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We study nodal sets of Steklov eigenfunctions in a bounded domain with $\mathcal{C}^2$ boundary. Our first result is a lower bound for the Hausdorff measure of the nodal set: we show that, for u_λ a Steklov eigenfunction with eigenvalue $\lambda \neq 0$, we have $\mathcal{H}^{d-1}(\{u_\lambda = 0\}) \geq c_\Omega$, where c_Ω is independent of λ. We also prove an almost sharp upper bound, namely, $\mathcal{H}^{d-1}(\{u_\lambda = 0\}) \leq C_\Omega \lambda \log(\lambda + e)$.

1. Introduction

Let Ω a bounded domain in $\mathbb{R}^d$, where $d \geq 2$. A Steklov eigenfunction $u_\lambda \in H^1(\Omega)$ is a solution of

$$\begin{cases} \Delta u_\lambda = 0 & \text{in } \Omega, \\ \partial_\nu u_\lambda = \lambda u_\lambda & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here and throughout the paper we denote by ∂_ν the outward normal derivative. A number λ for which a solution to (1) exists is called a Steklov eigenvalue, and it is well known that Steklov eigenvalues form a discrete sequence accumulating to infinity. Moreover, Steklov eigenvalues coincide with the eigenvalues of the Dirichlet-to-Neumann operator, which is the operator that maps a function on $\partial\Omega$ to the normal derivative of its harmonic extension in Ω, and a Steklov eigenfunction restricted to $\partial\Omega$ is an eigenfunction of the Dirichlet-to-Neumann operator. For a survey on the Steklov problem outlining many results and open questions see [Girouard and Polterovich 2017].

Inspired by a famous conjecture of Yau on the Hausdorff measure of nodal sets of Laplace eigenfunctions, an analogous question has been asked for nodal sets of Steklov eigenfunctions (it is stated explicitly in [Girouard and Polterovich 2017], for example); the conjecture can be formulated both for interior and boundary nodal sets. For the interior nodal set, the question is as follows:

- Is it true that there exist positive constants c and C, depending only on Ω, such that

$$c\lambda \leq \mathcal{H}^{d-1}(\{u_\lambda = 0\}) \leq C\lambda? \tag{2}$$

Similarly, for the boundary nodal set (which is the nodal set of an eigenfunction of the Dirichlet-to-Neumann operator) one can ask:

- Is it true that there exist positive constants c' and C', depending only on Ω, such that

$$c'\lambda \leq \mathcal{H}^{d-2}(\{u_\lambda = 0\} \cap \partial\Omega) \leq C'\lambda? \tag{3}$$

Here we do not deal with question (3) and just note that the upper bound was proved in [Zelditch 2015] when $\partial\Omega$ is real-analytic. About question (2), a polynomial upper bound was proved in [Georgiev and Roy-Fortin 2019], following the corresponding polynomial upper bound in the Laplace–Beltrami eigenfunction case proved in [Logunov 2018a]. On real-analytic surfaces (that is, real-analytic metric in the interior and real-analytic boundary), the full conjecture (2) was established in [Polterovich et al. 2019]. Again in the real-analytic category, the upper bound was recently obtained in any dimension in [Zhu 2020]. Concerning lower bounds, as far as we know, the best result was contained in [Sogge et al. 2016], where the bound $\mathcal{H}^{d-1}(\{u_\lambda = 0\}) \geq c\lambda^{(2-d)/2}$ is obtained for Ω a domain with $\mathcal{C}^\infty$ boundary (actually, a smooth Riemannian manifold with smooth boundary). The first contribution of the present article is an improvement on the lower bound; we show that the Hausdorff measure of the interior nodal set is bounded below by a constant independent of λ (so the result is really an improvement over [Sogge et al. 2016] if $d \geq 3$).

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^d$ with $\mathcal{C}^2$-smooth boundary, and let u_λ be a solution of* (1) *in Ω, $\lambda \neq 0$. Then there exists a constant $c_\Omega > 0$ independent of λ such that*

$$\mathcal{H}^{d-1}(\{u_\lambda = 0\}) \geq c_\Omega. \tag{4}$$

In the previous work [Decio 2022] we established a density property of the zero set near the boundary, under weaker hypothesis on the boundary regularity: we transcribe the result below.

Theorem A [Decio 2022]. *Let Ω be a Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, and let u_λ be a solution of* (1), *where we assume $\lambda \neq 0$. There exists a constant $C = C(\Omega)$ such that*

$$\{u_\lambda = 0\} \cap B \neq \varnothing \tag{5}$$

for any ball B in $\mathbb{R}^d$ of radius C/λ centered at a point in $\partial\Omega$.

The proof of Theorem 1 involves a combination of Theorem A and the recent breakthrough by Logunov [2018b] on Yau's conjecture. We cannot apply the results of [Logunov 2018b] directly and have to do some work to modify the necessary arguments. The fact that we are one power of λ away from the optimal result is a consequence of the deficiency of the density result, which we can only prove very close to the boundary, and not of the arguments in [Logunov 2018b].

Remark. It will be apparent from the proof that Theorem 1 extends without much difficulty to the case of manifolds equipped with a $\mathcal{C}^2$-smooth Riemannian metric and $\mathcal{C}^2$ boundary.

The conjectured upper bound in (2) would be sharp, as the example of a ball shows; the second main contribution of this article is an almost sharp upper bound for Euclidean domains with $\mathcal{C}^2$ boundary.

Theorem 2. *Let Ω be a bounded domain in $\mathbb{R}^d$ with $\mathcal{C}^2$-smooth boundary, and let u_λ be a solution of* (1) *in Ω. Then there is a constant $C_\Omega > 0$ independent of λ such that*

$$\mathcal{H}^{d-1}(\{u_\lambda = 0\}) \leq C_\Omega \lambda \log(\lambda + e). \tag{6}$$

Remark. The proof of Theorem 2 uses the sharp bounds of Donnelly and Fefferman [1988] in the interior of the domain and a multiscale induction argument at the boundary, which is based on a version of

the hyperplane lemma of [Logunov 2018a; Logunov et al. 2021]. While, as remarked above, the proof of the lower bound can be extended almost *verbatim* to smooth Riemannian manifolds with boundary, for Theorem 2 we rely heavily on the fact that Ω is a Euclidean domain, or at least we have to require that the metric inside Ω is real analytic; this is because the results of [Donnelly and Fefferman 1988] require real analyticity. Our theorem lies in between previous results on the upper bound: the multiscale argument at the boundary allows for C^2-regularity of the boundary only, as opposed to real analyticity as in the aforementioned paper [Zhu 2020]; on the other hand, if the metric inside is assumed to be only C^2 (or C^∞), the best result attainable with these methods is still the polynomial upper bound of [Georgiev and Roy-Fortin 2019].

Plan of the paper. We prove Theorem 1 in Sections 2 and 3; in Section 2 we discuss a procedure for extending a Steklov eigenfunction across the boundary, which gives rise to an auxiliary equation for which a statement very similar to Logunov's theorem [2018b] holds (see Theorem 3), and we use this together with Theorem A to prove the lower bound. Section 3 is quite long and contains the proof of Theorem 3, which requires us to review Logunov's argument carefully and use a combination of classical elliptic estimates and frequency function techniques. Section 4 is dedicated to the proof of Theorem 2.

2. Lower bound on nodal sets

Here we deduce Theorem 1 using Theorem A and ideas stemming from Logunov's solution [2018b] of a conjecture of Nadirashvili on nodal sets of harmonic functions. In order to do this, we transform a solution to (1) into a solution of an elliptic equation in the interior of a domain. To the best of our knowledge, this idea was introduced first in [Bellová and Lin 2015] and then also applied successfully in [Georgiev and Roy-Fortin 2019; Zhu 2015].

We now describe this extension procedure, which requires $\partial\Omega$ to be of class C^2; we follow [Bellová and Lin 2015] very closely. There is a $\delta > 0$ such that the map $\partial\Omega \times (-\delta, \delta) \ni (y, t) \to y + t\nu(y)$ is one-to-one onto a neighborhood of $\partial\Omega$ in $\mathbb{R}^d$. We set $d(x) = \text{dist}(x, \partial\Omega)$, and for $\rho \leq \delta$ we define $\Omega_\rho = \{x \in \Omega : d(x) < \rho\}$ and $\Omega'_\rho = \{x \in \mathbb{R}^d : d(x) < \rho\} \setminus \overline{\Omega}$. Let now u_λ be a solution of (1), and for $x \in \Omega_\delta \cup \partial\Omega$ define

$$v(x) = u_\lambda(x)\exp(\lambda d(x)); \tag{7}$$

an easy computation shows that v satisfies

$$\begin{cases} \text{div}(A\nabla v) + b(x) \cdot \nabla v + c(x)v = 0 & \text{in } \Omega_\delta, \\ \partial_\nu v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $A = I$, $b = -2\lambda\nabla d$ and $c = \lambda^2 - \lambda\Delta d$. Consider now the reflection map $\Psi : \Omega_\delta \to \Omega'_\delta$ given by $\Psi(y + t\nu(y)) = y - t\nu(y)$, where $y \in \partial\Omega$; since v satisfies a Neumann boundary condition on $\partial\Omega$, we can extend it "evenly" across the boundary, i.e., set $v(\Psi(x)) = v(x)$ for $x \in \Omega_\delta$. Write $\Psi(x) = x'$. Another easy computation shows that on Ω'_δ the extended function (which we still call v) satisfies the equation

$$\text{div}(\tilde{A}\nabla v) + \tilde{b} \cdot \nabla v + \tilde{c}v = 0,$$

where

$$\tilde{A}(x') = \nabla\Psi(x)(\nabla\Psi(x))^T, \quad \tilde{b}^i(x') = -\sum_j \partial_{x'_j}\tilde{a}^{ij}(x') + \Delta\Psi^i(x) + \nabla\Psi^i(x)\cdot b(x), \quad \tilde{c}(x') = c(x).$$

Consider now $D = \Omega_\delta \cup \partial\Omega \cup \Omega'_\delta$; we abuse notation and denote by A, b and c the functions that are equal to the previous A, b and c in Ω_δ and equal to $\tilde{A}$, $\tilde{b}$ and $\tilde{c}$ in Ω'_δ. In [Bellová and Lin 2015] it is shown that A is Lipschitz across $\partial\Omega$ with Lipschitz constant depending only on Ω, and A is uniformly positive definite, again with constant depending only on Ω. Pasting together the pieces, one obtains that v is a strong solution of the uniformly elliptic equation

$$\text{div}(A\nabla v) + b\cdot\nabla v + cv = 0 \tag{8}$$

in D, with A Lipschitz, $\|A\|_{L^\infty(D)} \leq C$, $\|b\|_{L^\infty(D)} \leq C\lambda$ and $\|c\|_{L^\infty(D)} \leq C\lambda^2$.

We want to study (8) at wavelength scale. In order to deal with its zero set we use the theorem below, which is just an extension to more general equations of the aforementioned theorem of Logunov on harmonic functions [2018b]; its proof, which merely consists of a tedious but necessary verification that Logunov's argument carries over in this slightly more general setting, is relegated to the next section. We warn the reader that below and in the rest of the paper we do not explicitly indicate dependence of the constants on the dimension.

Theorem 3. *Consider a strong solution of the equation*

$$Lu = \text{div}(A\nabla u) + b\cdot\nabla u + cu = 0 \tag{9}$$

in $B = B(0,1) \subset \mathbb{R}^d$, with the following assumptions on the coefficients:

(i) *A is a uniformly positive definite matrix; that is, $A(x)\xi\cdot\xi \geq \alpha|\xi|^2$ for any $\xi \in \mathbb{R}^d$.*

(ii) *A is Lipschitz; that is, $\sum_{i,j}|a^{ij}(x) - a^{ij}(y)| \leq \gamma|x-y|$.*

(iii) *$\sum_{i,j}\|a^{ij}\|_{L^\infty(B)} + \sum_i\|b^i\|_{L^\infty(B)} \leq K$.*

(iv) *$c \geq 0$ and $\|c\|_{L^\infty(B)} \leq \varepsilon_0$, where ε_0 is a small enough constant depending on α, γ, K.*

Then there exist $r_0 = r_0(\alpha, \gamma, K) < 1$ and $c_0 = c_0(\alpha, \gamma, K)$ such that, for any solution u of (9) and any ball $B(x,r) \subset B(0,r_0)$ for which $u(x) = 0$, we have the lower measure bound

$$\mathcal{H}^{d-1}(\{u=0\}\cap B(x,r)) \geq c_0 r^{d-1}. \tag{10}$$

Assume now that λ is large enough depending on Ω and consider a ball $B(x_0, \varepsilon/\lambda) \subset D$, where ε is a small enough constant, with smallness depending only on Ω. We set $v_{x_0,\lambda}(x) = v(x_0 + \varepsilon x/\lambda)$ for $x \in B = B(0,1)$; note that $v_{x_0,\lambda}$ satisfies the equation

$$\text{div}(A_{x_0,\lambda}\nabla v_{x_0,\lambda}) + b_{x_0,\lambda}\cdot\nabla v_{x_0,\lambda} + c_{x_0,\lambda}v_{x_0,\lambda} = 0, \tag{11}$$

where the ellipticity constant of $A_{x_0,\lambda}$ is the same as that of A and the Lipschitz constant is the same if not better, and the coefficients satisfy $\|A_{x_0,\lambda}\|_{L^\infty(B)} \leq C$, $\|b_{x_0,\lambda}\|_{L^\infty(B)} \leq C\varepsilon$ and $\|c_{x_0,\lambda}\|_{L^\infty(B)} \leq C\varepsilon^2$. Note that if λ is large enough then $c_{x_0,\lambda} \geq 0$. If we then take ε small enough, $v_{x_0,\lambda}$ satisfies (9) and assumptions (i)–(iv) with constants α, γ, K depending only on Ω. By Theorem A, any ball centered

at $\partial\Omega$ of radius C/λ contains a zero of the Steklov eigenfunction u_λ and hence of v. We can reduce the radius of the balls and take a maximal disjoint subcollection of balls $B(x_i, C_1/\lambda) \subset D$, $x_i \in \overline{\Omega}$, such that $v(x_i) = 0$ and consider the corresponding rescaled functions $v_{x_i,\lambda}$; we can assume that $C_1 < r_0$, so that by Theorem 3 we obtain

$$\mathcal{H}^{d-1}(\{v_{x_i,\lambda} = 0\} \cap B(0, C_1)) \geq c C_1^{d-1}. \tag{12}$$

Note also that

$$\mathcal{H}^{d-1}\left(\{u_\lambda = 0\} \cap B\left(x_i, \frac{C_1}{\lambda}\right) \cap \Omega\right) \sim \mathcal{H}^{d-1}\left(\{v = 0\} \cap B\left(x_i, \frac{C_1}{\lambda}\right)\right)$$
$$\sim \varepsilon^{d-1}\lambda^{1-d}\mathcal{H}^{d-1}(\{v_{x_i,\lambda} = 0\} \cap B(0, C_1)) \geq \widetilde{C}\lambda^{1-d},$$

where $\widetilde{C}$ depends on Ω only. Since there are $\sim \lambda^{d-1}$ such balls $B(x_i, C_1/\lambda)$, we obtain

$$\mathcal{H}^{d-1}(\{u_\lambda = 0\}) \geq c_\Omega,$$

and Theorem 1 is proved.

Remark. If one could improve the result of Theorem A by showing that every ball of radius C/λ centered at any point in a corona of fixed (independent of λ) size around the boundary contains a zero of u_λ, the optimal lower bound $\mathcal{H}^{d-1}(\{u_\lambda = 0\}) \gtrsim \lambda$ would follow immediately by the preceding argument (actually more easily, since one could directly apply Logunov's result without the need to go through Theorem 3).

3. Proof of Theorem 3

This entire long section is dedicated to the proof of Theorem 3. We follow essentially the arguments of [Logunov 2018b], which carry through in this setting with few changes; the difference is that we have to use more general elliptic estimates, such as a weaker form of the maximum principle, and a frequency function that takes into account the lower-order terms in the equation. In Sections 3.1 and 3.2 we introduce the main tools we need in the proof, namely, classical elliptic estimates and the monotonicity of the frequency function. Section 3.3 will serve as a break from technicalities: here we try to convey an idea of the scheme of the proof to the reader. Sections 3.4–3.8 contain the actual body of the proof with full details.

Throughout the section we consider the operator L defined by (9) satisfying conditions (i)–(iv). It will be convenient to denote by $L_1 = L - cI$ the operator without the zeroth-order term.

3.1. _Elliptic estimates._ We first recall some standard elliptic estimates for L, paraphrasing the results in [Gilbarg and Trudinger 1983] in our notation. Note that whenever we consider a bounded domain we can assume for our purposes that it is contained in the unit ball, so we can ignore the dependency of the constants on the diameter of Ω and on the radius of balls contained in Ω. We start with the weak maximum principle.

Theorem 4 [Gilbarg and Trudinger 1983, Theorem 9.1]. *Let $L_1 u \geq -\delta$ in a bounded domain Ω. Then*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C|\delta|,$$

where $C = C(\alpha, \gamma, K)$.

Corollary 5. *Let $Lu = 0$ in a bounded domain Ω, with ε_0 in* (iv) *small enough. Then*

$$\sup_{\Omega} u \leq 2 \sup_{\partial\Omega} u^+. \tag{13}$$

Proof. We can assume $\sup_{\Omega} u \geq 0$. Since $Lu = 0$, we have $L_1 u = -cu \geq -\varepsilon_0 \sup_{\Omega} u$ using assumption (iv). By Theorem 4, then $\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C\varepsilon_0 \sup_{\Omega} u$, and the corollary follows as soon as $C\varepsilon_0 \leq \frac{1}{2}$. $\square$

The next theorem is a local pointwise estimate for subsolutions.

Theorem 6 [Gilbarg and Trudinger 1983, Theorem 9.20]. *Let $Lu \geq -\delta$ in Ω. Then for any ball $B(x, 2R) \subset \Omega$ and any $p > 0$ we have*

$$\sup_{B(x,R)} u \leq C_1 \left\{ \fint_{B(x,2R)} (u^+)^p \right\}^{1/p} + C_2 |\delta|, \tag{14}$$

where C_1 and C_2 depend on α, K and p.

Remark. In Theorem 9.20 in [Gilbarg and Trudinger 1983], the constants depend on R. However, they get worse as R increases and improve as R decreases; in this work we will only be concerned with small R, so that we can ignore the dependency on it.

We now come to the weak Harnack inequality and then the full Harnack inequality.

Theorem 7 [Gilbarg and Trudinger 1983, Theorem 9.22]. *Let $Lu \leq \delta$ in Ω, and suppose that u is nonnegative in a ball $B(x, 2R) \subset \Omega$. Then*

$$\left\{ \fint_{B(x,R)} u^p \right\}^{1/p} \leq C(\inf_{B(x,R)} u + |\delta|), \tag{15}$$

where p and C are positive numbers depending on α and K.

Theorem 8 [Gilbarg and Trudinger 1983, Corollary 9.25]. *Let $Lu = 0$ in Ω, and suppose that u is nonnegative in a ball $B(x, 2R) \subset \Omega$. Then*

$$\sup_{B(x,R)} u \leq C \inf_{B(x,R)} u, \tag{16}$$

where $C = C(\alpha, K)$.

Corollary 9. *Let $Lu = 0$ in Ω. If $u(x_0) \geq 0$ and $B(x_0, R) \subset \Omega$, then the inequality*

$$\sup_{B(x_0, 2R/3)} |u| \leq C \sup_{B(x_0, R)} u \tag{17}$$

holds for $C = C(\alpha, K)$.

Proof. Call $M = \sup_{B(x_0, R)} u$ and consider the function $h = M - u$, which is nonnegative in $B(x_0, R)$. Note that $Lh = cM$, so that $|Lh| \leq \varepsilon M$. By applying to h Theorem 6 and then Theorem 7 with $\delta = \varepsilon M$, one gets that

$$\sup_{B(x_0, 2R/3)} (M - u) \leq C_1 \left\{ \fint_{B(x_0, 3R/4)} u^p \right\}^{1/p} + C_2 \varepsilon M \leq C_3 \inf_{B(x_0, 3R/4)} (M - u) + C_4 \varepsilon M \leq C_5 M,$$

where the last inequality holds because $u(x_0) \geq 0$. Hence we obtain $\sup_{B(x_0, 2R/3)}(-u) \leq CM$. Since clearly we have that $\sup_{B(x_0, 2R/3)} u \leq M$, the corollary is proved. $\square$

3.2. *Frequency function and doubling index.* The frequency function, which as far as we know was used first by Almgren and then subsequently developed in the works of Garofalo and Lin [1986; 1987], is a powerful tool in the study of unique continuation and zero sets of elliptic PDEs. We are now going to define it for operators of the form (9) and state some of its properties, following mainly [Garofalo and Lin 1987; Han and Lin].

Let $u \in W_{\mathrm{loc}}^{1,2}(B)$ be a solution of (9). In [Garofalo and Lin 1987] and [Han and Lin] a metric $g(x) = \sum_{i,j} g_{ij}(x)\, dx_i \otimes dx_j$ is introduced in the following way: let first

$$\bar{g}_{ij}(x) = a^{ij}(x)(\det A)^{1/(d-2)},$$

where, as customary, a^{ij} denote the entries of the matrix A^{-1}. To define $\bar{g}_{ij}$ we assume here $d \geq 3$; if $d = 2$, we can just add a "mute" variable. Next, one defines

$$r(x)^2 = \sum_{i,j} \bar{g}_{ij}(0)x_i x_j \quad \text{and} \quad \eta(x) = \sum_{k,l} \bar{g}^{kl}(x)\frac{\partial r}{\partial x_k}(x)\frac{\partial r}{\partial x_l}(x).$$

Finally, one sets

$$g_{ij}(x) = \eta(x)\bar{g}_{ij}(x).$$

Note that η is a positive Lipschitz function with Lipschitz constant depending on α, γ and K. Let G be the matrix (g_{ij}) and define $|g| = \det(G)$. We can now write (9) as

$$\mathrm{div}_g(\mu(x)\nabla_g u) + b_g(x) \cdot \nabla_g u + c_g(x)u = 0,$$

where $\mu = \eta^{-(d-2)/2}$ is a Lipschitz function in B with $C_1 \leq \mu(x) \leq C_2$, $b_g = Gb/\sqrt{|g|}$ and $c_g = c/\sqrt{|g|}$. Note that, since $|g|^{-1/2}$ is a Lipschitz function bounded above and below by constants depending on α, γ and K only, b_g and c_g satisfy analogous bounds to b and c in (9). The following quantities are then introduced, where the integrals are with respect to the measure induced by the metric g:

$$H(x,r) = \int_{\partial B(x,r)} \mu u^2, \quad D(x,r) = \int_{B(x,r)} \mu |\nabla_g u|^2, \quad I(x,r) = \int_{B(x,r)} \mu |\nabla_g u|^2 + u b_g \cdot \nabla_g u + c_g u^2.$$

The frequency function is finally defined as

$$\beta(x,r) = \frac{2r I(x,r)}{H(x,r)}. \tag{18}$$

Compared with the definition in [Garofalo and Lin 1987] and [Han and Lin] there is an extra factor of 2 for aesthetic reasons in later formulas. More often than not, we will forget about the point x and only write the dependance on the radius r. The key property of the frequency function is the following almost monotonicity:

Theorem 10. *There are constants r_0, c_1 and c_2 depending on α, γ and K such that*

$$\beta(x,r) \leq c_1 + c_2\beta(x,r_0) \tag{19}$$

for $r \in (0, r_0)$. Moreover, c_2 can be chosen to be $1 + \varepsilon$ for any $\varepsilon > 0$ if $r_0 = r_0(\varepsilon)$ is small enough.

Remark. The statement of Theorem 10 is implicit in [Garofalo and Lin 1987], and the proof is contained there; in [Han and Lin] the theorem is stated as it is here, and the proof given is essentially the one of [Garofalo and Lin 1987]. The second assertion is not explicitly stated in [Garofalo and Lin 1987] or [Han and Lin] and needs some justification. In both papers, the strategy to prove the theorem is the following: one defines $\Omega_{r_0} = \{r \in (0, r_0) : \beta(r) > \max(1, \beta(r_0))\}$ and proves that it is an open subset of $\mathbb{R}$ and therefore it can be decomposed as $\Omega_{r_0} = \bigcup_{j=1}^{+\infty} (a_j, b_j)$ with a_j and b_j not belonging to Ω_{r_0}; it is then showed that $\beta'(r)/\beta(r) \geq -C$ for any $r \in \Omega_{r_0}$. By integration, one has that $\beta(r) \leq \beta(b_j) \exp(C(b_j - r))$ for any $r \in (a_j, b_j)$. Since $b_j \notin \Omega_{r_0}$, this implies that the constant c_2 can be chosen to be $\exp(Cr_0)$, which is close to 1 if r_0 is small.

In the course of the proof of Theorem 10 in [Garofalo and Lin 1987] and [Han and Lin] the differentiation formula

$$H'(r) = \left(\frac{d-1}{r} + O(1)\right) H(r) + 2I(r)$$

is obtained; the formula can be rewritten as

$$\frac{d}{dr}\left(\log \frac{H(r)}{r^{d-1}}\right) = O(1) + \frac{\beta(r)}{r}. \tag{20}$$

The next statement is an immediate consequence of this formula.

Proposition 11. *There is a constant C depending on α, γ and K such that the function $e^{Cr} H(r)/r^{d-1}$ is increasing for $r \in (0, r_0)$.*

From (20) and almost monotonicity (19), by integration one obtains the following:

Proposition 12. *The two-sided inequality*

$$c\left(\frac{r_2}{r_1}\right)^{c_2^{-1}\beta(r_1)-c_3} \leq \frac{H(r_2)}{H(r_1)} \leq C\left(\frac{r_2}{r_1}\right)^{c_2\beta(r_2)+c_3} \tag{21}$$

holds, where again c_2 can be chosen to be $1 + \varepsilon$ if r_0 is small enough.

From now on we denote with letters c, C, $c_1, \ldots$ constants which may vary from line to line and that depend only on α, γ and K without explicitly saying so every time. Additional dependencies will be indicated. We now define a quantity related to the frequency function: the doubling index.

Definition 13. For $B(x, 2r) \subset B$, the doubling index $\mathcal{N}(x, r)$ is defined by

$$2^{\mathcal{N}(x,r)} = \frac{\sup_{B(x,2r)} |u|}{\sup_{B(x,r)} |u|}. \tag{22}$$

The doubling index and the frequency function are comparable in the following sense:

Lemma 14. *Let $\varepsilon > 0$ be sufficiently small, and let r_0 be small enough that the constant c_2 in (21) is $1 + \varepsilon$; then, for $4r < r_0$,*

$$\beta(x, r(1+\varepsilon))(1 - 100\varepsilon) - c \leq \mathcal{N}(x, r) \leq \beta(x, 2r(1+\varepsilon))(1 + 100\varepsilon) + c.$$

The proof of Lemma 14 is an easy computation using the elliptic estimate (14), Proposition 11 and inequality (21); in fact, by (14),

$$\sup_{B(x,r)} |u|^2 \le C_\varepsilon \fint_{B(x,(1+\varepsilon)r)} |u|^2,$$

and further

$$\fint_{B(x,(1+\varepsilon)r)} |u|^2 \le C \frac{H((1+\varepsilon)r)}{r^{d-1}}$$

by integration and Proposition 11. From here on the computation is identical to the one in [Logunov 2018a, Lemma 7.1]. Using this, one can derive a scaling property for the doubling index; see [Logunov 2018a, Lemmas 7.2 and 7.3] for details of the computation.

Lemma 15. *Given any $\varepsilon \in (0, 1)$, there exist $r_0(\varepsilon) > 0$ and $C(\varepsilon) > 0$ such that, for $u \in W^{1,2}(B)$ a solution of (9) and any $0 < 2r_1 \le r_2 \le r_0$, we have*

$$\left(\frac{r_2}{r_1}\right)^{\mathcal{N}(x,r_1)(1-\varepsilon)-C} \le \frac{\sup_{B(x,r_2)} |u|}{\sup_{B(x,r_1)} |u|} \le \left(\frac{r_2}{r_1}\right)^{\mathcal{N}(x,r_2)(1+\varepsilon)+C}. \tag{23}$$

As a consequence, the doubling index is also almost monotonic in the sense that

$$\mathcal{N}(x, r_1)(1 - \varepsilon) - C \le \mathcal{N}(x, r_2)(1 + \varepsilon) + C.$$

3.3. *An informal outline of the proof.* We include here a brief discussion of the scheme of the proof avoiding details and technicalities; the latter are all included in the next subsections. Let us first note that in dimension 2 Theorem 3 is an easy consequence of the weak maximum principle (Corollary 5): if u vanishes at the center of a ball, the weak maximum principle tells us that there can be no small loops of zeros containing the center and therefore the nodal component containing the center must exit the ball, implying that its length must be greater than the diameter of the ball.

In higher dimensions, this simple argument does not give any lower measure bound because a priori the nodal set could be a very thin tube crossing the ball. However, a slightly more sophisticated argument, still using essentially only the maximum principle, does give a nonoptimal lower bound: we prove in Proposition 16 that if $u(x) = 0$,

$$\mathcal{H}^{d-1}(\{u = 0\} \cap B(x, r)) \ge cr^{d-1}N^{2-d},$$

where N is an upper bound for the doubling index $\mathcal{N}(x, \frac{1}{2}r)$. Note that when $d = 2$ this is already optimal, as it should be. If $d \ge 3$, this naive lower bound gets worse as the doubling index gets larger. This however contradicts intuition, since we are dealing with solutions of elliptic PDEs: if the doubling index is large, meaning that there is strong growth of u, then there should be many zeros. This suggests that one could use induction on N to promote the naive lower bound to the optimal one. The key to achieving this is Proposition 23, which shows that if the doubling index is comparable to $N \gg 1$ in balls of radii $\frac{1}{4}r$ to r (we call this "stable growth", see Definition 22), there are many zeros in the ball of radius r; more precisely, there are at least $[\sqrt{N}]^{d-1} f(N)$, with $f(N) \to \infty$ as $N \to \infty$, disjoint balls of radius $r/\sqrt{N}$

such that u vanishes at the center. The fact that $f(N)$ grows with N essentially shows that indeed there are more zeros as the doubling index increases, and it is needed to close the induction in Section 3.8.

The proof of Proposition 23 uses crucially Theorem 19, which tells us that if a cube is partitioned into some large number B^d of subcubes, the number of subcubes which have doubling indices dropping by an amount increasing with B compared to the doubling index of the original cube form the vast majority of the subcubes. The argument goes as follows: since the doubling index is comparable to N on scales $\frac{1}{4}r$ to r, we can assume that in the ball of radius $\frac{1}{4}r$, $|u| \leq 1$, while in the ball of radius $\frac{1}{2}r$, $|u| \geq 2^{cN}$. We then connect points where u is small to points where u is large by many chains of cubes (called "tunnels" later): since there is considerable growth of u from one endpoint of the tunnel to the other, the Harnack inequality tells us that there must be zeros and the growth happens in the cubes with zeros; an application of Theorem 19 gives us that most of the cubes in the tunnel have doubling index much smaller than N, so that the growth from one endpoint to the other cannot be realized in very few cubes, and hence each tunnel must have many cubes with zeros. The formal proof is a matter of quantifying what "small", "large", "few" and "many" mean.

The only issue remaining is ensuring that there are balls of stable growth: this is done in Claim 3, and the proof uses the estimates in Section 3.6 which are consequences of the almost monotonicity of the frequency function.

Let us emphasize once again that the proof scheme described above is due to Aleksandr Logunov, and it appeared first in [Logunov 2018b]. In our case we have to adapt it to elliptic equations with lower-order terms, but the more general estimates that we need are collected above in Sections 3.1 and 3.2, and using those estimates the proof runs in the same way as for harmonic functions.

3.4. *Local asymmetry.* We now derive a lower estimate for the relative volume of the set $\{u > 0\}$ in balls centered at zeros of u, and consequently a nonoptimal lower estimate for the measure of the zero set. The estimate and the proof are analogous to the Laplace–Beltrami eigenfunctions case, for which see, for example, [Logunov and Malinnikova 2018; Mangoubi 2008]. For the reader's convenience, we reproduce here essentially the same proof as [Logunov and Malinnikova 2018].

Proposition 16. *Let $B(x, r) \subset B$ and u be a solution of (9) such that $u(x) = 0$. Suppose that $\mathcal{N}(x, \frac{1}{2}r) \leq N$, where N is a positive integer. Then the lower measure bound*

$$\mathcal{H}^{d-1}(\{u = 0\} \cap B(x, r)) \geq cr^{d-1}N^{2-d} \tag{24}$$

holds for some $c > 0$.

Proof. For notational simplicity we assume $x = 0$ and write $B_r = B(0, r)$. We can also safely assume that $N \geq 4$, say. Note that by (17) and (13) we have $\sup_{B_{r/2}} |u| \leq C \max_{\partial B_{3r/4}} u$, so that

$$\frac{\max_{\partial B_r} u}{\max_{\partial B_{3r/4}} u} \leq C_1 \frac{\sup_{B_r} |u|}{\sup_{B_{r/2}} |u|} \leq C_1 2^N.$$

Let now $r_j = r\left(\frac{3}{4} + j/(4N)\right)$ for $j = 0, 1, \ldots, N$, and consider the concentric spheres $S_j = \{|x| = r_j\}$. Write $m_j^+ = \max_{S_j} u$ and $m_j^- = \min_{S_j} u$. From the weak maximum principle (13) (applied to u as well

as $-u$), it follows that

$$m_j^+ > 0, \quad m_j^- < 0, \quad m_j^+ \le 2m_{j+1}^+ \quad \text{and} \quad |m_j^-| \le 2|m_{j+1}^-|.$$

For $j = 0, 1, \ldots, N - 1$, define

$$\tau_j^+ = m_{j+1}^+ / m_j^+ \quad \text{and} \quad \tau_j^- = |m_{j+1}^-|/|m_j^-|;$$

from the above, $\tau_j^{+/-} \ge \frac{1}{2}$. Moreover, we have that

$$\tau_0^+ \cdots \tau_{N-1}^+ = \frac{\max_{\partial B_r} u}{\max_{\partial B_{3r/4}} u} \le C_1 2^N,$$

so at most $\frac{1}{4}N$, say, of the τ_j^+ are greater than some C independent of N. The same holds for the τ_j^-, so that for at least $\frac{1}{2}N$ indices there holds $m_{k+1}^+ \le Cm_k^+$ and $|m_{k+1}^-| \le C|m_k^-|$. Consider each such k and let $x_0 \in S_k$ be such that $u(x_0) = m_k^+$. Denote by b the ball centered at x_0 of radius $r/(8N)$; then by (13) and the choice of k,

$$\sup_b u \le \sup_{\{|x| \le r_{k+1}\}} u \le 2m_{k+1}^+ \le Cm_k^+.$$

Applying (17), we then get that $\sup_{b/2} |u| \le Cm_k^+$. We now use this last inequality and the elliptic gradient estimate (see, for instance, [Gilbarg and Trudinger 1983, Theorem 8.32])

$$\sup_{B(y,s/2)} |\nabla u| \le \left(\frac{C}{s}\right) \sup_{B(y,s)} |u|$$

for $y = x_0$ and $s = r/(16N)$ to get, for $x \in B(x_0, \theta r/N)$ with θ a sufficiently small number,

$$u(x) \ge u(x_0) - |x - x_0| \sup_{b/4} |\nabla u| \ge m_k^+ - C\theta m_k^+ \ge 0.$$

We thus found a ball centered on S_k of radius $\theta r/N$ where u is positive, call it b_+. Replace now u with $-u$, which is also a solution of (9): repeating the argument above with m_k^- and τ_k^- instead of m_k^+ and τ_k^+ gives us a ball centered on S_k of radius $\theta r/N$ where u is negative, call it b_-. Now consider the sections of the two balls with hyperplanes through the origin that contain the center of the balls: any path within the annulus $\{r_{k-1} < |x| < r_{k+1}\}$ that connects these two sections contains a zero of u, since u is positive on b_+ and negative on b_-. This implies that the measure of the zero set is greater than the measure of the section of the balls, that is to say,

$$\mathcal{H}^{d-1}(\{x : r_{k-1} < |x| < r_{k+1}, u(x) = 0\}) \ge c\left(\frac{r}{N}\right)^{d-1}.$$

The above holds for all indices k for which $m_{k+1}^+ \le Cm_k^+$ and $|m_{k+1}^-| \le C|m_k^-|$, and recall that there are at least $\frac{1}{2}N$ such indices. Summing the inequality above over those indices, we see that (24) holds. □

Remark. Note that the argument above also shows that

$$\frac{\text{Vol}(\{u > 0\} \cap B(x, r))}{\text{Vol}(B(x, r))} \ge \frac{c}{N^{d-1}}$$

if $u(x) = 0$, which is analogous to the best known lower bound (when $d \ge 3$) for the local asymmetry of Laplace eigenfunctions [Mangoubi 2008].

3.5. *Counting doubling indices.* We now recall some very useful results from [Logunov 2018a; 2018b; Logunov and Malinnikova 2018] that allow us to find many small cubes with better doubling index than the original ball (or cube). The proofs are combinatorial in nature. First we define a version of the doubling index for cubes, which are more suitable for partitioning than balls. Given a cube Q and a solution u of (9), we define the doubling index $N(Q)$ as

$$N(Q) = \sup_{\{x \in Q,\, r < \operatorname{diam}(Q)\}} \log \frac{\sup_{B(x,10dr)} |u|}{\sup_{B(x,r)} |u|}.$$

The constant $10d$ is there for technical reasons and the reader should not worry about it. It is clear that with this definition $N(Q_1) \le N(Q_2)$ if $Q_1 \subset Q_2$. Theorem 18 was proved in [Logunov 2018a], and then extended in [Georgiev and Roy-Fortin 2019] to the more general equation (9); the proof combines an accumulation of growth result ([Logunov 2018a, Lemma 2.1] and [Georgiev and Roy-Fortin 2019, Proposition 3.1], called the simplex lemma), and a propagation of smallness result ([Logunov 2018a, Lemma 4.1] and [Georgiev and Roy-Fortin 2019, Proposition 3.2], called the hyperplane lemma). The hyperplane lemma is a consequence of quantitative Cauchy uniqueness, which we state in a simple version below; it can be obtained from a very general result in [Alessandrini et al. 2009] (Theorem 1.7). See also [Lin 1991].

Proposition 17. *Let D be a bounded domain with $\mathcal{C}^2$ boundary, and let B be a ball of radius $\rho < 1$. Let u be a solution of (9) in $D \cap B$, $u \in \mathcal{C}^1(\overline{D} \cap B)$. There exist $\beta = \beta(\alpha, \gamma, K, D, \rho) \in (0, 1)$ and $C = C(\alpha, \gamma, K, D, \rho) > 0$ such that, if $|u| \le 1$ and $|\nabla u| \le \rho^{-1}$ in $D \cap B$, and $|u| \le \eta$ and $|\nabla u| \le \eta \rho^{-1}$ on $\partial D \cap B$, where η is a real number, then*

$$|u(x)| \le C\eta^{\beta}$$

for any $x \in D \cap \frac{1}{2}B$.

Remark. In [Logunov 2018a; Georgiev and Roy-Fortin 2019], Proposition 17 is applied when ∂D is flat; this is sufficient to prove the theorem below. We will use the proposition in the nonflat case later in Section 4, to prove a version of the hyperplane lemma.

Theorem 18 [Logunov 2018a, Theorem 5.1; Georgiev and Roy-Fortin 2019, Theorem 4.1]. *There exist a constant $c > 0$ and an integer $A > 1$ depending on the dimension only, and positive numbers $N_0 = N_0(\alpha, \gamma, K)$ and $R_0 = R_0(\alpha, \gamma, K)$ such that for any cube $Q \subset B(0, R_0)$ the following holds: if Q is partitioned into A^d equal subcubes, then the number of subcubes with doubling index greater than $\max(N(Q)/(1 + c), N_0)$ is less than $\frac{1}{2} A^{d-1}$.*

Starting from Theorem 18, in [Logunov 2018b] an iterated version is proved, which is the one decisively used in the proof of the lower bound on zero sets. We state it below and refer to [Logunov 2018b] for the proof.

Theorem 19 [Logunov 2018b, Theorem 5.3]. *There exist positive constants c_1, c_2, C and an integer $B_0 > 1$ depending on the dimension only, and positive numbers $N_0 = N_0(\alpha, \gamma, K)$ and $R_0 = R_0(\alpha, \gamma, K)$ such that for any cube $Q \subset B(0, R_0)$ the following holds: if Q is partitioned into B^d equal subcubes, where $B > B_0$ is an integer, then the number of subcubes with doubling index greater than $\max(N(Q)2^{-c_1 \log B/\log\log B}, N_0)$ is less than $C B^{d-1-c_2}$.*

3.6. *Estimates in a spherical shell.* In the following we always indicate by u a solution of (9); the frequency function and doubling index are relative to u. Consider a ball $B(p, s) \subset B\left(0, \frac{1}{4}r_0\right)$; we are going to establish some estimates for the growth of u near a point of maximum. Let $x \in \partial B(p, s)$ be a point where the maximum of $|u|$ on $\overline{B(p, s)}$ is almost attained, in the sense that $\sup_{B(p,s)} |u| \leq 2|u(x)|$; the existence of such an x is guaranteed by Corollary 5. Write $M = |u(x)|$. In the next two lemmas we will assume that there is a large enough number N and

$$\delta \in \left(\frac{1}{\log^{100} N}, \frac{1}{8}\right)$$

such that

$$\frac{1}{10} N \leq \beta(p, t) \leq 10^4 N \tag{25}$$

for $t \in I := (s(1 - \delta), s(1 + \delta))$.

Lemma 20 (variation on [Logunov 2018b, Lemma 4.1]). *Let (25) be satisfied. There exist positive constants C and c such that*

$$\sup_{B(p,s(1-\delta))} |u| \leq C M 2^{-c\delta N}, \tag{26}$$

$$\sup_{B(p,s(1+\delta))} |u| \leq C M 2^{C\delta N}. \tag{27}$$

Proof. Let us prove (26) only. By (21) and (25), we have that

$$\left(\frac{t_2}{t_1}\right)^{N/30} \leq \frac{H(p, t_2)}{H(p, t_1)} \leq C \left(\frac{t_2}{t_1}\right)^{10^5 N} \tag{28}$$

for $t_1 < t_2 \in I$, where we assume that r_0 is small enough to take $c_2 = 2$ in (21). We estimate

$$M^2 \geq C_1 s^{-d+1} H(p, s) \geq C_1 s^{-d+1} H\left(p, s\left(1 - \frac{1}{2}\delta\right)\right)\left(1 + \frac{1}{2}\delta\right)^{N/30},$$

where the first inequality is just the estimate of the L^2-norm by the L^∞-norm and the second inequality comes from (28). By integration and Proposition 11 we have

$$s H\left(p, s\left(1 - \frac{1}{2}\delta\right)\right) = s \int_{\partial B(p, s(1-\delta/2))} |u|^2 \geq C_2 \int_{B(p, s(1-\delta/2))} |u|^2.$$

Let now $\tilde{x}$ be a point on $\partial B(p, s(1 - \delta))$ where the supremum of $|u|$ on $B(p, s(1 - \delta))$ is almost attained, i.e., $\sup_{B(p,s(1-\delta))} |u| \leq 2|u(\tilde{x})|$, and write $\widetilde{M} = |u(\tilde{x})|$. Note now that

$$\int_{B(p, s(1-\delta/2))} |u|^2 \geq \int_{B(\tilde{x}, \delta s/2)} |u|^2 \geq C_3 (\delta s)^d \fint_{B(\tilde{x}, \delta s/2)} |u|^2;$$

moreover, by (14) we have

$$\widetilde{M}^2 \leq C_4 \fint_{B(\tilde{x}, \delta s/2)} |u|^2.$$

Combining the estimates we obtain

$$M^2 \geq C_5 \delta^d \left(1 + \frac{1}{2}\delta\right)^{N/30} \widetilde{M}^2.$$

Since $\log\left(1 + \frac{1}{2}\delta\right) \geq \frac{1}{4}\delta$, it follows easily from the above and $\delta \gtrsim 1/\log^{100} N$ that $M^2 \geq C_6 \exp\left(\frac{1}{100} N\delta\right)\widetilde{M}^2$, from which one obtains (26) recalling the definitions of M and $\widetilde{M}$. □

Using the properties of the doubling index, we now derive some estimates on small balls close to x; we keep on denoting by x the point on $\partial B(p, s)$ where the maximum of $|u|$ on $\overline{B(p, s)}$ is almost attained.

Lemma 21 (variation on [Logunov 2018b, Lemma 4.2]). *Let (25) be satisfied. There exists $C > 0$ such that*

$$\sup_{B(x,\delta s)} |u| \leq M 2^{C\delta N + C} \tag{29}$$

and, for any $\tilde{x}$ with $d(x, \tilde{x}) \leq \frac{1}{4}\delta s$,

$$\mathcal{N}(\tilde{x}, \tfrac{1}{4}\delta s) \leq C\delta N + C, \tag{30}$$

$$\sup_{B(\tilde{x},\delta s/10N)} |u| \geq M 2^{-C\delta N \log N - C}. \tag{31}$$

Proof. Note that since $B(x, \delta s) \subset B(p, s(1 + \delta))$, the first estimate (29) is an immediate consequence of (27). By definition of doubling index and (29) we have that

$$2^{\mathcal{N}(\tilde{x},\delta s/4)} \leq \frac{\sup_{B(\tilde{x},\delta s/2)} |u|}{\sup_{B(\tilde{x},\delta s/4)} |u|} \leq \frac{\sup_{B(x,\delta s)} |u|}{M} \leq 2^{C\delta N + C},$$

and (30) is proved. Now recall the scaling properties (23); by those and (30) we obtain

$$\frac{\sup_{B(\tilde{x},\delta s/4)} |u|}{\sup_{B(\tilde{x},\delta s/10N)} |u|} \leq (40N)^{2\mathcal{N}(\tilde{x},\delta s/4)+C_1} \leq (40N)^{C_1\delta N + C_1} \leq 2^{C_2\delta N \log N + C_2 \log N} \leq 2^{C_3\delta N \log N + C_3},$$

where the last inequality holds because $\delta \gtrsim 1/\log^{100} N$. Since, by the distance condition, $\sup_{B(\tilde{x},\delta s/4)} |u| \geq |u(x)| = M$, (31) follows. $\qquad\square$

3.7. *Finding many balls around the zero set.*
We follow the arguments in Section 6 of [Logunov 2018b], in the reformulation contained in [Logunov and Malinnikova 2020]; the estimates in the spherical shell will be used together with the combinatorial results on doubling indices. We use the notion of "stable growth", which is taken from [Logunov and Malinnikova 2020] and was not present in [Logunov 2018b].

Definition 22. We say that u has a stable growth of order N in a ball $B(y, s)$ if $\mathcal{N}(y, \frac{1}{4}s) \geq N$ and $\mathcal{N}(y, s) \leq 1000N$.

The number 1000 does not have any special meaning, it is just a large enough numerical constant. The following result is the key to the proof of the lower bound.

Proposition 23 (variation on [Logunov 2018b, Proposition 6.1]). *Let $B(p, 2r) \subset B(0, r_0)$. There exists a number $N_0 > 0$ large enough such that, for $N > N_0$ and any solution u of (9) that has stable growth of order N in $B(p, r)$, the following holds: there exist at least $[\sqrt{N}]^{d-1} 2^{c_1 \log N / \log \log N}$ disjoint balls $B(x_i, r/\sqrt{N}) \subset B(p, r)$ such that $u(x_i) = 0$.*

Proof. Assume without loss of generality that $\sup_{B(p,r/4)} |u| = 1$. The stable growth assumption then implies that

$$\sup_{B(p,r/2)} |u| \geq 2^N \quad \text{and} \quad \sup_{B(p,2r)} |u| \leq 2^{CN}.$$

We denote by x the point on $\partial B\left(p, \frac{1}{2}r\right)$ where the maximum over $\overline{B\left(p, \frac{1}{2}r\right)}$ is almost attained, so that by the above $|u(x)| \geq 2^{N-1}$. We now divide the ball $B(p, 2r)$ into cubes q_i of side length $cr/\sqrt{N}$ and organize these cubes into tunnels in the following way: the centers of the cubes in each tunnel lie on a line parallel to the segment that connects p and x. A tunnel contains at most $C\sqrt{N}$ cubes. Let us call a cube q_i "good" if

$$N(q_i) \leq \max\left(\frac{N}{2^{c \log N / \log \log N}}, N_0\right) \tag{32}$$

for some constant c. We will call a tunnel "good" if it contains only good cubes; by Theorem 19, most of the cubes are good and most of the tunnels are good. Another application of Theorem 19 gives the following:

Claim 1. *The number of good tunnels containing at least one cube with distance from x less than $r/\log^2 N$ is greater than $c(\sqrt{N}/\log^2 N)^{d-1}$.*

The proof of the proposition is then completed with the help of the next claim.

Claim 2. *Any good tunnel that contains at least one cube with distance from x less than $r/\log^2 N$ also contains at least $2^{c_2 \log N / \log \log N}$ cubes with zeros of u.*

Proof. Take one such tunnel T. Note that T contains at least one cube $q_a \subset B\left(p, \frac{1}{4}r\right)$, so that $\sup_{q_a} |u| \leq 1$. Call q_b a cube in T with distance from x less than $r/\log^2 N$; we want to show that the supremum of $|u|$ over q_b is large. To this end, we apply Lemma 21 with $\delta \sim 1/\log^2 N$ and $\tilde{x}$ being the center x_b of the cube q_b. By the stable growth assumption and the comparability of the doubling index and frequency function (Lemma 14), inequality (25) is satisfied for N large enough. Then (31) gives us

$$\sup_{B(x_b, \delta r/10N)} |u| \geq |u(x)| 2^{-CN/\log N - C},$$

and hence, recalling that $|u(x)| \geq 2^{N-1}$,

$$\sup_{q_b/2} |u| \geq 2^{cN}.$$

We now follow T from q_a to q_b and find many zeros. The proof is at this point identical to the one given in [Logunov 2018b]; for completeness we provide the details. We enumerate the cubes q_i from q_a to q_b such that q_a is the first and q_b is the last. Since T is a good tunnel, by (32) we have that for any two adjacent cubes

$$\log \frac{\sup_{q_{i+1}/2} |u|}{\sup_{q_i/2} |u|} \leq \log \frac{\sup_{4q_i} |u|}{\sup_{q_i/2} |u|} \leq \frac{N}{2^{c_3 \log N / \log \log N}}.$$

We split the set of indices S into two sets S_1 and S_2, where S_1 is the set of i such that u does not change sign in $\overline{q}_i \cup \overline{q}_{i+1}$ and $S_2 = S \setminus S_1$. The advantage of this is the possibility to use the Harnack inequality on S_1; the aim is to get a lower bound on the cardinality of S_2. In fact, for $i \in S_1$, by (16) we have that

$$\log \frac{\sup_{q_{i+1}/2} |u|}{\sup_{q_i/2} |u|} \leq C_1.$$

We then estimate

$$\log \frac{\sup_{q_b/2} |u|}{\sup_{q_a/2} |u|} = \sum_{S_1} \log \frac{\sup_{q_{i+1}/2} |u|}{\sup_{q_i/2} |u|} + \sum_{S_2} \log \frac{\sup_{q_{i+1}/2} |u|}{\sup_{q_i/2} |u|} \leq |S_1| C_1 + |S_2| \frac{N}{2^{c_3 \log N / \log \log N}};$$

on the other hand, recall that

$$\log \frac{\sup_{q_b/2} |u|}{\sup_{q_a/2} |u|} \geq cN.$$

Combining the two estimates one obtains

$$|S_1| C_1 + |S_2| \frac{N}{2^{c_3 \log N / \log \log N}} \geq cN,$$

and noting that $|S_1| C_1 \leq C_1 \sqrt{N} \leq \frac{1}{2} cN$ we conclude

$$|S_2| \geq c_3 2^{c_3 \log N / \log \log N}.$$

The last quantity is larger than $2^{c_2 \log N / \log \log N}$ if N is large enough, and the claim is proved. $\square$

It is now a straightforward matter to finish the proof of Proposition 23: by Claim 1 there are at least $c(\sqrt{N}/\log^2 N)^{d-1}$ tunnels satisfying the hypothesis of Claim 2, and hence there are at least $c(\sqrt{N}/\log^2 N)^{d-1} 2^{c_2 \log N / \log \log N}$ cubes that contain zeros of u; the last quantity can be made larger than $(\sqrt{N})^{d-1} 2^{c_1 \log N / \log \log N}$, and then one replaces cubes by balls. $\square$

3.8. *Proof of the lower bound.* We take r_0 small enough that (19), (21), Lemma 14 and (23) hold. Writing $N(0, r_0) = \sup_{\{B(x,r) \subset B(0,r_0)\}} \mathcal{N}(x, r)$, we define

$$F(N) = \inf \frac{\mathcal{H}^{d-1}(\{u = 0\} \cap B(x, \rho))}{\rho^{d-1}},$$

where the infimum is taken over all balls $B(x, \rho) \subset B(0, r_0)$ and all solutions u of (9) such that $u(x) = 0$ and $N(0, r_0) \leq N$. Theorem 3 then follows immediately from the following:

Theorem 24. $F(N) \geq c$, *where c is independent of N.*

Proof. Let u be a solution of (9) in competition for the infimum in the definition of $F(N)$; let $F(N)$ be almost attained on u, in the sense that

$$\frac{\mathcal{H}^{d-1}(\{u = 0\} \cap B(x, \rho))}{r^{d-1}} \leq 2F(N) \tag{33}$$

for some $B(x, r) \subset B(0, r_0)$ with $u(x) = 0$. Recall the easy bound (24):

$$\frac{\mathcal{H}^{d-1}(\{u = 0\} \cap B(x, r))}{r^{d-1}} \geq \frac{c_1}{\mathcal{N}(x, r/4)^{d-2}} \geq \frac{c_1}{N^{d-2}}. \tag{34}$$

Estimate (34) already finishes the proof if $\mathcal{N}(x, \frac{1}{4}r)$ is bounded uniformly in N; let us then argue by contradiction and assume that $\mathcal{N}(x, \frac{1}{4}r)$ is large enough. Denote $\widetilde{N} = \mathcal{N}(x, \frac{1}{4}r)$ and suppose first that u

has stable growth of order $\widetilde{N}$. We can then apply Proposition 23 and find at least $[\sqrt{\widetilde{N}}]^{d-1} 2^{c \log \widetilde{N} / \log \log \widetilde{N}}$ disjoint balls $B(x_i, r/\sqrt{\widetilde{N}}) \subset B(x, r)$ with $u(x_i) = 0$. By definition of $F(N)$, there holds:

$$\mathcal{H}^{d-1}\left(\{u = 0\} \cap B\left(x_i, \frac{r}{\sqrt{\widetilde{N}}}\right)\right) \geq F(N)\left(\frac{r}{\sqrt{\widetilde{N}}}\right)^{d-1}.$$

Summing the inequality over all the balls, we obtain

$$\mathcal{H}^{d-1}(\{u = 0\} \cap B(x, \rho)) \geq [\sqrt{\widetilde{N}}]^{d-1} 2^{c \log \widetilde{N} / \log \log \widetilde{N}} F(N)\left(\frac{r}{\sqrt{\widetilde{N}}}\right)^{d-1};$$

the quantity on the right can be made larger than $2F(N)r^{d-1}$ if $\widetilde{N}$ is large enough, which is a contradiction with (33). Therefore we would be done if we knew a priori that u has stable growth of order $\widetilde{N}$ in $B(x, r)$, but this is not necessarily the case; fortunately we can find a smaller ball where u has stable growth.

Claim 3. *If* $\mathcal{N}\left(x, \frac{1}{4}r\right)$ *is large enough, there is a number* $N_1 \gtrsim \mathcal{N}\left(x, \frac{1}{4}r\right)$ *and a ball* $B_1 \subset B(x, r)$ *with radius* $r_1 \sim r/\log^2 N_1$ *such that* u *has stable growth of order* $N_1/\log^2 N_1$ *in* B_1.

Proof. Let us define a modified frequency function as

$$\tilde{\beta}(p, r) = \sup_{t \in (0, r]} \beta(p, t) + c_1,$$

so that $\tilde{\beta}(p, r)$ is a positive monotonic increasing function. Note that by (19) we have

$$\beta(p, r) \leq \tilde{\beta}(p, r) \leq c_3 + 2\beta(p, r),$$

and the rightmost expression is less than $3\beta(p, r)$ if $\beta(p, r) \geq c_3$. We use the following claim:

Claim 4 [Logunov 2018b, Lemma 3.1]. *Let* f *be a nonnegative, monotonic nondecreasing function in* $[a, b]$, *and assume* $f \geq e$. *Then there exist* $x \in \left[a, \frac{1}{2}(a+b)\right)$ *and a number* $N_1 \geq e$ *such that*

$$N_1 \leq f(t) \leq eN_1 \quad \text{for any } t \in \left(x - \frac{b-a}{20 \log^2 f(x)}, x + \frac{b-a}{20 \log^2 f(x)}\right) \subset [a, b].$$

We apply Claim 4 to $\tilde{\beta}(p, \cdot)$ and hence identify a spherical shell of width $\sim r/\log^2 N_1$ about $s \in \left(\frac{2}{3}r, \frac{3}{4}r\right)$ where $\tilde{\beta}(p, \cdot)$ is comparable to N_1. Since $\mathcal{N}\left(x, \frac{1}{4}r\right)$ is large, by Lemma 14 and almost monotonicity $\beta(x, t)$ is large for $t > \frac{1}{2}r$ and then also $\beta(x, \cdot)$ is comparable to N_1 in the spherical shell. In other words, (25) holds with N_1 and $\delta \sim 1/\log^2 N$. Let now $y \in \partial B_s$ be a point where the maximum is almost attained, as in Lemmas 20 and 21. Take a ball B_1 of radius $\sim s/\log^2 N_1$ such that $\frac{1}{4}B_1 \subset B_{s(1-\delta)}$ and $y \in \frac{1}{2}B_1$; then (26) implies that

$$\mathcal{N}\left(\tfrac{1}{4}B_1\right) \geq c\frac{N_1}{\log^2 N_1}$$

and (27) implies that

$$\mathcal{N}(B_1) \leq C\frac{N_1}{\log^2 N_1},$$

which means that u has stable growth of order $N_1/\log^2 N_1$ in B_1, and the claim is proved. $\qquad \square$

Claim 3 gives an order of stable growth that is again large enough to get a contradiction with (33) if $\mathcal{N}\left(x, \frac{1}{4}r\right)$ and hence N_1 is large enough. This means that $\mathcal{N}\left(x, \frac{1}{4}r\right)$ is bounded from above by some N_0 independently of N, and therefore by (33) and (34) we obtain

$$F(N) \geq \frac{\mathcal{H}^{d-1}(\{u = 0\} \cap B(x, r))}{2r^{d-1}} \geq \frac{c_3}{(N_0)^{d-2}} \geq c, \tag{35}$$

which concludes the proof of the theorem. $\qquad\square$

4. Upper bound

Here we give the proof of Theorem 2. Throughout this section $\partial\Omega$ is assumed to be of class $\mathcal{C}^2$. As remarked in the introduction, the proof uses the Donnelly–Fefferman bound [1988] in the interior of the domain and a multiscale induction argument at the boundary. As will be apparent from the proof, the result with a $\mathcal{C}^\infty$-metric inside Ω would follow from an upper bound for zero sets of elliptic PDEs with smooth coefficients that is linear in the frequency; the best we have thus far is polynomial in the frequency [Logunov 2018a].

We introduce now a version of the doubling index that takes into account the boundary. Namely, for $x \in \overline{\Omega}$ and $u \in \mathcal{C}(\overline{\Omega})$ a harmonic function, we let

$$2^{\mathcal{N}_u^*(x,r)} = \frac{\sup_{B(x,2r)\cap\Omega} |u|}{\sup_{B(x,r)\cap\Omega} |u|}. \tag{36}$$

Note that if v is the extension across the boundary of the Steklov eigenfunction u_λ as in Section 2 and $\text{dist}(x, \partial\Omega) \lesssim 1/\lambda$, $r \lesssim 1/\lambda$, we have that $\mathcal{N}_{u_\lambda}^*(x, r) \sim \mathcal{N}_v(x, r)$, where $\mathcal{N}_v(x, r)$ is defined as in (22); this will allow us to use the almost monotonicity property (23). It was proved in [Zhu 2015] (using the extension v) that for any $r < r_0(\Omega)$

$$\mathcal{N}_{u_\lambda}^*(x, r) \leq C\lambda, \tag{37}$$

mirroring a corresponding statement for Laplace eigenfunctions proved by Donnelly and Fefferman. It will once again be convenient to define a maximal version of the doubling index for cubes; for $Q \subset \mathbb{R}^d$ a cube such that $Q \cap \Omega \neq \varnothing$, we set

$$N_u^*(Q) = \sup_{\substack{x \in Q \cap \overline{\Omega} \\ r \leq \text{diam}(Q)}} \mathcal{N}_u^*(x, r).$$

Definition 25. A Whitney cube in Ω is any cube Q such that $c_1 \text{dist}(Q, \partial\Omega) \leq s(Q) \leq c_2 \text{dist}(Q, \partial\Omega)$, where $s(Q)$ is the side length of Q and c_1 and c_2 are positive dimensional constants.

With this notation, we state the following important result of [Donnelly and Fefferman 1988].

Theorem 26. *Let u be a harmonic function in Ω. Then there is $C > 0$, independent of u, such that*

$$\mathcal{H}^{d-1}(\mathcal{Z}_u \cap Q) \leq C(N_u^*(Q) + 1)s(Q)^{d-1} \tag{38}$$

for any Whitney cube Q.

From now on, we will denote by u a Steklov eigenfunction with eigenvalue λ. We will first use the theorem above to bound the measure of the zero set of u in the interior, up to a distance from the boundary comparable to $1/\lambda$. We will assume $\lambda > \lambda_0$. As in the previous section, write $d(x) = \text{dist}(x, \partial\Omega)$; Let c_0 be a small constant depending only on Ω. We write the decomposition

$$\Omega = \text{In} \cup \text{Mid} \cup \text{Bd},$$

where $\text{In} = \{x \in \Omega : d(x) \geq c_0\}$, $\text{Mid} = \{x \in \Omega : c_0/\lambda < d(x) < c_0\}$ and $\text{Bd} = \{x \in \Omega : d(x) \leq c_0/\lambda\}$. It follows easily from Theorem 26 and (37) that

$$\mathcal{H}^{d-1}(\mathcal{Z}_u \cap \text{In}) \leq C\lambda, \tag{39}$$

with C depending on Ω only. The next lemma estimates the contribution of the nodal set in Mid.

Lemma 27. *There is $C > 0$ depending only on Ω such that*

$$\mathcal{H}^{d-1}(\mathcal{Z}_u \cap \text{Mid}) \leq C\lambda \log \lambda. \tag{40}$$

Proof. We set $M_k = \{x \in \Omega : c_0 2^{k-1}/\lambda < d(x) < c_0 2^k/\lambda\}$, and we have

$$\text{Mid} = \bigcup_{k=1}^{c \log \lambda} M_k.$$

We perform a decomposition of Ω into Whitney cubes with disjoint interior (the statement that this is possible is usually called the Whitney covering lemma). Define

$$\mathcal{Q}_k = \{\text{Whitney cubes intersecting } M_k\}.$$

In the following lines we will denote by $|\cdot|$ both the cardinality of a discrete collection and the Lebesgue measure of cubes; it should cause no confusion. Note that if $Q \in \mathcal{Q}_k$, then

$$|Q| \sim \frac{2^{kd}}{\lambda^d};$$

it follows that $|\mathcal{Q}_k| \lesssim 2^{-kd}\lambda^{d-1}$. We can then estimate, using Theorem 26 and (37),

$$\mathcal{H}^{d-1}(\mathcal{Z}_u \cap \text{Mid}) = \sum_{k=1}^{c \log \lambda} \mathcal{H}^{d-1}(\mathcal{Z}_u \cap M_k) \leq \sum_{k=1}^{c \log \lambda} \sum_{Q \in \mathcal{Q}_k} \mathcal{H}^{d-1}(\mathcal{Z}_u \cap Q)$$

$$\lesssim \lambda \sum_{k=1}^{c \log \lambda} \sum_{Q \in \mathcal{Q}_k} s(Q)^{d-1} \lesssim \lambda \sum_{k=1}^{c \log \lambda} |\mathcal{Q}_k| \frac{2^{kd}}{\lambda^{d-1}} \lesssim \lambda \log \lambda. \qquad \square$$

To prove Theorem 2 the only thing left is to estimate $\mathcal{H}^{d-1}(\mathcal{Z}_u \cap \text{Bd})$. We cover Bd with $\sim \lambda^{d-1}$ cubes q_λ centered at $\partial\Omega$ of side length $s(q_\lambda) = 4c_0/\lambda$; then Theorem 2 follows from (37) and the following proposition:

Proposition 28. *Let q_λ be one of the cubes above, and suppose $N_u^*(4q_\lambda) \leq N$. Then*

$$\mathcal{H}^{d-1}(\mathcal{Z}_u \cap q_\lambda) \leq C(\Omega)(N+1)s(q_\lambda)^{d-1}. \tag{41}$$

Remark. In the following we will rescale

$$h(x) = u\left(\frac{x}{\lambda}\right), \tag{42}$$

so that q_λ becomes a cube Q of side length $s < 1$, where s is small enough depending on Ω but independent of λ, and h satisfies $\Delta h = 0$ in $10Q \cap \Omega$ and $\partial_\nu h = h$ on $\partial\Omega \cap \overline{10Q}$. Note that the doubling index is unchanged under this rescaling. Proposition 28 will follow from

$$\mathcal{H}^{d-1}(\mathcal{Z}_h \cap Q) \leq C(\Omega)(N+1). \tag{43}$$

The main ingredient in the proof of Proposition 28 is a version of the hyperplane lemma of [Logunov 2018a] with cubes touching the boundary, the proof of which uses quantitative Cauchy uniqueness as stated in Proposition 17. The proof is very similar to the one contained in [Logunov et al. 2021], we reproduce it here for the reader's convenience.

Lemma 29. *Let h be as in* (42) *and Q a cube of side length s as in the remark above. There exist k and N_0 large enough depending on s and Ω such that if $Q \cap \partial\Omega$ is covered by $2^{k(d-1)}$ cubes q_j with disjoint interior centered at $\partial\Omega$ of side length $2^{-k}s$, and $N_h^*(Q) = N > N_0$, then there exists q_{j_0} such that $N_h^*(q_{j_0}) \leq \frac{1}{2}N$.*

Proof. We note first that since $\partial\Omega$ is of class $\mathcal{C}^2$, h is harmonic in $10Q \cap \Omega$ and $\partial_\nu h = h$ on $\partial\Omega \cap \overline{10Q}$, we can use the extension-across-the-boundary trick described in Section 4, namely, consider $v(x) = e^{d(x)}h(x)$; recall that the coefficients of the second-order term in the equation satisfied by v are at least Lipschitz. This gives us access to elliptic estimates that hold up to the boundary for h. In particular, we will use the gradient estimate

$$\sup_{B(y,r)\cap\overline{\Omega}} |\nabla h| \lesssim \frac{1}{r} \sup_{B(y,2r)\cap\overline{\Omega}} |h|, \tag{44}$$

where the implied constant depends on s and Ω. Denote now by $x_Q \in \partial\Omega$ the center of the cube Q. Consider a ball B centered at x_Q such that $2Q \subset B$, and let $M = \sup_{B\cap\Omega}|h|$. By contradiction, suppose that $N_h^*(q_j) > \frac{1}{2}N$ for any j; by definition, this implies that for any j there is $x_j \in q_j \cap \Omega$ and $r_j \leq 2^{-k}\sqrt{d}s =: r_0$ such that $\mathcal{N}_h^*(x_j, r_j) > \frac{1}{2}N$. Assuming N large enough, we use (23) to get

$$\sup_{B(x_j,2r_0)\cap\Omega} |h| \leq (C2^{-k})^{N/10} \sup_{B\cap\Omega} |h| \leq Me^{-cNk}$$

if k is large enough. Using (44), we get

$$\sup_{B(x_j,r_0)\cap\Omega} |\nabla h| \lesssim \frac{1}{r_0} Me^{-cNk},$$

with the implied constant depending on s and Ω. Note that since $q_j \subset B(x_j, r_0)$ the two estimates above give bounds for the Cauchy data of h on $\partial\Omega \cap Q$. On the other hand, if B' is the ball centered at x_Q such that $4B' \subset Q$ we have $\sup_{2B'\cap\Omega}|h| \leq M$ and $\sup_{2B'\cap\Omega}|\nabla h| \lesssim M/s$. Recalling that $r_0 = 2^{-k}\sqrt{d}s$, we can then apply Proposition 17 with $\eta = 2^k e^{-cNk}$ to get

$$\sup_{B'\cap\Omega} |h| \leq C(s,\Omega) 2^{\beta k c_d} e^{-c\beta Nk} M.$$

But then

$$\mathcal{N}_h^*(x_Q, \sqrt{d}s) \geq C_d \log \frac{\sup_{B \cap \Omega} |h|}{\sup_{B' \cap \Omega} |h|} \geq C_d(c\beta Nk - c_d\beta k - C),$$

and the rightmost expression is larger than N if k and N are large enough depending on s and Ω; this is a contradiction with $N_h^*(Q) = N$. □

We are now ready to prove Proposition 28, or actually (43). The argument is an iteration at the boundary; it originates in [Logunov et al. 2021].

Proof of (43). First, we consider again $v(x) = e^{d(x)}h(x)$ and its even extension across the boundary (which we still call v). Recall from Section 2 that v satisfies an elliptic PDE with Lipschitz second-order coefficients and bounded lower-order coefficients. The results of [Hardt and Simon 1989] then apply to this situation. Let Q be any cube with $s(Q) < s_0$ small enough. By [Hardt and Simon 1989, Theorem 1.7], we have that

$$\mathcal{H}^{d-1}(\mathcal{Z}_v \cap B(x, \rho)) \leq C N_v(Q)\rho^{d-1}$$

for any ball $B(x, \rho) \subset Q$ where $v(x) = 0$ and $\rho < \rho_0(N_v(Q))$. Covering $\mathcal{Z}_h \cap Q$ with balls of such small radius and summing the estimate above over all those balls, it follows that there is a function $\tilde{A} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\mathcal{H}^{d-1}(\mathcal{Z}_h \cap Q) \leq \tilde{A}(N_h^*(Q))s(Q)^{d-1}. \tag{45}$$

Let now Q be as above a cube centered at $\partial\Omega$ of side s, with s small enough depending on Ω. Fix a large number N_0; if $N_h^*(Q) < N_0$, (45) already implies the result. Otherwise, cover $Q \cap \Omega$ with smaller cubes of side length $2^{-k}s$, where $k = k(\Omega)$ is given by Lemma 29, in the following way: first $Q \cap \partial\Omega$ is covered by cubes $q \in \mathcal{B}$ centered at $\partial\Omega$ with disjoint interior, and then the rest of $Q \cap \Omega$ is covered by cubes $q \in \mathcal{I}$ with $\text{dist}(q, \partial\Omega) > cs(q)$ for some constant $c > 0$ independent of k. Cubes in $\mathcal{B}$ will be called boundary cubes and cubes in $\mathcal{I}$ will be called inner cubes; inner cubes are allowed to overlap, while boundary cubes are not. Write $N_h^*(Q) = N$. By (38) and almost monotonicity,

$$\mathcal{H}^{d-1}\left(\mathcal{Z}_h \cap \left(\bigcup_{q \in \mathcal{I}} q\right)\right) \leq C(k)Ns^{d-1}.$$

By Lemma 29, there is a boundary cube, call it q_0, such that $N_h^*(q_0) < \frac{1}{2}N$. The other cubes in $\mathcal{B}$ will be enumerated from 1 to $2^{k(d-1)}-1$. We have that

$$\frac{\mathcal{H}^{d-1}(\mathcal{Z}_h \cap Q)}{s^{d-1}} \leq CN + \frac{\mathcal{H}^{d-1}(\mathcal{Z}_h \cap q_0)}{s^{d-1}} + \sum_{j=1}^{2^{k(d-1)}-1} \frac{\mathcal{H}^{d-1}(\mathcal{Z}_h \cap q_j)}{s^{d-1}}.$$

We define

$$A(N) = \sup \frac{\mathcal{H}^{d-1}(\mathcal{Z}_h \cap q)}{s(q)^{d-1}},$$

where the supremum is taken over all harmonic functions h in $2Q$ with $\partial_\nu h = h$ on $\partial\Omega \cap 2Q$, $N_h^*(Q) \leq N$ and all cubes $q \subset Q$. By (45), $A(N) < +\infty$. From the inequality above, we get

$$A(N) \leq C(k)N + A\left(\tfrac{1}{2}N\right)2^{-k(d-1)} + (2^{k(d-1)} - 1)A(N)2^{-k(d-1)},$$

from which

$$A(N) < C(k)N + A\left(\tfrac{1}{2}N\right).$$

(Beware that $C(k)$ changes value from line to line and depends also on Ω). Iterating the last inequality until $\tfrac{1}{2}N < N_0$, we obtain

$$A(N) < C(k)N + A(N_0) < C(k)(N+1),$$

which concludes the proof. $\square$

Theorem 2 now follows by combining (39), (40), (41) and (37). We believe that the extra $\log \lambda$ factor is not necessary and is an artificial feature of the proof; it appears in the proof of (40) and it is due to the necessity of getting to cubes of side length $\sim \lambda^{-1}$.

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