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NONCOMMUTATIVE MAXIMAL OPERATORS WITH ROUGH KERNELS

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This paper is devoted to the study of noncommutative maximal operators with rough kernels. More precisely, we prove the weak-type (1, 1) boundedness for noncommutative maximal operators with rough kernels. The proof of the weak-type (1, 1) estimate is based on the noncommutative Calderón–Zygmund decomposition. To deal with the rough kernel, we use the microlocal decomposition in the proofs of both the bad and good functions.

1. Introduction and state of main result

In recent years, there has been extensive research on noncommutative harmonic analysis, especially on noncommutative Calderón–Zygmund theory; see, e.g., [Parcet 2009; Mei and Parcet 2009; Cadilhac 2018; Chen et al. 2013]. The main content of this topic is focused on investigating the boundedness property of various operators in harmonic analysis on the noncommutative L_p space. Due to the lack of commutativity (i.e., $ab = ba$ may not hold in general case), many problems in the study of noncommutative Calderón–Zygmund theory seem to be more difficult, for instance the weak-type (1, 1) bound of integral operators.

It is well known that the real-variable theory of classical harmonic analysis was initiated by A. P. Calderón and A. Zygmund [1952]. One of the remarkable techniques in [Calderón and Zygmund 1952] is the so-called Calderón–Zygmund decomposition, which is now a widely used method in harmonic analysis. This technique not only gives a real-variable method to show weak-type (1, 1) bounds of singular integrals, but also provides a basic idea of stopping-time arguments for many topics in harmonic analysis, such as the theory of Hardy and BMO spaces; see, e.g., [Grafakos 2014a; 2014b; Stein 1993]. The noncommutative Calderón–Zygmund decomposition was recently established in [Parcet 2009] via the theory of noncommutative martingales. With this tool, the weak-type (1, 1) bound theory of the standard Calderón–Zygmund operator was developed there. It was pointed out in [Parcet 2009] that the noncommutative Calderón–Zygmund decomposition and the related method should open a door to work for a more general class of operators. For the subsequent works related to weak-type (1, 1) problem and noncommutative Calderón–Zygmund decomposition, see [Mei and Parcet 2009; Cadilhac 2018; Caspers et al. 2018; 2019; Hong and Xu 2021; Hong et al. 2023; Cadilhac et al. 2022].

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On the other hand, the classical theory of singular integral operators tells us that to ensure the weak-type (1, 1) bound of the Calderón–Zygmund operator, the regularity condition of kernel can be relaxed to the so-called Hörmander condition; see, e.g., [Hörmander 1960; Grafakos 2014a]. Moreover, Calderón and Zygmund [1956] further studied the singular integral operator with a rough homogeneous kernel defined by

$$\text{p.v.} \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d} f(y)\, dy \tag{1-1}$$

and established its L_p boundedness for all $1 < p < \infty$. For its weak-type (1, 1) boundedness, it was quite later showed by S. Hofmann [1988] (and independently by M. Christ and Rubio de Francia [1988]) in two dimensions and by A. Seeger [1996] in higher dimensions (see further results in [Tao 1999]). Therefore a natural question inspired by [Parcet 2009] is whether can we weaken the Lipschitz regularity of kernel to the Hörmander condition or even rough homogeneous kernel. This problem has been open since then. The purpose of this paper is to develop some theory in this aspect for a class of rough operators. We consider the most fundamental operator: the maximal operator with a rough kernel which is defined by (in the sense of classical harmonic analysis)

$$M_\Omega f(x) = \sup_{r>0} |M_r f(x)|, \qquad M_r f(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} \Omega(x-y) f(y)\, dy, \tag{1-2}$$

where $B(x,r)$ is a ball in $\mathbb{R}^d$ with center x and radius r, the kernel Ω is a homogeneous function defined on $\mathbb{R}^d \setminus \{0\}$ with degree zero, that is,

$$\Omega(rx') = \Omega(x') \quad \text{for any } r > 0 \text{ and } x' \in \boldsymbol{S}^{d-1}. \tag{1-3}$$

Notice that the maximal operator M_Ω is a generalization of the Hardy–Littlewood maximal operator (by setting Ω as a constant, M_Ω is exactly the Hardy–Littlewood maximal operator). M_Ω is very important in the theory of rough singular integrals since it could be used to control many operators with rough kernels, just like the Hardy–Littlewood maximal operator plays an important role in analysis. By the method of rotation, it is easy to see that M_Ω is bounded on $L_p(\mathbb{R}^d)$ for all $1 < p \le \infty$ if $\Omega \in L_1(\boldsymbol{S}^{d-1})$; see, e.g., [Grafakos 2014a]. However, the weak-type (1, 1) boundedness of M_Ω is quite challenging. It was proved by Christ [1988] that M_Ω is of weak-type (1, 1) if $\Omega \in L_q(\boldsymbol{S}^1)$ with $1 < q \le \infty$ in two dimensions. Later Christ and Rubio de Francia [1988] showed in higher dimensions M_Ω is weak-type (1, 1) bounded if $\Omega \in L \log^+ L(\boldsymbol{S}^{d-1})$ by a depth investigation of the geometry in Euclidean space. For more topics, including open problems related to the maximal operator M_Ω, we refer to the reader to [Stein 1998; Grafakos and Stefanov 1999; Grafakos et al. 2017].

The noncommutative version of M_Ω should be important in the theory of noncommutative rough singular integral operators as expected. For instance, the noncommutative M_Ω will play a crucial role in the study of the noncommutative maximal operator of truncated operator in (1-1). In this paper, we will study the boundedness of M_Ω on the noncommutative L_p space for $1 \le p \le \infty$. In a special case Ω is a constant (i.e., M_Ω is the Hardy–Littlewood maximal operator), T. Mei [2007] investigated its noncommutative L_p, $1 < p \le \infty$, and weak-type (1, 1) boundedness. For general kernel Ω, there is no proper theory for the noncommutative M_Ω. To illustrate our noncommutative result of M_Ω, we should first give some basic notation.

Let us first introduce the noncommutative L_p space. Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a normal semifinite faithful (n.s.f. for short) trace τ. We consider the algebra $\mathcal{A}_B$ of essentially bounded $\mathcal{M}$-valued functions

$$\mathcal{A}_B = \Big\{ f : \mathbb{R}^d \to \mathcal{M} \mid f \text{ is strong measurable such that } \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \|f(x)\|_{\mathcal{M}} < \infty \Big\}.$$

equipped with the n.s.f. trace $\varphi(f) = \int_{\mathbb{R}^d} \tau(f(x))\,dx$. Let $\mathcal{A}$ be the weak-operator closure of $\mathcal{A}_B$. Then $\mathcal{A}$ is a von Neumann algebra. For $1 \le p < \infty$, define $L_p(\mathcal{M})$ as the noncommutative L_p space associated to the pairs $(\mathcal{M}, \tau)$ with the L_p norm given by $\|x\|_{L_p(\mathcal{M})} = (\tau(|x|^p))^{1/p}$. The space $L_p(\mathcal{A})$ is defined as the closure of $\mathcal{A}_B$ with respect to the norm

$$\|f\|_{L_p(\mathcal{A})} = \left(\int_{\mathbb{R}^d} \tau(|f(x)|^p)\,dx \right)^{1/p}, \tag{1-4}$$

which means that $L_p(\mathcal{A})$ is the noncommutative L_p space associated to the pairs $(\mathcal{A}, \varphi)$. On the other hand, from (1-4) we see that $L_p(\mathcal{A})$ is isometric to the Bochner L_p space with values in $L_p(\mathcal{M})$. For convenience, we set $L_\infty(\mathcal{M}) = \mathcal{M}$ and $L_\infty(\mathcal{A}) = \mathcal{A}$ equipped with the operator norm. The lattices of projections are written as $\mathcal{M}_\pi$ and $\mathcal{A}_\pi$, while $1_{\mathcal{M}}$ and $1_{\mathcal{A}}$ stand for the unit elements. Let $L_p^+(\mathcal{A})$ be the positive part of $L_p(\mathcal{A})$. A lot of basic properties of classical L_p spaces, such as Minkowski's inequality, Hölder's inequality, the dual property, real and complex interpolation, etc., have been transferred to this noncommutative setting. We refer to the very detailed introduction in [Parcet 2009] or the survey article [Pisier and Xu 2003] for more about the noncommutative L_p space, the noncommutative $L_{1,\infty}$ space and related topics.

We next define a noncommutative analogue of M_Ω. For two general elements belong to a von Neumann algebra, they may not be comparable (i.e., neither $a < b$ nor $a \ge b$ holds for $a, b \in \mathcal{A}$). Hence it is difficult to define the noncommutative maximal function directly. This obstacle could be overcome by straightforwardly defining the maximal weak-type (1, 1) norm or L_p norm. We adopt the definition of the noncommutative maximal norm introduced by G. Pisier [1998] and M. Junge [2002].

Definition 1.1. For any index set I, we define $L_p(\mathcal{M}; \ell_\infty(I))$, the space of all sequences $x = \{x_n\}_{n\in I}$ in $L_p(\mathcal{M})$ which admit a factorization of the following form: there exist $a, b \in L_{2p}(\mathcal{M})$ and a bounded sequence $y = \{y_n\}_{n\in I}$ in $L_\infty(\mathcal{M})$ such that $x_n = a y_n b$ for all $n \in I$. The norm of x in $L_p(\mathcal{M}; \ell_\infty(I))$ is given by

$$\|\{x_k\}_{k\in I}\|_{L_p(\mathcal{M};\ell_\infty(I))} = \inf\Big\{ \|a\|_{L_{2p}(\mathcal{M})} \sup_{n\in I} \|y_n\|_{L_\infty(\mathcal{M})} \|b\|_{L_{2p}(\mathcal{M})} \Big\},$$

where the infimum is taken over all factorizations of x as above. We define a sequence $x = \{x_k\}_{k\in I}$ in $L_{1,\infty}(\mathcal{M})$ with quasinorm given by

$$\|\{x_k\}_{k\in I}\|_{\Lambda_{1,\infty}(\mathcal{M};\ell_\infty(I))} = \sup_{\lambda>0} \lambda \inf_{e\in\mathcal{M}_\pi} \{\tau(e^\perp) : \|e x_k e\|_\infty \le \lambda \text{ for all } k \in I\}.$$

If $x = \{x_n\}_{n\in I}$ is a sequence of positive elements, then $x \in L_p(\mathcal{M}; \ell_\infty(I))$ if and only if there exists a positive element $a \in L_p(\mathcal{M})$ such that $0 < x_n \le a$, and

$$\|x\|_{L_p(\mathcal{M};\ell_\infty(I))} = \inf\{\|a\|_{L_p(\mathcal{M})} : 0 < x_n \le a \text{ for all } n \in I\}, \tag{1-5}$$

$$\|(x_k)_{k\in I}\|_{\Lambda_{1,\infty}(\mathcal{M};\ell_\infty(I))} = \sup_{\lambda>0} \lambda \inf_{e\in\mathcal{M}_\pi} \{\tau(e^\perp) : e x_n e \le \lambda \text{ for all } n\}. \tag{1-6}$$

Now we can state our main result as follows.

Theorem 1.2. *Suppose that* Ω *satisfies* (1-3) *and* $\Omega \in L(\log^+ L)^2(S^{d-1})$. *Then the operator sequence* $\{M_r\}_{r>0}$ *is of maximal weak-type* $(1, 1)$, *i.e.*,

$$\|\{M_r f\}_{r>0}\|_{\Lambda_{1,\infty}(\mathcal{A},\ell_\infty(0,\infty))} \lesssim \mathcal{C}_\Omega \|f\|_{L_1(\mathcal{A})},$$

where $\mathcal{C}_\Omega$ *is a constant depending only on the dimension* d *and* Ω. *Equivalently, for every* $f \in L_1(\mathcal{A})$ *and* $\lambda > 0$, *there exists a projection* $e \in \mathcal{A}_\pi$ *such that*

$$\sup_{r>0} \|eM_r fe\|_{L_\infty(\mathcal{A})} \le \lambda \quad \textit{and} \quad \lambda\varphi(1_{\mathcal{A}} - e) \lesssim \mathcal{C}_\Omega \|f\|_{L_1(\mathcal{A})}.$$

It is very easy to show that $\{M_r\}_{r>0}$ is of maximal strong type (p, p) for $1 < p \le \infty$ by the method of rotation. For completeness, we give a proof in the Appendix for this result.

The strategy in the proof of Theorem 1.2 is as follows. Firstly we convert the study of the maximal operator to a linearized singular integral operator T with a rough kernel (see Section 2). Secondly, to prove the weak-type $(1, 1)$ bound of this singular integral operator T, we use the noncommutative Calderón–Zygmund decomposition to split the function f into two parts: good functions and bad functions (see Section 3A). Roughly speaking, the proof is reduced to obtain some decay estimates for the good and bad functions separately. For the proof related to the bad functions, since the kernel Ω is rough, we will use a further decomposition, the so-called microlocal decomposition, to the operator T_j. Then we apply the L_2 norm and the L_1 norm to control the weak-type estimate (see Lemmas 3.4 and 3.5), where vector-valued Plancherel's theorem and an orthogonal geometric argument are involved in the proof of the L_2 estimate and the stationary phase method is used in the L_1 estimate (see Section 3B and the proofs in Section 4). For the proof of good functions we use the so-called pseudolocalization arguments to obtain some decay estimate for the L_2 norm of the singular integral operator T outside the support of functions on which it acts. To get such decay estimates, we adopt a similar method (the microlocal decomposition) from the proof of bad functions (see Section 3C).

In the classical Calderón–Zygmund decomposition, one can easily deal with the good function by the L_2 estimate. However, the proof of good functions from the noncommutative Calderón–Zygmund is much elaborated as showed in the case of smooth kernel by J. Parcet [2009]. In this paper, to overcome the nonsmoothness of kernel, we use the microlocal decomposition in the proofs of both bad and good functions. To the best knowledge of the author, this method seems to be new in the noncommutative Calderón–Zygmund theory. We should point out that the proof of bad functions is quite different from that in the classical case of[Christ and Rubio de Francia 1988], where they used the TT^* argument to obtain some regularity of the kernel $T_jT_j^*$ by some depth geometry but without using the Fourier transform. However, our method presented in this paper heavily depends on the Fourier transform where Plancherel's theorem and the stationary phase method are involved. These ideas are mainly inspired by [Seeger 1996]. Recall the following important pointwise property is crucial in the classical TT^* argument: $|Q|^{-1} \int_Q |b_Q(y)|\, dy \lesssim \lambda$, where b_Q is a bad function from the Calderón–Zygmund decomposition which is supported in a cube Q. Since in the noncommutative setting such kind of inequality may not hold for the off-diagonal terms of bad functions, our noncommutative TT^* argument is more complicated than that of the classical case. In fact only one pointwise property holds in the noncommutative Calderón–Zygmund decomposition: $q_k f_k q_k \lesssim \mathcal{C}_\Omega^{-1}\lambda q_k$ (see Lemma 3.1) and all pointwise estimates in the proof should finally be transferred to this property (see Section 4B for the details in the proof of the L_2 estimate).

This paper is organized as follows. First the study for maximal operator of M_r is reduced to a linearized singular integral operator in Section 2. In Section 3, by the noncommutative Calderón–Zygmund decomposition and microlocal decomposition, we finish the proof of our main theorem based on the estimates of bad and good functions. The proofs of lemmas related to the bad functions are all presented in Section 4. In Section 5, we give all proofs of lemmas related to the good functions. Finally in the Appendix, we give a proof of strong type (p, p), $1 < p \le \infty$, for $\{M_r\}_{r>0}$.

Further remark. After we finished this manuscript, L. Cadilhac [2022] found a more efficient noncommutative Calderón–Zygmund decomposition (see also [Hong et al. 2023]) so that the off-diagonal terms of the good functions vanish and the argument for the pseudolocalization can be avoided. Of course using this new Calderón–Zygmund decomposition, we only need to apply the L_2 estimate to deal with the good function and the proof related to the good functions in this paper can be greatly shortened. However, we point out that using this new method, the proof for the bad functions will be significantly more complicated than our arguments presented in this paper. So our proof in this paper still has its own interest. Nevertheless, we hope to show this in the study of weak-type (1, 1) boundedness for singular integral operators with rough kernels (1-1) which is our ongoing work.

Notation. Throughout this paper, we only consider the dimension $d \ge 2$ and the letter C stands for a positive finite constant which is independent of the essential variables, not necessarily the same one in each occurrence. $A \lesssim B$ means $A \le CB$ for some constant C. By the notation C_ε we mean that the constant depends on the parameter ε, $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$, $\mathbb{Z}_+$ denotes the set of all nonnegative integers and

$$\mathbb{Z}_+^d = \underbrace{\mathbb{Z}_+ \times \cdots \times \mathbb{Z}_+}_{d}.$$

For $\alpha \in \mathbb{Z}_+^d$ and $x \in \mathbb{R}^d$, we define $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$ and $|x|$ denotes the ℓ_2 norm. For all $s \in \mathbb{R}_+$, $[s]$ denotes the integer part of s. For any set A with finite elements, we define $\operatorname{card}(A)$ or $\#(A)$ as the number of elements in A. Let $s \ge 0$, we define

$$\|\Omega\|_{L(\log^+ L)^s} := \int_{\boldsymbol{S}^{d-1}} |\Omega(\theta)| [\log(2 + |\Omega(\theta)|)]^s \, d\sigma(\theta),$$

where $d\sigma(\theta)$ denotes the sphere measure of $\boldsymbol{S}^{d-1}$. When $s = 0$, we use the standard notation $\|\Omega\|_1 := \|\Omega\|_{L(\log^+ L)^0}$.

Define $\mathcal{F} f$ (or $\hat{f}$) and $\mathcal{F}^{-1} f$ (or $\check{f}$) the Fourier transform and the inverse Fourier transform of f by

$$\mathcal{F} f(\xi) = \int_{\mathbb{R}^d} e^{-i\langle x, \xi\rangle} f(x)\, dx, \quad \mathcal{F}^{-1} f(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle x, \xi\rangle} f(x)\, dx.$$

Let $\mathcal{Q}$ be the set of all dyadic cubes in $\mathbb{R}^d$. For any $Q \in \mathcal{Q}$, denote by $\ell(Q)$ the side length of the cube Q. Let sQ be the cube with the same center of Q such that $\ell(sQ) = s\ell(Q)$. Given an integer $k \in \mathbb{Z}$,

$\mathcal{Q}_k$ will be defined as the set of dyadic cubes of side length 2^{-k}. Let $|Q|$ be the volume of the cube Q. If $Q \in \mathcal{Q}$ and $f : \mathbb{R}^d \to \mathcal{M}$ is integrable on Q, we define its average as $f_Q = |Q|^{-1} \int_Q f(y)\,dy$.

For $k \in \mathbb{Z}$, set σ_k as the k-th dyadic σ-algebra, i.e., σ_k is generated by the dyadic cubes with side lengths equal to 2^{-k}. Let E_k be the conditional expectation which is associated to the classical dyadic filtration σ_k on $\mathbb{R}^d$. We also use E_k for the tensor product $\mathsf{E}_k \otimes \mathrm{id}_{\mathcal{M}}$ acting on $\mathcal{A}$. Then for $1 \le p < \infty$ and $f \in L_p(\mathcal{A})$, we get

$$\mathsf{E}_k(f) = \sum_{Q \in \mathcal{Q}_k} f_Q \chi_Q,$$

where χ_Q is the characteristic function of Q. Similarly, $\{\mathcal{A}_k\}_{k\in\mathbb{Z}}$ will stand for the corresponding filtration, i.e., $\mathcal{A}_k = \mathsf{E}_k(\mathcal{A})$. For simplicity, we will define the conditional expectation $f_k := \mathsf{E}_k(f)$ and the martingale difference $\Delta_k(f) := f_k - f_{k-1} =: df_k$.

2. Reduction to singular integral operator

In this section, we reduce the study of maximal operator of M_r to a singular integral operator with a rough kernel. This will be done by several steps as follows.

Step 1: By decomposing the functions Ω and f into four parts (i.e., real positive part, real negative part, imaginary positive part, imaginary negative part), together with the quasitriangle inequality for the quasi-norm $\|\cdot\|_{\Lambda_{1,\infty}(\mathcal{A},\ell_\infty(0,\infty))}$, we only consider the case that Ω is a positive function and f is positive in $\mathcal{A}$. Then by (1-6), it is enough to show that for any $f \in L_1^+(\mathcal{A})$ and $\lambda > 0$ there exists a projection $e \in \mathcal{A}_\pi$ such that

$$eM_r f e \le \lambda \quad \text{for all } r > 0 \qquad \text{and} \qquad \lambda\varphi(1_{\mathcal{A}} - e) \lesssim C_\Omega \|f\|_{L_1(\mathcal{A})}. \tag{2-1}$$

Step 2: Next we show that the study of M_r can be reduced to a dyadic smooth operator. More precisely, let ϕ be a $C_c^\infty(\mathbb{R}^d)$, radial, positive function which is supported in $\left\{x \in \mathbb{R}^d : \frac{1}{2} \le |x| \le 2\right\}$ and $\sum_{i\in\mathbb{Z}} \phi_j(x) = 1$ for all $x \in \mathbb{R}^d \setminus \{0\}$, where $\phi_j(x) = \phi(2^{-j}x)$. Define an operator $\mathfrak{M}_j$ by

$$\mathfrak{M}_j f(x) = \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d} \phi_j(x-y) f(y)\,dy.$$

We will prove that the maximal operator of $\mathfrak{M}_j$ is of weak-type $(1,1)$ below and (2-1) follows from it.

Theorem 2.1. *Let Ω be a positive function satisfying (1-3) and $\Omega \in L(\log^+ L)^2(S^{d-1})$. For any $f \in L_1^+(\mathcal{A})$, $\lambda > 0$, there exists a projection $e \in \mathcal{A}_\pi$ such that*

$$\sup_{j\in\mathbb{Z}} \|e\mathfrak{M}_j f e\|_{L_\infty(\mathcal{A})} \lesssim \lambda, \quad \lambda\varphi(1_{\mathcal{A}} - e) \lesssim C_\Omega \|f\|_{L_1(\mathcal{A})},$$

where the constant C_Ω only depends on Ω and the dimension.

The proof of Theorem 2.1 will be given later. We apply Theorem 2.1 to show (2-1). Let Ω be a positive function and f be positive in $L_1^+(\mathcal{A})$. Then by our choice of ϕ_j, for any $r > 0$, we have

$$\begin{aligned} M_r f(x) &= \frac{1}{|B(x,r)|} \int_{B(x,r)} \Omega(x-y) f(y)\,dy \\ &= \frac{C_d}{r^d} \sum_{j \le [\log r]+1} \int_{|x-y|\le r} \phi_j(x-y)\Omega(x-y) f(y)\,dy \lesssim \frac{1}{r^d} \sum_{j\le[\log r]+1} 2^{jd}\mathfrak{M}_j f(x). \end{aligned}$$

Notice that Ω is positive and $f \in L_1^+(\mathcal{A})$; thus the inequality $e\mathfrak{M}_j f e \le \lambda$ is equivalent to $\|e\mathfrak{M}_j f e\|_{\mathcal{A}} = \|e\mathfrak{M}_j f e\|_{L_\infty(\mathcal{A})} \le \lambda$. By Theorem 2.1, there exists a projection $e \in \mathcal{A}_\pi$ such that

$$e\mathfrak{M}_j f e \lesssim \lambda \quad \text{for all } j \in \mathbb{Z}, \qquad \lambda\varphi(1_{\mathcal{A}} - e) \lesssim C_\Omega \|f\|_{L_1(\mathcal{A})}.$$

Then it is easy to see that, for any $r > 0$,

$$eM_r f e \lesssim \frac{1}{r^d} \sum_{j \le [\log r]+1} 2^{jd} e\mathfrak{M}_j f e \lesssim \lambda.$$

Step 3: We will reduce the study of the maximal operator of $\mathfrak{M}_j$ to a class of square functions. Notice that the kernel Ω of $\mathfrak{M}_j$ has no cancellation. Formally we cannot study the operator $\big(\sum_j |\mathfrak{M}_j|^2\big)^{1/2}$ directly since it may not even be L_2 bounded. To avoid such case, we define a new operator T_j which is a modified version of the operator $\mathfrak{M}_j$

$$T_j f(x) = \int_{\mathbb{R}^d} \phi_j(x-y) \frac{\widetilde{\Omega}(x-y)}{|x-y|^d} f(y)\, dy, \tag{2-2}$$

where

$$\widetilde{\Omega}(x) = \Omega(x) - \frac{1}{\sigma_{d-1}} \int_{S^{d-1}} \Omega(\theta)\, d\sigma(\theta)$$

and σ_{d-1} is measure of the unit sphere. Then it is easy to see that $\widetilde{\Omega}$ has mean value zero over $\boldsymbol{S}^{d-1}$. Then formally the study of the maximal operator of $\mathfrak{M}_j$ may follow from that of the square function $\big(\sum_j |T_j|^2\big)^{1/2}$ and the maximal operator. In the following we use rigorous noncommutative language to explain how to do it. To define a noncommutative square function, we should first introduce the so-called column and row function space. Let $\{f_j\}_j$ be a finite sequence in $L_p(\mathcal{A})$, $1 \le p \le \infty$. Define

$$\|\{f_j\}_j\|_{L_p(\mathcal{A};\ell_2^r)} = \Big\| \Big(\sum |f_j^*|^2\Big)^{1/2} \Big\|_{L_p(\mathcal{A})}, \qquad \|(f_j)\|_{L_p(\mathcal{A};\ell_2^c)} = \Big\| \Big(\sum |f_j|^2\Big)^{1/2} \Big\|_{L_p(\mathcal{A})}.$$

This procedure is also used to define the spaces $L_{1,\infty}(\mathcal{A};\ell_2^r)$ and $L_{1,\infty}(\mathcal{A};\ell_2^c)$; i.e.,

$$\|\{f_j\}_j\|_{L_{1,\infty}(\mathcal{A};\ell_2^r)} = \Big\| \Big(\sum |f_j^*|^2\Big)^{1/2} \Big\|_{L_{1,\infty}(\mathcal{A})}, \qquad \|\{f_j\}\|_{L_{1,\infty}(\mathcal{A};\ell_2^c)} = \Big\| \Big(\sum |f_j|^2\Big)^{1/2} \Big\|_{L_{1,\infty}(\mathcal{A})}.$$

Let $L_{1,\infty}(\mathcal{A}, \ell_2^{rc})$ space be the weak-type square function of $\{T_j\}_j$ defined as

$$\|\{T_j\}_j\|_{L_{1,\infty}(\mathcal{A},\ell_2^{rc})} = \inf_{T_j f = g_j + h_j} \{\|\{g_j\}_j\|_{L_{1,\infty}(\mathcal{A},\ell_2^c)} + \|\{h_j\}_j\|_{L_{1,\infty}(\mathcal{A};\ell_2^r)}\}.$$

We have the following weak-type (1, 1) estimate of square function of $\{T_j\}_j$.

Theorem 2.2. *Suppose that Ω satisfies (1-3) and $\Omega \in L(\log^+ L)^2(\boldsymbol{S}^{d-1})$. Let T_j be defined in (2-2). Then we have*

$$\|\{T_j\}_j\|_{L_{1,\infty}(\mathcal{A},\ell_2^{rc})} \lesssim C_\Omega \|f\|_{L_1(\mathcal{A})},$$

where the constant C_Ω only depends on Ω and the dimension.

In the following we use Theorem 2.2 to prove Theorem 2.1. Our goal is to find a projection $e \in \mathcal{A}_\pi$ such that

$$\sup_{j\in\mathbb{Z}} \|e\mathfrak{M}_j f e\|_{L_\infty(\mathcal{A})} \lesssim \lambda, \quad \lambda\varphi(1_{\mathcal{A}} - e) \lesssim C_\Omega \|f\|_{L_1(\mathcal{A})}.$$

We first decompose $\mathfrak{M}_j f$ into two parts

$$T_j f(x) + \frac{1}{\sigma_{d-1}} \int_{S^{d-1}} \Omega(\theta)\, d\sigma(\theta) \int_{\mathbb{R}^d} \frac{\phi_j(x-y)}{|x-y|^d} f(y)\, dy =: T_j f(x) + \widetilde{M}_j f(x).$$

Notice that $(1/\sigma_{d-1}) \int_{S^{d-1}} \Omega(\theta)\, d\sigma(\theta)$ is a harmless constant which is bounded by $\|\Omega\|_1$. By using the fact that the noncommutative Hardy–Littlewood maximal operator is of weak-type (1, 1) (see, e.g., [Mei 2007]), it is not difficult to see that the maximal operator of $\widetilde{M}_j$ is of weak-type (1, 1). Thus we can find a projection $e_1 \in \mathcal{A}_\pi$ such that

$$\sup_{j\in\mathbb{Z}} \|e_1 \widetilde{M}_j f e_1\|_{L_\infty(\mathcal{A})} \le \lambda \quad \text{and} \quad \lambda\varphi(1_{\mathcal{A}} - e_1) \lesssim \|\Omega\|_1 \|f\|_{L_1(\mathcal{A})}.$$

Next we utilize Theorem 2.2 to construct other projection. By the definition of infimum, there exists a decomposition $T_j f = g_j + h_j$ satisfying

$$\|\{g_j\}\|_{L_{1,\infty}(\mathcal{A};\ell_2^c)} + \|\{h_j\}\|_{L_{1,\infty}(\mathcal{A};\ell_2^r)} \le \tfrac{1}{2}\mathcal{C}_\Omega \|f\|_{L_1(\mathcal{A})}.$$

We now take $e_2 = \chi_{(0,\lambda]}\big(\big(\sum_{j\in\mathbb{Z}} |g_j|^2\big)^{1/2}\big)$ and $e_3 = \chi_{(0,\lambda]}\big(\big(\sum_{j\in\mathbb{Z}} |h_j^*|^2\big)^{1/2}\big)$. Then

$$\bigg\| \bigg(\Big(\sum_{j\in\mathbb{Z}} |g_j|^2 \Big)^{1/2} \bigg) e_2 \bigg\|_{L_\infty(\mathcal{A})} \le \lambda \quad \text{and} \quad \lambda\varphi(1_{\mathcal{A}} - e_2) \lesssim \mathcal{C}_\Omega \|f\|_{L_1(\mathcal{A})}.$$

Also for e_3, we have

$$\bigg\| \bigg(\Big(\sum_{j\in\mathbb{Z}} |h_j^*|^2 \Big)^{1/2} \bigg) e_3 \bigg\|_{L_\infty(\mathcal{A})} \le \lambda \quad \text{and} \quad \lambda\varphi(1_{\mathcal{A}} - e_3) \lesssim \mathcal{C}_\Omega \|f\|_{L_1(\mathcal{A})}.$$

Let $e = e_1 \wedge e_2 \wedge e_3$. Then it is easy to see that

$$\sup_{j\in\mathbb{Z}} \|e \widetilde{M}_j f e\|_{L_\infty(\mathcal{A})} \le \lambda, \quad \lambda\varphi(1_{\mathcal{A}} - e) \lesssim \mathcal{C}_\Omega \|f\|_{L_1(\mathcal{A})}.$$

Hence to finish the proof of Theorem 2.1, it is sufficient to show

$$\sup_{j\in\mathbb{Z}} \|e T_j f e\|_{L_\infty(\mathcal{A})} \lesssim \lambda.$$

Recall the definition of $L_\infty(\mathcal{A})$, $\|f\|_{L_\infty(\mathcal{A})} = \|f\|_{\mathcal{A}}$. Then we get

$$\|e T_j f e\|_{L_\infty(\mathcal{A})} \le \|e g_j e\|_{\mathcal{A}} + \|e h_j e\|_{\mathcal{A}} = \|e g_j e\|_{\mathcal{A}} + \|e h_j^* e\|_{\mathcal{A}}.$$

Now using polar decomposition $g_j = u_j |g_j|$ and $h_j^* = v_j |h_j^*|$, we continue to estimate the above term as follows:

$$\begin{aligned} \|e u_j |g_j| e\|_{\mathcal{A}} + \|e v_j |h_j^*| e\|_{\mathcal{A}} &\le \||g_j| e\|_{\mathcal{A}} + \||h_j^*| e\|_{\mathcal{A}} = \|e |g_j|^2 e\|_{\mathcal{A}}^{1/2} + \|e |h_j^*|^2 e\|_{\mathcal{A}}^{1/2} \\ &\le \Big\| e \sum_{j\in\mathbb{Z}} |g_j|^2 e \Big\|_{\mathcal{A}}^{1/2} + \Big\| e \sum_{j\in\mathbb{Z}} |h_j^*|^2 e \Big\|_{\mathcal{A}}^{1/2} \\ &= \bigg\| \Big(\sum_{j\in\mathbb{Z}} |g_j|^2 \Big)^{1/2} e_2 e \bigg\|_{\mathcal{A}} + \bigg\| \Big(\sum_{j\in\mathbb{Z}} |h_j^*|^2 \Big)^{1/2} e_3 e \bigg\|_{\mathcal{A}} \lesssim \lambda. \end{aligned}$$

Hence we finish the proof of Theorem 2.1.

Step 4: We reduce the study of the square function to a linear operator. To simplify the notation, we still use Ω in (2-2), i.e.,

$$T_j f(x) = \int_{\mathbb{R}^d} K_j(x-y)\Omega(x-y)f(y)\,dy, \quad \text{with } K_j(x) = \phi_j(x)|x|^{-d}, \tag{2-3}$$

but we suppose that Ω satisfies the cancellation property $\int_{S^{d-1}} \Omega(\theta)\,d\sigma(\theta) = 0$. To linearize the square function, we use the following noncommutative Khintchine's inequality in $L_{1,\infty}(\mathcal{A}, \ell_2^{rc})$ which was recently established in [Cadilhac 2019].

Lemma 2.3. *Let $\{\varepsilon_j\}_j$ be a Rademacher sequence on a probability space $(\mathfrak{m}, P)$. Suppose that $f = \{f_j\}_j$ is a finite sequence in $L_{1,\infty}(\mathcal{A})$. Then we have*

$$\Big\| \sum_{j\in\mathbb{Z}} f_j \varepsilon_j \Big\|_{L_{1,\infty}(L_\infty(\mathfrak{m})\overline{\otimes}\mathcal{A})} \approx \|\{f_j\}_j\|_{L_{1,\infty}(\mathcal{A};\ell_2^{rc})}.$$

Now by the preceding lemma, Theorem 2.2 immediately follows from the result below.

Theorem 2.4. *Suppose that Ω satisfies (1-3), $\Omega \in L(\log^+ L)^2(S^{d-1})$ and the cancellation property $\int_{S^{d-1}} \Omega(\theta)\,d\sigma(\theta) = 0$. Let T_j be defined in (2-3). Assume $\{\varepsilon_j\}_j$ is the Rademacher sequence on a probability space $(\mathfrak{m}, P)$. Define $Tf(x,z) = \sum_j T_j f(x)\varepsilon_j(z)$ and the tensor trace $\tilde{\varphi} = \int_{\mathfrak{m}} \otimes \varphi$. Then T maps $L_1(\mathcal{A})$ to $L_{1,\infty}(L_\infty(\mathfrak{m})\,\overline{\otimes}\,\mathcal{A})$, i.e., for any $\lambda > 0$, $f \in L_1(\mathcal{A})$,*

$$\lambda\tilde{\varphi}\{|Tf| > \lambda\} \lesssim C_\Omega \|f\|_{L_1(\mathcal{A})},$$

where the constant C_Ω only depends on Ω and the dimension.

At present our main result Theorem 1.2 is reduced to Theorem 2.4. In the rest of this paper, we give effort to the proof of Theorem 2.4.

3. Proof of Theorem 2.4

In this section we give the proof of Theorem 2.4 based on some lemmas; their proofs will be given in Sections 4 and 5. We first introduce the noncommutative Calderón–Zygmund decomposition.

3A. ***Noncommutative Calderón–Zygmund decomposition.*** By the standard density argument, we only need to consider the following dense class of $L_1(\mathcal{A})$:

$$\mathcal{A}_{c,+} = \{f : \mathbb{R}^d \to \mathcal{M} \mid f \in \mathcal{A}_+,\ \overrightarrow{\operatorname{supp}} f \text{ is compact}\}.$$

Here $\overrightarrow{\operatorname{supp}} f$ represents the support of f as an operator-valued function in $\mathbb{R}^d$, which means that $\overrightarrow{\operatorname{supp}} f = \{x \in \mathbb{R}^d : \|f(x)\|_{\mathcal{M}} \neq 0\}$. Let $\Omega \in L(\log^+ L)^2(S^{d-1})$. Set a constant

$$C_\Omega = \|\Omega\|_{L(\log^+ L)^2} + \int_{S^{d-1}} |\Omega(\theta)| \bigg(1 + \bigg[\log^+\bigg(\frac{|\Omega(\theta)|}{\|\Omega\|_1}\bigg)\bigg]^2\bigg)\,d\sigma(\theta), \tag{3-1}$$

where $\log^+ a = 0$ if $0 < a < 1$ and $\log^+ a = \log a$ if $a \geq 1$. Since $\|\Omega\|_{L(\log^+ L)^2} < +\infty$, one can easily check that C_Ω is a finite constant. Now we fix $f \in \mathcal{A}_{c,+}$, and set $f_k = \mathsf{E}_k f$ for all $k \in \mathbb{Z}$. Then the sequence $\{f_k\}_{k\in\mathbb{Z}}$ is a positive dyadic martingale in $L_1(\mathcal{A})$. Applying the so-called Cuculescu construction introduced in [Parcet 2009, Lemma 3.1] at level $\lambda\mathcal{C}_\Omega{}^{-1}$, we get the following result.

Lemma 3.1. *There exists a decreasing sequence $\{q_k\}_{k\in\mathbb{Z}}$ depending on f and $\lambda\mathcal{C}_\Omega{}^{-1}$, where q_k is a projection in $\mathcal{A}_\pi$ satisfying the following conditions*:

(i) *q_k commutes with $q_{k-1}f_kq_{k-1}$ for every $k\in\mathbb{Z}$.*

(ii) *q_k belongs to $\mathcal{A}_k$ for every $k\in\mathbb{Z}$ and $q_kf_kq_k\le\lambda\mathcal{C}_\Omega{}^{-1}q_k$.*

(iii) *Set $q=\bigwedge_{k\in\mathbb{Z}}q_k$. We have the inequality*

$$\varphi(1_\mathcal{A}-q)\le\lambda^{-1}\mathcal{C}_\Omega\|f\|_{L_1(\mathcal{A})}.$$

(iv) *The expression of q_k can be written as follows*: *for some negative integer $m\in\mathbb{Z}$*

$$q_k=\begin{cases}1_\mathcal{A} & \text{if } k<m,\\ \chi_{(0,\lambda\mathcal{C}_\Omega^{-1}]}(f_k) & \text{if } k=m,\\ \chi_{(0,\lambda\mathcal{C}_\Omega^{-1}]}(q_{k-1}f_kq_{k-1}) & \text{if } k>m.\end{cases}$$

Below we introduce another expression of the projection q_k given in the previous lemma as done in [Parcet 2009]. We point out that such kind of expression will be quite helpful when we give some estimates to the terms related to q_k. In fact we can write $q_k=\sum_{Q\in\mathcal{Q}_k}\xi_Q\chi_Q$ for all $k\in\mathbb{Z}$, where ξ_Q is a projection in $\mathcal{M}$ which satisfies the following conditions:

(i) ξ_Q has the following explicit expression: $\widehat{Q}$ below is the father dyadic cube of Q,

$$\xi_Q=\begin{cases}1_\mathcal{M} & \text{if } k<m,\\ \chi_{(0,\lambda\mathcal{C}_\Omega^{-1}]}(f_Q) & \text{if } k=m,\\ \chi_{(0,\lambda\mathcal{C}_\Omega^{-1}]}(\xi_{\widehat{Q}}f_Q\xi_{\widehat{Q}}) & \text{if } k>m.\end{cases}$$

(ii) $\xi_Q\in\mathcal{M}_\pi$ and $\xi_Q\le\xi_{\widehat{Q}}$.

(iii) ξ_Q commutes with $\xi_{\widehat{Q}}f_Q\xi_{\widehat{Q}}$ and $\xi_Qf_Q\xi_Q\le\mathcal{C}_\Omega^{-1}\lambda\xi_Q$.

Define the projection $p_k=q_{k-1}-q_k$. By applying the above more explicit expression, we see that p_k equals $\sum_{Q\in\mathcal{Q}_k}(\xi_{\widehat{Q}}-\xi_Q)\chi_Q=:\sum_{Q\in\mathcal{Q}_k}\pi_Q\chi_Q$, where $\pi_Q=\xi_{\widehat{Q}}-\xi_Q$. Then it is easy to see that all p_k's are pairwise disjoint and $\sum_{k\in\mathbb{Z}}p_k=1_\mathcal{A}-q$.

Now we define the associated good functions and bad functions related to f as follows:

$$f=g+b,\quad g=\sum_{i,j\in\widehat{\mathbb{Z}}}p_if_{i\vee j}p_j,\quad b=\sum_{i,j\in\widehat{\mathbb{Z}}}p_i(f-f_{i\vee j})p_j,$$

where we set $p_\infty=q$, $\widehat{\mathbb{Z}}=\mathbb{Z}\cup\{\infty\}$ and $i\vee j=\max(i,j)$. If i or j is infinite, $i\vee j$ is just ∞ and $f_\infty=f$ by definition. We further decompose g as the diagonal terms and the off-diagonal terms:

$$g_d=qfq+\sum_{k\in\mathbb{Z}}p_kf_kp_k,\quad g_{\text{off}}=\sum_{i\neq j}p_if_{i\vee j}p_j+qf(1_\mathcal{A}-q)+(1_\mathcal{A}-q)fq.$$

The proofs for diagonal terms g_d and off-diagonal terms g_{off} will be different as we shall see below. For the bad function b, we can deal with the diagonal and off-diagonal terms uniformly. So it is unnecessary for us to decompose it as we did for the good functions. By the linearity of T, we get

$$\tilde{\varphi}(|Tf|>\lambda)\le\tilde{\varphi}\Big(|Tg|>\frac{\lambda}{2}\Big)+\tilde{\varphi}\Big(|Tb|>\frac{\lambda}{2}\Big).$$

In the following we give estimates for the good and bad functions, respectively. Before that we state a lemma to construct a projection in $\mathcal{A}$ such that the proof can be reduced to the case that the operators are restricted on this projection.

Lemma 3.2. *There exists a projection $\zeta\in\mathcal{A}_\pi$ which satisfies the following conditions*:

(i) $\lambda\varphi(1_{\mathcal{A}}-\zeta)\lesssim\mathcal{C}_\Omega\|f\|_{L_1(\mathcal{A})}$.

(ii) *If $Q_0\in\mathcal{Q}$ and $x\in(2^{101}+1)Q_0$, then $\zeta(x)\le 1_{\mathcal{M}}-\xi_{\widehat{Q}_0}+\xi_{Q_0}$ and $\zeta(x)\le\xi_{Q_0}$.*

The proof of this lemma can be easily modified from that of [Parcet 2009, Lemma 4.2]. Here the exact value of $2^{101}+1$ above is not essential and the reason we choose this value is just for convenience in a later calculation (see (3-3)). Now let us consider the bad functions first since our method presented here is also needed for the good functions.

3B. ***Estimates for the bad functions.*** We first use Lemma 3.2 to reduce the study of the operator T to that of $\zeta T\zeta$. Split Tb into four terms as follows:

$$(1_{\mathcal{A}}-\zeta)Tb(1_{\mathcal{A}}-\zeta)+\zeta Tb(1-\zeta)+(1-\zeta)Tb\zeta+\zeta Tb\zeta.$$

By the property (i) in Lemma 3.2, we get

$$\tilde{\varphi}\Big(|Tb|>\frac{\lambda}{2}\Big)\lesssim\varphi(1_{\mathcal{A}}-\zeta)+\tilde{\varphi}\Big(|\zeta Tb\zeta|>\frac{\lambda}{4}\Big)\lesssim\lambda^{-1}\mathcal{C}_\Omega\|f\|_{L_1(\mathcal{A})}+\tilde{\varphi}\Big(|\zeta Tb\zeta|>\frac{\lambda}{4}\Big).$$

Therefore it is enough to show that the term $\tilde{\varphi}(|\zeta Tb\zeta|>\lambda/4)$ satisfies our desired estimate. Recall the bad function

$$b=\sum_{k\in\mathbb{Z}}p_k(f-f_k)p_k+\sum_{s\ge1}\sum_{k\in\mathbb{Z}}p_k(f-f_{k+s})p_{k+s}+p_{k+s}(f-f_{k+s})p_k=:\sum_{s=0}\sum_{k\in\mathbb{Z}}b_{k,s},$$

where

$$b_{k,0}=p_k(f-f_k)p_k,\quad b_{k,s}=p_k(f-f_{k+s})p_{k+s}+p_{k+s}(f-f_{k+s})p_k. \tag{3-2}$$

By the definition of T, we further rewrite Tb as follows: for any $x\in\mathbb{R}^d$ and $z\in\mathfrak{m}$,

$$Tb(x,z)=\sum_{j\in\mathbb{Z}}T_j\Big[\sum_{s\ge0}\sum_{n\in\mathbb{Z}}b_{n-j,s}\Big](x)\varepsilon_j(z)=\sum_{s\ge0}\sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}T_jb_{n-j,s}(x)\varepsilon_j(z).$$

For any $Q\in\mathcal{Q}_{n-j+s}$, set $Q_{n-j}\in\mathcal{Q}_{n-j}$ as the s-th ancestor of Q. Consider x in the support of ζ (i.e., $\zeta(x)\ne0$) and let $n<100$. Then we get that, for all s, $\zeta(x)T_jb_{n-j,s}(x)\zeta(x)$ equals

$$\begin{aligned}&\sum_{Q\in\mathcal{Q}_{n-j+s}}\zeta(x)\int_Q K_j(x-y)b_{n-j,s}(y)\,dy\,\zeta(x)\\&\quad=\sum_{Q\in\mathcal{Q}_{n-j+s}}\zeta(x)\chi_{((2^{101}+1)Q_{n-j})^c}(x)\int_Q K_j(x-y)[\pi_{Q_{n-j}}(f(y)-f_Q)\pi_Q+\pi_Q(f(y)-f_Q)\pi_{Q_{n-j}}]\,dy\,\zeta(x)\\&\quad=0,\end{aligned}\tag{3-3}$$

where in the first equality we apply $\zeta(x)\pi_{Q_{n-j}}=0$ if $x\in(2^{101}+1)Q_{n-j}$ by the property (ii) of ζ in Lemma 3.2 and the second inequality follows from the fact $x\in((2^{101}+1)Q_{n-j})^c$ and $y\in Q$ implies that $|x-y|\ge 2^{100+j-n}$, which is a contradiction with the support of K_j and $n<100$. Therefore we get

$$\zeta Tb\zeta=\zeta\sum_{n\ge 100}\sum_{s\ge 0}\sum_{j\in\mathbb{Z}}T_jb_{n-j,s}\varepsilon_j\zeta.$$

Hence, to finish the proof related to the bad functions, it suffices to verify the following estimate:

$$\tilde{\varphi}\left(\left|\zeta\sum_{n\ge 100}\sum_{s\ge 0}\sum_{j\in\mathbb{Z}}T_jb_{n-j,s}\varepsilon_j\zeta\right|>\frac{\lambda}{4}\right)\lesssim\lambda^{-1}\mathcal{C}_\Omega\|f\|_{L_1(\mathcal{A})}. \tag{3-4}$$

Some important decompositions play key roles in the proof of (3-4). We present them by some lemmas, which will be proved in Section 4. It should be pointed out that the methods used here also work for the good functions, which will be clear in the next subsection.

The first lemma shows that (3-4) holds if Ω is restricted in some subset of $\boldsymbol{S}^{d-1}$. More precisely, for fixed $n\ge 100$ and $s\ge 0$, define

$$D^\iota=\{\theta\in\boldsymbol{S}^{d-1}:|\Omega(\theta)|\ge 2^{\iota(n+s)}\|\Omega\|_1\},$$

where $\iota>0$ will be chosen later. Let $T^{n,s}_{j,\iota}$ be defined by

$$T^{n,s}_{j,\iota}h(x)=\int_{\mathbb{R}^d}\Omega\chi_{D^\iota}\left(\frac{x-y}{|x-y|}\right)K_j(x-y)\cdot h(y)\,dy. \tag{3-5}$$

Lemma 3.3. *Suppose $\Omega\in L(\log^+L)^2(\boldsymbol{S}^{d-1})$. With all the notation above, we get*

$$\tilde{\varphi}\left(\left|\zeta\sum_{n\ge 100}\sum_{s\ge 0}\sum_{j\in\mathbb{Z}}T^{n,s}_{j,\iota}b_{n-j,s}\varepsilon_j\zeta\right|>\frac{\lambda}{8}\right)\lesssim\lambda^{-1}\mathcal{C}_\Omega\|f\|_{L_1(\mathcal{A})}.$$

Thus, by Lemma 3.3, to finish the proof for bad functions, it suffices to verify (3-4) under the condition that for fixed $n\ge 100$ and $s\ge 0$ the kernel function Ω satisfies $\|\Omega\|_\infty\le 2^{\iota(n+s)}\|\Omega\|_1$ in each T_j.

In the following, we introduce the *microlocal decomposition* of kernel. To do this, we give a partition of unity on the unit surface $\boldsymbol{S}^{d-1}$. Let $k\ge 100$. Choose $\{e^k_v\}_{v\in\Theta_k}$ be a collection of unit vectors on $\boldsymbol{S}^{d-1}$ which satisfies the following two conditions:

(a) $|e^k_v-e^k_{v'}|\ge 2^{-k\gamma-4}$ if $v\ne v'$.

(b) If $\theta\in\boldsymbol{S}^{d-1}$, there exists an e^k_v such that $|e^k_v-\theta|\le 2^{-k\gamma-3}$.

The constant $0<\gamma<1$ in (a) and (b) will be chosen later. To choose such an $\{e^k_v\}_{v\in\Theta_k}$, we simply take a maximal collection $\{e^k_v\}_v$ for which (a) holds and then (b) holds automatically by the maximality. Notice that there are $C2^{k\gamma(d-1)}$ elements in the collection $\{e^k_v\}_v$. For every $\theta\in\boldsymbol{S}^{d-1}$, there only exist finite e^k_v such that $|e^k_v-\theta|\le 2^{-k\gamma-4}$. Now we can construct an associated partition of unity on the unit surface $\boldsymbol{S}^{d-1}$. Let η be a smooth, nonnegative, radial function with $\eta(u)=1$ for $|u|\le\frac{1}{2}$ and $\eta(u)=0$ for $|u|>1$. Define

$$\widetilde{\Gamma}^k_v(u)=\eta\left(2^{k\gamma}\left(\frac{u}{|u|}-e^k_v\right)\right),\qquad \Gamma^k_v(u)=\widetilde{\Gamma}^k_v(u)\left(\sum_{v\in\Theta_k}\widetilde{\Gamma}^k_v(u)\right)^{-1}.$$

Then it is easy to see that Γ_v^k is homogeneous of degree 0 with $\sum_{v\in\Theta_k}\Gamma_v^k(u)=1$ for all $u\neq 0$ and all k. Now we define the operator $T_j^{n,s,v}$ by

$$T_j^{n,s,v}h(x)=\int_{\mathbb{R}^d}\Omega(x-y)\Gamma_v^{n+s}(x-y)\cdot K_j(x-y)\cdot h(y)\,dy. \tag{3-6}$$

Then it is easy to see that $T_j=\sum_{v\in\Theta_{n+s}}T_j^{n,s,v}$.

In the sequel, we will use the Fourier transform since we need to separate the phase in frequency space into different directions. Hence we define a Fourier multiplier operator by

$$\widehat{G_{k,v}h}(\xi)=\Phi\Big(2^{k\gamma}\Big\langle e_v^k,\frac{\xi}{|\xi|}\Big\rangle\Big)\hat{h}(\xi),$$

where $\hat{h}$ is the Fourier transform of h and Φ is a smooth, nonnegative, radial function such that $0\le\Phi(x)\le 1$ and $\Phi(x)=1$ on $|x|\le 2$, $\Phi(x)=0$ on $|x|>4$. Now we can split $T_j^{n,s,v}$ into two parts: $T_j^{n,s,v}=G_{n+s,v}T_j^{n,s,v}+(I-G_{n+s,v})T_j^{n,s,v}$.

The following lemma gives the L^2 estimate involving $G_{n+s,v}T_j^{n,s,v}$, which will be proved in Section 4.

Lemma 3.4. *Let $n\ge 100$ and $s\ge 0$. Suppose $\|\Omega\|_\infty\le 2^{\iota(n+s)}\|\Omega\|_1$ in each T_j. With all the notation above, we get the estimate*

$$\sum_j\Big\|\sum_{v\in\Theta_{n+s}}G_{n+s,v}T_j^{n,s,v}b_{n-j,s}\Big\|_{L_2(\mathcal{A})}^2\lesssim 2^{-(n+s)\gamma+2(n+s)\iota}\lambda\mathcal{C}_\Omega\|f\|_{L_1(\mathcal{A})}.$$

The terms involving $(I-G_{n+s,v})T_j^{n,s,v}$ are more complicated. For convenience, we set $L_j^{n,s,v}=(I-G_{n+s,v})T_j^{n,s,v}$. In Section 4, we shall prove the following lemma.

Lemma 3.5. *Let $n\ge 100$ and $s\ge 0$. Suppose $\|\Omega\|_\infty\le 2^{\iota(n+s)}\|\Omega\|_1$ in each T_j. With all the notation above, then there exists a positive constant α such that*

$$\sum_j\sum_{v\in\Theta_{n+s}}\|L_j^{n,s,v}b_{n-j,s}\|_{L_1(\mathcal{A})}\lesssim 2^{-(n+s)\alpha}\mathcal{C}_\Omega\|f\|_{L_1(\mathcal{A})}.$$

We now complete the proof of (3-4). It is sufficient to prove (3-4) under the condition that for all fixed $n\ge 100$ and $s\ge 0$ we have $\|\Omega\|_\infty\le 2^{\iota(n+s)}\|\Omega\|_1$ in T_j. By Chebyshev's inequality and the triangle inequality, we get

$$\begin{aligned}\tilde{\varphi}\Big(\Big|\zeta\sum_{n\ge 100}\sum_{s\ge 0}\sum_{j\in\mathbb{Z}}T_jb_{n-j,s}\varepsilon_j\zeta\Big|>\frac{\lambda}{8}\Big)&\\ \lesssim\lambda^{-2}\Big\|\zeta\sum_{n\ge 100}\sum_{s\ge 0}\sum_j\sum_{v\in\Theta_{n+s}}G_{n+s,v}T_j^{n,s,v}b_{n-j,s}\varepsilon_j\zeta\Big\|_{L_2(L_\infty\mathfrak{m}\bar{\otimes}\mathcal{A})}^2&\\ +\lambda^{-1}\sum_{n\ge 100}\sum_{s\ge 0}\sum_j\sum_{v\in\Theta_{n+s}}\|\zeta L_j^{n,s,v}b_{n-j,s}\varepsilon_j\zeta\|_{L_1(L_\infty(\mathfrak{m})\bar{\otimes}\mathcal{A})}&\\ =:I+II.&\end{aligned}$$

First we consider the term I. Recall that $\{\varepsilon_j\}_j$ is a Rademacher sequence on a probability space $(\mathfrak{m},P)$. So we have the orthogonal equality

$$\Big\|\sum_{j\in\mathbb{Z}}\varepsilon_ja_j\Big\|_{L_2(L_\infty(\mathfrak{m})\bar{\otimes}\mathcal{A})}^2=\sum_j\|a_j\|_{L_2(\mathcal{A})}^2. \tag{3-7}$$

Choose $0<\iota<\frac{\gamma}{2}<\frac{1}{2}$. By the triangle inequality, the above orthogonal equality, using Hölder's inequality to remove ζ since ζ is a projection in $\mathcal{A}$, and finally by Lemma 3.4, we get

$$\begin{aligned} I &\lesssim \lambda^{-2}\Big(\sum_{n\geq 100}\sum_{s\geq 0}\Big\|\sum_{j}\varepsilon_j\zeta\sum_{v\in\Theta_{n+s}}G_{n+s,v}T_j^{n,s,v}b_{n-j,s}\zeta\Big\|_{L_2(L_\infty(\mathfrak{m})\overline{\otimes}\mathcal{A})}\Big)^2\\ &\lesssim \lambda^{-2}\Big(\sum_{n\geq 100}\sum_{s\geq 0}\Big(\sum_{j}\Big\|\zeta\sum_{v\in\Theta_{n+s}}G_{n+s,v}T_j^{n,s,v}b_{n-j,s}\zeta\Big\|^2_{L_2(\mathcal{A})}\Big)^{1/2}\Big)^2\\ &\lesssim \lambda^{-2}\Big(\sum_{n\geq 100}\sum_{s\geq 0}(2^{-(n+s)\gamma+2(n+s)\iota}\mathcal{C}_\Omega\lambda\|f\|_{L_1()})^{\frac{1}{2}}\Big)^2\lesssim\lambda^{-1}\mathcal{C}_\Omega\|f\|_{L_1(\mathcal{A})}. \end{aligned}$$

For the term *II*, by the fact that $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ is a bounded sequence, using Hölder's inequality to remove ζ and by Lemma 3.5, we get

$$\begin{aligned} II &\lesssim \lambda^{-1}\sum_{n\geq 100}\sum_{s\geq 0}\sum_{j}\sum_{v\in\Theta_{n+s}}\|L_j^{n,s,v}b_{n-j,s}\|_{L_1(\mathcal{A})}\\ &\lesssim \lambda^{-1}\sum_{n\geq 100}\sum_{s\geq 0}2^{-(n+s)\alpha}\mathcal{C}_\Omega\|f\|_{L_1(\mathcal{A})}\lesssim\mathcal{C}_\Omega\lambda^{-1}\|f\|_{L_1(\mathcal{A})}. \end{aligned}$$

Hence we complete the proof of (3-4) based on Lemmas 3.3–3.5. Their proofs will be given in Section 4.

3C. ***Estimates for the good functions.*** Now we turn to the estimates for good functions. The proofs of diagonal terms and off-diagonal terms will be quite different. We first consider the diagonal terms, which are simpler since they behave similar to those in the classical Calderón–Zygmund decomposition. Following the classical strategy, we should first establish the L_2 boundedness of T. In this situation, the condition for the kernel Ω in fact can be relaxed to $\Omega\in L(\log^+ L)^{1/2}(\boldsymbol{S}^{d-1})$.

Lemma 3.6. *Suppose that* Ω *satisfy* (1-3), $\Omega\in L(\log^+ L)^{1/2}(\boldsymbol{S}^{d-1})$ *and the cancellation property* $\int_{\boldsymbol{S}^{d-1}}\Omega(\theta)\,d\sigma(\theta)=0$. *Then we have*

$$\|Tf\|_{L_2(L_\infty(\mathfrak{m})\overline{\otimes}\mathcal{A})}\lesssim\|f\|_{L_2(\mathcal{A})},$$

where the implicit constant above depends only on the dimension and Ω.

Remark. It should be pointed out that the cancellation condition $\int_{\boldsymbol{S}^{d-1}}\Omega(\theta)\,d\theta=0$ in Theorem 2.4 is only used in this lemma to guarantee the L_2 boundedness of T.

The proof of Lemma 3.6 will be given in Section 5. Based on this lemma, we could prove required bound for the diagonal term g_d of good functions as follows. By using the property of q_k's in Lemma 3.1, Parcet [2009] obtained the following basic property of g_d:

$$\|g_d\|_{L_1(\mathcal{A})}\lesssim\|f\|_{L_1(\mathcal{A})},\quad \|g_d\|_{L_\infty(\mathcal{A})}\lesssim\lambda\mathcal{C}_\Omega^{-1}. \tag{3-8}$$

By Lemma 3.6, it is not difficult to see that the L_2 norm of T is bounded by $\mathcal{C}_\Omega$ (see the details in Section 5A for its proof). Therefore we get the estimate for g_d as follows:

$$\begin{aligned} \tilde{\varphi}\Big(|Tg_d|>\frac{\lambda}{4}\Big) &\lesssim\lambda^{-2}\|Tg_d\|^2_{L_2(L_\infty(\mathfrak{m})\overline{\otimes}\mathcal{A})}\lesssim\lambda^{-2}\mathcal{C}_\Omega^2\|g_d\|^2_{L_2(\mathcal{A})}\\ &\lesssim\lambda^{-2}\mathcal{C}_\Omega^2\|g_d\|_{L_1(\mathcal{A})}\|g_d\|_{L_\infty(\mathcal{A})}\lesssim\lambda^{-1}\mathcal{C}_\Omega\|f\|_{L_1(\mathcal{A})}, \end{aligned}$$

where in the first inequality we use Chebyshev's inequality, the second inequality follows from Lemma 3.6, the third and fourth inequalities just follow from (3-8).

In the remaining parts of this subsection, we give effort to the estimate of g_{off}. We first use Lemma 3.2 to reduce the proof to the case $\zeta T g_{\mathrm{off}}\zeta$. In fact

$$Tg_{\mathrm{off}} = (1_{\mathcal{A}}-\zeta)Tg_{\mathrm{off}}(1_{\mathcal{A}}-\zeta)+\zeta Tg_{\mathrm{off}}(1-\zeta)+(1-\zeta)Tg_{\mathrm{off}}\zeta+\zeta Tg_{\mathrm{off}}\zeta.$$

By Lemma 3.2 and the same argument as used for the bad functions, it is sufficient to consider the last term $\zeta T g_{\mathrm{off}}\zeta$ above. Thus our goal is to prove

$$\tilde{\varphi}\Big(|\zeta Tg_{\mathrm{off}}\zeta|>\frac{\lambda}{8}\Big)\lesssim \lambda^{-1}\mathcal{C}_\Omega\|f\|_{L_1(\mathcal{A})}. \tag{3-9}$$

Next we introduce another expression of the off-diagonal terms g_{off} and related estimates which were proved in [Parcet 2009].

Lemma 3.7. *Let df_s be martingale difference. We can rewrite g_{off} as*

$$g_{\mathrm{off}}=\sum_{s\geq 1}\sum_{k\in\mathbb{Z}} p_k df_{k+s}q_{k+s-1}+q_{k+s-1}df_{k+s}p_k =: \sum_{s\geq 1}\sum_{k\in\mathbb{Z}} g_{k,s} =: \sum_{s\geq 1} g_{(s)}.$$

The martingale difference sequence of $g_{(s)}$ satisfies $dg_{(s)k+s}=g_{k,s}$ and $\operatorname{supp}^* g_{k,s}\leq p_k\leq 1_{\mathcal{A}}-q_k$, *where* supp^* *is weak support projection defined by* $\operatorname{supp}^* a = 1_{\mathcal{A}}-\boldsymbol{q}$, *with $\boldsymbol{q}$ is the greatest projection satisfying $\boldsymbol{q}a\boldsymbol{q}=0$. Meanwhile, we have the estimates*

$$\sup_{s\geq 1}\|g_{(s)}\|^2_{L_2(\mathcal{A})}=\sup_{s\geq 1}\sum_{k\in\mathbb{Z}}\|g_{k,s}\|^2_{L_2(\mathcal{A})}\lesssim \lambda\mathcal{C}_\Omega^{-1}\|f\|_{L_1(\mathcal{A})}.$$

The strategy to deal with the off-diagonal terms g_{off} is similar to that we use in the proof for the bad functions, although the technical proofs may be different. By the expression of g_{off} in Lemma 3.7 and the formula $f=\sum_{n\in\mathbb{Z}} df_n$, we can write

$$\begin{aligned}\zeta Tg_{\mathrm{off}}\zeta &= \zeta\sum_{s\geq 1}\sum_{j\in\mathbb{Z}}\varepsilon_j T_j g_{(s)}\zeta = \zeta\sum_{s\geq 1}\sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\varepsilon_j T_j d(g_{(s)})_{n-j+s}\zeta\\ &= \zeta\sum_{s\geq 1}\sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\varepsilon_j T_j g_{n-j,s}\zeta = \zeta\sum_{s\geq 1}\sum_{n\geq 100}\sum_{j\in\mathbb{Z}}\varepsilon_j T_j g_{n-j,s}\zeta,\end{aligned}$$

where the last equality follows from the fact if $\zeta(x)\neq 0$ and $n<100$, we get $T_j g_{n-j,s}(x)=0$ for all $s\geq 1$ by property in (ii) of Lemma 3.2, $\operatorname{supp}^* g_{k,s}\leq p_k\leq 1_{\mathcal{A}}-q_k$ in Lemma 3.7 and the similar arguments in (3-3).

By Chebyshev's inequality, the triangle inequality, ζ is a projection in $\mathcal{A}$ and the orthogonal equality (3-7), we then get

$$\begin{aligned}\tilde{\varphi}(|\zeta Tg_{\mathrm{off}}\zeta|>\lambda) &\lesssim \lambda^{-2}\|\zeta Tg_{\mathrm{off}}\zeta\|^2_{L_2(L_\infty(\mathrm{m})\overline{\otimes}\mathcal{A})}\\ &\lesssim \lambda^{-2}\bigg(\sum_{s\geq 1}\sum_{n\geq 100}\Big(\sum_j\|T_j g_{n-j,s}\|^2_{L_2(\mathcal{A})}\Big)^{1/2}\bigg)^2.\end{aligned}$$

Hence to finish the proof for the off-diagonal terms g_{off}, it is sufficient to show that

$$\sum_{s\geq 1}\sum_{n\geq 100}\Big(\sum_j\|T_j g_{n-j,s}\|^2_{L_2(\mathcal{A})}\Big)^{1/2}\lesssim (\mathcal{C}_\Omega\lambda\|f\|_{L_1(\mathcal{A})})^{1/2}. \tag{3-10}$$

As done in the proof for the bad functions, we first show that (3-10) holds if Ω is restricted in $D^\iota=\{\theta\in \mathbf{S}^{d-1}:|\Omega(\theta)|\ge 2^{\iota(n+s)}\|\Omega\|_1\}$, where $\iota\in(0,1)$. Recall the definition of $T^{n,s}_{j,\iota}$ in (3-5). Then we have the following lemma.

Lemma 3.8. *Suppose $\Omega\in L(\log^+L)^2(\mathbf{S}^{d-1})$. With all the notation above, we get*

$$\sum_{s\ge1}\sum_{n\ge100}\Big(\sum_j\|T^{n,s}_{j,\iota}g_{n-j,s}\|^2_{L_2(\mathcal{A})}\Big)^{1/2}\lesssim(\mathcal{C}_\Omega\lambda\|f\|_{L_1(\mathcal{A})})^{1/2}.$$

The proof of Lemma 3.8 will be given in Section 5. By Lemma 3.8, to prove (3-10), we only need to show (3-10) under the condition that the kernel function Ω satisfies $\|\Omega\|_\infty\le 2^{\iota(n+s)}\|\Omega\|_1$ in each T_j. For each fixed $s\ge1$ and $n\ge100$, we make a microlocal decomposition of T_j as follows:

$$T_j=\sum_{v\in\Theta_{n+s}}T_j^{n,s,v},\qquad T_j^{n,s,v}=G_{n+s,v}T_j^{n,s,v}+(I-G_{n+s,v})T_j^{n,s,v}.$$

Here the notation $T_j^{n,s,v}$, $G_{n+s,v}$ is the same as those in the proof of the bad functions.

Lemma 3.9. *Let $n\ge100$ and $s\ge1$. Suppose that $\|\Omega\|_\infty\le2^{\iota(n+s)}\|\Omega\|_1$ in each T_j. Then we get the estimate*

$$\Big\|\sum_{v\in\Theta_{n+s}}G_{n+s,v}T_j^{n,s,v}g_{n-j,s}\Big\|^2_{L_2(\mathcal{A})}\lesssim2^{-(n+s)(\gamma-2\iota)}\|\Omega\|_1^2\|g_{n-j,s}\|^2_{L_2(\mathcal{A})}.$$

Lemma 3.10. *Let $n\ge100$ and $s\ge1$. Suppose $\|\Omega\|_\infty\le2^{\iota(n+s)}\|\Omega\|_1$ in each T_j. There exists a constant $\kappa>0$ such that*

$$\sum_{v\in\Theta_{n+s}}\|(I-G_{n+s,v})T_j^{n,s,v}g_{n-j,s}\|_{L_2(\mathcal{A})}\lesssim2^{-(n+s)\kappa}\|\Omega\|_1\|g_{n-j,s}\|_{L_2(\mathcal{A})}.$$

The proofs of Lemmas 3.9 and 3.10 will be given in Section 5. Now we use Lemmas 3.9 and 3.10 to prove (3-10) as follows:

$$\begin{aligned}\sum_{s\ge1}\sum_{n\ge100}\Big(\sum_j\|T_jg_{n-j,s}\|^2_{L_2(\mathcal{A})}\Big)^{1/2}&\le\sum_{s\ge1}\sum_{n\ge100}\Big(\sum_j\Big(\sum_{v\in\Theta_{n+s}}\|(I-G_{n+s,v})T_j^{n,s,v}g_{n-j,s}\|_{L_2(\mathcal{A})}\Big)^2\Big)^{1/2}\\&\quad+\sum_{s\ge1}\sum_{n\ge100}\Big(\sum_j\Big\|\sum_{v\in\Theta_{n+s}}G_{n+s,v}T_j^{n,s,v}g_{n-j,s}\Big\|^2_{L_2(\mathcal{A})}\Big)^{1/2}\\&\lesssim(\mathcal{C}_\Omega\lambda\|f\|_{L_1(\mathcal{A})})^{1/2},\end{aligned}$$

where in the second inequality we use Lemmas 3.9 and 3.10, with the fact that $\sum_j\|g_{n-j,s}\|^2_{L_2(\mathcal{A})}\lesssim\lambda\mathcal{C}_\Omega^{-1}\|f\|_{L_1(\mathcal{A})}$ for all $s\ge1$ in Lemma 3.7. Thus to finish the proof for good functions, it remains to show Lemmas 3.6, 3.8, 3.9 and 3.10, which are all given in Section 5.

Remark 3.11. At present, it is easy to see that the proofs for off-diagonal terms of good functions are similar to that of bad functions. Notice that for the bad functions, we can deal with the diagonal terms (i.e., $s=0$) and the off-diagonals terms (i.e., $s>1$) in a unified way. However this cannot be done for the good functions; thus we prove the diagonal terms g_d and the off-diagonal terms g_{off} using different methods. The main reason comes from the fact that g_{off} has the following property: for all $Q\in\mathcal{Q}_{n-j+s-1}$,

$\int_Q g_{n-j,s}(y)\,dy = 0$. Such kind of cancellation property is crucial in the proof of Lemma 3.10. But the diagonal terms g_d do not have this cancellation property.

4. Proofs of lemmas related to the bad functions

In this section, we begin to prove all the lemmas for the bad functions in Section 3B. Before that we introduce some lemmas needed in our proof. We first state Schur's Lemma which will be used later.

Lemma 4.1 (Schur's lemma). *Suppose that T is an operator with the kernel $K(x, y)$. Thus*

$$Tf(x) = \int_{\mathbb{R}^d} K(x, y) f(y)\,dy.$$

Then T is bounded on $L_2(\mathcal{A})$ with bound $\sqrt{c_1c_2}$, where

$$c_1 = \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |K(x, y)|\,dy, \quad c_2 = \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} |K(x, y)|\,dx.$$

The proof of this lemma could be found in [Parcet 2009; Grafakos 2014a]. We also need the following convexity inequality (or the Cauchy–Schwarz-type inequality) for the operator-valued function; see [Mei 2007, page 9]. Let $(\mathfrak{m}, \mu)$ be a measure space. Suppose that $f : \mathfrak{m} \to \mathcal{M}$ is a weak-* integrable function and $g : \mathfrak{m} \to \mathbb{C}$ is an integrable function. Then

$$\left|\int_{\mathfrak{m}} f(x)g(x)\,d\mu(x)\right|^2 \le \int_{\mathfrak{m}} |f(x)|^2\,d\mu(x) \int_{\mathfrak{m}} |g(x)|^2\,d\mu(x). \tag{4-1}$$

Below we introduce some basic properties of the bad functions that we will use in our proof.

Lemma 4.2. *Let $b_{k,s}$ be defined in (3-2). Fix any $s \ge 0$. Then we have the following properties for the bad functions $b_{k,s}$:*

(i) *The L_1 estimate $\sum_{k\in\mathbb{Z}} \|b_{k,s}\|_{L_1(\mathcal{A})} \lesssim \|f\|_{L_1(\mathcal{A})}$ holds.*

(ii) *For all $k \in \mathbb{Z}$ and $Q \in \mathcal{Q}_{k+s}$, the cancellation property $\int_Q b_{k,s}(y)\,dy = 0$ holds.*

The proof of Lemma 4.2 can be found in [Cadilhac 2018; Parcet 2009]. Now we start to prove Lemmas 3.3, 3.4 and 3.5.

4A. ***Proof of Lemma 3.3.*** Denote the kernel of the operator $T^{n,s}_{j,\iota}$ by

$$K^{n,s}_{j,\iota}(x-y) := \Omega\chi_{D^\iota}\left(\frac{x-y}{|x-y|}\right) K_j(x-y). \tag{4-2}$$

By the support of K_j, it is easy to see that

$$\|K^{n,s}_{j,\iota}\|_{L_1(\mathbb{R}^d)} \lesssim \int_{D^\iota} \int_{2^{j-1}}^{2^{j+1}} |\Omega(\theta)| r^{d-1} 2^{-jd}\,dr\,d\sigma(\theta) \lesssim \int_{D^\iota} |\Omega(\theta)|\,d\sigma(\theta).$$

Therefore by Chebyshev's inequality and the triangle inequality, using Hölder's inequality to remove the projection ζ, we get

$$\tilde{\varphi}\bigg(\bigg|\zeta\sum_{n\geq 100}\sum_{s\geq 0}\sum_{j\in\mathbb{Z}}T^{n,s}_{j,\iota}b_{n-j,s}\varepsilon_j\zeta\bigg|>\lambda\bigg)\leq\lambda^{-1}\sum_{n\geq 100}\sum_{s\geq 0}\sum_{j\in\mathbb{Z}}\|T^{n,s}_{j,\iota}b_{n-j,s}\varepsilon_j\|_{L_1(L_\infty(\mathfrak{m})\overline{\otimes}\mathcal{A})}.$$

Since $\{\varepsilon_j\}_j$ is the Rademacher sequence, $\{\varepsilon_j\}_j$ is a bounded sequence. Then from above we have

$$\begin{aligned}\lambda^{-1}\sum_{n\geq 100}\sum_{s\geq 0}\sum_{j\in\mathbb{Z}}\|T^{n,s}_{j,\iota}b_{n-j,s}\|_{L_1(\mathcal{A})}&\lesssim\lambda^{-1}\sum_{n\geq 100}\sum_{s\geq 0}\sum_{j\in\mathbb{Z}}\|K^{n,s}_{j,\iota}\|_{L_1(\mathbb{R}^d)}\|b_{n-j,s}\|_{L_1(\mathcal{A})}\\&\lesssim\lambda^{-1}\sum_{n\geq 100}\sum_{s\geq 0}\int_{D^\iota}|\Omega(\theta)|\,d\sigma(\theta)\sum_j\|b_{n-j,s}\|_{L_1(\mathcal{A})}.\end{aligned}$$

Now applying the property (i) in Lemma 4.2, the above estimate is bounded by

$$\begin{aligned}&\lambda^{-1}\|f\|_{L_1(\mathcal{A})}\int_{S^{d-1}}\#\bigg\{(n,s):n\geq 100,\,s\geq 0,\,2^{\iota(n+s)}\leq\frac{|\Omega(\theta)|}{\|\Omega\|_1}\bigg\}|\Omega(\theta)|\,d\sigma(\theta)\\&\qquad\lesssim\lambda^{-1}\|f\|_{L_1(\mathcal{A})}\int_{S^{d-1}}|\Omega(\theta)|\bigg(\bigg(\log^+\frac{|\Omega(\theta)|}{\|\Omega\|_1}\bigg)^2\bigg)\,d\sigma(\theta)\lesssim\lambda^{-1}\mathcal{C}_\Omega\|f\|_{L_1(\mathcal{A})}.\qquad\square\end{aligned}$$

4B. ***Proof of Lemma 3.4.*** The proof of Lemma 3.4 is based on the following observation of some orthogonality of the support of $\mathcal{F}(G_{k,v})$: For a fixed $k\geq 100$, we have

$$\sup_{\xi\neq 0}\sum_{v\in\Theta_k}\bigg|\Phi^2\bigg(2^{k\gamma}\bigg\langle e^k_v,\frac{\xi}{|\xi|}\bigg\rangle\bigg)\bigg|\lesssim 2^{k\gamma(d-2)}.\tag{4-3}$$

In fact, by the homogeneity of $\Phi^2(2^{k\gamma}\langle e^k_v,\xi/|\xi|\rangle)$, it suffices to take the supremum over the surface S^{d-1}. For $|\xi|=1$ and $\xi\in\operatorname{supp}\Phi^2(2^{k\gamma}\langle e^k_v,\xi/|\xi|\rangle)$, denote by $\xi^\perp$ the hyperplane perpendicular to ξ. Then it is easy to see that

$$\operatorname{dist}(e^k_v,\xi^\perp)\lesssim 2^{-k\gamma}.\tag{4-4}$$

Since the mutual distance of e^k_v's is bounded by $2^{-k\gamma-4}$, there are at most $2^{k\gamma(d-2)}$ vectors satisfying (4-4). We hence get (4-3).

Notice that $L_2(\mathcal{M})$ is a Hilbert space; then the following vector-valued Plancherel's theorem holds:

$$\|\mathcal{F}f\|_{L^2(\mathcal{A})}=(2\pi)^{d/2}\|f\|_{L^2(\mathcal{A})}=(2\pi)^d\|\mathcal{F}^{-1}f\|_{L^2(\mathcal{A})}.$$

By applying this Plancherel's theorem, the convex inequality for the operator-valued function (4-1), the fact (4-3) and finally Plancherel's theorem again, we get

$$\begin{aligned}&\bigg\|\sum_{v\in\Theta_{n+s}}G_{n+s,v}T^{n,s,v}_j b_{n-j,s}\bigg\|^2_{L_2(\mathcal{A})}\\&\quad=(2\pi)^{-d/2}\int_{\mathbb{R}^d}\tau\bigg(\bigg|\sum_{v\in\Theta_{n+s}}\Phi\bigg(2^{(n+s)\gamma}\bigg\langle e^{n+s}_v,\frac{\xi}{|\xi|}\bigg\rangle\bigg)\mathcal{F}(T^{n,s,v}_j b_{n-j,s})(\xi)\bigg|^2\bigg)\,d\xi\\&\quad\lesssim\int_{\mathbb{R}^d}\sum_{v\in\Theta_{n+s}}\Phi^2\bigg(2^{(n+s)\gamma}\bigg\langle e^{n+s}_v,\frac{\xi}{|\xi|}\bigg\rangle\bigg)\tau\bigg(\sum_{v\in\Theta_{n+s}}|\mathcal{F}(T^{n,s,v}_j b_{n-j,s})(\xi)|^2\bigg)\,d\xi\\&\quad\lesssim 2^{(n+s)\gamma(d-2)}\sum_{v\in\Theta_{n+s}}\|T^{n,s,v}_j b_{n-j,s}\|^2_{L_2(\mathcal{A})}.\end{aligned}\tag{4-5}$$

Once it is showed that, for a fixed e_v^{n+s},

$$\sum_j \|T_j^{n,s,v} b_{n-j,s}\|_{L_2(\mathcal{A})}^2 \lesssim 2^{-2(n+s)\gamma(d-1)+2(n+s)\iota} \lambda \|\Omega\|_1 \|f\|_{L_1(\mathcal{A})}, \tag{4-6}$$

by $\operatorname{card}(\Theta_{n+s}) \lesssim 2^{(n+s)\gamma(d-1)}$, and applying (4-5) and (4-6) we get

$$\sum_j \Big\| \sum_{v\in\Theta_{n+s}} G_{n+s,v} T_j^{n,s,v} b_{n-j,s} \Big\|_{L_2(\mathcal{A})}^2 \lesssim 2^{-(n+s)\gamma+2(n+s)\iota} \lambda \mathcal{C}_\Omega \|f\|_{L_1(\mathcal{A})},$$

which is the asserted bound of Lemma 3.4. Thus, to finish the proof of Lemma 3.4, it is sufficient to show (4-6).

Recall the definition of $b_{n-j,s}$ in (3-2). By using triangle inequality, to prove (4-6), it is enough to prove the four terms

$$\sum_j \|T_j^{n,s,v} p_{n-j} f p_{n-j+s}\|_{L_2(\mathcal{A})}^2, \quad \sum_j \|T_j^{n,s,v} p_{n-j} f_{n-j+s} p_{n-j+s}\|_{L_2(\mathcal{A})}^2,$$
$$\sum_j \|T_j^{n,s,v} p_{n-j+s} f p_{n-j}\|_{L_2(\mathcal{A})}^2, \quad \sum_j \|T_j^{n,s,v} p_{n-j+s} f_{n-j+s} p_{n-j}\|_{L_2(\mathcal{A})}^2$$

satisfy the desired bound in (4-6). In the following we will only give the detailed proofs of the first and the second terms above, since the proofs of the third and the fourth terms are similar.

We first consider the second term, which involves $p_{n-j} f_{n-j+s} p_{n-j+s}$. Set the kernel of $T_j^{n,s,v}$ as

$$K_j^{n,s,v}(x) = \Gamma_v^{n+s}(x)\Omega(x)\phi_j(x)|x|^{-d}.$$

By Young's inequality, we get

$$\|T_j^{n,s,v} p_{n-j} f_{n-j+s} p_{n-j+s}\|_{L_2(\mathcal{A})}^2 \lesssim \|K_j^{n,s,v}\|_{L_1(\mathbb{R}^d)}^2 \|p_{n-j} f_{n-j+s} p_{n-j+s}\|_{L_2(\mathcal{A})}^2. \tag{4-7}$$

Below we give some estimates for the bound in (4-7). Recall that $|\Omega(\theta)| \le 2^{(n+s)\iota}\|\Omega\|_1$ and the definition of Γ_v^{n+s} in Section 3B. Then by some elementary calculation, we get

$$\|K_j^{n,s,v}\|_{L_1(\mathbb{R}^d)} \lesssim 2^{-(n+s)\gamma(d-1)+(n+s)\iota}\|\Omega\|_1. \tag{4-8}$$

Notice that f is positive in $\mathcal{A}$. By some basic properties of trace φ, we write

$$\begin{aligned}\|p_{n-j} f_{n-j+s} p_{n-j+s}\|_{L_2(\mathcal{A})}^2 &= \varphi(|p_{n-j} f_{n-j+s} p_{n-j+s}|^2)\\ &= \varphi(|p_{n-j+s} f_{n-j+s} p_{n-j}|^2) = \varphi(p_{n-j} f_{n-j+s} p_{n-j+s} f_{n-j+s} p_{n-j})\\ &\le \varphi(p_{n-j} f_{n-j+s}^{1/2} f_{n-j+s}^{1/2} p_{n-j}) \cdot \|f_{n-j+s}^{1/2} p_{n-j+s} f_{n-j+s}^{1/2}\|_{\mathcal{A}}.\end{aligned} \tag{4-9}$$

By the trace invariance and modularity of conditional expectations, the first term in the last line above has the trace-preserving property

$$\varphi(p_{n-j} f_{n-j+s} p_{n-j}) = \varphi(p_{n-j} f p_{n-j}) = \varphi(p_{n-j} f). \tag{4-10}$$

Applying the basic property of C^* algebra, $\|aa^*\|_{\mathcal{A}} = \|a^*a\|_{\mathcal{A}}$, we get

$$\begin{aligned}\|f_{n-j+s}^{1/2} p_{n-j+s} f_{n-j+s}^{1/2}\|_{\mathcal{A}} &= \|p_{n-j+s} f_{n-j+s} p_{n-j+s}\|_{\mathcal{A}} \\ &= \|p_{n-j+s} q_{n-j+s-1} f_{n-j+s} q_{n-j+s-1} p_{n-j+s}\|_{\mathcal{A}} \\ &\le 2^d \|p_{n-j+s} q_{n-j+s-1} f_{n-j+s-1} q_{n-j+s-1} p_{n-j+s}\|_{\mathcal{A}} \\ &\lesssim \lambda \mathcal{C}_{\Omega}^{-1}, \end{aligned} \tag{4-11}$$

where the second equality follows from the identity $p_k = p_k q_{k-1}$ by the definition of p_k and the last inequality follows from $q_k f_k q_k \le \lambda \mathcal{C}_{\Omega}^{-1} q_k$, property (ii) in Lemma 3.1. Now combining (4-7)–(4-11), we get

$$\begin{aligned}\sum_j \|T_j^{n,s,v} p_{n-j} f_{n-j+s} p_{n-j+s}\|_{L_2(\mathcal{A})}^2 &\lesssim \mathcal{C}_{\Omega}^{-1} \lambda 2^{-2(n+s)\gamma(d-1)+2(n+s)\iota} \|\Omega\|_1^2 \sum_j \varphi(p_{n-j} f) \\ &\lesssim \lambda 2^{-2(n+s)\gamma(d-1)+2(n+s)\iota} \|\Omega\|_1 \|f\|_{L_1(\mathcal{A})},\end{aligned}$$

which is the required estimate in (4-6).

Next we give an estimate of the term corresponding to $p_{n-j} f p_{n-j+s}$. Notice that there is no average of f in this case and the crucial property $q_k f_k q_k \le \lambda \mathcal{C}_{\Omega}^{-1} q_k$ cannot be applied in the estimate (4-11). Our strategy here is to *add* an average of f. In the following we first reduce the proof to the case that the kernel is positive. To do that, we first take the decomposition

$$K_j^{n,s,v} = (K_j^{n,s,v})^+ - (K_j^{n,s,v})^-,$$

where $(K_j^{n,s,v})^+$ and $(K_j^{n,s,v})^-$ are positive functions. Then by using triangle inequality, we get

$$\begin{aligned}\sum_j \|T_j^{n,s,v} p_{n-j} f p_{n-j+s}\|_{L_2(\mathcal{A})}^2 \lesssim \sum_j \Big\| \int (K_j^{n,s,v}(\cdot - y))^+ p_{n-j} f p_{n-j+s}(y)\, dy \Big\|_{L_2(\mathcal{A})}^2 \\ + \sum_j \Big\| \int (K_j^{n,s,v}(\cdot - y))^- p_{n-j} f p_{n-j+s}(y)\, dy \Big\|_{L_2(\mathcal{A})}^2.\end{aligned}$$

Therefore we need to consider the terms related to $(K_j^{n,s,v})^+$ and $(K_j^{n,s,v})^-$, respectively. We only consider the term related to $(K_j^{n,s,v})^+$ since the proof of the other term is similar. For convenience, in the remaining part of this section we still use the abused notation $K_j^{n,s,v}$ to represent $(K_j^{n,s,v})^+$.

Denote the support of $K_j^{n,s,v}$ by $E_j^{n,s,v}$. Then it is not difficult to see

$$\begin{aligned}E_j^{n,s,v} &\subset \Big\{ x \in \mathbb{R}^d : \Big| \frac{x}{|x|} - e_v^{n+s} \Big| \le 2^{-(n+s)\gamma},\ 2^{j-1} \le |x| \le 2^{j+1} \Big\} \\ &\subset \{ x \in \mathbb{R}^d : |\langle x, e_v^{n+s}\rangle| \le 2^{j+1},\ |x - \langle x, e_v^{n+s}\rangle e_v^{n+s}| \le 2^{j+1-(n+s)\gamma} \}.\end{aligned}$$

For any $Q \in \mathcal{Q}_{n-j+s}$, let $Q_{n-j} \in \mathcal{Q}_{n-j}$ be the s-th ancestor of Q. By the definition of p_k, we may write

$$\begin{aligned}T_j^{n,s,v}(p_{n-j} f p_{n-j+s})(x) &= \int_{\mathbb{R}^d} K_j^{n,s,v}(x-y)(p_{n-j} f p_{n-j+s})(y)\, dy \\ &= \sum_{\substack{Q \in \mathcal{Q}_{n-j+s} \\ Q \cap \{x - E_j^{n,s,v}\} \ne \varnothing}} \pi_{Q_{n-j}} \Big(\int_Q K_j^{n,s,v}(x-y) f(y)\, dy \Big) \pi_Q\end{aligned}$$

$$= \sum_{\substack{Q\in\mathcal{Q}_{n-j+s}\\ Q\cap\{x-E_j^{n,s,v}\}\neq\varnothing}} \int_Q [p_{n-j}(K_j^{n,s,v}(x-\cdot)f(\cdot))_{n-j+s}p_{n-j+s}](z)\,dz$$
$$= \int_{E_j^{n,s,v}(x)} [p_{n-j}(K_j^{n,s,v}(x-\cdot)f(\cdot))_{n-j+s}p_{n-j+s}](z)\,dz,$$

where we use the notation

$$E_j^{n,s,v}(x) = \bigcup_{\substack{Q\in\mathcal{Q}_{n-j+s}\\ Q\cap\{x-E_j^{n,s,v}\}\neq\varnothing}} Q.$$

By the support of $E_j^{n,s,v}$, we see that $E_j^{n,s,v}$ is contained in a rectangle with one sidelength at most 2^{j+1} and $d-1$ sidelength at most $2^{j+1-(n+s)\gamma}$. Since for any $Q\in\mathcal{Q}_{n-j+s}$, the sidelength satisfies $l(Q)=2^{j-(n+s)}\le 2^{j+1-(n+s)\gamma}$. So we get $E_j^{n,s,v}(x)$ is contained in a rectangle with one sidelength at most 2^{j+2} and $d-1$ sidelength at most $2^{j+2-(n+s)\gamma}$. Therefore we have the estimate

$$|E_j^{n,s,v}(x)| \lesssim 2^{jd-(n+s)\gamma(d-1)}.$$

Next by using the convexity inequality for the operator-valued function (4-1) and the preceding inequality, we get

$$|T_j^{n,s,v}(p_{n-j}fp_{n-j+s})(x)|^2 \lesssim 2^{jd-(n+s)\gamma(d-1)} \int_{E_j^{n,s,v}(x)} |p_{n-j}(K_j^{n,s,v}(x-\cdot)f(\cdot))_{n-j+s}p_{n-j+s}(z)|^2\,dz.$$

Combining the above estimates, we get

$$\|T_j^{n,s,v}p_{n-j}fp_{n-j+s}\|^2_{L_2(\mathcal{A})} \lesssim 2^{jd-(n+s)\gamma(d-1)} \int_{\mathbb{R}^d}\int_{E_j^{n,s,v}(x)} \tau\big(|p_{n-j}(K_j^{n,s,v}(x-\cdot)f(\cdot))_{n-j+s}p_{n-j+s}(z)|^2\big)\,dz\,dx. \tag{4-12}$$

Since $K_j^{n,s,v}$ is a positive function and f is a positive operator-valued function in $\mathcal{A}$, we see that $K(x-\cdot)f(\cdot)$ is positive in $\mathcal{A}$. Therefore

$$(K_j^{n,s,v}(x-\cdot)f(\cdot))_{n-j+s} = \sum_{Q\in\mathcal{Q}_{n-j+s}} \frac{1}{|Q|}\int_Q K_j^{n,s,v}(x-y)f(y)\,dy\,\chi_Q$$
$$\lesssim \sum_{Q\in\mathcal{Q}_{n-j+s}} \frac{1}{|Q|}\int_Q f(y)\,dy\,\chi_Q 2^{-jd+(n+s)\iota}\|\Omega\|_1$$
$$= 2^{-jd+(n+s)\iota}\|\Omega\|_1 f_{n-j+s}.$$

Now applying the above estimate and using the same idea in the estimates of (4-9) and (4-11), we get

$$\begin{aligned}
&\tau\big(|p_{n-j}(K_j^{n,s,v}(x-\cdot)f(\cdot))_{n-j+s}p_{n-j+s}(z)|^2\big)\\
&\quad= \tau\big(p_{n-j}(K_j^{n,s,v}(x-\cdot)f(\cdot))_{n-j+s}p_{n-j+s}(K_j^{n,s,v}(x-\cdot)f(\cdot))_{n-j+s}p_{n-j}(z)\big)\\
&\quad\le \tau\big(p_{n-j}(K_j^{n,s,v}(x-\cdot)f(\cdot))_{n-j+s}p_{n-j}(z)\big)\|p_{n-j+s}(K_j^{n,s,v}(x-\cdot)f(\cdot))_{n-j+s}p_{n-j+s}(z)\|_{\mathcal{M}}\\
&\quad\lesssim 2^{-2jd+2(n+s)\iota}\|\Omega\|_1^2\tau(p_{n-j}f_{n-j+s}p_{n-j}(z))\|p_{n-j+s}f_{n-j+s}p_{n-j+s}\|_{\mathcal{A}}\\
&\quad\lesssim 2^{-2jd+2(n+s)\iota}\|\Omega\|_1\lambda\tau(p_{n-j}f_{n-j+s}p_{n-j}(z)).
\end{aligned} \tag{4-13}$$

By the definition of $E_j^{n,s,v}(x)$, for any fixed $z \in \mathbb{R}^d$, we have the estimate

$$\left| \int_{\{x:\, E_j^{n,s,v}(x) \ni z\}} dx \right| \lesssim 2^{jd-(n+s)\gamma(d-1)}. \tag{4-14}$$

Plugging (4-13) into (4-12), then applying Fubini's theorem with (4-14), and finally using the trace-preserving property (4-10), we get

$$\begin{aligned}\sum_j \|T_j^{n,s,v} p_{n-j} f p_{n-j+s}\|^2_{L_2(\mathcal{A})} &\lesssim 2^{-2(n+s)\gamma(d-1)+2(n+s)\iota} \|\Omega\|_1 \lambda \sum_{j\in\mathbb{Z}} \varphi(p_{n-j} f_{n-j+s} p_{n-j}) \\ &\lesssim 2^{-2(n+s)\gamma(d-1)+2(n+s)\iota} \|\Omega\|_1 \lambda \|f\|_{L_1(\mathcal{A})}.\end{aligned}$$

Hence, we complete the proof of Lemma 3.4. □

4C. *Proof of Lemma 3.5.* To prove Lemma 3.5, we have to face some oscillatory integrals which come from $L_j^{n,s,v}$. Before stating the proof of Lemma 3.5, let us first give some notation. We introduce the Littlewood–Paley decomposition. Let ψ be a radial C^∞ function such that $\psi(\xi)=1$ for $|\xi|\le 1$, $\psi(\xi)=0$ for $|\xi|\ge 2$ and $0\le\psi(\xi)\le 1$ for all $\xi\in\mathbb{R}^d$. Define $\beta_k(\xi)=\psi(2^k\xi)-\psi(2^{k+1}\xi)$. Then β_k is supported in $\{\xi : 2^{-k-1}\le|\xi|\le 2^{-k+1}\}$. Choose $\tilde\beta$ be a radial C^∞ function such that $\tilde\beta(\xi)=1$ for $\frac12\le|\xi|\le 2$, $\tilde\beta$ is supported in $\left\{\xi : \frac14\le|\xi|\le 4\right\}$ and $0\le\tilde\beta(\xi)\le 1$ for all $\xi\in\mathbb{R}^d$. Set $\tilde\beta_k(\xi)=\tilde\beta(2^k\xi)$. Then it is easy to see $\beta_k=\tilde\beta_k\beta_k$. Define the convolution operators Λ_k and $\tilde\Lambda_k$ with the Fourier multipliers β_k and $\tilde\beta_k$, respectively. That is,

$$\widehat{\Lambda_k f}(\xi)=\beta_k(\xi)\hat f(\xi), \qquad \widehat{\tilde\Lambda_k f}(\xi)=\tilde\beta_k(\xi)\hat f(\xi).$$

Then by the construction of β_k and $\tilde\beta_k$, we have $\Lambda_k=\tilde\Lambda_k\Lambda_k,\ I=\sum_{k\in\mathbb{Z}}\Lambda_k$.

Write

$$L_j^{n,s,v}=\sum_k (I-G_{n+s,v})\Lambda_k T_j^{n,s,v}.$$

Then triangle inequality gives us

$$\|L_j^{n,s,v} b_{n-j,s}\|_{L_1(\mathcal{A})} \le \sum_{k\in\mathbb{Z}} \|(I-G_{n+s,v})\Lambda_k T_j^{n,s,v} b_{n-j,s}\|_{L_1(\mathcal{A})}.$$

In the remaining part of this subsection, we show that two different estimates can be established for $\|(I-G_{n+s,v})\Lambda_k T_j^{n,s,v} b_{n-j,s}\|_{L_1(\mathcal{A})}$, which will give Lemma 3.5 by taking a sum over $k\in\mathbb{Z}$ with these two different estimates.

Lemma 4.3. *With all the notation above. Then there exists $N>0$ such that the following estimate holds*:

$$\|(I-G_{n+s,v})\Lambda_k T_j^{n,s,v} b_{n-j,s}\|_{L_1(\mathcal{A})} \lesssim 2^{-(n+s)\gamma(d-1)+(n+s)\iota+(k-j)+(n+s)\gamma(1+2N)} \|\Omega\|_1 \|b_{n-j,s}\|_{L_1(\mathcal{A})}. \tag{4-15}$$

Proof. Applying Fubini's theorem, we may write

$$(I-G_{n+s,v})\Lambda_k T_j^{n,s,v} b_{n-j,s}(x) =: \int_{\mathbb{R}^d} D_k^{n,s,v}(x-y) b_{n-j,s}(y)\,dy, \tag{4-16}$$

where $D_k^{n,s,v}(x)$ is defined as the kernel of the operator $(I-G_{n+s,v})\Lambda_k T_j^{n,s,v}$. More precisely, $D_k^{n,s,v}$ can be written as

$$D_k^{n,s,v}(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}h_{k,n,s,v}(\xi)\int_{\mathbb{R}^d} e^{-i\xi\cdot\omega}\Omega(\omega)\Gamma_v^{n+s}(\omega)K_j(\omega)\,d\omega\,d\xi, \tag{4-17}$$

where $h_{k,n,s,v}(\xi)=(1-\Phi(2^{(n+s)\gamma}\langle e_v^{n+s},\xi/|\xi|\rangle))\beta_k(\xi)$. Using Young's inequality, we get

$$\|(I-G_{n+s,v})\Lambda_k T_j^{n,s,v}b_{n-j,s}\|_{L_1(\mathcal{A})}\le\|D_k^{n,s,v}\|_{L_1(\mathbb{R}^d)}\|b_{n-j,s}\|_{L_1(\mathcal{A})}.$$

Hence in the following we only need to give an L_1 estimate of $D_k^{n,s,v}$. In order to separate the rough kernel, we make a change of variable $\omega=r\theta$. By Fubini's theorem, $D_k^{n,s,v}(x)$ can be written as

$$\frac{1}{(2\pi)^d}\int_{S^{d-1}}\Omega(\theta)\Gamma_v^{n+s}(\theta)\left\{\int_{\mathbb{R}^d}\int_0^\infty e^{i\langle x-r\theta,\xi\rangle}h_{k,n,s,v}(\xi)K_j(r)r^{d-1}\,dr\,d\xi\right\}d\sigma(\theta). \tag{4-18}$$

Concerning the support of K_j, we have $2^{j-1}\le r\le 2^{j+1}$. Integrating by parts with r, the integral involving r can be rewritten as

$$\int_0^\infty e^{-i\langle r\theta,\xi\rangle}(i\langle\theta,\xi\rangle)^{-1}\partial_r[K_j(r)r^{d-1}]\,dr.$$

Since $\theta\in\operatorname{supp}\Gamma_v^{n+s}$, we have $|\theta-e_v^{n+s}|\le 2^{-(n+s)\gamma}$. By the support of Φ, we see $|\langle e_v^{n+s},\xi/|\xi|\rangle|\ge 2^{1-(n+s)\gamma}$. Thus,

$$\left|\left\langle\theta,\frac{\xi}{|\xi|}\right\rangle\right|\ge\left|\left\langle e_v^{n+s},\frac{\xi}{|\xi|}\right\rangle\right|-\left|\left\langle e_v^{n+s}-\theta,\frac{\xi}{|\xi|}\right\rangle\right|\ge 2^{-(n+s)\gamma}. \tag{4-19}$$

After integrating by parts with r, integrating by parts N times with ξ, the integral in (4-18) can be rewritten as

$$\frac{1}{(2\pi)^d}\int_{S^{d-1}}\Omega(\theta)\Gamma_v^{n+s}(\theta)\int_{\mathbb{R}^d}\int_0^\infty e^{i\langle x-r\theta,\xi\rangle}\partial_r[K_j(r)r^{d-1}] \\ \times\frac{(I-2^{-2k}\Delta_\xi)^N}{(1+2^{-2k}|x-r\theta|^2)^N}(h_{k,n,s,v}(\xi)(i\langle\theta,\xi\rangle)^{-1})\,dr\,d\xi\,d\sigma(\theta). \tag{4-20}$$

In the following, we give explicit estimates of all terms in (4-20). We show that the following estimate holds:

$$|(I-2^{-2k}\Delta_\xi)^N[\langle\theta,\xi\rangle^{-1}h_{k,n,s,v}(\xi)]|\lesssim 2^{(n+s)\gamma+k+2(n+s)\gamma N}. \tag{4-21}$$

Firstly we prove (4-21) when $N=0$. By (4-19), we have

$$|(-i\langle\theta,\xi\rangle)^{-1}\cdot h_{k,n,s,v}(\xi)|\lesssim|\langle\theta,\xi\rangle|^{-1}\lesssim 2^{(n+s)\gamma+k}.$$

Next we consider $N=1$ in (4-21). By using product rule and some elementary calculation, we get

$$|\partial_{\xi_i}h_{k,n,s,v}(\xi)|\le\left|-\partial_{\xi_i}\left[\Phi\left(2^{(n+s)\gamma}\left\langle e_v^{n+s},\frac{\xi}{|\xi|}\right\rangle\right)\right]\cdot\beta_k(\xi)\right|+\left|\partial_{\xi_i}\beta_k(\xi)\cdot\left(1-\Phi\left(2^{(n+s)\gamma}\left\langle e_v^{n+s},\frac{\xi}{|\xi|}\right\rangle\right)\right)\right| \\ \lesssim 2^{(n+s)\gamma+k}.$$

Therefore by induction, we have $|\partial_\xi^\alpha h_{k,n,s,v}(\xi)| \lesssim 2^{((n+s)\gamma+k)|\alpha|}$ for any multi-indices $\alpha \in \mathbb{Z}_+^n$. By using product rule again and (4-19), we have

$$\begin{aligned}|\partial_{\xi_i}^2(\langle\theta,\xi\rangle)^{-1}h_{k,n,s,v}(\xi))| &\le |2\langle\theta,\xi\rangle^{-3}\cdot\theta_i^2\cdot h_{k,n,s,v}(\xi)|+|2\langle\theta,\xi\rangle^{-2}\cdot\theta_i\partial_{\xi_i}h_{k,n,s,v}(\xi)|+|\langle\theta,\xi\rangle^{-1}\partial_{\xi_i}^2 h_{k,n,s,v}(\xi)|\\ &\lesssim 2^{3((n+s)\gamma+k)}.\end{aligned}$$

Hence we conclude that $2^{-2k}|\Delta_\xi[(\langle\theta,\xi\rangle)^{-1}h_{k,n,s,v}(\xi)]| \lesssim 2^{(n+s)\gamma+k+2(n+s)\gamma}$. Proceeding by induction, we get (4-21).

By the definition of K_j and using product rule, it is not difficult to get

$$|\partial_r(K_j(r)r^{d-1})| \lesssim 2^{-2j}. \tag{4-22}$$

Now we choose $N = [d/2]+1$. Since we need to get the L^1 estimate of (4-20), by the support of $h_{k,n,s,v}$, $|\xi| \approx 2^{-k}$,

$$\int_{|\xi|\approx 2^{-k}} \int_{\mathbb{R}^d} (1+2^{-2k}|x-r\theta|^2)^{-N}\,dx\,d\xi \le C.$$

Now combine (4-22), (4-21) and above estimates. Next integrating with r, we get a bound 2^j. Note that we suppose that $\|\Omega\|_\infty \le 2^{(n+s)\iota}\|\Omega\|_1$. Then integrating with θ, we get a bound $2^{-(n+s)\gamma(d-1)+(n+s)\iota}\|\Omega\|_1$. So we finally get

$$\begin{aligned}\|D_k^{n,s,v}\|_{L_1(\mathbb{R}^d)} &\lesssim 2^{-2j+(n+s)\gamma+k+2(n+s)\gamma N+j-(n+s)\gamma(d-1)+(n+s)\iota}\|\Omega\|_1\\ &= 2^{-(n+s)\gamma(d-1)+(n+s)\iota-j+k+(n+s)\gamma(1+2N)}\|\Omega\|_1.\end{aligned} \tag{4-23}$$

Hence we complete the proof of Lemma 4.3 with $N=[d/2]+1$. □

Lemma 4.4. *With all the notation above, the following estimate holds*:

$$\|(I-G_{n+s,v})\Lambda_k T_j^{n,s,v}b_{n-j,s}\|_{L_1(\mathcal{A})} \lesssim 2^{-(n+s)\gamma(d-1)-(n+s)+j-k+(n+s)\iota}\|\Omega\|_1\|b_{n-j,s}\|_{L_1(\mathcal{A})}.$$

Proof. Using $\Lambda_k = \Lambda_k\tilde{\Lambda}_k$, we write

$$\begin{aligned}\|(I-G_{n+s,v})\Lambda_k T_j^{n,s,v}b_{n-j,s}\|_{L_1(\mathcal{A})} &= \|(I-G_{n+s,v})\tilde{\Lambda}_k\Lambda_k T_j^{n,s,v}b_{n-j,s}\|_{L_1(\mathcal{A})}\\ &\lesssim \|(I-G_{n+s,v})\tilde{\Lambda}_k\|_{L_1(\mathcal{A})\to L_1(\mathcal{A})}\|\Lambda_k T_j^{n,s,v}b_{n-j,s}\|_{L_1(\mathcal{A})}.\end{aligned}$$

Then it is easy to see that the proof of this lemma follows from the two estimates

$$\|(I-G_{n+s,v})\tilde{\Lambda}_k\|_{L_1(\mathcal{A})\to L_1(\mathcal{A})} \lesssim 1 \tag{4-24}$$

and

$$\|\Lambda_k T_j^{n,s,v}b_{n-j,s}\|_{L_1(\mathcal{A})} \lesssim 2^{-(n+s)\gamma(d-1)-(n+s)+j-k+(n+s)\iota}\|\Omega\|_1\|b_{n-j,s}\|_{L_1(\mathcal{A})}. \tag{4-25}$$

We first consider the estimate (4-24). The kernel of $(I-G_{n+s,v})\tilde{\Lambda}_k$ is the inverse Fourier transform of $\tilde{h}_{k,n,s,v}(\xi) = [1-\Phi(2^{(n+s)\gamma}\langle e_v^{n+s},\xi/|\xi|\rangle)]\tilde{\beta}_k(\xi)$. So

$$\|(I-G_{n+s,v})\tilde{\Lambda}_k\|_{L_1(\mathcal{A})\to L_1(\mathcal{A})} \lesssim \|\mathcal{F}(\tilde{h}_{k,n,s,v})\|_{L^1(\mathbb{R}^d)} = \|\mathcal{F}[\tilde{h}_{k,n,s,v}(A_k^{n,s,v}\,\cdot)]\|_{L_1(\mathbb{R}^d)},$$

where $A_k^{n,s,v}$ is an invertible linear transform such that $A_k^{n,s,v} e_v^{n+s} = 2^{-(n+s)\gamma-k} e_v^{n+s}$ and $A_k^{n,s,v} y = 2^{-k} y$ if $\langle y, e_v^{n+s}\rangle = 0$. For all $\alpha \in \mathbb{Z}_+^d$, it is straightforward to check that

$$\|\partial^\alpha[\tilde{h}_{k,n,s,v}(A_k^{n,s,v}\,\cdot)]\|_{L_2(\mathbb{R}^d)} \lesssim C_\alpha$$

uniformly with k, n, s, v; see [Seeger 1996, page 100]. Therefore splitting the following integral into two parts and using Plancherel's theorem, we get

$$\begin{aligned}
&\|\mathcal{F}[\tilde{h}_{k,n,s,v}(A_k^{n,s,v}\,\cdot)]\|_{L^1(\mathbb{R}^d)} \\
&= \biggl(\int_{|\xi|\ge 1} + \int_{|\xi|<1}\biggr) |\mathcal{F}[\tilde{h}_{k,n,s,v}(A_k^{n,s,v}\,\cdot)](\xi)|\,d\xi \\
&\lesssim \biggl(\int_{|\xi|\ge 1} \frac{d\xi}{|\xi|^{2([d/2]+1)}}\biggr)^{1/2} \sum_{|\alpha|=[d/2]+1} \biggl(\int_{\mathbb{R}^d} |\xi^\alpha \mathcal{F}[\tilde{h}_{k,n,s,v}(A_k^{n,s,v}\,\cdot)](\xi)|^2\,d\xi\biggr)^{1/2} + \|\mathcal{F}[\tilde{h}_{k,n,s,v}(A_k^{n,s,v})]\|_{L_2(\mathbb{R}^d)} \\
&\lesssim \sum_{|\alpha|=[d/2]+1} \|\partial^\alpha[\tilde{h}_{k,n,s,v}(A_k^{n,s,v}\,\cdot)]\|_{L_2(\mathbb{R}^d)} + \|\tilde{h}_{k,n,s,v}(A_k^{n,s,v}\,\cdot)\|_{L_2(\mathbb{R}^d)} \lesssim 1,
\end{aligned}$$

which completes the proof of (4-24).

Now we turn to another estimate (4-25). Write

$$\Lambda_k T_j^{n,s,v} b_{n-j,s} = \check{\beta}_k * K_j^{n,s,v} * b_{n-j,s} = K_j^{n,s,v} * \check{\beta}_k * b_{n-j,s}.$$

Then by the estimate (4-8) of $K_j^{n,s,v}$, we get

$$\begin{aligned}
\|\Lambda_k T_j^{n,s,v} b_{n-j,s}\|_{L_1(\mathcal{A})} &\le \|K_j^{n,s,v}\|_{L_1(\mathbb{R}^d)} \|\check{\beta}_k * b_{n-j,s}\|_{L_1(\mathcal{A})} \\
&\lesssim 2^{-(n+s)(\gamma(d-1)-\iota)} \|\Omega\|_1 \|\check{\beta}_k * b_{n-j,s}\|_{L_1(\mathcal{A})}.
\end{aligned} \tag{4-26}$$

Note that $\beta_k(\xi) = \beta(2^k\xi)$; we get $\check{\beta}_k(x) = 2^{-kd}\check{\beta}(2^{-k}x)$. Therefore we see

$$\int_{\mathbb{R}^d} |\nabla[\check{\beta}_k](x)|\,dx = 2^{-k(d+1)} \int_{\mathbb{R}^d} |\nabla(\check{\beta})(2^{-k}x)|\,dx = 2^{-k} \int_{\mathbb{R}^d} |\nabla(\check{\beta})(x)|\,dx. \tag{4-27}$$

Using the cancellation property (ii) in Lemma 4.2, we see that, for all $Q \in \mathcal{Q}_{n-j+s}$, $\int_Q b_{n-j,s}(y)\,dy = 0$. Let y_Q be the center of Q. Notice that, for all $y \in Q$, $|y - y_Q| \lesssim 2^{j-n-s}$. Using this cancellation property, we then get

$$\begin{aligned}
\|\check{\beta}_k * b_{n-j,s}\|_{L_1(\mathcal{A})} &= \int_{\mathbb{R}^d} \tau\biggl(\biggl|\sum_{Q\in\mathcal{Q}_{n-j+s}} \int_Q [\check{\beta}_k(x-y) - \check{\beta}_k(x-y_Q)] b_{n-j,s}(y)\,dy\biggr|\biggr)\,dx \\
&\le \int_{\mathbb{R}^d} \sum_{Q\in\mathcal{Q}_{n-j+s}} \int_Q \biggl|\int_0^1 \langle y-y_Q, \nabla[\check{\beta}_k](x-\rho y-(1-\rho)y_Q)\rangle\,d\rho\biggr| \tau(|b_{n-j,s}(y)|)\,dy\,dx \\
&\lesssim 2^{j-n-s-k} \|b_{n-j,s}\|_{L_1(\mathcal{A})},
\end{aligned}$$

where in the second inequality we just use the mean value formula. Combining this inequality with (4-26) yields the estimate (4-25). Hence we finish the proof of this lemma. □

Now we conclude the proof of Lemma 3.5 as follows. Let $\varepsilon_0 \in (0,1)$ be a constant which will be chosen later. Notice that $\mathrm{card}(\Theta_{n+s}) \lesssim 2^{(n+s)\gamma(d-1)}$. Then by Lemma 4.3 with $N = [d/2]+1$, Lemma 4.4 and the property $\sum_j \|b_{n-j,s}\|_{L_1(\mathcal{A})} \lesssim \|f\|_{L_1(\mathcal{A})}$ in Lemma 4.2, we get

$$\begin{aligned}\sum_j \sum_{v\in\Theta_{n+s}} \|L_j^{n,s,v} b_{n-j,s}\|_{L_1(\mathcal{A})} &\le \sum_j \sum_{v\in\Theta_{n+s}} \Bigg(\sum_{k\le j-[(n+s)\varepsilon_0]} + \sum_{k\ge j-[(n+s)\varepsilon_0]} \Bigg) \|(I-G_{n+s,v})\Lambda_k T_j^{n,s,v} b_{n-j,s}\|_{L_1(\mathcal{A})} \\ &\lesssim \sum_j (2^{-(n+s)(\varepsilon_0-\gamma(3+2[d/2])-\iota)} + 2^{-(n+s)(1-\varepsilon_0-\iota)})\|b_{n-j,s}\|_{L_1(\mathcal{A})} \lesssim 2^{-(n+s)\alpha}\|f\|_{L_1(\mathcal{A})},\end{aligned}$$

where we choose the constants $0 < \iota \ll \gamma \ll \varepsilon_0 \ll 1$ such that the constant α is defined by

$$\alpha = \min\Big\{\varepsilon_0 - \gamma\Big(3+2\Big[\frac{d}{2}\Big]\Big) - \iota,\ 1-\varepsilon_0-\iota\Big\} > 0. \qquad \square$$

5. Proofs of lemmas related to the good functions

In this section, we begin to prove all lemmas for the good functions in Section 3C. The proofs for off-diagonal terms are similar to those for bad functions in Section 4, so we shall be brief and only indicate necessary changes in the proofs of off-diagonal terms. We first consider the proofs of diagonal terms.

5A. ***Proof of Lemma 3.6.*** Recall the definition of T. Let $\boldsymbol{K}_j$ be the kernel of the operator T_j, i.e., $\boldsymbol{K}_j(x) = \Omega(x)\phi_j(x)|x|^{-d}$. Notice that $\{\varepsilon_j\}_j$ is a Rademacher sequence on a probability space $(\mathfrak{m}, P)$; then applying the equality (3-7), we can write

$$\|Tf\|^2_{L_2(L_\infty(\mathfrak{m}\bar{\otimes}\mathcal{A}))} = \sum_{j\in\mathbb{Z}} \|T_j f\|^2_{L_2(\mathcal{A})} = (2\pi)^{-d/2}\int_{\mathbb{R}^d}\sum_{j\in\mathbb{Z}} |\widehat{\boldsymbol{K}}_j(\xi)|^2 \tau(|\hat{f}(\xi)|^2)\,d\xi,$$

where the second equality follows from Plancherel's theorem since $L_2(\mathcal{M})$ is a Hilbert space. In the following we show that

$$\sum_{j\in\mathbb{Z}} |\widehat{\boldsymbol{K}}_j(\xi)|^2 < \infty \tag{5-1}$$

holds for almost every $\xi \in \mathbb{R}^d$. Once we prove the inequality (5-1), Lemma 3.6 follows from Plancherel's theorem. Now we fix $\xi \neq 0$. By the cancellation property of Ω, $\int_{\boldsymbol{S}^{d-1}} \Omega(\theta)\,d\sigma(\theta) = 0$, we get

$$|\widehat{\boldsymbol{K}}_j(\xi)| = \Big|\int_{\mathbb{R}^d} K_j(x)(e^{-i\xi x}-1)\,dx\Big| \lesssim 2^j|\xi|\|\Omega\|_1.$$

Therefore the sum over all j's satisfying $2^j|\xi| \le 1$ is convergent.

Now we turn to the case $2^j|\xi| > 1$. We split the kernel $\Omega(\theta)$ into two parts:

$$\Omega_1(\theta) = \Omega(\theta)\chi_{\{\theta\in\boldsymbol{S}^{d-1}:|\Omega(\theta)|\le 2^{j\nu}|\xi|^\nu\|\Omega\|_1\}} \quad \text{and} \quad 1-\Omega_1(\theta)$$

for some constant $\nu \in \big(0, \frac{1}{2}\big)$. We first consider Ω_1. By making a change of variable $x = r\theta$, we get

$$|\widehat{\boldsymbol{K}}_j(\xi)| \le \int_{\boldsymbol{S}^{d-1}} |\Omega_1(\theta)| \Big|\int_{\mathbb{R}} e^{-ir\langle\theta,\xi\rangle}\phi_j(r)r^{-1}\,dr\Big|\,d\sigma(\theta). \tag{5-2}$$

It is easy to see that $\left|\int_{\mathbb{R}} e^{-ir\langle\theta,\xi\rangle}\phi_j(r)r^{-1}\,dr\right|$ is finite. By integrating by parts with the variable r, we get

$$\left|\int_{\mathbb{R}} e^{-ir\langle\theta,\xi\rangle}\phi_j(r)r^{-1}\,dr\right| = \left|\int_{\mathbb{R}} e^{-ir\langle\theta,\xi\rangle}\langle\theta,\xi\rangle^{-1}\partial_r[\phi_j(r)r^{-1}]\,dr\right| \lesssim (2^j|\xi|)^{-1}|\langle\theta,\xi'\rangle|^{-1},$$

where $\xi' = \xi/|\xi|$. Interpolating these two estimates we get, that, for any $\delta \in \left(\frac{1}{2}, 1\right)$,

$$\left|\int_{\mathbb{R}} e^{-ir\langle\theta,\xi\rangle}\phi_j(r)r^{-1}\,dr\right| \lesssim (2^j|\xi|)^{-\delta}|\langle\theta,\xi'\rangle|^{-\delta}.$$

Plugging the above estimate into (5-2) with the fact $\int_{\boldsymbol{S}^{d-1}}|\langle\theta,\xi'\rangle|^{-\delta}\,d\sigma(\theta) < \infty$, we hence get

$$|\widehat{\boldsymbol{K}}_j(\xi)| \lesssim (2^j|\xi|)^{-\delta+\nu}\|\Omega\|_1,$$

which is sufficient for us taking a sum over all j's satisfying $2^j|\xi| > 1$. Consider the other term $1-\Omega_1$. Then we get

$$\begin{aligned}
\sum_{j:2^j|\xi|>1} &|\widehat{\boldsymbol{K}}_j(\xi)|^2 \\
&\lesssim \sum_{j:2^j|\xi|>1}\left(\int_{\{\theta\in\boldsymbol{S}^{d-1}:|\Omega(\theta)|\geq(2^j|\xi|)^{\nu}\|\Omega\|_1\}}|\Omega(\theta)|\,d\sigma(\theta)\right)^2 \\
&= \int_{\boldsymbol{S}^{d-1}\times\boldsymbol{S}^{d-1}}\#\left\{j : 1 < 2^j|\xi| \leq \min\left\{\left(\frac{|\Omega(\theta)|}{\|\Omega\|_1}\right)^{1/\nu}, \left(\frac{|\Omega(\alpha)|}{\|\Omega\|_1}\right)^{1/\nu}\right\}\right\}|\Omega(\theta)||\Omega(\alpha)|\,d\sigma(\theta)\,d\sigma(\alpha) \\
&\lesssim \left(\int_{\boldsymbol{S}^{d-1}}|\Omega(\theta)|\left(1+\left[\log^+\frac{|\Omega(\theta)|}{\|\Omega\|_1}\right]^{1/2}\right)d\sigma(\theta)\right)^2 < \infty,
\end{aligned}$$

where the last inequality just follows from $\Omega \in L(\log^+ L)^{1/2}(\boldsymbol{S}^{d-1})$. Hence we complete the proof. □

5B. *Proof of Lemma 3.8.* Recall the definition of the kernel $K_{j,\iota}^{n,s}$ in (4-2). By Young's inequality, it is easy to see that

$$\|T_{j,\iota}^{n,s}g_{n-j,s}\|_{L_2(\mathcal{A})} \leq \|K_{j,\iota}^{n,s}\|_{L_1(\mathbb{R}^d)}\|g_{n-j,s}\|_{L_2(\mathcal{A})} \lesssim \int_{D^\iota}|\Omega(\theta)|\,d\sigma(\theta)\|g_{n-j,s}\|_{L_2(\mathcal{A})}.$$

Now applying $\sum_j \|g_{n-j,s}\|_{L_2(\mathcal{A})}^2 \lesssim \lambda\mathcal{C}_\Omega^{-1}\|f\|_{L_1(\mathcal{A})}$ in Lemma 3.7 and the above estimate, we get

$$\begin{aligned}
\sum_{s\geq1}\sum_{n\geq100}&\left(\sum_j\|T_{j,\iota}^{n,s}g_{n-j,s}\|_{L_2(\mathcal{A})}^2\right)^{1/2} \\
&\lesssim \sum_{s\geq1}\sum_{n\geq100}\int_{D^\iota}|\Omega(\theta)|\,d\sigma(\theta)(\lambda\mathcal{C}_\Omega^{-1}\|f\|_{L_1(\mathcal{A})})^{1/2} \\
&\lesssim \int_{\boldsymbol{S}^{d-1}}\#\{(s,n) : s\geq1, n\geq100, |\Omega(\theta)|\geq2^{(n+s)\iota}\|\Omega\|_1\}|\Omega(\theta)|\,d\sigma(\theta)(\lambda\mathcal{C}_\Omega^{-1}\|f\|_{L_1(\mathcal{A})})^{1/2} \\
&\lesssim (\mathcal{C}_\Omega\lambda\|f\|_{L_1(\mathcal{A})})^{1/2},
\end{aligned}$$

which is our desired estimate. Hence we complete the proof. □

5C. ***Proof of Lemma 3.9.*** By applying Plancherel's theorem, the convex inequality for the operator-valued function (4-1), the fact (4-3) and finally Plancherel's theorem again, we get

$$\begin{aligned}\Big\|\sum_{v\in\Theta_{n+s}} G_{n+s,v}T_j^{n,s,v}g_{n-j,s}\Big\|^2_{L_2(\mathcal{A})} &= (2\pi)^{-d/2}\int_{\mathbb{R}^d}\tau\bigg(\Big|\sum_{v\in\Theta_{n+s}}\Phi\Big(2^{(n+s)\gamma}\Big\langle e_v^{n+s},\frac{\xi}{|\xi|}\Big\rangle\Big)\mathcal{F}(T_j^{n,s,v}g_{n-j,s})(\xi)\Big|^2\bigg)\,d\xi\\ &\lesssim \int_{\mathbb{R}^d}\sum_{v\in\Theta_{n+s}}\Phi^2\Big(2^{(n+s)\gamma}\Big\langle e_v^{n+s},\frac{\xi}{|\xi|}\Big\rangle\Big)\sum_{v\in\Theta_{n+s}}\tau(|\mathcal{F}(T_j^{n,s,v}g_{n-j,s})(\xi)|^2)\,d\xi\\ &\lesssim 2^{(n+s)\gamma(d-2)}\sum_{v\in\Theta_{n+s}}\|T_j^{n,s,v}g_{n-j,s}\|^2_{L_2(\mathcal{A})}.\end{aligned}\tag{5-3}$$

Using Young's inequality and (4-8), we get that $\|T_j^{n,s,v}g_{n-j,s}\|^2_{L_2(\mathcal{A})}$ is bounded by

$$\|K_j^{n,s,v}\|^2_{L_1(\mathbb{R}^d)}\|g_{n-j,s}\|^2_{L_2(\mathcal{A})}\lesssim 2^{2(n+s)(-\gamma(d-1)+\iota)}\|\Omega\|_1^2\|g_{n-j,s}\|^2_{L_2(\mathcal{A})}.\tag{5-4}$$

Now plugging (5-4) into (5-3) and using the fact $\mathrm{card}(\Theta_{n+s})\lesssim 2^{(n+s)\gamma(d-1)}$, we get

$$\Big\|\sum_{v\in\Theta_{n+s}} G_{n+s,v}T_j^{n,s,v}g_{n-j,s}\Big\|^2_{L_2(\mathcal{A})}\lesssim 2^{(n+s)(-\gamma+2\iota)}\|\Omega\|_1^2\|g_{n-j,s}\|^2_{L_2(\mathcal{A})},$$

which is just our desired estimate. □

5D. ***Proof of Lemma 3.10.*** Using $I=\sum_k\Lambda_k$ and the triangle inequality, we get

$$\|(I-G_{n+s,v})T_j^{n,s,v}g_{n-j,s}\|_{L_2(\mathcal{A})}\le\sum_k\|(I-G_{n+s,v})\Lambda_kT_j^{n,s,v}g_{n-j,s}\|_{L_2(\mathcal{A})}.$$

Let $\varepsilon_0\in(0,1)$ be a constant which will be chosen later. Separating the above sum into two parts, we will prove that

$$\sum_{k\le j-[(n+s)\varepsilon_0]}\|(I-G_{n+s,v})\Lambda_kT_j^{n,s,v}g_{n-j,s}\|_{L_2(\mathcal{A})}\lesssim 2^{-(n+s)(\gamma(d-1)+\varepsilon_0-\gamma(3+2[d/2])-\iota)}\|\Omega\|_1\|g_{n-j,s}\|_{L_2(\mathcal{A})}\tag{5-5}$$

and

$$\sum_{k>j-[(n+s)\varepsilon_0]}\|(I-G_{n+s,v})\Lambda_kT_j^{n,s,v}g_{n-j,s}\|_{L_2(\mathcal{A})}\lesssim 2^{-(n+s)(\gamma(d-1)+1-\varepsilon_0-\iota)}\|\Omega\|_1\|g_{n-j,s}\|_{L_2(\mathcal{A})}.\tag{5-6}$$

Based on (5-5), (5-6) and the fact $\mathrm{card}(\Theta_{n+s})\lesssim 2^{(n+s)\gamma(d-1)}$, we finish the proof of this lemma by choosing the constants $0<\iota\ll\gamma\ll\varepsilon_0\ll 1$ such that the constant κ is defined by

$$\kappa=\min\Big\{\varepsilon_0-\gamma\Big(3+2\Big[\frac{d}{2}\Big]\Big)-\iota,\,1-\varepsilon_0-\iota\Big\}>0.$$

Now we give the proof of (5-5) and (5-6). Consider (5-5) first. Recall that $D_k^{n,s,v}(x)$ is defined as the kernel of the operator $(I-G_{n+s,v})\Lambda_kT_j^{n,s,v}$ in (4-17). Applying Young's inequality and the estimate of $D_k^{n,s,v}$ in (4-23), we get

$$\begin{aligned}\|(I-G_{n+s,v})\Lambda_k T_j^{n,s,v} g_{n-j,s}\|_{L_2(\mathcal{A})} &\le \|D_k^{n,s,v}\|_{L_1(\mathbb{R}^d)}\|g_{n-j,s}\|_{L_2(\mathcal{A})}\\ &\lesssim 2^{-(n+s)(\gamma(d-1)-\iota-(3+2[d/2])\gamma)-j+k}\|\Omega\|_1\|g_{n-j,s}\|_{L_2(\mathcal{A})}.\end{aligned}$$

Taking a sum over $k \le j-[(n+s)\varepsilon_0]$ yields (5-5).

Next we turn to the proof of (5-6). By Plancherel's theorem, we see that

$$\|(I-G_{n+s,v})\Lambda_k T_j^{n,s,v} g_{n-j,s}\|_{L_2(\mathcal{A})} \lesssim \|\Lambda_k T_j^{n,s,v} g_{n-j,s}\|_{L_2(\mathcal{A})}. \tag{5-7}$$

Write

$$\Lambda_k T_j^{n,s,v} g_{n-j,s} = \check{\beta}_k * K_j^{n,s,v} * g_{n-j,s} = K_j^{n,s,v} * \check{\beta}_k * g_{n-j,s}.$$

Then by Young's inequality and (4-8), we get

$$\begin{aligned}\|\Lambda_k T_j^{n,s,v} g_{n-j,s}\|_{L_2(\mathcal{A})} &\le \|K_j^{n,s,v}\|_{L_1(\mathbb{R}^d)}\|\check{\beta}_k * g_{n-j,s}\|_{L_2(\mathcal{A})}\\ &\lesssim 2^{-(n+s)(\gamma(d-1)-\iota)}\|\Omega\|_1\|\check{\beta}_k * g_{n-j,s}\|_{L_2(\mathcal{A})}.\end{aligned} \tag{5-8}$$

Recall the definition of $g_{n-j,s}$; we have the following cancellation property: for all $s \ge 1$ and $Q \in \mathcal{Q}_{n-j+s-1}$, we have $\int_Q g_{n-j,s}(y)\,dy = 0$. Let y_Q be the center of Q. Using this cancellation property, we get

$$\begin{aligned}\check{\beta}_k * g_{n-j,s}(x) &= \int_{\mathbb{R}^d} \sum_{Q\in\mathcal{Q}_{n-j+s-1}} [\check{\beta}_k(x-y)-\check{\beta}_k(x-y_Q)]\chi_Q(y) g_{n-j,s}(y)\,dy\\ &=: \int_{\mathbb{R}^d} K_k(x,y) g_{n-j,s}(y)\,dy,\end{aligned}$$

with $K_k(x,y)=\sum_{Q\in\mathcal{Q}_{n-j+s-1}}[\check{\beta}_k(x-y)-\check{\beta}_k(x-y_Q)]\chi_Q(y)$. Below we will apply Schur's lemma to give an estimate of $\|\check{\beta}_k * g_{n-j,s}\|_{L_2(\mathcal{A})}$. We first consider $K_k(x,y)$ as follows: For any y, there exists a unique cube $Q\in\mathcal{Q}_{n-j+s-1}$ such that $y\in Q$. Then by (4-27),

$$\int_{\mathbb{R}^d} |K_k(x,y)|\,dx \le \int_{\mathbb{R}^d} |y-y_Q| \int_0^1 |\nabla[\check{\beta}_k](x-\rho y-(1-\rho)y_Q)|\,d\rho\,dx \lesssim 2^{j-n-s-k}. \tag{5-9}$$

For any $x\in\mathbb{R}^d$, we have the estimate

$$\begin{aligned}\int_{\mathbb{R}^d} |K_k(x,y)|\,dy &\le \sum_{Q\in\mathcal{Q}_{n-j+s-1}} \int_Q |y-y_Q| \int_0^1 |\nabla[\check{\beta}_k](x-\rho y-(1-\rho)y_Q)|\,d\rho\,dy\\ &\lesssim 2^{j-n-s-k}\int_0^1 \sum_{Q\in\mathcal{Q}_{n-j+s-1}} 2^{-kd}\int_Q |\nabla[\check{\beta}](2^{-k}(x-\rho y-(1-\rho)y_Q))|\,dy\,d\rho\\ &\lesssim 2^{j-n-s-k}\end{aligned} \tag{5-10}$$

once we can show that the estimate below holds uniformly in x, ρ, k

$$\sum_{Q\in\mathcal{Q}_{n-j+s-1}} 2^{-kd}\int_Q |\nabla[\check{\beta}](2^{-k}(x-\rho y-(1-\rho)y_Q))|\,dy \lesssim 1. \tag{5-11}$$

In the following we prove (5-11). Making a change of variables $\tilde{y} = 2^{-k}y$, the integral now integrates over all cubes $Q \in \mathcal{Q}_{n-j+s-1+k}$ with $\tilde{y}_Q = 2^{-k}y_Q$ the center of this cube Q, which is rewritten as

$$\begin{aligned}\sum_{Q\in\mathcal{Q}_{n-j+s-1+k}} &\int_Q |\nabla[\check{\beta}](2^{-k}x - \rho\tilde{y} - (1-\rho)\tilde{y}_Q)|\,d\tilde{y}\\ &= \bigg(\sum_{\operatorname{dist}(Q,2^{-k}x)\le 2} + \sum_{l=1}^{\infty}\sum_{2^l<\operatorname{dist}(Q,2^{-k}x)\le 2^{l+1}}\bigg)\int_Q |\nabla[\check{\beta}](2^{-k}x - \rho\tilde{y} - (1-\rho)\tilde{y}_Q)|\,d\tilde{y}\\ &=: I + II,\end{aligned}$$

where in the second line we split the sum $\sum_{Q\in\mathcal{Q}_{n-j+s-1+k}}$ into two parts. Notice that the sidelength of $Q \in \mathcal{Q}_{n-j+s-1+k}$ is $2^{-n+j-s+1-k}$, which is less than 1 since we only consider the sum over $k > j-[(n+s)\varepsilon_0]$ and $0 < \varepsilon_0 \ll 1$. For I, note that the cubes belonging in $\mathcal{Q}_{n-j+s-1+k}$ are disjoint with interior; therefore the sum $\sum_{\operatorname{dist}(Q,2^{-k}x)\le 2}$ over these cubes is supported in $B(2^{-k}x, 2+\sqrt{d})$, a ball with center $2^{-k}x$ and radius $2+\sqrt{d}$. Thus we get

$$|I| \lesssim \sum_{\operatorname{dist}(Q,2^{-k}x)\le 2} |Q| \le |B(2^{-k}x, 2+\sqrt{d})| \le C.$$

Consider II. Since $\tilde{y}$ lies in a cube $Q \in \mathcal{Q}_{n-j+s-1+k}$ and $\tilde{y}_Q$ is the center of this cube, we get $\rho\tilde{y}+(1-\rho)\tilde{y}_Q$ lies in a line segment which is started at $\tilde{y}_Q$ and ended at $\tilde{y}$. So we have $\rho\tilde{y} + (1-\rho)\tilde{y}_Q \in Q$ for any $\rho \in [0, 1]$. Because of $2^l < \operatorname{dist}(Q, 2^{-k}x) \le 2^{l+1}$ and $l(Q) \le 1$, we get $|2^{-k}x - \rho\tilde{y} - (1-\rho)\tilde{y}_Q| \approx 2^l$. Combining the above estimates, we get

$$|II| \lesssim \sum_{l=1}^{\infty}\sum_{2^l<\operatorname{dist}(Q,2^{-k}x)\le 2^{l+1}} |Q|2^{-(d+1)l} \lesssim \sum_{l=1}^{\infty} 2^{-l} \le C,$$

where in the first inequality we also use the fact $\nabla[\check{\beta}]$ is a Schwartz function which decays fast away from the origin, while the second inequality follows from the fact that the sum over all cubes $2^l < \operatorname{dist}(Q, 2^{-k}x) \le 2^{l+1}$ is supported in a ball with center $2^{-k}x$ and approximate radius 2^l. Hence we finish the proof of (5-11).

Now utilizing Schur's lemma in Lemma 4.1 with (5-9) and (5-10), we get

$$\|\check{\beta}_k * g_{n-j,s}\|_{L_2(\mathcal{A})} \lesssim 2^{j-n-s-k}\|g_{n-j,s}\|_{L_2(\mathcal{A})}.$$

Plugging this inequality into (5-8) and later (5-7) , we get

$$\|(I - G_{n+s,v})\Lambda_k T_j^{n,s,v} g_{n-j,s}\|_{L_2(\mathcal{A})} \lesssim 2^{j-k-(n+s)(\gamma(d-1)+1-\iota)}\|\Omega\|_1\|g_{n-j,s}\|_{L_2(\mathcal{A})}.$$

Taking a sum of the above estimate over $k > j-[(n+s)\varepsilon_0]$ yields (5-6). Hence we complete the proof. □

Appendix: Strong (p, p) bound for $\{M_r\}_{r>0}$

Theorem A.1. *Suppose that Ω satisfies (1-3) and $\Omega \in L_1(S^{d-1})$. Then the operator sequence $\{M_r\}_{r>0}$ is of maximal strong type (p, p) for $1 < p \le \infty$, i.e.,*

$$\|\{M_r f\}_{r>0}\|_{L_p(\mathcal{A},\ell_\infty(0,\infty))} \lesssim \|\Omega\|_1\|f\|_{L_p(\mathcal{A})}.$$

Proof. By decomposing the functions Ω and f into four parts (i.e., real positive part, real negative part, imaginary positive part, imaginary negative part), together with triangle inequality for the norm $\|\cdot\|_{L_p(\mathcal{A},\ell_\infty(0,\infty))}$, we only consider the case that Ω is a positive function and f is positive in $\mathcal{A}$. Then by (1-5), it is enough to show that for any $f \in L_p^+(\mathcal{A})$ there exists a positive function $F \in L_p^+(\mathcal{A})$ such that

$$M_r f \le F \quad \text{for all } r > 0 \qquad \text{and} \qquad \|F\|_{L_p(\mathcal{A})} \lesssim \|\Omega\|_1 \|f\|_{L_p(\mathcal{A})}. \tag{A-1}$$

We will use the method of rotation. Let $f \in L_p^+(\mathcal{A})$, by making a change of variables $x - y = r\theta$, we get

$$\begin{aligned} M_r f(x) &= \frac{1}{|B(x,r)|} \int_{B(x,r)} \Omega(x-y) f(y)\,dy \\ &= \frac{1}{v_n} \int_{\boldsymbol{S}^{d-1}} \Omega(\theta) \frac{1}{r^d} \int_0^r f(x - s\theta) s^{d-1}\,ds\,d\sigma(\theta) \\ &\lesssim \int_{\boldsymbol{S}^{d-1}} \Omega(\theta) \biggl(\frac{1}{r} \int_0^r f(x-s\theta)\,ds \biggr) d\sigma(\theta). \end{aligned}$$

For a fixed $\theta \in \boldsymbol{S}^{d-1}$, we define the directional Hardy–Littlewood average operator as

$$\mathfrak{M}_r^\theta f(x) = \frac{1}{r} \int_0^r f(x - s\theta)\,ds.$$

We will prove at the end of this section the result

$$\|\{\mathfrak{M}_r^\theta f\}_{r>0}\|_{L_p(\mathcal{A},\ell_\infty(0,\infty))} \lesssim \|f\|_{L_p(\mathcal{A})}. \tag{A-2}$$

Assuming (A-2) and using (1-5), there exists a positive function $F_\theta \in L_p^+(\mathcal{A})$ such that

$$\mathfrak{M}_r^\theta f \le F_\theta \quad \text{for all } r > 0 \qquad \text{and} \qquad \|F_\theta\|_{L_p(\mathcal{A})} \lesssim \|f\|_{L_p(\mathcal{A})}.$$

Now if set $F(x) = \int_{\boldsymbol{S}^{d-1}} \Omega(\theta) F_\theta(x)\,d\sigma(\theta)$, then $M_r f(x) \lesssim F(x)$ and

$$\|F\|_{L_p(\mathcal{A})} \lesssim \int_{\boldsymbol{S}^{d-1}} \Omega(\theta) \|F_\theta\|_{L_p(\mathcal{A})}\,d\sigma(\theta) \lesssim \|\Omega\|_1 \|f\|_{L_p(\mathcal{A})}.$$

Thus F is the desired function satisfying (A-1).

It remains to show (A-2). Let $e_1 = (1, 0, \dots, 0)$ be the unit vector. Now, for any orthogonal matrix A, we have

$$\mathfrak{M}_r^{A(e_1)} f(x) = \mathfrak{M}_r^{e_1} (f \circ A)(A^{-1}x), \tag{A-3}$$

which implies that the L_p boundedness of $\{\mathfrak{M}_r^\theta\}_{r>0}$ can be reduced to that of $\{\mathfrak{M}_r^{e_1}\}_{r>0}$. Let $f \in L_p(L_\infty(\mathbb{R}^d) \overline{\otimes} \mathcal{M})$. Without loss of generality, we may assume that f is positive. Fixing $x_2, \dots, x_d \in \mathbb{R}$, we consider $f(\cdot, x_2, \dots, x_d)$ as a function in $L_p(L_\infty(\mathbb{R}) \overline{\otimes} \mathcal{M})_+$. By the strong-type (p, p) boundedness of noncommutative Hardy–Littlewood maximal operator (see [Mei 2007]), we know that, for $1 < p \le \infty$,

$$\|\{\mathfrak{M}_r^{e_1} f(\cdot, x_2, \dots, x_d)\}_{r>0}\|_{L_p(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M},\ell_\infty(0,\infty))} \lesssim \|f(\cdot, x_2, \dots, x_d)\|_{L_p(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})}.$$

By (1-5), there exists a positive function $F(\cdot, x_2, \ldots, x_d) \in L_p(L_\infty(\mathbb{R}) \overline{\otimes} \mathcal{M})$ such that, for any $r > 0$, $\mathfrak{M}_r^{e_1} f(x) \le F(x)$ and

$$\|F(\cdot, x_2, \ldots, x_d)\|_{L_p(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})} \lesssim \|f(\cdot, x_2, \ldots, x_d)\|_{L_p(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})}.$$

Then it is easy to see that

$$\|F\|_{L_p(L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M})} \lesssim \|f\|_{L_p(L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M})}.$$

Therefore, we conclude that $\{\mathfrak{M}_r^{e_1}\}_{r>0}$ is of strong-type (p, p). □

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References

[Cadilhac 2018] L. Cadilhac, "Weak boundedness of Calderón–Zygmund operators on noncommutative L_1-spaces", *J. Funct. Anal.* **274**:3 (2018), 769–796. MR Zbl

[Cadilhac 2019] L. Cadilhac, "Noncommutative Khintchine inequalities in interpolation spaces of L_p-spaces", *Adv. Math.* **352** (2019), 265–296. MR Zbl

[Cadilhac et al. 2022] L. Cadilhac, J. M. Conde-Alonso, and J. Parcet, "Spectral multipliers in group algebras and noncommutative Calderón–Zygmund theory", *J. Math. Pures Appl.* (9) **163** (2022), 450–472. MR Zbl

[Calderón and Zygmund 1952] A. P. Calderón and A. Zygmund, "On the existence of certain singular integrals", *Acta Math.* **88** (1952), 85–139. MR Zbl

[Calderón and Zygmund 1956] A. P. Calderón and A. Zygmund, "On singular integrals", *Amer. J. Math.* **78** (1956), 289–309. MR Zbl

[Caspers et al. 2018] M. Caspers, F. Sukochev, and D. Zanin, "Weak type operator Lipschitz and commutator estimates for commuting tuples", *Ann. Inst. Fourier (Grenoble)* **68**:4 (2018), 1643–1669. MR Zbl

[Caspers et al. 2019] M. Caspers, D. Potapov, F. Sukochev, and D. Zanin, "Weak type commutator and Lipschitz estimates: resolution of the Nazarov–Peller conjecture", *Amer. J. Math.* **141**:3 (2019), 593–610. MR Zbl

[Chen et al. 2013] Z. Chen, Q. Xu, and Z. Yin, "Harmonic analysis on quantum tori", *Comm. Math. Phys.* **322**:3 (2013), 755–805. MR Zbl

[Christ 1988] M. Christ, "Weak type $(1, 1)$ bounds for rough operators", *Ann. of Math.* (2) **128**:1 (1988), 19–42. MR Zbl

[Christ and Rubio de Francia 1988] M. Christ and J. L. Rubio de Francia, "Weak type $(1, 1)$ bounds for rough operators, II", *Invent. Math.* **93**:1 (1988), 225–237. MR Zbl

[Grafakos 2014a] L. Grafakos, *Classical Fourier analysis*, 3rd ed., Grad. Texts in Math. **249**, Springer, 2014. MR Zbl

[Grafakos 2014b] L. Grafakos, *Modern Fourier analysis*, 3rd ed., Grad. Texts in Math. **250**, Springer, 2014. MR Zbl

[Grafakos and Stefanov 1999] L. Grafakos and A. Stefanov, "Convolution Calderón–Zygmund singular integral operators with rough kernels", pp. 119–143 in *Analysis of divergence* (Orono, ME, 1997), edited by W. O. Bray, Birkhäuser, Boston, 1999. MR Zbl

[Grafakos et al. 2017] L. Grafakos, D. Oliveira e Silva, M. Pramanik, A. Seeger, and B. Stovall, "Some problems in harmonic analysis", preprint, 2017. arXiv 1701.06637

[Hofmann 1988] S. Hofmann, "Weak $(1, 1)$ boundedness of singular integrals with nonsmooth kernel", *Proc. Amer. Math. Soc.* **103**:1 (1988), 260–264. MR Zbl

[Hong and Xu 2021] G. Hong and B. Xu, "A noncommutative weak type $(1, 1)$ estimate for a square function from ergodic theory", *J. Funct. Anal.* **280**:9 (2021), art. id. 108959. MR Zbl

[Hong et al. 2023] G. Hong, X. Lai, and B. Xu, "Maximal singular integral operators acting on noncommutative L_p-spaces", *Math. Ann.* **386**:1-2 (2023), 375–414. MR Zbl

[Hörmander 1960] L. Hörmander, "Estimates for translation invariant operators in L^p spaces", *Acta Math.* **104** (1960), 93–140. MR Zbl

[Junge 2002] M. Junge, "Doob's inequality for non-commutative martingales", *J. Reine Angew. Math.* **549** (2002), 149–190. MR Zbl

[Mei 2007] T. Mei, *Operator valued Hardy spaces*, Mem. Amer. Math. Soc. **881**, Amer. Math. Soc., Providence, RI, 2007. MR Zbl

[Mei and Parcet 2009] T. Mei and J. Parcet, "Pseudo-localization of singular integrals and noncommutative Littlewood–Paley inequalities", *Int. Math. Res. Not.* **2009**:8 (2009), 1433–1487. MR Zbl

[Parcet 2009] J. Parcet, "Pseudo-localization of singular integrals and noncommutative Calderón–Zygmund theory", *J. Funct. Anal.* **256**:2 (2009), 509–593. MR Zbl

[Pisier 1998] G. Pisier, *Non-commutative vector valued L_p-spaces and completely p-summing maps*, Astérisque **247**, Math. Soc. France, Paris, 1998. MR Zbl

[Pisier and Xu 2003] G. Pisier and Q. Xu, "Non-commutative L^p-spaces", pp. 1459–1517 in *Handbook of the geometry of Banach spaces*, *II*, edited by W. B. Johnson and J. Lindenstrauss, North-Holland, Amsterdam, 2003. MR Zbl

[Seeger 1996] A. Seeger, "Singular integral operators with rough convolution kernels", *J. Amer. Math. Soc.* **9**:1 (1996), 95–105. MR Zbl

[Stein 1993] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Math. Series **43**, Princeton Univ. Press, 1993. MR Zbl

[Stein 1998] E. M. Stein, "Singular integrals: the roles of Calderón and Zygmund", *Not. Amer. Math. Soc.* **45**:9 (1998), 1130–1140. MR Zbl

[Tao 1999] T. Tao, "The weak-type (1, 1) of $L \log L$ homogeneous convolution operator", *Indiana Univ. Math. J.* **48**:4 (1999), 1547–1584. MR Zbl

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