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STRUCTURE OF SETS WITH NEARLY MAXIMAL FAVARD LENGTH

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Let $E \subset B(1) \subset \mathbb{R}^2$ be an $\mathcal{H}^1$ measurable set with $\mathcal{H}^1(E) < \infty$, and let $L \subset \mathbb{R}^2$ be a line segment with $\mathcal{H}^1(L) = \mathcal{H}^1(E)$. It is not hard to see that $\mathrm{Fav}(E) \le \mathrm{Fav}(L)$. We prove that in the case of near equality, that is,

$$\mathrm{Fav}(E) \ge \mathrm{Fav}(L) - \delta,$$

the set E can be covered by an ϵ-Lipschitz graph, up to a set of length ϵ. The dependence between ϵ and δ is polynomial: in fact, the conclusions hold with $\epsilon = C\delta^{1/70}$ for an absolute constant $C > 0$.

1. Introduction

Let $E \subset \mathbb{R}^2$ be $\mathcal{H}^1$ measurable with $\mathcal{H}^1(E) < \infty$. We recall the definition of Favard length:

$$\mathrm{Fav}(E) = \int_0^{\pi} \mathcal{H}^1(\pi_\theta(E))\, d\theta.$$

Here $\pi_\theta : \mathbb{R}^2 \to \mathbb{R}$ is the orthogonal projection $\pi_\theta(x) = x \cdot (\cos\theta, \sin\theta)$. The definition of $\mathrm{Fav}(E)$ can be posed without the assumption $\mathcal{H}^1(E) < \infty$, but this hypothesis will be crucial for most of the statements below, and it will be assumed unless otherwise stated. A fundamental result in geometric measure theory is the Besicovitch projection theorem [1939] which relates Favard length and rectifiability: $\mathrm{Fav}(E) > 0$ if and only if $\mathcal{H}^1(E \cap \Gamma) > 0$ for some Lipschitz graph $\Gamma \subset \mathbb{R}^2$ — in other words, E is not purely 1-unrectifiable.

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The proof of the Besicovitch projection theorem is famous for being difficult to quantify, partly because of its reliance on the Lebesgue differentiation theorem: it is hard to decipher from the argument just how large the intersection $E \cap \Gamma$ is, and what the Lipschitz constant of Γ is. In fact, it is nontrivial to even find the right question: for example, if $E \subset B(1)$, $\mathcal{H}^1(E) = 1$, and $\mathrm{Fav}(E) \geq \delta$ for some small but fixed constant $\delta > 0$, then it is not true that $\mathcal{H}^1(E \cap \Gamma) \geq \epsilon$ for some ϵ^{-1}-Lipschitz graph $\Gamma \subset \mathbb{R}^2$, where $\epsilon = \epsilon(\delta) > 0$. We construct a relevant counterexample in Section 6.

In Theorem 1.1, we show that similar counterexamples are no longer possible if the assumption "$\mathrm{Fav}(E) \geq \delta$" is upgraded to "$\mathrm{Fav}(E) \geq 2\,\mathcal{H}^1(E) - \delta$" for a sufficiently small constant $\delta > 0$. The number 2 comes from the fact that $\mathrm{Fav}([0, 1] \times \{0\}) = 2$ and that $[0, 1] \times \{1\}$ has the maximal Favard length among sets of length unity (see (2.4)).

Theorem 1.1. *For every $\epsilon > 0$ there exists $\delta > 0$ such that the following holds: Let $E \subset B(1)$ be an $\mathcal{H}^1$ measurable set with $\mathcal{H}^1(E) < \infty$, and assume that*

$$\mathrm{Fav}(E) \geq \mathrm{Fav}(L) - \delta, \tag{1.2}$$

where $L \subset \mathbb{R}^2$ is a line segment with $\mathcal{H}^1(L) = \mathcal{H}^1(E)$. Then, there exists an ϵ-Lipschitz graph $\Gamma \subset \mathbb{R}^2$ such that $\mathcal{H}^1(E \cap \Gamma) \geq \mathcal{H}^1(E) - \epsilon$. One can take $\delta = \epsilon^{70}/C$ for an absolute constant $C > 1$.

By an *ϵ-Lipschitz graph* we mean a set of the form $R(\mathrm{Graph}_f)$, where $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation, and $\mathrm{Graph}_f = \{(x, f(x)) : x \in \mathbb{R}\}$ is the graph of an ϵ-Lipschitz function $f\colon \mathbb{R} \to \mathbb{R}$. This means that

$$|f(x) - f(y)| \leq \epsilon |x - y|$$

for all $x, y \in \mathbb{R}$. It is easy to check that the intersection of an ϵ-Lipschitz graph with $B(1)$ is contained in the 2ϵ-neighborhood of some line $\ell \subset \mathbb{R}^2$, so in particular the same is true of $E \cap \Gamma$ (as in Theorem 1.1).

Theorem 1.1 shows that if $\mathrm{Fav}(E)$ is nearly maximal, the Besicovitch projection theorem can be quantified in a very strong way, whereas the example constructed in Section 6 shows that any similar conclusion fails completely if we make the weaker assumption $\mathrm{Fav}(E) \geq \delta$. However, it remains plausible that the assumption $\mathrm{Fav}(E) \geq \delta$ is sufficient to guarantee a quantitative version of Besicovitch's theorem under the additional assumption that E is 1-Ahlfors regular, or satisfies other *multiscale* 1*-dimensionality* hypotheses. For recent partial results, and more discussion on this question; see [Davey and Taylor 2022; Martikainen and Orponen 2018; Orponen 2021; Tao 2009]. The problem is closely related to Vitushkin's conjecture [1967] on the connection between analytic capacity and Favard length; see [Chang and Tolsa 2020; Dąbrowski and Villa 2022].

We briefly mention another closely related topic: if $E \subset \mathbb{R}^2$ is self-similar and purely 1-unrectifiable, then $\mathrm{Fav}(E) = 0$ by the Besicovitch projection theorem. It is an interesting and very popular question to attempt quantifying the (sharp) rate of decay at which $\mathrm{Fav}(E_n) \to 0$, where E_n is the *n-th iteration* of the self-similar set. For recent developments; see [Bateman and Volberg 2010; Bond et al. 2014; Bond and Volberg 2010; 2012; Bongers and Taylor 2023; Cladek et al. 2022; Łaba and Zhai 2010; Łaba 2015; Łaba and Marshall 2022; Nazarov et al. 2010; Peres and Solomyak 2002].

It is tempting to consider the following scale-invariant version of Theorem 1.1: *for any* $\epsilon_1, \epsilon_2 > 0$ *there exists* $\delta > 0$ *such that if* $E \subset B(1)$ *satisfies* $\mathcal{H}^1(E) < \infty$ *and*

$$\mathrm{Fav}(E) \geq (1-\delta)\,\mathrm{Fav}(L),$$

then there exists an ϵ_1*-Lipschitz graph* $\Gamma \subset \mathbb{R}^2$ *such that* $\mathcal{H}^1(E \setminus \Gamma) \leq \epsilon_2 \mathcal{H}^1(E)$. Note that for sets E with $\mathcal{H}^1(E) \sim 1$ this statement is equivalent to Theorem 1.1; however, in general, the statement is false. Consider a set E_n consisting of four horizontal segments of length $1/n$ placed in the corners of $[0,1]^2$. Clearly, one may cover at most half of E_n using a single 1-Lipschitz graph. At the same time, $\mathrm{Fav}(E_n)/\mathrm{Fav}(L_n) \to 1$, where $L_n = [0, 4/n] \times \{0\}$. To see this, let $\mathcal{B}_n := \{\theta \in [0,\pi) : \pi_\theta \text{ is not injective on } E_n\}$. Note that $\mathcal{H}^1(\mathcal{B}_n) \to 0$, and at the same time for $\theta \notin \mathcal{B}_n$ we have $\mathcal{H}^1(\pi_\theta(E_n)) = \mathcal{H}^1(\pi_\theta(L_n))$. It follows easily that $\mathrm{Fav}(E_n)/\mathrm{Fav}(L_n) \to 1$.

1A. ***Outline of the paper.*** A quick outline of the article is as follows: In Section 2 we introduce Crofton's formula and prove that line segments maximize Favard length. In Section 3 we prove Theorem 1.1 using two main propositions, Proposition 3.3 and Proposition 3.11. The moral of these propositions is discussed at the beginning of Section 3. These two propositions are then proven in Section 4 and Section 5, respectively. Section 6 contains the counterexample mentioned above to the scale-invariant version of Theorem 1.1. Finally, in the Appendix we give an exact formula for the measure of lines spanned by two rectifiable curves — this is used in Section 5 but it might be of independent interest.

2. Measure-theoretic preliminaries

2A. ***Notation.*** For $x \in \mathbb{R}^d$ and $r > 0$, the notation $B(x, r)$ stands for a closed ball of radius r centered at x. For $A \subset \mathbb{R}^d$, we denote the cardinality of A by $\#A$, and we write $A(r) := \{x \in \mathbb{R}^d : \mathrm{dist}(x, A) \leq r\}$, where "dist" is Euclidean distance. For $f, g \geq 0$, we write $f \lesssim g$ if there exists an absolute constant $C > 0$ such that $f \leq Cg$. The notation $f \gtrsim g$ means the same as $g \lesssim f$, and $f \sim g$ is shorthand for $f \lesssim g \lesssim f$. If the constant $C > 0$ is allowed to depend on some parameter p, we signify this by writing $f \lesssim_p g$.

2B. ***Integralgeometry and Crofton's formula.*** One of the main tools is Crofton's formula for rectifiable sets, which states the following: if $E \subset \mathbb{R}^2$ is an $\mathcal{H}^1$ measurable 1-rectifiable set with $\mathcal{H}^1(E) < \infty$, then

$$\mathcal{H}^1(E) = \frac{1}{2}\int_0^\pi \int_{\mathbb{R}} \#(E \cap \pi_\theta^{-1}\{t\})\,dt\,d\theta. \tag{2.1}$$

Equation (2.1) is false without the rectifiability assumption, but the inequality "$\geq$" remains valid in this case. This formula (and the inequality) is a special case of a more general relation between Hausdorff measure and integralgeometric measure for n-rectifiable sets in $\mathbb{R}^d$; see [Federer 1947, Theorem 9.7; 1969, Theorem 3.2.26]. We next rephrase the formula (2.1) in slightly more abstract terms. We define the following measure η on the family $\mathcal{A} := \mathcal{A}(2,1)$ of all affine lines in $\mathbb{R}^2$:

$$\eta(\mathcal{L}) = \int_0^\pi \mathcal{H}^1(\{t \in \mathbb{R} : \pi_\theta^{-1}\{t\} \in \mathcal{L}\})\,d\theta, \quad \mathcal{L} \subset \mathcal{A}.$$

With this notation, the Crofton formula (2.1) can be rewritten as

$$\mathcal{H}^1(E) = \frac{1}{2}\int_{\mathcal{L}(E)} \#(E\cap \ell)\, d\eta(\ell), \tag{2.2}$$

where

$$\mathcal{L}(E) := \{\ell \in \mathcal{A} : E\cap \ell \neq \varnothing\}.$$

Lemma 2.3 (the line segment maximizes Favard length). *If $E \subset \mathbb{R}^2$ is $\mathcal{H}^1$ measurable, $\mathcal{H}^1(E) < \infty$, and $L \subset \mathbb{R}^2$ is a line segment with $\mathcal{H}^1(E) = \mathcal{H}^1(L)$, then*

$$\mathrm{Fav}(E) \leq \mathrm{Fav}(L) \tag{2.4}$$

and

$$\mathrm{Fav}(L) - \mathrm{Fav}(E) \geq \int_{\mathcal{L}(E)} (\#(E\cap\ell) - 1)\, d\eta(\ell). \tag{2.5}$$

If E is rectifiable, then equality holds in (2.5).

Proof. Suppose $E \subset \mathbb{R}^2$ is $\mathcal{H}^1$ measurable, $\mathcal{H}^1(E) < \infty$, and $L \subset \mathbb{R}^2$ is a line segment with $\mathcal{H}^1(E) = \mathcal{H}^1(L)$. Then

$$\mathrm{Fav}(E) = \eta(\mathcal{L}(E)) = \int_{\mathcal{L}(E)} 1\, d\eta(\ell) \leq \int_{\mathcal{L}(E)} \#(E\cap\ell)\, d\eta(\ell) \leq 2\mathcal{H}^1(E). \tag{2.6}$$

If we replace E with the line segment L, then equality holds in both inequalities above. Thus, $\mathrm{Fav}(L) = 2\mathcal{H}^1(L) = 2\mathcal{H}^1(E)$, which combined with (2.6) (for E) proves (2.5).

Next, (2.4) follows from the fact that the right-hand side of (2.5) is nonnegative. Finally, if E is rectifiable, then the second inequality in (2.6) becomes an equality, which implies that equality holds in (2.5). □

2C. *Coarea formula.* We now record another tool in the proof of Theorem 1.1. It is closely related to Crofton's formula, but only considers the intersections with lines in a fixed direction. The price to pay is that the tangent of the rectifiable set enters the formula. It is a generalization of the following standard fact: if $f : [a,b] \to \mathbb{R}$ is α-Lipschitz, then

$$\mathcal{H}^1\big(\{(t, f(t)) : t\in[a,b]\}\big) = \int_a^b \sqrt{1+f'(t)^2}\, dt \leq \sqrt{1+\alpha^2}\,(b-a).$$

Lemma 2.7 (coarea formula). *Let $\alpha > 0$. Let $E \subset \mathbb{R}^2$ be a countable union of α-Lipschitz graphs over the x-axis. Then,*

$$\mathcal{H}^1(A) \leq \sqrt{1+\alpha^2}\int_{\mathbb{R}} \#(A\cap \pi_0^{-1}\{t\})\, dt \tag{2.8}$$

for all $\mathcal{H}^1$ measurable subsets $A \subset E$. (Recall that $\pi_0 : \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the x-axis.)

Proof. This follows from the coarea formula for rectifiable sets. (See, e.g., [Federer 1969, Theorem 3.2.22] or [Krantz and Parks 2008, Theorem 5.4.9].) □

3. Proof of Theorem 1.1 in two main steps

In this section we prove Theorem 1.1 using Propositions 3.3 and 3.11 introduced below. Proposition 3.3 says roughly the following: Assume a priori that E is a union of line segments (we reduce matters to something like this in Section 3A), fix a small *angle* $\alpha > 0$, and let $E_{\ell,\alpha}$ be the union of those segments which make an angle $\leq \alpha$ with some given line $\ell \subset \mathbb{R}^2$. Evidently E can be expressed as the union of $\sim 1/\alpha$ sets of the form $E_{\ell,\alpha}$. Proposition 3.3 says that if the parameter δ in our hypothesis $\mathrm{Fav}(E) \geq \mathrm{Fav}(L) - \delta$ is sufficiently small, then each of the sets $E_{\ell,\alpha}$ can be (almost) covered by a single ($\sim\alpha$)-Lipschitz graph over ℓ. After this step, we know that E can be (almost) covered by a union of $\sim 1/\alpha$ Lipschitz graphs with constant $\sim\alpha$. Thereafter, to complete the proof of Theorem 1.1, it remains to show that only one of these graphs can have a nontrivial intersection with E. This uses the hypothesis $\mathrm{Fav}(E) \geq \mathrm{Fav}(L) - \delta$ once more, and is accomplished in Proposition 3.11 (and the discussion right below).

3A. ***Step 1: first reductions.*** Let $E \subset \mathbb{R}^2$ be a Borel set with $\mathcal{H}^1(E) < \infty$. We start with the following simple lemma:

Lemma 3.1. *It suffices to prove Theorem 1.1 under the additional assumption that E is a finite union of disjoint C^1 curves.*

Proof. We may assume that $E \subset B(1)$ is rectifiable, because by the Besicovitch projection theorem, the rectifiable part of E continues to satisfy all the assumptions of Theorem 1.1 (with the same constant $\delta > 0$). By this assumption, $\mathcal{H}^1$ almost all of E can be covered by a countable union of C^1-curves. Decomposing the curves further, we may assume that they are disjoint, and for any given $\eta > 0$ we may write

$$E = \bigcup_{j=1}^{M_1} (\gamma_j \cap E) \cup S,$$

where $\mathcal{H}^1(S) \leq \eta$, and $\mathcal{H}^1(E \cap \gamma_j) \geq (1-\eta)\mathcal{H}^1(\gamma_j)$. Now, the set $\bar{E} := \bigcup_{j=1}^{M_1} \gamma_j$ satisfies

$$\mathcal{H}^1(\bar{E}) \leq (1-\eta)^{-1}\mathcal{H}^1(E) \quad \text{and} \quad \mathrm{Fav}(\bar{E}) \geq \mathrm{Fav}(E) - \eta$$

and is additionally a finite union of disjoint C^1-curves. If Theorem 1.1 is already known under this additional assumption, we may now infer that $\mathcal{H}^1(\bar{E} \setminus \Gamma) \leq \epsilon$, where Γ is an ϵ-Lipschitz graph. But then also $\mathcal{H}^1(E \setminus \Gamma) \leq \mathcal{H}^1(E \setminus \bar{E}) + \mathcal{H}^1(\bar{E} \setminus \Gamma) \leq \eta + \epsilon$, and Theorem 1.1 follows for E by choosing the parameters ϵ, η appropriately. □

3B. ***Step 2: minigraphs and how to merge them.*** By Lemma 3.1, we may assume that E is a finite union of disjoint C^1-curves $\gamma_1, \dots, \gamma_{M_1}$. We further chop up each curve γ_j into connected pieces whose tangent varies by less than α, where α is a small constant depending on ϵ fixed later on (see (3.5)). At this point, we have managed to write E as a finite union of disjoint α-Lipschitz graphs $\gamma_1, \dots, \gamma_{M_1'}$, where $M_1 \leq M_1' < +\infty$. At this point we have no quantitative control on the constant M_1'. Each of the graphs γ_j will be called a *minigraph*, and their collection is denoted $\mathcal{E}$. The main tasks in Theorem 1.1 are to combine the minigraphs into roughly $1/\alpha$ bigger graphs, and to show that nearly all of E lies on just one of these bigger graphs.

To begin with, let $M_2 = \lceil \pi\alpha^{-1} \rceil \sim \alpha^{-1}$. We would like to divide the collection of minigraphs $\mathcal{E}$ into M_2 subcollections $\mathcal{E}_1, \dots \mathcal{E}_{M_2}$, each of them containing the minigraphs with roughly the same direction. To do this, we consider M_2 vectors of the form

$$v_k := (\cos(k\pi/M_2), \sin(k\pi/M_2)) \quad \text{for } 1 \le k \le M_2 \sim \alpha^{-1}.$$

Observe that for each minigraph $\gamma \in \mathcal{E}$ there exists $k \in \{1, \dots, M_2\}$ such that γ is a 2α-Lipschitz graph over the line $\mathrm{span}(v_k)$. The vector v_k will be called the *direction* of the minigraph (if there are several suitable vectors for one minigraph, fix any one of them; we will only need to know that each minigraph is a 2α-Lipschitz graph over the line spanned by its direction). Statements about the (relative) angles of minigraphs should always be interpreted as statements about the relative angles of the direction vectors v_k.

For $k \in \{1, \dots, M_2\}$ fixed, we define $\mathcal{E}_k \subset \mathcal{E}$ as the collection of minigraphs with direction v_k. We suggest that the reader visualize the minigraphs as line segments I with $\angle(I, \mathrm{span}(v_k)) \le \alpha$. It seems likely that Theorem 1.1 could be reduced to the case where E is a finite union of line segments, but employing the minigraphs seems to spare us some unnecessary steps.

We write $E_k := \bigcup \mathcal{E}_k$. Thus

$$E = E_1 \cup \dots \cup E_{M_2}. \tag{3.2}$$

It turns out that, except for a small error, each set E_k is covered by a single Lipschitz graph with constant $\sim\alpha$ over $\mathrm{span}(v_k)$. Indeed, note that Lemma 2.3 and (1.2) together imply $\int_{\mathcal{L}(E)} \#(E \cap \ell) - 1 \, d\eta(\ell) \le \delta$. Since for each $k \in \{1, \dots, M_2\}$ we have $E_k \subset E$, one sees immediately that $\mathcal{L}(E_k) \subset \mathcal{L}(E)$ and $\#(E_k \cap \ell) \le \#(E \cap \ell)$, so that we also get $\int_{\mathcal{L}(E_k)} \#(E_k \cap \ell) - 1 \, d\eta(\ell) \le \delta$. Then, the desired Lipschitz graph Γ covering most of E_k is constructed in the following proposition, whose proof will be carried out in Section 4:

Proposition 3.3. *There exist absolute constants $C_0, \alpha_0 \in (0, 1)$ and $\boldsymbol{C}_{\mathrm{lip}} > 1$ such that the following holds: Let $\delta, \epsilon \in (0, 1)$ and $\alpha \in (0, \alpha_0)$ be such that $\delta \le C_0 \alpha^3 \epsilon^2$. Let $E \subset B(1)$ be a set with $\mathcal{H}^1(E) < \infty$ of the form*

$$E = \bigcup_{\gamma \in \mathcal{E}} \gamma,$$

where $\mathcal{E}$ is a finite collection of disjoint α-Lipschitz graphs over a fixed line $L \subset \mathbb{R}^2$. Assume further that E satisfies

$$\int_{\mathcal{L}(E)} (\#(E \cap \ell) - 1) \, d\eta(\ell) \le \delta. \tag{3.4}$$

Then, there exists a Lipschitz graph Γ over L, with Lipschitz constant at most $\boldsymbol{C}_{\mathrm{lip}} \cdot \alpha$, such that

$$\mathcal{H}^1(E \setminus \Gamma) \le \epsilon.$$

We remark that the absolute constants α_0 and $\boldsymbol{C}_{\mathrm{lip}}$ are such that $\alpha_0 \le \boldsymbol{C}_{\mathrm{lip}}^{-1}$. In particular, the $\boldsymbol{C}_{\mathrm{lip}}\alpha$-Lipschitz graph Γ from above has a Lipschitz constant bounded by 1.

The proof of Proposition 3.3 recycles most of the ideas from Besicovitch's original proof of the Besicovitch projection theorem [1939]. Indeed, we first use (in Lemma 4.1) the assumption (3.4) to show that E must have arbitrarily low conical density in arbitrarily wide cones centered at most points $x \in E$, whose axis is perpendicular to the line L. The quantifications of *arbitrarily low* and *arbitrarily wide* can be made stronger by reducing the value of the constants α and δ. After this step, we use Besicovitch's *two cones* argument (quantified in Lemma 4.18) to show that most of E can be contained on a Lipschitz graph over L.

3C. *Step 3*: *there can only be one graph.* In Proposition 3.3 we managed to pack a majority of each set E_j (as defined in (3.2)) to a Lipschitz graph of constant $\sim\alpha$, up to errors which tend to zero as $\delta \to 0$ in the main assumption (1.2). However, at this point there might be up to $\sim\alpha^{-1}$ distinct Lipschitz graphs, and to prove Theorem 1.1, we would (roughly speaking) like to reduce their number to one. That this should be possible is not hard to believe: if E consists of several distinct Lipschitz graphs of substantial measure, which nevertheless cannot be fit into a single Lipschitz graph, then $\mathrm{Fav}(E)$ cannot possibly be maximal.

We turn to the details. We recall the *given* constant $\epsilon > 0$ from the statement of Theorem 1.1, and we set

$$\delta := \frac{\epsilon^{70}}{\boldsymbol{C}_{\mathrm{thm}}}$$

for a sufficiently large absolute constant $\boldsymbol{C}_{\mathrm{thm}} > 1$. We define also

$$\alpha := \left(\frac{\epsilon}{\boldsymbol{C}_{\mathrm{alp}}}\right)^{10} \tag{3.5}$$

for some universal $\boldsymbol{C}_{\mathrm{alp}} > 1$. The universal constant $\boldsymbol{C}_{\mathrm{thm}}$ will depend on $\boldsymbol{C}_{\mathrm{alp}}$, whereas $\boldsymbol{C}_{\mathrm{alp}}$ depends only on $\boldsymbol{C}_{\mathrm{lip}}$ and another constant $\boldsymbol{C}_{\mathrm{sep}}$, which is introduced below. The additional constant $\boldsymbol{C}_{\mathrm{alp}}$ will make it easier for us to ensure that the Lipschitz graph Γ obtained from the application of Proposition 3.3 has Lipschitz constant smaller than ϵ; see the discussion around (3.8). We record that

$$\alpha^7 = \boldsymbol{C}_{\mathrm{alp}}^{-70}\epsilon^{70} = \boldsymbol{C}_{\mathrm{thm}}\boldsymbol{C}_{\mathrm{alp}}^{-70} \cdot \delta. \tag{3.6}$$

Recall, once more, the decompositions $\mathcal{E} = \mathcal{E}_0 \cup \cdots \cup \mathcal{E}_{M_2}$ and $E = E_0 \cup \cdots \cup E_{M_2}$ from the previous subsection: this decomposition depends on the parameter α fixed above. In addition to the decomposition $E = E_0 \cup \cdots \cup E_{M_2}$, we will also need another, coarser, decomposition of E in this section. Write $\kappa := \frac{1}{10}$, fix $M_3 \sim \alpha^{-\kappa}$, and decompose $\mathcal{E} = \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_{M_3}$ in such a way that

- each $\mathcal{F}_k$ is a union of finitely many consecutive families $\mathcal{E}_j$, and
- $\mathcal{F}_k$ contains those minigraphs whose direction makes an angle no larger than α^κ with $w_k = (\cos(k\pi/M_3), \sin(k\pi/M_3))$ for $0 \le k \le M_3$.

We write

$$F_k := \bigcup \mathcal{F}_k, \quad 0 \le k \le M_3 \sim \alpha^{-\kappa}.$$

At this point, we consider two distinct cases. Let $\boldsymbol{C}_{\mathrm{sep}}$ be a large constant depending only on the absolute constant $\boldsymbol{C}_{\mathrm{lip}}$ appearing in Proposition 3.3 (the letters sep stand for *separation*). Thus, the constant $\boldsymbol{C}_{\mathrm{sep}}$ is also absolute, and we may (and will) assume that $\boldsymbol{C}_{\mathrm{alp}}$ is large relative to $\boldsymbol{C}_{\mathrm{sep}}$.

Case 1. Given the constant $\epsilon > 0$ from Theorem 1.1, the first case is that we can find consecutive sets $F_k, F_{k+1}, \dots, F_{k+\boldsymbol{C}_{\mathrm{sep}}}$ with the property

$$\mathcal{H}^1(E \setminus (F_k \cup \dots \cup F_{k+\boldsymbol{C}_{\mathrm{sep}}})) \le \epsilon. \tag{3.7}$$

In this case we note that $F := F_k \cup \dots \cup F_{k+\boldsymbol{C}_{\mathrm{sep}}}$ is a union of minigraphs whose directions are within $\lesssim \boldsymbol{C}_{\mathrm{sep}}\alpha^{\kappa}$ of the fixed vector w_k. In particular, F can be expressed as a union of finitely many disjoint $\bar{\alpha}$-Lipschitz graphs over the line $\mathrm{span}(w_k)$, with $\bar{\alpha} \sim \boldsymbol{C}_{\mathrm{sep}}\alpha^{\kappa}$. This will place us in a position to use Proposition 3.3 (with E replaced by F and α replaced by $\bar{\alpha}$). Of course also

$$\int_{\mathcal{L}(F)} (\#(F \cap \ell) - 1)\, d\eta(\ell) \le \int_{\mathcal{L}(E)} (\#(E \cap \ell) - 1)\, d\eta(\ell) \le \delta,$$

so the analogue of the assumption (3.4) is valid for F in place of E. We also note that

$$\delta = \boldsymbol{C}_{\mathrm{thm}}^{-1}\epsilon^{70} \le \boldsymbol{C}_{\mathrm{thm}}^{-1}\boldsymbol{C}_{\mathrm{alp}}^{3} \cdot (\epsilon/\boldsymbol{C}_{\mathrm{alp}})^3 \cdot \epsilon^2 = (\boldsymbol{C}_{\mathrm{thm}}^{-1}\boldsymbol{C}_{\mathrm{alp}}^{3}) \cdot \alpha^{3\kappa}\epsilon^2 \sim (\boldsymbol{C}_{\mathrm{thm}}^{-1}\boldsymbol{C}_{\mathrm{alp}}^{3}\boldsymbol{C}_{\mathrm{sep}}^{-3}) \cdot \bar{\alpha}^3\epsilon^2,$$

so if $\boldsymbol{C}_{\mathrm{thm}}$ is sufficiently large relative to $\boldsymbol{C}_{\mathrm{alp}}$, then the hypothesis in Proposition 3.3 on the relation between δ, $\bar{\alpha}$, and ϵ is satisfied (the constant $\boldsymbol{C}_{\mathrm{sep}}$ is large, so it can be safely ignored here). Consequently, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ of constant $\lesssim \boldsymbol{C}_{\mathrm{lip}}\boldsymbol{C}_{\mathrm{sep}} \cdot \alpha^{\kappa} = \boldsymbol{C}_{\mathrm{lip}}\boldsymbol{C}_{\mathrm{sep}} \cdot \epsilon/\boldsymbol{C}_{\mathrm{alp}}$ with the property

$$\mathcal{H}^1(F \setminus \Gamma) \le \epsilon, \tag{3.8}$$

and consequently $\mathcal{H}^1(E \setminus \Gamma) \le 2\epsilon$. By choosing $\boldsymbol{C}_{\mathrm{alp}}$ sufficiently large relative to $\boldsymbol{C}_{\mathrm{sep}}$ and $\boldsymbol{C}_{\mathrm{lip}}$, we may ensure that Γ is an ϵ-Lipschitz graph, as desired.

Case 2. We then move to consider the other option, where E cannot be exhausted, up to measure ϵ, by a constant number of consecutive sets $F_k, F_{k+1}, \dots, F_{k+\boldsymbol{C}_{\mathrm{sep}}}$. Since (3.7) fails for every k, we may find an index pair $k, l \in \{0, \dots, M_3\}$ with $|k - l| \ge \boldsymbol{C}_{\mathrm{sep}}$ such that

$$\mathcal{H}^1(F_k) \ge \alpha^{2\kappa} \quad \text{and} \quad \mathcal{H}^1(F_l) \ge \alpha^{2\kappa}. \tag{3.9}$$

This follows immediately from the pigeonhole principle, recalling that the cardinality of the pieces F_k is $\lesssim \alpha^{-\kappa}$, and also that α^{κ} is much smaller than ϵ by (3.5).

Remark 3.10. Recall that the *separation* constant $\boldsymbol{C}_{\mathrm{sep}}$ above has been chosen to be large relative to the constant $\boldsymbol{C}_{\mathrm{lip}}$ in Proposition 3.3: morally, if Γ_1, Γ_2 are two $\boldsymbol{C}_{\mathrm{lip}}\alpha^{\kappa}$-Lipschitz graphs over lines L_1, L_2 with $\angle(L_1, L_2) \ge \boldsymbol{C}_{\mathrm{sep}}\alpha^{\kappa}$, we need to know that Γ_1 and Γ_2 are still *transversal* (their tangents form angles $\ge \frac{1}{2}\boldsymbol{C}_{\mathrm{sep}}\alpha^{\kappa}$ with each other).

The next key proposition will imply that Case 2 cannot happen:

Proposition 3.11. *Suppose that $\boldsymbol{C}_{\mathrm{sep}} > 0$ is sufficiently large, and suppose that there are $k, l \in \{0, \dots, M_3\}$ with $|k - l| \ge \boldsymbol{C}_{\mathrm{sep}}$ such that*

$$\mathcal{H}^1(F_k) \ge \alpha^{2\kappa} \quad \textit{and} \quad \mathcal{H}^1(F_l) \ge \alpha^{2\kappa}.$$

Then

$$\int_{\mathcal{L}(E)} (\#(E \cap \ell) - 1)\, d\eta(\ell) \gtrsim \alpha^7. \tag{3.12}$$

As we recorded in (3.6), we have $\alpha^7 = \boldsymbol{C}_{\text{thm}}\boldsymbol{C}_{\text{alp}}^{-70} \cdot \delta$. Thus, if $\boldsymbol{C}_{\text{thm}}$ is chosen sufficiently large relative to $\boldsymbol{C}_{\text{alp}}$ and the implicit absolute constants in (3.12), then (3.12) would lead to the contradiction

$$\delta \geq \int_{\mathcal{L}(E)} (\#(E \cap \ell) - 1)\, d\eta(\ell) > \delta.$$

(For the first inequality, recall (2.5) and our main assumption (1.2).) Thus, with the choices of constants specified in this section, Case 2 cannot occur. This concludes the proof of Theorem 1.1.

In the next two sections we prove the two key results used above, Propositions 3.3 and 3.11.

4. Proof of Proposition 3.3

Let $E \subset \mathbb{R}^2$ be as in the proposition. With no loss of generality, we may assume that L is the x-axis, so the minigraphs in $\mathcal{E}$ are roughly horizontal. We introduce further notation. We write

$$\mathcal{C}_\beta := \{(x, y) \in \mathbb{R}^2 : |y| \geq \beta|x|\}, \quad \beta > 0.$$

Thus, the smaller the β, the wider the cone. We also write

$$\mathcal{C}_\beta(x) := x + \mathcal{C}_\beta \quad \text{and} \quad \mathcal{C}_\beta(x, r) := \mathcal{C}_\beta(x) \cap B(x, r).$$

With this notation, if a set $\Gamma \subset \mathbb{R}^2$ satisfies $\Gamma \cap \mathcal{C}_\beta(x) = \{x\}$ for all $x \in \Gamma$, then Γ is (a subset of) a β-Lipschitz graph. Thus, in view of Proposition 3.3, it would be desirable to show that $E \cap \mathcal{C}_{\boldsymbol{C}_{\text{lip}}\alpha}(x) = \{x\}$ for all $x \in E$. In reality, we will prove a similar statement about a subset of E (of nearly full length). It is worth noting that a toy version of these statements is already present in our hypotheses: each minigraph $\gamma \in \mathcal{E}$ is an α-Lipschitz graph over the x-axis.

Define the maximal conical density

$$\Theta^*_{E,\beta}(x) = \sup_{r>0} \frac{\mathcal{H}^1(\mathcal{C}_\beta(x, r) \cap E)}{r}.$$

Lemma 4.1 says that points of high conical density are negligible, whereas Lemma 4.18 says that points of low conical density can be mostly contained in a Lipschitz graph.

Lemma 4.1 (high conical density points are negligible). *Let $E \subset B(1)$, $\alpha \in (0, \alpha_0)$ and $\delta \in (0, 1)$ be as in Proposition 3.3, so that in particular (3.4) holds. Let $\varepsilon > 0$. If the absolute constant $\boldsymbol{C}_{\text{lip}} > 0$ is chosen sufficiently large, then*

$$\mathcal{H}^1(\{x \in E : \Theta^*_{E,\alpha'}(x) \geq \varepsilon\}) \lesssim \frac{\delta}{\varepsilon\alpha^2}, \tag{4.2}$$

where $\alpha' := \boldsymbol{C}_{\text{lip}}\alpha/2$.

Write $\ell_{x,\theta} := \pi_\theta^{-1}\{\pi_\theta(x)\}$ for $\theta \in [0, \pi)$, so that $\ell_{0,\theta} = \operatorname{span}(\cos\theta, \sin\theta)^\perp$. Let $J(\beta) \subset [0, \pi)$ be the set of directions in the cone $\mathcal{C}_\beta$, i.e.,

$$J(\beta) = \{\theta \in [0, \pi) : \ell_{0,\theta} \subset \mathcal{C}_\beta\} = \{\theta \in [0, \pi) : \operatorname{span}(\cos\theta, \sin\theta)^\perp \subset \mathcal{C}_\beta\}.$$

If ℓ is a line, we let $\ell(w)$ denote the tube that is the w-neighborhood of ℓ. For a tube $T = \ell(w)$, we write $w(T) = w$.

To prove Lemma 4.1, we rely on the following lemma:

Lemma 4.3 (the Besicovitch alternative). *Let $E \subset \mathbb{R}^2$ and $\beta \leq 1$. Then for all $x \in E$ and $H \geq 1$, at least one of the following two alternatives holds*:

(A1) *There exists a set $I_x \subset J(\beta)$ of measure $\mathcal{H}^1(I_x) \geq H^{-1}$ such that*

$$\#(E \cap \ell_{x,\theta}) \geq 2, \quad \theta \in I_x.$$

(A2) *There exists a set $J_x \subset J(\beta)$ of measure $\mathcal{H}^1(J_x) \gtrsim H^{-1}$ with the following property*: *for every $\theta \in J_x$, there is a tube $T = T_{x,\theta} = \ell_{x,\theta}(w(T))$ centered around $\ell_{x,\theta}$ such that*

$$\mathcal{H}^1(E \cap T) \gtrsim \Theta^*_{E,\beta}(x) \cdot H \cdot w(T).$$

We call this lemma the *Besicovitch alternative*, because its proof is part of Besicovitch's original argument [1939] for his projection theorem. For a more recent presentation; see [Falconer 1986, Lemma 6.11] or [Mattila 1995, Lemma 18.7]. Neither the hypotheses nor the conclusion of Falconer's lemma are exactly the same as ours, but the reader can easily convince himself that the proof of Lemma 4.3 heavily draws inspiration from his proof.

Proof of Lemma 4.3. Let E, x, β, H be as in the statement of the lemma. Let $\varepsilon := \frac{1}{2}\Theta^*_{E,\beta}(x)$, so that there exists an $r > 0$ such that $\mathcal{H}^1(\mathcal{C}_\beta(x,r) \cap E) \geq \varepsilon r$. We set also $J := J(\beta)$.

If the alternative (A1) fails, then

$$\mathcal{H}^1\big(\{\theta \in J : \#(\mathcal{C}_\beta(x,r) \cap E \cap \ell_{x,\theta}) \geq 2\}\big) \leq \mathcal{H}^1(\{\theta \in J : \#(E \cap \ell_{x,\theta}) \geq 2\}) \leq H^{-1}.$$

Since evidently $x \in \mathcal{C}_\beta(x,r) \cap E \cap \ell_{x,\theta}$, this implies that most of the lines $\ell_{x,\theta}$ do not intersect the set $\mathcal{C}_\beta(x,r) \cap E$ outside x. Consequently, $\mathcal{C}_\beta(x,r) \cap E$ is contained in a union of narrow cones $\mathcal{C}_1, \mathcal{C}_2, \ldots$ which are centered around certain lines ℓ_{x,θ_j} with $\theta_j \in J$, and whose opening angles $\beta_1, \beta_2, \ldots$ satisfy $\sum \beta_j \leq 2H^{-1}$. We may arrange that the cones have the form

$$\mathcal{C}_j := \mathcal{C}(I_j) := \bigcup\{\ell_{x,\theta} : \theta \in I_j\},$$

where $I_j \subset J$ is a dyadic interval, $|I_j| = \beta_j$, and $\theta_j \in J$ is the midpoint of I_j. We may also assume that the dyadic intervals I_j are disjoint, so the sets $\mathcal{C}_j \setminus \{x\}$ are disjoint.

To use these cones to arrive at alternative (A2), recall that $\mathcal{H}^1(\mathcal{C}_\beta(x,r) \cap E) \geq \varepsilon r$, where $\varepsilon = \frac{1}{2}\Theta^*_{E,\beta}(x)$. Now, we throw away cones which are not *heavy*: we call a cone *heavy* if it satisfies

$$\mathcal{H}^1(\mathcal{C}_j \cap B(x,r) \cap E) \geq \tfrac{1}{4} \cdot \varepsilon H |I_j| \cdot r. \tag{4.4}$$

The total length of $\mathcal{C}_\beta(x,r) \cap E$ contained in the nonheavy cones is bounded from above by

$$\tfrac{1}{4}\varepsilon H r \sum_{j \in \mathbb{N}} |I_j| \leq \tfrac{1}{2}\varepsilon r \leq \tfrac{1}{2}\mathcal{H}^1(\mathcal{C}_\beta(x,r) \cap E),$$

so at least half of the length in $\mathcal{C}_\beta(x,r) \cap E$ is contained in the union of the heavy cones. In the sequel, we assume that all the cones $\mathcal{C}_j$ are heavy.

Next, we would like to prove that $\sum \beta_j = \sum |I_j| \gtrsim H^{-1}$. This would be easy if the heavy cones also satisfied an upper bound roughly matching the lower bound in (4.4). If we knew this, then we could estimate

$$\sum_{j \in \mathbb{N}} |I_j| \gtrsim (\varepsilon H r)^{-1} \sum_{j \in \mathbb{N}} \mathcal{H}^1(\mathcal{C}_j \cap B(x,r) \cap E) \gtrsim H^{-1}. \tag{4.5}$$

This desired upper bound in (4.4) need not be true to begin with, but can be easily arranged. Fix a heavy cone $\mathcal{C}(I_j)$, and perform the following stopping time argument: the dyadic interval I_j is successively replaced by its parent $\hat{I}_j$ until either the upper bound

$$\mathcal{H}^1(\mathcal{C}(\hat{I}_j) \cap B(x,r) \cap E) \leq \varepsilon H |\hat{I}_j| \cdot r \tag{4.6}$$

holds, or then $\hat{I}_j = J$. This procedure gives rise to a new collection of cones $\mathcal{C}(\hat{I}_j)$ which are evidently still heavy, and whose union covers the union of the initial heavy cones. Since the intervals $\hat{I}_j$ are dyadic, we may arrange that the new heavy cones are disjoint outside $\{x\}$ without violating the previous two properties.

At this point, either $\hat{I}_j = J$ for some index j, in which case (4.5) is trivially true (using $|J| \sim 1$), or then the upper bound (4.6) holds for all the heavy cones. In this case the lower bound (4.5) holds by the very calculation shown in (4.5).

We are now fully equipped to establish alternative (A2). Consider a line $\ell_{x,\theta}$ contained in the union of the heavy cones. According to (4.5), the set of angles $\theta \in J$ of such lines has length $\gtrsim H^{-1}$. This set of angles is the set $J_x \subset J$ whose existence is claimed in (A2). It remains to associate the tube $T_{x,\theta}$ to each line $\ell_{x,\theta}$ with $\theta \in J_x$. Let $\mathcal{C}(I_j) = \mathcal{C}_j \supset \ell_{x,\theta}$ be the (unique) heavy cone containing $\ell_{x,\theta}$. The opening angle of $\mathcal{C}_j$ is $\beta_j = |I_j| \in (0, |J|]$, and it follows by elementary geometry that

$$\mathcal{C}_j \cap B(x,r) \subset \ell_{x,\theta}(2\beta_j r) =: T_{x,\theta}.$$

Finally,

$$\mathcal{H}^1(E \cap T_{x,\theta}) \geq \mathcal{H}^1(\mathcal{C}_j \cap B(x,r) \cap E) \gtrsim \varepsilon H \beta_j \cdot r \sim \varepsilon H \cdot w(T),$$

as claimed in alternative (A2). □

Proof of Lemma 4.1. Recall that E is a union of finitely many disjoint α-Lipschitz minigraphs $\gamma \in \mathcal{E}$, all defined over the x-axis. The main geometric observation is the following: every minigraph in $\mathcal{E}$ is an α^{-1}-Lipschitz graph over every line $L_\theta := \operatorname{span}(\cos\theta, \sin\theta) = \ell_{0,\theta}^{\perp}$ with $\theta \in J(\alpha')$ (recall that $\alpha' = \boldsymbol{C}_{\mathrm{lip}}\alpha/2$). This is simply because the minigraphs in $\mathcal{E}$ are α-Lipschitz graphs over the x-axis, but for all $\theta \in J(\alpha')$, the lines L_θ form an angle $\gtrsim \alpha$ with the y-axis. See Figure 1. Thus, E is a union of finitely many α^{-1}-Lipschitz graphs over L_θ, for every $\theta \in J(\alpha')$. This places us in a position to use the coarea formula (2.8): for every $\theta \in J(\alpha')$ and every $\mathcal{H}^1$ measurable subset $E' \subset E$ we have

$$\int_{\pi_\theta(E')} \#(E' \cap \pi_\theta^{-1}\{t\})\, dt \gtrsim \alpha \mathcal{H}^1(E'). \tag{4.7}$$

Let

$$R = \{x \in E : \Theta^*_{E,\alpha'}(x) \geq \varepsilon\}.$$

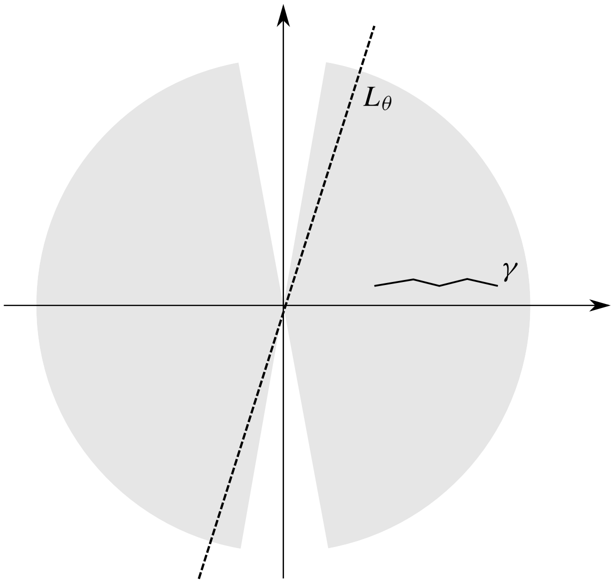

Figure 1. Every minigraph $\gamma \in \mathcal{E}$ is an α^{-1}-Lipschitz graph over every line L_θ with $\theta \in J(\alpha')$.

Fix $H \geq 1$. (We will eventually choose $H \sim 1/(\alpha\varepsilon)$; see (4.16) below.) By Lemma 4.3 (with $\beta = \alpha'$), we can write $R = R_1 \cup R_2$, where alternative (A1) holds on R_1 and (A2) holds on R_2. To prove (4.2), it suffices to show

$$\mathcal{H}^1(R_i) \lesssim \frac{\delta}{\varepsilon\alpha^2} \quad \text{for } i = 1, 2. \tag{4.8}$$

We first consider R_1. Recall the sets $I_x \subset J(\alpha')$ defined in (A1). Since E is a union of finitely many compact Lipschitz graphs, there are no measurability issues, and we may freely use Fubini's theorem:

$$H^{-1}\mathcal{H}^1(R_1) \leq \int_{R_1} \mathcal{H}^1(I_x)\, d\mathcal{H}^1(x) = \int_{J(\alpha')} \mathcal{H}^1(\{x \in R_1 : \theta \in I_x\})\, d\theta. \tag{4.9}$$

For $\theta \in J(\alpha')$ fixed, abbreviate $R'_\theta := \{x \in R_1 : \theta \in I_x\}$. Write also

$$E'_\theta := \bigcup_{t \in \pi_\theta(R'_\theta)} (E \cap \pi_\theta^{-1}\{t\}),$$

so certainly $R'_\theta \subset E'_\theta$. Note that if $t \in \pi_\theta(E'_\theta)$, then $t = \pi_\theta(x)$ for some $x \in R'_\theta$. Thus $\theta \in I_x$ by definition, so

$$\#(E'_\theta \cap \pi_\theta^{-1}\{t\}) = \#(E \cap \ell_{x,\theta}) \geq 2.$$

Therefore

$$\#(E'_\theta \cap \pi_\theta^{-1}\{t\}) - 1 \sim \#(E'_\theta \cap \pi_\theta^{-1}\{t\}), \quad t \in \pi_\theta(E'_\theta). \tag{4.10}$$

We may now deduce from (4.7) applied to $E' := E'_\theta$, and (4.10), that

$$\int_{\pi_\theta(E'_\theta)} (\#(E'_\theta \cap \pi_\theta^{-1}\{t\}) - 1)\, dt \sim \int_{\pi_\theta(E'_\theta)} \#(E'_\theta \cap \pi_\theta^{-1}\{t\})\, dt \gtrsim \alpha\mathcal{H}^1(E'_\theta) \geq \alpha\mathcal{H}^1(R'_\theta),$$

and finally

$$\int_{\mathcal{L}(E)} (\#(E \cap \ell) - 1)\, d\eta(\ell) \geq \int_{J(\alpha')} \int \#(E'_\theta \cap \pi_\theta^{-1}\{t\}) - 1 \, dt \, d\theta \overset{(4.9)}{\geq} \alpha H^{-1} \mathcal{H}^1(R_1).$$

By (3.4) the left-hand side is bounded from above by δ, so

$$\mathcal{H}^1(R_1) \lesssim \frac{\delta H}{\alpha}. \tag{4.11}$$

Recalling that we promised to choose $H \sim 1/(\alpha\varepsilon)$ in the end, the bound above implies (4.8) for R_1.

Next, we tackle R_2. This time we define $R'_\theta := \{x \in R_2 : \theta \in J_x\} \subset E$, and we deduce exactly as in (4.9) that

$$H^{-1} \mathcal{H}^1(R_2) \lesssim \int_{J(\alpha')} \mathcal{H}^1(R'_\theta)\, d\theta. \tag{4.12}$$

Fix $\theta \in J(\alpha')$ with $R'_\theta \neq \varnothing$. For each $x \in R'_\theta$, by definition, there exists a tube $T = T_{x,\theta}$ centered around $\ell_{x,\theta}$ with the property

$$\mathcal{H}^1(E \cap T) \gtrsim \varepsilon H \cdot w(T). \tag{4.13}$$

The tubes $\{T_{x,\theta} : x \in R'_\theta\}$ may overlap, but they are all parallel. By the Besicovitch covering theorem (e.g., [Mattila 1995, Theorem 2.7]) applied to the projections $\pi_\theta(T_{x,\theta}) \subset \mathbb{R}$, there exists a countable subcollection $\mathcal{T}_\theta \subset \{T_{x,\theta} : x \in R'_\theta\}$, with the properties

$$R'_\theta \subset \bigcup_{x \in R'_\theta} T_{x,\theta} \subset \bigcup_{T \in \mathcal{T}_\theta} T \quad \text{and} \quad \sum_{T \in \mathcal{T}_\theta} \mathbf{1}_T \lesssim 1. \tag{4.14}$$

Fix $T \in \mathcal{T}_\theta$, and let $\mathrm{Bad}(E \cap T) \subset E \cap T$ consist of those points $x \in E \cap T$ with $\#(\ell_{x,\theta} \cap E) = 1$. We apply the coarea formula (2.8) to the set $A := \mathrm{Bad}(E \cap T) \subset E$. Recalling that for every $\theta \in J(\alpha')$ the set E is a union of finitely many α^{-1}-Lipschitz graphs over L_θ (see the remark above (4.7)) we get that

$$\mathcal{H}^1(\mathrm{Bad}(E \cap T)) \lesssim \frac{1}{\alpha} \int_{\pi_\theta(T)} 1 \, dt = \frac{w(T)}{\alpha}. \tag{4.15}$$

Now, for a suitable choice $H \sim 1/(\alpha\varepsilon)$, a combination of (4.13) and (4.15) shows that

$$\mathcal{H}^1((E \cap T) \setminus \mathrm{Bad}(E \cap T)) \geq \tfrac{1}{2} \mathcal{H}^1(E \cap T). \tag{4.16}$$

At this point, we simplify notation by setting

$$E_\theta := \bigcup_{T \in \mathcal{T}_\theta} (E \cap T) \setminus \mathrm{Bad}(E \cap T) \subset E.$$

By the definition of the sets $\mathrm{Bad}(E \cap T)$, if $x \in E_\theta$, then $\#(E \cap \ell_{x,\theta}) \geq 2$, and therefore

$$\#(E \cap \pi_\theta^{-1}\{t\}) - 1 \sim \#(E \cap \pi_\theta^{-1}\{t\}) \geq \#(E_\theta \cap \pi_\theta^{-1}\{t\}), \quad t \in \pi_\theta(E_\theta). \tag{4.17}$$

It follows that

$$\begin{aligned}\int_{\mathcal{L}(E)} \#(E\cap\ell)-1\,d\eta(\ell) &\geq \int_{J(\alpha')}\int \#(E\cap\pi_\theta^{-1}\{t\})-1\,dt\,d\theta\\ &\overset{(4.17)}{\gtrsim} \int_{J(\alpha')}\int_{\pi_\theta(E_\theta)} \#(E_\theta\cap\pi_\theta^{-1}\{t\})\,dt\,d\theta\\ &\overset{(4.14)}{\gtrsim} \int_{J(\alpha')}\sum_{T\in\mathcal{T}_\theta}\int_{\pi_\theta(E_\theta\cap T)} \#(E_\theta\cap\pi_\theta^{-1}\{t\})\,dt\,d\theta\\ &\overset{(4.7)}{\gtrsim} \alpha\int_{J(\alpha')}\sum_{T\in\mathcal{T}_\theta}\mathcal{H}^1(E_\theta\cap T)\,d\theta\\ &\overset{(4.16)}{\geq} \frac{\alpha}{2}\int_{J(\alpha')}\sum_{T\in\mathcal{T}_\theta}\mathcal{H}^1(E\cap T)\,d\theta\\ &\overset{(4.14)}{\geq} \alpha\int_{J(\alpha')}\mathcal{H}^1(R_\theta')\,d\theta\\ &\overset{(4.12)}{\geq} \frac{\alpha}{H}\cdot\mathcal{H}^1(R_2).\end{aligned}$$

Recalling once again from (3.4) that the left-hand side above is $\leq\delta$, we deduce that

$$\mathcal{H}^1(R_2)\lesssim\frac{\delta H}{\alpha}\sim\frac{\delta}{\varepsilon\alpha^2},$$

which is (4.8) for R_2. The proof of Lemma 4.1 is complete. □

Next, repeating the classical *two cones* argument of Besicovitch (e.g., [Mattila 1995, Lemma 15.14]), we show that we can pack most of points of low conical density into a single Lipschitz graph:

Lemma 4.18 (most low conical density points fit into a Lipschitz graph). *Let $E\subset B(1)\subset\mathbb{R}^2$ and let $\varepsilon\in(0,1)$, $\beta\in\left(0,\frac{1}{2}\right)$. Then, there exists a 2β-Lipschitz graph $\Gamma\subset\mathbb{R}^2$ over the x-axis such that*

$$\mathcal{H}^1(\{x\in E:\Theta^*_{E,\beta}(x)\leq\varepsilon\}\setminus\Gamma)\lesssim\varepsilon/\beta.$$

Proof. Let $G=\{x\in E:\Theta^*_{E,\beta}(x)\leq\varepsilon\}$. Our task is to find a subset $\Gamma\subset G$ with $\mathcal{H}^1(G\setminus\Gamma)\lesssim\varepsilon/\beta$ and the property $\mathcal{C}_{2\beta}(x)\cap\Gamma=\{x\}$ for all $x\in\Gamma$. Then Γ extends to a 2β-Lipschitz graph, as desired.

Let B be the set of points $x\in G$ with the "bad" property that there exists a point $y\in G\cap\mathcal{C}_{2\beta}(x)$ with $y\neq x$. The goal is to show that $\mathcal{H}^1(B)\lesssim\varepsilon/\beta$. For each $x\in B$, let $r(x)=\sup\{|x-y|:y\in G\cap\mathcal{C}_{2\beta}(x)\}$, so

$$B\cap\mathcal{C}_{2\beta}(x)\subset B(x,r(x)),\quad x\in B. \tag{4.19}$$

See Figure 2 for an illustration.

Let T_x be the tube around the vertical line passing through x with $w(T_x):=\frac{1}{10}\beta r(x)$. Then

$$T_x\setminus B\left(x,\tfrac{1}{2}\beta r(x)\right)\subset\mathcal{C}_1(x)\subset\mathcal{C}_{2\beta}(x)\subset\mathcal{C}_\beta(x). \tag{4.20}$$

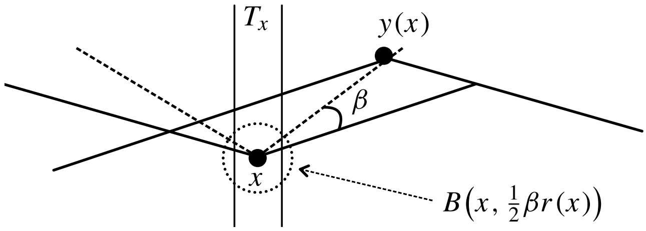

Figure 2. Containing the tube T_x in the union of the cones $\mathcal{C}_\beta(x)$ and $\mathcal{C}_\beta(y(x))$. The dotted cone illustrates $\mathcal{C}_{2\beta}(x) \ni y(x)$.

(Recall that $2\beta \le 1$.) In particular, (4.20) implies $T_x \setminus B(x, r(x)) \subset \mathcal{C}_{2\beta}(x)$. Using this, we observe that

$$\begin{aligned} B \cap T_x &\subset B(x, r(x)) \cup [(B \cap T_x) \setminus B(x, r(x))] \\ &= B(x, r(x)) \cup [B \cap (T_x \setminus B(x, r(x)))] \subset B(x, r(x)) \cup [B \cap \mathcal{C}_{2\beta}(x)] \overset{(4.19)}{\subset} B(x, r(x)). \end{aligned} \tag{4.21}$$

Choose a point $y(x) \in G \cap \mathcal{C}_{2\beta}(x)$ such that $|x - y(x)| \ge \frac{9}{10} r(x)$. A slightly more delicate geometric fact is that

$$T_x \subset \mathcal{C}_\beta(x) \cup \mathcal{C}_\beta(y(x)).$$

This is an exercise in elementary geometry; see Figure 2 (or the proof in [Mattila 1995, Lemma 15.14] for a more formal argument): the disc $B\big(x, \frac{1}{2}\beta r(x)\big)$, and in particular the intersection $T_x \cap B\big(x, \frac{1}{2}\beta r(x)\big)$, is contained in the cone $\mathcal{C}_\beta(y(x))$, whereas the rest of T_x is contained in $\mathcal{C}_\beta(x)$, as already noted in (4.20). Consequently, using (4.21), the trivial inclusion $B(x, r(x)) \subset B(y(x), 2r(x))$, and $x, y(x) \in G$, we have

$$\mathcal{H}^1(B \cap T_x) \le \mathcal{H}^1(\mathcal{C}_\beta(y(x), 2r(x)) \cap E) + \mathcal{H}^1(\mathcal{C}_\beta(x, r(x)) \cap E) \le 2\varepsilon r(x) + \varepsilon r(x) \le 30(\varepsilon/\beta) \cdot w(T_x).$$

We have now shown that every point $x \in B$ is contained on the central line of a vertical tube T_x satisfying the estimate above. By the Besicovitch covering theorem, as in the proof of Lemma 4.1, we may then find a countable, boundedly overlapping subfamily $\mathcal{T}$ of these tubes which still cover B. All the tubes intersect $B(1) \supset B$, so $\sum_{T \in \mathcal{T}} w(T) \lesssim 1$. It follows that

$$\mathcal{H}^1(B) \le \sum_{T \in \mathcal{T}} \mathcal{H}^1(B \cap T) \le \frac{30\varepsilon}{\beta} \sum_{T \in \mathcal{T}} w(T) \lesssim \frac{\varepsilon}{\beta}.$$

This completes the proof of Lemma 4.18. □

We are then ready to prove Proposition 3.3:

Proof of Proposition 3.3. Fix $\epsilon > 0$ as in the statement of the proposition, and set $\alpha' = \boldsymbol{C}_{\mathrm{lip}}\alpha/2$. Define $\epsilon_1 := \alpha\epsilon/C$ for a suitable absolute constant $C > 0$. By Lemma 4.1 applied to $\varepsilon = \epsilon_1$, we know that the set $R \subset E$ of bad points $x \in E$ with

$$\Theta^*_{E_\alpha, \alpha'}(x) \ge \epsilon_1$$

satisfies

$$\mathcal{H}^1(R) \lesssim \delta \cdot \epsilon_1^{-1} \alpha^{-2} = C\delta \cdot \epsilon^{-1}\alpha^{-3}.$$

Since $\delta \le C_0\epsilon^2\alpha^3$, taking $C_0 = C^{-2}$ gives $\mathcal{H}^1(R) \le \epsilon/2$ (assuming that $C > 0$ was large enough).

The set $G := E \setminus R$ satisfies the hypotheses of Lemma 4.18 (with $\beta = \alpha' = \boldsymbol{C}_{\text{lip}}\alpha/2$ and $\varepsilon = \epsilon_1$), so there exists a $\boldsymbol{C}_{\text{lip}}\alpha$-Lipschitz graph $\Gamma \subset \mathbb{R}$ over the x-axis such that $\mathcal{H}^1(G \setminus \Gamma) \lesssim \epsilon_1/\alpha = \epsilon/C$. If the constant $C > 0$ was chosen large enough, we see that

$$\mathcal{H}^1(E \setminus \Gamma) \le \mathcal{H}^1(R) + \mathcal{H}^1(G \setminus \Gamma) \le \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.$$

This concludes the proof of Proposition 3.3. □

5. Proof of Proposition 3.11

In this section we prove Proposition 3.11. Recall that we are assuming to be in Case 2; that is, E cannot be exhausted, up to measure ϵ, by a constant number of consecutive sets $F_k, F_{k+1}, \dots, F_{k+\boldsymbol{C}_{\text{sep}}}$ (recall this notation from Section 3C). More precisely, this means that

$$\mathcal{H}^1(E \setminus (F_k \cup \dots \cup F_{k+\boldsymbol{C}_{\text{sep}}})) \le \epsilon \tag{5.1}$$

fails for every k; thus we find an index pair $k, l \in \{0, \dots, M_3\}$ with $|k - l| \ge \boldsymbol{C}_{\text{sep}}$ such that

$$\mathcal{H}^1(F_k) \ge \alpha^{2\kappa} \quad \text{and} \quad \mathcal{H}^1(F_l) \ge \alpha^{2\kappa}. \tag{5.2}$$

Recall that all the minigraphs in $\mathcal{F}_k$ make an angle $\le \alpha^\kappa$ with

$$L_k := \operatorname{span}(w_k) = \operatorname{span}(\cos(k\pi/M_3), \sin(k\pi/M_3)),$$

and similarly all the minigraphs in $\mathcal{F}_l$ make an angle $\le \alpha^\kappa$ with $L_l = \operatorname{span}(w_l)$.

The existence of F_k and F_l will imply a configuration such as the one depicted in Figure 3. A more precise definition is given in the lemma below.

Lemma 5.3. *If the inequalities in (5.2) hold, then there exists an absolute constant $C \sim \boldsymbol{C}_{\text{lip}}$ (the constant from Proposition 3.3) such that the following objects exist:*

(1) *affine lines ℓ_k and ℓ_l with $\angle(\ell_k, L_k) \le \alpha^\kappa$ and $\angle(\ell_l, L_l) \le \alpha^\kappa$,*

(2) *tubes $T_k' := \ell_k(C\alpha)$ and $T_k := \ell_k(\alpha^{1/2})$,*

(3) *tubes $T_l' := \ell_l(C\alpha)$ and $T_l := \ell_l(\alpha^{1/2})$,*

(4) *$\boldsymbol{C}_{\text{lip}}\alpha$-Lipschitz graphs γ_k, γ_l over the lines ℓ_k, ℓ_l, respectively such that*

$$\gamma_k \cap B(1) \subset T_k' \quad \textit{and} \quad \gamma_l \cap B(1) \subset T_l',$$

(5) *compact subsets*

$$G_k \subset (E \cap \gamma_k) \setminus T_l \subset B(1) \quad \textit{and} \quad G_l \subset (E \cap \gamma_l) \setminus T_k \subset B(1) \tag{5.4}$$

of measure $\mathcal{H}^1(G_k) \ge \alpha^3/C$ and $\mathcal{H}^1(G_l) \ge \alpha^3/C$.

Once the objects in Lemma 5.3 are found, it follows from a relatively simple geometric argument, presented below, that positively many lines intersect E twice (the lines in question are depicted in red in Figure 3):

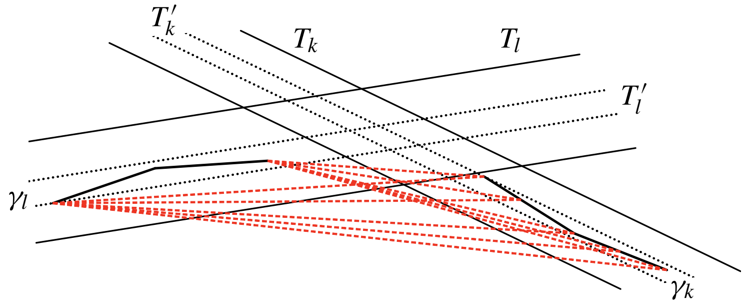

Figure 3. A configuration where positively many lines hit E twice.

Lemma 5.5. *There exists a set of lines $\mathcal{L}(G_k, G_l)$ of measure $\eta(\mathcal{L}(G_k, G_l)) \gtrsim \alpha^7$ such that $\ell \cap G_k \neq \varnothing$ and $\ell \cap G_l \neq \varnothing$ for all $\ell \in \mathcal{L}(G_k, G_l)$. In particular, since $G_k, G_l \subset E$ are disjoint,*

$$\int_{\mathcal{L}(E)} (\#(E \cap \ell) - 1)\, d\eta(\ell) \gtrsim \eta(\mathcal{L}(G_k, G_l)) \gtrsim \alpha^7. \tag{5.6}$$

Proposition 3.11 follows immediately by Lemma 5.5. We will next derive Lemma 5.5 from Lemma 5.3. (See Remark 5.10 and the Appendix for an alternative proof of Lemma 5.5.)

Proof. The key geometric observation is the following: if $\ell \subset \mathbb{R}^2$ is any line with

$$G_k \cap \ell \neq \varnothing \neq G_l \cap \ell,$$

then ℓ must make an angle $\gtrsim \alpha^{1/2}$ with both ℓ_k and ℓ_l; see Figure 3: indeed, if for example $\angle(\ell, \ell_l) \ll \alpha^{1/2}$ and $\ell \cap G_l \neq \varnothing$, then $\ell \cap B(1) \subset T_l$, and hence $\ell \cap G_k = \varnothing$ by (5.4). It follows that both ℓ_k, ℓ_l are $C\alpha^{-1/2}$-graphs over $\ell^\perp$, for any line ℓ connecting G_k and G_l. But since γ_k, γ_l were by definition $\boldsymbol{C}_{\mathrm{lip}}\alpha$-Lipschitz graphs over ℓ_k, ℓ_l, it follows that also γ_k, γ_l are $C\alpha^{-1/2}$-Lipschitz graphs over $\ell^\perp$ (assuming that $\alpha > 0$ is small enough).

To prove the lower bound (5.6), start by fixing $x \in G_l \subset \gamma_l$, recall that $\ell_{x,\theta} := \pi_\theta^{-1}\{\pi_\theta(x)\}$, and consider the set of directions

$$\Theta(x, G_k) := \{\theta \in [0, \pi) : \ell_{x,\theta} \cap G_k \neq \varnothing\}.$$

With this notation, we claim that

$$\mathcal{H}^1(\Theta(x, G_k)) \gtrsim \alpha^{1/2} \mathcal{H}^1(G_k), \qquad x \in G_l. \tag{5.7}$$

Indeed, if $\{B(\theta_j, r_j)\}_{j \in \mathbb{N}}$ is an arbitrary cover of $\Theta(x, G_k)$, then the tubes $\ell_{x,\theta_j}(Cr_j)$ cover G_k, where $C > 0$ is an absolute constant. This is because G_k is covered by the cones $C_j := \bigcup\{\ell_{x,\theta} : \theta \in B(\theta_j, r_j)\}$ by definition, and each intersection $G_k \cap C_j \subset B(1) \cap C_j$ is further covered by a tube of the form $\ell_{x,\theta_j}(Cr_j)$. Now recall that $\gamma_k \supset G_k$ is an $\alpha^{-1/2}$-Lipschitz graph over each line $\ell_{x,\theta_j}^\perp$: this gives

$$\alpha^{-1/2} \sum_{j \in \mathbb{N}} r_j \gtrsim \sum_{j \in \mathbb{N}} \mathcal{H}^1(G_k \cap \ell_{x,\theta_j}(r_j)) \geq \mathcal{H}^1(G_k),$$

which implies (5.7).

We now infer from (5.7) and Fubini's theorem that

$$\int_0^{\pi} \mathcal{H}^1(\{x \in G_l : \theta \in \Theta(x, G_k)\})\, d\theta = \int_{G_l} \mathcal{H}^1(\Theta(x, G_k))\, d\mathcal{H}^1(x) \gtrsim \alpha^{1/2}\mathcal{H}^1(G_k)\mathcal{H}^1(G_l). \tag{5.8}$$

To proceed, write $G_l(\theta) := \{x \in G_l : \theta \in \Theta(x, G_k)\}$. We claim that

$$\mathcal{H}^1(G_l(\theta)) \neq 0 \quad \Longrightarrow \quad \mathcal{H}^1\big(\pi_\theta(G_l(\theta))\big) \gtrsim \alpha^{1/2}\mathcal{H}^1(G_l(\theta)), \quad \theta \in [0, \pi). \tag{5.9}$$

This will complete the proof of the corollary, because (5.8) then implies

$$\int_0^{\pi} \mathcal{H}^1(\pi_\theta(G_l(\theta))\, d\theta \overset{(5.8)}{\gtrsim} \alpha\mathcal{H}^1(G_k)\mathcal{H}^1(G_l) \overset{\text{Lem. } 5.3}{\gtrsim} \alpha^7,$$

and the left-hand side above is a lower bound for $\eta(\mathcal{L}(G_k, G_l))$.

Finally, let us prove (5.9). If $\mathcal{H}^1(G_l(\theta)) \neq 0$, then $\theta \in \Theta(x, \gamma_k)$ for at least one $x \in G_l$, which means that $\ell_{x,\theta} = \pi_\theta^{-1}\{\pi_\theta(x)\}$ intersects both G_k and G_l. Thus, γ_l is a $C\alpha^{-1/2}$-Lipschitz graph over the line $\ell_{x,\theta}^{\perp}$. Consequently, the relation $\mathcal{H}^1(\pi_\theta(H)) \gtrsim \alpha^{1/2}\mathcal{H}^1(H)$ holds for all $\mathcal{H}^1$ measurable subsets $H \subset \gamma_l$, in particular for $H := G_l(\theta)$. □

Remark 5.10. In fact, we have an exact expression for $\eta(\mathcal{L}(G_k, G_l))$:

$$\eta(\mathcal{L}(G_k, G_l)) = \iint_{G_k \times G_l} \frac{|\pi_{\theta(x_k,x_l)}(\tau_k(x_k))|\,|\pi_{\theta(x_k,x_l)}(\tau_l(x_l))|}{|x_k - x_l|}\, d(\mathcal{H}^1 \times \mathcal{H}^1)(x_k, x_l). \tag{5.11}$$

In (5.11), $\tau_k(x)$ denotes the unit tangent vector to γ_k at $x \in \gamma_k$, and $\tau_l(x)$ is defined similarly. For distinct $x, x' \in \mathbb{R}^2$, $\theta(x, x')$ denotes the angle θ such that $\pi_\theta(x) = \pi_\theta(x')$.

Now we show how (5.11) implies Lemma 5.5. By the key geometric observation in the first paragraph of the proof of Lemma 5.5 and the fact that $G_k, G_l \subset B(1)$, the integrand in (5.11) is $\gtrsim \alpha^{1/2}\alpha^{1/2}/1 = \alpha$. Thus, $\eta(\mathcal{L}(G_k, G_l)) \gtrsim \alpha\mathcal{H}^1(G_k)\mathcal{H}^1(G_l) \gtrsim \alpha^7$.

We state and prove a more general form of (5.11) in the Appendix.

The remainder of this section is devoted to constructing the objects listed in Lemma 5.3. This is based on the assumption (3.9), that is, $\mathcal{H}^1(F_k) \geq \alpha^{2\kappa}$ and $\mathcal{H}^1(F_l) \geq \alpha^{2\kappa}$. Recall also that F_k, F_l were the unions of the minigraphs in $\mathcal{F}_k$ and $\mathcal{F}_l$. The minigraphs in $\mathcal{F}_k$ make an angle $\leq \alpha^\kappa$ with L_k, while the minigraphs in $\mathcal{F}_l$ make an angle $\leq \alpha^\kappa$ with L_l. Furthermore, $\angle(L_k, L_l) \geq \boldsymbol{C}_{\text{sep}}\alpha^\kappa$, so the minigraphs from $\mathcal{F}_k$ and $\mathcal{F}_l$ point in quantitatively different directions. We also recall that $\mathcal{F}_k$ (respectively $\mathcal{F}_l$) can be expressed as a union of certain consecutive families $\mathcal{E}_i$:

$$\mathcal{F}_k = \mathcal{E}_s \cup \mathcal{E}_{s+1} \cup \cdots \cup \mathcal{E}_{s+m} \quad \text{and} \quad \mathcal{F}_l = \mathcal{E}_t \cup \cdots \cup \mathcal{E}_{t+m}. \tag{5.12}$$

Some of these families may be empty, but not all, according to (5.2). Of course

$$m \lesssim \alpha^{-1}, \tag{5.13}$$

since there were no more than α^{-1} of the families $\mathcal{E}_j$ altogether.

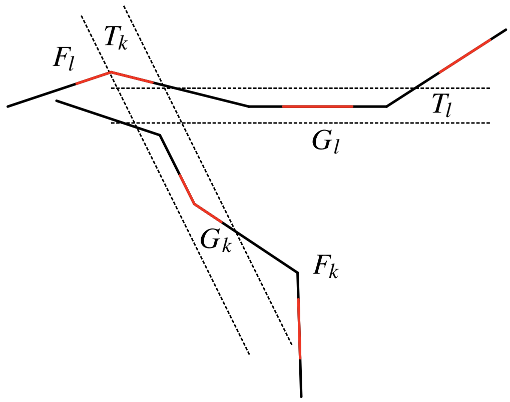

Figure 4. Finding the graphs and tubes claimed by Lemma 5.3.

5A. ***Sketch of the proof.*** We now explain the proof strategy with a picture. In Figure 4, we have depicted the sets F_k and F_l, which are roughly speaking α^κ-Lipschitz graphs over the lines L_k, L_l by Proposition 3.3 (details will follow). Both F_k and F_l are, moreover, tiled by $\lesssim \alpha^{-1}$ of the sets E_j. Most of sets E_j are (individually) contained on α-Lipschitz graphs γ_j, by another application of Proposition 3.3. The red sets shown in Figure 4 illustrate sets of the form

$$G_j = E_j \cap \gamma_j \cap B_j,$$

where B_j is some ball of radius α with the property that $\mathcal{H}^1(G_j) \sim_\alpha \mathcal{H}^1(E_j)$. Each G_j is contained in a tube T_j of width $\alpha^{1/2}$ (or even a tube of width α, which was also required in Lemma 5.3). So, picking $G_k \subset F_k$ and $G_l \subset F_l$ arbitrarily, we would satisfy all the points (1)-(5) in Lemma 5.3, except for the inclusions (5.4).

The problem is that if we pick $G_k \subset F_k$ and $G_l \subset F_l$ arbitrarily, the tube T_k associated with G_k might intersect G_l, or vice versa, violating (5.4). To satisfy (5.4), we need to pick G_k, G_l in such a way that the G_k-tube avoids G_l and the G_l-tube avoids G_k. To achieve this, we roughly choose three well-separated sets $G_1^l, G_2^l, G_3^l \subset F_l$, and two further well-separated sets $G_1^k, G_2^k \subset F_k$.

Then, we use the *transversality* of the graphs F_k, F_l to deduce the following: each G_i^k-tube can intersect at most one of the sets G_j^l, and vice versa. At this point, we may deduce from the pigeonhole principle that there must exists a pair (G_i^k, G_j^l) such that the G_i^k-tube does not intersect G_j^l, and the G_j^l-tube does not intersect G_i^k. Indeed, there are six pairs (G_i^k, G_j^l), but only five tubes. This will complete the proof.

5B. ***Proof.*** We turn to the details. First, we apply Proposition 3.3 to the sets F_k, F_l, each of which can be written as a finite union of α^κ-Lipschitz minigraphs over the lines L_k, L_l, respectively. It follows from the choice of constants $\delta = \epsilon^{70}/\boldsymbol{C}_{\mathrm{thm}}$ and $\alpha = (\epsilon/\boldsymbol{C}_{\mathrm{alp}})^{10}$ made in Section 3C that $\delta \ll \alpha^{5\kappa}$, assuming that $\boldsymbol{C}_{\mathrm{thm}}$ is chosen sufficiently small compared to the absolute constant $\boldsymbol{C}_{\mathrm{alp}}$. Writing $\alpha^{5\kappa} = (\alpha^\kappa)^3\alpha^{2\kappa}$, this means that the main hypothesis of Proposition 3.3 is valid with constants α^κ and $\frac{1}{2}\alpha^{2\kappa}$ in place of α and ϵ. It follows that there exist $\boldsymbol{C}_{\mathrm{lip}}\alpha^\kappa$-Lipschitz graphs Γ_k, Γ_l over L_k, L_l, respectively, which cover most of F_k and F_l in the sense

$$\mathcal{H}^1(F_k \setminus \Gamma_k) \le \tfrac{1}{2}\alpha^{2\kappa} \overset{(3.9)}{\le} \tfrac{1}{2}\mathcal{H}^1(F_k) \quad \text{and} \quad \mathcal{H}^1(F_l \setminus \Gamma_l) \le \tfrac{1}{2}\mathcal{H}^1(F_l).$$

We write $F_k' := F_k \cap \Gamma_k$ and $F_l' := F_l \cap \Gamma_l$. Next, recall from (5.12) that

$$F_k = E_s \cup \cdots \cup E_{s+m} \quad \text{and} \quad F_l = E_t \cup \cdots \cup E_{t+m},$$

and each E_j is a finite union of α-Lipschitz minigraphs $\mathcal{E}_j$ over a certain line (which makes an angle $\le \alpha^\kappa$ with L_k). Applying Proposition 3.3 again, for each E_j with either $j \in \{s, \dots, s+m\}$ or $j \in \{t, \dots, t+m\}$, we find Lipschitz graphs γ_j with constant $\le \boldsymbol{C}_{\mathrm{lip}}\alpha$ and the property

$$\mathcal{H}^1(E_j \setminus \gamma_j) \lesssim \alpha^2, \quad s \le j \le s+m \text{ or } t \le j \le t+m.$$

For this application of Proposition 3.3 to be legitimate, we need $\delta \ll \alpha^3(\alpha^2)^2 = \alpha^7$, which also follows from our choice of constants recalled above, taking $\boldsymbol{C}_{\mathrm{thm}} \gg \boldsymbol{C}_{\mathrm{alp}}^{70}$. We write $E_j' := E_j \cap \gamma_j$. With these choices, a major part of F_k' is covered by the union of the graphs γ_j: indeed since $F_k' \subset F_k \subset (E_s \cup \cdots \cup E_{s+m})$, we have

$$\mathcal{H}^1\Big(F_k' \setminus \bigcup_{j=1}^m E_{s+j}'\Big) \le \sum_{j=1}^m \mathcal{H}^1(E_{s+j} \setminus \gamma_{s+j}) \lesssim \sum_{j=1}^m \alpha^2 \overset{(5.13)}{\lesssim} \alpha.$$

Since $\mathcal{H}^1(F_k') \gtrsim \mathcal{H}^1(F_k) \ge \alpha^{2\kappa}$, and $\kappa = \frac{1}{10}$, we infer that at least half of F_k' is covered by the (subsets of) α-Lipschitz graphs E_j' with $s \le j \le s+m$. The same conclusion *mutatis mutandis* holds for F_l' and the sets E_j' with $t \le j \le t+m$. We finally redefine

$$F_k := F_k' \cap \bigcup_{j=1}^m E_{s+j}' \quad \text{and} \quad F_l := F_l' \cap \bigcup_{j=1}^m E_{t+j}'.$$

This should cause no confusion, since the original sets F_k, F_l will no longer be used. We list all the properties of F_k, F_l we will need in the sequel:

- $F_k, F_l \subset E$ and $\mathcal{H}^1(F_k) \gtrsim \alpha^{2\kappa}$ and $\mathcal{H}^1(F_l) \gtrsim \alpha^{2\kappa}$ (compare with (3.9)).
- F_k is covered by the Lipschitz graph Γ_k over L_k with constant $\le \boldsymbol{C}_{\mathrm{lip}}\alpha^\kappa$.
- F_l is covered by the Lipschitz graph Γ_l over L_l with constant $\le \boldsymbol{C}_{\mathrm{lip}}\alpha^\kappa$.
- F_k is covered by the union of $\lesssim \alpha^{-1}$ Lipschitz graphs $\gamma_s, \dots, \gamma_{s+m}$ with constant $\le \boldsymbol{C}_{\mathrm{lip}}\alpha$ over certain lines ℓ_{s+j} making an angle $\le \alpha^\kappa$ with L_k.
- F_l is covered by the union of $\lesssim \alpha^{-1}$ Lipschitz graphs $\gamma_t, \dots, \gamma_{t+m}$ with constant $\le \boldsymbol{C}_{\mathrm{lip}}\alpha$ over certain lines ℓ_{t+j} making an angle $\le \alpha^\kappa$ with L_l.

We have now defined carefully the objects F_k and F_l in Figure 4. In defining the objects E_k and E_l in the same picture, there is the technical problem that the *initial* sets E_j need not be localized, as the picture suggests. This will be easily fixed by intersecting the initial sets E_j with balls. First, using that $\mathcal{H}^1(F_k) \gtrsim \alpha^{2\kappa}$, we choose two special points $x_1, x_2 \in \bar{F}_k$ with the properties

$$|x_1 - x_2| \gtrsim \alpha^{2\kappa} \quad \text{and} \quad \mathcal{H}^1(F_k \cap B(x_j, \alpha)) \ge \alpha^2 \quad \text{for } j \in \{1, 2\}. \tag{5.14}$$

This can be arranged, because the set of points $x \in F_k$ with $\mathcal{H}^1(F_k \cap B(x,\alpha)) \le \alpha^2$ has total length at most $\lesssim \alpha \ll \mathcal{H}^1(F_k)$. Thus, the admissible points for the second condition in (5.14) have total length $\ge \frac{1}{2}\mathcal{H}^1(F_k) \gtrsim \alpha^{2\kappa}$. Then, to finish the selection, it remains to pick two of these points with separation $\alpha^{2\kappa}$: this is possible because F_k lies on a Lipschitz graph with constant ≤ 1, so in particular $\mathcal{H}^1(F_k \cap B(x,r)) \lesssim r$ for all $r > 0$.

Next, we move attention from F_k to F_l. This time we pick three special points $y_1, y_2, y_3 \in F_l$ with properties similar to those in (5.14):

$$|y_i - y_j| \gtrsim \alpha^{2\kappa} \text{ for } i \neq j \qquad \text{and} \qquad \mathcal{H}^1(F_l \cap B(y_j,\alpha)) \ge \alpha^2 \quad \text{for } j \in \{1,2,3\}. \tag{5.15}$$

The details of the selection are the same as we have seen above.

Next, recall that both F_k and F_l can be written as a finite union of (subsets of) $\boldsymbol{C}_{\mathrm{lip}}\alpha$-Lipschitz graphs: the covering graphs for F_k were denoted $\gamma_s, \dots, \gamma_{s+m}$ and the covering graphs for F_l were denoted $\gamma_t, \dots, \gamma_{t+m}$, where $m \lesssim \alpha^{-1}$. Since $\mathcal{H}^1(F_k \cap B(x_1,\alpha)) \ge \alpha^2$, at least one of the graphs $\gamma_s, \dots, \gamma_{s+m}$ must have large intersection with $F_k \cap B(x_1,\alpha)$. We denote this graph by γ_1^k; then we have

$$\mathcal{H}^1(F_k \cap \gamma_1^k \cap B(x_1,\alpha)) \gtrsim \alpha^3. \tag{5.16}$$

We find similarly a graph $\gamma_2^k \in \{\gamma_s, \dots, \gamma_{s+m}\}$ such that $\mathcal{H}^1(F_k \cap \gamma_2^k \cap B(x_2,\alpha)) \gtrsim \alpha^3$. Then, we also repeat the argument for the three balls $B(y_j,\alpha)$: we find three graphs $\gamma_1^l, \gamma_2^l, \gamma_3^l \in \{\gamma_t, \dots, \gamma_{t+m}\}$ with the property

$$\mathcal{H}^1(F_l \cap B(y_j,\alpha) \cap \gamma_j^l) \gtrsim \alpha^3, \quad 1 \le j \le 3. \tag{5.17}$$

The sets

$$G_i^k := F_k \cap \gamma_i^k \cap B(x_i,\alpha), \quad i = 1,2, \qquad \text{and} \qquad G_j^l := F_l \cap \gamma_j^l \cap B(y_j,\alpha), \quad j = 1,2,3, \tag{5.18}$$

are the ones we informally discussed below Figure 4.

Next, we associate the lines and tubes (required by Lemma 5.3) to the sets G_i^k, G_j^l. We associate to each graph γ_i^k or γ_j^l an affine line ℓ_i^k or ℓ_j^l with the following properties:

- γ_i^k is a $\boldsymbol{C}_{\mathrm{lip}}\alpha$-Lipschitz graph over ℓ_i^k for $i \in \{1,2\}$.
- γ_j^l is a $\boldsymbol{C}_{\mathrm{lip}}\alpha$-Lipschitz graph over ℓ_j^l for $j \in \{1,2,3\}$.
- The lines are chosen so that

  $$G_j^k \subset \ell_i^k(C\alpha) \text{ for } i \in \{1,2\} \quad \text{and} \quad G_j^l \subset \ell_j^l(C\alpha) \quad \text{for } j \in \{1,2,3\},$$

  where $C \sim \boldsymbol{C}_{\mathrm{lip}}$.

We now define

$$(T_i^k)' := \ell_i^k(C\alpha) \quad \text{and} \quad T_i^k := \ell_i^k(\alpha^{1/2})$$

for $i \in \{1,2\}$, and similarly

$$(T_j^l)' := \ell_j^l(C\alpha) \quad \text{and} \quad T_j^l := \ell_j^l(\alpha^{1/2})$$

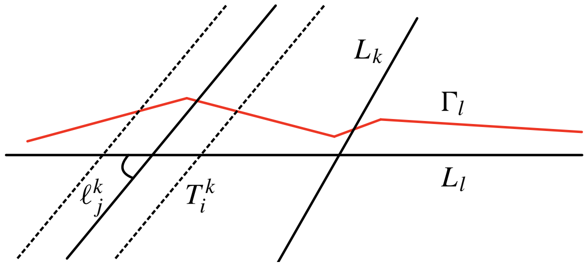

Figure 5. Transversality of T_i^k and Γ_l. The angle between ℓ_j^k and L_l is $\gtrsim C\alpha^\kappa$.

for $j \in \{1,2,3\}$. Thus, $G_i^k \subset (T_i^k)' \subset T_i^k$ and $G_j^l \subset (T_j^l)' \subset T_j^l$. Since moreover $\mathcal{H}^1(G_i^k) \gtrsim \alpha^3$ and $\mathcal{H}^1(G_j^l) \gtrsim \alpha^3$ by (5.16)–(5.17), *any* pair (G_i^k, G_j^l) (with associated lines and tubes) would now satisfy all the requirements of Lemma 5.3, except perhaps the inclusions (5.4).

We will now use the pigeonhole principle to show that at least one of the pairs (G_i^k, G_j^l) also satisfies the inclusions (5.4). The main geometric observation is

$$\operatorname{diam}(T_i^k \cap \Gamma_l) \lesssim \alpha^{1/2-\kappa} \quad \text{and} \quad \operatorname{diam}(T_j^l \cap \Gamma_k) \lesssim \alpha^{1/2-\kappa}. \tag{5.19}$$

The first inequality holds for $i \in \{1,2\}$, the second for $j \in \{1,2,3\}$. The proof of (5.19) is contained in Figure 5. Recall that T_i^k is an $\alpha^{1/2}$-tube around a certain line ℓ_i^k with $\angle(\ell_i^k, L_k) \le \alpha^\kappa$. On the other hand, $\angle(L_k, L_l) \ge \boldsymbol{C}_{\mathrm{sep}}\alpha^\kappa$, so also $\angle(\ell_i^k, L_l) \ge (\boldsymbol{C}_{\mathrm{sep}} - 1)\alpha^\kappa$. Finally, Γ_l is a $\boldsymbol{C}_{\mathrm{lip}}\alpha^\kappa$-Lipschitz graph over L_l, so every tangent of Γ_l makes an angle $\gtrsim \boldsymbol{C}_{\mathrm{sep}}\alpha^\kappa$ with ℓ_i^k, since we chose $\boldsymbol{C}_{\mathrm{sep}}$ much larger than $\boldsymbol{C}_{\mathrm{lip}}$ in Section 3C. Thus Γ_l is an $\alpha^{-\kappa}$-Lipschitz graph over $(\ell_j^k)^\perp$. It follows that

$$\operatorname{diam}(T_i^k \cap \Gamma_l) \le \mathcal{H}^1(T_i^k \cap \Gamma_l) \lesssim \alpha^{1/2-\kappa}.$$

Now that we have proved (5.19), recall from (5.15) the three balls $B(y_j, \alpha)$, all of which were centered at $y_j \in F_l \subset \Gamma_l$, and whose centers y_j had pairwise separation $\gtrsim \alpha^{2\kappa}$. Since $\kappa = \frac{1}{10}$, we have $\alpha^{1/2-\kappa} \ll \alpha^{2\kappa}$ for $\alpha > 0$ small enough (or in other words assuming that the constant $\boldsymbol{C}_{\mathrm{alp}} > 0$ is chosen large enough), and therefore (5.19) implies that

$$\#\{j \in \{1,2,3\} : T_i^k \cap B(y_j, \alpha) \ne \varnothing\} \le 1, \quad i \in \{1,2\}. \tag{5.20}$$

By a similar argument,

$$\#\{i \in \{1,2\} : T_j^l \cap B(x_i, \alpha) \ne \varnothing\} \le 1, \quad j \in \{1,2,3\}. \tag{5.21}$$

We finally claim, as a consequence of (5.20)- (5.21) and the pigeonhole principle, that there exists a pair of balls $(B(x_{i_0}, \alpha), B(y_{j_0}, \alpha))$, for some $i_0 \in \{1,2\}$ and $j_0 \in \{1,2,3\}$ with the property

$$T_{i_0}^k \cap B(y_{j_0}, \alpha) = \varnothing \quad \text{and} \quad T_{j_0}^l \cap B(x_{i_0}, \alpha) = \varnothing. \tag{5.22}$$

This, by definition, yields

$$G_{i_0}^k \overset{(5.18)}{\subset} B(x_{i_0}, \alpha) \setminus T_{j_0}^l \quad \text{and} \quad G_{j_0}^l \overset{(5.18)}{\subset} B(y_{j_0}, \alpha) \setminus T_{i_0}^k,$$

which (combined with (5.18)) completes the proof of the inclusions (5.4), and Lemma 5.3.

To prove (5.22), consider the bipartite graph with 5 vertices $\{v_1, v_2\} \cup \{w_1, w_2, w_3\}$ and the following edge set:

- For $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$, the edge (v_i, w_j) is included if $T_i^k \cap B(y_j, \alpha) \neq \varnothing$.
- For $j \in \{1, 2, 3\}$ and $i \in \{1, 2\}$, the edge (w_j, v_i) is included if $T_j^l \cap B(x_i, \alpha) \neq \varnothing$.

Now, (5.20)–(5.21) can be restated as follows: for v_i fixed, there can be at most one edge (v_i, w_j), and for w_i fixed, there can be at most one edge (w_i, v_j). Thus, the edge set contains at most five edges. On the other hand, the product set $\{v_1, v_2\} \times \{w_1, w_2, w_3\}$ contains six elements, so there must be a pair $\{v_i, w_j\}$ so that neither (v_i, w_j) nor (w_j, v_i) lies in the edge set. This is equivalent to (5.22). This completes the proof of Lemma 5.3.

6. The grid example

In this section we provide an example showing that Theorem 1.1 is optimal in the sense that the assumption $\mathrm{Fav}(E) \geq \mathrm{Fav}(L) - \delta$ cannot be relaxed to $\mathrm{Fav}(E) \geq \delta$.

Proposition 6.1. *There exists an absolute constant $\delta > 0$ and a sequence of compact rectifiable sets $E_n \subset [0, 1]^2 \subset \mathbb{R}^2$ such that*

(1) $\mathcal{H}^1(E_n) = 1$,

(2) $\mathrm{Fav}(E_n) \geq \delta$,

(3) *for any $\alpha \in [2n^{-2}, 1)$ and any curve Γ with $\mathcal{H}^1(\Gamma \cap E_n) \geq \alpha$ we have $\mathcal{H}^1(\Gamma) \gtrsim \alpha n$.*

In particular, property (3) *implies that if $M \geq 1$, then for any M-Lipschitz graph Γ, $\mathcal{H}^1(\Gamma \cap E_n) \lesssim M n^{-1}$.*

We begin the construction. Fix an integer $n \geq 2$, and let $[n] := \{1, \dots, n\}$. For any $j = (k, l) \in [n]^2$ set

$$x_j = \left(\frac{k}{n+1}, \frac{l}{n+1}\right) \tag{6.2}$$

and

$$B_j = B\left(x_j, \frac{1}{2\pi n^2}\right).$$

Note that $B_j \subset [0, 1]^2$ and if $i, j \in [n]^2$, $i \neq j$, then

$$\mathrm{dist}(B_i, B_j) \geq \frac{1}{n+1} - \frac{2}{2\pi n^2} \geq \frac{1}{2n}. \tag{6.3}$$

Define $S_j = \partial B_j$, and observe that $\mathcal{H}^1(S_j) = n^{-2}$.

We define the set E_n as

$$E_n := \bigcup_{j \in [n]^2} S_j.$$

Since $\mathcal{H}^1(S_j) = n^{-2}$, we have $\mathcal{H}^1(E_n) = 1$. This verifies property (1) for E_n. It is also clear that E_n is compact and rectifiable.

Now we check property (3). We will use the following result:

Lemma 6.4 [Schul 2007, Lemma 3.7]. *Any compact connected set $\Gamma \subset \mathbb{R}^2$ with $\mathcal{H}^1(\Gamma) < \infty$ can be parametrized with $\gamma : [0, 1] \to \mathbb{R}^2$ such that $\gamma([0, 1]) = \Gamma$ and* $\mathrm{Lip}(\gamma) \le 32\,\mathcal{H}^1(\Gamma)$.

Lemma 6.5. *For any $\alpha \in [2n^{-2}, 1)$ and any curve Γ with $\mathcal{H}^1(\Gamma \cap E_n) \ge \alpha$ we have $\mathcal{H}^1(\Gamma) \gtrsim \alpha n$.*

Proof. Suppose that $\alpha \in [2n^{-2}, 1)$ and let Γ be a curve with $\mathcal{H}^1(\Gamma \cap E_n) \ge \alpha$. Since each circle S_j comprising E_n has length n^{-2}, we get that Γ intersects at least αn^2 different circles. Let $J_0 \subset [n]^2$ be the set of indices such that for $j \in J_0$ we have $\Gamma \cap S_j \neq \varnothing$, so that

$$N := \#J_0 \ge \alpha n^2. \tag{6.6}$$

To estimate $\mathcal{H}^1(\Gamma)$, we are going to use (6.6) together with the fact that the circles S_j are centered on a well-separated grid (6.2), (6.3). We provide the details below:

Let γ be the parametrization of the curve Γ given by Lemma 6.4. Without loss of generality, we may assume that the curve Γ begins and ends on E_n, i.e., $\gamma(0), \gamma(1) \in \Gamma \cap E_n$. For all $j \in J_0$ we choose a point $y_j \in \Gamma \cap S_j$, and let $t_j \in [0, 1]$ be such that $\gamma(t_j) = y_j$ (γ might be noninjective, in which case t_j is nonunique, but in this case we pick t_j arbitrarily among the admissible options). The only constraint we make on our choice of $\{y_j\}_{j \in J_0}$ is so that $\gamma(0), \gamma(1) \in \{y_j\}_{j \in J_0}$. For convenience, we relabel the points t_j in "ascending order": for all $i \in \{1, \dots, N\}$ we set $t_i := t_j$ for some $j \in J_0$, in such a way that $t_1 < t_2 < \dots < t_N$. We relabel in a similar way y_j and S_j.

Recalling that the circles S_j are centered on a grid (6.2), it follows from the separation property (6.3) that, for any $i \in \{1, \dots, N\}$,

$$\frac{1}{2n} \le |y_{i+1} - y_i| = |\gamma(t_{i+1}) - \gamma(t_i)| \le \mathrm{Lip}(\gamma) \cdot |t_{i+1} - t_i| = \mathrm{Lip}(\gamma) \cdot (t_{i+1} - t_i).$$

Summing over $i \in \{1, \dots, N-1\}$ we get

$$\frac{N-1}{2n} \le \mathrm{Lip}(\gamma) \cdot (t_N - t_1) \le 32\,\mathcal{H}^1(\Gamma) \cdot (t_N - t_1).$$

Since we assumed $\gamma(0), \gamma(1) \in \{y_j\}_{j \in J_0}$, we get that $t_N = 1$ and $t_1 = 0$. Thus,

$$32\,\mathcal{H}^1(\Gamma) \ge \frac{N-1}{2n} \overset{(6.6)}{\ge} \frac{\alpha n^2 - 1}{2n} \ge \frac{\alpha n}{4}.$$

This completes the proof of the lemma. $\square$

It remains to prove the property (2), that is, $\mathrm{Fav}(E_n) \ge \delta$. Let

$$G_n = \bigcup_{j \in [n]^2} B_j,$$

so that $E_n = \partial G_n$. Note that $\mathrm{Fav}(E_n) = \mathrm{Fav}(G_n)$. We define an auxiliary measure

$$\mu = \mu_n = \frac{1}{\mathcal{L}^2(G_n)} \mathcal{L}^2|_{G_n}.$$

Recall that the 1-energy of μ is defined as

$$I_1(\mu) = \iint \frac{1}{|x-y|}\, d\mu(x) d\mu(y).$$

Lemma 6.7. *We have*

$$I_1(\mu) \lesssim 1.$$

As a consequence,

$$\mathrm{Fav}(E_n) = \mathrm{Fav}(G_n) \gtrsim 1. \tag{6.8}$$

Proof. We write

$$\begin{aligned}
I_1(\mu) &= \iint \frac{1}{|x-y|}\, d\mu(x) d\mu(y) \\
&= \sum_{i,j\in[n]^2} \int_{B_i}\int_{B_j} \frac{1}{|x-y|}\, d\mu(x) d\mu(y) \\
&= \sum_{i\in[n]^2} \int_{B_i}\int_{B_i} \frac{1}{|x-y|}\, d\mu(x) d\mu(y) + \sum_{\substack{i,j\in[n]^2\\ i\neq j}} \int_{B_i}\int_{B_j} \frac{1}{|x-y|}\, d\mu(x) d\mu(y) \\
&= A_1 + A_2.
\end{aligned}$$

To estimate A_1 we note that for any $i \in [n]^2$ and any fixed $x \in B_i$

$$\begin{aligned}
\int_{B_i} \frac{1}{|x-y|}\, d\mu(y) &\leq \sum_{k=\lfloor \log_2 n^2 \rfloor}^{\infty} \int_{B(x,2^{-k})\setminus B(x,2^{-k-1})} \frac{1}{|x-y|}\, d\mu(y) \\
&\sim \sum_{k=\lfloor \log_2 n^2 \rfloor}^{\infty} 2^k \mu(B(x,2^{-k}) \setminus B(x,2^{-k-1})) \\
&\lesssim \frac{1}{\mathcal{L}^2(G_n)} \sum_{k=\lfloor \log_2 n^2 \rfloor}^{\infty} 2^k \mathcal{L}^2(B(x,2^{-k})) \sim n^2 \sum_{k=\lfloor \log_2 n^2 \rfloor}^{\infty} 2^k \cdot 2^{-2k} \sim 1.
\end{aligned}$$

Hence,

$$A_1 = \sum_{i\in[n]^2} \int_{B_i}\int_{B_i} \frac{1}{|x-y|}\, d\mu(x)\, d\mu(y) \lesssim \sum_{i\in[n]^2} \mu(B_i) = 1.$$

We move on to estimating A_2. Let Q_j denote the square centered at x_j with sidelength $1/(n+1)$. Note that $B_j \subset Q_j$, and the squares Q_j, $j \in [n]^2$ are pairwise disjoint. If $x \in B_i$ and $y \in B_j$, with $i \neq j$, then $|x-y| \sim \mathrm{dist}(B_i, B_j) \sim |x-z|$ for any $z \in Q_j$. It follows that for a fixed $x \in B_i$

$$\int_{B_j} \frac{1}{|x-y|}\, d\mu(y) \sim \mathrm{dist}(B_i,B_j)^{-1}\, \mu(B_j) \sim \mathrm{dist}(B_i,B_j)^{-1}\, \mathcal{L}^2(Q_j) \sim \int_{Q_j} \frac{1}{|x-z|}\, d\mathcal{L}^2(z).$$

Summing over $j \in [n]^2 \setminus \{i\}$ yields

$$\begin{aligned}
\sum_{j\in[n]^2\setminus\{i\}} \int_{B_j} \frac{1}{|x-y|}\, d\mu(y) \sim \sum_{j\in[n]^2\setminus\{i\}} \int_{Q_j} \frac{1}{|x-z|}\, d\mathcal{L}^2(z) &\leq \int_{[-1,2]^2} \frac{1}{|x-z|}\, d\mathcal{L}^2(z) \\
&\lesssim \sum_{k=-1}^{\infty} \int_{B(x,2^{-k})\setminus B(x,2^{-k-1})} 2^k\, d\mathcal{L}^2(z) \lesssim 1.
\end{aligned}$$

Thus,

$$A_2 = \sum_{i\in[n]^2} \int_{B_i} \Big(\sum_{j\in[n]^2\setminus\{i\}} \int_{B_j} \frac{1}{|x-y|}\, d\mu(y) \Big)\, d\mu(x) \lesssim \sum_{i\in[n]^2} \mu(B_i) = 1.$$

It follows that $I_1(\mu) \lesssim 1$.

To see (6.8), we use Theorem 4.3 from [Mattila 2015] to conclude that

$$\mathrm{Fav}(E_n) = \mathrm{Fav}(G_n) \gtrsim \frac{1}{I_1(\mu)} \gtrsim 1.$$

This concludes the proof of Proposition 6.1. □

Appendix: Lines spanned by rectifiable curves

We state and prove a generalization of (5.11), which was mentioned in Remark 5.10:

Lemma A.1. *Let $\gamma_1, \gamma_2 \subset \mathbb{R}^2$ be rectifiable curves. For $\mathcal{H}^1$ almost every $x \in \gamma_i$, let $\tau_i(x)$ denote the unit tangent vector to γ_i at x. (The choice of direction is irrelevant.) Then for any $G_1 \subset \gamma_1$ and $G_2 \subset \gamma_2$, we have*

$$\int_{\mathcal{A}} \#\{(x_1, x_2) \in G_1 \times G_2 : x_1 \neq x_2 \text{ and } x_1, x_2 \in \ell\}\, d\eta(\ell)$$
$$= \iint_{G_1 \times G_2} \frac{|\pi_{\theta(x_1,x_2)}(\tau_1(x_1))|\,|\pi_{\theta(x_1,x_2)}(\tau_2(x_2))|}{|x_1 - x_2|}\, d(\mathcal{H}^1 \times \mathcal{H}^1)(x_1, x_2),$$

where $\theta(x_1, x_2)$ denotes the angle θ such that $\pi_\theta(x_1) = \pi_\theta(x_2)$.

Proof. Let $\phi_i(s)$ be a parametrization of γ_i by arclength. Consider the map $\Psi : (s_1, s_2) \mapsto (\theta, t)$ defined implicitly by

$$\pi_\theta(\phi_1(s_1)) = \pi_\theta(\phi_2(s_2)) = t. \tag{A.2}$$

By the change of variables formula,

$$\int_{\mathcal{A}} \#\{(x_1, x_2) \in G_1 \times G_2 : x_1 \neq x_2 \text{ and } x_1, x_2 \in \ell\}\, d\eta(\ell)$$
$$= \int_{[0,\pi]\times\mathbb{R}} \#\{(x_1, x_2) \in G_1 \times G_2 : x_1 \neq x_2 \text{ and } x_1, x_2 \in \pi_\theta^{-1}(t)\}\, d\mathcal{H}^2(\theta, t)$$
$$= \iint_{s_1 \in \phi_1^{-1}(G_1),\, s_2 \in \phi_2^{-1}(G_2)} J\Psi(s_1, s_2)\, ds_1\, ds_2,$$

where $J\Psi$ denotes the Jacobian determinant of Ψ. (Note that the set $\{(s_1, s_2) : \phi_1(s_1) = \phi_2(s_2)\}$ has $\mathcal{H}^2$-measure zero.)

We now prove that

$$J\Psi(s_1, s_2) := \mathrm{abs}\begin{vmatrix} \partial_{s_1}\theta & \partial_{s_2}\theta \\ \partial_{s_1}t & \partial_{s_2}t \end{vmatrix} = \frac{|\pi_{\theta(s_1,s_2)}(\gamma_1'(s_1))|\,|\pi_{\theta(s_1,s_2)}(\gamma_2'(s_2))|}{|\gamma_1(s_1) - \gamma_2(s_2)|}. \tag{A.3}$$

Note that this would finish the proof of the lemma. To show (A.3), define $e_\theta = (\cos\theta, \sin\theta)$ and $e_\theta^\perp = d/d\theta\, e_\theta = (-\sin\theta, \cos\theta)$. By differentiating (A.2) with respect to s_1 and s_2, we obtain

$$e_\theta \cdot \phi_1'(s_1) + e_\theta^\perp \cdot \phi_1(s_1)\partial_{s_1}\theta = e_\theta^\perp \cdot \phi_2(s_2)\partial_{s_1}\theta = \partial_{s_1}t,$$
$$e_\theta \cdot \phi_2'(s_2) + e_\theta^\perp \cdot \phi_2(s_2)\partial_{s_2}\theta = e_\theta^\perp \cdot \phi_1(s_1)\partial_{s_2}\theta = \partial_{s_2}t.$$

The two equalities on the left give

$$|\partial_{s_i}\theta| = \frac{|e_\theta \cdot \phi_i'(s_i)|}{|e_\theta^\perp \cdot (\phi_1(s_1) - \phi_2(s_2))|} \quad \text{for } i = 1, 2,$$

which, when combined with the two equalities on the right, give

$$J\Psi(s_1, s_2) = |\partial_{s_1}\theta|\,|\partial_{s_2}\theta|\,|e_\theta^\perp \cdot (\phi_1(s_1) - \phi_2(s_2))| = \frac{|e_\theta \cdot \phi_1'(s_1)|\,|e_\theta \cdot \phi_2'(s_2)|}{|e_\theta^\perp \cdot (\phi_1(s_1) - \phi_2(s_2))|}.$$

Finally, observe that $e_\theta \cdot (\phi_1(s_1) - \phi_2(s_2)) = 0$ by the definition of Ψ, which implies $|e_\theta^\perp \cdot (\phi_1(s_1) - \phi_2(s_2))| = |\phi_1(s_1) - \phi_2(s_2)|$. This completes the proof of (A.3). □

By using the coarea formula for rectifiable sets (e.g., [Krantz and Parks 2008, Theorem 5.4.9]), it is not hard to show that Lemma A.1 can be generalized to Lemma A.4, below. We omit the details.

Lemma A.4. *Let $E \subset \mathbb{R}^2$ be a 1-rectifiable set. For $\mathcal{H}^1$ almost every $x \in E$, let $\tau(x)$ denote the unit tangent vector to E at x. (The choice of direction is irrelevant.) Then for any $G \subset (E \times E) \setminus \{(x, x) : x \in E\}$, we have*

$$\int_{\mathcal{A}} \#\{(x_1, x_2) \in G : x_1, x_2 \in \ell\}\, d\eta(\ell) = \iint_G \frac{|\pi_{\theta(x_1,x_2)}(\tau(x_1))|\,|\pi_{\theta(x_1,x_2)}(\tau(x_2))|}{|x_1 - x_2|}\, d(\mathcal{H}^1 \times \mathcal{H}^1)(x_1, x_2), \qquad \text{(A.5)}$$

where $\theta(x_1, x_2)$ denotes the angle θ such that $\pi_\theta(x_1) = \pi_\theta(x_2)$.

A version of Lemma A.4 was discovered independently by Steinerberger [2024]; see the sixth displayed equation in Section 1.2.

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