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THE STABILITY OF SIMPLE PLANE-SYMMETRIC SHOCK FORMATION FOR THREE-DIMENSIONAL COMPRESSIBLE EULER FLOW WITH VORTICITY AND ENTROPY

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Consider a one-dimensional simple small-amplitude solution $(\varrho_{(\mathrm{bkg})}, v^1_{(\mathrm{bkg})})$ to the isentropic compressible Euler equations which has smooth initial data, coincides with a constant state outside a compact set, and forms a shock in finite time. Viewing $(\varrho_{(\mathrm{bkg})}, v^1_{(\mathrm{bkg})})$ as a plane-symmetric solution to the full compressible Euler equations in three dimensions, we prove that the shock-formation mechanism for the solution $(\varrho_{(\mathrm{bkg})}, v^1_{(\mathrm{bkg})})$ is stable against all sufficiently small and compactly supported perturbations. In particular, these perturbations are allowed to break the symmetry and have nontrivial vorticity and variable entropy.

Our approach reveals the full structure of the set of blowup-points at the first singular time: within the constant-time hypersurface of first blowup, the solution's first-order Cartesian coordinate partial derivatives blow up precisely on the zero level set of a function that measures the inverse foliation density of a family of characteristic hypersurfaces. Moreover, relative to a set of geometric coordinates constructed out of an acoustic eikonal function, the fluid solution and the inverse foliation density function remain smooth up to the shock; the blowup of the solution's Cartesian coordinate partial derivatives is caused by a degeneracy between the geometric and Cartesian coordinates, signified by the vanishing of the inverse foliation density (i.e., the intersection of the characteristics).

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Keywords: compressible Euler equations, shock formation, stable singularity formation, wave breaking, vectorfield method, characteristics, eikonal function, null condition, null hypersurface, null structure.

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1. Introduction

It is classically known — going back to the work of Riemann — that the compressible Euler equations admit solutions for which singularities develop from smooth initial data. Indeed, such examples can already be found in the plane symmetric isentropic case. In this case, the compressible Euler equations reduce to a 2×2 hyperbolic system in $1+1$-dimensions, which can be analyzed using Riemann invariants. In particular, it is easy to show that simple plane-symmetric solutions — solutions with one vanishing Riemann invariant — obey a Burgers-type equation, and that a shock can form in finite time. By a shock, we mean that the solution remains bounded but its first-order partial derivative with respect to the standard spatial coordinate blows up, and that the blowup is tied to the intersection of the characteristics.

In this article, we prove that a class of simple plane-symmetric isentropic small-amplitude shock-forming solutions to the compressible Euler equations are *stable* under small perturbations which break the symmetry and admit variable vorticity and entropy. In particular, the perturbed solutions develop a shock singularity in finite time. This provides the details of the argument sketched in [37; 52] and completes the program that we have initiated (partly joint also with Gustav Holzegel and Willie Wai-Yeung Wong) in [36; 37; 50; 52].

We will consider the spatial domain[1] $\Sigma \doteq \mathbb{R} \times \mathbb{T}^2 = \mathbb{R} \times (\mathbb{R}/\mathbb{Z})^2$ and a time interval I. Our unknowns are the density $\varrho : I \times \Sigma \to \mathbb{R}_{>0}$, the velocity $v : I \times \Sigma \to \mathbb{R}^3$, and the entropy $s : I \times \Sigma \to \mathbb{R}$. Relative to the standard Cartesian coordinates (t, x^1, x^2, x^3) on $I \times \mathbb{R} \times \mathbb{T}^2$, the compressible Euler equations can be expressed as

$$(\partial_t + v^a \partial_a)\varrho = -\varrho \operatorname{div} v, \tag{1-1}$$

$$(\partial_t + v^a \partial_a)v^j = -\frac{1}{\varrho}\delta^{ja}\partial_a p, \qquad j = 1, 2, 3, \tag{1-2}$$

$$(\partial_t + v^a \partial_a)s = 0, \tag{1-3}$$

where (from now on) δ^{ij} denotes the Kronecker delta, $\operatorname{div} v \doteq \partial_a v^a$ is the Euclidean divergence of v, repeated lowercase Latin indices are summed over $i, j = 1, 2, 3$, and the pressure p relates to ϱ and s by a prescribed smooth *equation of state* $p = p(\varrho, s)$. In other words, the right-hand side of (1-2) can be expressed as

$$-\frac{1}{\varrho}\delta^{ja}\,\partial_a p = -\frac{1}{\varrho}p_{;\varrho}\delta^{ja}\,\partial_a \varrho - \frac{1}{\varrho}p_{;s}\delta^{ja}\,\partial_a s,$$

where $p_{;\varrho}$ denotes[2] the partial derivative of the equation of state with respect to the density at fixed s, and analogously for $p_{;s}$.

For the remainder of the paper:

(1) We fix a constant $\bar{\varrho} > 0$ and a constant solution $(\varrho, v^i, s) = (\bar{\varrho}, 0, 0)$ to (1-1)–(1-3).

(2) We fix an equation of state $p = p(\varrho, s)$ such that[3] $(\partial p/\partial\varrho)(\bar{\varrho}, 0) = 1$.

[1] It is only for technical convenience that we chose the spatial topology $\mathbb{R} \times \mathbb{T}^2$. Similar results also hold, for instance, on $\mathbb{R}^3$.

[2] Later in the paper, we will take the partial derivative of various quantities with respect to the logarithmic density ρ. If f is a function of the fluid unknowns, then $f_{;\rho}$ will denote the partial derivative of f with respect to ρ when the other fluid variables are held fixed. Similarly, $f_{;s}$ denotes the partial derivative of f with respect to s when the other fluid variables are held fixed.

[3] This normalization can always be achieved by a change of variables as long as $(\partial p/\partial\varrho)(\bar{\varrho}, 0) > 0$; see [36, footnote 19].

For notational convenience, we define the logarithmic density $\rho \doteq \log(\varrho/\bar{\varrho})$ and the speed of sound $c(\rho, s) \doteq \sqrt{\partial p/\partial \varrho}(\varrho, s)$. We will from now on think of c as a function of (ρ, s).

We will study perturbations of a shock-forming background solution $(\varrho_{(\mathrm{bkg})}, v^i_{(\mathrm{bkg})}, s_{(\mathrm{bkg})})$ arising from smooth initial data such that the following hold:

(1) The background solution is plane-symmetric and isentropic, i.e., $v^2_{(\mathrm{bkg})} = v^3_{(\mathrm{bkg})} = s_{(\mathrm{bkg})} = 0$, and $(\varrho_{(\mathrm{bkg})}, v^1_{(\mathrm{bkg})})$ are functions only of t and x^1.

(2) The background solution is simple, i.e., the Riemann invariant $\mathcal{R}^{(\mathrm{bkg})}_{(-)}$, satisfies

$$\mathcal{R}^{(\mathrm{bkg})}_{(-)} \doteq v^1_{(\mathrm{bkg})} - \int_0^{\rho_{(\mathrm{bkg})}} c(\rho', 0)\, d\rho' = 0.$$

(3) The background solution is initially compactly supported in an x^1-interval of length $\leq 2\mathring{\sigma}$, i.e., outside this interval, $(\varrho_{(\mathrm{bkg})}, v^i_{(\mathrm{bkg})}, s_{(\mathrm{bkg})}) \restriction_{t=0} = (\bar{\varrho}, 0, 0)$.

(4) At time 0 (and hence throughout the evolution), the Riemann invariant $\mathcal{R}^{(\mathrm{bkg})}_{(+)} \doteq v^1_{(\mathrm{bkg})} + \int_0^{\rho_{(\mathrm{bkg})}} c(\rho', 0)\, d\rho'$ has small $\leq \mathring{\alpha}$ amplitude.

(5) At time 0, the Cartesian spatial derivatives of $\mathcal{R}^{(\mathrm{bkg})}_{(+)}$ up to the third order[4] are bounded above pointwise by $\leq \mathring{\delta}^{(\mathrm{bkg})}$ (where $\mathring{\delta}^{(\mathrm{bkg})}$ is not necessarily small).

(6) The quantity[5] $\mathring{\delta}^{(\mathrm{bkg})}_*$ (where $\mathring{\delta}^{(\mathrm{bkg})}_*$ is not necessarily small) that controls the blowup-time satisfies[6]

$$\mathring{\delta}^{(\mathrm{bkg})}_* \doteq \frac{1}{2} \sup_{\{t=0\}} \left[\frac{1}{c} \left\{ \frac{1}{c} \frac{\partial c}{\partial \rho}(\rho_{(\mathrm{bkg})}, 0) + 1 \right\} (\partial_1 \mathcal{R}^{(\mathrm{bkg})}_{(+)}) \right]_+ > 0,$$

and the solution forms a shock at time[7] $T^{(\mathrm{bkg})}_{(\mathrm{sing})} = (\mathring{\delta}^{(\mathrm{bkg})}_*)^{-1}$.

The analysis for plane-symmetric solutions can be carried out easily using Riemann invariants. It is then straightforward to check that there exists a large class of plane-symmetric solutions satisfying (1)–(6) above.

We now provide a rough version of our main theorem; see Section 4B for a more precise statement.

Theorem 1.1 (main theorem, rough version). *Consider a plane-symmetric, shock-forming background solution* $(\varrho_{(\mathrm{bkg})}, v^i_{(\mathrm{bkg})}, s_{(\mathrm{bkg})})$ *satisfying* (1)–(6) *above, where the parameter* $\mathring{\alpha}$ *from point* (4) *is small. Consider a small perturbation of the initial data of this background solution satisfying the following assumptions* (*see Section 4A for the precise assumptions*):

[4] In the one-dimensional case, one only needs information about the data's first derivative to close a proof of blowup for a simple plane wave. However, when studying perturbations in three dimensions, we need estimates on these derivatives up to third order in order to close the proof. For example, the proof of the bound (8-23c) relies on having control of up to these third-order derivatives (as is provided by (8-20b)–(8-20c)), and we use the bound (8-23c) in the proof of Lemma 14.2 as well as in the proof of the energy estimates in the Appendix.

[5] One can check that this rules out the Chaplygin gas, whose speed of sound (after normalization) is given by $c(\rho, s) = \exp(-\rho)$. One can also check that for any other equation of state, it is possible to choose $\bar{\varrho}$ appropriately so that $\mathring{\delta}^{(\mathrm{bkg})}_* > 0$.

[6] Here, $[\,\cdot\,]_+$ denotes the positive part.

[7] In the plane-symmetric, isentropic, simple case, $\mathcal{R}^{(\mathrm{bkg})}_{(+)}$ satisfies the transport equation $\partial_t \mathcal{R}^{(\mathrm{bkg})}_{(+)} + (v^1_{(\mathrm{bkg})} + c(\rho_{(\mathrm{bkg})})) \cdot \partial_1 \mathcal{R}^{(\mathrm{bkg})}_{(+)} = 0$, and the blowup-time of $\partial_1 \mathcal{R}^{(\mathrm{bkg})}_{(+)}$ can easily be computed explicitly by commuting this transport equation with ∂_1 to obtain a Riccati-type ODE in $\partial_1 R^{(\mathrm{bkg})}_{(+)}$ along the integral curves of $\partial_t + (v^1_{(\mathrm{bkg})} + c(\rho_{(\mathrm{bkg})}))\partial_1$.

- *The perturbation is compactly supported in a region of* x^1*-length* $\leq 2\mathring{\sigma}$.
- *The perturbation belongs to a high-order Sobolev space, where the required Sobolev regularity is independent of the background solution and equation of state.*
- *The perturbation is small, where the smallness is captured by the small parameter* $0 < \mathring{\epsilon} \ll 1$, *and the required smallness depends on the order of the Sobolev space, the equation of state, and the parameters of the background solution.*

Then the corresponding unique perturbed solution satisfies the following:

(1) *The solution is initially smooth, but it becomes singular at a time* $T_{(\text{sing})}$, *which is a small perturbation of the background blowup-time* $(\mathring{\delta}_*^{(\text{bkg})})^{-1}$.

(2) *Defining*[8] $\mathcal{R}_{(+)} \doteq v^1 + \int_0^{\rho} c(\rho', s)\, d\rho'$, *we have the singular behavior*

$$\limsup_{t \to T_{(\text{sing})}^-} \sup_{\{t\}\times\Sigma} |\partial_1 \mathcal{R}_{(+)}| = +\infty. \tag{1-4}$$

(3) *Relative to a geometric coordinate system* (t, u, x^2, x^3), *where* u *is an eikonal function, the solution remains smooth, all the way up to time* $T_{(\text{sing})}$. *In particular, the partial derivatives of the solution with respect to the geometric coordinates do not blow up.*

(4) *The blowup at time* $T_{(\text{sing})}$ *is characterized by the vanishing of the inverse foliation density* μ (*see Definition 2.15*) *of a family of acoustically null hypersurfaces defined to be the level sets of* u.

(5) *In particular, the set of blowup-points at time* $T_{(\text{sing})}$ *is characterized by*

$$\left\{(u, x^2, x^3) \in \mathbb{R} \times \mathbb{T}^2 : \limsup_{(\tilde{t},\tilde{u},\tilde{x}^2,\tilde{x}^3) \to (T_{(\text{sing})}^-, u, x^2, x^3)} |\partial_1 \mathcal{R}_{(+)}|(\tilde{t}, \tilde{u}, \tilde{x}^2, \tilde{x}^3) = \infty\right\} \\ = \{(u, x^2, x^3) \in \mathbb{R} \times \mathbb{T}^2 : \mu(T_{(\text{sing})}, u, x^2, x^3) = 0\},$$

where $|\partial_1 \mathcal{R}_{(+)}|(\tilde{t}, \tilde{u}, \tilde{x}^2, \tilde{x}^3)$ *denotes the absolute value of the Cartesian partial derivative* $\partial_1 \mathcal{R}_{(+)}$ *evaluated at the point with geometric coordinates* $(\tilde{t}, \tilde{u}, \tilde{x}^2, \tilde{x}^3)$.

(6) *At the same time, as* $T_{(\text{sing})}$ *is approached from below, the fluid variables* ϱ, v^i, s *all remain bounded, as do the specific vorticity* $\Omega^i \doteq (\text{curl}\, v)^i/(\varrho/\bar{\varrho})$ *and the entropy gradient* $S \doteq \nabla s$.

The proof of Theorem 1.1 relies on two main ingredients: (i) Christodoulou's geometric theory of shock formation for irrotational and isentropic solutions, in which case the dynamics reduces to the study of quasilinear wave equations and (ii) a (re-)formulation of the compressible Euler equations as a quasilinear system of wave-transport equations, which was derived in [50], following the earlier works [36; 37] in the barotropic[9] case. This formulation exhibits remarkable null structures and regularity properties, which in total allow us to perturbatively control the vorticity and entropy gradient all the way up to the singular

[8] In higher dimensions or in the presence of dynamic entropy, $\mathcal{R}_{(+)}$ is *not* a Riemann invariant because its dynamics is not determined purely by a transport equation. Nonetheless, for comparison purposes, we continue to use the symbol $\mathcal{R}_+$ to denote this quantity.

[9] A barotropic fluid is such that the equation of state for the pressure is a function of the density alone, as opposed to being a function of the pressure and entropy.

time — even though generically, their first-order Cartesian partial derivatives blow up at the singularity. See Section 1A for further discussion of the proof.

Some remarks are in order.

Remark 1.2. Note that even though the rough Theorem 1.1 is formulated in terms of plane-symmetric background solutions, we do not actually "subtract off a background" in the proof. See Theorem 4.3 for the precise formulation.

Remark 1.3 (results building up towards Theorem 1.1).

- Concerning stability of simple plane-symmetric shock-forming solutions to the compressible Euler equations, the first result was our joint work with G. Holzegel and W. Wong [52], which proved the analog[10] of Theorem 1.1 in the case[11] where the perturbation is irrotational and isentropic (i.e., $\Omega \equiv 0$, $S \equiv 0$).

- In [36], we proved the first stable shock formation result without symmetry assumptions for the compressible Euler equations for open sets of initial data that can have nontrivial specific vorticity Ω. Specifically, in [36], we treated the two-dimensional barotropic compressible Euler equations (see footnote 9). One of the key points in [36] was our reformulation of equations into a system of quasilinear wave-transport equations which has favorable nonlinear null structures. This allowed us to use the full power of the geometric vectorfield method on the wave part of the system while treating the vorticity perturbatively.

- In [37], we considered three-dimensional barotropic compressible Euler flow and derived a similar reformulation of the equations that allowed for nonzero vorticity. In contrast to the two-dimensional case, the transport equation satisfied by the specific vorticity Ω featured vorticity-stretching source terms (of the schematic form $\Omega \cdot \partial v$). In order to handle the vorticity-stretching source terms in the framework of [36], we also showed in [37] that Ω satisfies a div-curl-transport system with source terms that are favorable from the point of view of regularity and from the point of view of null structure. We refer to Section 1A6 for further discussion of this point.

- To incorporate thermodynamic effects into compressible fluid flow, one must look beyond the family of barotropic equations of state, e.g., consider equations of state in which the pressure depends on the density and entropy.[12] Fortunately, in [50], it was shown that a similar good reformulation of the compressible Euler equations holds under an arbitrary equation of state (in which the pressure is a function of the density and the entropy) in the presence of vorticity and variable entropy. In the present paper, we use this reformulation to prove our main results; we recall it below as Theorem 5.1. The analysis in [50] is substantially more complicated compared to the barotropic case, and the basic setup requires the

[10] We remark that while [52] only explicitly stated a theorem in two spatial dimensions, the analogous result in three (or indeed higher) dimensions can be proved using similar arguments; see [52, Remarks 1.4,1.11].

[11] The main theorem in [52] is stated for general quasilinear wave equations. Particular applications to the relativistic compressible Euler equations in the irrotational and isentropic regime can be found in [52, Appendix B]. It applies equally well to the nonrelativistic case.

[12] Incorporating entropy into the analysis is expected to be especially important for studying weak solutions after the shock (see Section 1B4 for further discussion), since formal calculations [16] suggest that the entropy (even if initially zero) should jump across the shock hypersurface, which in turn should induce a jump in vorticity.

observation of some new structures tied to elliptic estimates for Ω and S, such as good regularity and null structures tied to the modified fluid variables from Definition 2.7.

This paper completes the program described above by giving the analytic details already sketched in [37; 50]. Chief among the analytic novelties in the present paper are the elliptic estimates for Ω and S at the top-order; see [37, Sections 1.3, 4.2.7], [50, Section 4.3] and Section 1A6. We also point out that there are other related works, which we discuss in Section 1B.

Remark 1.4 (blowup and boundedness of quantities involving higher derivatives). For generic perturbations, derivatives of fluid variables other than $\mathcal{R}_+$ (whose blowup was highlighted in (1-4)) can also blow up. In particular, while the ∂_2 and ∂_3 derivatives of the fluid variables are identically 0 for the plane-symmetric background solutions, for the perturbed solution, $\partial_2 v^i$, say, is generically unbounded at the singularity. This is because the perturbation changes the geometry of the solution, and the regular directions no longer align with the Cartesian directions.

On the other hand, there are indeed higher derivatives of the fluid variables that remain bounded up to the singular time. These include the specific vorticity and the entropy gradient that we already mentioned explicitly in Theorem 1.1. Moreover, any null-hypersurface-tangential geometric derivatives (see further discussions in Section 1A) of the fluid variables are also bounded up to the singular time. This is not just a curiosity, but rather is a fundamental aspect of the proof.

Remarkably, there are additionally quantities, denoted by $\mathcal{C}$ and $\mathcal{D}$ (these variables were identified in [50], see (2-5a)–(2-5b)), which are special combinations of up-to-second-order Cartesian coordinate derivatives of the fluid variables, which remain uniformly bounded up to the singularity (as do their derivatives in directions tangent to a family of null hypersurfaces); $\mathcal{C}$ and $\mathcal{D}$ are precisely the modified fluid variables mentioned in Remark 1.3. The existence of such regular higher-order quantities is not only an interesting fact, but is also quite helpful in controlling the solution up to the first singularity; see Section 1A.

Finally, as a comparison with our two-dimensional work [36], note that in the two-dimensional case, we proved that the specific vorticity remains Lipschitz (in Cartesian coordinates) up to the first singular time. This is no longer the case in three dimensions. Indeed, in the language of this paper, the improved regularity for the specific vorticity in [36] stems from the fact that in two dimensions, the Cartesian coordinate derivatives of the specific vorticity Ω coincide with $\mathcal{C}$.

Remark 1.5 (additional information on subclasses of solutions). Within the solution regime we study, we are able to derive additional information about the solution by making further assumptions on the data. For instance, there are open subsets of data such that the vorticity/entropy gradient are nonvanishing at the first singularity, and also open subsets of data such that the fluid variables remain Hölder[13] $C^{1/3}$ up to the singularity. See Section 4B for details.

Remark 1.6 (the maximal smooth development). The approach we take here allows us to analyze the solution up to the first singular time, and our main results yield a complete description of the set of blowup-points at that time (see, for example, conclusions (4)–(5) of Theorem 1.1). However, since

[13] The Hölder estimates hold only for an open subset of data satisfying certain nondegeneracy assumptions. They were not announced in [37; 50]. We were instead inspired by [9; 11] to include such estimates.

the compressible Euler equations are a hyperbolic system, it is desirable to go beyond our results by deriving a full description of the maximal smooth development of the initial data, in analogy with [15]. Understanding the maximal smooth development is particularly important for the shock development problem; see Section 1B4 below.

Our methods, at least on their own, are not enough to construct the maximal smooth development.[14] This is in part because our approach here relies on spatially global elliptic estimates on constant-t hypersurfaces; the point is that a full description of the smooth maximal development would require spatially localized estimates. On the other hand, the recent preprint [1] discovered an integral identity that allows the elliptic estimates to be localized, and thus gives hope that Theorem 1.1 can be extended to derive the structure of the full maximal smooth development.

Remark 1.7 (no universal blowup-profile). One of the main advantages of our geometric framework is that it works for many kinds of singular solutions, not just those exhibiting a specific blowup-profile. In particular, the solutions featured in Theorem 1.1 do not exhibit a universal blowup-profile. Although we do not rigorously study the full class of blowup-profiles exhibited by the solutions from Theorem 1.1, the full class is likely quite complicated to describe. This can already be seen in model case of Burgers' equation, where there are a continuum of possible blowup-profiles and corresponding blowup-rates [27] (recall that we work in the near plane-symmetric regime and our work includes, as special cases, plane-symmetric solutions, which are analogs of Burgers' equation solutions). A related issue is that at the time of first singularity formation, the set of blowup-points can be complicated and/or of infinite cardinality (as one can already see in the special case of plane-symmetric solutions, viewed as solutions in three dimensions with symmetry).

Remark 1.8 (the relativistic case). While our present work treats only the nonrelativistic case, it is likely that the relativistic case can also be treated in the same way. This is because the relativistic compressible Euler equations also admit a similar reformulation as we consider here, and likewise the variables in the reformulation also exhibit a very similar null structure [25].

In the remainder of the Introduction, we will first discuss the proof in Section 1A and then discuss some related works in Section 1B. We will end the introduction with an outline of the remainder of the paper.

1A. *Ideas of the proof.*

1A1. *The Christodoulou theory.* The starting point of our proof is the work of Christodoulou [15] on shock formation for quasilinear wave equations.[15] Consider the following model quasilinear covariant wave equation for the scalar function Ψ: $\square_{g(\Psi)}\Psi = 0$, where the Cartesian component functions $g_{\alpha\beta}$ are given (nonlinear in general) functions of Ψ, i.e., $g_{\alpha\beta} = g_{\alpha\beta}(\Psi)$. Our study of compressible Euler flow in

[14] Notice that in our earlier result [36] for the isentropic Euler equations in two spatial dimensions, we also only solved the equations up to the first singular time. However, there is an important difference. In the two-dimensional case, there does not seem to be a philosophical obstruction in extending [36] to provide a complete description of the maximal smooth development. In contrast, in the three-dimensional case it seems that ideas in [1] would be needed in a fundamental way.

[15] Strictly speaking, [15] is only concerned with the irrotational isentropic relativistic Euler equations. However, its methods apply to much more general quasilinear wave equations; see further discussions in [30; 48].

this paper essentially amounts to studying a system of similar equations with source terms and showing that the source terms do not radically distort the dynamics. This is possible only because the source terms have remarkable null structure, described below.

A key insight for studying the formation of shocks, going back to [15], is that it is advantageous to study the shock formation via a system of geometric coordinates. The point is that when appropriately constructed, such coordinates regularize the problem, which allows one to treat the problem of shock formation as if it were a standard local existence problem. More precisely, one constructs geometric coordinates, adapted to the flow, such that the solution remains regular relative to them.[16] However, the geometric coordinates *degenerate* relative to the Cartesian ones, and the blowup of the solution's first-order Cartesian coordinate partial derivatives can be derived as a consequence of this degeneracy.

To carry out this strategy, one must use the Lorentzian geometry associated to the acoustical metric g (see Definition 2.9). The following geometric objects are of central importance in implementing this program:

- A foliation by constant-u characteristic hypersurfaces $\mathcal{F}_u$ (where $g^{-1}(du, du) = 0$; see (2-13)). The function u is known as an "acoustic eikonal function".

- The inverse foliation density μ ($\doteq -1/g^{-1}(dt, du)$), where μ^{-1} measures the density of $\mathcal{F}_u$ with respect to the constant-t hypersurfaces.

- A frame of vectorfields $\{L, X, Y, Z\}$, where $\{L, Y, Z\}$ are tangent to $\mathcal{F}_u$ (with L being its null generator) and X is transversal to $\mathcal{F}_u$; see Figure 1, where we have suppressed the Z-direction.

- $\{L, X, Y, Z\}$ is a frame that is "comparable" to the Cartesian frame $\{\partial_t, \partial_1, \partial_2, \partial_3\}$, by which we mean the coefficients relating the frames to each other are size $\mathcal{O}(1)$.

- However, in the analysis, uniform boundedness estimates are generally available for the derivatives of quantities with respect to only the *rescaled* frame elements $\{L, \breve{X} \doteq \mu X, Y, Z\}$.

The analysis simultaneously yields control of the derivatives of Ψ with respect to the rescaled frame and gives also quantitative estimates on the geometry. In this geometric picture, the blowup is completely captured by $\mu \to 0$. The connection between the vanishing of μ and the blowup of some Cartesian coordinate partial derivative of Ψ can be understood as follows: one proves an estimate of the form $|\breve{X}\Psi| \approx 1$ (which is consistent with the uniform boundedness estimates mentioned above). In view of the relation $\breve{X} = \mu X$, this estimate implies that $|X\Psi|$ blows up like $1/\mu$ as $\mu \to 0$.

We now give a more detailed description of the behavior of the solution, with a focus on how it behaves at different derivative levels.

- As our discussion above suggested, at the lower derivative levels, derivatives of quantities with respect to the *rescaled* frame are *regular*, e.g., $L\Psi$, $\breve{X}\Psi$, $Y\Psi$, $Z\Psi$, $\ldots$, $L^3\breve{X}Y\Psi$, etc. are uniformly bounded.

[16] It should be emphasized that it is only at the low derivative levels that the solution is regular. The high-order geometric energies can still blow up, even though the low-order energies remain bounded. The possible growth of the high-order energies is one of the central technical difficulties in the problem, and we will discuss it below in more detail.

• As we highlighted above, the formation of the shock corresponds to $\mu \to 0$ in finite time, and moreover, the *nonrescaled* first-order derivative $X\Psi$ *blows up* in finite time, exactly at points where μ vanishes.

• The main difficulty in the proof is that the only known approach to the solution's regularity theory *with respect to the rescaled frame derivatives* that is able to avoid a loss of derivatives allows for the following possible scenario: the energy estimates are such that the high-order geometric energies might blow up when the shock forms. This leads to severe difficulties in the proof, especially considering that one needs to show that the low-order derivatives of the solution remain bounded in order to derive the singular high-order energy estimates.[17]

In [15], Christodoulou showed that the maximum possible blowup-rate of the high-order energies is of the form $\mu_\star^{-2P}(t)$, where P is a universal positive constant and $\mu_\star(t) \doteq \min\{1, \min_{\Sigma_t} \mu\}$. To reconcile this possible high-order energy blowup with the regular behavior at the lower derivative levels, one is forced to derive a hierarchy of energy estimates of the form, where M_* is a universal[18] positive integer:

$$\mathbb{E}_{N_{\text{top}}}(t) \lesssim \mathring{\epsilon}^2 \mu_\star^{-2M_*+1.8}(t), \quad \mathbb{E}_{N_{\text{top}}-1}(t) \lesssim \mathring{\epsilon}^2 \mu_\star^{-2M_*+3.8}(t), \quad \mathbb{E}_{N_{\text{top}}-2}(t) \lesssim \mathring{\epsilon}^2 \mu_\star^{-2M_*+5.8}(t), \quad \ldots, \tag{1-5}$$

where $\mathbb{E}_N$ denotes the energy after N commutations and all energies are by assumption initially of small size $\mathring{\epsilon}^2$. In other words, the energy estimates become less singular by two powers of $\mu_\star$ for each descent below the top derivative level. Importantly, despite the possible blowup at higher orders, all the sufficiently low-order energies are bounded, which, by Sobolev embedding, is what allows one to show the uniform pointwise boundedness of the solution's lower-order derivatives:[19]

$$\sum_{N=1}^{N_{\text{top}}-M_*} \mathbb{E}_N(t) \lesssim \mathring{\epsilon}^2. \tag{1-6}$$

1A2. *The nearly simple plane-symmetric regime.* Christodoulou's work [15] concerned compactly supported[20] initial data in $\mathbb{R}^3$, a regime in which dispersive effects dominate for a long time before the singularity formation processes eventually take over. In a joint work with Holzegel and Wong [52], we adapted the Christodoulou theory to the almost simple plane symmetric regime. The important point is that the commutators $\{L, Y, Z\}$, in addition to being regular derivatives near the singularity, also simultaneously capture the fact that the solution is "almost simple plane symmetric." Moreover, the following analytical considerations were fundamental to the philosophy of the proof in [52]:

[17] The possible high-order energy blowup has its origins in the presence of some difficult factors of $1/\mu$ in the top-order energy identities, where one must work hard to avoid a loss of derivatives. To close the energy estimates, one commutes the wave equation many times with the $\mathcal{F}_u$-tangent subset $\{L, Y, Z\}$ of the rescaled frame. The most difficult terms in the commuted wave equation are top-order terms in which all the derivatives fall onto the components of $\{L, Y, Z\}$. It turns out that due to the way the rescaled frame is constructed, the corresponding difficult error terms depend on the top-order derivatives of the eikonal function u. In Proposition A.4, we identify these difficult commutator terms. To avoid the loss of derivatives, one must work with modified quantities and use elliptic estimates. It is in this process that one creates difficult factors of $1/\mu$.

[18] Our proof of the universality of M_* in the presence of vorticity and entropy requires some new observations, described below (1-11).

[19] The lowest-order energy $\mathbb{E}_N(t)$ is excluded from this estimate because it is not of small size $\mathring{\epsilon}^2$, owing to the largeness of $\breve{X}\mathcal{R}_{(+)}$.

[20] More precisely, his work addressed compactly supported irrotational perturbations of constant, nonvacuum fluid solutions.

• All energy estimates can be closed by commuting only with tangential derivatives $\{L, Y, Z\}$ (and *without* $\breve{X}$). This is a slightly different strategy than we used in our paper [36] in the two-dimensional case, in which we closed the energy estimates by commuting the equations with strings of tangential derivatives $\{L, Y, Z\}$, as well as strings that contain up to one factor of $\breve{X}$. In [36], we also could have closed the energy estimates by commuting only with tangential derivatives $\{L, Y, Z\}$, but we would have had to work with the modified fluid variable $\mathcal{C}$ (which, though fundamental in three dimensions, was not needed in [36] due to the absence of the vorticity-stretching term) or to treat the Cartesian gradients $\partial_\alpha \Omega^i$ as independent unknowns.

• After being commuted with (at least one of) L, Y, Z, the wave equation solutions are *small*. In particular, we can capture the smallness from "nearly simple plane-symmetric" data without explicitly subtracting the simple plane-symmetric background solution; see also Remark 1.2.

1A3. *The reformulation of the equations.* In order to extend Christodoulou's theory so that it can be applied to the compressible Euler equations, a crucial first step is to reformulate the compressible Euler equations as a system of quasilinear wave equations and transport equations. Here, the transport part of the system refers to the vorticity and the entropy, and the intention is to handle them perturbatively.

As we mentioned earlier, the reformulation has been carried out in [36; 37; 50]. Here we highlight the main features and philosophy of the reformulation, and explain how we derived it.

(1) To the extent possible, formulate compressible Euler flow as a perturbation of a system of quasilinear wave equations.

(a) We compute $\square_g v^i$, $\square_g \rho$, and $\square_g s$, where $\square_g$ is the covariant wave operator associated to the acoustical metric (see (2-7)). Then using the compressible Euler equations (1-1)–(1-3), we eliminate and re-express many terms.

(b) We find that v^i, ρ, and s do not exactly satisfy wave equations; instead, the right-hand sides contain second derivatives of the fluid variables, which we will show to be *perturbative*, despite their appearance of being principal order in terms of the number of derivatives.

(2) The "perturbative" terms mentioned above are equal to *good transport variables* that we identify, specifically $(\Omega, S, \mathcal{C}, \mathcal{D})$. These variables behave better than what one might naïvely expect, from the points of view of their regularity and their singularity strength.

(a) While both $\Omega^i \doteq (\operatorname{curl} v)^i/(\varrho/\bar{\varrho})$ and $S \doteq \nabla s$ are derivatives of the fluid variables, they play a distinguished role since they satisfy independent transport equations, and obey better bounds than generic first derivatives of the fluid variables.

(b) We have introduced the *modified fluid variables* $\mathcal{C}^i$ and $\mathcal{D}$ (see Definition 2.7), which, up to lower-order correction terms, are equal to $(\operatorname{curl} \Omega)^i$ and $\Delta s = \operatorname{div} S$ respectively. These quantities satisfy better estimates than generic first derivatives of Ω and S, which is crucial for our proof.

1A4. *The remarkable null structure of the reformulation.* In the reformulation of compressible Euler flow, we consider the unknowns to be all of $(v^i, \rho, s, \Omega^i, S^i, \mathcal{C}^i, \mathcal{D})$. Note that these include not only the fluid variables, but also higher-order variables which can be derived from the fluid variables.

The equations satisfied by these variables take the following schematic form (see Theorem 5.1 for the precise equations):[21]

$$\square_g(v,\rho,s)=\overbrace{\partial(v,\rho)\cdot\partial(v,\rho)}^{\doteq I}+(\Omega,S)\cdot\partial(v,\rho)+(\mathcal{C},\mathcal{D}), \tag{1-7}$$

$$B(\Omega,S)=(\Omega,S)\cdot\partial(v,\rho), \tag{1-8}$$

$$B(\mathcal{C},\mathcal{D})=\underbrace{\partial(v,\rho)\cdot\partial(\Omega,S)}_{\doteq II}+\underbrace{(\Omega,S)\cdot\partial(v,\rho)\cdot\partial(v,\rho)}_{\doteq III}+S\cdot S\cdot\partial(v,\rho). \tag{1-9}$$

Here, $\square_g$ is the covariant wave operator associated to the acoustical metric (see (2-7)) and $B\doteq\partial_t+v^a\partial_a$ is the transport operator associated with the material derivative (cf. (1-1)–(1-3)).

Although it is not apparent from the way we have written it, the system of equations (1-7)–(1-9) has a remarkable null structure! Importantly, the terms I, II and III are g-null forms: when decomposed in the $\{L,X,Y,Z\}$ frame, we do not have $X(v^i,\rho)\cdot X(v^i,\rho)$ in I and III, nor do we have $X(v,\rho)\cdot X(\Omega,S)$ in II.

Because $X(v^i,\rho)$ is the only derivative that blows up (while $\breve{X}(v^i,\rho)$ is bounded), it follows that given a g-null form $\mathcal{Q}$ in the fluid variables (see Definition 8.1 concerning g-null forms), such as $\mathcal{Q}(\partial v^i,\partial v^j)$, the quantity $\mu\mathcal{Q}(\partial v^i,\partial v^j)$ remains bounded up to the singularity, while a generic quadratic nonlinearity $\mathcal{Q}_{\text{bad}}$ would be such that $\mu\mathcal{Q}_{\text{bad}}(\partial v^i,\partial v^j)$ blows up when μ vanishes.

As is already observed in [48], a null form I on the right-hand side of the wave equation allows all the wave estimates in Section 1A1 to be proved. As we will discuss below, the null forms II and III in (1-9) will also be important for estimating the full system.

1A5. *Estimates for the transported variables.* To control solutions to the system (1-7)–(1-9), we in particular need to estimate the transport variables $(\Omega,S,\mathcal{C},\mathcal{D})$ and understand how they interact with the wave variables (v,ρ,s) on the left-hand side of (1-7). Here, we will discuss the estimates at the low derivative levels. We will discuss the difficult technical issues of a potential loss of derivatives and the blowup of the higher-order energies in Sections 1A6 and 1A7 respectively.

We begin with two basic — but crucial — properties regarding the transport operator for the compressible Euler system, which were already observed in [36]:

- *The transport vectorfield B is* ***transversal*** *to the null hypersurfaces $\mathcal{F}_u$*; see Figure 1, where some integral curves of B are depicted. As a result, one *gains* a power of μ by integrating along B; i.e., for solutions ϕ to $B\phi=\mathfrak{F}$, we have $\|\phi\|_{L^\infty}\lesssim\|\mu\mathfrak{F}\|_{L^\infty}$.
- μB *is a* ***regular*** *vectorfield in the (t,u,x^2,x^3) differential structure.* Thus, if $B\phi=\mathfrak{F}$ and $\mu\mathfrak{F}$ has bounded $\{L,Y,Z\}$ derivatives, then ϕ also has bounded $\{L,Y,Z\}$ derivatives.

We now apply these observations to (1-8) and (1-9):

- Even though $\partial(v,\rho)$ blows up as the shock forms, $\mu\partial(v,\rho)$ remains regular. This is because $\mu\partial$ can be written as a linear combination of the rescaled frame vectorfields $\{\mu X,L,Y,Z\}$ (see Section 1A1) with

[21] Here, our notation above the brackets is such that $\partial(v,\rho)\cdot\partial(v,\rho)$ may contain all of $\partial v^i\partial v^j$, $\partial v^i\partial\rho$ and $\partial\rho\partial\rho$. A similar convention applies for other terms.

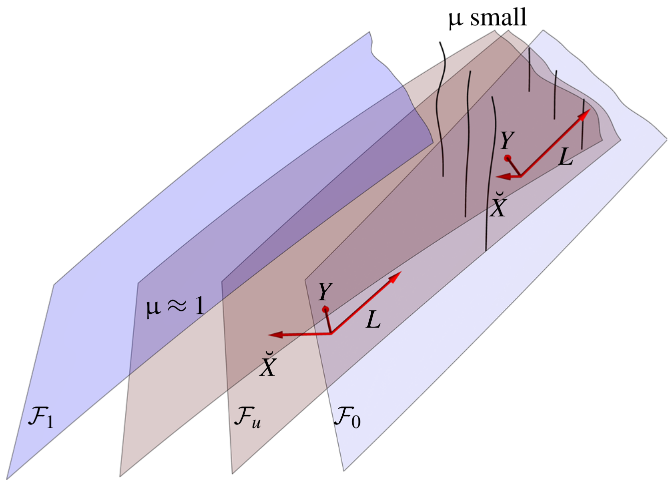

Figure 1. The dynamic vectorfield frame at two distinct points on $\mathcal{F}_u$ with the Z-direction suppressed, and the integral curves of the transport operator B for the specific vorticity and entropy.

coefficients that are $\mathcal{O}(1)$ or $\mathcal{O}(\upmu)$. Hence, the above observations imply that $(\varOmega, S)$ and their $\{L, Y, Z\}$ derivatives are bounded.

- The null structure and the bounds for the wave variables and $(\varOmega, S)$ together imply that the right-hand side of (1-9) is $\mathcal{O}(\upmu^{-1})$. Thus, $\mathcal{C}$, $\mathcal{D}$ and their $\{L, Y, Z\}$ derivatives are also bounded.

1A6. *Elliptic estimates for the vorticity and the entropy gradient.* Despite the favorable structure of (1-7)–(1-9), there is apparently a potential loss of derivatives. To see this, consider the following simple derivative count. Suppose we bound $(v, \uprho, s)$ with $N_{\mathrm{top}} + 1$ derivatives. Equation (1-7) dictates[22] that we should control $(\mathcal{C}, \mathcal{D})$ with N_{top} derivatives. If we rely only on (1-8), then we can only bound N_{top} derivatives of $(\varOmega, S)$. However, this is insufficient: plugging this into (1-9) and using only transport estimates, we are only able to control $N_{\mathrm{top}} - 1$ derivatives of $(\mathcal{C}, \mathcal{D})$, which is not enough.

The key to handling this difficulty is the observation that in fact, $\mathcal{C}$ and $\mathcal{D}$ can be used in conjunction with elliptic estimates to control one derivative of $\varOmega$ and S. This is because up to lower-order terms, $\mathcal{C} \approx \operatorname{curl} \varOmega$ and $\mathcal{D} \approx \operatorname{div} S$, while at the same time, by the definitions of $\varOmega$ and S — precisely that $\varOmega$ is almost a curl of a vectorfield and $S = \nabla s$ is an exact gradient — $\operatorname{div} \varOmega$ and $\operatorname{curl} S$ are of lower order in terms of the number of derivatives. It follows that we can control *all* first-order spatial derivatives of $\varOmega$ and S, including $\mathcal{C}$ and $\mathcal{D}$, using elliptic estimates.

1A7. *L^2 estimates for the transport variables and the high-order blowup-rate.* We end this section with a few comments on the L^2 energy estimates for the transport variables $(\varOmega, S)$ (and $(\mathcal{C}, \mathcal{D})$), with a focus on how to handle the degeneracies tied to the vanishing of $\upmu$.

First, due to the eventual vanishing of $\upmu$ and the corresponding blowup of the wave variables, we need to incorporate $\upmu$ weights into our analysis of the transport variables $(\varOmega, S)$ (and $(\mathcal{C}, \mathcal{D})$). In particular, we

[22] We use here the fact that inverting the wave operator gains one derivative.

need to incorporate μ weights into the transport equations and energies so that the wave terms appearing as inhomogeneous terms in the energy estimates for the transport variables are regular. Importantly, despite the need to rely on μ weights in some parts of the analysis, the "transport energy" that we construct controls a nondegenerate energy flux (i.e., an energy flux without μ weights) on constant-u hypersurfaces $\mathcal{F}_u$. That this energy flux is bounded can be thought of as another manifestation of the transversality of the transport operator and $\mathcal{F}_u$. More precisely, with Σ_t denoting constant-t hypersurfaces, we have, roughly, L^2 estimates of the following form, where $\mathcal{P}^N$ is an order-N differential operator corresponding to repeated differentiation with respect to the $\mathcal{F}_u$-tangent vectorfields $\{L, Y, Z\}$:

$$\sup_{t'\in[0,t)} \|\sqrt{\mu}\mathcal{P}^N(\varOmega, S)\|^2_{L^2(\Sigma_{t'})} + \sup_{u'\in[0,u)} \|\mathcal{P}^N(\varOmega, S)\|^2_{L^2(\mathcal{F}_{u'})} \lesssim \text{data terms } + \text{regular wave terms } + \int_{u'=0}^{u'=u} \|\mathcal{P}^N(\varOmega, S)\|^2_{L^2(\mathcal{F}_{u'})}\, du'. \quad (1\text{-}10)$$

Here, the nondegenerate energy flux (i.e., the energy along $\mathcal{F}_{u'}$ on the left-hand side of (1-10), which does not have a μ-weight) allows one to absorb the last term on the right-hand side of (1-10) using Grönwall's inequality[23] in u (as opposed to Grönwall's inequality in t which has a loss in μ). For the lower-order energies, the "regular wave terms" are indeed bounded (see (1-6)), which in total allows us to prove that the transport energies on the left-hand side of (1-10) are also bounded at the lower derivative levels.

Second, since the higher-order energies of the wave variables (v, ρ, s) can blow up as $\mu_\star(t) \to 0$ (even in the absence of inhomogeneous terms; see (1-5)), (1-10) allows for the possibility that the higher-order energies of the transport variables $(\varOmega, S)$ (and $(\mathcal{C}, \mathcal{D})$) might also blow up. Hence, one needs to verify that there is consistency between the blowup-rates (with respect to powers of $\mu_\star^{-1}$) associated to the different kinds of solution variables. That is, using (1-10) and the wave energy blowup-rates from (1-5), one needs to compute the expected blowup-rate of the transport variables and then plug these back into the energy estimates for the wave variables to confirm that the transport terms have an expected singularity strength that is consistent with wave energy blowup-rates. See, for example, the proof of Proposition 12.7.

Third, due to issues mentioned in Section 1A6, the transport estimates at the top-order are necessarily coupled with elliptic estimates. By their nature, the elliptic estimates treat derivatives in all spatial directions on the same footing. This clashes with the philosophy of bounding the solution with respect to the rescaled frame (which would mean that derivatives in the Y and Z frame directions should be more regular than those in the X-direction), and it leads to estimates that are singular in $\mu_\star^{-1}$. To illustrate the difficulties and our approach to overcoming them, we first note that, suppressing many error terms, we can derive a top-order inequality of the following form, with $\underline{\partial}$ denoting Cartesian spatial derivatives and A denoting a constant depending on the equation of state:

$$\|\sqrt{\mu}\underline{\partial}\mathcal{P}^{N_{\text{top}}}(\varOmega, S)\|_{L^2(\Sigma_t)} \leq C\mathring{\epsilon}^{3/2}\mu^{-2M_*+2.8}(t) + A\int_{t'=0}^{t'=t} \mu_\star^{-1}(t')\|\sqrt{\mu}\underline{\partial}\mathcal{P}^{N_{\text{top}}}(\varOmega, S)\|_{L^2(\Sigma_{t'})}\, dt' + \cdots. \quad (1\text{-}11)$$

[23] Our analysis takes place in regions of bounded u width, so that factors of e^{Cu} which arise in our Grönwall estimates can be bounded by a constant.

To apply Grönwall's inequality to (1-11), one must quantitatively control the behavior of the crucial "Grönwall factor" $\int_{t'=0}^{t'=t} A/\upmu_\star(t')\,dt'$. A fundamental aspect of our analysis is that $\upmu_\star(t)$ tends to 0 linearly[24] in t towards the blowup-time. It follows that one can at best prove an estimate of the form $\int_{t'=0}^{t'=t} \upmu_\star^{-1}(t')\,dt' \lesssim \log(\upmu_\star^{-1})(t)$ (recall that $\upmu_\star(t) = \min\{1, \min_{\Sigma_t} \upmu\}$, and see Proposition 8.11 for related estimates). Using only this estimate and applying Grönwall's inequality to (1-11), we find (ignoring the error terms "$\cdots$") that $\|\underline{\partial}\mathcal{P}^{N_{\mathrm{top}}}(\varOmega, S)\|_{L^2(\Sigma_t)} \lesssim \mathring{\epsilon}^{3/2}\upmu_\star^{-\max\{\mathcal{O}(A),\, 2M_*-2.8\}}(t)$. Notice that unless A is small, the dominant blowup-rate in the problem would be the one corresponding to these elliptic estimates for $(\varOmega, S)$, which could in principle be much larger than the blowup-rates corresponding to the irrotational and isentropic case.[25]

However, we can prove a better result: we can show that the blowup-rates are not dominated by the top-order elliptic estimates for the transport variables, but rather by the blowup-rates for the wave variables.[26] The key to showing this is to replace the estimate (1-11) with a related L^2 estimate that features *weights in the eikonal function u*; see Proposition 11.4. Thanks to the u weights, the corresponding constant A in this analog of (1-11) can be chosen to be arbitrarily *small*, and thus the main contribution to the blowup-rate comes from the wave variables error terms, which are present in the "$\cdots$" on the right-hand side of (1-11). That this can be done is related to the fact that we have good flux estimates for top derivatives of $\mathcal{C}$ and $\mathcal{D}$ on $\mathcal{F}_u$. We refer to Propositions 11.2, 11.10, and 11.11 for the details.

1B. *Related works.*

1B1. *Shock formation in one spatial dimension.* One-dimensional shock formation has a long tradition starting from [45]. See the works of Lax [34], John [31], Liu [35], and Christodoulou and Raoul Perez [20], as well as the surveys [12; 24] for details.

1B2. *Multidimensional shock formation for quasilinear wave equations.* Multidimensional shock formation for quasilinear wave equations was first proven in Alinhac's groundbreaking papers [3; 4; 5]. Alinhac's methods allowed him to prove the formation of nondegenerate shock singularities which, roughly speaking, are shock singularities that are isolated within the constant-time hypersurface of first blowup. The problem was revisited in Christodoulou's monumental book [15], which concerned the quasilinear wave equations of irrotational and isentropic relativistic fluid mechanics. In this book, Christodoulou introduced methods that apply to a more general class of shock singularities than the nondegenerate ones treated by Alinhac and, for a large open subset of these solutions, are able to yield a complete description of the maximal smooth development, up to the boundary. This was the starting point of his follow-up breakthrough monograph [16] on the restricted shock development problem.

[24] The linear vanishing rate is crucial for the proof of Proposition 8.11 and for the Grönwall-type estimates for the energies that we carry out in Proposition 12.7 and in the Appendix. See (14-1) for a precise description of how $\upmu_\star$ goes to 0.

[25] In principle, the largeness of A would not be an obstruction to closing the estimates. It would just mean that the number of derivatives needed to close the problem would increase in the presence of vorticity and entropy. We refer readers to the technical estimates in Section A9 for clarification on the role that the sizes of various constants play in determining the blowup-rates in the problem, as well as the number of derivatives needed to close the proof.

[26] In other words, our approach yields the same maximum possible high-order energy blowup-rates for the wave variables in the general case as it does for irrotational and isentropic solutions.

For quasilinear wave equations, there are many extensions, variations, and simplifications of [15], some of which adapted Christodoulou's geometric framework to other solution regimes. See, for instance, [14; 18; 19; 30; 41; 42; 48; 52].

1B3. *Multidimensional shock formation for the compressible Euler equations.* Multidimensional singularity formation for the compressible Euler equations without symmetry assumptions was first discovered by Sideris [47] via an indirect argument. A constructive proof of stable shock formation in a symmetry-reduced regime for which multidimensional phenomena (such as dispersion and vorticity) are present was given by Alinhac in [2]. See also [10; 11].

All the works in Section 1B2 on quasilinear wave equations can be used to obtain an analogous result for the compressible Euler equations in the irrotational and isentropic regime, where the dynamics reduces to a single, scalar quasilinear wave equation for a potential function. The regime of small, compact, irrotational perturbations of nonvacuum constant fluid states was treated in Christodoulou's aforementioned breakthrough work [15] in the relativistic case, and later in [19] in the nonrelativistic case.

Shock formation beyond the irrotational and isentropic regime was first proven in [36; 37; 50]. These are already discussed above; see Remark 1.3.

In very interesting recent works [10; 11], Buckmaster, Shkoller and Vicol provided a philosophically new proof of stable singularity formation without symmetry assumptions in three dimensions under adiabatic equations of state in a solution regime with vorticity and/or dynamic entropy for initial data such that precisely one singular point forms at the first singular time; these are analogs of the nondegenerate singularities that Alinhac studied [3; 4; 5] in the case of quasilinear wave equations. Moreover, in their regime (compare with Remark 1.7), they proved that the singularity is a perturbation of a self-similar Burgers shock. See also the two-dimensional precursor work [9] in symmetry, and the recent work [7], which, in two dimensions in azimuthal symmetry, constructed a set of shock-forming solutions whose cusp-like spatial behavior at the singularity is unstable (nongeneric).

1B4. *Shock development problem.* In the one-dimensional case, the theory of global solutions of small bounded variation (BV) norms [6; 28] allows one to study solutions that form shocks, as well as the subsequent interactions of the shocks in the corresponding weak solutions. In higher dimensions, the compressible Euler equations are ill-posed in BV spaces [44]. Nonetheless, in two or three dimensions, one still hopes to develop a theory that allows one to uniquely extend the solution as a piecewise smooth weak solution beyond the first shock singularity and to prove that the resulting solution has a propagating shock hypersurface. This is known as *the shock development problem.*

Even though the shock development problem for the compressible Euler equations in its full generality is open in higher dimensions, it has been solved under spherical symmetry in three dimensions, or in azimuthal symmetry in two dimensions. See [18; 55] and, most recently, [8].

In the irrotational and isentropic regime, the restricted shock development problem was solved in the recent monumental work [16] of Christodoulou *without any symmetry assumptions*. Here, the word "restricted" means that the approach of [16] does not exactly construct a weak solution to the compressible Euler equations, but instead yields a weak solution to a closely related hyperbolic PDE system such that the solution was "forced" to remain irrotational and isentropic. Nonetheless, this gives hope that under an arbitrary equation of state for the compressible Euler equations in three dimensions, one could construct a unique weak solution with a propagating shock hypersurface, starting from the first singular time exhibited in Theorem 1.1. To solve this problem would in particular require extending the ideas in [16] beyond the irrotational and isentropic regime. This is an outstanding open problem.

1B5. *Other singularities for the compressible Euler equations.* It has been known since [29; 46] that the compressible Euler equations admit self-similar solutions. Recently, this has been revisited by Merle, Raphaël, Rodnianski and Szeftel [39] to show that singularities more severe than shocks can arise in three dimensions starting from smooth initial data. See also [40; 38] for some spectacular applications.

1B6. *Singularity formation in related models.* For shock formation results concerning some other multi-speed hyperbolic problems, see [49; 51] by the second author.

Interestingly, there are also *nonhyperbolic* models with stable self-similar blowup-profiles modeled on a self-similar Burgers shock. Examples include the Burgers equation with transverse viscosity [23], the Burgers–Hilbert equations [54], and the fractal Burgers equation [13], as well as general dispersive or dissipative perturbations of the Burgers equation [43]. See also [21; 22].

1B7. *Other works.* The framework we introduced in [36; 37; 50] is useful in other low-regularity settings. See for example results on improved regularity for vorticity/entropy in [25], and results on local existence with rough data in [26; 53; 56].

1C. ***Structure of the paper.*** The remainder of the paper is structured as follows.

Sections 2–4 are introductory sections. We introduce the basic setup in Section 2, and we define the norms and energies in Section 3. The setup is similar to the setups in [36; 52]. Then in Section 4, we state our precise assumptions on the initial data and give a precise statement of our main results, which we split into several theorems and corollaries.

In Section 5, we recall the results of [50] on the reformulation of the equations, which is important for the remainder of the paper.

The bulk of paper is devoted to proving the main a priori estimates, which we state in Section 6 as Theorem 6.3. The proof of Theorem 6.3, which we provide in Section 14, relies on a set of bootstrap assumptions that we also state in Section 6. Next, after an easy (but crucial) finite-speed-of-propagation argument in Section 7, in Section 8, we cite various straightforward pointwise and L^∞ estimates for geometric quantities found in [52], and we complement these results with a few related ones that allow us to handle the transport variables.

We then turn to the main estimates in this paper. In Section 9, we carry out the transport estimates, specifically L^∞ estimates and energy estimates, for Ω, S and their derivatives. In Section 10, we prove analogous transport estimates for $\mathcal{C}$, $\mathcal{D}$, and their derivatives, except we delay the proof of the top-order estimates until the next section. In Section 11, we derive the top-order estimates for $\mathcal{C}$ and $\mathcal{D}$, which, as we described in Section 1A7, requires elliptic estimates in addition to transport estimates. In total, these estimates for the transport variables can be viewed as the main new contribution of the paper.

Next, in Section 12, we derive energy estimates for the fluid wave variables. For convenience, we have organized the wave equation estimates so that they rely on an auxiliary proposition, namely

Proposition 12.1, that provides estimates for solutions to the fluid wave equations in terms of various norms of their inhomogeneous terms, which for purposes of the proposition, we simply denote by $\mathfrak{G}$. To prove the final a priori energy estimates for the wave equations, which are located in Proposition 12.7, we must use the bounds for $\mathfrak{G}$ that we obtained in the previous sections, including the bounds for the transport variables. Since the auxiliary result Proposition 12.1 does not rely on the precise structure of $\mathfrak{G}$, it can be proved using essentially same arguments that have been used in previous works on shock formation for wave equations. For this reason, and to aid the flow of the paper, we delay the proof of Proposition 12.1 until the Appendix.

Next, in Section 13, we use the energy estimates to derive L^∞ estimates for the wave variables. In particular, these estimates yield improvements of the L^∞ bootstrap assumptions that we made in Section 6.

In Section 14, we combine the results of the previous sections to provide the proof of the main a priori estimates as well as the main theorems and their corollaries.

Finally, in the Appendix, we provide the details behind the proof of the auxiliary result Proposition 12.1. The proof relies on small modifications to the proofs of [36; 52] that account for the third spatial dimension (note that three dimensions wave equations were also handled in [15; 48]), as well as the presence of the inhomogeneous terms $\mathfrak{G}$ in the wave equations.

2. Geometric setup

In this section, we construct most of the geometric objects that we use to study shock formation and exhibit their basic properties.

2A. *Notational conventions and remarks on constants.*

The precise definitions of some of the concepts referred to here are provided later in the article.

- Lowercase Greek spacetime indices α, β, etc. correspond to the Cartesian spacetime coordinates (see Section 2C) and vary over 0, 1, 2, 3. Lowercase Latin spatial indices a,b, etc. correspond to the Cartesian spatial coordinates and vary over 1, 2, 3. Uppercase Latin spatial indices A,B, etc. correspond to the coordinates on $\ell_{t,u}$ and vary over 2, 3. All lowercase Greek indices are lowered and raised with the acoustical metric g and its inverse g^{-1}, and *not with the Minkowski metric*. We use Einstein's summation convention in that repeated indices are summed.

- By "$\cdot$" we denote the natural contraction between two tensors. For example, if ξ is a spacetime one-form and V is a spacetime vectorfield, then $\xi \cdot V \doteq \xi_\alpha V^\alpha$.

- If ξ is an $\ell_{t,u}$-tangent one-form (as defined in Section 2J), then $\xi^\#$ denotes its $g\!\!\!/$-dual vectorfield, where $g\!\!\!/$ is the Riemannian metric induced on $\ell_{t,u}$ by g. Similarly, if ξ is a symmetric type-$\binom{0}{2}$ $\ell_{t,u}$-tangent tensor, then $\xi^\#$ denotes the type-$\binom{1}{1}$ $\ell_{t,u}$-tangent tensor formed by raising one index with $g\!\!\!/^{-1}$ and $\xi^{\#\#}$ denotes the type-$\binom{2}{0}$ $\ell_{t,u}$-tangent tensor formed by raising both indices with $g\!\!\!/^{-1}$.

- If V is an $\ell_{t,u}$-tangent vectorfield, then $V_\flat$ denotes its $g\!\!\!/$-dual one-form.

- If V and W are vectorfields, then $V_W \doteq V^\alpha W_\alpha = g_{\alpha\beta} V^\alpha W^\beta$.

• If ξ is a one-form and V is a vectorfield, then $\xi_V \doteq \xi_\alpha V^\alpha$. We use similar notation when contracting higher-order tensorfields against vectorfields. For example, if ξ is a type-$\binom{0}{2}$ tensorfield and V and W are vectorfields, then $\xi_{VW} \doteq \xi_{\alpha\beta} V^\alpha W^\beta$.

• Unless otherwise indicated, all quantities in our estimates that are not explicitly under an integral are viewed as functions of the geometric coordinates (t, u, x^2, x^3). Unless otherwise indicated, integrands have the functional dependence established below in Definition 3.1.

• $[Q_1, Q_2] = Q_1 Q_2 - Q_2 Q_1$ denotes the commutator of the operators Q_1 and Q_2.

• $A \lesssim B$ means that there exists $C > 0$ such that $A \leq CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. $A = \mathcal{O}(B)$ means that $|A| \lesssim |B|$.

• The constants C are free to vary from line to line. *These constants, and implicit constants as well, are allowed to depend on the equation of state, the background* $\bar{\varrho}$, *the maximum number of times* N_{top} *that we commute the equations, and the parameters* $\mathring{\sigma}$, $\mathring{\delta}$ *and* $\mathring{\delta}_*^{-1}$ from Section 4A.

• Constants $C_\blacklozenge$ are also allowed to vary from line to line, but unlike C, the $C_\blacklozenge$ are *only allowed to depend on the equation of state and the background* $\bar{\varrho}$.

• In the Appendix, there appear *absolute constants* M_{abs}, which can be chosen to be independent of the equation of state and all other parameters in the problem.

• For our proof to close, the high-order energy blowup-rate parameter M_* needs to be chosen to be large in a manner that depends only on M_{abs}; hence, M_* can also be chosen to be an absolute constant.

• The integer N_{top} denotes the maximum number of times we need to commute the equations to close the estimates. For our proof to close, N_{top} needs to be chosen to be large in a manner that depends only on M_*. N_{top} could be chosen to be an absolute constant, but we choose to think of it as a parameter that we are free to adjust so that we can study solutions with arbitrary sufficiently large regularity.

• For our proof to close, the data-size parameters $\mathring{\alpha}$ and $\mathring{\epsilon}$ must be chosen to be sufficiently small, where the required smallness is clarified in Theorem 6.3. We always assume that $\mathring{\epsilon}^{1/2} \leq \mathring{\alpha}$.

• $A \lesssim_\blacklozenge B$ means that $A \leq C_\blacklozenge B$, with $C_\blacklozenge$ as above. Similarly, $A = \mathcal{O}_\blacklozenge(B)$ means that $|A| \leq C_\blacklozenge |B|$.

• For example, $\mathring{\delta}_*^{-2} = \mathcal{O}(1)$, $2 + \mathring{\alpha} + \mathring{\alpha}^2 = \mathcal{O}_\blacklozenge(1)$, $\mathring{\alpha}\mathring{\epsilon} = \mathcal{O}(\mathring{\epsilon})$, $C_\blacklozenge \mathring{\alpha}^2 = \mathcal{O}_\blacklozenge(\mathring{\alpha})$, $N!\,\mathring{\epsilon} = \mathcal{O}(\mathring{\epsilon})$, and $C\mathring{\alpha} = \mathcal{O}(1)$. Some of these examples are nonoptimal; e.g., we actually have $\mathring{\alpha}\mathring{\epsilon} = \mathcal{O}_\blacklozenge(\mathring{\epsilon})$.

• $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ respectively denote the standard floor and ceiling functions.

2B. ***Caveats on citations.*** Before we introduce our geometric setup, we should say that our setup is essentially the same as that in [36; 52], except for some small differences. We will therefore cite whenever possible the computations in [36; 52], except we will need to take into account the following differences:

• The work [52] allows for very general metrics, while in the present paper, we are only concerned with the acoustical metric for the compressible Euler equations. In citing [52], we sometimes adjust formulas to take into account the explicit form of the Cartesian metric components $g_{\alpha\beta}$ stated in Definition 2.9.

- The papers [36; 52] concern two spatial dimensions (with ambient manifold $\Sigma = \mathbb{R} \times \mathbb{T}$), while in the present paper, we are concerned with three spatial dimensions (with $\Sigma = \mathbb{R} \times \mathbb{T}^2$).
- In [52], the metric components $g_{\alpha\beta}$ were functions of a scalar function Ψ, as opposed to the array $\vec{\Psi}$ (defined in (2-3)). For this reason, we must make minor adjustments to many of the formulas from [52] to account for the fact that in the present article, $\vec{\Psi}$ is an array.

In all cases, our minor adjustments can easily be verified by examining the proof in [52].

2C. ***Basic setup and ambient manifold.*** We recall again the setup from the Introduction. We will work on the spacetime manifold $I \times \Sigma$ (with $I \subseteq \mathbb{R}$ a time interval and $\Sigma \doteq \mathbb{R} \times \mathbb{T}^2$ the spatial domain). We fix a standard Cartesian coordinate system $\{x^\alpha\}_{\alpha=0,1,2,3}$ on $I \times \Sigma$, where $t \doteq x^0 \in I$ is the time coordinate and $x \doteq (x^1, x^2, x^3) \in \mathbb{R} \times \mathbb{T}^2$ are the spatial coordinates.[27] We use the notation $\{\partial_\alpha\}_{\alpha=0,1,2,3}$ (or $\partial_t \doteq \partial_0$) to denote the Cartesian coordinate partial derivative vectorfields.

In this coordinate system, the *plane-symmetric* solutions are exactly those whose fluid variables are independent of (x^2, x^3).

2D. ***Fluid variables and new variables useful for the reformulation.*** As we already discussed in Section 1A3, at the heart of our approach is a reformulation of the compressible Euler equations in terms of new variables. We introduce these new variables in this subsection; see Definitions 2.3 and 2.7.

The basic fluid variables are (ϱ, v^i, s) (see the Introduction). We fix an equation of state $p = p(\varrho, s)$ and a constant $\bar{\varrho} > 0$ such that $p_{;\varrho}(\bar{\varrho}, 0) = 1$.

Definition 2.1. Define the *logarithmic density* ρ and the *speed of sound* $c(\rho, s)$ by

$$\rho = \log\left(\frac{\varrho}{\bar{\varrho}}\right), \quad c(\rho, s) = \sqrt{\frac{\partial p}{\partial \varrho}(\varrho, s)}.$$

Remark 2.2. As is suggested by our notation, we will consider $c(\rho, s)$ as a function of (ρ, s). The normalization of $p_{;\varrho}$ that we stated above is equivalent to

$$c(0, 0) = 1. \tag{2-1}$$

Definition 2.3 (the fluid variables arrays).

(1) Define the *almost Riemann invariants*[28] $\mathcal{R}_{(\pm)}$ as follows (recall Definition 2.1):

$$\mathcal{R}_{(\pm)} \doteq v^1 \pm F(\rho, s), \quad F(\rho, s) \doteq \int_0^\rho c(\rho', s)\, d\rho'. \tag{2-2}$$

[27] While the coordinates x^2, x^3 on $\mathbb{T}^2$ are only locally defined, the corresponding partial derivative vectorfields ∂_2, ∂_3 can be extended so as to form a global smooth frame on $\mathbb{T}^2$. Similar remarks apply to the one-forms dx^2, dx^3 These simple observations are relevant for this paper because when we derive estimates, the coordinate functions x^2, x^3 themselves are never directly relevant; what matters are estimates for the components of various tensorfields with respect to the frame $\{\partial_t, \partial_1, \partial_2, \partial_3\}$ and the basis dual coframe $\{dt, dx^1, dx^2, dx^3\}$, which are everywhere smooth.

[28] $\mathcal{R}_{(\pm)}$ coincide with the well-known *Riemann invariants* in the plane-symmetric isentropic case. Even though they are no longer "invariant" in our case, they are useful in capturing smallness.

(2) Define the *array of wave variables*:[29]

$$\vec{\Psi} \doteq (\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) \doteq (\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s). \tag{2-3}$$

Remark 2.4. We sometimes use the simpler notation Ψ in place of $\vec{\Psi}$ when there is no danger of confusion. At other times, we use the notation Ψ to denote a generic element of $\vec{\Psi}$. The precise meaning of the symbol Ψ will be clear from context.

Remark 2.5 (clarification on our approach to estimating ρ and v^1). Recall that we have introduced $\mathcal{R}_{(\pm)}$ to allow us to capture the fact that our solutions are perturbations of simple plane waves (for which only $\mathcal{R}_{(+)}$ is nonvanishing). In the one-dimensional isentropic case, $\{\mathcal{R}_{(+)}, \mathcal{R}_{(-)}\}$ can be taken to be the unknowns in place of $\{\rho, v^1\}$. A similar remark holds in the present three-dimensional case as well, provided we take into account the entropy. Specifically, from (2-1) and Definition 2.3, it follows that $v^1 = \frac{1}{2}(\mathcal{R}_{(+)} + \mathcal{R}_{(-)})$, and that when ρ, v^1, and s are sufficiently small (as is captured by the smallness parameters $\mathring{\alpha}$ and $\mathring{\epsilon}$ described at the beginning of Section 4A), we have (via the implicit function theorem) $\rho = (\mathcal{R}_{(+)} - \mathcal{R}_{(-)}) \cdot \widetilde{F}(\mathcal{R}_{(+)} - \mathcal{R}_{(-)}, s)$, where $\widetilde{F}$ is a smooth function. This allows us to control ρ and v^1 in terms of $\mathcal{R}_{(+)}$, $\mathcal{R}_{(-)}$, and s. Throughout the article, we use this observation without explicitly pointing it out. In particular, even though many of the equations we cite explicitly involve ρ and v^1, it should be understood that we always estimate these quantities in terms of the wave variables $\mathcal{R}_{(+)}$, $\mathcal{R}_{(-)}$, and s, which are featured in the array (2-3).

Definition 2.6 (Euclidean divergence and curl). Denote by[30] div and curl the Euclidean spatial divergence and curl operator. That is, given a Σ_t-tangent vectorfield $V = V^a \partial_a$, define

$$\operatorname{div} V \doteq \partial_a V^a, \quad (\operatorname{curl} V)^i \doteq \epsilon_{iab} \partial_a V^b, \tag{2-4}$$

where ϵ_{iab} is the fully antisymmetric symbol normalized by $\epsilon_{123} = 1$.

Definition 2.7 (the higher-order variables).

(1) Define the *specific vorticity* to be the Σ_t-tangent vectorfield with the Cartesian spatial components

$$\varOmega^i \doteq \frac{(\operatorname{curl}\, v)^i}{\varrho/\bar{\varrho}} = \frac{(\operatorname{curl}\, v)^i}{\exp(\rho)}.$$

(2) Define the *entropy gradient* to be the Σ_t-tangent vectorfield with the Cartesian spatial components

$$S^i \doteq \partial_i s.$$

(3) Define the *modified fluid variables* by

$$\mathcal{C}^i \doteq \exp(-\rho)(\operatorname{curl} \varOmega)^i + \exp(-3\rho) c^{-2} \frac{p_{;s}}{\bar{\varrho}} S^a \partial_a v^i - \exp(-3\rho) c^{-2} \frac{p_{;s}}{\bar{\varrho}} (\partial_a v^a) S^i, \tag{2-5a}$$

$$\mathcal{D} \doteq \exp(-2\rho) \operatorname{div} S - \exp(-2\rho) S^a \partial_a \rho. \tag{2-5b}$$

We think of $\mathcal{C}$ as a Σ_t-tangent vectorfield with Cartesian spatial components given by (2-5a).

[29] Throughout, we consider $\vec{\Psi}$ as an array of scalar functions; we will not attribute any tensorial structure to the labeling index ι of Ψ_ι besides simple contractions, denoted by $\diamond$, corresponding to the chain rule; see Definition 2.13.

[30] This is in contrast to $\operatorname{d\!\!/iv}$; see Definition 2.33.

2E. ***The acoustical metric and related objects in Cartesian coordinates.*** Hidden within compressible Euler flow lies a geometric structure captured by the acoustical metric, which governs the dynamics of the sound waves. We introduce in this subsection the acoustical metric g in Cartesian coordinates.

Definition 2.8 (material derivative vectorfield). We define the *material derivative vectorfield* as follows relative to the Cartesian coordinates:

$$B \doteq \partial_t + v^a \partial_a. \tag{2-6}$$

Definition 2.9 (the acoustical metric). Define the *acoustical metric* g (in Cartesian coordinates) by

$$g \doteq -dt \otimes dt + c^{-2} \sum_{a=1}^{3} (dx^a - v^a\, dt) \otimes (dx^a - v^a\, dt). \tag{2-7}$$

The following lemma follows from straightforward computations.

Lemma 2.10 (the inverse acoustical metric). *The inverse of the acoustical metric g from* (2-7) *can be expressed as*

$$g^{-1} = -B \otimes B + c^2 \sum_{a=1}^{3} \partial_a \otimes \partial_a. \tag{2-8}$$

Remark 2.11 (closeness to the Minkowski metric). In our analysis, v and $c-1$ will be small, where the smallness is captured by the parameters $\mathring{\alpha}$ and $\mathring{\varepsilon}$ described at the beginning of Section 4A. Recalling (2-7), we see that g will be L^∞-close to the Minkowski metric. It is therefore convenient to introduce the decomposition

$$g_{\alpha\beta}(\vec{\Psi}) = m_{\alpha\beta} + g_{\alpha\beta}^{(\text{small})}(\vec{\Psi}), \quad m_{\alpha\beta} \doteq \operatorname{diag}(-1, 1, 1, 1), \tag{2-9}$$

where m is the Minkowski metric and $g_{\alpha\beta}^{(\text{small})}(\vec{\Psi})$ is a smooth function of $\vec{\Psi}$ such that

$$g_{\alpha\beta}^{(\text{small})}(\vec{\Psi} = 0) = 0. \tag{2-10}$$

Definition 2.12 ($\vec{\Psi}$-derivatives of $g_{\alpha\beta}$). For $\alpha, \beta = 0, \dots, 3$ and $\iota = 1, \dots, 5$, we define

$$G_{\alpha\beta}^{\iota}(\vec{\Psi}) \doteq \frac{\partial}{\partial \Psi_\iota} g_{\alpha\beta}(\vec{\Psi}), \quad \vec{G}_{\alpha\beta} = \vec{G}_{\alpha\beta}(\vec{\Psi}) \doteq \big(G_{\alpha\beta}^{1}(\vec{\Psi}), G_{\alpha\beta}^{2}(\vec{\Psi}), G_{\alpha\beta}^{3}(\vec{\Psi}), G_{\alpha\beta}^{4}(\vec{\Psi}), G_{\alpha\beta}^{5}(\vec{\Psi})\big). \tag{2-11}$$

For each fixed $\iota \in \{1, \dots, 5\}$, we think of $\{G_{\alpha\beta}^{\iota}\}_{\alpha,\beta=0,\dots,3}$, as the Cartesian components of a spacetime tensorfield. Similarly, we think of $\{\vec{G}_{\alpha\beta}\}_{\alpha,\beta=0,\dots,3}$ as the Cartesian components of an array-valued spacetime tensorfield.

Definition 2.13 (operators involving $\vec{\Psi}$). Let U_1, U_2, V be vectorfields. We define

$$V\vec{\Psi} \doteq (V\Psi_1, V\Psi_2, V\Psi_3, V\Psi_4, V\Psi_5), \quad \vec{G}_{U_1U_2} \diamond V\vec{\Psi} \doteq \sum_{\iota=1}^{5} G_{\alpha\beta}^{\iota} U_1^\alpha U_2^\beta V\Psi_\iota. \tag{2-12}$$

We use similar notation with other differential operators in place of vectorfield differentiation. For example, $\vec{G}_{U_1U_2} \diamond \not{\Delta}\vec{\Psi} \doteq \sum_{\iota=1}^{5} G_{\alpha\beta}^{\iota} U_1^\alpha U_2^\beta \not{\Delta}\Psi_\iota$ (where $\not{\Delta}$ is defined in Definition 2.33).

2F. ***The acoustic eikonal function and related constructions.*** To control the solution up to the shock, we will crucially rely on an eikonal function for the acoustical metric.

Definition 2.14 (acoustic eikonal function). The acoustic eikonal function (eikonal function for short) u solves the eikonal equation initial value problem

$$(g^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0, \quad \partial_t u > 0, \quad u\restriction_{t=0} = \mathring{\sigma} - x^1, \tag{2-13}$$

where $\mathring{\sigma} > 0$ is the constant controlling the initial support (recall Theorem 1.1).

Definition 2.15 (inverse foliation density). Define the inverse foliation density μ by

$$\mu \doteq \frac{-1}{(g^{-1})^{\alpha\beta}(\vec{\Psi})\partial_\alpha t \partial_\beta u} > 0. \tag{2-14}$$

Note that $1/\mu$ measures the density of the level sets of u relative to the constant-time hypersurfaces Σ_t. For the data that we will consider, we have $\mu\restriction_{\Sigma_0} \approx 1$. When μ vanishes, the level sets of u intersect and, as it turns out, $\max_{\alpha=0,1,2,3}|\partial_\alpha u|$ and $\max_{\alpha=0,1,2,3}|\partial_\alpha \mathcal{R}_{(+)}|$, blow up.

The following quantities, tied to μ, play an important role in our description of the singular behavior of our high-order energies.

Definition 2.16. Define $\mu_\star(t,u)$ and $\mu_\star(t)$ by[31]

$$\mu_\star(t,u) \doteq \min\{1, \min_{u' \le u} \mu(t,u')\}, \quad \mu_\star(t) \doteq \min\{1, \min_{\Sigma_t} \mu\}.$$

2G. ***Subsets of spacetimes.***

Definition 2.17 (subsets of spacetime). For $0 \le t'$ and $0 \le u'$, define

$$\Sigma_{t'} \doteq \{(t,x) \in \mathbb{R} \times (\mathbb{R} \times \mathbb{T}^2) \mid t = t'\}, \tag{2-15a}$$

$$\Sigma_{t'}^{u'} \doteq \{(t,x) \in \mathbb{R} \times (\mathbb{R} \times \mathbb{T}^2) \mid t = t',\ 0 \le u(t,x) \le u'\}, \tag{2-15b}$$

$$\mathcal{F}_{u'} \doteq \{(t,x) \in \mathbb{R} \times (\mathbb{R} \times \mathbb{T}^2) \mid u(t,x) = u'\}, \tag{2-15c}$$

$$\mathcal{F}_{u'}^{t'} \doteq \{(t,x) \in \mathbb{R} \times (\mathbb{R} \times \mathbb{T}^2) \mid 0 \le t \le t',\ u(t,x) = u'\}, \tag{2-15d}$$

$$\ell_{t',u'} \doteq \mathcal{F}_{u'}^{t'} \cap \Sigma_{t'}^{u'} = \{(t,x) \in \mathbb{R} \times (\mathbb{R} \times \mathbb{T}^2) \mid t = t',\ u(t,x) = u'\}, \tag{2-15e}$$

$$\mathcal{M}_{t',u'} \doteq \bigcup_{u \in [0,u']} \mathcal{F}_u^{t'} \cap \{(t,x) \in \mathbb{R} \times (\mathbb{R} \times \mathbb{T}^2) \mid 0 \le t < t'\}. \tag{2-15f}$$

We refer to the Σ_t and Σ_t^u as "constant time slices," the $\mathcal{F}_u$ and $\mathcal{F}_u^t$ as "null hyperplanes," "null hypersurfaces," "characteristics," or "acoustic characteristics," and the $\ell_{t,u}$ as "tori." Note that $\mathcal{M}_{t,u}$ is "open-at-the-top" by construction.

[31] By definition, $\mu_\star(t,u) \ge \mu_\star(t)$ for all $u \in \mathbb{R}$. Note that by the localization lemma (Lemma 7.1) we prove below, we have $\mu_\star(t) = \mu_\star(t, U_0)$. In most of the proof, it suffices to consider the function $\mu_\star(t)$ without considering $\mu_\star(t,u)$. The more refined definition for $\mu_\star(t,u)$ will only be referred to in the Appendix, so that the formulas take the same forms as their counterparts in [36; 52].

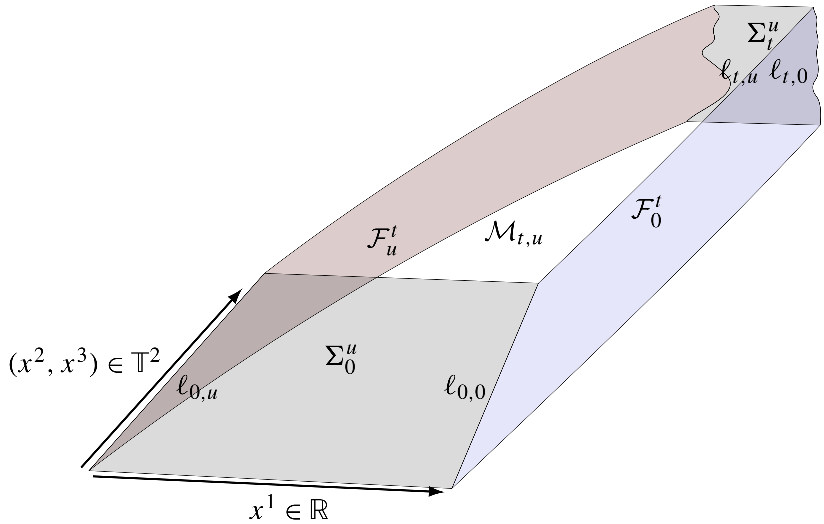

Figure 2. The spacetime region and various subsets. The (unlabeled and uncolored) flat front and back surfaces should be identified.

2H. *Important vectorfields, the rescaled frame, and the nonrescaled frame.*

Definition 2.18 (important vectorfields). (1) Define the *geodesic null vectorfield* by

$$L^{\nu}_{(\mathrm{Geo})} \doteq -(g^{-1})^{\nu\alpha}\, \partial_\alpha u. \tag{2-16}$$

(2) Define the *rescale null vectorfield* (recall the definition of μ in (2-14)) by

$$L \doteq \mu L_{(\mathrm{Geo})}. \tag{2-17}$$

(3) Define X to be the unique vectorfield that is Σ_t-tangent, g-orthogonal to the $\ell_{t,u}$, and normalized by

$$g(L, X) = -1. \tag{2-18}$$

Define the "rescaled" vectorfield $\breve{X}$ by

$$\breve{X} \doteq \mu X. \tag{2-19}$$

(4) Define Y and Z respectively to be the g-orthogonal projection[32] of the Cartesian partial derivative vectorfields ∂_2 and ∂_3 to the tangent space of $\ell_{t,u}$, i.e.,

$$Y \doteq \partial_2 - g(\partial_2, X)X, \quad Z \doteq \partial_3 - g(\partial_3, X)X. \tag{2-20}$$

(5) We will use vectorfields in $\mathscr{P} \doteq \{L, Y, Z\}$ for commutation, and we therefore refer to them as *commutation vectorfields*. An element of $\mathscr{P}$ will often be denoted schematically by $\mathcal{P}$ (see also Definition 3.4).

We collect some basic properties of these vectorfields; see [52, (2.12), (2.13) and Lemma 2.1] for proofs.

[32]To see that Y and Z are tangent to $\ell_{t,u}$, one can use (2-18), (2-23), the fact that B is g-orthogonal to Σ_t, and the fact that ∂_i is tangent to Σ_t. Alternatively, see (2-30b).

Lemma 2.19 (basic properties of the vectorfields).

(1) $L_{(\mathrm{Geo})}$ *is geodesic and null, i.e.*,

$$g(L_{(\mathrm{Geo})}, L_{(\mathrm{Geo})}) = 0, \quad \mathscr{D}_{L_{(\mathrm{Geo})}} L_{(\mathrm{Geo})} = 0,$$

where $\mathscr{D}$ *is the Levi-Civita connection associated to* g.

(2) *The following identities hold*:

$$Lu = 0, \quad Lt = L^0 = 1, \qquad \breve{X}u = 1, \quad \breve{X}t = \breve{X}^0 = 0, \tag{2-21}$$

$$g(X, X) = 1, \quad g(\breve{X}, \breve{X}) = \mu^2, \quad g(L, X) = -1, \quad g(L, \breve{X}) = -\mu. \tag{2-22}$$

(3) *The vectorfield* B (*see* (2-6)) *is future-directed*, g*-orthogonal to* Σ_t, *and is normalized by* $g(B,B)=-1$. *Moreover*,

$$B = \partial_t + v^a \partial_a = L + X, \tag{2-23}$$

$$B_\alpha = -\delta_\alpha^0, \tag{2-24}$$

where δ_α^β *is the Kronecker delta.*

2I. *Transformations.* Having introduced various vectorfields in Section 2H, we now derive some related transformation formulas that we will use later on.

Definition 2.20 (coordinate vectorfields in geometric (t, u, x^2, x^3)-coordinates). Define $(\partial\!\!\!/_t, \partial\!\!\!/_u, \partial\!\!\!/_2, \partial\!\!\!/_3)$ to be the coordinate partial derivative vectorfields in the geometric (t, u, x^2, x^3)-coordinate system.

Definition 2.21 (Cartesian components of geometric vectorfields).

(1) Define L^i and X^i to be the Cartesian i-th components of L and X respectively. (Note $L^i + X^i - v^i = 0$; see (2-23).)

(2) Define[33] $L_{(\mathrm{small})}$ and $X_{(\mathrm{small})}$ by

$$L^1_{(\mathrm{small})} \doteq L^1 - 1, \quad L^2_{(\mathrm{small})} \doteq L^2, \quad L^3_{(\mathrm{small})} \doteq L^3, \tag{2-25a}$$

$$X^1_{(\mathrm{small})} \doteq X^1 + 1, \quad X^2_{(\mathrm{small})} \doteq X^2, \quad X^3_{(\mathrm{small})} \doteq X^3. \tag{2-25b}$$

Lemma 2.22 (relations between $\{\partial_\alpha\}_{\alpha=0,1,2,3}$ and $\{L, X, Y, Z\}$). *The following identities hold*:

$$\partial_t \doteq \partial_0 = L + X - v^a \partial_a, \tag{2-26a}$$

$$\partial_1 = c^{-2} X^1 X - \frac{X^2}{X^1} Y - \frac{X^3}{X^1} Z, \tag{2-26b}$$

$$\partial_2 = Y + (c^{-2} X^2) X, \quad \partial_3 = Z + (c^{-2} X^3) X. \tag{2-26c}$$

Proof. Equation (2-26a) is simply a restatement of (2-23), and (2-26c) follows from (2-20) and $g(\partial_A, X) = c^{-2} X^A$ for $A = 2, 3$ (see (2-7)). Finally, to obtain (2-26b), we write $X = X^a \partial_a$ and use (2-26c) to obtain

$$\partial_1 = \frac{1}{X^1}[1 - c^{-2}((X^2)^2 + (X^3)^2)]X - \frac{X^2}{X^1} Y - \frac{X^3}{X^1} Z.$$

This then implies (2-26b) since $\sum_{a=1}^3 (X^a)^2 = c^2$ by $g(X, X) = 1$ (see (2-22)) and (2-7). $\square$

[33] The notation is suggestive of the fact that these quantities are of size $\mathcal{O}(\mathring{\alpha})$ (and hence small).

Lemma 2.23 (relation between $\{\partial\!\!\!/_\alpha\}$ and $\{L, X, Y, Z\}$). *The following identities hold*, *where repeated capital Latin indices are summed over* $A = 2, 3$:

$$L = \partial\!\!\!/_t + L^A \partial\!\!\!/_A, \qquad \breve{X} = \partial\!\!\!/_u + \mu X^A \partial\!\!\!/_A, \tag{2-27a}$$

$$Y = (1 - c^{-2}(X^2)^2)\partial\!\!\!/_2 - c^{-2} X^2 X^3 \partial\!\!\!/_3, \qquad Z = (1 - c^{-2}(X^3)^2)\partial\!\!\!/_3 - c^{-2} X^2 X^3 \partial\!\!\!/_2. \tag{2-27b}$$

Proof. Equation (2-27a) is an immediate consequence of (2-21) (and (2-19)).

To derive the first equation in (2-27b), simply note that $Yx^2 = 1 - c^{-2}(X^2)^2$ and $Yx^3 = -cX^2X^3$ by (2-26c), and that $Yt = Yu = 0$ since Y is $\ell_{t,u}$-tangent. The second equation in (2-27b) follows from similar reasoning. □

Lemma 2.24 (relation[34] between $\{\partial_a\}_{a=1,2,3}$, $\{\partial\!\!\!/_u, \partial\!\!\!/_2, \partial\!\!\!/_3\}$, and $\{\breve{X}, Y, Z\}$). *The following identities hold*:

$$\partial\!\!\!/_u = \frac{\mu c^2}{X^1}\partial_1 = \breve{X} - \mu c^2 \frac{X^2}{(X^1)^2} Y - \mu c^2 \frac{X^3}{(X^1)^2} Z, \tag{2-28a}$$

$$\partial\!\!\!/_2 = \partial_2 - \frac{X^2}{X^1}\partial_1 = \left\{1 + \left(\frac{X^2}{X^1}\right)^2\right\} Y + \frac{X^2 X^3}{(X^1)^2} Z, \tag{2-28b}$$

$$\partial\!\!\!/_3 = \partial_3 - \frac{X^3}{X^1}\partial_1 = \frac{X^2 X^3}{(X^1)^2} Y + \left\{1 + \left(\frac{X^3}{X^1}\right)^2\right\} Z. \tag{2-28c}$$

Proof. It suffices to derive the identities

$$\partial\!\!\!/_u x^1 = \frac{\mu c^2}{X^1}, \quad \partial\!\!\!/_2 x^1 = -\frac{X^2}{X^1}, \quad \partial\!\!\!/_3 x^1 = -\frac{X^3}{X^1}; \tag{2-29}$$

it is straightforward to see that the first identities in each of (2-28a)–(2-28c) follow from (2-29); the second identities in (2-28a)–(2-28c) then follow from the first ones and Lemma 2.22. To prove (2-29), we invert (2-27b) to obtain (with the help of the identity $\sum_{a=1}^3 (X^a)^2 = c^2$, which follows from (2-22) and (2-7)):

$$\partial\!\!\!/_2 = \left\{\frac{c^2}{(X^1)^2} - \left(\frac{X^3}{X^1}\right)^2\right\} Y + \frac{X^2 X^3}{(X^1)^2} Z, \quad \partial\!\!\!/_3 = \frac{X^2 X^3}{(X^1)^2} Y + \left\{\frac{c^2}{(X^1)^2} - \left(\frac{X^2}{X^1}\right)^2\right\} Z.$$

On the other hand, by (2-26c), $Yx^1 = -c^{-2} X^2 X^1$ and $Zx^1 = -c^{-2} X^3 X^1$. Hence,

$$\partial\!\!\!/_2 x^1 = -\frac{X^2}{X^1}, \quad \partial\!\!\!/_3 x^1 = -\frac{X^3}{X^1}.$$

Plugging back into the second identity in (2-27a), we obtain

$$\partial\!\!\!/_u x^1 = \mu X^1 - \sum_{A=2}^{3} \mu X^A \partial\!\!\!/_A x^1 = \mu X^1 + \sum_{A=2}^{3} \mu \frac{(X^A)^2}{X^1} = \frac{\mu c^2}{X^1},$$

where we again used $\sum_{i=1}^3 (X^i)^2 = c^2$. □

[34]We could also obtain $\partial\!\!\!/_t = \partial_t + (L^1 + (X^2 L^2 + X^3 L^3)/X_1)\partial_1$. Since this will not be explicitly needed, we will not prove it.

2J. *Projection tensorfields, $\vec{G}_{(\mathrm{frame})}$, and projected Lie derivatives.*

Definition 2.25 (projection tensorfields). We define the Σ_t projection tensorfield[35] $\underline{\Pi}$ and the $\ell_{t,u}$ projection tensorfield $\not{\Pi}$ relative to Cartesian coordinates as

$$\underline{\Pi}_\nu{}^\mu \doteq \delta_\nu{}^\mu + B_\nu B^\mu = \delta_\nu{}^\mu - \delta_\nu{}^0 L^\mu - \delta_\nu{}^0 X^\mu, \tag{2-30a}$$

$$\not{\Pi}_\nu{}^\mu \doteq \delta_\nu{}^\mu + X_\nu L^\mu + L_\nu (L^\mu + X^\mu) = \delta_\nu{}^\mu - \delta_\nu{}^0 L^\mu + L_\nu X^\mu. \tag{2-30b}$$

In (2-30a)–(2-30b), $\delta_\nu{}^\mu$ is the standard Kronecker delta. The last equalities in (2-30a) and (2-30b) follow from (2-23)–(2-24).

Definition 2.26 (projections of tensorfields). Given any type-$\binom{m}{n}$ spacetime tensorfield ξ, we define its Σ_t projection $\underline{\Pi}\xi$ and its $\ell_{t,u}$ projection $\not{\Pi}\xi$ as

$$(\underline{\Pi}\xi)^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} \doteq \underline{\Pi}_{\widetilde{\mu}_1}^{\mu_1} \cdots \underline{\Pi}_{\widetilde{\mu}_m}^{\mu_m} \underline{\Pi}_{\nu_1}^{\widetilde{\nu}_1} \cdots \underline{\Pi}_{\nu_n}^{\widetilde{\nu}_n} \xi^{\widetilde{\mu}_1\cdots\widetilde{\mu}_m}_{\widetilde{\nu}_1\cdots\widetilde{\nu}_n}, \tag{2-31a}$$

$$(\not{\Pi}\xi)^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} \doteq \not{\Pi}_{\widetilde{\mu}_1}^{\mu_1} \cdots \not{\Pi}_{\widetilde{\mu}_m}^{\mu_m} \not{\Pi}_{\nu_1}^{\widetilde{\nu}_1} \cdots \not{\Pi}_{\nu_n}^{\widetilde{\nu}_n} \xi^{\widetilde{\mu}_1\cdots\widetilde{\mu}_m}_{\widetilde{\nu}_1\cdots\widetilde{\nu}_n}. \tag{2-31b}$$

We say that a spacetime tensorfield ξ is Σ_t-tangent (respectively $\ell_{t,u}$-tangent) if $\underline{\Pi}\xi = \xi$ (respectively if $\not{\Pi}\xi = \xi$). Alternatively, we say that ξ is a Σ_t tensor (respectively $\ell_{t,u}$ tensor).

Definition 2.27 ($\ell_{t,u}$ projection notation). If ξ is a spacetime tensor, then $\not{\xi} \doteq \not{\Pi}\xi$.

If ξ is a symmetric type-$\binom{0}{2}$ spacetime tensor and V is a spacetime vectorfield, then $\not{\xi}_V \doteq \not{\Pi}(\xi_V)$, where ξ_V is the spacetime one-form with Cartesian components $\xi_{\alpha\nu}V^\alpha$, $(\nu = 0, 1, 2, 3)$.

Remark 2.28 (clarification of the symbols $(\not{\partial}_t, \not{\partial}_u, \not{\partial}_2, \not{\partial}_3)$). We caution that the coordinate partial derivative vectorfields $(\not{\partial}_t, \not{\partial}_u, \not{\partial}_2, \not{\partial}_3)$ from Definition 2.20 are not $\ell_{t,u}$ projections of other vectorfields; i.e., for $(\not{\partial}_t, \not{\partial}_u, \not{\partial}_2, \not{\partial}_3)$, we are not using the "slash conventions" of Definition 2.27.

Throughout, $\mathcal{L}_V\xi$ denotes the Lie derivative of the tensorfield ξ with respect to the vectorfield V. We often use the Lie bracket notation $[V, W] \doteq \mathcal{L}_V W$ when V and W are vectorfields.

Definition 2.29 (Σ_t- and $\ell_{t,u}$-projected Lie derivatives). If ξ is a tensorfield and V is a vectorfield, we define the Σ_t-projected Lie derivative $\underline{\mathcal{L}}_V\xi$ and the $\ell_{t,u}$-projected Lie derivative $\not{\mathcal{L}}_V\xi$ as

$$\underline{\mathcal{L}}_V\xi \doteq \underline{\Pi}\mathcal{L}_V\xi, \qquad \not{\mathcal{L}}_V\xi \doteq \not{\Pi}\mathcal{L}_V\xi. \tag{2-32}$$

Definition 2.30 (components of $\vec{G}$ relative to the nonrescaled frame). We define

$$\vec{G}_{(\mathrm{frame})} \doteq \{\vec{G}_{LL}, \vec{G}_{LX}, \vec{G}_{XX}, \vec{\not{G}}_L, \vec{\not{G}}_X, \vec{\not{G}}\}, \tag{2-33}$$

where $\vec{G}_{\alpha\beta}$ is defined in (2-11).

Our convention is that derivatives of $\vec{G}_{(\mathrm{frame})}$ form a new array consisting of the differentiated components. For example,

$$\not{\mathcal{L}}_L \vec{G}_{(\mathrm{frame})} \doteq \{L(\vec{G}_{LL}), L(\vec{G}_{LX}), \ldots, \not{\mathcal{L}}_L \vec{\not{G}}\},$$

[35] In (2-30a), we have corrected a sign error that occurred in [52, Definition 2.8].

where

$$L(\vec{G}_{LL}) \doteq \{L(G^1_{LL}), L(G^2_{LL}), \dots, L(G^5_{LL})\},$$
$$\not{\mathcal{L}}_L(\vec{\not{G}}_X) \doteq \{\not{\mathcal{L}}_L(\not{G}^1_X), \not{\mathcal{L}}_L(\not{G}^2_X), \dots, \not{\mathcal{L}}_L(\not{G}^5_X)\},$$

etc.

2K. *First and second fundamental forms and covariant differential operators.*

Definition 2.31 (first fundamental forms). Let $\underline{\Pi}$ and $\not{\Pi}$ be as in Definition 2.27. We define the first fundamental form $\underline{g}$ of Σ_t and the first fundamental form $\not{g}$ of $\ell_{t,u}$ as

$$\underline{g} \doteq \underline{\Pi} g, \quad \not{g} \doteq \not{\Pi} g. \tag{2-34}$$

We define the inverse first fundamental forms by raising the indices with g^{-1}:

$$(\underline{g}^{-1})^{\mu\nu} \doteq (g^{-1})^{\mu\alpha}(g^{-1})^{\nu\beta}\underline{g}_{\alpha\beta}, \quad (\not{g}^{-1})^{\mu\nu} \doteq (g^{-1})^{\mu\alpha}(g^{-1})^{\nu\beta}\not{g}_{\alpha\beta}, \tag{2-35}$$

where $\underline{g}$ is the Riemannian metric on Σ_t induced by g, while $\not{g}$ is the Riemannian metric on $\ell_{t,u}$ induced by g. Simple calculations imply that $(\underline{g}^{-1})^{\mu\alpha}\underline{g}_{\alpha\nu} = \underline{\Pi}_\nu{}^\mu$ and $(\not{g}^{-1})^{\mu\alpha}\not{g}_{\alpha\nu} = \not{\Pi}_\nu{}^\mu$.

Lemma 2.32 (identities for induced metrics). *In the (t, u, x^2, x^3)-coordinate system, we have*

$$\underline{g} = \frac{\mu^2 c^2}{(X^1)^2} du \otimes du - \mu \sum_{A=2}^{3} \frac{X^A}{(X^1)^2}(dx^A \otimes du + du \otimes dx^A) + \not{g}, \quad \not{g} = \sum_{A,B=2}^{3} c^{-2}\Big(\delta_{AB} + \frac{X^A X^B}{(X^1)^2}\Big) dx^A \otimes dx^B.$$

Moreover,

$$\not{g}^{-1} = \sum_{A,B=2}^{3} (c^2\delta^{AB} - X^A X^B)\not{\partial}_A \otimes \not{\partial}_B.$$

Proof. The identities for $\underline{g}$ and $\not{g}$ follow easily from Lemma 2.24 and the fact that $\underline{g}_{ij} = c^{-2}\delta_{ij}$ in Cartesian coordinates (see (2-7)). The identity for $\not{g}^{-1}$ follows from inverting the matrix $(\not{g}_{AB})_{A,B=2,3}$ and using the identity $\sum_{i=1}^3 (X^i)^2 = c^2$, which follows from the first identity in (2-22) and (2-7). □

Definition 2.33 (differential operators associated to the metrics).

- $\mathscr{D}$ denotes the Levi-Civita connection of the acoustical metric g.
- $\not{\nabla}$ denotes the Levi-Civita connection of $\not{g}$.
- If f is a scalar function on $\ell_{t,u}$, then $\not{d} f \doteq \not{\nabla} f = \not{\Pi}\mathscr{D} f$, where $\mathscr{D} f$ is the gradient one-form associated to f.
- If ξ is an $\ell_{t,u}$-tangent one-form, then $\not{\mathrm{div}}\,\xi$ is the scalar function $\not{\mathrm{div}}\,\xi \doteq \not{g}^{-1} \cdot \not{\nabla}\xi$.
- Similarly, if V is an $\ell_{t,u}$-tangent vectorfield, then $\not{\mathrm{div}}\,V \doteq \not{g}^{-1} \cdot \not{\nabla} V_\flat$, where $V_\flat$ is the one-form $\not{g}$-dual to V.
- If ξ is a symmetric type-$\binom{0}{2}$ $\ell_{t,u}$-tangent tensorfield, then $\not{\mathrm{div}}\,\xi$ is the $\ell_{t,u}$-tangent one-form $\not{\mathrm{div}}\,\xi \doteq \not{g}^{-1} \cdot \not{\nabla}\xi$, where the two contraction indices in $\not{\nabla}\xi$ correspond to the operator $\not{\nabla}$ and the first index of ξ.
- $\not{\Delta} \doteq \not{g}^{-1} \cdot \not{\nabla}^2$ denotes the covariant Laplacian corresponding to $\not{g}$.

2L. ***Ricci coefficients.***

Definition 2.34 (Ricci coefficients).

(1) Define the *second fundamental form* k of Σ_t and the *null second fundamental form* χ of $\ell_{t,u}$ as

$$k \doteq \tfrac{1}{2}\underline{\mathcal{L}}_B \underline{g}, \quad \chi \doteq \tfrac{1}{2}{\not{\mathcal{L}}}_L {g\mkern-8.5mu/}. \tag{2-36}$$

(2) Define ζ to be the $\ell_{t,u}$-tangent one-form whose components are given by

$$\zeta(\not{\partial}_A) \doteq g(\mathscr{D}_{\not{\partial}_A} L, X) = \mu^{-1} g(\mathscr{D}_{\not{\partial}_A} L, \breve{X}), \quad A = 2, 3. \tag{2-37}$$

(3) Given any symmetric type-$\binom{0}{2}$ $\ell_{t,u}$-tangent tensorfield ξ, define its *trace* by

$$\mathrm{tr}_{g\mkern-8.5mu/}\, \xi \doteq ({g\mkern-8.5mu/}^{-1})^{AB} \xi_{AB}.$$

Lemma 2.35 (useful identities for the Ricci coefficients). *The following identities hold*:[36]

$$\chi = g_{ab}({\not{d}} L^a) \otimes ({\not{d}} x^b) + \tfrac{1}{2}\vec{{G\mkern-12mu/}} \diamond L\vec{\Psi} + \tfrac{1}{2}{\not{d}}\vec{\Psi} \overset{\diamond}{\otimes} \vec{{G\mkern-12mu/}}_L - \tfrac{1}{2}\vec{{G\mkern-12mu/}}_L \overset{\diamond}{\otimes} {\not{d}}\vec{\Psi}, \tag{2-38a}$$

$$\mathrm{tr}_{g\mkern-8.5mu/}\, \chi = g_{ab} {g\mkern-8.5mu/}^{-1} \cdot \{({\not{d}} L^a) \otimes ({\not{d}} x)^b\} + \tfrac{1}{2}{g\mkern-8.5mu/}^{-1} \cdot \vec{{G\mkern-12mu/}} \diamond L\vec{\Psi}, \tag{2-38b}$$

$${k\mkern-8mu/} = \tfrac{1}{2}\mu^{-1}\vec{{G\mkern-12mu/}} \diamond \breve{X}\vec{\Psi} + \tfrac{1}{2}\vec{{G\mkern-12mu/}} \diamond L\vec{\Psi} - \tfrac{1}{2}\vec{{G\mkern-12mu/}}_L \overset{\diamond}{\otimes} {\not{d}}\vec{\Psi} - \tfrac{1}{2}{\not{d}}\vec{\Psi} \overset{\diamond}{\otimes} \vec{{G\mkern-12mu/}}_L - \tfrac{1}{2}\vec{{G\mkern-12mu/}}_X \overset{\diamond}{\otimes} {\not{d}}\vec{\Psi} - \tfrac{1}{2}{\not{d}}\vec{\Psi} \overset{\diamond}{\otimes} \vec{{G\mkern-12mu/}}_X, \tag{2-38c}$$

$$\zeta = -\tfrac{1}{2}\mu^{-1}\vec{{G\mkern-12mu/}}_L \diamond \breve{X}\vec{\Psi} + \tfrac{1}{2}\vec{{G\mkern-12mu/}}_X \diamond L\vec{\Psi} - \tfrac{1}{2}\vec{G}_{LX} \diamond {\not{d}}\vec{\Psi} - \tfrac{1}{2}\vec{G}_{XX} \diamond {\not{d}}\vec{\Psi}. \tag{2-38d}$$

Proof. This is the same as [52, Lemmas 2.13, 2.15] except for small modifications incorporating the third dimension. □

2M. ***Pointwise norms.*** We always measure the magnitude of $\ell_{t,u}$ tensors[37] using ${g\mkern-8.5mu/}$.

Definition 2.36 (pointwise norms). For any type-$\binom{m}{n}$ $\ell_{t,u}$ tensor $\xi_{\nu_1\cdots\nu_n}^{\mu_1\cdots\mu_m}$, we define

$$|\xi| \doteq \sqrt{{g\mkern-8.5mu/}_{\mu_1\widetilde{\mu}_1} \cdots {g\mkern-8.5mu/}_{\mu_m\widetilde{\mu}_m} ({g\mkern-8.5mu/}^{-1})^{\nu_1\widetilde{\nu}_1} \cdots ({g\mkern-8.5mu/}^{-1})^{\nu_n\widetilde{\nu}_n} \xi_{\nu_1\cdots\nu_n}^{\mu_1\cdots\mu_m} \xi_{\widetilde{\nu}_1\cdots\widetilde{\nu}_n}^{\widetilde{\mu}_1\cdots\widetilde{\mu}_m}}. \tag{2-39}$$

2N. ***Transport equations for the eikonal function quantities.*** The next lemma provides the transport equations that, in conjunction with (2-38b), we use to estimate the eikonal function quantities μ, $L^i_{(\mathrm{small})}$, and $\mathrm{tr}_{g\mkern-8.5mu/}\, \chi$ below top order.

Lemma 2.37 ([52, Lemma 2.12] the transport equations satisfied by μ and $L^i_{(\mathrm{small})}$). *The following transport equations hold*:

$$L\mu = \tfrac{1}{2}\vec{G}_{LL} \diamond \breve{X}\vec{\Psi} - \tfrac{1}{2}\mu\vec{G}_{LL} \diamond L\vec{\Psi} - \mu\vec{G}_{LX} \diamond L\vec{\Psi}, \tag{2-40}$$

$$LL^i = \tfrac{1}{2}\vec{G}_{LL} \diamond (L\vec{\Psi})X^i - \vec{{G\mkern-12mu/}}_L^{\#} \diamond (L\vec{\Psi}) \cdot {\not{d}} x^i + \tfrac{1}{2}\vec{G}_{LL} \diamond ({\not{d}}^{\#}\vec{\Psi}) \cdot {\not{d}} x^i. \tag{2-41}$$

[36] Here, $\vec{{G\mkern-12mu/}}_L \overset{\diamond}{\otimes} {\not{d}}\vec{\Psi} \doteq \sum_{\iota=1}^{5} {G\mkern-12mu/}_L^{\iota} \otimes {\not{d}}\Psi_\iota$, and similarly for the other terms involving $\overset{\diamond}{\otimes}$.

[37] Note that in contrast, for Σ_t tensors, we measure their magnitude using the Euclidean metric or an equivalent norm; see, for example, Definition 11.1.

2O. ***Calculations connected to the failure of the null condition.*** Many important estimates are tied to the coefficients $\vec{G}_{LL}$. In the next two lemmas, we derive expressions for $\vec{G}_{LL}$ and $\frac{1}{2}\vec{G}_{LL} \diamond \breve{X}\vec{\Psi}$. This presence of the latter term on the right-hand side (2-40) is tied to the failure of Klainerman's null condition [32] and thus one expects that the product must be nonzero for shocks to form; this is explained in more detail in the survey article [30] in a slightly different context.

Lemma 2.38 (formula for $\frac{1}{2}\vec{G}_{LL} \diamond \breve{X}\vec{\Psi}$). *Let F be the smooth function of* (ρ, s) *from* (2-2), *and let $F_{;s}$ denote its partial derivative with respect to s at fixed ρ. For solutions to* (1-1)–(1-3), *we have*

$$\frac{1}{2}\vec{G}_{LL} \diamond \breve{X}\vec{\Psi} = -\frac{1}{2}c^{-1}(c^{-1}c_{;\rho}+1)\{\breve{X}\mathcal{R}_{(+)} - \breve{X}\mathcal{R}_{(-)}\} - \frac{1}{2}\mu c^{-2}X^1\{L\mathcal{R}_{(+)} + L\mathcal{R}_{(-)}\} \\ - \mu c^{-2}(X^2 Lv^2 + X^3 Lv^3) - \mu c^{-1}c_{;s}X^a S^a + \mu c^{-1}(c^{-1}c_{;\rho}+1)F_{;s}X^a S^a. \tag{2-42}$$

Proof. This is the same as [36, Lemmas 2.45, 2.46], except for minor modifications incorporating the third dimension and the entropy (via the $c_{;s}$-dependent and $F_{;s}$-dependent products). □

3. Volume forms and energies

In this section, we first define geometric integration forms and corresponding integrals. We then define the energies and null fluxes which we will use in the remainder of the paper to derive a priori L^2-type estimates.

3A. ***Geometric forms and related integrals.*** We define our geometric integrals in terms of area and volume forms that remain nondegenerate relative to the geometric coordinates throughout the evolution (i.e., all the way up to the shock).

Definition 3.1 (geometric forms and related integrals). Define the area form $d\lambda_{g\!\!/}$ on $\ell_{t,u}$, the area form $d\underline{\varpi}$ on Σ_t^u, the area form $d\overline{\varpi}$ on $\mathcal{F}_u^t$, and the volume form $d\varpi$ on $\mathcal{M}_{t,u}$ as follows (relative to the (t, u, x^2, x^3)-coordinates):

$$d\lambda_{g\!\!/} = d\lambda_{g\!\!/}(t,u,x^2,x^3) \doteq \frac{dx^2\,dx^3}{c|X^1|}, \qquad d\underline{\varpi} = d\underline{\varpi}(t,u',x^2,x^3) \doteq d\lambda_{g\!\!/}(t,u',x^2,x^3)\,du',$$
$$d\overline{\varpi} = d\overline{\varpi}(t',u,x^2,x^3) \doteq d\lambda_{g\!\!/}(t',u,x^2,x^3)\,dt', \quad d\varpi = d\varpi(t',u',x^2,x^3) \doteq d\lambda_{g\!\!/}(t',u',x^2,x^3)\,du'\,dt'.$$

It is understood that unless we explicitly indicate otherwise, all integrals are defined with respect to the forms of Definition 3.1. Moreover, in our notation, we often suppress the variables with respect to which we integrate; i.e., we write $\int_{\ell_{t,u}} f\,d\lambda_{g\!\!/} \doteq \int_{(x^2,x^3)\in\mathbb{T}^2} f(t,u,x^2,x^3)\,d\lambda_{g\!\!/}(t,u,x^2,x^3)$, etc.

The following lemma clarifies the geometric and analytic significance of the forms from Definition 3.1.

Lemma 3.2 (identities concerning the forms).

(1) *$d\lambda_{g\!\!/}$ is the volume measure induced by $g\!\!/$ on $\ell_{t,u}$.*

(2) *$\mu\,d\underline{\varpi}$ is the volume measure induced by $\underline{g}$ on Σ_t^u.*

(3) *Let dx be the standard Euclidean volume measure on Σ_t^u, i.e., $dx = dx^1\,dx^2\,dx^3$ relative to the Cartesian spatial coordinates. Then*

$$dx = \mu c^3\,d\underline{\varpi}. \tag{3-1}$$

Proof. A computation based on Lemma 2.32 and the identity $\sum_{a=1}^{3}(X^a)^2 = c^2$ (which follows from (2-22) and (2-7)) yields that $\det \not{g} = 1/(c^2(X^1)^2)$. Since $d\lambda_{\not{g}} = \sqrt{\det \not{g}}\, dx^2\, dx^3$, we thus obtain (1).

Next, we again use Lemma 2.32 and the identity $\sum_{a=1}^{3}(X^a)^2 = c^2$ to compute that relative to the (u, x^2, x^3)-coordinates, we have $\det \underline{g} = \mu^2/(c^2(X^1)^2)$. Taking the square root, we see that the volume measure induced by $\underline{g}$ on Σ_t^u is given in the (u, x^2, x^3)-coordinates by $\mu/(c|X^1|)\, du\, dx^2\, dx^3$, which gives (2).

Finally, we obtain (3) from (2) via (2-7), which implies that relative to the Cartesian spatial coordinates, the canonical volume form induced by $\underline{g}$ on Σ_t is $c^{-3} dx^1\, dx^2\, dx^3$. □

3B. *The definitions of the energies and null fluxes.*

3B1. *Forms and conventions.*

Definition 3.3 (volume forms for L^p norms). For $p \in \{1, 2\}$, we define L^p norms with respect to the volume forms introduced in Definition 3.1. That is, for scalar functions or $\ell_{t,u}$-tangent tensorfields ξ, we define

$$\|\xi\|_{L^p(\ell_{t,u})} \doteq \left(\int_{\ell_{t,u}} |\xi|^p\, d\lambda_{\not{g}}\right)^{1/p}, \quad \|\xi\|_{L^p(\mathcal{F}_u^t)} \doteq \left(\int_{\mathcal{F}_u^t} |\xi|^p\, d\overline{\varpi}\right)^{1/p},$$
$$\|\xi\|_{L^p(\Sigma_t^u)} \doteq \left(\int_{\Sigma_t^u} |\xi|^p\, d\underline{\varpi}\right)^{1/p}, \quad \|\xi\|_{L^p(\Sigma_t)} \doteq \left(\int_{\Sigma_t} |\xi|^p\, d\underline{\varpi}\right)^{1/p},$$
$$\|\xi\|_{L^p(\mathcal{M}_{t,u})} \doteq \left(\int_{\mathcal{M}_{t,u}} |\xi|^p\, d\varpi\right)^{1/p}.$$

Definition 3.4 (conventions with variable arrays and differentiated quantities).

(1) Given the fluid variable array $\vec{\Psi}$ in Definition 2.3, define

$$|\vec{\Psi}| = |\Psi| \doteq \max_{\iota=1,\dots,5} |\Psi_\iota|.$$

We also set

$$|\Omega| \doteq \max_{a=1,2,3} |\Omega^a|,$$

and similarly for the other Σ_t-tangent tensorfields such as S and $\mathcal{C}$ that correspond to the transport variables. For $p = 2$ or $p = \infty$, define also

$$\|\Psi\|_{L^p(\ell_{t,u})} \doteq \max_{\iota=1,\dots,5} \|\Psi_\iota\|_{L^p(\ell_{t,u})},$$

and similarly for $L^p(\Sigma_t^u)$, $L^p(\Sigma_t)$, $L^p(\mathcal{F}_u^t)$, and $L^p(\mathcal{M}_{t,u})$. Similarly, we set

$$\|\Omega\|_{L^p(\ell_{t,u})} \doteq \max_{a=1,2,3} \|\Omega^a\|_{L^p(\ell_{t,u})},$$

and we analogously define L^p norms of other Σ_t-tangent tensorfields that correspond to the transport variables, such as S and $\mathcal{C}$.

(2) When estimating multiple solution variables simultaneously, we use the following convention (for $p = 2$ or $p = \infty$):

$$\|(\Omega, S)\|_{L^p(\ell_{t,u})} \doteq \max\{\|\Omega\|_{L^p(\ell_{t,u})},\ \|S\|_{L^p(\ell_{t,u})}\},$$

and similarly for $L^p(\Sigma_t^u)$, $L^p(\Sigma_t)$, $L^p(\mathcal{F}_u^t)$, and $L^p(\mathcal{M}_{t,u})$.

(3) Let $\mathscr{P} \doteq \{L, Y, Z\}$ be the set of commutation vectorfields and

$$\mathscr{P}^{(N)} \doteq \{\mathcal{P}_1\mathcal{P}_2 \cdots \mathcal{P}_N \mid \mathcal{P}_i \in \mathscr{P} \text{ for } 1 \leq i \leq N\}.$$

For any smooth scalar function ϕ, define

$$|\mathcal{P}^N \phi| \doteq \max_{\mathcal{P}_1,\dots,\mathcal{P}_N \in \mathscr{P}^{(N)}} |\mathcal{P}_1 \cdots \mathcal{P}_N \phi|.$$

For $p = 2$ or $p = \infty$, the L^p norms are defined similarly, with

$$\|\mathcal{P}^N \phi\|_{L^p(\ell_{t,u})} \doteq \||\mathcal{P}^N \phi|\|_{L^p(\ell_{t,u})}, \quad \text{etc.}$$

Moreover, we let $\mathcal{P}^N \Omega$ denote the Σ_t-tangent vectorfield with Cartesian spatial components $\mathcal{P}^N \Omega^i$, and we define

$$|\mathcal{P}^N \Omega| \doteq \max_{\mathcal{P}_1 \cdots \mathcal{P}_N \in \mathscr{P}^{(N)}} \max_{a=1,2,3} |\mathcal{P}_1 \cdots \mathcal{P}_N \Omega^a|,$$
$$\|\mathcal{P}^N \Omega\|_{L^p(\ell_{t,u})} \doteq \||\mathcal{P}^N \Omega|\|_{L^p(\ell_{t,u})}, \quad \text{etc.},$$

and similarly for other Σ_t-tangent tensorfields that correspond to the transport variables, such as S and $\mathcal{C}$.

(4) Similarly, we let $\not{\mathscr{P}} \doteq \{Y, Z\}$ be the set of $\ell_{t,u}$-tangent commutation vectorfields and define

$$\not{\mathscr{P}}^{(N)} \doteq \{\not{\mathcal{P}}_1\not{\mathcal{P}}_2 \cdots \not{\mathcal{P}}_N \mid \not{\mathcal{P}}_i \in \not{\mathscr{P}} \text{ for } 1 \leq i \leq N\},$$
$$|\not{\mathcal{P}}^N \phi| \doteq \max_{\not{\mathcal{P}}_1,\dots,\not{\mathcal{P}}_N \in \not{\mathscr{P}}^{(N)}} |\not{\mathcal{P}}_1 \cdots \not{\mathcal{P}}_N \phi|, \quad \text{etc.}$$

The importance of distinguishing the subset of $\ell_{t,u}$-tangent commutation vectorfields from the full set $\mathscr{P}$ will be made clear in the Appendix.[38]

(5) We use the following conventions for sums:

$$|\mathcal{P}^{[N_1,N_2]} \phi| \doteq \sum_{N'=N_1}^{N_2} |\mathcal{P}^{N'} \phi|, \quad |\mathcal{P}^{\leq N} \phi| \doteq |\mathcal{P}^{[0,N]} \phi|,$$
$$|\not{\mathcal{P}}^{[N_1,N_2]} \phi| \doteq \sum_{N'=N_1}^{N_2} |\not{\mathcal{P}}^{N'} \phi|, \quad |\not{\mathcal{P}}^{\leq N} \phi| \doteq |\not{\mathcal{P}}^{[0,N]} \phi|.$$

(6) We will combine the above conventions. For instance,

$$|\mathcal{P}^N(\Omega, S)| \doteq \max_{\mathcal{P}_1 \cdots \mathcal{P}_N \in \mathscr{P}^{(N)}} \max\{|\mathcal{P}_1 \cdots \mathcal{P}_N \Omega|, \ |\mathcal{P}_1 \cdots \mathcal{P}_N S|\},$$
$$|\mathcal{P}^N \Psi| \doteq \max_{\iota=1,\dots,5} \max_{\mathcal{P}_1 \cdots \mathcal{P}_N \in \mathscr{P}^{(N)}} |\mathcal{P}_1 \cdots \mathcal{P}_N \Psi_\iota|.$$

3B2. *Definitions of the energies.* We are now ready to introduce the main energies we use to control the solution.

[38]As in the two-dimensional case, the most difficult error terms in the wave equation energy estimates are commutator terms involving the pure $\ell_{t,u}$-tangent derivatives of the null mean curvature of the $\ell_{t,u}$.

Wave energies:

$$\mathbb{E}_N(t,u) \doteq \sup_{t'\in[0,t]} (\|\breve{X}\mathcal{P}^N\Psi\|^2_{L^2(\Sigma^u_{t'})} + \|\sqrt{\upmu}\mathcal{P}^{N+1}\Psi\|^2_{L^2(\Sigma^u_{t'})}), \tag{3-2a}$$

$$\mathbb{F}_N(t,u) \doteq \sup_{u'\in[0,u]} (\|L\mathcal{P}^N\Psi\|^2_{L^2(\mathcal{F}^t_{u'})} + \|\sqrt{\upmu}{d\!\!/}\mathcal{P}^N\Psi\|^2_{L^2(\mathcal{F}^t_{u'})}), \tag{3-2b}$$

$$\mathbb{K}_N(t,u) \doteq \|{d\!\!/}\mathcal{P}^N\Psi\|^2_{L^2(\mathcal{M}_{t,u}\cap\{\upmu\le 1/4\})}, \tag{3-2c}$$

$$\mathbb{Q}_N(t,u) \doteq \mathbb{E}_N(t,u) + \mathbb{F}_N(t,u), \tag{3-2d}$$

$$\mathbb{W}_N(t,u) \doteq \mathbb{E}_N(t,u) + \mathbb{F}_N(t,u) + \mathbb{K}_N(t,u). \tag{3-2e}$$

Specific vorticity energies:

$$\mathbb{V}_N(t,u) \doteq \sup_{t'\in[0,t]} \|\sqrt{\upmu}\mathcal{P}^N\varOmega\|^2_{L^2(\Sigma^u_{t'})} + \sup_{u'\in[0,u]} \|\mathcal{P}^N\varOmega\|^2_{L^2(\mathcal{F}^t_{u'})}, \tag{3-3a}$$

$$\mathbb{C}_N(t,u) \doteq \sup_{t'\in[0,t]} \|\sqrt{\upmu}\mathcal{P}^N\mathcal{C}\|^2_{L^2(\Sigma^u_{t'})} + \sup_{u'\in[0,u]} \|\mathcal{P}^N\mathcal{C}\|^2_{L^2(\mathcal{F}^t_{u'})}. \tag{3-3b}$$

Entropy gradient energies:

$$\mathbb{S}_N(t,u) \doteq \sup_{t'\in[0,t]} \|\sqrt{\upmu}\mathcal{P}^N S\|^2_{L^2(\Sigma^u_{t'})} + \sup_{u'\in[0,u]} \|\mathcal{P}^N S\|^2_{L^2(\mathcal{F}^t_{u'})}, \tag{3-4a}$$

$$\mathbb{D}_N(t,u) \doteq \sup_{t'\in[0,t]} \|\sqrt{\upmu}\mathcal{P}^N\mathcal{D}\|^2_{L^2(\Sigma^u_{t'})} + \sup_{u'\in[0,u]} \|\mathcal{P}^N\mathcal{D}\|^2_{L^2(\mathcal{F}^t_{u'})}. \tag{3-4b}$$

Definition 3.5 (important conventions for energies).

(1) We define the following convention for sums (cf. Definition 3.4(3)):

$$\mathbb{E}_{\le N}(t,u) \doteq \sum_{N'=0}^{N} \mathbb{E}_{N'}(t,u), \quad \mathbb{E}_{[1,N]}(t,u) \doteq \sum_{N'=1}^{N} \mathbb{E}_{N'}(t,u),$$

and similarly for other energies.

(2) Abusing notation slightly, if we write an energy as a function of only t (instead of a function of (t,u)), then it is understood that we take supremum in u, e.g.,

$$\mathbb{E}_N(t) \doteq \sup_{u\in\mathbb{R}} \mathbb{E}_N(t,u).$$

4. Assumptions on the data and statement of the main theorems

4A. ***Assumptions on the data of the fluid variables.*** We now introduce the assumptions on the data for our main theorem. We have five parameters (see Theorem 1.1), denoted by $\mathring{\sigma}$, $\mathring{\delta}_*$, $\mathring{\delta}$, $\mathring{\alpha}$ and $\mathring{\epsilon}$:

- $\mathring{\sigma}$ measures the size of the initial support in x^1.
- $\mathring{\delta}_*$ gives a lower bound on the quantity that controls the blowup, and in particular determines the time interval for which we need to control our solution before a singularity forms.

- $\mathring{\delta}$, $\mathring{\alpha}$ and $\mathring{\varepsilon}$ are parameters that control the sizes of various norms of the solution. The parameter $\mathring{\delta}$ measures the L^∞ size of the transversal derivatives of $\mathcal{R}_{(+)}$, and it *can be large*, while $\mathring{\alpha}$ limits the size of the amplitude of the fluid variables, is *small* depending on the equation of state and the background density $\bar{\varrho} > 0$, and is used to control basic features of the Lorentzian geometry. The parameter $\mathring{\varepsilon}$ is *small* depending on the equation of state and all the other parameters. In particular, $\mathring{\varepsilon}$ controls the size of solution in "directions that break the simple plane symmetry," and it provides the most crucial smallness that we exploit in the analysis.
- We assume that $\mathring{\varepsilon}^{1/2} \leq \mathring{\alpha}$.

Here are the assumptions on the initial data.[39]

In what follows, N_{top} and M_* denote large positive integers that are constrained in particular by $N_{\mathrm{top}} \geq 2M_* + 10$. In our proof of Proposition 12.1, we will show that our estimates close with M_* chosen to be a universal positive integer. The restriction $N_{\mathrm{top}} \geq 2M_* + 10$ is further explained in Remark 6.1. See also the discussion in Section 2A.

Compact support assumptions:

$$\text{If } |x^1| \geq \mathring{\sigma}, \text{ then } (\rho, v, s) = (0, 0, 0). \tag{4-1}$$

By (2-13), when $t = 0$, the data are supported on the set where $u \in [0, 2\mathring{\sigma}]$. This explains why in some of the data assumptions stated below, we only consider regions in which $u \in [0, 2\mathring{\sigma}]$.

Lower bound for the quantity that controls the blowup-time:[40]

$$\mathring{\delta}_* \doteq \sup_{\Sigma_0} \tfrac{1}{2}[c^{-1}(c^{-1}c_{;\rho} + 1)(\breve{X}\mathcal{R}_{(+)})]_+ > 0. \tag{4-2}$$

Remark 4.1 (nondegeneracy assumption on the factor $c^{-1}(c^{-1}c_{;\rho}+1)$). Recall the factor $c^{-1}(c^{-1}c_{;\rho}+1)$ in (4-2) can be viewed as a function of (ρ, s). For the solutions under study, we are assuming that $c^{-1}(c^{-1}c_{;\rho} + 1)$ is nonvanishing when evaluated at the trivial background solution $(\rho, s) \equiv (0, 0)$ (recall that this background corresponds to a state with constant density $\varrho \equiv \bar{\varrho} > 0$). One can check that for any smooth equation of state except that of a Chaplygin gas, there are always open sets of $\bar{\varrho} > 0$ such that the nonvanishing condition holds; see the end of [36, Section 2.16] for further discussion. We also point out that for the Chaplygin gas, it is not expected that shocks will form.

Assumptions on the amplitude and transversal derivatives of the wave variables:

$$\|\mathcal{R}_{(+)}\|_{L^\infty(\Sigma_0)} \leq \mathring{\alpha}, \tag{4-3a}$$

$$\|\breve{X}^{[1,3]}\mathcal{R}_{(+)}\|_{L^\infty(\Sigma_0)} \leq \mathring{\delta}, \tag{4-3b}$$

$$\|\breve{X}^{\leq 3}(\mathcal{R}_{(-)}, v^2, v^3, s)\|_{L^\infty(\Sigma_0)} \leq \mathring{\varepsilon}, \tag{4-3c}$$

$$\|L\breve{X}\breve{X}\breve{X}\Psi\|_{L^\infty(\Sigma_0)} \leq \mathring{\varepsilon}. \tag{4-3d}$$

[39] Of course, we are only allowed to prescribe (ϱ, v^i, s) without explicitly specifying their derivatives transversal to Σ_0. Nevertheless, using (1-1)–(1-3), we can compute the traces of all derivatives on Σ_0. The derivative assumptions that we specify here are to be understood in this sense. Notice that all the assumptions are satisfied by the data of exactly simple plane-symmetric solutions with $\mathring{\varepsilon} = 0$. Thus, they are also satisfied by small perturbations of them.

[40] Here, $z_+ \doteq \max\{z, 0\}$.

Smallness assumptions for good derivatives of the wave variables:

$$\|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-2]}\Psi\|_{L^\infty(\Sigma_0)},\quad \|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-4]}\breve{X}\Psi\|_{L^\infty(\Sigma_0)},\quad \|\mathcal{P}^{[1,2]}\breve{X}\breve{X}\Psi\|_{L^\infty(\Sigma_0)},$$
$$\sup_{u\in[0,2\mathring{\sigma}]}\|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*]}\Psi\|_{L^2(\ell_{0,u})},\quad \|\mathcal{P}^{[1,N_{\mathrm{top}}+1]}\Psi\|_{L^2(\Sigma_0)},\quad \|\mathcal{P}^{[1,N_{\mathrm{top}}]}\breve{X}\Psi\|_{L^2(\Sigma_0)}\leq\mathring{\varepsilon}. \tag{4-4}$$

Smallness assumptions for the specific vorticity and entropy gradient:

$$\|\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-2}(\varOmega,S)\|_{L^\infty(\Sigma_0)},\quad \sup_{u\in[0,2\mathring{\sigma}]}\|\mathcal{P}^{\leq N_{\mathrm{top}}-M_*}(\varOmega,S)\|_{L^2(\ell_{0,u})},\quad \|\mathcal{P}^{\leq N_{\mathrm{top}}}(\varOmega,S)\|_{L^2(\Sigma_0)}\leq\mathring{\varepsilon}^{3/2}. \tag{4-5}$$

Smallness assumptions for the modified fluid variables:

$$\|\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-3}(\mathcal{C},\mathcal{D})\|_{L^\infty(\Sigma_0)},\quad \sup_{u\in[0,2\mathring{\sigma}]}\|\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-1}(\mathcal{C},\mathcal{D})\|_{L^2(\ell_{0,u})},\quad \|\mathcal{P}^{\leq N_{\mathrm{top}}}(\mathcal{C},\mathcal{D})\|_{L^2(\Sigma_0)}\leq\mathring{\varepsilon}^{3/2}. \tag{4-6}$$

4B. *Statement of the main theorem.* We are now ready to give a precise statement of Theorem 1.1 (see Theorems 4.2 and 4.3 below), as well as the corollaries in interesting subregimes of solutions discussed in Remark 1.5 (see Corollaries 4.4 and 4.5).

We first discuss Theorem 1.1. It will be convenient to think of Theorem 1.1 as two theorems. The first, which is the much harder theorem, is a *regularity* statement, stating — with precise estimates — that in the region under study, the only possible singularity is that of a shock, i.e., one that is associated with the vanishing of μ. This is the content of Theorem 4.2. Once Theorem 4.2 has been proved, the proof that a shock indeed occurs is much easier. This is the content of Theorem 4.3.

Theorem 4.2 (regularity unless shock occurs). *Let $\mathring{\sigma}$, $\mathring{\delta}$, $\mathring{\delta}_*>0$. There exists a large integer M_* that is absolute in the sense that it is independent of the equation of state, $\bar{\varrho}$, $\mathring{\sigma}$, $\mathring{\delta}$, and $\mathring{\delta}_*^{-1}$ such that the following hold. Assume that*:

- *The integer N_{top} satisfies $N_{\mathrm{top}}\geq 2M_*+10$ (see Remark 6.1 regarding the size of N_{top}).*
- *$\mathring{\alpha}>0$ is sufficiently small in a manner that depends only on the equation of state and $\bar{\varrho}$.*
- *$\mathring{\varepsilon}>0$ satisfies*[41] *$\mathring{\varepsilon}^{1/2}\leq\mathring{\alpha}$ and is sufficiently small in a manner that depends only on the equation of state, N_{top}, $\bar{\varrho}$, $\mathring{\sigma}$, $\mathring{\delta}$, and $\mathring{\delta}_*^{-1}$.*
- *The initial data satisfy the support-size and norm-size assumptions*[42] (4-1)–(4-6).

Then the corresponding solution $(\varrho, v^1, v^2, v^3, s)$ to the compressible Euler equations (1-1)–(1-3) *exhibits the following properties.*

Suppose $T\in(0,2\mathring{\delta}_*^{-1}]$, and assume that there is a smooth solution such that the following two conditions hold:

- *The change of variables map $(t,u,x^2,x^3)\to(t,x^1,x^2,x^3)$ from geometric to Cartesian coordinates is a diffeomorphism from $[0,T)\times\mathbb{R}\times\mathbb{T}^2$ onto $[0,T)\times\Sigma$.*
- *$\mu>0$ in $[0,T)\times\Sigma$.*

[41] The assumption $\mathring{\varepsilon}^{1/2}\leq\mathring{\alpha}$ allows us to simplify the presentation of various estimates, for example, by allowing us to write $\mathcal{O}(\mathring{\alpha})$ instead of $\mathcal{O}(\mathring{\varepsilon}^{1/2})+\mathcal{O}(\mathring{\alpha})$.

[42] Note that our plane-symmetric background solutions satisfy these assumptions with $\mathring{\varepsilon}=0$.

Then the following estimates hold for every $t \in [0, T)$, where the implicit constants in $\lesssim_{\blacklozenge}$ depend only on the equation of state and $\bar{\varrho}$, while the implicit constants in $\lesssim$ depend only on the equation of state, N_{top}, $\bar{\varrho}$ $\mathring{\sigma}$, $\mathring{\delta}$, and $\mathring{\delta}_*^{-1}$ (in particular, all implicit constants are independent of t and T).

(1) *The following energy estimates hold (where the energies are defined in* (3-2a)–(3-4b) *and $\upmu_\star(t)$ is as in Definition 2.16):*

$$\mathbb{W}_N(t) \lesssim \mathring{\epsilon}^2 \max\{1, \upmu_\star^{-2M_*+2N_{\text{top}}-2N+1.8}(t)\} \quad \text{for } 1 \le N \le N_{\text{top}}, \tag{4-7a}$$

$$\mathbb{V}_N(t),\ \mathbb{S}_N(t) \lesssim \mathring{\epsilon}^3 \max\{1, \upmu_\star^{-2M_*+2N_{\text{top}}-2N+2.8}(t)\} \quad \text{for } 0 \le N \le N_{\text{top}}, \tag{4-7b}$$

$$\mathbb{C}_N(t),\ \mathbb{D}_N(t) \lesssim \mathring{\epsilon}^3 \max\{1, \upmu_\star^{-2M_*+2N_{\text{top}}-2N+0.8}(t)\} \quad \text{for } 0 \le N \le N_{\text{top}}. \tag{4-7c}$$

(2) *The following L^∞ estimates hold:*

$$\|\mathcal{P}^{[1,N_{\text{top}}-M_*-2]}\Psi\|_{L^\infty(\Sigma_t)},\quad \|\mathcal{P}^{[1,N_{\text{top}}-M_*-4]}\breve{X}\Psi\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}, \tag{4-8a}$$

$$\|\mathcal{R}_{(+)}\|_{L^\infty(\Sigma_t)} \lesssim_{\blacklozenge} \mathring{\alpha},\quad \|(\mathcal{R}_{(-)}, v^2, v^3, s)\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}, \tag{4-8b}$$

$$\|\breve{X}\mathcal{R}_{(+)}\|_{L^\infty(\Sigma_t)} \le 2\mathring{\delta},\quad \|\breve{X}(\mathcal{R}_{(-)}, v^2, v^3, s)\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}, \tag{4-8c}$$

$$\begin{aligned}\|\mathcal{P}^{[1,N_{\text{top}}-M_*-2]}(\varOmega, S)\|_{L^\infty(\Sigma_t)},\quad \|\mathcal{P}^{[1,N_{\text{top}}-M_*-3]}(\mathcal{C},\mathcal{D})\|_{L^\infty(\Sigma_t)},&\\ \|\mathcal{P}^{[1,N_{\text{top}}-M_*-4]}\breve{X}(\varOmega, S)\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}^{3/2}&.\end{aligned} \tag{4-8d}$$

In addition, the solution can be smoothly extended to $[0, T] \times \mathbb{R} \times \mathbb{T}^2$ as a function of the geometric coordinates (t, u, x^2, x^3).

Finally, if $\inf_{t\in[0,T)} \upmu_\star(t) > 0$, then the solution can be smoothly extended to a Cartesian slab $[0, T+\epsilon] \times \Sigma$ for some $\epsilon > 0$ such that the map $(t, u, x^2, x^3) \to (t, x^1, x^2, x^3)$ is a diffeomorphism from $[0, T+\epsilon] \times \mathbb{R} \times \mathbb{T}^2$ onto $[0, T+\epsilon] \times \Sigma$. In particular, on the extended region, the solution is a smooth function of the geometric coordinates and the Cartesian coordinates.

Theorem 4.3 (complete description of the shock formation at the first singular time). *Under the assumptions of Theorem 4.2 — perhaps taking $\mathring{\alpha}$ and $\mathring{\epsilon}$ smaller in a manner that depends on the same quantities stated in the theorem — there exists $T_{(\text{sing})} \in [0, 2\mathring{\delta}_*^{-1}]$ satisfying the estimate*[43]

$$T_{(\text{sing})} = \{1 + \mathcal{O}_{\blacklozenge}(\mathring{\alpha}) + \mathcal{O}(\mathring{\epsilon})\}\mathring{\delta}_*^{-1} \tag{4-9}$$

such that the following hold:

(1) *The solution variables are smooth functions of the Cartesian coordinates (t, x^1, x^2, x^3) in $[0, T_{(\text{sing})}) \times \Sigma$.*

(2) *The solution variables extend as smooth functions of the geometric coordinates (t, u, x^2, x^3) to $[0, T_{(\text{sing})}] \times \mathbb{R} \times \mathbb{T}^2$.*

(3) *The inverse foliation density tends to zero at $T_{(\text{sing})}$, i.e., $\liminf_{t\uparrow T_{(\text{sing})}^-} \upmu_\star(t) = 0$.*

(4) *$\partial_1\mathcal{R}_{(+)}$ blows up as $t \uparrow T_{(\text{sing})}^-$, i.e., $\limsup_{t\uparrow T_{(\text{sing})}^-} \sup_{\Sigma_t} |\partial_1\mathcal{R}_{(+)}| = \infty$.*

[43] See Section 2A regarding our use of the notation $\mathcal{O}_{\blacklozenge}(\,\cdot\,)$, $\mathcal{O}(\,\cdot\,)$, etc.

(5) *Moreover, let*[44]

$$\mathscr{S}_{\text{blowup}} \doteq \Big\{(u, x^2, x^3) \in \mathbb{R} \times \mathbb{T}^2 : \limsup_{(\tilde{t},\tilde{u},\tilde{x}^2,\tilde{x}^3)\to(T^-_{(\text{sing})},u,x^2,x^3)} |\partial_1 \mathcal{R}_{(+)}|(\tilde{t}, \tilde{u}, \tilde{x}^2, \tilde{x}^3) = \infty\Big\},$$

$$\mathscr{S}_{\text{vanish}} \doteq \{(u, x^2, x^3) \in \mathbb{R} \times \mathbb{T}^2 : \mu(T_{(\text{sing})}, u, x^2, x^3) = 0\},$$

and

$$\mathscr{S}_{\text{regular}} \doteq \Big\{(u, x^2, x^3) \in \mathbb{R} \times \mathbb{T}^2 : \textit{all solution variables extend to be } C^1 \textit{ functions of the geometric and Cartesian coordinates in a neighborhood of the point with geometric coordinates } (T_{(\text{sing})}, u, x^2, x^3), \textit{ intersected with the half-space} \{t \le T_{(\text{sing})}\}\Big\}.$$

Then $\mathscr{S}_{\text{blowup}}$ *and* $\mathscr{S}_{\text{vanish}}$ *are nonempty, and*

$$\mathscr{S}_{\text{blowup}} = \mathscr{S}_{\text{vanish}} = \mathbb{R} \times \mathbb{T}^2 \setminus \mathscr{S}_{\text{regular}}.$$

The proofs of both Theorems 4.2 and 4.3 are located in Section 14B.

The next two corollaries concern some refined conclusions one can make with *additional* assumptions on the initial data.

Corollary 4.4 (nonvanishing of the vorticity and entropy at the blowup-points). *Assume the hypotheses and conclusions of Theorem 4.2, but perhaps taking* $\mathring{\alpha}$ *and* $\mathring{\epsilon}$ *smaller in a manner that depends on the same quantities stated in the theorem. Assume in addition that,*[45] *for all* $(x^2, x^3) \in \mathbb{T}^2$,

$$\tfrac{1}{2}[c^{-1}(c^{-1}c_{;\rho} + 1)(\breve{X}\mathcal{R}_{(+)})]_+(t = 0, u, x^2, x^3) \le \tfrac{1}{2}\mathring{\delta}_*^{-1} \quad \textit{when } |u - \mathring{\sigma} + \mathring{\delta}_*^{-1}| \ge 3\mathring{\alpha}\mathring{\delta}_*^{-1}, \tag{4-10}$$

and

$$\tfrac{1}{2}\mathring{\epsilon}^2 \le |\varOmega(t = 0, u, x^2, x^3)| \le \mathring{\epsilon}^2, \quad \tfrac{1}{2}\mathring{\epsilon}^3 \le |S(t = 0, u, x^2, x^3)| \le \mathring{\epsilon}^3 \quad \textit{when } |u - \mathring{\sigma}| \le \mathring{\alpha}^{1/2}. \tag{4-11}$$

Then $\varOmega$ *and* S *are nonvanishing near the singular set; i.e., for any* $(u, x^2, x^3) \in \mathscr{S}_{\text{blowup}}$ *(as in Theorem 4.3), we have* $\varOmega(T_{(\text{sing})}, u, x^2, x^3) \ne 0$ *and* $S(T_{(\text{sing})}, u, x^2, x^3) \ne 0$.

The proof of Corollary 4.4 is located in Section 14C.

Corollary 4.5 (the spatial Hölder regularity of the solution relative to the Cartesian coordinates). *Let* $\mathring{\beta} > 0$ *be a constant, and assume that the following hold:*

(1) *For all u such that* $|u - \mathring{\sigma}| \ge \mathring{\sigma}/4$ *and all* $(x^2, x^3) \in \mathbb{T}^2$, *we have*

$$\tfrac{1}{2}[c^{-1}(c^{-1}c_{;\rho} + 1)(\breve{X}\mathcal{R}_{(+)})]_+(t = 0, u, x^2, x^3) \le \tfrac{1}{4}\mathring{\delta}_*.$$

(2) *For all*[46] $u \in [\mathring{\sigma}/2, 3\mathring{\sigma}/2]$ *and all* $(x^2, x^3) \in \mathbb{T}^2$,

$$\tfrac{1}{2}\breve{X}\breve{X}\{(c^{-1}c_{;\rho} + 1)(\breve{X}\mathcal{R}_{(+)})\}(t = 0, u, x^2, x^3) \le -3\mathring{\delta}_*\mathring{\beta} < 0. \tag{4-12}$$

[44] For definiteness, in the definition of the subset $\mathscr{S}_{\text{regular}}$, we have made statements only about the boundedness of the solution's C^1 norm. However, our proof shows that on $\mathscr{S}_{\text{regular}}$, the solution inherits the full regularity enjoyed by the initial data.

[45] Recall the initial condition (2-13) for u, which shows that $u\restriction_{\Sigma_0} = \mathring{\sigma} - x^1$.

[46] This is a nondegeneracy condition in the sense that it guarantees that for every $(x^2, x^3) \in \mathbb{T}^2$, the quantity $(c^{-1}c_{;\rho} + 1)(\breve{X}\mathcal{R}_{(+)})\restriction_{\Sigma_0}$, when viewed as a one-variable function of u, has a nondegenerate maximum. (Note also that $(c^{-1}c_{;\rho} + 1)(\breve{X}\mathcal{R}_{(+)})\restriction_{\Sigma_0}$ is related to the quantity in (4-2), whose reciprocal controls the blowup-time.)

Also assume the hypotheses and conclusions of Theorems 4.2 and 4.3, but perhaps taking $\mathring{\alpha}$ *and* $\mathring{\varepsilon}$ *smaller in a manner that depends on* $\mathring{\beta}$ *and the same quantities stated in Theorem 4.2. Then the spatial* $C^{1/3}$ *norms* (*i.e., the standard* $C^{1/3}$ *Hölder norms with respect to the Cartesian spatial coordinates*) *of all of the fluid variables and higher-order variables* ρ, v^i, Ω^i, S^i, $\mathcal{C}^i$ *and* $\mathcal{D}$ *are uniformly bounded up to the first singular time.*

The proof of Corollary 4.5 is located in Section 14D.

5. Reformulation of the equations and the remarkable null structure

We recall in this section the main result in [50], which is of crucial importance for our analysis.

Theorem 5.1 (the geometric wave-transport-divergence-curl formulation of the compressible Euler equations). *Consider a smooth solution to the compressible Euler equations* (1-1)–(1-3) *under an equation of state* $p = p(\varrho, s)$ *and constant* $\bar{\varrho} > 0$ *such that the normalization condition* (2-1) *holds. Then the scalar-valued functions* v^i, $\mathcal{R}_{(\pm)}$, Ω^i, s, S^i, $\operatorname{div}\Omega$, $\mathcal{C}^i$, $\mathcal{D}$, *and* $(\operatorname{curl} S)^i$, $i = 1, 2, 3$, (*see Definitions 2.3 and 2.7*) *obey the following system of equations* (*where the Cartesian component functions* v^i *are treated as scalar-valued functions under covariant differentiation on the left-hand side of* (5-1a)):

<u>*Covariant wave equations*</u>:

$$\square_g v^i = -c^2 \exp(2\rho)\mathcal{C}^i + \mathfrak{Q}^i_{(v)} + \mathfrak{L}^i_{(v)}, \tag{5-1a}$$

$$\square_g \mathcal{R}_{(\pm)} = -c^2 \exp(2\rho)\mathcal{C}^1 \pm \left\{ F_{;s} c^2 \exp(2\rho) - c \exp(\rho) \frac{p_{;s}}{\bar{\varrho}} \right\} \mathcal{D} + \mathfrak{Q}_{(\pm)} + \mathfrak{L}_{(\pm)}, \tag{5-1b}$$

$$\square_g s = c^2 \exp(2\rho)\mathcal{D} + \mathfrak{L}_{(s)}. \tag{5-1c}$$

<u>*Transport equations*</u>:

$$B\Omega^i = \mathfrak{L}^i_{(\Omega)}, \tag{5-2a}$$

$$Bs = 0, \tag{5-2b}$$

$$BS^i = \mathfrak{L}^i_{(S)}. \tag{5-2c}$$

<u>*Transport-divergence-curl system for the specific vorticity*</u>:

$$\operatorname{div}\Omega = \mathfrak{L}_{(\operatorname{div}\Omega)}, \tag{5-3a}$$

$$B\mathcal{C}^i = \mathfrak{M}^i_{(\mathcal{C})} + \mathfrak{Q}^i_{(\mathcal{C})} + \mathfrak{L}^i_{(\mathcal{C})}. \tag{5-3b}$$

<u>*Transport-divergence-curl system for the entropy gradient*</u>:

$$B\mathcal{D} = \mathfrak{M}_{(\mathcal{D})} + \mathfrak{Q}_{(\mathcal{D})}, \tag{5-4a}$$

$$(\operatorname{curl} S)^i = 0. \tag{5-4b}$$

Above, the main terms in the transport equations for the modified fluid variables take the form

$$\begin{aligned}\mathfrak{M}^i_{(\mathcal{C})} \doteq{}& -2\delta_{jk}\epsilon_{iab}\exp(-\rho)(\partial_a v^j)\partial_b\Omega^k + \epsilon_{ajk}\exp(-\rho)(\partial_a v^i)\partial_j\Omega^k \\ &+ \exp(-3\rho)c^{-2}\frac{p_{;s}}{\bar{\varrho}}\{(BS^a)\partial_a v^i - (Bv^i)\partial_a S^a\} + \exp(-3\rho)c^{-2}\frac{p_{;s}}{\bar{\varrho}}\{(Bv^a)\partial_a S^i - (\partial_a v^a)BS^i\},\end{aligned} \tag{5-5a}$$

$$\mathfrak{M}_{(\mathcal{D})} \doteq 2\exp(-2\rho)\{(\partial_a v^a)\partial_b S^b - (\partial_a v^b)\partial_b S^a\} + \exp(-\rho)\delta_{ab}(\operatorname{curl}\Omega)^a S^b. \tag{5-5b}$$

The terms $\mathfrak{Q}^i_{(v)}$, $\mathfrak{Q}_{(\pm)}$, $\mathfrak{Q}^i_{(\mathcal{C})}$, *and* $\mathfrak{Q}_{(\mathcal{D})}$ *are the null forms relative to* g *defined by*

$$\mathfrak{Q}^i_{(v)} \doteq -\{1 + c^{-1}c_{;\rho}\}(g^{-1})^{\alpha\beta}(\partial_\alpha\rho)\partial_\beta v^i, \tag{5-6a}$$

$$\mathfrak{Q}_{(\pm)} \doteq \mathfrak{Q}^i_{(v)} \mp 2c_{;\rho}(g^{-1})^{\alpha\beta}\partial_\alpha\rho\partial_\beta\rho \pm c\{(\partial_a v^a)(\partial_b v^b) - (\partial_a v^b)\partial_b v^a\}, \tag{5-6b}$$

$$\begin{aligned}\mathfrak{Q}^i_{(\mathcal{C})} \doteq{}& \exp(-3\rho)c^{-2}\frac{p_{;s}}{\bar{\varrho}}S^i\{(\partial_a v^b)\partial_b v^a - (\partial_a v^a)\partial_b v^b\} \\ &+ \exp(-3\rho)c^{-2}\frac{p_{;s}}{\bar{\varrho}}\{(\partial_a v^a)S^b\partial_b v^i - (S^a\partial_a v^b)\partial_b v^i\} \\ &+ 2\exp(-3\rho)c^{-2}\frac{p_{;s}}{\bar{\varrho}}\{(S^a\partial_a\rho)Bv^i - (B\rho)S^a\partial_a v^i\} \\ &+ 2\exp(-3\rho)c^{-3}c_{;\rho}\frac{p_{;s}}{\bar{\varrho}}\{(S^a\partial_a\rho)Bv^i - (B\rho)S^a\partial_a v^i\} \\ &+ \exp(-3\rho)c^{-2}\frac{p_{;s;\rho}}{\bar{\varrho}}\{(B\rho)S^a\partial_a v^i - (S^a\partial_a\rho)Bv^i\} \\ &+ \exp(-3\rho)c^{-2}\frac{p_{;s;\rho}}{\bar{\varrho}}S^i\{(Bv^a)\partial_a\rho - (\partial_a v^a)B\rho\} \\ &+ 2\exp(-3\rho)c^{-2}\frac{p_{;s}}{\bar{\varrho}}S^i\{(\partial_a v^a)B\rho - (Bv^a)\partial_a\rho\} \\ &+ 2\exp(-3\rho)c^{-3}c_{;\rho}\frac{p_{;s}}{\bar{\varrho}}S^i\{(\partial_a v^a)B\rho - (Bv^a)\partial_a\rho\},\end{aligned} \tag{5-6c}$$

$$\mathfrak{Q}_{(\mathcal{D})} \doteq 2\exp(-2\rho)\{(S^a\partial_a v^b)\partial_b\rho - (\partial_a v^a)S^b\partial_b\rho\}. \tag{5-6d}$$

In addition, the terms $\mathfrak{L}^i_{(v)}$, $\mathfrak{L}_{(\pm)}$, $\mathfrak{L}_{(s)}$, $\mathfrak{L}^i_{(\Omega)}$, $\mathfrak{L}^i_{(S)}$, $\mathfrak{L}_{(\operatorname{div}\Omega)}$, *and* $\mathfrak{L}^i_{(\mathcal{C})}$, *which are at most linear in the derivatives of the unknowns, are defined as*

$$\begin{aligned}\mathfrak{L}^i_{(v)} \doteq{}& 2\exp(\rho)\epsilon_{iab}(Bv^a)\Omega^b - \frac{p_{;s}}{\bar{\varrho}}\epsilon_{iab}\Omega^a S^b \tfrac{1}{2}\exp(-\rho)\frac{p_{;\rho;s}}{\bar{\varrho}}S^a\partial_a v^i \\ &- 2\exp(-\rho)c^{-1}c_{;\rho}\frac{p_{;s}}{\bar{\varrho}}(B\rho)S^i + \exp(-\rho)\frac{p_{;s;\rho}}{\bar{\varrho}}(B\rho)S^i,\end{aligned} \tag{5-7a}$$

$$\begin{aligned}\mathfrak{L}_{(\pm)} \doteq{}& \mathfrak{L}^i_{(v)} \pm F_{;s}\mathfrak{L}_{(s)} \mp \tfrac{5}{2}c\exp(-\rho)\frac{p_{;s;\rho}}{\bar{\varrho}}S^a\partial_a\rho \pm 2c^2c_{;s}S^a\partial_a\rho \\ &\mp c\exp(-\rho)\frac{p_{;s;s}}{\bar{\varrho}}\delta_{ab}S^aS^b \pm F_{;s;s}c^2\delta_{ab}S^aS^b,\end{aligned} \tag{5-7b}$$

$$\mathfrak{L}_{(s)} \doteq c^2S^a\partial_a\rho - cc_{;\rho}S^a\partial_a\rho - cc_{;s}\delta_{ab}S^aS^b, \tag{5-7c}$$

$$\mathfrak{L}^i_{(\Omega)} \doteq \Omega^a\partial_a v^i - \exp(-2\rho)c^{-2}\frac{p_{;s}}{\bar{\varrho}}\epsilon_{iab}(Bv^a)S^b, \tag{5-7d}$$

$$\mathfrak{L}^i_{(S)} \doteq -S^a\partial_a v^i + \epsilon_{iab}\exp(\rho)\Omega^aS^b, \tag{5-7e}$$

$$\mathfrak{L}_{(\operatorname{div}\Omega)} \doteq -\Omega^a\partial_a\rho, \tag{5-7f}$$

$$\begin{aligned}\mathfrak{L}^i_{(\mathcal{C})} \doteq{}& 2\exp(-3\rho)c^{-3}c_{;s}\frac{p_{;s}}{\bar{\varrho}}(Bv^i)\delta_{ab}S^aS^b - 2\exp(-3\rho)c^{-3}c_{;s}\frac{p_{;s}}{\bar{\varrho}}\delta_{ab}S^a(Bv^b)S^i \\ &+ \exp(-3\rho)c^{-2}\frac{p_{;s;s}}{\bar{\varrho}}(Bv^i)\delta_{ab}S^aS^b - \exp(-3\rho)c^{-2}\frac{p_{;s;s}}{\bar{\varrho}}\delta_{ab}(Bv^a)S^bS^i.\end{aligned} \tag{5-7g}$$

Proof. The equations are copied from [50, Theorem 1], except we have replaced the wave equations for ρ, v^1 from [50, Theorem 1] with equivalent wave equations for $\mathcal{R}_{(\pm)}$ with the help of the identity

$$\square_g\mathcal{R}_{(\pm)} = \square_g v^1 \pm \{c\,\square_g\,\rho + c_{;\rho}(g^{-1})^{\alpha\beta}\partial_\alpha\rho\partial_\beta\rho + 2c^2c_{;s}S^a\partial_a\rho + F_{;s;s}c^2\delta_{ab}S^aS^b + F_{;s}\,\square_g\,s\},$$

which follows from (2-2), the chain rule, the expression (2-8) for g^{-1}, and the transport equation $Bs = 0$, i.e., (1-3). □

6. The bootstrap assumptions and statement of the main a priori estimates

We prove our theorem with a bootstrap argument. In this section, we state the precise bootstrap assumptions, as well as a theorem that features our main a priori estimates. The proof of the theorem occupies Sections 7–14A.

6A. *Bootstrap assumptions.* We now introduce our bootstrap assumptions. In the context of Theorem 6.3 below, we assume that the bootstrap assumptions in the next two subsubsections hold for $t \in [0, T_{(\mathrm{Boot})})$, where $T_{(\mathrm{Boot})} \in [0, 2\mathring{\delta}_*^{-1}]$ is a "bootstrap time."

6A1. *Soft bootstrap assumptions.*

(1) We assume that the change-of-variables map $(t, u, x^2, x^3) \to (t, x^1, x^2, x^3)$ from geometric to Cartesian coordinates is a C^1 diffeomorphism from $[0, T_{(\mathrm{Boot})}) \times \mathbb{R} \times \mathbb{T}^2$ onto $[0, T_{(\mathrm{Boot})}) \times \Sigma$.

(2) We assume that $\mu > 0$ on $[0, T_{(\mathrm{Boot})}) \times \mathbb{R} \times \mathbb{T}^2$.

The first of these "soft bootstrap assumptions" allows us, in particular, to switch back and forth between viewing tensorfields as a function of the geometric coordinates (which is the dominant view we take throughout the analysis) and the Cartesian coordinates. The second soft bootstrap assumption guarantees that there are no shocks present in the bootstrap region (though it allows for the possibility that a shock will form precisely at time $T_{(\mathrm{Boot})}$).

6A2. *Quantitative bootstrap assumptions.* Let $M_* \in \mathbb{N}$ be the absolute constant appearing in the statements of Theorem 4.2 above and Proposition 12.1 below. Moreover, as we stated already in Section 4A, N_{top} denotes any fixed positive integer satisfying $N_{\mathrm{top}} \geq 2M_* + 10$.

Remark 6.1 (rationale behind our choice $N_{\mathrm{top}} \geq 2M_* + 10$). Later on, our assumption $N_{\mathrm{top}} \geq 2M_* + 10$ and the bootstrap assumptions will allow us to control $\leq N_{\mathrm{top}}$ derivatives of nonlinear products by bounding all terms in L^∞ except perhaps the one factor hit by the most derivatives. Roughly,[47] the reason is that our derivative count will be such that any factor that is hit by $\leq N_{\mathrm{top}} - M_* - 4$ or fewer derivatives is bounded in L^∞. We will often avoid explicitly pointing out this aspect of our derivative count.

<u>L^2 bootstrap assumptions for the wave variables</u>: For $N_{\mathrm{top}} - M_* + 1 \leq N \leq N_{\mathrm{top}}$,[48] we assume the following bounds, where the energies $\mathbb{W}_N$ are defined in Section 3B2 and $\mu_\star(t)$ is defined in Definition 2.16:

$$\mathbb{W}_N(t) \leq \mathring{\epsilon} \mu_\star^{-2M_* + 2N_{\mathrm{top}} - 2N + 1.8}(t). \tag{6-1}$$

For $1 \leq N \leq N_{\mathrm{top}} - M_*$,

$$\mathbb{W}_N(t) \leq \mathring{\epsilon}. \tag{6-2}$$

[47] In reality, the different solution variables that we have to track, such as Ψ, Ω^i, L^i, μ, etc., exhibit slightly different amounts of L^∞ regularity.

[48] Equivalently, for $0 \leq K \leq M_* - 1$, we have $\mathbb{W}_{N_{\mathrm{top}} - K}(t) \leq \mathring{\epsilon} \mu_\star^{-2M_* + 2K + 1.8}(t)$.

L^∞ bootstrap assumptions for the wave variables:

$$\|\mathcal{R}_{(+)}\|_{L^\infty(\Sigma_t)} \leq \mathring{\alpha}^{1/2}, \quad \|\breve{X}\mathcal{R}_{(+)}\|_{L^\infty(\Sigma_t)} \leq 3\mathring{\delta}, \tag{6-3}$$

$$\|(\mathcal{R}_{(-)}, v^2, v^3, s)\|_{L^\infty(\Sigma_t)}, \quad \|\breve{X}(\mathcal{R}_{(-)}, v^2, v^3, s)\|_{L^\infty(\Sigma_t)} \leq \mathring{\epsilon}^{1/2}, \tag{6-4}$$

$$\|\mathcal{P}^{[1,N_{\text{top}}-M_*-2]}\Psi\|_{L^\infty(\Sigma_t)} \leq \mathring{\epsilon}^{1/2}, \quad \|\mathcal{P}^{[1,N_{\text{top}}-M_*-4]}\breve{X}\Psi\|_{L^\infty(\Sigma_t)} \leq \mathring{\epsilon}^{1/2}. \tag{6-5}$$

L^∞ bootstrap assumptions for the specific vorticity:

$$\|\mathcal{P}^{\leq N_{\text{top}}-M_*-2}\Omega\|_{L^\infty(\Sigma_t)} + \|\mathcal{P}^{\leq N_{\text{top}}-M_*-4}\breve{X}\Omega\|_{L^\infty(\Sigma_t)} \leq \mathring{\epsilon}. \tag{6-6}$$

L^∞ bootstrap assumptions for the entropy gradient:

$$\|\mathcal{P}^{\leq N_{\text{top}}-M_*-2}S\|_{L^\infty(\Sigma_t)} + \|\mathcal{P}^{\leq N_{\text{top}}-M_*-4}\breve{X}S\|_{L^\infty(\Sigma_t)} \leq \mathring{\epsilon}. \tag{6-7}$$

L^∞ bootstrap assumptions for the modified fluid variables:

$$\|\mathcal{P}^{\leq N_{\text{top}}-M_*-3}(\mathcal{C}, \mathcal{D})\|_{L^\infty(\Sigma_t)} \leq \mathring{\epsilon}. \tag{6-8}$$

Remark 6.2 (the main large quantity in the problem). From the discussion of the parameters at the beginning of Section 4A and (6-3)–(6-8) we see that the main large quantity in the problem is $\breve{X}\mathcal{R}_{(+)}$; all other terms exhibit smallness that is controlled by $\mathring{\alpha}$ and $\mathring{\epsilon}$. This, of course, is tied to the kind of initial data we treat here.

6B. ***Statement of the main a priori estimates.*** We now state the theorem that yields our main a priori estimates. Its proof will be the content of Sections 7–14A.

Theorem 6.3 (the main a priori estimates). *Let $T_{(\text{Boot})} \in [0, 2\mathring{\delta}_*^{-1}]$. Suppose that*:

(1) *The assumptions on the initial data stated in Section 4A hold.* (*Note that these assumptions involve N_{top}, M_*, $\mathring{\sigma}$, $\mathring{\delta}_*$, $\mathring{\delta}$, $\mathring{\alpha}$, and $\mathring{\epsilon}$.*)

(2) *The bootstrap assumptions* (6-1)–(6-8) *all hold for all $t \in [0, T_{(\text{Boot})})$* (*where we recall that in the bootstrap assumptions, N_{top} is any integer satisfying $N_{\text{top}} \geq 2M_* + 10$, where $M_* \in \mathbb{N}$ is the absolute constant appearing in the statements of Theorem 4.2 and Proposition 12.1*).

(3) *In* (6-3), *the parameter $\mathring{\alpha}$ is sufficiently small in a manner that depends only on the equation of state and $\bar{\varrho}$.*

(4) *The parameter $\mathring{\epsilon} > 0$ in* (6-1)–(6-8) *satisfies $\mathring{\epsilon}^{1/2} \leq \mathring{\alpha}$ and is sufficiently small in a manner that depends only on the equation of state, N_{top}, $\bar{\varrho}$, $\mathring{\sigma}$, $\mathring{\delta}$, and $\mathring{\delta}_*^{-1}$.*

(5) *The soft bootstrap assumptions stated in Section 6A1 hold* (*including $\mu > 0$ in $[0, T_{(\text{Boot})}) \times \mathbb{R} \times \mathbb{T}^2$*).

Then there exists a constant $C_\blacklozenge > 0$ depending only on the equation of state and $\bar{\varrho}$, and a constant $C > 0$ depending on the equation of state, N_{top}, $\bar{\varrho}$, $\mathring{\sigma}$, $\mathring{\delta}$, and $\mathring{\delta}_*^{-1}$ such that the following holds for all $t \in [0, T_{(\text{Boot})})$:

(1) (6-1) *and* (6-2) *hold with $\mathring{\epsilon}$ replaced by $C\mathring{\epsilon}^2$.*

(2) *The two inequalities in* (6-3) *hold with $\mathring{\alpha}^{1/2}$ replaced by $C_\blacklozenge\mathring{\alpha}$ and $3\mathring{\delta}$ replaced by $2\mathring{\delta}$ respectively.*

(3) *The inequalities in* (6-4) *and* (6-5) *hold with* $\mathring{\epsilon}^{1/2}$ *replaced by* $C\mathring{\epsilon}$.

(4) *The inequalities* (6-6)–(6-8) *all hold with* $\mathring{\epsilon}$ *replaced by* $C\mathring{\epsilon}^{3/2}$.

Sections 7–13 will be devoted to the proof of Theorem 6.3. See Section 14A for the conclusion of the proof.

From now on, we will use the conventions for constants stated in Section 2A and Theorem 6.3.

7. A localization lemma via finite speed of propagation

We work under the assumptions of Theorem 6.3.

Lemma 7.1 (a localization lemma). *Let* $U_0 \doteq 2\mathring{\sigma} + 4\mathring{\delta}_*^{-1}$. *Then, for all* $t \in [0, T_{(\text{Boot})})$,

$$(\rho, v, s) = (0, 0, 0), \quad \textit{whenever } u \notin (0, U_0).$$

Proof. Recall that we have normalized (see (2-1)) $c(0, 0) = 1$, and (by (4-1)) the data are compactly supported in the region where $|x^1| \leq \mathring{\sigma}$. Hence, by a standard finite speed of propagation argument, we see that $(\rho, v, s) = (0, 0, 0)$ whenever $|x^1| \geq \mathring{\sigma} + t$. More precisely, this can be proved by applying standard energy methods to the first-order formulation of the compressible Euler equations provided by [16, equation (1.201)], where the relevant energy identities can be obtained with the help of the "energy current" vectorfields defined by [16, equations (1.204), (1.205)]. Since $t < T_{(\text{Boot})} \leq 2\mathring{\delta}_*^{-1}$,

$$\{(t, x) \in [0, T_{(\text{Boot})}) \times \Sigma : t - x^1 \geq \mathring{\sigma} + 4\mathring{\delta}_*^{-1}\} \subseteq \overbrace{\{(t, x) \in [0, T_{(\text{Boot})}) \times \Sigma : x^1 \leq -\mathring{\sigma} - t\}}^{\text{solution is trivial here}}.$$

In particular, this implies

$$(\rho, v, s) = (0, 0, 0) \quad \text{unless } -\mathring{\sigma} < t - x^1 < \mathring{\sigma} + 4\mathring{\delta}_*^{-1}. \tag{7-1}$$

Observe now that since $u \restriction_{\{t=0\}} = \mathring{\sigma} - x^1$, in the set $\{(t, x) \in [0, T_{(\text{Boot})}) \times \Sigma : |x^1| \geq \mathring{\sigma} + t\}$ (where the solution is trivial), we have $u = t + \mathring{\sigma} - x^1$. In particular, $\{u = 0\} = \{t - x^1 = -\mathring{\sigma}\}$ and $\{u = U_0\} = \{t - x^1 = \mathring{\sigma} + 4\mathring{\delta}_*^{-1}\}$. The conclusion thus follows from (7-1). □

For the rest of the paper, $U_0 > 0$ *denotes the constant appearing in the statement of Lemma 7.1.*

8. Estimates for the geometric quantities associated to the acoustical metric

We continue to work under the assumptions of Theorem 6.3.

In this section, we collect some estimates of the geometric quantities μ, $L^i_{(\text{small})}$ (see Definition 2.21), under the bootstrap assumptions on the fluid variables. These estimates are the same as those appearing in [36; 52]. Our analysis will therefore be somewhat brief in some spots, and we will refer the reader to [36; 52] for details.

We highlight the following point, which is crucial for the subsequent analysis: the bounds for μ, $L^i_{(\text{small})}$ and the wave variables Ψ control all the other geometric quantities, including the transformation coefficients between different sets of vectorfields, as well as the commutators of vectorfields.

8A. ***Some preliminary geoanalytic identities.*** In this section, we provide some geoanalytic identities that we will use throughout our analysis.

We start by recalling the definition of a null form with respect to the acoustical metric ("g-null form" for short).

Definition 8.1 (g-null forms). Let $\phi^{(1)}$ and $\phi^{(2)}$ be scalar functions. We use the notation $\mathcal{Q}^{(g)}(\partial\phi^{(1)}, \partial\phi^{(2)})$ to denote any derivative-quadratic term of the form

$$\mathcal{Q}^{(g)}(\partial\phi^{(1)}, \partial\phi^{(2)}) = \mathrm{f}(L^i, \Psi)(g^{-1})^{\alpha\beta}\partial_\alpha\phi^{(1)}\partial_\beta\phi^{(2)}, \tag{8-1}$$

where $\mathrm{f}(\cdot)$ is a smooth function.

We use the notation $\mathcal{Q}_{\alpha\beta}(\partial\phi^{(1)}, \partial\phi^{(2)})$ to denote any derivative-quadratic term of the form

$$\mathcal{Q}_{\alpha\beta}(\partial\phi^{(1)}, \partial\phi^{(2)}) = \mathrm{f}(L^i, \Psi)\{\partial_\alpha\phi^{(1)}\partial_\beta\phi^{(2)} - \partial_\beta\phi^{(1)}\partial_\alpha\phi^{(2)}\}, \tag{8-2}$$

where $\mathrm{f}(\cdot)$ is a smooth function.

Lemma 8.2 (crucial structural properties of null forms). *Let $\mathcal{Q}(\partial\phi^{(1)}, \partial\phi^{(2)})$ be a g-null form of type* (8-1) *or* (8-2). *Then there exist smooth functions, all schematically denoted by* f (*and which are different from the* f *in Definition 8.1*), *such that the following identity holds*:

$$\upmu\mathcal{Q}(\partial\phi^{(1)}, \partial\phi^{(2)}) = \mathrm{f}(L^i, \Psi)\breve{X}\phi^{(1)}\cdot\mathcal{P}\phi^{(2)} + \mathrm{f}(L^i, \Psi)\breve{X}\phi^{(2)}\cdot\mathcal{P}\phi^{(1)} + \upmu\mathrm{f}(L^i, \Psi)\mathcal{P}\phi^{(1)}\cdot\mathcal{P}\phi^{(2)}. \tag{8-3}$$

In particular, decomposing all differentiations in the null form with respect to the $\{L, X, Y, Z\}$ frame leads to the absence of all $X\phi^{(1)}\cdot X\phi^{(2)}$ terms on the right-hand side of (8-3).

Proof. For null forms of type (8-2), (8-3) follows from Lemma 2.22 and the fact that the Cartesian component functions X^1, X^2, X^3 are smooth functions of the L^i and Ψ (see (2-23)). For null forms of type (8-1), (8-3) follows from the basic identity $g^{-1} = -L\otimes L - (L\otimes X + X\otimes L) + \not{g}^{-1}$ (see, e.g., [52, (2.40b)]) and Lemma 2.32. □

Lemma 8.3 (expressions for the transversal derivatives of the transport variables in terms of tangential derivatives). *There exist smooth functions, all schematically denoted by* "f", *such that the following identities hold*:

$$\breve{X}\Omega^i = -\upmu L\Omega^i + (\Omega, S)\cdot\mathrm{f}(\Psi, L^i, \upmu, \breve{X}\Psi, \mathcal{P}\Psi), \tag{8-4}$$

$$\breve{X}S^i = -\upmu LS^i + (\Omega, S)\cdot\mathrm{f}(\Psi, L^i, \upmu, \breve{X}\Psi, \mathcal{P}\Psi), \tag{8-5}$$

$$\breve{X}\mathcal{C}^i = -\upmu L\mathcal{C}^i + (\Omega, S, \mathcal{P}\Omega, \mathcal{P}S)\cdot\mathrm{f}(\Psi, L^i, \upmu, \breve{X}\Psi, \mathcal{P}\Psi), \tag{8-6}$$

$$\breve{X}\mathcal{D}^i = -\upmu L\mathcal{D}^i + (\Omega, S, \mathcal{P}\Omega, \mathcal{P}S)\cdot\mathrm{f}(\Psi, L^i, \upmu, \breve{X}\Psi, \mathcal{P}\Psi). \tag{8-7}$$

Proof. Equations (8-4) and (8-5) follow from the transport equations (5-2a) and (5-2c), (2-23) (which implies that $\upmu B = \breve{X} + \upmu L$), and Lemma 2.22.

Equations (8-6) and (8-7) follow from a similar argument based the transport equations (5-3b) and (5-4a), where we use Lemma 8.2 to decompose the null form source terms and (8-4)–(8-5) to re-express all $\breve{X}$ derivatives of (Ω, S). □

Lemma 8.4 (identity for $\breve{X}L^i$). *There exist smooth functions, all schematically denoted by* f, *such that*

$$\breve{X}L^i = \mathrm{f}(\Psi, L^i)\breve{X}\Psi + \upmu\mathrm{f}(\Psi, L^i)\mathcal{P}\Psi + \mathrm{f}(\Psi, L^i)\mathcal{P}\upmu. \tag{8-8}$$

Proof. This was proved as [52, (2.71)] (which holds in the present context with obvious modifications such as replacing $G_{LL}\breve{X}\Psi$ with $\vec{G}_{LL} \diamond \breve{X}\vec{\Psi}$, etc.), where we have used that the Cartesian component functions X^1, X^2, X^3 are smooth functions of the L^i and Ψ (see (2-23)). □

Lemma 8.5 (simple commutator identities). *For each pair $\mathcal{P}_1, \mathcal{P}_2 \in \{L, Y, Z\}$, there exist smooth functions, all schematically denoted by "f", such that the following identity holds*:

$$[\mathcal{P}_1, \mathcal{P}_2] = \mathrm{f}(L^i, \Psi, \mathcal{P}L^i, \mathcal{P}\Psi)Y + \mathrm{f}(L^i, \Psi, \mathcal{P}L^i, \mathcal{P}\Psi)Z. \tag{8-9}$$

Moreover, for each $\mathcal{P} \in \{L, Y, Z\}$, there exist smooth functions, all schematically denoted by "f", such that the following identity holds:

$$[\mathcal{P}, \breve{X}] = \mathrm{f}(\mu, L^i, \Psi, \mathcal{P}\mu, \breve{X}\Psi, \mathcal{P}\Psi)Y + \mathrm{f}(\mu L^i, \Psi, \mathcal{P}\mu, \breve{X}\Psi, \mathcal{P}\Psi)Z. \tag{8-10}$$

Proof. We first prove (8-10). Lemma 2.23 implies that $[\mathcal{P}, \breve{X}]$ is $\ell_{t,u}$-tangent, i.e., that $[\mathcal{P}, \breve{X}]t = [\mathcal{P}, \breve{X}]u = 0$. Hence, (2-28b)–(2-28c) imply that this commutator can be written as a linear combination of Y, Z. Since the Cartesian component functions X^1, X^2, X^3 are smooth functions of the L^i and Ψ (see (2-23)), the same holds for the component functions $\mathcal{P}^0$, $\mathcal{P}^1$, $\mathcal{P}^2$, $\mathcal{P}^3$ (this is obvious for $\mathcal{P} = L$, while see Lemmas 2.23–2.24 for $\mathcal{P} = Y, Z$). Also using that $\breve{X}^i = \mu\breve{X}^i$, we conclude (8-10) by computing relative to the Cartesian coordinates, using Lemma 2.22 to express Cartesian coordinate partial derivatives in terms of derivatives with respect to Y, Z (the X- and L-derivative components of the commutator must vanish since $[\mathcal{P}, \breve{X}]$ is $\ell_{t,u}$-tangent), and using (8-8) to substitute for $\breve{X}L^i$ factors.

The identity (8-9) can be proved through similar but simpler arguments that do not involve factors of μ or $\breve{X}$ differentiations. □

8B. *The easy L^∞ estimates.*

Proposition 8.6 (L^∞ estimates for the acoustical geometry). *The following estimates hold for all $t \in [0, T_{(\mathrm{Boot})})$*:

$$\|\mu\|_{L^\infty(\Sigma_t)} + \|L\mu\|_{L^\infty(\Sigma_t)} \lesssim 1, \quad \|L^i_{(\mathrm{small})}\|_{L^\infty(\Sigma_t)} \lesssim_\blacklozenge \mathring{\alpha}, \quad \|Y\mu\|_{L^\infty(\Sigma_t)} + \|Z\mu\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}^{1/2},$$
$$\|\mathcal{P}^{[2,N_{\mathrm{top}}-M_*-4]}\mu\|_{L^\infty(\Sigma_t)} + \|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-3]}L^i\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}^{1/2}.$$

Proof. These can be proved using the transport equations (2-40) and (2-41) (commuted with $\mathcal{P}^N$), the initial data size-assumptions (4-3a)–(4-4), and the bootstrap assumptions (6-3)–(6-5). See [52, Proposition 8.10] for details of this argument. We note these estimates lose a slight amount of regularity compared to Ψ because the transport equations (2-40) and (2-41) depend on the derivatives of Ψ. □

Our analysis also relies on the following L^∞ estimates.

Proposition 8.7 (L^∞ estimates for other geometric quantities). *The following estimates hold for all $t \in [0, T_{(\mathrm{Boot})})$, where c denotes the speed of sound*:

$$\|X^i_{(\mathrm{small})}\|_{L^\infty(\Sigma_t)} \lesssim_\blacklozenge \mathring{\alpha}^{1/2}, \qquad \|c-1\|_{L^\infty(\Sigma_t)} \lesssim_\blacklozenge \mathring{\alpha}^{1/2},$$
$$\|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-3]}X^i\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}^{1/2}, \qquad \|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-2]}c\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}^{1/2}.$$

Proof. The estimates for $X_{(\mathrm{small})}$ follow from (2-25a)–(2-25b), (2-26a), the bootstrap assumptions (6-3)–(6-5), and Proposition 8.6.

The estimates for c follow from the bootstrap assumptions (6-3)–(6-5) and the fact that c is a smooth function of ρ and s with $c(0,0)=1$ (see (2-1)). □

The estimates in Propositions 8.6 and 8.7 also imply the following bounds for the commutators.

Proposition 8.8 (pointwise bounds for vectorfield commutators). *All the commutators* $[L,\breve{X}]$, $[L,Y]$, $[L,Z]$, $[\breve{X},Y]$, $[\breve{X},Z]$ *and* $[Y,Z]$ *are* $\ell_{t,u}$*-tangent.*

Moreover, if ϕ *is a scalar function, then for* $0\le N\le N_{\mathrm{top}}$ *iterated commutators can be bounded pointwise as follows*:

$$\begin{aligned}
|[L,\mathcal{P}^N]\phi| &\lesssim \mathring{\epsilon}^{1/2}|\mathcal{P}^{[1,N]}\phi| + \sum_{\substack{N_1+N_2\le N+1\\ N_1,N_2\le N}} |\mathcal{P}^{[2,N_1]}(L^i,\Psi)|\,|\mathcal{P}^{[1,N_2]}\phi|,\\
|[\breve{X},\mathcal{P}^N]\phi| &\lesssim |\mathcal{P}^{[1,N]}\phi| + \sum_{\substack{N_1+N_2\le N+1\\ N_1,N_2\le N}} |\mathcal{P}^{[2,N_1]}(\mu,L^i,\Psi)|\,|\mathcal{P}^{[1,N_2]}\phi| + \sum_{\substack{N_1+N_2\le N\\ N_1\le N-1}} |\mathcal{P}^{[2,N_1]}\breve{X}\Psi|\,|\mathcal{P}^{[1,N_2]}\phi|.
\end{aligned}\tag{8-11}$$

In particular,

$$\begin{aligned}
|[L,\mathcal{P}^N]\phi| &\lesssim \mathring{\epsilon}^{1/2}|\mathcal{P}^{[1,N]}\phi| \quad \textit{if } 0\le N\le N_{\mathrm{top}}-M_*-3,\\
|[\breve{X},\mathcal{P}^N]\phi| &\lesssim |\mathcal{P}^{[1,N]}\phi| \qquad\quad\; \textit{if } 0\le N\le N_{\mathrm{top}}-M_*-4.
\end{aligned}\tag{8-12}$$

Proof. All the commutators can be read off from Lemma 2.23 (and using that coordinate vectorfields commute). In particular, since the coefficient of $\not\partial_t$ in L and the coefficient of $\not\partial_u$ in $\breve{X}$ both are equal to 1, all the stated commutators are $\ell_{t,u}$-tangent.

We first prove (8-11) for $|[L,\mathcal{P}^N]\phi|$. By Lemma 2.23 and the fact $L^i+X^i-v^i=0$ (by (2-26a)),

$$|[L,\mathcal{P}^N]\phi| \lesssim \sum_{k=2}^{N}\sum_{\substack{N_1+\cdots+N_k=N+1\\ 1\le N_k\le N}} \underbrace{|\mathcal{P}^{N_1}(L^i,\Psi)|\cdots|\mathcal{P}^{N_{k-1}}(L^i,\Psi)|\,|\mathcal{P}^{[1,N_k]}\phi|}_{\doteq(*)}.\tag{8-13}$$

By (6-3)–(6-5), Propositions 8.6, 8.7 (and $N\le N_{\mathrm{top}}$), either $|\mathcal{P}^{N_j}(L^i,\Psi)|\lesssim\mathring{\epsilon}^{1/2}$ for $1\le j\le k-1$ (in which case $(*)\lesssim\mathring{\epsilon}^{1/2}|\mathcal{P}^{[1,N]}\phi|$), or else there is exactly one factor $|\mathcal{P}^{N_j}(L^i,\Psi)|$ with $N_j>N_{\mathrm{top}}-M_*-3$ not bounded by $\lesssim\mathring{\epsilon}^{1/2}$, in which case

$$(*)\lesssim \sum_{\substack{N_1+N_2\le N+1\\ N_1,N_2\le N}} |\mathcal{P}^{[2,N_1]}(L^i,\Psi)|\,|\mathcal{P}^{[1,N_2]}\phi|.$$

Hence, (8-13) is bounded above by the right-hand side of the first inequality in (8-11).

To bound $[\breve{X},\mathcal{P}^N]\phi$, we note that according to Lemma 2.23, there is, in addition to (8-13), the terms[49]

$$\sum_{k=2}^{N}\sum_{\substack{N_1+\cdots+N_k=N\\ N_{k-1}\le N-1\\ 1\le N_k\le N}} |\mathcal{P}^{N_1}(L^i,\Psi)|\cdots|\mathcal{P}^{N_{k-2}}(L^i,\Psi)|\,|\mathcal{P}^{N_{k-1}}\breve{X}(L^i,\Psi)|\,|\mathcal{P}^{N_k}\phi|,\tag{8-14}$$

[49]Importantly, one checks from Lemma 2.23 that there are no terms of the form $|\mathcal{P}^{N_{k-1}}\breve{X}\mu|$!

$$\sum_{k=2}^{N} \sum_{\substack{N_1+\cdots+N_k=N+1 \\ 1\le N_k\le N}} |\mathcal{P}^{N_1}(L^i,\Psi)|\cdots|\mathcal{P}^{N_{k-2}}(L^i,\Psi)||\mathcal{P}^{N_{k-1}}\upmu||\mathcal{P}^{N_k}\phi|. \tag{8-15}$$

Hence, with the help of (8-8), we can substitute for the terms $\breve{X}L^i$ on the right-hand side of (8-14), and thus the right-hand side of (8-14) can be bounded above by the right-hand side of (8-13) plus (8-15) and

$$\sum_{k=2}^{N} \sum_{\substack{N_1+\cdots+N_k=N \\ N_{k-1}\le N-1 \\ 1\le N_k\le N}} |\mathcal{P}^{N_1}(L^i,\Psi)|\cdots|\mathcal{P}^{N_{k-2}}(L^i,\Psi)||\mathcal{P}^{N_{k-1}}\breve{X}\Psi||\mathcal{P}^{N_k}\phi|, \tag{8-16}$$

both of which, by arguments similar to the ones we used to prove (8-13), can be bounded above by the right-hand side of the second inequality in (8-11).

To get from (8-11) to (8-12), we use the L^∞ bounds in (6-3)–(6-5) and Propositions 8.6 and 8.7, which are applicable in the sense that they control a sufficient number of derivatives of all relevant quantities in L^∞. □

In the rest of the paper, we will often silently use the following simple lemma.

Lemma 8.9 (the norm of the $\ell_{t,u}$-tangent commutator vectorfields and simple comparison estimates). *The $\ell_{t,u}$-tangent commutator vectorfields $\{Y, Z\}$ satisfy the following pointwise bounds on $\mathcal{M}_{T_{(\mathrm{Boot})},U_0}$:*

$$|Y| \lesssim 1, \quad |Z| \lesssim 1. \tag{8-17}$$

Moreover, for any $\ell_{t,u}$-tangent tensorfield ξ the following pointwise bounds hold on $\mathcal{M}_{T_{(\mathrm{Boot})},U_0}$:

$$|\nabla\!\!\!/\,\xi| \approx |\nabla\!\!\!/_Y\xi| + |\nabla\!\!\!/_Z\xi|. \tag{8-18}$$

Proof. To prove (8-17), we use Lemmas 2.23 and 2.32 and the fact that the Cartesian component functions X^1, X^2, X^3 are smooth functions of the L^i and Ψ (see (2-23)) to deduce that $|Y|^2 = g\!\!\!/_{AB}Y^AY^B = \mathrm{f}(L^i,\Psi)$, where f is a smooth function. Similar remarks hold for $|Z|^2$. The desired estimates in (8-17) therefore follow from the bootstrap assumptions (6-3)–(6-4) and Proposition 8.6.

To prove (8-18), we note that the $g\!\!\!/$-Cauchy–Schwarz inequality and (8-17) imply that $|\nabla\!\!\!/_Y\xi| + |\nabla\!\!\!/_Z\xi| \lesssim |\nabla\!\!\!/\,\xi|$. We will show how to obtain the reverse inequality when ξ is a scalar function; the case of an arbitrary $\ell_{t,u}$-tangent tensorfield can be handled using the same arguments, which will complete the proof. To proceed, we note that for scalar functions ξ we have $|\nabla\!\!\!/\,\xi|^2 = (g\!\!\!/^{-1})^{AB}(\partial\!\!\!/_A\xi)(\partial\!\!\!/_B\xi)$. We now use Lemmas 2.24 and 2.32 and the fact that X^1, X^2, X^3 are smooth functions of L^i and Ψ (as noted above) to deduce that there exist smooth functions, all schematically denoted by f, such that $(g\!\!\!/^{-1})^{AB}(\partial\!\!\!/_A\xi)(\partial\!\!\!/_B\xi) = \mathrm{f}(L^i,\Psi)(Y\xi)^2 + \mathrm{f}(L^i,\Psi)(Y\xi)(Z\xi) + \mathrm{f}(L^i,\Psi)(Z\xi)^2$. Also using the bootstrap assumptions (6-3)–(6-4), Young's inequality, and Proposition 8.6, we conclude that $|\nabla\!\!\!/\,\xi|^2 \lesssim |Y\xi|^2 + |Z\xi|^2 = |\nabla\!\!\!/_Y\xi|^2 + |\nabla\!\!\!/_Z\xi|^2$ as desired. □

8C. *L^∞ estimates involving higher transversal derivatives.* Some aspects of our main results rely on having L^∞ estimates for the higher transversal derivatives of various solution variables. We provide these estimates in the next proposition. The proofs are similar to the proofs of related estimates in [52].

Proposition 8.10 (L^∞ estimates involving higher transversal derivatives). *The following estimates hold*[50] *for all* $t \in [0, T_{(\mathrm{Boot})})$ *and* $u \in [0, U_0]$, *where in* (8-22b), $\not{\mathcal{P}} \in \{Y, Z\}$:

<u>L^∞ *estimates involving two or three transversal derivatives of the wave variables*</u>:

$$\|L\mathcal{P}^{\leq 2}\breve{X}\breve{X}\Psi\|_{L^\infty(\mathcal{M}_{t,u})} \leq C\mathring{\epsilon}^{1/2}, \tag{8-19a}$$

$$\|\mathcal{P}^{[1,2]}\breve{X}\breve{X}\Psi\|_{L^\infty(\mathcal{M}_{t,u})} \leq C\mathring{\epsilon}^{1/2}, \tag{8-19b}$$

$$\|\breve{X}\breve{X}\mathcal{R}_{(+)}\|_{L^\infty(\mathcal{M}_{t,u})} \leq \|\breve{X}\breve{X}\mathcal{R}_{(+)}\|_{L^\infty(\Sigma_0)} + C\mathring{\epsilon}^{1/2}, \tag{8-19c}$$

$$\|\breve{X}\breve{X}(\mathcal{R}_{(-)}, v^1, v^2, s)\|_{L^\infty(\mathcal{M}_{t,u})} \leq C\mathring{\epsilon}^{1/2}, \tag{8-19d}$$

$$\|L\breve{X}\breve{X}\breve{X}\Psi\|_{L^\infty(\mathcal{M}_{t,u})} \leq C\mathring{\epsilon}^{1/2}, \tag{8-20a}$$

$$\|\breve{X}\breve{X}\breve{X}\mathcal{R}_{(+)}\|_{L^\infty(\mathcal{M}_{t,u})} \leq \|\breve{X}\breve{X}\breve{X}\mathcal{R}_{(+)}\|_{L^\infty(\Sigma_0)} + C\mathring{\epsilon}^{1/2}, \tag{8-20b}$$

$$\|\breve{X}\breve{X}\breve{X}(\mathcal{R}_{(-)}, v^1, v^2, s)\|_{L^\infty(\mathcal{M}_{t,u})} \leq C\mathring{\epsilon}^{1/2}. \tag{8-20c}$$

<u>L^∞ *estimates involving one or two transversal derivatives of* $\upmu$</u>:

$$\|L\breve{X}\upmu\|_{L^\infty(\mathcal{M}_{t,u})} \leq \tfrac{1}{2}\|\breve{X}(\vec{G}_{LL} \diamond \breve{X}\vec{\Psi})\|_{L^\infty(\Sigma_0)} + C\mathring{\epsilon}^{1/2}, \tag{8-21a}$$

$$\|\breve{X}\upmu\|_{L^\infty(\mathcal{M}_{t,u})} \leq \|\breve{X}\upmu\|_{L^\infty(\Sigma_0)} + \mathring{\delta}_*^{-1}\|\breve{X}(\vec{G}_{LL} \diamond \breve{X}\vec{\Psi})\|_{L^\infty(\Sigma_0)} + C\mathring{\epsilon}^{1/2}, \tag{8-21b}$$

$$\|L\breve{X}\mathcal{P}\upmu\|_{L^\infty(\mathcal{M}_{t,u})}, \quad \|L\breve{X}\mathcal{P}^2\upmu\|_{L^\infty(\mathcal{M}_{t,u})} \leq C\mathring{\epsilon}^{1/2}, \tag{8-22a}$$

$$\|\breve{X}\not{\mathcal{P}}\upmu\|_{L^\infty(\mathcal{M}_{t,u})}, \quad \|\breve{X}\mathcal{P}^2\upmu\|_{L^\infty(\mathcal{M}_{t,u})} \leq C\mathring{\epsilon}^{1/2}, \tag{8-22b}$$

$$\|LL\breve{X}\breve{X}\upmu\|_{L^\infty(\mathcal{M}_{t,u})} \leq C\mathring{\epsilon}^{1/2}, \tag{8-23a}$$

$$\|L\breve{X}\breve{X}\upmu\|_{L^\infty(\mathcal{M}_{t,u})} \leq \tfrac{1}{2}\|\breve{X}\breve{X}(\vec{G}_{LL} \diamond \breve{X}\vec{\Psi})\|_{L^\infty(\Sigma_0)} + C\mathring{\epsilon}^{1/2}, \tag{8-23b}$$

$$\|\breve{X}\breve{X}\upmu\|_{L^\infty(\mathcal{M}_{t,u})} \leq \|\breve{X}\breve{X}\upmu\|_{L^\infty(\Sigma_0)} + \mathring{\delta}_*^{-1}\|\breve{X}\breve{X}(\vec{G}_{LL} \diamond \breve{X}\vec{\Psi})\|_{L^\infty(\Sigma_0)} + C\mathring{\epsilon}^{1/2}. \tag{8-23c}$$

<u>L^∞ *estimates involving one or two transversal derivatives of* L^i</u>:

$$\|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-5]}\breve{X}L^i\|_{L^\infty(\mathcal{M}_{t,u})} \leq C\mathring{\epsilon}^{1/2}, \tag{8-24a}$$

$$\|\breve{X}L^i\|_{L^\infty(\mathcal{M}_{t,u})} \leq C, \tag{8-24b}$$

$$\|L\mathcal{P}\breve{X}\breve{X}L^i\|_{L^\infty(\mathcal{M}_{t,u})} \leq C\mathring{\epsilon}^{1/2}, \tag{8-25a}$$

$$\|\mathcal{P}\breve{X}\breve{X}L^i\|_{L^\infty(\mathcal{M}_{t,u})} \leq C\mathring{\epsilon}^{1/2}, \tag{8-25b}$$

$$\|\breve{X}\breve{X}L^i\|_{L^\infty(\mathcal{M}_{t,u})} \leq C. \tag{8-25c}$$

<u>L^∞ *estimates involving transversal derivatives of the transported variables*</u>:

$$\begin{aligned} &\|\mathcal{P}^{\leq 3}\breve{X}^{\leq 1}(\Omega, S)\|_{L^\infty(\mathcal{M}_{t,u})} + \|\mathcal{P}^{\leq 2}\breve{X}\breve{X}(\Omega, S)\|_{L^\infty(\mathcal{M}_{t,u})} + \|\breve{X}^{\leq 3}(\Omega, S)\|_{L^\infty(\mathcal{M}_{t,u})} \\ &\qquad + \|\mathcal{P}^{\leq 2}\breve{X}^{\leq 1}(\mathcal{C}, \mathcal{D})\|_{L^\infty(\mathcal{M}_{t,u})} + \|\breve{X}^{\leq 2}(\mathcal{C}, \mathcal{D})\|_{L^\infty(\mathcal{M}_{t,u})} \leq C\mathring{\epsilon}. \end{aligned} \tag{8-26}$$

[50] Based on our assumptions on the data (see Section 4A), we could obtain L^∞ control over additional $\mathcal{F}_u$-tangential derivatives of the quantities stated in the proposition — but not! additional $\breve{X}$ differentiations. However, for convenience, in the proposition, we have only derived control of a sufficient number of derivatives so that the estimates close and so that we can use the results in our proof of Lemma 14.2 and in the Appendix.

Finally,

$$\textit{we can permute the vectorfield operators on the left-hand sides of } \text{(8-19a)–(8-25c)} \\ \textit{up to error terms of } L^\infty \textit{ size } \mathcal{O}(\mathring{\varepsilon}^{1/2}), \tag{8-27}$$

$$\textit{and on the left-hand side of } \text{(8-26)} \textit{ up to error terms of } L^\infty \textit{ size } \mathcal{O}(\mathring{\varepsilon}). \tag{8-28}$$

Proof. To prove the lemma, we make the "new bootstrap assumption" that the estimates in (8-26) hold for $t \in [0, T_{(\mathrm{Boot})})$ with the $C\mathring{\varepsilon}$-term on the right-hand side replaced by $\mathring{\varepsilon}^{1/2}$, and also that (8-28) holds with $\mathcal{O}(\mathring{\varepsilon})$ replaced by $\mathring{\varepsilon}^{1/2}$. Given this new bootstrap assumption, to obtain (8-19a)–(8-25c) and (8-27), we can simply repeat[51] the proof of [52, Lemma 9.3], which relies on transport-type estimates that lose derivatives (in particular, one uses the transport equations (2-40)–(2-41) and also treats the wave equation as a derivative-losing transport equation $L\breve{X}\Psi = \cdots$ by using (13-13)). The only difference between the estimates derived in [52, Lemma 9.3] and the estimates we need to derive is that our wave equations (5-1a)–(5-1c), when weighted with a factor of μ (so that the decomposition (13-13) of $\mu\Box_g$ can be employed), feature some new inhomogeneous terms compared to [52, Lemma 9.3], specifically, some of the ones depending on $(\mathcal{C}, \mathcal{D}, \varOmega, S)$ and the first derivatives of $(\varOmega, S)$. The key point is that our new bootstrap assumption implies that the new inhomogeneous terms are all bounded in L^∞ by $\lesssim \mathring{\varepsilon}^{1/2}$, which is compatible with the $\mathcal{O}(\mathring{\varepsilon}^{1/2})$-size bounds that one is aiming to prove; i.e., our new $\mathcal{O}(\mathring{\varepsilon}^{1/2})$-sized error terms are harmless in the context of the proof. From this logic, it follows that the estimates (8-19a)–(8-25c) and (8-27) hold for all $t \in [0, T_{(\mathrm{Boot})})$. We clarify that the estimates (8-23a) and (8-25a) were not explicitly stated in [52, Lemma 9.3]. However (8-23a) follows from commuting the transport equation (2-40) with $L\breve{X}\breve{X}$ via Lemma 8.5 and bounding the resulting algebraic expression for $LL\breve{X}\breve{X}\mu$ using the fact that the Cartesian component functions X^1, X^2, X^3 are smooth functions of the L^i and Ψ (see (2-23)), the bootstrap assumptions (6-3)–(6-7), Proposition 8.6, and the estimates in (8-19a)–(8-25c) and (8-27) besides (8-23a) and (8-25a) . Similarly, (8-25a) follows from commuting the transport equation (2-41) with $\mathcal{P}\breve{X}\breve{X}$.

To complete the proof, it only remains for us to prove (8-26) and (8-28) (with the help of the already established bounds (8-19a)–(8-25c) and (8-27)); for if $\mathring{\varepsilon}$ is sufficiently small, this yields a strict improvement of the new bootstrap assumption mentioned at the beginning of the proof, and the conclusions of the proposition then follow from a standard continuity argument. We start by noting that the bounds in (8-26) for the pure $\mathcal{F}_u$-tangential derivatives of $(\varOmega, S)$ are included in the bootstrap assumptions (6-6)–(6-7), as are the bounds

$$\|\mathcal{P}^{\leq 3}\breve{X}(\varOmega, S)\|_{L^\infty(\mathcal{M}_{t,u})} \lesssim \mathring{\varepsilon}. \tag{8-29}$$

Next, we use Lemma 8.5, the bootstrap assumptions (6-3)–(6-7), Proposition 8.6, the estimates (8-19a)–(8-25c) and (8-27), and the bounds (8-29) to deduce that the estimate (8-29) also holds for all permutations of the vectorfield operators on the left-hand side.

[51] We clarify that the bootstrap parameter "ε" from [52] should be identified with the quantity $\mathring{\varepsilon}^{1/2}$ in our bootstrap assumptions (6-4)–(6-8).

We next show that

$$\|\mathcal{P}^{\leq 2}\breve{X}\breve{X}(\varOmega, S)\|_{L^\infty(\mathcal{M}_{t,u})} \lesssim \mathring{\epsilon}. \tag{8-30}$$

This estimate follows from differentiating the identities (8-4)–(8-5) with $\mathcal{P}^{\leq 2}\breve{X}$ and using the bootstrap assumptions (6-3)–(6-7), Proposition 8.6, the estimates (8-19a)–(8-25c) and (8-27), the estimate (8-29), and the analog of (8-29) for all permutations of the vectorfield operators on the left-hand side. (Notice that we can indeed prove (8-30) with a strict improvement of our new bootstrap assumptions because the terms arising from differentiating (8-4)–(8-5) by $\mathcal{P}^{\leq 2}\breve{X}$ contain at least one factor of $(\varOmega, S)$ differentiated with at most one $\breve{X}$ derivative, and such factors have already been shown to bounded in the norm $\|\cdot\|_{L^\infty(\mathcal{M}_{t,u})}$ by $\lesssim \mathring{\epsilon}$.) Again using Lemma 8.5 to commute vectorfield derivatives, we also deduce that the estimate (8-30) also holds for all permutations of the vectorfield operators on the left-hand side.

We next show that

$$\|\breve{X}\breve{X}\breve{X}(\varOmega, S)\|_{L^\infty(\mathcal{M}_{t,u})} \lesssim \mathring{\epsilon}. \tag{8-31}$$

This estimate follows from differentiating the identities (8-4)–(8-5) with $\breve{X}\breve{X}$ and using the bootstrap assumptions (6-3)–(6-7), Proposition 8.6, the estimates (8-19a)–(8-25c) and (8-27), the estimates (8-29)–(8-30), and the analogs of (8-29)–(8-30) for all permutations of the vectorfield operators on the left-hand sides.

Similarly, we can first prove

$$\|\mathcal{P}^{\leq 2}\breve{X}^{\leq 1}(\mathcal{C}, \mathcal{D})\|_{L^\infty(\mathcal{M}_{t,u})} \lesssim \mathring{\epsilon} \tag{8-32}$$

and then

$$\|\breve{X}^{\leq 2}(\mathcal{C}, \mathcal{D})\|_{L^\infty(\mathcal{M}_{t,u})} \lesssim \mathring{\epsilon} \tag{8-33}$$

(and that (8-32) holds for all permutations of the vectorfield operators on the left-hand side all permutations of the vectorfield operators on the left-hand side) by using the identities (8-6)–(8-7) and arguing as above, using in addition the bootstrap assumption (6-8) and the already proven estimates for $(\varOmega, S)$.

We have therefore established (8-26) and (8-28), which completes the proof of the proposition. □

8D. ***Sharp estimates for*** $\upmu_\star$**.** Recall the definition of $\upmu_\star(t)$ in Definition 2.16. In this subsection, in Propositions 8.11 and 8.12, we provide some estimates for $\upmu_\star(t)$ that were proved in [52]. We will simply cite the relevant estimates, noting that their proof relies only on the L^∞ bounds for (lower-order derivatives of) the wave variables and the geometric quantities that we have already established. Moreover, we remark that these estimates capture that $\upmu_\star(t)$ tends to 0 linearly, a fact that is crucial for bounding the maximum possible singularity strength of our high-order geometric energies (i.e., for controlling the blowup-rate of the energies in, for example, (6-1)).

Thanks to our bootstrap assumptions and the estimates of Proposition 8.6, the following estimates for $\upmu_\star(t)$ can be proved exactly as in [52, (10.36), (10.39)]:

Proposition 8.11 (control of integrals of $\upmu_\star$). *Let $M_* \in \mathbb{N}$ be the absolute constant appearing in the statements of Theorem 4.2 and Proposition 12.1 below. For $1 < b \leq 100M_*$, the quantities $\upmu_\star(t, u)$ and $\upmu_\star(t)$ from Definition 2.16 obey the following estimates for every $(t, u) \in [0, T_{(\text{Boot})}) \times [0, U_0]$:*

$$\int_{t'=0}^{t'=t} \upmu_\star^{-b}(t', u)\, dt' \lesssim \left(1 + \frac{1}{b-1}\right)\upmu_\star^{-b+1}(t, u), \quad \int_{t'=0}^{t'=t} \upmu_\star^{-b}(t')\, dt' \lesssim \left(1 + \frac{1}{b-1}\right)\upmu_\star^{-b+1}(t). \tag{8-34}$$

Moreover, for all $t \in [0, T_{(\text{Boot})})$,

$$\int_{t'=0}^{t'=t} \upmu_{\star}^{-0.9}(t', u)\, dt' \lesssim 1, \quad \int_{t'=0}^{t'=t} \upmu_{\star}^{-0.9}(t')\, dt' \lesssim 1. \tag{8-35}$$

Thanks to our bootstrap assumptions and the estimates of Proposition 8.6, the following "almost-monotonicity" of $\upmu_{\star}$ can be proved as in [52, (10.23)]:

Proposition 8.12 (the approximate monotonicity of $\upmu_{\star}$). *For* $0 \leq s_1 \leq s_2 < T_{(\text{Boot})}$,

$$\upmu_{\star}^{-1}(s_1) \leq 2\upmu_{\star}^{-1}(s_2).$$

8E. ***L^2 estimates for the geometric quantities.*** We start with a simple lemma that provides L^2 estimates for solutions to transport equations along the integral curves of L.

Lemma 8.13 (L^2 estimate for solutions to L-transport equations). *Let F and f be smooth scalar functions on* $[0, T_{(\text{Boot})}) \times [0, U_0] \times \mathbb{T}^2$. *Assume that* $LF(t, u, x^2, x^3) = f(t, u, x^2, x^3)$ *with initial data* $F(0, u, x^2, x^3)$ *for every* $(t, u, x^2, x^3) \in [0, T_{(\text{Boot})}) \times [0, U_0] \times \mathbb{T}^2$. *Then the following estimate holds for every* $(t, u) \in [0, T_{(\text{Boot})}) \times [0, U_0]$:

$$\|F\|_{L^2(\Sigma_t^u)} \leq (1 + C\mathring{\varepsilon}^{1/2})\|F\|_{L^2(\Sigma_0^u)} + (1 + C\mathring{\varepsilon}^{1/2}) \int_{t'=0}^{t'=t} \|f\|_{L^2(\Sigma_{t'}^u)}\, dt'. \tag{8-36}$$

Proof. Thanks to our bootstrap assumptions and the estimates of Proposition 8.6, (8-36) can be proved using essentially the same arguments used in the proof of [52, Lemmas 12.2, 12.3, 13.2]. The only differences are that we have to use the bootstrap assumptions (6-3)–(6-8) in place of the similar bootstrap assumptions from [52], and that different coordinates along $\ell_{t,u}$ were used in [52] (this is irrelevant in the sense that the estimate (8-36) is independent of the coordinates on $\ell_{t,u}$). We clarify that the bootstrap parameter "ε" from [52] should be identified with the quantity $\mathring{\varepsilon}^{1/2}$ in our bootstrap assumptions (6-3)–(6-8). ☐

Proposition 8.14 (easy L^2 estimates for the acoustical geometry). *For* $1 \leq N \leq N_{\text{top}}$, *the following estimates hold for all* $t \in [0, T_{(\text{Boot})})$:

$$\|\mathcal{P}^{[2,N]}\upmu\|^2_{L^2(\Sigma_t)},\ \|\mathcal{P}^{[1,N]}L^i\|^2_{L^2(\Sigma_t)} \lesssim \mathring{\varepsilon} \max\{1, \upmu_{\star}^{-2M_*+2N_{\text{top}}-2N+2.8}(t)\}.$$

Proof. In an identical manner as [52, Lemma 14.3], based on the transport equations (2-40)–(2-41) and (8-36), we obtain

$$\|\mathcal{P}^{[2,N]}\upmu\|_{L^2(\Sigma_t)},\ \|\mathcal{P}^{[1,N]}L^i\|_{L^2(\Sigma_t)} \lesssim \mathring{\varepsilon} + \int_{s=0}^{s=t} \frac{\mathbb{W}^{1/2}_{[1,N]}(s)}{\upmu_{\star}^{1/2}(s)}\, ds.$$

(Recall our notation in Definition 3.4, (3-2e) and Definition 3.5.) Also using our bootstrap assumptions (6-1) and (6-2) and Proposition 8.11, we arrive at the desired conclusion. ☐

In the next proposition, with the help of Proposition 8.14, we derive L^2 estimates for commutators.

Proposition 8.15 (L^2 estimates for commutator terms). *Let ϕ be a scalar function. For $1 \leq N \leq N_{\mathrm{top}}$, the following estimates hold for all $(t, u) \in [0, T_{(\mathrm{Boot})}) \times [0, U_0]$:*

$$\begin{aligned} &\|[L, \mathcal{P}^N]\phi\|^2_{L^2(\Sigma_t^u)}, \ \|[\breve{X}, \mathcal{P}^N]\phi\|^2_{L^2(\Sigma_t^u)}, \ \|[\mu B, \mathcal{P}^N]\phi\|^2_{L^2(\Sigma_t^u)} \\ &\qquad \lesssim \|\mathcal{P}^{[1,N]}\phi\|^2_{L^2(\Sigma_t^u)} + \mathring{\epsilon}\max\{1, \mu_\star^{-2M_*+2N_{\mathrm{top}}-2N+2.8}(t)\}\|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-5]}\phi\|^2_{L^\infty(\Sigma_t^u)}. \end{aligned} \tag{8-37}$$

Moreover, we also have

$$\|\mathcal{P}^N \breve{X}\Psi\|^2_{L^2(\Sigma_t^u)} \lesssim \mathring{\epsilon}\max\{1, \mu_\star^{-2M_*+2N_{\mathrm{top}}-2N+1.8}(t)\}. \tag{8-38}$$

Proof. Recall the pointwise estimate (8-11). For each of the sums in (8-11), either $N_2 > N_1$, in which case by (6-5) and Proposition 8.6, we have $|\mathcal{P}^{[2,N_1]}(\mu, L^i, \Psi)|$, $|\mathcal{P}^{[2,N_1]}\breve{X}\Psi| \lesssim 1$; or else $N_2 \leq N_1$, in which case (since $N \leq N_{\mathrm{top}}$) $|\mathcal{P}^{[1,N_2]}\phi| \lesssim |\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-5]}\phi|$. Hence,

$$\begin{aligned} &|[L, \mathcal{P}^N]\phi|, \ |[\breve{X}, \mathcal{P}^N]\phi| \\ &\quad \lesssim |\mathcal{P}^{[1,N]}\phi| + \big\{|\mathcal{P}^{[2,N]}(\mu, L^i, \Psi)| + |\mathcal{P}^{[2,N-1]}\breve{X}\Psi|\big\}|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-5]}\phi| \\ &\quad \lesssim |\mathcal{P}^{[1,N]}\phi| + \big\{|\mathcal{P}^{[2,N]}(\mu, L^i, \Psi)| + |\breve{X}\mathcal{P}^{[2,N-1]}\Psi| + |[\breve{X}, \mathcal{P}^{[2,N-1]}]\Psi|\big\}|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-5]}\phi|. \end{aligned} \tag{8-39}$$

We first apply (8-39) to $\phi = \Psi$. Taking the $L^2(\Sigma_t^u)$ norm and introducing an induction argument in N which uses (6-1)–(6-5) and Proposition 8.14, we obtain

$$\|[\breve{X}, \mathcal{P}^N]\Psi\|^2_{L^2(\Sigma_t^u)} \lesssim \mathring{\epsilon}\max\{1, \mu_\star^{-2M_*+2N_{\mathrm{top}}-2N+2.8}(t)\}. \tag{8-40}$$

Taking the $L^2(\Sigma_t^u)$ norm in (8-39), plugging in the estimate (8-40), and using (6-1), (6-2), and Proposition 8.14, we deduce the desired estimates in (8-37) for $[L, \mathcal{P}^N]\phi$ and $[\breve{X}, \mathcal{P}^N]\phi$.

To obtain the $[\mu B, \mathcal{P}^N]\phi$ estimate in (8-37), we first note that, by (2-23),

$$[\mu B, \mathcal{P}^N]\phi = \mu[L, \mathcal{P}^N]\phi + [\mu, \mathcal{P}^N]L\phi + [\breve{X}, \mathcal{P}^N]\phi.$$

The first and last terms can be controlled by combining the commutator estimates we just established with the simple bound $\|\mu\|_{L^\infty(\Sigma_t)} \lesssim 1$ from Proposition 8.6, while the second term can be controlled simply using the product rule and Propositions 8.6 and 8.14. We have therefore established (8-37).

Finally, we have (8-38) thanks to (6-1), (6-2) and (8-40). □

9. Transport estimates for the specific vorticity and the entropy gradient

We continue to work under the assumptions of Theorem 6.3.

In this section, we use the transport equations (5-2a) and (5-2c) to bound $\mathcal{P}^N\Omega$ and $\mathcal{P}^N S$ for $N \leq N_{\mathrm{top}}$. We clarify that the "true" top-order estimates for the vorticity and entropy are found in Section 11; those estimates are more involved and rely on the modified fluid variables as well as elliptic estimates.

We will start by deriving energy estimates for general transport equations (which will also be useful in the next section). In particular, this will reduce the derivation of the energy estimates for $\mathcal{P}^N\Omega$ and $\mathcal{P}^N S$ to controlling the inhomogeneous terms in the transport equations and their derivatives, which we will carry out in Section 9B. The final estimates for $\mathcal{P}^N\Omega$ and $\mathcal{P}^N S$ are located in Section 9C.

9A. ***Estimates for general transport equations.***

Proposition 9.1 (L^2 estimates for solutions to B-transport equations). *Let ϕ be a scalar function satisfying*

$$\mu B\phi = \mathfrak{F},$$

with both ϕ and $\mathfrak{F}$ being compactly supported in $[0, U_0] \times \mathbb{T}^2$ *for every* $t \in [0, T_{(\mathrm{Boot})})$.

Then the following estimate holds for every $(t, u) \in [0, T_{(\mathrm{Boot})}) \times [0, U_0]$:

$$\sup_{t' \in [0,t)} \|\sqrt{\mu}\phi\|^2_{L^2(\Sigma^u_{t'})} + \sup_{u' \in [0,u)} \|\phi\|^2_{L^2(\mathcal{F}^t_{u'})} \lesssim \|\sqrt{\mu}\phi\|^2_{L^2(\Sigma^u_0)} + \|\mathfrak{F}\|^2_{L^2(\mathcal{M}_{t,u})}.$$

Proof. In an identical manner as [36, Proposition 3.5], we have, for any $(t', u') \in [0, t) \times [0, u)$, the identity

$$\int_{\Sigma^{u'}_{t'}} \mu\phi^2 \, d\underline{\varpi} + \int_{\mathcal{F}^{t'}_{u'}} \phi^2 \, d\overline{\varpi} = \int_{\Sigma^{u'}_0} \mu\phi^2 \, d\underline{\varpi} + \underbrace{\int_{\mathcal{F}^{t'}_0} \phi^2 \, d\overline{\varpi}}_{0 \text{ by support assumptions}} + \int_{\mathcal{M}_{t',u'}} \{2\phi\mathfrak{F} + (L\mu + \mu \operatorname{tr}_{\not g} \not k)\phi^2\} \, d\varpi. \tag{9-1}$$

Using (2-38c), (2-40), Lemma 2.32, (6-3)–(6-5), and Propositions 8.6 and 8.7, we have $|L\mu|$, $|\mu \operatorname{tr}_{\not g} \not k| \lesssim 1$. Thus, applying also the Cauchy–Schwarz inequality to the $2\phi\mathfrak{F}$ term, we have

$$\sup_{t' \in [0,t)} \|\sqrt{\mu}\phi\|^2_{L^2(\Sigma^u_{t'})} + \sup_{u' \in [0,u)} \|\phi\|^2_{L^2(\mathcal{F}^t_{u'})} \lesssim \|\sqrt{\mu}\phi\|^2_{L^2(\Sigma^u_0)} + \int_{u'=0}^{u'=u} \|\phi\|^2_{L^2(\mathcal{F}^t_{u'})} \, du' + \|\mathfrak{F}\|^2_{L^2(\mathcal{M}_{t,u})}.$$

The conclusion follows from applying Grönwall's inequality in u. □

Proposition 9.2 (higher-order L^2 estimates for solutions to transport equations). *Let ϕ be a scalar function satisfying*

$$\mu B\phi = \mathfrak{F},$$

with both ϕ and $\mathfrak{F}$ being compactly supported in $[0, U_0] \times \mathbb{T}^2$ *for every* $t \in [0, T_{(\mathrm{Boot})})$.

Then the following estimate holds for every $(t, u) \in [0, T_{(\mathrm{Boot})}) \times [0, U_0]$ *and* $0 \le N \le N_{\mathrm{top}}$:

$$\begin{aligned}\sup_{t' \in [0,t)} \|\sqrt{\mu}\mathcal{P}^{\le N}\phi\|^2_{L^2(\Sigma^u_{t'})} &+ \sup_{u' \in [0,u)} \|\mathcal{P}^{\le N}\phi\|^2_{L^2(\mathcal{F}^t_{u'})} \\ &\lesssim \|\mathcal{P}^{\le N}\phi\|^2_{L^2(\Sigma_0)} + \|\mathcal{P}^{\le N}\mathfrak{F}\|^2_{L^2(\mathcal{M}_{t,u})} + \mathring{\varepsilon} \max\{1, \mu_\star^{-2M_*+2N_{\mathrm{top}}-2N+3.8}(t)\} \|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-5]}\phi\|^2_{L^\infty(\mathcal{M}_{t,u})}.\end{aligned}$$

Proof. Take $0 \le N' \le N$. We write

$$\mu B \mathcal{P}^{N'}\phi = \mathcal{P}^{N'}\mathfrak{F} + [\mu B, \mathcal{P}^{N'}]\phi.$$

Therefore, by Proposition 9.1,

$$\begin{aligned}\sup_{t' \in [0,t)} \|\sqrt{\mu}\mathcal{P}^{N'}\phi\|^2_{L^2(\Sigma^u_{t'})} &+ \sup_{u' \in [0,u)} \|\mathcal{P}^{N'}\phi\|^2_{L^2(\mathcal{F}^t_{u'})} \\ &\lesssim \|\mathcal{P}^{N'}\phi\|^2_{L^2(\Sigma_0)} + \|\mathcal{P}^{N'}\mathfrak{F}\|^2_{L^2(\mathcal{M}_{t,u})} + \|[\mu B, \mathcal{P}^{N'}]\phi\|^2_{L^2(\mathcal{M}_{t,u})}.\end{aligned} \tag{9-2}$$

Using Proposition 8.15 and then Proposition 8.11, we obtain

$$\begin{aligned}
&\|[\upmu B, \mathcal{P}^{N'}]\phi\|^2_{L^2(\mathcal{M}_{t,u})} \\
&\quad\lesssim \int_{t'=0}^{t'=t} \|[\upmu B, \mathcal{P}^{N'}]\phi\|^2_{L^2(\Sigma^u_{t'})}\,dt' \\
&\quad\lesssim \|\mathcal{P}^{[1,N']}\phi\|^2_{L^2(\mathcal{M}_{t,u})} + \mathring{\epsilon}\|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-5]}\phi\|^2_{L^\infty(\mathcal{M}_{t,u})}\int_{t'=0}^{t'=t}\max\{1,\, \upmu_\star^{-2M_*+2N_{\mathrm{top}}-2N'+2.8}(t')\}\,dt' \\
&\quad\lesssim \int_{u'=0}^{u'=u} \|\mathcal{P}^{\le N}\phi\|^2_{L^2(\mathcal{F}^t_{u'})}\,du' + \mathring{\epsilon}\max\{1,\, \upmu_\star^{-2M_*+2N_{\mathrm{top}}-2N'+3.8}(t)\}\|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-5}\phi\|^2_{L^\infty(\mathcal{M}_{t,u})}.
\end{aligned}\tag{9-3}$$

Plugging (9-3) into (9-2) and summing over all $0 \le N' \le N$, we obtain

$$\begin{aligned}
&\sup_{t'\in[0,t)} \|\sqrt{\upmu}\mathcal{P}^{\le N}\phi\|^2_{L^2(\Sigma^u_{t'})} + \sup_{u'\in[0,u)} \|\mathcal{P}^{\le N}\phi\|^2_{L^2(\mathcal{F}^t_{u'})} \\
&\qquad\lesssim \|\mathcal{P}^{\le N}\phi\|^2_{L^2(\Sigma_0)} + \|\mathcal{P}^{\le N}\mathfrak{F}\|^2_{L^2(\mathcal{M}_{t,u})} + \int_{u'=0}^{u'=u}\|\mathcal{P}^{\le N}\phi\|^2_{L^2(\mathcal{F}^t_{u'})}\,du' \\
&\qquad\qquad + \mathring{\epsilon}\max\{1,\, \upmu_\star^{-2M_*+2N_{\mathrm{top}}-2N+3.8}(t)\}\|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-5}\phi\|^2_{L^\infty(\mathcal{M}_{t,u})}.
\end{aligned}\tag{9-4}$$

Applying Grönwall's inequality in u, we arrive at the desired estimate. □

9B. *Controlling the inhomogeneous terms.*

Proposition 9.3 (estimates tied to the inhomogeneous terms in the transport equations for $\varOmega$ and S). *For* $0 \le N \le N_{\mathrm{top}}$, *the following hold for every* $(t, u) \in [0, T_{(\mathrm{Boot})}) \times [0, U_0]$:

$$\begin{aligned}
&\|\mathcal{P}^N(\upmu B\varOmega)\|^2_{L^2(\mathcal{M}_{t,u})} + \|\mathcal{P}^N(\upmu BS)\|^2_{L^2(\mathcal{M}_{t,u})} \\
&\qquad\lesssim \mathring{\epsilon}^3\max\{1,\, \upmu_\star^{-2M_*+2N_{\mathrm{top}}-2N+2.8}(t)\} + \int_{u'=0}^{u'=u}(\mathbb{V}_{\le N}(t,u') + \mathbb{S}_{\le N}(t,u'))\,du'
\end{aligned}\tag{9-5}$$

and

$$\begin{aligned}
&\|(\mathcal{P}^N\varOmega)\mathcal{P}^N(\upmu B\varOmega)\|_{L^1(\mathcal{M}_{t,u})} + \|(\mathcal{P}^N S)\mathcal{P}^N(\upmu BS)\|_{L^1(\mathcal{M}_{t,u})} \\
&\qquad\lesssim \mathring{\epsilon}^3\max\{1,\, \upmu_\star^{-2M_*+2N_{\mathrm{top}}-2N+2.8}(t)\} + \int_{u'=0}^{u'=u}(\mathbb{V}_{\le N}(t,u') + \mathbb{S}_{\le N}(t,u'))\,du'.
\end{aligned}\tag{9-6}$$

Proof. Step 1: basic pointwise estimates. We claim that the derivatives of the $\upmu$-weighted inhomogeneous terms $\upmu\mathfrak{L}^i_{(\varOmega)}$ and $\upmu\mathfrak{L}^i_{(S)}$, which are defined respectively in (5-7d) and (5-7e), obey the following pointwise bounds:

$$\begin{aligned}
&|\mathcal{P}^N(\upmu\mathfrak{L}^i_{(\varOmega)})| + |\mathcal{P}^N(\upmu\mathfrak{L}^i_{(S)})| \\
&\qquad\lesssim \underbrace{|\mathcal{P}^{\le N}(\varOmega, S)|}_{\doteq I} + \underbrace{\mathring{\epsilon}(|\upmu\mathcal{P}^{[2,N+1]}\Psi| + |\mathcal{P}^{[1,N]}\breve{X}\Psi|)}_{\doteq II} + \underbrace{\mathring{\epsilon}|\mathcal{P}^{[2,N]}(\upmu,\ L^i,\ \Psi)|}_{\doteq III}.
\end{aligned}\tag{9-7}$$

Since this is the first instance of these kind of estimates (and we will derive similar estimates later), we give some details on how to obtain (9-7).

(1) By Lemma 2.22 and the fact that the Cartesian component functions X^1, X^2, X^3 are smooth functions of the L^i and Ψ (see (2-23)), the weighted Cartesian coordinate vectorfield $\mu\partial_i$ and the transport vectorfield μB can be decomposed regularly (i.e., with coefficients being smooth functions of μ, L^i and Ψ) in terms of $\breve{X}$, μY, μZ and μL.

(2) Therefore, $\mathcal{P}^N(\mu\mathfrak{L}^i_{(\Omega)})$ and $\mathcal{P}^N(\mu\mathfrak{L}^i_{(S)})$ can be bounded as follows:

$$
\begin{aligned}
&|\mathcal{P}^N(\mu\mathfrak{L}^i_{(\Omega)})|+|\mathcal{P}^N(\mu\mathfrak{L}^i_{(S)})| \\
&\quad\lesssim \sum_{k=0}^{N}\sum_{N_1+\cdots+N_k+n_1+n_2=N}(1+|\mathcal{P}^{N_1}(\mu,L^i,\Psi)|)\cdots(1+|\mathcal{P}^{N_k}(\mu,L^i,\Psi)|)\times|\mathcal{P}^{n_1}(\Omega,S)|\times|\mathcal{P}^{n_2}(\mu\mathcal{P}\Psi,\breve{X}\Psi)| \\
&\quad\doteq \sum_{k=0}^{N}\mathrm{Error}_{N_1,\dots,N_k,n_1,n_2}.
\end{aligned}
\tag{9-8}
$$

We now bound the right-hand side of (9-8).

(3) If $N_1,\ \dots,\ N_k \le N_{\mathrm{top}}-M_*-5$ and $n_2 \le N_{\mathrm{top}}-M_*-5$, we bound the terms $(1+|\mathcal{P}^{N_j}(\mu,L^i,\Psi)|)$ (for all $j=1,\dots,k$) and $|\mathcal{P}^{n_2}(\mu\mathcal{P}\Psi,\breve{X}\Psi)|$ in L^∞ by $\lesssim 1$ using (6-3)–(6-5) and Proposition 8.6, which yields

$$\mathrm{Error}_{N_1,\dots,N_k,n_1,n_2} \lesssim |\mathcal{P}^{\le N}(\Omega,S)|. \tag{9-9}$$

(4) If $N_j > N_{\mathrm{top}}-M_*-5$ for some[52] j, then all the terms $(1+|\mathcal{P}^{N_{j'}}(\mu,L^i,\Psi)|)$, when $j'\ne j$, and $|\mathcal{P}^{n_2}(\mu\mathcal{P}\Psi,\breve{X}\Psi)|$ can be bounded in L^∞ by $\lesssim 1$ using (6-3)–(6-5) and Proposition 8.6. Moreover, since it must also hold that $n_1 \le N_{\mathrm{top}}-M_*-5$, we also have $|\mathcal{P}^{n_1}(\Omega,S)| \lesssim \mathring{\epsilon}$ by the bootstrap assumptions (6-6) and (6-7). Hence, we have

$$
\begin{aligned}
\mathrm{Error}_{N_1,\dots,N_k,n_1,n_2} &\lesssim (1+|\mathcal{P}^{[2,N_j]}(\mu,\ L^i,\ \Psi)|)|\mathcal{P}^{\le n_1}(\Omega,S)| \\
&\lesssim |\mathcal{P}^{\le N}(\Omega,S)|+\mathring{\epsilon}|\mathcal{P}^{[2,N]}(\mu,\ L^i,\ \Psi)|.
\end{aligned}
\tag{9-10}
$$

(5) When $n_2 > N_{\mathrm{top}}-M_*-5$, we can argue as above to see that $(1+|\mathcal{P}^{N_j}(\mu,L^i,\Psi)|)\lesssim 1$ for all j, and $|\mathcal{P}^{n_1}(\Omega,S)|\lesssim\mathring{\epsilon}$. Notice further that since $n_2 > N_{\mathrm{top}}-M_*-5$, by (6-5) and Proposition 8.6 we have

$$|\mathcal{P}^{n_2}(\mu\mathcal{P}\Psi)| \lesssim |\mu\mathcal{P}^{[2,n_2+1]}\Psi|+|\mathcal{P}^{[2,n_2]}\Psi|+|\mathcal{P}^{[2,n_2]}\mu|.$$

Hence, we have

$$\mathrm{Error}_{N_1,\dots,N_k,n_1,n_2} \lesssim \mathring{\epsilon}(\mu|\mathcal{P}^{N+1}\Psi|+|\mathcal{P}^{[2,N]}\Psi|+|\mathcal{P}^{[1,N]}\breve{X}\Psi|+|\mathcal{P}^{[2,n_2]}\mu|). \tag{9-11}$$

Finally, it is easy to check that (9-9)–(9-11) are all bounded above by the right-hand side of (9-7).

Step 2: proof of (9-5). To derive (9-5), we control each term in (9-7) in the $L^2(\mathcal{M}_{t,u})$ norm.

We begin with the term I in (9-7), which we estimate using the definition of the $\mathbb{V}_{\le N}$ and $\mathbb{S}_{\le N}$ energies (see Section 3B2):

$$\|\mathcal{P}^{\le N}(\Omega,S)\|^2_{L^2(\mathcal{M}_{t,u})} \lesssim \int_{u'=0}^{u'=u}[\mathbb{V}_{\le N}+\mathbb{S}_{\le N}](t,u')\,du'. \tag{9-12}$$

[52] Note that there can be at most one such j.

We control term II in (9-7) by the $\mathbb{E}_{[1,N]}$ norm, and use the bootstrap assumptions (6-1), (6-2), the bound (8-38), and Proposition 8.11 to obtain

$$\begin{aligned}\mathring{\epsilon}^2\|\mu\mathcal{P}^{[2,N+1]}\Psi\|^2_{L^2(\mathcal{M}_{t,u})} + \mathring{\epsilon}^2\|\mathcal{P}^{[1,N]}\breve{X}\Psi\|^2_{L^2(\mathcal{M}_{t,u})} \\ &\lesssim \mathring{\epsilon}^2\int_{t'=0}^{t'=t}\big[\mathbb{E}_{[1,N]}(t') + \mathring{\epsilon}^2\max\{1, \mu_\star^{-2M_*+2N_{\text{top}}-2N+1.8}(t')\}\big]\,dt' \\ &\lesssim \mathring{\epsilon}^3\max\{1, \mu_\star^{-2M_*+2N_{\text{top}}-2N+2.8}(t)\}. \qquad (9\text{-}13)\end{aligned}$$

Finally, for the term III, we use the control for $\mathbb{K}_{[1,N-1]}$ and $\mathbb{F}_{[1,N-1]}$ provided by the bootstrap assumptions (6-1) and (6-2), the bounds in Proposition 8.14, and Proposition 8.11 to obtain

$$\begin{aligned}&\mathring{\epsilon}^2\|\mathcal{P}^{[2,N]}(\mu, L^i, \Psi)\|^2_{L^2(\mathcal{M}_{t,u})} \\ &\quad\lesssim \mathring{\epsilon}^2\mathbb{K}_{[1,N-1]}(t,u) + \mathring{\epsilon}^2\int_{u'=0}^{u'=u}\mathbb{F}_{[1,N-1]}(t,u')\,du' + \mathring{\epsilon}^3\int_{t'=0}^{t'=t}\max\{1, \mu_\star^{-2M_*+2N_{\text{top}}-2N+2.8}(t')\}\,dt' \\ &\quad\lesssim \mathring{\epsilon}^3\max\{1, \mu_\star^{-2M_*+2N_{\text{top}}-2N+3.8}(t)\}. \qquad (9\text{-}14)\end{aligned}$$

Combining (9-7) with (9-12)–(9-14), we arrive at the desired bound (9-5).

Step 3: proof of (9-6). The estimate (9-6) follows as a simple consequence of the already obtained bound (9-5) and the Cauchy–Schwarz inequality. □

9C. *Putting everything together.*

Proposition 9.4 (estimates for the specific vorticity and entropy gradient). *For* $0 \le N \le N_{\text{top}}$, *the following holds for all* $t \in [0, T_{(\text{Boot})}) \times [0, U_0]$:

$$\mathbb{V}_N(t,u) + \mathbb{S}_N(t,u) \lesssim \mathring{\epsilon}^3\max\{1, \mu_\star^{-2M_*+2N_{\text{top}}-2N+2.8}(t)\}.$$

Proof. Using Proposition 9.2 for $\phi = \Omega^i, S^i$, the initial data size assumptions in (4-5), the bootstrap assumptions (6-6)–(6-7), and the inhomogeneous term estimates in Proposition 9.3 for the terms on right-hand sides of the transport equations (5-2a) and (5-2c), we deduce

$$\mathbb{V}_{\le N}(t,u) + \mathbb{S}_{\le N}(t,u) \lesssim \mathring{\epsilon}^3\max\{1, \mu_\star^{-2M_*+2N_{\text{top}}-2N+2.8}(t)\} + \int_{u'=0}^{u'=u}(\mathbb{V}_{\le N}(t,u') + \mathbb{S}_{\le N}(t,u'))\,du'.$$

The desired estimate now follows from applying Grönwall's inequality in u. □

10. Lower-order transport estimates for the modified fluid variables

We continue to work under the assumptions of Theorem 6.3.

In this section, we derive the energy estimates for the modified fluid variables $\mathcal{C}$ and $\mathcal{D}$ *except for the top-order*. (We will derive the top-order estimates in the next section.) Thanks to Proposition 9.2, to obtain the desired estimates, it remains only for us to bound the inhomogeneous terms in the transport equations (5-3b) and (5-4a). Before we estimate the inhomogeneous terms, we will first control the $\breve{X}$ derivative of Ω and S in Section 10A, and give general bounds for null forms in Section 10B. (The null forms will also be useful later on, in Section 12.) We will combine these results to control the inhomogeneous terms in Section 10C. We provide the final estimate in Section 10D.

10A. *Preliminaries.* A priori, the norms $\mathbb{V}_N$ and $\mathbb{S}_N$ do not control the $\breve{X}$ derivatives of Ω or S. Nonetheless, we can obtain such control in terms of the norms $\mathbb{V}_N$ and $\mathbb{S}_N$ by using the transport equations (5-2a) and (5-2c).

Proposition 10.1 (L^2 control of the transversal derivatives of the Ω and S). *For* $1 \le N \le N_{\mathrm{top}}$, *the following holds for all* $(t,u) \in [0, T_{(\mathrm{Boot})}) \times [0, U_0]$:

$$\|\mathcal{P}^{N-1}\breve{X}(\Omega, S)\|^2_{L^2(\mathcal{M}_{t,u})} \lesssim \mathring{\epsilon}^3 \max\{1, \upmu_\star^{-2M_*+2N_{\mathrm{top}}-2N+2.8}(t)\}.$$

Proof. Recalling (2-23), we have

$$\mathcal{P}^{N-1}\breve{X}\Omega = \mathcal{P}^{N-1}(\upmu B\Omega) - \mathcal{P}^{N-1}(\upmu L\Omega), \quad \mathcal{P}^{N-1}\breve{X}S = \mathcal{P}^{N-1}(\upmu BS) - \mathcal{P}^{N-1}(\upmu LS). \tag{10-1}$$

The terms $\mathcal{P}^{N-1}(\upmu B\Omega)$ and $\mathcal{P}^{N-1}(\upmu BS)$ can be bounded as follows using (9-5) and Proposition 9.4:

$$\|\mathcal{P}^{N-1}(\upmu B\Omega)\|^2_{L^2(\mathcal{M}_{t,u})} + \|\mathcal{P}^{N-1}(\upmu BS)\|^2_{L^2(\mathcal{M}_{t,u})} \lesssim \mathring{\epsilon}^3 \max\{1, \upmu_\star^{-2M_*+2N_{\mathrm{top}}-2N+2.8}(t)\}. \tag{10-2}$$

By (6-6), (6-7), Propositions 8.6, 8.12, 8.14, and 9.4, we have

$$\begin{aligned}
&\|\mathcal{P}^{N-1}(\upmu L\Omega)\|^2_{L^2(\mathcal{M}_{t,u})} + \|\mathcal{P}^{N-1}(\upmu LS)\|^2_{L^2(\mathcal{M}_{t,u})} \\
&\qquad \lesssim \|\mathcal{P}^{\le N}(\Omega, S)\|^2_{L^2(\mathcal{M}_{t,u})} + \mathring{\epsilon}^2 \|\mathcal{P}^{[2,N-1]}\upmu\|^2_{L^2(\mathcal{M}_{t,u})} \\
&\qquad \lesssim \int_0^u [\mathbb{V}_{\le N} + \mathbb{S}_{\le N}](t,u')\,du' + \mathring{\epsilon}^3 \int_{t'=0}^{t'=t} \max\{1, \upmu_\star^{-2M_*+2N_{\mathrm{top}}-2N+4.8}(t')\}\,dt' \\
&\qquad \lesssim \mathring{\epsilon}^3 \max\{1, \upmu_\star^{-2M_*+2N_{\mathrm{top}}-2N+2.8}(t)\}.
\end{aligned} \tag{10-3}$$

Therefore, combining (10-1)–(10-3), we obtain the desired conclusion. ☐

10B. *General estimates for null forms.*

Lemma 10.2 (pointwise estimates for null forms). *Suppose*

(1) $\mathcal{Q}(\partial\phi^{(1)}, \partial\phi^{(2)})$ *is a g-null form, as in Definition 8.1*; *and*

(2) $\phi^{(1)}$ *and* $\phi^{(2)}$ *obey the following L^∞ estimates for some* $\eth^{(1,1)} \gtrsim \eth^{(1,2)}$, $\eth^{(2,1)} \gtrsim \eth^{(2,2)}$ *for all* $t \in [0, T_{(\mathrm{Boot})})$:

$$\begin{aligned}
\|\mathcal{P}^{\le N_{\mathrm{top}}-M_*-5}\breve{X}\phi^{(1)}\|_{L^\infty(\Sigma_t)} &\le \eth^{(1,1)}, \\
\|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-5]}\phi^{(1)}\|_{L^\infty(\Sigma_t)} &\le \eth^{(1,2)}, \\
\|\mathcal{P}^{\le N_{\mathrm{top}}-M_*-5}\breve{X}\phi^{(2)}\|_{L^\infty(\Sigma_t)} &\le \eth^{(2,1)}, \\
\|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-5]}\phi^{(2)}\|_{L^\infty(\Sigma_t)} &\le \eth^{(2,2)}.
\end{aligned} \tag{10-4}$$

Then, for any $0 \le N \le N_{\mathrm{top}}$, *the following pointwise estimate holds on* $[0, T_{(\mathrm{Boot})}) \times \Sigma$:

$$\begin{aligned}
&|\mathcal{P}^N[\upmu\mathcal{Q}(\partial\phi^{(1)}, \partial\phi^{(2)})]| \\
&\quad \lesssim \eth^{(2,1)}|\mathcal{P}^{[1,N+1]}\phi^{(1)}| + \eth^{(2,2)}|\mathcal{P}^{[1,N]}\breve{X}\phi^{(1)}| + \eth^{(1,1)}|\mathcal{P}^{[1,N+1]}\phi^{(2)}| + \eth^{(1,2)}|\mathcal{P}^{[1,N]}\breve{X}\phi^{(2)}| \\
&\qquad + \max\{\eth^{(1,1)}\eth^{(2,2)}, \eth^{(1,2)}\eth^{(2,1)}\}|\mathcal{P}^{[2,N]}(\upmu, L^i, \Psi)|,
\end{aligned} \tag{10-5}$$

and, *for any* $1 \le N \le N_{\mathrm{top}}$, *we have*

$$\begin{aligned}
&|\mathcal{P}^{[1,N]}[\upmu \mathcal{Q}(\partial\phi^{(1)}, \partial\phi^{(2)})]| \\
&\quad\lesssim \mathfrak{d}^{(2,1)}|\mathcal{P}^{[2,N+1]}\phi^{(1)}| + \mathfrak{d}^{(2,2)}|\mathcal{P}^{[1,N]}\breve{X}\phi^{(1)}| + \mathfrak{d}^{(1,1)}|\mathcal{P}^{[2,N+1]}\phi^{(2)}| + \mathfrak{d}^{(1,2)}|\mathcal{P}^{[1,N]}\breve{X}\phi^{(2)}| \\
&\qquad\qquad + \underbrace{\mathring{\epsilon}^{1/2}(\mathfrak{d}^{(2,1)}|\mathcal{P}\phi^{(1)}| + \mathfrak{d}^{(1,1)}|\mathcal{P}\phi^{(2)}|)}_{\doteq \mathscr{A}} + \underbrace{\mathfrak{d}^{(2,2)}|\mathcal{P}\phi^{(1)}| + \mathfrak{d}^{(1,2)}|\mathcal{P}\phi^{(2)}|}_{\doteq \mathscr{B}} \\
&\qquad\qquad\qquad + \max\{\mathfrak{d}^{(1,1)}\mathfrak{d}^{(2,2)},\ \mathfrak{d}^{(1,2)}\mathfrak{d}^{(2,1)}\}|\mathcal{P}^{[2,N]}(\upmu, L^i, \Psi)|. \qquad (10\text{-}6)
\end{aligned}$$

Proof. Throughout this proof, $\mathrm{f}(\cdot)$ denotes a smooth function of its arguments that is free to vary from line to line. By (8-3), we need to control

$$\underbrace{\mathcal{P}^N[\mathrm{f}(L^i, \Psi)\upmu(\mathcal{P}\phi^{(1)})(\mathcal{P}\phi^{(2)})]}_{\doteq I}, \quad \underbrace{\mathcal{P}^N[\mathrm{f}(L^i, \Psi)(\mathcal{P}\phi^{(1)})(\breve{X}\phi^{(2)})]}_{\doteq II}, \quad \underbrace{\mathcal{P}^N[\mathrm{f}(L^i, \Psi)(\breve{X}\phi^{(1)})(\mathcal{P}\phi^{(2)})]}_{\doteq III}.$$

We first prove (10-5). Consider term II. Arguing as in the proof of (9-7) and then using (10-4), we obtain

$$\begin{aligned}
&|\mathcal{P}^N[\mathrm{f}(L^i, \Psi, \upmu)(\mathcal{P}\phi^{(1)})(\breve{X}\phi^{(2)})]| \\
&\quad\lesssim |\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-5]}\phi^{(1)}||\mathcal{P}^{[1,N]}\breve{X}\phi^{(2)}| + |\mathcal{P}^{[1,N+1]}\phi^{(1)}||\mathcal{P}^{\le N_{\mathrm{top}}-M_*-5}\breve{X}\phi^{(2)}| \\
&\qquad\qquad + |\mathcal{P}^{\le N_{\mathrm{top}}-M_*-5}\phi^{(1)}||\mathcal{P}^{\le N_{\mathrm{top}}-M_*-5}\breve{X}\phi^{(2)}||\mathcal{P}^{[2,N]}(\upmu, L^i, \Psi)| \\
&\quad\lesssim \mathfrak{d}^{(1,2)}|\mathcal{P}^{[1,N]}\breve{X}\phi^{(2)}| + \mathfrak{d}^{(2,1)}|\mathcal{P}^{[1,N+1]}\phi^{(1)}| + \mathfrak{d}^{(1,2)}\mathfrak{d}^{(2,1)}|\mathcal{P}^{[2,N]}(\upmu, L^i, \Psi)|,
\end{aligned}$$

which is bounded from above by the right-hand side of (10-5).

Next, we observe that the term III can be handled just like term II, after we interchange the roles of $\phi^{(1)}$ and $\phi^{(2)}$. Moreover, the term I is even easier to handle because $\mathfrak{d}^{(1,1)} \gtrsim \mathfrak{d}^{(1,2)}$ and $\mathfrak{d}^{(2,1)} \gtrsim \mathfrak{d}^{(2,2)}$.

We finally turn to the proof of (10-6), in which we need to show an improvement compared to (10-5) using the fact that on the left-hand side of the estimate, the $\upmu$-weighted null form is differentiated by *at least one* $\mathcal{P}$. More precisely, we need to improve $\mathfrak{d}^{(1,1)}|\mathcal{P}^{[1,N+1]}\phi^{(2)}|$ and $\mathfrak{d}^{(1,2)}|\mathcal{P}^{[1,N]}\breve{X}\phi^{(2)}|$ to $\mathfrak{d}^{(1,1)}|\mathcal{P}^{[2,N+1]}\phi^{(2)}|$ and $\mathfrak{d}^{(1,2)}|\mathcal{P}^{[2,N]}\breve{X}\phi^{(2)}|$, at the expense of incurring terms of the type $\mathscr{A}$ and $\mathscr{B}$ in (10-6).

It is straightforward to use the arguments given in the previous paragraph to confirm that if $N \ge 2$, then $\mathfrak{d}^{(1,1)}|\mathcal{P}^{[1,N+1]}\phi^{(2)}|$ and $\mathfrak{d}^{(1,2)}|\mathcal{P}^{[1,N]}\breve{X}\phi^{(2)}|$ on the right-hand side of (10-5) can be replaced by $\mathfrak{d}^{(1,1)}|\mathcal{P}^{[2,N+1]}\phi^{(2)}|$ and $\mathfrak{d}^{(1,2)}|\mathcal{P}^{[2,N]}\breve{X}\phi^{(2)}|$. We are thus only concerned with the following terms in the case when $N = 1$:

$$\underbrace{[\mathcal{P}(\mathrm{f}(L^i, \Psi)\upmu)](\mathcal{P}\phi^{(1)})(\mathcal{P}\phi^{(2)})}_{\doteq I'}, \quad \underbrace{[\mathcal{P}\mathrm{f}(L^i, \Psi)](\mathcal{P}\phi^{(1)})(\breve{X}\phi^{(2)})}_{\doteq II'}, \quad \underbrace{[\mathcal{P}\mathrm{f}(L^i, \Psi)](\breve{X}\phi^{(1)})(\mathcal{P}\phi^{(2)})}_{\doteq III'}.$$

Next, we observe that for the terms II' and III', when the $\mathcal{P}$ derivative falls on $\mathrm{f}(L^i, \Psi)$, (6-5) and Proposition 8.6 yield a smallness factor of $\mathring{\epsilon}^{1/2}$. Thus, II' and III' can be bounded by $\mathscr{A}$. Finally, to handle the term I', we can control either $\mathcal{P}\phi^{(1)}$ or $\mathcal{P}\phi^{(2)}$ in L^∞, which allows us to bound I' by $\mathscr{B}$. $\square$

10C. ***Estimates of the inhomogeneous terms in the transport equations for $\mathcal{C}$ and $\mathcal{D}$.***

Proposition 10.3 (below-top-order estimates for the main inhomogeneous terms in the transport equations for the modified fluid variables). *For*[53] $0 \le N \le N_{\text{top}} - 1$, *the main terms* $\mathfrak{M} \in \{\mathfrak{M}^i_{(\mathcal{C})}, \mathfrak{M}_{(\mathcal{D})}\}$ (*see* (5-5a)–(5-5b)) *can be estimated as follows for every* $(t, u) \in [0, T_{(\text{Boot})}) \times [0, U_0]$:

$$\|\mathcal{P}^N(\mu\mathfrak{M})\|^2_{L^2(\mathcal{M}_{t,u})} \lesssim \mathring{\epsilon}^3 \max\{1, \mu_\star^{-2M_*+2N_{\text{top}}-2N+0.8}(t)\}. \tag{10-7}$$

Proof. Note that $\mathfrak{M}^i_{(\mathcal{C})}$ consists of null forms (see Definition 8.1) $\mathcal{Q}(\partial\Psi, \partial\Omega)$, $\mathcal{Q}(\partial\Psi, \partial S)$. Therefore, by Lemma 10.2 (with $\phi^{(1)} = \Omega^i, S^i$, $\phi^{(2)} = \Psi$, $\mathfrak{d}^{(1,1)} = \mathfrak{d}^{(1,2)} \doteq \mathring{\epsilon}$, $\mathfrak{d}^{(2,2)} \doteq \mathring{\epsilon}^{1/2}$, and $\mathfrak{d}^{(2,1)} = \mathcal{O}(1)$ by virtue of the bootstrap assumptions (6-3)–(6-7)),[54] we have

$$|\mathcal{P}^N(\mu\mathfrak{M}^i_{(\mathcal{C})})| \lesssim \mathring{\epsilon}^{3/2} + \underbrace{|\mathcal{P}^{\le N+1}(\Omega, S)| + |\mathcal{P}^{\le N}\breve{X}(\Omega, S)|}_{\doteq I} + \underbrace{\mathring{\epsilon}(|\mathcal{P}^{[2,N+1]}\Psi| + |\mathcal{P}^{[1,N]}\breve{X}\Psi|)}_{\doteq II} + \underbrace{\mathring{\epsilon}|\mathcal{P}^{[2,N]}(\mu, L^i, \Psi)|}_{\doteq III}. \tag{10-8}$$

We recall the expression for $\mathfrak{M}_{(\mathcal{D})}$ given by (5-5b). The term $2\exp(-2\rho)\{(\partial_a v^a)\partial_b S^b - (\partial_a v^b)\partial_b S^a\}$ is a null form of type $\mathcal{Q}(\partial\Psi, \partial S)$. Thus, using the same arguments we gave when handling $\mathfrak{M}^i_{(\mathcal{C})}$, we can pointwise bound its $\mathcal{P}^N(\mu\,\cdot)$ derivatives by the right-hand side of (10-8).

Moreover, using the same arguments given below (9-7), we see that the $\mathcal{P}^N$ derivatives of the term $\mu\exp(-\rho)\delta_{ab}(\operatorname{curl}\Omega)^a S^b$ can be pointwise bounded by the right-hand side of (10-8). From now on, it therefore suffices to consider the terms on the right-hand side of (10-8).

The term I can be controlled using Propositions 9.4 and 10.1 so that

$$\|\mathcal{P}^{\le N+1}(\Omega, S)\|^2_{L^2(\mathcal{M}_{t,u})} + \|\mathcal{P}^{\le N}\breve{X}(\Omega, S)\|^2_{L^2(\mathcal{M}_{t,u})} \lesssim \mathring{\epsilon}^3 \max\{1, \mu_\star^{-2M_*+2N_{\text{top}}-2N+0.8}(t)\}. \tag{10-9}$$

For the term II in (10-8), we use the bootstrap assumptions (6-1), (6-2), and (6-5) and the estimates of Propositions 8.12 and 8.15 to obtain

$$\begin{aligned}
&\mathring{\epsilon}^2\|\mathcal{P}^{[2,N+1]}\Psi\|^2_{L^2(\mathcal{M}_{t,u})} + \mathring{\epsilon}^2\|\mathcal{P}^{[1,N]}\breve{X}\Psi\|^2_{L^2(\mathcal{M}_{t,u})} \\
&\lesssim \mathring{\epsilon}^2\mathbb{K}_{[1,N]}(t,u) + \mathring{\epsilon}^2\int_{u'=0}^{u'=u}\mathbb{F}_{[1,N]}(t,u')\,du' + \mathring{\epsilon}^2\int_{t'=0}^{t'=t}\mathbb{E}_{[1,N]}(t',u)\,dt' + \mathring{\epsilon}^3\max\{1, \mu_\star^{-2M_*+2N_{\text{top}}-2N+1.8}(t)\} \\
&\lesssim \mathring{\epsilon}^3\max\{1, \mu_\star^{-2M_*+2N_{\text{top}}-2N+1.8}(t)\}.
\end{aligned} \tag{10-10}$$

The term III in (10-8) is the same as the term III in (9-7), and can be bounded as in the proof of Proposition 9.3, which, when combined with Proposition 9.4, implies that it is bounded by $\lesssim \mathring{\epsilon}^3\max\{1, \mu_\star^{-2M_*+2N_{\text{top}}-2N+1.8}(t)\}$.

Combining the above estimates, we conclude the desired estimate (10-7). □

[53] Note that in the case $N = N_{\text{top}}$, the error terms on the right-hand side involving $\mathbb{V}_{\le N+1}$ and $\mathbb{S}_{\le N+1}$ have *not* been estimated in Section 9A. It is for this reason that we only consider $0 \le N \le N_{\text{top}} - 1$ at this point.

[54] Note that by Lemma 10.2, there is also a term $\mathring{\epsilon}|\mathcal{P}\Psi|$, which we bound by $\lesssim \mathring{\epsilon}^{3/2}$ using (6-5).

Proposition 10.4 (L^2 control of some null forms in the modified fluid variable transport equations). *For* $0 \leq N \leq N_{\text{top}}$, *the terms* $\mathfrak{Q} \in \{\mathfrak{Q}^i_{(\mathcal{C})}, \mathfrak{Q}_{(\mathcal{D})}\}$ (*see* (5-6c)–(5-6d)) *can be estimated as follows for all* $(t, u) \in [0, T_{(\text{Boot})}) \times [0, U_0]$:

$$\|\mathcal{P}^N(\mu\mathfrak{Q})\|^2_{L^2(\mathcal{M}_{t,u})} \lesssim \mathring{\epsilon}^3 \max\{1, \mu_\star^{-2M_*+2N_{\text{top}}-2N+0.8}(t)\}. \tag{10-11}$$

Proof. The $\mathfrak{Q}$ terms can all be expressed as S multiplied by a null form $\mathcal{Q}(\partial\Psi, \partial\Psi)$. We control the null form using (10-5) with $\mathfrak{d}^{(1,1)}$, $\mathfrak{d}^{(1,2)}$, $\mathfrak{d}^{(2,1)}$, $\mathfrak{d}^{(2,2)} \lesssim 1$ (justified by (6-3)–(6-5)) so that

$$\begin{aligned}
|\mathcal{P}^N(\mu\mathfrak{Q})| &\lesssim \sum_{N_1+N_2 \leq N} |\mathcal{P}^{\leq N_1}(\varOmega, S)|(|\mathcal{P}^{[1,N_2+1]}\Psi| + |\mathcal{P}^{[1,N_2]}\breve{X}\Psi| + |\mathcal{P}^{[2,N_2]}(\mu, L^i, \Psi)|) \\
&\lesssim \underbrace{|\mathcal{P}^{\leq N}(\varOmega, S)|}_{\doteq I} + \underbrace{\mathring{\epsilon}(|\mathcal{P}^{[2,N+1]}\Psi| + |\mathcal{P}^{[1,N]}\breve{X}\Psi|)}_{\doteq II} + \underbrace{\mathring{\epsilon}|\mathcal{P}^{[2,N]}(\mu, L^i, \Psi)|}_{\doteq III},
\end{aligned} \tag{10-12}$$

where in the last line, we used the L^∞ estimates (6-6), (6-7) for $(\varOmega, S)$ if $N_1 \leq N_{\text{top}} - M_* - 5$, and otherwise, we used the L^∞ estimates (6-3)–(6-5) and Proposition 8.6 for Ψ, μ, and L^i.

Next, we observe that the terms II and III are exactly the same as II and III in (10-8) in Proposition 10.3. We can therefore argue exactly as in Proposition 10.3 to show that these terms in $\|\cdot\|^2_{L^2(\mathcal{M}_{t,u})}$ are bounded above by $\mathring{\epsilon}^3 \max\{1, \mu_\star^{-2M_*+2N_{\text{top}}-2N+1.8}(t)\}$. Notice in particular that while Proposition 10.3 was only stated for $0 \leq N \leq N_{\text{top}} - 1$, the bounds for these two terms in fact also hold (and can be proved in the same way) for $N = N_{\text{top}}$.

It thus remains to consider the term I in (10-12). Importantly, notice that term I in (10-12) is *better* than the corresponding term I in (10-8) because it has up to N, as opposed to $N+1$ derivatives. We control this term using the definition of $\mathbb{V}_{\leq N}$, $\mathbb{S}_{\leq N}$ and Proposition 9.4 as follows:

$$\|\mathcal{P}^{\leq N}(\varOmega, S)\|^2_{L^2(\mathcal{M}_{t,u})} \lesssim \int_{u'=0}^{u'=u} [\mathbb{V}_{\leq N} + \mathbb{S}_{\leq N}](t, u')\, du' \lesssim \mathring{\epsilon}^3 \max\{1, \mu_\star^{-2M_*+2N_{\text{top}}-2N+2.8}(t)\}.$$

Combining the above estimates, we conclude the proposition. □

Proposition 10.5 (L^2 control of some easy terms in the transport equation for $\mathcal{C}$). *For* $0 \leq N \leq N_{\text{top}}$, *the term* $\mathfrak{L}^i_{(\mathcal{C})}$ (*see* (5-7g)) *can be estimated as follows for all* $(t, u) \in [0, T_{(\text{Boot})}) \times [0, U_0]$:

$$\|\mathcal{P}^N(\mu\mathfrak{L}^i_{(\mathcal{C})})\|_{L^2(\mathcal{M}_{t,u})} \lesssim \mathring{\epsilon}^3 \max\{1, \mu_\star^{-2M_*+2N_{\text{top}}-2N+0.8}(t)\}.$$

Proof. We begin with the pointwise estimate

$$|\mathcal{P}^{\leq N}(\mu\mathfrak{L}^i_{(\mathcal{C})})| \lesssim \mathring{\epsilon}|\mathcal{P}^{\leq N}(\varOmega, S)| + \mathring{\epsilon}^2(|\mathcal{P}^{[2,N+1]}\Psi| + |\mathcal{P}^{[1,N]}\breve{X}\Psi|) + \mathring{\epsilon}^2|\mathcal{P}^{[2,N]}(\mu, L^i)|,$$

which can be derived by using the same arguments we used to obtain (9-7). Notice that all the above terms can be bounded above by the right-hand of (10-12). They can therefore be bounded in the norm $\|\cdot\|_{L^2(\mathcal{M}_{t,u})}$ via exactly the same arguments we used in the proof of Proposition 10.4. This yields the desired conclusion. □

10D. ***Below top-order estimates for*** $\mathbb{C}$ ***and*** $\mathbb{D}$***.***

Proposition 10.6 (below top-order estimates for the modified fluid variables). *For* $0 \leq N \leq N_{\text{top}} - 1$, *the following holds for* $(t, u) \in [0, T_{(\text{Boot})}) \times [0, U_0]$:

$$\mathbb{C}_N(t, u) + \mathbb{D}_N(t, u) \lesssim \mathring{\epsilon}^3 \max\{1, \mu_\star^{-2M_*+2N_{\text{top}}-2N+0.8}(t)\}.$$

Proof. This follows from combining Proposition 9.2 for $\phi = \mathcal{C}^i, \mathcal{D}^i$ with the initial data size assumptions in (4-6), the bootstrap assumptions (6-8), and the inhomogeneous term estimates (in Propositions 10.3–10.5) for the terms on the right-hand sides of the transport equations (5-3b) and (5-4a). □

11. Top-order transport and elliptic estimates for the specific vorticity and the entropy gradient

We continue to work under the assumptions of Theorem 6.3.

In this section, we derive top-order estimates for the modified fluid variables $\mathcal{C}$ and $\mathcal{D}$. The key difference with the lower-order estimates (which we derived in Proposition 10.6) is that we *cannot* bound the top-order derivatives of Ω and S using the $\mathbb{V}$ and $\mathbb{S}$ norms; that approach would lead to a loss of a derivative, which is not permissible at the top-order. To avoid losing a derivative, we rely on the following additional ingredient: *weighted* elliptic estimates for the specific vorticity and entropy gradient (recall Sections 1A6, 1A7).

In Section 11A, we derive top-order transport estimates. The estimates are similar to the ones we derived in Section 10, except there are some top-order inhomogeneous terms. We derive the elliptic estimates in Sections 11B and 11C. For the final estimate, see Section 11D.

In our analysis, we rely on elliptic estimates relative to the *Cartesian* spatial coordinates. In deriving these estimates, we will use the "Cartesian pointwise norms" from the following definition.

Definition 11.1. Denote by $\underline{\partial}$ the gradient with respect to the Cartesian spatial coordinates. For a scalar function f and a one-form ϕ, define respectively

$$|\underline{\partial} f|^2 \doteq \sum_{i=1}^{3} |\partial_i f|^2, \quad |\underline{\partial}\phi|^2 \doteq \sum_{i,j=1}^{3} |\partial_i \phi_j|^2.$$

11A. ***Top-order transport estimates for*** $\mathbb{C}_{N_{\text{top}}}$ ***and*** $\mathbb{D}_{N_{\text{top}}}$***.***

Proposition 11.2 (preliminary top-order L^2 estimates for the modified fluid variables). *Let* $\varsigma \in (0, 1]$. *There exists a constant* $C > 0$ *independent of* ς *and a constant* $\mathfrak{c}_\varsigma > 0$ (*depending on* ς) *such that whenever* $\mathfrak{c} \geq \mathfrak{c}_\varsigma$ *the following estimate holds for every* $(t, u) \in [0, T_{(\text{Boot})}) \times [0, U_0]$ (*with* u' *denoting the* u*-value of the integrand*):

$$\begin{aligned}\|e^{-\mathfrak{c}u'/2}\sqrt{\mu}\mathcal{P}^{N_{\text{top}}}(\mathcal{C}, \mathcal{D})\|^2_{L^2(\Sigma^u_t)} + \|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\text{top}}}(\mathcal{C}, \mathcal{D})\|^2_{L^2(\mathcal{F}^t_u)} + \frac{\mathfrak{c}}{2}\|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\text{top}}}(\mathcal{C}, \mathcal{D})\|^2_{L^2(\mathcal{M}_{t,u})} \\ \leq C\mathring{\epsilon}^3\mu_\star^{-2M_*+0.8}(t) + \varsigma\int_{t'=0}^{t'=t} \frac{1}{\mu_\star(t')}\|e^{-\mathfrak{c}u'/2}\sqrt{\mu}\underline{\partial}\mathcal{P}^{N_{\text{top}}}(\Omega, S)\|^2_{L^2(\Sigma^u_{t'})}\, dt'. \quad (11\text{-}1)\end{aligned}$$

Proof. Let $\varsigma', \mathfrak{c} > 0$ be constants to be specified later. It is crucial that all explicit constants $C > 0$ and implicit constants in this proof are *independent* of ς' and $\mathfrak{c}$. At the end of the proof, there will be a large constant C such that we will choose ς' to satisfy $\varsigma = C\varsigma'$, where $\varsigma > 0$ is the constant from the statement of the proposition.

<u>Step 1</u>: transport estimate in the weighted norms. Since $\upmu Bu = 1$ by (2-21), (2-23), we have

$$\upmu B(e^{-\mathfrak{c}u/2}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}) = -\frac{\mathfrak{c}}{2}e^{-\mathfrak{c}u/2}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C} + e^{-\mathfrak{c}u/2}\upmu B(\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}), \tag{11-2}$$

$$\upmu B(e^{-\mathfrak{c}u/2}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{D}) = -\frac{\mathfrak{c}}{2}e^{-\mathfrak{c}u/2}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{D} + e^{-\mathfrak{c}u/2}\upmu B(\mathcal{P}^{N_{\mathrm{top}}}\mathcal{D}). \tag{11-3}$$

Starting with (11-2) and (11-3), we now argue using the identity (9-1) with $\phi \doteq (\mathcal{C}^i, \mathcal{D}^i)$, except now, unlike in the proof of Proposition 9.1, we do not use Grönwall's inequality but instead take advantage of the good terms associated with the terms $-(\mathfrak{c}/2)e^{-\mathfrak{c}u/2}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}$ and $-(\mathfrak{c}/2)e^{-\mathfrak{c}u/2}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{D}$ on the right-hand sides (11-2)–(11-3). We thus obtain, for any $\varsigma' > 0$ (here, u' denotes the u-value of the integrand),

$$\begin{aligned}
&\|e^{-\mathfrak{c}u'/2}\sqrt{\upmu}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}\|^2_{L^2(\Sigma^u_t)} + \|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}\|^2_{L^2(\mathcal{F}^t_u)} + \mathfrak{c}\|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}\|^2_{L^2(\mathcal{M}_{t,u})} \\
&\qquad + \|e^{-\mathfrak{c}u'/2}\sqrt{\upmu}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{D}\|^2_{L^2(\Sigma^u_t)} + \|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{D}\|^2_{L^2(\mathcal{F}^t_u)} + \mathfrak{c}\|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{D}\|^2_{L^2(\mathcal{M}_{t,u})} \\
&\quad \lesssim \|e^{-\mathfrak{c}u'/2}\sqrt{\upmu}\mathcal{P}^{N_{\mathrm{top}}}(\mathcal{C},\mathcal{D})\|^2_{L^2(\Sigma^u_0)} + \|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\mathrm{top}}}(\mathcal{C},\mathcal{D})\|^2_{L^2(\mathcal{M}_{t,u})} \\
&\qquad\qquad + \|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\mathrm{top}}}(\mathcal{C},\mathcal{D})\|_{L^2(\mathcal{M}_{t,u})}\|e^{-\mathfrak{c}u'/2}\upmu B\mathcal{P}^{N_{\mathrm{top}}}(\mathcal{C},\mathcal{D})\|_{L^2(\mathcal{M}_{t,u})} \\
&\quad \lesssim \|e^{-\mathfrak{c}u'/2}\sqrt{\upmu}\mathcal{P}^{N_{\mathrm{top}}}(\mathcal{C},\mathcal{D})\|^2_{L^2(\Sigma^u_0)} + (1+(\varsigma')^{-1})\|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\mathrm{top}}}(\mathcal{C},\mathcal{D})\|^2_{L^2(\mathcal{M}_{t,u})} \\
&\qquad\qquad + \varsigma'\|e^{-\mathfrak{c}u'/2}\upmu B\mathcal{P}^{N_{\mathrm{top}}}(\mathcal{C},\mathcal{D})\|^2_{L^2(\mathcal{M}_{t,u})}.
\end{aligned} \tag{11-4}$$

<u>Step 2</u>: estimating the easy terms. We now consider the terms on the right-hand side of (11-4). First, the assumptions (4-6) on the initial data and the simple bound $\|\upmu\|_{L^\infty(\Sigma_0)} \lesssim 1$ from Proposition 8.6 give

$$\|e^{-\mathfrak{c}u'/2}\sqrt{\upmu}\mathcal{P}^{N_{\mathrm{top}}}(\mathcal{C},\mathcal{D})\|^2_{L^2(\Sigma^u_0)} \lesssim \mathring{\epsilon}^3. \tag{11-5}$$

Recalling the transport equations (5-3b), (5-4a), we notice that the terms $\|e^{-\mathfrak{c}u'/2}\upmu B\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}\|_{L^2(\mathcal{M}_{t,u})}$ and $\|e^{-\mathfrak{c}u'/2}\upmu B\mathcal{P}^{N_{\mathrm{top}}}\mathcal{D}\|_{L^2(\mathcal{M}_{t,u})}$ have essentially been estimated in Propositions 10.3–10.5 (using $e^{-\mathfrak{c}u'/2} \le 1$). Crucially, however, unlike in Proposition 10.3, we have not yet bounded the following terms in (10-9):

$$\|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\mathrm{top}}+1}(\Omega, S)\|^2_{L^2(\mathcal{M}_{t,u})} + \|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\mathrm{top}}}\breve{X}(\Omega, S)\|^2_{L^2(\mathcal{M}_{t,u})}$$

(since this is one more derivative than $\mathbb{V}_{N_{\mathrm{top}}}$ and $\mathbb{S}_{N_{\mathrm{top}}}$ control). In other words, simply repeating the argument in Propositions 10.3–10.5 and separating the error terms that depend on $N_{\mathrm{top}}+1$ derivatives of (Ω, S), we obtain

$$\begin{aligned}
&\|e^{-\mathfrak{c}u'/2}\upmu B\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}\|^2_{L^2(\mathcal{M}_{t,u})} + \|e^{-\mathfrak{c}u'/2}\upmu B\mathcal{P}^{N_{\mathrm{top}}}\mathcal{D}\|^2_{L^2(\mathcal{M}_{t,u})} \\
&\qquad \lesssim \mathring{\epsilon}^3\upmu_\star^{-2M_*+0.8}(t) + \|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\mathrm{top}}+1}(\Omega, S)\|^2_{L^2(\mathcal{M}_{t,u})} + \|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\mathrm{top}}}\breve{X}(\Omega, S)\|^2_{L^2(\mathcal{M}_{t,u})}.
\end{aligned} \tag{11-6}$$

<u>Step 3</u>: controlling the top-order terms. We now consider the terms on the right-hand side of (11-6). First, using the commutator estimates (8-37), Proposition 9.4, and the bootstrap assumptions (6-6)–(6-7)

to control $[\mathcal{P}^{N_{\text{top}}}, \breve{X}](\varOmega, S)$ (and using $e^{-\mathfrak{c}u'/2} \le 1$), we see that

$$\begin{aligned}
&\|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\text{top}}+1}(\varOmega, S)\|^2_{L^2(\mathcal{M}_{t,u})} + \|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\text{top}}}\breve{X}(\varOmega, S)\|^2_{L^2(\mathcal{M}_{t,u})} \\
&\quad\lesssim \mathring{\epsilon}^3 \mu_\star^{-2M_*+2.8}(t) + \|e^{-\mathfrak{c}u'/2}\breve{X}\mathcal{P}^{N_{\text{top}}}(\varOmega, S)\|^2_{L^2(\mathcal{M}_{t,u})} + \|e^{-\mathfrak{c}u'/2}L\mathcal{P}^{N_{\text{top}}}(\varOmega, S)\|^2_{L^2(\mathcal{M}_{t,u})} \\
&\qquad\qquad + \|e^{-\mathfrak{c}u'/2}Y\mathcal{P}^{N_{\text{top}}}(\varOmega, S)\|^2_{L^2(\mathcal{M}_{t,u})} + \|e^{-\mathfrak{c}u'/2}Z\mathcal{P}^{N_{\text{top}}}(\varOmega, S)\|^2_{L^2(\mathcal{M}_{t,u})} \\
&\quad\lesssim \mathring{\epsilon}^3 \mu_\star^{-2M_*+2.8}(t) + \|e^{-\mathfrak{c}u'/2}B\mathcal{P}^{N_{\text{top}}}(\varOmega, S)\|^2_{L^2(\mathcal{M}_{t,u})} + \|e^{-\mathfrak{c}u'/2}\underline{\partial}\mathcal{P}^{N_{\text{top}}}(\varOmega, S)\|^2_{L^2(\mathcal{M}_{t,u})},
\end{aligned} \tag{11-7}$$

where we have replaced $L\mathcal{P}^{N_{\text{top}}}(\varOmega, S) = B\mathcal{P}^{N_{\text{top}}}(\varOmega, S) - X\mathcal{P}^{N_{\text{top}}}(\varOmega, S)$ (by (2-23)) and also used Lemmas 2.23 and 2.24 to express (X, Y, Z) in terms of the Cartesian coordinate spatial partial derivative vectorfields, and Propositions 8.6 and 8.7 to bound the coefficients in the expressions by $\lesssim 1$. Moreover, using the commutator identity $B\mathcal{P}^{N_{\text{top}}}(\varOmega, S) = \mu^{-1}\mathcal{P}^{N_{\text{top}}}[\mu B(\varOmega, S)] + \mu^{-1}[\mu B, \mathcal{P}^{N_{\text{top}}}](\varOmega, S)$, the commutator estimates of Proposition 8.15 with $\phi \doteq (\varOmega^i, S^i)$, the bootstrap assumptions (6-6)–(6-7), Proposition 9.4, the estimate (9-5), and Proposition 8.11, we deduce (also using $e^{-\mathfrak{c}u'/2} \le 1$) that $\|e^{-\mathfrak{c}u'/2}B\mathcal{P}^{N_{\text{top}}}(\varOmega, S)\|^2_{L^2(\mathcal{M}_{t,u})} \lesssim \mathring{\epsilon}^3 \mu_\star^{-2M_*+0.8}(t)$. Combining the above results, we deduce

$$\begin{aligned}
&\|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\text{top}}+1}(\varOmega, S)\|^2_{L^2(\mathcal{M}_{t,u})} + \|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\text{top}}}\breve{X}(\varOmega, S)\|^2_{L^2(\mathcal{M}_{t,u})} \\
&\qquad\qquad \lesssim \mathring{\epsilon}^3 \mu_\star^{-2M_*+0.8}(t) + \int_{t'=0}^{t'=t} \frac{1}{\mu_\star(t')} \|e^{-\mathfrak{c}u'/2}\sqrt{\mu}\underline{\partial}\mathcal{P}^{N_{\text{top}}}(\varOmega, S)\|^2_{L^2(\Sigma^u_{t'})}\, dt'.
\end{aligned} \tag{11-8}$$

<u>Step 4</u>: putting everything together. Using (11-5), (11-6) and (11-8) to control the terms on the right-hand side of (11-4), we see that there is a $C > 0$ such that

$$\begin{aligned}
&\|e^{-\mathfrak{c}u'/2}\sqrt{\mu}\mathcal{P}^{N_{\text{top}}}\mathcal{C}\|^2_{L^2(\Sigma^u_t)} + \|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\text{top}}}\mathcal{C}\|^2_{L^2(\mathcal{F}^t_u)} + \mathfrak{c}\|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\text{top}}}\mathcal{C}\|^2_{L^2(\mathcal{M}_{t,u})} \\
&\qquad + \|e^{-\mathfrak{c}u'/2}\sqrt{\mu}\mathcal{P}^{N_{\text{top}}}\mathcal{D}\|^2_{L^2(\Sigma^u_t)} + \|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\text{top}}}\mathcal{D}\|^2_{L^2(\mathcal{F}^t_u)} + \mathfrak{c}\|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\text{top}}}\mathcal{D}\|^2_{L^2(\mathcal{M}_{t,u})} \\
&\quad \le C(1+\varsigma')\mathring{\epsilon}^3 \mu_\star^{-2M_*+0.8}(t) + C(1+(\varsigma')^{-1})\|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\text{top}}}(\mathcal{C}, \mathcal{D})\|^2_{L^2(\mathcal{M}_{t,u})} \\
&\qquad\qquad + C\varsigma' \int_{t'=0}^{t'=t} \frac{1}{\mu_\star(t')} \|e^{-\mathfrak{c}u'/2}\sqrt{\mu}\underline{\partial}\mathcal{P}^{N_{\text{top}}}(\varOmega, S)\|^2_{L^2(\Sigma^u_{t'})}\, dt'.
\end{aligned} \tag{11-9}$$

Finally, relabeling the coefficients $C\varsigma'$ on the right-hand side of (11-9) by setting $\varsigma \doteq C\varsigma'$, bounding the data term $C(1+\varsigma')\mathring{\epsilon}^3 \mu_\star^{-2M_*+0.8}(t)$ by a new constant C times $\mathring{\epsilon}^3 \mu_\star^{-2M_*+0.8}(t)$ via the assumption $\varsigma \in (0, 1]$, taking $\mathfrak{c}_\varsigma$ sufficiently large (depending on ς) so that

$$C(1+(\varsigma')^{-1})\|e^{-\mathfrak{c}u'/2}\mathcal{P}^{N_{\text{top}}}(\mathcal{C}, \mathcal{D})\|^2_{L^2(\mathcal{M}_{t,u})} \le \frac{\mathfrak{c}_\varsigma}{2} \int_{\mathcal{M}_{t,u}} e^{-\mathfrak{c}u'}[|\mathcal{P}^{N_{\text{top}}}\mathcal{C}|^2 + |\mathcal{P}^{N_{\text{top}}}\mathcal{D}|^2]\, d\varpi,$$

now allowing $\mathfrak{c}$ to be any constant such that $\mathfrak{c} \ge \mathfrak{c}_\varsigma$, and subtracting $(\mathfrak{c}/2)\int_{\mathcal{M}_{t,u}} e^{-\mathfrak{c}u'}[|\mathcal{P}^{N_{\text{top}}}\mathcal{C}|^2 + |\mathcal{P}^{N_{\text{top}}}\mathcal{D}|^2]\, d\varpi$ from both sides of (11-9), we obtain the desired inequality (11-1). □

11B. *General elliptic estimates on* $\mathbb{R}\times\mathbb{T}^2$. We begin with a standard weighted *Euclidean* elliptic estimate on $\mathbb{R}\times\mathbb{T}^2$ in Proposition 11.3. We then apply this in our geometric setting for general one-forms in Proposition 11.4.

Proposition 11.3 (weighted Euclidean elliptic estimates). *Let $w : \mathbb{R}\times\mathbb{T}^2 \to \mathbb{R}_{>0}$ be a smooth, strictly positive, bounded weight function.*

The following inequality holds for all one-forms $\phi = \phi_a dx^a \in C_c^2(\mathbb{R}\times\mathbb{T}^2)$:

$$\begin{aligned}&\|\sqrt{w}\underline{\partial}\phi\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)}\\ &\qquad\le 4\|\sqrt{w}\,\mathrm{curl}\,\phi\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)} + 4\|\sqrt{w}\,\mathrm{div}\,\phi\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)} + 3\|\underline{\partial}\log w\|^2_{L^\infty(\mathbb{R}\times\mathbb{T}^2)}\|\sqrt{w}\phi\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)},\end{aligned}$$

where $\underline{\partial}$ is as in Definition 11.1, $\|\xi\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)} \doteq \int_{\mathbb{R}\times\mathbb{T}^2}|\xi|_e^2\,dx$ for tensorfields ξ, $|\xi|_e$ denotes the standard Euclidean pointwise norm of ξ, and $dx = dx^1\,dx^2\,dx^3$.

Proof. Integrating by parts and using Hölder's inequality, we find that

$$\begin{aligned}
&\|\sqrt{w}\underline{\partial}\phi\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)}\\
&= \sum_{i,j=1}^3 \int_{\mathbb{R}\times\mathbb{T}^2} w(\partial_i\phi_j)^2\,dx\\
&= -\sum_{i,j=1}^3 \left\{\int_{\mathbb{R}\times\mathbb{T}^2} w\phi_j(\partial^2_{ii}\phi_j)\,dx + \int_{\mathbb{R}\times\mathbb{T}^2}(\partial_i w)\phi_j(\partial_i\phi_j)\,dx\right\}\\
&= -\sum_{i,j=1}^3 \int_{\mathbb{R}\times\mathbb{T}^2} w\phi_j\partial^2_{ij}\phi_i\,dx + \sum_{i,j=1}^3\int_{\mathbb{R}\times\mathbb{T}^2} w\phi_j\partial_i(\partial_j\phi_i - \partial_i\phi_j)\,dx - \sum_{i,j=1}^3\int_{\mathbb{R}\times\mathbb{T}^2}(\partial_i w)\phi_j(\partial_i\phi_j)\,dx\\
&= \sum_{i,j=1}^3 \int_{\mathbb{R}\times\mathbb{T}^2} w(\partial_j\phi_j)(\partial_i\phi_i)\,dx - \sum_{i,j=1}^3\int_{\mathbb{R}\times\mathbb{T}^2} w(\partial_i\phi_j)(\partial_j\phi_i - \partial_i\phi_j)\,dx\\
&\quad + \sum_{i,j=1}^3 \int_{\mathbb{R}\times\mathbb{T}^2}(\partial_j w)\phi_j(\partial_i\phi_i)\,dx - \sum_{i,j=1}^3\int_{\mathbb{R}\times\mathbb{T}^2}(\partial_i w)\phi_j(\partial_j\phi_i - \partial_i\phi_j)\,dx - \sum_{i,j=1}^3\int_{\mathbb{R}\times\mathbb{T}^2}(\partial_i w)\phi_j(\partial_i\phi_j)\,dx\\
&\le \|\sqrt{w}\ \mathrm{div}\ \phi\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)} + \|\sqrt{w}\,\underline{\partial}\phi\|_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)}\|\sqrt{w}\,\mathrm{curl}\ \phi\|_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)}\\
&\quad + \|\underline{\partial}\log w\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)}\|\sqrt{w}\phi\|_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)}\{\|\sqrt{w}\ \mathrm{div}\ \phi\|_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)} + \|\sqrt{w}\,\mathrm{curl}\ \phi\|_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)}\}\\
&\qquad + \|\underline{\partial}\log w\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)}\|\sqrt{w}\phi\|_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)}\|\sqrt{w}\,\underline{\partial}\phi\|_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)}.
\end{aligned} \tag{11-10}$$

Using $|ab| \le a^2/4 + b^2$, we find that

$$\begin{aligned}\|\sqrt{w}\underline{\partial}\phi\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)} &\le \tfrac{1}{2}\|\sqrt{w}\,\underline{\partial}\phi\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)} + 2\|\sqrt{w}\,\mathrm{div}\,\phi\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)}\\ &\quad + 2\|\sqrt{w}\,\mathrm{curl}\,\phi\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)} + \tfrac{3}{2}\|\underline{\partial}\log w\|^2_{L^\infty(\mathbb{R}\times\mathbb{T}^2)}\|\sqrt{w}\phi\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)}.\end{aligned} \tag{11-11}$$

The conclusion of the lemma follows from subtracting $\frac{1}{2}\|\sqrt{w}\,\underline{\partial}\phi\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2,dx)}$ from both sides of (11-11). □

Proposition 11.4 (Euclidean elliptic estimates with u-weights). *Let $\phi = \phi_a dx^a$ be a smooth compactly supported one-form on Σ_t. Then for each $\mathfrak{c} > 0$ and each $t \in [0, T_{(\mathrm{Boot})})$, the following elliptic estimate holds, where the implicit constants are independent of $\mathfrak{c}$:*

$$\|e^{-\mathfrak{c}u/2}\sqrt{\mu}\,\underline{\partial}\phi\|_{L^2(\Sigma_t)} \lesssim \|e^{-\mathfrak{c}u/2}\sqrt{\mu}\,\mathrm{div}\,\phi\|_{L^2(\Sigma_t)} + \|e^{-\mathfrak{c}u/2}\sqrt{\mu}\,\mathrm{curl}\,\phi\|_{L^2(\Sigma_t)} + \mathfrak{c}\mu_\star^{-1}(t)\|e^{-\mathfrak{c}u/2}\sqrt{\mu}\phi\|_{L^2(\Sigma_t)}.$$

Proof. In this proof, the implicit constants in $\lesssim$ are *independent* of $\mathfrak{c}$.

We apply Proposition 11.3 with $w \doteq e^{-\mathfrak{c}u}$. By Lemma 2.22, (2-21), and Proposition 8.7, we have $\|\underline{\partial} \log w\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)} \lesssim \mathfrak{c}\mu_\star^{-1}(t)$. Hence,

$$\|e^{-\mathfrak{c}u/2}\underline{\partial}\phi\|_{L^2(\Sigma_t,dx)} \lesssim \|e^{-\mathfrak{c}u/2} \operatorname{div} \phi\|_{L^2(\Sigma_t,dx)} + \|e^{-\mathfrak{c}u/2} \operatorname{curl} \phi\|_{L^2(\Sigma_t,dx)} + \mathfrak{c}\mu_\star^{-1}(t)\|e^{-\mathfrak{c}u/2} \phi\|_{L^2(\Sigma_t,dx)}.$$

The conclusion thus follows from the fact that the volume measures $\mu\, dx$ and $d\underline{\varpi}$ are comparable, which in turn follows from (3-1) and Proposition 8.7. □

11C. ***Top-order elliptic estimates for Ω and S.*** In this section, we derive top-order elliptic estimates for Ω and S.

There are four main steps. Ultimately, our goal is to exploit the preliminary energy inequality for $(\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}, \mathcal{P}^{N_{\mathrm{top}}}\mathcal{D})$ that we derived in Proposition 11.2, and to do this, we have to control the integrand term $\|e^{-\mathfrak{c}u'/2}\sqrt{\mu}\underline{\partial}\mathcal{P}^{N_{\mathrm{top}}}(\Omega, S)\|^2_{L^2(\Sigma^u_{t'})}$ on the right-hand side of (11-1) with the help of elliptic estimates. To achieve this, we first commute the top-order operators $\mathcal{P}^{N_{\mathrm{top}}}$ through the Euclidean operators div and curl. To avoid uncontrollable commutator terms, we introduce a μ weight into the commutators. In the second step, we have to control $(\operatorname{div}\mathcal{P}^{N_{\mathrm{top}}}\Omega, \operatorname{div}\mathcal{P}^{N_{\mathrm{top}}}S)$ and $(\operatorname{curl}\mathcal{P}^{N_{\mathrm{top}}}\Omega, \operatorname{curl}\mathcal{P}^{N_{\mathrm{top}}}S)$ in terms of the modified fluid variables $(\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}, \mathcal{P}^{N_{\mathrm{top}}}\mathcal{D})$ from (2-5a)–(2-5b) plus simpler error terms. The first and second steps are carried out in Lemmas 11.6–11.9.

Next, in Proposition 11.10, we use the weighted elliptic estimates on Σ_t provided by Proposition 11.4 and the results of the first two steps to obtain

$$\|e^{-\mathfrak{c}u/2}\sqrt{\mu}(\underline{\partial}\mathcal{P}^{N_{\mathrm{top}}}\Omega, \underline{\partial}\mathcal{P}^{N_{\mathrm{top}}}S)\|^2_{L^2(\Sigma_t)} \lesssim \|e^{-\mathfrak{c}u/2}\sqrt{\mu}(\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}, \mathcal{P}^{N_{\mathrm{top}}}\mathcal{D})\|^2_{L^2(\Sigma_t)} + \cdots,$$

where "$\cdots$" denotes simpler error terms for which we already have an independent bound. Finally, in Proposition 11.11, we combine all of these results to obtain our main L^2 estimate[55] for $(\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}, \mathcal{P}^{N_{\mathrm{top}}}\mathcal{D})$.

11C1. *Controlling* $\operatorname{curl}\mathcal{P}^{N_{\mathrm{top}}}\Omega$ *and* $\operatorname{div}\mathcal{P}^{N_{\mathrm{top}}}\Omega$. We start with a simple commutation lemma.

Lemma 11.5 (commuting geometric vectorfields with μ-weighted Cartesian vectorfields). *Let ϕ be a smooth function such that*

$$\|\mathcal{P}^{\le N_{\mathrm{top}}-M_*-5}\phi\|_{L^\infty(\Sigma_t)} \le \mathring{\varepsilon}, \quad \|\mathcal{P}^{\le N_{\mathrm{top}}-M_*-5}\breve{X}\phi\|_{L^\infty(\Sigma_t)} \le \mathring{\varepsilon}$$

for all $t \in [0, T_{(\mathrm{Boot})})$.

Then, for $0 \le N \le N_{\mathrm{top}}$, the following holds in $\mathcal{M}_{T_{(\mathrm{Boot})},U_0}$:

$$|[\mu\partial_i, \mathcal{P}^N]\phi| \lesssim |\mathcal{P}^{[1,N]}\phi| + |\mathcal{P}^{\le N-1}\breve{X}\phi| + \mathring{\varepsilon}(|\mathcal{P}^{[2,N]}(\mu, L^i, \Psi)| + |\mathcal{P}^{[2,N-1]}\breve{X}\Psi|).$$

Proof. We first use Lemma 2.22 to express $\mu\partial_i$ in terms of the geometric vectorfields and then argue as in Proposition 8.8. □

[55] We clarify that although the estimate for $(\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}, \mathcal{P}^{N_{\mathrm{top}}}\mathcal{D})$ and the aforementioned estimates $\|e^{-\mathfrak{c}u/2}\sqrt{\mu}(\underline{\partial}\mathcal{P}^{N_{\mathrm{top}}}\Omega, \underline{\partial}\mathcal{P}^{N_{\mathrm{top}}}S)\|^2_{L^2(\Sigma_t)} \lesssim \|e^{-\mathfrak{c}u/2}\sqrt{\mu}(\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}, \mathcal{P}^{N_{\mathrm{top}}}\mathcal{D})\|^2_{L^2(\Sigma_t)} + \cdots$ together imply a top-order L^2 estimate for $(\underline{\partial}\mathcal{P}^{N_{\mathrm{top}}}\Omega, \underline{\partial}\mathcal{P}^{N_{\mathrm{top}}}S)$, we do not explicitly state such an estimate in the paper because we do not need it for our main results.

Lemma 11.6 (L^2 estimates for the Euclidean curl of the derivatives of $\varOmega$ in terms of the derivatives of $\mathcal{C}$). *Let $\mathfrak{c} \geq 0$ be a real number. The following estimate holds for all $t \in [0, T_{(\text{Boot})})$, where the implicit constants are independent of $\mathfrak{c}$*:

$$\|e^{-\mathfrak{c}u'/2}\sqrt{\mu}\ \text{curl}\ \mathcal{P}^{N_{\text{top}}}\varOmega\|^2_{L^2(\Sigma_t)} \lesssim \mathring{\epsilon}^3\mu_\star^{-2M_*+0.8}(t) + \|e^{-\mathfrak{c}u'/2}\sqrt{\mu}\,\mathcal{P}^{N_{\text{top}}}\mathcal{C}\|^2_{L^2(\Sigma_t)}.$$

Proof. We first compute the commutator $[\mu\,\text{curl}, \mathcal{P}^{N_{\text{top}}}]$ using Lemma 11.5 and the bootstrap assumption (6-6):

$$|[\mu\ \text{curl}, \mathcal{P}^{N_{\text{top}}}]\varOmega| \lesssim |\mathcal{P}^{\leq N_{\text{top}}}\varOmega| + |\mathcal{P}^{\leq N_{\text{top}}-1}\breve{X}\varOmega| + \mathring{\epsilon}(|\mathcal{P}^{[2,N_{\text{top}}]}(\mu, L^i, \Psi)| + |\mathcal{P}^{[2,N_{\text{top}}-1]}\breve{X}\Psi|). \quad (11\text{-}12)$$

On the other hand, by (2-5a), Lemma 2.22, the bootstrap assumptions (6-3)–(6-8), and Propositions 8.6 and 8.7, we have

$$\begin{aligned}
|\mathcal{P}^{N_{\text{top}}}(\mu\ \text{curl}\ \varOmega)| &= \left|\mathcal{P}^{N_{\text{top}}}\left\{\mu\left[\exp(\rho)\mathcal{C} - \exp(-2\rho)c_s^{-2}\frac{p_{;s}}{\bar{\varrho}}S^a\partial_a v + \exp(-2\rho)c_s^{-2}\frac{p_{;s}}{\bar{\varrho}}(\partial_a v^a)S\right]\right\}\right| \\
&\lesssim \mu|\mathcal{P}^{N_{\text{top}}}\mathcal{C}| + |\mathcal{P}^{\leq N_{\text{top}}-1}\mathcal{C}| + |\mathcal{P}^{\leq N_{\text{top}}}S| \\
&\quad + \mathring{\epsilon}\big(|\mathcal{P}^{[2,N_{\text{top}}]}(\mu, L^i)| + \mu|\mathcal{P}^{N_{\text{top}}+1}\Psi| + |\mathcal{P}^{[2,N_{\text{top}}]}\Psi| + |\mathcal{P}^{[1,N_{\text{top}}]}\breve{X}\Psi|\big). \quad (11\text{-}13)
\end{aligned}$$

We stress that on the right-hand side of (11-13), it is important that the top-order terms $\mathcal{P}^{N_{\text{top}}}\mathcal{C}$ and $\mathcal{P}^{N_{\text{top}}+1}\Psi$ are accompanied by a factor of μ.

We can therefore use (11-12) and (11-13) (to write $\mu\,\text{curl}\,\mathcal{P}^{N_{\text{top}}}\varOmega = [\mu\ \text{curl}, \mathcal{P}^{N_{\text{top}}}]\varOmega + \mathcal{P}^{N_{\text{top}}}(\mu\ \text{curl}\ \varOmega)$), multiply by $e^{-\mathfrak{c}u/2}\mu^{-1/2}$, take the $L^2(\Sigma_t)$ norm, and then use $e^{-\mathfrak{c}u/2} \leq 1$ to obtain

$$\begin{aligned}
&\|e^{-\mathfrak{c}u'/2}\sqrt{\mu}\,\text{curl}(\mathcal{P}^{N_{\text{top}}}\varOmega)\|_{L^2(\Sigma_t)} \\
&\quad\lesssim \|e^{-\mathfrak{c}u'/2}\sqrt{\mu}\mathcal{P}^{N_{\text{top}}}\mathcal{C}\|_{L^2(\Sigma_t)} + \mu_\star^{-1}(t)\|\sqrt{\mu}\mathcal{P}^{\leq N_{\text{top}}-1}\mathcal{C}\|_{L^2(\Sigma_t)} + \mu_\star^{-1}(t)\|\sqrt{\mu}\mathcal{P}^{\leq N_{\text{top}}}(\varOmega, S)\|_{L^2(\Sigma_t)} \\
&\qquad + \mu_\star^{-1}(t)\|\sqrt{\mu}\mathcal{P}^{\leq N_{\text{top}}-1}\breve{X}\varOmega\|_{L^2(\Sigma_t)} + \mathring{\epsilon}\mu_\star^{-1/2}(t)\|\mathcal{P}^{[2,N_{\text{top}}]}(\mu, L^i)\|_{L^2(\Sigma_t)} \\
&\qquad\quad + \mathring{\epsilon}(\|\sqrt{\mu}\mathcal{P}^{N_{\text{top}}+1}\Psi\|_{L^2(\Sigma_t)} + \mu_\star^{-1/2}(t)\|\mathcal{P}^{[1,N_{\text{top}}]}\breve{X}\Psi\|_{L^2(\Sigma_t)} + \mu_\star^{-1}(t)\|\sqrt{\mu}\mathcal{P}^{[2,N_{\text{top}}]}\Psi\|_{L^2(\Sigma_t)}) \\
&\quad\lesssim \|e^{-\mathfrak{c}u'/2}\sqrt{\mu}\mathcal{P}^{N_{\text{top}}}\mathcal{C}\|_{L^2(\Sigma_t)} + \mathring{\epsilon}^{3/2}\mu_\star^{-M_*+0.4}(t),
\end{aligned} \quad (11\text{-}14)$$

where we have used Proposition 10.6 to bound $\mu_\star^{-1}(t)\|\sqrt{\mu}\mathcal{P}^{\leq N_{\text{top}}-1}\mathcal{C}\|_{L^2(\Sigma_t)}$, Proposition 9.4 to bound $\mu_\star^{-1}(t)\|\sqrt{\mu}\mathcal{P}^{\leq N_{\text{top}}}(\varOmega, S)\|_{L^2(\Sigma_t)}$, Proposition 10.1 to bound $\mu_\star^{-1}(t)\|\sqrt{\mu}\mathcal{P}^{\leq N_{\text{top}}-1}\breve{X}\varOmega\|_{L^2(\Sigma_t)}$, Proposition 8.14 to bound $\mathring{\epsilon}\mu_\star^{-1/2}(t)\|\mathcal{P}^{[2,N_{\text{top}}]}(\mu, L^i)\|_{L^2(\Sigma_t)}$, and the bootstrap assumptions (6-1), (6-2), and (8-38) to estimate all the remaining terms. (We remark that the worst terms are $\mu_\star^{-1}(t)\|\sqrt{\mu}\mathcal{P}^{\leq N_{\text{top}}-1}\mathcal{C}\|_{L^2(\Sigma_t)}$, $\mu_\star^{-1}(t)\|\sqrt{\mu}\mathcal{P}^{\leq N_{\text{top}}}(\varOmega, S)\|_{L^2(\Sigma_t)}$, $\mu_\star^{-1}(t)\|\sqrt{\mu}\mathcal{P}^{\leq N_{\text{top}}-1}\breve{X}\varOmega\|_{L^2(\Sigma_t)}$, and $\mu_\star^{-1/2}(t)\|\mathcal{P}^{[1,N_{\text{top}}]}\breve{X}\Psi\|_{L^2(\Sigma_t)}$, which determine the blowup-exponent $-M_* + 0.4$ for $\mu_\star$ on the right-hand side of (11-14)). Squaring (11-14), we arrive at the desired result. □

Lemma 11.7 (L^2 estimates for the Euclidean divergence of the derivatives of $\varOmega$). *Let $\mathfrak{c} \geq 0$ be a real number. The following estimate holds for all $t \in [0, T_{(\text{Boot})})$, where the implicit constant is independent of $\mathfrak{c}$*:

$$\|e^{-\mathfrak{c}u'/2}\sqrt{\mu}\,\text{div}\ \mathcal{P}^{N_{\text{top}}}\varOmega\|^2_{L^2(\Sigma_t)} \lesssim \mathring{\epsilon}^3\mu_\star^{-2M_*+0.8}(t).$$

Proof. The commutator $[\upmu\,\mathrm{div}, \mathcal{P}^{N_{\mathrm{top}}}]\varOmega$ can be computed exactly as (11-12). Thus, we have

$$|[\upmu\,\mathrm{div}, \mathcal{P}^{N_{\mathrm{top}}}]\varOmega| \lesssim \text{the right-hand side of (11-12).} \tag{11-15}$$

We also use Lemma 2.22, the fact that the Cartesian component functions X^1, X^2, X^3 are smooth functions of the L^i and Ψ (see (2-23)), (5-3a), and the L^∞ bounds in (6-3)–(6-6) and Proposition 8.6 to deduce

$$\begin{aligned}|\mathcal{P}^{N_{\mathrm{top}}}(\upmu\ \mathrm{div}\ \varOmega)| &= |\mathcal{P}^{N_{\mathrm{top}}}(\upmu\,\varOmega^a\partial_a\uprho)| \\ &\lesssim |\mathcal{P}^{\leq N_{\mathrm{top}}}\varOmega| + \mathring{\upepsilon}(|\mathcal{P}^{[2,N_{\mathrm{top}}]}\upmu| + \upmu|\mathcal{P}^{N_{\mathrm{top}}+1}\Psi| + |\mathcal{P}^{[2,N_{\mathrm{top}}]}\Psi| + |\mathcal{P}^{[1,N_{\mathrm{top}}]}\breve{X}\Psi|).\end{aligned} \tag{11-16}$$

Notice that every term on the right-hand side of (11-16) has already appeared on the right-hand sides of (11-12) and (11-13). Hence, with the help of the simple identity

$$\upmu\,\mathrm{div}\,\mathcal{P}^{N_{\mathrm{top}}}\varOmega = \mathcal{P}^{N_{\mathrm{top}}}(\upmu\,\mathrm{div}\,\varOmega) + [\upmu\,\mathrm{div}, \mathcal{P}^{N_{\mathrm{top}}}]\varOmega$$

and the estimates obtained above, we can argue exactly as in Lemma 11.6 to obtain the same estimate. (Note that here there are no $\mathcal{C}$ terms and so we do not have the term $\|e^{-\mathfrak{c}u'/2}\sqrt{\upmu}\,\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}\|^2_{L^2(\Sigma_t)}$.) □

11C2. *Controlling* $\mathrm{curl}\,\mathcal{P}^{N_{\mathrm{top}}}S$ *and* $\mathrm{div}\,\mathcal{P}^{N_{\mathrm{top}}}S$.

Lemma 11.8 (L^2 estimates for the Euclidean curl of the derivatives of S). *Let* $\mathfrak{c} \geq 0$ *be a real number. The following estimate holds for all* $t \in [0, T_{(\mathrm{Boot})})$, *where the implicit constant is independent of* $\mathfrak{c}$:

$$\|e^{-\mathfrak{c}u'/2}\sqrt{\upmu}\,\mathrm{curl}\,\mathcal{P}^{N_{\mathrm{top}}}S\|^2_{L^2(\Sigma_t)} \lesssim \mathring{\upepsilon}^3\upmu_\star^{-2M_*+0.8}(t).$$

Proof. By (5-4b), $\mathrm{curl}\,S = 0$. Hence, using Lemma 11.5 and the bootstrap assumption (6-7),

$$\begin{aligned}|\upmu\,\mathrm{curl}\,\mathcal{P}^{N_{\mathrm{top}}}S| &= |[\upmu\ \mathrm{curl}, \mathcal{P}^{N_{\mathrm{top}}}]S| \\ &\lesssim |\mathcal{P}^{\leq N_{\mathrm{top}}}S| + |\mathcal{P}^{\leq N_{\mathrm{top}}-1}\breve{X}S| + \mathring{\upepsilon}(|\mathcal{P}^{[2,N_{\mathrm{top}}]}(\upmu, L^i, \Psi)| + |\mathcal{P}^{[2,N_{\mathrm{top}}-1]}\breve{X}\Psi|).\end{aligned} \tag{11-17}$$

The only new terms here compared to (11-12) and (11-13) are $|\mathcal{P}^{\leq N_{\mathrm{top}}}S|$ and $|\mathcal{P}^{\leq N_{\mathrm{top}}-1}\breve{X}S|$, which can be handled using Propositions 9.4 and 10.1 in the same way that we handled the corresponding terms $\upmu_\star^{-1}\|\sqrt{\upmu}\mathcal{P}^{\leq N_{\mathrm{top}}}\varOmega\|_{L^2(\Sigma_t)}$ and $\upmu_\star^{-1}\|\sqrt{\upmu}\mathcal{P}^{\leq N_{\mathrm{top}}-1}\breve{X}\varOmega\|_{L^2(\Sigma_t)}$ in the proof of Lemma 11.6. □

Lemma 11.9 (L^2 estimates for the Euclidean divergence of the derivatives of S in terms of the derivatives of $\mathcal{D}$). *Let* $\mathfrak{c} \geq 0$ *be a real number. The following estimate holds for all* $t \in [0, T_{(\mathrm{Boot})})$, *where the implicit constants are independent of* $\mathfrak{c}$:

$$\|e^{-\mathfrak{c}u'/2}\sqrt{\upmu}\,\mathrm{div}\,\mathcal{P}^{N_{\mathrm{top}}}S\|^2_{L^2(\Sigma_t)} \lesssim \mathring{\upepsilon}^3\upmu_\star^{-2M_*+0.8} + \|e^{-\mathfrak{c}u'/2}\sqrt{\upmu}\,\mathcal{P}^{N_{\mathrm{top}}}\mathcal{D}\|^2_{L^2(\Sigma_t)}.$$

Proof. Using Lemma 11.5 and the bootstrap assumption (6-7), we find that

$$|[\upmu\,\mathrm{div}, \mathcal{P}^{N_{\mathrm{top}}}]S| \lesssim \text{the right-hand side of (11-17).}$$

Therefore, we can therefore handle $|[\upmu\ \mathrm{div}, \mathcal{P}^{N_{\mathrm{top}}}]S|$ by using the same arguments we gave in the proof of Lemma 11.8.

We then express $\operatorname{div} S$ in terms of $\mathcal{D}$ using (2-5b) and use Lemma 2.22, the fact that the Cartesian component functions $X^1,\ X^2,\ X^3$ are smooth functions of the L^i and Ψ (see (2-23)), and the L^∞ bounds in (6-3)–(6-5), (6-7), (6-8), and Proposition 8.6 to deduce

$$\begin{aligned}|\mathcal{P}^{N_{\mathrm{top}}}(\mu \operatorname{div} S)| &\le |\mathcal{P}^{N_{\mathrm{top}}}(\mu\exp(2\rho)\mathcal{D})| + |\mathcal{P}^{N_{\mathrm{top}}}(\mu\exp(2\rho)S^a\partial_a\rho)| \\ &\lesssim \mu|\mathcal{P}^{N_{\mathrm{top}}}\mathcal{D}| + |\mathcal{P}^{\le N_{\mathrm{top}}-1}\mathcal{D}| + |\mathcal{P}^{\le N_{\mathrm{top}}}S| \\ &\qquad + \mathring{\epsilon}(|\mathcal{P}^{[2,N_{\mathrm{top}}]}(\mu, L^i)| + \mu|\mathcal{P}^{N_{\mathrm{top}}+1}\Psi| + |\mathcal{P}^{[2,N_{\mathrm{top}}]}\Psi| + |\mathcal{P}^{[1,N_{\mathrm{top}}]}\breve{X}\Psi|).\end{aligned}$$

The new terms here compared to (11-12) and (11-13) are $|\mathcal{P}^{\le N_{\mathrm{top}}}S|$, which we handled just below (11-17), and $\mu|\mathcal{P}^{N_{\mathrm{top}}}\mathcal{D}|$ and $|\mathcal{P}^{\le N_{\mathrm{top}}-1}\mathcal{D}|$, which can be treated using the same arguments we used to handle the terms $\mu|\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}|$ and $|\mathcal{P}^{\le N_{\mathrm{top}}-1}\mathcal{C}|$ in our proof of Lemma 11.6. Hence, the weighted, squared $L^2(\Sigma_t)$ norms corresponding to these new terms are bounded above by $\mathring{\epsilon}^3\mu_\star^{-2M_*+0.8} + \|e^{-\mathfrak{c}u'/2}\sqrt{\mu}\,\mathcal{P}^{N_{\mathrm{top}}}\mathcal{D}\|^2_{L^2(\Sigma_t)}$. □

11C3. *Proving the elliptic estimates.* We now combine Lemmas 11.6–11.9 and the elliptic estimates in Proposition 11.4 to obtain the following proposition.

Proposition 11.10 (preliminary top-order elliptic estimates for Ω and S). *Let* $\mathfrak{c} \ge 0$ *be a real number. The following estimates hold for all* $t \in [0, T_{(\mathrm{Boot})})$, *where the implicit constants are independent of* $\mathfrak{c}$:

$$\|e^{-\mathfrak{c}u/2}\sqrt{\mu}\underline{\partial}\mathcal{P}^{N_{\mathrm{top}}}\Omega\|^2_{L^2(\Sigma_t)} \lesssim \mathring{\epsilon}^3(1+\mathfrak{c}^2)\mu_\star^{-2M_*+0.8}(t) + \|e^{-\mathfrak{c}u/2}\sqrt{\mu}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}\|^2_{L^2(\Sigma_t)}, \tag{11-18}$$

$$\|e^{-\mathfrak{c}u/2}\sqrt{\mu}\underline{\partial}\mathcal{P}^{N_{\mathrm{top}}}S\|^2_{L^2(\Sigma_t)} \lesssim \mathring{\epsilon}^3(1+\mathfrak{c}^2)\mu_\star^{-2M_*+0.8}(t) + \|e^{-\mathfrak{c}u/2}\sqrt{\mu}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{D}\|^2_{L^2(\Sigma_t)}. \tag{11-19}$$

Proof. Applying first Proposition 11.4, and then Lemmas 11.6, 11.7, Proposition 9.4 (and using $e^{-\mathfrak{c}u/2} \le 1$), we obtain

$$\begin{aligned}&\|e^{-\mathfrak{c}u/2}\sqrt{\mu}\underline{\partial}\mathcal{P}^{N_{\mathrm{top}}}\Omega\|^2_{L^2(\Sigma_t)} \\ &\quad\lesssim \|e^{-\mathfrak{c}u/2}\sqrt{\mu}\operatorname{div}\mathcal{P}^{N_{\mathrm{top}}}\Omega\|^2_{L^2(\Sigma_t)} + \|e^{-\mathfrak{c}u/2}\sqrt{\mu}\operatorname{curl}\mathcal{P}^{N_{\mathrm{top}}}\Omega\|^2_{L^2(\Sigma_t)} + \mathfrak{c}^2\mu_\star^{-2}(t)\|e^{-\mathfrak{c}u/2}\sqrt{\mu}\mathcal{P}^{N_{\mathrm{top}}}\Omega\|^2_{L^2(\Sigma_t)} \\ &\quad\lesssim \mathring{\epsilon}^3(1+\mathfrak{c}^2)\mu_\star^{-2M_*+0.8}(t) + \|e^{-\mathfrak{c}u/2}\sqrt{\mu}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{C}\|^2_{L^2(\Sigma_t)},\end{aligned}$$

which proves (11-18). The proof of (11-19) is similar, except we use Lemmas 11.8, 11.9 instead of Lemmas 11.6, 11.7. □

11D. *Putting everything together.*

Proposition 11.11 (the main top-order estimates for the modified fluid variables). *The following estimate holds for every* $(t, u) \in [0, T_{(\mathrm{Boot})}) \times [0, U_0]$:

$$\mathbb{C}_{N_{\mathrm{top}}}(t, u) + \mathbb{D}_{N_{\mathrm{top}}}(t, u) \lesssim \mathring{\epsilon}^3\mu_\star^{-2M_*+0.8}(t).$$

Proof. Step 1: controlling $\|e^{-\mathfrak{c}u/2}\sqrt{\mu}\underline{\partial}\mathcal{P}^{N_{\mathrm{top}}}(\Omega, S)\|^2_{L^2(\Sigma_t)}$ via Grönwall-type argument. Given $\varsigma > 0$, we first apply Proposition 11.10 and then use[56] Proposition 11.2 (for[57] $u = U_0$) to deduce that if $\mathfrak{c} > 0$ is

[56] Here, we again relabeled the ς from Proposition 11.2

[57] Note that in view of the fact that Ω, S are compactly supported in $u \in [0, U_0]$ (by Lemma 7.1), it follows that the integral on $\Sigma_t^{U_0}$ is the same as the integral on Σ_t.

sufficiently large (depending on ς), then the following estimate holds, where the constants $C > 0$ and $C_* > 0$ are independent of $\mathfrak{c}$ and ς:

$$\begin{aligned}
&\|e^{-\mathfrak{c}u/2}\sqrt{\upmu}\underline{\partial}\mathcal{P}^{N_{\mathrm{top}}}(\varOmega, S)\|^2_{L^2(\Sigma_t)} \\
&\qquad \le C\mathring{\mathfrak{e}}^3(1+\mathfrak{c}^2)\upmu_\star^{-2M_*+0.8}(t) + C\|e^{-\mathfrak{c}u/2}\sqrt{\upmu}\mathcal{P}^{N_{\mathrm{top}}}(\mathcal{C}, \mathcal{D})\|^2_{L^2(\Sigma_t)} \\
&\qquad \le C_*\mathring{\mathfrak{e}}^3(1+\mathfrak{c}^2)\upmu_\star^{-2M_*+0.8}(t) + \varsigma\int_{t'=0}^{t'=t}\frac{1}{\upmu_\star(t')}\|e^{-\mathfrak{c}u/2}\sqrt{\upmu}\underline{\partial}\mathcal{P}^{N_{\mathrm{top}}}(\varOmega, S)\|^2_{L^2(\Sigma_{t'})}\,dt'. \qquad (11\text{-}20)
\end{aligned}$$

We clarify that it is only for notational convenience for the argument in (11-21)–(11-23) below that we have used the symbol $C_* > 0$ to denote the fixed constant on the last line of (11-20).

We now argue by a continuity argument to show that, after choosing ς smaller and $\mathfrak{c}$ larger if necessary, (11-20) implies the estimate

$$\|e^{-\mathfrak{c}u/2}\sqrt{\upmu}\underline{\partial}\mathcal{P}^{N_{\mathrm{top}}}(\varOmega, S)\|^2_{L^2(\Sigma_t)} \le 2C_*\mathring{\mathfrak{e}}^3(1+\mathfrak{c}^2)\upmu_\star^{-2M_*+0.8}(t). \qquad (11\text{-}21)$$

If it is not the case that (11-21) holds on $[0, T_{(\mathrm{Boot})})$, then by continuity, there exists $T_* \in [0, T_{(\mathrm{Boot})})$ such that (11-21) holds for all $t \in [0, T_*]$ and such that

$$\|e^{-\mathfrak{c}u/2}\sqrt{\upmu}\underline{\partial}\mathcal{P}^{N_{\mathrm{top}}}(\varOmega, S)\|^2_{L^2(\Sigma_{T_*})} = 2C_*\mathring{\mathfrak{e}}^3(1+\mathfrak{c}^2)\upmu_\star^{-2M_*+0.8}(T_*). \qquad (11\text{-}22)$$

However, plugging the estimate (11-21) (which by assumption holds for $t \in [0, T_*]$) into the integral in (11-20), using Proposition 8.11 (and $M_* \ge 1$) to integrate away a negative power of $\upmu_\star$, and finally choosing ς sufficiently small, we obtain that for $t \in [0, T_*]$, we have

$$\|e^{-\mathfrak{c}u/2}\sqrt{\upmu}\underline{\partial}\mathcal{P}^{N_{\mathrm{top}}}(\varOmega, S)\|^2_{L^2(\Sigma_t)} \le \tfrac{3}{2}C_*\mathring{\mathfrak{e}}^3(1+\mathfrak{c}^2)\upmu_\star^{-2M_*+0.8}(t), \qquad (11\text{-}23)$$

which obviously contradicts (11-22) when $t = T_*$. It therefore follows that our desired estimate (11-21) holds for all $t \in [0, T_{(\mathrm{Boot})})$.

Step 2: deducing the estimates for $\mathbb{C}_{N_{\mathrm{top}}}(t, u)$ and $\mathbb{D}_{N_{\mathrm{top}}}(t, u)$. At this point, we can fix the constants $\mathfrak{c}$, ς, which we will absorb into the ensuring generic constants C. Moreover, since $u \in [0, U_0]$ on the support of $\varOmega$ and S (by Lemma 7.1), we will also absorb the weights $e^{-\mathfrak{c}u/2}$ into the constants. Hence, plugging (11-21) into the right-hand side of (11-1) and then using Proposition 8.11, we obtain

$$\begin{aligned}
\mathbb{C}_{N_{\mathrm{top}}}(t, u) + \mathbb{D}_{N_{\mathrm{top}}}(t, u) &\lesssim \mathring{\mathfrak{e}}^3\upmu_\star^{-2M_*+0.8}(t) + \int_{t'=0}^{t'=t}\frac{1}{\upmu_\star(t')}\|e^{-\mathfrak{c}u'/2}\sqrt{\upmu}\underline{\partial}\mathcal{P}^{N_{\mathrm{top}}}(\varOmega, S)\|^2_{L^2(\Sigma_{t'})}\,dt' \\
&\lesssim \mathring{\mathfrak{e}}^3\upmu_\star^{-2M_*+0.8}(t) + \mathring{\mathfrak{e}}^3\int_{t'=0}^{t'=t}\upmu_\star^{-2M_*-0.2}(t')\,dt' \lesssim \mathring{\mathfrak{e}}^3\upmu_\star^{-2M_*+0.8}(t), \qquad (11\text{-}24)
\end{aligned}$$

as desired. □

12. Wave estimates for the fluid variables

We continue to work under the assumptions of Theorem 6.3.

In this section, we derive a priori energy estimates for the wave variables, which will in particular yield strict improvements of the bootstrap assumptions (6-1)–(6-2). In Section 12A, we start by providing a somewhat general[58] "auxiliary" proposition, which yields energy estimates for solutions to *inhomogeneous* quasilinear wave equations *in terms of norms of the inhomogeneity*. The difficult aspect of the proof is that we have to close the estimates even though μ can be tending towards 0, that is, even though the shock may be forming. We delay discussing the proof of the auxiliary proposition until the Appendix; as we will explain, modulo small modifications based on established techniques, the proposition was proved as [36, Proposition 14.1] (see also [52, Proposition 14.1]). Then, in Section 12B, we bound the specific inhomogeneous terms that are relevant for our main results, that is, the inhomogeneous terms on the right-hand sides of the fluid wave equations (5-1a)–(5-1c). Finally, in Section 12C, we prove the final a priori energy estimates.

12A. ***The main estimates for inhomogeneous covariant wave equations.*** In this section, we state the "auxiliary" Proposition 12.1, which yields energy estimates for solutions to the fluid wave equations. In this section, we ignore the precise structure of the inhomogeneous terms and simply denote them by $\mathfrak{G}$. That is, we state the estimates of Proposition 12.1 in terms of various norms of $\mathfrak{G}$. Later on, in Proposition 12.7, we will control the relevant norms of $\mathfrak{G}$ to obtain our final a priori energy estimates for the wave variables. Proposition 12.1 is of independent interest in the sense that with small modifications, it could be used to study shock formation for compressible Euler flow with given smooth forcing terms.

Proposition 12.1 (the main estimates for the inhomogeneous geometric wave equations). *Let* $\vec{\Psi} \doteq (\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) \doteq (\mathcal{R}_{(+)}, \mathcal{R}_{(-)}, v^2, v^3, s)$, *as in* (2-3). *Recall that the* $\Psi_\imath$ *are solutions to the inhomogeneous covariant wave system*

$$\mu \square_{g(\vec{\Psi})} \Psi_\imath = \mathfrak{G}_\imath,$$

where $\vec{\mathfrak{G}} = (\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3, \mathfrak{G}_4, \mathfrak{G}_5)$ *is the array whose entries are the product of* μ *and the inhomogeneous terms on the right-hand sides of the five scalar wave equations* (5-1a)–(5-1c). *Assume that the following smallness bound holds*:[59]

$$\|\mathcal{P}^{\leq \lceil N_{\mathrm{top}}/2 \rceil} \vec{\mathfrak{G}}\|_{L^\infty(\mathcal{M}_{t,u})} \leq \mathring{\epsilon}^{1/2}. \tag{12-1}$$

Then there exists an absolute constant $M_* \in \mathbb{N}$, *independent of the equation of state and all other parameters in the problem, such that the following hold. As in Theorem 6.3, let* $T_{(\mathrm{Boot})} \in [0, 2\mathring{\delta}_*^{-1}]$, *and assume that*:

(1) *The bootstrap assumptions* (6-1)–(6-8) *all hold for all* $t \in [0, T_{(\mathrm{Boot})})$, *where we recall that in the bootstrap assumptions*, N_{top} *is any integer satisfying* $N_{\mathrm{top}} \geq 2M_* + 10$.

(2) *In* (6-3), *the parameter* $\mathring{\alpha}$ *is sufficiently small in a manner only on the equation of state and* $\bar{\varrho}$.

[58] Using a slight reorganization of the paper, these estimates could be upgraded so that they are "black box" estimates for inhomogeneous wave equations. Given the setup of this paper, they are not quite black box estimates because the proofs rely on the estimates of Section 8, some of which (e.g., some of the estimates in Proposition 8.10) depend on the structure of the inhomogeneous terms in the wave equations.

[59] We clarify that in our main results, in the proof of Proposition 12.7, we will show that the smallness assumption (12-1) is satisfied for the particular inhomogeneous terms $\vec{\mathfrak{G}}$ stated in the hypotheses of the proposition. However, for the purposes of proving Proposition 12.1, the precise structure of $\vec{\mathfrak{G}}$ is not important.

(3) *The parameter* $\mathring{\epsilon} > 0$ *in* (6-1)–(6-8) *satisfies* $\mathring{\epsilon}^{1/2} \leq \mathring{\alpha}$ *and is sufficiently small in a manner that depends only on the equation of state*, N_{top}, $\bar{\varrho}$, $\mathring{\sigma}$, $\mathring{\delta}$, *and* $\mathring{\delta}_*^{-1}$.

(4) *The soft bootstrap assumptions stated in Section 6A1 hold* (*including* $\mu > 0$ *in* $[0, T_{(\text{Boot})}) \times \mathbb{R} \times \mathbb{T}^2$).

Then the following estimates hold for every $(t, u) \in [0, T_{(\text{Boot})}) \times [0, U_0]$, *where* $\mu_\star$ *is defined in Definition 2.16*:

(1) *The top- and penultimate-order wave energies defined in* (3-2e) *obey the estimates*

$$\begin{aligned}\sup_{\hat{t}\in[0,t]} \mu_\star^{2M_*-1.8}(\hat{t})\mathbb{W}_{[1,N_{\text{top}}]}(\hat{t}, u) &+ \sup_{\hat{t}\in[0,t]} \mu_\star^{2M_*-3.8}(\hat{t})\mathbb{W}_{[1,N_{\text{top}}-1]}(\hat{t}, u) \\ &\lesssim \mathring{\epsilon}^2 + \sup_{\hat{t}\in[0,t]} \mu_\star^{2M_*-1.8}(\hat{t}) \int_{t'=0}^{t'=\hat{t}} \mu_\star^{-3/2}(t') \left\{ \int_{s=0}^{s=t'} \|\mathcal{P}^{[1,N_{\text{top}}]}\vec{\mathfrak{G}}\|_{L^2(\Sigma_s^u)}\, ds \right\}^2 dt' \\ &\quad + \sup_{\hat{t}\in[0,t]} \mu_\star^{2M_*-1.8}(\hat{t}) \|(|L\mathcal{P}^{[1,N_{\text{top}}]}\Psi| + |\breve{X}\mathcal{P}^{[1,N_{\text{top}}]}\Psi|)|\mathcal{P}^{[1,N_{\text{top}}]}\vec{\mathfrak{G}}|\|_{L^1(\mathcal{M}_{\hat{t},u})} \\ &\quad + \sup_{\hat{t}\in[0,t]} \mu_\star^{2M_*-3.8}(\hat{t}) \|(|L\mathcal{P}^{[1,N_{\text{top}}-1]}\Psi| + |\breve{X}\mathcal{P}^{[1,N_{\text{top}}-1]}\Psi|)|\mathcal{P}^{[1,N_{\text{top}}-1]}\vec{\mathfrak{G}}|\|_{L^1(\mathcal{M}_{\hat{t},u})}. \end{aligned} \tag{12-2}$$

(2) *For* $1 \leq N \leq N_{\text{top}} - 1$, *the lower-order wave energies* $\mathbb{W}_{[1,N]}$ *defined in* (3-2c) *obey the estimates*

$$\begin{aligned}\mathbb{W}_{[1,N]}(t, u) \lesssim \mathring{\epsilon}^2 + \max\{1, \mu_\star^{-2M_*+2N_{\text{top}}-2N+1.8}(t)\}\Big(\sup_{s\in[0,t]} \min\{1, \mu_\star^{2M_*-2N_{\text{top}}+2N+0.2}(s)\}\mathbb{Q}_{[1,N+1]}(s)\Big) \\ + \|(|L\mathcal{P}^{[1,N]}\Psi| + |\breve{X}\mathcal{P}^{[1,N]}\Psi|)|\mathcal{P}^{[1,N]}\vec{\mathfrak{G}}|\|_{L^1(\mathcal{M}_{t,u})}.\end{aligned} \tag{12-3}$$

Remark 12.2. The proof of Proposition 12.1 follows from almost exactly the same arguments used in the proof of [36, Proposition 14.1]. The only differences are the following two changes:

(1) We have to track the influence of the inhomogeneous terms $\mathfrak{G}$ on the estimates.

(2) In three dimensions, the second fundamental form of the null hypersurfaces of the acoustical metric has three (as opposed to one) independent components. This necessitates an additional elliptic estimate that was not needed in the two-dimensional case treated in [36]. This elliptic estimate is standard; see [15; 17; 33].

These differences necessitate minor modifications to the proof of [36, Proposition 14.1]. We will sketch them in the Appendix.

Remark 12.3 (additional term in the top-order estimate). In Proposition 12.1, the inhomogeneous term $\vec{\mathfrak{G}}$ makes an additional appearance in the top- and penultimate-order estimates as compared to the estimates of all the lower orders. By "additional appearance," we are referring to the double time integral, which comes from a difficult top-order commutator term that depends on the acoustic geometry; this difficult term has to be controlled by first integrating a transport equation, which explains the double time-integration; see the Appendix.

12B. ***Estimates for the inhomogeneous terms.*** We start by controlling the null forms in the wave equations.

Proposition 12.4 (control of wave equation error terms involving null forms). *For* $\mathfrak{Q} \in \{\mathfrak{Q}^i_{(v)}, \mathfrak{Q}_{(\pm)}\}$ (*see* (5-6a), (5-6b)) *and* $1 \leq N \leq N_{\mathrm{top}}$, *the following hold for all* $(t,u) \in [0, T_{(\mathrm{Boot})}) \times [0, U_0]$ *and for all* $\varsigma \in (0,1]$, *where the implicit constants are independent of* ς:

$$\begin{aligned}\|(|L\mathcal{P}^{[1,N]}\Psi| + |\breve{X}\mathcal{P}^{[1,N]}\Psi|)\mathcal{P}^{[1,N]}(\upmu\mathfrak{Q})\|_{L^1(\mathcal{M}_{t,u})} &\lesssim \mathring{\epsilon}^2 \max\{1, \upmu_\star^{-2M_*+2N_{\mathrm{top}}-2N+1.8}(t)\} \\ &\quad + \varsigma\mathbb{K}_{[1,N]}(t,u) + (1+\varsigma^{-1})\left(\int_{u'=0}^{u'=u} \mathbb{F}_{[1,N]}(t,u')\,du' + \int_{t'=0}^{t'=t} \mathbb{E}_{[1,N]}(t',u)\,dt'\right) \qquad (12\text{-}4)\end{aligned}$$

and

$$\begin{aligned}&\int_{t'=0}^{t'=t} \upmu_\star^{-3/2}(t')\left\{\int_{s=0}^{s=t'} \|\mathcal{P}^{[1,N_{\mathrm{top}}]}(\upmu\mathfrak{Q})\|_{L^2(\Sigma_s^u)}\,ds\right\}^2 dt' \\ &\qquad \lesssim \mathring{\epsilon}^2\upmu_\star^{-2M_*+1.8}(t) + \int_{t'=0}^{t'=t} \upmu_\star^{-3/2}(t')\left\{\int_{s=0}^{s=t'} \upmu_\star^{-1/2}(s)\mathbb{E}^{1/2}_{[1,N_{\mathrm{top}}]}(s)\,ds\right\}^2 dt'. \qquad (12\text{-}5)\end{aligned}$$

Proof. Step 1: proof of (12-4). To bound the left-hand side of (12-4), we use the Cauchy–Schwarz and the Young inequalities to obtain, for any $\varsigma > 0$,

$$\begin{aligned}&\|(|L\mathcal{P}^{[1,N]}\Psi| + |\breve{X}\mathcal{P}^{[1,N]}\Psi|)\mathcal{P}^{[1,N]}(\upmu\mathfrak{Q})\|_{L^1(\mathcal{M}_{t,u})} \\ &\qquad \lesssim (1+\varsigma^{-1})\left(\int_{u'=0}^{u'=u} \mathbb{F}_{[1,N]}(t,u')\,du' + \int_{t'=0}^{t'=t} \mathbb{E}_{[1,N]}(t',u)\,dt'\right) + \varsigma\|\mathcal{P}^{[1,N]}(\upmu\mathfrak{Q})\|^2_{L^2(\mathcal{M}_{t,u})}. \qquad (12\text{-}6)\end{aligned}$$

By inspection, it can be checked that $\mathfrak{Q}$ is a g-null form (see Definition 8.1) that is quadratic in the wave variables. Hence, applying (10-6) with $\phi^{(1)}, \phi^{(2)} = \Psi$, $\mathfrak{d}^{(1,1)}, \mathfrak{d}^{(2,1)} \lesssim 1$, $\mathfrak{d}^{(1,2)}, \mathfrak{d}^{(2,2)} \lesssim \mathring{\epsilon}^{1/2}$ (which is justified by the bootstrap assumptions (6-3)–(6-5)), we obtain

$$|\mathcal{P}^{[1,N]}(\upmu\mathfrak{Q})| \lesssim |\mathcal{P}^{[2,N+1]}\Psi| + \mathring{\epsilon}^{1/2}\{|\mathcal{P}^{[1,N]}\breve{X}\Psi| + |\mathcal{P}\Psi| + |\mathcal{P}^{[2,N]}(\upmu, L^i)|\}. \qquad (12\text{-}7)$$

To bound (12-7) in $L^2(\mathcal{M}_{t,u})$, we control $|\mathcal{P}^{[2,N+1]}\Psi|$ by the energies (3-2a)–(3-2c), control $|\mathcal{P}^{[1,N]}\breve{X}\Psi|$ by (8-38), bound $|\mathcal{P}\Psi|$ by (6-5), and $|\mathcal{P}^{[2,N]}(\upmu, L^i)|$ by Proposition 8.14. We thus obtain the following bound for any $\varsigma \in (0,1]$, where the implicit constants are independent of ς:

$$\begin{aligned}\varsigma\|\mathcal{P}^{[1,N]}(\upmu\mathfrak{Q})\|^2_{L^2(\mathcal{M}_{t,u})} &\lesssim \varsigma\left\{\mathbb{K}_{[1,N]}(t,u) + \int_{u'=0}^{u'=u} \mathbb{F}_{[1,N]}(t,u')\,du' + \int_{t'=0}^{t'=t} \mathbb{E}_{[1,N]}(t',u)\,dt'\right\} \\ &\qquad + \mathring{\epsilon}^2\int_{t'=0}^{t'=t} \max\{1, \upmu_\star^{-2M_*+2N_{\mathrm{top}}-2N+1.8}(t')\}\,dt' + \mathring{\epsilon}^2 \\ &\lesssim \mathring{\epsilon}^2 \max\{1, \upmu_\star^{-2M_*+2N_{\mathrm{top}}-2N+1.8}(t)\} \\ &\qquad + \varsigma\left\{\mathbb{K}_{[1,N]}(t,u) + \int_{u'=0}^{u'=u} \mathbb{F}_{[1,N]}(t,u')\,du' + \int_{t'=0}^{t'=t} \mathbb{E}_{[1,N]}(t',u)\,dt'\right\}, \qquad (12\text{-}8)\end{aligned}$$

where in the last line, we have used Proposition 8.12.

Putting (12-6)–(12-8) together, we obtain (12-4).

Step 2: proof of (12-5). We begin with (12-7) when $N = N_{\mathrm{top}}$. Notice that unlike in Step 1, we now have to control $|\mathcal{P}^{[2,N+1]}\Psi|$ only with the $\mathbb{E}$ (but not $\mathbb{F}$ and $\mathbb{K}$) energy (since we need an estimate on a fixed-t

hypersurface). This gives a $\upmu_\star^{-1/2}$ degeneration. The other terms can be controlled by using arguments similar to the ones we used in Step 1. In total, for $0 \le s \le t' \le t$, we have

$$\|\mathcal{P}^{[1,N_{\mathrm{top}}]}(\upmu\mathfrak{Q})\|_{L^2(\Sigma_s^u)} \lesssim \upmu_\star^{-1/2}(s)\mathbb{E}^{1/2}_{[1,N_{\mathrm{top}}]}(s) + \mathring{\epsilon}\max\{1, \upmu_\star^{-M_*+0.9}(s)\}. \tag{12-9}$$

Finally, integrating with respect to time and using Proposition 8.11, we obtain (12-5). □

Next, we control the easy linear terms in the wave equations.

Proposition 12.5 (control of wave equation error terms involving easy linear inhomogeneous terms). *For* $\mathfrak{L} \in \{\mathfrak{L}^i_{(v)}, \mathfrak{L}_{(\pm)}, \mathfrak{L}_{(s)}\}$ *(see* (5-7a), (5-7b), (5-7c)) *and* $1 \le N \le N_{\mathrm{top}}$, *the following holds for all* $(t, u) \in [0, T_{(\mathrm{Boot})}) \times [0, U_0]$:

$$\|(|L\mathcal{P}^{[1,N]}\Psi| + |\breve{X}\mathcal{P}^{[1,N]}\Psi|)\mathcal{P}^{[1,N]}(\upmu\mathfrak{L})\|_{L^1(\mathcal{M}_{t,u})} \lesssim \textit{the right-hand side of } (12\text{-}4), \tag{12-10}$$

and

$$\int_{t'=0}^{t'=t} \upmu_\star^{-3/2}(t')\left\{\int_{s=0}^{s=t'} \|\mathcal{P}^{[1,N_{\mathrm{top}}]}(\upmu\mathfrak{L})\|_{L^2(\Sigma_s^u)}\,ds\right\}^2 dt' \lesssim \textit{the right-hand side of } (12\text{-}5). \tag{12-11}$$

Proof. We first pointwise bound $\upmu\mathfrak{L} \in \{\upmu\mathfrak{L}^i_{(v)}, \upmu\mathfrak{L}_{(\pm)}, \upmu\mathfrak{L}_{(s)}\}$ in a similar manner[60] to (12-7):

$$|\mathcal{P}^{[1,N]}(\upmu\mathfrak{L})| \lesssim |\mathcal{P}^{\le N}(\varOmega, S)| + \text{terms already in (12-7)}. \tag{12-12}$$

Proof of (12-10). The terms in (12-12) that are already in (12-7) can of course be controlled as in Proposition 12.4. We therefore focus on $|\mathcal{P}^{\le N}(\varOmega, S)|$, for which we have the following estimate using the Cauchy–Schwarz and Hölder inequalities and Proposition 9.4:

$$\begin{aligned}
&\|(|L\mathcal{P}^{[1,N]}\Psi| + |\breve{X}\mathcal{P}^{[1,N]}\Psi|)\mathcal{P}^{\le N}(\varOmega, S)\|_{L^1(\mathcal{M}_{t,u})} \\
&\qquad \lesssim \|L\mathcal{P}^{[1,N]}\Psi\|^2_{L^2(\mathcal{M}_{t,u})} + \|\breve{X}\mathcal{P}^{[1,N]}\Psi\|^2_{L^2(\mathcal{M}_{t,u})} + \int_0^u \|\mathcal{P}^{\le N}(\varOmega, S)\|^2_{L^2(\mathcal{F}^t_{u'})}\,du' \\
&\qquad \lesssim \|L\mathcal{P}^{[1,N]}\Psi\|^2_{L^2(\mathcal{M}_{t,u})} + \|\breve{X}\mathcal{P}^{[1,N]}\Psi\|^2_{L^2(\mathcal{M}_{t,u})} + \mathring{\epsilon}^3\max\{1, \upmu_\star^{-2M_*+2N_{\mathrm{top}}+2N+2.8}(t)\},
\end{aligned} \tag{12-13}$$

which can indeed be bounded above by the right-hand side of (12-4) as claimed.

Proof of (12-11). Again, we only focus on the $|\mathcal{P}^{\le N_{\mathrm{top}}}(\varOmega, S)|$ term in (12-12). Using the definitions of the $\mathbb{V}$ and $\mathbb{S}$ norms and Propositions 8.11 and 9.4, we deduce

$$\begin{aligned}
&\int_{t'=0}^{t'=t} \upmu_\star^{-3/2}(t')\left\{\int_{s=0}^{s=t'} \|\mathcal{P}^{\le N_{\mathrm{top}}}(\varOmega, S)\|_{L^2(\Sigma_s^u)}\,ds\right\}^2 dt' \\
&\qquad \lesssim \int_{t'=0}^{t'=t} \upmu_\star^{-3/2}(t')\left\{\int_{s=0}^{s=t'} \upmu_\star^{-1/2}(s)[\mathbb{V}^{1/2}_{\le N_{\mathrm{top}}}(s) + \mathbb{S}^{1/2}_{\le N_{\mathrm{top}}}(s)]\,ds\right\}^2 dt' \\
&\qquad \lesssim \mathring{\epsilon}^3 \int_{t'=0}^{t'=t} \upmu_\star^{-3/2}(t')\left\{\int_{s=0}^{s=t'} \upmu_\star^{-M_*+0.9}(s)\,ds\right\}^2 dt' \\
&\qquad \lesssim \mathring{\epsilon}^3\max\{1, \upmu_\star^{-2M_*+3.3}(t)\} \lesssim \mathring{\epsilon}^2\max\{1, \upmu_\star^{-2M_*+1.8}(t)\},
\end{aligned} \tag{12-14}$$

which can indeed be bounded above by the right-hand side of (12-5) as claimed. □

[60] In fact, we can even do better than terms in (12-7) because of the extra smallness in $\mathring{\epsilon}$ we have from the bootstrap assumptions. However, we do not need this improvement for our proof.

Finally, we consider the linear terms involving $\mathcal{C}$ and $\mathcal{D}$.

Proposition 12.6 (control of wave equation error terms involving $\mathcal{C}$ and $\mathcal{D}$). *For*

$$\mathfrak{M} \in \left\{ c^2 \exp(2\rho)\mathcal{C}^i,\ c\exp(\rho)\frac{p_{;s}}{\bar{\varrho}}\mathcal{D},\ c^2\exp(2\rho)\mathcal{D},\ F_{;s}c^2\exp(2\rho)\mathcal{D} \right\}$$

(*cf. main terms in* (5-1a)–(5-1c)) *and* $1 \le N \le N_{\mathrm{top}}$, *the following hold for all* $(t,u) \in [0, T_{(\mathrm{Boot})}) \times [0, U_0]$:

$$\|(|L\mathcal{P}^{[1,N]}\Psi| + |\breve{X}\mathcal{P}^{[1,N]}\Psi|)\mathcal{P}^{[1,N]}(\mu\mathfrak{M})\|_{L^1(\mathcal{M}_{t,u})} \lesssim \textit{the right-hand side of } (12\text{-}4), \tag{12-15}$$

$$\int_{t'=0}^{t'=t} \mu_\star^{-3/2}(t') \left\{ \int_{s=0}^{s=t'} \|\mathcal{P}^{[1,N_{\mathrm{top}}]}(\mu\mathfrak{M})\|_{L^2(\Sigma_s^u)}\, ds \right\}^2 dt' \lesssim \textit{the right-hand side of } (12\text{-}5). \tag{12-16}$$

Proof. We first use the bootstrap assumptions (6-3)–(6-5) and (6-8) and Proposition 8.6 to deduce

$$|\mathcal{P}^N(\mu\mathfrak{M})| \lesssim \underbrace{\mu|\mathcal{P}^N(\mathcal{C},\mathcal{D})|}_{\doteq I} + \underbrace{|\mathcal{P}^{\le N-1}(\mathcal{C},\mathcal{D})|}_{\doteq II} + \text{terms already in (12-7)}. \tag{12-17}$$

<u>Step 1</u>: proof of (12-15). The terms already in (12-7) were handled in the proof of (12-4), so we only have to handle I and II in (12-17). We will use slightly different arguments for each of these two terms. For I, we have[61] by the Cauchy–Schwarz inequality, Propositions 10.6, 11.11, the bootstrap assumptions (6-1), (6-2), and Propositions 8.6 and 8.11 that

$$\begin{aligned}
&\|(|L\mathcal{P}^{[1,N]}\Psi| + |\breve{X}\mathcal{P}^{[1,N]}\Psi|)\mu\mathcal{P}^N(\mathcal{C},\mathcal{D})\|_{L^1(\mathcal{M}_{t,u})} \\
&\qquad \lesssim \int_{t'=0}^{t'=t} \mathbb{E}_{[1,N]}^{1/2}(t',u)[\mathbb{C}_{\le N}^{1/2} + \mathbb{D}_{\le N}^{1/2}](t',u)\, dt' \\
&\qquad \lesssim \mathring{\epsilon}^{1/2}\mathring{\epsilon}^{3/2} \int_{t'=0}^{t'=t} \max\{1, \mu_\star^{-M_*+N_{\mathrm{top}}-N+0.9}(t')\} \max\{1, \mu_\star^{-M_*+N_{\mathrm{top}}-N+0.4}(t')\}\, dt' \\
&\qquad \lesssim \mathring{\epsilon}^2 \max\{1, \mu_\star^{-2M_*+2N_{\mathrm{top}}-2N+2.3}(t)\}.
\end{aligned} \tag{12-18}$$

For II in (12-17), we use Cauchy–Schwarz and Proposition 10.6 to obtain

$$\begin{aligned}
&\|(|L\mathcal{P}^{[1,N]}\Psi| + |\breve{X}\mathcal{P}^{[1,N]}\Psi|)\mathcal{P}^{\le N-1}(\mathcal{C},\mathcal{D})\|_{L^1(\mathcal{M}_{t,u})} \\
&\qquad \lesssim \|L\mathcal{P}^{[1,N]}\Psi\|_{L^2(\mathcal{M}_{t,u})}^2 + \|\breve{X}\mathcal{P}^{[1,N]}\Psi\|_{L^2(\mathcal{M}_{t,u})}^2 + \|\mathcal{P}^{\le N-1}(\mathcal{C},\mathcal{D})\|_{L^2(\mathcal{M}_{t,u})}^2 \\
&\qquad \lesssim \|L\mathcal{P}^{[1,N]}\Psi\|_{L^2(\mathcal{M}_{t,u})}^2 + \|\breve{X}\mathcal{P}^{[1,N]}\Psi\|_{L^2(\mathcal{M}_{t,u})}^2 + \int_{u'=0}^{u'=u} [\mathbb{C}_{\le N-1} + \mathbb{D}_{\le N-1}](t,u')\, du' \\
&\qquad \lesssim \|L\mathcal{P}^{[1,N]}\Psi\|_{L^2(\mathcal{M}_{t,u})}^2 + \|\breve{X}\mathcal{P}^{[1,N]}\Psi\|_{L^2(\mathcal{M}_{t,u})}^2 + \mathring{\epsilon}^3 \max\{1, \mu_\star^{-2M_*+2N_{\mathrm{top}}-2N+2.8}(t)\}.
\end{aligned} \tag{12-19}$$

Finally, we observe that the right-hand side of (12-18) and the right-hand side of (12-19) are less than or equal to the right-hand side of (12-4). We have therefore proved (12-15).

<u>Step 2</u>: proof of (12-16). Returning to (12-17), we again note that we only have to consider terms not already controlled in Proposition 12.4. Applying Propositions 8.6, 8.11, 10.6, and 11.11, we have

[61] Note that it is only at the top $N = N_{\mathrm{top}}$ level that $\mathbb{C}_{\le N}^{1/2}$ and $\mathbb{D}_{\le N}^{1/2}$ is only bounded by $\mu_\star^{-M_*+N_{\mathrm{top}}-N+0.4}(t')$. For $N < N_{\mathrm{top}}$, we have the stronger estimates in Proposition 10.6, which in principle would allow us to avoid controlling the term I separately.

$$\begin{aligned}
&\int_{t'=0}^{t'=t} \upmu_\star^{-3/2}(t')\left\{\int_{s=0}^{s=t'}[\|\upmu\mathcal{P}^{\le N_{\mathrm{top}}}(\mathcal{C},\mathcal{D})\|_{L^2(\Sigma_s)}+\|\mathcal{P}^{\le N_{\mathrm{top}}-1}(\mathcal{C},\mathcal{D})\|_{L^2(\Sigma_s)}]\,ds\right\}^2 dt' \\
&\quad\lesssim \int_{t'=0}^{t'=t}\upmu_\star^{-3/2}(t')\left\{\int_{s=0}^{s=t'}[\mathbb{C}^{1/2}_{\le N_{\mathrm{top}}}+\mathbb{D}^{1/2}_{\le N_{\mathrm{top}}}](s)+\frac{1}{\upmu_\star^{1/2}(s)}[\mathbb{C}^{1/2}_{\le N_{\mathrm{top}}-1}+\mathbb{D}^{1/2}_{\le N_{\mathrm{top}}-1}](s)\,ds\right\}^2 dt' \\
&\quad\lesssim \mathring{\epsilon}^3\int_{t'=0}^{t'=t}\upmu_\star^{-3/2}(t')\left\{\int_{s=0}^{s=t'}\upmu_\star^{-M_*+0.4}(s)\,ds\right\}^2 dt' \lesssim \mathring{\epsilon}^3\upmu_\star^{-2M_*+2.3}(t)\lesssim \mathring{\epsilon}^2\upmu_\star^{-2M_*+1.8}(t),
\end{aligned}$$

which is therefore bounded above by the right-hand side of (12-5). □

12C. *Putting everything together.*

Proposition 12.7 (main L^2 estimates for the wave variables). *For $1\le N\le N_{\mathrm{top}}$, the following holds for all $(t,u)\in[0,T_{(\mathrm{Boot})})\times[0,U_0]$:*

$$\mathbb{W}_{[1,N]}(t,u)\lesssim \mathring{\epsilon}^2\max\{1,\upmu_\star^{-2M_*+2N_{\mathrm{top}}-2N+1.8}(t)\}. \tag{12-20}$$

Proof. We first use the pointwise bounds (12-7), (12-12), (12-17), the bootstrap assumptions (6-5)–(6-8), and Proposition 8.6 to deduce that the assumption (12-1) in Proposition 12.1 on the inhomogeneous terms $\vec{\mathfrak{G}}$, i.e., the terms on the right-hand sides of (5-1a)–(5-1c), is satisfied. Hence, the results of Proposition 12.1 are valid, and we will use them throughout the rest of this proof. We will also silently use the basic fact that $\upmu_\star(t,u)\le 1$ and $\upmu_\star(t)\le 1$; see Definition 2.16.

Step 1: $N=N_{\mathrm{top}}$. By the top- and penultimate-order general wave estimates (12-2) in Proposition 12.1, the initial data assumptions in (4-1), (4-3a)–(4-4), and the bounds for the inhomogeneous terms in Propositions 12.4–12.6, we obtain the following bound for any $\varsigma\in(0,1]$ (with implicit constants that are independent of ς):

$$\begin{aligned}
&\sup_{\hat{t}\in[0,t]}\upmu_\star^{2M_*-1.8}(\hat{t})\big(\mathbb{E}_{[1,N_{\mathrm{top}}]}(\hat{t},u)+\mathbb{F}_{[1,N_{\mathrm{top}}]}(\hat{t},u)+\mathbb{K}_{[1,N_{\mathrm{top}}]}(\hat{t},u)\big) \\
&\qquad+\sup_{\hat{t}\in[0,t]}\upmu_\star^{2M_*-3.8}(\hat{t})\big(\mathbb{E}_{[1,N_{\mathrm{top}}-1]}(\hat{t},u)+\mathbb{F}_{[1,N_{\mathrm{top}}-1]}(\hat{t},u)+\mathbb{K}_{[1,N_{\mathrm{top}}-1]}(\hat{t},u)\big) \\
&\lesssim \mathring{\epsilon}^2+\sup_{\hat{t}\in[0,t]}\upmu_\star^{2M_*-1.8}(\hat{t})\int_{t'=0}^{t'=\hat{t}}\upmu_\star^{-3/2}(t')\left\{\int_{s=0}^{s=t'}\|\mathcal{P}^{[1,N_{\mathrm{top}}]}\vec{\mathfrak{G}}\|_{L^2(\Sigma_s)}\,ds\right\}^2 dt' \\
&\quad+\sup_{\hat{t}\in[0,t]}\upmu_\star^{2M_*-1.8}(\hat{t})\|(|L\mathcal{P}^{[1,N_{\mathrm{top}}]}\Psi|+|\breve{X}\mathcal{P}^{[1,N_{\mathrm{top}}]}\Psi|)|\mathcal{P}^{[1,N_{\mathrm{top}}]}\vec{\mathfrak{G}}|\|_{L^1(\mathcal{M}_{\hat{t},u})} \\
&\quad+\sup_{\hat{t}\in[0,t]}\upmu_\star^{2M_*-3.8}(\hat{t})\|(|L\mathcal{P}^{[1,N_{\mathrm{top}}-1]}\Psi|+|\breve{X}\mathcal{P}^{[1,N_{\mathrm{top}}-1]}\Psi|)|\mathcal{P}^{[1,N_{\mathrm{top}}-1]}\vec{\mathfrak{G}}|\|_{L^1(\mathcal{M}_{\hat{t},u})} \\
&\lesssim \mathring{\epsilon}^2+\sup_{\hat{t}\in[0,t]}\upmu_\star^{2M_*-1.8}(\hat{t})\int_{t'=0}^{t'=t}\upmu_\star^{-3/2}(t',u)\left\{\int_{s=0}^{s=t'}\upmu_\star^{-1/2}(s)\mathbb{E}^{1/2}_{[1,N_{\mathrm{top}}]}(s)\,ds\right\}^2 dt' \\
&\quad+\sup_{\hat{t}\in[0,t]}\upmu_\star^{2M_*-1.8}(\hat{t})\Bigg\{\varsigma\mathbb{K}_{[1,N_{\mathrm{top}}]}(\hat{t},u) \\
&\qquad\qquad+(1+\varsigma^{-1})\bigg(\int_{t'=0}^{t'=\hat{t}}\mathbb{E}_{[1,N_{\mathrm{top}}]}(t',u)\,dt'+\int_{u'=0}^{u'=u}\mathbb{F}_{[1,N_{\mathrm{top}}]}(\hat{t},u')\,du'\bigg)\Bigg\} \\
&\quad+\sup_{\hat{t}\in[0,t]}\upmu_\star^{2M_*-3.8}(\hat{t})\Bigg\{\varsigma\mathbb{K}_{[1,N_{\mathrm{top}}-1]}(\hat{t},u) \\
&\qquad\qquad+(1+\varsigma^{-1})\bigg(\int_{t'=0}^{t'=\hat{t}}\mathbb{E}_{[1,N_{\mathrm{top}}-1]}(t',u)\,dt'+\int_{u'=0}^{u'=u}\mathbb{F}_{[1,N_{\mathrm{top}}-1]}(\hat{t},u')\,du'\bigg)\Bigg\}.
\end{aligned}\tag{12-21}$$

We now argue as follows using (12-21):

- We choose $\varsigma > 0$ sufficiently small and absorb the terms

$$\varsigma \sup_{\hat{t}\in[0,t]} \upmu_\star^{2M_*-1.8}(\hat{t})\mathbb{K}_{[1,N_{\mathrm{top}}]}(\hat{t},u), \quad \varsigma \sup_{\hat{t}\in[0,t]} \upmu_\star^{2M_*-3.8}(\hat{t})\mathbb{K}_{[1,N_{\mathrm{top}}-1]}(\hat{t},u)$$

appearing on the right-hand side by the terms

$$\sup_{\hat{t}\in[0,t]} \upmu_\star^{2M_*-1.8}(\hat{t})\mathbb{K}_{[1,N_{\mathrm{top}}]}(\hat{t},u), \quad \sup_{\hat{t}\in[0,t]} \upmu_\star^{2M_*-3.8}(\hat{t})\mathbb{K}_{[1,N_{\mathrm{top}}-1]}(\hat{t},u)$$

on the left-hand side.

- We then apply Proposition 8.12 (using that the exponents $2M_* - 1.8$ and $2M_* - 3.8$ are positive) and Grönwall's inequality to handle the terms involving the integrals of $\mathbb{E}$ and $\mathbb{F}$.

This leads to the following estimate (where on the left-hand side, we have dropped the below-top-order energies):

$$\begin{aligned}\sup_{\hat{t}\in[0,t]} \upmu_\star^{2M_*-1.8}(\hat{t})\big(\mathbb{E}_{[1,N_{\mathrm{top}}]}(\hat{t},u)+\mathbb{F}_{[1,N_{\mathrm{top}}]}(\hat{t},u)+\mathbb{K}_{[1,N_{\mathrm{top}}]}(\hat{t},u)\big)&\\ \lesssim \mathring{\epsilon}^2 + \sup_{\hat{t}\in[0,t]} \upmu_\star^{2M_*-1.8}(\hat{t})\int_{t'=0}^{t'=t} \upmu_\star^{-3/2}(t',u)\bigg\{\int_{s=0}^{s=t'} \upmu_\star^{-1/2}(s)\mathbb{E}^{1/2}_{[1,N_{\mathrm{top}}]}(s)\,ds\bigg\}^2\,dt'.& \quad (12\text{-}22)\end{aligned}$$

We will now apply a further Grönwall-type argument to (12-22). Define

$$\iota(t) \doteq \exp\bigg(\int_{s=0}^{s=t} \upmu_\star^{-0.9}(s)\,ds\bigg),$$

and, for a large $\mathfrak{C} > 0$ to be chosen later,

$$H(t) \doteq \sup_{\hat{t}\in[0,t]} \iota^{-2\mathfrak{C}}(\hat{t})\upmu_\star^{2M_*-1.8}(\hat{t})\mathbb{E}_{[1,N_{\mathrm{top}}]}(\hat{t}).$$

From the definitions of $\mathbb{E}_{[1,N_{\mathrm{top}}]}$, ι, and H, the fact that ι is increasing, and the estimate (12-22), we find that there exists a constant[62] $C_{**} > 0$ *independent of* $\mathfrak{C} > 0$ so that

$$H(t) \le C_{**}\bigg(\mathring{\epsilon}^2 + \sup_{\hat{t}\in[0,t]} \upmu_\star^{2M_*-1.8}(\hat{t})\iota^{-2\mathfrak{C}}(\hat{t})\int_{t'=0}^{t'=\hat{t}} \upmu_\star^{-3/2}(t')\bigg\{\int_{s=0}^{s=t'} \upmu_\star^{-1/2}(s)\mathbb{E}^{1/2}_{[1,N_{\mathrm{top}}]}(s)\,ds\bigg\}^2\,dt'\bigg). \quad (12\text{-}23)$$

Before we proceed, note that for $n = 1, 2$ an easy change of variables gives

$$\int_{s=0}^{s=t'} \iota^{n\mathfrak{C}}(s)\upmu_\star^{-0.9}(s)\,ds = \int_{y=0}^{y=\int_{\tau=0}^{\tau=t'} \upmu_\star^{-0.9}(\tau)\,d\tau} e^{n\mathfrak{C}y}\,dy \le \frac{1}{n\mathfrak{C}}\iota^{n\mathfrak{C}}(t'). \quad (12\text{-}24)$$

Fix $t \in [0, T_{(\mathrm{Boot})})$ and $\hat{t} \in [0, t]$. Since $\iota^{-\mathfrak{C}}$ is decreasing and $\upmu_\star$ is almost decreasing by Proposition 8.12, we have, using (12-24) and the estimate (12-23) for H, the following bound for the terms under the sup

[62] We call the constant C_{**} so as to make the notation clearer later in the proof.

on right-hand side of (12-23):

$$
\begin{aligned}
&\mu_\star^{2M_*-1.8}(\hat{t})\iota^{-2\mathfrak{C}}(\hat{t})\int_{t'=0}^{t'=\hat{t}}\mu_\star^{-3/2}(t')\left\{\int_{s=0}^{s=t'}\mu_\star^{-1/2}(s)\mathbb{E}^{1/2}_{[1,N_{\mathrm{top}}]}(s)\,ds\right\}^2 dt' \\
&\quad\le \mu_\star^{2M_*-1.8}(\hat{t})\iota^{-2\mathfrak{C}}(\hat{t})\int_{t'=0}^{t'=\hat{t}}\mu_\star^{-3/2}(t')\left\{\int_{s=0}^{s=t'}\mu_\star^{-M_*+1.3}(s)\right. \\
&\qquad\qquad\qquad\qquad \left.\times[\iota^{\mathfrak{C}}(s)\mu_\star^{-0.9}(s)][\iota^{-\mathfrak{C}}(s)\mu_\star^{M_*-0.9}(s)\mathbb{E}^{1/2}_{[1,N_{\mathrm{top}}]}(s)]\,ds\right\}^2 dt' \\
&\quad\le 2^{2M_*-2.6}\mu_\star^{2M_*-1.8}(\hat{t})\iota^{-2\mathfrak{C}}(\hat{t})\int_{t'=0}^{t'=\hat{t}}\mu_\star^{-2M_*+1.1}(t')\left\{\int_{s=0}^{s=t'}\iota^{\mathfrak{C}}(s)\mu_\star^{-0.9}(s)H^{1/2}(s)\,ds\right\}^2 dt' \\
&\quad\le 2^{2M_*-2.6}\mu_\star^{2M_*-1.8}(\hat{t})\iota^{-2\mathfrak{C}}(\hat{t})H(t)\int_{t'=0}^{t'=\hat{t}}\mu_\star^{-2M_*+1.1}(t')\left\{\int_{s=0}^{s=t'}\iota^{\mathfrak{C}}(s)\mu_\star^{-0.9}(s)\,ds\right\}^2 dt' \\
&\quad\le 2^{2M_*-2.6}\mu_\star^{2M_*-1.8}(\hat{t})\iota^{-2\mathfrak{C}}(\hat{t})\frac{H(t)}{\mathfrak{C}^2}\int_{t'=0}^{t'=\hat{t}}[\iota^{2\mathfrak{C}}(t')\mu_\star^{-0.9}(t')]\mu_\star^{-2M_*+2}(t')\,dt' \\
&\quad\le 2^{4M_*-4.6}\mu_\star^{0.2}(\hat{t})\iota^{-2\mathfrak{C}}(\hat{t})\frac{H(t)}{\mathfrak{C}^2}\int_{t'=0}^{t'=\hat{t}}\iota^{2\mathfrak{C}}(t')\mu_\star^{-0.9}(t')\,dt' \\
&\quad\le 2^{4M_*-5.6}\mu_\star^{0.2}(\hat{t})\frac{H(t)}{\mathfrak{C}^3}\le 2^{4M_*-5.6}\frac{H(t)}{\mathfrak{C}^3}.
\end{aligned}
\tag{12-25}
$$

Plugging (12-25) into (12-23), we obtain

$$
H(t)\le C_{**}\left\{\mathring{\epsilon}^2+2^{4M_*-5.6}\frac{H(t)}{\mathfrak{C}^3}\right\}. \tag{12-26}
$$

Choosing $\mathfrak{C}>0$ sufficiently large such that $2^{4M_*-5.6}/\mathfrak{C}^3\le\frac{1}{2}$, we immediately infer from (12-26) that $H(t)\le 2C_{**}\mathring{\epsilon}^2$. From this estimate, (12-25) the definition of $\iota(t)$, and the estimate (8-35), we find that the right-hand side of (12-22) is at most $C\mathring{\epsilon}^2$, where C is allowed to depend on $\mathfrak{C}$. From this estimate and the definition of $\mathbb{W}_{[1,N]}(t,u)$, we conclude (12-20) in the case $N=N_{\mathrm{top}}$.

<u>Step 2</u>: $1\le N\le N_{\mathrm{top}}-1$. Let $1\le N\le N_{\mathrm{top}}-1$. Arguing like we did at the beginning of Step 1, except for using (12-3) instead of (12-2), we obtain

$$
\begin{aligned}
&\mathbb{E}_{[1,N]}(t,u)+\mathbb{F}_{[1,N]}(t,u)+\mathbb{K}_{[1,N]}(t,u) \\
&\qquad\lesssim \mathring{\epsilon}^2\max\{1,\mu_\star^{-2M_*+2N_{\mathrm{top}}-2N+1.8}(t)\} \\
&\qquad\quad+\max\{1,\mu_\star^{-2M_*+2N_{\mathrm{top}}-2N+1.8}(t)\}\Big(\sup_{s\in[0,t]}\min\{1,\mu_\star^{2M_*-2N_{\mathrm{top}}+2N+0.2}(s)\}\mathbb{Q}_{[1,N+1]}(s)\Big) \\
&\qquad\qquad+\varsigma\mathbb{K}_{[1,N]}+(1+\varsigma^{-1})\left(\int_{t'=0}^{t'=t}\mathbb{E}_{[1,N]}(t',u)\,dt'+\int_{u'=0}^{u'=u}\mathbb{F}_{[1,N]}(t,u')\,du'\right) \\
&\qquad\lesssim \mathring{\epsilon}^2\max\{1,\mu_\star^{-2M_*+2N_{\mathrm{top}}-2N+1.8}(t)\} \\
&\qquad\quad+\max\{1,\mu_\star^{-2M_*+2N_{\mathrm{top}}-2N+1.8}(t)\}\Big(\sup_{s\in[0,t]}\min\{1,\mu_\star^{2M_*-2N_{\mathrm{top}}+2N+0.2}(s)\}\mathbb{Q}_{[1,N+1]}(s)\Big),
\end{aligned}
\tag{12-27}
$$

where to obtain the last inequality, we first took ς to be sufficiently small to absorb $\varsigma\mathbb{K}_{[1,N]}$, and then used Grönwall's inequality.

Using (12-27), we easily obtain (12-20) by induction in decreasing N. Notice in particular that the base case $N = N_{\mathrm{top}}$ has already been proven in Step 1. □

13. Proving the L^∞ estimates

We continue to work under the assumptions of Theorem 6.3.

In this section, we derive L^∞ estimates that in particular yield an improvement over the bootstrap assumptions we made in Section 6A. This is the final section in which we derive PDE estimates that are needed for the proof of Theorem 6.3; aside from the Appendix, the rest of the paper (i.e., Section 14) entails deriving consequences of the estimates and assembling the logic of the proof.

We first bound (in Propositions 13.2, 13.3) the L^∞ norm of the fluid variables, specific vorticity, entropy gradient and modified fluid variables and their $\mathcal{P}$ derivatives using the energy estimates we have already obtained and Sobolev embedding (Lemma 13.1). Then, in Propositions 13.3 and 13.4, we control derivatives of these variables that involve one factor of $\breve{X}$ by combining the just-obtained L^∞-estimates for $\mathcal{P}$-derivatives with the (wave or transport) equations.

Lemma 13.1 (Sobolev embedding estimates). *Suppose ϕ is a smooth function with u-support in* $[0, U_0]$. *Then*, *for every $t \in [0, T_{(\mathrm{Boot})})$*, *we have the estimate*

$$\|\phi\|_{L^\infty(\Sigma_t)} \lesssim \sup_{u\in[0,U_0]} \|\mathcal{P}^{\le 2}\phi\|_{L^2(\ell_{0,u})} + \sup_{u\in[0,U_0]} \|L\mathcal{P}^{\le 2}\phi\|_{L^2(\mathcal{F}_u^t)}. \tag{13-1}$$

Proof. First, using standard Sobolev embedding on $\mathbb{T}^2$, using (2-28b)–(2-28c) to express $\mathring{\partial}_2$, $\mathring{\partial}_3$ in terms of derivatives with respect to $\{Y, Z\}$, comparing the volume forms using Definition 3.1, and using the estimates of Proposition 8.7, we deduce

$$\|\phi\|_{L^\infty(\ell_{t,u})} \lesssim \sum_{i+j\le 2} \left(\int_{\ell_{t,u}} |\mathring{\partial}_2^i \mathring{\partial}_3^j \phi|^2 \, dx^2 \, dx^3\right)^{\frac{1}{2}} \lesssim \left(\int_{\ell_{t,u}} |\mathcal{P}^{\le 2}\phi|^2 \, d\lambda_{\not g}\right)^{\frac{1}{2}} \doteq \|\mathcal{P}^{\le 2}\phi\|_{L^2(\ell_{t,u})}. \tag{13-2}$$

To complete the proof of (13-1), it remains only for us to control the right-hand side of (13-2) by showing that for any smooth function φ (where the role of φ will be played by $\mathcal{P}^{\le 2}\phi$), we have

$$\|\varphi\|_{L^2(\ell_{t,u})} \le C\|\varphi\|_{L^2(\ell_{0,u})} + C\|L\varphi\|_{L^2(\mathcal{F}_u^t)}. \tag{13-3}$$

To prove (13-3), we start by using the identity $\mathring{\partial}_t = L - L^A \mathring{\partial}_A$ (see (2-27a)) to deduce that

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\ell_{t,u}} \varphi^2 \, dx^2 \, dx^3 &= 2\int_{\ell_{t,u}} \varphi \mathring{\partial}_t \varphi \, dx^2 \, dx^3 = 2\int_{\ell_{t,u}} \varphi L\varphi \, dx^2 \, dx^3 - 2\int_{\ell_{t,u}} \varphi L^A \mathring{\partial}_A \varphi \, dx^2 \, dx^3 \\ &= 2\int_{\ell_{t,u}} \varphi L\varphi \, dx^2 \, dx^3 + \int_{\ell_{t,u}} \varphi^2 (\mathring{\partial}_A L^A) \, dx^2 \, dx^3, \end{aligned} \tag{13-4}$$

where in the last step, we integrated the geometric coordinate partial derivatives $\mathring{\partial}_A$ by parts (and we recall that capital Latin indices vary over 2, 3). Again using (2-28b)–(2-28c) to express $\mathring{\partial}_2$, $\mathring{\partial}_3$ in terms of derivatives with respect to $\{Y, Z\}$, and using the estimates of Propositions 8.6 and 8.7, we find that

$|\partial\!\!\!/_A L^A| \leq C$. From this estimate, (13-4), and Young's inequality, we deduce that

$$\left| \frac{\partial}{\partial t} \int_{\ell_{t,u}} \varphi^2 \, dx^2 \, dx^3 \right| \leq C \int_{\ell_{t,u}} |L\varphi|^2 \, dx^2 \, dx^3 + C \int_{\ell_{t,u}} \varphi^2 \, dx^2 \, dx^3. \tag{13-5}$$

Integrating (13-5) with respect to time, using the fundamental theorem of calculus, and then applying Grönwall's inequality, we find that

$$\int_{\ell_{t,u}} \varphi^2 \, dx^2 \, dx^3 \leq C \int_{\ell_{0,u}} \varphi^2 \, dx^2 \, dx^3 + C \int_{t'=0} \int_{\ell_{t',u}} |L\varphi|^2 \, dx^2 \, dx^3 \, dt'. \tag{13-6}$$

Again comparing the volume forms using Definition 3.1 and using the estimates of Proposition 8.7, we arrive at the desired bound (13-3). □

Proposition 13.2. *The following L^∞ estimates hold for all $t \in [0, T_{(\mathrm{Boot})})$:*

$$\|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-2]}\Psi\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}, \tag{13-7}$$

$$\|\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-2}(\varOmega, S)\|_{L^\infty(\Sigma_t)} + \|\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-3}(\mathcal{C}, \mathcal{D})\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}^{3/2}. \tag{13-8}$$

Proof. These two estimates follow as immediate consequences of the energy estimates (respectively for $(\mathbb{V}, \mathbb{S})$, $(\mathbb{C}, \mathbb{D})$ and $\mathbb{W}$) in Propositions 9.4, 10.6, and 12.7, Lemma 13.1, and the initial data size-assumptions (4-4)–(4-6). □

Proposition 13.3. *The following L^∞ estimates hold for all $t \in [0, T_{(\mathrm{Boot})})$:*

$$\|\mathcal{R}_{(+)}\|_{L^\infty(\Sigma_t)} \lesssim_\blacklozenge \mathring{\alpha}, \qquad \|(\mathcal{R}_{(-)}, v^2, v^3, s)\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}, \tag{13-9a}$$

$$\|\breve{X}\mathcal{R}_{(+)}\|_{L^\infty(\Sigma_t)} \leq 2\mathring{\delta}, \quad \|\breve{X}(\mathcal{R}_{(-)}, v^2, v^3, s)\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}, \tag{13-9b}$$

$$\|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-4]}\breve{X}\Psi\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}. \tag{13-9c}$$

Proof. Step 1: proof of (13-9a). Since $Lt = 1$, we can apply the fundamental theorem of calculus along the integral curves of L to deduce that for any scalar function ϕ, we have

$$\|\phi\|_{L^\infty(\Sigma_t)} \leq \|\phi\|_{L^\infty(\Sigma_0)} + \int_{t'=0}^{t} \|L\phi\|_{L^\infty(\Sigma_{t'})} \, dt'. \tag{13-10}$$

By Proposition 13.2, we have $\|L\Psi\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}$. From this estimate, the data assumptions (4-3a) and (4-3c), and (13-10) with $\phi \doteq \Psi$, we conclude the desired bounds in (13-9a).

Step 2: an auxiliary estimate for $\mathrm{tr}_{g\!\!\!/}\chi$. We need an auxiliary estimate before proving (13-9b). To start, we note that the same arguments used to prove Proposition 8.6, based on the transport equation[63] (2-41), but now with the estimate (13-7) in place of the L^∞ bootstrap assumptions for $\|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-2]}\Psi\|_{L^\infty(\Sigma_t)}$ in (4-4), yield the estimate

$$\|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-3]}L^i\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}. \tag{13-11}$$

We next use Lemmas 2.23 and 2.32, and the fact that the Cartesian component functions X^1, X^2, X^3 are smooth functions of the L^i and Ψ (see (2-23)) to write the identity (2-38b) in the following form, where f

[63]Note importantly that the right-hand side of (2-41) does not contain an $\breve{X}\Psi$ term!

schematically denotes smooth functions: $\mathrm{tr}_{\not g}\chi = \mathrm{f}(L^i,\Psi)\mathcal{P}L^i + \mathrm{f}(L^i,\Psi)\mathcal{P}\Psi$. From this equation, the estimates of Proposition 13.2, (13-9a), and (13-11), we obtain the desired auxiliary estimate:

$$\|\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-4}\,\mathrm{tr}_{\not g}\chi\|_{L^\infty(\Sigma_t)} \lesssim \mathring{\epsilon}. \tag{13-12}$$

Step 3: controlling $\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-4}L\breve{X}\Psi$. By [52, Proposition 2.16], the wave operator is given by[64]

$$\mu\square_{g(\vec{\Psi})} f = -L(\mu Lf + 2\breve{X}f) + \mu\not\Delta f - \mathrm{tr}_{\not g}\chi\breve{X}f - \mu\,\mathrm{tr}_{\not g}\not k Lf - 2\mu\zeta^{\#}\cdot\not d f. \tag{13-13}$$

Consider now the wave equations (5-1a)–(5-1c). We will now bound the inhomogeneous terms in these equations. For $\mathfrak{Q}\in\{\mathfrak{Q}^i_{(v)},\mathfrak{Q}_{(\pm)},\mathfrak{Q}_{(s)}\}$, we first apply (10-5) with $\phi^{(1)}$, $\phi^{(2)}=\Psi$, $\mathfrak{d}^{(1,1)}$, $\mathfrak{d}^{(2,1)}\lesssim 1$, $\mathfrak{d}^{(1,2)}$, $\mathfrak{d}^{(2,2)}\lesssim\mathring{\epsilon}$ (which is justified by Proposition 13.2 and the bootstrap assumptions (6-3)–(6-5)), and then use (6-3)–(6-5) and Propositions 8.6 and 13.2 to obtain

$$|\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-4}(\mu\mathfrak{Q})| \lesssim |\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-3]}\Psi| + \mathring{\epsilon}\{|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-4]}\breve{X}\Psi| + |\mathcal{P}^{[2,N_{\mathrm{top}}-M_*-4]}(\mu,L^i)|\} \lesssim \mathring{\epsilon}. \tag{13-14}$$

For $\mathfrak{L}\in\{\mathfrak{L}^i_{(v)},\ \mathfrak{L}_{(\pm)},\ \mathfrak{L}_{(s)}\}$ and

$$\mathfrak{M}\in\left\{c^2\exp(2\rho)\mathcal{C}^i,\ c\exp(\rho)\frac{p_{;s}}{\bar{\varrho}}\mathcal{D},\ c^2\exp(2\rho)\mathcal{D}\right\},$$

we use the pointwise bounds (12-12), (12-17) together with (6-3)–(6-8) and Propositions 8.6 and 13.2 to obtain

$$|\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-4}(\mu\mathfrak{L})| + |\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-4}(\mu\mathfrak{M})| \lesssim \mathring{\epsilon}. \tag{13-15}$$

Combining (13-14) and (13-15), we thus obtain

$$|\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-4}(\mu\square_g\Psi)| \lesssim \mathring{\epsilon}. \tag{13-16}$$

We now use (13-16) together with (13-13) to control $\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-4}L\breve{X}\Psi$. The key point is that every term in $\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-4}$(13-13) except for $\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-4}(-2L\breve{X}\Psi)$ is already known to be bounded in L^∞ by $\mathcal{O}(\mathring{\epsilon})$. More precisely, we express the Ricci coefficients on the right-hand side of (13-13) using (2-38b)–(2-38d) and $\not\Delta$ using Lemmas 2.24 and 2.32. We also use the transport equation (2-40) to express[65] the factor of $L\mu$ on the right-hand side of (13-13) as the right-hand side of (2-40). Then using Propositions 8.6, 8.7, and 13.2, the estimates (13-9a) and (13-11)–(13-12), and the bootstrap assumptions (6-3)–(6-5) (to control all $\breve{X}\Psi$-involving products on the right-hand side of (13-13) except $-2L\breve{X}\Psi$), we obtain $|\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-4}L\breve{X}\Psi| \lesssim \mathring{\epsilon}$. Also using the first commutator estimate in (8-12) with $\phi \doteq \breve{X}\Psi$ and the bootstrap assumption (6-5), we further deduce that

$$\|L\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-4}\breve{X}\Psi\|_{L^\infty(\Sigma_t)} \lesssim |\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-4}L\breve{X}\Psi| + \mathring{\epsilon}^{1/2}|\mathcal{P}^{[1,N_{\mathrm{top}}-M_*-4]}\breve{X}\Psi| \lesssim \mathring{\epsilon}. \tag{13-17}$$

Step 4: proof of (13-9b) and (13-9c). We finally conclude (13-9b) and (13-9c) using (13-10) and (13-17), together with the initial data bounds (4-3b), (4-3c) and (4-4). □

[64] Here, $\not\Delta$ is the Laplace–Beltrami operator on $\ell_{t,u}$, which can be expressed as a second order differential operator in Y and Z with regular coefficients.

[65] This step is needed to avoid having to control $N_{\mathrm{top}}-M_*-3$ $\mathcal{P}$-derivatives of μ in L^∞, since Proposition 8.6 does not yield L^∞ control of that many derivatives of μ.

Proposition 13.4. *The following L^∞ estimates hold for all $t \in [0, T_{(\mathrm{Boot})})$:*

$$\|\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-4}\breve{X}(\Omega, S)\|_{L^\infty(\mathcal{M}_{t,u})} \lesssim \mathring{\epsilon}^{3/2}.$$

Proof. We apply $\mathcal{P}^{\leq N_{\mathrm{top}}-M_*-4}$ to (8-4)–(8-5) and then bound all terms on the right-hand side in L^∞ using Propositions 8.6, 13.2, and 13.3. □

14. Putting everything together

This is the concluding section. First, in Section 14A, we use the estimates derived in Sections 7–13 to conclude our main a priori estimates, i.e., to prove Theorem 6.3.

With the help of Theorem 6.3, all of the main results stated in Section 4B are quite easy to prove. We will prove Theorems 4.2 and 4.3 in Section 14B, Corollary 4.4 in Section 14C, and finally, Corollary 4.5 in Section 14D.

14A. *Proof of the main a priori estimates.*

Proof of Theorem 6.3. We prove each of the four conclusions asserted by Theorem 6.3.

(1) By Proposition 12.7, for $1 \leq N \leq N_{\mathrm{top}}$, the following wave estimates hold:

$$\mathbb{W}_N(t) \lesssim \mathring{\epsilon}^2 \max\{1, \mu_\star^{-2M_*+2N_{\mathrm{top}}-2N+1.8}(t)\}.$$

Hence, the inequalities in (6-1)–(6-2) hold with $\mathring{\epsilon}$ replaced by $C\mathring{\epsilon}^2$.

(2) By (13-9a)–(13-9b), the inequalities in (6-3) hold with $\mathring{\alpha}^{1/2}$ replaced by $C_\blacklozenge\mathring{\alpha}$ and $3\mathring{\delta}$ replaced by $2\mathring{\delta}$.

(3) By (13-7) and (13-9a)–(13-9c), the inequalities in (6-4)–(6-5) hold with $\mathring{\epsilon}^{1/2}$ replaced by $C\mathring{\epsilon}$.

(4) By (13-8) and Proposition 13.4, the inequalities (6-6)–(6-8) hold with $\mathring{\epsilon}$ replaced by $C\mathring{\epsilon}^{3/2}$. □

14B. *Proof of the main theorems.*

Proof of the regularity theorem (*Theorem 4.2*). By the main a priori estimates (Theorem 6.3) and a standard continuity argument, all the estimates established in the proof of Theorem 6.3 hold on $[0, T) \times \Sigma$. As a consequence, the energy estimates (4-7a), (4-7b) and (4-7c) follow from Propositions 12.7, 9.4, 10.6, and 11.11. As for the L^∞ estimates, (4-8a) holds thanks to (13-7) and (13-9c); (4-8b) and (4-8c) hold thanks to (13-9a) and (13-9b) respectively; and (4-8d) holds thanks to (13-8) and Proposition 13.4.

Moreover, Lemma 2.24, the identity $\partial\!\!\!/_t = L - L^A \partial\!\!\!/_A$ (see (2-27a)), and the L^∞ estimates mentioned above, together with those of Propositions 8.6, 8.7, and 8.10, imply that the solution can be smoothly extended[66] to $[0, T] \times \mathbb{R} \times \mathbb{T}^2$ as a function of the geometric coordinates (t, u, x^2, x^3).

It remains for us to show that the solution can be extended as a smooth solution of both the geometric and the Cartesian coordinates as long as $\inf_{t\in[0,T)} \mu_\star(t) > 0$. Now the estimates (4-8a)–(4-8c), Lemma 2.22, and the assumed lower bound on $\mu_\star$ together imply that the fluid variables and their first partial derivatives with respect to the Cartesian coordinates remain bounded. Standard local existence results/continuation criteria

[66]Note that these estimates imply that the $\partial\!\!\!/_t$ derivatives of many geometric coordinate partial derivatives of the solution are uniformly bounded on $[0, T] \times \mathbb{R} \times \mathbb{T}^2$, which leads to their extendibility to $[0, T] \times \mathbb{R} \times \mathbb{T}^2$.

then imply that the solution can be smoothly extended in the Cartesian coordinates to a Cartesian slab $[0, T+\epsilon]\times\Sigma$ for some $\epsilon>0$. Finally, within this Cartesian slab, one can solve the eikonal equation (2-13) such that the map $(t,u,x^2,x^3)\to(t,x^1,x^2,x^3)$ is a diffeomorphism from $[0,T+\epsilon]\times\mathbb{R}\times\mathbb{T}^2$ onto $[0,T+\epsilon]\times\Sigma$; the diffeomorphism property of this map follows easily from the identity $\not{\partial}_u x^1=\mu c^2/X^1$ (see (2-28a)) and the fact that $\mu c^2/X^1<0$ in $[0,T+\epsilon]\times\mathbb{R}\times\mathbb{T}^2$ whenever ϵ is small enough, thanks to $\mu>0$, (2-25b), and the estimates of Proposition 8.7 for $X^i_{(\mathrm{small})}$ and $c-1$. This implies that the solution can also be smoothly extended in the geometric coordinates (t,u,x^2,x^3). □

Proof of the shock formation theorem (*Theorem 4.3*). Step 1: vanishing of $\mu_\star$. First, we will show that

$$\mu_\star(t)=1+\mathcal{O}_\blacklozenge(\mathring{\alpha})+\mathcal{O}(\mathring{\epsilon})-\mathring{\delta}_* t. \tag{14-1}$$

To prove (14-1), we start by using (2-40), (2-42), and the L^∞ estimates established in Propositions 8.6 and 8.7 and Theorem 4.2 to deduce that

$$L\mu=-\tfrac{1}{2}c^{-1}(c^{-1}c_{;\rho}+1)\breve{X}\mathcal{R}_{(+)}+\mathcal{O}(\mathring{\epsilon}) \tag{14-2}$$

and

$$L\left\{\tfrac{1}{2}c^{-1}(c^{-1}c_{;\rho}+1)\right\}=\mathcal{O}(\mathring{\epsilon}),\quad L\left\{\tfrac{1}{2}c^{-1}(c^{-1}c_{;\rho}+1)\breve{X}\mathcal{R}_{(+)}\right\}=\mathcal{O}(\mathring{\epsilon}). \tag{14-3}$$

Moreover, from (2-13), (2-14), and our data assumptions (4-3a) and (4-3c), we have the following initial condition estimate for μ:

$$\mu\restriction_{\Sigma_0}=1+\mathcal{O}_\blacklozenge(\mathring{\alpha})+\mathcal{O}(\mathring{\epsilon}). \tag{14-4}$$

From (14-2)–(14-4), (4-2), and the fundamental theorem of calculus along the integral curves of L (and recalling that $Lt=1$), we conclude (14-1).

Step 2: proof of (1), (2), and (3). Define

$$T_{(\mathrm{sing})}\doteq\sup\{T\in[0,2\mathring{\delta}_*^{-1}]:\text{a smooth solutions exists with }\mu>0\text{ on }[0,T)\times\Sigma\}. \tag{14-5}$$

From Theorem 4.2, it follows that either $T_{(\mathrm{sing})}=2\mathring{\delta}_*^{-1}$ or $\liminf_{t\to T^-_{(\mathrm{sing})}}\mu_\star(t)=0$.

Using (14-2), we infer that $\mu_\star(t)$ first vanishes at a time equal to $\{1+\mathcal{O}_\blacklozenge(\mathring{\alpha})+\mathcal{O}(\mathring{\epsilon})\}\mathring{\delta}_*^{-1}$. From this fact, the definition of $T_{(\mathrm{sing})}$, and the above discussion, it follows that this time of first vanishing of $\mu_\star(t)$ is equal to $T_{(\mathrm{sing})}$, which implies (4-9). Using Theorem 4.2 again, we have therefore proved parts (1), (2) and (3) of Theorem 4.3.

Step 3: proof of (4). In the next step, we will show that the vanishing of $\mu_\star$ along $\Sigma_{T_{(\mathrm{sing})}}$ coincides with the blowup of $|\partial_1\mathcal{R}_{(+)}|$ at one or more points in $\Sigma_{T_{(\mathrm{sing})}}$; that will show that $T_{(\mathrm{sing})}$ is indeed the time of first singularity formation and in particular yields the conclusion (4) stated in Theorem 4.3.

Step 4: proof of (5). We now prove that $\mathscr{S}_{\mathrm{blowup}}=\mathscr{S}_{\mathrm{vanish}}$. This in particular also implies the blowup-claim in conclusion (4) of Theorem 4.3. We first prove $\mathscr{S}_{\mathrm{blowup}}\subseteq\mathscr{S}_{\mathrm{vanish}}$. If $(u,x^2,x^3)\notin\mathscr{S}_{\mathrm{vanish}}$, then μ has a lower bound away from 0 near $(T_{(\mathrm{sing})},u,x^2,x^3)$ and thus the estimates in Theorem 4.2 (and Lemma 2.22) imply that the fluid variables are C^1 functions of the geometric coordinates and the Cartesian coordinates near the point with geometric coordinates $(T_{(\mathrm{sing})},u,x^2,x^3)$, i.e., $(u,x^2,x^3)\notin\mathscr{S}_{\mathrm{blowup}}$.

To show $\mathscr{S}_{\text{blowup}} \supseteq \mathscr{S}_{\text{vanish}}$, suppose $(u, x^2, x^3) \in \mathscr{S}_{\text{vanish}}$. Let $\beta(t)$ denote the t-parametrized integral curve of L emanating from $(T_{(\text{sing})}, u, x^2, x^3)$. Note in particular that $\upmu \circ \beta(T_{(\text{sing})}) \doteq \upmu(T_{(\text{sing})}, u, x^2, x^3) = 0$, and recall that $Lt = 1$. We next use (14-2)–(14-4), (4-2), (4-9), and the fundamental theorem of calculus along the integral curve $\beta(t)$ to deduce that, for $0 \leq t \leq T_{(\text{sing})}$, we have

$$\tfrac{1}{2}|c^{-1}(c^{-1}c_{;\rho} + 1)| \circ \beta(0) \times |\breve{X}\mathcal{R}_{(+)}| \circ \beta(t) \geq \tfrac{1}{4}3\mathring{\delta}_*$$

(for otherwise, $\upmu \circ \beta(T_{(\text{sing})}) = 0$ would not be possible). Also using (2-26b), Propositions 8.6, 8.7, and the L^∞ estimates of Theorem 6.3, we find that the following estimate holds for $0 \leq t \leq T_{(\text{sing})}$:

$$\tfrac{1}{2}|c^{-1}(c^{-1}c_{;\rho} + 1)| \circ \beta(0) \times |\upmu\partial_1\mathcal{R}_{(+)}| \circ \beta(t) \geq \tfrac{1}{2}\mathring{\delta}_*.$$

In particular, also considering Remark 4.1, we deduce that

$$\limsup_{t \uparrow T^-_{(\text{sing})}} |\partial_1\mathcal{R}_{(+)})| \circ \beta(t) \geq \frac{\mathring{\delta}_*}{2|c^{-1}(c^{-1}c_{;\rho} + 1)| \circ \beta(0)} \limsup_{t \uparrow T^-_{(\text{sing})}} \frac{1}{\upmu \circ \beta(t)} = \infty.$$

Hence $(u, x^2, x^3) \in \mathscr{S}_{\text{blowup}}$, which finishes the proof that $\mathscr{S}_{\text{blowup}} = \mathscr{S}_{\text{vanish}}$.

Finally, we prove that $\mathscr{S}_{\text{vanish}} = \mathbb{R} \times \mathbb{T}^2 \setminus \mathscr{S}_{\text{regular}}$. The direction $\subseteq$ holds since $\mathscr{S}_{\text{vanish}} = \mathscr{S}_{\text{blowup}}$ and obviously $\mathscr{S}_{\text{blowup}} \subseteq \mathbb{R} \times \mathbb{T}^2 \setminus \mathscr{S}_{\text{regular}}$. We now show the direction $\supseteq$. Suppose that $(u, x^2, x^3) \notin \mathscr{S}_{\text{blowup}}$, i.e., $\upmu(T_{(\text{sing})}, u, x^2, x^3) > 0$. Then the estimates with respect to the geometric vectorfields established in Theorem 4.2 and Lemma 2.22 imply that in a neighborhood of $(T_{(\text{sing})}, u, x^2, x^3)$ intersected with $\{t \leq T_{(\text{sing})}\}$, the fluid variables remain C^1 functions of the geometric coordinates and Cartesian coordinates. We have therefore proved part (5) of Theorem 4.3, which completes its proof. □

14C. *Nontriviality of Ω and S (Proof of Corollary 4.4).*

Proof of Corollary 4.4. Using equations (14-2)–(14-4), we deduce (recalling that $\mathring{\epsilon}^{1/2} \leq \mathring{\alpha}$ by assumption) that along any t-parametrized integral curve $\beta(t)$ of L emanating from Σ_0 (i.e., $\beta^0(0) = 0$, where β^α denotes the Cartesian components of β), we have $\upmu \circ \beta(t) = 1 - \frac{1}{2}t[c^{-1}(c^{-1}c_{;\rho} + 1)\breve{X}\mathcal{R}_{(+)}] \circ \beta(0) + \mathcal{O}_{\blacklozenge}(\mathring{\alpha})$. From this bound, (4-9) (which implies that $0 \leq t \leq T_{(\text{sing})} = \{1 + \mathcal{O}_{\blacklozenge}(\mathring{\alpha}) + \mathcal{O}(\mathring{\epsilon})\}\mathring{\delta}_*^{-1}$), (4-2), and the assumption (4-10), we see that if $|u \circ \beta(0) - \mathring{\sigma} + \mathring{\delta}_*^{-1}| \geq 3\mathring{\alpha}\mathring{\delta}_*^{-1}$ (where $u \circ \beta(0)$ is the value of the u-coordinate at $\beta(0)$), then $\upmu \circ \beta(t) \geq \frac{3}{8}$ for $0 \leq t \leq T_{(\text{sing})}$ (assuming that $\mathring{\alpha}$ and $\mathring{\epsilon}$ are sufficiently small).

Now fix any $(u_*, x^2_*, x^3_*) \in \mathscr{S}_{\text{vanish}}$ (that is, $\upmu(T_{(\text{sing})}, u_*, x^2_*, x^3_*) = 0$). We will show that under the assumptions of the corollary, there is a constant $C > 1$ such that

$$C^{-1}\mathring{\epsilon}^3 \leq |S(T_{(\text{sing})}, u_*, x^2_*, x^3_*)| \leq C\mathring{\epsilon}^3, \quad C^{-1}\mathring{\epsilon}^2 \leq |\Omega(T_{(\text{sing})}, u_*, x^2_*, x^3_*)| \leq C\mathring{\epsilon}^2. \tag{14-6}$$

Clearly, the bounds (14-6) imply the desired conclusion of the corollary.

To initiate the proof of (14-6), we let $\beta_{(\text{sing})}(t)$ denote the t-parametrized integral curve of L passing through $(T_{(\text{sing})}, u_*, x^2_*, x^3_*)$. Then since (2-21) implies that the coordinate function u is constant along $\beta_{(\text{sing})}$ (and thus $u \circ \beta_{(\text{sing})}(0) = u_*$), the results derived two paragraphs above guarantee that $|u_* - \mathring{\sigma} + \mathring{\delta}_*^{-1}| \leq 3\mathring{\alpha}\mathring{\delta}_*^{-1}$. In particular, in view of the initial condition (2-13) for u along Σ_0, we see that $|\beta^1_{(\text{sing})}(0) - \mathring{\delta}_*^{-1}| \leq 3\mathring{\alpha}\mathring{\delta}_*^{-1}$, where $\beta^1_{(\text{sing})}(0) \doteq x^1 \circ \beta_{(\text{sing})}(0)$ is the x^1-coordinate of the point $\beta_{(\text{sing})}(0) \in \Sigma_0$.

Then, since Proposition 8.6 yields that $(d/dt)\beta^1 = L\beta^1 = L^1 = 1 + L^1_{(\text{small})} = 1 + \mathcal{O}_{\blacklozenge}(\mathring{\alpha})$, we can integrate in time and use (4-9) to deduce that

$$\beta^1(T_{(\text{sing})}) = \beta^1(0) + T_{(\text{sing})} + \mathcal{O}_{\blacklozenge}(\mathring{\alpha})T_{(\text{sing})} = -\mathring{\delta}_*^{-1} + T_{(\text{sing})} + \mathcal{O}_{\blacklozenge}(\mathring{\alpha})T_{(\text{sing})} = \mathcal{O}_{\blacklozenge}(\mathring{\alpha})\mathring{\delta}_*^{-1}.$$

That is, the x^1-coordinate of the singular point $(T_{(\text{sing})}, u_*, x^2_*, x^3_*)$ is of size $\mathcal{O}_{\blacklozenge}(\mathring{\alpha})\mathring{\delta}_*^{-1}$.

Let now $\gamma_{(\text{sing})}$ be the integral curve of B passing through the singular point $(T_{(\text{sing})}, u_*, x^2_*, x^3_*)$ as above. Since (2-23) and (4-8b) imply that $B = \partial_t + \mathcal{O}_{\blacklozenge}(\mathring{\alpha})\partial$, we can integrate with respect to time along $\gamma_{(\text{sing})}$ and use (4-9) and the bound on the x^1-coordinate of the singular point $(T_{(\text{sing})}, u_*, x^2_*, x^3_*)$ proved above to deduce that $\gamma_{(\text{sing})}$ intersects Σ_0 at a point q with x^1-coordinate q^1 of size $q^1 = \mathcal{O}_{\blacklozenge}(\mathring{\alpha})\mathring{\delta}_*^{-1}$. In view of the initial condition (2-13) for u along Σ_0, we see that the u-coordinate of q^1, which we denote by $u|_q$, satisfies $|u|_q - \mathring{\sigma}| = \mathcal{O}_{\blacklozenge}(\mathring{\alpha})\mathring{\delta}_*^{-1}$. From this bound and the assumption (4-11), we see that

$$\tfrac{1}{2}\mathring{\epsilon}^2 \le |\Omega|_q| \le \mathring{\epsilon}^2, \quad \tfrac{1}{2}\mathring{\epsilon}^3 \le |S|_q| \le \mathring{\epsilon}^3. \tag{14-7}$$

To complete the proof, we need to use (14-7) to prove (14-6). To this end, we find it convenient to parametrize $\gamma_{(\text{sing})}$ by the eikonal function. Since (2-23) and (2-21) guarantee that $\upmu Bu = 1$, this is equivalent to studying integral curves of $\upmu B$. That is, we slightly abuse notation by denoting the reparametrized integral curve by the same symbol $\gamma_{(\text{sing})}$; i.e., $\gamma_{(\text{sing})}$ solves the integral curve ODE $(d/du)\gamma_{(\text{sing})}(u) = \upmu B \circ \gamma_{(\text{sing})}(u)$. To proceed, we multiply the transport equations (5-2a) and (5-2c) by $\upmu$ and use (2-23), (2-21), Lemma 2.22, Propositions 8.6, 8.7, and the L^∞ estimates of Theorem 6.3 to deduce that along $\gamma_{(\text{sing})}$, (5-2a) and (5-2c) imply the following evolution equations, expressed in schematic form:

$$\frac{d}{du}\Omega \circ \gamma_{(\text{sing})}(u) = \mathcal{O}(1)\Omega \circ \gamma_{(\text{sing})}(u) + \mathcal{O}(1)S \circ \gamma_{(\text{sing})}(u), \tag{14-8}$$

$$\frac{d}{du}S \circ \gamma_{(\text{sing})}(u) = \mathcal{O}(1)S \circ \gamma_{(\text{sing})}(u). \tag{14-9}$$

From the evolution equations (14-8)–(14-9), the initial conditions (14-7), and the fact that $0 \le u \le U_0$ in the support of the solution (see Section 7), we conclude that if $\mathring{\epsilon}$ is sufficiently small, then there is a $C > 1$ such that (14-6) holds. □

14D. ***Hölder estimates (proof of Corollary 4.5).*** Throughout this section, we work under the assumptions of Corollary 4.5.

Lemma 14.1 (a simple calculus lemma). *Let $J \subseteq \mathbb{R}$ be an interval. Suppose $f : J \to \mathbb{R}$ is a C^3 function such that*:

(1) *f is increasing, i.e., $f' \ge 0$.*

(2) *There exists $\mathring{b} > 0$ such that $f^{(3)}(y) \ge \mathring{b}$ for every $y \in J$, where $f^{(3)}$ denotes the third derivative of f.*

Then for any $y_1, y_2 \in J$, the following estimate holds:

$$|f(y_1) - f(y_2)| \ge \frac{\mathring{b}}{48}|y_1 - y_2|^3.$$

Proof. First, note that the assumption on $f^{(3)}$ implies that f'' is strictly increasing. In particular, f'' can at most change sign once.

Without loss of generality, assume $y_1 \neq y_2$. We consider three cases: the first two are such that $f''(y_1)$ and $f''(y_2)$ are of the same sign, while the third is such that they have opposite sign.

Case 1: $y_1 < y_2$ and $f''(y_1) < f''(y_2) \leq 0$. By Taylor's theorem,

$$f(y_1) = f(y_2) - f'(y_2)(y_2 - y_1) + \frac{1}{2} f''(y_2)(y_2 - y_1)^2 \\ - \frac{1}{2}(y_2 - y_1)^3 \int_0^1 (1-\tau)^2 f^{(3)}(y_2 + \tau(y_1 - y_2))\, d\tau \leq f(y_2) - \frac{\mathring{b}}{6}(y_2 - y_1)^3,$$

where we have used $f'(y_2) \geq 0$, $f''(y_2) \leq 0$ and $f^{(3)}(y) \geq \mathring{b}$.

Therefore,

$$|f(y_1) - f(y_2)| = f(y_2) - f(y_1) \geq \frac{\mathring{b}}{6}(y_2 - y_1)^3.$$

Case 2: $y_2 < y_1$ and $f''(y_1) > f''(y_2) \geq 0$. This can be treated in the same way as Case 1 so that we have

$$|f(y_1) - f(y_2)| = f(y_1) - f(y_2) \geq \frac{\mathring{b}}{6}(y_1 - y_2)^3.$$

Case 3: $y_1 < y_2$, $f''(y_1) < 0 < f''(y_2)$. Since f'' is strictly increasing, there exists a unique $z \in (y_1, y_2)$ such that $f''(z) = 0$. Therefore, using Case 1 (for y_1 and z) and Case 2 (for y_2 and z), we have

$$|f(y_1) - f(y_2)| = f(y_2) - f(z) + f(z) - f(y_1) \geq \frac{\mathring{b}}{6}(|y_2 - z|^3 + |y_1 - z|^3) \geq \frac{\mathring{b}}{2^3 \cdot 6}(y_2 - y_1)^3,$$

where in the very last inequality we have used $y_2 - y_1 \leq 2\max\{|y_1 - z|,\ |y_2 - z|\}$.

Combining all three cases, we conclude the desired inequality. □

Lemma 14.2 (quantitative negativity of $\breve{\partial}_u^3 x^1$). *Under the assumptions of Corollary 4.5, the following holds at all points such that* $(t, u) \in [3T_{(\mathrm{sing})}/4, T_{(\mathrm{sing})}) \times [\mathring{\sigma}/2, 3\mathring{\sigma}/2]$:

$$\breve{\partial}_u^3 x^1 \leq -\mathring{\upbeta}.$$

Proof. In this proof, we will silently use the fact that the Cartesian component functions X^1, X^2, X^3 are smooth functions of the L^i and Ψ (see (2-23)) and the fact that c is a smooth function of Ψ.

By (2-29), to prove the lemma, we need to estimate $\breve{\partial}_u^3 x^1 = \breve{\partial}_u^2(\upmu c^2/X^1)$. To proceed, we use (2-28a) (in particular, the fact that $\breve{\partial}_u - \breve{X}$ is $\ell_{t,u}$-tangent) and the L^∞ estimates of Propositions 8.6, 8.7, and 8.10 and Theorem 6.3 to deduce that

$$\breve{\partial}_u^3 x^1 = \breve{X}\breve{X}\left(\frac{\upmu c^2}{X^1}\right) + \mathcal{O}(\mathring{\epsilon}). \tag{14-10}$$

We will now estimate the term $\breve{X}\breve{X}(\upmu c^2/X^1)$ on the right-hand side of (14-10). We start by noting that the L^∞ estimates of Propositions 8.6, 8.7, and 8.10 and Theorem 6.3 together imply that $|LL\breve{X}\breve{X}(\upmu c^2/X^1)| = \mathcal{O}(\mathring{\epsilon})$. Therefore, letting $\gamma(t)$ be any integral curve of L parametrized by Cartesian time t (with $\gamma(0)\in\Sigma_0$) and recalling that $Lt=1$, we integrate this estimate twice in time to deduce that for $t\in[0,T_{(\mathrm{sing})})$, we have

$$\begin{aligned}\breve{X}\breve{X}\left(\frac{\mu c^2}{X^1}\right)\circ\gamma(t)&=\left[\breve{X}\breve{X}\left(\frac{\mu c^2}{X^1}\right)\right]\circ\gamma(0)+t\left[L\breve{X}\breve{X}\left(\frac{\mu c^2}{X^1}\right)\right]\circ\gamma(0)+\mathcal{O}(\mathring{\epsilon})\\&=\left[\breve{X}\breve{X}\left(\frac{\mu c^2}{X^1}\right)\right]\circ\gamma(0)+t\left[\breve{X}\breve{X}L\left(\frac{\mu c^2}{X^1}\right)\right]\circ\gamma(0)+\mathcal{O}(\mathring{\epsilon}),\end{aligned}\tag{14-11}$$

where to deduce the last equality, we used in particular (8-27).

Next, using the transport equation (2-40), (2-42), the fact that $X\restriction_{\Sigma_0}=-c\partial_1$ (by (2-7), (2-13), (2-26b), and the normalization condition $g(X,X)=1$), and the L^∞ estimates mentioned above, we deduce that

$$\begin{aligned}\left[\breve{X}\breve{X}L\left(\frac{\mu c^2}{X^1}\right)\right]\circ\gamma(0)&=\left[\breve{X}\breve{X}\left\{(L\mu)\frac{c^2}{X^1}\right\}\right]\circ\gamma(0)+\mathcal{O}(\mathring{\epsilon})\\&=\frac{1}{2}[\breve{X}\breve{X}\{(c^{-1}c_{;\rho}+1)(\breve{X}\mathcal{R}_{(+)})\}]\circ\gamma(0)+\mathcal{O}(\mathring{\epsilon}).\end{aligned}\tag{14-12}$$

Next, using that $X\restriction_{\Sigma_0}=-c\partial_1$, and using that $\mu\restriction_{\Sigma_0}=1/c$ (this follows from the initial condition in (2-13) and the fact that (2-21) implies that $Xu=1/\mu$), we deduce

$$\breve{X}\breve{X}\left(\frac{\mu c^2}{X^1}\right)\restriction_{\Sigma_0}=-\breve{X}\breve{X}(1)=0.\tag{14-13}$$

Combining (14-11)–(14-13), we find that

$$\breve{X}\breve{X}\left(\frac{\mu c^2}{X^1}\right)\circ\gamma(t)=\frac{t}{2}[\breve{X}\breve{X}\{(c^{-1}c_{;\rho}+1)(\breve{X}\mathcal{R}_{(+)})\}]\circ\gamma(0)+\mathcal{O}(\mathring{\epsilon}).\tag{14-14}$$

From (14-14) and our assumption (4-12), we deduce that at any point whose corresponding u-coordinate[67] satisfies $u\in[\mathring{\sigma}/2,3\mathring{\sigma}/2]$, we have

$$\breve{X}\breve{X}\left(\frac{\mu c^2}{X^1}\right)\circ\gamma(t)\le-2t\mathring{\delta}_*\mathring{\beta}+\mathcal{O}(\mathring{\epsilon}).\tag{14-15}$$

In particular, for points whose corresponding u- and t-coordinates satisfy, respectively, $u\in[\mathring{\sigma}/2,3\mathring{\sigma}/2]$ and $t\in[3T_{(\mathrm{sing})}/4,T_{(\mathrm{sing})})$, we have, in view of (4-9), the estimate

$$\breve{X}\breve{X}\left(\frac{\mu c^2}{X^1}\right)\circ\gamma(t)\le-\frac{3\mathring{\beta}}{2}+\mathcal{O}_{\blacklozenge}(\mathring{\alpha})\mathring{\delta}_*\mathring{\beta}+\mathcal{O}(\mathring{\epsilon}).\tag{14-16}$$

Combining (14-10) and (14-16), we conclude the lemma. □

Lemma 14.3 (the main Hölder estimate for the eikonal function). *Under the assumptions of Corollary 4.5, the following holds for* $t\in[3T_{(\mathrm{sing})}/4,T_{(\mathrm{sing})})$:

$$\sup_{\substack{p_1,p_2\in\Sigma_t,\,p_1\neq p_2\\u(p_i)\in[\mathring{\sigma}/2,3\mathring{\sigma}/2]}}\frac{|u(p_1)-u(p_2)|}{\mathrm{dist}_{\mathrm{Euc}}(p_1,p_2)^{1/3}}\le5\mathring{\beta}^{-1/3}.$$

Above, $u(p_i)$ *denotes the value of the eikonal function at* p_i, $x(p_i)$ *denotes the Cartesian spatial coordinates of* p_i, *and* $\mathrm{dist}_{\mathrm{Euc}}(p_1,p_2)$ *denotes the Euclidean distance in* Σ_t *between* p_1 *and* p_2.

[67] Recall that $u\restriction_{\Sigma_0}=\mathring{\sigma}-x^1$ and the u-value is constant along the integral curves of L by virtue of the first equation in (2-21).

Proof. Step 1: estimating $\min_{u(p_i)=u_i} \mathrm{dist}_{\mathrm{Euc}}(p_1, p_2)$ by carefully choosing two points. Consider two distinct values u_1, u_2 which obey $u_i \in [\mathring{\sigma}/2, 3\mathring{\sigma}/2]$. By compactness of the constant-u hypersurfaces in Σ_t, there exist points $\boldsymbol{p}_1, \boldsymbol{p}_2 \in \Sigma_t$ with $u(\boldsymbol{p}_i) = u_i$ and $\mathrm{dist}_{\mathrm{Euc}}(\boldsymbol{p}_1, \boldsymbol{p}_2) = \min_{u(p_i)=u_i} \mathrm{dist}_{\mathrm{Euc}}(p_1, p_2)$. In particular, $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are connected by a Euclidean straight line $L_{\boldsymbol{p}_1,\boldsymbol{p}_2}$ which is Euclidean-perpendicular to $\{u = u_i\}$ at the point $\boldsymbol{p}_i$ for $i = 1, 2$.

Now by Lemma 2.22 and (2-21), the Euclidean gradient of u satisfies

$$\upmu\partial_i u = c^{-2}X^i, \quad i = 1, 2, 3. \tag{14-17}$$

Recalling (by Proposition 8.7 and conclusions (2) and (3) of Theorem 6.3) that $c^{-2}X^1 = -1 + \mathcal{O}_{\blacklozenge}(\mathring{\alpha})$, $c^{-2}X^2$, $c^{-2}X^3 = \mathcal{O}_{\blacklozenge}(\mathring{\alpha})$, we deduce from (14-17) that $L_{\boldsymbol{p}_1,\boldsymbol{p}_2}$ makes a Euclidean angle of $\mathcal{O}_{\blacklozenge}(\mathring{\alpha})$ with respect to ∂_1. Therefore, using (14-17) again (which implies that constant-u hypersurfaces in Σ_t make an angle $\mathcal{O}(\mathring{\alpha})$ with constant-x^1 planes), we infer that there exist[68] $\mathfrak{p}_1$, $\mathfrak{p}_2$ such that:

(1) $u(\mathfrak{p}_i) = u_i$.

(2) ∂_1 is tangent to the Euclidean line $\mathfrak{L}$ connecting $\mathfrak{p}_1$ and $\mathfrak{p}_2$.

(3) $\min_{u(p_i)=u_i} \mathrm{dist}_{\mathrm{Euc}}(p_1, p_2) = \mathrm{dist}_{\mathrm{Euc}}(\boldsymbol{p}_1, \boldsymbol{p}_2) \geq \frac{1}{2} \mathrm{dist}_{\mathrm{Euc}}(\mathfrak{p}_1, \mathfrak{p}_2) = \frac{1}{2}|x^1(\mathfrak{p}_1) - x^1(\mathfrak{p}_2)|$.

We fix such a choice of $(\mathfrak{p}_1, \mathfrak{p}_2)$ for any given (u_1, u_2) (with $u_1 \neq u_2$).

Step 2: estimating $|x^1(\mathfrak{p}_1) - x^1(\mathfrak{p}_2)|$. By (2-29), Proposition 8.7, and conclusions (2) and (3) of Theorem 6.3, we have

$$\breve{X}_u x^1 = \upmu(-1 + \mathcal{O}_{\blacklozenge}(\mathring{\alpha})).$$

Hence, for every fixed (x^2, x^3), x^1 is a strictly decreasing function in u. Moreover, by Lemma 14.2, $\breve{X}_u^3 x^1 \leq -\mathring{\upbeta}$. Hence, we are exactly in the setting to apply Lemma 14.1 (for the one-variable function $f(u) = -x^1(u)$, where (x^2, x^3) is fixed, and $\mathring{b} = \mathring{\upbeta}$) to obtain

$$|x^1(\mathfrak{p}_1) - x^1(\mathfrak{p}_2)| \geq \frac{\mathring{\upbeta}}{48}|u_1 - u_2|^3. \tag{14-18}$$

In view of our choice of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in Step 1, we conclude from (14-18) that

$$\begin{aligned}
\sup_{\substack{p_1,p_2\in\Sigma_t,p_1\neq p_2\\ u(p_i)\in[\mathring{\sigma}/2,3\mathring{\sigma}/2]}} \frac{|u(p_1) - u(p_2)|}{\mathrm{dist}_{\mathrm{Euc}}(p_1, p_2)^{1/3}} &\leq \sup_{\substack{u_1\neq u_2\\ u_i\in[\mathring{\sigma}/2,3\mathring{\sigma}/2]}} \frac{|u_1 - u_2|}{\inf_{p_1,p_2\in\Sigma_t,u(p_i)=u_i} \mathrm{dist}_{\mathrm{Euc}}(p_1, p_2)^{1/3}} \\
&\leq 2^{1/3} \sup_{\substack{p_1,p_2\in\Sigma_t,p_1\neq p_2\\ u(p_i)\in[\mathring{\sigma}/2,3\mathring{\sigma}/2]}} \frac{|u_1 - u_2|}{|x^1(\mathfrak{p}_1) - x^1(\mathfrak{p}_2)|^{1/3}} \leq 96^{1/3}\mathring{\upbeta}^{-1/3} \leq 5\mathring{\upbeta}^{-1/3}. \qquad \square
\end{aligned}$$

We are now ready to conclude the proof of Corollary 4.5.

Proof of Corollary 4.5. Our starting point is the observation that the estimates in Theorem 4.2 guarantee that, for at each fixed t with $0 \leq t \leq T_{(\mathrm{sing})}$, the fluid variables and higher-order variables ρ, v^i, Ω^i, S^i, $\mathcal{C}^i$, and $\mathcal{D}$ are all uniformly Lipschitz *when viewed as functions of the* (u, x^2, x^3)*-coordinates*. Therefore, the key to proving Corollary 4.5 is to understand the regularity of the map $(x^1, x^2, x^3) \mapsto (u, x^2, x^3)$.

[68] We can, for instance, take $\mathfrak{p}_1 = \boldsymbol{p}_1$ and let $\mathfrak{p}_2$ be the unique point in both the level set $\{u = u_2\}$ and the line passing through $\mathfrak{p}_1$ with tangent vector everywhere equal to ∂_1.

To this end, we first note that by the assumption (1) in Corollary 4.5, the equations (14-2)–(14-4), (4-9), and the arguments given in the proof of Corollary 4.4, it follows that away from $u \in [3\mathring{\sigma}/4, 5\mathring{\sigma}/4]$, we have $\mu > \frac{1}{2}$. From this lower bound, Lemma 2.22, and the estimates of Proposition 8.7, we see that when $u \notin [3\mathring{\sigma}/4, 5\mathring{\sigma}/4]$, the map $(x^1, x^2, x^3) \mapsto (u, x^2, x^3)$ remains uniformly Lipschitz (in fact, we could prove that it is even more regular). Combined with the aforementioned fact that ρ, v^i, Ω^i, S^i, $\mathcal{C}^i$ and $\mathcal{D}$ are uniformly Lipschitz in the (u, x^2, x^3)-coordinates, we see that at each fixed t, with $0 \le t \le T_{(\mathrm{sing})}$, ρ, v^i, Ω^i, S^i, $\mathcal{C}^i$, and $\mathcal{D}$ are also uniformly Lipschitz in the (x^1, x^2, x^3)-coordinates away from $u \in [3\mathring{\sigma}/4, 5\mathring{\sigma}/4]$. Moreover, (14-1) guarantees that in the region $\{0 \le t \le 3T_{(\mathrm{sing})}/4\}$, we have $\mu > \frac{1}{8}$. Thus, for the same reasons given above, the map $(x^1, x^2, x^3) \mapsto (u, x^2, x^3)$ is uniformly Lipschitz in $\{0 \le t \le 3T_{(\mathrm{sing})}/4\}$, and thus ρ, v^i, Ω^i, S^i, $\mathcal{C}^i$, and $\mathcal{D}$ also remain uniformly Lipschitz in the (x^1, x^2, x^3)-coordinates in this region.

It remains for us to consider the difficult region in which $u \in [3\mathring{\sigma}/4, 5\mathring{\sigma}/4] \subseteq [\mathring{\sigma}/2, 3\mathring{\sigma}/2]$ and $t \in [3T_{(\mathrm{sing})}/4, T_{(\mathrm{sing})})$. Using Lemma 14.3, we see that the map $(x^1, x^2, x^3) \mapsto (u, x^2, x^3)$ is uniformly $C^{1/3}$ in this difficult region. Hence, ρ, v^i, Ω^i, S^i, $\mathcal{C}^i$, and $\mathcal{D}$ all have uniformly bounded Cartesian spatial $C^{1/3}$ norms in this region as well. □

Appendix: Proof of the wave estimates

In this appendix, we sketch the proof of the wave equation estimates, that is, of Proposition 12.1. As we already discussed in Section 12A, although the wave equation estimates that we need are almost identical to the ones derived in [36], there are two differences:

(1) The wave equations in Proposition 12.1 feature the inhomogeneous terms $\vec{\mathfrak{G}}$, and we need to track the influence of these inhomogeneous terms on the estimates. Recall that the precise inhomogeneous terms are located on the right-hand sides of (5-1a)–(5-1c), but for purposes of proving Proposition 12.1, we do not need to know their precise structure.

(2) Recall that our commutation vectorfields $\{L, Y, Z\}$ are constructed out of the acoustic eikonal function u, and hence the commuted wave equations feature error terms that depend on the acoustic geometry. In three dimensions, some additional arguments are needed (compared to the two-dimensional case treated in [36]) to control the top-order derivatives of some of these error terms.

The issue (2) is tied to the fact that the null second fundamental form of null hypersurfaces in 1+3 dimensions has now three independent components, which stands in contrast to the case of 1+2 dimensions, where it has only a single component (i.e., it is trace-free in 1+2 dimensions). This issue is by now very well-understood, and it can be resolved by using an elliptic estimate. For completeness, we will nonetheless sketch the main points needed for the argument in this appendix.

We now further discuss the issue (2). In 1+2 dimensions, $\mathrm{tr}_{g\!\!/}\,\chi$ satisfies a transport equation known as the Raychaudhuri equation[69] (see [52, (6.2.5)]):

$$\mu L\, \mathrm{tr}_{g\!\!/}\,\chi = (L\mu)\, \mathrm{tr}_{g\!\!/}\,\chi - \mu(\mathrm{tr}_{g\!\!/}\,\chi)^2 - \mu\, \mathrm{Ric}_{LL}, \tag{A-1}$$

[69] Note that this is a purely differential geometric identity that is independent of the compressible Euler equations.

where Ric is the Ricci curvature of the acoustical metric g and $\mathrm{Ric}_{LL} \doteq \mathrm{Ric}_{\alpha\beta} L^\alpha L^\beta$. In contrast, in 1+3 dimensions, the right-hand side of (A-1) features some additional terms. Specifically, in 1+3 dimensions, the Raychaudhuri equation takes the following form (see [48, (11.23)]):

$$\mu L \operatorname{tr}_{g\!\!/} \chi = (L\mu) \operatorname{tr}_{g\!\!/} \chi - \mu|\chi|^2 - \mu \mathrm{Ric}_{LL} = (L\mu) \operatorname{tr}_{g\!\!/} \chi - \mu(\operatorname{tr}_{g\!\!/} \chi)^2 - \mu|\hat{\chi}|^2 - \mu \mathrm{Ric}_{LL}, \qquad \text{(A-2)}$$

where $\hat{\chi}$ is the traceless part of χ, i.e., it can be defined by imposing the identity $\chi = \hat{\chi} + \frac{1}{2}(\operatorname{tr}_{g\!\!/} \chi) g\!\!/$. In other words, (A-2) has an additional $-\mu|\hat{\chi}|^2$ term compared to (A-1), and this additional term cannot be bounded using the only the transport equation (A-2), (since the left-hand side of (A-2) features a transport operator acting only on the component $\operatorname{tr}_{g\!\!/} \chi$, as opposed to the full second fundamental form χ). The saving grace, however, as already noticed in [15] (see also [17; 33]), is that one can use geometric identities (specifically, the famous Codazzi equation) and elliptic estimates to control $\nabla\!\!\!/ \hat{\chi}$ in terms of $d\!\!/ \operatorname{tr}_{g\!\!/} \chi$ plus simpler error terms. A top-order version of this kind of argument allows one to control the difficult top-order derivatives of the term $-\mu|\hat{\chi}|^2$ on the right-hand side of (A-2); see Section A5 for the details. We remark that for the solutions under study, the $-\mu|\hat{\chi}|^2$ term is quadratically small and, as it turns out, it does not have much effect on the dynamics.

A1. ***Running assumptions in the appendix and the dependence of constants and parameters.*** Throughout the entire appendix, we work in the setting of Proposition 12.1. In particular, we make the same assumptions as we did in Theorem 6.3 (which provides the main a priori estimates), as well as the smallness assumption (12-1) for the inhomogeneous terms $\vec{\mathfrak{G}}$.

Our analysis involves various constants and parameters that play distinct roles in the proof. We have already introduced these quantities earlier in the article. For the reader's convenience, we again provide a brief description of these quantities in order to help the reader understand their role in our subsequent arguments in the appendix.

- The background density constant $\bar{\varrho} > 0$ was fixed at the beginning of the paper. The parameters $\mathring{\sigma}$, $\mathring{\delta}_*$, $\mathring{\delta}$, $\mathring{\alpha}$ and $\mathring{\epsilon}$ measure the size of the x^1-support and various norms of the initial data; see Section 4A.
- As in the rest of the paper, the positive integer N_{top} denotes the maximum number of times that we commute the equations for the purpose of obtaining L^2-type energy estimates.
- M_{abs} denotes an *absolute constant*, that is, a constant that can be chosen to be independent of N_{top}, the equation of state, $\bar{\varrho}$, $\mathring{\sigma}$, $\mathring{\delta}$, and $\mathring{\delta}_*^{-1}$, as long as $\mathring{\alpha}$ and $\mathring{\epsilon}$ are sufficiently small. The constants M_{abs} arise as numerical coefficients that multiply the borderline energy error integrals; see in particular the right-hand side of (A-37). The universality of the M_{abs} is crucial since, as the next two points clarify, they drive the blowup-rate of the top-order energies, which in turn controls the size of largeness of N_{top} needed to close the proof.
- As in the rest of the paper, the positive integer M_* controls the blowup-rate of the high-order energies. The following point is crucial: *for the proof to close we need to choose M_* to be sufficiently large in a manner that depends only on the absolute constants M_{abs}. In particular, M_* does not depend on N_{top}.*
- Once M_* has been chosen to be sufficiently large (as described in the previous point), for the proof to close *we need to choose N_{top} to be sufficiently large in a manner that depends only on the integer M_* fixed in the previous step.*

- Once N_{top} has been chosen to be sufficiently large (as described in the previous point), to close the proof, we must choose $\mathring{\epsilon}$ to be sufficiently small in a manner that is allowed to depend on all other parameters and constants. We must also choose $\mathring{\alpha}$ to be sufficiently small in a manner that depends only on the equation of state and $\bar{\varrho}$. We always assume that $\mathring{\epsilon}^{1/2} \leq \mathring{\alpha}$.
- In contrast to M_{abs}, the constants C' are less delicate and are allowed to depend on the equation of state, $\bar{\varrho}$, $\mathring{\sigma}$, $\mathring{\delta}$, and $\mathring{\delta}_*^{-1}$. We use the notation C' to emphasize that these constants multiply difficult, borderline energy estimate error terms, but we could have just as well denoted these constants by C (where C has the properties described in the next point), and the proof would go through.
- Unless otherwise stated, "general" constants C are allowed to depend on N_{top}, M_{abs}, the equation of state, $\bar{\varrho}$, $\mathring{\sigma}$, $\mathring{\delta}$, and $\mathring{\delta}_*^{-1}$. When we write $A \lesssim B$, it means that there exists a $C > 0$ with the above dependence properties such that $A \leq CB$. Moreover, $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

A2. ***An outline of the rest of the appendix.*** In Sections A3–A8, we will derive the estimates we need to prove Proposition 12.1. The conclusion of the proof of Proposition 12.1 is located in Section A9.

Proposition 12.1 is an analog of the similar result [36, Proposition 14.1]. In fact, in our proof of the proposition, we will exactly follow the strategy from [36]. For this reason, we will only focus on terms which did not already appear in [36]. We begin by identifying the most difficult wave equation error terms in Section A3. As in [36; 52], these hardest terms are commutator terms involving the top-order derivatives of $\mathrm{tr}_{\not{g}}\,\chi$, which we control using the following steps:

- In Section A4, we write down the transport equations satisfied by the important *modified quantities*. The modified quantities are special combinations of solution variables involving $\mathrm{tr}_{\not{g}}\,\chi$. With the help of the Raychaudhuri equation (A-2), the modified quantities will allow us to avoid the loss of a derivative at the top order and/or allow us to avoid fatal borderline error integrals.
- In Section A5, we use elliptic estimates on $\ell_{t,u}$ to control the top-order derivatives of $\hat{\chi}$ in terms of the modified quantities.
- In Section A6, we define partial energies, which are similar to the energies we defined in Section 3B, but they control all wave variables *except for* the "difficult" one $\mathcal{R}_{(+)}$ (which is such that $|\partial_1 \mathcal{R}_{(+)}|$ blows up as the shock forms). As in [36], the partial energies play an important role in allowing us to close the proof using a universal number of derivatives, that is, a number N_{top} that is independent of the equation of state and all parameters in the problem; the role of these partial energies will be made clear in Section A9.
- In Section A7, we use the transport equations in Section A4 and the estimates in Section A5 to obtain the bounds for the top-order derivatives of $\mathrm{tr}_{\not{g}}\,\chi$.

At this point in the proof, we will have obtained all of the main new estimates we need to prove Proposition 12.1. In Section A8, we use our estimates for the top-order derivatives of $\mathrm{tr}_{\not{g}}\,\chi$ to derive preliminary energy integral inequalities for the wave equation solutions. These are the same integral inequalities that were derived in [36, Proposition 14.3], except they include the new terms generated by the inhomogeneous terms $\vec{\mathfrak{G}}$ featured in the statement of Proposition 12.1. Finally, in Section A9, we use these integral inequalities and a slightly modified version of the Grönwall-type argument used in the proof of [36, Proposition 14.1], carefully tracking the different kinds of constants, thereby obtaining a priori estimates for the energies and concluding the proof of Proposition 12.1.

We close this section with three remarks to help the reader understand how we use cite/use results that were proved in [36].

Remark A.1 (implicit reliance on results we have already proved). The estimates in this appendix rely, in addition to the bootstrap assumptions, on many of the estimates that we independently derived in Section 8, such as the results of Propositions 8.6, 8.7, 8.10, 8.11, 8.12, and 8.14. Many of the results that we cite from [36] rely on these propositions, and we will not always explicitly indicate the dependence of the results of [36] on these propositions.

Remark A.2 (ε vs. $\mathring{\epsilon}^{1/2}$). The bootstrap smallness parameter ε from [36] should be identified with the quantity $\mathring{\epsilon}^{1/2}$ in our bootstrap assumptions (6-4)–(6-8). For this reason, various error terms from [36] reappear in the present paper, but with the factors of ε replaced by $\mathring{\epsilon}^{1/2}$. This minor point has no substantial effect on our analysis, and we will often avoid explicitly pointing out that the error terms from [36] need to be modified as such.

Remark A.3 (vorticity terms have been absorbed into $\vec{\mathfrak{G}}$). Many error terms in the estimates of [36] involve vorticity terms that are generated by the vorticity terms on the right-hand side of the wave equations. However, in this appendix, we have absorbed these error terms into our definition of the inhomogeneous terms $\vec{\mathfrak{G}}$ in Proposition 12.1. For this reason, it is to be understood that many of the estimates cited from [36] have to be modified so that these vorticity terms are absent and are instead replaced with analogous error terms that depend on $\vec{\mathfrak{G}}$ (where throughout the appendix, we carefully explain how the term $\vec{\mathfrak{G}}$ appears in various estimates).

A3. ***The top-order commutator terms that require the modified quantities.*** To begin, we recall that $\{Y, Z\}$ denotes the commutation vectorfields tangent to $\ell_{t,u}$, and that we use the notation $\not{\mathcal{P}}$ to denote a generic element of this set. In the following proposition, we identify the most difficult error terms in the top-order commuted wave equations.

Proposition A.4 (identifying the most difficult commutator terms). *Let $\mathfrak{G}$ denote the inhomogeneous terms in the wave equations from Proposition 12.1. Then solutions to the wave equations of Proposition 12.1 satisfy the following top-order wave equations* (*which identify the most difficult commutator terms*):

$$\mu\,\square_g\,(\not{\mathcal{P}}^{N_{\mathrm{top}}-1}L\Psi) = (\not{d}^{\sharp}\Psi)(\mu\not{d}\not{\mathcal{P}}^{N_{\mathrm{top}}-1}\operatorname{tr}_{\not{g}}\chi) + \not{\mathcal{P}}^{N_{\mathrm{top}}-1}L\mathfrak{G} + \textit{Harmless}, \tag{A-3}$$

$$\mu\,\square_g\,(\not{\mathcal{P}}^{N_{\mathrm{top}}-1}Y\Psi) = (\breve{X}\Psi)(\not{\mathcal{P}}^{N_{\mathrm{top}}-1}Y\operatorname{tr}_{\not{g}}\chi) + c^{-2}X^2(\not{d}^{\sharp}\Psi)(\mu\not{d}\not{\mathcal{P}}^{N_{\mathrm{top}}-1}\operatorname{tr}_{\not{g}}\chi) + \not{\mathcal{P}}^{N_{\mathrm{top}}-1}Y\mathfrak{G} + \textit{Harmless}, \tag{A-4}$$

$$\mu\,\square_g\,(\not{\mathcal{P}}^{N_{\mathrm{top}}-1}Z\Psi) = (\breve{X}\Psi)(\not{\mathcal{P}}^{N_{\mathrm{top}}-1}Z\operatorname{tr}_{\not{g}}\chi) + c^{-2}X^3(\not{d}^{\sharp}\Psi)(\mu\not{d}\not{\mathcal{P}}^{N_{\mathrm{top}}-1}\operatorname{tr}_{\not{g}}\chi) + \not{\mathcal{P}}^{N_{\mathrm{top}}-1}Z\mathfrak{G} + \textit{Harmless}. \tag{A-5}$$

Above, the terms "Harmless" are precisely the $\mathrm{Harmless}^{\le N_{\mathrm{top}}}_{(\mathrm{Wave})}$ *terms defined in* [36, Definition 13.1], *except here we do not need to allow for the presence of vorticity-involving terms in the definition of* $\mathrm{Harmless}^{\le N_{\mathrm{top}}}_{(\mathrm{Wave})}$ *because we have absorbed these terms into our definition of the wave equation inhomogeneous term* $\mathfrak{G}$.

Moreover, for any other top-order operator $\mathcal{P}^{N_{\text{top}}}$ (*i.e., a top-order operator featuring at least two copies of L or featuring only a single L but in an order different from* (A-3)), *there are no difficult commutator terms in the sense that the following equation holds*:

$$\mu \square_g (\mathcal{P}^{N_{\text{top}}}\Psi) = \mathcal{P}^{N_{\text{top}}}\mathfrak{G} + \mathit{Harmless}. \tag{A-6}$$

Proof. This is exactly the same as [36, Proposition 13.2] with the obvious modifications: we have $\{L, Y, Z\}$ (as opposed to just $\{L, Y\}$) as commutation vectorfields, and we have accounted for the presence of the inhomogeneous terms $\mathfrak{G}$. We stress that even in three spatial dimensions, the top-order derivatives of χ that appear on the right-hand sides of (A-3)–(A-5) only involve its trace-part $\mathrm{tr}_{\not{g}}\,\chi$, as opposed to involving the full tensor χ. Roughly speaking, this follows from three basic facts: all of these top-order terms are generated when all $N_{\text{top}}+1$ derivatives (including the two coming from $\square_g$) on the left-hand sides fall on the components $\mathcal{P}^i$ (where $\mathcal{P} \in \{L, Y, Z\}$); all $\mathcal{P}^i$ can be expressed as functions Ψ and L^1, L^2, L^3; and Lemma 2.19 and (13-13) with $f \doteq u$ together imply that $\mu\square_g u = -\,\mathrm{tr}_{\not{g}}\,\chi$. Hence, considering also (2-14), we have, schematically, that $\mu\square_g \partial u = -\partial\,\mathrm{tr}_{\not{g}}\,\chi + \cdots$, where "$\cdots$" denotes terms that involve lower-order derivatives (i.e., up to second-order derivatives) of the eikonal function u and/or derivatives of Ψ. Thus, (2-14), (2-16), (2-17) imply that the scalar functions $\mathcal{P}^i$ satisfy, schematically,[70] $\square_g\mathcal{P}^i = \partial\,\mathrm{tr}_{\not{g}}\,\chi + \cdots$. □

Remark A.5. Notice that in [36, Proposition 13.2], there is an additional difficult commutator term coming from (in the language of the present paper) the commutation with $\breve{X}$. Since in this paper, we use only the subset of energy estimates in [36] that *avoid commutations with* $\breve{X}$, an added benefit of our approach here is that we do not need to handle these additional terms.[71]

A4. ***The modified quantities and the additional terms in the transport equations.*** In order to control the top-order commutator terms from Proposition A.4, the idea from [15] is to introduce modified quantities, which are corrected versions of $\mathrm{tr}_{\not{g}}\,\chi$. The "fully modified quantities" solve transport equations with source terms that enjoy improved regularity, thus allowing us to avoid a loss of regularity at the top order. The "partially modified quantities" lead to cancellations in the energy identities that allow us to avoid error integrals whose singularity strength would have been too severe for us to control.

Definition A.6 (modified versions of the derivatives of $\mathrm{tr}_{\not{g}}\,\chi$). We define, for every[72] fixed string of order-N commutators $\mathcal{P}^N \in \mathscr{P}^{(N)}$, the *fully modified quantity* ${}^{(\mathcal{P}^N)}\mathscr{X}$ as

$${}^{(\mathcal{P}^N)}\mathscr{X} \doteq \mu\mathcal{P}^N\,\mathrm{tr}_{\not{g}}\,\chi + \mathcal{P}^N\mathfrak{X}, \tag{A-7a}$$

$$\mathfrak{X} \doteq -\vec{G}_{LL} \diamond \breve{X}\vec{\Psi} - \tfrac{1}{2}\mu\,\mathrm{tr}_{\not{g}}\,\vec{\not{G}} \diamond L\vec{\Psi} - \tfrac{1}{2}\mu\vec{G}_{LL} \diamond L\vec{\Psi} + \mu\vec{\not{G}}_L^{\#} \diamond \not{d}\vec{\Psi}. \tag{A-7b}$$

[70] Of course, careful geometric decompositions are needed to obtain the precise form of the terms on the right-hand sides of (A-3)–(A-5); here we are simply emphasizing that the dependence of the top-order terms is through the derivatives of $\mathrm{tr}_{\not{g}}\,\chi$.

[71] Of course, even if these terms had been present in our work here, we could have handled them in the same way they were handled in [36].

[72] In practice, we need these quantities only to handle the difficult terms from Proposition A.4, which involve purely $\ell_{t,u}$-tangential derivatives of $\mathrm{tr}_{\not{g}}\,\chi$. Put differently, in practice, we only need to use the quantities ${}^{(\mathscr{P}^N)}\mathscr{X}$.

We define, for every[73] fixed string of order $\mathcal{P}^N \in \mathscr{P}^{(N)}$, the *partially modified quantity* ${}^{(\mathcal{P}^N)}\widetilde{\mathscr{X}}$ as

$$ {}^{(\mathcal{P}^N)}\widetilde{\mathscr{X}} \doteq \mathcal{P}^N \operatorname{tr}_{\not{g}} \chi + {}^{(\mathcal{P}^N)}\widetilde{\mathfrak{X}}, \tag{A-8a}$$

$$ {}^{(\mathcal{P}^N)}\widetilde{\mathfrak{X}} \doteq -\tfrac{1}{2} \operatorname{tr}_{\not{g}} \vec{\not{G}} \diamond L\mathcal{P}^N \vec{\Psi} + \vec{\not{G}}_L^{\#} \diamond \not{d}\mathcal{P}^N \vec{\Psi}. \tag{A-8b}$$

Proposition A.7 (transport equations satisfied by the modified quantities). *The fully modified quantities solve the following modified version of equation* [36, (6.9)], *where* $\vec{\mathfrak{G}}$ *denotes the array of inhomogeneous terms in the wave equations from Proposition 12.1*:

$$\begin{aligned} L^{(\mathcal{P}^{N_{\mathrm{top}}})}\mathscr{X} &- \left(2\frac{L\mu}{\mu} - 2\operatorname{tr}_{\not{g}}\chi\right)^{(\mathcal{P}^{N_{\mathrm{top}}})}\mathscr{X} \\ &= \textit{non-vorticity-involving terms in } [36, (6.9)] - \mathcal{P}^{N_{\mathrm{top}}}(\mu|\hat{\chi}|^2) + \tfrac{1}{2}\mathcal{P}^{N_{\mathrm{top}}}(\vec{G}_{LL} \diamond \vec{\mathfrak{G}}). \end{aligned} \tag{A-9}$$

Moreover, *the partially modified quantities solve the following modified version of equation* [36, (6.10)]:

$$ L^{(\mathcal{P}^{N_{\mathrm{top}}-1})}\widetilde{\mathscr{X}} = \textit{terms in } [36, (6.10)] - \mathcal{P}^{N_{\mathrm{top}}-1}(|\hat{\chi}|^2). \tag{A-10}$$

Remark A.8. We clarify that the vorticity-involving terms in [36, (6.9)] are absent from the right-hand side of (A-9) because we have absorbed these terms into our definition of the wave equation inhomogeneous term $\vec{\mathfrak{G}}$.

Proof of Proposition A.7. The key point is that the derivations of both [36, (6.9), (6.10)] used the Raychaudhuri transport equation satisfied by $\operatorname{tr}_{\not{g}}\chi$, and thus we need to take into account the additional $-\mu|\hat{\chi}|^2$ term in (A-2) as compared to (A-1).

The derivation of [36, (6.9)] consists of two steps. First, in [36, Lemma 6.1], one expresses $\mu\,\mathrm{Ric}_{LL}$ in terms of a sum of two terms: one term is a total L derivative, and the other term is of lower order; see [36, (6.1)]. Step 1 in particular uses the wave equations $\mu\square_{g(\vec{\Psi})}\Psi_\iota = \cdots$. In the second step, one combines the result of [36, Lemma 6.1] with the 1+2-dimensional Raychaudhuri equation (A-1) and then commutes the resulting equation to obtain [36, (6.9)]. In our setting, each step requires a small modification.

- In the first step, instead of $\mu\square_{g(\vec{\Psi})}\Psi_\iota = \cdots$, we have $\mu\square_{g(\vec{\Psi})}\Psi_\iota = \mathfrak{G}_\iota$. Thus, we get an additional term $\frac{1}{2}\mathcal{P}^{N_{\mathrm{top}}}(\vec{G}_{LL} \diamond \vec{\mathfrak{G}})$ on the right-hand side of (A-9).
- In the second step, we need to use the 1+3-dimensional Raychaudhuri equation (A-2) instead of (A-1) and get the extra term $-\mathcal{P}^{N_{\mathrm{top}}}(\mu|\hat{\chi}|^2)$ on the right-hand side of (A-9).

We thus obtain (A-9).

The derivation of [36, (6.10)] is simpler because its proof relies only on the 1+2-dimensional Raychaudhuri equation (A-1) (in particular, it does not rely on the wave equations $\mu\square_{g(\vec{\Psi})}\Psi_\iota = \cdots$). Thus, to obtain (A-10), we simply replace the application of (A-1) from [36, (6.10)] by an application of (A-2). The additional term in (A-10) is a result of the extra $-\mu|\hat{\chi}|^2$ term in (A-2) compared to (A-1). ☐

[73]As in footnote 72, in practice, we only need to use the quantities ${}^{(\mathcal{P}^N)}\widetilde{\mathscr{X}}$.

A5. ***Control of the geometry of $\ell_{t,u}$ and the elliptic estimates for $\hat{\chi}$.*** The following elliptic estimate is standard; see [15, Lemma 8.8].

Lemma A.9 (elliptic estimate for symmetric, trace-free tensorfields). *Let $(\mathcal{M}_2, \gamma\!\!/)$ be a closed, orientable Riemannian manifold, and let μ be a nonnegative function on $\mathcal{M}_2$. Then the following estimate holds for all trace-free symmetric covariant 2-tensorfields ξ belonging to $W^{1,2}(\mathcal{M}_2, \gamma\!\!/)$:*

$$\int_{\mathcal{M}_2} \mu^2\big(\tfrac{1}{2}|\nabla\!\!\!/\,\xi|^2_{\gamma\!\!/} + 2\mathfrak{K}_{\gamma\!\!/}|\xi|^2_{\gamma\!\!/}\big)\, d\mathrm{A}_{\gamma\!\!/} \le 3\int_{\mathcal{M}_2} \mu^2 |\mathrm{d}\!\!/\mathrm{iv}_{\gamma\!\!/}\,\xi|^2_{\gamma\!\!/}\, d\mathrm{A}_{\gamma\!\!/} + 3\int_{\mathcal{M}_2} |\nabla\!\!\!/\,\mu|^2_{\gamma\!\!/}|\xi|^2_{\gamma\!\!/}\, d\mathrm{A}_{\gamma\!\!/}, \tag{A-11}$$

where $\nabla\!\!\!/$, $\mathrm{d}\!\!/\mathrm{iv}_{\gamma\!\!/}$, $\mathfrak{K}_{\gamma\!\!/}$ and $\mathrm{dA}_{\gamma\!\!/}$ are respectively the Levi-Civita connection, divergence operator, Gaussian curvature and induced area measure associated with $\gamma\!\!/$.

In order to use Lemma A.9, we need an L^∞ estimate for the Gaussian curvature of the tori $(\ell_{t,u}, g\!\!/)$. We provide this basic estimate in the following proposition.

Proposition A.10. *The Gaussian curvature $\mathfrak{K}_{g\!\!/}$ of $(\ell_{t,u}, g\!\!/)$ satisfies the following estimate for every $(t,u) \in [0, T_{(\mathrm{Boot})}) \times [0, U_0]$:*

$$\|\mathfrak{K}_{g\!\!/}\|_{L^\infty(\mathcal{M}_{t,u})} \lesssim \mathring{\epsilon}^{1/2}.$$

Proof. It is a standard fact that at fixed (t,u), $\mathfrak{K}_{g\!\!/}$ can be expressed in terms of the components of $g\!\!/$, $g\!\!/^{-1}$ with respect to the coordinate system (x^2, x^3) on $\ell_{t,u}$ and their first and second partial derivatives with respect to the geometric coordinate vectorfields $\partial\!\!\!/_2$, $\partial\!\!\!/_3$. Schematically, we have

$$\mathfrak{K}_{g\!\!/} = g\!\!/^{-1}\cdot g\!\!/^{-1}\cdot \partial\!\!\!/^2 g\!\!/ + g\!\!/^{-1}\cdot g\!\!/^{-1}\cdot g\!\!/^{-1}\cdot \partial\!\!\!/ g\!\!/ \cdot \partial\!\!\!/ g\!\!/,$$

where $\partial\!\!\!/ \in \{\partial\!\!\!/_2, \partial\!\!\!/_3\}$.

Recalling the expression for the induced metric $g\!\!/$ in Lemma 2.32 and the relations between the vectorfields in Lemma 2.24, we see that the desired estimate for $\mathfrak{K}_{g\!\!/}$ follows from Proposition 8.7. □

We now apply the elliptic estimate in Lemma A.9 to control the top-order derivatives of $\hat{\chi}$ in terms of the top-order pure $\ell_{t,u}$-tangential derivatives of $\mathrm{tr}_{g\!\!/}\,\chi$.

Proposition A.11. *The following estimate holds for[74] the N_{top}-th $\ell_{t,u}$-tangential derivatives of $\hat{\chi}$ for every $(t,u) \in [0, T_{(\mathrm{Boot})}) \times [0, U_0]$:*

$$\|\upmu(\not{\mathcal{L}}_{\mathcal{P}\!\!\!/})^{N_{\mathrm{top}}}\hat{\chi}\|_{L^2(\Sigma_t^u)} \lesssim \|\upmu \mathcal{P}\!\!\!/^{N_{\mathrm{top}}}\,\mathrm{tr}_{g\!\!/}\,\chi\|_{L^2(\Sigma_t^u)} + \mathring{\epsilon}^{1/2}\upmu_\star^{-M_*+0.9}(t).$$

Proof. Step 0: preliminaries. Throughout the proof, we will silently use the following observations, valid for $\mathcal{P} \in \{L, Y, Z\}$ and $\mathcal{P}\!\!\!/ \in \{Y, Z\}$, where $\mathrm{f}(\cdot)$ denotes a generic smooth function of its arguments that is allowed to vary from line to line.

- The component functions X^1, X^2, X^3 are smooth functions of the L^i and Ψ; see (2-23). The same holds for the component functions $\mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3$; this is obvious for $\mathcal{P} = L$, while see Lemma 2.23 for $\mathcal{P} = Y, Z$. Similarly, the geometric coordinate component functions $g\!\!/_{AB}$ and $(g\!\!/^{-1})^{AB}$ are smooth functions of the L^i and Ψ; see Lemma 2.32.

[74] Recall that $\not{\mathcal{L}}_{\mathcal{P}\!\!\!/}$ denotes Lie differentiation with respect to elements $\mathcal{P}\!\!\!/ \in \{Y, Z\}$, followed by projection onto $\ell_{t,u}$.

- For $\not\partial \in \{\not\partial_2, \not\partial_3\}$, we have the following schematic identity: $\not\partial = \mathrm{f}(L^i, \Psi)Y + \mathrm{f}(L^i, \Psi)Z$; see Lemma 2.24.
- For $\ell_{t,u}$-tangent one-forms ξ, we have $|\xi| \approx \sum_{A=2,3} |\xi_A| \approx |\xi_Y| + |\xi_Z| \doteq |\xi(Y)| + |\xi(Z)|$; this follows from the discussion in the previous two points, the bootstrap assumptions (6-3)–(6-5), and the L^∞ estimates for $L^i_{(\mathrm{small})}$ from Proposition 8.6. In particular, for scalar functions ϕ, we have $|\nabla\!\!\!/ \phi| \approx \sum_{A=2,3} |\not\partial_A \phi| \approx |Y\phi| + |Z\phi|$. Analogous estimates hold for $\ell_{t,u}$-tangent tensorfields of any order.
- For type-$\binom{0}{n}$ tensorfields, we have the following covariant identity, expressed schematically: $[\nabla\!\!\!/, \not\mathcal{L}_{\mathcal{P}}]\xi = (\nabla\!\!\!/ \not\mathcal{L}_{\mathcal{P}} g\!\!\!/) \cdot \xi$. It is straightforward to check that $\not\mathcal{L}_{\mathcal{P}} g\!\!\!/$ is in fact equal to the $\ell_{t,u}$-projection of the deformation tensor of $\mathcal{P}$ (the deformation tensor itself is equal to $\mathcal{L}_{\mathcal{P}} g$, where g is the acoustical metric).
- Relative to the geometric coordinates (t, u, x^2, x^3), we have $\not\mathcal{L}_{\mathcal{P}} g\!\!\!/ = \mathrm{f}(L^i, \Psi)(\mathcal{P}L^i, \mathcal{P}\Psi)$ (where the $\mathcal{P}$'s on the left- and right-hand sides can be different).
- For $\ell_{t,u}$-tangent tensorfields ξ, we have the following schematic identity, valid relative to the geometric coordinates, where $\not\partial \in \{\not\partial_2, \not\partial_3\}$: $\nabla\!\!\!/ \xi - \not\partial \xi = \mathrm{f}(L^i, \Psi)\xi \cdot (\not\mathcal{P} L^i, \not\mathcal{P}\Psi)$; this follows from expressing $\nabla\!\!\!/$ in terms of geometric coordinate partial derivatives and the Christoffel symbols of $g\!\!\!/$ and then expressing $\not\partial = \mathrm{f}(L^i, \Psi)\not\mathcal{P}$ on the right-hand side.
- For $\ell_{t,u}$-tangent tensorfields ξ, we have the following schematic identity, valid relative to the geometric coordinates, where $\not\partial \in \{\not\partial_2, \not\partial_3\}$: $\not\mathcal{L}_{\not\mathcal{P}} \xi = \not\mathcal{P}^A \not\partial_A \xi + \mathrm{f}(L^i, \Psi)\xi \cdot (\not\mathcal{P} L^i, \not\mathcal{P}\Psi)$ (where the $\not\mathcal{P}$'s on the left- and right-hand sides can be different); this formula is straightforward to verify relative to the geometric coordinates.
- For $\ell_{t,u}$-tangent tensorfields ξ, we have the following schematic identity, valid relative to the geometric coordinates, where $\not\partial \in \{\not\partial_2, \not\partial_3\}$: $\nabla\!\!\!/_{\not\mathcal{P}} \xi - \not\mathcal{L}_{\not\mathcal{P}} \xi = \mathrm{f}(L^i, \vec{\Psi})\xi \cdot (\not\mathcal{P} L^i, \not\mathcal{P}\Psi)$ (where the $\not\mathcal{P}$'s on the left- and right-hand sides can be different); this formula is straightforward to verify relative to the geometric coordinates.
- If f is a scalar function, then $\mathcal{L}_{\mathcal{P}} d\!\!\!/ f = d\!\!\!/ \mathcal{P} f$, where $d\!\!\!/$ denotes $\ell_{t,u}$-gradient of f; this formula is straightforward to verify relative to the geometric coordinates.

Step 1: Codazzi equation.[75] We compute $(\not\mathcal{L}_{\mathcal{P}})^{N_{\mathrm{top}}-1} \nabla\!\!\!/^A \chi_{BA}$ by differentiating (2-38a) with the operator $(\not\mathcal{L}_{\mathcal{P}})^{N_{\mathrm{top}}-1} \mathrm{d}\!\!\!/\mathrm{iv}_{g\!\!\!/}$ and treating all capital Latin indices as tensorial indices, while treating all lowercase Latin indices as corresponding to scalar functions. We clarify that the tensor on the left-hand sides of (2-38a) is symmetric, while the first, third, and fourth products on the right-hand side of (2-38a) are not. Hence, for clarity, we emphasize that when we write "differentiating (2-38a) with $(\not\mathcal{L}_{\mathcal{P}})^{N_{\mathrm{top}}-1} \nabla\!\!\!/^A \chi_{BA}$," it is to be understood that the corresponding first term on the right-hand side is an $\ell_{t,u}$-tangent one-form with index B whose top-order part (in the sense of the number of derivatives that fall on L^a) is $(\not\mathcal{L}_{\mathcal{P}})^{N_{\mathrm{top}}-1}(g_{ab}((g\!\!\!/^{-1})^{AC} \nabla\!\!\!/_C \not\partial_B L^a) \otimes \not\partial_A x^b)) = (\not\mathcal{L}_{\mathcal{P}})^{N_{\mathrm{top}}-1}(g_{ab}((g\!\!\!/^{-1})^{AC} \nabla\!\!\!/_B \not\partial_C L^a) \otimes \not\partial_A x^b))$, where to obtain the last equality, we used the commutation identity $\nabla\!\!\!/_C \not\partial_B L^a = \nabla\!\!\!/_B \not\partial_C L^a$, which is a consequence of the torsion-free property of $\nabla\!\!\!/$ and the fact that we are viewing the Cartesian components L^a as scalar functions. Notice that unless all the N_{top} derivatives fall on the factor $d\!\!\!/ L^a$ in the first product on the

[75] We use the phrase "Codazzi equation" because the equations we use in this analysis are closely related to the classical Codazzi equation, which links $\mathrm{d}\!\!\!/\mathrm{iv}\chi$, $\nabla\!\!\!/ \mathrm{tr}_{g\!\!\!/} \chi$, and the curvature components of the acoustical metric.

right-hand side of (2-38a), the expression involves at most N_{top} derivatives on L and Ψ, and we can control such terms using the bounds we have obtained thus far. In total, using the symmetry property $\chi_{BA} = \chi_{AB}$, isolating the terms featuring the top-order derivatives of the components L^a, and estimating the remaining terms with (6-1)–(6-5) and Propositions 8.6 and 8.14, we obtain

$$\begin{aligned} &\|\mu(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}\not\nabla^A\chi_{AB} - \mu(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}\{g_{ab}(\not{g}^{-1})^{AC}(\not\nabla_C\not\partial_B L^a)(\not\partial_A x^b)\}\|_{L^2(\Sigma_t^u)} \\ &\qquad\qquad \lesssim \|\mu\mathcal{P}^{N_{\text{top}}+1}\Psi\|_{L^2(\Sigma_t^u)} + \|\mu\mathcal{P}^{[1,N_{\text{top}}]}(L^i,\Psi)\|_{L^2(\Sigma_t^u)} \lesssim \mathring{\epsilon}^{1/2}\mu_\star^{-M_*+0.9}(t). \end{aligned} \tag{A-12}$$

We then compute $(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}\not\partial_B \operatorname{tr}_{\not{g}}\chi$ in a similar manner using (2-38b) to obtain

$$\|\mu(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}\not\partial_B \operatorname{tr}_{\not{g}}\chi - \mu(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}\{g_{ab}(\not{g}^{-1})^{AC}(\not\nabla_B\not\partial_C L^a)(\not\partial_A x^b)\}\|_{L^2(\Sigma_t^u)} \lesssim \mathring{\epsilon}^{1/2}\mu_\star^{-M_*+0.9}(t). \tag{A-13}$$

In view of the commutation identity $\not\nabla_C\not\partial_B L^a = \not\nabla_B\not\partial_C L^a$ mentioned above (which implies that the second terms on left-hand sides of (A-12) and (A-13) coincide), we can use (A-12), (A-13), and the triangle inequality to obtain

$$\begin{aligned} \|\mu(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}\not\nabla^A\chi_{AB}\|_{L^2(\Sigma_t^u)} &\lesssim \|\mu(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}\not\partial_B \operatorname{tr}_{\not{g}}\chi\|_{L^2(\Sigma_t^u)} + \mathring{\epsilon}^{1/2}\mu_\star^{-M_*+0.9}(t) \\ &\lesssim \|\mu\not{\mathcal{P}}^{N_{\text{top}}}\operatorname{tr}_{\not{g}}\chi\|_{L^2(\Sigma_t^u)} + \mathring{\epsilon}^{1/2}\mu_\star^{-M_*+0.9}(t), \end{aligned} \tag{A-14}$$

where to obtain the last line, we used the commutation identity $(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}\not\partial_B \operatorname{tr}_{\not{g}}\chi = \not\partial_B\not{\mathcal{P}}^{N_{\text{top}}-1}\operatorname{tr}_{\not{g}}\chi$ (in which we are thinking of both sides as $\ell_{t,u}$-tangent one-forms with components corresponding to the index B), the schematic identity $\not\partial = \mathrm{f}(L^i,\Psi)Y + \mathrm{f}(L^i,\Psi)Z$, and Proposition 8.6.

Now since $(\operatorname{d\!\!/iv}_{\not{g}}\hat{\chi})_B = (\operatorname{d\!\!/iv}_{\not{g}}\chi)_B - \frac{1}{2}\not\partial_B \operatorname{tr}_{\not{g}}\chi = \not\nabla^A\chi_{AB} - \frac{1}{2}\not\partial_B \operatorname{tr}_{\not{g}}\chi$, we deuce from the estimate (A-14) that

$$\|\mu(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}\operatorname{d\!\!/iv}_{\not{g}}\hat{\chi}\|_{L^2(\Sigma_t^u)} \lesssim \|\mu\not{\mathcal{P}}^{N_{\text{top}}}\operatorname{tr}_{\not{g}}\chi\|_{L^2(\Sigma_t^u)} + \mathring{\epsilon}^{1/2}\mu_\star^{-M_*+0.9}(t). \tag{A-15}$$

<u>Step 2</u>: commuting $\operatorname{d\!\!/iv}_{\not{g}}$ with $\not{\mathcal{L}}_{\not{\mathcal{P}}}$ derivatives. We now deduce from (A-15) an estimate for $\operatorname{d\!\!/iv}_{\not{g}}(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}\hat{\chi}$. For this, we simply note that the commutator $[\operatorname{d\!\!/iv}_{\not{g}}, (\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}]\hat{\chi}$ can be controlled by up to N_{top} $\not{\mathcal{P}}$ derivatives of Ψ and L^i. Hence, by (A-15), (6-1)–(6-5), and Propositions 8.6 and 8.14, we have

$$\begin{aligned} \|\mu\operatorname{d\!\!/iv}_{\not{g}}(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}\hat{\chi}\|_{L^2(\Sigma_t^u)} &\lesssim \|\mu(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}\operatorname{d\!\!/iv}_{\not{g}}\hat{\chi}\|_{L^2(\Sigma_t^u)} + \|\mu\not{\mathcal{P}}^{[1,N_{\text{top}}]}(\Psi,L^i)\|_{L^2(\Sigma_t^u)} \\ &\lesssim \|\mu\not{\mathcal{P}}^{N_{\text{top}}}\operatorname{tr}_{\not{g}}\chi\|_{L^2(\Sigma_t^u)} + \mathring{\epsilon}^{1/2}\mu_\star^{-M_*+0.9}(t). \end{aligned} \tag{A-16}$$

<u>Step 3</u>: bounding the trace-part of $(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}\hat{\chi}$. By definition, $\operatorname{tr}_{\not{g}}\hat{\chi} = 0$. Note that the commutator $[\not{g}^{-1}, (\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}]\hat{\chi}$ can be controlled by up to $N_{\text{top}} - 1$ $\not{\mathcal{P}}$ derivatives of Ψ and L^i. Hence, this commutator can be treated in the same way we treated the commutator term in Step 2, which yields the bound

$$\|\operatorname{tr}_{\not{g}}(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}\hat{\chi}\|_{L^2(\Sigma_t^u)} \lesssim \|[\not{g}^{-1}, (\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}]\hat{\chi}\|_{L^2(\Sigma_t^u)} \lesssim \mathring{\epsilon}^{1/2}\mu_\star^{-M_*+0.9}(t). \tag{A-17}$$

Moreover, we can take a further $\not{\mathcal{P}}$-derivative of $\operatorname{tr}_{\not{g}}(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}\hat{\chi}$, and the resulting term can be controlled by up to N_{top} $\not{\mathcal{P}}$ derivatives of Ψ and L^i. Therefore, using (6-1)–(6-5) and Propositions 8.6 and 8.14, we obtain

$$\|\mu\not\nabla(\operatorname{tr}_{\not{g}}(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\text{top}}-1}\hat{\chi})\|_{L^2(\Sigma_t^u)} \lesssim \|\mu\not{\mathcal{P}}^{[1,N_{\text{top}}]}(\Psi,L^i)\|_{L^2(\Sigma_t^u)} \lesssim \mathring{\epsilon}^{1/2}\mu_\star^{-M_*+0.9}(t). \tag{A-18}$$

Step 4: elliptic estimates. Define ξ to be the $\not{g}$-trace-free part of $(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\mathrm{top}}-1}\hat{\chi}$, i.e.,

$$\xi_{AB} \doteq (\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\mathrm{top}}-1}\hat{\chi}_{AB} - \tfrac{1}{2}\not{g}_{AB}\,\mathrm{tr}_{\not{g}}(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\mathrm{top}}-1}\hat{\chi}. \tag{A-19}$$

The term $(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\mathrm{top}}-1}\hat{\chi}_{AB}$ on the right-hand side of (A-19) can be written using (2-38a), (2-38b) as an expression of up to N_{top} $\mathcal{P}$ derivatives of Ψ and L^i. Hence, by (2-38a), (2-38b), (6-1)–(6-5), and Propositions 8.6 and 8.14, we obtain

$$\|(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\mathrm{top}}-1}\hat{\chi}\|_{L^2(\Sigma_t^u)} \lesssim \|\mathcal{P}^{[1,N_{\mathrm{top}}]}(\Psi, L^i)\|_{L^2(\Sigma_t^u)} \lesssim \mathring{\epsilon}^{1/2}\mu_\star^{-M_*+0.9}(t). \tag{A-20}$$

Combining (A-20) with (A-17), we find that

$$\|\xi\|_{L^2(\Sigma_t^u)} \lesssim \mathring{\epsilon}^{1/2}\mu_\star^{-M_*+0.9}(t). \tag{A-21}$$

Moreover, in view of the algebraic relation

$$\not{\mathrm{div}}_{\not{g}}\,\xi = \not{\mathrm{div}}_{\not{g}}(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\mathrm{top}}-1}\hat{\chi} - \tfrac{1}{2}\not{\nabla}(\mathrm{tr}_{\not{g}}(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\mathrm{top}}-1}\hat{\chi})$$

and the estimates (A-16) and (A-18), we have

$$\|\mu\not{\mathrm{div}}_{\not{g}}\,\xi\|_{L^2(\Sigma_t^u)} \lesssim \|\mu\not{\mathcal{P}}^{N_{\mathrm{top}}}\,\mathrm{tr}_{\not{g}}\,\chi\|_{L^2(\Sigma_t^u)} + \mathring{\epsilon}^{1/2}\mu_\star^{-M_*+0.9}(t). \tag{A-22}$$

Therefore, applying the elliptic estimates in Lemma A.9 on $\ell_{t,u}$ with ξ as in (A-19) and $\mu = \mu$, integrating over $u \in [0, U_0]$, and using (A-18), (A-20), (A-21), and (A-22), as well as the Gauss curvature estimate in Proposition A.10 and the estimates of Proposition 8.6 (including the bound $|\not{\nabla}\mu| \lesssim \mathring{\epsilon}^{1/2}$ that it implies), we obtain

$$\begin{aligned}
\|\mu(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\mathrm{top}}}\hat{\chi}\|_{L^2(\Sigma_t^u)} &\lesssim \|\mu\not{\nabla}(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\mathrm{top}}-1}\hat{\chi}\|_{L^2(\Sigma_t^u)} + \|\mu(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\mathrm{top}}-1}\hat{\chi}\|_{L^2(\Sigma_t^u)} \\
&\lesssim \|\mu\not{\nabla}\xi\|_{L^2(\Sigma_t^u)} + \mathring{\epsilon}^{1/2}\mu_\star^{-M_*+0.9}(t) \\
&\lesssim \|\mu\not{\mathrm{div}}_{\not{g}}\,\xi\|_{L^2(\Sigma_t^u)} + (\|\mathfrak{K}_{\not{g}}\|_{L^\infty(\Sigma_t)}^{1/2} + \|\not{\nabla}\mu\|_{L^\infty(\Sigma_t)})\|\xi\|_{L^2(\Sigma_t^u)} + \mathring{\epsilon}^{1/2}\mu_\star^{-M_*+0.9}(t) \\
&\lesssim \|\mu\not{\mathcal{P}}^{N_{\mathrm{top}}}\,\mathrm{tr}_{\not{g}}\,\chi\|_{L^2(\Sigma_t^u)} + \mathring{\epsilon}^{1/2}\mu_\star^{-M_*+0.9}(t),
\end{aligned}$$

which is what we wanted to prove. □

A6. *The partial energies.* To derive our top-order energy estimates for the wave equations, we will use the approach of [36], which relies on distinguishing the "full energies" featured in definitions (3-2a)–(3-2e) (which control all wave variables) from the "partial energies," which are captured by the next definition. The main point is that the partial energies do not control the difficult almost Riemann invariant $\mathcal{R}_{(+)}$ (it is difficult in the sense that the shock formation is driven by the relative largeness of $|\breve{X}\mathcal{R}_{(+)}|$), and it turns out that this leads to easier error terms in the corresponding energy identities. Importantly, we need to distinguish the partial energies from the full energies in order to close the proof using a uniform number of derivatives[76] N_{top}, that is, a number derivatives that does not depend on the equation of state or any parameters in the problem; see the arguments in Section A9 for clarification on the role played by the

[76]We could close the proof without introducing the partial energies, but those simpler, less precise arguments would allow for the possibility that the number of derivatives needed to close the estimates might depend on the equation of state, $\bar{\varrho}$, $\mathring{\sigma}$, $\mathring{\delta}$ and $\mathring{\delta}_*^{-1}$.

partial energies in allowing us to close the proof using a number of derivatives that is independent of the equation of state and all parameters in the problem.

Definition A.12 (the partial energies). At the top-order, we define the partial energy by

$$\mathbb{E}^{(\text{Partial})}_{N_{\text{top}}}(t,u) \doteq \sup_{t'\in[0,t)} \sum_{\widetilde{\Psi}\in\{\mathcal{R}_{(-)},v^2,v^3,s\}} \left(\|\breve{X}\mathcal{P}^{N_{\text{top}}}\widetilde{\Psi}\|^2_{L^2(\Sigma^u_{t'})} + \|\sqrt{\upmu}\mathcal{P}^{N_{\text{top}}+1}\widetilde{\Psi}\|^2_{L^2(\Sigma^u_{t'})} \right).$$

Similarly, we separate the contribution of $\mathcal{R}_{(+)}$ from that of other components of Ψ and define $\mathbb{F}^{(\text{Partial})}_{N_{\text{top}}}$, $\mathbb{K}^{(\text{Partial})}_{N_{\text{top}}}$, $\mathbb{Q}^{(\text{Partial})}_{N_{\text{top}}}$ in an analogous way, that is, as in Section 3B, but without the $\mathcal{R}_{(+)}$-involving terms.

A7. L^2 estimates for the top-order derivatives of $\mathrm{tr}_{\not g}\,\chi$ tied to the modified quantities.

Proposition A.13 (L^2 estimates for the top-order derivatives of $\mathrm{tr}_{\not g}\,\chi$ tied to the fully modified quantities). *There exists an absolute positive constant $M_{\text{abs}} \in \mathbb{N}$, a positive constant $C' \in \mathbb{N}$, and a constant $C > 0$ (each having the properties described in Section A1) such that the following estimates (whose right-hand sides involve the wave energies* (3-2a)–(3-2e) *as well as the partial energies of Definition A.12) holds for every $(t,u) \in [0, T_{(\text{Boot})}) \times [0, U_0]$:*

$$\begin{aligned}
&\|(\breve{X}\mathcal{R}_{(+)})\not{\mathcal{P}}^{N_{\text{top}}}\,\mathrm{tr}_{\not g}\,\chi\|_{L^2(\Sigma^u_t)} \\
&\qquad \le \textit{non-vorticity-involving terms on the right-hand side of } [36, (14.27)] \\
&\qquad\quad \textit{with the boxed constants replaced by } \boxed{M_{\text{abs}}} \textit{ and the constant } C_* \textit{ replaced by } \boxed{C'} \\
&\qquad\qquad + C\mathring{\epsilon}\upmu_\star^{-M_*+0.9}(t) + C\upmu_\star^{-1}(t)\int_{t'=0}^{t'=t} \|\mathcal{P}^{[1,N_{\text{top}}]}\vec{\mathfrak{G}}\|_{L^2(\Sigma^u_{t'})}\,dt', \qquad \text{(A-23)}
\end{aligned}$$

and

$$\begin{aligned}
&\|\upmu\not{\mathcal{P}}^{N_{\text{top}}}\,\mathrm{tr}_{\not g}\,\chi\|_{L^2(\Sigma^u_t)} \\
&\qquad \lesssim \textit{non-vorticity-involving terms on the right-hand side of } [36, (14.28)] \\
&\qquad\qquad + \mathring{\epsilon}\upmu_\star^{-M_*+1.9}(t) + \int_{t'=0}^{t'=t} \|\mathcal{P}^{[1,N_{\text{top}}]}\vec{\mathfrak{G}}\|_{L^2(\Sigma^u_{t'})}\,dt'. \qquad \text{(A-24)}
\end{aligned}$$

Remark A.14. We clarify that in the proofs of [36, (14.27)] and [36, (14.28)], the vorticity-involving inhomogeneous terms in the wave equations led to error integrals on the right-hand sides of [36, (14.27)] and [36, (14.28)] that involved the vorticity energies; in contrast, on the right-hand sides of (A-23)–(A-24), the vorticity-involving terms are not explicitly indicated because we have absorbed them into our definition of the wave equation inhomogeneous term $\vec{\mathfrak{G}}$.

Proof. The proofs of both estimates are similar. We first discuss the proof of (A-24) in Steps 1–2. In Step 3, we describe the changes we need in order to obtain (A-23). Throughout this proof, we freely use the observations made in Step 0 of the proof of Proposition A.11.

Following [36; 52], in order to bound $\upmu\not{\mathcal{P}}^{N_{\text{top}}}\,\mathrm{tr}_{\not g}\,\chi$, we first control the fully modified quantity (recall the definition in (A-7a)), and then bound the difference of $\upmu\not{\mathcal{P}}^{N_{\text{top}}}\,\mathrm{tr}_{\not g}\,\chi$ and the fully modified quantity. See the corresponding estimates in [36, Lemma 13.9, Proposition 13.11, Lemma 14.14].

Step 1: controlling the inhomogeneous terms in (A-9). We first estimate the two new terms on the right-hand sides of (A-9) in the following norms (recall that here we are assuming that in (A-9), $\mathcal{P}^{N_{\mathrm{top}}}$ is equal to a pure $\ell_{t,u}$-tangential operator $\not{\mathcal{P}}^{N_{\mathrm{top}}}$):

$$\int_{t'=0}^{t'=t} \|\not{\mathcal{P}}^{N_{\mathrm{top}}}(\mu|\hat{\chi}|^2)\|_{L^2(\Sigma_{t'}^u)}\,dt', \qquad \int_{t'=0}^{t'=t} \|\not{\mathcal{P}}^{N_{\mathrm{top}}}(\vec{G}_{LL}\diamond\vec{\mathfrak{G}})\|_{L^2(\Sigma_{t'}^u)}\,dt'. \tag{A-25}$$

Step 1(a): the $\not{\mathcal{P}}^{N_{\mathrm{top}}}(\mu|\hat{\chi}|^2)$ term. For the first term in (A-25), the most (and indeed only) difficult contribution arises when all operators $\not{\mathcal{P}}^{N_{\mathrm{top}}}$ fall on one factor of $\hat{\chi}$. For the lower-order terms, we use the identities (2-38a), (2-38b), and $\hat{\chi}_{AB} = \chi_{AB} - \frac{1}{2}\not{g}_{AB}\operatorname{tr}_{\not{g}}\chi$, (6-3)–(6-5), and Proposition 8.6 to obtain the pointwise estimates

$$|\not{\mathcal{P}}^{N_{\mathrm{top}}}(\mu|\hat{\chi}|^2) - 2\mu\hat{\chi}^{\sharp\sharp}(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\mathrm{top}}}\hat{\chi}| \lesssim \mathring{\epsilon}^{1/2}|\not{\mathcal{P}}^{[1,N_{\mathrm{top}}]}(\Psi, L^i, \mu)|. \tag{A-26}$$

From (6-1), (6-2) and Proposition 8.14, and the estimate (A-26), we see that

$$\|\not{\mathcal{P}}^{N_{\mathrm{top}}}(\mu|\hat{\chi}|^2) - 2\mu\hat{\chi}^{\sharp\sharp}(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\mathrm{top}}}\hat{\chi}\|_{L^2(\Sigma_t^u)} \lesssim \mathring{\epsilon}^{1/2}\|\not{\mathcal{P}}^{[1,N_{\mathrm{top}}]}(\Psi, L^i, \mu)\|_{L^2(\Sigma_t^u)} \lesssim \mathring{\epsilon}\mu_\star^{-M_*+1.4}(t). \tag{A-27}$$

On the other hand, the top-order derivative $\mu(\not{\mathcal{L}}_{\not{\mathcal{P}}})^{N_{\mathrm{top}}}\hat{\chi}$ can be bounded using Proposition A.11, while the low-order factor $\hat{\chi}^{\sharp\sharp}$ can be bounded in L^∞ by $\lesssim \mathring{\epsilon}^{1/2}$ by virtue of the bootstrap assumptions (6-3)–(6-5) and the estimates of Proposition 8.6. Therefore, combining (A-27) and Proposition A.11, and then using Proposition 8.11, we bound the first term in (A-25) as

$$\begin{aligned}\int_{t'=0}^{t'=t} \|\not{\mathcal{P}}^{N_{\mathrm{top}}}(\mu|\hat{\chi}|^2)\|_{L^2(\Sigma_{t'}^u)}\,dt' &\lesssim \mathring{\epsilon}^{1/2}\int_{t'=0}^{t'=t}\|\mu\not{\mathcal{P}}^{N_{\mathrm{top}}}\operatorname{tr}_{\not{g}}\chi\|_{L^2(\Sigma_{t'}^u)}\,dt' + \mathring{\epsilon}\int_{t'=0}^{t'=t}\mu_\star^{-M_*+0.9}(t')\,dt'\\ &\lesssim \mathring{\epsilon}^{1/2}\int_{t'=0}^{t'=t}\|\mu\not{\mathcal{P}}^{N_{\mathrm{top}}}\operatorname{tr}_{\not{g}}\chi\|_{L^2(\Sigma_{t'}^u)}\,dt' + \mathring{\epsilon}\mu_\star^{-M_*+1.9}(t).\end{aligned} \tag{A-28}$$

Step 1(b): the $\not{\mathcal{P}}^{N_{\mathrm{top}}}(\vec{G}_{LL}\diamond\vec{\mathfrak{G}})$ term. To handle the second term in (A-25), we simply use Hölder's inequality together with (6-1)–(6-5), Propositions 8.6, 8.14, the assumption (12-1), and Proposition 8.11 to obtain the bound

$$\begin{aligned}&\int_{t'=0}^{t'=t} \|\not{\mathcal{P}}^{N_{\mathrm{top}}}(\vec{G}_{LL}\diamond\vec{\mathfrak{G}})\|_{L^2(\Sigma_{t'}^u)}\,dt'\\ &\quad\lesssim \int_{t'=0}^{t'=t}\left\{\|\not{\mathcal{P}}^{[2,N_{\mathrm{top}}]}(\Psi, L^i)\|_{L^2(\Sigma_{t'}^u)}\|\not{\mathcal{P}}^{\le\lceil N_{\mathrm{top}}/2\rceil}\vec{\mathfrak{G}}\|_{L^\infty(\Sigma_{t'})} + \|\not{\mathcal{P}}^{[1,N_{\mathrm{top}}]}\vec{\mathfrak{G}}\|_{L^2(\Sigma_{t'}^u)}\right\}dt'\\ &\quad\lesssim \int_{t'=0}^{t'=t}\left\{\mathring{\epsilon}\mu_\star^{-M_*+1.4}(t') + \|\not{\mathcal{P}}^{[1,N_{\mathrm{top}}]}\vec{\mathfrak{G}}\|_{L^2(\Sigma_{t'}^u)}\right\}dt'\\ &\quad\lesssim \mathring{\epsilon}\mu_\star^{-M_*+2.4}(t) + \int_{t'=0}^{t'=t}\|\mathcal{P}^{[1,N_{\mathrm{top}}]}\vec{\mathfrak{G}}\|_{L^2(\Sigma_{t'}^u)}\,dt'.\end{aligned} \tag{A-29}$$

Step 2: bounding the fully modified quantity. The fully modified quantity ${}^{(\not{\mathcal{P}}^{N_{\mathrm{top}}})}\mathscr{X}$ satisfies the transport equation (A-9) in the L-direction. We use the arguments given in [36, Proposition 13.11] to integrate the transport equation to obtain a pointwise estimate for ${}^{(\not{\mathcal{P}}^{N_{\mathrm{top}}})}\mathscr{X}$. On the right-hand side of the pointwise estimate there appears, in particular, the time integral of the new terms $\mathscr{P}^{N_{\mathrm{top}}}(\mu|\hat{\chi}|^2)$ and $\frac{1}{2}\mathscr{P}^{N_{\mathrm{top}}}(\vec{G}_{LL}\diamond\vec{\mathfrak{G}})$ on the right-hand side of (A-9). We then take the $L^2(\Sigma_t^u)$ norm of the resulting pointwise inequality, as in the proof of [36, Lemma 14.14]. This yields an $L^2(\Sigma_t^u)$ estimate for ${}^{(\mathscr{P}^{N_{\mathrm{top}}})}\mathscr{X}$. We next use (A-7a) to algebraically express $\mu\mathscr{P}^{N_{\mathrm{top}}}\operatorname{tr}_{\not{g}}\chi$ in terms of ${}^{(\mathscr{P}^{N_{\mathrm{top}}})}\mathscr{X}$ plus a remainder term, and then use the triangle inequality to obtain an $L^2(\Sigma_t^u)$ estimate for $\mu\mathscr{P}^{N_{\mathrm{top}}}\operatorname{tr}_{\not{g}}\chi$. One of the remainder terms is $\mathscr{P}^{N_{\mathrm{top}}}\mathfrak{X}$, and it can be estimated exactly as in [36, Lemma 14.14]. In total, we find that

$$
\begin{aligned}
&\|\mu\mathscr{P}^{N_{\mathrm{top}}}\operatorname{tr}_{\not{g}}\chi\|_{L^2(\Sigma_t^u)}\\
&\quad\lesssim \|{}^{(\mathscr{P}^{N_{\mathrm{top}}})}\mathscr{X}\|_{L^2(\Sigma_t^u)}+\|\mathscr{P}^{N_{\mathrm{top}}}\mathfrak{X}\|_{L^2(\Sigma_t^u)}\\
&\quad\lesssim \text{terms on the right-hand side of [36, (14.28)]}\\
&\qquad\qquad+\int_{t'=0}^{t'=t}\|\mathscr{P}^{N_{\mathrm{top}}}(\mu|\hat{\chi}|^2)\|_{L^2(\Sigma_{t'}^u)}\,dt'+\int_{t'=0}^{t'=t}\|\mathscr{P}^{N_{\mathrm{top}}}(\vec{G}_{LL}\diamond\vec{\mathfrak{G}})\|_{L^2(\Sigma_{t'}^u)}\,dt'\\
&\quad\lesssim \text{terms on the right-hand side of [36, (14.28)]}\\
&\qquad\qquad+\mathring{\epsilon}\mu_\star^{-M_*+1.9}(t)+\mathring{\epsilon}^{1/2}\int_{t'=0}^{t'=t}\|\mu\mathscr{P}^{N_{\mathrm{top}}}\operatorname{tr}_{\not{g}}\chi\|_{L^2(\Sigma_{t'}^u)}\,dt'+\int_{t'=0}^{t'=t}\|\mathscr{P}^{[1,N_{\mathrm{top}}]}\vec{\mathfrak{G}}\|_{L^2(\Sigma_{t'}^u)}\,dt'\\
&\quad\lesssim \text{terms on the right-hand side of [36, (14.28)]}+\mathring{\epsilon}\mu_\star^{-M_*+1.9}(t)+\int_{t'=0}^{t'=t}\|\mathcal{P}^{[1,N_{\mathrm{top}}]}\vec{\mathfrak{G}}\|_{L^2(\Sigma_{t'}^u)}\,dt',
\end{aligned}
$$

where to obtain the next-to-last line, we used the estimates (A-28) and (A-29), and to obtain the last line, we used Grönwall's inequality to eliminate the factor $\mathring{\epsilon}^{1/2}\int_{t'=0}^{t'=t}\|\mu\mathscr{P}^{N_{\mathrm{top}}}\operatorname{tr}_{\not{g}}\chi\|_{L^2(\Sigma_{t'}^u)}\,dt'$ on the right-hand side. We have therefore proved (A-24).

Step 3: proof of (A-23). Estimate (A-23) can be proved using arguments that are very similar to the ones we used in the proof of (A-24), except that we need to keep track of the constants in the borderline terms, i.e., the absolute constant M_{abs} (whose precise value we do not bother to estimate here) and the parameter-dependent constant C'. This can be done exactly as in the proof of [36, (14.27)]. The only terms which are not already present in [36, (14.27)] are exactly those we encountered already in Steps 1–2. These new terms can be treated exactly as in the proof of (A-24), since we do not have to keep track of the sharp constants for these new terms (we instead allow a general constant C). □

Proposition A.15 (L^2 estimates for the partially modified quantities). *There exists an absolute positive constant $M_{\mathrm{abs}}\in\mathbb{N}$, a positive constant $C'\in\mathbb{N}$, and a constant $C>0$ (each having the properties described in Section A1) such that the partially modified quantity ${}^{(\mathscr{P}^{N_{\mathrm{top}}-1})}\widetilde{\mathscr{X}}$ obeys the following estimates (whose right-hand sides involve the wave energies* (3-2a)–(3-2e) *as well as the partial energies of Definition A.12) for every $(t,u)\in[0,T_{(\mathrm{Boot})})\times[0,U_0]$:*

$$
\begin{aligned}
&\left\|\frac{1}{\sqrt{\mu}}(\breve{X}\mathcal{R}_{(+)})L^{(\mathscr{P}^{N_{\mathrm{top}}-1})}\widetilde{\mathscr{X}}\right\|_{L^2(\Sigma_t^u)}\\
&\quad\le \textit{terms on the right-hand side of } \text{[36, (14.32a)]} \textit{ with the boxed constants replaced by } \boxed{M_{\mathrm{abs}}}\\
&\qquad\qquad\qquad\qquad\textit{and the constant } C_* \textit{ replaced by } \boxed{C'}+C\mathring{\epsilon}\mu_\star^{-M_*+0.9}(t),
\end{aligned}
\tag{A-30}
$$

$$\left\| \frac{1}{\sqrt{\mu}} (\breve{X}\mathcal{R}_{(+)})^{(\mathcal{P}^{N_{\mathrm{top}}-1})}\widetilde{\mathscr{X}} \right\|_{L^2(\Sigma_t^u)}$$

$\leq$ *terms on the right-hand side of* [36, (14.32b)] *with the boxed constants replaced by* $\boxed{M_{\mathrm{abs}}}$ *and the constant* C_* *replaced by* $\boxed{C'} + C\mathring{\epsilon}\mu_\star^{-M_*+1.9}(t)$, (A-31)

$$\|L^{(\mathcal{P}^{N_{\mathrm{top}}-1})}\widetilde{\mathscr{X}}\|_{L^2(\Sigma_t^u)} \lesssim \textit{terms in } [36, (14.33a)] + \mathring{\epsilon}\mu_\star^{-M_*+1.4}(t), \tag{A-32}$$

$$\|^{(\mathcal{P}^{N_{\mathrm{top}}-1})}\widetilde{\mathscr{X}}\|_{L^2(\Sigma_t^u)} \lesssim \textit{terms in } [36, (14.33b)] + \mathring{\epsilon}\mu_\star^{-M_*+2.4}(t). \tag{A-33}$$

Proof. To control $L^{(\mathcal{P}^{N_{\mathrm{top}}-1})}\widetilde{\mathscr{X}}$, we bound the terms on the right-hand side of the transport equation (A-10). Note that for this estimate, the only term not already found in [36] is the term $-\mathcal{P}^{N_{\mathrm{top}}-1}(|\hat{\chi}|^2)$. Compared to the estimates for the fully modified quantity that we derived in Proposition A.13, the estimates for the partially modified quantity is simpler in two ways: the transport equation (A-10) does not feature the wave equation inhomogeneous term $\vec{\mathfrak{G}}$, and the additional term only has up to $N_{\mathrm{top}}-1$ derivatives of $\hat{\chi}$, and thus elliptic estimates are not necessary to control this term.

We now estimate $-\mathcal{P}^{N_{\mathrm{top}}-1}(|\hat{\chi}|^2)$. By (2-38a), (6-1)–(6-5), and Propositions 8.6 and 8.14, we have

$$\|\mathcal{P}^{N_{\mathrm{top}}-1}(|\hat{\chi}|^2)\|_{L^2(\Sigma_t^u)} \lesssim \mathring{\epsilon}^{1/2}\|\mathcal{P}^{[1,N_{\mathrm{top}}]}(\Psi, L^i)\|_{L^2(\Sigma_t^u)} \lesssim \mathring{\epsilon}\mu_\star^{-M_*+1.4}(t). \tag{A-34}$$

We now recall (A-10). The terms that are already in terms in [36, (6.10)] can be treated using the same arguments that were used to prove [36, (14.32a)] and [36, (14.33a)], except here we do not bother to estimate the absolute constant M_{abs} that arises in the arguments, and we have renamed the constant C_* as $\boxed{C'}$. From this fact, the estimate (A-34), and the bootstrap assumption (6-3) for $\breve{X}\mathcal{R}_{(+)}$, we deduce (A-30) and (A-32).

To obtain (A-33), we use the transport equation estimate provided by Lemma 8.13, the estimate (A-32) for the source term, Proposition 8.11, and the initial data bound $\|^{(\mathcal{P}^{N_{\mathrm{top}}-1})}\widetilde{\mathscr{X}}(0,\cdot)\|_{L^2(\Sigma_t^u)} \lesssim \mathring{\epsilon}$ obtained in the proof of [36, (14.33b)].

Similarly, (A-31) can be proved using the same arguments used in the proof of [36, (14.32b)]. The estimate is based on integrating the transport equation (A-10) along the integral curves of L and using Lemma 8.13. The only new term we have to handle comes from the $-\mathcal{P}^{N_{\mathrm{top}}-1}(|\hat{\chi}|^2)$ term on the right-hand side of (A-10), and by Lemma 8.13, this term leads to the following additional term that has to be controlled:

$$\frac{1}{\sqrt{\mu_\star(t)}}\|\breve{X}\mathcal{R}_{(+)}\|_{L^\infty(\Sigma_t^u)} \int_{t'=0}^{t'=t} \|\mathcal{P}^{N_{\mathrm{top}}-1}(|\hat{\chi}|^2)\|_{L^2(\Sigma_{t'}^u)}\,dt'.$$

In view of the bootstrap assumption (6-3), the estimate (A-34), and Proposition 8.11, we bound this additional term by $\lesssim \mu_\star^{-M_*+1.9}(t)$, which is less than or equal to the right-hand side of (A-31) as desired. □

A8. ***The main integral inequalities for the energies.*** Our main goal in this section is to prove Proposition A.17, which provides integral inequalities for the various wave energies at various derivative levels. Most of the analysis is the same as in [36]. In the next definition, we highlight the error terms in the energy estimates that are new in the present paper compared to [36]. The new terms stem from the inhomogeneous term $\vec{\mathfrak{G}}$ in the wave equations as well as the $-\upmu|\hat{\chi}|^2$ term on the right-hand side of the three-dimensional Raychaudhuri equation (A-2).

Definition A.16 (new energy estimate error terms). We use the notation $\text{NewError}_{N_{\text{top}}}^{(\text{Top})}(t,u)$ to denote any term that obeys the following bound for every $(t,u) \in [0, T_{(\text{Boot})}) \times [0, U_0]$:

$$\begin{aligned}\text{NewError}_{N_{\text{top}}}^{(\text{Top})}(t,u) \leq C\mathring{\epsilon}^2\upmu_\star^{-2M_*+1.8}(t) + C\int_{t'=0}^{t'=t}\upmu_\star^{-3/2}(t')\left\{\int_{s=0}^{s=t'}\|\mathcal{P}^{[1,N_{\text{top}}]}\vec{\mathfrak{G}}\|_{L^2(\Sigma_s^u)}\,ds\right\}^2 dt' \\ + C\|(|L\mathcal{P}^{[1,N_{\text{top}}]}\Psi| + |\breve{X}\mathcal{P}^{[1,N_{\text{top}}]}\Psi|)|\mathcal{P}^{[1,N_{\text{top}}]}\vec{\mathfrak{G}}|\|_{L^1(\mathcal{M}_{t,u})}, \qquad \text{(A-35)}\end{aligned}$$

where $C > 0$ is a constant of the type described in Section A1.

Similarly, we use the notation $\text{NewError}_{N-1}^{(\text{Below}-\text{Top})}(t,u)$ to denote any term that obeys the following bound for every $(t,u) \in [0, T_{(\text{Boot})}) \times [0, U_0]$:

$$\text{NewError}_{N-1}^{(\text{Below}-\text{Top})}(t,u) \leq C\|(|L\mathcal{P}^{[1,N-1]}\Psi| + |\breve{X}\mathcal{P}^{[1,N-1]}\Psi|)|\mathcal{P}^{[1,N-1]}\vec{\mathfrak{G}}|\|_{L^1(\mathcal{M}_{t,u})}. \qquad \text{(A-36)}$$

Proposition A.17 (the main integral inequalities for the energies). *Let* $\mathbb{Q}_{[1,N]}(t,u)$, $\mathbb{K}_{[1,N]}(t,u)$ *be the wave energies from Section 3B2, and let* $\mathbb{Q}_{[1,N]}^{(\text{Partial})}(t,u)$, $\mathbb{K}_{[1,N]}^{(\text{Partial})}(t,u)$ *be the partial wave energies from Section A6. There exist an absolute constant* $M_{\text{abs}} \in \mathbb{N}$ *and a constant* $C' \in \mathbb{N}$ *depending on the equation of state,* $\bar{\varrho}$, $\mathring{\sigma}$, $\mathring{\delta}$ *and* $\mathring{\delta}_*^{-1}$ *such that the following estimate, which is a modified version of* [36, (14.3)], *hold for every* $(t,u) \in [0, T_{(\text{Boot})}) \times [0, U_0]$:

$$\begin{aligned}&\max\{\mathbb{Q}_{[1,N_{\text{top}}]}(t,u), \mathbb{K}_{[1,N_{\text{top}}]}(t,u)\} \\
&\leq \boxed{M_{\text{abs}}}\int_{t'=0}^{t}\frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_{t'}^u)}}{\upmu_\star(t',u)}\mathbb{Q}_{[1,N_{\text{top}}]}(t',u)\,dt' \\
&\quad + \boxed{M_{\text{abs}}}\int_{t'=0}^{t'=t}\frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_{t'}^u)}}{\upmu_\star(t',u)}\sqrt{\mathbb{Q}_{[1,N_{\text{top}}]}}(t',u)\int_{s=0}^{s=t'}\frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_s^u)}}{\upmu_\star(s,u)}\sqrt{\mathbb{Q}_{[1,N_{\text{top}}]}}(s,u)\,ds\,dt' \\
&\quad + \boxed{M_{\text{abs}}}\frac{\|L\upmu\|_{L^\infty({}^{(-)}\Sigma_{t;t}^u)}}{\upmu_\star^{1/2}(t,u)}\sqrt{\mathbb{Q}_{[1,N_{\text{top}}]}}(t,u)\int_{t'=0}^{t'=t}\frac{1}{\upmu_\star^{1/2}(t',u)}\sqrt{\mathbb{Q}_{[1,N_{\text{top}}]}}(t',u)\,dt' \\
&\quad + \boxed{C'}\int_{t'=0}^{t}\frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_{t'}^u)}}{\upmu_\star(t',u)}\sqrt{\mathbb{Q}_{[1,N_{\text{top}}]}}(t',u)\sqrt{\mathbb{Q}_{[1,N_{\text{top}}]}^{(\text{Partial})}}(t',u)\,dt' \\
&\quad + \boxed{C'}\int_{t'=0}^{t'=t}\frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_{t'}^u)}}{\upmu_\star(t',u)}\sqrt{\mathbb{Q}_{[1,N_{\text{top}}]}}(t',u)\int_{s=0}^{s=t'}\frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_s^u)}}{\upmu_\star(s,u)}\sqrt{\mathbb{Q}_{[1,N_{\text{top}}]}^{(\text{Partial})}}(s,u)\,ds\,dt' \\
&\quad + \boxed{C'}\frac{\|L\upmu\|_{L^\infty({}^{(-)}\Sigma_{t;t}^u)}}{\upmu_\star(t,u)^{1/2}}\sqrt{\mathbb{Q}_{[1,N_{\text{top}}]}}(t,u)\int_{t'=0}^{t'=t}\frac{1}{\upmu_\star^{1/2}(t',u)}\sqrt{\mathbb{Q}_{[1,N_{\text{top}}]}^{(\text{Partial})}}(t',u)\,dt' \\
&\quad + \textit{the error terms } \text{Error}_{N_{\text{top}}}^{(\text{Top})}(t,u) \textit{ defined by } [36, (14.4)] \\
&\quad + \textit{the error terms } \text{NewError}_{N_{\text{top}}}^{(\text{Top})}(t,u) \textit{ defined by } \text{(A-35)}. \qquad \text{(A-37)}\end{aligned}$$

The set ${}^{(-)}\Sigma_{t;t}^u$ *appearing on the right-hand side of* (A-37) *is defined in*[77] [36, Definition 10.4].

[77] We have no need to state its precise definition here; later, we will simply quote the relevant estimates from [36] that are tied to this set.

Moreover, the partial wave energies obey the following estimate, which is a modified version of [36, (14.5)]:

$$\max\{\mathbb{Q}^{(\mathrm{Partial})}_{[1,N_{\mathrm{top}}]}(t,u),\mathbb{K}^{(\mathrm{Partial})}_{[1,N_{\mathrm{top}}]}(t,u)\} \leq \textit{the error terms } \mathrm{Error}^{(\mathrm{Top})}_{N_{\mathrm{top}}}(t,u) \textit{ defined by } [36, (14.4)] \\ + \textit{the error terms } \mathrm{NewError}^{(\mathrm{Top})}_{N_{\mathrm{top}}}(t,u) \textit{ defined by } (\text{A-35}). \quad (\text{A-38})$$

Finally, we have the following below-top-order estimate, which is a modified version[78] *of* [36, (14.6)]:

$$\max\{\mathbb{Q}_{[1,N-1]}(t,u),\mathbb{K}_{[1,N-1]}(t,u)\} \\ \leq C\int_{t'=0}^{t}\frac{1}{\upmu_\star^{1/2}(t',u)}\sqrt{\mathbb{Q}_{[1,N-1]}}(t',u)\int_{s=0}^{t'}\frac{1}{\upmu_\star^{1/2}(s,u)}\sqrt{\mathbb{Q}_{[1,N]}}(s,u)\,ds\,dt' \\ + \textit{the error terms } \mathrm{Error}^{(\mathrm{Below-Top})}_{N-1}(t,u) \textit{ defined by } [36, (14.7)] \\ + \textit{the error terms } \mathrm{NewError}^{(\mathrm{Below-Top})}_{N-1}(t,u) \textit{ defined by } (\text{A-36}). \quad (\text{A-39})$$

Proof. Step 1: proof of (A-39). We begin with (A-39), which is the easier estimate since it is below top-order. Here, we use that [36, (14.6)] is proved by differentiating the wave equation $\upmu\,\square_{g(\vec{\Psi})}\Psi = \cdots$ with $\mathcal{P}^{N'}$, computing the commutator $[\upmu\square_{g(\vec{\Psi})},\mathcal{P}^{N'}]$, multiplying the commuted equation by $(1+2\upmu)L\mathcal{P}^{N'}\Psi+\breve{X}\mathcal{P}^{N'}\Psi$, and then integrating (with respect to the volume form $d\varpi$ Definition 3.1) by parts over the spacetime region $\mathcal{M}_{t,u}$ (for $1\leq N'\leq N-1$). Hence, to prove (A-39), we repeat the argument in [36], except that here we simply denote all of the inhomogeneous terms in the wave equations as $\mathfrak{G}$. That is, we start with the wave equations $\upmu\square_{g(\vec{\Psi})}\Psi_\iota=\mathfrak{G}_\iota$ and commute them to obtain the wave equations $\upmu\square_{g(\vec{\Psi})}\mathcal{P}^{N'}\Psi_\iota=[\upmu\square_{g(\vec{\Psi})},\mathcal{P}^{N'}]\Psi_\iota+\mathcal{P}^{N'}\mathfrak{G}_\iota$. The main point is that for the below-top-order estimates, all commutator terms $[\upmu\square_{g(\vec{\Psi})},\mathcal{P}^{N'}]\Psi_\iota$ can be handled exactly as in [36]. These commutator terms lead to the presence of the first term on the right-hand side of (A-39), as well as the error term $\mathrm{Error}^{(\mathrm{Below-Top})}_{N-1}(t,u)$ on the right-hand side of (A-39). We clarify that in the proof of [36, (14.6)], the vorticity-involving inhomogeneous terms in the wave equation led to error integrals on the right-hand side of [36, (14.6)] that involved the vorticity energies; in contrast, on the right-hand side of (A-39), the vorticity-involving terms are not explicitly indicated because we have absorbed them into our definition of $\mathfrak{G}_\iota$. Thus, to complete the proof of (A-39), we only have to discuss the contribution of the inhomogeneous term $\mathfrak{G}_\iota$. From the above discussion, it follows that we only have to show that the following energy identity error integrals are bounded above in magnitude by the right-hand side of (A-39) when $1\leq N'\leq N-1$ and $(t,u)\in[0,T_{(\mathrm{Boot})})\times[0,U_0]$:

$$\int_{\mathcal{M}_{t,u}}\{(1+2\upmu)L\mathcal{P}^{N'}\Psi+\breve{X}\mathcal{P}^{N'}\Psi\}\mathcal{P}^{N'}\mathfrak{G}\,d\varpi.$$

[78] Note that the lower-order estimate [36, (14.6)] is easier and has fewer additional terms. This is because to obtain the top-order estimates [36, (14.3), (14.5)], one needs to bound all of the commutator terms, including the difficult ones identified in Proposition A.4, without losing derivatives. In contrast, to obtain the lower-order estimates [36, (14.6)], one is allowed to lose a derivative, as is manifested by the double-time-integral term on the right-hand side of (A-39). This double-time-integral will eventually be responsible for the coupling between the energies of different orders; see in particular the estimates (12-3) in the statement of Proposition 12.1.

The desired estimate is simple; in view of the L^∞ estimates for $\upmu$ provided by Proposition 8.6, we see that these error integrals are all bounded by $C\|(|L\mathcal{P}^{[1,N-1]}\Psi| + |\breve{X}\mathcal{P}^{[1,N-1]}\Psi|)|\mathcal{P}^{[1,N-1]}\mathfrak{G}|\|_{L^1(\mathcal{M}_{t,u})}$, which are exactly the error terms we have defined in (A-36).

Step 2: proof of (A-37).

Step 2(a): preliminaries. As in our proof of (A-39), to prove (A-37), the only new step compared to [36] is tracking the contribution of the wave equation inhomogeneous terms $\mathfrak{G}_\iota$ to the energy estimates. As in Step 1, one way in which this inhomogeneous term contributes to the energy estimates is through the error terms $C\|(|L\mathcal{P}^{[1,N_{\text{top}}]}\Psi| + |\breve{X}\mathcal{P}^{[1,N_{\text{top}}]}\Psi|)|\mathcal{P}^{[1,N_{\text{top}}]}\vec{\mathfrak{G}}|\|_{L^1(\mathcal{M}_{t,u})}$, which are found on the right-hand side of (A-35). However, in the top-order case, there is a second way in which $\mathfrak{G}_\iota$ contributes to the top-order energy estimates. To explain this contribution, we first note that, as in the proof of [36, (14.3)], we have to handle some additional difficult top-order commutator terms involving the top-order derivatives of $\mathrm{tr}_{\not g}\,\chi$. Specifically, these difficult top-order commutator terms are explicitly listed on the right-hand sides of (A-3)–(A-5). Recalling that we multiply the wave equation by $(1+2\upmu)L\mathcal{P}^{N'}\Psi + \breve{X}\mathcal{P}^{N'}\Psi$ to derive the wave equation energy estimates at level N', we see that up to harmless factors that are $\mathcal{O}(1)$ by virtue of the estimates of Proposition 8.7, these difficult commutator terms lead to the following three error integrals in the top-order energy estimates:

$$\int_{\mathcal{M}_{t,u}} (\breve{X}\mathcal{P}^{N_{\text{top}}}\Psi)(\breve{X}\Psi)\not{\mathcal{P}}^{N_{\text{top}}}\,\mathrm{tr}_{\not g}\,\chi\,d\varpi,$$
$$\int_{\mathcal{M}_{t,u}} \{(1+2\upmu)L\mathcal{P}^{N_{\text{top}}}\Psi\}(\breve{X}\Psi)\not{\mathcal{P}}^{N_{\text{top}}}\,\mathrm{tr}_{\not g}\,\chi\,d\varpi,$$
$$\int_{\mathcal{M}_{t,u}} \{(1+2\upmu)L\mathcal{P}^{N_{\text{top}}}\Psi + \breve{X}\mathcal{P}^{N_{\text{top}}}\Psi\}(\not{\mathcal{P}}\Psi)\upmu\not{\mathcal{P}}^{N_{\text{top}}}\,\mathrm{tr}_{\not g}\,\chi\,d\varpi.$$

We will control these three terms, respectively, in Steps 2(b)–(d) below.

Step 2(b): contributions from $\int_{\mathcal{M}_{t,u}} (\breve{X}\mathcal{P}^{N_{\text{top}}}\Psi)(\breve{X}\Psi)\not{\mathcal{P}}^{N_{\text{top}}}\,\mathrm{tr}_{\not g}\,\chi\,d\varpi$. We first consider the case $\Psi = \mathcal{R}_{(+)}$, which is by far the most difficult case. Using Hölder's inequality and the estimate (A-23) in Proposition A.13, we deduce that

$$\begin{aligned}
&\left|\int_{\mathcal{M}_{t,u}} (\breve{X}\mathcal{P}^{N_{\text{top}}}\mathcal{R}_{(+)})(\breve{X}\mathcal{R}_{(+)})\not{\mathcal{P}}^{N_{\text{top}}}\,\mathrm{tr}_{\not g}\,\chi\,d\varpi\right| \\
&\le \int_{t'=0}^{t'=t} \|\breve{X}\mathcal{P}^{N_{\text{top}}}\mathcal{R}_{(+)}\|_{L^2(\Sigma_{t'}^u)} \|(\breve{X}\mathcal{R}_{(+)})\not{\mathcal{P}}^{N_{\text{top}}}\,\mathrm{tr}_{\not g}\,\chi\|_{L^2(\Sigma_{t'}^u)}\,dt' \\
&\le \text{terms on the right-hand sides of [36, (14.3)] with the boxed constants replaced by } \boxed{M_{\text{abs}}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{and the constant } C_* \text{ replaced by } \boxed{C'} \\
&\quad + \underbrace{C\int_{t'=0}^{t'=t} \|\breve{X}\mathcal{P}^{N_{\text{top}}}\mathcal{R}_{(+)}\|_{L^2(\Sigma_{t'}^u)}\mathring{\epsilon}\upmu_\star^{-M_*+0.9}(t')\,dt'}_{\doteq \mathrm{I}} \\
&\quad + C\overbrace{\int_{t'=0}^{t'=t} \upmu_\star^{-1}(t)\|\breve{X}\mathcal{P}^{N_{\text{top}}}\mathcal{R}_{(+)}\|_{L^2(\Sigma_{t'}^u)}\left\{\int_{s=0}^{s=t'} \|\mathcal{P}^{[1,N_{\text{top}}]}\mathfrak{G}\|_{L^2(\Sigma_s^u)}\,ds\right\}dt'}^{\doteq \mathrm{II}}.
\end{aligned} \tag{A-40}$$

We clarify that Remark A.14 also applies to the terms on the right-hand sides of [36, (14.3)] (some of which also appear on the right-hand side of (A-40)).

To handle the term I in (A-40), we use Cauchy–Schwarz inequality in t' and Proposition 8.11 to deduce

$$
\begin{aligned}
&\int_{t'=0}^{t'=t} \|\breve{X}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{R}_{(+)}\|_{L^2(\Sigma_{t'}^u)}\mathring{\epsilon}\mu_\star^{-M_*+0.9}(t')\,dt' \\
&\qquad\qquad \lesssim \int_{t'=0}^{t'=t} \mu_\star^{-1/2}(t)\|\breve{X}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{R}_{(+)}\|^2_{L^2(\Sigma_{t'}^u)}\,dt' + \mathring{\epsilon}^2\int_{t'=0}^{t'=t}\mu_\star^{-2M_*+2.3}(t')\,dt' \\
&\qquad\qquad \lesssim \int_{t'=0}^{t'=t} \mu_\star^{-1/2}(t)\|\breve{X}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{R}_{(+)}\|^2_{L^2(\Sigma_{t'}^u)}\,dt' + \mathring{\epsilon}^2\mu_\star^{-M_*+3.3}(t).
\end{aligned}
\tag{A-41}
$$

For the term II in (A-40), we apply first the Cauchy–Schwarz inequality in t' and then Young's inequality to obtain

$$
\begin{aligned}
&\int_{t'=0}^{t'=t} \mu_\star^{-1}(t)\|\breve{X}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{R}_{(+)}\|_{L^2(\Sigma_{t'}^u)}\left\{\int_{s=0}^{s=t'}\|\mathcal{P}^{[1,N_{\mathrm{top}}]}\mathfrak{G}\|_{L^2(\Sigma_s^u)}\,ds\right\}dt' \\
&\lesssim \int_{t'=0}^{t'=t} \mu_\star^{-1/2}(t)\|\breve{X}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{R}_{(+)}\|^2_{L^2(\Sigma_{t'}^u)}\,dt' + \int_{t'=0}^{t'=t}\mu_\star^{-3/2}(t')\left\{\int_{s=0}^{s=t'}\|\mathcal{P}^{[1,N_{\mathrm{top}}]}\mathfrak{G}\|_{L^2(\Sigma_s^u)}\,ds\right\}^2 dt'.
\end{aligned}
\tag{A-42}
$$

Notice that the term

$$
\int_{t'=0}^{t'=t} \mu_\star^{-1/2}(t)\|\breve{X}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{R}_{(+)}\|^2_{L^2(\Sigma_{t'}^u)}\,dt'
$$

appearing on the right-hand sides of both (A-41) and (A-42) is bounded above by

$$
\int_{t'=0}^{t'=t} \mu_\star^{-1/2}(t',u)\mathbb{Q}_{N_{\mathrm{top}}}(t',u)\,dt',
$$

which is among the error terms $\mathrm{Error}^{(\mathrm{Top})}_{N_{\mathrm{top}}}(t,u)$ defined by [36, (14.4)]. Therefore, combining (A-40)–(A-42) and taking into account (A-35), we obtain that

$$
\|(\breve{X}\mathcal{P}^{N_{\mathrm{top}}}\mathcal{R}_{(+)})(\breve{X}\mathcal{R}_{(+)})\not{\mathcal{P}}^{N_{\mathrm{top}}}\operatorname{tr}_{\not{g}}\chi\|_{L^1(\mathcal{M}_{t,u})} \leq \text{the right-hand side of (A-37)} \tag{A-43}
$$

as desired.

We also need to bound the integral

$$
\int_{\mathcal{M}_{t,u}} (\breve{X}\mathcal{P}^{N_{\mathrm{top}}}\Psi)(\breve{X}\Psi)\not{\mathcal{P}}^{N_{\mathrm{top}}}\operatorname{tr}_{\not{g}}\chi\,d\varpi
$$

in the remaining cases $\Psi \in \{R_{(-)}, v^2, v^3, s\}$. As we further explain below in Step 3, a similar argument allows us to bound these error integrals by exploiting one crucial simplifying feature: these error integrals are bounded by the right-hand side of (A-37), but *without the difficult boxed-constant-involving integrals* on the right-hand side. The difference is that we can take advantage of the smallness of the factor $\|\breve{X}\Psi\|_{L^\infty(\Sigma_t)} \leq \mathring{\epsilon}^{1/2}$ (valid for $\Psi \in \{\mathcal{R}_{(-)}, v^2, v^3, s\}$ — but not for $\mathcal{R}_{(+)}$!), which is provided by the bootstrap assumption (6-4); this allows us to avoid the error terms with large boxed constants and thus allows us to relegate the contribution of these error integrals to the error term $\mathrm{Error}^{(\mathrm{Top})}_{N_{\mathrm{top}}}(t,u)$ on the right-hand side of (A-37); we refer to [36, pg. 154] for further details.

Step 2(c): contributions from $\int_{\mathcal{M}_{t,u}}\{(1+2\mu)L\mathcal{P}^{N_{\mathrm{top}}}\Psi\}(\breve{X}\Psi)\not{\mathcal{P}}^{N_{\mathrm{top}}}\operatorname{tr}_{\not{g}}\chi\,d\varpi$. We first consider the case $\Psi = \mathcal{R}_{(+)}$, which is by far the most difficult case. Unlike the error integral we controlled in Step 2(b), as in [36], this error integral can be controlled by first using the definition (A-8a) of the partially modified quantities to algebraically replace the factor $\not{\mathcal{P}}^{N_{\mathrm{top}}} \mathrm{tr}_{\not{g}}\, \chi$ with a $\not{\mathcal{P}}$ derivative of ${}^{(\not{\mathcal{P}}^{N_{\mathrm{top}}-1})}\widetilde{\mathscr{X}}$ plus remainder terms (that one controls separately), and then using integration by parts to swap the L and $\not{\mathcal{P}}$ derivatives. Notice that by Proposition A.15, the partially modified quantity obeys the same bounds as in [36, Lemma 14.19], except the estimates of Proposition A.15 feature $\mathring{\epsilon}$-multiplied terms such as $C\mathring{\epsilon}\mu_{\star}^{-M_*+1.4}(t)$ on the right-hand sides, which can be handled using arguments of the type we used to control the error term (A-41). In particular, the right-hand sides of the estimates in Proposition A.15 do not involve the wave equation inhomogeneity $\mathfrak{G}$. Hence, the error integral $\int_{\mathcal{M}_{t,u}} \{(1+2\mu)L\mathcal{P}^{N_{\mathrm{top}}}\mathcal{R}_{(+)}\}(\breve{X}\mathcal{R}_{(+)})\not{\mathcal{P}}^{N_{\mathrm{top}}} \mathrm{tr}_{\not{g}}\, \chi\, d\varpi$ can be bounded using exactly the same arguments given in [36, Lemma 14.17] and [52, Lemma 14.12], except with the boxed constants from [36] replaced by $\boxed{M_{\mathrm{abs}}}$ and the constant C_* from [36] replaced by $\boxed{C'}$. As a consequence, the error integral under consideration is bounded above in magnitude by the right-hand side of (A-37).

To bound the integral

$$\int_{\mathcal{M}_{t,u}} \{(1+2\mu)L\mathcal{P}^{N_{\mathrm{top}}}\Psi\}(\breve{X}\Psi)\not{\mathcal{P}}^{N_{\mathrm{top}}} \mathrm{tr}_{\not{g}}\, \chi\, d\varpi$$

in the remaining cases $\Psi \in \{R_{(-)}, v^2, v^3, s\}$, we can again (as in Step 2(b)) take advantage of the smallness $\|\breve{X}\Psi\|_{L^\infty(\Sigma_t)} \leq \mathring{\epsilon}^{1/2}$ (valid for $\Psi \in \{\mathcal{R}_{(-)}, v^2, v^3, s\}$ — but not for $\mathcal{R}_{(+)}$!), which is provided by the bootstrap assumption (6-4). This again allows us to relegate the contribution of these integrals to the error term $\mathrm{Error}_{N_{\mathrm{top}}}^{(\mathrm{Top})}(t,u)$ on the right-hand side of (A-37); see [36, p. 154] for further details.

<u>Step 2(d)</u>: contributions from $\int_{\mathcal{M}_{t,u}} \{(1+2\mu)L\mathcal{P}^{N_{\mathrm{top}}}\Psi + \breve{X}\mathcal{P}^{N_{\mathrm{top}}}\Psi\}(\not{\mathcal{P}}\Psi)\mu\not{\mathcal{P}}^{N_{\mathrm{top}}} \mathrm{tr}_{\not{g}}\, \chi\, d\varpi$. This error integral is similar to the one we treated in Step 2(b), but easier. Here are the differences:

- There is an additional μ factor.
- There is a $L\mathcal{P}^{N_{\mathrm{top}}}\Psi$ term, in addition to a $\breve{X}\mathcal{P}^{N_{\mathrm{top}}}\Psi$ term.
- There is a factor of $\not{\mathcal{P}}\Psi$ instead of $\breve{X}\Psi$.

Notice that due to the additional factor of μ, we can control the $L^2(\Sigma_t^u)$ norm of $\sqrt{\mu}L\mathcal{P}^{N_{\mathrm{top}}}\Psi$ by the $\mathbb{Q}_{N_{\mathrm{top}}}$ energy (recall the definition (3-2a) for the energy). Moreover, comparing (6-5) with (6-3), we see that the factor $\not{\mathcal{P}}\Psi$ gives an additional $\mathring{\epsilon}^{1/2}$ L^∞-smallness factor compared to $\breve{X}\mathcal{R}_{(+)}$. Therefore, we can use Hölder's inequality, (6-5), the L^∞ bound for μ in Proposition 8.6, (A-24) in Proposition A.13, and Proposition 8.11 and argue as in Step 2(b) (taking into account (A-35)) to obtain

$$\begin{aligned}
&\left| \int_{\mathcal{M}_{t,u}} \{(1+2\mu)L\mathcal{P}^{N_{\mathrm{top}}}\Psi + \breve{X}\mathcal{P}^{N_{\mathrm{top}}}\Psi\}(\not{\mathcal{P}}\Psi)\mu\not{\mathcal{P}}^{N_{\mathrm{top}}} \mathrm{tr}_{\not{g}}\, \chi\, d\varpi \right| \\
&\quad \lesssim \mathring{\epsilon}^{1/2} \int_{t'=0}^{t'=t} \mu_{\star}^{-1/2}(t',u)\big(\|\breve{X}\mathcal{P}^{N_{\mathrm{top}}}\Psi\|_{L^2(\Sigma_{t'}^u)} + \|\sqrt{\mu}L\mathcal{P}^{N_{\mathrm{top}}}\Psi\|_{L^2(\Sigma_{t'}^u)}\big)\|\mu\not{\mathcal{P}}^{N_{\mathrm{top}}} \mathrm{tr}_{\not{g}}\, \chi\|_{L^2(\Sigma_{t'}^u)}\, dt' \\
&\quad \lesssim \mathring{\epsilon}^{1/2} \int_{t'=0}^{t'=t} \mu_{\star}^{-1/2}(t',u)\mathbb{Q}_{N_{\mathrm{top}}}(t',u)\, dt' + \mathring{\epsilon}^{5/2}\mu_{\star}^{-2M_*+4.3}(t) \\
&\qquad\qquad + \mathring{\epsilon}^{1/2} \int_{t'=0}^{t'=t} \mu_{\star}^{-1/2}(t',u)\left\{\int_{s=0}^{s=t'} \|\mathcal{P}^{[1,N_{\mathrm{top}}]}\mathfrak{G}\|_{L^2(\Sigma_s^u)}\, ds\right\}^2 dt' \\
&\quad \leq \text{non-boxed-constant-involving terms on the right-hand side of (A-37).}
\end{aligned}$$

Combining Steps 2(a)–2(d), we arrive at the desired bound (A-37).

Step 3: proof of (A-38). In this step, we only have to derive top-order energy estimates for $\mathcal{R}_{(-)}$, v^2, v^3, s. This is in contrast to Step 2, in which we also had to derive energy estimates for $\mathcal{R}_{(+)}$. The proof of (A-38) is the same as the proof of [36, (14.5)], except we have to account for the contribution of the inhomogeneous terms $\mathfrak{G}_\iota$ in the wave equations satisfied by $\widetilde{\Psi} \in \{\mathcal{R}_{(-)}, v^2, v^3, s\}$. For the same reason as in Step 2, these inhomogeneous terms lead to error integrals that are controlled by the terms $\mathrm{NewError}_{N_{\mathrm{top}}}^{(\mathrm{Top})}(t, u)$ on the right-hand side of (A-38). We clarify that the proof of (A-38) requires that we control the difficult error integrals

$$\int_{\mathcal{M}_{t,u}} (\breve{X}\mathcal{P}^{N_{\mathrm{top}}}\widetilde{\Psi})(\breve{X}\widetilde{\Psi})\not{\mathcal{P}}^{N_{\mathrm{top}}} \operatorname{tr}_{\not{g}} \chi \, d\varpi,$$

$$\int_{\mathcal{M}_{t,u}} \{(1+2\mu)L\mathcal{P}^{N_{\mathrm{top}}}\widetilde{\Psi}\}(\breve{X}\widetilde{\Psi})\not{\mathcal{P}}^{N_{\mathrm{top}}} \operatorname{tr}_{\not{g}} \chi \, d\varpi,$$

$$\int_{\mathcal{M}_{t,u}} \{(1+2\mu)L\mathcal{P}^{N_{\mathrm{top}}}\widetilde{\Psi} + \breve{X}\mathcal{P}^{N_{\mathrm{top}}}\widetilde{\Psi}\}(\mathcal{P}\widetilde{\Psi})\mu\not{\mathcal{P}}^{N_{\mathrm{top}}} \operatorname{tr}_{\not{g}} \chi \, d\varpi,$$

as in Step 2. In Step 2, the first two of these error integrals led to error terms that are controlled by the boxed-constant-involving terms on the right-hand side of (A-37). However, in Step 3, we can take advantage of the smallness of the factors $\breve{X}\widetilde{\Psi}$ in these integrals. That is, we can exploit the smallness estimate $\|\breve{X}\widetilde{\Psi}\|_{L^\infty(\Sigma_t)} \leq \mathring{\varepsilon}^{1/2}$ (valid for $\widetilde{\Psi} \in \{\mathcal{R}_{(-)}, v^2, v^3, s\}$ — but not for $\mathcal{R}_{(+)}$!), which is provided by the bootstrap assumption (6-4); this allows us to avoid the error terms with large boxed constants (which are found on the right-hand of (A-37)), and allow us to relegate the contribution of the corresponding error integrals to the error term $\mathrm{Error}_{N_{\mathrm{top}}}^{(\mathrm{Top})}(t, u)$ on the right-hand side of (A-38). See [36, p. 154] for further details. □

A9. *Sketch of the proof of Proposition 12.1.* The argument here is the same as in the proof of [36, Proposition 14.1], except we have to handle the additional terms in Proposition A.17.

Sketch of proof of Proposition 12.1. Step 1: the top- and penultimate- orders (proof of (12-2)). It turns out that the top-order energies are heavily coupled to the penultimate-order energies. In turn, this forces us to perform a Grönwall-type argument that simultaneously handles the top- and penultimate-order energy estimates at the same time. For these reasons, we follow the notation of [36, Proposition 14.1] and define[79]

$$F(t,u) \doteq \sup_{(\hat{t},\hat{u})\in[0,t]\times[0,u]} \iota_F^{-1}(\hat{t},\hat{u}) \max\{\mathbb{Q}_{[1,N_{\mathrm{top}}]}(\hat{t},\hat{u}),\ \mathbb{K}_{[1,N_{\mathrm{top}}]}(\hat{t},\hat{u})\}, \tag{A-44}$$

$$G(t,u) \doteq \sup_{(\hat{t},\hat{u})\in[0,t]\times[0,u]} \iota_G^{-1}(\hat{t},\hat{u}) \max\{\mathbb{Q}_{[1,N_{\mathrm{top}}]}^{(\mathrm{Partial})}(\hat{t},\hat{u}),\ \mathbb{K}_{[1,N_{\mathrm{top}}]}^{(\mathrm{Partial})}(\hat{t},\hat{u})\}, \tag{A-45}$$

$$H(t,u) \doteq \sup_{(\hat{t},\hat{u})\in[0,t]\times[0,u]} \iota_H^{-1}(\hat{t},\hat{u}) \max\{\mathbb{Q}_{[1,N_{\mathrm{top}}-1]}(\hat{t},\hat{u}),\ \mathbb{K}_{[1,N_{\mathrm{top}}-1]}(\hat{t},\hat{u})\}, \tag{A-46}$$

where

$$\iota_1(t) \doteq \int_{t'=0}^{t'=t} \frac{1}{\sqrt{T_{(\mathrm{Boot})}-t'}}\,dt', \quad \iota_F(t,u) = \iota_G(t,u) \doteq \mu_\star^{-2M_*+1.8}(t)\iota_1^c(t)\iota_2^c(t)e^{ct}e^{cu},$$

$$\iota_2(t) \doteq \int_{t'=0}^{t'=t} \mu_\star^{-0.9}(t')\,dt', \qquad \iota_H(t,u) \doteq \mu_\star^{-2M_*+3.8}(t)\iota_1^c(t)\iota_2^c(t)e^{ct}e^{cu}.$$

[79] For easy comparisons with the proof of [36, Proposition 14.1], we are using the notation F, G, and H here. The reader should be careful to distinguish these functions from the different functions F and G in Definitions 2.3 and 2.12.

Following exactly the same[80] argument[81] used in the proof of [36, Proposition 14.1] (see in particular [36, (14.64)–(14.66)]), but taking into account the additional terms in Proposition A.17, we can choose $M_* \in \mathbb{N}$ and $c > 0$ sufficiently large depending on the absolute constant M_{abs} in Proposition A.17 so that the following hold[82] for every $(\hat{t}, \hat{u}) \in [0, t] \times [0, u]$:

$$\begin{aligned} F(\hat{t}, \hat{u}) &\leq C\mathring{\epsilon}^2 + \alpha_1 F(t,u) + \alpha_2 H(t,u) + \alpha_3 G(t,u) \\ &\quad + C\iota_F^{-1}(\hat{t}, \hat{u}) \int_{t'=0}^{t'=\hat{t}} \|(|L\mathcal{P}^{[1,N_{\text{top}}]}\Psi| + |\breve{X}\mathcal{P}^{[1,N_{\text{top}}]}\Psi|)\mathcal{P}^{[1,N_{\text{top}}]}\vec{\mathfrak{G}}\|_{L^1(\Sigma_{t'}^u)}\, dt' \\ &\quad + C\iota_F^{-1}(\hat{t}, \hat{u}) \int_{t'=0}^{t'=\hat{t}} \mu_\star^{-3/2}(t') \left\{ \int_{s=0}^{s=t'} \|\mathcal{P}^{[1,N_{\text{top}}]}\vec{\mathfrak{G}}\|_{L^2(\Sigma_s^u)}\, ds \right\}^2 dt', \end{aligned} \tag{A-47}$$

$$\begin{aligned} G(\hat{t}, \hat{u}) &\leq C\mathring{\epsilon}^2 + \beta_1 F(t,u) + \beta_2 H(t,u) \\ &\quad + C\iota_G^{-1}(\hat{t}, \hat{u}) \int_{t'=0}^{t'=\hat{t}} \|(|L\mathcal{P}^{[1,N_{\text{top}}]}\Psi| + |\breve{X}\mathcal{P}^{[1,N_{\text{top}}]}\Psi|)\mathcal{P}^{[1,N_{\text{top}}]}\mathfrak{G}\|_{L^1(\Sigma_{t'}^u)}\, dt' \\ &\quad + C\iota_G^{-1}(\hat{t}, \hat{u}) \int_{t'=0}^{t'=\hat{t}} \mu_\star^{-3/2}(t') \left\{ \int_{s=0}^{s=t'} \|\mathcal{P}^{[1,N_{\text{top}}]}\mathfrak{G}\|_{L^2(\Sigma_s^u)}\, ds \right\}^2 dt', \end{aligned} \tag{A-48}$$

$$\begin{aligned} H(\hat{t}, \hat{u}) &\leq C\mathring{\epsilon}^2 + \gamma_1 F(t,u) + \gamma_2 H(t,u) \\ &\quad + C\iota_H^{-1}(\hat{t}, \hat{u}) \int_{t'=0}^{t'=\hat{t}} \|(|L\mathcal{P}^{[1,N_{\text{top}}-1]}\Psi| + |\breve{X}\mathcal{P}^{[1,N_{\text{top}}-1]}\Psi|)\mathcal{P}^{[1,N_{\text{top}}-1]}\mathfrak{G}\|_{L^1(\Sigma_{t'}^u)}\, dt', \end{aligned} \tag{A-49}$$

where $C > 0$ is a constant, while α_1, α_2, α_3, β_1, β_2, γ_1 and γ_2 are constants that obey the following smallness conditions (as long as $M_* \in \mathbb{N}$ and $c > 0$ are sufficiently large):

$$\begin{gathered} \alpha_1 + 4\alpha_2\gamma_1 + \alpha_3\beta_1 + 4\alpha_3\beta_2\gamma_1 < 1, \\ \gamma_2 < \tfrac{3}{4}. \end{gathered} \tag{A-50}$$

At this point we fix $c > 0$ and $M_* \in \mathbb{N}$. From now on, we allow the general constants $C > 0$ to depend on these particular fixed choices of c and M_*.

For each of the three integrals on the right-hand sides of (A-47)–(A-49), we absorb $\iota_1^c(\hat{t})\iota_2^c(\hat{t})e^{c\hat{t}}e^{c\hat{u}}$ into the general constant C, and then take the supremum with respect to $\hat{t}$. For instance, for the first

[80]Here we note one minor difference compared to [36, Proposition 14.1]: that proposition was more precise with respect to u in the sense that it yielded a priori estimates in terms of powers of $\mu_\star(t,u)$, rather than $\mu_\star(t)$ (see Definition 2.16). For this reason, in the proof [36, Proposition 14.1], the definition of the analog of ι_2 involved $\mu_\star(t,u)$, and similarly for the $\mu_\star$-dependent factors on the right-hand sides of the analogs of ι_F, ι_G, and ι_H. The change we have made in this paper has no substantial effect on the analysis; at the relevant points in the proof of [36, Proposition 14.1], all of the needed estimates hold true with $\mu_\star(t)$ in place of $\mu_\star(t,u)$.

[81]The detailed argument relies on some extensions and sharpened versions of the estimates of Proposition 8.11. Given the estimates of Section 8, such as Propositions 8.6, 8.7, and 8.10, the needed estimates can be proved using the same arguments given in [36].

[82]The inequality [36, (14.64)] featured a term $CF^{1/2}(t,u)G^{1/2}(t,u)$ on the right-hand side. We used Young's inequality to bound this term by $\leq aF(t,u) + \alpha_3 G(t,u)$, where $\alpha_3 \doteq C^2/a$ and we have chosen a to be small, which allows us to absorb $aF(t,u)$ into the term $\alpha_1 F(t,u)$.

integral on the right-hand side of (A-47), we deduce that for $(\hat{t}, \hat{u}) \in [0, t] \times [0, u]$, we have

$$\iota_F^{-1}(\hat{t}, \hat{u}) \int_{t'=0}^{t'=\hat{t}} \|(|L\mathcal{P}^{[1,N_{\mathrm{top}}]}\Psi| + |\breve{X}\mathcal{P}^{[1,N_{\mathrm{top}}]}\Psi|)\mathcal{P}^{[1,N_{\mathrm{top}}]}\vec{\mathfrak{G}}\|_{L^1(\Sigma_{t'}^u)}\, dt' \lesssim \sup_{\hat{t}'\in[0,t]} \upmu_\star^{2M_*-1.8}(\hat{t}') \int_{t'=0}^{t'=\hat{t}'} \|(|L\mathcal{P}^{[1,N_{\mathrm{top}}]}\Psi| + |\breve{X}\mathcal{P}^{[1,N_{\mathrm{top}}]}\Psi|)\mathcal{P}^{[1,N_{\mathrm{top}}]}\vec{\mathfrak{G}}\|_{L^1(\Sigma_{t'}^u)}\, dt'.$$

We perform the same operation on the other integrals. Since we have taken a supremum, the right-hand sides are independent of $(\hat{t}, \hat{u})$. We then take supremum over $(\hat{t}, \hat{u}) \in [0, t] \times [0, u]$ on the left-hand sides of (A-47)–(A-49) to obtain, with the same constants α_1, α_2, α_3, β_1, β_2, γ_1 and γ_2, but with a different constant C, the inequalities

$$\begin{aligned} F(t,u) \le{}& C\mathring{\epsilon}^2 + \alpha_1 F(t,u) + \alpha_2 H(t,u) + \alpha_3 G(t,u) \\ &+ C \sup_{\hat{t}\in[0,t]} \upmu_\star^{2M_*-1.8}(\hat{t}) \int_{t'=0}^{t'=\hat{t}} \|(|L\mathcal{P}^{[1,N_{\mathrm{top}}]}\Psi| + |\breve{X}\mathcal{P}^{[1,N_{\mathrm{top}}]}\Psi|)\mathcal{P}^{[1,N_{\mathrm{top}}]}\vec{\mathfrak{G}}\|_{L^1(\Sigma_{t'}^u)}\, dt' \\ &+ C \sup_{\hat{t}\in[0,t]} \upmu_\star^{2M_*-1.8}(\hat{t}) \int_{t'=0}^{t'=\hat{t}} \upmu_\star^{-3/2}(t') \left\{ \int_{s=0}^{s=t'} \|\mathcal{P}^{[1,N_{\mathrm{top}}]}\vec{\mathfrak{G}}\|_{L^2(\Sigma_s^u)}\, ds \right\}^2 dt', \end{aligned} \tag{A-51}$$

$$\begin{aligned} G(t,u) \le{}& C\mathring{\epsilon}^2 + \beta_1 F(t,u) + \beta_2 H(t,u) \\ &+ C \sup_{\hat{t}\in[0,t]} \upmu_\star^{2M_*-1.8}(\hat{t}) \int_{t'=0}^{t'=\hat{t}} \|(|L\mathcal{P}^{[1,N_{\mathrm{top}}]}\Psi| + |\breve{X}\mathcal{P}^{[1,N_{\mathrm{top}}]}\Psi|)\mathcal{P}^{[1,N_{\mathrm{top}}]}\vec{\mathfrak{G}}\|_{L^1(\Sigma_{t'}^u)}\, dt' \\ &+ C \sup_{\hat{t}\in[0,t]} \upmu_\star^{2M_*-1.8}(\hat{t}) \int_{t'=0}^{t'=\hat{t}} \upmu_\star^{-3/2}(t') \left\{ \int_{s=0}^{s=t'} \|\mathcal{P}^{[1,N_{\mathrm{top}}]}\vec{\mathfrak{G}}\|_{L^2(\Sigma_s^u)}\, ds \right\}^2 dt', \end{aligned} \tag{A-52}$$

$$\begin{aligned} H(t,u) \le{}& C\mathring{\epsilon}^2 + \gamma_1 F(t,u) + \gamma_2 H(t,u) \\ &+ C \sup_{\hat{t}\in[0,t]} \upmu_\star^{2M_*-3.8}(\hat{t}) \int_{t'=0}^{t'=\hat{t}} \|(|L\mathcal{P}^{[1,N_{\mathrm{top}}-1]}\Psi| + |\breve{X}\mathcal{P}^{[1,N_{\mathrm{top}}-1]}\Psi|)\mathcal{P}^{[1,N_{\mathrm{top}}-1]}\vec{\mathfrak{G}}\|_{L^1(\Sigma_{t'}^u)}\, dt'. \end{aligned} \tag{A-53}$$

The main point is the smallness conditions (A-50) on the constants $\alpha_1, \ldots, \gamma_2$ allow us to solve the inequalities (A-51)–(A-53) using a reductive approach. More precisely, using that $\gamma_2 < \frac{3}{4}$, we absorb the $\gamma_2 H(t,u)$ term on the right-hand side of (A-53) back into the left-hand side to isolate $H(t,u)$, at the expense of enlarging C and replacing γ_1 with $4\gamma_1$. We then insert this estimate for $H(t,u)$ into the right-hand side of (A-52) to obtain an estimate for $G(t,u)$, and then insert these estimates for $H(t,u)$ and $G(t,u)$ into the right-hand side of (A-51) to obtain the inequality

$$\begin{aligned} &F(t,u) \\ &\le C\mathring{\epsilon}^2 + \{\alpha_1 + 4\alpha_2\gamma_1 + \alpha_3\beta_1 + 4\alpha_3\beta_2\gamma_1\} F(t,u) \\ &\quad + C \sup_{\hat{t}\in[0,t]} \upmu_\star^{2M_*-1.8}(\hat{t}) \int_{t'=0}^{t'=\hat{t}} \|(|L\mathcal{P}^{[1,N_{\mathrm{top}}]}\Psi| + |\breve{X}\mathcal{P}^{[1,N_{\mathrm{top}}]}\Psi|)\mathcal{P}^{[1,N_{\mathrm{top}}]}\vec{\mathfrak{G}}\|_{L^1(\Sigma_{t'}^u)}\, dt' \\ &\quad + C \sup_{\hat{t}\in[0,t]} \upmu_\star^{2M_*-1.8}(\hat{t}) \int_{t'=0}^{t'=\hat{t}} \upmu_\star^{-3/2}(t') \left\{ \int_{s=0}^{s=t'} \|\mathcal{P}^{[1,N_{\mathrm{top}}]}\vec{\mathfrak{G}}\|_{L^2(\Sigma_s^u)}\, ds \right\}^2 dt' \\ &\quad + C \sup_{\hat{t}\in[0,t]} \upmu_\star^{2M_*-3.8}(\hat{t}) \int_{t'=0}^{t'=\hat{t}} \|(|L\mathcal{P}^{[1,N_{\mathrm{top}}-1]}\Psi| + |\breve{X}\mathcal{P}^{[1,N_{\mathrm{top}}-1]}\Psi|)\mathcal{P}^{[1,N_{\mathrm{top}}-1]}\vec{\mathfrak{G}}\|_{L^1(\Sigma_{t'}^u)}\, dt'. \end{aligned} \tag{A-54}$$

From the smallness condition $\alpha_1+4\alpha_2\gamma_1+\alpha_3\beta_1+4\alpha_3\beta_2\gamma_1<1$ featured in (A-50), it follows that we can absorb the terms $\{\alpha_1+4\alpha_2\gamma_1+\alpha_3\beta_1+4\alpha_3\beta_2\gamma_1\}F(t,u)$ on the right-hand side of (A-54) back into the left-hand side of (A-54) to isolate $F(t,u)$, at the expense of increasing the constant C. We therefore deduce the inequality

$$\begin{aligned}F(t,u)\lesssim \mathring{\epsilon}^2 &+\sup_{\hat{t}\in[0,t]}\mu_\star^{2M_*-1.8}(\hat{t})\int_{t'=0}^{t'=\hat{t}}\|(|L\mathcal{P}^{[1,N_{\mathrm{top}}]}\Psi|+|\breve{X}\mathcal{P}^{[1,N_{\mathrm{top}}]}\Psi|)\mathcal{P}^{[1,N_{\mathrm{top}}]}\vec{\mathfrak{G}}\|_{L^1(\Sigma_{t'}^u)}\,dt'\\ &+\sup_{\hat{t}\in[0,t]}\mu_\star^{2M_*-1.8}(\hat{t})\int_{t'=0}^{t'=\hat{t}}\mu_\star^{-3/2}(t')\left\{\int_{s=0}^{s=t'}\|\mathcal{P}^{[1,N_{\mathrm{top}}]}\vec{\mathfrak{G}}\|_{L^2(\Sigma_s^u)}\,ds\right\}^2dt'\\ &+\sup_{\hat{t}\in[0,t]}\mu_\star^{2M_*-3.8}(\hat{t})\int_{t'=0}^{t'=\hat{t}}\|(|L\mathcal{P}^{[1,N_{\mathrm{top}}-1]}\Psi|+|\breve{X}\mathcal{P}^{[1,N_{\mathrm{top}}-1]}\Psi|)\mathcal{P}^{[1,N_{\mathrm{top}}-1]}\vec{\mathfrak{G}}\|_{L^1(\Sigma_{t'}^u)}\,dt'.\end{aligned}\tag{A-55}$$

Then from (A-55) and the arguments given above, we deduce that $G(t,u)$ and $H(t,u)$ are also bounded above by the right-hand side of (A-55) (where we enlarge C if necessary).

Recalling the definitions of F,G, and H in (A-44)–(A-46), we see that (A-55) and the similar bounds for $G(t,u)$ and $H(t,u)$ collectively imply (12-2).

Step 2: the lower orders (proof of (12-3)). To prove the lower-order energy estimates, we start by considering the energy inequality given by the below-top-order estimate from Proposition A.17, i.e., the estimate (A-39), which features the additional term (A-36) compared to [36, (14.6)].

Observe that on the right-hand side of (A-39), except for

$$\int_{t'=0}^{t'=t}\frac{\mathbb{Q}^{1/2}_{[1,N-1]}(t',u)}{\mu_\star^{1/2}(t',u)}\left\{\int_{s=0}^{s=t'}\frac{\mathbb{Q}^{1/2}_{[1,N]}(s,u)}{\mu_\star^{1/2}(s,u)}\,ds\right\}dt',$$

every other term can be treated directly by Grönwall's inequality (using Proposition 8.11), as in [36]. It thus follows that

$$\begin{aligned}&\sup_{t'\in[0,t]}\max\{\mathbb{Q}_{[1,N-1]}(t',u),\ \mathbb{K}_{[1,N-1]}(t',u)\}\\ &\quad\le C\mathring{\epsilon}^2+C\int_{t'=0}^{t'=t}\frac{\mathbb{Q}^{1/2}_{[1,N-1]}(t',u)}{\mu_\star^{1/2}(t',u)}\left\{\int_{s=0}^{s=t'}\frac{\mathbb{Q}^{1/2}_{[1,N]}(s,u)}{\mu_\star^{1/2}(s,u)}\,ds\right\}dt'\\ &\qquad\qquad+C\|(|L\mathcal{P}^{[1,N-1]}\Psi|+|\breve{X}\mathcal{P}^{[1,N-1]}\Psi|)|\mathcal{P}^{[1,N-1]}\vec{\mathfrak{G}}|\|_{L^1(\mathcal{M}_{t,u})}.\end{aligned}\tag{A-56}$$

To proceed, we analyze the double time-integral term on the right-hand side of (A-56). For any $\varsigma>0$, we have

$$\begin{aligned}&\int_{t'=0}^{t'=t}\frac{\mathbb{Q}^{1/2}_{[1,N-1]}(t',u)}{\mu_\star^{1/2}(t',u)}\left\{\int_{s=0}^{s=t'}\frac{\mathbb{Q}^{1/2}_{[1,N]}(s,u)}{\mu_\star^{1/2}(s,u)}\,ds\right\}dt'\\ &\quad\le\big(\sup_{t'\in[0,t]}\mathbb{Q}^{1/2}_{[1,N-1]}(t')\big)\times\big(\sup_{s\in[0,t]}\min\{1,\mu_\star^{M_*-N_{\mathrm{top}}+N-0.9}(s)\}\mathbb{Q}^{1/2}_{[1,N]}(s)\big)\\ &\qquad\qquad\times\int_{t'=0}^{t'=t}\frac{1}{\mu_\star^{1/2}(t')}\left\{\int_{s=0}^{s=t'}\frac{\max\{1,\mu_\star^{-M_*+N_{\mathrm{top}}-N+0.9}(s)\}}{\mu_\star^{1/2}(s)}\,ds\right\}dt'\\ &\quad\le\varsigma\sup_{t'\in[0,t]}\mathbb{Q}_{[1,N-1]}(t')\\ &\qquad+C\varsigma^{-1}\max\{1,\mu_\star^{-2M_*+2N_{\mathrm{top}}-2N+3.8}(t)\}\big(\sup_{s\in[0,t]}\min\{1,\mu_\star^{2M_*-2N_{\mathrm{top}}+2N-1.8}(s)\}\mathbb{Q}_{[1,N]}(s)\big),\end{aligned}\tag{A-57}$$

where to obtain the last inequality, we have used Young's inequality and the following estimate, which follows from Proposition 8.11:

$$\int_{t'=0}^{t'=t} \frac{1}{\mu_\star^{1/2}(t')}\left\{\int_{s=0}^{s=t'} \frac{\max\{1, \mu_\star^{-M_*+N_{\text{top}}-N+0.9}(s)\}}{\mu_\star^{1/2}(s)}\,ds\right\}dt' \lesssim \int_{t'=0}^{t'=t} \frac{\max\{1, \mu_\star^{-M+N_{\text{top}}-N+1.4}(t')\}}{\mu_\star^{1/2}(t')}\,dt' \lesssim \max\{1, \mu_\star^{-M_*+N_{\text{top}}-N+1.9}(t)\}.$$

Inserting (A-57) into (A-56) and fixing $\varsigma > 0$ to be sufficiently small, we can absorb the term $C\varsigma(\sup_{t'\in[0,t]} \mathbb{Q}_{[1,N-1]}(t'))$ back into the left-hand side of (A-56). Thus, for this fixed value of ς, we obtain

$$\begin{aligned}\sup_{t'\in[0,t]} \max\{\mathbb{Q}_{[1,N-1]}(t',u),\ \mathbb{K}_{[1,N-1]}(t',u)\} &\lesssim \mathring{\epsilon}^2 + \max\{1, \mu_\star^{-2M+2N_{\text{top}}-2N+3.8}(t)\}\Big(\sup_{s\in[0,t]} \min\{1, \mu_\star^{2M_*-2N_{\text{top}}+2N-1.8}(s)\}\mathbb{Q}_{[1,N]}(s)\Big) \\ &\quad + \|(|L\mathcal{P}^{[1,N-1]}\Psi| + |\breve{X}\mathcal{P}^{[1,N-1]}\Psi|)|\mathcal{P}^{[1,N-1]}\vec{\mathfrak{G}}|\|_{L^1(\mathcal{M}_{t,u})}.\end{aligned}$$

After changing the index N to $N+1$, we conclude the estimate (12-3). □

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