# ANALYSIS & PDEVolume 17No. 42024

MICHAEL STRUWE

PLATEAU FLOW OR THE HEAT FLOW FOR HALF-HARMONIC MAPS





# PLATEAU FLOW OR THE HEAT FLOW FOR HALF-HARMONIC MAPS

MICHAEL STRUWE

Using the interpretation of the half-Laplacian on  $S^1$  as the Dirichlet-to-Neumann operator for the Laplace equation on the ball B, we devise a classical approach to the heat flow for half-harmonic maps from  $S^1$  to a closed target manifold  $N \subset \mathbb{R}^n$ , recently studied by Wettstein, and for arbitrary finite-energy data we obtain a result fully analogous to the author's 1985 results for the harmonic map heat flow of surfaces and in similar generality. When N is a smoothly embedded, oriented closed curve  $\Gamma \subset \mathbb{R}^n$ , the half-harmonic map heat flow may be viewed as an alternative gradient flow for a variant of the Plateau problem of disc-type minimal surfaces.

# 1. Background and results

*Half-harmonic maps and their heat flow.* Let  $N \subset \mathbb{R}^n$  be a closed submanifold, that is, compact and without boundary. The concept of a half-harmonic map  $u: S^1 \to N \subset \mathbb{R}^n$  was introduced by Da Lio and Rivière [2011], who together with Martinazzi [Da Lio et al. 2015, Theorem 2.9] also made the interesting observation that the harmonic extension of a half-harmonic map yields a free boundary minimal surface supported by N, a fact which also was noticed by Millot and Sire [2015, Remark 4.28].

In his Ph.D. thesis, Wettstein [2021; 2022; 2023], studied the corresponding heat flow given by

$$d\pi_N(u)(u_t + (-\Delta)^{1/2}u) = 0 \quad \text{on } S^1 \times [0, \infty[, \tag{1-1})$$

where  $u_t = \partial_t u$  and where  $\pi_N \colon N_\rho \to N$  is the smooth nearest-neighbor projection on a  $\rho$ -neighborhood  $N_\rho$  of the given target manifold to N, and, with the help of a fine analysis of the fractional differential operators involved, he showed global existence for initial data of small energy.

Moser [2011] and Millot and Sire [2015] contributed important results to the study of half-harmonic maps by exploiting the fact that for any smooth  $u: S^1 \to \mathbb{R}^n$  we can represent the half-Laplacian classically in the form

$$(-\Delta)^{1/2}u = \partial_r U, \tag{1-2}$$

where  $U: B \to \mathbb{R}^n$  is the harmonic extension of *u* to the unit disc B.<sup>1</sup> Here, using the identity (1-2), we are able to remove the smallness assumption in Wettstein's work and show the existence of a "global"

I thank Amélie Loher and the anonymous referee for careful reading of the manuscript and useful suggestions.

MSC2020: primary 35K55, 35K65, 53E99; secondary 53A10.

Keywords: half-harmonic maps, harmonic map heat flow, Plateau problem.

<sup>&</sup>lt;sup>1</sup>The classical formula (1-2) is a special case of a much more general result due to Caffarelli and Silvestre [2007], who pointed out that many nonlocal problems involving fractional powers of the Laplacian can be related to a local, possibly degenerate, elliptic equation via a suitable extension of the solution to a half-space.

<sup>© 2024</sup> MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

weak solution to the heat flow (1-1) for data of arbitrarily large (but finite) energy, which is defined for all times and smooth away from finitely many "blow-up points" where energy concentrates, and whose energy is nonincreasing. The solution is unique in this class in exact analogy with the classical result by the author [Struwe 1985] on the harmonic map heat flow for maps from a closed surface to a closed target manifold  $N \subset \mathbb{R}^n$ ; see Theorem 1.2 below.

In order to describe our work in more detail, let

$$H^{1/2}(S^1; N) = \{ u \in H^{1/2}(S^1; \mathbb{R}^n) : u(z) \in N \text{ for almost every } z \in S^1 \}.$$

Interpreting  $S^1 = \partial B$ , where  $B = B_1(0; \mathbb{R}^2)$ , and tacitly identifying a map  $u \in H^{1/2}(S^1; N)$  with its harmonic extension  $U \in H^1(B; \mathbb{R}^n)$ , for a given function  $u_0 \in H^{1/2}(S^1; N)$  we then seek to find a family of harmonic functions  $u(t) \in H^1(B; \mathbb{R}^n)$  with traces  $u(t) \in H^{1/2}(S^1; N)$  for t > 0, solving the equation

$$d\pi_N(u)(u_t + \partial_r u) = u_t + d\pi_N(u)\partial_r u = 0 \quad \text{on } S^1 \times [0, \infty[, \tag{1-3})$$

with initial data

$$u|_{t=0} = u_0 \in H^{1/2}(S^1; N).$$
(1-4)

Energy. The half-harmonic heat flow may be regarded as the heat flow for the half-energy

$$E_{1/2}(u) = \frac{1}{2} \int_{S^1} |(-\Delta)^{1/4} u|^2 \, d\phi$$

of a map  $u \in H^{1/2}(S^1; N)$ . Note that the half-energy of u equals the standard Dirichlet energy

$$E(u) = \frac{1}{2} \int_{B} |\nabla u|^2 \, dz$$

of its harmonic extension  $u \in H^1(B; \mathbb{R}^n)$ . Indeed, integrating by parts we have

$$\int_{B} |\nabla u|^{2} dz = \int_{S^{1}} u \partial_{r} u \, d\phi = \int_{S^{1}} u (-\Delta)^{1/2} u \, d\phi = \int_{S^{1}} |(-\Delta)^{1/4} u|^{2} \, d\phi, \tag{1-5}$$

where we use the identity (1-2) and where the last identity easily follows from the representation of the operators  $(-\Delta)^{1/2}$  and  $(-\Delta)^{1/4}$  in Fourier space with symbols  $|\xi|$  and  $\sqrt{|\xi|}$ , respectively, and Parceval's identity.<sup>2</sup> Therefore, in the following for convenience we may always work with the classically defined Dirichlet energy. Moreover, we may interpret the half-harmonic heat flow as the heat flow for the Dirichlet energy in the class of harmonic functions with trace in  $H^{1/2}(S^1; N)$ ; see Section 2 below for details.

**Results.** Identifying  $\mathbb{R}^2$  with  $\mathbb{C}$ , we denote by *M* the three dimensional Möbius group of conformal transformations of the unit disc, given by

$$M = \left\{ \Phi(z) = e^{i\theta} \frac{z+a}{1+\bar{a}z} \in C^{\infty}(\bar{B}; \bar{B}) : |a| < 1, \ \theta \in \mathbb{R} \right\}.$$

Observe that the Dirichlet energy is invariant under conformal transformations, and we have  $E(u) = E(u \circ \Phi)$  for any  $u \in H^1(B; \mathbb{R}^n)$  and any  $\Phi \in M$ .

<sup>&</sup>lt;sup>2</sup>Conversely, via Fourier expansion we also can prove (1-5) directly. Computing the first variations of *E* and  $E_{1/2}$ , respectively, we then obtain (1-2).

For smooth data we then have the following result.

**Theorem 1.1.** Let  $N \subset \mathbb{R}^n$  be a closed, smooth submanifold of  $\mathbb{R}^n$ , and suppose that the normal bundle  $T^{\perp}N$  is parallelizable. Then the following holds:

(i) For any smooth  $u_0 \in H^{1/2}(S^1; N)$  there exists a time  $T_0 \leq \infty$  and a unique smooth solution u = u(t) of (1-3), and hence of (1-1), with data (1-4) for  $0 < t < T_0$ .

(ii) If  $T_0 < \infty$ , we have concentration in the sense that, for some  $\delta > 0$  and any R > 0,

$$\sup_{\substack{z_0\in B\\0$$

and for suitable  $t_k \uparrow T_0$  there exist finitely many points  $z_k^{(1)}, \ldots, z_k^{(i_0)}$  and conformal maps  $\Phi_k^{(i)} \in M$  with  $z_k^{(i)} \to z^{(i)} \in \overline{B}$  and  $\Phi_k^{(i)} \to \Phi_{\infty}^{(i)} \equiv z^{(i)}$  weakly in  $H^1(B)$  such that  $u(t_k) \circ \Phi_k^{(i)} \to \overline{u}^{(i)}$  weakly in  $H^1(B)$  as  $k \to \infty$ , where  $\overline{u}^{(i)}$  is nonconstant and conformal and satisfies

$$d\pi_N(\bar{u}^{(i)})\partial_r \bar{u}^{(i)} = 0, \quad 1 \le i \le i_0.$$
(1-6)

Moreover, there exists  $\delta = \delta(N) > 0$  such that  $E(\bar{u}^{(i)}) \ge \delta$ , and  $i_0 \le E(u_0)/\delta$ . Finally,  $u(t_k)$  smoothly converges to a limit  $u_1 \in H^{1/2}(S^1; N)$  on  $\overline{B} \setminus \{z^{(1)}, \ldots, z^{(i_0)}\}$ .

(iii) If  $T_0 = \infty$ , then, as  $t \to \infty$  suitably, u(t) smoothly converges to a half-harmonic limit map away from at most finitely many concentration points where nonconstant half-harmonic maps "bubble off" as in (ii).

By the Da Lio–Rivière interpretation of (1-6), the "bubbles"  $\bar{u}^{(i)}$  as well as the limit  $u_{\infty}$  of the flow conformally parametrize minimal surfaces with free boundary on N, meeting N orthogonally along their free boundaries.

The hypothesis regarding the target manifold N in particular is fulfilled if N is a closed, orientable hypersurface of codimension 1 in  $\mathbb{R}^n$ , or if N is a smoothly embedded, closed curve  $\Gamma \subset \mathbb{R}^3$ .

It would be interesting to find examples of initial data for which the flow blows up in finite time, as in the work of Chang, Ding, and Ye [Chang et al. 1992] on the harmonic map heat flow.

For data in  $H^{1/2}(S^1; N)$  the following global existence result holds, which is our main result.

**Theorem 1.2.** For  $N \subset \mathbb{R}^n$  as in Theorem 1.1 the following holds:

(i) For any  $u_0 \in H^{1/2}(S^1; N)$  there exists a unique global weak solution of (1-3) with data (1-4) as in Definition 6.3, whose energy is nonincreasing and which is smooth for positive time away from finitely many points in space-time where nontrivial half-harmonic maps "bubble off" in the sense of Theorem 1.1(ii).

(ii) As  $t \to \infty$  suitably, u(t) smoothly converges to a half-harmonic limit map away from at most finitely many concentration points where nonconstant half-harmonic maps "bubble off" as in Theorem 1.1(iii).

Note that uniqueness is only asserted within the class of partially regular weak solutions with nonincreasing energy, as in the case of the harmonic map heat flow. It would be interesting to find out if the latter condition suffices, as in the work of Freire [1995], and, conversely, to explore the possibility of "backward bubbling" in (1-3), as in the examples of Topping [2002] for the latter flow.

*Key features of the proof and related flow equations.* In our approach, in a similar vein as [Lenzmann and Schikorra 2020], we uncover and exploit surprising regularity properties of the normal component  $d\pi_N^{\perp}(u)\partial_r u$  for the harmonic extension of u, likely related to the fractional commutator estimates for the normal projection in the work of Da Lio and Rivière [2011] or the regularity estimates of Da Lio and Pigati [2020], Mazowiecka and Schikorra [2018], and others.

The use of the Dirichlet-to-Neumann map for the harmonic extension  $u: B \to \mathbb{R}^n$  of u instead of the half-Laplacian, and the simple identity (3-2) as well as (3-5) allow us to perform the analysis using only local, classically defined operators, avoiding fractional calculus almost entirely.

Note that (1-3) is similar to the equation governing the (scalar) evolution problem for conformal metrics  $e^{2u}g_{\mathbb{R}^2}$  of prescribed geodesic boundary curvature and vanishing Gauss curvature on the unit disc *B*, studied for instance by Brendle [2002] or Gehrig [2020]. In contrast to the latter flows, due to the presence of the projection operator mapping  $u_r$  to its tangent component, the flow (1-3) at first sight appears to be degenerate. However, surprisingly, within our framework we are able to obtain similar smoothing properties as in the case of the harmonic map heat flow of surfaces.

A different heat flow associated with half-harmonic maps, using the half-heat operator  $(\partial_t - \Delta)^{1/2}$  instead of (1-1), was suggested by Hyder et al. [2022], and they obtained global existence of partially regular, but possibly nonunique, weak solutions for their flow, with a possibly large singular set of measure zero.

Applications to the Plateau problem. In the case when N is a smoothly embedded, oriented closed curve  $\Gamma \subset \mathbb{R}^3$ , the half-harmonic heat flow (1-3) may furnish an alternative gradient flow for the Plateau problem of minimal surfaces of the type of the disc, which has a long and famous tradition in geometric analysis.

Posed in the 1890's, Plateau's problem was finally solved independently by Douglas [1931] and Radó [1930]. In order to analyze the set of *all* minimal surfaces solving the Plateau problem, including saddle points of the Dirichlet integral, thereby building on Douglas' ideas, Morse and Tompkins [1939] proposed a critical point theory for Plateau's problem in the sense of [Morse 1937], attempting to characterize nonminimizing solutions as "homotopy-critical" points of Dirichlet's integral. However, Tromba [1984; 1985] pointed out that it was not even clear that all smooth, nondegenerate minimal surfaces would be "homotopy-critical" in the sense of [Morse and Tompkins 1939]. To overcome this problem, Tromba developed a version of degree theory that could be applied in this case and which yielded at least a proof of the "last" Morse inequality, which is an identity for the total degree.

Finally, this author [Struwe 1984] recast the Plateau problem as a variational problem on a closed convex set and was able to develop a version of the Palais–Smale type critical point theory for the problem within this frame-work, which allowed him to obtain all Morse inequalities in a rigorous fashion; see [Struwe 1988] and [Imbusch and Struwe 1999] for further details. In [Struwe 1986] and [Jost and Struwe 1990], the approach was extended to the case of multiple boundaries and/or higher genus.

A key element of critical point theory for a variational problem is the construction of a pseudogradient flow for the problem at hand. In [Struwe 1984] this was achieved in an ad-hoc way. However, starting with the work of Eells and Sampson [1964] on the harmonic map heat flow, it is now an established approach in geometric analysis to study the (negative)  $(L^2$ -)gradient flow related to a variational problem, similar

to the standard heat equation. For Plateau's problem, such a flow was obtained by Chang and Liu [2005] within the frame-work laid out by Struwe [1984] in the form of a parabolic variational inequality, for which Chang and Liu obtained a solution of class  $H^2$  by means of a time-discrete minimization scheme. Rupflin [2017] and Rupflin and Schrecker [2018] studied the analogous parabolic variational inequality in the case of an annulus, which again had previously been studied in [Struwe 1986] by means of an ad-hoc pseudogradient flow.

In view of the much better regularity properties of the flow equation (1-3) it would be tempting to regard this as the correct definition of the canonical gradient flow for the Plateau problem, but an important issue still needs to be addressed.

*Monotonicity.* Recall that in the classical Plateau problem u(t) is required to induce a (weakly) monotone parametrization of  $\Gamma$  for each t > 0. Even though it may seem likely that — at least for curves  $\Gamma$  on the boundary of a convex body in  $\mathbb{R}^3$  — this Plateau boundary condition will be preserved along the flow (1-3) whenever it is satisfied initially, at this moment even for a strictly convex planar curve  $\Gamma \subset \mathbb{R}^2$  it is not clear whether this actually happens. However, the results that we obtain also seem to be of interest if we drop the Plateau condition. In particular, our results motivate the study of smooth minimal surfaces with continuous trace covering only a part of the given boundary curve  $\Gamma$ ; dropping the monotonicity condition also brings the parametric approach to the Plateau problem closer to the approach via geometric measure theory or level sets.

**Plateau flow.** It should be straightforward to extend our results to the case when the disc *B* is replaced by a surface  $\Sigma$  of higher genus with boundary  $\partial \Sigma \cong S^1$ , if for given initial data  $u_0 \in H^{1/2}(S^1; N)$  we consider a family u = u(t) in  $H^{1/2}(S^1; N)$  solving (1-3), that is,

$$u_t + d\pi_N(u)\partial_\nu u = 0$$

instead of (1-1), where for each time we harmonically extend u(t) to  $\Sigma$  and denote by  $\partial_{\nu}u$  the outward normal derivative of u along  $\partial \Sigma$ , as was proposed and analyzed by Da Lio and Pigati [2020] in the time-independent case. Similarly, one might study the flow (1-3) on a domain  $\Sigma$  with multiple boundaries. Of course, in order for the flow to converge to a minimal surface in the case of higher genus or higher connectivity it will be necessary to couple the flow (1-3) with a corresponding evolution equation for the conformal structure on  $\Sigma$ , as in the work of Rupflin and Topping [2019] on minimal immersions. Note that on a general domain  $\Sigma$  the flow equations (1-1) and (1-3) no longer agree. In order to clearly distinguish the flow equation (1-3) from the equation (1-1) defining the half-harmonic map heat flow, we therefore propose to say that (1-3) defines the "Plateau flow".

*Outline.* After a brief discussion of energy estimates in Section 2, in Section 3 we present the analytic core of the argument for higher regularity in Section 4 and for the blow-up analysis, later presented in Section 8. These tools are also instrumental in proving uniqueness of partially regular weak solutions in Section 7. The  $L^2$ -bounds for higher and higher derivatives which we establish in Section 4, assuming that energy does not concentrate, may be of particular interest. These bounds either concern estimates for  $\nabla \partial_{\phi}^{k} u$  on *B* or on  $\partial B$ , and we view the latter bounds as stronger by an order of  $\frac{1}{2}$ . These bounds may

be iterated interlaced, as we later do in Section 6, to prove uniform smooth estimates, locally in time, for smooth flows with smooth initial data converging in  $H^{1/2}(N; S^1)$ . Since the latter data are dense in  $H^{1/2}(N; S^1)$ , we thus not only obtain existence of weak solutions for arbitrary data  $u_0 \in H^{1/2}(N; S^1)$ but also can show their smoothness for positive time and hence are able to derive Theorem 1.2 from Theorem 1.1. A peculiar feature is that one set of regularity estimates can only be obtained globally, that is on all of *B*, whereas the other set of estimates may be localized using cut-off functions. Similar estimates for a regularized version of (1-3) are employed in Section 5 to prove local existence of smooth solutions of (1-3) for smooth data (1-4). Finally, in Section 9 the large-time behavior of smooth solutions to (1-3) is discussed, finishing the proof of Theorem 1.1.

**Notation.** The letter C is used throughout to denote a generic constant, possibly depending on the "target" N and the initial energy  $E(u_0)$ .

Moreover, since  $T^{\perp}N$  by assumption is parallelizable and compact, there exists  $\rho > 0$  such that the representation

$$T: N \times B_{\rho}(0; \mathbb{R}^m) \ni (p, y) \rightarrow p + \sum_{i=1}^m y^i \nu_i(p) \in N_{\rho}$$

of the tubular neighborhood  $N_{\rho} = \bigcup_{p \in N} B_{\rho}(p)$  of N is a diffeomorphism, where  $v_1, \ldots, v_m$  is a suitable smooth orthonormal frame along N and where we let  $y = (y^1, \ldots, y^m) \in \mathbb{R}^m$ . For  $q \in N_{\rho}$ , then  $T^{-1}(q) = (p, h)$  with  $p = \pi_N(q)$  defines a (vector-valued) signed distance function  $h = h(q) = (h^1(q), \ldots, h^m(q))$ with  $h^i(q) = v_i(p) \cdot (q - \pi_N(q))$  for each  $1 \le i \le m$ . Fixing a smooth function  $\eta : \mathbb{R} \to \mathbb{R}$  such that  $\eta(s) = s$  for  $|s| < \frac{1}{2}\rho$ , and with  $\eta(s) = 0$  for  $|s| \ge \frac{3}{4}\rho$ , we then let

$$\operatorname{dist}_N(q) = (\operatorname{dist}_N^1(q), \dots, \operatorname{dist}_N^m(q)),$$

with

$$\operatorname{dist}_{N}^{i}(q) = \eta(h^{i}(q)) \text{ for } q \in N_{\rho}, \text{ otherwise } \operatorname{dist}_{N}^{i}(q) = 0, \quad 1 \le i \le m.$$

Then for any smooth  $u \in H^{1/2}(S^1; N)$  with harmonic extension  $u \in H^1(B; \mathbb{R}^n)$  we have

$$\sum_{i=1}^{m} v_i(u) \partial_r \operatorname{dist}_N^i(u) = \sum_{i=1}^{m} v_i(u) v_i(u) \cdot u_r = d\pi_N^{\perp}(u) u_r \quad \text{on } \partial B = S^1,$$
(1-7)

where for each  $p \in N$  we denote by  $d\pi_N^{\perp}(p) = 1 - d\pi_N(p) \colon \mathbb{R}^n \to T_p^{\perp}N$  the orthogonal projection. In the sequel, we abbreviate

$$\sum_{i=1}^{m} v_i(u)v_i(u) \cdot u_r =: v(u)v(u) \cdot u_r = v(u)\partial_r \operatorname{dist}_N(u);$$

moreover, we extend the vector fields  $v_i$  to the whole ambient space by letting  $v_i(q) = \nabla \operatorname{dist}_N^i(q)$  for  $q \in \mathbb{R}^n$ ,  $1 \le i \le m$ .

Finally, we fix a smooth cut-off function  $\varphi \in C_c^{\infty}(B)$  satisfying  $0 \le \varphi \le 1$  with  $\varphi \equiv 1$  on  $B_{1/2}(0)$ , and for any  $z_0 \in B$  and any 0 < R < 1 we scale

$$\varphi_{z_0,R}(z) = \varphi\left(\frac{(z-z_0)}{R}\right) \in C_c^{\infty}(B_R(z_0)).$$

# 2. Energy inequality and first consequences

The half-harmonic heat flow may be regarded as the heat flow for the Dirichlet energy in the class  $H^{1/2}(S^1; N)$ . Indeed, let u(t) be a smooth solution of (1-3) and (1-4) for  $0 < t < T_0$ . Then we have the following result.

**Lemma 2.1.** *For any*  $0 \le S < T < T_0$ ,

$$E(u(T)) + \int_{S}^{T} \int_{\partial B} |u_t|^2 \, d\phi \, dt \le E(u(S)).$$

Proof. Integrating by parts and using (1-3) we compute

$$\frac{d}{dt}E(u) = \int_{B} \nabla u \nabla u_t \, dz = \int_{\partial B} u_r \cdot u_t \, d\phi = -\int_{\partial B} |d\pi_N(u)u_r|^2 \, d\phi = -\int_{\partial B} |u_t|^2 \, d\phi$$

for any  $0 < t < T_0$ . The claim follows by integration.

Moreover, there holds a localized version of this energy inequality.

**Lemma 2.2.** There exists a constant C > 0 such that, for any  $z_0 \in B$ , any 0 < R < 1, any  $\varepsilon > 0$ , and any  $0 < t_0 < t_1 \le t_0 + \varepsilon R < T_0$ ,

$$\int_{B} |\nabla u(t_1)|^2 \varphi_{z_0,R}^2 \, dz + 4 \int_{t_0}^{t_1} \int_{\partial B} |u_t|^2 \varphi_{z_0,R}^2 \, d\phi \, dt \le 4 \int_{B} |\nabla u(t_0)|^2 \varphi_{z_0,R}^2 \, dz + C \varepsilon E(u_0).$$

*Proof.* Writing  $\varphi = \varphi_{z_0,R}$  for brevity, integrating by parts, and using Young's inequality, similar to the proof of Lemma 2.1 for any  $0 < t < T_0$  we have

$$\begin{aligned} \frac{d}{dt} \left( \frac{1}{2} \int_{B} |\nabla u|^{2} \varphi^{2} dz \right) &= \int_{\partial B} u_{t} \cdot u_{r} \varphi^{2} d\phi - \int_{B} u_{t} \operatorname{div}(\nabla u \varphi^{2}) dz \\ &= -\int_{\partial B} |d\pi_{N}(u)u_{r}|^{2} \varphi^{2} d\phi - 2 \int_{B} u_{t} \nabla u \varphi \nabla \varphi dz \\ &\leq -\int_{\partial B} |u_{t}|^{2} \varphi^{2} d\phi + (8\varepsilon R)^{-1} \int_{B} |\nabla u|^{2} \varphi^{2} dz + 8\varepsilon R \int_{B} |u_{t}|^{2} |\nabla \varphi|^{2} dz. \end{aligned}$$
(2-1)

Letting

$$A = \sup_{t_0 < t < t_1} \left( \frac{1}{2} \int_B |\nabla u(t)|^2 \varphi^2 \, dz \right),$$

then upon integration we find

$$A + \int_{t_0}^{t_1} \int_{\partial B} |u_t|^2 \varphi^2 \, d\phi \, dt \le \int_{B} |\nabla u(t_0)|^2 \varphi^2 \, dz + \frac{t_1 - t_0}{2\varepsilon R} A + C\varepsilon R^{-1} \int_{t_0}^{t_1} \int_{B_R(z_0) \cap B} |u_t|^2 \, dz \, dt.$$

But with u = u(t), then also  $u_t = u_t(t)$  is harmonic for each t. Expanding

$$u_t(re^{i\phi}) = \sum_{k\geq 0} a_k r^k e^{ik\phi}$$

in a Fourier series, we see that the map

$$r \mapsto \int_{\partial B_r(0)} |u_t|^2 ds = 2\pi \sum_{k \ge 0} |a_k|^2 r^{2k+1},$$

with ds denoting the element of length along  $\partial B_r(0)$ , is nondecreasing. Thus, for any  $z_0 \in B$ , any 0 < R < 1, and any  $t_0 < t < t_1$ ,

$$\int_{B_R(z_0)\cap B} |u_t|^2 dz \le 2R \int_{\partial B} |u_t|^2 d\phi, \qquad (2-2)$$

and we may use Lemma 2.1 to conclude.

# 3. A regularity estimate

To illustrate the key ideas that later will allow us to prove higher regularity and analyze blow-up of solutions of (1-3), we first consider smooth solutions  $u \in H^{1/2}(S^1; N)$  of the equation

$$d\pi_N(u)\partial_r u + f = 0 \quad \text{on } \partial B = S^1, \tag{3-1}$$

where  $f \in L^2(S^1)$ . We prove the following a-priori estimate, where we use classical estimates similar to [Wettstein 2022, Lemma 3.4], which in turn is a fractional version of a result by Rivière [1993, Chapter 4, pp. 96-104]. Note that with the truncated signed distance function  $\operatorname{dist}_N : \mathbb{R}^n \to \mathbb{R}^m$  we have the orthogonal decomposition

$$\partial_r u = d\pi_N(u)\partial_r u + d\pi_N^{\perp}(u)\partial_r u = d\pi_N(u)\partial_r u + \nu(u)\partial_r(\operatorname{dist}_N(u))$$
(3-2)

on  $\partial B = S^1$ , where we recall that we use the shorthand notation

$$v(u)\partial_r(\operatorname{dist}_N(u)) = \sum_{i=1}^m v_i(u)\partial_r(\operatorname{dist}_N^i(u)) = \sum_{i=1}^m v_i(u)v_i(u)\cdot\partial_r u$$

and extend  $v_i(p) = \nabla \operatorname{dist}^i_N(p), \ p \in \mathbb{R}^n$ .

**Proposition 3.1.** There exist constants C and  $\delta_0 = \delta_0(N) > 0$  such that, for any smooth solution  $u \in H^{1/2}(S^1; N)$  of (3-1) with  $E(u) \le \delta^2 < \delta_0^2$ ,

$$\int_{S^1} |\partial_{\phi} u|^2 \, d\phi \le C \, \|f\|_{L^2(S^1)}^2. \tag{3-3}$$

*Proof.* Multiplying (3-2) with  $\partial_r u$ , we find the Pythagorean identity

$$|\partial_r u|^2 = |d\pi_N(u)\partial_r u|^2 + |d\pi_N^{\perp}(u)\partial_r u|^2 = |d\pi_N(u)\partial_r u|^2 + |\partial_r(\operatorname{dist}_N(u))|^2.$$
(3-4)

Note that  $\operatorname{dist}_N(u) \in H_0^1(B)$ ; moreover, for each  $1 \le i \le m$  we have  $\nabla(\operatorname{dist}_N^i(u)) = \nu_i(u) \cdot \nabla u$ , and there holds the equation

$$\Delta(\operatorname{dist}^{i}_{N}(u)) = \operatorname{div}(v_{i}(u) \cdot \nabla u) = \nabla u \cdot dv_{i}(u) \nabla u \quad \text{in } B.$$
(3-5)

1404

The divergence theorem now gives

$$\begin{aligned} \|\partial_r(\operatorname{dist}_N(u))\|_{L^2(S^1)}^2 &= (\nabla(\operatorname{dist}_N(u)), \nabla(\operatorname{dist}_N(u))_r)_{L^2(B)} + (\Delta(\operatorname{dist}_N(u)), (\operatorname{dist}_N(u))_r)_{L^2(B)} \\ &\leq C \|\nabla u\|_{L^2(B)} \|\nabla^2(\operatorname{dist}_N(u))\|_{L^2(B)} \leq C\delta \|\nabla^2(\operatorname{dist}_N(u))\|_{L^2(B)}, \end{aligned}$$

where the basic  $L^2$ -theory for the Laplace equation (3-5) yields the bound

$$\|\nabla^{2}(\operatorname{dist}_{N}(u))\|_{L^{2}(B)} \leq C \|\Delta(\operatorname{dist}_{N}^{i}(u))\|_{L^{2}(B)} \leq C \|\nabla u\|_{L^{4}(B)}^{2}$$

With Sobolev's embedding  $H^{1/2}(B) \hookrightarrow L^4(B)$  we then conclude

$$\|\partial_r(\operatorname{dist}_N(u))\|_{L^2(S^1)}^2 \le C\delta \|\nabla u\|_{H^{1/2}(B)}^2$$

Thus from (3-4) and (3-1) we have

$$\|\partial_{r}u\|_{L^{2}(S^{1})}^{2} \leq \|f\|_{L^{2}(S^{1})}^{2} + \|\partial_{r}(\operatorname{dist}_{N}(u))\|_{L^{2}(S^{1})}^{2} \leq \|f\|_{L^{2}(S^{1})}^{2} + C\delta\|\nabla u\|_{H^{1/2}(B)}^{2}.$$
(3-6)

But Fourier expansion of the harmonic function u gives

$$\|\partial_{\phi}u\|_{L^{2}(S^{1})}^{2} = \|\partial_{r}u\|_{L^{2}(S^{1})}^{2} = \frac{1}{2}\|\nabla u\|_{L^{2}(S^{1})}^{2}$$
(3-7)

as well as the bound

$$\|\nabla u\|_{H^{1/2}(B)}^2 \le C \|\nabla u\|_{L^2(S^1)}^2,$$

and from (3-6) we obtain

$$\|\partial_r u\|_{L^2(S^1)}^2 \le \|f\|_{L^2(S^1)}^2 + C\delta \|\nabla u\|_{H^{1/2}(B)}^2 \le \|f\|_{L^2(S^1)}^2 + C\delta \|\partial_r u\|_{L^2(S^1)}^2,$$

which for sufficiently small  $\delta > 0$  by (3-7) yields the claim.

In particular, from Proposition 3.1 we obtain a positive energy threshold for nonconstant solutions of (1-6).

**Corollary 3.2.** Suppose  $u \in H^{1/2}(S^1; N)$  smoothly solves (1-6). Then, either u is constant or  $E(u) \ge \delta_0^2$ , with  $\delta_0 = \delta_0(N) > 0$  given by Proposition 3.1.

Combining the ideas in the proof of the previous result with ideas from the classical proof of the Courant–Lebesgue lemma in minimal surface theory, we can obtain the following local version of Proposition 3.1.

**Proposition 3.3.** There exists a constant  $\delta > 0$  with the following property. Given any smooth solution  $u \in H^{1/2}(S^1; N)$  of (3-1) with harmonic extension  $u \in H^1(B)$ , any  $z_0 \in \partial B$ , and any  $0 < R \le \frac{1}{2}$  such that

$$\int_{B_R(z_0)\cap B} |\nabla u|^2 \, dz < \delta^2,\tag{3-8}$$

with a constant C = C(R) > 0 there holds

$$\int_{B_{R^2}(z_0)\cap S^1} |\partial_{\phi}u|^2 d\phi \le C \|f\|_{L^2(B_R(z_0)\cap S^1)}^2 + CE(u)$$

*Proof.* Fix any  $z_0 \in \partial B$  and  $0 < R \le \frac{1}{2}$  such that (3-8) holds. For suitable  $\rho \in [R^2, R]$ , with *s* denoting arc-length along the curve  $C_{\rho} = \{z_0 + \rho e^{i\theta} \in B : \theta \in \mathbb{R}\}$  with end-points  $z_j = z_0 + \rho e^{i\theta_j} = e^{i\phi_j} \in \partial B$ , j = 1, 2, we have

$$\rho \int_{C_{\rho}} |\nabla u|^2 \, ds \leq 2 \inf_{R^2 < \rho' < R} \left( \rho' \int_{C_{\rho'}} |\nabla u|^2 \, ds \right)$$

We can bound the latter infimum by the average over  $\rho \in [R^2, R]$  with respect to the measure with density  $\rho^{-1}$  to obtain the bound

$$\rho \int_{C_{\rho}} |\nabla u|^2 \, ds \le \frac{2 \int_{R^2}^{R} \int_{C_{\rho}} |\nabla u|^2 \, ds \, d\rho}{\int_{R^2}^{R} \rho^{-1} \, d\rho} \le \frac{2 \int_{B} |\nabla u|^2 \, dz}{|\log(R)|} = \frac{4E(u)}{|\log(R)|}.$$
(3-9)

Let  $\Phi_0: B \to B$  be the conformal map fixing the circular arc  $C_\rho$  and mapping the point  $z_0$  to the point  $-z_0$ , obtained as the composition  $\Phi_0 = \pi_0^{-1} \circ \Psi_0 \circ \pi_0$  of a conformal diffeomorphism  $\pi_0: B \to \mathbb{R}^2_+$  mapping the points  $z_0$  and  $-z_0$  to the origin and infinity, respectively, and the reflection  $\Psi_0: \mathbb{R}^2_+ \to \mathbb{R}^2_+$  of the upper half-plane  $\mathbb{R}^2_+$  in the half-circle  $\pi_0(C_\rho)$ . Replacing *u* by the map  $u \circ \Phi_0$  in  $B \setminus B_\rho(z_0)$  we obtain a piecewise smooth map  $v_1: B \to \mathbb{R}^n$  which is harmonic on  $B \setminus C_\rho$  and continuous on all of *B*. Let  $v_0 \in H^1(B)$  be harmonic with  $w := v_1 - v_0 \in H^1_0(B)$ . Note that by the variational characterization of harmonic functions and conformal invariance of the Dirichlet integral we have

$$E(v_0) \le E(v_1) \le \int_{B_R(z_0) \cap B} |\nabla u|^2 \, dz \le \delta^2.$$
(3-10)

Moreover, for any smooth  $\varphi \in H_0^1(B)$ , by (3-9) we can estimate

$$\left| \int_{B} \nabla w \nabla \varphi \, dz \right| = \left| \int_{B} \nabla v_1 \nabla \varphi \, dz \right| = \left| \int_{C_{\rho}} [\partial_{\nu} v_1] \varphi \, ds \right| \le \left( \int_{C_{\rho}} |\nabla u|^2 \, ds \right)^{1/2} \left( \int_{C_{\rho}} |\varphi|^2 \, ds \right)^{1/2} \le C(R) E(u)^{1/2} \|\varphi\|_{H^{1/2}(B)},$$

where  $[\partial_{\nu}v_1]$  denotes the difference of the outer and inner normal derivatives of  $v_1$  along  $C_{\rho}$ . Thus we have  $\Delta w \in H^{-1/2}(B)$ , and the basic  $L^2$ -theory for the Laplace equation gives  $w \in H^{3/2} \cap H_0^1(B)$  with

$$\|w\|_{H^{3/2}(B)} \le \sup_{\substack{\varphi \in H_0^1(B) \\ \|\varphi\|_{H^{1/2}(B)} \le 1}} \left( \int_B \nabla w \nabla \varphi \, dz \right) \le C(R) E(u)^{1/2},$$

and then also

$$\|\partial_r w\|_{L^2(S^1)}^2 \le C \|w\|_{H^{3/2}(B)}^2 \le C(R)E(u).$$
(3-11)

In view of (3-10), for sufficiently small  $\delta > 0$ , from Proposition 3.1 we obtain the estimate

$$\|\partial_{\phi} v_0\|_{L^2(S^1)}^2 \le C \|d\pi_N(v_0)\partial_r v_0\|_{L^2(S^1)}^2.$$
(3-12)

Observe that since  $v_0 = v_1$  on  $\partial B = S^1$  and since we also have  $v_1 = u$  on  $B \cap B_\rho(z_0)$  and  $v_1 = u \circ \Phi_0$  on  $B \setminus B_\rho(z_0)$ , respectively, we can bound

$$\|d\pi_N(v_0)\partial_r v_0\|_{L^2(S^1)}^2 = \|d\pi_N(v_1)\partial_r v_0\|_{L^2(S^1)}^2 \le 2\|d\pi_N(v_1)\partial_r v_1\|_{L^2(S^1)}^2 + 2\|\partial_r w\|_{L^2(S^1)}^2$$

and

$$\|d\pi_N(v_1)\partial_r v_1\|_{L^2(S^1)}^2 \le C(R) \|d\pi_N(u)\partial_r u\|_{L^2(S^1 \cap B_{\rho}(z_0))}^2$$

Thus from (3-11) we obtain

$$\begin{aligned} \|d\pi_N(v_0)\partial_r v_0\|_{L^2(S^1)}^2 &\leq C(R) \|d\pi_N(u)\partial_r u\|_{L^2(S^1 \cap B_\rho(z_0))}^2 + C \|\partial_r w\|_{L^2(S^1)}^2 \\ &\leq C(R) \|f\|_{L^2(S^1 \cap B_\rho(z_0))}^2 + C(R)E(u), \end{aligned}$$

and from (3-12) there results the bound

$$\begin{aligned} \|\partial_{\phi}u\|_{L^{2}(S^{1}\cap B_{\rho}(z_{0}))}^{2} &= \|\partial_{\phi}v_{0}\|_{L^{2}(S^{1}\cap B_{\rho}(z_{0}))}^{2} \leq \|\partial_{\phi}v_{0}\|_{L^{2}(S^{1})}^{2} \\ &\leq C\|d\pi_{N}(v_{0})\partial_{r}v_{0}\|_{L^{2}(S^{1})}^{2} \leq C(R)\|f\|_{L^{2}(S^{1}\cap B_{R}(z_{0})))}^{2} + C(R)E(u), \end{aligned}$$

$$\end{aligned}$$

as claimed.

The local estimate Proposition 3.3 also implies the following global bound.

**Proposition 3.4.** There exists a constant  $\delta > 0$  with the following property. Given any smooth solution  $u \in H^{1/2}(S^1; N)$  of (3-1) and any  $0 < R \le \frac{1}{2}$  with

$$\sup_{z_0 \in B} \int_{B_R(z_0) \cap B} |\nabla u|^2 dz < \delta^2, \tag{3-13}$$

there holds

$$\int_{S^1} |\partial_{\phi} u|^2 d\phi \leq C(R) \|f\|_{L^2(S^1)}^2 + C(R)E(u).$$

*Proof.* Covering  $\partial B$  with balls  $B_{R^2}(z_i)$ ,  $1 \le i \le i_0$ , from Proposition 3.3 we obtain the claim. **Remark 3.5.** The proofs of the above propositions only require  $u \in H^1(S^1; N)$  with harmonic extension  $u \in H^{3/2}(B)$ .

#### 4. Higher regularity

Again let u(t) be a smooth solution of the half-harmonic heat flow (1-3) for  $0 < t < T_0$  with smooth initial data (1-4). We show that as long as the flow does not concentrate energy in the sense of Theorem 1.1(ii) the solution remains smooth and can be a-priori bounded in any  $H^k$ -norm in terms of the data.

*H*<sup>2</sup>-*bound*. In a first step we show an  $L^2$ -bound in space-time for the second derivatives of our solution to the flow (1-3). Recall that by harmonicity, writing u = u(t),  $\partial_{\phi} u = u_{\phi}$ , and so on, for any  $0 < t < T_0$  we have (3-7), that is,

$$\int_{\partial B} |u_{\phi}|^2 d\phi = \int_{\partial B} |u_r|^2 d\phi,$$

as Fourier expansion shows, with similar identities for partial derivatives of u of higher order. Indeed, writing

$$\Delta u = \frac{1}{r} (r u_r)_r + \frac{1}{r^2} u_{\phi\phi} \tag{4-1}$$

we see that  $\partial_{\phi}^{j} u$  and then also  $\nabla^{k-j} \partial_{\phi}^{j} u$  is harmonic for any  $j \le k$  in  $\mathbb{N}_{0}$ , where  $\nabla u = (u_{x}, u_{y})$  in Euclidean coordinates z = x + iy. Thus by induction we obtain

$$\int_{\partial B} |\nabla^k u|^2 d\phi = 2 \int_{\partial B} |\nabla^{k-1} u_{\phi}|^2 d\phi = \dots = 2^k \int_{\partial B} |\partial^k_{\phi} u|^2 d\phi$$
(4-2)

for any  $k \in \mathbb{N}$ . Similarly, for any  $\frac{1}{4} < r < 1$  with uniform constants C > 0 we have

$$\int_{\partial B_r(0)} |\nabla^k u|^2 dz \le C \int_{\partial B_r(0)} |\nabla^{k-1} u_{\phi}|^2 dz \le \dots \le C \int_{\partial B_r(0)} |\partial_{\phi}^k u|^2 dz$$

Integrating and using the mean value property of harmonic functions together with (4-2) to bound

$$\sup_{B_{1/4}(0)} |\nabla^k u|^2 \leq C \int_{B \setminus B_{1/4}(0)} |\nabla^k u|^2 dz \leq C \int_B |\nabla \partial_{\phi}^{k-1} u|^2 dz,$$

in particular, for any  $k \in \mathbb{N}$ , we have the bound

$$\int_{B} |\nabla^{k} u|^{2} dz \leq C \int_{B} |\nabla \partial_{\phi}^{k-1} u|^{2} dz$$
(4-3)

with an absolute constant C > 0.

The following lemma is strongly reminiscent of analogous results for the harmonic map heat flow in two space dimensions.

**Lemma 4.1.** With a constant C > 0 depending only on N,

$$\frac{d}{dt}\left(\int_{\partial B} |u_{\phi}|^2 \, d\phi\right) + \int_{B} |\nabla u_{\phi}|^2 \, dz \le C \int_{B} |\nabla u|^2 |u_{\phi}|^2 \, dz$$

*Proof.* Writing  $d\pi_N(u) = 1 - d\pi_N^{\perp}(u)$  with

$$d\pi_N^{\perp}(u)X = v(u)v(u) \cdot X = \sum_{i=1}^m v_i(u)v_i(u) \cdot X$$

for any  $X \in \mathbb{R}^n$ , we compute

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\partial B}|u_{\phi}|^{2}d\phi\right) = \int_{\partial B}u_{\phi}\cdot u_{\phi,t}\,d\phi = -\int_{\partial B}u_{\phi\phi}\cdot u_{t}\,d\phi$$
$$= \int_{\partial B}u_{\phi\phi}\cdot d\pi_{N}(u)u_{r}\,d\phi = -\int_{\partial B}(u_{\phi}\cdot u_{r\phi} - u_{\phi}\cdot\partial_{\phi}(v(u)v(u)\cdot u_{r}))\,d\phi$$
$$= -\frac{1}{2}\int_{\partial B}\partial_{r}(|u_{\phi}|^{2})\,d\phi - \int_{\partial B}u_{\phi}\cdot dv(u)u_{\phi}v(u)\cdot u_{r}\,d\phi,$$

where we use orthogonality  $u_{\phi} \cdot v_i(u) = 0$  on  $\partial B$ ,  $1 \le i \le m$ , in the last step. But  $u_{\phi}$  is harmonic. So with  $\Delta |u_{\phi}|^2 = 2|\nabla u_{\phi}|^2$ , from Gauss' theorem we obtain

$$\frac{1}{2}\int_{\partial B}\partial_r(|u_{\phi}|^2)\,d\phi = \int_B |\nabla u_{\phi}|^2\,dz.$$

On the other hand, by Young's inequality we can estimate

$$\begin{split} \int_{\partial B} u_r \cdot v(u) u_{\phi} \cdot dv(u) u_{\phi} \, d\phi &= \int_B \nabla u \cdot \nabla (v(u) u_{\phi} \cdot dv(u) u_{\phi}) \, dz \\ &\leq C \int_B |\nabla u_{\phi}| \, |\nabla u| \, |u_{\phi}| \, dz + C \int_B |\nabla u|^2 |u_{\phi}|^2 \, dz \\ &\leq \frac{1}{2} \int_B |\nabla u_{\phi}|^2 \, dz + C \int_B |\nabla u|^2 |u_{\phi}|^2 \, dz, \end{split}$$
  
m follows.

and our claim follows.

Combining the previous result with a quantitative bound for the concentration of energy, we obtain a space-time bound for the second derivatives of u. Note that since u is smooth by assumption, for any  $\delta > 0$  and any  $T < T_0$ , there exists a number R = R(T, u) > 0 such that

$$\sup_{\substack{z_0 \in B\\0 < t < T}} \int_{B_R(z_0) \cap B} |\nabla u(t)|^2 dz < \delta.$$
(4-4)

**Proposition 4.2.** There exist constants  $\delta = \delta(N) > 0$  and C > 0 such that, for any  $T < T_0$  with R > 0 as in (4-4),

$$\sup_{0 < t < T} \int_{\partial B} |u_{\phi}(t)|^2 d\phi + \int_0^T \int_B |\nabla u_{\phi}|^2 dx \, dt \le C \int_{\partial B} |u_{0,\phi}|^2 d\phi + CT R^{-2} E(u_0).$$
(4-5)

*Proof.* For given  $T < T_0$  and  $\delta > 0$  to be determined, we fix R > 0 such that (4-4) holds. Let  $B_{R/2}(z_i)$ ,  $1 \le i \le i_0$ , be a cover of B such that any point  $z_0 \in B$  belongs to at most L of the balls  $B_R(z_i)$ , where  $L \in \mathbb{N}$  is independent of R > 0. We then use the decomposition

$$\int_{B} |\nabla u|^{2} |u_{\phi}|^{2} dz \leq \sum_{i=1}^{i_{0}} \int_{B_{R/2}(z_{i})} |\nabla u|^{4} dz \leq \sum_{i=1}^{i_{0}} \int_{B} |\nabla u\varphi_{z_{i},R}|^{4} dz$$

Using the multiplicative inequality (A-2) in the Appendix, for each i we can bound

$$\int_{B} |\nabla u \varphi_{z_i,R}|^4 dz \le C\delta \int_{B_R(z_i)} (|\nabla^2 u|^2 + R^{-2} |\nabla u|^2) dz$$

Summing over  $1 \le i \le i_0$ , we thus obtain the bound

$$\int_{B} |\nabla u|^{2} |u_{\phi}|^{2} dz \leq CL\delta \int_{B} |\nabla^{2}u|^{2} dz + CL\delta R^{-2} E(u) \leq CL\delta \int_{B} |\nabla u_{\phi}|^{2} dz + CL\delta R^{-2} E(u_{0}),$$
  
for sufficiently small  $\delta > 0$  we obtain the claim from Lemma 4.1.

and for sufficiently small  $\delta > 0$  we obtain the claim from Lemma 4.1.

With the help of Proposition 4.2 we can now bound u in  $H^2(B)$  also uniformly in time. For this, we first note the following estimate, which also will be useful later for bounding higher-order derivatives.

**Lemma 4.3.** For any  $k \in \mathbb{N}$ , with a constant C > 0 depending only on k and N, for the solution u = u(t)*to* (1-3) *and* (1-4) *for any*  $0 < t < T_0$ 

$$\frac{d}{dt}(\|\nabla\partial_{\phi}^{k}u\|_{L^{2}(B)}^{2})+\|\partial_{\phi}^{k}u_{r}\|_{L^{2}(S^{1})}^{2}\leq C\sum_{\substack{1\leq j_{i}\leq k+1\\\Sigma_{i}j_{i}\leq k+2}}\|\nabla\partial_{\phi}^{k}u\|_{L^{2}(B)}\left\|\prod_{i}\nabla^{j_{i}}u\right\|_{L^{2}(B)}.$$

*Proof.* For any  $k \in \mathbb{N}$  we use harmonicity of  $\partial_{\phi}^{2k} u$  to compute

$$\frac{1}{2} \frac{d}{dt} (\|\nabla \partial_{\phi}^{k} u\|_{L^{2}(B)}^{2}) = (-1)^{k} \int_{B} \nabla \partial_{\phi}^{2k} u \nabla u_{t} dx$$

$$= (-1)^{k} (\partial_{\phi}^{2k} u_{r}, u_{t})_{L^{2}(S^{1})} = (-1)^{k+1} (\partial_{\phi}^{2k} u_{r}, d\pi_{N}(u)u_{r})_{L^{2}(S^{1})}$$

$$= -(\partial_{\phi}^{k} u_{r}, \partial_{\phi}^{k} u_{r})_{L^{2}(S^{1})} + (\partial_{\phi}^{k} u_{r}, \partial_{\phi}^{k}(v(u)v(u) \cdot u_{r}))_{L^{2}(S^{1})} = -\|\partial_{\phi}^{k} u_{r}\|_{L^{2}(S^{1})}^{2} + I, \quad (4-6)$$

where we use the decomposition  $I = \sum_{j=0}^{k} {k \choose j} I_j$  with

$$I_j = (\partial_{\phi}^k u_r, \partial_{\phi}^j (\nu(u)\nu(u))\partial_{\phi}^{k-j} u_r)_{L^2(S^1)} = (\nabla \partial_{\phi}^k u, \nabla (\partial_{\phi}^j (\nu(u)\nu(u)) \cdot \partial_{\phi}^{k-j} u_r))_{L^2(B)}$$

Hence for any  $1 \le j \le k$  we can bound

$$\begin{split} |I_{j}| &\leq C \sum_{0 \leq i \leq j} \|\nabla \partial_{\phi}^{k} u\|_{L^{2}(B)} \|\nabla \partial_{\phi}^{j-i} v(u) \partial_{\phi}^{i} v(u) \partial_{\phi}^{k-j} u_{r}\|_{L^{2}(B)} \\ &+ C \sum_{0 \leq i \leq j} \|\nabla \partial_{\phi}^{k} u\|_{L^{2}(B)} \|\partial_{\phi}^{j-i} v(u) \partial_{\phi}^{i} v(u) \nabla \partial_{\phi}^{k-j} u_{r}\|_{L^{2}(B)} \\ &\leq C \sum_{\substack{1 \leq j_{i} \leq k+1 \\ \Sigma_{i} j_{i} = k+2}} \|\nabla \partial_{\phi}^{k} u\|_{L^{2}(B)} \left\| \prod_{i} \nabla^{j_{i}} u \right\|_{L^{2}(B)}, \end{split}$$

as claimed. It remains to bound the term  $I_0 = \|\partial_{\phi}^k u_r \cdot v(u)\|_{L^2(S^1)}^2$ . With the signed distance function we can express

$$v(u) \cdot u_{\phi r} = (v(u) \cdot u_r)_{\phi} - u_r \cdot dv(u)u_{\phi} = (\operatorname{dist}_N(u))_{\phi r} - u_r \cdot dv(u)u_{\phi},$$

so that

$$I_0 = \|\partial_{\phi}^k u_r \cdot \nu(u)\|_{L^2(S^1)}^2 = (\partial_{\phi}^k u_r \cdot \nu(u), \partial_{\phi}^k (\operatorname{dist}_N(u))_r)_{L^2(S^1)} + II$$
$$= (\nabla \partial_{\phi}^k u, \nabla (\nu(u) \partial_{\phi}^k (\operatorname{dist}_N(u))_r))_{L^2(B)} + II,$$

where all terms in *II* can be dealt with as in the case  $1 \le j \le k$ . Finally, we have

$$\begin{aligned} (\nabla \partial_{\phi}^{k} u, \nabla(\nu(u) \partial_{\phi}^{k}(\operatorname{dist}_{N}(u))_{r}))_{L^{2}(B)} \\ & \leq \|\nabla \partial_{\phi}^{k} u\|_{L^{2}(B)}(\|\nabla^{2} \partial_{\phi}^{k}(\operatorname{dist}_{N}(u))\|_{L^{2}(B)} + \|\nabla \nu(u) \partial_{\phi}^{k}(\operatorname{dist}_{N}(u))_{r}\|_{L^{2}(B)}). \end{aligned}$$

But by the chain rule we can bound

$$\|\nabla v(u)\partial_{\phi}^{k}(\operatorname{dist}_{N}(u))_{r}\|_{L^{2}(B)} \leq C \|\nabla u\nabla^{k+1}(\operatorname{dist}_{N}(u))\|_{L^{2}(B)} \leq C \sum_{\substack{1 \leq j_{i} \leq k+1 \\ \Sigma_{i}, j_{i}=k+2}} \left\|\prod_{i} \nabla^{j_{i}} u\right\|_{L^{2}(B)}.$$

Moreover, by (3-5) and elliptic regularity theory,

$$\begin{aligned} \|\nabla^{k+2}(\operatorname{dist}_{N}(u))\|_{L^{2}(B)}^{2} &\leq C \|\Delta(\operatorname{dist}_{N}(u))\|_{H^{k}(B)}^{2} \\ &\leq C \|\nabla u \cdot dv_{i}(u)\nabla u\|_{H^{k}(B)}^{2} \leq C \sum_{\substack{1 \leq j_{i} \leq k+1 \\ \Sigma_{i} j_{i} \leq k+2}} \left\|\prod_{i} \nabla^{j_{i}} u\right\|_{L^{2}(B)}, \end{aligned}$$

which gives the claim.

For k = 1, from Proposition 4.2 we now easily derive a uniform  $L^2$ -bound for the second derivatives of the flow.

**Proposition 4.4.** For any smooth  $u_0 \in H^{1/2}(S^1; N)$  and any  $T < T_0$  with R > 0 as in Proposition 4.2 with a constant  $C_1 = C_1(T, R, u_0) > 0$  depending on the right-hand side of (4-5),

$$\sup_{0 < t < T} \int_{B} |\nabla u_{\phi}(t)|^{2} dz + \int_{0}^{T} \int_{\partial B} |u_{\phi r}|^{2} d\phi dt \leq C_{1} \int_{B} |\nabla u_{0,\phi}|^{2} dz + C_{1}.$$

*Proof.* For k = 1, by Lemma 4.3 we need to bound the term

$$J = \sum_{\substack{1 \le j_i \le 2\\ \Sigma_i \, j_i \le 3}} \left\| \prod_i \nabla^{j_i} u \right\|_{L^2(B)} \le C \| |\nabla^2 u| |\nabla u| + |\nabla u|^3 \|_{L^2(B)} + J_1,$$

where  $J_1$  contains all terms of lower order. By the maximum principle and Sobolev's embedding  $H^1(\partial B) \hookrightarrow L^{\infty}(\partial B)$  we can estimate

$$\|\nabla u\|_{L^{\infty}(B)}^{2} \leq \|\nabla u\|_{L^{\infty}(\partial B)}^{2} \leq C \|\nabla u\|_{H^{1}(\partial B)}^{2} \leq C \|u_{\phi r}\|_{L^{2}(\partial B)}^{2} + C_{1},$$

where we have also used (3-7) and Proposition 4.2. Also bounding

$$\|\nabla u\|_{L^{6}(B)}^{3} \leq \|\nabla u\|_{L^{4}(B)}^{2} \|\nabla u\|_{L^{\infty}(B)} \leq C(\|\nabla^{2}u\|_{L^{2}(B)}\|\nabla u\|_{L^{2}(B)} + E(u))\|\nabla u\|_{L^{\infty}(B)}$$

via (A-2), and again using (3-7) (and with similar, but simpler bounds for  $J_1$ ), we arrive at the estimate

$$J \leq C \| |\nabla^2 u| |\nabla u| + |\nabla u|^3 \|_{L^2(B)} + C_1 \leq C (\|\nabla^2 u\|_{L^2(B)} + E(u)) \|\nabla u\|_{L^\infty(B)} + C_1$$
  
$$\leq C (1 + \|\nabla u_\phi\|_{L^2(B)} + E(u_0)) (\|u_{\phi r}\|_{L^2(\partial B)} + C_1).$$

With Lemma 4.3 and Young's inequality we then have

$$\frac{d}{dt}(1 + \|\nabla u_{\phi}\|_{L^{2}(B)}^{2}) + \|u_{\phi r}\|_{L^{2}(S^{1})}^{2} \leq C \|\nabla u_{\phi}\|_{L^{2}(B)}(\|\nabla u_{\phi}\|_{L^{2}(B)} + E(u_{0}))(\|u_{\phi r}\|_{L^{2}(\partial B)} + C_{1}) \\
\leq \frac{1}{2} \|u_{\phi r}\|_{L^{2}(\partial B)}^{2} + C(1 + \|\nabla u_{\phi}\|_{L^{2}(B)}^{2})(\|\nabla u_{\phi}\|_{L^{2}(B)}^{2} + C_{1}). \quad (4-7)$$

Absorbing the term  $\frac{1}{2} \|u_{\phi r}\|_{L^2(\partial B)}^2$  into the left-hand side of this inequality and dividing by  $1 + \|\nabla u_{\phi}\|_{L^2(B)}^2$  we obtain

$$\frac{d}{dt}(\log(1+\|\nabla u_{\phi}\|_{L^{2}(B)}^{2})) \leq C\|\nabla u_{\phi}\|_{L^{2}(B)}^{2}+C_{1},$$

and from Proposition 4.2 we obtain the bound

$$\sup_{0 < t < T} \|\nabla u_{\phi}(t)\|_{L^{2}(B)}^{2} \le C_{1}(1 + \|\nabla u_{0,\phi}\|_{L^{2}(B)}^{2}).$$

The claim then follows from (4-7).

 $H^3$ -bounds. The derivation of a-priori  $L^2$ -bounds for third derivatives of the solution u to the flow (1-3), (1-4) requires special care, which is why we highlight this case.

**Proposition 4.5.** For any smooth  $u_0 \in H^{1/2}(S^1; N)$  and any  $T < T_0$ ,

$$\sup_{0 < t < T} \int_{B} |\nabla u_{\phi\phi}(t)|^{2} dz + \int_{0}^{T} \int_{\partial B} |u_{\phi\phi r}|^{2} d\phi dt \leq C_{2} \int_{B} |\nabla u_{0,\phi\phi}|^{2} dz + C_{2},$$

where we denote by  $C_2 = C_2(T, R, u_0) > 0$  a constant bounded by the terms on the right-hand side in the statements of Propositions 4.2 and 4.4.

*Proof.* For k = 2, by Lemma 4.3 we need to bound the term

$$J = \sum_{\substack{1 \le j_i \le 3\\ \Sigma_i j_i = 4}} \left\| \prod_i \nabla^{j_i} u \right\|_{L^2(B)} \le C \||\nabla u|^4 + |\nabla u|^2 |\nabla^2 u| + |\nabla^2 u|^2 + |\nabla u| |\nabla^3 u|\|_{L^2(B)}$$

and corresponding terms involving at most three derivatives in total, which we will omit.

For the first term, by the multiplicative inequality (A-2) and Sobolev's embedding  $H^2(B) \hookrightarrow L^{\infty}(B)$ , we can estimate

$$\begin{aligned} \|\nabla u\|_{L^{8}(B)}^{4} &\leq \|\nabla u\|_{L^{4}(B)}^{2} \|\nabla u\|_{L^{\infty}(B)}^{2} \leq C \|\nabla u\|_{H^{1}(B)} \|\nabla u\|_{L^{2}(B)} \|\nabla u\|_{L^{\infty}(B)}^{2} \\ &\leq C(\|\nabla^{2} u\|_{L^{2}(B)}^{2} + E(u)) \|\nabla u\|_{L^{\infty}(B)}^{2} \leq C_{2} \|\nabla u\|_{L^{\infty}(B)}^{2} \\ &\leq C_{2}(\|\nabla^{3} u\|_{L^{2}(B)} + \|\nabla u\|_{L^{2}(B)}) \|\nabla u\|_{L^{\infty}(B)} \end{aligned}$$

with a constant  $C_2 = C_2(T, R, u_0) > 0$  as in the statement of the proposition. Similarly,

$$\|\nabla^{2}u\|_{L^{4}(B)}^{2} \leq C \|\nabla^{2}u\|_{H^{1}(B)} \|\nabla^{2}u\|_{L^{2}(B)} \leq \|\nabla^{3}u\|_{L^{2}(B)} \|\nabla^{2}u\|_{L^{2}(B)} + \|\nabla^{2}u\|_{L^{2}(B)}^{2} \leq C_{2}(1 + \|\nabla^{3}u\|_{L^{2}(B)})$$

Hence we can also bound

$$\||\nabla u|^{2}|\nabla^{2}u|\|_{L^{2}(B)} \leq \|\nabla u\|_{L^{8}(B)}^{4} + \|\nabla^{2}u\|_{L^{4}(B)}^{2} \leq C_{2}(1 + \|\nabla^{3}u\|_{L^{2}(B)})(1 + \|\nabla u\|_{L^{\infty}(B)})$$

Finally, we estimate

$$\||\nabla u||\nabla^3 u|\|_{L^2(B)} \le \|\nabla^3 u\|_{L^2(B)} \|\nabla u\|_{L^{\infty}(B)}$$

to obtain

$$U \le C_2(1 + \|\nabla^3 u\|_{L^2(B)})(1 + \|\nabla u\|_{L^\infty(B)})$$

But with the inequality

$$\|f\|_{L^{\infty}(B)} \le C \|f\|_{H^{1}(B)} \left(1 + \log^{1/2} \left(1 + \frac{\|f\|_{H^{2}(B)}}{\|f\|_{H^{1}(B)}}\right)\right)$$

for  $f \in H^2(B)$  due to Brezis and Gallouet [1980] (see also [Brézis and Wainger 1980] for a more general version), we have

$$\|\nabla u\|_{L^{\infty}(B)}^{2} \leq C \|\nabla u\|_{H^{1}(B)}^{2} \left(1 + \log\left(1 + \frac{\|\nabla u\|_{H^{2}(B)}}{\|\nabla u\|_{H^{1}(B)}}\right)\right) \leq C_{2}(1 + \log(1 + \|\nabla^{3}u\|_{L^{2}(B)})),$$

and Lemma 4.3 yields the differential inequality

$$\frac{d}{dt}(\|\nabla\partial_{\phi}^{2}u\|_{L^{2}(B)}^{2}) + \|u_{\phi\phi r}\|_{L^{2}(\partial B)}^{2} \le C_{2}\|\nabla\partial_{\phi}^{2}u\|_{L^{2}(B)}(1 + \|\nabla^{3}u\|_{L^{2}(B)})(1 + \log(1 + \|\nabla^{3}u\|_{L^{2}(B)})).$$

Simplifying, and recalling that  $\|\nabla^3 u\|_{L^2(B)}^2 \leq C \|\nabla \partial_{\phi}^2 u\|_{L^2(B)}^2$  by (4-3), we then find

$$\frac{d}{dt}(1 + \|\nabla\partial_{\phi}^{2}u\|_{L^{2}(B)}) \leq C_{2}(1 + \|\nabla\partial_{\phi}^{2}u\|_{L^{2}(B)})(1 + \log(1 + \|\nabla\partial_{\phi}^{2}u\|_{L^{2}(B)}));$$

that is, we have

$$\frac{d}{dt}(1 + \log(1 + \|\nabla\partial_{\phi}^2 u\|_{L^2(B)})) \le C_2(1 + \log(1 + \|\nabla\partial_{\phi}^2 u\|_{L^2(B)})).$$

Arguing as in the proof of Proposition 4.4 we then obtain the claim.

1412

**Proposition 4.6.** For any  $k \ge 3$ , any smooth  $u_0 \in H^{1/2}(S^1; N)$ , and any  $T < T_0$ ,

$$\sup_{0 < t < T} \int_{B} |\nabla \partial_{\phi}^{k}(t)|^{2} dz + \int_{0}^{T} \int_{\partial B} |\partial_{\phi}^{k}u_{r}|^{2} d\phi dt \leq C_{k} \int_{B} |\nabla \partial_{\phi}^{k}u_{0}|^{2} dz + C_{k}$$

where we denote by  $C_k = C_k(T, R, u_0) > 0$  a constant bounded by the terms on the right-hand side in the statement of the proposition for k - 1.

*Proof.* By Proposition 4.5 the claimed result holds true for k = 2. Suppose the claim holds true for some  $k_0 \ge 2$  and let  $k = k_0 + 1$ . Note that by Sobolev's embedding  $H^2(B) \hookrightarrow W^{1,4} \cap C^0(\overline{B})$  and (4-3) for  $0 \le t < T$  we then have the uniform bounds

$$\|\nabla^{k_0+1}u\|_{L^2(B)}^2 + \|\nabla^{k_0}u\|_{L^4(B)}^2 + \sum_{1 \le j \le k_0-1} \|\nabla^j u\|_{L^\infty(B)}^2 \le C_{k_0} \|\nabla^{k_0+1}u_0\|_{L^2(B)}^2 + C_{k_0} \le C_k < \infty \quad (4-8)$$

with a constant of the type  $C_k$ , as defined above.

By Lemma 4.3 again we only need to bound the term

$$J = \sum_{\substack{1 \le j_i \le k+1 \\ \Sigma_i j_i \le k+2}} \left\| \prod_i \nabla^{j_i} u \right\|_{L^2(B)}.$$

Clearly we have

$$\begin{split} J &\leq \|\nabla^{k+1}u\|_{L^{2}(B)} \|\nabla u\|_{L^{\infty}(B)} + \|\nabla^{k}u\|_{L^{2}(B)} \|\nabla u\|_{L^{\infty}(B)}^{2} + \|\nabla^{k}u\nabla^{2}u\|_{L^{2}(B)} \\ &\quad + \|\nabla^{k-1}u\nabla^{3}u\|_{L^{2}(B)} + \|\nabla^{k-1}u\nabla^{2}u\|_{L^{2}(B)} \|\nabla u\|_{L^{\infty}(B)} + C_{k} \\ &\leq C_{k} \|\nabla^{k+1}u\|_{L^{2}(B)} + \|\nabla^{k}u\nabla^{2}u\|_{L^{2}(B)} + \|\nabla^{k-1}u\nabla^{3}u\|_{L^{2}(B)} + C_{k}. \end{split}$$

We now distinguish the following cases: If  $k - 1 = k_0 \ge 3$ , by (4-8) we can bound

$$\|\nabla^{k} u \nabla^{2} u\|_{L^{2}(B)} \leq \|\nabla^{k} u\|_{L^{2}(B)} \|\nabla^{2} u\|_{L^{\infty}(B)} \leq C_{k_{0}} \|\nabla^{k_{0}+1} u\|_{L^{2}(B)}^{2} + C_{k_{0}} \leq C_{k_{0}}$$

as well as

$$\|\nabla^{k-1}u\nabla^{3}u\|_{L^{2}(B)} \leq \|\nabla^{k-1}u\|_{L^{4}(B)}\|\nabla^{3}u\|_{L^{4}(B)} \leq C_{k_{0}}\|\nabla^{k_{0}}u\|_{L^{4}(B)}^{2} + C_{k_{0}} \leq C_{k}$$

to obtain the estimate

$$J \le C_k \|\nabla^{k+1} u\|_{L^2(B)} + C_k.$$

If, on the other hand,  $k_0 = k - 1 = 2$ , by our induction hypothesis (4-8) we have

$$\begin{aligned} \|\nabla^{k-1}u\nabla^{3}u\|_{L^{2}(B)} &= \|\nabla^{2}u\nabla^{k}u\|_{L^{2}(B)}^{2} \leq \|\nabla^{k}u\|_{L^{4}(B)} \|\nabla^{2}u\|_{L^{4}(B)} \\ &\leq C_{k}\|\nabla^{k}u\|_{H^{1}(B)} + C_{k} \leq C_{k}\|\nabla^{k+1}u\|_{L^{2}(B)} + C_{k}, \end{aligned}$$

and we find

$$J \le C_k \|\nabla^{k+1} u\|_{L^2(B)} + C_k$$

as before.

In any case, inequality (4-3) and Lemma 4.3 now may be invoked to obtain

$$\frac{d}{dt}(\|\nabla\partial_{\phi}^{k}u\|_{L^{2}(B)}) \leq C_{k}\|\nabla\partial_{\phi}^{k}u\|_{L^{2}(B)} + C_{k},$$

and our claim follows.

*Local*  $H^k$ -*bounds.* The bounds established so far all require the initial data to be sufficiently smooth for the estimate at hand and do not yet allow to show smoothing of the flow. For the latter purpose we next prove a second set of "intermediate" estimates that in combination with the first set of estimates later will allow boot-strapping. Moreover, in contrast to the estimates in Lemma 4.3, the following estimates may be localized. This will be important for showing regularity of the flow at blow-up times away from concentration points of the energy on  $\partial B$ .

For the localized estimates, fix a point  $z_0 \in \partial B$  and some radius  $0 < R_0 < \frac{1}{4}$  and for  $k \in \mathbb{N}$  set  $R_k = 2^{-k}R_0$  and  $\varphi_k = \varphi_{z_0, R_k}$ . Set  $\varphi_k = 1$  for each  $k \in \mathbb{N}$  for the analogous global bounds.

We first establish the following localized version of Lemma 4.1.

**Lemma 4.7.** With a constant C > 0 depending only on N,

$$\frac{d}{dt} \left( \int_{\partial B} |u_{\phi}|^2 \varphi_1^2 \, d\phi \right) + \int_{B} |\nabla u_{\phi}|^2 \varphi_1^2 \, dz \le C \int_{B} |\nabla u|^2 |u_{\phi}|^2 \varphi_1^2 \, dz + C R_0^{-2} E(u_0)$$

Proof. Similar to the proof of Lemma 4.1, we compute

$$\begin{split} \frac{1}{2} \frac{d}{dt} \left( \int_{\partial B} |u_{\phi}|^2 \varphi_1^2 \, d\phi \right) &= \int_{\partial B} u_{\phi} \cdot u_{\phi,t} \varphi_1^2 \, d\phi = -\int_{\partial B} \partial_{\phi} (u_{\phi} \varphi_1^2) \cdot u_t \, d\phi \\ &= \int_{\partial B} \partial_{\phi} (u_{\phi} \varphi_1^2) \cdot d\pi_N (u) u_r \, d\phi = -\int_{\partial B} (u_{\phi} \cdot u_{r\phi} - u_{\phi} \cdot \partial_{\phi} (v(u)v(u) \cdot u_r)) \varphi_1^2 \, d\phi \\ &= -\frac{1}{2} \int_{\partial B} \partial_r (|u_{\phi}|^2) \varphi_1^2 \, d\phi - \int_{\partial B} u_{\phi} \cdot dv(u) u_{\phi} v(u) \cdot u_r \varphi_1^2 \, d\phi. \end{split}$$

With  $\Delta |u_{\phi}|^2 = 2|\nabla u_{\phi}|^2$  we obtain

$$\frac{1}{2} \int_{\partial B} \partial_r (|u_{\phi}|^2) \varphi_1^2 \, d\phi = \int_B |\nabla u_{\phi}|^2 \varphi_1^2 \, dz + \int_B \nabla |u_{\phi}|^2 \varphi_1 \nabla \varphi_1 \, dz,$$

where

$$\left| \int_{B} \nabla |u_{\phi}|^{2} \varphi_{1} \nabla \varphi_{1} dz \right| \leq \frac{1}{4} \int_{B} |\nabla u_{\phi}|^{2} \varphi_{1}^{2} dz + C \int_{B} |u_{\phi}|^{2} |\nabla \varphi_{1}|^{2} dz$$

by Young's inequality. Finally, we can bound

$$\begin{split} \int_{\partial B} u_r \cdot v(u) u_{\phi} \cdot dv(u) u_{\phi} \varphi_1^2 \, d\phi &= \int_B \nabla u \cdot \nabla (v(u) u_{\phi} \cdot dv(u) u_{\phi} \varphi_1^2) \, dz \\ &\leq C \int_B (|\nabla u_{\phi}| |\nabla u| |u_{\phi}| + |\nabla u|^2 |u_{\phi}|^2) \varphi_1^2 \, dz + C \int_B |\nabla u| |\nabla \varphi_1| |u_{\phi}|^2 \varphi_1 \, dz \\ &\leq \frac{1}{4} \int_B |\nabla u_{\phi}|^2 \varphi_1^2 \, dz + C \int_B |\nabla u|^2 |u_{\phi}|^2 \varphi_1^2 \, dz + C \int_B |\nabla u|^2 |\nabla \varphi_1|^2 \, dz, \end{split}$$

and our claim follows.

We need a substitute for the global bound (4-3). For this, we note that (4-1) also implies the pointwise bound

$$|u_{rr}|^2 \le \frac{2|u_{\phi\phi}|^2}{r^4} + \frac{2|u_r|^2}{r^2};$$

hence we have

$$|\nabla^2 u|^2 \le C(|\nabla u_{\phi}|^2 + |\nabla u|^2)$$
 in  $B_{R_0}(z_0)$ 

with an absolute constant C > 0, uniformly in  $z_0 \in \partial B$  and  $0 < R_0 < \frac{1}{4}$ . By induction then, similarly, we have

$$|\nabla^{k+1}u|^{2} \le C(|\nabla^{k}\partial_{\phi}u|^{2} + |\nabla^{k}u|^{2}) \le C\sum_{j=0}^{k} |\nabla\partial_{\phi}^{j}u|^{2} \quad \text{in } B_{R_{0}}(z_{0})$$
(4-9)

with an absolute constant C = C(k) > 0, uniformly in  $z_0 \in \partial B$  and  $0 < R_0 < \frac{1}{4}$  for any  $k \in \mathbb{N}$ .

Likewise, as a substitute for the global nonconcentration condition (4-4) we now suppose that  $z_0 \in \partial B$  is not a concentration point in the sense that for suitably chosen  $\delta > 0$  to be determined in the sequel and some  $0 < R_0 < \frac{1}{4}$  as above,

$$\sup_{0 < t < T_0} \int_{B_{R_0}(z_0) \cap B} |\nabla u(t)|^2 \, dz < \delta. \tag{4-10}$$

We then obtain the following localized version of Proposition 4.2.

**Proposition 4.8.** There exist constants  $\delta > 0$  and C > 0 independent of  $R_0 > 0$  such that whenever (4-10) holds then for any  $T \le T_0$  we have

$$\sup_{0 < t < T} \int_{\partial B} |u_{\phi}(t)|^{2} \varphi_{1}^{2} d\phi + \int_{0}^{T} \int_{B} |\nabla u_{\phi}|^{2} \varphi_{1}^{2} dz dt \leq 2 \int_{\partial B} |u_{0,\phi}|^{2} \varphi_{1}^{2} d\phi + CT R_{0}^{-2} E(u_{0}).$$

Proof. With the help of inequality (A-1) in the Appendix we can bound

$$\int_{B} |\nabla u|^{4} \varphi_{1}^{2} dz \leq C \delta \int_{B_{R}(z_{i})} |\nabla^{2} u|^{2} \varphi_{1}^{2} dz + C \delta R_{0}^{-2} \int_{B_{R}(z_{i})} |\nabla u|^{2} dz.$$

Thus, for sufficiently small  $\delta > 0$  our claim follows from Lemma 4.7.

The next lemma again prepares for a proposition that later will allow us to obtain higher-derivative bounds by induction. Note the differences to Lemma 4.3.

**Lemma 4.9.** For any  $k \ge 2$ , with a constant C > 0 depending only on k and N, for the solution u = u(t) to (1-3) and (1-4) for any  $0 < t < T_0$ ,

$$\frac{d}{dt}(\|\partial_{\phi}^{k}u\varphi_{k}\|_{L^{2}(\partial B)}^{2}) + \|\nabla\partial_{\phi}^{k}u\varphi_{k}\|_{L^{2}(B)}^{2}$$

$$\leq C \sum_{\substack{1 \leq j_{i} \leq k \\ \Sigma_{i} j_{i} \leq 2k+2}} \left\|\prod_{i} \nabla^{j_{i}}u\varphi_{k}^{2}\right\|_{L^{1}(B)} + C \sum_{\substack{1 \leq j_{0}, j_{i} \leq k \\ \Sigma_{i\geq 0} j_{i} \leq k+1}} \left\|\prod_{i>0} \nabla^{j_{i}}u\nabla^{j_{0}}\varphi_{k}\right\|_{L^{2}(B)}^{2} + CR_{0}^{-2k}E(u_{0}).$$

Proof. Fix  $k \ge 2$ . With  $\Delta |\partial_{\phi}^{k}u|^{2} = 2|\nabla\partial_{\phi}^{k}u|^{2}$  we compute  $\frac{1}{2}\frac{d}{dt}(||\partial_{\phi}^{k}u\varphi_{k}||_{L^{2}(\partial B)}^{2}) = (-1)^{k}\int_{\partial B}\partial_{\phi}^{k}(\partial_{\phi}^{k}u\varphi_{k}^{2}) \cdot u_{t} d\phi = (-1)^{k+1}\int_{\partial B}\partial_{\phi}^{k}(\partial_{\phi}^{k}u\varphi_{k}^{2}) \cdot (u_{r} - v(u)v(u) \cdot u_{r}) d\phi$   $= -\frac{1}{2}\int_{\partial B}\partial_{r}(|\partial_{\phi}^{k}u|^{2})\varphi_{k}^{2} d\phi + \int_{\partial B}\partial_{\phi}^{k}u \cdot \partial_{\phi}^{k}(v(u)v(u) \cdot u_{r})\varphi_{k}^{2} d\phi$   $= -\int_{B}|\nabla\partial_{\phi}^{k}u|^{2}\varphi_{k}^{2} dz - \int_{B}\nabla(|\partial_{\phi}^{k}u|^{2})\varphi_{k} \nabla\varphi_{k} dz + I,$ 

where the term  $\int_B \nabla (|\partial_{\phi}^k u|^2) \varphi_k \nabla \varphi_k dz$  can be bounded as claimed. We use the decomposition

$$I = \int_{\partial B} \partial_{\phi}^{k} u \cdot \partial_{\phi}^{k} (v(u)v(u) \cdot u_{r}) \varphi_{k}^{2} d\phi = \sum_{j=0}^{k} {\binom{k}{j}} I_{j}$$

with

$$I_j = (\partial_{\phi}^k u \cdot \partial_{\phi}^j (\nu(u)\nu(u))\varphi_k^2, \partial_{\phi}^{k-j} u_r)_{L^2(\partial B)} = (\nabla(\partial_{\phi}^k u \cdot \partial_{\phi}^j (\nu(u)\nu(u))\varphi_k^2), \nabla\partial_{\phi}^{k-j} u)_{L^2(B)}, \quad 0 \le j \le k.$$

For  $1 \le j \le k$  we bound

$$\begin{split} |I_{j}| &\leq C \sum_{0 \leq i \leq j} \|\nabla \partial_{\phi}^{k} u \varphi_{k}\|_{L^{2}(B)} \|\partial_{\phi}^{j-i} v(u) \partial_{\phi}^{i} v(u) \nabla \partial_{\phi}^{k-j} u \varphi_{k}\|_{L^{2}(B)} \\ &+ C \sum_{0 \leq i \leq j} \|\partial_{\phi}^{k} u \cdot \nabla (\partial_{\phi}^{j-i} v(u) \partial_{\phi}^{i} v(u) \varphi_{k}^{2}) \cdot \nabla \partial_{\phi}^{k-j} u\|_{L^{1}(B)}. \end{split}$$

By the chain rule then for  $1 \le j \le k$  we have

$$\begin{split} |I_{j}| &\leq C \sum_{\substack{1 \leq j_{i} \leq k \\ \Sigma_{i} j_{i}=k+1}} \|\nabla \partial_{\phi}^{k} u \varphi_{k}\|_{L^{2}(B)} \left\| \prod_{i} \nabla^{j_{i}} u \varphi_{k} \right\|_{L^{2}(B)} \\ &+ C \sum_{\substack{1 \leq j_{i} \leq k \\ \Sigma_{i} j_{i}=k+2}} \left\| \partial_{\phi}^{k} u \cdot \prod_{i} \nabla^{j_{i}} u \varphi_{k}^{2} \right\|_{L^{1}(B)} + C \sum_{\substack{1 \leq j_{i} \leq k \\ \Sigma_{i} j_{i}=k+1}} \left\| \partial_{\phi}^{k} u \cdot \prod_{i} \nabla^{j_{i}} u \varphi_{k} \nabla \varphi_{k} \right\|_{L^{1}(B)}. \end{split}$$

By Cauchy-Schwarz and Young's inequalities then we can bound

$$\begin{split} \sum_{1 \le j \le k} |I_j| \le \frac{1}{4} \|\nabla \partial_{\phi}^k u \varphi_k\|_{L^2(B)}^2 + C \sum_{\substack{1 \le j_i \le k \\ \Sigma_i j_i = k+1}} \left\| \prod_i \nabla^{j_i} u \varphi_k \right\|_{L^2(B)}^2 \\ + C \sum_{\substack{1 \le j_i \le k \\ \Sigma_i j_i = 2k+2}} \left\| \prod_i \nabla^{j_i} u \varphi_k^2 \right\|_{L^1(B)} + C \|\partial_{\phi}^k u \nabla \varphi_k\|_{L^2(B)}^2 \\ \le \frac{1}{4} \|\nabla \partial_{\phi}^k u \varphi_k\|_{L^2(B)}^2 + C \sum_{\substack{1 \le j_i \le k \\ \Sigma_i j_i = 2k+2}} \left\| \prod_i \nabla^{j_i} u \varphi_k^2 \right\|_{L^1(B)} + C \|\partial_{\phi}^k u \nabla \varphi_k\|_{L^2(B)}^2, \end{split}$$

as claimed. Finally, with

$$v(u) \cdot u_{\phi r} = (\operatorname{dist}_N(u))_{\phi r} - u_r \cdot dv(u)u_{\phi}$$

as in the proof of Lemma 4.3, for j = 0 we can write

$$\nu(u) \cdot \partial_{\phi}^{k} u_{r} = \partial_{\phi}^{k-1}(\nu(u) \cdot u_{\phi r}) + II = \partial_{\phi}^{k}(\operatorname{dist}_{N}(u))_{r} + III,$$

1417

where the terms in II and III involve products of at least two derivatives of orders between 1 and k of u. Thus we have

$$I_0 = (\partial_{\phi}^k u \cdot v(u)\varphi_k^2, v(u) \cdot \partial_{\phi}^k u_r)_{L^2(\partial B)} = (\partial_{\phi}^k u \cdot v(u)\varphi_k^2, \partial_{\phi}^k(\operatorname{dist}_N(u))_r)_{L^2(\partial B)} + II_0$$

with a term  $II_0$  that can be dealt with in the same way as the terms  $I_j$ ,  $1 \le j \le k$ .

Using the divergence theorem and integrating by parts we can write the leading term as

$$\begin{split} \hat{I}_0 &:= (\partial_{\phi}^k u \cdot v(u)\varphi_k^2, \partial_{\phi}^k (\operatorname{dist}_N(u))_r)_{L^2(\partial B)} \\ &= (\nabla(\partial_{\phi}^k u \cdot v(u)\varphi_k^2), \nabla\partial_{\phi}^k (\operatorname{dist}_N(u)))_{L^2(B)} + (\partial_{\phi}^k u \cdot v(u)\varphi_k^2, \Delta\partial_{\phi}^k (\operatorname{dist}_N(u)))_{L^2(B)} \\ &= (\nabla(\partial_{\phi}^k u \cdot v(u)\varphi_k^2), \nabla\partial_{\phi}^k (\operatorname{dist}_N(u)))_{L^2(B)} - (\partial_{\phi}(\partial_{\phi}^k u \cdot v(u)\varphi_k^2), \Delta\partial_{\phi}^{k-1} (\operatorname{dist}_N(u)))_{L^2(B)} \end{split}$$

to see that this term may be bounded:

$$|\hat{I}_0| \le C \| (|\nabla \partial_{\phi}^k u| + |\partial_{\phi}^k u \nabla u|) \varphi_k + |\partial_{\phi}^k u \nabla \varphi_k| \|_{L^2(B)} \|\nabla^{k+1} (\operatorname{dist}_N(u)) \varphi_k\|_{L^2(B)}.$$

But by elliptic regularity we again have

$$\begin{aligned} \|\nabla^{k+1}(\operatorname{dist}_{N}(u))\varphi_{k}\|_{L^{2}(B)} &\leq \|\nabla^{k+1}(\operatorname{dist}_{N}(u)\varphi_{k})\|_{L^{2}(B)} + C\sum_{1\leq j\leq k+1} \|\nabla^{k+1-j}(\operatorname{dist}_{N}(u))\nabla^{j}\varphi_{k}\|_{L^{2}(B)} \\ &\leq C\|\Delta(\operatorname{dist}_{N}(u))\varphi_{k}\|_{H^{k-1}(B)} + C\sum_{1\leq j\leq k+1} \|\nabla^{k+1-j}(\operatorname{dist}_{N}(u))\nabla^{j}\varphi_{k}\|_{L^{2}(B)}. \end{aligned}$$

where from (3-5) we can bound the first term on the right:

$$\|\Delta(\operatorname{dist}_N(u))\varphi_k\|_{H^{k-1}(B)} \leq \sum_{0 \leq j < k} \|\nabla^j (\nabla u \cdot d\nu(u)\nabla u\varphi_k)\|_{L^2(B)} \leq C \sum_{\substack{0 \leq j_0 < k \\ 1 \leq j_i \leq k \\ \sum_{i>0} j_i \leq k+1}} \left\|\prod_i \nabla^{j_i} u\nabla^{j_0}\varphi_k\right\|_{L^2(B)}.$$

Moreover, using that  $dist_N(u) = 0$  on  $\partial B$ , with the help of Poincaré's inequality we find the bound

$$\|\operatorname{dist}_{N}(u)\nabla^{k+1}\varphi_{k}\|_{L^{2}(B)}^{2} \leq CR_{k}^{-2k}\|\nabla(\operatorname{dist}_{N}(u))\|_{L^{2}(B_{R_{k}}(z_{0}))}^{2} \leq CR_{0}^{-2k}E(u).$$

The remaining terms for  $1 \le j \le k$  can be estimated as

$$\|\nabla^{k+1-j}(\operatorname{dist}_{N}(u))\nabla^{j}\varphi_{k}\|_{L^{2}(B)} \leq C \sum_{\substack{1 \leq j_{i} \leq k \\ \Sigma_{i}j_{i}=k+1-j}} \left\|\prod_{i} \nabla^{j_{i}} u \nabla^{j} \varphi_{k}\right\|_{L^{2}(B)}$$

via the chain rule. Thus, finally, we obtain the bound

$$\|\nabla^{k+1}(\operatorname{dist}_{N}(u))\varphi_{k}\|_{L^{2}(B)}^{2} \leq C \sum_{\substack{1 \leq j_{0}, j_{i} \leq k \\ \Sigma_{i \geq 0} j_{i} \leq k+1}} \left\|\prod_{i>0} \nabla^{j_{i}} u \nabla^{j_{0}} \varphi_{k}\right\|_{L^{2}(B)}^{2} + C R_{0}^{-2k} E(u_{0}).$$

By Cauchy-Schwarz and Young's inequalities thus we can bound

$$|\hat{I}_{0}| \leq \frac{1}{4} \|\nabla \partial_{\phi}^{k} u \varphi_{k}\|_{L^{2}(B)}^{2} + C \|\partial_{\phi}^{k} u \nabla u \varphi_{k}\|_{L^{2}(B)}^{2} + C \sum_{\substack{1 \leq j_{0}, j_{i} \leq k \\ \Sigma_{i \geq 0} j_{i} \leq k+1}} \left\|\prod_{i>0} \nabla^{j_{i}} u \nabla^{j_{0}} \varphi_{k}\right\|_{L^{2}(B)}^{2} + C R_{0}^{-2k} E(u_{0}),$$

and together with our above estimate for the terms  $I_j$ ,  $j \ge 1$ , our claim follows.

**Proposition 4.10.** There exists a constant  $\delta > 0$  independent of  $R_0 > 0$  such that whenever (4-10) holds then for any  $T \leq T_0$  with a constant  $C_2 = C_2(T, R, u_0) > 0$  bounded by the terms on the right-hand side in the statement of Proposition 4.8 there holds the estimate

$$\sup_{0 < t < T} \int_{\partial B} |u_{\phi\phi}(t)|^2 \varphi_2^2 \, d\phi + \int_0^T \int_B |\nabla u_{\phi\phi}|^2 \varphi_2^2 \, dz \, dt \le C_2 \int_{\partial B} |u_{0,\phi\phi}|^2 \varphi_2^2 \, d\phi + C_2.$$

*Proof.* For k = 2, with the help of Young's inequality we can bound

$$J_{1} = \sum_{\substack{1 \le j_{i} \le k \\ \Sigma_{i} j_{i} \le 2k+2}} \left\| \prod_{i} \nabla^{j_{i}} u \varphi_{k}^{2} \right\|_{L^{1}(B)} \le C \| (|\nabla^{2} u|^{3} + |\nabla^{2} u|^{2} |\nabla u|^{2} + |\nabla^{2} u| |\nabla u|^{4} + |\nabla u|^{6} + 1) \varphi_{2}^{2} \|_{L^{1}(B)} \le C \| (|\nabla^{2} u|^{3} + |\nabla u|^{6} + 1) \varphi_{2}^{2} \|_{L^{1}(B)}$$

and

$$J_{2} = \sum_{\substack{1 \le j_{0}, j_{i} \le k \\ \Sigma_{i \ge 0} j_{i} \le k+1}} \left\| \prod_{i>0} \nabla^{j_{i}} u \nabla^{j_{0}} \varphi_{2} \right\|_{L^{2}(B)}^{2} \le C \| (|\nabla^{2} u|^{2} + |\nabla u|^{4} + 1) |\nabla \varphi_{2}|^{2} + (|\nabla u|^{2} + 1) |\nabla^{2} \varphi_{2}|^{2} \|_{L^{1}(B)}.$$

Observing that  $\varphi_1 = 1$  on the support of  $\varphi_2$ , by (A-2) for the first term in  $J_1$  we have

$$\begin{aligned} \||\nabla^{2}u|^{3}\varphi_{2}^{2}\|_{L^{1}(B)} &\leq \|\nabla^{2}u\varphi_{2}\|_{L^{4}(B)}^{2}\|\nabla^{2}u\varphi_{1}\|_{L^{2}(B)} \leq C\|\nabla^{2}u\varphi_{2}\|_{H^{1}(B)}\|\nabla^{2}u\varphi_{2}\|_{L^{2}(B)}\|\nabla^{2}u\varphi_{1}\|_{L^{2}(B)} \\ &\leq C(\|\nabla^{3}u\varphi_{2}\|_{L^{2}(B)} + \|\nabla^{2}u\varphi_{1}\|_{L^{2}(B)})\|\nabla^{2}u\varphi_{2}\|_{L^{2}(B)}\|\nabla^{2}u\varphi_{1}\|_{L^{2}(B)}. \end{aligned}$$

Moreover, arguing as in (A-1) for the function  $|\nabla u|^6 \varphi_2^2$  in place of  $|v|^4 \varphi^2$ , we can bound

$$\begin{split} \int_{B} |\nabla u|^{6} \varphi_{2}^{2} dz &\leq C \bigg( \int_{B} (|\nabla^{2} u| |\nabla u|^{2} \varphi_{2} + |\nabla u|^{3} |\nabla \varphi_{2}|) dz \bigg)^{2} \\ &\leq C \bigg( \int_{B} |\nabla^{2} u|^{3} \varphi_{2}^{2} dz \bigg)^{2/3} \bigg( \int_{B} |\nabla u|^{3} \varphi_{2}^{1/2} dz \bigg)^{4/3} + C \bigg( \int_{B} |\nabla u|^{3} |\nabla \varphi_{2}| dz \bigg)^{2}, \end{split}$$

where by Hölder's inequality we have

$$\int_{B} |\nabla u|^{3} \varphi_{2}^{1/2} dz \leq \left( \int_{B} |\nabla u|^{6} \varphi_{2}^{2} dz \right)^{1/4} \left( \int_{B} |\nabla u|^{2} \varphi_{1}^{2} dz \right)^{3/4},$$

so that with Young's inequality we obtain

$$\begin{split} \int_{B} |\nabla u|^{6} \varphi_{2}^{2} \, dz &\leq C \delta \bigg( \int_{B} |\nabla^{2} u|^{3} \varphi_{2}^{2} \, dz \bigg)^{2/3} \bigg( \int_{B} |\nabla u|^{6} \varphi_{2}^{2} \, dz \bigg)^{1/3} + C \bigg( \int_{B} |\nabla u|^{3} |\nabla \varphi_{2}| \, dz \bigg)^{2} \\ &\leq \frac{1}{2} \int_{B} |\nabla u|^{6} \varphi_{2}^{2} \, dz + C \int_{B} |\nabla^{2} u|^{3} \varphi_{2}^{2} \, dz + C \bigg( \int_{B} |\nabla u|^{3} |\nabla \varphi_{2}| \, dz \bigg)^{2}. \end{split}$$

With Young's inequality for suitable  $\varepsilon > 0$ , and using (4-9), we then can bound

$$\begin{split} J_{1} &\leq C \| (|\nabla^{2}u|^{3} + 1)\varphi_{2}^{2} \|_{L^{1}(B)} + C \| |\nabla u|^{3} |\nabla \varphi_{2}| \|_{L^{1}(B)}^{2} \\ &\leq \varepsilon \| \nabla^{3}u\varphi_{2} \|_{L^{2}(B)}^{2} + C(1 + \|\nabla^{2}u\varphi_{2}\|_{L^{2}(B)}^{2}) \| \nabla^{2}u\varphi_{1} \|_{L^{2}(B)}^{2} + C \| |\nabla u|^{3} |\nabla \varphi_{2}| \|_{L^{1}(B)}^{2} \\ &\leq \frac{1}{2} \| \nabla \partial_{\phi}^{2}u\varphi_{2} \|_{L^{2}(B)}^{2} + C(1 + \|\nabla \partial_{\phi}u\varphi_{2}\|_{L^{2}(B)}^{2}) \| \nabla \partial_{\phi}u\varphi_{1} \|_{L^{2}(B)}^{2} + C, \end{split}$$

where we also have estimated

$$\begin{aligned} \||\nabla u|^{3}|\nabla \varphi_{2}|\|_{L^{1}(B)}^{2} &\leq C \|\nabla u\varphi_{1}\|_{L^{4}(B)}^{4} \|\nabla u\varphi_{1}\|_{L^{2}(B)}^{2} \\ &\leq C(\|\nabla^{2}u\varphi_{1}\|_{L^{2}(B)}^{2} + E(u))\|\nabla u\varphi_{1}\|_{L^{2}(B)}^{4} \leq C \|\nabla \partial_{\phi}u\varphi_{1}\|_{L^{2}(B)}^{2} + C. \end{aligned}$$

Similarly, with (A-2) we have

$$J_2 \le C \|\nabla^2 u\varphi_1\|_{L^2(B)}^2 + C.$$

Thus, from Lemma 4.9 we obtain

$$\frac{d}{dt}(\|\partial_{\phi}^{2}u\varphi_{2}\|_{L^{2}(\partial B)}^{2}) + \frac{1}{2}\|\nabla\partial_{\phi}^{2}u\varphi_{2}\|_{L^{2}(B)}^{2} \le C(1 + \|\nabla\partial_{\phi}u\varphi_{2}\|_{L^{2}(B)}^{2})\|\nabla\partial_{\phi}u\varphi_{1}\|_{L^{2}(B)}^{2} + C.$$
(4-11)

Denote by  $C_1 = C_1(T, R, u_0) > 0$  a constant bounded by the terms on the right-hand side in the statement of Proposition 4.8. By elliptic regularity, using that  $|\Delta(u\varphi_2)| \le 2|\nabla u \nabla \varphi_2| + C$  we can bound

$$\begin{split} \|\nabla^{2} u\varphi_{2}\|_{L^{2}(B)}^{2} &\leq \|u\varphi_{2}\|_{H^{2}(B)}^{2} + C\|\nabla u\nabla\varphi_{2}\|_{L^{2}(B)}^{2} + C \\ &\leq C\|u\varphi_{2}\|_{H^{2}(\partial B)}^{2} + \|\Delta(u\varphi_{2})\|_{L^{2}(B)}^{2} + C\|\nabla u\nabla\varphi_{2}\|_{L^{2}(B)}^{2} + C \\ &\leq C\|\partial_{\phi}^{2} u\varphi_{2}\|_{L^{2}(\partial B)}^{2} + CE(u) + C_{1}. \end{split}$$

From (4-11) we then obtain the differential inequality

$$\frac{d}{dt}(1+\|\partial_{\phi}^{2}u\varphi_{2}\|_{L^{2}(\partial B)}^{2}) \leq C_{1}(1+\|\partial_{\phi}^{2}u\varphi_{2}\|_{L^{2}(\partial B)}^{2})\|\nabla\partial_{\phi}u\varphi_{1}\|_{L^{2}(B)}^{2}+C_{1};$$

that is,

$$\frac{d}{dt}(\log(1+\|\partial_{\phi}^{2}u\varphi_{2}\|_{L^{2}(\partial B)}^{2})) \leq C_{1}\|\nabla\partial_{\phi}u\varphi_{1}\|_{L^{2}(B)}^{2}+C_{1},$$

and the right-hand side is integrable in time by Proposition 4.8. The claim follows.

We continue by induction.

**Proposition 4.11.** There exists a constant  $\delta > 0$  independent of  $R_0 > 0$  with the following property. Whenever (4-10) holds, then, for any  $k \ge 3$ , any smooth  $u_0 \in H^{1/2}(S^1; N)$ , and any  $T \le T_0$ ,

$$\sup_{0 < t < T} \int_{\partial B} |\partial_{\phi}^{k} u(t)|^{2} \varphi_{k}^{2} d\phi + \int_{0}^{T} \int_{B} |\nabla \partial_{\phi}^{k} u|^{2} \varphi_{k}^{2} dz dt \leq C_{k} \int_{\partial B} |\partial_{\phi}^{k} u_{0}|^{2} \varphi_{k}^{2} d\phi + C_{k},$$

where we denote by  $C_k = C_k(T, R, u_0) > 0$  a constant bounded by the terms on the right-hand side in the statement of the proposition for k - 1.

*Proof.* By Proposition 4.10 the claimed result holds true for k = 2. Suppose the claim holds true for some  $k_0 \ge 2$  and let  $k = k_0+1$ . Note that by elliptic regularity, as in the proof of Proposition 4.10, we can bound

$$\begin{split} \|\nabla^{k} u\varphi_{k}\|_{L^{2}(B)}^{2} &\leq \|u\varphi_{k}\|_{H^{k}(B)}^{2} + C\sum_{j < k} \|\nabla^{j} u\nabla^{k-j}\varphi_{k}\|_{L^{2}(B)}^{2} \\ &\leq C \|u\varphi_{k}\|_{H^{k}(\partial B)}^{2} + C \|\Delta(u\varphi_{k})\|_{H^{k-2}(B)}^{2} + C\sum_{j < k} \|\nabla^{j} u\nabla^{k-j}\varphi_{k}\|_{L^{2}(B)}^{2} \\ &\leq C \|\partial_{\phi}^{k} u\varphi_{k}\|_{L^{2}(\partial B)}^{2} + C\sum_{j < k} \|\nabla^{j} u\nabla^{k-j}\varphi_{k}\|_{L^{2}(B)}^{2} + C_{k}. \end{split}$$

By the induction hypothesis and Sobolev's embedding  $H^2(B) \hookrightarrow W^{1,4} \cap C^0(\overline{B})$  for  $0 \le t < T$ , we then have the uniform bounds

$$\|\nabla^{k_0} u\varphi_{k_0}\|_{L^2(B)}^2 + \|\nabla^{k_0-1} u\varphi_{k_0}\|_{L^4(B)}^2 + \sum_{j=1}^{k_0-2} \|\nabla^j u\varphi_{k_0}\|_{L^\infty(B)}^2 \le C_k$$

and it follows that

$$\|\nabla^{k}u\varphi_{k}\|_{L^{2}(B)}^{2} + \|\nabla^{k_{0}}u\varphi_{k}\|_{L^{4}(B)}^{2} + \|\nabla^{k_{0}-1}u\varphi_{k}\|_{L^{\infty}(B)}^{2} \le C \|\partial_{\phi}^{k}u\varphi_{k}\|_{L^{2}(\partial B)}^{2} + C_{k}.$$

Again let

$$J_{1} := \sum_{\substack{1 \le j_{i} \le k \\ \Sigma_{i} j_{i} = 2k+2}} \left\| \prod_{i} \nabla^{j_{i}} u \varphi_{k}^{2} \right\|_{L^{1}(B)}$$
  
$$\leq \| (|\nabla^{k} u|^{2} (|\nabla^{2} u| + |\nabla u|^{2}) + |\nabla^{k} u| |\nabla^{k_{0}} u| |\nabla^{3} u| + \dots + |\nabla u|^{2k+2}) \varphi_{k}^{2} \|_{L^{1}(B)}$$

and set

$$J_2 = \sum_{\substack{1 \le j_0, j_i \le k \\ \Sigma_i \ge 0, j_i \le k+1}} \left\| \prod_{i>0} \nabla^{j_i} u \nabla^{j_0} \varphi_k \right\|_{L^2(B)}^2$$

Suppose  $k_0 = 2$ . Recalling that  $\varphi_k = \varphi_k \varphi_{k_0}$ , we can bound the terms

$$\begin{split} \||\nabla^{3}u|^{2}(|\nabla^{2}u|+|\nabla u|^{2})\varphi_{3}^{2}\|_{L^{1}(B)} &\leq \|\nabla^{3}u\varphi_{3}\|_{L^{4}(B)}^{2}(\|\nabla^{2}u\varphi_{2}\|_{L^{2}(B)} + \|\nabla u\varphi_{2}\|_{L^{4}(B)}^{2}) \\ &\leq C_{3}\|\nabla\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(B)}\|\nabla^{3}u\varphi_{3}\|_{L^{2}(B)} + C_{3}\|\nabla^{3}u\varphi_{2}\|_{L^{2}(B)}^{2} + C_{3} \\ &\leq C_{3}\|\nabla\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(B)}(\|\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(\partial B)} + 1) + C_{3}\|\nabla\partial_{\phi}^{2}u\varphi_{2}\|_{L^{2}(B)}^{2} + C_{3} \\ &\leq \varepsilon\|\nabla\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(B)}^{2} + C_{3}\|\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(\partial B)}^{2} + C_{3}\|\nabla\partial_{\phi}^{2}u\varphi_{2}\|_{L^{2}(B)}^{2} + C_{3} \end{split}$$

and

$$\||\nabla u|^{8}\varphi_{3}^{2}\|_{L^{1}(B)} \leq \|\nabla u\varphi_{3}\|_{L^{\infty}(B)}^{2}\|\nabla u\varphi_{2}\|_{L^{6}(B)}^{6} \leq C_{3}\|\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(\partial B)}^{2} + C_{3}$$

from the estimate of  $J_1$ . Here we also have used (A-1) and (A-2) to bound

$$\begin{split} \|\nabla u\varphi_2\|_{L^6(B)}^3 &\leq \|\nabla (|\nabla u|^3\varphi_2^3)\|_{L^1(B)} \leq C \|(|\nabla^2 u|\varphi_2 + |\nabla u||\nabla\varphi_2|)|\nabla u|^2\varphi_2^2\|_{L^1(B)} \\ &\leq C (\|\nabla^2 u\varphi_2\|_{L^2(B)} + \|\nabla u\nabla\varphi_2\|_{L^2(B)})\|\nabla u\varphi_2\|_{L^4(B)}^2 \\ &\leq C (\|\nabla^2 u\varphi_2\|_{L^2(B)} + \|\nabla u\nabla\varphi_2\|_{L^2(B)})^2\|\nabla u\varphi_2\|_{L^2(B)} \leq C_3. \end{split}$$

Similarly, we can bound the remaining terms and the terms in  $J_2$  to obtain

$$\frac{d}{dt}(\|\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(\partial B)}^{2}) + \frac{1}{2}\|\nabla\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(B)}^{2} \le C_{3}(1 + \|\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(\partial B)}^{2})(1 + \|\nabla\partial_{\phi}^{2}u\varphi_{2}\|_{L^{2}(B)}^{2}) + C_{3}(1 + \|\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(\partial B)}^{2})(1 + \|\nabla\partial_{\phi}^{2}u\varphi_{3}\|_{L^{2}(B)}^{2}) + C_{3}(1 + \|\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(\partial B)}^{2})(1 + \|\nabla\partial_{\phi}^{2}u\varphi_{3}\|_{L^{2}(B)}^{2}) + C_{3}(1 + \|\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(\partial B)}^{2})(1 + \|\nabla\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(B)}^{2}) + C_{3}(1 + \|\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(\partial B)}^{2})(1 + \|\nabla\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(B)}^{2})(1 + \|\nabla\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(B)}^{2})(1 + \|\nabla\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(B)}^{2}) + C_{3}(1 + \|\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(\partial B)}^{2})(1 + \|\nabla\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}(B)}^{2})(1 + \|\nabla\partial_{\phi}^{3}u\varphi_{3}\|_{L^{2}($$

from Lemma 4.9 and then

$$\frac{d}{dt}(\log(1+\|\partial_{\phi}^{3}u\varphi_{2}\|_{L^{2}(\partial B)}^{2})) \leq C_{3}(1+\|\nabla\partial_{\phi}^{2}u\varphi_{2}\|_{L^{2}(B)}^{2}),$$

where the right-hand side is integrable in time by Proposition 4.10. The claim for k = 3 thus follows.

For  $k \ge 4$  the analysis is similar (but simpler) and may be left to the reader.

#### 5. Local existence

In order to show local existence we approximate the flow equation (1-3) by the equation

$$u_t = -(\varepsilon + d\pi_N(u))u_r \quad \text{on } \partial B, \tag{5-1}$$

where  $\varepsilon > 0$  and where we smoothly extend the nearest-neighbor projection  $\pi_N$ , originally defined only in the  $\rho$ -neighborhood  $N_{\rho}$  of N, to the whole ambient  $\mathbb{R}^n$ . Our aim then is to show that for given smooth initial data  $u_0$  the evolution problem (5-1), (1-4) admits a smooth solution  $u_{\varepsilon}$  which remains uniformly smoothly bounded on a uniform time interval as  $\varepsilon \downarrow 0$ . Fixing some  $0 < \varepsilon < \frac{1}{2}$ , we show existence for the problem (5-1) with data (1-4) by means of a fixed-point argument.

To set up the argument, fix smooth initial data  $u_0: S^1 \to N$  with harmonic extension  $u_0 \in C^{\infty}(\overline{B}; \mathbb{R}^n)$ and some  $k \ge 2$ . For suitable T > 0 to be determined let

$$X = L^{\infty}([0, T]; H^{k+1}(B; \mathbb{R}^n)) \cap H^1(S^1 \times [0, T]; \mathbb{R}^n)$$

and set

(

$$V = \left\{ v \in X : v(0) = u_0, \ \Delta v(t) = 0 \text{ in } B \text{ for } 0 \le t \le T, \\ \|v\|_X^2 = \sup_{0 \le t \le T} \|v(t)\|_{H^{k+1}(B)}^2 + \int_0^T \int_{S^1} |v_t|^2 d\phi \, dt \le 4R_0^2 \right\},$$

where  $R_0 = ||u_0||_{H^{k+1}(B)}$ . We endow the space V with the metric derived from the seminorm

$$|v|_X^2 = \sup_{0 \le t \le T} \|\nabla v(t)\|_{L^2(B)}^2 + \int_0^T \int_{S^1} |v_t|^2 \, d\phi \, dt.$$

Note that this metric is positive definite on V in view of the initial condition that we impose.

Lemma 5.1. V is a complete metric space.

*Proof.* Let  $(v_m)_{m \in \mathbb{N}} \subset V$  with  $|v_l - v_m|_X \to 0$   $(l, m \to \infty)$ . By the theorem of Banach–Alaoglu a subsequence  $v_m \to v$  weakly-\* in  $L^{\infty}([0, T]; H^{k+1}(B))$  with  $v_{m,t} \to v_t$  weakly in  $L^2([0, T] \times S^1)$ , and by weak lower semicontinuity of the norm

$$||v||_X^2 \le \limsup_{m\to\infty} ||v_m||_X^2 \le 4R_0^2.$$

Moreover, we have  $\Delta v(t) = 0$  for all  $0 \le t \le T$  and  $v(0) = u_0$  by compactness of the trace operator  $H^1(S^1 \times [0, T]) \ni u \mapsto u(0) \in L^2(S^1)$ . Hence  $v \in V$ .

Moreover, we have

$$|v_l - v|_X \le \limsup_{m \to \infty} |v_l - v_m|_X \to 0 \quad \text{as } l \to \infty.$$

**Lemma 5.2.** There is  $T_2 > 0$  such that for any  $T \le T_2$  and any  $v \in V$  there is a solution  $u = \Phi(v) \in V$  of *the equation* 

$$u_t = -(\varepsilon + d\pi_N(v))u_r \quad on \ \partial B \times [0, T_2[ \tag{5-2})$$

satisfying (1-4).

*Proof.* For  $v \in V$  we construct a solution  $u = \Phi(v) \in X$  of (5-2) via Galerkin approximation. For this let  $(\varphi_l)_{l \in \mathbb{N}_0}$  be Steklov eigenfunctions of the Laplacian, satisfying

$$\Delta \varphi_l = 0$$
 in *B*

with boundary condition

$$\partial_r \varphi_l = \lambda_l \varphi_l \quad \text{on } \partial B, \qquad l \in \mathbb{N}_0.$$

Note that the Steklov eigenvalues are given by  $\lambda_0 = 0$  and  $\lambda_{2l-1} = \lambda_{2l} = l$ ,  $l \in \mathbb{N}$ . In fact, we may choose  $\varphi_0 \equiv 1/\sqrt{2\pi}$  and

$$\varphi_{2l-1}(re^{i\theta}) = \frac{1}{\sqrt{\pi}} r^l \sin(l\theta), \quad \varphi_{2l}(re^{i\theta}) = \frac{1}{\sqrt{\pi}} r^l \cos(l\theta), \qquad l \in \mathbb{N}$$
(5-3)

to obtain an orthonormal basis for  $L^2(S^1)$  consisting of these functions. Given  $m \in \mathbb{N}$  then let

$$u^{(m)}(t,z) = \sum_{l=0}^{m} a_l^{(m)}(t)\varphi_l(z)$$

solve the system of equations

$$\partial_t a_l^{(m)} = (\varphi_l, u_t^{(m)})_{L^2(S^1)} = -(\varphi_l, (\varepsilon + d\pi_N(v))u_r^{(m)})_{L^2(S^1)}$$
$$= -\sum_{j=0}^m a_j^{(m)} \lambda_j (\varphi_l, (\varepsilon + d\pi_N(v))\varphi_j)_{L^2(S^1)}, \quad 0 \le l \le m.$$
(5-4)

Since for any  $m \in \mathbb{N}$  the coefficients  $\lambda_j(\varphi_l, (\varepsilon + d\pi_N(v))\varphi_j)_{L^2(S^1)}$  of this system are uniformly bounded for any  $v \in V$ , for any  $m \in \mathbb{N}$  there exists a unique solution  $a^{(m)} = (a_l^{(m)})_{0 \le l \le m}$  of (5-4) on [0, T] with initial data  $a_l^{(m)}(0) = a_{l0} = (u_0, \varphi_l)_{L^2(S^1)}, \ 0 \le l \le m$ .

Note that for any  $m \in \mathbb{N}$  and any  $j \in \mathbb{N}_0$  we have

$$\partial_{\phi}^{2j}(ru_r^{(m)}) \in \operatorname{span}\{\varphi_l : 0 \le l \le m\},\$$

and the function  $\partial_{\phi}^{2j} u^{(m)}$  is harmonic. In particular, for j = 0 we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u^{(m)}\|_{L^{2}(B)}^{2}) = \int_{B} \nabla u^{(m)} \nabla u_{t}^{(m)} dz = (u_{r}^{(m)}, u_{t}^{(m)})_{L^{2}(S^{1})} = -(u_{r}^{(m)}, (\varepsilon + d\pi_{N}(\upsilon))u_{r}^{(m)})_{L^{2}(S^{1})}$$
$$= -\varepsilon \|u_{r}^{(m)}\|_{L^{2}(S^{1})}^{2} - \|d\pi_{N}(\upsilon)u_{r}^{(m)}\|_{L^{2}(S^{1})}^{2} \le -\frac{1}{2}\|u_{t}^{(m)}\|_{L^{2}(S^{1})}^{2} \le 0,$$
(5-5)

and we find the uniform  $H^1$ -bound

$$\sup_{t \ge 0} \|\nabla u^{(m)}(t)\|_{L^{2}(B)}^{2} + \varepsilon \|u_{r}^{(m)}\|_{L^{2}([0,\infty[\times S^{1})}^{2} + \|u_{t}^{(m)}\|_{L^{2}([0,\infty[\times S^{1})}^{2} \le 2\|\nabla u^{(m)}(0)\|_{L^{2}(B)}^{2}$$

$$\le 2\|\nabla u_{0}\|_{L^{2}(B)}^{2} \le 2R_{0}^{2}.$$
(5-6)

Moreover, for  $j = k \in \mathbb{N}$  as in the definition of *X*, upon integrating by parts we find

$$\frac{1}{2} \frac{d}{dt} (\|\nabla \partial_{\phi}^{k} u^{(m)}\|_{L^{2}(B)}^{2}) = (-1)^{k} \int_{B} \nabla \partial_{\phi}^{2k} u^{(m)} \nabla u_{t}^{(m)} dz = (-1)^{k} (\partial_{\phi}^{2k} u_{r}^{(m)}, u_{t}^{(m)})_{L^{2}(S^{1})} 
= (-1)^{k+1} (\partial_{\phi}^{2k} u_{r}^{(m)}, (\varepsilon + d\pi_{N}(\upsilon)) u_{r}^{(m)})_{L^{2}(S^{1})} 
= -\varepsilon \|\partial_{\phi}^{k} u_{r}^{(m)}\|_{L^{2}(S^{1})}^{2} - \|d\pi_{N}(\upsilon)\partial_{\phi}^{k} u_{r}^{(m)}\|_{L^{2}(S^{1})}^{2} + I,$$
(5-7)

where  $I = \sum_{j=1}^{k} {k \choose j} I_j$  with

$$I_j = -(\partial_{\phi}^k u_r^{(m)}, \partial_{\phi}^j (d\pi_N(v)) \partial_{\phi}^{k-j} u_r^{(m)})_{L^2(S^1)}$$

similar to the proof of Lemma 4.3. However, now we simply bound

$$|I_{j}| \leq C \sum_{\Sigma_{i} j_{i}=j} \|\partial_{\phi}^{k} u_{r}^{(m)}\|_{L^{2}(S^{1})} \left\| \prod_{i} \partial_{\phi}^{j_{i}} v \partial_{\phi}^{k-j} u_{r}^{(m)} \right\|_{L^{2}(S^{1})}, \quad 1 \leq j \leq k.$$

Note that by compactness of Sobolev's embedding  $H^1(S^1) \hookrightarrow L^{\infty}(S^1)$  and Ehrlich's lemma for any number  $1 \le j \le k$ , any  $\delta > 0$  we can bound

$$\begin{aligned} \|\partial_{\phi}^{k-j}u_{r}^{(m)}\|_{L^{\infty}(S^{1})} &\leq \delta \|\partial_{\phi}^{k-j+1}u_{r}^{(m)}\|_{L^{2}(S^{1})} + C(\delta) \|\partial_{\phi}^{k-j}u_{r}^{(m)}\|_{L^{2}(S^{1})} \\ &\leq 2\delta \|\partial_{\phi}^{k}u_{r}^{(m)}\|_{L^{2}(S^{1})} + C(\delta) \|u_{r}^{(m)}\|_{L^{2}(S^{1})}. \end{aligned}$$

On the other hand, for any  $v \in V$  by the trace theorem we have

$$\|\partial_{\phi}^{k}v\|_{L^{2}(S^{1})} \leq C \|\partial_{\phi}^{k}v\|_{H^{1}(B)} \leq C \|v\|_{H^{k+1}(B)} \leq CR_{0},$$

and we therefore also can bound

$$\|\partial_{\phi}^{j}v\|_{L^{\infty}(S^{1})} \leq C \|\partial_{\phi}^{k}v\|_{L^{2}(S^{1})} + \|\partial_{\phi}^{j}v\|_{L^{2}(S^{1})} \leq C \|v\|_{H^{k+1}(B)} \leq CR_{0}$$

for any  $1 \le j < k$ .

Thus, for sufficiently small  $\delta > 0$  with a constant C > 0 depending on  $\varepsilon > 0$  and  $R_0$ ,

$$|I| \leq \frac{1}{2}\varepsilon \|\partial_{\phi}^{k} u_{r}^{(m)}\|_{L^{2}(S^{1})}^{2} + C \|u_{r}^{(m)}\|_{L^{2}(S^{1})}^{2},$$

and from (5-7) with the help of (3-7) we obtain the inequality

$$\begin{aligned} \frac{d}{dt} (\|\nabla\partial_{\phi}^{k}u^{(m)}\|_{L^{2}(B)}^{2}) &\leq C \|u_{r}^{(m)}\|_{L^{2}(S^{1})}^{2} = C \|u_{\phi}^{(m)}\|_{L^{2}(S^{1})}^{2} \leq C \|u_{\phi}^{(m)}\|_{H^{1}(B)}^{2} \\ &\leq C \|\nabla\partial_{\phi}^{k}u^{(m)}\|_{L^{2}(B)}^{2} + C \|\nabla u^{(m)}\|_{L^{2}(B)}^{2} \leq C(1 + \|\nabla\partial_{\phi}^{k}u^{(m)}\|_{L^{2}(B)}^{2}), \end{aligned}$$

where we recall (5-6) for the last conclusion.

It follows that for suitably small T > 0 there holds  $||u^{(m)}||_X^2 \le 4R_0^2$  for all  $m \in \mathbb{N}$ . Thus, there is a sequence  $m \to \infty$  such that  $u^{(m)} \to u$  weakly-\* in  $L^{\infty}([0, T]; H^{k+1}(B))$  with  $u_t^{(m)} \to u_t$  weakly in  $L^2([0, T] \times S^1)$ , where  $u =: \Phi(v) \in V$  solves (5-2).

**Lemma 5.3.** There is T > 0 such that, for  $v_1, v_2 \in V$ ,

$$|\Phi(v_1) - \Phi(v_2)|_X \le \frac{1}{2}|v_1 - v_2|_X.$$

*Proof.* Let  $T_2 > 0$  be as determined in Lemma 5.2 and fix some  $0 < T \le T_2$ . For  $v_1, v_2 \in V$  then we have  $u_i =: \Phi(v_i) \in V$ , i = 1, 2. Set  $w = u_1 - u_2$  and  $v = v_1 - v_2$ , and compute

$$w_t = -(\varepsilon + d\pi_N(v_1))w_r - (d\pi_N(v_1) - d\pi_N(v_2))u_{2,r} \quad \text{on } \partial B = S^1.$$
(5-8)

Multiplying with  $w_r$  and integrating we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\nabla w\|_{L^{2}(B)}^{2}) = \int_{B} \nabla w \nabla w_{t} \, dx = (w_{r}, w_{t})_{L^{2}(S^{1})}$$
$$= -\varepsilon \|w_{r}\|_{L^{2}(S^{1})}^{2} - \|d\pi_{N}(v_{1})w_{r}\|_{L^{2}(S^{1})}^{2} - (w_{r}, (d\pi_{N}(v_{1}) - d\pi_{N}(v_{2}))u_{2,r})_{L^{2}(S^{1})},$$

where with  $||u_{2,r}||_{L^{\infty}(S^1)} \le C ||u_2||_{H^3(B)} \le CR_0$  we can bound

$$\begin{aligned} |(w_r, (d\pi_N(v_1) - d\pi_N(v_2))u_{2,r})_{L^2(S^1)}| &\leq C ||w_r||_{L^2(S^1)} ||v||_{L^2(S^1)} ||u_{2,r}||_{L^{\infty}(S^1)} \\ &\leq C ||w_r||_{L^2(S^1)} ||v||_{L^2(S^1)} \leq \frac{1}{2}\varepsilon ||w_r||_{L^2(S^1)}^2 + C(\varepsilon) ||v||_{L^2(S^1)}^2. \end{aligned}$$

Thus, with a constant  $C = C(\varepsilon) > 0$  we find

$$\frac{d}{dt} \|\nabla w\|_{L^2(B)}^2 + \varepsilon \|w_r\|_{L^2(S^1)}^2 \le C \|v\|_{L^2(S^1)}^2.$$
(5-9)

Similarly, from (5-8) we can bound

$$\|w_t\|_{L^2(S^1)}^2 \le C \|w_r\|_{L^2(S^1)}^2 + C \|v\|_{L^2(S^1)}^2.$$
(5-10)

Integrating over  $0 \le t \le T$  and observing that we have

$$\sup_{0 \le t \le T} \|v(t)\|_{L^2(S^1)}^2 \le \left(\int_0^T \|v_t(t)\|_{L^2(S^1)} dt\right)^2 \le T \int_0^T \|v_t(t)\|_{L^2(S^1)}^2 dt,$$

from (5-9) we first obtain

$$\sup_{0 \le t \le T} \|\nabla w(t)\|_{L^2(B)}^2 + \varepsilon \|w_r\|_{L^2([0,T] \times S^1)}^2 \le CT \sup_{0 \le t \le T} \|v(t)\|_{L^2(S^1)}^2 \le CT^2 |v|_X^2,$$

which we may use together with (5-10) to bound

$$\|w\|_X^2 = \sup_{0 \le t \le T} \|\nabla w(t)\|_{L^2(B)}^2 + \|w_t\|_{L^2([0,T] \times S^1)}^2 \le CT^2 \|v\|_X^2.$$

For sufficiently small T > 0 then our claim follows.

Thus, by Banach's fixed-point theorem, for any  $\varepsilon > 0$  and any smooth  $u_0 \in H^{1/2}(S^1; N)$ , there exists T > 0 and a solution  $u = u(t) \in V$  of the initial value problem (5-1), (1-4). We now show that the number T > 0 may be chosen uniformly as  $\varepsilon \downarrow 0$ . Indeed, we have the following result.

**Lemma 5.4.** There exists a constant C > 0 such that, for any  $k \ge 2$ , any smooth  $u_0 \in H^{1/2}(S^1; N)$ , and any  $0 < \varepsilon \le \frac{1}{2}$  for the solution u to (5-1) with  $u(0) = u_0$ ,

$$\frac{d}{dt}(\|\nabla\partial_{\phi}^{k}u\|_{L^{2}(B)}^{2}) \leq C(1+\|\nabla u\|_{L^{2}(B)}^{2}+\|\nabla\partial_{\phi}^{k}u\|_{L^{2}(B)})^{k+3}.$$

*Proof.* Similar to the proof of Lemma 5.2, for given  $2 \le k \in \mathbb{N}$  we compute

$$\frac{1}{2} \frac{d}{dt} (\|\nabla \partial_{\phi}^{k} u\|_{L^{2}(B)}^{2}) = (-1)^{k} \int_{B} \nabla \partial_{\phi}^{2k} u \nabla u_{t} \, dx = (-1)^{k} (\partial_{\phi}^{2k} u_{r}, u_{t})_{L^{2}(S^{1})} 
= (-1)^{k+1} (\partial_{\phi}^{2k} u_{r}, (\varepsilon + d\pi_{N}(u))u_{r})_{L^{2}(S^{1})} 
\leq -\|d\pi_{N}(u)\partial_{\phi}^{k} u_{r}\|_{L^{2}(S^{1})}^{2} - I,$$
(5-11)

where we now drop the term  $\varepsilon \|\partial_{\phi}^{k} u_{r}\|_{L^{2}(S^{1})}^{2}$  from (5-7). Again we use the decomposition  $I = \sum_{j=1}^{k} {k \choose j} I_{j}$  with

$$I_j = (\partial_{\phi}^k u_r, \partial_{\phi}^j (d\pi_N(u)) \partial_{\phi}^{k-j} u_r)_{L^2(S^1)} = (\nabla \partial_{\phi}^k u, \nabla (\partial_{\phi}^j (d\pi_N(u)) \partial_{\phi}^{k-j} u_r))_{L^2(B)},$$

but now we bound these terms as in the proof of Lemma 4.3 via

$$\begin{split} I_{j}| &\leq C \|\nabla \partial_{\phi}^{k} u\|_{L^{2}(B)} (\|\nabla \partial_{\phi}^{j}(d\pi_{N}(u))\partial_{\phi}^{k-j} u_{r}\|_{L^{2}(B)} + \|\partial_{\phi}^{j}(d\pi_{N}(u))\nabla \partial_{\phi}^{k-j} u_{r}\|_{L^{2}(B)}) \\ &\leq C \sum_{\substack{1 \leq j_{i} \leq k+1 \\ \Sigma_{i} j_{i} = k+2}} \|\nabla \partial_{\phi}^{k} u\|_{L^{2}(B)} \left\|\prod_{i} \nabla^{j_{i}} u\right\|_{L^{2}(B)}. \end{split}$$

Using that for any  $k \ge 2$  by Sobolev's embedding  $H^2(B) \hookrightarrow W^{1,4} \cap C^0(\overline{B})$  we can bound

$$\sum_{\substack{1 \le j_i \le k+1\\ \Sigma_i \, j_i = k+2}} \left\| \prod_i \nabla^{j_i} u \right\|_{L^2(B)} \le C (1 + \|\nabla u\|_{L^2(B)} + \|\nabla^{k+1} u\|_{L^2(B)})^{k+2}$$

and also using (4-3), we obtain the claim.

We now are able to conclude.

**Proposition 5.5.** For any  $k \ge 2$  and any smooth  $u_0 \in H^{1/2}(S^1; N)$  there exist T > 0 and a solution  $u \in V$  to (1-3) on [0, T] with initial data  $u(0) = u_0$ .

*Proof.* In view of Lemma 5.4, there exists a uniform number T > 0 such that, with V as defined above, for any  $0 < \varepsilon \le \frac{1}{2}$  there exists a solution  $u_{\varepsilon} \in V$  to (5-1) on [0, T]. By definition of V, as  $\varepsilon \downarrow 0$  suitably, we have  $u_{\varepsilon} \to u$  weakly-\* in  $L^{\infty}([0, T]; H^{k+1}(B)) \cap H^1(S^1 \times [0, T])$ . But this suffices to pass to the limit  $\varepsilon \downarrow 0$  in (5-1), and  $u \in V$  solves (1-3) with  $u(0) = u_0$ .

*Proof of Theorem 1.1*(i). By Proposition 5.5 for any smooth  $u_0 \in H^{1/2}(S^1; N)$  and any  $k \ge 2$  there exists T > 0 and a solution  $u \in V$  of (1-3), (1-4) for 0 < t < T. Alternatingly employing Propositions 4.11 and 4.6, we then obtain smoothness of u for  $0 < t \le T$ , including the final time T. (This argument later appears in more detail in Section 6 after Lemma 6.2.) Iterating, the solution u may be extended smoothly until some maximal time  $T_0$  where condition (4-4) ceases to hold. Uniqueness (even within a much larger class of competing functions) is established in Section 7.

### 6. Weak solutions

Given  $u_0 \in H^{1/2}(S^1; N)$ , there are smooth functions  $u_{0k} \in H^{1/2}(S^1; N)$  with  $u_{0k} \to u_0$  in  $H^1(B)$  as  $k \to \infty$ . Indeed, similar to an argument of Schoen and Uhlenbeck [1982, Theorem 3.1], with a standard mollifying sequence  $(\rho_k)_{k\in\mathbb{N}}$  for the mollified functions  $v_{0k} := u_0 * \rho_k$  we have  $\operatorname{dist}_N(v_{0k}) \to 0$  uniformly, and  $u_{0k} := \pi_N(v_{0k}) \to u_0 \in H^{1/2}(S^1; N)$  as  $k \to \infty$ .

Let  $u_k$  be the corresponding solutions of (1-4) with initial data  $u_k(0) = u_{0k}$ , defined on a maximal time interval [0,  $T_k$ [,  $k \in \mathbb{N}$ . We claim that each function  $u_k$  can be smoothly extended to a uniform time interval [0, T[ for some T > 0. To see this, we first establish the following nonconcentration result.

1425

**Lemma 6.1.** For any  $\delta > 0$  there exists a number R > 0 and a time  $T_0 > 0$  such that

$$\sup_{\substack{z_0\in B\\0$$

*Proof.* Given  $\delta > 0$ , by absolute continuity of the Lebesgue integral and  $H^1$ -convergence  $u_{0k} \to u_0$   $(k \to \infty)$  we can find R > 0 such that

$$\sup_{z_0 \in B} \int_{B_{2R}(z_0) \cap B} |\nabla u_{0k}|^2 \, dz < \delta \quad \text{for all } k \in \mathbb{N}.$$

Choosing  $T_0 = \delta R$ , by Lemma 2.2 then we have

$$\sup_{\substack{z_0 \in B \\ 0 < t < T_0}} \int_{B_R(z_0) \cap B} |\nabla u_k(t)|^2 \, dz < 4\delta + C\delta E(u_{k0}) < L\delta$$

with a uniform constant L > 0 for all  $k \in \mathbb{N}$ . The claim follows if we replace  $\delta$  with  $\delta/L$ .

In view of Proposition 3.4, from Lemmas 6.1 and 2.1 we obtain the following bound for  $u_k$  in  $H^1(S^1)$ .

**Lemma 6.2.** There exist a time  $T_0 > 0$  and constants C > 0,  $C_0 = C_0(E(u_0)) > 0$  such that

$$\int_0^{T_0} \int_{S^1} |\partial_{\phi} u_k(t)|^2 \, d\phi \, dt \le C E(u_{k0}) \le C_0 \quad \text{for all } k \in \mathbb{N}$$

From Lemma 6.2 we obtain locally in time uniform smooth bounds for  $(u_k)$  for t > 0 by iteratively applying our previous regularity results. More precisely, Fatou's lemma and Lemma 6.2 first yield the bound

$$\int_0^{T_0} \liminf_{k \to \infty} \left( \int_{S^1} |\partial_{\phi} u_k(t)|^2 \, d\phi \right) dt \le C_0.$$

Thus, for almost every  $0 < t_0 < T_0$ ,

$$\liminf_{k\to\infty}\int_{S^1}|\partial_{\phi}u_k(t_0)|^2\,d\phi<\infty.$$

For any such  $0 < t_0 < T_0$ , if  $\delta > 0$  is sufficiently small, from Proposition 4.2 with another appeal to Fatou's lemma we may conclude

$$\int_{t_0}^{T_0} \liminf_{k \to \infty} \int_B |\nabla \partial_{\phi} u_k|^2 \, dz \, dt \le \liminf_{k \to \infty} \int_{t_0}^{T_0} \int_B |\nabla \partial_{\phi} u_k|^2 \, dz \, dt \le C_1$$

for some  $C_1 > 0$ , so that now we even have

$$\liminf_{k\to\infty}\int_B |\nabla\partial_\phi u_k(t_1)|^2\,dz<\infty$$

for almost every  $t_0 < t_1 < T_0$ . Hence we may next invoke Proposition 4.4 and (4-2) to obtain the bound

$$\liminf_{k\to\infty}\int_{t_1}^{T_0}\int_{\partial B}|\nabla\partial_{\phi}u_k|^2\,dz\,dt<\infty$$

for any such  $t_0 < t_1 < T_0$ , and Fatou's lemma gives

$$\liminf_{k\to\infty}\int_{\partial B}|\nabla\partial_{\phi}u_k(t_2)|^2\,d\phi<\infty$$

for almost every  $t_1 < t_2 < T_0$ . Now Proposition 4.10 may be applied with  $\varphi_0 = 1$ , and we obtain

$$\liminf_{k\to\infty}\int_{t_2}^{T_0}\int_B|\nabla\partial_\phi^2 u_k|^2\,dz\,dt<\infty$$

for any such  $t_1 < t_2 < T_0$ . Another application of Fatou's lemma gives

$$\liminf_{k\to\infty}\int_B|\nabla\partial_\phi^2 u_k(t_3)|^2\,dz<\infty$$

for almost every  $t_2 < t_3 < T_0$ , and Proposition 4.5 yields

$$\liminf_{k\to\infty}\int_{t_3}^{T_0}\int_{\partial B}|\nabla\partial_{\phi}^2 u_k|^2\,d\phi\,dt<\infty$$

for any such  $t_2 < t_3 < T_0$ . We may then iterate, using (3-7) and alternatingly employing Propositions 4.11 and 4.6 for  $3 \le k \in \mathbb{N}$ , to find a subsequence  $(u_k)$  satisfying uniform smooth bounds on  $]t_0, T_0]$  for any  $t_0 > 0$ . Passing to the limit as  $k \to \infty$  for this subsequence we obtain a weak solution to (1-3), (1-4) of energy-class in the following sense.

**Definition 6.3.** A function  $u \in H^1([0, T_0] \times S^1; N) \cap L^{\infty}([0, T_0]; H^{1/2}(S^1; N))$  with harmonic extension u = u(t) for each t is a weak solution of (1-3), (1-4) of energy-class, if (1-3) is satisfied in the weak sense, that is, if

$$\int_0^{T_0} \int_{\partial B} (u_t + d\pi_N(u)u_r) \cdot \varphi \, d\phi \, dt = \int_0^{T_0} \int_{\partial B} u_t \cdot \varphi \, d\phi \, dt + \int_0^{T_0} \int_B \nabla u \cdot \nabla (d\pi_N(u)\varphi) \, dz \, dt = 0 \quad (6-1)$$

for all  $\varphi \in C_c^{\infty}(\overline{B} \times ]0, T_0[$ ), and if there holds the energy inequality

$$E(u(T)) + \int_{S}^{T} \int_{\partial B} |u_t|^2 \, d\phi \, dt \le E(u(S)) \tag{6-2}$$

for any  $0 \le S < T < T_0$ , with the initial data  $u_0 \in H^{1/2}(S^1; N)$  being attained in the sense of traces.

We then may summarize our results as follows.

**Proposition 6.4.** For any  $u_0 \in H^{1/2}(S^1; N)$  there exists  $T_0 > 0$  and a weak solution u to (1-3), (1-4) on  $[0, T_0]$  of energy-class, which is smooth for t > 0.

*Proof.* For any open  $U \subset [0, T_0[$  we have uniform smooth bounds for  $u_k$  on U; thus a suitable subsequence  $u_k$  approaches u smoothly locally as  $k \to \infty$ . Equation (6-1) follows from the corresponding identities for  $u_k$ .

Moreover, (6-2) follows from the energy identity, Lemma 2.1, for  $u_k$ , where we also use  $H^1$ -convergence  $u_{0k} \rightarrow u_0$  as well as weak lower semicontinuity of the energy and of the  $L^2$ -norm.

Finally, with error  $o(1) \rightarrow 0$  as  $k \rightarrow \infty$  for  $0 < t < T_0$  we can estimate

$$\begin{aligned} \|u(t) - u_0\|_{L^2(\partial B)}^2 &\leq \|u_k(t) - u_{0k}\|_{L^2(\partial B)}^2 + o(1) \leq \left(\int_0^t \|\partial_t u_k(t')\|_{L^2(\partial B)} dt'\right)^2 + o(1) \\ &\leq t \int_0^t \|\partial_t u_k(t')\|_{L^2(\partial B)}^2 dt' + o(1) \leq t E(u_0) + o(1), \end{aligned}$$

and  $u(t) \to u_0$  weakly in  $H^{1/2}(S^1; N) \cap H^1(B; \mathbb{R}^n)$  as  $t \downarrow 0$ . In fact, by (6-2) we then even have strong convergence.

# 7. Uniqueness

With the help of the tools developed in Section 3 we can show uniqueness of partially regular weak energy-class solutions as in Proposition 6.4.

**Theorem 7.1.** Let  $u_0 \in H^{1/2}(S^1; N)$ . Suppose u and v both are weak energy-class solutions of (1-3), (1-4) on  $[0, T_0]$  for some  $T_0 > 0$  with initial data  $u_0$ , and suppose u and v are smooth for t > 0. Then u = v.

*Proof.* Using (3-2) for u and v, for the function w = u - v for almost every  $0 < t < T_0$  we have

$$\partial_t w + \partial_r w = \nu(u)\partial_r(\operatorname{dist}_N(u)) - \nu(v)\partial_r(\operatorname{dist}_N(v))$$
  
=  $(\nu(u) - \nu(v))\partial_r(\operatorname{dist}_N(u)) + \nu(v)\partial_r(\operatorname{dist}_N(u) - \operatorname{dist}_N(v))$  (7-1)

on  $\partial B = S^1$ . From (3-5), moreover, we obtain

$$|\Delta(\operatorname{dist}_{N}(u) - \operatorname{dist}_{N}(v))| = |\nabla u \cdot dv(u)\nabla u - \nabla v \cdot dv(v)\nabla v|$$
  
$$\leq C(|w||\nabla u|^{2} + (|\nabla u| + |\nabla v|)|\nabla w|) \quad \text{in } B.$$
(7-2)

Observing that

$$|\operatorname{dist}_N(u) - \operatorname{dist}_N(v)| \le C|w|,$$

upon multiplying (7-2) with the function  $(\text{dist}_N(u) - \text{dist}_N(v)) \in H_0^1(B)$ , integrating by parts, and using Young's inequality, for any  $\varepsilon > 0$  we obtain

$$\begin{aligned} \|\nabla(\operatorname{dist}_{N}(u) - \operatorname{dist}_{N}(v))\|_{L^{2}(B)}^{2} &\leq C \int_{B} (|w|^{2} |\nabla u|^{2} + (|\nabla u| + |\nabla v|) |\nabla w| |w|) \, dz \\ &\leq \varepsilon \|\nabla w\|_{L^{2}(B)}^{2} + C(\varepsilon) \|w\|_{L^{4}(B)}^{2} (\|\nabla u\|_{L^{4}(B)}^{2} + \|\nabla v\|_{L^{4}(B)}^{2}). \end{aligned}$$
(7-3)

On the other hand, for any  $0 < t_0 < T \le T_0$ , multiplying (7-1) with w and integrating by parts on  $S^1 \times [t_0, T]$ , upon letting  $t_0 \downarrow 0$  we find

$$\begin{split} \sup_{0 < t < T} \|w(t)\|_{L^{2}(\partial B)}^{2} + \int_{0}^{T} \int_{B} |\nabla w|^{2} dz dt \\ &\leq C \int_{0}^{T} \int_{\partial B} (\partial_{t} w + \partial_{r} w) w \, d\phi \, dt \\ &= C \int_{0}^{T} \int_{\partial B} w(v(u) - v(v)) \partial_{r} (\operatorname{dist}_{N}(u)) \, d\phi \, dt + C \int_{0}^{T} \int_{\partial B} wv(v) \partial_{r} (\operatorname{dist}_{N}(u) - \operatorname{dist}_{N}(v)) \, d\phi \, dt \\ &=: C \int_{0}^{T} (I + II) \, dt. \end{split}$$

We first estimate the term

$$I = I(t) = \int_{\partial B} w(v(u) - v(v))\partial_r(\operatorname{dist}_N(u)) d\phi$$
  
= 
$$\int_B \nabla(w(v(u) - v(v)))\nabla(\operatorname{dist}_N(u)) dz + \int_B w(v(u) - v(v))\Delta(\operatorname{dist}_N(u)) dz.$$

Using

$$\begin{aligned} |\nabla(w(v(u) - v(v)))| &\leq C |\nabla w| |w| + |w((dv(u) - dv(v))\nabla u + dv(v)\nabla w)| \\ &\leq C(|\nabla w| |w| + |w|^2 |\nabla u|) \end{aligned}$$

we can bound

$$\left| \int_{B} \nabla (w(\nu(u) - \nu(v))) \nabla (\operatorname{dist}_{N}(u)) \, dz \right| \leq C \int_{B} (|\nabla w| |w| + |w|^{2} |\nabla u|) |\nabla u| \, dz$$
$$\leq \varepsilon \|\nabla w\|_{L^{2}(B)}^{2} + C(\varepsilon) \|w\|_{L^{4}(B)}^{2} \|\nabla u\|_{L^{4}(B)}^{2}$$

for each t. Also using (3-5), we can moreover estimate

$$\left| \int_{B} w(v(u) - v(v)) \Delta(\operatorname{dist}_{N}(u)) \, dz \right| \le C \|w\|_{L^{4}(B)}^{2} \|\nabla u\|_{L^{4}(B)}^{2}$$

for almost every 0 < t < T to obtain

$$|I| \le \varepsilon \|\nabla w\|_{L^{2}(B)}^{2} + C(\varepsilon) \|w\|_{L^{4}(B)}^{2} \|\nabla u\|_{L^{4}(B)}^{2}.$$

Similarly, we estimate the term

$$II = II(t) = \int_{\partial B} wv(v)\partial_r (\operatorname{dist}_N(u) - \operatorname{dist}_N(v)) \, d\phi$$
$$= \int_B \nabla(wv(v))\nabla(\operatorname{dist}_N(u) - \operatorname{dist}_N(v)) \, dz + \int_B wv(v)\Delta(\operatorname{dist}_N(u) - \operatorname{dist}_N(v)) \, dz.$$

Noting that with (7-3) we can bound

$$\left| \int_{B} \nabla(wv(v)) \nabla(\operatorname{dist}_{N}(u) - \operatorname{dist}_{N}(v)) dz \right| \leq C(\|\nabla w\|_{L^{2}(B)} + \|w\nabla v\|_{L^{2}(B)}) \|\nabla(\operatorname{dist}_{N}(u) - \operatorname{dist}_{N}(v))\|_{L^{2}(B)}$$
$$\leq \varepsilon \|\nabla w\|_{L^{2}(B)}^{2} + C(\varepsilon) \|w\|_{L^{4}(B)}^{2} (\|\nabla u\|_{L^{4}(B)}^{2} + \|\nabla v\|_{L^{4}(B)}^{2})$$

and that with (7-2) we have

$$\begin{aligned} \left| \int_{B} wv(v) \Delta(\operatorname{dist}_{N}(u) - \operatorname{dist}_{N}(v)) \, dz \right| &\leq C \int_{B} (|w|^{2} |\nabla u|^{2} + |w| |\nabla w| (|\nabla u| + |\nabla v|)) \, dz \\ &\leq \varepsilon \|\nabla w\|_{L^{2}(B)}^{2} + C(\varepsilon) \|w\|_{L^{4}(B)}^{2} (\|\nabla u\|_{L^{4}(B)}^{2} + \|\nabla v\|_{L^{4}(B)}^{2}), \end{aligned}$$

we find the estimate

$$|II| \le \varepsilon \|\nabla w\|_{L^{2}(B)}^{2} + C(\varepsilon) \|w\|_{L^{4}(B)}^{2} (\|\nabla u\|_{L^{4}(B)}^{2} + \|\nabla v\|_{L^{4}(B)}^{2})$$

for almost every 0 < t < T.

But Sobolev's embedding  $H^{1/2}(B) \hookrightarrow L^4(B)$  and Fourier expansion give the bound

$$\|w\|_{L^4(B)}^2 \le C \|w\|_{H^{1/2}(B)}^2 \le C \|w\|_{L^2(\partial B)}^2$$

and similar bounds for  $\nabla u$  as well as  $\nabla v$ . Moreover, since by the energy inequality (6-2) we have  $u(t), v(t) \rightarrow u_0$  strongly in  $H^1(B)$  as  $t \downarrow 0$ , there exist a radius  $0 < R \le \frac{1}{2}$  and a time  $0 < T < T_0$  such that condition (3-13) in Proposition 3.4 holds true on [0, T] for both u and v, allowing us to bound

$$\begin{aligned} \int_0^T \|\nabla u(t)\|_{L^4(B)}^2 \, dt &\leq C \int_0^T \|\nabla u(t)\|_{L^2(\partial B)}^2 \, dt \leq C \int_0^T \|\partial_{\phi} u(t)\|_{L^2(\partial B)}^2 \, dt \\ &\leq C \int_0^T \int_{\partial B} |u_t|^2 \, d\phi \, dt + C(R) T E(u_0) \leq C(R) (1+T_0) E(u_0) \end{aligned}$$

with the help of (3-7), and similarly for  $|\nabla v|$ . Choosing  $\varepsilon = \frac{1}{4}$ , for sufficiently small  $0 < T < T_0$  by absolute continuity of the integral we thus can estimate

$$\begin{split} \sup_{0 < t < T} \|w(t)\|_{L^{2}(\partial B)}^{2} + \int_{0}^{T} \int_{B} |\nabla w|^{2} dz dt \\ &\leq \frac{1}{2} \|\nabla w\|_{L^{2}(B \times [0,T])}^{2} + C \sup_{0 < t < T} \|w(t)\|_{L^{2}(\partial B)}^{2} \int_{0}^{T} (\|\nabla u\|_{L^{4}(B)}^{2} + \|\nabla v\|_{L^{4}(B)}^{2}) dt \\ &\leq \frac{1}{2} \left( \sup_{0 < t < T} \|w(t)\|_{L^{2}(\partial B)}^{2} + \int_{0}^{T} \int_{B} |\nabla w|^{2} dz dt \right), \end{split}$$

and it follows that w = 0, as claimed.

*Proof of Theorem 1.2.* Existence for short time and uniqueness of a partially regular weak solution to (1-3), (1-4) for given data  $u_0 \in H^{1/2}(S^1; N)$  follow from Proposition 6.4 and Theorem 7.1, respectively. Since by Proposition 6.4 our weak solution is smooth for t > 0, the remaining assertions follow from Theorem 1.1.

Note that at any blow-up time  $T_{i-1}$ ,  $i \ge 1$ , of the flow as in Theorem 1.1(ii) there exists a unique weak limit  $u_i = \lim_{t \uparrow T_{i-1}} u(t) \in H^{1/2}(S^1; N)$ , and we may uniquely continue the flow using Proposition 6.4.  $\Box$ 

#### 8. Blow-up

Preparing for the proof of part (ii) of Theorem 1.1, suppose now that for the solution constructed in part (i) of that theorem there holds  $T_0 < \infty$ . Then, as we shall see in more detail below, by the results in Section 4 condition (4-4) must be violated for  $T = T_0$  and there exist  $\delta > 0$  and points  $z_k \in B$  as well as radii  $r_k \downarrow 0$  as  $k \to \infty$  such that, for suitable  $t_k \uparrow T_0$ ,

$$\int_{B_{r_k}(z_k)\cap B} |\nabla u(t_k)|^2 dz = \sup_{\substack{z_0\in B\\t\leq t_k}} \int_{B_{r_k}(z_0)\cap B} |\nabla u(t)|^2 dz = \delta.$$

We may later choose a smaller constant  $\delta > 0$ , if necessary. Moreover, for later use from now on we consider local concentrations in the sense that, for some  $z_0 \in B$  and some fixed radius  $r_0 > 0$  for a

1430

sequence of points  $z_k \in B$  with  $z_k \to z_0$  and radii  $r_k \downarrow 0$  for suitable  $t_k \uparrow T_0$  as  $k \to \infty$ ,

$$\int_{B_{r_k}(z_k)\cap B} |\nabla u(t_k)|^2 dz = \sup_{\substack{z'\in B_{r_0}(z_0)\\t\leq t_k}} \int_{B_{r_k}(z')\cap B} |\nabla u(t)|^2 dz = \delta.$$

Scale

$$u_k(z,t) = u(z_k + r_k z, t_k + r_k t)$$

for

$$z \in \Omega_k = \{z : z_k + r_k z \in B\}, \quad t \in I_k = \{t : 0 \le t_k + tr_k < T_0\}$$

Note that then

$$\int_{B_1(0)\cap\Omega_k} |\nabla u_k(0)|^2 dz = \sup_{\substack{z_k + r_k z' \in B_{r_0}(z_0) \\ -t_k/r_k \le t < 0}} \int_{B_1(z')\cap\Omega_k} |\nabla u_k(t)|^2 dz = \delta.$$
(8-1)

Passing to a subsequence we may assume that the domains  $\Omega_k$  exhaust a limit domain  $\Omega_{\infty} \subset \mathbb{R}^2$ , which either is the whole space  $\mathbb{R}^2$  or a half-space *H*.

By the energy inequality Lemma 2.1 for  $t \in I_k$ ,

$$\int_{\Omega_k} |\nabla u_k(t)|^2 \, dz = \int_B |\nabla u(t_k + r_k t)|^2 \, dz \le 2E(u_0), \tag{8-2}$$

and for any  $t_0 < 0$  and sufficiently large  $k \in \mathbb{N}$  we have

$$\int_{t_0}^{0} \int_{\partial \Omega_k} |\partial_t u_k|^2 \, ds \, dt = \int_{t_0}^{0} \int_{\partial \Omega_k} |d\pi_N(u_k)\partial_{\nu_k} u_k|^2 \, ds \, dt$$
$$= \int_{t_k+r_k t_0}^{t_k} \int_{\partial B} |u_t|^2 \, d\phi \, dt \le \int_{t_k+r_k t_0}^{T_0} \int_{\partial B} |u_t|^2 \, d\phi \, dt \to 0$$
(8-3)

as  $k \to \infty$ , where ds is the element of length and where  $v_k$  is the outward unit normal along  $\partial \Omega_k$ . Expressing the harmonic functions  $\partial_t u_k(t)$  in Fourier series for each t < 0, it then also follows that  $\partial_t u_k \to 0$  locally in  $L^2$  on  $\Omega_{\infty} \times ]-\infty$ , 0[. Finally, again using the fact that  $u_k(t)$  for each t is harmonic, by the maximum principle we have the uniform bound  $|u_k| \leq \sup_{p \in N} |p|$  as well as uniform smooth bounds locally away from the boundary of  $\Omega_{\infty}$ .

Hence we may assume that as  $k \to \infty$  we have  $u_k \to u_\infty$  weakly locally in  $H^1$  on  $\Omega_\infty \times ]-\infty, 0[$ , where  $u_\infty(z, t) = u_\infty(z)$  is independent of time, harmonic, and bounded. Moreover, we have smooth convergence away from  $\partial \Omega_\infty$ . Thus, if we assume that  $\Omega_\infty = \mathbb{R}^2$ , by (8-1) it follows that

$$\int_{B_1(0)} |\nabla u_\infty|^2 \, dz = \delta.$$

But any function  $v \colon \mathbb{R}^2 \to \mathbb{R}$  which is bounded and harmonic must be constant, which rules out this possibility. Hence  $\Omega_{\infty}$  can only be a half-space.

After a suitable rotation of the domain *B* and shift of coordinates in  $\mathbb{R}^2 \cong \mathbb{C}$  we may then assume that  $z_k = (0, -y_k)$  with  $1 - y_k \leq Mr_k$  for some  $M \in \mathbb{N}$  and that  $\Omega_{\infty} = \{(x, y) : y > y_0\}$  for some  $y_0$ . Finally,

replacing  $r_k > 0$  with  $(M + 1)r_k$  and  $z_k$  with  $z_k = (0, -1)$ , if necessary, we may assume that

$$\Omega_k \subset \mathbb{R}^2_+ = \{(x, y) : y > 0\}$$

is the ball of radius  $1/r_k$  around the point  $(0, 1/r_k)$  with  $0 \in \partial \Omega_k$ , while from (8-1) with a uniform number  $L \in \mathbb{N}$  we have

$$L \int_{B_1(0)\cap\Omega_k} |\nabla u_k(0)|^2 dz \ge L\delta \ge \sup_{\substack{|z'| \le r_0/r_k \\ -t_k/r_k \le t < 0}} \int_{B_1(z')\cap\Omega_k} |\nabla u_k(t)|^2 dz$$
(8-4)

for any  $k \in \mathbb{N}$ . Let  $\Phi_k \colon \mathbb{R}^2_+ \to \Omega_k$  be the conformal maps given by

$$\Phi_k(z) = \frac{2z}{1 - ir_k z}, \quad z \in \mathbb{R}^2_+, \ k \in \mathbb{N},$$

with  $\Phi_k \to 2 \cdot id$  locally uniformly on  $\mathbb{R}^2 \cong \mathbb{C}$  as  $k \to \infty$ .

Let  $v_k = u_k \circ \Phi_k$  with  $k \in \mathbb{N}$ . By conformal invariance of the Dirichlet energy, from (8-2) for any *t* we have

$$\int_{\mathbb{R}^2_+} |\nabla v_k(t)|^2 \, dz = \int_{\Omega_k} |\nabla u_k(t)|^2 \, dz \le 2E(u_0), \tag{8-5}$$

and by (8-4) with a uniform number  $L_1 \in \mathbb{N}$  there holds

$$L_1 \int_{B_2^+(0)} |\nabla v_k(0)|^2 dz \ge L_1 \delta \ge \sup_{\substack{|z'| \le r_0/r_k \\ -t_k/r_k \le t < 0}} \int_{B_1^+(z')} |\nabla v_k(t)|^2 dz,$$
(8-6)

where

$$B_r^+(z) = B_r(z) \cap \mathbb{R}^2_+$$

for any r > 0 and any  $z = (x, y) \in \mathbb{R}^2$ . Moreover, from (8-3) for any  $t_0 < 0$  and any R > 0 for the integral over ]-R,  $R[ \times \{0\} \subset \partial \mathbb{R}^2_+$  we obtain

$$\int_{t_0}^0 \int_{-R}^R |\partial_t v_k|^2 \, dx \, dt \le C \int_{t_0}^0 \int_{-R}^R |d\pi_N(v_k)\partial_y v_k|^2 \, dx \, dt \to 0 \quad \text{as } k \to \infty, \tag{8-7}$$

and  $\partial_t v_k \to 0$  locally in  $L^2$  on  $\mathbb{R}^2_+ \times ]-\infty$ , 0[. In addition, from our choice of  $(u_k)$  it follows that  $v_k \to v_\infty$  weakly locally in  $H^1$  on  $\mathbb{R}^2_+ \times ]-\infty$ , 0[ as  $k \to \infty$ , where  $v_\infty(z, t) =: w_\infty(z)$  is harmonic and bounded.

For a suitable sequence of times  $t_0 < s_k < 0$ , we also have locally weak convergence  $w_k := v_k(s_k) \rightarrow w_{\infty}$ in  $H^1$  on  $\overline{\mathbb{R}}^2_+$  and, in addition,

$$d\pi_N(w_k)\partial_y w_k \to 0 \quad \text{in } L^2_{\text{loc}}(\partial \mathbb{R}^2_+) \quad \text{as } k \to \infty.$$
 (8-8)

Thus, for sufficiently small  $\delta > 0$ , by Proposition 3.3 applied to the functions  $w_k \circ \Psi$ , where  $\Psi : B \to \mathbb{R}^2_+$ is a suitable conformal map, we also have uniform local  $L^2$ -bounds for  $\partial_x w_k$  on  $\partial \mathbb{R}^2_+$ , and we may assume that  $w_k \to w_\infty$  locally uniformly and weakly locally in  $H^1$  on  $\partial \mathbb{R}^2_+$  as  $k \to \infty$ . Since  $w_k$  is harmonic, we then also have locally strong  $H^1$ -convergence  $w_k \to w_\infty$  on  $\overline{\mathbb{R}}^2_+$ .

To see that  $w_{\infty}$  is nonconstant, let  $\varphi_k = \varphi_{z_0,4r_k}$ ,  $k \in \mathbb{N}$ . Integrating the identity (2-1) from the proof of Lemma 2.2 in time, with error  $o(1) \to 0$  and suitable numbers  $\varepsilon_k \downarrow 0$  as  $k \to \infty$  in view of (8-3), we find

$$\frac{1}{2} \left| \int_{B} |\nabla u(t_{k})|^{2} \varphi_{k}^{2} dz - \int_{B} |\nabla u(t_{k} + r_{k} s_{k})|^{2} \varphi_{k}^{2} dz \right| \\
\leq \int_{t_{k} + r_{k} s_{k}}^{t_{k}} \int_{\partial B} |u_{t}|^{2} \varphi_{k}^{2} d\phi dt + 2 \int_{t_{k} + r_{k} s_{k}}^{t_{k}} \int_{B} |u_{t} \nabla u \varphi_{k} \nabla \varphi_{k}| dz dt \\
\leq o(1) + 8\varepsilon_{k} r_{k} \int_{t_{k} + r_{k} s_{k}}^{t_{k}} \int_{B} |\nabla u|^{2} |\nabla \varphi_{k}|^{2} dz dt + (8\varepsilon_{k} r_{k})^{-1} \int_{t_{k} + r_{k} s_{k}}^{t_{k}} \int_{B} |u_{t}|^{2} \varphi_{k}^{2} dz dt. \quad (8-9)$$

With the help of (2-2) and (8-3) for suitable  $\varepsilon_k \downarrow 0$  we can bound

$$(8\varepsilon_k r_k)^{-1} \int_{t_k+r_k s_k}^{t_k} \int_B |u_t|^2 \varphi_k^2 dz dt \leq C\varepsilon_k^{-1} \int_{t_k+r_k s_k}^{t_k} \int_{\partial B} |u_t|^2 dz dt \to 0.$$

Since for any choice  $t_0 < s_k < 0$  we also can estimate

$$8\varepsilon_k r_k \int_{t_k+r_k s_k}^{t_k} \int_B |\nabla u|^2 |\nabla \varphi_k|^2 \, dz \, dt \leq C\varepsilon_k |t_0| E(u_0) \to 0,$$

from (8-9) and (8-6) it follows that with error  $o(1) \rightarrow 0$  as  $k \rightarrow \infty$  we have

$$\int_{B_{4}^{+}(0)} |\nabla w_{k}|^{2} dz + o(1) = \int_{B_{4}^{+}(0)} |\nabla v_{k}(s_{k})|^{2} dz + o(1) \ge \int_{B} |\nabla u(t_{k} + r_{k}s_{k})|^{2} \varphi_{k}^{2} dz + o(1)$$
$$\ge \int_{B} |\nabla u(t_{k})|^{2} \varphi_{k}^{2} dz \ge \int_{B_{2}^{+}(0)} |\nabla v_{k}(0)|^{2} dz \ge \delta.$$
(8-10)

Finally, in view of locally uniform convergence  $w_k \to w_\infty$  and weak local  $L^2$ -convergence of the traces  $\nabla w_k \to \nabla w_\infty$  on  $\partial \mathbb{R}^2_+$ , we may pass to the limit  $k \to \infty$  in (8-8) to conclude that  $w_\infty : \partial \mathbb{R}^2_+ \to N$  with

$$d\pi_N(w_\infty)\partial_y w_\infty = 0 \quad \text{on } \partial \mathbb{R}^2_+. \tag{8-11}$$

Since  $w_{\infty}$  is harmonic, the Hopf differential

$$f = |\partial_x w_{\infty}|^2 - |\partial_y w_{\infty}|^2 - 2i \partial_x w_{\infty} \cdot \partial_y w_{\infty}$$

defines a holomorphic function  $f \in L^1(\mathbb{R}^2_+, \mathbb{C})$ . Moreover,  $w_{\infty} \in H^{3/2}_{\text{loc}}(\overline{\mathbb{R}}^2_+)$  with trace  $\nabla w_{\infty} \in L^2_{\text{loc}}(\partial \mathbb{R}^2_+)$ ; thus also the trace of f is well-defined on  $\partial \mathbb{R}^2_+$ . By (8-11) now the trace of f is real-valued; thus  $f \equiv c$ for some constant  $c \in \mathbb{R}$ . But  $\nabla w_{\infty} \in L^2(\mathbb{R}^2_+)$ ; hence  $f \in L^1(\mathbb{R}^2_+)$ . It follows that c = 0, and  $w_{\infty}$  is conformal.

With a conformal diffeomorphism  $\Phi: B \to \mathbb{R}^2_+$  mapping a point  $z_0 \in \partial B$  to infinity, define the map  $\bar{u} = w_\infty \circ \Phi \in H^{1/2}(S^1; N)$ . By conformal invariance,  $\bar{u}$  again is harmonic with finite Dirichlet integral and satisfies (1-6) on  $\partial B \setminus \{z_0\}$ ; since the point  $\{z_0\}$  has vanishing  $H^1$ -capacity,  $\bar{u}$  then is stationary in the sense of [Grüter et al. 1981]. Moreover,  $\bar{u}$  is conformal. For such mappings, smooth regularity on  $\bar{B}$  was shown by Grüter, Hildebrandt, and Nitsche [Grüter et al. 1981]; thus condition (1-6) holds everywhere on

 $\partial B$  in the pointwise sense, and  $\bar{u}$  parametrizes a minimal surface of finite area supported by N which meets N orthogonally along its boundary.

*Proof of Theorem 1.1*(ii). For given smooth data  $u_0 \in H^{1/2}(S^1; N)$  let u be the unique solution to (1-3), (1-4) guaranteed by part (i) of the theorem, and suppose that the maximal time of existence  $T_0$  is less than  $\infty$ . Then condition (4-4) must fail as  $t \uparrow T_0$ ; else from Propositions 4.11 and 4.6 we obtain smooth bounds for u(t) as  $t \uparrow T_0$  and there exists a smooth trace  $u_1 = \lim_{t \uparrow T_0} u(t)$ . But by the first part of the theorem there is a smooth solution to the initial value problem for (1-3) with initial data  $u_1$  at time  $T_0$ , and this solution extends the original solution u to an interval  $[0, T_1[$  for some  $T_1 > T_0$ , contradicting the maximality of  $T_0$ .

Let  $z^{(i)} \in B$ ,  $1 \le i \le i_0$ , such that, for some number  $\delta > 0$  and suitable  $t_k^{(i)} \uparrow T_0$ ,  $z_k^{(i)} \to z^{(i)}$ ,  $r_k^{(i)} \to 0$  as  $k \to \infty$ ,

$$\liminf_{k\to\infty}\int_{B_{r_k^{(i)}}(z_k^{(i)})\cap B}|\nabla u(t_k^{(i)})|^2\,dz\geq\delta.$$

By the argument following (8-9), for a suitable sequence of radii  $0 < r_k^{(0)} \to 0$  such that  $r_k^{(i)}/r_k^{(0)} \to 0$  as well as  $(T_0 - t_k^{(i)})/r_k^{(0)} \to 0$ , then with error  $o(1) \to 0$  as  $k \to \infty$ ,

$$\int_{B_{r_k^{(0)}(z^{(i)})\cap B}} |\nabla u(t)|^2 \, dz + o(1) \ge \int_{B_{r_k^{(i)}(z_k^{(i)})\cap B}} |\nabla u(t_k^{(i)})|^2 \, dz \ge \delta$$

for all  $T_0 - r_k^{(0)} < t < T_0$ , uniformly in  $1 \le i \le i_0$ . For sufficiently large  $k \in \mathbb{N}$  such that

$$r_k^{(0)} < \inf_{i < j} \frac{1}{4} |z^{(i)} - z^{(j)}|$$

it follows that  $i_0 \le E(u_0)/\delta$ , and we may fix  $r_0 > 0$  and redefine  $t_k^{(i)}$ ,  $r_k^{(i)}$ , and  $z_k^{(i)}$ , if necessary, such that, for each  $1 \le i \le i_0$ ,

$$\int_{B_{r_k^{(i)}(z_k^{(i)})\cap B}} |\nabla u(t_k^{(i)})|^2 \, dz = \sup_{\substack{z' \in B_{r_0}(z^{(i)})\\0 < t \le t_k^{(i)}}} \int_{B_{r_k^{(i)}(z')\cap B}} |\nabla u(t)|^2 \, dz = \delta.$$

Moreover, we may assume that  $\delta < \delta_0$  as defined in Proposition 3.1. The characterization of the concentration points as in Theorem 1.2(ii) via solutions  $\bar{u}^{(i)}$  of (1-6) then follows from our above analysis.

In addition, Corollary 3.2 yields the uniform lower bound

$$\liminf_{r_0 \downarrow 0} \liminf_{t \uparrow T} \int_{B_{r_0}(z^{(i)}) \cap B} |\nabla u(t)|^2 \, dz \ge 2E(\bar{u}^{(i)}) \ge 2\delta_0^2$$

for the concentration energy quanta, which gives the claimed upper bound for the total number of concentration points.

Finally, with the help of Proposition 4.11 we can smoothly extend the solution u to  $B \setminus \{z^{(1)}, \ldots, z^{(i_0)}\}$  at time  $t = T_0$ .

## 9. Asymptotics

Suppose next that the solution *u* to (1-3), (1-4) exists for all time  $0 < t < \infty$ . Then *u* either concentrates for suitable  $t_k \uparrow \infty$  in the sense that condition (4-4) does not hold true uniformly in time, or *u* satisfies uniform smooth bounds, as shown in Section 4.

In the latter case, the claim made in Theorem 1.1(iii) easily follows.

**Proposition 9.1.** Suppose that for any  $\delta > 0$  there exists R > 0 such that condition (4-4) holds true for all  $0 < t < \infty$ . Then there exists a smooth solution  $u_{\infty} \in H^{1/2}(S^1; N)$  of (1-6) such that  $u(t) \to u_{\infty}$  smoothly as  $t \to \infty$  suitably, and  $u_{\infty}$  parametrizes a minimal surface of finite area supported by N which meets N orthogonally along its boundary.

*Proof.* For sufficiently small  $\delta > 0$  and for any  $j \in \mathbb{N}$ , by iterative reference to Propositions 4.2, 4.4–4.6, and 4.10, 4.11 as in Section 6 we can find constants  $C_j > 0$  such that  $||u(t)||_{H^j(B)} \leq C_j$  for all t > 1. Moreover, by the energy inequality Lemma 2.1 for a suitable sequence  $t_k \to \infty$  there holds  $u_t(t_k) \to 0$  in  $L^2(\partial B)$  as  $k \to \infty$ . Then for any  $j \in \mathbb{N}$  a subsequence  $u(t_k)$  approaches  $u_\infty$  in  $H^j(B)$ , and a diagonal subsequence converges smoothly, where  $u_\infty$  solves (1-6). By the argument after (8-11) in Section 8,  $u_\infty$  is conformal and  $u_\infty$  parametrizes a minimal surface with free boundary on N which meets N orthogonally along its boundary.

In the remaining case that for some  $\delta > 0$  condition (4-4) fails to hold, there exists a sequence  $t_k \uparrow \infty$  and points  $z^{(1)}, \ldots, z^{(i_0)}$  such that, for sequences  $z_k^{(i)} \to z^{(i)}$  and radii  $r_k^{(i)} \to 0$  as  $k \to \infty$ ,

$$\liminf_{k \to \infty} \int_{B_{r_t^{(i)}(z_k^{(i)}) \cap B}} |\nabla u(t_k)|^2 \, dz \ge \delta, \quad 1 \le i \le i_0$$

By Lemma 2.1 there holds the a-priori bound  $i_0 \le E(u_0)/\delta$  for the number of concentration points. By the argument leading to (8-10) then, for a suitable number

$$0 < r_0 \le \inf_{i < j} \frac{1}{4} |z^{(i)} - z^{(j)}|$$

with error  $o(1) \to 0$  as  $k \to \infty$  and with some constant  $L \in \mathbb{N}$  for all  $1 \le i \le i_0$ ,

$$L \int_{B_{2r_k^{(i)}}(z_k^{(i)})\cap B} |\nabla u(t_k)|^2 \, dz + o(1) \ge \sup_{\substack{z_0 \in B_{r_0}(z_k^{(i)})\\t_k - r_0 \le t \le t_k}} \int_{B_{r_k^{(i)}}(z_0)\cap B} |\nabla u(t)|^2 \, dz \ge \delta.$$

Fixing any index  $1 \le i \le i_0$  and renaming  $z_k^{(i)} =: z_k$  and  $r_k^{(i)} =: r_k$ , we then scale

$$u_k(z,t) = u(z_k + r_k z, t_k + r_k t), \quad z \in \Omega_k = \{z : z_k + r_k z \in B\}, \quad -t_k/r_k \le t \le 0,$$

as before and observe that, for any  $t_0 < 0$ ,

$$\int_{t_0}^0 \int_{\partial\Omega_k} |\partial_t u_k|^2 \, ds \, dt = \int_{t_0}^0 \int_{\partial\Omega_k} |d\pi_N(u_k)\partial_{\nu_k} u_k|^2 \, ds \, dt$$
$$= \int_{t_k+r_k t_0}^{t_k} \int_{\partial B} |u_t|^2 \, d\phi \, dt \le \int_{t_k+r_k t_0}^\infty \int_{\partial B} |u_t|^2 \, d\phi \, dt \to 0 \tag{9-1}$$

as  $k \to \infty$ , where  $\nu_k$  is the outward unit normal along  $\partial \Omega_k$ . Just as in Section 8 for suitable  $t_0 < s_k < 0$ we then obtain local uniform and  $H^1$ -convergence of a subsequence of the conformally rescaled maps  $w_k = u_k(s_k) \circ \Phi_k \in H^1_{loc}(\mathbb{R}^2_+)$  to a smooth, nonconstant, harmonic and conformal limit  $w_\infty$  with finite energy and continuously mapping  $\partial \mathbb{R}^2_+$  to N, inducing a solution  $\bar{u}_\infty = w_\infty \circ \Phi \in H^{1/2}(S^1; N)$  of (1-6) corresponding to a minimal surface with free boundary on N. This ends the proof of Theorem 1.1(iii).

#### Appendix

In this section, for the convenience of the reader we derive two interpolation inequalities that play a crucial role in our arguments.

Let  $v \in H^1(B)$ , and for r > 0 let  $(z_i)_{1 \le i \le i_0}$  be such that the collection of balls  $B_{r/2}(z_i)$ ,  $1 \le i \le i_0$ , covers  $\overline{B}$  with at most L balls  $B_r(z_i)$  overlapping at any  $z \in B$ , with  $L \in \mathbb{N}$  independent of r > 0. We may assume  $r < \frac{1}{8}$  so that for any  $1 \le i \le i_0$  there is a pair of orthogonal vectors  $e_{1,i}$  and  $e_{2,i}$  such that for any  $z \in B_r(z_i)$  there holds  $z + se_{1,i} + te_{2,i} \in B$  for any  $0 \le s, t \le 2r$ . After a rotation of coordinates, we may assume that  $e_{1,i} = (1, 0)$  and  $e_{2,i} = (0, 1)$  are the standard basis vectors. Writing  $\varphi$  for  $\varphi_{z_i,r}$ , by arguing as Ladyzhenskaya [1963] for any  $z = (x, y) \in B_r(z_i)$  and using that

$$(v^2\varphi)(x+2r, y) = 0 = (v^2\varphi)(x, y+2r)$$

then we can estimate

$$v^{4}(z) = |(v^{2}\varphi)(z)|^{2} \leq \int_{0}^{2r} |\partial_{x}(v^{2}\varphi)(x+s, y)| \, ds \cdot \int_{0}^{2r} |\partial_{y}(v^{2}\varphi)(x, y+t)| \, dt$$
  
$$\leq \int_{\{s:(s,y)\in B\}} |\partial_{x}(v^{2}\varphi)(s, y)| \, ds \cdot \int_{\{t:(x,t)\in B\}} |\partial_{y}(v^{2}\varphi)(x, t)| \, dt, \qquad (A-1)$$

and with the help of Fubini's theorem we find

$$\begin{split} \int_{B_{r/2}(z_i)} |v|^4 dz &\leq \int_B |v|^4 \varphi^2 dz \leq \int_{-\infty}^{\infty} \left( \int_{\{x:(x,y)\in B\}} |(v^2\varphi)(x,y)|^2 dx \right) dy \\ &\leq \int_{-\infty}^{\infty} \int_{\{s:(s,y)\in B\}} |\partial_x (v^2\varphi)(s,y)| \, ds \, dy \cdot \int_{-\infty}^{\infty} \int_{\{t:(x,t)\in B\}} |\partial_y (v^2\varphi)(x,t)| \, dt \, dx \\ &\leq \left( \int_B |\nabla (v^2\varphi)| \, dz \right)^2 \\ &\leq \left( \int_B (2|\nabla v| |v\varphi| + v^2 |\nabla \varphi|) \, dz \right)^2 \\ &\leq C \left( \int_{B_r(z_i)} |\nabla v|^2 \, dz + r^{-2} \int_{B_r(z_i)} v^2 \, dz \right) \int_{B_r(z_i)} v^2 \, dz. \end{split}$$

Fixing  $r = \frac{1}{10}$  and summing over  $1 \le i \le i_0$ , with an absolute constant C > 0 we obtain the bound

$$\|v\|_{L^{4}(B)}^{4} \le C \|v\|_{H^{1}(B)}^{2} \|v\|_{L^{2}(B)}^{2}$$
(A-2)

for any  $v \in H^1(B)$ .

#### References

[Brendle 2002] S. Brendle, "Curvature flows on surfaces with boundary", Math. Ann. 324:3 (2002), 491-519. MR Zbl

- [Brézis and Gallouet 1980] H. Brézis and T. Gallouet, "Nonlinear Schrödinger evolution equations", *Nonlinear Anal.* **4**:4 (1980), 677–681. MR Zbl
- [Brézis and Wainger 1980] H. Brézis and S. Wainger, "A note on limiting cases of Sobolev embeddings and convolution inequalities", *Comm. Partial Differential Equations* **5**:7 (1980), 773–789. MR Zbl
- [Caffarelli and Silvestre 2007] L. Caffarelli and L. Silvestre, "An extension problem related to the fractional Laplacian", *Comm. Partial Differential Equations* **32**:7-9 (2007), 1245–1260. MR Zbl
- [Chang and Liu 2005] K.-C. Chang and J.-Q. Liu, "Boundary flow for the minimal surfaces in  $\mathbb{R}^n$  with Plateau boundary condition", *Proc. Roy. Soc. Edinburgh Sect. A* **135**:3 (2005), 537–562. MR Zbl
- [Chang et al. 1992] K.-C. Chang, W. Y. Ding, and R. Ye, "Finite-time blow-up of the heat flow of harmonic maps from surfaces", *J. Differential Geom.* **36**:2 (1992), 507–515. MR Zbl
- [Da Lio and Pigati 2020] F. Da Lio and A. Pigati, "Free boundary minimal surfaces: a nonlocal approach", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **20**:2 (2020), 437–489. MR Zbl
- [Da Lio and Rivière 2011] F. Da Lio and T. Rivière, "Three-term commutator estimates and the regularity of  $\frac{1}{2}$ -harmonic maps into spheres", *Anal. PDE* **4**:1 (2011), 149–190. MR Zbl
- [Da Lio et al. 2015] F. Da Lio, L. Martinazzi, and T. Rivière, "Blow-up analysis of a nonlocal Liouville-type equation", *Anal. PDE* **8**:7 (2015), 1757–1805. MR Zbl
- [Douglas 1931] J. Douglas, "Solution of the problem of Plateau", Trans. Amer. Math. Soc. 33:1 (1931), 263–321. MR Zbl
- [Eells and Sampson 1964] J. Eells, Jr. and J. H. Sampson, "Harmonic mappings of Riemannian manifolds", *Amer. J. Math.* **86** (1964), 109–160. MR Zbl
- [Freire 1995] A. Freire, "Uniqueness for the harmonic map flow from surfaces to general targets", *Comment. Math. Helv.* **70**:2 (1995), 310–338. Correction in **71**:2 (1996), 330-337. MR Zbl
- [Gehrig 2020] M. I. Gehrig, *Prescribed curvature on the boundary of the disc*, Ph.D. thesis, ETH Zurich, 2020, available at https://doi.org/10.3929/ethz-b-000445412.
- [Grüter et al. 1981] M. Grüter, S. Hildebrandt, and J. C. C. Nitsche, "On the boundary behavior of minimal surfaces with a free boundary which are not minima of the area", *Manuscripta Math.* **35**:3 (1981), 387–410. MR Zbl
- [Hyder et al. 2022] A. Hyder, A. Segatti, Y. Sire, and C. Wang, "Partial regularity of the heat flow of half-harmonic maps and applications to harmonic maps with free boundary", *Comm. Partial Differential Equations* **47**:9 (2022), 1845–1882. MR Zbl
- [Imbusch and Struwe 1999] C. Imbusch and M. Struwe, "Variational principles for minimal surfaces", pp. 477–498 in *Topics in nonlinear analysis*, edited by J. Escher and G. Simonett, Progr. Nonlinear Differential Equations Appl. 35, Birkhäuser, Basel, 1999. MR Zbl
- [Jost and Struwe 1990] J. Jost and M. Struwe, "Morse–Conley theory for minimal surfaces of varying topological type", *Invent. Math.* **102**:3 (1990), 465–499. MR Zbl
- [Ladyzhenskaya 1963] O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Gordon and Breach, New York, 1963. MR Zbl
- [Lenzmann and Schikorra 2020] E. Lenzmann and A. Schikorra, "Sharp commutator estimates via harmonic extensions", *Nonlinear Anal.* **193** (2020), art. id. 111375. MR Zbl
- [Mazowiecka and Schikorra 2018] K. Mazowiecka and A. Schikorra, "Fractional div-curl quantities and applications to nonlocal geometric equations", *J. Funct. Anal.* **275**:1 (2018), 1–44. MR Zbl
- [Millot and Sire 2015] V. Millot and Y. Sire, "On a fractional Ginzburg–Landau equation and 1/2-harmonic maps into spheres", *Arch. Ration. Mech. Anal.* **215**:1 (2015), 125–210. MR Zbl
- [Morse 1937] M. Morse, "Functional topology and abstract variational theory", Ann. of Math. (2) **38**:2 (1937), 386–449. MR Zbl
- [Morse and Tompkins 1939] M. Morse and C. Tompkins, "The existence of minimal surfaces of general critical types", *Ann. of Math.* (2) **40**:2 (1939), 443–472. MR Zbl

[Moser 2011] R. Moser, "Intrinsic semiharmonic maps", J. Geom. Anal. 21:3 (2011), 588-598. MR Zbl

- [Radó 1930] T. Radó, "On Plateau's problem", Ann. of Math. (2) 31:3 (1930), 457-469. MR
- [Rivière 1993] T. Rivière, *Applications harmoniques entre variétés*, Ph.D. thesis, Université Paris 6, 1993, available at https:// people.math.ethz.ch/~triviere/pdf/pub/flot-harmonique-93.pdf.
- [Rupflin 2017] M. Rupflin, "Teichmüller harmonic map flow from cylinders", Math. Ann. 368:3-4 (2017), 1227–1276. MR Zbl
- [Rupflin and Schrecker 2018] M. Rupflin and M. R. I. Schrecker, "Analysis of boundary bubbles for almost minimal cylinders", *Calc. Var. Partial Differential Equations* **57**:5 (2018), art. id. 121. MR Zbl
- [Rupflin and Topping 2019] M. Rupflin and P. M. Topping, "Global weak solutions of the Teichmüller harmonic map flow into general targets", *Anal. PDE* **12**:3 (2019), 815–842. MR Zbl
- [Schoen and Uhlenbeck 1982] R. Schoen and K. Uhlenbeck, "A regularity theory for harmonic maps", *J. Differential Geom.* **17**:2 (1982), 307–335. MR Zbl
- [Struwe 1984] M. Struwe, "On a critical point theory for minimal surfaces spanning a wire in  $\mathbb{R}^n$ ", J. Reine Angew. Math. 349 (1984), 1–23. MR Zbl
- [Struwe 1985] M. Struwe, "On the evolution of harmonic mappings of Riemannian surfaces", *Comment. Math. Helv.* **60**:4 (1985), 558–581. MR Zbl
- [Struwe 1986] M. Struwe, "A Morse theory for annulus-type minimal surfaces", J. Reine Angew. Math. 368 (1986), 1–27. MR Zbl
- [Struwe 1988] M. Struwe, *Plateau's problem and the calculus of variations*, Math. Notes **35**, Princeton Univ. Press, 1988. MR Zbl
- [Topping 2002] P. Topping, "Reverse bubbling and nonuniqueness in the harmonic map flow", *Int. Math. Res. Not.* **2002**:10 (2002), 505–520. MR Zbl
- [Tromba 1984] A. J. Tromba, "Degree theory on oriented infinite-dimensional varieties and the Morse number of minimal surfaces spanning a curve in  $\mathbb{R}^n$ ", *Manuscripta Math.* **48**:1-3 (1984), 139–161. MR Zbl
- [Tromba 1985] A. J. Tromba, "Degree theory on oriented infinite-dimensional varieties and the Morse number of minimal surfaces spanning a curve in  $\mathbb{R}^n$ , I:  $n \ge 4$ ", *Trans. Amer. Math. Soc.* **290**:1 (1985), 385–413. MR Zbl
- [Wettstein 2021] J. Wettstein, "Existence, uniqueness and regularity of the fractional harmonic gradient flow in general target manifolds", preprint, 2021. arXiv 2109.11458
- [Wettstein 2022] J. Wettstein, "Uniqueness and regularity of the fractional harmonic gradient flow in  $S^{n-1}$ ", *Nonlinear Anal.* **214** (2022), art. id. 112592. MR Zbl
- [Wettstein 2023] J. D. Wettstein, "Half-harmonic gradient flow: aspects of a non-local geometric PDE", *Math. Eng.* **5**:3 (2023), art. id. 058. MR

Received 3 Feb 2022. Revised 14 Jul 2022. Accepted 19 Aug 2022.

MICHAEL STRUWE: michael.struwe@math.ethz.ch Departement Mathematik, ETH Zürich, Zürich, Switzerland

# **Analysis & PDE**

msp.org/apde

#### EDITOR-IN-CHIEF

#### Clément Mouhot Cambridge University, UK c.mouhot@dpmms.cam.ac.uk

#### BOARD OF EDITORS

Massimiliano Berti	Scuola Intern. Sup. di Studi Avanzati, Italy berti@sissa.it	William Minicozzi II	Johns Hopkins University, USA minicozz@math.jhu.edu
Zbigniew Błocki	Uniwersytet Jagielloński, Poland zbigniew.blocki@uj.edu.pl	Werner Müller	Universität Bonn, Germany mueller@math.uni-bonn.de
Charles Fefferman	Princeton University, USA cf@math.princeton.edu	Igor Rodnianski	Princeton University, USA irod@math.princeton.edu
David Gérard-Varet	Université de Paris, France david.gerard-varet@imj-prg.fr	Yum-Tong Siu	Harvard University, USA siu@math.harvard.edu
Colin Guillarmou	Université Paris-Saclay, France colin.guillarmou@universite-paris-saclay.fr	Terence Tao	University of California, Los Angeles, USA tao@math.ucla.edu
Ursula Hamenstaedt	Universität Bonn, Germany ursula@math.uni-bonn.de	Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA met@math.unc.edu
Peter Hintz	ETH Zurich, Switzerland peter.hintz@math.ethz.ch	Gunther Uhlmann	University of Washington, USA gunther@math.washington.edu
Vadim Kaloshin	Institute of Science and Technology, Austria vadim.kaloshin@gmail.com	András Vasy	Stanford University, USA andras@math.stanford.edu
Izabella Laba	University of British Columbia, Canada ilaba@math.ubc.ca	Dan Virgil Voiculescu	University of California, Berkeley, USA dvv@math.berkeley.edu
Anna L. Mazzucato	Penn State University, USA alm24@psu.edu	Jim Wright	University of Edinburgh, UK j.r.wright@ed.ac.uk
Richard B. Melrose	Massachussets Inst. of Tech., USA rbm@math.mit.edu	Maciej Zworski	University of California, Berkeley, USA zworski@math.berkeley.edu
Frank Merle	Université de Cergy-Pontoise, France merle@ihes.fr		

#### PRODUCTION

production@msp.org Silvio Levy, Scientific Editor

# See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2024 is US \$440/year for the electronic version, and \$690/year (+\$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow<sup>®</sup> from MSP.

PUBLISHED BY



nonprofit scientific publishing http://msp.org/

© 2024 Mathematical Sciences Publishers

# ANALYSIS & PDE

Volume 17 No. 4 2024

The singular strata of a free-boundary problem for harmonic measure SEAN MCCURDY	1127
On complete embedded translating solitons of the mean curvature flow that are of finite genus GRAHAM SMITH	1175
Hausdorff measure bounds for nodal sets of Steklov eigenfunctions STEFANO DECIO	1237
On full asymptotics of real analytic torsions for compact locally symmetric orbifolds BINGXIAO LIU	1261
The Landau equation as a gradient Flow JOSÉ A. CARRILLO, MATIAS G. DELGADINO, LAURENT DESVILLETTES and JEREMY SH. WU	1331
Degenerating hyperbolic surfaces and spectral gaps for large genus YUNHUI WU, HAOHAO ZHANG and XUWEN ZHU	1377
Plateau flow or the heat flow for half-harmonic maps MICHAEL STRUWE	1397
Noncommutative maximal operators with rough kernels XUDONG LAI	1439
Structure of sets with nearly maximal Favard length ALAN CHANG, DAMIAN DABROWSKI, TUOMAS ORPONEN and MICHELE VILLA	1473