

ANALYSIS & PDE

Volume 17

No. 4

2024

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Using the interpretation of the half-Laplacian on S^1 as the Dirichlet-to-Neumann operator for the Laplace equation on the ball B , we devise a classical approach to the heat flow for half-harmonic maps from S^1 to a closed target manifold $N \subset \mathbb{R}^n$, recently studied by Wettstein, and for arbitrary finite-energy data we obtain a result fully analogous to the author’s 1985 results for the harmonic map heat flow of surfaces and in similar generality. When N is a smoothly embedded, oriented closed curve $\Gamma \subset \mathbb{R}^n$, the half-harmonic map heat flow may be viewed as an alternative gradient flow for a variant of the Plateau problem of disc-type minimal surfaces.

1. Background and results

Half-harmonic maps and their heat flow. Let $N \subset \mathbb{R}^n$ be a closed submanifold, that is, compact and without boundary. The concept of a half-harmonic map $u : S^1 \rightarrow N \subset \mathbb{R}^n$ was introduced by Da Lio and Rivière [2011], who together with Martinazzi [Da Lio et al. 2015, Theorem 2.9] also made the interesting observation that the harmonic extension of a half-harmonic map yields a free boundary minimal surface supported by N , a fact which also was noticed by Millot and Sire [2015, Remark 4.28].

In his Ph.D. thesis, Wettstein [2021; 2022; 2023], studied the corresponding heat flow given by

$$d\pi_N(u)(u_t + (-\Delta)^{1/2}u) = 0 \quad \text{on } S^1 \times [0, \infty[, \quad (1-1)$$

where $u_t = \partial_t u$ and where $\pi_N : N_\rho \rightarrow N$ is the smooth nearest-neighbor projection on a ρ -neighborhood N_ρ of the given target manifold to N , and, with the help of a fine analysis of the fractional differential operators involved, he showed global existence for initial data of small energy.

Moser [2011] and Millot and Sire [2015] contributed important results to the study of half-harmonic maps by exploiting the fact that for any smooth $u : S^1 \rightarrow \mathbb{R}^n$ we can represent the half-Laplacian classically in the form

$$(-\Delta)^{1/2}u = \partial_r U, \quad (1-2)$$

where $U : B \rightarrow \mathbb{R}^n$ is the harmonic extension of u to the unit disc B .¹ Here, using the identity (1-2), we are able to remove the smallness assumption in Wettstein’s work and show the existence of a “global”

I thank Amélie Lohér and the anonymous referee for careful reading of the manuscript and useful suggestions.

MSC2020: primary 35K55, 35K65, 53E99; secondary 53A10.

Keywords: half-harmonic maps, harmonic map heat flow, Plateau problem.

¹The classical formula (1-2) is a special case of a much more general result due to Caffarelli and Silvestre [2007], who pointed out that many nonlocal problems involving fractional powers of the Laplacian can be related to a local, possibly degenerate, elliptic equation via a suitable extension of the solution to a half-space.

weak solution to the heat flow (1-1) for data of arbitrarily large (but finite) energy, which is defined for all times and smooth away from finitely many “blow-up points” where energy concentrates, and whose energy is nonincreasing. The solution is unique in this class in exact analogy with the classical result by the author [Struwe 1985] on the harmonic map heat flow for maps from a closed surface to a closed target manifold $N \subset \mathbb{R}^n$; see Theorem 1.2 below.

In order to describe our work in more detail, let

$$H^{1/2}(S^1; N) = \{u \in H^{1/2}(S^1; \mathbb{R}^n) : u(z) \in N \text{ for almost every } z \in S^1\}.$$

Interpreting $S^1 = \partial B$, where $B = B_1(0; \mathbb{R}^2)$, and tacitly identifying a map $u \in H^{1/2}(S^1; N)$ with its harmonic extension $U \in H^1(B; \mathbb{R}^n)$, for a given function $u_0 \in H^{1/2}(S^1; N)$ we then seek to find a family of harmonic functions $u(t) \in H^1(B; \mathbb{R}^n)$ with traces $u(t) \in H^{1/2}(S^1; N)$ for $t > 0$, solving the equation

$$d\pi_N(u)(u_t + \partial_r u) = u_t + d\pi_N(u)\partial_r u = 0 \quad \text{on } S^1 \times [0, \infty[, \tag{1-3}$$

with initial data

$$u|_{t=0} = u_0 \in H^{1/2}(S^1; N). \tag{1-4}$$

Energy. The half-harmonic heat flow may be regarded as the heat flow for the half-energy

$$E_{1/2}(u) = \frac{1}{2} \int_{S^1} |(-\Delta)^{1/4} u|^2 d\phi$$

of a map $u \in H^{1/2}(S^1; N)$. Note that the half-energy of u equals the standard Dirichlet energy

$$E(u) = \frac{1}{2} \int_B |\nabla u|^2 dz$$

of its harmonic extension $u \in H^1(B; \mathbb{R}^n)$. Indeed, integrating by parts we have

$$\int_B |\nabla u|^2 dz = \int_{S^1} u \partial_r u d\phi = \int_{S^1} u(-\Delta)^{1/2} u d\phi = \int_{S^1} |(-\Delta)^{1/4} u|^2 d\phi, \tag{1-5}$$

where we use the identity (1-2) and where the last identity easily follows from the representation of the operators $(-\Delta)^{1/2}$ and $(-\Delta)^{1/4}$ in Fourier space with symbols $|\xi|$ and $\sqrt{|\xi|}$, respectively, and Parseval’s identity.² Therefore, in the following for convenience we may always work with the classically defined Dirichlet energy. Moreover, we may interpret the half-harmonic heat flow as the heat flow for the Dirichlet energy in the class of harmonic functions with trace in $H^{1/2}(S^1; N)$; see Section 2 below for details.

Results. Identifying \mathbb{R}^2 with \mathbb{C} , we denote by M the three dimensional Möbius group of conformal transformations of the unit disc, given by

$$M = \left\{ \Phi(z) = e^{i\theta} \frac{z+a}{1+\bar{a}z} \in C^\infty(\bar{B}; \bar{B}) : |a| < 1, \theta \in \mathbb{R} \right\}.$$

Observe that the Dirichlet energy is invariant under conformal transformations, and we have $E(u) = E(u \circ \Phi)$ for any $u \in H^1(B; \mathbb{R}^n)$ and any $\Phi \in M$.

²Conversely, via Fourier expansion we also can prove (1-5) directly. Computing the first variations of E and $E_{1/2}$, respectively, we then obtain (1-2).

For smooth data we then have the following result.

Theorem 1.1. *Let $N \subset \mathbb{R}^n$ be a closed, smooth submanifold of \mathbb{R}^n , and suppose that the normal bundle $T^\perp N$ is parallelizable. Then the following holds:*

- (i) *For any smooth $u_0 \in H^{1/2}(S^1; N)$ there exists a time $T_0 \leq \infty$ and a unique smooth solution $u = u(t)$ of (1-3), and hence of (1-1), with data (1-4) for $0 < t < T_0$.*
- (ii) *If $T_0 < \infty$, we have concentration in the sense that, for some $\delta > 0$ and any $R > 0$,*

$$\sup_{\substack{z_0 \in B \\ 0 < t < T_0}} \int_{B_R(z_0) \cap B} |\nabla u(t)|^2 dz \geq \delta,$$

and for suitable $t_k \uparrow T_0$ there exist finitely many points $z_k^{(1)}, \dots, z_k^{(i_0)}$ and conformal maps $\Phi_k^{(i)} \in M$ with $z_k^{(i)} \rightarrow z^{(i)} \in \bar{B}$ and $\Phi_k^{(i)} \rightarrow \Phi_\infty^{(i)} \equiv z^{(i)}$ weakly in $H^1(B)$ such that $u(t_k) \circ \Phi_k^{(i)} \rightarrow \bar{u}^{(i)}$ weakly in $H^1(B)$ as $k \rightarrow \infty$, where $\bar{u}^{(i)}$ is nonconstant and conformal and satisfies

$$d\pi_N(\bar{u}^{(i)})\partial_r \bar{u}^{(i)} = 0, \quad 1 \leq i \leq i_0. \tag{1-6}$$

Moreover, there exists $\delta = \delta(N) > 0$ such that $E(\bar{u}^{(i)}) \geq \delta$, and $i_0 \leq E(u_0)/\delta$. Finally, $u(t_k)$ smoothly converges to a limit $u_1 \in H^{1/2}(S^1; N)$ on $\bar{B} \setminus \{z^{(1)}, \dots, z^{(i_0)}\}$.

- (iii) *If $T_0 = \infty$, then, as $t \rightarrow \infty$ suitably, $u(t)$ smoothly converges to a half-harmonic limit map away from at most finitely many concentration points where nonconstant half-harmonic maps “bubble off” as in (ii).*

By the Da Lio–Rivière interpretation of (1-6), the “bubbles” $\bar{u}^{(i)}$ as well as the limit u_∞ of the flow conformally parametrize minimal surfaces with free boundary on N , meeting N orthogonally along their free boundaries.

The hypothesis regarding the target manifold N in particular is fulfilled if N is a closed, orientable hypersurface of codimension 1 in \mathbb{R}^n , or if N is a smoothly embedded, closed curve $\Gamma \subset \mathbb{R}^3$.

It would be interesting to find examples of initial data for which the flow blows up in finite time, as in the work of Chang, Ding, and Ye [Chang et al. 1992] on the harmonic map heat flow.

For data in $H^{1/2}(S^1; N)$ the following global existence result holds, which is our main result.

Theorem 1.2. *For $N \subset \mathbb{R}^n$ as in Theorem 1.1 the following holds:*

- (i) *For any $u_0 \in H^{1/2}(S^1; N)$ there exists a unique global weak solution of (1-3) with data (1-4) as in Definition 6.3, whose energy is nonincreasing and which is smooth for positive time away from finitely many points in space-time where nontrivial half-harmonic maps “bubble off” in the sense of Theorem 1.1(ii).*
- (ii) *As $t \rightarrow \infty$ suitably, $u(t)$ smoothly converges to a half-harmonic limit map away from at most finitely many concentration points where nonconstant half-harmonic maps “bubble off” as in Theorem 1.1(ii).*

Note that uniqueness is only asserted within the class of partially regular weak solutions with nonincreasing energy, as in the case of the harmonic map heat flow. It would be interesting to find out if the latter condition suffices, as in the work of Freire [1995], and, conversely, to explore the possibility of “backward bubbling” in (1-3), as in the examples of Topping [2002] for the latter flow.

Key features of the proof and related flow equations. In our approach, in a similar vein as [Lenzmann and Schikorra 2020], we uncover and exploit surprising regularity properties of the normal component $d\pi_N^\perp(u)\partial_r u$ for the harmonic extension of u , likely related to the fractional commutator estimates for the normal projection in the work of Da Lio and Rivière [2011] or the regularity estimates of Da Lio and Pigati [2020], Mazowiecka and Schikorra [2018], and others.

The use of the Dirichlet-to-Neumann map for the harmonic extension $u: B \rightarrow \mathbb{R}^n$ of u instead of the half-Laplacian, and the simple identity (3-2) as well as (3-5) allow us to perform the analysis using only local, classically defined operators, avoiding fractional calculus almost entirely.

Note that (1-3) is similar to the equation governing the (scalar) evolution problem for conformal metrics $e^{2u}g_{\mathbb{R}^2}$ of prescribed geodesic boundary curvature and vanishing Gauss curvature on the unit disc B , studied for instance by Brendle [2002] or Gehrig [2020]. In contrast to the latter flows, due to the presence of the projection operator mapping u_r to its tangent component, the flow (1-3) at first sight appears to be degenerate. However, surprisingly, within our framework we are able to obtain similar smoothing properties as in the case of the harmonic map heat flow of surfaces.

A different heat flow associated with half-harmonic maps, using the half-heat operator $(\partial_t - \Delta)^{1/2}$ instead of (1-1), was suggested by Hyder et al. [2022], and they obtained global existence of partially regular, but possibly nonunique, weak solutions for their flow, with a possibly large singular set of measure zero.

Applications to the Plateau problem. In the case when N is a smoothly embedded, oriented closed curve $\Gamma \subset \mathbb{R}^3$, the half-harmonic heat flow (1-3) may furnish an alternative gradient flow for the Plateau problem of minimal surfaces of the type of the disc, which has a long and famous tradition in geometric analysis.

Posed in the 1890's, Plateau's problem was finally solved independently by Douglas [1931] and Radó [1930]. In order to analyze the set of *all* minimal surfaces solving the Plateau problem, including saddle points of the Dirichlet integral, thereby building on Douglas' ideas, Morse and Tompkins [1939] proposed a critical point theory for Plateau's problem in the sense of [Morse 1937], attempting to characterize nonminimizing solutions as "homotopy-critical" points of Dirichlet's integral. However, Tromba [1984; 1985] pointed out that it was not even clear that all smooth, nondegenerate minimal surfaces would be "homotopy-critical" in the sense of [Morse and Tompkins 1939]. To overcome this problem, Tromba developed a version of degree theory that could be applied in this case and which yielded at least a proof of the "last" Morse inequality, which is an identity for the total degree.

Finally, this author [Struwe 1984] recast the Plateau problem as a variational problem on a closed convex set and was able to develop a version of the Palais–Smale type critical point theory for the problem within this frame-work, which allowed him to obtain all Morse inequalities in a rigorous fashion; see [Struwe 1988] and [Imbusch and Struwe 1999] for further details. In [Struwe 1986] and [Jost and Struwe 1990], the approach was extended to the case of multiple boundaries and/or higher genus.

A key element of critical point theory for a variational problem is the construction of a pseudogradient flow for the problem at hand. In [Struwe 1984] this was achieved in an ad-hoc way. However, starting with the work of Eells and Sampson [1964] on the harmonic map heat flow, it is now an established approach in geometric analysis to study the (negative) (L^2) -gradient flow related to a variational problem, similar

to the standard heat equation. For Plateau’s problem, such a flow was obtained by Chang and Liu [2005] within the frame-work laid out by Struwe [1984] in the form of a parabolic variational inequality, for which Chang and Liu obtained a solution of class H^2 by means of a time-discrete minimization scheme. Rupflin [2017] and Rupflin and Schrecker [2018] studied the analogous parabolic variational inequality in the case of an annulus, which again had previously been studied in [Struwe 1986] by means of an ad-hoc pseudogradient flow.

In view of the much better regularity properties of the flow equation (1-3) it would be tempting to regard this as the correct definition of the canonical gradient flow for the Plateau problem, but an important issue still needs to be addressed.

Monotonicity. Recall that in the classical Plateau problem $u(t)$ is required to induce a (weakly) monotone parametrization of Γ for each $t > 0$. Even though it may seem likely that — at least for curves Γ on the boundary of a convex body in \mathbb{R}^3 — this Plateau boundary condition will be preserved along the flow (1-3) whenever it is satisfied initially, at this moment even for a strictly convex planar curve $\Gamma \subset \mathbb{R}^2$ it is not clear whether this actually happens. However, the results that we obtain also seem to be of interest if we drop the Plateau condition. In particular, our results motivate the study of smooth minimal surfaces with continuous trace covering only a part of the given boundary curve Γ ; dropping the monotonicity condition also brings the parametric approach to the Plateau problem closer to the approach via geometric measure theory or level sets.

Plateau flow. It should be straightforward to extend our results to the case when the disc B is replaced by a surface Σ of higher genus with boundary $\partial\Sigma \cong S^1$, if for given initial data $u_0 \in H^{1/2}(S^1; N)$ we consider a family $u = u(t)$ in $H^{1/2}(S^1; N)$ solving (1-3), that is,

$$u_t + d\pi_N(u)\partial_\nu u = 0$$

instead of (1-1), where for each time we harmonically extend $u(t)$ to Σ and denote by $\partial_\nu u$ the outward normal derivative of u along $\partial\Sigma$, as was proposed and analyzed by Da Lio and Pigati [2020] in the time-independent case. Similarly, one might study the flow (1-3) on a domain Σ with multiple boundaries. Of course, in order for the flow to converge to a minimal surface in the case of higher genus or higher connectivity it will be necessary to couple the flow (1-3) with a corresponding evolution equation for the conformal structure on Σ , as in the work of Rupflin and Topping [2019] on minimal immersions. Note that on a general domain Σ the flow equations (1-1) and (1-3) no longer agree. In order to clearly distinguish the flow equation (1-3) from the equation (1-1) defining the half-harmonic map heat flow, we therefore propose to say that (1-3) defines the “Plateau flow”.

Outline. After a brief discussion of energy estimates in Section 2, in Section 3 we present the analytic core of the argument for higher regularity in Section 4 and for the blow-up analysis, later presented in Section 8. These tools are also instrumental in proving uniqueness of partially regular weak solutions in Section 7. The L^2 -bounds for higher and higher derivatives which we establish in Section 4, assuming that energy does not concentrate, may be of particular interest. These bounds either concern estimates for $\nabla\partial_\phi^k u$ on B or on ∂B , and we view the latter bounds as stronger by an order of $\frac{1}{2}$. These bounds may

be iterated interlaced, as we later do in [Section 6](#), to prove uniform smooth estimates, locally in time, for smooth flows with smooth initial data converging in $H^{1/2}(N; S^1)$. Since the latter data are dense in $H^{1/2}(N; S^1)$, we thus not only obtain existence of weak solutions for arbitrary data $u_0 \in H^{1/2}(N; S^1)$ but also can show their smoothness for positive time and hence are able to derive [Theorem 1.2](#) from [Theorem 1.1](#). A peculiar feature is that one set of regularity estimates can only be obtained globally, that is on all of B , whereas the other set of estimates may be localized using cut-off functions. Similar estimates for a regularized version of (1-3) are employed in [Section 5](#) to prove local existence of smooth solutions of (1-3) for smooth data (1-4). Finally, in [Section 9](#) the large-time behavior of smooth solutions to (1-3) is discussed, finishing the proof of [Theorem 1.1](#).

Notation. The letter C is used throughout to denote a generic constant, possibly depending on the “target” N and the initial energy $E(u_0)$.

Moreover, since $T^\perp N$ by assumption is parallelizable and compact, there exists $\rho > 0$ such that the representation

$$T : N \times B_\rho(0; \mathbb{R}^m) \ni (p, y) \rightarrow p + \sum_{i=1}^m y^i v_i(p) \in N_\rho$$

of the tubular neighborhood $N_\rho = \bigcup_{p \in N} B_\rho(p)$ of N is a diffeomorphism, where v_1, \dots, v_m is a suitable smooth orthonormal frame along N and where we let $y = (y^1, \dots, y^m) \in \mathbb{R}^m$. For $q \in N_\rho$, then $T^{-1}(q) = (p, h)$ with $p = \pi_N(q)$ defines a (vector-valued) signed distance function $h = h(q) = (h^1(q), \dots, h^m(q))$ with $h^i(q) = v_i(p) \cdot (q - \pi_N(q))$ for each $1 \leq i \leq m$. Fixing a smooth function $\eta : \mathbb{R} \rightarrow \mathbb{R}$ such that $\eta(s) = s$ for $|s| < \frac{1}{2}\rho$, and with $\eta(s) = 0$ for $|s| \geq \frac{3}{4}\rho$, we then let

$$\text{dist}_N(q) = (\text{dist}_N^1(q), \dots, \text{dist}_N^m(q)),$$

with

$$\text{dist}_N^i(q) = \eta(h^i(q)) \text{ for } q \in N_\rho, \quad \text{otherwise } \text{dist}_N^i(q) = 0, \quad 1 \leq i \leq m.$$

Then for any smooth $u \in H^{1/2}(S^1; N)$ with harmonic extension $u \in H^1(B; \mathbb{R}^n)$ we have

$$\sum_{i=1}^m v_i(u) \partial_r \text{dist}_N^i(u) = \sum_{i=1}^m v_i(u) v_i(u) \cdot u_r = d\pi_N^\perp(u) u_r \quad \text{on } \partial B = S^1, \tag{1-7}$$

where for each $p \in N$ we denote by $d\pi_N^\perp(p) = 1 - d\pi_N(p) : \mathbb{R}^n \rightarrow T_p^\perp N$ the orthogonal projection. In the sequel, we abbreviate

$$\sum_{i=1}^m v_i(u) v_i(u) \cdot u_r =: v(u) v(u) \cdot u_r = v(u) \partial_r \text{dist}_N(u);$$

moreover, we extend the vector fields v_i to the whole ambient space by letting $v_i(q) = \nabla \text{dist}_N^i(q)$ for $q \in \mathbb{R}^n$, $1 \leq i \leq m$.

Finally, we fix a smooth cut-off function $\varphi \in C_c^\infty(B)$ satisfying $0 \leq \varphi \leq 1$ with $\varphi \equiv 1$ on $B_{1/2}(0)$, and for any $z_0 \in B$ and any $0 < R < 1$ we scale

$$\varphi_{z_0, R}(z) = \varphi\left(\frac{(z - z_0)}{R}\right) \in C_c^\infty(B_R(z_0)).$$

2. Energy inequality and first consequences

The half-harmonic heat flow may be regarded as the heat flow for the Dirichlet energy in the class $H^{1/2}(S^1; N)$. Indeed, let $u(t)$ be a smooth solution of (1-3) and (1-4) for $0 < t < T_0$. Then we have the following result.

Lemma 2.1. *For any $0 \leq S < T < T_0$,*

$$E(u(T)) + \int_S^T \int_{\partial B} |u_t|^2 d\phi dt \leq E(u(S)).$$

Proof. Integrating by parts and using (1-3) we compute

$$\frac{d}{dt} E(u) = \int_B \nabla u \nabla u_t dz = \int_{\partial B} u_r \cdot u_t d\phi = - \int_{\partial B} |d\pi_N(u)u_r|^2 d\phi = - \int_{\partial B} |u_t|^2 d\phi$$

for any $0 < t < T_0$. The claim follows by integration. □

Moreover, there holds a localized version of this energy inequality.

Lemma 2.2. *There exists a constant $C > 0$ such that, for any $z_0 \in B$, any $0 < R < 1$, any $\varepsilon > 0$, and any $0 < t_0 < t_1 \leq t_0 + \varepsilon R < T_0$,*

$$\int_B |\nabla u(t_1)|^2 \varphi_{z_0,R}^2 dz + 4 \int_{t_0}^{t_1} \int_{\partial B} |u_t|^2 \varphi_{z_0,R}^2 d\phi dt \leq 4 \int_B |\nabla u(t_0)|^2 \varphi_{z_0,R}^2 dz + C\varepsilon E(u_0).$$

Proof. Writing $\varphi = \varphi_{z_0,R}$ for brevity, integrating by parts, and using Young’s inequality, similar to the proof of Lemma 2.1 for any $0 < t < T_0$ we have

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \int_B |\nabla u|^2 \varphi^2 dz \right) &= \int_{\partial B} u_t \cdot u_r \varphi^2 d\phi - \int_B u_t \operatorname{div}(\nabla u \varphi^2) dz \\ &= - \int_{\partial B} |d\pi_N(u)u_r|^2 \varphi^2 d\phi - 2 \int_B u_t \nabla u \varphi \nabla \varphi dz \\ &\leq - \int_{\partial B} |u_t|^2 \varphi^2 d\phi + (8\varepsilon R)^{-1} \int_B |\nabla u|^2 \varphi^2 dz + 8\varepsilon R \int_B |u_t|^2 |\nabla \varphi|^2 dz. \end{aligned} \tag{2-1}$$

Letting

$$A = \sup_{t_0 < t < t_1} \left(\frac{1}{2} \int_B |\nabla u(t)|^2 \varphi^2 dz \right),$$

then upon integration we find

$$A + \int_{t_0}^{t_1} \int_{\partial B} |u_t|^2 \varphi^2 d\phi dt \leq \int_B |\nabla u(t_0)|^2 \varphi^2 dz + \frac{t_1 - t_0}{2\varepsilon R} A + C\varepsilon R^{-1} \int_{t_0}^{t_1} \int_{B_R(z_0) \cap B} |u_t|^2 dz dt.$$

But with $u = u(t)$, then also $u_t = u_t(t)$ is harmonic for each t . Expanding

$$u_t(r e^{i\phi}) = \sum_{k \geq 0} a_k r^k e^{ik\phi}$$

in a Fourier series, we see that the map

$$r \mapsto \int_{\partial B_r(0)} |u_t|^2 ds = 2\pi \sum_{k \geq 0} |a_k|^2 r^{2k+1},$$

with ds denoting the element of length along $\partial B_r(0)$, is nondecreasing. Thus, for any $z_0 \in B$, any $0 < R < 1$, and any $t_0 < t < t_1$,

$$\int_{B_R(z_0) \cap B} |u_t|^2 dz \leq 2R \int_{\partial B} |u_t|^2 d\phi, \tag{2-2}$$

and we may use [Lemma 2.1](#) to conclude. □

3. A regularity estimate

To illustrate the key ideas that later will allow us to prove higher regularity and analyze blow-up of solutions of (1-3), we first consider smooth solutions $u \in H^{1/2}(S^1; N)$ of the equation

$$d\pi_N(u)\partial_r u + f = 0 \quad \text{on } \partial B = S^1, \tag{3-1}$$

where $f \in L^2(S^1)$. We prove the following a-priori estimate, where we use classical estimates similar to [\[Wettstein 2022, Lemma 3.4\]](#), which in turn is a fractional version of a result by Rivière [\[1993, Chapter 4, pp. 96-104\]](#). Note that with the truncated signed distance function $\text{dist}_N: \mathbb{R}^n \rightarrow \mathbb{R}^m$ we have the orthogonal decomposition

$$\partial_r u = d\pi_N(u)\partial_r u + d\pi_N^\perp(u)\partial_r u = d\pi_N(u)\partial_r u + \nu(u)\partial_r(\text{dist}_N(u)) \tag{3-2}$$

on $\partial B = S^1$, where we recall that we use the shorthand notation

$$\nu(u)\partial_r(\text{dist}_N(u)) = \sum_{i=1}^m v_i(u)\partial_r(\text{dist}_N^i(u)) = \sum_{i=1}^m v_i(u)v_i(u) \cdot \partial_r u$$

and extend $v_i(p) = \nabla \text{dist}_N^i(p)$, $p \in \mathbb{R}^n$.

Proposition 3.1. *There exist constants C and $\delta_0 = \delta_0(N) > 0$ such that, for any smooth solution $u \in H^{1/2}(S^1; N)$ of (3-1) with $E(u) \leq \delta^2 < \delta_0^2$,*

$$\int_{S^1} |\partial_\phi u|^2 d\phi \leq C \|f\|_{L^2(S^1)}^2. \tag{3-3}$$

Proof. Multiplying (3-2) with $\partial_r u$, we find the Pythagorean identity

$$|\partial_r u|^2 = |d\pi_N(u)\partial_r u|^2 + |d\pi_N^\perp(u)\partial_r u|^2 = |d\pi_N(u)\partial_r u|^2 + |\partial_r(\text{dist}_N(u))|^2. \tag{3-4}$$

Note that $\text{dist}_N(u) \in H_0^1(B)$; moreover, for each $1 \leq i \leq m$ we have $\nabla(\text{dist}_N^i(u)) = v_i(u) \cdot \nabla u$, and there holds the equation

$$\Delta(\text{dist}_N^i(u)) = \text{div}(v_i(u) \cdot \nabla u) = \nabla u \cdot dv_i(u)\nabla u \quad \text{in } B. \tag{3-5}$$

The divergence theorem now gives

$$\begin{aligned} \|\partial_r(\text{dist}_N(u))\|_{L^2(S^1)}^2 &= (\nabla(\text{dist}_N(u)), \nabla(\text{dist}_N(u))_r)_{L^2(B)} + (\Delta(\text{dist}_N(u)), (\text{dist}_N(u))_r)_{L^2(B)} \\ &\leq C\|\nabla u\|_{L^2(B)}\|\nabla^2(\text{dist}_N(u))\|_{L^2(B)} \leq C\delta\|\nabla^2(\text{dist}_N(u))\|_{L^2(B)}, \end{aligned}$$

where the basic L^2 -theory for the Laplace equation (3-5) yields the bound

$$\|\nabla^2(\text{dist}_N(u))\|_{L^2(B)} \leq C\|\Delta(\text{dist}_N^i(u))\|_{L^2(B)} \leq C\|\nabla u\|_{L^4(B)}.$$

With Sobolev’s embedding $H^{1/2}(B) \hookrightarrow L^4(B)$ we then conclude

$$\|\partial_r(\text{dist}_N(u))\|_{L^2(S^1)}^2 \leq C\delta\|\nabla u\|_{H^{1/2}(B)}^2.$$

Thus from (3-4) and (3-1) we have

$$\|\partial_r u\|_{L^2(S^1)}^2 \leq \|f\|_{L^2(S^1)}^2 + \|\partial_r(\text{dist}_N(u))\|_{L^2(S^1)}^2 \leq \|f\|_{L^2(S^1)}^2 + C\delta\|\nabla u\|_{H^{1/2}(B)}^2. \tag{3-6}$$

But Fourier expansion of the harmonic function u gives

$$\|\partial_\phi u\|_{L^2(S^1)}^2 = \|\partial_r u\|_{L^2(S^1)}^2 = \frac{1}{2}\|\nabla u\|_{L^2(S^1)}^2 \tag{3-7}$$

as well as the bound

$$\|\nabla u\|_{H^{1/2}(B)}^2 \leq C\|\nabla u\|_{L^2(S^1)}^2,$$

and from (3-6) we obtain

$$\|\partial_r u\|_{L^2(S^1)}^2 \leq \|f\|_{L^2(S^1)}^2 + C\delta\|\nabla u\|_{H^{1/2}(B)}^2 \leq \|f\|_{L^2(S^1)}^2 + C\delta\|\partial_r u\|_{L^2(S^1)}^2,$$

which for sufficiently small $\delta > 0$ by (3-7) yields the claim. □

In particular, from Proposition 3.1 we obtain a positive energy threshold for nonconstant solutions of (1-6).

Corollary 3.2. *Suppose $u \in H^{1/2}(S^1; N)$ smoothly solves (1-6). Then, either u is constant or $E(u) \geq \delta_0^2$, with $\delta_0 = \delta_0(N) > 0$ given by Proposition 3.1.*

Combining the ideas in the proof of the previous result with ideas from the classical proof of the Courant–Lebesgue lemma in minimal surface theory, we can obtain the following local version of Proposition 3.1.

Proposition 3.3. *There exists a constant $\delta > 0$ with the following property. Given any smooth solution $u \in H^{1/2}(S^1; N)$ of (3-1) with harmonic extension $u \in H^1(B)$, any $z_0 \in \partial B$, and any $0 < R \leq \frac{1}{2}$ such that*

$$\int_{B_R(z_0) \cap B} |\nabla u|^2 dz < \delta^2, \tag{3-8}$$

with a constant $C = C(R) > 0$ there holds

$$\int_{B_{R^2}(z_0) \cap S^1} |\partial_\phi u|^2 d\phi \leq C\|f\|_{L^2(B_R(z_0) \cap S^1)}^2 + CE(u).$$

Proof. Fix any $z_0 \in \partial B$ and $0 < R \leq \frac{1}{2}$ such that (3-8) holds. For suitable $\rho \in [R^2, R]$, with s denoting arc-length along the curve $C_\rho = \{z_0 + \rho e^{i\theta} \in B : \theta \in \mathbb{R}\}$ with end-points $z_j = z_0 + \rho e^{i\theta_j} = e^{i\phi_j} \in \partial B$, $j = 1, 2$, we have

$$\rho \int_{C_\rho} |\nabla u|^2 ds \leq 2 \inf_{R^2 < \rho' < R} \left(\rho' \int_{C_{\rho'}} |\nabla u|^2 ds \right).$$

We can bound the latter infimum by the average over $\rho \in [R^2, R]$ with respect to the measure with density ρ^{-1} to obtain the bound

$$\rho \int_{C_\rho} |\nabla u|^2 ds \leq \frac{2 \int_{R^2}^R \int_{C_\rho} |\nabla u|^2 ds d\rho}{\int_{R^2}^R \rho^{-1} d\rho} \leq \frac{2 \int_B |\nabla u|^2 dz}{|\log(R)|} = \frac{4E(u)}{|\log(R)|}. \tag{3-9}$$

Let $\Phi_0 : B \rightarrow B$ be the conformal map fixing the circular arc C_ρ and mapping the point z_0 to the point $-z_0$, obtained as the composition $\Phi_0 = \pi_0^{-1} \circ \Psi_0 \circ \pi_0$ of a conformal diffeomorphism $\pi_0 : B \rightarrow \mathbb{R}_+^2$ mapping the points z_0 and $-z_0$ to the origin and infinity, respectively, and the reflection $\Psi_0 : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$ of the upper half-plane \mathbb{R}_+^2 in the half-circle $\pi_0(C_\rho)$. Replacing u by the map $u \circ \Phi_0$ in $B \setminus B_\rho(z_0)$ we obtain a piecewise smooth map $v_1 : B \rightarrow \mathbb{R}^n$ which is harmonic on $B \setminus C_\rho$ and continuous on all of B . Let $v_0 \in H^1(B)$ be harmonic with $w := v_1 - v_0 \in H_0^1(B)$. Note that by the variational characterization of harmonic functions and conformal invariance of the Dirichlet integral we have

$$E(v_0) \leq E(v_1) \leq \int_{B_R(z_0) \cap B} |\nabla u|^2 dz \leq \delta^2. \tag{3-10}$$

Moreover, for any smooth $\varphi \in H_0^1(B)$, by (3-9) we can estimate

$$\begin{aligned} \left| \int_B \nabla w \nabla \varphi dz \right| &= \left| \int_B \nabla v_1 \nabla \varphi dz \right| = \left| \int_{C_\rho} [\partial_\nu v_1] \varphi ds \right| \leq \left(\int_{C_\rho} |\nabla u|^2 ds \right)^{1/2} \left(\int_{C_\rho} |\varphi|^2 ds \right)^{1/2} \\ &\leq C(R) E(u)^{1/2} \|\varphi\|_{H^{1/2}(B)}, \end{aligned}$$

where $[\partial_\nu v_1]$ denotes the difference of the outer and inner normal derivatives of v_1 along C_ρ . Thus we have $\Delta w \in H^{-1/2}(B)$, and the basic L^2 -theory for the Laplace equation gives $w \in H^{3/2} \cap H_0^1(B)$ with

$$\|w\|_{H^{3/2}(B)} \leq \sup_{\substack{\varphi \in H_0^1(B) \\ \|\varphi\|_{H^{1/2}(B)} \leq 1}} \left(\int_B \nabla w \nabla \varphi dz \right) \leq C(R) E(u)^{1/2},$$

and then also

$$\|\partial_r w\|_{L^2(S^1)}^2 \leq C \|w\|_{H^{3/2}(B)}^2 \leq C(R) E(u). \tag{3-11}$$

In view of (3-10), for sufficiently small $\delta > 0$, from Proposition 3.1 we obtain the estimate

$$\|\partial_\phi v_0\|_{L^2(S^1)}^2 \leq C \|d\pi_N(v_0) \partial_r v_0\|_{L^2(S^1)}^2. \tag{3-12}$$

Observe that since $v_0 = v_1$ on $\partial B = S^1$ and since we also have $v_1 = u$ on $B \cap B_\rho(z_0)$ and $v_1 = u \circ \Phi_0$ on $B \setminus B_\rho(z_0)$, respectively, we can bound

$$\|d\pi_N(v_0) \partial_r v_0\|_{L^2(S^1)}^2 = \|d\pi_N(v_1) \partial_r v_0\|_{L^2(S^1)}^2 \leq 2 \|d\pi_N(v_1) \partial_r v_1\|_{L^2(S^1)}^2 + 2 \|\partial_r w\|_{L^2(S^1)}^2$$

and

$$\|d\pi_N(v_1)\partial_r v_1\|_{L^2(S^1)}^2 \leq C(R)\|d\pi_N(u)\partial_r u\|_{L^2(S^1 \cap B_\rho(z_0))}^2.$$

Thus from (3-11) we obtain

$$\begin{aligned} \|d\pi_N(v_0)\partial_r v_0\|_{L^2(S^1)}^2 &\leq C(R)\|d\pi_N(u)\partial_r u\|_{L^2(S^1 \cap B_\rho(z_0))}^2 + C\|\partial_r w\|_{L^2(S^1)}^2 \\ &\leq C(R)\|f\|_{L^2(S^1 \cap B_\rho(z_0))}^2 + C(R)E(u), \end{aligned}$$

and from (3-12) there results the bound

$$\begin{aligned} \|\partial_\phi u\|_{L^2(S^1 \cap B_\rho(z_0))}^2 &= \|\partial_\phi v_0\|_{L^2(S^1 \cap B_\rho(z_0))}^2 \leq \|\partial_\phi v_0\|_{L^2(S^1)}^2 \\ &\leq C\|d\pi_N(v_0)\partial_r v_0\|_{L^2(S^1)}^2 \leq C(R)\|f\|_{L^2(S^1 \cap B_R(z_0))}^2 + C(R)E(u), \end{aligned}$$

as claimed. □

The local estimate [Proposition 3.3](#) also implies the following global bound.

Proposition 3.4. *There exists a constant $\delta > 0$ with the following property. Given any smooth solution $u \in H^{1/2}(S^1; N)$ of (3-1) and any $0 < R \leq \frac{1}{2}$ with*

$$\sup_{z_0 \in B} \int_{B_R(z_0) \cap B} |\nabla u|^2 dz < \delta^2, \tag{3-13}$$

there holds

$$\int_{S^1} |\partial_\phi u|^2 d\phi \leq C(R)\|f\|_{L^2(S^1)}^2 + C(R)E(u).$$

Proof. Covering ∂B with balls $B_{R^2}(z_i)$, $1 \leq i \leq i_0$, from [Proposition 3.3](#) we obtain the claim. □

Remark 3.5. The proofs of the above propositions only require $u \in H^1(S^1; N)$ with harmonic extension $u \in H^{3/2}(B)$.

4. Higher regularity

Again let $u(t)$ be a smooth solution of the half-harmonic heat flow (1-3) for $0 < t < T_0$ with smooth initial data (1-4). We show that as long as the flow does not concentrate energy in the sense of [Theorem 1.1\(ii\)](#) the solution remains smooth and can be a-priori bounded in any H^k -norm in terms of the data.

H^2 -bound. In a first step we show an L^2 -bound in space-time for the second derivatives of our solution to the flow (1-3). Recall that by harmonicity, writing $u = u(t)$, $\partial_\phi u = u_\phi$, and so on, for any $0 < t < T_0$ we have (3-7), that is,

$$\int_{\partial B} |u_\phi|^2 d\phi = \int_{\partial B} |u_r|^2 d\phi,$$

as Fourier expansion shows, with similar identities for partial derivatives of u of higher order. Indeed, writing

$$\Delta u = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\phi\phi} \tag{4-1}$$

we see that $\partial_\phi^j u$ and then also $\nabla^{k-j} \partial_\phi^j u$ is harmonic for any $j \leq k$ in \mathbb{N}_0 , where $\nabla u = (u_x, u_y)$ in Euclidean coordinates $z = x + iy$. Thus by induction we obtain

$$\int_{\partial B} |\nabla^k u|^2 d\phi = 2 \int_{\partial B} |\nabla^{k-1} u_\phi|^2 d\phi = \dots = 2^k \int_{\partial B} |\partial_\phi^k u|^2 d\phi \tag{4-2}$$

for any $k \in \mathbb{N}$. Similarly, for any $\frac{1}{4} < r < 1$ with uniform constants $C > 0$ we have

$$\int_{\partial B_r(0)} |\nabla^k u|^2 dz \leq C \int_{\partial B_r(0)} |\nabla^{k-1} u_\phi|^2 dz \leq \dots \leq C \int_{\partial B_r(0)} |\partial_\phi^k u|^2 dz.$$

Integrating and using the mean value property of harmonic functions together with (4-2) to bound

$$\sup_{B_{1/4}(0)} |\nabla^k u|^2 \leq C \int_{B \setminus B_{1/4}(0)} |\nabla^k u|^2 dz \leq C \int_B |\nabla \partial_\phi^{k-1} u|^2 dz,$$

in particular, for any $k \in \mathbb{N}$, we have the bound

$$\int_B |\nabla^k u|^2 dz \leq C \int_B |\nabla \partial_\phi^{k-1} u|^2 dz \tag{4-3}$$

with an absolute constant $C > 0$.

The following lemma is strongly reminiscent of analogous results for the harmonic map heat flow in two space dimensions.

Lemma 4.1. *With a constant $C > 0$ depending only on N ,*

$$\frac{d}{dt} \left(\int_{\partial B} |u_\phi|^2 d\phi \right) + \int_B |\nabla u_\phi|^2 dz \leq C \int_B |\nabla u|^2 |u_\phi|^2 dz.$$

Proof. Writing $d\pi_N(u) = 1 - d\pi_N^\perp(u)$ with

$$d\pi_N^\perp(u)X = v(u)v(u) \cdot X = \sum_{i=1}^m v_i(u)v_i(u) \cdot X$$

for any $X \in \mathbb{R}^n$, we compute

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\int_{\partial B} |u_\phi|^2 d\phi \right) &= \int_{\partial B} u_\phi \cdot u_{\phi,t} d\phi = - \int_{\partial B} u_{\phi\phi} \cdot u_t d\phi \\ &= \int_{\partial B} u_{\phi\phi} \cdot d\pi_N(u)u_r d\phi = - \int_{\partial B} (u_\phi \cdot u_{r\phi} - u_\phi \cdot \partial_\phi(v(u)v(u) \cdot u_r)) d\phi \\ &= -\frac{1}{2} \int_{\partial B} \partial_r(|u_\phi|^2) d\phi - \int_{\partial B} u_\phi \cdot dv(u)u_\phi v(u) \cdot u_r d\phi, \end{aligned}$$

where we use orthogonality $u_\phi \cdot v_i(u) = 0$ on ∂B , $1 \leq i \leq m$, in the last step. But u_ϕ is harmonic. So with $\Delta|u_\phi|^2 = 2|\nabla u_\phi|^2$, from Gauss' theorem we obtain

$$\frac{1}{2} \int_{\partial B} \partial_r(|u_\phi|^2) d\phi = \int_B |\nabla u_\phi|^2 dz.$$

On the other hand, by Young’s inequality we can estimate

$$\begin{aligned} \int_{\partial B} u_r \cdot v(u)u_\phi \cdot dv(u)u_\phi \, d\phi &= \int_B \nabla u \cdot \nabla(v(u)u_\phi \cdot dv(u)u_\phi) \, dz \\ &\leq C \int_B |\nabla u_\phi| |\nabla u| |u_\phi| \, dz + C \int_B |\nabla u|^2 |u_\phi|^2 \, dz \\ &\leq \frac{1}{2} \int_B |\nabla u_\phi|^2 \, dz + C \int_B |\nabla u|^2 |u_\phi|^2 \, dz, \end{aligned}$$

and our claim follows. □

Combining the previous result with a quantitative bound for the concentration of energy, we obtain a space-time bound for the second derivatives of u . Note that since u is smooth by assumption, for any $\delta > 0$ and any $T < T_0$, there exists a number $R = R(T, u) > 0$ such that

$$\sup_{\substack{z_0 \in B \\ 0 < t < T}} \int_{B_R(z_0) \cap B} |\nabla u(t)|^2 \, dz < \delta. \tag{4-4}$$

Proposition 4.2. *There exist constants $\delta = \delta(N) > 0$ and $C > 0$ such that, for any $T < T_0$ with $R > 0$ as in (4-4),*

$$\sup_{0 < t < T} \int_{\partial B} |u_\phi(t)|^2 \, d\phi + \int_0^T \int_B |\nabla u_\phi|^2 \, dx \, dt \leq C \int_{\partial B} |u_{0,\phi}|^2 \, d\phi + CTR^{-2}E(u_0). \tag{4-5}$$

Proof. For given $T < T_0$ and $\delta > 0$ to be determined, we fix $R > 0$ such that (4-4) holds. Let $B_{R/2}(z_i)$, $1 \leq i \leq i_0$, be a cover of B such that any point $z_0 \in B$ belongs to at most L of the balls $B_R(z_i)$, where $L \in \mathbb{N}$ is independent of $R > 0$. We then use the decomposition

$$\int_B |\nabla u|^2 |u_\phi|^2 \, dz \leq \sum_{i=1}^{i_0} \int_{B_{R/2}(z_i)} |\nabla u|^4 \, dz \leq \sum_{i=1}^{i_0} \int_B |\nabla u \varphi_{z_i,R}|^4 \, dz.$$

Using the multiplicative inequality (A-2) in the Appendix, for each i we can bound

$$\int_B |\nabla u \varphi_{z_i,R}|^4 \, dz \leq C\delta \int_{B_R(z_i)} (|\nabla^2 u|^2 + R^{-2}|\nabla u|^2) \, dz.$$

Summing over $1 \leq i \leq i_0$, we thus obtain the bound

$$\int_B |\nabla u|^2 |u_\phi|^2 \, dz \leq CL\delta \int_B |\nabla^2 u|^2 \, dz + CL\delta R^{-2}E(u) \leq CL\delta \int_B |\nabla u_\phi|^2 \, dz + CL\delta R^{-2}E(u_0),$$

and for sufficiently small $\delta > 0$ we obtain the claim from Lemma 4.1. □

With the help of Proposition 4.2 we can now bound u in $H^2(B)$ also uniformly in time. For this, we first note the following estimate, which also will be useful later for bounding higher-order derivatives.

Lemma 4.3. *For any $k \in \mathbb{N}$, with a constant $C > 0$ depending only on k and N , for the solution $u = u(t)$ to (1-3) and (1-4) for any $0 < t < T_0$*

$$\frac{d}{dt} (\|\nabla \partial_\phi^k u\|_{L^2(B)}^2) + \|\partial_\phi^k u_r\|_{L^2(S^1)}^2 \leq C \sum_{\substack{1 \leq j_i \leq k+1 \\ \sum j_i \leq k+2}} \|\nabla \partial_\phi^k u\|_{L^2(B)} \left\| \prod_i \nabla^{j_i} u \right\|_{L^2(B)}.$$

Proof. For any $k \in \mathbb{N}$ we use harmonicity of $\partial_\phi^{2k} u$ to compute

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\nabla \partial_\phi^k u\|_{L^2(B)}^2) &= (-1)^k \int_B \nabla \partial_\phi^{2k} u \nabla u_t \, dx \\ &= (-1)^k (\partial_\phi^{2k} u_r, u_t)_{L^2(S^1)} = (-1)^{k+1} (\partial_\phi^{2k} u_r, d\pi_N(u)u_r)_{L^2(S^1)} \\ &= -(\partial_\phi^k u_r, \partial_\phi^k u_r)_{L^2(S^1)} + (\partial_\phi^k u_r, \partial_\phi^k (v(u)v(u) \cdot u_r))_{L^2(S^1)} = -\|\partial_\phi^k u_r\|_{L^2(S^1)}^2 + I, \end{aligned} \tag{4-6}$$

where we use the decomposition $I = \sum_{j=0}^k \binom{k}{j} I_j$ with

$$I_j = (\partial_\phi^k u_r, \partial_\phi^j (v(u)v(u)) \partial_\phi^{k-j} u_r)_{L^2(S^1)} = (\nabla \partial_\phi^k u, \nabla (\partial_\phi^j (v(u)v(u)) \cdot \partial_\phi^{k-j} u_r))_{L^2(B)}.$$

Hence for any $1 \leq j \leq k$ we can bound

$$\begin{aligned} |I_j| &\leq C \sum_{0 \leq i \leq j} \|\nabla \partial_\phi^k u\|_{L^2(B)} \|\nabla \partial_\phi^{j-i} v(u) \partial_\phi^i v(u) \partial_\phi^{k-j} u_r\|_{L^2(B)} \\ &\quad + C \sum_{0 \leq i \leq j} \|\nabla \partial_\phi^k u\|_{L^2(B)} \|\partial_\phi^{j-i} v(u) \partial_\phi^i v(u) \nabla \partial_\phi^{k-j} u_r\|_{L^2(B)} \\ &\leq C \sum_{\substack{1 \leq j_i \leq k+1 \\ \sum_i j_i = k+2}} \|\nabla \partial_\phi^k u\|_{L^2(B)} \left\| \prod_i \nabla^{j_i} u \right\|_{L^2(B)}, \end{aligned}$$

as claimed. It remains to bound the term $I_0 = \|\partial_\phi^k u_r \cdot v(u)\|_{L^2(S^1)}^2$. With the signed distance function we can express

$$v(u) \cdot u_{\phi r} = (v(u) \cdot u_r)_\phi - u_r \cdot dv(u)u_\phi = (\text{dist}_N(u))_{\phi r} - u_r \cdot dv(u)u_\phi,$$

so that

$$\begin{aligned} I_0 &= \|\partial_\phi^k u_r \cdot v(u)\|_{L^2(S^1)}^2 = (\partial_\phi^k u_r \cdot v(u), \partial_\phi^k (\text{dist}_N(u))_r)_{L^2(S^1)} + II \\ &= (\nabla \partial_\phi^k u, \nabla (v(u) \partial_\phi^k (\text{dist}_N(u))_r))_{L^2(B)} + II, \end{aligned}$$

where all terms in II can be dealt with as in the case $1 \leq j \leq k$. Finally, we have

$$\begin{aligned} &(\nabla \partial_\phi^k u, \nabla (v(u) \partial_\phi^k (\text{dist}_N(u))_r))_{L^2(B)} \\ &\leq \|\nabla \partial_\phi^k u\|_{L^2(B)} (\|\nabla^2 \partial_\phi^k (\text{dist}_N(u))\|_{L^2(B)} + \|\nabla v(u) \partial_\phi^k (\text{dist}_N(u))_r\|_{L^2(B)}). \end{aligned}$$

But by the chain rule we can bound

$$\|\nabla v(u) \partial_\phi^k (\text{dist}_N(u))_r\|_{L^2(B)} \leq C \|\nabla v(u) \nabla^{k+1} (\text{dist}_N(u))\|_{L^2(B)} \leq C \sum_{\substack{1 \leq j_i \leq k+1 \\ \sum_i j_i = k+2}} \left\| \prod_i \nabla^{j_i} u \right\|_{L^2(B)}.$$

Moreover, by (3-5) and elliptic regularity theory,

$$\begin{aligned} \|\nabla^{k+2} (\text{dist}_N(u))\|_{L^2(B)}^2 &\leq C \|\Delta (\text{dist}_N(u))\|_{H^k(B)}^2 \\ &\leq C \|\nabla u \cdot dv_i(u) \nabla u\|_{H^k(B)}^2 \leq C \sum_{\substack{1 \leq j_i \leq k+1 \\ \sum_i j_i \leq k+2}} \left\| \prod_i \nabla^{j_i} u \right\|_{L^2(B)}, \end{aligned}$$

which gives the claim. □

For $k = 1$, from Proposition 4.2 we now easily derive a uniform L^2 -bound for the second derivatives of the flow.

Proposition 4.4. *For any smooth $u_0 \in H^{1/2}(S^1; N)$ and any $T < T_0$ with $R > 0$ as in Proposition 4.2 with a constant $C_1 = C_1(T, R, u_0) > 0$ depending on the right-hand side of (4-5),*

$$\sup_{0 < t < T} \int_B |\nabla u_\phi(t)|^2 dz + \int_0^T \int_{\partial B} |u_{\phi r}|^2 d\phi dt \leq C_1 \int_B |\nabla u_{0,\phi}|^2 dz + C_1.$$

Proof. For $k = 1$, by Lemma 4.3 we need to bound the term

$$J = \sum_{\substack{1 \leq j_i \leq 2 \\ \sum_i j_i \leq 3}} \left\| \prod_i \nabla^{j_i} u \right\|_{L^2(B)} \leq C \|\nabla^2 u\| \|\nabla u\| + \|\nabla u\|^3 \|_{L^2(B)} + J_1,$$

where J_1 contains all terms of lower order. By the maximum principle and Sobolev’s embedding $H^1(\partial B) \hookrightarrow L^\infty(\partial B)$ we can estimate

$$\|\nabla u\|_{L^\infty(B)}^2 \leq \|\nabla u\|_{L^\infty(\partial B)}^2 \leq C \|\nabla u\|_{H^1(\partial B)}^2 \leq C \|u_{\phi r}\|_{L^2(\partial B)}^2 + C_1,$$

where we have also used (3-7) and Proposition 4.2. Also bounding

$$\|\nabla u\|_{L^6(B)}^3 \leq \|\nabla u\|_{L^4(B)}^2 \|\nabla u\|_{L^\infty(B)} \leq C (\|\nabla^2 u\|_{L^2(B)} \|\nabla u\|_{L^2(B)} + E(u)) \|\nabla u\|_{L^\infty(B)}$$

via (A-2), and again using (3-7) (and with similar, but simpler bounds for J_1), we arrive at the estimate

$$\begin{aligned} J &\leq C \|\nabla^2 u\| \|\nabla u\| + \|\nabla u\|^3 \|_{L^2(B)} + C_1 \leq C (\|\nabla^2 u\|_{L^2(B)} + E(u)) \|\nabla u\|_{L^\infty(B)} + C_1 \\ &\leq C (1 + \|\nabla u_\phi\|_{L^2(B)} + E(u_0)) (\|u_{\phi r}\|_{L^2(\partial B)} + C_1). \end{aligned}$$

With Lemma 4.3 and Young’s inequality we then have

$$\begin{aligned} \frac{d}{dt} (1 + \|\nabla u_\phi\|_{L^2(B)}^2) + \|u_{\phi r}\|_{L^2(S^1)}^2 &\leq C \|\nabla u_\phi\|_{L^2(B)} (\|\nabla u_\phi\|_{L^2(B)} + E(u_0)) (\|u_{\phi r}\|_{L^2(\partial B)} + C_1) \\ &\leq \frac{1}{2} \|u_{\phi r}\|_{L^2(\partial B)}^2 + C (1 + \|\nabla u_\phi\|_{L^2(B)}^2) (\|\nabla u_\phi\|_{L^2(B)}^2 + C_1). \end{aligned} \tag{4-7}$$

Absorbing the term $\frac{1}{2} \|u_{\phi r}\|_{L^2(\partial B)}^2$ into the left-hand side of this inequality and dividing by $1 + \|\nabla u_\phi\|_{L^2(B)}^2$ we obtain

$$\frac{d}{dt} (\log(1 + \|\nabla u_\phi\|_{L^2(B)}^2)) \leq C \|\nabla u_\phi\|_{L^2(B)}^2 + C_1,$$

and from Proposition 4.2 we obtain the bound

$$\sup_{0 < t < T} \|\nabla u_\phi(t)\|_{L^2(B)}^2 \leq C_1 (1 + \|\nabla u_{0,\phi}\|_{L^2(B)}^2).$$

The claim then follows from (4-7). □

H^3 -bounds. The derivation of a-priori L^2 -bounds for third derivatives of the solution u to the flow (1-3), (1-4) requires special care, which is why we highlight this case.

Proposition 4.5. *For any smooth $u_0 \in H^{1/2}(S^1; N)$ and any $T < T_0$,*

$$\sup_{0 < t < T} \int_B |\nabla u_{\phi\phi}(t)|^2 dz + \int_0^T \int_{\partial B} |u_{\phi\phi r}|^2 d\phi dt \leq C_2 \int_B |\nabla u_{0,\phi\phi}|^2 dz + C_2,$$

where we denote by $C_2 = C_2(T, R, u_0) > 0$ a constant bounded by the terms on the right-hand side in the statements of Propositions 4.2 and 4.4.

Proof. For $k = 2$, by [Lemma 4.3](#) we need to bound the term

$$J = \sum_{\substack{1 \leq j_i \leq 3 \\ \sum_i j_i = 4}} \left\| \prod_i \nabla^{j_i} u \right\|_{L^2(B)} \leq C \left(\|\nabla u\|^4 + |\nabla u|^2 |\nabla^2 u| + |\nabla^2 u|^2 + |\nabla u| |\nabla^3 u| \right)_{L^2(B)}$$

and corresponding terms involving at most three derivatives in total, which we will omit.

For the first term, by the multiplicative inequality [\(A-2\)](#) and Sobolev’s embedding $H^2(B) \hookrightarrow L^\infty(B)$, we can estimate

$$\begin{aligned} \|\nabla u\|_{L^8(B)}^4 &\leq \|\nabla u\|_{L^4(B)}^2 \|\nabla u\|_{L^\infty(B)}^2 \leq C \|\nabla u\|_{H^1(B)} \|\nabla u\|_{L^2(B)} \|\nabla u\|_{L^\infty(B)}^2 \\ &\leq C (\|\nabla^2 u\|_{L^2(B)}^2 + E(u)) \|\nabla u\|_{L^\infty(B)}^2 \leq C_2 \|\nabla u\|_{L^\infty(B)}^2 \\ &\leq C_2 (\|\nabla^3 u\|_{L^2(B)} + \|\nabla u\|_{L^2(B)}) \|\nabla u\|_{L^\infty(B)} \end{aligned}$$

with a constant $C_2 = C_2(T, R, u_0) > 0$ as in the statement of the proposition. Similarly,

$$\|\nabla^2 u\|_{L^4(B)}^2 \leq C \|\nabla^2 u\|_{H^1(B)} \|\nabla^2 u\|_{L^2(B)} \leq \|\nabla^3 u\|_{L^2(B)} \|\nabla^2 u\|_{L^2(B)} + \|\nabla^2 u\|_{L^2(B)}^2 \leq C_2 (1 + \|\nabla^3 u\|_{L^2(B)}).$$

Hence we can also bound

$$\|\|\nabla u\|^2 |\nabla^2 u|\|_{L^2(B)} \leq \|\nabla u\|_{L^8(B)}^4 + \|\nabla^2 u\|_{L^4(B)}^2 \leq C_2 (1 + \|\nabla^3 u\|_{L^2(B)}) (1 + \|\nabla u\|_{L^\infty(B)}).$$

Finally, we estimate

$$\|\|\nabla u\| |\nabla^3 u|\|_{L^2(B)} \leq \|\nabla^3 u\|_{L^2(B)} \|\nabla u\|_{L^\infty(B)}$$

to obtain

$$J \leq C_2 (1 + \|\nabla^3 u\|_{L^2(B)}) (1 + \|\nabla u\|_{L^\infty(B)}).$$

But with the inequality

$$\|f\|_{L^\infty(B)} \leq C \|f\|_{H^1(B)} \left(1 + \log^{1/2} \left(1 + \frac{\|f\|_{H^2(B)}}{\|f\|_{H^1(B)}} \right) \right)$$

for $f \in H^2(B)$ due to Brezis and Gallouet [\[1980\]](#) (see also [\[Brézis and Wainger 1980\]](#) for a more general version), we have

$$\|\nabla u\|_{L^\infty(B)}^2 \leq C \|\nabla u\|_{H^1(B)}^2 \left(1 + \log \left(1 + \frac{\|\nabla u\|_{H^2(B)}}{\|\nabla u\|_{H^1(B)}} \right) \right) \leq C_2 (1 + \log(1 + \|\nabla^3 u\|_{L^2(B)})),$$

and [Lemma 4.3](#) yields the differential inequality

$$\frac{d}{dt} (\|\nabla \partial_\phi^2 u\|_{L^2(B)}^2) + \|u_{\phi\phi r}\|_{L^2(\partial B)}^2 \leq C_2 \|\nabla \partial_\phi^2 u\|_{L^2(B)} (1 + \|\nabla^3 u\|_{L^2(B)}) (1 + \log(1 + \|\nabla^3 u\|_{L^2(B)})).$$

Simplifying, and recalling that $\|\nabla^3 u\|_{L^2(B)} \leq C \|\nabla \partial_\phi^2 u\|_{L^2(B)}$ by [\(4-3\)](#), we then find

$$\frac{d}{dt} (1 + \|\nabla \partial_\phi^2 u\|_{L^2(B)}) \leq C_2 (1 + \|\nabla \partial_\phi^2 u\|_{L^2(B)}) (1 + \log(1 + \|\nabla \partial_\phi^2 u\|_{L^2(B)}));$$

that is, we have

$$\frac{d}{dt} (1 + \log(1 + \|\nabla \partial_\phi^2 u\|_{L^2(B)})) \leq C_2 (1 + \log(1 + \|\nabla \partial_\phi^2 u\|_{L^2(B)})).$$

Arguing as in the proof of [Proposition 4.4](#) we then obtain the claim. □

H^m -bounds, $m \geq 4$. In view of [Proposition 4.5](#) we can now use induction to prove the following result.

Proposition 4.6. *For any $k \geq 3$, any smooth $u_0 \in H^{1/2}(S^1; N)$, and any $T < T_0$,*

$$\sup_{0 < t < T} \int_B |\nabla \partial_\phi^k(t)|^2 dz + \int_0^T \int_{\partial B} |\partial_\phi^k u_r|^2 d\phi dt \leq C_k \int_B |\nabla \partial_\phi^k u_0|^2 dz + C_k,$$

where we denote by $C_k = C_k(T, R, u_0) > 0$ a constant bounded by the terms on the right-hand side in the statement of the proposition for $k - 1$.

Proof. By [Proposition 4.5](#) the claimed result holds true for $k = 2$. Suppose the claim holds true for some $k_0 \geq 2$ and let $k = k_0 + 1$. Note that by Sobolev's embedding $H^2(B) \hookrightarrow W^{1,4} \cap C^0(\bar{B})$ and [\(4-3\)](#) for $0 \leq t < T$ we then have the uniform bounds

$$\|\nabla^{k_0+1} u\|_{L^2(B)}^2 + \|\nabla^{k_0} u\|_{L^4(B)}^2 + \sum_{1 \leq j \leq k_0-1} \|\nabla^j u\|_{L^\infty(B)}^2 \leq C_{k_0} \|\nabla^{k_0+1} u_0\|_{L^2(B)}^2 + C_{k_0} \leq C_k < \infty \quad (4-8)$$

with a constant of the type C_k , as defined above.

By [Lemma 4.3](#) again we only need to bound the term

$$J = \sum_{\substack{1 \leq j_i \leq k+1 \\ \sum_i j_i \leq k+2}} \left\| \prod_i \nabla^{j_i} u \right\|_{L^2(B)}.$$

Clearly we have

$$\begin{aligned} J &\leq \|\nabla^{k+1} u\|_{L^2(B)} \|\nabla u\|_{L^\infty(B)} + \|\nabla^k u\|_{L^2(B)} \|\nabla u\|_{L^\infty(B)}^2 + \|\nabla^k u \nabla^2 u\|_{L^2(B)} \\ &\quad + \|\nabla^{k-1} u \nabla^3 u\|_{L^2(B)} + \|\nabla^{k-1} u \nabla^2 u\|_{L^2(B)} \|\nabla u\|_{L^\infty(B)} + C_k \\ &\leq C_k \|\nabla^{k+1} u\|_{L^2(B)} + \|\nabla^k u \nabla^2 u\|_{L^2(B)} + \|\nabla^{k-1} u \nabla^3 u\|_{L^2(B)} + C_k. \end{aligned}$$

We now distinguish the following cases: If $k - 1 = k_0 \geq 3$, by [\(4-8\)](#) we can bound

$$\|\nabla^k u \nabla^2 u\|_{L^2(B)} \leq \|\nabla^k u\|_{L^2(B)} \|\nabla^2 u\|_{L^\infty(B)} \leq C_{k_0} \|\nabla^{k_0+1} u\|_{L^2(B)}^2 + C_{k_0} \leq C_k$$

as well as

$$\|\nabla^{k-1} u \nabla^3 u\|_{L^2(B)} \leq \|\nabla^{k-1} u\|_{L^4(B)} \|\nabla^3 u\|_{L^4(B)} \leq C_{k_0} \|\nabla^{k_0} u\|_{L^4(B)}^2 + C_{k_0} \leq C_k$$

to obtain the estimate

$$J \leq C_k \|\nabla^{k+1} u\|_{L^2(B)} + C_k.$$

If, on the other hand, $k_0 = k - 1 = 2$, by our induction hypothesis [\(4-8\)](#) we have

$$\begin{aligned} \|\nabla^{k-1} u \nabla^3 u\|_{L^2(B)} &= \|\nabla^2 u \nabla^k u\|_{L^2(B)}^2 \leq \|\nabla^k u\|_{L^4(B)} \|\nabla^2 u\|_{L^4(B)} \\ &\leq C_k \|\nabla^k u\|_{H^1(B)} + C_k \leq C_k \|\nabla^{k+1} u\|_{L^2(B)} + C_k, \end{aligned}$$

and we find

$$J \leq C_k \|\nabla^{k+1} u\|_{L^2(B)} + C_k$$

as before.

In any case, inequality (4-3) and Lemma 4.3 now may be invoked to obtain

$$\frac{d}{dt}(\|\nabla \partial_\phi^k u\|_{L^2(B)}) \leq C_k \|\nabla \partial_\phi^k u\|_{L^2(B)} + C_k,$$

and our claim follows. □

Local H^k -bounds. The bounds established so far all require the initial data to be sufficiently smooth for the estimate at hand and do not yet allow to show smoothing of the flow. For the latter purpose we next prove a second set of “intermediate” estimates that in combination with the first set of estimates later will allow boot-strapping. Moreover, in contrast to the estimates in Lemma 4.3, the following estimates may be localized. This will be important for showing regularity of the flow at blow-up times away from concentration points of the energy on ∂B .

For the localized estimates, fix a point $z_0 \in \partial B$ and some radius $0 < R_0 < \frac{1}{4}$ and for $k \in \mathbb{N}$ set $R_k = 2^{-k} R_0$ and $\varphi_k = \varphi_{z_0, R_k}$. Set $\varphi_k = 1$ for each $k \in \mathbb{N}$ for the analogous global bounds.

We first establish the following localized version of Lemma 4.1.

Lemma 4.7. *With a constant $C > 0$ depending only on N ,*

$$\frac{d}{dt} \left(\int_{\partial B} |u_\phi|^2 \varphi_1^2 d\phi \right) + \int_B |\nabla u_\phi|^2 \varphi_1^2 dz \leq C \int_B |\nabla u|^2 |u_\phi|^2 \varphi_1^2 dz + C R_0^{-2} E(u_0).$$

Proof. Similar to the proof of Lemma 4.1, we compute

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\int_{\partial B} |u_\phi|^2 \varphi_1^2 d\phi \right) &= \int_{\partial B} u_\phi \cdot u_{\phi,t} \varphi_1^2 d\phi = - \int_{\partial B} \partial_\phi (u_\phi \varphi_1^2) \cdot u_t d\phi \\ &= \int_{\partial B} \partial_\phi (u_\phi \varphi_1^2) \cdot d\pi_N(u) u_r d\phi = - \int_{\partial B} (u_\phi \cdot u_r \varphi - u_\phi \cdot \partial_\phi (v(u)v(u) \cdot u_r)) \varphi_1^2 d\phi \\ &= -\frac{1}{2} \int_{\partial B} \partial_r (|u_\phi|^2) \varphi_1^2 d\phi - \int_{\partial B} u_\phi \cdot dv(u) u_\phi v(u) \cdot u_r \varphi_1^2 d\phi. \end{aligned}$$

With $\Delta |u_\phi|^2 = 2|\nabla u_\phi|^2$ we obtain

$$\frac{1}{2} \int_{\partial B} \partial_r (|u_\phi|^2) \varphi_1^2 d\phi = \int_B |\nabla u_\phi|^2 \varphi_1^2 dz + \int_B \nabla |u_\phi|^2 \varphi_1 \nabla \varphi_1 dz,$$

where

$$\left| \int_B \nabla |u_\phi|^2 \varphi_1 \nabla \varphi_1 dz \right| \leq \frac{1}{4} \int_B |\nabla u_\phi|^2 \varphi_1^2 dz + C \int_B |u_\phi|^2 |\nabla \varphi_1|^2 dz$$

by Young’s inequality. Finally, we can bound

$$\begin{aligned} \int_{\partial B} u_r \cdot v(u) u_\phi \cdot dv(u) u_\phi \varphi_1^2 d\phi &= \int_B \nabla u \cdot \nabla (v(u) u_\phi \cdot dv(u) u_\phi \varphi_1^2) dz \\ &\leq C \int_B (|\nabla u_\phi| |\nabla u| |u_\phi| + |\nabla u|^2 |u_\phi|^2) \varphi_1^2 dz + C \int_B |\nabla u| |\nabla \varphi_1| |u_\phi|^2 \varphi_1 dz \\ &\leq \frac{1}{4} \int_B |\nabla u_\phi|^2 \varphi_1^2 dz + C \int_B |\nabla u|^2 |u_\phi|^2 \varphi_1^2 dz + C \int_B |\nabla u|^2 |\nabla \varphi_1|^2 dz, \end{aligned}$$

and our claim follows. □

We need a substitute for the global bound (4-3). For this, we note that (4-1) also implies the pointwise bound

$$|u_{rr}|^2 \leq \frac{2|u_{\phi\phi}|^2}{r^4} + \frac{2|u_r|^2}{r^2};$$

hence we have

$$|\nabla^2 u|^2 \leq C(|\nabla u_{\phi}|^2 + |\nabla u|^2) \quad \text{in } B_{R_0}(z_0)$$

with an absolute constant $C > 0$, uniformly in $z_0 \in \partial B$ and $0 < R_0 < \frac{1}{4}$. By induction then, similarly, we have

$$|\nabla^{k+1} u|^2 \leq C(|\nabla^k \partial_{\phi} u|^2 + |\nabla^k u|^2) \leq C \sum_{j=0}^k |\nabla \partial_{\phi}^j u|^2 \quad \text{in } B_{R_0}(z_0) \tag{4-9}$$

with an absolute constant $C = C(k) > 0$, uniformly in $z_0 \in \partial B$ and $0 < R_0 < \frac{1}{4}$ for any $k \in \mathbb{N}$.

Likewise, as a substitute for the global nonconcentration condition (4-4) we now suppose that $z_0 \in \partial B$ is not a concentration point in the sense that for suitably chosen $\delta > 0$ to be determined in the sequel and some $0 < R_0 < \frac{1}{4}$ as above,

$$\sup_{0 < t < T_0} \int_{B_{R_0}(z_0) \cap B} |\nabla u(t)|^2 dz < \delta. \tag{4-10}$$

We then obtain the following localized version of Proposition 4.2.

Proposition 4.8. *There exist constants $\delta > 0$ and $C > 0$ independent of $R_0 > 0$ such that whenever (4-10) holds then for any $T \leq T_0$ we have*

$$\sup_{0 < t < T} \int_{\partial B} |u_{\phi}(t)|^2 \varphi_1^2 d\phi + \int_0^T \int_B |\nabla u_{\phi}|^2 \varphi_1^2 dz dt \leq 2 \int_{\partial B} |u_{0,\phi}|^2 \varphi_1^2 d\phi + CT R_0^{-2} E(u_0).$$

Proof. With the help of inequality (A-1) in the Appendix we can bound

$$\int_B |\nabla u|^4 \varphi_1^2 dz \leq C\delta \int_{B_R(z_i)} |\nabla^2 u|^2 \varphi_1^2 dz + C\delta R_0^{-2} \int_{B_R(z_i)} |\nabla u|^2 dz.$$

Thus, for sufficiently small $\delta > 0$ our claim follows from Lemma 4.7. □

The next lemma again prepares for a proposition that later will allow us to obtain higher-derivative bounds by induction. Note the differences to Lemma 4.3.

Lemma 4.9. *For any $k \geq 2$, with a constant $C > 0$ depending only on k and N , for the solution $u = u(t)$ to (1-3) and (1-4) for any $0 < t < T_0$,*

$$\begin{aligned} & \frac{d}{dt} (\|\partial_{\phi}^k u \varphi_k\|_{L^2(\partial B)}^2) + \|\nabla \partial_{\phi}^k u \varphi_k\|_{L^2(B)}^2 \\ & \leq C \sum_{\substack{1 \leq j_i \leq k \\ \sum_i j_i \leq 2k+2}} \left\| \prod_i \nabla^{j_i} u \varphi_k^2 \right\|_{L^1(B)} + C \sum_{\substack{1 \leq j_0, j_i \leq k \\ \sum_{i \geq 0} j_i \leq k+1}} \left\| \prod_{i > 0} \nabla^{j_i} u \nabla^{j_0} \varphi_k \right\|_{L^2(B)}^2 + C R_0^{-2k} E(u_0). \end{aligned}$$

Proof. Fix $k \geq 2$. With $\Delta|\partial_\phi^k u|^2 = 2|\nabla\partial_\phi^k u|^2$ we compute

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\partial_\phi^k u \varphi_k\|_{L^2(\partial B)}^2) &= (-1)^k \int_{\partial B} \partial_\phi^k (\partial_\phi^k u \varphi_k^2) \cdot u_t \, d\phi = (-1)^{k+1} \int_{\partial B} \partial_\phi^k (\partial_\phi^k u \varphi_k^2) \cdot (u_r - v(u)v(u) \cdot u_r) \, d\phi \\ &= -\frac{1}{2} \int_{\partial B} \partial_r (|\partial_\phi^k u|^2) \varphi_k^2 \, d\phi + \int_{\partial B} \partial_\phi^k u \cdot \partial_\phi^k (v(u)v(u) \cdot u_r) \varphi_k^2 \, d\phi \\ &= - \int_B |\nabla\partial_\phi^k u|^2 \varphi_k^2 \, dz - \int_B \nabla (|\partial_\phi^k u|^2) \varphi_k \nabla \varphi_k \, dz + I, \end{aligned}$$

where the term $\int_B \nabla (|\partial_\phi^k u|^2) \varphi_k \nabla \varphi_k \, dz$ can be bounded as claimed. We use the decomposition

$$I = \int_{\partial B} \partial_\phi^k u \cdot \partial_\phi^k (v(u)v(u) \cdot u_r) \varphi_k^2 \, d\phi = \sum_{j=0}^k \binom{k}{j} I_j$$

with

$$I_j = (\partial_\phi^k u \cdot \partial_\phi^j (v(u)v(u)) \varphi_k^2, \partial_\phi^{k-j} u_r)_{L^2(\partial B)} = (\nabla(\partial_\phi^k u \cdot \partial_\phi^j (v(u)v(u)) \varphi_k^2), \nabla \partial_\phi^{k-j} u)_{L^2(B)}, \quad 0 \leq j \leq k.$$

For $1 \leq j \leq k$ we bound

$$\begin{aligned} |I_j| \leq C \sum_{0 \leq i \leq j} \|\nabla \partial_\phi^k u \varphi_k\|_{L^2(B)} \|\partial_\phi^{j-i} v(u) \partial_\phi^i v(u) \nabla \partial_\phi^{k-j} u \varphi_k\|_{L^2(B)} \\ + C \sum_{0 \leq i \leq j} \|\partial_\phi^k u \cdot \nabla(\partial_\phi^{j-i} v(u) \partial_\phi^i v(u) \varphi_k^2) \cdot \nabla \partial_\phi^{k-j} u\|_{L^1(B)}. \end{aligned}$$

By the chain rule then for $1 \leq j \leq k$ we have

$$\begin{aligned} |I_j| \leq C \sum_{\substack{1 \leq j_i \leq k \\ \Sigma_i j_i = k+1}} \|\nabla \partial_\phi^k u \varphi_k\|_{L^2(B)} \left\| \prod_i \nabla^{j_i} u \varphi_k \right\|_{L^2(B)} \\ + C \sum_{\substack{1 \leq j_i \leq k \\ \Sigma_i j_i = k+2}} \left\| \partial_\phi^k u \cdot \prod_i \nabla^{j_i} u \varphi_k^2 \right\|_{L^1(B)} + C \sum_{\substack{1 \leq j_i \leq k \\ \Sigma_i j_i = k+1}} \left\| \partial_\phi^k u \cdot \prod_i \nabla^{j_i} u \varphi_k \nabla \varphi_k \right\|_{L^1(B)}. \end{aligned}$$

By Cauchy–Schwarz and Young’s inequalities then we can bound

$$\begin{aligned} \sum_{1 \leq j \leq k} |I_j| &\leq \frac{1}{4} \|\nabla \partial_\phi^k u \varphi_k\|_{L^2(B)}^2 + C \sum_{\substack{1 \leq j_i \leq k \\ \Sigma_i j_i = k+1}} \left\| \prod_i \nabla^{j_i} u \varphi_k \right\|_{L^2(B)}^2 \\ &\quad + C \sum_{\substack{1 \leq j_i \leq k \\ \Sigma_i j_i = 2k+2}} \left\| \prod_i \nabla^{j_i} u \varphi_k^2 \right\|_{L^1(B)} + C \|\partial_\phi^k u \nabla \varphi_k\|_{L^2(B)}^2 \\ &\leq \frac{1}{4} \|\nabla \partial_\phi^k u \varphi_k\|_{L^2(B)}^2 + C \sum_{\substack{1 \leq j_i \leq k \\ \Sigma_i j_i = 2k+2}} \left\| \prod_i \nabla^{j_i} u \varphi_k^2 \right\|_{L^1(B)} + C \|\partial_\phi^k u \nabla \varphi_k\|_{L^2(B)}^2, \end{aligned}$$

as claimed. Finally, with

$$v(u) \cdot u_{\phi r} = (\text{dist}_N(u))_{\phi r} - u_r \cdot dv(u)u_\phi$$

as in the proof of [Lemma 4.3](#), for $j = 0$ we can write

$$v(u) \cdot \partial_\phi^k u_r = \partial_\phi^{k-1} (v(u) \cdot u_{\phi r}) + II = \partial_\phi^k (\text{dist}_N(u))_r + III,$$

where the terms in II and III involve products of at least two derivatives of orders between 1 and k of u . Thus we have

$$I_0 = (\partial_\phi^k u \cdot \nu(u)\varphi_k^2, \nu(u) \cdot \partial_\phi^k u_r)_{L^2(\partial B)} = (\partial_\phi^k u \cdot \nu(u)\varphi_k^2, \partial_\phi^k(\text{dist}_N(u))_r)_{L^2(\partial B)} + II_0$$

with a term II_0 that can be dealt with in the same way as the terms I_j , $1 \leq j \leq k$.

Using the divergence theorem and integrating by parts we can write the leading term as

$$\begin{aligned} \hat{I}_0 &:= (\partial_\phi^k u \cdot \nu(u)\varphi_k^2, \partial_\phi^k(\text{dist}_N(u))_r)_{L^2(\partial B)} \\ &= (\nabla(\partial_\phi^k u \cdot \nu(u)\varphi_k^2), \nabla\partial_\phi^k(\text{dist}_N(u)))_{L^2(B)} + (\partial_\phi^k u \cdot \nu(u)\varphi_k^2, \Delta\partial_\phi^k(\text{dist}_N(u)))_{L^2(B)} \\ &= (\nabla(\partial_\phi^k u \cdot \nu(u)\varphi_k^2), \nabla\partial_\phi^k(\text{dist}_N(u)))_{L^2(B)} - (\partial_\phi(\partial_\phi^k u \cdot \nu(u)\varphi_k^2), \Delta\partial_\phi^{k-1}(\text{dist}_N(u)))_{L^2(B)} \end{aligned}$$

to see that this term may be bounded:

$$|\hat{I}_0| \leq C\|(|\nabla\partial_\phi^k u| + |\partial_\phi^k u \nabla u|)\varphi_k + |\partial_\phi^k u \nabla\varphi_k|\|_{L^2(B)}\|\nabla^{k+1}(\text{dist}_N(u))\varphi_k\|_{L^2(B)}.$$

But by elliptic regularity we again have

$$\begin{aligned} \|\nabla^{k+1}(\text{dist}_N(u))\varphi_k\|_{L^2(B)} &\leq \|\nabla^{k+1}(\text{dist}_N(u))\varphi_k\|_{L^2(B)} + C \sum_{1 \leq j \leq k+1} \|\nabla^{k+1-j}(\text{dist}_N(u))\nabla^j\varphi_k\|_{L^2(B)} \\ &\leq C\|\Delta(\text{dist}_N(u))\varphi_k\|_{H^{k-1}(B)} + C \sum_{1 \leq j \leq k+1} \|\nabla^{k+1-j}(\text{dist}_N(u))\nabla^j\varphi_k\|_{L^2(B)}, \end{aligned}$$

where from (3-5) we can bound the first term on the right:

$$\|\Delta(\text{dist}_N(u))\varphi_k\|_{H^{k-1}(B)} \leq \sum_{0 \leq j < k} \|\nabla^j(\nabla u \cdot d\nu(u)\nabla u\varphi_k)\|_{L^2(B)} \leq C \sum_{\substack{0 \leq j_0 < k \\ 1 \leq j_i \leq k \\ \Sigma_{i \geq 0} j_i \leq k+1}} \left\| \prod_i \nabla^{j_i} u \nabla^{j_0} \varphi_k \right\|_{L^2(B)}.$$

Moreover, using that $\text{dist}_N(u) = 0$ on ∂B , with the help of Poincaré's inequality we find the bound

$$\|\text{dist}_N(u)\nabla^{k+1}\varphi_k\|_{L^2(B)}^2 \leq CR_k^{-2k}\|\nabla(\text{dist}_N(u))\|_{L^2(B_{R_k}(z_0))}^2 \leq CR_0^{-2k}E(u).$$

The remaining terms for $1 \leq j \leq k$ can be estimated as

$$\|\nabla^{k+1-j}(\text{dist}_N(u))\nabla^j\varphi_k\|_{L^2(B)} \leq C \sum_{\substack{1 \leq j_i \leq k \\ \Sigma_i j_i = k+1-j}} \left\| \prod_i \nabla^{j_i} u \nabla^j \varphi_k \right\|_{L^2(B)}$$

via the chain rule. Thus, finally, we obtain the bound

$$\|\nabla^{k+1}(\text{dist}_N(u))\varphi_k\|_{L^2(B)}^2 \leq C \sum_{\substack{1 \leq j_0, j_i \leq k \\ \Sigma_{i \geq 0} j_i \leq k+1}} \left\| \prod_{i>0} \nabla^{j_i} u \nabla^{j_0} \varphi_k \right\|_{L^2(B)}^2 + CR_0^{-2k}E(u_0).$$

By Cauchy–Schwarz and Young's inequalities thus we can bound

$$|\hat{I}_0| \leq \frac{1}{4}\|\nabla\partial_\phi^k u\varphi_k\|_{L^2(B)}^2 + C\|\partial_\phi^k u \nabla u\varphi_k\|_{L^2(B)}^2 + C \sum_{\substack{1 \leq j_0, j_i \leq k \\ \Sigma_{i \geq 0} j_i \leq k+1}} \left\| \prod_{i>0} \nabla^{j_i} u \nabla^{j_0} \varphi_k \right\|_{L^2(B)}^2 + CR_0^{-2k}E(u_0),$$

and together with our above estimate for the terms I_j , $j \geq 1$, our claim follows. \square

Proposition 4.10. *There exists a constant $\delta > 0$ independent of $R_0 > 0$ such that whenever (4-10) holds then for any $T \leq T_0$ with a constant $C_2 = C_2(T, R, u_0) > 0$ bounded by the terms on the right-hand side in the statement of Proposition 4.8 there holds the estimate*

$$\sup_{0 < t < T} \int_{\partial B} |u_{\phi\phi}(t)|^2 \varphi_2^2 d\phi + \int_0^T \int_B |\nabla u_{\phi\phi}|^2 \varphi_2^2 dz dt \leq C_2 \int_{\partial B} |u_{0,\phi\phi}|^2 \varphi_2^2 d\phi + C_2.$$

Proof. For $k = 2$, with the help of Young’s inequality we can bound

$$\begin{aligned} J_1 &= \sum_{\substack{1 \leq j_i \leq k \\ \sum_i j_i \leq 2k+2}} \left\| \prod_i \nabla^{j_i} u \varphi_k^2 \right\|_{L^1(B)} \leq C \| (|\nabla^2 u|^3 + |\nabla^2 u|^2 |\nabla u|^2 + |\nabla^2 u| |\nabla u|^4 + |\nabla u|^6 + 1) \varphi_2^2 \|_{L^1(B)} \\ &\leq C \| (|\nabla^2 u|^3 + |\nabla u|^6 + 1) \varphi_2^2 \|_{L^1(B)} \end{aligned}$$

and

$$J_2 = \sum_{\substack{1 \leq j_0, j_i \leq k \\ \sum_i j_i \leq k+1}} \left\| \prod_{i>0} \nabla^{j_i} u \nabla^{j_0} \varphi_2 \right\|_{L^2(B)}^2 \leq C \| (|\nabla^2 u|^2 + |\nabla u|^4 + 1) |\nabla \varphi_2|^2 + (|\nabla u|^2 + 1) |\nabla^2 \varphi_2|^2 \|_{L^1(B)}.$$

Observing that $\varphi_1 = 1$ on the support of φ_2 , by (A-2) for the first term in J_1 we have

$$\begin{aligned} \| |\nabla^2 u|^3 \varphi_2^2 \|_{L^1(B)} &\leq \| \nabla^2 u \varphi_2 \|_{L^4(B)}^2 \| \nabla^2 u \varphi_1 \|_{L^2(B)} \leq C \| \nabla^2 u \varphi_2 \|_{H^1(B)} \| \nabla^2 u \varphi_2 \|_{L^2(B)} \| \nabla^2 u \varphi_1 \|_{L^2(B)} \\ &\leq C (\| \nabla^3 u \varphi_2 \|_{L^2(B)} + \| \nabla^2 u \varphi_1 \|_{L^2(B)}) \| \nabla^2 u \varphi_2 \|_{L^2(B)} \| \nabla^2 u \varphi_1 \|_{L^2(B)}. \end{aligned}$$

Moreover, arguing as in (A-1) for the function $|\nabla u|^6 \varphi_2^2$ in place of $|v|^4 \varphi^2$, we can bound

$$\begin{aligned} \int_B |\nabla u|^6 \varphi_2^2 dz &\leq C \left(\int_B (|\nabla^2 u| |\nabla u|^2 \varphi_2 + |\nabla u|^3 |\nabla \varphi_2|) dz \right)^2 \\ &\leq C \left(\int_B |\nabla^2 u|^3 \varphi_2^2 dz \right)^{2/3} \left(\int_B |\nabla u|^3 \varphi_2^{1/2} dz \right)^{4/3} + C \left(\int_B |\nabla u|^3 |\nabla \varphi_2| dz \right)^2, \end{aligned}$$

where by Hölder’s inequality we have

$$\int_B |\nabla u|^3 \varphi_2^{1/2} dz \leq \left(\int_B |\nabla u|^6 \varphi_2^2 dz \right)^{1/4} \left(\int_B |\nabla u|^2 \varphi_1^2 dz \right)^{3/4},$$

so that with Young’s inequality we obtain

$$\begin{aligned} \int_B |\nabla u|^6 \varphi_2^2 dz &\leq C \delta \left(\int_B |\nabla^2 u|^3 \varphi_2^2 dz \right)^{2/3} \left(\int_B |\nabla u|^6 \varphi_2^2 dz \right)^{1/3} + C \left(\int_B |\nabla u|^3 |\nabla \varphi_2| dz \right)^2 \\ &\leq \frac{1}{2} \int_B |\nabla u|^6 \varphi_2^2 dz + C \int_B |\nabla^2 u|^3 \varphi_2^2 dz + C \left(\int_B |\nabla u|^3 |\nabla \varphi_2| dz \right)^2. \end{aligned}$$

With Young’s inequality for suitable $\varepsilon > 0$, and using (4-9), we then can bound

$$\begin{aligned} J_1 &\leq C \| (|\nabla^2 u|^3 + 1) \varphi_2^2 \|_{L^1(B)} + C \| |\nabla u|^3 |\nabla \varphi_2| \|_{L^1(B)}^2 \\ &\leq \varepsilon \| \nabla^3 u \varphi_2 \|_{L^2(B)}^2 + C (1 + \| \nabla^2 u \varphi_2 \|_{L^2(B)}^2) \| \nabla^2 u \varphi_1 \|_{L^2(B)}^2 + C \| |\nabla u|^3 |\nabla \varphi_2| \|_{L^1(B)}^2 \\ &\leq \frac{1}{2} \| \nabla \partial_\phi^2 u \varphi_2 \|_{L^2(B)}^2 + C (1 + \| \nabla \partial_\phi u \varphi_2 \|_{L^2(B)}^2) \| \nabla \partial_\phi u \varphi_1 \|_{L^2(B)}^2 + C, \end{aligned}$$

where we also have estimated

$$\begin{aligned} \|\nabla u\|^3 \|\nabla \varphi_2\|_{L^1(B)}^2 &\leq C \|\nabla u \varphi_1\|_{L^4(B)}^4 \|\nabla u \varphi_1\|_{L^2(B)}^2 \\ &\leq C (\|\nabla^2 u \varphi_1\|_{L^2(B)}^2 + E(u)) \|\nabla u \varphi_1\|_{L^2(B)}^4 \leq C \|\nabla \partial_\phi u \varphi_1\|_{L^2(B)}^2 + C. \end{aligned}$$

Similarly, with (A-2) we have

$$J_2 \leq C \|\nabla^2 u \varphi_1\|_{L^2(B)}^2 + C.$$

Thus, from Lemma 4.9 we obtain

$$\frac{d}{dt} (\|\partial_\phi^2 u \varphi_2\|_{L^2(\partial B)}^2) + \frac{1}{2} \|\nabla \partial_\phi^2 u \varphi_2\|_{L^2(B)}^2 \leq C (1 + \|\nabla \partial_\phi u \varphi_2\|_{L^2(B)}^2) \|\nabla \partial_\phi u \varphi_1\|_{L^2(B)}^2 + C. \tag{4-11}$$

Denote by $C_1 = C_1(T, R, u_0) > 0$ a constant bounded by the terms on the right-hand side in the statement of Proposition 4.8. By elliptic regularity, using that $|\Delta(u\varphi_2)| \leq 2|\nabla u \nabla \varphi_2| + C$ we can bound

$$\begin{aligned} \|\nabla^2 u \varphi_2\|_{L^2(B)}^2 &\leq \|u \varphi_2\|_{H^2(B)}^2 + C \|\nabla u \nabla \varphi_2\|_{L^2(B)}^2 + C \\ &\leq C \|u \varphi_2\|_{H^2(\partial B)}^2 + \|\Delta(u\varphi_2)\|_{L^2(B)}^2 + C \|\nabla u \nabla \varphi_2\|_{L^2(B)}^2 + C \\ &\leq C \|\partial_\phi^2 u \varphi_2\|_{L^2(\partial B)}^2 + C E(u) + C_1. \end{aligned}$$

From (4-11) we then obtain the differential inequality

$$\frac{d}{dt} (1 + \|\partial_\phi^2 u \varphi_2\|_{L^2(\partial B)}^2) \leq C_1 (1 + \|\partial_\phi^2 u \varphi_2\|_{L^2(\partial B)}^2) \|\nabla \partial_\phi u \varphi_1\|_{L^2(B)}^2 + C_1;$$

that is,

$$\frac{d}{dt} (\log(1 + \|\partial_\phi^2 u \varphi_2\|_{L^2(\partial B)}^2)) \leq C_1 \|\nabla \partial_\phi u \varphi_1\|_{L^2(B)}^2 + C_1,$$

and the right-hand side is integrable in time by Proposition 4.8. The claim follows. □

We continue by induction.

Proposition 4.11. *There exists a constant $\delta > 0$ independent of $R_0 > 0$ with the following property. Whenever (4-10) holds, then, for any $k \geq 3$, any smooth $u_0 \in H^{1/2}(S^1; N)$, and any $T \leq T_0$,*

$$\sup_{0 < t < T} \int_{\partial B} |\partial_\phi^k u(t)|^2 \varphi_k^2 d\phi + \int_0^T \int_B |\nabla \partial_\phi^k u|^2 \varphi_k^2 dz dt \leq C_k \int_{\partial B} |\partial_\phi^k u_0|^2 \varphi_k^2 d\phi + C_k,$$

where we denote by $C_k = C_k(T, R, u_0) > 0$ a constant bounded by the terms on the right-hand side in the statement of the proposition for $k - 1$.

Proof. By Proposition 4.10 the claimed result holds true for $k = 2$. Suppose the claim holds true for some $k_0 \geq 2$ and let $k = k_0 + 1$. Note that by elliptic regularity, as in the proof of Proposition 4.10, we can bound

$$\begin{aligned} \|\nabla^k u \varphi_k\|_{L^2(B)}^2 &\leq \|u \varphi_k\|_{H^k(B)}^2 + C \sum_{j < k} \|\nabla^j u \nabla^{k-j} \varphi_k\|_{L^2(B)}^2 \\ &\leq C \|u \varphi_k\|_{H^k(\partial B)}^2 + C \|\Delta(u\varphi_k)\|_{H^{k-2}(B)}^2 + C \sum_{j < k} \|\nabla^j u \nabla^{k-j} \varphi_k\|_{L^2(B)}^2 \\ &\leq C \|\partial_\phi^k u \varphi_k\|_{L^2(\partial B)}^2 + C \sum_{j < k} \|\nabla^j u \nabla^{k-j} \varphi_k\|_{L^2(B)}^2 + C_k. \end{aligned}$$

By the induction hypothesis and Sobolev’s embedding $H^2(B) \hookrightarrow W^{1,4} \cap C^0(\bar{B})$ for $0 \leq t < T$, we then have the uniform bounds

$$\|\nabla^{k_0} u \varphi_{k_0}\|_{L^2(B)}^2 + \|\nabla^{k_0-1} u \varphi_{k_0}\|_{L^4(B)}^2 + \sum_{j=1}^{k_0-2} \|\nabla^j u \varphi_{k_0}\|_{L^\infty(B)}^2 \leq C_k,$$

and it follows that

$$\|\nabla^k u \varphi_k\|_{L^2(B)}^2 + \|\nabla^{k_0} u \varphi_k\|_{L^4(B)}^2 + \|\nabla^{k_0-1} u \varphi_k\|_{L^\infty(B)}^2 \leq C \|\partial_\phi^k u \varphi_k\|_{L^2(\partial B)}^2 + C_k.$$

Again let

$$\begin{aligned} J_1 &:= \sum_{\substack{1 \leq j_i \leq k \\ \sum_i j_i = 2k+2}} \left\| \prod_i \nabla^{j_i} u \varphi_k^2 \right\|_{L^1(B)} \\ &\leq \|(|\nabla^k u|^2 (|\nabla^2 u| + |\nabla u|^2) + |\nabla^k u| |\nabla^{k_0} u| |\nabla^3 u| + \dots + |\nabla u|^{2k+2}) \varphi_k^2\|_{L^1(B)} \end{aligned}$$

and set

$$J_2 = \sum_{\substack{1 \leq j_0, j_i \leq k \\ \sum_i j_i \geq 0, \sum_i j_i \leq k+1}} \left\| \prod_{i>0} \nabla^{j_i} u \nabla^{j_0} \varphi_k \right\|_{L^2(B)}^2.$$

Suppose $k_0 = 2$. Recalling that $\varphi_k = \varphi_k \varphi_{k_0}$, we can bound the terms

$$\begin{aligned} \| |\nabla^3 u|^2 (|\nabla^2 u| + |\nabla u|^2) \varphi_3^2 \|_{L^1(B)} &\leq \|\nabla^3 u \varphi_3\|_{L^4(B)}^2 (\|\nabla^2 u \varphi_2\|_{L^2(B)} + \|\nabla u \varphi_2\|_{L^4(B)}^2) \\ &\leq C_3 \|\nabla \partial_\phi^3 u \varphi_3\|_{L^2(B)} \|\nabla^3 u \varphi_3\|_{L^2(B)} + C_3 \|\nabla^3 u \varphi_2\|_{L^2(B)}^2 + C_3 \\ &\leq C_3 \|\nabla \partial_\phi^3 u \varphi_3\|_{L^2(B)} (\|\partial_\phi^3 u \varphi_3\|_{L^2(\partial B)} + 1) + C_3 \|\nabla \partial_\phi^2 u \varphi_2\|_{L^2(B)}^2 + C_3 \\ &\leq \varepsilon \|\nabla \partial_\phi^3 u \varphi_3\|_{L^2(B)}^2 + C_3 \|\partial_\phi^3 u \varphi_3\|_{L^2(\partial B)}^2 + C_3 \|\nabla \partial_\phi^2 u \varphi_2\|_{L^2(B)}^2 + C_3 \end{aligned}$$

and

$$\| |\nabla u|^8 \varphi_3^2 \|_{L^1(B)} \leq \|\nabla u \varphi_3\|_{L^\infty(B)}^2 \|\nabla u \varphi_2\|_{L^6(B)}^6 \leq C_3 \|\partial_\phi^3 u \varphi_3\|_{L^2(\partial B)}^2 + C_3$$

from the estimate of J_1 . Here we also have used (A-1) and (A-2) to bound

$$\begin{aligned} \|\nabla u \varphi_2\|_{L^6(B)}^3 &\leq \|\nabla (|\nabla u|^3 \varphi_2^3)\|_{L^1(B)} \leq C \|(|\nabla^2 u| \varphi_2 + |\nabla u| |\nabla \varphi_2|) |\nabla u|^2 \varphi_2^2 \|_{L^1(B)} \\ &\leq C (\|\nabla^2 u \varphi_2\|_{L^2(B)} + \|\nabla u \nabla \varphi_2\|_{L^2(B)}) \|\nabla u \varphi_2\|_{L^4(B)}^2 \\ &\leq C (\|\nabla^2 u \varphi_2\|_{L^2(B)} + \|\nabla u \nabla \varphi_2\|_{L^2(B)})^2 \|\nabla u \varphi_2\|_{L^2(B)} \leq C_3. \end{aligned}$$

Similarly, we can bound the remaining terms and the terms in J_2 to obtain

$$\frac{d}{dt} (\|\partial_\phi^3 u \varphi_3\|_{L^2(\partial B)}^2) + \frac{1}{2} \|\nabla \partial_\phi^3 u \varphi_3\|_{L^2(B)}^2 \leq C_3 (1 + \|\partial_\phi^3 u \varphi_3\|_{L^2(\partial B)}^2) (1 + \|\nabla \partial_\phi^2 u \varphi_2\|_{L^2(B)}^2) + C_3$$

from Lemma 4.9 and then

$$\frac{d}{dt} (\log(1 + \|\partial_\phi^3 u \varphi_3\|_{L^2(\partial B)}^2)) \leq C_3 (1 + \|\nabla \partial_\phi^2 u \varphi_2\|_{L^2(B)}^2),$$

where the right-hand side is integrable in time by Proposition 4.10. The claim for $k = 3$ thus follows.

For $k \geq 4$ the analysis is similar (but simpler) and may be left to the reader. □

5. Local existence

In order to show local existence we approximate the flow equation (1-3) by the equation

$$u_t = -(\varepsilon + d\pi_N(u))u_r \quad \text{on } \partial B, \tag{5-1}$$

where $\varepsilon > 0$ and where we smoothly extend the nearest-neighbor projection π_N , originally defined only in the ρ -neighborhood N_ρ of N , to the whole ambient \mathbb{R}^n . Our aim then is to show that for given smooth initial data u_0 the evolution problem (5-1), (1-4) admits a smooth solution u_ε which remains uniformly smoothly bounded on a uniform time interval as $\varepsilon \downarrow 0$. Fixing some $0 < \varepsilon < \frac{1}{2}$, we show existence for the problem (5-1) with data (1-4) by means of a fixed-point argument.

To set up the argument, fix smooth initial data $u_0 : S^1 \rightarrow N$ with harmonic extension $u_0 \in C^\infty(\bar{B}; \mathbb{R}^n)$ and some $k \geq 2$. For suitable $T > 0$ to be determined let

$$X = L^\infty([0, T]; H^{k+1}(B; \mathbb{R}^n)) \cap H^1(S^1 \times [0, T]; \mathbb{R}^n)$$

and set

$$V = \left\{ v \in X : v(0) = u_0, \Delta v(t) = 0 \text{ in } B \text{ for } 0 \leq t \leq T, \right. \\ \left. \|v\|_X^2 = \sup_{0 \leq t \leq T} \|v(t)\|_{H^{k+1}(B)}^2 + \int_0^T \int_{S^1} |v_t|^2 d\phi dt \leq 4R_0^2 \right\},$$

where $R_0 = \|u_0\|_{H^{k+1}(B)}$. We endow the space V with the metric derived from the seminorm

$$|v|_X^2 = \sup_{0 \leq t \leq T} \|\nabla v(t)\|_{L^2(B)}^2 + \int_0^T \int_{S^1} |v_t|^2 d\phi dt.$$

Note that this metric is positive definite on V in view of the initial condition that we impose.

Lemma 5.1. *V is a complete metric space.*

Proof. Let $(v_m)_{m \in \mathbb{N}} \subset V$ with $|v_l - v_m|_X \rightarrow 0$ ($l, m \rightarrow \infty$). By the theorem of Banach–Alaoglu a subsequence $v_m \rightharpoonup v$ weakly-* in $L^\infty([0, T]; H^{k+1}(B))$ with $v_{m,t} \rightarrow v_t$ weakly in $L^2([0, T] \times S^1)$, and by weak lower semicontinuity of the norm

$$\|v\|_X^2 \leq \limsup_{m \rightarrow \infty} \|v_m\|_X^2 \leq 4R_0^2.$$

Moreover, we have $\Delta v(t) = 0$ for all $0 \leq t \leq T$ and $v(0) = u_0$ by compactness of the trace operator $H^1(S^1 \times [0, T]) \ni u \mapsto u(0) \in L^2(S^1)$. Hence $v \in V$.

Moreover, we have

$$|v_l - v|_X \leq \limsup_{m \rightarrow \infty} |v_l - v_m|_X \rightarrow 0 \quad \text{as } l \rightarrow \infty. \quad \square$$

Lemma 5.2. *There is $T_2 > 0$ such that for any $T \leq T_2$ and any $v \in V$ there is a solution $u = \Phi(v) \in V$ of the equation*

$$u_t = -(\varepsilon + d\pi_N(v))u_r \quad \text{on } \partial B \times [0, T_2] \tag{5-2}$$

satisfying (1-4).

Proof. For $v \in V$ we construct a solution $u = \Phi(v) \in X$ of (5-2) via Galerkin approximation. For this let $(\varphi_l)_{l \in \mathbb{N}_0}$ be Steklov eigenfunctions of the Laplacian, satisfying

$$\Delta \varphi_l = 0 \quad \text{in } B$$

with boundary condition

$$\partial_r \varphi_l = \lambda_l \varphi_l \quad \text{on } \partial B, \quad l \in \mathbb{N}_0.$$

Note that the Steklov eigenvalues are given by $\lambda_0 = 0$ and $\lambda_{2l-1} = \lambda_{2l} = l$, $l \in \mathbb{N}$. In fact, we may choose $\varphi_0 \equiv 1/\sqrt{2\pi}$ and

$$\varphi_{2l-1}(r e^{i\theta}) = \frac{1}{\sqrt{\pi}} r^l \sin(l\theta), \quad \varphi_{2l}(r e^{i\theta}) = \frac{1}{\sqrt{\pi}} r^l \cos(l\theta), \quad l \in \mathbb{N} \tag{5-3}$$

to obtain an orthonormal basis for $L^2(S^1)$ consisting of these functions. Given $m \in \mathbb{N}$ then let

$$u^{(m)}(t, z) = \sum_{l=0}^m a_l^{(m)}(t) \varphi_l(z)$$

solve the system of equations

$$\begin{aligned} \partial_t a_l^{(m)} &= (\varphi_l, u_t^{(m)})_{L^2(S^1)} = -(\varphi_l, (\varepsilon + d\pi_N(v))u_r^{(m)})_{L^2(S^1)} \\ &= -\sum_{j=0}^m a_j^{(m)} \lambda_j (\varphi_l, (\varepsilon + d\pi_N(v))\varphi_j)_{L^2(S^1)}, \quad 0 \leq l \leq m. \end{aligned} \tag{5-4}$$

Since for any $m \in \mathbb{N}$ the coefficients $\lambda_j (\varphi_l, (\varepsilon + d\pi_N(v))\varphi_j)_{L^2(S^1)}$ of this system are uniformly bounded for any $v \in V$, for any $m \in \mathbb{N}$ there exists a unique solution $a^{(m)} = (a_l^{(m)})_{0 \leq l \leq m}$ of (5-4) on $[0, T]$ with initial data $a_l^{(m)}(0) = a_{l0} = (u_0, \varphi_l)_{L^2(S^1)}$, $0 \leq l \leq m$.

Note that for any $m \in \mathbb{N}$ and any $j \in \mathbb{N}_0$ we have

$$\partial_\phi^{2j}(ru_r^{(m)}) \in \text{span}\{\varphi_l : 0 \leq l \leq m\},$$

and the function $\partial_\phi^{2j}u^{(m)}$ is harmonic. In particular, for $j = 0$ we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\nabla u^{(m)}\|_{L^2(B)}^2) &= \int_B \nabla u^{(m)} \nabla u_t^{(m)} dz = (u_r^{(m)}, u_t^{(m)})_{L^2(S^1)} = -(u_r^{(m)}, (\varepsilon + d\pi_N(v))u_r^{(m)})_{L^2(S^1)} \\ &= -\varepsilon \|u_r^{(m)}\|_{L^2(S^1)}^2 - \|d\pi_N(v)u_r^{(m)}\|_{L^2(S^1)}^2 \leq -\frac{1}{2} \|u_t^{(m)}\|_{L^2(S^1)}^2 \leq 0, \end{aligned} \tag{5-5}$$

and we find the uniform H^1 -bound

$$\begin{aligned} \sup_{t \geq 0} \|\nabla u^{(m)}(t)\|_{L^2(B)}^2 + \varepsilon \|u_r^{(m)}\|_{L^2([0, \infty[\times S^1)}^2 + \|u_t^{(m)}\|_{L^2([0, \infty[\times S^1)}^2 &\leq 2\|\nabla u^{(m)}(0)\|_{L^2(B)}^2 \\ &\leq 2\|\nabla u_0\|_{L^2(B)}^2 \leq 2R_0^2. \end{aligned} \tag{5-6}$$

Moreover, for $j = k \in \mathbb{N}$ as in the definition of X , upon integrating by parts we find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\nabla \partial_\phi^k u^{(m)}\|_{L^2(B)}^2) &= (-1)^k \int_B \nabla \partial_\phi^{2k} u^{(m)} \nabla u_t^{(m)} dz = (-1)^k (\partial_\phi^{2k} u_r^{(m)}, u_t^{(m)})_{L^2(S^1)} \\ &= (-1)^{k+1} (\partial_\phi^{2k} u_r^{(m)}, (\varepsilon + d\pi_N(v))u_r^{(m)})_{L^2(S^1)} \\ &= -\varepsilon \|\partial_\phi^k u_r^{(m)}\|_{L^2(S^1)}^2 - \|d\pi_N(v) \partial_\phi^k u_r^{(m)}\|_{L^2(S^1)}^2 + I, \end{aligned} \tag{5-7}$$

where $I = \sum_{j=1}^k \binom{k}{j} I_j$ with

$$I_j = -(\partial_\phi^k u_r^{(m)}, \partial_\phi^j (d\pi_N(v)) \partial_\phi^{k-j} u_r^{(m)})_{L^2(S^1)}$$

similar to the proof of Lemma 4.3. However, now we simply bound

$$|I_j| \leq C \sum_{\Sigma_i j_i=j} \|\partial_\phi^k u_r^{(m)}\|_{L^2(S^1)} \left\| \prod_i \partial_\phi^{j_i} v \partial_\phi^{k-j} u_r^{(m)} \right\|_{L^2(S^1)}, \quad 1 \leq j \leq k.$$

Note that by compactness of Sobolev's embedding $H^1(S^1) \hookrightarrow L^\infty(S^1)$ and Ehrlich's lemma for any number $1 \leq j \leq k$, any $\delta > 0$ we can bound

$$\begin{aligned} \|\partial_\phi^{k-j} u_r^{(m)}\|_{L^\infty(S^1)} &\leq \delta \|\partial_\phi^{k-j+1} u_r^{(m)}\|_{L^2(S^1)} + C(\delta) \|\partial_\phi^{k-j} u_r^{(m)}\|_{L^2(S^1)} \\ &\leq 2\delta \|\partial_\phi^k u_r^{(m)}\|_{L^2(S^1)} + C(\delta) \|u_r^{(m)}\|_{L^2(S^1)}. \end{aligned}$$

On the other hand, for any $v \in V$ by the trace theorem we have

$$\|\partial_\phi^k v\|_{L^2(S^1)} \leq C \|\partial_\phi^k v\|_{H^1(B)} \leq C \|v\|_{H^{k+1}(B)} \leq C R_0,$$

and we therefore also can bound

$$\|\partial_\phi^j v\|_{L^\infty(S^1)} \leq C \|\partial_\phi^k v\|_{L^2(S^1)} + \|\partial_\phi^j v\|_{L^2(S^1)} \leq C \|v\|_{H^{k+1}(B)} \leq C R_0$$

for any $1 \leq j < k$.

Thus, for sufficiently small $\delta > 0$ with a constant $C > 0$ depending on $\varepsilon > 0$ and R_0 ,

$$|I| \leq \frac{1}{2} \varepsilon \|\partial_\phi^k u_r^{(m)}\|_{L^2(S^1)}^2 + C \|u_r^{(m)}\|_{L^2(S^1)}^2,$$

and from (5-7) with the help of (3-7) we obtain the inequality

$$\begin{aligned} \frac{d}{dt} (\|\nabla \partial_\phi^k u^{(m)}\|_{L^2(B)}^2) &\leq C \|u_r^{(m)}\|_{L^2(S^1)}^2 = C \|u_\phi^{(m)}\|_{L^2(S^1)}^2 \leq C \|u_\phi^{(m)}\|_{H^1(B)}^2 \\ &\leq C \|\nabla \partial_\phi^k u^{(m)}\|_{L^2(B)}^2 + C \|\nabla u^{(m)}\|_{L^2(B)}^2 \leq C(1 + \|\nabla \partial_\phi^k u^{(m)}\|_{L^2(B)}^2), \end{aligned}$$

where we recall (5-6) for the last conclusion.

It follows that for suitably small $T > 0$ there holds $\|u^{(m)}\|_X^2 \leq 4R_0^2$ for all $m \in \mathbb{N}$. Thus, there is a sequence $m \rightarrow \infty$ such that $u^{(m)} \rightharpoonup u$ weakly-* in $L^\infty([0, T]; H^{k+1}(B))$ with $u_t^{(m)} \rightharpoonup u_t$ weakly in $L^2([0, T] \times S^1)$, where $u =: \Phi(v) \in V$ solves (5-2). □

Lemma 5.3. *There is $T > 0$ such that, for $v_1, v_2 \in V$,*

$$|\Phi(v_1) - \Phi(v_2)|_X \leq \frac{1}{2} |v_1 - v_2|_X.$$

Proof. Let $T_2 > 0$ be as determined in Lemma 5.2 and fix some $0 < T \leq T_2$. For $v_1, v_2 \in V$ then we have $u_i =: \Phi(v_i) \in V$, $i = 1, 2$. Set $w = u_1 - u_2$ and $v = v_1 - v_2$, and compute

$$w_t = -(\varepsilon + d\pi_N(v_1))w_r - (d\pi_N(v_1) - d\pi_N(v_2))u_{2,r} \quad \text{on } \partial B = S^1. \tag{5-8}$$

Multiplying with w_r and integrating we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\nabla w\|_{L^2(B)}^2) &= \int_B \nabla w \nabla w_t \, dx = (w_r, w_t)_{L^2(S^1)} \\ &= -\varepsilon \|w_r\|_{L^2(S^1)}^2 - \|d\pi_N(v_1)w_r\|_{L^2(S^1)}^2 - (w_r, (d\pi_N(v_1) - d\pi_N(v_2))u_{2,r})_{L^2(S^1)}, \end{aligned}$$

where with $\|u_{2,r}\|_{L^\infty(S^1)} \leq C \|u_2\|_{H^3(B)} \leq CR_0$ we can bound

$$\begin{aligned} |(w_r, (d\pi_N(v_1) - d\pi_N(v_2))u_{2,r})_{L^2(S^1)}| &\leq C \|w_r\|_{L^2(S^1)} \|v\|_{L^2(S^1)} \|u_{2,r}\|_{L^\infty(S^1)} \\ &\leq C \|w_r\|_{L^2(S^1)} \|v\|_{L^2(S^1)} \leq \frac{1}{2}\varepsilon \|w_r\|_{L^2(S^1)}^2 + C(\varepsilon) \|v\|_{L^2(S^1)}^2. \end{aligned}$$

Thus, with a constant $C = C(\varepsilon) > 0$ we find

$$\frac{d}{dt} \|\nabla w\|_{L^2(B)}^2 + \varepsilon \|w_r\|_{L^2(S^1)}^2 \leq C \|v\|_{L^2(S^1)}^2. \tag{5-9}$$

Similarly, from (5-8) we can bound

$$\|w_t\|_{L^2(S^1)}^2 \leq C \|w_r\|_{L^2(S^1)}^2 + C \|v\|_{L^2(S^1)}^2. \tag{5-10}$$

Integrating over $0 \leq t \leq T$ and observing that we have

$$\sup_{0 \leq t \leq T} \|v(t)\|_{L^2(S^1)}^2 \leq \left(\int_0^T \|v_t(t)\|_{L^2(S^1)} \, dt \right)^2 \leq T \int_0^T \|v_t(t)\|_{L^2(S^1)}^2 \, dt,$$

from (5-9) we first obtain

$$\sup_{0 \leq t \leq T} \|\nabla w(t)\|_{L^2(B)}^2 + \varepsilon \|w_r\|_{L^2([0,T] \times S^1)}^2 \leq CT \sup_{0 \leq t \leq T} \|v(t)\|_{L^2(S^1)}^2 \leq CT^2 |v|_X^2,$$

which we may use together with (5-10) to bound

$$|w|_X^2 = \sup_{0 \leq t \leq T} \|\nabla w(t)\|_{L^2(B)}^2 + \|w_t\|_{L^2([0,T] \times S^1)}^2 \leq CT^2 |v|_X^2.$$

For sufficiently small $T > 0$ then our claim follows. □

Thus, by Banach’s fixed-point theorem, for any $\varepsilon > 0$ and any smooth $u_0 \in H^{1/2}(S^1; N)$, there exists $T > 0$ and a solution $u = u(t) \in V$ of the initial value problem (5-1), (1-4). We now show that the number $T > 0$ may be chosen uniformly as $\varepsilon \downarrow 0$. Indeed, we have the following result.

Lemma 5.4. *There exists a constant $C > 0$ such that, for any $k \geq 2$, any smooth $u_0 \in H^{1/2}(S^1; N)$, and any $0 < \varepsilon \leq \frac{1}{2}$ for the solution u to (5-1) with $u(0) = u_0$,*

$$\frac{d}{dt} (\|\nabla \partial_\phi^k u\|_{L^2(B)}^2) \leq C(1 + \|\nabla u\|_{L^2(B)}^2 + \|\nabla \partial_\phi^k u\|_{L^2(B)}^2)^{k+3}.$$

Proof. Similar to the proof of Lemma 5.2, for given $2 \leq k \in \mathbb{N}$ we compute

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\nabla \partial_\phi^k u\|_{L^2(B)}^2) &= (-1)^k \int_B \nabla \partial_\phi^{2k} u \nabla u_t \, dx = (-1)^k (\partial_\phi^{2k} u_r, u_t)_{L^2(S^1)} \\ &= (-1)^{k+1} (\partial_\phi^{2k} u_r, (\varepsilon + d\pi_N(u))u_r)_{L^2(S^1)} \\ &\leq -\|d\pi_N(u) \partial_\phi^k u_r\|_{L^2(S^1)}^2 - I, \end{aligned} \tag{5-11}$$

where we now drop the term $\varepsilon \|\partial_\phi^k u_r\|_{L^2(S^1)}^2$ from (5-7). Again we use the decomposition $I = \sum_{j=1}^k \binom{k}{j} I_j$ with

$$I_j = (\partial_\phi^k u_r, \partial_\phi^j (d\pi_N(u)) \partial_\phi^{k-j} u_r)_{L^2(S^1)} = (\nabla \partial_\phi^k u, \nabla (\partial_\phi^j (d\pi_N(u)) \partial_\phi^{k-j} u_r))_{L^2(B)},$$

but now we bound these terms as in the proof of Lemma 4.3 via

$$\begin{aligned} |I_j| &\leq C \|\nabla \partial_\phi^k u\|_{L^2(B)} (\|\nabla \partial_\phi^j (d\pi_N(u)) \partial_\phi^{k-j} u_r\|_{L^2(B)} + \|\partial_\phi^j (d\pi_N(u)) \nabla \partial_\phi^{k-j} u_r\|_{L^2(B)}) \\ &\leq C \sum_{\substack{1 \leq j_i \leq k+1 \\ \sum_i j_i = k+2}} \|\nabla \partial_\phi^k u\|_{L^2(B)} \left\| \prod_i \nabla^{j_i} u \right\|_{L^2(B)}. \end{aligned}$$

Using that for any $k \geq 2$ by Sobolev’s embedding $H^2(B) \hookrightarrow W^{1,4} \cap C^0(\bar{B})$ we can bound

$$\sum_{\substack{1 \leq j_i \leq k+1 \\ \sum_i j_i = k+2}} \left\| \prod_i \nabla^{j_i} u \right\|_{L^2(B)} \leq C(1 + \|\nabla u\|_{L^2(B)} + \|\nabla^{k+1} u\|_{L^2(B)})^{k+2}$$

and also using (4-3), we obtain the claim. □

We now are able to conclude.

Proposition 5.5. *For any $k \geq 2$ and any smooth $u_0 \in H^{1/2}(S^1; N)$ there exist $T > 0$ and a solution $u \in V$ to (1-3) on $[0, T]$ with initial data $u(0) = u_0$.*

Proof. In view of Lemma 5.4, there exists a uniform number $T > 0$ such that, with V as defined above, for any $0 < \varepsilon \leq \frac{1}{2}$ there exists a solution $u_\varepsilon \in V$ to (5-1) on $[0, T]$. By definition of V , as $\varepsilon \downarrow 0$ suitably, we have $u_\varepsilon \rightarrow u$ weakly- $*$ in $L^\infty([0, T]; H^{k+1}(B)) \cap H^1(S^1 \times [0, T])$. But this suffices to pass to the limit $\varepsilon \downarrow 0$ in (5-1), and $u \in V$ solves (1-3) with $u(0) = u_0$. □

Proof of Theorem 1.1(i). By Proposition 5.5 for any smooth $u_0 \in H^{1/2}(S^1; N)$ and any $k \geq 2$ there exists $T > 0$ and a solution $u \in V$ of (1-3), (1-4) for $0 < t < T$. Alternatingly employing Propositions 4.11 and 4.6, we then obtain smoothness of u for $0 < t \leq T$, including the final time T . (This argument later appears in more detail in Section 6 after Lemma 6.2.) Iterating, the solution u may be extended smoothly until some maximal time T_0 where condition (4-4) ceases to hold. Uniqueness (even within a much larger class of competing functions) is established in Section 7. □

6. Weak solutions

Given $u_0 \in H^{1/2}(S^1; N)$, there are smooth functions $u_{0k} \in H^{1/2}(S^1; N)$ with $u_{0k} \rightarrow u_0$ in $H^1(B)$ as $k \rightarrow \infty$. Indeed, similar to an argument of Schoen and Uhlenbeck [1982, Theorem 3.1], with a standard mollifying sequence $(\rho_k)_{k \in \mathbb{N}}$ for the mollified functions $v_{0k} := u_0 * \rho_k$ we have $\text{dist}_N(v_{0k}) \rightarrow 0$ uniformly, and $u_{0k} := \pi_N(v_{0k}) \rightarrow u_0 \in H^{1/2}(S^1; N)$ as $k \rightarrow \infty$.

Let u_k be the corresponding solutions of (1-4) with initial data $u_k(0) = u_{0k}$, defined on a maximal time interval $[0, T_k[$, $k \in \mathbb{N}$. We claim that each function u_k can be smoothly extended to a uniform time interval $[0, T[$ for some $T > 0$. To see this, we first establish the following nonconcentration result.

Lemma 6.1. *For any $\delta > 0$ there exists a number $R > 0$ and a time $T_0 > 0$ such that*

$$\sup_{\substack{z_0 \in B \\ 0 < t < T_0}} \int_{B_R(z_0) \cap B} |\nabla u_k(t)|^2 dz < \delta \quad \text{for all } k \in \mathbb{N}.$$

Proof. Given $\delta > 0$, by absolute continuity of the Lebesgue integral and H^1 -convergence $u_{0k} \rightarrow u_0$ ($k \rightarrow \infty$) we can find $R > 0$ such that

$$\sup_{z_0 \in B} \int_{B_{2R}(z_0) \cap B} |\nabla u_{0k}|^2 dz < \delta \quad \text{for all } k \in \mathbb{N}.$$

Choosing $T_0 = \delta R$, by [Lemma 2.2](#) then we have

$$\sup_{\substack{z_0 \in B \\ 0 < t < T_0}} \int_{B_R(z_0) \cap B} |\nabla u_k(t)|^2 dz < 4\delta + C\delta E(u_{k0}) < L\delta$$

with a uniform constant $L > 0$ for all $k \in \mathbb{N}$. The claim follows if we replace δ with δ/L . □

In view of [Proposition 3.4](#), from [Lemmas 6.1](#) and [2.1](#) we obtain the following bound for u_k in $H^1(S^1)$.

Lemma 6.2. *There exist a time $T_0 > 0$ and constants $C > 0$, $C_0 = C_0(E(u_0)) > 0$ such that*

$$\int_0^{T_0} \int_{S^1} |\partial_\phi u_k(t)|^2 d\phi dt \leq CE(u_{k0}) \leq C_0 \quad \text{for all } k \in \mathbb{N}.$$

From [Lemma 6.2](#) we obtain locally in time uniform smooth bounds for (u_k) for $t > 0$ by iteratively applying our previous regularity results. More precisely, Fatou’s lemma and [Lemma 6.2](#) first yield the bound

$$\int_0^{T_0} \liminf_{k \rightarrow \infty} \left(\int_{S^1} |\partial_\phi u_k(t)|^2 d\phi \right) dt \leq C_0.$$

Thus, for almost every $0 < t_0 < T_0$,

$$\liminf_{k \rightarrow \infty} \int_{S^1} |\partial_\phi u_k(t_0)|^2 d\phi < \infty.$$

For any such $0 < t_0 < T_0$, if $\delta > 0$ is sufficiently small, from [Proposition 4.2](#) with another appeal to Fatou’s lemma we may conclude

$$\int_{t_0}^{T_0} \liminf_{k \rightarrow \infty} \int_B |\nabla \partial_\phi u_k|^2 dz dt \leq \liminf_{k \rightarrow \infty} \int_{t_0}^{T_0} \int_B |\nabla \partial_\phi u_k|^2 dz dt \leq C_1$$

for some $C_1 > 0$, so that now we even have

$$\liminf_{k \rightarrow \infty} \int_B |\nabla \partial_\phi u_k(t_1)|^2 dz < \infty$$

for almost every $t_0 < t_1 < T_0$. Hence we may next invoke [Proposition 4.4](#) and (4-2) to obtain the bound

$$\liminf_{k \rightarrow \infty} \int_{t_1}^{T_0} \int_{\partial B} |\nabla \partial_\phi u_k|^2 dz dt < \infty$$

for any such $t_0 < t_1 < T_0$, and Fatou's lemma gives

$$\liminf_{k \rightarrow \infty} \int_{\partial B} |\nabla \partial_\phi u_k(t_2)|^2 d\phi < \infty$$

for almost every $t_1 < t_2 < T_0$. Now Proposition 4.10 may be applied with $\varphi_0 = 1$, and we obtain

$$\liminf_{k \rightarrow \infty} \int_{t_2}^{T_0} \int_B |\nabla \partial_\phi^2 u_k|^2 dz dt < \infty$$

for any such $t_1 < t_2 < T_0$. Another application of Fatou's lemma gives

$$\liminf_{k \rightarrow \infty} \int_B |\nabla \partial_\phi^2 u_k(t_3)|^2 dz < \infty$$

for almost every $t_2 < t_3 < T_0$, and Proposition 4.5 yields

$$\liminf_{k \rightarrow \infty} \int_{t_3}^{T_0} \int_{\partial B} |\nabla \partial_\phi^2 u_k|^2 d\phi dt < \infty$$

for any such $t_2 < t_3 < T_0$. We may then iterate, using (3-7) and alternately employing Propositions 4.11 and 4.6 for $3 \leq k \in \mathbb{N}$, to find a subsequence (u_k) satisfying uniform smooth bounds on $]t_0, T_0]$ for any $t_0 > 0$. Passing to the limit as $k \rightarrow \infty$ for this subsequence we obtain a weak solution to (1-3), (1-4) of energy-class in the following sense.

Definition 6.3. A function $u \in H^1([0, T_0] \times S^1; N) \cap L^\infty([0, T_0]; H^{1/2}(S^1; N))$ with harmonic extension $u = u(t)$ for each t is a weak solution of (1-3), (1-4) of energy-class, if (1-3) is satisfied in the weak sense, that is, if

$$\int_0^{T_0} \int_{\partial B} (u_t + d\pi_N(u)u_r) \cdot \varphi d\phi dt = \int_0^{T_0} \int_{\partial B} u_t \cdot \varphi d\phi dt + \int_0^{T_0} \int_B \nabla u \cdot \nabla (d\pi_N(u)\varphi) dz dt = 0 \quad (6-1)$$

for all $\varphi \in C_c^\infty(\bar{B} \times]0, T_0[)$, and if there holds the energy inequality

$$E(u(T)) + \int_S^T \int_{\partial B} |u_t|^2 d\phi dt \leq E(u(S)) \quad (6-2)$$

for any $0 \leq S < T < T_0$, with the initial data $u_0 \in H^{1/2}(S^1; N)$ being attained in the sense of traces.

We then may summarize our results as follows.

Proposition 6.4. For any $u_0 \in H^{1/2}(S^1; N)$ there exists $T_0 > 0$ and a weak solution u to (1-3), (1-4) on $[0, T_0]$ of energy-class, which is smooth for $t > 0$.

Proof. For any open $U \subset]0, T_0[$ we have uniform smooth bounds for u_k on U ; thus a suitable subsequence u_k approaches u smoothly locally as $k \rightarrow \infty$. Equation (6-1) follows from the corresponding identities for u_k .

Moreover, (6-2) follows from the energy identity, Lemma 2.1, for u_k , where we also use H^1 -convergence $u_{0k} \rightarrow u_0$ as well as weak lower semicontinuity of the energy and of the L^2 -norm.

Finally, with error $o(1) \rightarrow 0$ as $k \rightarrow \infty$ for $0 < t < T_0$ we can estimate

$$\begin{aligned} \|u(t) - u_0\|_{L^2(\partial B)}^2 &\leq \|u_k(t) - u_{0k}\|_{L^2(\partial B)}^2 + o(1) \leq \left(\int_0^t \|\partial_t u_k(t')\|_{L^2(\partial B)} dt' \right)^2 + o(1) \\ &\leq t \int_0^t \|\partial_t u_k(t')\|_{L^2(\partial B)}^2 dt' + o(1) \leq tE(u_0) + o(1), \end{aligned}$$

and $u(t) \rightarrow u_0$ weakly in $H^{1/2}(S^1; N) \cap H^1(B; \mathbb{R}^n)$ as $t \downarrow 0$. In fact, by (6-2) we then even have strong convergence. □

7. Uniqueness

With the help of the tools developed in Section 3 we can show uniqueness of partially regular weak energy-class solutions as in Proposition 6.4.

Theorem 7.1. *Let $u_0 \in H^{1/2}(S^1; N)$. Suppose u and v both are weak energy-class solutions of (1-3), (1-4) on $[0, T_0]$ for some $T_0 > 0$ with initial data u_0 , and suppose u and v are smooth for $t > 0$. Then $u = v$.*

Proof. Using (3-2) for u and v , for the function $w = u - v$ for almost every $0 < t < T_0$ we have

$$\begin{aligned} \partial_t w + \partial_r w &= v(u) \partial_r(\text{dist}_N(u)) - v(v) \partial_r(\text{dist}_N(v)) \\ &= (v(u) - v(v)) \partial_r(\text{dist}_N(u)) + v(v) \partial_r(\text{dist}_N(u) - \text{dist}_N(v)) \end{aligned} \tag{7-1}$$

on $\partial B = S^1$. From (3-5), moreover, we obtain

$$\begin{aligned} |\Delta(\text{dist}_N(u) - \text{dist}_N(v))| &= |\nabla u \cdot dv(u) \nabla u - \nabla v \cdot dv(v) \nabla v| \\ &\leq C(|w| |\nabla u|^2 + (|\nabla u| + |\nabla v|) |\nabla w|) \quad \text{in } B. \end{aligned} \tag{7-2}$$

Observing that

$$|\text{dist}_N(u) - \text{dist}_N(v)| \leq C|w|,$$

upon multiplying (7-2) with the function $(\text{dist}_N(u) - \text{dist}_N(v)) \in H^1_0(B)$, integrating by parts, and using Young's inequality, for any $\varepsilon > 0$ we obtain

$$\begin{aligned} \|\nabla(\text{dist}_N(u) - \text{dist}_N(v))\|_{L^2(B)}^2 &\leq C \int_B (|w|^2 |\nabla u|^2 + (|\nabla u| + |\nabla v|) |\nabla w| |w|) dz \\ &\leq \varepsilon \|\nabla w\|_{L^2(B)}^2 + C(\varepsilon) \|w\|_{L^4(B)}^2 (\|\nabla u\|_{L^4(B)}^2 + \|\nabla v\|_{L^4(B)}^2). \end{aligned} \tag{7-3}$$

On the other hand, for any $0 < t_0 < T \leq T_0$, multiplying (7-1) with w and integrating by parts on $S^1 \times [t_0, T]$, upon letting $t_0 \downarrow 0$ we find

$$\begin{aligned} \sup_{0 < t < T} \|w(t)\|_{L^2(\partial B)}^2 + \int_0^T \int_B |\nabla w|^2 dz dt &\leq C \int_0^T \int_{\partial B} (\partial_t w + \partial_r w) w d\phi dt \\ &= C \int_0^T \int_{\partial B} w(v(u) - v(v)) \partial_r(\text{dist}_N(u)) d\phi dt + C \int_0^T \int_{\partial B} wv(v) \partial_r(\text{dist}_N(u) - \text{dist}_N(v)) d\phi dt \\ &=: C \int_0^T (I + II) dt. \end{aligned}$$

We first estimate the term

$$\begin{aligned} I &= I(t) = \int_{\partial B} w(v(u) - v(v)) \partial_r(\text{dist}_N(u)) \, d\phi \\ &= \int_B \nabla(w(v(u) - v(v))) \nabla(\text{dist}_N(u)) \, dz + \int_B w(v(u) - v(v)) \Delta(\text{dist}_N(u)) \, dz. \end{aligned}$$

Using

$$\begin{aligned} |\nabla(w(v(u) - v(v)))| &\leq C(|\nabla w| |w| + |w|(dv(u) - dv(v)) \nabla u + dv(v) \nabla w) \\ &\leq C(|\nabla w| |w| + |w|^2 |\nabla u|) \end{aligned}$$

we can bound

$$\begin{aligned} \left| \int_B \nabla(w(v(u) - v(v))) \nabla(\text{dist}_N(u)) \, dz \right| &\leq C \int_B (|\nabla w| |w| + |w|^2 |\nabla u|) |\nabla u| \, dz \\ &\leq \varepsilon \|\nabla w\|_{L^2(B)}^2 + C(\varepsilon) \|w\|_{L^4(B)}^2 \|\nabla u\|_{L^4(B)}^2 \end{aligned}$$

for each t . Also using (3-5), we can moreover estimate

$$\left| \int_B w(v(u) - v(v)) \Delta(\text{dist}_N(u)) \, dz \right| \leq C \|w\|_{L^4(B)}^2 \|\nabla u\|_{L^4(B)}^2$$

for almost every $0 < t < T$ to obtain

$$|I| \leq \varepsilon \|\nabla w\|_{L^2(B)}^2 + C(\varepsilon) \|w\|_{L^4(B)}^2 \|\nabla u\|_{L^4(B)}^2.$$

Similarly, we estimate the term

$$\begin{aligned} II &= II(t) = \int_{\partial B} wv(v) \partial_r(\text{dist}_N(u) - \text{dist}_N(v)) \, d\phi \\ &= \int_B \nabla(wv(v)) \nabla(\text{dist}_N(u) - \text{dist}_N(v)) \, dz + \int_B wv(v) \Delta(\text{dist}_N(u) - \text{dist}_N(v)) \, dz. \end{aligned}$$

Noting that with (7-3) we can bound

$$\begin{aligned} \left| \int_B \nabla(wv(v)) \nabla(\text{dist}_N(u) - \text{dist}_N(v)) \, dz \right| &\leq C(\|\nabla w\|_{L^2(B)} + \|w \nabla v\|_{L^2(B)}) \|\nabla(\text{dist}_N(u) - \text{dist}_N(v))\|_{L^2(B)} \\ &\leq \varepsilon \|\nabla w\|_{L^2(B)}^2 + C(\varepsilon) \|w\|_{L^4(B)}^2 (\|\nabla u\|_{L^4(B)}^2 + \|\nabla v\|_{L^4(B)}^2) \end{aligned}$$

and that with (7-2) we have

$$\begin{aligned} \left| \int_B wv(v) \Delta(\text{dist}_N(u) - \text{dist}_N(v)) \, dz \right| &\leq C \int_B (|w|^2 |\nabla u|^2 + |w| |\nabla w| (|\nabla u| + |\nabla v|)) \, dz \\ &\leq \varepsilon \|\nabla w\|_{L^2(B)}^2 + C(\varepsilon) \|w\|_{L^4(B)}^2 (\|\nabla u\|_{L^4(B)}^2 + \|\nabla v\|_{L^4(B)}^2), \end{aligned}$$

we find the estimate

$$|II| \leq \varepsilon \|\nabla w\|_{L^2(B)}^2 + C(\varepsilon) \|w\|_{L^4(B)}^2 (\|\nabla u\|_{L^4(B)}^2 + \|\nabla v\|_{L^4(B)}^2)$$

for almost every $0 < t < T$.

But Sobolev’s embedding $H^{1/2}(B) \hookrightarrow L^4(B)$ and Fourier expansion give the bound

$$\|w\|_{L^4(B)}^2 \leq C \|w\|_{H^{1/2}(B)}^2 \leq C \|w\|_{L^2(\partial B)}^2$$

and similar bounds for ∇u as well as ∇v . Moreover, since by the energy inequality (6-2) we have $u(t), v(t) \rightarrow u_0$ strongly in $H^1(B)$ as $t \downarrow 0$, there exist a radius $0 < R \leq \frac{1}{2}$ and a time $0 < T < T_0$ such that condition (3-13) in Proposition 3.4 holds true on $[0, T]$ for both u and v , allowing us to bound

$$\begin{aligned} \int_0^T \|\nabla u(t)\|_{L^4(B)}^2 dt &\leq C \int_0^T \|\nabla u(t)\|_{L^2(\partial B)}^2 dt \leq C \int_0^T \|\partial_\phi u(t)\|_{L^2(\partial B)}^2 dt \\ &\leq C \int_0^T \int_{\partial B} |u_t|^2 d\phi dt + C(R)TE(u_0) \leq C(R)(1 + T_0)E(u_0) \end{aligned}$$

with the help of (3-7), and similarly for $|\nabla v|$. Choosing $\varepsilon = \frac{1}{4}$, for sufficiently small $0 < T < T_0$ by absolute continuity of the integral we thus can estimate

$$\begin{aligned} \sup_{0 < t < T} \|w(t)\|_{L^2(\partial B)}^2 + \int_0^T \int_B |\nabla w|^2 dz dt \\ \leq \frac{1}{2} \|\nabla w\|_{L^2(B \times [0, T])}^2 + C \sup_{0 < t < T} \|w(t)\|_{L^2(\partial B)}^2 \int_0^T (\|\nabla u\|_{L^4(B)}^2 + \|\nabla v\|_{L^4(B)}^2) dt \\ \leq \frac{1}{2} \left(\sup_{0 < t < T} \|w(t)\|_{L^2(\partial B)}^2 + \int_0^T \int_B |\nabla w|^2 dz dt \right), \end{aligned}$$

and it follows that $w = 0$, as claimed. □

Proof of Theorem 1.2. Existence for short time and uniqueness of a partially regular weak solution to (1-3), (1-4) for given data $u_0 \in H^{1/2}(S^1; N)$ follow from Proposition 6.4 and Theorem 7.1, respectively. Since by Proposition 6.4 our weak solution is smooth for $t > 0$, the remaining assertions follow from Theorem 1.1.

Note that at any blow-up time T_{i-1} , $i \geq 1$, of the flow as in Theorem 1.1(ii) there exists a unique weak limit $u_i = \lim_{t \uparrow T_{i-1}} u(t) \in H^{1/2}(S^1; N)$, and we may uniquely continue the flow using Proposition 6.4. □

8. Blow-up

Preparing for the proof of part (ii) of Theorem 1.1, suppose now that for the solution constructed in part (i) of that theorem there holds $T_0 < \infty$. Then, as we shall see in more detail below, by the results in Section 4 condition (4-4) must be violated for $T = T_0$ and there exist $\delta > 0$ and points $z_k \in B$ as well as radii $r_k \downarrow 0$ as $k \rightarrow \infty$ such that, for suitable $t_k \uparrow T_0$,

$$\int_{B_{r_k}(z_k) \cap B} |\nabla u(t_k)|^2 dz = \sup_{\substack{z_0 \in B \\ t \leq t_k}} \int_{B_{r_k}(z_0) \cap B} |\nabla u(t)|^2 dz = \delta.$$

We may later choose a smaller constant $\delta > 0$, if necessary. Moreover, for later use from now on we consider local concentrations in the sense that, for some $z_0 \in B$ and some fixed radius $r_0 > 0$ for a

sequence of points $z_k \in B$ with $z_k \rightarrow z_0$ and radii $r_k \downarrow 0$ for suitable $t_k \uparrow T_0$ as $k \rightarrow \infty$,

$$\int_{B_{r_k}(z_k) \cap B} |\nabla u(t_k)|^2 dz = \sup_{\substack{z' \in B_{r_0}(z_0) \\ t \leq t_k}} \int_{B_{r_k}(z') \cap B} |\nabla u(t)|^2 dz = \delta.$$

Scale

$$u_k(z, t) = u(z_k + r_k z, t_k + r_k t)$$

for

$$z \in \Omega_k = \{z : z_k + r_k z \in B\}, \quad t \in I_k = \{t : 0 \leq t_k + r_k t < T_0\}.$$

Note that then

$$\int_{B_1(0) \cap \Omega_k} |\nabla u_k(0)|^2 dz = \sup_{\substack{z_k + r_k z' \in B_{r_0}(z_0) \\ -t_k/r_k \leq t < 0}} \int_{B_1(z') \cap \Omega_k} |\nabla u_k(t)|^2 dz = \delta. \tag{8-1}$$

Passing to a subsequence we may assume that the domains Ω_k exhaust a limit domain $\Omega_\infty \subset \mathbb{R}^2$, which either is the whole space \mathbb{R}^2 or a half-space H .

By the energy inequality [Lemma 2.1](#) for $t \in I_k$,

$$\int_{\Omega_k} |\nabla u_k(t)|^2 dz = \int_B |\nabla u(t_k + r_k t)|^2 dz \leq 2E(u_0), \tag{8-2}$$

and for any $t_0 < 0$ and sufficiently large $k \in \mathbb{N}$ we have

$$\begin{aligned} \int_{t_0}^0 \int_{\partial \Omega_k} |\partial_t u_k|^2 ds dt &= \int_{t_0}^0 \int_{\partial \Omega_k} |d\pi_N(u_k) \partial_{v_k} u_k|^2 ds dt \\ &= \int_{t_k + r_k t_0}^{t_k} \int_{\partial B} |u_t|^2 d\phi dt \leq \int_{t_k + r_k t_0}^{T_0} \int_{\partial B} |u_t|^2 d\phi dt \rightarrow 0 \end{aligned} \tag{8-3}$$

as $k \rightarrow \infty$, where ds is the element of length and where v_k is the outward unit normal along $\partial \Omega_k$. Expressing the harmonic functions $\partial_t u_k(t)$ in Fourier series for each $t < 0$, it then also follows that $\partial_t u_k \rightarrow 0$ locally in L^2 on $\Omega_\infty \times]-\infty, 0[$. Finally, again using the fact that $u_k(t)$ for each t is harmonic, by the maximum principle we have the uniform bound $|u_k| \leq \sup_{p \in N} |p|$ as well as uniform smooth bounds locally away from the boundary of Ω_∞ .

Hence we may assume that as $k \rightarrow \infty$ we have $u_k \rightarrow u_\infty$ weakly locally in H^1 on $\Omega_\infty \times]-\infty, 0[$, where $u_\infty(z, t) = u_\infty(z)$ is independent of time, harmonic, and bounded. Moreover, we have smooth convergence away from $\partial \Omega_\infty$. Thus, if we assume that $\Omega_\infty = \mathbb{R}^2$, by [\(8-1\)](#) it follows that

$$\int_{B_1(0)} |\nabla u_\infty|^2 dz = \delta.$$

But any function $v : \mathbb{R}^2 \rightarrow \mathbb{R}$ which is bounded and harmonic must be constant, which rules out this possibility. Hence Ω_∞ can only be a half-space.

After a suitable rotation of the domain B and shift of coordinates in $\mathbb{R}^2 \cong \mathbb{C}$ we may then assume that $z_k = (0, -y_k)$ with $1 - y_k \leq M r_k$ for some $M \in \mathbb{N}$ and that $\Omega_\infty = \{(x, y) : y > y_0\}$ for some y_0 . Finally,

replacing $r_k > 0$ with $(M + 1)r_k$ and z_k with $z_k = (0, -1)$, if necessary, we may assume that

$$\Omega_k \subset \mathbb{R}_+^2 = \{(x, y) : y > 0\}$$

is the ball of radius $1/r_k$ around the point $(0, 1/r_k)$ with $0 \in \partial\Omega_k$, while from (8-1) with a uniform number $L \in \mathbb{N}$ we have

$$L \int_{B_1(0) \cap \Omega_k} |\nabla u_k(0)|^2 dz \geq L\delta \geq \sup_{\substack{|z'| \leq r_0/r_k \\ -t_k/r_k \leq t < 0}} \int_{B_1(z') \cap \Omega_k} |\nabla u_k(t)|^2 dz \tag{8-4}$$

for any $k \in \mathbb{N}$. Let $\Phi_k : \mathbb{R}_+^2 \rightarrow \Omega_k$ be the conformal maps given by

$$\Phi_k(z) = \frac{2z}{1 - ir_k z}, \quad z \in \mathbb{R}_+^2, \quad k \in \mathbb{N},$$

with $\Phi_k \rightarrow 2 \cdot \text{id}$ locally uniformly on $\mathbb{R}^2 \cong \mathbb{C}$ as $k \rightarrow \infty$.

Let $v_k = u_k \circ \Phi_k$ with $k \in \mathbb{N}$. By conformal invariance of the Dirichlet energy, from (8-2) for any t we have

$$\int_{\mathbb{R}_+^2} |\nabla v_k(t)|^2 dz = \int_{\Omega_k} |\nabla u_k(t)|^2 dz \leq 2E(u_0), \tag{8-5}$$

and by (8-4) with a uniform number $L_1 \in \mathbb{N}$ there holds

$$L_1 \int_{B_2^+(0)} |\nabla v_k(0)|^2 dz \geq L_1\delta \geq \sup_{\substack{|z'| \leq r_0/r_k \\ -t_k/r_k \leq t < 0}} \int_{B_1^+(z')} |\nabla v_k(t)|^2 dz, \tag{8-6}$$

where

$$B_r^+(z) = B_r(z) \cap \mathbb{R}_+^2$$

for any $r > 0$ and any $z = (x, y) \in \mathbb{R}^2$. Moreover, from (8-3) for any $t_0 < 0$ and any $R > 0$ for the integral over $] -R, R[\times \{0\} \subset \partial\mathbb{R}_+^2$ we obtain

$$\int_{t_0}^0 \int_{-R}^R |\partial_t v_k|^2 dx dt \leq C \int_{t_0}^0 \int_{-R}^R |d\pi_N(v_k) \partial_y v_k|^2 dx dt \rightarrow 0 \quad \text{as } k \rightarrow \infty, \tag{8-7}$$

and $\partial_t v_k \rightarrow 0$ locally in L^2 on $\overline{\mathbb{R}_+^2} \times] -\infty, 0[$. In addition, from our choice of (u_k) it follows that $v_k \rightarrow v_\infty$ weakly locally in H^1 on $\overline{\mathbb{R}_+^2} \times] -\infty, 0[$ as $k \rightarrow \infty$, where $v_\infty(z, t) =: w_\infty(z)$ is harmonic and bounded.

For a suitable sequence of times $t_0 < s_k < 0$, we also have locally weak convergence $w_k := v_k(s_k) \rightarrow w_\infty$ in H^1 on $\overline{\mathbb{R}_+^2}$ and, in addition,

$$d\pi_N(w_k) \partial_y w_k \rightarrow 0 \quad \text{in } L_{\text{loc}}^2(\partial\mathbb{R}_+^2) \quad \text{as } k \rightarrow \infty. \tag{8-8}$$

Thus, for sufficiently small $\delta > 0$, by Proposition 3.3 applied to the functions $w_k \circ \Psi$, where $\Psi : B \rightarrow \mathbb{R}_+^2$ is a suitable conformal map, we also have uniform local L^2 -bounds for $\partial_x w_k$ on $\partial\mathbb{R}_+^2$, and we may assume that $w_k \rightarrow w_\infty$ locally uniformly and weakly locally in H^1 on $\partial\mathbb{R}_+^2$ as $k \rightarrow \infty$. Since w_k is harmonic, we then also have locally strong H^1 -convergence $w_k \rightarrow w_\infty$ on $\overline{\mathbb{R}_+^2}$.

To see that w_∞ is nonconstant, let $\varphi_k = \varphi_{z_0, 4r_k}$, $k \in \mathbb{N}$. Integrating the identity (2-1) from the proof of Lemma 2.2 in time, with error $o(1) \rightarrow 0$ and suitable numbers $\varepsilon_k \downarrow 0$ as $k \rightarrow \infty$ in view of (8-3), we find

$$\begin{aligned} & \frac{1}{2} \left| \int_B |\nabla u(t_k)|^2 \varphi_k^2 dz - \int_B |\nabla u(t_k + r_k s_k)|^2 \varphi_k^2 dz \right| \\ & \leq \int_{t_k+r_k s_k}^{t_k} \int_{\partial B} |u_t|^2 \varphi_k^2 d\phi dt + 2 \int_{t_k+r_k s_k}^{t_k} \int_B |u_t \nabla u \varphi_k \nabla \varphi_k| dz dt \\ & \leq o(1) + 8\varepsilon_k r_k \int_{t_k+r_k s_k}^{t_k} \int_B |\nabla u|^2 |\nabla \varphi_k|^2 dz dt + (8\varepsilon_k r_k)^{-1} \int_{t_k+r_k s_k}^{t_k} \int_B |u_t|^2 \varphi_k^2 dz dt. \end{aligned} \tag{8-9}$$

With the help of (2-2) and (8-3) for suitable $\varepsilon_k \downarrow 0$ we can bound

$$(8\varepsilon_k r_k)^{-1} \int_{t_k+r_k s_k}^{t_k} \int_B |u_t|^2 \varphi_k^2 dz dt \leq C \varepsilon_k^{-1} \int_{t_k+r_k s_k}^{t_k} \int_{\partial B} |u_t|^2 dz dt \rightarrow 0.$$

Since for any choice $t_0 < s_k < 0$ we also can estimate

$$8\varepsilon_k r_k \int_{t_k+r_k s_k}^{t_k} \int_B |\nabla u|^2 |\nabla \varphi_k|^2 dz dt \leq C \varepsilon_k |t_0| E(u_0) \rightarrow 0,$$

from (8-9) and (8-6) it follows that with error $o(1) \rightarrow 0$ as $k \rightarrow \infty$ we have

$$\begin{aligned} \int_{B_4^+(0)} |\nabla w_k|^2 dz + o(1) &= \int_{B_4^+(0)} |\nabla v_k(s_k)|^2 dz + o(1) \geq \int_B |\nabla u(t_k + r_k s_k)|^2 \varphi_k^2 dz + o(1) \\ &\geq \int_B |\nabla u(t_k)|^2 \varphi_k^2 dz \geq \int_{B_2^+(0)} |\nabla v_k(0)|^2 dz \geq \delta. \end{aligned} \tag{8-10}$$

Finally, in view of locally uniform convergence $w_k \rightarrow w_\infty$ and weak local L^2 -convergence of the traces $\nabla w_k \rightarrow \nabla w_\infty$ on $\partial \mathbb{R}_+^2$, we may pass to the limit $k \rightarrow \infty$ in (8-8) to conclude that $w_\infty : \partial \mathbb{R}_+^2 \rightarrow N$ with

$$d\pi_N(w_\infty) \partial_y w_\infty = 0 \quad \text{on } \partial \mathbb{R}_+^2. \tag{8-11}$$

Since w_∞ is harmonic, the Hopf differential

$$f = |\partial_x w_\infty|^2 - |\partial_y w_\infty|^2 - 2i \partial_x w_\infty \cdot \partial_y w_\infty$$

defines a holomorphic function $f \in L^1(\mathbb{R}_+^2, \mathbb{C})$. Moreover, $w_\infty \in H_{\text{loc}}^{3/2}(\overline{\mathbb{R}_+^2})$ with trace $\nabla w_\infty \in L_{\text{loc}}^2(\partial \mathbb{R}_+^2)$; thus also the trace of f is well-defined on $\partial \mathbb{R}_+^2$. By (8-11) now the trace of f is real-valued; thus $f \equiv c$ for some constant $c \in \mathbb{R}$. But $\nabla w_\infty \in L^2(\mathbb{R}_+^2)$; hence $f \in L^1(\mathbb{R}_+^2)$. It follows that $c = 0$, and w_∞ is conformal.

With a conformal diffeomorphism $\Phi : B \rightarrow \mathbb{R}_+^2$ mapping a point $z_0 \in \partial B$ to infinity, define the map $\bar{u} = w_\infty \circ \Phi \in H^{1/2}(S^1; N)$. By conformal invariance, \bar{u} again is harmonic with finite Dirichlet integral and satisfies (1-6) on $\partial B \setminus \{z_0\}$; since the point $\{z_0\}$ has vanishing H^1 -capacity, \bar{u} then is stationary in the sense of [Grüter et al. 1981]. Moreover, \bar{u} is conformal. For such mappings, smooth regularity on \bar{B} was shown by Grüter, Hildebrandt, and Nitsche [Grüter et al. 1981]; thus condition (1-6) holds everywhere on

∂B in the pointwise sense, and \bar{u} parametrizes a minimal surface of finite area supported by N which meets N orthogonally along its boundary.

Proof of Theorem 1.1(ii). For given smooth data $u_0 \in H^{1/2}(S^1; N)$ let u be the unique solution to (1-3), (1-4) guaranteed by part (i) of the theorem, and suppose that the maximal time of existence T_0 is less than ∞ . Then condition (4-4) must fail as $t \uparrow T_0$; else from Propositions 4.11 and 4.6 we obtain smooth bounds for $u(t)$ as $t \uparrow T_0$ and there exists a smooth trace $u_1 = \lim_{t \uparrow T_0} u(t)$. But by the first part of the theorem there is a smooth solution to the initial value problem for (1-3) with initial data u_1 at time T_0 , and this solution extends the original solution u to an interval $[0, T_1[$ for some $T_1 > T_0$, contradicting the maximality of T_0 .

Let $z^{(i)} \in B$, $1 \leq i \leq i_0$, such that, for some number $\delta > 0$ and suitable $t_k^{(i)} \uparrow T_0$, $z_k^{(i)} \rightarrow z^{(i)}$, $r_k^{(i)} \rightarrow 0$ as $k \rightarrow \infty$,

$$\liminf_{k \rightarrow \infty} \int_{B_{r_k^{(i)}}(z_k^{(i)}) \cap B} |\nabla u(t_k^{(i)})|^2 dz \geq \delta.$$

By the argument following (8-9), for a suitable sequence of radii $0 < r_k^{(0)} \rightarrow 0$ such that $r_k^{(i)}/r_k^{(0)} \rightarrow 0$ as well as $(T_0 - t_k^{(i)})/r_k^{(0)} \rightarrow 0$, then with error $o(1) \rightarrow 0$ as $k \rightarrow \infty$,

$$\int_{B_{r_k^{(0)}}(z^{(i)}) \cap B} |\nabla u(t)|^2 dz + o(1) \geq \int_{B_{r_k^{(i)}}(z_k^{(i)}) \cap B} |\nabla u(t_k^{(i)})|^2 dz \geq \delta$$

for all $T_0 - r_k^{(0)} < t < T_0$, uniformly in $1 \leq i \leq i_0$. For sufficiently large $k \in \mathbb{N}$ such that

$$r_k^{(0)} < \inf_{i < j} \frac{1}{4} |z^{(i)} - z^{(j)}|,$$

it follows that $i_0 \leq E(u_0)/\delta$, and we may fix $r_0 > 0$ and redefine $t_k^{(i)}$, $r_k^{(i)}$, and $z_k^{(i)}$, if necessary, such that, for each $1 \leq i \leq i_0$,

$$\int_{B_{r_k^{(i)}}(z_k^{(i)}) \cap B} |\nabla u(t_k^{(i)})|^2 dz = \sup_{\substack{z' \in B_{r_0}(z^{(i)}) \\ 0 < t \leq t_k^{(i)}}} \int_{B_{r_k^{(i)}}(z') \cap B} |\nabla u(t)|^2 dz = \delta.$$

Moreover, we may assume that $\delta < \delta_0$ as defined in Proposition 3.1. The characterization of the concentration points as in Theorem 1.2(ii) via solutions $\bar{u}^{(i)}$ of (1-6) then follows from our above analysis.

In addition, Corollary 3.2 yields the uniform lower bound

$$\lim_{r_0 \downarrow 0} \liminf_{t \uparrow T} \int_{B_{r_0}(z^{(i)}) \cap B} |\nabla u(t)|^2 dz \geq 2E(\bar{u}^{(i)}) \geq 2\delta_0^2$$

for the concentration energy quanta, which gives the claimed upper bound for the total number of concentration points.

Finally, with the help of Proposition 4.11 we can smoothly extend the solution u to $B \setminus \{z^{(1)}, \dots, z^{(i_0)}\}$ at time $t = T_0$. □

9. Asymptotics

Suppose next that the solution u to (1-3), (1-4) exists for all time $0 < t < \infty$. Then u either concentrates for suitable $t_k \uparrow \infty$ in the sense that condition (4-4) does not hold true uniformly in time, or u satisfies uniform smooth bounds, as shown in Section 4.

In the latter case, the claim made in Theorem 1.1(iii) easily follows.

Proposition 9.1. *Suppose that for any $\delta > 0$ there exists $R > 0$ such that condition (4-4) holds true for all $0 < t < \infty$. Then there exists a smooth solution $u_\infty \in H^{1/2}(S^1; N)$ of (1-6) such that $u(t) \rightarrow u_\infty$ smoothly as $t \rightarrow \infty$ suitably, and u_∞ parametrizes a minimal surface of finite area supported by N which meets N orthogonally along its boundary.*

Proof. For sufficiently small $\delta > 0$ and for any $j \in \mathbb{N}$, by iterative reference to Propositions 4.2, 4.4–4.6, and 4.10, 4.11 as in Section 6 we can find constants $C_j > 0$ such that $\|u(t)\|_{H^j(B)} \leq C_j$ for all $t > 1$. Moreover, by the energy inequality Lemma 2.1 for a suitable sequence $t_k \rightarrow \infty$ there holds $u_t(t_k) \rightarrow 0$ in $L^2(\partial B)$ as $k \rightarrow \infty$. Then for any $j \in \mathbb{N}$ a subsequence $u(t_k)$ approaches u_∞ in $H^j(B)$, and a diagonal subsequence converges smoothly, where u_∞ solves (1-6). By the argument after (8-11) in Section 8, u_∞ is conformal and u_∞ parametrizes a minimal surface with free boundary on N which meets N orthogonally along its boundary. □

In the remaining case that for some $\delta > 0$ condition (4-4) fails to hold, there exists a sequence $t_k \uparrow \infty$ and points $z^{(1)}, \dots, z^{(i_0)}$ such that, for sequences $z_k^{(i)} \rightarrow z^{(i)}$ and radii $r_k^{(i)} \rightarrow 0$ as $k \rightarrow \infty$,

$$\liminf_{k \rightarrow \infty} \int_{B_{r_k^{(i)}}(z_k^{(i)}) \cap B} |\nabla u(t_k)|^2 dz \geq \delta, \quad 1 \leq i \leq i_0.$$

By Lemma 2.1 there holds the a-priori bound $i_0 \leq E(u_0)/\delta$ for the number of concentration points. By the argument leading to (8-10) then, for a suitable number

$$0 < r_0 \leq \inf_{i < j} \frac{1}{4} |z^{(i)} - z^{(j)}|$$

with error $o(1) \rightarrow 0$ as $k \rightarrow \infty$ and with some constant $L \in \mathbb{N}$ for all $1 \leq i \leq i_0$,

$$L \int_{B_{2r_k^{(i)}}(z_k^{(i)}) \cap B} |\nabla u(t_k)|^2 dz + o(1) \geq \sup_{\substack{z_0 \in B_{r_0}(z_k^{(i)}) \\ t_k - r_0 \leq t \leq t_k}} \int_{B_{r_k^{(i)}}(z_0) \cap B} |\nabla u(t)|^2 dz \geq \delta.$$

Fixing any index $1 \leq i \leq i_0$ and renaming $z_k^{(i)} =: z_k$ and $r_k^{(i)} =: r_k$, we then scale

$$u_k(z, t) = u(z_k + r_k z, t_k + r_k t), \quad z \in \Omega_k = \{z : z_k + r_k z \in B\}, \quad -t_k/r_k \leq t \leq 0,$$

as before and observe that, for any $t_0 < 0$,

$$\begin{aligned} \int_{t_0}^0 \int_{\partial \Omega_k} |\partial_t u_k|^2 ds dt &= \int_{t_0}^0 \int_{\partial \Omega_k} |d\pi_N(u_k) \partial_{v_k} u_k|^2 ds dt \\ &= \int_{t_k+r_k t_0}^{t_k} \int_{\partial B} |u_t|^2 d\phi dt \leq \int_{t_k+r_k t_0}^\infty \int_{\partial B} |u_t|^2 d\phi dt \rightarrow 0 \end{aligned} \tag{9-1}$$

as $k \rightarrow \infty$, where ν_k is the outward unit normal along $\partial\Omega_k$. Just as in Section 8 for suitable $t_0 < s_k < 0$ we then obtain local uniform and H^1 -convergence of a subsequence of the conformally rescaled maps $w_k = u_k(s_k) \circ \Phi_k \in H^1_{\text{loc}}(\mathbb{R}^2_+)$ to a smooth, nonconstant, harmonic and conformal limit w_∞ with finite energy and continuously mapping $\partial\mathbb{R}^2_+$ to N , inducing a solution $\bar{u}_\infty = w_\infty \circ \Phi \in H^{1/2}(S^1; N)$ of (1-6) corresponding to a minimal surface with free boundary on N . This ends the proof of Theorem 1.1(iii).

Appendix

In this section, for the convenience of the reader we derive two interpolation inequalities that play a crucial role in our arguments.

Let $v \in H^1(B)$, and for $r > 0$ let $(z_i)_{1 \leq i \leq i_0}$ be such that the collection of balls $B_{r/2}(z_i)$, $1 \leq i \leq i_0$, covers \bar{B} with at most L balls $B_r(z_i)$ overlapping at any $z \in B$, with $L \in \mathbb{N}$ independent of $r > 0$. We may assume $r < \frac{1}{8}$ so that for any $1 \leq i \leq i_0$ there is a pair of orthogonal vectors $e_{1,i}$ and $e_{2,i}$ such that for any $z \in B_r(z_i)$ there holds $z + se_{1,i} + te_{2,i} \in B$ for any $0 \leq s, t \leq 2r$. After a rotation of coordinates, we may assume that $e_{1,i} = (1, 0)$ and $e_{2,i} = (0, 1)$ are the standard basis vectors. Writing φ for $\varphi_{z_i,r}$, by arguing as Ladyzhenskaya [1963] for any $z = (x, y) \in B_r(z_i)$ and using that

$$(v^2\varphi)(x + 2r, y) = 0 = (v^2\varphi)(x, y + 2r)$$

then we can estimate

$$\begin{aligned} v^4(z) &= |(v^2\varphi)(z)|^2 \leq \int_0^{2r} |\partial_x(v^2\varphi)(x + s, y)| ds \cdot \int_0^{2r} |\partial_y(v^2\varphi)(x, y + t)| dt \\ &\leq \int_{\{s:(s,y) \in B\}} |\partial_x(v^2\varphi)(s, y)| ds \cdot \int_{\{t:(x,t) \in B\}} |\partial_y(v^2\varphi)(x, t)| dt, \end{aligned} \tag{A-1}$$

and with the help of Fubini’s theorem we find

$$\begin{aligned} \int_{B_{r/2}(z_i)} |v|^4 dz &\leq \int_B |v|^4 \varphi^2 dz \leq \int_{-\infty}^\infty \left(\int_{\{x:(x,y) \in B\}} |(v^2\varphi)(x, y)|^2 dx \right) dy \\ &\leq \int_{-\infty}^\infty \int_{\{s:(s,y) \in B\}} |\partial_x(v^2\varphi)(s, y)| ds dy \cdot \int_{-\infty}^\infty \int_{\{t:(x,t) \in B\}} |\partial_y(v^2\varphi)(x, t)| dt dx \\ &\leq \left(\int_B |\nabla(v^2\varphi)| dz \right)^2 \\ &\leq \left(\int_B (2|\nabla v| |v\varphi| + v^2 |\nabla\varphi|) dz \right)^2 \\ &\leq C \left(\int_{B_r(z_i)} |\nabla v|^2 dz + r^{-2} \int_{B_r(z_i)} v^2 dz \right) \int_{B_r(z_i)} v^2 dz. \end{aligned}$$

Fixing $r = \frac{1}{10}$ and summing over $1 \leq i \leq i_0$, with an absolute constant $C > 0$ we obtain the bound

$$\|v\|_{L^4(B)}^4 \leq C \|v\|_{H^1(B)}^2 \|v\|_{L^2(B)}^2 \tag{A-2}$$

for any $v \in H^1(B)$.

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Received 3 Feb 2022. Revised 14 Jul 2022. Accepted 19 Aug 2022.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow[®] from MSP.

PUBLISHED BY

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Volume 17 No. 4 2024

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