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ON COMPLETE EMBEDDED TRANSLATING SOLITONS OF THE MEAN CURVATURE FLOW THAT ARE OF FINITE GENUS

GRAHAM SMITH

We desingularise the union of three Grim paraboloids along Costa–Hoffman–Meeks surfaces in order to obtain complete embedded translating solitons of the mean curvature flow with three ends and arbitrary finite genus.

1. Introduction

1.1. ***Main result.*** For the purposes of this paper, a *mean curvature flow* (*MCF*) *soliton* is a complete surface in $\mathbb{R}^3$ whose evolution under the mean curvature flow is given by translation. In other words, up to rescalings and rigid motions of the ambient spacetime, it is a solution of what we will call the *MCFS equation*

$$H + \langle N, e_z \rangle = 0, \tag{1-1}$$

where H here denotes the mean curvature of the surface, N its unit normal vector field, and e_z the unit vector in the direction of the z-axis. We refer the reader to the review of Martín, Savas-Halilaj and Smoczyk [Martín et al. 2015] for a good overview of the theory of MCF solitons at the time of writing.

We use surgery to construct embedded MCF solitons with three ends and arbitrary finite genus. Before stating our result, we describe the two components of our construction. First, given a positive integer g, the *Costa–Hoffman–Meeks* (*CHM*) *surface* of genus g, denoted by C_g, is a properly embedded minimal surface in $\mathbb{R}^3$ with three ends, each of which may be taken to be a graph over an unbounded annulus in $\mathbb{R}^2$; see [Hoffman and Meeks 1990; Weber 2005]. For $0 \leq k \leq g$, this surface is invariant under reflection

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in the vertical plane making an angle of $k\pi/(g+1)$ with the x-axis at the origin. We call the group of symmetries of $\mathbb{R}^3$ generated by these reflections the group of *horizontal symmetries* of C_g.[1] Next, the *Grim paraboloid* (also known as the *bowl soliton*) is defined to be the unique simply connected MCF soliton which is symmetric under revolution about the z-axis. It is known (see [Clutterbuck et al. 2007]) that this surface is asymptotic at infinity to a vertical translate of the graph of

$$\tfrac{1}{2}r^2 - \log(r),$$

where r here denotes the distance in $\mathbb{R}^2$ to the origin.

Theorem A. *For all $g \in \mathbb{N}$ and for all sufficiently small ϵ, there exists a complete, embedded MCF soliton $\Sigma_{g,\epsilon}$ of genus g with three ends. Furthermore, letting $R := \epsilon^{-1/\lambda}$ for some $\lambda \in]4,5[$, we may suppose*:

(1) $\Sigma_{g,\epsilon} \setminus (B(\epsilon R) \times \mathbb{R})$ *consists of three connected components, each of which converges towards the same Grim paraboloid as ϵ tends to* 0.

(2) *Upon rescaling by a factor of* $1/\epsilon$, $\Sigma_{g,\epsilon} \cap (B(\epsilon R) \times \mathbb{R})$ *converges to C_g as ϵ tends to* 0.

(3) $\Sigma_{g,\epsilon}$ *is invariant under the group of horizontal symmetries of C_g.*

Remark. Theorem A follows immediately from Theorems 7.1.3 and 7.1.4, below.

Remark. All notation and terminology used in this paper is explained in detail in Appendix A. Recall, in particular, that, by elliptic regularity, all standard modes of convergence of smooth, embedded solutions of parametric elliptic functionals to smooth, embedded solutions of the same functionals are equivalent over any compact region.

Remark. The constants that appear in Theorem A have the following geometric significance. The quantity ϵ determines the scaling factor of the CHM surface. Roughly speaking, it is the "neck radius" of $\Sigma_{g,\epsilon}$. The quantity R determines how far along the end of the CHM surface the surgery is carried out. For the construction to work, ϵ and R must converge in tandem to 0 and infinity respectively, hence the condition $R^\lambda \epsilon = 1$. Distinct values of ϵ ought to yield distinct surfaces. Indeed, a refinement of our result ought to yield a continuous family $(\Sigma_{g,\epsilon})_{\epsilon < r_0}$ of distinct embedded MCF solitons with neck radii converging to 0. However, our current argument, which uses the Schauder fixed-point theorem, does not yield such fine control over the surfaces constructed.

1.2. *Techniques.* The proof of Theorem A follows the standard desingularisation construction for minimal surfaces originally described in [Kapouleas 1990; 1995; 1997]. In simple terms, we first use surgery to construct an approximate MCF soliton $\widehat{\Sigma}_{g,\epsilon}$ and then apply a fixed-point argument to prove the existence of an actual MCF soliton lying nearby in some suitable function space. As in all singular perturbation constructions, this is much easier said than done, and the main challenge lies in deriving the many nontrivial analytic estimates required to make the perturbation argument work.

[1] Hoffman and Meeks showed that the complete symmetry group of C_g is the dihedral group generated by the elements A and B, where A is reflection in the $(x-z)$-plane and B is rotation by an angle of $k\pi/(g+1)$ about the z-axis followed by reflection in the $(x-y)$-plane.

The use of CHM surfaces in Kapouleas' construction presents particular difficulties on account of their low orders of rotational symmetry. Indeed, rotational symmetries often serve in such constructions to improve decay rates and thus in turn provide stronger estimates. This phenomenon is well illustrated by the case of bounded solutions $u : S^1 \times [0, \infty[\to \mathbb{R}$ of Laplace's equation $\Delta u = 0$. By separation of variables, all such solutions have the form

$$u(\theta, t) = \sum_{m\in\mathbb{Z}} a_m e^{im\theta} e^{-|m|t}.$$

In particular, when u has n-th order rotational symmetry, all of its coefficients of order $0 < |m| < n$ vanish, so that the difference $(u - a_0)$ decays like e^{-nt}. Since this argument does not apply when CHM surfaces are used, we obtain our estimates by introducing instead, in Section 4.2, what we call the *hybrid norm*. This functional norm, which is a combination of Hölder and Sobolev norms, encapsulates the singular nature of our construction as ϵ tends to 0. Its main properties, established in Lemma 4.3.1, follow readily from the Sobolev embedding theorem and classical interpolation inequalities, and play a key role in the derivation of various estimates throughout the rest of the paper.

Finally, before reviewing our argument, it is worth highlighting an ingenious variant of the desingularisation construction for CHM surfaces used in [Hauswirth and Pacard 2007; Mazzeo and Pacard 2001; Morabito 2009]. In each of these papers, it is observed that the Jacobi operator $\hat{J}_{g,\epsilon}$ of the approximate minimal surface $\widehat{\Sigma}_{g,\epsilon}$ is, modulo a conformal transformation when necessary, *intrinsically* close to the Jacobi operator of the original CHM surface. A direct perturbation argument then yields a priori estimates for the norm of its Green's operator, thereby bypassing one of the main technical challenges of the perturbation part of the construction. In addition, in these works, the initial surgery is carried out in a different manner than in [Kapouleas 1990; 1995; 1997], more pleasing to the geometric eye, though it is not clear to us whether this actually leads to simpler estimates. Regardless, their argument cannot be applied in the present case where the Jacobi operator of the joined surface is not intrinsically of the correct type.

The proof is organised as follows.

1.2.1. *Rotationally symmetric Grim surfaces.* We will desingularise the join of a CHM surface with three rotationally symmetric Grim ends, that is, unbounded, rotationally symmetric MCF solitons in $\mathbb{R}^3$. The geometry of CHM surfaces is well understood (see, for example, [Hoffman and Meeks 1990; Weber 2005]) and the large-scale geometry of rotationally symmetric Grim ends has been studied by Clutterbuck, Schnürer and Schulze [Clutterbuck et al. 2007]. In Section 2, we study the small-scale geometry of rotationally symmetric Grim ends, which has not previously been addressed in the literature.

Rotationally symmetric Grim ends exhibit a dual nature over the region of interest to us. Indeed, they are roughly catenoidal towards the lower end of this region, and roughly parabolic towards its upper end. This presents us with our first main challenge, which we address via the following algebraic trick. We introduce two abstract variables, representing respectively the catenoidal part and the parabolic part of the Grim end. We then construct formal solutions to the MCFS equation in terms of these variables, and obtain the desired formulae upon applying the contraction mapping theorem to their partial sums.

The main results of this section are Theorems 2.1.1 and 2.2.1, which provide asymptotic formulae for the profiles of rotationally symmetric Grim ends over the large and small scales respectively.

1.2.2. *Green's operators.* Our perturbation argument requires estimates for the norm of a Green's operator of the MCFS Jacobi operator of the approximate MCF soliton. These are in turn derived from estimates of the norms of the corresponding operators of CHM surfaces and rotationally symmetric Grim ends. Green's operators of Jacobi operators of CHM surfaces are well understood; see, for example, [Hauswirth and Pacard 2007; Morabito 2009; Nayatani 1993; Pacard 2008]. In Sections 3 and 4, we study the Green's operators of the MCFS Jacobi operators, first of Grim paraboloids, then of rotationally symmetric Grim ends. The former are relatively straightforward, but the latter present us with our second main challenge, namely, to address the singularities that catenoids produce as their neck radii tend to 0. This simple phenomenon, which we call the *vanishing neck problem*, will be responsible for the introduction of a number of technical constructions, as we now proceed to explain.

To begin with, in Section 4.1, we modify the Jacobi operator in two ways. First, we introduce the *modified MCFS Jacobi operator*, which measures the first-order variation of mean curvature arising from first-order perturbations of the surface in the direction of a suitable modification of the unit normal vector field. At this stage, this modification serves to reduce the divergence rates of the coefficients of the Green's operators as the neck radii vanish. We underline that, since different modifications are made on different scales, the precise definition of this operator varies according to context (the general framework, unifying these definitions, is described in Section 5.4). Next, we introduce *canonical extensions* of operators across the region within the neck, which allow the modified MCFS Jacobi operators of distinct rotationally symmetric Grim ends to be compared as if they were all defined over $\mathbb{R}^2$.

Notably, even with these modifications, the vanishing neck problem still imposes restrictions on the way in which ϵ and R tend respectively to 0 and infinity. Indeed, it is precisely at this stage that we require that ϵR^5 tend to infinity in the statement of Theorem A, for otherwise we could not guarantee decay in Lemmas 4.2.1 and 4.2.2.

The main result of these two sections is Theorem 4.1.1, which provides the required uniform estimates for the Green's operators of the modified MCFS Jacobi operators of rotationally symmetric Grim ends. We prove this result using a perturbation argument. To this end, we examine the differences between the modified MCFS Jacobi operators of Grim paraboloids and those of rotationally symmetric Grim ends. We decompose these differences into regular and singular components. In Section 4.2, we show that the operator norms of the regular components tend to 0 as ϵ tends to 0, and in Section 4.3, making use of the hybrid norm described above, we prove the same result for the singular components. In particular, we see that an adequate treatment of the vanishing neck problem already calls for the hybrid norm, which will play a larger role later on in the construction.

1.2.3. *Surgery and the deformation family.* In Section 5, we describe the surgery operation used to construct the approximate MCF soliton $\widehat{\Sigma}_{g,\epsilon}$ as well as the deformation family around this surface within which the actual MCF soliton $\Sigma_{g,\epsilon}$ will be found. The surgery operation is straightforward and is described in Section 5.1. Simply put, the ends of the CHM surface are amputated, suitably chosen rescaled rotationally symmetric Grim ends are grafted in their place, and the join is smoothed out using cut-off functions. The construction of the deformation family about $\widehat{\Sigma}_{g,\epsilon}$ is more technical and is carried out in Section 5.2.

The challenge in understanding (and explaining!) this construction arises from the fact that four different families of deformations must be considered. The first concerns deformations in the direction of a suitable modification of the unit normal vector field. We refer to the resulting first-order perturbations of the surface as *microscopic perturbations*, since they decay at infinity. The remaining three families involve variations of the logarithmic parameters of the ends, starting far inside the locus of surgery, and vertical translations of the ends, starting far inside and far outside the locus of surgery respectively. We refer to the resulting first-order perturbations as *macroscopic perturbations*, since they remain large at infinity.

We associate to each of the four classes of perturbation described above the operator of first-order variation of the MCFS functional about $\widehat{\Sigma}_{g,\epsilon}$. We denote these operators by $\hat{J}_\epsilon$, X_ϵ, Y_ϵ and Z_ϵ respectively. Understanding their analytic properties is key to estimating the norm of the Green's operator of the modified MCFS Jacobi operator of $\widehat{\Sigma}_{g,\epsilon}$, and we conclude this section by deriving preliminary estimates in Sections 5.3, 5.4 and 5.5.

1.2.4. *Constructing the Green's operator.* In Section 6, we construct a Green's operator of the modified MCFS Jacobi operator of $\widehat{\Sigma}_{g,\epsilon}$, together with estimates of its operator norm. This section constitutes the hardest technical part of the paper. The determination of sufficiently strong estimates is made possible, on the one hand, by the correct choice of functional norms over the different components of $\widehat{\Sigma}_{g,\epsilon}$ and, on the other, by the use of the hybrid norm described above.

The estimates for the norm of the Green's operator are obtained in Sections 6.3, 6.4 and 6.5 via a classical iteration process which we call the "ping-pong" argument. This process, which is common to all singular perturbation constructions, involves passing successive error terms back and forth over the join region. From a conformal perspective, the join region consists of cylinders which become very long as ϵ tends to 0. More explicitly, if $R = \epsilon^{-1/\lambda}$, then these cylinders are roughly of length $(\lambda - 1)\,\mathrm{Ln}(R)$. The estimates we require to ensure the convergence of the iteration process then follow from the fact that bounded harmonic functions decay exponentially over long cylinders. In particular, we maximise decay by choosing λ as large as possible. We have already seen in Section 1.2.2, above, that λ must be less than 5. It turns out that $\lambda \in \,]4, 5[$ is sufficient for our purposes, thus explaining the condition imposed in the statement of Theorem A. We believe that the ideas underlying this technique are best illustrated by the simplest version of this construction, used in the theory of Morse homology, and described in detail in Section 2.5 of [Schwarz 1993].

The first main results of this section are Theorems 6.3.1 and 6.4.1, which provide estimates for the norms of the operators used in the two stages of the iteration process. In addition, Theorems 6.5.2, 6.5.3 and 6.5.4 provide estimates for the norms of the different components of the Green's operator that we construct.

1.2.5. *Existence and embeddedness.* Finally, in Section 7 we prove Theorem A by applying the Schauder fixed-point theorem to the MCFS functional about the approximate MCF soliton $\widehat{\Sigma}_{g,\epsilon}$. First, we determine estimates for the MCFS functional up to second order about $\widehat{\Sigma}_{g,\epsilon}$. Then, using the estimates obtained in Section 6, we prove existence in Theorem 7.1.3, and we prove embeddedness in Theorem 7.1.4 using a straightforward geometric argument.

1.3. ***Notation.*** In order not to be overwhelmed by a deluge of constants, throughout the paper we use the following notation, which we have found to be of great help. First, given two variable quantities a and b, we will write

$$a \lesssim b \tag{1-2}$$

to mean that there exists a constant C, which for our purposes we consider universal, such that

$$a \leq Cb.$$

Next, given a function f and a sequence of functions (g_m), we will write

$$f = \mathrm{O}(g_m) \tag{1-3}$$

to mean that there exists a sequence (C_m) of constants, which for our purposes we again consider universal, such that the relation

$$|D^m f| \leq C_m g_m$$

holds pointwise for all m. The indexing variable of the sequence (g_m) should be clear from the context. In certain cases, every element of this sequence may be the same. It should also be clear from the context when this occurs. All other notation and terminology is explained in detail in Appendix A.

2. Rotationally symmetric Grim surfaces

2.1. ***The large scale.*** We define a *Grim surface* to be any unit-speed MCF soliton which is a graph over some domain in $\mathbb{R}^2$. We define a *Grim end* to be a Grim surface which is defined over some unbounded annulus $A(a, \infty)$. These will be studied in more detail in Section 4. In this section, we study rotationally symmetric Grim surfaces defined over some annulus $A(a, b)$. We first recall the general formula for such surfaces. Let u be a twice differentiable function defined over some closed interval $[a, b]$ and let Σ be the surface of revolution generated by rotating its graph about the z-axis. The principal curvatures of Σ in the radial and angular directions are respectively

$$c_r = \frac{-u_{rr}}{\sqrt{1+u_r^2}^3}, \quad c_\theta = \frac{-u_r}{r\sqrt{1+u_r^2}}, \tag{2-1}$$

where r here denotes the radial distance in $A(a, b)$ from the origin, and the subscript r denotes differentiation with respect to this variable. The vertical component of the upward-pointing, unit normal vector over Σ is

$$\langle N_\Sigma, e_z \rangle = \frac{1}{\sqrt{1+u_r^2}}, \tag{2-2}$$

so that, by (1-1), Σ is a rotationally symmetric Grim surface whenever

$$r u_{rr} + (u_r - r)(1 + u_r^2) = 0. \tag{2-3}$$

Solutions of this equation have no straightforward closed form. However, it will be sufficient for our purposes to obtain approximations by semiconvergent, that is, asymptotic, series. We first derive an asymptotic formula which is valid as r tends to infinity.

Theorem 2.1.1. *If* $u :]a, \infty[\to \mathbb{R}$ *is a solution of* (2-3) *then*, *as* $r \to +\infty$,

$$u = \tfrac{1}{2}r^2 - \log(r) + a + \mathrm{O}(r^{-(k+2)}) \tag{2-4}$$

for some real constant a.[2]

Theorem 2.1.1 follows immediately upon integrating (2-13), below. A similar result has already been obtained in [Clutterbuck et al. 2007]. However, we consider it worth deriving (2-4) in full, not only because we use different techniques, but also because we believe it serves as good preparation for the more subtle small-scale asymptotic estimates that will be studied in the following sections.

Define the nonlinear operator $\mathcal{G}$ by

$$\mathcal{G}v := rv_r + (v - r)(1 + v^2), \tag{2-5}$$

and observe that v solves $\mathcal{G}v = 0$ if and only if its integral is the profile of a rotationally symmetric Grim surface. We first derive formal solutions to (2-5). To this end, we define a *Laurent series* in the formal variable R to be a formal power series of the form

$$V := \sum_{m=-\infty}^{k} V_m R^m, \tag{2-6}$$

where, for all m, V_m is a real number and k is some finite integer, which we henceforth call the *order* of V. Since the operations of formal multiplication and formal differentiation are well-defined over the space of Laurent series, the operator $\mathcal{G}$ also has a well-defined action over this space.

Lemma 2.1.2. *There exists a unique Laurent series* V *such that* $\mathcal{G}V = 0$. *Furthermore*:

(1) V *has order* 1.

(2) $V_1 = 1$, $V_{-1} = -1$.

(3) *If* m *is even*, *then* $V_m = 0$.

(4) *If* $\widehat{V}_n := \sum_{m=1-2n}^{1} V_m R^m$ *denotes the* n*-th partial sum of* V, *then* $\mathcal{G}\widehat{V}_n$ *is a finite Laurent series of order* $(1 - 2n)$.

Proof. Consider the ansatz (2-6). If $k \leq -1$, then the highest-order term in $\mathcal{G}V$ is $(-R)$, if $k = 0$, then it is $(-R)(1 + V_0^2)$, and if $k \geq 2$, then it is $V_k^3 R^{3k}$. Since none of these vanish, it follows that V must be of order 1. In this case, the highest-order term in $\mathcal{G}V$ is $V_1^2(V_1 - 1)R^3$ so that, in order to have nontrivial solutions, we require $V_1 = 1$. We now have

$$R\frac{dV}{dR} + (V - R)(1 + V^2) = R + \sum_{m=-\infty}^{0} (m+1)V_m R^m + \sum_{m=-\infty}^{2} \Bigg(\sum_{\substack{p+q+r=m \\ p \leq 0,\ q,r \leq 1}} V_p V_q V_r \Bigg) R^m.$$

In particular, setting the respective coefficients of R^2 and R equal to 0 yields

$$V_0 = 0, \quad V_{-1} = -1.$$

[2]We refer the reader to Section 1.3 and Appendix A for a detailed review of the notation used here and throughout the sequel.

For all $m \leq -2$, setting the coefficient of R^{m+2} equal to 0 now yields

$$V_m + \Bigg(\sum_{\substack{p+q+r=m+2 \\ m+1 \leq p \leq -1 \\ m+2 \leq q,r \leq 1}} V_p V_q V_r \Bigg) + (m+3) V_{m+2} = 0. \tag{2-7}$$

The existence and uniqueness of V now follow from this recurrence relation. Furthermore, if $p+q+r = m+2$, and if m is even, then at least one of p, q and r is also even, and since $V_0 = 0$, it follows by induction that $V_m = 0$ for all even m. In addition, by (2-7), for all n, and for all $m \geq (3-2n)$, the coefficient of R^m in $\mathcal{G}\widehat{V}_n$ is equal to 0. However, since $V_{-2n} = 0$, by (2-7) again, the coefficient of R^{2-2n} in $\mathcal{G}\widehat{V}_n$ is also equal to 0, so that $\mathcal{G}\widehat{V}_n$ is a finite Laurent series of order $(1-2n)$, as desired. □

For all n, define the n-th partial sum $v_n :]0, \infty[\to \mathbb{R}$ by

$$v_n(r) := \sum_{m=1-2n}^{1} V_m r^m. \tag{2-8}$$

We now show that the sequence (v_n) yields successively better approximations over the large scale of the exact solutions of $\mathcal{G}v = 0$. We first derive zeroth order bounds.

Lemma 2.1.3. *If* $v : [a, \infty[\to \mathbb{R}$ *solves* (2-5) *then*, *for large* r,

$$|v_0 - v| \lesssim \frac{1}{r}. \tag{2-9}$$

Proof. Consider the family of polynomials $p_t(y) = (y-1)(t^2 + y^2)$. For all $t > 0$, $y = 1$ is the unique real root of p_t. Since $y = 0$ is the unique local maximum of p_0, for sufficiently small t, the unique local maximum of p_t is also near 0, and the value of p_t at this point is less than $-t^2/2$. Since p_0 is convex over the interval $\left[\frac{1}{3}, \infty\right[$, for $\frac{1}{3} < y < 1$ we have $p_0(y) \leqslant \frac{3}{2}(1-y)p_0\left(\frac{1}{3}\right) = \frac{1}{9}(y-1)$ and so, for sufficiently small t, over the smaller interval $\left[\frac{1}{2}, 1\right]$, $p_t(y) \leqslant \frac{1}{18}(y-1)$.

Now let v be a solution of $\mathcal{G}v = 0$. In particular, using a dot to denote differentiation with respect to r, we have $\dot{v} = -r^2 p_{1/r}(v/r)$. Suppose, furthermore, that $r \gg 1$ so that the estimates of the preceding paragraph hold for $p_{1/r}$. When $v \geq r$, we have $\dot{v} - \dot{r} = \dot{v} - 1 \leq -1$, so that, for sufficiently large r, $v(r) \leqslant r$. If $v \leqslant \frac{1}{2}r$, then $\dot{v} - \frac{1}{2}\dot{r} \geqslant \frac{1}{2}r - \frac{1}{2}$, so that, for sufficiently large r, $v(r) \geqslant \frac{1}{2}r$. Finally, if $\frac{1}{2}r \leqslant v \leqslant r$ then, by the preceding discussion, $\dot{v} \geqslant \frac{1}{18}r(r-v)$. It follows that if $w := r(v_0 - v) = r(r-v)$, then $w > 0$ and $\dot{w} = 2r - v - r\dot{v} \leqslant r + w/r - \frac{1}{18}rw$. Since this is negative for $w \geqslant 36$ and $r > 6$, the function w is bounded, and the result follows. □

Lemma 2.1.4. *If* $v : [a, \infty[\to \mathbb{R}$ *solves* (2-5) *then*, *for all* n, *and for large* r,

$$|v_n - v| \lesssim r^{-(2n+1)}. \tag{2-10}$$

Proof. For all n, let $w_n := r^{2n-1}(v_n - v)$ be the rescaled error. We prove by induction that $|w_n| \lesssim r^{-2}$ for all n. Indeed, the case $n = 0$ follows from (2-9). We suppose therefore that $n \geq 1$. Since $w_n = r^2 w_{n-1} + V_{1-2n}$, it follows by the inductive hypothesis that w_n is bounded. Now let $P(a, b)$ denote any polynomial in the variables a and b. Since $\mathcal{G}v_n$ is a finite Laurent polynomial of order $(1-2n)$, using a dot to denote differentiation with respect to r, we have

$$\begin{aligned}\dot{w}_n &= \frac{(2n-1)}{r}w_n + r^{2n-2}(r\dot{v}_n - r\dot{v})\\ &= \frac{1}{r}P\Big(\frac{1}{r},\, w_n\Big) - r^{2n-2}\big((v_n-r)(1+v_n^2)-(v-r)(1+v^2)\big)\\ &= \frac{1}{r}P\Big(\frac{1}{r},\, w_n\Big) - \frac{1}{r}w_n(1-r(v_n+v)+(v_n^2+v_nv+v^2)).\end{aligned}$$

Since $v = v_n - r^{-(2n-1)}w_n$ and since $(v_n - r)$ is also a polynomial in r^{-1} with no constant term, this yields

$$\dot{w}_n = \frac{1}{r}P\Big(\frac{1}{r},\, w_n\Big) - rw_n.$$

Since w_n is bounded, there therefore exists a constant $B > 0$ such that, for all $r \geqslant 1$,

$$|\dot{w}_n + rw_n| \leqslant Br^{-1}. \tag{2-11}$$

In particular, for $r \geq 2$ and $r^2 w_n \geq 2B$,

$$\frac{d}{dr}r^2 w_n = r^2(\dot{w}_n + rw_n) + (2r - r^3)w_n \leq Br - \tfrac{1}{2}r^3 w_n \leq 0,$$

so that $r^2 w_n$ is bounded from above. Since $(-w_n)$ also satisfies (2-11), we see that $r^2 w_n$ is bounded from below, and this completes the proof. □

Lemma 2.1.5. *If* $v : [a, \infty[\to \mathbb{R}$ *solves* (2-5) *then*, *for all* n,

$$v_n - v = \mathrm{O}(r^{-(k+2n+1)}). \tag{2-12}$$

In particular,

$$v = r - \frac{1}{r} + \mathrm{O}(r^{-(k+3)}). \tag{2-13}$$

Proof. For all n, define $w_n := (v_n - v)$. As in the proof of Lemma 2.1.4, we obtain

$$\dot{w}_n = P_1\Big(\frac{1}{r},\, w_n\Big)rw_n + \frac{1}{r}\mathcal{G}v_n,$$

where P_1 is some polynomial. Since $\mathcal{G}v_n$ is a finite Laurent polynomial of order $(1-2n)$, it follows by induction that, for all k,

$$\frac{d^k w_n}{dr^k} = P_k\Big(\frac{1}{r},\, w_n\Big)r^k w_n + Q_k\Big(\frac{1}{r},\, w_n\Big)r^{k-(2n+1)},$$

where P_k and Q_k are polynomials. It follows by (2-10) that, for all k,

$$\left|\frac{d^k}{dr^k}(v_n - v)\right| = \left|\frac{d^k w_n}{dr^k}\right| \lesssim r^{k-(2n+1)}.$$

However, since $(v_{n+k} - v_n)$ is a finite Laurent series of order $-(2n+1)$, for all k,

$$\left|\frac{d^k}{dr^k}(v_n - v)\right| \leq \left|\frac{d^k}{dr^k}(v_n - v_{n+k})\right| + \left|\frac{d^k}{dr^k}(v_{n+k} - v)\right| \lesssim r^{-(k+2n+1)},$$

and the result follows. □

2.2. ***The small scale: formal solutions.*** We now study solutions to (2-5) over the small scale. We fix positive constants $K \gg 1$ and $\eta \ll 1$ which we henceforth consider to be universal. Let Λ be a large, positive real number, and let ϵ, $R > 0$ and $c \in \mathbb{R}$ be such that

$$\left(\epsilon R^{4+\eta} + \frac{1}{R^{1-\eta}}\right) \leq \frac{1}{\Lambda}, \quad \epsilon R^{5-\eta} \geq \Lambda, \quad |c| \leq K. \tag{2-14}$$

These conditions will be used repeatedly throughout the paper. Observe, in particular, that (2-14) implies that ϵ becomes small and R becomes large as Λ tends to infinity. We will prove:

Theorem 2.2.1. *For all sufficiently large Λ, and for all R, ϵ satisfying* (2-14), *there exists a smooth function $\sigma[\epsilon, R] : \mathbb{R} \to \mathbb{R}$ such that, for all $c \in [-K, K]$, if $v : [\epsilon R, \epsilon R^4] \to \mathbb{R}$ solves $\mathcal{G}v = 0$ with initial value*

$$v(\epsilon R) = \frac{1}{R}\sigma[\epsilon, R](c) + \frac{\epsilon R}{2}, \tag{2-15}$$

then

$$v(r) = \frac{1}{2}r + \frac{c\epsilon}{r} + \mathrm{O}\left(\left[1 + \log\left(\frac{r}{\epsilon R}\right)\right]\frac{1}{r^k}\left(r + \frac{\epsilon}{r}\right)^3\right). \tag{2-16}$$

Furthermore, the function $\sigma[\epsilon, R]$ converges to the identity in the C^∞_{loc} sense as Λ tends to $+\infty$.

Remark. We leave the reader to verify that the same conclusion also holds over the interval $[\epsilon R, C\epsilon R^4]$ for any constant C not depending on ϵ or R.

The function $\sigma[\epsilon, R]$ will be defined explicitly in Section 2.3, below, and Theorem 2.2.1 will follow immediately from Lemma 2.4.2, below. The constant c will henceforth be referred to as the *logarithmic parameter* of the function v. Observe that, up to a small perturbation, it is related to the initial value of v by a linear function. This perturbation is required in order to guarantee good estimates over the whole interval. Indeed, replacing $\sigma[\epsilon, R](c)$ by c in (2-15) would increase the error in (2-16), making it then of order (ϵ/r).

In order to appreciate Theorem 2.2.1 and the argument that follows, we find it helpful to first recall the geometric properties of the function v over the interval $[\epsilon R, \epsilon R^4]$. Indeed, by definition, its integral u is the profile of some rotationally symmetric Grim surface. However, it is known (see [Clutterbuck et al. 2007]) that, near the lower end of this interval, the first term in the MCFS equation (1-1) dominates, so that the graph of u is close to some minimal catenoid in $\mathbb{R}^3$ and the function u is itself approximately logarithmic. On the other hand, near the upper end of this interval, it is the second term in the MCFS equation which dominates, and the function u is approximately quadratic, in accordance with the asymptotic formula obtained in the preceding section. These two contrasting behaviours are reflected in (2-16) by the ϵ/r terms and the r terms respectively.

In order to derive an asymptotic formula for u that simultaneously describes these two behaviours, we introduce two abstract variables M and N, where M measures its quadratic behaviour, and N measures its logarithmic behaviour. By expressing the equation $\mathcal{G}v = 0$ in terms of these new variables, the asymptotic formula for v is then obtained in the same manner as in Section 2.1 namely, by first determining formal solutions which then serve as approximations for exact solutions.

Upon applying the change of variables $r := \epsilon R e^x$ we obtain

$$\mathcal{G}v = \mathcal{D}v - \epsilon R e^x + (v - \epsilon R e^x)v^2, \tag{2-17}$$

where the operator $\mathcal{D}$ is defined by

$$\mathcal{D}v := v_x + v, \tag{2-18}$$

and the subscript x here denotes differentiation with respect to this variable. Now let $\mathbb{R}[X, M, N]$ be the ring of polynomials with real coefficients in the variables X, M and N. We consider a general element V of $\mathbb{R}[X, M, N]$ as a sum of the form

$$V = \sum_{p,q \leq k} V_{p,q}(X) M^p N^q, \tag{2-19}$$

where, for all p, q, $V_{p,q}$ is a polynomial in the variable X and k is some finite, nonnegative integer which we henceforth refer to as the *order* of V. There is a natural correspondence sending $\mathbb{R}[X, M, N]$ into the space of continuous functions over $[0, 3\log(R)]$ given by

$$V \mapsto v(x) := \sum_{p,q \leq k} V_{p,q}(x)(\epsilon R e^x)^p \Big(\frac{c}{R} e^{-x}\Big)^q. \tag{2-20}$$

In other words, this correspondence is the unique $\mathbb{R}[X]$-ring homomorphism which sends M to $\epsilon R e^x$ and N to $(c/R)e^{-x}$. Although this homomorphism is not injective, it keeps track of the parameters ϵ, R and c, which is the reason why it serves our purposes. Operators $\mathcal{G}$ and $\mathcal{D}$ are also defined over $\mathbb{R}[X, M, N]$ by

$$\begin{aligned} \mathcal{G}V &:= \mathcal{D}V - M + (V - M)V^2, \\ (\mathcal{D}V)_{p,q} &:= \Big(\frac{d}{dX} + 1 + (p - q)\Big) V_{p,q}, \end{aligned} \tag{2-21}$$

where $d/(dX)$ here denotes the operator of formal differentiation with respect to the variable X. In particular, $\mathcal{G}$ and $\mathcal{D}$ both map through the above correspondence to the operators given in (2-17) and (2-18) respectively, thereby justifying this notation. Observe, furthermore, that $\mathcal{D}$ defines a surjective linear map from $\mathbb{R}[X, M, N]$ to itself and that its kernel consists of finite sums of the form

$$V = \sum_{p \leq k} a_p M^p N^{p+1},$$

where $a_0, \ldots, a_k$ are real constants.

Let $\mathbb{R}[X][\![M, N]\!]$ be the ring of formal power series over the variables M and N with coefficients that are polynomials in the variable X. Observe that the operators $\mathcal{G}$ and $\mathcal{D}$ naturally extend again to well-defined operators over this space.

Lemma 2.2.2. *There exists a unique formal power series V in $\mathbb{R}[X][\![M, N]\!]$ such that*

(1) $V_{0,1} = 1$,

(2) $V_{p,p+1}(0) = 0$ *for all* $p \geq 1$, *and*

(3) $\mathcal{G}V = 0$.

Furthermore,

(4) $V_{1,0} = \frac{1}{2}$,

(5) *if* $p + q$ *is even*, *then* $V_{p,q} = 0$, *and*

(6) *if* $p + q = 2k + 1$ *is odd*, *then* $V_{p,q}$ *has order at most* k *in* X.

Finally, if we define

$$\widehat{V}_k := \sum_{p+q\leq 2k+1} V_{p,q}(X)M^pN^q,$$

then,

(7) *if* $(p+q)\leq(2k+1)$, *then the coefficient of* M^pN^q *in* $\mathcal{G}\widehat{V}_k$ *vanishes,*

(8) *if* $(p+q)>(2k+1)$ *is even, then the coefficient of* M^pN^q *in* $\mathcal{G}\widehat{V}_k$ *vanishes, and*

(9) *if* $(p+q)>(2k+1)$ *is odd, then the coefficient of* M^pN^q *in* $\mathcal{G}\widehat{V}_k$ *has order at most* $\frac{1}{2}(p+q-3)$ *in* X.

Proof. Let $V=\sum_{p,q}V_{p,q}(X)M^pN^q$ be an element of $\mathbb{R}[X][[M,N]]$ which solves $\mathcal{G}V=0$. For all (p,q), equating the coefficient of M^pN^q in $\mathcal{G}V$ to 0, we obtain

$$\left(\frac{d}{dX}+(1+(p-q))\right)V_{p,q}=\delta_{p1}\delta_{q0}-\sum_{\substack{p_1+p_2+p_3=p\\ q_1+q_2+q_3=q}}V_{p_1,q_1}V_{p_2,q_2}V_{p_3,q_3}+\sum_{\substack{p_1+p_2=p-1\\ q_1+q_2=q}}V_{p_1,q_1}V_{p_2,q_2}. \tag{2-22}$$

In particular,

$$\frac{dV_{0,0}}{dX}+V_{0,0}(1+V_{0,0}^2)=0,$$

and since there exists no nontrivial polynomial solution to this equation, it follows that $V_{0,0}=0$. From this it follows that the two summations on the right-hand side of (2-22) only involve terms of order at most $p+q-2$ in (M,N). In particular, $V_{0,1}$ satisfies

$$\frac{dV_{0,1}}{dX}=0.$$

It is thus constant, and we henceforth set it equal to 1. It now follows by induction that there exists a unique sequence of polynomials $(V_{p,q})$ satisfying (2-22) such that $V_{0,1}=1$ and $V_{p,p+1}(0)=0$ for all $p\geq 1$.

To prove (4), observe that $V_{1,0}$ satisfies $dV_{1,0}/(dX)+2V_{1,0}=1$ so that, since it is a polynomial, $V_{1,0}=\frac{1}{2}$, as desired. To prove (5), observe that if $p+q$ is even, then every summand on the right-hand side in (2-22) involves at least one term of the form $V_{p',q'}$, where $p'+q'$ is an even number no greater than $p+q-2$. Since $V_{0,0}=0$, it follows by induction that $V_{p,q}=0$ whenever $p+q$ is even, as desired. To prove (6), suppose that for all $l<k$, and for $p+q=2l+1$, the polynomial $V_{p,q}$ has order at most l in X. By (2-22), for all $p+q=2k+1$, the polynomial $V_{p,q}$ is obtained by integrating terms of order at most $(k-1)$ in X, and it follows by induction that $V_{p,q}$ has order at most k in X, as desired.

Finally, observe that, by (2-22), the term $V_{p,q}$ is defined by setting the coefficient of M^pN^q equal to 0 in $\mathcal{G}V$, and (7) follows. Furthermore, for $p+q>(2k+1)$, the coefficient of M^pN^q in $\mathcal{G}V$ is equal to the right-hand side of (2-22). Items (8) and (9) now follow by similar arguments used to prove (5) and (6), above, and this completes the proof. □

2.3. *The small scale: exact solutions, I.* Let V be the formal power series constructed in Lemma 2.2.2. For ϵ, R satisfying (2-14), for $c\in\mathbb{R}$, and for nonnegative, integer k, let $v_{k,c}$ be the k-th partial sum of V with logarithmic parameter c, that is,

$$v_{k,c}(x):=\sum_{p+q\leq 2k+1}V_{p,q}(x)(\epsilon Re^x)^p\left(\frac{c}{R}e^{-x}\right)^q. \tag{2-23}$$

Define the function $\sigma[\epsilon, R, k] : \mathbb{R} \to \mathbb{R}$ by

$$\sigma[\epsilon, R, k](c) := R v_{k,c}(0) - \frac{\epsilon R^2}{2}. \tag{2-24}$$

Trivially, if $v : [0, 3\log(R)] \to \mathbb{R}$ satisfies

$$v(0) = \frac{1}{R}\sigma[\epsilon, R, k](c) + \frac{\epsilon R}{2},$$

then v has the same initial value as $v_{k,c}$. Observe that $\sigma[\epsilon, R, k]$ is a polynomial in c with coefficients that depend on ϵ, R and k and, for all k, $\sigma[\epsilon, R, k]$ converges to the identity in the C^∞_{loc} sense as Λ tends to infinity. We will see presently that the estimates we require follow when k is at least 9, and we therefore define

$$\sigma[\epsilon, R](c) := \sigma[\epsilon, R, 9](c). \tag{2-25}$$

This is the function that appears in the statement of Theorem 2.2.1.

As in Section 2.1, we now determine zeroth order bounds for the difference between $v_{k,c}$ and an exact solution with the same initial value. We achieve this via the contraction mapping theorem. We first introduce the required analytic framework. For $T \in [0, 3\log(R)]$, let $C^0([0, T])$ be the Banach space of continuous functions over the interval $[0, T]$ furnished with the uniform norm and let $C^1_0([0, T])$ be the Banach space of continuously differentiable functions over this interval with initial value 0, furnished with the norm

$$\|w\|_{C^1_0} := \|w_x\|_{C^0}, \tag{2-26}$$

where the subscript x here denotes differentiation with respect to this variable. Observe that, for all $w \in C^1_0([0, T])$,

$$\|w\|_{C^0} \leq T\|w\|_{C^1_0}. \tag{2-27}$$

Lemma 2.3.1. *The operator $\mathcal{D}$ defines a linear isomorphism from $C^1_0([0, T])$ into $C^0([0, T])$. Furthermore, the operator norms of $\mathcal{D}$ and its inverse satisfy*

$$\|\mathcal{D}\| \leq 1 + T, \quad \|\mathcal{D}^{-1}\| \leq 2. \tag{2-28}$$

Proof. First, bearing in mind (2-27),

$$\|\mathcal{D}w\|_{C^0} \leq \|w_x\|_{C^0} + \|w\|_{C^0} \leq (1 + T)\|w\|_{C^1_0},$$

so that $\|\mathcal{D}\| \leq 1 + T$. By inspection, for all w,

$$(\mathcal{D}^{-1}w)(x) = e^{-x}\int_0^x e^y w(y)\,dy.$$

In particular,

$$\|\mathcal{D}^{-1}w\|_{C^0} \leq \|w\|_{C^0}.$$

Thus,

$$\|\mathcal{D}^{-1}w\|_{C^1_0} = \|(\mathcal{D}^{-1}w)_x\|_{C^0} \leq \|\mathcal{D}\mathcal{D}^{-1}w\|_{C^0} + \|\mathcal{D}^{-1}w\|_{C^0} \leq 2\|w\|_{C^0},$$

so that $\|\mathcal{D}^{-1}\| \leq 2$. ☐

Consider now the functional $\mathcal{H} : C^1_0([0, T]) \to C^0([0, T])$ given by

$$\mathcal{H}(w) := \mathcal{G}(v_k + w). \tag{2-29}$$

Its Fréchet derivative at w is

$$D\mathcal{H}(w) f := \mathcal{D} f + \mathcal{E}(w) f, \tag{2-30}$$

where

$$\mathcal{E}(w) f := 3(v_{k,c} + w)^2 f - 2\epsilon Re^x (v_{k,c} + w) f. \tag{2-31}$$

Lemma 2.3.2. *For all $w \in C^1_0([0, T])$, the operator norm of $\mathcal{E}(w)$, considered as a linear map from $C^1_0([0, T])$ into $C^0([0, T])$, satisfies*

$$\|\mathcal{E}(w)\| \lesssim T\Big((\epsilon Re^T)^2 + \frac{1}{R^2} + T^2\|w\|^2_{C^1_0}\Big). \tag{2-32}$$

Proof. Indeed, over $[0, T]$,

$$\|\epsilon Re^x\|_{C^0} \le \epsilon Re^T, \qquad \Big\|\frac{c}{R}e^{-x}\Big\|_{C^0} \le \frac{c}{R}.$$

Thus, by Lemma 2.2.2 and (2-14),

$$\|v_k\|_{C^0} \lesssim \sum_{i=0}^{k}(1 + T^i)\Big(\epsilon Re^T + \frac{1}{R}\Big)^{2i+1} \lesssim \epsilon Re^T + \frac{1}{R},$$

so that, by (2-27) and (2-31),

$$\begin{aligned}\|\mathcal{E}(w) f\|_{C^0} &\lesssim \Big((\epsilon Re^T)^2 + \frac{1}{R^2} + \|w\|^2_{C^0}\Big)\|f\|_{C^0}\\ &\lesssim T\Big((\epsilon Re^T)^2 + \frac{1}{R^2} + T^2\|w\|^2_{C^1_0}\Big)\|f\|_{C^1_0},\end{aligned}$$

as desired. □

Define the map $\Phi : C^1_0([0, T]) \to C^1_0([0, T])$ by

$$\Phi(w) := w - \mathcal{D}^{-1}\mathcal{H}(w). \tag{2-33}$$

Lemma 2.3.3. *For $w, \bar{w} \in C^1_0([0, T])$,*

$$\|\Phi(w) - \Phi(\bar{w})\|_{C^1_0} \lesssim T\Big((\epsilon Re^T)^2 + \frac{1}{R^2} + T^2\|w\|^2_{C^1_0} + T^2\|\bar{w}\|^2_{C^1_0}\Big)\|w - \bar{w}\|_{C^1_0}. \tag{2-34}$$

Proof. Indeed, for $w, \bar{w} \in C^1_0([0, T])$, using (2-30),

$$\begin{aligned}\Phi(w) - \Phi(\bar{w}) &= w - \bar{w} - \mathcal{D}^{-1}(\mathcal{H}(w) - \mathcal{H}(\bar{w}))\\ &= -\mathcal{D}^{-1}(\mathcal{H}(w) - \mathcal{H}(\bar{w}) - \mathcal{D}(w - \bar{w}))\\ &= -\mathcal{D}^{-1}\Big(\int_0^1 \mathcal{E}(tw + (1-t)\bar{w})\, dt\Big)(w - \bar{w}).\end{aligned}$$

Thus, by (2-28) and (2-32),

$$\|\Phi(w) - \Phi(\bar{w})\|_{C^1_0} \lesssim T\Big((\epsilon Re^T)^2 + \frac{1}{R^2} + T^2\|w\|^2_{C^1_0} + T^2\|\bar{w}\|^2_{C^1_0}\Big)\|w - \bar{w}\|_{C^1_0},$$

as desired. □

Applying the contraction mapping theorem now yields:

Lemma 2.3.4. *For sufficiently large Λ, if $v_{k,c}$ is the k-th partial sum of V with logarithmic parameter c, and if $v : [0, 3\log(R)] \to \mathbb{R}$ solves $\mathcal{G}v = 0$ with initial value*

$$v(0) = \frac{1}{R}\sigma[\epsilon, R, k](c) + \frac{\epsilon R}{2}, \tag{2-35}$$

then

$$\|v - v_{k,c}\|_{C^0} \lesssim (1 + T^{k+1})\Big(\epsilon R e^T + \frac{1}{R}\Big)^{2k+3}. \tag{2-36}$$

Proof. By Lemma 2.2.2,

$$\|\mathcal{G}v_k\|_{C^0} \lesssim (1 + T^k)\Big(\epsilon R e^T + \frac{1}{R}\Big)^{2k+3}.$$

By (2-28), there therefore exists $B > 0$, which we may consider to be universal, such that

$$\|\Phi(0)\|_{C^1_0} = \|\mathcal{D}^{-1}\mathcal{G}v_k\|_{C^1_0} \le B(1 + T^k)\Big(\epsilon R e^T + \frac{1}{R}\Big)^{2k+3}.$$

Let X be the closed ball of radius $2B(1 + T^k)(\epsilon R e^T + R^{-1})^{2k+3}$ about 0 in $C^1_0([0, T])$. By (2-14), if $w, \bar{w} \in X$ then, in particular,

$$T\|w\|_{C^1_0}, T\|\bar{w}\|_{C^1_0} \lesssim \Big(\epsilon R e^T + \frac{1}{R}\Big),$$

so that, by (2-34) and (2-14) again,

$$\|\Phi(w) - \Phi(\bar{w})\|_{C^1_0} \lesssim \frac{1}{\Lambda}\|w - \bar{w}\|_{C^1_0}.$$

The map Φ thus defines a contraction from X to itself, and there therefore exists $w \in X$ such that $\Phi(w) = w$. In particular $\mathcal{H}w = 0$, and

$$\|w\|_{C^0} \le T\|w\|_{C^1_0} \lesssim (1 + T^{k+1})\Big(\epsilon R e^T + \frac{1}{R}\Big)^{2k+3}.$$

Finally, by the definition of the function $\sigma[\epsilon, R, k]$, we have $v(0) = v_{k,c}(0)$ so that, by uniqueness of solutions to ODEs with prescribed initial values, $v - v_{k,c} = w$, and the result follows. □

2.4. *The small scale: exact solutions, II.* The final step in proving Theorem 2.2.1 involves extending the estimates obtained in Lemma 2.3.4 to derivatives of all orders.

Lemma 2.4.1. *If $v_{k,c}$ and v are as in Lemma 2.3.4, then*

$$v = v_{k,c} + \mathrm{O}\Big((1 + T^{k+1})\Big(\epsilon R e^T + \frac{1}{R}\Big)^{2k+3}\Big). \tag{2-37}$$

Proof. Define $w := v - v_{k,c}$. Since $v_{k,c}$ is a polynomial in $\epsilon R e^x$ and $(c/R)e^{-x}$ with coefficients in $\mathbb{R}[X]$, as in the proof of Lemma 2.1.4,

$$w_x = P_1\Big(w, \epsilon R e^x, \frac{c}{R}e^{-x}\Big)w + \mathcal{G}v_{k,c}$$

for some polynomial P_1 with coefficients in $\mathbb{R}[X]$. Since $\mathcal{G}v_{k,c}$ is also a polynomial in ϵRe^x and $(c/R)e^{-x}$ with coefficients in $\mathbb{R}[X]$, it follows by induction that, for all l,

$$\frac{d^l}{dx^l}w = P_l\Big(w, \epsilon Re^x, \frac{c}{R}e^{-x}\Big)w + \sum_{p=0}^{l-1} Q_{p,l}\Big(w, \epsilon Re^x, \frac{c}{R}\Big)\frac{d^p}{dx^p}\mathcal{G}v_{k,c} \tag{2-38}$$

for suitable polynomials P_l and $(Q_{p,l})_{0\leq p\leq l-1}$ also with coefficients in $\mathbb{R}[X]$. However, by (2-36),

$$\|w\|_{C^0} \lesssim (1+T^{k+1})\Big(\epsilon Re^T + \frac{1}{R}\Big)^{2k+3}.$$

Thus, by (2-14),

$$\Big\|P_l\Big(w, \epsilon Re^x, \frac{c}{R}e^{-x}\Big)\Big\|_{C^0}, \Big\|Q_{p,l}\Big(\epsilon Re^x, \frac{c}{R}e^{-x}\Big)\Big\|_{C^0} \lesssim 1.$$

Finally, Lemma 2.2.2 and (2-14) again,

$$\Big\|\frac{d^{l-1}}{dx^{l-1}}\mathcal{G}v_k\Big\|_{C^0} \lesssim (1+T^k)\Big(\epsilon Re^T + \frac{1}{R}\Big)^{2k+3},$$

and the result follows upon combining these relations. □

Lemma 2.4.2. *If $v_{k,c}$ is the k-th partial sum of V with logarithmic parameter c, and if $v : [0, 3\log(R)] \to \mathbb{R}$ solves $\mathcal{G}v = 0$ with initial value*

$$v(0) = \frac{1}{R}\sigma[\epsilon, R, 4k+9](c) + \frac{\epsilon R}{2}, \tag{2-39}$$

then, for sufficiently large Λ,

$$v = v_{k,c} + \mathrm{O}\Big((1+x^{k+1})\Big(\epsilon Re^x + \frac{1}{R}e^{-x}\Big)^{2k+3}\Big). \tag{2-40}$$

Remark. Since $r = \epsilon Re^x$, by the chain rule,

$$\frac{d}{dr} = \frac{1}{r}\frac{d}{dx},$$

so that Theorem 2.2.1 follows immediately from (2-40) upon setting $k = 0$.

Proof. For nonnegative, integer l, if $v : [0, 3\log(R)] \to \mathbb{R}$ solves $\mathcal{G}v = 0$ with initial value as in (2-35) then, since (2-37) holds for all $T \in [0, 3\log(R)]$,

$$v = v_{l,c} + \mathrm{O}\Big((1+x^{l+1})\Big(\epsilon Re^x + \frac{1}{R}\Big)^{2l+3}\Big).$$

In particular, if $v : [0, 3\log(R)] \to \mathbb{R}$ now solves $\mathcal{G}v = 0$ with initial value given by (2-39), then, bearing in mind (2-14),

$$v = v_{4k+9,c} + \mathrm{O}\Big((1+x^{k+1})\Big(\epsilon Re^x + \frac{1}{R}e^{-x}\Big)^{2k+3}\Big).$$

However, by Lemma 2.2.2 and (2-14) again,

$$v_{4k+9,c} = v_{k,c} + \mathrm{O}\Big((1+x^{k+1})\Big(\epsilon Re^x + \frac{1}{R}e^{-x}\Big)^{2k+3}\Big),$$

and the result follows. □

2.5. *The small scale: solutions of the linearised equation.* We conclude this section by studying how solutions of the equation $\mathcal{G}v = 0$ vary with the logarithmic parameter c.

Theorem 2.5.1. *For sufficiently large* Λ *and for all* R, ϵ *satisfying* (2-14), *if*, *for all* $c \in [-K, K]$, *the function* $v_c : [\epsilon R, \epsilon R^4] \to \mathbb{R}$ *solves* $\mathcal{G}v_c = 0$ *with initial value*

$$v_c(\epsilon R) = \frac{1}{R}\sigma[\epsilon, R](c) + \frac{\epsilon R}{2},$$

then

$$\frac{dv_c}{dc}(r) = \frac{\epsilon}{r} + \mathrm{O}\Big(\Big[1 + \log\Big(\frac{r}{\epsilon R}\Big)\Big]\frac{1}{r^k}\Big(r + \frac{\epsilon}{r}\Big)^3\Big). \tag{2-41}$$

Theorem 2.5.1 follows from (2-49), below, via reasoning similar to that used in Section 2.4. It suffices to study solutions of the linearisation of $\mathcal{G}$ about v, the asymptotic properties of which are readily derived from the analysis of the previous sections. Indeed, let $\mathbb{R}[X][[M, N]]$ be as in Section 2.2 and define the operator ∂_N over this space by

$$(\partial_N V)_{p,q} := (q+1)V_{p,q+1}. \tag{2-42}$$

In other words, ∂_N is simply the operator of formal differentiation with respect to N. By explicit calculation, $N\partial_N$ commutes with $\mathcal{D}$. Now let V be the formal power series constructed in Lemma 2.2.2 and define

$$W := N\partial_N V. \tag{2-43}$$

Applying $N\partial_N$ to the relation $\mathcal{G}V = 0$ yields

$$\mathcal{D}W + 3V^2W - 2MVW = 0, \tag{2-44}$$

so that W is a formal solution to the linearisation of $\mathcal{G}$ about the formal series V.

Fix a nonnegative integer k, let $\widehat{V}_k$ be as in Lemma 2.2.2 and define

$$\widehat{W}_k := \sum_{p+q\leq 2k+1} W_{p,q}(X)M^pN^q. \tag{2-45}$$

By (2-44),

$$\mathcal{D}\widehat{W}_k + 3\widehat{V}_k^2\widehat{W}_k - 2M\widehat{V}_k\widehat{W}_k = \mathrm{O}((M+N)^{2k+3}). \tag{2-46}$$

Consider now $\Lambda, K > 0$, let $\epsilon, R > 0$ and $c \in \mathbb{R}$ satisfy (2-14), and let $v_{k,c}$ and $w_{k,c}$ be the functions corresponding to $\widehat{V}_k$ and $\widehat{W}_k$ respectively. By (2-46), for all k,

$$\mathcal{D}w_{k,c} + 3v_{k,c}^2 w_{k,c} - 2(\epsilon Re^x)v_{k,c}w_{k,c} = \mathrm{O}\Big(x^{k+1}\Big(\epsilon Re^x + \frac{1}{R}e^{-x}\Big)^{2k+3}\Big). \tag{2-47}$$

Lemma 2.5.2. *For sufficiently large* Λ *and for all* $T \in [0, 3\log(R)]$, *if* $v : [0, T] \to \mathbb{R}$ *solves* $\mathcal{G}v = 0$ *with initial value*

$$v(0) = \frac{1}{R}\sigma[\epsilon, R, k](c) + \frac{\epsilon R}{2},$$

and if $w : [0, T] \to \mathbb{R}$ *solves*

$$\mathcal{D}w + 3v^2w - 2\epsilon Re^x vw = 0 \tag{2-48}$$

with initial value $w(0) = w_{k,c}(0)$, *then*

$$\|w - w_{k,c}\|_{C^1_0} \lesssim (1+T)^{k+1}\Big(\epsilon Re^T + \frac{1}{R}\Big)^{2k+3}. \tag{2-49}$$

Proof. Indeed, by (2-47),

$$\|\mathcal{D}w_{k,c} + 3v_{k,c}^2 w_{k,c} - 2(\epsilon Re^x)v_{k,c}w_{k,c}\| \lesssim (1+T)^{k+1}\Big(\epsilon Re^T + \frac{1}{R}\Big)^{2k+3}.$$

Observe that

$$\|v\|_{C^0},\ \|v_{k,c}\|_{C^0},\ \|w_{k,c}\|_{C^0} \lesssim 1.$$

Thus, by (2-36),

$$\begin{aligned}\|(3v_{k,c}^2 - 3v^2)w_{k,c}\|_{C^0} &= 3\|(v_{k,c} - v)(v_{k,c} + v)w_{k,c}\|_{C^0}\\ &\lesssim (1+T)^{k+1}\Big(\epsilon Re^T + \frac{1}{R}\Big)^{2k+3}.\end{aligned}$$

Likewise

$$\|(2(\epsilon Re^x)v_{k,c} - 2(\epsilon Re^x)v)w_{k,c}\|_{C^0} \lesssim (1+T)^{k+1}\Big(\epsilon Re^T + \frac{1}{R}\Big)^{2k+3}.$$

Thus

$$\begin{aligned}\|\mathcal{D}(w_{k,c}-w)+3v^2(w_{k,c}-w)-2(\epsilon Re^x)v(w_{k,c}-w)\|_{C^0} &= \|\mathcal{D}w_{k,c}+3v^2w_{k,c}-2(\epsilon Re^x)vw_{k,c}\|_{C^0}\\ &\lesssim (1+T)^{k+1}\Big(\epsilon Re^T+\frac{1}{R}\Big)^{2k+3}.\end{aligned} \tag{2-50}$$

Observe now that, for all $\phi : [0, T] \to \mathbb{R}$,

$$3v^2\phi - 2\epsilon Re^x\phi = \mathcal{E}(v - v_{k,c})\phi,$$

where $\mathcal{E}$ is given by (2-31). In particular, by (2-14), (2-32) and (2-36), the operator norm of $\mathcal{E}(v - v_{k,c})$ considered as a map from $C^1_0([0, T])$ into $C^0([0, T])$ satisfies

$$\|\mathcal{E}(v - v_{k,c})\| \lesssim T\Big((\epsilon Re^T)^2 + \frac{1}{R^2}\Big).$$

Thus, by (2-28), for sufficiently large Λ, the operator $\mathcal{D} + \mathcal{E}(v - v_{k,c})$ defines an invertible map from $C^1_0([0, T])$ into $C^0([0, T])$ and the result now follows by (2-50). □

Theorem 2.5.1 now follows as indicated above. In addition, a further iteration of this process also yields:

Theorem 2.5.3. *With the same hypotheses as in Theorem 2.5.1*,

$$\frac{d^2v_c}{dc^2}(r) = \mathrm{O}\Big(\Big[1 + \log\Big(\frac{r}{\epsilon R}\Big)\Big]\frac{1}{r^k}\Big(r + \frac{\epsilon}{r}\Big)^3\Big). \tag{2-51}$$

3. The Grim paraboloid

3.1. *The MCFS Jacobi operator.* The *Grim paraboloid*, which we henceforth denote by G_0, is defined to be the unique rotationally symmetric MCF soliton which is a graph over the whole of $\mathbb{R}^2$. Put differently, using the notation of Section 2, there is a unique solution v to the ODE $\mathcal{G}v = 0$ which is defined over the whole interval $]0,\infty[$. This solution tends to 0 as x tends to 0, and the Grim paraboloid is the surface of revolution generated by rotating the graph of its integral about the z-axis.

Let J be the MCFS Jacobi operator of the Grim paraboloid as defined in Section A2. In this section, we show that this operator defines a linear isomorphism over suitably weighted Sobolev and Hölder spaces. We first describe the spaces of interest to us (see Section A4 for details). Let g denote the metric induced over $\mathbb{R}^2$ by the graph G_0, that is,

$$g := (1+v^2)\,dr^2 + r^2\,d\theta^2. \tag{3-1}$$

For all nonnegative, integer m, let $\|\cdot\|_{H^m(G)}$ denote the *Sobolev norm* of order m of functions over $\mathbb{R}^2$ with respect to this metric. Likewise, for all nonnegative, integer m, and, for all $\alpha \in [0,1]$, let $\|\cdot\|_{C^{m,\alpha}(G)}$ denote the *Hölder norm* of order (m,α) of functions over $\mathbb{R}^2$ with respect to this metric. Observe that, by (2-4), these Sobolev and Hölder norms are uniformly equivalent to the Sobolev and Hölder norms defined with respect to the more straightforward metric

$$g' := (1+r^2)\,dr^2 + r^2\,d\theta^2. \tag{3-2}$$

For all nonnegative, integer m, let $H^m(G)$ denote the *Sobolev space* of measurable functions f over $\mathbb{R}^2$ whose distributional derivatives up to and including order m are locally square integrable and which satisfy $\|f\|_{H^m(G)} < \infty$. Likewise, for all nonnegative, integer m, and, for all $\alpha \in [0,1]$, let $C^{m,\alpha}(G)$ denote the *Hölder space* of m-times differentiable functions f over $\mathbb{R}^2$ which satisfy $\|f\|_{C^{m,\alpha}(G)} < \infty$. Recall that both $H^m(G)$ and $C^{m,\alpha}(G)$, furnished with the above norms, are Banach spaces.

For all real γ, define $\phi_\gamma : \mathbb{R}^2 \to \mathbb{R}$ by

$$\phi_\gamma := e^{(1+\gamma)u/2}. \tag{3-3}$$

where u here denotes the integral of v with initial value 0. For all nonnegative, integer m, for all $\alpha \in [0,1]$ and for all real γ, define the *weighted Sobolev and Hölder norms* of weight γ over $\mathbb{R}^2$ by

$$\begin{aligned} \|f\|_{H^m_\gamma(G)} &:= \|\phi_\gamma f\|_{H^m(G)},\\ \|f\|_{C^{m,\alpha}_\gamma(G)} &:= \|\phi_\gamma f\|_{C^{m,\alpha}(G)}. \end{aligned} \tag{3-4}$$

Observe that, by (2-4) again, these *weighted Sobolev and Hölder norms* are uniformly equivalent to the weighted norms defined using instead of ϕ_γ the more straightforward weight function

$$\phi'_\gamma := e^{(1+\gamma)r^2/4}. \tag{3-5}$$

For all nonnegative, integer m, for all $\alpha \in [0,1]$, and for all real γ, define the *weighted Sobolev* and *Hölder spaces* of *weight* γ over $\mathbb{R}^2$ by

$$\begin{aligned} H^m_\gamma(G) &:= \{f \mid \phi_\gamma f \in H^m(G)\},\\ C^{m,\alpha}_\gamma(G) &:= \{f \mid \phi_\gamma f \in C^{m,\alpha}(G)\}. \end{aligned} \tag{3-6}$$

These spaces, furnished with the weighted Sobolev and Hölder norms are trivially also Banach spaces.

Since G_0 is a graph over $\mathbb{R}^2$, its MCFS Jacobi operator may be thought of as an operator acting on functions over $\mathbb{R}^2$. In particular, as we will see presently, for all $\alpha \in [0, 1]$, and for all real γ, J defines bounded linear maps from $H^2_\gamma(G)$ into $H^0_\gamma(G)$ and from $C^{2,\alpha}_\gamma(G)$ into $C^{0,\alpha}_\gamma(G)$. We show:

Theorem 3.1.1. (1) *For all sufficiently small γ, J defines a linear isomorphism from $H^2_\gamma(G)$ into $H^0_\gamma(G)$.*

(2) *For all $\alpha \in]0, 1[$ and for all sufficiently small γ, J defines a linear isomorphism from $C^{2,\alpha}_\gamma(G)$ into $C^{0,\alpha}_\gamma(G)$.*

Theorem 3.1.1 will follow from Lemmas 3.2.6 and 3.3.4 below. Before proceeding, we first observe that, for all γ, the function ϕ_γ is strictly positive so that, for all nonnegative, integer m, and for all $\alpha \in [0, 1]$, the operator of multiplication by this function, which we denote by M_γ, defines linear isomorphisms from $H^m_\gamma(G)$ into $H^m(G)$ and from $C^{m,\alpha}_\gamma(G)$ into $C^{m,\alpha}(G)$. For all real γ, we therefore define

$$J_\gamma := M_\gamma J M_\gamma^{-1}. \tag{3-7}$$

This operator is none other than the ϕ_γ-Jacobi operator of the Grim paraboloid, which has been studied in detail in [Cheng and Zhou 2015; Cheng et al. 2014; 2015a; 2015b]. Trivially, J defines linear isomorphisms from $H^2_\gamma(G)$ into $H^0_\gamma(G)$ and from $C^{2,\alpha}_\gamma(G)$ into $C^{0,\alpha}_\gamma(G)$ if and only if J_γ defines linear isomorphisms from $H^2(G)$ into $H^0(G)$ and from $C^{2,\alpha}(G)$ into $C^{0,\alpha}(G)$ respectively.

Lemma 3.1.2. *For all real γ,*

$$J_\gamma f = \Delta^{G_0} f - \gamma \langle e_z, \nabla^{G_0} f\rangle + \frac{(\gamma^2-1)}{4} f - \frac{(1+\gamma)^2}{4}\langle e_z, N_{G_0}\rangle^2 f + \mathrm{Tr}(A^2_{G_0}) f. \tag{3-8}$$

Proof. By (A-3),

$$\nabla^{G_0}\phi_\gamma^{-1} = -\frac{(1+\gamma)}{2\phi_\gamma}\pi^{G_0}(e_z),$$
$$\mathrm{Hess}^{G_0}\phi_\gamma^{-1} = \frac{(1+\gamma)^2}{4\phi_\gamma} dz\otimes dz + \frac{(1+\gamma)}{2\phi_\gamma}\langle e_z, N_{G_0}\rangle \mathrm{II}^{G_0}.$$

However, since G_0 is a mean curvature flow soliton, $H_{G_0} = -\langle e_z, N_{G_0}\rangle$, and taking the trace therefore yields

$$\Delta^{G_0}\phi_\gamma^{-1} = \frac{(1+\gamma)^2}{4\phi_\gamma} - \frac{(1+\gamma)(3+\gamma)}{4\phi_\gamma}\langle e_z, N_{G_0}\rangle^2.$$

Thus, by (A-2),

$$\phi_\gamma J_0 \phi_\gamma^{-1} = \frac{(\gamma^2-1)}{4} - \frac{(1+\gamma)^2}{4}\langle e_z, N_{G_0}\rangle^2 + \mathrm{Tr}(A_{G_0})^2.$$

The result now follows by (A-4). □

By (A-6) and (2-13),

$$\begin{aligned}\langle e_z, N_{G_0}\rangle^2 &= \mathrm{O}(r^{-(2+k)}),\\ \mathrm{Tr}(A^2_{G_0}) &= \mathrm{O}(r^{-(2+k)}).\end{aligned} \tag{3-9}$$

It follows that, as γ tends to 0, the family (J_γ) converges to J_0 in every operator norm of relevance to us. Since invertibility is stable under small perturbations, it is therefore sufficient to consider only the case $\gamma = 0$ where, in particular, J_0 is self-adjoint.

We now derive a formula for J_0 which is better adapted to our purposes. First, let $c :]0, \infty[\to \mathbb{R}$ be such that, for all r, $c(r)$ is the geodesic curvature of the circle $C(r)$ with respect to the metric induced by the graph G_0 over $\mathbb{R}^2$.

Lemma 3.1.3. *The function c is given by*

$$c = \frac{1}{r}\langle e_z, N_{G_0}\rangle. \tag{3-10}$$

In particular, for large values of r,

$$c = \mathrm{O}(r^{-(2+k)}). \tag{3-11}$$

Proof. Let D denote the Levi-Civita covariant derivative of the Euclidean metric over $\mathbb{R}^3$. Think of $C(r)$ as a horizontal circle in $\mathbb{R}^3$ at height $u(r)$, where u here denotes the integral of v with initial value 0. In particular, $D_{e_\theta} e_\theta = (1/r)e_r$, where e_θ and e_r denote respectively the unit, horizontal vector fields in the angular and radial directions about the z-axis. Since the geodesic curvature of $C(r)$ with respect to the induced metric over G_0 is equal to the length of the tangential component of this vector, the function c is given by

$$c = \frac{1}{r}\sqrt{1 - \langle e_r, N_{G_0}\rangle^2} = \frac{1}{r}\langle e_z, N_{G_0}\rangle,$$

as desired. Equation (3-11) now follows from (3-9), and this completes the proof. □

Let $\rho :]0, \infty[\to \mathbb{R}$ be such that, for all r, $\rho(r)$ is the intrinsic distance along G_0 of any point on the circle $C(r)$ from the origin. Since ρ is obtained by integrating $\sqrt{1+v^2}$, by (2-4) again, for large values of r,

$$\begin{aligned} \rho_r &= r + \mathrm{O}(r^{-(k+1)}), \\ r_\rho &= \frac{1}{r} + \mathrm{O}(r^{-(k+3)}), \end{aligned} \tag{3-12}$$

where the subscripts r and ρ here denote differentiation with respect to the variables r and ρ respectively.

Lemma 3.1.4. *Away from the z-axis,*

$$J_0 f = f_{\rho\rho} + f_{\theta\theta} + c f_\rho - \tfrac{1}{4} f + \psi f, \tag{3-13}$$

where the subscripts ρ and θ denote differentiation along the unit radial and unit angular directions in G_0 and, for large values of ρ,

$$|\psi| \lesssim \rho^{-1}. \tag{3-14}$$

Proof. Indeed, away from the z-axis,

$$\Delta^{G_0} f = f_{\rho\rho} + f_{\theta\theta} + c f_\rho,$$

so that (3-13) follows by (3-8) and (3-9) with

$$|\psi| \lesssim r^{-2}.$$

Finally, integrating (3-12), yields $\rho \lesssim r^2$, so that $r^{-2} \lesssim \rho^{-1}$ and the result follows. □

3.2. ***Invertibility over Sobolev spaces.*** We now obtain the invertibility of J_0 for Sobolev spaces. The main technical difficulty here arises from the noncompactness of the ambient space. This is compensated for by the following estimate.

Lemma 3.2.1. *There exist B, $R>0$ such that, for all f in $H^2(G)$,*

$$\|f|_{A(R,\infty)}\|_{L^2(G)} \leq B\big(\|f|_{A(R-1,R+1)}\|_{L^2(G)} + \|J_0 f|_{A(R-1,\infty)}\|_{L^2(G)}\big). \tag{3-15}$$

Proof. Since $C_0^\infty(G)$ is dense in $H^2(G)$, it suffices to prove the result when f is smooth and has compact support. Set $g := J_0 f$ and define $\alpha, \beta :]0,\infty[\to \mathbb{R}$ by

$$\alpha(\rho) := \int_{C(\rho)} f^2\,dl, \quad \beta(\rho) := \int_{C(\rho)} g^2\,dl,$$

where $C(\rho)$ here denotes the circle of points lying at intrinsic distance ρ along G_0 from the origin. Twice differentiating α yields

$$\begin{aligned}\alpha_\rho &= \int_{C(\rho)} 2ff_\rho + f^2 c\,dl,\\ \alpha_{\rho\rho} &= \int_{C(\rho)} 2f_\rho^2 + 2ff_{\rho\rho} + 4ff_\rho c + f^2 c_\rho + f^2c^2\,dl,\end{aligned}$$

where the subscript ρ here denotes differentiation with respect to this variable. By (3-13),

$$\alpha_{\rho\rho} = \int_{C(\rho)} 2f_\rho^2 - 2ff_{\theta\theta} + \tfrac{1}{2}f^2 - 2\psi f^2 + 2fg + 2ff_\rho c + f^2c_\rho + f^2c^2\,dl.$$

Integrating the term $2ff_{\theta\theta}$ by parts and applying the algebraic-geometric mean inequality now yields

$$\alpha_{\rho\rho} \geq \int_{C(\rho)} \big(\tfrac{1}{4} - 2\psi + c_\rho - c^2\big)f^2 - 4g^2\,dl.$$

However, by (3-11), (3-12) and (3-14), c, $c_\rho = c_r r_\rho$ and ψ all tend to 0 as ρ tends to $+\infty$ so that, for sufficiently large ρ,

$$\alpha_{\rho\rho} \geq \tfrac{1}{8}\alpha - 4\beta.$$

Since f has compact support, upon integrating this relation we obtain, for sufficiently large R,

$$\|f|_{A(R,\infty)}\|_{L^2(G)}^2 = \int_R^\infty \alpha\,d\rho \leq 32\int_R^\infty \beta\,d\rho - 8\alpha_\rho(R) = 32\|\hat{J}_0 f|_{A(R,\infty)}\|_{L^2(G)}^2 - 8\alpha_\rho(R).$$

However, by the Sobolev trace formula and classical elliptic estimates,

$$\begin{aligned}\alpha_\rho(R) &\leq B_1\|f|_{A(R-1/2,R+1/2)}\|_{H^2(G)}^2\\ &\leq B_2(\|f|_{A(R-1,R+1)}\|_{L^2(G)}^2 + \|J_0 f|_{A(R-1,R+1)}\|_{L^2(G)}^2)\end{aligned}$$

for suitable constants B_1 and B_2. The result now follows upon combining the last two relations. □

Combining Lemma 3.2.1 with classical elliptic estimates yields:

Lemma 3.2.2. *There exist B, $R>0$ such that, for all f in $H^2(G)$,*

$$\|f\|_{H^2(G)} \leq B(\|f|_{B(R)}\|_{L^2(G)} + \|J_0 f\|_{L^2(G)}). \tag{3-16}$$

Proof. Observe that G_0 is of bounded geometry in the sense that, as x tends to infinity in G_0, the geodesic ball of unit radius about x in this surface converges in the pointed Cheeger–Gromov sense to the unit ball about the origin in $\mathbb{R}^2$. It thus follows by classical elliptic theory (see [Gilbarg and Trudinger 1983]) that there exists $B>0$ such that

$$\|f\|_{H^2(G)} \leq B(\|f\|_{L^2(G)} + \|J_0 f\|_{L^2(G)}).$$

The result follows upon combining this relation with (3-15). □

Since, J_0 is self-adjoint, standard arguments of the theory of elliptic operators now yield

Lemma 3.2.3. *J_0 defines a Fredholm map from $H^2(G)$ into $L^2(G)$ of Fredholm index equal to* 0.

Proof. Since $B(R)$ is relatively compact, it follows by Rellich's compactness theorem that the restriction map sending $H^2(G)$ into $L^2(B(R))$ is also compact. Thus, by (3-16), J_0 satisfies an elliptic estimate, as defined in Section A5, so that, by Theorem A5.1, J_0 has finite-dimensional kernel and closed image. Observe now that J_0 is self adjoint, so that $\mathrm{Ker}(J_0)$ is contained within the orthogonal complement of $\mathrm{Im}(J_0)$ in $L^2(G)$. We claim that these two spaces coincide. Indeed, let u be an element of the orthogonal complement of $\mathrm{Im}(J_0)$. In particular, $J_0 u=0$ in the distributional sense. Thus, bearing in mind that G_0 is of bounded geometry, it follows by classical elliptic regularity that u is an element of $H^2(G)$. In particular, u is therefore an element of $\mathrm{Ker}(J_0)$, so that $\mathrm{Ker}(J_0)$ coincides with the orthogonal complement of $\mathrm{Im}(J_0)$ in $L^2(G)$, as asserted. It immediately follows that J_0 is a Fredholm map of Fredholm index equal to 0, and this completes the proof. □

It remains only to prove that J_0 has trivial kernel in $H^2(G)$. We obtain a slightly more general result which will serve also for the Hölder space case of the following section.

Lemma 3.2.4. *There exists no nontrivial, bounded function $f : G_0 \to \mathbb{R}$ such that $J_0 f = 0$.*

Proof. Indeed, suppose that there exists a nontrivial bounded function $f : G_0 \to \mathbb{R}$ such that $J_0 f = 0$. Upon multiplying by (-1), we may suppose that f is positive at some point. Now, since all vertical translates of G_0 are also mean curvature flow solitons, the function $\mu = \langle e_z, N_G\rangle$ is a Jacobi field over this surface, that is,

$$J_0\phi_0\mu = \phi_0 J\mu = 0.$$

Since G_0 is a graph, the function μ is everywhere strictly positive. It follows that $\phi_0\mu$ is also positive, so that the quotient $f/\phi_0\mu$ is smooth. Since $\phi_0 \gtrsim e^{r^2/4}$ and $\mu = \mathrm{O}(r^{-1})$, the function $\phi_0\mu$ tends to infinity as r tends to infinity, and so $f/\phi_0\mu$ attains its maximum value at some point x, say, of G_0. In particular, upon rescaling, we may suppose that $f/\phi_0\mu \leq 1$ and that $f(x)/\phi_0(x)\mu(x) = 1$.

Bearing in mind that μ is positive, we define the operator $J_\mu := M_\mu^{-1} J M_\mu$, where M_μ here denotes the operator of multiplication by μ. Since $J\mu = 0$, by (A-4), this operator has no zeroth order term. Thus, since $J_\mu(f/\mu\phi_0) = (1/\mu\phi_0)J_0 f = 0$, it follows by the strong maximum principle that $f/\phi_0\mu$ is constant and equal to 1. However, since $\phi_0\mu$ is unbounded, this is absurd, and the result follows. □

Corollary 3.2.5. *J_0 has trivial kernel in $H^2(G)$.*

Proof. Indeed, by the Sobolev embedding theorem, every element of $H^2(G)$ is bounded, and the result now follows by Lemma 3.2.4. □

The above results together with a perturbation argument now yield

Lemma 3.2.6. *For sufficiently small γ, J defines a linear isomorphism from $H^2_\gamma(G)$ into $H^0_\gamma(G)$.*

3.3. ***Invertibility over Hölder spaces.*** We prove the invertibility of J_0 over $C^{2,\alpha}(G)$ in essentially the same manner. We first require the following preliminary result.

Lemma 3.3.1. *Let α and β be positive constants. If $\phi : [0,\infty[\to]0,\infty[$ is a bounded, positive function such that $\phi'' \geq \alpha^2\phi - \beta$ in the viscosity sense, then, for all t,*

$$\phi(t) \leq \mathrm{Max}\left(\phi(0) - \frac{\beta}{\alpha^2}, 0\right)e^{-\alpha t} + \frac{\beta}{\alpha^2}. \tag{3-17}$$

Proof. Let $A = \mathrm{Max}(\phi(0) - \beta/\alpha^2, 0)$ and let $B = \mathrm{Sup}_{t\in[0,\infty[}\,\phi(t)$. Fix $T > 0$ and define

$$f = \frac{Be^{\alpha T} - A}{e^{2\alpha T} - 1}e^{\alpha t} + \frac{A - Be^{-\alpha T}}{1 - e^{-2\alpha T}}e^{-\alpha t} + \frac{\beta}{\alpha^2}.$$

In other words, f is the unique solution of the ODE problem $f_{tt} = \alpha^2 f - \beta$ with boundary values $f(0) = A + \beta/\alpha^2 \geq \phi(0)$ and $f(T) = B + \beta/\alpha^2 \geq \phi(T)$. Let C be the minimum value of $f - \phi$ over $[0, T]$ and let $t \in [0, T]$ be the point at which this minimum is attained. If t is a boundary point of this interval, then $C \geq 0$. Otherwise, $f - C \geq \phi$ and $f(t) - C = \phi(t)$. Thus, since ϕ is a viscosity solution of $\phi'' \geq \alpha^2\phi - \beta$, at this point, we have

$$\alpha^2 f - \beta = (f - C)_{tt} \geq \alpha^2(f - C) - \beta$$

so that, once again, $C \geq 0$. In each case, we therefore obtain

$$\phi \leq f = \frac{Be^{\alpha T} - A}{e^{2\alpha T} - 1}e^{\alpha t} + \frac{A - Be^{-\alpha T}}{1 - e^{-2\alpha T}}e^{-\alpha t} + \frac{\beta}{\alpha^2},$$

and the result follows upon taking the limit as T tends to $+\infty$. □

As in the Sobolev case, the noncompactness of the ambient space is compensated for by the following estimate.

Lemma 3.3.2. *There exist $B, R > 0$ such that, for all f in $C^{2,\alpha}(G)$,*

$$\|f|_{A(R,\infty)}\|_{C^0(G)} \leq B(\|f|_{C(R)}\|_{C^0(G)} + \|J_0 f|_{A(R-1,\infty)}\|_{C^0(G)}). \tag{3-18}$$

Proof. Define $\alpha :]0,\infty[\to \mathbb{R}$ by

$$\alpha(\rho) := \mathrm{Sup}_{x\in C(\rho)}\, f(x)^2,$$

where $C(\rho)$ here denotes the circle of points lying at intrinsic distance ρ along G_0 from the origin. Denote $g := J_0 f$, and define $B \geqslant 0$ by

$$B := \|g^2|_{A(R,\infty)}\|_{C^0(G)}.$$

Choose $x \in C(\rho)$ maximising f^2, and observe that $f f_{\theta\theta}$ is nonpositive at this point. Thus, bearing in mind (3-13),

$$\begin{aligned}(f^2)_{\rho\rho} &= 2f_\rho^2 + 2ff_{\rho\rho},\\ &\geq 2f_\rho^2 + 2fg - 2cf_\rho + \tfrac{1}{2}f^2 - 2\psi f^2,\\ &\geq \left(\tfrac{1}{4} - \tfrac{1}{2}c^2 - 2\psi\right)f^2 - 4g^2.\end{aligned}$$

By (3-11) and (3-14), for sufficiently large ρ

$$(f^2)_{\rho\rho} \geq \tfrac{1}{8}f^2 - 4g^2.$$

Since α is the envelope of the restriction of $f(x)^2$ to each radial line, it follows that over $[R,\infty[$,

$$\alpha_{\rho\rho} \geq \tfrac{1}{8}\alpha - 4B,$$

in the viscosity sense. Thus, by Lemma 3.3.1,

$$\mathrm{Sup}_{x\in A(R,\infty)} f^2(x) = \mathrm{Sup}_{\rho\geq R}\,\alpha(\rho) \leq \mathrm{Max}(\|f^2|_{C(R)}\|_{C^0} - 32B, 0) + 32B,$$

and the result follows. □

Using classical elliptic estimates again, this yields

Lemma 3.3.3. *There exist B, $R > 0$ such that for all f in $C^{2,\alpha}(G)$,*

$$\|f\|_{C^{2,\alpha}(G)} \leq B(\|f|_{B(R)}\|_{C^0(G)} + \|J_0 f\|_{C^{0,\alpha}(G)}). \tag{3-19}$$

Proof. Recall that G_0 is of bounded geometry in the sense that, as x tends to infinity in G_0, the geodesic ball of unit radius about x in this surface converges in the pointed Cheeger–Gromov sense to the unit ball about the origin in $\mathbb{R}^2$. It thus follows by classical elliptic theory (see [Gilbarg and Trudinger 1983]) that there exists $B > 0$ such that

$$\|f\|_{C^{2,\alpha}(G)} \leq B(\|f\|_{C^0(G)} + \|J_0 f\|_{C^{0,\alpha}(G)}),$$

and the result now follows upon combining this relation with (3-18). □

As before, this yields the desired invertibility result.

Lemma 3.3.4. *For all α and for all sufficiently small γ, J defines a linear isomorphism from $C^{2,\alpha}_\gamma(G)$ into $C^{0,\alpha}_\gamma(G)$.*

Proof. Recall that this is equivalent to showing that, for sufficiently small γ, J_γ defines a linear isomorphism from $C^{2,\alpha}(G)$ into $C^{0,\alpha}(G)$. Furthermore, by (3-8) and (3-9), J_γ converges to J_0 in the operator norm as γ tends to 0, so that it suffices to prove the result for J_0.

Since $B(R)$ is a relatively compact subset of G_0, it follows by the Arzelà–Ascoli theorem that the restriction map of $C^{2,\alpha}(G)$ into $C^0(B(R))$ is compact. Thus, by (3-19), J_0 satisfies an elliptic estimate, as defined in Section A5. By Theorem A5.1, the image of J_0 is closed and, in particular, is a Banach subspace of $C^{0,\alpha}(G)$. Furthermore, by Lemma 3.2.4, the kernel of J_0 in $C^{2,\alpha}(G)$ is trivial, so that, by the closed graph theorem, J_0 defines a linear isomorphism from $C^{2,\alpha}(G)$ into its image. In particular, there exists a constant $B > 0$ such that, for all $u \in C^{2,\alpha}(G)$,

$$\|u\|_{2,\alpha} \leq B\|J_0 u\|_{0,\alpha}. \tag{3-20}$$

It remains only to prove surjectivity. Choose $v \in C^{0,\alpha}(G)$ and let (v_m) be a sequence of smooth functions of compact support in $\mathbb{R}^2$ which is bounded in $C^{0,\alpha}(G)$ and which converges to v in the $C^{0,\beta}_{\text{loc}}$ sense for all $\beta < \alpha$. For all m, since v_m is a smooth function with compact support, it is an element of $L^2(G)$ so that, by Lemma 3.2.6, there exists an element u_m of $H^2(G)$ such that $J_0 u_m = v_m$. Since G_0 is of bounded geometry, it follows by classical elliptic regularity that, for all m, u_m is in fact an element of $C^{2,\alpha}(G)$. In particular, by (3-20), for all m,

$$\|u_m\|_{C^{2,\alpha}(G)} \leq B\|v_m\|_{C^{0,\alpha}(G)}.$$

Since the sequence (u_m) is uniformly bounded in $C^{2,\alpha}(G)$, it follows by the Arzelà–Ascoli theorem there exists $u \in C^{2,\alpha}(G)$ towards which (u_m) subconverges in the $C^{2,\beta}_{\text{loc}}$-topology for all $\beta < \alpha$. By continuity, $J_0 u = v$ and surjectivity follows. □

4. Rotationally symmetric Grim ends

4.1. ***The modified MCFS Jacobi operator.*** We now consider the case of rotationally symmetric Grim ends. Let Λ be a large, positive real number, let $K > 0$ be fixed, and let $\epsilon, R > 0$ and $c \in \mathbb{R}$ satisfy (2-14). Let $v : [\epsilon R, \infty[\to \mathbb{R}$ solve (2-5) with logarithmic parameter c so that, by (2-16), over the interval $[\epsilon R, \epsilon R^4]$,

$$v = \frac{1}{2}r + \frac{c\epsilon}{r} + O\Big(\Big[1 + \log\Big(\frac{r}{\epsilon R}\Big)\Big]\frac{1}{r^k}\Big(r + \frac{\epsilon}{r}\Big)^3\Big). \tag{4-1}$$

Let $u : [\epsilon R, \infty[\to \mathbb{R}$ be a primitive of v, let G be the Grim end generated by rotating the graph of u about the z-axis, and let J be its MCFS Jacobi operator, as defined in Section A2.

Since G is a graph over $A(\epsilon R, \infty)$, J may again be thought of as an operator acting on functions over this annulus. For all nonnegative, integer m, for all $\alpha \in [0, 1]$, and for all real γ, we define the norms $\|\cdot\|_{H^m_\gamma(G)}$ and $\|\cdot\|_{C^{m,\alpha}_\gamma(G)}$ as in Section 3. For all nonnegative integer m, for all $\alpha \in [0, 1]$ and for all real γ, we define the *hybrid norm* with weight γ of functions over $\mathbb{R}^m$ by

$$\|f\|_{m,\alpha,\gamma} := \|f\|_{C^{m,\alpha}_\gamma(G)} + \frac{1}{(\epsilon R)}\|f\|_{H^m_\gamma(G)}. \tag{4-2}$$

As we will see in Section 6, this norm encapsulates the asymptotic behaviour of J as Λ tends to infinity. Let $\mathcal{L}^{m,\alpha}_\gamma(G)$ denote the Banach space of m-times differentiable functions f over $\mathbb{R}^2$ with finite hybrid norm. In this section, we show that, for sufficiently small γ, and for sufficiently large Λ, the operator J *more or less* defines linear isomorphisms from $\mathcal{L}^{2,\alpha}_\gamma(G)$ into $\mathcal{L}^{0,\alpha}_\gamma(G)$ and, furthermore, that the norms of this isomorphism and its inverse are uniformly bounded as Λ tends to infinity. In order to properly formalise these assertions, we now apply the following two modifications.

First, on account of the vanishing neck problem, discussed in the Introduction, the zeroth-order coefficient of J diverges rapidly over the annulus $A(\epsilon R, \epsilon R^4)$ as Λ tends to infinity. We address this by introducing what we call the modified MCFS Jacobi operator. Recall that different modifications are applied at different scales, so that the definition of this operator varies according to context, and the general framework will be discussed in Section 5.4, below. In the present case, the modified MCFS Jacobi operator is defined as follows. Let χ_1 be the cut-off function of the transition region $A(1,2)$ as defined in Section A1 and define $\psi : A(\epsilon R, \infty) \to \mathbb{R}$ by

$$\psi(r) = \chi_1 \langle e_z, N_G\rangle + (1-\chi_1), \tag{4-3}$$

where N_G here denotes the upward-pointing unit normal vector field over G. Bearing in mind that ψ is always positive, the *modified MCFS Jacobi operator* of G is now defined by

$$\hat{J} := M_\psi^{-1} J M_\psi, \tag{4-4}$$

where M_ψ here denotes the operator of multiplication by ψ.

Next, observe that $\hat{J}$ is in fact only defined over the annulus $A(\epsilon R, \infty)$. We thus extend it to an operator defined over the whole of $\mathbb{R}^2$ as follows. Given a function $\phi : A(\epsilon R, \infty) \to \mathbb{R}$, we define its *canonical extension* $\tilde{\phi} : \mathbb{R}^2 \to \mathbb{R}$ such that $\tilde{\phi}(x) = \phi(x)$ over $A(\epsilon R, \infty)$, $\tilde{\phi}(0)$ is equal to the mean value of ϕ over the circle $C(\epsilon R)$, and $\tilde{\phi}$ restricts to a linear function over every radial line in $B(\epsilon R)$. In particular, if ϕ is Lipschitz, then so too is $\tilde{\phi}$, and

$$\|\tilde{\phi}\|_{C^{0,1}} \leq \tfrac{\pi}{2} \|\phi\|_{C^{0,1}}.$$

Now, given a linear operator L over $A(\epsilon R, \infty)$, we define its *canonical extension* $\widetilde{L}$ to be the operator over $\mathbb{R}^2$ whose coefficients are the canonical extensions of each of the coefficients of L. We henceforth identify all operators with their canonical extensions over $\mathbb{R}^2$. Observe, in particular, that if L has any rotational symmetries, then so too does its canonical extension.

Theorem 4.1.1. *For all sufficiently small $\alpha \in]0,1[$ and for all sufficiently large Λ, $\hat{J}$ defines a linear isomorphism from $\mathcal{L}^{2,\alpha}_\gamma(G)$ into $\mathcal{L}^{0,\alpha}_\gamma(G)$. Furthermore, the operator norms of $\hat{J}$ and its inverse are uniformly bounded independent of Λ.*

Theorem 4.1.1 follows from Theorem 3.1.1 by a perturbation argument and Lemmas 4.2.7 and 4.3.4. We conclude this section by deriving formulae for $\hat{J}$ over different regions.

Lemma 4.1.2. *Over $A(\epsilon R, 1)$, the modified MCFS Jacobi operator of G is given by*

$$\hat{J} f = g^{ij} f_{ij} - 2\mu g^{ip} g^{jq} u_{pq} u_j f_i. \tag{4-5}$$

Proof. First observe that, for every tangent vector X over G,

$$\langle \nabla^G \psi, X\rangle = X\psi = X\langle N_G, e_z\rangle = \langle D_X N_G, e_z\rangle = \langle A_G X, e_z\rangle = \langle X, A_G \pi^G(e_z)\rangle,$$

and so,

$$\nabla^G \psi = A_G \pi^G(e_z).$$

Since every vertical translate of G is also a rotationally symmetric Grim end, $J\langle e_z, N_G\rangle = 0$, and so, by (A-4),

$$\hat{J}f = \Delta^G f + \langle e_z, \nabla^G f\rangle + 2\psi^{-1}\langle A_G \nabla^G f, e_z\rangle.$$

By (A-3),

$$\mathrm{Hess}^G f = \mathrm{Hess}(f)\circ\pi - \langle D(f\circ\pi), N\rangle \mathrm{II}_G.$$

Furthermore, since $D(f\circ\pi)$ is horizontal

$$\langle D(f\circ\pi), N_G\rangle = -\frac{1}{\langle N_G, e_z\rangle}\langle D(f\circ\pi), e_z - \langle N_G, e_z\rangle N_G\rangle = -\frac{1}{\langle N_G, e_z\rangle}\langle \nabla^G f, e_z\rangle.$$

Taking the trace therefore yields

$$\Delta^G f = g^{ij} f_{ij} + \frac{1}{\langle N_G, e_z\rangle}\langle \nabla^G f, e_z\rangle H_G.$$

However, since G is a mean curvature flow soliton, $H_G = -\langle N, e_z\rangle$, and so

$$\Delta^G f = g^{ij} f_{ij} - \langle \nabla^G f, e_z\rangle.$$

We conclude that

$$\hat{J}f = g^{ij} f_{ij} + 2\psi^{-1}\langle A_G \nabla^G f, e_z\rangle,$$

and the result now follows by (A-6). □

Lemma 4.1.3. *Over $A(\epsilon R, 2\epsilon R^4)$, the modified MCFS Jacobi operator of G satisfies*

$$\hat{J}f = \Delta f - \left(\frac{1}{2} + \frac{c\epsilon}{r^2}\right)^2 x^i x^j f_{ij} - \left(\frac{1}{2} - \frac{2c^2\epsilon^2}{r^4}\right) x^i f_i + \mathcal{E}_G f, \tag{4-6}$$

where $\mathcal{E}_G f := a^{ij} f_{ij} + b^i f_i$, and a and b satisfy

$$\begin{aligned} a &= \mathrm{O}\Big(\Big[1 + \log\Big(\frac{r}{\epsilon R}\Big)\Big]\frac{1}{r^k}\Big(r + \frac{\epsilon}{r}\Big)^4\Big), \\ b &= \mathrm{O}\Big(\Big[1 + \log\Big(\frac{r}{\epsilon R}\Big)\Big]\frac{1}{r^{k+1}}\Big(r + \frac{\epsilon}{r}\Big)^4\Big). \end{aligned} \tag{4-7}$$

Proof. Indeed, by (4-1),

$$u_i = \frac{1}{2}x_i + \frac{c\epsilon}{r^2}x_i + \mathrm{O}\Big(\Big[1 + \log\Big(\frac{r}{\epsilon R}\Big)\Big]\frac{1}{r^k}\Big(r + \frac{\epsilon}{r}\Big)^3\Big).$$

Thus, by (A-6),

$$\begin{aligned} \mu^2 &= 1 - \Big(\frac{r}{2} + \frac{c\epsilon}{r}\Big)^2 + \mathrm{O}\Big(\Big[1 + \log\Big(\frac{r}{\epsilon R}\Big)\Big]\frac{1}{r^k}\Big(r + \frac{\epsilon}{r}\Big)^4\Big), \\ g^{ij} &= \delta_{ij} - \Big(\frac{1}{2} + \frac{c\epsilon}{r^2}\Big)^2 x^i x^j + \mathrm{O}\Big(\Big[1 + \log\Big(\frac{r}{\epsilon R}\Big)\Big]\frac{1}{r^{k+1}}\Big(r + \frac{\epsilon}{r}\Big)^4\Big). \end{aligned}$$

It follows that

$$g^{ij} f_{ij} = \Delta f - \Big(\frac{1}{2} + \frac{c\epsilon}{r^2}\Big)^2 x^i x^j f_{ij} + a^{ij} f_{ij},$$

where $a = \mathrm{O}([1+\log(r/\epsilon R)]r^{-k}(r+\epsilon/r)^4)$, and since $r^{-1}(r+\epsilon/r)^4$ bounds $(r+\epsilon/r)^3$,

$$-2\mu g^{ip}g^{jq}u_{pq}u_i f_j = -\left(\frac{1}{2} - \frac{2\epsilon^2 c^2}{r^4}\right)x^i f_i + b^i f_i,$$

where $b = \mathrm{O}([1+\log(r/\epsilon R)]r^{-(k+1)}(r+\epsilon/r)^4)$. The result follows. □

4.2. ***The regular component.*** Theorem 4.1.1 is derived from Theorem 3.1.1 by a perturbation argument. First, let $v_p :]0,\infty[\to \mathbb{R}$ denote the unique solution of (2-5) which is defined over the whole positive half-line, as in Section 3. Let u_p denote its primitive with initial value 0 so that its graph is a Grim paraboloid. Let $\hat{J}_p$ denote its modified MCFS Jacobi operator, as defined in Section 4.1. Over the ball $B(2\epsilon R^4)$,

$$v_p(r) = \tfrac{1}{2}r + O(r^{3-k}), \tag{4-8}$$

so that, as in Lemma 4.1.3, over $B(0, 2\epsilon R^4)$,

$$\hat{J}_p f = \Delta f - \tfrac{1}{2}x^i x^j f_{ij} - \tfrac{1}{2}x^i f_i + \mathcal{E}_p f, \tag{4-9}$$

where $\mathcal{E}_p f := a^{ij} f_{ij} + b^i f_i$ and

$$a = \mathrm{O}(r^{4-k}), \quad b = \mathrm{O}(r^{3-k}). \tag{4-10}$$

Define

$$\hat{J}_\gamma := M_\gamma^{-1}\hat{J}M_\gamma, \tag{4-11}$$

where M_γ here denotes the operator of multiplication by $\chi_2 + (1-\chi_2)\phi_\gamma$, ϕ_γ is given by (3-3), and χ_2 is the cut-off function of the transition region $A(2,4)$ as defined in Section A1. Observe that, since ϕ_γ and ψ only depend on v and its integral u, it follows by (A-2) that the coefficients of $\hat{J}_\gamma$ are functions of u, v and v_r only. Finally, define

$$\hat{J}_{p,\gamma} := M_\gamma^{-1}\hat{J}_p M_\gamma. \tag{4-12}$$

A straightforward modification of Theorem 3.1.1 shows that, for all $\alpha \in]0,1[$, and for all sufficiently small γ, $\hat{J}_{p,\gamma}$ defines a linear isomorphism from $\mathcal{L}^{2,\alpha}_\gamma(G)$ into $\mathcal{L}^{0,\alpha}_\gamma(G)$ whose Green's operator has norm uniformly bounded independent of Λ.

It will suffice to show that the difference $\hat{J}_{p,0} - \hat{J}_0$ converges to 0 with respect to the hybrid norm as Λ tends to $+\infty$. This is, in fact, a nontrivial result, since the coefficients of this operator diverge. However, the region over which they diverge itself converges to a point; the relative rates of convergence are such that the coefficients converge in the mean, which will be sufficient for us to conclude. Formally, we define the operators D and E over $A(\epsilon R, \infty)$ by

$$\begin{aligned} Df &:= (\hat{J}_0 - E)f - \hat{J}_{p,0}f,\\ Ef &:= \chi\frac{2c^2\epsilon^2}{r^4}x^i f_i, \end{aligned} \tag{4-13}$$

where χ here denotes the cut-off function of the transition region $A(\epsilon R^4, 2\epsilon R^4)$. We then extend these operators canonically to operators over the whole of $\mathbb{R}^2$, as in Section 4.1. By definition,

$$\hat{J}_0 := \hat{J}_{p,0} + D + E. \tag{4-14}$$

We call D and E respectively the *regular component* and the *singular component* of the difference. We now show that the coefficients of the regular component tend to 0 in all norms that concern us as Λ tends to infinity. We will study the singular component in the next section.

By (4-6), (4-7), (4-9), (4-10) and (4-13),

$$Df = a^{ij} f_{ij} + b^i f_i,$$

where, over $A(\epsilon R, 2\epsilon R^4)$,

$$\begin{aligned} a^{ij} &= -\frac{c\epsilon}{r^2}x^i x^j - \frac{c^2\epsilon^2}{r^4}x^i x^j + \mathrm{O}\Big(\Big[1+\log\Big(\frac{r}{\epsilon R}\Big)\Big]\frac{1}{r^k}\Big(r+\frac{\epsilon}{r}\Big)^4\Big),\\ b^i &= (1-\chi)\frac{2c^2\epsilon^2}{r^4}x^i + \mathrm{O}\Big(\Big[1+\log\Big(\frac{r}{\epsilon R}\Big)\Big]\frac{1}{r^{k+1}}\Big(r+\frac{\epsilon}{r}\Big)^4\Big).\end{aligned} \tag{4-15}$$

Lemma 4.2.1. *For sufficiently small α,*

$$\|a|_{B(\epsilon R)}\|_{C^{0,\alpha}},\ \|b|_{B(\epsilon R)}\|_{C^{0,\alpha}} \to 0, \tag{4-16}$$

as Λ tends to infinity.

Proof. Indeed, by (4-15), since χ equals 1 near $C(\epsilon R)$, over this circle,

$$\begin{aligned} a &= \mathrm{O}\Big(\frac{1}{(\epsilon R)^k}\Big(\epsilon + \frac{1}{R^2} + (\epsilon R)^4 + \frac{1}{R^4}\Big)\Big),\\ b &= \mathrm{O}\Big(\frac{1}{(\epsilon R)^{k+1}}\Big((\epsilon R)^4 + \frac{1}{R^4}\Big)\Big).\end{aligned}$$

Since the Lipschitz seminorms of the canonical extensions of a and b over $B(\epsilon R)$ are controlled by their Lipschitz seminorms over $C(\epsilon R)$, by (A-10), for all $\alpha \in [0, 1]$,

$$\begin{aligned} \|a|_{B(\epsilon R)}\|_{C^{0,\alpha}} &\lesssim \frac{\epsilon^{1-\alpha}}{R^\alpha} + \frac{1}{\epsilon^\alpha R^{2+\alpha}} + (\epsilon R)^{4-\alpha} + \frac{1}{\epsilon^\alpha R^{4+\alpha}},\\ \|b|_{B(\epsilon R)}\|_{C^{0,\alpha}} &\lesssim (\epsilon R)^{3-\alpha} + \frac{1}{\epsilon^{1+\alpha}R^{5+\alpha}}.\end{aligned}$$

By (2-14), for sufficiently small α, these both tend to 0 as Λ tends to infinity, as desired. □

Lemma 4.2.2. *For sufficiently small α,*

$$\|a|_{A(\epsilon R, 2\epsilon R^4)}\|_{C^{0,\alpha}},\ \ \|b|_{A(\epsilon R, 2\epsilon R^4)}\|_{C^{0,\alpha}} \to 0, \tag{4-17}$$

as Λ tends to infinity.

Proof. Indeed, by (4-15), over $A(\epsilon R, 2\epsilon R^4)$,

$$a = \mathrm{O}\Big(\frac{1}{r^k}\Big(\epsilon + \frac{\epsilon^2}{r^2}\Big)\Big) + \mathrm{O}\Big(\Big[1+\log\Big(\frac{r}{\epsilon R}\Big)\Big]\frac{1}{r^k}\Big(r+\frac{\epsilon}{r}\Big)^4\Big),$$

and $b = b_1 + b_2$, where

$$\begin{aligned} b_1 &= \mathrm{O}\Big(\Big[1+\log\Big(\frac{r}{\epsilon R}\Big)\Big]\frac{1}{r^{k+1}}\Big(r^4 + \frac{\epsilon^4}{r^4}\Big)\Big),\\ b_2 &= (1-\chi)\frac{2c^2\epsilon^2}{r^4}x^i.\end{aligned}$$

Thus, by (A-10) and (A-20), for all $\alpha \in [0, 1]$,

$$\|a|_{A(\epsilon R,2\epsilon R^4)}\|_{C^{0,\alpha}} \lesssim \frac{\epsilon^{1-\alpha}}{R^\alpha} + \frac{1}{\epsilon^\alpha R^{2+\alpha}} + \log(R)(\epsilon R^4)^{4-\alpha} + \frac{1}{\epsilon^\alpha R^{4+\alpha}},$$
$$\|b_1|_{A(\epsilon R,2\epsilon R^4)}\|_{C^{0,\alpha}} \lesssim \log(R)(\epsilon R^4)^{3-\alpha} + \frac{1}{\epsilon^{1+\alpha} R^{5+\alpha}}.$$

By (2-14), for sufficiently small α, these both tend to 0 as Λ tends to infinity. Finally, over $A(\epsilon R^4, 2\epsilon R^4)$,

$$b_2 = O(\epsilon^2 r^{-(k+3)}),$$

so that, by (A-10),

$$\|b_2|_{A(\epsilon R^4,2\epsilon R^4)}\|_{C^{0,\alpha}} \lesssim \frac{1}{\epsilon^{1+\alpha} R^{12+4\alpha}}.$$

By (2-14), for sufficiently small α, this also tends to 0 as Λ tends to infinity, and the result follows. ☐

Lemma 4.2.3. *If $\epsilon R < s < t < \sqrt{2}$, then*

$$|v(t) - v_p(t)| \leq |v(s) - v_p(s)|. \tag{4-18}$$

Proof. Indeed, by (2-5), using a dot to denote differentiation with respect to r, we have

$$r(\dot{v} - \dot{v}_p) = -(v - v_p)(1 - r(v + v_p) + (v^2 + vv_p + v_p^2)).$$

However,

$$1 - r(v + v_p) + (v^2 + vv_p + v_p^2) \geq 1 - \frac{r^2}{2}.$$

Thus, for $r \leq \sqrt{2}$, $|v - v_p|$ is decreasing, as desired. ☐

Lemma 4.2.4. *For all $\alpha \in]0, 1]$,*

$$\|a|_{A(\epsilon R^4,1)}\|_{C^1}, \|b|_{A(\epsilon R^4,1)}\|_{C^1} \to 0, \tag{4-19}$$

as Λ tends to infinity.

Proof. By (4-1) and (4-8), over $C(2\epsilon R^4)$,

$$|v - v_p| \lesssim \frac{1}{R^4} + \log(R)(\epsilon R^4)^3 + \log(R)\frac{1}{R^{12}}.$$

By Lemma 4.2.3, this inequality continues to hold over the whole of $A(2\epsilon R^4, 1)$. Since v and v_p both solve (2-5), it follows that, over this annulus,

$$v - v_p = \mathrm{O}\bigg(\frac{1}{(\epsilon R^4)^k}\bigg(\frac{1}{R^4} + \log(R)(\epsilon R^4)^3 + \log(R)\frac{1}{R^{12}}\bigg)\bigg).$$

Thus,

$$\|(v - v_p)|_{[2\epsilon R^4,1]}\|_{C^2} \lesssim \frac{1}{\epsilon^2 R^{12}} + \log(R)\epsilon R^4 + \log(R)\frac{1}{\epsilon^2 R^{20}},$$

so that, by (2-14),

$$\|(v - v_p)|_{[2\epsilon R^4,1]}\|_{C^2} \to 0,$$

as Λ tends to infinity. However, by (4-5), over $A(\epsilon R^4, 1)$, the coefficients a and b only depend on the first derivatives of v and v_p, so that

$$\|a|_{A(2\epsilon R^4,1)}\|_{C^1},\ \|b|_{A(2\epsilon R^4,1)}\|_{C^1} \to 0,$$

as Λ tends to infinity, as desired. □

Lemma 4.2.5. *For all* $R_0 > 1$,

$$\|a|_{A(1,R_0)}\|_{C^1}, \|b|_{A(1,R_0)}\|_{C^1} \to 0, \tag{4-20}$$

as Λ *tends to infinity.*

Proof. By (4-1), (4-8) and (4-18), over $C(1)$,

$$|v - v_p| \lesssim \frac{1}{R^4} + \log(R)(\epsilon R^4)^3 + \log(R)\frac{1}{R^{12}}.$$

Since solutions of first-order ODEs vary smoothly with their parameters,

$$\|(v - v_p)|_{[1,R_0]}\|_{C^2} \to 0,$$

as Λ tends to ∞. However, over $A(1, R_0)$, a and b only depend on v and v_p and their derivatives up to order 2, and the result follows. □

Lemma 4.2.6. *For all* $\epsilon > 0$, *there exists* $R_0 > 0$ *such that if* $|v(1) - v_p(1)| \leq 1$, *then*

$$\|a|_{A(R_0,\infty)}\|_{C^1(G)}, \|b|_{A(R_0,\infty)}\|_{C^1(G)} \leq \epsilon. \tag{4-21}$$

Proof. Indeed, over $A(4, \infty)$, both $\hat{J}_0$ and $\hat{J}_{p,0}$ are given by (3-8). The result now follows by local uniform dependence of the estimates in (3-9) on the initial value. □

Combining these results yields:

Lemma 4.2.7. (1) *The operator norm of* D, *considered as a map from* $H^2(G)$ *into* $L^2(G)$ *converges to* 0 *as* Λ *tends to infinity.*

(2) *For sufficiently small* α, *the operator norm of* D, *considered as a map from* $C^{2,\alpha}(G)$ *into* $C^{0,\alpha}(G)$ *converges to* 0 *as* Λ *tends to infinity.*

Proof. Indeed, by (4-16), (4-17), (4-19), (4-21) and (4-20), for sufficiently small α, both $\|a\|_{C^{0,\alpha}(G)}$ and $\|b\|_{C^{0,\alpha}(G)}$ converge to 0 as Λ tends to infinity, and the result follows. □

4.3. ***The singular component.*** We now write

$$Ef =: a^i f_i. \tag{4-22}$$

Since E is defined by canonical extension, over the ball $B(\epsilon R)$,

$$a^i = \frac{2c^2}{\epsilon^2 R^4} x^i. \tag{4-23}$$

At this stage we require the following key estimate, which reveals the significance of the hybrid norm.

Lemma 4.3.1. *For sufficiently small α and for sufficiently small γ,*

$$\|f\|_{C^{1,\alpha}_\gamma(G)} \lesssim (\epsilon R)^{1-2\alpha}\|f\|_{2,\alpha,\gamma}. \tag{4-24}$$

Remark. It will be useful to observe that this relation is also valid for spaces of functions defined over an unbounded annulus.

Proof. Indeed, by the Sobolev embedding theorem, for all $\beta < 1$,

$$\|f\|_{C^{0,\beta}_\gamma(G)} \lesssim \|f\|_{H^2_\gamma(G)} \lesssim (\epsilon R)\|f\|_{2,\alpha,\gamma}.$$

Setting $\beta = (1-\alpha)$ and using (A-10) and (A-11), we obtain

$$\|f\|_{C^{1,\alpha}_\gamma(G)} \lesssim (\epsilon R)^{1/(1+2\alpha)}\|f\|_{2,\alpha,\gamma} \lesssim (\epsilon R)^{1-2\alpha}\|f\|_{2,\alpha,\gamma},$$

as desired. □

Lemma 4.3.2. *For sufficiently small $\alpha \in [0, 1]$ and for sufficiently small γ, the operator norm of E, considered as a map from $\mathcal{L}^{2,\alpha}_\gamma(G)$ into $C^{0,\alpha}_\gamma(G)$ tends to 0 as Λ tends to infinity.*

Proof. Indeed, over $A(\epsilon R, 2\epsilon R^4)$,

$$a^i = \mathrm{O}\left(\frac{\epsilon^2}{r^{3+k}}\right),$$

so that

$$\|a^i|_{A(\epsilon R,2\epsilon R^4)}\|_{C^0} \lesssim \frac{1}{\epsilon R^3} \quad \text{and} \quad [a^i|_{A(\epsilon R,2\epsilon R^4)}]_1 \lesssim \frac{1}{\epsilon^2 R^4}.$$

Since a^i is extended canonically over $B(\epsilon R)$, these inequalities also hold over the whole of $B(2\epsilon R^4)$ so that, by (A-10), for all $\alpha \in [0, 1]$,

$$[a^i]_\alpha \lesssim \frac{1}{(\epsilon R)^\alpha \epsilon R^3}.$$

It follows by (4-24) and (A-12) that

$$\|Ef\|_{C^{0,\alpha}_\gamma(G)} \lesssim \frac{1}{(\epsilon R)^\alpha \epsilon R^3}\|f\|_{C^{1,\alpha}_\gamma(G)},$$

and the result follows by (2-14) and (4-24). □

Lemma 4.3.3. *For sufficiently small $\alpha \in [0, 1]$ and for sufficiently small γ, the operator norm of $(\epsilon R)^{-1}E$ considered as a map from $\mathcal{L}^{2,\alpha}_\gamma(G)$ into $H^0_\gamma(G)$ tends to 0 as Λ tends to infinity.*

Proof. Indeed, a direct calculation yields

$$\|a^i\|_{L^2_\gamma(G)} \lesssim \frac{1}{R^2}.$$

Thus, bearing in mind (4-24),

$$\begin{aligned}\|(\epsilon R)^{-1}Ef\|_{L^2_\gamma(G)} &\lesssim (\epsilon R)^{-1}\|a^i\|_{L^2_\gamma(G)}\|Df\|_{L^\infty(G)} \\ &\lesssim (\epsilon R)^{-1}\|a^i\|_{L^2_\gamma(G)}\|f\|_{C^{1,\alpha}_\gamma(G)} \lesssim \frac{1}{(\epsilon R)^{2\alpha}R^2}\|f\|_{2,\alpha,\gamma},\end{aligned}$$

and the result follows by (2-14). □

Combining these results yields:

Lemma 4.3.4. *For sufficiently small* $\alpha \in [0, 1]$ *and for sufficiently small* γ*, the operator norm of* E *considered as a map from* $\mathcal{L}^{2,\alpha}_{\gamma}(G)$ *into* $\mathcal{L}^{0,\alpha}_{\gamma}(G)$ *tends to* 0 *as* Λ *tends to infinity.*

5. Surgery and the perturbation family

5.1. ***The basic surgery operation.*** Recall that our strategy for proving Theorem A consists of two stages. The first involves a surgery operation in which approximate MCF solitons are constructed out of properly embedded minimal surfaces and rotationally symmetric Grim ends. The second involves a fixed-point argument in which these approximate MCF solitons are perturbed into actual MCF solitons. In this section, we describe the surgery operation and in Section 5.2, we describe the family of deformations of the approximate MCF soliton in which the actual MCF soliton will be found. Though conceptually simple, our construction is inevitably rather technical. However, we believe that a careful reading of the following two sections will be rewarded by a clear understanding of the essence of this paper.

Consider first a properly embedded surface C in $\mathbb{R}^3$, minimal outside of some compact set, and with finitely many ends, all of which are horizontal. Let $R_0 > 0$ be such that every component of $C \cap (A(R_0, \infty) \times \mathbb{R})$ is a minimal graph over $A(R_0, \infty)$. Let $F : A(R_0, \infty) \to \mathbb{R}$ be the profile of one of these minimal ends. In Appendix B, we show how the Weierstrass representation yields

$$F = a + c\log(r) + \mathrm{O}(r^{-(1+k)})$$

for some real constants a and c, which will henceforth be referred to respectively as the *constant term* and the *logarithmic parameter* of the minimal end. In particular, planar ends are simply catenoidal ends with vanishing logarithmic parameters. We will only be concerned with minimal ends invariant under reflection in at least two distinct vertical planes. In this case, the above asymptotic series contains no terms of order (-1), so that

$$F = a + c\log(r) + \mathrm{O}(r^{-(2+k)}). \tag{5-1}$$

This asymptotic formula will be used repeatedly throughout the sequel.

Let Λ be a large, positive number, let $K > 0$ be a fixed constant, and choose ϵ, $R > 0$ and $|c| < K$ as in (2-14). Let $G : A(R/4, \infty) \to \mathbb{R}$ be the profile of a rotationally symmetric Grim end with constant term a, logarithmic parameter c and speed ϵ. Rescaling and integrating (2-16) we obtain, over the annulus $A(R/4, 2R^4)$,

$$G = a + c\log(r) + \frac{1}{4}\epsilon r^2 + \mathrm{O}\Big(\Big[1 + \log\Big(\frac{r}{R}\Big)\Big] r^{1-k}\Big(\epsilon r + \frac{1}{r}\Big)^3\Big). \tag{5-2}$$

Let χ_c be the cut-off function of the central transition region $A(R, 2R)$, as defined in Section A1, and define the function H over $A(R_0, \infty)$ by

$$H := \chi_c F + (1 - \chi_c)G. \tag{5-3}$$

Its graph will be called the *joined end*. Observe that H is entirely determined by F and the parameters ϵ and R. Furthermore, over the annuli $A(R_0, R)$ and $A(2R, \infty)$, H simply coincides with F and G

respectively whilst, over the annulus $A(R, 2R)$, by (5-1), (5-2) and the fact that $\chi_c = O(r^{-k})$,

$$H = a + c\log(r) + \tfrac{1}{4}\epsilon(1-\chi_c)r^2 + O(r^{-(2+k)}). \tag{5-4}$$

5.2. ***The deformation family.*** Continuing to use the notation of Section 5.1, let S denote the surface obtained by replacing each of the ends of C with their respective joined ends. We now construct a family of deformations of S out of which the actual MCF soliton will be selected when Λ is large. We first describe how the logarithmic parameters of C and S are varied. Let n denote the number of ends of C, and, for each $1 \leq i \leq n$, let $a_{0,i}$ and $c_{0,i}$ denote respectively the constant term and the logarithmic parameter of the i-th end. Let U be a neighbourhood of $(c_{0,1}, \ldots, c_{0,n})$ in $\mathbb{R}^n$ and let $(C_c)_{c\in U}$ be a smoothly varying family of immersed surfaces in $\mathbb{R}^3$ such that $C_{c_0} = C$ and, for all $c \in U$ and for all $1 \leq i \leq n$, the i-th component of $C_c \cap (A(R_0, \infty) \times \mathbb{R})$ is a horizontal, minimal end with constant term $a_{0,i}$ and logarithmic parameter c_i. Finally, for all $c \in U$, let S_c denote the surface obtained by replacing each end of C_c with its corresponding joined end, as described in Section 5.1.

Let $E : U \times S \to \mathbb{R}^3$ be a smooth function such that

(1) for all $c \in U$, $E_c := E(c, \cdot)$ parametrises S_c, and

(2) for all $c \in U$, and for all $p \in S \cap (A(R_0, +\infty) \times \mathbb{R})$, the point $E_c(p)$ lies vertically above or below the point p.

Let χ_0, χ_0', χ_ϵ' and χ_ϵ be the cut-off functions of the transition regions $A(R_0, 2R_0)$, $A(2R_0, 4R_0)$, $A(1/(2\epsilon), 1/\epsilon)$ and $A(1/\epsilon, 2/\epsilon)$ respectively, as defined in Section A1. By composing with vertical projections onto $\mathbb{R}^2$, we think of these functions also as functions defined over S. For all $c \in U$, let N_c denote the unit normal vector field over S_c. For all $1 \leq i \leq n$, let $\mathbb{I}_i : S \to \{0, 1\}$ denote the indicator function of the i-th component of $S_c \cap (A(R_0, \infty) \times \mathbb{R})$. Observe that, since this intersection is a union of graphs, every component is transverse to the unit vertical vector e_z. For all $1 \leq i \leq n$, let $\epsilon_i \in \{\pm 1\}$ be such that $\epsilon_i e_z$ lies on the same side of the i-th component as N_c. For all $c \in U$, define the *modified normal vector field* over S_c by

$$\widehat{N}_c := (\chi_\epsilon - \chi_0)\epsilon_i e_z + (1 - (\chi_\epsilon - \chi_0))N_c. \tag{5-5}$$

Observe that, over the regions $S_c \cap (B(R_0) \times \mathbb{R})$ and $S_c \cap (A(2/\epsilon, \infty) \times \mathbb{R})$, this vector field coincides with N_c whilst, over the region $S_c \cap (A(2R_0, 1/\epsilon) \times \mathbb{R})$, it coincides with $\pm e_z$. Now let V and W be neighbourhoods of 0 in $\mathbb{R}^n$ and define $\widetilde{E} : U \times V \times W \times C^\infty(S) \to C^\infty(S, \mathbb{R}^3)$ by

$$\widetilde{E}_{c,a,b,f}(p) := E_c(p) + f(p)\widehat{N}_c(p) + \sum_{i=1}^{n} \epsilon_i \mathbb{I}_i(p)\big(a_i(1-\chi_0'(p)) + b_i(1-\chi_\epsilon'(p))\big)e_z. \tag{5-6}$$

Upon reducing U, V and W if necessary, there trivially exists $\delta > 0$, which is independent of Λ, ϵ and R, such that, for all $(c, a, b) \in U \times V \times W$, and, for all $\|f\|_{C^0} < \delta$, the function $\widetilde{E}_{c,a,b,f}$ defines an immersion of S into $\mathbb{R}^3$. This concludes the description of the deformation family in which the actual MCF soliton will be found.

5.3. ***Microscopic and macroscopic perturbations.*** Continuing to use the notation of Sections 5.1 and 5.2, we consider now the first-order perturbations of S defined by the above deformation family. We classify these perturbations into two main types. Those in the direction of $C^\infty(S)$ will be called *microscopic perturbations*, and those in the directions of U, V and W will be called *macroscopic perturbations*. We now describe the first-order variations of the MCFS functional resulting from macroscopic perturbations. The first-order variations resulting from microscopic perturbations will be studied in the next section.

Recall that, as in Section A2, the MCFS functional with speed ϵ of an immersion $E : S \to \mathbb{R}^3$ is given by

$$M_E := H_E + \epsilon\langle N_E, e_z\rangle, \tag{5-7}$$

where H_E here denotes the mean curvature function of E, and N_E here denotes its unit normal vector field. We define $M_\epsilon : U \times V \times W \to C_0^\infty(S)$ such that, for all $(c, a, b) \in U \times V \times W$, and, for all $p \in S$, $M_{\epsilon,c,a,b}(p)$ is the value of this functional for the immersion $E_{c,a,b}$ at the point p. We define the operators $X_\epsilon, Y_\epsilon, Z_\epsilon : \mathbb{R}^n \to C_0^\infty(S)$ by

$$\begin{aligned}(X_\epsilon u)(p) &:= \frac{1}{\langle \widehat{N}_S, N_S\rangle}\frac{d}{dt}M_{\epsilon,c_0+tu,0,0}(p)\bigg|_{t=0},\\ (Y_\epsilon v)(p) &:= \frac{1}{\langle \widehat{N}_S, N_S\rangle}\frac{d}{dt}M_{\epsilon,c_0,tv,0}(p)\bigg|_{t=0},\\ (Z_\epsilon w)(p) &:= \frac{1}{\langle \widehat{N}_S, N_S\rangle}\frac{d}{dt}M_{\epsilon,c_0,0,tw}(p)\bigg|_{t=0}.\end{aligned} \tag{5-8}$$

These are the first-order variations of the MCFS functional arising from the three types of macroscopic perturbation. In particular, since $M_{\epsilon,c,0,0}$ vanishes over $S \cap (A(2R, +\infty) \times \mathbb{R})$ for all $c \in V$, for all $u \in \mathbb{R}^d$, Xu is supported over $S \cap (B(2R) \times \mathbb{R})$. Likewise, for all $v, w \in \mathbb{R}^n$, Yv and Zw are supported over $S \cap (A(2R_0, 4R_0) \times \mathbb{R})$ and $S \cap (A(1/(2\epsilon), 1/\epsilon) \times \mathbb{R})$ respectively. In later sections, when no ambiguity arises, the subscript ϵ will be suppressed, and these operators will be denoted simply by X, Y and Z respectively.

5.4. ***Modified Jacobi operators.*** The operator of first-order variation of the MCFS functional resulting from microscopic perturbations is none other than the modified MCFS Jacobi operator. In this section, we determine asymptotic formulae for its coefficients over different regions. We recall that, since different modifications are made on different scales, the precise definition of the modified MCFS Jacobi operator varies with context. We now describe the framework which unifies these different definitions. We will then study three different cases corresponding to, in order, CHM surfaces, rotationally symmetric Grim ends, and joined surfaces.

Consider first a general immersed surface Σ in $\mathbb{R}^3$ such that, for some $R_0 > 0$, every component of $\Sigma \cap (A(R_0, \infty) \times \mathbb{R})$ is a graph over $A(R_0, \infty)$. Let $\Lambda > 0$ be a large, positive number, let $\epsilon, R > 0$ be as in (2-14), and let $\widehat{N}_\Sigma$ be the modified normal vector field over Σ as defined in (5-5). We define $E : C_0^\infty(\Sigma) \to C^\infty(\Sigma, \mathbb{R}^3)$ by

$$E_f(p) := p + f(p)\widehat{N}_\Sigma(p).$$

Observe that if f is sufficiently small, then E_f is an immersion. Define $M : C_0^\infty(\Sigma) \to C^\infty(\Sigma)$ such that, for all such f, and for all $p \in \Sigma$, $M_f(p)$ is the value of the MCFS functional (5-7) with speed ϵ for the immersion E_f at the point p. The *modified MCFS Jacobi operator* of Σ with speed ϵ is now defined by

$$(\hat{J}_{\Sigma,\epsilon} f)(p) := \frac{1}{\langle \widehat{N}_\Sigma, N_\Sigma \rangle} \frac{d}{dt} M_{tf}(p)\bigg|_{t=0}. \tag{5-9}$$

In later sections, when no ambiguity arises, the subscript ϵ will be suppressed, and this operator will be denoted simply by $\hat{J}_\Sigma$.

Over the annulus $A(R/4, 1/\epsilon)$, since $\widehat{N}_\Sigma$ here coincides with e_z, the operator $\hat{J}_{\Sigma,\epsilon}$ is simply $\langle N_\Sigma, e_z \rangle^{-1}$ times the linearisation of the MCFS functional for graphs. Consequently, if $F : A(R/4, 1/\epsilon) \to \mathbb{R}$ is the profile of a component of $\Sigma \cap (A(R/4, 1/\epsilon) \times \mathbb{R})$ then, upon differentiating (A-7) we obtain, over this annulus,

$$\hat{J}_{\Sigma,\epsilon} f = g^{ij} f_{ij} - \mu^2 g^{ij} F_{ij} F_k f_k + 2\mu^4 F_i F_j F_k F_{ij} f_k - 2\mu^2 F_{ij} F_i f_j - \epsilon \mu^2 F_i f_i. \tag{5-10}$$

In particular, for all $v, w \in \mathbb{R}^n$, and for all $p \in S$,

$$\begin{aligned} (Yv)(p) &= -\sum_{i=1}^n \mathbb{I}_i(p) v_i (\hat{J}_{\Sigma,\epsilon} \chi_0')(p), \\ (Zw)(p) &= -\sum_{i=1}^n \mathbb{I}_i(p) w_i (\hat{J}_{\Sigma,\epsilon} \chi_\epsilon')(p). \end{aligned} \tag{5-11}$$

Now let C be a minimal end over the annulus $A(R_0, \infty)$ satisfying (5-1) and let $\hat{J}_{C,\epsilon}$ be its modified MCFS Jacobi operator with speed ϵ.

Lemma 5.4.1. *Over $A(R/4, 2R^4)$,*

$$\hat{J}_{C,\epsilon} f = \Delta f - \frac{c^2}{r^4} x^i x^j f_{ij} - \frac{\epsilon c}{r^2} x^i f_i + \frac{2c^2}{r^4} x^i f_i + \mathcal{E}_{C,\epsilon} f, \tag{5-12}$$

where $\mathcal{E}_{C,\epsilon} f := a^{ij} f_{ij} + b^i f_i$ and a and b satisfy

$$a = \mathrm{O}(r^{-(k+4)}), \quad b = \mathrm{O}\Big(r^{-(k+4)}\Big(\epsilon r + \frac{1}{r}\Big)\Big). \tag{5-13}$$

Proof. By (5-1),

$$F_i = \frac{c}{r^2} x^i + \mathrm{O}(r^{-(k+3)}).$$

Thus, by (A-6),

$$\begin{aligned} \mu^2 &= 1 - \frac{c^2}{r^2} + \mathrm{O}(r^{-(k+4)}), \\ g^{ij} &= \delta_{ij} - \frac{c^2}{r^4} x^i x^j + \mathrm{O}(r^{-(k+4)}). \end{aligned}$$

Therefore,

$$g^{ij} f_{ij} = \Delta f - \frac{c^2}{r^4} x^i x^j f_{ij} + a^{ij} f_{ij},$$

where $a = \mathrm{O}(r^{-(k+4)})$. Likewise,

$$\begin{aligned}\mu^2 g^{ij} F_{ij} F_k f_k &= b_1^i f_i,\\ 2\mu^4 F_i F_j F_k F_{ij} f_k &= b_2^i f_i,\\ -2\mu^2 F_{ij} F_i f_j &= \frac{2c^2}{r^4} x^i f_i + b_3^i f_i,\end{aligned}$$

where $b_1^i, b_2^i, b_3^i = \mathrm{O}(r^{-(k+5)})$. Finally,

$$\epsilon\mu^2 F_i f_i = \frac{\epsilon c}{r^2} x^i f_i + b_4^i,$$

where $b_4^i = \mathrm{O}(\epsilon r^{-(k+3)})$. The result follows. □

Next let G be a rotationally symmetric Grim end of speed ϵ over the annulus $A(R/4, \infty)$ and let $\hat{J}_{G,\epsilon}$ be its modified MCFS Jacobi operator with speed ϵ. Define $\psi : G \to \mathbb{R}$ by

$$\psi := \langle \widehat{N}_G, N_G\rangle = \chi_\epsilon \langle e_z, N_G\rangle + (1 - \chi_\epsilon), \tag{5-14}$$

and denote by M_ψ the operator of multiplication by ψ.

Lemma 5.4.2. *Over $A(R/4, \infty)$,*

$$\hat{J}_{G,\epsilon} := M_\psi^{-1} J_{G,\epsilon} M_\psi, \tag{5-15}$$

where $J_{G,\epsilon}$ denotes the MCFS Jacobi operator with speed ϵ of G, as defined in Section A2.

Remark. In particular, in the case of rotationally symmetric Grim ends, the modified MCFS Jacobi operator as defined above coincides, up to rescaling, with the modified MCFS Jacobi operator as defined in Section 4.1.

Proof. Indeed, more generally, with $M := M_0$ defined as at the beginning of this section, for all $f \in C_0^\infty(\Sigma)$,

$$\hat{J}_{\Sigma,\epsilon} f = M_\psi^{-1} J_{\Sigma,\epsilon} M_\psi f + M_\psi^{-1}\langle X, \nabla M\rangle f,$$

where X here denotes the tangential component of the vector field $\widehat{N}_\Sigma$. The result now follows since M vanishes identically over G. □

In particular, rescaling (4-6) and (4-7) immediately yields:

Lemma 5.4.3. *Over $A(R/4, \infty)$,*

$$\hat{J}_{G,\epsilon} f = \Delta f - \left(\frac{\epsilon}{2} + \frac{c}{r^2}\right)^2 x^i x^j f_{ij} - \left(\frac{\epsilon^2}{2} - \frac{2c^2}{r^4}\right) x^i f_i + \mathcal{E}_G f. \tag{5-16}$$

where $\mathcal{E}_{G,\epsilon} f := a^{ij} f_{ij} + b^i f_i$, and a and b satisfy

$$\begin{aligned} a &= \mathrm{O}\Big(\Big[1 + \log\Big(\frac{r}{R}\Big)\Big]\frac{1}{r^k}\Big(\epsilon r + \frac{1}{r}\Big)^4\Big),\\ b &= \mathrm{O}\Big(\Big[1 + \log\Big(\frac{r}{R}\Big)\Big]\frac{1}{r^{k+1}}\Big(\epsilon r + \frac{1}{r}\Big)^4\Big).\end{aligned} \tag{5-17}$$

Finally, let S be a joined end, as constructed in Section 5.1, and let $\hat{J}_{S,\epsilon}$ denote its modified MCFS Jacobi operator with speed ϵ.

Lemma 5.4.4. *Over $A(R, 2R)$,*

$$(\hat{J}_{S,\epsilon} - \hat{J}_{C,\epsilon})f = a_1^{ij} f_{ij} + b_1^i f_i,$$
$$(\hat{J}_{S,\epsilon} - \hat{J}_{G,\epsilon})f = a_2^{ij} f_{ij} + b_2^i f_i,$$

where a_1, a_2, b_1 and b_2 satisfy

$$\begin{aligned} a_1, a_2 &= \mathrm{O}(r^{-(4+k)}), \\ b_1, b_2 &= \mathrm{O}(r^{-(5+k)}). \end{aligned} \tag{5-18}$$

Proof. By (5-1), (5-4) and (2-14), over $A(R, 2R)$,

$$H_i - F_i = \mathrm{O}(r^{-(3+k)}),$$
$$F_i, H_i = \mathrm{O}(r^{-(1+k)}).$$

Thus, by (A-6),

$$\mu_H - \mu_F = \mathrm{O}(r^{-(4+k)}),$$
$$g_H^{ij} - g_F^{ij} = \mathrm{O}(r^{-(4+k)}).$$

The result follows for $(\hat{J}_{S,\epsilon} - \hat{J}_{C,\epsilon})$ by (5-10). The result for $(\hat{J}_{S,\epsilon} - \hat{J}_{G,\epsilon})$ follows in a similar manner, and this completes the proof. ☐

We conclude this section by studying commutators of modified Jacobi operators with certain multiplication operators. Indeed, let $[\hat{J}_{C,\epsilon}, \chi_l]$ denote the commutator of $\hat{J}_{C,\epsilon}$ with the operator of multiplication by the cut-off function χ_l of the lower transition region $A(R/4, R/2)$. Likewise, let $[\hat{J}_{G,\epsilon}, \chi_u]$ denote the commutator of $\hat{J}_{G,\epsilon}$ with the operator of multiplication by the cut-off function χ_u of the upper transition region $A(R^4, 2R^4)$. Observe that these operators are supported over the annuli $A(R/4, R/2)$ and $A(R^4, 2R^4)$ respectively.

Lemma 5.4.5. $$[\hat{J}_{C,\epsilon}, \chi_l]f = a_1^i f_i + b_1 f, \quad [\hat{J}_{G,\epsilon}, \chi_u]f = a_2^i f_i + b_2 f,$$

where a_1, a_2, b_1 and b_2 satisfy,

$$a_1, a_2 = \mathrm{O}(r^{-(k+1)}), \quad b_1, b_2 = \mathrm{O}(r^{-(k+2)}). \tag{5-19}$$

Proof. Indeed, since $\chi_l, \chi_u = \mathrm{O}(r^{-k})$, the result follows by (5-12), (5-13), (5-16) and (5-17). ☐

5.5. *Controlling macroscopic perturbations.* We conclude this section by studying the first-order variation of the MCFS functional resulting from the first macroscopic perturbation. Recall that, for all $u \in \mathbb{R}^n$, Xu vanishes outside $B(2R)$. Inside this ball, we have:

Lemma 5.5.1. *For $u \in \mathbb{R}^d$ such that $\|u\| = 1$, over $A(2R_0, R)$,*

$$Xu = \mathrm{O}(\epsilon r^{-(2+k)}), \tag{5-20}$$

and over $A(R, 2R)$,

$$Xu = \mathrm{O}(r^{-(4+k)}). \tag{5-21}$$

Proof. For notational convenience, we suppose that C and S each only have one end and, in particular, that $u = 1$. Let C_c and S_c be smooth families of immersed surfaces as in Section 5.1. For all t, let $F_t : A(2R_0, \infty) \to \mathbb{R}$ and $H_t : A(2R_0, \infty) \to \mathbb{R}$ denote the profiles of $C_{c_0+t} \cap (A(2R_0, \infty) \times \mathbb{R})$ and $S_{c_0+t} \cap (A(2R_0, \infty) \times \mathbb{R})$ respectively. Define

$$Z := \frac{d}{dt} F_t|_{t=0}, \quad W := \frac{d}{dt} H_t|_{t=0},$$

and observe that, over $A(2R_0, 2R)$,

$$Xu = \hat{J}_{S,\epsilon} W.$$

Now, by (5-1),

$$Z = \log(r) + \mathrm{O}(r^{-(2+k)}).$$

Next, by (2-41) and (5-3), and bearing in mind that $\chi_c = \mathrm{O}(r^{-k})$, over $A(R, 2R)$, we have

$$W = \log(r) + \mathrm{O}(r^{-(2+k)}) = Z + \mathrm{O}(r^{-(2+k)}), \tag{5-22}$$

and since $Z = W$ over $A(2R_0, R)$, (5-22) in fact holds over the whole of $A(2R_0, 2R)$. We now write

$$Xu = \hat{J}_{C,\epsilon} Z + (\hat{J}_{S,\epsilon} - \hat{J}_{C,\epsilon}) Z + \hat{J}_{S,\epsilon}(W - Z).$$

The second and third terms are supported over $A(R, 2R)$, and by (5-12) and (5-18),

$$(\hat{J}_{S,\epsilon} - \hat{J}_{C,\epsilon}) Z = \mathrm{O}(r^{-(6+k)}),$$
$$\hat{J}_{S,\epsilon}(W - Z) = \mathrm{O}(r^{-(4+k)}).$$

Finally, since the graph of F_t is minimal for all t, by (A-6) and (A-7),

$$\hat{J}_{C,\epsilon} Z = -\epsilon \mu^2 F_{0,i} Z_i = \mathrm{O}(\epsilon r^{-(2+k)}),$$

and the result follows by (2-14). □

6. Constructing the Green's operator

6.1. ***The cylindrical, Grim and hybrid norms.*** We now prepare the ground for the perturbation argument that will be used to construct actual MCF solitons out of the approximate MCF solitons constructed in Section 5.1. In this section, we construct the Green's operator of the modified MCFS Jacobi operator of the approximate MCF soliton together with estimates of its operator norm. It is the determination of suitable estimates, requiring a careful and lengthy analysis, which constitutes the hardest part of this paper. We will see presently that sufficiently strong estimates are made possible by the correct choice of functional norms over the different components of the approximate MCF soliton, as well as the use of the hybrid norm, already mentioned in the Introduction and Section 4. Throughout this section, we will make use of (2-14) without comment.

We first study the analytic properties of Green's operators over CHM surfaces. Thus, for g a positive integer, let $C := C_g$ be the CHM surface of genus g. Observe that functions over $C \cap (A(R_0, \infty) \times \mathbb{R})$ may be considered as functions over three copies of $A(R_0, \infty)$. In defining norms over spaces of functions, we will pass between these two perspectives without comment. Consider now the triplet $(X, Y, \hat{J}_C)$, where X and Y are the operators constructed in Section 5.1 and $\hat{J}_C$ is the modified MCFS Jacobi operator of C as constructed in Section 5.4. We now construct a right inverse for this operator when Λ is large. We first gather various basic results that will be of use to us. Let D denote the total differentiation operator over $\mathbb{R}^2$ and define

$$D_{\mathrm{SF}} := rD, \tag{6-1}$$

where r here denotes the radial distance from the origin. Likewise, for $\alpha \in [0, 1]$ and for $f : \mathbb{R}^2 \to \mathbb{R}$, define

$$\delta^{\alpha}_{\mathrm{SF}} f(r) := r^{\alpha}[f|_{A(r/2,2r)}]_{\alpha}. \tag{6-2}$$

For all nonnegative integer m, for all $\alpha \in [0, 1]$ and for all real δ, define the *scale-free weighted Hölder norm* of any m-times differentiable function $f : A(R_0, \infty) \to \mathbb{R}$ by

$$\|f\|_{C^{m,\alpha}_{\delta,\mathrm{SF}}(A(R_0,\infty))} := \sum_{i=0}^{m} \|r^{\delta} D^{i}_{\mathrm{SF}} f\|_{C^0(A(R_0,\infty))} + \|r^{\delta}\delta^{\alpha}_{\mathrm{SF}} D^{m}_{\mathrm{SF}} f\|_{C^0(]2R_0,\infty[)}. \tag{6-3}$$

For nonnegative, integer m, for all $\alpha \in [0, 1]$, for all real δ and for any m-times differentiable function $f : C \to \mathbb{R}$, define

$$\|f\|_{C^{m,\alpha}_{\delta,\mathrm{SF}}(C)} := \|f|_{C\cap(B(2R_0)\times\mathbb{R})}\|_{C^{m,\alpha}} + \|f|_{C\cap(A(R_0,\infty)\times\mathbb{R})}\|_{C^{m,\alpha}_{\delta,\mathrm{SF}}(A(R_0,\infty))}. \tag{6-4}$$

For all such m, α and δ, let $C^{m,\alpha}_{\delta,\mathrm{SF},g}(C)$ denote the space of m-times differentiable functions f over C which satisfy $\|f\|_{C^{m,\alpha}_{\delta,\mathrm{SF}}(C)} < \infty$ and which also satisfy $f \circ \sigma = f$ for every horizontal symmetry σ of C. Observe in particular that, since each of X and Y has compact support, we may also think of them as taking values in $C^{0,\alpha}_{\delta+2,\mathrm{SF},g}(C)$.

Recall that, with the above symmetries imposed, for all $\delta \in]1, 2[$, and for all $\alpha \in]0, 1[$, the *Jacobi operator* J_C of C defines an injective Fredholm map of Fredholm index (-3) from $C^{2,\alpha}_{\delta,\mathrm{SF},g}(C)$ into $C^{0,\alpha}_{\delta+2,\mathrm{SF},g}(C)$; see [Hauswirth and Pacard 2007; Morabito 2009; Nayatani 1993; Pacard 2008]).[3]

Lemma 6.1.1. *For all $\alpha \in]0, 1[$, for all $\delta \in]1, 2[$, for all $R_0 > 0$ sufficiently large, and for all $\Lambda > 0$ sufficiently large, the triplet $(X, Y, \hat{J}_C)$ defines a surjective Fredholm map from $\mathbb{R}^3 \oplus \mathbb{R}^3 \oplus C^{2,\alpha}_{\delta,\mathrm{SF},g}(C)$ into $C^{0,\alpha}_{2+\delta,\mathrm{SF},g}(C)$ of Fredholm index 3. Furthermore, the right inverse (U, V, Φ) can be chosen in such a manner that its norm is uniformly bounded, independent of Λ.*

Remark. In the sequel, R_0 will be chosen large enough for Lemma 6.1.1 to hold for all large values of Λ. It will then be fixed once and for all, and Λ will be made to tend to $+\infty$.

Proof. For all $c \in U$, where U is a suitable open subset of $\mathbb{R}^3$, let C_c be as in Section 5.1 and suppose in addition that C_c is also invariant under all the horizontal symmetries of C. Let $E : U \times C \to \mathbb{R}^3$ be a smooth function such that

(1) for all $c \in U$, E_c parametrises C_c,

[3] We aim to include an overview of the perturbation theory of the Costa–Hoffman–Meeks surfaces in forthcoming work, as we are not aware of any readily accessible account in the literature.

(2) for all $c \in U$ and for all $p \in C \cap (A(R_0, +\infty) \times \mathbb{R})$, the point $E_c(p)$ lies vertically above or below the point p, and

(3) for all $c \in U$, $E_c := E(c, \cdot)$ is equivariant under all the horizontal symmetries of C.

Let V be a neighbourhood of 0 in $\mathbb{R}^3$ and define $\widetilde{E} : U \times V \times C \to \mathbb{R}^3$ such that, for all $(c, a) \in U \times V$, and, for all $p \in C$,

$$\widetilde{E}_{c,a}(p) = E_c(p) + \sum_{i=1}^{3} \epsilon_i \mathbb{I}_i(p) a_i (1 - \chi_0'(p)) e_z,$$

where $(\epsilon_i)_{1 \le i \le 3}$, $(\mathbb{I}_i)_{1 \le i \le 3}$ and χ_0' are defined as in Section 5.1. Define $H : U \times V \times C \to \mathbb{R}^3$ such that, for all $(c, a) \in U \times V$, and for all $p \in C$, $H_{c,a}(p)$ is the mean curvature of the immersion $\widetilde{E}_{c,a}$ at the point p. Define the operators $X_0, Y_0 : \mathbb{R}^3 \to C_0^\infty(C)$ by

$$(X_0 u)(p) := \frac{d}{dt} H_{c_0 + tu, 0}(p)|_{t=0},$$
$$(Y_0 v)(p) := \frac{d}{dt} H_{c_0, tv}(p)|_{t=0}.$$

By the perturbation theory of CHM surfaces (see [Hauswirth and Pacard 2007]), (X_0, Y_0, J_C) defines a surjective Fredholm map of Fredholm index 3 from $\mathbb{R}^3 \oplus \mathbb{R}^3 \oplus C^{2,\alpha}_{\delta,\mathrm{SF},g}(C)$ into $C^{0,\alpha}_{\delta+2,\mathrm{SF},g}(C)$.

Let N and $\widehat{N}$ be respectively the unit normal vector field and the modified normal vector field over C. Observe that, as Λ and R_0 tend to $+\infty$, the difference $(\widehat{N} - N)$ tends to 0 in the C^k sense for all k so that the difference $(\hat{J}_C - J_C)$ tends to 0 in the operator norm. Next, it is straightforward to show that, considered as an operator from $\mathbb{R}^3$ into $C^{0,\alpha}_{\delta+2,\mathrm{SF},g}(C)$, $\|Y - Y_0\| \lesssim \epsilon$. Finally, by (2-14), (5-20) and (5-21), considered as another operator between these two spaces, $\|X - X_0\| \lesssim R^{\delta-2}$. Since these both tend to 0 as Λ tends to $+\infty$, the result follows by the stability of surjectivity of Fredholm maps under small perturbations. □

We now review the analytic properties of rotationally symmetric Grim ends. Let G be a rotationally symmetric Grim end of speed ϵ over the annulus $A(R/4, +\infty)$. For all nonnegative, integer m, for all $\alpha \in [0, 1]$, for all $\gamma \in \mathbb{R}$ and for all $\epsilon > 0$, define the following weighted Hölder and Sobolev norms for functions over $\mathbb{R}^2$,

$$\begin{aligned} \|f\|_{C^{m,\alpha}_{\gamma,\epsilon}(G)} &:= \|f(\cdot/\epsilon)\|_{C^{m,\alpha}_{\gamma}(G)}, \\ \|f\|_{H^{m}_{\gamma,\epsilon}(G)} &:= \|f(\cdot/\epsilon)\|_{H^{m}_{\gamma}(G)}, \end{aligned} \tag{6-5}$$

and define the *hybrid norm* by

$$\|f\|_{m,\alpha,\gamma,\epsilon} := \|f\|_{C^{m,\alpha}_{\gamma,\epsilon}(G)} + \frac{1}{\epsilon R}\|f\|_{H^{m}_{\gamma,\epsilon}(G)}. \tag{6-6}$$

For all such m, α, γ, let $\mathcal{L}^{m,\alpha}_{\gamma,\epsilon,g}(G)$ denote the space of m-times differentiable functions with finite hybrid norm. Let $\hat{J}_G$ denote the modified MCFS Jacobi operator of G, as defined in Sections 4.1 and 5.4. Upon rescaling, Theorem 4.1.1 immediately yields

Lemma 6.1.2. *For all $\alpha \in]0, 1[$, for all sufficiently small γ, and for sufficiently large Λ, the operator $\epsilon^2 \hat{J}_G$ defines a linear isomorphism from $\mathcal{L}^{2,\alpha}_{\gamma,\epsilon,g}(G)$ into $\mathcal{L}^{0,\alpha}_{\gamma,\epsilon,g}(G)$. Furthermore, we may suppose that the operator norm of its inverse is uniformly bounded independent of Λ.*

We conclude this section by describing an alternative form of (6-5), more amenable to calculations. We define operators D_G and δ_G^α by

$$D_G := \frac{1}{\epsilon} D, \tag{6-7}$$

$$\delta_G^\alpha f(x) := \frac{1}{\epsilon^\alpha} [f|_{B(x,1/\epsilon)}]_\alpha. \tag{6-8}$$

Up to uniform equivalence, for any function f supported in $A(R/4, 2R^4)$,

$$\|f\|_{C^{m,\alpha}_{\gamma,\epsilon}(G)} = \sum_{i=0}^{m} \|D_G^i f\|_{C^0} + \|\delta_G^\alpha D_G^m f\|_{C^0}. \tag{6-9}$$

Likewise, let dVol denote the canonical volume form of $\mathbb{R}^2$ and, in analogy to (6-1), (6-2), (6-7) and (6-8), define

$$\mathrm{dVol}_{\mathrm{SF}} := \frac{1}{r^2}\,\mathrm{dVol}, \quad \mathrm{dVol}_G := \epsilon^2\,\mathrm{dVol}. \tag{6-10}$$

In particular, a formula similar to (6-9) also holds for $\|f\|_{H^m_{\gamma,\epsilon}(G)}$ when f is supported over the annulus $A(R/4, 2R^4)$. It is these forms of the norms introduced in (6-5) that we will use in the sequel.

Comparing (6-1) and (6-7) reveals a key phenomenon that must be addressed in order to obtain good estimates. Indeed, over the transition region $A(R/4, 2R)$, the respective differentiation operators of the CHM surface and the Grim ends are approximately related to one another by

$$D_G \simeq \frac{1}{\epsilon R} D_{\mathrm{SF}}, \tag{6-11}$$

so that, whenever a function is transferred from the CHM surface to one of the Grim ends, each order of differentiation introduces a factor of roughly $1/(\epsilon R)$ into the norm. This factor, which is inevitably large, would be ruinous for our estimates unless correctly addressed, and it is in order to do so that we adopt the following two measures. Firstly, we use norms of the least possible order, and likewise take α to be arbitrarily small (see Theorems 6.4.1, 6.5.2 and 6.5.3). In particular, any term involving an exponent of α may be considered heuristically to be close to 1 (see, for example, (6-15), (6-16), (6-19), and so on). Secondly, and more significantly, it is precisely in order to tame this phenomenon that the hybrid norm is introduced. To see how this works, recall that the Sobolev embedding theorem states that, for all m, the Sobolev norm of order m is roughly comparable to the Hölder norm of order $(m-1)$. That is, although the second-order Sobolev norm depends on the second derivative, from a scaling perspective, it behaves more like a first derivative. It is precisely for this reason that the introduction of the factor of $1/(\epsilon R)$ in (6-6) yields a norm which scales, roughly, like a second derivative whilst furnishing, via the Sobolev embedding theorem, stronger information about the first derivative than we would have obtained by working with the Hölder norm alone.

6.2. ***Ping-pong*: *overview*.** We now describe the iteration process used to construct the Green's operator of the approximate MCF soliton. As before, for g a positive integer, let $C := C_g$ denote the CHM surface of genus g and let $S := S_g$ denote the surface obtained by replacing each of its ends with their respective joined ends, as described in Section 5.1. Since there is a natural diffeomorphism from C to S which maps points in the ends of C vertically upwards or downwards, functions over C may equally well be considered as functions over S and vice versa. As before, we will pass between these two perspectives without comment.

Before proceeding, it is worth reviewing the role played by each component within the iteration process that we will apply. We first recall from the previous section that a CHM surface C has been joined to the union $G := G_1 \cup G_2 \cup G_3$ of three Grim ends to yield an approximate soliton S. The surgery is carried out above the annulus $A(R, 2R)$, which we call the *central transition region*. However, these surfaces also all overlap over the larger annulus $A(R/4, 2R^4)$. Consequently, functions supported above $B(2R^4)$ are viewed as functions over C, functions supported over $A(R/4, \infty)$ are viewed as functions over G, and functions supported over $A(R/4, 2R^4)$ are viewed alternately, at different stages of the process, as functions over C and G.

Our aim is to construct a right inverse of the modified Jacobi operator $\hat{J}_S$ of S, using the right inverses of the respective modified Jacobi operators $\hat{J}_C$ and $\hat{J}_G$ of C and G. Ignoring for the moment the finite-dimensional components X, Y, Z and W, we proceed as follows. First let $e : S \to \mathbb{R}$ be a function supported above $B(2R)$. Let χ_u denote the cut-off function of the annulus $A(R^4, 2R^4)$, which we call the *upper transition region*. Viewing e as a function over C, we view $\chi_u(\hat{J}_C^{-1})e$ as an approximator for $(\hat{J}_S^{-1})e$, the cut-off function being here necessary to yield a function supported over $B(2R^4)$, which we may view as a function over S. The error of this approximation is measured by the function $f := \hat{J}_S \chi_u(\hat{J}_C^{-1})e$. Since $\hat{J}_S$ coincides with $\hat{J}_C$ above $B(R)$, this function is supported above $A(R, \infty)$, and we may thus view it as a function over G. In this manner, we have concluded the "upward" stage of the process. Repeating the process in the "downward" direction then yields a function e' supported above $B(2R)$, and the process may then be iterated indefinitely.

Proceeding in this manner, we obtain two sequences $(e_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ of successive errors which should ideally both converge to 0. In this and the next section, estimates for these functions will be obtained in a pointwise manner via the definitions of the norms. In this process, we will encounter some phenomena driving growth and others driving decay. Convergence is ensured upon choosing parameters in such a manner that the latter dominate. The main contributor to growth is the large norm (6-23) of $\hat{J}_G^{-1}$ resulting from the rescaling of the Grim ends. The main contributor to decay is the tendency of bounded harmonic functions to decay over long cylinders, already outlined in Section 1.2, and here encoded implicitly in the weighted Hölder norm introduced in Section 6.1. Roughly speaking, if the radii of the lower and upper transition region are respectively proportional to $R/2$ and R^λ, then the two will be separated by an annulus conformally equivalent to a cylinder roughly of length $(\lambda-1)\,\mathrm{Ln}(R)$. We thus choose λ as large as possible in order to maximise decay. We have already seen in Section 4 that the strict upper bound $\lambda < 5$ is required in order to obtain uniform estimates for the norms of the Green's operators of the Grim ends (see the proofs of Lemmas 4.2.1 and 4.2.2), and it turns out that $\lambda \in \,]4, 5[$ is sufficient for our purposes.

It remains only to explain the finite-dimensional components in (6-13) and (6-24). It is common in singular perturbation constructions for the Green's operators used to have singular subspaces over which divergence occurs more rapidly than over the rest of the space. This can be understood as a consequence of the existence of a "kernel at infinity", itself often associated to symmetries of the construction, such as vertical translations and dilatations (or, equivalently, variations of the logarithmic parameter). It is thus common to introduce "geometric" terms which, by eliminating the kernel at infinity, allow us to focus on the essential asymptotic behaviours of the Green's operators used, and this is the role played by these finite-dimensional components. Finally, we observe that infinitesimal vertical translations can in fact be introduced in two different ways. Indeed, they can be introduced either in the "upward" stage, as infinitesimal vertical translations of the ends of the CHM surface, or in the "downward" stage as infinitesimal vertical translations of the Grim ends. The former addresses the kernel at infinity of the Green's operator of the CHM surface, whilst the latter addresses the kernel at infinity of the Grim ends. Thus, despite their superficial equivalence, they play distinct roles in the construction, and are both required for it to work.

6.3. *Ping-pong*: *batting up*. For notational convenience, we will henceforth work as if C and S had only one end. Consider now the following seminorms for functions over S:

$$\begin{aligned} \|f\|_{m,C} &:= \|f|_{B(0,4R)}\|_{C^{m,\alpha}_{(2-m)+\delta,\mathrm{SF}}(C)}, & \|f\|_{m,G,S} &:= \|f|_{A(R,\infty)}\|_{H^m_{\gamma,\epsilon}(G)}, \\ \|f\|_{m,G,H} &:= \|f|_{A(R,\infty)}\|_{C^{m,\alpha}_{\gamma,\epsilon}(G)}, & \|f\|_{m,G} &:= \|f\|_{m,G,H} + \frac{1}{\epsilon R}\|f\|_{m,G,S}. \end{aligned} \tag{6-12}$$

Let $\mathcal{E}$ denote the closure with respect to $\|\cdot\|_{0,C}$ of the space of functions supported over $S\cap(B(4R)\times\mathbb{R})$ which are invariant under every horizontal symmetry of the CHM surface C. Likewise, let $\mathcal{F}$ denote the closure with respect to $\|\cdot\|_{0,G}$ of the space of functions supported over $S\cap(A(R,\infty)\times\mathbb{R})$ that are also invariant under these symmetries.

We define the operator $A:\mathcal{E}\to\mathcal{F}$ by

$$Ae := \hat{J}_S\chi_u\Phi e + XUe + YVe - e, \tag{6-13}$$

where χ_u is the cut-off function of the upper transition region $A(R^4, 2R^4)$, and (U, V, Φ) is defined as in Lemma 6.1.1. This operator measures the extent to which $(U, V, \chi_u\Phi)$ fails to be a Green's operator of $(X, Y, \hat{J}_S)$ for functions in $\mathcal{E}$. In particular, since $\hat{J}_S$ coincides with $\hat{J}_C$ over $B(0, R)$, Ae is supported in the interior of $A(R,\infty)$ making it indeed an element of $\mathcal{F}$. In addition, by the definition of $\hat{J}_S$, and bearing in mind that X and Y are both supported in $B(2R)$,

$$Ae = [\hat{J}_G, \chi_u]\Phi e + \chi_u(\hat{J}_S - \hat{J}_C)\Phi e. \tag{6-14}$$

In this section, we prove:

Theorem 6.3.1. *For all* $\delta > 1$,

$$\|Ae\|_{0,G} \lesssim \frac{1}{(\epsilon R)^{2\alpha}}\frac{1}{R^{6+\delta}}\|e\|_{0,C}. \tag{6-15}$$

Theorem 6.3.1 follows immediately from (6-14) together with (6-16), (6-18), (6-19) and (6-21), below, and the fact that

$$\|\chi_u\|_{C^{0,\alpha}_{\gamma,\epsilon}(G)} \lesssim \frac{1}{(\epsilon R^4)^\alpha} \lesssim \frac{1}{(\epsilon R)^\alpha}.$$

For convenience, we now define $\phi := \Phi e$.

Lemma 6.3.2.
$$\|(\hat{J}_S-\hat{J}_C)\Phi e|_{A(R,2R^4)}\|_{C^{0,\alpha}_{\gamma,\epsilon}(G)}\lesssim\frac{1}{(\epsilon R)^\alpha}\frac{1}{R^{6+\delta}}\|e\|_{0,C}. \tag{6-16}$$

Proof. Indeed, by (6-1), for $k\in\{0,1,2\}$, over $A(R,2R^4)$,
$$|D^k\phi|\lesssim\frac{1}{r^{k+\delta}}\|\phi\|_{C^{2,\alpha}_{\delta,\mathrm{SF}}(C)}\lesssim\frac{1}{r^{k+\delta}}\|e\|_{0,C}.$$
Likewise, by (6-2), for all $r\in[2R,R^4]$,
$$|\delta^\alpha(D^2\phi|_{A(r/2,2r)})|\lesssim\frac{1}{r^{k+\alpha+\delta}}\|e\|_{0,C}.$$
Thus, by (5-12), (5-13), (5-16), (5-17) and (5-18), over $A(R,2R^4)$,
$$|(\hat{J}_S-\hat{J}_C)\phi|\lesssim\left(\frac{\epsilon}{r^{2+\delta}}+\frac{\epsilon^2}{r^\delta}+\left[1+\log\left(\frac{r}{R}\right)\right]\epsilon^4r^{2-\delta}+\left[1+\log\left(\frac{r}{R}\right)\right]\frac{1}{r^{6+\delta}}\right)\|e\|_{0,C}, \tag{6-17}$$
so that, by (A-20),
$$|(\hat{J}_S-\hat{J}_C)\phi|_{A(R,2R^4)}|\lesssim\frac{1}{R^{6+\delta}}\|e\|_{0,C}.$$
Likewise, using also (A-10) and (A-12), for $r\in[2R,R^4]$,
$$|\delta^\alpha((\hat{J}_S-\hat{J}_C)\phi|_{A(r/2,2r)})|\lesssim\frac{1}{r^\alpha}\left(\frac{\epsilon}{r^{2+\delta}}+\frac{\epsilon^2}{r^\delta}+\left[1+\log\left(\frac{r}{R}\right)\right]\epsilon^4r^{2-\delta}+\left[1+\log\left(\frac{r}{R}\right)\right]\frac{1}{r^{6+\delta}}\right)\|e\|_{0,C},$$
so that, by (6-8), for $r\in[2R,R^4]$,
$$|\delta^\alpha_G((\hat{J}_S-\hat{J}_C)\phi|_{A(r/2,2r)})|\lesssim\frac{1}{(\epsilon r)^\alpha}\left(\frac{\epsilon}{r^{2+\delta}}+\frac{\epsilon^2}{r^\delta}+\left[1+\log\left(\frac{r}{R}\right)\right]\epsilon^4r^{2-\delta}+\left[1+\log\left(\frac{r}{R}\right)\right]\frac{1}{r^{6+\delta}}\right)\|e\|_{0,C}.$$
Thus, by (A-14) and (A-20),
$$|\delta^\alpha_G((\hat{J}_S-\hat{J}_C)\phi|_{A(R,2R^4)})|\lesssim\frac{1}{(\epsilon R)^\alpha}\frac{1}{R^{6+\delta}}\|e\|_{0,C}.$$
The result follows upon combining the above relations. □

Lemma 6.3.3. *For all* $\delta>1$,
$$\|(\hat{J}_S-\hat{J}_C)\Phi e|_{A(R,2R^4)}\|_{H^0_{\gamma,\epsilon}(G)}\lesssim\frac{(\epsilon R)}{R^{6+\delta}}\|e\|_{0,C}. \tag{6-18}$$

Proof. By (6-10) and (6-17), over $A(R,2R^4)$,
$$|(\hat{J}_S-\hat{J}_C)\phi|^2\,\mathrm{dVol}_G$$
$$\lesssim\left(\frac{\epsilon^4}{r^{2+2\delta}}+\epsilon^6r^{2-2\delta}+\left[1+\log\left(\frac{r}{R}\right)^2\right]\epsilon^{10}r^{6-2\delta}+\left[1+\log\left(\frac{r}{R}\right)^2\right]\frac{\epsilon^2}{r^{10+2\delta}}\right)\|e\|^2_{0,C}\,\mathrm{dVol}_{\mathrm{SF}},$$
so that, by (A-21),
$$\int_{A(R,2R^4)}|(\hat{J}_S-\hat{J}_C)\phi|^2\,\mathrm{dVol}_G\lesssim\frac{(\epsilon R)^2}{R^{12+2\delta}}\|e\|^2_{0,C},$$
and the result follows. □

Lemma 6.3.4. *For all* $\delta > 1$,

$$\|[\hat{J}_G, \chi_u]\Phi e\|_{C^{0,\alpha}_{\gamma,\epsilon}(G)} \lesssim \frac{1}{(\epsilon R^4)^\alpha}\frac{1}{R^{8+4\delta}}\|e\|_{0,C}. \tag{6-19}$$

Proof. By (6-1) and (6-3) for $k \in \{0, 1, 2\}$, over $A(R^4, 2R^4)$,

$$|D^k\phi| \lesssim \frac{1}{R^{4k+4\delta}}\|\phi\|_{C^{2,\alpha}_{\delta,\mathrm{SF}}(C)} \lesssim \frac{1}{R^{4k+4\delta}}\|e\|_{0,C}.$$

It follows by (5-19) that, for $k \in \{0, 1\}$, over this annulus,

$$|D^k[\hat{J}_G, \chi_u]\phi| \lesssim \frac{1}{R^{8+4k+4\delta}}\|e\|_{0,C}. \tag{6-20}$$

Thus, by (6-7), for $k \in \{0, 1\}$, over this annulus,

$$|D^k_G[\hat{J}_G, \chi_u]\phi| \lesssim \frac{1}{(\epsilon R^4)^k}\frac{1}{R^{8+4\delta}}\|e\|_{0,C},$$

and the result follows by (A-10). □

Lemma 6.3.5.

$$\|[\hat{J}_G, \chi_u]\Phi e\|_{H^0_{\gamma,\epsilon}(G)} \lesssim \frac{(\epsilon R)}{R^{5+4\delta}}\|e\|_{0,C}. \tag{6-21}$$

Proof. By (6-20) and (6-10), over $A(R^4, 2R^4)$,

$$|[\hat{J}_G, \chi_u]\phi|^2 \,\mathrm{dVol}_G \lesssim \frac{\epsilon^2}{R^{8+8\delta}}\|e\|^2_{0,C}\,\mathrm{dVol}_{\mathrm{SF}},$$

so that, by (A-21),

$$\int_{A(R^4,2R^4)} |[\hat{J}_G, \chi_u]\phi|^2 \,\mathrm{dVol}_G \lesssim \frac{\epsilon^2}{R^{8+8\delta}}\|e\|_{0,C},$$

and the result follows. □

These estimates prove Theorem 6.3.1. In addition, the following estimate will also be of use later.

Lemma 6.3.6. *For all* $\delta > 1$,

$$\|\chi_u\Phi e\|_{2,G} \lesssim \frac{1}{(\epsilon R)^\alpha}\frac{1}{\epsilon^2 R^{2+\delta}}\|e\|_{0,C}. \tag{6-22}$$

Proof. Indeed, since $\chi_u = O(r^{-k})$, we have $\|\chi_u\|_{C^{2,\alpha}_{0,\mathrm{SF}}(C)} \lesssim 1$. Thus

$$\|\chi_u\phi\|_{C^{2,\alpha}_{\delta,\mathrm{SF}}(C)} \lesssim \|\phi\|_{C^{2,\alpha}_{\delta,\mathrm{SF}}(C)} \lesssim \|e\|_{C^{0,\alpha}_{2+\delta,\mathrm{SF}}(C)} = \|e\|_{0,C}.$$

Thus, by (6-1), (6-3) and (6-7), for $k \in \{0, 1, 2\}$, over $A(R, 2R^4)$,

$$|D^k_G\chi_u\phi| \lesssim \frac{1}{(\epsilon r)^k}\frac{1}{r^\delta}\|e\|_{0,C}.$$

Likewise, by (6-2), (6-3) and (6-8), for all $r \in [2R, R^4]$,

$$|\delta^\alpha_G(D^2_G\chi_u\phi|_{A(r/2,2r)})| \lesssim \frac{1}{(\epsilon r)^{2+\alpha}}\frac{1}{r^\delta}\|e\|_{0,C},$$

so that, by (A-14),

$$|\delta_G^\alpha(D_G^2\chi_u\phi|_{A(R,2R^4)})|\lesssim\frac{1}{(\epsilon R)^{2+\alpha}}\frac{1}{R^\delta}\|e\|_{0,C}.$$

Combining the above relations yields

$$\|\chi_u\phi\|_{2,G,H}\lesssim\frac{1}{(\epsilon R)^\alpha}\frac{1}{\epsilon^2R^{2+\delta}}\|e\|_{0,C}.$$

Likewise, by (6-10), for all k, over $A(R,2R^4)$,

$$|D_G^k\chi_u\phi|^2\,\mathrm{dVol}_G\lesssim\frac{1}{(\epsilon r)^{2k}}\frac{1}{r^{2\delta}}\|e\|_{0,C}^2(\epsilon r)^2\,\mathrm{dVol}_{\mathrm{SF}},$$

and since $\delta>1$, it follows by (A-21) that

$$\|\chi_u\phi\|_{2,G,S}\lesssim\frac{1}{R^\delta}\|e\|_{0,C}\lesssim\frac{1}{\epsilon R^{1+\delta}}\|e\|_{0,C}.$$

The result follows. □

6.4. *Ping-pong*: *batting down.* By Lemma 6.1.2, there exists a linear map $\Psi: C^{0,\alpha}_{\gamma,\epsilon,g}(G)\cap H^0_{\gamma,\epsilon,g}(G)\to C^{2,\alpha,g}_{\gamma,\epsilon,g}(G)\cap H^2_{\gamma,\epsilon,g}(G)$ such that, for all $f\in\mathcal{F}$,

$$f=\hat{J}_G\Psi f,$$

and

$$\|\Psi f\|_{2,\alpha,\gamma,\epsilon}\lesssim\frac{1}{\epsilon^2}\|f\|_{0,\alpha,\gamma,\epsilon}. \tag{6-23}$$

Define the operators $B:\mathcal{F}\to\mathcal{E}$ and $W:\mathcal{F}\to\mathbb{R}^3$ by

$$\begin{aligned}Bf&:=\hat{J}_S(1-\chi_l)(\Psi f-\chi'_\epsilon(Wf))-ZWf-f,\\ Wf&:=(\Psi f)(0),\end{aligned} \tag{6-24}$$

where χ_l is the cut-off function of the lower transition region $A(R/4,R/2)$, and χ'_ϵ is the cut-off function of the transition region $A(1/2\epsilon,1/\epsilon)$, as in Section 5.1. As before, B measures the extent to which $(-W,(1-\chi_l)(\Psi-\chi'_\epsilon W))$ fails to be a Green's operator of $(Z,\hat{J}_S)$ for functions in $\mathcal{F}$. In particular, by (5-11) together with the fact that $\hat{J}_S$ coincides with $\hat{J}_G$ over $A(2R,\infty)$, Bf is supported in $B(4R)$, and is thus indeed an element of $\mathcal{E}$. In addition, since $\chi'_\epsilon=1$ over $B(4R)$, over this ball, we have

$$Bf=-[\hat{J}_C,\chi_l](\Psi f-(\Psi f)(0))+(1-\chi_l)(\hat{J}_S-\hat{J}_G)\Psi f. \tag{6-25}$$

In this section, we prove:

Theorem 6.4.1. *For sufficiently small α,*

$$\|Bf\|_{0,C}\lesssim\frac{R^2}{(\epsilon R)}\|f\|_{0,G}. \tag{6-26}$$

Theorem 6.4.1 follows immediately from (6-25) together with (6-28) and (6-30), below, and the fact that

$$\|(1-\chi_l)\|_{C^{0,\alpha}_{0,\mathrm{SF}}(C)}\lesssim 1.$$

For convenience, we now define $\psi:=\Psi f$.

Lemma 6.4.2.
$$\|Wf\| \lesssim \frac{R^2}{(\epsilon R)}\|f\|_{0,G}. \tag{6-27}$$

Proof. Indeed, by the Sobolev embedding theorem,

$$\|Wf\| \lesssim \|\Psi f\|_{H^2_{\gamma,\epsilon}(G)} \lesssim (\epsilon R)\|\Psi f\|_{2,\alpha,\gamma,\epsilon}.$$

Thus, by (6-23),

$$\|Wf\| \lesssim \frac{R}{\epsilon}\|f\|_{0,\alpha,\gamma,\epsilon} \lesssim \frac{R^2}{(\epsilon R)}\|f\|_{0,G},$$

as desired. □

Lemma 6.4.3.
$$\|(\hat{J}_S - \hat{J}_G)\Psi f|_{A(R/4,2R)}\|_{C^{0,\alpha}_{2+\delta,\mathrm{SF}}(C)} \lesssim \frac{1}{(\epsilon R)}\frac{1}{R^{2-\delta}}\|f\|_{0,G}. \tag{6-28}$$

Proof. Indeed, by (6-7), for $k \in \{0, 1, 2\}$, over $A(R/4, 2R)$,

$$|D^k\psi| \lesssim \epsilon^k\|\psi\|_{C^{2,\alpha}_{\gamma,\epsilon}(G)} \lesssim \frac{1}{\epsilon^{2-k}}\|f\|_{0,G},$$

and so, by (5-12), (5-13), (5-16), (5-17) and (5-18), over $A(R/4, 2R)$,

$$|(\hat{J}_S - \hat{J}_G)\psi| \lesssim \frac{1}{(\epsilon R)}\left(\epsilon + \epsilon^2R^2 + \frac{1}{R^4}\right)\|f\|_{0,G} \lesssim \frac{1}{(\epsilon R)}\frac{1}{R^4}\|f\|_{0,G}.$$

Likewise, by (6-8),

$$|\delta^\alpha(D^2\psi|_{A(R/4,2R)})| \lesssim \epsilon^\alpha\|f\|_{0,G} \lesssim \frac{1}{R^\alpha}\|f\|_{0,G}.$$

Thus, by (6-2), using also (A-10) and (A-12),

$$|\delta^\alpha_{\mathrm{SF}}((\hat{J}_S - \hat{J}_G)\psi|_{A(R/4,2R)})| \lesssim \frac{1}{(\epsilon R)}\frac{1}{R^4}\|f\|_{0,G},$$

and the result follows. □

Lemma 6.4.4.
$$\|\Psi f - (\Psi f)(0)|_{A(R/4,2R)}\|_{2,C} \lesssim \frac{R^{2+\delta}}{(\epsilon R)^{2\alpha}}\|f\|_{0,G}. \tag{6-29}$$

Proof. Bearing in mind (6-8) and the Sobolev embedding theorem, over $A(R/4, 2R)$,

$$[\psi_0] \lesssim (\epsilon R)^{1-\alpha}\|\psi\|_{C^{0,1-\alpha}_{\gamma,\epsilon}(G)} \lesssim (\epsilon R)^{1-\alpha}\|\psi\|_{H^2_{\gamma,\epsilon}(G)} \lesssim (\epsilon R)^{2-\alpha}\|\psi\|_{2,G}.$$

Consequently, by (6-23),

$$[\psi_0] \lesssim \frac{R^2}{(\epsilon R)^\alpha}\|f\|_{0,G}.$$

Likewise, by (4-24) and the subsequent remark, over this annulus,

$$|D_G\psi| \lesssim (\epsilon R)^{1-2\alpha}\|\psi\|_{2,G} \lesssim \frac{1}{(\epsilon R)^{2\alpha}}\frac{R}{\epsilon}\|f\|_{0,G}.$$

Finally, over this annulus,

$$|D^2_G\psi| \lesssim \|\psi\|_{C^{2,\alpha}_{\gamma,\epsilon}(G)} \lesssim \frac{1}{\epsilon^2}\|f\|_{0,G},$$

and

$$|\delta_G^\alpha(D_G^2\psi|_{A(R/4,2R)})| \lesssim \|\phi\|_{C^{2,\alpha}_{\gamma,\epsilon}(G)} \lesssim \frac{1}{\epsilon^2}\|f\|_{0,G}.$$

The result now follows by (6-1), (6-2), (6-3), (6-7), (6-8) and (2-14). □

Lemma 6.4.5. *For sufficiently small* α,

$$\|[\hat{J}_C,\chi_l](\Psi f-(\Psi f)(0))\|_{C^{0,\alpha}_{2+\delta,\mathrm{SF}}(C)} \lesssim \frac{1}{(\epsilon R)^{2\alpha}}R^{2+\delta}\|f\|_{0,G}. \tag{6-30}$$

Proof. This follows from (5-19) and (6-29). □

6.5. *Ping-pong: iteration.* By (6-15) and (6-26), for $\delta\in\,]1,2[$ and for sufficiently small α, the operator norms of the products AB and BA satisfy

$$\|AB\|,\|BA\| \lesssim \frac{1}{(\epsilon R)^{2\alpha}}\frac{1}{\epsilon R^{5+\delta}} \lesssim \frac{1}{\Lambda}.$$

We therefore define $Q_E:\mathcal{E}\to\mathcal{E}$ and $Q_F:\mathcal{F}\to\mathcal{F}$ by

$$Q_E := \sum_{m=0}^{\infty}(BA)^m,\quad Q_F := \sum_{m=0}^{\infty}(AB)^m. \tag{6-31}$$

In particular, the operator norms of both Q_E and Q_F are uniformly bounded for large values of Λ. We now define

$$\begin{aligned}
U_C e &:= UQ_E e, & U_G f &:= -UBQ_F f,\\
V_C e &:= VQ_E e, & V_G f &:= -VBQ_F f,\\
W_C e &:= WAQ_E e, & W_G f &:= -WQ_F f,
\end{aligned}$$

$$\begin{aligned}
P_C e &:= \chi_u\Phi Q_E e-(1-\chi_l)(\Psi AQ_E e-\chi_\epsilon'(WAQ_E e)),\\
P_G f &:= -\chi_u\Phi BQ_F f+(1-\chi_l)(\Psi Q_F f-\chi_\epsilon'(WQ_F f)).
\end{aligned} \tag{6-32}$$

Lemma 6.5.1. *For all* $e\in\mathcal{E}$ *and for all* $f\in\mathcal{F}$,

$$\begin{aligned}
\hat{J}_S P_C e+XU_C e+YV_C e+ZW_C e &= e,\\
\hat{J}_S P_G f+XU_G f+YV_G f+ZW_G f &= f.
\end{aligned} \tag{6-33}$$

Proof. Indeed, bearing in mind (6-13) and (6-24),

$$\begin{aligned}
&\hat{J}_S P_C e+XU_C e+YV_C e+ZW_C e\\
&\qquad= \hat{J}_S\chi_u\Phi Q_E e+XUQ_E e+YVQ_E e-\hat{J}_S(1-\chi_l)(\Psi AQ_E e-\chi_\epsilon'(WAQ_E e))+ZWAQ_E e\\
&\qquad= AQ_E e+Q_E e-BAQ_E e-AQ_E e=e.
\end{aligned}$$

The second relation follows in a similar manner, and this completes the proof. □

Now let χ be the cut-off function of the transition region $A(2R,4R)$. Since $\chi=O(r^{-k})$, for all f,

$$\|\chi f\|_{0,C}\lesssim\|f\|_{0,C},\quad \|(1-\chi)f\|_{0,G}\lesssim\frac{1}{(\epsilon R)^\alpha}\|f\|_{0,G}. \tag{6-34}$$

Define

$$\begin{aligned}\widehat{U}f &:= U_C\chi f + U_G(1-\chi)f, & \widehat{W}f &:= W_C\chi f + W_G(1-\chi)f,\\ \widehat{V}f &:= V_C\chi f + V_G(1-\chi)f, & \widehat{P}f &:= P_C\chi f + P_G(1-\chi)f.\end{aligned} \tag{6-35}$$

In particular, by (6-33),

$$\hat{J}_S\widehat{P}f + X\widehat{U}f + Y\widehat{V}f + Z\widehat{W}f = f, \tag{6-36}$$

so that $(\widehat{U}, \widehat{V}, \widehat{W}, \widehat{P})$ defines a Green's operator for $(X, Y, Z, \hat{J}_S)$. We conclude this section by determining the norms of its different components. First, since the operator norms of U and V are uniformly bounded, by (6-26), (6-32) and (6-34)

$$\begin{aligned}\|\widehat{U}f\| &\lesssim \|f\|_{0,C} + \frac{R^2}{(\epsilon R)^{1+\alpha}}\|f\|_{0,G},\\ \|\widehat{V}f\| &\lesssim \|f\|_{0,C} + \frac{R^2}{(\epsilon R)^{1+\alpha}}\|f\|_{0,G}.\end{aligned} \tag{6-37}$$

Theorem 6.5.2. *For sufficiently small α, for all $\delta \in]1, 2[$, and for all f,*

$$\|\widehat{W}f\| \lesssim \|f\|_{0,C} + \frac{R^2}{(\epsilon R)^{1+\alpha}}\|f\|_{0,G}. \tag{6-38}$$

Proof. For $e \in \mathcal{E}$, by (6-15) and (6-27),

$$\|W_C e\| = \|WAQ_E e\| \lesssim \frac{R^2}{(\epsilon R)}\|AQ_E e\|_{0,G} \lesssim \frac{1}{(\epsilon R)^{2\alpha}}\frac{1}{\epsilon R^{5+\delta}}\|e\|_{0,C} \lesssim \|e\|_{0,C}.$$

For $f \in \mathcal{F}$, by (6-27),

$$\|W_G f\| = \|WQ_F f\| \lesssim \frac{R^2}{(\epsilon R)}\|f\|_{0,G}.$$

The result now follows by (6-34). □

Theorem 6.5.3. *For sufficiently small α, for all $\delta \in]1, 2[$, and for all f,*

$$\|\widehat{P}f\|_{2,C} \lesssim \|f\|_{0,C} + \frac{R^2}{(\epsilon R)^{1+\alpha}}\|f\|_{0,G}. \tag{6-39}$$

Proof. Consider $e \in \mathcal{E}$. Observe that, over $B(4R)$,

$$P_C e = \Phi Q_E e - (1-\chi_l)(\Psi AQ_E e - \Psi AQ_E e(0)).$$

Now,

$$\|\Phi Q_E e\|_{2,C} \lesssim \|e\|_{0,C},$$

and by (6-15) and (6-29),

$$\|(1-\chi_l)(\Psi AQ_E e - (\Psi AQ_E e)(0))\|_{2,C} \lesssim \frac{1}{(\epsilon R)^{4\alpha}R^4}\|e\|_{0,C} \lesssim \|e\|_{0,C},$$

so that

$$\|P_C e\|_{2,C} \lesssim \|e\|_{0,C}.$$

Now consider $f \in \mathcal{F}$. Over $B(4R)$,

$$P_G f = -\Phi B Q_F f - (1-\chi_l)(\Psi Q_F f - \Psi Q_F f(0)).$$

By (6-26),

$$\|\Phi B Q_F f\|_{2,C} \lesssim \frac{R^2}{(\epsilon R)} \|f\|_{0,G},$$

and, by (6-29),

$$\|(1-\chi_l)(\Psi Q_F f - (\Psi Q_F f)(0))\|_{2,C} \lesssim \frac{R^{2+\delta}}{(\epsilon R)^{2\alpha}} \|f\|_{0,G},$$

so that,

$$\|P_G f\|_{2,C} \lesssim \frac{R^2}{(\epsilon R)} \|f\|_{0,G}.$$

The result now follows by (6-34) and (6-35). ☐

Theorem 6.5.4. *For sufficiently small α, for all $\delta \in]1,2[$, and for all f,*

$$\|\widehat{P} f\|_{2,G} \lesssim \frac{1}{(\epsilon R)^\alpha} \frac{1}{\epsilon^2 R^{2+\delta}} \left(\|f\|_{0,C} + \frac{R^2}{(\epsilon R)^{1+\alpha}} \|f\|_{0,G} \right). \tag{6-40}$$

Proof. Consider $e \in \mathcal{E}$. Observe that, over $S \cap (A(R,\infty) \times \mathbb{R})$,

$$P_C e = \chi_u \Phi Q_E e - \Psi A Q_E e + (W A Q_E e)\chi'_\epsilon.$$

By (6-22),

$$\|\chi_u \Phi Q_E e\|_{2,G} \lesssim \frac{1}{(\epsilon R)^\alpha} \frac{1}{\epsilon^2 R^{2+\delta}} \|e\|_{0,C}.$$

By (6-15) and (6-23),

$$\|\Psi A Q_E e\|_{2,G} \lesssim \frac{1}{(\epsilon R)^{2\alpha}} \frac{1}{\epsilon^2 R^{6+\delta}} \|e\|_{0,C}.$$

By (6-15) and (6-27)

$$\|W A Q_E e\| \lesssim \frac{R^2}{(\epsilon R)} \|A Q_E e\|_{0,G} \lesssim \frac{1}{(\epsilon R)^{2\alpha}} \frac{1}{\epsilon R^{5+\delta}} \|e\|_{0,C}.$$

However,

$$\|\chi'_\epsilon\|_{2,G} \lesssim \frac{1}{(\epsilon R)},$$

and so

$$\|(W A Q_E e)\chi'_\epsilon\|_{2,G} \lesssim \frac{1}{(\epsilon R)^{2\alpha}} \frac{1}{\epsilon^2 R^{6+\delta}} \|e\|_{0,C}.$$

Combining these relations yields

$$\|P_C e\|_{2,G} \lesssim \frac{1}{(\epsilon R)^\alpha} \frac{1}{\epsilon^2 R^{2+\delta}} \|e\|_{0,C}.$$

Consider now $f \in \mathcal{E}$. Over $S \cap (A(R,\infty) \times \mathbb{R})$,

$$P_G f = -\chi_u \Phi B Q_F f + \Psi Q_F f - (W Q_F f)\chi'_\epsilon.$$

By (6-22) and (6-26),

$$\|\chi_u \Phi B Q_F f\|_{2,G} \lesssim \frac{1}{(\epsilon R)^\alpha} \frac{1}{\epsilon^2 R^\delta(\epsilon R)} \|f\|_{0,G}.$$

By (6-23),

$$\|\Psi Q_F f\|_{2,G} \lesssim \frac{1}{\epsilon^2} \|f\|_{0,G}.$$

By (6-27),

$$\|W Q_F f\| \lesssim \frac{R^2}{(\epsilon R)} \|f\|_{0,G},$$

so that

$$\|(W Q_F f)\chi'_\epsilon\|_{2,G} \lesssim \frac{1}{\epsilon^2} \|f\|_{0,G}.$$

Combining these relations yields

$$\|P_G f\|_{2,G} \lesssim \frac{1}{(\epsilon R)^\alpha} \frac{1}{\epsilon^2 R^\delta(\epsilon R)} \|f\|_{0,G}.$$

The result now follows by (6-34). □

7. Existence and embeddedness

7.1. ***The Schauder fixed-point theorem.*** It remains only to perturb the approximate MCF solitons constructed in Section 5 into actual MCF solitons. This perturbation will be carried out using the Schauder fixed-point theorem. It will first be convenient to modify slightly the norms introduced in (6-12). We thus define

$$\begin{aligned} \|f\|'_{m,G,H} &:= \|f|_{A(2R,\infty)}\|_{C^{m,\alpha}_{\gamma,\epsilon}(G)}, \\ \|f\|'_{m,G,S} &:= \|f|_{A(2R,\infty)}\|_{H^m_{\gamma,\epsilon}(G)}, \\ \|f\|'_{m,G} &:= \|f\|'_{m,G,H} + \frac{1}{(\epsilon R)} \|f\|'_{m,G,S}. \end{aligned} \tag{7-1}$$

By (6-35), this does not affect (6-37), (6-38), (6-39) and (6-40). In addition, we will also ignore the factor $\langle \widehat{N}_S, N_S\rangle^{-1}$ used in the definitions (5-8) and (5-9) of $(X, Y, Z, \hat{J}_S)$. Indeed, we readily show that the operator of multiplication by this function is uniformly bounded, independent of Λ, with respect to the norms $\|\cdot\|_{0,C}$ and $\|\cdot\|_{0,G}$, for which reason it also does not affect the above estimates.

For all nonnegative, integer m, for all $\alpha \in [0, 1]$ and for all real γ, let $E_{m,\alpha,\gamma}$ be the space of m-times differentiable functions $f : S \to \mathbb{R}$ which are invariant under all horizontal symmetries of C and which satisfy

$$\|f\|_{m,C}, \|f\|'_{m,G} < \infty.$$

Observe that $E_{m,\alpha,\gamma}$ furnished with these norms is a Fréchet space. Now let

$$M : U \oplus V \oplus W \oplus E_{2,\alpha,\gamma} \to E_{0,\alpha,\gamma}$$

be the MCFS functional about S, as defined in Sections 5.1 and 5.4. It only remains to study how M varies up to second order about S. As before, throughout this section, we apply (2-14) without comment.

Lemma 7.1.1. $$\|M(0, 0, 0, 0)\|_{0,C} \lesssim R^{\delta-2}, \quad \|M(0, 0, 0, 0)\|'_{0,G} = 0. \tag{7-2}$$

Proof. Define $\psi := M(0,0,0,0)$. Since C is minimal, over $B(R)$,

$$\psi = \epsilon\mu.$$

Thus, by (5-1) and (A-6),

$$\|\psi|_{B(R)}\|_{C^{0,\alpha}_{2+\delta,\mathrm{SF}}(C)} \lesssim \epsilon R^{2+\delta} \lesssim R^{\delta-2}.$$

By (5-4), over $A(R,2R)$,

$$H_i = \frac{cx^i}{r^2} + \mathrm{O}(R^{-(3+k)}),$$
$$H_{ij} = \frac{c}{r^2}\left(\delta_{ij} - \frac{x^i x^j}{2r^2}\right) + \mathrm{O}(R^{-(4+k)}).$$

Thus, by (A-6), over this annulus,

$$\mu = 1 + \mathrm{O}(R^{-2+k}), \quad g^{ij} = \delta_{ij} + \mathrm{O}(R^{-2+k}),$$

so that, by (A-7),

$$\psi = \mathrm{O}(R^{-(4+k)}).$$

Consequently,

$$\|\psi|_{A(R,2R)}\|_{C^{0,\alpha}_{2+\delta,\mathrm{cyl}}(C)} \lesssim R^{\delta-2},$$

and the first estimate follows upon combining these relations. Finally, by construction, ψ vanishes over $A(2R,\infty)$, so that $\|\psi\|'_{0,G} = 0$, and this completes the proof. □

It is straightforward to show that for $\|u\|$, $\|v\|$, $\|w\|$ and $\|f\|_{2,C}$ sufficiently small, independent of Λ,

$$\|M(u,v,w,f) - M(0,0,0,0) - \hat{J}_S f - \widehat{X}u - \widehat{Y}v\|_{0,C} \lesssim \|f\|^2_{2,C} + \|u\|^2 + \|v\|^2. \tag{7-3}$$

The corresponding estimate over rotationally symmetric Grim ends is more subtle.

Lemma 7.1.2. *There exists $\eta > 0$ such that, for sufficiently large Λ, if $\epsilon(\epsilon R)^{1-2\alpha}\|f\|'_{2,G} < \eta$, then*

$$\|M(u,v,w,f) - M(0,0,0,0) - \hat{J}_S f - \widehat{Z}w\|'_{0,G}$$
$$\lesssim \frac{\epsilon^2}{R}\|u\|^2 + \frac{\epsilon^2}{R}\|v\|^2 + \frac{\epsilon^2}{R}\|w\|^2 + \epsilon^3(\epsilon R)^{1-2\alpha}(\|f\|'_{2,G})^2. \tag{7-4}$$

Remark. Before continuing, it is worth reflecting on the terms that will appear in the following proof. First, on the scale of the rotationally symmetric Grim end, the perturbation that we make is of order ϵ so that, since this perturbation is quadratic, it introduces a factor of ϵ^2. Second, returning to the scale of the joined surface introduces a further factor of ϵ, thus explaining the factor of ϵ^3 in the formulae below.

Proof. Since M is a second-order quasilinear functional, upon rescaling, we obtain, for all u, for all v, and for all g with $\|\epsilon g\|'_{1,G,H}$ sufficiently small,

$$\|M(u,v,0,g) - M(u,v,0,0) - \hat{J}_{S,u,v}g\|'_{0,G} \lesssim \epsilon^3\|g\|'_{1,G,H}\|g\|'_{2,G}$$
$$\lesssim \frac{\epsilon^2}{R}(\|g\|'_{1,G,H})^2 + \epsilon^3(\epsilon R)(\|g\|'_{2,G})^2.$$

Next, for all sufficiently small u and v, and for all g,

$$\|(\hat{J}_{S,u,v} - \hat{J}_S)g\|'_{0,G} \lesssim \epsilon^3(\|u\| + \|v\|)\|g\|'_{2,G}$$
$$\lesssim \frac{\epsilon^2}{R}\|u\|^2 + \frac{\epsilon^2}{R}\|v\|^2 + \epsilon^3(\epsilon R)(\|g\|'_{2,G})^2.$$

Now, bearing in mind the definition of the macroscopic perturbation in the direction of w,

$$\begin{aligned}
&\|M(u,v,w,f)-M(0,0,0,0)-\hat{J}_S f-\widehat{Z}w\|'_{0,G}\\
&\qquad\lesssim \|M(u,v,w,f)-M(u,v,0,0)-\hat{J}_S f-\widehat{Z}w\|'_{0,G}\\
&\qquad\lesssim \|M(u,v,0,f+w(1-\chi'_\epsilon))-M(u,v,0,0)-\hat{J}_S(f+w(1-\chi_{\epsilon'}))\|'_{0,G}\\
&\qquad\lesssim \|M(u,v,0,f+w(1-\chi'_\epsilon))-M(u,v,0,0)-\hat{J}_{S,u,v}(f+w(1-\chi_{\epsilon'}))\|'_{0,G}\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\|(\hat{J}_{S,u,v}-\hat{J}_S)(f+w(1-\chi_{\epsilon'}))\|'_{0,G}\\
&\qquad\lesssim \frac{\epsilon^2}{R}\|u\|^2+\frac{\epsilon^2}{R}\|v\|^2+\frac{\epsilon^2}{R}(\|f+w(1-\chi'_\epsilon)\|'_{1,G,H})^2+\epsilon^3(\epsilon R)(\|f+w(1-\chi'_\epsilon)\|'_{2,G})^2.
\end{aligned}$$

Finally,

$$\begin{aligned}
\|(1-\chi'_\epsilon)|_{A(1/(2\epsilon),1/\epsilon)}\|'_{1,G,H}&\lesssim 1,\\
\|(1-\chi'_\epsilon)|_{A(1/(2\epsilon),1/\epsilon)}\|'_{2,G}&\lesssim \frac{1}{\epsilon R},
\end{aligned}$$

and the result now follows by Lemma 4.3.1 and the subsequent remark. □

This concludes our analysis of M up to second order about S. We are now ready to prove existence.

Theorem 7.1.3. *For γ sufficiently small, for all $\delta\in\,]1,2[$, for $\alpha\in\,]0,1[$ sufficiently small, and for Λ sufficiently large, there exist u, v, w and f such that*

$$M(u,v,w,f)=0.$$

Furthermore,

$$\|u\|,\ \|v\|,\ \|w\|,\ \|f\|_{2,C}\lesssim R^{\delta-2},\ \ \|f\|_{2,G}\lesssim\frac{1}{(\epsilon R)^\alpha\epsilon^2R^4}. \tag{7-5}$$

Proof. Fix $\gamma\ll 1$, $\delta\in\,]1,2[$ and $\alpha\in\,]0,1[$ small. Set $\psi_0:=M(0,0,0,0)$ and define

$$(u_0,v_0,w_0,f_0):=\phi_0:=-(\widehat{U}\psi_0,\widehat{V}\psi_0,\widehat{W}\psi_0,\widehat{P}\psi_0).$$

By (6-37), (6-38), (6-39), (6-40) and (7-2), there exists a constant $B>0$, such that, for all large Λ,

$$\|u_0\|,\ \|v_0\|,\ \|w_0\|,\ \|f_0\|_{2,C}\le BR^{\delta-2},\quad \|f_0\|'_{2,G}\le\frac{B}{(\epsilon R)^\alpha\epsilon^2R^4}.$$

Define $\Omega\subseteq\mathbb{R}^3\oplus\mathbb{R}^3\oplus\mathbb{R}^3\oplus E_{2,\alpha,\gamma}$ to be the set of all quadruplets (u,v,w,f) such that

$$\|u\|,\ \|v\|,\ \|w\|,\ \|f\|_{2,C}\le 2BR^{\delta-2},\quad \|f\|'_{2,G}\le\frac{2B}{(\epsilon R)^\alpha\epsilon^2R^4}.$$

Observe that Ω is convex and, by the Arzelà–Ascoli theorem, for all $\alpha'<\alpha$ and $\gamma'<\gamma$, Ω is a compact subset of $\mathbb{R}^3\oplus\mathbb{R}^3\oplus\mathbb{R}^3\oplus E_{2,\alpha',\gamma'}$. For $\phi:=(u,v,w,f)$ in Ω, define

$$\Phi(\phi):=\phi_0-(\widehat{U}\psi,\widehat{V}\psi,\widehat{W}\psi,\widehat{P}\psi),$$

where

$$\psi:=M(u,v,w,f)-M(0,0,0,0)-\hat{J}_S f-\widehat{X}u-\widehat{Y}v-\widehat{Z}w.$$

By (7-3), (7-4) and (2-14),

$$\|\psi\|_{0,C} \lesssim R^{2\delta-4}, \ \ \|\psi\|'_{0,G} \lesssim \frac{1}{(\epsilon R)^{6\alpha} R^7},$$

so that, by (6-37), (6-38), (6-39) and (6-40), for sufficiently large Λ, Φ maps Ω to itself. Furthermore, for all $\alpha' < \alpha$ and $\gamma' < \gamma$, Φ is continuous with respect to the topology of $E_{2,\alpha',\gamma'}$. It follows by the Schauder fixed-point theorem (see [Gilbarg and Trudinger 1983]) that there exists a fixed point ϕ of Φ in Ω. We readily verify that $M(\phi) = 0$, and this completes the proof. □

Theorem 7.1.4. *Let (u, v, w, f) be as in Theorem 7.1.3. For sufficiently large Λ, the surface $\widetilde{E}(u, v, w, f)$ is embedded.*

Proof. We denote the joined surface by S, we denote the image of $\widetilde{E}(u, v, w, f)$ by S', and we rescale both S and S' by ϵ. Observe that the intersection of S with $A(2\epsilon R, \infty) \times \mathbb{R}$ consists of three distinct rotationally symmetric Grim ends, which we denote by G_+, G_0 and G_- respectively. Let u_+, u_0 and u_- be the respective profiles of these ends, and let v_+, v_0 and v_- be the respective derivatives of these functions in the radial direction. Observe that

$$u_+(\epsilon R) > u_0(\epsilon R) > u_-(\epsilon R),$$
$$v_+(\epsilon R) > v_0(\epsilon R) > v_-(\epsilon R).$$

Since v_+, v_0 and v_- are all solutions of the same first-order ODE, it follows that $v_+(r) > v_0(r) > v_-(r)$ for all r. In particular, the ends G_+, G_0 and G_+ are separated vertically by a distance of no less than η, where $\eta \sim \epsilon \log(R)$. Let Ω_+, Ω_0 and Ω_- denote the open sets of points lying at a vertical distance of no more than $\eta/2$ from G_+, G_0 and G_- respectively. Observe, in particular, that these three sets are disjoint.

Now let G'_+, G'_0 and G'_- be the three ends of S'. Over the annulus $A(\epsilon R, 2\epsilon R)$, by (7-5),

$$\|\epsilon f|_{A(R,2R)}\|_{C^0} \lesssim \epsilon R^{-\delta} \|f|_{(A(R,2R)}\|_{2,C} \lesssim \epsilon R^{-2},$$

so that, over this annulus, G'_+ lies strictly above G'_0, and G'_0 lies strictly above G'_-. However, by Lemma 4.3.1 and the subsequent remark and (7-5) again,

$$\|\epsilon f\|'_{1,G,H} \lesssim \frac{1}{(\epsilon R)^{3\alpha} R^3}.$$

Bearing in mind the definition of the norm $\|\cdot\|_{1,G,H}$, it follows that for sufficiently large Λ, G'_+, G'_0 and G'_- are all graphs over $A(\epsilon R, \infty)$. Furthermore, for some large R', the intersections of G'_+, G'_0 and G'_- with $A(R', \infty) \times \mathbb{R}$ are contained in Ω_+, Ω_0 and $\Omega-$ respectively. In particular, outside $B(R') \times \mathbb{R}$, G'_+ lies strictly above G'_0 and G'_0 lies strictly above G'_-. Since vertical translates of mean curvature flow solitons are also mean curvature flow solitons, it now follows by the strong maximum principle that, over the whole of $A(\epsilon R, \infty)$, G'_+ lies strictly above G'_0 and G'_0 lies strictly above G_-. □

Appendix A: Terminology, conventions and standard results

A1. ***General definitions.*** Let $\mathbb{R}^2$ and $\mathbb{R}^3$ denote respectively 2- and 3-dimensional Euclidean space. We consider $\mathbb{R}^2$ as the $(x-y)$ plane in $\mathbb{R}^3$. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ denote the canonical projection. Let r denote a smooth positive function over $\mathbb{R}^2$ which is equal to the distance to the origin outside some (suitably large) compact set. We denote the composition of r with π also by r. Let e_x, e_y and e_z denote the vectors of the canonical basis of $\mathbb{R}^3$. Let e_r, e_θ denote respectively the unit radial and unit angular vector fields about the origin over $\mathbb{R}^2$ and about the z-axis over $\mathbb{R}^3$. Let D denote the canonical differentiation operator over $\mathbb{R}^2$ and $\mathbb{R}^3$. Let Δ denote the canonical Laplacian over $\mathbb{R}^2$ (not to be confused with Δ^Σ, defined below). Let $C(a)$ denote the circle of radius a about the origin in $\mathbb{R}^2$. Let $B(a)$ denotes the closed disk of radius a about the origin in $\mathbb{R}^2$. Let $A(a, b)$ denote the closed annulus of inner radius a and outer radius b about the origin in $\mathbb{R}^2$. Let $\chi : [0, \infty[\to \mathbb{R}$ be a nonnegative, nonincreasing function such that $\chi = 1$ over $[0, 1]$ and $\chi = 0$ over $[2, \infty[$. For all a, define $\chi_a : \mathbb{R}^2 \to \mathbb{R}$ by $\chi_a(x) = \chi(\|x\|/a)$. We call χ_a the *cut-off function* of the *transition region* $A(a, 2a)$. Composing with π, we likewise consider χ_a as a function over $\mathbb{R}^3$.

A2. ***Surface geometry.*** Let Σ be an embedded surface in $\mathbb{R}^3$. Let N_Σ denote the unit normal vector field over Σ. Let π^Σ denote the orthogonal projection onto the tangent space of Σ. Let ∇^Σ denote the gradient operator as well as the Levi-Civita covariant derivative of Σ. Let Hess^Σ denote the intrinsic Hessian operator of Σ. Let Δ^Σ denote the intrinsic Laplacian of Σ. Let II^Σ denote the second fundamental form of Σ. Let A_Σ denote the shape operator of Σ. Let H_Σ denote the mean curvature of Σ (taken to be the *sum* of the principle curvatures, or the trace of the shape operator). Let M_Σ denote the MCFS operator of Σ (with speed ϵ). It is given by

$$M_\Sigma := H_\Sigma + \epsilon\langle N_\Sigma, e_z\rangle. \tag{A-1}$$

Let J_Σ denote the MCFS Jacobi operator (with speed ϵ) of Σ. That is, J_Σ is the linearisation of the MCFS operator of Σ. It is given by

$$J_\Sigma f = \Delta^\Sigma f + \mathrm{Tr}(A_\Sigma^2) f + \epsilon\langle \nabla^G f, e_z\rangle. \tag{A-2}$$

Finally, we recall the following elementary relations. For any function f defined over a neighbourhood of Σ,

$$\begin{aligned} \nabla^\Sigma f &= Df - \langle Df, N_\Sigma\rangle N_\Sigma, \\ \mathrm{Hess}^\Sigma(f) &= \mathrm{Hess}(f) - \langle Df, N_\Sigma\rangle \mathrm{II}^\Sigma. \end{aligned} \tag{A-3}$$

Given any positive function ϕ defined over Σ, if $\hat{J}_\Sigma := M_\phi^{-1} J_\Sigma M_\phi$ denotes the conjugate of J_Σ with the operator of multiplication by ϕ, then

$$\hat{J}_\Sigma f = \Delta^\Sigma f + 2\phi^{-1}\langle \nabla^\Sigma \phi, \nabla^\Sigma f\rangle + \epsilon\langle \nabla^\Sigma f, e_z\rangle + (\phi^{-1} J_\Sigma \phi) f. \tag{A-4}$$

A3. ***Surface geometry of graphs.*** If Σ is the graph of a function u over a subset of $\mathbb{R}^2$, then we call u the *profile* of Σ. In this case, π defines a coordinate chart of Σ in $\mathbb{R}^2$. It will be more convenient to work, sometimes over Σ, and sometimes over $\mathbb{R}^2$, and we will move freely between these two perspectives. Let g_{ij} denote the intrinsic metric of Σ. Its inverse is denoted by g^{ij}. Let Γ_{ij}^k denote the Christoffel symbols of the Levi-Civita covariant derivative of g_{ij}. Setting

$$\mu := \langle e_z, N_\Sigma\rangle, \tag{A-5}$$

we readily verify the following relations:

$$\begin{aligned} \mu &= \frac{1}{\sqrt{1+\|Du\|^2}}, & \Delta^\Sigma(f) &= g^{ij}f_{ij} - g^{ij}g^{kp}u_{ij}u_p f_k, \\ g_{ij} &= \delta_{ij} + u_i u_j, & \mathrm{II}^\Sigma_{ij} &= -\mu u_{ij}, \\ g^{ij} &= \delta_{ij} - \mu^2 u^i u^j, & (A^\Sigma)^i_j &= -\mu g^{ip}u_{pj}, \\ \Gamma^k_{ij} &= g^{kp}u_{ij}u_p, & H^\Sigma &= -\mu g^{ij}u_{ij}, \\ \mathrm{Hess}^\Sigma(f)_{ij} &= f_{ij} - g^{kp}u_{ij}u_p f_k, & \pi^T(e_z)_i &= \mu^2 u_i. \end{aligned} \tag{A-6}$$

Finally, when Σ is a graph, the MCFS functional is given by

$$M_\Sigma = -\mu g^{ij}u_{ij} + \epsilon\mu. \tag{A-7}$$

A4. ***Function spaces.*** Let X be a metric space. For all $\alpha \in [0, 1]$, we define the *Hölder seminorm* of order α over X by

$$[f]_\alpha := \mathrm{Sup}_{x\neq y\in X} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}. \tag{A-8}$$

Observe that $[f]_0$ measures the *total oscillation* of f. In particular,

$$[f]_0 \leq 2\|f\|_{C^0}. \tag{A-9}$$

For all $\alpha \in [0, 1]$,

$$[f]_\alpha \leq [f]_0^{1-\alpha}[f]_1^\alpha \leq 2^{1-\alpha}\|f\|_{C^0}^{1-\alpha}[f]_1^\alpha. \tag{A-10}$$

If X is a complete manifold, and if f is differentiable over X, then, for all $\alpha \in [0, 1[$ and for all $\beta \in]0, 1]$,

$$\|Df\|_{C^0} \leq 2[f]_\alpha^{\beta/(1+(\beta-\alpha))}[Df]_\beta^{(1-\alpha)/(1+(\beta-\alpha))}. \tag{A-11}$$

For all α,

$$[fg]_\alpha \leq \|f\|_{C^0}[g]_\alpha + [f]_\alpha\|g\|_{C^0}. \tag{A-12}$$

Finally, if $X = X_1 \cup \cdots \cup X_m$, then, for all α,

$$[f]_\alpha \leq m^{1-\alpha}\,\mathrm{Sup}_{1\leq k\leq m}[f|_{X_i}]_\alpha. \tag{A-13}$$

If, in particular, $X = [0, m+1] \times S^1$ is a cylinder and $X_i = [i, i+1] \times S^1$ for all i, then (A-13) refines to

$$[f]_\alpha \leq \sum_{i=1}^m [f|_{X_i}]_\alpha. \tag{A-14}$$

For a continuous function f over X, for all α, we define

$$\delta^\alpha f(x) := [f|_{B_1(x)}]_\alpha. \tag{A-15}$$

Now suppose that X is a smooth Riemannian manifold. For all k, α, we define the $C^{k,\alpha}$-Hölder norm over $C^\infty(M)$ by

$$\|f\|_{C^{k,\alpha}} := \sum_{i=0}^k \|D^i f\|_{C^0} + \|\delta^\alpha D^k f\|_{C^0}. \tag{A-16}$$

We define the space $C^{k,\alpha}(X)$ to be the closure of $C^\infty(X)$ with respect to this norm. For all p, we define the L^p-norm over $C_0^\infty(M)$ by

$$\|f\|_{L^p}^p := \int_X |f|^p \,\mathrm{dVol}. \tag{A-17}$$

We define the space $L^p(X)$ to be the closure of $C_0^\infty(X)$ with respect to this norm. For all k, we define the H^k-Sobolev norm over $C_0^\infty(M)$ by

$$\|f\|_{H^k} := \sum_{i=0}^{k} \|D^i f\|_{L^2}. \tag{A-18}$$

The reader may verify that all surfaces studied in this paper are sufficiently regular at infinity for the Sobolev embedding theorem to hold. That is for all l, and for all $k+\alpha < l-1$,

$$\|f\|_{C^{k,\alpha}} \lesssim \|f\|_{H^l}. \tag{A-19}$$

The following formulae are readily verified:

$$\mathrm{Sup}_{t\in[1,T]} \log(t)t^\alpha \lesssim \begin{cases} \log(T)T^\alpha & \text{if } \alpha > 0, \\ 1 & \text{if } \alpha < 0 \end{cases} \tag{A-20}$$

and

$$\int_{A(1,T)} \log(r)^m r^\alpha \,\mathrm{dVol}_{\mathrm{SF}} \lesssim \begin{cases} \log(T)^m T^\alpha & \text{if } \alpha > 0, \\ 1 & \text{if } \alpha < 0. \end{cases} \tag{A-21}$$

A5. *Elliptic estimates.* Let E and F be Banach spaces and let $A : E \to F$ be a bounded linear map. We say that A satisfies an *elliptic estimate* whenever there exists a normed vector space G, a compact map $K : E \to G$, and a constant C such that, for all e in E,

$$\|e\| \le C(\|Ke\| + \|Ae\|). \tag{A-22}$$

The following straightforward result plays an important role in Fredholm theory.

Theorem A5.1. *If A satisfies an elliptic estimate, then the kernel of A is finite-dimensional and its image is a closed subset of F.*

Appendix B: Catenoidal minimal ends

In this appendix, we use the Weierstrass representation to determine the asymptotics of horizontal, catenoidal minimal ends. This is used in Sections 5 and 6 to model the asymptotics of CHM surfaces.

Let C be a horizontal, catenoidal minimal end. Its intrinsic metric is biholomorphic to the punctured disk which, for the purposes of this appendix, it is useful to view as the complement of the closed unit disk in $\mathbb{C}$, that is,

$$\Delta^* := \{\zeta \in \mathbb{C} \mid |\zeta| > 1\}. \tag{B-1}$$

The Weierstrass representation (see [Weber 2005]) is a parametrisation of C by a function $\Phi : \Delta^* \to \mathbb{R}^3$ of the form

$$\Phi(\zeta) := \mathrm{Re}\left(\int^\zeta \left(\frac{1}{2}\left(G - \frac{1}{G}\right), \frac{1}{2i}\left(G + \frac{1}{G}\right), 1\right) h \, d\zeta\right) \tag{B-2}$$

for some holomorphic functions $G, h : \Delta^* \to \mathbb{C}$. These functions are interpreted geometrically as follows. Setting $\Phi := (\Phi_1, \Phi_2, \Phi_3)$, we readily show that

$$h = 2\partial_\zeta \Phi_3. \tag{B-3}$$

That is, $hd\zeta$ is twice the holomorphic part of the derivative of the height function of C. The geometric significance of G is more subtle, but with some work we can show that it is the image under the stereographic projection of the unit normal vector field over C.

Define $\rho := |\zeta|$. Since C is a horizontal catenoidal end, Φ_3 is asymptotic to $a + c\log(\rho)$ for some constants a and c, and it follows that

$$h = \sum_{k=-\infty}^{-1} h_k \zeta^k. \tag{B-4}$$

Meanwhile, since the normal of C is asymptotically vertical, G may be chosen to vanish at infinity, so that

$$G = \sum_{k=-\infty}^{-1} G_k \zeta^k. \tag{B-5}$$

In addition, since C is a single-valued graph over some neighbourhood of infinity in $\mathbb{R}^2$, the functions h and G together satisfy a vanishing holonomy condition around the puncture at infinity. In terms of their Laurent coefficients, this holonomy condition is

$$h_{-1}G_{-2} - h_{-2}G_{-1} = 0. \tag{B-6}$$

This condition ensures, in particular, that the first two components of Φ contain no logarithmic terms. Thus, defining $\zeta =: \xi + i\nu$ and rotating and rescaling if necessary, we obtain, near infinity

$$\Phi(\xi, \nu) = \left(\xi + \alpha\left(\frac{\xi}{\rho^2}, \frac{\nu}{\rho^2}\right), \nu + \beta\left(\frac{\xi}{\rho^2}, \frac{\nu}{\rho^2}\right), a + b\log(\rho) + \gamma\left(\frac{\xi}{\rho^2}, \frac{\nu}{\rho^2}\right)\right), \tag{B-7}$$

where α, β and γ are analytic functions of their arguments defined in a neighbourhood of the origin which, furthermore, vanish at this point.

We now define

$$(x, y) := (\Phi_1(\xi, \nu), \Phi_2(\xi, \nu)) \quad \text{and} \quad r^2 := x^2 + y^2. \tag{B-8}$$

That is, (x, y) is the composition of the parametrisation Φ with the projection onto the horizontal plane. Trivially, near infinity, C is the graph of some function F defined over the (x, y)-plane. We now use (B-7) to determine the asymptotic structure of this function. First, upon observing that

$$\frac{1}{\rho^2} = \frac{\xi^2}{\rho^4} + \frac{\nu^2}{\rho^4}, \tag{B-9}$$

we find that

$$\left(\frac{x}{r^2}, \frac{y}{r^2}\right) = \Psi\left(\frac{\xi}{\rho^2}, \frac{\nu}{\rho^2}\right) \tag{B-10}$$

for some analytic function Ψ, defined in a neighbourhood of the origin, such that $\Psi(0,0)=0$ and $D\Psi(0,0)=\mathrm{Id}$. Upon applying the implicit function theorem for analytic functions, we deduce that

$$F(x,y)=a+b\log(r)+\delta\left(\frac{x}{r^2},\frac{y}{r^2}\right) \tag{B-11}$$

for some analytic function δ, vanishing at the origin. In particular, with the notation of Section 1.3,

$$F(x,y)=a+c\log(r)+\mathrm{O}(r^{-(1+k)}), \tag{B-12}$$

thus confirming the first formula of Section 5.1.

It remains only to verify (5-1). However, by (B-11),

$$F(x,y)=a+c\log(r)+\frac{\phi(x,y)}{r^2}+\mathrm{O}(r^{-(2+k)}) \tag{B-13}$$

for some linear form ϕ. Let e_z denote the unit vector in the direction of the positive z-axis. Let u be any nonzero, horizontal vector. If C is symmetric under reflection in the plane spanned by e_z and u, then ϕ annihilates the line orthogonal to u. Consequently, if C is symmetric under reflection in two distinct planes of this type, then ϕ vanishes, so that

$$F(x,y)=a+c\log(r)+\mathrm{O}(r^{-(2+k)}), \tag{B-14}$$

thus confirming (5-1).

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