{n}{2}} \left(\frac{x'}{x'+h} \right)^{-1} \in \mathcal{A}^{\frac{n}{2}+1-\epsilon, n-\epsilon, \frac{n}{2}-1-\epsilon, 0,0,\infty}(X^2_{c\hbar}; {}^{\mathrm{b}}\Omega^{\frac{1}{2}}). \tag{5-17}$$

Since this kernel is bounded section of ${}^{\mathrm{b}}\Omega^{1/2}$ (all indices being ≥ 0), this is a consequence of Schur's lemma.

The operator acting on u in the second term on the right in (5-16) has Schwartz kernel supported in $x' \geq c > 0$ and $|x - x'| > c > 0$ (since $\operatorname{supp} \chi \cap \operatorname{supp} \mathrm{d}\tilde{\chi} = \varnothing$), hence lies in $\mathcal{A}^{1-\epsilon,\infty,\infty,\infty,\infty,\infty}(X^2_{c\hbar})$; therefore, the second term in (5-16) can be bounded by $h^N \|\tilde{\chi}^\sharp u\|$ for any N, where $\tilde{\chi}^\sharp = 1$ on $\operatorname{supp} \tilde{\chi}$. Altogether, forgetting the cutoff $\tilde{\chi}$ and renaming $\tilde{\chi}^\sharp$ as $\tilde{\chi}$, we have proved

$$\|\chi u_1\|_{H^{N,1,0,0}_{\mathrm{c},h}} \lesssim h^{-1} \|(x+h) \tilde{\chi} P u\|_{H^{-N,-1,0,0}_{\mathrm{c},h}} + \|E u\|_{H^{-N,-N,-N,0}_{\mathrm{c},h}} + h^{\frac{1}{2}} \|\tilde{\chi} u\|_{H^{-N,0,0,0}_{\mathrm{c},h}} \tag{5-18}$$

for any $s, N \in \mathbb{R}$.

It remains to control χu_2, where

$$u_2 := u - u_1 = u - \tilde{\chi} \Phi \tilde{\chi} u.$$

Let $\phi^\flat \in \mathcal{C}^\infty_{\mathrm{c}}((-3,3))$ be identically 1 on $[-2,2]$, and let $\Phi^\flat = \phi^\flat(h^2 \Delta_g)$. Then χu_2 is localized near high frequencies, in the sense that its localization to low frequencies

$$\Phi^\flat(\chi u_2) = \Phi^\flat(\chi u) + \Phi^\flat [\Phi, \chi] \tilde{\chi} u - \Phi^\flat \Phi \chi \tilde{\chi} u = \Phi^\flat [\Phi, \chi] \tilde{\chi} u \tag{5-19}$$

(using $\Phi^\flat \Phi = \Phi^\flat$ and $\chi \tilde{\chi} = \chi$) is $\mathcal{O}(h^\infty)$ (due to the presence of $[\Phi, \chi]$) near $x = 0$ and vanishes to an order h more than u near $\operatorname{supp} \mathrm{d}\chi \subset X^\circ$. Moreover, χu_2 satisfies the equation

$$P(\chi u_2) = (\chi - \chi \Phi \tilde{\chi}) P u + \big([P, \chi] u - [P, \chi] \Phi \tilde{\chi} u - \chi \Phi [P, \tilde{\chi}] u \big), \tag{5-20}$$

Altogether, if we put

$$P^\sharp := P + 2\Phi^\flat,$$

then we have

$$P^\sharp(\chi u_2) = f_2 := (\chi - \chi \Phi \tilde{\chi}) P u + [P, \chi] u - \big([P, \chi] \Phi \tilde{\chi} u + \chi \Phi [P, \tilde{\chi}] u - 2\Phi^\flat [\Phi, \chi] \tilde{\chi} u \big). \tag{5-21}$$

We moreover have $P^\sharp = p^\sharp(h^2 \Delta_g)$, where $p^\sharp(\sigma) := (\sigma - 1) + 2\phi^\flat \geq \frac{1}{2}(\sigma + 1)$ for $\sigma \geq 0$; hence we can invert $P^\sharp$ using the functional calculus for Δ_g by $(P^\sharp)^{-1} = q^\sharp(h^2 \Delta)$ where $q^\sharp(\sigma) = 1/p^\sharp(\sigma)$ is equal to $(\sigma-1)^{-1}$ for large σ. One can then show, by a combination of the arguments leading to Lemma 5.5 and [Hintz 2022, Theorem 5.2], that

$$(P^\sharp)^{-1} \in \left(\frac{x}{x+h}\right)^2 \Psi_{c\hbar}^{-2}(X) + \Psi_{c\hbar}^{-\infty,\mathcal{E}}(X)$$

where the collection $\mathcal{E}$ of index sets is equal to $\mathcal{E}(-1)$ in the notation of [Hintz 2022, Theorem 6.1]. Therefore, using (5-21), the mapping properties of elements of $(x/(x+h))^2\Psi_{c\hbar}^{-2}(X)$, and estimating the smoothing contribution in the space $\Psi_{c\hbar}^{-\infty,\mathcal{E}}(X)$ to $(P^\sharp)^{-1}$ by means of Schur's lemma, we have

$$\|\chi u_2\|_{H_{c,h}^{s,1,0,0}} = \|(P^\sharp)^{-1} f_2\|_{H_{c,h}^{s,1,0,0}} \lesssim \|\tilde{\chi} P u\|_{H_{c,h}^{s-2,-1,0,0}} + \|\tilde{\chi}[P,\chi]u\|_{H_{c,h}^{s-2,-1,0,0}} + h\|\tilde{\chi} u\|_{H_{c,h}^{-N,-N,0,0}}. \tag{5-22}$$

Here, the first term on the right comes from the first term in (5-21) and the boundedness of $\chi - \chi\Phi\tilde{\chi}^\sharp$ (with $\tilde{\chi}^\sharp \equiv 1$ on $\operatorname{supp}\tilde{\chi}$) on $H_{c,h}^{s-2,-1,0,0}$; this boundedness follows from the boundedness of the Schwartz kernel of

$$x^{\frac{n}{2}} \frac{x}{x+h} \Phi \left(\frac{x'}{x'+h}\right)^{-1} (x')^{-\frac{n}{2}} \in \mathcal{A}^{\frac{n}{2}+2-\epsilon, n+1-\epsilon, \frac{n}{2}-1-\epsilon, 0,0,\infty}(X_{c\hbar}^2; {}^{\mathrm{b}}\Omega^{\frac{1}{2}})$$

similarly to the discussion of Ψ in (5-17). The final term in (5-22) comes from the big parenthesis in (5-21), every term of which involves the localizer Φ to low frequencies as well as a commutator with a cutoff χ or $\tilde{\chi}$. But $\tilde{\chi}[P,\chi] \in \Psi_{c\hbar}^{1,-\infty,-\infty,-1}(X)$; hence the second term on the right is bounded from above by $\|\tilde{\chi} u\|_{H_{c,h}^{s-1,-N,-N,-1}}$ for any N. By elliptic regularity at infinite semiclassical cone frequencies, this can be bounded by $C(\|\tilde{\chi}^\sharp P u\|_{H_{c,h}^{s-3,-N-2,-N,-1}} + \|\tilde{\chi}^\sharp u\|_{H_{c,h}^{-N,-N,-N,-1}})$. Combining the resulting estimate with (5-18) proves the theorem for $H_{c,h}$-spaces. The proof of the more general statement for $H_{cb,h}$-spaces requires only notational changes which are left to the reader. □

To complete the proof, it remains to prove Lemma 5.5.

Proof of Lemma 5.5. This can be proved using the Helffer–Sjöstrand formula [1989] similarly to [Vasy 2000, Lemma 10.1 and Proposition 10.2]. Choosing a compactly supported almost analytic extension $\tilde{\phi} \in \mathcal{C}_c^\infty(\mathbb{C})$ of ϕ (that is, $\tilde{\phi}|_{\mathbb{R}} = \phi$ and $|\partial_{\bar{z}}\tilde{\phi}| = \mathcal{O}(|\operatorname{Im} z|^N)$ for all N), we have

$$\phi(h^2\Delta_g) = \frac{1}{2\pi i}\int \partial_{\bar{z}}\tilde{\phi}(z)(h^2\Delta_g - z)^{-1}\, d\bar{z} \wedge dz. \tag{5-23}$$

For $z \notin \mathbb{R}$, [Hintz 2022, Theorem 3.10 and §6.1] gives

$$(h^2\Delta_g - z)^{-1} \in \left(\frac{x}{x+h}\right)^2 \Psi_{c\hbar}^{-2}(X) + \Psi_{c\hbar}^{-\infty,\mathcal{E}}(X), \tag{5-24}$$

where $\mathcal{E} = (\mathcal{E}_{\mathrm{lb}}, \mathcal{E}_{\mathrm{ff}}, \mathcal{E}_{\mathrm{rb}}, \mathcal{E}_{\mathrm{tf}})$, with $\operatorname{Re} z \geq 0$ for $(z,k) \in \mathcal{E}_{\mathrm{lb}}$; $\operatorname{Re} z \geq n$ for $(z,k) \in \mathcal{E}_{\mathrm{rb}}$; $\operatorname{Re} z \geq 2$ for $(z,k) \in \mathcal{E}_{\mathrm{ff}}$; and $\operatorname{Re} z \geq 0$ for $(z,k) \in \mathcal{E}_{\mathrm{tf}}$. Using that the principal symbol of $h^2\Delta_g$ is real-valued and its normal operator $\Delta_{\hat{g}}$ is self-adjoint, we claim that any fixed seminorm (5-24) is moreover bounded by $|\operatorname{Im} z|^{-k}$ for some k (depending on the seminorm).

To justify this claim, it is instructive to first consider the corresponding statement for $(\Delta - z)^{-1} \in \Psi^{-2}$, $z \in \mathbb{C}\setminus\mathbb{R}$, where Δ is the Laplacian on a *closed* Riemannian manifold: for any N, the construction of a symbolic parametrix of $\Delta - z$ of order N gives $Q_{N,z}, Q'_{N,z} \in \Psi^{-2}$, with seminorms bounded by $|\mathrm{Im}\, z|^{-C}$ for some C depending on N and the seminorm, so that $(\Delta - z)Q_{N,z} = I - R_{N,z}$ and $Q'_{N,z}(\Delta - z) = I - R'_{N,z}$, where $R_{N,z}$, $R'_{N,z} \in \Psi^{-N}$ obey such bounds as well. But then $(\Delta - z)^{-1} = Q_{N,z} + Q'_{N,z}R_{N,z} + R'_{N,z}(\Delta - z)^{-1}R_{N,z}$, where the first two summands as pseudodifferential operators and the third summand (the "remainder operator") $R'_{N,z}(\Delta - z)^{-1}R_{N,z}$ as a map $H^{-N} \to H^N$ obey such bounds; this uses that $\|(\Delta - z)^{-1}\| = |\mathrm{Im}\, z|^{-1}$ as an operator on L^2. But any seminorm on Ψ^{-2} is continuous on the space of bounded operators $H^{-N} \to H^N$ for sufficiently large N. This implies that any Ψ^{-2}-seminorm of $(\Delta - z)^{-1}$ is bounded by $|\mathrm{Im}\, z|^{-k}$ for some k. (In this simple example, the claim follows also directly from Beals' Theorem [1977].)

Analogous arguments can be used to control the inverse of the normal operator $\Delta_{\hat{g}} - z$: in addition to carrying out N steps of the symbolic parametrix construction, one uses the inverse of the b-normal operator (which is independent of z) to solve away to order N at the left/right boundary (for the left/right parametrix) and the b-front face of $\mathrm{tf}^2_{\mathrm{b,sc}}$; by virtue of taking only finitely many steps, one ensures the validity of $|\mathrm{Im}\, z|^{-k}$ bounds on seminorms. The true inverse $(\Delta_{\hat{g}} - z)^{-1}$ then obeys such bounds as well since any seminorm on the space of large b-scattering ps.d.o.s in which $(\Delta_{\hat{g}} - z)^{-1}$ lies (see [Hintz 2022, equation (3.28)] for a general statement) is continuous on the space of remainder operators $H^{-N,l_1,-N}_{\mathrm{b,sc}} \to H^{N,l_2,N}_{\mathrm{b,sc}}$ (for appropriate orders l_1, l_2, with $l_2 = -N$, resp. $l_1 = -N$ in the case of the left, resp. right parametrix construction) for sufficiently large N.

The $|\mathrm{Im}\, z|^{-k}$ bounds for seminorms of (5-24) can then be proved by completely analogous means, namely by constructing a parametrix which is accurate to some finite order, and observing that any fixed seminorm on the space (5-24) is continuous on the relevant space of remainder operators.

Plugging these into (5-23), we conclude that $\phi(h^2\Delta_g)$ is of the same class as the resolvent. To improve the orders, let $m \in \mathbb{N}$ and write $\phi(\sigma) = (\sigma + C)^{-m}\phi_m(\sigma)$ with $C > -\inf\operatorname{supp}\phi$. Then

$$\phi(h^2\Delta_g) = (h^2\Delta_g + C)^{-m}\phi_m(h^2\Delta_g).$$

Applying the previous discussion to ϕ_m and using [Hintz 2022, Theorem 6.3(3)] (with $w = -m$) to control $(h^2\Delta_g + C)^{-m}$ implies, upon letting $m \to \infty$, that $\phi(h^2\Delta_g) \in \Psi^{-\infty,-\infty,0,0}_{c\hbar}(X) + \mathcal{A}^{-\epsilon,n-\epsilon,n-\epsilon,0,\infty,\infty}(X^2_{c\hbar})$. This gives (5-8). □

5B. ***Scattering by potentials with inverse square singularities.*** Complex absorption is a somewhat drastic method for gaining microlocal control along incoming directions. As a more natural setting, let us thus consider scattering by potentials on $\mathbb{R}^n$, $n \geq 2$, which are singular at the origin $0 \in \mathbb{R}^n$, as in Theorem 1.1. (Working on more general conic manifolds requires only minor modifications.) That is, the underlying spatial manifold is

$$X = [\mathbb{R}^n_x; \{0\}] \cong [0, \infty)_r \times \mathbb{S}^{n-1}, \qquad g = dx^2 = dr^2 + r^2 g_{\mathbb{S}^{n-1}}. \tag{5-25}$$

We write $\Delta \equiv \Delta_g = \sum D^2_{x^j}$ for the (nonnegative) Laplacian. Let $N \in \mathbb{N}$, and denote by Δ the Laplacian acting componentwise on $\mathbb{C}^N$-valued functions. We consider scattering by matrix-valued potentials

$$V(x) = |x|^{-2}V_0(x), \qquad V_0 \in \mathcal{C}^\infty_c(X; \mathbb{C}^{N\times N}).$$

The assumption of compact support of V can of course be relaxed considerably, but since our interest lies in understanding the effect of the singularity at $r=0$, we shall not concern ourselves with more general conditions on V at infinity here.

We are interested in high-energy estimates for the resolvent of $\Delta+V$; concretely, we shall consider $\Delta+V-\sigma^2$, where $\operatorname{Im}\sigma>0$ is bounded and $|\operatorname{Re}\sigma|\gg 1$. Upon introducing

$$h=|\sigma|^{-1},\qquad z=(h\sigma)^2=1+\mathcal{O}(h), \tag{5-26}$$

we define

$$P_{h,z}:=h^2(\Delta+V-\sigma^2)=h^2\Delta-z+h^2r^{-2}V_0. \tag{5-27}$$

This is admissible in the sense of Definition 4.1, with $Q_{1,z}=V_0$ and $q_{0,z}=0$. Since $Q_{1,z}$ has differential order 0, the threshold quantities in Definition 4.7 are $r_{\mathrm{in}}=r_{\mathrm{out}}=-\frac{1}{2}$. The normal operator of $P_{h,z}$ is computed by passing to $\hat r=\frac{r}{h}$ and setting $h=0$:

$$\begin{gathered} N(P)=\Delta_{\hat g}-1+\hat r^{-2}V_\partial,\\ \hat g:=\mathrm{d}\hat r^2+\hat r^2 g_{\mathbb{S}^{n-1}},\qquad V_\partial:=V_0|_{\partial X}\in\mathcal{C}^\infty(\partial X;\mathbb{C}^{N\times N}). \end{gathered} \tag{5-28}$$

(Thus $V_\partial(\omega)=V_0(0,\omega)$ in the coordinates $(r,\omega)\in[0,\infty)\times\mathbb{S}^2$ on X.)

Theorem 5.7 (potential scattering). *Assume that the operator $N(P)$ defined in (5-28) is invertible at weight $l\in\mathbb{R}$ (in the sense of Definition 4.6(3)). Let $C>0$, and let $\chi_0\in\mathcal{C}^\infty_{\mathrm{c}}(X)$ be identically 1 near $r=0$. Then there exists $C'>0$ so that for $0<\operatorname{Im}\sigma<C$ and $|\operatorname{Re}\sigma|>C'$, the operator $\Delta+V-\sigma^2$ is invertible as a map*

$$\begin{aligned} \Delta+V-\sigma^2:\{u\in H^2_{\mathrm{loc}}(X^\circ):\chi_0u\in r^lH^2_{\mathrm{b}}(X),\ (1-\chi_0)u\in H^2(\mathbb{R}^n)\}\\ \to\{f\in L^2_{\mathrm{loc}}(X^\circ):\chi_0f\in r^{l-2}L^2(X),\ (1-\chi_0)f\in L^2(\mathbb{R}^n)\}. \end{aligned} \tag{5-29}$$

Moreover, in the notation (5-26)–(5-27), the following uniform estimate holds for all $\epsilon,\delta>0$, a suitable constant $C_{\epsilon,\delta}>0$, and all $0<\operatorname{Im}\sigma<C$, $|\operatorname{Re}\sigma|>C'$:

$$\begin{aligned} &\|\chi_0u\|_{H^{s,l,1/2+\epsilon,0}_{\mathrm{c},h}}+\|(1-\chi_0)u\|_{H^{s,-1/2-\delta}_{\mathrm{sc},h}}\\ &\qquad\le C_{\epsilon,\delta}h^{-1-2\epsilon}(\|\chi_0P_{h,z}u\|_{H^{s-2,l-2,-1/2-\epsilon,0}_{\mathrm{c},h}}+\|(1-\chi_0)P_{h,z}u\|_{H^{s-2,1/2+\delta}_{\mathrm{sc},h}}); \end{aligned} \tag{5-30}$$

here, $H^{s,\gamma}_{\mathrm{sc}}=H^{s,\gamma}_{\mathrm{sc}}(\overline{\mathbb{R}^n})=\langle r\rangle^{-\gamma}\mathcal{F}^{-1}(\langle hD\rangle^{-s}L^2(\mathbb{R}^n))$ is the semiclassical scattering Sobolev space. In particular, for $l_1\le\min\bigl(l,\frac{1}{2}+\epsilon\bigr)$, $l_2\ge\max\bigl(l-2,-\frac{1}{2}-\epsilon\bigr)$, and for any fixed $\chi\in\mathcal{C}^\infty_{\mathrm{c}}(X)$ we have

$$\|\chi(\Delta+V-\sigma^2)^{-1}\chi f\|_{r^{l_1}L^2}\le C_{\epsilon,\chi}|\sigma|^{-1+2\epsilon}\|f\|_{r^{l_2}L^2}. \tag{5-31}$$

We recall that for $V(x)=|x|^{-2}V_0(x)$, with real-valued V_0 satisfying $V_0(0)>-((n-2)/2)^2$ (and indeed under relaxed regularity requirements on V_0, and allowing for the presence of several such inverse square singularities), Duyckaerts [2006] obtained cutoff resolvent estimates of the form (5-31) without the 2ϵ-loss. It is an interesting question — which we do not address here — whether in this setting, or perhaps even in the general setting of Theorem 5.7, one can prove a lossless estimate using a global commutator argument similar to the one used in Section 5A.

Remark 5.8 (meromorphic continuation). For V with compact support as above, the resolvent $(\Delta+V-\sigma^2)^{-1}$ can be meromorphically continued to the complex plane when n is odd, and the logarithmic cover of $\mathbb{C}^\times$ when n is even; the estimate (5-31) holds in strips of bounded $\operatorname{Im}\sigma$ for large $|\operatorname{Re}\sigma|$. The construction of this continuation can be accomplished along the lines of black box scattering [Sjöstrand and Zworski 1991] (see also [Dyatlov and Zworski 2019, §4]), with those estimates in the references in $\operatorname{Im}\sigma \gg 1$ relying on self-adjointness replaced by estimates on the off-spectrum resolvent that follow from [Hintz 2022, Theorem 3.10]. For applications of such estimates to expansions of scattered waves for n odd, we refer the reader to [Dyatlov and Zworski 2019, §3.2.2].

Remark 5.9 (vector bundles). One may more generally consider potentials valued in endomorphisms of a vector bundle $E \to X$, with Δ denoting an operator acting on sections of E with scalar principal symbol given by the dual metric function. The main difference to the case of a trivial bundle is that the threshold quantities $r_{\mathrm{in}}, r_{\mathrm{out}}$ depend on subprincipal terms of Δ (and their calculation requires the choice of a fiber inner product on E; see Remark 4.11). In the special case of tensor bundles E, and with Δ denoting the tensor Laplacian, the fiber inner product on E induced by g does give $r_{\mathrm{in}} = r_{\mathrm{out}} = -\frac{1}{2}$.

Proof of Theorem 5.7. Semiclassical propagation estimates near infinity of $\mathbb{R}^n$ are standard, see, e.g., [Vasy and Zworski 2000, Theorem 1] (following [Melrose 1994]) in a general geometric setting, and can be combined with the propagation estimates through the singularity at $r = 0$ given in Theorem 4.10 (where we shall take $\alpha = \frac{1}{2} + \epsilon$, $\mathsf{b} = 2\epsilon$ near $\mathcal{R}_{\mathrm{in}}$, and $\mathsf{b} = 0$ near $\mathcal{R}_{\mathrm{out}}$). Altogether, upon simplifying to constant orders, we obtain, for any $\delta > 0$, and for $0 < h < h_0$ with $h_0 > 0$ sufficiently small,

$$\|\chi_0 u\|_{H^{s,l,1/2+\epsilon,0}_{\mathrm{c},h}} + \|(1-\chi_0)u\|_{H^{s,-1/2-\delta}_{\mathrm{sc},h}} \lesssim \|\chi_0 P_{h,z} u\|_{H^{s-2,l-2,1/2+\epsilon,1+2\epsilon}_{\mathrm{c},h}} + h^{-1-2\epsilon}\|(1-\chi_0)P_{h,z}u\|_{H^{s-2,1/2+\delta}_{\mathrm{sc},h}},$$

which is the estimate (5-30). (The loss of $h^{-2\epsilon}$ in the second term on the right is due to the fact that the propagation through $r = 0$ comes with this loss, which then gets propagated out to infinity.) This estimate also gives the injectivity of $P_{h,z}$ for small $h > 0$ and $|z-1| < Ch$, with surjectivity following from the analogous estimate for the adjoint; this proves the first part of the theorem, albeit on function spaces with weights $\langle r\rangle^{\pm(1/2+\delta)}$ at infinity. But for any fixed σ with $\sigma^2 \notin [0,\infty)$, $\Delta + V - \sigma^2$ is an elliptic scattering operator near infinity; hence these weights can be removed. (It is only in the high-energy limit $|\operatorname{Re}\sigma| \to \infty$ with $|\operatorname{Im}\sigma/\operatorname{Re}\sigma| \to 0$ that one loses uniform (in σ) ellipticity.)

The simplified estimate (5-31) follows by setting $s = 2$ in

$$h^{-1-2\epsilon}\|\chi P_{h,z}u\|_{H^{s-2,l-2,-1/2-\epsilon,0}_{\mathrm{c},h}} \le h^{1-2\epsilon}\|\chi(\Delta + V - \sigma^2)u\|_{H^{s-2,l_2,l_2,0}_{\mathrm{c},h}},$$

$$\|\chi u\|_{H^{s,l,1/2+\epsilon,0}_{\mathrm{c},h}} \le \|\chi u\|_{H^{s,l_1,l_1,0}_{\mathrm{c},h}}. \qquad \square$$

We describe a few scenarios in which the invertibility assumption on $N(P)$ can be verified:

(1) The operator $N(P)$ is invertible for an open set of $V_\partial \in \mathcal{C}^\infty(\mathbb{S}^{n-1}; \mathbb{C}^{N\times N})$. In particular, it holds when $n \ge 3$, $l \in (1-(n-2)/2, 1+(n-2)/2)$, and $V_\partial \equiv 0$ by Lemma 5.1, and therefore also when $\|V_\partial\|_{\mathcal{C}^k}$ is sufficiently small (depending on l) for some sufficiently large k.

(2) Consider V_∂ which depends holomorphically on a parameter $w \in \Omega$, where $\Omega \subset \mathbb{C}$ is open and contains 0. (For example, this is the case when $V_\partial(w) = wV_{\partial,0}$.) Let us write $N(P_w)$ for the w-dependent

normal operator, and assume that $N(P_0)$ is invertible at weight l_0; assume moreover that there is a continuous function $l:\Omega\to\mathbb{R}$ with $l(0)=l_0$ so that $l(w)\notin -\operatorname{Im}\operatorname{spec}_{\mathrm{b}}(N(P_w))$. Then there exists a discrete set $D\subset\Omega$ so that $N(P_w)$ is invertible at weight $l(w)$ for $w\in\Omega\setminus D$. This follows from analytic Fredholm theory in w; we leave the details to the reader.

A very concrete third scenario is the following:

Lemma 5.10 (scalar inverse square potentials). *Let $n\geq 2$, consider the scalar case $N=1$, and suppose $V_\partial=\mathsf{Z}\in\mathbb{C}\setminus(-\infty,-((n-2)/2)^2]$ is a constant (so $V(x)=\mathsf{Z}/|x|^2+\mathcal{O}(|x|^{-1})$). Then $N(P)$ is invertible at weight l if and only if $|l-1|<\operatorname{Re}\sqrt{((n-2)/2)^2+\mathsf{Z}}$.*

Proof of Lemma 5.10. The boundary spectrum of $N(P)$ can be computed, via expansion into spherical harmonics, as

$$\operatorname{spec}_{\mathrm{b}}(N(P))=\left\{i\left(\frac{n-2}{2}\pm\sqrt{\left(\frac{n-2}{2}+\ell\right)^2+\mathsf{Z}}\right):\ell\in\mathbb{N}_0\right\}.$$

The condition on Z ensures that

$$\operatorname{Re}\sqrt{\left(\frac{n-2}{2}\right)^2+\mathsf{Z}}>0,$$

and thus for l as in the statement of the lemma, one has $l-n/2\notin -\operatorname{Im}\operatorname{spec}_{\mathrm{b}}(N(P))$.

Expanding an outgoing solution u of $N(P)u=0$ at weight l into spherical harmonics, $u(\hat r,\omega)=\sum_{|m|\leq\ell}u_{\ell m}(\hat r)Y_{\ell m}(\omega)$, the coefficient $u_{\ell m}$ satisfies a Bessel ODE

$$-u_{\ell m}''-\frac{n-1}{\hat r}u_{\ell m}'+\frac{\ell(\ell+n-2)+\mathsf{Z}}{\hat r^2}u_{\ell m}-u_{\ell m}=0,$$

and hence is a linear combination of $\hat r^{-(n-2)/2}H^{(1)}_{\nu_\ell}(\hat r)$ and $\hat r^{-(n-2)/2}H^{(2)}_{\nu_\ell}(\hat r)$ where we set

$$\nu_\ell=\sqrt{\left(\frac{n-2}{2}+l\right)^2+\mathsf{Z}}.$$

The outgoing condition can only be satisfied if $u_{\ell m}$ is a multiple of $\hat r^{-(n-2)/2}H^{(1)}_{\nu_\ell}(\hat r)$. But for $0<\hat r\ll 1$, one has

$$|\hat r^{-\frac{n-2}{2}}H^{(1)}_{\nu_\ell}(\hat r)|\geq c_\ell\hat r^{-\frac{n-2}{2}-\operatorname{Re}\nu_\ell}\geq c_\ell\hat r^{-\frac{n-2}{2}-\operatorname{Re}\nu_0},$$

with $c_\ell>0$, which lies in $\hat r^{l'}L^2(\hat r^{n-1}\,d\hat r)$ if and only if $l'<1-\operatorname{Re}\nu_0$, which is violated for $l'=l$. Hence necessarily $u_{\ell m}=0$. This proves that $N(P)$ is injective at weight l on outgoing functions; the injectivity of $N(P)^*$ at weight $-l+2$ on incoming functions is proved similarly. □

Theorem 1.1 follows from Theorem 5.7 and Lemma 5.10 upon taking $\sigma=\sqrt{\lambda}$ and $l=2$, which allows for the choice $l_1=l_2=0$. Note that for $l=2$, the target space in (5-29) in $L^2(X)=L^2(\mathbb{R}^n)$, and the domain is $H_0^2(\mathbb{R}^n\setminus\{0\})$ for $n\geq 5$ by Hardy's inequality.

Remark 5.11 (multiple scatterers). By exploiting the diffractive improvement obtained in Section 4F as in [Baskin and Wunsch 2013], it is conceivable that one can generalize (up to ϵ-losses) Duyckaerts' high-energy resolvent estimates [2006] for scattering by a finite number of real-valued inverse square potentials and analyze the complex-valued case $\mathsf{Z}_j \in \mathbb{C} \setminus (-\infty, -((n-2)/2)^2]$ (or more generally the case of finitely many matrix-valued inverse square potentials). However, the study of this problem exceeds the scope of this paper.

Remark 5.12 (operators with inverse square singularities arising in the study of wave equations). Following [Baskin and Marzuola 2022], consider a static metric $g = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 k$, where k is a Riemannian metric on a closed manifold Z; e.g., $(Z, k) = (\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$, in which case g is the Minkowski metric. In the region $t > 1$, $r/t < \frac{1}{2}$, we introduce $T = 1/t$, $R = r/t$ (in the notation of the reference: ρ, x) and compute

$$t^2 \Box_g = -(T D_T + R D_R)^2 + i(T D_T + R D_R) + D_R^2 - \frac{i(n-1)}{R} D_R + R^{-2} \Delta_k.$$

Restricting the coefficients to $T = 0$ (as a b-operator) and formally passing to the Mellin transform by replacing $T D_T$ with multiplication by $\sigma \in \mathbb{C}$ gives

$$P_\sigma = \Delta_G - (R D_R + \sigma)^2 + i(R D_R + \sigma) = \Delta_G - \sigma^2 + Q, \quad G := \mathrm{d}R^2 + R^2 k,$$

where $Q = i\sigma - (2\sigma - i) R D_R$. When $|\operatorname{Im} \sigma|$ is bounded and $|\operatorname{Re} \sigma| \to \infty$, we let $h = |\sigma|^{-1}$; then the rescaling

$$P_{h,z} = h^2 P_{h^{-1} z} = h^2 \Delta_G - z^2 + h^2 Q \tag{5-32}$$

is an admissible operator on $\left[0, \frac{1}{2}\right)_R \times Z$ in the sense of Definition 4.1. More generally, consider the coupling of $\Box_g$ with a potential $V = V(r, z)$ which asymptotes to an inverse square potential as $r \to \infty$, i.e., $V(r, z) = r^{-2} V_0(z) + \mathcal{O}(r^{-2-\delta})$ with $V_0 \in \mathcal{C}^\infty(Z)$. Then we have $t^2(\Box_g + V) = t^2 \Box_g + R^{-2} V_0(z) + \mathcal{O}(T^\delta R^{-2-\delta})$, and therefore the rescaling (5-32) has an additional $h^2 R^{-2} V_0$ term, as studied in the present section; that is, a stationary *asymptotically* inverse square potential on $[0, \infty)_r \times Z$ gives rise to an inverse square *singularity* of $P_{h,z}$ at $R = 0$.[16] (If $V = r^{-2} V_0(z)$ is an exact inverse square potential, then $P_{h,z}$ has the same additional term.) Operators of this type, acting on sections of vector bundles without natural positive definite fiber inner products (and correspondingly without symmetry conditions on V_0), appear in the study of the equations of linearized gravity on stationary and asymptotically flat spacetimes in certain gauges, and indeed this was the author's original motivation for the investigations in the present paper; the details will appear elsewhere.

5C. ***Scattering for the Dirac–Coulomb equation.*** Motivated by [Baskin and Wunsch 2023], we consider the stationary scattering theory for the Dirac–Coulomb equation on Minkowski space at high energies. As discussed in Section 1, our framework allows us to deal directly with the associated matrix-valued Klein–Gordon operator — which has nonsymmetric leading-order terms at the Coulomb singularity — albeit with an arbitrarily small loss upon propagation through the singularity. Moreover, our results include a larger range of Coulomb charges $\mathsf{Z} \in \mathbb{R}$ than [Baskin and Wunsch 2023] (which requires $|\mathsf{Z}| < \frac{1}{2}$ for technical reasons); we can even allow for Z which $|\mathsf{Z}| > \frac{\sqrt{3}}{2}$, in which case the Dirac–Coulomb Hamiltonian is not essentially self-adjoint.

[16] After the original version of the present paper appeared, this has been worked out in detail by the author in the preprint [Hintz 2023].

The underlying spatial manifold is again given by (5-25), now with $n=3$. We recall relevant notation from [Baskin and Wunsch 2023]. Denote the Pauli matrices by

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Put further

$$\beta:=\gamma^0:=\begin{pmatrix}I&0\\0&-I\end{pmatrix},\quad \gamma^j:=\begin{pmatrix}0&\sigma_j\\-\sigma_j&0\end{pmatrix},$$
$$\alpha^j:=\beta\gamma^j=\begin{pmatrix}0&\sigma_j\\\sigma_j&0\end{pmatrix},\quad \alpha_r:=\sum_{j=1}^3\frac{x_j}{r}\alpha_j.$$

The equation governing a massive Dirac field (with mass $m\in\mathbb{R}$) minimally coupled to an electromagnetic potential $\boldsymbol{A}=(A_0,A_1,A_2,A_3)$ is

$$(i\not{\partial}_{\boldsymbol{A}}-m)\psi=0,\quad \not{\partial}_{\boldsymbol{A}}:=\gamma^\mu(\partial_\mu+iA_\mu),$$

where ψ takes values in $\mathbb{C}^4$. We now take

$$A_0=\frac{\mathsf{Z}}{r}+V,\quad V\in\mathcal{C}^\infty(X),\quad A_j\in\mathcal{C}^\infty(X),\tag{5-33}$$

with $\mathsf{Z}\in\mathbb{R}$ the charge of the Coulomb field. As shown in [Baskin and Wunsch 2023, §4.3], the operator $-(i\not{\partial}_{\boldsymbol{A}}+m)(i\not{\partial}_{\boldsymbol{A}}-m)$ is then of the form

$$P=-\Big(D_t+\frac{\mathsf{Z}}{r}\Big)^2+\Delta+m^2+i\frac{\mathsf{Z}}{r^2}\alpha_r+r^{-1}\boldsymbol{R},\quad \boldsymbol{R}\in\mathrm{Diff}^1_{\mathrm{b}}(X;\mathbb{C}^4).\tag{5-34}$$

Let us pass to a fixed temporal frequency $\sigma\in\mathbb{C}$, thus replacing D_t in (5-34) by $-\sigma$, resulting in the operator family $\widehat{P}(\sigma)$. Introducing $h=|\sigma|^{-1}$ and $z=(h\sigma)^2$ and multiplying $\widehat{P}(\sigma)=\widehat{P}(h^{-1}\sqrt{z})$ by h^2 gives

$$P_{h,z}=h^2\Delta-\Big(\sqrt{z}-\frac{h\mathsf{Z}}{r}\Big)^2+\frac{h^2}{r^2}(i\mathsf{Z}\alpha_r+r^2m^2)+\frac{h^2}{r^2}r\boldsymbol{R}=h^2\Delta-z+h^2r^{-2}Q_{1,z}+hr^{-1}q_{0,z},\tag{5-35}$$
$$Q_{1,z}=-\mathsf{Z}^2+i\mathsf{Z}\alpha_r+r^2m^2+r\boldsymbol{R},\quad q_{0,z}=2\sqrt{z}\mathsf{Z}.$$

When $\operatorname{Im}\sigma$ is bounded while $|\operatorname{Re}\sigma|\to\infty$, one has $z=1+\mathcal{O}(h)$; thus, $P_{h,z}$ is an admissible operator in the sense of Definition 4.1. The threshold quantities in Definition 4.7 take the values

$$r_{\mathrm{in}}=r_{\mathrm{out}}=-\tfrac{1}{2}$$

since $q_{0,1}$ is real and the principal symbol of $Q_{1,z}$ (as a first-order b-differential operator) vanishes at $r=0$.

The normal operator of $P_{h,z}$ is obtained by passing to $\hat{r}=r/h$ and restricting to $h=0$, giving in polar coordinates $(\hat{r},\omega)\in[0,\infty]\times\mathbb{S}^2$

$$N(P)=D_{\hat{r}}^2-\frac{2i}{\hat{r}}D_{\hat{r}}-\Big(1-\frac{\mathsf{Z}}{\hat{r}}\Big)^2+\hat{r}^{-2}(\not{\Delta}+i\mathsf{Z}\alpha_r(\omega)).\tag{5-36}$$

For $\mathsf{Z}=0$, the operator $N(P)$ is equal to $(\Delta_{\hat{g}}-1)\otimes\mathrm{Id}_{\mathbb{C}^4}$, where $\hat{g}=d\hat{r}^2+\hat{r}^2g_{\mathbb{S}^2}$, and hence is invertible at weight $l\in\big(\frac{1}{2},\frac{3}{2}\big)$ by Lemma 5.1. For fixed l, this will remain true for Z in a small neighborhood of 0.

The determination of the largest set of Z for which $N(P)$ is invertible at some weight requires explicit calculations:

Lemma 5.13 (invertibility of $N(P)$). *Let $\mathsf{Z} \in \mathbb{R}$ be such that $|\mathsf{Z}| \neq \sqrt{\kappa^2 - \frac{1}{4}}$ for all $\kappa \in \mathbb{N}$. Then the operator $N(P)$ is invertible at weight $l = 1$.*

The conclusion of the Lemma in particular holds in the range $|\mathsf{Z}| < \frac{\sqrt{3}}{2}$; this is related to the essential self-adjointness of Dirac operators; see [Weidmann 1971; Lesch 1997; Baskin and Wunsch 2023, §4.1].

Proof of Lemma 5.13. We begin by separating into spinor spherical harmonics following [Baskin and Wunsch 2023, §2.1]: For

$$\kappa \in \mathbb{Z} \setminus \{0\}, \quad \mu \in \left\{-|\kappa| + \tfrac{1}{2}, \dots, |\kappa| - \tfrac{1}{2}\right\},$$

define the $\mathbb{C}^2$-valued function on $\mathbb{S}^2$

$$\Omega_{\kappa,\mu}(\omega) = \begin{pmatrix} -\operatorname{sgn}(\kappa)\big((\kappa + \frac{1}{2} - \mu)/(2\kappa+1)\big)^{1/2} Y_{l,\mu-1/2}(\omega) \\ \big((\kappa + \frac{1}{2} + \mu)/(2\kappa+1)\big)^{1/2} Y_{l,\mu+1/2}(\omega) \end{pmatrix}, \quad l = \left|\kappa + \tfrac{1}{2}\right| - \tfrac{1}{2}.$$

Thus $\not{\Delta}\Omega_{\kappa,\mu} = \kappa(\kappa+1)\Omega_{\kappa,\mu}$. Moreover,

$$\alpha_r \begin{pmatrix} a\Omega_{\kappa,\mu} \\ b\Omega_{-\kappa,\mu'} \end{pmatrix} = \begin{pmatrix} -b\Omega_{\kappa,\mu'} \\ -a\Omega_{-\kappa,\mu} \end{pmatrix},$$

as follows from [Baskin and Wunsch 2023, equations (3), (4), (9)] or [Szmytkowski 2007, equation (3.1.3)]. Thus, the action of the spherical operator $\not{\Delta} + i\mathsf{Z}\alpha_r \in \mathrm{Diff}^2(\mathbb{S}^2; \mathbb{C}^4)$ appearing in (5-36) on the 2-dimensional space with basis $(\Omega_{\kappa,\mu}, 0)$ and $(0, \Omega_{-\kappa,\mu})$ is given by the matrix

$$\begin{pmatrix} \kappa(\kappa+1) & -i\mathsf{Z} \\ -i\mathsf{Z} & \kappa(\kappa-1) \end{pmatrix}. \tag{5-37}$$

This can be diagonalized when $|\mathsf{Z}| \neq |\kappa|$, and it has eigenvalue $\lambda_\kappa^\pm = \kappa^2 \pm \sqrt{\kappa^2 - \mathsf{Z}^2}$ on the eigenspace spanned by

$$\mathcal{Y}^\pm_{\kappa,\mu}(\omega) := \begin{pmatrix} i\mathsf{Z}\Omega_{\kappa,\mu}(\omega) \\ (\kappa \mp \sqrt{\kappa^2 - \mathsf{Z}^2})\Omega_{-\kappa,\mu}(\omega) \end{pmatrix}, \quad \mu \in \left\{-|\kappa| + \tfrac{1}{2}, \dots, |\kappa| - \tfrac{1}{2}\right\}.$$

Thus, the action of $N(P)$ on separated functions of the form $u(\hat{r})\mathcal{Y}^\pm_{\kappa,\mu}(\omega)$ is given by the action on u of the differential operator

$$N_\kappa^\pm = D_{\hat{r}}^2 - \frac{2i}{\hat{r}} D_{\hat{r}} - \left(1 - \frac{\mathsf{Z}}{\hat{r}}\right)^2 + \hat{r}^{-2}\lambda_\kappa^\pm.$$

The Mellin-transformed normal operator of $\hat{r}^2 N_\kappa^\pm$ at $\hat{r} = 0$ is the polynomial

$$\lambda^2 - i\lambda - \mathsf{Z}^2 + \lambda_\kappa^\pm, \quad \lambda \in \mathbb{C};$$

for its roots, we have

$$-(\operatorname{Im}\lambda) + \tfrac{3}{2} \in \left\{1 - \left(\tfrac{1}{2} \pm \sqrt{\kappa^2 - \mathsf{Z}^2}\right),\ 1 + \left(\tfrac{1}{2} \pm \sqrt{\kappa^2 - \mathsf{Z}^2}\right)\right\}.$$

Now if $\mathsf{Z}^2 > \kappa^2$, then these two roots have real parts $\frac{1}{2}$ and $\frac{3}{2}$, whereas if $\mathsf{Z}^2 < \kappa^2$, they are disjoint from an open interval $(1-\delta, 1+\delta)$ around 1 due to the assumption that $\kappa^2 - \mathsf{Z}^2 \neq \frac{1}{4}$.

An outgoing solution of $N(P)$ at weight $l = 1$, expanded into the spherical eigenfunctions $\mathcal{Y}^{\pm}_{\kappa,\mu}$, is an outgoing solution of $N^{\pm}_{\kappa}$; one easily finds $u = u_\infty \hat{r}^{-1-iZ} e^{i\hat{r}} + \mathcal{O}(\hat{r}^{-2})$ as $\hat{r} \to \infty$, where $u_\infty \in \mathbb{C}$, and the $\mathcal{O}(\hat{r}^{-2})$ term is conormal at $\hat{r} = 0$; near $\hat{r} = 0$ on the other hand, we have $u = \mathcal{A}^{-1/2+\delta}([0, 1)_{\hat{r}})$. A boundary-pairing argument, i.e., the evaluation of

$$0 = \lim_{\epsilon\to 0} \int_\epsilon^{1/\epsilon} ((N^{\pm}_{\kappa} u)\bar{u} - u\overline{N^{\pm}_{\kappa} u})\, \hat{r}^2 \, d\hat{r} = 2i \lim_{\epsilon \to 0} \operatorname{Im}(\hat{r}^2 u \overline{u'}|_{\epsilon}^{1/\epsilon}) = -2i|u_\infty|^2,$$

gives $u_\infty = 0$, and thus $u \equiv 0$ by standard ODE analysis near $\hat{r} = \infty$. This shows that $N(P)$ is injective at weight $l = 1$ on outgoing solutions. Since $(N^{\pm}_{\kappa})^* = N^{\mp}_{\kappa}$ with respect to the $L^2(\hat{r}^2\, d\hat{r})$ inner product, the injectivity of $N(P)^*$ at weight $-l+2$ on incoming functions is proved similarly. This completes the proof when Z is not a nonzero integer.

When $\mathsf{Z} \in \mathbb{Z} \setminus \{0\}$ and κ satisfies $|\mathsf{Z}| = |\kappa|$, then the action of $\not{\Delta} + i\mathsf{Z}\alpha_r$ on the span of $(\Omega_{\kappa,\mu}, 0)$ and $(0, \Omega_{-\kappa,\mu'})$ is not diagonalizable anymore. By inspection of (5-37), it still has the eigenvalue κ^2 with eigenspace spanned by $\mathcal{Y}^{+}_{\kappa,\mu} = \mathcal{Y}^{-}_{\kappa,\mu}$. Let $\widetilde{\mathcal{Y}}_{\kappa,\mu} = (\Omega_{\kappa,\mu}, 0)$, then an outgoing solution $u = u_1 \mathcal{Y}^{+}_{\kappa,\mu} + u_2 \widetilde{\mathcal{Y}}_{\kappa,\mu}$ of $N(P)$ satisfies a lower triangular ODE system, with a decoupled equation for u_1 which implies $u_1 \equiv 0$ by the previous arguments, whence u_2 is now an outgoing solution to the same equation as u_1 and must therefore also vanish. □

If we cut $\boldsymbol{A}$ off via multiplication by a cutoff $\chi \in \mathcal{C}^\infty_c(X)$, the operator $P_{h,z}$ is equal to $h^2\Delta - z$ near infinity and can thus be analyzed as in Section 5B. In this setting, we thus obtain invertibility and quantitative estimates for $P_{h,z}$:

Theorem 5.14 (high-energy estimates for the Dirac–Coulomb equation). *Suppose $\boldsymbol{A} = (A_0, A_1, A_2, A_3)$ as in (5-33) has compact support. Let $\mathsf{Z} \in \mathbb{R}$ be such that $|\mathsf{Z}| \neq \sqrt{\kappa^2 - \frac{1}{4}}$ for all $\kappa \in \mathbb{N}$. Then for $l = 1$ (and indeed for l sufficiently close to 1), $0 < \operatorname{Im} z < Ch$, and for all sufficiently small $h > 0$, the operator $P_{h,z} = h^2 \widehat{P}(h^{-1}z)$ defined in (5-34)–(5-35) is invertible as a map between the spaces (5-29) and satisfies the uniform bound (5-30) as well as the bound (5-31) (with $\Delta_g + V - \sigma^2$ replaced by $\widehat{P}(\sigma)$, where $\sigma = h^{-1}z$) for $l_1 = l_2 = 0$.*

Remark 5.15 (complex charges). One can also analyze the case of nonreal $\mathsf{Z} \in \mathbb{C}$, in which case $r_{\text{in}} = -\frac{1}{2} + \operatorname{Im} \mathsf{Z}$ and $r_{\text{out}} = -\frac{1}{2} - \operatorname{Im} \mathsf{Z}$. The difference $r_{\text{out}} - r_{\text{in}} = -2 \operatorname{Im} \mathsf{Z}$ results in an additional $2 \operatorname{Im} \mathsf{Z}$ loss of powers of the semiclassical parameter h when propagating through the singularity at $r = 0$. Nonetheless, the invertibility of $N(P)$ automatically holds for values of Z close to those allowed in Theorem 5.14, as discussed prior to Lemma 5.10.

Appendix A: A class of examples with sharp semiclassical loss

Note that the semiclassical order b in Theorem 4.10 must decrease from $\mathcal{R}_{\text{in}}$ to $\mathcal{R}_{\text{out}}$ by more than

$$D = \max(r_{\text{in}} - r_{\text{out}}, 0); \tag{A-1}$$

thus, the estimate (4-27) controls u in L^2, say, microlocally near the flow-out of $\mathcal{R}_{\mathrm{out}}$ by $h^{-D-\epsilon}$ (for any $\epsilon > 0$) times the L^2-norm of the microlocalization of u near the flow-in of $\mathcal{R}_{\mathrm{in}}$. While in many natural settings, such as those discussed in Section 5, one has $D = 0$, it is easy to construct examples where $D > 0$. The following example (placed into a general context at the end of this appendix) shows that a loss of h^{-D} typically *does* occur, whence our estimates are sharp up to an ϵ-loss. This ϵ-loss may be avoidable, though we are not able to prove or disprove this here.

Consider $X = [0, 2)_r$, $\mu = |\mathrm{d}r|$, and

$$P_{h,z} = h^2\left(-\partial_r^2 - \frac{2}{r}\partial_r\right) - z + \frac{2ih}{r}q, \quad z = 1 + \mathcal{O}(h),$$

where $q \in \mathbb{C}$ is a parameter. (The term in parentheses is the radial part of the Laplacian on $\mathbb{R}^3$ in polar coordinates.) The normal operator is

$$N(P) = -\partial_{\hat r}^2 - \frac{2}{\hat r}\partial_{\hat r} - 1 + \frac{2i}{\hat r}q.$$

For $q = 0$, the kernel of $N(P)$ is spanned by $\hat r^{-1}e^{\pm i\hat r}$; since $\hat r^{-1}$ barely fails to lie in $\hat r^{-1/2}L^2([0,1)_{\hat r}, |\mathrm{d}\hat r|)$, it is easy to see that $N(P)$ is invertible at weight l in the sense of Definition 4.6 for $l \in \left(-\frac{1}{2}, \frac{1}{2}\right)$; this persists for small values of $q \in \mathbb{C}$. (The boundary spectrum of $N(P)$ at $\hat r = 0$ is independent of q.) In the notation of Definition 4.7, we have

$$r_{\mathrm{in}} = \tfrac{1}{2} + \operatorname{Re} q, \quad r_{\mathrm{out}} = \tfrac{1}{2} - \operatorname{Re} q, \tag{A-2}$$

so $D = \max(2\operatorname{Re} q, 0)$. The quantities (A-2) correspond precisely to the L^2-decay rates of incoming and outgoing solutions $\hat u_{\mathrm{in}}, \hat u_{\mathrm{out}} \in \ker N(P)$, which have the asymptotic behavior

$$\hat u_{\mathrm{in}} \sim \hat r^{-1-\operatorname{Re} q} e^{-i\hat r}, \quad \hat u_{\mathrm{out}} \sim \hat r^{-1+\operatorname{Re} q} e^{i\hat r}, \quad \hat r \to \infty. \tag{A-3}$$

(We omit the explicit expressions involving confluent hypergeometric functions.)

We can now construct an element $\hat u \in \ker N(P)$ which lies in $\hat r^l L^2$, $l \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, near $\hat r = 0$ and which is uniquely specified by requiring its incoming data at $\hat r = \infty$ to be given by $\hat u_{\mathrm{in}}$. Indeed, $\hat u$ is necessarily of the form

$$\hat u(\hat r) = \hat u_{\mathrm{in}} + c\hat u_{\mathrm{out}},$$

where $c \in \mathbb{C}$ is the "scattering matrix"; necessarily $c \neq 0$ (since $\hat u_{\mathrm{in}}$ fails to lie in $\hat r^l L^2$ near $\hat r = 0$). But then

$$P_{h,1}u_h(r) = 0, \quad u_h(r) := \hat u(r/h).$$

(This exact formula uses the invariance of $P_{h,1}$ under dilations in (h, r).) Considering a neighborhood of $r = 1$ then, the asymptotics (A-3) for

$$u_{\bullet,h}(r) := u_\bullet(r/h), \quad \bullet = \mathrm{in}, \mathrm{out},$$

imply

$$u_h = u_{\mathrm{in,h}} + cu_{\mathrm{out,h}},$$
$$u_{\mathrm{in,h}} \sim h^{1+\operatorname{Re} q} e^{-ir/h}, \quad u_{\mathrm{out,h}} \sim h^{1-\operatorname{Re} q} e^{ir/h} \quad (\text{near } r = 1).$$

This demonstrates the loss of $h^{-2\operatorname{Re} q}$ between the amplitudes $h^{1+\operatorname{Re} q}$, resp. $h^{1-\operatorname{Re} q}$ of the incoming, resp. outgoing pieces of u_h. (The fact that there is in fact a gain when $\operatorname{Re} q < 0$ is a peculiar feature of the 1-dimensional situation considered here: the characteristic set of $P_{h,z}$ has two connected components, with the incoming and outgoing radial sets lying in different components, and the monotonicity requirement in Theorem 4.10 does not relate the two components.)

The same idea can applied to produce many more examples with sharp loss D. Indeed, when $N(P)$ is invertible at weight l, then the solution $\hat{u} = \hat{u}(\hat{r}, y)$ (with y denoting points on ∂X) of $N(P)\hat{u} = 0$, where $\hat{u}$ has prescribed incoming data and lies in $\hat{r}^l L^2$ near $\hat{r} = 0$, gives rise to a solution $u_h(r, y) = \hat{u}(r/h, y)$ of $P_{h,1} u_h(r, y) = 0$, where $P_{h,1} = N(P)$ (upon changing coordinates $\hat{r} = r/h$). The relative decay rates of incoming/outgoing solutions of $N(P)$ are then directly reflected in the relative semiclassical orders of u_h near the flow-in/flow-out of the cone point. (Since the characteristic set of $P_{h,1}$ is connected when $\dim X \geq 2$, the loss is at least 0, see (A-1); after all, even away from the cone point, semiclassical regularity cannot improve under real-principal-type propagation.)

Appendix B: Sobolev spaces and pseudodifferential calculi

This section consists entirely of figures which illustrate the orders of weighted Sobolev spaces as well as of spaces of pseudodifferential operators, with references to the original definitions. Concretely, labeling a boundary hypersurface H by "l" means that the order l of some weighted Sobolev space $H^{\cdots,l,\cdots}_{\cdots}$ refers to ρ_H^l decay at H of its elements, where ρ_H is a defining function of H, or l orders of regularity when H is a boundary hypersurface at fiber infinity of some compactified phase space. For spaces of pseudodifferential operators on the other hand, the same label "l" refers to a ρ_H^{-l} bound of the full symbol of the operator, or of its Schwartz kernel at the hypersurface of the double space corresponding to H.

See Figure 13 for b- and scattering Sobolev spaces (or operators), Figure 14 for b, sc-Sobolev spaces, Figure 15 for semiclassical cone Sobolev spaces, and Figure 16 for cb$\hbar$-Sobolev spaces.

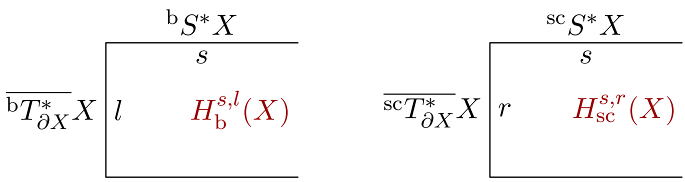

Figure 13. X is a manifold with boundary ∂X. Left: the orders of $H_{\mathrm{b}}^{s,l}(X)$; see Section 2A. Right: the orders of $H_{\mathrm{sc}}^{s,r}(X)$; see Section 2B.

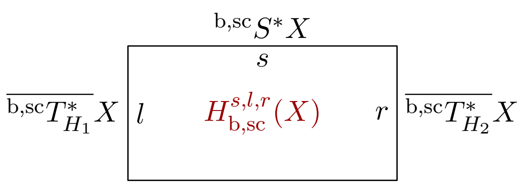

Figure 14. X is a manifold with two connected and embedded boundary hypersurfaces $\partial X = H_1 \sqcup H_2$. Indicated are the orders for $H_{\mathrm{b,sc}}^{s,l,r}(X)$.

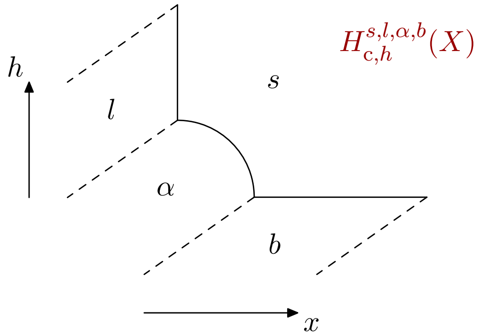

Figure 15. X is a manifold with boundary ∂X. Indicated are the orders for $H^{s,l,\alpha,b}_{c,h}(X)$; see Section 3B.

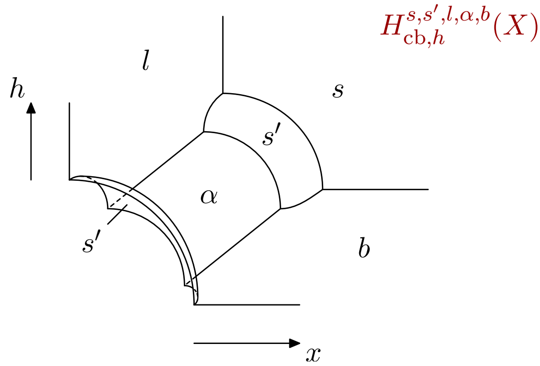

Figure 16. This is essentially a repetition of Figure 9. Indicated are the orders for $H^{s,s',l,\alpha,b}_{cb,h}(X)$; see Section 3D.

Acknowledgments

I am grateful to András Vasy and Jared Wunsch for helpful conversations, and to two anonymous referees for their careful reading of the paper and for excellent comments and suggestions. I gratefully acknowledge support from a Sloan Research Fellowship and from the NSF under grant no. DMS-1955614.

References

[Baskin and Marzuola 2019] D. Baskin and J. L. Marzuola, "Locating resonances on hyperbolic cones", *Math. Res. Lett.* **26**:2 (2019), 365–381. MR Zbl

[Baskin and Marzuola 2022] D. Baskin and J. L. Marzuola, "The radiation field on product cones", *Adv. Math.* **408** (2022), art. id. 108589. MR Zbl

[Baskin and Wunsch 2013] D. Baskin and J. Wunsch, "Resolvent estimates and local decay of waves on conic manifolds", *J. Differential Geom.* **95**:2 (2013), 183–214. MR Zbl

[Baskin and Wunsch 2023] D. Baskin and J. Wunsch, "Diffraction for the Dirac–Coulomb propagator", *Ann. Henri Poincaré* **24**:8 (2023), 2607–2659. MR Zbl

[Baskin et al. 2015] D. Baskin, A. Vasy, and J. Wunsch, "Asymptotics of radiation fields in asymptotically Minkowski space", *Amer. J. Math.* **137**:5 (2015), 1293–1364. MR Zbl

[Baskin et al. 2018] D. Baskin, A. Vasy, and J. Wunsch, "Asymptotics of scalar waves on long-range asymptotically Minkowski spaces", *Adv. Math.* **328** (2018), 160–216. MR Zbl

[Beals 1977] R. Beals, "Characterization of pseudodifferential operators and applications", *Duke Math. J.* **44**:1 (1977), 45–57. MR Zbl

[Bony 1986] J.-M. Bony, "Second microlocalization and propagation of singularities for semilinear hyperbolic equations", pp. 11–49 in *Hyperbolic equations and related topics* (Katata/Kyoto, 1984), edited by S. Mizohata, Academic Press, Boston, MA, 1986. MR Zbl

[Burq et al. 2003] N. Burq, F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh, "Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential", *J. Funct. Anal.* **203**:2 (2003), 519–549. MR Zbl

[Cheeger and Taylor 1982a] J. Cheeger and M. Taylor, "On the diffraction of waves by conical singularities, I", *Comm. Pure Appl. Math.* **35**:3 (1982), 275–331. MR Zbl

[Cheeger and Taylor 1982b] J. Cheeger and M. Taylor, "On the diffraction of waves by conical singularities, II", *Comm. Pure Appl. Math.* **35**:4 (1982), 487–529. MR Zbl

[Chin 2022] X. Chin, "The semiclassical resolvent on conic manifolds and application to Schrödinger equations", *Commun. Math. Phys.* **390**:2 (2022), 757–826. Zbl

[Datchev and Vasy 2012] K. Datchev and A. Vasy, "Gluing semiclassical resolvent estimates via propagation of singularities", *Int. Math. Res. Not.* **2012**:23 (2012), 5409–5443. MR Zbl

[Duistermaat and Hörmander 1972] J. J. Duistermaat and L. Hörmander, "Fourier integral operators, II", *Acta Math.* **128**:3-4 (1972), 183–269. MR Zbl

[Duyckaerts 2006] T. Duyckaerts, "Inégalités de résolvante pour l'opérateur de Schrödinger avec potentiel multipolaire critique", *Bull. Soc. Math. France* **134**:2 (2006), 201–239. MR Zbl

[Dyatlov and Zworski 2019] S. Dyatlov and M. Zworski, *Mathematical theory of scattering resonances*, Grad. Stud. in Math. **200**, Amer. Math. Soc., Providence, RI, 2019. MR Zbl

[Friedlander 1958] F. G. Friedlander, *Sound pulses*, Cambridge Univ. Press, 1958. MR Zbl

[Galkowski 2017] J. Galkowski, "A quantitative Vainberg method for black box scattering", *Comm. Math. Phys.* **349**:2 (2017), 527–549. MR Zbl

[Gannot and Wunsch 2023] O. Gannot and J. Wunsch, "Semiclassical diffraction by conormal potential singularities", *Ann. Sci. École Norm. Sup.* (4) **56**:3 (2023), 713–800. MR

[Gell-Redman et al. 2020] J. Gell-Redman, A. Hassell, J. Shapiro, and J. Zhang, "Existence and asymptotics of nonlinear Helmholtz eigenfunctions", *SIAM J. Math. Anal.* **52**:6 (2020), 6180–6221. MR Zbl

[Grieser 2001] D. Grieser, "Basics of the b-calculus", pp. 30–84 in *Approaches to singular analysis* (Berlin, 1999), edited by J. Gil et al., Oper. Theory Adv. Appl. **125**, Birkhäuser, Basel, 2001. MR Zbl

[Guillarmou et al. 2013] C. Guillarmou, A. Hassell, and A. Sikora, "Resolvent at low energy, III: The spectral measure", *Trans. Amer. Math. Soc.* **365**:11 (2013), 6103–6148. MR Zbl

[Haber and Vasy 2013] N. Haber and A. Vasy, "Propagation of singularities around a Lagrangian submanifold of radial points", pp. 113–116 in *Microlocal methods in mathematical physics and global analysis*, edited by D. Grieser et al., Birkhäuser, Basel, 2013. MR Zbl

[Hassell et al. 2008] A. Hassell, R. Melrose, and A. Vasy, "Microlocal propagation near radial points and scattering for symbolic potentials of order zero", *Anal. PDE* **1**:2 (2008), 127–196. MR Zbl

[Helffer and Sjöstrand 1989] B. Helffer and J. Sjöstrand, "Équation de Schrödinger avec champ magnétique et équation de Harper", pp. 118–197 in *Schrödinger operators* (Sønderborg, Denmark, 1988), edited by H. Holden and A. Jensen, Lecture Notes in Phys. **345**, Springer, 1989. MR Zbl

[Hillairet and Wunsch 2020] L. Hillairet and J. Wunsch, "On resonances generated by conic diffraction", *Ann. Inst. Fourier (Grenoble)* **70**:4 (2020), 1715–1752. MR Zbl

[Hintz 2022] P. Hintz, "Resolvents and complex powers of semiclassical cone operators", *Math. Nachr.* **295**:10 (2022), 1990–2035. MR Zbl

[Hintz 2023] P. Hintz, "Linear waves on asymptotically flat spacetimes, I", preprint, 2023. arXiv 2302.14647

[Hirsch et al. 1977] M. W. Hirsch, C. C. Pugh, and M. Shub, *Invariant manifolds*, Lecture Notes in Math. **583**, Springer, 1977. MR Zbl [de Hoop et al. 2015] M. de Hoop, G. Uhlmann, and A. Vasy, "Diffraction from conormal singularities", *Ann. Sci. École Norm. Sup.* (4) **48**:2 (2015), 351–408. MR Zbl

[Hörmander 1971] L. Hörmander, "Fourier integral operators, I", *Acta Math.* **127**:1-2 (1971), 79–183. MR Zbl

[Keeler and Marzuola 2022] B. Keeler and J. L. Marzuola, "Pointwise dispersive estimates for Schrödinger operators on product cones", *J. Differential Equations* **320** (2022), 419–468. MR Zbl

[Keller 1985] J. B. Keller, "One hundred years of diffraction theory", *IEEE Trans. Antennas and Propagation* **33**:2 (1985), 123–126. MR Zbl

[Lebeau 1997] G. Lebeau, "Propagation des ondes dans les variétés à coins", *Ann. Sci. École Norm. Sup.* (4) **30**:4 (1997), 429–497. MR Zbl

[Lesch 1997] M. Lesch, *Operators of Fuchs type, conical singularities, and asymptotic methods*, Teubner-Texte Math. **136**, Teubner, Stuttgart, 1997. MR Zbl

[Loya 2002] P. Loya, "On the resolvent of differential operators on conic manifolds", *Comm. Anal. Geom.* **10**:5 (2002), 877–934. MR Zbl

[Mazzeo 1991] R. Mazzeo, "Elliptic theory of differential edge operators, I", *Comm. Partial Differential Equations* **16**:10 (1991), 1615–1664. MR Zbl

[Melrose 1975] R. B. Melrose, "Microlocal parametrices for diffractive boundary value problems", *Duke Math. J.* **42**:4 (1975), 605–635. MR Zbl

[Melrose 1993] R. B. Melrose, *The Atiyah–Patodi–Singer index theorem*, Res. Notes in Math. **4**, Peters, Wellesley, MA, 1993. MR Zbl

[Melrose 1994] R. B. Melrose, "Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces", pp. 85–130 in *Spectral and scattering theory* (Sanda, Japan, 1992), edited by M. Ikawa, Lecture Notes in Pure Appl. Math. **161**, Dekker, New York, 1994. MR Zbl

[Melrose 1995] R. B. Melrose, *Geometric scattering theory*, Cambridge Univ. Press, 1995. MR Zbl

[Melrose 1996] R. B. Melrose, "Differential analysis on manifolds with corners", book in preparation, 1996, available at https://math.mit.edu/~rbm/book.html.

[Melrose and Sjöstrand 1978] R. B. Melrose and J. Sjöstrand, "Singularities of boundary value problems, I", *Comm. Pure Appl. Math.* **31**:5 (1978), 593–617. MR Zbl

[Melrose and Sjöstrand 1982] R. B. Melrose and J. Sjöstrand, "Singularities of boundary value problems, II", *Comm. Pure Appl. Math.* **35**:2 (1982), 129–168. MR Zbl

[Melrose and Taylor 2018] R. B. Melrose and M. E. Taylor, "Boundary problems for wave equations with grazing and gliding rays", preprint, 2018, available at http://mtaylor.web.unc.edu/wp-content/uploads/sites/16915/2018/04/glide.pdf.

[Melrose and Wunsch 2004] R. Melrose and J. Wunsch, "Propagation of singularities for the wave equation on conic manifolds", *Invent. Math.* **156**:2 (2004), 235–299. MR Zbl

[Melrose and Zworski 1996] R. Melrose and M. Zworski, "Scattering metrics and geodesic flow at infinity", *Invent. Math.* **124**:1-3 (1996), 389–436. MR Zbl

[Melrose et al. 2008] R. Melrose, A. Vasy, and J. Wunsch, "Propagation of singularities for the wave equation on edge manifolds", *Duke Math. J.* **144**:1 (2008), 109–193. MR Zbl

[Melrose et al. 2013] R. Melrose, A. Vasy, and J. Wunsch, *Diffraction of singularities for the wave equation on manifolds with corners*, Astérisque **351**, Soc. Math. France, Paris, 2013. MR Zbl

[Mourre 1980/81] E. Mourre, "Absence of singular continuous spectrum for certain selfadjoint operators", *Comm. Math. Phys.* **78**:3 (1980/81), 391–408. MR Zbl

[Nonnenmacher and Zworski 2009] S. Nonnenmacher and M. Zworski, "Quantum decay rates in chaotic scattering", *Acta Math.* **203**:2 (2009), 149–233. MR Zbl

[Planchon et al. 2003] F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh, "Dispersive estimate for the wave equation with the inverse-square potential", *Discrete Contin. Dyn. Syst.* **9**:6 (2003), 1387–1400. MR Zbl

[Qian 2009] R. Z. Qian, *Diffractive theorems for the wave equation with inverse square potentials*, Ph.D. thesis, Northwestern University, 2009, available at https://www.proquest.com/docview/304971420.

[Sjöstrand and Zworski 1991] J. Sjöstrand and M. Zworski, "Complex scaling and the distribution of scattering poles", *J. Amer. Math. Soc.* **4**:4 (1991), 729–769. MR Zbl

[Sjöstrand and Zworski 2007] J. Sjöstrand and M. Zworski, "Fractal upper bounds on the density of semiclassical resonances", *Duke Math. J.* **137**:3 (2007), 381–459. MR Zbl

[Sommerfeld 1896] A. Sommerfeld, "Mathematische Theorie der Diffraction", *Math. Ann.* **47**:2-3 (1896), 317–374. MR Zbl

[Szmytkowski 2007] R. Szmytkowski, "Recurrence and differential relations for spherical spinors", *J. Math. Chem.* **42**:3 (2007), 397–413. MR Zbl

[Taylor 1976] M. E. Taylor, "Grazing rays and reflection of singularities of solutions to wave equations", *Comm. Pure Appl. Math.* **29**:1 (1976), 1–38. MR Zbl

[Vasy 2000] A. Vasy, *Propagation of singularities in three-body scattering*, Astérisque **262**, Soc. Math. France, Paris, 2000. MR Zbl

[Vasy 2001] A. Vasy, "Propagation of singularities in many-body scattering", *Ann. Sci. École Norm. Sup.* (4) **34**:3 (2001), 313–402. MR Zbl

[Vasy 2008] A. Vasy, "Propagation of singularities for the wave equation on manifolds with corners", *Ann. of Math.* (2) **168**:3 (2008), 749–812. MR Zbl

[Vasy 2013] A. Vasy, "Microlocal analysis of asymptotically hyperbolic and Kerr–de Sitter spaces", *Invent. Math.* **194**:2 (2013), 381–513. MR Zbl

[Vasy 2018] A. Vasy, "A minicourse on microlocal analysis for wave propagation", pp. 219–374 in *Asymptotic analysis in general relativity* (Grenoble, France, 2014), edited by T. Daudé et al., Lond. Math. Soc. Lect. Note Ser. **443**, Cambridge Univ. Press, 2018. MR Zbl

[Vasy 2021a] A. Vasy, "Limiting absorption principle on Riemannian scattering (asymptotically conic) spaces, a Lagrangian approach", *Comm. Partial Differential Equations* **46**:5 (2021), 780–822. MR Zbl

[Vasy 2021b] A. Vasy, "Resolvent near zero energy on Riemannian scattering (asymptotically conic) spaces", *Pure Appl. Anal.* **3**:1 (2021), 1–74. MR Zbl

[Vasy 2021c] A. Vasy, "Resolvent near zero energy on Riemannian scattering (asymptotically conic) spaces, a Lagrangian approach", *Comm. Partial Differential Equations* **46**:5 (2021), 823–863. MR Zbl

[Vasy and Wunsch 2009] A. Vasy and J. Wunsch, "Semiclassical second microlocal propagation of regularity and integrable systems", *J. Anal. Math.* **108** (2009), 119–157. Correction in **115** (2011), 389–391. MR Zbl

[Vasy and Zworski 2000] A. Vasy and M. Zworski, "Semiclassical estimates in asymptotically Euclidean scattering", *Comm. Math. Phys.* **212**:1 (2000), 205–217. MR Zbl

[Wang 2020] J. Wang, "Sharp radial estimates in Besov spaces", preprint, 2020. arXiv 2003.11218

[Weidmann 1971] J. Weidmann, "Oszillationsmethoden für Systeme gewöhnlicher Differentialgleichungen", *Math. Z.* **119** (1971), 349–373. MR Zbl

[Wunsch and Zworski 2011] J. Wunsch and M. Zworski, "Resolvent estimates for normally hyperbolic trapped sets", *Ann. Henri Poincaré* **12**:7 (2011), 1349–1385. MR Zbl

[Yang 2022] M. Yang, "Propagation of polyhomogeneity, diffraction, and scattering on product cones", *J. Spectr. Theory* **12**:2 (2022), 381–415. MR Zbl

Received 28 May 2022. Revised 14 May 2023. Accepted 18 Jul 2023.

PETER HINTZ: phintz@mit.edu, peter.hintz@math.ethz.ch
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, United States

and

Department of Mathematics, ETH Zürich, Zürich, Switzerland

ANALYSIS AND PDE
Vol. 17 (2024), No. 10, pp. 3551–3588
DOI: 10.2140/apde.2024.17.3551

SMALL CAP DECOUPLING FOR THE MOMENT CURVE IN $\mathbb{R}^3$

LARRY GUTH AND DOMINIQUE MALDAGUE

We prove sharp small cap decoupling estimates for the moment curve in $\mathbb{R}^3$. Our formulation of the small caps is motivated by a conjecture of Demeter, Guth, and Wang about L^p estimates for exponential sums.

1. Introduction

We use high-/low-frequency methods to prove small cap decoupling inequalities for the moment curve $\mathcal{M}^3 = \{(t, t^2, t^3) : t \in [0, 1]\}$ in $\mathbb{R}^3$. We begin by describing the problem and our results in terms of exponential sums. The motivation for this paper is to prove Conjecture 2.5 with $n = 3$ from [Demeter et al. 2020], which we state now. We use the standard notation $e(t) = e^{2\pi i t}$.

Conjecture 1. *For each $N \geq 1$, $0 \leq \sigma \leq 2$, and $s \geq 1$,*

$$\int_{[0,1]^2\times[0,1/N^\sigma]} \left|\sum_{k=1}^{N} e(kx_1 + k^2x_2 + k^3x_3)\right|^{2s} dx \leq C_\varepsilon N^\varepsilon [N^{s-\sigma} + N^{2s-6}].$$

The $s = 1$ and $s = \infty$ versions of this conjecture are easily verified using L^2-orthogonality and the triangle inequality, respectively. When $\sigma = 0$, this is Vinogradov's mean value theorem, solved in three dimensions by Wooley [2016] and using decoupling for the moment curve by Bourgain, Demeter, and Guth [Bourgain et al. 2016]. The case of $\sigma = 2$ was proven by Bombieri and Iwaniec [1986] and by Bourgain [2017b] using a different argument. In [Demeter et al. 2020], they prove a slightly more general statement which implies Conjecture 1 in the range $0 \leq \sigma \leq \frac{3}{2}$. We prove the following general exponential sum estimate which implies Conjecture 1 for the full range of σ.

Theorem 2. *For each $N \geq 1$, $0 \leq \sigma \leq 2$, interval H of length $1/N^\sigma$, and $s \geq 1$,*

$$\int_{[0,1]^2\times H} \left|\sum_{k=1}^{N} a_k e(kx_1 + k^2x_2 + k^3x_3)\right|^{2s} dx \leq C_\varepsilon N^\varepsilon [N^{s-\sigma} + N^{2s-6}]$$

for any $a_k \in \mathbb{C}$ satisfying $|a_k| \lesssim 1$.

The terms in the upper bound come from two examples. The upper bound $N^{s-\sigma}$ follows from taking random $a_\xi \in \{\pm 1\}$, by Khintchine's inequality. The upper bound N^{2s-6} follows from the example $a_\xi = 1$ and noting that the integrand is $\gtrsim N^{2s}$ on roughly the box $[0, 1/N] \times [0, 1/N^2] \times [0, 1/N^3]$. Theorem 2 is an estimate for the moments of exponential sums over subsets smaller than the full domain of periodicity (i.e., N^3 in the x_3-variable). Bourgain [2017a; 2017b] investigated examples of this type of inequality.

MSC2020: primary 42B10; secondary 11L07, 53A04.
Keywords: exponential sum estimates, decoupling inequalities, moment curve.

Theorem 2 is a corollary of a small cap decoupling problem for $\mathcal{M}^3$, which we now describe. For $R \geq 1$, and small cap parameter $\beta \in [\frac{1}{3}, 1]$, consider the anisotropic small cap neighborhood

$$\mathcal{M}^3(R^\beta, R) = \{(\xi_1, \xi_2, \xi_3) : \xi_1 \in [0, 1],\ |\xi_2 - \xi_1^2| \leq R^{-2\beta},\ |\xi_3 - 3\xi_1\xi_2 + 2\xi_1^3| \leq R^{-1}\}.$$

This is the anisotropic neighborhood of $\mathcal{M}^3$ at scale R^β (for which canonical decoupling for the moment curve applies) plus a vertical interval of length R^{-1}. Next we define small caps γ, which form a partition of $\mathcal{M}^3(R^\beta, R)$ and are defined precisely in Section 2.3. Each γ has the form

$$\gamma = \{(\xi_1, \xi_2, \xi_3) : lR^{-\beta} \leq \xi_1 < (l+1)R^{-\beta},\ |\xi_2 - \xi_1^2| \leq R^{-2\beta},\ |\xi_3 - 3\xi_1\xi_2 + 2\xi_1^3| \leq R^{-1}\} \tag{1}$$

for some integer l, $0 \leq l < R^\beta$. If $\beta = \frac{1}{3}$, then γ coincides with canonical $R^{-1/3} \times R^{-2/3} \times R^{-1}$ moment curve blocks. In the range $\frac{1}{3} \leq \beta \leq \frac{1}{2}$, γ is essentially equivalent to the R^{-1}-neighborhood of a canonical $R^{-\beta} \times R^{-2\beta} \times R^{-3\beta}$ moment curve block. In the range $\frac{1}{2} \leq \beta \leq 1$, γ looks like a canonical $R^{-\beta} \times R^{-2\beta} \times R^{-3\beta}$ moment curve block plus a vertical R^{-1}-interval. In each case, γ has dimensions $R^{-\beta} \times R^{-2\beta} \times R^{-1}$. Our definition of small caps using the vertical R^{-1} neighborhood is motivated by Theorem 2, which we explain further in Section 1.1. See the paragraph following (2) for some remarks about the decoupling problem associated to small caps which are the (3-dimensional) R^{-1}-neighborhood of canonical $R^{-\beta} \times R^{-2\beta} \times R^{-3\beta}$ blocks.

The small cap decoupling theorem we obtain is:

Theorem 3. *Let $\frac{1}{3} \leq \beta \leq 1$ and $p \geq 2$. Then*

$$\|f\|^p_{L^p(\mathbb{R}^3)} \leq C_\varepsilon R^\varepsilon (R^{\beta(p/2-1)} + R^{\beta(p-4)-1}) \sum_\gamma \|f_\gamma\|^p_{L^p(\mathbb{R}^3)}$$

for any Schwartz function $f : \mathbb{R}^3 \to \mathbb{C}$ with Fourier transform supported in $\mathcal{M}^3(R^\beta, R)$.

The only other result of this form that we are aware of is in [Jung 2023], which essentially proves the $\beta = \frac{1}{2}$ case of Theorem 3. The proof of Theorem 3 uses the same framework as the high-low argument from [Guth et al. 2024]. We require a crucial new ingredient, which is small cap decoupling for the cone established in [Guth and Maldague 2022]. See Section 1.2 for some discussion of the role of small cap decoupling for the cone in the proof of Theorem 3. Modulo some minor adaptations, our high-low proof of Theorem 3 with $\beta = \frac{1}{3}$ actually yields the stronger (ℓ^2, L^p)-decoupling estimates from [Bourgain et al. 2016] rather than the (ℓ^p, L^p) inequalities stated in Theorem 3. See Section 1.3 for a sketch of the necessary adaptations. The powers of R in the upper bound of Theorem 3 come from considering two natural sharp examples for the ratio $\|f\|_p^p / \big(\sum_\gamma \|f_\gamma\|_p^q\big)^{p/q}$. The first is the square root cancellation example, where $|f_\gamma| \sim \chi_{B_{R^{\max(2\beta,1)}}}$ for all γ and $f = \sum_\gamma e_\gamma f_\gamma$, where e_γ are ± 1 signs chosen (using Khintchine's inequality) so that $\|f\|_p^p \sim R^{\beta p/2} R^{3\max(2\beta,1)}$ and

$$\frac{\|f\|_p^p}{\sum_\gamma \|f_\gamma\|_p^p} \gtrsim \frac{R^{\beta(p/2)} R^{3\max(2\beta,1)}}{R^\beta R^{3\max(2\beta,1)}} \sim R^{\beta(p/2-1)}.$$

The second example is the constructive interference example. Let $f_\gamma = R^{\beta+2\beta+1}\check{\eta}_\gamma$, where η_γ is a smooth bump function approximating χ_γ. Since $|f| = \big|\sum_\gamma f_\gamma\big|$ is approximately constant on unit balls and

$|f(0)| \sim R^\beta$, we have

$$\frac{\|f\|_p^p}{\sum_\gamma \|f_\gamma\|_p^p} \gtrsim \frac{R^{\beta p}}{R^\beta R^{\beta+2\beta+1}} \sim R^{\beta(p-4)-1}. \tag{2}$$

We remark that the arguments in this paper could also be used to analyze small cap decoupling problem with R^{-1} neighborhoods of canonical blocks. These are different from taking the vertical R^{-1} neighborhood in the range $\frac{1}{2} \le \beta \le 1$. For example, if we consider the constructive interference example for the R^{-1}-neighborhood small caps, then each f_γ is equal to ~ 1 on a dual set of size $R^\beta \times R \times R$, which leads to the lower bound

$$\frac{\|f\|_p^p}{\sum_\gamma \|f_\gamma\|_p^p} \gtrsim \frac{R^{\beta p}}{R^\beta R^{\beta+1+1}} \sim R^{\beta(p-2)-2}.$$

If $p \ge 6 + 2/\beta$ and $\frac{1}{2} \le \beta \le 1$, this is larger than the factors of R in the upper bound from Theorem 3, so optimal small cap estimates for the R^{-1} neighborhood would not have the same upper bound as in Theorem 3. In the R^{-1} set-up, there is also a third type of example which dominates for certain parameters β and p: the block example $f = \check{\eta}_\theta$, with θ a canonical $R^{-1/3} \times R^{-2/3} \times R^{-1}$ block. This leads to extra cases and a more complicated proof that we do not present here.

An immediate corollary of Theorem 3 is the following general exponential sum estimate.

Corollary 4. *For each* $\frac{1}{3} \le \beta \le 1$, $2 \le p \le 6 + 2/\beta$, *and* $r \ge R^{\max(2\beta, 1)}$,

$$|Q_r|^{-1} \int_{Q_r} \Big| \sum_{\xi \in \Xi} a_\xi e(x \cdot (\xi, \xi^2, \xi^3)) \Big|^p \, dx \lesssim_\varepsilon R^{\beta(p/2)+\varepsilon}$$

for any r*-cube* Q_r *and any collection* $\Xi \subset [0, 1]$ *with* $|\Xi| \sim R^\beta$ *consisting of* $\sim R^{-\beta}$*-separated points and* $a_\xi \in \mathbb{C}$ *with* $|a_\xi| \lesssim 1$.

Note that the corresponding corollary of canonical decoupling $\mathcal{M}^3$ only holds in the range $r \ge R^{3\beta}$.

For $a, b > 0$, the notation $a \lesssim b$ means that $a \le Cb$, where $C > 0$ is a universal constant whose definition varies from line to line, but which only depends on fixed parameters of the problem. Also, $a \sim b$ means $C^{-1} b \le a \le Cb$ for a universal constant C, and $a \lesssim_\varepsilon b$ means that the implicit constant depends on $\varepsilon > 0$.

The paper is organized as follows. We explain the implications of Theorem 3 in Section 1.1 and give some intuition for the proof of Theorem 3 in Section 1.2. Then in Section 2, we develop multiscale high-/low-frequency tools and lemmas. Some of these tools are very similar to those developed in [Guth et al. 2024], but the high-frequency analysis uses the geometry of the moment curve and relies on small cap decoupling estimates for the cone recently established in [Guth and Maldague 2022]. We use these tools in Section 3 to prove a weak (superlevel set) version of Theorem 3 for the critical exponent $p_c = 6 + 2/\beta$. Then in Section 3.2, we perform a sequence of pigeonholing steps analogous to those in Section 5 of [Guth et al. 2024] to show that Theorem 3 follows from the superlevel set version with the critical exponent.

1.1. *Implications of Theorem 3.*

Corollary 4 follows from Theorem 3. Let ϕ_{Q_r} be a nonnegative Schwartz function satisfying $\phi_{Q_r} \gtrsim 1$ on Q_r, $\operatorname{supp}\hat{\phi}_{Q_r} \subset B_{r^{-1}}$, and $\int |\phi_{Q_r}|^p \sim_p |Q_r|$. Then the function

$$f(x) = \sum_{\xi \in \Xi} a_\xi e(x \cdot (\xi, \xi^2, \xi^3))\phi_{Q_r}(x)$$

satisfies the hypotheses of Theorem 3. Using the triangle inequality, we may split the indexing set Ξ into $O(1)$ many subsets Ξ' so that each $\xi \in \Xi'$ is identified with a unique small cap γ which completely contains the r^{-1}-neighborhood of (ξ, ξ^2, ξ^3). This is possible because $r \geq R^{\max(2\beta,1)}$, so a ball of radius r^{-1} can be completely contained in an $R^{-\beta} \times R^{-2\beta} \times R^{-1}$ small cap γ, whose geometry is described in detail in Section 2.3. Applying Theorem 3 in the range $2 \leq p \leq 6 + 2/\beta$ gives

$$\int_{Q_r} |f|^p \lesssim_\varepsilon R^{\beta(p/2-1)+\varepsilon} \sum_{\xi \in \Xi} \|a_\xi e(\,\cdot\,(\xi, \xi^2, \xi^3))\phi_{Q_R}\|_p^p \sim R^{\beta(p/2)+\varepsilon}|Q_r|. \qquad \square$$

Theorem 2 follows from Theorem 3. Begin with the integral on the left-hand side of Theorem 2. Perform the change of variables $(x_1, x_2, x_3) = (y_1/N, y_2/N^2, y_3/N^3)$:

$$\int_{[0,1]^2 \times H} \left| \sum_{k=1}^{N} a_k e(x \cdot (k, k^2, k^3)) \right|^{2s} dx = N^{-6} \int_{[0,N]\times[0,N^2]\times N^3 H} \left| \sum_{k=1}^{N} a_k e\left(y \cdot \left(\frac{k}{N}, \frac{k^2}{N^2}, \frac{k^3}{N^3} \right) \right) \right|^{2s} dy.$$

Using the periodicity of the exponential sum in the first two variables,

$$\int_{[0,N]\times[0,N^2]\times N^3 H} \left| \sum_{k=1}^{N} a_k e\left(y \cdot \left(\frac{k}{N}, \frac{k^2}{N^2}, \frac{k^3}{N^3} \right) \right) \right|^{2s} dy$$
$$= N^{-3} \int_{[0,N^3]^2 \times N^3 H} \left| \sum_{k=1}^{N} a_k e\left(y \cdot \left(\frac{k}{N}, \frac{k^2}{N^2}, \frac{k^3}{N^3} \right) \right) \right|^{2s} dy.$$

Let ϕ_H be a bump function which satisfies $\phi_H \gtrsim 1$ on $[0, N^3]^2 \times N^3 H$, $\operatorname{supp} \hat{\phi}_H \subset [0, N^{-3}]^2 \times [0, N^{\sigma-3}]$, and $\int |\phi_H|^p \sim_p N^{9-\sigma}$. Then

$$\int_{[0,N^3]^2 \times N^3 H} \left| \sum_{k=1}^{N} a_k e\left(y \cdot \left(\frac{k}{N}, \frac{k^2}{N^2}, \frac{k^3}{N^3} \right) \right) \right|^{2s} dy \lesssim \int_{\mathbb{R}^3} \left| \sum_{k=1}^{N} a_k e\left(y \cdot \left(\frac{k}{N}, \frac{k^2}{N^2}, \frac{k^3}{N^3} \right) \right) \phi_H(y) \right|^{2s} dy.$$

Then apply Theorem 3 with $p = 2s$, $R = N^{3-\sigma}$, and β defined by $R^\beta = N$, which means that $\beta = 1/(3-\sigma) \in \left[\frac{1}{3}, 1\right]$ (since $\sigma \in [0, 2]$), giving

$$\int_{\mathbb{R}^3} \left| \sum_{k=1}^{N} a_k e\left(y \cdot \left(\frac{k}{N}, \frac{k^2}{N^2}, \frac{k^3}{N^3} \right) \right) \phi_H(y) \right|^{2s} dy \lesssim_\varepsilon R^\varepsilon [R^{\beta(s-1)} + R^{\beta(2s-4)-1}] \sum_{k=1}^{N} |a_k|^{2s} \|\phi_H\|_{2s}^{2s}.$$

Incorporate the extra factors from the substitution and the periodicity steps, and use the assumption $|a_k| \lesssim 1$ and the property $\|\phi_H\|_{2s}^{2s} \sim_s N^{9-\sigma}$ to get the bound

$$\int_{[0,1]^2 \times H} \left| \sum_{k=1}^{N} a_k e(x \cdot (k, k^2, k^3)) \right|^{2s} dx \lesssim_\varepsilon N^{-9} R^\varepsilon [R^{\beta(s-1)} + R^{\beta(2s-4)-1}] N N^{9-\sigma}.$$

Finally, using the relationship between R, N, β, and σ, the upper bound simplifies to

$$N^{\varepsilon}[N^{(s-1)}+N^{(2s-4)-(3-\sigma)}]N^{1-\sigma}=N^{\varepsilon}[N^{s-\sigma}+N^{2s-6}],$$

as desired. □

1.2. ***Some intuition behind the proof of Theorem 3.*** Here we describe one of the cases from the proof of Theorem 3 which illustrates the role of small cap decoupling for the cone. After a series of standard reductions which are also used in [Guth et al. 2024], to prove Theorem 3 it suffices to show that

$$\alpha^{6+2/\beta}|\{x\in B_{R^{\max(2\beta,1)}}:\alpha\le|f(x)|\}|\lesssim_{\varepsilon} R^{\varepsilon}R^{2\beta+1}\sum_{\gamma}\|f_\gamma\|_2^2, \tag{3}$$

where $\alpha>0$, $B_{R^{\max(2\beta,1)}}$ is a ball of radius $R^{\max(2\beta,1)}$, and we have the extra assumption that $\|f_\gamma\|_\infty\lesssim 1$ for all γ. The spatial localization to a ball of radius $R^{\max(2\beta,1)}$ is natural since this is the smallest size of ball that contains an $R^\beta\times R^{2\beta}\times R$ wave packet dual to each γ^*. Consider the special case of maximal α, so $\alpha\sim\#\gamma\sim R^\beta$, and call $\{x\in B_{R^{\max(2\beta,1)}}:R^\beta\sim|f(x)|\}$ the high set H. Using a local trilinear restriction estimate for the moment curve, recorded below in Proposition 27, we show roughly that

$$(R^\beta)^6|H|\lesssim\int_{\mathcal{N}_{R^\beta}(H)}\Big|\sum_\gamma|f_\gamma|^2(x)\Big|^3\,dx.$$

Suppose that on most of $\mathcal{N}_{R^\beta}(H)$, we have $\sum_\gamma|f_\gamma|^2(x)\lesssim\big|\sum_\gamma|f_\gamma|^2*\check{\eta}_{>R^{-\beta}/2}(x)\big|$, where $\eta_{>R^{-\beta}/2}$ is a smooth approximation of the characteristic function of the set $\frac12R^{-\beta}\le|\xi|\le 2R^{-\beta}$. Each $\widehat{|f_\gamma|^2}$ is supported in $\gamma-\gamma$. Writing $\mathfrak{m}(t)=(t,t^2,t^3)$ and using the definition (1), the support of each $\widehat{|f_\gamma|^2}\eta_{>R^{-\beta}/2}$ is approximately contained in

$$\big\{A\mathfrak{m}'(lR^{-\beta})+B\mathfrak{m}''(lR^{-\beta})+C\mathfrak{m}'''(lR^{-\beta}):\tfrac12R^{-\beta}\le A\le R^{-\beta},\ |B|\le R^{-2\beta},\ |C|\le R^{-1}\big\}.$$

In Section 2.3, we show that these sets are disjoint for distinct $l\in\{1,\dots,R^\beta\}$, and each of the above sets is contained in the $R^{-\beta}$-dilation of a conical small cap. Note that this is not exactly true when $\beta=1$, which is why we use cylinders instead of balls to cut out the low set in the actual argument. Ignoring this technicality, this means that we may apply a small cap decoupling theorem for the cone to bound the integral

$$\int_{\mathcal{N}_{R^\beta}(H)}\Big|\sum_\gamma|f_\gamma|^2*\check{\eta}_{>R^{-\beta}/2}\Big|^3.$$

Finally, the functions $\sum_\gamma|f_\gamma|^2$ and $\big|\sum_\gamma|f_\gamma|^2*\check{\eta}_{>R^{-\beta}/2}\big|$ are roughly constant on R^β balls, which implies that for any $p\ge0$, we have

$$(R^\beta)^6|H|\lesssim\frac{1}{R^{\beta p}}\int_{\mathcal{N}_{R^\beta}(H)}\Big|\sum_\gamma|f_\gamma|^2*\check{\eta}_{>R^{-\beta}/2}(x)\Big|^{3+p}\,dx.$$

This is an important observation since we have more factors of R^β in the denominator on the right-hand side and we may choose p so that $3+p$ is the critical exponent for the scale of conical small caps that we have, thus using the full strength of the small cap decoupling theorem for the cone. Our argument shows that each of these steps can be sharp, which leads to the upper bound (3).

1.3. ***Canonical*** (ℓ^2, L^p) ***decoupling in the case*** $\beta = \frac{1}{3}$**.** In this section, we sketch a small variation of our argument which recovers the sharp (ℓ^2, L^p)-decoupling estimates for $\mathcal{M}^3$ of [Bourgain et al. 2016].

As in [Guth et al. 2024] for the parabola, pigeonholing arguments combined with a version of interpolation with L^2 and L^∞ estimates may be used to show that proving the critical (ℓ^2, L^{12})-decoupling inequality implies sharp (ℓ^2, L^p)-decoupling inequalities for all other $p \geq 2$. It further suffices to prove the following level-set version of the inequality:

$$\alpha^{12}|U_\alpha| \lesssim_\varepsilon R^\varepsilon \Bigl(\sum_\gamma \|f_\gamma\|^2_{L^{12}(w_{B_R})}\Bigr)^6, \tag{4}$$

where $B_R \subset \mathbb{R}^3$ is any ball of radius R, $U_\alpha := \{x \in B_R : |f(x)| \sim \alpha\}$, and w_{B_R} is a weight function adapted to B_R. Via pigeonholing steps similar to those in [Guth et al. 2024], we may assume that each f_γ is either identically equal to 1 or has a wave packet decomposition $f_\gamma = \sum_{T_\gamma} \psi_{T_\gamma}$, where the T_γ are a subset of a tiling of $\mathbb{R}^3$ by disjoint translates of the dual set γ^* (which has dimensions $R^{1/3} \times R^{2/3} \times R$), each function ψ_{T_γ} approximately satisfies $|\psi_{T_\gamma}| \sim \chi_{T_\gamma}$ (χ_{T_γ} the characteristic function of T_γ), each T_γ intersects B_R, and the number of T_γ which appear in the sum $f_\gamma = \sum_{T_\gamma} \psi_{T_\gamma}$ is $\sim A$ for some constant A that is independent of γ. Notice then that each L^{12} norm appearing on the right-hand side of (4) is essentially

$$\|f_\gamma\|^{12}_{L^{12}(w_{B_R})} \sim A \cdot R^{1/3+2/3+1},$$

and so is uniform in the nonzero f_γ. We also have $\|f_\gamma\|^p_{L^p(w_{B_R})} \sim_p A \cdot R^2$ for any $2 \leq p < \infty$. Since each f_γ is made up of wave packets which all have height 1, $\|f_\gamma\|_\infty \lesssim 1$.

In the proof of Theorem 3, we bound $|U_\alpha|$ by dividing U_α into $O(\varepsilon^{-1})$ many subsets and bounding each subset separately. Those subsets are H, Ω_k, and L and are defined in Definition 13 below (there are no Λ_k when $\beta = \frac{1}{3}$). We replace the R^β factor which appears in each set by $\#\{\gamma : f_\gamma \not\equiv 0\} =: \#\gamma$. The only further modification needed is to replace R^β in the pruning process by $\#\gamma$. Then each $F^k_{\tau_k}$ satisfies $\|F^k_{\tau_k}\|_\infty \lesssim_\varepsilon R^\varepsilon(\#\gamma/\alpha)$. Considering the bound for $|\Lambda_k|$, for example, the argument then yields

$$\alpha^6|\Lambda_k| \lesssim_\varepsilon R^\varepsilon \frac{(\#\gamma)^3}{\alpha^6}\Bigl(\sum_\gamma \|f_\gamma\|^{1/3}_{L^6(B_R)}\Bigr)^3.$$

The right-hand side (without the $C_\varepsilon R^\varepsilon$ factor) is essentially

$$\alpha^{-6}\#\gamma^6 A R^2 \sim \alpha^{-6}\Bigl(\sum_\gamma \|f_\gamma\|^2_{L^{12}(w_{B_R})}\Bigr)^6,$$

so we have the desired L^{12} estimate.

2. Tools for the high/low approach to $\mathcal{M}^3$

We perform a high/low frequency analysis of square functions at various scales, incorporating the pruning process for wave packets analogous to [Guth et al. 2024]. We develop language to discuss canonical caps and small caps of various scales, associated wave packets, and averaged versions of functions which satisfy useful locally constant properties. Then we write a series of key lemmas to analyze the high/low frequency portions of averaged, pruned square functions at various scales.

Begin by fixing some notation. Fix a ball $B_{R^{\max(2\beta,1)}}$ of radius $R^{\max(2\beta,1)}$. The parameter $\alpha > 0$ describes the superlevel set

$$U_\alpha = \{x \in B_{R^{\max(2\beta,1)}} : |f(x)| \geq \alpha\}.$$

Fix $\beta \in \left[\frac{1}{3}, 1\right]$ and $R \geq 2$. Let $\varepsilon > 0$ be given and consider scales $R_k \in 8^{\mathbb{N}}$ closest to $R^{k\varepsilon}$ for $R^{-1/3} \leq R_k^{-1/3} \leq 1$, and scales $r_k \in 2^{\mathbb{N}}$ closest to $R^{1/3+k\varepsilon}$ for $R^{-\beta} \leq r_k^{-1} \leq R^{-1/3}$. Let N distinguish the index so that R_N is closest to R. Since R and R_N differ at most by a factor of R^ε, we will ignore the distinction between R_N and R in the rest of the argument. Similarly, assume that $r_M = R^\beta$ for some index $M \in \mathbb{N}$. The relationship between the parameters is

$$1 = R_0 \leq R_k^{1/3} \leq R_{k+1}^{1/3} \leq R_N^{1/3} = r_0 \leq r_m \leq r_{m+1} \leq r_M = R^\beta.$$

Next we fix notation for moment curve blocks and small caps of various sizes. For the explicit definitions, see Section 2.3 below.

(1) $\{\gamma\}$ are small caps associated to R^β and R, meaning $\sim R^{-\beta} \times R^{-2\beta} \times R^{-3\beta}$ moment curve blocks plus the set $\{(0, 0, z) : |z| \leq R^{-1}\}$.

(2) $\{\gamma_k\}$ are small caps associated to r_k and R, meaning $\sim r_k^{-1} \times r_k^{-2} \times r_k^{-3}$ moment curve blocks plus the set $\{(0, 0, z) : |z| \leq R^{-1}\}$.

(3) $\{\theta\}$ are canonical $\sim R^{-1/3} \times R^{-2/3} \times R^{-1}$ moment curve blocks.

(4) $\{\tau_k\}$ are canonical $R_k^{-1/3} \times R_k^{-2/3} \times R_k^{-1}$ moment curve blocks.

The specific definitions of $\gamma, \gamma_k, \theta, \tau_k$ in Section 2.3 provide the additional property that if $\gamma_k \cap \gamma_{k+m} \neq \varnothing$, then $\gamma_{k+m} \subset \gamma_k$ (and similarly for the τ_k).

We assume throughout this section (actually until Section 3.2) that the f_γ satisfy the extra condition that

$$\tfrac{1}{2} \leq \|f_\gamma\|_{L^\infty(\mathbb{R}^3)} \leq 2 \quad \text{or} \quad \|f_\gamma\|_{L^\infty(\mathbb{R}^3)} = 0. \tag{5}$$

2.1. ***A pruning step.*** Here we define wave packets for blocks γ_k, τ_k, and prune the wave packets associated to f_{γ_k}, f_{τ_k} according to their amplitudes.

For each γ_k, fix a dual block γ_k^* with dimensions $r_k^{-1} \times r_k^{-2} \times R$ which is comparable to the convex set

$$\{x \in \mathbb{R}^3 : |x \cdot \xi| \leq 1 \text{ for all } \xi \in \gamma_k - \gamma_k\}.$$

For each τ_k, fix a dual block τ_k^* of dimensions $R_k^{1/3} \times R_k^{2/3} \times R_k$ which is comparable to the convex set

$$\{x \in \mathbb{R}^3 : |x \cdot \xi| \leq 1 \text{ for all } \xi \in \tau_k - \tau_k\}.$$

The main difference between dual small caps γ_k^* and dual canonical caps τ_k^* is that for each k we have $\gamma_k^* = \tilde{\gamma}_k^*$ if $\gamma_k, \tilde{\gamma}_k \subset \theta$, whereas the τ_k^* are all distinct.

We will describe wave packet decompositions for small caps $\{\gamma_k\}$ and for canonical caps $\{\tau_k\}$ in parallel. Let $\mathbb{T}_{\gamma_k}, \mathbb{T}_{\tau_k}$ be the collection of tubes T_{γ_k}, T_{τ_k} which are dual to γ_k, τ_k, contain γ_k^*, τ_k^*, and which tile $\mathbb{R}^3$, respectively. Next, define associated partitions of unity $\psi_{T_{\gamma_k}}, \psi_{T_{\tau_k}}$. Let $\varphi(\xi)$ be a bump function supported in $\left[-\frac{1}{4}, \frac{1}{4}\right]^3$. For each $m \in \mathbb{Z}^3$, let

$$\psi_m(x) = c \int_{[-1/2,1/2]^3} |\check{\varphi}|^2(x-y-m)\, dy,$$

where c is chosen so that $\sum_{m\in\mathbb{Z}^3} \psi_m(x) = c\int_{\mathbb{R}^3} |\check{\varphi}|^2 = 1$. Since $|\check{\varphi}|$ is a rapidly decaying function, for any $n \in \mathbb{N}$, there exists $C_n > 0$ such that

$$\psi_m(x) \le c \int_{[-1/2,1/2]^3} \frac{C_n}{(1+|x-y-m|^2)^n}\, dy \le \frac{\tilde{C}_n}{(1+|x-m|^2)^n}.$$

Define the partitions of unity $\psi_{T_{\gamma_k}}, \psi_{T_{\tau_k}}$ associated to γ_k, τ_k to be $\psi_{T_{\gamma_k}} = \psi_m \circ A_{\gamma_k}$ $\psi_{T_{\tau_k}}(x) = \psi_m \circ A_{\tau_k}$, where A_{γ_k}, A_{τ_k} are linear transformations taking γ_k^*, τ_k^* to $\left[-\frac{1}{2}, \frac{1}{2}\right]^3$ and $A_{\gamma_k}(T_{\gamma_k}) = m + \left[-\frac{1}{2}, \frac{1}{2}\right]^3$, $A_{\tau_k}(T_{\tau_k}) = m + \left[-\frac{1}{2}, \frac{1}{2}\right]^3$. The important properties of $\psi_{T_{\gamma_k}}, \psi_{T_{\tau_k}}$ are rapid decay off of T_{γ_k}, T_{τ_k} and Fourier support contained in γ_k, τ_k translated to the origin.

To prove upper bounds for the size of U_α, we will actually bound the sizes of $\sim \varepsilon^{-1}$ many subsets which will be denoted by $U_\alpha \cap H$, $U_\alpha \cap \Lambda_k$, $U_\alpha \cap \Omega_k$, and $U_\alpha \cap L$. The pruning process sorts between important and unimportant wave packets on each of these subsets, as described in Lemma 16 below.

In the following definition, $A_\varepsilon \gg 1$ is a large enough (determined by Lemma 16) constant depending on ε which also satisfies $A_\varepsilon \ge D_\varepsilon$, where D_ε is given by Lemma 14. We partition the wave packets $\mathbb{T}_{\gamma_k} = \mathbb{T}^g_{\gamma_k} \sqcup \mathbb{T}^b_{\gamma_k}$ and $\mathbb{T}_{\tau_k} = \mathbb{T}^g_{\tau_k} \sqcup \mathbb{T}^b_{\tau_k}$ into "good" and "bad" sets, and define corresponding versions of f, as follows.

Remark. In the following definitions, let $K \ge 1$ be a large parameter which will be used to define the broad set in Proposition 28.

Definition 5 (pruning with respect to γ_k). Let $f^M_\gamma = f_\gamma$ and $f^M_{\gamma_{M-1}} = f_{\gamma_{M-1}}$. For each $1 \le k < M$, let

$$\mathbb{T}^g_{\gamma_k} = \left\{ T_{\gamma_k} \in \mathbb{T}_{\gamma_k} : \|\psi_{T_{\gamma_k}} f^{k+1}_{\gamma_k}\|_{L^\infty(\mathbb{R}^3)} \le K^3 A_\varepsilon^{M-k+1} \frac{R^\beta}{\alpha} \right\},$$

$$f^k_{\gamma_k} = \sum_{T_{\gamma_k} \in \mathbb{T}^g_{\gamma_k}} \psi_{T_{\gamma_k}} f^{k+1}_{\gamma_k} \quad \text{and} \quad f^k_{\gamma_{k-1}} = \sum_{\gamma_k \subset \gamma_{k-1}} f^k_{\gamma_k}.$$

Recall that $\gamma_0 = \theta = \tau_N$. Once the wave packets corresponding to all of the small caps have been pruned, we have $f^1 = \sum_{\gamma_1} f^1_{\gamma_1}$.

Definition 6 (pruning with respect to τ_k). Let $F^{N+1} = f^1$, $F^{N+1}_{\tau_N} = f^1_\theta$. For each $1 \le k \le N$, let

$$\mathbb{T}^g_{\tau_k} = \left\{ T_{\tau_k} \in \mathbb{T}_{\tau_k} : \|\psi_{T_{\tau_k}} F^{k+1}_{\tau_k}\|_{L^\infty(R^3)} \le K^3 A_\varepsilon^{M+N-k+1} \frac{R^\beta}{\alpha} \right\},$$

$$F^k_{\tau_k} = \sum_{T_{\tau_k} \in \mathbb{T}^g_{\tau_k}} \psi_{T_{\tau_k}} F^{k+1}_{\tau_k} \quad \text{and} \quad F^k_{\tau_{k-1}} = \sum_{\tau_k \subset \tau_{k-1}} F^k_{\tau_k}.$$

For each k, define the k-th versions of f, F to be $f^k = \sum_{\gamma_k} f^k_{\gamma_k}$ and $F^k = \sum_{\tau_k} F^k_{\tau_k}$.

Lemma 7 (properties of f^k and F^k). (1) $|f^k_{\gamma_k}(x)| \le |f^{k+1}_{\gamma_k}(x)| \lesssim \#\gamma \subset \gamma_k$ *and* $|F^k_{\tau_k}(x)| \le |F^{k+1}_{\tau_k}(x)| \lesssim \#\gamma \subset \tau_k$.

(2) $\|f^k_{\gamma_k}\|_{L^\infty(\mathbb{R}^3)} \le K^3 A_\varepsilon^{M-k+1} R^{3\varepsilon} R^\beta/\alpha$ *and* $\|F^k_{\tau_k}\|_{L^\infty(\mathbb{R}^3)} \le K^3 A_\varepsilon^{M+N-k+1} R^{3\varepsilon} R^\beta/\alpha$.

(3) *There is some constant* $\underline{C}_\varepsilon \lesssim \varepsilon^{-2}$ *so that* $\operatorname{supp} \widehat{f^{k+1}_{\gamma_k}} \subset \operatorname{supp} \widehat{f^k_{\gamma_k}} \subset \underline{C}_\varepsilon \gamma_k$ *and* $\operatorname{supp} \widehat{F^{k+1}_{\tau_k}} \subset \operatorname{supp} \widehat{f^k_{\tau_k}} \subset \underline{C}_\varepsilon \tau_k$.

Proof. For the first property, recall that $\sum_{T_{\gamma_k}\in\mathbb{T}_{\gamma_k}} \psi_{T_{\gamma_k}}$, $\sum_{T_{\tau_k}\in\mathbb{T}_{\tau_k}} \psi_{T_{\tau_k}}$ are partitions of unity so we may iterate the inequalities

$$|F^k_{\tau_k}| \le |F^{k+1}_{\tau_k}| \le \sum_{\tau_{k+1}\subset\tau_k} |F^{k+1}_{\tau_{k+1}}| \le \cdots \le \sum_{\tau_N\subset\tau_k} |F^N_{\tau_N}| \le \sum_{\gamma_1\subset\tau_k} |f^1_{\gamma_1}|,$$

$$|f^1_{\gamma_1}| \le |f^2_{\gamma_1}| \le \sum_{\gamma_2\subset\gamma_1} |f^2_{\gamma_2}| \le \cdots \le \sum_{\gamma_N\subset\gamma_1} |f^N_{\gamma_N}| \le \sum_{\gamma\subset\gamma_1} \|f_\gamma\|_{L^\infty(\mathbb{R}^3)}.$$

Then use the assumption that each $\|f_\gamma\|_{L^\infty(\mathbb{R}^3)} \lesssim 1$. Now consider the L^∞ bound in the second property. We write

$$f^k_{\gamma_k}(x) = \sum_{\substack{T_{\gamma_k}\in\mathbb{T}^g_{\gamma_k}\\ x\in R^\varepsilon T_{\gamma_k}}} \psi_{T_{\gamma_k}} f^{k+1}_{\gamma_k} + \sum_{\substack{T_{\gamma_k}\in\mathbb{T}^g_{\gamma_k}\\ x\notin R^\varepsilon T_{\gamma_k}}} \psi_{T_{\gamma_k}} f^{k+1}_{\gamma_k}.$$

The first sum has at most $R^{3\varepsilon}$ terms, and each term has norm bounded by $K^3 A_\varepsilon^{N-k} R^\beta/\alpha$, by the definition of $\mathbb{T}^g_{\gamma_k}$. By the first property, we may trivially bound $f^{k+1}_{\tau_k}$ by $\#\gamma \subset \tau_k \max_\gamma \|f_\gamma\|_\infty \lesssim R$. But if $x \notin R^\varepsilon T_{\gamma_k}$, then $\psi_{T_{\gamma_k}}(x) \le R^{-1000}$. Thus

$$\Bigg|\sum_{\substack{T_{\gamma_k}\in\mathbb{T}^h_{\gamma_k}\\ x\notin R^\varepsilon T_{\gamma_k}}} \psi_{T_{\gamma_k}} f^{k+1}_{\gamma_k}\Bigg| \le \sum_{\substack{T_{\gamma_k}\in\mathbb{T}^h_{\gamma_k}\\ x\notin R^\varepsilon T_{\gamma_k}}} R^{-500} \psi^{1/2}_{T_{\gamma_k}}(x) \|f^{k+1}_{\gamma_k}\|_\infty \le R^{-250} \max_\gamma \|f_\gamma\|_\infty.$$

Since $\alpha \lesssim |f(x)| \lesssim \sum_\gamma \|f_\gamma\|_\infty \lesssim R^\beta$, we certainly have $R^{-250} \le R^\beta/\alpha$. The argument for $\|F^k_{\tau_k}\|_{L^\infty(\mathbb{R}^3)}$ is analogous.

The third property depends on the Fourier supports of $\psi_{T_{\gamma_k}}$, $\psi_{T_{\tau_k}}$, which are contained in γ_k, τ_k shifted to the origin. If each $f^{k+1}_{\gamma_k}$ has Fourier support in $C\gamma_k$ (that is, a dilated copy of γ_k by a factor of C, taken with respect to its centroid), then $\operatorname{supp} \widehat{f^k_{\gamma_k}}$ is contained in $(1+C)\gamma_k$. The same type of argument is true for the claims about $F^k_{\tau_k}$ and $F^{k+1}_{\tau_k}$. ◻

Definition 8. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth function supported in $\left[-\frac{1}{4}, \frac{1}{4}\right]^3$. Define

$$w_0(t) = \sum_{k\in\mathbb{Z}} \frac{1}{(1+k^2)^{100}} (|\check{\phi}|^2(t-k)).$$

Let $w(t_1, t_2, t_3) = w_0(t_1)w_0(t_2)w_0(t_3)$ and let $Q = \left[-\frac{1}{2}, \frac{1}{2}\right]^3$ denote the unit cube centered at the origin. For any set $U = T(B)$, where T is an affine transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, define

$$w_U(x) = |U|^{-1} w(T^{-1}(x)).$$

For γ_k, τ_k, let A_{γ_k}, A_{τ_k} be affine transformations taking γ_k^*, τ_k^* to $\left[-\frac{1}{2},\frac{1}{2}\right]^3$ and define ω_{γ_k}, ω_{τ_k} by

$$\omega_{\gamma_k}(x)=|\gamma_k^*|^{-1}w(A_{\gamma_k}(x)) \quad \text{and} \quad \omega_{\tau_k}(x)=|\tau_k^*|^{-1}w(A_{\tau_k}(x)).$$

Let the capital-W version of weight functions denote the L^∞-normalized (as opposed to L^1-normalized) versions, so for example, for any cube Q_s of sidelength s, $W_{Q_s}(x)=|Q_s|w_{Q_s}(x)$. If a weight function has subscript which is only a scale, say s, then the functions w_s, W_s are weight function localized to the s-cube centered at the origin. We will ignore the distinction between an s-ball and an s-cube.

Remark. Note the additional property that $\hat{w}(\xi_1,\xi_2,\xi_3)$ is supported in $\left[-\frac{1}{2},\frac{1}{2}\right]^3$, so w_s is Fourier supported in an s^{-1}-cube at the origin. Similarly, ω_{γ_k} and ω_{τ_k} are Fourier supported in γ_k and τ_k translated to the origin, respectively. The same is true for the W_{B_s}, $W_{\gamma_k^*}$, $W_{\tau_k^*}$ weight functions. Finally, note that if $S_1=T_1(Q)$ and $S_2=T_2(Q)$, where T_i are anisotropic dilations with respect to the standard basis and $S_1\subset S_2$, then $w_{S_1}*w_{S_2}\lesssim w_{S_2}$.

The weights ω_{τ_k}, $\omega_\theta=\omega_{\tau_N}$, and w_s are useful when we invoke the locally constant property. By locally constant property, we mean generally that if a function f has Fourier transform supported in a convex set A, then, for a bump function $\varphi_A\equiv 1$ on A, $f=f*\check{\varphi}_A$. Since $|\check{\varphi}_A|$ is an L^1-normalized function which is positive on a set dual to A, $|f|*|\check{\varphi}_A|$ is an averaged version of $|f|$ over a dual set A^*. We record some of the specific locally constant properties we need in the following lemma.

Lemma 9 (locally constant property). *For each γ_k, τ_k and $T_{\gamma_k}\in\mathbb{T}_{\gamma_k}$, $T_{\tau_k}\in\mathbb{T}_{\tau_k}$ respectively,*

$$\|f_{\gamma_k}\|^2_{L^\infty(T_{\gamma_k})}\lesssim|f_{\gamma_k}|^2*\omega_{\gamma_k}(x) \quad \textit{for any } x\in T_{\gamma_k},$$
$$\|f_{\tau_k}\|^2_{L^\infty(T_{\tau_k})}\lesssim|f_{\tau_k}|^2*\omega_{\tau_k}(x) \quad \textit{for any } x\in T_{\tau_k}.$$

Also, for any r_k-ball B_{r_k} or $R_k^{1/3}$-ball $B_{R_k^{1/3}}$,

$$\Big\|\sum_{\gamma_k}|f_{\gamma_k}|^2\Big\|_{L^\infty(B_{r_k})}\lesssim\sum_{\gamma_k}|f_{\gamma_k}|^2*w_{B_{r_k}}(x) \quad \textit{for any } x\in B_{r_k},$$
$$\Big\|\sum_{\tau_k}|f_{\tau_k}|^2\Big\|_{L^\infty(B_{R_k^{1/3}})}\lesssim|f_{\tau_k}|^2*w_{B_{R_k^{1/3}}}(x) \qquad \textit{for any } x\in B_{R_k^{1/3}}.$$

Because the pruned versions of f, f_{γ_k}, and f_{τ_k} have Fourier supports similar to those of the unpruned versions (see Lemma 7), the locally constant lemma applies to the pruned versions as well.

Proof of Lemma 9. For the first claim, we write the argument for f_{τ_k} in detail (the argument for the f_{γ_k} is analogous). Let ρ_{τ_k} be a bump function equal to 1 on τ_k and supported in $2\tau_k$. Then using Fourier inversion and Hölder's inequality,

$$|f_{\tau_k}(y)|^2=|f_{\tau_k}*\check{\rho}_{\tau_k}(y)|^2\le\|\check{\rho}_{\tau_k}\|_1|f_{\tau_k}|^2*|\check{\rho}_{\tau_k}|(y).$$

Since ρ_{τ_k} may be taken to be an affine transformation of a standard bump function adapted to the unit ball, $\|\check{\rho}_{\tau_k}\|_1$ is a constant. The function $\check{\rho}_{\tau_k}$ decays rapidly off of τ_k^*, so $|\check{\rho}_{\tau_k}|\lesssim w_{\tau_k}$. Since for any $T_{\tau_k}\in\mathbb{T}_{\tau_k}$,

$\omega_{\tau_k}(y)$ is comparable for all $y \in T_{\tau_k}$, we have

$$\sup_{x\in T_{\tau_k}} |f_{\tau_k}|^2 * \omega_{\tau_k}(x) \le \int |f_{\tau_k}|^2(y) \sup_{x\in T_{\tau_k}} \omega_{\tau_k}(x-y)\,dy \sim \int |f_{\tau_k}|^2(y)\omega_{\tau_k}(x-y)\,dy \quad \text{for all } x \in T_{\tau_k}.$$

For the second part of the lemma, repeat analogous steps as above, except begin with ρ_{r_k}, which is identically 1 on a ball of radius $2r_k^{-1}$ containing $\gamma_k - \gamma_k$ (which is the Fourier support of $|f_{\gamma_k}|^2$). Then

$$\sum_{\gamma_k} |f_{\gamma_k}(y)|^2 = \left|\sum_{\gamma_k} |f_{\gamma_k}|^2 * \check{\rho}_{r_k}(y)\right| \lesssim \sum_{\gamma_k} |f_{\gamma_k}|^2 * |\check{\rho}_{r_k}|(y).$$

The rest of the argument is analogous to the first part. The argument for $\sum_{\tau_k} |f_{\tau_k}|^2$ is the same. □

For ease of future reference, we record the following standard local and global L^2-orthogonality lemma. For $U \subset \mathbb{R}^3$, let $U^* = \{\xi \in \mathbb{R}^3 : |\xi \cdot x| \le 1 \text{ for all } x \in U - U\}$.

Lemma 10 (local and global L^2 orthogonality). *Let $U = T(Q)$, where Q is the unit ball centered at the origin and $T : \mathbb{R}^3 \to \mathbb{R}^3$ is an affine transformation. Let $h : \mathbb{R}^3 \to \mathbb{C}$ be a Schwartz function with Fourier transform supported in a disjoint union $X = \bigsqcup_k X_k$, where $X_k \subset B$ are Lebesgue measurable. If the maximum overlap of the sets $U^* + X_k$ is L, then*

$$\int |h_X|^2 w_U \lesssim L \sum_{X_k} \int |h_{X_k}|^2 w_U,$$

where $h_{X_k} = \int_{X_k} \widehat{h}(\xi) e^{2\pi i x\cdot\xi}\,d\xi$. The corresponding global statement is

$$\int |h_X|^2 = \sum_{X_k} \int |h_{X_k}|^2.$$

Proof. The global statement is just Plancherel's theorem. For the local statement, we have

$$\int |h_X|^2 w_U = \int h_X \overline{h_X w_U} = \int \hat{h}_X \overline{\hat{h}_X * \hat{w}_U}$$

by Plancherel's theorem again. Next we used the definition of $\hat{h}_X$ and $\hat{h}_{X_k}$ to write

$$\int \hat{h}_X \overline{\hat{h}_X * \hat{w}_U} = \sum_{X_k} \sum_{X_k'} \int \hat{h}_{X_k} \overline{\hat{h}_{X_k'} * \hat{w}_U}.$$

The function $\hat{h}_{X_k}$ is supported in X_k and the function $\hat{h}_{X_k'} * \hat{w}_U$ is supported in $X_k' + U^*$. Write $X_k' \sim X_k$ to denote the property that $(X_k + U^*) \cap (X_k' + U^*) \ne \varnothing$. By hypothesis, for each X_k, there are at most L many X_k' such that $X_k' \sim X_k$. Since $X_k \cap (X_k' + U^*) \subset (X_k + U^*) \cap (X_k' + U^*)$, this leads to the bound

$$\sum_{X_k} \sum_{X_k'} \int \hat{h}_{X_k} \overline{\hat{h}_{X_k'} * \hat{w}_U} = \sum_{X_k} \sum_{X_k' \sim X_k} \int h_{X_k} \overline{h_{X_k'}} w_U \le \sum_{X_k} \sum_{X_k' \sim X_k} \int (|h_{X_k}|^2 + |h_{X_k'}|^2) w_U$$
$$\le \sum_{X_k} \sum_{X_k' \sim X_k} \int (|h_{X_k}|^2 + |h_{X_k'}|^2) w_U \le 2L \sum_{X_k} \int |h_{X_k}|^2 w_U. \qquad \square$$

Definition 11 (auxiliary functions). For $i = 1, 2$, let $\varphi_i : \mathbb{R}^i \to [0, \infty)$ be a radial, smooth bump function satisfying $\varphi_i(x) = 1$ on the unit ball in $\mathbb{R}^i$ and supported in the ball of radius 2. Then for each $s > 0$, let $\rho_{\le s^{-1}} : \mathbb{R}^3 \to [0, \infty)$ be defined by

$$\rho_{\le s^{-1}}(\xi_1, \xi_2, \xi_3) = \varphi_2(s\xi_1, s\xi_2)\varphi_1(\xi_3).$$

Write $\mathcal{C}_{s^{-1}}$ for the set where $\rho_{\le s^{-1}} = 1$.

We will sometimes abuse the notation from the previous definition by writing $h * \check{\rho}_{>s^{-1}} = h - h * \check{\rho}_{\le s^{-1}}$.

Definition 12. Let $g_M(x) = \sum_\gamma |f_\gamma|^2 * \omega_\gamma(x)$. For $1 \le k \le M - 1$, let

$$g_k(x) = \sum_{\gamma_k} |f_{\gamma_k}^{k+1}|^2 * \omega_{\gamma_k}, \quad g_k^\ell(x) = g_k * \check{\rho}_{\le r_{k+1}^{-1}}, \quad \text{and} \quad g_k^h = g_k - g_k^\ell.$$

For $1 \le k \le N$, let

$$G_k(x) = \sum_{\tau_k} |F_{\tau_k}^{k+1}|^2 * \omega_{\tau_k}, \quad G_k^\ell(x) = G_k * \check{\rho}_{\le R_{k+1}^{-1/3}}, \quad \text{and} \quad G_k^h(x) = G_k - G_k^\ell.$$

In the following definition, $A_\varepsilon \gg 1$ is the same ε-dependent constant from the pruning definition of f^k and F^k.

Definition 13. Define the high set by

$$H = \{x \in B_{R^{\max(2\beta,1)}} : A_\varepsilon R^\beta \le g_{M-1}(x)\}.$$

For each $k = 1, \dots, M - 2$, let $H = \Lambda_{M-1}$ and let

$$\Lambda_k = \left\{ x \in B_{R^{\max(2\beta,1)}} \setminus \bigcup_{l=k+1}^{M-1} \Lambda_l : (A_\varepsilon)^{(M-k)} R^\beta \le g_k(x) \right\}.$$

For each $k = 1, \dots, N$, let $\Omega_{N+1} = \bigcup_{l=1}^{M-1} \Lambda_l$ and let

$$\Omega_k = \left\{ x \in B_{R^{\max(2\beta,1)}} \setminus \bigcup_{l=k+1}^{N+1} \Omega_l : (A_\varepsilon)^{(M+N-k)} R^\beta \le G_k(x) \right\}.$$

Define the low set to be

$$L = B_{R^{\max(2\beta,1)}} \setminus \left[\left(\bigcup_{l=1}^{N+1} \Omega_N \right) \cup \left(\bigcup_{k=1}^{M-1} \Lambda_k \right) \right].$$

2.2. *Lemmas related to the pruning process for wave packets.*

Lemma 14 (low lemma). *There is a constant $D = D_\varepsilon > 0$ depending on ε so that, for each x, we have $|g_k^\ell(x)| \le D_\varepsilon g_{k+1}(x)$ and $|G_k^\ell(x)| \le D_\varepsilon G_{k+1}(x)$.*

Proof. Prove the claim in detail for g_k^ℓ since the argument for G_k^ℓ is analogous. We perform a pointwise version of the argument in the proof of local/global L^2-orthogonality (Lemma 10). For each γ_k^{k+1}, using

Plancherel's theorem,

$$|f_{\gamma_k}^{k+1}|^2 * \check{\rho}_{\le r_{k+1}^{-1}}(x) = \int_{\mathbb{R}^3} |f_{\gamma_k}^{k+1}|^2(x-y)\check{\rho}_{\le r_{k+1}^{-1}}(y)\,dy$$
$$= \int_{\mathbb{R}^3} \widehat{f_{\gamma_k}^{k+1}} * \widehat{\overline{f_{\gamma_k}^{k+1}}}(\xi) e^{2\pi i x\cdot\xi} \rho_{\le r_{k+1}^{-1}}(\xi)\,d\xi$$
$$= \sum_{\gamma_{k+1},\gamma'_{k+1}\subset\gamma_k} \int_{\mathbb{R}^3} e^{2\pi i x\cdot\xi}\, \widehat{f_{\gamma_{k+1}}^{k+1}} * \widehat{\overline{f_{\gamma'_{k+1}}^{k+1}}}(\xi)\rho_{\le r_{k+1}^{-1}}(\xi)\,d\xi.$$

The integrand is supported in $(\underline{C}_\varepsilon\gamma_{k+1} - \underline{C}_\varepsilon\gamma'_{k+1}) \cap (2\mathcal{C}_{r_{k+1}^{-1}})$, where $\underline{C}_\varepsilon$ comes from (3) of Lemma 7 and $2\mathcal{C}_{r_{k+1}^{-1}}$ contains the support of $\rho_{\le r_{k+1}^{-1}}$. The set $\mathcal{C}_{r_{k+1}^{-1}}$ is contained in a cylinder with a vertical axis, centered at the origin and of radius $2r_{k+1}^{-1}$. The distance between the sets $\underline{C}_\varepsilon\gamma_{k+1}$ and $\underline{C}_\varepsilon\gamma'_{k+1}$ is controlled by the distance of their projections to the (ξ_1,ξ_2)-plane. This means that the final integral displayed above vanishes unless γ_{k+1} is within $\sim \underline{C}_\varepsilon r_{k+1}^{-1}$ of γ'_{k+1}, in which case we write $\gamma_{k+1}\sim\gamma'_{k+1}$. Then

$$\sum_{\gamma_{k+1},\gamma'_{k+1}\subset\gamma_k} \int_{\mathbb{R}^3} e^{2\pi i x\cdot\xi}\, \hat{f}_{\gamma_{k+1}}^{k+1} * \widehat{\bar{f}_{\gamma'_{k+1}}^{k+1}}(\xi)\rho_{\le r_{k+1}^{-1}}(\xi)\,d\xi = \sum_{\substack{\gamma_{k+1},\gamma'_{k+1}\subset\gamma_k\\ \gamma_{k+1}\sim\gamma'_{k+1}}} \int_{\mathbb{R}^3} e^{2\pi i x\cdot\xi}\, \hat{f}_{\gamma_{k+1}}^{k+1} * \widehat{\bar{f}_{\gamma'_{k+1}}^{k+1}}(\xi)\rho_{\le r_{k+1}^{-1}}(\xi)\,d\xi.$$

Use Plancherel's theorem again to return to a convolution in x and conclude that

$$|g_k * \check{\rho}_{\le r_{k+1}^{-1}}(x)| = \Bigg| \sum_{\substack{\gamma_{k+1},\gamma'_{k+1}\subset\gamma_k\\ \gamma_{k+1}\sim\gamma'_{k+1}}} (f_{\gamma_{k+1}}^{k+1}\overline{f_{\gamma'_{k+1}}^{k+1}}) * \omega_{\tau_k} * \check{\rho}_{\le r_{k+1}^{-1}}(x)\Bigg| \lesssim \underline{C}_\varepsilon \sum_{\gamma_k}\sum_{\gamma_{k+1}\subset\gamma_k} |f_{\gamma_{k+1}}^{k+1}|^2 * \omega_{\tau_k} * |\check{\rho}_{\le r_{k+1}^{-1}}|(x).$$

By the locally constant property (Lemma 9) and (1) of Lemma 7,

$$\sum_{\gamma_k}\sum_{\gamma_{k+1}\subset\gamma_k} |f_{\gamma_{k+1}}^{k+1}|^2 * \omega_{\tau_k} * |\check{\rho}_{\le r_{k+1}^{-1}}|(x) \lesssim \sum_{\gamma_k}\sum_{\gamma_{k+1}\subset\gamma_k} |f_{\gamma_{k+1}}^{k+2}|^2 * w_{\gamma_{k+1}} * \omega_{\tau_k} * |\check{\rho}_{\le r_{k+1}^{-1}}|(x) \lesssim g_{k+1}(x).$$

It remains to note that

$$w_{\gamma_{k+1}} * \omega_{\gamma_k} * |\check{\rho}_{\le r_{k+1}^{-1}}|(x) \lesssim w_{\gamma_{k+1}}(x)$$

since γ_k^* is comparable to a dilation of γ_{k+1}^* and $\check{\rho}_{\le r_{k+1}^{-1}}$ is an L^1-normalized function that is rapidly decaying away from $B_{r_{k+1}}$ (actually, it decays rapidly away from the small set $B_{r_{k+1}}^{(2)}(0)\times B_1^{(1)}(0)$). □

Corollary 15 (high-dominance on Λ_k,Ω_k). *For R large enough depending on ε,*

$$g_k(x) \le 2|g_k^h(x)| \quad \textit{for all } x\in\Lambda_k \qquad \textit{and} \qquad G_k(x) \le 2|G_k^h(x)| \quad \textit{for all } x\in\Omega_k.$$

Proof. This follows directly from Lemma 14. Indeed, since $g_k(x) = g_k^\ell(x) + g_k^h(x)$, the inequality $g_k(x) > 2|g_k^h(x)|$ implies that $g_k(x) < 2|g_k^\ell(x)|$. Then by Lemma 14, $|g_k(x)| < 2D_\varepsilon g_{k+1}(x)$. Since $x\in\Lambda_k$, $g_{k+1}(x) \le A_\varepsilon^{M-k-1}R^\beta$, or in the case that $k = M-1$,

$$g_M(x) = \sum_\gamma |f_\gamma|^2 * \omega_\gamma(x) \lesssim \Bigg\|\sum_\gamma |f_\gamma|^2\Bigg\|_\infty \lesssim R^\beta$$

using the assumption that $\|f_\gamma\|_\infty \lesssim 1$ for all γ. Altogether gives the upper bound

$$g_k(x) \le 2D_\varepsilon A_\varepsilon^{M-k-1} R^\beta.$$

The contradicts the property that on Λ_k we have $A_\varepsilon^{M-k} R^\beta \le g_k(x)$ for A_ε sufficiently larger than D_ε, which finishes the proof. The argument for G_k on Ω_k is analogous. □

Lemma 16 (pruning lemma). *If A_ε is a large enough constant depending on ε, then, for any τ,*

$$\Big|\sum_{\gamma_k\subset\tau} f_{\gamma_k} - \sum_{\gamma_k\subset\tau} f_{\gamma_k}^{k+1}(x)\Big| \le \frac{\alpha}{A_\varepsilon^{1/2}K^3} \quad \textit{for all } x\in\Lambda_k,$$
$$\Big|\sum_{\tau_k\subset\tau} f_{\tau_k} - \sum_{\tau_k\subset\tau} F_{\tau_k}^{k+1}(x)\Big| \le \frac{\alpha}{A_\varepsilon^{1/2}K^3} \quad \textit{for all } x\in\Omega_k,$$
$$\Big|\sum_{\tau_1\subset\tau} f_{\tau_1} - \sum_{\tau_1\subset\tau} F_{\tau_1}^{1}(x)\Big| \le \frac{\alpha}{A_\varepsilon^{1/2}K^3} \quad \textit{for all } x\in L.$$

Proof. Begin by proving the claim about Λ_k. By the definition of the pruning process, we have

$$f_\tau = f_\tau^{M-1} + (f_\tau^M - f_\tau^{M-1}) = \cdots = f_\tau^{k+1}(x) + \sum_{m=k+1}^{M-1} (f_\tau^{m+1} - f_\tau^m), \tag{6}$$

where here, the subscript τ means $f_\tau = \sum_{\gamma\subset\tau} f_\gamma$ and $f_\tau^m = \sum_{\gamma_m\subset\tau} f_{\gamma_m}^m$. We will show that each difference in the sum is much smaller than α. For each $M-1\ge m\ge k+1$ and γ_m, use the notation $\mathbb{T}_{\gamma_m}^b = \mathbb{T}_{\gamma_m}\setminus\mathbb{T}_{\gamma_m}^g$ and write

$$\begin{aligned}
|f_{\gamma_m}^m(x) - f_{\gamma_m}^{m+1}(x)| &= \Big|\sum_{T_{\gamma_m}\in\mathbb{T}_{\gamma_m}^b} \psi_{T_{\gamma_m}}(x) f_{\gamma_m}^{m+1}(x)\Big| = \sum_{T_{\gamma_m}\in T_{\gamma_m}^b} |\psi_{T_{\gamma_m}}^{1/2}(x) f_{\gamma_m}^{m+1}(x)| \psi_{T_{\gamma_m}}^{1/2}(x) \\
&\le \sum_{T_{\gamma_m}\in\mathbb{T}_{\gamma_m}^b} K^{-3} A_\varepsilon^{-(M-m+1)} \frac{\alpha}{R^\beta} \|\psi_{T_{\gamma_m}} f_{\gamma_m}^{m+1}\|_{L^\infty(\mathbb{R}^3)} \|\psi_{T_{\gamma_m}}^{1/2} f_{\gamma_m}^{m+1}\|_{L^\infty(\mathbb{R}^3)} \psi_{T_{\gamma_m}}^{1/2}(x) \\
&\lesssim K^{-3} A_\varepsilon^{-(M-m+1)} \frac{\alpha}{R^\beta} \sum_{T_{\gamma_m}\in\mathbb{T}_{\gamma_m}^b} \|\psi_{T_{\gamma_m}}^{1/2} f_{\gamma_m}^{m+1}\|_{L^\infty(\mathbb{R}^3)}^2 \psi_{T_{\gamma_m}}^{1/2}(x) \\
&\lesssim K^{-3} A_\varepsilon^{-(M-m+1)} \frac{\alpha}{R^\beta} \sum_{T_{\gamma_m}\in\mathbb{T}_{\gamma_m}^b} \sum_{\widetilde{T}_{\gamma_m}} \|\psi_{T_{\gamma_m}} |f_{\gamma_m}^{m+1}|^2\|_{L^\infty(\widetilde{T}_{\gamma_m})} \psi_{T_{\gamma_m}}^{1/2}(x) \\
&\lesssim K^{-3} A_\varepsilon^{-(M-m+1)} \frac{\alpha}{R^\beta} \sum_{T_{\gamma_m},\widetilde{T}_{\gamma_m}\in\mathbb{T}_{\gamma_m}} \|\psi_{T_{\gamma_m}}\|_{L^\infty(\widetilde{T}_{\gamma_m})} \||f_{\gamma_m}^{m+1}|^2\|_{L^\infty(\widetilde{T}_{\gamma_m})} \psi_{T_{\gamma_m}}^{1/2}(x).
\end{aligned}$$

Let $c_{\widetilde{T}_{\gamma_m}}$ denote the center of $\widetilde{T}_{\gamma_m}$ and note the pointwise inequality

$$\sum_{T_{\gamma_m}} \|\psi_{T_{\gamma_m}}\|_{L^\infty(\widetilde{T}_{\gamma_m})} \psi_{T_{\gamma_m}}^{1/2}(x) \lesssim |\gamma_m^*| \omega_{\gamma_m}(x - c_{\widetilde{T}_{\gamma_m}}),$$

which means that

$$|f_{\gamma_m}^m(x) - f_{\gamma_m}^{m+1}(x)| \lesssim K^{-3}A_\varepsilon^{-(M-m+1)}\frac{\alpha}{R^\beta}|\gamma_m^*|\sum_{\widetilde{T}_{\gamma_m}\in\mathbb{T}_{\gamma_m}}\omega_{\gamma_m}(x-c_{\widetilde{T}_{\gamma_m}})\||f_{\gamma_m}^{m+1}|^2\|_{L^\infty(\widetilde{T}_{\gamma_m})}$$
$$\lesssim_\varepsilon K^{-3}A_\varepsilon^{-(M-m+1)}\frac{\alpha}{R^\beta}|\gamma_m^*|\sum_{\widetilde{T}_{\gamma_m}\in\mathbb{T}_{\gamma_m}}\omega_{\gamma_m}(x-c_{\widetilde{T}_{\gamma_m}})|f_{\gamma_m}^{m+1}|^2 * \omega_{\gamma_m}(c_{\widetilde{T}_{\gamma_m}})$$
$$\lesssim_\varepsilon K^{-3}A_\varepsilon^{-(M-m+1)}\frac{\alpha}{R^\beta}|f_{\gamma_m}^{m+1}|^2 * \omega_{\gamma_m}(x),$$

where we used the locally constant property in the second-to-last inequality. The last inequality is justified by the fact that $\omega_{\gamma_m}(x-c_{\widetilde{T}_{\gamma_m}}) \sim \omega_{\gamma_m}(x-y)$ for any $y\in\widetilde{T}_{\gamma_m}$, and we have the pointwise relation $\omega_{\gamma_m} * \omega_{\gamma_m} \lesssim \omega_{\gamma_m}$. The last two inequalities incorporate a dependence on $\underline{C}_\varepsilon$ from Lemma 7 since the locally constant property uses that $|\widehat{f_{\gamma_m}^{m+1}}|^2$ is supported in the $\underline{C}_\varepsilon$-dilation of $\gamma_m-\gamma_m$. It is important to note that $\underline{C}_\varepsilon$ is a combinatorial factor that does not depend on A_ε. Then

$$\Bigg|\sum_{\gamma_m\subset\tau} f_{\gamma_m}^m(x) - f_{\gamma_m}^{m+1}(x)\Bigg| \lesssim_\varepsilon K^{-3}A_\varepsilon^{-(M-m+1)}\frac{\alpha}{R^\beta}\sum_{\gamma_m\subset\tau}|f_{\gamma_m}^{m+1}|^2 * \omega_{\gamma_m}(x) \sim_\varepsilon K^{-3}A_\varepsilon^{-(M-m+1)}\frac{\alpha}{R^\beta}g_m(x).$$

At this point, choose A_ε large enough so that if $g_m(x)\le A_\varepsilon^{M-m}R^\beta$, then the above inequality implies that

$$\Bigg|\sum_{\gamma_m\subset\tau} f_{\gamma_m}^m(x) - f_{\gamma_m}^{m+1}(x)\Bigg| \le \varepsilon K^{-3}A_\varepsilon^{-1/2}\alpha.$$

This finishes the proof since $M+N\lesssim\varepsilon^{-1}$, so the number of steps from (6) is controlled. The argument for the pruning on Ω_k and on L is analogous. □

2.3. ***Geometry related to the high-frequency parts of square functions.*** We have seen in Corollary 15 that on Λ_k and Ω_k, g_k and G_k are high-dominated. In this subsection, we describe the geometry of the Fourier supports of g_k^h and G_k^h, which will allow us to apply certain decoupling theorems for the cone in Section 2.4. We begin with the precise definitions of canonical blocks and small cap blocks (which we also call "small caps") of the moment curve.

Definition 17 (canonical moment curve blocks). For $S\in 2^{\mathbb{N}}$, $S\ge 10$, consider the anisotropic neighborhood

$$\mathcal{M}^3(S)=\{(\xi_1,\xi_2,\xi_3):\xi_1\in[0,1],\ |\xi_2-\xi_1^2|\le S^{-2},\ |\xi_3-3\xi_1\xi_2+2\xi_1^3|\le S^{-3}\}.$$

Define canonical moment curve blocks at scale S which partition $\mathcal{M}^3(S)$ as follows:

$$\bigsqcup_{l=0}^{S-1}\{(\xi_1,\xi_2,\xi_3): lS^{-1}\le\xi_1<(l+1)S^{-1},\ |\xi_2-\xi_1^2|\le S^{-2},\ |\xi_3-3\xi_1\xi_2+2\xi_1^3|\le S^{-3}\}.$$

Definition 18 ("small caps" of the moment curve). Let $R\ge 10$ and let $S\in 2^{\mathbb{N}}$ satisfy $R^{-1}\le S^{-1}\le R^{-1/3}$. Consider the anisotropic small cap neighborhood

$$\mathcal{M}^3(S,R)=\{(\xi_1,\xi_2,\xi_3):\xi_1\in[0,1],\ |\xi_2-\xi_1^2|\le S^{-2},\ |\xi_3-3\xi_1\xi_2+2\xi_1^3|\le R^{-1}\}.$$

Define small caps γ associated to the parameters S and R by

$$\bigsqcup \gamma = \bigsqcup_{l=0}^{S-1} \{(\xi_1, \xi_2, \xi_3) : lS^{-1} \le \xi_1 < (l+1)S^{-1},\ |\xi_2 - \xi_1^2| \le S^{-2},\ |\xi_3 - 3\xi_1\xi_2 + 2\xi_1^3| \le R^{-1}\}. \quad (7)$$

Note that the small caps γ are essentially canonical moment curve blocks at scale S plus a vertical (ξ_3-direction) R^{-1}-neighborhood.

To analyze g_k^h, we need to understand the Fourier support of $\sum_{\gamma_k} |f_{\gamma_k}^{k+1}|^2$ outside of a cylinder of radius r_{k+1}^{-1}. By (3) of Lemma 7, the support of $\widehat{|f_{\gamma_k}^{k+1}|^2}$ is $\underline{C}_\varepsilon \gamma_k - \underline{C}_\varepsilon \gamma_k$. Suppose that γ_k is the l-th piece, meaning that

$$\gamma_k = \{(\xi_1, \xi_2, \xi_3) : lr_k^{-1} \le \xi_1 < (l+1)r_k^{-1},\ |\xi_2 - \xi_1^2| \le r_k^{-2},\ |\xi_3 - 3\xi_1\xi_2 + 2\xi_1^3| \le R^{-1}\},$$

where $l \in \{0, \ldots, r_k - 1\}$. The small cap γ_k is comparable to the set

$$\underline{\gamma_k} = \{\mathfrak{m}(lr_k^{-1}) + A\mathfrak{m}'(lr_k^{-1}) + B\mathfrak{m}''(lr_k^{-1}) + C\mathfrak{m}'''(lr_k^{-1}) : 0 \le A \le r_k^{-1},\ |B| \le r_k^{-2},\ |C| \le R^{-1}\}$$

in the sense that $\frac{1}{20}\underline{\gamma_k} \subset \gamma_k \subset 20\underline{\gamma_k}$ (where the dilations are taken with respect to the centroid of γ_k). Then $\gamma_k - \gamma_k$ is contained in

$$\{A\mathfrak{m}'(lr_k^{-1}) + B\mathfrak{m}''(lr_k^{-1}) + C\mathfrak{m}'''(lr_k^{-1}) : |A| \lesssim r_k^{-1},\ |B| \lesssim r_k^{-2},\ |C| \lesssim R^{-1}\}.$$

Recall that $1 - \rho_{\le r_{k+1}^{-1}}$ is supported outside $\mathcal{C}_{r_{k+1}^{-1}} \supseteq \{(\xi_1, \xi_2, \xi_3) : |(\xi_1, \xi_2)| \le r_{k+1}^{-1},\ |\xi_3| \le 1\}$. Intersecting $\underline{C}_\varepsilon \gamma_k - \underline{C}_\varepsilon \gamma_k$ with the support of $1 - \rho_{\le r_{k+1}^{-1}}$ forces the relation $A^2 + (A2(lr_k^{-1}) + 2B)^2 \ge r_{k+1}^{-2}$. Using the upper bounds $|A| \lesssim \underline{C}_\varepsilon r_k^{-1}$ and $|B| \lesssim \underline{C}_\varepsilon r_k^{-2}$, it follows that for R large enough depending on ε, the support of the high-frequency part of $\widehat{|f_{\gamma_k}^{k+1}|^2}$ is contained in

$$\tilde{\gamma}_k := \left\{A\mathfrak{m}'(lr_k^{-1}) + B\mathfrak{m}''(lr_k^{-1}) + C\mathfrak{m}'''(lr_k^{-1}) : \tfrac{1}{2}r_{k+1}^{-1} \le |A| \lesssim \underline{C}_\varepsilon r_k^{-1},\ |B| \lesssim \underline{C}_\varepsilon r_k^{-2},\ |C| \lesssim \underline{C}_\varepsilon R^{-1}\right\}. \quad (8)$$

Our "high lemmas" will require geometric properties that are recorded in the following propositions.

Proposition 19. *The sets $\tilde{\gamma}_k$, varying over γ_k, are $\le C_\varepsilon R^\varepsilon$-overlapping.*

Proof. Suppose that a point corresponding to parameters A, B, C, l and A', B', C', l' respectively is in the intersection of two sets as in (8). By analyzing the first coordinate, we must have $A = A'$. By analyzing the second coordinate, we must have

$$|A2lr_k^{-1} - A2l'r_k^{-1}| \lesssim \underline{C}_\varepsilon r_k^{-2}.$$

Therefore, since $A \gtrsim r_{k+1}^{-1}$, we have $|l - l'| \lesssim \underline{C}_\varepsilon R^\varepsilon$. □

Next we describe the geometry of a small cap partition for the cone. Let $\beta_1 \in \left[\frac{1}{2}, 1\right]$ and $\rho \ge 1$. Let $S \in 2^{\mathbb{N}}$ a dyadic number closest to ρ^{β_1}. For the (truncated) cone $\Gamma = \left\{\xi : \xi_1^2 + \xi_2^2 = \xi_3^2,\ \frac{1}{2} \le \xi_3 \le 1\right\}$, divide $[0, 2\pi)$ into S many intervals I_S of length $2\pi/S$ and define the small cap partition

$$\mathcal{N}_{S^{-1}}(\Gamma) = \bigsqcup_{I_S} \mathcal{N}_{S^{-1}}(\Gamma) \cap \{(\rho\cos\zeta, \rho\sin\zeta, z) : \zeta \in I_S)\}$$

corresponding to parameters β_1 and $\beta_2 = 0$, as in Theorem 3 from [Guth and Maldague 2022]. After a linear transformation, we will identify the high parts of sets $\gamma_k - \gamma_k$ as subsets of conical small caps.

Proposition 20. *Let $r^{-1} \in [r_{k+1}^{-1}, 20\underline{C}_\varepsilon r_k^{-1}]$ be a dyadic value and write $\{\xi_3 \sim r^{-1}\} := \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \frac{1}{2}r^{-1} \le \xi_3 \le r^{-1}\}$. There is an affine transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ so that the following hold*:

(1) *If $r_k^{-1} \le R^{-1/2}$, then the collection of γ_k may be partitioned into $\lesssim_\varepsilon R^{2\varepsilon}$ many subsets $\mathcal{S}_i$ which satisfy the following. For each $\mathcal{S}_i$, there is a conical small cap partition of $\sim 1 \times \underline{C}_\varepsilon r/R \times \underline{C}_\varepsilon r/R$ blocks so that, for each $\gamma_k \in \mathcal{S}_i$, $r[T(\tilde{\gamma}_k) \cap \{\xi_3 \sim r^{-1}\}]$ is completely contained in one of the conical small caps. Collections of $r_k^2 R^{-1}$ many neighboring γ_k are identified with the same conical small cap.*

(2) *If $R^{-1/2} \le r_k^{-1}$ and $(Rr_k^{-1})^{-\beta_1} = r_k^{-1}$ for some $\beta_1 \in [\frac{1}{2}, 1]$, then the collection of γ_k may be partitioned into $\lesssim_\varepsilon R^{2\varepsilon}$ many subsets $\mathcal{S}_i$ which satisfy the following. For each $\mathcal{S}_i$, there is a conical small cap partition of $\sim 1 \times \underline{C}_\varepsilon (r/R)^{\beta_1} \times \underline{C}_\varepsilon^{\beta_1^{-1}} r/R$ blocks so that each $r[T(\tilde{\gamma}_k) \cap \{\xi_3 \sim r^{-1}\}]$, where $\gamma_k \subset \mathcal{S}_i$, is completely contained in one of the conical small caps. Each $\gamma_k \in \mathcal{S}_i$ is assigned to its own conical small cap.*

Proof. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the affine transformation

$$T(x, y, z) := \left(\frac{y}{2}, \frac{x - z/6}{\sqrt{2}}, \frac{x + z/6}{\sqrt{2}}\right).$$

The image of the set (8) under T is

$$T(\tilde{\gamma}_k) = \Bigg\{A\left(lr_k^{-1}, \frac{1 - l^2 r_k^{-2}/2}{\sqrt{2}}, \frac{1 + l^2 r_k^{-2}/2}{\sqrt{2}}\right) + B\left(1, \frac{-lr_k^{-1}}{\sqrt{2}}, \frac{lr_k^{-1}}{\sqrt{2}}\right) + C\left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) : \frac{1}{2}r_{k+1}^{-1} \le |A| \lesssim \underline{C}_\varepsilon r_k^{-1},\ |B| \lesssim \underline{C}_\varepsilon r_k^{-2},\ |C| \lesssim \underline{C}_\varepsilon R^{-1}\Bigg\}.$$

Defining $\omega \in [\frac{\pi}{4}, \frac{\pi}{2}]$ by

$$(\cos\omega, \sin\omega) = \left(\frac{2\sqrt{2}lr_k^{-1}}{2 + l^2 r_k^{-2}}, \frac{2 - l^2 r_k^{-2}}{2 + l^2 r_k^{-2}}\right),$$

the set $T(\tilde{\gamma}_k)$ is contained in

$$\{A(\cos\omega, \sin\omega, 1) + B(\sin\omega, -\cos\omega, 0) + C(\cos\omega, \sin\omega, -1) : r_{k+1}^{-1} \le |A| \lesssim \underline{C}_\varepsilon r_k^{-1},\ |B| \lesssim \underline{C}_\varepsilon (r_k^{-2} + R^{-1}),\ |C| \lesssim \underline{C}_\varepsilon R^{-1}\}. \quad (9)$$

Suppose that $r_k^{-1} \le R^{-1/2}$. Then

$$T(\tilde{\gamma}_k) \cap \{\xi_3 \sim r^{-1}\} \subset \{A(\cos\omega, \sin\omega, 1) + B(\sin\omega, -\cos\omega, 0) + C(\cos\omega, \sin\omega, -1) : \tfrac{1}{2}r^{-1} \le |A| \le r^{-1},\ |B| \lesssim \underline{C}_\varepsilon R^{-1},\ |C| \lesssim \underline{C}_\varepsilon R^{-1}\}. \quad (10)$$

The $\omega = \omega(\gamma_k)$ in (9) form an $\sim r_k^{-1}$-separated subset of $[\frac{\pi}{4}, \frac{\pi}{2}]$. For a dyadic S closest to $\underline{C}_\varepsilon R/r$, we may sort the $\omega(\gamma_k)$ into different intervals $I_S \subset [0, 2\pi)$ of length S^{-1} and note that the r dilation of $T(\tilde{\gamma}_k) \cap \{\xi_3 \sim r^{-1}\}$ for $\omega(\gamma_k) \in I_S$ is contained in a single $\sim 1 \times S^{-1} \times S^{-1}$ conical small cap. If γ_k and γ_k' are within $\sim r_k/R$ of one another, then $\omega(\gamma_k)$ and $\omega(\gamma_k')$ are assigned to the same I_S.

Now suppose that $R^{-1/2} \le r_k^{-1} \le R^{-1/3}$. Then

$$T(\tilde{\gamma}_k) \cap \{\xi_3 \sim r^{-1}\} \subset \{A(\cos\omega, \sin\omega, 1) + B(\sin\omega, -\cos\omega, 0) + C(\cos\omega, \sin\omega, -1) : \tfrac{1}{2}r^{-1} \le |A| \le r^{-1},\ |B| \lesssim \underline{C}_\varepsilon r_k^{-2},\ |C| \lesssim \underline{C}_\varepsilon R^{-1}\}. \quad (11)$$

Let $S\in 2^{\mathbb{N}}$ be chosen so $S^{-\beta_1}$ is the smallest dyadic number satisfying $\underline{C}_\varepsilon R^\varepsilon r_k^{-1}\le S^{-\beta_1}$ (recalling that β_1 is defined by $(Rr_k^{-1})^{-\beta_1}=r_k^{-1}$ in the proposition statement). Then $\underline{C}_\varepsilon^{\beta_1^{-1}} R^{\varepsilon\beta_1^{-1}} r_k R^{-1}\le S^{-1}$ and so each r-dilation of $T(\tilde{\gamma}_k)\cap\{\xi_3\sim r^{-1}\}$ is contained in a single approximate $1\times S^{-\beta_1}\times S^{-1}$ conical small cap. If γ_k and γ_k' are conical small caps which are a distance Cr_k^{-1} from one another, then their corresponding angles $\omega(\gamma_k)$ and $\omega(\gamma_k)$ are also a distance $\gtrsim Cr_k^{-1}$ and make the sets on the right-hand side of (11) distinct. □

To analyze G_k^h, we need to understand the Fourier support of $\sum_{\tau_k}|F_{\tau_k}^{k+1}|^2$ outside of a low set $\mathcal{C}_{R_{k+1}^{-1/3}}$. By (3) of Lemma 7, the support of $\widehat{|F_{\gamma_k}^{k+1}|^2}$ is contained in $\underline{C}_\varepsilon\tau_k-\underline{C}_\varepsilon\tau_k$.

Proposition 21. *Let r be a dyadic value, $R_{k+1}^{-1/3}\le r^{-1}\le \underline{C}_\varepsilon R_k^{-1/3}$. There is an affine transformation $T:\mathbb{R}^3\to\mathbb{R}^3$ so that the following holds. We may partition the τ_k into $\lesssim_\varepsilon R^\varepsilon$ many sets $\mathcal{S}_i$ which satisfy: there is a canonical partition of the cone into approximate $1\times \underline{C}_\varepsilon r R_k^{-2/3}\times \underline{C}_\varepsilon^2 r^2 R_k^{-4/3}$ blocks so that, for each $\tau_k\in\mathcal{S}_i$, the r-dilation of the sets $T[(\underline{C}_\varepsilon\tau_k-\underline{C}_\varepsilon\tau_k)\setminus B_{R_{k+1}^{-1/3}}]\cap\{\xi_3\sim r^{-1}\}$ is contained in one of the canonical cone blocks.*

Proof. Suppose that τ_k is the l-th piece, meaning that

$$\tau_k=\{(\xi_1,\xi_2,\xi_3): lR_k^{-1/3}\le\xi_1<(l+1)R_k^{-1/3},\ |\xi_2-\xi_1^2|\le R_k^{-2/3},\ |\xi_3-3\xi_1\xi_2+2\xi_1^3|\le R_k^{-1}\},$$

where $l\in\{0,\ldots,R_k^{1/3}-1\}$. Let T be the affine transformation from the proof of Proposition 20. Then $T[(\underline{C}_\varepsilon\tau_k-\underline{C}_\varepsilon\tau_k)\setminus B_{R_{k+1}^{-1/3}}]\cap\{\xi_3\sim r^{-1}\}$ is contained in the set

$$\Big\{A(\cos\omega,\sin\omega,1)+B(\sin\omega,-\cos\omega,0)+C(\cos\omega,\sin\omega,-1):\\ \tfrac12 r^{-1}\le|A|\le r^{-1},\ |B|\lesssim\underline{C}_\varepsilon R_k^{-2/3},\ |C|\lesssim\underline{C}_\varepsilon R_k^{-1}\Big\}.$$

where $\omega\in\left[\frac{\pi}{4},\frac{\pi}{2}\right]$ is defined by

$$(\cos\omega,\sin\omega)=\left(\frac{2\sqrt{2}lR_k^{-1/3}}{2+l^2R_k^{-2/3}},\frac{2-l^2R_k^{-2/3}}{2+l^2R_k^{-2/3}}\right).$$

Since the $\omega=\omega(\tau_k)$ form an $\sim R_k^{-1/3}$-separated set, the r-dilation of each displayed set above is contained in a canonical cone block of approximate dimensions $1\times\underline{C}_\varepsilon rR_k^{-2/3}\times\underline{C}_\varepsilon^2 r^2R_k^{-4/3}$. □

2.4. ***Lemmas related to the high-frequency parts of square functions.*** First we recall the small cap decoupling theorem for the cone from [Guth and Maldague 2022]. Subdivide the R^{-1} neighborhood of the truncated cone $\Gamma=\left\{(\xi_1,\xi_2,\xi_3):\xi_1^2+\xi_2^2=\xi_3^2,\ \frac12\le\xi_3\le1\right\}$ into $R^{-\beta_2}\times R^{-\beta_1}\times R^{-1}$ small caps γ, where $\beta_1\in\left[\frac12,1\right]$ and $\beta_2\in[0,1]$. Here, $R^{-\beta_2}$ corresponds to the flat direction of the cone and $R^{-\beta_1}$ corresponds to the angular direction. The (ℓ^p,L^p) small cap theorem for Γ is the following.

Theorem 22 [Guth and Maldague 2022, Theorem 3]. *Let $\beta_1\in\left[\frac12,1\right]$ and $\beta_2\in[0,1]$. For $p\ge2$,*

$$\int_{\mathbb{R}^3}|f|^p\le C_\varepsilon R^\varepsilon\big(R^{(\beta_1+\beta_2)(p/2-1)}+R^{(\beta_1+\beta_2)(p-2)-1}+R^{(\beta_1+\beta_2-1/2)(p-2)}\big)\sum_\gamma\|f_\gamma\|_{L^p(\mathbb{R}^3)}^p$$

for any Schwartz function $f:\mathbb{R}^3\to\mathbb{C}$ with Fourier transform supported in $\mathcal{N}_{R^{-1}}(\Gamma)$.

Lemma 23 (high lemma I). *Suppose that $R^{-\beta} \le r_k^{-1} \le R^{-1/2}$. Then*

$$\int |g_k^h|^4 \le C_\varepsilon R^\varepsilon r_k^{-1} R \sum_\zeta \Big\| \sum_{\gamma_k \subset \zeta} |f_{\gamma_k}^{k+1}|^2 * \omega_{\gamma_k} * \check{\rho}_{>r_{k+1}^{-1}} \Big\|_{L^4(\mathbb{R}^3)}^4,$$

where the ζ are disjoint collections of $r_k^2 R^{-1}$ many adjacent γ_k.

Proof. Let T be the affine transformation from Proposition 20 and write $Tx = Ax + b$ for a 3×3 invertible matrix A and $b \in \mathbb{R}^3$. Then

$$g_k^h(x) = |\det A|^{-1} e^{-2\pi i x\cdot b} \widehat{\widehat{g_k^h} \circ T^{-1}}((A^{-1})^* x). \tag{12}$$

Perform the change of variables $x \mapsto A^* x$ to get

$$\int |g_k^h(x)|^4\, dx = |\det A|^{-3} \int |\widehat{\widehat{g_k^h} \circ T^{-1}}(x)|^4\, dx.$$

Let r be a dyadic parameter in the range $r_{k+1}^{-1} \le r^{-1} \le \underline{C}_\varepsilon r_k^{-1}$. Let $\eta_r : \mathbb{R}^3 \to [0,\infty)$ be a smooth function with compact support in the set $\{(\xi_1,\xi_2,\xi_3) : \frac{1}{2} r^{-1} \le \xi_3 \le r^{-1}\} =: \{\xi_3 \sim r^{-1}\}$ and satisfying the property that the sum of η_r over dyadic r is identically 1 on the support of $\widehat{g_k^h} \circ T^{-1}$. By dyadic pigeonholing, there is an r so that

$$|\det A|^{-3} \int |\widehat{\widehat{g_k^h} \circ T^{-1}}(x)|^4\, dx \le C_\varepsilon (\log R)^4 |\det A|^{-3} \int |\widehat{(\widehat{g_k^h} \circ T^{-1})\eta_r}(x)|^4\, dx.$$

Finally, perform the change of variables $x \mapsto rx$ to get

$$|\det A|^{-3} r^3 \int |\widehat{(\widehat{g_k^h} \circ T^{-1})\eta_r}(rx)|^4\, dx.$$

Now, note that

$$\widehat{(\widehat{g_k^h} \circ T^{-1})\eta_r}(rx) = \sum_{\gamma_k} [(\widehat{|f_{\gamma_k}^{k+1}|^2} \hat{\omega}_{\gamma_k}(1-\rho_{\le r_{k+1}^{-1}})) \circ T^{-1} \cdot \eta_r]\check{}\,(rx)$$
$$= \sum_i \sum_{\gamma_k \in \mathcal{S}_i} [(\widehat{|f_{\gamma_k}^{k+1}|^2} \hat{\omega}_{\gamma_k}(1-\rho_{\le r_{k+1}^{-1}})) \circ T^{-1} \cdot \eta_r]\check{}\,(rx),$$

where $\mathcal{S}_i$ is one of the $\lesssim_\varepsilon R^\varepsilon$ many sets partitioning the γ_k from (1) of Proposition 20. Apply the triangle inequality in the first sum over i and then apply Theorem 22 with parameters $\underline{C}_\varepsilon^{-1}(R/r)$, $\beta_1 = 1$, and $\beta_2 = 0$ to obtain

$$\int |g_k^h|^4 \lesssim_\varepsilon (\log R) R^{6\varepsilon} (r_k^{-1} R) |\det A|^{-3} r^3 \sum_\zeta \int \Big| \sum_{\gamma_k \subset \zeta} [(\widehat{|f_{\gamma_k}^{k+1}|^2} \hat{\omega}_{\gamma_k}(1-\rho_{\le r_{k+1}^{-1}})) \circ T^{-1} \cdot \eta_r]\check{}\,(rx) \Big|^4 dx,$$

where ζ are disjoint collections of $\sim r_k^2 R^{-1}$ many neighboring γ_k. This number comes about since one has r_k many γ_k's and they get sorted into $\sim R/r_k$ many conical small caps, so each conical small cap contains $\sim r_k/(R/r_k) = r_k^2 R^{-1}$ many γ_k's. It remains to undo the initial steps which allowed us to apply small cap decoupling for the cone. First do the change of variables $x \mapsto r^{-1}x$:

$$r^3 \sum_\zeta \int \Big| \sum_{\gamma_k \subset \zeta} [(\widehat{|f_{\gamma_k}^{k+1}|^2} \hat{\omega}_{\gamma_k}(1-\rho_{\le r_{k+1}^{-1}})) \circ T^{-1} \cdot \eta_r]\check{}\,(rx) \Big|^4 dx$$
$$= \sum_\zeta \int \Big| \sum_{\gamma_k \subset \zeta} [(\widehat{|f_{\gamma_k}^{k+1}|^2} \hat{\omega}_{\gamma_k}(1-\rho_{\le r_{k+1}^{-1}})) \circ T^{-1} \cdot \eta_r]\check{}\,(x) \Big|^4 dx.$$

By Young's convolution inequality (since multiplication on the Fourier side by η_r is equivalent to convolution on the spatial side by $\check{\eta}_r$, which is L^1-normalized),

$$\sum_{\zeta}\int\Big|\sum_{\gamma_k\subset\zeta}[(\widehat{|f_{\gamma_k}^{k+1}|^2}\hat{\omega}_{\gamma_k}(1-\rho_{\le r_{k+1}^{-1}}))\circ T^{-1}\cdot\eta_r]^{\check{}}\Big|^4\lesssim\sum_{\zeta}\int\Big|\sum_{\gamma_k\subset\zeta}[(\widehat{|f_{\gamma_k}^{k+1}|^2}\hat{\omega}_{\gamma_k}(1-\rho_{\le r_{k+1}^{-1}}))\circ T^{-1}]^{\check{}}\Big|^4.$$

Perform the change of variables $x\mapsto(A^{-1})^*x$ and use (12) to get

$$|\det A|^3\sum_{\zeta}\int\Big|\sum_{\gamma_k\subset\zeta}[(\widehat{|f_{\gamma_k}^{k+1}|^2}\hat{\omega}_{\gamma_k}(1-\rho_{\le r_{k+1}^{-1}}))\circ T^{-1}]^{\check{}}\Big|^4\lesssim\sum_{\zeta}\int\Big|\sum_{\gamma_k\subset\zeta}[\widehat{|f_{\gamma_k}^{k+1}|^2}\hat{\omega}_{\gamma_k}(1-\rho_{\le r_{k+1}^{-1}})]^{\check{}}\Big|^4,$$

which finishes the proof. ☐

Lemma 24 (high lemma II). *Suppose that* $\max(R^{-\beta},R^{-1/2})\le r_k^{-1}\le R^{-1/3}$. *Then*

$$\int|g_k^h|^{2+2/\beta_1}\le C_\varepsilon R^{14\varepsilon}r_k^{-1}R\sum_{\gamma_k}\|f_{\gamma_k}^{k+1}\|_{L^{4+4/\beta_1}(\mathbb{R}^3)}^{4+4/\beta_1},$$

where $\beta_1\in\left[\frac{1}{2},1\right]$ *satisfies* $(r_kR^{-1})^{-\beta_1}=r_k$.

Proof. Repeat the argument from the proof of Lemma 23, using (2) in place of (1) from Proposition 20 and applying Theorem 22 with β_1 as in the hypothesis of the lemma and $\beta_2=0$. The result is

$$\int|g_k^h|^{2+2/\beta_1}\lesssim_\varepsilon R^{14\varepsilon}(r_k^{-1}R)\sum_{\gamma_k}\int|[\widehat{|f_{\gamma_k}^{k+1}|^2}\hat{\omega}_{\gamma_k}(1-\rho_{\le r_{k+1}^{-1}})]^{\check{}}|^{2+2/\beta_1}.$$

The $R^{14\varepsilon}$ factor is to account for the fact that the Fourier support of g_k^h may only be identified up to some R^ε factors with small caps of the cone. Since $1-\rho_{\le r_{k+1}^{-1}}=\rho_{\le C_\varepsilon}-\rho_{\le r_{k+1}^{-1}}$ on the support of $\widehat{|f_{\gamma_k}^{k+1}|^2}$, by Young's convolution inequality, we have

$$\int|[\widehat{|f_{\gamma_k}^{k+1}|^2}\hat{\omega}_{\gamma_k}(1-\rho_{\le r_{k+1}^{-1}})]^{\check{}}|^{2+2/\beta_1}\lesssim\int|(\widehat{|f_{\gamma_k}^{k+1}|^2})^{\check{}}|^{2+2/\beta_1}=\int|f_{\gamma_k}^{k+1}|^{4+4/\beta_1}.$$ ☐

Lemma 25. *For each* m, $1\le m\le N$,

$$\int|G_m^h|^6\le C_\varepsilon R^\varepsilon\Big(\sum_{\tau_m}\|F_{\tau_m}^{m+1}\|_{L^{12}(\mathbb{R}^3)}^4\Big)^3.$$

Proof. Repeat the argument from the proof of Lemma 23, using Proposition 21 in place of Proposition 20 and applying canonical L^6 cone decoupling [Bourgain and Demeter 2015] instead of small cap decoupling. The result is

$$\int|G_m^h|^6\lesssim_\varepsilon R^{8\varepsilon}\sum_{\tau_m}\int|[\widehat{|F_{\tau_m}^{m+1}|^2}\widehat{\omega_{\tau_m}}(1-\rho_{\le R_{m+1}^{-1/3}})]^{\check{}}|^6.$$

Since $1-\rho_{\le R_{m+1}^{-1/3}}=\rho_{\le C_\varepsilon}-\rho_{\le R_{m+1}^{-1/3}}$ on the support of $\widehat{|F_{\tau_m}^{m+1}|^2}$, by Young's convolution inequality, we have

$$\int|[\widehat{|F_{\tau_m}^{m+1}|^2}\widehat{\omega_{\tau_m}}(1-\rho_{\le R_{m+1}^{-1/3}})]^{\check{}}|^6\lesssim\int|(\widehat{|F_{\tau_m}^{m+1}|^2})^{\check{}}|^6=\int|F_{\tau_m}^{m+1}|^{12}.$$ ☐

Theorem 26 (cylindrical decoupling over $\mathbb{P}^1$). *Let $\mathbb{P}^1 = \{(t, t^2) : 0 \le t \le 1\}$ and for $\delta > 0$, let $\mathcal{N}_\delta(\mathbb{P}^1)$ denote the δ-neighborhood of $\mathcal{P}^1$ in $\mathbb{R}^2$. If $h : \mathbb{R}^3 \to \mathbb{C}$ is a Schwartz function with Fourier transform supported in $\mathcal{N}_\delta(\mathbb{P}^1) \times \mathbb{R}$, then, for each $4 \le p \le 6$,*

$$\int_{\mathbb{R}^3} |h|^p \lesssim_\varepsilon \delta^{-\varepsilon} \Big(\sum_\zeta \|h_\zeta\|^2_{L^p(\mathbb{R}^3)} \Big)^{p/2},$$

where the ζ are products of $\sim \delta^{1/2} \times \delta$ rectangles that partition $\mathcal{N}_\delta(\mathbb{P}^1)$ with $\mathbb{R}$.

Proof. Begin by using Fourier inversion to write

$$h(x', x_3) = \int_{\mathcal{N}_\delta(\mathbb{P}^1)} \int_{\mathbb{R}} \widehat{h}(\xi', \xi_3) e^{2\pi i \xi \cdot x'} e^{2\pi i \xi_3 x_3} \, d\xi_3 \, d\xi'.$$

For each x_3, the function

$$x' \mapsto \int_{\mathcal{N}_\delta(\mathbb{P}^1)} \int_{\mathbb{R}} \widehat{h}(\xi', \xi_3) e^{2\pi i \xi_3 x_3} \, d\xi_3 e^{2\pi i \xi \cdot x'} \, d\xi'$$

satisfies the hypotheses of the decoupling theorem for $\mathbb{P}^1$. Use Fubini's theorem to apply the ℓ^2-decoupling theorem for $\mathbb{P}^1$ from [Bourgain and Demeter 2015] to the inner integral

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} |h(x', x_3)|^p \, dx' \, dx_3 \lesssim_\varepsilon \int_{\mathbb{R}} \delta^{-\varepsilon} \bigg(\sum_\nu \bigg(\int_{\mathbb{R}^2} \bigg| \int_\nu \int_{\mathbb{R}} \widehat{h}(\xi', \xi_3) e^{2\pi i \xi \cdot x'} e^{2\pi i \xi_3 x_3} \, d\xi_3 \, d\xi' \bigg|^p dx' \bigg)^{2/p} \bigg)^{p/2} dx_3,$$

where $\{\nu\}$ form a partition of $\mathcal{N}_\delta(\mathbb{P}^1)$ into $\sim \delta^{1/2} \times \delta$ blocks. By the triangle inequality, the right-hand side above (omitting $C_\varepsilon \delta^{-\varepsilon}$) is bounded by

$$\bigg(\sum_\nu \bigg(\int_{\mathbb{R}} \int_{\mathbb{R}^2} \bigg| \int_\nu \int_{\mathbb{R}} \widehat{h}(\xi', \xi_3) e^{2\pi i \xi \cdot x'} e^{2\pi i \xi_3 x_3} \, d\xi_3 \, d\xi' \bigg|^p dx' \, dx_3 \bigg)^{2/p} \bigg)^{p/2}.$$

The sets $\nu \times \mathbb{R}$ are the ζ in the statement of the lemma. □

Remark. The implicit upper bound in the statement of Theorem 26 is uniform in $4 \le p \le 6$. For the specific exponent $p = 4$, the implicit $C_\varepsilon \delta^{-\varepsilon}$ upper bound may be replaced by an absolute constant B which does not depend on δ.

2.5. *Local trilinear restriction for $\mathcal{M}^3$.*

The weight function W_{B_r} in the following theorem decays by a factor of 10 off of the ball B_r. It is specifically defined in Definition 8.

Proposition 27. *Let $s \ge 10r \ge 10$ and let $f : \mathbb{R}^3 \to \mathbb{C}$ be a Schwartz function with Fourier transform supported in $\mathcal{N}_{r^{-1}}(\mathcal{M}^3)$. Suppose that $\tau_1^1, \tau_1^2, \tau_1^3$ are canonical moment curve blocks at scale $R_1^{1/3}$ which satisfy $(\tau_1^i, \tau_1^j) \ge s^{-1}$ for $i \ne j$. Then*

$$\int_{B_r} |f_{\tau_1^1} f_{\tau_1^2} f_{\tau_1^3}|^2 \lesssim s^3 |B_r|^{-2} \Big(\int |f_{\tau_1^1}|^2 W_{B_r} \Big) \Big(\int |f_{\tau_1^2}|^2 W_{B_r} \Big) \Big(\int |f_{\tau_1^3}|^2 W_{B_r} \Big).$$

The weight function W_{B_r} is the generic ball weight defined in Definition 8.

Proof. Let $\mathfrak{m}(t)=(t,t^2,t^3)$ and let $B_{r^{-1}}$ be the ball of radius r^{-1} in $\mathbb{R}^3$ centered at the origin. Then

$$\begin{aligned}W_{B_r}(x)f_{\tau_1^i}(x)&=\int_{\tau_1^i+B_{r^{-1}}}\widehat{W}_{B_r}*\hat{f}_{\tau_1^i}(\xi^i)e^{2\pi ix\cdot\xi^i}\,d\xi^i\\&=\int_{\tau_1^i+B_{r^{-1}}}\widehat{W}_{B_r}*\hat{f}_{\tau_1^i}(\xi_1^i,\xi_2^i,\xi_3^i)e^{2\pi ix\cdot(\xi_1^i,\xi_2^i,\xi_2^i)}\,d\xi_1^i\xi_2^i\xi_3^i\\&=\int_{|\{\omega_i\in\mathbb{R}^2:|\omega_i|\le 2r^{-1}\}}\int_{I_i}\widehat{W}_{B_r}*\hat{f}_{\tau_1^i}(\mathfrak{m}(\xi_1^i)+(0,\omega_i))e^{2\pi ix\cdot(\mathfrak{m}(\xi_1^i)+(0,\omega_i))}\,d\xi_1^i\,d\omega^i,\end{aligned}$$

where $B_{r^{-1}}+\operatorname{supp} f_{\tau_1^i}\subset\{\mathfrak{m}(\xi_1^i)+(0,\omega_i):\xi_1^i\in I_1,\ |\omega_i|\le r^{-1}\}$. Let $\{\omega_i\in\mathbb{R}^2:|\omega_i|\le 2r^{-1}\}=B^{(2)}_{r^{-1}}$. Then for $\omega=(\omega_1,\omega_2,\omega_3)$, we have

$$\begin{aligned}&\int|W_{B_r}(x)f_{\tau_1^1}(x)W_{B_r}(x)f_{\tau_1^2}(x)W_{B_r}(x)f_{\tau_1^3}(x)|^2\,dx\\&\quad=\int_{B_r}\left|\prod_{i=1}^3\int_{B^{(2)}_{r^{-1}}}\int_{I_i}\widehat{W}_{B_r}*\hat{f}_{\tau_1^i}(\mathfrak{m}(\xi_1^i)+(0,\omega_i))e^{2\pi ix\cdot(\mathfrak{m}(\xi_1^i)+(0,\omega_i))}\,d\xi_1^i\,d\omega_i\right|^2dx\\&\quad\le\int_{B_r}\left|\int_{(B^{(2)}_{r^{-1}})^3}\left|\prod_{i=1}^3\int_{I_i}\widehat{W}_{B_r}*\hat{f}_{\tau_1^i}(\mathfrak{m}(\xi_1^i)+(0,\omega_i))e^{2\pi ix\cdot\mathfrak{m}(\xi_1^i)}\,d\xi_1^i\right|d\omega\right|^2dx\\&\quad\le\left(\int_{(B^{(2)}_{r^{-1}})^3}\left(\int_{B_r}\left|\prod_{i=1}^3\int_{I_i}\widehat{W}_{B_r}*\hat{f}_{\tau_1^i}(\mathfrak{m}(\xi_1^i)+(0,\omega_i))e^{2\pi ix\cdot\mathfrak{m}(\xi_1^i)}\,d\xi_1^i\right|^2dx\right)^{1/2}d\omega\right)^2.\qquad(13)\end{aligned}$$

For each $\omega\in(B^{(2)}_{r^{-1}})^3$, analyze the inner integral in x. Use the abbreviation $\widehat{W}_{B_r}*\hat{f}_{\tau_1^i}(\,\cdot\,+(0,\omega_i))=\hat{f}^{\omega_i}_{\tau_1^i}(\,\cdot\,)$ and further manipulate the innermost integral as a function of x:

$$\begin{aligned}&\prod_{i=1}^3\int_{I_i}\widehat{W}_{B_r}*\hat{f}_{\tau_i}(\mathfrak{m}(\xi_1^i)+(0,\omega_i))e^{2\pi ix\cdot\mathfrak{m}(\xi_1^i)}\,d\xi_1^i\\&\qquad=\int_{I_1\times I_2\times I_3}\hat{f}^{\omega_1}_{\tau_1^1}(\mathfrak{m}(\xi_1^1))\hat{f}^{\omega_2}_{\tau_1^2}(\mathfrak{m}(\xi_1^2))\hat{f}^{\omega_3}_{\tau_1^3}(\mathfrak{m}(\xi_1^3))e^{2\pi ix\cdot[\mathfrak{m}(\xi_1^i)+\mathfrak{m}(\xi_1^2)+\mathfrak{m}(\xi_1^3)]}\,d\xi_1,\end{aligned}$$

where $\xi_1=(\xi_1^1,\xi_1^2,\xi_1^3)$. Perform the change of variables $\tilde{\xi}=\mathfrak{m}(\xi_1^1)+\mathfrak{m}(\xi_1^2)+\mathfrak{m}(\xi_1^3)$. The Jacobian factor is $1/|\det J|$, where $\det J$ is defined explicitly in terms of ξ_1 by

$$\det\begin{bmatrix}1&1&1\\2\xi_1^1&2\xi_1^2&2\xi_1^3\\3(\xi_1^1)^2&3(\xi_1^2)^2&3(\xi_1^3)^2\end{bmatrix}=6(\xi_2-\xi_1)(\xi_3-\xi_1)(\xi_3-\xi_2),$$

using the formula for the determinant of a Vandermonde matrix. Note that since $(I_i,I_j)\ge s^{-1}-2r^{-1}>0$, $|\det J|$ is nonzero. The change of variables yields

$$\int_{\mathfrak{m}(I_1)+\mathfrak{m}(I_2)+\mathfrak{m}(I_3)}\hat{f}^{\omega_1}_{\tau_1^1}(\mathfrak{m}(\xi_1^1))\hat{f}^{\omega_2}_{\tau_1^2}(\mathfrak{m}(\xi_1^2))\hat{f}^{\omega_3}_{\tau_1^3}(\mathfrak{m}(\xi_1^3))e^{2\pi ix\cdot\tilde{\xi}}\frac{1}{|\det J(\xi_1)|}\,d\tilde{\xi},\qquad(14)$$

where we interpret ξ_1 in the integrand as implicitly depending on $\tilde{\xi}$. Define $F^\omega(\tilde{\xi})$ by

$$\chi_{\mathfrak{m}(I_1)+\mathfrak{m}(I_2)+\mathfrak{m}(I_3)}(\tilde{\xi})\hat{f}^{\omega_1}_{\tau_1^1}(\mathfrak{m}(\xi_1^1))\hat{f}^{\omega_2}_{\tau_1^2}(\mathfrak{m}(\xi_1^2))\hat{f}^{\omega_3}_{\tau_1^3}(\mathfrak{m}(\xi_1^3))\frac{1}{|\det J(\xi_1)|}$$

so that we may view the integral in (14) as the inverse Fourier transform of F^ω. The summary of the inequality so far, picking up from (13) and using the change of variables and the definition of F^ω, is

$$\int_{B_r} |f_{\tau_1^1}(x) f_{\tau_1^2}(x) f_{\tau_1^3}(x)|^2\,dx \lesssim \biggl(\int_{(B^{(2)}_{r-1})^3} \biggl(\int |\check{F}^\omega(x)|^2\,dx\biggr)^{1/2} d\omega\biggr)^2.$$

By Plancherel's theorem, the right-hand side above equals

$$\biggl(\int_{(B^{(2)}_{r-1})^3} \biggl(\int |F^\omega(\tilde{\xi})|^2\,d\tilde{\xi}\biggr)^{1/2} d\omega\biggr)^2.$$

By Cauchy–Schwarz, this is bounded above by

$$|(B^{(2)}_{r-1})^3| \int_{(B^{(2)}_{r-1})^3} \int |F^\omega(\tilde{\xi})|^2\,d\tilde{\xi}\,d\omega \sim r^{-6} \int_{(B^{(2)}_{r-1})^3} \int |F^\omega(\tilde{\xi})|^2\,d\tilde{\xi}\,d\omega.$$

Undo the change of variables, again writing $\tilde{\xi} = \mathfrak{m}(\xi_1^1) + \mathfrak{m}(\xi_1^2) + \mathfrak{m}(\xi_1^3)$ to get

$$r^{-6} \int_{(B^{(2)}_{r-1})^3} \int_{I_1\times I_2\times I_3} |\hat{f}^{\omega_1}_{\tau_1^1}(\mathfrak{m}(\xi_1^1))\,\hat{f}^{\omega_2}_{\tau_1^2}(\mathfrak{m}(\xi_1^2))\,\hat{f}^{\omega_3}_{\tau_1^3}(\mathfrak{m}(\xi_1^3))|^2\,|\det J(\xi_1)|^{-1}\,d\xi_1\,d\omega.$$

Note that $|\det J(\xi_1)| \gtrsim s^{-3}$, so the previous line is bounded by

$$r^{-6}s^3 \int_{(B^{(2)}_{r-1})^3} \int |\hat{f}^{\omega_1}_{\tau_1^1}(\mathfrak{m}(\xi_1^1))\,\hat{f}^{\omega_2}_{\tau_1^2}(\mathfrak{m}(\xi_1^2))\,\hat{f}^{\omega_3}_{\tau_1^3}(\mathfrak{m}(\xi_1^3))|^2\,d\xi_1\,d\omega \sim r^{-6}s^3 \prod_{i=1}^3 \int_{\mathcal{N}_{r-1}(\tau_i)} |\widehat{W}_{B_r} * \hat{f}_{\tau_1^i}(\xi)|^2\,d\xi.$$

By Plancherel's theorem, this is bounded by

$$r^{-6}s^3 \prod_{i=1}^3 \int_{\mathbb{R}^3} |f_{\tau_1^i}(x)|^2 W_{B_r}\,dx. \qquad \square$$

3. A weak version of Theorem 3 for the critical exponent

3.1. ***The broad part of U_α.*** For three canonical blocks $\tau_1^1, \tau_1^2, \tau_1^3$ (with dimensions $\sim R_1^{-1/3}\times R_1^{-2/3}\times R_1^{-1}$) which are pairwise $\ge 10\underline{C}_\varepsilon R^{-\varepsilon/3}$-separated, where $\underline{C}_\varepsilon$ is from Lemma 7, define the broad part of U_α to be

$$\mathrm{Br}_\alpha^K = \bigl\{x \in U_\alpha : \alpha \le K|f_{\tau_1^1}(x) f_{\tau_1^2}(x) f_{\tau_1^3}(x)|^{1/3},\ \max_{\tau_1^i} |f_{\tau_1^i}(x)| \le \alpha\bigr\}.$$

We bound the broad part of U_α in the following proposition.

Proposition 28. *Let $R, K \ge 1$. Suppose that $\|f_\gamma\|_{L^\infty(\mathbb{R}^3)} \le 2$ for all γ. Then*

$$\alpha^{6+2/\beta}|\mathrm{Br}_\alpha^K| \lesssim_\varepsilon K^{50} R^{10\varepsilon} A_\varepsilon^{10(M+N)} R^{2\beta+1} \sum_\gamma \|f_\gamma\|^2_{L^2(\mathbb{R}^3)}.$$

Proof of Proposition 28. Begin by observing that we may assume that $R^\beta \le \alpha^2$. Indeed, if $\alpha^2 \le R^\beta$, then we have

$$\alpha^{6+2/\beta}|U_\alpha| \le R^{2\beta+1}\|f\|^2_{L^2(\mathbb{R}^3)} \le R^{2\beta+1}\sum_\gamma \|f_\gamma\|_2^2$$

using L^2-orthogonality. Assume for the remainder of the argument that $R^\beta \le \alpha^2$.

We bound each of the sets $\mathrm{Br}_\alpha^K \cap \Lambda_k$, $\mathrm{Br}_\alpha^K \cap \Omega_m$, and $\mathrm{Br}_\alpha^K \cap L$ in separate cases. It suffices to consider the case that R is at least some constant depending on ε since if $R \le C_\varepsilon$, we may prove the proposition using trivial inequalities.

Case 1: bounding $|\mathrm{Br}_\alpha^K \cap \Lambda_k|$. By Lemma 16,

$$|\mathrm{Br}_\alpha^K \cap \Lambda_k| \le \big|\big\{x \in U_\alpha \cap \Lambda_k : \alpha \lesssim K |f_{\tau_1^1}^{k+1}(x) f_{\tau_1^2}^{k+1}(x) f_{\tau_1^3}^{k+1}(x)|^{1/3},\ \max_{\tau_1^i} |f_{\tau_1^i}(x)| \le \alpha\big\}\big|.$$

By Lemma 7, the Fourier supports of $f_{\tau_1^1}^{k+1}$, $f_{\tau_1^2}^{k+1}$, $f_{\tau_1^3}^{k+1}$ are contained in the $\underline{C}_\varepsilon r_k^{-1}$-neighborhood of $\underline{C}_\varepsilon \tau_1^1, \underline{C}_\varepsilon \tau_1^2, \underline{C}_\varepsilon \tau_1^3$ respectively, which are $\ge \underline{C}_\varepsilon R^{-\varepsilon/3}$-separated blocks of the moment curve. Let $\{B_{r_k}\}$ be a finitely overlapping cover of $\mathrm{Br}_\alpha^K \cap \Lambda_k$ by r_k-balls. For R large enough depending on ε, apply Proposition 27 to get

$$\int_{B_{r_k}} |f_{\tau_1^1}^{k+1} f_{\tau_1^2}^{k+1} f_{\tau_1^3}^{k+1}|^2 \lesssim_\varepsilon R^\varepsilon |B_{r_k}|^{-2} \Big(\int |f_{\tau_1^1}^{k+1}|^2 W_{B_{r_k}}\Big)\Big(\int |f_{\tau_1^2}^{k+1}|^2 W_{B_{r_k}}\Big)\Big(\int |f_{\tau_1^3}^{k+1}|^2 W_{B_{r_k}}\Big).$$

Using local L^2-orthogonality (Lemma 10), each integral on the right-hand side above is bounded by

$$C_\varepsilon \int \sum_{\tau_k} |f_{\gamma_k}^{k+1}|^2 W_{B_{r_k}}.$$

If $x \in \mathrm{Br}_\alpha^K \cap \Lambda_k \cap B_{r_k}$, then the above integral is bounded by

$$C_\varepsilon \int \sum_{\gamma_k} |f_{\gamma_k}^{k+1}|^2 * \omega_{\gamma_k} W_{B_{r_k}} \lesssim C_\varepsilon |B_{r_k}| \sum_{\gamma_k} |f_{\gamma_k}^{k+1}|^2 * \omega_{\gamma_k}(x)$$

by the locally constant property (Lemma 9) and properties of the weight functions. The summary of the inequalities so far is that

$$\alpha^6 |\mathrm{Br}_\alpha^K \cap \Lambda_k \cap B_{r_k}| \lesssim_\varepsilon K^6 \int_{B_{r_k}} |f_{\tau_1^1}^{k+1} f_{\tau_1^2}^{k+1} f_{\tau_1^3}^{k+1}|^2 \lesssim_\varepsilon R^\varepsilon K^6 |B_{r_k}| g_k(x)^3,$$

where $x \in \mathrm{Br}_\alpha^K \cap \Lambda_k \cap B_{r_k}$.

Recall that since $x \in \Lambda_k$, we have the lower bound $A_\varepsilon^{M-k} R^\beta \le g_k(x)$ (where A_ε is from Definition 13), which leads to the inequality

$$\alpha^6 |\mathrm{Br}_\alpha^K \cap \Lambda_k \cap B_{r_k}| \lesssim_\varepsilon K^6 R^\varepsilon \frac{1}{(A_\varepsilon^{M-k} R^\beta)^p} |B_{r_k}| g_k(x)^{3+p}$$

for any $p \ge 0$. By Corollary 15, we also have the upper bound $|g_k(x)| \le 2|g_k^h(x)|$, so that

$$\alpha^6 |\mathrm{Br}_\alpha^K \cap \Lambda_k \cap B_{r_k}| \lesssim_\varepsilon K^6 R^\varepsilon \frac{1}{(A_\varepsilon^{M-k} R^\beta)^p} |B_{r_k}| |g_k^h(x)|^{3+p}$$

for any $p \ge 0$. By the locally constant property applied to g_k^h, $|g_k^h|^{3+p} \lesssim_\varepsilon |g_k^h * w_{B_{r_k}}|^{3+p}$ and by Cauchy–Schwarz, $|g_k^h * w_{B_{r_k}}|^{3+p} \lesssim |g_k^h|^{3+p} * w_{B_{r_k}}$. Combine this with the previous displayed inequality to get

$$\alpha^6 |\mathrm{Br}_\alpha^K \cap \Lambda_k \cap B_{r_k}| \lesssim_\varepsilon K^6 R^\varepsilon \frac{1}{(A_\varepsilon^{M-k} R^\beta)^p} \int |g_k^h|^{3+p} W_{B_{r_k}}.$$

Summing over the balls B_{r_k} in our finitely overlapping cover of $\mathrm{Br}_\alpha^K \cap \Lambda_k$, we conclude that

$$\alpha^6 |\mathrm{Br}_\alpha^K \cap \Lambda_k| \lesssim_\varepsilon K^6 R^\varepsilon \frac{1}{(A_\varepsilon^{M-k} R^\beta)^p} \int_{\mathbb{R}^3} |g_k^h|^{3+p}. \tag{15}$$

We are done using the properties of the set $\mathrm{Br}_\alpha^K \cap \Lambda_k$, which is why we now integrate over all of $\mathbb{R}^3$ on the right-hand side. We will choose different $p > 0$ and analyze the high part g_k^h in two subcases which depend on the size of r_k.

Subcase 1a: $R^{-\beta} \le r_k^{-1} \le R^{-1/2}$. This case only appears if $\frac{1}{2} \le \beta$. Choose $p = 1$ in (15) and use Lemma 23 to obtain

$$\alpha^6 |\mathrm{Br}_\alpha^K \cap \Lambda_k| \lesssim_\varepsilon K^6 R^\varepsilon \frac{1}{A_\varepsilon^{M-k} R^\beta} C_\varepsilon R^\varepsilon r_k^{-1} R \sum_\zeta \Big\| \sum_{\gamma_k \subset \zeta} |f_{\gamma_k}^{k+1}|^2 * \omega_{\gamma_k} * \check{\rho}_{>r_{k+1}^{-1}} \Big\|_{L^4(\mathbb{R}^3)}^4,$$

where ζ are collections of $r_k^2 R^{-1}$ many adjacent γ_k.

The Fourier supports of the terms in the L^4 norm are still approximately disjoint (actually C_ε-overlapping, see Proposition 19), so by Plancherel's theorem and L^2-orthogonality, we have

$$\Big\| \sum_{\gamma_k \subset \zeta} |f_{\gamma_k}^{k+1}|^2 * \omega_{\gamma_k} * \check{\rho}_{>r_{k+1}^{-1}} \Big\|_{L^4(\mathbb{R}^3)}^4 \lesssim_\varepsilon R^\varepsilon \Big\| \sum_{\gamma_k \subset \zeta} |f_{\gamma_k}^{k+1}|^2 * \omega_{\gamma_k} * \check{\rho}_{>r_{k+1}^{-1}} \Big\|_{L^\infty(\mathbb{R}^3)}^2 \sum_{\gamma_k \subset \zeta} \| |f_{\gamma_k}^{k+1}|^2 * \omega_{\gamma_k} * \check{\rho}_{>r_{k+1}^{-1}} \|_{L^2(\mathbb{R}^3)}^2 \tag{16}$$

for each ζ. First bound the L^∞ norm by

$$\Big\| \sum_{\gamma_k \subset \zeta} |f_{\gamma_k}^{k+1}|^2 * \omega_{\gamma_k} * \check{\rho}_{>r_{k+1}^{-1}} \Big\|_{L^\infty(\mathbb{R}^3)}^2 \lesssim (\#\gamma_k \subset \zeta)^2 \max_{\gamma_k} \| f_{\gamma_k}^{k+1} \|_{L^\infty(\mathbb{R}^3)}^4 \lesssim (r_k^2 R^{-1})^2 \max_{\gamma_k} \| f_{\gamma_k}^{k+1} \|_{L^\infty(\mathbb{R}^3)}^4,$$

where we used that $\|\omega_k * \check{\rho}_{>r_{k+1}^{-1}}\|_1 \sim 1$. To bound each of the L^2 norms in (16), we use cylindrical L^4-decoupling the parabola (Theorem 26) and unravel the pruning process using properties from Lemma 7:

$$\begin{aligned} &\| |f_{\gamma_k}^{k+1}|^2 * \omega_{\gamma_k} * \check{\rho}_{>r_{k+1}^{-1}} \|_{L^2(\mathbb{R}^3)}^2 \\ &\quad \lesssim \| f_{\gamma_k}^{k+1} \|_{L^4(\mathbb{R}^3)}^4 && \text{(by Young's inequality)} \\ &\quad \lesssim_\varepsilon R^{\varepsilon^2} \Big(\sum_{\gamma_{k+1} \subset \gamma_k} \| f_{\gamma_{k+1}}^{k+1} \|_{L^4(\mathbb{R}^3)}^2 \Big)^2 && \text{(by cylindrical } L^4\text{-decoupling for } \mathbb{P}^1) \\ &\quad \lesssim \Big(\sum_{\gamma_{k+1} \subset \gamma_k} \| f_{\gamma_{k+1}}^{k+2} \|_{L^4(\mathbb{R}^3)}^2 \Big)^2 && \text{(by (1) from Lemma 7)} \\ &\quad \lesssim \cdots \lesssim \Big(\sum_{\gamma_N \subset \gamma_k} \| f_{\gamma_N}^N \|_{L^4(\mathbb{R}^3)}^2 \Big)^2 \lesssim \Big(\sum_{\gamma \subset \gamma_k} \| f_\gamma \|_{L^4(\mathbb{R}^3)}^2 \Big)^2 && \text{(by iterating the previous two inequalities).} \end{aligned}$$

Note that each application of L^4-decoupling involves an explicit constant B in the upper bound, so it does not depend on a scale R. The accumulated constant in the unwinding-the-pruning process above is $B^{C\varepsilon^{-1}}$

since there are fewer than $\sim \varepsilon^{-1}$ many different scales of γ_k until we arrive at γ. Use Cauchy–Schwarz to bound the expression in the final upper bound above by

$$\#\gamma \subset \gamma_k \sum_{\gamma\subset\gamma_k} \|f_\gamma\|_{L^4(\mathbb{R}^3)}^4 \lesssim (r_k^{-1}R^\beta) \sum_{\gamma\subset\gamma_k} \|f_\gamma\|_{L^4(\mathbb{R}^3)}^4.$$

Using the assumption $\|f_\gamma\|_\infty \lesssim 1$ for each γ, we have $\|f_\gamma\|_{L^4(\mathbb{R}^4)}^4 \lesssim \|f_\gamma\|_{L^2(\mathbb{R}^3)}^2$. The summary of the argument in this case so far is that

$$\begin{aligned}\alpha^6|\mathrm{Br}_\alpha^K \cap \Lambda_k| &\lesssim_\varepsilon K^6R^{2\varepsilon}R^{-\beta}r_k^{-1}R\sum_\zeta (r_k^2R^{-1})^2 \max_{\gamma_k}\|f_{\gamma_k}^{k+1}\|_\infty^4 (r_k^{-1}R^\beta)\sum_{\gamma_k\subset\zeta}\|f_\gamma\|_2^2\\ &\lesssim_\varepsilon K^6R^{2\varepsilon}r_k^2R^{-1}\max_{\gamma_k}\|f_{\gamma_k}^{k+1}\|_\infty^4\sum_\gamma\|f_\gamma\|_2^2.\end{aligned}$$

For the remainder of the proof, we use the notation $\lessapprox$ to mean $\lesssim_\varepsilon R^{8\varepsilon}$. It now suffices to verify that $r_k^2R^{-1}\max_{\gamma_k}\|f_{\gamma_k}^{k+1}\|_\infty^4 \lessapprox R^{2\beta+1}\alpha^{-2/\beta}$. We will use the upper bounds

$$\|f_{\gamma_k}^{k+1}\|_\infty \lesssim \min\left(r_k^{-1}R^\beta,\, K^3A_\varepsilon^{M-k}\frac{R^\beta}{\alpha}\right)$$

(from (1) and (2) in Lemma 7). Suppose that $r_k < \alpha$. Use $\|f_{\gamma_k}^{k+1}\|_\infty \lesssim K^3A_\varepsilon^{M-k}R^\beta/\alpha$ and $\beta \geq \frac{1}{2}$ to check

$$\begin{aligned}(r_k)^{2/\beta-2}\leq(R^\beta)^{2/\beta-2} \quad &\Longrightarrow \quad r_k^2R^{-1+4\beta}\leq R^{2\beta+1}r_k^{4-2/\beta}\\ &\Longrightarrow \quad r_k^2R^{-1}\left(\frac{R^\beta}{\alpha}\right)^4 \leq R^{2\beta+1}\alpha^{-2/\beta}\\ &\Longrightarrow \quad r_k^2R^{-1}\max_{\gamma_k}\|f_{\gamma_k}^{k+1}\|_\infty^4 \lesssim A_\varepsilon^{4(M-k)}R^{2\beta+1}\alpha^{-2/\beta},\end{aligned}$$

as desired. Now suppose that $r_k \geq \alpha$. Then use $\|f_{\gamma_k}^{k+1}\|_\infty \lesssim r_k^{-1}R^\beta$ and check

$$\begin{aligned}(r_k)^{2/\beta-2}\leq(R^\beta)^{2/\beta-2} \quad &\Longrightarrow \quad r_k^2R^{-1}(r_k^{-1}R^\beta)^4\leq R^{2\beta+1}(r_k)^{-2/\beta}\\ &\Longrightarrow \quad r_k^2R^{-1}\max_{\gamma_k}\|f_{\gamma_k}^{k+1}\|_\infty^4 \lesssim R^{2\beta+1}(\alpha)^{-2/\beta},\end{aligned}$$

which finishes this subcase.

Subcase 1b: $\max(R^{-\beta}, R^{-1/2}) \leq r_k^{-1} \leq R^{-1/3}$. In this case, let $\beta_1 \in \left[\frac{1}{2}, 1\right]$ satisfy $(r_k^{-1}R)^{-\beta_1} = r_k^{-1}$ and take $p = 2/\beta_1 - 1$ in (15). Then by Lemma 24

$$\alpha^6|\mathrm{Br}_\alpha^K\cap\Lambda_k| \lesssim_\varepsilon K^6R^\varepsilon\frac{1}{R^{\beta(2/\beta_1-1)}}C_\varepsilon R^\varepsilon r_k^{-1}R\sum_{\gamma_k}\|f_{\gamma_k}^{k+1}\|_{L^{4+4/\beta_1}(\mathbb{R}^3)}^{4+4/\beta_1}.$$

Majorize each L^{4+4/β_1} norm by a combination of L^∞ and L^6 norms to get

$$\alpha^6|\mathrm{Br}_\alpha^K\cap\Lambda_k| \lesssim_\varepsilon K^6R^{2\varepsilon}\frac{1}{R^{\beta(2/\beta_1-1)}}r_k^{-1}R\sum_{\gamma_k}\max_{\gamma_k}\|f_{\gamma_k}^{k+1}\|_\infty^{4/\beta_1-2}\|f_{\gamma_k}^{k+1}\|_{L^6(\mathbb{R}^3)}^6.$$

Repeat the unwinding-the-pruning argument from Subcase 1a to obtain

$$\|f_{\gamma_k}^{k+1}\|_{L^6(\mathbb{R}^3)}^6 \lesssim B_{\varepsilon^5} R^{\varepsilon^4}\Big(\sum_{\gamma\subset\gamma_k}\|f_\gamma\|_{L^6(\mathbb{R}^3)}^2\Big)^3 \lesssim B_{\varepsilon^5} R^{\varepsilon^4}(r_k^{-1}R^\beta)^2\sum_{\gamma\subset\gamma_k}\|f_\gamma\|_{L^2(\mathbb{R}^3)}^2,$$

where we used Cauchy–Schwarz and the assumption $\|f_\gamma\|_\infty \lesssim 1$ in the final inequality. Note that we have the additional constant $B_{\varepsilon^5}^{\varepsilon^{-1}} R^{\varepsilon^4}$ due the accumulation of $\le \varepsilon^{-1}$ many factors of the upper bound $B_{\varepsilon^5} R^{\varepsilon^5}$ for L^6-decoupling of the parabola with small parameter ε^5. In summary,

$$\alpha^6|\mathrm{Br}_\alpha^K\cap\Lambda_k| \lesssim_\varepsilon K^6R^{3\varepsilon}\frac{1}{R^{\beta(2/\beta_1-1)}}r_k^{-1}R\sum_{\gamma_k}\max_{\gamma_k}\|f_{\gamma_k}^{k+1}\|_\infty^{4/\beta_1-2}(r_k^{-1}R^\beta)^2\sum_{\gamma\subset\gamma_k}\|f_\gamma\|_{L^2(\mathbb{R}^3)}^2.$$

It suffices to check that

$$\frac{1}{R^{\beta(2/\beta_1-1)}}r_k^{-1}R\max_{\gamma_k}\|f_{\gamma_k}^{k+1}\|_\infty^{4/\beta_1-2}(r_k^{-1}R^\beta)^2 \lessapprox R^{2\beta+1}\alpha^{-2/\beta},$$

which simplifies to

$$R^{\beta(1-2/\beta_1)}r_k^{-3}\max_{\gamma_k}\|f_{\gamma_k}\|_\infty^{4/\beta_1-2}\lessapprox \alpha^{-2/\beta}.$$

Using $\|f_{\gamma_k}^{k+1}\|_\infty \le K^3A_\varepsilon^{(M-k)}R^\beta/\alpha$, it further suffices to verify the inequality $r_k^{-3}R^{\beta(2/\beta_1-1)} \lessapprox \alpha^{4/\beta_1-2-2/\beta}$.

Suppose that the exponent $4/\beta_1-2-2/\beta\ge 0$. Use $r_k^{-1}\le R^{-1/3}$ and $R^\beta\le\alpha^2$ to verify

$$(R^\beta)^{2/\beta_1-1-1/\beta}\le(\alpha^2)^{2/\beta_1-1-1/\beta} \quad\Longrightarrow\quad r_k^{-3}R^{\beta(2/\beta_1-1)}\le\alpha^{4/\beta_1-2-2/\beta}.$$

Now suppose that the exponent $4/\beta_1-2-2/\beta<0$. Using Cauchy–Schwarz, the locally constant property, and the definition of Λ_k, for $x\in U_\alpha\cap\Lambda_k$, we have

$$\alpha^2\lesssim \#\gamma_{k+1}\sum_{\gamma_{k+1}}|f_{\gamma_{k+1}}^{k+2}|^2\lesssim R^\varepsilon r_k g_{k+1}(x)\lesssim R^\varepsilon r_k A_\varepsilon^{(M-k-1)}R^\beta.$$

Also use $r_k^{1/\beta_1}=r_k^{-1}R$ to verify

$$\begin{aligned} R^{-1}\le r_k^{-1/\beta} \quad&\Longrightarrow\quad r_k^{-3}R\le (r_k^{-1}R)^2r_k^{-1-1/\beta}\\ &\Longrightarrow\quad r_k^{-3}R\le r_k^{2/\beta_1-1-1/\beta}\\ &\Longrightarrow\quad r_k^{-3}R(R^\varepsilon A_\varepsilon^{(M-k-1)}R^\beta)^{2/\beta_1-1-1/\beta}\le(\alpha^2)^{2/\beta_1-1-1/\beta}\\ &\Longrightarrow\quad r_k^{-3}R^{\beta(2/\beta_1-1)}\le (R^\varepsilon A_\varepsilon^{(M-k-1)})^8\alpha^{4/\beta_1-2-2/\beta},\end{aligned}$$

as desired.

Case 2: bounding $|\mathrm{Br}_\alpha^K\cap\Omega_m|$. Repeat the reasoning at the beginning of Case 1. By Lemma 16,

$$|\mathrm{Br}_\alpha^K\cap\Omega_m|\le\big|\big\{x\in U_\alpha\cap\Omega_m:\alpha\lesssim K|F_{\tau_1^1}^{m+1}(x)F_{\tau_1^2}^{m+1}(x)F_{\tau_1^3}^{m+1}(x)|^{1/3},\ \max_{\tau_1^i}|f_{\tau_1^i}(x)|\le\alpha\big\}\big|.$$

Let $\{B_{R_m^{1/3}}\}$ be a finitely overlapping cover of $\mathrm{Br}_\alpha^K\cap\Omega_m$ by $R_m^{1/3}$-balls. Then by Proposition 27, for R large enough depending on ε,

$$\int_{B_{R_m^{1/3}}}|F_{\tau_1^1}^{m+1}F_{\tau_1^2}^{m+1}F_{\tau_1^3}^{m+1}|^2\lesssim_\varepsilon R^\varepsilon|B_{R_m^{1/3}}|^{-2}\Big(\int|F_{\tau_1^1}^{m+1}|^2W_{B_{R_m^{1/3}}}\Big)\Big(\int|F_{\tau_1^2}^{m+1}|^2W_{B_{R_m^{1/3}}}\Big)\Big(\int|F_{\tau_1^3}^{m+1}|^2W_{B_{R_m^{1/3}}}\Big).$$

The integrals on the right-hand side are bounded by

$$C_\varepsilon \int \sum_{\tau_m} |F_{\tau_m}^{m+1}|^2 W_{B_{R_m^{1/3}}}$$

using local L^2-orthogonality (Lemma 10). If $x \in \mathrm{Br}_\alpha^K \cap \Omega_m \cap B_{R_m^{1/3}}$, then the above integral is bounded by

$$C_\varepsilon \int \sum_{\tau_m} |F_{\tau_m}^{m+1}|^2 * \omega_{\tau_m} W_{B_{R_m^{1/3}}} \lesssim C_\varepsilon \sum_{\tau_m} |F_{\tau_m}^{m+1}|^2 * \omega_{\tau_m}(x) = C_\varepsilon G_m(x)$$

by the locally constant property. Recall that since $x \in \Omega_m$, we have the lower bound $A_\varepsilon^{M+N-m} R^\beta \le G_m(x)$. Also, by Corollary 15, $G_m(x) \le 2|G_m^h(x)|$. Combining the information so far yields

$$\alpha^6 |\mathrm{Br}_\alpha^K \cap \Omega_m \cap B_{R_m^{1/3}}| \lesssim_\varepsilon K^6 R^\varepsilon \frac{1}{(A_\varepsilon^{M+N-m} R^\beta)^3} |B_{R_m^{1/3}}| \, |G_m^h(x)|^6.$$

Use the locally constant property for G_m^h and sum over all $B_{R_m^{1/3}}$ to get

$$\alpha^6 |\mathrm{Br}_\alpha^K \cap \Omega_m| \lesssim_\varepsilon K^6 R^\varepsilon \frac{1}{R^{3\beta}} \int_{\mathbb{R}^3} |G_m^h|^6.$$

Note that we dropped the unnecessary factors of $A_\varepsilon^{M+N-m} \ge 1$ and that we are done using the properties of the set $\mathrm{Br}_\alpha^{R_m^{1/3}}(\tau, \tau', \tau'')$, which is why we now integrate over all of $\mathbb{R}^3$ on the right-hand side.

By Lemma 25,

$$\int_{\mathbb{R}^3} |G_m^h|^6 \lesssim_\varepsilon R^\varepsilon \Big(\sum_{\tau_m} \|F_{\tau_m}^{m+1}\|_{L^{12}(\mathbb{R}^3)}^4 \Big)^3.$$

Use Cauchy–Schwarz and then (2) (with $F_{\tau_{m+1}}^{m+1}$) of Lemma 7 to bound the L^{12} norm by a combination of L^∞ and L^6 norms:

$$\Big(\sum_{\tau_m} \|F_{\tau_m}^{m+1}\|_{L^{12}(\mathbb{R}^3)}^4 \Big)^3 \le R^\varepsilon K^6 \Big(K^3 A_\varepsilon^{M+N-m} \frac{R^\beta}{\alpha} \Big)^6 \Big(\sum_{\tau_{m+1}} \|F_{\tau_{m+1}}^{m+1}\|_{L^6(\mathbb{R}^3)}^2 \Big)^3.$$

Next, we use cylindrical L^6-decoupling over the parabola to unwind the pruning process. For each τ_{m+1}, we have

$$\begin{aligned} \|F_{\tau_{m+1}}^{m+1}\|_{L^6(\mathbb{R}^3)}^6 &\le \|F_{\tau_{m+1}}^{m+2}\|_{L^6(\mathbb{R}^3)}^6 && \text{(by (1) of Lemma 7)} \\ &\le B_{\varepsilon^5} R^{\varepsilon^5} \Big(\sum_{\tau_{m+2} \subset \tau_{m+1}} \|f_{\tau_{m+2}}^{m+2}\|_{L^6(\mathbb{R}^3)}^2 \Big)^3 && \text{(by cylindrical } L^6\text{-decoupling for } \mathbb{P}^1) \\ &\le \cdots \le (B_{\varepsilon^5} R^{\varepsilon^5})^N \Big(\sum_{\tau_N \subset \tau_{m+1}} \|f_{\tau_N}^{N+1}\|_{L^6(\mathbb{R}^3)}^2 \Big)^3 && \text{(by iterating the previous two inequalities).} \end{aligned}$$

Note that $\{\tau_N\}$ are canonical blocks of the moment curve. Our goal is to have an expression involving the small caps γ. We defined the γ so that they lie in the cylindrical region over canonical $R^{-\beta} \times R^{-2\beta}$ blocks of $\mathbb{P}^1$. Therefore, we may continue unwinding the pruning process using Theorem 26, ultimately obtaining

$$\Big(\sum_{\tau_{m+1}}\|F^{m+1}_{\tau_{m+1}}\|^2_{L^6(\mathbb{R}^3)}\Big)^3 \le (B_{\varepsilon^5}R^{\varepsilon^5})^{M+N}\Big(\sum_{\gamma}\|f_\gamma\|^2_{L^6(\mathbb{R}^3)}\Big)^3.$$

By Cauchy–Schwarz and using the assumption $\|f_\gamma\|_\infty \lesssim 1$, we have

$$\Big(\sum_{\gamma}\|f_\gamma\|^2_{L^6(\mathbb{R}^3)}\Big)^3 \le \#\gamma^2\sum_\gamma\|f_\gamma\|^6_{L^6(\mathbb{R}^3)} \lesssim R^{2\beta}\sum_\gamma\|f_\gamma\|^2_{L^2(\mathbb{R}^3)}.$$

The summary in this case is that

$$\alpha^6|\mathrm{Br}^K_\alpha\cap\Omega_m| \lesssim_\varepsilon K^{30}R^{3\varepsilon}A_\varepsilon^{10(M+N)}\frac{1}{R^{3\beta}}\Big(\frac{R^\beta}{\alpha}\Big)^6(R^{2\beta})\sum_\gamma\|f_\gamma\|^2_{L^2(\mathbb{R}^3)}.$$

It suffices to verify that $R^{5\beta}\alpha^{-6}\le R^{2\beta+1}\alpha^{-2/\beta}$. This follows immediately from the relation $R^\beta\le\alpha^2$.

Case 3: bounding $|U_\alpha\cap L|$. Begin by using Lemma 16 to bound

$$\alpha^{6+2/\beta}|\mathrm{Br}^K_\alpha\cap L| \lesssim K^{12}\int_{U_\alpha\cap L}|f|^2|F_1|^{4+2/\beta}.$$

Then use Cauchy–Schwarz and the locally constant property for G_1 to get

$$\int_{U_\alpha\cap L}|f|^2|F_1|^{4+2/\beta} \lesssim_\varepsilon R^\varepsilon\int_{U_\alpha\cap L}|f|^2G_1^{2+1/\beta}.$$

Using the definition of L, we bound the factors of G_1 by

$$\int_{U_\alpha\cap L}|f|^2(A_\varepsilon^{M+N}R^\beta)^{2+1/\beta}.$$

Finally, use L^2 orthogonality to conclude

$$\alpha^{6+2/\beta}|\mathrm{Br}^K_\alpha\cap L| \lesssim_\varepsilon K^{12}R^{2\varepsilon}A_\varepsilon^{10(M+N)}R^{2\beta+1}\sum_\gamma\|f_\gamma\|^2_{L^2(\mathbb{R}^3)}. \qquad \square$$

3.2. ***Wave packet decomposition and pigeonholing.*** To prove Theorem 3, it suffices to prove a local version presented in the next lemma.

Lemma 29. *Let $\frac13\le\beta\le1$ and $p\ge2$. Then, for any $R^{\max(2\beta,1)}$-ball $B_{R^{\max(2\beta,1)}}\subset\mathbb{R}^3$, suppose that*

$$\|f\|^p_{L^p(B_{R^{\max(2\beta,1)}})} \le C_\varepsilon R^\varepsilon(R^{\beta(p/2-1)}+R^{\beta(p-4)-1})\sum_\gamma\|f_\gamma\|^p_{L^p(\mathbb{R}^3)}$$

for any Schwartz function $f:\mathbb{R}^3\to\mathbb{C}$ with Fourier transform supported in $\mathcal{M}^3(R^\beta,R)$. Then Theorem 3 is true.

Proof. Write

$$\|f\|^p_{L^p(\mathbb{R}^3)} \lesssim \sum_{B_{R^{\max(2\beta,1)}}}\int_{B_{R^{\max(2\beta,1)}}}|f|^p,$$

where the sum is over a finitely overlapping cover of $\mathbb{R}^3$ by $R^{\max(2\beta,1)}$-balls. Let ϕ_B be a weight function decaying by order 100 away from $B_{R^{\max(2\beta,1)}}$, satisfying $\phi_B \gtrsim 1$ on $B_{R^{\max(2\beta,1)}}$, and with Fourier transform supported in an $R^{-\max(2\beta,1)}$ neighborhood of the origin. The Fourier support of each $f_\gamma\phi_B$ is contained in a $2R^{-\beta}\times 4R^{-2\beta}\times 2^{1/\beta}R^{-1}$ small cap. By the triangle inequality, there is a subset $\mathcal{S}$ of the small caps γ so that for each $\gamma\in\mathcal{S}$, the Fourier support of $f_\gamma\phi_B$ is contained in a unique small cap and

$$\|f\|^p_{L^p(B_{R^{\max(2\beta,1)}})} \lesssim \Big\|\sum_{\gamma\in\mathcal{S}} f_\gamma\phi_B\Big\|^p_{L^p(B_{R^{\max(2\beta,1)}})}.$$

Then by applying the hypothesized local version of small cap decoupling,

$$\Big\|\sum_{\gamma\in\mathcal{S}} f_\gamma\phi_B\Big\|^p_{L^p(B_{R^{\max(2\beta,1)}})} \le C_\varepsilon R^\varepsilon (R^{\beta(p/2-1)} + R^{\beta(p-4)-1}) \sum_{\gamma\in\mathcal{S}} \|f_\gamma\phi_B\|^p_{L^p(\mathbb{R}^3)}.$$

It remains to note that $\sum_{B_{R^{\max(2\beta,1)}}} \int |f_\gamma|^p\phi_B^p \lesssim \int |f_\gamma|^p$. □

It further suffices to prove a weak, level-set version of Theorem 3.

Lemma 30. *Let $p\ge 2$. For each B_{R^2} and Schwartz function $f:\mathbb{R}^3\to\mathbb{C}$ with Fourier transform supported in $\mathcal{M}^3(R^\beta, R)$, there exists $\alpha>0$ such that*

$$\|f\|^p_{L^p(B_{R^{\max(2\beta,1)}})} \lesssim_p (\log R)\alpha^p |\{x\in B_{R^{\max(2\beta,1)}} : \alpha \le |f(x)|\}| + R^{-500p}\sum_\gamma \|f_\gamma\|^p_{L^p(\mathbb{R}^3)}.$$

Proof. Split the integral as

$$\int_{B_{R^{\max(2\beta,1)}}} |f|^p = \sum_{R^{-1000}\le\lambda\le 1} \int_{\{x\in B_{R^{\max(2\beta,1)}} : \lambda\|f\|_{L^\infty(B_{R^{\max(2\beta,1)}})} \le |f(x)| \le 2\lambda \|f\|_{L^\infty(B_{R^{\max(2\beta,1)}})}\}} |f|^p \\ + \int_{\{x\in B_{R^{\max(2\beta,1)}} : |f(x)|\le R^{-1000}\|f\|_{L^\infty(R^{\max(2\beta,1)})}\}} |f|^p,$$

in which λ varies over dyadic values in the range $[R^{-1000}, 1]$. If one of the $\lesssim \log R$ many terms in the first sum dominates, then we are done. Suppose instead that the second expression dominates:

$$\int_{B_{R^{\max(2\beta,1)}}} |f|^p \le 2\int_{\{x\in B_{R^{\max(2\beta,1)}} : |f(x)|\le R^{-1000}\|f\|_{L^\infty(R^{\max(2\beta,1)})}\}} |f|^p \lesssim R^3 R^{-1000p}\|f\|^p_{L^\infty(B_{R^{\max(2\beta,1)}})}.$$

Then by Hölder's inequality, we have

$$\int_{B_{R^{\max(2\beta,1)}}} |f|^p \lesssim R^3 R^{-1000p+(p-1)} \sum_\gamma \|f_\gamma\|^p_{L^\infty(B_{R^{\max(2\beta,1)}})}.$$

Finally, by the locally constant property and Hölder's inequality,

$$\|f_\gamma\|^p_{L^\infty(B_{R^{\max(2\beta,1)}})} \lesssim \||f_\gamma| * \omega_{\gamma^*}\|^p_{L^\infty(B_{R^{\max(2\beta,1)}})} \lesssim_p \||f_\gamma|^p * \omega_{\gamma^*}\|_{L^\infty(B_{R^{\max(2\beta,1)}})} \lesssim \int_{\mathbb{R}^3} |f_\gamma|^p.$$ □

Use the notation

$$U_\alpha = \{x\in B_{R^{\max(2\beta,1)}} : \alpha\le |f(x)|\}.$$

We will show that to estimate the size of U_α, it suffices to replace f with a version whose wave packets have been pigeonholed. Write

$$f = \sum_{\gamma} \sum_{T\in\mathbb{T}_\gamma} \psi_T f_\gamma, \tag{17}$$

where, for each γ, $\{\psi_T\}_{T\in\mathbb{T}_\gamma}$ is the partition of unity from a partition of unity from Section 2.1. If

$$\alpha \le C_\varepsilon (\log R) R^{-500} \max_{\gamma} \|f_\gamma\|_\infty,$$

then by an argument analogous to the one dealing with the low integral over $\{x : |f(x)| \le R^{-1000}\|f\|_\infty\}$ in the proof of Lemma 30, bounding $\alpha^p |U_\alpha|$ by the right-hand side of the small cap decoupling theorem is trivial. Let ϕ_B be the weight function from Lemma 29.

Proposition 31 (wave packet decomposition). *Let* $\alpha > C_\varepsilon (\log R) R^{-100} \max_\gamma \|f_\gamma\|_{L^\infty(\mathbb{R}^3)}$. *There exist subsets* $\mathcal{S} \subset \{\gamma\}$ *and* $\tilde{\mathbb{T}}_\gamma \subset \mathbb{T}_\gamma$, *as well as a constant* $A > 0$ *with the following properties*:

$$|U_\alpha| \lesssim (\log R)\left|\left\{x \in U_\alpha : \alpha \lesssim \left|\sum_{\gamma\in\mathcal{S}} \sum_{T\in\tilde{\mathbb{T}}_\gamma} \psi_T(x)\phi_B(x) f_\gamma(x)\right|\right\}\right|, \tag{18}$$

$$\left\|\sum_{T\in\tilde{\mathbb{T}}_\gamma} \psi_T \phi_B f_\gamma\right\|_{L^\infty(\mathbb{R}^3)} \sim A \quad \textit{for all } \gamma \in \mathcal{S}, \tag{19}$$

$$\#\tilde{\mathbb{T}}_\gamma A^p R^{\beta+2\beta+1} \lesssim \left\|\sum_{T\in\tilde{\mathbb{T}}_\gamma} \psi_T \phi_B f_\gamma\right\|_{L^p(\mathbb{R}^3)} \lesssim R^{3p\varepsilon} \#\tilde{\mathbb{T}}_\gamma A^p R^{\beta+2\beta+1} \quad \textit{for all } \gamma \in \mathcal{S}. \tag{20}$$

Proof. Split the sum (17) into

$$\phi_B f = \sum_{\gamma} \sum_{T\in\mathbb{T}_\gamma^c} \psi_T \phi_B f_\gamma + \sum_{\gamma} \sum_{T\in\mathbb{T}_\gamma^f} \psi_T \phi_B f_\gamma, \tag{21}$$

where the close set is

$$\mathbb{T}_\gamma^c := \{T \in \mathbb{T}_\gamma : T \cap R^{10} B_{R^{\max(2\beta,1)}} \ne \varnothing\}$$

and the far set is

$$\mathbb{T}_\gamma^f := \{T \in \mathbb{T}_\gamma : T \cap R^{10} B_{R^{\max(2\beta,1)}} = \varnothing\}.$$

Using decay properties of the partition of unity, for each $x \in B_{R^{\max(2\beta,1)}}$,

$$\left|\sum_{\gamma} \sum_{T\in\mathbb{T}_\gamma^f} \psi_T(x)\phi_B(x) f_\gamma(x)\right| \lesssim R^{-1000} \max_{\gamma} \|\phi_B f_\gamma\|_{L^\infty(B_{R^{\max(2\beta,1)}})}.$$

Therefore, using the assumption that α is at least $R^{-100} \max_\gamma \|f_\gamma\|_{L^\infty(B_{R^{\max(2\beta,1)}})}$,

$$|U_\alpha| \le 2\left|\left\{x \in U_\alpha : \alpha \le 2\left|\sum_{\gamma} \sum_{T\in\tilde{\mathbb{T}}_\gamma^c} \psi_T(x)\phi_B(x) f_\gamma(x)\right|\right\}\right|.$$

The close set has cardinality $|\mathbb{T}_\gamma^c| \le R^{33}$. Let

$$M = \max_{\gamma} \max_{T\in\mathbb{T}_\gamma^c} \|\psi_T \phi_B f_\gamma\|_{L^\infty(\mathbb{R}^3)}. \tag{22}$$

Split the remaining wave packets into

$$\sum_{\gamma}\sum_{T\in\mathbb{T}_\gamma^c}\psi_T\phi_B f_\gamma=\sum_{\gamma}\sum_{R^{-10^3}\le\lambda\le 1}\sum_{T\in\mathbb{T}^c_{\gamma,\lambda}}\psi_T\phi_B f_\gamma+\sum_{\gamma}\sum_{T\in\mathbb{T}^c_{\gamma,s}}\psi_T\phi_B f_\gamma, \tag{23}$$

where λ is a dyadic number in the range $[R^{-10^3}, 1]$,

$$\mathbb{T}^c_{\gamma,\lambda}:=\{T\in\mathbb{T}^c_\gamma:\|\psi_T\phi_B f_\gamma\|_{L^\infty(\mathbb{R}^3)}\sim\lambda M\},$$
$$\mathbb{T}^c_{\gamma,s}:=\{T\in\mathbb{T}^c_\gamma:\|\psi_T\phi_B f_\gamma\|_{L^\infty(\mathbb{R}^3)}\le R^{-1000}M\}.$$

Again using the lower bound for α, the small wave packets cannot dominate and we have

$$|U_\alpha|\le 4\left|\left\{x\in U_\alpha:\alpha\le 4\left|\sum_{\gamma}\sum_{R^{-10^3}\le\lambda\le 1}\sum_{T\in\mathbb{T}^c_{\gamma,\lambda}}\psi_T(x)\phi_B(x)f_\gamma(x)\right|\right\}\right|.$$

By dyadic pigeonholing, for some $\lambda\in[R^{-1000}, 1]$,

$$|U_\alpha|\lesssim(\log R)\left|\left\{x\in U_\alpha:\alpha\lesssim(\log R)\left|\sum_{\gamma}\sum_{T\in\mathbb{T}^c_{\gamma,\lambda}}\psi_T(x)\phi_B(x)f_\gamma(x)\right|\right\}\right|.$$

Finally, we analyze the L^p norm for each $p\ge 2$ and each γ. Note that we have the pointwise inequality

$$\begin{aligned}\left|\sum_{T\in\mathbb{T}^c_{\gamma,\lambda}}\psi_T(x)\phi_B(x)f_\gamma(x)\right|&=\left|\sum_{\substack{T\in\mathbb{T}^c_{\gamma,\lambda}\\ x\in R^\varepsilon T}}\psi_T(x)\phi_B(x)f_\gamma(x)\right|+\left|\sum_{\substack{T\in\mathbb{T}^c_{\gamma,\lambda}\\ x\notin R^\varepsilon T}}\psi_T(x)\phi_B(x)f_\gamma(x)\right|\\&\le\left|\sum_{\substack{T\in\mathbb{T}^c_{\gamma,\lambda}\\ x\in R^\varepsilon T}}\psi_T(x)\phi_B(x)f_\gamma(x)\right|+C_\varepsilon R^{-1000}|\phi_B(x)f_\gamma(x)|.\end{aligned}$$

Let $\mathcal{S}'$ be the subset of $\{\gamma\}$ for which

$$\left\|\sum_{T\in\mathbb{T}^c_{\gamma,\lambda}}\psi_T\phi_B f_\gamma\right\|_{L^\infty(\mathbb{R}^3)}\ge C_\varepsilon R^{-500}\max_\gamma\|\phi_B f_\gamma\|_{L^\infty(\mathbb{R}^3)}.$$

Using the lower bound for α, we then have

$$|U_\alpha|\lesssim(\log R)\left|\left\{x\in U_\alpha:\alpha\lesssim(\log R)\left|\sum_{\gamma\in\mathcal{S}'}\sum_{T\in\mathbb{T}^c_{\gamma,\lambda}}\psi_T(x)\phi_B(x)f_\gamma(x)\right|\right\}\right|.$$

It follows from the pointwise inequality above that, for each $\gamma\in\mathcal{S}'$,

$$\lambda M\lesssim\left\|\sum_{T\in\mathbb{T}^c_{\gamma,\lambda}}\psi_T\phi_B f_\gamma\right\|_{L^\infty(\mathbb{R}^3)}\lesssim R^{3\varepsilon}\lambda M.$$

Perform one more dyadic pigeonholing step to obtain a dyadic $\mu\in[1, R^\varepsilon]$ for which

$$|U_\alpha|\lesssim(\log R)^2\left|\left\{x\in U_\alpha:\alpha\lesssim(\log R)^2\left|\sum_{\gamma\in\mathcal{S}}\sum_{T\in\mathbb{T}^c_{\gamma,\lambda}}\psi_T(x)\phi_B(x)f_\gamma(x)\right|\right\}\right|,$$

where $\mathcal{S}$ is the set of γ satisfying $\|\sum_{T\in\mathbb{T}^c_{\gamma,\lambda}}\psi_T\phi_B f_\gamma\|_{L^\infty(\mathbb{R}^3)}\sim\mu M$.

It remains to check the property about the L^p norms. For each $\gamma \in \mathcal{S}$, using the locally constant property, we have

$$\begin{aligned}\#\mathbb{T}^c_{\gamma,\lambda} R^{\beta+2\beta+1}(\mu M)^p &\lesssim \sum_{T\in\mathbb{T}^c_{\gamma,\lambda}} \int |\psi_T \phi_B f_\gamma|^p \lesssim \int \Big| \sum_{T\in\mathbb{T}^c_{\gamma,\lambda}} \psi_T \phi_B f_\gamma \Big|^p \\ &\lesssim \int \Big| \sum_{\substack{T\in\mathbb{T}^c_{\gamma,\lambda}\\ x\in R^\varepsilon T}} \psi_T(x) f_\gamma(x)\Big|^p dx + C_\varepsilon R^{-1000p} \|\phi_B f_\gamma\|^p_{L^p(\mathbb{R}^3)} \\ &\lesssim R^{3p\varepsilon} \#\mathbb{T}^c_{\gamma,\lambda} R^{\beta+2\beta+1}(\mu M)^p + C_\varepsilon R^{-1000p}\|\phi_B f_\gamma\|^p_{L^p(\mathbb{R}^3)}.\end{aligned}$$

By construction, we have $M \geq C_\varepsilon R^{-501} \max_\gamma \|f_\gamma\|_{L^\infty(\mathbb{R}^3)}$. It follows that

$$C_\varepsilon R^{-1000p} \|\phi_B f_\gamma\|^p_{L^p(\mathbb{R}^3)} \lesssim R^{-100} \#\mathbb{T}^c_{\gamma,\lambda} R^{\beta+2\beta+1}(\mu M)^p,$$

which concludes the proof. □

3.3. *Trilinear reduction.* We will present a broad/narrow analysis to show that Proposition 28 implies the following level set version of Theorem 3 for the critical $p = 6 + 2/\beta$.

Theorem 32. *For any $R \geq 2$, $\frac{1}{3} \leq \beta \leq 1$, and $\alpha > 0$,*

$$\alpha^{6+2/\beta} |U_\alpha| \lesssim_\varepsilon R^{O(\varepsilon)} R^{2\beta+1} \sum_\gamma \|f_\gamma\|_2^2$$

for any Schwartz function $f : \mathbb{R}^3 \to \mathbb{C}$ with Fourier transform supported in $\mathcal{M}^3(R^\beta, R)$ and satisfying $\|f_\gamma\|_\infty \leq 2$ for all γ.

Proposition 28 implies Theorem 32. We present an algorithm incorporating a broad-narrow argument. For each k, $1 \leq k \leq N$, recall that $\{\tau_k\}$ is a collection of canonical $\sim R_k^{-1/3} \times R_k^{-2/3} \times R_k^{-1}$ moment curve blocks. Write $\ell(\tau) = r^{-1}$ to denote that τ is a canonical $r^{-1} \times r^{-2} \times r^{-3}$ moment curve block.

Step 1 of the algorithm is as follows. Let E_ε be a constant we choose to be larger than $10\underline{C}_\varepsilon$, where $\underline{C}_\varepsilon$ is from Lemma 7. We have the broad/narrow inequality

$$|f(x)| \leq 4E_\varepsilon \max_{\tau_1} |f_{\tau_1}(x)| + R^{2\varepsilon} \max_{d(\tau_1^i, \tau_1^j) \geq E_\varepsilon R_1^{-1/3}} |f_{\tau_1^1}(x) f_{\tau_1^2}(x) f_{\tau_1^3}(x)|^{1/3}, \tag{24}$$

where the second term is the maximum over 3-tuples of τ_1 which are pairwise $\geq E_\varepsilon R_1^{-1/3}$-separated. Indeed, suppose that the set $\{\tau_1 : |f_{\tau_1}(x)| \geq R_1^{-1/3} \max_{\tau_1'} |f_{\tau_1'}(x)|\}$ has at least $3E_\varepsilon$ elements. Then we can find three $\tau_1^1, \tau_1^2, \tau_1^3$ which are pairwise $\geq E_\varepsilon R_1^{-1/3}$-separated and satisfy $|f(x)| \leq R^{2\varepsilon} |f_{\tau_1^1}(x) f_{\tau_1^2}(x) f_{\tau_1^3}(x)|^{1/3}$. If there are fewer than $3E_\varepsilon$ elements, then $|f(x)| \leq 3E_\varepsilon \max_{\tau_1'} |f_{\tau_1'}(x)| + \max_{\tau_1'} |f_{\tau_1'}(x)|$.

Suppose that

$$|U_\alpha| \leq 2 \Big| \Big\{ x \in U_\alpha : \max_{\tau_1} |f_{\tau_1}(x)| \leq \alpha \Big\} \Big|.$$

If this does not hold, then proceed to Step 2 of the algorithm. Further suppose that there are blocks τ_1^i which satisfy $d(\tau_1^i,\tau_1^j)\ge E_\varepsilon R_1^{-1/3}$ and

$$|U_\alpha|\lesssim R^{3\varepsilon}\big|\big\{x\in U_\alpha:\alpha\le 2R^{2\varepsilon}|f_{\tau_1^1}(x)f_{\tau_1^2}(x)f_{\tau_1^3}(x)|^{1/3},\ \max_{\tau_1}|f_{\tau_1}(x)|\le\alpha\big\}\big|. \tag{25}$$

If (25) does not hold, then proceed to Step 2 of the algorithm. Assuming (25), apply Proposition 28 to get the inequality

$$\alpha^{6+2/\beta}|U_\alpha|\lesssim_\varepsilon R^{O(\varepsilon)}R^{2\beta+1}\sum_\gamma\|f_\gamma\|_2^2,$$

which terminates the algorithm.

Next, we describe step k of the algorithm for $k\ge 2$ and $R_{k-1}^{2/3}\le R^{1-\beta}$. The input for step k is

$$|U_\alpha|\lesssim_\varepsilon(\log R)^{k-1}\big|\big\{x\in U_\alpha:\alpha\lesssim(\log R)^{k-1}\max_{\tau_{k-1}}|f_{\tau_{k-1}}(x)|\big\}\big|. \tag{26}$$

For each τ_{k-1}, we have the broad-narrow inequality

$$|f_{\tau_{k-1}}(x)|\le 2E_\varepsilon\max_{\tau_k\subset\tau_{k-1}}|f_{\tau_k}(x)|+R^{2\varepsilon}\max_{\substack{\tau_k^i\subset\tau_{k-1}\\ d(\tau_k^i,\tau_k^j)\ge E_\varepsilon R_k^{-1/3}}}|f_{\tau_k^1}(x)f_{\tau_k^2}(x)f_{\tau_k^3}(x)|^{1/3}.$$

Either proceed to Step $k+1$ or assume that

$$|U_\alpha|\lesssim(\log R)^{k-1}\big|\big\{x\in U_\alpha:\alpha\lesssim(\log R)^{k-1}\max_{\tau_{k-1}}|f_{\tau_{k-1}}(x)|,\ \max_{\tau_k}|f_{\tau_k}(x)|\le\alpha\big\}\big|.$$

Again, either proceed to Step $k+1$ or assume further that there are $\tau_k^i\subset\tau_{k-1}$ which are pairwise $\ge E_\varepsilon R_k^{-1/3}$-separated and satisfy

$$|U_\alpha|\le(\log R)^kR^{3\varepsilon}\sum_{\tau_{k-1}}\big|\big\{x\in U_\alpha:\alpha\lesssim(\log R)^{k-1}R^\varepsilon|f_{\tau_k^1}(x)f_{\tau_k^2}(x)f_{\tau_k^3}(x)|^{1/3},\ \max_{\tau_k}|f_{\tau_k}(x)|\le\alpha\big\}\big|.$$

By rescaling for the moment curve, there exists a linear transformation T so that $|f_{\tau_k^i}\circ T|=|g_{\underline{\tau}_k^i}|$, where the $\underline{\tau}_k^i$ are pairwise $\gtrsim E_\varepsilon R_1^{-1/3}$-separated blocks and g is Fourier supported in the anisotropic neighborhood $\mathcal{M}^3(R_{k-1}^{-1/3}R^\beta,R_{k-1}^{-1}R)$. Indeed, suppose that τ_{k-1} is the l-th piece

$$\tau_{k-1}=\{(\xi_1,\xi_2,\xi_3):lR_{k-1}^{-1/3}\le\xi_1<(l+1)R_{k-1}^{-1/3},\ |\xi_2-\xi_1^2|\le R_{k-1}^{-2/3},\ |\xi_3-3\xi_1\xi_2+2\xi_1^3|\le R_{k-1}^{-1}\}.$$

Since the Fourier support of f is in $\mathcal{M}^3(R^\beta,R)$ by hypothesis, the Fourier support of $f_{\tau_{k-1}}$ is in $\tau_{k-1}\cap\mathcal{M}^3(R^\beta,R)$. Define the affine transformation $L(\xi_1,\xi_2,\xi_3)$ by

$$\begin{aligned}
\xi_1&\mapsto R_{k-1}^{1/3}(\xi_1-lR_{k-1}^{-1/3}),\\
\xi_2&\mapsto R_{k-1}^{2/3}(\xi_2-l^2R_{k-1}^{-2/3})-2lR_{k-1}^{1/3}(\xi_1-lR_{k-1}^{-1/3}),\\
\xi_3&\mapsto R_{k-1}(\xi_3-l^3R_{k-1}^{-1})-3lR_{k-1}^{2/3}(\xi_2-l^2R_{k-1}^{-2/3})+3l^2R_{k-1}^{1/3}(\xi_1-lR_{k-1}^{-1/3}).
\end{aligned}$$

This affine map satisfies $L(\tau_{k-1}\cap\mathcal{M}^3(R^\beta,R))=\mathcal{M}^3(R_{k-1}^{-1/3}R^\beta,R_{k-1}^{-1}R)$. If we write $L^{-1}(\xi_1,\xi_2,\xi_3)=A(\xi_1,\xi_2,\xi_3)+b$, where A is a linear map, then the rescaling map T above is equal to $(A^{-1})^*$. In this step, we have assumed that $R_{k-1}R^{-1}\le R_{k-1}^{1/3}R^{-\beta}$. One may then verify that $L(\gamma)=\underline{\gamma}$ are

$\sim R_{k-1}^{1/3}R^{-\beta}\times R_{k-1}^{2/3}R^{-2\beta}\times R_{k-1}R^{-1}$ small caps partitioning $\mathcal{M}^3(R_{k-1}^{-1/3}R^{\beta}, R_{k-1}^{-1}R)$. Apply Proposition 28 to the rescaled functions to obtain the inequality

$$\alpha^{6+2/\beta'}\big|\big\{x\in U_\alpha:\alpha\lesssim(\log R)^{k-1}R^{\varepsilon}|g_{\underline{\tau}_k^1}(x)g_{\underline{\tau}_k^2}(x)g_{\underline{\tau}_k^3}(x)|^{1/3},\ \max_{\underline{\tau}_k\subset\underline{\tau}_{k-1}}|g_{\underline{\tau}_k}(x)|\le\alpha\big\}\big| \lesssim_\varepsilon R^{3\varepsilon+10\varepsilon}(R_{k-1}^{-1}R)^{2\beta'+1}\sum_{\underline{\gamma}}\|g_{\underline{\gamma}}\|_2^2,$$

where $\beta'\in\left[\frac{1}{3},1\right]$ is defined by $(R_{k-1}R^{-1})^{\beta'}=R_{k-1}^{1/3}R^{-\beta}$. By undoing the rescaling change of variables and summing over τ_{k-1}, this implies

$$\alpha^{6+2/\beta'}|U_\alpha|\lesssim_\varepsilon R^{13\varepsilon}(R_{k-1}^{-1}R)^{2\beta'+1}\sum_{\gamma}\|f_\gamma\|_2^2.$$

It suffices to verify that $(R_{k-1}^{-1}R)^{2\beta'+1}\lessapprox R^{2\beta+1}/\alpha^{2/\beta-2/\beta'}$. Use the upper bound $\alpha\lessapprox R_{k-1}^{-1/3}R^{\beta}$ from the step we are considering so that it suffices to verify $(R_{k-1}^{-1}R)^{2\beta'+1}(R_{k-1}^{-1/3}R^{\beta})^{2/\beta-2/\beta'}\lessapprox R^{2\beta+1}$, which simplifies to $R_{k-1}^{-2\beta'-1-2/3\beta+2/3\beta'}\lessapprox R^{2\beta-2\beta'-2+2\beta/\beta'}$. Using the definition of β', this further simplifies to $R_{k-1}^{-2\beta'-1-2/3\beta+2/3\beta'}\lessapprox R_{k-1}^{(-\beta'+1/3)(2+2/\beta')}$, which is true since $\beta\le 2$. In this case, the algorithm terminates.

Next, we describe step k with $k\ge 2$ and $R_{k-1}^{2/3}\ge R^{1-\beta}$. The input for step k is

$$|U_\alpha|\le(\log R)^{k-1}\big|\big\{x\in U_\alpha:\alpha\lesssim(\log R)^{k-1}\max_{\tau_{k-1}}|f_{\tau_{k-1}}(x)|\big\}\big|. \tag{27}$$

Let $\{\zeta\}$ be a partition of $\mathcal{M}^3(R^{\beta}, R)$ into $\sim R_{k-1}^{2/3}R^{-1}\times R_{k-1}^{4/3}R^{-2}\times R^{-1}$ small caps. By Proposition 31, we may assume that there are versions $\tilde f_{\tau_{k-1}}$ of the $f_{\tau_{k-1}}$ whose wave packets corresponding to ζ have been localized and pigeonholed and which satisfy

$$|U_\alpha|\lesssim(\log R)^{k}\big|\big\{x\in U_\alpha:\alpha\lesssim(\log R)^{k}\max_{\tau_{k-1}}|\tilde f_{\tau_{k-1}}(x)|\big\}\big|.$$

As in the previous case, either we proceed to Step $k+1$ or we have

$$|U_\alpha|\le(\log R)^{k}R^{3\varepsilon}\sum_{\tau_{k-1}}\big|\big\{x\in U_\alpha:\alpha\lesssim(\log R)^{k}|\tilde f_{\tau_k^1}(x)\tilde f_{\tau_k^2}(x)\tilde f_{\tau_k^3}(x)|^{1/3},\ \max_{\tau_k\subset\tau_{k-1}}|\tilde f_{\tau_k}(x)|\le\alpha\big\}\big|.$$

By the same rescaling argument as above, let T be the linear transformation so that $|\tilde f_{\tau_k^i}\circ T|=|g_{\underline{\tau}_k^i}|$ and the $\underline{\tau}_k^i$ are pairwise $\gtrsim E_\varepsilon R_1^{-1/3}$-separated blocks and g is Fourier supported in the anisotropic neighborhood $\mathcal{M}^3(R_{k-1}^{-1/3}R^{\beta}, R_{k-1}^{-1}R)$. Note that each $|\tilde f_\zeta\circ T|=|g_{\underline{\zeta}}|$, where $\underline{\zeta}$ is an $R_{k-1}R^{-1}\times R_{k-1}^{2}R^{-2}\times R_{k-1}R^{-1}$ small cap. Apply Proposition 28 to the rescaled functions $(\max_{\underline{\zeta}}\|g_{\underline{\zeta}}\|_\infty)^{-1}(g_{\underline{\tau}_k^1}+g_{\underline{\tau}_k^2}+g_{\underline{\tau}_k^3})$ to obtain the inequality

$$\alpha^{8}\big|\big\{x\in U_\alpha:\alpha\le(\log R)^{k}|g_{\underline{\tau}_k^1}(x)g_{\underline{\tau}_k^2}(x)g_{\underline{\tau}_k^3}(x)|^{1/3},\ \max_{\underline{\tau}_k}|g_{\underline{\tau}_k}(x)|\le\alpha\big\}\big| \lesssim_\varepsilon R^{10\varepsilon}(R_{k-1}^{-1}R)^{2(1)+1}\max_{\underline{\zeta}}\|g_{\underline{\zeta}}\|_\infty^{6}\sum_{\underline{\zeta}}\|g_{\underline{\zeta}}\|_2^2.$$

By undoing the rescaling change of variables and summing over τ_{k-1}, this implies

$$\alpha^{8}|U_\alpha|\lesssim_\varepsilon R^{10\varepsilon}(R_{k-1}^{-1}R)^{3}(\max_{\zeta}\|\tilde f_\zeta\|_\infty)^{6}\sum_{\zeta}\|\tilde f_\zeta\|_2^2.$$

By properties of the pigeonholing lemma, for each ζ, $(\max_\zeta \|\tilde{f}_\zeta\|_\infty)^6 \|\tilde{f}_\zeta\|_2^2 \lesssim_\varepsilon R^{3\varepsilon}(R_{k-1}^{2/3}R^{-1}R^\beta)^2\|f_\zeta\|_6^6$. By cylindrical L^6-decoupling (Theorem 26), for each ζ,

$$\|f_\zeta\|_6^6 \lesssim_\varepsilon R^\varepsilon \Big(\sum_{\gamma\subset\zeta}\|f_\gamma\|_6^2\Big)^3 \lesssim_\varepsilon R^\varepsilon (R_{k-1}^{2/3}R^{-1}R^\beta)^2 \sum_{\gamma\subset\zeta}\|f_\gamma\|_2^2.$$

The summary of step k in this case is that

$$\alpha^8|U_\alpha| \lesssim_\varepsilon R^{3\varepsilon+20\varepsilon}(R_{k-1}^{-1}R)^3(R_{k-1}^{2/3}R^{-1}R^\beta)^4\sum_\gamma\|\tilde{f}_\gamma\|_2^2.$$

It remains to verify that $R_{k-1}^{-1/3}R^{4\beta-1} \lessapprox R^{2\beta+1}/\alpha^{2/\beta-2}$. This is true since $R_{k-1}^{1/3}\ge 1$ and $\alpha\le R^\beta$. The algorithm terminates in this case.

The final step, if the algorithm has not terminated yet, gives the case

$$|U_\alpha| \lesssim (\log R)^N \big|\{x\in U_\alpha : \alpha \lesssim (\log R)^N \max_{\tau_N}|f_{\tau_N}(x)|\}\big|.$$

Write $\tau_N=\theta$ and use trivial inequalities:

$$\begin{aligned}\alpha^{6+2/\beta}\big|\{x\in U_\alpha : \alpha\lesssim(\log R)^N\max_\theta|f_\theta(x)|\}\big| &\lesssim_\varepsilon (\log R)^N\sum_\theta\int|f_\theta|^{6+2/\beta}\\ &\lesssim_\varepsilon (\log R)^N\sum_\theta\max_\theta\|f_\theta\|_\infty^{4+2/\beta}\int|f_\theta|^2\\ &\lesssim_\varepsilon (\log R)^N\sum_\theta\max_\theta(\#\gamma\subset\theta)^{4+2/\beta}\int\sum_{\gamma\subset\theta}|f_\gamma|^2\\ &\lesssim_\varepsilon (\log R)^N R^{(\beta-1/2)(4+2/\beta)}\sum_\gamma\|f_\gamma\|_2^2,\end{aligned}$$

where we used Lemma 7 for the L^∞ bound. Technically, our algorithm could give us a version of f whose wave packets have been pigeonholed at a few scales. In that case, we incorporate a process analogous to that of "unwinding the pruning" from the proof of Proposition 28 into the trivial argument above. Noting that $N\sim\varepsilon^{-1}$, and $(\log R)^N(\log R)^N\lesssim_\varepsilon R^\varepsilon$, we are done since $(\beta-1/2)(4+2/\beta)\le 2\beta+1$, which is equivalent to $\beta\le 1$. □

3.4. *Proof that Theorem 32 implies Theorem 3.* We divide the work into two propositions. First, in Proposition 33, we show that Theorem 32 implies the critical exponent $p=6+2/\beta$ version of Theorem 3. Then, we show that the general Theorem 3 follows from the critical exponent case.

Proposition 33. *Theorem 3 holds for the critical exponent $p=6+2/\beta$.*

Proof. Fix $p=6+2/\beta$. By Lemma 29, it suffices to bound the L^p norm of f on a fixed ball $B_{R^{\max(2\beta,1)}}$. By Lemma 30, there is a constant $\alpha>0$ (which we may assume is $\ge C_\varepsilon(\log R)R^{-100}\max_\gamma\|f_\gamma\|_\infty$) so that it suffices to bound $\alpha^p|U_\alpha|$ for $U_\alpha=\{x\in B_{R^{\max(2\beta,1)}} : \alpha\le|f(x)|\}$. Finally, by Proposition 31, we may replace f by a pigeonholed and localized version $\tilde{f}$. One of the properties of the pigeonholed version is that, for all γ, either $\|\tilde{f}_\gamma\|_\infty\sim A$ or $\|\tilde{f}_\gamma\|_\infty=0$ for some constant A.

Apply Theorem 32 to the function $\tilde{f}/A$ to obtain the inequality

$$(\alpha/A)^p|U_\alpha| \lesssim_\varepsilon R^{20\varepsilon}R^{2\beta+1}\sum_\gamma \|\tilde{f}_\gamma/A\|^2_{L^2(\mathbb{R}^3)}.$$

It remains to note that by (20) from the pigeonholing proposition,

$$A^{p-2}\|\tilde{f}_\gamma\|^2_{L^2(R^{\max(2\beta,1)})} \lesssim R^{6\varepsilon}A^p\#\tilde{\mathbb{T}}_\gamma R^{\beta+2\beta+1} \lesssim R^{6\varepsilon}\|\tilde{f}_\gamma\|^p_{L^p(\mathbb{R}^3)}.$$

Since $|\tilde{f}_\gamma| \lesssim |f_\gamma|$ for each γ, this concludes the proof. □

Next, we show that Theorem 3 for general p follows from Theorem 3 at the critical exponent $p=6+2/\beta$ via an interpolation argument with L^2 and L^∞ estimates.

Proof of Theorem 3. Let $p \geq 2$. Repeat the initial steps in the proof of Proposition 33 so that it suffices to prove

$$\alpha^p|U_\alpha| \lesssim_\varepsilon R^\varepsilon(R^{\beta(p/2-1)} + R^{\beta(p-4)-1})\sum_\gamma \|f_\gamma\|^p_{L^p(\mathbb{R}^3)},$$

where f has been pigeonholed and localized as in Proposition 31. First suppose that $2 \leq p \leq 6+2/\beta$. By Proposition 33, we have

$$\alpha^{6+2/\beta}|U_\alpha| \lesssim_\varepsilon R^\varepsilon R^{2\beta+1}\sum_\gamma \|f_\gamma\|^{6+2/\beta}_{L^{6+2/\beta}(\mathbb{R}^3)}.$$

Write $A \sim \max_\gamma \|f_\gamma\|_\infty$. We would be done if $R^{2\beta+1}A^{6+2/\beta-p} \lesssim R^{\beta(p/2-1)}\alpha^{6+2/\beta-p}$, which simplifies to $R^{\beta/2}A \lesssim \alpha$. If this does not hold, then using L^2 orthogonality,

$$\alpha^p|U_\alpha| \lesssim R^{\beta(p/2-1)}A^{p-2}\sum_\gamma \|f_\gamma\|_2^2.$$

By (20), $A^{p-2}\|f_\gamma\|_2^2 \lesssim R^{3\varepsilon}\|f_\gamma\|_p^p$, which finishes this case.

Next, assume that $6+2/\beta \leq p$. Then by Proposition 33,

$$\alpha^p|U_\alpha| \lesssim_\varepsilon R^\varepsilon R^{2\beta+1}\sum_\gamma \alpha^{p-6-2/\beta}\|f_\gamma\|^{6+2/\beta}_{6+2/\beta}.$$

We would be done if $R^{2\beta+1}\alpha^{p-6-2/\beta} \lesssim R^{\beta(p-4)-1}A^{p-6-2/\beta}$, which simplifies to $\alpha \lesssim R^\beta A$. Since $\alpha \lesssim |f(x)| = \left|\sum_\gamma f_\gamma(x)\right|$ and $\#\gamma \lesssim R^\beta$, this is true. □

Acknowledgements

Thank you to Hong Wang for helpful conversations related to this work. In particular, she suggested to us the idea of using decoupling in place of orthogonality within the high/low method.

Guth is supported by a Simons Investigator grant. Maldague is supported by the National Science Foundation under Award No. 2103249.

References

[Bombieri and Iwaniec 1986] E. Bombieri and H. Iwaniec, "Some mean-value theorems for exponential sums", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **13**:3 (1986), 473–486. MR Zbl

[Bourgain 2017a] J. Bourgain, "Decoupling, exponential sums and the Riemann zeta function", *J. Amer. Math. Soc.* **30**:1 (2017), 205–224. MR Zbl

[Bourgain 2017b] J. Bourgain, "Decoupling inequalities and some mean-value theorems", *J. Anal. Math.* **133** (2017), 313–334. MR Zbl

[Bourgain and Demeter 2015] J. Bourgain and C. Demeter, "The proof of the l^2 decoupling conjecture", *Ann. of Math.* (2) **182**:1 (2015), 351–389. MR Zbl

[Bourgain et al. 2016] J. Bourgain, C. Demeter, and L. Guth, "Proof of the main conjecture in Vinogradov's mean value theorem for degrees higher than three", *Ann. of Math.* (2) **184**:2 (2016), 633–682. MR Zbl

[Demeter et al. 2020] C. Demeter, L. Guth, and H. Wang, "Small cap decouplings", *Geom. Funct. Anal.* **30**:4 (2020), 989–1062. MR Zbl

[Guth and Maldague 2022] L. Guth and D. Maldague, "Amplitude dependent wave envelope estimates for the cone in $\mathbb{R}^3$", preprint, 2022. arXiv 2206.01093

[Guth et al. 2024] L. Guth, D. Maldague, and H. Wang, "Improved decoupling for the parabola", *J. Eur. Math. Soc.* (*JEMS*) **26**:3 (2024), 875–917. MR Zbl

[Jung 2023] H. Jung, "A sharp L^{10} decoupling for the twisted cubic", *J. Anal. Math.* **149**:2 (2023), 563–609. MR Zbl

[Wooley 2016] T. D. Wooley, "The cubic case of the main conjecture in Vinogradov's mean value theorem", *Adv. Math.* **294** (2016), 532–561. MR Zbl

Received 19 Jul 2022. Revised 21 Apr 2023. Accepted 18 Jul 2023.

LARRY GUTH: lguth@math.mit.edu
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, United States

DOMINIQUE MALDAGUE: dmal@mit.edu
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, United States

ANALYSIS AND PDE
Vol. 17 (2024), No. 10, pp. 3589–3621
DOI: 10.2140/apde.2024.17.3589

THE QUERMASSINTEGRAL-PRESERVING MEAN CURVATURE FLOW IN THE SPHERE

ESTHER CABEZAS-RIVAS AND JULIAN SCHEUER

We introduce a mean curvature flow with global term of convex hypersurfaces in the sphere, for which the global term can be chosen to keep any quermassintegral fixed. Then, starting from a strictly convex initial hypersurface, we prove that the flow exists for all times and converges smoothly to a geodesic sphere. This provides a workaround to an issue present in the volume-preserving mean curvature flow in the sphere introduced by Huisken (1987). We also classify solutions for some constant curvature-type equations in space forms, as well as solitons in the sphere and in the upper branch of the De Sitter space.

1. Introduction and statement of main results

Let $n \geq 2$, and let $M^n \subset \mathbb{M}_K^{n+1}$ be a smooth, closed, embedded hypersurface in a simply connected space form $\mathbb{M}_K^{n+1}$ of constant curvature $K \in \mathbb{R}$, given by the embedding x_0. We consider a family of embeddings $x = x(t, \cdot\,)$ satisfying the mean curvature-type flow with a global forcing term

$$\partial_t x = (\mu(t)\mathrm{c}_K(r) - H)\nu, \tag{1-1}$$

which has initial condition $x(0, \cdot\,) = x_0$. Here H is the mean curvature and ν the outward unit normal of the evolving hypersurfaces M_t. For convex hypersurfaces (i.e., with $\kappa_1 \geq 0$, where $\kappa_1 \leq \cdots \leq \kappa_n$ denote the principal curvatures), the sign conventions are taken so that $-H\nu$ points inwards. Moreover, let r denote the radial distance to a given point $\mathcal{O} \in \mathbb{M}_K^{n+1}$, which we call the *origin* in the sequel. This means that the flow (1-1) depends on the choice of the origin and, in fact, along the flow we will change the origin in a controlled way. We use the notation

$$\mathrm{c}_K(r) = \mathrm{s}_K'(r), \quad \text{where } \mathrm{s}_K(r) = \begin{cases} K^{-1/2}\sin(\sqrt{K}r) & \text{if } K > 0, \\ r & \text{if } K = 0, \\ |K|^{-1/2}\sinh(\sqrt{|K|}r) & \text{if } K < 0. \end{cases} \tag{1-2}$$

If σ_ℓ represents the ℓ-th elementary symmetric function, we define the time-dependent term by

$$\mu(t) = \frac{\int_M H\sigma_\ell \, dV_t}{\int_M \mathrm{c}_K \sigma_\ell \, dV_t} \tag{1-3}$$

The work of Cabezas-Rivas is partially supported by the projects PID2019-105019GB-C21 and PID2022-136589NB-I00, both funded by MCIN/AEI/10.13039/501100011033 and by ERDF A way of making Europe; by the project AICO/2021/252 funded by Conselleria d'Educació, Universitats i Ocupació; and by the network RED2022-134077 funded by MCIN/AEI/10.13039/501100011033 and by ERDF A way of making Europe.
MSC2020: 53C21, 53E10.
Keywords: volume-preserving mean curvature flow, spherical geometry.

for each $\ell = 0, 1, \dots, n$, where dV_t denotes the volume element of M_t. This choice guarantees that (1-1) gives a family of globally constrained mean curvature flows, where μ can be chosen to preserve any of the $n+1$ quermassintegrals $W_\ell(\Omega_t)$ of the evolving hypersurfaces M_t (see Section 2.1 for a review of the quermassintegrals). Here Ω_t denotes the convex region enclosed by $M_t = x(t, \mathbb{S}^n)$ — note that we may assume that the common domain of the embeddings is $\mathbb{S}^n$ due to convexity.

Let us stress that, for $\ell = 0$ and $K = 0$, the flow (1-1) with nonlocal term as in (1-3) coincides with the *volume-preserving mean curvature flow* (VPMCF) introduced by Huisken [1987]. He proved that strictly convex hypersurfaces in $\mathbb{R}^{n+1}$ remain convex and embedded under the flow, and the solution exists for all times and converges to a round sphere smoothly as $t \to \infty$. Since then it was still an open question of extending the result to an $(n+1)$-dimensional sphere $\mathbb{S}^{n+1}_K$, $K > 0$, where convexity can be lost under VPMCF, as pointed out in [Huisken 1987, p. 38].

Our main result settles this question by proposing the flow (1-1) as the most natural generalization of the VPMCF to a space form with positive curvature. Indeed, such a definition preserves convexity under the flow, and allows us to prove the following version of Huisken's original result within the half sphere, where, for a point $p \in \mathbb{S}^{n+1}_K$, we denote by $\mathcal{H}(p)$ the open hemisphere around p.

Theorem 1.1. *Let $n \geq 2$, and let $M_0 \subset \mathbb{S}^{n+1}_K$ be a strictly convex hypersurface enclosing a domain Ω_0. Then there exists a finite system of origins $(\mathcal{O}_i)_{0\leq i\leq m}$ and numbers $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = \infty$ such that the problem*

$$\begin{aligned} \partial_t x &= (\mu_i(t)\mathrm{c}_K(r_i) - H)\nu, \quad t \in [t_i, t_{i+1}), \quad 0 \leq i \leq m, \\ x(0, M) &= M_0, \\ x(t_i, M) &= \lim_{t \nearrow t_i} M_t, \quad 1 \leq i \leq m, \end{aligned}$$

where r_i is the distance to $\mathcal{O}_i$ and μ_i is given as in (1-3) to keep the quermassintegral $W_\ell(\Omega_t)$ fixed for any $\ell = 0, 1, \dots, n$, has a solution

$$x : [0, \infty) \times \mathbb{S}^n \to \mathbb{S}^{n+1}_K.$$

For every $t \geq 0$, the embeddings $x(t, \cdot)$ smoothly map $\mathbb{S}^n$ to strictly convex hypersurfaces, with

$$\mathcal{O}_i \in \Omega_t \quad \text{and} \quad M_t \subset \mathcal{H}(\mathcal{O}_i) \qquad \text{for all } t \in [t_i, t_{i+1}),$$

and satisfy spatial C^∞-estimates which are uniform in time. The restriction

$$x : [t_m, \infty) \times \mathbb{S}^n \to \mathbb{S}^{n+1}_K$$

is smooth and converges for $t \to \infty$ in C^∞ to a geodesic sphere around $\mathcal{O}_m$ with radius determined by $W_\ell(B_r) = W_\ell(\Omega_0)$.

At this stage we should mention that Guan and Li [2015] invented a *purely local* mean curvature-type flow in the sphere, which is volume-preserving and drives star-shaped hypersurfaces to geodesic spheres. There is also a flow of Guan–Li-type that preserves $W_{\ell-1}(\Omega_t)$ and decreases $W_\ell(\Omega_t)$, which has so far refused to allow curvature estimates (see [Chen et al. 2022] for an overview of known results).

However, notice that nonlocal flows are more challenging than their corresponding local counterparts, as the evolution depends heavily on the global shape of the hypersurface M_t and the presence of the term $\mu(t)$ in all the relevant evolution equations causes a plethora of extra complications; e.g., comparison principles and preservation of key properties fail (see [Cabezas-Rivas and Miquel 2016]), and embedded hypersurfaces may develop self-intersections (see [Mayer and Simonett 2000]).

In this framework, our flow (1-1) is, to the best of our knowledge, the first known curvature flow in the sphere which preserves any desired quermassintegral by a suitable choice of the global term μ and which enjoys smooth convergence to a geodesic sphere.

Elliptic counterpart*: *rigidity results. As a byproduct of the pinching estimates (see Proposition 4.3) that we require to prove Theorem 1.1, we can also classify hypersurfaces in a space form $\mathbb{M}_K^{n+1}$ which have a rotationally symmetric curvature function under suitable assumptions on the sectional curvature.

With this goal, we work with curvature functions more general than H. Let $\Gamma \subset \mathbb{R}^n$ be a symmetric, open cone containing the positive cone

$$\Gamma_+ = \{\kappa \in \mathbb{R}^n : \kappa_i > 0, \forall i = 1, \dots, n\},$$

and consider a symmetric function $f \in C^2(\Gamma)$. Let

$$F(A) = f(\kappa_1, \dots, \kappa_n)$$

be the corresponding operator-dependent function, where A denotes the Weingarten or shape operator. We assume the following:

Conditions 1.2. Let $f(\kappa) = F(A)$ be a C^2 symmetric function defined on an open, symmetric cone $\Gamma \supset \Gamma_+$. We ask further that

(a) f is strictly increasing in each argument,

(b) f is homogeneous of degree 1,

(c) f is normalized so that $f(1, \dots, 1) = n$.

Notice that (a) implies that F defines a strictly elliptic operator on M, as proved in [Huisken and Polden 1999]. We say that f is *inverse concave/convex* if the dual function

$$\tilde{f}(\kappa_1, \dots, \kappa_n) = f(\kappa_1^{-1}, \dots, \kappa_n^{-1})$$

is concave/convex (see Section 8.1 for a more detailed introduction).

A classical result by Alexandrov [1962] says that if a compact hypersurface embedded in $\mathbb{R}^{n+1}$ has H equal to a constant, it must be a round sphere. Later on, Ros [1987] extended this result to the constancy of higher-order symmetric functions σ_ℓ. Hypersurfaces in a model space $\mathbb{M}_K^{n+1}$ which have a constant curvature function F were often called *Weingarten hypersurfaces* in the previous literature. It was shown in [Espinar et al. 2009, Theorem 28] that round spheres are the only examples of compact Weingarten hypersurfaces in the hyperbolic space $\mathbb{H}^{n+1}$. In this spirit, we obtain similar rigidity results for F radially symmetric instead of constant.

Theorem 1.3. *Let $n \geq 2$, $K \in \mathbb{R}$ and $|\alpha| \geq 1$, and let $M^n \subset \mathbb{M}^{n+1}_K$ be a convex hypersurface, which is located in the northern hemisphere for $K > 0$, such that*

$$\mathrm{Sec}_M \geq -\alpha K.$$

Suppose that F is a convex function satisfying Conditions 1.2, which is a solution to

$$F = \gamma \mathrm{c}_K^\alpha$$

for some constant γ. Then M is a geodesic sphere which is centered at the origin provided $K \neq 0$. If $\alpha = 1$ the convexity assumption on M can be dropped, while if $\alpha = -1$ the convexity of F may be replaced by inverse concavity.

Such results have been obtained for σ_ℓ by integral methods: for instance, [Wu and Xia 2014] studies constant linear combinations of higher-order mean curvatures, [Wu 2016] analyses the constancy of $\mathrm{c}_K \sigma_\ell$ in $\mathbb{H}^{n+1}$, and [Kwong et al. 2018] deals with hypersurfaces having radially symmetric higher-order mean curvatures in general $\mathbb{M}^{n+1}_K$ under mild convexity assumptions. But those integral techniques are restricted to the σ_ℓ because they are divergence-free in space forms. Our maximum principle approach enables us to relax the assumptions on the curvature functions at the cost of having to impose a condition on the sectional curvature of the hypersurface. However, notice that if $(1+\alpha)K > 0$ this assumption is weaker than convexity, while if this product is ≤ 0 the condition already implies convexity.

Classification of solitons. In the study of singularity formation along curvature flows, especially the mean curvature flow, the class of *self-shrinking solutions*, simply called *solitons* subsequently, plays an important role. For the mean curvature flow in Euclidean space, they arise as blow-up limits of type-I singularities (see [Huisken 1990]), and they satisfy the elliptic equation

$$H = \langle x, \nu \rangle.$$

Huisken [1990] showed that the only compact mean-convex solitons are spheres.

A similar recent result with H replaced by the Gauss curvature K (see [Brendle et al. 2017]) settled the long-standing open problem of whether the flow by certain powers of the Gauss curvature of n-dimensional hypersurfaces, $n \geq 3$, converges to a round sphere; the convergence to a soliton had already been proved in [Andrews et al. 2016].

The study of solitons for more general curvature functions has received plenty of attention, as well as in space forms; see, e.g., [Gao and Ma 2019; Gao et al. 2018; 2022; McCoy 2011]. Here one considers the general equation

$$F^\beta = u, \tag{1-4}$$

where $\beta \in \mathbb{R}$, F is a function of the principal curvatures with suitable assumptions, and

$$u = \mathrm{s}_K(r)\langle \partial_r, \nu \rangle \tag{1-5}$$

is the generalized support function. From a well-known duality relation by means of the Gauss map for hypersurfaces of the sphere and itself, from Theorem 1.3 we can deduce a new classification result for convex solitons in the sphere $\mathbb{S}^{n+1}_1$, and similarly, from a duality relation between hypersurfaces of the hyperbolic and De Sitter space, from Theorem 1.3 we can deduce a new classification result for convex solitons in the upper branch of the $(n+1)$-dimensional De Sitter space $\mathbb{S}^{n,1}$ with sectional curvature $K=1$, i.e.,

$$\mathbb{S}^{n,1}=\left\{y\in\mathbb{R}^{n+2}:-(y^0)^2+\sum_{i=1}^{n+1}(y^i)^2=1,\,y^0>0\right\}.$$

More precisely, with the notation

$$\operatorname{sgn}(\mathbb{M})=\begin{cases}1, & \mathbb{M}=\mathbb{S}^{n+1}_1,\\ -1, & \mathbb{M}=\mathbb{S}^{n,1},\end{cases}\tag{1-6}$$

we prove the following result.

Corollary 1.4. *Let $n\geq 2$, $|\beta|\leq 1$, $\beta\neq 0$, and let $\mathbb{M}$ be either $\mathbb{S}^{n+1}_1$ or $\mathbb{S}^{n,1}$. Consider $M^n\subset\mathbb{M}$ a closed strictly convex hypersurface, and, in the case $((1-\beta)/\beta)\operatorname{sgn}(\mathbb{M})>0$, we assume further that*

$$\operatorname{Sec}_M\leq\frac{\operatorname{sgn}(\mathbb{M})}{1-\beta}.$$

If F is an inverse convex function satisfying Conditions 1.2, which is a solution to the soliton equation (1-4), then M is a geodesic sphere centered at the origin. In the case $\beta=1$, the inverse convexity may be replaced by concavity.

Remark 1.5. Corollary 1.4 is remarkable in several ways. Firstly, to our knowledge this is the first such result, where the β-regime ranges down to zero. This is surprising, as in the Euclidean space, for $F=K^{1/n}$ and $\beta\leq n/(n+2)$ the result is false; see [Andrews 2000; Brendle et al. 2017]. Note however that $K^{1/n}$ is not inverse convex. Secondly, in all of the previous results of this type, the inverse concavity of F was exploited crucially. The duality approach allows us to deal with a further class of curvature functions, which could not be treated by earlier methods.

Notice that, while Weingarten hypersurfaces are known to be geodesic spheres in $\mathbb{S}^{n,1}$ (see [Roldán 2022]), we are not aware of rigidity results for solitons in this setting. On the other hand, to have some model examples in mind, $F=|A|$ satisfies the assumptions of Theorem 1.3, and the harmonic mean curvature is suitable for Corollary 1.4.

The problem of extending a nonlocal flow to curved spaces. As said before, Huisken [1987] introduced the VPMCF of convex hypersurfaces in the Euclidean space:

$$\partial_t x=(\mu(t)-H)\nu,\tag{1-7}$$

where the global term is the average mean curvature $\mu=\overline{H}=\fint_{M_t}H$. Taking μ as in (1-3) for $K=0$, McCoy [2004] obtained convergence of convex hypersurfaces in $\mathbb{R}^{n+1}$ to round spheres under a flow that preserves any quermassintegral (which in the Euclidean case coincide with the mixed volumes; see Section 2.1).

Huisken already pointed out that an interesting problem is to extend his result to non-Euclidean ambient spaces, with the warning that the generalization will not be straightforward because (1-7) does not preserve convexity in general Riemannian manifolds, due to terms with an unfavorable sign in the evolution equation of the second fundamental form. In particular, for hypersurfaces in $\mathbb{M}^{n+1}_K$, the Weingarten matrix h^i_j evolves according to

$$(\partial_t - \Delta)h^i_j = (|A|^2 - nK)h^i_j + 2KH\delta^i_j - \mu(h^i_\ell h^\ell_j + K\delta^i_j).$$

Notice that for $K < 0$ the bad term is $2KH\delta^i_j$, which comes from the background geometry and causes that convexity is not preserved in general. This failure is independent of the nonlocal nature of the flow; indeed, if we replace convexity by h-convexity ($\kappa_1 > |K|$), Miquel and the first author [Cabezas-Rivas and Miquel 2007] proved that h-convex hypersurfaces can be deformed under (1-7) to a geodesic sphere; this was extended by Andrews and Wei [2018] for a class of quermassintegral-preserving flows. The curvature condition was relaxed to positive sectional curvature ($\kappa_1\kappa_2 > |K|$) by Andrews, Chen and Wei [Andrews et al. 2021] in the volume-preserving case.

Notice that the complication for $K > 0$ is of a completely different nature, since the fatal term is now $-\mu K\delta^i_j$, and thus comes directly from the global term. Indeed, Huisken [1987] illustrated this with an intuitive example: if the flow starts with a convex hypersurface of $\mathbb{S}^{n+1}$ with a portion M^* C^2-close to the equator, then in this region $\overline{H} \gg H$ and hence M^* moves in the outward direction crossing the equator, and thus the evolving hypersurface becomes nonconvex.

This obstruction to the preservation of convexity in an ambient sphere supports the claim that the flow (1-7) is, geometrically, not the most natural generalization of the same flow in the Euclidean case to the spherical ambient space. Indeed, our alternative flow (1-1) does preserve pinching of the principal curvatures, and hence, it succeeds in driving any convex initial hypersurface to a geodesic sphere. Notice that Huisken's example is actually the motivation for the definition of (1-1), as the effect of multiplying the global term by $\mathrm{c}_K(r)$ is to slow down the motion as the hypersurfaces approach the equator.

In short, to extend Huisken's results to the hyperbolic space one needs to strengthen the notion of convexity, whereas for the ambient sphere we propose a different generalization of the flow (notice that (1-1) and (1-7) coincide for the Euclidean space), which works for convex hypersurfaces.

The isoperimetric nature of the flow. In addition, under (1-7) the surface area is nonincreasing, and hence Huisken's theorem provides an alternative proof of the isoperimetric inequality for convex hypersurfaces of $\mathbb{R}^{n+1}$. An interesting side effect of the extra term in (1-1) is that this flow is no longer of isoperimetric nature in the classical sense, because if we choose μ to preserve enclosed volume, the surface area is no longer decreasing necessarily.

However, the flow (1-1), with global term chosen to preserve the weighted volume $\int_{\Omega_t} \mathrm{c}_K$, has decreasing surface area, which suggests that in principle it is the right flow to prove the isoperimetric type inequality

$$\int_{\Omega_0} \mathrm{c}_K \le \phi(|M_0|),$$

with equality if and only if Ω_0 is a ball centered at the origin. Here ϕ is a function that gives equality on the slices. This was originally shown in [Girão and Pinheiro 2017, Proposition 4] by other means, and hence we do not pursue any further investigation in this matter here.

This reinforces the idea that our new flow has a geometric meaning beyond the generalization of Huisken's result, and we hope that in the future some interesting new applications will follow.

Structure of the paper. The contents of this paper are organized as follows. We first introduce in Section 2 the basic notation and evolution equations that ensure that our flow preserves the quermassintegrals, while Section 3 gathers new estimates for strictly convex hypersurfaces in the sphere, which may be of independent interest, like a refined outradius bound (Theorem 3.3) or inradius control in terms of pinching (Corollary 3.2). Then in Section 4 we prove that the *pinching deficit* decreases exponentially under the flow as time evolves, which is the key to get convergence of the evolving hypersurfaces. To achieve upper curvature bounds, we perform a technically intricate process in Section 5, which includes a delicate iterative changing of origin to ensure an optimal configuration that enables us to gain some uniform bound on the global term for some controlled time interval. This is a novel method, providing an alternative to proving initial value-independent curvature bounds after a waiting time. To finish the proof of Theorem 1.1, in Section 6 we establish long-time existence, and convergence to a geodesic sphere is done in Section 7. Finally, the elliptic results are proved in Section 8.

2. Notation, conventions and preliminary results

Hypersurfaces in space forms. Let $x : M \hookrightarrow \mathbb{M}^{n+1}_K$ be the embedding of a smooth hypersurface in a simply connected space form $\mathbb{M}^{n+1}_K$ enclosing a bounded domain Ω. Then the metric in polar coordinates is given by

$$\bar{g} = dr^2 + \mathrm{s}^2_K(r)\sigma,$$

where r is the radial distance to a fixed point $\mathcal{O} \in \mathbb{M}^{n+1}_K$ and σ is the round metric on $\mathbb{S}^n$.

The trigonometric functions in (1-2) satisfy the computational rules

$$\mathrm{c}'_K = -K\mathrm{s}_K, \quad \mathrm{c}^2_K + K\mathrm{s}^2_K = 1.$$

We will also use the related notation $\mathrm{co}_K(r) = \mathrm{c}_K(r)/\mathrm{s}_K(r)$.

For the outward pointing unit normal ν, we define the second fundamental form $h = (h_{ij})$ by

$$\overline{\nabla}_X Y = \nabla_X Y - h(X, Y)\nu,$$

where $\overline{\nabla}$ is the Levi-Civita connection of the metric $\bar{g} = \langle \cdot\,, \cdot \rangle$ on $\mathbb{M}^{n+1}_K$, and X and Y are vector fields on M. We adopt the summation convention throughout, and latin indices indicate components with respect to a coordinate frame $(\partial_i)_{1\le i\le n}$ on the domain of the embedding x.

If the induced metric on M is denoted by g, then we write Δ for its Laplace–Beltrami operator and define the Weingarten operator $A = (h^i_j)$ via

$$h_{ij} = g(A(\partial_i), \partial_j) = g_{ik}h^k_j.$$

Recall that the symmetry of h and the *Codazzi equations*

$$\nabla_i h_{jk} = \nabla_j h_{ik}$$

imply that the tensor ∇A is totally symmetric. Moreover, one can relate the geometry of a hypersurface M with the ambient manifold $\mathbb{M}_K^{n+1}$ by means of the *Gauss equation*

$$R_{ijk\ell} = h_{ik}h_{j\ell} - h_{i\ell}h_{jk} + K(g_{ik}g_{j\ell} - g_{i\ell}g_{jk}). \tag{2-1}$$

On the other hand, if

$$\kappa_1 \le \cdots \le \kappa_n$$

denote the eigenvalues of the operator A, that is, the principal curvatures of M, we consider the normalized mean curvatures H_ℓ defined as

$$H_\ell = \binom{n}{\ell}^{-1} \sigma_\ell, \quad \text{with } \sigma_\ell = \sum_{1 \le i_1 < \cdots < i_\ell \le n} \kappa_{i_1} \cdots \kappa_{i_\ell}.$$

In particular, $H_1 = H/n$ and H_n equals the Gauss curvature. We use the convention that $H_0 = 1$. For convex hypersurfaces, these symmetric functions satisfy the *Newton–MacLaurin* inequalities [Wang and Xia 2014]

$$H_{\ell-1}H_k \ge H_\ell H_{k-1} \quad \text{for } 1 \le k < \ell \le n. \tag{2-2}$$

We will also use the *Hsiung–Minkowski* identities [Guan and Li 2015]

$$(\ell+1)\int_M u\sigma_{\ell+1} = (n-\ell)\int_M \mathrm{c}_K \sigma_\ell \tag{2-3}$$

for $\ell = 0, \dots, n-1$.

Later on, we need to control the support function from below, given that there is a uniform ball enclosed by the evolving domain. Fortunately, for strictly convex domains, such control is easy to obtain. We furnish quantities like r and u with a hat if their reference point is not the origin. The right reference point will then be displayed as a subscript, and in cases where the reference point is clear by context, it is suppressed.

Lemma 2.1. *Let $\Omega \subset \mathbb{M}_K^{n+1}$ be a strictly convex domain with $p \in \Omega$ and $M = \partial\Omega$. Then the support function*

$$\hat{u}_p = \mathrm{s}_K(\hat{r}_p)\langle \partial_{\hat{r}_p}, \nu\rangle,$$

where $\hat{r}_p$ is the distance to the point p, satisfies

$$\hat{u}_p \ge \min_M \hat{u}_p = \min_M \mathrm{s}_K(\hat{r}_p) = \mathrm{s}_K(\operatorname{dist}(p, M)).$$

Proof. At a global minimum of the support function, we have $\nabla \hat{u}_p = 0$. It is well known (see [Guan and Li 2015]) that

$$\nabla_i \hat{u}_p = h_i^j \nabla_j\left(\frac{1-\mathrm{c}_K}{K}\right) = -h_i^j \frac{\mathrm{c}_K'}{K}\nabla_j \hat{r}_p = \mathrm{s}_K(\hat{r}_p) h_i^j \nabla_j \hat{r}_p. \tag{2-4}$$

Accordingly, due to the invertibility of A, we also have $\nabla \hat{r}_p = 0$. Hence, at such a point and for $K > 0$,

$$\hat{u}_p = \mathrm{s}_K(\hat{r}_p) \ge \min\bigl(\mathrm{s}_K\bigl(\min_M \hat{r}_p\bigr), \mathrm{s}_K\bigl(\max_M \hat{r}_p\bigr)\bigr)$$

due to the concavity of s_K within the interval $[0, \pi/\sqrt{K}]$. In the case $K \le 0$, we have that s_K is increasing, so we are done. Now suppose that

$$\mathrm{s}_K\big(\max_M \hat{r}_p\big) < \mathrm{s}_K\big(\min_M \hat{r}_p\big). \tag{2-5}$$

Due to the symmetries of the sine function we must then have

$$\min_M \hat{r}_p > \frac{\pi}{\sqrt{K}} - \max_M \hat{r}_p.$$

At a point $\xi \in M$, where $\hat{r}_p$ is maximized, the geodesic which connects p and ξ intersects M in another point, say $\zeta \in M$. Due to the convexity of M, we have

$$\frac{\pi}{\sqrt{K}} > \operatorname{dist}(\zeta, \xi) = \hat{r}_p(\zeta) + \hat{r}_p(\xi) \ge \hat{r}_p(\zeta) + \frac{\pi}{\sqrt{K}} - \min_M \hat{r}_p \ge \frac{\pi}{\sqrt{K}},$$

a contradiction. Hence (2-5) cannot be valid and the proof is complete. □

2.1. ***Mixed volumes and quermassintegrals.*** We define the *curvature integrals* or *mixed volumes* as

$$V_{n-\ell}(\Omega) = \int_M H_\ell \, dV \quad \text{for } \ell = 0, \dots, n.$$

On the other hand, for any connected domain $\Omega \subset \mathbb{M}_K^{n+1}$ bounded by a compact hypersurface of class C^3, the *quermassintegrals* are given by (see [Solanes 2006] or [Santaló 1976, Chapter 17])

$$W_\ell(\Omega) = \frac{(n+1-\ell)\omega_{\ell-1}\cdots\omega_0}{(n+1)\omega_{n-1}\cdots\omega_{n-\ell}} \int_{\mathcal{L}_\ell} \chi(L_\ell \cap \Omega)\, dL_\ell, \quad \ell = 1, \dots, n. \tag{2-6}$$

Here $\mathcal{L}_\ell$ represents the space of ℓ-dimensional totally geodesic subspaces L_ℓ in $\mathbb{M}_K^{n+1}$, where one can define a natural invariant measure dL_ℓ, and $\omega_n = |\mathbb{S}^n|$ is the area of the n-dimensional unit sphere in $\mathbb{R}^{n+1}$. If Ω is a convex set, then the function χ is equal to 1 if $L_\ell \cap \Omega \ne \varnothing$ and 0 otherwise.

One typically sets

$$W_0(\Omega) = |\Omega| \quad \text{and} \quad W_{n+1}(\Omega) = \frac{\omega_n}{n+1}.$$

Moreover, using the Cauchy–Crofton formula (see [Santaló 1976]), we recover the area of the hypersurface

$$|\partial\Omega| = (n+1)W_1(\Omega).$$

Accordingly, volume- and area-preserving flows can be regarded as particular cases of quermassintegral-preserving flows.

Mixed volumes and quermassintegrals are related (see [Solanes 2006, Proposition 7]) in a space of constant curvature $\mathbb{M}_K^{n+1}$ by means of

$$\begin{aligned} \frac{1}{n+1} V_{n-\ell}(\Omega) &= W_{\ell+1}(\Omega) - K \frac{\ell}{n+2-\ell} W_{\ell-1}(\Omega), \quad \ell = 1, \dots, n, \\ V_n(\Omega) &= (n+1)W_1(\Omega) = |\partial\Omega|. \end{aligned} \tag{2-7}$$

Notice that in $\mathbb{R}^{n+1}$ the mixed volumes coincide with the quermassintegrals, up to a constant factor.

The next result gathers the evolution equations of the quantities defined above under a normal variation.

Lemma 2.2. *If M_t is a hypersurface of $\mathbb{M}_K^{n+1}$ evolving along a flow given by $\partial_t x = \varphi \nu$, then*

(a) $\partial_t \operatorname{Vol}(\Omega_t) = \int_M \varphi \, dV_t$ *and* $\partial_t |M_t| = \int_M \varphi H \, dV_t$,

(b) $\partial_t \int_M H_\ell \, dV_t = \int_M \varphi((n-\ell)H_{\ell+1} - K\ell H_{\ell-1}) \, dV_t, \quad \ell = 0, \dots, n,$

(c) $\partial_t W_\ell(\Omega_t) = \dfrac{n+1-\ell}{n+1} \int_M \varphi H_\ell \, dV_t, \quad \ell = 0, \dots, n,$

(d) $\partial_t g_{ij} = 2\varphi h_{ij}$,

(e) $\partial_t h^i_j = -g^{ik}\nabla^2_{kj}\varphi - \varphi h^i_k h^k_j - K\varphi\delta^i_j$.

Proof. The formulas in (a) and (b) were deduced in [Reilly 1973]. The evolution in (c) follows by induction on ℓ and using the relation (2-7); see [Wang and Xia 2014, Proposition 3.1] for $K = -1$. The evolution for the metric and the Weingarten operator are standard, e.g., [Gerhardt 2006, Chapter 2]. □

Corollary 2.3. *If the global term in (1-1) is chosen as in (1-3), then the quermassintegral $W_\ell(\Omega_t)$ is constant along the flow (1-1).*

The fact that $\mu(t) > 0$ for strictly convex hypersurfaces is heavily used within the proof of Theorem 1.1.

Remark 2.4. Notice that a global term given by

$$\mu(t) = \frac{\int_M H((n-\ell)H_{\ell+1} - K\ell H_{\ell-1}) \, dV_t}{\int_M \mathrm{c}_K((n-\ell)H_{\ell+1} - K\ell H_{\ell-1}) \, dV_t}$$

leads to a flow that preserves the mixed volume $V_{n-\ell}(\Omega_t)$. Unlike the quermassintegral-preserving case, this term does not have a sign for convex hypersurfaces if $K > 0$. For $K < 0$ this difficulty disappears, but another type of mixed volume-preserving curvature flows for h-convex hypersurfaces in the hyperbolic space was already studied in [Makowski 2012].

We use the following conventions for constants. Indexed letters C, i.e., C_0, C_1, etc. will retain a specific meaning throughout the whole paper, while the letter C denotes a generic constant, which is always allowed to change from line to line and depends on the quantities listed in the formulation of the lemma or theorem. Capital letters also stand for "large" constants. A similar convention holds for lower case letters, which stand for "small" constants. The only exception from this convention concerns the use of various versions of the letter t, like T, τ, $\hat{\tau}$, etc. Those always refer in some way to time and t denotes the time variable, while T, τ, $\hat{\tau}$, etc. will, once defined, not change value.

3. Geometry and location of strictly convex hypersurfaces in the sphere

This section presents some geometric results for strictly convex hypersurfaces of the sphere, which are required to prove Theorem 1.1. In particular, we obtain inradius estimates in terms of pinching, as well as a suitable outball configuration in terms of pinching and the value of any given $W_\ell(\Omega)$. *Throughout Sections 3–5, we make the standing assumption that $M \subset \mathbb{S}_K^{n+1}$ is a strictly convex hypersurface enclosing a domain Ω.*

Let B_r denote a geodesic ball of radius r in $\mathbb{S}^{n+1}_K$. The outer radius of Ω is given by

$$\rho_+(\Omega) = \inf\{R > 0 : \Omega \subset B_R(q) \text{ for some } q \in \mathbb{S}^{n+1}_K\},$$

and the inner radius is given by

$$\rho_-(\Omega) = \sup\{\rho > 0 : B_\rho(p) \subset \Omega \text{ for some } p \in \mathbb{S}^{n+1}_K\}.$$

In [Andrews 1994a], it was shown that pinched hypersurfaces of the Euclidean space satisfy a uniform control of the outer radius by inner radius, and a version for a positive ambient space can be found in [Gerhardt 2015, Section 6]. The proof of this version relied on uniform positivity of the smallest principal curvature, which is insufficient for our purposes. Hence we provide a more general version in the following proposition.

Proposition 3.1. *If, for some number $C_0 > 0$, we have the pinching estimate $\kappa_n \le C_0\kappa_1$ in M, then the outer radius is estimated from above according to*

$$\rho_+(\Omega) \le C_1\rho_-(\Omega)$$

for some positive constant $C_1 = C_1(n, K, C_0)$.

Proof. For simplicity but without loss of generality, we assume $K = 1$. Due to a classical result [do Carmo and Warner 1970],

$$0 < \rho_+(\Omega) < \tfrac{1}{2}\pi$$

because M lies in some open hemisphere. Hence, there is a center q such that

$$\Omega \subset B_{\rho_+}(q),$$

and it is true that $q \in \bar{\Omega}$ (see [Santaló 1946, p. 455]). By moving q slightly inwards, we can achieve $M \subset B_{\pi/2}(q)$, and that M is star-shaped around q.

Now consider the stereographic projection from the antipodal point $-q$, where q is mapped to the origin $0 \in \mathbb{R}^{n+1}$. It follows (see [Gerhardt 2015, (6.15)]) that the metric $\bar{g}$ of $\mathbb{S}^{n+1}$ is conformal to the Euclidean metric; more precisely,

$$\bar{g} = e^{2\psi}(dr^2 + r^2\sigma), \quad \text{with } \psi(r) = -\ln\bigl(1 + \tfrac{1}{4}r^2\bigr).$$

Hereafter, we denote by tilde the Euclidean geometric quantities. On $\bar{B}_{\pi/2}(q)$ the metric $\bar{g}$ is uniformly equivalent to the Euclidean metric.

Next, from [Gerhardt 2006, (1.1.51)], we get

$$e^{\psi}\kappa_i = \tilde{\kappa}_i + d\psi(\tilde{\nu})$$

and hence, from our pinching assumption,

$$0 < C_0^{-1} \le \frac{\kappa_1}{\kappa_n} = \frac{\tilde{\kappa}_1 + d\psi(\tilde{\nu})}{\tilde{\kappa}_n + d\psi(\tilde{\nu})} \le 1.$$

Thus

$$\tilde{\kappa}_1 + d\psi(\tilde{\nu}) \ge C_0^{-1}(\tilde{\kappa}_n + d\psi(\tilde{\nu}))$$

and

$$\tilde{\kappa}_1 \ge C_0^{-1}\tilde{\kappa}_n + (C_0^{-1} - 1)\psi'(r)\langle\partial_r, \tilde{\nu}\rangle \ge C_0^{-1}\tilde{\kappa}_n$$

because $\psi' < 0$ and $\widetilde{M}$ is star-shaped, i.e., $\langle \partial_r, \tilde{\nu} \rangle > 0$. Therefore, the Euclidean hypersurface $\widetilde{M} \subset \widetilde{B}_2(0)$ is pinched, which from [Andrews 1994a, Lemma 5.4] leads to

$$\rho_+(\Omega) \le C\tilde{\rho}_+(\Omega) \le C\tilde{\rho}_-(\Omega) \le C_1\rho_-(\Omega),$$

where we have used the uniform equivalence of the ambient metrics. □

Corollary 3.2. *If, for some number $C_0 > 0$, we have the pinching estimate $\kappa_n \le C_0\kappa_1$ in M, then one can find positive constants d_1 and C_2, depending on n, K, C_0 and $W_\ell(\Omega)$, such that*

$$d_1 \le \rho_-(\Omega) \le C_2 < \frac{\pi}{2\sqrt{K}}.$$

Proof. By the definition of inner and outer radius, we can find points $p, q \in \mathbb{S}^{n+1}_K$ such that

$$B_{\rho_-(\Omega)}(p) \subset \Omega \subset B_{\rho_+(\Omega)}(q).$$

From (2-6), the quermassintegrals W_ℓ are clearly monotone under the inclusion of convex domains, and hence

$$W_\ell(B_{\rho_-(\Omega)}(p)) \le W_\ell(\Omega) \le W_\ell(B_{\rho_+(\Omega)}(q)).$$

We obtain, with Proposition 3.1,

$$C_2 := f_\ell^{-1}(W_\ell(\Omega)) \ge \rho_-(\Omega) \ge C_1^{-1}\rho_+(\Omega) \ge C_1^{-1} f_\ell^{-1}(W_\ell(\Omega)) =: d_1,$$

where f_ℓ denotes the increasing function given by $f_\ell(r) = W_\ell(B_r)$. □

The trivial outer radius estimate

$$\rho_+(\Omega) < \frac{\pi}{2\sqrt{K}}$$

is not good enough for our purposes. Now we present a refined estimate which should be of independent interest in the future.

Theorem 3.3. *The outer radius satisfies*

$$\rho_+(\Omega) \le \frac{\pi}{2\sqrt{K}} - \frac{\log W_\ell(\mathcal{H}) - \log W_\ell(\Omega)}{(n+1-\ell)\max_M H} =: \frac{\pi}{2\sqrt{K}} - \frac{d_2}{\max_M H},$$

where $\mathcal{H}$ is an open hemisphere.

Proof. From the initial hypersurface $M_0 = M$, we start the curvature flow

$$x : [0, T^*) \times \mathbb{S}^n \to \mathbb{S}^{n+1}_K, \quad \partial_t x = \frac{H_{\ell-1}}{H_\ell}\nu. \tag{3-1}$$

Then Lemma 2.2 (c) ensures that W_ℓ evolves in time according to

$$\partial_t W_\ell(\Omega_t) = \frac{n+1-\ell}{n+1} V_{n-\ell+1}(\Omega_t) \le (n+1-\ell)W_\ell(\Omega_t),$$

where the inequality follows from (2-7).

From [Gerhardt 2015; Makowski and Scheuer 2016], we know that (3-1) preserves the strict convexity and the solution converges smoothly to an equator, while we also have the estimate

$$W_\ell(\Omega_t) \le W_\ell(\Omega)e^{(n+1-\ell)t}.$$

Hence the maximal existence time of (3-1) is at least

$$T^* \ge \frac{1}{n+1-\ell}(\log W_\ell(\mathcal{H}) - \log W_\ell(\Omega))$$

because we know that $W_\ell(\Omega)$ must converge to $W_\ell(\mathcal{H})$ at T^*.

Now [Makowski and Scheuer 2016, Lemma 4.7] leads to the curvature bound

$$\max_{M_t} H \le \max_{M_0} H \quad \text{for all } 0 \le t < T^*.$$

Due to the convexity this implies a full second fundamental form bound, as well as a bound

$$\frac{H_\ell}{H_{\ell-1}} \le H_1 \le \max_{M_0} H, \tag{3-2}$$

which follows by application of (2-2) for $k = 1$.

Next, let $E = \partial\mathcal{H}$ be the limiting equator of the flow, and let r be the radial distance from the center of $\mathcal{H}$, which contains all M_t. Define

$$\tilde{r}(t) = \max_{\mathbb{S}^n} r(t,\cdot) = r(t,\xi_t),$$

where ξ_t is chosen to be any point where the maximum is realized. The function $\tilde{r}$ is Lipschitz and hence differentiable almost everywhere. At each time t, where $\tilde{r}$ is differentiable, we have

$$\frac{d}{dt}\tilde{r}(t) = \frac{H_{\ell-1}}{H_\ell}(t,\xi_t),$$

where we used that $\nu(t,\xi_t) = \partial_r$. Integration and (3-2) yield

$$\frac{\pi}{2\sqrt{K}} - \tilde{r}(0) = \tilde{r}(T^*) - \tilde{r}(0) = \int_0^{T^*} \frac{d}{dt}\tilde{r} \ge \frac{\log W_\ell(\mathcal{H}) - \log W_\ell(\Omega)}{(n+1-\ell)\max_{M_0} H}.$$

Hence

$$\max_{M_0} r = \tilde{r}(0) \le \frac{\pi}{2\sqrt{K}} - \frac{\log W_\ell(\mathcal{H}) - \log W_\ell(\Omega)}{(n+1-\ell)\max_{M_0} H}.$$

Accordingly, M_0 fits into a neighborhood of the origin of size given by the right-hand side of the latter inequality, and therefore the outer radius is controlled by the very same quantity. □

Remark 3.4. A lune, i.e., the intersection of two hemispheres, shows that an outer radius bound in terms of $W_\ell(\Omega)$ cannot be independent of $\max H$.

Theorem 3.3 enables us to find, for a given strictly convex hypersurface of the sphere, a suitable origin, which allows a ball of controlled size within Ω and at the same time ensures a controlled positive distance of M to the equator.

Lemma 3.5. *There exists an origin $\mathcal{O} \in \Omega$ such that, with the constant d_2 from Theorem 3.3, we have*

$$B_{4\epsilon}(\mathcal{O}) \subset \Omega \quad \text{and} \quad \max_M r \le \frac{\pi}{2\sqrt{K}} - 4\epsilon \tag{3-3}$$

for all

$$\epsilon \le \frac{1}{4} \min\left(\frac{d_2}{2 \max_M H}, \frac{\frac{1}{2}\pi - \tan^{-1}(\max_M H/\sqrt{K})}{2\sqrt{K}} \right).$$

Proof. If $B_{\rho_+(\Omega)}$ denotes an outball for Ω with center $\mathcal{O}$, then we know that $\mathcal{O} \in \bar{\Omega}$. The distance of focal points from M can be calculated from the evolution of the Weingarten operator in Lemma 2.2 along the normal variation with speed $\varphi = -1$. Then the largest principal curvature is controlled by the solution to the ODE

$$y' = y^2 + K,$$
$$y(0) = \max_M H,$$

which exists for all $t < t_0$, with

$$t_0 := \frac{\pi}{2\sqrt{K}} - \frac{\tan^{-1}(\max_M H/\sqrt{K})}{\sqrt{K}}.$$

Hence, around all points belonging to the set

$$\left\{ x \in \Omega : \operatorname{dist}(x, M) = \tfrac{1}{2} t_0 \right\},$$

there exists an interior ball of radius $\frac{1}{2}t_0$. In addition, if we shift $\mathcal{O}$ by a distance of $d_2/(2 \max_M H)$ in any direction, there still is the same amount of space between M and the new equator. Therefore, if we shift $\mathcal{O}$ into Ω along a perpendicular geodesic only by the amount

$$\epsilon \le \epsilon_0 = \frac{1}{4} \min\left(\frac{d_2}{2 \max_M H}, \frac{t_0}{2} \right),$$

then (3-3) holds. ☐

4. Monotonicity of the pinching deficit

The geometric results from Section 3 depend on the quality of the pinching and the size of the quermassintegral. In the following we investigate how these quantities behave under the flow (1-1). As this flow is defined to be quermassintegral-preserving, the key ingredient for proving Theorem 1.1 is the pinching estimate to be proven in this section.

Again, we assume $K > 0$, unless stated otherwise. As a first step, we need the following evolution equations.

Lemma 4.1. *For every choice of origin $\mathcal{O}$, along (1-1), the induced metric g and second fundamental form h satisfy the evolution equations*

$$\partial_t g_{ij} = 2(\mu c_K - H) h_{ij},$$
$$\partial_t h^i_j = \Delta h^i_j + (|A|^2 - nK) h^i_j + 2K H \delta^i_j - \mu (c_K h^i_k h^k_j + K u h^i_j),$$

where u is the generalized support function in (1-5) of M_t with respect to the origin $\mathcal{O}$.

Proof. The evolution of the metric comes from Lemma 2.2 (d). For the evolution of A, we depart from the standard evolution equation

$$\partial_t h_{ij} = \nabla^2_{ij}(H - \mu c_K) + (\mu c_K - H)(h_{ik}h^k_j - K g_{ij})$$

(see [Andrews 1994b, Theorem 3-15]) and use the Simons-type identity

$$\nabla^2_{ij} H = \Delta h_{ij} + (|A|^2 - nK)h_{ij} + H(K g_{ij} - h^k_i h_{jk}).$$

Now we expand the second derivatives of c_K:

$$-\nabla^2_{ij} c_K = d c_K(\nu) h_{ij} - \overline{\nabla}^2 c_K(x_i, x_j) = c'_K \langle \partial_r, \nu\rangle h_{ij} + K c_K g_{ij}, \tag{4-1}$$

where we have used

$$\overline{\nabla}^2 c_K = -K c_K \bar{g}.$$

The proof is complete, using the evolution of the metric to revert to h^i_j. □

Remark 4.2. Notice that the strong maximum principle for tensors applied to the evolution of h^i_j already implies that the property of strict convexity is preserved for all times. Accordingly, $H > 0$, and the quotient κ_1/H is well defined as long as the flow exists.

Next we deduce an evolution equation that is the key to convergence of the flow.

Proposition 4.3. *Let κ_1 be the smallest eigenvalue of A. Then, for every choice of origin $\mathcal{O}$, under the flow (1-1) with initial data M, the function*

$$p = \frac{\kappa_1}{H}$$

is a supersolution to the following evolution equation in the viscosity sense, as long as the flow exists:

$$\partial_t p - \Delta p = \frac{2}{H}\sum_{k=1}^{n}\sum_{j>D}\frac{(\nabla_k h^1_j)^2}{\kappa_j - \kappa_1} + 2\,dp(\nabla \log H) + \frac{\mu}{H} c_K |\mathring{A}|^2 p + \mu c_K \kappa_1\Big(\frac{1}{n} - p\Big) + 2nK\Big(\frac{1}{n} - p\Big),$$

where D is the multiplicity of κ_1.

Proof. Assume that the flow is defined on a maximal time interval $[0, T)$. Let $(t_0, \xi_0) \in (0, T) \times M$, and let η be a smooth lower support of p at (t_0, ξ_0), i.e., η is defined on a spacetime neighborhood $\mathcal{U}$ of (t_0, ξ_0), and we have

$$\eta(t_0, \xi_0) = p(t_0, \xi_0), \qquad \eta \le p_{|\mathcal{U}}.$$

Hence $\varphi = H\eta$ is a smooth lower support for κ_1.

Now we take coordinates with the properties

$$g_{ij} = \delta_{ij}, \qquad h^i_j = \kappa_j \delta^i_j \quad \text{at } (t_0, \xi_0).$$

If we denote by D the multiplicity of $\kappa_1(t_0, \xi_0)$, then at the point (t_0, ξ_0) and for all $1 \le i,\ j \le D$, we have (see [Brendle et al. 2017, Lemma 5])

$$\partial_t h^i_j = \delta^i_j \partial_t \varphi, \qquad \nabla_k h^i_j = \delta^i_j \nabla_k \varphi$$

and

$$\nabla^2_{kk}\varphi \le \nabla^2_{kk}h^1_1 - 2\sum_{j>D}\frac{(\nabla_k h^1_j)^2}{\kappa_j-\kappa_1}. \tag{4-2}$$

Next, write

$$G^i_j = \partial_t h^i_j - \Delta h^i_j,$$

and compute at (t_0, ξ_0)

$$\begin{aligned}\partial_t\eta = \frac{\partial_t h^1_1}{H} - \frac{\varphi}{H^2}\partial_t H &= \frac{G^1_1 + g^{kl}\nabla^2_{kl}h^1_1}{H} - \frac{\varphi}{H^2}\partial_t H\\ &\ge \frac{1}{H}\sum_{k=1}^n\left(\nabla^2_{kk}\varphi + 2\sum_{j>D}\frac{(\nabla_k h^1_j)^2}{\kappa_j-\kappa_1}\right) + \frac{G^1_1}{H} - \frac{\varphi}{H^2}\partial_t H\\ &= \frac{2}{H}\sum_{k=1}^n\sum_{j>D}\frac{(\nabla_k h^1_j)^2}{\kappa_j-\kappa_1} + \frac{G^1_1}{H} + \Delta\eta + 2\,d\eta(\nabla\log H) - \frac{\eta}{H}G^k_k.\end{aligned}$$

On the other hand, we know by Lemma 4.1 that

$$G^i_j = (|A|^2 - nK)h^i_j + 2KH\delta^i_j - \mu(\mathrm{c}_K h^i_k h^k_j + Kuh^i_j).$$

Accordingly, we get

$$G^1_1 - \eta G^k_k = -\mu\mathrm{c}_K\kappa_1^2 + \eta\mu\mathrm{c}_K|A|^2 + 2KH(1-n\eta).$$

Finally, by means of

$$|A|^2 = |\mathring{A}|^2 + \frac{1}{n}H^2,$$

we obtain

$$G^1_1 - \eta G^k_k = \eta\mu\mathrm{c}_K|\mathring{A}|^2 + \mu\mathrm{c}_K\kappa_1 H\left(\frac{1}{n}-\eta\right) + 2KH(1-n\eta). \qquad \square$$

Corollary 4.4. *For every choice of origin $\mathcal{O}$ for which $M \subset \mathcal{H}(\mathcal{O})$, the flow (1-1) with initial data M stays in $\mathcal{H}(\mathcal{O})$ and improves every pinching, i.e., the* ***pinching deficit***

$$\omega(t) = \frac{1}{n} - \min_{M_t}\frac{\kappa_1}{H}$$

is exponentially decreasing:

$$\omega(t) \le \omega(s)e^{-2nK(t-s)} \quad \textit{for all } 0 \le s \le t < T,$$

where T is the maximal time of existence of the flow with initial data M.

Proof. We need the evolution equation of c_K. From (1-1) and (4-1), we obtain

$$\partial_t\mathrm{c}_K = \mathrm{c}'_K\partial_t r = -K\mathrm{s}_K\,dr(\partial_t x) = Ku(H-\mu\mathrm{c}_K) = \Delta\mathrm{c}_K + K\mathrm{c}_K(n-\mu u).$$

The preservation of $\mathrm{c}_K > 0$ follows immediately by the strong maximum principle, as long as the flow exists.

The strict convexity is also preserved from Remark 4.2. The statement about the pinching deficit follows from the strong maximum principle for viscosity solutions, e.g., see [Da Lio 2004], and from the fact that

$$\partial_t \omega \le -2nK\omega,$$

where we can discard the terms including μ because $c_K > 0$ and $\mu(t) > 0$ by convexity. □

In particular, for $s = 0$, we reach a pinching relation between the biggest and smallest principal curvatures of M_t:

$$\kappa_1 \ge \left(\min_{M_0} \frac{\kappa_1}{H}\right) H \ge C_0^{-1} \kappa_n, \tag{4-3}$$

provided that an origin is chosen such that the strictly convex initial hypersurface is contained in the open hemisphere centered at that origin.

5. Upper curvature bounds

Notice that, unlike in previous treatments of quermassintegral-preserving curvature flows, an upper bound for the global term does not come automatically from an upper bound for H, since the c_K in the denominator of μ is not uniformly bounded away from zero, at least not without further work. Moreover, we need some uniform control of μ to get bounds for H.

To overcome these difficulties, the idea is to choose the origin such that a configuration as in Lemma 3.5 is achieved, which will allow us to deduce uniform bounds on the curvature and the global term in a short but controlled interval $[0, \tau(\epsilon)]$ (Lemma 5.4). Then, since the pinching is at least as good as at the beginning, we can repeat this process as often as needed, in order to keep the flow going as long as we like (Lemma 6.1). During this evolution, the pinching improves exponentially and at some point will be strong enough that M_t is very close to a sphere. From here the flow is very easy to estimate and no further shifting of the origin is necessary. Now we will implement all the required steps to make this argument rigorous.

A key idea to obtain a curvature bound is to adapt a well-known trick from [Tso 1985], which consists in a suitable combination of the generalized support function with the mean curvature. For this we need control on the size of inballs during the flow.

A lower bound for the support function of an arbitrary inball. For a domain Ω and a point $p \in \Omega$, we say that B is an *inball at* p if p is the center of B and B has maximal radius with the property that $B \subset \Omega$. In the sequel we are going to prove that, along the flow, the radii of inballs at p don't decrease too quickly. We give a quantitative estimate.

We need to be careful because now we are dealing with two different support functions: we denote by u the support function with respect to the origin $\mathcal{O}$ that is implicit in the flow equation, while for a given point $p \in \Omega$,

$$\hat{u} \equiv \hat{u}_p = s_K(\hat{r}_p)\langle \partial_{\hat{r}_p}, \nu \rangle$$

takes another interior point p as the origin of distances. Accordingly, r and $\hat{r}$ mean distance from $\mathcal{O}$ and p, respectively. Similar notations will apply to the corresponding trigonometric functions, i.e.,

$$\hat{\mathrm{s}}_K := \mathrm{s}_K(\hat{r}_p) \quad \text{and} \quad \hat{\mathrm{c}}_K := \mathrm{c}_K(\hat{r}_p).$$

Note that for brevity we suppress the dependence on the point p within the notation $\hat{u}$ and $\hat{r}$.

Lemma 5.1. *For every choice of origin, along (1-1), the evolution equations of the mean curvature H and the support function $\hat{u} = \hat{u}_p$ are given by*

$$\partial_t H = \Delta H + H(|A|^2 + Kn) - \mu(\mathrm{c}_K|A|^2 + uKH),$$
$$\partial_t \hat{u} = \Delta\hat{u} + \hat{u}|A|^2 + (\mu\mathrm{c}_K - 2H)\hat{\mathrm{c}}_K + \mu K\mathrm{s}_K\hat{\mathrm{s}}_K\langle\nabla r, \nabla\hat{r}\rangle.$$

Proof. The formula for H follows directly by taking the trace in the evolution equation for h^i_j from Lemma 4.1. On the other hand, a standard calculation leads to

$$\begin{aligned}\partial_t\hat{u} &= \langle\overline{\nabla}_t(\hat{\mathrm{s}}_K\partial_{\hat{r}}), \nu\rangle + \langle\hat{\mathrm{s}}_K\partial_{\hat{r}}, \overline{\nabla}_t\nu\rangle \\ &= (\mu\mathrm{c}_K - H)\hat{\mathrm{c}}_K + \hat{\mathrm{s}}_K\langle\partial_{\hat{r}}, \nabla(H - \mu\mathrm{c}_K)\rangle \\ &= \Delta\hat{u} + (\mu\mathrm{c}_K - 2H)\hat{\mathrm{c}}_K + \hat{u}|A|^2 - \mu\,\hat{\mathrm{s}}_K\langle\partial_{\hat{r}}, \nabla\mathrm{c}_K\rangle,\end{aligned}$$

where we applied well-known formulas for $\Delta\hat{u}$ and $\overline{\nabla}_t(\hat{\mathrm{s}}_K\partial_{\hat{r}})$ on $\mathbb{M}^{n+1}_K$ (see [Cabezas-Rivas and Miquel 2007, (4.6) and (4.11)]). The stated formula follows by realizing that

$$\langle\partial_{\hat{r}}, \nabla\mathrm{c}_K\rangle = -K\mathrm{s}_K\langle\partial_{\hat{r}}, \nabla r\rangle. \qquad \square$$

Proposition 5.2. *For every choice of origin $\mathcal{O}$ for which $M \subset \mathcal{H}(\mathcal{O})$, and for every $p \in \Omega$ and radius ρ with the property $B_\rho(p) \subset \Omega$, the solution $M_t = \partial\Omega_t$ of (1-1), with initial data M and maximal existence time $T > 0$, satisfies the following:*

(i) *There is a positive constant $\tilde{\tau} = \tilde{\tau}(n, K, \rho)$ with the property*

$$B_{\rho/4}(p) \subset \Omega_t \quad \textit{for all } t \in [0, \min(\tilde{\tau}, T)).$$

(ii) *One can find positive constants $d_3 \le 1/(2\sqrt{K})$ and τ, depending on $n,\ K,\ C_0$ and $W_\ell(\Omega)$, with the property*

$$\hat{u}_{p_\Omega} - 2d_3 \ge 2d_3 > 0 \quad \textit{for all } t \in [0, \min(\tau, T)),$$

where p_Ω is the center of an inball corresponding to the inradius $\rho_-(\Omega)$.

Proof. (i) Let us first obtain the evolution of the distance $\hat{r} = \hat{r}_p$ from the fixed point p to the points on M_t under the flow (1-1):

$$\partial_t\hat{r} = d\hat{r}(\partial_t x) = (\mu\mathrm{c}_K - H)\langle\nu, \partial_{\hat{r}}\rangle. \tag{5-1}$$

On the other hand, $\mathtt{r}(t)$ denotes the radius of a geodesic sphere centered at p that moves under the ordinary mean curvature flow starting at $\mathtt{r}(0) = \frac{1}{2}\rho$, that is,

$$\mathtt{r}'(t) = -n\mathrm{co}_K(\mathtt{r}(t)),$$

whose solution is given by

$$\mathrm{c}_K(\mathtt{r}(t)) = e^{Knt}\mathrm{c}_K\left(\tfrac{1}{2}\rho\right) \quad \text{for } t \geq 0.$$

As c_K is a decreasing function,

$$\mathtt{r}(t) \geq \tfrac{1}{4}\rho \quad \Longleftrightarrow \quad e^{Knt}\mathrm{c}_K\left(\tfrac{1}{2}\rho\right) \leq \mathrm{c}_K\left(\tfrac{1}{4}\rho\right),$$

meaning that

$$\mathtt{r}(t) \geq \tfrac{1}{4}\rho \quad \Longleftrightarrow \quad t \leq \frac{1}{Kn}\log\frac{\mathrm{c}_K\left(\frac{1}{4}\rho\right)}{\mathrm{c}_K\left(\frac{1}{2}\rho\right)} =: \tilde{\tau}.$$

Set $f(t,\cdot) = \hat{r}(t,\cdot) - \mathtt{r}(t)$ for $t \in [0, \min(\tau, T))$. Then $f(0,\cdot) > 0$, and f evolves according to

$$\partial_t f = (\mu\mathrm{c}_K - H)\langle \nu, \partial_{\hat{r}}\rangle + n\mathrm{co}_K(\mathtt{r}(t)).$$

If there exists a first time t_1 such that the geodesic sphere $B_{\mathtt{r}(t_1)}$ touches the hypersurface M_{t_1} at some point x_1, then at this first minimum for f we have $H(x_1,t_1) \leq n\mathrm{co}_K(\mathtt{r}(t_1))$, $\langle \partial_{\hat{r}}, \nu\rangle = 1$ and $\partial_t f(x_1,t_1) \leq 0$. Consequently, taking into account that $\mathrm{c}_K > 0$ and strict convexity is preserved, we have

$$\partial_t f(x_1,t_1) \geq \mu(t_1)\mathrm{c}_K(r(t_1)) > 0,$$

which is a contradiction, and hence the statement follows.

(ii) Apply (i) with $p = p_\Omega$ and $\rho = \rho_-(\Omega)$, and obtain the desired τ due to Corollary 3.2. Using Lemma 2.1, we get

$$\hat{u}_{p_\Omega} \geq \mathrm{s}_K\left(\tfrac{1}{4}\rho_-(\Omega)\right) \geq \mathrm{s}_K\left(\tfrac{1}{4}d_1\right) =: 4d_3, \tag{5-2}$$

where we applied the lower bound in Corollary 3.2. □

An upper bound for the mean curvature. To estimate the mean curvature along a solution with suitably located initial data, we use the well-known auxiliary function

$$\Phi_p = \frac{H}{\hat{u}_p - d_3},$$

which, after choosing the origin $\mathcal{O}$ as in Proposition 5.2, is well defined for a while for some suitable $p \in \Omega$. Routine computations lead to the evolution of Φ, where we suppress the dependence on p within the notation.

Lemma 5.3. *Under the assumptions of Proposition 5.2, along the flow (1-1), the function Φ evolves according to*

$$\partial_t\Phi = \Delta\Phi + \frac{2}{\hat{u}-d_3}\langle\nabla\Phi,\nabla\hat{u}\rangle + \Phi\left(nK - \frac{d_3}{\hat{u}-d_3}|A|^2\right) + 2\Phi^2\hat{\mathrm{c}}_K$$
$$- \mu K\Phi u - \frac{\mu}{\hat{u}-d_3}(\mathrm{c}_K|A|^2 + \Phi(K\mathrm{s}_K\hat{\mathrm{s}}_K\langle\nabla r,\nabla\hat{r}\rangle + \mathrm{c}_K\hat{\mathrm{c}}_K)).$$

Unlike in previous literature, we cannot neglect all the terms including μ, as some do not have a sign. Nor is it known at this point that μ is bounded. The novelty of our approach is to make use of Lemma 3.5, a configuration which enables us to gain some control on μ and then get an estimate on Φ and H.

Another complication arises from the necessity of using an iterative change of origin. The configuration of Lemma 3.5 depends on curvature. Hence we need a very precise estimate of curvature as the flow progresses, and it is insufficient to estimate the curvature by a multiple of its initial value, as then our time interval, along which $B_\epsilon(\mathcal{O}) \subset \Omega_t$ is valid, would decrease and we would not be able to prove long-time existence.

For this reason, we introduce a novel method, which also gives an interesting alternative to proving initial value-independent curvature bounds after a waiting time, as for example in [McCoy 2004, Equation (17)]. It provides a bound on Φ, which is uniform in p lying within a certain region.

With this purpose, we define a modified auxiliary function

$$\Psi : [0, T) \times \mathbb{S}^n \times \mathbb{S}^{n+1}_K \to \mathbb{R}, \quad (t, \xi, p) \mapsto \begin{cases} (\operatorname{dist}(p, M_t) - 2d_3)\Phi_p, & p \in V_t, \\ 0, & \text{otherwise}, \end{cases}$$

with

$$V_t := \{p \in \Omega_t : \operatorname{dist}(p, M_t) > 2d_3\},$$

and where for $p \notin \overline{\Omega}$ the distance to M_t is defined to be negative. In short,

$$\Psi(t, \xi, p) = \max\bigl(0, \min_{M_t} \hat{r}_p - 2d_3\bigr)\Phi_p.$$

Note that Ψ is Lipschitz, because when $\min_{M_t} \hat{r}_p = \operatorname{dist}(p, M_t) \geq 2d_3$, then, by Lemma 2.1,

$$\hat{u}_p \geq \mathrm{s}_K\bigl(\min_{M_t} \hat{r}_p\bigr) \geq \mathrm{s}_K(2d_3)$$

due to $\min_{M_t} \hat{r}_p < \pi/(2\sqrt{K})$. Furthermore,

$$\hat{u}_p \geq \mathrm{s}_K(2d_3) = \frac{1}{\sqrt{K}} \sin(2d_3\sqrt{K}) \geq \tfrac{6}{5} d_3,$$

where we used $\sin x \geq \frac{3}{5}x$ for $x \in \bigl[0, \frac{1}{2}\pi\bigr]$. Hence Φ_p is well defined for $p \in \overline{V}_t$. In the sequel we write, for brevity,

$$\max_{M_t} \Psi = \max_{\mathbb{S}^n \times \mathbb{S}^{n+1}_K} \Psi(t, \cdot).$$

Lemma 5.4. *There exists an origin $\mathcal{O} \in \Omega$, a constant $C_3(n, K, C_0, W_\ell(\Omega))$, and constants τ and ϵ_1, depending on n, K, C_0, $W_\ell(\Omega)$, $\max(C_3, \max_{M_0} \Psi)$, such that, for the solution $M_t = \partial\Omega_t$ of (1-1) with initial data $M_0 = M$ and maximal existence time $T > 0$, we have*:

(i) *$M_t \subset \mathcal{H}(\mathcal{O})$ for all $t \in [0, \min(\tau, T)]$.*

(ii) *$\hat{u}_{p_\Omega} \geq 4d_3$ and $B_{\epsilon_1}(\mathcal{O}) \subset \Omega_t$ for all $t \in [0, \min(\tau, T))$.*

(iii) *The function Ψ satisfies*

$$\max\bigl(C_3, \max_{M_t} \Psi\bigr) \leq \max\bigl(C_3, \max_{M} \Psi\bigr) \quad \textit{for all } t \in [0, \min(\tau, T)).$$

Proof. Define the functions

$$\epsilon_0(y) = \frac{1}{4}\min\left(\frac{d_2 d_3\sqrt{K}}{2y}, \frac{\frac{1}{2}\pi - \tan^{-1}(y/d_3 K)}{2\sqrt{K}}\right) \quad \text{for } y > 0$$

and

$$q(y) = \frac{n\pi C_0^{\ell+1}}{\sqrt{K}}\left(1 + \frac{5\pi}{\sqrt{K}d_3}\right)\frac{y}{\epsilon_0(y)} + \frac{n\pi^2}{4}y + \frac{\pi}{\sqrt{K}}y^2 - \frac{d_3^2}{n}y^3,$$

where d_3 and C_0 are the constants from Proposition 5.2 and (4-3), respectively. It is clear that $q(y)$ converges to $-\infty$ as $y \to \infty$, and hence it has a largest zero $\bar{y}$, which only depends on n, K, C_0 and $W_\ell(\Omega)$. Let $\psi_0 = \max_{t=0}\Psi$, and define

$$\epsilon_1 = \epsilon_0(\max(\bar{y}, \psi_0)).$$

(i) & (ii) With p_Ω from Proposition 5.2 (ii) and by definition of d_3 in (5-2), along $M_0 = M$ we have

$$H = (\hat{u}_{p_\Omega} - d_3)\Phi_{p_\Omega} \le \frac{\Phi_{p_\Omega}}{\sqrt{K}} \le \frac{\mathrm{s}_K(\min_M \hat{r}_{p_\Omega}) - 2d_3}{2d_3}\frac{\Phi_{p_\Omega}}{\sqrt{K}} \le \frac{1}{d_3\sqrt{K}}\psi_0,$$

and hence

$$\epsilon_1 \le \epsilon_0\big(d_3\sqrt{K}\max_M H\big) < \frac{\pi}{2\sqrt{K}},$$

where the latter estimate is due to the definition of ϵ_0. This is exactly the threshold required to apply Lemma 3.5.

Then we can apply Lemma 3.5 with $\epsilon = \epsilon_1$ in order to obtain a suitable origin $\mathcal{O} \in \Omega$ with the property (3-3). From the first part of Proposition 5.2 applied to $p = \mathcal{O}$ and $\rho = 4\epsilon_1$, as well as from the second part of Proposition 5.2, we obtain $\tau = \tau(n, K, C_0, W_\ell(\Omega), \epsilon_1)$, up to which the claimed properties of (ii) are satisfied. Property (i) is then clear from the fact that, at the equator, $\mathrm{c}_K = 0$, and hence $(d/dt)\max r < 0$.

(iii) Next, we bound the function Ψ. Suppose Ψ attains a positive maximum over the set $[0, \bar{t}] \times \mathbb{S}^n \times \mathbb{S}^{n+1}_K$ at some $(\bar{t}, \bar{\xi}, \bar{p})$. Define the Lipschitz function

$$\overline{\Psi}_{\bar{p}}(t) = \max_{\mathbb{S}^n}\Psi(t, \cdot, \bar{p}),$$

which is positive in some small interval $J := [\bar{t} - \delta, \bar{t}]$. Thus we have

$$\mathrm{dist}(\bar{p}, M_t) > 2d_3 \quad \text{for all } t \in J.$$

Hence in J the function $\Phi_{\bar{p}}$ is smooth, and

$$\overline{\Psi}_{\bar{p}}(t) = \big(\min_{M_t}\hat{r}_{\bar{p}} - 2d_3\big)\max_{M_t}\Phi_{\bar{p}}$$

is differentiable almost everywhere in J.

For almost every $t \in J$, using (5-1) and $\langle \partial_{\hat{r}_{\bar{p}}}, \nu \rangle > 0$ because $\bar{p} \in \Omega_t$ for all $t \in J$, we have

$$\begin{aligned}\frac{d}{dt}\overline{\Psi}_{\bar{p}} &= \frac{d}{dt}\min_{M_t}\hat{r}_{\bar{p}}\max_{M_t}\Phi_{\bar{p}} + \bigl(\min_{M_t}\hat{r}_{\bar{p}} - 2d_3\bigr)\frac{d}{dt}\max_{M_t}\Phi_{\bar{p}} \\ &= \bigl(\mathrm{c}_K(r)\mu - H\bigl(\bar{t}, x\bigl(\bar{t}, \operatorname{argmin}_{M_{\bar{t}}}\hat{r}_{\bar{p}}\bigr)\bigr)\bigr)\langle \partial_{\hat{r}_{\bar{p}}}, \nu\rangle \max_{M_t}\Phi_{\bar{p}} + \bigl(\min_{M_{\bar{t}}}\hat{r}_{\bar{p}} - 2d_3\bigr)\frac{d}{dt}\max_{M_t}\Phi_{\bar{p}} \\ &\le \mu\max_{M_t}\Phi_{\bar{p}} + \bigl(\min_{M_{\bar{t}}}\hat{r}_{\bar{p}} - 2d_3\bigr)\frac{d}{dt}\max_{M_t}\Phi_{\bar{p}}.\end{aligned}$$

Inserting the evolution equation from Lemma 5.3, discarding two good terms, and using $\nabla\Phi_{\bar{p}}(t, \xi_t) = 0$ at all maximizers ξ_t, we obtain

$$\begin{aligned}\frac{d}{dt}\overline{\Psi}_{\bar{p}} \le \mu\max_{M_t}\Phi_{\bar{p}} + \bigl(\min_{M_{\bar{t}}}\hat{r}_{\bar{p}} - 2d_3\bigr)\Biggl(\max_{M_t}\Phi_{\bar{p}}\Bigl(nK - \frac{d_3}{\hat{u}_{\bar{p}} - d_3}|A|^2\Bigr) + 2\max_{M_t}\Phi_{\bar{p}}^2 \\ - \frac{\mu}{\hat{u}_{\bar{p}} - d_3}\Bigl(\max_{M_t}\Phi_{\bar{p}}(K\mathrm{s}_K\hat{\mathrm{s}}_K\langle\nabla r, \nabla\hat{r}_{\bar{p}}\rangle + \mathrm{c}_K\hat{\mathrm{c}}_K)\Bigr)\Biggr),\end{aligned}$$

where we used that u and c_K are positive due to $M_t \subset \mathcal{H}(\mathcal{O})$ and $\mathcal{O} \in \Omega_t$. Using $\mathrm{s}_K \le K^{-1/2}$ and

$$\frac{\min_{M_{\bar{t}}}\hat{r}_{\bar{p}} - 2d_3}{\hat{u}_{\bar{p}} - d_3}(K\mathrm{s}_K\hat{\mathrm{s}}_K\langle\nabla r, \nabla\hat{r}_{\bar{p}}\rangle + \mathrm{c}_K\hat{\mathrm{c}}_K) \le 2\cdot\frac{\pi/(2\sqrt{K})}{\frac{1}{5}d_3} = \frac{5\pi}{d_3\sqrt{K}},$$

we get

$$\frac{d}{dt}\overline{\Psi}_{\bar{p}} \le \Bigl(1 + \frac{5\pi}{\sqrt{K}d_3}\Bigr)\mu\max_{M_t}\Phi_{\bar{p}} + nK\overline{\Psi}_{\bar{p}} + 2\overline{\Psi}_{\bar{p}}\max_{M_t}\Phi_{\bar{p}} - \frac{d_3^2}{n}\overline{\Psi}_{\bar{p}}\max_{M_t}\Phi_{\bar{p}}^2.$$

By means of the Hsiung–Minkowski identity (2-3) and Lemma 2.1, we estimate, for all $0 \le t \le \tau$,

$$\mu(t) \le n\binom{n}{\ell+1}\frac{\int_{M_t}\kappa_n^{\ell+1}}{\int_{M_t}u\sigma_{\ell+1}} \le \frac{n}{\mathrm{s}_K(\epsilon_1)}\frac{\int_{M_t}\kappa_n^{\ell+1}}{\int_{M_t}\kappa_1^{\ell+1}} \le \frac{2nC_0^{\ell+1}}{\epsilon_1}, \tag{5-3}$$

where the constant C_0 comes from the pinching (4-3). Hence

$$\begin{aligned}\frac{d}{dt}\overline{\Psi}_{\bar{p}} &\le 2nC_0^{\ell+1}\Bigl(1 + \frac{5\pi}{\sqrt{K}d_3}\Bigr)\frac{\max_{M_t}\Phi_{\bar{p}}}{\epsilon_0(\max(\bar{y}, \psi_0))} + nK\overline{\Psi}_{\bar{p}} + 2\overline{\Psi}_{\bar{p}}\max_{M_t}\Phi_{\bar{p}} - \frac{d_3^2}{n}\overline{\Psi}_{\bar{p}}\max_{M_t}\Phi_{\bar{p}}^2 \\ &= \frac{2nC_0^{\ell+1}}{\min_{M_{\bar{t}}}\hat{r}_{\bar{p}} - 2d_3}\Bigl(1 + \frac{5\pi}{\sqrt{K}d_3}\Bigr)\frac{\overline{\Psi}_{\bar{p}}}{\epsilon_0(\max(\bar{y}, \psi_0))} + nK\overline{\Psi}_{\bar{p}} \\ &\qquad + \frac{2}{\min_{M_{\bar{t}}}\hat{r}_{\bar{p}} - 2d_3}\overline{\Psi}_{\bar{p}}^2 - \frac{d_3^2}{n(\min_{M_{\bar{t}}}\hat{r}_{\bar{p}} - 2d_3)^2}\overline{\Psi}_{\bar{p}}^3.\end{aligned}$$

Multiplication with $(\min_{M_{\bar{t}}}\hat{r}_{\bar{p}} - 2d_3)^2$ gives, for almost every $t \in J$,

$$\bigl(\min_{M_{\bar{t}}}\hat{r}_{\bar{p}} - 2d_3\bigr)^2\frac{d}{dt}\overline{\Psi}_{\bar{p}} \le \frac{n\pi C_0^{\ell+1}}{\sqrt{K}}\Bigl(1 + \frac{5\pi}{\sqrt{K}d_3}\Bigr)\frac{\overline{\Psi}_{\bar{p}}}{\epsilon_0(\max(\bar{y}, \psi_0))} + \frac{n\pi^2}{4}\overline{\Psi}_{\bar{p}} + \frac{\pi}{\sqrt{K}}\overline{\Psi}_{\bar{p}}^2 - \frac{d_3^2}{n}\overline{\Psi}_{\bar{p}}^3,$$

which is strictly negative whenever $\max(\bar{y}, \psi_0) < \overline{\Psi}_{\bar{p}}(t)$. To conclude the argument, suppose that

$$\Psi(\bar{t}, \bar{\xi}, \bar{p}) = \max_{[0,\bar{t}]\times\mathbb{S}^n\times\mathbb{S}^{n+1}_K} \Psi > \max(\bar{y}, \psi_0).$$

We know that

$$\overline{\Psi}_{\bar{p}}(t) \le \Psi(\bar{t}, \bar{\xi}, \bar{p}),$$

with equality at $t = \bar{t}$. But previously we showed that, for t close to $\bar{t}$, we have $(d/dt)\overline{\Psi}_{\bar{p}} < 0$ almost everywhere, which is impossible. Hence we obtain the desired estimate with $C_3 = \bar{y}$. □

Higher-order curvature bounds. We use the estimate from Lemma 5.4 to control the global term and estimate the derivatives of curvature.

Lemma 5.5. *For the origin $\mathcal{O} \in \Omega$ from Lemma 5.4 and the solution $M_t = \partial\Omega_t$ of (1-1) with initial data M and maximal existence time $T > 0$ and for all $m \in \mathbb{N}$, there exists*

$$C_4 = C_4\big(n, m, K, C_0, W_\ell(\Omega), \max\big(C_3, \max_M \Psi\big)\big)$$

with the property

$$\mu(t) + |\nabla^m A| \le C_4 \quad \text{for all } t \in [0, \min(\tau, T)),$$

where τ is the number from Lemma 5.4. In particular, we have $T \ge \tau$, and the flow exists smoothly on $[0, \tau]$.

Proof. Up to the time $\min(\tau, T)$, (5-3) holds, i.e., with the notation from the proof of Lemma 5.4, we have

$$\mu(t) \le \frac{2nC_0^{\ell+1}}{\epsilon_1} = \frac{2nC_0^{\ell+1}}{\epsilon_0(\max(C_3, \psi_0))}.$$

The curvature derivative bound can be proved by a well-known induction argument, as for example in [Huisken 1984]. First, due to convexity and Lemma 5.4 (ii) and (iii),

$$|A|^2 \le H^2 \le \frac{\Phi^2_{p\Omega}}{K} \le \frac{1}{d_3^2 K}\big(\max\big(C_3, \max_M \Psi\big)\big)^2.$$

Assuming that all derivatives up to order $m-1$ are bounded by a constant of the form C_4, we obtain the evolution equation of $|\nabla^m A|$:

$$\partial_t|\nabla^m A|^2 \le \Delta|\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 + C(\mu u + K)|\nabla^m A|^2 + C \sum_{i+j+k=m} (\mu|\nabla^i \mathrm{c}_K| + |\nabla^i A|)|\nabla^j A||\nabla^k A||\nabla^m A|,$$

where we used that $\nabla u = A * \nabla \mathrm{c}_K$; see (2-4). Here $S * T$ denotes any linear combination of tensors formed by contracting S and T by means of g.

Then we claim

$$|\nabla^m A|^2 \le C \quad \text{for all } t \in [0, \min(\tau, T)). \tag{5-4}$$

As μ and u are bounded, we get

$$(\partial_t - \Delta)|\nabla^m A|^2 \le -2|\nabla^{m+1} A|^2 + C|\nabla^m A|^2 + C\Big(\sum_{i=0}^{m} |\nabla^i \mathrm{c}_K| + 1\Big)|\nabla^m A|.$$

Notice that $\mathrm{c}_K(r)$ and $|\nabla \mathrm{c}_K|$ are bounded as well. Moreover, from (4-1), one has, for $\ell \ge 0$, the covariant derivatives

$$\nabla^{\ell+2}\mathrm{c}_K = \nabla^\ell \mathrm{c}_K * K + u * \nabla^\ell A + \sum_{i+j+k=\ell} \nabla^i \mathrm{c}_K * \nabla^j A * \nabla^k A,$$

which are controlled by uniform constants arguing by induction. In short, we reach

$$(\partial_t - \Delta)|\nabla^m A|^2 \le -2|\nabla^{m+1} A|^2 + C|\nabla^m A|(1 + |\nabla^m A|),$$

which leads to (5-4) by standard maximum principle arguments, as for example in the proof of [Huisken 1987, Theorem 4.1].

As the right-hand side of the flow equation and all higher derivatives of the curvature remain uniformly bounded, we conclude (as in [Huisken 1984, pp. 257 ff.]) that, if $T < \tau$, then M_t converges (as $t \to T$ in the C^∞-topology) to a unique, smooth and strictly convex hypersurface.[1] Now we can apply short-time existence to continue the solution after T, contradicting the maximality of T. Hence the solution of (1-1) starting at a strictly convex hypersurface exists on $[0, \tau)$. On this interval we have uniform smooth estimates, and hence the flow also exists on $[0, \tau]$. □

6. Construction of a global solution

In the previous section we achieved existence and uniform estimates of any solution to (1-1) with strictly convex initial data M on a time interval $[0, \tau]$ from Lemma 5.4, the length of which only depends on preserved data of the problem. Those are, in particular, the hemisphere $\mathcal{H}(\mathcal{O})$, the pinching constant C_0, the quermassintegral $W_\ell(\Omega)$ and the number $\max(C_3, \max_M \Psi)$. The full curvature derivative bounds also only depend on those quantities.

Hence we can start an iteration process and shift, at time $i\tau$ with $i \in \mathbb{N}$, the origin according to Lemma 5.4 applied to the new strictly convex initial hypersurface $M_{i\tau}$. The constant C_4 from Lemma 5.5 is then uniform among the integers i, because it only depends on quantities which are always preserved. The following lemma makes this precise.

Lemma 6.1. *Let $M_0 \subset \mathbb{S}^{n+1}_K$ be a strictly convex hypersurface enclosing a domain Ω. Then there exists a sequence of origins $(\mathcal{O}_i)_{i\in\mathbb{N}\cup\{0\}}$ and positive numbers τ_0, $\epsilon_1(0)$ depending only on n, K, C_0, $W_\ell(\Omega_0)$,* $\max(C_3, \max_{M_0} \Psi)$, *such that the problem*

$$\begin{aligned} \partial_t x &= (\mu_i(t)\mathrm{c}_K(r_i) - H)\nu \quad \textit{for all } t \in [i\tau_0, (i+1)\tau_0), \\ x(0, \mathbb{S}^n) &= M_0, \\ x((i+1)\tau_0, \mathbb{S}^n) &= \lim_{t \nearrow (i+1)\tau_0} M_t, \end{aligned}$$

[1] Note that, due to the bound on μ, it can be seen from Lemma 5.1 that H is bounded from below on every finite time interval.

where r_i is the distance to $\mathcal{O}_i$ and μ_i is given as in (1-3) to keep the quermassintegral $W_\ell(\Omega_t)$ fixed for any $\ell = 0, 1, \dots, n$, has a solution

$$x : [0, \infty) \times \mathbb{S}^n \to \mathbb{S}^{n+1}_K.$$

For every $t \geq 0$, M_t is strictly convex and we have

$$B_{\epsilon_1(0)}(\mathcal{O}_i) \subset \Omega_t \quad \textit{and} \quad M_t \subset \mathcal{H}(\mathcal{O}_i) \quad \textit{for all } t \in [i\tau_0, (i+1)\tau_0). \tag{6-1}$$

The mappings $x(t, \cdot)$ satisfy spatial C^∞-estimates which are uniform in time.

Proof. For M_0, pick $\mathcal{O}_0$ according to Lemma 5.4. From Lemma 5.5 we conclude that the solution M_t of (1-1) with initial data M_0 exists on $[0, \tau_0]$, where τ_0 and $\epsilon_1(0)$ depend on $n,\ K,\ C_0,\ W_\ell(\Omega_0)$, and $\max(C_3, \max_{M_0} \Psi)$. The derivatives of A are bounded by

$$C_4 = C_4\big(n, m, K, C_0, W_\ell(\Omega_0), \max\big(C_3, \max_{M_0} \Psi\big)\big).$$

Now suppose that, for $i \geq 0$, the hypersurface $M_{i\tau_0}$, the origin $\mathcal{O}_i$ and the solution $(M_t)_{t\in[i\tau_0,(i+1)\tau_0)}$ were already constructed such that (6-1),

$$\max\big(C_3, \max_{M_t} \Psi\big) \leq \max\big(C_3, \max_{M_0} \Psi\big),$$

as well as

$$\mu_i(t) + |\nabla^m A| \leq C_4\big(n, m, K, C_0, W_\ell(\Omega_0), \max\big(C_3, \max_{M_0} \Psi\big)\big) \tag{6-2}$$

all hold for all $t \in [i\tau_0, (i+1)\tau_0]$. Then apply Lemma 5.4 to the initial hypersurface $M_{(i+1)\tau_0}$, and obtain an origin $\mathcal{O}_{i+1}$ such that the solution M_t of (1-1) with initial data $M_{(i+1)\tau_0}$ satisfies (6-1),

$$\max\big(C_3, \max_{M_t} \Psi\big) \leq \max\big(C_3, \max_{M_{(i+1)\tau_0}} \Psi\big) \leq \max\big(C_3, \max_{M_0} \Psi\big)$$

and

$$\mu_{i+1}(t) + |\nabla^m A| \leq C_4\big(n, m, K, C_0, W_\ell(\Omega_0), \max\big(C_3, \max_{M_0} \Psi\big)\big)$$

during the interval $[(i+1)\tau_0, (i+2)\tau_0]$ and with the same $\epsilon_1(0)$. Here we also used that C_0 and W_ℓ are preserved. This means that the construction can be carried out infinitely often to obtain the desired long-time solution. □

7. Asymptotic estimates and convergence to a spherical cap

In the previous sections we have put ourselves into a position where we have a strictly convex flow $(M_t)_{0\leq t<\infty}$ in the sphere. This flow is not necessarily smooth in time, but it satisfies spatial C^k-estimates which are uniform with respect to time and has a uniformly bounded global term, due to the proof of Lemma 6.1.

Additionally, by means of Corollary 4.4 and the curvature bounds, we get

$$\sum_{i=1}^{n} (\kappa_i - \kappa_1) = H - n\kappa_1 \leq nH\omega(t) \leq nCe^{-2nKt},$$

which implies exponential decay of the traceless second fundamental form

$$|\mathring{A}|^2 = \frac{1}{n}\sum_{i<j}(\kappa_j - \kappa_i)^2 \le Ce^{-4nKt}. \tag{7-1}$$

Using this property, in the following we are going to apply some recent estimates of almost-umbilical type due to De Rosa and Gioffré [2021], to show that the process of picking new origins actually terminates after finitely many steps and that the flow will then converge to a geodesic sphere of a uniquely determined radius. The crucial ingredient is the following result.

Theorem 7.1 [De Rosa and Gioffrè 2021, Theorem 1.3]. *Let $n \ge 2$, let Σ be a closed hypersurface in $\mathbb{R}^{n+1}$ and let $p > n$ be given. We assume that there exists $c_0 > 0$ such that Σ satisfies the conditions*

$$|\Sigma| = |\mathbb{S}^n|, \quad \|A\|_{L^p(\Sigma)} \le c_0.$$

There exist positive numbers $\delta, C > 0$, depending only on $n,\ p,\ c_0$, with the following property: if

$$\|\mathring{A}\|_{L^p(\Sigma)} \le \delta,$$

then there exists a vector $c = c(\Sigma)$, such that $\Sigma - c$ is a graph over the sphere; namely, there exists a parametrization

$$\psi : \mathbb{S}^n \to \Sigma, \quad \psi(x) = e^{f(x)}x + c,$$

and f satisfies the estimate

$$\|f\|_{W^{2,p}(\mathbb{S}^n)} \le C\|\mathring{A}\|_{L^p(\Sigma)}.$$

In the following we will use this result to prove that the surfaces become exponentially C^2-close to geodesic spheres and that the necessity to pick new origins vanishes.

Lemma 7.2. *In the situation of Lemma 6.1, there exists an integer $m > 0$ depending on $n,\ K$ and M, such that, in Lemma 6.1, the origins $\mathcal{O}_i,\ i > m$, may be chosen constantly equal to $\mathcal{O}_m$.*

Proof. Let m be a positive integer to be specified during the proof. Let $\mathcal{O}_m$ be the flow origin associated to the interval $I_m := [m\tau_0, (m+1)\tau_0)$. By stereographic projection from the antipodal point of $\mathcal{O}_m$, the family $(M_t)_{t\in I_m}$ can be viewed as a flow in the Euclidean space, which we denote by $(\widetilde{M}_t)_{t\in I_m}$. Geometric quantities of this flow, denoted by a tilde, are related to the original ones as follows, see [Gerhardt 2006, Proposition 1.1.11], where $e^{2\varphi}$ is the conformal factor:

$$g = e^{2\varphi}\tilde{g}, \qquad \nu = e^{-\varphi}\tilde{\nu},$$
$$e^{\varphi}A = \tilde{A} + d\varphi(\tilde{\nu})\,\mathrm{id}, \quad e^{\varphi}H = \widetilde{H} + n\,d\varphi(\tilde{\nu}).$$

In particular, we obtain

$$\mathring{\tilde{A}} = e^{\varphi}\mathring{A}.$$

The surface areas of M_t and $\widetilde{M}_t$ are related by

$$|M_t| = \int_{M_t} 1 = \int_{\widetilde{M}_t} e^{n\varphi},$$

and hence

$$C^{-1}|M_t| \leq |\widetilde{M}_t| \leq |M_t|,$$

where C depends on $|\varphi|_{C^0(M_t)}$. Now define the scaled hypersurface

$$\widehat{M}_t = \lambda \widetilde{M}_t, \quad \text{with } \lambda^n = \frac{|\mathbb{S}^n|}{|\widetilde{M}_t|},$$

so that $|\widehat{M}_t| = |\mathbb{S}^n|$. Now the associated Weingarten operator is

$$\hat{A} = \lambda^{-1}\tilde{A} = \frac{e^{\varphi}}{\lambda} A - \lambda^{-1}\, d\varphi(\tilde{\nu})\,\mathrm{id},$$

and similarly for the traceless Weingarten operator. Hence

$$\|\hat{A}\|_{L^\infty(\widehat{M}_t)} \leq C(\|A\|_{L^\infty(M_t)} + 1)$$

and

$$\|\mathring{\hat{A}}\|_{L^\infty(\widehat{M}_t)} \leq C\|\mathring{A}\|_{L^\infty(M_t)},$$

where C depends on n, $|\varphi|_{C^1(M_t)}$ and $|M_t|$.

Then $\|\hat{A}\|_{L^\infty(\widehat{M}_t)}$ is bounded by Lemma 5.5 and (7-1) ensures that $\|\mathring{\hat{A}}\|_{L^\infty}$ is as small as needed for m big enough. Therefore, we can apply Theorem 7.1 for sufficiently large m to get a function $\hat{f}$ which, from the embedding theorems of Sobolev spaces into Hölder spaces, satisfies

$$\|\hat{f}\|_{C^1(\mathbb{S}^n)} \leq C\|\mathring{\hat{A}}\|_{L^\infty(\widehat{M}_t)} \leq C\|\mathring{A}\|_{L^\infty(M_t)} \leq Ce^{-2nKt} \quad \text{for all } t \in I_m.$$

Then, due to our curvature bounds (5-4), we have full C^k-bounds on $\hat{f}$ for all k. By iteration of interpolation arguments for C^k bounds (see [Gerhardt 2011, Corollary 6.2]), this implies that

$$\|\hat{f}\|_{C^k(\mathbb{S}^n)} \leq Ce^{-2nKt} \quad \text{for all } t \in I_m.$$

In other words, $\widehat{M}_t$ is exponentially C^k-close to a sphere $\hat{\mathcal{S}}_t$ for all $k \in \mathbb{N}$ and for all $t \in I_m$. As the area along the M_t is uniformly bounded above and below by Corollary 3.2, we get C^k bounds for the conformal factor as well, and this property of closeness to a sphere $\mathcal{S}_t$ carries over to the original flow in $\mathbb{S}^{n+1}_K$. Note that the radii of the spheres $\mathcal{S}_t$ converge to a well-defined limit, which is strictly less than $\pi/(2\sqrt{K})$, determined by the initial value of $W_\ell(\Omega_0)$. Hence the curvature of M_t is uniformly bounded from below.

On the other hand, the radial distance to the origin $\mathcal{O}_m$ satisfies

$$\partial_t r = (\mu \mathrm{c}_K - H)v^{-1}, \quad \text{with } v^2 = 1 + \mathrm{s}_K^{-2}|dr|^2_\sigma. \tag{7-2}$$

Hence, for an error δ_m that converges to zero when $m \to \infty$,

$$\partial_t r = \left(\frac{\int_{M_t} \sigma_\ell H}{\int_{M_t} \sigma_\ell \mathrm{c}_K}\mathrm{c}_K - H\right)v^{-1} \leq \left(\frac{\mathrm{c}_K}{⨍_{\mathcal{S}_t} \mathrm{c}_K} - 1\right)\frac{H_{\mathcal{S}_t}}{v} + \delta_m, \tag{7-3}$$

where $H_{\mathcal{S}_t}$ is the mean curvature of the sphere $\mathcal{S}_t$. At points which maximize r, we have that c_K is minimized. At such points the first term on the right-hand side of (7-3) is strictly negative if c_K is not constant. Hence, for large m, if $\mathcal{S}_t$ is uniformly off-center, $\max_{M_t} r$ is decreasing and a similar estimate shows that $\min_{M_t} r$ is increasing. Hence, from then on, there is no longer a need to adjust the origin. □

Convergence to a spherical cap. To complete the proof of Theorem 1.1, it only remains to show that the immortal solution coming from Lemma 6.1 actually converges to a limit geodesic sphere. After m-fold picking of a new origin, we now may, without loss of generality, assume that origins have not been changed at all. We will exploit the C^∞-estimates for the flow hypersurfaces M_t coming from (5-4). We already know from Lemma 7.2 that every limit point of the flow must be a round sphere.

Now we prove that only the sphere centered at the origin can arise as a limit. Notice that the radius R of any limit sphere is determined by the initial hypersurfaces by means of the equality $W_\ell(B_R) = W_\ell(\Omega_0)$. Denote by H_R the mean curvature of such a sphere S_R. Hence, from the evolution (7-2) of the radial distance and for an error δ that converges to zero when $M_{t_k} \to S_R$, we get

$$\partial_t r = \left(\frac{c_K}{⨍_M c_K} - 1\right)\frac{H_R}{v} + \delta. \tag{7-4}$$

As above, at points which maximize r, we have that c_K is minimized, thus at such points the first term on the right-hand side of (7-4) is strictly negative if c_K is not constant. Therefore the function $\max_{M_t} r$ is strictly decreasing in sufficiently small C^2-neighborhoods of any noncentered sphere, which excludes those as limits. Thus subsequential limits are unique and the whole flow must converge.

8. The elliptic case: rigidity results

In order to prove Theorem 1.3, we first have to get the elliptic viscosity equation of the pinching deficit for general curvature function F. Let us first gather some prerequisites about these functions.

8.1. ***Symmetric curvature functions.*** If M is a hypersurface of $\mathbb{M}_K^{n+1}$, then we set

$$F(x) = f(\kappa_1(x), \ldots, \kappa_n(x)),$$

which can be alternatively seen as a function defined on the diagonalizable endomorphisms, $F = F(A)$, or as a function of a symmetric and a positive definite bilinear form, $F = F(g, h)$. In the latter case, we write

$$F^{ij} = \frac{\partial F}{\partial h_{ij}}, \qquad F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij}\partial h_{kl}}.$$

With these conventions, the covariant derivatives are given by

$$\nabla_i F = F^{jk}\nabla_i h_{jk}. \tag{8-1}$$

On strictly convex hypersurfaces M, we can define the so-called *inverse curvature function* by

$$\widetilde{F}(A) = \frac{1}{F(A^{-1})}.$$

A curvature function F is called *inverse concave* if $\widetilde{F}$ is concave. Notice that concavity/convexity with respect to the matrix variables is equivalent to the same property with respect to the eigenvalues (see [Andrews 2007; Gerhardt 2006] for further information). Next we gather several useful properties.

Lemma 8.1. (a) *If F is inverse concave and M is strictly convex, then*

$$F^{ij,kl}\nabla_1 h_{ij}\nabla_1 h_{kl} + 2\sum_j \frac{F^{ii}}{\kappa_j}(\nabla_1 h_{ij})^2 \geq \frac{2}{F}(\nabla_1 F)^2. \tag{8-2}$$

(b) *Under Conditions 1.2, we have:*

(i) *If F is convex, then it is inverse concave* (*see* [Gerhardt 2006, Lemmas 2.2.12 and 2.2.14]).

(ii) *Euler's formula $F^{ii}\kappa_i = F$ implies that F is strictly positive.*

Rigidity for radial curvature functions. We start with a result that contains an elliptic version of Proposition 4.3.

Lemma 8.2. *Suppose F satisfies Conditions 1.2. Then:*

(i) *The Weingarten operator satisfies the elliptic equation*

$$-F^{rs}\nabla^2_{rs}h_{ij} = F^{pq,rs}\nabla_i h_{pq}\nabla_j h_{rs} - \nabla^2_{ij}F + F^{rs}h_{ms}h^m_r h_{ij} + KFg_{ij} - Fh_{mj}h^m_i - KF^{rs}g_{rs}h_{ij}.$$

(ii) *For the function $p = \kappa_1/F$, we have*

$$-FF^{kl}\nabla^2_{kl}p \geq 2\sum_{j>D}\frac{F^{kk}(\nabla_k h^1_j)^2}{\kappa_j-\kappa_1} + F^{pq,rs}\nabla_1 h_{pq}\nabla_1 h_{rs} + 2F^{kl}\nabla_k p\nabla_l F + pF^{kl}\nabla_{kl}F - \nabla_{11}F + F^{ii}\operatorname{Sec}_{i1}(\kappa_i-\kappa_1)$$

in the viscosity sense, where D is the multiplicity of κ_1.

Proof. (i) We differentiate (8-1) to get

$$\nabla^2_{ij}F = F^{pq,rs}\nabla_i h_{pq}\nabla_j h_{rs} + F^{rs}\nabla^2_{ij}h_{rs}.$$

Now combine the commutator formula for second covariant derivatives with the Gauss equation (2-1) to deduce

$$\begin{aligned}F^{rs}\nabla^2_{ij}h_{rs} &= F^{rs}(\nabla^2_{rs}h_{ij} + R^m_{isj}h_{mr} + R^m_{rsj}h_{mi})\\ &= F^{rs}\nabla^2_{rs}h_{ij} + F^{rs}(h_{ms}h_{ij} - h_{mj}h_{is} + Kg_{ms}g_{ij} - Kg_{mj}g_{is})h^m_r\\ &\qquad + F^{rs}(h_{ms}h_{rj} - h_{mj}h_{rs} + Kg_{ms}g_{rj} - Kg_{mj}g_{rs})h^m_i\\ &= F^{rs}(\nabla^2_{rs}h_{ij} + h_{ms}h^m_r h_{ij}) + KFg_{ij} - Fh_{mj}h^m_i - KF^{rs}g_{rs}h_{ij}.\end{aligned}$$

(ii) As in the proof of Proposition 4.3, let η be a smooth lower support of p at $\xi_0 \in M$, and define $\varphi = \eta F$. Then, in coordinates with

$$g_{ij} = \delta_{ij}, \quad h_{ij} = \kappa_i\delta_{ij}, \tag{8-3}$$

we have inequality (4-2), which combined with (i) leads to

$$
\begin{aligned}
-FF^{kl}\nabla^2_{kl}\eta &= -F^{kl}\nabla^2_{kl}\varphi + \frac{\varphi}{F}F^{kl}\nabla^2_{kl}F + 2F^{kl}\nabla_k\eta\nabla_l F \\
&\geq 2F^{kl}\nabla_k\eta\nabla_l F + 2\sum_{j>D}\frac{F^{kk}(\nabla_k h^1_j)^2}{\kappa_j-\kappa_1} + F^{pq,rs}\nabla_1 h_{pq}\nabla_1 h_{rs} \\
&\qquad + \eta F^{kl}\nabla^2_{kl}F - \nabla^2_{11}F + F^{ii}(\kappa_i\kappa_1 + K)(\kappa_i-\kappa_1). \qquad \square
\end{aligned}
$$

In order to prove Theorem 1.3, let $F=\gamma \mathrm{c}_K^{\alpha}$. Then taking derivatives and using (4-1), we get

$$
\nabla^2_{kl}F = \alpha\gamma \mathrm{c}_K^{\alpha-1}\nabla^2_{kl}\mathrm{c}_K + \alpha(\alpha-1)\gamma \mathrm{c}_K^{\alpha-2}\nabla_k \mathrm{c}_K\nabla_l \mathrm{c}_K = K\Big(\frac{u}{\mathrm{c}_K}h_{kl} - g_{kl}\Big)\alpha F + \frac{2\epsilon}{F}\nabla_k F\nabla_l F,
$$

where

$$
\epsilon = \frac{\alpha-1}{2\alpha} = \frac{1}{2} - \frac{1}{2\alpha}. \tag{8-4}
$$

This implies

$$
\begin{aligned}
pF^{kl}\nabla^2_{kl}F - \nabla^2_{11}F &= p\alpha KF\Big(\frac{u}{\mathrm{c}_K}F - F^{kl}g_{kl}\Big) - \alpha KF\Big(\frac{u}{\mathrm{c}_K}\kappa_1 - 1\Big) + \frac{2\epsilon p}{F}F^{kl}\nabla_k F\nabla_l F - \frac{2\epsilon}{F}(\nabla_1 F)^2 \\
&= K\alpha F^{ii}(\kappa_i-\kappa_1) + \frac{2\epsilon}{F}(pF^{kl}\nabla_k F\nabla_l F - (\nabla_1 F)^2).
\end{aligned} \tag{8-5}
$$

Here we have used Euler's relation, and computations are done in the coordinates (8-3).

Proof of Theorem 1.3. As $|\alpha|\geq 1$, we have that $\epsilon\in[0,1]$ for ϵ defined as in (8-4). Then using that M is convex, we can estimate

$$
2\sum_{j>D}\frac{F^{kk}(\nabla_k h^1_j)^2}{\kappa_j-\kappa_1} \geq 2\epsilon\sum_{j=1}^{n}\frac{F^{kk}(\nabla_k h^1_j)^2}{\kappa_j} - 2\epsilon\frac{p}{F}F^{kl}\nabla_k F\nabla_l F + T*\nabla p,
$$

where we also used the following (see [Brendle et al. 2017, Lemma 5]):

$$
\nabla_k\varphi\delta_{ij} = \nabla_k h_{ij} \quad \text{for all } 1\leq i,\ j\leq D.
$$

Plugging this into Lemma 8.2 (ii) and using the convexity of F, we get

$$
\begin{aligned}
-FF^{kl}\nabla^2_{kl}p &\geq 2\epsilon\sum_{j=1}^{n}\frac{F^{kk}(\nabla_k h^1_j)^2}{\kappa_j} - 2\epsilon\frac{p}{F}F^{kl}\nabla_k F\nabla_l F + \epsilon F^{pq,rs}\nabla_1 h_{pq}\nabla_1 h_{rs} \\
&\qquad + pF^{kl}\nabla_{kl}F - \nabla_{11}F + F^{ii}\,\mathrm{Sec}_{i1}(\kappa_i-\kappa_1) + T*\nabla p \\
&\underset{(8\text{-}5)}{\geq} 2\epsilon\sum_{j=1}^{n}\frac{F^{kk}(\nabla_k h^1_j)^2}{\kappa_j} + \epsilon F^{pq,rs}\nabla_1 h_{pq}\nabla_1 h_{rs} - \frac{2\epsilon}{F}(\nabla_1 F)^2 \\
&\qquad + F^{ii}(\mathrm{Sec}_{i1} + \alpha K)(\kappa_i-\kappa_1) + T*\nabla p.
\end{aligned}
$$

Notice that in the case $\alpha=-1$, we have $\epsilon=1$ and therefore we do not need the convexity of F in the first inequality above.

By Lemma 8.1 (b) our F is inverse concave, and hence (8-2) leads to

$$-FF^{kl}\nabla^2_{kl}p + T * \nabla p \geq F^{ii}(\mathrm{Sec}_{i1} + \alpha K)(\kappa_i - \kappa_1).$$

This completes the proof using the strong maximum principle for viscosity solutions. We conclude that M has to be centered at the origin provided $K \neq 0$, since only in this case can c_K be constant. □

Remark 8.3. Note that this approach also provides a direct maximum principle proof of Liebmann's soap bubble theorem (the convex case of Alexandrov's theorem); see [Liebmann 1900].

Solitons. We complete this paper by proving Corollary 1.4.

Proof of Corollary 1.4. For the given hypersurface M, there is a dual hypersurface $\widetilde{M} \subset \mathbb{M}^{n+1}_K$, where $K = \mathrm{sgn}(\mathbb{M})$ as in (1-6) and with the properties

$$\tilde{\kappa}_i = \frac{1}{\kappa_i}, \quad \tilde{\mathrm{c}}_K = u;$$

see [Gerhardt 2006, Theorems 10.4.4 and 10.4.9] and [Scheuer 2021]. Hence $\widetilde{M}$ satisfies the equation

$$\tilde{c}_K^{1/\beta} = F(\kappa_i) = F(\tilde{\kappa}_i^{-1}) = \frac{1}{\widetilde{F}(\tilde{\kappa}_i)},$$

i.e., with $\alpha = -1/\beta$, we have

$$\widetilde{F}_{|\widetilde{M}} = \tilde{\mathrm{c}}_K^{\alpha},$$

where $\widetilde{F}$ is the inverse curvature function of F. Therefore to complete the proof it only remains to check the conditions of Theorem 1.3 for $\widetilde{M}$, which hold provided that, for any $\tilde{g}$-orthonormal frame, we have

$$\widetilde{\mathrm{Sec}}_{ij} \geq -\alpha K.$$

In coordinates that diagonalize $\tilde{A}$, the Gauss equation (2-1) for $\widetilde{M}$ gives

$$\widetilde{R}_{ijij} + \alpha K = \tilde{h}_{ii}\tilde{h}_{jj} - \tilde{h}_{ij}\tilde{h}_{ij} + (1+\alpha)K = \frac{1}{\kappa_i\kappa_j} + (1+\alpha)K \geq 0,$$

provided

$$\frac{1}{\kappa_i\kappa_j} \geq \frac{1-\beta}{\beta}K.$$

Notice that if $((1-\beta)/\beta)K \leq 0$, this is guaranteed by convexity of M; otherwise, the inequality follows by the assumption on Sec_M. Hence the statement follows by direct application of Theorem 1.3. □

References

[Alexandrov 1962] A. D. Alexandrov, "A characteristic property of spheres", *Ann. Mat. Pura Appl.* (4) **58** (1962), 303–315. MR Zbl

[Andrews 1994a] B. Andrews, "Contraction of convex hypersurfaces in Euclidean space", *Calc. Var. Partial Differential Equations* **2**:2 (1994), 151–171. MR Zbl

[Andrews 1994b] B. Andrews, "Contraction of convex hypersurfaces in Riemannian spaces", *J. Differential Geom.* **39**:2 (1994), 407–431. MR Zbl

[Andrews 2000] B. Andrews, "Motion of hypersurfaces by Gauss curvature", *Pacific J. Math.* **195**:1 (2000), 1–34. MR Zbl

[Andrews 2007] B. Andrews, "Pinching estimates and motion of hypersurfaces by curvature functions", *J. Reine Angew. Math.* **608** (2007), 17–33. MR Zbl

[Andrews and Wei 2018] B. Andrews and Y. Wei, "Quermassintegral preserving curvature flow in hyperbolic space", *Geom. Funct. Anal.* **28**:5 (2018), 1183–1208. MR Zbl

[Andrews et al. 2016] B. Andrews, P. Guan, and L. Ni, "Flow by powers of the Gauss curvature", *Adv. Math.* **299** (2016), 174–201. MR Zbl

[Andrews et al. 2021] B. Andrews, X. Chen, and Y. Wei, "Volume preserving flow and Alexandrov–Fenchel type inequalities in hyperbolic space", *J. Eur. Math. Soc.* **23**:7 (2021), 2467–2509. MR Zbl

[Brendle et al. 2017] S. Brendle, K. Choi, and P. Daskalopoulos, "Asymptotic behavior of flows by powers of the Gaussian curvature", *Acta Math.* **219**:1 (2017), 1–16. MR Zbl

[Cabezas-Rivas and Miquel 2007] E. Cabezas-Rivas and V. Miquel, "Volume preserving mean curvature flow in the hyperbolic space", *Indiana Univ. Math. J.* **56**:5 (2007), 2061–2086. MR Zbl

[Cabezas-Rivas and Miquel 2016] E. Cabezas-Rivas and V. Miquel, "Non-preserved curvature conditions under constrained mean curvature flows", *Differential Geom. Appl.* **49** (2016), 287–300. MR Zbl

[do Carmo and Warner 1970] M. P. do Carmo and F. W. Warner, "Rigidity and convexity of hypersurfaces in spheres", *J. Differential Geom.* **4**:2 (1970), 133–144. MR Zbl

[Chen et al. 2022] C. Chen, P. Guan, J. Li, and J. Scheuer, "A fully-nonlinear flow and quermassintegral inequalities in the sphere", *Pure Appl. Math. Q.* **18**:2 (2022), 437–461. MR Zbl

[Da Lio 2004] F. Da Lio, "Remarks on the strong maximum principle for viscosity solutions to fully nonlinear parabolic equations", *Commun. Pure Appl. Anal.* **3**:3 (2004), 395–415. MR Zbl

[De Rosa and Gioffrè 2021] A. De Rosa and S. Gioffrè, "Absence of bubbling phenomena for non-convex anisotropic nearly umbilical and quasi-Einstein hypersurfaces", *J. Reine Angew. Math.* **780** (2021), 1–40. MR Zbl

[Espinar et al. 2009] J. M. Espinar, J. A. Gálvez, and P. Mira, "Hypersurfaces in $\mathbb{H}^{n+1}$ and conformally invariant equations: the generalized Christoffel and Nirenberg problems", *J. Eur. Math. Soc.* **11**:4 (2009), 903–939. MR Zbl

[Gao and Ma 2019] S. Gao and H. Ma, "Self-similar solutions of curvature flows in warped products", *Differential Geom. Appl.* **62** (2019), 234–252. MR Zbl

[Gao et al. 2018] S. Gao, H. Li, and H. Ma, "Uniqueness of closed self-similar solutions to σ_k^α-curvature flow", *Nonlinear Differential Equations Appl.* **25**:5 (2018), art. id. 45. MR Zbl

[Gao et al. 2022] S. Gao, H. Li, and X. Wang, "Self-similar solutions to fully nonlinear curvature flows by high powers of curvature", *J. Reine Angew. Math.* **783** (2022), 135–157. MR Zbl

[Gerhardt 2006] C. Gerhardt, *Curvature problems*, Ser. Geom. Topol. **39**, Int. Press, Somerville, MA, 2006. MR Zbl

[Gerhardt 2011] C. Gerhardt, "Inverse curvature flows in hyperbolic space", *J. Differential Geom.* **89**:3 (2011), 487–527. MR Zbl

[Gerhardt 2015] C. Gerhardt, "Curvature flows in the sphere", *J. Differential Geom.* **100**:2 (2015), 301–347. MR Zbl

[Girão and Pinheiro 2017] F. Girão and N. M. Pinheiro, "An Alexandrov–Fenchel-type inequality for hypersurfaces in the sphere", *Ann. Global Anal. Geom.* **52**:4 (2017), 413–424. MR Zbl

[Guan and Li 2015] P. Guan and J. Li, "A mean curvature type flow in space forms", *Int. Math. Res. Not.* **2015**:13 (2015), 4716–4740. MR Zbl

[Huisken 1984] G. Huisken, "Flow by mean curvature of convex surfaces into spheres", *J. Differential Geom.* **20**:1 (1984), 237–266. MR Zbl

[Huisken 1987] G. Huisken, "The volume preserving mean curvature flow", *J. Reine Angew. Math.* **382** (1987), 35–48. MR Zbl

[Huisken 1990] G. Huisken, "Asymptotic behavior for singularities of the mean curvature flow", *J. Differential Geom.* **31**:1 (1990), 285–299. MR Zbl

[Huisken and Polden 1999] G. Huisken and A. Polden, "Geometric evolution equations for hypersurfaces", pp. 45–84 in *Calculus of variations and geometric evolution problems* (Cetraro, Italy, 1996), edited by S. Hildebrandt and M. Struwe, Lecture Notes in Math. **1713**, Springer, 1999. MR Zbl

[Kwong et al. 2018] K.-K. Kwong, H. Lee, and J. Pyo, "Weighted Hsiung–Minkowski formulas and rigidity of umbilical hypersurfaces", *Math. Res. Lett.* **25**:2 (2018), 597–616. MR Zbl

[Liebmann 1900] H. Liebmann, "Ueber die Verbiegung der geschlossenen Flächen positiver Krümmung", *Math. Ann.* **53**:1-2 (1900), 81–112. MR Zbl

[Makowski 2012] M. Makowski, "Mixed volume preserving curvature flows in hyperbolic space", preprint, 2012. arXiv 1208.1898

[Makowski and Scheuer 2016] M. Makowski and J. Scheuer, "Rigidity results, inverse curvature flows and Alexandrov–Fenchel type inequalities in the sphere", *Asian J. Math.* **20**:5 (2016), 869–892. MR Zbl

[Mayer and Simonett 2000] U. F. Mayer and G. Simonett, "Self-intersections for the surface diffusion and the volume-preserving mean curvature flow", *Differential Integral Equations* **13**:7-9 (2000), 1189–1199. MR Zbl

[McCoy 2004] J. A. McCoy, "The mixed volume preserving mean curvature flow", *Math. Z.* **246**:1-2 (2004), 155–166. MR Zbl

[McCoy 2011] J. A. McCoy, "Self-similar solutions of fully nonlinear curvature flows", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **10**:2 (2011), 317–333. MR Zbl

[Reilly 1973] R. C. Reilly, "Variational properties of functions of the mean curvatures for hypersurfaces in space forms", *J. Differential Geom.* **8**:3 (1973), 465–477. MR Zbl

[Roldán 2022] Á. Roldán, "Hypersurfaces with prescribed curvatures in the de Sitter space", *Rev. Mat. Iberoam.* **38**:1 (2022), 269–294. MR Zbl

[Ros 1987] A. Ros, "Compact hypersurfaces with constant higher order mean curvatures", *Rev. Mat. Iberoam.* **3**:3-4 (1987), 447–453. MR Zbl

[Santaló 1946] L. A. Santaló, "Convex regions on the n-dimensional spherical surface", *Ann. of Math.* (2) **47** (1946), 448–459. MR Zbl

[Santaló 1976] L. A. Santaló, *Integral geometry and geometric probability*, Encycl. Math. Appl. **1**, Addison-Wesley, Reading, MA, 1976. MR Zbl

[Scheuer 2021] J. Scheuer, "The Minkowski inequality in de Sitter space", *Pacific J. Math.* **314**:2 (2021), 425–449. MR Zbl

[Solanes 2006] G. Solanes, "Integral geometry and the Gauss–Bonnet theorem in constant curvature spaces", *Trans. Amer. Math. Soc.* **358**:3 (2006), 1105–1115. MR Zbl

[Tso 1985] K. Tso, "Deforming a hypersurface by its Gauss–Kronecker curvature", *Comm. Pure Appl. Math.* **38**:6 (1985), 867–882. MR Zbl

[Wang and Xia 2014] G. Wang and C. Xia, "Isoperimetric type problems and Alexandrov–Fenchel type inequalities in the hyperbolic space", *Adv. Math.* **259** (2014), 532–556. MR Zbl

[Wu 2016] J. Wu, "A new characterization of geodesic spheres in the hyperbolic space", *Proc. Amer. Math. Soc.* **144**:7 (2016), 3077–3084. MR Zbl

[Wu and Xia 2014] J. Wu and C. Xia, "On rigidity of hypersurfaces with constant curvature functions in warped product manifolds", *Ann. Global Anal. Geom.* **46**:1 (2014), 1–22. MR Zbl

Received 5 Dec 2022. Accepted 31 Aug 2023.

ESTHER CABEZAS-RIVAS: esther.cabezas-rivas@uv.es
Departament de Matemàtiques, Universitat de València, València, Spain

JULIAN SCHEUER: scheuer@math.uni-frankfurt.de
Institut für Mathematik, Goethe-Universität, Frankfurt, Germany

ANALYSIS AND PDE
Vol. 17 (2024), No. 10, pp. 3623–3670
DOI: 10.2140/apde.2024.17.3623

msp

UPPER BOUND ON THE NUMBER OF RESONANCES FOR EVEN ASYMPTOTICALLY HYPERBOLIC MANIFOLDS WITH REAL-ANALYTIC ENDS

MALO JÉZÉQUEL

We prove a polynomial upper bound on the number of resonances in a disk whose radius tends to $+\infty$ for even asymptotically hyperbolic manifolds with real-analytic ends. Our analysis also gives a similar upper bound on the number of quasinormal frequencies for Schwarzschild–de Sitter spacetimes.

1. Introduction

The purpose of this work is to prove an upper bound for the number of resonances for even asymptotically hyperbolic manifolds with real-analytic (but a priori not exactly hyperbolic) ends. Let us recall that an asymptotically hyperbolic manifold is a Riemannian manifold (M, g) such that M is the interior of a compact manifold with boundary $\overline{M}$ and there is an identification of a neighborhood of $\partial\overline{M}$ with $[0, \epsilon[_{y_1} \times \partial\overline{M}_{y'}$ that puts the metric g into the form

$$g = \frac{\mathrm{d}y_1^2 + g_1(y_1, y', \mathrm{d}y')}{y_1^2}, \tag{1}$$

where $g_1(y_1, y', \mathrm{d}y')$ is a family of metrics on $\partial\overline{M}$ depending on y_1. We say that (M, g) is even if g_1 is a smooth function of y_1^2. We refer to [Dyatlov and Zworski 2019, §5.1] for a detailed discussion of this notion.

Letting Δ denote the (nonpositive) Laplace operator on an even asymptotically hyperbolic manifold (M, g) of dimension n, one commonly introduces the family of operators, depending on the complex parameter λ,

$$\left(-\Delta - \tfrac{1}{4}(n-1)^2 - \lambda^2\right)^{-1} : L^2(M) \to L^2(M), \quad \operatorname{Im}\lambda > 0. \tag{2}$$

MSC2020: 58J50.
Keywords: scattering, resonances, asymptotically hyperbolic manifolds, real-analytic.

Since the essential spectrum of $-\Delta$ is $[(n-1)^2/4, +\infty[$, this family of operators is well-defined and meromorphic for $\operatorname{Im}\lambda > 0$, with maybe a finite number of poles between 0 and $i(n-1)/2$ on the imaginary axis, corresponding to the eigenvalues of $-\Delta$ in $]0, (n-1)^2/4[$. Notice that the residues of these poles have finite ranks.

The *scattering resolvent* of (M, g) is then defined as the meromorphic continuation of (2), as provided by the following result.

Theorem 1 [Mazzeo and Melrose 1987; Guillarmou 2005]. *Let (M, g) be an even asymptotically hyperbolic manifold of dimension n. Then the resolvent (2) admits a meromorphic extension $R_{\mathrm{scat}}(\lambda)$ to $\mathbb{C}$ as an operator from $C_c^\infty(M)$ to $\mathcal{D}'(M)$, with residues of finite rank.*

In the case of manifolds that are exactly hyperbolic near infinity, one may also refer to [Guillopé and Zworski 1995a]. Notice that we do not use here the same spectral parameter as in [Mazzeo and Melrose 1987; Guillarmou 2005; Guillopé and Zworski 1995a]. The spectral parameter from these references is given in terms of our λ as $\zeta = (n-1)/2 - i\lambda$. Another proof of Theorem 1 has been given by Vasy [2013a] (see also [Vasy 2013b; Zworski 2016; Dyatlov and Zworski 2019, Chapter 5]).

The poles of the scattering resolvent (the meromorphic continuation of (2)) are called the resonances of (M, g). If $\mu \neq 0$ is a scattering resonance for (M, g) then we define the multiplicity of μ as the rank of the operator

$$\frac{i}{\pi} \int_\gamma \lambda R_{\mathrm{scat}}(\lambda)\, \mathrm{d}\lambda, \tag{3}$$

where γ is a small positively oriented circle around μ (so that the index of μ with respect to γ is 1, and the index of any other resonance is zero). That this operator has finite rank follows from the fact that the residues of $R_{\mathrm{scat}}(\lambda)$ have finite ranks. Another definition for the multiplicity of resonances may be found for instance in [Guillopé and Zworski 1997, Definition 1.2], but it coincides with the one we gave when μ is nonzero (see [Guillopé and Zworski 1997, Proposition 2.11]). The definition of the multiplicity of 0 as a resonance is more subtle (and will not matter in our case), see the discussion after Theorem 1 in [Zworski 1997]. Notice that in [Mazzeo and Melrose 1987; Vasy 2013a], the scattering resolvent $R_{\mathrm{scat}}(\lambda)$ is constructed as an operator from the space $\dot{C}^\infty(M)$ of smooth functions on $\overline{M}$ that vanish at infinite orders on $\partial\overline{M}$ to its dual. Since $C_c^\infty(M)$ is contained in $\dot{C}^\infty(M)$, we stated in Theorem 1 a weaker result. Notice however that, since $C_c^\infty(M)$ is dense in $\dot{C}^\infty(M)$, this simplification does not modify the notion of multiplicity of a resonance.

Our main result is an upper bound on the number of resonances for even asymptotically hyperbolic manifolds with real-analytic ends (as defined in Section 4.1).

Theorem 2. *Let (M, g) be an even asymptotically hyperbolic manifold real-analytic near infinity (as defined in Section 4.1) of dimension n. For $r > 0$, let $N(r)$ denote the number of resonances of (M, g) of modulus less than r, counted with multiplicities. Then*

$$N(r) \underset{r \to +\infty}{=} \mathcal{O}(r^n). \tag{4}$$

This upper bound is natural, since it is coherent with the asymptotic for the number of eigenvalues for the Laplacian on a closed Riemannian manifold given by Weyl law. There are also noncompact examples for which the bound (4) is optimal; see the lower bounds from [Guillopé and Zworski 1997; Borthwick 2008] discussed below.

There is a long tradition of studies of such counting problems in scattering theory, going back to the work of Tullio Regge [1958]. Results similar to Theorem 2 have been established in the context of scattering (e.g., by a compactly supported potential or by certain black boxes) on odd-dimensional *Euclidean* spaces [Melrose 1984; Zworski 1989; Sjöstrand and Zworski 1991; Vodev 1992]. In the context of asymptotically hyperbolic manifolds, the bound (4) is known for manifolds with *exactly* hyperbolic ends [Guillopé and Zworski 1995b; Cuevas and Vodev 2003; Borthwick 2008]. Still in the case of manifolds with exactly hyperbolic ends, we also have some lower bounds available: in the case of surfaces Guillopé and Zworski [1997] proved that $r^2 = \mathcal{O}(N(r))$, which implies that (4) is optimal in that case. In higher dimension n, Borthwick [2008] proved a similar lower bound $r^n = \mathcal{O}(N^{\mathrm{sc}}(r))$ for compact perturbations of conformally compact hyperbolic manifolds (a stronger assumption than just having exactly hyperbolic ends). This lower bound is obtained for the counting function $N^{\mathrm{sc}}(r)$ associated to a larger set of resonances than $N(r)$, and that also satisfies (4). However, a few cases in which the same lower bound for $N(r)$ follows are given in [Borthwick 2008]. Finally, a lower bound for $N(r)$ of the form

$$\limsup_{r \to +\infty} \frac{\log N(r)}{\log r} = n$$

is proven for generic compact perturbations of a manifold with exactly hyperbolic ends in [Borthwick et al. 2011].

Leaving the context of manifolds with exactly hyperbolic ends, much less is known on the asymptotic of the counting function $N(r)$. The bound (4) was established by Borthwick and Philipp [2014] in the case of asymptotically hyperbolic manifolds with warped-product ends, that is, for which the coordinates (y_1, y') in (1) may be chosen so that $g_1(y_1, y', \mathrm{d}y') = g_1(y', \mathrm{d}y')$ does not depend on y_1. The proof of a similar bound is sketched in [Froese and Hislop 2000] for a class of asymptotically hyperbolic manifolds with ends that are asymptotically warped. Wang [2019] established, for certain real-analytic asymptotically hyperbolic metrics on $\mathbb{R}^3$, a polynomial bound $\mathcal{O}(r^6)$ for the number of resonances in a sector of the form

$$\left\{ z \in \mathbb{C} : \epsilon < |z| < r, -\tfrac{1}{2}\pi + \epsilon < \arg z < \tfrac{3}{2}\pi - \epsilon \right\} \tag{5}$$

when r tends to $+\infty$ while $\epsilon > 0$ is fixed. The evenness assumption is not made in [Wang 2019], hence the necessity to count resonances in sectors of the form (5) rather than in disks (one has to avoid the essential singularities that can appear in the noneven case according to [Guillarmou 2005]). In the even case, our result, Theorem 2, improves the bound from [Wang 2019], not only because we can count resonances in a disk, but also because our result, valid in any dimension, gives a better exponent in the 3-dimensional case.

Let us point out that the upper bound (4) is also satisfied by the counting functions for the *Ruelle resonances* of a real-analytic Anosov flow, as follows from a result of Fried [1995] based on techniques introduced by Rugh [1992; 1996]. We gave a new proof of this result in [Bonthonneau and Jézéquel 2020], adapting techniques originally developed in [Helffer and Sjöstrand 1986; Sjöstrand 1996]. The tools of real-analytic microlocal analysis that we use in the present paper rely heavily on [Bonthonneau and Jézéquel 2020].

The main idea behind the proof of Theorem 2 is to adapt the method of Vasy [2013a] to construct the scattering resolvent, by introducing tools of real-analytic microlocal analysis. The method of Vasy does not only apply to even asymptotically hyperbolic manifolds, it may also be used to study resonances associated to the wave equation on *Schwarzschild–de Sitter spacetimes* (in this context, resonances are also called *quasinormal frequencies*). The interested reader may for instance refer to [Dafermos and Rodnianski 2013, §6] for a description of the geometry of Schwarzschild–de Sitter spacetimes. Consequently, our method also gives an upper bound on the number of resonances (or quasinormal frequencies) for Schwarzschild–de Sitter spacetimes.

Theorem 3. *The number of quasinormal frequencies of modulus less than r for a Schwarzschild–de Sitter spacetime is $\mathcal{O}(r^3)$ when r tends to $+\infty$.*

It is proven in [Sá Barreto and Zworski 1997] that the quasinormal frequencies for a Schwarzschild–de Sitter spacetime are well approximated by the pseudopoles

$$c\big(\pm\ell \pm \tfrac{1}{2} - i\big(k + \tfrac{1}{2}\big)\big),$$

for $k \in \mathbb{N}$ and $\ell \in \mathbb{N}^*$, the corresponding pole having multiplicity $2\ell + 1$. Here, c is a constant depending on the mass of the black hole and the cosmological constant. However, the approximation given in [loc. cit.] is only effective for a pseudopole μ such that $|\mu|$ tends to $+\infty$ while the imaginary part of μ remains bounded from below. Consequently, while Theorem 3 seems reasonable in view of the approximation result from [loc. cit.], these two results discuss two different asymptotics. The result from [loc. cit.] cannot be used to prove Theorem 3, nor to prove that Theorem 3 is sharp (even though it suggest that it should be the case).

It may be possible that the method of the proof of Theorems 2 and 3 generalizes to the case of slowly rotating Kerr–de Sitter black holes (as the method of Vasy [2013a, §6] also applies in this context). However, there are some additional technical difficulties that would probably arise in that case, due to the microlocal geometry being more complicated than in the Schwarzschild–de Sitter case. In particular, there are bicharacteristics that originate at the source above the event horizon, then enter the domain of outer communication and eventually leave it. Our strategy of proof would require the propagation of singularities along these bicharacteristics using real-analytic microlocal analysis. Consequently, in order to deal with Kerr–de Sitter spacetimes, one cannot use real-analytic tools only near the event and cosmological horizon, as it is the case in the proof of Theorem 3; see Remark 4. Since the coefficients of Kerr–de Sitter spacetimes are real-analytic on the whole domain of outer communication, it is not unlikely that this problem may be solved. In any case, we expect that one would need to use an escape function more carefully designed than in our analysis below.

Idea of the proof. As mentioned above, the proof of Theorems 2 and 3 is based on an adaptation of the method of Vasy [2013a] to construct the scattering resolvent, by introducing tools of real-analytic microlocal analysis. Our approach of the method of Vasy is mostly based on the exposition from [Dyatlov and Zworski 2019, Chapter 5].

The starting point of the proof of Theorems 2 and 3 is the following observation. When using the method of Vasy to construct the scattering resolvent, one will construct a meromorphic extension to (2) on a half plane of the form

$$\{\lambda \in \mathbb{C} : \operatorname{Im} \lambda > -C\} \tag{6}$$

for a given $C > 0$, by studying the action of a modified Laplacian on a functional space H_C *that depends on* C. The constant C may be chosen arbitrarily large, so that we get indeed a meromorphic continuation to $\mathbb{C}$, but this requires a change in the space on which the modified Laplacian is acting.

In the context of even asymptotically hyperbolic manifolds, the space H_C is constructed in the following manner: one embeds M as a relatively compact subset of a manifold X, and replaces the operator $-\Delta - (n-1)^2/4 - \lambda^2$ by a family of modified Laplacians. These modified Laplacians are elliptic on M but have a source/sink structure above the boundary of M in X. One can then set up a Fredholm theory for the modified Laplacians by using microlocal radial estimates (see for instance [Dyatlov and Zworski 2019, §E.4]). However, radial estimates in the C^∞ category are limited by a threshold condition. In our setting, it imposes choosing space H_C as a space of functions with a number of derivatives proportional to C in order to get a meromorphic continuation of (2) on the half-plane (6).

Consequently, working only with C^∞ tools, one will a priori only have access to bound on the number of resonances when restricting to a half-plane of the form (6). A natural idea to tackle this difficulty is to work with a space "H_∞" of functions that are smooth near the boundary of M in X (in our case, this would be real-analytic objects). If one is able to prove a real-analytic version of the radial estimates, it should be possible to bypass the threshold condition and construct directly the meromorphic continuation of (2) to the whole $\mathbb{C}$, working on a single space H_∞. One can then hope that this functional analytic setting can be used to prove a global bound on the number of resonances, without the need to restrict to a half-plane of the form (6). We will use the tools from [Bonthonneau and Jézéquel 2020], based on [Helffer and Sjöstrand 1986; Sjöstrand 1996], to prove an estimate that is in some sense a real-analytic version of a radial estimate (see also [Galkowski and Zworski 2022]). Notice that similar estimates are proved in [Galkowski and Zworski 2021; Guedes-Bonthonneau et al. 2024] in different geometric contexts, and with a focus more on the hypoellipticity statement that may be deduced from the radial estimates rather than on the functional analytic consequences. In some sense, the results on resonances for zeroth order pseudodifferential operators in [Galkowski and Zworski 2022] and the results on real-analytic and Gevrey Anosov flows from [Bonthonneau and Jézéquel 2020] are already implicitly based on real-analytic radial estimates.

There is an important technical difference between the idea of the proof of Theorems 2 and 3 as depicted above and the way the proof is actually written. Indeed, we cannot work with a space H_∞ of functions that are analytic everywhere on X (in particular because we do not want to assume that g is analytic everywhere in M). Due to the lack of real-analytic bump functions, it is not easy to construct a space of functions that are real-analytic somewhere but have (at most) finite differentiability somewhere else, and that can be used to construct the scattering resolvent. We solve this issue using a strategy that was already present in [Guedes-Bonthonneau et al. 2024]: we introduce a semiclassical parameter $h > 0$ and work with a space of distributions $\mathcal{H}$ on X that depends on h. Let us point out that the *space* $\mathcal{H}$ really depends on h, not only its norm. As h tends to 0, the elements of $\mathcal{H}$ are more and more regular near the boundary of M in X. We can then invert a rescaled modified Laplacian acting on $\mathcal{H}$ after the addition of a trace class operator whose trace class norm is controlled as h tends to 0, and the upper bound from Theorems 2 and 3 will follow.

Structure of the paper. In Section 2, we introduce a set of general assumptions that will allow us to deal simultaneously with the analysis in the context of Theorems 2 and 3. The point of these assumptions is not to cover a wide generality, but to avoid to write the same proof twice with only notational changes. We state in Section 2 a general result, Proposition 5, from which Theorems 2 and 3 will be deduced.

In Sections 3 and 4, we prove respectively Theorems 3 and 2.

In Section 5, we recall and extend some results from [Bonthonneau and Jézéquel 2020] that will be needed for the proof of Proposition 5.

Finally, Section 6 is the main technical part of the paper, as it contains the proof of Proposition 5.

2. A general statement

In order to deal with the cases of asymptotically hyperbolic manifolds and of Schwarzschild–de Sitter spacetimes simultaneously, we introduce here an abstract set of assumptions that are enough to make our analysis work.

2.1. ***General assumption.*** We will use the notion of semiclassical differential operator, so let us recall very briefly what it means (see [Zworski 2012] or [Dyatlov and Zworski 2019, Appendix E] for more details on semiclassical analysis). A semiclassical differential operator Q of order $m \in \mathbb{N}$ on a smooth manifold X is a differential operator on X, depending on a small, so-called semiclassical, implicit parameter $h > 0$, of the form

$$Q = \sum_{k=0}^{m} h^k Q_k,$$

where Q_k is a differential operator of order k on X that does not depend on h, for $k = 0, \dots, m$. With Q one may associate its (semiclassical) principal symbol $q : T^*X \to \mathbb{C}$, which is a polynomial of degree m in each fiber of T^*X. We may define q as the unique h-independent function on T^*X such that, for every smooth function $\varphi : M \to \mathbb{C}$ and $x \in X$, we have

$$e^{-i\frac{\varphi(x)}{h}} Q(e^{i\frac{\varphi}{h}})(x) \underset{h \to 0}{=} q(x, \mathrm{d}_x\varphi) + \mathcal{O}(h).$$

Notice that $q = \sum_{k=0}^{m} q_m$, where q_k denotes the (classical) homogeneous principal symbol of the differential operator Q_k for $k = 0, \dots, m$. In the applications from Sections 3 and 4, the introduction of the semiclassical parameter h will be somehow artificial, this is just a technical trick.

Now that this reminder is done, we are ready to state our set of general assumptions.

Let X be a closed real-analytic manifold of dimension n. We endow X with a real-analytic Riemannian metric (this is always possible; see [Morrey 1958]). Let Y be an open subset of X with real-analytic boundary ∂Y. Consider a family of differential operators

$$\mathcal{P}_h(\omega) = P_2 + \omega P_1 + \omega^2 P_0, \tag{7}$$

where $\omega \in \mathbb{C}$ and the operator P_j for $j \in \{0, 1, 2\}$ is a semiclassical differential operator (that does not depend on ω) on X of order j with principal symbol p_j. We assume that there is $\epsilon > 0$ and a neighborhood U of ∂Y with real-analytic coordinates $(x_1, x') : U \to]-\epsilon, \epsilon[\times \partial Y$ such that $\{x_1 = 0\} = \partial Y$ and $\{x_1 > 0\} = Y \cap U$. We require in addition that P_0, P_1 and P_2 have real-analytic coefficients in U and that the following properties hold:

(a) For $(x_1, x', \xi_1, \xi') \in T^*U \simeq T^*(]-\epsilon, \epsilon[\times \partial Y)$, we have $p_2(x_1, x', \xi_1, \xi') = w(x_1)\xi_1^2 + q_1(x_1, x', \xi')$ where q_1 is a homogeneous real-valued symbol of order 2 on $]-\epsilon, \epsilon[\times T^*\partial Y$ and $w :]-\epsilon, \epsilon[\to \mathbb{R}$ is a real-analytic function such that $w(0) = 0$ and $w'(0) > 0$.

(b) There is a constant $C > 0$ such that for $(x_1, x', \xi_1, \xi') \in T^*U$ we have $q_1(x_1, x', \xi') \geq C^{-1}|\xi'|^2$.

(c) The symbol p_1 is real-valued, $p_1(x_1, x', \xi_1, \xi') = p_1(x_1, \xi_1)$ does not depend on (x', ξ') for $(x_1, x', \xi_1, \xi') \in T^*U$, and there is $C > 0$ such that

$$\frac{p_1(x_1, \xi_1)}{\xi_1} \leq -C^{-1},$$

in particular the sign of $p_1(x_1, \xi_1)$ is the same as the sign of $-\xi_1$.

(d) The symbol p_2 is real-valued and positive on $T^*Y \setminus \{0\}$.

(e) The symbol p_0 is real-valued and negative on a neighborhood of $\overline{Y}$.

Remark 4. Let us explain the significance of these assumptions. In the context of the proof of Theorem 2, the manifold X will be an even extension for M, and Y will be M seen as a subset of the even extension X. In the context of Theorem 3, Y will be the domain of outer communication and ∂Y corresponds to the event and cosmological horizons. In both cases, $\mathcal{P}_h(\omega)$ will be a (semiclassically rescaled) family of modified operators. For instance, in the context of Theorem 2, we replace the operator $-h^2\Delta - h^2(n-1)^2/4 - \omega^2$ by a modified Laplacian $\mathcal{P}_h(\omega)$ (see for instance [Dyatlov and Zworski 2019, §5.3]). The new operator $\mathcal{P}_h(\omega)$ is defined on the whole X, and, for f smooth and compactly supported in Y, solving for u the equation $\mathcal{P}_h(\omega)u = f$ with u satisfying a regularity condition near ∂Y amounts to solving for $\tilde{u}$ the equation $(-h^2\Delta - h^2(n-1)^2/4 - \omega^2)\tilde{u} = \tilde{f}$ while imposing a certain behavior at infinity for $\tilde{u}$ (here $\tilde{f}$ depends on f and is smooth and compactly supported in M).

A method to construct the scattering resolvent is then to construct a meromorphic inverse $\mathcal{P}_h(\omega)^{-1}$ for $\mathcal{P}_h(\omega)$. In Proposition 5 below, we give a new construction of this meromorphic inverse (maybe after modifying $\mathcal{P}_h(\omega)$ away from $\overline{Y}$, which is harmless since we only care about what happens on Y). This new construction is inspired by the method of Vasy [2013a] (see also [Dyatlov and Zworski 2019, Chapter 5]) with the addition of tools of real-analytic microlocal analysis near ∂Y.

Let us explain very briefly how it works. The idea is to set up a Fredholm theory for $\mathcal{P}_h(\omega)$. Inside Y, the operator $\mathcal{P}_h(\omega)$ is elliptic (due to (d)), so there is no problem here. Outside of $\overline{Y}$, we are allowed to modify $\mathcal{P}_h(\omega)$, and we can consequently deal with this part of X by adding to $\mathcal{P}_h(\omega)$ a well-chosen elliptic operator. This is similar to the addition of a complex absorbing potential in [Vasy 2013a], and possible because of the hyperbolic structure of $\mathcal{P}_h(\omega)$ near ∂Y in $X \setminus \overline{Y}$. Hence, the most important point is to understand what happens at ∂Y, where $\mathcal{P}_h(\omega)$ stops being elliptic. At that place, the operator $\mathcal{P}_h(\omega)$ has a source/sink structure on its characteristic set (this is a consequence of the assumptions (a) and (b)), so that one can use radial estimates (see for instance to [Dyatlov and Zworski 2019, §E.4]) to set up a Fredholm theory for $\mathcal{P}_h(\omega)$. However, the C^∞ versions of the radial estimates are restricted by a threshold condition: they can be used to construct the scattering resolvent, but they do not give a bound on the number of resonances in disks as in Theorem 2. This is where real-analytic microlocal analysis becomes useful: using methods as in [Bonthonneau and Jézéquel 2020; Galkowski and Zworski 2022] (see also [Galkowski and Zworski 2021; Guedes-Bonthonneau et al. 2024]), we are able to get an estimate which is in some sense a C^ω version of a radial estimate and allows us to prove Theorem 2. This estimate corresponds to the fourth and fifth case in the proof of Lemma 26.

There are some technical reasons that make our set of assumptions very specific. The (e) is rather artificial, this is just a way to ensure that our family of Fredholm operators will be invertible at a point. The assumptions (a), (b) and (c) impose that the source/sink structure of $\mathcal{P}_h(\omega)$ on its characteristic set is very particular. This specific structure will allow us to work in the real-analytic category only near ∂Y, which is essential because we are not able to ensure that $\mathcal{P}_h(\omega)$ is analytic away from ∂Y. Concretely, this ensures that near ∂Y in $X \setminus \overline{Y}$, the projection on X of the bicharacteristics curve of $\mathcal{P}_h(\omega)$ that are contained in its characteristic set go either toward or away from ∂Y. This allows us to set up a propagation estimate by working on spaces weighted by $e^{\psi/h}$, where ψ is a function on X monotone along the projection to X of the bicharacteristics of $\mathcal{P}_h(\omega)$. This estimate does not require real-analytic coefficients, so it can be used to make the link between ∂Y (where we really need real-analytic machinery) and the place in $X \setminus \overline{Y}$ where $\mathcal{P}_h(\omega)$ is artificially made elliptic by the addition of a differential operator with C^∞ coefficients.

2.2. *General result.* The assumptions from Section 2.1 allow us to state an abstract result from which Theorems 2 and 3 follow.

Proposition 5. *Under the assumptions from Section 2.1, we may modify the operator $\mathcal{P}_h(\omega)$ away from $\overline{Y}$ into a new operator $P_h(\omega)$ so that the following holds. There are two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ (depending on h) and a constant $\kappa > 0$ (that does not depend on h) such that the following properties hold when h is small enough*:

(i) *For $j = 1, 2$, there are continuous inclusions $C^\infty(X) \subseteq \mathcal{H}_j \subseteq \mathcal{D}'(X)$.*

(ii) *For $j = 1, 2$, the elements of $\mathcal{H}_j$ are continuous on a neighborhood of ∂Y.*

(iii) *$P_h(\omega) : \mathcal{H}_1 \to \mathcal{H}_2$ is a holomorphic family of bounded operators.*

(iv) *There is $\nu > 0$ such that $P_h(i\nu) : \mathcal{H}_1 \to \mathcal{H}_2$ is invertible.*

(v) *For every open and relatively compact subset V of $\{z \in \mathbb{C} : \operatorname{Im} z > -\kappa\}$, if h is small enough then, for every $\omega \in V$, the operator $P_h(\omega) : \mathcal{H}_1 \to \mathcal{H}_2$ is Fredholm of index 0. Moreover, this operator has a meromorphic inverse $\omega \mapsto P_h(\omega)^{-1}$ on V with poles of finite rank.*

(vi) *If $\delta \in]0, \kappa[$, there is $C > 0$ such that for every h small enough, the number of ω in the disk of center 0 and radius δ such that $P_h(\omega) : \mathcal{H}_1 \to \mathcal{H}_2$ is not invertible (counted with null multiplicity) is less than Ch^{-n}.*

Remark 6. The notion of null multiplicity used in the statement of Proposition 5 is defined using Gohberg–Sigal theory (see for instance [Dyatlov and Zworski 2019, §C.4]). In our context, we can use the following definition: if ω_0 is such that the meromorphic inverse $\omega \mapsto P_h(\omega)$ is defined near ω_0, then the null multiplicity of $P_h(\omega)$ at ω_0 is the trace of the residue of $\omega \mapsto P_h(\omega)^{-1}\partial_\omega P_h(\omega)$ at ω_0 (which is a finite rank operator).

Remark 7. The modification of $\mathcal{P}_h(\omega)$ needed to get Proposition 5 will be obtained by modifying the coefficients of P_0, P_1 and P_2 away from $\overline{Y}$, so that the general assumption are still satisfied by $P_h(\omega)$ after this modification.

3. Schwarzschild–de Sitter spacetimes (proof of Theorem 3)

In this section, we explain how the general framework from Section 2 can be used to prove Theorem 3. We start with this case because the setting is slightly simpler than in Theorem 2 that we prove in Section 4 below. We recall a few basic facts about Schwarzschild–de Sitter spacetimes in Section 3.1 and then apply Proposition 5 in Section 3.2. Finally, in Section 3.3, we discuss the number of resonances for the operators obtained by decomposing functions on Schwarzschild–de Sitter spacetimes on spherical harmonics.

3.1. ***The model.*** We start by recalling the definition of Schwarzschild–de Sitter spacetimes and of the associated quasinormal frequencies. The interested reader may refer to [Dafermos and Rodnianski 2013] for the geometry of Schwarzschild–de Sitter spacetimes (and other notions from general relativity). For the definition of the resonances, one may refer to [Sá Barreto and Zworski 1997] or [Vasy 2013a]. Fix two constants

$$M_0 > 0 \quad \text{and} \quad 0 < \Lambda < \frac{1}{9M_0^2}.$$

The constant M_0 is called the *mass of the black hole* and Λ the *cosmological constant*. We define the function

$$G(r) = 1 - \frac{\Lambda r^2}{3} - \frac{2M_0}{r} \quad \text{for } r > 0.$$

Let then $r_- < r_+$ be the positive roots of the polynomial $rG(r)$. Define $M =]r_-, r_+[_r \times \mathbb{S}^2_y$ and $\widehat{M} = \mathbb{R}_t \times M$. Let g be the Lorentzian metric

$$g = -G\mathrm{d}t^2 + G^{-1}\mathrm{d}r^2 + r^2 g_S(y, \mathrm{d}y),$$

where g_S denotes the standard metric on $\mathbb{S}^2$. The Lorentzian manifold $(\widehat{M}, g)$ is called a Schwarzschild–de Sitter spacetime. The hypersurfaces $\{r_-\} \times \mathbb{S}^2$ and $\{r_+\} \times \mathbb{S}^2$ are called respectively the event and the cosmological horizons.

In order to understand the asymptotic of the solution to the wave equation on $(\widehat{M}, g)$, one studies the meromorphic continuation of the resolvent $(P_{\mathrm{SdS}} - \lambda^2)^{-1}$, where

$$P_{\mathrm{SdS}} = Gr^{-2} D_r (r^2 G) D_r - Gr^{-2} \Delta_{\mathbb{S}^2}.$$

Here, $D_r = -i\partial_r$ and $\Delta_{\mathbb{S}^2}$ is the (nonpositive) Laplace operator on the sphere $\mathbb{S}^2$. The operator P_{SdS} is self-adjoint and nonnegative on the Hilbert space $L^2(]r_-, r_+[\times \mathbb{S}^2; G^{-1} r^2 \, \mathrm{d}r \, \mathrm{d}y)$, where $\mathrm{d}y$ denotes the standard volume form on $\mathbb{S}^2$. Consequently, the operator $(P_{\mathrm{SdS}} - \lambda^2)^{-1}$ is well-defined on this space when $\operatorname{Im} \lambda > 0$. It is proven for instance in [Sá Barreto and Zworski 1997, §2] that $(P_{\mathrm{SdS}} - \lambda^2)^{-1}$ has a meromorphic continuation $R_{\mathrm{SdS}}(\lambda)$ to $\mathbb{C}$, with poles of finite rank, as an operator from $C_c^\infty(]r_-, r_+[\times \mathbb{S}^2)$ to $\mathcal{D}'(]r_-, r_+[\times \mathbb{S}^2)$. The poles of this meromorphic continuation are called the *quasinormal frequencies* for the Schwarzschild–de Sitter spacetime. If $\lambda_0 \neq 0$ is a quasinormal frequency, we define its multiplicity as the rank of the operator

$$\frac{i}{\pi} \int_\gamma \lambda R_{\mathrm{SdS}}(\lambda) \, \mathrm{d}\lambda,$$

where γ is a positively oriented circle around λ_0, small enough so that the index of any other quasinormal frequency with respect to γ is zero.

3.2. ***Upper bound on the number of quasinormal frequencies.*** Our proof of Theorem 3 is based on the method of Vasy [2013a] to construct the resolvent $R_{\mathrm{SdS}}(\lambda)$, following mostly the exposition from [Dyatlov and Zworski 2019, Exercise 16, p. 376]. We start with a standard modification of the operator $P_{\mathrm{SdS}} - \lambda^2$, with some minor changes that will be convenient to check the assumptions from Section 2.1.

Let us embed a neighborhood of $[r_-, r_+]$ in the circle $\mathbb{S}^1$ and set $X = \mathbb{S}^1 \times \mathbb{S}^2$ and $Y =]r_-, r_+[\times \mathbb{S}^2$. Let $\rho :]r_-, r_+[\to [-1, 1]$ be a C^∞ function, identically equal to ± 1 near $r_\pm$. Let then $F :]r_-, r_+[\to \mathbb{R}$ be a primitive of

$$F'(r) = \rho(r) \left(\frac{1}{G(r)} - \frac{1}{2(1 - (9M_0^2 \Lambda)^{1/3})} \right) \tag{8}$$

and introduce, for $\lambda \in \mathbb{C}$, the operator

$$G^{-1} e^{-i\lambda F(r)} (P_{\mathrm{SdS}} - \lambda^2) e^{i\lambda F(r)},$$

which is explicitly given by the formula

$$G D_r^2 - r^{-2} \Delta_{\mathbb{S}^2} + \left(2\lambda F' G - i \left(\frac{2G}{r} + G' \right) \right) D_r - i\lambda \left(\frac{2GF'}{r} + G'F' + GF'' \right) - \lambda^2 \frac{(1 - G^2 (F')^2)}{G}. \tag{9}$$

The coefficients of this differential operator extend as real-analytic functions near r_- and r_+. Indeed, the definition of F ensures that $F'G$ continues analytically passed r_- and r_+. Moreover, near $r_\pm$ a direct computation yields

$$G'F' + GF'' = \mp \frac{G'}{2(1 - (9M_0^2 \Lambda)^{1/3})}$$

and

$$\frac{1 - G^2 (F')^2}{G} = \frac{1}{1 - (9M_0^2 \Lambda)^{1/3}} - \frac{G}{4(1 - (9M_0^2 \Lambda)^{1/3})^2}.$$

Letting χ be a C^∞ function supported in a small neighborhood of $[r_-, r_+]$ and identically equal to 1 on a smaller neighborhood of $[r_-, r_+]$, we can define a family of operators on X by

$$\Theta(\lambda) = \chi(r) \times (9).$$

Finally, for $\omega \in \mathbb{C}$, we define the semiclassical differential operator

$$\mathcal{P}_h(\omega) := h^2 \Theta(h^{-1}\omega).$$

Notice that this operator depends on the implicit semiclassical parameter h as in Section 2. It is of the form (7) with

$$P_0 = \chi(r)\left(G h^2 D_r^2 - r^{-2} h^2 \Delta_{\mathbb{S}^2} - i\left(\frac{2G}{r} + G'\right) h^2 D_r\right),$$

$$P_1 = \chi(r)\left(2F' G h D_r - i h\left(\frac{2GF'}{r} + G'F' + GF''\right)\right),$$

$$P_2 = -\chi(r)\frac{1 - G^2 (F')^2}{G},$$

where it is understood that the factor in parentheses continues analytically in r passed r_- and r_+. Let us check that the general assumptions from Section 2.1 are satisfied by this family of operator.

We already mentioned that $\mathcal{P}_h(\omega)$ is of the form (7), and it follows from the expression for the P_j's given above that they are semiclassical differential operators of order j with analytic coefficients on a neighborhood of ∂Y. Moreover, the principal symbols of the P_j's are given on Y by

$$p_2(r, y, \rho, \eta) = G(r)\rho^2 + r^{-2}\eta^2, \quad p_1(r, y, \rho, \eta) = 2F'(r)G(r)\rho, \quad p_0(r, y) = -\frac{1 - G(r)^2 F'(r)^2}{G(r)}.$$

We get the values of these symbols on a neighborhood of $\overline{Y}$ by continuing these formulas analytically in r.

We can define the coordinates (x_1, x') near ∂Y by taking $x_1 = r - r_-$ (when r is near r_-) or $x_1 = r_+ - r$ (when r is near r_+) and $x' = y$. Beware here that this change of coordinates reverses the orientation of the real line near r_+. Then, we see that the (a) holds with $w(x_1) = G(r_\pm \mp x_1)$ and $q_1(x_1, y, \eta) = (r_\pm \mp x_1)^{-2}\eta^2$. In particular, we have $w'(0) = \mp G'(r_\pm) > 0$. The point (b) follows from the definition of q_1. To get (c), one only needs to notice that the value at $r_\pm$ of the real-analytic extension of $F'(r)G(r)$ is ± 1 (and that our change of variable reverses orientation near r_+). Since G is positive on $]r_-, r_+[$, we get (d). In order to check (e), write

$$p_0(r, y) = \frac{\rho(r)^2\left(1 - \dfrac{G(r)}{2(1-(9M_0^2\Lambda)^{1/3})}\right)^2 - 1}{G(r)}.$$

Since $1 - (9M_0^2\Lambda)^{1/3}$ is an upper bound for G on $]r_-, r_+[$, we find that $p_0(r, y) < 0$ for $r \in]r_-, r_+[$. Using that $\rho(r)^2$ is equal to 1 when r is near $r_\pm$, we find that

$$p_0(r_\pm, y) = -\frac{1}{1 - (9M_0^2\Lambda)^{1/3}} < 0,$$

and thus (e) holds.

Consequently, we can modify $\mathcal{P}_h(\omega)$ away from $\overline{Y}$ in order to get a family of operator $P_h(\omega)$ that satisfies Proposition 5. With κ as in Proposition 5, we let V be a connected, relatively compact and open subset of $\{z \in \mathbb{C} : \operatorname{Im} z > -\kappa\}$ that contains the closed disk of center 0 and radius $3\kappa/4$. Let ι_2 denote the injection of $C_c^\infty(Y)$ in $\mathcal{H}_2$ and ι_1 denote the map from $\mathcal{H}_1$ to $\mathcal{D}'(Y)$ obtained by composing the injection $\mathcal{H}_1 \to \mathcal{D}'(X)$ with the restriction map $\mathcal{D}'(X) \to \mathcal{D}'(Y)$.

If $\lambda \in h^{-1}V$, we define the resolvent

$$R_h(\lambda) = e^{i\lambda F(r)} h^2 \iota_1 P_h(\lambda h)^{-1} \iota_2 e^{-i\lambda F(r)} G^{-1} : C_c^\infty(Y) \to \mathcal{D}'(Y). \tag{10}$$

This is a meromorphic family of operators. We just got a new construction of the meromorphic continuation $R_{\mathrm{SdS}}(\lambda)$ of the L^2 resolvent $(P_{\mathrm{SdS}} - \lambda^2)^{-1}$, as we will now demonstrate.

Lemma 8. *If $\lambda \in h^{-1}V$ is such that $\operatorname{Im}\lambda > 0$, then $R_h(\lambda)$ is the restriction to $C_c^\infty(Y)$ of the L^2 resolvent $(P_{\mathrm{SdS}} - \lambda^2)^{-1}$. In particular $R_h(\lambda)$ does not depend on h.*

Proof. Let $\lambda \in h^{-1}V$ be such that $\operatorname{Im}\lambda > 0$. Let $u \in C_c^\infty(Y)$. Notice that

$$\begin{aligned}(P_{\mathrm{SdS}} - \lambda^2) R_h(\lambda) u &= G e^{i\lambda F(r)} G^{-1} e^{-i\lambda F(r)} (P_{\mathrm{SdS}} - \lambda^2) e^{i\lambda F(r)} h^2 \iota_1 P_h(\lambda h)^{-1} \iota_2 e^{-i\lambda F(r)} G^{-1} u \\ &= G e^{i\lambda F(r)} P_h(\lambda h) \iota_1 P_h(\lambda h)^{-1} \iota_2 e^{-i\lambda F(r)} G^{-1} u = u,\end{aligned}$$

where we used that $h^2 G^{-1} e^{-i\lambda F(r)} (P_{\mathrm{SdS}} - \lambda^2) e^{i\lambda F(r)} \iota_1 = P_h(\lambda h) \iota_1 = \iota_3 P_h(\lambda h)$, where ι_3 is the map obtained by composing the injection $\mathcal{H}_2 \to \mathcal{D}'(X)$ with the restriction map $\mathcal{D}'(X) \to \mathcal{D}'(Y)$. Consequently, we only need to prove that the distribution $R_h(\lambda)u$ belongs to the space $L^2(]r_-, r_+[\times \mathbb{S}^2; G^{-1} r^2 \,\mathrm{d}r \,\mathrm{d}y)$. Since P_{SdS} is elliptic, we know that u is smooth, and thus bounded on all compact subsets of Y. It remains to understand the behavior of u near ∂Y.

Notice that $R_h(\lambda)u = e^{i\lambda F(r)} v$, where v is the restriction to Y of an element of $\mathcal{H}_1$. In particular, since the elements of $\mathcal{H}_1$ are continuous near ∂Y, there is a compact subset K of Y such that v is continuous and bounded outside of K. Let us study for instance the behavior of u near $r = r_-$ (the behavior near r_+ is similar). From (8), we see that

$$F(r) \underset{r \to r_-}{=} -\frac{\ln |r - r_-|}{G'(r_-)} + \mathcal{O}(1).$$

Consequently, we have that $e^{i\lambda F(r)}$ is $\mathcal{O}(|r - r_-|^{\operatorname{Im}\lambda / G'(r_-)})$ when r tends to r_-. Working similarly near r_+, we find that u belongs to the Hilbert space $L^2(]r_-, r_+[\times \mathbb{S}^2; G^{-1} r^2 \,\mathrm{d}r \,\mathrm{d}y)$. □

Remark 9. It follows from Lemma 8 that $R_h(\lambda) = R_{\mathrm{SdS}}(\lambda)$ on $h^{-1}V$. In particular, $\lambda \in h^{-1}V$ is a quasinormal frequency if an only if it is a pole of $R_h(\lambda)$ and, if in addition $\lambda \neq 0$, its multiplicity is the rank of the operator

$$\frac{i}{\pi} \int_\gamma \mu R_h(\mu) \,\mathrm{d}\mu,$$

where γ is a small circle around λ.

With this new construction of the resolvent $R_{\mathrm{SdS}}(\lambda)$ at our disposal, we are ready to prove Theorem 3.

Proof of Theorem 3. Considering the bound on the number of points where $P_h(\omega)$ is not invertible given in Proposition 5, we only need to prove that if λ is a nonzero complex number of modulus less than $\kappa/(4h)$ then its multiplicity as a quasinormal frequency is less than the null multiplicity of $\omega \mapsto P_h(\omega)$ at λh.

Let us consider a quasinormal frequency λ of modulus less than $\kappa/(4h)$. Since $P_h(\omega)$ is a holomorphic family of operators with a meromorphic inverse near λh (because λh belongs to V), it follows from the Gohberg–Sigal theory [Dyatlov and Zworski 2019, Theorem C.10], that there are holomorphic families of invertible operators $U_1(\omega)$ and $U_2(\omega)$ for ω near λh, respectively on $\mathcal{H}_2$ and from $\mathcal{H}_1$ to $\mathcal{H}_2$, an integer $M \geq 0$, operators $P_0, \dots, P_M$ on $\mathcal{H}_2$ and nonzero integers $k_1, \dots, k_M$ such that

$$P_h(\omega) = U_1(\omega)\Big(P_0 + \sum_{m=1}^{M}(\omega - \lambda h)^{k_m} P_m\Big)U_2(\omega) \tag{11}$$

for ω near λh. Moreover, $P_1, \dots, P_M$ are rank 1 and $P_\ell P_m = \delta_{\ell,m} P_m$ for $0 \leq \ell, m \leq M$. We also have that $I = \sum_{m=0}^{M} P_m$, since $P_h(\omega)$ is invertible for $\omega \neq \lambda h$ near λh. Notice that the k_m's must be positive, since $P_h(\omega)$ is holomorphic in ω, and that the null multiplicity of $P_h(\omega)$ at λh is $\sum_{m=1}^{M} k_m$.

It follows from (11) that

$$P_h(\omega)^{-1} = U_2(w)^{-1}\Big(P_0 + \sum_{m=1}^{M}(\omega - \lambda h)^{-k_m} P_m\Big)U_1(\omega)^{-1}. \tag{12}$$

From (10) we get

$$R_h(\mu) = A_1(\mu) + A_2(\mu),$$

where A_1 and A_2 are obtained by replacing the inverse $P_h(\omega)^{-1}$ respectively by $U_2(\omega)^{-1} P_0 U_1(\omega)^{-1}$ and by $U_2(\omega)^{-1} \sum_{m=1}^{M} (\omega - \lambda h)^{-k_m} P_m U_1(\omega)^{-1}$ in (10), with $\omega = \mu h$. Notice that $A_1(\mu)$ is holomorphic in μ, so that

$$\int_\gamma \mu R_h(\mu)\, \mathrm{d}\mu = \int_\gamma \mu A_2(\mu)\, \mathrm{d}\mu. \tag{13}$$

The operator $\mu A_2(\mu)$ is of the form $B_1(\mu)\big(\sum_{k=1}^{M}(\mu - \lambda)^{-k_m} P_m\big)B_2(\mu)$, where $B_1(\mu)$ and $B_2(\mu)$ are holomorphic near λ. Writing the Taylor expansions for $B_1(\mu)$ and $B_2(\mu)$,

$$B_j(\mu) = \sum_{\ell \geq 0} (\mu - \lambda)^\ell C_{j,l},$$

we find that the residue of $\mu A_2(\mu)$ at λ is

$$\sum_{\substack{m,k,\ell \\ k+\ell = k_m - 1}} C_{1,k} P_m C_{2,\ell}.$$

This operator is the sum of $\sum_{m=1}^{M} k_m$ operators of rank at most 1, and thus is of rank at most $\sum_{m=1}^{M} k_m$. It follows then from Remark 9 and (13) that the multiplicity of λ as a scattering resonance is at most $\sum_{m=1}^{M} k_m$, which is the null multiplicity of $\omega \mapsto P_h(\omega)$ at λh. □

3.3. *Decomposition on spherical harmonics.* Notice that the Schwarzschild–de Sitter spacetime is radially symmetric. It is standard to use this kind of symmetry to study quasinormal frequencies by decomposing the operator P_{SdS} on spherical harmonics (see for instance [Sá Barreto and Zworski 1997] or [Hintz and Xie 2022]). Let $\ell \in \mathbb{N}$ and Y be a spherical harmonics satisfying $-\Delta_{\mathbb{S}^2} Y = \ell(\ell+1)Y$. The action of P_{SdS} on functions of the form $u(r)Y(y)$ is then equivalent to the action of the operator

$$P_{\mathrm{SdS}}^{\ell} = Gr^{-2}D_r(r^2G)D_r + Gr^{-2}\ell(\ell+1).$$

The operator $(P_{\mathrm{SdS}}^{\ell} - \lambda^2)^{-1}$ defined for $\operatorname{Im}\lambda > 0$ by the spectral theory on $L^2(]r_-, r_+[;\, G^{-1}r^2\,\mathrm{d}r)$ admits a meromorphic continuation to $\mathbb{C}$. The poles of this extension are quasinormal frequencies corresponding to angular momentum ℓ.

We can then apply Proposition 5 as in Section 3.2 to get:

Theorem 10. *The number of quasinormal frequencies corresponding to the angular momentum ℓ of modulus less than r is $\mathcal{O}(r)$ when r tends to $+\infty$.*

4. Scattering on asymptotically hyperbolic manifolds (proof of Theorem 2)

In this section, we specify the geometric assumptions from Theorem 2 and explain how one can use Proposition 5 to prove Theorem 2. In Section 4.1 we describe the class of asymptotically hyperbolic manifolds with real-analytic ends that we are going to study. In Sections 4.2 and 4.3, we check the assumptions from Section 2.1 in order to use Proposition 5 and prove Theorem 2 in Section 4.4.

Sections 4.1, 4.2 and 4.3 are based on the exposition in [Dyatlov and Zworski 2019, Chapter 5] of the method of Vasy [2013a] to construct the scattering resolvent, with a few additional technicalities required to deal with real-analytic ends and apply Proposition 5.

4.1. *Geometric assumptions.* We explain here how the definition of asymptotically hyperbolic manifold may be modified to obtain the definition of asymptotically hyperbolic manifolds with real-analytic ends that appears in Theorem 2. Let us consider a Riemannian manifold (M, g) where M is a real-analytic manifold but the metric g is a priori only C^∞. One could just say that (M, g) is asymptotically hyperbolic with real-analytic ends if M is the interior of a compact real-analytic manifold with boundary $\overline{M}$ such that g may be put into the form (1), with g_1 real-analytic, near $\partial\overline{M}$, using a real-analytic diffeomorphism between $[0, \epsilon[\, \times \partial\overline{M}$ and a neighborhood of $\partial\overline{M}$. This is for instance the assumption that is made in [Zuily 2017]. However, it may seem a priori too restrictive to assume the existence of such coordinates defined on a neighborhood of the whole $\partial\overline{M}$. Consequently, we will rather make a local assumption on g and then see that it implies that g takes the form (1) in real-analytic coordinates.

Definition 11. Let M be a real-analytic manifold and g be a smooth (C^∞) Riemannian metric on M. We assume that M is the interior of a compact real-analytic manifold with boundary $\overline{M}$. Assume that, for every $x_0 \in \partial\overline{M}$, there is a neighborhood U of x_0 in $\overline{M}$ and a real-analytic function y_1 from U to $\mathbb{R}$ such that

(i) $y_1 \geq 0$ on U and $\partial\overline{M} \cap U = \{y_1 = 0\}$;

(ii) $\mathrm{d}y_1(x) \neq 0$ for every $x \in \partial\overline{M} \cap U$;

(iii) $y_1^2 g$ extends to a real-analytic metric $\tilde{g}$ on U;

(iv) $|dy_1|_{\tilde{g}} = 1$ on $\partial \overline{M} \cap U$.

Then we say that (M, g) is an asymptotically hyperbolic manifold real-analytic near infinity.

A function that satisfies (i) and (ii) is called a boundary defining function for $\overline{M}$. Notice that if y_1 and $\tilde{y}_1$ are two real-analytic boundary defining functions, then there is a real-analytic real-valued function f, defined wherever y_1 and $\tilde{y}_1$ are both defined, and such that $\tilde{y}_1 = e^f y_1$. In particular, the validity of (iii) and (iv) does not depend on the choice of the boundary defining function y_1. One can check that if (M, g) is an asymptotically hyperbolic manifold real-analytic near infinity, then it is also an asymptotically hyperbolic manifold in the standard (C^∞) sense (see for instance [Dyatlov and Zworski 2019, Definition 5.2]).

Let us fix an asymptotically hyperbolic manifold real-analytic near infinity (M, g), and let $\overline{M}$ be as in Definition 11. The existence of a real-analytic boundary defining function defined on a neighborhood of $\partial \overline{M}$ does not seem obvious, and will be established in Lemma 13 below. However, notice that one easily shows that there are C^∞ boundary defining functions defined on the whole $\overline{M}$ and let us define the conformal class of Riemannian metrics on ∂M:

$$[g]_{\partial \overline{M}} = \left\{ (y_1^2 g)|_{\partial \overline{M}} : y_1 \in C^\infty(\overline{M}) \text{ is a boundary defining function} \right\}.$$

It will be convenient to know that:

Lemma 12. *The conformal class $[g]_{\partial \overline{M}}$ admits a real-analytic representative.*

Proof. Let g_0 be any C^∞ representative of $[g]_{\partial \overline{M}}$. Let $\hat{g}$ be a real-analytic Riemannian metric on $\partial \overline{M}$ (whose existence is guaranteed by [Morrey 1958]). For every $x \in \partial \overline{M}$, let $B(x)$ be the self-adjoint (for $\hat{g}(x)$) endomorphism of $T_x \partial M$ such that $g_0(x) = \hat{g}(x)(B(x) \cdot, \cdot)$. Let g_1 be the metric defined by $g_1(x) = g_0(x)/\|B(x)\|$, where the operator norm of $B(x)$ is defined using the metric $\hat{g}(x)$. From its very definition, g_1 is a representative of $[g]_{\partial M}$. Let us prove that g_1 is real-analytic.

Let $x_0 \in \partial \overline{M}$. From our assumption above (Definition 11), there is a neighborhood V of x_0 in $\partial \overline{M}$ and a real-analytic metric g_2 on V such that g_2 is conformal to g_0 on V. We have $g_0 = e^{2f} g_2$ for some C^∞ function f on V. For $x \in V$, we have

$$\begin{aligned} g_1(x) &= \frac{g_0(x)}{\|B(x)\|} = \hat{g}(x)\left(\frac{B(x)}{\|B(x)\|} \cdot, \cdot \right) \\ &= \hat{g}(x)\left(\frac{e^{-2f(x)} B(x)}{\|e^{-2f(x)} B(x)\|} \cdot, \cdot \right). \end{aligned}$$

On the other hand, for $x \in V$, we have

$$g_2(x) = \hat{g}(x)(e^{-2f(x)} B(x) \cdot, \cdot).$$

Since g_2 and $\hat{g}$ are real-analytic, it follows that $x \mapsto e^{-2f(x)} B(x)$ is real-analytic on V, and thus so is g_1. □

We can then establish the existence of a real-analytic diffeomorphism on a neighborhood of $\partial \overline{M}$ that puts the metric g into the form (1) (this is also known as a canonical product structure). The C^∞ version of this result is standard; see for instance [Dyatlov and Zworski 2019, Theorem 5.4].

Lemma 13. *Let g_0 be a real-analytic representative of $[g]_{\partial \overline{M}}$. Then there is a real-analytic boundary function y_1 defined on a neighborhood U of $\partial \overline{M}$ such that*

$$|\mathrm{d}y_1|_{y_1^2 g} = 1 \text{ on a neighborhood of } \partial \overline{M} \text{ and } g_0 = (y_1^2 g)|_{\partial \overline{M}}. \tag{14}$$

Moreover, there is a real-analytic map y' from U to $\partial \overline{M}$ such that y' is the identity on $\partial \overline{M}$, the map $\Psi = (y_1, y')$ is a diffeomorphism from U to $[0, \epsilon[\times \partial \overline{M}$ for some $\epsilon > 0$, and the pushforward of g under this map has the form

$$(\Psi^{-1})^* g = \frac{\mathrm{d}y_1^2 + g_1(y_1, y', \mathrm{d}y')}{y_1^2},$$

where $g_1(y_1, y', \mathrm{d}y')$ is a real-analytic family of Riemannian metrics on $\partial \overline{M}$.

Proof. We start by constructing y_1 locally. Let $x_0 \in \partial \overline{M}$. Let $\tilde{y}_1$ be a real-analytic boundary function defined on a neighborhood U of x_0 as in Definition 11. Up to multiplying $\tilde{y}_1$ by a real-analytic function, we may assume that $(\tilde{y}_1^2 g)_{|\partial \overline{M} \cap U} = g_0$. We want to construct y_1 on a neighborhood of x_0 of the form $y_1 = e^f \tilde{y}_1$ with f real-analytic that vanishes on $\partial \overline{M}$. The condition $|\mathrm{d}y_1|_{y_1^2 g} = 1$ may be rewritten as an eikonal equation, $F(x, \mathrm{d}f(x)) = 0$, noncharacteristic with respect to $\partial \overline{M}$, like in [Dyatlov and Zworski 2019, (5.1.11)–(5.1.12)], which in our case has real-analytic coefficients. We can then use [Taylor 2011, Theorem 1.15.3] to find a (unique) solution f to this equation near x_0, which happens to be real-analytic. Thus, we constructed a boundary defining function y_1 that satisfies (14) near x_0.

Notice that if y_1 and y_2 are boundary defining functions that satisfy (14) on open sets U_1 and U_2 of $\overline{M}$, then y_1 and y_2 coincide on all the connected components of $U_1 \cap U_2$ that intersect $\partial \overline{M}$. Indeed, we can write $y_1 = e^f y_2$ with f that satisfies an eikonal equation as above and vanishes on $\partial \overline{M}$, and there is only one solution to this equation near $\partial \overline{M}$. We get the coincidence of y_1 and y_2 on the whole connected component of $U_1 \cap U_2$ by analytic continuation.

We can consequently glue the local solutions to (14) to get a solution defined on a neighborhood of the whole $\partial \overline{M}$.

Finally, we construct the normal coordinates (y_1, y') by integrating the gradient vector field $\nabla^{y_1^2 g} y_1$ starting on $\partial \overline{M}$ as in the proof of [Dyatlov and Zworski 2019, Theorem 5.4]. □

Definition 14. Using the notation from Lemma 13, we say that (M, g) is even if for every integer k, we have

$$\partial_{y_1}^{2k+1} g_1(0, y', \mathrm{d}y') = 0. \tag{15}$$

From now on, we will always assume that (M, g) satisfies the evenness assumption Definition 14. Notice that Definitions 11 and 14 together are the hypotheses from Theorem 2. It is also worth noticing that the evenness assumption (15) does not depend on the choice of the canonical product structure; see [Dyatlov and Zworski 2019, Theorem 5.6].

4.2. ***Even extension.*** We define an even extension X for M in the following way. We fix a canonical product structure (y_1, y') on a neighborhood $U \simeq [0, \epsilon[\times \partial \overline{M}$ of $\partial \overline{M}$, as in Lemma 13. Let us define the real-analytic diffeomorphisms

$$\psi_+ : U \cap M \to]0, \epsilon^2[\times \partial \overline{M}, \qquad x \mapsto (y_1(x)^2, y'(x)),$$
$$\psi_- : U \cap M \to]-1-\epsilon^2, -1[\times \partial \overline{M}, \quad x \mapsto (-1 - y_1(x)^2, y'(x)).$$

We let X be the closed real-analytic manifold obtained by gluing $]-1-\epsilon^2, \epsilon^2[\times \partial \overline{M}$ with two distinct copies of M using the maps ψ_- and ψ_+. We let x_1 be the function on X given by the first coordinate in $]-1-\epsilon^2, \epsilon^2[\times \partial \overline{M}$. Up to making ϵ smaller, we extend x_1 to a smooth function on X, real-analytic on $]-1-\epsilon^2, \epsilon^2[\times \partial \overline{M}$, and such that $]-1-\epsilon^2, \epsilon^2[\times \partial \overline{M} = \{-1-\epsilon^2 < x_1 < \epsilon^2\}$.

The features of the even extension X of M in $\{x_1 < 0\}$ are somehow irrelevant: we are only concerned by the analysis in $\{x_1 \geq 0\}$ (but it is more convenient to work on a closed real-analytic manifold). In particular, we will identify $Y := \{x_1 > 0\}$ with M. We will never do that with $\{x_1 < -1\}$. Notice however that $\overline{Y} \subseteq X$ does not have the same smooth structure as $\overline{M}$ as defined above (the manifold $\overline{Y}$ is the even compactification of M).

Notice that the diffeomorphism $\psi_+ : U \cap M \to]0, \epsilon^2[_{x_1} \times \partial \overline{M}_{x'}$ puts the metric g into the form

$$(\psi_+^{-1})^* g = \frac{\mathrm{d}x_1^2}{4x_1^2} + \frac{g_1(\sqrt{x_1}, x', \mathrm{d}x')}{x_1}.$$

It follows from our evenness assumption, Definition 14, that the family $x_1 \mapsto g_1(\sqrt{x_1}, x', \mathrm{d}x')$ of real-analytic metrics on $\partial \overline{M}$ has a real-analytic extension to $\{-\zeta < x_1 < \zeta\}$ for some $\zeta > 0$.

4.3. ***The modified Laplacian.*** Let $\eta > 0$ be smaller than $\zeta/2, \epsilon^2/2$ and 1 (where ζ and ϵ are defined in the previous section), and choose a function $\rho : \mathbb{R} \to \mathbb{R}$ such that $\rho(x) = x$ for $|x| \leq \eta$ and $\rho(x) = \pm 3\eta/2$ for $|x| \geq 2\eta$ (where $\pm$ is the sign of x). Notice that we can choose ρ such that $\rho'(x)x/\rho(x) \leq 1$ for positive x. Define then the function

$$\tilde{x}_1 = \rho\left(\frac{4x_1}{(1+x_1)^2}\right)$$

on X. For $\lambda \in \mathbb{C}$, let us consider the operator on $M \simeq Y$

$$\tilde{x}_1^{\frac{i\lambda}{2} - \frac{n+3}{4}} \left(-\Delta_g - \frac{(n-1)^2}{4} - \lambda^2\right) \tilde{x}_1^{\frac{n-1}{4} - \frac{i\lambda}{2}}, \tag{16}$$

where Δ_g is the (nonpositive) Laplacian on M. Using ψ_+ to identify the set $\{0 < x_1 < \eta\}$ with $]0, \eta[_{x_1} \times \partial \overline{M}_{x'}$, we see that the operator (16) takes the form

$$-x_1(1+x_1)^2 \partial_{x_1}^2 - \frac{(1+x_1)^2}{4} \Delta_{g_1} + (1+x_1)\big((n-2-i\lambda)x_1 + i\lambda - 1 - \gamma x_1(1+x_1)\big)\partial_{x_1}$$
$$- \left(\frac{n-1}{2} - i\lambda\right)\left(x_1 \frac{n-1}{2} + i\lambda - 1 - \gamma \frac{(1+x_1)(1-x_1)}{2}\right) \tag{17}$$

there. Here Δ_{g_1} is the Laplacian for the metric $g_1(\sqrt{x_1}, x', \mathrm{d}x')$ on $\partial \overline{M}$, the function γ is the logarithmic derivative $J^{-1} \frac{\partial J}{\partial x_1}$ with respect to x_1 of the Jacobian J of the metric $g_1(\sqrt{x_1}, x', \mathrm{d}x')$ on $\partial \overline{M}$. The

Jacobian J may be defined by taking local coordinates on $\partial\overline{M}$. While J depends on the choice of coordinates, the logarithmic derivative γ does not. It follows from our evenness assumption that γ extends to a real-analytic function on $\{-\eta < x_1 < \eta\} \subseteq X$. Notice that the expression (17) extends real-analytically to $\{-\eta < x_1 < \eta\} \subseteq X$.

Remark 15. Here, we differ from the exposition in [Dyatlov and Zworski 2019, Chapter 5] where, instead of (16), the operator

$$x_1^{\frac{i\lambda}{2}-\frac{n+3}{4}}\left(-\Delta_g - \frac{(n-1)^2}{4} - \lambda^2\right)x_1^{\frac{n-1}{4}-\frac{i\lambda}{2}} \tag{18}$$

is considered. This is an artificial modification that we introduce in order to be able to check (e) from Section 2.1. The formula (17) for (16) can be deduced from the formula for (18) given in [Dyatlov and Zworski 2019, Lemma 5.10].

Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth function such that $\chi(t) = 0$ for $t \le -2\eta/3$ and $\chi(t) = 1$ for $t \ge -\eta/3$. Define then for $\lambda \in \mathbb{C}$ the differential operator $P(\lambda)$ on X by

$$P(\lambda) = \begin{cases} \tilde{x}_1^{\frac{i\lambda}{2}-\frac{n+3}{4}}\left(-\Delta_g - \frac{(n-1)^2}{4} - \lambda^2\right)\tilde{x}_1^{\frac{n-1}{4}-\frac{i\lambda}{2}} & \text{on } Y \simeq M, \\ \chi(x_1) \times (17) & \text{on } \{-\eta < x_1 < \eta\}, \\ 0 & \text{on } \{x_1 < -2\eta/3\}. \end{cases}$$

Notice that the differential operator $P(\lambda)$ has real-analytic coefficients on the set $\{-\eta/3 < x_1 < \eta\}$. Let us define for $\omega \in \mathbb{C}$ and $h > 0$ the semiclassical operator

$$\mathcal{P}_h(\omega) = h^2 P(\omega/h).$$

Let us check that this family of operators satisfy the general assumptions from Section 2.1. We recall that the manifold X and its open subset Y have been defined at the end of Section 4.2. It follows from (17) that $\mathcal{P}_h(\omega)$ is of the form (7) with P_0, P_1 and P_2 that have real-analytic coefficients in the neighborhood $\{-\eta/3 < x_1 < \eta\}$ of ∂Y.

Let p_j denote the principal symbol of P_j for $j = 0, 1, 2$. For x in the interior of Y, we have

$$\begin{aligned} p_2(x,\xi) &= \frac{(1+x_1)^2}{4x_1}|\xi|^2_{g(x)}, \\ p_1(x,\xi) &= -\frac{(1+x_1)^2}{4x_1}\left\langle \xi, \frac{\mathrm{d}\tilde{x}_1}{\tilde{x}_1}\right\rangle_{g(x)}, \\ p_0(x) &= \frac{(1+x_1)^2}{4x_1}\left(\left|\frac{\mathrm{d}\tilde{x}_1}{2\tilde{x}_1}\right|^2_{g(x)} - 1\right). \end{aligned}$$

Near ∂Y, we can express these symbols in the (x_1, x') coordinates to find

$$\begin{aligned} p_2(x_1, x, \xi_1, \xi') &= x_1(1+x_1)^2\xi_1^2 + \frac{(1+x_1)^2}{4}|\xi'|^2_{g_1(\sqrt{x_1},x')}, \\ p_1(x_1, x, \xi_1, \xi') &= -(1+x_1)(1-x_1)\xi_1, \\ p_0(x_1, x') &= -1. \end{aligned}$$

We are now in position to check that the assumptions from Section 2.1 are satisfied. We see that (a) holds with $w(x_1) = x_1(1+x_1)$ and $q_1(x_1, x', \xi') = \frac{1}{4}(1+x_1)|\xi'|^2_{g_1(\sqrt{x_1},x')}$. It is clear from the definition of q_1 that (b) also holds. The validity of (c) and (d) follows immediately from the formulae for $p_1(x_1, x, \xi_1, \xi')$ and $p_2(x, \xi)$ above.

It remains to prove (e), that is, that p_0 is negative on a neighborhood of $\overline{Y}$. It is clear that p_0 is negative on a neighborhood of ∂Y from the formula above, so that we only need to check that

$$\left|\frac{\mathrm{d}\tilde{x}_1}{2\tilde{x}_1}\right|_{g(x)} < 1$$

on the interior of Y.

Notice that we have

$$\frac{\mathrm{d}\tilde{x}_1}{2\tilde{x}_1} = \frac{\rho'\left(\frac{4x_1}{(1+x_1)^2}\right)}{\rho\left(\frac{4x_1}{(1+x_1)^2}\right)} \frac{4x_1}{(1+x_1)^2} \frac{1-x_1}{1+x_1} \frac{\mathrm{d}x_1}{2x_1}.$$

Since $\left|\frac{\mathrm{d}x_1}{2x_1}\right|_{g(x)} = 1$ when $0 < x_1 < 2\eta$, we get

$$\left|\frac{\mathrm{d}\tilde{x}_1}{2\tilde{x}_1}\right|_{g(x)} = \left|\frac{\rho'\left(\frac{4x_1}{(1+x_1)^2}\right)}{\rho\left(\frac{4x_1}{(1+x_1)^2}\right)} \frac{4x_1}{(1+x_1)^2}\right| \frac{1-x_1}{1+x_1} \leq \frac{1-x_1}{1+x_1},$$

and the validity of the (e) follows.

4.4. ***Upper bound on the number of resonances.*** Since the assumptions from Section 2.1 are satisfied by the operator $\mathcal{P}_h(\omega)$ introduced in Section 4.3, we may modify $\mathcal{P}_h(\omega)$ to get an operator $P_h(\omega)$ that satisfies Proposition 5.

From here, the strategy to prove Theorem 2 is the same as in Section 3.2. We let κ be as in Proposition 5 and choose a connected, relatively compact and open subset V of $\{z \in \mathbb{C} : \operatorname{Im} z > -\kappa\}$ that contains the closed disk of center 0 and radius $3\kappa/4$. We write ι_2 for the inclusion of $C_c^\infty(M)$ in $\mathcal{H}_2$ and ι_1 for the map obtained by composition of the inclusion of $\mathcal{H}_1$ in $\mathcal{D}'(X)$ and the restriction map $\mathcal{D}'(X) \to \mathcal{D}'(M)$.

For $\lambda \in h^{-1}V$, define the resolvent

$$R_h(\lambda) = \tilde{x}_1^{\frac{n-1}{4}-\frac{i\lambda}{2}} \iota_1 h^2 P_h(h\lambda)^{-1} \iota_2 \tilde{x}_1^{\frac{i\lambda}{2}-\frac{n+3}{4}} : C_c^\infty(M) \to \mathcal{D}'(M).$$

As in Section 3.2, we get:

Lemma 16. *If h is small enough, λ is in $h^{-1}V$ and* $\operatorname{Im}\lambda > 0$, *then $R_h(\lambda)$ coincides with the inverse of $-\Delta_g - \frac{1}{4}(n-1)^2 - \lambda^2$ on $L^2(M)$. In particular, $R_h(\lambda)$ does not depend on h for $\lambda \in h^{-1}K$.*

Proof. The proof is the same as for Lemma 8. One just needs to notice that if $\operatorname{Im}\lambda > 0$ then the function $\tilde{x}_1^{(n-1)/4-i\lambda/2}$ belongs to $L^2(M)$. □

Notice that Lemma 16 implies that for $\lambda \in h^{-1}V$ the scattering resolvent $R_{\mathrm{scat}}(\lambda)$ coincides with $R_h(\lambda)$. With Proposition 5 and Lemma 16 at our disposal, the proof of Theorem 2 follows exactly the same lines as the proof of Theorem 3 given in Section 3.2. Consequently, we do not repeat it.

5. Real-analytic Fourier–Bros–Iagolnitzer transform

In this section, we detail the tools of real-analytic microlocal analysis that will be used in the proof of Proposition 5 in Section 6. The main ingredient that we need is a real-analytic Fourier–Bros–Iagolnitzer transform as we studied in [Bonthonneau and Jézéquel 2020].

In Section 5.1, we recall the main feature of such an FBI transform, and prove a slight generalization, Proposition 18, of [Bonthonneau and Jézéquel 2020, Proposition 2.10]. In Section 5.2, we give a description, Proposition 20, of the dual of a Hilbert space defined in Section 5.1. This result will be useful to construct the injection of the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{D}'(X)$ in the proof of Proposition 5 (see Proposition 21) and to reuse results from [Guedes-Bonthonneau et al. 2024] in Section 5.3, where we study the specificities of certain spaces defined using FBI transform and logarithmic weights (rather than weight of order 1 as in [Bonthonneau and Jézéquel 2020]).

5.1. ***Generality.*** Let us recall the tools from [Bonthonneau and Jézéquel 2020] that we need for the proof of Proposition 5. As in Section 2, we let X be a closed real-analytic manifold, and we endow it with a real-analytic metric g_X (which is possible due to [Morrey 1958]). We endow T^*X with an associated metric g_{KN} which is given, using the decomposition into horizontal and vertical direction

$$T_\alpha(T^*X) \simeq T_{\alpha_x}X \oplus T^*_{\alpha_x}X \simeq T_{\alpha_x}X \oplus T_{\alpha_x}X$$

for $\alpha = (\alpha_x, \alpha_\xi) \in T^*M$, by the formula

$$g_{\mathrm{KN},\alpha}((u, v), (u, v)) = g_{X,\alpha_x}(u, u) + \frac{g_{X,\alpha_x}(v, v)}{1 + g_{X,\alpha_x}(\alpha_\xi, \alpha_\xi)}$$

for $(u, v) \in T_{\alpha_x}X \oplus T_{\alpha_x}X$. This metric can be used to give a characterization of Kohn–Nirenberg symbols (see for instance [Bonthonneau and Jézéquel 2020, Remark 2.5]), and we will consequently call it a Kohn–Nirenberg metric. Let $\widetilde{X}$ be a complexification of X (endowed with any smooth distance) and $T^*\widetilde{X}$ its cotangent bundle. If $r > 0$ is small, we let $(X)_r$ denote the Grauert tube (see for instance [Guillemin and Stenzel 1991; 1992]) of size r for X, that is, the image of

$$\{(x, v) \in TX : g_{X,x}(v, v) \leq r^2\} \tag{19}$$

by the map

$$(x, v) \mapsto \exp_x(iv),$$

which is well-defined on (19) if r is small enough (here we use the holomorphic extension of the exponential map for g_X). We define similarly the Grauert tube $(T^*X)_r \subseteq T^*\widetilde{X}$ by using the Kohn–Nirenberg metric on T^*X. Because of the noncompactness of T^*X, it is not clear a priori that $(T^*X)_r$ is well-defined. However, one can reduce the study of the Kohn–Nirenberg metric on T^*X to its study near the zero section and the study of its pullbacks by the dilations $(\alpha_x, \alpha_\xi) \mapsto (\alpha_x, \lambda\alpha_\xi)$ for $\lambda \geq 1$ on a bounded subset of T^*X (for instance the space between the spheres of radii 1 and 2 in each fiber). Since these pullbacks are uniformly analytic and positive definite, we see in particular that $(T^*X)_r$ is well-defined when r is small enough.

Working in the holomorphic extension of real-analytic coordinates on X, we get a holomorphic trivialization $(\tilde{x}, \tilde{\xi}) = (x + iy, \xi + i\eta)$ of $T^*\widetilde{X}$ in which T^*X is described by $\{y = \eta = 0\}$. Using the same dilation trick as above, one may then check that, for every compact subset K of the domain of the coordinate patch $\tilde{x}$, there is $C > 0$ such that, for every $r > 0$ small enough, the image of $(T^*X)_r$ above K in this trivialization is intermediate between

$$T_K^*\widetilde{X} \cap \{|y| \leq C^{-1}r, |\eta| \leq C^{-1}(1+|\xi|)r\}$$

and

$$T_K^*\widetilde{X} \cap \{|y| \leq Cr, |\eta| \leq C(1+|\xi|)r\}.$$

Here, we write $T_K^*\widetilde{X}$ for the reciprocal image of K by the canonical projection $T^*\widetilde{X} \to \widetilde{X}$.

If m is a real number, $r > 0$ is small and a is a smooth function on $(T^*X)_r$, we say that $a \in S^m_{\mathrm{KN}}((T^*X)_r)$ is a Kohn–Nirenberg symbol of order m on $(T^*X)_r$ if, for every compact subset of the domain of a coordinate patch as above and every $k, k', \ell, \ell' \in \mathbb{N}^n$ there is a constant $C > 0$ such that on the image of $T_K^*\widetilde{X} \cap (T^*X)_r$ by the trivialization of $T^*\widetilde{X}$ associated to the coordinate patch, we have

$$|\partial_x^k \partial_y^{k'} \partial_\xi^\ell \partial_\eta^{\ell'} a(\tilde{x}, \tilde{\xi})| \leq C(1+|\xi|)^m (1+|\xi|)^{-|\ell|-|\ell'|}.$$

We define similarly symbols of logarithmic order by replacing $(1+|\xi|)^m$ by $\log(2+|\xi|)$.

Let us fix a *real* C^∞ metric $\tilde{g}$ on the vector bundle $T^*\widetilde{X} \to X$ (seen as a real vector bundle) and define for $\alpha = (x, \xi) \in T^*\widetilde{X}$ the Japanese bracket

$$\langle|\alpha|\rangle = \sqrt{2 + \tilde{g}_x(\xi)}.$$

This is just a more convenient way to denote the size of α than taking the norm of ξ directly, notice in particular that $\langle|\alpha|\rangle$ and $\log\langle|\alpha|\rangle$ are bounded from below. Notice that if $r > 0$ is small enough, then the function $\alpha \mapsto \langle|\alpha|\rangle$ is a Kohn–Nirenberg symbol of order 1 on $(T^*X)_r$, as defined above.

It will also be useful to endow $T^*\widetilde{X}$ with a distance adapted to Kohn–Nirenberg symbols. One way to do that is to endow $T\widetilde{X}$ with a smooth Hermitian metric, which gives an identification of $T^*\widetilde{X}$ with $T\widetilde{X}$. Then, one may define a Kohn–Nirenberg metric on $T\widetilde{X}$ as above when $\widetilde{X}$, seen as a real manifold, is endowed with a smooth Riemannian metric (e.g., the real part of the Hermitian metric). We let d_{KN} denote the associated distance. Restricting to a compact subset K of $\widetilde{X}$, one may check that $\alpha, \beta \in T_K^*\widetilde{X}$ are close for d_{KN} if their position variables are close to each other and, in local coordinates, their momentum variables have the same order of magnitude and the Euclidean distance between them is small with respect to this order of magnitude. This can be proved using a rescaling argument as described above.

For $R \gg 1$, so that $(X)_{1/R}$ is defined, we let $\widetilde{E}_R(X)$ denote the space of bounded holomorphic functions on the interior of $(X)_{1/R}$, endowed with the supremum norm. Then, we let $E_R(X)$ denote the closure of $\widetilde{E}_{R'}(X)$ in $E_R(X)$ for any $R' < R$ large enough so that $(X)_{1/R'}$ is well-defined. It follows from the Oka–Weil theorem [Forstnerič 2017, Theorems 2.3.1 and 2.5.2] that the space $E_R(X)$ does not depend on the choice of R'. Let $E'_R(X)$ denote the dual of $E_R(X)$, and notice that if $R > R'$ are such that $(X)_{1/R}$ and $(X)_{1/R'}$ are well-defined, then the injection of $E_{R'}(X)$ in $E_R(X)$ has dense image (because it contains $\widetilde{E}_{R''}(X)$ for some $R'' < R'$), so that the adjoint of this map defines an injection of $E'_R(X)$ into $E'_{R'}(X)$.

We choose a real-analytic FBI transform $T : \mathcal{D}'(X) \to C^\infty(T^*X)$ on X, as defined in [Bonthonneau and Jézéquel 2020, Definition 2.1]. This is a transform defined by a real-analytic kernel K_T:

$$Tu(\alpha) = \int_X K_T(\alpha, x)u(x)\,\mathrm{d}x$$

for $u \in \mathcal{D}'(X)$ and $\alpha \in T^*X$. Here, $\mathrm{d}x$ denotes the Lebesgue density associated to the Riemannian metric g_X on X. The kernel K_T, and thus T, depends on the implicit semiclassical parameter $h > 0$ introduced in the beginning of Section 2.1. Unless the opposite is explicitly stated, all the estimates below will be uniform in h. The fact that T is a real-analytic FBI transform [loc. cit., Definition 2.1] means that the kernel K_T has a holomorphic extension to $(T^*X)_r \times (X)_r$ for some small $r > 0$, which satisfies the following properties:

- For every $\delta > 0$, there is $r' > 0$ such that if $(\alpha, x) \in (T^*X)_{r'} \times (X)_{r'}$ are such that $d(\alpha_x, x) \geq \delta$ then
$$|K_T(\alpha, x)| \leq (r')^{-1} \exp\left(-r' \frac{\langle|\alpha|\rangle}{h}\right). \tag{20}$$
- There is $\delta > 0$ and $r' > 0$ such that if $(\alpha, x) \in (T^*X)_{r'} \times (X)_{r'}$ are such that $d(\alpha_x, x) \leq \delta$ then
$$|K_T(\alpha, x) - e^{i\frac{\Phi_T(\alpha,x)}{h}} a(\alpha, x)| \leq (r')^{-1} \exp\left(-r' \frac{\langle|\alpha|\rangle}{h}\right). \tag{21}$$

Here, $a(\alpha, x)$ is an analytic symbol defined near the diagonal, elliptic in the class of $h^{-3n/4}\langle|\alpha|\rangle^{n/4}$, meaning that for $r', \delta > 0$ small enough, there is a constant $C > 0$ such that $a(\alpha, x)$ is holomorphic in $\{(\alpha, x) \in (T^*X)_{r'} \times (X)_{r'} : d(\alpha_x, x) < \delta\}$ and satisfies on that set the estimate

$$C^{-1} h^{-\frac{3n}{4}} \langle|\alpha|\rangle^{\frac{n}{4}} \leq |a(\alpha, x)| \leq C h^{-\frac{3n}{4}} \langle|\alpha|\rangle^{\frac{n}{4}}.$$

The phase $\Phi_T(\alpha, x)$ from (21) is an analytic symbol of order 1 on the set

$$\{(\alpha, x) \in (T^*X)_{r'} \times (X)_{r'} : d(\alpha_x, x) < \delta\}$$

(it is holomorphic and bounded by $C\langle\alpha\rangle$ for some $C > 0$), which satisfies in addition the following properties:

- For $\alpha \in T^*X$, we have $\Phi_T(\alpha, \alpha_x) = 0$.
- For $\alpha \in T^*X$, we have $\mathrm{d}_x \Phi_T(\alpha, \alpha_x) = -\alpha_\xi$.
- There is $C > 0$ such that, if $(\alpha, x) \in T^*X \times X$ and $d(\alpha_x, x) < \delta$, then
$$\mathrm{Im}(\Phi_T(\alpha, x)) \geq C^{-1} \langle|\alpha|\rangle d(\alpha_x, x)^2. \tag{22}$$

According to [loc. cit., Theorem 6], such a FBI transform exists. Moreover, if we endow T^*X with the volume associated to the canonical symplectic form, then we may assume that the formal adjoint $S := T^*$ of T is a left inverse for T, i.e., that T is an isometry on its image. Notice that S has a real-analytic kernel K_S that satisfies for α and x real

$$K_S(x, \alpha) = \overline{K_T(\alpha, x)}.$$

In particular, K_S is negligible away from the diagonal, and may be described near the diagonal in a similar fashion as K_T.

Let us fix some small $r > 0$, and let G_0 be a Kohn–Nirenberg symbol of order 1 on $(T^*X)_r$ and set $G = \tau G_0$ for some small $\tau > 0$ (the function G is sometimes called an escape function). We let $\Lambda = \Lambda_G$ be the submanifold of $(T^*X)_r$ defined by

$$\Lambda = e^{H_G^{\omega_I}} T^*X, \tag{23}$$

where $H_G^{\omega_I}$ is the Hamiltonian vector field of G for the symplectic form $\omega_I = \operatorname{Im}\omega$, where ω denotes the canonical complex symplectic form on $T^*\widetilde{X}$. By taking τ small, we ensure that Λ is C^∞ close to T^*X (this statement can be made uniform by pulling back Λ to a bounded subset of $T^*\widetilde{X}$ using dilation in the fibers as above). Notice that in [Bonthonneau and Jézéquel 2020, Definition 2.2], the symbol G_0 was assumed to be supported in $(T^*X)_{r'}$ for some $r' < r$. The only reason for that was to ensure that the flow of $H_G^{\omega_I}$ is complete, which implies that (23) makes sense. However, taking τ small (which we will always do) is enough to ensure that (23) is well-defined. Moreover, we see that Λ only depends on the values of G on $(T^*X)_{r'}$ for some $r' < r$, so that the assumption on the support of G_0 from [loc. cit., Definition 2.2] may be lifted without harm.

We will say that a smooth function a on Λ is a symbol of order $m \in \mathbb{R}$, and write $a \in S^m_{\mathrm{KN}}(\Lambda)$, if the function $a \circ e^{H_G^{\omega_I}}$ is a symbol of order m, in the standard Kohn–Nirenberg class on T^*X. We define similarly symbols on $\Lambda \times \Lambda$.

On Λ, we can construct a real-valued symbol H of order 1 such that $\mathrm{d}H = -\operatorname{Im}\theta$ where θ denotes the canonical complex 1-form on $T^*\widetilde{X}$ (see [loc. cit., §2.1.1], in particular equation (2.9) there). Notice also that $\omega_R = \operatorname{Re}\omega$ is a symplectic form on Λ if τ is small enough. We let $\mathrm{d}\alpha = \omega_R^n/n!$ denote the associated volume form.

Notice that if $u \in E'_R(X)$ with R large enough, then Tu is well-defined and holomorphic on $(T^*X)_r$ for some small $r > 0$, so that if τ is small enough, Tu is defined on Λ. We can consequently define the FBI transform T_Λ associated to Λ by restriction $T_\Lambda u = (Tu)_{|\Lambda}$. Notice that since the kernel of S is holomorphic, we also have an operator S_Λ that is a left inverse for T_Λ (see [loc. cit., Lemma 2.7]). We will work with the spaces

$$L^2_k(\Lambda) := L^2(\Lambda, \langle|\alpha|\rangle^{2k} e^{-\frac{2H}{h}} \mathrm{d}\alpha) \qquad \text{for } k \in \mathbb{R},$$

$$\mathcal{H}^k_\Lambda := \{u \in E'_R(X) : T_\Lambda u \in L^2_k(\Lambda)\}.$$

Here, R needs to be large enough so that $E_R(X)$ is well-defined, and τ small enough depending on R (but the particular choice of R is irrelevant when τ is small). According to [loc. cit., Corollary 2.2], we know that $\mathcal{H}^k_\Lambda$ is a Hilbert space. We let also $\mathcal{H}^k_{\Lambda,\mathrm{FBI}} \subseteq L^2_k(\Lambda)$ denote the (closed) image of $\mathcal{H}^k_\Lambda$ by T_Λ. The structure of the projector $\Pi_\Lambda := T_\Lambda S_\Lambda$ on the image of T_Λ has been studied in [loc. cit., §2.2]. The orthogonal projector B_Λ on $\mathcal{H}^0_{\Lambda,\mathrm{FBI}}$ in $L^2_0(\Lambda)$ is studied in [loc. cit., §2.3]. Notice that in order to prove Proposition 5, we will work with a symbol G_0 which is of logarithmic order. As explained in Section 5.3 (see also [Guedes-Bonthonneau et al. 2024]), it implies that $\mathcal{H}^k_\Lambda$ is in fact a space of distributions. Consequently, we could have worked from the beginning only with distributions (and avoid the introduction of the space $E'_R(X)$). However, we decided to start from the context of [Bonthonneau and Jézéquel 2020] and then specify to the case of logarithmic weights in Section 5.3. This is because we will need some extensions of the results from [loc. cit.] that are not made easier by assuming that G_0 is of logarithmic order. It is also useful to see the case of logarithmic weights as a particular case of [loc. cit.], as it allows us to use the results from this reference.

Assume that $A(\alpha, \beta)$ is a smooth function on $\Lambda \times \Lambda$ and let A be the associated operator

$$Au(\alpha) = \int_\Lambda A(\alpha, \beta) u(\beta)\, d\beta \quad \text{for } \alpha \in \Lambda.$$

The operator A may be defined for instance as an operator from the space of smooth compactly supported functions u on Λ to the space of smooth functions on Λ. In order to understand the action of A on $L^2_0(\Lambda)$, one has to study the reduced kernel of A:

$$A_{\text{red}}(\alpha, \beta) = A(\alpha, \beta) e^{\frac{H(\beta) - H(\alpha)}{h}}.$$

To study the action of A from $L^2_k(\Lambda)$ to $L^2_\ell(\Lambda)$, one can study the kernel $A_{\text{red}}(\alpha, \beta)\langle|\beta|\rangle^{-2k}\langle|\alpha|\rangle^{2\ell}$. We will say that the kernel A is negligible if

$$A_{\text{red}}(\alpha, \beta) = \mathcal{O}_{C^\infty}(h^\infty(\langle|\alpha|\rangle + \langle|\beta|\rangle)^{-\infty}). \tag{24}$$

Here, the C^∞ estimates may be understood by identifying Λ with T^*X using $e^{H_G^{\omega_I}}$, taking a trivialization for T^*X and then asking for all partial derivatives of A_{red} to be $\mathcal{O}(h^\infty(\langle|\alpha|\rangle + \langle|\beta|\rangle)^{-\infty})$. We do not need to ask for symbolic estimates in that case, as it is automatic for something that decays that fast. Notice that an operator whose reduced kernel satisfies (24) is bounded from $L^2_k(\Lambda)$ to $L^2_\ell(\Lambda)$ for every $k, \ell \in \mathbb{R}$, with norm $\mathcal{O}(h^\infty)$. An operator whose reduced kernel satisfy (24) will be called a negligible operator.

Recall the phase $\Phi_{TS}(\alpha, \beta)$ from [Bonthonneau and Jézéquel 2020, §2.2], which is the critical value of $y \mapsto \Phi_T(\alpha, y) + \Phi_S(y, \beta)$. Here, Φ_S is the phase that appear when describing the kernel $K_S(y, \beta)$ of S locally as we do for K_T in (21). That is, $\Phi_S(y, \beta) = -\overline{\Phi_T(\bar{\beta}, \bar{y})}$. The following fact follows from the analysis in [loc. cit.].

Lemma 17. *Let $\delta > 0$ be small enough. Assume that $\tau > 0$ and $h > 0$ are small enough. Assume that $A(\alpha, \beta)$ is a smooth function on $\Lambda \times \Lambda$ and let A be the associated operator. Let $m \in \mathbb{R}$. Assume that there is a symbol $a \in S^m_{\text{KN}}(\Lambda \times \Lambda)$ supported in $\{(\alpha, \beta) \in \Lambda \times \Lambda, d_{\text{KN}}(\alpha, \beta) < \delta\}$ such that*

$$A_{\text{red}}(\alpha, \beta) = \frac{1}{(2\pi h)^n} e^{\frac{H(\beta) + i\Phi_{TS}(\alpha, \beta) - H(\alpha)}{h}} a(\alpha, \beta) + \mathcal{O}_{C^\infty}(h^\infty(\langle|\alpha|\rangle + \langle|\beta|\rangle)^{-\infty}). \tag{25}$$

Then, A is bounded from $L^2_k(\Lambda)$ to $L^2_{k-m}(\Lambda)$ for every $k \in \mathbb{R}$, and there is a symbol $\sigma \in S^m_{\text{KN}}(\Lambda)$ such that the operators $B_\Lambda A B_\Lambda$ and $B_\Lambda \sigma B_\Lambda$ differ by a negligible operator.

Moreover, σ coincides with $\alpha \mapsto a(\alpha, \alpha)$ up to $\mathcal{O}(h)$ in $S^{m-1}_{\text{KN}}(\Lambda)$.

Indeed, the boundedness statement follows from the proof of [loc. cit., Proposition 2.4]. Our assumption on the kernel of A implies that A belongs to the class of FIO from [loc. cit., Definition 2.5], and thus the proof of [loc. cit., Proposition 2.10] may be rewritten replacing the operator "$fT_\Lambda PS_\Lambda$" by the operator A.

This gives the symbol σ such that $B_\Lambda A B_\Lambda - B_\Lambda \sigma B_\Lambda$ is a negligible operator. The proof gives that σ coincides with $\alpha \mapsto g_0(\alpha)a(\alpha,\alpha)$ for a symbol g_0 of order 0 that does not depend on A. To see that one can take $g_0 = 1$, just notice that the operator $\Pi_\Lambda = T_\Lambda S_\Lambda$ satisfies the hypotheses from Lemma 17 with $\alpha \mapsto a(\alpha,\alpha)$ identically equal to 1 up to $\mathcal{O}(h)$ in $S^{-1}_{\mathrm{KN}}(\Lambda)$, according to [Bonthonneau and Jézéquel 2020, Lemma 2.10], and that $B_\Lambda \Pi_\Lambda B_\Lambda = B_\Lambda B_\Lambda$. Moreover, one may retrieve the leading part of a symbol σ from restriction to the diagonal of the kernel of the operator $B_\Lambda \sigma B_\Lambda$ (the kernel may be computed by the stationary phase method as in [loc. cit., Lemma 2.16]).

We need to extend certain results from [loc. cit.] to a slightly more general context in order to prove Proposition 5. Let P be a semiclassical differential operator of order m with C^∞ coefficients and let p be the principal symbol of P. We make the following assumption:

$$\begin{aligned} &\text{for every } x \in X \text{ either } G_0(y,\xi) = 0 \text{ for every } y \text{ near } x \\ &\text{and } \xi \in T^*_y X, \text{ or } P \text{ has real-analytic coefficients near } x. \end{aligned} \tag{26}$$

Notice that under the assumption (26) the principal symbol p of P may be restricted to Λ provided τ is small enough. Indeed, for every $x \in X$, either p has a holomorphic extension near $T^*_x X$ or Λ coincides with T^*X near $T^*_x X$. We let p_Λ denote this restriction. If P is an operator that satisfies (26), we may define $T_\Lambda P S_\Lambda$ as the operator with kernel

$$T_\Lambda P S_\Lambda(\alpha,\beta) = \int_M K_T(\alpha,y) P_y(K_S(y,\beta))\,dy. \tag{27}$$

The reason for which we use this definition is because since P is a priori not an operator with real-analytic coefficients, it is not straightforward to define the action of P on elements of $E'_R(X)$. Notice that the following result allows to define P as an operator from $\mathcal{H}^k_\Lambda$ to $\mathcal{H}^{k-m}_\Lambda$. When we will specify to the case of logarithmic weights in Section 5.3, the spaces $\mathcal{H}^k_\Lambda$'s will be included in $\mathcal{D}'(M)$, and the natural relation $T_\Lambda P u = T_\Lambda P S_\Lambda T_\Lambda u$ will be satisfied; see Lemma 23.

Proposition 18. *Under the assumption (26), if τ is small enough, then the operator $T_\Lambda P S_\Lambda$ is bounded from $L^2_k(\Lambda)$ to $L^2_{k-m}(\Lambda)$. Moreover, if $\ell \in \mathbb{R}$ and $f \in S^\ell_{\mathrm{KN}}(\Lambda)$, there is a symbol $\sigma \in S^{m+\ell}_{\mathrm{KN}}(\Lambda)$ and an operator L with negligible kernel such that*

$$B_\Lambda f T_\Lambda P S_\Lambda B_\Lambda = B_\Lambda \sigma B_\Lambda + L.$$

In addition, σ coincides with $f p_\Lambda$ up to $\mathcal{O}(h\langle|\alpha|\rangle^{m+\ell-1})$.

The proof of Proposition 18 is based on applications of the stationary and nonstationary phase methods with complex phase. We will apply both the C^∞ and the holomorphic versions of these methods. We are not aware of a reference stating the C^∞ version of the nonstationary phase method with complex phase that would cover all the cases we are going to consider (for the stationary phase method, see [Melin and Sjöstrand 1974], and for a standard version of the non stationary phase method with complex phase, see [Hörmander 1983, Theorem 7.7.1]), so that we prove here a statement adapted to our needs. This result and its proof should be no surprise for specialists.

Lemma 19. *Let m, n be integer. Let U, V be open subsets respectively of $\mathbb{R}^m$ and $\mathbb{R}^n$. Let $\Phi : U \times V \to \mathbb{C}$ be a C^∞ function. Let K_1 and K_2 be compact subsets respectively of U and V. Assume that for every $(x, y) \in K_1 \times K_2$ we have $\operatorname{Im} \Phi(x, y) \geq 0$ and $\mathrm{d}_y \Phi(x, y) \neq 0$. Then, for every $L, N > 0$, there are constants $k \in \mathbb{N}$ and $\lambda_0 > 0$ such that, for every $\lambda \geq \lambda_0$, every C^k function u supported in K_2 and every $x \in U$ such that $d(x, K_1) \leq L \log \lambda / \lambda$, we have*

$$\left| \int_V e^{i\lambda\Phi(x,y)} u(y) \, \mathrm{d}y \right| \leq \lambda^{-N} \|u\|_{C^k}.$$

Proof. Let $x \in U$ be such that $d(x, K_1) \leq L \log \lambda / \lambda$. From our nonstationary assumption, we see that if λ is large enough then $\mathrm{d}_y \Phi(x, y) \neq 0$ for every $y \in K_2$. We can consequently introduce the differential operator

$$L_x = -i \sum_{j=1}^{n} \frac{\overline{\partial_{y_j} \Phi(x, y)}}{|\nabla_y \Phi(x, y)|^2} \partial_{y_j},$$

and notice that $L_x(e^{i\lambda\Phi(x,y)}) = \lambda e^{i\lambda\Phi(x,y)}$. Letting k be a large integer and ${}^t L_x$ denote the formal adjoint of L_x, we find that

$$\int_V e^{i\lambda\Phi(x,y)} u(y) \, \mathrm{d}y = \lambda^{-k} \int_V e^{i\lambda\Phi(x,y)} {}^t L_x^k u(y) \, \mathrm{d}y.$$

Then, we notice that the L^∞ norm of ${}^t L_x^k u$ is controlled by the C^k norm of u. Moreover, since $d(x, K_1) \leq L \log \lambda / \lambda$, we find that for every $y \in K_2$, if λ is large enough, we have $\operatorname{Im} \Phi(x, y) \geq -C_\Phi L \log \lambda / \lambda$ for some constant C_Φ that does not depend on k nor u. Consequently, for λ large, we have

$$\left| \int_V e^{i\lambda\Phi(x,y)} u(y) \, \mathrm{d}y \right| \leq C \lambda^{-k + C_\Phi L} \|u\|_{C^k}.$$

Here the constant C may depend on k and Φ, but not on λ nor u. Taking k large enough, we ensure that $k - C_\Phi L > N$ and the result follows. □

We have now at our disposal all the tools to prove Proposition 18.

Proof of Proposition 18. We want to apply Lemma 17 to the operators $T_\Lambda P S_\Lambda$ and $f T_\Lambda P S_\Lambda$. Let us introduce the open sets

$$U_1 = \{x \in X : G_0(y, \xi) = 0 \text{ for every } y \text{ near } x \text{ and } \xi \in T_y^* X\},$$
$$U_2 = \{x \in X : P \text{ has real-analytic coefficients near } x\}.$$

By assumption $X = U_1 \cup U_2$. We start by proving that for every $\delta > 0$, provided τ is small enough, we have

$$T_\Lambda P S_\Lambda(\alpha, \beta) e^{\frac{H(\beta) - H(\alpha)}{h}} = \mathcal{O}(h^\infty (\langle|\alpha|\rangle + \langle|\beta|\rangle)^{-\infty}) \tag{28}$$

whenever $\alpha, \beta \in \Lambda$ are such that $d_{\mathrm{KN}}(\alpha, \beta) \geq \delta$. Let us write $\alpha = e^{H_G^{\omega_I}}(x, \xi)$ and $\beta = e^{H_G^{\omega_I}}(y, \eta)$ where (x, ξ) and (y, η) are in $T^* X$. Assume first that x and y are at distance larger than δ / L for some large constant $L \gg 1$. We can then write

$$T_\Lambda P S_\Lambda(\alpha, \beta) = \left(\int_{D(x,\delta/10L)} + \int_{D(y,\delta/10L)} + \int_{X \setminus (D(x,\delta/10L) \cup D(y,\delta/10L))} \right) K_T(\alpha, z) P_z(K_S(z, \beta)) \, \mathrm{d}z. \tag{29}$$

We write $D(w,r)$ for the ball of center w and radius r in X. Notice that, provided τ is small enough, the third integral in (29) is $\mathcal{O}(\exp(-(\langle|\alpha|\rangle+\langle|\beta|\rangle)/Ch))$ since the kernel K_T and K_S are negligible away from the diagonal (20). Since $e^{(H(\beta)-H(\alpha))/h}$ is $\mathcal{O}(\exp(C\tau(\langle|\alpha|\rangle+\langle|\beta|\rangle)))$, we see that for τ small enough we have

$$\begin{aligned} e^{\frac{H(\beta)-H(\alpha)}{h}}\int_{X\setminus(D(x,\delta/10L)\cup D(y,\delta/10L))} K_T(\alpha,z)P_z(K_S(z,\beta))\,\mathrm{d}z &= \mathcal{O}\left(\exp\left(-\frac{\langle|\alpha|\rangle+\langle|\beta|\rangle}{Ch}\right)\right) \\ &= \mathcal{O}(h^\infty(\langle|\alpha|\rangle+\langle|\beta|\rangle)^{-\infty}), \end{aligned}$$

and we only need to care about the two other terms.

Let us deal with the first term in (29). Up to a negligible term, it is given by

$$\int_{D(x,\delta/10L)} e^{i\frac{\Phi_T(\alpha,z)}{h}}a(\alpha,z)P_z(K_S(z,\beta))\,\mathrm{d}z. \tag{30}$$

By taking L large enough, we have either $D(x,\delta/10L)\subseteq U_1$ or $D(x,\delta/10L)\subseteq U_2$.

Let us begin with the case of $D(x,\delta/10L)\subseteq U_1$. In that case, the differential operator P has a priori only C^∞ coefficients on $D(x,\delta/10L)$ so that we find that $P_z(K_S(z,\beta))$ is $\mathcal{O}(\exp(-\langle|\beta|\rangle/Ch))$ in C^∞. Notice also that $\mathrm{d}_y\Phi_T(\alpha,\alpha_x)=-\alpha_\xi$ and that the imaginary part of $\Phi_T(\alpha,z)$ is nonnegative when $z\in D(x,\delta/10L)$. Hence, provided L is large enough, we can use the C^∞ nonstationary phase method (apply Lemma 19 with a rescaling argument) to find that (30) is

$$\mathcal{O}(h^\infty\langle|\alpha|\rangle^{-\infty}\exp(-\langle|\beta|\rangle/Ch)).$$

Here, the integrand is not supported away from the boundary of the domain of integration, but since the imaginary part of the phase is larger than $C^{-1}\langle|\alpha|\rangle/h$ near the boundary of the domain of integration, we may just introduce a bump function to fix that. The same trick allows to remove the dependence on x of the domain of integration. Using that $x\in U_1$, we find that $\alpha=(x,\xi)$ and that $H(\alpha)=0$ (see [Bonthonneau and Jézéquel 2020, (2.9)]), so that

$$e^{\frac{H(\beta)-H(\alpha)}{h}}=e^{\frac{H(\beta)}{h}}=\mathcal{O}\left(\exp\left(C\tau\frac{\langle|\beta|\rangle}{h}\right)\right).$$

Hence, for τ small enough, we find that

$$e^{\frac{H(\beta)-H(\alpha)}{h}}\int_{D(x,\delta/10L)} e^{i\frac{\Phi_T(\alpha,z)}{h}}a(\alpha,z)P_z(K_S(z,\beta))\,\mathrm{d}z=\mathcal{O}(h^\infty(\langle|\alpha|\rangle+\langle|\beta|\rangle)^{-\infty}). \tag{31}$$

When $D(x,\delta/10L)\subseteq U_2$, the coefficients of P are analytic, and $P_z(K_S(z,\beta))$ is $\mathcal{O}(\exp(-\langle|\beta|\rangle/Ch))$ as a real-analytic function. Hence, provided L is large enough, we can use the holomorphic nonstationary phase method (see for instance [loc. cit., Proposition 1.1], and use a rescaling argument) as in the proof of [loc. cit., Lemma 2.9] to see that (30) is $\mathcal{O}(\exp(-(\langle|\alpha|\rangle+\langle|\beta|\rangle)/Ch))$, provided τ is small enough. Hence, if τ is small enough, this is enough to beat the potential growth of the factor $e^{(H(\beta)-H(\alpha))/h}$, so that we also have (31) in that case.

We deal similarly with the second term in (29), distinguishing the cases $D(y,\delta/10L)\subseteq U_1$ and $D(y,\delta/10L)\subseteq U_2$.

Let us now prove (28) when the distance between x and y is less than δ/L (and consequently ξ and η are away from each other in a trivialization of T^*X). As above, we can discard the z's that are away from x (and thus from y) and write up to a negligible term the kernel of $T_\Lambda P S_\Lambda$ as (the error term coming from the approximation (21) is dealt with by an application of the nonstationary phase method as in the previous case)

$$\int_{D(x,10\delta/L)} e^{i\frac{\Phi_T(\alpha,z)+\Phi_S(z,\beta)}{h}} a(\alpha,z)\tilde{b}(z,\beta)\,\mathrm{d}z, \tag{32}$$

where the symbol $\tilde{b}$ is defined by

$$\tilde{b}(z,\beta)=e^{-i\frac{\Phi_S(z,\beta)}{h}}P_z\big(e^{i\frac{\Phi_S(z,\beta)}{h}}b(z,\beta)\big).$$

Notice that the phase in (32) is holomorphic and nonstationary. Indeed, working in coordinates and assuming that L is large enough, we find that, for some $C>0$ and every $z\in D(x,10\delta/L)$,

$$\begin{aligned}|\nabla_z(\Phi_T(\alpha,z)+\Phi_S(z,\beta))| &= |\beta_\xi-\alpha_\xi|+\mathcal{O}\bigg(\frac{\max(\langle|\alpha|\rangle,\langle|\beta|\rangle)}{L}\bigg)\\ &\geq C^{-1}\max(\langle|\alpha|\rangle,\langle|\beta|\rangle).\end{aligned}$$

Moreover, provided τ is small enough, the imaginary part of the phase is larger than $C^{-1}\max(\langle|\alpha|\rangle,\langle|\beta|\rangle)$ when z is on the boundary of $D(x,10\delta/L)$ (because z is away from α_x and β_x), and is always nonnegative when $D(x,10\delta/L)\subseteq U_1$. We can apply the C^∞ nonstationary phase method when $D(x,10\delta/L)\subseteq U_1$ and the holomorphic nonstationary phase method when $D(x,10\delta/L)\subseteq U_2$ (for this second case, see the similar computation in the proof of [Bonthonneau and Jézéquel 2020, Lemma 2.9]). Indeed, in the latter case $\tilde{b}$ is holomorphic in z, while in the first case it is only C^∞. In the first case, we get that (32) is $\mathcal{O}(h^\infty(\langle|\alpha|\rangle+\langle|\beta|\rangle)^{-\infty})$ and in the second case that it is $\mathcal{O}(\exp(-(\langle|\alpha|\rangle+\langle|\beta|\rangle)/Ch))$. Noticing that in the first case $H(\alpha)=H(\beta)=0$, we find that (28) holds.

Notice that differentiating the kernel of K_T or of K_S (in a local trivialization of T^*X) amount to replace the symbols a and b by symbols of higher orders (in terms of α, β and h). Thus, all the estimates that we established when α and β are away from each other actually hold in C^∞.

We must now understand what happens when α and β are close to each other. We write as above $\alpha=e^{H_G^{\omega_I}}(x,\xi)$ and $\beta=e^{H_G^{\omega_I}}(y,\eta)$ where (x,ξ) and (y,η) are in T^*X. Then, up to negligible terms, the kernel of $T_\Lambda P S_\Lambda$ at (α,β) is given as above, for some small $\delta>0$, by

$$\int_{D(x,\delta)} e^{i\frac{\Phi_T(\alpha,z)+\Phi_S(z,\beta)}{h}} a(\alpha,z)\tilde{b}(z,\beta)\,\mathrm{d}z.$$

As above, the error coming from the approximation (21) is dealt with by an application of the nonstationary phase method. The asymptotic of this integral when $\langle|\alpha|\rangle/h$ tends to $+\infty$ is given by the stationary phase method. Indeed, when $\alpha=\beta$, the rescaled phase $y\mapsto(\Phi_T(\alpha,y)+\Phi_S(y,\beta))/\langle|\alpha|\rangle$ has a uniformly nondegenerate critical point at $y=\alpha_x=\beta_x$, as a consequence of (22). Moreover, when $D(x,\delta)\subseteq U_1$, the imaginary part of this phase is nonnegative on $D(x,\delta)$, provided the distance between α_x and β_x is way smaller than δ. When $D(x,\delta)\subseteq U_2$, we may ensure that the imaginary part of the (rescaled) phase is uniformly positive on the boundary of $D(x,\delta)$ by taking τ small enough. As above, we apply the stationary phase method in the C^∞ category (see [Melin and Sjöstrand 1974, §2]) when $D(x,\delta) \subseteq U_1$ and in the C^ω category when $D(x,\delta) \subseteq U_2$ (see [Sjöstrand 1982, §2] for the general method and the proof of [Bonthonneau and Jézéquel 2020, Lemma 2.10] in the case $s=1$, page 111, for the details of the computation in our particular setting). In both cases, we can use the fact that the imaginary part of the phase is positive on the boundary of the domain of integration to remove the dependence of this domain on x. In the first case we get an expansion with an error term of the form $\mathcal{O}(h^\infty \langle |\alpha| \rangle^{-\infty})$ and in the second case of the form $\mathcal{O}(\exp(-\langle|\alpha|\rangle/Ch))$. Since in the first case we have $H(\alpha)=H(\beta)=0$, we see that in both cases we get the desired expansion (25) for the reduced kernel of $T_\Lambda P S_\Lambda$, with an error term of the required size.

We can then apply Lemma 17 to end the proof. Indeed, we just saw that the kernel of $f T_\Lambda P S_\Lambda$ is of the form (25). Moreover, it follows from the application of the stationary phase method that, up to $\mathcal{O}(h)$ in $S^{m-1}_{\mathrm{KN}}(\Lambda)$, the symbol $\alpha \mapsto a(\alpha,\alpha)$ coincides with $f p_\Lambda g_0$, where g_0 is a symbol of order 0 that does not depend on P. Thus, the operator $f T_\Lambda P S_\Lambda - f p_\Lambda \Pi_\Lambda$ is also of the form (25) but with an a such that $a \mapsto a(\alpha,\alpha)$ is $\mathcal{O}(h)$ in $S^{m+\ell-1}_{\mathrm{KN}}(\Lambda)$. Consequently, there is a symbol $\tilde{\sigma} \in h S^{m+\ell-1}_{\mathrm{KN}}(\Lambda)$ such that $B_\Lambda(f T_\Lambda P S_\Lambda - f p_\Lambda \Pi_\Lambda) B_\Lambda - B_\Lambda \tilde{\sigma} B_\Lambda = B_\Lambda f T_\Lambda P S_\Lambda B_\Lambda - B_\Lambda (f p_\Lambda + \tilde{\sigma}) B_\Lambda$ is a negligible operator. We get the announced result with $\sigma = f p_\Lambda + \tilde{\sigma}$. □

5.2. *Duality statement.* In [Bonthonneau and Jézéquel 2020, Lemma 2.24], an identification between $\mathcal{H}^{-k}_\Lambda$ and the dual of $\mathcal{H}^k_\Lambda$ is given. However, the pairing used to define this identification is not the L^2 pairing. We explain here how to describe the dual of $\mathcal{H}^k_\Lambda$ using the L^2 pairing. This will allow us in particular to reuse results from [Guedes-Bonthonneau et al. 2024] in Section 5.3.

Let us first recall that there is an antiholomorphic involution $\alpha \mapsto \bar{\alpha}$ on $(T^*X)_r$ such that

$$\{\alpha \in (T^*X)_r : \alpha = \bar{\alpha}\} = T^*X;$$

see [Guillemin and Stenzel 1991]. Let G be a symbol of order 1 on $(T^*X)_r$ as above (of the form $G = \tau G_0$ with τ small) and Λ be defined by (23). Let us introduce a new symbol $G^*(\alpha) = -G(\bar{\alpha})$, and notice that the Lagrangian associated to G^* by (23) is $\overline{\Lambda}$, that is, the image of Λ by the involution $\alpha \mapsto \bar{\alpha}$. Notice also that changing G to G^*, we have to replace H by the function H^* on $\overline{\Lambda}$ given by $H^*(\alpha) = -H(\bar{\alpha})$.

Consequently, if $u \in \mathcal{H}^k_\Lambda$ and $v \in \mathcal{H}^{-k}_{\overline{\Lambda}}$, we may define the pairing

$$\langle u, v \rangle = \int_\Lambda T_\Lambda u(\alpha) \overline{T_{\overline{\Lambda}} v(\bar{\alpha})}\, \mathrm{d}\alpha, \tag{33}$$

for which we can prove:

Proposition 20. *Let $R \gg 1$. Assume that τ is small enough. The pairing (33) induces an identification between $\mathcal{H}^{-k}_{\overline{\Lambda}}$ and the dual of $\mathcal{H}^k_\Lambda$. Moreover, if u or v belongs to $E_R(X)$ then (33) is just the natural (sesquilinear) pairing between elements of $E_R(X)$ and $E'_R(X)$.*

Proof. Assume that u is in $E_R(X)$ and that $v \in \mathcal{H}^{-k}_{\overline{\Lambda}}$. Since T is an isometry on its image, we know that

$$\int_X u \bar{v}\, \mathrm{d}x = \int_{T^*X} T u \overline{T v}\, \mathrm{d}\alpha. \tag{34}$$

Notice that the function $\alpha \mapsto Tu(\alpha)\overline{Tv(\bar{\alpha})}$ is holomorphic on $(T^*X)_r$. Moreover, from [Bonthonneau and Jézéquel 2020, Lemmas 2.4 and 2.5, Corollary 2.2], we see that, provided τ is small enough, there is $r > 0$ such that $Tu(\alpha)\overline{Tv(\bar{\alpha})}$ decays exponentially fast in $(T^*X)_r$. This allows us to shift contour in (34) to find that $\int_X u\bar{v}\,dx$ coincides with (33), provided τ is small enough. By symmetry, we have the same equality when v is assumed to belong to $E_R(X)$.

Consequently, the (antilinear) map from $\mathcal{H}^{-k}_{\bar{\Lambda}}$ to the dual of $\mathcal{H}^{k}_{\Lambda}$ induced by the pairing (33) is injective. Let us prove that it is surjective. Let l be a continuous linear form on $\mathcal{H}^{k}_{\Lambda}$. It follows from [loc. cit., Proposition 2.4] that S_Λ is bounded from $L^2_k(\Lambda)$ to $\mathcal{H}^{k}_{\Lambda}$, and we can thus define a linear form $\tilde{l}$ on $L^2_k(\Lambda)$ by the formula $\tilde{l}(w) = l(S_\Lambda w)$. Notice that if $u \in \mathcal{H}^{k}_{\Lambda}$ then $l(u) = \tilde{l}(T_\Lambda u)$. Let then h_1 be the element of $L^2_k(\Lambda)$ such that

$$\tilde{l}(w) = \int_\Lambda w(\alpha)\overline{h_1(\alpha)}\langle|\alpha|\rangle^{2k} e^{-\frac{2H(\alpha)}{h}}\,d\alpha$$

for every $w \in L^2_k(\Lambda)$. Let us define the function h_2 on $\bar{\Lambda}$ by

$$h_2(\alpha) = h_1(\bar{\alpha})\langle|\bar{\alpha}|\rangle^{2k} e^{-\frac{2H(\bar{\alpha})}{h}},$$

and notice that h_2 belongs to $L^2_{-k}(\bar{\Lambda})$, so that $v := S_{\bar{\Lambda}} h_2$ belongs to $\mathcal{H}^{-k}_{\bar{\Lambda}}$. Let $u \in E_R(X)$, then with the pairing above, we have

$$\langle u, v\rangle = \int_\Lambda T_\Lambda u(\alpha)\overline{\Pi_{\bar{\Lambda}} h_2(\bar{\alpha})}\,d\alpha.$$

Notice that the kernel of the operators Π_Λ and $\Pi_{\bar{\Lambda}}$ are obtained by restricting respectively to $\Lambda \times \Lambda$ and $\bar{\Lambda} \times \bar{\Lambda}$ the holomorphic kernel of the operator $\Pi = TS$. We write $\Pi(\alpha, \beta)$ for this kernel. Since S is the adjoint of T, we find by analytic continuation that $\overline{\Pi(\bar{\alpha}, \bar{\beta})} = \Pi(\beta, \alpha)$. It follows then from Fubini's theorem that

$$\begin{aligned}\langle u, v\rangle &= \int_\Lambda \Pi_\Lambda T_\Lambda u(\alpha)\overline{h_2(\bar{\alpha})}\,d\alpha \\ &= \int_\Lambda T_\Lambda u(\alpha)\overline{h_1(\alpha)}\langle|\alpha|\rangle^{2k} e^{-\frac{2H(\alpha)}{h}}\,d\alpha = l(u).\end{aligned}$$

The equality on the first line can be proved first by replacing h_2 by a rapidly decaying function and then using an approximation argument. It follows from [loc. cit., Corollary 2.3] and the Oka–Weil theorem that $E_R(X)$ is dense in $\mathcal{H}^{k}_{\Lambda}$ and the result follows. □

5.3. ***Particularity of logarithmic weights.*** When applying the FBI transform techniques that we describe here in Section 6, the weight G_0 will be of logarithmic order. This is a strategy that we already applied in [Guedes-Bonthonneau et al. 2024]. It amounts to doing C^∞ microlocal analysis with respect to the large parameter $\langle|\alpha|\rangle$ but real-analytic microlocal analysis with respect to the small parameter h.

Using a logarithmic weight allows us to construct spaces that are intermediate between $C^\infty(X)$ and $\mathcal{D}'(X)$.

Proposition 21. *Assume that G_0 has logarithmic order. Assume that τ and h are small enough. Then, for every $k \in \mathbb{R}$, there are continuous injections $C^\infty(X) \subseteq \mathcal{H}^k_\Lambda \subseteq \mathcal{D}'(X)$. Moreover, these injections are natural in the following sense: the diagram*

$$\begin{array}{ccccc} C^\infty(X) & \longrightarrow & \mathcal{H}^k_\Lambda(X) & \longrightarrow & E'_R(X) \\ & \searrow & \downarrow & \nearrow & \\ & & \mathcal{D}'(X) & & \end{array}$$

is commutative, with R as in the definition of $\mathcal{H}^k_\Lambda$. The arrows that are not given by the proposition are the standard injections.

Proof. It follows from Lemma 19, using for instance [Bonthonneau and Jézéquel 2020, (2.9)] to bound H, that $C^\infty(X)$ is contained in $\mathcal{H}^k_\Lambda$, where we identify an element of $C^\infty(X)$ with an element of $E'_R(X)$ using the L^2 pairing (see also [Guedes-Bonthonneau et al. 2024, Lemma 4.10]). The proof of this result actually proves that the injection is continuous (even if the estimates are not uniform in h). Notice that $C^\infty(X)$ is dense in $\mathcal{H}^k_\Lambda$ as a consequence of [Bonthonneau and Jézéquel 2020, Corollary 2.3]. Replacing G by G^*, we find that $C^\infty(X)$ is also a dense subset of $\mathcal{H}^{-k}_{\overline{\Lambda}}$, with continuous injection. Consequently, the pairing (33) induces a continuous injection of $\mathcal{H}^k_\Lambda$ into $\mathcal{D}'(X)$ according to Proposition 20. Since the pairing (33) coincides with the L^2 pairing when u or v is in $E_R(X)$, we see that the diagram above is indeed commutative. □

Remark 22. It follows from Propositions 20 and 21 that if $u \in \mathcal{H}^k_\Lambda$ and $v \in \mathcal{H}^{-k}_{\overline{\Lambda}}$ are such that u or v is in $C^\infty(X)$, then the pairing (33) coincides with the natural pairing between a smooth function and a distribution.

When G_0 is of logarithmic order, we may identify the $\mathcal{H}^k_\Lambda$'s with spaces of distributions, and consequently it makes sense to let a differential operator P with C^∞ coefficients act on the elements of the $\mathcal{H}^k_\Lambda$'s. In the following lemma, we see that under the assumption (26) we can relate the action of P on these spaces with the action of the operator $T_\Lambda P S_\Lambda$ that we studied in Proposition 18.

Lemma 23. *Assume that G_0 has logarithmic order. Let P be a semiclassical operator of order $m \in \mathbb{N}$ that satisfy (26). Assume that τ is small enough. Then, for every $k \in \mathbb{R}$, the operator P is bounded from $\mathcal{H}^k_\Lambda$ to $\mathcal{H}^{k-m}_\Lambda$ and for every $u \in \mathcal{H}^k_\Lambda$ we have*

$$T_\Lambda P u = (T_\Lambda P S_\Lambda) T_\Lambda u,$$

where we recall that $T_\Lambda P S_\Lambda$ is the operator with kernel (27).

Proof. For $\alpha \in \Lambda$, we have by definition

$$T_\Lambda P u(\alpha) = \int_M P u(y) K_T(\alpha, y)\, \mathrm{d}y = \int_M u(y) {}^t P_y(K_T(\alpha, y))\, \mathrm{d}y, \tag{35}$$

where ${}^tP$ denotes the adjoint of P for the bilinear (rather than sesquilinear) L^2 pairing on M. Notice that for $\alpha \in \Lambda$, the function $h_\alpha : y \mapsto {}^tP_y(K_T(\alpha, y))$ is C^∞. Consequently, one may use the C^∞ nonstationary phase method, Lemma 19, to find that ${}^tSh_\alpha(\beta)$ decays faster than the inverse of any polynomial when β becomes large while its imaginary part remains bounded (from the Kohn–Nirenberg point of view) by $L\log\langle|\beta|\rangle/\langle|\beta|\rangle$ (for any large constant L). Notice however that this estimate is not uniform in h (we apply Lemma 19 with h fixed and $\langle|\beta|\rangle$, rather than $\langle|\beta|\rangle/h$, as a large parameter). Consequently, we can shift contour in the integral equality ${}^tT\,{}^tSh_\alpha = {}^t(ST)h_\alpha = h_\alpha$ to find

$$\begin{aligned} h_\alpha(x) &= \int_\Lambda K_T(\beta,x)\left(\int_M K_S(y,\beta)h_\alpha(y)\,\mathrm{d}y\right)\mathrm{d}\beta \\ &= \int_\Lambda K_T(\beta,x)T_\Lambda P S_\Lambda(\alpha,\beta)\,\mathrm{d}\beta. \end{aligned}$$

Using the fast decay of ${}^tSh_\alpha$, we see that this integral actually converges in $C^\infty(X)$, and plugging this equality into (35), we get $T_\Lambda Pu = (T_\Lambda P S_\Lambda)T_\Lambda u$. It follows from Proposition 18 that $T_\Lambda Pu \in L^2_{k-m}(\Lambda)$, that is, $Pu \in \mathcal{H}^{k-m}_\Lambda(\Lambda)$. □

The following result will be used in the demonstration of Proposition 5 to prove that the elements of the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are bounded near ∂Y.

Proposition 24. *Let K be a compact subset of X. Assume that G_0 has logarithmic order and that there is $C > 0$ such that if $\alpha \in T^*_K X$ is large enough then*

$$G_0(\alpha) \leq -C^{-1}\log\langle|\alpha|\rangle.$$

Assume that τ is small enough. Then, for every $k \in \mathbb{R}$, if h is small enough then the elements of $\mathcal{H}^k_\Lambda$ are continuous on a neighborhood of K.

Proof. Let $N > n$. It follows from [Guedes-Bonthonneau et al. 2024, Lemmas 4.2 and 4.9] that, for h small enough, there is a neighborhood U of K such that if v is in $H^{-N}(X)$ and supported in U then v belongs to $\mathcal{H}^{-k}_{\overline{\Lambda}}$ and its norm in this space is less than $C\|v\|_{H^{-N}}$, where the constant C may depend on h but not on v.

Let $u \in \mathcal{H}^k_\Lambda$. If χ is a C^∞ function supported in the intersection of U with a coordinates patch, then we see that in these coordinates the Fourier transform of χu decays faster than $\langle\xi\rangle^{-N}$. Indeed, the H^{-N} norm of the functions given in coordinates by $\chi(x)e^{ix\xi}$ decays like $\langle\xi\rangle^{-N}$ when ξ tends to $+\infty$. Thus, the same is true for the norm of these functions in $\mathcal{H}^{-k}_{\overline{\Lambda}}$. It follows then from Remark 22 that $\widehat{\chi u}(\xi)$, which is the L^2 pairing of u with one of these functions, decays like $\langle\xi\rangle^{-N}$ when ξ tends to $+\infty$. Consequently, the distribution χu is a continuous function, and the result follows by a partition of unity argument. □

6. General construction (proof of Proposition 5)

The aim of this section is to prove Proposition 5. We will use the notation that we introduced in Section 2.1.

In Section 6.1, we fix the value of certain parameters that play an important role in the proof of Proposition 5 and define the modification $P_h(\omega)$ of $\mathcal{P}_h(\omega)$. In Section 6.2, we define the spaces that will be $\mathcal{H}_1$ and $\mathcal{H}_2$ in Proposition 5, and explain how the action of $P_h(\omega)$ on these spaces is related to the values of a certain symbol (Proposition 25). In Section 6.3, we prove ellipticity estimates on this symbol (Lemmas 26 and 28). In Section 6.4, we use these estimates to study the functional analytic properties of $P_h(\omega)$ acting on the spaces defined in Section 6.3: we prove that $P_h(\omega)$ is Fredholm by proving that it is invertible after perturbation by a compact operator (Lemmas 30 and 32 and Proposition 36), and that $P_h(i\nu)$ is invertible for some $\nu > 0$ (Lemmas 31 and 33 and Proposition 36). In Section 6.5, we prove the crucial point (vi) from Proposition 5 (from which our upper bounds on resonances, Theorems 2 and 3, follow). This is done by evaluating the trace class norm of the compact perturbation that we use to make $P_h(\omega)$ invertible (Lemmas 37 and 38). Finally, in Section 6.6, we put all these information together in order to get a full proof of Proposition 5.

Notice that, in most of this section, we are not working directly with the operator $P_h(\omega)$, but rather with an operator $\widetilde{P}_h(\omega)$, defined in Section 6.1, which is conjugated to $P_h(\omega)$, but simpler to apprehend.

6.1. ***Choice of parameters and modification of the operator.*** We use the notation from Section 2.1. Up to making ϵ smaller, we may assume that $w'(x_1) > \epsilon$ for every $x_1 \in]-\epsilon, \epsilon[$ and $p_0(x) < -\epsilon$ for every $x \in U \cup Y$ (this second point is a consequence of assumption (e)). We may also assume that x_1 extends to a smooth function on the whole X (analytic on U) such that $U = \{-\epsilon < x_1 < \epsilon\}$, $Y = \{x_1 > 0\}$ and $X \setminus \overline{Y} = \{x_1 < 0\}$.

Let us introduce on $T^*U \simeq T^*(]-\epsilon, \epsilon[)_{(x_1,\xi_1)} \times T^*\partial Y_{(x',\xi')}$ the symbol of logarithmic order

$$G_1(x_1, x', \xi_1, \xi') = \log(2 + \xi_1^2 + |\xi'|^2)$$

and denote by H_{G_1} the Hamiltonian flow of G_1 for the canonical symplectic form on T^*U. Here, the quantity $|\xi'|^2$ is computed using any smooth Riemannian metric on ∂Y, e.g., the restriction of g_X. Let us compute $H_{G_1} p_2$ where we recall that p_2 is the principal symbol of the order 2 differential operator P_2 from (7). Using local coordinates on ∂Y, we find that

$$\begin{aligned} H_{G_1} p_2(x_1, x', \xi_1, \xi') = \frac{2\xi_1}{2 + \xi_1^2 + |\xi'|^2} w'(x_1)\xi_1^2 + \frac{2\xi_1}{2 + \xi_1^2 + |\xi'|^2} \frac{\partial q_1}{\partial x_1}(x_1, x', \xi') \\ + \frac{\nabla_{\xi'}(|\xi'|^2)}{2 + \xi_1^2 + |\xi'|^2} \cdot \nabla_{x'} q_1(x_1, x', \xi') - \frac{\nabla_{x'}(|\xi'|^2)}{2 + \xi_1^2 + |\xi'|^2} \cdot \nabla_{\xi'} q_1(x_1, x', \xi'). \end{aligned}$$

Since $w'(x_1) > \epsilon$, the first term on the right-hand side is elliptic of order 1 whenever ξ_1 is larger than a fixed proportion of $|\xi'|$. Moreover, this term has the same sign as ξ_1. The other terms are also of order 1, and they can be made arbitrarily small by assuming that ξ_1 is much larger than ξ'. Hence, there is some small $\epsilon_1 \in]0, \epsilon[$ such that if $(x_1, x', \xi_1, \xi') \in T^*U$ and $|\xi_1| \geq \epsilon_1^{-1}(1 + |\xi'|)$ we have

$$\frac{H_{G_1} p_2(x_1, x', \xi_1, \xi')}{\xi_1} \geq C^{-1} \tag{36}$$

for some constant $C > 0$.

Let then C_0 be a bound for the derivative of w on $]-\epsilon, \epsilon[$. We choose $\epsilon_0 \in]0, \epsilon[$ small enough so that if $(x_1, x', \xi_1, \xi') \in T^*U$ and $|\xi_1| \leq 2\epsilon_1^{-1}(1 + |\xi'|)$ we have, with $\xi = (\xi_1, \xi')$,

$$-C_0\epsilon_0\xi_1^2 + |\xi|^2 + 1 \geq C^{-1}(1 + |\xi|^2), \tag{37}$$

$$-C_0\epsilon_0\xi_1^2 + q_1(x_1, x', \xi') + 1 \geq C^{-1}(1 + |\xi|^2). \tag{38}$$

Here, $|\xi|^2$ is defined using the metric g_X on X, and we used the ellipticity condition on q_1 (assumption (b)).

Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth function such that $\chi(t) = 0$ for $t \leq -\epsilon_0$ and $\chi(t) = 1$ for $t \geq -5\epsilon_0/6$. Let ψ be a real-analytic function from $\mathbb{R}$ to $\mathbb{R}$ such that $t\psi'(t) \leq 0$ for $t \in \mathbb{R}$ and $t\psi'(t) < 0$ for $t \neq 0$. One can take for instance $\psi(t) = -t^2/2$.

Let Q be a semiclassical differential operator of order 2 with principal symbol q. We assume that Q has the following properties:

- The coefficients of Q are supported in $\{x_1 < -\epsilon_0/2\}$.
- The principal symbol of Q is
$$q(x, \xi) = \chi_1(x_1)(1 + |\xi|^2),$$
where $\chi_1 : \mathbb{R} \to [0, 1]$ is a smooth function supported in $]-\infty, -\epsilon_0/2]$ and that takes value 1 on $]-\infty, -2\epsilon_0/3]$. For instance, one can take $Q = \chi_1(I - h^2\Delta_{g_X})$.

The modification $P_h(\omega)$ of the operator $\mathcal{P}_h(\omega)$ for which Proposition 5 will be established is

$$P_h(\omega) = \chi(x_1)\mathcal{P}_h(\omega) + e^{-\frac{\psi(x_1)}{h}} Q e^{\frac{\psi(x_1)}{h}}, \tag{39}$$

but we will rather study the conjugated operator

$$\widetilde{P}_h(\omega) = e^{\frac{\psi(x_1)}{h}} \chi(x_1)\mathcal{P}_h(\omega) e^{-\frac{\psi(x_1)}{h}} + Q.$$

For $(x, \xi) = (x_1, x', \xi_1, \xi') \in T^*U \simeq T^*(]-\epsilon, \epsilon[\times \partial Y)$, the principal symbol of $\tilde{p}(x, \xi; \omega)$ of $\widetilde{P}_h(\omega)$ is given by

$$\begin{aligned}\tilde{p}(x, \xi; \omega) = \chi(x_1)\big(&w(x_1)\xi_1^2 + q_1(x_1, x', \xi') + 2iw(x_1)\psi'(x_1)\xi_1 + \omega p_1(x_1, \xi_1) \\ &+ i\omega p_1(x_1, \psi'(x_1)) + \omega^2 p_0(x) - \psi'(x_1)^2 w(x_1)\big) + \chi_1(x_1)(1 + |\xi|^2).\end{aligned} \tag{40}$$

Finally, let ϕ be a C^∞ function from $\mathbb{R}$ to $[0, 1]$, supported in $]-\epsilon_0/3, \epsilon_0[$, such that $\phi(t) = 1$ for $t \in [-\epsilon_0/6, 2\epsilon_0/3]$ and $t\phi'(t) \leq 0$ for every $t \in \mathbb{R}$.

Our choices of parameters are summed up in Figure 1, where the black line represent the x_1-axis. The colored zones in this drawing correspond to places where we will use different mechanisms to prove the Fredholm property for $\widetilde{P}_h(\omega)$. In the purple zone (which is compactly contained in Y), we will use the ellipticity of $\widetilde{P}_h(\omega)$, which follows from our assumption (d) in Section 2.1. In the green zone (which is away from $\overline{Y}$), the operator $\widetilde{P}_h(\omega)$ is also elliptic, but this is just because Q is. Finally, the most interesting part is the blue zone, where two phenomena occur: in some places $\widetilde{P}_h(\omega)$ is elliptic and in other places we need to use propagation and radial estimates to get the Fredholm property. See Section 6.3 for the details on how Figure 1 can be turned into actual estimates.

6.2. ***Definition of the spaces.*** We define the symbol G_0 on T^*X by

$$G_0(x, \xi) = -\phi(x_1)G_1(x, \xi) \quad \text{for } (x, \xi) \in T^*X.$$

Then, for some small $r > 0$, we extend G_0 to $(T^*X)_r$ as a symbol of logarithmic order. The particular features of the extension are irrelevant as soon as we have symbolic estimates, and that G_0 is identically equal to 0 away from a small neighborhood of the support of ϕ, so that all derivatives of G_0 vanish at any

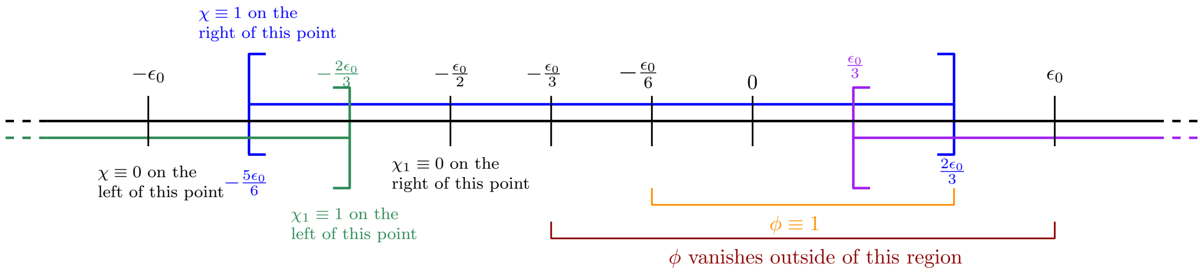

Figure 1. Some relevant places near ∂Y.

point of T^*X such that $x_1 \leq -\epsilon_0/3$ or $x_1 \geq \epsilon_0$ (even derivatives in directions that are not tangent to T^*X). As above we define the escape function $G = \tau G_0$ for some small $\tau > 0$. We let $\Lambda = e^{H_G^{\omega I}} T^*X$ be the associated Lagrangian deformation and $(\mathcal{H}_\Lambda^k)_{k\in\mathbb{R}}$ the associated family of Hilbert spaces (see Section 5). Notice that these are spaces of distributions according to Proposition 21. For $k \in \mathbb{R}$, define the Hilbert space

$$\mathcal{F}_k = \{u \in \mathcal{D}'(X) : e^{\frac{\psi}{h}} u \in \mathcal{H}_\Lambda^k\}, \tag{41}$$

where we recall that ψ is defined in Section 6.1. The spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ in Proposition 5 will be respectively $\{u \in \mathcal{F}_k : P_h(0)u \in \mathcal{F}_{k-1}\}$ and $\mathcal{F}_{k-1}$.

Notice that it is equivalent to study $P_h(\omega)$ acting on the $\mathcal{F}_k$'s or $\widetilde{P}_h(\omega)$ acting on the $\mathcal{H}_\Lambda^k$'s. Also, we can write $\widetilde{P}_h(\omega) = \widetilde{P}_2 + \omega \widetilde{P}_1 + \omega^2 \widetilde{P}_0$ where the $\widetilde{P}_j$'s are semiclassical differential operator with analytic coefficients near the support of ϕ. Consequently, these operators satisfy the assumption (26) and it makes sense to restrict their principal symbols (and thus the principal symbol $\tilde{p}(\,\cdot\,;\omega)$ of $\widetilde{P}_h(\omega)$) to Λ, see the remark below (26). Applying Proposition 18 and Lemma 23 to the operators $\widetilde{P}_0$, $\widetilde{P}_1$ and $\widetilde{P}_2$, we find:

Proposition 25. *Assume that τ and h are small enough. Let $m \in \mathbb{R}$ and f be a symbol of order m on Λ. Let $\omega \in \mathbb{C}$. Let k_1 and k_2 be such that $k_1 + k_2 = m + 1$. Then, there is a constant C such that for every $u, v \in \mathcal{H}_\Lambda^\infty$, we have*

$$\left| \int_\Lambda f T_\Lambda \widetilde{P}_h(\omega) u \overline{T_\Lambda v} e^{-\frac{2H}{h}} \, d\alpha - \int_\Lambda f(\alpha) \tilde{p}(\alpha;\omega) T_\Lambda u \overline{T_\Lambda v} e^{-\frac{2H}{h}} \, d\alpha \right| \leq Ch \|u\|_{\mathcal{H}_\Lambda^{k_1}} \|v\|_{\mathcal{H}_\Lambda^{k_2}}.$$

Here, the constant C depends continuously on ω and $\tilde{p}(\,\cdot\,;\omega)$ denotes the principal symbol of $\widetilde{P}_h(\omega)$. We also wrote $\mathcal{H}_\Lambda^\infty$ for $\bigcap_{k\in\mathbb{R}} \mathcal{H}_\Lambda^k$.

Another consequence of Proposition 18 and Lemma 23 that it will be useful to remember is that, under the assumptions of Proposition 25, for every $k \in \mathbb{R}$ the family $\omega \mapsto \widetilde{P}_h(\omega) - \widetilde{P}_h(0)$ is a holomorphic family of bounded operators from $\mathcal{H}_\Lambda^k$ to $\mathcal{H}_\Lambda^{k-1}$.

6.3. ***Ellipticity estimates.*** In order to use Proposition 25, let us introduce the following subsets of T^*X:

$$V_R = \{x_1 \leq -2\epsilon_0/3\} \cup \{x_1 \geq \epsilon_0/3\} \cup (\{-5\epsilon_0/6 \leq x_1 \leq 2\epsilon_0/3\} \cap \{|\xi_1| \leq 2\epsilon_1^{-1}(1+|\xi'|)\}),$$
$$V_+ = \{-5\epsilon_0/6 \leq x_1 \leq 2\epsilon_0/3\} \cap \{\xi_1 \geq \epsilon_1^{-1}(1+|\xi'|)\},$$
$$V_- = \{-5\epsilon_0/6 \leq x_1 \leq 2\epsilon_0/3\} \cap \{\xi_1 \leq -\epsilon_1^{-1}(1+|\xi'|)\}.$$

Here, we see the function $x_1 : X \to \mathbb{R}$ as a function on T^*X by composition by the canonical projection $T^*X \to X$, and the constant ϵ_1 has been defined in Section 6.1. Notice that a point of T^*X for which x_1 is between $-\epsilon$ and ϵ is in $T^*U \simeq T^*(]-\epsilon,\epsilon[) \times T^*(\partial Y)$, and may consequently be written as (x_1, x', ξ_1, ξ') with $(x_1,\xi_1) \in T^*(]-\epsilon,\epsilon[)$ and $(x',\xi') \in T^*\partial Y$. This is how we make sense of ξ' in the equation above. We use the same metric on $T^*\partial Y$ as in Section 6.1.

We let also W_R, W_+ and W_- denote the images respectively of V_R, V_+ and V_- by $e^{H_G^{\omega I}}$. Notice that $T^*X = V_R \cup V_+ \cup V_-$ so that $\Lambda = W_R \cup W_+ \cup W_-$. We are going to prove two ellipticity estimates, Lemmas 26 and 28, that will be used in Section 6.4 below to prove that $\widetilde{P}_h(\omega)$ is Fredholm for a certain range of ω's and invertible for at least one of these ω's.

Lemma 26. *Let $\tau > 0$ be small and fixed. There is $\kappa > 0$ (depending on τ) such that the following holds. For every compact subset K of $\{z \in \mathbb{C} : \operatorname{Im} z \geq -\kappa\}$, there is a constant C_K such that for $\omega \in K$ and $\alpha \in \Lambda$ such that $\langle|\alpha|\rangle \geq C_K$ we have*

$$\begin{aligned} &\operatorname{Re} \tilde{p}(\alpha;\omega) \geq C_K^{-1}\langle|\alpha|\rangle && \text{if } \alpha \in W_R, \\ &\operatorname{Im} \tilde{p}(\alpha;\omega) \leq -C_K^{-1}\langle|\alpha|\rangle && \text{if } \alpha \in W_+, \\ &\operatorname{Im} \tilde{p}(\alpha;\omega) \geq C_K^{-1}\langle|\alpha|\rangle && \text{if } \alpha \in W_-. \end{aligned}$$

Proof. Let us write $\alpha = e^{H_G^{\omega I}}(x,\xi)$ for $(x,\xi) \in T^*X$. We will distinguish the different cases that appear in the definitions of V_R, V_+ and V_-.

<u>First case</u>: $(x,\xi) \in \{x_1 \leq -2\epsilon_0/3\}$. In that case, we see that G_0 is null on a neighborhood of (x,ξ) so that $\alpha = (x,\xi)$. Moreover $\chi_1(x_1) = 1$, so that $q(x,\xi) = 1 + |\xi|^2$. Using that $\chi(x_1) = 0$ for $x_1 \leq -\epsilon_0$, we find from (40) that the real part of $\tilde{p}(x,\xi;\omega)$ is greater than

$$-C_0\epsilon_0\xi_1^2 - C(1+|\xi_1|) + |\xi|^2. \tag{42}$$

Here, $C > 0$ is some constant that depends continuously on ω (and does not depend on α). Thanks to our assumption (37) on ϵ_0, we see that (42) is larger than $C^{-1}\langle|\alpha|\rangle^2$ and hence that $C^{-1}\langle|\alpha|\rangle$ when α is large enough.

<u>Second case</u>: $(x,\xi) \in \{x_1 \geq \epsilon_0/3\}$. Notice that $H_G^{\omega I}\tilde{p}(\alpha;h)$ is $\mathcal{O}(\langle|\alpha|\rangle \log\langle|\alpha|\rangle)$, with symbolic estimates (it follows from the fact that G has logarithmic order). Consequently, we have

$$\begin{aligned} \tilde{p}(\alpha;h) &= \tilde{p}(x,\xi;\omega) + \mathcal{O}(\langle|\alpha|\rangle \log\langle|\alpha|\rangle) \\ &= p(x,\xi + i\psi'(x_1)\mathrm{d}x_1;\omega) + \mathcal{O}(\langle|\alpha|\rangle \log\langle|\alpha|\rangle) \\ &= p_2(x,\xi) + \mathcal{O}(\langle|\alpha|\rangle \log\langle|\alpha|\rangle). \end{aligned}$$

Thanks to our assumption (d) of ellipticity on p_2 in Section 2.1, we see that this quantity is larger than $C^{-1}\langle|\alpha|\rangle^2$ and hence that $C^{-1}\langle|\alpha|\rangle$ when α is large enough.

<u>Third case</u>: $(x,\xi) \in \{-5\epsilon_0/6 \leq x_1 \leq 2\epsilon_0/3\} \cap \{|\xi_1| \leq 2\epsilon_1^{-1}(1+|\xi'|)\}$. In that case, we notice that $\chi(x_1) = 1$ and that $\tilde{p}(\alpha;\omega) = \tilde{p}(x,\xi;\omega) + \mathcal{O}(\langle|\alpha|\rangle \log\langle|\alpha|\rangle)$ as above. Using (40), we find that

$$\operatorname{Re}\tilde{p}(\alpha;\omega) \geq q_1(x_1,x',\xi') - C_0\epsilon_0\xi_1^2 - C\langle|\alpha|\rangle \log\langle|\alpha|\rangle.$$

Using that $|\xi_1| \leq 2\epsilon_1^{-1}(1+|\xi'|)$ and our assumption (38) on ϵ_0, we see that for α large enough, this real part is larger than $C^{-1}\langle|\alpha|\rangle^2$ and hence that $C^{-1}\langle|\alpha|\rangle$.

Fourth case: $(x,\xi) \in V_+$. Notice that we have

$$\operatorname{Im} \tilde{p}(\alpha;\omega) = \operatorname{Im} \tilde{p}(x,\xi;\omega) + \tau H_{G_0}^{\omega_I} \operatorname{Im} \tilde{p}(x,\xi;\omega) + \mathcal{O}((\log\langle|\alpha|\rangle)^2). \tag{43}$$

We want to estimate $H_{G_0}^{\omega_I} \operatorname{Im} \tilde{p}(x,\xi;\omega)$. First, notice that $\tilde{p}(\,\cdot\,;\omega) - p_2$ is a symbol of order 1 on a neighborhood of the support of G_0, so that

$$\begin{aligned} H_{G_0}^{\omega_I} \operatorname{Im} \tilde{p}(x,\xi;\omega) &= H_{G_0}^{\omega_I} \operatorname{Im} p_2(x,\xi) + \mathcal{O}(\log\langle|\alpha|\rangle) \\ &= -H_{\operatorname{Im} p_2}^{\omega_I} G_0(x,\xi) + \mathcal{O}(\log\langle|\alpha|\rangle). \end{aligned}$$

Notice that the symbol $\operatorname{Im} p_2$ vanishes on the real cotangent bundle T^*X, which is a Lagrangian submanifold for the symplectic form ω_I. Consequently, the Hamiltonian vector field $H_{\operatorname{Im} p_2}^{\omega_I}$ is tangent to T^*X (this is why we only care about the value of G_0 on T^*X). Recall that ω_R denotes the real part of the canonical symplectic form ω on $(T^*X)_r$. For u tangent to T^*X, we have

$$\begin{aligned} \omega_R(u, H_{\operatorname{Im} p_2}^{\omega_I}) &= \operatorname{Im}(i\omega(u, H_{\operatorname{Im} p_2}^{\omega_I})) = \operatorname{Im}(\omega(iu, H_{\operatorname{Im} p_2}^{\omega_I})) \\ &= \mathrm{d}(\operatorname{Im} p_2)\cdot(iu) = \mathrm{d}(\operatorname{Re} p_2)\cdot u \\ &= \omega_R(u, H_{p_2}), \end{aligned} \tag{44}$$

where H_{p_2} is the Hamiltonian vector field of p_2 for the (real) canonical symplectic form on the real cotangent bundle T^*X. We used the Cauchy–Riemann equation on the second line of (44). On the last line, we used the fact that p_2 is real-valued on T^*X and that the pullback of ω_R on T^*X is the canonical symplectic form on T^*X. Since ω_R is symplectic on T^*X and the vector fields H_{p_2} and $H_{\operatorname{Im} p_2}^{\omega_I}$ are parallel to T^*X, we find that $H_{\operatorname{Im} p_2}^{\omega_I}$ coincides with H_{p_2} on T^*X. It follows that

$$\begin{aligned} H_{G_0}^{\omega_I} \operatorname{Im} \tilde{p}(x,\xi;\omega) &= H_{G_0} p_2(x,\xi) + \mathcal{O}(\log\langle|\alpha|\rangle) \\ &= -\phi(x_1) H_{G_1} p_2(x,\xi) + 2w(x_1)\phi'(x_1) G_1(x,\xi)\xi_1 + \mathcal{O}(\log\langle|\alpha|\rangle) \\ &\leq -C^{-1}\phi(x_1)\xi_1 + C_\omega \log\langle|\alpha|\rangle \end{aligned}$$

for some constant $C_\omega > 0$ that depends continuously on ω and some constant C that does not depend on ω. Here, we used (36), which is valid thanks to the assumption $(x,\xi)\in V_+$, and the fact that $w(x_1)\phi'(x_1) \leq 0$. Then, we plug this estimate into (43) to find that

$$\begin{aligned} \operatorname{Im} \tilde{p}(\alpha;\omega) &\leq 2w(x_1)\psi'(x_1)\xi_1 + \operatorname{Im} \omega p_1(x_1,\xi_1) - C^{-1}\tau\phi(x_1)\xi_1 + C_\omega(\log\langle|\alpha|\rangle)^2 \\ &\leq -C^{-1}(-w(x_1)\psi'(x_1) + \tau\phi(x_1) + \operatorname{Im}\omega)\xi_1 + C_\omega(\log\langle|\alpha|\rangle)^2, \end{aligned}$$

where the constants C may change from one line to another but still does not depend on ω. We used here that $p_1(x_1,\xi_1)$ is elliptic of order 1 with the same sign as $-\xi_1$, that is, our (c) from Section 2.1. Notice that $w(x_1)\psi'(x_1)$ has the same sign as $x_1\psi'(x_1)$, and consequently there is a constant $\kappa > 0$ such that $-w(x_1)\psi'(x_1) + \tau\phi(x_1) > \kappa$ for $-5\epsilon_0/6 < x_1 < 2\epsilon_0/3$. Hence, if $\operatorname{Im}\omega > -\kappa$, we see that $\operatorname{Im}\tilde{p}(\alpha;\omega)$ is less than $-C^{-1}\langle|\alpha|\rangle$ when α is large.

Fifth case: $(x,\xi) \in V_-$. This is the same as the fourth case up to a few sign flips. ☐

Remark 27. Let us point out how the five cases in the proof of Lemma 26 correspond to different places in Figure 1. The first and the second cases correspond respectively to the green and the purple zone. The last three cases correspond to the blue zone (to distinguish these cases one need to consider the momentum variable which is not represented on Figure 1).

Lemma 28. *Assume that ν is large enough. Assume that τ is small enough* (*depending on ν*). *Then there is a constant $C > 0$ such that for every $\alpha \in \Lambda$ we have*

$$\begin{aligned} \operatorname{Re} \tilde{p}(\alpha; i\nu) &\geq C^{-1}\langle|\alpha|\rangle && \text{if } \alpha \in W_R, \\ \operatorname{Im} \tilde{p}(\alpha; i\nu) &\leq -C^{-1}\langle|\alpha|\rangle && \text{if } \alpha \in W_+, \\ \operatorname{Im} \tilde{p}(\alpha; i\nu) &\geq C^{-1}\langle|\alpha|\rangle && \text{if } \alpha \in W_-. \end{aligned}$$

Proof. We write as above $\alpha = e^{H_G^{\omega_I}}(x,\xi)$ for $(x,\xi) \in T^*X$. We review the same five cases as in the proof of Lemma 26, with the additional assumption that $\omega = i\nu$ with $\nu > 0$ large.

First case: $(x,\xi) \in \{x_1 \leq -2\epsilon_0/3\}$. The symbol $q(x,\xi)$ is still $1 + |\xi|^2$ in that case. Notice that we have here $p_0(x) < -\epsilon$ (see the beginning of Section 6.1). Looking at (40), we find that, for some constant $C > 0$, the real part of $\tilde{p}(x,\xi; i\nu)$ is larger than

$$\chi(x_1)(-C_0\epsilon_0\xi_1^2 + \epsilon\nu^2 - C(1+\nu)) + (1 + |\xi|^2) \geq -C_0\epsilon_0\xi_1^2 + (1 + |\xi|^2),$$

provided ν is large enough so that $\epsilon\nu^2 - C(1+\nu) \geq 0$. Using our assumption (37), we see that the real part of $\tilde{p}(\alpha; i\nu)$ is indeed larger than $C^{-1}\langle|\alpha|\rangle$.

Second case: $(x,\xi) \in \{x_1 \geq \epsilon_0/3\}$. Notice that $\tilde{p}(\cdot\,; i\nu) = \tilde{p}_2 + i\nu\tilde{p}_1 - \nu^2\tilde{p}_0$, where for $j = 0, 1, 2$, the principal symbol $\tilde{p}_j$ of $\widetilde{P}_j$ is a symbol of order j that does not depend on ν. It follows that $H_G^{\omega_I}\tilde{p}(\cdot\,; i\nu)$ is $\mathcal{O}(\tau(\langle|\alpha|\rangle\log\langle|\alpha|\rangle + \nu^2\log\langle|\alpha|\rangle/\langle|\alpha|\rangle))$, uniformly in ν and τ and with symbolic estimates.

Consequently, we have in this second case

$$\begin{aligned} \operatorname{Re} \tilde{p}(\alpha; i\nu) &= \operatorname{Re} \tilde{p}(x,\xi; i\nu) + \mathcal{O}\left(\tau\left(\langle|\alpha|\rangle\log\langle|\alpha|\rangle + \nu^2\frac{\log\langle|\alpha|\rangle}{\langle|\alpha|\rangle}\right)\right) \\ &= \operatorname{Re} p(x, \xi + i\psi'(x_1)\,\mathrm{d}x_1; i\nu) + \mathcal{O}(\tau\nu^2\langle|\alpha|\rangle\log\langle|\alpha|\rangle) \\ &= p_2(x,\xi) - \nu^2 p_0(x) + \mathcal{O}(\nu + \tau\nu^2\langle|\alpha|\rangle\log\langle|\alpha|\rangle), \end{aligned}$$

uniformly in τ and ν. We start by taking ν large enough so that $p_2(x,\xi) - \nu^2 p_0(x) + \mathcal{O}(\nu)$ is larger than $C^{-1}\langle|\alpha|\rangle^2$ (which is possible by our ellipticity assumptions on p_2 and p_0). Then, by taking τ small enough, we ensure that $\mathcal{O}(\tau\nu^2\langle|\alpha|\rangle\log\langle|\alpha|\rangle)$ is smaller than $C^{-1}\langle|\alpha|\rangle^2$, which gives the required estimate. Let us point out here that how small τ needs to be depend on ν, but how large ν has to be does not depend on τ.

Third case: $(x,\xi) \in \{-5\epsilon_0/6 \leq x_1 \leq 2\epsilon_0/3\} \cap \{|\xi_1| \leq 2\epsilon_1^{-1}(1 + |\xi'|)\}$. As in the previous case, we notice that

$$\tilde{p}(\alpha; i\nu) = \tilde{p}(x,\xi; i\nu) + \mathcal{O}(\tau\nu^2\langle|\alpha|\rangle\log\langle|\alpha|\rangle).$$

Then, we use (40) to find that

$$\operatorname{Re} \tilde{p}(x,\xi;i\nu) \geq -C_0\epsilon_0\xi_1^2 + q_1(x_1,x',\xi') + \epsilon\nu^2 - C(1+\nu),$$

for some $C>0$ that does not depend on ν, nor τ. Using (38), we find that if ν is large enough we have

$$\operatorname{Re} \tilde{p}(x,\xi;i\nu) \geq C^{-1}(1+|\xi|^2).$$

Consequently, we have

$$\operatorname{Re} \tilde{p}(\alpha;i\nu) \geq C^{-1}(1+|\xi|^2) - C\tau\nu^2\langle|\alpha|\rangle\log\langle|\alpha|\rangle,$$

where the constant $C>0$ may have changed, but still does not depend on ν, nor τ. Taking τ small enough (depending on ν), we get rid of the term $-C\tau\nu^2\langle|\alpha|\rangle\log\langle|\alpha|\rangle$. Thus, we get the required estimate. As above, it is crucial here that how small τ needs to be depend on ν, but how large ν has to be does not depend on τ.

Fourth case: $(x,\xi)\in V_+$. Writing $\tilde{p}(\cdot\,;i\nu) = \tilde{p}_2 + i\nu\tilde{p}_1 - \nu^2\tilde{p}_0$, we find that

$$\operatorname{Im}\tilde{p}(\alpha;i\nu) = \operatorname{Im}\tilde{p}(x,\xi;i\nu) + \tau H_{G_0}^{\omega_I}\operatorname{Im}\tilde{p}(x,\xi;i\nu) + \mathcal{O}(\tau^2\nu^2(\log\langle|\alpha|\rangle)^2).$$

Then, we notice that on a neighborhood of the support of G_0, the symbol $\tilde{p}(\cdot\,;i\nu) - p_2$ is the sum of a symbol of order 1, a symbol of order 1 multiplied by ν and a symbol of order 0 multiplied by ν^2. Consequently, we have

$$H_{G_0}^{\omega_I}\operatorname{Im}\tilde{p}(x,\xi;i\nu) = H_{G_0}^{\omega_I}\operatorname{Im} p_2(x,\xi) + \mathcal{O}(\nu^2\log\langle|\alpha|\rangle).$$

Using (36) as in the proof of Lemma 26, we see that $H_{G_0}^{\omega_I}\operatorname{Im} p_2(x,\xi)$ is nonpositive. We recall (40) and the fact that $w(x_1)\psi'(x_1)$ is nonpositive, to find, for some $C>0$ that does not depend on τ nor α, that

$$\operatorname{Im}\tilde{p}(\alpha;i\nu) \leq \nu p_1(x_1,\xi_1) + C\tau\nu^2(\log\langle|\alpha|\rangle)^2. \tag{45}$$

Since $(x,\xi)\in V_+$, we know that ξ_1 is nonzero. Moreover, $p_1(x_1,\xi_1)$ is negative and elliptic. Thus, we only need to take τ small enough to get rid of the last term in (45) and the required estimate follows. Once again here, see that τ depends on ν, but ν does not depend on τ.

Fifth case: $(x,\xi)\in V_-$. This is the same as the fourth case up to a few sign flips. □

6.4. ***Invertibility and Fredholm properties.*** With the estimates from Section 6.3, we are now ready to study the functional analytic properties of $\widetilde{P}_h(\omega)$ acting on suitable spaces.

Let ν be large enough and τ be small enough so that Lemma 28 and Proposition 25 hold. Let then κ be as in Lemma 26. Let $\delta\in\,]0,\kappa[$ and V be a relatively compact open subset of $\{z\in\mathbb{C} : \operatorname{Im} z > -\kappa\}$. Without loss of generality, we may assume that V is connected and contains the compact set

$$\{z\in\mathbb{C} : |\operatorname{Re} z| \leq \nu+\kappa,\ -\delta \leq \operatorname{Im} z \leq 2\nu+\kappa\}.$$

Let then C_K be the constant from Lemma 26 applied with $K=\overline{V}$. We shall always assume that h is small enough so that Proposition 25 holds. Let k be any real number.

Let a be a compactly supported smooth function from Λ to $\mathbb{R}_+$ such that

$$\inf_{\omega \in K} \inf_{\substack{\alpha \in \Lambda \\ \langle |\alpha| \rangle \leq 2C_K}} a(\alpha) + \operatorname{Re} \tilde{p}(\alpha; \omega) > 0. \tag{46}$$

We let then A be the operator

$$A := S_\Lambda B_\Lambda a B_\Lambda T_\Lambda, \tag{47}$$

where we recall that S_Λ is a left inverse for T_Λ, and B_Λ is the orthogonal projector on $\mathcal{H}^0_{\Lambda,\mathrm{FBI}}$ in $L^2_0(\Lambda)$ (see page 3645). The operator $A : C^\infty(X) \to \mathcal{D}'(X)$ extends to a bounded operator from $\mathcal{H}^m_\Lambda$ to $\mathcal{H}^\ell_\Lambda$ for every $m, \ell \in \mathbb{R}$, see for instance [Bonthonneau and Jézéquel 2020, Proposition 2.4 and Remark 2.20].

Let us define the domain of $\widetilde{P}_h(\omega)$ on $\mathcal{H}^k_\Lambda$ as

$$D_k = \{u \in \mathcal{H}^k_\Lambda : \widetilde{P}_h(0)u \in \mathcal{H}^{k-1}_\Lambda\}.$$

We put a Hilbert space structure on D_k by endowing it with the norm

$$\|u\|^2_{D_k} = \|u\|^2_{\mathcal{H}^k_\Lambda} + \|\widetilde{P}_h(0)u\|^2_{\mathcal{H}^{k-1}_\Lambda}.$$

We will need the following approximation result.

Lemma 29. *Let $u \in D_k$. Then $\widetilde{P}_h(\omega)u$ belongs to $\mathcal{H}^{k-1}_\Lambda$ and there is a sequence $(u_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{H}^\infty_\Lambda$ such that $(u_n)_{n\in\mathbb{N}}$ tends to u in $\mathcal{H}^k_\Lambda$ and $(\widetilde{P}_h(\omega)u_n)_{n\in\mathbb{N}}$ converges to $\widetilde{P}_h(\omega)u$ in $\mathcal{H}^{k-1}_\Lambda$.*

Proof. Start by noticing that

$$\widetilde{P}_h(\omega)u = (\widetilde{P}_h(\omega) - \widetilde{P}_h(0))u + \widetilde{P}_h(0)u.$$

Since $\widetilde{P}_h(\omega) - \widetilde{P}_h(0)$ is bounded from $\mathcal{H}^k_\Lambda$ to $\mathcal{H}^{k-1}_\Lambda$, we see that $\widetilde{P}_h(\omega)u$ belongs to $\mathcal{H}^{k-1}_\Lambda$ when u belongs to D_k.

Let then I_ϵ be the operator

$$I_\epsilon = S_\Lambda B_\Lambda s_\epsilon B_\Lambda T_\Lambda,$$

where s_ϵ is a symbol on Λ defined by $s_\epsilon(\alpha) = \theta(\epsilon\langle|\alpha|\rangle)$, where θ is a compactly supported function in $\mathbb{R}$, identically equal to 1 near 0. It follows for instance from [loc. cit., Proposition 2.4 and Remark 2.20] that if $u \in \mathcal{H}^k_\Lambda$ then $I_\epsilon u \in \mathcal{H}^\infty_\Lambda$. We see that for $u \in \mathcal{H}^k_\Lambda$, we have

$$\begin{aligned} \|I_\epsilon u - u\|_{\mathcal{H}^k_\Lambda} &= \|\Pi_\Lambda B_\Lambda s_\epsilon T_\Lambda u - T_\Lambda u\|_{L^2_k(\Lambda)} \\ &= \|B_\Lambda(s_\epsilon - 1)T_\Lambda u\|_{L^2_k(\Lambda)} \leq C\|(s_\epsilon - 1)T_\Lambda u\|_{L^2_k(\Lambda)}. \end{aligned}$$

It follows that $I_\epsilon u$ converges to u in $\mathcal{H}^k_\Lambda$ when ϵ tends to 0.

If u belongs to D_k, we see that

$$\widetilde{P}_h(\omega)I_\epsilon u = I_\epsilon \widetilde{P}_h(\omega)u + [\widetilde{P}_h(\omega), I_\epsilon]u.$$

From the analysis above, we have that $I_\epsilon \widetilde{P}_h(\omega)u$ converges to $\widetilde{P}_h(\omega)u$ in $\mathcal{H}^{k-1}_\Lambda$ when ϵ tends to 0. Notice that the symbol s_ϵ is uniformly bounded as a symbol of order 0 on Λ. Hence, it follows from [loc. cit., Proposition 2.12], as in the proof of [loc. cit., Lemma 3.4], that the operator $[\widetilde{P}_h(\omega), I_\epsilon]$ is uniformly bounded from $\mathcal{H}_\Lambda^k$ to $\mathcal{H}_\Lambda^{k-1}$ when ϵ tends to 0. If u is in $\mathcal{H}_\Lambda^\infty$, the analysis above implies that $[\widetilde{P}_h(\omega), I_\epsilon]u$ tends to 0 in $\mathcal{H}_\Lambda^{k-1}$. Thanks to the uniform boundedness of $[\widetilde{P}_h(\omega), I_\epsilon]$ when ϵ tends to 0, we see that the same holds when u is only in $\mathcal{H}_\Lambda^{k-1}$. □

We first use Lemma 26 to find:

Lemma 30. *There is $C > 0$ such that for h small enough and every $\omega \in V$ and $u \in D_k$ we have*

$$\|u\|_{\mathcal{H}_\Lambda^k} \leq C\|(\widetilde{P}_h(\omega) + A)u\|_{\mathcal{H}_\Lambda^{k-1}}.$$

Proof. Thanks to Lemma 29, we only need to prove this estimate for $u \in \mathcal{H}_\Lambda^\infty$. Let f_+, f_-, f_R and f_a be symbols of order 0 on Λ such that $f_+ + f_- + f_R + f_a = 1$. Moreover, we assume that f_+, f_- and f_R are supported in the intersection of $\{\langle|\alpha|\rangle \geq C_K\}$ respectively with W_+, W_- and W_R and that f_a is supported in $\{\langle|\alpha|\rangle \leq 2C_K\}$.

For $u \in \mathcal{H}_\Lambda^\infty$ and $\omega \in K$, we have, from Proposition 25,

$$\begin{aligned}\operatorname{Re}\left(\int_\Lambda f_R(\alpha)\langle|\alpha|\rangle^{2k-1} T_\Lambda(\widetilde{P}_h(\omega) + A)u\overline{T_\Lambda u}\, e^{-\frac{2H(\alpha)}{h}}\,\mathrm{d}\alpha\right)\\ \geq \int_\Lambda f_R(\alpha)\langle|\alpha|\rangle^{2k-1}\operatorname{Re}(\tilde{p}(\alpha;\omega) + a(\alpha))|T_\Lambda u(\alpha)|^2 e^{-\frac{2H(\alpha)}{h}}\,\mathrm{d}\alpha - Ch\|u\|_{\mathcal{H}_\Lambda^k}^2\\ \geq C^{-1}\int_\Lambda f_R(\alpha)\langle|\alpha|\rangle^{2k}|T_\Lambda u(\alpha)|^2 e^{-\frac{2H(\alpha)}{h}}\,\mathrm{d}\alpha - Ch\|u\|_{\mathcal{H}_\Lambda^k}^2,\end{aligned}$$

where we used Lemma 26 in the last line (since a takes positive values it does not harm the positivity of the real part of $\tilde{p}$). From Cauchy–Schwarz inequality, we find that

$$\int_\Lambda f_R(\alpha)\langle|\alpha|\rangle^{2k}|T_\Lambda u(\alpha)|^2 e^{-\frac{2H(\alpha)}{h}}\,\mathrm{d}\alpha \leq C\|(\widetilde{P}_h(\omega) + A)u\|_{\mathcal{H}_\Lambda^{k-1}}\|u\|_{\mathcal{H}_\Lambda^k} + Ch\|u\|_{\mathcal{H}_\Lambda^k}^2.$$

Replacing the real part by an imaginary part, and varying the sign, we get the same estimates with f_R replaced by f_+ and f_-. Using (46), we get the same estimates with f_R replaced by f_a. Summing these four estimates, we find that

$$\|u\|_{\mathcal{H}_\Lambda^k}^2 \leq C\|(\widetilde{P}_h(\omega) + A)u\|_{\mathcal{H}_\Lambda^{k-1}}\|u\|_{\mathcal{H}_\Lambda^k} + Ch\|u\|_{\mathcal{H}_\Lambda^k}^2.$$

When h is small enough, we can get rid of the second term in the right-hand side. Dividing by $\|u\|_{\mathcal{H}_\Lambda^k}$ the result follows (the result is trivial when $u = 0$). □

The same proof using Lemma 28 instead of Lemma 26 gives:

Lemma 31. *There is $C > 0$ such that for h small enough and every $u \in D_k$ we have*

$$\|u\|_{\mathcal{H}_\Lambda^k} \leq C\|\widetilde{P}_h(i\nu)u\|_{\mathcal{H}_\Lambda^{k-1}}.$$

Applying Proposition 18 as in the justification of Proposition 25, we find that, for every $\omega \in V$, there is a symbol σ_ω of order 2 on Λ and an operator Z with negligible kernel on $\Lambda \times \Lambda$ such that

$$B_\Lambda T_\Lambda \widetilde{P}_h(\omega) S_\Lambda B_\Lambda = B_\Lambda \sigma_\omega B_\Lambda + Z = B_\Lambda \sigma_\omega B_\Lambda + B_\Lambda Z B_\Lambda.$$

Let us identify the dual of $\mathcal{H}_\Lambda^k$ with $\mathcal{H}_\Lambda^{-k}$ as in [Bonthonneau and Jézéquel 2020, Lemma 2.24], that is using the pairing

$$\langle u, v\rangle_\Lambda := \int_\Lambda T_\Lambda u \overline{T_\Lambda v} e^{-\frac{2H}{h}} \, d\alpha. \tag{48}$$

Notice that it is a priori not the L^2 pairing (recall that the L^2 pairing identifies the dual of $\mathcal{H}_\Lambda^k$ with $\mathcal{H}_{\overline{\Lambda}}^{-k}$, see Section 5.2). Under this identification, the formal adjoint of $\widetilde{P}_h(\omega)$ may be defined as

$$\widetilde{P}_h(\omega)^* = S_\Lambda (B_\Lambda \bar{\sigma}_\omega B_\Lambda + B_\Lambda Z^* B_\Lambda) T_\Lambda$$

By this, we just mean that if $u, v \in \mathcal{H}_\Lambda^\infty$ then

$$\langle \widetilde{P}_h(\omega) u, v\rangle_\Lambda = \langle u, \widetilde{P}_h(\omega)^* v\rangle_\Lambda.$$

Notice that we do not claim that $\widetilde{P}_h(\omega)^*$ is the adjoint of $\widetilde{P}_h(\omega)$ for a Hilbert space structure. We define the domain of $\widetilde{P}_h(\omega)^*$ as

$$D_{-k}^* = \{u \in \mathcal{H}_\Lambda^{-k+1} : \widetilde{P}_h(0)^* u \in \mathcal{H}_\Lambda^{-k}\}.$$

Notice that we have $\bar{\sigma}_\omega(\alpha) = \overline{\tilde{p}(\alpha;\omega)} + \mathcal{O}(h\langle|\alpha|\rangle)$. Hence, the operator $\widetilde{P}_h(\omega)^*$ satisfies Proposition 25 with $\tilde{p}$ replaced by $\bar{\tilde{p}}$. In order to introduce the symbol f, one may use [loc. cit., Proposition 2.12]. Consequently, we can use Lemmas 26 and 28, as in the proofs of Lemmas 30 and 31, to get:

Lemma 32. *There is $C > 0$ such that for h small enough and every $\omega \in V$ and $u \in D_{-k}^*$ we have*

$$\|u\|_{\mathcal{H}_\Lambda^{-k+1}} \le C \|(\widetilde{P}_h(\omega) + A)^* u\|_{\mathcal{H}_\Lambda^{-k}}.$$

In this statement, $(\widetilde{P}_h(\omega) + A)^* = \widetilde{P}_h(\omega)^* + A$ is the formal adjoint of $\widetilde{P}_h(\omega) + A$ for the pairing (48).

Lemma 33. *There is $C > 0$ such that for h small enough and every $u \in D_{-k}^*$ we have*

$$\|u\|_{\mathcal{H}_\Lambda^{-k+1}} \le C \|\widetilde{P}_h(i\nu)^* u\|_{\mathcal{H}_\Lambda^{-k}}.$$

Remark 34. Here, we used (48) rather than the L^2 pairing to describe the dual of $\mathcal{H}_\Lambda^k$ because this identification makes A self-adjoint, so that we can reuse directly the estimates from Lemmas 26 and 28. We expect however that the L^2 pairing studied in Section 5.2 would allow to get similar estimates that we could also use in the proofs below.

From Lemmas 30, 31, 32 and 33, we deduce:

Proposition 35. *There is $C > 0$ such that for h small enough and $\omega \in V$ the operators $\widetilde{P}_h(\omega) + A$ and $\widetilde{P}_h(i\nu)$ are invertible as operators from D_k to $\mathcal{H}_\Lambda^{k-1}$. Moreover, the operator norms of their inverses is bounded by C.*

Proof. From Lemma 30, we find that $\widetilde{P}_h(\omega) + A$ is injective on D_k and that its image is closed in $\mathcal{H}_\Lambda^{k-1}$.

Let us consider and element $v \in \mathcal{H}_\Lambda^{-k+1}$ such that $\langle u, v\rangle_\Lambda = 0$ for every $u \in \mathcal{H}_\Lambda^{k-1}$ in the image of $\widetilde{P}_h(\omega) + A$. In particular, if $u \in \mathcal{H}_\Lambda^\infty$, we have $\langle (\widetilde{P}_h(\omega) + A)u, v\rangle = 0$. Notice that $\mathcal{H}_\Lambda^\infty$ is dense in $\mathcal{H}_\Lambda^{-k+1}$ (for instance because it contains all real-analytic functions due to Proposition 20, and they form a dense subset of $\mathcal{H}_\Lambda^{-k+1}$ according to [Bonthonneau and Jézéquel 2020, Corollary 2.3], one can also work as in Lemma 29). Consequently, we have

$$\langle (\widetilde{P}_h(\omega)+A)u, v\rangle_\Lambda = \langle u, (\widetilde{P}_h(\omega)+A)^* v\rangle_\Lambda$$

for every $u \in \mathcal{H}_\Lambda^\infty$, since this equality holds when $v \in \mathcal{H}_\Lambda^\infty$. Hence, we have $\langle u, (\widetilde{P}_h(\omega)+A)^* v\rangle = 0$ for every $u \in \mathcal{H}_\Lambda^\infty$, and thus $(\widetilde{P}_h(\omega)+A)^* v = 0$. It follows from Lemma 32 that $v = 0$.

We just proved that the image of $\widetilde{P}_h(\omega)+A$ is dense in $\mathcal{H}_\Lambda^{k-1}$, and thus $\widetilde{P}_h(\omega)+A$ is invertible. The estimate on the operator norm of the inverse immediately follows from Lemma 30.

The argument to invert $\widetilde{P}_h(i\nu)$ is the same using Lemmas 31 and 33 instead of Lemmas 30 and 32. □

The analytic Fredholm theory then implies that:

Proposition 36. *Assume that h is small enough. For every $\omega \in V$, the operator $\widetilde{P}_h(\omega) : D_k \to \mathcal{H}_\Lambda^{k-1}$ is Fredholm of index 0. Moreover, the operator $\widetilde{P}_h(\omega) : D_k \to \mathcal{H}_\Lambda^{k-1}$ has a meromorphic inverse $\omega \mapsto \widetilde{P}_h(\omega)^{-1}$ with poles of finite rank on V.*

Proof. From [loc. cit., Proposition 21.3] or Lemma 38 below, we find that A is a compact operator from D_k to $\mathcal{H}_\Lambda^{k-1}$. Hence, it follows from Proposition 35 that $\widetilde{P}_h(\omega) : D_k \to \mathcal{H}_\Lambda^{k-1}$ is Fredholm for $\omega \in V$.

Since $\widetilde{P}_h(\omega) - \widetilde{P}_h(0)$ is a holomorphic family of bounded operators from $\mathcal{H}_\Lambda^k$ to $\mathcal{H}_\Lambda^{k-1}$, we see that $\widetilde{P}_h(\omega)$ is a holomorphic family of operators from D_k to $\mathcal{H}_\Lambda^{k-1}$, for ω in V. Since this operator is invertible for $\omega = i\nu$ and V is connected, we find that the index of $\widetilde{P}_h(\omega)$ is 0. Finally, the analytic Fredholm theorem [Dyatlov and Zworski 2019, Theorem C.8] implies the existence of the meromorphic inverse $\omega \mapsto \widetilde{P}_h(\omega)^{-1}$, with poles of finite rank. □

6.5. *Counting resonances.* We will now use the functional analytic framework from Section 6.4 to prove the point (vi) in Proposition 5. The bounds on the number of resonances from Theorems 2 and 3 ultimately come from the following lemma.

Lemma 37. *Recall that $\delta \in \,]0, \kappa[$. There is $C > 0$ such that, for every h small enough, the number of ω's in the disk of center 0 and radius δ such that $\widetilde{P}_h(\omega) : D_k \to \mathcal{H}_\Lambda^{k-1}$ is not invertible (counted with null multiplicity) is less than Ch^{-n}.*

Before being able to prove Lemma 37, we need to establish a bound on the trace class operator norm of A, which is defined by (47).

Lemma 38. *The operator $A : D_k \to \mathcal{H}_\Lambda^{k-1}$ is trace class, with trace class norm $\mathcal{O}(h^{-n})$.*

Proof. We only need to prove that the operator $\widetilde{A} = B_\Lambda a B_\Lambda$ is trace class from $L_k^2(\Lambda)$ to $L_{k-1}^2(\Lambda)$, with trace class norm $\mathcal{O}(h^{-n})$.

For every $N > 0$, introduce the operator $\Box_N := B_\Lambda \langle |\alpha|\rangle^N B_\Lambda$. Using [Bonthonneau and Jézéquel 2020, Proposition 2.12] to make a parametrix construction, we see that there is a symbol σ_N of order $-N$ on Λ such that $\Box_N B_\Lambda \sigma_N B_\Lambda - B_\Lambda$ and $B_\Lambda \sigma_N B_\Lambda \Box_N - B_\Lambda$ are negligible operators, in particular they are $\mathcal{O}(h^\infty)$ as operators from $L_{s_1}^2(\Lambda) \to L_{s_2}^2(\Lambda)$ for any $s_1, s_2 \in \mathbb{R}$. Hence, for h small enough, we get an inverse $\Box_N^{-1} : \mathcal{H}_{\Lambda,\mathrm{FBI}}^0 \to \mathcal{H}_{\Lambda,\mathrm{FBI}}^N$ for $\Box_N$, which is bounded uniformly in h and satisfies the equation

$$\Box_N^{-1} B_\Lambda = B_\Lambda \sigma_N B_\Lambda + B_\Lambda \sigma_N B_\Lambda (B_\Lambda - \Box_N B_\Lambda \sigma_N B_\Lambda) + (B_\Lambda - B_\Lambda \sigma_N B_\Lambda \Box_N)\Box_N^{-1}(B_\Lambda - \Box_N B_\Lambda \sigma_N B_\Lambda).$$

Thus, we see that $\square_N^{-1} B_\Lambda$ is equal to $B_\Lambda \sigma_N B_\Lambda$ up to a negligible operator. Let us recall that $\mathcal{H}^k_{\Lambda,\mathrm{FBI}}$ is the image of $\mathcal{H}^k_\Lambda$ by T_Λ (which is also the image of $L^2_k(\Lambda)$ by B_Λ).

Fix $N > n$. Notice that $\widetilde{A}$ is bounded, uniformly in h, as an operator from $L^2_k(\Lambda)$ to $L^2_{k+N}(\Lambda)$ (since B_Λ is bounded on $L^2_k(\Lambda)$ and on $L^2_{k+N}(\Lambda)$). We can then write

$$\widetilde{A} = \iota \square_k^{-1} \square_N^{-1} B_\Lambda \square_N \square_k \widetilde{A}. \tag{49}$$

On the left-hand side, $\widetilde{A}$ is seen as an operator from $L^2_k(\Lambda)$ to $L^2_{k-1}(\Lambda)$. On the right-hand side, $\widetilde{A}$ sends $L^2_k(\Lambda)$ into $\mathcal{H}^{k+N}_{\Lambda,\mathrm{FBI}}$, the operator $\square_k$ sends $\mathcal{H}^{k+N}_{\Lambda,\mathrm{FBI}}$ into $\mathcal{H}^{N}_{\Lambda,\mathrm{FBI}}$, the operator $\square_N$ sends $\mathcal{H}^{N}_{\Lambda,\mathrm{FBI}}$ into $\mathcal{H}^{0}_{\Lambda,\mathrm{FBI}}$, the operator $\square_N^{-1} B_\Lambda$ sends $\mathcal{H}^{0}_{\Lambda,\mathrm{FBI}}$ into $\mathcal{H}^{0}_{\Lambda,\mathrm{FBI}}$, the operator $\square_k^{-1}$ sends $\mathcal{H}^{0}_{\Lambda,\mathrm{FBI}}$ into $\mathcal{H}^{k}_{\Lambda,\mathrm{FBI}}$ and ι is the inclusion of $\mathcal{H}^{k}_{\Lambda,\mathrm{FBI}}$ into $L^2_{k-1}(\Lambda)$. With these mapping properties, the operators $\widetilde{A}$, $\square_k$, $\square_N$, $\square_k^{-1}$ and ι on the right-hand side of (49) are bounded uniformly in h. From [Bonthonneau and Jézéquel 2020, Lemma 2.25], we see that $\square_N^{-1} B_\Lambda$ is trace class on $L^2_0(\Lambda)$ (since $B_\Lambda \sigma_N B_\Lambda$ is). Moreover, its trace is given by the integral of its kernel on the diagonal, which is $\mathcal{O}(h^{-n})$. Indeed, $\square_N^{-1} B_\Lambda$ is a "complex FIO associated to Δ_Λ of order $-N$" in the sense of [loc. cit., Definition 2.5] as a consequence of [loc. cit., Lemmas 2.16 and 2.23]. Since $\mathcal{H}^{0}_{\Lambda,\mathrm{FBI}}$ is a closed subset of $L^2_0(\Lambda)$, we see that $\square_N^{-1} B_\Lambda$ is also a trace class operator from $\mathcal{H}^{0}_{\Lambda,\mathrm{FBI}}$ to itself, with the same trace. Moreover, $\square_N^{-1} B_\Lambda$ is a positive self adjoint operator on $\mathcal{H}^{0}_{\Lambda,\mathrm{FBI}}$ with h small enough (because $\langle|\alpha|\rangle^N$ is positive), so that its trace class norm coincides with its trace. This ends the proof of the lemma. $\square$

Proof of Lemma 37. For $\omega \in V$, let us introduce the spectral determinant

$$f_h(\omega) = \det(I - (\widetilde{P}_h(\omega) + A)^{-1} A).$$

Since $\widetilde{P}_h(\omega) - \widetilde{P}_h(0)$ is a holomorphic family of bounded operators from $\mathcal{H}^k_\Lambda$ to $\mathcal{H}^{k-1}_\Lambda$, we see that $\widetilde{P}_h(\omega) + A$ is a holomorphic family of operators from D_k to $\mathcal{H}^{k-1}_\Lambda$. From Proposition 35, the operators $(\widetilde{P}_h(\omega) + A)^{-1} : \mathcal{H}^{k-1}_\Lambda \to D_k$ are bounded uniformly in $\omega \in V$, and thus it is a holomorphic family of operators in V. Consequently, the spectral determinant $f_h(\omega)$ is holomorphic in V.

The logarithmic derivative of f_h is given by

$$\begin{aligned}\frac{f_h'(\omega)}{f_h(\omega)} &= \operatorname{tr}\big((I - (\widetilde{P}_h(\omega) + A)^{-1} A)^{-1} (\widetilde{P}_h(\omega) + A)^{-1} \partial_\omega \widetilde{P}_h(\omega) (\widetilde{P}_h(\omega) + A)^{-1} A\big) \\ &= \operatorname{tr}\big((\widetilde{P}_h(\omega) + A)^{-1} A (I - (\widetilde{P}_h(\omega) + A)^{-1} A)^{-1} (\widetilde{P}_h(\omega) + A)^{-1} \partial_\omega \widetilde{P}_h(\omega)\big).\end{aligned}$$

Let us then write

$$\begin{aligned}&(\widetilde{P}_h(\omega) + A)^{-1} A (I - (\widetilde{P}_h(\omega) + A)^{-1} A)^{-1} (\widetilde{P}_h(\omega) + A)^{-1} \partial_\omega \widetilde{P}_h(\omega) \\ &\qquad = ((I - (\widetilde{P}_h(\omega) + A)^{-1} A)^{-1} - I)(\widetilde{P}_h(\omega) + A)^{-1} \partial_\omega \widetilde{P}_h(\omega) \\ &\qquad = \widetilde{P}_h(\omega)^{-1} \partial_\omega \widetilde{P}_h(\omega) - (\widetilde{P}_h(\omega) + A)^{-1} \partial_\omega \widetilde{P}_h(\omega)\end{aligned} \tag{50}$$

Hence, if ω_0 is in V, the residue of the family of operators (50) at ω_0 is the same as the residue of the family of operators $\omega \mapsto \widetilde{P}_h(\omega)^{-1} \partial_\omega \widetilde{P}_h(\omega)$. Consequently, the order of annulation of f_h at ω_0 coincides with the null multiplicity of $\omega \mapsto \widetilde{P}_h(\omega)$ at ω_0.

Since V is open, there is $\eta > 0$ such that the closed disk of center $i\nu$ and radius $\nu+\delta+2\eta$ is contained in V. Since the poles of $\widetilde{P}_h(\omega)^{-1}$ are isolated, we may choose $0 \leq \eta' \leq \eta$ such that there is no poles of $\widetilde{P}_h(\omega)^{-1}$ on the circle of center $i\nu$ and radius $\nu+\delta+\eta+\eta'$. For $r \geq 0$, let $n_h(r)$ denote the number of zeros of f_h in the disk of center $i\nu$ and radius r. Notice that

$$n_h(\nu+\delta) \leq \frac{\nu+\delta+\eta}{\eta} \int_{\nu+\delta}^{\nu+\delta+\eta} \frac{n_h(r)}{r}\, \mathrm{d}r \leq \frac{\nu+\delta+\eta}{\eta} \int_0^{\nu+\delta+\eta+\eta'} \frac{n_h(r)}{r}\, \mathrm{d}r. \tag{51}$$

From Jensen's formula, we know that

$$\int_0^{\nu+\delta+\eta+\eta'} \frac{n_h(r)}{r}\, \mathrm{d}r \leq -\log|f_h(i\nu)| + \sup_{|\omega-i\nu|=\nu+\delta+\eta+\eta'} \log|f_h(\omega)|. \tag{52}$$

From Proposition 35 and Lemma 38, we know that the trace class norm of the operator $(\widetilde{P}_h(\omega)+A)^{-1}A$ is $\mathcal{O}(h^{-n})$ uniformly in h and in ω on the circle of center $i\nu$ and radius $\nu+\delta+\eta+\eta'$. Then, from [Gohberg et al. 2000, Theorem IV.5.2], we find that

$$\sup_{|\omega-i\nu|=\nu+\delta+\eta+\eta'} \log|f_h(\omega)| \leq Ch^{-n}, \tag{53}$$

for some $C > 0$ and h small enough. In order to estimate $|f_h(i\nu)|$ from below, let us write

$$(I-(\widetilde{P}_h(i\nu)+A)^{-1}A)^{-1} = I + \widetilde{P}_h(i\nu)^{-1}A.$$

From Proposition 35 and Lemma 38, we see that the trace class operator norm of $(I-(\widetilde{P}_h(i\nu)+A)^{-1})^{-1}-I$ is $\mathcal{O}(h^{-n})$. Since

$$f_h(i\nu)^{-1} = \det((I-(\widetilde{P}_h(i\nu)+A)^{-1})^{-1}),$$

we find using [Gohberg et al. 2000, Theorem IV.5.2] again that

$$-\log|f_h(i\nu)| \leq Ch^{-n} \tag{54}$$

for some $C > 0$ and h small enough. From (51), (52), (53) and (54), we find that $n_h(\nu+\delta)$ is $\mathcal{O}(h^{-n})$. The result follows since the disk of center 0 and radius δ is contained in the disk of center $i\nu$ and radius $\nu+\delta$. □

6.6. *Summary.* Let us put together the definitions from Sections 6.1 and 6.2 and the results from Sections 6.4 and 6.5 to check that Proposition 5 holds.

Proof of Proposition 5. We just need to collect facts that we already proved. We recall that the modification $P_h(\omega)$ of $\mathcal{P}_h(\omega)$ is given by (39). Recalling (41), we let $\mathcal{H}_1 = \{u \in \mathcal{D}'(X) : e^{\psi/h}u \in D_k\} = \{u \in \mathcal{F}_k : P_h(0)u \in \mathcal{F}_{k-1}\}$ and $\mathcal{H}_2 = \mathcal{F}_{k-1}$ (for any value of $k \in \mathbb{R}$).

The inclusions $C^\infty(X) \subseteq \mathcal{H}_j \subseteq \mathcal{D}'(X)$ for $j = 1, 2$ are given by Proposition 21. The fact that the elements of $\mathcal{H}_j$ are continuous near ∂Y follows from Proposition 24.

All the properties needed for $P_h(\omega) : \mathcal{H}_1 \to \mathcal{H}_2$ follow from the same properties for $\widetilde{P}_h(\omega) : D_k \to \mathcal{H}_\Lambda^{k-1}$. The holomorphic dependence on ω follows from the remark after Proposition 25. The invertibility for $\omega = i\nu$ with a $\nu > 0$ is given by Proposition 35. Point (v) follows from Proposition 36.

Finally, the counting bound (vi) is given by Lemma 37. □

Acknowledgement

I would like to thank Maciej Zworski and Semyon Dyatlov for useful discussions about resonances for asymptotically hyperbolic manifolds. This work was done while the author was working at Massachusetts Institute of Technology. The author benefits from the support of the French government "Investissements d'Avenir" program integrated to France 2030, under the reference ANR-11-LABX-0020-01.

References

[Bonthonneau and Jézéquel 2020] Y. G. Bonthonneau and M. Jézéquel, "FBI transform in Gevrey classes and Anosov flows", preprint, 2020. arXiv 2001.03610

[Borthwick 2008] D. Borthwick, "Upper and lower bounds on resonances for manifolds hyperbolic near infinity", *Comm. Partial Differential Equations* **33**:7-9 (2008), 1507–1539. MR Zbl

[Borthwick and Philipp 2014] D. Borthwick and P. Philipp, "Resonance asymptotics for asymptotically hyperbolic manifolds with warped-product ends", *Asymptot. Anal.* **90**:3-4 (2014), 281–323. MR Zbl

[Borthwick et al. 2011] D. Borthwick, T. J. Christiansen, P. D. Hislop, and P. A. Perry, "Resonances for manifolds hyperbolic near infinity: optimal lower bounds on order of growth", *Int. Math. Res. Not.* **2011**:19 (2011), 4431–4470. MR Zbl

[Cuevas and Vodev 2003] C. Cuevas and G. Vodev, "Sharp bounds on the number of resonances for conformally compact manifolds with constant negative curvature near infinity", *Comm. Partial Differential Equations* **28**:9-10 (2003), 1685–1704. MR Zbl

[Dafermos and Rodnianski 2013] M. Dafermos and I. Rodnianski, "Lectures on black holes and linear waves", pp. 97–205 in *Evolution equations* (Zürich, 2008), edited by D. Ellwood et al., Clay Math. Proc. **17**, Amer. Math. Soc., Providence, RI, 2013. MR Zbl

[Dyatlov and Zworski 2019] S. Dyatlov and M. Zworski, *Mathematical theory of scattering resonances*, Grad. Stud. in Math. **200**, Amer. Math. Soc., Providence, RI, 2019. MR Zbl

[Forstnerič 2017] F. Forstnerič, *Stein manifolds and holomorphic mappings: the homotopy principle in complex analysis*, 2nd ed., Ergebnisse der Math. (3) **56**, Springer, 2017. MR Zbl

[Fried 1995] D. Fried, "Meromorphic zeta functions for analytic flows", *Comm. Math. Phys.* **174**:1 (1995), 161–190. MR Zbl

[Froese and Hislop 2000] R. G. Froese and P. D. Hislop, "On the distribution of resonances for some asymptotically hyperbolic manifolds", exposé VII in *Journées équations aux dérivées partielles* (La Chapelle sur Erdre, France, 2000), Univ. Nantes, 2000. MR Zbl

[Galkowski and Zworski 2021] J. Galkowski and M. Zworski, "Analytic hypoellipticity of Keldysh operators", *Proc. Lond. Math. Soc.* (3) **123**:5 (2021), 498–516. MR Zbl

[Galkowski and Zworski 2022] J. Galkowski and M. Zworski, "Viscosity limits for zeroth-order pseudodifferential operators", *Comm. Pure Appl. Math.* **75**:8 (2022), 1798–1869. MR Zbl

[Gohberg et al. 2000] I. Gohberg, S. Goldberg, and N. Krupnik, *Traces and determinants of linear operators*, Operator Theory Adv. Appl. **116**, Birkhäuser, Basel, 2000. MR Zbl

[Guedes-Bonthonneau et al. 2024] Y. Guedes-Bonthonneau, C. Guillarmou, and M. Jézéquel, "Scattering rigidity for analytic metrics", *Camb. J. Math.* **12**:1 (2024), 165–222. MR

[Guillarmou 2005] C. Guillarmou, "Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds", *Duke Math. J.* **129**:1 (2005), 1–37. MR Zbl

[Guillemin and Stenzel 1991] V. Guillemin and M. Stenzel, "Grauert tubes and the homogeneous Monge–Ampère equation", *J. Differential Geom.* **34**:2 (1991), 561–570. MR Zbl

[Guillemin and Stenzel 1992] V. Guillemin and M. Stenzel, "Grauert tubes and the homogeneous Monge–Ampère equation, II", *J. Differential Geom.* **35**:3 (1992), 627–641. MR Zbl

[Guillopé and Zworski 1995a] L. Guillopé and M. Zworski, "Polynomial bounds on the number of resonances for some complete spaces of constant negative curvature near infinity", *Asymptot. Anal.* **11**:1 (1995), 1–22. MR Zbl

[Guillopé and Zworski 1995b] L. Guillopé and M. Zworski, "Upper bounds on the number of resonances for non-compact Riemann surfaces", *J. Funct. Anal.* **129**:2 (1995), 364–389. MR Zbl

[Guillopé and Zworski 1997] L. Guillopé and M. Zworski, "Scattering asymptotics for Riemann surfaces", *Ann. of Math.* (2) **145**:3 (1997), 597–660. MR Zbl

[Helffer and Sjöstrand 1986] B. Helffer and J. Sjöstrand, *Résonances en limite semi-classique*, Mém. Soc. Math. France (N.S.) **24-25**, Soc. Math. France, Paris, 1986. MR Zbl

[Hintz and Xie 2022] P. Hintz and Y. Xie, "Quasinormal modes of small Schwarzschild–de Sitter black holes", *J. Math. Phys.* **63**:1 (2022), art. id. 011509. MR Zbl

[Hörmander 1983] L. Hörmander, *The analysis of linear partial differential operators, I: Distribution theory and Fourier analysis*, Grundl. Math. Wissen. **256**, Springer, 1983. MR Zbl

[Mazzeo and Melrose 1987] R. R. Mazzeo and R. B. Melrose, "Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature", *J. Funct. Anal.* **75**:2 (1987), 260–310. MR Zbl

[Melin and Sjöstrand 1974] A. Melin and J. Sjöstrand, "Fourier integral operators with complex-valued phase functions", pp. 120–223 in *Fourier integral operators and partial differential equations* (Nice, 1974), edited by J. Chazarain, Lecture Notes in Math. **459**, Springer, 1974. MR Zbl

[Melrose 1984] R. Melrose, "Polynomial bound on the distribution of poles in scattering by an obstacle", art. id. 3 in *Journées équations aux dérivées partielles* (Saint-Jean-de-Monts, France, 1984), Soc. Math. France, Paris, 1984. Zbl

[Morrey 1958] C. B. Morrey, Jr., "The analytic embedding of abstract real-analytic manifolds", *Ann. of Math.* (2) **68** (1958), 159–201. MR Zbl

[Regge 1958] T. Regge, "Analytic properties of the scattering matrix", *Nuovo Cimento* (10) **8**:5 (1958), 671–679. MR Zbl

[Rugh 1992] H. H. Rugh, "The correlation spectrum for hyperbolic analytic maps", *Nonlinearity* **5**:6 (1992), 1237–1263. MR Zbl

[Rugh 1996] H. H. Rugh, "Generalized Fredholm determinants and Selberg zeta functions for Axiom A dynamical systems", *Ergodic Theory Dynam. Systems* **16**:4 (1996), 805–819. MR Zbl

[Sá Barreto and Zworski 1997] A. Sá Barreto and M. Zworski, "Distribution of resonances for spherical black holes", *Math. Res. Lett.* **4**:1 (1997), 103–121. MR Zbl

[Sjöstrand 1982] J. Sjöstrand, "Singularités analytiques microlocales", pp. 1–166 in Astérisque **95**, Soc. Math. France, Paris, 1982. MR Zbl

[Sjöstrand 1996] J. Sjöstrand, "Density of resonances for strictly convex analytic obstacles", *Canad. J. Math.* **48**:2 (1996), 397–447. MR Zbl

[Sjöstrand and Zworski 1991] J. Sjöstrand and M. Zworski, "Complex scaling and the distribution of scattering poles", *J. Amer. Math. Soc.* **4**:4 (1991), 729–769. MR Zbl

[Taylor 2011] M. E. Taylor, *Partial differential equations, I: Basic theory*, 2nd ed., Appl. Math. Sci. **115**, Springer, 2011. MR Zbl

[Vasy 2013a] A. Vasy, "Microlocal analysis of asymptotically hyperbolic and Kerr–de Sitter spaces", *Invent. Math.* **194**:2 (2013), 381–513. MR Zbl

[Vasy 2013b] A. Vasy, "Microlocal analysis of asymptotically hyperbolic spaces and high-energy resolvent estimates", pp. 487–528 in *Inverse problems and applications: inside out, II*, edited by G. Uhlmann, Math. Sci. Res. Inst. Publ. **60**, Cambridge Univ. Press, 2013. MR Zbl

[Vodev 1992] G. Vodev, "On the distribution of scattering poles for perturbations of the Laplacian", *Ann. Inst. Fourier* (*Grenoble*) **42**:3 (1992), 625–635. MR Zbl

[Wang 2019] Y. Wang, "Distribution of resonances for analytic asymptotically hyperbolic spaces", *J. Spectr. Theory* **9**:2 (2019), 651–675. MR Zbl

[Zuily 2017] C. Zuily, "Real analyticity of radiation patterns on asymptotically hyperbolic manifolds", *Appl. Math. Res. Express.* **2017**:2 (2017), 386–401. MR Zbl

[Zworski 1989] M. Zworski, "Sharp polynomial bounds on the number of scattering poles", *Duke Math. J.* **59**:2 (1989), 311–323. MR Zbl

[Zworski 1997] M. Zworski, "Poisson formulae for resonances", exposé XIII in *Séminaire sur les équations aux dérivées partielles*, 1996/1997, École Polytech., Palaiseau, France, 1997. MR Zbl

[Zworski 2012] M. Zworski, *Semiclassical analysis*, Grad. Stud. in Math. **138**, Amer. Math. Soc., Providence, RI, 2012. MR Zbl

[Zworski 2016] M. Zworski, "Resonances for asymptotically hyperbolic manifolds: Vasy's method revisited", *J. Spectr. Theory* **6**:4 (2016), 1087–1114. MR Zbl

Received 29 Dec 2022. Revised 26 Apr 2023. Accepted 18 Jul 2023.

MALO JÉZÉQUEL: malo.jezequel@math.cnrs.fr
CNRS, Univ. Brest, UMR 6205, Laboratoire de Mathématiques de Bretagne Atlantique, Brest, France

Guidelines for Authors

Authors may submit manuscripts in PDF format on-line at the Submission page at msp.org/apde.

Originality. Submission of a manuscript acknowledges that the manuscript is original and and is not, in whole or in part, published or under consideration for publication elsewhere. It is understood also that the manuscript will not be submitted elsewhere while under consideration for publication in this journal.

Language. Articles in APDE are usually in English, but articles written in other languages are welcome.

Required items. A brief abstract of about 150 words or less must be included. It should be self-contained and not make any reference to the bibliography. If the article is not in English, two versions of the abstract must be included, one in the language of the article and one in English. Also required are keywords and subject classifications for the article, and, for each author, postal address, affiliation (if appropriate), and email address.

Format. Authors are encouraged to use LaTeX but submissions in other varieties of TeX, and exceptionally in other formats, are acceptable. Initial uploads should be in PDF format; after the refereeing process we will ask you to submit all source material.

References. Bibliographical references should be complete, including article titles and page ranges. All references in the bibliography should be cited in the text. The use of BibTeX is preferred but not required. Tags will be converted to the house format, however, for submission you may use the format of your choice. Links will be provided to all literature with known web locations and authors are encouraged to provide their own links in addition to those supplied in the editorial process.

Figures. Figures must be of publication quality. After acceptance, you will need to submit the original source files in vector graphics format for all diagrams in your manuscript: vector EPS or vector PDF files are the most useful.

Most drawing and graphing packages (Mathematica, Adobe Illustrator, Corel Draw, MATLAB, etc.) allow the user to save files in one of these formats. Make sure that what you are saving is vector graphics and not a bitmap. If you need help, please write to graphics@msp.org with details about how your graphics were generated.

White space. Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Proofs. Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

ANALYSIS & PDE

Volume 17 No. 10 2024