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SMOOTH EXTENSIONS FOR INERTIAL MANIFOLDS OF SEMILINEAR PARABOLIC EQUATIONS

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The paper is devoted to a comprehensive study of smoothness of inertial manifolds (IMs) for abstract semilinear parabolic problems. It is well known that in general we cannot expect more than $C^{1,\varepsilon}$-regularity for such manifolds (for some positive, but small ε). Nevertheless, as shown in the paper, under natural assumptions, the obstacles to the existence of a C^n-smooth inertial manifold (where $n \in \mathbb{N}$ is any given number) can be removed by increasing the dimension and by modifying properly the nonlinearity outside of the global attractor (or even outside the $C^{1,\varepsilon}$-smooth IM of a minimal dimension). The proof is strongly based on the Whitney extension theorem.

1. Introduction

It is believed that in many cases the long-time behaviour of infinite-dimensional dissipative dynamical systems generated by evolutionary PDEs (at least in bounded domains) can be effectively described by finitely many parameters (the so-called order parameters in the terminology of I. Prigogine) which obey a system of ODEs. This system of ODEs (if it exists) is usually referred as an inertial form (IF) of the considered PDE; see [Hale 1988; Robinson 2001; 2011; Temam 1988; Zelik 2014] and references therein for more details. However, despite the fundamental significance of this reduction from both theoretical and applied points of view and big interest during the last 50 years, the nature of such a reduction and its rigorous justification remains a mystery.

Indeed, it is well understood now that the key question of the theory is how smooth the desired IF can/should be. For instance, in the case of *Hölder* continuous IFs, there is a highly developed machinery for constructing them based on the theory of global attractors and the Mañé projection theorem. We recall that, by definition, a global attractor is a compact invariant set in the phase space of the dissipative system

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considered which attracts as time goes to infinity the images of bounded sets under the evolutionary semigroup related to the considered problem. Thus, on the one hand, a global attractor (if it exists) contains all of the nontrivial dynamics and, on the other hand, it is usually essentially "smaller" than the initial phase space and this second property allows us to speak about the reduction of degrees of freedom in the limit dynamics. In particular, one of the main results of the attractors theory tells us that, under relatively weak assumptions on a dissipative PDE (in a bounded domain), the global attractor exists and has finite Hausdorff and fractal dimensions. In turn, due to the Mañé projection theorem, this finite-dimensionality guarantees that this attractor can be projected one-to-one to a generic finite-dimensional plane of the phase space and that the inverse map is Hölder continuous. Finally, this scheme gives us an IF with Hölder continuous vector field defined on some compact set of $\mathbb{R}^N$ which is treated as a rigorous justification of the above-mentioned finite-dimensional reduction. This approach works, for instance, for 2-dimensional Navier–Stokes equations, reaction-diffusion systems, pattern formation equations, damped wave equations, etc.; see [Babin and Vishik 1992; Ben-Artzi et al. 1993; Chepyzhov and Vishik 2002; Hale 1988; Henry 1981; Hunt and Kaloshin 1999; Miranville and Zelik 2008; Robinson 2011; Sell and You 2002; Temam 1988].

However, the above-described scheme has a very essential intrinsic drawback which prevents us from treating it as a satisfactory solution of the finite-dimensional reduction problem. Namely, the vector field in the IF thus constructed is Hölder continuous *only* and there is no way in general to get even its Lipschitz continuity. As a result, we may lose the uniqueness of solutions for the obtained IF and have to use the initial infinite-dimensional system at least in order to select the correct solution of the reduced IF. Another drawback is that the Mañé projection theorem is not constructive, so it is not clear how to choose this "generic" plane for projection in applications; in addition, the IF constructed in such a way is defined only on a complicated compact set (the image of the attractor under the projection) and it is not clear how to extend it on the whole $\mathbb{R}^N$ preserving the dynamics (surprisingly, this is also a deep open problem; a partial solution of it is given in [Robinson 1999]).

It is also worth noting that the restriction for IF to be only Hölder continuous is far from being just a technical problem here. As relatively simple counterexamples show (see [Eden et al. 2013; Kostianko and Zelik 2018; Mallet-Paret et al. 1993; Romanov 2000; Zelik 2014]) the fractal dimension of the global attractor may be finite and not big, but the attractor cannot be embedded into any finite-dimensional Lipschitz (or even log-Lipschitz) finite-dimensional submanifold of the phase space. Even more importantly, the dynamics on this attractor does not look finite-dimensional at all (despite the existence of a *Hölder continuous* (with the Hölder exponent arbitrarily close to 1) IF provided by the Mañé projection theorem). For instance, it may contain limit cycles with superexponential rate of attraction, decaying travelling waves in Fourier space and other phenomena which are impossible in the classical dynamics generated by smooth ODEs. These examples suggest that, in contradiction to the widespread paradigm, Hölder continuous IF is probably not an appropriate tool for distinguishing between finite and infinite-dimensional limit behaviour and, as a result, fractal-dimension is not so good for estimating the number of degrees of freedom for the reduced dynamics; see [Eden et al. 2013; Kostianko and Zelik 2018; Zelik 2014] for more details.

An alternative, probably more transparent approach to the finite-dimensional reduction problem which has been suggested in [Foias et al. 1988] is related to the concept of an inertial manifold (IM). By definition, an IM is a finite-dimensional smooth (at least Lipschitz) invariant submanifold of the phase space which is globally exponentially stable and possesses the so-called exponential tracking property (that is, existence of asymptotic phase). Usually this manifold is $C^{1,\varepsilon}$-smooth for some positive ε and is normally hyperbolic, so the exponential tracking is an immediate corollary of normal hyperbolicity. Then the corresponding IF is just a restriction of the initial PDE to IM and is also $C^{1,\varepsilon}$-smooth. However, being a sort of centre manifold, an IM requires a separation of the dependent variable to the "slow" and "fast" components and this, in turn, leads to extra rather restrictive assumptions which are usually formulated in terms of *spectral gap* conditions. Namely, let us consider the following abstract semilinear parabolic equation in a real Hilbert space H:

$$\partial_t u + Au = F(u), \quad u|_{t=0} = u_0, \tag{1-1}$$

where $A : D(A) \to H$ is a self-adjoint positive operator such that A^{-1} is compact and $F : H \to H$ is a given nonlinearity which is globally Lipschitz in H with Lipschitz constant L. Let also $0 < \lambda_1 \le \lambda_2 \le \cdots$ be the eigenvalues of A enumerated in the nondecreasing order and $\{e_n\}_{n=1}^{\infty}$ be the corresponding eigenvectors. Then, the sufficient condition for the existence of an N-dimensional IM reads

$$\lambda_{N+1} - \lambda_N > 2L. \tag{1-2}$$

If this condition is satisfied, the desired IM $\mathcal{M}_N$ is actually a graph of a Lipschitz function $M_N : H_N \to (H_N)^{\perp}$, where $H_N = \operatorname{span}\{e_1, \dots, e_N\}$ is a spectral subspace spanned by the first N eigenvectors, and the corresponding IF has the form

$$\frac{d}{dt} u_N + Au_N = \boldsymbol{P}_N F(u_N + M_N(u_N)), \quad u_N \in H_N \sim \mathbb{R}^N, \tag{1-3}$$

where $\boldsymbol{P}_N$ is the orthoprojector to H_N; see [Chow et al. 1992; Constantin et al. 1989; Foias et al. 1988; Kokscn 1998; Miklavčič 1991; Romanov 1993; Rosa and Temam 1996; Zelik 2014] and also Section 2 below.

We see that, in contrast to the IF constructed via the Mañé projection theorem, the IF which corresponds to the IM is explicit (uses the spectral projections) and is as smooth as the functions F and M_N are. We mention that although the spectral gap condition (1-2) is rather restrictive (e.g., in the case where A is a Laplacian in a bounded domain, it is satisfied in 1-dimensional case only) and is known to be sharp in the class of abstract semilinear parabolic equations (see [Eden et al. 2013; Miklavčič 1991; Romanov 1993; Zelik 2014] for more details), it can be relaxed for some concrete classes of PDEs. For instance, for scalar 3-dimensional reaction-diffusion equations (using the so-called spatial averaging principle, see [Mallet-Paret and Sell 1988]), for 1-dimensional reaction-diffusion-advection systems (using the proper integral transforms, see [Kostianko and Zelik 2017; 2018]), for 3-dimensional Cahn–Hilliard equations and various modifications of 3-dimensional Navier–Stokes equations (using various modifications of spatial-averaging, see [Gal and Guo 2018; Kostianko 2018; Kostianko and Zelik 2015; Li and Sun 2020]), for the 3-dimensional complex Ginzburg–Landau equation (using the so-called spatiotemporal averaging, see [Kostianko 2020]), etc. Note also that the global Lipschitz continuity assumption for the nonlinearity F is not an essential extra restriction since usually one proves the well-posedness and dissipativity of the PDE under consideration *before* constructing the IM. Cutting off the nonlinearity outside the absorbing ball does not affect the limit dynamics, but reduces the case of locally Lipschitz continuous nonlinearity (satisfying the proper dissipativity restrictions) to the model case where the nonlinearity is globally Lipschitz continuous. Of course, this cut-off procedure is not unique and, as we will see below, choosing it correctly is extremely important in the theory of IMs.

The main aim of the present paper is to study the smoothness of the IFs for semilinear parabolic equations (1-1) in the ideal situation where the nonlinearity F is smooth and the spectral gap condition (1-2) is satisfied. As we have already mentioned, in this case we have a $C^{1,\varepsilon}$-smooth IM $\mathcal{M}_N$ for some $\varepsilon > 0$ and the associated IF (1-3) which is also $C^{1,\varepsilon}$-smooth; see [Zelik 2014]. But, unfortunately, the exponent $\varepsilon > 0$ here is usually very small (depending on the spectral gap) and in a more or less general situation, we cannot expect even the C^2-regularity of the IM. The spectral gap condition for C^2-regular IM is

$$\lambda_{N+1} - 2\lambda_N > 3L \tag{1-4}$$

and such exponentially big spectral gaps are not available if A is a finite-order elliptic operator in a bounded domain. The corresponding counterexamples were given in [Chow et al. 1992]; see also Example 3.11 below. Thus, the existing IM theory does not allow us, even in the ideal situation, to construct more regular than $C^{1,\varepsilon}$ IFs (where $\varepsilon > 0$ is small). This looks to be an essential drawback for at least two reasons:

(1) The lack of regularity prevents us from using higher-order methods for numerical simulations of the reduced IF (as a result, direct simulations for the initial *smooth* PDE using the standard methods may be more effective than simulations based on the reduced nonsmooth ODEs).

(2) $C^{1,\varepsilon}$-regularity is not enough to build up normal forms and/or study the bifurcations properly (for instance, the simplest saddle-node bifurcation requires C^2-smoothness, the Hopf bifurcation needs C^3, etc.; see [Katok and Hasselblatt 1995; Kielhöfer 2004] for more details) and, therefore, we need to return back to the initial PDE to study these bifurcations.

Thus, the natural question,

> "*Is it possible to construct a smooth* (C^k*-smooth for any finite* k) *or to extend the existing* $C^{1,\varepsilon}$*-smooth IF to a more regular one*?"

becomes crucial for the theory of inertial manifolds.

Here we give an affirmative answer to this question under the slightly stronger spectral gap assumption

$$\limsup_{N\to\infty}(\lambda_{N+1} - \lambda_N) = \infty. \tag{1-5}$$

In contrast to (1-4), this assumption does not require exponentially big spectral gaps (and is satisfied for most of the examples where the IMs exist), but guarantees the existence of infinitely many spectral gaps of size larger than $2L$ and, consequently, the existence of an infinite tower of the embedded IMs

$$\mathcal{M}_{N_1} \subset \mathcal{M}_{N_2} \subset \cdots \subset \mathcal{M}_{N_n} \subset \cdots \tag{1-6}$$

and the corresponding IFs

$$\frac{d}{dt}u_{N_n} + Au_{N_n} = \boldsymbol{P}_{N_n} F(u_{N_n} + M_{N_n}(u_{N_n})), \quad u_{N_n} \in H_{N_n}. \tag{1-7}$$

Let $n \in \mathbb{N}$ be given. We say that a $C^{n,\varepsilon}$-smooth submanifold $\widetilde{\mathcal{M}}_{N_n}$ of the phase space H (which is a graph of $C^{n,\varepsilon}$-smooth $\widetilde{M}_{N_n} : H_{N_n} \to (H_{N_n})^{\perp}$) is a $C^{n,\varepsilon}$-smooth extension of the initial IM $\mathcal{M}_{N_1}$ for some $\varepsilon > 0$ if

(1) $\mathcal{M}_{N_1} \subset \widetilde{\mathcal{M}}_{N_n}$,

(2) the manifold $\widetilde{\mathcal{M}}_{N_n}$ is C^1_b-close to the IM $\mathcal{M}_{N_n}$.

Then, the first condition guarantees that the $C^{n,\varepsilon}$-smooth system of ODEs

$$\frac{d}{dt}u_{N_n} + Au_{N_n} = \boldsymbol{P}_{N_n} F(u_{N_n} + \widetilde{M}_{N_n}(u_{N_n})), \quad u_{N_n} \in H_{N_n}, \tag{1-8}$$

will possess the initial IM $\boldsymbol{P}_{N_n}\mathcal{M}_{N_1}$ as an invariant submanifold. The second condition together with the robustness theorem for normally hyperbolic manifolds ensures that this manifold will be globally exponentially stable and normally hyperbolic (in particular, it will possess an exponential tracking property in H_{N_n}). In this case we refer to the system (1-8) as a $C^{n,\varepsilon}$-smooth extension of the corresponding IF (1-3); see Section 3 for more details. Thus, the extended IF is C^n-smooth on the one hand and, on the other hand, its limit dynamics *coincides* with the dynamics of the IF which corresponds to the IM $\mathcal{M}_{N_1}$ and, in turn, coincides with the limit dynamics of the initial abstract parabolic problem (1-1). Note that the manifold $\widetilde{\mathcal{M}}_{N_n}$ is *not necessarily* invariant under the solution semigroup $S(t)$ generated by the initial equation (1-1) and this allows us to overcome the standard obstacles to the smoothness of an invariant manifold (e.g., such as resonances, see Examples 3.11 and 5.6 below).

The main result of the paper is the following theorem which suggests a solution of the smoothness problem for IMs.

Theorem 1.1. *Let the nonlinearity $F \in C^\infty_b(H, H)$ and let the operator A satisfy the spectral gap condition (1-5). Let also $N_1 \in \mathbb{N}$ be the smallest number for which the spectral gap condition (1-2) is satisfied and $\mathcal{M}_{N_1}$ be the corresponding IM. Then, for every $n \in \mathbb{N}$, one can find $\varepsilon = \varepsilon_n > 0$ for which there exists a $C^{n,\varepsilon}$-smooth extension of the IM $\mathcal{M}_{N_1}$ as well as the $C^{n,\varepsilon}$-smooth extension of the corresponding IF in the sense described above.*

The proof of this theorem is given in Section 4 and the Appendix. To construct the desired extension $\widetilde{M}_{N_n}$, we first define it on the manifold $\boldsymbol{P}_{N_n}\mathcal{M}_{N_1}$ only in a natural way $\widetilde{M}_{N_n}(p) = (1 - \boldsymbol{P}_{N_n})M_{N_1}(\boldsymbol{P}_{N_1} p)$. Then, we present an explicit construction of Taylor jets of order n for this function via an inductive procedure; see Section 4. Finally, we check (in the Appendix) the compatibility conditions for the constructed Taylor jets and get the desired extension by the Whitney extension theorem.

Our main result can be reformulated in the following way.

Corollary 1.2. *Let the assumptions of Theorem 1.1 hold. Then, for every $n \in \mathbb{N}$, there exists $\varepsilon = \varepsilon_n > 0$ and a $C^{n-1,\varepsilon}$-smooth "correction" $\widetilde{F}_n(u)$ of the initial nonlinearity F such that*:

(1) *$\widetilde{F}_n(u) = F(u)$ for all $u \in \mathcal{M}_{N_1}$ and $\mathcal{M}_{N_1}$ is an IM for the modified equation*

$$\partial_t u + Au = \widetilde{F}_n(u), \quad u|_{t=0} = u_0, \tag{1-9}$$

as well. In particular, the dynamics of (1-9) *on* $\mathcal{M}_{N_1}$ *coincides with the initial dynamics* (*generated by* (1-1)) *and* $\mathcal{M}_{N_1}$ *possesses an exponential tracking property for solutions of* (1-9).

(2) *The extended manifold* $\widetilde{\mathcal{M}}_{N_n}$ *constructed in Theorem 1.1 is an IM* (*of smoothness* $C^{n,\varepsilon}$) *for the modified equation* (1-9); *see Corollary 5.4 below.*

In this interpretation, the modified nonlinearity $\widetilde{F}_n$ can be considered as a "cut-off" version of the initial function F and the main result claims that all obstacles for the existence of C^n-smooth IM can be removed by increasing the dimension of the IM and using a properly chosen cut-off procedure.

To conclude, we note that the main aim of this paper is to verify the principal possibility to get smooth extensions of an IM rather than to obtain the optimal bounds for the dimensions N_n of the constructed extensions. For this reason, the obtained bounds look far from being optimal, but we believe that they can be essentially improved; see Remark 5.7 for the discussion of this problem.

The paper is organized as follows. In Section 2 we recall the standard facts about smooth functions in Banach spaces, their Taylor jets, direct and converse Taylor theorems and the Whitney extension theorem, which is the main technical tool for what follows. In Section 3 we collect basic facts about the construction of IMs for semilinear parabolic equations via the Perron method and discuss known facts about the smoothness of these IMs. The main result (Theorem 1.1) is presented in Section 4. The proof of it is also given there by modulo of compatibility conditions for Whitney extension theorem which are verified in the Appendix. Finally, the applications of the proved theorem as well as a discussion of open problems and related topics are given in Section 5.

2. Preliminaries, I: Taylor expansions and the Whitney extension theorem

In this section we briefly recall the standard results on Taylor expansions of smooth functions in Banach spaces and the related Whitney extension theorem, as well as prepare some technical tools which will be used later. We start with some basic facts from multilinear algebra; see, e.g., [Hájek and Johanis 2014] for a more detailed exposition. Let X and Y be two normed spaces. For any $n \in \mathbb{N}$, we denote by $\mathcal{L}_s(X^n, Y)$ the space of multilinear continuous symmetric maps from X^n to Y endowed by the standard norm

$$\|M\|_{\mathcal{L}_s(X^n,Y)} := \sup_{\xi_i \in X,\, \xi_i \neq 0} \left\{ \frac{\|M(\xi_1, \dots, \xi_n)\|}{\|\xi_1\| \cdots \|\xi_n\|} \right\}.$$

Every element $M \in \mathcal{L}_s(X^n, Y)$ defines a homogeneous continuous polynomial P_M of order n on X with values in Y via

$$P_M(\xi) := M(\{\xi\}^n), \quad \text{where } \{\xi\}^n := \underbrace{\xi, \dots, \xi}_{n\text{-times}}.$$

Vice versa, the multilinear symmetric map $M = M_P$ can be restored in a unique way if the corresponding homogeneous polynomial is known via the polarization equality:

$$M_P(\xi_1, \dots, \xi_n) = \frac{1}{2^n n!} \sum_{\varepsilon_i = \pm 1, i=1,\dots,n} \varepsilon_1 \cdots \varepsilon_n P\Bigl(a + \sum_{j=1}^n \varepsilon_j \xi_j\Bigr)$$

for all $a, \xi_1, \dots, \xi_n \in X$; see, e.g., [Hájek and Johanis 2014]. Thus, there is a one-to-one correspondence between homogeneous polynomials and multilinear symmetric maps. Moreover, if we introduce the norm

$$\|P\|_{\mathcal{P}_n(X,Y)} := \sup_{\xi\neq 0}\left\{\frac{\|P(\xi)\|}{\|\xi\|^n}\right\}$$

on the space $\mathcal{P}_n(X, Y)$ of n-homogeneous polynomials, this correspondence becomes an isometry. For this reason, we will identify below multilinear forms and the corresponding homogeneous polynomials where this does not lead to misunderstandings. We also mention here the generalization of the Newton binomial formula; namely, for any $P \in \mathcal{P}_n(X, Y)$ and $\xi, \eta \in X$, we have

$$P(\xi+\eta) = \sum_{j=0}^{n} C_n^j P(\{\xi\}^j, \{\eta\}^{n-j}), \quad C_n^j := \frac{n!}{j!(n-j)!}; \tag{2-1}$$

see, e.g., [Hájek and Johanis 2014]. Finally, we denote by $\mathcal{P}^n(X, Y)$ the space of all continuous polynomials of order less than or equal to n on X with values in Y, i.e., $P(\xi) \in \mathcal{P}^n(X, Y)$ if

$$P(\xi) = \sum_{j=0}^{n} \frac{1}{j!} P_j(\xi), \quad P_j(\xi) \in \mathcal{P}_j(X, Y).$$

The following standard result is crucial for our purposes.

Lemma 2.1. *For every $n \in \mathbb{N}$ there exist real numbers $a_{kj} \in \mathbb{R}$, $k, j \in \{0, \dots, n\}$, such that for every $P = \sum_{k=0}^{n} \frac{1}{k!} P_k$, $P_k \in \mathcal{P}_k(X, Y)$ and every $k \in \{0, \dots, n\}$, we have*

$$P_k(\xi) = \sum_{j=0}^{n} a_{kj} P\left(\frac{j}{n}\xi\right) \tag{2-2}$$

and, therefore,

$$\|P_k(\xi)\| \le K_{n,k} \max_{j=0,\dots,n} \left\| P\left(\frac{j}{n}\xi\right)\right\| \tag{2-3}$$

for some constants $K_{n,k}$ which are independent of P.

For the proof of this lemma, see [Hájek and Johanis 2014].

Corollary 2.2. *Let $P(\xi, \delta) \in P^n(X, Y)$ be a family of polynomials of ξ depending on a parameter $\delta \in B$, where B is a set in X containing zero. Assume that*

$$\|P(\xi, \delta)\| \le C(\|\xi\| + \|\delta\|)^{n+\alpha}, \quad \xi \in X,\ \delta \in B, \tag{2-4}$$

for some $\alpha \ge 0$. Then, for any $k \in \{0, \dots, n\}$,

$$\|P_k(\,\cdot\,, \delta)\|_{\mathcal{P}_k(X,Y)} \le C_k \|\delta\|^{n-k+\alpha} \tag{2-5}$$

for some constants C_k depending on C, n and k.

Proof. Indeed, according to (2-3) and (2-4), we have

$$\|P_k(\xi, \delta)\| \le C'(\|\xi\| + \|\delta\|)^{n+\alpha}.$$

Assuming that $\delta \neq 0$ (there is nothing to prove otherwise), replacing ξ by $\|\delta\|\xi$ and using that P_k is homogeneous of order k, we get

$$\|P_k(\xi,\delta)\| \le C'(1+\|\xi\|)^{n+\alpha}\|\delta\|^{n-k+\alpha}.$$

Using once more that P_k is homogeneous of order k in ξ, we finally arrive at

$$\|P_k(\xi,\delta)\| \le C'\|\xi\|^k(1+\|\xi/\|\xi\|\|^{n+\alpha})\|\delta\|^{n-k+\alpha} \le C''\|\xi\|^k\|\delta\|^{n-k+\alpha},$$

which gives (2-5) and finishes the proof. □

Let now $U \subset X$ be an open set and let $F : U \to Y$ be a map. As usual, for any $u \in U$, we denote by $F'(u) \in \mathcal{L}(X, Y)$ the Fréchet derivative of F at u (if it exists). Analogously, for any $n \in \mathbb{N}$, we denote by $F^{(n)}(u) \in \mathcal{L}_s(X^n, Y)$ its n-th Fréchet derivative. The space of all functions $F : U \to Y$ such that $F^{(n)}(u)$ exists and is continuous as a function from U to $\mathcal{L}_s(X^n, Y)$ is denoted by $C^n(U, Y)$. For any $\alpha \in (0, 1]$, we denote by $C^{n,\alpha}(U, Y)$ the space of functions $F \in C^n(U, Y)$ such that $F^{(n)}$ is Hölder continuous with exponent α on U. The action of $F^{(n)}(u)$ to vectors $\xi_1, \dots, \xi_n \in X$ is denoted by $F^{(n)}(u)[\xi_1, \dots, \xi_n]$. The Taylor jet of length $n+1$ of the function F at the point u and vector $\xi \in X$ will be denoted by $J^n_\xi F(u)$:

$$J^n_\xi F(u) := F(u) + \frac{1}{1!}F'(u)\xi + \frac{1}{2!}F''(u)[\xi,\xi] + \cdots + \frac{1}{n!}F^{(n)}(u)[\{\xi\}^n]. \tag{2-6}$$

Obviously, the function $\xi \to J^n_\xi F(u)$ is in $\mathcal{P}^n(X, Y)$ for every $u \in U$. We will also systematically use the truncated Taylor jets

$$j^n_\xi F(u) := \frac{1}{1!}F'(u)\xi + \frac{1}{2!}F''(u)[\xi,\xi] + \cdots + \frac{1}{n!}F^{(n)}(u)[\{\xi\}^n], \tag{2-7}$$

which do not contain zero-order terms.

Theorem 2.3 (direct Taylor theorem). *Let $F \in C^n(U, Y)$ and take $u_1, u_2 \in U$ such that $u_t := tu_1 + (1-t)u_2 \in U$ for all $t \in [0, 1]$. Let also $\xi := u_2 - u_1$. Then*

$$F(u_2) = J^n_\xi F(u_1) + \frac{1}{n!}\int_0^1 (1-s)^{n-1}(F^{(n)}(u_1+s\xi) - F^{(n)}(u_1))\,ds[\{\xi\}^n]. \tag{2-8}$$

In particular, if $F \in C^{n,\alpha}(U, Y)$, then

$$\|F(u_2) - J^n_\xi F(u_1)\| \le C\|\xi\|^{n+\alpha} \tag{2-9}$$

for some positive C.

For the proof of this classical result; see, e.g., [Hájek and Johanis 2014]. We also mention that in terms of truncated jets formula (2-9) reads

$$F(u_2) - F(u_1) = j^n_\xi F(u_1) + O(\|\xi\|^{n+\alpha}), \quad \xi := u_2 - u_1. \tag{2-10}$$

The above theorem can be inverted as follows.

Theorem 2.4 (converse Taylor theorem). *Let F be a function such that, for any $u \in U$, there exists a polynomial $\xi \to P(\xi, u) \in \mathcal{P}^n(X, Y)$ such that, for all $u_1, u_2 \in U$,*

$$\|F(u_2) - P(\xi, u_1)\| \le C\|\xi\|^{n+\alpha}, \quad \xi := u_2 - u_1, \tag{2-11}$$

for some $C > 0$ and $\alpha \in (0, 1]$. Then, $F \in C^n(U, Y)$,

$$P(\xi, u) = J^n_\xi F(u)$$

for all $u \in U$ and $F^{(n)}(u)$ is locally Hölder continuous in U with exponent α. If, in addition, U is convex, then $F \in C^n(U, Y)$ and

$$\|F^{(n)}(u_2) - F^{(n)}(u_1)\| \le C'\|u_2 - u_1\|^\alpha,$$

where C' depends only on n, α and the constant C from (2-11).

For the proof of this theorem, see [Hájek and Johanis 2014].

Keeping in mind the Whitney extension problem, we recall that an arbitrarily chosen set of polynomials $P(\xi, u)$, $u \in U$, does not define in general a $C^{n,\alpha}$- smooth function, but some compatibility conditions must be satisfied for that. Indeed, let $u_1 \in U$ and let $\delta, \xi \in X$ be such that $u_2 := u_1 + \delta \in U$ and $u_3 := u_1 + \delta + \xi = u_2 + \xi \in X$. Then, from (2-9), we have

$$\|F(u_3) - P(\xi + \delta, u_1)\| \le C\|\xi + \delta\|^{n+\alpha},$$
$$\|F(u_3) - P(\xi, u_1 + \delta)\| \le C\|\xi\|^{n+\alpha}.$$

Therefore,

$$\|P(\xi + \delta, u_1) - P(\xi, u_1 + \delta)\| \le C_1(\|\xi\| + \|\delta\|)^{n+\alpha}. \tag{2-12}$$

These are the desired compatibility conditions. In other words, if we are given a set $V \subset X$ and a family of polynomials

$$\{P(\xi, u) : u \in V\} \subset P^n(X, Y)$$

and want to find a function $F \in C^{n,\alpha}(X, Y)$ such that $J^n_\xi F(u) = P(\xi, u)$ for all $u \in V$, then the compatibility condition (2-12) must be satisfied for all $u_1, u_1 + \delta \in V$ and all $\xi \in X$.

Inequality (2-12) can be rewritten in a more standard form, which usually appears in the statement of the Whitney extension theorem. Namely, using (2-1), we see that

$$P(\xi + \delta, u_1) = \sum_{l=0}^{n} \frac{1}{l!} \sum_{k=l}^{n} \frac{1}{(k-l)!} P_k([\{\xi\}^l, \{\delta\}^{k-l}], u_1),$$

where $P(\xi, u_1) = \sum_{l=0}^n (1/l!) P_l([\{\xi\}^l], u_1)$, $P_l(\,\cdot\,, u_1) \in \mathcal{P}_l(X, Y)$. Applying now Corollary 2.2 to (2-12), we get the desired alternative form of the compatibility conditions:

$$\left\| P_l(\{\xi\}^l, u_1 + \delta) - \sum_{k=0}^{n-l} \frac{1}{k!} P_{l+k}([\{\xi\}^l, \{\delta\}^k], u_1) \right\| \le C\|\xi\|^l \|\delta\|^{n-l+\alpha} \tag{2-13}$$

for $l = \{0, \dots, n\}$. The compatibility condition (2-13) has a natural interpretation: if $P_k(\{\xi\}, u_1) = F^{(k)}(u_1)[\{\xi\}^k]$ as we expect, then (2-13) is nothing more than Taylor expansions of $F^{(l)}(u_1+\delta)[\{\xi\}^l]$ at u_1.

The next theorem shows that the introduced compatibility conditions are sufficient for the existence of F in the case when X is finite-dimensional.

Theorem 2.5 (Whitney extension theorem). *Let* $\dim X < \infty$ *and let* V *be an arbitrary subset of* X. *Assume also that we are given a family of polynomials* $\{P(\xi, u) : u \in V\} \subset P^n(X, Y)$ *which satisfies the compatibility condition* (2-12) *with some* $\alpha \in (0, 1]$. *Then, there exists a function* $F \in C^{n,\alpha}(X, Y)$ *such that* $J^n_\xi F(u) = P(\xi, u)$ *for all* $u \in V$.

For the proof of this theorem, see [Stein 1970; Fefferman 2005]. Note that the theorem fails if the dimension of X is infinite, but there are no restrictions on the dimension of the space Y; see [Wells 1973].

3. Preliminaries, II: Spectral gaps and the construction of an inertial manifold

In this section we briefly discuss the classical theory of inertial manifolds for semilinear parabolic equations; see, e.g., [Zelik 2014] for a more detailed exposition.

Let H be an infinite-dimensional real Hilbert space. Let us consider an abstract parabolic equation in H:

$$\partial_t u + Au = F(u), \quad u|_{t=0} = u_0, \tag{3-1}$$

where $A : D(A) \to H$ is a linear self-adjoint positive operator in H with compact inverse and $F \in C^\infty_b(H, H)$ is a smooth bounded function on H such that all its derivatives are also bounded on H.

It is well known that under the above assumptions (3-1) is globally well-posed for any $u_0 \in H$ in the class of solutions $u \in C([0, T], H)$ for all $T > 0$ and, therefore, generates a semigroup in H:

$$S(t) : H \to H, \quad t \geq 0, \quad S(t)u_0 := u(t). \tag{3-2}$$

Moreover, the solution operators $S(t)$ are in $C^\infty(H, H)$ for every fixed $t \geq 0$; see [Henry 1981; Zelik 2014] for the details.

Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of the operator A enumerated in the nondecreasing order and let $\{e_n\}_{n=1}^\infty$ be the corresponding orthonormal system of eigenvectors. Then, by the Parseval equality, for every $u \in H$, we have

$$\|u\|^2_H = \sum_{n=1}^\infty (u, e_n)^2, \quad u = \sum_{n=1}^\infty (u, e_n)e_n,$$

where $(\cdot, \cdot)$ is an inner product in H. For a given $N \in \mathbb{N}$, we denote by $\boldsymbol{P}_N$ and $\boldsymbol{Q}_N$ the orthoprojectors on the first N and the rest of eigenvectors of A respectively:

$$\boldsymbol{P}_N u := \sum_{n=1}^N (u, e_n)e_n, \quad \boldsymbol{Q}_N u := \sum_{n=N+1}^\infty (u, e_n)e_n.$$

We are now ready to introduce the main object of study in this paper — an inertial manifold (IM).

Definition 3.1. A set $\mathcal{M} = \mathcal{M}_N$ is an inertial manifold of dimension N for problem (3-1) (with the base $H_N := \boldsymbol{P}_N H$) if

(1) $\mathcal{M}$ is invariant with respect to the semigroup $S(t)$: $S(t)\mathcal{M} = \mathcal{M}$.

(2) $\mathcal{M}$ is a graph of a Lipschitz continuous function $M : H_N \to \boldsymbol{Q}_N H$:

$$\mathcal{M} = \{p + M(p) : p \in H_N\}.$$

(3) $\mathcal{M}$ possesses an exponential tracking property, namely, for every trajectory $u(t)$ of (3-1) there exists a trace solution $\bar{u}(t) \in \mathcal{M}$ such that

$$\|u(t) - \bar{u}(t)\| \le Ce^{-\theta t}, \quad t \ge 0, \tag{3-3}$$

for some $\theta > \lambda_N$ and constant $C = C_u$ which depends on u.

Note that, although only Lipschitz continuity is traditionally required in the definition, usually IMs are $C^{1,\varepsilon}$-smooth for some $\varepsilon > 0$ (see the discussion below) and are normally hyperbolic. Then the exponential tracking property (that is, existence of an asymptotic phase), as well as robustness with respect to perturbations, are the standard corollaries of this normal hyperbolicity; see [Bates et al. 1999; Fenichel 1972; Katok and Hasselblatt 1995; Rosa and Temam 1996] for the details. We also mention that these results can be obtained without formally referring to normal hyperbolicity; see, e.g., [Foias et al. 1988], as well as Theorem 3.2 and Corollary 4.4 below.

Note also the dynamics of (3-1) restricted to the IM $\mathcal{M}$ is governed by the system of ODEs

$$\frac{d}{dt}u_N + Au_N = \boldsymbol{P}_N F(u_N + M(u_N)), \quad u_N := \boldsymbol{P}_N u \in \mathbb{R}^N, \tag{3-4}$$

which is called an inertial form (IF) associated with (3-1). In the case where the spectral subspace H_N is used as a base for IM (like in Definition 3.1), the regularity of the corresponding vector field in the IF is determined by the regularity of the IM only.

The following theorem is the key result in the theory of IMs.

Theorem 3.2. *Let the function F in (3-1) be globally Lipschitz continuous with Lipschitz constant L and let, for some $N \in \mathbb{N}$, the spectral gap condition*

$$\lambda_{N+1} - \lambda_N > 2L \tag{3-5}$$

be satisfied. Then (3-1) possesses an IM $\mathcal{M}_N$ of dimension N.

Proof. Although this statement is classical, see, e.g., [Foias et al. 1988; Miklavčič 1991; Romanov 1993; Zelik 2014], the elements of its proof will be crucially used in what follows, so we sketch them below.

To construct the IM, we will use the so-called Perron method; namely, we will prove that, for every $p \in H_N$, the problem

$$\partial_t u + Au = F(u), \quad t \le 0, \quad \boldsymbol{P}_N u|_{t=0} = p \tag{3-6}$$

possesses a unique backward solution $u(t) = V(p, t)$, $t \le 0$, belonging to an appropriately weighted space, and then define the desired map $M : H_N \to \boldsymbol{Q}_N H$ via

$$M(p) := \boldsymbol{Q}_N V(p, 0). \tag{3-7}$$

To solve (3-6) we use the Banach contraction theorem treating the nonlinearity F as a perturbation. To this end we need the following two lemmas.

Lemma 3.3. *Let $\theta\in(\lambda_N,\lambda_{N+1})$ and let us consider the equation*

$$\partial_t v+Av=h(t),\quad t\in\mathbb{R},\ h\in L^2_{e^{\theta t}}(\mathbb{R},H),\tag{3-8}$$

where the space $L^2_{e^{\theta t}}(\mathbb{R},H)$ is defined via the weighted norm

$$\|h\|^2_{L^2_{e^{\theta t}}(\mathbb{R},H)}:=\int_{t\in\mathbb{R}}e^{2\theta t}\|h(t)\|^2\,dt<\infty.\tag{3-9}$$

Then, problem (3-8) possesses a unique solution $u\in L^2_{e^{\theta t}}(\mathbb{R},H)$ and the solution operator $\mathcal{T}:L^2_{e^{\theta t}}\to L^2_{e^{\theta t}}$, $\mathcal{T}:h\mapsto u$ satisfies

$$\|\mathcal{T}\|_{\mathcal{L}(L^2_{e^{\theta t}},L^2_{e^{\theta t}})}=\frac{1}{\min\{\theta-\lambda_N,\lambda_{N+1}-\theta\}}.\tag{3-10}$$

The proof of this identity is just a straightforward calculation based on decomposition of the solution $u(t)$ with respect to the base $\{e_n\}_{n=1}^\infty$ and solving the corresponding ODEs; see [Zelik 2014].

The second lemma gives the analogue of this formula for the linear equation on a negative semiaxis.

Lemma 3.4. *Let $\theta\in(\lambda_N,\lambda_{N+1})$. Then, for any $p\in H_N$ and any $h\in L^2_{e^{\theta t}}(\mathbb{R}_-,H)$, the problem*

$$\partial_t v+Av=h(t),\quad t\le 0,\quad \boldsymbol{P}_N v|_{t=0}=p\tag{3-11}$$

possesses a unique solution $v\in L^2_{e^{\theta t}}(\mathbb{R}_-,H)$. This solution can be written in the form

$$v=\mathcal{T}h+\mathcal{H}p,$$

where $\mathcal{T}$ is exactly the solution operator constructed in Lemma 3.3 applied to the extension of the function $h(t)$ by zero for $t\ge 0$ and $\mathcal{H}:H_N\to L^2_{e^{\theta t}}(\mathbb{R}_-,H)$ is a solution operator for the problem with zero right-hand side:

$$\mathcal{H}(p,t):=\sum_{n=1}^{N}(p,e_n)e^{-\lambda_n t}.$$

Indeed, this lemma is an easy corollary of Lemma 3.3; see [Zelik 2014].

We are now ready to prove the theorem. To this end, we fix the optimal value $\theta=(\lambda_{N+1}+\lambda_N)/2$ and write (3-6) as a fixed-point problem

$$u=\mathcal{T}\circ F(u)+\mathcal{H}(p)\tag{3-12}$$

in the space $L^2_{e^{\theta t}}(\mathbb{R}_-,H)$. Since the norm of the operator $\mathcal{T}$ is equal to $2/(\lambda_{N+1}-\lambda_N)$ and the Lipschitz constant of F is L, the spectral gap condition (3-5) guarantees that the right-hand side of (3-12) is a contraction for every $p\in H_N$. Thus, by the Banach contraction theorem, for every $p\in H_N$, there exists a unique solution $u(t)=V(p,t)$ of problem (3-6) belonging to $L^2_{e^{\theta t}}(\mathbb{R}_-,H)$ and the map $p\mapsto V(p,\cdot)$ is Lipschitz continuous. Due to the parabolic smoothing property, we know that

$$\|u(0)\|\le C(1+\|u\|_{L^2([-1,0],H)})\quad\text{and}\quad\|u(0)-w(0)\|\le C\|u\|_{L^2([-1,0],H)}$$

for any two backward solutions u, w of (3-1); see, e.g., [Zelik 2014]. In particular, these formulas show that the solution $V(p,t)$ is continuous in time ($V(p,\cdot)\in C_{e^{\theta t}}(\mathbb{R}_-,H)$, where the weighted space of continuous functions is defined analogously to (3-9)) and the map $p\mapsto V(p,\cdot)$ is Lipschitz continuous as a map from H_N to $C_{e^{\theta t}}(\mathbb{R}_-, H)$. Thus, formula (3-7), defines indeed a Lipschitz manifold of dimension N over the base H_N as graph of Lipschitz continuous function $M : H_N \to \boldsymbol{Q}_N H$.

The invariance of this manifold follows by the construction, so we only need to verify the exponential tracking property.

Let $u(t) = S(t)u_0$ be an arbitrary solution of problem (3-1) and let $\phi(t) \in C^\infty(R)$ be a cut-off function such that $\phi(t) \equiv 0$ for $t \le 0$ and $\phi(t) \equiv 1$ for $t \ge 1$. Then the function $\phi(t)u(t)$ is defined for all $t \in \mathbb{R}$. We seek for the desired solution $\bar{u}(t) \in \mathcal{M}$ (by the construction of $\mathcal{M}$ such solutions are defined for all $t \in \mathbb{R}$) in the form

$$\bar{u}(t) = \phi(t)u(t) + v(t). \tag{3-13}$$

Inserting this anzatz to (3-1), we end up with the following equation for $v(t)$:

$$\partial_t v + Av = F(\phi u + v) - \phi F(u) - \phi' u. \tag{3-14}$$

Let $v \in L^2_{e^{\theta t}}(\mathbb{R}, H)$ be a solution of this equation. Then, since $\bar{u} = v$ for $t \le 0$, we necessarily have $\bar{u} \in \mathcal{M}$ by the construction of the IM. On the other hand, for $t \ge 1$, we have $v = \bar{u} - u \in L^2_{e^{\theta t}}([1, \infty), H)$ and using the parabolic smoothing again, we get the desired estimate (3-3). Thus, we only need to find such a solution $v(t)$. To this end, we invert the linear part of (3-14) to get the fixed-point equation

$$v = \mathcal{T}(F(\phi u + v) - \phi F(u) - \phi' u). \tag{3-15}$$

It is straightforward to verify using Lemma 3.3 that the right-hand side of (3-15) is a contraction on the space $L^2_{e^{\theta t}}(\mathbb{R}, H)$ if the spectral gap condition holds; see [Zelik 2014]. Thus, the Banach contraction theorem finishes the proof of exponential tracking. □

Remark 3.5. It is well known that the spectral gap condition (3-5) is sharp in the sense that if it is violated for some N and L, one can find a nonlinearity F such that (3-1) does not possess an IM of dimension N with base H_N; see [Romanov 1993].

More recent examples show that if the condition

$$\sup_{N \in \mathbb{N}} \{\lambda_{N+1} - \lambda_N\} < 2L$$

is violated for all N, one can construct a smooth nonlinearity F such that (3-1) does not possess any Lipschitz or even log-Lipschitz finite-dimensional manifold (not necessarily invariant) which contains the global attractor; see [Eden et al. 2013; Zelik 2014].

Remark 3.6. Theorem 3.2 guarantees the existence of an IM $\mathcal{M}_N$ for *every* N such that the spectral gap condition (3-5) is satisfied. Typically, this N is not unique, instead, we have a whole sequence $\{N_k\}_{k=1}^\infty$ of N's satisfying the spectral gap condition. Therefore, according to the theorem, we will have a sequence of IMs $\{\mathcal{M}_{N_k}\}_{k=1}^\infty$ of increasing dimensions: $N_1 < N_2 < N_3 < \cdots$. Moreover, from the explicit description of an IM using backward solutions of (3-6), we see that

$$\mathcal{M}_{N_1} \subset \mathcal{M}_{N_2} \subset \mathcal{M}_{N_3} \subset \cdots; \tag{3-16}$$

see [Foias et al. 1988] for more details. In this case it can be also proved that $\mathcal{M}_{N_{k-1}}$ is a normally hyperbolic submanifold of $\mathcal{M}_{N_k}$.

Let us now discuss the further regularity of the IM $\mathcal{M}$. To this end, we need one more auxiliary statement.

Proposition 3.7. *Let the spectral gap condition (3-5) hold and let $u(t) \in C(R_-, H)$ be an arbitrary function. Let also the exponent $\theta \in (\lambda_N, \lambda_{N+1})$ satisfy*

$$\theta_- := L + \lambda_N < \theta < \lambda_{N+1} - L := \theta_+. \tag{3-17}$$

Then, for any $h \in L^2_{e^{\theta t}}(\mathbb{R}_-, H)$ and every $p \in H_N$, the corresponding equation of variations

$$\partial_t v + Av - F'(u(t))v = h(t), \quad t \le 0, \quad \boldsymbol{P}_N v|_{t=0} = p \tag{3-18}$$

possesses a unique solution $v \in L^2_{e^{\theta t}}(\mathbb{R}_-, H) \cap C_{e^{\theta t}}(\mathbb{R}_-, H)$ and the following estimate holds:

$$\|v\|_{C_{e^{\theta t}}(\mathbb{R}_-,H)} \le C\|v\|_{L^2_{e^{\theta t}}(\mathbb{R}_-,H)} \le C_{L,\theta}(\|h\|_{L^2_{e^{\theta t}}(\mathbb{R}_-,H)} + \|p\|), \tag{3-19}$$

where the constant $C_{L,\theta}$ is independent of u, h and p.

Indeed, (3-18) can be solved via the Banach contraction theorem treating the term $F'(u)v$ as a perturbation analogously to the nonlinear case. Inequalities (3-17) guarantee that the map $\mathcal{T}F'(u)v$ is a contraction on $L^2_{e^{\theta t}}(\mathbb{R}_-, H)$, due to (3-10).

Corollary 3.8. *Let the assumptions of Theorem 3.2 hold and let, in addition, the exponent $\varepsilon \in (0, 1]$ be such that*

$$\lambda_{N+1} - (1+\varepsilon)\lambda_N > (2+\varepsilon)L. \tag{3-20}$$

Assume also that $F \in C^{1,\varepsilon}(H, H)$. Then the associated IM $\mathcal{M}_N$ is $C^{1,\varepsilon}$-smooth, for any $p, \xi \in H_N$, the derivative $M'(p)\xi$ can be found as the value of the $\boldsymbol{Q}_N$ projection of $V'(t) = V'(p, t)\xi$ at $t = 0$, where the function V' solves the equation of variations

$$\partial_t V' + AV' - F'(u(t))V' = 0, \quad t \le 0, \quad \boldsymbol{P}_N V'|_{t=0} = \xi, \quad u(t) := V(p, t), \tag{3-21}$$

and

$$\|M'(p_1) - M'(p_2)\|_{\mathcal{L}(H_N,H)} \le C\|p_1 - p_2\|^{\varepsilon}$$

for some constant C independent of $p_1, p_2 \in H_N$.

Proof. Let $p_1, p_2 \in H_N$ and $u_i(t) := V(p_i, t)$ be the corresponding trajectories belonging to the IM. Let also $v(t) := u_1(t) - u_2(t)$ and $\xi := p_1 - p_2$. Then v solves

$$\partial_t v + Av - L_{u_1,u_2}(t)v = 0, \quad t \le 0, \quad \boldsymbol{P}_N v|_{t=0} = \xi, \tag{3-22}$$

where $L_{u_1,u_2}(t) := \int_0^1 F'(su_1(t) + (1-s)u_2(t))\,ds$. Since the norm of $L_{u_1,u_2}(t)$ does not exceed L, Proposition 3.7 is applicable to (3-22) and, therefore, for every θ satisfying (3-17), we have the estimate

$$\|v\|_{C_{e^{\theta t}}(\mathbb{R}_-,H)} \le C\|v\|_{L^2_{e^{\theta t}}(\mathbb{R}_-,H)} \le C_\theta\|p_1 - p_2\|. \tag{3-23}$$

Note also that the function $V'(p, t)\xi$ is well-defined for all $p, \xi \in H_N$ due to Proposition 3.7 and satisfies the analogue of (3-23). Let $w(t) := v(t) - V'(p_1, t)\xi$, with $\xi := p_1 - p_2$. Then, this function solves

$$\partial_t w + Aw - F'(u_1)w = F(u_1) - F(u_2) - F'(u_1)v := h_{u_1,u_2}(t), \quad \boldsymbol{P}_N w|_{t=0} = 0. \tag{3-24}$$

Since $F\in C^{1,\varepsilon}(H,H)$, by the Taylor theorem, we have

$$\|h_{u_1,u_2}(t)\|\le C\|v(t)\|^{1+\varepsilon},$$

which, due to (3-23), gives

$$\|h_{u_1,u_2}\|_{L^2_{e^{(1+\varepsilon)\theta t}}(\mathbb{R}_-,H)}\le C\|v\|_{L^2_{e^{\theta t}}(\mathbb{R}_-,H)}\|v\|^{\varepsilon}_{C_{e^{\theta t}}(\mathbb{R}_-,H)}\le C'\|\xi\|^{1+\varepsilon}.$$

Fixing now θ in such a way that $\theta>\theta_-$ and $(1+\varepsilon)\theta<\theta_+$ (this is possible to do due to assumption (3-20)) and applying Proposition 3.7 to (3-24), we finally arrive at

$$\|M(p_2)-M(p_1)-M'(p_1)\xi\|=\|w(0)\|\le C_1\|w\|_{L^2_{e^{(1+\varepsilon)\theta t}}(\mathbb{R}_-,H)}\le C_2\|\xi\|^{1+\varepsilon}$$

and the converse Taylor theorem finishes the proof of the corollary. ☐

The next corollary claims that the constructed manifold $\mathcal{M}$ actually lives in a more regular space $H^2:=D(A)$.

Corollary 3.9. *Let the assumptions of Corollary 3.8 hold. Then the manifold $\mathcal{M}$ is simultaneously a $C^{1,\varepsilon}$-smooth IM for (3-1) in the phase space $H^2=D(A)$.*

Proof. This is an almost immediate corollary of the parabolic smoothing property. Indeed, let us first check that $\mathcal{M}\in H^2$. To this end, it is enough to check that the backward solution (3-6) actually belongs to $C_{e^{\theta t}}(\mathbb{R}_-,H^2)$. First, using the $L^2(H^2)$-maximal regularity for the solutions of a linear parabolic equation

$$\partial_t v+Av=h(t),\qquad t\le 0,\tag{3-25}$$

namely, that

$$\|v\|_{C^\alpha(-1,0;H)}+\|\partial_t v\|_{L^2(-1,0;H)}+\|Av\|_{L^2(-1,0;H)}\le C_\alpha(\|h\|_{L^2(-2,0,H)}+\|v\|_{L^2(-2,0,H)}),\tag{3-26}$$

where $\alpha\in\left(0,\frac12\right)$, we end up with the estimate

$$\|u\|_{C^\alpha(-1,0;H)}\le C_\alpha(\|F(u)\|_{L^2(-2,0;H)}+\|u\|_{L^2(-2,0;H)})\le C_{\alpha,\theta}(1+\|u\|_{L^2_{e^{\theta t}}(\mathbb{R}_-,H)})\le C(1+\|p\|),\tag{3-27}$$

where $\alpha\in\left(0,\frac12\right)$. Second, using the $C^\alpha(H)$-maximal regularity for solutions of (3-25) and the obvious estimate

$$\|F(u)\|_{C^\alpha(-2,0;H)}\le\|F\|_{C^\alpha(H,H)}(1+\|u\|^\alpha_{C^\alpha(-2,0;H)}),$$

we arrive at

$$\begin{aligned}\|\partial_t u\|_{C^\alpha(-1,0;H)}+\|Au\|_{C^\alpha(-1,0;H)}&\le C(\|F(u)\|_{C^\alpha(-2,0;H)}+\|u\|_{C^\alpha(-2,0;H)})\\&\le C_1(1+\|u\|_{C^\alpha(-2,0;H)})\le C_2(1+\|p\|)\end{aligned}\tag{3-28}$$

and the fact that $M(p)$ belongs to H^2 is proved. The fact that M is $C^{1,\varepsilon}$-smooth as a map from H_N to H^2 can be verified analogously and the corollary is proved. ☐

Remark 3.10. The analogue of Corollary 3.8 holds for higher derivatives as well. For instance, if we want to have a $C^{n,\varepsilon}$-smooth IM, we need to require that

$$\lambda_{N+1}-(n+\varepsilon)\lambda_N>(n+1+\varepsilon)L.\tag{3-29}$$

To verify this, we just need to define the higher-order Taylor jets for the IM $\mathcal{M}$ using second, third, etc., equations of variations for (3-6) and use again Proposition 3.7. For instance, the second derivative $V'' = V''(p,t)[\xi,\xi]$ solves

$$\partial_t V'' + AV'' - F'(u(t))V'' = F''(u(t))[V'(p,t)\xi, V'(p,t)\xi], \quad \boldsymbol{P}_N V''|_{t=0} = 0, \quad u(t) := V(p,t). \tag{3-30}$$

According to Proposition 3.7, in order to be able to solve this equation, we need $\theta_+ > 2\theta_-$ (since $V' \in L^2_{e^{\theta t}}$ with $\theta > \theta_-$ and the right-hand side $F''(u)[V', V'] \in L^2_{e^{2\theta t}}$), which gives (3-29) for $n = 2$.

We believe that sufficient condition (3-29) for the existence of a $C^{n,\varepsilon}$-smooth IM is sharp for any n and ε, but we restrict ourselves by recalling below the classical counterexample of G. Sell to the existence of C^2-smooth IM which demonstrates the sharpness of (3-29) for $n = 2$; see [Chow et al. 1992].

Example 3.11. Let $H := l^2$ (the space of square summable sequences with the standard inner product) and let us consider the following particular case of (3-1):

$$\frac{d}{dt}u_1 + u_1 = 0, \quad \frac{d}{dt}u_n + 2^{n-1}u_n = u_{n-1}^2, \quad n = 2, 3, \dots. \tag{3-31}$$

Here $\lambda_n = 2^{n-1}$ and we have a set of resonances $2\lambda_n = \lambda_{n+1}$ which prevent the existence of any finite-dimensional invariant local manifold of dimension greater than zero which is C^2-smooth and contains zero. Note that the nonlinearity here is locally smooth near zero and since we are interested in *local* invariant manifolds near zero, the behaviour of it outside the small neighbourhood of zero is not important (we may always cut-off it outside of the neighbourhood to get global Lipschitz continuity). Moreover, since $F'(0) = 0$, decreasing the size of the neighbourhood we may make the Lipschitz constant L as small as we want. Thus, according to Corollary 3.8, for any $N \in \mathbb{N}$, there exists a local invariant manifold $\mathcal{M}_N$ of dimension N with the base H_N which is $C^{1,\varepsilon}$-smooth for any $\varepsilon < 1$.

Let us check that a C^2-smooth invariant local manifold does not exist. Indeed, let $\mathcal{M}_N$ be such a manifold of dimension N. Then, since the tangent plane $T\mathcal{M}_N(0)$ to this manifold at zero is invariant with respect to A (due to the fact that $F'(0) = 0$), we must have

$$H_N' := T\mathcal{M}_N(0) = \operatorname{span}\{e_{n_1}, \dots, e_{n_N}\}$$

for some $n_1 < n_2 < \dots < n_N$. Thus, the manifold $\mathcal{M}_N$ can be presented locally near zero as a graph of a C^2-function $M : H_N' \to (H_N')^\perp$ such that $M(0) = M'(0) = 0$. In particular, expanding M in Taylor series near zero, we have

$$u_{n_N+1} = (M(u_{n_1}, \dots, u_{n_N}), e_{n_N+1}) = cu_{n_N}^2 + \cdots.$$

Let us try to compute the constant c. Inserting this in the (n_N+1)-th equation and using the invariance, we get

$$\begin{aligned} \partial_t u_{n_N+1} + 2^{n_N}u_{n_N+1} &= 2c\partial_t u_{n_N}u_{n_N} + 2^{n_N}cu_{n_N}^2 + \cdots \\ &= -2c2^{n_N-1}u_{n_N}^2 + 2^{n_N}cu_{n_N}^2 + \cdots = 0 + \cdots = u_{n_N}^2, \end{aligned} \tag{3-32}$$

which gives $0 = 1$. Thus, the manifold $\mathcal{M}_N$ cannot be C^2-smooth.

Remark 3.12. Note that in the case where A is an elliptic operator of order $2k$ in a bounded domain Ω of $\mathbb{R}^d$, we have $\lambda_n \sim Cn^{2k/d}$ due to the Weyl asymptotic. Thus, one may expect in general only gaps of the size

$$\lambda_{N+1}-\lambda_N \sim CN^{2k/d-1} \sim C'\lambda_N^{1-d/(2k)}, \tag{3-33}$$

which is much weaker than (3-29) with $n>1$. Sometimes the exponent in the right-hand side of (3-33) may be improved due to multiplicity of eigenvalues (e.g., for the Laplace–Beltrami operator on a sphere S^d, we have $\lambda_N^{1/2}$ instead of $\lambda_N^{1-d/(2k)}$ in the right-hand side of (3-33) for infinitely many values of N, no matter how big the dimension d is), but this exponent is always *less than one* in all more or less realistic examples. Thus, the existence of C^n-smooth IMs with $n>1$ looks unrealistic and could be obtained in general, but only for bifurcation problems where, e.g., $\lambda_1,\dots,\lambda_N$ are close to zero, λ_{N+1} is of order 1 and L is small.

In contrast to this, if the spectral gap condition (3-5) is satisfied for some N, i.e., $\lambda_{N+1}-\lambda_N>2L$, we always can find a small positive $\varepsilon=\varepsilon_N$ such that $\lambda_{N+1}-(1+\varepsilon)\lambda_N>(2+\varepsilon)N$ and, therefore, (3-20) will be also satisfied. Thus, if the nonlinearity F is smooth enough, we automatically get a $C^{1,\varepsilon}$-smooth IM for some small ε depending on N and L.

Remark 3.13. Let $\bar u(t)$ be a trajectory of (3-1) belonging to the IM, i.e.,

$$\boldsymbol{Q}_N\bar u(t)\equiv M_N(\boldsymbol{P}_N\bar u(t))$$

and let $\bar u_N:=\boldsymbol{P}_N\bar u(t)$. Then, we may write a linearization near the trajectory $\bar u(t)$ in two natural ways. First, we may just linearize (3-1) without using the fact that $\bar u\in\mathcal{M}_N$. This gives the equation

$$\partial_t v+Av-F'(\bar u)v=h(t), \tag{3-34}$$

which we have used above to get the existence of the IM, its smoothness and exponential tracking.

Alternatively, we may linearize the reduced ODEs (3-4):

$$\partial_t v_N+Av_N-F'(\bar u)(v_N+M_N'(\bar u)v_N)=h_N(t). \tag{3-35}$$

Of course, these two equations are closely related. Namely, if $v_N(t)$ solves (3-35), then the function

$$v(t):=v_N(t)+M_N'(\bar u(t))v_N(t) \tag{3-36}$$

solves (3-34) with

$$h(t):=h_N(t)+M_N'(\bar u(t))h_N(t). \tag{3-37}$$

Vice versa, if $h(t)$ satisfies (3-37) and the solution $v(t)$ of (3-34) satisfies (3-36) for some t, then it satisfies (3-36) for all t and $v_N(t):=\boldsymbol{P}_N v(t)$ solves (3-35).

This equivalence is a straightforward corollary of the invariance of the manifold $\mathcal{M}_N$ and we leave its rigorous proof to the reader.

4. Main result

In this section we develop an alternative approach for constructing C^n-smooth IFs which does not require huge spectral gaps. The key idea is to require instead the existence of *many* spectral gaps and to use the second spectral gap in order to solve (3-30) for the second derivative, the third gap to solve the appropriate equation for the third derivative, etc. Of course, this will not allow us to construct a C^n-smooth IM (we know that it may not exist for $n > 1$, see Example 3.11). Instead, for every $p \in \mathcal{M}_{N_2}$ and the corresponding trajectory $u = V(p,t)$, we construct the corresponding Taylor jet $J^n_\xi V(p,t)$ of length $n+1$ belonging to the space $\mathcal{P}^n(H_{N_n}, H)$ for all $t \le 0$, where N_k is the dimension of the IM $\mathcal{M}_{N_k}$ built up on the k-th spectral gap. These jets must be constructed in such a way that the compatibility conditions are satisfied. Then, the Whitney embedding theorem will give us the desired smooth extension of the initial IM. To be more precise, we give the following definition of such a smooth extension.

Definition 4.1. Let (3-1) possess at least two spectral gaps which correspond to the dimensions K_1 and K_2 and let $\varepsilon > 0$ be a small number. Denote the corresponding IMs by $\mathcal{M}_{K_1}$ and $\mathcal{M}_{K_2}$ respectively; the corresponding $C^{1,\varepsilon}$-functions generating these manifolds are denoted by M_{K_1} and M_{K_2} respectively. A $C^{n,\varepsilon}$-smooth submanifold $\widetilde{\mathcal{M}}_{K_2}$ (not necessarily invariant) of dimension K_2 is called a C^n-extension of the IM $\mathcal{M}_{K_1}$ if the following conditions hold:

(1) $\widetilde{\mathcal{M}}_{K_2}$ is a graph of a $C^{n,\varepsilon}$-smooth function $\widetilde{M}_{K_2} : \boldsymbol{P}_{K_2}H \to \boldsymbol{Q}_{K_2}H$.

(2) $\widetilde{M}_{K_2}|_{\boldsymbol{P}_{K_2}\mathcal{M}_{K_1}} = \boldsymbol{Q}_{K_2}M_{K_1}$ and therefore $\mathcal{M}_{K_1} \subset \widetilde{\mathcal{M}}_{K_2}$.

(3) $\widetilde{M}_{K_2}$ is μ-close in the C^1_b-norm to M_{K_2} for a sufficiently small μ.

Remark 4.2. The $C^{n,\varepsilon}$ dynamics on the extended IM $\widetilde{\mathcal{M}}_{K_2}$ is naturally defined via

$$\partial_t u_{K_2} + Au_{K_2} = \boldsymbol{P}_{K_2}F(u_{K_2} + \widetilde{M}_{K_2}(u_{K_2})), \quad u_{K_2} \in H_{K_2}, \tag{4-1}$$

and $u(t) := u_{K_2}(t) + \widetilde{M}_{K_2}(u_{K_2}(t))$. Obviously, the manifold $\widetilde{\mathcal{M}}_{K_2}$ is invariant with respect to the dynamical system thus defined. Moreover, due to the second condition of Definition 4.1, the $C^{1,\varepsilon}$-submanifold $\boldsymbol{P}_{K_2}\mathcal{M}_{K_1} \subset H_{K_2}$ is invariant with respect to (4-1) and the restriction of (4-1) coincides with the initial IF (3-4) generated by the IM $\mathcal{M}_{K_1}$. Thus, system of ODEs (4-1) is indeed a smooth extension of the IF (3-4).

Finally, the third condition of Definition 4.1 guarantees that $\boldsymbol{P}_{K_2}\mathcal{M}_{K_1}$ is a normally hyperbolic stable invariant manifold for (4-1) (since it is so for the IF generated by the function M_{K_2}). This means that $\boldsymbol{P}_{K_2}\mathcal{M}_{K_1}$ also possesses an exponential tracking property. Thus, the limit dynamics generated by the extended IF coincides with the one generated by the initial abstract parabolic equation (3-1).

We are now ready to state the main result of the paper.

Theorem 4.3. *Let the nonlinearity $F : H \to H$ in (3-1) be smooth and all its derivatives be globally bounded. Let also the following form of spectral gap conditions be satisfied*:

$$\limsup_{N\to\infty}(\lambda_{N+1} - \lambda_N) = \infty. \tag{4-2}$$

Then, for any $n \in \mathbb{N}$ and any $\mu > 0$, equation (3-1) possesses a $C^{n,\varepsilon}$-smooth extension $\widetilde{\mathcal{M}}_{N_n}$ of the initial IM $\mathcal{M}_{N_1}$ (where N_1 is the first N which satisfies the spectral gap condition (3-5) and $\varepsilon > 0$ is small enough) such that $\widetilde{\mathcal{M}}_{N_n}$ is μ-close to the IM $\mathcal{M}_{N_n}$ in the C^1_b-norm.

Proof for $n = 2$. Let N_1 be the first N for which the spectral gap condition (3-5) is satisfied with $L := \|F'\|_{C_b(H,\mathcal{L}(H,H))}$ and let the corresponding $\mathcal{M}_1$ be the $C^{1,\varepsilon}$-smooth IM which exists due to Theorem 3.2 and Corollary 3.8. Recall that for any $p \in H$, we have a solution $V(p,t)$ of problem (3-6) (where p is replaced by $\boldsymbol{P}_{N_1}p$) and its Fréchet derivative $V'_\xi(t) := V'(p,t)\xi$ in p satisfies the equation of variations

(3-21), such that both functions $V(p,\cdot)$ and $V'_\xi(\cdot)$ and belong to the space $L^2_{e^{\theta_1 t}}(\mathbb{R}_-,H)$ for any θ_1 satisfying (3-17), i.e., $\lambda_{N_1}+L<\theta_1<\lambda_{N_1}-L$. Moreover, for any other $p_1\in H$, we have the estimate

$$\|V(p_1,t)-V(p,t)-V'_\xi(t)\|_{L^2_{e^{\theta_1(1+\varepsilon)t}}(\mathbb{R}_-,H)}\le C\|\boldsymbol{P}_{N_1}(p-p_1)\|^{1+\varepsilon}, \tag{4-3}$$

where $\varepsilon>0$, $\xi:=p_1-p$ and C is independent of p and p_1.

Let now $N_2>N_1$ be the first N which satisfies

$$\lambda_{N_2+1}-\lambda_{N_2}-\lambda_{N_1}>3L \tag{4-4}$$

(such N exists due to condition (4-2)). Then, we have the corresponding $C^{1,\varepsilon}$-smooth IM $\mathcal{M}_{N_2}$. Let us denote by $W(p,t)$, $p\in H$, the corresponding solution of (3-6) (where N is replaced by N_2 and p is replaced by $\boldsymbol{P}_{N_2}p$). This solution belongs to $L^2_{e^{\theta_2 t}}(\mathbb{R}_-,H)$ with θ_2 satisfying (3-17) (with N replaced by N_2). Moreover, analogously to (4-3), we have

$$\|W(p_1,t)-W(p,t)-W'_\xi(t)\|_{L^2_{e^{\theta_2(1+\varepsilon)t}}(\mathbb{R}_-,H)}\le C\|\boldsymbol{P}_{N_2}(p-p_1)\|^{1+\varepsilon}, \tag{4-5}$$

where $W'_\xi(t)=W'(p,t)\xi$ solves (3-21) with N replaced by N_2. We also know that $V(p,t)=W(p,t)$ if $p\in\mathcal{M}_{N_1}$ and, therefore, due to (4-3) and (4-5),

$$\|V'(p,\cdot)\xi-W'(p,\cdot)\xi\|_{L^2_{e^{\theta_2(1+\varepsilon)t}}(\mathbb{R}_-,H)}\le C\|\boldsymbol{P}_{N_2}\xi\|^{1+\varepsilon},\quad \xi=p_1-p,\ \ p,p_1\in\mathcal{M}_{N_1}. \tag{4-6}$$

Let us define for every $p\in\mathcal{M}_{N_1}$ and every $\xi\in H$ the "second derivative" $W''_\xi=W''(p,t)[\xi,\xi]$ of the trajectory $u(t)=W(p,t)=V(p,t)$ as a solution of the problem

$$\partial_t W''_\xi+AW''_\xi-F'(V(p,t))W''_\xi=2F''(V(p,t))[V'_\xi,W'_\xi]-F''(V(p,t))[V'_\xi,V'_\xi],\quad \boldsymbol{P}_{N_2}W''_\xi|_{t=0}=0. \tag{4-7}$$

Note that the right-hand side of this equation belongs to the weighted space $L^2_{e^{(\theta_1+\theta_2)t}}(\mathbb{R}_-,H)$, where the exponents θ_1 and θ_2 satisfy assumption (3-17) with $N=N_1$ and $N=N_2$ respectively, i.e.,

$$\lambda_{N_1}+L<\theta_1<\lambda_{N_1}-L,\quad \lambda_{N_2}+L<\theta_2<\lambda_{N_2}-L.$$

Moreover, due to assumption (4-4), it is possible to fix θ_1 and θ_2 in such a way that the exponent $\theta_1+\theta_2$ still satisfies (3-17) with $N=N_2$. Thus, by Proposition 3.7, there exists a unique solution of (4-7) belonging to the space $L^2_{e^{(\theta_1+\theta_2)t}}(\mathbb{R}_-,H)$ and the function W''_ξ is well-defined and satisfies

$$\|W''_\xi\|_{C_{e^{(\theta_1+\theta_2)t}}(\mathbb{R}_-,H)}\le C\|W''_\xi\|_{L^2_{e^{(\theta_1+\theta_2)t}}(\mathbb{R}_-,H)}\le C^2\|\xi\|^2,$$

where C is independent of p.

Let us define the desired quadratic polynomial $\xi\to J^2_\xi W(p,t)$, $p\in\mathcal{M}_{N_1}$, as

$$J^2_\xi W(p,t):=V(p,t)+W'(p,t)\xi+\tfrac{1}{2}W''(p,t)[\xi,\xi],\quad \xi\in H. \tag{4-8}$$

We need to verify the compatibility conditions for these "Taylor jets" on $p\in\mathcal{M}_{N_1}$. It is straightforward to check using $F\in C^{2,\varepsilon}$, $V,W\in C^{1,\varepsilon}$ and Proposition 3.7 that

$$\|W''(p_1,\cdot)[\xi,\xi]-W''(p,\cdot)[\xi,\xi]\|_{L^2_{e^{(\theta_1+\theta_2+\varepsilon)t}}(\mathbb{R}_-,H)}\le C\|\xi\|^2\|p-p_1\|^{\varepsilon}$$

for $p, p_1 \in \mathcal{M}_{N_1}$. This gives us the desired compatibility condition for the second derivative; see (2-13) for $n = l = 2$.

Let us now verify the compatibility conditions for the first derivative ($l = 1$, $n = 2$ in (2-13)). To this end, we need to expand the difference $w(t) := W'(p_1, t)\xi - W'(p, t)\xi,\ p, p_1 \in \mathcal{M}_{N_1}$, in terms of $\delta = p - p_1$. By the definition of W', this function satisfies the equation

$$\begin{aligned}\partial_t w + Aw - F'(V(p,t))w &= (F'(V(p_1,t)) - F'(V(p,t)))W'(p_1,t)\xi \\ &= F''(V(p,t))[V'(p,t)\delta, W'(p,t)\xi] + h(t), \quad \boldsymbol{P}_{N_2} w|_{t=0} = 0, \end{aligned} \tag{4-9}$$

where the reminder h satisfies

$$\|h\|_{L^2_{e^{(\theta_1+\theta_2+\varepsilon)t}}(\mathbb{R}_-, H)} \le C\|\delta\|^{1+\varepsilon}\|\xi\|$$

for sufficiently small positive ε (this also follows from the fact that F is smooth and $V, W \in C^{1,\varepsilon}$). Thus, the remainder h in the right-hand side of (4-9) is of higher order in δ and, for this reason, is not essential, so we need to study the bilinear form (with respect to δ, ξ) in the right-hand side. Note that, in contrast to the case where the IM is C^2, this form is even not symmetric, so it should be corrected. Namely, we write the identity

$$\begin{aligned} &F''(V(p,t))[V'(p,t)\delta, W'(p,t)\xi] \\ &\quad = \big\{F''(V(p,t))[V'(p,t)\delta, W'(p,t)\xi] + F''(V(p,t))[V'(p,t)\xi, W'(p,t)\delta] \\ &\qquad\qquad - F''(V(p,t))[V'(p,t)\delta, V'(p,t)\xi]\big\} \\ &\qquad\qquad - F''(V(p,t))[V'(p,t)\xi, W'(p,t)\delta - V'(p,t)\delta] \end{aligned} \tag{4-10}$$

and note that the first term in the right-hand side is nothing more than the symmetric bilinear form which corresponds to the quadratic form

$$2F''(V(p,t))[V'(p,t)\xi, W'(p,t)\xi] - F''(V(p,t))[V'(p,t)\xi, V'(p,t)\xi]$$

used in (4-7) to define W'' and the second term is of order $\|\delta\|^{1+\varepsilon}\|\xi\|$ due to estimate (4-6) (where ξ is replaced by δ) and the growth rate of this term does not exceed $e^{-(\theta_1+\theta_2+\varepsilon)t}$ as $t \to -\infty$. Thus, by Proposition 3.7, we have

$$\|w - W''(p, \cdot)[\delta, \xi]\|_{L^2_{e^{(\theta_1+\theta_2+\varepsilon)t}}(\mathbb{R}_-, H)} \le C\|\delta\|^{1+\varepsilon}\|\xi\|$$

and the compatibility condition for $l = 1$ is verified.

Finally, let us check the zero-order compatibility condition ($l = 0$, $n = 2$ in (2-13)). Let

$$R(t) := V(p_1, t) - V(p,t) - W'(p,t)\delta - \frac{1}{2!}W''(p,t)[\delta, \delta].$$

Then, as elementary computations show, this function satisfies the equation

$$\begin{aligned} &\partial_t R + AR - F'(V(p,t))R \\ &\quad = \{F(V(p_1,t)) - F(V(p,t)) - F'(V(p,t))(V(p_1,t) - V(p,t))\} \\ &\qquad\qquad - \frac{1}{2!}\big(2F''(V(p,t))[V'(p,t)\delta, W'(p,t)\delta] \\ &\qquad\qquad\qquad - F''(V(p,t))[V'(p,t)\delta, V'(p,t)\delta]\big), \quad \boldsymbol{P}_{N_2}|_{t=0} R = 0. \end{aligned} \tag{4-11}$$

Since $F \in C^{2,\varepsilon}$ and $V \in C^{1,\varepsilon}$, the first term in the right-hand side equals

$$\frac{1}{2!}F''(V(p,t))[V'(p,t)\delta, V'(p,t)\delta] \tag{4-12}$$

up to the controllable in $L^2_{e^{(\theta_1+\theta_2+\varepsilon)t}}(\mathbb{R}_-, H)$-norm remainder of order $\|\delta\|^{2+\varepsilon}$. The second term can be simplified using (4-6) and also equals (4-12) up to higher-order terms. Thus, the right-hand side of (4-11) vanishes up to terms of order $\|\delta\|^{2+\varepsilon}$ and Proposition 3.7 gives us that

$$\|R\|_{L^2_{e^{(\theta_1+\theta_2+\varepsilon)t}}(\mathbb{R}_-,H)} \le C\|\delta\|^{2+\varepsilon} \tag{4-13}$$

for some positive ε. This finishes the verification of the compatibility conditions.

We are now ready to use the Whitney extension theorem. To this end, we first recall that the IM $\mathcal{M}_{N_2}$ is a graph of the $C^{1,\varepsilon}$-function $M_{N_2} : \boldsymbol{P}_{N_2}H \to \boldsymbol{Q}_{N_2}H$, which is defined via $M_{N_2}(p) := \boldsymbol{Q}_{N_2}W(p,0)$, $p \in \boldsymbol{P}_{N_2}H = H_{N_2}$ (all functions V, W, W', W'' defined above depend only on $\boldsymbol{P}_{N_2}$-component of $p \in H$, so without loss of generality we may assume that $p, \xi, \delta \in H_{N_2}$ (we took them from H in order to simplify the notation only). Thus, projecting the constructed Taylor jets to $t = 0$ and $\boldsymbol{Q}_{N_2}H$, we get the $C^{1,\varepsilon}$-function $M_{N_2}(p)$ restricted to the invariant set $p \in \boldsymbol{P}_{N_2}\mathcal{M}_{N_1}$ and a family of quadratic polynomials

$$J^2_\xi M_{N_2}(p) := \boldsymbol{Q}_{N_2}J^2_\xi W(p,0),$$

which satisfy the compatibility conditions on $p \in \boldsymbol{P}_{N_2}\mathcal{M}_{N_1}$. Therefore, since H_{N_2} is finite-dimensional, the Whitney extension theorem gives the existence of a $C^{2,\varepsilon}$-function $\widehat{M}_{N_2} : \boldsymbol{P}_{N_2}H \to \boldsymbol{Q}_{N_2}H$ such that

$$J^2_\xi \widehat{M}_{N_2}(p) = J^2_\xi M_{N_2}(p), \quad p \in \boldsymbol{P}_{N_2}\mathcal{M}_{N_1}.$$

Thus, the desired $C^{2+\varepsilon}$-extension of the IM $\mathcal{M}_{N_1}$ is "almost" constructed. It only remains to take care of the closeness in the C^1-norm. To this end, for any small $\nu > 0$, we introduce a cut-off function $\rho_\nu \in C^\infty(H_{N_2}, \mathbb{R})$ such that $\rho(p) \equiv 0$ if p belongs to the ν-neighbourhood $\mathcal{O}_\nu$ of $\boldsymbol{P}_{N_2}\mathcal{M}_{N_1}$ and $\rho(p) \equiv 1$ if $p \notin \mathcal{O}_{2\nu}$. Moreover, since $\boldsymbol{P}_{N_2}\mathcal{M}_{N_1}$ is $C^{1,\varepsilon}$-smooth, we may require also that

$$|\nabla_p \rho(p)| \le C\nu^{-1}, \tag{4-14}$$

where the constant C is independent of ν. Finally, we define

$$\widetilde{M}_{N_2}(p) := (1-\rho_\nu(p))\widehat{M}_{N_2}(p) + \rho_\nu(p)(\mathbb{S}_{\nu^2}M_{N_2})(p), \tag{4-15}$$

where $\mathbb{S}_\mu$ is a standard mollifying operator,

$$(\mathbb{S}_\mu f)(p) := \int_{\mathbb{R}^{N_2}} \beta_\mu(p-q)f(q)\,dq,$$

and the kernel $\beta_\mu(p)$ satisfies $\beta_\mu(p) = (1/\mu^{N_2})\beta_1(p/\mu)$ and $\beta_1(p)$ is a smooth, nonnegative function with compact support satisfying $\int_{\mathbb{R}^{N_2}} \beta_1(p)\,dp = 1$.

We claim that $\widetilde{M}_{N_2}$ is a desired extension. Indeed, $\widetilde{M}_{N_2}(p) \equiv \widehat{M}_{N_2}(p)$ in $\mathcal{O}_\nu$ and therefore $\widetilde{M}_{N_2}$ and M_{N_2} coincide on $\boldsymbol{P}_{N_2}\mathcal{M}_{N_1}$. Obviously, $\widetilde{M}_{N_2}$ is $C^{2,\varepsilon}$-smooth. To verify closeness, we note that

$$\widetilde{M}_{N_2}(p) - M_{N_2}(p) = (1-\rho_\nu(p))(\widehat{M}_{N_2}(p) - M_{N_2}(p)) + \rho_\nu(p)((\mathbb{S}_{\nu^2}M_{N_2})(p) - M_{N_2}(p)). \tag{4-16}$$

Using the fact that $M_{N_2} \in C^{1,\varepsilon}$ together with the standard estimates for the mollifying operator, we get

$$\|(\mathbb{S}_{\nu^2} M_{N_2})(p) - M_{N_2}(p)\| \le C\nu^2, \quad \|\nabla_p(\mathbb{S}_{\nu^2} M_{N_2})(p) - \nabla_p M_{N_2}(p)\| \le C\nu^{2\varepsilon},$$

which together with (4-14) shows that the C^1-norm of the second term in the right-hand side of (4-16) is of order $\nu^{2\varepsilon}$. To estimate the first term, we use that both functions $\widehat{M}_{N_2}(p)$ and $M_{N_2}(p)$ are at least $C^{1,\varepsilon}$-smooth and

$$\widehat{M}_{N_2}(p) = M_{N_2}(p), \quad \nabla_p \widehat{M}_{N_2}(p) = \nabla_p M_{N_2}(p), \quad p \in \boldsymbol{P}_{N_2}\mathcal{M}_{N_1}.$$

For this reason,

$$\|\widehat{M}_{N_2}(p) - M_{N_2}(p)\| \le C\nu^{1+\varepsilon}, \quad \|\nabla_p \widehat{M}_{N_2}(p) - \nabla_p M_{N_2}(p)\| \le C\nu^{\varepsilon}$$

for all $p \in \mathcal{O}_{2\nu}$. Thus, using (4-14) again, we see that

$$\|\widetilde{M}_{N_2}(\,\cdot\,) - M_{N_2}(\,\cdot\,)\|_{C^1_b(H_{N_2},H)} \le C\nu^{\varepsilon}.$$

This finishes the proof of the theorem for the case $n = 2$. □

Proof for general $n \in \mathbb{N}$. We will proceed by induction with respect to n. Assume that, for some $n \in \mathbb{N}$, we have already constructed the $C^{1,\varepsilon}$-smooth inertial manifold $\mathcal{M}_{N_n}$ which is a graph of a map $M_{N_n} : \boldsymbol{P}_{N_n}H \to \boldsymbol{Q}_{N_n}H$ and this map is constructed via the solution $V(p,t)$, $t \le 0$, $p \in H$, of the backward problem (3-6), where N is replaced by N_n. Recall that this manifold is constructed using the n-th spectral gap. Assume also that, for every $p \in \boldsymbol{P}_{N_n}\mathcal{M}_{N_1}$, we have already constructed the n-th Taylor jet $J^n_\xi V(p,t)$ such that the compatibility conditions up to order n are satisfied. In contrast to the proof for the case $n = 2$, it is convenient for us to write these conditions in the form of (2-12):

$$\|J^n_\xi V(p_1,\cdot) - J^n_{\xi+\delta} V(p,\cdot)\|_{L^2_{e^{(\theta_n+(n-1)\theta_{n-1}+\varepsilon)t}}(\mathbb{R}_-,H)} \le C(\|\delta\| + \|\xi\|)^{n+\varepsilon}. \tag{4-17}$$

Here $\xi \in H$ is arbitrary, $\delta := p_1 - p$, $\varepsilon > 0$ and $\theta_1 < \theta_2 \cdots < \theta_n$ are the exponents which satisfy condition (3-17) for $N = N_1, \dots, N_n$. In order to simplify the notation, we will write below

$$J^n_\xi V(p_1) - J^n_{\xi+\delta} V(p) = O_{\theta_n+(n-1)\theta_{n-1}+\varepsilon}((\|\delta\| + \|\xi\|)^{n+\varepsilon}) \tag{4-18}$$

instead of (4-17) and likewise in similar situations. Rewriting (4-18) in terms of truncated jets with the help of (2-10) (where ξ is replaced by δ), we have

$$j^n_\xi V(p_1) + j^n_\delta V(p) - j^n_{\xi+\delta} V(p) = O_{n\theta_n+\varepsilon}((\|\delta\| + \|\xi\|)^{n+\varepsilon}), \tag{4-19}$$

where we have used that $\theta_{n-1} < \theta_n$. We also need the induction assumption that (4-19) holds for every $m \le n$, namely,

$$J^m_\xi V(p_1) - J^m_{\xi+\delta} V(p) = O_{m\theta_n+\varepsilon}((\|\delta\| + \|\xi\|)^{m+\varepsilon}). \tag{4-20}$$

Let us now consider the $(n+1)$-th spectral gap at $N = N_{n+1}$ which is the first N satisfying

$$\lambda_{N_{n+1}} + L + n(\lambda_{N_n+1} - L) < \lambda_{N_{n+1}+1} - L. \tag{4-21}$$

Let $\mathcal{M}_{N_{n+1}}$ be the corresponding IM which is generated by the backward solution $W(p,t)$ of problem (3-6) with N replaced by N_{n+1}. We need to define the $(n+1)$-th Taylor jet $J^{n+1}_\xi W(p,t)$ for the function $W(p,t)$,

$$J^{n+1}_\xi W(p,t) = W(p,t) + \sum_{k=1}^{n+1} \frac{1}{k!} W^{(k)}(p,t)[\{\xi\}^k], \tag{4-22}$$

$\xi \in H$ and $p \in \boldsymbol{P}_{N_{n+1}}\mathcal{M}_{N_1}$ and to verify the compatibility conditions of order $n+1$. Keeping in mind the already-considered cases $n=1$ and $n=2$, we introduce the required jet (4-22) as a backward solution of the equation

$$\partial_t J^{n+1}_\xi W(p) + A J^{n+1}_\xi W(p) = F^{[n+1]}(p,\xi), \quad \boldsymbol{P}_{N_{n+1}} J^{n+1}_\xi(p)|_{t=0} = \boldsymbol{P}_{N_{n+1}}(p+\xi), \tag{4-23}$$

where

$$\begin{aligned} F^{[n+1]}(p,\xi,t) := {} & F(W(p,t)) + F'(W(p,t)) j^{n+1}_\xi W(p,t) \\ & + \sum_{k=2}^{n+1} \frac{1}{k!}\Big(k F^{(k)}(W(p,t))[\{j^n_\xi V(p,t)\}^{k-1}, j^n_\xi W(p,t)] \\ & \qquad\qquad - (k-1) F^{(k)}(W(p,t))[\{j^n_\xi V(p,t)\}^k]\Big). \end{aligned} \tag{4-24}$$

The symbol "$[n+1]$" means that we have dropped out all terms of order greater than $n+1$ from the right-hand side, so $F^{[n+1]}$ is a polynomial of order $n+1$ in $\xi \in H$. Alternatively, the dropping out procedure means that we use the substitution

$$\{j^n_\xi V(p)\}^k \to \sum_{\substack{n_1+\dots+n_k \le n+1 \\ n_i \in \mathbb{N}}} B_{n_1,\dots,n_k}\{j^{n_1}_\xi V(p), \dots, j^{n_k}_\xi V(p)\}, \tag{4-25}$$

where the numbers $B_{n_1,\dots,n_k} \in \mathbb{R}$ are chosen in such a way that polynomials in the left- and right-hand sides of (4-25) coincide up to order $\{\xi\}^{n+1}$ inclusively and the term $[\{j^n_\xi V(p,t)\}^{k-1}, j^n_\xi W(p,t)]$ is treated analogously. The explicit expressions for these coefficients can be found using the formulas for the higher-order chain rule (Faà di Bruno-type formulas; see, e.g., [Roman 1980; Hájek and Johanis 2014]), but these expressions are lengthy and not essential for what follows, so we omit them.

Note also that the truncated jets $j^n_\xi V(p,t)$ are taken from the induction assumption. We seek the solution of (4-23) belonging to $L^2_{e^{n\theta_n+\theta_{n+1}}}(\mathbb{R}_-, H)$ for some θ_{n+1} satisfying (3-17) with N replaced by N_{n+1}. Expanding (4-24) in series with respect to ξ, we get the recurrent equations for finding the "derivatives" $W^{(k)}_\xi(p,t) := W^{(k)}(p,t)[\{\xi\}^k]$:

$$\partial_t W^{(k)}_\xi + A W^{(k)}_\xi - F'(W(p)) W^{(k)}_\xi = \Phi(j^{k-1}_\xi W, j^{k-1}_\xi V), \quad \boldsymbol{P}_{N_{n+1}} W^{(k)}_\xi|_{t=0} = 0, \tag{4-26}$$

for $k \ge 2$, where Φ is polynomial of order k in ξ which does not contain $W^{(l)}_\xi$, with $l \ge k$. Thus, the functions $W^{(k)}_\xi$ can be, indeed, found recursively. Moreover, the spectral gap assumption (4-21) guarantees that we can find θ_{n+1} satisfying (3-17) with $N = N_{n+1}$ such that $\theta_{n+1} + n\theta_n$ also satisfies this condition. Therefore, Proposition 3.7 guarantees the existence and uniqueness of the homogeneous polynomials $W^{(k)}_\xi(p)$ satisfying

$$\|W^{(k)}_\xi(p)\|_{L^2_{e^{(\theta_{n+1}+k\theta_n)t}}(\mathbb{R}_-,H)} \le C\|\xi\|^k \tag{4-27}$$

for $k = 1, \dots, n+1$.

To complete the proof of the theorem, we only need to verify that the jet $J_\xi W(p,t)$ satisfies the compatibility conditions of order $n+1$. If this is verified, the rest of the proof coincides with the one given above for the case $n=2$. We postpone this verification till the next section. Thus, the theorem is proved by modulo of compatibility conditions. □

Corollary 4.4. *Let the assumptions of Theorem 4.3 hold with $\mu > 0$ being small enough. Then the invariant manifold $\boldsymbol{P}_{N_n}\mathcal{M}_{N_1}$ of the extended IF (4-1) possesses an exponential tracking property in H_{N_n}; i.e., for every solution $u_{N_n}(t)$ of (4-1) there exists the corresponding solution $\bar u_{N_n}$ belonging to this manifold such that*

$$\|u_{N_n}(t) - \bar u_{N_n}(t)\| \le Ce^{-\theta_1 t} \tag{4-28}$$

for some positive C and θ_1.

Proof. As we have already mentioned, this is the standard corollary of the fact that $\mathcal{M}_{N_1}$ is normally hyperbolic and, therefore, persists under small C^1-perturbations; see [Bates et al. 1999; Fenichel 1972; Hirsch et al. 1977; Katok and Hasselblatt 1995]. Nevertheless, for the convenience of the reader, we now sketch a direct proof that does not use the normal hyperbolicity explicitly.

We first construct an invariant manifold $\overline{\mathcal{M}}_{N_1}$ with the base H_{N_1} in H_{N_n} for the extended IF. We do this exactly as in the proof of Theorem 3.2 by solving the backward problem

$$\partial_t u_{N_n} + Au_{N_n} - \boldsymbol{P}_{N_n}F(u_{N_n} + \widetilde{M}_{N_n}(u_{N_n})) = 0, \quad \boldsymbol{P}_{N_1}u_{N_n} = p \tag{4-29}$$

in the space $L^2_{e^{\theta t}}(\mathbb{R}_-, H_{N_n})$ with $\theta = (\lambda_{N_1} + \lambda_{N_1+1})/2$. This equation is $(C\mu)$-closed to

$$\partial_t \bar u_{N_n} + A\bar u_{N_n} - \boldsymbol{P}_{N_n}F(\bar u_{N_n} + M_{N_n}(\bar u_{N_n})) = 0, \quad \boldsymbol{P}_{N_1}\bar u_{N_n} = p \tag{4-30}$$

in the C^1-norm (since $\widetilde{M}_{N_n}$ is μ-closed to M_{N_n} due to Theorem 4.3). Thus, using Remark 3.13 and the Banach contraction theorem, we can construct a unique solution $u_{N_n}(t)$ of (4-29) in the $(C\mu)$-neighbourhood of the corresponding solution $\bar u_{N_n}$ of problem (4-30) and vice versa. This gives us the existence of the manifold $\overline{\mathcal{M}}_{N_1}$ which is generated by all backward solutions of (4-30) belonging to the space $L^2_{e^{\theta t}}(\mathbb{R}_-, H_{N_n})$. Since the solutions belonging to the invariant manifold $\boldsymbol{P}_{N_n}\mathcal{M}_{N_1}$ satisfy exactly the same property, we conclude that $\overline{\mathcal{M}}_{N_1} = \boldsymbol{P}_{N_n}\mathcal{M}_{N_1}$.

It remains to verify that the manifold $\overline{\mathcal{M}}_{N_1}$ possesses an exponential tracking property. This can be done as in the proof of Theorem 3.2 by considering the analogue of (3-14) for system (4-1) and using again that $\widetilde{M}_{N_n}$ is close to M_{N_n} in the C^1-norm. This finishes the proof of the corollary. □

Corollary 4.5. *Arguing as in Corollary 3.9, we check that the extended IM $\widetilde{\mathcal{M}}_{N_n}$ is also a $C^{n,\varepsilon}$-submanifold of $H^2 := D(A)$.*

5. Examples and concluding remarks

We now give several examples of our main theorem, as well as its reinterpretations, and state some interesting problems for further study. We start with the application to the 1-dimensional reaction-diffusion equation.

Example 5.1. Let us consider the following reaction-diffusion system in a 1-dimensional domain $\Omega = (-\pi, \pi)$:

$$\partial_t u = a\partial_x^2 u - f(u), \quad u|_{\Omega} = 0, \quad u|_{t=0} = u_0, \tag{5-1}$$

where u is an unknown function, $a > 0$ is a given viscosity parameter, and $f(u)$ is a given smooth function satisfying $f(0) = 0$ and some dissipativity conditions, for instance,

$$f(u)u \geq -C + \alpha|u|^2, \quad u \in \mathbb{R}.$$

for some C and $\alpha > 0$ (e.g., $f(u) = u^3 - u$ as in the case of real Ginzburg–Landau equation). Then, due to the maximum principle, we have the following dissipative estimate for the solutions of (5-1):

$$\|u(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} e^{-\alpha t} + C_*, \tag{5-2}$$

where the constant C_* is independent of u_0; see, e.g., [Babin and Vishik 1992; Chepyzhov and Vishik 2002; Temam 1988]. Thus, the associated solution semigroup $S(t)$ acting in the phase space $H := H_0^1(\Omega)$ possesses an absorbing set in $C(\overline{\Omega})$, and cutting-off the nonlinearity outside of this ball, we may assume without loss of generality that $f \in C_0^\infty(\mathbb{R})$.

After this transformation, (5-1) can be considered as an abstract parabolic equation (3-1) in the Sobolev space $H = H_0^1(\Omega)$. Since this space is an algebra with respect to pointwise multiplication (since we have only one spatial variable), the corresponding nonlinearity $F(u)(x) := f(u(x))$ is C^∞-smooth and all its derivatives are globally bounded.

Finally, the linear operator A in this example is $A = -a\partial_x^2$ endowed with the Dirichlet boundary conditions. Obviously, this operator is self-adjoint, positive definite and its inverse is compact. Moreover, its eigenvalues

$$\lambda_k = ak^2, \quad k \in \mathbb{N},$$

satisfy (4-2). Thus, our main Theorem 4.3 is applicable here and, therefore, problem (5-1) possesses an IM $\mathcal{M}_{N_1}$ of smoothness $C^{1,\varepsilon}$ for some $\varepsilon > 0$ and, for every $n \in \mathbb{N}$, this IM can be extended to a manifold $\widetilde{\mathcal{M}}_{N_n}$ of regularity C^{n,ε_n}, $\varepsilon_n > 0$, in the sense of Definition 4.1.

Remark 5.2. Our general theorem is applicable not only for a scalar reaction-diffusion equation (5-1), but also for systems where the analogue of (5-2) is known, for instance, for the case of 1-dimensional complex Ginzburg–Landau equation. However, one should be careful in the case where the diffusion matrix is not self-adjoint and especially when it contains nontrivial Jordan cells. In this case, even Lipschitz IM may not exist; see [Kostianko and Zelik 2022] for more details.

A bit unusual choice of the phase space $H = H_0^1(\Omega)$ (instead of the natural one $H = L^2(\Omega)$) is related to the fact that we need H to be an algebra in order to define Taylor jets for the nonlinearity F and to verify that it is C^∞. This, however, may be relaxed in applications since backward solutions of (3-4) and (3-18) are usually smooth in space and time if the nonlinearity f is smooth, so the Taylor jets for $V(p, t)$ will be well-defined even if we consider $L^2(\Omega)$ as a phase space and the theory works with minimal changes. This observation may be useful if we want to remove the assumption $f(0) = 0$ in (5-1), but in order to avoid technicalities, we prefer not to go further in this direction here.

The restriction to the 1-dimensional case is motivated by the fact that the spectral gap condition (4-2) is naturally satisfied by the Laplacian in 1-dimensional case only (it is an open problem already in the 2-dimensional case).

If we consider higher-order operators, say bi-Laplacian then the analogous result holds also in three dimensions. The typical example here is given by the Swift–Hohenberg equation in a bounded domain $\Omega \subset \mathbb{R}^3$:

$$\partial_t u = -(\Delta + 1)^2 u + u - u^2, \quad u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0,$$

where the spectral gap condition (4-2) is also satisfied; see [Zelik 2014]. We also note that although our main theorem is stated and proved for the case where F maps H to H, it can be generalized in a very straightforward way to the case where the operator F decreases smoothness and maps H to $H^{-s} := D(A^{-s/2})$ for some $s \in (0, 2)$. The spectral gap assumption (4-2) should be replaced by

$$\limsup_{n\to\infty} \left\{ \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}^{s/2} + \lambda_n^{s/2}} \right\} = \infty.$$

After this extension, our theorem becomes applicable to equations which contain spatial derivatives in the nonlinearity. A typical example of such applications is the 1-dimensional Kuramoto–Sivashinsky equation

$$\partial_t u + a\partial_x^2 u + \partial_x^4 u + u\partial_x u = 0, \quad \Omega = (-\pi, \pi), \quad a > 0,$$

endowed with Dirichlet or periodic boundary conditions; see [Zelik 2014] for more details.

Remark 5.3. As we mentioned in the Introduction, there is some significant recent progress in constructing IMs for concrete classes of parabolic equations which do not satisfy the spectral gap conditions (such as scalar reaction-diffusion equations in higher dimensions, 3-dimensional Cahn–Hilliard or complex Ginzburg–Landau equations, various modifications of Navier–Stokes systems, 1-dimensional reaction-diffusion-advection systems, etc.). The techniques developed in the present paper are not directly applicable to such problems (in particular, our technique is strongly based on the Perron method of constructing the IMs and it is not clear how to use the Perron method here since we do not have the so-called absolute normal hyperbolicity in the most part of equations mentioned above; see [Kostianko 2018; Kostianko and Zelik 2015] for more details). However, we believe that the proper modification of our method would allow us to cover these cases as well. We return to this problem elsewhere.

We now give an alternative (probably more transparent and more elegant) formulation of Theorem 4.3. We recall that in Theorem 4.3, we have directly constructed a smooth extended IF (4-1) for the initial equation (3-1). This extended IF captures all nontrivial dynamics of (3-1), but the associated smooth extended IM $\mathcal{M}_n$ is not associated with the "true" IM of any system of the form (3-1). This drawback can be easily corrected in a more or less standard way which leads to the following reformulation of our main result.

Corollary 5.4. *Let the assumptions of Theorem 4.3 be satisfied and let $\mathcal{M}_{N_1}$ be the C^{1,ε_1}-smooth IM of (3-1) which corresponds to the first spectral gap. Then, for every $n \in \mathbb{N}$, $n > 1$, there exists a modified nonlinearity $\widetilde{F} : H \to H$ which belongs to $C_b^{n-1,\varepsilon_n}(H, H)$ for some $\varepsilon_n > 0$ such that:*

(1) *The initial IM $\mathcal{M}_{N_1}$ is simultaneously an IM for the modified equation*

$$\partial_t u + Au = \widetilde{F}_n(u). \tag{5-3}$$

(2) *Equation (5-3) possesses a C^{n,ε_n}-smooth IM $\widetilde{\mathcal{M}}_{N_n}$ of dimension N_n such that the initial IM $\mathcal{M}_1$ is a normally hyperbolic globally stable submanifold of $\widetilde{\mathcal{M}}_{N_n}$.*

(3) *The nonlinearity $\widetilde{F}_n(u)$ depends on the variable $u_{N_n} := \boldsymbol{P}_{N_n}u$ only and the IF associated with the IM $\widetilde{\mathcal{M}}_{N_n}$ is given by (4-1) where K_2 is replaced by N_n.*

Proof. Indeed, we take the manifold $\widetilde{\mathcal{M}}_{N_n}$ constructed in Theorem 4.3 and define the desired function $\widetilde{F}_n$ as

$$\boldsymbol{P}_{N_n}\widetilde{F}_n(u) := \boldsymbol{P}_{N_n}F(u_{N_n} + \widetilde{M}_{N_n}(u_{N_n})) \tag{5-4}$$

and

$$\boldsymbol{Q}_{N_n}\widetilde{F}_n(u) := \widetilde{M}'_{N_n}(u_{N_n})[-A\widetilde{M}_{N_n}(u_{N_n}) + \boldsymbol{P}_{N_n}F(u_{N_n} + \widetilde{M}_{N_n}(u_{N_n}))] + A\widetilde{M}_{N_n}(u_{N_n}). \tag{5-5}$$

Then, due to the choice of $\boldsymbol{P}_{N_n}$-component of $\widetilde{F}_n(u)$, the equation for u_{N_n} is decoupled from the equation for the $\boldsymbol{Q}_{N_n}$-component and coincides with the extended IF for (5-3) constructed in Theorem 4.3. On the other hand, the $\boldsymbol{Q}_{N_n}$-component of $\widetilde{F}_n$ is chosen in a form which guarantees that $\widetilde{\mathcal{M}}_{N_n}$ is an invariant manifold for (5-3). Moreover, if $u(t)$ solves (5-3) with such a nonlinearity and $v(t) := u(t) - \boldsymbol{P}_{N_n}u(t) - \widetilde{M}_{N_n}(u_{N_n}(t))$, then this function satisfies

$$\partial_t v + Av = 0, \quad \boldsymbol{P}_{N_n}v(t) \equiv 0,$$

and, therefore,

$$\|v(t)\|_H \le \|v(0)\|_H e^{-\lambda_{N_n+1}t}.$$

Thus, $\widetilde{M}_{N_n}$ is indeed an IM for problem (5-3) and we only need to check the regularity of the modified function $\widetilde{F}_n$.

The $\boldsymbol{P}_{N_n}$ component (5-4) is clearly C^{n,ε_n}-smooth, but the situation with the $\boldsymbol{Q}_{N_n}$ is a bit more delicate due to the presence of terms $A\widetilde{M}_{N_n}(u_{N_n})$ and $\widetilde{M}'_{N_n}(u_{N_n})$. The first term is not dangerous since we know that $\widetilde{M}_{N_n}$ is C^{n,ε_n}-smooth as the map from H_{N_n} to H^2. The second term is worse and decreases the smoothness of the $\widetilde{F}_n$ till C^{n-1,ε_n}. Thus, the corollary is proved. □

Remark 5.5. The modified nonlinearity $\widetilde{F}_n(u)$ can be interpreted as a "clever" cut-off of the initial nonlinearity $F(u)$ outside of the global attractor (even outside of the IM of minimal dimension). In this sense we may say that all obstacles for the existence of $C^{n,\varepsilon}$-smooth IMs can be removed by appropriately cutting off the nonlinearity outside of the global attractor, which does not affect the dynamics of the initial problem. This demonstrates the importance of finding the proper cut off procedure in the theory of IMs.

Example 5.6. We now return to the model example of G. Sell introduced in Example 3.11 and show how the problem of smoothness of an invariant manifold can be resolved. Since the nonlinearity for this system is not *globally* Lipschitz continuous, the above-developed theory is formally not applicable and we need to cut-off the nonlinearity first. We overcome this problem by considering only *local* manifolds in a small neighbourhood of the origin.

Indeed, it is not difficult to see that system (3-31) has an explicit particular solution

$$u_1(t) = \pm e^{-t}, \quad u_{n+1}(t) = C_n e^{-2^n t} t^{2^n - 1}, \quad n > 1,$$

where the coefficients C_n satisfy the recurrence relation

$$C_{n+1} = \frac{1}{2^n - 1} C_n^2, \quad C_0 = 1.$$

This solution determines a 1-dimensional local invariant manifold

$$\mathcal{M}_1 = \{p + M(p) : p \in H_1 = \mathbb{R},\ |p| < \beta\},$$

where $M : \mathbb{R} \to H$ is defined by $M = (0, M_1(p), M_2(p), \dots)$ and

$$M_{n+1}(p) = C_n p^{2^n} \left(\ln \frac{1}{|p|} \right)^{2^n - 1}, \quad n \in \mathbb{N},$$

which is a 1-dimensional IM for system (3-31) and β is a sufficiently small positive number. Indeed, since $C_n \le 2^{-\alpha 2^n}$ for some positive α, this manifold is well-defined as a local submanifold of $H = l_2$ (if $\beta > 0$ is small enough) and is $C^{1,\varepsilon}$-smooth for any $\varepsilon \in (0, 1)$. Moreover, we see that $M_2(p)$ is only $C^{1,\varepsilon}$-smooth and higher components are more regular; in particular, $M_n(p)$ is $C^{2^{n-1}-1,\varepsilon}$-smooth. This shows us how to define the extended manifolds of an arbitrary finite smoothness. Namely, let us fix some $n \in \mathbb{N}$ and consider the manifold

$$\widetilde{\mathcal{M}}_n := \{p + \widetilde{M}_n(p) : p \in H_n,\ |p_1| < \beta\}, \quad \widetilde{M}_n(p) := (\{0\}^n, M_{n+1}(p_1), M_{n+2}(p_1), M_{n+3}(p_1), \dots). \tag{5-6}$$

Clearly $\widetilde{\mathcal{M}}_n$ is $C^{2^n-1,\varepsilon}$-smooth and $\mathcal{M}_1$ is a submanifold of $\widetilde{\mathcal{M}}_n$. Moreover, if we define the modified nonlinearity $\widetilde{F}_n(u)$ as

$$\widetilde{F}_n(u) = (0, u_1^2, u_2^2, \dots, u_{n-1}^2, M_{n+1}(u_1), M_{n+2}(u_1), \dots), \tag{5-7}$$

then it will be $C^{2^n-1,\varepsilon}$-smooth and the extended manifold $\widetilde{\mathcal{M}}_n$ will be an IM for the corresponding modified equation (5-3). Finally, the normal hyperbolicity of $\mathcal{M}_1$ in $\widetilde{\mathcal{M}}_n$ follows from the fact that any solution on $\mathcal{M}_1$ decays to zero no faster than e^{-t} due to the nonzero first component, if we look to the transversal directions, the smallest decay rate is determined by the second component and this decay is at least as $t^3 e^{-2t}$. Since our model system is explicitly solvable, we leave verifying this normal hyperbolicity to the reader. We also note that the extended IF in this case reads

$$\frac{d}{dt} u_1 + u_1 = 0, \quad \frac{d}{dt} u_k + 2^{k-1} u_k = u_{k-1}^2, \quad k = 2, \dots, n,$$

which is nothing more than the Galerkin approximation system to (3-31).

Remark 5.7. We see that, in the toy example of (3-31), we can find the desired extension of the initial IM explicitly without using the Whitney extension theorem (and even without assuming the global boundedness of F and its derivatives). Moreover, the dependence of smoothness of the extended IM on its dimension is very nice; namely, if we want to have a C^n-smooth IM, it is enough to take $\dim \widetilde{M} \sim \log_2 n$. Of course, this is partially related to good exponentially growing spectral gaps, but the main reason is that we have an extra regularity property for the initial IM, namely, that the smoothness of projections $\boldsymbol{Q}_k M(p)$ grows with k. Unfortunately, this is not true in a more or less general case, which makes the extension construction much more involved. In particular, we do not know how to gain more than one unit of smoothness from one spectral gap and have to use n different spectral gaps to get n units of smoothness. This, in turn, leads to extremely fast growth of the dimension of the manifold with respect to the regularity (as not difficult to see, in Example 5.1, the dimension of $\widetilde{\mathcal{M}}_{N_n}$ grows as a double exponent with respect to n).

We believe that this problem is technical and the estimates for the dimension can be essentially improved. Indeed, if we would be able to get n units of extra regularity using one extra (sufficiently large) gap the above-mentioned growth of the dimension would become linear in n in Example 5.1. We expect that this linear growth is optimal, and we are even able to construct the corresponding Taylor jets. But these jets do not satisfy the compatibility conditions and we do not know how to correct them properly.

Appendix: Verifying the compatibility conditions

The aim of this appendix is to show that the jets $J^{n+1}_\xi W(p,t),\ p\in \boldsymbol{P}_{N_{n+1}}H$, constructed via (4-23), satisfy the compatibility conditions up to order $n+1$ and, thus, to complete the proof of Theorem 4.3. We will proceed by induction with respect to the order $m\le n+1$.

Indeed, the first-order compatibility conditions are trivially satisfied since the functions $W(p,t)$ are $C^{1,\varepsilon}$-smooth. Assume that the m-th order conditions are satisfied for some $m\le n+1$, and for all $m_1\le m$

$$J^{m_1}_\xi W(p_1)-J^{m_1}_{\delta+\xi}W(p)=O_{\theta_{n+1}+(m_1-1)\theta_n}((\|\delta\|+\|\xi\|)^{m_1+\varepsilon}) \tag{A-1}$$

for all $\xi\in H,\ p_1,p\in \boldsymbol{P}_{N_{n+1}}\mathcal{M}_{N_1},\ \varepsilon>0,\ \delta:=p_1-p$ and some constant C which is independent of p,p_1. Using the fact that $V(p,t)=W(p,t)$ for all $p\in \boldsymbol{P}_{N_{n+1}}\mathcal{M}_{N_1}$ together with the analogue of (A-1) for the already constructed jets $J^m_\xi V(p,t)$, we end up with

$$\begin{aligned}V(p_1)=W(p_1)&=V(p)+j^{m_1}_\delta V(p)+O_{m_1\theta_n+\varepsilon}(\|\delta\|^{m_1+\varepsilon})\\&=W(p)+j^{m_1}_\delta W(p)+O_{\theta_{n+1}+(m_1-1)\theta_n+\varepsilon}(\|\delta\|^{m_1+\varepsilon})\end{aligned} \tag{A-2}$$

for all $p_1,p\in \boldsymbol{P}_{N_{n+1}}\mathcal{M}_{N_1},\ \delta:=p_1-p$ and, therefore $v(t):=V(p_1,t)-V(p,t)$ satisfies

$$\begin{gathered}v=j^{m_1}_\delta V(p)+O_{m_1\theta_n+\varepsilon}(\|\delta\|^{m_1+\varepsilon})=j^{m_1}_\delta W(p)+O_{\theta_{n+1}+(m_1-1)\theta_n+\varepsilon}(\|\delta\|^{m_1+\varepsilon}),\\ j^{m_1}_\delta V(p)-j^{m_1}_\delta W(p)=O_{\theta_{n+1}+(m_1-1)\theta_n+\varepsilon}(\|\delta\|^{m_1+\varepsilon}).\end{gathered} \tag{A-3}$$

We now turn to the $(m+1)$-th jets and start with the following lemma which gives the compatibility conditions in the particular case $\xi=0$.

Lemma A.1. *Let the above assumptions hold. Then*

$$v=W(p_1)-W(p)=j^{m+1}_\delta W(p)+O_{\theta_{n+1}+m\theta_n+\varepsilon}(\|\delta\|^{m+1+\varepsilon}) \tag{A-4}$$

for all $p_1,p\in \boldsymbol{P}_{N_{n+1}}\mathcal{M}_{N_1}$ and $\delta:=p_1-p$. Moreover,

$$F(V(p_1))=F^{[m+1]}(p,\delta)+O_{\theta_{n+1}+m\theta_n+\varepsilon}(\|\delta\|^{m+1+\varepsilon}) \tag{A-5}$$

for some $\varepsilon>0$.

Proof. Let $R := v - j_\delta^{m+1}W(p)$. Then, by the definition (4-23), this function solves
$$\partial_t R + AR = F(V(p_1)) - F^{[m+1]}(p,\delta), \qquad \boldsymbol{P}_{N_{n+1}}R|_{t=0} = 0. \tag{A-6}$$
Let us study the term $F^{[m+1]}(p,\delta)$ at the right-hand side (which is defined by (4-24)). Using (A-3) and the trick (4-25), we may replace $j_\delta^m V(p)$ and $j_\delta^m W(p)$ by v in all terms in (4-24) which contain the second and higher derivatives of F (the error will be of order $\|\delta\|^{m+1+\varepsilon}$). Actually, we cannot do this in the term with the first derivative at the moment since this requires (A-3) for W of order $m+1$, which we are now verifying. This, gives
$$F^{[m+1]}(p,\delta) = F(V(p)) + F'(V(p))j_\delta^{m+1}W(p) \\ + \sum_{k=2}^{m+1} \frac{1}{k!} F^{(k)}(V(p))[\{v\}^k] + O_{\theta_{n+1}+m\theta_n+\varepsilon}(\|\delta\|^{m+1+\varepsilon}). \tag{A-7}$$
Indeed, let us consider the terms in (A-7) containing $j_\delta^m W$ only (the terms without it are analogous, but simpler). Using the analogue of (4-25),
$$[\{j_\delta^m V(p)\}^{k-1}, j_\delta^m W(p)] \to \sum_{\substack{n_1+\cdots+n_k\le m+1\\ n_i\in\mathbb{N}}} B'_{n_1,\dots,n_k}\{j_\delta^{n_1}V(p),\dots,j_\delta^{n_{k-1}}V(p), j_\delta^{n_k}W(p)\}, \tag{A-8}$$
the growth exponent of the remainder does not exceed
$$(n_1+\cdots+n_{k-1})\theta_n + \theta_{n+1} + (n_k-1)\theta_n + \varepsilon \le \theta_{n+1} + m\theta_n + \varepsilon,$$
where we have implicitly used our induction assumptions (A-3) and decreased the exponent ε if necessary.

Using now the Taylor theorem for $F \in C^{m+1,\varepsilon}$ together with estimate (3-23) for v, we infer that
$$F(V(p_1)) - F^{[m+1]}(p,\delta) = F'(V(p))R + O_{\theta_{n+1}+m\theta_n+\varepsilon}(\|\delta\|^{m+1+\varepsilon})$$
and, therefore, the function R solves
$$\partial_t R + AR - F'(V(p))R = O_{\theta_{n+1}+m\theta_n+\varepsilon}(\|\delta\|^{m+1+\varepsilon}), \qquad \boldsymbol{P}_{N_{n+1}}R|_{t=0} = 0. \tag{A-9}$$
Since by the induction assumption $\theta_n < \lambda_{N_n+1} - L$, assumption (4-21) guarantees the existence of θ_{n+1} and $\varepsilon > 0$ such that $\theta_{n+1} + m\theta_n + \varepsilon$ satisfies (3-17) with N replaced by N_{n+1}. Thus, Proposition 3.7 gives the estimate
$$\|R\|_{L^2_{e^{(\theta_{n+1}+m\theta_n+\varepsilon)t}}(\mathbb{R}_-,H)} \le C\|\delta\|^{m+1+\varepsilon}$$
and (A-4) is proved. Estimate (A-5) is now a straightforward corollary of (A-7) and the Taylor theorem (since we are now allowed to replace $j_\delta^{m+1}W$ by v). Thus, the lemma is proved. □

We now turn to the general case $\xi \ne 0$. To this end we need the following key lemma.

Lemma A.2. *Let the above assumptions hold. Then, the following formula is satisfied*:
$$F^{[m+1]}(p_1,\xi) - F^{[m+1]}(p,\xi+\delta) = F'(V(p))\big(j_\delta^{m+1}W(p) + j_\xi^{m+1}W(p_1) - j_{\xi+\delta}^{m+1}W(p)\big) \\ + O_{\theta_{n+1}+m\theta_n+\varepsilon}((\|\delta\|+\|\xi\|)^{m+1+\varepsilon}), \tag{A-10}$$
where $\xi \in H$, $p_1, p \in \boldsymbol{P}_{N_{n+1}}\mathcal{M}_{N_1}$ *and* $\delta = p_1 - p$.

Proof. Indeed, according to the definition (4-24) and formula (A-5), we have

$$\begin{aligned}F^{[m+1]}(p_1,\xi) &= F^{[m+1]}(p,\delta)+F'(V(p_1))j_\xi^{m+1}W(p_1)\\ &\quad+\sum_{l=2}^{m+1}\frac{1}{l!}\big(lF^{(l)}(V(p_1))[j_\xi^m W(p_1),\{j_\xi^m V(p_1)\}^{l-1}]-(l-1)F^{(l)}(V(p_1))[\{j_\xi^m V(p_1)\}^l]\big)\\ &\quad+O_{\theta_{n+1}+m\theta_n+\varepsilon}((\|\xi\|+\|\delta\|)^{m+1+\varepsilon}).\end{aligned}\tag{A-11}$$

We recall that, according to our agreement and formula (4-25), the right-hand side does not contain the terms of order larger than $m+1$. Expanding now the derivatives $F^{(l)}(V(p_1))$ into Taylor series around $V(p)$ and using (A-3), we get

$$\begin{aligned}F^{[m+1]}(p_1,\xi) &= F^{[m+1]}(p,\delta)+F'(V(p))(j_\xi^{m+1}W(p_1)-j_\xi^m W(p_1))\\ &\quad+\sum_{l=1}^{m+1}\sum_{k=l}^{m+1}\frac{1}{l!(k-l)!}\big(lF^{(k)}(V(p))[\{j_\delta^m V(p)\}^{k-l},j_\xi^m W(p_1),\{j_\xi^m V(p_1)\}^{l-1}]\\ &\qquad\qquad-(l-1)F^{(k)}(V(p))[\{j_\delta^m V(p)\}^{k-l},\{j_\xi^m V(p_1)\}^l]\big)\\ &\quad+O_{\theta_{n+1}+m\theta_n+\varepsilon}((\|\xi\|+\|\delta\|)^{m+1+\varepsilon}).\end{aligned}\tag{A-12}$$

Finally, changing the order of summation, we arrive at

$$\begin{aligned}F^{[m+1]}(p_1,\xi) &= F^{[m+1]}(p,\delta)+F'(V(p))j_\xi^{m+1}W(p_1)\\ &\quad+\sum_{k=2}^{m+1}\frac{1}{k!}\sum_{l=1}^{k}C_k^l\big(lF^{(k)}(V(p))[\{j_\delta^m V(p)\}^{k-l},j_\xi^m W(p_1),\{j_\xi^m V(p_1)\}^{l-1}]\\ &\qquad\qquad-(l-1)F^{(k)}(V(p))[\{j_\delta^m V(p)\}^{k-l},\{j_\xi^m V(p_1)\}^l]\big)\\ &\quad+O_{\theta_{n+1}+m\theta_n+\varepsilon}((\|\xi\|+\|\delta\|)^{m+1+\varepsilon}).\end{aligned}\tag{A-13}$$

Let us now look to the term $F^{[m+1]}(p,\xi+\delta)$. According to (4-24), we have

$$\begin{aligned}&F^{[m+1]}(p,\xi+\delta)\\ &\quad= F(V(p))+F'(V(p))j_{\xi+\delta}^{m+1}W(p)\\ &\quad+\sum_{k=2}^{m+1}\frac{1}{k!}\big(kF^{(k)}(V(p))[j_{\xi+\delta}^m W(p),\{j_{\xi+\delta}^m V(p)\}^{k-1}]-(k-1)F^{(k)}(V(p))[\{j_{\xi+\delta}^m V(p)\}^k]\big).\end{aligned}\tag{A-14}$$

From the induction assumption, the compatibility assumptions (A-1) hold for $j_{\xi+\delta}^{m_1}W$ and give

$$j_{\xi+\delta}^{m_1}W(p)=j_\delta^{m_1}W(p)+j_\xi^{m_1}W(p_1)+O_{\theta_{n+1}+(m_1-1)\theta_n+\varepsilon}((\|\delta\|+\|\xi\|)^{m_1+\varepsilon})$$

for all $m_1\le m$ and the analogous identities hold also for $j_{\xi+\delta}^{m_1}V$:

$$j_{\xi+\delta}^{m_1}V(p)=j_\delta^{m_1}V(p)+j_\xi^{m_1}V(p_1)+O_{m_1\theta_n+\varepsilon}((\|\delta\|+\|\xi\|)^{m_1+\varepsilon}).$$

Moreover, using (A-3), we may also get

$$j_{\xi+\delta}^{m_1}W(p)=j_\delta^{m_1}V(p)+j_\xi^{m_1}W(p_1)+O_{\theta_{n+1}+(m_1-1)\theta_n+\varepsilon}((\|\delta\|+\|\xi\|)^{m_1+\varepsilon})$$

for all $m_1 \le m$. Inserting these formulas to (A-14), we arrive at

$$
\begin{aligned}
F^{[m+1]}(p,\xi+\delta) = F(V(p)) + F'(V(p))\, j^{m+1}_{\xi+\delta} W(p)\\
+ \sum_{k=2}^{m+1} \frac{1}{k!}\Big(kF^{(k)}(V(p))[j^m_\delta V(p) + j^m_\xi W(p_1), \{j^m_\delta V(p) + j^m_\xi V(p_1)\}^{k-1}]\\
- (k-1)F^{(k)}(V(p))[\{j^m_\delta V(p) + j^m_\xi V(p_1)\}^k]\Big)\\
+ O_{\theta_{n+1}+m\theta_n+\varepsilon}((\|\delta\| + \|\xi\|)^{m+1+\varepsilon}).
\end{aligned}
\tag{A-15}
$$

Using the binomial formula (2-1), we arrive at

$$
\begin{aligned}
F^{[m+1]}(p,\xi+\delta) = F(V(p)) + F'(V(p))\, j^{m+1}_{\xi+\delta} W(p)\\
+ \sum_{k=2}^{m+1} \frac{1}{k!}\Bigg(\sum_{l=1}^{k} kC^{l-1}_{k-1} F^{(k)}(V(p))[j^m_\xi W(p_1), \{j^m_\delta V(p)\}^{k-l}, \{j^m_\xi V(p_1)\}^{l-1}]\\
+ \sum_{l=0}^{k-1} kC^{l}_{k-1} F^{(k)}(V(p))[j^m_\delta V(p), \{j^m_\delta V(p)\}^{k-l-1}, \{j^m_\xi V(p_1)\}^{l}]\\
- \sum_{l=0}^{k} (k-1)C^{l}_{k} F^{(k)}(V(p))[\{j^m_\delta V(p)\}^{k-l}, \{j^m_\xi V(p_1)\}^{l}]\Bigg)\\
+ O_{\theta_{n+1}+m\theta_n+\varepsilon}.((\|\delta\| + \|\xi\|)^{m+1+\varepsilon}).
\end{aligned}
\tag{A-16}
$$

We need to compare (A-13) and (A-16). To this end, we first note that

$$
lC^l_k = kC^{l-1}_{k-1}
$$

and, therefore, the terms containing the jets of W in these two formulas coincide. Thus, we only need to look at the terms without jets of W. In the case $l = k$, we have only one term in the right-hand side of (A-16), which obviously coincides with the analogous term in (A-13). Let us now look at the terms with $l = 1, \dots, k-1$. Due to the obvious identity

$$
-(l-1)C^l_k = kC^l_{k-1} - (k-1)C^l_k,
$$

these terms again coincide. Thus, it remains to look at the extra terms which correspond to $l = 0$ in (A-16) and which are absent in the sums of (A-13). Finally, using (A-3) and (A-5), we get the following identity involving these extra terms:

$$
\begin{aligned}
F(V(p)) + \sum_{k=2}^{m+1} \frac{1}{k!} F^{(k)}(V(p))[\{j^m_\delta V(p)\}^k]\\
= F^{[m+1]}(p,\delta) - F'(V(p))\, j^{m+1}_\delta W(p) + O_{\theta_{n+1}+m\theta_n+\varepsilon}((\|\delta\| + \|\xi\|)^{m+1+\varepsilon}).
\end{aligned}
\tag{A-17}
$$

This gives the identity

$$
\begin{aligned}
F^{[m+1]}(p_1,\xi) - F'(V(p))\, j^{m+1}_\xi W(p_1)\\
= F^{[m+1]}(p,\xi+\delta) - F'(V(p))(j^{m+1}_{\xi+\delta} W(p) - j^{m+1}_\delta W(p)) + O_{\theta_{n+1}+m\theta_n+\varepsilon}((\|\delta\| + \|\xi\|)^{m+1+\varepsilon})
\end{aligned}
\tag{A-18}
$$

and finishes the proof of the lemma. □

We are now ready to finish the check of the compatibility conditions. Note that, due to (A-4), we have

$$J^{m+1}_{\xi}W(p_1) - J^{m+1}_{\xi+\delta}W(p) \\ = j^{m+1}_{\delta}W(p) + j^{m+1}_{\xi}W(p_1) - j^{m+1}_{\xi+\delta}W(p) + O_{\theta_{n+1}+m\theta_n+\varepsilon}((\|\delta\| + \|\xi\|)^{m+1+\varepsilon}). \quad \text{(A-19)}$$

Let finally $U(t) := J^{m+1}_{\xi}W(p_1) - J^{m+1}_{\xi+\delta}W(p)$. Then, according to definition (4-22), Lemma A.2 and the fact that $\delta = p_1 - p$, this function solves the equation

$$\partial_t U + AU - F'(V(p))U = O_{\theta_{n+1}+m\theta_n+\varepsilon}(\|\delta\| + \|\xi\|)^{m+1+\varepsilon}), \quad \boldsymbol{P}_{N_{n+1}}U|_{t=0} = 0, \quad \text{(A-20)}$$

and by Proposition 3.7, we arrive at

$$J^{m+1}_{\xi}W(p_1) - J^{m+1}_{\xi+\delta}W(p) = O_{\theta_{n+1}+m\theta_n+\varepsilon}((\|\delta\| + \|\xi\|)^{m+1+\varepsilon}). \quad \text{(A-21)}$$

Thus, the $(m+1)$-th order compatibility conditions for $J^{m+1}_{\xi}W(p)$ are verified. The induction with respect to m gives us that $J^{n+1}_{\xi}W(p)$ also satisfies the compatibility conditions (of course, we cannot take $m > n$ since we need the compatibility conditions of order m for $J^{m}_{\xi}V(p)$ to proceed). This completes the proof of our main Theorem 4.3.

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