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THE SINGULAR STRATA OF A FREE-BOUNDARY PROBLEM FOR HARMONIC MEASURE

SEAN MCCURDY

We obtain *quantitative* estimates on the fine structure of the singular set of the mutual boundary $\partial\Omega^\pm$ for pairs of complementary domains $\Omega^+, \Omega^- \subset \mathbb{R}^n$ which arise in a class of two-sided free boundary problems for harmonic measure. These estimates give new insight into the structure of the mutual boundary $\partial\Omega^\pm$.

1. Introduction

The focus of this paper is the study of a class of two-phase free boundary problems for harmonic measure. For $n \geq 3$, let $\Omega^+ \subset \mathbb{R}^n$ and $\Omega^- = \overline{\Omega}^{+c}$ be unbounded nontangentially accessible (NTA) domains (see Definition 2.1), let $\omega^\pm$ be their associated harmonic measures, and let $u^\pm$ be the associated Green's functions with poles at infinity. Let $\omega^- \ll \omega^+ \ll \omega^-$, and let $h = d\omega^-/d\omega^+$ satisfy $\ln(h) \in C^{0,\alpha}$ for some $0 < \alpha < 1$. We obtain new results on the structure of the geometric singular set of the boundary $\partial\Omega^\pm$.

This problem was introduced without the regularity assumption on $\omega^\pm$ by Kenig, Preiss, and Toro [Kenig et al. 2009], with other work under the assumption that $\ln(h) \in \mathrm{VMO}(\partial\Omega^\pm)$ by Kenig and Toro [2006], Badger [2011; 2013], and Badger, Engelstein, and Toro [Badger et al. 2017]. Questions about the structure of the free boundary and the singular set when $\ln(h) \in C^{0,\alpha}$ for $0 < \alpha < 1$ have been addressed by Engelstein [2016] and Badger, Engelstein, Toro [Badger et al. 2020], respectively. Engelstein [2016]

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shows that under the additional assumption that the boundary is sufficiently flat in the sense of Reifenberg, the boundary is locally $C^{1,\alpha}$. In [Badger et al. 2020], the authors remove the assumption of flatness and prove that the geometric singular set is contained in countably many $C^{1,\beta}$ submanifolds of the appropriate dimension. See [Kenig et al. 2009] for an overview of this problem in lower dimensions and [Badger et al. 2017; 2020] for further background.

Until recently, almost all work on the two-sided free boundary problem for harmonic measure in higher dimensions has operated under the assumption that $\Omega^{\pm}$ are NTA domains because the NTA conditions allow for scale-invariant estimates of harmonic measure. However, Azzam, Mourgoglou, Tolsa, and Volberg [Azzam et al. 2019] proved, among other things, that if we relax the assumption that the domains are NTA, then $\omega^- \ll \omega^+ \ll \omega^-$ on $G \subset \partial\Omega^{\pm}$ implies that G can be decomposed into $G = R \cup B$, where R is $(n-1)$-rectifiable and $\omega^{\pm}(B) = 0$. However, we shall work under the assumption that $\Omega^{\pm}$ are NTA domains.

Based upon [Badger et al. 2020], we know that when $\ln(h) \in C^{0,\alpha}$, the singular set of $\partial\Omega^{\pm}$ is countably $C^{1,\beta}$-rectifiable where β depends on but is not equal to α. This leaves open the question of whether or not the singular set is dense, or more generally how it sits in space. In this paper, we answer the question of how the singular set "sits in space". In particular, we provide upper Minkowski content bounds upon the quantitative strata of the singular set (see Theorem 2.15). The main approach will be to follow [Engelstein 2016] and consider jump functions $v = u^+ - u^-$ which are *almost* harmonic, and employ the Almgren frequency function and geometric techniques as in [Cheeger et al. 2015; Han and Lin 1994] in conjunction with the powerful quantitative differentiation techniques of [De Lellis et al. 2018; Naber and Valtorta 2017]. While these tools are common for problems in calculus of variations, it is important to note that the jump functions v are not minimizers of any energy, nor do they satisfy any global PDE.

2. Definitions and statement of main results

2A. ***Domains and their Green's functions.*** Nontangentially accessible (NTA) domains were formally introduced by Jerison and Kenig [1982] to study the boundary behavior of PDEs on nonsmooth domains.

Definition 2.1. A domain $\Omega \subset \mathbb{R}^n$ is a nontangentially accessible (NTA) domain if there exist constants $M > 1$ and $R_0 > 0$ such that the following holds:

(1) Ω satisfies the *corkscrew condition*. That is, for any $Q \in \partial\Omega$ and $0 < r < R_0$, there exists a point $A_r(Q) \in \Omega$ with the following two properties:

$$|A_r(Q) - Q| < r \quad \text{and} \quad B_{r/M}(A_r(Q)) \subset \Omega.$$

(2) $\overline{\Omega}^c$ also satisfies the *corkscrew condition*.

(3) Ω satisfies the *Harnack chain condition*. That is, for any $\epsilon > 0$ and $Q \in \partial\Omega$, if

$$x_1, x_2 \in \Omega \cap B_{R_0/4}(Q) \setminus B_\epsilon(\partial\Omega) \quad \text{and} \quad |x_1 - x_2| \leq 2^k \epsilon,$$

then there exists a "Harnack chain" of balls $\{B_{r_i}(y_i)\}_{i=1}^N$ satisfying:

(a) $x_1 \in B_{r_i}(y_1)$ and $x_1 \in B_{r_N}(y_N)$.

(b) $B_{r_i}(y_i) \subset \Omega$ for all $i = 1, \dots, N$.
(c) $B_{r_i}(y_i) \cap B_{r_{i+1}}(y_{i+1}) \neq \varnothing$ for all $i = 1, \dots, N-1$.
(d) $N \leq Mk$.
(e) For all $i = 1, \dots, N$,

$$\frac{1}{2M} \min_{i=1,2,\dots,N} \{\operatorname{dist}(x_i, \partial\Omega)\} \leq r_i \leq \operatorname{dist}(y_i, \partial\Omega).$$

Note that by increasing the radii if necessary, we may assume that $r_i \sim_M \operatorname{dist}(y_i, \partial\Omega)$.

We say that Ω^+ is a *two-sided NTA domain* if both Ω^+ and $\Omega^- := \overline{\Omega}^c$ are NTA domains. We shall refer to the complementary pair $\Omega^\pm$ of domains as complementary two-sided NTA domains and denote their mutual boundary by $\partial\Omega^\pm$.

In this paper, we shall only deal with unbounded two-sided NTA domains. That is, we shall assume that $R_0 = \infty$. However, the results are essentially local.

Definition 2.2 (Green's functions). For $\Omega^\pm \subset \mathbb{R}^n$ a pair of complementary two-sided NTA domains, we shall use $u^\pm$ to denote the *Green's function with pole at infinity* corresponding to $\Omega^\pm$, respectively.

Recall that $u^\pm$ are unique up to scalar multiplication and that to each $u^\pm$ is associated the *harmonic measure* $\omega^\pm$, defined by the property that, for all $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\int \Delta\phi u^\pm \, dV = \int \phi \, d\omega^\pm.$$

See [Garnett and Marshall 2005] for more details about harmonic measures.

Observe that if ω^+ is the harmonic measure associated to u^+, then $c\omega^+$ is the harmonic measure associated to cu^+ for any $c > 0$.

If $f \in C^{0,\alpha}(\mathbb{R}^n)$, we shall use $\|f\|_\alpha$ to denote the local norm:

$$\|f\|_\alpha := \sup_{B_2(0)} |f| + \sup_{x \neq y \in B_2(0)} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

Definition 2.3. We define the class $\mathcal{D}(n, \alpha, M_0)$ to be the collection of domains $\Omega^\pm \subset \mathbb{R}^n$ such that $\Omega^\pm$ are complementary unbounded two-sided NTA domains for which $M < M_0$, $\omega^- \ll \omega^+ \ll \omega^-$, the Radon–Nikodym derivative $h = d\omega^-/d\omega^+$ satisfies $\ln(h) \in C^{0,\alpha}(\partial\Omega)$, and $0 \in \partial\Omega^\pm$.

Note that if $\Omega^\pm \in \mathcal{D}(n, \alpha, M_0)$ and $Q \in \partial\Omega^\pm$, then $\Omega^\pm - Q \in \mathcal{D}(n, \alpha, M_0)$.

2B. *A class of functions and their rescalings.*

Definition 2.4. Let $\Omega^\pm \subset \mathbb{R}^n$ be a pair of complementary two-sided NTA domains with mutual boundary $\partial\Omega^\pm$. For any $Q \in \partial\Omega^\pm$ and any Green's functions $u^\pm$ we define the *jump function*

$$v^Q(x) := h(Q)u^+(x) - u^-(x). \tag{2-1}$$

The scaling $h(Q)u^+$ normalizes the Radon–Nikodym derivative of the harmonic measure associated to $h(Q)u^+$ and u^- at $Q \in \partial\Omega^\pm$.

Definition 2.5. Let $\Omega^\pm \in \mathcal{D}(n, \alpha, M_0)$ and $Q \in \partial\Omega^\pm$. For scales $0 < r$, we define the rescaling of the function v^Q at scale r at the point $Q' \in \partial\Omega^\pm$ by

$$v^Q_{Q',r}(x) := v^Q(rx + Q')\frac{r^{n-2}}{\omega^-(B_r(Q'))}$$

and the corresponding rescaled measure as

$$\omega^\pm_{Q',r}(E) := \frac{\omega^\pm(rE + Q')}{\omega^\pm(B_r(Q'))}. \tag{2-2}$$

The rescalings $v^Q_{Q,r}$ were first introduced by Kenig and Toro [2006]. In this paper, we shall employ the following results by Kenig, Toro, Badger, and Engelstein.

Theorem 2.6 [Badger 2011; Engelstein 2016; Kenig and Toro 2006]. *For $v^Q_{Q,r}$, $\omega^\pm_{Q,r}$ as in Definition 2.5*:

(1) *Subsequential limits as $r \to 0$ of the functions $v^Q_{Q,r}$ converge to harmonic polynomials. Furthermore, the degree of these polynomials is bounded and depends only upon the NTA constant*, M_0. [Kenig and Toro 2006]

(2) *Subsequential limits as $r \to 0$ of the functions $v^Q_{Q,r}$ converge to* ***homogeneous*** *harmonic polynomials. Furthermore, the degree of homogeneity is unique along blow-ups.* [Badger 2011]

(3) *The rescalings $v^Q_{Q,r}$ are uniformly locally Lipschitz with Lipschitz constant that only depends upon M_0.* [Engelstein 2016]

(4) *The measures $\omega^\pm_{Q,r}$ are locally uniformly bounded.* [Engelstein 2016]

In addition to the $v^Q_{Q',r}$ rescalings, we shall also use a different kind of rescaling.

Definition 2.7 [Cheeger et al. 2015]. Let $f : B_1(0) \to \mathbb{R}$ be a function in $C(\mathbb{R}^n)$. We define the rescaled function $T_{x,r}f$ of f at a point $x \in B_{1-r}(0)$ at scale $0 < r < 1$ by

$$T_{x,r}f(y) := \frac{f(x+ry) - f(x)}{\left(⨍_{\partial B_1(0)}(f(x+rz) - f(x))^2\, d\sigma(z)\right)^{1/2}}. \tag{2-3}$$

In the case that the denominator is zero, we define $T_{x,r}f = \infty$. We denote the limit as $r \to 0$ by

$$T_x f(y) := \lim_{r\to 0} T_{x,r} f(y).$$

Definition 2.8. Let $\mathcal{A}(n, \alpha, M_0)$ be the set of functions $v : \mathbb{R}^n \to \mathbb{R}$ such that

$$v := v^0 = h(0)u^+ - u^-,$$

where $u^\pm$ are the Green's functions with poles at infinity associated to a two-sided NTA domain $\Omega^\pm \in \mathcal{D}(n, \alpha, M_0)$ and $h = d\omega^-/d\omega^+$, where $\omega^\pm$ are the harmonic measures associated to $u^\pm$.

Remark 2.9. For any fixed domain $\Omega^\pm \in \mathcal{D}(n, \alpha, M_0)$ there is a one-parameter family of associated functions $v \in \mathcal{A}(n, \alpha, M_0)$ with $\{v = 0\} = \partial\Omega^\pm$. Indeed, $cv^0_{0,1} \in \mathcal{A}(n, \alpha, M_0)$ for all $c > 0$. To avoid degeneracy because of this degree of freedom within the family $\mathcal{A}(n, \alpha, M_0)$, we shall make extensive use of the normalizations in Definitions 2.5 and 2.7 in the arguments to come.

Finally, note that in general the functions $v^Q_{Q',r}$ will not belong to $\mathcal{A}(n, \alpha, M_0)$ if $Q \neq 0$ and/or $Q' \neq Q$.

2C. *Quantitative symmetry.* The geometry we wish to capture with the blow-ups $T_x f$ is encoded in their translational symmetries.

Definition 2.10. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. We say f is 0-*symmetric* if

$$f(x) := cP^+(x) - P^-(x) \tag{2-4}$$

for some $c > 0$, where $P^\pm$ are the positive and negative parts of a homogeneous harmonic polynomial P. We will say that f is k-*symmetric* if f is 0-symmetric and there exists a k-dimensional subspace V such that $f(x+y) = f(x)$ for all $x \in \mathbb{R}^n$ and all $y \in V$.

The constant $c > 0$ is there to allow for the function to "hinge" along its zero set. We must allow this kind of "hinging" to accommodate for the "nonalignment" issue in the blow-ups at $Q \in \partial\Omega^\pm \setminus \{0\}$. See Remark 3.2.

We now define a quantitative version of symmetry.

Definition 2.11. For any $f \in C(\mathbb{R}^n)$, f will be called (k, ϵ, r, p)-*symmetric* if there exists a k-symmetric function P such that

(1) $\fint_{\partial B_1(0)} |P|^2 \, dV = 1$,

(2) $\fint_{B_1(0)} |T_{p,r} f - P|^2 \, dV < \epsilon$.

Sometimes, we shall refer to a function f as being (k, ϵ)-symmetric in the ball $B_r(p)$ to mean f is (k, ϵ, r, p)-symmetric.

This quantitative control allows us to define a quantitative stratification following [Cheeger and Naber 2013].

Definition 2.12 (quantitative singular strata). Let $v \in \mathcal{A}(n, \alpha, M_0)$ and $0 < r \le 1$. We denote the (k, ϵ, r)-*singular stratum* of v by $\mathcal{S}^k_{\epsilon,r}(v)$, and we define it by

$$\mathcal{S}^k_{\epsilon,r}(v) := \{x \in \partial\Omega^\pm : v \text{ is not } (k+1, \epsilon, s, x)\text{-symmetric for all } r \le s \le 1\}. \tag{2-5}$$

We shall also use the notation $\mathcal{S}^k_\epsilon(v)$ for $\mathcal{S}^k_{\epsilon,0}(v)$. It is immediate from the definitions that $\mathcal{S}^k_{\epsilon,r}(v) \subset \mathcal{S}^{k'}_{\epsilon',r'}(v)$ if $k \le k'$, $\epsilon' \le \epsilon$, $r \le r'$.

We can recover the *qualitative* stratification

$$\mathcal{S}^k(v) := \{x \in \partial\Omega^\pm : T_x v \text{ is not } (k+1)\text{-symmetric}\} = \bigcup_\eta \bigcap_r \mathcal{S}^k_{\eta,r}(v).$$

The set $\mathcal{S}^k(v)$ is the k-th stratum of $\mathcal{S}^{n-2}(v) = \operatorname{sing}(\partial\Omega^\pm)$. Furthermore, if $x \in \mathcal{S}^k(v)$, then there exists an $\epsilon > 0$ such that $x \in \mathcal{S}^k_\epsilon(v)$.

Remark 2.13. Note that the singular set and its strata are all stable under the operations

$$\mathcal{S}^k(v) = \mathcal{S}^k(cv) \quad \text{and} \quad \mathcal{S}^k(v) = \mathcal{S}^k(cv^+ - v^-)$$

for all $c \ne 0$. The former is a trivial consequence of the fact that $T_{p,r} f = T_{p,r}(cf)$. The latter follows from Definition 2.10 and Theorem 2.6. In particular, for all $v \in \mathcal{A}(n, \alpha, M_0)$, we have $\mathcal{S}^k(v) = \mathcal{S}^k(v^Q)$ for all $Q \in \partial\Omega^\pm$.

Previous results on the singular set are summed up in the following theorem.

Theorem 2.14 [Badger et al. 2017; 2020; Engelstein 2016]. *For $v \in \mathcal{A}(n, \alpha, M_0)$, the following hold*:

(1) $(\mathcal{S}^{n-2}(v) \setminus \mathcal{S}^{n-3}(v)) \cap \partial\Omega^{\pm} = \varnothing$. [Badger et al. 2020, Remark 7.2]

(2) *There exists an* $\epsilon > 0$ *such that* $\mathrm{sing}(\partial\Omega^{\pm}) = \mathcal{S}^{n-3}(v) \cap \partial\Omega^{\pm} \subset \mathcal{S}^{n-2}_{\epsilon}(v)$. [Engelstein 2016, Theorem 1.1]

(3) $\overline{\dim}_{\mathcal{M}}(\mathrm{sing}(\partial\Omega^{\pm})) \le n-3$. [Badger et al. 2017, Theorem 7.5]

2D. *Main results and outline of the proof.* In this paper, we prove volume bounds on tubular neighborhoods around the $S^k_{\epsilon,r}(v)$. We are able to show the following estimates.

Theorem 2.15. *Let $v \in \mathcal{A}(n, \alpha, M_0)$ with $\|\ln(h)\|_{\alpha} \le \Gamma$. For every $0 < \epsilon$ and $0 \le k \le n-2$ there is an $r_0(n, \alpha, M_0, \Gamma, \epsilon) > 0$ such that, for all $0 < r < r_0$ and any $r \le R \le 1$,*

$$\mathrm{Vol}(B_R(B_{1/4}(0) \cap \mathcal{S}^k_{\epsilon,r}(v))) \le C(n, \alpha, M_0, \Gamma, \epsilon) R^{n-k}. \tag{2-6}$$

We have the following immediate corollary.

Corollary 2.16. *Let $v \in \mathcal{A}(n, \alpha, M_0)$ and $0 \le k \le n-2$. For every $0 < \epsilon$,*

$$\overline{\dim}_{\mathcal{M}}(\mathcal{S}^k_{\epsilon}(v)) \le k, \tag{2-7}$$

and there exists a constant such that

$$\mathcal{M}^{*,k}(\mathcal{S}^k_{\epsilon}(v) \cap B_{1/4}(0)) \le C(n, \alpha, M_0, \Gamma, \epsilon). \tag{2-8}$$

Thanks to an ϵ-regularity result due to [Engelstein 2016] we are able to strengthen the conclusion of Theorem 2.15 when we consider the full singular set.

Corollary 2.17. *Let $v \in \mathcal{A}(n, \alpha, M_0)$ with $\|\ln(h)\|_{\alpha} \le \Gamma$. Recall that*

$$\mathrm{sing}(\partial\Omega^{\pm}) = \mathcal{S}^{n-3} \cap \partial\Omega^{\pm}.$$

There exists $0 < \epsilon = \epsilon(M_0, \Gamma, \alpha)$ such that $\mathrm{sing}(\partial\Omega^{\pm}) \subset \mathcal{S}^{n-3}_{\epsilon}$; see Lemma 12.1. Thus, there is a constant $C = C(n, \alpha, M_0, \Gamma) < \infty$ such that

$$\mathcal{M}^{*,n-3}(\mathrm{sing}(\partial\Omega^{\pm}) \cap B_{1/4}(0)) \le C(n, \alpha, M_0, \Gamma). \tag{2-9}$$

Proof. This follows immediately from Lemma 12.1 and Theorem 2.15. □

2E. *Outline of the proof of Theorem 2.15.* In order to prove a theorem of this kind, we must build a cover of $\mathcal{S}^k_{\epsilon,r}(v)$, and we must count how many balls we use. Therefore two things are critical: getting geometric information about $\mathcal{S}^k_{\epsilon,r}(v)$ and keeping track of how the balls pack.

The overall strategy of proof is similar to that of [De Lellis et al. 2018; Edelen and Engelstein 2019]. However, there are several major differences. First, the functions $v \in \mathcal{A}(n, \alpha, M_0)$ considered here are not harmonic functions or minimizers of an energy. Sections 3–5 are devoted to showing that the relevant analogs of harmonic results (e.g., compactness, almost monotonicity of the Almgren frequency, local uniform boundedness of the Almgren frequency, quantitative rigidity for the Almgren frequency, cone-splitting, etc.) hold for $v \in \mathcal{A}(n, \alpha, M_0)$. In particular, we prove an estimate on the nondegeneracy of the almost monotonicity for Almgren frequency in Lemma 4.9. Local geometric control on $\mathcal{S}^k_\epsilon(v)$ is obtained in Section 6.

However, geometric control is not enough to obtain Theorem 2.15. To obtain finite upper Minkowski content bounds we need the discrete Reifenberg theorem from [Naber and Valtorta 2017]; see Theorem 9.1. This requires that we prove a "frequency pinching" result (Lemma 8.2) in which we connect the drop in the Almgren frequency over small scales with the β-numbers. The main challenge is to connect the lower bound on the derivative of the Almgren frequency (Lemma 4.9) and employ the techniques of [De Lellis et al. 2018] to obtain the necessary estimates on $N(Q, r, v) - N(Q', r, v)$; see Section 7.

In Section 9, we obtain the necessary packing estimates, following the framework of [Naber and Valtorta 2017] to accommodate the estimates of Section 8. Sections 10 and 11 construct the covering which proves the theorem according to the program laid out by [Naber and Valtorta 2017]. These are included for completeness.

3. Compactness

The main goal of this section is to show that $\mathcal{A}(n, \alpha, M_0)$ enjoys sufficient compactness to allow for limit-compactness arguments. Namely, we wish to establish that, for any sequence $v_i \in \mathcal{A}(n, \alpha, M_0)$, we can extract a subsequence which converges to a function v_∞ and that $N(p, r, v_i) \to N(p, r, v_\infty)$; see Corollary 4.3. This requires strong convergence in $W^{1,2}_{\text{loc}}(\mathbb{R}^n)$; see Lemmas 3.10 and 3.6.

On a technical level, we must extend the compactness implied by Theorem 2.6 for $v^Q_{Q,r}$ to $v^Q_{Q',r}$ and $T_{Q',r}v$. Throughout, we shall make essential use of "standard NTA results" such as the doubling of harmonic measure and various comparability results, all of which may be found in [Jerison and Kenig 1982].

Remark 3.1. Recall that for $E \subset \partial\Omega^\pm$

$$\omega^+(E) = \int \chi_E \, d\omega^+ \quad \text{and} \quad \omega^-(E) = \int \chi_E h \, d\omega^+.$$

Furthermore, if $\ln(h) \in C^{0,\alpha}$ with $\|\ln(h)\|_\alpha \le \Gamma$, then for all $Q, Q' \in \partial\Omega^\pm$

$$e^{-\Gamma|Q-Q'|^\alpha} h(Q') \le h(Q) \le e^{\Gamma|Q-Q'|^\alpha} h(Q'). \tag{3-1}$$

Using (3-1) in the above integral equations implies that in any compact set K, if $v \in \mathcal{A}(n, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \le \Gamma$, there is a constant $C(K, \Gamma, \alpha) > 1$ such that for any $E \subset K \cap \partial\Omega^\pm$

$$C^{-1} \le \frac{\omega^-(E)}{\omega^+(E)} \le C.$$

Remark 3.2. By Theorem 2.6, we know that subsequential limits as $r \to 0$ of the functions $v^Q_{Q,r}$ converge to homogeneous harmonic polynomials. However, for $Q, Q' \in \partial\Omega^\pm$ and $Q \neq Q'$, it is not true in general that $v^Q_{Q',r}$ converges to a homogeneous harmonic polynomial. As $r \to 0$, the function $v^Q_{Q',r}$ will converge to a 0-symmetric function (see Definition 2.10) where $c = h(Q)$.

Definition 3.3. We shall abuse the notation $T_{Q,r}$ from Definition 2.7 to denote translated and scaled versions of various objects. For example, for sets this is the usual push-forward

$$T_{Q,r}\Omega^{\pm} := \frac{\Omega^{\pm} - Q}{r}, \quad T_{Q,r}\partial\Omega^{\pm} := \frac{\partial\Omega^{\pm} - Q}{r}.$$

However, for the measures $\omega^{\pm}$, we will denote by $T_{Q,r}\omega^{\pm}$ the harmonic measures associated to the positive and negative parts of $T_{Q,r}v$. The corkscrew points $A_R^{\pm}(Q)$ will always denote the corkscrew point associated to Q at scale R in the appropriate domain $\Omega^{\pm}$. We shall use $T_{Q,r}A_{r'}^{\pm}(Q')$ to denote the corkscrew point associated to $T_{Q,r}Q' = (Q' - Q)/r \in T_{Q,r}\partial\Omega^{\pm}$ at the scale r'/r.

Lemma 3.4 (local Lipschitz bounds). *Let $v \in \mathcal{A}(n, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \le \Gamma$. For all $Q \in \partial\Omega^{\pm} \cap B_2(0)$ and all radii $0 < r \le 2$, the function $T_{Q,r}v$ is locally Lipschitz with uniform constants depending only upon M_0, Γ, α.*

Proof. Recall that by Definition 2.5,

$$v_{Q,r} = v^0_{Q,r} = \frac{r^{n-2}}{\omega^-(B_r(Q))} v(rx + Q).$$

By NTA estimates, for all $0 < r$, we have $|v(A_r^-(Q))| \sim \omega^-(B_r(Q))/r^{n-2}$ by constants which only depend upon M_0. Thus, $v_{Q,r}(T_{Q,r}A_r^-(Q))$ is bounded above and below by constants which only depend upon M_0. By constructing Harnack chains from $T_{Q,r}A_r^-(Q)$ to $T_{Q,r}A^-_{M_0 r}(Q)$ we can find a point $y \in \partial B_1(0)$ such that $y \in B_{r_i}(y_i) \subset T_{Q,r}\Omega^-$ and $\mathrm{dist}(y_i, T_{Q,r}\partial\Omega^{\pm}) \ge (2M_0^2)^{-1}$. Applying Harnack's inequality to the function $-v$ in a chain of balls which connect $T_{Q,r}A_r^-(Q)$ and y in Ω^-, we have $|v_{Q,r}(y)| \sim_{M_0} |v_{Q,r}(T_{Q,r}A_r^-(Q))|$. That is, $|v_{Q,r}(y)|$ is bounded above and below by constants that only depend upon M_0. Thus, by the uniform Lipschitz property of $v_{Q,r}$ guaranteed by Theorem 2.6, we can find a ball of radius $0 < c$ such that $|v_{Q,r}| \ge c(M_0)$ on $\partial B_1(0) \cap B_c(y)$. Thus, $H(0, 1, v_{Q,r}) \ge c(M_0)$. Now, recalling Definition 2.7 and the fact that $T_{0,1}v = T_{0,1}(cv)$ for any constant $c > 0$, we have $T_{Q,r}v = T_{0,1}v_{Q,r}$. Since we assumed $\|\ln(h)\|_\alpha \le \Gamma$, $Q \in B_2(0)$, and $0 < r \le 2$, the $v_{Q,r}$ are locally uniformly Lipschitz by Theorem 2.6. Thus $H(0, 1, v_{Q,r}) \ge c(M_0)$ implies $T_{0,1}v_{Q,r} = T_{Q,r}v$ is also locally uniformly Lipschitz. □

Lemma 3.5 (local nondegeneracy). *Let $Q \in \partial\Omega^{\pm}$ and $0 < r < \infty$. Let $v \in \mathcal{A}(n, \alpha, M_0)$ be such that $\|\ln(h)\|_\alpha \le \Gamma$. The rescaling $T_{Q,r}v$ satisfies the following minimum growth conditions. For all $0 < \epsilon$, there is a constant $C = C(M_0, \alpha, \Gamma, \epsilon, R)$ such that, if $p \in B_R(0)$ with $\mathrm{dist}(p, \{T_{Q,r}\partial\Omega^{\pm}\} \cap B_R(0)) > \epsilon$,*

$$|T_{Q,r}v(p)| > C.$$

Proof. As in Lemma 3.4, $T_{Q,r}v(T_{Q,r}A_r^-(Q))$ is bounded above and below by constants that only depend upon the NTA constant M_0, Γ, and R. Thus, by Harnack chains between $T_{Q,r}A_r^-(Q)$ and $p \in T_{Q,r}\Omega^- \cap B_R(0)$ such that $\mathrm{dist}(p, T_{Q,r}\partial\Omega^{\pm} \cap B_R(0)) > \epsilon$, Harnack's inequality applied to $-T_{Q,r}v$ implies that $|T_{Q,r}v(p)| > C$. Note that C only depends upon R, M_0, and ϵ.

To get the same inequality for $p \in T_{Q,r}\Omega^+ \cap B_R(0)$, we recall that standard NTA results compare $T_{Q,r}v(T_{Q,r}A_r^+(Q))$ to $T_{Q,r}\omega^+(B_1(0))$. By Remark 3.1, $T_{Q,r}\omega^+(B_1(0)) \sim T_{Q,r}\omega^-(B_1(0))$ by constants which only depend upon R, Γ, α, and the NTA constants in the definition of the class $\mathcal{A}(n, \alpha, M_0)$. Applying the same Harnack chain and Harnack inequality argument as above gives the lemma. □

Lemma 3.6 (compactness). *Let $\{v_i\}$ be a sequence of functions in $\mathcal{A}(n, \alpha, M_0)$ such that $\|\ln(h)\|_\alpha \le \Gamma$. Let $\{Q_i\} \subset \partial\Omega_i^\pm \cap B_1(0)$ and $0 < r_i \le 1$. There is a subsequence $\{v_j\}$ and a Lipschitz function $v_\infty \in W^{1,2}_{\text{loc}}$ such that $T_{Q_j,r_j} v_j \to v_\infty$ in the following senses*:

(1) $T_{Q_j,r_j} v_j \to v_\infty$ *in* $C_{\text{loc}}(\mathbb{R}^n)$.

(2) $T_{Q_j,r_j} v_j \to v_\infty$ *in* $L^2_{\text{loc}}(\mathbb{R}^n)$.

(3) $\nabla T_{Q_j,r_j} v_j \rightharpoonup \nabla v_\infty$ *in* $L^2_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$.

Proof. To see (1), we recall Lemma 3.4 and the fact that $T_{Q_i,r_i} v_i(0) = 0$. By the Arzelà–Ascoli theorem there exists a subsequence such that $T_{Q_j,r_j} v_j \to v_\infty$ in $C_{\text{loc}}(\mathbb{R}^n)$. This implies convergence in $L^2_{\text{loc}}(\mathbb{R}^n)$. Being uniformly locally Lipschitz and uniformly bounded also implies that the functions $\{T_{Q_j,r_j} v_i\}$ are bounded in $W^{1,2}_{\text{loc}}(\mathbb{R}^n)$. By Rellich compactness, there exists a further subsequence such that $\nabla T_{Q_j,r_j} v_j \rightharpoonup \nabla v_\infty$ in $L^2_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$. □

Before we can prove the strong convergence $T_{Q_j,r_j} v_j \to v_\infty$ in $W^{1,2}_{\text{loc}}(\mathbb{R}^n)$, we need to control the upper Minkowski dimension of $\{v_\infty = 0\}$.

Lemma 3.7. *Under the assumptions of Lemma 3.6, if $T_{Q_i,r_i} v_i \to v_\infty$ in $C_{\text{loc}}(\mathbb{R}^n)$, then $T_{Q_i,r_i} \partial\Omega^\pm \to \{v_\infty = 0\}$ locally in the Hausdorff metric on compact subsets.*

Proof. We argue by contradiction. Suppose that there exists an $\epsilon > 0$, a radius $0 < R$, and a sequence of functions $T_{Q_i,r_i} v_i$ for which we can find a sequence of points $x_i \in B_R(0) \cap \{T_{Q_i,r_i} v_i = 0\}$ such that $\text{dist}(x_i, \{v_\infty = 0\}) > \epsilon$. Taking a subsequence which converges in $C_{\text{loc}}(\mathbb{R}^n)$, we may assume that $x_i \to x_\infty \in \overline{B_R(0)} \setminus B_\epsilon(\{v_\infty = 0\})$. Now, convergence in $C_{\text{loc}}(\mathbb{R}^n)$ implies that $T_{Q_i,r_i} v_i(x_\infty) \to v_\infty(x_\infty)$. Furthermore, since the $T_{Q_i,r_i} v_i$ are uniformly locally Lipschitz, $x_i \to x_\infty$, and $x_i \in \{T_{Q_i,r_i} v_i = 0\}$, we have

$$T_{Q_i,r_i} v_i(x_\infty) \to 0.$$

This implies $x_\infty \in \{v_\infty = 0\}$, which contradicts our previous assertion that $x_\infty \in \overline{B_R(0)} \setminus B_\epsilon(\{v_\infty = 0\})$.

The other direction goes the same way. Suppose that we could find a subsequence of $T_{Q_i,r_i} v_i \to v_\infty$ such that there was a point, $x \in \{v_\infty = 0\} \cap B_R(0)$, for which

$$\text{dist}(x, \{T_{Q_i,r_i} v_i = 0\} \cap B_R(0)) > \epsilon$$

for all $i = 1, 2, \dots$. By Lemma 3.5, we know that $T_{Q_i,r_i} v_i(x) > C$. This contradicts convergence in $C_{\text{loc}}(\mathbb{R}^n)$, however, since $v_\infty(x) = 0$. □

Theorem 3.8 [Kenig and Toro 2006, Theorem 4.1]. *In general, if $\partial\Omega_i^\pm \in \mathcal{D}(n, \alpha, M_0)$ converge to a closed set A locally in the Hausdorff metric on compact subsets, then A divides $\mathbb{R}^n$ into two unbounded, two-sided NTA domains with NTA constant bounded by $2M_0$.*

We must now bound the upper Minkowski dimension of $A = \{v_\infty = 0\}$. We do so crudely, using only that A is the mutual boundary of a pair of two-sided NTA domains. That is, using the machinery of porous sets we are able to prove the following lemma.

Lemma 3.9. *Let $\Sigma \subset \mathbb{R}^n$ be the mutual boundary of a pair of unbounded two-sided NTA domains with NTA constant $1 < M_0$. Then, there exists $0 < \epsilon = \epsilon(M_0, n)$ such that $\overline{\dim}_{\mathcal{M}}(E) \le n - \epsilon$.*

This is an elementary fact which seems to be omitted in the literature. We defer the proof to the Appendix. We now prove strong convergence.

Lemma 3.10 (strong compactness). *Let $\{v_i\}$ be a sequence of functions in $\mathcal{A}(n,\alpha,M_0)$ such that $\|\ln(h)\|_\alpha \le \Gamma$. Let $\{Q_i\} \subset \partial\Omega_i^\pm \cap B_1(0)$ and $0 < r_i < 1$. There is a subsequence $\{v_j\}$ and a Lipschitz function $v_\infty \in W^{1,2}_{\mathrm{loc}}$ such that $T_{Q_j,r_j}v_j \to v_\infty$ in the following senses*:

(1) $T_{Q_j,r_j}v_j \to v_\infty$ *in* $C_{\mathrm{loc}}(\mathbb{R}^n)$.

(2) $T_{Q_j,r_j}v_j \to v_\infty$ *in* $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. The only new claim is that $\nabla T_{Q_j,r_j}v_j \to \nabla v_\infty$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$. By Lemma 3.7, Theorem 3.8, and Lemma 3.9, we have that $\overline{\dim}_{\mathcal{M}}(\{v_\infty=0\}) \le n-\epsilon$. In particular, then, $\mathcal{H}^n(B_r(\{v_\infty=0\}\cap B_R(0))) \to 0$ as $r \to 0$ (see [Mattila 1995] for fundamental facts about Minkowski content, dimension and Hausdorff measure). Thus, for any $\theta > 0$ we can find an $r(\theta) > 0$ such that $\mathcal{H}^n(B_r(\{v_\infty=0\}\cap B_R(0))) \le \theta$. This allows us to estimate

$$\begin{aligned}\limsup_{j\to\infty}&\|\nabla T_{Q_j,r_j}v_j\|^2_{L^2(B_R(0))}\\ &= \limsup_{j\to\infty}\left(\int_{B_R(0)\cap B_r(\{v_\infty=0\})}|\nabla T_{Q_j,r_j}v_j|^2\,dV + \int_{B_R(0)\setminus B_r(\{v_\infty=0\})}|\nabla T_{Q_j,r_j}v_j|^2\,dV\right)\\ &\le \lim_{j\to\infty}\int_{B_R(0)\setminus B_r(\{v_\infty=0\})}|\nabla T_{Q_j,r_j}v_j|^2\,dV + C\theta\\ &\le \|\nabla v_\infty\|^2_{L^2(B_R(0))} + C\theta,\end{aligned}$$

where the penultimate inequality uses the fact that v_j are uniformly Lipschitz, and the last equality follows from convergence in $C(B_{R+r}(0)\setminus B_{r/2}(\{v_\infty=0\}))$ implying $C^\infty(B_R(0)\setminus B_r(\{v_\infty=0\}))$ convergence because the $T_{Q_j,r_j}v_j$ are harmonic functions in this region. Since $\theta > 0$ was arbitrary, we have that $\limsup_{j\to\infty}\|\nabla T_{Q_j,r_j}v_j\|^2_{B_R(0)} \le \|\nabla v_\infty\|^2_{L^2(B_R(0))}$. The other inequality follows from the same trick or from lower semicontinuity. Therefore, we have the equality

$$\lim_{j\to\infty}\|\nabla T_{Q_j,r_j}v_j\|^2_{L^2(B_R(0))} = \|\nabla v_\infty\|^2_{L^2(B_R(0))}.$$

Thus, by weak convergence and norm convergence we have

$$\begin{aligned}\lim_j&\|\nabla T_{Q_j,r_j}v_j - \nabla v_\infty\|^2_{L^2(B_R(0))}\\ &= \lim_j\int_{B_R(0)}|\nabla T_{Q_j,r_j}v_j - \nabla v_\infty|^2\,dV\\ &= \lim_j\|\nabla T_{Q_j,r_j}v_j\|^2_{L^2(B_R(0))} + \|\nabla v_\infty\|^2_{L^2(B_R(0))} - 2\lim_j\langle\nabla T_{Q_j,r_j}v_j,\nabla v_\infty\rangle_{L^2(B_R(0))}\\ &= 2\|\nabla v_\infty\|^2_{L^2(B_R(0))} - 2\|\nabla v_\infty\|^2_{L^2(B_R(0))} = 0.\end{aligned}$$

□

Because the functions $v^Q_{p,r}$ are merely Lipschitz, we will often need to work with a mollified version of them. We will use the convention that $v_\epsilon = v\star\phi_\epsilon$ for $\phi \in C^\infty$ a mollifying function (meaning $\mathrm{spt}(\phi) \subset B_1$ and $\int\phi\,dV = 1$).

Corollary 3.11. *Let $v \in \mathcal{A}(n, \alpha, M_0)$ and $v_\epsilon = v \star \phi_\epsilon$ be a mollification of v. By standard mollification results,*

$$v_\epsilon \to v \quad \text{in } W^{1,2}_{\text{loc}}(\mathbb{R}^n),\ C_{\text{loc}}(\mathbb{R}^n) \qquad \text{as } \epsilon \to 0.$$

4. Almost monotonicity of the Almgren frequency function

One of the key tools of this paper will be the Almgren frequency function (introduced in [Almgren 1979]).

Definition 4.1 (Almgren frequency function). For any Lipschitz function $v : \mathbb{R}^n \to \mathbb{R}$, radius $r > 0$, and point $Q \in \partial\Omega^\pm$, the *Almgren frequency function* is defined as

$$N(p, r, v) = r\frac{D(p, r, v)}{H(p, r, v)}, \tag{4-1}$$

where

$$H(p, r, v) = \int_{\partial B_r(p)} |v|^2 \, d\sigma, \quad D(p, r, v) = \int_{B_r(p)} |\nabla v|^2 \, dV. \tag{4-2}$$

Remark 4.2. The Almgren frequency function is invariant in the following senses. For $a, b \in \mathbb{R}$ with $a, b \neq 0$, if $w(x) = av(bx)$, then $N(0, r, v) = N(0, b^{-1}r, w)$.

If u is harmonic then $N(p, r, u)$ is monotonically nondecreasing. If additionally one assumes that $u(p) = 0$ then $\lim_{r\to 0} N(p, r, u) = N(p, 0, u) \geq 1$ is the degree of the leading homogeneous harmonic polynomial in the Taylor expansion of u at the point p.

4A. *Consequences of Section 3 for the Almgren frequency function.*

Before turning to the main results of this section, we note that the results of Section 3 immediately imply the following corollaries.

Corollary 4.3. *Under the hypotheses of Lemma 3.6, there exists a subsequence such that, for all $r \in (0, 2]$,*

$$N(0, r, T_{Q_j, r_j} v_j) \to N(0, r, v_\infty).$$

Moreover, if $v_\epsilon = v \star \phi$ for a mollifier ϕ as in Corollary 3.11 then, for all $Q \in B_1(0) \cap \partial\Omega^\pm$ and $0 < r \leq 1$,

$$\lim_{\epsilon\to 0} N(Q, r, v_\epsilon) = N(Q, r, v).$$

Proof. This follows from the convergence of the numerator and the denominator; the former follows from Lemma 3.6 (2) and the latter from Lemma 3.6 (1). For the convolution, both follow from Corollary 3.11. □

Corollary 4.4. *Let $v \in \mathcal{A}(n, \alpha, M_0)$ as above. There is a function $C(\alpha, \Gamma, M_0)$ such that, if $\|\ln(h)\|_\alpha \leq \Gamma$ then for all $Q \in B_1(0) \cap \partial\Omega^\pm$ and all $r \in (0, 1]$,*

$$N(Q, r, v) \leq C(\Gamma, \alpha, M_0). \tag{4-3}$$

Proof. We recall that the Almgren frequency function is invariant under rescalings of the function v. Therefore, $N(0, 1, v_{Q,r}) = D(0, 1, T_{Q,r} v)$ is bounded by Lemma 3.4 and the constant only depends upon M_0, Γ, and α. □

4B. *Quantitative almost monotonicity.* This section is dedicated to providing a quantitative version of the following result of Engelstein [2016].

Lemma 4.5 [Engelstein 2016]. *Let $v \in \mathcal{A}(n, \alpha, M_0)$ and $Q \in K \Subset \partial\Omega^{\pm}$. There exists a constant $C < \infty$ (which can be taken uniformly over K and $r \in (0, 1]$) such that*

$$\liminf_{\epsilon\to 0} N(Q, r, v_\epsilon^Q) - N(Q, 0, v_\epsilon^Q) > -Cr^\alpha.$$

The quantitative version of this result which we prove below in Lemma 4.9 is essential for connecting the Almgren frequency to Jones' beta numbers in the "frequency pinching" result later in Lemma 8.2. It comes from examining the derivative of the Almgren frequency function in the r variable.

Throughout this section, we shall use the notation $(v_\epsilon)_\nu(y) = \nabla v_\epsilon(y) \cdot \nu(y)$, where $\nu(y)$ is the unit normal to $\partial B_r(Q)$ at y. By differentiation (see [Engelstein 2016, Section 5.1] for details of the derivation),

$$\begin{aligned} H(Q, r, v_\epsilon)^2 \frac{d}{dr} N(Q, r, v_\epsilon) = 2r\bigg(\int_{\partial B_r(Q)} (v_\epsilon)_\nu^2\, d\sigma \int_{\partial B_r(Q)} |v_\epsilon|^2\, d\sigma - \bigg[\int_{\partial B_r(Q)} v_\epsilon (v_\epsilon)_\nu\, d\sigma\bigg]^2\bigg) \\ + 2r\bigg(\int_{B_r(Q)} v_\epsilon \Delta v_\epsilon\, dV\bigg)\bigg(\int_{\partial B_r(Q)} v_\epsilon (v_\epsilon)_\nu\, d\sigma\bigg) \\ - 2H(Q, r, v_\epsilon) \int_{B_r(Q)} \langle x - Q, \nabla v_\epsilon\rangle \Delta v_\epsilon\, dV. \end{aligned} \tag{4-4}$$

We write the decomposition $\frac{d}{dr} N(Q, r, v_\epsilon) = N_1'(Q, r, v_\epsilon) + N_2'(Q, r, v_\epsilon)$ with

$$N_1'(Q, r, v_\epsilon) := H(Q, r, v_\epsilon)^{-2} 2r\bigg(\int_{\partial B_r(Q)} (v_\epsilon)_\nu^2\, d\sigma \int_{\partial B_r(Q)} |v_\epsilon|^2\, d\sigma - \bigg[\int_{\partial B_r(Q)} v_\epsilon (v_\epsilon)_\nu\, d\sigma\bigg]^2\bigg).$$

We call what remains $N_2'(Q, r, v_\epsilon)$:

$$\begin{aligned} N_2'(Q, r, v_\epsilon) := H(Q, r, v_\epsilon)^{-2}\bigg[2r\bigg(\int_{B_r(Q)} v_\epsilon \Delta v_\epsilon\, dV\bigg)\bigg(\int_{\partial B_r(Q)} v_\epsilon (v_\epsilon)_\nu\, d\sigma\bigg) \\ -2H(Q, r, v_\epsilon) \int_{B_r(Q)} \langle x - Q, \nabla v_\epsilon\rangle \Delta v_\epsilon\, dV(x)\bigg]. \end{aligned}$$

Note that by the Cauchy–Schwarz inequality, $N_1'(Q, r, v_\epsilon) \geq 0$.

Lemma 4.6. *Let $v \in \mathcal{A}(n, \alpha, M_0)$, $Q \in \partial\Omega^{\pm} \cap B_1(0)$ and $0 < r \leq 1$. Then, if $C = \mathrm{Lip}(v|_{B_2(0)})$,*

$$\begin{aligned} N_1'(Q, r, v) &= 2\int_{\partial B_r(Q)} \frac{|\nabla v \cdot (y - Q) - N(Q, r, v) v|^2}{H(Q, |y - Q|, v)|y - Q|}\, d\sigma(y) \\ &\geq \frac{2}{C} \int_{\partial B_r(Q)} \frac{|\nabla v \cdot (y - Q) - N(Q, r, v) v|^2}{|y - Q|^{n+2}}\, d\sigma(y). \end{aligned} \tag{4-5}$$

Proof. Recall that for the Cauchy–Schwarz inequality, we have, for $\lambda = \langle u, v\rangle / \|v\|^2$,

$$\|v\|^2 \|u - \lambda v\|^2 = \|u\|^2 \|v\|^2 - |\langle u, v\rangle|^2.$$

Choosing

$$u = \nabla v_\epsilon \cdot \left(\frac{y-Q}{|y-Q|}\right) \quad \text{and} \quad v = v_\epsilon,$$

using the divergence theorem on λ, and letting $\epsilon \to 0$, we have

$$N_1'(Q,r,v) = H(Q,r,v)^{-1} 2r \left(\int_{\partial B_r(Q)} \left| (v)_\nu - \frac{1}{r} N(Q,r,v) v \right|^2 d\sigma \right).$$

This proves the equality. To prove the lower bound, we let $C = \mathrm{Lip}(T_{0,1} v|_{B_2(0)})$ and observe that $H(Q,r,T_{0,1}v_\epsilon) \le Cr^{n+1}$. Plugging this into the above equation, we get the desired inequality

$$N_1'(Q,r,v) \ge \frac{2}{C} \int_{\partial B_r(Q)} \frac{|\nabla T_{0,1} v(y) \cdot (y-Q) - N(Q,r,v) T_{0,1} v(y)|^2}{|y-Q|^{n+2}} \, d\sigma(y). \qquad \square$$

In order to bound the parts of $N_2'(Q,r,v_\epsilon)$, we recall some estimates from [Engelstein 2016].

Lemma 4.7 [Engelstein 2016, Lemmata 5.4, 5.5, and 5.6]. *Let $v \in \mathcal{A}(n,\alpha,M_0)$ with $\|\ln(h)\|_\alpha \le \Gamma$, and let $Q \in B_1(0) \cap \partial\Omega^\pm$. For any $0 < s$ and $\epsilon \ll s$,*

$$\int_{\partial B_s(Q)} |v_\epsilon|^2 \, d\sigma \ge C(M_0) \frac{\omega^-(B_s(Q))^2}{s^{n-3}}, \tag{4-6}$$

$$\left| \int_{B_s(Q)} v_\epsilon \Delta v_\epsilon \, dV \right| \le C \|\ln(h)\|_\alpha s^\alpha \frac{\omega^-(B_s(Q))^2}{s^{n-2}}, \tag{4-7}$$

$$\left| \int_{B_s(Q)} \langle \nabla v_\epsilon, x-Q \rangle \Delta v_\epsilon \, dV(x) \right| \le C \|\ln(h)\|_\alpha s^\alpha \frac{\omega^-(B_s(Q))^2}{s^{n-2}}, \tag{4-8}$$

$$\left| \int_{\partial B_s(Q)} v_\epsilon (v_\epsilon)_\nu \, d\sigma \right| \le C \frac{\omega^-(B_s(Q))^2}{s^{n-1}}, \tag{4-9}$$

where $C = C(\alpha, M_0, \Gamma)$.

Proof. Let $v \in \mathcal{A}(n,\alpha,M_0)$ be given. Recall that $v = v^0$. Engelstein [2016, Lemmata 5.4, 5.5, and 5.6] proves the claim for the functions $v^Q_{Q,1}$. Hence, for any such v and any such Q, the integral estimates hold for $u(x) = v^Q(x+Q)$ as well. However, in general, such $v^0(\cdot + Q)$ are not in $\mathcal{A}(n,\alpha,M_0)$ because $h(0)$ may not be 0. But,

$$v^0(x+Q) = cu^+(x+Q) - u^-(x+Q)$$

is an element of $\mathcal{A}(n,\alpha,M_0)$ for some constant $e^{-\Gamma|Q|^\alpha} \le c \le e^{\Gamma|Q|^\alpha}$ as in (3-1). Using this identity and following the proofs of [Engelstein 2016, Lemmata 5.4, 5.5, and 5.6] gives the claim. $\square$

Remark 4.8. Recalling our expansion of $\frac{d}{dr} N(r,p,v_\epsilon)$ in (4-4) and the bounds contained in Lemma 4.7 we have that, for $v \in \mathcal{A}(n,\alpha,M_0)$ with $\|\ln(h)\|_\alpha \le \Gamma$, $\epsilon \ll r$, and $Q \in B_1(0) \cap \partial\Omega^\pm$,

$$|N_2'(Q,r,v_\epsilon)| \le C_1 \|\ln(h)\|_\alpha r^{\alpha-1}, \tag{4-10}$$

where $C_1 = C(\alpha, M_0, \Gamma)$.

We now state the main result of this section.

Lemma 4.9. *Let $v \in \mathcal{A}(n, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \le \Gamma$, and let $Q \in B_1(0) \cap \partial\Omega^\pm$. For any $0 \le s < S \le 1$,*

$$\frac{2}{C}\int_{A_{s,S}(Q)} \frac{|\nabla T_{0,1}v(y)\cdot(y-Q) - N(Q,|y-Q|,T_{0,1}v)T_{0,1}v(y)|^2}{|y-Q|^{n+2}}\,dV(y)$$
$$\le 2\int_{A_{s,S}(Q)} \frac{|\nabla v(y)\cdot(y-Q) - N(Q,|y-Q|,v)v(y)|^2}{H(Q,|y-Q|,v)|y-Q|}\,dV(y)$$
$$\le N(Q,S,v) - N(Q,s,v) + C_1\|\ln(h)\|_\alpha S^\alpha, \tag{4-11}$$

where $C_1 = C_1(\alpha, M_0, \Gamma)$ and $C(M_0, \Gamma, \alpha) = \mathrm{Lip}(T_{0,1}v|_{B_2(0)})$.

Proof. We begin by normalizing v. Since $N(r,p,v) = N(r,p,cv)$ for any $c \neq 0$, we may work with $T_{0,1}v$. Note that by Remark 4.8 and (4-5), $N(Q,r,v)$ is continuous in r and hence we may find an $0 \le s < s_1$ such that

$$|N(Q,s,v) - N(Q,s_1,v)| \le \|\ln(h)\|_\alpha S^\alpha.$$

By Corollaries 3.11 and 4.3 we can find an $\epsilon \ll s$ small enough that

$$|N(Q,s_1,v_\epsilon) - N(Q,s_1,v)| < \|\ln(h)\|_\alpha S^\alpha \quad\text{and}\quad |N(Q,S,v_\epsilon) - N(Q,S,v)| < \|\ln(h)\|_\alpha S^\alpha.$$

Thus, we reduce to estimating $N(Q,S,T_{0,1}v_\epsilon) - N(Q,s_1,T_{0,1}v_\epsilon)$:

$$N(Q,S,T_{0,1}v_\epsilon) - N(Q,s_1,T_{0,1}v_\epsilon) = \int_{s_1}^{S} \frac{d}{dr}N(Q,r,T_{0,1}v_\epsilon)\,dr$$
$$= \int_{s_1}^{S} N_1'(Q,r,T_{0,1}v_\epsilon)\,dr + \int_{s_1}^{S} N_2'(Q,r,T_{0,1}v_\epsilon)\,dr.$$

Recalling Remark 4.8, Lemma 4.6, and letting $\epsilon \to 0$ gives the lemma. □

Using these estimates it is possible to control the drop across scales from the total drop.

Lemma 4.10. *If $v \in \mathcal{A}(n, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \le \Gamma$ and $Q \in B_1(0) \cap \partial\Omega^\pm$, then for any $0 \le r \le s < S \le R$*

$$N(Q,S,v) - N(Q,s,v) \le 2C_1\|\ln(h)\|_\alpha R^\alpha + |N(Q,R,v) - N(Q,r,v)|.$$

Proof. This is essentially a "rays of the sun" argument. To wit,

$$N(Q,S,v) - N(Q,s,v) = \int_s^S N_1'(Q,\rho,v) + N_2'(Q,\rho,v)\,d\rho$$
$$\le \int_s^S N_1'(Q,\rho,v) + |N_2'(Q,\rho,v)|\,d\rho$$
$$\le \int_r^R N_1'(Q,\rho,v) + |N_2'(Q,\rho,v)|\,d\rho$$
$$\le 2\int_r^R |N_2'(Q,\rho,v)|\,d\rho + |N(Q,R,v) - N(Q,r,v)|.$$

The bounds in Remark 4.8 give the desired statement. □

We now turn our attention to proving a "doubling" property for $H(p,r,v)$. This is an analog of classical harmonic results for the Almgren frequency function, modified for our almost harmonic functions $v \in \mathcal{A}(n,\alpha,M_0)$.

Lemma 4.11 ($H(r,p,v_\epsilon)$ is almost doubling). *Let $v \in \mathcal{A}(n,\alpha,M_0)$ with $\|\ln(h)\|_\alpha \le \Gamma$, and let $Q \in B_{1/2}(0)\cap\partial\Omega^\pm$. For any $0 < s < S \le 1$, if $\epsilon \ll s$ is sufficiently small,*

$$H(Q,S,v_\epsilon) \le \left(\frac{S}{s}\right)^{(n-1)+2(N(Q,S,v_\epsilon)+CS^\alpha)} \exp\left(\frac{2C}{\alpha}[S^\alpha - s^\alpha]\right) H(Q,s,v_\epsilon), \tag{4-12}$$

where $C = \|\ln(h)\|_\alpha C_1(M_0,\alpha,\Gamma)$ and C_1 is as in Remark 4.8.

Proof. First, observe that

$$\frac{d}{dr}H(Q,r,v_\epsilon) = \frac{n-1}{r}\int_{\partial B_r(Q)} |v_\epsilon|^2\,d\sigma + 2\int_{B_r(Q)} |\nabla v_\epsilon|^2\,dV + 2\int_{B_r(Q)} v_\epsilon \Delta v_\epsilon\,dV.$$

Next, we consider the identity

$$\begin{aligned}\ln\left(\frac{H(Q,S,v_\epsilon)}{H(Q,s,v_\epsilon)}\right) &= \ln(H(Q,S,v_\epsilon)) - \ln(H(Q,s,v_\epsilon))\\ &= \int_s^S \frac{H'(Q,r,v_\epsilon)}{H(Q,r,v_\epsilon)}\,dr = \int_s^S \frac{n-1}{r} + \frac{2}{r}N(Q,r,v_\epsilon) + 2\left(\frac{\int_{B_r(Q)} v_\epsilon \Delta v_\epsilon dV}{\int_{\partial B_r(Q)} (v_\epsilon)^2 d\sigma}\right) dr.\end{aligned}$$

We bound $N(r,Q,v_\epsilon)$ by Lemma 4.9. We bound the last term using Lemma 4.7. Plugging in these bounds, we have, for $\epsilon \ll s$,

$$\ln\left(\frac{H(Q,S,v_\epsilon)}{H(Q,s,v_\epsilon)}\right) \le [(n-1) + 2(N(Q,S,v_\epsilon) + CS^\alpha)]\ln(r)\Big|_s^S + \frac{2C}{\alpha}r^\alpha\Big|_s^S.$$

Evaluating and exponentiating gives the desired result. □

Remark 4.12. Because $H(Q,r,v_\epsilon) \to H(Q,r,v)$ as $\epsilon \to 0$ and $N(Q,r,v_\epsilon) \to N(Q,r,v)$ as $\epsilon \to 0$ (a consequence of Corollary 3.11), we have the following inequality. For all $v \in \mathcal{A}(n,\alpha,M_0)$ with $\|\ln(h)\|_\alpha \le \Gamma$, $Q \in B_{1/2}(0)\cap\partial\Omega^\pm$, and $0 < s < S \le 1$,

$$H(Q,S,v) \le \left(\frac{S}{s}\right)^{(n-1)+2(N(Q,S,v)+CS^\alpha)} \exp\left(\frac{2C}{\alpha}[S^\alpha - s^\alpha]\right) H(Q,s,v). \tag{4-13}$$

5. Quantitative rigidity

Throughout the rest of the paper, we shall need to use limit-compactness arguments. The key will be that $v \to u$ for some harmonic function u as $\|\ln(h)\|_\alpha \to 0$. We make this rigorous in the following lemma.

Lemma 5.1 (convergence to harmonic functions). *Let $v_i \in \mathcal{A}(n,\alpha,M_0)$ with $\|\ln(h_i)\|_\alpha \to 0$. Assume that $Q_i \in B_1(0)\cap\partial\Omega_i^\pm$ and $\{r_i\} \subset (0,1]$. Then there exists a function v_∞ and a subsequence v_j such that $T_{Q_j,r_j}v_j \to v_\infty$ in the sense of Lemma 3.10 and v_∞ is harmonic.*

Proof. Lemma 3.10 gives a subsequence $T_{Q_j,r_j} v_j$ which converges strongly in $W^{1,2}_{\text{loc}}(\mathbb{R}^n)$ to a function v_∞. We claim that v_∞ is harmonic. To see this, we investigate the behavior of its mollifications $v_{\infty,\epsilon} = v_\infty \star \phi_\epsilon$. Observe that by Young's inequality,

$$\|T_{Q_j,r_j} v_{j,\epsilon} - v_{\infty,\epsilon}\|_{L^2(B_2(0))} \le \|\phi_\epsilon\|_{L^1(B_2(0))} \|T_{Q_j,r_j} v_j - v_\infty\|_{L^2(B_2(0))}.$$

Thus, for any $\epsilon > 0$ we have $T_{Q_j,r_j} v_{j,\epsilon} \to v_{\infty,\epsilon}$ as $j \to \infty$ strongly in $L^2(B_2(0))$. By a similar argument applied to $\nabla T_{Q_j,r_j} v_{j,\epsilon}$, we also have that $\nabla T_{Q_j,r_j} v_{j,\epsilon} \to \nabla v_{\infty,\epsilon}$ in $L^2(B_2(0); \mathbb{R}^n)$ as $j \to \infty$. Furthermore, by our uniform Lipschitz bounds, $T_{Q_j,r_j} v_{j,\epsilon} \to v_{\infty,\epsilon}$ as $j \to \infty$ in $C(B_2(0))$ as well.

We will show that for $\epsilon \ll 1$ the function $v_{\infty,\epsilon}$ is harmonic. First, for any test function $\xi \in C^\infty_c(B_2(0))$, we have

$$\left| \int_{B_2(0)} \xi (\Delta T_{Q_j,r_j} v_{j,\epsilon} - \Delta v_{\infty,\epsilon}) \, dV \right| = \left| \int_{B_2(0)} \Delta\xi (T_{Q_j,r_j} v_{j,\epsilon} - v_{\infty,\epsilon}) \, dV \right|$$
$$\le \|\Delta\xi\|_{L^2(B_2(0))} \|T_{Q_j,r_j} v_{j,\epsilon} - v_{\infty,\epsilon}\|_{L^2(B_2(0))}.$$

Since $T_{Q_j,r_j} v_{j,\epsilon} \to v_{\infty,\epsilon}$ strongly in $L^2(B_2(0))$, we have $\Delta T_{Q_j,r_j} v_{j,\epsilon} \rightharpoonup \Delta v_{\infty,\epsilon}$ in $L^2(B_2(0))$.

However, by assumption, we also have

$$\left| \int_{B_2(0)} \xi \Delta T_{Q_j,r_j} v_{j,\epsilon} \, dV \right| \le \int_{B_2(0)} |\xi_\epsilon| \left| \frac{h_j(0)}{h_j(x)} - 1 \right| dT_{Q_j,r_j} \omega^- \le C \max_{B_2(0)} |\xi| \cdot \|\ln(h_j)\|_\alpha T_{Q_j,r_j} \omega^-(B_3(0)),$$

where $T_{Q_j,r_j}\omega^\pm$ are the interior and exterior harmonic measures associated to $T_{Q_j,r_j} v_j$. Note that $T_{Q_j,r_j}\omega^- \neq \omega^-_{Q_j,r_j}$, but, by Definitions 2.4 and 2.7 and Lemma 3.4, there is a constant $c' = c'(M_0)$ such that $T_{Q_j,r_j}\omega^- = c\omega^-_{Q_j,r_j}$ and $c \le c'$. Since $\omega^-_{r_j,Q_j}(B_3(0))$ are uniformly bounded by Theorem 2.6, the $T_{Q_j,r_j}\omega^-(B_3(0))$ are, too. Thus, as $j \to \infty$, we have that $\Delta T_{Q_j,r_j} v_{j,\epsilon} \rightharpoonup 0$ in $L^2(B_2(0))$ as well. Thus, $\Delta v_{\infty,\epsilon} = 0$ weakly in $L^2(B_2(0))$. Since $v_{\infty,\epsilon} \in C^\infty(B_2(0))$, we have that $v_{\infty,\epsilon}$ is harmonic.

Since v_∞ is Lipschitz continuous, $v_{\infty,\epsilon} \to v_\infty$ in $C(B_R(0))$ as $\epsilon \to 0$. Thus, for all $x \in B_R(0)$ we have both that $v_{\infty,\epsilon}(x) \to v_\infty(x)$ as $\epsilon \to 0$ and that

$$\fint_{B_r(x)} v_{\infty,\epsilon}(y) \, dV(y) \to \fint_{B_r(x)} v_\infty(y) \, dV(y)$$

as $\epsilon \to 0$. Thus, v_∞ must satisfy the mean value property and is therefore harmonic. □

Now that we have Lemma 5.1, we can prove a quantitative rigidity result. Loosely speaking, it says that if a function $v \in \mathcal{A}(n, \alpha, M_0)$ behaves like a homogeneous harmonic polynomial with respect to the Almgren frequency (in the sense that it has small drop across scales), then it must be close to being a homogeneous harmonic polynomial. This will connect the behavior of the Almgren frequency to our quantitative stratification.

Lemma 5.2 (quantitative rigidity). *Let $v \in \mathcal{A}(n, \alpha, M_0)$, as above. Let $Q \in B_1(0) \cap \partial\Omega^\pm$. For every $\delta > 0$, there is an $\gamma = \gamma(n, \alpha, M_0, \delta) > 0$ such that if $\|\ln(h)\|_\alpha \le \gamma$ and*

$$N(Q, 1, v) - N(Q, \gamma, v) \le \gamma,$$

then v is $(0, \delta, 1, Q)$-symmetric.

Proof. We argue by contradiction. Assume that there exists a $\delta > 0$ such that there is a sequence of functions $v_i \in \mathcal{A}(n, \alpha, M_0)$ with $\|\ln(h_i)\|_\alpha \le 2^{-i}$ for which there exists a point $Q_i \in B_1(0) \cap \partial\Omega_i^\pm$ with

$$N(Q_i, 1, v_i) - N(Q_i, 2^{-i}, v_i) \le 2^{-i},$$

but where no v_i is $(0, \delta, 1, Q_i)$-symmetric.

By Lemma 5.1 there exists a subsequence $T_{Q_j,1}v_j$ which converges strongly in $W^{1,2}_{\text{loc}}$ to a harmonic function v_∞. Therefore $N(Q, r, v_\infty)$ is monotone increasing. Further, by Corollary 4.3 we know that $\lim_{j\to\infty} N(0, r, T_{Q_j,1}v_j) = N(0, r, v_\infty)$ for all $0 < r \le 1$. By Lemma 4.10 and the aforementioned convergence, we have that

$$N(0, 1, v_\infty) - N(0, 0, v_\infty) = 0.$$

This implies that v_∞ is a homogeneous harmonic polynomial (see, for example, the proof of [Han and Lin 1994, Theorem 2.2.3]). Thus, we arrive at our contradiction, since the $T_{Q_j,1}v_j$ were assumed to stay away from all such functions in $L^2(B_1(0))$. □

Remark 5.3. Since $N(Q, r, v)$ is scale-invariant, Lemma 5.2 is also scale-invariant in the sense that if $N(Q, r, v) - N(Q, \gamma r, v) \le \gamma$ and $\|\ln(h)\|_\alpha \le \gamma$, then v is $(0, \delta, r, Q)$-symmetric.

6. A dichotomy

The proof technique in the rest of the paper is an adaptation of techniques developed by Naber and Valtorta [2017].

This section is dedicated to proving a lemma that gives us geometric information on the quantitative strata. Roughly, it says that if we can find $(k+1)$ points that are well-separated and the Almgren frequency has very small drop at these points, then the quantitative strata is contained in a neighborhood of the affine k-plane which contains them and we have control on the Almgren frequency for all points in that neighborhood. This is a quantitative analog of the following classical result.

Proposition 6.1. *Let $P : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous harmonic polynomial. Let $0 \le k \le n-2$. If P is translation-invariant with respect to some k-dimensional subspace V and P is homogeneous with respect to some point $x \notin V$, then P is $(k+1)$-symmetric with respect to* $\text{span}\{x, V\}$.

See [Cheeger et al. 2015, Proposition 2.11] or [Han and Lin 1994, proof of Theorem 4.1.3].

We shall use the notation $\langle y_0, \dots, y_k\rangle$ to denote the k-dimensional affine linear subspace which passes through $y_0, \dots, y_k$.

Lemma 6.2. *Let $v \in \mathcal{A}(n, \alpha, M_0)$ and $0 < \epsilon$ be fixed. Let $\gamma, \eta', \rho > 0$ be fixed, then there exist constants $0 < \eta_0(n, \alpha, E_0, \epsilon, \eta', \gamma, \rho) \ll \rho$ and $0 < \beta(n, \alpha, E_0, \epsilon, \eta', \rho) < 1$ such that, if*

(1) $E = \sup_{Q\in B_1(0)\cap\partial\Omega^\pm} N(Q, 2, v) \in [0, E_0]$,

(2) *there exist points $\{y_0, y_1, \dots, y_k\} \subset B_1(0) \cap \partial\Omega^\pm$ satisfying $y_i \notin B_\rho(\langle y_0, \dots, y_{i-1}, y_{i+1}, \dots, y_k\rangle)$ and*

$$N(y_i, \gamma\rho, v) \ge E - \eta_0$$

for all $i = 0, 1, \dots, k$, and

(3) $\|\ln(h)\|_\alpha \leq \eta_0$,

then, writing $\langle y_0, \dots, y_k\rangle = L$, *for all* $Q \in B_\beta(L) \cap B_1(0) \cap \partial\Omega^\pm$,

$$N(Q, \gamma\rho, v) \geq E - \eta'$$

and

$$\mathcal{S}^k_{\epsilon,\eta_0} \cap B_1(0) \subset B_\beta(L).$$

Proof. There are two conclusions. We argue by contradiction for both. Suppose that the first claim fails. That is, assume that there exist constants $\gamma, \rho, \eta' > 0$ for which there exists a sequence $v_i \in \mathcal{A}(n, \alpha, M_0)$ with $\sup_{Q\in B_1(0)} N(Q, 2, v_i) = E_i \in [0, E_0]$ and points $\{y_{i,j}\}_j$ satisfying (2) above, with $\|\ln(h_i)\|_\alpha \leq 2^{-i}$, $\eta_0 < 2^{-i}$, and a sequence $\beta_i \leq 2^{-i}$ such that, for each i, there exists a point $x_i \in B_{\beta_i}(L_i) \cap B_1(0) \cap \partial\Omega^\pm_i$ for which $N(x_i, \gamma\rho, v_i) < E - \eta'$.

By Lemma 5.1, there exists a subsequence v_i such that $T_{0,1}v_i$ converges to a harmonic function v_∞ in the senses outlined in the lemma. Further, by the compactness of $[0, E_0]$, $\overline{B_1(0)}$, and the Grassmannian, we may assume that

$$E_i \to E, \quad y_{i,j} \to y_j, \quad L_i \to L, \quad x_i \to x_\infty \in \overline{B_1(0)} \cap \partial\Omega^\pm_\infty,$$

where $\partial\Omega^\pm_\infty = \{v_\infty = 0\}$ is a two-sided NTA domain with constant $2M_0$ by Theorem 3.8. Note that the convergence given by Lemma 5.1 implies

$$\sup_{Q\in B_1(0)} N(Q, 2, v_\infty) \leq E, \qquad N(x_\infty, \gamma\rho, v_\infty) < E - \eta',$$

and

$$N(y_j, \gamma\rho, v_\infty) \geq E$$

for all $j = 0, 1, \dots, k$. Because v_∞ is harmonic, $N(p, r, v_\infty)$ is nondecreasing in r for all $p \in B_2(0)$. Therefore, $N(y_i, r, v_\infty) = E$ for all y_i and all $r \in [\gamma\rho, 2]$. Thus, v_∞ is a 0-symmetric function in $B_2(y_j) \setminus B_{\gamma\rho}(y_j)$ for each y_j. By unique continuation, v_∞ is 0-symmetric with respect to y_j for each j. Because the $y_j \in \overline{B_1(0)}$ are in general position, by Proposition 6.1, v_∞ is translation-invariant along L in $B_2(0)$. Since $x_\infty \in L \cap \overline{B_1(0)}$, this implies that $N(x_\infty, 0, v_\infty) = E$. But this contradicts $N(x_\infty, \gamma\rho, v_\infty) < E - \eta'$, since $N(x_\infty, r, v_\infty)$ must be nondecreasing in r. This proves the first claim.

Now assume that the second claim fails. That is, fix $\beta > 0$ and assume that there is a sequence $v_i \in \mathcal{A}(n, \alpha, M_0)$ with $\sup_{Q\in B_1(0)} N(Q, 2, v_i) = E_i \in [0, E_0]$ and points $\{y_{i,j}\}_j$ satisfying (2) above, with $\|\ln(h_i)\|_\alpha \leq 2^{-i}$ and a sequence $\eta_i \to 0$ such that for each i there exists a point $x_i \in \mathcal{S}^k_{\epsilon,\eta_i}(v_i) \cap B_1(0) \setminus B_\beta(L_i)$.

Again, we extract a subsequence as above. The function v_∞ will be harmonic and k-symmetric in $B_{1+\delta}(0)$, as above, and $x_i \to x \in \overline{B_1(0)} \setminus B_\beta(L)$. Note that by our definition of $\mathcal{S}^k_{\epsilon,\eta_i}(v_i)$ and the convergence in Lemma 5.1, $x \in \mathcal{S}^k_{\epsilon/2}(v_\infty)$.

Since v_∞ is k-symmetric and L is its k-dimensional spine, every blow-up at a point in $\overline{B_1(0)} \setminus B_\beta(L)$ will be $(k+1)$-symmetric. Thus, there must exist a radius r for which v_∞ is $\left(k+1, \frac{1}{4}\epsilon, r, x\right)$-symmetric. This contradicts the conclusion that $x \in \mathcal{S}^k_{\epsilon/2}(v_\infty)$. ☐

Consider the following dichotomy: either we can find $(k+1)$ well-separated points y_{ij} with very small drop in frequency or we cannot. In the former case, Lemma 6.2 implies that the Almgren frequency has small drop on all of $\mathcal{S}^k_{\epsilon,\eta}(v)$ (and we also get good geometric control). In the latter case, the set on which the Almgren frequency has small drop is close to a $(k-1)$-plane. In this case, even though we have no geometric control on $\mathcal{S}^k_{\epsilon,\eta}(v)$, we have very good packing control on the part with small drop in frequency. We make this formal in the following corollary.

Corollary 6.3 (key dichotomy). *Let $\gamma, \rho, \eta' \in (0,1)$ and $0 < \epsilon$ be fixed. There exist*

$$0 < \beta(n, \alpha, E_0, \epsilon, \eta', \rho) < 1 \quad \textit{and} \quad 0 < \eta_0 = \eta_0(n, \alpha, E_0, \epsilon, \eta', \gamma, \rho) \ll \rho$$

such that the following holds. For all $v \in \mathcal{A}(n, \alpha, M_0)$ with $\sup_{Q \in B_1(0)} N(Q, 2, v) \leq E \in [0, E_0]$, if $\eta \leq \eta_0$ and $\|\ln(h)\|_\alpha \leq \eta$, then one of the following possibilities must occur:

(1) *$N(Q, \gamma\rho, v) \geq E - \eta'$ on $S^k_{\epsilon,\eta_0}(v) \cap B_1(0)$ and*

$$\mathcal{S}^k_{\epsilon,\eta_0} \cap B_1(0) \subset B_\beta(L).$$

(2) *There exists a $(k-1)$-dimensional affine plane L^{k-1} such that*

$$\{Q \in \partial\Omega^\pm : N(Q, 2\eta, v) \geq E - \eta_0\} \cap B_1(0) \subset B_\rho(L^{k-1}).$$

Remark 6.4. The former case is simply the conclusion of Lemma 6.2. In the latter case of the dichotomy, we know that all points in $\partial\Omega^\pm \cap B_1(0) \setminus B_\rho(L^{k-1})$ must have $N(Q, 2\eta, v) < E - \eta_0$. Since $N(Q, r, v)$ is almost monotonic and uniformly bounded, this can happen for each Q only finitely many times.

7. Spatial derivatives of the Almgren frequency

The main result of this section is Corollary 7.7, in which we estimate the difference between the Almgren frequency at nearby points. First, we need a preliminary estimate which extends one of the results of Lemma 4.7 to points $p \in B_1(0) \setminus \partial\Omega^\pm$.

Lemma 7.1. *Let $v \in \mathcal{A}(n, \alpha, M_0)$, and let $0 < s \leq 1$, $Q \in \partial\Omega^\pm \cap B_1(0)$, and $p \in B_{s/3}(Q)$. Then we have the estimate*

$$H(p, s, v_\epsilon) \geq C(n, M_0) \frac{\omega^-(B_s(Q))^2}{s^{n-3}}.$$

Furthermore, for all $0 < s \leq \frac{1}{2}$ and all $\frac{1}{2}s \leq r \leq 2s$,

$$H(p, r, v_\epsilon) \approx_{n,\alpha,M_0,\Gamma} H(Q, 2s, v_\epsilon) \approx_{n,\alpha,M_0,\Gamma} H\big(Q, \tfrac{1}{2}s, v_\epsilon\big).$$

Proof. Let $x_{\max}(p, s)^\pm$ denote the point in $\partial B_s(p) \cap \Omega^\pm$ which maximizes $|v|$ on $\partial B_s(p) \cap \Omega^\pm$.

If we can show that, for all p and all $0 < s \leq \frac{1}{2}$,

$$|v(x_{\max}(Q, s)^-)| \sim_{M_0} \frac{\omega^-(B_s(Q))}{s^{n-2}}$$

and that $\operatorname{dist}(x_{\max}(p,s), \partial\Omega^{\pm}) \geq \delta(M_0) > 0$, then

$$\int_{\partial B_s(p)} |v|^2\, d\sigma \geq \int_{\partial B_s(p)\cap B_{\delta s}(x_{\max}(p,s))} |v|^2\, d\sigma \geq C(M_0)|v(x_{\max}(p,s))|^2(\delta s)^{n-1} \geq C(n, M_0)\frac{\omega^-(B_s(Q))^2}{s^{n-3}}.$$

If this can be shown, then recalling the doubling of harmonic measure on NTA domains, the above string of inequalities proves that $H(p,s,v_\epsilon)$, $H(Q,s,v_\epsilon)$, and $H\big(Q, \frac{1}{2}s, v_\epsilon\big)$ share a common lower bound.

The common upper bound follows from a similar argument using Remark 3.1. That is, if we can show, for all p and all $0 < s \leq \frac{1}{2}$, that $\operatorname{dist}(x_{\max}(p,s), \partial\Omega^{\pm}) \geq \delta(\alpha, M_0, \Gamma) > 0$, then by Harnack chains we know that

$$|v(x_{\max}(Q,s)^+)| \sim_{\alpha, M_0, \Gamma} \frac{\omega^+(B_s(Q))}{s^{n-2}}$$

and that

$$\begin{aligned}\int_{\partial B_s(Q)} |v|^2\, d\sigma &\leq |v(x_{\max}(Q,s))^-|^2 r^{n-1} + |v(x_{\max}(Q,s))^+|^2 r^{n-1} \\ &\leq C(M_0)(|v(A_s(Q)^-)|^2 r^{n-1} + |v(A_s(Q)^+)|^2 r^{n-1}) \leq C(n,\alpha,M_0,\Gamma)\frac{\omega^-(B_s(Q))^2}{s^{n-3}}.\end{aligned}$$

Recalling the doubling of harmonic measure on NTA domains, the above string of inequalities proves that $H(p,s,v_\epsilon)$, $H(Q,s,v_\epsilon)$, and $H\big(Q, \frac{1}{2}s, v_\epsilon\big)$ share a common lower bound. This would prove the lemma.

Let Q, p, and s be given. By the maximum principle for harmonic functions applied to v^- in Ω^-, we have $|v(x_{\max}(p,s)^{\pm})| \geq \big|v\big(x_{\max}\big(Q, \frac{1}{2}s\big)^{\pm}\big)\big|$. By NTA estimates [Engelstein 2016, Lemma 5.4], we have

$$\frac{\omega^{\pm}(B_{1/2s}(Q))}{\big(\frac{1}{2}s\big)^{n-2}} \sim_{M_0} |v(A^{\pm}_{s/2}(Q))| \leq \big|v\big(x_{\max}\big(Q, \tfrac{1}{2}s\big)^{\pm}\big)\big| \leq |v(x_{\max}(p,s)^{\pm})|.$$

Therefore, by the uniform Lipschitz estimates of Theorem 2.6 and Remark 3.1 we infer that

$$\operatorname{dist}(x_{\max}(p,s)^{\pm}, \partial\Omega^{\pm}) \gtrsim_{M_0,\Gamma,\alpha} s.$$

Therefore, we may use Harnack chains and estimate

$$|v(x_{\max}(p,s)^{\pm})| \leq |v(A^{\pm}_{2s}(Q))| \sim_{M_0} \frac{\omega^{\pm}(B_{2s}(Q))}{(2s)^{n-2}}.$$

Thus, by the doubling of harmonic measure on NTA domains (see [Jerison and Kenig 1982]), we infer that $|v(x_{\max}(p,s)^{\pm})| \sim_{M_0} |v(A_{2s}(Q)^{\pm})|$. This proves the lemma. □

Remark 7.2. As a consequence of Lemma 7.1 and Corollary 4.4, we observe that if $v \in \mathcal{A}(n,\alpha,M_0)$ then, for every $0 < r \leq \frac{1}{2}$ and every point $p \in B_1(0)$ such that $\operatorname{dist}(p, \partial\Omega^{\pm}) \leq \frac{1}{3}r$ and for any $0 < \epsilon \ll r$,

$$N(p,r,v) \leq C(n,\alpha,M_0,\Gamma) \quad \text{and} \quad N(p,r,v_\epsilon) \leq C(n,\alpha,M_0,\Gamma).$$

Lemma 7.3. *Let $v \in \mathcal{A}(n,\alpha,M_0)$ with $\|\ln(h)\|_\alpha \leq \Gamma$, and let $Q \in B_1(0)\cap\partial\Omega^{\pm}$, $0 < s \leq 1$, and $\epsilon \ll s$. For all $p \in B_{s/3}(Q)\cap\overline{\Omega}^-$ and all vectors $|\vec{v}| \leq r$,*

$$\left|\int_{B_s(p)} \langle\nabla v_\epsilon, \vec{v}\rangle \Delta v_\epsilon\, dV(x)\right| \leq C\|\ln(h)\|_\alpha s^\alpha \frac{\omega^-(B_s(Q))^2}{s^{n-2}}|\vec{v}|, \tag{7-1}$$

where $C = C(M_0)$.

Proof. Let p, Q, and s be as above. For sufficiently small $0 < \epsilon$,

$$\left|\int_{B_s(p)} \langle \nabla v_\epsilon, \vec{v}\rangle \Delta v_\epsilon \, dV(x)\right|$$
$$= \left|\int_{B_s(p)} \langle \nabla v_\epsilon, \vec{v}\rangle_\epsilon \Delta v(x)\right| \le \int_{B_s(p)} |(\langle \nabla v_\epsilon, \vec{v}\rangle)_\epsilon| \left|\frac{h(0)}{h(x)} - 1\right| d\omega^- \le \int_{B_{2s}(Q)} |(\langle \nabla v_\epsilon, \vec{v}\rangle)_\epsilon| \left|\frac{h(0)}{h(x)} - 1\right| d\omega^-$$
$$\le \|\ln(h)\|_\alpha (2s)^\alpha \int_{B_{2s}(Q)} |(\langle \nabla v_\epsilon, \vec{v}\rangle)_\epsilon| \, d\omega^- \le \|\ln(h)\|_\alpha (2s)^{\alpha+1} |\vec{v}| \int_{B_{2s}(Q)} |\nabla v_\epsilon|_\epsilon \, d\omega^-.$$

Chasing through the change of variables $x = ry + Q$, we see that

$$\nabla_x v(x) = \frac{1}{r} \nabla_y v(ry + Q) = \frac{\omega^-(B_r(Q))}{r^{n-1}} \nabla_y v_r(y).$$

Thus, we calculate that for the change of variables $x = 2sy + Q$,

$$\left|\int_{B_s(p)} \langle \nabla v_\epsilon, \vec{v}\rangle \Delta v_\epsilon \, dV(x)\right| \le \|\ln(h)\|_\alpha (2s)^{\alpha+1} |\vec{v}| \frac{\omega^-(B_{2s}(Q))^2}{(2s)^{n-1}} \int_{B_1(0)} |\nabla v_{Q,2s} \star \phi_{\epsilon/(2s)}| \star \phi_{\epsilon/(2s)} \, d\omega^-_{Q,2s}$$
$$\le \|\ln(h)\|_\alpha C s^{\alpha+1} |\vec{v}| \frac{\omega^-(B_{2s}(Q))^2}{(s)^{n-1}} \omega^-_{Q,2s}(B_2(0))$$
$$\le \|\ln(h)\|_\alpha C s^\alpha \frac{\omega^-(B_s(Q))^2}{s^{n-2}} |\vec{v}|,$$

where the last two inequalities are because the $v_{Q,r}$ are uniformly locally Lipschitz, $1 + \epsilon/r < 2$, the $\omega^-_{Q,r}(B_2(0))$ are uniformly bounded for $Q \in B_1(0)$ and $r < 2$, and the doubling of harmonic measure on NTA domains. □

Lemma 7.4. *Let $v \in \mathcal{A}(n, \alpha, M_0)$, $Q \in \partial\Omega^\pm \cap B_1(0)$, and $0 < r$. Then for $p \in B_{r/3}(Q)$ and $\vec{v} \in \mathbb{R}^n$ such that $|\vec{v}| \le r$, we calculate the spatial directional derivatives as follows*:

$$\frac{\partial}{\partial \vec{v}} H(p, r, v_\epsilon) = 2 \int_{\partial B_r(p)} v_\epsilon \nabla v_\epsilon \cdot \vec{v} \, d\sigma, \tag{7-2}$$

$$\frac{\partial}{\partial \vec{v}} D(p, r, v_\epsilon) = 2 \int_{\partial B_r(p)} (\nabla v_\epsilon \cdot \vec{v})(\nabla v_\epsilon \cdot \eta) \, d\sigma - \int_{B_r(p)} \frac{\partial}{\partial \vec{v}} v_\epsilon \Delta v_\epsilon \, dV, \tag{7-3}$$

$$\frac{\partial}{\partial \vec{v}} N(p, r, v_\epsilon) = \frac{2}{H(p, r, v_\epsilon)} \left(\int_{\partial B_r(p)} (r \nabla v_\epsilon \cdot \eta - N(p, r, v_\epsilon) v_\epsilon)(\nabla v_\epsilon \cdot \vec{v}) \, d\sigma \right) - \frac{r \int_{B_r(p)} \frac{\partial}{\partial \vec{v}} v_\epsilon \Delta v_\epsilon \, dV}{H(p, r, v_\epsilon)}. \tag{7-4}$$

Proof. Equation (7-4) follows immediately from the preceding equations. The spatial derivative for $H(Q, r, u)$ follows from differentiating inside the integral. To obtain the spatial derivative for $D(Q, r, v)$, we recall the divergence theorem:

$$\frac{\partial}{\partial \vec{v}} D(p, r, v) = \frac{\partial}{\partial \vec{v}} \left(\int_{\partial B_r(p)} v \nabla v \cdot \eta \, d\sigma(x) - \int_{B_r(p)} v_\epsilon \Delta v_\epsilon \, dV \right)$$
$$= \int_{\partial B_r(p)} (\nabla v_\epsilon \cdot \vec{v})(\nabla v_\epsilon \cdot \eta) \, d\sigma + \int_{\partial B_r(p)} v_\epsilon \frac{\partial}{\partial \vec{v}} (\nabla v_\epsilon \cdot \eta) \, d\sigma.$$

Now, we focus upon the last term. Recalling Green's theorem and the fact that partial derivatives of harmonic functions are themselves harmonic,

$$\begin{aligned}\int_{\partial B_r(p)} v_\epsilon \frac{\partial}{\partial \vec{v}}(\nabla v_\epsilon \cdot \eta)\, d\sigma &= \int_{\partial B_r(p)} v_\epsilon \nabla\left(\frac{\partial}{\partial \vec{v}} v_\epsilon\right)\cdot \eta\, d\sigma = \int_{\partial B_r(p)} \nabla v_\epsilon \cdot \eta \frac{\partial}{\partial \vec{v}} v_\epsilon\, d\sigma - \int_{B_r(p)} \frac{\partial}{\partial \vec{v}} v_\epsilon \Delta v_\epsilon\, dV \\ &= \int_{\partial B_r(p)} (\nabla v_\epsilon \cdot \vec{v})(\nabla v_\epsilon \cdot \eta)\, d\sigma - \int_{B_r(p)} \frac{\partial}{\partial \vec{v}} v_\epsilon \Delta v_\epsilon\, dV. \qquad \square\end{aligned}$$

Definition 7.5. For the sake of concision, we define the following notation for $v \in \mathcal{A}(n, \alpha, M_0)$, $y \in \overline{\Omega}$, and radii $0 < r, R \leq 2$.

$$E_y(z) := \nabla v_\epsilon(z)\cdot(z-y) - N(y, |z-y|, v_\epsilon)v_\epsilon(z), \tag{7-5}$$

$$W_{r,R}(y) := N(y, R, v_\epsilon) - N(y, r, v_\epsilon). \tag{7-6}$$

Lemma 7.6. *Let $v \in \mathcal{A}(n, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \leq \Gamma$. Let $Q \in \partial\Omega^\pm \cap B_1(0)$ and $0 < r \leq 1$. Let $p \in [Q, Q']$ with $Q' \in \partial\Omega^\pm \cap B_{r/3}(Q)$. Then, for $\vec{v} = Q' - Q$ and $0 < \epsilon \ll r$,*

$$\begin{aligned}\left|\frac{\partial}{\partial \vec{v}} N(p, r, v_\epsilon)\right| \lesssim_{n,\alpha,M_0,\Gamma} &\ 2(W_{r/2,2r}(Q) + W_{r/2,2r}(Q') + C\Gamma r^\alpha)\left(r\left(\frac{\int_{\partial B_r(p)} |\nabla v_\epsilon|^2\, d\sigma}{H(p, r, v_\epsilon)}\right)^{\frac12} + 1\right) \\ &+ \frac{2}{H(p, r, v_\epsilon)}\left(\int_{\partial B_r(p)} |E_Q(z)|^2 + |E_{Q'}(z)|^2\, d\sigma\right)^{\frac12}\left(\int_{\partial B_r(p)} (\nabla v_\epsilon(z)\cdot(z-p))^2\, d\sigma\right)^{\frac12} \\ &+ C(n, \Lambda)\left(\frac{1}{H(p, r, v_\epsilon)}\int_{\partial B_r(p)} |E_Q(z)|^2 + |E_{Q'}(z)|^2\, d\sigma\right)^{\frac12} \\ &+ 2(N(Q, r, v_\epsilon) - N(Q', r, v_\epsilon))\left(\frac{\int_{B_r(p)} v_\epsilon \Delta v_\epsilon\, dV}{H(p, r, v_\epsilon)}\right) + r^\alpha\Gamma.\end{aligned}$$

Proof. We begin by noting that Lemmas 7.3 and 7.1 give

$$\left|\frac{r\int_{B_r(p)} \frac{\partial}{\partial \vec{v}} v_\epsilon \Delta v_\epsilon\, dV}{H(p, r, v_\epsilon)}\right| \leq C(n, \alpha, M_0, \Gamma) r^\alpha \Gamma.$$

Now, we write the decomposition

$$\begin{aligned}\nabla v_\epsilon\cdot(Q-Q') &= \nabla v_\epsilon\cdot(z-Q') - \nabla v_\epsilon\cdot(z-Q) \\ &= (N(Q, |z-Q|, v_\epsilon) - N(Q', |z-Q'|, v_\epsilon))v_\epsilon + (E_Q(z) - E_{Q'}(z)).\end{aligned}$$

Therefore, plugging this into (7-4), we obtain for $v = Q' - Q$,

$$\begin{aligned}&\frac{2}{H(p, r, v_\epsilon)}\left(\int_{\partial B_r(p)} \Big(r\nabla v_\epsilon\cdot\eta - N(p, r, v_\epsilon)v_\epsilon\Big)(\nabla v_\epsilon\cdot\vec{v})\, d\sigma\right) \\ &= \frac{2}{H(p, r, v_\epsilon)}\left(\int_{\partial B_r(p)} E_p(z)([N(Q, |z-Q|, v_\epsilon) - N(Q', |z-Q'|, v_\epsilon)]v_\epsilon + (E_Q(z) - E_{Q'}(z)))\, d\sigma\right) \\ &= A + B - C,\end{aligned}$$

where for the purposes of this lemma

$$A := \frac{2}{H(p,r,v_\epsilon)} \int_{\partial B_r(p)} E_p(z)(N(Q,|z-Q|,v_\epsilon) - N(Q',|z-Q'|,v_\epsilon))v_\epsilon \, d\sigma,$$
$$B := \frac{2}{H(p,r,v_\epsilon)} \int_{\partial B_r(p)} \nabla v_\epsilon(z)\cdot(z-p)(E_Q(z) - E_{Q'}(z))\, d\sigma,$$
$$C := \frac{2}{H(p,r,v_\epsilon)} \int_{\partial B_r(p)} N(p,|z-p|,v_\epsilon)v_\epsilon(z)(E_Q(z) - E_{Q'}(z))\, d\sigma.$$

We begin by estimating A. We rewrite

$$N(Q,|z-Q|,v_\epsilon) - N(Q',|z-Q'|,v_\epsilon) = N(Q,r,v_\epsilon) - N(Q',r,v_\epsilon) - W_{|z-Q|,r}(Q) + W_{|z-Q'|,r}(Q').$$

Note that if $|Q-Q'| \le \frac{1}{3}r$ and $p \in [Q,Q']$, then $\frac{1}{2}r \le |z-x_i| \le 2r$ for $x_i \in \{Q,Q'\}$ and all $z \in \partial B_r(p)$. Therefore, by Lemma 4.10, for all $z \in \partial B_r(p)$,

$$|W_{|z-Q|,r}(Q)| \le W_{r/2,2r}(Q) + 2C_1\Gamma(2r)^\alpha,$$
$$|W_{|z-Q'|,r}(Q')| \le W_{r/2,2r}(Q') + 2C_1\Gamma(2r)^\alpha.$$

Furthermore, we estimate by the divergence theorem

$$\begin{aligned}\int_{\partial B_r(p)} & E_p(z)(N(Q,r,v_\epsilon) - N(Q',r,v_\epsilon))v_\epsilon \, d\sigma \\ &= (N(Q,r,v_\epsilon) - N(Q',r,v_\epsilon)) \int_{\partial B_r(p)} v_\epsilon \nabla v_\epsilon(z)\cdot(z-y) - N(p,|z-p|,v_\epsilon)v_\epsilon^2 \, d\sigma \\ &= (N(Q,r,v_\epsilon) - N(Q',r,v_\epsilon)) \cdot \left(\int_{B_r(p)} v_\epsilon \Delta v_\epsilon \, dV\right).\end{aligned}$$

Thus, we may give the following preliminary estimate on A:

$$\begin{aligned}|A| \le (W_{r/2,2r}(Q) + W_{r/2,2r}(Q') + C\Gamma r^\alpha) \frac{2}{H(p,r,v_\epsilon)} \int_{\partial B_r(p)} |E_p(z)||v_\epsilon|\, d\sigma \\ + 2(N(Q,r,v_\epsilon) - N(Q',r,v_\epsilon))\left(\frac{\int_{B_r(p)} v_\epsilon \Delta v_\epsilon \, dV}{H(p,r,v_\epsilon)}\right).\end{aligned}$$

Focusing upon the term

$$\frac{2}{H(p,r,v_\epsilon)} \int_{\partial B_r(p)} |E_p(z)||v_\epsilon|\, d\sigma,$$

using Remark 7.2 and Cauchy–Schwartz we estimate

$$\begin{aligned}\frac{2}{H(p,r,v_\epsilon)} \int_{\partial B_r(p)} |E_p(z)||v_\epsilon|\, d\sigma &\le \frac{2}{H(p,r,v_\epsilon)} \int_{\partial B_r(p)} r|v_\epsilon \nabla v_\epsilon \cdot \eta| + N(p,r,v_\epsilon)|v_\epsilon|^2 \, d\sigma \\ &\le 2C\left(r\left(\frac{\int_{\partial B_r(p)} |\nabla v_\epsilon|^2 \, d\sigma}{H(p,r,v_\epsilon)}\right)^{\frac{1}{2}} + 1\right).\end{aligned}$$

Now we estimate B using Cauchy–Schwartz:

$$\left(\frac{2}{H(p,r,v_\epsilon)}\int_{\partial B_r(p)}\nabla v_\epsilon(z)\cdot(z-p)(E_Q(z)-E_{Q'}(z))\,d\sigma\right)^2$$
$$\le\frac{4}{H(p,r,v_\epsilon)^2}\int_{\partial B_r(p)}(E_Q(z)-E_{Q'}(z))^2\,d\sigma\int_{\partial B_r(p)}(\nabla v_\epsilon(z)\cdot(z-p))^2\,d\sigma$$
$$\le\frac{8}{H(p,r,v_\epsilon)^2}\int_{\partial B_r(p)}|E_Q(z)|^2+|E_{Q'}(z)|^2\,d\sigma\left(\int_{\partial B_r(p)}(\nabla v_\epsilon(z)\cdot(z-p))^2\,d\sigma\right).$$

For $|C|$, the same Cauchy–Schwartz argument plus Corollary 4.4 shows that

$$|C|\lesssim_{n,\alpha,M_0,\Gamma}\left(\frac{1}{H(p,r,v_\epsilon)}\int_{\partial B_r(p)}|E_Q(z)|^2+|E_{Q'}(z)|^2\,d\sigma\right)^{\frac12}.$$

This proves the pointwise estimate

$$\frac{\partial}{\partial v}N(p,r,v_\epsilon)\lesssim_{n,\alpha,M_0,\Gamma}(W_{r/2,2r}(Q)+W_{r/2,2r}(Q')+C\Gamma r^\alpha)2\left(r\left(\frac{\int_{\partial B_r(p)}|\nabla v_\epsilon|^2\,d\sigma}{H(p,r,v_\epsilon)}\right)^{\frac12}+1\right)$$
$$+\frac{2}{H(p,r,v_\epsilon)}\left(\int_{\partial B_r(p)}|E_Q(z)|^2+|E_{Q'}(z)|^2\,d\sigma\right)^{\frac12}\left(\int_{\partial B_r(p)}(\nabla v_\epsilon(z)\cdot(z-p))^2\,d\sigma\right)^{\frac12}$$
$$+\left(\frac{1}{H(p,r,v_\epsilon)}\int_{\partial B_r(p)}|E_Q(z)|^2+|E_{Q'}(z)|^2\,d\sigma\right)^{\frac12}$$
$$+2(N(Q,r,v_\epsilon)-N(Q',r,v_\epsilon))\left(\frac{\int_{B_r(p)}v_\epsilon\Delta v_\epsilon\,dV}{H(p,r,v_\epsilon)}\right)+r^\alpha\Gamma.$$

We prove the lemma by reversing the roles of Q and Q'. □

Corollary 7.7. *Let $v\in\mathcal{A}(n,\alpha,M_0)$ with $\|\ln(h)\|_\alpha\le\Gamma$, let $Q\in\partial\Omega^\pm\cap B_1(0)$ and $0<r\le 1$, and let $Q'\in\partial\Omega^\pm\cap B_{r/3}(Q)$. Then*

$$|N(Q',r,v)-N(Q,r,v)|\lesssim_{n,\alpha,M_0,\Gamma}W_{r/2,2r}(Q)+W_{r/2,2r}(Q')+C\Gamma r^\alpha+1+r^{-1/2}.$$

Proof. First, since for any $0\neq c$ we have $N(Q,r,v)=N(Q,r,cv)$, we shall assume for the purposes of this lemma that $v=T_{0,1}v$. We shall show that $|N(Q',r,v_\epsilon)-N(Q,r,v_\epsilon)|$ satisfies a corresponding inequality, and let $\epsilon\to 0$. Since $N(Q',r,v_\epsilon)\to N(Q',r,v)$ as $\epsilon\to 0$, this will prove the claim.

Let $\vec{v}=Q'-Q$ and $p_t:=Q+t\vec{v}$. Then we calculate

$$|N(Q',r,v_\epsilon)-N(Q,r,v_\epsilon)|\le\int_0^1\left|\frac{\partial}{\partial t}N(p_t,r,v_\epsilon)\right|dt\lesssim_{n,\alpha,M_0,\Gamma}A+B+C+D+E,$$

where

$$A:=\int_0^1(W_{r/2,2r}(Q)+W_{r/2,2r}(Q')+C\Gamma r^\alpha)2\left(r\left(\frac{\int_{\partial B_r(p_t)}|\nabla v_\epsilon|^2\,d\sigma}{H(p_t,r,v_\epsilon)}\right)^{\frac12}+1\right)dt,$$
$$B:=\int_0^1\frac{2}{H(p_t,r,v_\epsilon)}\left(\int_{\partial B_r(p_t)}|E_Q(z)|^2+|E_{Q'}(z)|^2\,d\sigma\right)^{\frac12}\left(\int_{\partial B_r(p_t)}(\nabla v_\epsilon(z)\cdot(z-p_t))^2\,d\sigma\right)^{\frac12}dt,$$

$$C := \int_0^1 \left(\frac{1}{H(p_t, r, v_\epsilon)} \int_{\partial B_r(p_t)} |E_Q(z)|^2 + |E_{Q'}(z)|^2 \, d\sigma \right)^{\frac{1}{2}} dt,$$

$$D := 2(N(Q, r, v_\epsilon) - N(Q', r, v_\epsilon)) \int_0^1 \left(\frac{\int_{B_r(p_t)} v_\epsilon \Delta v_\epsilon \, dV}{H(p_t, r, v_\epsilon)} \right) dt,$$

$$E := C\Gamma r^\alpha.$$

We estimate each term separately.

Bounding A. We begin by rewriting A:

$$A = 2(W_{r/2,2r}(Q) + W_{r/2,2r}(Q') + C\Gamma r^\alpha) \int_0^1 \left(r \left(\frac{\int_{\partial B_r(p_t)} |\nabla v_\epsilon|^2 \, d\sigma}{H(p_t, r, v_\epsilon)} \right)^{\frac{1}{2}} + 1 \right) dt.$$

Observe that by Lemma 7.1 and Remark 4.12 we may using Hölder's inequality to estimate

$$\int_0^1 r \left(\frac{\int_{\partial B_r(p_t)} |\nabla v_\epsilon|^2 \, d\sigma}{H(p_t, r, v_\epsilon)} \right)^{\frac{1}{2}} dt \lesssim_{n,\alpha,M_0,\Gamma} \int_0^1 r \left(\frac{\int_{\partial B_r(p_t)} |\nabla v_\epsilon|^2 \, d\sigma}{H(Q, r, v_\epsilon)} \right)^{\frac{1}{2}} dt$$
$$\lesssim_{n,\alpha,M_0,\Gamma} r \frac{2}{H(Q, r, v_\epsilon)^{1/2}} \left(\int_0^1 \int_{\partial B_r(p_t)} |\nabla v_\epsilon|^2 \, d\sigma \, dt \right)^{\frac{1}{2}}.$$

Now, divide the spheres as follows: $\partial B_r(p_t) = \partial B_r(p_t)^+ \cup \partial B_r(p_t^-)$, where

$$\partial B_r(p_t)^- = \{x \in \partial B_r(p_t) : (x - p_t) \cdot \vec{v} < 0\} \quad \text{and} \quad \partial B_r(p_t)^+ = \{x \in \partial B_r(p_t) : (x - p_t) \cdot \vec{v} \geq 0\}.$$

Notice that

$$\max_{z \in \bigcup_{t \in [0,1]} \partial B_r(p_t)} \#\{t \in [0, 1] : z \in \partial B_r(p_t)^+ \text{ or } z \in \partial B_r(p_t^-)\} = 2.$$

Then, use the coarea formula for the function $\phi^\pm : \bigcup_{t \in [0,1]} \partial B_r(p_t)^\pm \to \mathbb{R}$ defined by $\phi|_{\partial B_r(p_t)^\pm} = t$. Note that if we write $L := Q + \operatorname{span}\{\vec{v}\}$ and $\operatorname{dist}(z, L) = \delta$, then

$$J\phi^\pm(z) = |\nabla \phi(z)| = \frac{r}{|Q - Q'|\sqrt{r^2 - \delta^2}} = \frac{1}{|Q - Q'| \cos(\theta(z))},$$

where

$$\theta(z) = \frac{z - p_t}{|z - p_t|} \cdot \frac{\vec{v}}{|\vec{v}|} \quad \text{for } z \in \partial B_r(p_t)^\pm.$$

Thus, we obtain

$$\int_0^1 r \left(\frac{\int_{\partial B_r(p_t)} |\nabla v_\epsilon|^2 \, d\sigma}{H(p_t, r, v_\epsilon)} \right)^{\frac{1}{2}} dt \lesssim_{n,\alpha,M_0,\Gamma} \frac{2r}{H(Q, r, v_\epsilon)^{1/2}} \left(\int_{\bigcup_{t \in [0,1]} \partial B_r(p_t)} \frac{2|\nabla v_\epsilon|^2}{|Q - Q'||\cos(\theta(z))|} \, dV \right)^{\frac{1}{2}}.$$

Note that a simple calculation gives, for any $1 \leq p < 2$,

$$\int_{\bigcup_{t \in [0,1]} \partial B_r(p_t)} |Q - Q'|^{-1} |\cos(\theta(z))|^{-p} \, dV = \int_0^r \int_{\mathscr{C}_\delta} |Q - Q'|^{-1} |\cos(\theta(z))|^{-p} \, d\mathcal{H}^{n-1} \, d\delta$$
$$\leq c(n) \int_0^r \frac{r^p \delta^{n-2}}{(r^2 - \delta^2)^{p/2}} \, d\delta \leq \frac{c(n)}{1 - \frac{1}{2}p} r^{n-1}. \tag{7-7}$$

Since $T_{Q,r}v_\epsilon$ is uniformly locally Lipschitz by Lemma 3.4, recalling Definition 2.7 and choosing $p=1$ above, we see

$$\begin{aligned} r\frac{2}{H(Q,r,v_\epsilon)^{1/2}}&\left(2\int_{\bigcup_{t\in[0,1]}\partial B_r(p_t)}\frac{|\nabla v_\epsilon|^2}{|Q-Q'||\cos(\theta(z))|}\,dV\right)^{\frac12}\\ &=2\sqrt{2}\left(r^{1-n}\int_{\bigcup_{t\in[0,1]}\partial B_1(T_{Q,r}p_t)}\frac{|\nabla T_{Q,r}v_\epsilon|^2}{|Q-Q'||\cos(\theta(z))|}\,dV\right)^{\frac12}\\ &=2\sqrt{2}C\left(r^{1-n}\int_{\bigcup_{t\in[0,1]}\partial B_r(p_t)}|Q-Q'|^{-1}|\cos(\theta(z))|^{-1}\,dV\right)^{\frac12}\\ &\le C(n,\alpha,M_0,\Gamma). \end{aligned}$$

Thus

$$|A|\lesssim_{n,\alpha,M_0,\Gamma}(W_{r/2,2r}(Q)+W_{r/2,2r}(Q')+C\Gamma r^\alpha).$$

<u>*Bounding C*</u>. By Hölder's inequality (or Jensen's inequality for concave functions) and Lemma 7.1, we may reduce to considering

$$\begin{aligned} \int_0^1\frac{1}{H(p_t,r,v_\epsilon)}&\int_{\partial B_r(p_t)}|\nabla v_\epsilon(z)\cdot(z-Q)-N(Q,|z-Q|,v_\epsilon)v_\epsilon(z)|^2\,d\sigma(z)\,dt\\ &\lesssim_{n,\alpha,M_0,\Gamma}\int_0^1\frac{1}{H\big(Q,\frac12 r,v_\epsilon\big)}\int_{\partial B_r(p_t)}|\nabla v_\epsilon(z)\cdot(z-Q)-N(Q,|z-Q|,v_\epsilon)v_\epsilon(z)|^2\,d\sigma(z)\,dt. \end{aligned}$$

Now, we change variables using Definition 2.7 and Lemma 7.1, and use Young's inequality to get

$$\begin{aligned} \int_0^1\frac{1}{H\big(Q,\frac12 r,v_\epsilon\big)}&\int_{\partial B_r(p_t)}|\nabla v_\epsilon(z)\cdot(z-Q)-N(Q,|z-Q|,v_\epsilon)v_\epsilon(z)|^2\,d\sigma(z)\,dt\\ &\le 2\int_{\bigcup_{t\in[0,1]}\partial B_r(p_t)}\frac{|\nabla v_\epsilon(z)\cdot(z-Q)-N(Q,|z-Q|,v_\epsilon)v_\epsilon(z)|^2}{H\big(Q,\frac12 r,v_\epsilon\big)|Q-Q'||\cos(\theta(z))|}\,dV\\ &\lesssim_{n,\alpha,M_0,\Gamma}2r^n\int_{\bigcup_{t\in[0,1]}\partial B_1(T_{Q,r}p_t)}\frac{|\nabla T_{Q,r}v_\epsilon(z)\cdot z-N(0,|z|,T_{Q,r}v_\epsilon)T_{Q,r}v_\epsilon(z)|^2}{r^{n-1}\big|\frac{Q-Q'}{r}\big||\cos(\theta(z))|}\,dV\\ &\le 4\int_{\bigcup_{t\in[0,1]}\partial B_1(T_{Q,r}p_t)}\frac{|\nabla T_{Q,r}v_\epsilon(z)\cdot z|^2+|N(0,|z|,T_{Q,r}v_\epsilon)T_{Q,r}v_\epsilon(z)|^2}{\big|\frac{Q-Q'}{r}\big||\cos(\theta(z))|}\,dV. \end{aligned}$$

Now, by Corollary 4.4 and Lemma 3.4, the numerator is bounded by a constant. Whence, by a calculation similar to (7-7), we obtain

$$\int_0^1\frac{1}{H\big(Q,\frac12 r,v_\epsilon\big)}\int_{\partial B_r(p_t)}|\nabla v_\epsilon(z)\cdot(z-Q)-N(Q,|z-Q|,v_\epsilon)v_\epsilon(z)|^2\,d\sigma(z)\,dt\lesssim_{n,\alpha,M_0,\Gamma}1.$$

An identical argument holds for Q' in the place of Q. Thus, we have that $|C|\lesssim_{n,\alpha,M_0,\Gamma}1$.

Bounding B. Using Cauchy–Schwartz, Lemma 7.1, and the estimates of the term $|C|$ above, we obtain

$$B = \int_0^1 \frac{2}{H(p_t, r, v_\epsilon)} \left(\int_{\partial B_r(p_t)} |E_Q(z)|^2 + |E_{Q'}(z)|^2 \, d\sigma \right)^{\frac{1}{2}} \left(\int_{\partial B_r(p_t)} (\nabla v_\epsilon(z) \cdot (z - p_t))^2 \, d\sigma \right)^{\frac{1}{2}} dt$$
$$\lesssim_{n,\alpha,M_0,\Gamma} \left(\frac{r^2}{H(Q, r, v_\epsilon)} \int_{\bigcup_{t\in[0,1]} \partial B_1((p_t - Q_i)/r_i)} \frac{|\nabla v_\epsilon(z)|^2}{|Q - Q'| \, |\cos(\theta_{\vec{v}})|} \, dV \right)^{\frac{1}{2}}$$
$$\lesssim_{n,\alpha,M_0,\Gamma} r^{-1/2} \left(\int_{\bigcup_{t\in[0,1]} \partial B_1((p_t - Q_i)/r_i)} \frac{|\nabla T_{Q,r} v_\epsilon(z)|^2}{\left|\frac{Q-Q'}{r}\right| \, |\cos(\theta_{\vec{v}})|} \, dV \right)^{\frac{1}{2}}.$$

Thus, by Lemma 3.4 and a calculation identical to that of (7-7), we obtain $|B| \lesssim_{n,\alpha,M_0,\Gamma} r^{-1/2}$.

Bounding D. Note that

$$\left| \int_{B_r(p_t)} v_\epsilon \Delta v_\epsilon \, dV(x) \right| \le \int_{B_r(p)} |(v_\epsilon)_\epsilon| \left| \frac{h(0)}{h(x)} - 1 \right| d\omega^- \le \int_{B_{2r}(Q)} |(v_\epsilon)_\epsilon| \left| \frac{h(0)}{h(x)} - 1 \right| d\omega^-$$
$$\le \|\ln(h)\|_\alpha (2r)^\alpha \int_{B_{2r}(Q)} |(v_\epsilon)_\epsilon| \, d\omega^- \le \|\ln(h)\|_\alpha (2r)^\alpha C \int_{B_{2r}(Q)} |(v_\epsilon)_\epsilon| \, d\omega^-.$$

Chasing through the change of variables $x = ry + Q$, we see that

$$\nabla_x v(x) = \frac{1}{r} \nabla_y v(ry + Q) = \frac{\omega^-(B_r(Q))}{r^{n-1}} \nabla_y v_r(y).$$

Thus, we calculate that, for the change of variables $x = 2ry + Q$,

$$\left| \int_{B_r(p)} v_\epsilon \Delta v_\epsilon \, dV(x) \right| \le \|\ln(h)\|_\alpha (2r)^\alpha \frac{\omega^-(B_{2r}(0))^2}{(2r)^{n-1}} \int_{B_1(0)} |v_{Q,2r} \star \phi_{\epsilon/(2r)}| \star \phi_{\epsilon/(2r)} \, d\omega^-_{2r}$$
$$\le \|\ln(h)\|_\alpha C r^\alpha \frac{\omega^-(B_{2r}(0))^2}{r^{n-1}} C\left(\frac{\epsilon}{r}\right) \omega^-_{Q,2r}(B_2(0))$$
$$\le \|\ln(h)\|_\alpha C r^\alpha C\left(\frac{\epsilon}{r}\right) \frac{\omega^-(B_r(0))^2}{r^{n-2}},$$

where the last two inequalities are because the $v_{Q,r}$ are uniformly locally Lipschitz, $1 + \epsilon/r < 2$, the $\omega^-_{Q,r}(B_2(0))$ are uniformly bounded for $Q \in B_1(0)$ and $r < 2$, and the doubling of harmonic measure on NTA domains.

Thus, by Lemma 7.1 we have

$$2(N(Q, r, v_\epsilon) - N(Q', r, v_\epsilon)) \int_0^1 \left(\frac{\int_{B_r(p_t)} v_\epsilon \Delta v_\epsilon \, dV}{H(p_t, r, v_\epsilon)} \right) dt \le C(n, \alpha, M_0, \Gamma) \Gamma r^{\alpha-1} \left(\frac{\epsilon}{r}\right).$$

Letting $\epsilon \to 0$, we see that D vanishes.

Thus, putting together the estimates for A, B, C, D we have

$$|N(Q', r, v) - N(Q, r, v)| \le \lim_{\epsilon\to 0} |N(Q', r, v_\epsilon) - N(Q, r, v_\epsilon)|$$
$$\lesssim_{n,\alpha,M_0,\Gamma} W_{r/2,2r}(Q) + W_{r/2,2r}(Q') + \Gamma r^\alpha + 1 + r^{-1/2}.$$

This proves the lemma. □

8. Frequency pinching

In this section, we prove a "frequency pinching" result (Lemma 8.2) in the style of [De Lellis et al. 2018]. This kind of result relates Jones' beta numbers to the drop in Almgren frequency.

Definition 8.1 (Jones' beta numbers). For μ a Borel measure, we define $\beta^k_{\mu,2}(Q,r)^2$ as follows:

$$\beta^k_{\mu,2}(Q,r)^2 = \inf_{L^k} \frac{1}{r^k} \int_{B_r(p)} \frac{\operatorname{dist}(x,L)^2}{r^2}\, d\mu(x),$$

where the infimum is taken over all affine k-planes.

Taking the *infimum* here — as opposed to the *minimum* — is a convention. The space of admissible planes is compact, so a minimizing plane exists. Let $V^k_\mu(Q,r)$ denote a k-plane which minimizes the *infimum* in the definition of $\beta^k_\mu(Q,r)^2$. Note that this k-plane is not a priori unique.

Lemma 8.2 (frequency pinching). *There exists a constant $\delta_0 = \delta_0(n,\alpha,M_0,\Gamma) > 0$ such that, for any $0 < \delta \le \delta_0$, if $v \in \mathcal{A}(n,\alpha,M_0)$ with $\|\ln(h)\|_\alpha \le \Gamma$ then, for any $Q \in \partial\Omega \cap B_1(0)$ and $0 < r \le \frac{1}{16}$, if v is $(0,\delta,8r,Q)$-symmetric but not $(k+1,\epsilon,8r,Q)$-symmetric, then, for any finite Borel measure μ supported in $B_r(Q)\cap\partial\Omega$,*

$$\beta^k_{\mu,2}(Q,r)^2 \lesssim_{n,\alpha,M_0,\Gamma,\epsilon} \frac{r^2}{r^k}\left(\int_{B_r(Q)} W_{r/2,16r}(y)\,d\mu(y)\right) + \frac{r^2}{r^k}\int_{B_r(Q)} W_{r/2,16r}(y)^2\,d\mu(y)$$
$$+ r^2(W_{r/2,16r}(Q)^2 + \Gamma^2 r^{2\alpha} + 1 + r^{-1})\frac{\mu(B_r(Q))}{r^k}. \quad (8\text{-}1)$$

Before proving Lemma 8.2, we prove a few preliminary lemmas. We begin by noting that for any finite Borel measure μ and any $B_r(Q)$ we can define the μ center of mass by $X = ⨍_{B_r(Q)} x\,d\mu(x)$ and define the covariance matrix of the mass distribution in $B_r(Q)$ by

$$\Sigma = ⨍_{B_r(Q)} (y-X)(y-X)^\perp\,d\mu(y).$$

With this matrix, we may naturally define a symmetric, nonnegative bilinear form

$$Q(v,w) = v^\perp \Sigma w = ⨍_{B_r(Q)} (v\cdot(y-X))(w\cdot(y-X))\,d\mu(y).$$

Let $\vec{v}_1,\dots,\vec{v}_n$ be an orthonormal eigenbasis and $\lambda_1 \ge \dots \ge \lambda_n \ge 0$ their associated eigenvalues. These objects enjoy the relationships

$$V^k_{\mu,2}(Q,r) = X + \operatorname{span}\{\vec{v}_1,\dots,\vec{v}_k\} \quad \text{and} \quad \beta^k_{\mu,2}(x,r)^2 = \frac{\mu(B_r(Q))}{r^k}(\lambda_{k+1}+\dots+\lambda_n).$$

See [Hochman 2015, Section 4.2] or [Naber and Valtorta 2017, Section 7.2].

Lemma 8.3. *Let $v \in \mathcal{A}(n,\alpha,M_0)$, and let $Q \in \partial\Omega\cap B_1(0)$ and $0 < r \le \frac{1}{4}$. Let μ, Q, λ_i, $\vec{v}_i$ be defined as above. For any i and any scalar $c \in \mathbb{R}$,*

$$\lambda_i \int_{A_{3r,4r}(Q)} (\vec{v}_i\cdot\nabla v(z))^2\,dz \le ⨍_{B_r(Q)}\left(\int_{A_{3r,4r}(y)} |cv(z) - \nabla v(z)\cdot(z-y)|^2\,dz\right) d\mu(y). \quad (8\text{-}2)$$

Proof. Observe that by the definition of center of mass,

$$\fint_{B_r(Q)} \vec{w} \cdot (y - X)\, d\mu(y) = 0$$

for any $\vec{w} \in \mathbb{R}^n$. Therefore, for any z for which $\nabla v(z)$ is defined,

$$\begin{aligned}
\lambda_i(\vec{v}_i \cdot \nabla v(z)) &= Q(\vec{v}_i, \nabla v(z)) \\
&= \fint_{B_r(Q)} (\vec{v}_i \cdot (y - X))(\nabla v(z) \cdot (y - X))\, d\mu(y) \\
&= \fint_{B_r(Q)} (\vec{v}_i \cdot (y - X))(\nabla v(z) \cdot (y - X))\, d\mu(y) + \fint_{B_r(Q)} cv(z)(\vec{v}_i \cdot (y - X))\, d\mu(y) \\
&= \fint_{B_r(Q)} (\vec{v}_i \cdot (y - X))(cv(z) - \nabla v(z) \cdot (X - z + z - y))\, d\mu(y) \\
&= \fint_{B_r(Q)} (\vec{v}_i \cdot (y - X))(cv(z) - \nabla v(z) \cdot (z - y))\, d\mu(y) \\
&\le \lambda_i^{1/2} \left(\fint_{B_r(Q)} |cv(z) - \nabla v(z) \cdot (z - y)|^2\, d\mu(y) \right)^{\frac{1}{2}}.
\end{aligned}$$

Squaring both sides and integrating over $A_{r,R}(Q) = B_R(Q) \setminus B_r(Q)$ gives the result. □

Lemma 8.4. *Let $v \in \mathcal{A}(n, \alpha, M_0)$ with $\|\ln(h)\|_\alpha \le \Gamma$ and $0 \le k \le n - 2$. Let $Q \in \partial\Omega^\pm \cap B_{1/4}(0)$ and $0 < r \le \frac{1}{32}$. Then, for any $Q' \in B_r(Q) \cap \partial\Omega^\pm$,*

$$\begin{aligned}
&\int_{A_{3r,4r}(Q')} \frac{|N(Q, 7r, v)v(z) - \nabla v(z) \cdot (z - Q')|^2}{H(Q', |z - Q'|, v)}\, dz \\
&\qquad \lesssim_{n,\alpha,M_0,\Gamma} \int_{A_{2r,7r}(Q')} \frac{|N(Q', |z - Q'|, v)v(z) - \nabla v(z) \cdot (z - Q')|^2}{H(Q', |z - Q'|, v)}\, dz \\
&\qquad\qquad + (W_{r/2,16r}(Q)^2 + W_{r/2,16r}(Q')^2 + \Gamma^2 r^{2\alpha} + 1 + r^{-1})r.
\end{aligned}$$

Proof. First, we observe that

$$N(Q, 7r, v) = N(Q', |z - Q'|, v) + W_{|z-Q'|,7r}(Q) + [N(Q, |z - Q'|, v) - N(Q', |z - Q'|, v)].$$

Therefore, by the triangle inequality,

$$\begin{aligned}
&|N(Q, 7r, v)v(z) - \nabla v(z) \cdot (z - Q')|^2 \\
&\quad \le \big(|(W_{|z-Q'|,7r}(Q) + N(Q, |z - Q'|, v) - N(Q', |z - Q'|, v))v(z)| \\
&\qquad\qquad + |(N(Q', |z - Q'|, v))v(z) - \nabla v(z) \cdot (z - Q')|\big)^2 \\
&\quad \le 2|(W_{|z-Q'|,7r}(Q) + N(Q, |z - Q'|, v) - N(Q', |z - Q'|, v))v(z)|^2 \\
&\qquad\qquad + 2|(N(Q', |z - Q'|, v))v(z) - \nabla v(z) \cdot (z - Q')|^2.
\end{aligned}$$

Now, using Corollary 7.7 at scale $r = |z - Q'|$ and the almost monotonicity of the Almgren frequency, we estimate

$$\int_{A_{2r,7r}(Q')} \frac{|(W_{|z-Q'|,7r}(Q) + N(Q,|z-Q'|,v) - N(Q',|z-Q'|,v))v(z)|^2}{H(Q',|z-Q'|,v)}\,dz \lesssim_{n,\alpha,M_0,\Gamma} \mathscr{C}_r(Q,Q') \int_{A_{2r,7r}(Q')} \frac{|v(z)|^2}{H(Q',|z-Q'|,v)}\,dz,$$

where the term $\mathscr{C}_r(Q,Q')$ is defined by

$$\mathscr{C}_r(Q,Q') := W_{r/2,16r}(Q)^2 + W_{r/2,16r}(Q')^2 + \Gamma^2 r^{2\alpha} + 1 + r^{-1}.$$

We finish the proof by observing that

$$\int_{A_{2r,7r}(Q')} \frac{|v(z)|^2}{H(Q',|z-Q'|,v)}\,dz \le 7r. \qquad \square$$

Lemma 8.5. *Let $v \in \mathcal{A}(n,\alpha,M_0)$ with $\|\ln(h)\|_\alpha \le \Gamma$ and $0 \le k \le n-2$. Let $Q \in B_1(0) \cap \partial\Omega^\pm$, $0 < r \le \frac{1}{16}$. Let $0 < \epsilon$ be fixed. There exists a constant $\delta = \delta_0(n,\alpha,M_0,\Gamma,\epsilon) > 0$ and a constant $0 < C(n,\alpha,M_0,\Gamma,\epsilon)$ such that if v is $(0,\delta,8r,Q)$-symmetric but not $(k+1,\epsilon,8r,Q)$-symmetric, then, for any orthonormal vectors $\vec{v}_1,\dots,\vec{v}_{k+1}$,*

$$\frac{1}{C} \le \int_{A_{3r,4r}(Q)} \frac{r}{H(Q,r,v)} \sum_{i=1}^{k+1} (\vec{v}_i \cdot \nabla v(z))^2\,dz.$$

Proof. We argue by contradiction. Assume that there is a sequence of functions $v_i \in \mathcal{A}(n,\alpha,M_0)$, $Q_i \in B_{1/16}(0) \cap \partial\Omega_i^\pm$, and $0 < r_i \le \frac{1}{16}$ such that v_i is $(0,2^{-j},8r_i,Q_i)$-symmetric but not $(k+1,\epsilon,8r_i,Q_i)$-symmetric. And, for each i, there exists an orthonormal collection of vectors $\{\vec{v}_{ij}\}$ such that

$$\int_{A_{3,4}(0)} \sum_{j=1}^{k+1} (\vec{v}_{ij} \cdot \nabla T_{Q,r} v_i(z))^2\,dz \le 2^{-i}.$$

By Lemma 3.6, we may extract a subsequence $T_{Q_j,r_j} v_j$ for which $T_{Q_j,r_j} v_j$ converges to a nondegenerate function v_∞. Similarly, $\{\vec{v}_{ij}\}$ converges to an orthonormal collection $\{\vec{v}_i\}$. Given the assumptions above, v_∞ is also 0-symmetric in $B_8(0)$ and $\nabla v_\infty \cdot \vec{v}_i = 0$ for all $i = 1,\dots,k+1$. Thus, v_∞ is $(k+1,0)$-symmetric in $B_8(0)$. But, this is a contradiction, since the $T_{Q_j,r_j} v_j$ were supposed to stay away from $(k+1)$-symmetric functions in $L^2(B_1(0))$. $\square$

8A. *The proof of Lemma 8.2.* By Lemma 8.5 and properties of the Jones' beta numbers, we have, for $\{\vec{v}_i\}$ the orthonormal basis and λ_i the associated eigenvalues of the quadratic form in Lemma 8.3,

$$\beta^k_{\mu,2}(Q,r)^2 \le \frac{\mu(B_r(Q))}{r^k} n\lambda_{k+1} \le \frac{\mu(B_r(Q))}{r^k} nC\lambda_{k+1} \frac{r}{H(Q,r,v)} \sum_{i=1}^{k+1} \int_{A_{3r,4r}(Q)} (\vec{v}_i \cdot \nabla v(z))^2\,dz$$
$$\le \frac{\mu(B_r(Q))}{r^k} nC \frac{r}{H(Q,r,v)} \sum_{i=1}^{k+1} \lambda_i \int_{A_{3r,4r}(Q)} (\vec{v}_i \cdot \nabla v(z))^2\,dz.$$

By choosing $c = N(Q, 7r, v)$ in Lemma 8.3 and recalling Lemma 8.4, we have

$$\begin{aligned}
&\frac{r}{H(Q,r,v)}\lambda_i \int_{A_{3r,4r}(Q)} (\vec{v}_i \cdot \nabla v(z))^2\, dz \\
&\quad \lesssim_{n,\alpha,M_0,\Gamma} \frac{r}{H(Q,r,v)} \fint_{B_r(Q)} \left(\int_{A_{3r,4r}(y)} |N(Q,7r,v)v(z) - \nabla v(z)\cdot(z-y)|^2\, dz \right) d\mu(y) \\
&\quad \lesssim_{n,\alpha,M_0,\Gamma} r \fint_{B_r(Q)} \left(\int_{A_{2r,7r}(y)} \frac{|N(y,|z-y|,v)v(z) - \nabla v(z)\cdot(z-y)|^2}{H(Q,|z-y|,v)}\, dz \right) d\mu(y) \\
&\qquad\qquad + r^2 \fint_{B_r(Q)} (W_{r,16r}(Q)^2 + W_{r,16r}(y)^2 + \Gamma^2 r^{2\alpha} + 1 + r^{-1})\, d\mu(y).
\end{aligned}$$

Therefore, collecting constants and using Lemmas 4.9 and 7.1, we have

$$\begin{aligned}
\beta^k_{\mu,2}(Q,r)^2 &\lesssim_{n,\alpha,M_0,\Gamma} \frac{r^2}{r^k}\left(\int_{B_r(Q)} N(y,8r,v) - N(y,r,v)\, d\mu(y) \right) \\
&\quad + r^2 (W_{r/2,16r}(Q)^2 + \Gamma^2 r^{2\alpha} + 1 + r^{-1}) \frac{\mu(B_r(Q))}{r^k} + \frac{r^2}{r^k} \int_{B_r(Q)} W_{r/2,16r}(y)^2\, d\mu(y). \qquad \square
\end{aligned}$$

9. Packing

The following theorem of Naber and Valtorta [2017] is a powerful tool which links the sum of the $\beta^k_\mu(Q,r)^2$ over all points and scales to packing estimates.

Theorem 9.1 [Naber and Valtorta 2017, discrete Reifenberg]. *Let $\{B_{\tau_i}(x_i)\}_i$ be a collection of disjoint balls such that, for all $i = 1, 2, \dots$, we have $\tau_i \le 1$. Let $\epsilon_k > 0$ be fixed. Define a measure*

$$\mu := \sum_i \tau_i^k \delta_{x_i},$$

and suppose that, for any $x \in B_2(0)$ and any scale $l \in \{0, 1, 2, \dots\}$, if $B_{r_l}(x) \subset B_2(0)$ and $\mu(B_{r_l}(x)) \ge \epsilon_k r_l^k$ then

$$\sum_{i \ge l} \int_{B_{2r_l}(x)} \beta^k_\mu(z, 16r_i)^2\, d\mu(z) < r_l^k \delta^2.$$

Then there exists a $\delta_0 = \delta_0(n, \epsilon_k) > 0$ such that if $\delta \le \delta_0$,

$$\mu(B_1(0)) = \sum_{i \text{ s.t. } x_i \in B_1(0)} \tau_i^k \le C(n).$$

Now we are ready to prove the crucial packing lemma.

Lemma 9.2. *Fix $0 < \epsilon$, and let $v \in \mathcal{A}(n, \alpha, M_0)$ satisfy $\|\ln(h)\|_\alpha \le \eta$ and $\sup_{Q \in B_1(0)\cap\partial\Omega^\pm} N(Q, 2, v) = E$. There is an $\eta_1(n, \alpha, M_0, \epsilon) > 0$ such that if $\eta \le \eta_1$, then for any $r > 0$ if $\{B_{2r_{Q'}}(Q')\}$ is a collection of disjoint balls satisfying*

$$N(p, \eta r_{Q'}, v) \ge E - \eta_1, \quad Q' \in S^k_{\eta_1, r}, \quad r \le r_{Q'} \le 1, \tag{9-1}$$

we have the packing estimate

$$\sum_{Q'} r^k_{Q'} \le C_2(n, \alpha, M_0, \epsilon). \tag{9-2}$$

Proof. Choose $\delta_0(n, \alpha, M_0, \epsilon)$ as in Lemma 8.2, and $\gamma(n, \alpha, M_0, \delta_0)$ as in Lemma 5.2. Note that we may assume without loss of generality that $\eta_1 \leq 1$, and so for $C_1(\alpha, M_0, 1)$ the constant in Lemma 4.9, let

$$\eta_1 \leq \frac{\min\{\delta_0, \gamma\}}{2C_1 + 1}.$$

We will employ the convention that $r_i = 2^{-i}$. For each $i \in \mathbb{N}$, define the truncated measure

$$\mu_i = \sum_{r_{Q'} \leq r_i} r_{Q'}^k \delta_{Q'}.$$

We will write $\beta_i(x, r) = \beta^k_{\mu_i,2}(x, r)$. Observe that the β_i enjoy the following properties. First, because the balls are disjoint, for all $j \geq i$,

$$\beta_i(x, r_j) = \begin{cases} \beta_j(x, r_j) & \text{if } x \in \operatorname{supp}(\mu_j), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, for $r_i \leq 2^{-4}$, recalling Lemma 4.10 our assumption of the Almgren frequency gives that $N(16r_i, Q, v) - N(r_Q, Q, v) \leq (2C_1 + 1)\eta \leq \max\{\delta_0, \gamma\} \leq 1$ and

$$|W_{r_j/2,16r_j}(Q')| \leq \eta + C(\alpha, M_0, \eta)\eta(16r_j)^\alpha.$$

Thus, for $0 < \eta$ small enough depending only upon α and M_0, we have $|W_{r_j/2,16r_j}(Q')| \leq 1$. Therefore

$$W_{r_j/2,16r_j}(Q')^2 \leq |W_{r_j/2,16r_j}(Q')|.$$

In particular, by Lemmas 5.2 and 8.2 and our choice of $\eta \leq \eta_1$,

$$\beta^k_{\mu_i,2}(Q, r_i)^2 \lesssim_{n,\alpha,M_0,\Gamma,\epsilon} \frac{1}{r^k}\left(\int_{B_r(Q)} |W_{r/2,16r}(y)|\, d\mu(y)\right) + (|W_{r/2,16r}(Q)| + \Gamma^2 r^{2\alpha+2} + r^2 + r)\frac{\mu(B_r(Q))}{r^k}.$$

The claim of the lemma is that $\mu_0(B_1(0)) \leq C(n, \alpha, M_0, \epsilon)$. We prove the claim inductively. That is, we shall argue that there is a fixed scale $0 < R = 2^{-\ell}$ (depending only upon $n,\ \alpha,\ M_0,\ \epsilon$) such that, for $r_i \leq R$ and all $x \in B_1(0)$,

$$\mu_i(B_{r_i}(x)) \leq C_{DR}(n) r_i^k.$$

Observe that since $r_{Q'} \geq r > 0$, for $r_i < r$ the claim is trivially satisfied because $\mu_i = 0$. Assume, then, that the inductive hypothesis holds for all $j \geq i + 1$. Let $x \in B_1(0)$. We consider $\mu_i(B_{4r_i}(x))$. Observe that we can get a course bound

$$\mu_j(B_{4r_j}(x)) \leq \Gamma(n) r_j^k \quad \text{for all } j \geq i - 2 \text{ for all } x \in B_1(0)$$

by writing $\mu_j(B_{4r_j}(x)) = \mu_{j+2}(B_{4r_j}(x)) + \sum r^k_{Q'}$, where the sum is taken over all $Q' \in B_{4r_j}(x)$ with $r_{j+2} < r_{Q'} \leq r_j$. Since the balls $B_{r_{Q'}}(Q')$ are disjoint, there is a dimensional constant $c(n)$ which bounds the number of such points. Thus, we may take $\Gamma(n) = c(n) C_{DR}$.

Now, we calculate

$$\begin{aligned}
\sum_{r_j<2r_i}\int_{B_{2r_i}(x)}\beta_i(z,r_j)^2\,d\mu_i(z)
&=\sum_{r_j<2r_i}\int_{B_{2r_i}(x)}\beta_j(z,r_j)^2\,d\mu_j(z)\\
&\le C\sum_{r_j<2r_i}\frac{1}{r_j^k}\int_{B_{2r_i}(x)}\left(\int_{B_{r_j}(z)}|W_{r_j/2,16r_j}(y)|d\mu_j(y)\right)d\mu_j(z)\\
&\qquad+C\sum_{r_j<2r_i}\int_{B_{2r_i}(x)}\left((|W_{r_j/2,16r_j}(z)|+\eta r_j^{2\alpha+2}+r_j^2+r_j)\frac{\mu(B_{r_j}(z))}{r_j^k}\right)d\mu_j(z)\\
&\le C\sum_{r_j<2r_i}\int_{B_{2r_i+r_j}(x)}\frac{\mu_j(B_{r_j}(y))}{r_j^k}|W_{r_j/2,16r_j}(y)|\,d\mu_j(y)\\
&\qquad+C\sum_{r_j<2r_i}\int_{B_{2r_i}(x)}\left((|W_{r_j/2,16r_j}(z)|+\eta r_j^{2\alpha+2}+r_j^2+r_j)\frac{\mu(B_{r_j}(z))}{r_j^k}\right)d\mu_j(z)\\
&\le 2C\Gamma(n)\int_{B_{4r_i}(x)}\left(\sum_{r_j<2r_i}|W_{r_j/2,16r_j}(y)|\right)d\mu_j(y)+C\Gamma(n)\sum_{r_j<2r_i}(\eta r_j^{2\alpha+2}+r_j^2+r_j)\mu_i(B_{4r_i}(x)).
\end{aligned}$$

Therefore, recalling $r_i=2^{-i}$ we see that

$$\sum_{j=i-1}^{N}|W_{r_j/2,16r_j}(Q')|\le 6\,\mathrm{var}_{r\in[r_{Q'},r_{i-1}]}N(r,Q',v)\le 12C(\alpha,M_0)\eta(r_{i-1}^\alpha-r_{Q'}^\alpha)+6\eta.$$

Therefore

$$\begin{aligned}
\sum_{r_j<2r_i}\int_{B_{2r_i}(x)}\beta_i(z,r_j)^2\,d\mu_i(z)&\le C\Gamma(n)\mu_i(B_{4r_i}(x))\Big(6\eta+12C_1r_{i-1}^\alpha\eta\Big)+C\Gamma^2(n)\Big(\sum_{r_j<2r_i}\eta r_j^{2\alpha}+r_j^2+r_j\Big)r_i^k\\
&\le C\Gamma^2(n)(1+C(\alpha))\eta r_i^k+C\Gamma^2(n)r_i^k\sum_{r_j<2r_i}r_j^2+r_j.
\end{aligned}$$

Thus, for $\eta\le\eta_1(n,\alpha,M_0,\epsilon)$ sufficiently small and $r_i\le R(n,\alpha,M_0,\epsilon)=2^{-\ell}$ sufficiently small,

$$C\Gamma(n)^2(1+C(\alpha))\eta<\tfrac12\delta_{DR}\quad\text{and}\quad C\Gamma^2(n)\sum_{r_j<2r_i}r_j^2+r_j<\tfrac12\delta_{DR}.$$

For such i and μ_i satisfying the hypotheses of Theorem 9.1,

$$\sum_{r_j<2r_i}\int_{B_{2r_i}(x)}\beta_i(z,r_j)^2\,d\mu_i(z)\le\delta_{DR}r_i^k.$$

The discreet Reifenberg theorem therefore implies that $\mu_i(B_{r_i}(x))\le C_{DR}r_i^k$. Thus, by induction, the claim holds for $r_i\le R=2^{-\ell}$. We may use a packing argument using balls of radius $2^{-\ell}$ to obtain estimates at larger scales. That is, $\mu_0(B_1(0))\le C_{DR}C(n,\ell)$. □

10. Tree construction

In this section, we detail two procedures for inductively refined covering schemes. We will use these covering schemes in the next section to generate the actual cover which proves Theorem 2.15. First, we fix our constants.

10A. ***Fixing constants and a definition.*** In this section, we fix our constants as follows. Fix $0 < \epsilon$, and let $v \in \mathcal{A}(n, \alpha, M_0)$. Let $E = \sup_{Q \in B_1(0) \cap \partial\Omega^{\pm}} N(Q, 2, v)$, and fix the scale of the covering we wish to construct as $R \in (0, 1]$.

We will let ρ denote the inductive scale at which we will refine our cover. For convenience, we will use the convention $r_i = \rho^{-i}$. Let $\rho < \frac{1}{10}$ be small enough that

$$2C_2(n, \alpha, M_0, \epsilon)c_2(n)\rho < \tfrac{1}{2},$$

where $C_2(n, \alpha, M_0, \epsilon)$ is as in Lemma 9.2 and $c_2(n)$ is a dimensional constant which will be given in the following lemmas.

Let $\delta(n, \alpha, M_0, \epsilon)$ be as in Lemma 8.2 and $\gamma(n, \alpha, M_0, \delta)$ as in Lemma 5.2. We also let $\eta_1(n, \alpha, M_0, \epsilon)$ be as in Lemma 9.2, and let

$$\gamma_0 = \eta' = \tfrac{1}{20}\eta_1.$$

Note that while $\gamma_0 \leq \gamma$, Lemma 5.2 still holds with γ_0 in place of γ. As in Corollary 6.3, we then let $\eta = \eta_0(n, \alpha, E+1, \epsilon, \eta', \gamma_0, \rho)$. We shall assume that v satisfies

$$\|\ln(h)\|_\alpha \leq \frac{1}{2C_1+1}\eta.$$

The sorting principle for our covering comes from Corollary 6.3. To formalize this, we make the following definition.

Definition 10.1. For $Q' \in B_2(0) \cap \partial\Omega^{\pm}$ and $0 < R < r < 2$, the ball $B_r(Q)$ will be called "good" if

$$N(Q, \gamma\rho r, v) \geq E - \eta' \quad \text{for all } Q \in S^k_{\epsilon,\eta R}(v) \cap B_r(Q').$$

We will say that $B_r(Q')$ is "bad" if it is not good.

Remark 10.2. By Corollary 6.3, with $E + \frac{1}{2}\eta_0$ in place of E — which is admissible by monotonicity and our choice of $\|\ln(h)\|_\alpha \leq \eta/(2C_1+1)$ — in any bad ball $B_r(Q')$ there exists a $(k-1)$-dimensional affine plane L^{k-1} such that

$$\{N(Q, \gamma\rho r, v) \geq E - \tfrac{1}{2}\eta_0\} \cap B_r(Q') \subset B_{\rho r}(L^{k-1}).$$

10B. ***Good trees.*** Let $x \in B_1(0) \cap \partial\Omega^{\pm}$ and $B_{r_A}(x)$ be a good ball for $A \geq 0$. We will detail the inductive construction of a good tree based at $B_{r_A}(x)$. The induction will build a successively refined covering $B_{r_A}(x) \cap S^k_{\epsilon,\eta R}(v)$. We will terminate the process and have a cover which consists of a collection of bad balls with packing estimates and a collection of stop balls whose radii are comparable to R. We shall use the notation $\mathcal{G}_i$ to denote the collection of centers of good balls of scale r_i, and $\mathcal{B}_i$ shall denote the collection of centers of bad balls of scale r_i.

Because $B_{r_A}(x)$ is a good ball, at scale $i = A$, we set $\mathcal{G}_A = x$. We let $\mathcal{B}_A = \varnothing$. Now the inductive step. Suppose that we have constructed our collections of good and bad balls down to scale $j-1 \geq A$. Let $\{z\}_{J_i}$ be a maximal $\frac{2}{5}r_j$-net in

$$B_{r_A}(x) \cap S^k_{\epsilon,\eta R}(v) \cap B_{r_{j-1}}(\mathcal{G}_{j-1}) \setminus \bigcup_{i=A}^{j-1} B_{r_i}(\mathcal{B}_i).$$

We then sort these points into $\mathcal{G}_j$ and $\mathcal{B}_j$ depending on whether $B_{r_j}(z)$ is a good ball or a bad ball. If $r_j > R$, we proceed inductively. If $r_j \leq R$, then we stop the procedure. In this case, we let $\mathcal{S} = \mathcal{G}_j \cup \mathcal{B}_j$ and we call this the collection of "stop" balls.

The covering at which we arrive at the end of this process shall be called the "good tree at $B_{r_A}(x)$". We shall follow [Edelen and Engelstein 2019] and denote this by $\mathcal{T}_\mathcal{G} = \mathcal{T}_\mathcal{G}(B_{r_A}(x))$. We shall call the collection of "bad" ball centers $\bigcup_i \mathcal{B}_i$ the "leaves of the tree" and denote this collection by $\mathcal{F}(\mathcal{T}_\mathcal{G})$. We shall denote the collection of "stop" ball centers by $\mathcal{S}(\mathcal{T}_\mathcal{G}) = \mathcal{S}$.

For $b \in \mathcal{F}(\mathcal{T}_\mathcal{G})$ we let $r_b = r_i$ for i such that $b \in \mathcal{B}_i$. Similarly, if $s \in \mathcal{S}(\mathcal{T}_\mathcal{G})$, we let $r_s = r_j$ for the terminal j.

Theorem 10.3. *A good tree $\mathcal{T}_\mathcal{G}(B_{r_A}(x))$ enjoys the following properties*:

(A) *Tree-leaf packing*:

$$\sum_{b\in\mathcal{F}(\mathcal{T}_\mathcal{G})} r_b^k \leq C_2(n,\alpha,M_0,\epsilon) r_A^k.$$

(B) *Stop ball packing*:

$$\sum_{s\in\mathcal{S}(\mathcal{T}_\mathcal{G})} r_s^k \leq C_2(n,\alpha,M_0,\epsilon) r_A^k.$$

(C) *Covering control*:

$$\mathcal{S}^k_{\epsilon,\eta R}(v) \cap B_{r_A}(x) \subset \bigcup_{s\in\mathcal{S}(\mathcal{T}_\mathcal{G})} B_{r_s}(s) \cup \bigcup_{b\in\mathcal{F}(\mathcal{T}_\mathcal{G})} B_{r_b}(b).$$

(D) *Size control*: *for any $s \in \mathcal{S}(\mathcal{T}_\mathcal{G})$, we have $\rho R \leq r_s \leq R$.*

Proof. First, observe that by construction

$$\{B_{r_b/5}(b) : b \in \mathcal{F}(\mathcal{T}_\mathcal{G})\} \cup \{B_{r_s/5}(s) : s \in \mathcal{S}(\mathcal{T}_\mathcal{G})\}$$

is pairwise disjoint and centered in the set $\mathcal{S}^k_{\epsilon,\eta R}(v)$. Next, all bad balls and stop balls are centered in a good ball of the previous scale. By our definition of good balls, then, we have for all i

$$N(b,\gamma r_i, v) = N(b, \gamma\rho r_{i-1}, v) \geq E - \eta' \quad \text{for all } \in \mathcal{B}_i$$

and

$$N(s,\gamma r_s, v) \geq E - \eta' \quad \text{for all } s \in \mathcal{S}(\mathcal{T}_\mathcal{G}).$$

Since by monotonicity we have that $\sup_{p\in B_{r_A}(x)} N(Q, 2r_A, v) \leq E + \eta'$, we can apply Lemma 9.2 to $B_{r_A}(x)$ and get the packing estimates (A) and (B).

Covering control (C) follows from our choice of a maximal $\frac{2}{5}r_i$-net at each scale i. If i is the first scale at which a point $x \in \mathcal{S}^k_{\epsilon,\eta R}(v)$ was not contained in our inductively refined cover, it would violate the maximality assumption.

The last condition (D) follows because we stop only if j is the first scale for which $r_j \le R$. Since we decrease by a factor of ρ at each scale, (D) follows. □

10C. *Bad trees.* Let $B_{r_A}(x)$ be a bad ball. Note that for every bad ball, there is a $(k-1)$-dimensional affine plane L^{k-1} associated to it which satisfies the properties elaborated in Corollary 6.3. Our construction of bad trees will differ in several respects from our construction of good trees. The idea is still to define an inductively refined cover at decreasing scales of $B_{r_A}(x) \cap \mathcal{S}^k_{\epsilon,\eta R}(v)$. We shall again sort balls at each step into "good", "bad", and "stop" balls. But these balls will play slightly different roles and the "stop" balls will have different radii.

We reuse the notation $\mathcal{G}_i$ to denote the collection of centers of good balls of scale r_i, $\mathcal{B}_i$ to denote the collection of centers of bad balls of scale r_i, and $\mathcal{S}_i$ to denote the collection of centers of stop balls of scale r_i.

At scale $i = A$, we set $\mathcal{B}_A = x$, since $B_{r_A}(x)$ is a bad ball, and set $\mathcal{S}_A = \mathcal{G}_A = \varnothing$. Suppose, now that we have constructed good, bad, and stop balls for scale $i-1 \ge A$. If $r_i > R$, then define $\mathcal{S}_i$ to be a maximal $\frac{2}{5}\eta r_{i-1}$-net in

$$B_{r_A}(x) \cap \mathcal{S}^k_{\epsilon,\eta R}(v) \cap \bigcup_{b \in \mathcal{B}_{i-1}} B_{r_{i-1}}(b) \setminus B_{2\rho r_{i-1}}(L^{k-1}_b).$$

Note that $\eta \ll \rho$, so $\eta r_{i-1} < r_i$. We then let $\{z\}$ be a maximal $\frac{2}{5}r_i$-net in

$$B_{r_A}(x) \cap \mathcal{S}^k_{\epsilon,\eta R}(v) \cap \bigcup_{b \in \mathcal{B}_{i-1}} B_{r_{i-1}}(b) \cap B_{2\rho r_{i-1}}(L^{k-1}_b).$$

We then sort $\{z\}$ into the disjoint union $\mathcal{G}_i \cup \mathcal{B}_i$ depending on whether $B_{r_i}(z)$ is a good ball or a bad ball.

If $r_i \le R$, we terminate the process by defining $\mathcal{G}_i = \mathcal{B}_i = \varnothing$ and letting $\mathcal{S}_i$ be a maximal $\frac{2}{5}\eta r_{i-1}$-net in

$$B_{r_A}(x) \cap \mathcal{S}^k_{\epsilon,\eta R}(v) \cap B_{r_i}(\mathcal{B}_{i-1}).$$

The covering at which we arrive at the end of this process shall be called the "bad tree at $B_{r_A}(x)$". We shall follow [Edelen and Engelstein 2019] and denote this by $\mathcal{T}_{\mathcal{B}} = \mathcal{T}_{\mathcal{B}}(B_{r_A}(x))$. We shall call the collection of "good" ball centers, $\bigcup_i \mathcal{G}_i$, the "leaves of the tree" and denote this collection by $\mathcal{F}(\mathcal{T}_{\mathcal{B}})$. We shall denote the collection of "stop" ball centers by $\mathcal{S}(\mathcal{T}_{\mathcal{B}}) = \bigcup_i \mathcal{S}_i$.

As before, we shall use the convention that for $g \in \mathcal{F}(\mathcal{T}_{\mathcal{B}})$ we let $r_g = r_i$ for i such that $g \in \mathcal{G}_i$. However, note that now, if $s \in \mathcal{S}_i \subset \mathcal{S}(\mathcal{T}_{\mathcal{B}})$, we let $r_s = \eta r_{i-1}$.

Theorem 10.4. *A bad tree $\mathcal{T}_{\mathcal{B}}(B_{r_A}(x))$ enjoys the following properties*:

(A) *Tree-leaf packing*:

$$\sum_{g \in \mathcal{F}(\mathcal{T}_{\mathcal{B}})} r_g^k \le 2c_2(n)\rho r_A^k.$$

(B) *Stop ball packing*:

$$\sum_{s \in \mathcal{S}(\mathcal{T}_{\mathcal{B}})} r_s^k \le c(n,\eta) r_A^k.$$

(C) *Covering control*:

$$\mathcal{S}^k_{\epsilon,\eta R}(v) \cap B_{r_A}(x) \subset \bigcup_{s\in\mathcal{S}(\mathcal{T}_\mathcal{B})} B_{r_s}(s) \cup \bigcup_{g\in\mathcal{F}(\mathcal{T}_\mathcal{B})} B_{r_g}(g).$$

(D) *Size control*: *for any* $s \in \mathcal{S}(\mathcal{T}_\mathcal{B})$, *at least one of the following holds*:

$$\eta R \le r_s \le R \quad \text{or} \quad \sup_{Q\in B_{2r_s}(s)\cap\partial\Omega^\pm} N(Q, 2r_s, v) \le E - \tfrac{1}{2}\eta.$$

Proof. Conclusion (C) follows identically as in Theorem 10.3. Next we consider the packing estimates. Let $r_i > R$. Then, by construction, for any $b \in \mathcal{B}_{i-1}$, we have

$$\mathcal{G}_i \cup \mathcal{B}_i \cup B_{r_{i-1}}(b) \subset B_{2\rho r_{i-1}}(L^{k-1}_b).$$

Thus, since the points $\mathcal{G}_i \cup \mathcal{B}_i$ are $\frac{2}{5}r_i$ disjoint, we calculate

$$|\mathcal{G}_i \cup \mathcal{B}_i \cup B_{r_{i-1}}(b)| \le \omega_{k-1}\omega_{n-k+1}(3\rho)^{n-k+1}\frac{1}{\omega_n\left(\frac{1}{5}\rho\right)^n} \le c_2(n)\rho^{1-k}.$$

We can push this estimate up the scales as follows:

$$\begin{aligned}|\mathcal{G}_i \cup \mathcal{B}_i| r_i^k &\le c_2(n)\rho^1|\mathcal{B}_{i-1}|r_{i-1}^k\\ &\le c_2(n)\rho^1|\mathcal{B}_{i-1}\cup\mathcal{G}_{i-1}|r_{i-1}^k\\ &\vdots\\ &\le (c_2\rho)^{i-A}r_A^k.\end{aligned}$$

Summing over all $i \ge A$, then, we have that

$$\sum_{i=A+1}^{\infty} |\mathcal{B}_{i-1}\cup\mathcal{G}_{i-1}|r_i^k \le \sum_{i=A+1}^{\infty}(c_2\rho)^{i-A}r_A^k.$$

Since we chose $c_2\rho \le \frac{1}{2}$, we have that the sum converges and

$$\sum_{i=A+1}^{\infty} |\mathcal{B}_{i-1}\cup\mathcal{G}_{i-1}|r_i^k \le 2c_2\rho r_A^k.$$

This proves (A).

To see (B), we observe that for any given scale $i \ge A+1$, the collection of stop balls $\{B_{\eta r_{i-1}}(s)\}_{s\in\mathcal{S}_i}$ form a Vitali collection centered in $B_{r_{i-1}}(\mathcal{B}_{i-1})$. Thus, we have

$$|\{\mathcal{S}_i\}| \le \frac{10^n}{\eta^n}|\{\mathcal{B}_{i-1}\}|.$$

Since by construction there are no stop balls at the initial scale A, we compute that

$$\sum_{i=A+1}^{\infty}|\{\mathcal{S}_i\}|(\eta r_{i-1})^k \le 10^k\eta^{k-n}\sum_{i=A}^{\infty}|\{\mathcal{B}_i\}|r_i^k \le c(n,\eta)r_A^k.$$

This is (B).

We now argue (D). For $s \in \mathcal{S}_i$ where $r_i > R$, by construction $s \in B_{r_{i-1}}(b) \setminus B_{2\rho r_{i-1}}(L^{k-1})$ for some $b \in \mathcal{B}_{i-1}$. By Corollary 6.3, the construction, and our choice of $\eta \leq \frac{1}{2}\rho$, we have

$$\sup_{p \in B_{2r_s}(s)} N(Q, 2r_s, v) \leq \sup_{p \in B_{2\eta r_{i-1}}(s)} N(Q, 2\eta r_{i-1}, v) \leq E - \tfrac{1}{2}\eta.$$

On the other hand, if $r_i \leq R$, then $r_{i-1} > R$. Thus

$$R \geq \rho r_{i-1} \geq \eta r_{i-1} = r_s \geq \eta R.$$

This proves (D). □

11. The covering

Assuming that $\|\ln(h)\|_\alpha \leq \eta/(2C_1+1)$, for $0 < \eta \leq \eta_0(n, \alpha, E+1, \epsilon, \eta', \gamma_0, \rho)$ as in Section 10, we now wish to build the covering of $\mathcal{S}^k_{\epsilon,\eta R} \cap B_1(0)$ using the tree constructions above. Note that $B_1(0)$ is either a good ball or a bad ball. Therefore, we can construct a tree with $B_1(0)$ as the root. Then in each of the leaves, we construct either good trees or bad trees, depending upon the type of the leaf. Since in each construction we decrease the size of the leaves by a factor of $\rho < \frac{1}{10}$, we can continue alternating tree types until the process terminates in finite time.

Explicitly, we let $\mathcal{F}_0 = \{0\}$ and let $B_1(0)$ be the only leaf. We set $\mathcal{S}_0 = \varnothing$. Now, assume that we have defined the leaves and stop balls up to stage $i-1$. Since by hypothesis, the leaves in $\mathcal{F}_i$ are all good balls or bad balls, if they are good, we define for each $f \in \mathcal{F}_{i-1}$ the good tree $\mathcal{T}_{\mathcal{G}}(B_{r_f}(f))$. We then set

$$\mathcal{F}_i = \bigcup_{f \in \mathcal{F}_{i-1}} \mathcal{F}(\mathcal{T}_{\mathcal{G}}(B_{r_f}(f))) \quad \text{and} \quad \mathcal{S}_i = \mathcal{S}_{i-1} \cup \bigcup_{f \in \mathcal{F}_{i-1}} \mathcal{S}(\mathcal{T}_{\mathcal{G}}(B_{r_f}(f))).$$

Since all the leaves of good trees are bad balls, all the leaves of $\mathcal{F}_i$ are bad.

If, on the other hand, leaves of $\mathcal{F}_{i-1}$ are bad, then for each $f \in \mathcal{F}_{i-1}$ we construct a bad tree $\mathcal{T}_{\mathcal{B}}(B_{r_f}(f))$. In this case, we set

$$\mathcal{F}_i = \bigcup_{f \in \mathcal{F}_{i-1}} \mathcal{F}(\mathcal{T}_{\mathcal{B}}(B_{r_f}(f))) \quad \text{and} \quad \mathcal{S}_i = \mathcal{S}_{i-1} \cup \bigcup_{f \in \mathcal{F}_{i-1}} \mathcal{S}(\mathcal{T}_{\mathcal{B}}(B_{r_f}(f))).$$

Since all the leaves of bad trees are good balls, all the leaves of $\mathcal{F}_i$ are good.

This construction gives the following estimates.

Lemma 11.1. *For the construction described above, there is an $N \in \mathbb{N}$ such that $\mathcal{F}_N = \varnothing$ with the following properties*:

(A) *Leaf packing*:

$$\sum_{i=0}^{N-1} \sum_{f \in \mathcal{F}_i} r_f^k \leq c(n).$$

(B) *Stop ball packing*:

$$\sum_{s \in \mathcal{S}_N} r_s^k \leq c(n, \alpha, M_0, \epsilon).$$

(C) *Covering control*:

$$\mathcal{S}^k_{\epsilon,\eta R}(v) \cap B_1(0) \subset \bigcup_{s\in\mathcal{S}_N} B_{r_s}(s).$$

(D) *Size control*: *for any* $s \in \mathcal{S}_N$, *at least one of the following holds*:

$$\eta R \le r_s \le R \quad \text{or} \quad \sup_{Q\in B_{2r_s}(s)\cap\partial\Omega^\pm} N(Q, 2r_s, v) \le E - \tfrac{1}{2}\eta.$$

Proof. By construction, each of the leaves of a good or bad tree satisfy $r_f \le r_i$. Thus, there is an i sufficiently large such that $r_i < R$. Thus, N is finite.

To see (A), we use the previous theorems. That is, if the leaves $\mathcal{F}_i$ are good, then they are the leaves of bad trees rooted in $\mathcal{F}_{i-1}$. Thus, we calculate by Theorem 10.4

$$\sum_{f\in\mathcal{F}_i} r_f^k \le 2c_2(n)\rho \sum_{f'\in\mathcal{F}_{i-1}} r_{f'}^k.$$

On the other hand, if the leaves $\mathcal{F}_i$ are bad, then they are the leaves of good trees rooted in $\mathcal{F}_{i-1}$. Thus, we calculate by Theorem 10.3

$$\sum_{f\in\mathcal{F}_i} r_f^k \le C_2(n,\alpha,M_0,\epsilon) \sum_{f'\in\mathcal{F}_{i-1}} r_{f'}^k.$$

Concatenating the estimates, since we alternate between good and bad leaves, we have

$$\sum_{f\in\mathcal{F}_i} r_f^k \le c(n)(2C_2(n,\alpha,M_0,\epsilon)c_2(n)\rho)^{i/2}.$$

By our choice of ρ,

$$\sum_{f\in\mathcal{F}_i} r_f^k \le c(n)2^{-i/2}.$$

The estimate (A) follows immediately.

We now turn our attention to (B). Each stop ball $s \in \mathcal{S}_N$ is a stop ball coming from a good or a bad tree rooted in one of the leaves of a bad tree or good tree. We have the estimates from Theorems 10.3 and 10.4, which give bounds packing both leaves and stop balls. Combining these, we get

$$\sum_{s\in\mathcal{S}_N} r_s^k = \sum_{i=0}^{N}\sum_{s\in\mathcal{S}_i} r_s^k \le \sum_{i=0}^{N-1}\sum_{f\in\mathcal{F}_i} c(n,\eta) r_f^k \le C(n,\eta).$$

Recalling the dependencies of η gives the desired result.

Property (C) follows inductively from the analogous covering control in Theorems 10.3 and 10.4 applied to each tree constructed. Property (D) is immediate from these theorems as well. □

Corollary 11.2. *Fix* $0 < \epsilon$. *Let* $v \in \mathcal{A}(n,\alpha,M_0)$ *satisfy* $\sup_{p\in B_2(0)} N(Q,2,v) \le E$. *Fix* $0<\epsilon$. *There is an* $\eta_0(n,\alpha,M_0,\epsilon,E) > 0$ *such that if* $0 < \eta \le \eta_0$ *and* $\|\ln(h)\|_\alpha \le \eta/(2C_1+1)$ *then given any* $0 < R \le 1$ *there is a collection of balls* $\{B_{r_x}(x)\}_{x\in\mathcal{U}}$ *with centers* $x \in S^k_{\epsilon,\eta R}(v) \cap B_1(0)$. *Further*, $R \le r_x \le \frac{1}{10}$ *and the collection has the following properties*:

(A) *Packing*:

$$\sum_{x\in\mathcal{U}} r_x^k \le c(n,\alpha,M_0,E,\epsilon).$$

(B) *Covering control*:

$$\mathcal{S}^k_{\epsilon,\eta R}(v)\cap B_1(0)\subset \bigcup_{x\in\mathcal{U}} B_{r_x}(x).$$

(C) *Energy drop*: *for every* $x\in\mathcal{U}$, *either*

$$r_x = R \quad\text{or}\quad \sup_{Q\in B_{2r_s}(s)\cap\partial\Omega^\pm} N(Q,2r_s,v)\le E-\tfrac{1}{2}\eta_0.$$

This follows immediately from Lemma 11.1 with $\eta\le\eta_1$, $\mathcal{S}_N=\mathcal{U}$, and setting $r_x=\max\{R,r_s\}$.

11A. *Proof of Theorem 2.15.*

Lemma 11.3. *Let* $v\in\mathcal{A}(n,\alpha,M_0)$ *with* $\|\ln(h)\|_\alpha\le\Gamma$. *There exists a scale* $\kappa(n,\alpha,M_0,\Gamma,\epsilon)>0$ *such that, for all balls* $B_r(Q)$ *with* $0<r<\kappa$ *and* $Q\in B_{1/4}(0)\cap\partial\Omega^\pm$, *the function* $\tilde v(x)=v(rx+Q)$ *on* $B_1(0)$ *satisfies the following properties*:

$$\sup_{Q'\in B_1(0)\cap T_{Q,r}\partial\Omega^\pm} N(Q',2,\tilde v)\le C(\alpha,M_0,\Gamma)\quad \textit{and}\quad \|\ln(\tilde h)\|_{C^{0,\alpha}(B_1(0))}\le\frac{\eta_0}{2C_1+1},$$

where

$$\eta_0=\eta_0(n,\alpha,C(n,\alpha,M_0,\Gamma)+1,\eta',\epsilon,\gamma_0,\rho)=\eta_0(n,\alpha,M_0,\Gamma,\epsilon)$$

is as in Corollary 6.3 and $C(n,\alpha,M_0,\Gamma)$ *is as in Corollary 4.4.*

Proof. First, note that if $\ln(h)\in C^{0,\alpha}(B_1(0))$, then $\ln(\tilde h(x))=\ln(h(rx+Q))$ satisfies

$$|\ln(\tilde h(x))-\ln(\tilde h(z))|=|\ln(h(rx+Q))-\ln(h(rz+Q))|\le\Gamma|rx-rz|^\alpha=\Gamma r^\alpha|x-z|^\alpha.$$

Since $r^\alpha\to0$ as $r\to0$, there exists a $\kappa(n,\alpha,M_0,\Gamma,\epsilon)>0$ such that $\|\ln(\tilde h)\|_{C^{0,\alpha}}\le\Gamma\kappa^\alpha<\eta_0/(2C_1+1)$. By a similar calculation, we see that $\mathrm{Lip}(\tilde v)\le r\,\mathrm{Lip}(v)$. Thus, the fact that $H(Q,R,v)=H(0,R/r,\tilde v)$ for any $Q\in B_{1/4}(0)\cap\partial\Omega^\pm$ and $0<r\le2\kappa$, Lemma 3.4, and $Q'\in B_r(Q)\cap\partial\Omega^\pm$ yields

$$N(T_{Q,r}Q',2,\tilde v)=r^2N(Q,2r,v)\le r^2C(\alpha,M_0,\Gamma).\qquad\square$$

Theorem 11.4. *Let* $v\in\mathcal{A}(n,\alpha,M_0)$ *with* $\|\ln(h)\|_\alpha\le\Gamma$. *For all* $\epsilon>0$ *there exists an* $\eta_0(n,\alpha,M_0,\Gamma,\epsilon)>0$ *such that, for all* $0<R<1$ *and* $k=1,2,\dots,n-1$, *we can find a collection of balls* $\{B_R(x_i)\}_i$ *with the following properties*:

(1) $\mathcal{S}^k_{\epsilon,\eta_0R}(v)\cap B_{1/4}(0)\subset\bigcup_i B_R(x_i)$.

(2) $|\{x_i\}_i|\le c(n,\alpha,M_0,\Gamma,\epsilon)R^{-k}$.

Proof. Cover $\mathcal{S}^k_{\epsilon,\eta r}(v)\cap B_{1/4}(0)$ by balls $B_\kappa(Q_j)$, with $Q_j\in B_{1/4}(0)\cap\partial\Omega^\pm$, such that

$$B_{1/4}(0)\cap\partial\Omega^\pm\subset\bigcup_j B_\kappa(Q_j)$$

for $0 < \kappa(n, \alpha, M_0, \Gamma, \epsilon)$ the constant in Lemma 11.3. Note that we need at most $c(n, \alpha, M_0, \Gamma, \epsilon)$ such balls.

We now wish to apply Corollary 11.2 to the rescaled functions $\tilde{v}_i(x) = v(\kappa x + Q_i)$ in $B_1(0)$. However, a careful reader may object that $\tilde{v}_i$ is not in $\mathcal{A}(n, \alpha, M_0)$, since it is possible that $\tilde{h}(0) \neq 1$. However, $\tilde{v}_i(x) = c\tilde{h}(0)u^+(x\kappa + Q) - u^-(x\kappa + Q)$, where by Remark 3.1 we can control $0 < c < \infty$ by constants that only depend upon κ and α. Thus, by multiplying the positive part by a constant controlled by Γ, α, and M_0, we obtain a new function (which we also label $\tilde{v}_i$) which is in $\mathcal{A}(n, \alpha, M_0)$.

We now construct the desired covering in $B_1(0)$ for each $\tilde{v}_i$. Ensuring that $c(n, \alpha, M_0, \Gamma, \epsilon)$ is sufficiently large, we may reduce to arguing for $r < \eta$. We now use Corollary 11.2 to build a covering $\mathcal{U}_1$. If every r_x equals R, then the packing and covering estimates give the claim directly, since

$$R^{k-n}\operatorname{Vol}(B_R(\mathcal{S}^k_{\epsilon,\eta_0 R}(\tilde{v}_i) \cap B_1(0))) \leq \omega_n R^{k-n} \sum_{\mathcal{U}_1} (2R)^n = \omega_n 2^n \sum_{\mathcal{U}_1} r_x^k \leq c(n, \alpha, M_0, \Gamma, \epsilon).$$

If there exists an $r_x \neq R$, we use Corollary 11.2 to build a finite sequence of refined covers $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots$ such that, for each for each i, the covering satisfies the following properties:

(A_i) Packing:

$$\sum_{x \in \mathcal{U}_i} r_x^k \leq c(n, \alpha, M_0, \Gamma, \epsilon)\Big(1 + \sum_{x \in \mathcal{U}_{i-1}} r_x^k\Big).$$

(B_i) Covering control:

$$\mathcal{S}^k_{\epsilon,\eta_0 R}(\tilde{v}_i) \cap B_1(0) \subset \bigcup_{x \in \mathcal{U}_i} B_{r_x}(x).$$

(C_i) Energy drop: for every $x \in \mathcal{U}_i$, either

$$r_x = R \quad \text{or} \quad \sup_{Q \in B_{2r_s}(s) \cap \partial\Omega^\pm} N(Q, 2r_s, \tilde{v}_i) \leq C(n, \alpha, M_0, \Gamma) - i\left(\tfrac{1}{2}\eta_0\right).$$

(D_i) Radius control:

$$\sup_{x \in \mathcal{U}_i} r_x \leq 10^{-i}.$$

If we can construct such a sequence of covers, then we claim that this process will terminate in finite time, *independent of* R. Recall that blow-ups of $\tilde{v}_i$ are homogeneous harmonic polynomials. Therefore

$$N(Q, 0, \tilde{v}_i) = \lim_{r \to \infty} N(Q, r, \tilde{v}_i) \geq 1$$

for all $Q \in \partial\Omega^\pm$. By Remark 4.8 we have that, for all $0 < r \leq 1$,

$$N(Q, r, \tilde{v}_i) \geq 1 - C(n, \alpha, M_0, \Gamma, \epsilon)$$

for all $p \in B_1(0)$. Therefore, we know that, for i large enough that

$$i > (C(n, \alpha, M_0, \Gamma, \epsilon) + C(n, \alpha, M_0, \Gamma, \epsilon) - 1)\frac{2}{\eta_0},$$

it must be the case that $r_x = R$ for all $x \in \mathcal{U}_i$. In this case, we will have the claim with a bound of the form

$$R^{k-n}\operatorname{Vol}(B_R(\mathcal{S}^k_{\epsilon,\eta_0 R}(\tilde{v}_i) \cap B_1(0))) \leq c(n, \alpha, M_0, \Gamma, \epsilon)^{C(n,\alpha,M_0,\Gamma,\epsilon)}.$$

Thus, we reduce to inductively constructing the required covers. Suppose we have already constructed $\mathcal{U}_{i-1}$ as desired. For each $x \in \mathcal{U}_{i-1}$ with $r_x > R$, we apply Corollary 11.2 at scale $B_{r_x}(x)$ to obtain a new collection of balls $\mathcal{U}_{i,x}$. From the assumption that $r_x \leq \frac{1}{10}$ and the way that Hölder norms scale, it is clear that $\tilde{v}_i$ satisfies the hypotheses of Corollary 11.2 in $B_{r_x}(x)$ with the same constants. To check packing control, we have that

$$\sum_{y \in \mathcal{U}_{i,x}} r_y^k \leq c(n, \alpha, M_0, \Gamma, \epsilon) r_x^k.$$

Covering control follows immediately from the statement of Corollary 11.2. Similarly, from hypothesis (C_{i-1}), we have that $\sup_{p \in B_{2r_x}(x)} N(Q, 2r_x, \tilde{v}_i) \leq C(n, \alpha, M_0, \Gamma, \epsilon) - \frac{1}{2}(i-1)\eta_0$. Thus, the statement of Corollary 11.2 at scale $B_{r_x}(x)$ gives $\sup_{p \in B_{2r_y}(y)} N(Q, 2r_y, \tilde{v}_i) \leq C(n, \alpha, M_0, \Gamma, \epsilon) - \frac{1}{2} i \eta_0$ for all $y \in \mathcal{U}_{i,x}$ with $r_y > R$. Radius control follows immediately from the fact that $\sup_{y \in \mathcal{U}_{i,x}} r_y \leq \frac{1}{10} r_x \leq 10^{-i}$.

Thus, if we let

$$\mathcal{U}_i = \{x \in \mathcal{U}_{i-1} \mid r_x = R\} \cup \bigcup_{x \in \mathcal{U}_{i-1},\, r_x > R} \mathcal{U}_{i,x}$$

then $\mathcal{U}_i$ satisfies the inductive claim.

To obtain the cover which proves the theorem, we simply scale each covering of $\mathcal{S}^k_{\epsilon, \eta_0 R/\kappa}(\tilde{v}_i) \cap B_1(0)$ to a covering of $\mathcal{S}^k_{\epsilon, \eta_0 R}(v) \cap B_\kappa(y_i)$ and sum over the $c(n, \alpha, M_0, \Gamma, \epsilon)$ such balls which cover $\mathcal{S}^k_{\epsilon, \eta_0 R}(v) \cap B_{1/4}(0)$. This completes the proof. □

Proof of Theorem 2.15. By Theorem 11.4, we have

$$\mathrm{Vol}(B_R(\mathcal{S}^k_{\epsilon, \eta_0 R}(v) \cap B_{1/4}(0))) \leq C(n, \alpha, M_0, \Gamma, \epsilon) R^{n-k}.$$

Thus, let $r_0 = \eta_0$ and $r = \eta_0 R'$ for $0 < R' \leq 1$. For any $r \leq R \leq R'$, by containment, we have

$$B_R(\mathcal{S}^k_{\epsilon,r}(v) \cap B_{1/4}(0)) \subset B_{R'}(\mathcal{S}^k_{\epsilon,r}(v) \cap B_{1/4}(0)) \subset \bigcup_i B_{2R'}(x_i),$$

where $\{x_i\}$ are the centers of the balls in the covering constructed in Theorem 11.4. Therefore, the estimates in Theorem 11.4 give

$$\begin{aligned}\mathrm{Vol}(B_R(\mathcal{S}^k_{\epsilon,r}(v) \cap B_{1/4}(0))) &\leq C(n, \alpha, M_0, \Gamma, \epsilon) 2^n (R')^{n-k} \\ &\leq C(n, \alpha, M_0, \Gamma, \epsilon) 2^n \left(\frac{R}{\eta_0}\right)^{n-k} \leq C(n, \alpha, M_0, \Gamma, \epsilon) R^{n-k}\end{aligned}$$

by increasing our constant $C(n, \alpha, M_0, \Gamma, \epsilon)$.

For any $R' \leq R$, by containment, we have

$$B_R(\mathcal{S}^k_{\epsilon,r}(v) \cap B_{1/4}(0)) \subset \bigcup_i B_{2R}(x_i),$$

where $\{x_i\}$ are the centers of the balls in the covering constructed in Theorem 11.4. In this case

$$\mathrm{Vol}(B_R(\mathcal{S}^k_{\epsilon,r}(v) \cap B_{1/4}(0))) \leq C(n, \alpha, M_0, \Gamma, \epsilon) 2^n (R)^{n-k} \leq C(n, \alpha, M_0, \Gamma, \epsilon) R^{n-k}$$

by increasing our constant $C(n, \alpha, M_0, \Gamma, \epsilon)$. This concludes the proof of Theorem 2.15. □

12. Proof of Corollary 2.17

In this section, we prove that $\mathrm{sing}(\partial\Omega^{\pm}) \subset \mathcal{S}^{k-3}_{\epsilon}(v)$ for ϵ small enough.

Lemma 12.1. *Let* $v \in \mathcal{A}(n, \alpha, M_0)$ *with* $\|\ln(h)\|_{\alpha} \le \Gamma$. *Then there exists* $0 < \epsilon = \epsilon(M_0, \alpha, \Gamma)$ *such that* $\mathrm{sing}(\partial\Omega^{\pm}) \cap B_{1/4}(0) \subset \mathcal{S}^{n-3}_{\epsilon}(v)$.

Proof. We must argue that there is an $\epsilon > 0$ such that, for all $Q \in \mathrm{sing}(\partial\Omega^{\pm}) \cap B_1(0)$ and all radii $0 < r$,

$$\int_{B_1(0)} |T_{Q,r}v - P|^2 \, dV \ge \epsilon$$

for all $(n-2)$-symmetric functions P.

If P is $(n-2)$-symmetric, P only depends upon two variables. By complex analysis all homogeneous harmonic polynomials in two dimensions are of the form $q(z) = c(x+iy)^k$. By Theorem 2.14 (2), we need only consider $k \ge 2$. Hence, the zero set Σ_q of any $\mathrm{Re}(q)$ is the union of an even number of infinite rays equidistributed in angle. If we label the connected components of $\mathbb{R}^2 \setminus \Sigma_q$ as $\{U_i\}$, we see that by the maximum principle, the sign of q must change from one U_i to another contiguous U_j.

Thus, the zero set of P is $\Sigma_P = \Sigma_q \times \mathbb{R}^{n-2}$ for some homogeneous harmonic polynomial $\mathrm{Re}(q) : \mathbb{R}^2 \to \mathbb{R}$ of degree ≥ 2. We label the connected components of $\mathbb{R}^n \setminus \Sigma_q \times \mathbb{R}^{n-2}$ as $\{W_i\}$.

Now, we claim that there is a constant, $0 < c(M_0, \Gamma, \alpha) \le 1$, such that one of the following estimates must hold:

$$\mathcal{H}^n\Big(T_{Q,r}\Omega^- \cap \bigcup_i \{W_i : P > 0 \text{ on } W_i\} \cap B_1(0)\Big) \ge c, \qquad \text{(estimate 1)}$$

$$\mathcal{H}^n\Big(T_{Q,r}\Omega^+ \cap \bigcup_i \{W_i : P < 0 \text{ on } W_i\} \cap B_1(0)\Big) \ge c. \qquad \text{(estimate 2)}$$

Note that by Theorem 2.6 (2), we need only consider P with degree $\le d(M_0) < \infty$. Reducing to $\mathbb{R}^2$, since the rays of Σ_q are equidistributed, for q of degree k, the connected components occupy a sector of aperture π/k. Thus, if $B_{1/M_0}(A_1^{\pm}(0)) \subset T_{Q,r}\Omega^{\pm}$ is the ball guaranteed by the corkscrew condition, then, for $c = (4M_0^n)^{-1}$, there exists an integer $k(M_0)$ such that

$$\mathcal{H}^n(B_{1/M_0}(A_1^{\pm}(0)) \cap \{P \cdot T_{Q,r}v < 0\}) \ge c$$

for all P with degree $\ge k(M_0)$.

For P with degree $\le k(M_0)$, we argue by contradiction. Suppose that no such constant exists. Then there would be a sequence of functions $v_j \in \mathcal{A}(n, \alpha, M_0)$ with points $Q_j \in B_{1/4}(0)$ and radii $0 < r_j \le \frac{1}{2}$ and zero sets Σ_{P_j} for P_j satisfying $2 \le \mathrm{degree}(P_j) \le k(M_0)$ such that the scaled and translated mutual boundaries $T_{Q_j,r_j}\partial\Omega_j^{\pm}$ satisfy the properties

$$\mathcal{H}^n\Big(T_{Q_j,r_j}\Omega_j^- \cap \bigcup_i \{W_{i,j} : P_j > 0 \text{ on } W_{i,j}\} \cap B_1(0)\Big) \to 0,$$

$$\mathcal{H}^n\Big(T_{Q_j,r_j}\Omega_j^+ \cap \bigcup_i \{W_{i,j} : P_j < 0 \text{ on } W_{i,j}\} \cap B_1(0)\Big) \to 0.$$

By Lemma 3.6 there exists a subsequence for which $T_{Q_j,r_j}\partial\Omega_j^{\pm}$ converge locally in the Hausdorff metric to a limit set $A \subset \mathbb{R}^n$. By Theorem 3.8, A must be the mutual boundary of a pair of two-sided NTA domains $\Omega_\infty^{\pm}$ with constant $2M_0$. Furthermore, up to scaling and rotation, the space of homogeneous harmonic functions of two variables in $\mathbb{R}^n$ with $2 \le \text{degree}(P) \le k(M_0)$ is finite-dimensional. Since the space of rotations is compact, we may find a subsequence Σ_{P_j} which converges to Σ_{P_∞} locally in the Hausdorff metric for some $(n-2)$-symmetric P_∞. This implies that

$$\mathcal{H}^n\Big(\Omega_\infty^- \cap \bigcup_i \{W_{i,\infty} : P_\infty > 0 \text{ on } W_{i,\infty}\} \cap B_1(0)\Big) = 0, \tag{12-1}$$

$$\mathcal{H}^n\Big(\Omega_\infty^+ \cap \bigcup_i \{W_{i,\infty} : P_\infty < 0 \text{ on } W_{i,\infty}\} \cap B_1(0)\Big) = 0. \tag{12-2}$$

Indeed, if there were $p \in \bigcup_i \{W_{i,\infty} : P_\infty > 0 \text{ on } W_{i,\infty}\} \cap B_1(0)$ such that $p \in \Omega_\infty^-$, since $W_{i,\infty}$ and Ω^- are open, there would exist a ball $B_\delta(p) \subset \Omega^- \cap W_{i,\infty}$. Therefore, since $\Sigma_{P_j} \to \Sigma_{P_\infty}$ and $T_{Q_j,r_j}\partial\Omega_j^{\pm} \to A$ locally in the Hausdorff metric, for all j sufficiently large, $B_{\delta/2}(p) \subset W_{i,j} \cap T_{Q_j,r_j}\partial\Omega_j^-$. This is a contradiction. The other equation follows identically.

Now we claim that $A \cap B_1(0) = \Sigma_{P_\infty} \cap B_1(0)$. Suppose not, then there exists a point $p \in \Sigma_{P_\infty}$ with $p \notin A$ or there exists a point $Q \in A$ such that $Q \notin \Sigma_{P_\infty}$. In the former case, suppose $\text{dist}(Q, A) > \delta$. Then $B_\delta(p)$ must intersect at least two contiguous connected components, $W_{i,\infty}$ and $W_{j,\infty}$. Since they are contiguous, the sign of P_∞ must be positive on one and negative on the other. This contradicts (12-1). Similarly, if there exists a point $Q \in A$ such that $Q \notin \Sigma_{P_\infty}$ then there exists a ball $B_\delta(Q)$ which intersects both $\Omega_\infty^{\pm}$ but which is contained in a single $W_{i,\infty}$. This also contradicts (12-1).

However, if P_∞ is $(n-2)$-symmetric with degree ≥ 2, then Σ_{P_∞} does not divide $\mathbb{R}^n$ into two connected components. This contradicts our assumption that $A = \Sigma_{P_\infty}$ was the mutual boundary of a pair of two-sided NTA domains with constant $2M_0$. Therefore, such a constant $0 < c = c(M_0, \Gamma, \alpha)$ must exist.

Without loss of generality, we assume (estimate 1) holds. By Lemma A.2 we may find a radius $0 < r = r(M_0, \Gamma, \alpha)$ such that $\mathcal{H}^n(B_r(T_{Q,r}\partial\Omega^{\pm})) < \frac{1}{20}c(\alpha, M_0, \Gamma)$. Now, consider

$$p \in \bigcup \{W_i : P > 0 \text{ on } W_i\} \cap B_1(0) \setminus B_r(T_{Q,r}\partial\Omega^{\pm}).$$

By Lemma 3.5, $|T_{Q,r}v(p)| \ge c'$ for a constant $c' = c'(M_0, \Gamma, \alpha)$. Thus

$$\begin{aligned}\int_{B_1(0)} |T_{Q,r}v - P|^2\,dV &\ge \int_{B_1(0)\cap T_{Q,r}\partial\Omega^- \cap \bigcup_i \{W_i : P>0 \text{ on } W_i\}} |T_{Q,r}v - P|^2\,dV \\ &\ge \tfrac{19}{20}c(\alpha, M_0, \Gamma)c'(\alpha, M_0, \Gamma)^2.\end{aligned}$$

If (estimate 2) holds, an identical argument with signs switched proves the claim. □

Remark 12.2. The argument above can be modified to show that there is an $\epsilon' > 0$ such that if $Q \in \partial\Omega$ but $Q \notin \mathcal{S}^{n-3}_{\epsilon',r_0}$, then $Q \notin \mathcal{S}^{n-2}_{\epsilon',r_0}$. Indeed, if $Q \notin \mathcal{S}^{n-3}_{\epsilon',r_0}$, then there exists a radius $r_0 \le r$ and an $(n-2)$-symmetric function P such that $\|T_{Q,r}v - P\|^2_{L^2(B_1(0))} \le \epsilon'$. However, by taking $\epsilon' < \epsilon(\alpha, M_0, \Gamma)$ in Lemma 12.1, we see that P must be $(n-1)$-symmetric.

Appendix

The purpose of this section is to justify Lemma 3.9. We use the language of porous sets. For a nonempty set $E \subset \mathbb{R}^n$, $x \in E$, and radius $0 < r$, we write

$$P(E, x, r) = \sup\{0, h : h > 0, B_h(y) \subset B_r(x) \setminus E \text{ for some } y \in B_r(x)\}. \tag{A-1}$$

For $\alpha > 0$, we say that E is *α-porous* if

$$\liminf_{r \to 0} \frac{P(E, x, r)}{r} > \alpha \tag{A-2}$$

for all $x \in E$.

We shall say that E is *α-porous down to scale r_0* if

$$\frac{P(E, x, r)}{r} > \alpha \tag{A-3}$$

for all $x \in E$ and for all $r_0 \le r$.

Remark A.1. By definition, for $\Omega^\pm \in \mathcal{D}(n, \alpha, M_0)$, the boundary $\partial\Omega^\pm$ is $1/M_0$-porous. Similarly, $B_r(\partial\Omega^\pm)$ is $1/(2M_0)$-porous down to scale $r_0 = 2rM_0$.

Lemma A.2. *Let $E \subset \mathbb{R}^n$ be a nonempty, bounded set, $E \subset [0, 1]^n$ with $\mathbf{0} \in E$. If E is α-porous down to scale $r_0 \ll 1$, then there are $k = k(\alpha)$, $k' = k'(n)$, and $N \le -\log_2(r_0)/(k + k')$ such that*

$$\mathrm{Vol}(E) \le \left(1 - \frac{1}{2^{k+k'(n)}}\right)^N.$$

Moreover, there exists $0 < \epsilon = \epsilon(\alpha, n)$ and a constant $c(n, \alpha)$ such that

$$\mathcal{M}^{n-\epsilon}_{r_0}(E) \le (1 - c)^N.$$

Proof. Let $\{Q^i_j\}_j$ be the collection of dyadic subcubes $Q^i_j \subset [0, 1]^n$ with $\ell(Q^i_j) = 2^{-i}$. Let $k \in \mathbb{N}$ be the smallest number such that $2^{-k} \le \alpha$. Note that, for any $y \in [0, 1]^n$ with $B_{\alpha/2}(y) \subset [0, 1]^n$, there exists a dyadic cube $Q^{k+k'(n)}_j \subset B_{\alpha/4}(y)$ where $k'(x)$ is the smallest integer such that $k'(n) \ge 2 + \frac{1}{2}\log_2(n)$. Let $\frac{1}{2}Q^i_j$ denote an axis-parallel cube with the same center as Q^i_j but side length half that of Q^i_j.

Now we apply the standard argument. Tile $[0, 1]^n$ by $Q^{k+k'(n)}_j$. By our porosity assumption, there exists a $Q^{k+k'(n)}_{j'}$ which does not intersect E. Thus

$$\mathrm{Vol}(E) \le \sum_{j \ne j'} \mathrm{Vol}(Q^{k+k'(n)}_j) \le (2^{(k+k'(n))n} - 1)2^{(-k-k'(n))n} \le \left(1 - \frac{1}{2^{k+k'(n)}}\right).$$

Now, within each of the $Q^{k+k'(n)}_j$ which intersects E, either E intersects $\frac{1}{2}Q^{k+k'(n)}_j$ or it doesn't. If $E \cap \frac{1}{2}Q^{k+k'(n)}_j = \varnothing$, then we tile $Q^{k+k'(n)}_j$ by cubes $\{Q^{2(k+k'(n))}_\ell\}_\ell$ and overestimate

$$\begin{aligned}\mathrm{Vol}(E \cap Q^{k+k'(n)}_j) &\le \sum_{\ell : Q^{2(k+k'(n))}_\ell \cap (E \cap Q^{k+k'(n)}_j) \ne \varnothing} \mathrm{Vol}(Q^{2(k+k'(n))}_\ell) \\ &\le (2^{2(k+k'(n))n} - 1)2^{-2(k+k'(n))n}\,\mathrm{Vol}(Q^{k+k'(n)}_j) \le \left(1 - \frac{1}{2^{k+k'(n)}}\right)\mathrm{Vol}(Q^{k+k'(n)}_j).\end{aligned}$$

If $E \cap \frac{1}{2} Q_j^{k+k'(n)} \neq \varnothing$, then there exists a ball $B_{2^{-k-k'(n)-1}}(x) \subset Q_j^{k+k'(n)}$ centered on $x \in E$. By our porosity assumption and choice of $k'(n)$, we can still tile $Q_j^{k+k'(n)}$ by $Q_\ell^{2(k+k'(n))}$ and be guaranteed that at least one such subcube does not intersect $E \cap Q_j^{k+k'(n)}$. Thus, we overestimate in the same manner as above.

We can continue, inductively, only stopping at the first N such that $2^{-(N+1)(k+k'(n))} < r_0$. This gives the desired bound

$$\mathrm{Vol}(E) \leq \left(1 - \frac{1}{2^{k+k'(n)}}\right)^N .$$

Taking a bit more care, we can actually improve these estimates. Let $0 < \epsilon = \epsilon(\alpha, n)$ be such that

$$\left(1 - \frac{1}{2^{k+k'(n)}}\right) < \left(2^{\epsilon(k+k'(n))} - \frac{1}{2^{(k+k'(n))(n-\epsilon)}}\right) < 1.$$

Then we bound $\mathcal{M}_{r_0}^{n-\epsilon}(E)$ as follows:

$$\begin{aligned} \mathcal{M}_{r_0}^{n-\epsilon}(E) &= \inf\left\{ \sum_i r^{n-\epsilon} : x_i \in E,\ r_0 \leq r, E \subset \bigcup_i B_r(x_i) \right\} \\ &\leq \sum_j \ell(Q^{N(k+k'(n))})^{n-\epsilon} \leq \left(2^{\epsilon(k+k'(n))} - \frac{1}{2^{(k+k'(n))(n-\epsilon)}}\right)^N . \end{aligned}$$

□

As immediate corollaries, we have the following statements.

Corollary A.3. *If $E \subset \mathbb{R}^n$ is α-porous, then there exists $0 < \epsilon = \epsilon(\alpha, n)$ such that $\overline{\dim}_{\mathcal{M}}(E) \leq n - \epsilon$.*

Proof. Recall that $\overline{\dim}_{\mathcal{M}}(E) = \inf\{s : \mathcal{M}^{*,s}(E) = 0\}$ and that $\mathcal{M}^{*,s}(E) = \limsup_{r_0 \to 0} \mathcal{M}_{r_0}^{n-\epsilon}(E)$.

By taking $0 < \epsilon$ to be as in Lemma A.2, we have

$$\mathcal{M}_{r_0}^{n-\epsilon}(E) \leq \left(2^{\epsilon(k+k'(n))} - \frac{1}{2^{(k+k'(n))(n-\epsilon)}}\right)^N \leq (1-c)^N,$$

where $c = c(\alpha, n, \epsilon)$ and $N = N(\alpha, n, r_0)$, as in the previous lemma. Thus, letting $r_0 \to 0$ and $N \to \infty$ we have that $\mathcal{M}^{n-\epsilon}(E) = 0$. □

Recalling Remark A.1, Corollary A.3 gives Lemma 3.9.

Corollary A.4. *Let $\Sigma \subset \mathbb{R}^n$ be the mutual boundary of a pair of unbounded two-sided NTA domains with NTA constant $1 < M_0$. Then, there exists $0 < \epsilon = \epsilon(M_0, n)$ such that $\overline{\dim}_{\mathcal{M}}(E) \leq n - \epsilon$.*

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