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STRONG COSMIC CENSORSHIP IN THE PRESENCE OF MATTER: THE DECISIVE EFFECT OF HORIZON OSCILLATIONS ON THE BLACK HOLE INTERIOR GEOMETRY

CHRISTOPH KEHLE AND MAXIME VAN DE MOORTEL

Motivated by the strong cosmic censorship conjecture in the presence of matter, we study the Einstein equations coupled with a charged/massive scalar field with spherically symmetric characteristic data relaxing to a Reissner–Nordström event horizon. Contrary to the vacuum case, the relaxation rate is conjectured to be *slow* (nonintegrable), opening the possibility that the matter fields and the metric coefficients *blow up in amplitude* at the Cauchy horizon, not just in energy. We show that whether this blow-up in amplitude occurs or not depends on a novel *oscillation condition* on the event horizon which determines whether or not a resonance is excited dynamically:

- If the oscillation condition is satisfied, then the resonance is not excited and we show boundedness and continuous extendibility of the matter fields and the metric across the Cauchy horizon.
- If the oscillation condition is violated, then by the *combined effect of slow decay and the resonance being excited*, we show that the massive uncharged scalar field blows up in amplitude.
  In a companion paper, we will show that in that case a novel *null contraction singularity* forms at the Cauchy horizon, across which the metric is not continuously extendible in the usual sense.

Heuristic arguments in the physics literature indicate that the oscillation condition should be satisfied generically on the event horizon. If these heuristics are true, then *our result falsifies the C^0-formulation of strong cosmic censorship by means of oscillation*.

1. Introduction

Is general relativity a deterministic theory? This fundamental question can only be addressed in the context of the initial value problem for the Einstein equations (see (1-1)), which govern the dynamics of spacetime in general relativity. Well-posedness for the initial value problem was established in [Choquet-Bruhat and Geroch 1969] (see also [Fourès-Bruhat 1952]), proving that any suitably regular Cauchy data admit a unique maximal future development, the so-called *maximal globally hyperbolic development* (MGHD). With this dynamical formulation at hand, general relativity can be considered deterministic if the MGHD of *generic* Cauchy data for the Einstein equations is inextendible. The genericity stipulation is clearly necessary because the MGHD of Kerr Cauchy data [1963] (rotating black holes) and of Reissner–Nordström Cauchy data [Reissner 1916; Nordström 1918] (their charged analogs) admit a future boundary, the Cauchy horizon, across which the metric is smoothly extendible. Heuristics of Penrose [1968] however suggest the instability of the Kerr/Reissner–Nordström Cauchy horizons and these led him to his famous

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strong cosmic censorship conjecture [Penrose 1974] supporting the idea of determinism in general relativity. The most definitive and perhaps most desirable formulation of Penrose's strong cosmic censorship is the conjecture that the metric coefficients cannot be extended as continuous functions, namely:

Conjecture 1 (C^0-formulation of strong cosmic censorship). *The MGHD of generic asymptotically flat Cauchy data is inextendible as a continuous Lorentzian metric* (*we say the metric is C^0-inextendible*).

Conjecture 1 is related to the expectation that physical observers approaching the boundary of the MGHD of generic Cauchy data are destroyed. If Conjecture 1 is false, then one may still be able to prove a weaker version of inextendibility, but this would correspond to a weaker version of determinism.

Conjecture 1 is false in the absence of matter. In the celebrated work [Dafermos and Luk 2017], it is proved that, in vacuum, small perturbations of Kerr still admit a Cauchy horizon across which the spacetime is C^0-extendible — thus falsifying Conjecture 1 in the absence of matter. The key ingredient to their proof is an *integrable* inverse polynomial rate assumption for the decay of perturbations along the event horizon. Note, however, that a weaker H^1-formulation is still expected to hold [Christodoulou 2009; Dafermos and Luk 2017; Van de Moortel 2021]. If true, this would restore determinism at least in a weaker sense.

Can Conjecture 1 be salvaged in the presence of matter? In the present paper, we consider a nonvacuum model: the Einstein–Maxwell–Klein–Gordon (1-1)–(1-5) system in spherical symmetry governing the dynamics of gravitation coupled to a charged/massive scalar field. Arguments in the physics literature [Hod and Piran 1998; Koyama and Tomimatsu 2001; Konoplya and Zhidenko 2013; Burko and Khanna 2004; Oren and Piran 2003] suggest that perturbations of the exterior of Reissner–Nordström in this model settle down merely at a slow, *nonintegrable* rate (at least for massive and/or strongly charged perturbations), which is in stark contrast to the perturbations of Kerr in the vacuum case. As such, the methods of [Dafermos and Luk 2017] manifestly do not apply and the slow decay of perturbations may even raise hopes that for generic Cauchy data the metric is C^0-inextendible and thus, Conjecture 1 would be true after all for this matter model.

The question of C^0-extendibility across a future null boundary $\mathcal{CH}_{i^+}$. At first, it may appear that the slow decay in the above matter model in fact opens the possibility of a more drastic scenario where the singularity is everywhere spacelike inside the black hole. Notwithstanding, it was proven in [Van de Moortel 2018] that for this model black holes are bound to the future by a null boundary $\mathcal{CH}_{i^+} \neq \varnothing$ as depicted in Figure 1. We will continue using the term "Cauchy horizon" for $\mathcal{CH}_{i^+}$ by analogy with the Cauchy horizon of Reissner–Nordström, although the spacetime may or may not be C^0-extendible across the null boundary $\mathcal{CH}_{i^+}$. Therefore, although the future boundary is null and in particular not spacelike, the question of C^0-extendibility of the spacetime across $\mathcal{CH}_{i^+}$, i.e., Conjecture 1, remains open. This is the question that we shall now address.

Summary of our results. As we will show, the question of Conjecture 1 becomes unexpectedly subtle: In addition to the decay rates of perturbations on the exterior, it turns out that the validity of Conjecture 1 depends crucially on Fourier support properties of late-time perturbations due to a scattering resonance associated to the Cauchy horizon $\mathcal{CH}_{i^+}$. In our main Theorem I (i) we identify an *oscillation condition* on perturbations along the event horizon $\mathcal{H}^+$: *If the oscillation condition is satisfied by the perturbation*,

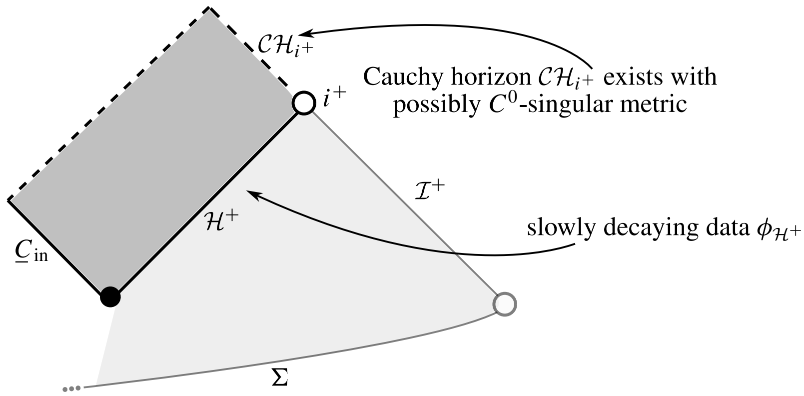

Figure 1. A Cauchy horizon $\mathcal{CH}_{i^+}$ exists for slowly decaying perturbations $\phi_{\mathcal{H}^+}$ as proven in [Van de Moortel 2018]; see Theorem A.

we show boundedness and continuous extendibility of the matter fields and the metric across the Cauchy horizon $\mathcal{CH}_{i^+}$ despite the obstruction created by slow decay. On the other hand, in Theorem I (ii) we show that *if the oscillation condition is violated on the event horizon* $\mathcal{H}^+$, *the resonance is excited and the uncharged scalar field blows up in amplitude*, namely $|\phi| \to +\infty$ at the Cauchy horizon $\mathcal{CH}_{i^+}$.

Heuristic and numerical arguments in the physics literature [Hod and Piran 1998; Koyama and Tomimatsu 2001; Konoplya and Zhidenko 2013; Burko and Khanna 2004; Oren and Piran 2003] suggest that the oscillation condition is indeed satisfied on $\mathcal{H}^+$ for generic perturbations of the black hole exterior. Assuming this, our result Theorem II *falsifies the* C^0*-formulation of strong cosmic censorship by means of oscillation*.

In Theorem III, we show that for both oscillating and nonoscillating perturbations,[1] *the scalar field blows up in the* $W^{1,1}_{\text{loc}}$*-norm at the Cauchy horizon* $\mathcal{CH}_{i^+}$, i.e., $\int |D_v\phi|\, \mathrm{d}v = +\infty$ schematically. This $W^{1,1}$ blow-up is in contrast to the vacuum case where the analogous statement is false [Dafermos and Luk 2017]. This shows that for both oscillating and nonoscillating perturbations, the Cauchy horizon $\mathcal{CH}_{i^+}$ is more singular in the presence of matter than in vacuum. Moreover, the blow-up of the scalar field in $W^{1,1}$ indicates that our result cannot be captured using only physical space techniques which have been used previously.

Finally, in our companion paper [Kehle and Van de Moortel $\geq$ 2024] we will prove Theorem IV, which shows that blow-up in amplitude of the scalar field indeed gives rise to a C^0-inextendibility statement on the metric within a spherically symmetric class. Theorem IV, in conjunction with Theorem I (ii), provides the first example of a dynamically formed singularity leading to a C^0-inextendibility statement of the metric across a null spacetime boundary (albeit within a restricted spherically symmetric class). Whether this statement can be upgraded to the full C^0-inextendibility of the spacetime remains open.[2]

Similarities with the $\Lambda < 0$ case. In the asymptotically AdS case ($\Lambda < 0$), solutions to the linear wave equation on AdS black holes also decay at a slow, nonintegrable rate [Holzegel and Smulevici 2014]. It turns out that in this context, oscillations also play a crucial role [Kehle 2020b; 2022] in addressing

[1]Up to a genericity condition in the charged scalar field case, which we can get rid of in the uncharged case; see Theorem III.

[2]Unrestricted C^0-inextendibility results (even for spacelike singularities) are known to be notoriously difficult to show; see, e.g., [Sbierski 2018] for the proof of C^0-inextendibility of the Schwarzschild solution across the spacelike singularity $\{r = 0\}$.

the question of the validity of the linear analog of Conjecture 1. The slow inverse logarithmic decay in the $\Lambda < 0$ case however arises from the superposition of infinitely many high ℓ angular modes. This is different from the present problem for $\Lambda = 0$, where the slow decay is inverse-polynomial (see Section 1A) and already occurs in spherical symmetry.

Outline of the Introduction. In Section 1A we introduce the Einstein–Maxwell–Klein–Gordon system and give a more detailed overview of our new results addressing the issue of strong cosmic censorship within this matter model in spherical symmetry. Further, we present a first version of our main theorems. In Section 1B we outline the important differences between the EMKG model and other models regarding the existence of a Cauchy horizon and the continuous extendibility of the metric. In Section 1C we mention previous results on the dynamical formation of weak null singularities at the Cauchy horizon, which we compare to the new singularities that dynamically form in our setting. In Section 1D we present previous results on scattering inside Reissner–Nordström black holes which are important for our proof. In Section 1E we elaborate on the interior of black holes with $\Lambda < 0$, in which oscillations turn out to play an important role as well. In Section 1F we briefly discuss the strategy of the proof.

1A. *Main results: first versions.*

1A1. *The EMKG system and existence of a Cauchy horizon for slowly decaying scalar fields.*

The EMKG model in spherical symmetry. We study the Einstein equations coupled to a charged massive scalar field: the Einstein–Maxwell–Klein–Gordon (EMKG) model in spherical symmetry

$$\mathrm{Ric}_{\mu\nu}(g) - \tfrac{1}{2}R(g)g_{\mu\nu} = \mathbb{T}^{EM}_{\mu\nu} + \mathbb{T}^{KG}_{\mu\nu}, \tag{1-1}$$

$$\mathbb{T}^{EM}_{\mu\nu} = 2\big(g^{\alpha\beta}F_{\alpha\nu}F_{\beta\mu} - \tfrac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu}\big), \tag{1-2}$$

$$\mathbb{T}^{KG}_{\mu\nu} = 2\big(\Re(D_\mu\phi\overline{D_\nu\phi}) - \tfrac{1}{2}(g^{\alpha\beta}D_\alpha\phi\overline{D_\beta\phi} + m^2|\phi|^2)g_{\mu\nu}\big), \tag{1-3}$$

$$\nabla^\mu F_{\mu\nu} = \frac{q_0}{2}i(\phi\overline{D_\nu\phi} - \bar{\phi}D_\nu\phi), \quad F = \mathrm{d}A, \tag{1-4}$$

$$g^{\mu\nu}D_\mu D_\nu\phi = m^2\phi, \quad D_\mu = \nabla_\mu + iq_0A_\mu \tag{1-5}$$

for a quintuplet $(\mathcal{M}, g, F, A, \phi)$, where $(\mathcal{M}, g)$ is a (3+1)-dimensional Lorentzian manifold, ϕ is a complex-valued scalar field, A is a real-valued 1-form, and F is a real-valued 2-form. Here $q_0 \in \mathbb{R}$ and $m \geq 0$ are fixed constants representing respectively the charge and the mass of the scalar field. The EMKG model describes self-gravitating matter and provides a setting for studying spherical gravitational collapse of charged and massive matter if $q_0 \neq 0$ and $m^2 \neq 0$ (see the discussion in Section 1C3). This model has attracted much attention in the literature [An and Lim 2022; Bizoń and Wasserman 2000; Dias et al. 2019; Kommemi 2013; Gajic and Luk 2019; Van de Moortel 2018; 2019; 2021; 2022]; see also [Yang and Yu 2019; Lindblad and Sterbenz 2006; Klainerman and Machedon 1994; Krieger et al. 2015; Oh and Tataru 2016; Rodnianski and Tao 2004] for work on the flat Minkowski background.

Setting of the problem. Consider the maximal globally hyperbolic development of suitably regular spherically symmetric Cauchy data prescribed on an asymptotically flat initial hypersurface Σ as depicted in Figure 1. General results for the EMKG model in spherical symmetry [Kommemi 2013] allow us to define null infinity $\mathcal{I}^+$ — a conformal boundary where idealized far away observers live, and the black hole interior region as the complement of the causal past of $\mathcal{I}^+$. If the black hole interior is nonempty, we also define the event horizon $\mathcal{H}^+$ as the past boundary of the black hole interior which separates the black hole interior from the black hole exterior.

In the current paper we will only be interested in the dynamics of the black hole interior. In particular, instead of studying the Cauchy problem with data on Σ, we will prescribe the scalar field ϕ and the metric on an ingoing cone $\underline{C}_{\text{in}}$ and on an outgoing cone $\mathcal{H}^+$ emulating the event horizon of an already-formed black hole. This setting corresponds to a characteristic initial value problem with data imposed on $\mathcal{H}^+ \cup \underline{C}_{\text{in}}$; see Figure 1. Our study of this characteristic initial value problem will be entirely self-contained. We will however continue to depict Σ on Figure 1 and subsequent figures for completeness. Our assumptions on the characteristic initial data on $\mathcal{H}^+ \cup \underline{C}_{\text{in}}$ will be made in accordance with the conjectured late-time tails on the event horizon $\mathcal{H}^+$ arising from generic Cauchy data on asymptotically flat Σ; see the discussion below.

Conjectured late-time asymptotics on the event horizon $\mathcal{H}^+$ and contrast with the vacuum case. Heuristic arguments regarding the black hole exterior in the physics literature (see [Hod and Piran 1998; Koyama and Tomimatsu 2001; Konoplya and Zhidenko 2013; Burko and Khanna 2004; Oren and Piran 2003]) indicate that (spherically symmetric) dynamical black holes arising from Cauchy data on Σ for the EMKG model relax to Reissner–Nordström along the event horizon $\mathcal{H}^+$ at a slow,[3] *nonintegrable* rate v^{-s}, $s \in \left(\frac{1}{2}, 1\right]$ for large v, in a standard Eddington–Finkelstein coordinate v. This is in contrast to the faster and integrable rate $s > 1$ proved in the uncharged massless case $m^2 = q_0 = 0$ [Dafermos and Rodnianski 2005], or assumed in vacuum in [Dafermos and Luk 2017] (see (1-21)). This fast, integrable rate v^{-s}, $s > 1$ in vacuum is indeed sufficient to prove the existence of a Cauchy horizon $\mathcal{CH}_{i^+}$, across which the spacetime is continuously extendible: this led to a *falsification of Conjecture 1 in vacuum without symmetry assumptions* [Dafermos and Luk 2017] (or for spherically symmetric models as in [Dafermos 2003; Van de Moortel 2018]); see Section 1B2.

Existence of a Cauchy horizon $\mathcal{CH}_{i^+}$ for slowly decaying scalar fields. Returning to the EMKG model, the first step in addressing Conjecture 1 is to understand whether, for slowly decaying characteristic data on the event horizon $\mathcal{H}^+$, the future boundary inside the black hole is null (a Cauchy horizon) or spacelike. In view of the slow decay on the event horizon $\mathcal{H}^+$, the spacelike singularity scenario is plausible and indeed desirable (if it were true, then Conjecture 1 would likely be valid). Despite the obstruction created by the slow decay of event horizon perturbations, it turns out however that the black hole future boundary has a nonempty null component $\mathcal{CH}_{i^+} \neq \varnothing$ emanating from i^+, see Figure 1, and is not everywhere spacelike as one might have hoped:

Theorem A ([Van de Moortel 2018], rough version; precise version recalled in Section 4A). *Consider spherically symmetric characteristic initial data for* (1-1)–(1-5) *on the event horizon* $\mathcal{H}^+$ (*and on an ingoing cone*). *Assume the following* ***slow decay*** *upper bound on the scalar field* $\phi_{\mathcal{H}^+}$ *on the event horizon* $\mathcal{H}^+ = [v_0, +\infty)$ *as*

$$|\phi_{\mathcal{H}^+}(v)| \leq C_0 v^{-s}, \quad |D_v \phi_{\mathcal{H}^+}| \leq C_0 v^{-s} \tag{1-6}$$

[3] Precisely, these slow rates hold conjecturally for a massive ($m^2 \neq 0$) scalar field and/or strongly charged $\left(|q_0 e| \geq \frac{1}{2}\right)$ one.

for all $v \geq v_0$ *in a standard Eddington–Finkelstein-type* v*-coordinate on* $\mathcal{H}^+ = [v_0, +\infty)$ *for some* $C_0 > 0$ *and some decay rate* $s > \frac{1}{2}$.

Then the spacetime is bound to the future by an ingoing null boundary $\mathcal{CH}_{i^+} \neq \varnothing$ (*the Cauchy horizon*) *foliated by spheres of positive radius and emanating from* i^+, *and the Penrose diagram is given by the dark gray region in Figure 1.*

Since by Theorem A the black hole future boundary is not everywhere spacelike and has a null component $\mathcal{CH}_{i^+} \neq \varnothing$, one may at first expect continuous extendibility across $\mathcal{CH}_{i^+}$. It turns out however that the spacetime of Theorem A may or may not be continuously extendible across $\mathcal{CH}_{i^+}$. This is perhaps unexpected, since all previous instances of black hole spacetimes with a null future boundary component are at least continuously extendible across that component [Dafermos and Luk 2017; Dafermos 2003; Luk and Oh 2019a]. Thus, Theorem A is not sufficient to fully address Conjecture 1 and the question of continuous extendibility across the null boundary $\mathcal{CH}_{i^+}$ has remained open.

The slow rate $s > \frac{1}{2}$ assumed in Theorem A is indeed *too slow* to prove the C^0-extendibility of spacetime across the Cauchy horizon $\mathcal{CH}_{i^+}$ using the same method as [Dafermos and Luk 2017] in vacuum. The method of [Dafermos and Luk 2017] requires the faster integrable decay assumption $s > 1$ and does not extend to the nonintegrable case $s \leq 1$, a failure that may even raise the attractive possibility that Conjecture 1 is true after all for the EMKG matter model. ***This could mean that determinism is in better shape in the presence of matter!***

1A2. *Theorem I*: *event horizon oscillations are decisive for the* C^0 *extendibility of the metric.* Our main result however shows that the situation is more subtle than one may first think: assuming that the scalar field ϕ oscillates sufficiently on the event horizon $\mathcal{H}^+$, we show in Theorem I (i) that ϕ *is uniformly bounded in the black hole interior and the metric is continuously extendible*. The event horizon oscillation assumption is sharp in the following sense: conversely assuming that the scalar field ϕ does not oscillate sufficiently on the event horizon $\mathcal{H}^+$, we show in Theorem I (ii) that ϕ *blows up in amplitude at the Cauchy horizon* $\mathcal{CH}_{i^+}$. It turns out that the oscillation condition on the event horizon $\mathcal{H}^+$, i.e., the main assumption of Theorem I (i), is conjecturally satisfied for generic Cauchy data on an asymptotically flat Σ, and thus, the *hope that determinism is in better shape in the presence of matter in the end does not come true*! (See Section 1A3.)

Theorems I (i) and I (ii) show that uniform boundedness or blow-up of the matter fields unexpectedly relies on fine properties of the scalar field ϕ on the event horizon $\mathcal{H}^+$ in both physical and Fourier space. At the heart of our novel oscillation condition lies the resonant frequency

$$\omega_{\mathrm{res}}(M, e, q_0) := \omega_-(M, e, q_0) - \omega_+(M, e, q_0), \tag{1-7}$$

where

$$\omega_- = \omega_-(M, e, q_0) := \frac{q_0 e}{r_-(M, e)}, \quad \omega_+ = \omega_+(M, e, q_0) := \frac{q_0 e}{r_+(M, e)}$$

for asymptotic black hole parameters $0 < |e| < M$.

In what follows we will give rough versions of Theorems I (i) and I (ii). For the precise versions we refer the reader to Sections 4B and 4C.

Theorem I (i) (boundedness (rough version; precise version in Section 4B)). *Consider spherically symmetric characteristic initial data for* (1-1)–(1-5) *on the event horizon* $\mathcal{H}^+$ (*and on an ingoing cone*). *Assume the following* ***slow decay*** *upper bound on the scalar field* $\phi_{\mathcal{H}^+}$ *on the event horizon* $\mathcal{H}^+ = [v_0, +\infty)$ *as*

$$|\phi_{\mathcal{H}^+}(v)| \leq C v^{-s}, \quad |D_v \phi_{\mathcal{H}^+}| \leq C v^{-s} \tag{1-8}$$

for all $v \geq v_0$ *in a standard Eddington–Finkelstein-type* v*-coordinate on* $\mathcal{H}^+ = [v_0, +\infty)$, *for* $v_0 > 1$ *sufficiently large and for some* $C > 0$ *and some* (*nonintegrable*) *decay rate*

$$\tfrac{3}{4} < s \leq 1. \tag{1-9}$$

By Theorem A, *the spacetime*, *i.e.*, *the dark gray region in Figure 1*, *is bound to the future by a null boundary* $\mathcal{CH}_{i^+} \neq \varnothing$ (*the Cauchy horizon*). *Then*, *in the gauge* $A_v = 0$, *the following hold true*:

- *If* $\phi_{\mathcal{H}^+}$ *satisfies the* ***qualitative oscillation condition*** *on* $\mathcal{H}^+ = [v_0, +\infty)$, *i.e.*, *if for all* $O(v^{1-2s})$ *functions*

$$\limsup_{\tilde{v} \to +\infty} \left| \int_{v_0}^{\tilde{v}} \phi_{\mathcal{H}^+}(v) e^{i\omega_{\mathrm{res}} v(1+O(v^{1-2s}))} \, \mathrm{d}v \right| < +\infty, \tag{1-10}$$

then the scalar field ϕ *is uniformly bounded in amplitude up to and including the Cauchy horizon* $\mathcal{CH}_{i^+}$.

- *If* $\phi_{\mathcal{H}^+}$ *satisfies the* ***strong qualitative oscillation condition*** *on* $\mathcal{H}^+ = [v_0, +\infty)$, *i.e.*, *if for all* $O(v^{1-2s})$ *functions*

$$\lim_{\tilde{v} \to +\infty} \left| \int_{v_0}^{\tilde{v}} \phi_{\mathcal{H}^+}(v) e^{i\omega_{\mathrm{res}} v(1+O(v^{1-2s}))} \, \mathrm{d}v \right| \text{ exists and is finite}, \tag{1-11}$$

then additionally the metric g *and the scalar field* ϕ *are continuously extendible across the Cauchy horizon* $\mathcal{CH}_{i^+}$.

- *If* $\phi_{\mathcal{H}^+}$ *satisfies the* ***quantitative oscillation condition*** *on* $\mathcal{H}^+ = [v_0, +\infty)$, *i.e.*, *if there exist* $E > 0$, $\epsilon > 1 - s$ *such that for all* $O(v^{1-2s})$ *functions*

$$\lim_{\tilde{v} \to +\infty} \left| \int_{v_1}^{\tilde{v}} \phi_{\mathcal{H}^+}(v) e^{i\omega_{\mathrm{res}} v(1+O(v^{1-2s}))} \, \mathrm{d}v \right| \leq E v_1^{-\epsilon} \quad \text{for all } v_1 \geq v_0, \tag{1-12}$$

then, *additionally the Maxwell field contraction* $F_{\mu\nu}F^{\mu\nu}$ *is uniformly bounded in amplitude and continuously extendible across the Cauchy horizon* $\mathcal{CH}_{i^+}$.

We refer to Figure 2 for an illustration of Theorem I (i).

In the uncharged case $q_0 = 0$, where $\omega_{\mathrm{res}} = 0$, we show that the qualitative oscillation condition (1-10) is sharp to obtain boundedness.

Theorem I (ii) (blow-up (rough version; precise version in Section 4C)). *Consider spherically symmetric characteristic initial data for* (1-1)–(1-5) *on the event horizon* $\mathcal{H}^+$ (*and on an ingoing cone*). *Assume the following* ***slow decay*** *upper bound on the scalar field on the event horizon* $\mathcal{H}^+$ (*i.e.*, $\phi_{\mathcal{H}^+}$ *satisfies* (1-8) *where* s *satisfies* (1-9)). *Assume additionally* $q_0 = 0$ *and let* $m^2 > 0$ *be generic.*

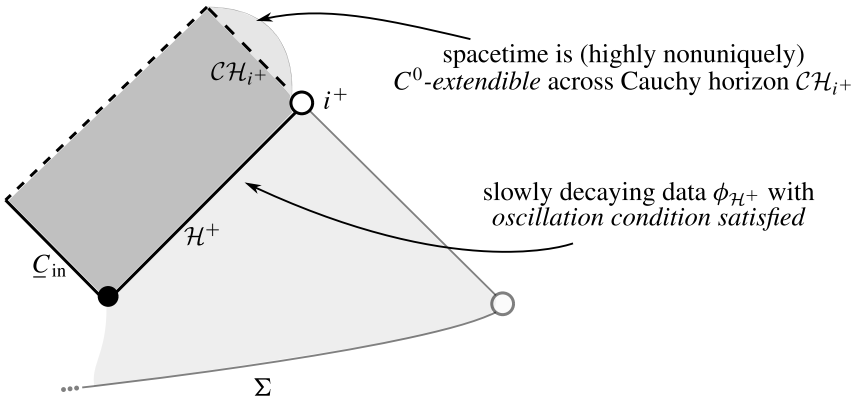

Figure 2. Theorem I (i): if the strong qualitative oscillation condition is satisfied, then the spacetime is C^0-extendible across the Cauchy horizon $\mathcal{CH}_{i^+}$.

Then, ϕ blows up in amplitude at every point on the Cauchy horizon $\mathcal{CH}_{i^+}$

$$\limsup_{(u,v)\to\mathcal{CH}_{i^+}} |\phi(u,v)| = +\infty \tag{1-13}$$

if and only if

$$\limsup_{\tilde{v}\to+\infty} \left| \int_{v_0}^{\tilde{v}} \phi_{\mathcal{H}^+}(v)\,\mathrm{d}v \right| = +\infty, \tag{1-14}$$

i.e., if and only if $\phi_{\mathcal{H}^+}$ violates the qualitative oscillation condition (1-10).

Further, in the case where the scalar field ϕ blows up at the Cauchy horizon $\mathcal{CH}_{i^+}$ as in (1-13), *a null contraction singularity forms at the Cauchy horizon $\mathcal{CH}_{i^+}$ as stated in Theorem IV and proved in* [Kehle and Van de Moortel ≥ 2024].

We refer to Figure 3 for an illustration of Theorem I (ii).

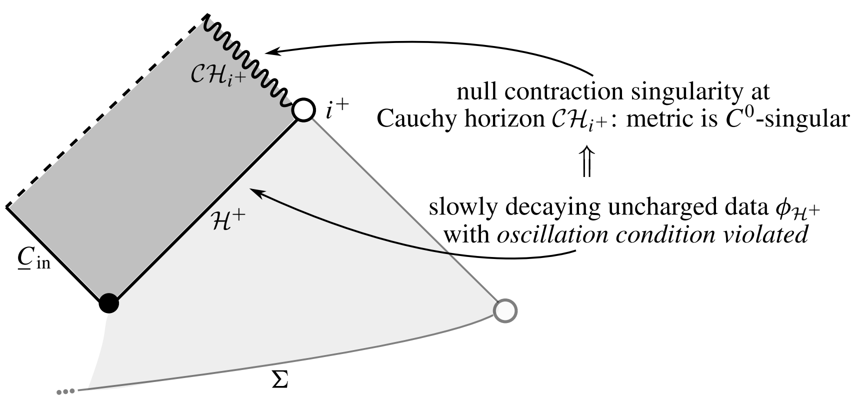

Figure 3. Theorem I (ii): If the oscillation condition is violated in the uncharged case, then a novel null contraction singularity forms at the Cauchy horizon $\mathcal{CH}_{i^+}$ and the metric is C^0-singular at $\mathcal{CH}_{i^+}$.

Theorem I (ii) also shows that it is impossible to prove boundedness of the scalar field ϕ only under the assumptions of Theorem A. This motivates a posteriori the introduction of the oscillation conditions (1-10), (1-11), (1-12), which are thus necessary to obtain boundedness and C^0 extendibility as claimed in Theorem I (i). Anticipating Section 1A5, we note that it is also impossible to prove the continuous extendibility of the metric in the usual sense only under the assumptions of Theorem A, by Theorem IV.

For concreteness, we will now give explicit examples of profiles $\phi_{\mathcal{H}^+}$ which satisfy (respectively violate) the oscillation condition (1-10), (1-11), (1-12) from above.

Example. For any fixed $\omega \neq \omega_{\text{res}}$ the profile $\phi_{\mathcal{H}^+} := e^{-i\omega v} v^{-s}$ satisfies the quantitative oscillation condition (1-12).

Nonexample. The profile $\phi_{\mathcal{H}^+} := e^{-i\omega_{\text{res}} v} v^{-s}$ violates the oscillation condition (1-10).

1A3. *Theorem II: the C^0-formulation of strong cosmic censorship is false.*

Slow decay on $\mathcal{H}^+$ for generic Cauchy data on Σ. We now return to Conjecture 1, which is formulated in terms of generic Cauchy data on an asymptotically flat Σ. First, the scalar field ϕ on the event horizon $\mathcal{H}^+$ is indeed expected to decay slowly for generic Cauchy data on Σ; i.e., $\phi_{\mathcal{H}^+}$ satisfies (1-8) only for $s \leq 1$, at least for almost every pair of parameters (m^2, q_0) (see Conjecture 2). This slow decay makes Theorems I (i) and I (ii) decisive to the study of Cauchy data on Σ as above, since the validity of Conjecture 1 now crucially depends on whether generic Cauchy data on Σ give rise to solutions for which the (slowly decaying) scalar field ϕ on the event horizon $\mathcal{H}^+$ satisfies or violates the oscillation condition (1-10) (or (1-11), its stronger analog).

Oscillations on $\mathcal{H}^+$ for generic Cauchy data on Σ. As it turns out, $\phi_{\mathcal{H}^+}$ is expected to satisfy the (even stronger) quantitative oscillation condition (1-12) for generic regular Cauchy data on Σ. This expectation is based on works in the physics literature relying on heuristic analysis [Hod and Piran 1998; Konoplya and Zhidenko 2013; Koyama and Tomimatsu 2001; 2002] or numerics [Burko and Khanna 2004; Oren and Piran 2003] giving precise asymptotic tails on the event horizon $\mathcal{H}^+$. We formulate this as the following conjecture, where $\phi_{\mathcal{H}^+}$ is the scalar field ϕ restricted to the event horizon $\mathcal{H}^+ = [v_0, +\infty)$, v is an Eddington–Finkelstein-type coordinate (see the gauge choice later defined in (3-6)), and electromagnetic gauge $A_v = 0$ (see (2-26)):

Conjecture 2. *Let $(\mathcal{M}, g, F, A, \phi)$ be a black hole solution of the system* (1-1)–(1-5) *arising from generic, spherically symmetric smooth Cauchy data on an asymptotically flat Σ. Then, the black hole exterior settles down to a Reissner–Nordström exterior with asymptotic mass M and asymptotic charge e satisfying $0 < |e| < M$. Moreover, the scalar field has the following late-time asymptotics on the event horizon $\mathcal{H}^+ = [v_0, +\infty)$:*

(1) *In the massive uncharged case, i.e., $m^2 > 0$, $q_0 = 0$,*

$$\phi_{\mathcal{H}^+}(v) = C(m \cdot M, D) \sin(mv + \omega_{\text{err}}(v)) \cdot v^{-5/6} + \phi_{\text{err}} \tag{1-15}$$

for fast decaying ϕ_{err} (i.e., ϕ_{err} satisfies (1-8) *for $s > 1$), a constant $C(m \cdot M, D) \neq 0$ depending on $m \cdot M$ and the initial data D, and a sublinear growing phase*

$$\omega_{\mathrm{err}}(v) = -\frac{3m}{2}(2\pi M)^{2/3} v^{1/3} + \omega(m \cdot M).$$

(2) *In the massless charged case, i.e., $m^2 = 0,\ q_0 \neq 0$,*

$$\phi_{\mathcal{H}^+}(v) = C_H(q_0 e, D) \cdot e^{(iq_0 e/r_+)v} \cdot v^{-1-\delta} + \phi_{\mathrm{err}}, \tag{1-16}$$

where $C_H(q_0 e, D) \neq 0$ is a constant depending on $q_0 e$ and the initial data D, $\delta(q_0 e) := \sqrt{1 - 4(q_0 e)^2} \in \mathbb{C}$, and ϕ_{err} is fast decaying (i.e., ϕ_{err} satisfies (1-8) *for $s > 1$).*

(3) *In the massive charged case, i.e., $m^2 > 0,\ q_0 \neq 0$,*

$$\phi_{\mathcal{H}^+}(v) = C(M \cdot m, D) \cdot e^{(iq_0 e/r_+)v} \cdot \sin(mv + \omega_{\mathrm{err}}(v)) \cdot v^{-5/6} + \phi_{\mathrm{err}}, \tag{1-17}$$

where all the quantities are as above and generically $|q_0 e| \neq r_-|m|$.

Falsification of Conjecture 1 assuming Conjecture 2. We will show that the conjectured profiles in (1-15), (1-16) and (1-17) indeed satisfy the quantitative oscillation (1-12). Thus, as a corollary of our main result Theorem I (i) we obtain a conditional, but otherwise definitive resolution of Conjecture 1:

Theorem II (rough version; precise version in Section 4D). *If $\phi_{\mathcal{H}^+}$ is as in Conjecture 2, then the metric g and the scalar field ϕ are continuously extendible across the Cauchy horizon $\mathcal{CH}_{i^+}$.*

In particular, if Conjecture 2 is true, then Conjecture 1 is false for the Einstein–Maxwell–Klein–Gordon system in spherical symmetry.

We refer to Section 4D for the precise statement of Theorem II.

The conjectured decay rates for $\phi_{\mathcal{H}^+}$ in Conjecture 2 are nonintegrable; i.e., $\phi_{\mathcal{H}^+}$ satisfies (1-8) with s in the range (1-9), except for the massless charged case with $|q_0 e| < \frac{1}{2}$. We also recall that nonintegrable decay of $\phi_{\mathcal{H}^+}$ is insufficient to prove continuous extendibility for g and ϕ by means of decay and indeed even leads to the blow-up of $|\phi|$ as shown in Theorem I (ii) in the case where the oscillation condition (1-10) is violated. In that sense, under the assumption of Conjecture 2, Theorem II shows that C^0-strong cosmic censorship for the EMKG model is false *only* by virtue of the oscillations of the scalar field ϕ on the event horizon $\mathcal{H}^+$.

Lack of oscillations for nongeneric Cauchy data on Σ. Having addressed the generic case in Conjecture 2, there remains still the possibility that there exist (nongeneric) Cauchy data for which the scalar field $\phi_{\mathcal{H}^+}$ on the event horizon $\mathcal{H}^+$ does not satisfy the (qualitative) oscillation condition (1-10). Indeed, on the basis of certain scattering arguments [Angelopoulos et al. 2020; Dafermos et al. 2018; Masaood 2022] we conjecture.[4]

Conjecture 3. *For any suitable finite-energy profile $\phi_{\mathcal{H}^+}$ there exist sufficiently regular Cauchy data on Σ for the EMKG system in spherical symmetry giving rise to a dynamical black hole for which the scalar field along the event horizon is given by $\phi_{\mathcal{H}^+}$.*

[4]We also note that Conjecture 3 is not specific to the EMKG system in spherical symmetry: similar conjectures can be made for a rather general class of models; see for instance [Angelopoulos et al. 2020; Dafermos et al. 2018].

In particular, if Conjecture 3 is true, this means that there exist Cauchy data on Σ for which the scalar field $\phi_{\mathcal{H}^+}$ on the event horizon $\mathcal{H}^+$ obeys (1-8) for $s > \frac{3}{4}$, but violates the oscillation condition (1-10); thus by Theorem I (ii), the scalar field ϕ blows up in amplitude at the Cauchy horizon $\mathcal{CH}_{i^+}$ (if $q_0 = 0$). Such (nongeneric) Cauchy data will be important in Section 1A5 as they will constitute examples of null contraction singularities at $\mathcal{CH}_{i^+}$; see Theorem IV. Finding the precise regularity (see [Dafermos and Shlapentokh-Rothman 2018; Dias et al. 2018b]) of such Cauchy data on Σ is also part of the resolution of Conjecture 3.

1A4. *Theorem III*: *$W^{1,1}$-blow-up along outgoing cones — a complete contrast with the vacuum case.* We remarked before that the falsification of the C^0-formulation of strong cosmic censorship in vacuum [Dafermos and Luk 2017] — the vacuum analog of Theorem II outside spherical symmetry — crucially relies on integrable decay along the event horizon $\mathcal{H}^+$ for perturbations and their derivatives (see (1-21)). Indeed, in their work, Dafermos and Luk propagate this integrable decay towards i^+ with suitable weighted energy estimates into the black hole interior. This integrable decay for outgoing derivatives is then used to show that the metric is actually $W^{1,1}$-extendible along outgoing null cones, i.e., with locally integrable Christoffel symbols. Note that this $W^{1,1}$-extendibility result of the metric is *strictly stronger* than the C^0-extendibility which subsequently follows by integrating. Mutatis mutandis, this robust physical space method of showing the stronger $W^{1,1}$-extendibility result as an intermediate step has been applied in various previous contexts to show C^0-extendibility; see, e.g., [Dafermos 2003; 2005a; Luk and Oh 2019a; Dafermos and Luk 2017], exploiting the null structure of the Einstein equations: in fact, this was the only known method to prove C^0-extendibility so far. For the EMKG model, however, only in the case $m^2 = 0$, $|q_0 e| < \frac{1}{2}$ do perturbations along the event horizon $\mathcal{H}^+$ decay at an *integrable* rate. For such integrable rates, the analog of Theorem II was shown already [Van de Moortel 2018] using the aforementioned physical space method and proving $W^{1,1}$-extendibility as an intermediate step (schematically $\int |\partial_v g| \, dv < \infty$):

Theorem [Van de Moortel 2018]. *Consider spherically symmetric characteristic initial data for* (1-1)–(1-5) *on the event horizon $\mathcal{H}^+$ (and on an ingoing cone). Let the scalar field $\phi_{\mathcal{H}^+}$ decay fast on the event horizon $\mathcal{H}^+$ (i.e., $\phi_{\mathcal{H}^+}$ satisfies* (1-8) *for $s > 1$). Then ϕ is uniformly bounded in amplitude and in $W^{1,1}$, i.e.,*

$$\sup_{(u,v)} |\phi|(u, v) < +\infty, \quad \sup_{u} \int_{v_0}^{+\infty} |D_v \phi|(u, v) \, dv < +\infty. \tag{1-18}$$

Moreover the metric g admits a $W^{1,1}$ extension $\tilde{g}$ across the Cauchy horizon $\mathcal{CH}_{i^+}$ and $\tilde{g}$ is C^0-admissible (Definition 2.1). In particular, g is C^0-extendible.

Note that the $W^{1,1}$-extendibility method provides a so-called C^0-admissible extension, which is a continuous extension also admitting null coordinates (a slightly stronger result than general C^0-extendibility).

Apart from the massless case $m^2 = 0$ with $|q_0 e| < \frac{1}{2}$, the scalar field ϕ on the event horizon $\mathcal{H}^+$ is expected to be *nonintegrable* along the event horizon $\mathcal{H}^+$ (Conjecture 2) and as such, the robust physical space methods of [Dafermos and Luk 2017; Luk and Oh 2019a; Dafermos 2003; Van de Moortel 2018] showing the intermediate and stronger $W^{1,1}_{\text{loc}}$-extendibility fail.

We show in Theorem III below that indeed for a *generic nonintegrable* scalar field $\phi_{\mathcal{H}^+}$ on the event horizon $\mathcal{H}^+$, the scalar field ϕ *blows up* in $W^{1,1}$ (i.e., $\int |D_v \phi| \, dv = \infty$) at the Cauchy horizon $\mathcal{CH}_{i^+}$.

This is yet another manifestation of the fact that the C^0-extendibility result for the nonintegrable perturbations is unexpectedly subtle and crucially relies on the precise oscillations of the perturbation on the event horizon $\mathcal{H}^+$. In this sense, our result cannot be captured solely in physical space — making our mixed physical space-Fourier space approach seemingly necessary.

We now give a rough version of Theorem III and refer to Section 4E for the precise formulation.

Theorem III ($W^{1,1}$-blow-up along outgoing cones (rough version; precise version in Section 4E)). *Consider spherically symmetric characteristic initial data for* (1-1)–(1-5) *on the event horizon* $\mathcal{H}^+ = [v_0, +\infty)$ (*and on an ingoing cone*). *Then the following hold true*:

- *Consider arbitrary* $q_0 \in \mathbb{R}$, $m^2 \geq 0$.

  Then, for generic $\phi_{\mathcal{H}^+}$ *satisfying* (1-8) *and* (1-9), *the scalar field* ϕ *blows up in* $W^{1,1}_{\mathrm{loc}}$ *at the Cauchy horizon* $\mathcal{CH}_{i^+}$; *i.e., for all* u
$$\int_{v_0}^{+\infty} |D_v\phi|(u, v)\,\mathrm{d}v = +\infty. \tag{1-19}$$

- *Consider either the small charge case* (*i.e.*, $0 < |q_0 e| < \epsilon(M, e, m^2)$ *for* $\epsilon(M, e, m^2) > 0$ *sufficiently small*, $m^2 \geq 0$) *or the uncharged case* $q_0 = 0$ *for almost every mass* $m^2 \in \mathbb{R}_{>0}$.

  Then, for all nonintegrable $\phi_{\mathcal{H}^+} \notin L^1$ *satisfying* (1-8) *and* (1-9), *the scalar field* ϕ *blows up in* $W^{1,1}_{\mathrm{loc}}$ *along outgoing cones at the Cauchy horizon* $\mathcal{CH}_{i^+}$: *i.e., for all* u
$$\int_{v_0}^{+\infty} |D_v\phi|(u, v)\,\mathrm{d}v = +\infty. \tag{1-20}$$

Theorem III shows that the Cauchy horizon $\mathcal{CH}_{i^+}$ is already more singular in the slowly decaying case (i.e., $\phi_{\mathcal{H}^+}$ obeys (1-8) for $s \leq 1$) than in the fast decaying case (i.e., $\phi_{\mathcal{H}^+}$ obeys (1-8) for $s > 1$) as the comparison with (1-18) illustrates.

Assuming that Conjecture 2 is true, as part of our novel Theorem III, we also show that the $W^{1,1}$ blow-up of ϕ given by (1-19) also occurs for generic and regular Cauchy data (for almost all parameters (q_0, m^2)).

Further Theorem III strongly suggests that generically the metric itself is also $W^{1,1}$-inextendible, i.e., does not admit locally integrable Christoffel symbols in any coordinate system. If true, this statement would be in dramatic contrast with the vacuum perturbations of Kerr considered in [Dafermos and Luk 2017] and the weak null singularities from [Luk 2018] (both enjoying the analog of fast decay on the event horizon $\mathcal{H}^+$; see Section 1B2) in which the metric is shown to be $W^{1,1}$-extendible across the Cauchy horizon $\mathcal{CH}_{i^+}$. Extending Theorem III to a full $W^{1,1}$-inextendibility result on the metric is however a difficult (albeit very interesting) open problem due to the geometric nature of such a statement; see [Dafermos and Luk 2017; Luk 2018; Sbierski 2018; 2022; Kehle and Van de Moortel $\geq$ 2024] for related discussions.

1A5. *Theorem IV: the null contraction singularity at the Cauchy horizon $\mathcal{CH}_{i^+}$ for perturbations violating the oscillation condition.* By Theorem I (ii), if $q_0 = 0$, then any scalar field $\phi_{\mathcal{H}^+}$ that violates on oscillation condition (1-10) on the event horizon $\mathcal{H}^+$ gives rise to ϕ that blows up in amplitude at the Cauchy horizon $\mathcal{CH}_{i^+}$. A natural question then emerges: how does this blow up of the matter field translate geometrically, i.e., does the metric admit a singularity?

This question is answered in the affirmative in our companion paper [Kehle and Van de Moortel $\geq$ 2024]: We show that the metric admits a novel type of C^0-singularity at the Cauchy horizon $\mathcal{CH}_{i^+}$ that we call a *null contraction singularity*. The main result of [Kehle and Van de Moortel $\geq$ 2024] is conditional: we show that the metric admits a null contraction singularity if $|\phi|$ blows up at the Cauchy horizon. Combining this result with Theorem I (ii) (if $q_0 = 0$) shows that a null contraction singularity is formed dynamically for a scalar field $\phi_{\mathcal{H}^+}$ violating the oscillation condition (1-10) on $\mathcal{H}^+$.

We emphasize that the null contraction singularity is a C^0-singularity and different (in particular stronger) from the usual blue-shift instability [Dafermos and Shlapentokh-Rothman 2018] for derivatives, which additionally occurs at the Cauchy horizon of dynamical EMKG black holes and triggers the blow up of curvature and of the Hawking mass (mass inflation); see [Van de Moortel 2018; 2021] and the discussion in Section 1C. Specifically, the null contraction singularity has the following novel characteristics.

Theorem IV [Kehle and Van de Moortel $\geq$ 2024]. *Consider spherically symmetric characteristic initial data for* (1-1)–(1-5) *on the event horizon* $\mathcal{H}^+$ (*and on an ingoing cone*). *Let the scalar field* $\phi_{\mathcal{H}^+}$ *decay slowly on the event horizon* $\mathcal{H}^+$ (*i.e.,* $\phi_{\mathcal{H}^+}$ *satisfies* (1-8), (1-9)). *Assume additionally that* ϕ *blows up in amplitude at the Cauchy horizon* $\mathcal{CH}_{i^+}$, *i.e., assume that* $\limsup_{(u,v)\to\mathcal{CH}_{i^+}} |\phi|(u,v) = +\infty$.

Then the metric g *admits a null contraction singularity in the following sense*:

(a) *The metric does not admit any* C^0*-admissible extension* (*as defined in Definition 2.1*) *across the Cauchy horizon* $\mathcal{CH}_{i^+}$.

(b) *The affine parameter time on ingoing null geodesics* (*with uniform but otherwise arbitrary normalization*) *between two radial causal curves with distinct endpoint at the Cauchy horizon* $\mathcal{CH}_{i^+}$ *tends to zero as the Cauchy horizon* $\mathcal{CH}_{i^+}$ *is approached.*

(c) *The angular tidal deformations of radial ingoing null geodesics* (*with uniform but otherwise arbitrary normalization*) *become arbitrarily large near the Cauchy horizon* $\mathcal{CH}_{i^+}$.

For the precise definitions of the terms employed in the statement of Theorem IV we refer the reader to [Kehle and Van de Moortel $\geq$ 2024]. Note that the null contraction singularity is named in reference to statement (b), the most emblematic: physically, it means that the (suitably renormalized) affine parameter time in the ingoing null direction between two observers tends to zero as both observers approach the Cauchy horizon $\mathcal{CH}_{i^+}$.

Theorem IV is the first instance of a *null contraction singularity*: statements (a)–(c) have only been shown to occur in the context of matter fields blowing up at the Cauchy horizon $\mathcal{CH}_{i^+}$, as we prove in [Kehle and Van de Moortel $\geq$ 2024]. In particular, statements (a)–(c) are all false on the exact Reissner–Nordström interior or on the spacetimes of Theorem I (i) for which ϕ is bounded.

In view of Theorem I (ii), we note that there exists a large class of characteristic data on $\mathcal{H}^+ \cup \underline{C}_{\text{in}}$ giving rise to a null contraction singularity at $\mathcal{CH}_{i^+}$; see Figure 3. Moreover, assuming Conjecture 3, we also note that there exist Cauchy data on asymptotically flat Σ which give rise to a null contraction singularity at $\mathcal{CH}_{i^+}$.

Finally, we note that statement (a) of Theorem IV is, to the best of the authors' knowledge, the first C^0-inextendibility result across a null boundary (in our case the Cauchy horizon $\mathcal{CH}_{i^+}$). The geometric statement (a) strongly suggests that the oscillation condition (1-10) is indeed crucial to falsify Conjecture 1. Note however that Theorem IV only proves the impossibility to extend the metric in a spherically symmetric C^0-class (also used in [Moschidis 2017]), where C^0 double null coordinates exist. It would be interesting to investigate whether statement (a) can be promoted to a full C^0-inextendibility statement. However such statements are notoriously difficult to obtain: even in the more singular case where the black hole boundary is spacelike,[5] the C^0-extendibility of the metric has only been proved for the Schwarzschild black hole [Sbierski 2018].

1B. ***Cauchy horizons in other models: a comparison with our results.*** Having introduced our main results on the EMKG model (1-1)–(1-5) in Section 1A, we will now mention selected results on the existence/regularity of Cauchy horizons and Conjecture 1 for different models, which will appear to be in dramatic contrast with the previous Theorem A and our new results given in Theorems I (i), I (ii), III and IV on the EMKG model in spherical symmetry.

1B1. *Spherically symmetric models with no Maxwell field: absence of a Cauchy horizon.* Before turning to models admitting Cauchy horizons emanating from i^+, it is useful to recall that there exist models for which such Cauchy horizons do not form. An example of such a model is given by the Einstein-scalar-field system (i.e., (1-1)–(1-5) with $F \equiv 0$, $m^2 = 0$) in spherical symmetry. This model was studied in the seminal series [Christodoulou 1991; 1993; 1999] where it is shown that the MGHD of generic spherically symmetric data is bound to the future by a spacelike boundary $\mathcal{S} = \{r=0\}$ (in particular, there exists no null component of the boundary) and observers approaching $\mathcal{S} = \{r=0\}$ experience infinite tidal deformations.

From [Christodoulou 1991], it follows that Conjecture 1 is *true* for the Einstein-scalar-field system in spherical symmetry in the sense that there exists no spherically symmetric C^0-extension of the metric.

1B2. *Stability of the Cauchy horizon and the downfall of Conjecture 1 for massless fields and in vacuum.*

The Einstein–Maxwell-uncharged-scalar-field in spherical symmetry. Christodoulou's spherically symmetric spacetimes however fail to capture the repulsive effect that angular momentum exerts on the geometry in nonspherical collapse. One way to model this repulsive effect while remaining in the realm of spherical symmetry is to add a Maxwell field to the Einstein-scalar-field equations: The electromagnetic force then plays the role of angular momentum in nonspherical collapse [Dafermos 2004]. The resulting Einstein–Maxwell-uncharged-scalar-field system, i.e., (1-1)–(1-5) with $m^2 = q_0 = 0$, admits a (spherically symmetric) stationary charged black hole, the Reissner–Nordström metric (for which $\phi \equiv 0$) whose MGHD is bound to the future by a smooth Cauchy horizon $\mathcal{CH}_{i^+}$; see Figure 4.

Falsification of Conjecture 1 for the Einstein–Maxwell-uncharged-scalar-field model in spherical symmetry. The interior dynamics[6] near i^+ for the Einstein–Maxwell-uncharged-scalar-field model were studied in the pioneering work [Dafermos 2003; 2005a], which proved that the interior of the black hole admits a Cauchy horizon $\mathcal{CH}_{i^+}$ across which the metric is continuously extendible, under the crucial

[5]A spacelike singularity is indeed widely associated to C^0-inextendibility, and viewed as a stronger singularity than a Cauchy horizon, notably because of the blow-up of tidal deformations experienced on timelike geodesics [Dafermos and Luk 2017; Sbierski 2018; 2022].

[6]For a discussion of the dynamics far away from i^+ in the context of gravitational collapse, see Section 1C3.

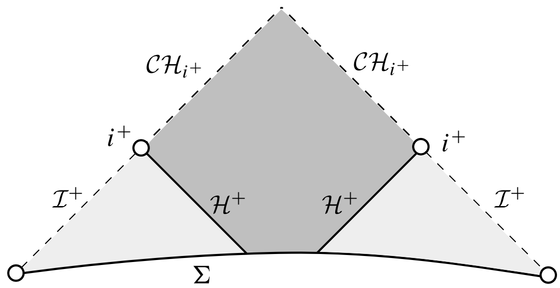

Figure 4. Penrose diagram of the subextremal Reissner–Nordström spacetime.

assumption of integrable decay of the scalar field on the event horizon $\mathcal{H}^+$. Integrable decay for the scalar field on the event horizon $\mathcal{H}^+$ (i.e., $\phi_{\mathcal{H}^+}$ satisfies (1-8) for $s > 1$) was later proved for sufficiently regular Cauchy data of [Dafermos and Rodnianski 2005]; therefore Conjecture 1 is *false* for the Einstein–Maxwell-uncharged-scalar-field model in spherical symmetry [Dafermos 2003; 2005a; Dafermos and Rodnianski 2005] *by means of fast decay* $s > 1$.

Moreover, for this spherically symmetric model, Dafermos [2014] characterized entirely the black hole future boundary for any small, two-ended perturbation of Reissner–Nordström. He indeed showed that the resulting dynamical black hole has no spacelike singularity: its maximal globally hyperbolic development is bound to the future by a null bifurcate Cauchy horizon $\mathcal{CH}_{i^+}$, and has the Penrose diagram of Figure 4.

Falsification of Conjecture 1 for the vacuum Einstein equations without symmetry. As we already mentioned in Section 1A, Conjecture 1 was also falsified in vacuum with no symmetry assumption in the celebrated work [Dafermos and Luk 2017]. In this case as well, the crucial assumption in [Dafermos and Luk 2017] is the fast decay of metric perturbations along the event horizon, i.e., schematically in a standard choice of v-coordinate

$$\|v^{s-1/2}(g - g_K)\|_{L^2(\mathcal{H}^+)} \leq \epsilon \quad \text{for some } s > 1, \tag{1-21}$$

where g_K is the Kerr metric and $\epsilon > 0$ is small. Note that (1-21) shows $|g - g_K|(v) \lesssim v^{-s}$ (at least along a sequence) and in that sense (1-21) is indeed the analog for $g - g_K$ of fast decay of the scalar field, i.e., (1-8) for $s > 1$.

The linear analog of (1-21) for the black hole exterior stability problem around Kerr has been established in [Dafermos et al. 2016; Shlapentokh-Rothman and Teixeira da Costa 2023]; see also the recent nonlinear work [Dafermos et al. 2021]. If (1-21) (and related estimates) are shown for the full Einstein equations in a neighborhood of Kerr, then the result of [Dafermos and Luk 2017] *unconditionally falsifies Conjecture 1 in vacuum*, by means of fast decay $s > 1$.

1C. ***Weak null singularities at the Cauchy horizon and a weaker formulation of strong cosmic censorship.*** In this section, we mention briefly other types of singularities at the Cauchy horizon $\mathcal{CH}_{i^+}$, and how they compare with the new singularities at the Cauchy horizon $\mathcal{CH}_{i^+}$ from Theorems III and IV.

1C1. *Weak null singularities and blue-shift instability.* As discussed earlier, our new results exhibit the first examples of Cauchy horizons $\mathcal{CH}_{i^+}$ singular at the C^0 level (for nonoscillating scalar fields at $\mathcal{H}^+$) and the $W^{1,1}$ level (for all slowly decaying scalar fields at $\mathcal{H}^+$). This new singularity at the Cauchy horizon $\mathcal{CH}_{i^+}$ is very different from the well-known weak null singularity at $\mathcal{CH}_{i^+}$ [Luk 2018; Van de Moortel 2023; Ori and Flanagan 1996; Brady et al. 1998; Burko and Ori 1998; Ori 1999], which corresponds to blow-up in the energy class (i.e., H^1 norm) at $\mathcal{CH}_{i^+}$ due to the celebrated blue-shift instability [Penrose 1968; McNamara 1978]. Blow-up in energy (i.e., H^1 norm in nondegenerate coordinate) at the Cauchy horizon of Kerr and Reissner–Nordström has indeed been proven to occur for the linear wave equation in [Dafermos and Shlapentokh-Rothman 2018; Luk and Sbierski 2016; Luk and Oh 2017]. Based on the blue-shift instability, Christodoulou suggested an alternative formulation of strong cosmic censorship that is weaker than Conjecture 1. Specifically, he conjectured in [Christodoulou 2009] that for generic asymptotically flat Cauchy data, the metric is H^1-inextendible i.e., admits no extension with square-integrable Christoffel symbols; see also [Chruściel 1991; Dafermos and Luk 2017].

More generally, we say that the Cauchy horizon $\mathcal{CH}_{i^+}$ is a weak null singularity if already the metric is C^2-inextendible across $\mathcal{CH}_{i^+}$, a property which is generally obtained from the blow-up of some curvature component in an appropriate frame [Van de Moortel 2021; Luk and Oh 2019a; Kommemi 2013].

1C2. *Dynamical formation of weak null singularities and known inextendibility results.* While examples of weak null singularities have been constructed in vacuum [Luk 2018], their dynamical formation from an "open set" of data with no symmetry assumption is still an open problem. Nevertheless, for the EMKG model in spherical symmetry, it was proven [Van de Moortel 2018; 2021] that *the Cauchy horizon $\mathcal{CH}_{i^+}$ of Theorem A is weakly singular*, i.e., the metric is C^2-inextendible across the Cauchy horizon $\mathcal{CH}_{i^+}$, under the assumptions of Theorem A and additional lower bounds on the scalar field consistent with Conjecture 2. In the uncharged massless model $q_0 = m^2 = 0$ of Section 1B2, the same result was previously proven unconditionally in [Luk and Oh 2019a; 2019b] for generic asymptotically flat two-ended Cauchy data. Both for the EMKG and the $q_0 = m^2 = 0$ model, the above C^2-inextendibility result was improved to a $C^{0,1}$-inextendibility statement in [Sbierski 2022].

1C3. *Weak null singularities in gravitational collapse.* We conclude this section by a brief discussion of the influence of a weak null singularity on the black hole geometry away from i^+. To study this question in the framework of gravitational collapse (i.e., one-ended spacetimes with a center Γ as in Figure 5), we cannot study the Einstein–Maxwell-uncharged-scalar-field model of Section 1B2 because of a well-known [Kommemi 2013; Van de Moortel 2023] topological obstruction caused by the scalar field being uncharged, i.e., $q_0 = 0$, forcing the initial data Σ to be two-ended [Dafermos 2014]. However in the EMKG model, where $q_0 \neq 0$, there is no such obstruction and one can study the one-ended global geometry of the black hole interior with a weak null singularity, even in spherical symmetry [Kommemi 2013]. The main known result in this context is that the *weak null singularity $\mathcal{CH}_{i^+}$ breaks down* [Van de Moortel 2023] before reaching the center: Consequently a so-called first singularity b_Γ is formed at the center Γ, as depicted in Figure 5. This is in complete contrast with the two-ended case where the future boundary is entirely null [Dafermos 2014] for a large class of spacetimes as we discussed in Section 1B2. In the conjecturally generic case where b_Γ is not a so-called *locally naked singularity* [Van de Moortel 2023; Kommemi 2013; Dafermos 2005b; Christodoulou 1999], then the breakdown of the weak null

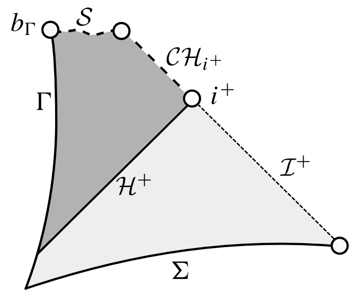

Figure 5. Conjectured Penrose diagram of a generic EMKG black hole with weakly singular $\mathcal{CH}_{i^+}$ [Van de Moortel 2023].

singularity $\mathcal{CH}_{i^+}$ proven [Van de Moortel 2023] implies that a stronger singularity $\mathcal{S} = \{r=0\}$ takes over and connects the weak null singularity $\mathcal{CH}_{i^+}$ to the center Γ as depicted in Figure 5.

1D. ***Scattering resonances associated to the Reissner–Nordström Cauchy horizon.*** We now turn to another result which is not directly concerned with the stability/instability of the Cauchy horizon but turns out to be important for the proofs of our main theorems: the finite-energy scattering theory for the linear wave equation on the interior of Reissner–Nordström developed in [Kehle and Shlapentokh-Rothman 2019]. A key insight to the result in that work was the absence of scattering resonances associated to the Killing generator of the Cauchy horizon, which is an exceptional feature of the massless and uncharged wave equation on exact Reissner–Nordström. Indeed, for the massive wave equation with generic masses $m^2 \in \mathbb{R}_{>0} - D(M, e)$ or for the charged equation, the scattering resonances are present and there does not exist an analogous scattering theory [Kehle and Shlapentokh-Rothman 2019]. As we will show, these scattering resonances are also the key sources of blow-up in amplitude of ϕ at the Cauchy horizon if the scalar field along the event horizon is nonoscillating and slowly decaying and thus, sufficiently resonant. In view of this, for the blow-up statement of Theorem I (ii) these exceptional masses for which the scattering resonances are absent have to be excluded. Refer also to [Mokdad 2022; Häfner et al. 2021] for a scattering theory of the Dirac equation on the interior of Reissner–Nordström and to [Bachelot 1994; Dimock and Kay 1987; Dafermos et al. 2018; Masaood 2022; Alford 2020] for scattering theories on the exterior.

1E. ***Connection to the linear analog of Conjecture 1 for negative cosmological constant* $\boldsymbol{\Lambda < 0}$.** In the discussion above we have studied the Einstein equations with cosmological constant $\Lambda = 0$. Analogously, for $\Lambda \neq 0$, the Reissner–Nordström–(anti-)de Sitter and Kerr–(anti-)de Sitter spacetimes admit a smooth Cauchy horizon and the issue of strong cosmic censorship analogously arises in this setting. In particular, the case $\Lambda < 0$ has some similarities with our case in the sense that linear perturbations also only decay at a nonintegrable (inverse logarithmic for $\Lambda < 0$) rate due to a stable trapping phenomenon [Holzegel and Smulevici 2013; 2014; Holzegel et al. 2020]. A difference to our result is however that only perturbations consisting of a superposition of infinitely many high ℓ angular modes decay slowly and thus, the problem for $\Lambda < 0$ cannot be reasonably studied in spherical symmetry. Nevertheless, as in our case, this nonintegrable rate of decay might raise hopes that, in the case of negative cosmological constant $\Lambda < 0$, Conjecture 1 holds true.

On the one hand, for Reissner–Nordström–AdS, since stable trapping is a high-frequency phenomenon and uniform boundedness (on the linear level) is associated to zero-frequency scattering resonances of the Cauchy horizon, it was shown in [Kehle 2020b] that these difficulties decouple on Reissner–Nordström–AdS. (This decoupling can be seen as the analog of the fact that the oscillation condition of (1-10) is satisfied.) As a consequence of this frequency decoupling, it is shown in [Kehle 2020b] that, despite slow nonintegrable decay on the exterior, linear perturbations remain uniformly bounded and extend continuously across the Reissner–Nordström–AdS Cauchy horizon. This falsifies the linear analog of Conjecture 1 for Reissner–Nordström–AdS.

On the other hand, for Kerr–AdS, in view of the rotation of the black hole, frequency mixing occurs and trapped high-frequency perturbations on the exterior can at the same time be low-frequency when frequency is measured with respect to the Killing generator of the Cauchy horizon. In [Kehle 2020a; 2022] it is shown that this frequency mixing gives rise to a resonance phenomenon and an associated small divisors problem. In particular, for a set of Baire-generic Kerr–AdS black hole parameters, which are associated to a Diophantine condition, it is shown that linear perturbations ϕ blow up in amplitude at the Cauchy horizon. This shows that the linear analog of Conjecture 1 holds true for Baire-generic Kerr–AdS black holes.

There is yet another possible scenario in which the exteriors of AdS black holes are nonlinearly unstable (see [Moschidis 2017; 2020; 2023; Bizoń and Rostworowski 2011]) and the question of strong cosmic censorship would be thrown even more open.

Let us finally also briefly mention the case of positive cosmological constant $\Lambda > 0$, where perturbations on the exterior of Reissner–Nordström/Kerr–de Sitter decay at an exponential rate as proved in [Dyatlov 2011; Mavrogiannis 2023] for the linear wave equation and in [Hintz and Vasy 2018] for the vacuum Einstein equations. In view of this rapid decay, the theorem of [Dafermos and Luk 2017] manifestly also applies and thus, Conjecture 1 is false for $\Lambda > 0$. However, in view of this exponential decay, even weaker formulations such as the H^1-formulation of strong cosmic censorship mentioned in Section 1C may fail. We refer to [Dafermos 2014; Hintz and Vasy 2017; Dias et al. 2018a; 2018b; 2019; Dafermos and Shlapentokh-Rothman 2018; Costa et al. 2018; Costa and Franzen 2017; Mo et al. 2018; Hollands et al. 2020; Cardoso et al. 2018] for details.

1F. ***Summary of the strategy of the proof.*** We now turn to an outline of our proof and begin with the obstructions and difficulties encountered when attempting to prove boundedness of the scalar field at the Cauchy horizon $\mathcal{CH}_{i^+}$ and continuous extendibility of the metric.

- The physical space estimates used to show $\mathcal{CH}_{i^+} \neq \varnothing$ in the proof of Theorem A, under the assumption of a slowly decaying $\phi_{\mathcal{H}^+}$ on $\mathcal{H}^+$, i.e., obeying (1-8) and (1-9), are *consistent with the blow-up of the scalar field* ϕ at the Cauchy horizon $\mathcal{CH}_{i^+}$ and the failure of $\partial_v \phi$ to be integrable in v. As our new result shows, these estimates from [Van de Moortel 2018] are sharp by Theorem III and blow-up in amplitude indeed occurs for some perturbations by Theorem I (ii).

- The estimates of the proof of Theorem A however suggest that, *if* $\partial_v\phi$ *oscillates infinitely towards the Cauchy horizon* $\mathcal{CH}_{i^+}$ then ϕ *is bounded* (see Section 4F1): the hope would be that, although $\partial_v\phi$

is not Lebesgue-integrable (i.e., $\int_{v_0}^{+\infty} |\partial_v \phi| \, dv = +\infty$), it has a semiconvergent Riemann integral (i.e., $\lim_{\tilde{v}\to+\infty} \left| \int_{v_0}^{\tilde{v}} \partial_v \phi \, dv \right| < +\infty$ exists). A natural approach is then to attempt to propagate the event horizon oscillations (1-10) satisfied by $\phi_{\mathcal{H}^+}$ towards the Cauchy horizon $\mathcal{CH}_{i^+}$ in a suitable sense and deduce the boundedness of ϕ. However, this is not easy to show in physical space and prompts a Fourier space approach for the linearized equation.

- A complete understanding of the linearized problem is however insufficient in itself to prove the boundedness of ϕ since the *nonlinear terms cannot be treated purely perturbatively* in view of the slow decay. Consequently the precise structure of these nonlinear terms has to be understood and plays an important role in the argument (in contrast to the fast decay case $s > 1$) (see Section 4F3).

- Even once ϕ is proven to be bounded in amplitude, there is no clear mechanism yielding the continuous extendibility of the metric, contrary to the fast decay case $s > 1$ in which the mechanism is given by the integrability of the Christoffel symbols [Dafermos and Luk 2017; Luk and Oh 2019a] in a suitable sense (see Section 4F4 for a discussion).

Strategy. To address and overcome these difficulties in order to prove our main theorems as stated in Section 1A, we proceed as follows:

(1) We take advantage on the one hand of the previous result of Theorem A, the future black hole boundary is null, i.e., $\mathcal{CH}_{i^+} \neq \varnothing$ and the Penrose diagram is given by Figure 1, and on the other hand of the nonlinear estimates (see Section 4F1) that were already proven in [Van de Moortel 2018] for slowly decaying $\phi_{\mathcal{H}^+}$.

(2) We consider the massive/charged linear wave equation $g_{\mathrm{RN}}^{\mu\nu} D_\mu^{\mathrm{RN}} D_\nu^{\mathrm{RN}} \phi_{\mathcal{L}} = m^2 \phi_{\mathcal{L}}$ on a fixed Reissner–Nordström background g_{RN}, which we view as the linearization of the EMKG system (1-1)–(1-5). Using Fourier methods and a scattering approach, we prove uniform boundedness (respectively blow-up in amplitude) of $\phi_{\mathcal{L}}$ at the Cauchy horizon $\mathcal{CH}_{i^+}$ for an oscillating scalar field $\phi_{\mathcal{H}^+}$ obeying (1-10) at $\mathcal{H}^+$ (respectively nonoscillating $\phi_{\mathcal{H}^+}$, i.e., $\phi_{\mathcal{H}^+}$ violates (1-10) at $\mathcal{H}^+$); see Section 4F2.

(3) Independently of step (2), we prove nonlinear difference estimates on $g - g_{\mathrm{RN}}$. Although these estimates are, in a sense, weaker[7] than the nonlinear estimates of step (1), they are crucial in our proof that, for all slowly decaying $\phi_{\mathcal{H}^+}$, the linear solution $\phi_{\mathcal{L}}$ is bounded if and only if the nonlinear ϕ is bounded (at least in the $q_0 = 0$ case). In the charged $q_0 \neq 0$ case, we follow a similar logic but additional difficulties arise from the nonlinear backreaction of the Maxwell field. This step will be discussed in Section 4F3.

(4) With the boundedness of ϕ at hand from the previous step, we prove the continuous extendibility of the metric for oscillating perturbations $\phi_{\mathcal{H}^+}$ satisfying (1-11). For the proof, we introduce a crucial new quantity Υ (see (4-38)) exploiting the exact algebraic[8] structure of the nonlinear terms in the Einstein equations; see Section 4F4.

[7] In the sense that these estimates alone are insufficient to show that $\mathcal{CH}_{i^+} \neq \varnothing$ as proven in [Van de Moortel 2018] (see Theorem A).

[8] In contrast, when the decay is integrable as in vacuum, the null structure of the Einstein equations is sufficient [Dafermos and Luk 2017; Luk and Oh 2019a].

The proofs of Theorems I (i) and I (ii) are finally obtained by combining steps (1), (2), (3), and (4). Theorem II follows immediately. The proof of Theorem III is also derived from the strategy given by the same steps (1)–(4); see the last paragraphs in Section 4F4. We refer to Section 4F for a more detailed outline of the strategy of the proof.

1G. ***Outline of the paper.*** In Section 2, we set out notation, definitions and the geometric setting for the solutions of (1-1)–(1-5) under spherical symmetry. In Section 3, for any arbitrary slowly decaying scalar field $\phi_{\mathcal{H}^+}$, we construct and set up spherically symmetric characteristic data on the event horizon $\mathcal{H}^+$ and an ingoing cone such that the scalar field is given by $\phi_{\mathcal{H}^+}$ on $\mathcal{H}^+$. In Section 4, we give the precise formulations of our main results Theorems I (i), I (ii), II, III and their assumptions. We end this section with a detailed outline of our proof in Section 4F. In Section 5, we develop the linear theory and show our main linear results in Section 5D. In Section 6, we develop the nonlinear theory and show the boundedness of the scalar field for the coupled (1-1)–(1-5) and the continuous extendibility of the metric. We first outline in Section 6A the estimates proved in [Van de Moortel 2018], which will be useful for the nonlinear EMKG system. Then in Section 6B, we establish the main estimates necessary for the continuous extendibility of the metric. In Section 6C, we prove difference estimates which we combine in Section 6D with the linear estimates from Section 5 to prove our main results Theorems I (i), I (ii), II, and III.

2. Preliminaries

2A. ***The Reissner–Nordström interior.*** Reissner–Nordström black holes constitute a 2-parameter family of spherically symmetric spacetimes, indexed by charge and mass (e, M), which satisfy the Einstein–Maxwell system ((1-1)–(1-5) with $\phi \equiv 0$) in spherical symmetry. We are interested in the interiors of subextremal Reissner–Nordström black holes satisfying $0 < |e| < M$. To define these spacetimes, we first set

$$\Omega^2_{\mathrm{RN}}(r_{\mathrm{RN}}) := -\left(1 - \frac{2M}{r_{\mathrm{RN}}} + \frac{e^2}{r^2_{\mathrm{RN}}}\right), \tag{2-1}$$

which is nonnegative between the zeros given by

$$r_+(M, e) = M + \sqrt{M^2 - e^2} > 0,$$
$$r_-(M, e) = M - \sqrt{M^2 - e^2} > 0.$$

Now, we define the smooth manifold $\mathring{\mathcal{M}}_{\mathrm{RN}}$ as a 4-dimensional smooth manifold diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^2$. Up to the well-known degeneracy of the spherical coordinates on $\mathbb{S}^2$, let $(r_{\mathrm{RN}}, t, \theta, \varphi) \in (r_-, r_+) \times \mathbb{R} \times \mathbb{S}^2$ be a global chart. In that chart we define the smooth Lorentzian metric g_{RN} and Maxwell 2-form F_{RN}

$$g_{\mathrm{RN}} := -\,\Omega^{-2}_{\mathrm{RN}}\,\mathrm{d}r^2_{\mathrm{RN}} + \Omega^2_{\mathrm{RN}}\,\mathrm{d}t^2 + r^2_{\mathrm{RN}}(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2), \tag{2-2}$$

$$F_{\mathrm{RN}} = \mathrm{d}A^{\mathrm{RN}} = \frac{e}{r^2}\,\mathrm{d}t \wedge \mathrm{d}r. \tag{2-3}$$

We time-orient the Lorentzian manifold such that vector field $-\nabla r_{\mathrm{RN}}$ is future-directed. Further, we define the tortoise coordinate r^* by $\mathrm{d}r^* = -\Omega^{-2}_{\mathrm{RN}}\,\mathrm{d}r_{\mathrm{RN}}$ or more explicitly by

$$r^* = r^*(r_{\mathrm{RN}}) = r_{\mathrm{RN}} + \frac{1}{4K_+}\log(r_+ - r_{\mathrm{RN}}) + \frac{1}{4K_-}\log(r_{\mathrm{RN}} - r_-), \tag{2-4}$$

where $K_+(M,e)$, $K_-(M,e)$ are the surface gravities associated to the event/Cauchy horizon defined as

$$K_+(M,e) = \frac{1}{2r_+^2}\left(M - \frac{e^2}{r_+}\right) = \frac{r_+ - r_-}{4r_+^2} > 0, \quad K_-(M,e) = \frac{1}{2r_-^2}\left(M - \frac{e^2}{r_-}\right) = \frac{r_- - r_+}{4r_-^2} < 0. \tag{2-5}$$

We further introduce the null coordinates $(u, v, \theta, \varphi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{S}^2$ on $\mathring{\mathcal{M}}_{\mathrm{RN}}$ as

$$v = r^*(r) + t, \quad u = r^*(r) - t, \quad \theta = \theta, \quad \varphi = \varphi. \tag{2-6}$$

In this coordinate system the metric g_{RN} has the form

$$g_{\mathrm{RN}} = -\frac{\Omega_{\mathrm{RN}}^2}{2}(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u) + r_{\mathrm{RN}}^2[\mathrm{d}\theta^2 + \sin(\theta)^2\,\mathrm{d}\varphi^2]. \tag{2-7}$$

Now, we attach the (right) event horizon $\mathcal{H}^+$, the past/future bifurcation sphere $\mathcal{B}_-$, $\mathcal{B}_+$, the left event horizon $\mathcal{H}^{+,L}$, the (right) Cauchy horizon $\mathcal{CH}_{i^+}$, and the left Cauchy horizon $\mathcal{CH}_{i_L^+}$ to our manifold, formally defined as

$$\begin{aligned} \mathcal{H}^+ &= \{u = -\infty, v \in \mathbb{R}\}, \quad \mathcal{CH}_{i_L^+} = \{u = +\infty, v \in \mathbb{R}\}, \quad \mathcal{B}_- = \{u = -\infty, v = -\infty\}, \\ \mathcal{H}^{+,L} &= \{v = -\infty, u \in \mathbb{R}\}, \quad \mathcal{CH}_{i^+} = \{v = +\infty, u \in \mathbb{R}\}, \quad \mathcal{B}_+ = \{u = +\infty, v = +\infty\}. \end{aligned}$$

A word of caution. In the linear theory of Section 5 we will indeed denote by $\mathcal{H}^+$ the Reissner–Nordström event horizon $\{u = -\infty, v \in \mathbb{R}\}$. However, in the other parts of the paper we denote by $\mathcal{H}^+$ the dynamical event horizon $\{u = -\infty, v \geq v_0\}$ in the nonlinear part of Section 6 (see also the set-up of the characteristic data in Section 3 and the main theorems stated in Section 4). We do similarly for the Cauchy horizon $\mathcal{CH}_{i^+}$. We also note that the left event and the left Cauchy horizon only play a minor role in the linear part of Section 5 and we often omit "right" when referring to $\mathcal{H}^+$ and $\mathcal{CH}_{i^+}$.

The metric g_{RN} extends smoothly to the boundary and the resulting spacetime is a time-oriented Lorentzian manifold $(\mathcal{M}_{\mathrm{RN}}, g_{\mathrm{RN}})$ with corners — the Reissner–Nordström interior. We remark that

$$\Omega_{\mathrm{RN}}^2 \underset{r \to r_+}{\sim} C_{e,M} e^{4K_+ r^*} = C_{e,M} e^{2K_+(u+v)}, \quad \Omega_{\mathrm{RN}}^2 \underset{r \to r_-}{\sim} C'_{e,M} e^{4K_- r^*} = C'_{e,M} e^{2K_-(u+v)} \tag{2-8}$$

for some $C_{e,M} > 0$, $C'_{e,M} > 0$. Further, we introduce regular coordinates (U, v) on $\mathring{\mathcal{M}}_{\mathrm{RN}} \cup \mathcal{H}^+$ as

$$\mathrm{d}U = \tfrac{1}{2}\Omega_{\mathrm{RN}}^2(u, v_0)\,\mathrm{d}u, \quad U(-\infty) = 0, \; v = v \tag{2-9}$$

and note that $\mathcal{H}^+ = \{U = 0\}$. Here $v_0 = v_0(M, e, D_1, s)$ will be determined in Proposition 3.2 later. In these coordinates we have obtained a different lapse function $(\Omega_{\mathrm{RN}}^2)_H = (\Omega_{\mathrm{RN}}^2)_H(U, v) = -2g_{\mathrm{RN}}(\partial_U, \partial_v)$ and the metric reads

$$g_{\mathrm{RN}} = -\frac{(\Omega_{\mathrm{RN}}^2)_H}{2}(\mathrm{d}U \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}U) + r_{\mathrm{RN}}^2[\mathrm{d}\theta^2 + \sin(\theta)^2\,\mathrm{d}\varphi^2]. \tag{2-10}$$

Of course we can invert the coordinate change (2-9) and obtain

$$u = u(U), \quad v = v. \tag{2-11}$$

We also remark that $T := \partial_t$ in $(r_{\mathrm{RN}}, t, \theta, \phi)$-coordinates is a Killing vector field which extends smoothly to $(\mathcal{M}_{\mathrm{RN}}, g_{\mathrm{RN}})$.

2B. ***Class of spacetimes, null coordinates, mass, charge.***

Spherically symmetric solution to the EMKG system. A smooth spherically symmetric solution of the EMGK system is described by a quintuplet $(\mathcal{M}, g, F, A, \phi)$, where $(\mathcal{M}, g)$ is a smooth (3+1)-dimensional Lorentzian manifold, ϕ is a smooth complex-valued scalar field, A is a smooth real-valued 1-form, and F is a smooth real-valued 2-form satisfying (1-1)–(1-5) and admitting a free SO(3) action on $(\mathcal{M}, g)$ which acts by isometry with spacelike 2-dimensional orbits (homeomorphic to $\mathbb{S}^2$) and which additionally leaves F, A and ϕ invariant.[9] In this case, the quotient $\mathcal{Q} = \mathcal{M}/\,\mathrm{SO}(3)$ is a 2-dimensional manifold with projection $\Pi : \mathcal{M} \to \mathcal{Q}$ taking a point of $\mathcal{M}$ into its spherical orbit. As SO(3) acts by isometry, $\mathcal{Q}$ inherits a natural metric, which we call $g_{\mathcal{Q}}$. The metric on $\mathcal{M}$ is then given by the warped product $g = g_{\mathcal{Q}} + r^2\,\mathrm{d}\sigma_{\mathbb{S}^2}$, where $r = \sqrt{\mathrm{Area}(\Pi^{-1}(p))/(4\pi)}$ for $p \in \mathcal{Q}$ is the area radius of the orbit and $\mathrm{d}\sigma_{\mathbb{S}^2}$ is the standard metric on the sphere. The Lorentzian metric $g_{\mathcal{Q}}$ over the smooth 2-dimensional manifold $\mathcal{Q}$ can be written in null coordinates (u, v) as a conformally flat metric

$$g_{\mathcal{Q}} := -\frac{\Omega^2}{2}(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u) \tag{2-12}$$

such that (in mild abuse of notation) we have upstairs

$$g = -\frac{\Omega^2}{2}(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u) + r^2\,\mathrm{d}\sigma_{\mathbb{S}^2}. \tag{2-13}$$

On $(\mathcal{Q}, g_{\mathcal{Q}})$, we now define the Hawking mass as

$$\rho := \frac{r}{2}(1 - g_{\mathcal{Q}}(\nabla r, \nabla r)), \tag{2-14}$$

as well as κ and ι as

$$\kappa := \frac{-\Omega^2}{2\partial_u r} \in \mathbb{R} \cup \{\pm\infty\}, \tag{2-15}$$

$$\iota := \frac{-\Omega^2}{2\partial_v r} \in \mathbb{R} \cup \{\pm\infty\}. \tag{2-16}$$

Electromagnetic fields on $\mathcal{Q}$. In what follows, we will abuse notation and denote by F the 2-form over $\mathcal{Q}$ that is the push-forward by Π of the electromagnetic 2-form originally on $\mathcal{M}$, and similarly for A and ϕ. In view of the SO(3) symmetry of the potential A we have (see [Kommemi 2013]) that F has the form

$$F = \frac{Q}{2r^2}\Omega^2 \mathrm{d}u \wedge \mathrm{d}v, \tag{2-17}$$

where Q is a scalar function called the electric charge. From $F = \mathrm{d}A$ we also obtain

$$[D_u, D_v] = iq_0 F_{uv} = \frac{iq_0 Q \Omega^2}{2r^2}.$$

Now we introduce the modified Hawking mass ϖ that involves the charge Q:

$$\varpi := \rho + \frac{Q^2}{2r}. \tag{2-18}$$

[9] Note that we assume that the SO(3) action is free, i.e., free of fixed points "$r = 0$" as we are interested in the region near i^+, i.e., away from $r = 0$.

An elementary computation relating geometric quantities (on the left) to coordinate-dependent ones (on the right) gives

$$1-\frac{2\rho}{r}=\frac{-4\partial_u r\,\partial_v r}{\Omega^2}=\frac{-\Omega^2}{\iota\kappa}=1-\frac{2\varpi}{r}+\frac{Q^2}{r^2}. \tag{2-19}$$

We also define the quantity

$$2K:=\frac{1}{r^2}\left(\varpi-\frac{Q^2}{r}\right), \tag{2-20}$$

and notice that, if $\varpi=M$ and $Q=e$, then $2K(r_\pm)=2K_\pm$. Further, we introduce the following notation, first used by Christodoulou:

$$\lambda=\partial_v r,\qquad \nu=\partial_u r.$$

Finally, note that (1-4)–(1-5) are invariant under electromagnetic gauge transformations (see Section 2C) and two solutions (ϕ, A) which differ by a gauge transformation represent the same physical behavior. An equivalent formulation to express this gauge freedom is to consider electromagnetism as a $U(1)$ gauge theory with principal $U(1)$-bundle $\pi\colon P\to M$: the charged scalar field is a global section of the associated complex line bundle $P\times_\rho\mathbb{C}$ through the representation ρ such that ϕ corresponds to an equivariant $\mathbb{C}$-valued map on P, i.e., $\phi(pg)=\rho(g)^{-1}\phi$. The representation $\rho: U(1)\to \mathrm{GL}(1,\mathbb{C})$ models the coupling of the scalar field and electromagnetic field. We refer to [Kommemi 2013, Section 1.1] and stick to our equivalent and more concrete formulation of the EMKG system.

C^0-admissible spacetimes and extensions. Lastly, we define the notion of a C^0-admissible extension of the metric (inspired from [Moschidis 2017, Definition A.3]). For the sake of brevity and concreteness we will give neither the most geometric nor the most general formulation and we refer to [Moschidis 2017; Kehle and Van de Moortel ≥ 2024] for further details.

Definition 2.1. We call $(\mathcal{M}, g)$ an admissible C^0 spherically symmetric spacetime if the following hold:

(1) $\mathcal{M}$ is a C^1-manifold diffeomorphic to $\mathcal{Q}\times\mathbb{S}^2$ for an open domain $\mathcal{Q}\subset\mathbb{R}^2$.

(2) g is an admissible C^0 spherically symmetric Lorentzian metric in the sense that for a diffeomorphism $\Phi\colon\mathcal{M}\to\mathcal{Q}\times\mathbb{S}^2$ there exist C^1-coordinates (u, v) on $\mathcal{Q}$ in which the metric $\Phi^*(g)$ on $\mathcal{Q}\times\mathbb{S}^2$ can be written as

$$\Phi^*(g)=-\frac{\Omega^2}{2}(\mathrm{d}u\otimes\mathrm{d}v+\mathrm{d}v\otimes\mathrm{d}u)+r^2 g_{\mathbb{S}^2}, \tag{2-21}$$

where $g_{\mathbb{S}^2}$ is the standard round metric on $\mathbb{S}^2$ and $\Omega^2, r^2\colon\mathcal{Q}\to(0,+\infty)$ are continuous.

(3) If $(\tilde{u}, \tilde{v})$ is another C^1-coordinate system such that (2-21) holds with $\widetilde{\Omega}^2$ in place of Ω^2, then $\tilde{u}=U(u)$ and $\tilde{v}=V(v)$ for some unique and strictly monotonic C^1-functions U, V.

Remark 2.2. The pair (u, v) as above is called a null coordinate system. In the case where the metric g is locally Lipschitz such null coordinates always exist. Since we merely consider C^0 metrics, in our definition of admissible C^0 metric we additionally impose the existence and uniqueness (up to rescaling) of such null coordinates.

Definition 2.3. Let $(\mathcal{M}, g)$ and $(\widetilde{\mathcal{M}}, \tilde{g})$ be time-oriented admissible C^0 spherically symmetric spacetimes. We say that $(\widetilde{\mathcal{M}}, \tilde{g})$ is an admissible C^0 spherically symmetric future extension if

(1) there exists a C^1 embedding $i : \mathcal{M} \to \widetilde{\mathcal{M}}$ which is also a time-orientation-preserving isometry,

(2) there exists $p \in \widetilde{\mathcal{M}} - i(\mathcal{M})$ which is to the future of $i(\mathcal{M})$.

2C. *Electromagnetic gauge choices.* As remarked above, for a fixed metric g, the Maxwell–Klein–Gordon system of equations (1-4)–(1-5) is invariant under the *gauge* transform

$$\phi \to \tilde{\phi} = e^{-iq_0 f}\phi, \tag{2-22}$$

$$A \to \tilde{A} = A + \mathrm{d}f, \tag{2-23}$$

where f is a smooth real-valued function. Notice that for $\widetilde{D} := \nabla + \tilde{A}$ we have

$$\widetilde{D}\tilde{\phi} = e^{-iq_0 f} D\phi.$$

Therefore the quantities $|\phi|$ and $|D\phi|$ are gauge-invariant. In Section 6, we will use that these gauge-invariant quantities satisfy the following estimates which are an immediate consequence of the fundamental theorem of calculus, see, e.g., [Gajic and Luk 2019, Lemma 2.1]. In any (u, v)-coordinate system and for $u \geq u_1$ and $v \geq v_1$,

$$|f(u, v)| \leq |f(u_1, v)| + \int_{u_1}^{u} |D_u f|(u', v)\, \mathrm{d}u, \tag{2-24}$$

$$|f(u, v)| \leq |f(u, v_1)| + \int_{v_1}^{v} |D_v f|(u, v')\, \mathrm{d}v' \tag{2-25}$$

for any sufficiently regular function $f(u, v)$.

Although we will mainly estimate gauge-invariant quantities, to set up the characteristic data it is useful to fix an electromagnetic gauge. For the analysis of the nonlinear system in Section 6 in double null coordinates (u, v) we will impose

$$A_v \equiv 0. \tag{2-26}$$

In this gauge, the condition $F = \mathrm{d}A$ from (1-4) can be written (in any (u, v)-coordinate system) as

$$\partial_v A_u = -\frac{Q\Omega^2}{2r^2}. \tag{2-27}$$

To estimate the dynamics of $A = A_u\, \mathrm{d}u$ in the coupled system it is useful to define a background electromagnetic field A^{RN} which is governed by the fixed Maxwell form $F = F_{\mathrm{RN}}$ as in (2-3) on a fixed Reissner–Nordström background with mass and charge (M, e). Using coordinates (u, v) as defined in (2-6) we impose the gauge

$$A_v^{\mathrm{RN}} \equiv 0 \tag{2-28}$$

such that $F_{\mathrm{RN}} = \mathrm{d}A_{\mathrm{RN}}$ becomes

$$\partial_v A_u^{\mathrm{RN}} = -\frac{e\Omega^2_{\mathrm{RN}}(u, v)}{2r^2_{\mathrm{RN}}(u, v)}. \tag{2-29}$$

Moreover, we choose the normalization for A^{RN} to obtain

$$A^{\mathrm{RN}} = \left(-\frac{e}{r_{\mathrm{RN}}} + \frac{e}{r_+}\right) \mathrm{d}u \tag{2-30}$$

such that the 1-form A^{RN} extends smoothly to the right event horizon $\mathcal{H}^+$ on Reissner–Nordström.

For the linear theory in Section 5 we will work with the t-Fourier transform. In that context it is useful to use a gauge which is *different* from (2-30) and which is given (see (5-1)) by

$$A'_{\mathrm{RN}} = \left(\frac{e}{r_{\mathrm{RN}}} - \frac{e}{r_+}\right) \mathrm{d}t = \left(\frac{e}{r_+} - \frac{e}{r_{\mathrm{RN}}}\right) \frac{\mathrm{d}u - \mathrm{d}v}{2}. \tag{2-31}$$

2D. ***The Einstein–Maxwell–Klein–Gordon system in null coordinates.*** We now express the EMKG system (1-1)–(1-5) in a double-coordinate system (u, v) on $\mathcal{Q}$ using the electromagnetic gauge (2-26). The unknown functions $(r, \Omega^2, A_u, Q, \phi)$ on $\mathcal{Q}$ are subject to the system

$$\partial_u \partial_v r = \frac{-\Omega^2}{4r} - \frac{\partial_u r\, \partial_v r}{r} + \frac{\Omega^2}{4r^3} Q^2 + \frac{m^2 r}{4} \Omega^2 |\phi|^2 = -\frac{\Omega^2}{2} \cdot 2K + \frac{m^2 r}{4} \Omega^2 |\phi|^2, \tag{2-32}$$

$$\partial_u \partial_v \log(\Omega^2) = -2\Re(D_u \phi \partial_v \bar{\phi}) + \frac{\Omega^2}{2r^2} + \frac{2\partial_u r\, \partial_v r}{r^2} - \frac{\Omega^2}{r^4} Q^2, \tag{2-33}$$

the Raychaudhuri equations

$$\partial_u \left(\frac{\partial_u r}{\Omega^2}\right) = \frac{-r}{\Omega^2} |D_u \phi|^2, \tag{2-34}$$

$$\partial_v \left(\frac{\partial_v r}{\Omega^2}\right) = \frac{-r}{\Omega^2} |\partial_v \phi|^2, \tag{2-35}$$

the charged and massive Klein–Gordon equation

$$\partial_u \partial_v \phi = -\frac{\partial_u \phi\, \partial_v r}{r} - \frac{\partial_u r\, \partial_v \phi}{r} + \frac{q_0 i \Omega^2}{4r^2} Q\phi - \frac{m^2 \Omega^2}{4} \phi - i q_0 A_u \frac{\phi\, \partial_v r}{r} - i q_0 A_u\, \partial_v \phi, \tag{2-36}$$

and the Maxwell equations

$$\partial_u Q = -q_0 r^2 \Im(\phi \overline{D_u \phi}), \tag{2-37}$$

$$\partial_v Q = q_0 r^2 \Im(\phi \overline{\partial_v \phi}). \tag{2-38}$$

Finally, $F = \mathrm{d}A$ reads

$$\partial_v A_u = \frac{-Q\Omega^2}{2r^2}. \tag{2-39}$$

Note that (2-37) and (2-38) can be equivalently formulated introducing the quantity $\psi := r\phi$ as

$$\partial_u Q = -q_0 \Im(\psi \overline{D_u \psi}), \tag{2-40}$$

$$\partial_v Q = q_0 \Im(\psi \overline{\partial_v \psi}). \tag{2-41}$$

Further, (2-32) is equivalent to

$$\partial_u (r\, \partial_v r) = \frac{-\Omega^2}{4} + \frac{\Omega^2}{4r^2} Q^2 + \frac{m^2 r^2}{4} \Omega^2 |\phi|^2. \tag{2-42}$$

We can also rewrite (2-36) to control $|\partial_v \phi|$ more easily:

$$D_u\, \partial_v \phi = e^{-i q_0 \int_{u_0}^{u} A_u}\, \partial_u (e^{i q_0 \int_{u_0}^{u} A_u}\, \partial_v \phi) = -\frac{\partial_v r\, D_u \phi}{r} - \frac{\partial_u r\, \partial_v \phi}{r} + \frac{q_0 i \Omega^2}{4r^2} Q\phi - \frac{m^2 \Omega^2}{4} \phi. \tag{2-43}$$

We also have (recalling the notation $\psi = r\phi$)

$$e^{-iq_0\int_{u_0}^{u} A_u}\,\partial_u(e^{iq_0\int_{u_0}^{u} A_u}\,\partial_v\psi)$$
$$= D_u(\partial_v\psi) = \frac{-\Omega^2\phi}{4r} - \frac{\partial_u r\,\partial_v r\cdot\phi}{r} - \frac{\Omega^2\phi}{4r^3}Q^2 + \frac{m^2 r}{4}\Omega^2\phi|\phi|^2 - \frac{m^2\Omega^2 r}{4}\phi - \frac{q_0 i\Omega^2}{4r}Q\phi \tag{2-44}$$

and

$$\partial_v(D_u\psi) = \frac{-\Omega^2\phi}{4r} - \frac{\partial_u r\,\partial_v r\cdot\phi}{r} + \frac{\Omega^2\phi}{4r^3}Q^2 + \frac{m^2 r}{4}\Omega^2\phi|\phi|^2 - \frac{m^2\Omega^2 r}{4}\phi - \frac{q_0 i\Omega^2}{4r}Q\phi. \tag{2-45}$$

3. Setup of the characteristic data and the oscillation condition

We first fix the arbitrary quantities

$$\text{subextremal charge and mass parameters } 0 < |e| < M, \tag{3-1}$$
$$\text{a decay rate } \tfrac{3}{4} < s \le 1, \tag{3-2}$$
$$\text{constants } D_1,\, D_2 > 0. \tag{3-3}$$

These quantities will be kept fixed from now onward.

3A. ***Characteristic cones $\underline{C}_{\mathrm{in}}$, $\mathcal{H}^+$ and underlying manifold $\mathcal{Q}^+$.*** Our yet-to-be-constructed spacetime of study will be the future domain of dependence $\mathcal{Q}^+$ of the characteristic set $\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+ \subset \mathbb{R}^{1+1}$, where $\mathcal{H}^+ := \{U = 0,\ v_0 \le v < +\infty\}$ and $\underline{C}_{\mathrm{in}} := \{0 \le U \le U_s,\ v = v_0\}$, which meet transversely at the common boundary point $p := \{U = 0,\ v = v_0\}$. Here, we use the convention that $f \in C^1(\mathcal{H}^+)$ means that $f \in C^1((v_0,\infty)) \cap C^0([v_0,\infty))$ with the property that $\partial_v f$ extends continuously to $v_0 = \partial\mathcal{H}^+$. Analogously, we define $C^1(\underline{C}_{\mathrm{in}})$. Moreover, we say that $f \in C^1(\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+)$ if f is continuous on $\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+$ and $f|_{\mathcal{H}^+} \in C^1(\mathcal{H}^+)$, $f|_{\underline{C}_{\mathrm{in}}} \in C^1(\underline{C}_{\mathrm{in}})$. In particular, note that if $f_1 \in C^1(\mathcal{H}^+)$ and $f_2 \in C^1(\underline{C}_{\mathrm{in}})$ satisfy $f_1(p) = f_2(p)$, then they define a function in $C^1(\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+)$. Analogously, we define C^k for $k \ge 2$. We define $\mathcal{Q}^+ := \{0 \le U \le U_s,\ v_0 \le v < +\infty\}$. Here $v_0 = v_0(M, e, s, D_1) \ge 1$ only depends on M, e, s, D_1 and $U_s = U_s(M, e, s, D_2, D_1)$ only depends on M, e, s, D_2, D_1 — both of which will be determined in Proposition 3.2 below.

A new coordinate u. We will make use of other coordinates (u, v) on $\mathcal{Q}^+ - \mathcal{H}^+$ given by $u := u(U)$, $v = v$, where $u(U)$ is the function given through the condition (2-9) and (M, e) are as in (3-1). We also define $u_s := u(U_s)$.

An additional electromagnetic gauge freedom. At this point we recall our global electromagnetic gauge choice $A_v \equiv 0$ in Section 2C. An additional electromagnetic gauge freedom we have is the specification of A_U (or equivalently A_u) on $\underline{C}_{\mathrm{in}}$. We impose that A_U on $\underline{C}_{\mathrm{in}} = \{0 \le U \le U_s,\ v = v_0\}$ satisfies

$$A_U(U, v_0) = \left(-\frac{e}{r_{\mathrm{RN}}(U, v_0)} + \frac{e}{r_+(e, M)}\right)\frac{\mathrm{d}u}{\mathrm{d}U}(U) = 2\left(-\frac{e}{r_{\mathrm{RN}}(U, v_0)} + \frac{e}{r_+(e, M)}\right)\Omega_{\mathrm{RN}}^{-2}(U, v_0), \tag{3-4}$$

where we used (2-9) for the second identity and thus

$$A_u(u, v_0) = -\frac{e}{r_{\mathrm{RN}}(u, v_0)} + \frac{e}{r_+(e, M)}. \tag{3-5}$$

Here, r_{RN} is the r-value on Reissner–Nordström with parameters (M, e) as given in (3-1) and $r_+(M, e) = M^2 + \sqrt{M^2 - e^2}$.

3B. ***Coordinate gauge conditions on $\mathcal{H}^+$ and $\underline{C}_{\mathrm{in}}$.*** On $\mathcal{H}^+ = \{U = 0,\ v_0 \leq v\}$ we will impose the gauge condition

$$\frac{\partial_U r(0, v)}{\Omega_H^2(0, v)} = -\frac{1}{2} \tag{3-6}$$

and on $\underline{C}_{\mathrm{in}} = \{0 \leq U \leq U_s,\ v = v_0\}$ we will impose

$$\partial_U r = -1. \tag{3-7}$$

3C. ***Free data $\phi \in C^1(\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+)$ with slow decay on $\mathcal{H}^+$ and construction of r, Q, Ω_H^2.*** Having set up the gauges we will now — in addition to the free prescription of $0 < |e| < M$ in (3-1) — freely prescribe data for ϕ on $\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+$. We recall (3-2) and (3-3) and define the class of slowly decaying data $\mathcal{SL}$ on the event horizon $\mathcal{H}^+$ in the following. In order to highlight that the definition does not depend on the gauge choice for the electromagnetic potential A, we formulate it in a gauge-invariant form (although we have already fixed the gauge $A_v \equiv 0$ in (2-26) and (3-5)).

Definition 3.1 (set of slowly decaying data $\mathcal{SL}$). We say that $\phi_{\mathcal{H}^+} \in C^1(\mathcal{H}^+, \mathbb{C})$ is *slowly decaying*, denoted by $\phi_{\mathcal{H}^+} \in \mathcal{SL}$, if

$$|\phi_{\mathcal{H}^+}|(v) + |D_v \phi_{\mathcal{H}^+}|(v) \leq D_1 v^{-s} \tag{3-8}$$

for all $v \in \mathcal{H}^+$, where we recall $\frac{3}{4} < s \leq 1$ was introduced and fixed in (3-2), and $D_1 > 0$ was introduced and fixed in (3-3).

Similarly, on $\underline{C}_{\mathrm{in}}$ we will also impose arbitrary (up to the corner condition) data $\phi_{\mathrm{in}} \in C^1(\underline{C}_{\mathrm{in}})$ satisfying

$$|D_U \phi_{\mathrm{in}}| \leq D_2. \tag{3-9}$$

We will now finally conclude the setup of the initial data, where we recall that we freely prescribed subextremal e, M and the scalar field ϕ on $\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+$. In particular, using standard results about ODEs (recall that $s > \frac{3}{4}$; actually $s > \frac{1}{2}$ is sufficient to prove Proposition 3.2) we obtain:

Proposition 3.2. *There exist $v_0(M, e, s, D_1) \geq 1$ sufficiently large and $U_s(M, e, s, D_2, D_1) > 0$ sufficiently small such that the following holds true. Let $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ and $\phi_{\mathrm{in}} \in C^1(\underline{C}_{\mathrm{in}})$ satisfying* (3-9) *with $\phi_{\mathcal{H}^+}(p) = \phi_{\mathrm{in}}(p)$ be arbitrary. Then, there exist unique solutions $r \in C^2(\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+)$, $\Omega_H \in C^1(\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+)$ and $Q \in C^1(\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+)$ of the ODE system consisting of the Raychaudhuri equation* (2-35), *equation* (2-38), *the equation* (2-32) *using* (3-6) *on $\mathcal{H}^+$ and the ODE system consisting of* (3-7), (2-34) *and* (2-37) *on $\underline{C}_{\mathrm{in}}$ such that*

$$\lim_{v \to +\infty} r(0, v) = r_+(M, e) = M + \sqrt{M^2 + e^2}, \tag{3-10}$$

$$\lim_{v \to +\infty} Q(0, v) = e. \tag{3-11}$$

Moreover, $\mathcal{H}^+$ is affine complete, i.e., $\int_{v_0}^{+\infty} \Omega_H^2(0, v)\, \mathrm{d}v = +\infty$.

This shows that our free data (e, M, ϕ) and the gauge conditions give rise to a full set of data (r, Q, Ω_H^2, ϕ) on $\underline{C}_{\text{in}} \cup_p \mathcal{H}^+$ satisfying the constraint equations.

Further, note that (3-6) implies that

$$\kappa_{|\mathcal{H}^+} \equiv 1$$

in view of (2-15).

Remark 3.3. We also associate to (u, v) a lapse function Ω^2 through

$$\Omega^2 := \Omega_H^2 \frac{\mathrm{d}U}{\mathrm{d}u} \tag{3-12}$$

such that $\Omega^2 = -2g(\partial_u, \partial_v)$ and $\Omega_H^2 = -2g(\partial_U, \partial_v)$ once the spacetime is constructed.

Remark 3.4. In Theorem III we will introduce generic properties of functions in $\mathcal{SL}$. We remark that $\mathcal{SL}$ is the ball of size D_1 in the Banach space

$$\mathcal{SL}_0 := \left\{ f \in C^1(\mathcal{H}^+; \mathbb{C}) : \sup_{v \geq v_0} (|v^s f| + |v^s D_v f|) < +\infty \right\}. \tag{3-13}$$

In Theorem III (more precisely in Corollary 5.27) we identify a (exceptional) subspace $H_0 \subset \mathcal{SL}_0$ of infinite codimension. We then call functions $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ generic if $\phi_{\mathcal{H}^+} \in \mathcal{SL} - H$, where $H := H_0 \cap \mathcal{SL}$.

3D. *Definitions of the oscillation spaces $\mathcal{O}$, $\mathcal{O}'$, $\mathcal{O}''$.* We now define the subsets $\mathcal{O}, \mathcal{O}', \mathcal{O}'' \subset \mathcal{SL}$ of slowly decaying data on the event horizon describing the oscillation conditions. In order to highlight that the definitions do not depend on the gauge choice for the electromagnetic potential A we formulate them in a gauge-invariant form (although we have already fixed the gauge $A_v \equiv 0$ in (2-26) and (3-5)).

Definition 3.5 (qualitative oscillation condition $\mathcal{O}$). A function $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ is said to satisfy the *qualitative oscillation condition*, denoted by $\phi_{\mathcal{H}^+} \in \mathcal{O}$, if the qualitative condition

$$\limsup_{v \to +\infty} \left| \int_{v_0}^{v} \phi_{\mathcal{H}^+}(v') e^{i(\omega_{\text{res}} v' + q_0 \sigma_{\text{br}}(v'))} e^{i q_0 \int_{v_0}^{v'} (A_v)_{|\mathcal{H}^+}(v'')\, \mathrm{d}v''} \,\mathrm{d}v' \right| < +\infty \tag{3-14}$$

holds for all $D_{\text{br}} > 0$ and all functions $\sigma_{\text{br}} \in C^2([v_0, +\infty), \mathbb{R})$ satisfying

$$|\sigma_{\text{br}}(v)| \leq D_{\text{br}} \cdot (v^{2-2s} 1_{s<1} + \log(1+v) 1_{s=1}), \tag{3-15}$$

$$|\sigma'_{\text{br}}(v)| + |\sigma''_{\text{br}}(v)| \leq D_{\text{br}} v^{1-2s} \tag{3-16}$$

for all $v \geq v_0$, where we recall that $v_0(M, e, s, D_1) > 1$.

We will also denote by $\mathcal{NO} := \mathcal{SL} - \mathcal{O}$ the space of $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ violating (3-14).

Definition 3.6 (strong qualitative oscillation condition $\mathcal{O}'$). A function $\phi_{\mathcal{H}^+} \in \mathcal{O}$ is said to satisfy the *strong qualitative oscillation condition*, denoted by $\phi_{\mathcal{H}^+} \in \mathcal{O}'$, if the limit

$$\lim_{v \to +\infty} \left| \int_{v_0}^{v} \phi_{\mathcal{H}^+}(v') e^{i(\omega_{\text{res}} v' + q_0 \sigma_{\text{br}}(v'))} e^{i q_0 \int_{v_0}^{v'} (A_v)_{|\mathcal{H}^+}(v'')\, \mathrm{d}v''} \,\mathrm{d}v' \right| \tag{3-17}$$

exists (and is finite) for all $D_{\text{br}} > 0$ and all functions $\sigma_{\text{br}} \in C^2([v_0, +\infty), \mathbb{R})$ satisfying (3-15) and (3-16).

Definition 3.7 (quantitative oscillation condition $\mathcal{O}''$). A function $\phi_{\mathcal{H}^+} \in \mathcal{O}'$ is said to satisfy the *quantitative oscillation condition*, denoted by $\phi_{\mathcal{H}^+} \in \mathcal{O}''$, if for all $D_{\mathrm{br}} > 0$ there exist $E_{\mathcal{O}''}(D_{\mathrm{br}}) > 0$, $\eta_0(D_{\mathrm{br}}) > 0$ such that

$$\left| \int_v^{+\infty} e^{i(\omega_{\mathrm{res}} \cdot v' + q_0 \sigma_{\mathrm{br}}(v'))} e^{iq_0 \int_{v_0}^{v'} (A_v)_{|\mathcal{H}^+}(v'')\, dv''} \phi_{\mathcal{H}^+}(v')\, dv' \right| \le E_{\mathcal{O}''} \cdot v^{s-1-\eta_0} \tag{3-18}$$

for all $v \ge v_0$ and all functions $\sigma_{\mathrm{br}} \in C^2([v_0, +\infty), \mathbb{R})$ satisfying (3-15) and (3-16).

Remark 3.8. Note that we have by definition the inclusions $\mathcal{O}'' \subset \mathcal{O}' \subset \mathcal{O} \subset \mathcal{SL}$. Moreover, note that $\mathcal{O}'' \not\subset L^1([v_0, +\infty))$; more generally, a generic function of $\mathcal{O}''$ is not in $L^1([v_0, +\infty))$.

Remark 3.9. The condition (3-14) and its stronger versions (3-17), (3-18) guarantee sufficiently robust nonresonant oscillations. These conditions are sufficient (our proof also suggests that they are necessary to some extent) to avoid that the backreaction of the Maxwell field (which, as we will show, creates unbounded but sublinear oscillations σ_{br} obeying (3-15), (3-16)) turns *linearly nonresonant* profiles into *nonlinearly resonant* profiles; see the last paragraph of Section 4F3 for a discussion.

Remark 3.10. In the uncharged case $q_0 = 0$, the backreaction of the electric field is absent. In this case note that (3-14) simplifies to a "finite average" condition.

4. Precise statements of the main theorems and outline of their proofs

4A. ***Existence of a Cauchy horizon $\mathcal{CH}_{i^+} \neq \emptyset$ and quantitative estimates in the black hole interior from*** **[Van de Moortel 2018].** In [Van de Moortel 2018], the second author proved (among other results) that spherically symmetric EMKG black holes converging to a subextremal Reissner–Nordström admit a null boundary $\mathcal{CH}_{i^+} \neq \emptyset$ that we still call a Cauchy horizon. The proof of this main result in [Van de Moortel 2018] required many quantitative estimates that will be useful in the analysis of the current paper.

Theorem B [Van de Moortel 2018]. *Consider the characteristic data on $\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+$ as described in Section 3 and fix the electromagnetic gauge* (2-26) *as in Section 2C. Let $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ be arbitrary, and let $\phi_{\mathrm{in}} \in C^1(\underline{C}_{\mathrm{in}})$ satisfying* (3-9) *with $\phi_{\mathrm{in}}(p) = \phi_{\mathcal{H}^+}(p)$ be arbitrary.*

Then, by choosing $U_s(M, e, s, D_2, D_1) > 0$ potentially smaller, the characteristic data give rise to the unique C^1 maximal globally hyperbolic development $(r, \Omega_H^2, A, Q, \phi)$ on $\mathcal{Q}^+$ solving the EMKG system of Section 2D. In addition, an (ingoing) null boundary $\mathcal{CH}_{i^+} \neq \emptyset$ (the Cauchy horizon) can be attached to $\mathcal{Q}^+$ on which r extends as a continuous function r_{CH} which remains bounded away from zero, depicted in the Penrose diagram in Figure 1. Note that $(r, \Omega_H^2, A, Q, \phi)$ on $\mathcal{Q}^+$ defines $(\mathcal{M}, g, A, F, \phi)$ which solves (1-1)–(1-5).

Moreover, all the quantitative estimates stated in Propositions 6.1, 6.2, 6.3, 6.4 and 6.5 are satisfied.

If we additionally assume fast decay (i.e., $\phi_{\mathcal{H}^+}$ satisfies (3-8) *for $s > 1$), then ϕ is in $W^{1,1}_{\mathrm{loc}} \cap L^\infty$ at the Cauchy horizon $\mathcal{CH}_{i^+}$ and extends as a continuous function across the Cauchy horizon $\mathcal{CH}_{i^+}$. Moreover, in this case, the metric admits a C^0-admissible extension $\tilde{g}$ across the Cauchy horizon $\mathcal{CH}_{i^+}$ in the sense of Definition 2.1 and $\tilde{g}$ has locally integrable Christoffel symbols.*

Remark 4.1. We note that the above Theorem B showing $\mathcal{CH}_{i^+} \neq \varnothing$, together with all the quantitative estimates stated in Propositions 6.1, 6.2, 6.3, 6.4 and 6.5, actually holds under the weaker assumption of decay rate $s > \frac{1}{2}$ as opposed to $s > \frac{3}{4}$; see [Van de Moortel 2018]. For the purpose of extendibility across the Cauchy horizon $\mathcal{CH}_{i^+}$ for oscillating data as stated in our main result below, the decay assumption $s > \frac{3}{4}$ is needed and appears to be crucial; see the discussion in Section 4F.

4B. ***Theorem I*(i): *scalar field boundedness and continuous extendibility for oscillating data.*** In this section we give the precise version of Theorem I (i), which is proved as Corollary 6.18 in Section 6D1.

Theorem I (i) (boundedness). *Let the assumptions of Theorem B hold.*

(1) *If $\phi_{\mathcal{H}^+}$ satisfies the qualitative oscillation condition $\phi_{\mathcal{H}^+} \in \mathcal{O}$* (*see Definition 3.5*), *then*

$$\sup_{(u,v)\in\mathcal{Q}^+} |\phi(u,v)| < +\infty. \tag{4-1}$$

(2) *If $\phi_{\mathcal{H}^+}$ satisfies the strong qualitative oscillation condition $\phi_{\mathcal{H}^+} \in \mathcal{O}'$* (*see Definition 3.6*), *then* (4-1) *is true and moreover ϕ admits a continuous extension to $\mathcal{CH}_{i^+}$ and g admits a C^0-admissible extension to $\mathcal{CH}_{i^+}$ in the sense of Definition 2.1. In particular*, *g is continuously extendible.*

(3) *If $\phi_{\mathcal{H}^+}$ satisfies the quantitative oscillation condition $\phi_{\mathcal{H}^+} \in \mathcal{O}''$* (*see Definition 3.7*), *then* (4-1) *is true*, *ϕ admits a continuous extension to $\mathcal{CH}_{i^+}$ and g admits a C^0-admissible extension to $\mathcal{CH}_{i^+}$. Moreover*, *Q is uniformly bounded on $\mathcal{Q}^+$ and admits a continuous extension to $\mathcal{CH}_{i^+}$. Further*, *there exists a constant $\widetilde{C} = \widetilde{C}(D_1, D_2, E_{\mathcal{O}''}, \eta_0, e, M, m^2, q_0, s) > 0$ such that for all $(u,v) \in \mathcal{LB} \subset \mathcal{Q}^+$*

$$|\phi|(u,v) \le \widetilde{C} \cdot |u|^{s-1-\eta_0}, \tag{4-2}$$

$$|Q - e|(u,v) \le \widetilde{C} \cdot |u|^{-\eta_0}, \tag{4-3}$$

where $E_{\mathcal{O}''} = E_{\mathcal{O}''}(D_{\mathrm{br}}) > 0$, $\eta_0 = \eta_0(D_{\mathrm{br}}) > 0$ are as in (3-18) *and $D_{\mathrm{br}} := D_{\mathrm{br}}(D_1, D_2, e, M, m^2, q_0, s) > 0$ is defined in the proof of Proposition 6.17. Here $\mathcal{LB}$ denotes the late blue-shift region* (*see Figure 7*), *a neighborhood of the Cauchy horizon which is defined in Section 6A.*

4C. ***Theorem I*(ii): *blow-up in amplitude of the uncharged scalar field for nonoscillating data.*** In this section we give the precise version of Theorem I (ii), which is proved as Corollary 6.20 in Section 6D2.

Theorem I (ii) (blow-up). *Let the assumptions of Theorem B hold and let $q_0 = 0$ and $m^2 \in \mathbb{R}_{>0} - D(M,e)$, where $D(M,e)$ is the discrete set of exceptional nonresonant masses as defined in* [Kehle and Shlapentokh-Rothman 2019, Theorem 7]. *In addition*, *assume that $\phi_{\mathcal{H}^+}$ violates the qualitative oscillation condition as in Definition 3.1*, *i.e.*, *assume that $\phi_{\mathcal{H}^+} \in \mathcal{NO} := \mathcal{SL} - \mathcal{O}$.*

Then, *for all $u \le u_s$*, *the scalar field blows up in amplitude at the Cauchy horizon $\mathcal{CH}_{i^+}$*:

$$\limsup_{v\to+\infty} |\phi|(u,v) = +\infty. \tag{4-4}$$

4D. ***Theorem II*: *falsification of C^0-formulation of strong cosmic censorship if Conjecture 2 is true.*** We now give the precise version of Theorem II which is proved as Corollary 6.23 in Section 6D3.

Theorem II. *Let the assumptions of Theorem B hold. Additionally assume that Conjecture 2 is true, i.e., $\phi_{\mathcal{H}^+}$ is given by* (1-15) (*if* $q_0 = 0,\ m^2 > 0$), (1-16) (*if* $q_0 \neq 0,\ m^2 = 0$), *or* (1-17) (*if* $q_0 \neq 0,\ m^2 > 0$) *in the v-coordinate defined by* (3-6) *and that the generic condition $|q_0 e| \neq r_-(M, e)|m|$ holds.*

Then $|\phi|$, Q and the metric g admit a continuous extension to $\mathcal{CH}_{i^+}$ and the extension of g can be chosen to be C^0-admissible.

In the above sense, assuming that Conjecture 2 is true, then Conjecture 1 is false for the Einstein–Maxwell–Klein–Gordon system in spherical symmetry.

4E. ***Theorem III*: $W^{1,1}$ *blow-up of the scalar field for nonintegrable data.*** In this section we give the precise version of Theorem III, which is proved in Section 6D4. To state the theorem we first define the set

$$\mathcal{Z}_{\mathfrak{t}}(M, e, q_0, m^2) := \{\omega \in \mathbb{R} \colon \mathfrak{t}(\omega, M, e, q_0, m^2) = 0\} \subset \mathbb{R}, \tag{4-5}$$

which is the zero set of the renormalized transmission coefficient $\mathfrak{t}(\omega)$ defined in (5-23). At this point we note that $\mathcal{Z}_{\mathfrak{t}}(M, e, q_0, m^2)$ is discrete and, depending on the parameters (M, e, q_0, m^2), possibly empty. For small $\delta > 0$ we also define the smeared out set $\mathcal{Z}^\delta_{\mathfrak{t}}(M, e, q_0, m^2) \subset \mathbb{R}$ as the set of all $\omega \in \mathbb{R}$ with $\operatorname{dist}(\omega, \mathcal{Z}_{\mathfrak{t}}(M, e, q_0, m^2)) < \delta$. We remark that $\mathcal{Z}^\delta_{\mathfrak{t}}(M, e, q_0, m^2) = \varnothing$ if $\mathcal{Z}_{\mathfrak{t}}(M, e, q_0, m^2) = \varnothing$.

Associated to $\mathcal{Z}^\delta_{\mathfrak{t}}(M, e, q_0, m^2)$ we now define a family (parametrized by $\delta > 0$) of Fourier projection operators $P_\delta \colon f \in L^2([v_0, +\infty)) \mapsto \mathcal{F}^{-1}[\chi_\delta \mathcal{F}[\tilde{f}]] \in L^2(\mathbb{R})$, where $\tilde{f} \in L^2(\mathbb{R})$ is the extension of f by the zero function for $v < v_0$. Here, $\chi_\delta(\omega)$ is a family (parametrized by $\delta > 0$) of smooth functions which are positive on $\mathcal{Z}^\delta_{\mathfrak{t}}(M, e, q_0, m^2)$ and vanish otherwise. In the case where $\mathcal{Z}^\delta_{\mathfrak{t}}(M, e, q_0, m^2) = \varnothing$, also $\chi_\delta \equiv 0$. Further, for the Fourier transform, we use the convention

$$\mathcal{F}[\tilde{f}](\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{f}(v) e^{i\omega v} \, \mathrm{d}v.$$

Finally, we are in the position to state Theorem III which is proved in Section 6D4. The first part is shown as Corollary 6.25; the second part is shown as Corollary 6.26.

Theorem III. *Let the assumptions of Theorem B hold.*

***Part* 1.** *Let $\phi_{\mathcal{H}^+} \in \mathcal{SL} - L^1([v_0, +\infty))$ and let at least one of the following assumptions hold*:

(a) $P_\delta \phi_{\mathcal{H}^+} \in L^1(\mathbb{R})$ *for some* $\delta > 0$,

(b) *or* $0 < |q_0 e| \leq \epsilon(M, e, m^2)$ *for some* $\epsilon(M, e, m^2) > 0$ *sufficiently small or* $q_0 = 0,\ m^2 \notin D(M, e)$.

Then, the scalar field ϕ blows up in $W^{1,1}$ along outgoing cones at the Cauchy horizon $\mathcal{CH}_{i^+}$ in the sense that for all $u \leq u_s$

$$\int_{v_0}^{+\infty} |D_v \phi|(u, v) \, \mathrm{d}v = +\infty. \tag{4-6}$$

In particular, for any $q_0 \in \mathbb{R}$ and $m^2 \geq 0$, the set H of data $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ for which (4-6) *is not satisfied for all $u \leq u_s$ is exceptional in the sense that $H = H_0 \cap \mathcal{SL}$, where $H_0 \subset \mathcal{SL}_0$ is a subspace of infinite codimension within $\mathcal{SL}_0$* (*recall the definition of $\mathcal{SL}_0$ from* (3-13)). *In the above sense, $\mathcal{SL} - H$ is a generic set and thus $W^{1,1}$-blow-up of the scalar field at the Cauchy horizon $\mathcal{CH}_{i^+}$ is a generic property of the data $\phi_{\mathcal{H}^+} \in \mathcal{SL}$.*

Part 2. *Assume that* $\phi_{\mathcal{H}^+}$ *is given by* (1-15) (*if* $q_0 = 0,\ m^2 > 0$), (1-16) (*if* $q_0 \neq 0,\ m^2 = 0$), *or* (1-17) (*if* $q_0 \neq 0,\ m^2 > 0$) *in the* v*-coordinate defined by* (3-6). *Assume the conditions*

$$\mathcal{Z}_{\mathrm{t}} \cap \Theta = \varnothing, \tag{4-7}$$

$$(m^2, |q_0 e|) \notin \{0\} \times \left[0, \tfrac{1}{2}\right), \tag{4-8}$$

where $\mathcal{Z}_{\mathrm{t}}(M, e, q_0, m^2)$ *is defined in* (4-5) *and where*

$$\Theta(M, e, q_0, m^2) := \begin{cases} \{-m, +m\} & \text{if } q_0 = 0, m^2 \neq 0, \\ \{-q_0 e/r_+\} & \text{if } |q_0 e| \geq \frac{1}{2},\ m^2 = 0, \\ \{-m - q_0 e/r_+, m - q_0 e/r_+\} & \text{if } q_0 \neq 0,\ m^2 \neq 0. \end{cases}$$

Then, the scalar field ϕ *blows up in* $W^{1,1}$ *along outgoing cones at the Cauchy horizon* $\mathcal{CH}_{i^+}$, *i.e.*, (4-6) *holds for all* $u \leq u_s$.

Moreover, (4-7) *is satisfied generically in the sense that for given parameters* $m^2 \geq 0,\ q_0 \in \mathbb{R}$, *with* $m^2 \neq q_0^2$, *the condition* (4-7) *is satisfied for*

$$(M, e) \in \{(M, e) \in \mathbb{R}^2 : 0 < |e| < M\} - E_{m^2, q_0},$$

where $E_{m^2, q_0} \subset \mathbb{R}^2$ *is the zero set of an analytic function.*

In particular, *for fixed* $m^2 \geq 0,\ q_0 \in \mathbb{R}$ *with* $m^2 \neq q_0^2$ *and* $(m^2, |q_0 e|) \notin \{0\} \times \left[0, \frac{1}{2}\right)$ *and for almost all parameters*

$$(M, e) \in \{(M, e) \in \mathbb{R}^2 : 0 < |e| < M\},$$

assuming $\phi_{\mathcal{H}^+}$ *is as above*, *then* (4-6) *holds for all* $u \leq u_s$.

Remark 4.2. Note that (4-6) also implies the blow-up of the spacetime $W^{1,1}$ norm in (u, v)-coordinates, i.e., for all $u_1 < u_2 \leq u_s$

$$\int_{u_1}^{u_2} \int_{v_0}^{+\infty} |D_v \phi|(u, v)\, \mathrm{d}v\, \mathrm{d}u = +\infty.$$

The precise formulation and the proof of Theorem IV will be given in our companion paper [Kehle and Van de Moortel $\geq$ 2024].

4F. ***Outline of the proofs.*** In this section, we elaborate on steps (1)–(4) originally presented in Section 1F. The reader may wish to come back to the current section while consulting the proofs given in Sections 5 and 6. For convenience, we will conclude this section with a guide for the reader; see Section 4F5.

4F1. *A first approach in physical space and the difficulties associated to slow decay* (*step* (1)).

Physical space estimates for the nonlinear problem. Theorem B proving $\mathcal{CH}_{i^+} \neq \varnothing$ also comes with many quantitative stability estimates (see Section 6A) for the nonlinear problem (1-1)–(1-5) under the assumption of slowly decaying $\phi_{\mathcal{H}^+}$ satisfying (1-8) on $\mathcal{H}^+$ $\left(\text{not only for } s > \frac{3}{4} \text{ but also } s > \frac{1}{2}\right)$. These estimates already proven in [Van de Moortel 2018] will be our starting point in Section 6. Although these estimates are sharp, they are however not sufficient to prove the boundedness of ϕ in amplitude, in view of the slow decay obstruction if $s \leq 1$ as we shall explain below. To illustrate our point, we start with one of the main estimates[10] obtained by physical space methods in [Van de Moortel 2018]:

$$|D_v\phi|(u,v) \lesssim v^{-s}. \tag{4-9}$$

Boundedness/continuous extendibility in the integrable case. In the integrable case $s > 1$, integrating (4-9) gives immediately boundedness

$$\|\phi\|_{L^\infty} \lesssim \text{data} + \sup_u \|D_v\phi(u,\cdot)\|_{L^1_v} \lesssim \text{data} + \|\langle v\rangle^{-s}\|_{L^1_v} < +\infty, \tag{4-10}$$

and also gives the $W^{1,1}$-extendibility of the metric (i.e., locally integrable Christoffel symbols). From the estimates giving the $W^{1,1}$-extendibility of the metric, one can immediately deduce the continuous extendibility of the metric (see the discussion in Section 4F4). All the known previous proofs of continuous extendibility of the metric indeed proceed via this method [Luk and Oh 2019a; Dafermos 2003; Dafermos and Luk 2017].

Slow decay obstruction in the nonintegrable case. In present paper we however have to deal with the nonintegrable case $s \leq 1$, where we note that the above method fails as $\|\langle v\rangle^{-s}\|_{L^1_v}$ (the right-hand side of (4-10)) is infinite, even suggesting that the left-hand side $\|D_v\phi(u,\cdot)\|_{L^1_v}$ could be infinite as well. Indeed, we prove *blow-up* of $\|D_v\phi(u,\cdot)\|_{L^1_v}$ (the so-called $W^{1,1}$ norm on outgoing cones) for generic data $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ (Theorem III, see Section 4F4 for a description of its proof), which illustrates the obstruction to proving boundedness by the standard method previously used in the $s > 1$ case.

Summary of the rate numerology. To summarize, square-integrable decay $\big($i.e., (3-8) with $s > \frac{1}{2}\big)$ is sufficient to show that the black hole boundary admits a null component $\mathcal{CH}_{i^+}$ (the Cauchy horizon) by Theorem B, but is in general insufficient for $W^{1,1}$ extendibility and boundedness of the matter fields and metric coefficients (for which integrable decay, i.e., (3-8) with $s > 1$, is sufficient). In the rest of the section, we explain how to deal with the broader range $\frac{3}{4} < s \leq 1$ ($s > \frac{3}{4}$ is important for the new nonlinear estimates; see Section 4F4 and Remark 4.3).

An ingoing derivative estimate. Yet another particularity of the nonintegrable case $s \leq 1$ is that $|D_u\phi|$ *may potentially* blow up in amplitude at the Cauchy horizon [Van de Moortel 2018] (there are indeed known examples for which $|D_u\phi|$ blows up; see [Van de Moortel 2021]). Nevertheless, assuming $s > \frac{3}{4}$, we show that $D_u(r\phi)$ is *uniformly bounded* (Proposition 6.6), although not integrable, i.e., we prove that for all $\phi_{\mathcal{H}^+}$ satisfying (3-8)

$$|D_u(r\phi)|(u,v) \lesssim |u|^{-s}. \tag{4-11}$$

Note that, consistently with our result that $|\phi|$ blows up for some data, (4-11) cannot be integrated in u.

Compensate the failure of integrability with oscillations. Slow decay of the data, as we explained, leads to a lack of integrability of the metric and fields derivatives which are roughly of the form, for $\frac{3}{4} < s \leq 1$,

$$|D_v\phi| \approx v^{-s}, \tag{4-12}$$

[10] The main difficulty in obtaining (4-9) is nonlinear in nature: its proof in [Van de Moortel 2018] exploits the structure of the Einstein equations to address the delicate issue of controlling the metric for a slow rate $s \leq 1$. In contrast, the null condition suffices if $s > 1$.

which is not integrable as $v \to +\infty$ (i.e., towards the Cauchy horizon $\mathcal{CH}_{i^+}$). Nevertheless, boundedness of ϕ could be obtained *by means of the oscillations*, i.e., if we could propagate an estimate of the form

$$D_v\phi \approx e^{i\omega v} \cdot v^{-s} \tag{4-13}$$

for some $\omega \in \mathbb{R} - \{0\}$. However, the propagation of such oscillations, if present on the event horizon characteristic data $\phi_{\mathcal{H}^+}$, requires further estimates in Fourier space that we introduce in the following section.

4F2. *The linear problem* (*step* (2)). In this section, we discuss how to prove boundedness or blow-up of $\phi_{\mathcal{L}}$ solving the linearized equation. This step corresponds to the proof of our main linear result Theorem V in Section 5.

Representation formula using the Fourier transform. For the linear (charged massive) wave equation $g_{\mathrm{RN}}^{\mu\nu} D_\mu^{\mathrm{RN}} D_\nu^{\mathrm{RN}} \phi_{\mathcal{L}} = m^2 \phi_{\mathcal{L}}$ on a fixed subextremal Reissner–Nordström interior metric (2-7), the physical space estimates of Section 4F1 also apply, but a Fourier approach is also possible, taking advantage of the Killing vector field ∂_t. Taking the Fourier transform in t, the wave equation then reduces to the so-called radial ODE (see (5-13)). Using this, we will view aspects of the interior propagation from the event horizon to the Cauchy horizon as a scattering problem mapping data on the event horizon to their evolution restricted to the Cauchy horizon; see [Kehle and Shlapentokh-Rothman 2019; Kehle 2022]. Formally, we have, in a suitable regular electromagnetic gauge at the Cauchy horizon:

$$\begin{aligned}\phi_{\mathcal{L}} \restriction_{\mathcal{CH}_{i^+}} (u) = {} & \frac{r_+}{\sqrt{2\pi} r_-} \,\mathrm{p.v.}\int_{\mathbb{R}} \frac{\mathfrak{r}(\omega)}{\omega - \omega_{\mathrm{res}}} \mathcal{F}[\phi_{\mathcal{H}^+}](\omega) e^{i(\omega-\omega_{\mathrm{res}})u}\, \mathrm{d}\omega \\ & + \lim_{v\to\infty} \frac{r_+}{\sqrt{2\pi} r_-} \,\mathrm{p.v.}\int_{\mathbb{R}} \frac{\mathfrak{t}(\omega)}{\omega - \omega_{\mathrm{res}}} \mathcal{F}[\phi_{\mathcal{H}^+}](\omega) e^{-i(\omega-\omega_{\mathrm{res}})v}\, \mathrm{d}\omega + \mathrm{Error},\end{aligned} \tag{4-14}$$

where Error is uniformly bounded by the energy of $\phi_{\mathcal{H}^+}$ along the event horizon $\mathcal{H}^+$ and $\omega_{\mathrm{res}}(M, e, q_0)$ is as in (1-7). Here, $\mathfrak{r}(\omega)$ and $\mathfrak{t}(\omega)$ are the (renormalized) scattering coefficients (see Definition 5.2).

Using that $\mathcal{F}[\mathrm{p.v.}(1/x)] = i\pi \,\mathrm{sgn}$ and $\mathfrak{t}(\omega_{\mathrm{res}}) = -\mathfrak{r}(\omega_{\mathrm{res}})$ (see (5-25)) we formally obtain

$$\phi_{\mathcal{L}} \restriction_{\mathcal{CH}_{i^+}} (u) = \frac{\sqrt{2\pi}\, i r_+}{r_-} \mathfrak{r}(\omega_{\mathrm{res}}) \lim_{v\to\infty} \int_{-u}^{v} \phi_{\mathcal{H}^+}(\tilde{v}) e^{i\omega_{\mathrm{res}}\tilde{v}}\, \mathrm{d}\tilde{v} + \mathrm{Error}. \tag{4-15}$$

Note that $\mathfrak{r}(\omega)$ is real-analytic and in the charged case when $\omega_{\mathrm{res}} \neq 0$, then always $\mathfrak{r}(\omega = \omega_{\mathrm{res}}) \neq 0$. In this charged case, the formal scattering operator (4-14) has a resonance at $\omega = \omega_{\mathrm{res}}$. However, in the uncharged case $q_0 = \omega_{\mathrm{res}} = 0$, there exists a discrete set of nonresonant masses $m^2 \in D(M, e)$ (particularly $0 \in D(M, e)$) such that $\mathfrak{r}(\omega = \omega_{\mathrm{res}} = 0) = 0$ for $m^2 \in D(M, e)$ as shown in [Kehle and Shlapentokh-Rothman 2019]. In that case, the scattering pole is absent and this can be seen as a key observation towards the T-energy scattering theory on the interior of Reissner–Nordström for the uncharged massless wave equation developed in [Kehle and Shlapentokh-Rothman 2019]. However, it is shown in that work that, for generic masses $m^2 \in \mathbb{R}_{>0} - D(M, e)$, the resonance is present and scattering fails.

A sharp condition for boundedness or blow-up at the Cauchy horizon. Restricting to parameters $q_0 \neq 0$ or $q_0 = 0$, $m^2 \in \mathbb{R}_{>0} - D(M, e)$, the resonance is present and from the formal computation and (4-15) we read off that $|\phi_{\mathcal{L}}| \leq C$ if the data $\phi_{\mathcal{H}^+}$ satisfy $\phi_{\mathcal{H}^+} \in L^1_v$ (in addition to having finite energy to control the error terms). Thus, in particular for fast decaying data (i.e., $\phi_{\mathcal{H}^+}$ satisfies (3-8) for $s > 1$), we formally obtain uniform boundedness of $\phi_{\mathcal{L}}$ at the Cauchy horizon.

For general $\phi_{\mathcal{H}^+} \in \mathcal{SL} - L^1$, the above reasoning does not hold, and blow-up in amplitude is possible. For concreteness, first consider the uncharged and massive case $q_0 = 0,\ m^2 \notin D(M, e)$. Then, $\omega_{\text{res}} = 0$ and, as we will show, $\phi_{\mathcal{L}}$ is uniformly bounded at the Cauchy horizon if and only if $\phi_{\mathcal{H}^+}$ satisfies

$$\sup_{v \in [v_0, +\infty)} \left| \int_{v_0}^{v} \phi_{\mathcal{H}^+}(v')\, \mathrm{d}v' \right| < +\infty. \tag{4-16}$$

For instance, (4-16) gives boundedness of $\phi_{\mathcal{L}}$ for data $\phi_{\mathcal{H}^+}$ of the form

$$\phi_{\mathcal{H}^+}(v) \approx e^{-i\omega v} \cdot v^{-s}, \tag{4-17}$$

where we recall $\frac{3}{4} < s \leq 1$, provided $\omega \in \mathbb{R} - \{0\}$: in this case, $\phi_{\mathcal{H}^+}$ obeys the quantitative oscillation condition $\phi_{\mathcal{H}^+} \in \mathcal{O}''$ as defined in Definition 3.7. If, however, $\omega = 0$ then $\phi_{\mathcal{H}^+}$ violates the oscillation condition, i.e., $\phi_{\mathcal{H}^+} \in \mathcal{NO} = \mathcal{SL} - \mathcal{O}$, and thus, $|\phi_{\mathcal{L}}|$ blows up at the Cauchy horizon $\mathcal{CH}_{i^+}$ in view of (4-16) (still assuming $q_0 = 0$).

In the charged case $q_0 \neq 0$, the resonance is always present and uniform boundedness of $\phi_{\mathcal{L}}$ at the Cauchy horizon is true for profiles satisfying the oscillation condition $\phi_{\mathcal{H}^+} \in \mathcal{O}$, e.g., profiles of the form

$$\phi_{\mathcal{H}^+} \approx e^{-i(\omega + \omega_{\text{res}}) \cdot v} \cdot v^{-s}, \tag{4-18}$$

where $\frac{3}{4} < s \leq 1$, provided $\omega \in \mathbb{R} - \{0\}$. If however $\omega = 0$ then $|\phi_{\mathcal{L}}|$ blows up at the Cauchy horizon $\mathcal{CH}_{i^+}$. We refer to Corollary 5.25 for a precise statement of the results of this paragraph.

Improved decay for $\phi_{\mathcal{H}^+} \in \mathcal{O}''$ to obtain the boundedness of the Maxwell field. Note that for the nonlinear EMKG system (1-1)–(1-5), the charge $Q(u, v)$ from (2-17) is a dynamical quantity (assuming $q_0 \neq 0$) that is nonlinearly coupled to ϕ and g, and hence the boundedness of Q is not guaranteed. Proving the boundedness of Q in amplitude indeed requires establishing further decay estimates proved in Corollary 5.25(3), whose proof we now outline. In the case where $\phi_{\mathcal{H}^+}$ satisfies the quantitative oscillation condition, i.e., $\phi_{\mathcal{H}^+} \in \mathcal{O}''$, the main term in (4-15) enjoys decay in $|u|$ as $u \to -\infty$ (corresponding to i^+ in Figure 1). In particular, for $\phi_{\mathcal{H}^+} \in \mathcal{O}''$ we will show (see Theorem V(B)) the quantitative control

$$|\phi_{\mathcal{L}}|(u, v) \lesssim |u|^{-1+s-\eta_0} \tag{4-19}$$

for some $\eta_0 > 0$. This (linear) quantitative estimate will be later useful to the boundedness proof of Q in the coupled case (see Section 4F4).

Towards the $W^{1,1}$-inextendibility. To illustrate the obstruction caused by slow decay explained in Section 4F1, we show in Theorem III that ϕ does not have locally outgoing integrable derivatives near the Cauchy horizon, i.e., $\int |D_v \phi|(u, v)\, \mathrm{d}v = +\infty$ for all u, consistently with the expectation given by (4-12). This blow-up in $W^{1,1}$ norm on outgoing cones justifies that, in the case where ϕ remains bounded, the reason is oscillation and not decay.

To show the $W^{1,1}$ blow-up in linear theory (see Corollary 5.27), we prove a representation formula for $\partial_v \phi_{\mathcal{L}}(u_0, v)$ (see (5-115)) and show that $\partial_v \phi_{\mathcal{L}}(u_0, v) \notin L^1_v$ for fixed u_0. Expressed in a regular gauge on the Cauchy horizon and neglecting error terms, we formally have

$$\partial_v \phi_{\mathcal{L}}(u_0, v) \approx -i \frac{r_+ e^{i\omega_{\text{res}} u_0}}{\sqrt{2\pi} r_-} \int_{\mathbb{R}} \mathcal{F}[\phi_{\mathcal{H}^+}](\omega)\, \mathfrak{t}(\omega) e^{-i(\omega-\omega_{\text{res}})v}\, \mathrm{d}\omega \tag{4-20}$$

close to the Cauchy horizon. We interpret (4-20) as a formal Fourier multiplication operator with multiplier $\mathfrak{t}(\omega)$, i.e., $T_{\mathfrak{t}}\colon \phi_{\mathcal{H}^+}(v) \mapsto \partial_v \phi_{\mathcal{L}}(u_0, v)$. Since our data $\phi_{\mathcal{H}^+}(v)$ are not integrable ($\phi_{\mathcal{H}^+} \notin L^1$) along the event horizon $\mathcal{H}^+$ and we aim to show that $\partial_v \phi_{\mathcal{L}}(u_0, v)$ is not in L^1_v, it is natural to consider to inverse operator $T_{\mathfrak{t}}^{-1} = T_{1/\mathfrak{t}}$ with Fourier multiplier $1/\mathfrak{t}(\omega)$. Formally, by Young's convolution inequality we have

$$\|\phi_{\mathcal{H}^+}\|_{L^1} = \|T_{\mathfrak{t}}^{-1}[\partial_v \phi_{\mathcal{L}}]\|_{L^1} = \|\mathcal{F}[\mathfrak{t}^{-1}] * \partial_v \phi_{\mathcal{L}}\|_{L^1} \le \|\mathcal{F}[\mathfrak{t}^{-1}]\|_{L^1} \|\partial_v \phi_{\mathcal{L}}\|_{L^1}. \tag{4-21}$$

Since our data $\phi_{\mathcal{H}^+}$ are assumed to be nonintegrable (i.e., $\phi_{\mathcal{H}^+} \notin L^1$), the above formal argument shows $W^{1,1}$ blow-up for $\phi_{\mathcal{L}}(u_0, \cdot\,)$ if $\mathcal{F}[\mathfrak{t}^{-1}] \in L^1$. The above formal computation is made rigorous in the proof of Theorem V(E). Further, we will prove that the only obstruction to $\mathcal{F}[\mathfrak{t}^{-1}] \in L^1$ is potential zeros of $\mathfrak{t}(\omega)$. In the uncharged case $q_0 = 0$, however, the ODE analog of the T-energy identity yields that

$$|\mathfrak{t}(\omega)|^2 = |\mathfrak{r}(\omega)|^2 + |\omega|^2. \tag{4-22}$$

Moreover, since we exclude nonresonant masses (i.e., $m^2 \in \mathbb{R}_{>0} - D(M, e)$), we have $\mathfrak{t}(0) \neq 0$ and as such, $\mathfrak{t}(\omega)$ is nowhere zero. As a result, we show $\mathcal{F}[\mathfrak{t}^{-1}] \in L^1$. For the uncharged case with resonant masses, this shows that all characteristic data $\phi_{\mathcal{H}^+}$ on the event horizon $\mathcal{H}^+$ that are not integrable give rise to solutions which blow up in $W^{1,1}$ along outgoing cones at the Cauchy horizon $\mathcal{CH}_{i^+}$.

In the charged case, however, the analog of (4-22) becomes

$$|\mathfrak{t}(\omega)|^2 = |\mathfrak{r}(\omega)|^2 + \omega(\omega - \omega_{\text{res}}) \tag{4-23}$$

such that $\mathfrak{t}(\omega)$ may have zeros for $\omega \in (0, \omega_{\text{res}})$ or $\omega \in (\omega_{\text{res}}, 0)$. For small charges, a perturbation argument shows that $\mathfrak{t}(\omega)$ does not have zeros but for general charges the set of zeros $\mathcal{Z}_t(M, e, q_0, m^2) = \{\omega \in \mathbb{R} \colon \mathfrak{t}(\omega, M, e, q_0, m^2) = 0\} \subset \{0 < |\omega| < |\omega_{\text{res}}|\}$ could be (and in general will be) nonempty. In view of this, for nonintegrable data (i.e., $\phi_{\mathcal{H}^+} \notin L^1$) which satisfy $P_\delta \phi_{\mathcal{H}^+} \in L^1$ (recall the definition of P_δ from Section 4E), we show that the arising solution blows up in $W^{1,1}$ along outgoing cones. It follows $\phi_{\mathcal{L}}$ blows up in $W^{1,1}$ along outgoing cones for all $\phi_{\mathcal{H}^+} \in \mathcal{SL} - H$, where $H \subset \mathcal{SL}$ is an exceptional subset first introduced in the statement of Theorem III.

4F3. *The nonlinear problem, I: physical space estimates of the difference* (*step* (3)). As we explained, the physical space method does not capture the oscillations of the field which are crucial to our proof. On the other hand, the (global) frequency analysis used for the linear equation Klein–Gordon equation on Reissner–Nordström (see (5-3) and as explained above) relies on two key properties: the existence of the Killing vector field ∂_t and the linearity of the equation — none of which extends to the coupled system (1-1)–(1-5).

In the present paper we overcome these limitations by controlling the difference between the nonlinear evolution and its linear counterpart in physical space (i.e., $g - g_{\text{RN}}$ and $\phi - \phi_{\mathcal{L}}$, see below). In the uncharged case $q_0 = 0$, this is exactly the strategy we adopt; see the first paragraph below. In the case $q_0 \neq 0$, unbounded backreaction oscillations of the Maxwell field however require a more sophisticated

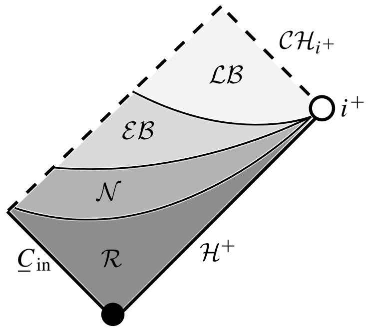

Figure 6. Division of a rectangular neighborhood of i^+ into four spacetime regions.

nonlinear scheme; see Section 4F4 and the second paragraph below. These unbounded backreaction oscillations motivate the precise definition of the oscillations spaces $\mathcal{O}$, $\mathcal{O}'$ and $\mathcal{O}''$ from Section 3D; see the third paragraph below.

The proof of the nonlinear differences estimates will be carried out in Section 6C and follows the splitting of spacetime into four different regions depicted in Figure 7 used already in [Van de Moortel 2018]; see Figure 6 (a similar splitting was first introduced in [Dafermos 2003] and subsequently used in [Franzen 2016; Dafermos and Luk 2017; Luk and Oh 2019a]). More specifically we refer the reader to Propositions 6.13–6.16.

It is important to note that the difference estimates described in this section (and proved in Section 6C) are completely independent of the estimates of Section 5 (whose description was outlined in Section 4F2), with the notable exception of the final formula (4-32) that uses the linear formula (4-15) "as a black box".

Difference estimates near i^+ for $q_0 = 0$. Near the Cauchy horizon $\mathcal{CH}_{i^+}$ and close to i^+ as in Figure 1 (i.e., for u close to $-\infty$) we obtain difference estimates of the schematic form

$$|\phi - \phi_{\mathcal{L}}|(u,v) + |u|^{-s} \cdot (|g - g_{\mathrm{RN}}| + |\partial_u(g - g_{\mathrm{RN}})|)(u,v) \lesssim |u|^{1-3s}, \tag{4-24}$$

$$\left|\partial_v(\phi - \phi_{\mathcal{L}})\right|(u,v) + v^{-s} \cdot |\partial_v(g - g_{\mathrm{RN}})|(u,v) \lesssim v^{1-3s}, \tag{4-25}$$

where (g, F, A, ϕ) solve (1-1)–(1-5) with data $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ and $\phi_{\mathcal{L}}$ solves (1-5) with same data $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ on a fixed Reissner–Nordström background (2-7) (corresponding to the one g is converging towards i^+). The key point is that $\phi - \phi_{\mathcal{L}}$, unlike ϕ, will turn out to be $\dot{W}^{1,1}$ along outgoing cones at $\mathcal{CH}_{i^+}$, namely (4-25) gives

$$\sup_{u,v} |\phi - \phi_{\mathcal{L}}|(u,v) \lesssim \sup_{u} \int_{v_0}^{+\infty} |\partial_v(\phi - \phi_{\mathcal{L}})|(u,v)\,\mathrm{d}v \lesssim \int_{v_0}^{+\infty} v^{1-3s} \lesssim v_0^{2-3s} < \infty$$

as $s > \frac{3}{4} > \frac{2}{3}$. Therefore $\phi - \phi_{\mathcal{L}}$ *is bounded.* In particular, in the uncharged case $q_0 = 0$, uniform boundedness of ϕ in the region of Figure 1 is equivalent to that of $\phi_{\mathcal{L}}$. As we will see below, this is no longer true if $q_0 \neq 0$.

Difference estimates near i^+ for $q_0 \neq 0$. If $q_0 \neq 0$, the metric differences are similar, but the scalar field difference is now impacted by the Maxwell backreaction. In particular, the first term of (4-25) is replaced by an estimate of the schematic form (in the gauge (2-26) where $A_v = A_v^{\mathrm{RN}} = 0$)

$$\left|e^{-i\sigma_{\mathrm{br}}(u,v)}\,\partial_v\phi - \partial_v\phi_{\mathcal{L}}\right|(u,v) \lesssim v^{1-3s}, \tag{4-26}$$

$$\sigma_{\mathrm{br}}(u,v) := \int_{u_\gamma(v)}^{u} \big((A_u)^{\mathrm{CH}}(u') - (A_u^{\mathrm{RN}})^{\mathrm{CH}}(u')\big)\,\mathrm{d}u', \tag{4-27}$$

where $u_\gamma(v) \sim -v$ and $(A_u)^{\mathrm{CH}}(u')$, $(A_u^{\mathrm{RN}})^{\mathrm{CH}}(u')$ are defined as the extensions of $A_u(u,v)$, $A_u^{\mathrm{RN}}(u,v)$ to $\mathcal{CH}_{i^+}$; see Proposition 6.16 for a precise statement. The difficulty is that σ_{br} is unbounded in general; nevertheless, we prove sublinear growth estimates (in Proposition 6.16 again)

$$|\sigma_{\mathrm{br}}(u,v)| \lesssim v^{2-2s}1_{s<1} + (1+\log(v))1_{s=1}, \tag{4-28}$$

$$|\partial_v\sigma_{\mathrm{br}}(u,v)| + |\partial_v^2\sigma_{\mathrm{br}}(u,v)| \lesssim v^{1-2s}. \tag{4-29}$$

Note that this is not a gauge issue: in fact, σ_{br} is a gauge-independent quantity obtained by the expression

$$\sigma_{\mathrm{br}}(u,v) := \iint_{[u_\gamma(v),u]\times[v_0,+\infty)} \left(\frac{\Omega^2 Q}{r^2} - \frac{\Omega^2_{\mathrm{RN}}e}{r^2_{\mathrm{RN}}}\right)\mathrm{d}u\,\mathrm{d}v, \tag{4-30}$$

assuming (3-5). As a consequence, *it is no longer true that $\phi - \phi_{\mathcal{L}}$ is uniformly bounded.* Instead, the consequence of (4-26) is that the following quantity is in $\dot{W}^{1,1}$ along outgoing cones and hence bounded:

$$\left|\phi(u,v) - \int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')}\,\partial_v\phi_{\mathcal{L}}(u,v')\,\mathrm{d}v'\right| \lesssim |u|^{2-3s}, \tag{4-31}$$

where $v_\gamma(u) \sim -u$. Therefore, boundedness of ϕ is now down to the boundedness of

$$\int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')}\,\partial_v\phi_{\mathcal{L}}(u,v')\,\mathrm{d}v'.$$

By our representation formula (4-15), this expression becomes, up to error, *an explicit integral of the data*

$$\phi(u,v) = \int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v') + i\omega_{\mathrm{res}}v'}\phi_{\mathcal{H}^+}(v')\,\mathrm{d}v' + O(|u|^{2-3s}). \tag{4-32}$$

Thus, the nonlinear representation formula (4-32) gives boundedness of ϕ up to and including the Cauchy horizon $\mathcal{CH}_{i^+}$ for characteristic event horizon data $\phi_{\mathcal{H}^+} \in \mathcal{O}$, one of the main goals of Theorem I (i) (see Section 6D1).

Further, (4-32) will also show blow-up of ϕ in amplitude at the Cauchy horizon $\mathcal{CH}_{i^+}$ for event horizon characteristic data $\phi_{\mathcal{H}^+} \notin \mathcal{O}$. We postpone the related discussion to the last paragraph of Section 4F4.

The motivation to introduce σ_{br} in the definition of the spaces $\mathcal{O}$, $\mathcal{O}'$, $\mathcal{O}''$. As explained above, the Maxwell field exerts a nontrivial backreaction with in general *unbounded oscillation* σ_{br} (recall (4-28)). Recalling that $\phi_{\mathcal{L}}$ is bounded if and only if the right-hand side of (4-15) is finite (where σ_{br} is as in (4-30)), and that ϕ is bounded if and only if the right-hand side of (4-32) is finite, it becomes clear that *the Maxwell backreaction may turn some linearly nonresonant profiles into nonlinearly resonant ones and vice versa* (a phenomenon which is absent in the uncharged case $q_0 = 0$ where the nonlinear estimates show that ϕ is bounded if and only if $\phi_{\mathcal{L}}$ is bounded).

Therefore, to ensure that our class of oscillating data $\phi_{\mathcal{H}^+} \in \mathcal{O}$ (and analogously $\mathcal{O}'$, $\mathcal{O}''$) gives rise to a bounded ϕ (and not only bounded $\phi_{\mathcal{L}}$), we must define $\phi_{\mathcal{H}^+} \in \mathcal{O}$ (and analogously $\mathcal{O}'$, $\mathcal{O}''$) as a stronger condition than the right-hand side of (4-15) being finite. This stronger condition is to impose *sufficiently robust oscillations* that yield finiteness of the right-hand side of (4-32) for all functions σ_{br} satisfying (3-15), (3-16). In particular, for σ_{br} given by the formula (4-30) (which obeys (3-15), (3-16), as we show, see (4-28), (4-29)), the condition $\phi_{\mathcal{H}^+} \in \mathcal{O}$ (and analogously $\mathcal{O}'$, $\mathcal{O}''$) shows that the oscillations in the initial data are sufficiently robust to not be over-powered by the nonlinear backreaction of the Maxwell field in evolution.

4F4. *The nonlinear problem II*: *boundedness/blow-up of matter fields and metric extendibility* (*step* (4)). Earlier we explained how the nonlinear difference estimates, culminating with (4-32), show that qualitatively oscillating $\phi_{\mathcal{H}^+} \in \mathcal{O}$ on the event horizon $\mathcal{H}^+$ give rise to uniformly bounded scalar field ϕ up to and including $\mathcal{CH}_{i^+}$. In this section, we outline the proof of the following results that conclude the proof of our main theorems:

- C^0-extendibility of the metric (within a certain spherically symmetric class) is *equivalent* to boundedness of $|\phi|$ in amplitude (first paragraph below; see also statements (A) and (B)). From the above equivalence given by (A) and (B), we deduce the main statement of Theorem I (i): the C^0-extendibility of the metric across $\mathcal{CH}_{i^+}$ holds under the strong qualitative oscillation condition $\phi_{\mathcal{H}^+} \in \mathcal{O}'$ on the event horizon $\mathcal{H}^+$ (see the proof in Section 6B). In our companion paper [Kehle and Van de Moortel $\geq$ 2024], the implication (B) that "blow-up of ϕ implies C^0-inextendibility" will be used to prove Theorem IV.
- The charge $Q(u, v)$ of the Maxwell field is bounded for quantitatively oscillating $\phi_{\mathcal{H}^+} \in \mathcal{O}''$ on the event horizon $\mathcal{H}^+$ (second paragraph below, proved in Section 6D1): one of the statements of Theorem I (i).
- The scalar field ϕ blows up in $W^{1,1}$, i.e., $\int |D_v\phi|(u, v)\,\mathrm{d}v = \infty$ for generic slowly decaying $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ on the event horizon $\mathcal{H}^+$ (third paragraph below, proved in Section 6D4): this is Theorem III.
- The scalar field ϕ blows up in L^∞, i.e., $\sup_{(u,v)}|\phi|(u, v) = \infty$ for nonoscillating $\phi_{\mathcal{H}^+} \in \mathcal{NO} = \mathcal{SL} - \mathcal{O}$ on the event horizon, assuming $q_0 = 0$ (fourth paragraph below, proved in Section 6D2): this is Theorem I (ii).

Continuous extendibility of the metric as a consequence of scalar field boundedness. We explained above how to prove boundedness/blow-up of the scalar field depending on the data $\phi_{\mathcal{H}^+}$. Now we explain how to prove that C^0-extendibility on the metric is in a sense equivalent to the boundedness of ϕ up to and including $\mathcal{CH}_{i^+}$, as it turns out! Combining this novel conditional result with the previously discussed boundedness theorem for ϕ will give the main result of Theorem I (i), i.e., the C^0-extendibility of the metric for any characteristic data $\phi_{\mathcal{H}^+} \in \mathcal{O}'$. The proof relies on a nonlinear scheme adapted to the slow decay of the solutions and taking advantage of the algebraic structure of the Einstein equations as explained below.

We begin by recalling from [Van de Moortel 2018] that the following estimates for $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ hold true near the Cauchy horizon $\mathcal{CH}_{i^+}$ and for some $\alpha > 0$ (see Section 6A for details)

$$\Omega^2(u, v) \lesssim e^{-\alpha v}, \tag{4-33}$$

$$|\partial_u \log(\Omega^2)| \lesssim |u|^{1-2s}, \tag{4-34}$$

$$|\partial_v \log(\Omega^2)| \lesssim v^{1-2s}, \tag{4-35}$$

$$|\partial_u r| \lesssim |u|^{-2s}, \tag{4-36}$$

$$|\partial_v r| \lesssim v^{-2s}. \tag{4-37}$$

Since the r estimates (4-36) (4-37) are integrable, it can be shown that $r(u_n, v_n)$ is a Cauchy sequence for any $u_n \to u,\ v_n \to +\infty$: Therefore, r extends to a continuous function. In contrast, the (conjecturally sharp) decay for $\log(\Omega^2)$ is too weak to adopt the same reasoning since $s \le 1$ ((4-34), (4-35) are nonintegrable).

Nevertheless, $\partial_u \partial_v \log(\Omega^2) + 2\Re(\overline{D_u\phi} D_v\phi)$ enjoys a better decay (see (6-54)), i.e., the weak decay from (4-34), (4-35) comes from a $\Re(\overline{D_u\phi} D_v\phi)$ term in the Einstein equations. It was first noticed by the second author in [Van de Moortel 2019] that it is useful to write the weakly decaying term as a *total derivative*, up to error

$$2\Re(\overline{D_u\phi} D_v\phi) = \partial_u \partial_v(|\phi|^2) + \cdots .$$

Exploiting the ideas of [Van de Moortel 2019], we introduce the following new quantity Υ, which is nonlinear and nonlocal:

$$\Upsilon(u, V) := \log(\Omega^2)(u, V) + |\phi|^2(u, V) + \int_u^{u_s} \frac{|\partial_u r|(u', V)}{r(u', V)} |\phi|^2(u', V)\,\mathrm{d}u', \tag{4-38}$$

where $\Omega^2 := -2g(\partial_u, \partial_V)$ for a suitably renormalized (u, V) coordinate system. We then prove that Υ is bounded and admits a continuous extension (see Section 6B2 for the proof).

Remark 4.3. To show that the right-hand side of (4-38) is bounded, we need the assumption $s > \frac{3}{4}$, which among other things, explains the numerology in the definition of $\mathcal{SL}$ (Definition 3.1); compare with Theorem A.

It turns out that the boundedness of Υ ultimately makes C^0-extendibility *equivalent* to the boundedness of ϕ in the following sense (see [Van de Moortel 2019]).

(A) If $|\phi|$ is bounded, then there exists a coordinate system (u, V) such that $\log(\Omega^2)$ is bounded.

(B) Conversely, if $|\phi|$ blows up, there *exists no coordinate system* (u, V) such that $\log(\Omega^2)$ is bounded.

Part (A) follows from the definition (4-38) and the (unconditional) boundedness of Υ (since $\partial_u r/r$ is also bounded). Moreover, because Υ is continuously extendible, if $|\phi|$ is continuously extendible, then $\log(\Omega^2)$ is also continuously extendible (hence so is Ω^2). In particular for data $\phi_{\mathcal{H}^+} \in \mathcal{O}'$, since we previously showed that $|\phi|$ is continuously extendible across $\mathcal{CH}_{i^+}$, we then obtain the continuous extendibility of g (see Section 6B3 for the proof), and a slightly improved statement: the existence of a C^0-admissible extension (Definition 2.1), i.e., a continuous extension admitting regular double null coordinates (u, V) given by the above pair (r, Ω^2).

Part (B) is more delicate and is proven in [Van de Moortel 2019, Theorem 2.3.5] (and used in [Kehle and Van de Moortel $\ge$ 2024] to prove Theorem IV): it implies that if $|\phi|$ blows up, then g does not admit any C^0-admissible extension.

Boundedness of the Maxwell field Q. We now outline the proof of the boundedness of the charge $Q(u, v)$ for $\phi_{\mathcal{H}^+} \in \mathcal{O}''$ given in Section 6D1. To prove boundedness of Q, we will actually need *decay as $u \to -\infty$ for ϕ* (in addition to its uniform boundedness already obtained assuming $\phi_{\mathcal{H}^+} \in \mathcal{O}$): this motivates the introduction of the space $\mathcal{O}'' \subset \mathcal{O}$ from Section 3D. We start taking advantage of the structure of the Maxwell equation:

$$\partial_u Q = r^2 \Im(\bar{\phi} D_u \phi) = \Im(\overline{r\phi} D_u(r\phi)).$$

Moreover, we use (4-11) to obtain the estimate (using also the boundedness of r):

$$|\partial_u Q| \lesssim |\phi| \cdot |D_u(r\phi)| \lesssim |\phi| \cdot |u|^{-s}.$$

To obtain boundedness, we integrate in u. For this, we take advantage of the quantitative $|u|$ decay of $|\phi|$ which is true if $\phi_{\mathcal{H}^+}$ satisfies the quantitative oscillation condition $\phi_{\mathcal{H}^+} \in \mathcal{O}''$. Combining both the linear estimate (4-19) on $\phi_{\mathcal{L}}$ and the nonlinear estimate (4-24) on $\phi - \phi_{\mathcal{L}}$, we obtain $|\phi| \lesssim |u|^{s-1-\eta_0}$ and thus

$$|\partial_u Q| \lesssim |u|^{-1-\eta_0},$$

which is integrable and thus sufficient to conclude the boundedness and continuous extendibility of Q.

$W^{1,1}$ blow-up of the scalar field. We now turn to the proof of $W^{1,1}$ blow-up on outgoing cones of ϕ for generic $\phi_{\mathcal{H}^+} \in \mathcal{SL} - L^1$(proof in Section 6D4). One of our nonlinear difference estimates gives near the Cauchy horizon $\mathcal{CH}_{i^+}$ and uniformly in u

$$\big| |D_v\phi|(u,v) - |D_v^{\mathrm{RN}}\phi_{\mathcal{L}}|(u,v) \big| \lesssim v^{1-3s},$$

which is integrable, since $s > \frac{3}{4} > \frac{2}{3}$. Therefore, $\|D_v\phi(u,\cdot)\|_{L^1} = +\infty$ if and only if $\|D_v^{\mathrm{RN}}\phi_{\mathcal{L}}(u,\cdot)\|_{L^1} = +\infty$. For $|q_0 e|$ small enough, (4-21) gives blow up of $\|D_v\phi(u,\cdot)\|_{L^1}$ for any $\phi_{\mathcal{H}^+} \in \mathcal{SL} - L^1$ (and for any $\phi_{\mathcal{H}^+} \in \mathcal{SL} - H$ in the case $q_0 \neq 0$, what we call the generic case, recalling the discussion at the end of Section 4F2).

Blow-up in amplitude of the scalar field ϕ if $\phi_{\mathcal{H}^+} \notin \mathcal{O}$. We now explain how the nonlinear representation formula (4-32) can be used to prove the blow-up in amplitude of $\phi_{\mathcal{H}^+}$ for $\phi_{\mathcal{H}^+} \in \mathcal{NO} = \mathcal{SL} - \mathcal{O}$ (see Section 6D2 for the proof). Recall indeed that (4-32) formally states that the uniform boundedness of ϕ up to and including the Cauchy horizon $\mathcal{CH}_{i^+}$ is equivalent to the finiteness of the characteristic data integral on the event horizon $\mathcal{H}^+$, i.e., for all $|u| \geq v_0$

$$\sup_v |\phi|(u,v) = \infty \iff \sup_v \left| \int_{-u}^{v} e^{i\sigma_{\mathrm{br}}(u,v') + i\omega_{\mathrm{res}}v'} \phi_{\mathcal{H}^+}(v')\, dv' \right| = \infty \tag{4-39}$$

for σ_{br} defined by (4-27) and in the gauge (2-26). If for given characteristic data $\phi_{\mathcal{H}^+} \in \mathcal{SL} - \mathcal{O}$ on the event horizon $\mathcal{H}^+$, the upper bounds (4-28), (4-29) also hold as *lower bounds* up to the Cauchy horizon $\mathcal{CH}_{i^+}$, (4-39) shows that ϕ *blows up at the Cauchy horizon* $\mathcal{CH}_{i^+}$: for instance, one can check that for $\frac{2}{3} < s < 1$,

$$\text{for the choice } \phi_{\mathcal{H}^+}(v) = e^{-iq_0\omega_{\mathrm{res}}v} v^{-s}, \quad \limsup_{v\to+\infty} \left| \int_{v_0}^{v} e^{iq_0(v')^{2-2s}} (v')^{-s}\, dv' \right| = +\infty.$$

Unfortunately, while we conjecture that such lower bounds are true[11] for *most solutions*, it seems that fine-tuned ones could violate them. When these lower bounds are violated and σ'_{br} or σ''_{br} decay faster, we

[11]The identity (4-30) indeed suggests that σ_{br} is comparable schematically to $|g - g_{\mathrm{RN}}|$ which is formally of order $\alpha\, v^{1-2s} + o(v^{1-2s})$ for some $\alpha \in \mathbb{R}$. The case $\alpha = 0$ is presumably nongeneric but leads to faster decay for σ'_{br} and σ''_{br} notably.

have a linearly resonant profile ($\phi_{\mathcal{H}^+} \notin \mathcal{O}$) become nonlinearly nonresonant (meaning ϕ is bounded at the Cauchy horizon) (for instance: if σ''_{br} decays faster, say $\sigma''_{\mathrm{br}}(v) = O(v^{-5s+3})$, then the right-hand side of (4-39) is finite for the choice $\phi_{\mathcal{H}^+}(v) = e^{-iq_0\omega_{\mathrm{res}}v}v^{-s}$). To sum up: the difficulty to control precisely these backreaction oscillations explains the absence of blowing-up examples for $q_0 \neq 0$ in the present paper, but not their plausibility!

In the case $q_0 = 0$, and for $m^2 \notin D(M, e)$, we obtain blow-up for all data $\phi_{\mathcal{H}^+} \in \mathcal{SL} - \mathcal{O}$. As mentioned before, the restriction of the mass parameter m^2 is due to "exceptional" so-called *nonresonant masses* (see [Kehle and Shlapentokh-Rothman 2019]) for which boundedness of the linearized $\phi_{\mathcal{L}}$ (hence of the EMKG-coupled scalar field ϕ, by our result) is true, even though $\phi_{\mathcal{H}^+} \notin \mathcal{O}$. Nevertheless, the set of nonresonant masses $D(M, e)$ is the zero set of a nontrivial analytic function as proved in [Kehle and Shlapentokh-Rothman 2019], and as such, it is discrete and of zero Lebesgue measure.

4F5. *Guide to the reader.* We conclude this section with a short guide to help the reader read through the proofs of Sections 5 and 6. While the above outline of the proof was organized thematically to highlight the resolution of various difficulties, for technical reasons the rest of the paper is organized slightly differently as follows:

(1) In Section 5 we study the solution $\phi_{\mathcal{L}}$ of the linear charged and massive Klein–Gordon equation $g^{\mu\nu}_{\mathrm{RN}} D^{\mathrm{RN}}_\mu D^{\mathrm{RN}}_\nu \phi_{\mathcal{L}} = m^2\phi_{\mathcal{L}}$ on a fixed Reissner–Nordström metric with slowly decaying characteristic data $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ on the event horizon $\mathcal{H}^+$. The approach is mostly focused on Fourier analysis, capturing the oscillations of $\phi_{\mathcal{L}}$ towards the Cauchy horizon $\mathcal{CH}_{i^+}$.

(a) In Section 5A, we set up the radial ODE satisfied by the Fourier transform of $\phi_{\mathcal{L}}$ associated to the timelike Killing vector field ∂_t on (2-7).

(b) In Section 5B, we first show the existence of a scattering resonance (i.e., a pole at the resonant frequency $\omega = \omega_{\mathrm{res}}$). Moreover, we show suitable resolvent estimates associated to the radial ODE. This allows us to prove properties of the (renormalized) scattering coefficients $\mathfrak{r}(\omega)$, $\mathfrak{t}(\omega)$.

(c) In Section 5C, we show a first representation formula involving $\mathfrak{r}(\omega)$ and $\mathfrak{t}(\omega)$ for $\phi_{\mathcal{L}}$ in terms of the event horizon data $\phi_{\mathcal{H}^+}$.

(d) In Section 5D, we take the limit of the representation formula to the Cauchy horizon of Reissner–Nordström which eventually yields our main linear result Theorem V.

(2) In Section 6 we estimate the solution (g, F, A, ϕ) of the nonlinear Einstein–Maxwell–Klein–Gordon system (1-1)–(1-5) with slowly decaying characteristic data $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ on the event horizon $\mathcal{H}^+$. The approach is mostly focused on physical space estimates, capturing the effect of ϕ on the metric g.

(a) In Section 6A we recall the nonlinear estimates from [Van de Moortel 2018]. They are essential to the analysis, both to show the continuous extendibility of g and for the nonlinear difference estimates; see below.

(b) In Section 6B, we show that, assuming ϕ is uniformly bounded, the metric g is continuously extendible. The proof exploits the special structure of the nonlinearity in the Einstein equations.

(c) In Section 6C, we estimate together the differences $g - g_{\mathrm{RN}}$ and $\phi - \phi_{\mathcal{L}}$. If $q_0 = 0$ this shows that boundedness of ϕ is equivalent to boundedness of $\phi_{\mathcal{L}}$. If $q_0 \neq 0$, we have (4-31) as a substitute.

(d) In Section 6D, we combine the results of Sections 5 and 6C to obtain the nonlinear representation formula (4-32). From (4-32) we can read off boundedness/blow-up of ϕ from the event horizon data $\phi_{\mathcal{H}^+}$. Combining with Section 6B gives the C^0-extendibility of g for oscillating event horizon data $\phi_{\mathcal{H}^+} \in \mathcal{O}'$ (Theorem I (i)). The other results follow from similar considerations.

5. Linear theory: the charged/massive Klein–Gordon equation on the Reissner–Nordström interior

We begin by studying the charged and massive scalar fields on the *fixed* subextremal Reissner–Nordström interior (2-7) with the subextremal parameters $0 < |e| < M$ from (3-1). In this section, the connection ∇ and the metric g_{RN} are the Reissner–Nordström connection and metric, respectively. As mentioned in Section 2C, we also use the electromagnetic gauge condition

$$A'_{\mathrm{RN}} = \left(\frac{e}{r} - \frac{e}{r_+}\right) \mathrm{d}t = \frac{1}{2}\left(\frac{e}{r} - \frac{e}{r_+}\right) \mathrm{d}v - \frac{1}{2}\left(\frac{e}{r} - \frac{e}{r_+}\right) \mathrm{d}u, \tag{5-1}$$

which satisfies $F_{\mathrm{RN}} = \mathrm{d}A'_{\mathrm{RN}}$ for

$$F_{\mathrm{RN}} = \frac{e}{2r^2}\Omega^2_{\mathrm{RN}}\, \mathrm{d}u \wedge \mathrm{d}v. \tag{5-2}$$

Note that F_{RN} satisfies the homogeneous Maxwell equations $\mathrm{d} * F_{\mathrm{RN}} = 0,\ \mathrm{d}F_{\mathrm{RN}} = 0$ and that (5-2) is the corresponding linear version of (2-17).

We now consider solutions $\phi'_{\mathcal{L}}$ of the charged Klein–Gordon equation (1-5), which reads

$$(\nabla_\mu + iq_0(A'_{\mathrm{RN}})_\mu)(\nabla^\mu + iq_0(A'_{\mathrm{RN}})^\mu)\phi'_{\mathcal{L}} - m^2\phi'_{\mathcal{L}} = 0, \tag{5-3}$$

where $q_0 \in \mathbb{R},\ m^2 \geq 0$, are the charge and mass parameters of the field. We also recall

$$\omega_r = \frac{q_0 e}{r}, \quad \omega_+ = \frac{q_0 e}{r_+}, \quad \omega_- = \frac{q_0 e}{r_-}, \quad \omega_{\mathrm{res}} = \omega_- - \omega_+. \tag{5-4}$$

Note that in the gauge (5-1), we have

$$D_v^{\mathrm{RN}} = \partial_v + iq_0(A'_{\mathrm{RN}})_v = \partial_v + \frac{i}{2}(\omega_r - \omega_+), \tag{5-5}$$

$$D_u^{\mathrm{RN}} = \partial_u + iq_0(A'_{\mathrm{RN}})_u = \partial_u - \frac{i}{2}(\omega_r - \omega_+) \tag{5-6}$$

such that for any C^1 function we have

$$e^{-i\omega_{\mathrm{res}}r^*}\, \partial_v(e^{i\omega_{\mathrm{res}}r^*} f) = \partial_v f + i(\omega_- - \omega_+)(\partial_v r^*) f = D_v^{\mathrm{RN}} f + \frac{i}{2}(\omega_- - \omega_r) f \tag{5-7}$$

and similarly for D_u^{RN}. For $q_0 = m^2 = 0$, the field is uncharged and massless, and (5-3) reduces to the well-known wave equation

$$\Box_{g_{\mathrm{RN}}}\phi'_{\mathcal{L}} = 0. \tag{5-8}$$

For $q_0 \neq 0,\ m^2 = 0$, the field is charged and massless and is governed by

$$(\nabla_\mu + iq_0(A'_{\mathrm{RN}})_\mu)(\nabla^\mu + iq_0(A'_{\mathrm{RN}})^\mu)\phi'_{\mathcal{L}} = 0. \tag{5-9}$$

Finally, for $q_0 = 0,\ m^2 \neq 0$, the field is uncharged and massive and governed by the Klein–Gordon equation

$$\Box_{g_{\mathrm{RN}}}\phi'_{\mathcal{L}} - m^2\phi'_{\mathcal{L}} = 0. \tag{5-10}$$

Notation. Throughout Section 5 we will use the following notation. If X and Y are two (typically nonnegative) quantities, we use $X \lesssim Y$ or $Y \lesssim X$ to denote that $X \leq C(M, e, m^2, q_0, s)Y$ for some constant $C(M, e, m^2, q_0, s)$ depending on the parameters (M, e, m^2, q_0, s). If C depends on an additional parameter p, we also use the notation $\lesssim_p$, $\gtrsim_p$. We also use $X = O(Y)$ for $|X| \lesssim Y$. We use $X \sim Y$ for $X \lesssim Y \lesssim X$. We also recall that throughout Section 5 we use the convention that $\mathcal{H}^+ = \mathcal{H}^+_R = \{u = -\infty,\ v \in \mathbb{R}\}$ as stated in Section 2A.

5A. ***Separation of variables and radial ODE.*** Since $T = \partial_t$ is a Killing field of the Reissner–Nordström spacetime and in view of the specific choice of electromagnetic gauge A'_{RN}, (5-3) admits a separation of variables. Formally, let $\phi'_{\mathcal{L}} = \phi'_{\mathcal{L}}(t, r)$ be a solution to (5-3). Then, we define the t-Fourier transform

$$\mathcal{F}[\phi'_{\mathcal{L}}](r, \omega) = \hat{\phi}'_{\mathcal{L}} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \phi'_{\mathcal{L}}(r, t)e^{i\omega t}\, \mathrm{d}t. \tag{5-11}$$

Formally, since $\phi'_{\mathcal{L}}$ solves (5-3), we have that

$$u(r^*) = u(\omega, r^*) := r(r^*)\mathcal{F}[\phi'_{\mathcal{L}}](r(r^*), \omega) \tag{5-12}$$

solves

$$-u'' - (\omega - (\omega_r - \omega_+))^2 u + Vu = 0, \tag{5-13}$$

where

$$V = -\Omega^2_{\mathrm{RN}}(r_*)\left(\frac{2M}{r^3} - \frac{2e^2}{r^4} + m^2\right). \tag{5-14}$$

The radial ODE (5-13) admits the following fundamental pairs of solution associated to the event horizon ($r^* \to -\infty$) and the Cauchy horizon ($r^* \to +\infty$).

Definition 5.1. Let $u_{\mathcal{H}_R}$, $u_{\mathcal{H}_L}$, $u_{\mathcal{CH}_R}$ and $u_{\mathcal{CH}_L}$ be the unique smooth solutions to (5-13) satisfying

$$u_{\mathcal{H}_R}(r^*) = e^{-i\omega r^*} + O(\Omega^2_{\mathrm{RN}}) \qquad \text{as } r^* \to -\infty, \tag{5-15}$$

$$u_{\mathcal{H}_L}(r^*) = e^{i\omega r^*} + O(\Omega^2_{\mathrm{RN}}) \qquad \text{as } r^* \to -\infty, \tag{5-16}$$

$$u_{\mathcal{CH}_R}(r^*) = e^{i(\omega-\omega_{\mathrm{res}})r^*} + O(\Omega^2_{\mathrm{RN}}) \qquad \text{as } r^* \to +\infty, \tag{5-17}$$

$$u_{\mathcal{CH}_L}(r^*) = e^{-i(\omega-\omega_{\mathrm{res}})r^*} + O(\Omega^2_{\mathrm{RN}}) \qquad \text{as } r^* \to +\infty \tag{5-18}$$

for $\omega \in \mathbb{R}$. The pairs $(u_{\mathcal{H}_R}, u_{\mathcal{H}_L})$ and $(u_{\mathcal{CH}_R}, u_{\mathcal{CH}_L})$ span the solution space of (5-13) for $\omega \in \mathbb{R} - \{0\}$ and $\omega \in \mathbb{R} - \{\omega_{\mathrm{res}}\}$, respectively.

Using the fact that the Wronskian

$$\mathfrak{W}(f,g) := fg' - f'g \tag{5-19}$$

of two solution of (5-13) is independent of r^*, we define transmission and reflection coefficients $\mathfrak{T}(\omega)$ and $\mathfrak{R}(\omega)$ as follows.

Definition 5.2. For $\omega \in \mathbb{R} - \{\omega_{\text{res}}\}$, we define the transmission and reflection coefficients $\mathfrak{T}$ and $\mathfrak{R}$ as

$$\mathfrak{T}(\omega) := \frac{\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_R})}{\mathfrak{W}(u_{\mathcal{CH}_L}, u_{\mathcal{CH}_R})} = \frac{\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_R})}{2i(\omega-\omega_{\text{res}})}, \tag{5-20}$$

$$\mathfrak{R}(\omega) := \frac{\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_L})}{\mathfrak{W}(u_{\mathcal{CH}_R}, u_{\mathcal{CH}_L})} = \frac{\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_L})}{-2i(\omega-\omega_{\text{res}})}, \tag{5-21}$$

where $u_{\mathcal{H}_R}$, $u_{\mathcal{H}_L}$, $u_{\mathcal{CH}_R}$ and $u_{\mathcal{CH}_L}$ are defined in Definition 5.1. Indeed, this allows us to write

$$u_{\mathcal{H}_R} = \mathfrak{T} u_{\mathcal{CH}_L} + \mathfrak{R} u_{\mathcal{CH}_R} \tag{5-22}$$

for $\omega \in \mathbb{R} - \{\omega_{\text{res}}\}$. Moreover, we define the normalized transmission and reflection coefficients as

$$\mathfrak{t}(\omega) = (\omega-\omega_{\text{res}})\mathfrak{T}(\omega) = \frac{\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_R})}{2i}, \tag{5-23}$$

$$\mathfrak{r}(\omega) = (\omega-\omega_{\text{res}})\mathfrak{R}(\omega) = \frac{\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_L})}{-2i}, \tag{5-24}$$

which manifestly satisfy

$$\mathfrak{t}(\omega_{\text{res}}) = -\mathfrak{r}(\omega_{\text{res}}). \tag{5-25}$$

Remark 5.3. Note that the radial ODE (5-13) depends analytically on ω. Thus, $u_{\mathcal{H}_R}$, $u_{\mathcal{H}_L}$, $u_{\mathcal{CH}_R}$ and $u_{\mathcal{CH}_L}$ are real-analytic functions for ω for fixed r^*. In particular, this means that the Wronskians $\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_R})$, $\mathfrak{W}(u_{\mathcal{CH}_R}, u_{\mathcal{CH}_L})$ etc. are real-analytic functions for $\omega \in \mathbb{R}$ which can be extended holomorphically to a neighborhood of the real line.

We will also define the renormalized functions.

Definition 5.4. We define

$$\widetilde{u_{\mathcal{H}_R}}(r^*,\omega) := e^{i\omega r^*} u_{\mathcal{H}_R}(r^*,\omega), \tag{5-26}$$

$$\widetilde{u_{\mathcal{H}_L}}(r^*,\omega) := e^{-i\omega r^*} u_{\mathcal{H}_L}(r^*,\omega), \tag{5-27}$$

$$\widetilde{u_{\mathcal{CH}_R}}(r^*,\omega) := e^{-i(\omega-\omega_{\text{res}})r^*} u_{\mathcal{CH}_R}(r^*,\omega), \tag{5-28}$$

$$\widetilde{u_{\mathcal{CH}_L}}(r^*,\omega) := e^{i(\omega-\omega_{\text{res}})r^*} u_{\mathcal{CH}_L}(r^*,\omega). \tag{5-29}$$

5B. *Analysis for the radial ODE.*

Proposition 5.5. *Let either of the following two assumptions hold true.*

- $q_0 \neq 0$.
- $q_0 = 0$ *but* $m^2 \notin D(M,e)$, *where* $D(M,e)$ *is the discrete set of* [Kehle and Shlapentokh-Rothman 2019, Theorem 7].

Then, the transition and reflection coefficients $\mathfrak{T}(\omega)$ *and* $\mathfrak{R}(\omega)$, *as defined in Definition 5.2, have (nonremovable) poles of first order at* $\omega = \omega_{\text{res}}$.

Proof. First, note that $(\mathrm{Im}(u'\bar{u}))' = 0$ holds true for any C^1 solution of (5-13). Applying this to $u_{\mathcal{H}_R}$ and expanding $u_{\mathcal{H}_R}$ as $u_{\mathcal{H}_R} = \mathfrak{T} u_{\mathcal{CH}_L} + \mathfrak{R} u_{\mathcal{CH}_R}$, we conclude the *ODE energy identity*

$$|\mathfrak{T}|^2 - |\mathfrak{R}|^2 = \frac{\omega}{\omega - \omega_{\mathrm{res}}}. \tag{5-30}$$

If $q_0 \neq 0$ and thus, $\omega_{\mathrm{res}} \neq 0$, we have $|\mathfrak{T}|^2 \geq \omega/(\omega - \omega_{\mathrm{res}})$ for $|\omega| > \omega_{\mathrm{res}}$. Sending $\omega \to \omega_{\mathrm{res}}$, we conclude that $\mathfrak{T}$ blows up and since $\mathfrak{T}$ is meromorphic in a complex neighborhood of ω_{res}, the claim follows. In particular, we have that $\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_R})(\omega = \omega_{\mathrm{res}}) \neq 0$ and $\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_L})(\omega = \omega_{\mathrm{res}}) \neq 0$. For $q_0 = 0$ and $m^2 \notin D(M, e)$, the claim follows from [Kehle and Shlapentokh-Rothman 2019, Theorem 7]. □

Proposition 5.6. *The solutions $u_{\mathcal{H}_R}$, $u_{\mathcal{CH}_L}$, $u_{\mathcal{CH}_R}$ and the renormalized functions $\widetilde{u_{\mathcal{H}_R}}$, $\widetilde{u_{\mathcal{CH}_L}}$, $\widetilde{u_{\mathcal{CH}_R}}$ as defined in Definitions 5.1 and 5.4, respectively, satisfy for $\omega \in \mathbb{R}$*

$$\sup_{r^* \in (-\infty, r_0^*]} |u_{\mathcal{H}_R}(\omega, r^*)| \lesssim_{r_0^*} 1, \tag{5-31}$$

$$\sup_{r^* \in (-\infty, r_0^*]} |u'_{\mathcal{H}_R}(\omega, r^*)| \lesssim_{r_0^*} |\omega| \tag{5-32}$$

for any fixed $r_0^* \in \mathbb{R}$ and

$$|\widetilde{u_{\mathcal{H}_R}}(\omega, r^*) - 1| \lesssim_{r_0^*} |\Omega^2_{\mathrm{RN}}(r^*)|, \tag{5-33}$$

$$|\widetilde{u'_{\mathcal{H}_R}}(\omega, r^*)| \lesssim_{r_0^*} |\Omega^2_{\mathrm{RN}}(r^*)| \tag{5-34}$$

uniformly for $r^* \leq r_0^*$. Moreover, for $\omega \in \mathbb{R}$ and any fixed $r_0^* \in \mathbb{R}$

$$\sup_{r^* \in [r_0^*, +\infty)} |u_{\mathcal{CH}_L}(\omega, r^*)| \lesssim_{r_0^*} 1, \tag{5-35}$$

$$\sup_{r^* \in [r_0^*, +\infty)} |u'_{\mathcal{CH}_L}(\omega, r^*)| \lesssim_{r_0^*} |\omega|, \tag{5-36}$$

$$\sup_{r^* \in [r_0^*, +\infty)} |u_{\mathcal{CH}_R}(\omega, r^*)| \lesssim_{r_0^*} 1, \tag{5-37}$$

$$\sup_{r^* \in [r_0^*, +\infty)} |u'_{\mathcal{CH}_R}(\omega, r^*)| \lesssim_{r_0^*} |\omega|, \tag{5-38}$$

and uniformly for $r^* \geq r_0^*$

$$|\widetilde{u_{\mathcal{CH}_L}}(\omega, r^*) - 1| \lesssim_{r_0^*} |\Omega^2_{\mathrm{RN}}(r^*)|, \tag{5-39}$$

$$|\widetilde{u'_{\mathcal{CH}_L}}(\omega, r^*)| \lesssim_{r_0^*} |\Omega^2_{\mathrm{RN}}|, \tag{5-40}$$

$$|\widetilde{u_{\mathcal{CH}_R}}(\omega, r^*) - 1| \lesssim_{r_0^*} |\Omega^2_{\mathrm{RN}}(r^*)|, \tag{5-41}$$

$$|\widetilde{u'_{\mathcal{CH}_R}}(\omega, r^*)| \lesssim_{r_0^*} |\Omega^2_{\mathrm{RN}}(r^*)|. \tag{5-42}$$

The transition and reflection coefficients as defined in Definition 5.2 satisfy

$$\sup_{|\omega - \omega_{\mathrm{res}}| \geq 1} (|\mathfrak{T}(\omega)| + |\mathfrak{R}(\omega)|) \lesssim 1. \tag{5-43}$$

Proof. It suffices to show the results for $u_{\mathcal{H}_R}$ and $\widetilde{u_{\mathcal{H}_R}}$ as the other cases follow completely analogously. We will consider the cases $|\omega| \leq \omega_0 := |\omega_{\mathrm{res}}| + 1$ and $|\omega| > \omega_0$ independently. First, for $|\omega| \leq \omega_0$, we note that $u_{\mathcal{H}_R}$ is the unique solution to the Volterra equation

$$u_{\mathcal{H}_R}(r^*,\omega) = e^{-i\omega r^*} + \int_{-\infty}^{r^*} \frac{\sin(\omega(r^*-y))}{\omega}\big(2\omega(\omega_r-\omega_+)-(\omega_r-\omega_+)^2+V(y)\big)u_{\mathcal{H}_R}(y,\omega)\,\mathrm{d}y. \tag{5-44}$$

For $\omega = 0$, we mean $\sin(\omega(r^*-y))/\omega = r^*-y$. Now, since

$$\int_{-\infty}^{r_0^*} \sup_{y\le r^*<r_0^*} |K(r^*,y)|\,\mathrm{d}y \lesssim \Omega^2_{\mathrm{RN}}(r_0^*), \tag{5-45}$$

where

$$K(r^*,y) = \frac{\sin(\omega(r^*-y))}{\omega}\big(2\omega(\omega_r-\omega_+)-(\omega_r-\omega_+)^2+V(y)\big), \tag{5-46}$$

we have by standard estimates on Volterra equations (e.g., [Kehle and Shlapentokh-Rothman 2019, Proposition 2.3] or [Olver 1974, §10]) that, for $|\omega|\le\omega_0$,

$$\|u_{\mathcal{H}_R}\|_{L^\infty(-\infty,r_0^*)} \lesssim_{r_0^*} 1, \tag{5-47}$$

as well as

$$|u_{\mathcal{H}_R} - e^{-i\omega r^*}| \lesssim |\Omega^2_{\mathrm{RN}}(r^*)| \tag{5-48}$$

uniformly for $r^*\le 0$. Similarly, we obtain

$$\|u'_{\mathcal{H}_R}\|_{L^\infty(-\infty,r_0^*)} \lesssim_{r_0^*} 1+|\omega| \lesssim_{r_0^*} 1. \tag{5-49}$$

Note that this also shows that, for $|\omega|\le\omega_0$, we have

$$\|\widetilde{u_{\mathcal{H}_R}}'\|_{L^\infty(-\infty,r_0^*)} \lesssim_{r_0^*} 1, \tag{5-50}$$

$$|\widetilde{u_{\mathcal{H}_R}} - 1| \lesssim |\Omega^2_{\mathrm{RN}}(r^*)| \tag{5-51}$$

uniformly for $r^*\le 0$.

Now, we consider the case $|\omega|\ge\omega_0$. Note that in this frequency regime, the frequency-dependent potential

$$W := -(\omega-(\omega_r-\omega_+))^2 \tag{5-52}$$

satisfies

$$-W \gtrsim \omega^2, \tag{5-53}$$

$$|W'/W| \lesssim \Omega^2_{\mathrm{RN}}/|\omega|, \tag{5-54}$$

$$|W''/W| \lesssim \Omega^2_{\mathrm{RN}}/|\omega|, \tag{5-55}$$

and the radial potential V satisfies

$$|V|, |V'|, |V''| \lesssim \Omega^2_{\mathrm{RN}} \tag{5-56}$$

uniformly on $r^*\in\mathbb{R}$.

Now we will use a WKB approximation for $u_{\mathcal{H}_R}$. First, we will estimate the total variation $\mathcal{V}_{-\infty,+\infty}$ associated to the error-control function

$$F_{u_{\mathcal{H}_R}}(r^*,\omega) := \int_{-\infty}^{r^*} \frac{1}{|W|^{1/4}}\frac{\mathrm{d}^2}{\mathrm{d}x^2}|W|^{-1/4} - \frac{V}{|W|^{1/2}}\,\mathrm{d}y. \tag{5-57}$$

In view of (5-53)–(5-55), we estimate

$$\mathcal{V}_{-\infty,+\infty}(F_{u_{\mathcal{H}_R}}) = \int_{-\infty}^{+\infty}\left|\frac{1}{|W|^{1/4}}\frac{\mathrm{d}^2}{\mathrm{d}x^2}|W|^{-1/4} - \frac{V}{|W|^{1/2}}\right|\mathrm{d}y \lesssim \frac{1}{|\omega|}. \tag{5-58}$$

Thus, applying [Olver 1974, Theorem 2.2, p. 196] we obtain

$$u_{\mathcal{H}_R}(r^*,\omega) = \frac{|\omega|^{1/2}}{|W(r^*,\omega)|^{1/4}} e^{-i\omega r^* + i\int_{-\infty}^{r^*} \omega_r - \omega_+ \mathrm{d}y}(1+\eta_{u_{\mathcal{H}_R}}), \tag{5-59}$$

where the error function $\eta_{u_{\mathcal{H}_R}}$ satisfies

$$|\eta_{u_{\mathcal{H}_R}}(r^*,\omega)| \lesssim \frac{1}{|\omega|}, \tag{5-60}$$

$$|\eta'_{u_{\mathcal{H}_R}}(r^*,\omega)| \lesssim |W(r^*,\omega)|^{1/2}\frac{1}{|\omega|} \lesssim 1 \tag{5-61}$$

uniformly for $r^* \in \mathbb{R}$ and $|\omega| \geq \omega_0$ as well as

$$|\eta_{u_{\mathcal{H}_R}}(r^*,\omega)| \lesssim \frac{\Omega^2_{\mathrm{RN}}}{|\omega|}, \tag{5-62}$$

$$|\eta'_{u_{\mathcal{H}_R}}(r^*,\omega)| \lesssim \Omega^2_{\mathrm{RN}} \tag{5-63}$$

uniformly for $r^* < 0$ and $|\omega| \geq \omega_0$. This shows that for $|\omega| \geq \omega_0$ we have

$$\|u_{\mathcal{H}_R}\|_{L^\infty(\mathbb{R})} \lesssim 1, \tag{5-64}$$

$$\|u'_{\mathcal{H}_R}\|_{L^\infty(\mathbb{R})} \lesssim |\omega|. \tag{5-65}$$

Note also that $\widetilde{u_{\mathcal{H}_R}} = e^{i\omega r^*} u_{\mathcal{H}_R}$ similarly satisfies

$$\|\widetilde{u_{\mathcal{H}_R}}\|_{L^\infty(\mathbb{R})} \lesssim 1, \tag{5-66}$$

$$\|\widetilde{u_{\mathcal{H}_R}}'\|_{L^\infty(\mathbb{R})} \lesssim 1 \tag{5-67}$$

and

$$|\widetilde{u_{\mathcal{H}_R}}(r^*,\omega) - 1| \lesssim_{r_0^*} \Omega^2_{\mathrm{RN}}, \tag{5-68}$$

$$|\widetilde{u_{\mathcal{H}_R}}'(r^*,\omega)| \lesssim_{r_0^*} \Omega^2_{\mathrm{RN}} \tag{5-69}$$

uniformly for $r^* \leq r_0^*$ and $\omega \in \mathbb{R}$. The other results for $u_{\mathcal{CH}_L}$ and $u_{\mathcal{CH}_R}$ are shown completely analogously.

Now, we will show the bounds on the transmission and reflection coefficients $\mathfrak{T}$ and $\mathfrak{R}$. The bound (5-43) follows from the fact that for $|\omega|$ sufficiently large, $|\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_R})|$, $|\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_L})| \lesssim |\omega|$ in view of (5-64), (5-65) and computing the Wronskian as $r^* \to +\infty$. For $|\omega|$ small, the bound follows from the continuity of $|\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_R})|$ and $|\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_L})|$. □

Lemma 5.7. *The bounds*

$$|\partial_\omega \widetilde{u_{\mathcal{CH}_R}}(\omega,r^*)| \lesssim \Omega^2_{\mathrm{RN}}, \tag{5-70}$$

$$|\partial_\omega \widetilde{u_{\mathcal{CH}_L}}(\omega,r^*)| \lesssim \Omega^2_{\mathrm{RN}} \tag{5-71}$$

and

$$|\partial_{r^*}\partial_\omega \widetilde{u_{\mathcal{CH}_R}}(\omega,r^*)| \lesssim \Omega^2_{\mathrm{RN}}\langle\omega\rangle, \tag{5-72}$$

$$|\partial_{r^*}\partial_\omega \widetilde{u_{\mathcal{CH}_L}}(\omega,r^*)| \lesssim \Omega^2_{\mathrm{RN}}\langle\omega\rangle \tag{5-73}$$

hold uniformly for $r^* \geq 0$ and $\omega \in \mathbb{R}$. (We recall that $\langle\omega\rangle := \sqrt{1+\omega^2}$).

Moreover,

$$|\partial_\omega \widetilde{u_{\mathcal{H}_R}}(\omega, r^*)| \lesssim \Omega_{\mathrm{RN}}^2, \tag{5-74}$$

$$|\partial_{r^*}\partial_\omega \widetilde{u_{\mathcal{H}_R}}(\omega, r^*)| \lesssim \Omega_{\mathrm{RN}}^2 \langle\omega\rangle \tag{5-75}$$

hold uniformly for $r^* \leq 0$ *and* $\omega \in \mathbb{R}$.

Proof. First, we consider the range $|\omega - \omega_{\mathrm{res}}| \leq 1$. First, note that $\widetilde{u_{\mathcal{CH}_R}}$ solves the Volterra integral equation

$$\widetilde{u_{\mathcal{CH}_R}}(r^*, \omega) = 1 + \int_{r^*}^{+\infty} \frac{\sin[(\omega-\omega_{\mathrm{res}})(r^*-y)]}{\omega-\omega_{\mathrm{res}}} e^{-i(\omega-\omega_{\mathrm{res}})(r^*-y)} \times [V(y) - (\omega_- - \omega_{r(y)})(2\omega + 2\omega_+ - \omega_- - \omega_{r(y)})] \widetilde{u_{\mathcal{CH}_R}}(\omega, y)\, \mathrm{d}y. \tag{5-76}$$

Thus, $\partial_\omega \widetilde{u_{\mathcal{CH}_R}}$ solves

$$\begin{aligned}\partial_\omega \widetilde{u_{\mathcal{CH}_R}}(r^*, \omega) = &\int_{r^*}^{+\infty} \frac{\sin[(\omega-\omega_{\mathrm{res}})(r^*-y)]}{\omega-\omega_{\mathrm{res}}} e^{-i(\omega-\omega_{\mathrm{res}})(r^*-y)} \\ &\qquad \times [V(y) - (\omega_- - \omega_{r(y)})(2\omega + 2\omega_+ - \omega_- - \omega_{r(y)})]\, \partial_\omega \widetilde{u_{\mathcal{CH}_R}}(\omega, y)\, \mathrm{d}y \\ &+ \int_{r^*}^{+\infty} \frac{\partial_\omega(\operatorname{sinc}[(\omega-\omega_{\mathrm{res}})(r^*-y)] e^{-i(\omega-\omega_{\mathrm{res}})(r^*-y)})}{r^*-y} (r^*-y)^2 \\ &\qquad \times [V(y) - (\omega_- - \omega_{r(y)})(2\omega + 2\omega_+ - \omega_- - \omega_{r(y)})] \widetilde{u_{\mathcal{CH}_R}}(\omega, y)\, \mathrm{d}y \\ &+ \int_{r^*}^{+\infty} \frac{\sin[(\omega-\omega_{\mathrm{res}})(r^*-y)]}{\omega-\omega_{\mathrm{res}}} e^{-i(\omega-\omega_{\mathrm{res}})(r^*-y)} \\ &\qquad \times 2[V(y) - (\omega_- - \omega_{r(y)}))] \widetilde{u_{\mathcal{CH}_R}}(\omega, y)\, \mathrm{d}y. \end{aligned} \tag{5-77}$$

Now, we have the following bounds uniformly for $r^* \geq 0$:

$$\left| \frac{\sin[(\omega-\omega_{\mathrm{res}})(r^*-y)]}{\omega-\omega_{\mathrm{res}}} e^{-i(\omega-\omega_{\mathrm{res}})(r^*-y)} \right| \lesssim (r^*-y), \tag{5-78}$$

$$\left| \frac{\partial_\omega(\operatorname{sinc}[(\omega-\omega_{\mathrm{res}})(r^*-y)] e^{-i(\omega-\omega_{\mathrm{res}})(r^*-y)})}{r^*-y} \right| \lesssim 1, \tag{5-79}$$

$$|V(y) - (\omega_- - \omega_{r(y)})(2\omega + 2\omega_+ - \omega_- - \omega_{r(y)})| \lesssim \Omega_{\mathrm{RN}}^2, \tag{5-80}$$

$$|V(y) - (\omega_- - \omega_{r(y)}))| \lesssim \Omega_{\mathrm{RN}}^2. \tag{5-81}$$

With these bounds, standard results (e.g., [Olver 1974, §10]) on estimates of solutions of Volterra integral equations show that

$$|\partial_\omega \widetilde{u_{\mathcal{CH}_R}}(r^*, \omega)| \lesssim \Omega_{\mathrm{RN}}^2 \tag{5-82}$$

uniformly for $r^* \geq 0$. Similarly, we have

$$|\partial_\omega \widetilde{u_{\mathcal{CH}_L}}(r^*, \omega)| \lesssim \Omega_{\mathrm{RN}}^2 \tag{5-83}$$

uniformly for $r^* \geq 0$.

Differentiation of (5-77) with respect to r^* also gives

$$|\partial_{r^*}\partial_\omega \widetilde{u_{\mathcal{CH}_R}}| \lesssim \Omega_{\mathrm{RN}}^2 \tag{5-84}$$

and analogously we obtain

$$|\partial_{r^*}\partial_\omega \widetilde{u_{\mathcal{CH}_L}}| \lesssim \Omega^2_{\mathrm{RN}}. \tag{5-85}$$

Now, we consider the range $|\omega-\omega_{\mathrm{res}}| \geq 1$. Then, for $r^* \geq 0$, we have the bounds

$$\left|\frac{\sin[(\omega-\omega_{\mathrm{res}})(r^*-y)]}{\omega-\omega_{\mathrm{res}}} e^{-i(\omega-\omega_{\mathrm{res}})(r^*-y)}\right| \lesssim \langle\omega\rangle^{-1}, \tag{5-86}$$

$$|\partial_\omega(\mathrm{sinc}[(\omega-\omega_{\mathrm{res}})(r^*-y)]e^{-i(\omega-\omega_{\mathrm{res}})(r^*-y)})| \lesssim \langle\omega\rangle^{-1}\frac{1+|r^*-y|}{|r^*-y|}, \tag{5-87}$$

$$|V(y)-(\omega_- - \omega_{r(y)})(2\omega+2\omega_+ - \omega_- - \omega_{r(y)})| \lesssim \Omega^2_{\mathrm{RN}}\langle\omega\rangle, \tag{5-88}$$

$$|V(y)-(\omega_- - \omega_{r(y)}))| \lesssim \Omega^2_{\mathrm{RN}}. \tag{5-89}$$

Thus, analogously to the above, this gives uniformly for $r^* \geq 0$

$$|\partial_\omega \widetilde{u_{\mathcal{CH}_R}}(r^*,\omega)| \lesssim \Omega^2_{\mathrm{RN}}, \tag{5-90}$$

$$|\partial_\omega \widetilde{u_{\mathcal{CH}_L}}(r^*,\omega)| \lesssim \Omega^2_{\mathrm{RN}}, \tag{5-91}$$

as well as

$$|\partial_{r^*}\partial_\omega \widetilde{u_{\mathcal{CH}_R}}| \lesssim \Omega^2_{\mathrm{RN}}\langle\omega\rangle, \tag{5-92}$$

$$|\partial_{r^*}\partial_\omega \widetilde{u_{\mathcal{CH}_L}}| \lesssim \Omega^2_{\mathrm{RN}}\langle\omega\rangle. \tag{5-93}$$

The result on $u_{\mathcal{H}_R}$ follows completely analogously. □

Corollary 5.8. *The normalized transmission and reflection coefficients satisfy*

$$|\mathfrak{t}(\omega)| + |\mathfrak{r}(\omega)| \lesssim 1+|\omega|. \tag{5-94}$$

Proof. This is a consequence of Propositions 5.5 and 5.6. □

Lemma 5.9. *We have*

$$|\partial_\omega \mathfrak{r}(\omega)| \lesssim \langle\omega\rangle, \tag{5-95}$$

$$|\partial_\omega \mathfrak{t}(\omega)| \lesssim \langle\omega\rangle. \tag{5-96}$$

Proof. We estimate

$$|\partial_\omega \mathfrak{r}| \lesssim |\partial_\omega \mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_R})| \lesssim |\mathfrak{W}(\partial_\omega u_{\mathcal{H}_R}, u_{\mathcal{CH}_R})(r^*=0)| + |\mathfrak{W}(u_{\mathcal{H}_R}, \partial_\omega u_{\mathcal{CH}_R})(r^*=0)| \lesssim \langle\omega\rangle \tag{5-97}$$

in view of Lemma 5.7 and Proposition 5.6. Analogously the same holds for $\mathfrak{t}$. □

Towards the $W^{1,1}$ inextendibility at the Cauchy horizon we need to analyze the zeros of the transmission coefficient $\mathfrak{t}$. To do so, we recall the definition of $\mathcal{Z}_{\mathfrak{t}}(M,e,q_0,m^2)$ from (4-5).

Lemma 5.10. (1) *Let $q_0 e \neq 0$. Then, $\mathcal{Z}_{\mathfrak{t}} \subset (0,\omega_{\mathrm{res}})$ if $q_0 e > 0$ or $\mathcal{Z}_{\mathfrak{t}} \subset (\omega_{\mathrm{res}},0)$ if $q_0 e < 0$.*

(2) *Let $0 < |q_0 e| < \epsilon(M,e,m^2)$ for some $\epsilon(M,e,m^2)$ sufficiently small. Then $\mathfrak{t}$ does not have any zeros, i.e., $\mathcal{Z}_{\mathfrak{t}} = \varnothing$.*

(3) *Let $q_0 = 0$ and let $m^2 \notin D(M,e)$, where $D(M,e)$ is the discrete set as in* [Kehle and Shlapentokh-Rothman 2019, Theorem 7]. *Then, $\mathfrak{t}(\omega)$ does not have any zeros, i.e., $\mathcal{Z}_{\mathfrak{t}}(M,e,0,m^2) = \varnothing$ if $m^2 \notin D(M,e)$.*

Proof. The first statement follows from the fact that $|\mathfrak{t}|^2 = |\mathfrak{r}|^2 + \omega(\omega-\omega_{\mathrm{res}}) \geq \omega(\omega-\omega_{\mathrm{res}})$, Proposition 5.5 and the fact that $\mathfrak{t}(\omega=0)\neq 0$. Indeed, if $\mathfrak{t}(\omega=0)=0$, then $\mathfrak{r}(\omega=0)=0$ and thus $\mathfrak{T}(\omega=0)=\mathfrak{R}(\omega=0)=0$. But this cannot be true, since otherwise $u_{\mathcal{H}_R} = \mathfrak{R}u_{\mathcal{CH}_R} + \mathfrak{T}u_{\mathcal{CH}_L}$ would be trivial. The second statement just follows from continuity of $\mathfrak{t}$ as a function of the parameters $q_0 e$. The third statement is shown in [Kehle and Shlapentokh-Rothman 2019, Theorem 7]. □

Remark 5.11. Note that for $q_0=0$ and $m^2=0$, we have that $\mathfrak{t}(\omega=0)=0$. This is a crucial observation for the existence of a T-energy scattering theory as established in [Kehle and Shlapentokh-Rothman 2019].

5C. *Representation formula.* We recall that throughout Section 5 we consider the event horizon $\mathcal{H}^+$ as the set $\{u=-\infty\}\times\{v\in\mathbb{R}\}$ as in Section 2A.

Definition 5.12. For $f\in L^2(\mathcal{H}^+)$ we define the Fourier transform along the event horizon as

$$\mathcal{F}_{\mathcal{H}^+}[f](\omega) := r_+\mathcal{F}[f](\omega) = \frac{r_+}{\sqrt{2\pi}}\int_{\mathbb{R}} f(\tilde{v})e^{i\omega\tilde{v}}\,\mathrm{d}\tilde{v} \tag{5-98}$$

in mild abuse of notation.

Lemma 5.13. *Let $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\in C^\infty(\mathcal{H}^+)$ be spherically symmetric smooth data on the event horizon and assume that $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}$ is supported away from the past bifurcation sphere. Assume vanishing data on the left event horizon and let $\phi'_{\mathcal{L}}$ be the arising smooth solution to* (5-3) *attaining that data. Then, for any fixed v_1 and any $u\in\mathbb{R}$, $v\leq v_1$, we have*

$$\phi'_{\mathcal{L}}(u,v) = \frac{1}{\sqrt{2\pi}r}\int \mathcal{F}_{\mathcal{H}^+}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v_1}](\omega)\widetilde{u_{\mathcal{H}_R}}(r^*(u,v),\omega)e^{-i\omega v}\,\mathrm{d}\omega \tag{5-99}$$

and

$$\partial_v(r\phi'_{\mathcal{L}}(u,v)) = \frac{1}{\sqrt{2\pi}}\int \mathcal{F}_{\mathcal{H}^+}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v_1}](\omega)\,\partial_v(\widetilde{u_{\mathcal{H}_R}}(r^*(u,v),\omega)e^{-i\omega v})\,\mathrm{d}\omega, \tag{5-100}$$

$$\partial_u(r\phi'_{\mathcal{L}}(u,v)) = \frac{1}{\sqrt{2\pi}}\int \mathcal{F}_{\mathcal{H}^+}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v_1}](\omega)\,\partial_u(\widetilde{u_{\mathcal{H}_R}}(r^*(u,v),\omega)e^{-i\omega v})\,\mathrm{d}\omega, \tag{5-101}$$

where $\chi_{\leq v_1}(v)=\chi_0(v-v_1)$ and $\chi_0\colon\mathbb{R}\to[0,1]$ is a smooth cut-off which satisfies $\chi_0(x)=1$ for $x\leq 0$ and $\chi_0(x)=0$ for $x\geq 1$.

By a standard density argument, (5-99), (5-100) *and* (5-101) *hold also for spherically symmetric data $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\in C^1(\mathcal{H}^+)$ with $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}$ supported away from the past bifurcation sphere.*

Proof. Fix any v_1 and let (u,v) with $v\leq v_1$ be arbitrary. By the domain of dependence property, we have that $\phi'_{\mathcal{L}}$ satisfies $\phi'_{\mathcal{L}}=\phi'_{\mathcal{L}\leq v_1}$ on (u,v) with $v\leq v_1$, where $\phi'_{\mathcal{L}\leq v_1}$ is the unique solution arising from data $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v_1}\in C_c^\infty(\mathcal{H}^+)$ on the right event horizon $\mathcal{H}^+$ together with vanishing data on the left event horizon. Now, since $\mathcal{F}_{\mathcal{H}^+}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v_1}]$ is Schwartz, $u_{\mathcal{H}_R}$ satisfies (5-13), and $u_{\mathcal{H}_R}$ obeys the bounds as in Proposition 5.6, we can differentiate under the integral sign on the right-hand side of (5-99) and conclude that indeed the right-hand side of (5-99) solves (5-3). Finally, to show that $\phi'_{\mathcal{L}}=\phi'_{\mathcal{L}\leq v_1}$ it suffices to show that the right-hand side assumes the data from which $\phi'_{\mathcal{L}\leq v_1}$ arises. But again, since $\mathcal{F}_{\mathcal{H}^+}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v_1}]$ is Schwartz, we immediately obtain that the right-hand side of (5-99) converges to $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v_1}$ towards the right event horizon, and — after an application of the Riemann–Lebesgue lemma — to 0 towards the left event horizon. Now, (5-99) follows from uniqueness of the characteristic initial value problem. The formulae (5-100) and (5-101) now follow from differentiating under the integral sign, which can be applied as $\mathcal{F}_{\mathcal{H}^+}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v_1}]$ is a Schwartz function. □

Note that the above proposition immediately implies:

Corollary 5.14. *Let $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}$ be as in Lemma 5.13 and assume vanishing data on the left horizon. Let $\phi'_{\mathcal{L}}$ be the arising smooth solution attaining that data. Then,*

$$\phi'_{\mathcal{L}}(u,v) = \frac{1}{\sqrt{2\pi}r}\int_{\mathbb{R}} \mathcal{F}_{\mathcal{H}^+}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v}](\omega)\widetilde{u_{\mathcal{H}_R}}(r(u,v),\omega)e^{-i\omega v}\,\mathrm{d}\omega \tag{5-102}$$

and

$$\partial_v(r\phi'_{\mathcal{L}}(u,v)) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{F}_{\mathcal{H}^+}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v}](\omega)\,\partial_v(\widetilde{u_{\mathcal{H}_R}}(r(u,v),\omega)e^{-i\omega v})\,\mathrm{d}\omega, \tag{5-103}$$

$$\partial_u(r\phi'_{\mathcal{L}}(u,v)) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{F}_{\mathcal{H}^+}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v}](\omega)\,\partial_u(\widetilde{u_{\mathcal{H}_R}}(r(u,v),\omega)e^{-i\omega v})\,\mathrm{d}\omega \tag{5-104}$$

for $u, v \in \mathbb{R}$, where $\chi_{\leq v}$ is as in Lemma 5.13.

Proof. Choosing $v = v_1$ in Lemma 5.13 yields the result. □

5D. ***Main results from the linear theory.*** Before we state the main proposition about the linear theory, we define the following norms for sufficiently regular functions:

$$E_1[f] := \left(\int_{\mathbb{R}} |f(v)|^2 + |\partial_v f(v)|^2\,\mathrm{d}v\right)^{1/2}, \tag{5-105}$$

$$E_1^{\beta}[f] := \left(\int_{\mathbb{R}} (|f(v)|^2 + |\partial_v f(v)|^2)\langle v\rangle^{2\beta}\,\mathrm{d}v\right)^{1/2}, \tag{5-106}$$

$$F^{\beta}[f] := \sup_{v\geq 0}\langle v\rangle^{\beta}\left|\int_v^{+\infty} f(\tilde{v})e^{i\omega_{\mathrm{res}}\tilde{v}}\,\mathrm{d}\tilde{v}\right|. \tag{5-107}$$

Further, for part (E) of the following proposition, we will use the Fourier projection operator P_δ defined in Section 4E. We will further state estimates in the so-called late blue-shift region $\mathcal{LB}$. This region is defined as

$$\mathcal{LB} = \left\{\Delta' + \frac{2s}{2|K_-|}\log(v) \leq u + v\right\}$$

for some $\Delta' \geq 0$ chosen in Section 6A. (Note that the estimate below involving $\mathcal{LB}$ actually holds true uniformly for all $\Delta' \geq 0$.) For given u, we also define $v_\gamma(u)$ to satisfy

$$\Delta' + \frac{2s}{2|K_-|}\log(v_\gamma(u)) = u + v_\gamma(u).$$

Note that the estimate $\Omega^2_{\mathrm{RN}}(u,v) \lesssim v^{-2s}$ is satisfied in $\mathcal{LB}$. We refer to Figure 6 for a visualization of the region $\mathcal{LB}$ near i^+. In fact, in the region $\mathcal{LB}$ all the following estimates apply and $\mathcal{LB}$ is also the region in which we will make use of the linear theory for the nonlinear theory.

Theorem V. *Let $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \in C^1(\mathcal{H}^+)$ be spherically symmetric and assume that $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}$ is supported away from the past bifurcation sphere. Assume further that $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}$ has finite energy along the event horizon, i.e., that*

$$E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] < +\infty. \tag{5-108}$$

Let $\phi'_{\mathcal{L}}$ be the arising solution on the black hole interior with no incoming radiation from the left event horizon.

(A) *Then, for $v \geq 0$ and $u \in \mathbb{R}$ with $r^* = \frac{1}{2}(u+v) \geq 0$, we have*

$$e^{i\omega_{\mathrm{res}} r^*} \phi'_{\mathcal{L}}(u,v) = \frac{\sqrt{2\pi}\, i r_+}{r} \mathfrak{r}_{\omega_{\mathrm{res}}}(0) e^{i\omega_{\mathrm{res}} u} \left(\int_{-u}^{v} (\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(\tilde{v}) e^{i\omega_{\mathrm{res}} \tilde{v}} \, \mathrm{d}\tilde{v} \right) + \phi_{\mathrm{r}}(u,v) + \phi_{\mathrm{err}}(u,v), \tag{5-109}$$

where $\phi_{\mathrm{r}}(u,v)$ and $\phi_{\mathrm{err}}(u,v)$ satisfy the quantitative bounds

$$|\phi_{\mathrm{r}}(u,v)| \lesssim E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}], \tag{5-110}$$

$$|\phi_{\mathrm{err}}(u,v)| \lesssim_\alpha E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \Omega^{2-\alpha}_{\mathrm{RN}}(u,v) \tag{5-111}$$

uniformly for $v \geq 0$, $u \in \mathbb{R}$, $2r^* = v+u \geq 2$ and any fixed $0 < \alpha < 2$. Further, $\phi_{\mathrm{r}}(u,v)$ and $\phi_{\mathrm{err}}(u,v)$ extend continuously to the right Cauchy horizon. In particular, $\lim_{n\to+\infty} \phi_{\mathrm{r}}(u_n, v_n)$ exists for any sequence $(u_n, v_n) \to (u, +\infty)$.

(B) *If additionally $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}$ satisfies*

$$E_1^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] < +\infty, \tag{5-112}$$

$$F^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] < +\infty \tag{5-113}$$

for some $0 < \beta \leq 1$, then

$$\langle u \rangle^\beta |\phi'_{\mathcal{L}}|(u,v) \lesssim \langle u \rangle^\beta \left| \mathfrak{r}_{\omega_{\mathrm{res}}}(0) \int_{-u}^{v} (\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(\tilde{v}) e^{i\omega_{\mathrm{res}} \tilde{v}} \, \mathrm{d}\tilde{v} \right| + E_1^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] + F^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \tag{5-114}$$

uniformly for all $v \geq 2$, $u \in \mathbb{R}$ such that $v \geq v_\gamma(u)$.

(C) *Moreover,*

$$\partial_v (r e^{i\omega_{\mathrm{res}} r^*} \phi'_{\mathcal{L}}(u,v)) = -i \frac{r_+ e^{i\omega_{\mathrm{res}} u}}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}} \cdot}](\omega) \mathfrak{t}_{\omega_{\mathrm{res}}}(\omega) e^{-i\omega v} \, \mathrm{d}\omega + \Phi_{\mathrm{error}}, \tag{5-115}$$

where Φ_{error} satisfy the quantitative bounds

$$|\Phi_{\mathrm{error}}|(u,v) \lesssim_\alpha E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \Omega^{2-\alpha}_{\mathrm{RN}}(u,v) \tag{5-116}$$

for any fixed $0 < \alpha < 2$ and every (u,v) such that $r^*(u,v) \geq 1$.

(D) *Additionally to the assumptions in parts* (A) *and* (B), *let $\sigma_{\mathrm{br}} = \sigma_{\mathrm{br}}(u,v) \in C^1_{u,v}$ with $|\partial_v \sigma_{\mathrm{br}}| \lesssim \langle v \rangle^{1-2s}$ be arbitrary. Assume further that*

$$G^s[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] := \| \langle v \rangle^s (\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \|_{L^\infty} + \| \langle v \rangle^s \, \partial_v (\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \|_{L^\infty} < +\infty. \tag{5-117}$$

Then, for all $v \geq v_\gamma(u)$

$$\left| \int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')} \, \partial_{v'} (e^{i\omega_{\mathrm{res}} r^*} r \phi'_{\mathcal{L}}(u,v')) \, \mathrm{d}v' \right| \\ \lesssim \left| \int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')} e^{i\omega_{\mathrm{res}} v'} (\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(v') \, \mathrm{d}v' \right| + \langle u \rangle^{2-3s} (G^s[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] + E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]). \tag{5-118}$$

(E) *Let $u \in \mathbb{R}$ be arbitrary and assume that $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}$ is such that $\|\partial_v(re^{i\omega_{\mathrm{res}}r^*}\phi'_{\mathcal{L}}(u,v))\|_{L^1_v} < +\infty$.*

- *Assume in addition that $P_\delta(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \in L^1_v(\mathbb{R})$ for some $\delta > 0$. Then,*
$$\|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v} \lesssim_\delta \|\partial_v(re^{i\omega_{\mathrm{res}}r^*}\phi'_{\mathcal{L}}(u,v))\|_{L^1_v} + E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] + \|P_\delta(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v}.$$
- *If $0 < |q_0 e| < \epsilon(M,e,m^2)$ or $(q_0,m^2) \in \{0\} \times \mathbb{R} - D(M,e)$ as in Lemma 5.10, then*
$$\|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v} \lesssim \|\partial_v(re^{i\omega_{\mathrm{res}}r^*}\phi'_{\mathcal{L}}(u,v))\|_{L^1_v} + E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}].$$

Proof of Theorem V. (A) We use the representation formula (5-99) in Lemma 5.13 and have
$$\phi'_{\mathcal{L}}(u,v) = \frac{r_+}{\sqrt{2\pi}r}\,\mathrm{p.v.}\int_{\mathbb{R}}\Bigg[\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v}](\omega) \times \frac{\mathfrak{r}(\omega)\widetilde{u_{\mathcal{CH}_R}}(\omega,r^*)e^{i(\omega-\omega_{\mathrm{res}})r^*}+\mathfrak{t}(\omega)\widetilde{u_{\mathcal{CH}_L}}(\omega,r^*)e^{-i(\omega-\omega_{\mathrm{res}})r^*}}{\omega-\omega_{\mathrm{res}}}e^{-i\omega t}\Bigg]\mathrm{d}\omega. \tag{5-119}$$
After a change of variables $\omega \mapsto \omega+\omega_{\mathrm{res}}$, we obtain
$$\phi'_{\mathcal{L}}(u,v) = \frac{r_+e^{-i\omega_{\mathrm{res}}r^*}e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}r}\,\mathrm{p.v.}\int_{\mathbb{R}}\Bigg[\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v}e^{i\omega_{\mathrm{res}}\cdot}](\omega) \times \frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)\widetilde{u_{\mathcal{CH}_R}}(\omega+\omega_{\mathrm{res}},r^*)e^{i\omega u}+\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)\widetilde{u_{\mathcal{CH}_L}}(\omega+\omega_{\mathrm{res}},r^*)e^{-i\omega v}}{\omega}\Bigg]\mathrm{d}\omega, \tag{5-120}$$
where $\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega) = \mathfrak{r}(\omega+\omega_{\mathrm{res}})$ and $\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega) = \mathfrak{t}(\omega+\omega_{\mathrm{res}})$.

We now expand the numerator and obtain
$$\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)\widetilde{u_{\mathcal{CH}_R}}(\omega+\omega_{\mathrm{res}},r^*) = \mathfrak{r}_{\omega_{\mathrm{res}}}(0) + (\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{r}_{\omega_{\mathrm{res}}}(0)) + \mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}}+\omega,r^*)-1) \tag{5-121}$$
$$= \mathfrak{r}_{\omega_{\mathrm{res}}}(0) \tag{5-122}$$
$$+ (\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{r}_{\omega_{\mathrm{res}}}(0)) \tag{5-123}$$
$$+ \mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}}+\omega,r^*)-\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)) \tag{5-124}$$
$$+ \mathfrak{r}_{\omega_{\mathrm{res}}}(0)(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)-1) \tag{5-125}$$
$$+ (\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{r}_{\omega_{\mathrm{res}}}(0))(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)-1), \tag{5-126}$$
as well as
$$\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)\widetilde{u_{\mathcal{CH}_L}}(\omega+\omega_{\mathrm{res}},r^*) = \mathfrak{t}_{\omega_{\mathrm{res}}}(0) + (\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{t}_{\omega_{\mathrm{res}}}(0)) + \mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)(\widetilde{u_{\mathcal{CH}_L}}(\omega_{\mathrm{res}}+\omega,r^*)-1) \tag{5-127}$$
$$= \mathfrak{t}_{\omega_{\mathrm{res}}}(0) \tag{5-128}$$
$$+ (\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{t}_{\omega_{\mathrm{res}}}(0)) \tag{5-129}$$
$$+ \mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)(\widetilde{u_{\mathcal{CH}_L}}(\omega_{\mathrm{res}}+\omega,r^*)-\widetilde{u_{\mathcal{CH}_L}}(\omega_{\mathrm{res}},r^*)) \tag{5-130}$$
$$+ \mathfrak{t}_{\omega_{\mathrm{res}}}(0)(\widetilde{u_{\mathcal{CH}_L}}(\omega_{\mathrm{res}},r^*)-1) \tag{5-131}$$
$$+ (\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{t}_{\omega_{\mathrm{res}}}(0))(\widetilde{u_{\mathcal{CH}_L}}(\omega_{\mathrm{res}},r^*)-1). \tag{5-132}$$
We write
$$\frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)}{\omega} = \frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(0)}{\omega} + \mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega), \qquad \frac{\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)}{\omega} = \frac{\mathfrak{t}_{\omega_{\mathrm{res}}}(0)}{\omega} + \mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega), \tag{5-133}$$

where

$$\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega) := \frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega) - \mathfrak{r}_{\omega_{\mathrm{res}}}(0)}{\omega}, \quad \mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega) := \frac{\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega) - \mathfrak{t}_{\omega_{\mathrm{res}}}(0)}{\omega} \tag{5-134}$$

are real-analytic.

In the following we will estimate each term from (5-122)–(5-132) independently. We start with the main term coming from (5-122).

Lemma 5.15. *We have*

$$e^{i\omega_{\mathrm{res}} r^*}\phi_{\mathrm{mainR}}(u,v) := \frac{r_+ e^{i\omega_{\mathrm{res}} u}}{\sqrt{2\pi} r}\,\mathrm{p.v.}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v} e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(0)}{\omega} e^{i\omega u}\,\mathrm{d}\omega \tag{5-135}$$

satisfies

$$e^{i\omega_{\mathrm{res}} r^*}\phi_{\mathrm{mainR}}(u,v) = i\pi\frac{r_+ e^{i\omega_{\mathrm{res}} u}\mathfrak{r}_{\omega_{\mathrm{res}}}(0)}{\sqrt{2\pi} r}\int_{\mathbb{R}} (\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(\tilde v)\chi_{\le v}(\tilde v) e^{i\omega_{\mathrm{res}} u}\,\mathrm{sgn}(\tilde v + u)\,\mathrm{d}\tilde v. \tag{5-136}$$

Proof. This follows directly from the fact that $\mathcal{F}[\mathrm{p.v.}(1/x)] = i\pi\,\mathrm{sgn}$. □

Lemma 5.16. *We have that*

$$\begin{aligned} e^{i\omega_{\mathrm{res}} r^*}\phi_{\mathrm{errorR1}}(u,v) :&= \frac{r_+ e^{i\omega_{\mathrm{res}} u}}{\sqrt{2\pi} r}\,\mathrm{p.v.}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v} e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega) - \mathfrak{r}_{\omega_{\mathrm{res}}}(0)}{\omega} e^{i\omega u}\,\mathrm{d}\omega & \text{(5-137)} \\ &= \frac{r_+ e^{i\omega_{\mathrm{res}} u}}{\sqrt{2\pi} r}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v} e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega) e^{i\omega u}\,\mathrm{d}\omega & \text{(5-138)} \end{aligned}$$

extends continuously to the Cauchy horizon and satisfies

$$|\phi_{\mathrm{errorR1}}(u,v)| \lesssim E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]. \tag{5-139}$$

If additionally, $E_1^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] < +\infty$ for some $0 < \beta \le 1$, we further have

$$|\langle u\rangle^\beta \phi_{\mathrm{errorR1}}(u,v)| \lesssim E_1^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \tag{5-140}$$

for all $r^* \ge 0$.

Proof. It suffices to show both claims for $\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v} e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega) e^{i\omega u}\,\mathrm{d}\omega$. We begin by showing (5-140) under the assumption $E_1^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] < \infty$. We will use the notation $\langle\partial_\omega\rangle^\beta$ to denote the Fourier multiplier with $(1+|u|^2)^{\beta/2}$, where u is the dual variable to ω. Using this, we estimate

$$\begin{aligned} &\left|\langle u\rangle^\beta \int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v} e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega) e^{i\omega u}\,\mathrm{d}\omega\right| \\ &\quad = \left|\int_{\mathbb{R}} \langle\partial_\omega\rangle^\beta\big(\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v} e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)\big) e^{i\omega u}\,\mathrm{d}\omega\right| \\ &\quad \le \|\langle\partial_\omega\rangle^\beta(\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v} e^{i\omega_{\mathrm{res}}\cdot}]\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}})\|_{L^1_\omega} \\ &\quad \le \|\langle\partial_\omega\rangle^\beta(\langle\omega\rangle\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v} e^{i\omega_{\mathrm{res}}\cdot}])\|_{L^2_\omega}\|\langle\omega\rangle^{-1}\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}\|_{L^2_\omega} \\ &\qquad\qquad + \|\langle\omega\rangle\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v} e^{i\omega_{\mathrm{res}}\cdot}]\|_{L^2_\omega}\|\langle\partial_\omega\rangle^\beta(\langle\omega\rangle^{-1}\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}})\|_{L^2_\omega} \\ &\quad \lesssim \|\langle v\rangle^\beta(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v} e^{i\omega_{\mathrm{res}}\cdot}\|_{L^2_v} + \|\langle v\rangle^\beta\partial_v((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v} e^{i\omega_{\mathrm{res}}\cdot})\|_{L^2_v} \\ &\quad \lesssim E_1^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \end{aligned} \tag{5-141}$$

in view of a Kato–Ponce inequality (see, e.g., [Grafakos and Oh 2014, Theorem 1]) and

$$\|\langle\omega\rangle^{-1}\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}\|_{L^2(\mathbb{R}_\omega)} \lesssim 1, \tag{5-142}$$

$$\|\langle\partial_\omega\rangle^{\beta}(\langle\omega\rangle^{-1}\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}})\|_{L^2(\mathbb{R}_\omega)} \lesssim 1, \tag{5-143}$$

which follow from the definition of $\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}$, $\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}$ as well as Lemma 5.9. Now, note that the previous estimates for $\beta = 0$ give (5-139).

For the continuous extendibility across the Cauchy horizon we need to show that for $(u_n, v_n) \to (u_0, +\infty)$, the limit

$$\lim_{n\to\infty}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v_n}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{i\omega u_n}\,\mathrm{d}\omega \tag{5-144}$$

exists and that the limiting function is continuous. In view of the triangle inequality we have

$$\begin{aligned}\bigg|\int_{\mathbb{R}} &\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v_n}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{i\omega u_n}\,\mathrm{d}\omega - \int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{i\omega u_0}\,\mathrm{d}\omega\bigg|\\ &\lesssim \int_{\mathbb{R}} |\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)|\,|e^{i\omega u_n}-e^{i\omega u_0}|\,\mathrm{d}\omega\\ &\qquad + \int_{\mathbb{R}} |\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(1-\chi_{\leq v_n})e^{i\omega_{\mathrm{res}}\cdot}](\omega)|\,|\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)|\,\mathrm{d}\omega.\end{aligned} \tag{5-145}$$

In the first term of (5-145) we apply dominated convergence to interchange the limit with the integral which is justified as

$$\begin{aligned}\int_{\mathbb{R}} &|\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)|\,|e^{i\omega u_n}-e^{i\omega u_0}|\,\mathrm{d}\omega\\ &\lesssim \int_{\mathbb{R}} |\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)|\,\mathrm{d}\omega \lesssim E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]\end{aligned} \tag{5-146}$$

in view of (5-142). For the second term in (5-145) we have that

$$\begin{aligned}\int_{\mathbb{R}} &|\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(1-\chi_{\leq v_n})e^{i\omega_{\mathrm{res}}\cdot}](\omega)|\,|\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)|\,\mathrm{d}\omega\\ &\lesssim \bigg(\int_{\mathbb{R}} |\partial_{\tilde{v}}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(\tilde{v})(1-\chi_{\leq v_n}(\tilde{v})))|^2\,\mathrm{d}\tilde{v}\bigg)^{1/2} \to 0\end{aligned} \tag{5-147}$$

as $n\to\infty$ since $E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] < +\infty$. That the limit is continuous also follows from (5-146). ☐

Lemma 5.17. *We have that*

$$\begin{aligned}e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{errorR2}}(u,v) := \frac{r_+e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}r}\,\mathrm{p.v.}\!\int_{\mathbb{R}} &\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\\ &\cdot\frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}}+\omega,r^*)-\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*))}{\omega}e^{i\omega u}\,\mathrm{d}\omega\end{aligned} \tag{5-148}$$

converges to zero towards the Cauchy horizon and satisfies the quantitative bound

$$|\phi_{\mathrm{errorR2}}(u,v)| \lesssim \Omega^2_{\mathrm{RN}}(u,v)E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \tag{5-149}$$

for $r^*\geq 1$.

Proof. We estimate

$$\left|\frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}}+\omega,r^*)-\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*))}{\omega}\right| \tag{5-150}$$

$$\lesssim \sup_{|\omega|\le 1}|\partial_\omega \widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}}+\omega,r^*)| + \sup_{|\omega|\ge 1}|\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}}+\omega,r^*)-\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)| \tag{5-151}$$

$$\lesssim \Omega^2_{\mathrm{RN}} \tag{5-152}$$

in view of Lemma 5.7 and Proposition 5.6. Now, (5-149) follows from a direct application of the Cauchy–Schwarz inequality. □

Lemma 5.18. *We have that*

$$e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{errorR3}}(u,v)$$

$$:= \frac{r_+e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}\,r}\,\mathrm{p.v.}\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(0)(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)-1)}{\omega}e^{i\omega u}\,\mathrm{d}\omega \tag{5-153}$$

$$= \mathfrak{r}_{\omega_{\mathrm{res}}}(0)(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)-1)\frac{r_+e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}\,r}\,\mathrm{p.v.}\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{1}{\omega}e^{i\omega u}\,\mathrm{d}\omega \tag{5-154}$$

converges to zero towards the Cauchy horizon and satisfies the quantitative bound

$$\begin{aligned}|\phi_{\mathrm{errorR3}}(u,v)| &\lesssim \Omega^2_{\mathrm{RN}}(u,v)\|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v+u}\|_{L^1_v}\\ &\lesssim \Omega^2_{\mathrm{RN}}(u,v)E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]\langle r^*\rangle^{1/2} \lesssim_\alpha \Omega^{2-\alpha}_{\mathrm{RN}}(u,v)E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]\end{aligned} \tag{5-155}$$

for $r^*\ge 1$ and any $\alpha>0$.

Proof. It suffices to control the principal value integral. A direct computation using that $\mathcal{F}[\mathrm{p.v.}(1/x)] = i\pi\,\mathrm{sgn}$ yields

$$\left|\mathrm{p.v.}\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{1}{\omega}e^{i\omega u}\,\mathrm{d}\omega\right|$$

$$\lesssim \int_{\mathbb{R}}|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(\tilde v-u)\chi_{\le v}(\tilde v-u)|\,\mathrm{d}\tilde v \le \|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v+u}\|_{L^1(\mathbb{R})}. \tag{5-156}$$

The second inequality in (5-155) is now a consequence of the Cauchy–Schwarz inequality. □

Now, we are in the position to control the last term as follows.

Lemma 5.19. *We have that*

$$e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{errorR4}}(u,v)$$

$$:= \frac{r_+e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}\,r}\,\mathrm{p.v.}\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{(\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{r}_{\omega_{\mathrm{res}}}(0))(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)-1)}{\omega}e^{i\omega u}\,\mathrm{d}\omega \tag{5-157}$$

$$= (\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)-1)e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{errorR1}} \tag{5-158}$$

converges to zero towards the Cauchy horizon and satisfies the quantitative bound

$$|\phi_{\mathrm{errorR4}}(u,v)| \lesssim \Omega^2_{\mathrm{RN}}(u,v)E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \tag{5-159}$$

for $r^*\ge 0$.

Proof. This follows immediately from Lemma 5.16. $\square$

Now, we turn to the terms arising from the transmission coefficient. Completely analogous to Lemma 5.15 we obtain:

Lemma 5.20. *We have that*

$$e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{mainT}}(u,v) := \frac{r_+ e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}r}\,\mathrm{p.v.}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{\mathfrak{t}_{\omega_{\mathrm{res}}}(0)}{\omega}e^{-i\omega v}\,\mathrm{d}\omega \tag{5-160}$$

satisfies

$$e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{mainT}}(u,v) = i\pi\frac{r_+ e^{i\omega_{\mathrm{res}}u}\mathfrak{t}_{\omega_{\mathrm{res}}}(0)}{\sqrt{2\pi}r}\int_{\mathbb{R}}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(\tilde{v})\chi_{\leq v}(\tilde{v})e^{i\omega_{\mathrm{res}}\tilde{v}}\,\mathrm{sgn}(\tilde{v}-v)\,\mathrm{d}\tilde{v}. \tag{5-161}$$

Lemma 5.21. *We have that*

$$\begin{aligned} e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{errorT1}}(u,v) &:= \frac{r_+ e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}r}\,\mathrm{p.v.}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{t}_{\omega_{\mathrm{res}}}(0)}{\omega}e^{-i\omega v}\,\mathrm{d}\omega \qquad (5\text{-}162)\\ &= \frac{r_+ e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}r}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega \qquad (5\text{-}163)\end{aligned}$$

extends continuously to zero at the right Cauchy horizon, i.e., for $v\to+\infty$ and $u\to u_0$. If in addition $E_1^{\beta}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]<\infty$, then we have the quantitative decay

$$|\phi_{\mathrm{errorT1}}(u,v)| \lesssim \langle v\rangle^{-\beta}E_1^{\beta}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]. \tag{5-164}$$

Proof. We first show the first claim without assuming that $E_1^{\beta}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]<+\infty$. Doing the analogous estimate as in (5-147) it suffices to show that

$$\left|\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde{v}}](\omega)\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega\right| \tag{5-165}$$

tends to zero as $v\to+\infty$. Thus, it suffices to show that $v\mapsto\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde{v}}](\omega)\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega$ is an H^1 function. This again follows from

$$\int_{\mathbb{R}}(1+\omega^2)|\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde{v}}](\omega)|^2|\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)|^2\,\mathrm{d}\omega \lesssim E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]\sup_{\omega\in\mathbb{R}}|\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)| \lesssim E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]. \tag{5-166}$$

We will now proceed to show the quantitative decay assuming $E^{\beta}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]<\infty$. In this case we have

$$\begin{aligned} &\left|\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v}e^{i\omega_{\mathrm{res}}\tilde{v}}](\omega)\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega\right|\\ &\quad= \left|\frac{1}{v}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v}e^{i\omega_{\mathrm{res}}\tilde{v}}](\omega)\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)\,\partial_\omega e^{-i\omega v}\,\mathrm{d}\omega\right|\\ &\quad\lesssim \left|\frac{1}{v}\int_{\mathbb{R}} \partial_\omega\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v}e^{i\omega_{\mathrm{res}}\tilde{v}}](\omega)\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega\right|\\ &\qquad\qquad+\left|\frac{1}{v}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v}e^{i\omega_{\mathrm{res}}\tilde{v}}](\omega)(\partial_\omega\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega))e^{-i\omega v}\,\mathrm{d}\omega\right| \end{aligned}$$

$$
\begin{aligned}
&\lesssim \left| \frac{1}{v} \int_{|\omega|\le 1} \mathcal{F}[\tilde{v}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \chi_{\le v} e^{i\omega_{\mathrm{res}}\tilde{v}}](\omega) \mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega) e^{-i\omega v}\, \mathrm{d}\omega \right| \\
&\qquad + \left| \frac{1}{v} \int_{|\omega|\le 1} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \chi_{\le v} e^{i\omega_{\mathrm{res}}\tilde{v}}](\omega) (\partial_\omega \mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)) e^{-i\omega v}\, \mathrm{d}\omega \right| \\
&\qquad\qquad + \left| \frac{1}{v} \int_{|\omega|\ge 1} \mathcal{F}[\tilde{v}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \chi_{\le v} e^{i\omega_{\mathrm{res}}\tilde{v}}](\omega) \mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega) e^{-i\omega v}\, \mathrm{d}\omega \right| \\
&\qquad\qquad\qquad + \left| \frac{1}{v} \int_{|\omega|\ge 1} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \chi_{\le v} e^{i\omega_{\mathrm{res}}\tilde{v}}](\omega) (\partial_\omega \mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)) e^{-i\omega v}\, \mathrm{d}\omega \right| \\
&\lesssim \frac{1}{v} \| \tilde{v}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \chi_{\le v} e^{i\omega_{\mathrm{res}}\tilde{v}} \|_{L^2_{\tilde{v}}} \| \partial_\omega \mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}} \|_{L^2_\omega[-1,1]} + \frac{1}{v} \| (\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \chi_{\le v} e^{i\omega_{\mathrm{res}}\tilde{v}} \|_{L^2_{\tilde{v}}} \| \mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}} \|_{L^2_\omega[-1,1]} \\
&\qquad + \frac{1}{v} \| \partial_{\tilde{v}} (\tilde{v}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \chi_{\le v} e^{i\omega_{\mathrm{res}}\tilde{v}}) \|_{L^2_{\tilde{v}}} \| \omega^{-1}\, \partial_\omega \mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}} \|_{L^2_\omega(\mathbb{R}-[-1,1])} \\
&\qquad\qquad + \frac{1}{v} \| \partial_{\tilde{v}} ((\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \chi_{\le v} e^{i\omega_{\mathrm{res}}\tilde{v}}) \|_{L^2_{\tilde{v}}} \| \omega^{-1} \mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}} \|_{L^2_\omega(\mathbb{R}-[-1,1])} \\
&\lesssim \frac{1}{v^\beta} E_1^\beta [(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]
\end{aligned}
$$

since

$$
\| \partial_\omega \mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}} \|_{L^2_\omega[-1,1]},\ \| \mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}} \|_{L^2_\omega[-1,1]},\ \| \omega^{-1}\, \partial_\omega \mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}} \|_{L^2_\omega(\mathbb{R}-[-1,1])},\ \| \omega^{-1} \mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}} \|_{L^2_\omega(\mathbb{R}-[-1,1])} \lesssim 1. \qquad \square
$$

Analogously to Lemma 5.17 we have:

Lemma 5.22. *We have that*

$$
\begin{aligned}
&e^{i\omega_{\mathrm{res}} r^*} \phi_{\mathrm{errorT2}}(u, v) \\
&:= \frac{r_+ e^{i\omega_{\mathrm{res}} u}}{\sqrt{2\pi} r} \mathrm{p.v.}\!\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \chi_{\le v} e^{i\omega_{\mathrm{res}} \cdot}](\omega) \frac{\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega) (\widetilde{u_{\mathcal{CH}_L}}(\omega_{\mathrm{res}}+\omega, r^*) - \widetilde{u_{\mathcal{CH}_L}}(\omega_{\mathrm{res}}, r^*))}{\omega} e^{-i\omega v}\, \mathrm{d}\omega \qquad (5\text{-}167)
\end{aligned}
$$

converges to zero towards the Cauchy horizon and satisfies the quantitative bound

$$
|\phi_{\mathrm{errorT2}}(u, v)| \lesssim \Omega^2_{\mathrm{RN}}(u, v) E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \qquad (5\text{-}168)
$$

for $r^* \ge 1$.

Analogously to Lemma 5.18 we further obtain:

Lemma 5.23. *We have that*

$$
\begin{aligned}
&e^{i\omega_{\mathrm{res}} r^*} \phi_{\mathrm{errorT3}}(u, v) \\
&:= \frac{r_+ e^{i\omega_{\mathrm{res}} u}}{\sqrt{2\pi} r} \mathrm{p.v.}\!\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \chi_{\le v} e^{i\omega_{\mathrm{res}} \cdot}](\omega) \frac{\mathfrak{t}_{\omega_{\mathrm{res}}}(0) (\widetilde{u_{\mathcal{CH}_L}}(\omega_{\mathrm{res}}, r^*) - 1)}{\omega} e^{-i\omega v}\, \mathrm{d}\omega \qquad (5\text{-}169) \\
&= \mathfrak{t}_{\omega_{\mathrm{res}}}(0) (\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}}, r^*) - 1) \frac{r_+ e^{i\omega_{\mathrm{res}} u}}{\sqrt{2\pi} r} \mathrm{p.v.}\!\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \chi_{\le v} e^{i\omega_{\mathrm{res}} \cdot}](\omega) \frac{1}{\omega} e^{-i\omega v}\, \mathrm{d}\omega \qquad (5\text{-}170)
\end{aligned}
$$

converges to zero towards the Cauchy horizon and satisfies the quantitative bound

$$
|\phi_{\mathrm{errorT3}}(u, v)| \lesssim_\alpha \Omega^{2-\alpha}_{\mathrm{RN}}(u, v) E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \qquad (5\text{-}171)
$$

for $r^* \ge 1$.

Finally, completely analogous to Lemma 5.19 we have:

Lemma 5.24. *We have that*

$$\begin{aligned} e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{errorT4}}(u,v) &:= \frac{r_+e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}\,r}\,\mathrm{p.v.}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v}e^{i\omega_{\mathrm{res}}\cdot}](\omega) \\ &\qquad\qquad \cdot \frac{(\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{t}_{\omega_{\mathrm{res}}}(0))(\widetilde{u_{\mathcal{CH}_L}}(\omega_{\mathrm{res}},r^*)-1)}{\omega}e^{-i\omega v}\,\mathrm{d}\omega \\ &= (\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)-1)e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{errorT1}} \end{aligned} \tag{5-172}$$

converges to zero towards the Cauchy horizon and satisfies the quantitative bound

$$|\phi_{\mathrm{errorR4}}(u,v)| \lesssim \Omega^2_{\mathrm{RN}}(u,v)E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \tag{5-173}$$

for $r^*\geq 1$.

Having estimated each term independently in the integral appearing in (5-120) and noting that

$$e^{i\omega_{\mathrm{res}}r^*}(\phi_{\mathrm{mainR}}+\phi_{\mathrm{mainT}})(u,v) = \frac{\sqrt{2\pi}\,ir_+}{r}\mathfrak{r}_{\omega_{\mathrm{res}}}(0)e^{i\omega_{\mathrm{res}}u}\left(\int_{-u}^{v}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(\tilde v)e^{i\omega_{\mathrm{res}}\tilde v}\,\mathrm{d}\tilde v\right) \tag{5-174}$$

in view of $\mathfrak{r}_{\omega_{\mathrm{res}}}(0)=-\mathfrak{t}_{\omega_{\mathrm{res}}}(0)$, we finally obtain (5-109) with

$$\phi_{\mathrm{r}} = e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{errorR1}} \tag{5-175}$$

and

$$\phi_{\mathrm{error}} = e^{i\omega_{\mathrm{res}}r^*}(\phi_{\mathrm{errorR2}}+\phi_{\mathrm{errorR3}}+\phi_{\mathrm{errorR4}}+\phi_{\mathrm{errorT1}}+\phi_{\mathrm{errorT2}}+\phi_{\mathrm{errorT3}}+\phi_{\mathrm{errorT4}}). \tag{5-176}$$

The bounds and continuity statement for ϕ_{r} and ϕ_{error} now follow from Lemma 5.16 and (5-172).

(B) In view of part (A) and the fact that Ω_{RN} decays exponentially in $r^*=\frac{1}{2}(u+v)$ towards the Cauchy horizon, it suffices to show that

$$\langle u\rangle^\beta\left|\int_{-u}^{v+1}\chi_{\leq v}(\tilde v)(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(\tilde v)e^{i\omega_{\mathrm{res}}\tilde v}\,\mathrm{d}\tilde v\right| + \langle u\rangle^\beta|\phi_{\mathrm{r}}(u,v)| \lesssim F^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]+E^\beta_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \tag{5-177}$$

as we consider the region $v\geq |u|+\log(v)/(2|K_-|)$ in which $\Omega^2_{\mathrm{RN}}(u,v)\lesssim\langle v\rangle^{-1}$. Now, the claim is a direct consequence of the second parts of Lemmas 5.16 and 5.21 together with the assumptions (5-113) and (5-112).

(C) We will now consider $\partial_v(re^{i\omega_{\mathrm{res}}r^*}\phi'_{\mathcal{L}})$. We use the second part of Lemma 5.13 and end up with

$$\begin{aligned} &\partial_v(e^{i\omega_{\mathrm{res}}r^*}r\phi'_{\mathcal{L}}(u,v)) \\ &= \frac{r_+e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}}\,\mathrm{p.v.}\int_{\mathbb{R}}\Bigg[\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v_1}e^{i\omega_{\mathrm{res}}\cdot}](\omega) \\ &\qquad \cdot\frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)\,\partial_v\widetilde{u_{\mathcal{CH}_R}}(\omega+\omega_{\mathrm{res}},r^*)e^{i\omega u}+\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)\,\partial_v(\widetilde{u_{\mathcal{CH}_L}}(\omega+\omega_{\mathrm{res}},r^*)e^{-i\omega v})}{\omega}\Bigg]\mathrm{d}\omega \end{aligned} \tag{5-178}$$

for $v_1>v$. Since $\partial_v\widetilde{u_{\mathcal{CH}_R}}$ and $\partial_v\widetilde{u_{\mathcal{CH}_L}}$ are bounded uniformly in absolute value by Ω^2_{RN} in view of Proposition 5.6, the terms of (5-178) which arise thereof are bounded by $\Omega^{2-\alpha}_{\mathrm{RN}}E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]$ for any $\alpha>0$

as in part (A). Similarly, $\widetilde{u_{\mathcal{CH}_L}} - 1$ is bounded by Ω^2_{RN} and thus, the main term arises from $\partial_v(e^{-i\omega v})$ and we obtain

$$\partial_v(e^{i\omega_{\mathrm{res}} r^*} r\phi'_{\mathcal{L}}(u,v)) = -i\frac{r_+ e^{i\omega_{\mathrm{res}} u}}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v_1} e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega + \Phi^{v_1}_{\mathrm{error}}, \tag{5-179}$$

where $|\Phi^{v_1}_{\mathrm{error}}| \lesssim_\alpha \Omega^{2-\alpha}_{\mathrm{RN}} E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]$. Note that $\Phi^{v_1}_{\mathrm{error}}$ depends on v_1 but the upper bound is uniform in v_1. Since $\langle\omega\rangle\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}] \in L^2_\omega$ and $\langle\omega\rangle^{-1}\mathfrak{t}_{\omega_{\mathrm{res}}} \in L^\infty_\omega$, we can take the limit $v_1 \to \infty$ and obtain

$$\partial_v(e^{i\omega_{\mathrm{res}} r^*} r\phi'_{\mathcal{L}}(u,v)) = -i\frac{r_+ e^{i\omega_{\mathrm{res}} u}}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega + \Phi_{\mathrm{error}}, \tag{5-180}$$

where $|\Phi_{\mathrm{error}}(u,v)| \lesssim_\alpha \Omega^{2-\alpha}_{\mathrm{RN}}(u,v)E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]$.

(D) Note that Φ_{error} as in part (C) decays proportional to $\Omega^{2-\alpha}_{\mathrm{RN}}$ for any $\alpha > 0$ and thus

$$\begin{aligned}\int_{v_\gamma(u)}^{v} |\Phi_{\mathrm{error}}|\,\mathrm{d}v' &\lesssim_\alpha (\Omega_{\mathrm{RN}})^{2-\alpha}(u, v_\gamma(u))E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \lesssim_\alpha \langle u\rangle^{-s(2-\alpha)}E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \\ &\lesssim \langle u\rangle^{2-3s}E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]\end{aligned} \tag{5-181}$$

choosing $\alpha > 0$ sufficiently small (recall that $s \le 1$ therefore $2s > 3s-2$). Thus, it suffices to show the result for the main part in (5-115). We further write

$$\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega) = \mathfrak{t}^0_{\omega_{\mathrm{res}}} + \omega\mathfrak{t}^1_{\omega_{\mathrm{res}}} + \omega^2\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}(\omega), \tag{5-182}$$

where we note that

$$|\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}| = \left|\frac{\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega) - \mathfrak{t}^0_{\omega_{\mathrm{res}}} - \omega\mathfrak{t}^1_{\omega_{\mathrm{res}}}}{\omega^2}\right| \lesssim \langle\omega\rangle^{-1}$$

and $|\partial_\omega\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}| \lesssim \langle\omega\rangle^{-1}$ in view of Corollary 5.8 and Lemma 5.9. Hence,

$$\langle v\rangle\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})(v) \in L^2(\mathbb{R}_v) \tag{5-183}$$

and thus, $\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}) \in L^1(\mathbb{R})$ by the Cauchy–Schwarz inequality.

Now, using (5-115) we obtain

$$\begin{aligned}&\left|\int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')}\,\partial_{v'}(e^{i\omega_{\mathrm{res}} r^*} r\phi'_{\mathcal{L}})\,\mathrm{d}v'\right| \\ &\quad\lesssim \left|\int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')} \int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{t}^0_{\omega_{\mathrm{res}}} e^{-i\omega v'}\,\mathrm{d}\omega\,\mathrm{d}v'\right| \\ &\qquad + \left|\int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')} \int \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega)\omega\mathfrak{t}^1_{\omega_{\mathrm{res}}} e^{-i\omega v'}\,\mathrm{d}\omega\,\mathrm{d}v'\right| \\ &\qquad + \left|\int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')} \int \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega)\omega^2\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}(\omega) e^{-i\omega v'}\,\mathrm{d}\omega\,\mathrm{d}v'\right|.\end{aligned} \tag{5-184}$$

For the first term we directly take the inverse Fourier transform and estimate

$$\begin{aligned}&\left|\int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')} \int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{t}^0_{\omega_{\mathrm{res}}} e^{-i\omega v'}\,\mathrm{d}\omega\,\mathrm{d}v'\right| \\ &\qquad\lesssim \left|\int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')} e^{i\omega_{\mathrm{res}} v'}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(v')\,\mathrm{d}v'\right|.\end{aligned} \tag{5-185}$$

Similarly, for the second term we integrate by parts and obtain

$$\begin{aligned}
&\left|\int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\omega \mathfrak{t}^1_{\omega_{\mathrm{res}}} e^{-i\omega v'}\,\mathrm{d}\omega\,\mathrm{d}v'\right| \\
&\qquad\lesssim \left|\int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')}\,\partial_{v'}(e^{i\omega_{\mathrm{res}}v'}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(v'))\,\mathrm{d}v'\right| \\
&\qquad\lesssim \langle u\rangle^{-s}\|\langle v\rangle^{s}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty} + \left|\int_{v_\gamma(u)}^{v} |\partial_{v'}\sigma_{\mathrm{br}}(u,v')|\,|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(v')|\,\mathrm{d}v'\right| \\
&\qquad\lesssim \langle u\rangle^{2-3s}\|v^{s}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty}.
\end{aligned}\tag{5-186}$$

Using the same method as above, the third term satisfies

$$\left|\int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')}\int \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\omega^2\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v'}\,\mathrm{d}\omega\,\mathrm{d}v'\right|$$

$$\lesssim \left|\int_{v_\gamma(u)}^{v} \partial_{v'}(e^{i\sigma_{\mathrm{br}}(u,v')})\int \mathcal{F}[\partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})](\omega)\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v'}\,\mathrm{d}\omega\,\mathrm{d}v'\right| \tag{5-187}$$

$$+\left|\int \mathcal{F}[\partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})](\omega)\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega\right| \tag{5-188}$$

$$+\left|\int \mathcal{F}[\partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})](\omega)\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}(\omega)e^{i\omega v_\gamma(u)}\,\mathrm{d}\omega\right|. \tag{5-189}$$

We will now estimate the three terms individually.

We start with integrand of (5-187) and note that the other terms (5-188) and (5-189) are treated analogously. We write

$$\begin{aligned}
\left|\int \mathcal{F}[\partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})](\omega)\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v'}\,\mathrm{d}\omega\right| &\lesssim \left|[\partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})]*\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})\right|(v') \\
&= \left|\int_{\mathbb{R}} \partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})(\tilde v)\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})(v'-\tilde v)\,\mathrm{d}\tilde v\right|.
\end{aligned}\tag{5-190}$$

To estimate the convolution, we note that for $v'\ge 2R$, either $|\tilde v|\ge R$ or $|\tilde v - v'|\ge R$. Thus,

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})(\tilde v)\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})(v'-\tilde v)\,\mathrm{d}\tilde v\right| \\
&\lesssim \left|\int_{|\tilde v|\ge R} \partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})(\tilde v)\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})(v'-\tilde v)\,\mathrm{d}\tilde v\right| + \left|\int_{|\tilde v - v'|\ge R} \partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})(\tilde v)\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})(v'-\tilde v)\,\mathrm{d}\tilde v\right| \\
&\lesssim R^{-s}\left|\int_{|\tilde v|\ge R} |\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})(v'-\tilde v)|\,\mathrm{d}\tilde v\right|(\|v^{s}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty} + \|v^{s}\,\partial_v(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty}) \\
&\qquad\qquad + R^{-1}\left|\int_{|\tilde v - v'|\ge R} |\partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})(\tilde v)|\,|v'-\tilde v|\,|\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})(v'-\tilde v)|\,\mathrm{d}\tilde v\right| \\
&\lesssim \langle v'\rangle^{-s}\|(\|v^{s}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty} + \|v^{s}\,\partial_v(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty})\|\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})\|_{L^1} + \langle\tilde v\rangle^{-1}E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]\|\langle v\rangle\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})\|_{L^2} \\
&\lesssim \langle v'\rangle^{-s}((\|v^{s}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty} + \|v^{s}\,\partial_v(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty}) + E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]),
\end{aligned}\tag{5-191}$$

where we used (5-183). Now, plugging these estimates in (5-187), (5-188) and (5-189) and using that $|\partial_v \sigma_{\mathrm{br}}| \lesssim \langle v\rangle^{1-2s}$, we obtain, since $\frac{3}{4} < s \leq 1$

$$\left|\int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')}\,\partial_{v'}(e^{i\omega_{\mathrm{res}} r^*} r\phi'_{\mathcal{L}})\right| \mathrm{d}v' \lesssim \langle u\rangle^{2-3s}(\|v^s(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty} + \|v^s\,\partial_v(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty} + E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]).$$

This shows (D).

(E) Assume that $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}$ is such that the arising solution $\phi'_{\mathcal{L}}$ satisfies $\partial_v(e^{i\omega_{\mathrm{res}} r^*} r\phi'_{\mathcal{L}})(u,\cdot) \in L^1_v$ on some constant u surface. Then, in view of (5-180), we have that

$$\begin{aligned}\left\|\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega\right\|_{L^1_v} &\lesssim \|\partial_v(re^{i\omega_{\mathrm{res}} r^*}\phi'_{\mathcal{L}}(u,v))\|_{L^1_v} + \|\Phi_{\mathrm{error}}\|_{L^1_v}\\ &\lesssim \|\partial_v(re^{i\omega_{\mathrm{res}} r^*}\phi'_{\mathcal{L}}(u,v))\|_{L^1_v} + E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]. \end{aligned} \tag{5-192}$$

We will first consider the cases for which $\mathfrak{t}_{\omega_{\mathrm{res}}}$ does not have any zeros (i.e., $\mathcal{Z}_{\mathfrak{t}} = \varnothing$); see Lemma 5.10. Then $1/\mathfrak{t}_{\omega_{\mathrm{res}}} \lesssim \langle\omega\rangle^{-1}$ since $|\mathfrak{t}|^2 = |\mathfrak{r}|^2 + \omega(\omega-\omega_{\mathrm{res}})$. For that, also recall $\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega) = \mathfrak{t}(\omega+\omega_{\mathrm{res}})$. Moreover, in this case, $\mathcal{F}^{-1}[1/\mathfrak{t}_{\omega_{\mathrm{res}}}] \in L^1_v$ since $1/\mathfrak{t}_{\omega_{\mathrm{res}}} \in L^2_\omega$, $\partial_\omega(1/\mathfrak{t}_{\omega_{\mathrm{res}}}) \in L^2_\omega$. Thus, $1/\mathfrak{t}_{\omega_{\mathrm{res}}}$ is a L^1 bounded Fourier multiplier. Hence, using that $1 = \mathfrak{t}_{\omega_{\mathrm{res}}}(1/\mathfrak{t}_{\omega_{\mathrm{res}}})$ and (5-192), we obtain

$$\begin{aligned}\|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v} &\lesssim \left\|\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega\right\|_{L^1_v}\\ &\lesssim \|\partial_v(re^{i\omega_{\mathrm{res}} r^*}\phi'_{\mathcal{L}}(u,v))\|_{L^1_v} + E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]. \end{aligned} \tag{5-193}$$

Now, we consider the case, where $\mathfrak{t}$ potentially has zeros, all of which have to lie in $\mathcal{Z}^\delta_{\mathfrak{t}}$. Then, by the inverse triangle inequality applied to (5-192) we obtain

$$\begin{aligned}&\|\partial_v(re^{i\omega_{\mathrm{res}} r^*}\phi'_{\mathcal{L}}(u,v))\|_{L^1_v} + E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]\\ &\quad\gtrsim \left\|\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega\right\|_{L^1_v}\\ &\quad\geq \left\|\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega)(1-\chi_\delta(\omega+\omega_{\mathrm{res}}))\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega\right\|_{L^1_v}\\ &\qquad - \left\|\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega)\chi_\delta(\omega+\omega_{\mathrm{res}})\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega\right\|_{L^1_v}, \end{aligned} \tag{5-194}$$

where we recall that χ_δ is supported in $\mathcal{Z}^\delta_{\mathfrak{t}}$. For the first term we use $|1/\mathfrak{t}| \lesssim_\delta \langle\omega\rangle^{-1}$ on $\mathbb{R} - \mathcal{Z}^\delta_{\mathfrak{t}}$ and obtain

$$\begin{aligned}\left\|\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega)(1-\chi_\delta(\omega+\omega_{\mathrm{res}}))\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega\right\|_{L^1_v} &\gtrsim_\delta \|(1-P_\delta)(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v}\\ &\geq \|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v} - \|P_\delta(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v}. \end{aligned} \tag{5-195}$$

For the second term we use $\mathfrak{t}\cdot\chi_\delta \in C^\infty_c$ and obtain

$$\left\|\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega)\chi_\delta(\omega+\omega_{\mathrm{res}})\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega\right\|_{L^1_v} \lesssim \|P_\delta(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v}. \tag{5-196}$$

Putting everything together yields

$$\|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v} \lesssim_\delta \|\partial_v(re^{i\omega_{\mathrm{res}}r^*}\phi'_{\mathcal{L}}(u,v))\|_{L^1_v} + E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] + \|P_\delta(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v}. \tag{5-197}$$

This shows part (E) and concludes the proof of Theorem V. □

To connect with the nonlinear theory and the various oscillation spaces from Section 3D we state the following corollaries from Theorem V. We will also introduce a smooth positive cut-off supported only on $v \geq v_0+2$ and such that $\chi_{\geq v_0+3} = 1$ for $v \geq v_0+3$. We assume that $|\partial_v \chi_{\geq v_0+3}| \leq 2$. We also recall the notation $\psi'_{\mathcal{L}} = r\phi'_{\mathcal{L}}$.

Corollary 5.25. *Let $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ be arbitrary and define $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(v) := \chi_{\geq v_0+3}(v)\phi_{\mathcal{H}^+}(v)$, which we trivially extend for $v \leq v_0$. Let $\phi'_{\mathcal{L}}$ be the unique solution of* (5-3) *with data $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}$ on $\mathcal{H}^+$ and no incoming data from the left event horizon. Note that by the definition of $\mathcal{SL}$ $\big($recalling $s \in (\frac{3}{4}, 1]\big)$ we have that for all $v \geq v_0$*

$$v^s(|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}|(v) + |\partial_v(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}|(v)) \leq 4D_1. \tag{5-198}$$

(1) *If $\phi_{\mathcal{H}^+} \in \mathcal{O}$, then*

$$\sup_{v \geq v_0, u_0 \leq u_s} \left| \int_{v_0}^{v} e^{iq_0\sigma_{\mathrm{br}}(v')} e^{iq_0 \int_{v_0}^{v'} (A'_{\mathrm{RN}})_v(u_0,v')\,dv'} D_v^{\mathrm{RN}} \psi'_{\mathcal{L}}(u_0,v')\,dv' \right| < +\infty \tag{5-199}$$

for all σ_{br} satisfying (3-15), (3-16).

(2) *If $\phi_{\mathcal{H}^+} \in \mathcal{O}'$, then additionally for all $u_0 \leq u_s$*

$$\lim_{v \to +\infty} \left| \int_{v_0}^{v} e^{iq_0\sigma_{\mathrm{br}}(v')} e^{iq_0 \int_{v_0}^{v} (A'_{\mathrm{RN}})_v(u_0,v')\,dv'} D_v^{\mathrm{RN}} \psi'_{\mathcal{L}}(u_0,v')\,dv' \right| \tag{5-200}$$

exists and is finite for all σ_{br} satisfying (3-15), (3-16).

(3) *If $\phi_{\mathcal{H}^+} \in \mathcal{O}''$, then additionally for all $D_{\mathrm{br}} > 0$ there exists $D' = D'(e, M, D_1, s, q_0, m^2, D_{\mathrm{br}}) > 0$ and $\tilde{\eta}_0(e, M, D_1, s, q_0, m^2, D_{\mathrm{br}}) > 0$ such that for all σ_{br} satisfying* (3-15), (3-16) *and for all $(u,v) \in \mathcal{LB}$*

$$\left| \int_{v_\gamma(u)}^{v} e^{iq_0\sigma_{\mathrm{br}}(v')} e^{iq_0 \int_{v_0}^{v} (A'_{\mathrm{RN}})_v(u,v')\,dv'} D_v^{\mathrm{RN}} \psi'_{\mathcal{L}}(u,v')\,dv' \right| \lesssim D' \cdot |u|^{s-1-\tilde{\eta}_0}. \tag{5-201}$$

(4) *Assume that $q_0 = 0$, $m^2 \notin D(M,e)$ and that $\phi_{\mathcal{H}^+} \in \mathcal{NO} = \mathcal{SL} - \mathcal{O}$. Then for all $u \in \mathbb{R}$*

$$\limsup_{v \to +\infty} |\phi'_{\mathcal{L}}|(u,v) = +\infty. \tag{5-202}$$

Remark 5.26. It should be noted that for the nonlinear problem we will impose nonzero data on $\underline{C}_{\mathrm{in}}$. For the difference estimates it however suffices if the linear data and the nonlinear data agree eventually on $\mathcal{H}^+$.

Proof. We begin by noting that $\phi_{\mathcal{H}^+} \in \mathcal{O}, \mathcal{O}', \mathcal{O}''$ if and only if $\frac{1}{4}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(v) = \frac{1}{4}\chi_{\geq v_0+3}(v)\phi_{\mathcal{H}^+}(v) \in \mathcal{O}, \mathcal{O}', \mathcal{O}''$, respectively.[12]

[12]The factor $\frac{1}{4}$ is just to make sure that $\frac{1}{4}\chi_{\geq v_0+3}(v)\phi_{\mathcal{H}^+}(v) \in \mathcal{SL}$ if $\phi_{\mathcal{H}^+} \in \mathcal{SL}$.

Now, the first statement is a consequence of part (D) of Theorem V, the expression for the gauge derivative in (5-7) and the fact that for some bounded function $f(u)$

$$\begin{aligned} q_0 \int_{v_0}^{v} (A'_{\mathrm{RN}})_{v'}(u, v')\, \mathrm{d}v' &= -\frac{1}{2}\int_{v_0}^{v} (\omega_- - \omega_r)\, \mathrm{d}v' + \frac{1}{2}\omega_{\mathrm{res}} \cdot (v - v_0) \\ &= -\frac{1}{2}\int_{v_0}^{+\infty} (\omega_- - \omega_r)\, \mathrm{d}v' + \frac{1}{2}\int_{v}^{+\infty} (\omega_- - \omega_r)\, \mathrm{d}v' + \frac{1}{2}\omega_{\mathrm{res}} \cdot (2r^* - u - v_0) \\ &= \omega_{\mathrm{res}} r^* + f(u) + O(\Omega^2_{\mathrm{RN}}(r^*)). \end{aligned} \tag{5-203}$$

The second statement follows completely analogously. For the third statement, we use part (D) of Theorem V, and that, defining $0 < \tilde{\eta}\eta_0 = \min\{\eta_0, \frac{1}{10}(3s-4)\}$ (where η_0 is as in the definition of $\mathcal{O}''$), we have $\min(1 - s + \tilde{\eta}_0, 2s - 3) = 1 - s + \tilde{\eta}_0$ for some $\tilde{\eta}_0 > 0$ as $s > \frac{3}{4}$.

Now, we proceed to the last statement. Indeed, under the assumption $q_0 = 0$ and $m^2 \notin D(M, e)$, we have that $\mathfrak{r}(\omega = 0) \neq 0$. Thus, from Theorem V(A), and the assumption $\phi_{\mathcal{H}^+} \in \mathcal{N}\mathcal{O}$, the claim follows. □

Moreover, we also deduce a result of $\dot{W}^{1,1}$ blow-up along outgoing cones for the linearized solution in the following sense. To state the following corollary we recall the definition of P_δ as in Section 4E.

Corollary 5.27. *Let the assumptions of Corollary 5.25 hold.*

(1) *Assume that $P_\delta(\phi_{\mathcal{H}^+}) \in L^1$ for some $\delta > 0$. Then, for all $u \leq u_s$, we have*

$$\int_{v_0}^{+\infty} |\phi_{\mathcal{H}^+}|(v')\, \mathrm{d}v' \lesssim_\delta \int_{v_0}^{+\infty} |D_v^{\mathrm{RN}} \psi'_{\mathcal{L}}|(u, v)\, \mathrm{d}v + \|P_\delta(\phi_{\mathcal{H}^+})\|_{L^1_v} + D_1, \tag{5-204}$$

recalling the definition $\psi'_{\mathcal{L}} = r_{\mathrm{RN}}\phi'_{\mathcal{L}}$. In particular, if

$$\phi_{\mathcal{H}^+} \in \mathcal{S}\mathcal{L} - L^1(\mathcal{H}^+) \quad \textit{with } P_\delta(\phi_{\mathcal{H}^+}) \in L^1(\mathbb{R}) \textit{ for some } \delta > 0, \tag{5-205}$$

then for all $u \leq u_s$,

$$\int_{v_0}^{+\infty} |D_v^{\mathrm{RN}} \psi'_{\mathcal{L}}|(u, v')\, \mathrm{d}v' = +\infty. \tag{5-206}$$

Thus, the set of data $\phi_{\mathcal{H}^+} \in \mathcal{S}\mathcal{L}$ leading to blow-up for each $u \leq u_s$ as in (5-206) *is generic in the sense that its complement H is the set $H = H_0 \cap \mathcal{S}\mathcal{L}$ for some vector space $H_0 \subset \mathcal{S}\mathcal{L}_0$ of infinite codimension in $\mathcal{S}\mathcal{L}_0$, where we recall* (3-13) *for the definition of $\mathcal{S}\mathcal{L}_0$.*

(2) *Assume $0 < |q_0 e| < \epsilon(M, e, m^2)$ or $q_0 = 0$ and $m^2 \notin D(M, e)$. Then, for all $u \leq u_s$, we have*

$$\int_{v_0}^{+\infty} |\phi_{\mathcal{H}^+}|(v')\, \mathrm{d}v' \lesssim \int_{v_0}^{+\infty} |D_v^{\mathrm{RN}} \psi'_{\mathcal{L}}|(u, v')\, \mathrm{d}v' + D_0. \tag{5-207}$$

In particular, if $\phi_{\mathcal{H}^+} \in \mathcal{S}\mathcal{L} - L^1(\mathcal{H}^+)$, then

$$\int_{v_0}^{+\infty} |D_v^{\mathrm{RN}} \psi'_{\mathcal{L}}|(u, v')\, \mathrm{d}v' = +\infty.$$

Proof. The statements follow from Theorem V(E). The genericity of $\mathcal{S}\mathcal{L} - H$ in the first statement is a direct consequence of (5-205). We have also used that $P_\delta((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}) \in L^1$ if and only if $P_\delta(\phi_{\mathcal{H}^+}) \in L^1$. □

6. Nonlinear estimates for the EMKG system and extendibility properties of the metric

We give a brief outline of Section 6:

(1) In Section 6A we recall the time-decay estimates that were established in the nonlinear setting by the second author in [Van de Moortel 2018] (see Theorem B). These estimates play a crucial role in the proof of the Cauchy horizon (in-)stability and will also be essential to the analysis of the present paper. Recall that the various gauges were defined in Sections 3 and 2C.

(2) In Sections 6B and 6B3, we provide some useful nonlinear estimates, and show how to deduce the continuous extendibility of the metric from the boundedness of the scalar field. To do so, we will in particular exploit the algebraic structure of the nonlinear terms in the Einstein equations.

(3) In Section 6C, we estimate the difference of the dynamical metric g with the Reissner–Nordström metric g_{RN} and the difference of the scalar field ϕ and its linear counterpart $\phi_{\mathcal{L}}$ ($\phi_{\mathcal{L}}$ differs from $\phi'_{\mathcal{L}}$ of Section 5 by a gauge change; see Section 6C). If $q_0=0$, we show that these differences are bounded, thus showing the coupled ϕ is bounded if and only if its linear counterpart $\phi_{\mathcal{L}}$ is bounded. If $q_0\neq 0$, the estimates are more involved and include a backreaction contribution from the Maxwell field; see Section 6C4.

(4) In Section 6D, we combine the results from the linear theory (Section 5) with the results above to prove Theorems I (i) (Section 6D1), I (ii) (Section 6D2), II (Section 6D3) and III (Section 6D4).

Throughout Section 6 we will work under the assumptions of Theorem B.

6A. ***The existence of a Cauchy horizon for the EMKG system and previously proven nonlinear estimates.*** We use five different regions which partition the domain $[-\infty, u_s]\times[v_0,+\infty]$; see Figure 7. To this effect, we first introduce the function $h(v)$ as in [Van de Moortel 2018, Proposition 4.4]; namely we define $h(v)$ by the relation

$$\Omega_H^2(U=0,v)=e^{2K_+\cdot(v+h(v)-v_0)}. \tag{6-1}$$

Note that $h(v_0)=0$ by gauges (3-7), (3-6). It is proven in [Van de Moortel 2018] that as $v\to+\infty$

$$h(v)=O(v^{2-2s})1_{s<1}+O(\log(v))1_{s=1},\qquad h'(v)=O(v^{1-2s}),\qquad h''(v)=O(v^{-2s}). \tag{6-2}$$

Now we can introduce the five regions partitioning our spacetime $\{0\le U\le U_s,\ v\ge v_0\}$:

(1) The event horizon $\mathcal{H}^+=\{u=-\infty\}=\{U=0\}$.

(2) The red-shift region $\mathcal{R}=\{u+v+h(v)\le-\Delta\}$.

(3) The no-shift region $\mathcal{N}:=\{-\Delta\le u+v+h(v)\le\Delta_N\}$.

(4) The early blue-shift region

$$\mathcal{EB}:=\left\{\Delta_N\le u+v+h(v)\le-\Delta'+\frac{2s}{2|K_-|}\log(v)\right\},$$

assuming that $|u_s|$is sufficiently large so that $\Delta_N+\Delta'<(2s/(2|K_-|))\log(v)$ in $\mathcal{EB}$.

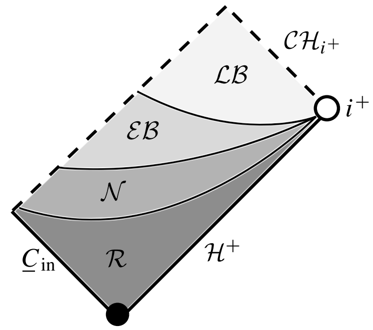

Figure 7. Division of a rectangular neighborhood of i^+ into five spacetime regions.

(5) The late blue-shift[13] region

$$\mathcal{LB} := \left\{ -\Delta' + \frac{2s}{2|K_-|} \log(v + h(v)) \le u + v + h(v) \right\}.$$

In the proof of Theorem B, it was shown that there exists a large constant $\Delta_0(M, e, q_0, m^2, s, D_1, D_2) > 0$ such that, if $\Delta, \Delta_N, \Delta' > \Delta_0$, the following estimates (as enumerated below) are true. In the course of the proof of the new result, we will implicitly always assume that $\Delta, \Delta_N, \Delta' > \Delta_0$ and choose when necessary $\Delta, \Delta_N, \Delta' > \Delta_1$ for some $\Delta_1(M, e, q_0, m^2, s, D_1, D_2) > \Delta_0$ that will be defined later.

Proposition 6.1 (nonlinear estimates on the event horizon $\mathcal{H}^+$ [Van de Moortel 2018]). *There exists a constant $D_H = D_H(M, e, q_0, m^2, s, D_1, D_2) > 0$ such that the following estimates hold true on $\mathcal{H}^+ = \{U = 0,\ v \ge v_0\}$:*

$$|Q(0, v) - e| \le D_H \cdot v^{1-2s}, \tag{6-3}$$

$$|\varpi(0, v) - M| \le D_H \cdot v^{1-2s}, \tag{6-4}$$

$$0 \le \lambda(0, v) \le D_H \cdot v^{-2s}, \tag{6-5}$$

$$0 \le r_+ - r(0, v) \le D_H \cdot v^{1-2s}, \tag{6-6}$$

$$|\partial_v \log(\Omega_H^2)(0, v) - 2K(0, v)| \le D_H \cdot v^{-2s}, \tag{6-7}$$

$$|2K_+ h'(v) + [2K_+ - 2K(0, v)]| \le D_H \cdot v^{-2s}, \tag{6-8}$$

$$|\partial_U \log(\Omega_H^2)|(0, v) \le D_H \cdot \Omega_H^2(0, v), \tag{6-9}$$

$$|\partial_U \phi|(0, v) \le D_H \cdot \Omega_H^2(0, v) \cdot v^{-s}, \tag{6-10}$$

$$|A_U|(0, v) \le D_H \cdot \Omega_H^2(0, v). \tag{6-11}$$

Proposition 6.2 (nonlinear estimates in the red-shift region $\mathcal{R}$ [Van de Moortel 2018]). *There exists a constant $D_R = D_R(M, e, q_0, m^2, s, D_1, D_2) > 0$ such that the following estimates hold true for all $(u, v) \in \mathcal{R}$:*

[13]Note that the late blue-shift differs slightly from [Van de Moortel 2018] where it was defined to be $\mathcal{LB} := \{-\Delta' + (2s/(2|K_-|)) \log(v) \le u + v + h(v)\}$.

$$|\phi|(u,v)+|D_v\phi|(u,v)\le D_R\cdot v^{-s}, \tag{6-12}$$
$$|D_u\phi|(u,v)\le D_R\cdot e^{2K_+\cdot(u+v+h(v))}\cdot v^{-s}, \tag{6-13}$$
$$|\log(\Omega^2(u,v))-2K_+\cdot(u+v+h(v))|\le D_R\cdot\Omega^2(u,v), \tag{6-14}$$
$$0\le 1-\kappa(u,v)\le D_R\cdot\Omega^2(u,v)\cdot v^{-2s}, \tag{6-15}$$
$$|\partial_u\log\Omega^2(u,v)|\le D_R\cdot\Omega^2(u,v), \tag{6-16}$$
$$|\partial_v\log(\Omega^2)(u,v)-2K(u,v)|\le D_R\cdot v^{-2s}, \tag{6-17}$$
$$0\le r_+-r(u,v)\le D_R\cdot\Omega^2(u,v)+v^{1-2s}, \tag{6-18}$$
$$|Q(u,v)-e|\le D_R\cdot v^{1-2s}, \tag{6-19}$$
$$|\varpi(u,v)-M|\le D_R\cdot v^{1-2s}, \tag{6-20}$$
$$|2K(u,v)-2K_+|\le D_R\cdot\Omega^2(u,v)+v^{1-2s}. \tag{6-21}$$

Proposition 6.3 (nonlinear estimates in the no-shift region $\mathcal{N}$ [Van de Moortel 2018]). *There exists a constant $D_N=D_N(M,e,q_0,m^2,s,D_1,D_2)>0$ such that the following estimates hold true for all $(u,v)\in\mathcal{N}$:*

$$|\phi(u,v)|+|D_v\phi(u,v)|\le D_N\cdot v^{-s}, \tag{6-22}$$
$$|D_u\phi(u,v)|\le D_N\cdot v^{-s}, \tag{6-23}$$
$$\left|\log\Omega^2(u,v)-\log\left(-\left(1-\frac{2M}{r(u,v)}+\frac{e^2}{r^2(u,v)}\right)\right)\right|\le D_N\cdot v^{1-2s}, \tag{6-24}$$
$$0\le 1-\kappa(u,v)\le D_N\cdot v^{-2s}, \tag{6-25}$$
$$|1-\iota(u,v)|\le D_N\cdot v^{1-2s}, \tag{6-26}$$
$$|\partial_u\log(\Omega^2)(u,v)-2K(u,v)|\le D_N\cdot v^{1-2s}, \tag{6-27}$$
$$|\partial_v\log(\Omega^2)(u,v)-2K(u,v)|\le D_N\cdot v^{-2s}, \tag{6-28}$$
$$|Q(u,v)-e|\le D_N\cdot v^{1-2s}. \tag{6-29}$$
$$|\varpi(u,v)-M|\le D_N\cdot v^{1-2s}. \tag{6-30}$$
$$|\log(\Omega^2)|(u,v)+|\log(r)|(u,v)\le D_N. \tag{6-31}$$

Moreover, denoting by $\gamma_N:=\{u+v+h(v)=\Delta_N\}$ the future boundary of $\mathcal{N}$, we have on γ_N

$$\Omega^2(u_\gamma(u),v)\le D_N\cdot e^{2K_-\cdot\Delta_N}. \tag{6-32}$$

Proposition 6.4 (nonlinear estimates in the early blue-shift region $\mathcal{EB}$ [Van de Moortel 2018]). *There exists a constant $D_E=D_E(M,e,q_0,m^2,s,D_1,D_2)>0$ such that the following estimates hold true for all $(u,v)\in\mathcal{EB}$:*

$$|\phi(u,v)|\le D_E\cdot v^{-s}\log(v), \tag{6-33}$$
$$|D_v\phi(u,v)|\le D_E\cdot v^{-s}, \tag{6-34}$$
$$|D_u\phi(u,v)|\le D_E\cdot v^{-s}, \tag{6-35}$$

$$|\log \Omega^2(u,v) - 2K_- \cdot (u+v+h(v))| \le D_E \cdot \Delta \cdot e^{-2K_+\Delta} < 1, \tag{6-36}$$

$$0 \le 1-\kappa(u,v) \le \tfrac{1}{3}, \tag{6-37}$$

$$|1-\iota(u,v)| \le \tfrac{1}{3}, \tag{6-38}$$

$$|\partial_u \log(\Omega^2)(u,v) - 2K(u,v)| \le D_E \cdot v^{1-2s} \log(v)^3, \tag{6-39}$$

$$|\partial_v \log(\Omega^2)(u,v) - 2K(u,v)| \le D_E \cdot v^{-2s} \log(v)^3, \tag{6-40}$$

$$|2K(u,v) - 2K_-| \le \tfrac{1}{1000}|K_-|, \tag{6-41}$$

$$|Q(u,v) - e| \le D_E \cdot v^{1-2s}, \tag{6-42}$$

$$|\varpi(u,v) - M| \le D_E \cdot v^{1-2s}, \tag{6-43}$$

$$|r(u,v) - r_-(M,e)| \le D_E \cdot (v^{1-2s} + \Omega^2(u,v)). \tag{6-44}$$

Moreover, denoting by $\gamma := \{u+v+h(v) = -\Delta' + (s/(2|K_-|))\log(v)\}$ *the future boundary of* $\mathcal{EB}$, *we have on* γ

$$\Omega^2(u_\gamma(v), v) \le D_E \cdot v^{-2s}. \tag{6-45}$$

Proposition 6.5 (nonlinear estimates in the late blue-shift region $\mathcal{LB}$ [Van de Moortel 2018]). *There exists a constant* $D_L = D_L(M, e, q_0, m^2, s, D_1, D_2) > 0$ *such that the following estimates hold true*: *for all* $\eta > 0$, *there exists* $C_\eta > 0$ *such that for all* $(u,v) \in \mathcal{LB}$

$$\Omega^{2\eta}(u,v)|\phi|(u,v) \le C_\eta \cdot v^{-s}, \tag{6-46}$$

$$\Omega^{2\eta}(u,v)|Q-e|(u,v) \le C_\eta \cdot v^{1-2s}, \tag{6-47}$$

$$|\phi|^2(u,v) + Q^2(u,v) \le D_L \cdot v^{2-2s} 1_{\{s<1\}} + D_L \cdot [\log(v)]^2 1_{\{s=1\}}, \tag{6-48}$$

$$|D_v\phi|(u,v) \le D_L \cdot v^{-s}, \tag{6-49}$$

$$|\partial_v \log(\Omega^2_{\mathrm{CH}})|(u,v) \le D_L \cdot v^{1-2s} 1_{\{s<1\}} + D_L \cdot \log(v) \cdot v^{-1} 1_{\{s=1\}}, \tag{6-50}$$

$$0 < \Omega^2(u,v) \le -\lambda(u,v) \le D_L \cdot v^{-2s}, \tag{6-51}$$

$$0 < -\nu(u,v) \le D_L \cdot |u|^{-2s}. \tag{6-52}$$

6B. *Nonlinear estimates exploiting the algebraic structure.* We emphasize that we *do not necessarily assume that* $\phi_{\mathcal{H}^+} \in \mathcal{O}$ in this section. The specific assumptions of this type are made in Section 6D only. In fact, we use many of these estimates in our companion paper [Kehle and Van de Moortel ≥ 2024] as well (where it is assumed that $\phi_{\mathcal{H}^+} \notin \mathcal{O}$). Throughout Sections 6B–6D we use the notation $|f(u,v)| \lesssim |g(u,v)|$ if there exists a constant $\Gamma(M, e, m^2, q_0, D_1, D_2, s) > 0$ such that $|f(u,v)| \le \Gamma \cdot |g(u,v)|$ for all (u,v) in the spacetime region of interest.

6B1. *Boundedness and continuous extendibility of* $D_u\psi$. To reach the goals of this section, we must first prove preliminary estimates on $D_u\psi$, where $\psi := r\phi$ is (what is called in the black hole exterior) the radiation field. Since r is upper and lower bounded in our region of interest, it may be very surprising to consider this quantity in the black hole *interior*. However, as it turns out, $D_u\psi$ is always bounded, while $D_u\phi$ is bounded if and only if ϕ is (providing $\liminf_{v\to+\infty} |\nu|(u,v) > 0$, which is conjecturally a generic condition; see [Van de Moortel 2021] for a discussion and proof of this result).

Proposition 6.6. *We have the following (gauge-independent) estimate for all $(u, v) \in \mathcal{LB}$:*

$$|D_u\psi|(u, v) \lesssim |u|^{-s}. \tag{6-53}$$

Moreover, in the gauge (2-26), both $D_u\psi$ and A_u admit a bounded extension to the Cauchy horizon, denoted by $(D_u\psi)_{\mathrm{CH}}$ and $(A_u)^{\mathrm{CH}}$, respectively.

Proof. Using (2-45) with the estimates of Proposition 6.5, we have

$$|\partial_v(D_u\psi)| \lesssim |\lambda| \cdot |\nu| \cdot |\phi| + \Omega^{1.99} \cdot v^{-s}.$$

Finally with (6-51) and (6-48) we get

$$|\partial_v(D_u\psi)| \lesssim v^{1-3s} \cdot |u|^{-2s} + \Omega^{1.99} \cdot v^{-s}.$$

Now the left-hand side is integrable in v since $s > \frac{2}{3}$ so $D_u\psi$ admits a bounded extension by integrability and integrating from γ we obtain the estimate, in view of the estimate on γ from Proposition 6.4. To conclude, the extendibility of A_u follows from (2-39) and the estimates of Proposition 6.5 that show that $|\partial_v A_u|$ is integrable in v. □

6B2. *Key estimates for a candidate coordinate system (u, V) for a continuous extension.* In this section, we construct an adequate coordinate system (u, V), in which the boundedness of the metric coefficient $\log(\Omega^2_{\mathrm{CH}})$ related to (u, V) by $\Omega^2_{\mathrm{CH}} = -2g(\partial_u, \partial_V)$ follows from the boundedness of the scalar field ϕ.

Proposition 6.7. *There exists a coordinate system (u, V) for which $V(v) < 1$, and $\lim_{v\to+\infty} V(v) = 1$ and for which, defining the metric coefficient $\Omega^2_{\mathrm{CH}}\, \mathrm{d}u\, \mathrm{d}V = \Omega^2\, \mathrm{d}u\, \mathrm{d}v$, we have for all $(u, v) \in \mathcal{LB}$:*

$$\left|\partial_v\left(\log(\Omega^2_{\mathrm{CH}})(u, v) + |\phi|^2(u, v) + \int_u^{u_s} \frac{|\nu|}{r}|\phi|^2(u', v)\, \mathrm{d}u'\right)\right| \lesssim v^{2-4s} + v^{-2s}|\log(v)|^3, \tag{6-54}$$

and

$$\left|\partial_v\partial_u\left(\log(\Omega^2_{\mathrm{CH}})(u, v) + |\phi|^2(u, v) + \int_u^{u_s} \frac{|\nu|}{r}|\phi|^2(u', v)\, \mathrm{d}u'\right)\right| \\ \lesssim |u|^{-2s} \cdot (v^{2-4s} + v^{-2s}|\log(v)|^3) + |u|^{-s} \cdot v^{1-3s}. \tag{6-55}$$

As a consequence, the quantity Υ defined as

$$\Upsilon(u, v) := \log(\Omega^2_{\mathrm{CH}}) + |\phi|^2 + \int_u^{u_s} \frac{|\nu|}{r}|\phi|^2\, \mathrm{d}u' \tag{6-56}$$

admits a continuous extension $\Upsilon_{\mathrm{CH}}(u)$ across $\mathcal{CH}_{i^+}$ and

$$\partial_u\Upsilon = \partial_u\left(\log(\Omega^2_{\mathrm{CH}}) + |\phi|^2 + \int_u^{u_s} \frac{|\nu|}{r}|\phi|^2\, \mathrm{d}u'\right) \tag{6-57}$$

admits a bounded extension across $\mathcal{CH}_{i^+}$.

Proof. We first use (2-43) to establish the two formulae

$$\frac{\partial_u\partial_v(r|\phi|^2)}{r} = \partial_u\partial_v(|\phi|^2) + \frac{\nu}{r}\,\partial_v(|\phi|^2) + \frac{1}{r}\partial_u(\lambda|\phi|^2),$$

$$-2\Re(D_u\phi\overline{\partial_v\phi}) = \frac{-\partial_u\partial_v(r|\phi|^2)}{r} + \left(\frac{\partial_u\partial_v r}{r} - \frac{m^2\Omega^2}{2}\right)|\phi|^2.$$

Now we define $2K_\gamma(v) := 2K(u_{\gamma(v)}, v)$ and we rewrite (2-33) using the two last formulae

$$\left|\partial_u(\partial_v \log(\Omega^2) - 2K_\gamma(v) + \partial_v(|\phi|^2)) + \frac{\nu}{r}\,\partial_v(|\phi|^2) + \frac{1}{r}\,\partial_u(\lambda|\phi|^2)\right| \lesssim |\lambda\nu|(1+|\phi|^2) + \Omega^2(1+Q^2+m^2|\phi|^2).$$

First note that the right-hand side is $O(|u|^{-2s}\cdot v^{2-4s} + |u|^{-2s}\cdot v^{-2s})$, using the estimates of Proposition 6.5. Using (2-32), (6-53) and the other estimates of Proposition 6.5 we get

$$|\partial_u(\lambda|\phi|^2)| = |\partial_u(r^{-2}\lambda|\psi|^2)| \lesssim |u|^{-2s}v^{2-4s} + |u|^{-s}\cdot v^{1-3s}.$$

This gives

$$\left|\partial_u(\partial_v \log(\Omega^2) - 2K_\gamma(v) + \partial_v(|\phi|^2)) + \frac{\nu}{r}\,\partial_v(|\phi|^2)\right| \lesssim |u|^{-2s}\cdot v^{-2s} + |u|^{-2s}v^{2-4s} + |u|^{-s}\cdot v^{1-3s}. \quad (6\text{-}58)$$

Now we want to integrate both sides on $[u_\gamma(v), u]$. Recall that on γ, $|\partial_v \log(\Omega^2)(u_\gamma(v), v) - 2K_\gamma(v)| \lesssim v^{-2s}|\log(v)|^3$ and $|\partial_v(\phi^2)| \lesssim v^{-2s}|\log(v)|$, as established in Proposition 6.4. Thus, we obtain

$$\left|\partial_v \log(\Omega^2) - 2K_\gamma(v) + \partial_v(|\phi|^2) + \int_{u_{\gamma(v)}}^{u} \frac{\nu}{r}\,\partial_v(|\phi|^2)\,\mathrm{d}u'\right| \lesssim v^{2-4s} + v^{-2s}|\log(v)|^3. \quad (6\text{-}59)$$

Now we write

$$\int_{u_{\gamma(v)}}^{u} \frac{\nu}{r}\,\partial_v(|\phi|^2)\,\mathrm{d}u' = \int_{u_{\gamma(v)}}^{u_s} \frac{\nu}{r}\,\partial_v(|\phi|^2)\,\mathrm{d}u' - \partial_v\left(\int_u^{u_s} \frac{\nu}{r}|\phi|^2\,\mathrm{d}u'\right) + \int_u^{u_s} \partial_v\left(\frac{\nu}{r}\right)|\phi|^2\,\mathrm{d}u'.$$

Using (2-32) and the estimates of Proposition 6.5 again, we see that

$$\left|\int_u^{u_s} \partial_v\left(\frac{\nu}{r}\right)|\phi|^2\,\mathrm{d}u'\right| \lesssim \int_u^{u_s} (|\nu|\,|\lambda| + \Omega^2(1+Q^2+|\phi|^2))|\phi|^2\,\mathrm{d}u' \lesssim v^{2-4s}.$$

Therefore we actually showed that

$$\left|\partial_v \log(\Omega^2) - 2K_\gamma(v) + \int_{u_\gamma(v)}^{u_s} \frac{\nu}{r}\,\partial_v(|\phi|^2)\,\mathrm{d}u' + \partial_v(|\phi|^2) - \partial_v\left(\int_u^{u_s} \frac{\nu}{r}|\phi|^2\,\mathrm{d}u'\right)\right| \lesssim v^{2-4s} + v^{-2s}|\log(v)|^3. \quad (6\text{-}60)$$

Note that the second and the third terms of the left-hand-side only depend on v and not on u.

We define a new coordinate system (u, V) with the equations

$$\frac{\mathrm{d}V}{\mathrm{d}v} = e^{f(v)}, \quad (6\text{-}61)$$

$$f'(v) = 2K_\gamma(v) + \int_{u_{\gamma(v)}}^{u_s} \frac{|\nu|}{r}\,\partial_v(|\phi|^2)(u', v)\,\mathrm{d}u'. \quad (6\text{-}62)$$

By the estimates of Proposition 6.5, note that $|f'(v) - 2K_-| \lesssim v^{1-2s}$ and we recall that $K_- < 0$; thus $V'(v)$ is integrable as $v \to +\infty$, and $V(v)$ increases towards a limit V_∞ which we can choose to be 1 without loss of generality. Therefore, we also have upon integration, as $v \to +\infty$:

$$1 - V(v) \approx e^{f(v)}.$$

We also denote by Ω^2_{CH} the metric coefficient in this system defined by $\Omega^2_{\mathrm{CH}} = -2g(\partial_u, \partial_V)$, i.e.,

$$\Omega^2_{\mathrm{CH}}\, \mathrm{d}u\, \mathrm{d}V = \Omega^2\, \mathrm{d}u\, \mathrm{d}v, \quad \text{hence} \quad \Omega^2_{\mathrm{CH}}(u, v) = \Omega^2(u, v)e^{-f(v)}.$$

We then have the claimed estimate (6-54)

$$\left|\partial_v\left(\log(\Omega^2_{\mathrm{CH}}) + |\phi|^2 + \int_u^{u_s} \frac{|\nu|}{r}|\phi|^2\, \mathrm{d}u'\right)\right| \lesssim v^{2-4s} + v^{-2s}|\log(v)|^3.$$

Clearly, (6-58) is a reformulation of (6-55). Since the right-hand sides of (6-54) and (6-55) are integrable in v for $s > \frac{3}{4}$, a standard Cauchy sequence argument shows that $\Upsilon(u, v)$ admits a continuous extension, and $\partial_u \Upsilon(u, v)$ has a (locally) bounded extension. □

6B3. *Metric extendibility conditional on the boundedness of the scalar field.* Now that we have built the quantity Υ and proven its extendibility, we will prove that the continuous extendibility of $|\phi|$ implies the continuous extendibility of the metric (conversely, the blow-up of $|\phi|$ implies that there exists no coordinate system (u, v) in which $\log(\Omega^2)$ is even bounded; see [Kehle and Van de Moortel $\geq$ 2024; Van de Moortel 2019]).

Lemma 6.8. *Assume that the function $(u, v) \in \mathcal{LB} \to |\phi|(u, v)$ extends continuously to $\mathcal{CH}_{i^+} \cap \{u \leq u_s\}$ as a continuous function $|\phi|_{\mathrm{CH}}(u)$. Then $\int_u^{u_s}(\nu/r)|\phi|^2(u', V)\, \mathrm{d}u'$ extends continuously to $\mathcal{CH}_{i^+} \cap \{u \leq u_s\}$ as a continuous function. Moreover, $\nu(u, v)$ extends to $\mathcal{CH}_{i^+} \cap \{u \leq u_s\}$ as a bounded function $\nu_{\mathrm{CH}}(u)$.*

Remark 6.9. In fact, we do not prove directly that ν extends as continuous function across the Cauchy horizon, as we do not control $\partial_u \nu$. However, even though ν_{CH} might not be continuous in u, it is clearly in L^1_{loc} (and even in $L^1(\mathcal{CH}_{i^+} \cap \{u \leq u_s\})$, as $|\nu_{\mathrm{CH}}| \lesssim |u|^{-2s}$) which is sufficient for our purpose.

Proof. Using the estimates of Proposition 6.5, we see that for $(u, v) \in \mathcal{LB}$

$$|\partial_v \nu|(u, v) \lesssim v^{-2s},$$

which shows, by integrability, that for all $u \leq u_s$ there exists $\nu_{\mathrm{CH}}(u)$ such that $\lim_{v\to+\infty} \nu(u, v) = \nu_{\mathrm{CH}}(u)$. Now take again $u_\infty < u_s$ and two sequences $u_i \to u_\infty$, $V_i \to 1$, $V_i < 1$ and write

$$\begin{aligned}
&\left|\int_{u_i}^{u_s} \frac{\nu}{r}|\phi|^2(u', V_i)\, \mathrm{d}u' - \int_{u_i}^{u_s} \frac{\nu_{\mathrm{CH}}(u')}{r_{\mathrm{CH}}(u')}|\phi|^2_{\mathrm{CH}}(u')\, \mathrm{d}u'\right| \\
&\quad \leq \left|\int_{u_i}^{u_\infty} \frac{\nu}{r}|\phi|^2(u', V_i)\, \mathrm{d}u'\right| + \left|\int_{u_i}^{u_\infty} \frac{\nu_{\mathrm{CH}}(u')}{r_{\mathrm{CH}}(u')}|\phi|^2_{\mathrm{CH}}(u')\, \mathrm{d}u'\right| \\
&\qquad\qquad + \left|\int_{u_\infty}^{u_s} \left(\frac{\nu}{r}|\phi|^2(u', V_i) - \frac{\nu_{\mathrm{CH}}(u')}{r_{\mathrm{CH}}(u')}|\phi|^2_{\mathrm{CH}}(u')\right) \mathrm{d}u'\right|.
\end{aligned}$$

Now both functions $(\nu/r)|\phi|^2(u, V)$ and $(\nu_{\mathrm{CH}}(u')/r_{\mathrm{CH}}(u'))|\phi|^2_{\mathrm{CH}}(u)$ are uniformly bounded in u and v on a set of the form $(u, V) \in [u_\infty - \epsilon, u_s] \times [1 - \epsilon, 1]$ and

$$\lim_{i\to+\infty} \frac{\nu}{r}|\phi|^2(u', V_i) = \frac{\nu_{\mathrm{CH}}(u')}{r_{\mathrm{CH}}(u')}|\phi|^2_{\mathrm{CH}}(u'),$$

so by the dominated convergence theorem, the last term tends to 0 as i tends to $+\infty$.

Moreover, the integrands of the first two terms are uniformly bounded, and thus these two terms tend to 0 as i tends to $+\infty$. This concludes the proof of the lemma. □

Corollary 6.10. *Assume that the function $(u, v) \in \mathcal{LB} \to |\phi|(u, v)$ extends continuously to $\mathcal{CH}_{i^+} \cap \{u \le u_s\}$ as a continuous function $|\phi|_{\mathrm{CH}}(u)$. Then the metric g admits a continuous extension $\tilde{g}$, which can be chosen to be C^0-admissible (Definition 2.1).*

Proof. It follows from Proposition 6.7 and Lemma 6.8 that Ω^2_{CH} extends continuously to $\mathcal{CH}_{i^+} \cap \{u \le u_s\} = \{u \le u_s\} \times \{V = 1\}$. We know already that r extends continuously to $\mathcal{CH}_{i^+} \cap \{u \le u_s\} = \{u \le u_s\} \times \{V = 1\}$; therefore, in view of the form of the metric (2-13), the corollary is proved. □

6C. ***Difference-type estimates on the scalar field and metric difference estimates.*** In this section, we carry out the nonlinear difference estimates. To do this, we have to introduce a new coordinate involving $h(v)$ defined in (6-1) (see the difference estimate (6-64), to compare with (6-7)):

$$\tilde{v}(v) := v + h(v). \tag{6-63}$$

Recalling (6-2), it is clear that $\tilde{v} = v \cdot (1 + O(v^{1-2s}))$ and $\partial_{\tilde{v}} f = \partial_v f \cdot (1 + O(v^{1-2s}))$ for all f. Note also

$$\widetilde{\Omega}^2(u, \tilde{v}(v)) = \frac{\Omega^2(u, v)}{1 + h'(v)} = (1 + O(v^{1-2s})) \cdot \Omega^2(u, v),$$

$$\partial_{\tilde{v}} \log(\widetilde{\Omega}^2)(u, \tilde{v}(v)) = \frac{\partial_v \log(\Omega^2)(u, v)}{1 + h'(v)} - \frac{h''(v)}{[1 + h'(v)]^2} = (1 + O(v^{1-2s})) \cdot \partial_v \log(\Omega^2)(u, v) + O(v^{-2s}),$$

where $\widetilde{\Omega}^2 := -2g(\partial_u, \partial_{\tilde{v}})$. Estimates from Section 6A can be easily translated into $(u, \tilde{v})$-coordinates:

Lemma 6.11. *Defining $\widetilde{\Omega}^2_H := -2g(\partial_U, \partial_{\tilde{v}})$, the estimate (6-7) on $\mathcal{H}^+$ is replaced by*

$$\left|\log\left(\frac{\widetilde{\Omega}^2(0, \tilde{v})}{\Omega^2_{\mathrm{RN}}(0, \tilde{v})}\right)\right| = \left|\log\left(\frac{\widetilde{\Omega}^2_H(0, \tilde{v})}{(\Omega^2_{\mathrm{RN}})_H(0, \tilde{v})}\right)\right| \lesssim \tilde{v}^{1-2s}, \quad |\partial_{\tilde{v}} \log(\widetilde{\Omega}^2_H)(0, \tilde{v}) - 2K_+| \lesssim \tilde{v}^{-2s}. \tag{6-64}$$

Moreover, (6-14), (6-17) are replaced by the following estimates valid in the spacetime region $\mathcal{R}$:

$$e^{2K_+(u+\tilde{v})} \lesssim \widetilde{\Omega}^2(U, \tilde{v}) \lesssim e^{2K_+(u+\tilde{v})}, \quad |\partial_{\tilde{v}} \log(\widetilde{\Omega}^2)(U, \tilde{v}) - 2K_+| \lesssim \tilde{v}^{-2s}. \tag{6-65}$$

Finally, (6-28) and (6-40) are replaced by the following (weaker) estimates in the regions $\mathcal{N} \cup \mathcal{EB}$:

$$|\partial_{\tilde{v}} \log(\widetilde{\Omega}^2)(U, \tilde{v}) - 2K(U, \tilde{v})| \lesssim \tilde{v}^{1-2s}. \tag{6-66}$$

All the others estimates of Section 6A are still valid replacing v by $\tilde{v}$, Ω^2 by $\widetilde{\Omega}^2$ and so on (adjusting the constants with no loss of generality, i.e., replacing D_H by $2D_H$, D_R by $2D_R$, $\frac{1}{3}$ by $\frac{2}{3}$ etc.).

Proof. This follows from the equation $\widetilde{\Omega}^2_H(0, \tilde{v}) = e^{2K_+ \cdot (\tilde{v} - v_0)}/(1 + h'(v))$ (using the identity (6-1)) and (6-8), (6-2). □

Notation. In view of Lemma 6.11, from now on and until the end of the paper, we make a mild abuse of notation and redefine v to be this new $\tilde{v}$ given by (6-63) with the necessary adjustments, i.e., λ becomes the notation for $\partial_{\tilde{v}} r$, Ω^2 the notation for $-2g(\partial_u, \partial_{\tilde{v}})$, etc. We will not use the old definition of v any longer in what follows.

The goal of this section is to take the difference between $\phi(u, v)$ and $\phi_{\mathcal{L}}(u, v)$ and estimate the quantity

$$\delta\phi(u, v) := \phi(u, v) - \phi_{\mathcal{L}}(u, v), \tag{6-67}$$

where $\phi_{\mathcal{L}}$ solves the linear equation

$$(\nabla_\mu + iq_0(A^{\mathrm{RN}})_\mu)(\nabla^\mu + iq_0(A^{\mathrm{RN}})^\mu)\phi_{\mathcal{L}} - m^2\phi_{\mathcal{L}} = 0 \tag{6-68}$$

on the fixed Reissner–Nordström background (2-7) in the gauge A^{RN} as in (2-30). More precisely, we will define data for $\phi_{\mathcal{L}}$ on $\mathcal{H}^+$ (data on $\underline{C}_{\mathrm{in}}$ is irrelevant) so as to match the data $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ for ϕ on $\mathcal{H}^+$ (see the paragraph immediately below): Our goal is then to prove that $\delta\phi$ is bounded and continuously extendible (for $q_0 = 0$), and similar estimates featuring nonlinear backreaction if $q_0 \neq 0$.

We now define $\phi_{\mathcal{L}}$ on $\mathcal{Q}^+$ as the unique solution of (6-68) on the fixed Reissner–Nordström metric (2-7) with parameters (M, e) and with data

$$\begin{aligned} \phi_{\mathcal{L}}(u, v_0) &\equiv 0 && \text{for all } u \in (-\infty, u_s], \\ (\phi_{\mathcal{L}})_{|\mathcal{H}^+}(v) &\equiv \chi_{\geq v_0+3}(v)\phi_{\mathcal{H}^+}(v) && \text{for all } v \in [v_0, +\infty), \end{aligned}$$

where $\chi_{v_0+\geq 3}$ is the smooth cut-off supported on $v \geq v_0 + 2$ and $\chi_{\geq v_0+3} = 1$ for $v \geq v_0 + 3$ as defined in Corollary 5.25.

Remark 6.12. Note that the unique solution $\phi'_{\mathcal{L}}$ arising from the above data in the gauge (2-31), which is used in Section 5, agrees with $\phi_{\mathcal{L}}$ up to a gauge transformation as the gauges agree for the initial data, in particular, $A_v^{\mathrm{RN}} = (A'_{\mathrm{RN}})_v = 0$ on the event horizon by construction.

Recall that $\phi_{\mathcal{L}}$ is also a solution of (2-36), (2-43), (2-45), (2-44) where $(r, \Omega^2, A, D, \phi)$ are all replaced by their Reissner–Nordström analogs $(r_{\mathrm{RN}}, \Omega^2_{\mathrm{RN}}, A^{\mathrm{RN}}, D^{\mathrm{RN}}, \phi_{\mathcal{L}})$. Similarly, $r_{\mathrm{RN}}, \Omega^2_{\mathrm{RN}}, A^{\mathrm{RN}}$ also satisfy the equations of Section 2D with $\phi \equiv 0$ (i.e., (2-7) satisfies the Einstein–Maxwell equations in spherical symmetry), a fact we will repetitively use.

The estimates of [Van de Moortel 2018], that are recalled in Section 6A and stated in Lemma 6.11 in our new coordinate system, are key to our new difference estimates. We will use these estimates throughout the argument, without necessarily referring to them explicitly.

6C1. *Difference estimates in the red-shift region.*

Proposition 6.13. *There exists $D'_H(M, e, q_0, m^2, s, D_1, D_2) > 0$ such that for all $(u, v) \in \mathcal{R}$*

$$\begin{aligned} &|r(u, v) - r_{\mathrm{RN}}(u, v)| + |\lambda(u, v) - \lambda_{\mathrm{RN}}(u, v)| \\ &\qquad + |Q(u, v) - e| + |\log(\Omega^2)(u, v) - \log(\Omega^2_{\mathrm{RN}})(u, v)| \leq D'_H \cdot v^{1-2s}, \end{aligned} \tag{6-69}$$

$$\begin{aligned} &|\partial_u \log(\Omega^2)(u, v) - \partial_u \log(\Omega^2_{\mathrm{RN}})(u, v)| + |\nu(u, v) - \nu_{\mathrm{RN}}(u, v)| \\ &\qquad + |A_u(u, v) - A_u^{\mathrm{RN}}(u, v)| \leq D'_H \cdot e^{2K_+(u+v)} \cdot v^{1-2s}, \end{aligned} \tag{6-70}$$

$$|\partial_u \delta\phi| \leq D'_H \cdot e^{2K_+(u+v)} \cdot v^{1-3s}, \tag{6-71}$$

$$|\delta\phi| + |\partial_v \delta\phi| \leq D'_H \cdot v^{1-3s}. \tag{6-72}$$

Proof. First, recall that $r_{\mathrm{RN}} \equiv r_+(M, e)$, $Q_{\mathrm{RN}} \equiv e$, $\varpi_{\mathrm{RN}} \equiv M$, and $\lambda_{\mathrm{RN}} \equiv 0$ on the event horizon $\mathcal{H}^+$, by definition. Lastly, recall that $A_u = A_u^{\mathrm{RN}}$ on $\underline{C}_{\mathrm{in}}$ by the gauge choice (3-5). Recalling that $D_H > 0$ is defined in Proposition 6.1, we bootstrap the estimates

$$|r(u, v) - r_{\mathrm{RN}}(u, v)| \leq 4D_H \cdot v^{1-2s}, \tag{6-73}$$

$$|\log(\Omega^2)(u, v) - \log(\Omega^2_{\mathrm{RN}})(u, v)| \leq 4B_H \cdot v^{1-2s} \tag{6-74}$$

for $B_H(M, e, q_0, m^2, D_1, D_2) > 0$ defined as the constant in (6-64) such that

$$\left|\log\left(\frac{\Omega^2(0, v)}{\Omega^2_{\mathrm{RN}}(0, v)}\right)\right| \leq B_H \cdot v^{1-2s}$$

in the new coordinate v. Plugging these bootstraps into (2-42) and using (6-10), (6-65), we find that

$$|\partial_u(r\,\partial_v r - r_{\mathrm{RN}}\,\partial_v r_{\mathrm{RN}})| = |\partial_u(r\lambda - r_{\mathrm{RN}}\lambda_{\mathrm{RN}})| \lesssim |\Omega^2 - \Omega^2_{\mathrm{RN}}| + \Omega^2 \cdot (|\phi|^2 + |r - r_{\mathrm{RN}}| + |Q - e|) \lesssim e^{2K_+(u+v)} \cdot v^{1-2s},$$

where we used

$$|\Omega^2 - \Omega^2_{\mathrm{RN}}| \lesssim \Omega^2 \cdot \left|\log\left(\frac{\Omega^2}{\Omega^2_{\mathrm{RN}}}\right)\right| \lesssim \Omega^2 \cdot v^{1-2s}.$$

This is also equivalent (recalling (3-12)) to

$$|\partial_U(r\lambda - r_{\mathrm{RN}}\lambda_{\mathrm{RN}})| \lesssim e^{2K_+ v} \cdot v^{1-2s}.$$

Integrating the above using (6-5) we get

$$|r\lambda - r\lambda_{\mathrm{RN}}| \lesssim v^{-2s} + \Omega^2 \cdot v^{1-2s}. \tag{6-75}$$

Writing now the difference for (2-32), taking advantage of (6-75) and the bootstraps gives

$$|\partial_v\partial_U(r - r_{\mathrm{RN}})| \lesssim |\lambda_{\mathrm{RN}}| \cdot |\partial_U r - \partial_U r_{\mathrm{RN}}| + e^{2K_+ v} v^{-2s} + e^{2K_+ v} v^{1-2s}.$$

Integrating in v using a Gronwall estimate and the boundedness of $\partial_U r$ on $\underline{C}_{\mathrm{in}}$ we get

$$|\partial_U r - \partial_U r_{\mathrm{RN}}| \lesssim 1 + e^{2K_+ v} \cdot v^{1-2s},$$

which, upon integrating in U this time and using (6-6) gives

$$|r - r_{\mathrm{RN}}| \leq D_H \cdot v^{1-2s} + D \cdot e^{2K_+(u+v)} e^{-2K_+ v} + D \cdot e^{2K_+(u+v)} \cdot v^{1-2s},$$

where $D(M, e, q_0, m^2, D_1, D_2) > 0$. Choosing Δ sufficiently large such that

$$e^{2K_+(u+v)} \leq e^{-2K_+\Delta} < D^{-1} \cdot D_H$$

allows us to retrieve bootstrap (6-73).

Similarly plugging (6-12), (6-13), (6-74) and the previously proven estimates into (2-33) we get

$$|\partial_u\partial_v(\log(\Omega^2) - \log(\Omega^2_{\mathrm{RN}}))| \lesssim |D_u\phi| \cdot |\partial_v\phi| + |\Omega^2 - \Omega^2_{\mathrm{RN}}| + \Omega^2 \cdot v^{1-2s} \lesssim \Omega^2 \cdot v^{1-2s},$$

or equivalently using (6-65)

$$|\partial_v \partial_U (\log(\Omega^2) - \log(\Omega^2_{\mathrm{RN}}))| \lesssim e^{2K_+ v} \cdot v^{1-2s}.$$

Integrating in v using the boundedness of $\partial_U \log(\Omega^2)$ and $\partial_U \log(\Omega^2_{\mathrm{RN}})$ on $\underline{C}_{\mathrm{in}}$ we get

$$|\partial_U (\log(\Omega^2) - \log(\Omega^2_{\mathrm{RN}}))| \lesssim 1 + e^{2K_+ v} \cdot v^{1-2s} \lesssim e^{2K_+ v} \cdot v^{1-2s},$$

from which we retrieve bootstrap (6-74), using the smallness of $e^{-2K_+\Delta}$ as we did above.

Now all bootstraps are closed and we continue with the proof of the claimed difference estimates. Taking the difference between (2-39) and its Reissner–Nordström version, and integrating in v using $A_u(u, v_0) - A_u^{\mathrm{RN}}(u, v_0) = 0$, we obtain

$$|A_u(u, v) - A_u^{\mathrm{RN}}(u, v)| \lesssim e^{2K_+ \cdot (u+v)} \cdot v^{1-2s}.$$

For $\delta\phi$, we introduce a new bootstrap assumption (completely independently from the other bootstrap assumptions that have already been retrieved), which is true on $\underline{C}_{\mathrm{in}}$ by assumption:

$$|\partial_u \delta\phi|(u, v) \le B_1 \cdot e^{2K_+(u+v)} \cdot v^{1-3s} \tag{6-76}$$

for some $B_1 > 0$ large enough to be chosen later. Integrating in u and using $|\delta\phi| \lesssim v^{1-3s}$ on the event horizon $\mathcal{H}^+$ (since $\delta\phi_{|\mathcal{H}^+} \equiv 0$ for $v \ge 3$) gives

$$|\delta\phi|(u, v) \lesssim (1 + B_1) \cdot e^{2K_+(u+v)} \cdot v^{1-3s} \lesssim (1 + B_1) \cdot e^{-2K_+\Delta} \cdot v^{1-3s}. \tag{6-77}$$

Now we take the difference of (2-36) obeyed by ϕ and the corresponding equation obeyed by $\phi_{\mathcal{L}}$, namely

$$\begin{aligned}\partial_u \partial_v(\delta\phi) = &-\frac{\partial_u\delta\phi\,\partial_v r}{r} - \frac{\partial_v\delta\phi\,\partial_u r}{r} + \frac{q_0 i\Omega^2}{4r^2} Q\,\delta\phi - \frac{m^2\Omega^2}{4}\delta\phi - iq_0 A_u \frac{\delta\phi\,\partial_v r}{r} - iq_0 A_u\,\partial_v\delta\phi \\ &- \partial_u\phi_{\mathcal{L}}\left[\frac{\partial_v r}{r} - \frac{\partial_v r_{\mathrm{RN}}}{r_{\mathrm{RN}}}\right] - \partial_v\phi_{\mathcal{L}}\left[\frac{\partial_u r}{r} - \frac{\partial_u r_{\mathrm{RN}}}{r_{\mathrm{RN}}}\right] + \left[\frac{q_0 i\Omega^2}{4r^2} Q - \frac{q_0 i\Omega^2_{\mathrm{RN}}}{4r^2_{\mathrm{RN}}} e\right]\phi_{\mathcal{L}} \\ &- \frac{m^2[\Omega^2 - \Omega^2_{\mathrm{RN}}]}{4}\phi_{\mathcal{L}} - iq_0\left[A_u\frac{\partial_v r}{r} - A_u^{\mathrm{RN}}\frac{\partial_v r_{\mathrm{RN}}}{r_{\mathrm{RN}}}\right]\phi_{\mathcal{L}} - iq_0[A_u - A_u^{\mathrm{RN}}]\partial_v\phi_{\mathcal{L}}.\end{aligned}$$

We get, using also (6-76), (6-77) and (6-65) (note that one can write $\phi_{\mathcal{L}} = \phi - \delta\phi$ and use (6-12), (6-13) to bound ϕ and (6-76), (6-77) to bound $\phi_{\mathcal{L}}$)

$$|\partial_u \partial_v \delta\phi| \lesssim e^{2K_+(u+v)} \cdot (1 + B_1) \cdot v^{1-3s} + v^{-s} \cdot e^{2K_+(u+v)} \cdot |\partial_v \delta\phi|. \tag{6-78}$$

Integrating in u and using Gronwall's estimate we get (recalling that $|\partial_v\delta\phi| \lesssim v^{1-3s}$ on $\mathcal{H}^+$) we get

$$|\partial_v \delta\phi| \lesssim (1 + B_1 \cdot e^{-2K_+\Delta}) \cdot v^{1-3s},$$

and using this in (6-78) we get

$$|\partial_v \partial_u \delta\phi| \lesssim e^{2K_+(u+v)} \cdot (1 + B_1) \cdot v^{1-3s} + B_1 \cdot e^{-2K_+\Delta} \cdot v^{1-4s} \cdot e^{2K_+(u+v)}.$$

Integrating in v this time, choosing B_1 appropriately and using the smallness of $e^{-2K_+\Delta}$, retrieves, together with another integration in u, bootstrap (6-76), gives the claimed estimates on $\delta\phi$ and concludes the proof. □

6C2. *Difference estimates in the no-shift region.*

Proposition 6.14. *There exists $C_N = C_N(M, e, q_0, m^2, s, D_1, D_2) > 0$ such that the following estimates are satisfied for all $(u, v) \in \mathcal{N}$:*

$$|r(u,v) - r_{\mathrm{RN}}(u,v)| + |Q(u,v) - e| + |\log(\Omega^2)(u,v) - \log(\Omega^2_{\mathrm{RN}})(u,v)| \le C_N \cdot v^{1-2s},$$

$$\begin{aligned}&|\partial_v \log(\Omega^2)(u,v) - \partial_v \log(\Omega^2_{\mathrm{RN}})(u,v)| + |\partial_u \log(\Omega^2)(u,v) - \partial_u \log(\Omega^2_{\mathrm{RN}})(u,v)| \\ &\qquad + |\lambda(u,v) - \lambda_{\mathrm{RN}}(u,v)| + |\nu(u,v) - \nu_{\mathrm{RN}}(u,v)| + |A_u(u,v) - A_u^{\mathrm{RN}}(u,v)| \le C_N \cdot v^{1-2s},\end{aligned}$$

$$|\delta\phi| + |\partial_u \delta\phi| + |\partial_v \delta\phi| \le C_N \cdot v^{1-3s}.$$

Proof. The proof consists of combination of the proof of Proposition 6.13 with that of in [Van de Moortel 2018, Proposition 4.7]: we partition $\mathcal{N}$ into smaller regions $\mathcal{N}_k := \{-\Delta + (k-1)\epsilon \le u + v \le -\Delta + k\epsilon\}$ for $k \in [\![1, N]\!]$ and $N \cdot \epsilon = \Delta'$. We will prove the result by finite induction on k. The induction hypothesis is that the following estimates hold in $\mathcal{N}_k$:

$$|r(u,v) - r_{\mathrm{RN}}(u,v)| + |\lambda(u,v) - \lambda_{\mathrm{RN}}(u,v)| + |\log(\Omega^2)(u,v) - \log(\Omega^2_{\mathrm{RN}})(u,v)| \le C_k \cdot v^{1-2s}, \tag{6-79}$$

$$|\nu(u,v) - \nu_{\mathrm{RN}}(u,v)| + |A_u(u,v) - A_u^{\mathrm{RN}}(u,v)| \le C_k \cdot v^{1-2s}, \tag{6-80}$$

where $C_k = 2^k \cdot B_N$ for a large enough constant $B_N > 0$ to be determined later. The estimates of Proposition 6.13 render the initialization of the induction true for B_N large enough. So we assume that (6-79), (6-80) hold for $\mathcal{N}_k$ and we prove them in $\mathcal{N}_{k+1}$. As before we bootstrap

$$|r(u,v) - r_{\mathrm{RN}}(u,v)| + |\lambda(u,v) - \lambda_{\mathrm{RN}}(u,v)| + |\log(\Omega^2)(u,v) - \log(\Omega^2_{\mathrm{RN}})(u,v)| \le 4C_k \cdot v^{1-2s}, \tag{6-81}$$

$$|\nu(u,v) - \nu_{\mathrm{RN}}(u,v)| + |A_u(u,v) - A_u^{\mathrm{RN}}(u,v)| \le 4C_k \cdot v^{1-2s}. \tag{6-82}$$

We treat one typical term, to show the specificity of the no-shift region $\mathcal{N}$ compared to $\mathcal{R}$: under the bootstraps and (6-31), (6-29) we have

$$|\partial_u \partial_v r| \lesssim (1 + C_k) \cdot v^{1-2s} \sim C_k \cdot |u|^{1-2s}.$$

Upon integration in the u direction, it gives, using (6-79) in the past, for some $E(M, e, q_0, m^2, D_1, D_2) > 0$

$$|\lambda - \lambda_{\mathrm{RN}}| \le C_k \cdot v^{1-2s} + E \cdot \epsilon \cdot C_k \cdot v^{1-2s};$$

thus for $\epsilon > 0$ sufficiently small, so that $E \cdot \epsilon < 1$, we close the part of bootstrap (6-81) relative to $\lambda - \lambda_{\mathrm{RN}}$. The other terms are addressed similarly, we omit the details. Such estimates allow us to retrieve bootstraps (6-81), (6-82) and prove the induction hypothesis. Once this is done, we can prove difference estimates for $\delta\phi$ exactly as in Proposition 6.13. □

6C3. *Difference estimates in the early blue-shift region.*

Proposition 6.15. *There exists a constant $C_E = C_E(M, e, q_0, m^2, s, D_1, D_2) > 0$ such that the following estimates are satisfied for all $(u, v) \in \mathcal{EB}$:*

$$|r(u,v)-r_{\mathrm{RN}}(u,v)| \le C_E \cdot v^{1-2s}, \tag{6-83}$$

$$|\nu(u,v)-\nu_{\mathrm{RN}}(u,v)|+|A_u(u,v)-A_u^{\mathrm{RN}}(u,v)|+|\lambda(u,v)-\lambda_{\mathrm{RN}}(u,v)| \le C_E \cdot v^{1-2s}, \tag{6-84}$$

$$|\partial_u \log(\Omega^2)(u,v)-\partial_u \log(\Omega^2_{\mathrm{RN}})(u,v)|+|\partial_v \log(\Omega^2)(u,v)-\partial_v \log(\Omega^2_{\mathrm{RN}})(u,v)| \le C_E \cdot v^{1-2s}, \tag{6-85}$$

$$|\partial_u \delta\phi|+|\partial_v \delta\phi| \le C_E \cdot v^{1-3s}, \tag{6-86}$$

$$|\delta\phi| \le C_E \cdot v^{1-3s} \cdot \log(v), \tag{6-87}$$

$$|\log(\Omega^2)(u,v)-\log(\Omega^2_{\mathrm{RN}})(u,v)| \le C_E \cdot v^{1-2s} \cdot \log(v). \tag{6-88}$$

Proof. Note that in $\mathcal{EB}$, as in $\mathcal{N}$, we have $v \sim |u|$ and that the size of the region is logarithmic, i.e., $u - u_{\gamma_N}(v) \lesssim \log(v) \sim \log(|u|)$ and $v - v_{\gamma_N}(u) \lesssim \log(|u|) \sim \log(v)$. As before, we start with bootstraps:

$$|\lambda(u,v) - \lambda_{\mathrm{RN}}(u,v)| + |\nu(u,v) - \nu_{\mathrm{RN}}(u,v)| \le 4C_N \cdot v^{1-2s}, \tag{6-89}$$

$$\Omega^2(u,v) \cdot |\log(\Omega^2)(u,v) - \log(\Omega^2_{\mathrm{RN}})(u,v)| \le 4C_N \cdot v^{1-3s}, \tag{6-90}$$

$$|r(u,v) - r_{\mathrm{RN}}(u,v)| \le B_N \cdot v^{1-2s} \tag{6-91}$$

for some $B_N > C_N$ to be determined later. The set of (u, v) for which these bootstraps are satisfied is nonempty by the estimates of Proposition 6.14.

Retrieving the bootstrap on $r - r_{\mathrm{RN}}$ is the most delicate. We use (2-19) and write the difference of the two identities below:

$$\lambda \cdot \kappa^{-1} = \nu \cdot \iota^{-1} = \frac{-2\lambda\nu}{\Omega^2} = \frac{1}{2} - \frac{\varpi}{r} + \frac{Q^2}{2r^2},$$

$$\nu_{\mathrm{RN}} = \lambda_{\mathrm{RN}} = -\frac{\Omega^2_{\mathrm{RN}}}{2} = \frac{1}{2} - \frac{M}{r_{\mathrm{RN}}} + \frac{e^2}{2r^2_{\mathrm{RN}}}.$$

Thus, we have

$$\begin{aligned}
&(\lambda - \lambda_{\mathrm{RN}}) \cdot \kappa^{-1} + \lambda_{\mathrm{RN}} \cdot (\kappa^{-1} - 1)\\
&\quad = (\lambda - \lambda_{\mathrm{RN}}) \cdot \kappa^{-1} + (\nu_{\mathrm{RN}} - \nu) + \nu \cdot (1 - e^{-\log(\Omega^2)+\log(\Omega^2_{\mathrm{RN}})}))\\
&\quad = -\frac{\varpi}{r} + \frac{Q^2}{2r^2} + \frac{M}{r_{\mathrm{RN}}} - \frac{e^2}{2r^2_{\mathrm{RN}}} \frac{M-\varpi}{r} + \frac{M}{r \cdot r_{\mathrm{RN}}} \cdot (r - r_{\mathrm{RN}}) + \frac{Q^2 - e^2}{2r^2} - \frac{e^2 \cdot (r + r_{\mathrm{RN}})}{2r^2 \cdot r^2_{\mathrm{RN}}} \cdot (r - r_{\mathrm{RN}});
\end{aligned}$$

hence, combined with the $(r - r_{\mathrm{RN}})$ terms, we have

$$\begin{aligned}
&\left(\frac{M}{r \cdot r_{\mathrm{RN}}} - \frac{e^2 \cdot (r + r_{\mathrm{RN}})}{2r^2 \cdot r^2_{\mathrm{RN}}}\right) \cdot (r - r_{\mathrm{RN}})\\
&\qquad = \frac{2M \cdot r \cdot r_{\mathrm{RN}} - e^2 \cdot (r + r_{\mathrm{RN}})}{2r^2 \cdot r^2_{\mathrm{RN}}} \cdot (r - r_{\mathrm{RN}})\\
&\qquad = (\lambda - \lambda_{\mathrm{RN}}) \cdot \kappa^{-1} + (\nu_{\mathrm{RN}} - \nu) + \nu \cdot (1 - e^{-\log(\Omega^2)(u,v)+\log(\Omega^2_{\mathrm{RN}})}) + \frac{-M + \varpi}{r} + \frac{-Q^2 + e^2}{2r^2}.
\end{aligned}$$

To conclude, we have to prove that the prefactor of the left-hand side $(2M\cdot r\cdot r_{\mathrm{RN}}-e^2\cdot(r+r_{\mathrm{RN}}))/(r^2\cdot r_{\mathrm{RN}}^2)$ is bounded away from zero: for this, notice that, since $0<|e|<M$, we have

$$r_-(M,e)=M-\sqrt{M^2-e^2}<e^2/M,$$

which is equivalent to

$$2M\cdot r_-^2<2e^2\cdot r_-.$$

By (6-44) and choosing Δ_N sufficiently large, there exists a small constant $\alpha(M,e)>0$ such that in $\mathcal{EB}$

$$\left|\frac{2M\cdot r\cdot r_{\mathrm{RN}}-e^2\cdot(r+r_{\mathrm{RN}})}{2r^2\cdot r_{\mathrm{RN}}^2}\right|>\alpha.$$

Thus, as a consequence of bootstrap (6-89) and (6-43), (6-42) and (6-37), there exists

$$C_N'(M,e,q_0,m^2,s,D_1,D_2)>0$$

such that

$$|r-r_{\mathrm{RN}}|\le C_N'\cdot v^{1-2s}+C_N'\cdot|\nu|\cdot|\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2)|\le C_N'\cdot v^{1-2s}+2C_N'\cdot 4C_N\cdot v^{1-2s}<B_N\cdot v^{1-2s},$$

where we chose $B_N=2C_N'+4C_N'\cdot 4C_N$ for the last inequality to be true. Therefore, bootstrap (6-91) is retrieved.

Now we turn to bootstrap (6-90), which is equally delicate (because we want to avoid a logarithmic loss). As in Proposition 6.13, we write the difference between (2-33) satisfied by Ω^2 and the analogous equation satisfied by Ω_{RN}^2. Using also (6-42) and bootstrap (6-89), (6-90), (6-91) we obtain

$$\begin{aligned}&|\partial_v\partial_u(\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2))|\\&\qquad\lesssim|D_u\phi|\cdot|\partial_v\phi|+\Omega^2\cdot(|Q-e|+|r-r_{\mathrm{RN}}|)+|\Omega^2-\Omega_{\mathrm{RN}}^2|+|\lambda|\cdot|\nu-\nu_{\mathrm{RN}}|+|\lambda-\lambda_{\mathrm{RN}}|\cdot|\nu|\\&\qquad\lesssim v^{-2s}+\Omega^2\cdot v^{1-2s}+\Omega^2\cdot\left|\log\left(\frac{\Omega^2}{\Omega_{\mathrm{RN}}^2}\right)\right|+\Omega^2\cdot|\nu-\nu_{\mathrm{RN}}|+\Omega^2\cdot|\lambda-\lambda_{\mathrm{RN}}|\lesssim v^{-2s}+\Omega^2\cdot v^{1-2s},\end{aligned}$$

where in the last line we have used (6-38), (6-37) as $|\lambda|,\ |\nu|\lesssim\Omega^2$ and the usual inequality

$$|\Omega^2-\Omega_{\mathrm{RN}}^2|\lesssim\Omega^2\cdot\left|\log\left(\frac{\Omega^2}{\Omega_{\mathrm{RN}}^2}\right)\right|$$

(which is true because $\Omega_{\mathrm{RN}}^2/\Omega^2\ge\frac{1}{10}$, an estimate which follows directly from (6-36)). Integrating in v (recall the v-difference is of size $\log(v)$), we get, using Proposition 6.14,

$$|\partial_u\log(\Omega^2)-\partial_u\log(\Omega_{\mathrm{RN}}^2)|(u,v)\lesssim v^{1-2s}.\tag{6-92}$$

Instead of integrating (6-92) directly (and incurring a logarithmic loss), we write an identity: for any $\eta>0$,

$$\partial_u[\Omega^\eta\cdot(\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2))]=\Omega^\eta\cdot\frac{\eta}{2}\cdot\partial_u\log(\Omega^2)\cdot(\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2))+\Omega^\eta\cdot\partial_u[\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2)],$$

from which we deduce, using also $\partial_u\log(\Omega^2)<0$ (see Proposition 6.4)

$$\begin{aligned}&\partial_u[\Omega^{2\eta}\cdot(\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2))^2]\\&\quad=2\eta\cdot\Omega^{2\eta}\cdot\partial_u\log(\Omega^2)\cdot(\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2))^2+2\Omega^\eta\cdot\partial_u[\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2)]\cdot(\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2))\\&\quad\le 2\Omega^{2\eta}\cdot|\partial_u[\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2)]|\cdot|\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2)|,\end{aligned}$$

which in turn implies, using (6-92)

$$\partial_u[\Omega^\eta \cdot |\log(\Omega^2) - \log(\Omega^2_{\mathrm{RN}})|] = \Omega^\eta \cdot |\partial_u[\log(\Omega^2) - \log(\Omega^2_{\mathrm{RN}})]| \lesssim \Omega^\eta \cdot v^{1-2s}.$$

Integrating the above in u using $\partial_u \log(\Omega^2) \in (3K_-, K_-)$ (see Proposition 6.4) and the bounds from Proposition 6.14, we get for some $E'(M, e, q_0, m^2, s, D_1, D_2) > 0$

$$\Omega^\eta(u,v) \cdot |\log(\Omega^2)(u,v) - \log(\Omega^2_{\mathrm{RN}})(u,v)| \le C_N \cdot v^{1-2s} + E' \cdot \eta^{-1} \cdot \Omega^\eta(u_{\gamma_{\mathcal{N}}}(v), v) \cdot v^{1-2s}. \tag{6-93}$$

Applying (6-93) for $\eta = 2$, choosing Δ_N large enough so that $\Omega^\eta(u_{\gamma_{\mathcal{N}}}(v), v) \approx e^{2K_- \Delta_N} < C_N/(10E')$ retrieves bootstrap (6-90).

Retrieving bootstrap (6-89) is done similarly: we integrate the difference between (2-32) satisfied by r and the analog satisfied by r_{RN}, using Proposition 6.14, and we prove

$$|\lambda(u,v) - \lambda_{\mathrm{RN}}(u,v)| + |\nu(u,v) - \nu_{\mathrm{RN}}(u,v)| \le 3C_N \cdot v^{1-2s},$$

which closes all the bootstrap assumptions.

Now we turn to the rest of the differences estimates claimed in the statement of the proposition. Integrating the differences into (2-39), (2-33) as we did in Proposition 6.13 gives straightforwardly

$$|A_u(u,v) - A_u^{\mathrm{RN}}(u,v)| \lesssim v^{1-2s},$$
$$|\partial_v \log(\Omega^2)(u,v) - \partial_v \log(\Omega^2_{\mathrm{RN}})(u,v)| \lesssim v^{1-2s},$$

where we also used that the size of the region of integration is logarithmic, i.e., $\int_{u_\gamma(v)}^u v^{-2s}\, du \lesssim v^{-2s}\log(v)$.

For $\delta\phi$, we proceed as in Proposition 6.13 and make the following bootstrap assumptions for some $B' > 0$:

$$\Omega(u,v) \cdot |\delta\phi|(u,v) \le B' \cdot v^{1-3s}, \tag{6-94}$$

$$|\partial_v \delta\phi|(u,v) \le B' \cdot v^{1-3s}. \tag{6-95}$$

Plugging differences into (2-36) satisfied by ϕ and the analogous equation satisfied by $\delta\phi$, we get, using (6-94), (6-95) and the previously proven difference estimates,

$$|\partial_u\partial_v \delta\phi| \lesssim B' \cdot \Omega \cdot v^{1-3s} + \Omega^2 \cdot |\partial_u \delta\phi|, \tag{6-96}$$

from which we deduce, upon integrating in v and using a Gronwall estimate,

$$|\partial_u \delta\phi|(u,v) \lesssim (1 + B' \cdot \Omega(u, v_{\gamma_{\mathcal{N}}}(u))) \cdot v^{1-3s}, \tag{6-97}$$

and plugging (6-97) into (6-96) and integrating in u this time we get

$$|\partial_v \delta\phi|(u,v) \lesssim (1 + B' \cdot [\Omega(u, v_{\gamma_{\mathcal{N}}}(u)) + \Omega(v_{\gamma_N}(v), v)]) \cdot v^{1-3s}, \tag{6-98}$$

which is sufficient to retrieve bootstrap (6-95) after an appropriate choice of B' and choosing also Δ_N large enough (to obtain a small constant from $\Omega(u_{\gamma_{\mathcal{N}}}(v), v)$ as we did above).

To retrieve bootstrap (6-94), we proceed as with $\partial_u \log(\Omega^2)$ earlier, with the identity

$$\partial_u(\Omega^\eta \delta\phi) = \frac{\eta}{2} \cdot \partial_u \log(\Omega^2) \cdot \Omega^\eta \cdot \delta\phi + \Omega^\eta \cdot \partial_u \delta\phi,$$

which also implies, using (6-97) and by the same reasoning as for $\partial_u \log(\Omega^2)$ above

$$\partial_u(\Omega^\eta|\delta\phi|) \le \Omega^\eta \cdot |\partial_u(\delta\phi)| \lesssim \Omega^\eta \cdot (1 + B' \cdot \Omega(u, v_{\gamma_\mathcal{N}}(u))) \cdot v^{1-3s}.$$

Integrating this inequality in u for $\eta = 1$, after an appropriate choice of B' and choosing also Δ_N large enough as we did above allows to retrieve bootstrap (6-94) and concludes the proof. □

6C4. *Difference estimates in the late blue-shift region.* In this section, we will not need to estimate metric differences anymore (although we will use the difference estimates from past sections); therefore, we do not require a bootstrap method and proceed directly.

Proposition 6.16. *There exists a constant $C_L = C_L(M, e, q_0, m^2, s, D_1, D_2) > 0$ such that the following are satisfied for all $(u, v) \in \mathcal{LB}$:*

$$|A_u(u, v) - A_u^{\mathrm{CH}}(u)| + |A_u^{\mathrm{RN}}(u, v) - (A_u^{\mathrm{RN}})^{\mathrm{CH}}(u)| \le C_L \cdot \Omega^2(u, v) \le C_L^2 \cdot v^{-2s}, \tag{6-99}$$

$$|A_u(u, v) - A_u^{\mathrm{RN}}(u, v)| + |A_u^{\mathrm{CH}}(u) - (A_u^{\mathrm{RN}})^{\mathrm{CH}}(u)| \le C_L \cdot |u|^{1-2s}, \tag{6-100}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}u}(A_u^{\mathrm{CH}} - (A_u^{\mathrm{RN}})^{\mathrm{CH}})\right|(u) \le C_L \cdot |u|^{1-2s}, \tag{6-101}$$

$$\begin{aligned}\big||D_v\psi|(u, v) - |D_v\psi_\mathcal{L}|(u, v)\big| \\ \le \left|e^{iq_0\int_{u_\gamma(v)}^u A_u^{\mathrm{CH}}(u')\,\mathrm{d}u'}\partial_v\psi(u, v) - e^{iq_0\int_{u_\gamma(v)}^u (A_u^{\mathrm{RN}})^{\mathrm{CH}}(u')\,\mathrm{d}u'}\partial_v\psi_\mathcal{L}(u, v)\right| \le C_L \cdot v^{1-3s},\end{aligned} \tag{6-102}$$

$$\left|\psi(u, v) - \int_{v_\gamma(u)}^v e^{iq_0\int_{u_\gamma(v')}^u [(A_u^{\mathrm{RN}})^{\mathrm{CH}} - A_u^{\mathrm{CH}}](u')\,\mathrm{d}u'}\partial_v\psi_\mathcal{L}(u, v')\,\mathrm{d}v'\right| \le C_L \cdot |u|^{2-3s}, \tag{6-103}$$

$$\big||D_u\psi|(u, v) - |D_u^{\mathrm{RN}}\psi_\mathcal{L}|(u, v)\big| \le \big|D_u\psi(u, v) - D_u^{\mathrm{RN}}\psi_\mathcal{L}(u, v)\big| \le C_L \cdot |u|^{1-3s} \cdot \log|u|. \tag{6-104}$$

Moreover, for every fixed $u < u_s$, there exists $f(u) \in \mathbb{C}$ such that

$$\lim_{v\to+\infty} \psi(u, v) - \int_{v_\gamma(u)}^v e^{iq_0\int_{u_\gamma(v')}^u [(A_u^{\mathrm{RN}})^{\mathrm{CH}} - A_u^{\mathrm{CH}}](u')\,\mathrm{d}u'}\partial_v\psi_\mathcal{L}(u, v')\,\mathrm{d}v' = f(u). \tag{6-105}$$

Proof. We start with estimates on the potentials: By (2-39) and (6-47) we have for $\eta = 0.01$

$$|\partial_v(A_u - A_u^{\mathrm{RN}})| \le |\partial_v A_u| + |\partial_v A_u^{\mathrm{RN}}| \lesssim \Omega^{2-\eta} + \Omega_{\mathrm{RN}}^{2-2\eta},$$

which we can integrate from the curve γ; using (6-45) and (6-50) using [Van de Moortel 2018, Lemma 4.1] as before, we obtain, using also Proposition 6.15, the bound

$$|A_u - A_u^{\mathrm{RN}}| \lesssim |u|^{1-2s}. \tag{6-106}$$

Moreover, recall that we proved in Proposition 6.6 that $A_u(u, v)$ and $A_u^{\mathrm{RN}}(u, v)$ extend to $\mathcal{CH}_{i^+}$ as bounded functions $(A_u)^{\mathrm{CH}}(u)$ and $(A_u^{\mathrm{RN}})^{\mathrm{CH}}(u)$, respectively. Integrating (2-39) towards the past from the Cauchy horizon $\mathcal{CH}_{i^+}$ we also obtain the following estimates for all $(u, v) \in \mathcal{LB}$:

$$|A_u(u, v) - A_u^{\mathrm{CH}}(u)| + |A_u^{\mathrm{RN}}(u, v) - (A_u^{\mathrm{RN}})^{\mathrm{CH}}(u)| \lesssim \Omega^2(u, v) \lesssim v^{-2s}, \tag{6-107}$$

$$\int_{u_\gamma(v)}^u |A_u(u', v) - A_u^{\mathrm{CH}}(u')|\,\mathrm{d}u' + \int_{u_\gamma(v)}^u |A_u^{\mathrm{RN}}(u', v) - (A_u^{\mathrm{RN}})^{\mathrm{CH}}(u')|\,\mathrm{d}u' \lesssim v^{-2s}. \tag{6-108}$$

To obtain (6-101), note the following identity obtained using (2-39) with (3-5) (note that $A_u(u,v_0)=A_u^{\mathrm{RN}}(u,v_0)$):

$$A_u^{\mathrm{CH}}(u)-(A_u^{\mathrm{RN}})^{\mathrm{CH}}(u)=\int_{v_0}^{+\infty}-\frac{\Omega^2(u,v')Q(u,v')}{r^2(u,v')}+\frac{\Omega^2_{\mathrm{RN}}(u,v')e}{r^2_{\mathrm{RN}}(u,v')}\,\mathrm{d}v'. \tag{6-109}$$

We now commute (2-39) with ∂_u to estimate $(d/du)(A_u^{\mathrm{CH}}(u)-(A_u^{\mathrm{RN}})^{\mathrm{CH}}(u))$ and we obtain a formula analogous to (6-109). Using the fact that $\partial_u\log(\Omega^2)\Omega^{0.1}$ is bounded (by Proposition 6.5) to estimate the parts of the integral lying in $\mathcal{LB}$, and we obtain an estimate only involving the regions strictly to the past of $\mathcal{LB}$:

$$\Big|\frac{\mathrm{d}}{\mathrm{d}u}(A_u^{\mathrm{CH}}(u)-(A_u^{\mathrm{RN}})^{\mathrm{CH}}(u))\Big|\le\Big|\int_{v_0}^{v_\gamma(u)}\partial_u\Big(-\frac{\Omega^2(u,v')Q(u,v')}{r^2(u,v')}+\frac{\Omega^2_{\mathrm{RN}}(u,v')e}{r^2_{\mathrm{RN}}(u,v')}\Big)\,\mathrm{d}v'\Big|+|u|^{-2s}. \tag{6-110}$$

Therefore, it is sufficient to control the above integral in $\mathcal{R}\cup\mathcal{N}\cup\mathcal{EB}$. Note that the differences $\Omega^2-\Omega^2_{\mathrm{RN}}$, $\partial_u\Omega^2-\partial_u\Omega^2_{\mathrm{RN}}$, $Q-e$, $\nu-\nu_{\mathrm{RN}}$ and $r-r_{\mathrm{RN}}$ have been controlled with $|u|^{1-2s}$ weights in Propositions 6.13, 6.14 and 6.15; this gives (6-101).

Now we turn to the ϕ estimates. We write (2-44) for $u_0=u_\gamma(v)$ and using the estimates from Proposition 6.5 (notably (6-47) and (6-46) with $\eta=0.1$) we obtain

$$\begin{aligned}
&|\partial_u(e^{iq_0\int_{u_\gamma(v)}^u A_u(u',v)\,\mathrm{d}u'}\partial_v\psi-e^{iq_0\int_{u_\gamma(v)}^u A_u^{\mathrm{RN}}(u',v)\,\mathrm{d}u'}\partial_v\psi_{\mathcal{L}})|\\
&\qquad\lesssim|\partial_u(e^{iq_0\int_{u_\gamma(v)}^u A_u(u',v)\,\mathrm{d}u'}\partial_v\psi)|+|\partial_u(e^{iq_0\int_{u_\gamma(v)}^u A_u^{\mathrm{RN}}(u,v)\,\mathrm{d}u'}\partial_v\psi_{\mathcal{L}})|\\
&\qquad\lesssim|u|^{-2s}\cdot v^{1-3s}+(\Omega^{1.9}+\Omega^{1.9}_{\mathrm{RN}})\cdot v^{-s}.
\end{aligned}$$

Integrating in u and using (6-45) with the usual integration rules (i.e., [Van de Moortel 2018, Lemma 4.1]) we obtain

$$\begin{aligned}
&|e^{iq_0\int_{u_\gamma(v)}^u A_u(u',v)\,\mathrm{d}u'}\partial_v\psi(u,v)-e^{iq_0\int_{u_\gamma(v)}^u A_u^{\mathrm{RN}}(u',v)\,\mathrm{d}u'}\partial_v\psi_{\mathcal{L}}(u,v)|\\
&\qquad\lesssim|\partial_v\psi(u_\gamma(v),v)-\partial_v\psi_{\mathcal{L}}(u_\gamma(v),v)|+|u|^{1-2s}\cdot v^{1-3s}+v^{-2.8s}\lesssim v^{1-3s},
\end{aligned}\tag{6-111}$$

where we also used (6-86). Then by (6-111), (6-108), we obtain

$$\begin{aligned}
&|\partial_v\psi(u,v)-e^{iq_0\int_{u_\gamma(v)}^u[(A_u^{\mathrm{RN}})^{\mathrm{CH}}(u')-A_u^{\mathrm{CH}}(u')]\,\mathrm{d}u'}\partial_v\psi_{\mathcal{L}}(u,v)|\\
&\quad\le|e^{iq_0\int_{u_\gamma(v)}^u A_u(u',v)\,\mathrm{d}u'}\partial_v\psi(u,v)-e^{iq_0\int_{u_\gamma(v)}^u A_u^{\mathrm{RN}}(u',v)\,\mathrm{d}u'}\partial_v\psi_{\mathcal{L}}(u,v)|\\
&\qquad\qquad+|e^{iq_0\int_{u_\gamma(v)}^u[A_u^{\mathrm{RN}}(u',v)-(A_u^{\mathrm{RN}})^{\mathrm{CH}}(u')-A_u(u',v)+A_u^{\mathrm{CH}}(u')]\,\mathrm{d}u'}-1|\cdot|\partial_v\psi_{\mathcal{L}}|(u,v)\\
&\quad\lesssim v^{1-3s}+|e^{iq_0\int_{u_\gamma(v)}^u[A_u^{\mathrm{RN}}(u',v)-(A_u^{\mathrm{RN}})^{\mathrm{CH}}(u')-A_u(u',v)+A_u^{\mathrm{CH}}(u')]\,\mathrm{d}u'}-1|\cdot|\partial_v\psi_{\mathcal{L}}|(u,v)\\
&\quad\lesssim v^{1-3s}+v^{-2s}\cdot v^{-s}\lesssim v^{1-3s},
\end{aligned}\tag{6-112}$$

where in the first line we multiplied by the phase $e^{iq_0\int_{u_\gamma(v)}^u A_u(u',v)\,\mathrm{d}u'}$ inside the absolute value and we used (6-49) (applied to $\phi_{\mathcal{L}}$) in the last line. This implies (6-102) (the first inequality is obtained by the reverse triangular inequality). Integrating in v from γ then gives (6-103) and (6-105), using also (6-34) to control the boundary term $|\psi(u,v_\gamma(u))|\lesssim|u|^{-s}\lesssim|u|^{2-3s}$ (recall that $s\le 1$).

For (6-104) we estimate (2-45) using the estimate of Proposition 6.5 (naively, without taking advantage of a difference structure) and $A_v = A_v^{\mathrm{RN}} = 0$, and we get

$$|\partial_v(D_u\psi - D_u^{\mathrm{RN}}\psi_{\mathcal{L}})| \lesssim |u|^{-2s}\cdot v^{1-3s} + (\Omega^{1.9}+\Omega_{\mathrm{RN}}^{1.9})\cdot v^{-s}.$$

Integrating in v, using the bounds of Proposition 6.15 and (6-106) (to control the difference on γ, similarly to what was done earlier in the proof) allows us to prove (6-104) thus concluding the proof. □

6D. *Combining the linear and the nonlinear estimates.* In this section, we combine the nonlinear difference estimates of Section 6C with the linear estimates on a fixed Reissner–Nordström background obtained in Section 5. This allows us to conclude the proof of the boundedness of ϕ if $\phi_{\mathcal{H}^+} \in \mathcal{O}$ and if $q_0 = 0$, blow up if $\phi_{\mathcal{H}^+} \notin \mathcal{O}$.

6D1. *Boundedness and extendibility of the matter fields for oscillating data and proof of Theorem I*(i).

Proposition 6.17. *Assume the following gauge-invariant condition*: *there exists $u_0 \le u_s$ such that*

$$\lim_{v\to+\infty}\int_{v_0}^{v} e^{iq_0\sigma_{\mathrm{br}}(v')}e^{iq_0\int_{v_0}^{v'}A_v(u_0,v'')\,\mathrm{d}v''}D_v\psi_{\mathcal{L}}(u_0,v')\,\mathrm{d}v' \tag{6-113}$$

exists and is finite for all σ_{br} satisfying (3-15), (3-16). *Then ϕ in the gauge* (2-26), (3-5) *admits a continuous extension to $\mathcal{CH}_{i^+}$. Moreover the gauge-independent quantities $|\phi|$ and the metric g also admit a continuous extension to $\mathcal{CH}_{i^+}$ and the extension of g can be chosen to be C^0-admissible as in Definition 2.1.*

If we additionally assume the following gauge-invariant condition: *for all $D_{\mathrm{br}} > 0$, there exists $\eta_0(D_{\mathrm{br}}) > 0$ such that for all σ_{br} satisfying* (3-15), (3-16) *and for all $(u,v)\in\mathcal{LB}$,*

$$\left|\int_{v_\gamma(u)}^{v} e^{iq_0\sigma_{\mathrm{br}}(v')}e^{iq_0\int_{v_0}^{v'}A_v(u_0,v'')\,\mathrm{d}v''}D_v\psi_{\mathcal{L}}(u_0,v')\,\mathrm{d}v'\right| \lesssim D'\cdot|u|^{s-1-\eta_0}, \tag{6-114}$$

then Q and ϕ are bounded and the following estimates are true for all $(u,v)\in\mathcal{LB}$:

$$|\phi|(u,v) \lesssim |u|^{-1+s-\eta_0}, \tag{6-115}$$

$$|Q-e|(u,v) \lesssim |u|^{-\eta_0}, \tag{6-116}$$

where the implicit constants are allowed to depend on $\eta_0 > 0$. Moreover, Q extends to a continuous function $Q_{\mathrm{CH}}(u)$ on $\mathcal{CH}_{i^+}$.

Proof. Applying the assumption to $\sigma_{\mathrm{br}}(v) = \int_{u_\gamma(v)}^{u_0}[(A_u^{\mathrm{RN}})^{\mathrm{CH}} - A_u^{\mathrm{CH}}](u')\,\mathrm{d}u'$ (which satisfies (3-15) and (3-16) by Proposition 6.16) we get by Proposition 6.16 that for ψ in the gauge (2-26) (note that $A_v \equiv 0$),

$$\lim_{v\to+\infty}\psi(u_0,v) := \psi_{\mathrm{CH}}(u_0)$$

exists and is finite. Recall also from Proposition 6.6 that $D_u\psi$ and A_u admit (in the gauge (2-26), (3-5)) a bounded extension to $\mathcal{CH}_{i^+}$ which we denoted respectively by $(D_u\psi)_{\mathrm{CH}}$ and $(A_u)^{\mathrm{CH}}$. Recall also that one can write for any $u_0\in\mathbb{R}$ the identity

$$\partial_u(e^{iq_0\int_{u_0}^{u}A_u(u',v)\,\mathrm{d}u'}\psi(u,v)) = e^{iq_0\int_{u_0}^{u}A_u(u',v)\,\mathrm{d}u'}D_u\psi(u,v),$$

which upon integration gives

$$\psi(u,v)=e^{-iq_0\int_{u_0}^{u}A_u(u',v)\,\mathrm{d}u'}\psi(u_0,v)+e^{-iq_0\int_{u_0}^{u}A_u(u',v)\,\mathrm{d}u'}\int_{u_0}^{u}e^{iq_0\int_{u_0}^{u'}A_u(u'',v)\,\mathrm{d}u''}D_u\psi(u',v)\,\mathrm{d}u'.$$

Now note by Proposition 6.16, $A_u\in L^\infty_{\mathrm{loc}}$; therefore by dominated convergence, the function $(u,v)\mapsto\int_{u_0}^{u}A_u(u',v)\,\mathrm{d}u'$ extends continuously to $\int_{u_0}^{u}(A_u)^{\mathrm{CH}}(u')\,\mathrm{d}u'$ at $\mathcal{CH}_{i^+}$. Since $D_u\psi\in L^\infty_{\mathrm{loc}}$ as well (by (6-53)), an other use of dominated convergence, together with the existence of the limit $\lim_{v\to+\infty}\psi(u_0,v)$ shows that $\psi(u,v)$ admits a continuous extension to $\mathcal{CH}_{i^+}$ denoted by $\psi_{\mathcal{CH}_{i^+}}(u)$. By Theorem B, r admits a continuous extension $r_{\mathcal{CH}_{i^+}}(u)$ to $\mathcal{CH}_{i^+}$ which is bounded away from zero. Therefore, $\phi(u,v)$ also admits a continuous extension to $\mathcal{CH}_{i^+}$ denoted by $\phi_{\mathcal{CH}_{i^+}}(u)$. The continuous extendibility of the metric g (and the C^0-admissible character of the extension) follows immediately as a consequence of Corollary 6.10.

Now we make the additional assumption (6-114). We define $(\sigma_{\mathrm{br}})_u(v):=\int_{u_\gamma(v)}^{u}[(A_u^{\mathrm{RN}})^{\mathrm{CH}}-A_u^{\mathrm{CH}}](u')\,\mathrm{d}u'$ for each $u\le u_s$. It follows from (6-100) and (6-101) that $(\sigma_{\mathrm{br}})_u$ satisfies (3-15), (3-16) with a constant $D_{\mathrm{br}}(M,e,q_0,m^2,s,D_1,D_2)>0$ *that is independent of* u. In view of this, (6-115) follows from (6-114) combined with (6-103) and the fact that $s>\frac{3}{4}$. Now we plug (6-115), the boundedness of r, and (6-53) into (2-40) to obtain the estimate in $\mathcal{LB}$:

$$|\partial_u Q|\lesssim|u|^{-1-\eta_0}.$$

Integrating this estimate from γ we obtain (6-116), in view of the estimate on γ from Proposition 6.4.

For the continuous extendibility of Q, we start integrating (2-40) to get for all $(u,v)\in\mathcal{LB}$

$$Q(u,v)=Q(u_\gamma(v),v)+q_0\int_{u_\gamma(v)}^{u}\Im(\bar{\psi}D_u\psi)(u',v)\,\mathrm{d}u'.$$

Note that the function $u\to\Im(\bar{\psi}D_u\psi)(u,v)$ is dominated by the integrable function $|u|^{-1-\eta_0}$ therefore by the dominated convergence theorem, $\int_{u_\gamma(v)}^{u}\Im(\bar{\psi}D_u\psi)(u',v)\,\mathrm{d}u'$ extends continuously to the function $\int_{-\infty}^{u}\Im(\bar{\psi}_{\mathrm{CH}}(D_u\psi)_{\mathrm{CH}})(u')\,\mathrm{d}u'$. Therefore, Q admits a continuous extension to $\mathcal{CH}_{i^+}$, which concludes the proof. □

Corollary 6.18. (1) *Assume that $\phi_{\mathcal{H}^+}\in\mathcal{O}$. Then ϕ is uniformly bounded on $\mathcal{LB}$ and thus* (4-1) *holds true.*

(2) *Assume additionally that $\phi_{\mathcal{H}^+}\in\mathcal{O}'$. Then $|\phi|$ and g are continuously extendible, and the extension of g can be chosen to be C^0-admissible.*

(3) *Assume additionally that $\phi_{\mathcal{H}^+}\in\mathcal{O}''$. Then* (6-115) *and* (6-116) *are true for all $(u,v)\in\mathcal{LB}$ and moreover Q admits a continuous extension to $\mathcal{CH}_{i^+}$.*

Proof. The first statement follows from (6-103) of Proposition 6.16 and Corollary 5.25. The others are direct applications of Proposition 6.17 and Corollary 5.25 (using that (6-113) and (6-114) are gauge-invariant conditions). □

In particular, Corollary 6.18 shows Theorem I (i).

6D2. *Blow-up of the scalar field for $\phi_{\mathcal{H}^+}\notin\mathcal{O}$ (nonoscillating data) if $q_0=0$ and proof of Theorem I*(ii).

Lemma 6.19. *Assume that there exists $u_0\le u_s$ such that*

$$\limsup_{v\to+\infty}|\phi|(u_0,v)=+\infty.$$

Then for all $u \leq u_s$ we have

$$\limsup_{v\to+\infty} |\phi|(u, v) = \limsup_{v\to+\infty} |\psi|(u, v) = +\infty.$$

Moreover we have the following bounds: *for all $u \leq u_s$, there exists $f(u) > 0$ for all $v > v_\gamma(u)$ such that*

$$\begin{aligned} \left| |\phi|(u, v) - \frac{r(u_0, v)}{r(u, v)} |\phi|(u_0, v) \right| &\leq f(u), \\ \liminf_{v\to+\infty} \frac{|\phi|(u, v)}{|\phi|(u_0, v)} = \frac{r_{CH}(u_0)}{r_{CH}(u)} &> 0. \end{aligned} \tag{6-117}$$

Proof. This is an immediate consequence of the integrating of (6-53) and the continuous extendibility of r to a function which is bounded away from zero (by the definition of $\mathcal{CH}_{i^+}$). □

We will not use (6-117) in the present work, but it is an important estimate for our companion paper [Kehle and Van de Moortel ≥ 2024].

Corollary 6.20. *Assume that $q_0 = 0$ and that $\phi_{\mathcal{H}^+} \in \mathcal{SL} - \mathcal{O}$. Then for all $u \leq u_s$ we have the blow-up*

$$\limsup_{v\to+\infty} |\phi|(u, v) = \limsup_{v\to+\infty} |\psi|(u, v) = +\infty,$$

and moreover the asymptotics (6-117) *are satisfied.*

Proof. The result follows from a combined application of Corollary 5.25 (using that $\phi'_{\mathcal{L}}$ and $\phi_{\mathcal{L}}$ relate by a gauge transformation; hence $|\phi'_{\mathcal{L}}| = |\phi_{\mathcal{L}}|$), (6-105) in Proposition 6.16 and Lemma 6.19. □

In particular, Corollary 6.20 shows Theorem I (ii).

6D3. *Proof of Theorem II.* Before turning to the proof of Theorem II, we prove the following.

Lemma 6.21. *Let $s > \frac{3}{4}$ and $\omega_1 \in \mathbb{R} - \{\omega_{\text{res}}\}$ and $\phi_{\mathcal{H}^+}$ be given by*

$$\phi_{\mathcal{H}^+}(v) = e^{-i\omega_1 v + \omega_{\text{err}}(v)} v^{-s} + \phi_{\text{err}} \tag{6-118}$$

for any $\phi_{\text{err}} \in C^1([v_0, +\infty), \mathbb{R})$ satisfying (1-8) *with $s > 1$ and any $\omega_{\text{err}} \in C^2([v_0, +\infty), \mathbb{R})$ such that $\omega'_{\text{err}}(v) \to 0$ as $v \to +\infty$ and such that $|\omega''_{\text{err}}|(v) \leq D \cdot v^{-2+2s-\eta_0}$ for $v \geq v_0$ and some constants $D > 0$ and $\eta_0 > 0$. Then $\phi_{\mathcal{H}^+} \in \mathcal{O}''$, where we assume without loss of generality that $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ (by choosing $D_1 > 0$ possibly larger).*

Proof. Since $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ (by possibly choosing $D_1 > 0$ larger) it suffices to check (3-18) independently for $e^{-i\omega_1 v + \omega_{\text{err}}(v)} v^{-s}$ and ϕ_{err}. First note that ϕ_{err} satisfies (3-18) since it satisfies (1-8) with $s > 1$.

For $e^{-i\omega_1 v + \omega_{\text{err}}(v)} v^{-s}$ we can assume with no loss of generality that $\frac{3}{4} < s \leq 1$ (since the case $s > 1$ follows immediately from integrability). It suffices to prove that there exists $\eta > 0$, $E > 0$ such that for all large enough $\tilde{v}$, v with $\tilde{v} < v$

$$\left| \int_{\tilde{v}}^{v} e^{i\omega_{\text{res}} v' - i\omega_1 v' + i\sigma_{\text{br}}(v') + i\omega_{\text{err}}(v')} (v')^{-s} \, dv' \right| \leq E \cdot \tilde{v}^{-1+s-\eta} \tag{6-119}$$

for all σ_{br} satisfying (3-15) and (3-16). For conciseness, we will introduce the notation $\omega = \omega_{\text{res}} - \omega_1 \neq 0$. We make use of integration by parts:

$$\begin{aligned}\int_{\tilde{v}}^{v} e^{i\omega v'+i\sigma_{\mathrm{br}}(v')+i\omega_{\mathrm{err}}(v')}(v')^{-s}\,\mathrm{d}v' &= -i\int_{\tilde{v}}^{v}\frac{\mathrm{d}}{\mathrm{d}v'}(e^{i\omega v'+i\sigma_{\mathrm{br}}(v')+i\omega_{\mathrm{err}}(v')})\frac{(v')^{-s}}{\omega+\sigma'_{\mathrm{br}}(v')+\omega'_{\mathrm{err}}(v')}\,\mathrm{d}v'\\ &= -i\,\frac{v^{-s}e^{i\omega v+i\sigma_{\mathrm{br}}(v)+i\omega_{\mathrm{err}}(v)}}{\omega+\sigma'_{\mathrm{br}}(v)+\omega'_{\mathrm{err}}(v)}+i\,\frac{\tilde{v}^{-s}e^{i\omega\tilde{v}+i\sigma_{\mathrm{br}}(\tilde{v})+i\omega_{\mathrm{err}}(\tilde{v})}}{\omega+\sigma'_{\mathrm{br}}(\tilde{v})+\omega'_{\mathrm{err}}(\tilde{v})}\\ &\quad -is\int_{\tilde{v}}^{v} e^{i\omega v'+i\sigma_{\mathrm{br}}(v')+i\omega_{\mathrm{err}}(v')}\frac{(v')^{-s-1}}{\omega+\sigma'_{\mathrm{br}}(v')+\omega'_{\mathrm{err}}(v')}\,\mathrm{d}v'\\ &\quad -i\int_{\tilde{v}}^{v} e^{i\omega v'+i\sigma_{\mathrm{br}}(v')+i\omega_{\mathrm{err}}(v')}\frac{(v')^{-s}\cdot(\sigma''_{\mathrm{br}}(v')+\omega''_{\mathrm{err}}(v'))}{(\omega+\sigma'_{\mathrm{br}}(v')+\omega'_{\mathrm{err}}(v'))^2}\,\mathrm{d}v'.\end{aligned}$$

Note that, using (3-16) and the decay assumption on ω'_{err}, we have $\omega+\sigma'_{\mathrm{br}}(v')+\omega'_{\mathrm{err}}(v')$ is bounded away from zero for $\tilde{v}$ large enough (since $\omega\neq 0$). The first two terms obviously obey (6-119) since $s>\frac{1}{2}$. Similarly, the third term can be integrated to show

$$\left|\int_{\tilde{v}}^{v} e^{i\omega v'+i\sigma_{\mathrm{br}}(v')+i\omega_{\mathrm{err}}(v')}\frac{(v')^{-s-1}}{\omega+\sigma'_{\mathrm{br}}(v')+\omega'_{\mathrm{err}}(v')}\,\mathrm{d}v'\right|\lesssim\tilde{v}^{-s}.$$

For the last term, we write using $|\omega''_{\mathrm{err}}(v)|\lesssim v^{-2+2s-\eta_0}$ and (3-16)

$$\begin{aligned}\left|\int_{\tilde{v}}^{v} e^{i\omega v'+i\sigma_{\mathrm{br}}(v')+i\omega_{\mathrm{err}}(v')}\frac{(v')^{-s}\cdot(\sigma''_{\mathrm{br}}(v')+\omega''_{\mathrm{err}}(v'))}{(\omega+\sigma'_{\mathrm{br}}(v')+\omega'_{\mathrm{err}}(v'))^2}\,\mathrm{d}v'\right| &\lesssim\int_{\tilde{v}}^{v}(v')^{-s}\cdot(v^{-2+2s-\eta_0}+v^{1-2s})\,\mathrm{d}v'\\ &\lesssim\tilde{v}^{-1+s-\eta_0}+\tilde{v}^{2-3s}\lesssim\tilde{v}^{-1+s-\eta_0}\end{aligned}$$

for some $\eta_0>0$, where to obtain this estimate, we used the fact that $s<1+\eta_0$ for some $\eta_0>0$ and also $2-3s<-1+s-\eta_0$ $\big($since we assumed $s>\frac{3}{4}\big)$. ☐

Proposition 6.22. *Assume that the parameters* (M,e,q_0,m^2) *are such that*

$$|q_0e|\neq r_-(M,e)|m|.$$

Let $\phi_{\mathcal{H}^+}$ be given by either the profile of (1-15) (*if* $m^2>0,\ q_0=0$) *or* (1-16) (*if* $m^2=0,\ q_0\neq 0$) *or* (1-17) (*if* $m^2>0,\ q_0\neq 0$). *Then* $\phi_{\mathcal{H}^+}\in\mathcal{O}''$, *where we again assume without loss of generality that* $\phi_{\mathcal{H}^+}\in\mathcal{SL}$ (*by choosing* $D_1>0$ *possibly larger*).

Proof. If $m^2=0$, $|q_0e|<\frac{1}{2}$, then $\phi_{\mathcal{H}^+}$ satisfies (3-8) for $s>1$ and thus $\phi_{\mathcal{H}^+}\in\mathcal{O}''$. Otherwise, we have three different cases:

(1) $q_0=0,\ m^2\neq 0$: It suffices to prove that $e^{\pm i(mv+\omega_{\mathrm{err}}(v))}\cdot v^{-5/6}\in\mathcal{O}''$, where $\omega_{\mathrm{err}}(v)=-\frac{3}{2}m(2\pi M)^{2/3}\cdot v^{1/3}+\omega(m\cdot M)$. Note that $\omega'_{\mathrm{err}}(v)\to 0$ as $v\to+\infty$ and such that $|\omega''_{\mathrm{err}}|(v)\lesssim v^{-5/3}\lesssim v^{-2+2\cdot(5/6)-\eta_0}$ for any $0<\eta_0<\frac{4}{3}$. Therefore by Lemma 6.21, $e^{\pm i(mv+\omega_{\mathrm{err}}(v))}\cdot v^{-5/6}\in\mathcal{O}''$.

(2) $|q_0e|\geq\frac{1}{2}$, $m^2=0$. Then $\delta=\pm i\sqrt{4(q_0e)^2-1}$ and $\phi_{\mathcal{H}^+}$ is of the form (6-118) with $\omega_1=-q_0e/r_+\neq\omega_{\mathrm{res}}$, $\omega_{\mathrm{err}}=-(\sqrt{4(q_0e)^2-1})\log(v)$ and $s=1$. Indeed we have $\omega'_{\mathrm{err}}(v)=o(1)$ and $|\omega''_{\mathrm{err}}|(v)\lesssim v^{-2}\lesssim v^{-2+2s-\eta_0}$ for $\eta_0>0$ since $2s-2=0$. Therefore, $\phi_{\mathcal{H}^+}\in\mathcal{O}''$ by Lemma 6.21.

(3) $q_0\neq 0,\ m^2\neq 0$: As in the case $q_0=0,\ m^2\neq 0$, we know $\phi_{\mathcal{H}^+}$ is a linear combination of two profiles of the form (6-118) with $\omega_1=\pm m-q_0e/r_+$. Since the parameters (M,e,q_0,m^2) do not satisfy $|q_0e|\neq r_-(M,e)|m|$, we know that $\omega_1\neq\omega_{\mathrm{res}}$. The rest of the argument follows as above. ☐

Corollary 6.23. *Assume that the parameters* (M, e, q_0, m^2) *are such that*

$$|q_0 e| \neq r_-(M, e)|m|.$$

Let $\phi_{\mathcal{H}^+}$ *by either the profile of* (1-15) (*if* $m^2 > 0,\ q_0 = 0$) *or* (1-16) (*if* $m^2 = 0,\ q_0 \neq 0$) *or* (1-17) (*if* $m^2 > 0,\ q_0 \neq 0$). *Then,* (6-115) *and* (6-116) *are true for all* $(u, v) \in \mathcal{LB}$. *Moreover,* $|\phi|$, Q *and the metric* g *admit a continuous extension to* $\mathcal{CH}_{i^+}$ *and the extension of* g *can be chosen to be* C^0*-admissible.*

Proof. This is an immediate application of Proposition 6.22 and Corollary 6.18 (using that $|\phi'_{\mathcal{L}}| = |\phi_{\mathcal{L}}|$ since $\phi'_{\mathcal{L}}$ and $\phi_{\mathcal{L}}$ only differ by gauge transformation). □

In particular, Corollary 6.23 shows Theorem II.

6D4. $\dot{W}^{1,1}_{\mathrm{loc}}$ *blow-up of the scalar field on outgoing cones: proof of Theorem III.*

Proposition 6.24. *Assume that for all* $u \le u_s$ *we have the blow up*

$$\int_{v_0}^{+\infty} |D_v^{\mathrm{RN}} \phi_{\mathcal{L}}|(u, v')\, \mathrm{d}v' = +\infty. \tag{6-120}$$

Then, for all $u \le u_s$,

$$\int_{v_0}^{+\infty} |D_v \phi|(u, v')\, \mathrm{d}v' = \int_{v_0}^{+\infty} |D_v \psi|(u, v')\, \mathrm{d}v' = +\infty. \tag{6-121}$$

Conversely, (6-121) *implies* (6-120).

Proof. Note that $D_v^{\mathrm{RN}} \psi_{\mathcal{L}} = r D_v^{\mathrm{RN}} \phi_{\mathcal{L}} - (\Omega^2_{\mathrm{RN}}/2)\phi_{\mathcal{L}}$. Since r is lower-bounded on $\mathcal{LB}$ and in view of (6-46) (which also applies to $\phi_{\mathcal{L}}$), for all $u \le u_s$

$$\int_{v_0}^{+\infty} |D_v^{\mathrm{RN}} \psi_{\mathcal{L}}|(u, v)\, \mathrm{d}v = +\infty.$$

Therefore, integrating (6-102) $\bigl($since $s > \frac{3}{4} > \frac{2}{3}\bigr)$ we also obtain, for all $u \le u_s$,

$$\int_{v_0}^{+\infty} |D_v \psi|(u, v)\, \mathrm{d}v = +\infty.$$

Since $D_v \psi = r D_v \phi + \lambda \phi$ and by (6-48), (6-51),

$$|\lambda \phi| \lesssim v^{1-3s}$$

is integrable; therefore, for all $u \le u_s$,

$$\int_{v_0}^{+\infty} |D_v \phi|(u, v)\, \mathrm{d}v = +\infty.$$

The above also shows that (6-121) implies (6-120). □

Corollary 6.25. *Assume* $\phi_{\mathcal{H}^+} \in \mathcal{SL} - H$ (*defined in the proof of Corollary 5.27*). *Then* (6-121) *holds true.*

In the particular case $|q_0 e| \le \epsilon(M, e, m^2)$ (*in particular if* $q_0 = 0$), *where* $\epsilon > 0$ *is defined in the proof of Corollary 5.27, for all* $\phi_{\mathcal{H}^+} \in \mathcal{SL} - L^1$ (6-121) *is satisfied.*

Proof. This follows from Corollary 5.27 (using that $\phi'_{\mathcal{L}}$ are $\phi_{\mathcal{L}}$ relate by a gauge transformation; hence $|\phi'_{\mathcal{L}}| = |\phi_{\mathcal{L}}|$ and $|D_v \phi'_{\mathcal{L}}| = |D_v \phi_{\mathcal{L}}|$) and Proposition 6.24. □

Corollary 6.25 thus concludes the proof of Part 1 of Theorem III. Now we turn to the proof of Part 2 of Theorem III.

Corollary 6.26. *Let $\phi_{\mathcal{H}^+}$ be given by either the profile of* (1-15) (*if* $m^2 > 0,\ q_0 = 0$) *or* (1-16) (*if* $m^2 = 0,\ q_0 \neq 0$) *or* (1-17) (*if* $m^2 > 0,\ q_0 \neq 0$). *Assume the condition $\mathcal{Z}_{\mathfrak{t}} \cap \Theta = \varnothing$. Then, there exists a $\delta(M, e, q_0, m^2) > 0$ sufficiently small such that $P_\delta \phi_{\mathcal{H}^+} \in L^1(\mathbb{R})$.*

Moreover, the condition $\mathcal{Z}_{\mathfrak{t}}(M, e, q_0, m^2) \cap \Theta(M, e, q_0, m^2) = \varnothing$ is generic in the sense that for given $m^2 \geq 0$, $q_0 \in \mathbb{R}$ with $m^2 \neq q_0^2$, the set of parameters (M, e) satisfying the conditions is the zero set of a nontrivial real-analytic function on $\{0 < |e| < M\}$. In particular, in view of Part 1 *of Theorem III, we obtain Part* 2 *of Theorem III.*

Proof. We start with the second claim. Fix $m^2 \geq 0,\ q_0 \in \mathbb{R}$ with $q_0^2 \neq m^2$. We define $f_{\pm,m^2,q_0}(M, e) := \mathfrak{t}(\pm m - q_0 e/r_+, M, e, q_0, m^2)$. By analyticity of $\mathfrak{t}$ (note that $\mathfrak{t}$ is the Wronskian of solutions to an ODE with analytic coefficients depending analytically on (ω, M, e)), we have that both $f_{\pm,m^2,q_0} \colon \{(M, e) \in \mathbb{R}^2 : 0 < |e| < M\} \to \mathbb{R}$ are analytic. It suffices to show that both $f_\pm$ are nontrivial. From the ODE energy identity, $|\mathfrak{t}|^2 = |\mathfrak{r}|^2 + \omega(\omega - \omega_{\text{res}}) \geq \omega(\omega - \omega_{\text{res}})$ we conclude

$$|f_\pm|^2 \geq \left(\pm m - \frac{q_0 e}{r_+}\right)\left(\pm m - \frac{q_0 e}{r_-}\right) \to \left(\pm m - \frac{q_0 e}{|e|}\right)^2 > 0$$

as $|e| \to M$. We used here that $m^2 \neq q_0^2$.

Now, fix $0 < \delta < \operatorname{dist}(\mathcal{Z}_{\mathfrak{t}}, \Theta)$. By Plancherel's theorem and the Cauchy–Schwarz inequality, it suffices to show that $\chi_\delta(\omega)\mathcal{F}(\phi_{\mathcal{H}^+})$ is in $H^{1/2+\tau}$ for some $\tau > 0$ (recalling the definition of $\chi_\delta(\omega)$ from Section 4E). Further, since χ_δ is smooth and has compact support ($\mathcal{Z}_{\mathfrak{t}}^\delta \subset [-|\omega_{\text{res}}| - \delta, |\omega_{\text{res}}| + \delta]$), and $\mathcal{F}(\phi_{\mathcal{H}^+}) \in L^2$, it suffices (e.g., by the Kato–Ponce inequality) to show that $\chi_\delta(\omega)\langle\partial_\omega\rangle^{1/2+\tau}\mathcal{F}(\phi_{\mathcal{H}^+})$ is in L^2. Thus, we need to show that $\mathcal{F}(\langle v\rangle^{1/2+\tau}\phi_{\mathcal{H}^+}) \in L^2(\mathcal{Z}_{\mathfrak{t}}^\delta)$ for some $\tau > 0$. We now fix $0 < \tau < s - \frac{1}{2}$. A direct adaption of the proofs of Lemma 6.21 and Proposition 6.22 then shows $\mathcal{F}(\langle v\rangle^{1/2+\tau}\phi_{\mathcal{H}^+}) \in L^\infty(\mathcal{Z}_{\mathfrak{t}}^\delta)$ from which the claim follows. □

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A SEMICLASSICAL BIRKHOFF NORMAL FORM FOR CONSTANT-RANK MAGNETIC FIELDS

LÉO MORIN

This paper deals with classical and semiclassical nonvanishing magnetic fields on a Riemannian manifold of arbitrary dimension. We assume that the magnetic field $B = dA$ has constant rank and admits a discrete well. On the classical part, we exhibit a harmonic oscillator for the Hamiltonian $H = |p - A(q)|^2$ near the zero-energy surface: the cyclotron motion. On the semiclassical part, we describe the semiexcited spectrum of the magnetic Laplacian $\mathcal{L}_\hbar = (i\hbar d + A)^*(i\hbar d + A)$. We construct a semiclassical Birkhoff normal form for $\mathcal{L}_\hbar$ and deduce new asymptotic expansions of the smallest eigenvalues in powers of $\hbar^{1/2}$ in the limit $\hbar \to 0$. In particular we see the influence of the kernel of B on the spectrum: it raises the energies at order $\hbar^{3/2}$.

1. Introduction

1A. ***Context.*** We consider the semiclassical magnetic Laplacian with Dirichlet boundary conditions

$$\mathcal{L}_\hbar = (i\hbar \mathrm{d} + A)^*(i\hbar \mathrm{d} + A)$$

on a d-dimensional oriented Riemannian manifold (M, g), which is either compact with boundary, or the Euclidean $\mathbb{R}^d$. A denotes a smooth 1-form on M, the magnetic potential. The magnetic field is the 2-form $B = \mathrm{d}A$.

The spectral theory of the magnetic Laplacian has given rise to many investigations, and appeared to have very various behaviors according to the variations of B and the geometry of M. We refer to the books and review [Helffer and Kordyukov 2014; Fournais and Helffer 2010; Raymond 2017] for a description of these works. Here we focus on the Dirichlet realization of $\mathcal{L}_\hbar$, and we give a description of semiexcited states, eigenvalues of order $\mathcal{O}(\hbar)$ in the semiclassical limit $\hbar \to 0$. As explained in the above references, the *magnetic intensity* has a great influence on these eigenvalues, and one can define it in the following way.

Using the isomorphism $T_qM \simeq T_qM^*$ given by the metric, one can define the following skew-symmetric operator $\boldsymbol{B}(q) : T_qM \to T_qM$ by

$$B_q(X, Y) = g_q(X, \boldsymbol{B}(q)Y) \quad \text{for all } X, Y \in T_qM, \text{ for all } q \in M. \tag{1-1}$$

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Since the operator $\boldsymbol{B}(q)$ is skew-symmetric with respect to the scalar product g_q, its eigenvalues are purely imaginary and symmetric with respect to the real axis. We denote these repeated eigenvalues by

$$\pm i\beta_1(q), \quad \dots, \quad \pm i\beta_s(q), \quad 0,$$

with $\beta_j(q) > 0$. In particular, the rank of $\boldsymbol{B}(q)$ is $2s$ and may depend on q. However, we will focus on the constant-rank case. We denote by k the dimension of the kernel of $\boldsymbol{B}(q)$, so that $d = 2s + k$. The magnetic intensity (or "trace+") is the scalar-valued function

$$b(q) = \sum_{j=1}^{s} \beta_j(q).$$

The function b is continuous on M, but nonsmooth in general. We are interested in discrete magnetic wells and nonvanishing magnetic fields.

Assumption 1. We assume that:

- The magnetic intensity is nonvanishing and admits a unique global minimum $b_0 > 0$ at $q_0 \in M \setminus \partial M$.
- The rank of $\boldsymbol{B}(q)$ is constant equal to $2s > 0$ on a neighborhood Ω of q_0.
- $\beta_i(q_0) \neq \beta_j(q_0)$ for every $1 \le i < j \le s$, and the minimum of b is nondegenerate.
- In the noncompact case $M = \mathbb{R}^d$,

$$b_\infty := \liminf_{|q| \to +\infty} b(q) > b_0$$

and there exists a $C > 0$ such that

$$|\partial_\ell \boldsymbol{B}_{ij}(q)| \le C(1 + |\boldsymbol{B}(q)|) \quad \text{for all } \ell, i, j, \text{ for all } q \in \mathbb{R}^d.$$

Remark 1.1. Since the nonzero eigenvalues of $\boldsymbol{B}$ are simple at q_0, the function b is smooth on a neighborhood of q_0. In particular, it is meaningful to say that the minimum of b is nondegenerate.

Under Assumption 1, the following useful inequality was proven in [Helffer and Mohamed 1996]. There is a $C_0 > 0$ such that, for $\hbar$ small enough,

$$(1 + \hbar^{1/4} C_0)\langle \mathcal{L}_\hbar u, u\rangle \ge \int_M \hbar(b(q) - \hbar^{1/4} C_0)|u(q)|^2 \, \mathrm{d}q \quad \text{for all } u \in \mathrm{Dom}(\mathcal{L}_\hbar). \tag{1-2}$$

Remark 1.2. Actually, one has the better inequality obtained replacing $\hbar^{1/4}$ by $\hbar$. This was proved in [Guillemin and Uribe 1988] in the case of a nondegenerate B, in [Borthwick and Uribe 1996] in the constant rank case, and in [Ma and Marinescu 2002] in a more general setting.

Remark 1.3. Using this inequality, one can prove Agmon-like estimates for the eigenfunctions of $\mathcal{L}_\hbar$. Namely, the eigenfunctions associated to an eigenvalue $< b_1\hbar$ are exponentially small outside $K_{b_1} = \{q : b(q) \le b_1\}$. We will use this result to localize our analysis to the neighborhood Ω of q_0. In particular, the greater b_1 is, the larger Ω must be.

Under Assumption 1, estimates on the ground states of $\mathcal{L}_\hbar$ in the semiclassical limit $\hbar \to 0$ were proven in several works, especially in dimensions $d = 2, 3$.

On $M = \mathbb{R}^2$, asymptotics for the j-th eigenvalue of $\mathcal{L}_\hbar$

$$\lambda_j(\mathcal{L}_\hbar) = b_0\hbar + (\alpha(2j-1) + c_1)\hbar^2 + o(\hbar^2) \tag{1-3}$$

with explicit α, $c_1 \in \mathbb{R}$ were proven in [Helffer and Morame 2001] (for $j = 1$) and [Helffer and Kordyukov 2011] ($j \geq 1$). Actually, this second paper contains a description of some higher eigenvalues. They proved that, for any integers n, $j \in \mathbb{N}$, there exist $\hbar_{jn} > 0$ and for $\hbar \in (0, \hbar_{jn})$ an eigenvalue $\lambda_{n,j}(\hbar) \in \mathrm{sp}(\mathcal{L}_\hbar)$ such that

$$\lambda_{n,j}(\hbar) = (2n-1)(b_0\hbar + ((2j-1)\alpha + c_n)\hbar^2) + o(\hbar^2)$$

for another explicit constant c_n. In particular, it gives a description of *some* semiexcited states (of order $(2n-1)b_0\hbar$). Finally, [Raymond and Vũ Ngọc 2015] (and [Helffer and Kordyukov 2015]) gives a description of the whole spectrum below $b_1\hbar$, for any fixed $b_1 \in (b_0, b_\infty)$. More precisely, they proved that this part of the spectrum is given by a family of effective operators $\mathcal{N}_\hbar^{[n]}$ ($n \in \mathbb{N}$) modulo $\mathcal{O}(\hbar^\infty)$. These effective operators are $\hbar$-pseudodifferential operators with principal symbol given by the function $\hbar(2n-1)b$. More interestingly, they explained why the two quantum oscillators

$$(2n-1)b_0\hbar \quad \text{and} \quad (2j-1)\alpha\hbar^2$$

appearing in the eigenvalue asymptotics correspond to two oscillatory motions in classical dynamics: the cyclotron motion and a rotation around the minimum point of b. The results of Raymond and Vũ Ngọc were generalized to an arbitrary d-dimensional Riemannian manifold in [Morin 2022b], under the assumption $k = 0$ ($\boldsymbol{B}(q)$ has full rank), proving in particular similar estimates (1-3) in a general setting. Actually, these eigenvalue estimates were proven simultaneously in [Kordyukov 2019] in the context of the Bochner Laplacian.

We are interested on the influence of the kernel of $\boldsymbol{B}$ ($k > 0$). Since the rank of $\boldsymbol{B}$ is even, this kernel always exists in odd dimensions: if $d = 3$, the kernel directions correspond to the usual field lines. On $M = \mathbb{R}^3$, Helffer and Kordyukov [2013] proved the existence of $\lambda_{nmj}(\hbar) \in \mathrm{sp}(\mathcal{L}_\hbar)$ such that

$$\lambda_{nmj}(\hbar) = (2n-1)b_0\hbar + (2n-1)^{1/2}(2m-1)\nu_0\hbar^{3/2} + ((2n-1)(2j-1)\alpha + c_{nm})\hbar^2 + \mathcal{O}(\hbar^{9/4})$$

for some $\nu_0 > 0$ and α, $c_{nm} \in \mathbb{R}$. Motivated by this result and the 2-dimensional case, Helffer, Kordyukov, Raymond and Vũ Ngọc [Helffer et al. 2016] gave a description of the whole spectrum below $b_1\hbar$, proving in particular the eigenvalue estimates

$$\lambda_j(\mathcal{L}_\hbar) = b_0\hbar + \nu_0\hbar^{3/2} + \alpha(2j-1)\hbar^2 + \mathcal{O}(\hbar^{5/2}). \tag{1-4}$$

Their results exhibit a new classical oscillatory motion in the directions of the field lines, corresponding to the quantum oscillator $(2m-1)\nu_0\hbar^{3/2}$.

The aim of this paper is to generalize the results of [Helffer et al. 2016] to an arbitrary Riemannian manifold M, under Assumption 1. In particular we describe the influence of the kernel of $\boldsymbol{B}$ in a general geometric and dimensional setting. Their approach, which we adapt, is based on a *semiclassical Birkhoff normal form*. The *classical* Birkhoff normal form has a long story in physics and goes back to [Delaunay 1860; Lindstedt 1883]. This formal normal form was the starting point of a lot of studies on stability near equilibrium, and KAM theory (after [Kolmogorov 1954; Arnold 1963; Moser 1962]). The name of this normal form comes from [Birkhoff 1927; Gustavson 1966]. We refer to the books [Moser 1968; Hofer and

Zehnder 1994] for precise statements. Our approach here relies on a quantization. Physicists and quantum chemists already noticed in the 1980s that a quantum analogue of the Birkhoff normal form could be used to compute energies of molecules [Delos et al. 1983; Jaffé and Reinhardt 1982; Marcus 1985; Shirts and Reinhardt 1982]. Joyeux and Sugny [2002] also used such techniques to describe the dynamics of excited states. Sjöstrand [1992] constructed a semiclassical Birkhoff normal form for a Schrödinger operator $-\hbar^2\Delta+V$ using the Weyl quantization, to make a mathematical study of semiexcited states. Raymond and Vũ Ngọc [2015] had the idea to adapt this method for $\mathcal{L}_\hbar$ on $\mathbb{R}^2$, and with Helffer and Kordyukov on $\mathbb{R}^3$ [Helffer et al. 2016]. This method is reminiscent of Ivrii's approach [2019].

1B. ***Main results.*** The first idea is to link the classical dynamics of a particle in the magnetic field B with the spectrum of $\mathcal{L}_\hbar$ using pseudodifferential calculus. Indeed, $\mathcal{L}_\hbar$ is an $\hbar$-pseudodifferential operator with principal symbol

$$H(q,p)=|p-A_q|^2 \quad \text{for all } p\in T_qM^*, \text{ for all } q\in M,$$

and H is the classical Hamiltonian associated to the magnetic field B. One can use this property to prove that, in the phase space T^*M, the eigenfunctions (with eigenvalue $< b_1\hbar$) are microlocalized on an arbitrarily small neighborhood of

$$\Sigma=H^{-1}(0)\cap T^*\Omega=\{(q,p)\in T^*\Omega : p=A_q\}.$$

Hence, the second main idea is to find a normal form for H on a neighborhood of Σ. Namely, we find canonical coordinates near Σ in which H has a "simple" form. The symplectic structure of Σ as a submanifold of T^*M is thus of great interest. One can see that the restriction of the canonical symplectic form $\mathrm{d}p\wedge\mathrm{d}q$ on T^*M to Σ is given by B (Lemma 2.1), and when B has constant rank, one can find Darboux coordinates $\varphi:\Omega'\subset\mathbb{R}^{2s+k}_{(y,\eta,t)}\to\Omega$ such that

$$\varphi^*B=\mathrm{d}\eta\wedge\mathrm{d}y,$$

up to shrinking Ω. We will start from these coordinates to get the following normal form for H.

Theorem 1.4. *Under Assumption 1, there exists a diffeomorphism*

$$\Phi_1 : U_1'\subset\mathbb{R}^{4s+2k}\to U_1\subset T^*M$$

between neighborhoods U_1' of 0 *and U_1 of Σ such that*

$$\widehat{H}(x,\xi,y,\eta,t,\tau):=H\circ\Phi_1(x,\xi,y,\eta,t,\tau)$$

satisfies (with the notation $\hat{\beta}_j=\beta_j\circ\varphi$)

$$\widehat{H}=\langle M(y,\eta,t)\tau,\tau\rangle+\sum_{j=1}^{s}\hat{\beta}_j(y,\eta,t)(\xi_j^2+x_j^2)+\mathcal{O}((x,\xi,\tau)^3)$$

uniformly with respect to (y,η,t) for some (y,η,t)-dependent positive definite matrix $M(y,\eta,t)$. Moreover,

$$\Phi_1^*(\mathrm{d}p\wedge\mathrm{d}q)=\mathrm{d}\xi\wedge\mathrm{d}x+\mathrm{d}\eta\wedge\mathrm{d}y+\mathrm{d}\tau\wedge\mathrm{d}t.$$

Remark 1.5. We will use the following notation for our canonical coordinates:

$$z = (x, \xi) \in \mathbb{R}^{2s}, \quad w = (y, \eta) \in \mathbb{R}^{2s}, \quad \tau = (t, \tau) \in \mathbb{R}^{2k}.$$

This theorem gives the Taylor expansion of H on a neighborhood of Σ. In particular $(x, \xi, \tau) \in \mathbb{R}^d$ measures the distance to Σ, whereas $(y, \eta, t) \in \mathbb{R}^d$ are canonical coordinates on Σ.

Remark 1.6. This theorem exhibits the harmonic oscillator $\xi_j^2 + x_j^2$ in the expansion of H. This oscillator, which is due to the nonvanishing magnetic field, corresponds to the well-known cyclotron motion.

Actually, one can use the *Birkhoff normal form* algorithm to improve the remainder. Using this algorithm, we can change the $\mathcal{O}((x, \xi)^3)$ remainder into an explicit function of $\xi_j^2 + x_j^2$, plus some smaller remainders $\mathcal{O}((x, \xi)^r)$. This remainder power r is restricted by resonances between the coefficients β_j. Thus, we take an integer $r_1 \in \mathbb{N}$ such that,

$$\text{for all } \alpha \in \mathbb{Z}^s, \quad 0 < |\alpha| < r_1 \quad \Longrightarrow \quad \sum_{j=1}^{s} \alpha_j \beta_j(q_0) \neq 0. \tag{1-5}$$

Here, $|\alpha| = \sum_j |\alpha_j|$. Moreover, we can use the pseudodifferential calculus to apply the Birkhoff algorithm to $\mathcal{L}_\hbar$, changing the classical oscillator $\xi_j^2 + x_j^2$ into the quantum harmonic oscillator

$$\mathcal{I}_\hbar^{(j)} = -\hbar^2 \partial_{x_j}^2 + x_j^2,$$

whose spectrum consists of the simple eigenvalues $(2n-1)\hbar$, $n \in \mathbb{N}$. Following this idea we construct a normal form for $\mathcal{L}_\hbar$ in Theorem 3.4. We also deduce a description of its spectrum.

Theorem 1.7. *Let $\varepsilon > 0$. Under Assumption 1, there exist $b_1 \in (b_0, b_\infty)$, an integer $N_{\max} > 0$ and a compactly supported function $f_1^\star \in \mathcal{C}^\infty(\mathbb{R}^{2s+2k} \times \mathbb{R}^s \times [0, 1))$ such that*

$$|f_1^\star(y, \eta, t, \tau, I, \hbar)| \lesssim \big((|I| + \hbar)^2 + |\tau|(|I| + \hbar) + |\tau|^3\big)$$

satisfying the following properties. For $n \in \mathbb{N}^s$, denote by $\mathcal{N}_\hbar^{[n]}$ the $\hbar$-pseudodifferential operator in (y, t) with symbol

$$N_\hbar^{[n]} = \langle M(y, \eta, t)\tau, \tau\rangle + \sum_{j=1}^{s} \hat{\beta}_j(y, \eta, t)(2n_j - 1)\hbar + f_1^\star(y, \eta, t, \tau, (2n-1)\hbar, \hbar).$$

For $\hbar \ll 1$, there exists a bijection

$$\Lambda_\hbar : \mathrm{sp}(\mathcal{L}_\hbar) \cap (-\infty, b_1\hbar) \to \bigcup_{|n| \leq N_{\max}} \mathrm{sp}(\mathcal{N}_\hbar^{[n]}) \cap (-\infty, b_1\hbar)$$

such that $\Lambda_\hbar(\lambda) = \lambda + \mathcal{O}(\hbar^{r_1/2-\varepsilon})$ uniformly with respect to λ.

Remark 1.8. In this theorem $\mathrm{sp}(\mathcal{A})$ denotes the *repeated* eigenvalues of an operator $\mathcal{A}$, so that there might be some multiple eigenvalues, but $\Lambda_\hbar$ preserves this multiplicity. We only consider self-adjoint operators with discrete spectrum.

Remark 1.9. One should care of how large b_1 can be. As mentioned above, the eigenfunctions of energy $< b_1\hbar$ are exponentially small outside $K_{b_1} = \{q \in M : b(q) \leq b_1\}$. Thus, we will chose b_1 such that $K_{b_1} \subset \Omega$, where Ω is some neighborhood of q_0. Hence the larger Ω is, the greater b_1 can be. However, there are three restrictions on the size of Ω:

- The rank of $\boldsymbol{B}(q)$ is constant on Ω.
- There exist canonical coordinates φ on Ω (i.e., such that $\varphi^* B = \mathrm{d}\eta \wedge \mathrm{d}y$).
- There is no resonance in Ω:

$$\text{for all } q \in \Omega, \text{ for all } \alpha \in \mathbb{Z}^s, \quad 0 < |\alpha| < r_1 \quad \Longrightarrow \quad \sum_{j=1}^{s} \alpha_j \beta_j(q) \neq 0.$$

Of course the last condition is the most restrictive. However, if we forget the second condition, which is of global geometric nature, given a magnetic field and an r_1 one can estimate an associated b_1 satisfying the third condition. In particular we can construct simple examples on $\mathbb{R}^d$ such that the threshold $b_1\hbar$ includes several Landau levels.

Remark 1.10. If $k = 0$ we recover the result of [Morin 2022b]. Here we want to study the influence of a nonzero kernel $k > 0$. This result generalizes the result of [Helffer et al. 2016], which corresponds to $d = 3,\ s = k = 1$ on the Euclidean $\mathbb{R}^3$. However, this generalization is not straightforward since the magnetic geometry is much more complicated in higher dimensions, in particular if $k > 1$. Moreover, there is a new phenomena in higher dimensions: resonances between the functions β_j (as in [Morin 2022b]).

The spectrum of $\mathcal{L}_\hbar$ in $(-\infty, b_1\hbar)$ is given by the operators $\mathcal{N}_\hbar^{[n]}$. Actually if we choose b_1 small enough, it is only given by the first operator $\mathcal{N}_\hbar^{[1]}$ (here we denote the multi-integer $1 = (1, \dots, 1) \in \mathbb{N}^s$). Hence in the second part of this paper, we study the spectrum $\mathcal{N}_\hbar^{[1]}$ using a second Birkhoff normal form. Indeed, the symbol of $\mathcal{N}_\hbar^{[1]}$ is

$$N_\hbar^{[1]}(w, t, \tau) = \langle M(w, t)\tau, \tau\rangle + \hbar \hat{b}(w, t) + \mathcal{O}(\hbar^2) + \mathcal{O}(\tau\hbar) + \mathcal{O}(\tau^3),$$

so if we denote by $s(w)$ the minimum point of $t \mapsto \hat{b}(w, t)$ (which is unique on a neighborhood of 0), we get the expansion

$$N_\hbar^{[1]}(w, t, \tau) = \langle M(w, s(w))\tau, \tau\rangle + \frac{\hbar}{2}\left\langle \frac{\partial^2 \hat{b}}{\partial t^2}(w, s(w)) \cdot (t - s(w)), t - s(w) \right\rangle + \cdots, \tag{1-6}$$

where we will show that the remaining terms are only perturbations. As explained in Section 5, in (1-6) we can recognize a harmonic oscillator with frequencies $\sqrt{\hbar}\nu_j(w)$ $(1 \le j \le k)$, where $(\nu_j^2(w))_{1 \le j \le k}$ are the eigenvalues of the symmetric matrix

$$M(w, s(w))^{1/2} \cdot \tfrac{1}{2}\partial_t^2 \hat{b}(w, s(w)) \cdot M(w, s(w))^{1/2}.$$

These frequencies are smooth nonvanishing functions of w on a neighborhood of 0, as soon as we assume that they are simple.

Assumption 2. For indices $1 \le i < j \le k$, we have $\nu_i(0) \neq \nu_j(0)$.

We fix an integer $r_2 \in \mathbb{N}$ such that,

$$\text{for all } \alpha \in \mathbb{Z}^k, \quad 0 < |\alpha| < r_2 \quad \Longrightarrow \quad \sum_{j=1}^{k} \alpha_j \nu_j(0) \neq 0,$$

and we construct a normal form for $\mathcal{N}_\hbar^{[1]}$ in Theorem 5.4. Again, we deduce a description of its spectrum.

Theorem 1.11. *Let $c > 0$ and $\delta \in \left(0, \frac{1}{2}\right)$. Under Assumptions 1 and 2, with $k > 0$, there exists a compactly supported function $f_2^\star \in \mathcal{C}^\infty(\mathbb{R}^{2s} \times \mathbb{R}^k \times [0, 1))$ such that*

$$|f_2^\star(y, \eta, J, \sqrt{\hbar})| \lesssim (|J| + \sqrt{\hbar})^2$$

satisfying the following properties. For $n \in \mathbb{N}^k$, denote by $\mathcal{M}_\hbar^{[n]}$ the $\hbar$-pseudodifferential operator in y with symbol

$$M_\hbar^{[n]}(y, \eta) = \hat{b}(y, \eta, s(y, \eta)) + \sqrt{\hbar} \sum_{j=1}^{k} \nu_j(y, \eta)(2n_j - 1) + f_2^\star(y, \eta, (2n - 1)\sqrt{\hbar}, \sqrt{\hbar}).$$

For $\hbar \ll 1$, there exists a bijection

$$\Lambda_\hbar : \mathrm{sp}(\mathcal{N}_\hbar^{[1]}) \cap (-\infty, (b_0 + c\hbar^\delta)\hbar) \to \bigcup_{n \in \mathbb{N}^k} \mathrm{sp}(\hbar \mathcal{M}_\hbar^{[n]}) \cap (-\infty, (b_0 + c\hbar^\delta)\hbar)$$

such that $\Lambda_\hbar(\lambda) = \lambda + \mathcal{O}(\hbar^{1+\delta r_2/2})$ uniformly with respect to λ.

Remark 1.12. The threshold $b_0 + c\hbar^\delta$ is needed to get microlocalization of the eigenfunctions of $\mathcal{N}_\hbar^{[1]}$ in an arbitrarily small neighborhood of $\tau = 0$.

Remark 1.13. This second harmonic oscillator (in variables (t, τ)) corresponds to a classical oscillation in the directions of the field lines. We see that this new motion, due to the kernel of $\boldsymbol{B}$, induces powers of $\sqrt{\hbar}$ in the spectrum.

As a corollary, we get a description of the low-lying eigenvalues of $\mathcal{L}_\hbar$ by the effective operator $\hbar\mathcal{M}_\hbar^{[1]}$.

Corollary 1.14. *Let $\varepsilon > 0$ and $c \in (0, \min_j \nu_j(0))$. Define $\nu(0) = \sum_j \nu_j(0)$ and $r = \min(2r_1, r_2 + 4)$. Under Assumptions 1 and 2, with $k > 0$, there exists a bijection*

$$\Lambda_\hbar : \mathrm{sp}(\mathcal{L}_\hbar) \cap (-\infty, \hbar b_0 + \hbar^{3/2}(\nu(0) + 2c)) \to \mathrm{sp}(\hbar\mathcal{M}_\hbar^{[1]}) \cap (-\infty, \hbar b_0 + \hbar^{3/2}(\nu(0) + 2c))$$

such that $\Lambda_\hbar(\lambda) = \lambda + \mathcal{O}(\hbar^{r/4-\varepsilon})$ uniformly with respect to λ.

We deduce the following eigenvalue asymptotics.

Corollary 1.15. *Under the assumptions of Corollary 1.14, for $j \in \mathbb{N}$, the j-th eigenvalue of $\mathcal{L}_\hbar$ admits an expansion*

$$\lambda_j(\mathcal{L}_\hbar) = \hbar \sum_{\ell=0}^{\lfloor r/2 \rfloor - 2} \alpha_{j\ell} \hbar^{\ell/2} + \mathcal{O}(\hbar^{r/4-\varepsilon}),$$

with coefficients $\alpha_{j\ell} \in \mathbb{R}$ such that

$$\alpha_{j,0} = b_0, \quad \alpha_{j,1} = \sum_{j=1}^{k} \nu_j(0), \quad \alpha_{j,2} = E_j + c_0,$$

where $c_0 \in \mathbb{R}$ and $\hbar E_j$ is the j-th eigenvalue of an s-dimensional harmonic oscillator.

Remark 1.16. Note $\hbar E_j$ is the j-th eigenvalue of a harmonic oscillator whose symbol is given by the Hessian at $w = 0$ of $\hat{b}(w, s(w))$. Hence, it corresponds to a third classical oscillatory motion: a rotation in the space of field lines.

Remark 1.17. The asymptotics

$$\lambda_j(\mathcal{L}_\hbar) = b_0\hbar + \nu(0)\hbar^{3/2} + (E_j + c_0)\hbar^2 + o(\hbar^2)$$

were unknown before, except in the special 3-dimensional case $M = \mathbb{R}^3$ in [Helffer et al. 2016].

1C. ***Related questions and perspectives.*** In this paper, we are restricted to energies $\lambda < b_1\hbar$, and as mentioned in Remark 1.9, the threshold $b_1 > b_0$ is limited by three conditions, including the nonresonance one:

$$\text{for all } q \in \Omega, \text{ for all } \alpha \in \mathbb{Z}^s, \quad 0 < |\alpha| < r_1 \quad \Longrightarrow \quad \sum_{j=1}^{s} \alpha_j \beta_j(q) \neq 0.$$

It would be interesting to study the influence of resonances between the functions β_j on the spectrum of $\mathcal{L}_\hbar$. Maybe the Grushin techniques could help, as in [Helffer and Kordyukov 2015] for instance. A Birkhoff normal form was given in [Charles and Vũ Ngọc 2008] for a Schrödinger operator $-\hbar^2\Delta + V$ with resonances, but the situation is somehow simpler, since the analogues of $\beta_j(q)$ are independent of q in this context.

We are also restricted by the existence of Darboux coordinates φ on (Σ, B) such that $\varphi^* B = \mathrm{d}\eta \wedge \mathrm{d}y$. Indeed, the coordinates (y, η) on Σ are necessary to use the Weyl quantization. To study the influence of the global geometry of B, one should consider another quantization method for the presymplectic manifold (Σ, B). In the symplectic case, for instance in dimension $d = 2$, a Toeplitz quantization may be useful. This quantization is linked to the complex structure induced by B on Σ, and the operator $\mathcal{L}_\hbar$ can be linked with this structure in the following way:

$$\mathcal{L}_\hbar = 4\hbar^2\Big(\bar{\partial} + \frac{i}{2\hbar}A\Big)^*\Big(\bar{\partial} + \frac{i}{2\hbar}A\Big) + \hbar B = 4\hbar^2\bar{\partial}_A^*\bar{\partial}_A + \hbar B,$$

with

$$A = A_1 + iA_2, \quad B = \partial_1 A_2 - \partial_2 A_1, \quad 2\bar{\partial} = \partial_1 + i\partial_2.$$

In [Tejero Prieto 2006], this is used to compute the spectrum of $\mathcal{L}_\hbar$ on a bidimensional Riemann surface M with constant curvature and constant magnetic field. See also [Charles 2020; Kordyukov 2022], where semiexcited states for constant magnetic fields in higher dimensions are considered.

If the 2-form B is not exact, we usually consider a Bochner Laplacian on the p-th tensor product of a complex line bundle L over M, with curvature B. This Bochner Laplacian Δ_p depends on $p \in \mathbb{N}$, and the limit $p \to +\infty$ is interpreted as the semiclassical limit. The Bochner Laplacian Δ_p is a good generalization of the magnetic Laplacian because *locally* it can be written $(1/\hbar^2)(i\hbar\nabla + \boldsymbol{A})^2$, where the potential $\boldsymbol{A}$ is a local primitive of B, and $\hbar = p^{-1}$. For details, we refer to [Kordyukov 2019; 2020; Marinescu and Savale 2018]. Kordyukov [2019] constructed quasimodes for Δ_p in the case of a symplectic B and discrete wells. He proved expansions

$$\lambda_j(\Delta_p) \sim \sum_{\ell \geq 0} \alpha_{j\ell} p^{-\ell/2}.$$

Our work also gives such expansions for Δ_p as explained in [Morin 2022a].

In this paper, we only mention the study of the eigenvalues of $\mathcal{L}_\hbar$: what about the eigenfunctions? WKB expansions for the j-th eigenfunction were constructed on $\mathbb{R}^2$ in [Bonthonneau and Raymond 2020] and on a 2-dimensional Riemannian manifold in [Bonthonneau et al. 2021a]. We do not know how to construct magnetic WKB solutions in higher dimensions. This article suggests that the directions corresponding to the kernel of B could play a specific role.

Another related question is the decreasing of the real eigenfunctions. Agmon estimates only give a $\mathcal{O}(e^{-c/\sqrt{\hbar}})$ decay outside any neighborhood of q_0, but the 2-dimensional WKB suggests a $\mathcal{O}(e^{-c/\hbar})$ decay. Recently Bonthonneau, Raymond and Vũ Ngọc [Bonthonneau et al. 2021b] proved this on $\mathbb{R}^2$ using the FBI transform to work on the phase space $T^*\mathbb{R}^2$. This kind of question is motivated by the study of the tunneling effect: the exponentially small interaction between two magnetic wells for example.

Finally, we only have investigated the spectral theory of the stationary Schrödinger equation with a pure magnetic field; it would be interesting to describe the long-time dynamics of the full Schrödinger evolution, as was done in the Euclidean 2-dimensional case in [Boil and Vũ Ngọc 2021].

1D. ***Structure of the paper.*** In Section 2 we prove Theorem 1.4, describing the symbol H of $\mathcal{L}_\hbar$ on a neighborhood of $\Sigma = H^{-1}(0)$. In Section 3 we construct the normal form, first in a space of formal series (Section 3B) and then the quantized version $\mathcal{N}_\hbar$ (Section 3C). In Section 4 we prove Theorem 1.7. For this we describe the spectrum of $\mathcal{N}_\hbar$ (Section 4A), then we prove microlocalization properties on the eigenfunctions of $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar$ (Section 4B), and finally we compare the spectra of $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar$ (Section 4C).

In Section 5 we focus on Theorem 1.11 which describes the spectrum of the effective operator $\mathcal{N}_\hbar^{[1]}$. In Section 5A we study its symbol, in Section 5B we construct a second formal Birkhoff normal form, and in Section 5C the quantized version $\mathcal{M}_\hbar$. In Section 5D we compare the spectra of $\mathcal{N}_\hbar^{[1]}$ and $\mathcal{M}_\hbar$.

Finally, Sections 6 and 7 are dedicated to the proofs of Corollaries 1.14 and 1.15 respectively.

2. Geometry of the classical Hamiltonian

2A. ***Notation.*** $\mathcal{L}_\hbar$ is an $\hbar$-pseudodifferential operator on M with principal symbol H:

$$H(q,p) = |p - A_q|^2_{g_q^*}, \quad p \in T_q^*M,\ q \in M.$$

Here, T^*M denotes the cotangent bundle of M, and $p \in T_q^*M$ is a linear form on T_qM. The scalar product g_q on T_qM induces a scalar product g_q^* on T_q^*M, and $|\cdot|_{g_q^*}$ denotes the associated norm. In this section we prove Theorem 1.4, thus describing H on a neighborhood of its minimum:

$$\Sigma = \{(q,p) \in T^*M : q \in \Omega,\ p = A_q\}.$$

Recall that Ω is a small neighborhood of $q_0 \in M \setminus \partial M$. We will construct canonical coordinates $(z, w, v) \in \mathbb{R}^{2d}$ on Ω, with

$$z = (x, \xi) \in \mathbb{R}^{2s}, \quad w = (y, \eta) \in \mathbb{R}^{2s}, \quad v = (t, \tau) \in \mathbb{R}^{2k}.$$

$\mathbb{R}^{2d}$ is endowed with the canonical symplectic form

$$\omega_0 = \mathrm{d}\xi \wedge \mathrm{d}x + \mathrm{d}\eta \wedge \mathrm{d}y + \mathrm{d}\tau \wedge \mathrm{d}t.$$

We will identify Σ with

$$\Sigma' = \{(x, \xi, y, \eta, t, \tau) \in \mathbb{R}^{2d} : x = \xi = 0,\ \tau = 0\} = \mathbb{R}^{2s+k}_{(y,\eta,t)} \times \{0\}.$$

We will use several lemmas to prove Theorem 1.4. Before constructing the diffeomorphism Φ_1^{-1} on a neighborhood U_1 of Σ, we will first define it on Σ. Thus we need to understand the structure of Σ induced by the symplectic structure on T^*M (Section 2B). Then we will construct Φ_1 and finally prove Theorem 1.4 (Section 2C).

2B. ***Structure of* $\boldsymbol{\Sigma}$.** Recall that on T^*M we have the Liouville 1-form α defined by

$$\alpha_{(q,p)}(\mathcal{V}) = p((\mathrm{d}\pi)_{(q,p)}\mathcal{V}) \quad \text{for all } (q, p) \in T^*M,\ \mathcal{V} \in T_{(q,p)}(T^*M),$$

where $\pi : T^*M \to M$ is the canonical projection: $\pi(q, p) = q$, and $\mathrm{d}\pi$ is its differential. T^*M is endowed with the symplectic form $\omega = \mathrm{d}\alpha$. Σ is a d-dimensional submanifold of T^*M which can be identified with Ω using

$$j : q \in \Omega \mapsto (q, A_q) \in \Sigma$$

and its inverse, which is π.

Lemma 2.1. *The restriction of ω to Σ is $\omega_\Sigma = \pi^*B$.*

Proof. Fix $q \in \Omega$ and $Q \in T_qM$. Then

$$(j^*\alpha)_q(Q) = \alpha_{j(q)}((\mathrm{d}j)Q) = A_q((\mathrm{d}\pi) \circ (\mathrm{d}j)Q) = A_q(Q),$$

because $\pi \circ j = \mathrm{Id}$. Thus $j^*\alpha = A$ and $\alpha_\Sigma = \pi^* j^*\alpha = \pi^*A$. Taking the exterior derivative we get

$$\omega_\Sigma = \mathrm{d}\alpha_\Sigma = \pi^*(\mathrm{d}A) = \pi^*B. \qquad \square$$

Since B is a closed 2-form with constant rank equal to $2s$, (Σ, π^*B) is a presymplectic manifold. It is equivalent to (Ω, B), using j. We recall the Darboux lemma, which states that such a manifold is locally equivalent to $(\mathbb{R}^{2s+k}, \mathrm{d}\eta \wedge \mathrm{d}y)$.

Lemma 2.2. *Up to shrinking Ω, there exists an open subset Σ' of $\mathbb{R}^{2s+k}_{(y,\eta,t)}$ and a diffeomorphism $\varphi : \Sigma' \to \Omega$ such that $\varphi^*B = \mathrm{d}\eta \wedge \mathrm{d}y$.*

One can always take any coordinate system on Ω. Up to working in these coordinates, it is enough to consider the case $M = \mathbb{R}^d$ with

$$H(q, p) = \sum_{k,\ell=1}^{d} g^{k\ell}(q)(p_k - A_k(q))(p_\ell - A_\ell(q)), \quad (q, p) \in T^*\mathbb{R}^d \simeq \mathbb{R}^{2d},$$

to prove Theorem 1.4. This is what we will do. In coordinates, ω is given by

$$\omega = \mathrm{d}p \wedge \mathrm{d}q = \sum_{j=1}^{d} \mathrm{d}p_j \wedge \mathrm{d}q_j$$

and Σ is the submanifold

$$\Sigma = \{(q, \boldsymbol{A}(q)) : q \in \Omega\} \subset \mathbb{R}^{2d},$$

and $j \circ \varphi : \Sigma' \to \Sigma$.

In order to extend $j \circ \varphi$ to a neighborhood of Σ' in $\mathbb{R}^{2d}$ *in a symplectic way*, it is convenient to split the tangent space $T_{j(q)}(\mathbb{R}^{2d})$ according to tangent and normal directions to Σ. This is the purpose of the following two lemmas.

Lemma 2.3. *Fix $j(q) = (q, \boldsymbol{A}(q)) \in \Sigma$. Then the tangent space to Σ is*

$$T_{j(q)}\Sigma = \{(Q, P) \in \mathbb{R}^{2d} : P = \nabla_q \boldsymbol{A} \cdot Q\}.$$

Moreover, the ω-orthogonal $T_{j(q)}\Sigma^\perp$ is

$$T_{j(q)}\Sigma^\perp = \{(Q, P) \in \mathbb{R}^{2d} : P = (\nabla_q \boldsymbol{A})^T \cdot Q\}.$$

Finally,

$$T_{j(q)}\Sigma \cap T_{j(q)}\Sigma^\perp = \operatorname{Ker}(\pi^* B).$$

Proof. Since Σ is the graph of $q \mapsto \boldsymbol{A}(q)$, its tangent space is the graph of the differential $Q \mapsto (\nabla_q \boldsymbol{A}) \cdot Q$. In order to characterize $T\Sigma^\perp$, note that the symplectic form $\omega = \mathrm{d}p \wedge \mathrm{d}q$ is defined by

$$\omega_{(q,p)}((Q_1, P_1), (Q_2, P_2)) = \langle P_2, Q_1\rangle - \langle P_1, Q_2\rangle, \tag{2-1}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product on $\mathbb{R}^d$. Thus,

$$\begin{aligned}
(Q, P) \in T_{j(q)}\Sigma^\perp \quad &\iff \quad \omega_{j(q)}((Q_0, \nabla_q \boldsymbol{A} \cdot Q_0), (Q, P)) = 0 \quad \text{for all } Q_0 \in \mathbb{R}^d \\
&\iff \quad \langle P, Q_0\rangle - \langle (\nabla_q \boldsymbol{A}) \cdot Q_0, Q\rangle = 0 \quad \text{for all } Q_0 \in \mathbb{R}^d \\
&\iff \quad \langle P - (\nabla_q \boldsymbol{A})^T \cdot Q, Q_0\rangle = 0 \quad \text{for all } Q_0 \in \mathbb{R}^d \\
&\iff \quad P = (\nabla_q \boldsymbol{A})^T \cdot Q.
\end{aligned}$$

Finally, with Lemma 2.1 we know that the restriction of ω to $T\Sigma$ is given by $\pi^* B$. Hence, $T_{j(q)}\Sigma \cap T_{j(q)}\Sigma^\perp$ is the set of $(Q, P) \in T_{j(q)}\Sigma$ such that

$$\pi^* B((Q, P), (Q_0, P_0)) = 0 \quad \text{for all } (Q_0, P_0) \in T_{j(q)}\Sigma.$$

It is the kernel of $\pi^* B$. □

Now we define specific basis of $T_{j(q)}\Sigma$ and its orthogonal. Since $\boldsymbol{B}(q)$ is skew-symmetric with respect to g, there exist orthonormal vectors

$$\boldsymbol{u}_1(q),\ \boldsymbol{v}_1(q),\ \ldots,\ \boldsymbol{u}_s(q),\ \boldsymbol{v}_s(q),\ \boldsymbol{w}_1(q),\ \ldots,\ \boldsymbol{w}_k(q) \in \mathbb{R}^d$$

such that

$$\begin{cases} \boldsymbol{B}\boldsymbol{u}_j = -\beta_j \boldsymbol{v}_j, & 1 \le j \le s, \\ \boldsymbol{B}\boldsymbol{v}_j = \beta_j \boldsymbol{u}_j, & 1 \le j \le s, \\ \boldsymbol{B}\boldsymbol{w}_j = 0, & 1 \le j \le k. \end{cases} \tag{2-2}$$

These vectors are smooth functions of q because the nonzero eigenvalues $\pm i\beta_j(q)$ are simple. They define a basis of $\mathbb{R}^d$. Define the following ω-orthogonal vectors to $T\Sigma$:

$$\begin{cases} \boldsymbol{f}_j(q) := (1/\sqrt{\beta_j(q)})(\boldsymbol{u}_j(q), (\nabla_q \boldsymbol{A})^T \cdot \boldsymbol{u}_j(q)), & 1 \le j \le s, \\ \boldsymbol{f}'_j(q) := (1/\sqrt{\beta_j(q)})(\boldsymbol{v}_j(q), (\nabla_q \boldsymbol{A})^T \boldsymbol{v}_j(q)), & 1 \le j \le s. \end{cases} \tag{2-3}$$

These vectors are linearly independent and

$$T_{j(q)}\Sigma^{\perp} = K \oplus F,$$

with

$$K = \operatorname{Ker}(\pi^* B), \qquad F = \operatorname{span}(\boldsymbol{f}_1, \boldsymbol{f}_1', \dots, \boldsymbol{f}_s, \boldsymbol{f}_s').$$

Similarly, the tangent space $T_{j(q)}\Sigma$ admits a decomposition

$$T_{j(q)}\Sigma = E \oplus K$$

defined as follows. The map $j \circ \varphi : \Sigma' \to \Sigma$ from Lemma 2.2 satisfies $(j \circ \varphi)^*(\pi^* B) = \mathrm{d}\eta \wedge \mathrm{d}y$. Thus its differential $\mathrm{d}(j \circ \varphi)$ maps the kernel of $\mathrm{d}\eta \wedge \mathrm{d}y$ on the kernel of $\pi^* B$:

$$K = \{\mathrm{d}(j \circ \varphi)_q(0, T) : T \in \mathbb{R}^k\}. \tag{2-4}$$

A complementary space of K in $T\Sigma$ is given by

$$E := \{\mathrm{d}(j \circ \varphi)_q(W, 0) : W \in \mathbb{R}^{2s}\}. \tag{2-5}$$

From all these considerations we deduce:

Lemma 2.4. *Fix $j(q) = (q, \boldsymbol{A}(q)) \in \Sigma$. Then we have the decomposition*

$$T_{j(q)}(\mathbb{R}^{2d}) = \underbrace{E \oplus \overbrace{K}^{}}_{T\Sigma}\!\!\overbrace{\;\oplus F}^{T\Sigma^{\perp}} \oplus L,$$

where L is any Lagrangian complement of K in $(E \oplus F)^{\perp}$.

Proof. We have $T\Sigma + T\Sigma^{\perp} = E \oplus K \oplus F$, and the restriction of $\omega = \mathrm{d}p \wedge \mathrm{d}q$ to this space has kernel $K = T\Sigma \cap T\Sigma^{\perp}$. Hence, the restriction $\omega_{E\oplus F}$ of ω to $E \oplus F$ is nondegenerate and its orthogonal $(E \oplus F)^{\perp}$ as well. Moreover $(E \oplus F)^{\perp}$ has dimension $2d - 4s = 2k$, and we have

$$T_{j(q)}\mathbb{R}^{2d} = (E \oplus F) \oplus (E \oplus F)^{\perp}.$$

K is a Lagrangian subspace of $(E \oplus F)^{\perp}$. Therefore it admits a complementary Lagrangian: a subspace L of $(E \oplus F)^{\perp}$ with dimension k such that $\omega_L = 0$ and $(E \oplus F)^{\perp} = K \oplus L$. □

Remark 2.5. From now on, we fix any choice of Lagrangian complement L. With this choice, we define a basis $(\boldsymbol{\ell}_j)$ of L as follows. First note that the decomposition $(E \oplus F)^{\perp} = K \oplus L$ yields a bijection between L and the dual K^*, which is $\ell \mapsto \omega(\ell, \cdot)$. We emphasize that this bijection *depends on the choice of L*. Using this bijection, we define $\boldsymbol{\ell}_j$ to be the unique vector in L satisfying

$$\omega(\boldsymbol{\ell}_j, \mathrm{d}(j \circ \varphi)(0, T)) = T_j \quad \text{for all } T \in \mathbb{R}^k. \tag{2-6}$$

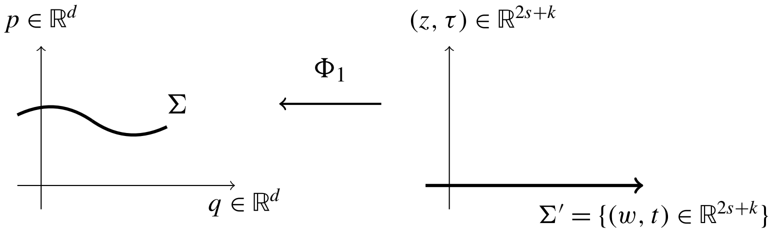

Figure 1. Using the canonical coordinates (w, t, τ, z), we identify Σ with Σ'.

2C. ***Construction of Φ_1 and proof of Theorem 1.4.*** We identified the "curved" manifold Σ with an open subset Σ' of $\mathbb{R}^{2s+k}$ using $j \circ \varphi$. Moreover, we did this in such a way that $(j \circ \varphi)^* \pi^* B = \mathrm{d}\eta \wedge \mathrm{d}y$. In this section we prove that we can identify a whole neighborhood of Σ in $\mathbb{R}^{2d}_{(q,p)}$ with a neighborhood of Σ' in $\mathbb{R}^{4s+2k}_{(z,w,v)}$, via a symplectomorphism Φ_1. See Figure 1.

Lemma 2.6. *There exists a diffeomorphism*

$$\Phi_1 : U_1' \subset \mathbb{R}^{2s+2k+2s}_{(w,t,\tau,z)} \to U_1 \subset \mathbb{R}^{2d}_{(q,p)}$$

between neighborhoods U_1 of Σ and U_1' of Σ' such that $\Phi_1^* \omega = \omega_0$ and $\Phi_1(w, t, 0, 0) = j \circ \varphi(w, t)$. Moreover its differential at $(w, t, \tau = 0, z = 0) \in \Sigma'$ is

$$\mathrm{d}\Phi_1(W, T, \mathcal{T}, Z) = \mathrm{d}_{(w,t)} j \circ \varphi(W, T) + \sum_{j=1}^{k} \mathcal{T}_j \hat{\boldsymbol{\ell}}_j(w, t) + \sum_{j=1}^{s} X_j \hat{\boldsymbol{f}}_j(w, t) + \Xi_j \hat{\boldsymbol{f}}_j'(w, t).$$

Remark 2.7. In this lemma we used the notation $Z = (X, \Xi)$ and $\hat{\boldsymbol{\ell}}_j = \boldsymbol{\ell}_j \circ \varphi$, $\hat{\boldsymbol{f}}_j = \boldsymbol{f}_j \circ \varphi$, and $\hat{\boldsymbol{f}}_j' = \boldsymbol{f}_j' \circ \varphi$.

Proof. We will first construct Φ such that $\Phi^* \omega_{|\Sigma'} = \omega_{0\ |\Sigma'}$ only on $\Sigma' = \Phi^{-1}(\Sigma)$. Then, we will use the Theorem B.2 to slightly change Φ into Φ_1 such that $\Phi_1^* \omega = \omega_0$ on a neighborhood of Σ'.

We define Φ by

$$\Phi(w, t, \tau, z) = j \circ \varphi(w, t) + \sum_{j=1}^{k} \tau_j \hat{\boldsymbol{\ell}}_j(w, t) + \sum_{j=1}^{s} x_j \hat{\boldsymbol{f}}_j(w, t) + \xi_j \hat{\boldsymbol{f}}_j'(w, t). \tag{2-7}$$

Its differential at $(w, t, 0, 0)$ has the desired form. Let us fix a point $(w, t, 0, 0) \in \Sigma'$ and compute $\Phi^* \omega$ at this point. By definition,

$$\Phi^* \omega_{(w,t,0,0)}(\cdot\,, \cdot) = \omega_{j(q)}((\mathrm{d}\Phi)\cdot\,, (\mathrm{d}\Phi)\cdot),$$

where $q = \varphi(w, t)$. Computing this 2-form in the canonical basis of $\mathbb{R}^{4s+2k}$ amounts to computing ω on the vectors $\boldsymbol{\ell}_j$, $\boldsymbol{f}_j$, $\boldsymbol{f}_j'$ and $\mathrm{d}(j \circ \varphi)(W, T)$. By (2-3) and (2-1) we have

$$\begin{aligned}
\omega(\boldsymbol{f}_i, \boldsymbol{f}_j) &= \frac{1}{\sqrt{\beta_i \beta_j}} \big(\langle (\nabla_q \boldsymbol{A})^{\perp} \cdot \boldsymbol{u}_j, \boldsymbol{u}_i \rangle - \langle (\nabla_q \boldsymbol{A})^{\perp} \cdot \boldsymbol{u}_i, \boldsymbol{u}_j \rangle \big) \\
&= \frac{1}{\sqrt{\beta_i \beta_j}} \langle ((\nabla_q \boldsymbol{A})^{\perp} - (\nabla_q \boldsymbol{A})) \cdot \boldsymbol{u}_j, \boldsymbol{u}_i \rangle \\
&= \frac{1}{\sqrt{\beta_i \beta_j}} B(\boldsymbol{u}_j, \boldsymbol{u}_i) = \frac{1}{\sqrt{\beta_i \beta_j}} g(\boldsymbol{u}_j, \boldsymbol{B}\boldsymbol{u}_i) = 0,
\end{aligned}$$

because $\boldsymbol{B}\boldsymbol{u}_i = -\beta_i \boldsymbol{v}_i$ is orthogonal to $\boldsymbol{u}_j$. Similarly we find

$$\omega(\boldsymbol{f}_i, \boldsymbol{f}_j') = \delta_{ij}, \quad \omega(\boldsymbol{f}_i', \boldsymbol{f}_j') = 0.$$

Moreover, $\boldsymbol{\ell}_i \in L \subset F^\perp$ so

$$\omega(\boldsymbol{\ell}_i, \boldsymbol{f}_j) = \omega(\boldsymbol{\ell}_i, \boldsymbol{f}_j') = 0.$$

Since L is Lagrangian we also have $\omega(\boldsymbol{\ell}_i, \boldsymbol{\ell}_j) = 0$. The vector $\mathrm{d}(j \circ \varphi)(W, T)$ is tangent to Σ and $\boldsymbol{f}_j, \boldsymbol{f}_j' \in T\Sigma^\perp$ so

$$\omega(\boldsymbol{f}_j, \mathrm{d}(j \circ \varphi)(W, T)) = \omega(\boldsymbol{f}_j', \mathrm{d}(j \circ \varphi)(W, T)) = 0.$$

Since $\boldsymbol{\ell}_i \in L \subset E^\perp$ and using (2-6), we have

$$\omega(\boldsymbol{\ell}_j, \mathrm{d}(j \circ \varphi)(W, T)) = \omega(\boldsymbol{\ell}_j, \mathrm{d}(j \circ \varphi)(0, T)) = T_j.$$

Finally, $(j \circ \varphi)^* \omega = \varphi^* B = \mathrm{d}\eta \wedge \mathrm{d}y$ so that

$$\omega(\mathrm{d}(j \circ \varphi)(W, T), \mathrm{d}(j \circ \varphi)(W', T')) = \mathrm{d}\eta \wedge \mathrm{d}y((W, T), (W', T')).$$

All these computations show that $(\Phi^*\omega)_{(w,t,0,0)}$ coincide with $\omega_0 = \mathrm{d}\xi \wedge \mathrm{d}x + \mathrm{d}\eta \wedge \mathrm{d}y + \mathrm{d}\tau \wedge \mathrm{d}t$. Thus $\Phi^*\omega = \omega_0$ on Σ. With Theorem B.2, we can change Φ into $\Phi_1(w, t, \tau, z) = \Phi(w, t, \tau, z) + \mathcal{O}((z, \tau)^2)$ such that $\Phi_1^*\omega = \omega_0$ on a neighborhood U_1' of Σ'. In particular, the differential of Φ_1 at $(w, t, 0, 0)$ coincides with the differential of Φ. □

Finally, the following lemma concludes the proof of Theorem 1.4.

Lemma 2.8. *The Hamiltonian $\widehat{H} = H \circ \Phi_1$ has the Taylor expansion*

$$\widehat{H}(w, t, \tau, x, \xi) = \tfrac{1}{2}\langle \partial_\tau^2 \widehat{H}(w, t, 0)\tau, \tau\rangle + \sum_{j=1}^{s} \hat{\beta}_j(w, t)(\xi_j^2 + x_j^2) + \mathcal{O}((\tau, x, \xi)^3).$$

Proof. Let us compute the differential and Hessian of

$$H(q, p) = \sum_{k,\ell=1}^{d} g^{k\ell}(q)(p_k - A_k(q))(p_\ell - A_\ell(q))$$

at a point $(q, \boldsymbol{A}(q)) \in \Sigma$. First,

$$\nabla_{(q,p)} H \cdot (Q, P) = \sum_{k,\ell=1}^{d} 2g^{k\ell}(q)(p_k - A_k(q))(P_\ell - \nabla_q A_\ell \cdot Q) + (p_k - A_k(q))(p_\ell - A_\ell(q))\nabla_q g \cdot Q, \quad (2\text{-}8)$$

and at $p = \boldsymbol{A}(q)$ the Hessian is

$$\langle \nabla_{j(q)}^2 H \cdot (Q, P), (Q', P')\rangle = 2\sum_{k,\ell=1}^{d} g^{k\ell}(q)(P_k - \nabla_q A_k \cdot Q)(P_\ell' - \nabla_q A_\ell \cdot Q'). \quad (2\text{-}9)$$

We can deduce a Taylor expansion of $\widehat{H}(w, t, \tau, z)$ with respect to (τ, z) (with fixed $q = \varphi(w, t)$). First,

$$\widehat{H}(w, t, 0, 0) = H(q, \boldsymbol{A}(q)) = 0.$$

Then we can compute the partial differential using Lemma 2.6,

$$\partial_{\tau,z}\widehat{H}(w,t,0,0)\cdot(W,T)=\nabla_{j(q)}H\cdot\partial_{\tau,z}\Phi_1(w,t,0,0)\cdot(W,T)=\nabla_{j(q)}H\cdot \mathrm{d}(j\circ\varphi)(W,T)=0,$$

because $\mathrm{d}(j\circ\varphi)(W,T)\in T_{j(q)}\Sigma$. The Taylor expansion of $\widehat{H}$ is thus

$$\widehat{H}(w,t,\tau,z)=\tfrac{1}{2}\langle\partial^2_{\tau,z}\widehat{H}(w,t,0)\cdot(\tau,z),(\tau,z)\rangle+\mathcal{O}((\tau,z)^3),$$

where $\partial^2_{\tau,z}\widehat{H}$ is the partial Hessian with respect to (τ,z). We have

$$\partial^2_{\tau,z}\widehat{H}=(\partial_{(\tau,z)}\Phi_1)^T\cdot\nabla^2_{j(q)}H\cdot(\partial_{(\tau,z)}\Phi_1),$$

and computing the Hessian matrix amounts to computing $\nabla^2_{j(q)}H$ on the vectors $\boldsymbol{g}_j$, $\boldsymbol{f}_j$, and $\boldsymbol{f}'_j$. If $(Q,P)\in T_{j(q)}\Sigma^\perp$, then $P=(\nabla_q A)^\perp\cdot Q$ so that, with (2-9),

$$\begin{aligned}\tfrac{1}{2}\nabla^2_{j(q)}H((Q,P),(Q',P'))&=\sum_{k,\ell,i,j=1}^{d}g^{k\ell}(q)(\partial_k A_j Q_j-\partial_j A_k Q_j)(\partial_\ell A_i Q'_i-\partial_i A_\ell Q'_i)\\&=\sum_{k,\ell,i,j}g^{k\ell}(q)B_{kj}Q_jB_{\ell i}Q'_i.\end{aligned}$$

But $\sum_k g^{k\ell}B_{kj}=\boldsymbol{B}_{\ell j}$ (by (1-1)) so

$$\tfrac{1}{2}\nabla^2_{j(q)}H((Q,P),(Q',P'))=\sum_{i,j,\ell}B_{\ell i}(\boldsymbol{B}_{\ell j}Q_j)Q'_i=B(\boldsymbol{B}\cdot Q,Q').$$

In the special case $(Q,P)=\boldsymbol{f}_j$ we have

$$\tfrac{1}{2}\nabla^2_{j(q)}H(\boldsymbol{f}_i,\boldsymbol{f}_j)=\frac{1}{\sqrt{\beta_i\beta_j}}B(\boldsymbol{B}\boldsymbol{u}_i,\boldsymbol{u}_j)=\frac{1}{\sqrt{\beta_i\beta_j}}g(\boldsymbol{B}\boldsymbol{u}_i,\boldsymbol{B}\boldsymbol{u}_j)=\sqrt{\beta_i\beta_j}\,g(\boldsymbol{v}_i,\boldsymbol{v}_j)=\sqrt{\beta_i\beta_j}\,\delta_{ij},$$

and similarly

$$\tfrac{1}{2}\nabla^2_{j(q)}H(\boldsymbol{f}'_i,\boldsymbol{f}'_j)=\sqrt{\beta_i\beta_j}\,\delta_{ij},\quad \tfrac{1}{2}\nabla^2_{j(q)}H(\boldsymbol{f}_i,\boldsymbol{f}'_j)=0.$$

Finally, it remains to prove

$$\nabla^2_{j(q)}H(\boldsymbol{\ell}_i,\boldsymbol{f}_j)=\nabla^2_{j(q)}H(\boldsymbol{\ell}_i,\boldsymbol{f}'_j)=0 \tag{2-10}$$

to conclude that the Hessian of $\widehat{H}$ is

$$\tfrac{1}{2}\partial^2_{\tau,z}\widehat{H}(w,t,0,0)=\begin{pmatrix}\tfrac{1}{2}\partial^2_\tau\widehat{H}(w,t,0,0)&&&&&\\&\beta_1&&&&\\&&\beta_1&&&\\&&&\ddots&&\\&&&&\beta_s&\\&&&&&\beta_s\end{pmatrix}.$$

Actually, (2-10) follows from the identity

$$L\subset F^\perp=(T\Sigma^\perp)^{\perp H}, \tag{2-11}$$

where $\perp H$ denotes the orthogonal with respect to the quadratic form $\nabla^2 H$ (which is different from the symplectic orthogonal $\perp$). Indeed, to prove (2-11) note that

$$\begin{aligned}
(Q,P)\in(T\Sigma^{\perp})^{\perp H} \quad &\Longrightarrow \quad \nabla^2 H((Q,P),(Q',(\nabla_q \boldsymbol{A})^T\cdot Q'))=0 \quad \text{for all } Q'\in\mathbb{R}^d\\
&\Longrightarrow \quad \sum_{k,\ell,j} g^{k\ell}(P_k-\nabla_q A_k\cdot Q)B_{\ell j}Q'_j=0 \quad \text{for all } Q'\in\mathbb{R}^d\\
&\Longrightarrow \quad \sum_{k,j}(P_k-\nabla_q A_k\cdot Q)\boldsymbol{B}_{kj}Q'_j=0 \quad \text{for all } Q'\in\mathbb{R}^d\\
&\Longrightarrow \quad \langle P-\nabla_q\boldsymbol{A}\cdot Q,\boldsymbol{B}Q'\rangle=0 \quad \text{for all } Q'\in\mathbb{R}^d\\
&\Longrightarrow \quad \langle P,\boldsymbol{B}Q'\rangle-\langle Q,(\nabla_q\boldsymbol{A})^T\cdot\boldsymbol{B}Q'\rangle=0 \quad \text{for all } Q'\in\mathbb{R}^d\\
&\Longrightarrow \quad \omega((Q,P),(\boldsymbol{B}Q',(\nabla_q\boldsymbol{A})^T\cdot\boldsymbol{B}Q'))=0 \quad \text{for all } Q'\in\mathbb{R}^d,
\end{aligned}$$

and we have

$$\begin{aligned}
F&=\{(V:(\nabla_q\boldsymbol{A})^T V),\ V\in\operatorname{span}(\boldsymbol{u}_1,\boldsymbol{v}_1,\dots,\boldsymbol{u}_s,\boldsymbol{v}_s)\}\\
&=\{(\boldsymbol{B}Q:(\nabla_q\boldsymbol{A})^T\boldsymbol{B}Q),\ Q\in\mathbb{R}^d\},
\end{aligned}$$

because the vectors $\boldsymbol{u}_j,\boldsymbol{v}_j$ span the range of $\boldsymbol{B}$. Hence we find

$$(Q,P)\in(T\Sigma^{\perp})^{\perp H} \quad\Longleftrightarrow\quad (Q,P)\in F^{\perp}. \qquad \square$$

3. Construction of the normal form $\mathcal{N}_\hbar$

3A. *Formal series.* Define $U=U_1'\cap\Sigma'\subset\mathbb{R}^{2s+k}_{(w,t)}\times\{0\}$. We construct the Birkhoff normal form in the space

$$\mathcal{E}_1=\mathcal{C}^\infty(U)[\![x,\xi,\tau,\hbar]\!].$$

It is a space of formal series in $(x,\xi,\tau,\hbar)$ with coefficients smoothly depending on (w,t). We see these formal series as Taylor series of symbols, which we quantize using the Weyl quantization. Given an $\hbar$-pseudodifferential operator $\mathcal{A}_\hbar=\operatorname{Op}_\hbar^w a_\hbar$ (with symbol $a_\hbar$ admitting an expansion in powers of $\hbar$ in some standard class), we denote by $[a_\hbar]$ or $\sigma^T(\mathcal{A}_\hbar)$ the Taylor series of $a_\hbar$ with respect to (x,ξ,τ) at $(x,\xi,\tau)=0$. Conversely, given a formal series $\rho\in\mathcal{E}_1$, we can find a bounded symbol $a_\hbar$ such that $[a_\hbar]=\rho$. This symbol is not uniquely defined, but any two such symbols differ by $\mathcal{O}((x,\xi,\hbar)^\infty)$, uniformly with respect to $(w,t)\in U$.

Remark 3.1. We prove below that the eigenfunctions of $\mathcal{L}_\hbar$ are microlocalized, where $(w,t)\in U$ and $|(x,\xi)|\lesssim\hbar^{1/2}$, so that the remainders $\mathcal{O}((x,\xi,\hbar)^\infty)$ are negligible.

• In order to make operations on Taylor series compatible with the Weyl quantization, we endow $\mathcal{E}_1$ with the Weyl–Moyal product $\star$, defined by $\operatorname{Op}_\hbar^w(a)\operatorname{Op}_\hbar^w(b)=\operatorname{Op}_\hbar^w(a\star b)$. This product satisfies

$$a_1\star a_2=\sum_{k=0}^{N}\frac{1}{k!}\Big(\frac{\hbar}{2i}\Box\Big)^k a_1(w,t,\tau,z)a_2(w',t',\tau',z')|_{w'=w,t'=t,\tau'=\tau,z'=z}+\mathcal{O}(\hbar^N),$$

where

$$\Box=\sum_{j=1}^{s}(\partial_{\eta_j}\partial_{y'_j}-\partial_{y_j}\partial_{\eta'_j})+\sum_{j=1}^{s}(\partial_{\xi_j}\partial_{x'_j}-\partial_{x_j}\partial_{\xi'_j})+\sum_{j=1}^{k}(\partial_{\tau_j}\partial_{t'_j}-\partial_{t_j}\partial_{\tau'_j}).$$

Note that to define such a product it is necessary to assume that our formal series depend smoothly on (w, t).

• The degree of a monomial is

$$\deg(x^\alpha \xi^{\alpha'} \tau^{\alpha''} \hbar^\ell) = |\alpha| + |\alpha'| + |\alpha''| + 2\ell. \tag{3-1}$$

We denote by $\mathcal{D}_N$ the $\mathcal{C}^\infty(U)$-module spanned by monomials of degree N, and

$$\mathcal{O}_N = \bigoplus_{n \geq N} \mathcal{D}_N, \tag{3-2}$$

which satisfies

$$\mathcal{O}_{N_1} \star \mathcal{O}_{N_2} \subset \mathcal{O}_{N_1+N_2}.$$

If $\rho_1, \rho_2 \in \mathcal{E}_1$, we denote their commutator by

$$[\rho_1, \rho_2] = \mathrm{ad}_{\rho_1} \rho_2 = \rho_1 \star \rho_2 - \rho_2 \star \rho_1,$$

and we have the formula

$$[\rho_1, \rho_2] = 2 \sinh\Big(\frac{\hbar}{2i} \square\Big) \rho_1 \rho_2. \tag{3-3}$$

In particular,

$$\text{for all } \rho_1 \in \mathcal{O}_{N_1}, \text{ for all } \rho_2 \in \mathcal{O}_{N_2}, \quad \frac{i}{\hbar}[\rho_1, \rho_2] \in \mathcal{O}_{N_1+N_2-2},$$

and $(i/\hbar)[\rho_1, \rho_2] = \{\rho_1, \rho_2\} + \mathcal{O}(\hbar^2)$. The Birkhoff normal form algorithm is based on the following lemma. We recall the definition (1-5) of r_1.

Lemma 3.2. *For $1 \leq j \leq s$, define $z_j = x_j + i\xi_j$ and $|z_j|^2 = x_j^2 + \xi_j^2$.*

(1) *Every series $\rho \in \mathcal{E}_1$ satisfies*

$$\frac{i}{\hbar} \mathrm{ad}_{|z_j|^2} \rho = \{|z_j|^2, \rho\}.$$

(2) *Let $0 \leq N < r_1$. For every $R_N \in \mathcal{D}_N$, there exist $\rho_N, K_N \in \mathcal{D}_N$ such that*

$$R_N = K_N + \sum_{j=1}^{s} \hat{\beta}_j(w, t) \frac{i}{\hbar} \mathrm{ad}_{|z_j|^2} \rho_N$$

and $[K_N, |z_j|^2] = 0$ for $1 \leq j \leq s$.

(3) *If $K \in \mathcal{E}_1$, then $[K, |z_j|^2] = 0$ for all $1 \leq j \leq s$ if and only if there exists a formal series $F \in \mathcal{C}^\infty(U)[\![I_1, \dots, I_s, \tau, \hbar]\!]$ such that*

$$K = F(|z_1|^2, \dots, |z_s|^2, \tau, \hbar).$$

Proof. The first statement is a simple computation. For the second and the third, it suffices to consider monomials $R_N = c(w, t) z^\alpha \bar{z}^{\alpha'} \tau^{\alpha''} \hbar^\ell$. Note that

$$\mathrm{ad}_{|z_j|^2}(c(w, t) z^\alpha \bar{z}^{\alpha'} \tau^{\alpha''} \hbar^\ell) = (\alpha'_j - \alpha_j) c(w, t) z^\alpha \bar{z}^{\alpha'} \tau^{\alpha''} \hbar^\ell,$$

so that R_N commutes with every $|z_j|^2$ ($1 \le j \le s$) if and only if $\alpha = \alpha'$, which amounts to saying that R_N is a function of $|z_j|^2$ and proves (3). Moreover,

$$\sum_j \hat{\beta}_j \operatorname{ad}_{|z_j|^2}(z^\alpha \bar{z}^{\alpha'} \tau^{\alpha''} \hbar^\ell) = \langle \alpha' - \alpha, \hat{\beta} \rangle z^\alpha \bar{z}^{\alpha'} \tau^{\alpha''} \hbar^\ell,$$

where $\langle \gamma, \hat{\beta} \rangle = \sum_{j=1}^s \gamma_j \hat{\beta}_j(w, t)$. Under the assumption $|\alpha| + |\alpha'| + |\alpha''| + 2\ell < r_1$, we have $|\alpha - \alpha'| < r_1$ and by the definition of r_1 the function $\langle \alpha' - \alpha, \hat{\beta}(w, t) \rangle$ cannot vanish for $(w, t) \in U$, unless $\alpha = \alpha'$. If $\alpha = \alpha'$, we choose $\rho_N = 0$ and $R_N = K_N$ commutes with $|z_j|^2$. If $\alpha \ne \alpha'$, we choose $K_N = 0$ and

$$\rho_N = \frac{c(w, t)}{\langle \alpha' - \alpha, \hat{\beta}(w, t) \rangle} z^\alpha \bar{z}^{\alpha'} \tau^{\alpha''} \hbar^\ell,$$

and this proves (2). □

3B. ***Formal Birkhoff normal form.*** In this section we construct the Birkhoff normal form at a formal level. We will work with the Taylor series of the symbol H of $\mathcal{L}_\hbar$, in the new coordinates Φ_1. According to Theorem 1.4, $\widehat{H} = H \circ \Phi_1$ defines a formal series

$$[\widehat{H}] = H_2 + \sum_{k \ge 3} H_k,$$

where $H_k \in \mathcal{D}_k$ and

$$H_2 = \langle M(w, t)\tau, \tau \rangle + \sum_{j=1}^s \hat{\beta}_j(w, t)|z_j|^2. \tag{3-4}$$

At a formal level, the normal form can be stated as follows.

Theorem 3.3. *For every $\gamma \in \mathcal{O}_3$, there are $\kappa, \rho \in \mathcal{O}_3$ such that*

$$e^{(i/\hbar)\operatorname{ad}_\rho}(H_2 + \gamma) = H_2 + \kappa + \mathcal{O}_{r_1},$$

where κ is a function of harmonic oscillators:

$$\kappa = F(|z_1|^2, \dots, |z_s|^2, \tau, \hbar), \quad \text{with some } F \in \mathcal{C}^\infty(U)[\![I_1, \dots, I_s, \tau, \hbar]\!].$$

Moreover, if γ has real-valued coefficients, then so do ρ, κ and the remainder $\mathcal{O}_{r_1}$.

Proof. We prove this by induction on an integer $N \ge 3$. Assume that we found $\rho_{N-1}, K_3, \dots, K_{N-1} \in \mathcal{O}_3$, with $[K_i, |z_j|^2] = 0$ for every (i, j) and $K_i \in \mathcal{D}_i$ such that

$$e^{(i/\hbar)\operatorname{ad}_{\rho_{N-1}}}(H_2 + \gamma) = H_2 + K_3 + \cdots + K_{N-1} + \mathcal{O}_N.$$

Rewriting the remainder as $R_N + \mathcal{O}_{N+1}$, with $R_N \in \mathcal{D}_N$, we have

$$e^{(i/\hbar)\operatorname{ad}_{\rho_{N-1}}}(H_2 + \gamma) = H_2 + K_3 + \cdots + K_{N-1} + R_N + \mathcal{O}_{N+1}.$$

We are looking for a $\rho' \in \mathcal{O}_N$. For such a ρ' we apply $e^{(i/\hbar)\operatorname{ad}_{\rho'}}$:

$$e^{(i/\hbar)\operatorname{ad}_{\rho_{N-1}+\rho'}}(H_2 + \gamma) = e^{(i/\hbar)\operatorname{ad}_{\rho'}}(H_2 + K_3 + \cdots + K_{N-1} + R_N + \mathcal{O}_{N+1}).$$

Since $(i/\hbar)\,\mathrm{ad}_{\rho'} : \mathcal{O}_k \to \mathcal{O}_{k+N-2}$ we have

$$e^{(i/\hbar)\,\mathrm{ad}_{\rho_{N-1}+\rho'}}(H_2+\gamma) = H_2 + K_3 + \cdots + K_{N-1} + R_N + \frac{i}{\hbar}\,\mathrm{ad}_{\rho'}(H_2) + \mathcal{O}_{N+1}. \tag{3-5}$$

The new term $(i/\hbar)\,\mathrm{ad}_{\rho'}(H_2) = -(i/\hbar)\,\mathrm{ad}_{H_2}(\rho')$ can still be simplified. Indeed by (3-4),

$$\frac{i}{\hbar}\,\mathrm{ad}_{H_2}(\rho') = \frac{i}{\hbar}[\langle M(w,t)\tau,\tau\rangle,\rho'] + \sum_{j=1}^{s}\Big(\hat{\beta}_j\frac{i}{\hbar}[|z_j|^2,\rho'] + |z_j|^2\frac{i}{\hbar}[\hat{\beta}_j,\rho']\Big), \tag{3-6}$$

with

$$\frac{i}{\hbar}[\hat{\beta}_j,\rho'] = \sum_{i=1}^{s}\left(\frac{\partial\hat{\beta}_j}{\partial y_i}\frac{\partial\rho'}{\partial\eta_i} - \frac{\partial\hat{\beta}_j}{\partial\eta_i}\frac{\partial\rho'}{\partial y_i}\right) + \sum_{i=1}^{k}\frac{\partial\hat{\beta}_j}{\partial t_i}\frac{\partial\rho'}{\partial\tau_i} + \mathcal{O}_{N-1} = \mathcal{O}_{N-1},$$

because a derivation with respect to (y,η,t) does not decrease the degree. Similarly,

$$\frac{i}{\hbar}[\langle M(w,t)\tau,\tau\rangle,\rho'] = \sum_{j=1}^{k}\left(\langle\partial_{t_j}M(w,t)\tau,\tau\rangle\frac{\partial\rho'}{\partial\tau_j} - \frac{\partial\langle M(w,t)\tau,\tau\rangle}{\partial\tau_j}\frac{\partial\rho'}{\partial t_j}\right) + \mathcal{O}_{N+1} = \mathcal{O}_{N+1},$$

and thus (3-6) becomes

$$\frac{i}{\hbar}\,\mathrm{ad}_{H_2}(\rho') = \sum_{j=1}^{s}\Big(\hat{\beta}_j\frac{i}{\hbar}\,\mathrm{ad}_{|z_j|^2}(\rho')\Big) + \mathcal{O}_{N+1}.$$

Using this formula in (3-5) we get

$$e^{(i/\hbar)\,\mathrm{ad}_{\rho_{N-1}+\rho'}}(H_2+\gamma) = H_2 + K_3 + \cdots + K_{N-1} + R_N - \sum_{j=1}^{s}\hat{\beta}_j\frac{i}{\hbar}\,\mathrm{ad}_{|z_j|^2}(\rho') + \mathcal{O}_{N+1}.$$

Thus, we are looking for $K_N,\rho' \in \mathcal{D}_N$ such that

$$R_N = K_N + \sum_{j=1}^{s}\hat{\beta}_j\frac{i}{\hbar}\,\mathrm{ad}_{|z_j|^2}(\rho'),$$

with $[K_N,|z_j|^2] = 0$. By Lemma 3.2, we can solve this equation provided $N < r_1$, and this concludes the proof. Moreover, $(i/\hbar)\,\mathrm{ad}_{|z_j|^2}$ is a real endomorphism, so we can solve this equation on $\mathbb{R}$. □

3C. ***Quantizing the normal form.*** We now construct the normal form $\mathcal{N}_\hbar$, quantizing Theorems 1.4 and 3.3. We denote by $\mathcal{I}_\hbar^{(j)}$ the harmonic oscillator with respect to x_j, defined by

$$\mathcal{I}_\hbar^{(j)} = \mathrm{Op}_\hbar^w(\xi_j^2 + x_j^2) = -\hbar^2\frac{\partial^2}{\partial x_j^2} + x_j^2.$$

We prove the following theorem.

Theorem 3.4. *There exist*

(1) *a microlocally unitary operator* $\mathrm{U}_\hbar : \mathrm{L}^2(\mathbb{R}^d_{x,y,t}) \to \mathrm{L}^2(M)$ *quantizing a symplectomorphism* $\tilde{\Phi}_1 = \Phi_1 + \mathcal{O}((x,\xi,\tau)^2)$, *microlocally on* $U_1' \times U_1$,

(2) *a function* $f_1^\star : \mathbb{R}^{2s+2k}_{y,\eta,t,\tau} \times \mathbb{R}^s_I \times [0,1]$ *which is* $\mathcal{C}^\infty$ *with compact support such that*

$$f_1^\star(y,\eta,t,\tau,I,\hbar) \le C\big((|I|+\hbar)^2 + |\tau|(|I|+\hbar) + |\tau|^3\big),$$

(3) *an* $\hbar$*-pseudodifferential operator* $\mathcal{R}_\hbar$, *whose symbol is* $\mathcal{O}((x,\xi,\tau,\hbar^{1/2})^{r_1})$ *on* U_1',

such that

$$\mathrm{U}_\hbar^* \mathcal{L}_\hbar \mathrm{U}_\hbar = \mathcal{N}_\hbar + \mathcal{R}_\hbar,$$

with

$$\mathcal{N}_\hbar = \mathrm{Op}_\hbar^w \langle M(w,t)\tau, \tau\rangle + \sum_{j=1}^{s} \mathcal{I}_\hbar^{(j)} \,\mathrm{Op}_\hbar^w \,\hat{\beta}_j(w,t) + \mathrm{Op}_\hbar^w \, f_1^\star(y, \eta, t, \tau, \mathcal{I}_\hbar^{(j)}, \dots, \mathcal{I}_\hbar^{(s)}, \hbar).$$

Remark 3.5. $\mathrm{U}_\hbar$ is a Fourier integral operator quantizing the symplectomorphism $\tilde{\Phi}_1$; see [Martinez 2002; Zworski 2012]. In particular, if $\mathcal{A}_\hbar$ is a pseudodifferential operator on M with symbol $a_\hbar = a_0 + \mathcal{O}(\hbar^2)$, then $\mathrm{U}_\hbar^* \mathcal{A}_\hbar \mathrm{U}_\hbar$ is a pseudodifferential operator on $\mathbb{R}^d$ with symbol

$$\sigma_\hbar = a_0 \circ \tilde{\Phi}_1 + \mathcal{O}(\hbar^2) \quad \text{on } U_1'.$$

Remark 3.6. Due to the parameters (y, η, t, τ) in the formal normal form, an additional quantization is needed, hence the $\mathrm{Op}_\hbar^w \, f_1^\star$-term. It is a quantization with respect to (y, η, t, τ) of an operator-valued symbol $f_1^\star(y, \eta, t, \tau, \mathcal{I}_\hbar^{(1)}, \dots, \mathcal{I}_\hbar^{(s)})$. Actually, this operator symbol is simple since one can diagonalize it explicitly. Denoting by $h_{n_j}^j(x_j)$ the n_j-th eigenfunction of $\mathcal{I}_\hbar^{(j)}$, associated to the eigenvalue $(2n_j - 1)\hbar$, we have for all $n \in \mathbb{N}^s$

$$f_1^\star(y, \eta, t, \tau, \mathcal{I}_\hbar^{(1)}, \dots, \mathcal{I}_\hbar^{(s)}, \hbar) h_n(x) = f_1^\star(y, \eta, \tau, (2n-1)\hbar, \hbar) h_n(x),$$

where $h_n(x) = h_{n_1}^1(x_1) \cdots h_{n_s}^s(x_s)$. Thus the operator $\mathrm{Op}_\hbar^w \, f_1^\star$ satisfies, for $u \in \mathrm{L}^2(\mathbb{R}^{s+k}_{(y,t)})$,

$$(\mathrm{Op}_\hbar^w \, f_1^\star) u \otimes h_n = \big(\mathrm{Op}_\hbar^w \, f_1^\star(y, \eta, t, \tau, (2n-1)\hbar, \hbar) u\big) \otimes h_n.$$

Proof. In order to prove Theorem 3.4, we first quantize Theorem 1.4. Using the Egorov theorem, there exists a microlocally unitary operator $V_\hbar : \mathrm{L}^2(\mathbb{R}^d) \to \mathrm{L}^2(M)$ quantizing the symplectomorphism $\Phi_1 : U_1' \to U_1$. Thus,

$$V_\hbar^* \mathcal{L}_\hbar V_\hbar = \mathrm{Op}_\hbar^w(\sigma_\hbar)$$

for some symbol $\sigma_\hbar$ such that

$$\sigma_\hbar = \widehat{H} + \mathcal{O}(\hbar^2) \quad \text{on } U_1'.$$

Then we use the following lemma to quantize the formal normal form and conclude. □

Lemma 3.7. *There exists a bounded pseudodifferential operator $\mathcal{Q}_\hbar$ with compactly supported symbol such that*

$$e^{(i/\hbar)\mathcal{Q}_\hbar} \,\mathrm{Op}_\hbar^w(\sigma_\hbar) e^{-(i/\hbar)\mathcal{Q}_\hbar} = \mathcal{N}_\hbar + \mathcal{R}_\hbar,$$

where $\mathcal{N}_\hbar$ and $\mathcal{R}_\hbar$ satisfy the properties stated in Theorem 3.4.

Remark 3.8. As explained below, the principal symbol Q of $\mathcal{Q}_\hbar$ is $\mathcal{O}((x, \xi, \tau)^3)$. Thus, the symplectic flow φ_t associated to the Hamiltonian Q is $\varphi_t(x, \xi, \tau) = (x, \xi, \tau) + \mathcal{O}((x, \xi, \tau)^2)$. Moreover, the Egorov theorem implies that $e^{-(i/\hbar)\mathcal{Q}_\hbar}$ quantizes the symplectomorphism φ_1. Hence, $V_\hbar e^{-(i/\hbar)\mathcal{Q}_\hbar}$ quantizes the symplectomorphism $\tilde{\Phi}_1 = \Phi_1 \circ \varphi_1 = \Phi_1 + \mathcal{O}((x, \xi, \tau)^2)$.

Proof. The proof of this lemma follows the exact same lines as in the case $k=0$ [Morin 2022b, Theorem 4.1]. Let us recall the main arguments. The symbol $\sigma_\hbar$ is equal to $\widehat{H}+\mathcal{O}(\hbar^2)$ on U_1'. Thus, its associated formal series is $[\sigma_\hbar]=H_2+\gamma$ for some $\gamma\in\mathcal{O}_3$. Using the Birkhoff normal form algorithm (Theorem 3.3), we get $\kappa,\ \rho\in\mathcal{O}_3$ such that

$$e^{(i/\hbar)\,\mathrm{ad}_\rho}(H_2+\gamma)=H_2+\kappa+\mathcal{O}_{r_1}.$$

If $Q_\hbar$ is a smooth compactly supported symbol with Taylor series $[Q_\hbar]=\rho$, then by the Egorov theorem the operator

$$e^{i\hbar^{-1}\operatorname{Op}_\hbar^w Q_\hbar}\operatorname{Op}_\hbar^w(\sigma_\hbar)e^{-i\hbar^{-1}\operatorname{Op}_\hbar^w Q_\hbar} \tag{3-7}$$

has a symbol with Taylor series $H_2+\kappa+\mathcal{O}_{r_1}$. Since κ commutes with the oscillator $|z_j|^2$, it can be written as

$$\kappa=\sum_{2|\alpha|+|\alpha'|+2\ell\geq 3}c_{\alpha\alpha'\ell}(w,t)|z_1|^{2\alpha_1}\cdots|z_s|^{2\alpha_s}\tau_1^{\alpha_1'}\cdots\tau_k^{\alpha_k'}\hbar^\ell.$$

We can reorder this formal series using the monomials $(|z_j|^2)^{\star\alpha_j}=|z_j|^2\star\cdots\star|z_j|^2$:

$$\kappa=\sum_{2|\alpha|+|\alpha'|+2\ell\geq 3}c^\star_{\alpha\alpha'\ell}(w,t)(|z_1|^2)^{\star\alpha_1}\cdots(|z_s|^2)^{\star\alpha_s}\tau_1^{\alpha_1'}\cdots\tau_k^{\alpha_k'}\hbar^\ell.$$

If $f_1^\star$ is a smooth compactly supported function with Taylor series

$$[f_1^\star]=\sum_{2|\alpha|+|\alpha'|+2\ell\geq 3}c^\star_{\alpha\alpha'\ell}(w,t)I_1^{\alpha_1}\cdots I_s^{\alpha_s}\tau_1^{\alpha_1'}\cdots\tau_k^{\alpha_k'}\hbar^\ell,$$

then the operator (3-7) is equal to

$$\mathcal{N}_\hbar=\operatorname{Op}_\hbar^w H_2+\operatorname{Op}_\hbar^w f_1^\star(y,\eta,t,\tau,\mathcal{I}_\hbar^{(1)},\dots,\mathcal{I}_\hbar^{(s)},\hbar)$$

modulo $\mathcal{O}_{r_1}$. □

4. Comparing the spectra of $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar$

4A. ***Spectrum of*** $\mathcal{N}_\hbar$**.** In this section we describe the spectral properties of $\mathcal{N}_\hbar$. We can use the properties of harmonic oscillators to diagonalize it in the following way. For $1\leq j\leq s$ and $n_j\geq 1$, we recall that the n_j-th Hermite function $h^j_{n_j}(x_j)$ is an eigenfunction of $\mathcal{I}_\hbar^{(j)}$,

$$\mathcal{I}_\hbar^{(j)}h^j_{n_j}=\hbar(2n_j-1)h^j_{n_j},$$

and the functions $(h_n)_{n\in\mathbb{N}^s}$ defined by

$$h_n(x)=h^1_{n_1}\otimes\cdots\otimes h^s_{n_s}(x)=h^1_{n_1}(x_1)\cdots h^s_{n_s}(x_s)$$

form a Hilbertian basis of $\mathrm{L}^2(\mathbb{R}^s_x)$. Thus, we can use this basis to decompose the space $\mathrm{L}^2(\mathbb{R}^{2s+k}_{x,y,t})$ on which $\mathcal{N}_\hbar$ acts:

$$\mathrm{L}^2(\mathbb{R}^{2s+k})=\bigoplus_{n\in\mathbb{N}^s}(\mathrm{L}^2(\mathbb{R}^{s+k}_{y,t})\otimes h_n).$$

$\mathcal{N}_\hbar$ preserves this decomposition and

$$\mathcal{N}_\hbar = \bigoplus_{n\in\mathbb{N}^s} \mathcal{N}_\hbar^{[n]},$$

where $\mathcal{N}_\hbar^{[n]}$ is the pseudodifferential operator with symbol

$$N_\hbar^{[n]} = \langle M(w,t)\tau,\tau\rangle + \sum_{j=1}^{s} \hat{\beta}_j(w,t)(2n_j+1)\hbar + f_1^\star(w,t,\tau,(2n-1)\hbar,\hbar). \tag{4-1}$$

In particular, the spectrum of $\mathcal{N}_\hbar$ is given by

$$\operatorname{sp}(\mathcal{N}_\hbar) = \bigcup_{n\in\mathbb{N}^s} \operatorname{sp}(\mathcal{N}_\hbar^{[n]}).$$

Moreover, as in the $k=0$ case, for any $b_1>0$ there is an $N_{\max}>0$ (independent of $\hbar$) such that

$$\operatorname{sp}(\mathcal{N}_\hbar)\cap(-\infty,b_1\hbar) = \bigcup_{|n|\le N_{\max}} \operatorname{sp}(\mathcal{N}_\hbar^{[n]})\cap(-\infty,b_1\hbar).$$

The reason is that the symbol $N_\hbar^{[n]}$ is greater than $b_1\hbar$ for n large enough. Finally, to prove our main result, Theorem 1.7, it remains to compare the spectra of $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar$.

4B. ***Microlocalization of the eigenfunctions.*** Here we prove microlocalization results for the eigenfunctions of $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar$. These results are needed to show that the remainders $\mathcal{O}((x,\xi,\tau)^{r_1})$ we got are small. More precisely, for each operator we need to prove that the eigenfunctions are microlocalized

- inside Ω (space localization),
- where $|(x,\xi,\tau)| \lesssim \hbar^\delta$ for $\delta\in\left(0,\frac{1}{2}\right)$ (i.e., close to Σ).

Fix $\tilde{b}_1$ such that

$$K_{\tilde{b}_1} = \{q\in M : b(q)\le \tilde{b}_1\} \Subset \Omega.$$

Lemma 4.1 (space localization for $\mathcal{L}_\hbar$). *Let $b_1\in(b_0,\tilde{b}_1)$ and $\chi_0\in\mathcal{C}_0^\infty(M)$ be a cutoff function such that $\chi_0=1$ on $K_{\tilde{b}_1}$. Then every normalized eigenfunction $\psi_\hbar$ of $\mathcal{L}_\hbar$ associated with an eigenvalue $\lambda_\hbar\le b_1\hbar$ satisfies*

$$\psi_\hbar = \chi_0\psi_\hbar + \mathcal{O}(\hbar^\infty),$$

where the $\mathcal{O}(\hbar^\infty)$ is independent of $(\lambda_\hbar,\psi_\hbar)$.

Proof. This follows from the Agmon estimates,

$$\|e^{d(q,K_{\tilde{b}_1})\hbar^{-1/4}}\psi_\hbar\| \le C\|\psi_\hbar\|^2, \tag{4-2}$$

as in the $k=0$ case (in [Morin 2022b]). Indeed, from (4-2) we deduce

$$\|(1-\chi_0)\psi\| \le Ce^{-\varepsilon\hbar^{-1/4}}\|\psi_\hbar\|,$$

as soon as $\chi_0=1$ on an ε-neighborhood of $K_{\tilde{b}_1}$. $\square$

Lemma 4.2 (microlocalization near Σ for $\mathcal{L}_\hbar$). *Let $\delta \in \left(0, \frac{1}{2}\right)$, $b_1 \in (b_0, \tilde{b}_1)$ and $\chi_1 \in \mathcal{C}^\infty(T^*M)$ be a cutoff function equal to* 1 *on a neighborhood of* Σ. *Then every eigenfunction $\psi_\hbar$ of $\mathcal{L}_\hbar$ associated with an eigenvalue $\lambda_\hbar \leq b_1\hbar$ satisfies*

$$\psi_\hbar = \mathrm{Op}_\hbar^w\, \chi_1(\hbar^{-\delta}(q, p))\psi_\hbar + \mathcal{O}(\hbar^\infty)\psi_\hbar,$$

where the $\mathcal{O}(\hbar^\infty)$ is in the space of bounded operators $\mathcal{L}(\mathrm{L}^2, \mathrm{L}^2)$ and independent of $(\lambda_\hbar, \psi_\hbar)$.

Proof. Let $g_\hbar \in \mathcal{C}_0^\infty(\mathbb{R})$ be such that

$$g_\hbar(\lambda) = \begin{cases} 1 & \text{if } \lambda \leq b_1\hbar, \\ 0 & \text{if } \lambda \geq \tilde{b}_1\hbar. \end{cases}$$

Then the eigenfunction $\psi_\hbar$ satisfies

$$\psi_\hbar = g_\hbar(\lambda_\hbar)\psi_\hbar = g_\hbar(\mathcal{L}_\hbar)\psi_\hbar.$$

With the notation $\chi = 1 - \chi_1$, we will prove that

$$\|\mathrm{Op}_\hbar^w\, \chi(\hbar^{-\delta}(q, p))g_\hbar(\mathcal{L}_\hbar)\|_{\mathcal{L}(\mathrm{L}^2,\mathrm{L}^2)} = \mathcal{O}(\hbar^\infty), \tag{4-3}$$

from which will follow $\psi_\hbar = \mathrm{Op}_\hbar^w\, \chi_1(\hbar^{-\delta}(q, p))\psi_\hbar + \mathcal{O}(\hbar^\infty)\psi_\hbar$, uniformly with respect to $(\lambda_\hbar, \psi_\hbar)$.

To lighten the notation, we define $\chi^w := \mathrm{Op}_\hbar^w\, \chi(\hbar^{-\delta}(q, p))$. For every $\psi \in \mathrm{L}^2(M)$ we define $\varphi = g_\hbar(\mathcal{L}_\hbar)\psi$. Then,

$$\langle \mathcal{L}_\hbar \chi^w\varphi, \chi^w\varphi\rangle = \langle \chi^w\mathcal{L}_\hbar\varphi, \chi^w\varphi\rangle + \langle [\mathcal{L}_\hbar, \chi^w]\varphi, \chi^w\varphi\rangle. \tag{4-4}$$

We will bound from above the right-hand side, and from below the left-hand side. First, since $g_\hbar(\lambda)$ is supported where $\lambda \leq \tilde{b}_1\hbar$, we have

$$\langle \chi^w\mathcal{L}_\hbar\varphi, \chi^w\varphi\rangle \leq \tilde{b}_1\hbar\|\chi^w\varphi\|^2. \tag{4-5}$$

Moreover, the commutator $[\mathcal{L}_\hbar, \chi^w]$ is a pseudodifferential operator of order $\hbar$, with symbol supported on $\operatorname{supp}\chi$. Hence, if $\underline{\chi}$ is a cutoff function having the same general properties of χ, such that $\underline{\chi} = 1$ on $\operatorname{supp}\chi$, we have

$$\langle [\mathcal{L}_\hbar, \chi^w]\varphi, \chi^w\varphi\rangle \leq C\hbar\|\underline{\chi}^w\varphi\|\,\|\chi^w\varphi\|. \tag{4-6}$$

Finally, the symbol of χ^w is equal to 0 on an $\hbar^\delta$-neighborhood of Σ, and thus the symbol $|p - A(q)|^2$ of $\mathcal{L}_\hbar$ is $\geq c\hbar^{2\delta}$ on the support of χ^w. Hence the Gårding inequality yields

$$\langle \mathcal{L}_\hbar \chi^w\varphi, \chi^w\varphi\rangle \geq c\hbar^{2\delta}\|\chi^w\varphi\|^2. \tag{4-7}$$

Using this last inequality in (4-4), and bounding the right-hand side with (4-5) and (4-6) we find

$$c\hbar^{2\delta}\|\chi^w\varphi\|^2 \leq \tilde{b}_1\hbar\|\chi^w\varphi\|^2 + C\hbar\|\underline{\chi}^w\varphi\|\,\|\chi^w\varphi\|,$$

and we deduce that

$$\|\chi^w\varphi\| \leq C\hbar^{1-2\delta}\|\underline{\chi}^w\varphi\|.$$

Iterating with $\underline{\chi}$ instead of χ, we finally get, for arbitrarily large $N > 0$,

$$\|\chi^w\varphi\| \leq C_N\hbar^N\|\varphi\|.$$

This is true for every ψ, with $\varphi = g_\hbar(\mathcal{L}_\hbar)\psi$, and thus $\|\chi^w g_\hbar(\mathcal{L}_\hbar)\| = \mathcal{O}(\hbar^\infty)$. ☐

Lemma 4.3 (microlocalization near Σ for $\mathcal{N}_\hbar$). *Let $\delta \in \left(0, \frac{1}{2}\right)$, $b_1 \in (b_0, \tilde{b}_1)$ and $\chi_1 \in \mathcal{C}_0^\infty(\mathbb{R}^{2s+k}_{x,\xi,\tau})$ be a cutoff function equal to* 1 *on a neighborhood of* 0. *Then every eigenfunction $\psi_\hbar$ of $\mathcal{N}_\hbar$ associated with an eigenvalue $\lambda_\hbar \leq b_1\hbar$ satisfies*

$$\psi_\hbar = \mathrm{Op}_\hbar^w\, \chi_1(\hbar^{-\delta}(x, \xi, \tau)) + \mathcal{O}(\hbar^\infty)\psi_\hbar,$$

where the $\mathcal{O}(\hbar^\infty)$ is in $\mathcal{L}(\mathrm{L}^2, \mathrm{L}^2)$ and independent of $(\lambda_\hbar, \psi_\hbar)$.

Proof. Just as in the previous lemma, it is enough to show that

$$\|\chi^w g_\hbar(\mathcal{N}_\hbar)\| = \mathcal{O}(\hbar^\infty),$$

where $\chi^w = \mathrm{Op}_\hbar^w(1 - \chi_1(\hbar^{-\delta}(x, \xi, \tau)))$. We prove this using the same method. If $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ and $\varphi = g_\hbar(\mathcal{N}_\hbar)\psi$,

$$\langle \mathcal{N}_\hbar \chi^w\varphi, \chi^w\varphi\rangle = \langle \chi^w\mathcal{N}_\hbar\varphi, \chi^w\varphi\rangle + \langle [\mathcal{N}_\hbar, \chi^w]\varphi, \chi^w\varphi\rangle. \tag{4-8}$$

The right-hand side can be bounded from above as before. On the left-hand side we find $\varepsilon > 0$ such that

$$\langle \mathcal{N}_\hbar \chi^w\varphi, \chi^w\varphi\rangle \geq (1-\varepsilon)\langle \mathcal{H}_2\chi^w\varphi, \chi^w\varphi\rangle, \tag{4-9}$$

with $\mathcal{H}_2 = \mathrm{Op}_\hbar^w\left(\langle M(w,t)\tau, \tau\rangle + \sum \hat{\beta}_j(w,t)|z_j|^2\right)$. The symbol of χ^w vanishes on an $\hbar^\delta$-neighborhood of $x = \xi = \tau = 0$. Thus we can bound from below the symbol of $\mathcal{H}_2$ and use the Gårding inequality:

$$\langle \mathcal{H}_2\chi^w\varphi, \chi^w\varphi\rangle \geq c\hbar^{2\delta}\|\chi^w\varphi\|^2.$$

We conclude the proof as in Lemma 4.2. □

Lemma 4.4 (space localization for $\mathcal{N}_\hbar$). *Let $b_1 \in (b_0, \tilde{b}_1)$ and $\chi_0 \in \mathcal{C}_0^\infty(\mathbb{R}^{2s+k}_{y,\eta,t})$ be a cutoff function equal to* 1 *on a neighborhood of $\{\hat{b}(y, \eta, t) \leq \tilde{b}_1\}$. Then every eigenfunction $\psi_\hbar$ of $\mathcal{N}_\hbar$ associated with an eigenvalue $\lambda_\hbar \leq b_1\hbar$ satisfies*

$$\psi_\hbar = \mathrm{Op}_\hbar^w\, \chi_0(w, t)\psi_\hbar + \mathcal{O}(\hbar^\infty)\psi_\hbar,$$

where the $\mathcal{O}(\hbar^\infty)$ is in $\mathcal{L}(\mathrm{L}^2, \mathrm{L}^2)$ and independent of $(\lambda_\hbar, \psi_\hbar)$.

Proof. Every eigenfunction of $\mathcal{N}_\hbar$ is given by $\psi_\hbar(x, y, t) = u_\hbar(y, t)h_n(x)$ for some Hermite function h_n with $|n| \leq N_{\max}$ and some eigenfunction $u_\hbar$ of $\mathcal{N}_\hbar^{[n]}$. Thus, it is enough to prove the lemma for the eigenfunctions of $\mathcal{N}_\hbar^{[n]}$. If $u_\hbar$ is such an eigenfunction, associated with an eigenvalue $\lambda_\hbar \leq b_1\hbar$, then

$$u_\hbar = g_\hbar(\mathcal{N}_\hbar^{[n]})u_\hbar.$$

We will prove that $\|\chi^w g_\hbar(\mathcal{N}_\hbar^{[n]})\| = \mathcal{O}(\hbar^\infty)$, with $\chi^w = \mathrm{Op}_\hbar^w(1-\chi_0)$, which is enough to conclude. If $u \in \mathrm{L}^2(\mathbb{R}^{k+s}_{y,t})$ and $\varphi = g_\hbar(\mathcal{N}_\hbar^{[n]})u$, then

$$\langle \mathcal{N}_\hbar^{[n]}\chi^w\varphi, \chi^w\varphi\rangle = \langle \chi^w\mathcal{N}_\hbar^{[n]}\varphi, \chi^w\varphi\rangle + \langle [\mathcal{N}_\hbar^{[n]}, \chi^w]\varphi, \chi^w\varphi\rangle. \tag{4-10}$$

We first have the bound

$$\langle \chi^w\mathcal{N}_\hbar^{[n]}\varphi, \chi^w\varphi\rangle \leq \tilde{b}_1\hbar\|\chi^w\varphi\|^2. \tag{4-11}$$

The commutator $[\mathcal{N}_\hbar^{[n]}, \chi^w]$ is a pseudodifferential operator of order $\hbar$ with symbol supported on $\operatorname{supp}\chi$. Moreover, its principal symbol is $\{N_\hbar^{[n]}, \chi\}$. From the definition of $N_\hbar^{[n]}$ we deduce

$$\langle[\mathcal{N}_\hbar^{[n]}, \chi^w]\varphi, \chi^w\varphi\rangle \leq C\hbar\langle\underline{\chi}^w|\tau|^w\varphi, \chi^w\varphi\rangle,$$

where $\underline{\chi}$ has the same general properties as χ, and is equal to 1 on $\operatorname{supp}\chi$. By Lemma 4.3, we can find a cutoff where $|\tau| \lesssim \hbar^\delta$ and we get

$$\langle[\mathcal{N}_\hbar^{[n]}, \chi^w]\varphi, \chi^w\varphi\rangle \leq C\hbar^{1+\delta}\|\underline{\chi}^w\varphi\|\,\|\chi^w\varphi\|. \tag{4-12}$$

Finally for $\varepsilon > 0$ small enough we have the lower bound

$$\langle\mathcal{N}_\hbar^{[n]}\chi^w\varphi, \chi^w\varphi\rangle \geq \hbar(\tilde{b}_1+\varepsilon)\|\chi^w\varphi\|^2,$$

because $N_\hbar^{[n]}(w,t) \geq \hbar\hat{b}(w,t)$ and χ vanishes on a neighborhood of $\{\hat{b}(w,t) \leq \tilde{b}_1\}$. Using this lower bound in (4-10), and bounding the right-hand side with (4-11) and (4-12) we get

$$\hbar(\tilde{b}_1+\varepsilon)\|\chi^w\varphi\|^2 \leq \hbar\tilde{b}_1\|\chi^w\varphi\|^2 + C\hbar^{1+\delta}\|\underline{\chi}^w\varphi\|\,\|\chi^w\varphi\|. \tag{4-13}$$

Thus

$$\varepsilon\|\chi^w\varphi\| \leq C\hbar^\delta\|\underline{\chi}^w\varphi\|,$$

and we can iterate with $\underline{\chi}$ instead of χ to conclude. □

4C. *Proof of Theorem 1.7.* To conclude the proof of Theorem 1.7, it remains to show that

$$\lambda_n(\mathcal{L}_\hbar) = \lambda_n(\mathcal{N}_\hbar) + \mathcal{O}(\hbar^{r_1/2-\varepsilon})$$

uniformly with respect to $n \in [1, N_\hbar^{\max}]$ with

$$N_\hbar^{\max} = \max\{n \in \mathbb{N} : \lambda_n(\mathcal{L}_\hbar) \leq b_1\hbar\}.$$

Here $\lambda_n(\mathcal{A})$ denotes the n-th eigenvalue of the self-adjoint operator $\mathcal{A}$, repeated with multiplicities.

Lemma 4.5. *One has*

$$\lambda_n(\mathcal{L}_\hbar) = \lambda_n(\mathcal{N}_\hbar) + \mathcal{O}(\hbar^{r_1/2-\varepsilon})$$

uniformly with respect to $n \in [1, N_\hbar^{\max}]$.

Proof. Let us focus on the "$\leq$" inequality. For $n \in [1, N_\hbar^{\max}]$, denote by $\psi_n^\hbar$ the normalized eigenfunction of $\mathcal{N}_\hbar$ associated with $\lambda_n(\mathcal{N}_\hbar)$, and

$$\varphi_n^\hbar = \mathrm{U}_\hbar\psi_n^\hbar,$$

where $\mathrm{U}_\hbar$ is given by Theorem 3.4. We will use $\varphi_n^\hbar$ as quasimode for $\mathcal{L}_\hbar$. Let $N \in [1, N_\hbar^{\max}]$ and

$$V_N^\hbar = \operatorname{span}\{\varphi_n^\hbar : 1 \leq n \leq N\}.$$

For $\varphi \in V_N^\hbar$ we use the notation $\psi = \mathrm{U}_\hbar^{-1}\varphi$. By Theorem 3.4, we have

$$\langle\mathcal{L}_\hbar\varphi, \varphi\rangle = \langle\mathcal{N}_\hbar\psi, \psi\rangle + \langle\mathcal{R}_\hbar\psi, \psi\rangle \leq \lambda_N(\mathcal{N}_\hbar)\|\psi\|^2 + \langle\mathcal{R}_\hbar\psi, \psi\rangle. \tag{4-14}$$

According to Lemmas 4.3 and 4.4, ψ is microlocalized, where $(w,t) \in \{\hat{b}(w,t) \le \tilde{b}_1\} \subset U$ and $|(x,\xi,\tau)| \le \hbar^{\delta}$. But the symbol of $\mathcal{R}_\hbar$ is such that $R_\hbar = \mathcal{O}((x,\xi,\tau,\hbar^{1/2})^{r_1})$ for $(w,t) \in U$, so

$$\langle \mathcal{R}_\hbar \psi, \psi \rangle = \mathcal{O}(\hbar^{\delta r_1}) = \mathcal{O}(\hbar^{r_1/2-\varepsilon}) \tag{4-15}$$

for suitable $\delta \in \left(0, \frac{1}{2}\right)$. By (4-14) and (4-15) we have

$$\langle \mathcal{L}_\hbar \varphi, \varphi \rangle \le (\lambda_N(\mathcal{N}_\hbar) + C\hbar^{r_1/2-\varepsilon}) \|\varphi\|^2 \quad \text{for all } \varphi \in V_N^\hbar.$$

Since $V_N^\hbar$ is N-dimensional, the minimax principle implies that

$$\lambda_N(\mathcal{L}_\hbar) \le \lambda_N(\mathcal{N}_\hbar) + C\hbar^{r_1/2-\varepsilon}. \tag{4-16}$$

The reversed inequality is proved in the same way: we take the eigenfunctions of $\mathcal{L}_\hbar$ as quasimodes for $\mathcal{N}_\hbar$, and we use the microlocalization lemma, Lemma 4.2. □

5. A second normal form in the case $k > 0$

In the previous sections, we compared the spectrum of $\mathcal{L}_\hbar$ and the spectrum of the normal form $\mathcal{N}_\hbar$. Moreover, if $b_1 > b_0$ is sufficiently close to b_0 the spectrum of $\mathcal{N}_\hbar$ in $(-\infty, b_1\hbar)$ is given by the spectrum of $\mathcal{N}_\hbar^{[1]}$, an $\hbar$-pseudodifferential operator on $\mathbb{R}^{s+k}_{(y,t)}$ with symbol

$$N_\hbar^{[1]} = \langle M(y,\eta,t)\tau, \tau \rangle + \hbar \hat{b}(y,\eta,t) + f_1^\star(y,\eta,t,\tau,\hbar). \tag{5-1}$$

In this section, we will construct a Birkhoff normal form again, to describe the spectrum of $\mathcal{N}_\hbar^{[1]}$ by an effective operator $\mathcal{M}_\hbar$ on $\mathbb{R}^s_y$. For that purpose, in Section 5A we will find new canonical variables $(\hat{t}, \hat{\tau})$ in which $N_\hbar^{[1]}$ is the perturbation of a harmonic oscillator. In Sections 5B and 5C we will construct the semiclassical Birkhoff normal form $\mathcal{M}_\hbar$. In Section 5D we will prove that the spectrum of $\mathcal{N}_\hbar^{[1]}$ is given by the spectrum of $\mathcal{M}_\hbar$.

Under Assumption 1 we know that $t \mapsto \hat{b}(w,t)$ admits a nondegenerate minimum at $s(w)$ for w in a neighborhood of 0, and we denote by $(\nu_1^2(w), \dots, \nu_k^2(w))$ the eigenvalues of the positive symmetric matrix

$$M(w, s(w))^{1/2} \cdot \tfrac{1}{2}\partial_t^2 \hat{b}(w, s(w)) \cdot M(w, s(w))^{1/2}.$$

The maps $\nu_1, \dots, \nu_k$ are smooth nonvanishing functions in a neighborhood of $w = 0$.

5A. *Geometry of the symbol* $\mathcal{N}_\hbar^{[1]}$.

We prove the following lemma.

Lemma 5.1. *There exists a canonical (symplectic) transformation $\Phi_2 : U_2 \to V_2$ between neighborhoods U_2, V_2 of $0 \in \mathbb{R}^{2s+2k}_{(y,\eta,t,\tau)}$ such that*

$$\widehat{N}_\hbar := N_\hbar^{[1]} \circ \Phi_2 = \hbar \hat{b}(w, s(w)) + \sum_{j=1}^{k} \nu_j(w)(\tau_j^2 + \hbar t_j^2) + \mathcal{O}(|t|^3|\tau|^2 + |t|^3\hbar + \hbar^2 + \hbar|\tau| + |\tau|^3 + |t||\tau|^2).$$

Proof. We want to expand $\mathcal{N}_\hbar^{[1]}$ near its minimum with respect to the variables $v = (t,\tau)$. First, from the Taylor expansion of $f_1^\star$ we deduce

$$N_\hbar^{[1]} = \langle M(w,t)\tau, \tau \rangle + \hbar \hat{b}(w,t) + \mathcal{O}(\hbar^2 + \tau\hbar + \tau^3).$$

We will Taylor-expand $t \mapsto \hat{b}(w,t)$ on a neighborhood of its minimum point $s(w)$. For that purpose, we define new variables $(\tilde{y}, \tilde{\eta}, \tilde{t}, \tilde{\tau}) = \tilde{\varphi}(y, \eta, t, \tau)$ by

$$\begin{cases} \tilde{y} = y - \sum_{j=1}^{k} \tau_j \nabla_\eta s_j(y,\eta), \\ \tilde{\eta} = \eta + \sum_{j=1}^{k} \tau_j \nabla_y s_j(y,\eta), \\ \tilde{t} = t - s(y,\eta), \\ \tilde{\tau} = \tau. \end{cases}$$

Then $\tilde{\varphi}^*\omega_0 = \omega_0 + \mathcal{O}(\tau)$. Using Theorem B.2, we can make $\tilde{\varphi}$ symplectic on a neighborhood of 0, up to a change of order $\mathcal{O}(\tau^2)$. In these new variables, the symbol $\widetilde{N}_\hbar := N_\hbar^{[1]} \circ \tilde{\varphi}^{-1}$ is

$$\begin{aligned} \widetilde{N}_\hbar &= \langle M[\tilde{w} + \mathcal{O}(\tilde{\tau}), \tilde{t} + s(\tilde{w} + \mathcal{O}(\tilde{\tau}))]\tilde{\tau}, \tilde{\tau}\rangle + \hbar\hat{b}[\tilde{y} + \mathcal{O}(\tilde{\tau}), \tilde{\eta} + \mathcal{O}(\tilde{\tau}), s(\tilde{y}, \tilde{\eta}) + \tilde{t} + \mathcal{O}(\tilde{\tau})] \\ &\qquad + \mathcal{O}(\hbar^2 + \hbar\tilde{\tau} + \tilde{\tau}^3) \\ &= \langle M(\tilde{w}, \tilde{t} + s(\tilde{w}))\tilde{\tau}, \tilde{\tau}\rangle + \hbar\hat{b}[\tilde{y}, \tilde{\eta}, s(\tilde{y}, \tilde{\eta}) + \tilde{t}] + \mathcal{O}(\hbar^2 + \hbar\tilde{\tau} + \tilde{\tau}^3). \end{aligned}$$

Then we remove the tildes and expand this symbol in powers of $t, \tau, \hbar$. We find

$$\widetilde{N}_\hbar = \langle M(w, s(w))\tau, \tau\rangle + \hbar\hat{b}(w, s(w)) + \frac{\hbar}{2}\langle \partial_t^2 \hat{b}(w, s(w))t, t\rangle + \mathcal{O}(|t|^3\hbar + \hbar^2 + \hbar|\tau| + |\tau|^3 + |t|\,|\tau|^2).$$

Now, we want to diagonalize the positive quadratic forms $M(w, s(w))$ and $\frac{1}{2}\partial_t^2\hat{b}[w, s(w)]$. The diagonalization of quadratic forms in orthonormal coordinates implies that there exists a matrix $P(w)$ such that

$${}^tP\ M^{-1}P = I \quad \text{and} \quad {}^tP\ \tfrac{1}{2}\partial_t^2\hat{b}\ P = \operatorname{diag}(\nu_1^2, \dots, \nu_k^2).$$

We define the new coordinates $(\check{y}, \check{\eta}, \check{t}, \check{\tau}) = \check{\varphi}(y, \eta, t, \tau)$ by

$$\begin{cases} \check{t} = P(w)^{-1}t, \\ \check{\tau} = {}^tP(w)\tau, \\ \check{y} = y + {}^t[\nabla_\eta(P^{-1}t)] \cdot {}^tP\tau, \\ \check{\eta} = \eta - {}^t[\nabla_y(P^{-1}t)] \cdot {}^tP\tau, \end{cases}$$

so that $\check{\varphi}^*\omega_0 - \omega_0 = \mathcal{O}(|t|^2 + |\tau|)$. Again, we can make it symplectic up to a change of order $\mathcal{O}(|t|^3 + |\tau|^2)$ by Theorem B.2. In these new variables, the symbol becomes (after removing the "checks")

$$\check{N}_\hbar = \hbar\hat{b}(w, s(w)) + \sum_{j=1}^{k}(\tau_j^2 + \hbar\nu_j(w)^2 t_j^2) + \mathcal{O}\big(|t|^3|\tau|^2 + |t|^3\hbar + \hbar^2 + \hbar|\tau| + |\tau|^3 + |t|\,|\tau|^2\big).$$

The last change of coordinates $(\hat{y}, \hat{\eta}, \hat{t}, \hat{\tau}) = \hat{\varphi}(y, \eta, t, \tau)$, defined by

$$\begin{cases} \hat{t}_j = \nu_j(w)^{1/2}t_j, \\ \hat{\tau}_j = \nu_j(w)^{-1/2}\tau_j, \\ \hat{y}_j = y_j + \sum_{i=1}^{k} \nu_i^{-1/2}\tau_i\partial_{\eta_j}\nu_i^{1/2}t_i, \\ \hat{\eta} = \eta - \sum_{i=1}^{k} \nu_i^{-1/2}\tau_i\partial_{y_j}\nu_i^{1/2}t_i, \end{cases}$$

is such that $\hat{\varphi}^*\omega_0 = \omega_0 + \mathcal{O}(\tau)$, so it can be corrected modulo $\mathcal{O}(|\tau|^2)$ to be symplectic, and we get the new symbol

$$\widehat{N}_\hbar = \hbar\hat{b}(w, s(w)) + \sum_{j=1}^{k} \nu_j(w)(\tau_j^2 + \hbar t_j^2) + \mathcal{O}\big(|t|^3|\tau|^2 + |t|^3\hbar + \hbar^2 + \hbar|\tau| + |\tau|^3 + |t|\,|\tau|^2\big),$$

which concludes the proof. □

5B. ***Second formal normal form.*** The harmonic oscillators appearing in $\widehat{N}_\hbar$ are

$$\mathcal{J}_\hbar^{(j)} = \mathrm{Op}_\hbar^w(\hbar^{-1}\tau_j^2 + t_j^2), \quad 1 \le j \le k.$$

If we define

$$h = \sqrt{\hbar},$$

the symbol of $\mathcal{J}_\hbar^{(j)}$ for the h-quantization is $\tilde{\tau}_j^2 + t_j^2$. This is why we use the mixed quantization

$$\mathrm{Op}_\sharp^w(\mathrm{a})u(y_0, t_0) = \frac{1}{(2\pi\hbar)^{n-k}(2\pi\sqrt{\hbar})^k} \int e^{(i/\hbar)\langle y_0 - y, \eta\rangle} e^{(i/\sqrt{\hbar})\langle t_0 - t, \tilde{\tau}\rangle} \mathrm{a}(\sqrt{\hbar}, y, \eta, t, \tilde{\tau})\,\mathrm{d}y\,\mathrm{d}\eta\,\mathrm{d}t\,\mathrm{d}\tilde{\tau}. \tag{5-2}$$

It is related to the $\hbar$-quantization by the relation

$$\tau = h\tilde{\tau}, \quad h = \sqrt{\hbar}.$$

In other words, if a is a symbol in some standard class $S(m)$, and if we define

$$\mathrm{a}(h, y, \eta, t, \tilde{\tau}) = a(h^2, y, \eta, t, h\tilde{\tau}),$$

then we have

$$\mathrm{Op}_\sharp^w(\mathrm{a}) = \mathrm{Op}_\hbar^w(a).$$

However, if we take $\mathrm{a} \in S(m)$, then $\mathrm{Op}_\sharp^w(\mathrm{a})$ is not necessarily an $\hbar$-pseudodifferential operator, since the associated a may not be bounded with respect to $\hbar$, and thus it does not belong to any standard class. For instance, we have

$$\partial_\tau a = \frac{1}{\sqrt{\hbar}} \partial_{\tilde{\tau}} \mathrm{a}.$$

But still $\mathrm{Op}_\sharp^w(\mathrm{a})$ is an h-pseudodifferential operator, with symbol

$$\mathfrak{a}(h, y, \tilde{\eta}, t, \tilde{\tau}) = \mathrm{a}(h, y, h\tilde{\eta}, t, \tilde{\tau}).$$

With this notation

$$\mathrm{Op}_\sharp^w(\mathrm{a}) = \mathrm{Op}_h^w(\mathfrak{a}).$$

Thus, in this sense, we can use the properties of $\hbar$-pseudodifferential and h-pseudodifferential operators to deal with our mixed quantization.

Remark 5.2. Operators of the form (5-2) are just special cases of the usual h-pseudodifferential operators for which the reader can refer to [Martinez 2002; Zworski 2012]. Moreover, our mixed quantization could be interpreted as a $\sqrt{\hbar}$-quantization with operator-valued symbols for which we refer to [Keraval 2018; Martinez 2007]. Indeed we can write

$$\mathrm{Op}_\sharp^w(\mathrm{a}) = \mathrm{Op}_h^w(\mathrm{Op}_\hbar^w\,\mathrm{a}), \tag{5-3}$$

where we first quantize with respect to (y, η) so that $\mathrm{Op}_\hbar^w\,\mathrm{a}$ is an operator-valued symbol which depends on $(t, \tilde{\tau})$. In the following we could have used this formalism, thus dealing with operator-valued symbols in $(t, \tilde{\tau})$ instead of real-valued symbols and mixed quantization.

In our case, we have

$$\mathrm{Op}^w_\sharp(\mathrm{N}_h) = \mathrm{Op}^w_h(\widehat{N}_\hbar),$$

with

$$\mathrm{N}_h = h^2 \hat{b}(w, s(w)) + h^2 \sum_{j=1}^{k} \nu_j(w)(\tilde{\tau}_j^2 + t_j^2) + \mathcal{O}(h^2|t|^3 + h^4 + h^3|\tilde{\tau}| + h^2|t||\tilde{\tau}|^2).$$

Let us construct a semiclassical Birkhoff normal form with respect to this quantization. We will work in the space of formal series

$$\mathcal{E}_2 := \mathcal{C}^\infty(U)[\![t, \tilde{\tau}, h]\!], \tag{5-4}$$

where $U = U_2 \cap \mathbb{R}^{2s}_w \times \{0\}$. This space is endowed with the star product $\star$ adapted to our mixed quantization. In other words

$$\mathrm{Op}^w_\sharp(\mathrm{a} \star \mathrm{b}) = \mathrm{Op}^w_\sharp(\mathrm{a})\, \mathrm{Op}^w_\sharp(\mathrm{b}).$$

The change of variable $\tau = h\tilde{\tau}$ between the usual $\hbar$-quantization and our mixed quantization yields the following formula for the star product:

$$\mathrm{a} \star \mathrm{b} = \sum_{k \geq 0} \frac{1}{k!} \Big(\frac{h}{2i}\Big)^k A_h(\partial)^k (\mathrm{a}(h, y_1, \eta_1, t_1, \tilde{\tau}_1) \mathrm{b}(h, y_2, \eta_2, t_2, \tilde{\tau}_2))_{|(t_1,\tau_1,y_1,\eta_1)=(t_2,\tau_2,y_2,\eta_2)}, \tag{5-5}$$

with

$$A_h(\partial) = \sum_{j=1}^{k} \frac{\partial}{\partial t_{1j}} \frac{\partial}{\partial \tilde{\tau}_{2j}} - \frac{\partial}{\partial t_{2j}} \frac{\partial}{\partial \tilde{\tau}_{1j}} + h \sum_{j=1}^{s} \frac{\partial}{\partial y_{1j}} \frac{\partial}{\partial \eta_{2j}} - \frac{\partial}{\partial y_{2j}} \frac{\partial}{\partial \eta_{1j}}.$$

The degree function on $\mathcal{E}_2$ is defined by

$$\deg(t^{\alpha_1} \tilde{\tau}^{\alpha_2} h^\ell) = |\alpha_1| + |\alpha_2| + 2\ell.$$

We denote by $\mathcal{D}_N$ the $\mathcal{C}^\infty(U)$-module spanned by monomials of degree N, and

$$\mathcal{O}_N = \bigoplus_{n \geq N} \mathcal{D}_n.$$

For $\tau_1, \tau_2 \in \mathcal{E}_2$, we define

$$\mathrm{ad}_{\tau_1}(\tau_2) = [\tau_1, \tau_2] = \tau_1 \star \tau_2 - \tau_2 \star \tau_1,$$

and if $\tau_1 \in \mathcal{O}_{N_1}$ and $\tau_2 \in \mathcal{O}_{N_2}$,

$$\frac{i}{h} \mathrm{ad}_{\tau_1}(\tau_2) \in \mathcal{O}_{N_1+N_2-2}.$$

We define

$$N_0 = \hat{b}(w, s(w)) \in \mathcal{D}_0 \quad \text{and} \quad N_2 = \sum_{j=1}^{k} \nu_j(w) |\tilde{v}_j|^2 \in \mathcal{D}_2,$$

with the notation $\tilde{v}_j = t_j + i\tilde{\tau}_j$, so that

$$\frac{1}{h^2} \mathrm{N}_h = N_0 + N_2 + \mathcal{O}_3.$$

Now we construct the following normal form. Recall that r_2 is an integer chosen such that,

$$\text{for all } \alpha \in \mathbb{Z}^k, \quad 0 < |\alpha| < r_2, \quad \sum_{j=1}^{s} \alpha_j \nu_j(0) \neq 0.$$

Moreover, this nonresonance relation at $w = 0$ can be extended to a small neighborhood of 0.

Lemma 5.3. *For any $\gamma \in \mathcal{O}_3$, there exist $\kappa, \tau \in \mathcal{O}_3$ and $\rho \in \mathcal{O}_{r_2}$ such that*

$$e^{(i/h)\,\mathrm{ad}_\tau}(N_0 + N_2 + \gamma) = N_0 + N_2 + \kappa + \rho, \tag{5-6}$$

and $[\kappa, |\tilde{v}_j|^2] = 0$ for $1 \le j \le k$.

Proof. We prove this result by induction. Assume that we have, for some $N > 0$, a $\tau \in \mathcal{O}_3$ such that

$$e^{(i/h)\,\mathrm{ad}_\tau}(N_0 + N_2 + \gamma) = N_0 + N_2 + K_3 + \cdots + K_{N-1} + R_N + \mathcal{O}_{N+1},$$

with $R_N \in \mathcal{D}_N$ and $K_i \in \mathcal{D}_i$ such that $[K_i, |\tilde{v}_j|^2] = 0$. We are looking for a $\tau_N \in \mathcal{D}_N$. For such a τ_N, $(i/h)\,\mathrm{ad}_{\tau_N} : \mathcal{O}_j \to \mathcal{O}_{N+j-2}$ so

$$e^{(i/h)\,\mathrm{ad}_{\tau+\tau_N}}(N_0 + N_2 + \gamma) = N_0 + N_2 + K_3 + \cdots + K_{N-1} + R_N + \frac{i}{h}\,\mathrm{ad}_{\tau_N}(N_0 + N_2) + \mathcal{O}_{N+1}.$$

Moreover N_0 does not depend on (t, τ) so the expansion (5-5) yields

$$\frac{i}{h}\,\mathrm{ad}_{\tau_N}(N_0) = h \sum_{j=1}^{s} \left(\frac{\partial \tau_N}{\partial y_j} \frac{\partial N_0}{\partial \eta_j} - \frac{\partial \tau_N}{\partial \eta_j} \frac{\partial N_0}{\partial y_j} \right) + \mathcal{O}_{N+6} = \mathcal{O}_{N+2},$$

and thus

$$e^{(i/h)\,\mathrm{ad}_{\tau+\tau_N}}(N_0 + N_2 + \gamma) = N_0 + N_2 + K_3 + \cdots + K_{N-1} + R_N + \frac{i}{h}\,\mathrm{ad}_{\tau_N}(N_2) + \mathcal{O}_{N+1}.$$

So we are looking for $\tau_N, K_N \in \mathcal{D}_N$ solving the equation

$$R_N = K_N + \frac{i}{h}\,\mathrm{ad}_{N_2} \tau_N + \mathcal{O}_{N+1}. \tag{5-7}$$

To solve this equation, we study the operator $(i/h)\,\mathrm{ad}_{N_2} : \mathcal{O}_N \to \mathcal{O}_N$,

$$\frac{i}{h}\,\mathrm{ad}_{N_2}(\tau_N) = \sum_{j=1}^{k} \left(\nu_j(w) \frac{i}{h}\,\mathrm{ad}_{|\tilde{v}_j|^2}(\tau_N) + \frac{i}{h}\,\mathrm{ad}_{\nu_j}(\tau_N) |\tilde{v}_j|^2 \right),$$

and since ν only depends on w, expansion (5-5) yields

$$\frac{i}{h}\,\mathrm{ad}_{\nu_i}(\tau_N) = \sum_{j=1}^{s} h \left(\frac{\partial \nu_i}{\partial y_j} \frac{\partial \tau_N}{\partial \eta_j} - \frac{\partial \nu_i}{\partial \eta_j} \frac{\partial \tau_N}{\partial y_j} \right) + \mathcal{O}_{N+6} = \mathcal{O}_{N+2}.$$

Hence,

$$\frac{i}{h}\,\mathrm{ad}_{N_2}(\tau_N) = \sum_{j=1}^{k} \nu_j(w) \frac{i}{h}\,\mathrm{ad}_{|\tilde{v}_j|^2}(\tau_N) + \mathcal{O}_{N+2},$$

and (5-7) becomes

$$R_N = K_N + \sum_{j=1}^{k} \nu_j(w) \frac{i}{h}\,\mathrm{ad}_{|\tilde{v}_j|^2}(\tau_N) + \mathcal{O}_{N+1}. \tag{5-8}$$

Moreover, $(i/h)\,\mathrm{ad}_{|\tilde{v}_j|^2}$ acts as

$$\sum_{j=1}^{k} \nu_j(w)\frac{i}{h}\,\mathrm{ad}_{|\tilde{v}_j|^2}(v^{\alpha_1}\bar{v}^{\alpha_2}h^{\ell}) = \langle \nu(w), \alpha_2-\alpha_1\rangle v^{\alpha_1}\bar{v}^{\alpha_2}h^{\ell}.$$

The definition of r_2 ensures that $\langle \nu(w), \alpha_2-\alpha_1\rangle$ does not vanish on a neighborhood of $w=0$ if $N=|\alpha_1|+|\alpha_2|+2\ell<r_2$ and $\alpha_1\neq\alpha_2$. Hence we can decompose every R_N as in (5-8), where K_N contains the terms with $\alpha_1=\alpha_2$. These terms are exactly the ones commuting with $|\tilde{v}_j|^2$ for $1\le j\le k$. $\square$

5C. ***Second quantized normal form.*** Now we can quantize Lemmas 5.1 and 5.3 to prove the following theorem.

Theorem 5.4. *There exist*

(1) *a unitary operator* $\mathrm{U}_{2,\hbar}:\mathrm{L}^2(\mathbb{R}^{s+k}_{(y,t)})\to\mathrm{L}^2(\mathbb{R}^{s+k}_{(y,t)})$ *quantizing a symplectomorphism* $\tilde{\Phi}_2=\Phi_2+\mathcal{O}((t,\tau)^2)$ *microlocally near* 0,

(2) *a function* $f_2^{\star}:\mathbb{R}^{2s}_w\times\mathbb{R}^k_J\times[0,1)\to\mathbb{R}$ *which is* $\mathcal{C}^\infty$ *with compact support such that*

$$|f_2^{\star}(w, J_1,\dots,J_k,\sqrt{\hbar})|\le C(|J|+\sqrt{\hbar})^2,$$

(3) *a* $\sqrt{\hbar}$*-pseudodifferential operator* $\mathcal{R}_{2,\hbar}$ *with symbol* $\mathcal{O}((t,\tilde{\tau},\hbar^{1/4})^{r_2})$ *on a neighborhood of* 0

such that

$$\mathrm{U}_{2,\hbar}^{*}\mathcal{N}_{\hbar}^{[1]}\mathrm{U}_{2,\hbar}=\hbar\mathcal{M}_{\hbar}+\hbar\mathcal{R}_{2,\hbar},$$

where $\mathcal{M}_\hbar$ *is the* $\hbar$*-pseudodifferential operator*

$$\mathcal{M}_{\hbar}=\mathrm{Op}_{\hbar}^{w}\,\hat{b}(w,s(w))+\sum_{j=1}^{k}\mathcal{J}_{\hbar}^{(j)}\,\mathrm{Op}_{\hbar}^{w}\,\nu_j+\mathrm{Op}_{\hbar}^{w}\,f_2^{\star}(w,\mathcal{J}_{\hbar}^{(1)},\dots,\mathcal{J}_{\hbar}^{(k)},\sqrt{\hbar}).$$

Proof. Lemma 5.1 provides us with a symplectomorphism Φ_2 such that

$$N_{\hbar}^{[1]}\circ\Phi_2=\hbar\hat{b}(w,s(w))+\sum_{j=1}^{k}\nu_j(w)(\tau_j^2+\hbar t_j^2)+\mathcal{O}(|t|^3|\tau|^2+|t|^3\hbar+\hbar^2+\hbar|\tau|+|\tau|^3+|t|\,|\tau|^2).$$

We can apply the Egorov theorem to get a Fourier integral operator $V_{2,\hbar}$ such that

$$V_{2,\hbar}^{*}\,\mathrm{Op}_{\hbar}^{w}(N_{\hbar}^{[1]})V_{2,\hbar}=\mathrm{Op}_{\hbar}^{w}(\widehat{N}_{\hbar}),$$

with $\widehat{N}_{\hbar}=N_{\hbar}^{[1]}\circ\Phi_2+\mathcal{O}(\hbar^2)$ on a neighborhood of $w=0$. We define

$$\mathrm{N}_h(y,\eta,t,\tilde{\tau})=\widehat{N}_{\hbar}(y,\eta,t,h\tilde{\tau}),$$

and following the notation of Section 5B, we have the associated formal series

$$\frac{1}{h^2}\mathrm{N}_h=N_0+N_2+\gamma,\quad \gamma\in\mathcal{O}_3.$$

We apply Lemma 5.3 and we get formal series κ, ρ such that

$$e^{(i/h)\,\mathrm{ad}_{\rho}}(N_0+N_2+\gamma)=N_0+N_2+\kappa+\mathcal{O}_{r_2}.$$

We take a compactly supported symbol $\mathrm{a}(h, w, t, \tilde{\tau})$ with Taylor series ρ. Then the operator

$$e^{ih^{-1}\operatorname{Op}_\sharp^w(\mathrm{a})}\operatorname{Op}_\sharp^w(h^{-2}\mathrm{N}_h)e^{-ih^{-1}\operatorname{Op}_\sharp^w(\mathrm{a})} \tag{5-9}$$

has a symbol with Taylor series $N_0 + N_2 + \kappa + \mathcal{O}_{r_2}$. Since $\kappa \in \mathcal{O}_3$ commutes with $|\tilde{v}_j|^2$, it can be written

$$\kappa = \sum_{2|\alpha|+2\ell\geq 3} c^\star_{\alpha\ell}(w)(|\tilde{v}_1|^2)^{\star\alpha_1}\cdots(|\tilde{v}_k|^2)^{\star\alpha_k}h^\ell.$$

If we take $f_2^\star(h, w, J_1, \ldots, J_k)$ a smooth compactly supported function with Taylor series

$$[f_2^\star] = \sum_{2|\alpha|+2\ell\geq 3} c^\star_{\alpha\ell}(w)J_1^{\alpha_1}\cdots J_k^{\alpha_k}h^\ell,$$

then the operator (5-9) is equal to

$$\operatorname{Op}_\sharp^w N_0 + \operatorname{Op}_\sharp^w N_2 + \operatorname{Op}_\hbar^w f_2^\star(h, w, \mathcal{J}_\hbar^{(1)}, \ldots, \mathcal{J}_\hbar^{(k)})$$

modulo $\mathcal{O}_{r_2}$. Multiplying by h^2, and getting back to the $\hbar$-quantization, we get

$$e^{ih^{-1}\operatorname{Op}_\sharp^w(\mathrm{a})}\operatorname{Op}_\hbar^w(\widehat{N}_\hbar)e^{-ih^{-1}\operatorname{Op}_\sharp^w(\mathrm{a})} = \hbar\mathcal{M}_\hbar + \hbar\mathcal{R}_\hbar,$$

with

$$\mathcal{M}_\hbar = \operatorname{Op}_\hbar^w \hat{b}(w, s(w)) + \sum_{j=1}^k \operatorname{Op}_\hbar^w \nu_j(w)\mathcal{J}_\hbar^{(j)} + \operatorname{Op}_\hbar^w f_2^\star(\sqrt{\hbar}, w, \mathcal{J}_\hbar^{(1)}, \ldots, \mathcal{J}_\hbar^{(k)}),$$

and $\mathcal{R}_\hbar$ a $\sqrt{\hbar}$-pseudodifferential operator with symbol $\mathcal{O}_{r_2}$. Note that $\mathcal{M}_\hbar$ is an $\hbar$-pseudodifferential operator whose symbol admits an expansion in powers of $\sqrt{\hbar}$. □

5D. *Proof of Theorem 1.11.* In order to prove Theorem 1.11, we need the following microlocalization lemma.

Lemma 5.5. *Let $\delta \in \left(0, \frac{1}{2}\right)$ and $c > 0$. Let $\chi_0 \in \mathcal{C}_0^\infty(\mathbb{R}^{2s}_{(y,\eta)})$ and $\chi_1 \in \mathcal{C}_0^\infty(\mathbb{R}^{2k}_{(t,\tilde{\tau})})$ both equal to* 1 *on a neighborhood of* 0. *Then every eigenfunction $\psi_\hbar$ of $\mathcal{N}_\hbar$ or $\hbar\mathcal{M}_\hbar$ associated to an eigenvalue $\lambda_\hbar \leq \hbar(b_0 + c\hbar^\delta)$ satisfies*

$$\psi_\hbar = \operatorname{Op}_{\sqrt{\hbar}}^w \chi_0(\sqrt{\hbar}^{-\delta}(t, \tilde{\tau}))\operatorname{Op}_\hbar^w \chi_1(y, \eta)\psi_\hbar + \mathcal{O}(\hbar^\infty)\psi_\hbar.$$

Proof. Using the mixed quantization and $h = \sqrt{\hbar}$, we have $\mathcal{N}_\hbar^{[1]} = \operatorname{Op}_\sharp^w \mathrm{N}_h^{[1]}$, with

$$\mathrm{N}_h^{[1]}(y, \eta, t, \tilde{\tau}) = h^2\langle M(y, \eta, t)\tilde{\tau}, \tilde{\tau}\rangle + h^2\hat{b}(w, t) + f_1^\star(y, \eta, t, h\tilde{\tau}, h^2).$$

The principal part of $\mathrm{N}_h^{[1]}$ is of order h^2, and implies a microlocalization of the eigenfunctions, where

$$h^2\langle M(w, t)\tilde{\tau}, \tilde{\tau}\rangle + h^2\hat{b}(w, t) \leq \lambda_h \leq h^2(b_0 + ch^{2\delta}).$$

Since $\hat{b}$ admits a unique and nondegenerate minimum b_0 at 0, this implies that w lies in an arbitrarily small neighborhood of 0, and that

$$|t|^2 \leq Ch^{2\delta}, \quad |\tilde{\tau}|^2 \leq Ch^{2\delta}.$$

The technical details follow the same ideas of Lemmas 4.2, 4.3 and 4.4. Now we can focus on $\mathcal{M}_\hbar$, whose principal symbol with respect to the $\mathrm{Op}_\sharp^w$-quantization is

$$\mathrm{M}_0(y,\eta,t,\tilde{\tau}) = \hat{b}(y,\eta,s(y,\eta)) + \sum_{j=1}^{k} \nu_j(y,\eta)(\tilde{\tau}_j^2 + t_j^2).$$

Hence its eigenfunctions are microlocalized where

$$\hat{b}(y,\eta,s(y,\eta)) + \sum_{j=1}^{k} \nu_j(y,\eta)(\tilde{\tau}_j^2 + t_j^2) \le b_0 + ch^{2\delta},$$

which implies again that w lies in an arbitrarily small neighborhood of 0 and that

$$|t|^2 \le Ch^{2\delta}, \quad |\tilde{\tau}|^2 \le Ch^{2\delta}. \qquad \square$$

Using the same method as before, we deduce from Theorem 5.4 and Lemma 5.5 a comparison of the spectra of $\mathcal{N}_\hbar^{[1]}$ and $\mathcal{M}_\hbar$. With the notation

$$N_\hbar^{\max}(c,\delta) = \max\{n \in \mathbb{N} : \lambda_n(\mathcal{N}_\hbar^{[1]}) \le \hbar(b_0 + c\hbar^\delta)\},$$

the following lemma concludes the proof of Theorem 1.11.

Lemma 5.6. *Let $\delta \in \left(0, \frac{1}{2}\right)$ and $c > 0$. We have*

$$\lambda_n(\mathcal{N}_\hbar^{[1]}) = \hbar\lambda_n(\mathcal{M}_\hbar) + \mathcal{O}(\hbar^{1+\delta r_2/2}),$$

uniformly with respect to $n \in [1, N_\hbar^{\max}(c,\delta)]$.

Proof. We use the same method as before (see Lemma 4.5). The remainder $\mathcal{R}_{2,\hbar}$ is $\mathcal{O}((t,\tilde{\tau},\sqrt{\hbar})^{r_2})$ and the eigenfunctions are microlocalized where $|t| + |\tilde{\tau}| \le C\hbar^{\delta/2}$. Hence the $\hbar\mathcal{R}_{2,\hbar}$ term yields an error in $\hbar^{1+\delta r_2/2}$. $\square$

6. Proof of Corollary 1.14

In this section we prove that the spectrum of $\mathcal{L}_\hbar$ below $\hbar b_0 + \hbar^{3/2}(\nu(0) + 2c)$ is given by the spectrum of $\hbar\mathcal{M}_\hbar^{[1]}$, up to $\mathcal{O}(\hbar^{r/4-\varepsilon})$. We recall that $c \in (0, \min_j \nu_j(0))$ and $r = \min(2r_1, r_2 + 4)$.

We can apply Theorem 1.7 for $b_1 > b_0$ arbitrarily close to b_0. Thus the spectrum of $\mathcal{L}_\hbar$ in $(-\infty, b_1\hbar)$ is given by the spectrum of $\bigoplus_{n\in\mathbb{N}^s} \mathcal{N}_\hbar^{[n]}$ modulo $\mathcal{O}(\hbar^{r_1/2-\varepsilon}) = \mathcal{O}(\hbar^{r/4-\varepsilon})$. Moreover, the symbol of $\mathcal{N}_\hbar^{[n]}$ for $n \neq (1,\dots,1)$ satisfies

$$N_\hbar^{[n]}(y,\eta,t,\tau) \ge \hbar(b_0 + 2\min \beta_j - C\hbar),$$

and we deduce from the Gårding inequality that

$$\langle \mathcal{N}_\hbar^{[n]}\psi, \psi\rangle \ge \hbar b_1 \|\psi\|^2 \quad \text{for all } \psi \in \mathrm{L}^2(\mathbb{R}^{s+k}),$$

if b_1 is close enough to b_0. Hence the spectrum of $\mathcal{L}_\hbar$ below $b_1\hbar$ is given by the spectrum of $\mathcal{N}_\hbar^{[1]}$. Then, we apply Theorem 1.11 for δ close enough to $\frac{1}{2}$, and we see that the spectrum of $\mathcal{N}_\hbar^{[1]}$ below $(b_0 + \hbar^\delta)\hbar$

is given by the spectrum of $\bigoplus_{n\in\mathbb{N}^k} \hbar\mathcal{M}_\hbar^{[n]}$ modulo $\mathcal{O}(\hbar^{1+r_2/4-\varepsilon}) = \mathcal{O}(\hbar^{r/4-\varepsilon})$. The symbol of $\mathcal{M}_\hbar^{[n]}$ for $n \neq 1$ satisfies

$$\mathcal{M}_\hbar^{[n]}(y,\eta) \geq b_0 + \hbar^{1/2}\sum_{j=1}^k \nu_j(y,\eta)(2n_j - 1) - C\hbar,$$

and the eigenfunctions of $\mathcal{M}_\hbar^{[n]}$ are microlocalized in an arbitrarily small neighborhood of $(y,\eta) = 0$ (Lemma 5.5), and $M_\hbar^{[n]}$ satisfies in this neighborhood

$$\begin{aligned} M_\hbar^{[n]}(y,\eta) &\geq b_0 + \hbar^{1/2}\sum_{j=1}^k \nu_j(0)(2n_j - 1) - \hbar^{1/2}\varepsilon - C\hbar \\ &\geq b_0 + \hbar^{1/2}(\nu(0) + 2\min_j \nu_j(0) - \varepsilon) - C\hbar. \end{aligned}$$

Using the Gårding inequality, the spectrum of $\mathcal{M}_\hbar^{[n]}$ $(n \neq 1)$ is thus $\geq b_0 + \hbar^{1/2}(\nu(0) + 2c)$ for ε and $\hbar$ small enough. It follows that the spectrum of $\mathcal{N}_\hbar^{[1]}$ below $\hbar b_0 + \hbar^{3/2}(\nu(0)+2c)$ is given by the spectrum of $\hbar\mathcal{M}_\hbar^{[1]}$.

7. Proof of Corollary 1.15

We explain here where the asymptotics for $\lambda_j(\mathcal{L}_\hbar)$ come from. First we use Corollary 1.14 so that the spectrum of $\mathcal{L}_\hbar$ below $\hbar b_0 + \hbar^{3/2}(\nu(0)+2c)$ is given by $\mathcal{M}_\hbar^{[1]}$, modulo $\mathcal{O}(\hbar^{r/4-\varepsilon})$. The symbol of $\mathcal{M}_\hbar^{[1]}$ has the expansion

$$M_\hbar^{[1]}(w) = \hat{b}(w, s(w)) + \hbar^{1/2}\nu(0) + \hbar^{1/2}\nabla\nu(0)\cdot w + \hbar\tilde{c}_0 + \mathcal{O}(\hbar w + \hbar^{3/2} + \hbar^{1/2}w^2),$$

with $\nu(w) = \sum_{j=1}^k \nu_j(w)$. The principal part admits a unique minimum at 0, which is nondegenerate. The asymptotics of the first eigenvalues of such an operator are well known. First one can make a linear change of canonical coordinates diagonalizing the Hessian of $\hat{b}$ and get a symbol of the form

$$\widehat{M}_\hbar^{[1]}(w) = b_0 + \sum_{j=1}^s \mu_j(\eta_j^2 + y_j^2) + \hbar^{1/2}\nu(0) + \hbar^{1/2}\nabla\nu(0)\cdot w + \hbar\tilde{c}_0 + \mathcal{O}(w^3 + \hbar w + \hbar^{3/2} + \hbar^{1/2}w^2).$$

One can factor the $\nabla\nu(0)\cdot w$ term to get

$$\begin{aligned} \widehat{M}_\hbar^{[1]}(w) = b_0 + \sum_{j=1}^s \mu_j\biggl(\Bigl(\eta_j + \frac{\partial_{\eta_j}\nu(0)}{2\mu_j}\hbar^{1/2}\Bigr)^2 + \Bigl(y_j + \frac{\partial_{y_j}\nu(0)}{2\mu_j}\hbar^{1/2}\Bigr)^2\biggr) &+ \hbar^{1/2}\nu(0) + \hbar c_0 \\ &+ \mathcal{O}(w^3 + \hbar w + \hbar^{3/2} + \hbar^{1/2}w^2), \end{aligned}$$

with a new $c_0 \in \mathbb{R}$. Conjugating $\mathrm{Op}_\hbar^w \widehat{M}_\hbar^{[1]}$ by the unitary operator $U_\hbar$,

$$U_\hbar v(x) = \exp\biggl(\frac{i}{\sqrt{\hbar}}\sum_{j=1}^s \frac{\partial_{\eta_j}\nu(0)}{2\mu_j} y_j\biggr) v\biggl(x - \sum_{j=1}^s \frac{\partial_{y_j}\nu(0)}{2\mu_j}\hbar^{1/2}\biggr),$$

amounts to making a phase-space translation and changes the symbol into

$$\widetilde{M}_\hbar^{[1]}(w) = b_0 + \sum_{j=1}^s \mu_j(\eta_j^2 + y_j^2) + \hbar^{1/2}\nu(0) + \hbar c_0 + \mathcal{O}(w^3 + \hbar w + \hbar^{3/2} + \hbar^{1/2}w^2).$$

For an operator with such symbol (i.e., harmonic oscillator + remainders) one can apply the results of [Charles and Vũ Ngọc 2008, Theorem 4.7] or [Helffer and Sjöstrand 1984] and deduce that the j-th eigenvalue $\lambda_j(\mathcal{M}_\hbar^{[1]})$ admits an asymptotic expansion in powers of $\hbar^{1/2}$ such that

$$\lambda_j(\mathcal{M}_\hbar^{[1]}) = b_0 + \hbar^{1/2}\nu(0) + \hbar(c_0 + E_j) + \hbar^{3/2}\sum_{m=0}^{\infty}\alpha_{j,m}\hbar^{m/2},$$

where $\hbar E_j$ is the j-th repeated eigenvalue of the harmonic oscillator with symbol $\sum_{j=1}^{s}\mu_j(\eta_j^2 + y_j^2)$.

Appendix A: Local coordinates

If we choose local coordinates $q = (q_1, \dots, q_d)$ on M, we get the corresponding vector field basis $(\partial_{q_1}, \dots, \partial_{q_d})$ on T_qM, and the dual basis $(\mathrm{d}q_1, \dots, \mathrm{d}q_d)$ on T_qM^*. In these bases, g_q can be identified with a symmetric matrix $(g_{ij}(q))$ with determinant $|g|$, and g_q^* is associated with the inverse matrix $(g^{ij}(q))$. We can write the 1-form A and the 2-form B in the coordinates:

$$A \equiv A_1\mathrm{d}q_1 + \dots + A_d\,\mathrm{d}q_d, \qquad B = \sum_{i<j} B_{ij}\mathrm{d}q_i \wedge \mathrm{d}q_j,$$

with $\boldsymbol{A} = (A_j)_{1\le j\le d} \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and

$$B_{ij} = \partial_i A_j - \partial_j A_i = ({}^t\mathrm{d}\boldsymbol{A} - \mathrm{d}\boldsymbol{A})_{ij}. \tag{A-1}$$

Let us denote by $(\boldsymbol{B}_{ij}(q))_{1\le i,j\le d}$ the matrix of the operator $\boldsymbol{B}(q) : T_qM \to T_qM$ in the basis $(\partial_{q_1}, \dots, \partial_{q_d})$. With this notation, (1-1) relating $\boldsymbol{B}$ to B can be rewritten,

$$\text{for all } Q, \widetilde{Q} \in \mathbb{R}^d, \quad \sum_{ijk} g_{kj}\boldsymbol{B}_{ki}Q_i\widetilde{Q}_j = \sum_{ij} B_{ij}Q_i\widetilde{Q}_j,$$

which means that,

$$\text{for all } i, j, \quad B_{ij} = \sum_k g_{kj}\boldsymbol{B}_{ki}. \tag{A-2}$$

Finally, in the coordinates, H is given by

$$H(q,p) = \sum_{i,j} g^{ij}(q)(p_i - A_i(q))(p_j - A_j(q)), \tag{A-3}$$

and $\mathcal{L}_\hbar$ acts as the differential operator:

$$\mathcal{L}_\hbar^{\mathrm{coord}} = \sum_{k,l=1}^{d} |g|^{-1/2}(i\hbar\partial_k + A_k)g^{kl}|g|^{1/2}(i\hbar\partial_l + A_l). \tag{A-4}$$

Appendix B: Darboux-Weinstein lemmas

We used the following presymplectic Darboux lemma.

Theorem B.1. *Let M be a d-dimensional manifold endowed with a closed constant-rank-2 form ω. We denote by $2s$ the rank of ω and by k the dimension of its kernel. For every $q_0 \in M$, there exist a* *neighborhood V of q_0, a neighborhood U of $0 \in \mathbb{R}^{2s+k}_{(y,\eta,t)}$, and a diffeomorphism*

$$\varphi : U \to V$$

such that

$$\varphi^*\omega = \mathrm{d}\eta \wedge \mathrm{d}y.$$

We also used the following Weinstein result; see [Weinstein 1971]. We follow the proof given in [Raymond and Vũ Ngọc 2015].

Theorem B.2. *Let ω_0 and ω_1 be two 2-forms on $\mathbb{R}^d$ which are closed and nondegenerate. Let us split $\mathbb{R}^d$ into $\mathbb{R}^k_x \times \mathbb{R}^{d-k}_y$. We assume that $\omega_0 = \omega_1 + \mathcal{O}(|x|^\alpha)$ for some $\alpha \geq 1$. Then there exists a neighborhood of $0 \in \mathbb{R}^d$ and a change of coordinates ψ on this neighborhood such that*

$$\psi^*\omega_1 = \omega_0 \quad \textit{and} \quad \psi = \mathrm{Id} + \mathcal{O}(|x|^{\alpha+1}).$$

Proof. First we recall how to find a 1-form σ on a neighborhood of $x = 0$ such that

$$\tau := \omega_1 - \omega_0 = \mathrm{d}\sigma \quad \text{and} \quad \sigma = \mathcal{O}(|x|^{\alpha+1}).$$

We define the family $(\phi_t)_{0 \leq t \leq 1}$ by

$$\phi_t(x, y) = (tx, y).$$

We have

$$\phi_0^*\tau = 0 \quad \text{and} \quad \phi_1^*\tau = \tau. \tag{B-1}$$

Let us denote by X_t the vector field associated with ϕ_t,

$$X_t = \frac{\mathrm{d}\phi_t}{\mathrm{d}t} \circ \phi_t^{-1} = t^{-1}(x, 0).$$

The Lie derivative of τ along X_t is given by $\phi_t^*\mathcal{L}_{X_t}\tau = (\mathrm{d}/\mathrm{d}t)\phi_t^*\tau$. From the Cartan formula we have

$$\mathcal{L}_{X_t}\tau = \iota(X_t)\mathrm{d}\tau + \mathrm{d}(\iota(X_t)).$$

Since τ is closed, $\mathrm{d}\tau = 0$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^*\tau = \mathrm{d}(\phi_t^*\iota(X_t)\tau). \tag{B-2}$$

We choose the following 1-form (where (e_j) denotes the canonical basis of $\mathbb{R}^d$):

$$\sigma_t := \phi_t^*\iota(X_t)\tau = \sum_{j=1}^k x_j \tau_{\phi_t(x,y)}(e_j, \nabla\phi_t(.)) = \mathcal{O}(|x|^{\alpha+1}).$$

Equation (B-2) shows that $t \mapsto \phi_t^*\tau$ is smooth on $[0, 1]$. Thus, we can define $\sigma = \int_0^1 \sigma_t \,\mathrm{d}t$. From (B-2) and (B-1) we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^*\tau = \mathrm{d}\sigma_t \quad \text{and} \quad \tau = \mathrm{d}\sigma.$$

Then we use the Moser deformation argument. For $t \in [0, 1]$, we let $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$. The 2-form ω_t is closed and nondegenerate on a small neighborhood of $x = 0$. We look for ψ_t such that

$$\psi_t^*\omega_t = \omega_0.$$

For that purpose, let us determine the associated vector field Y_t,

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = Y_t(\psi_t).$$

The Cartan formula yields

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\psi_t^*\omega_t = \psi_t^*\Big(\frac{\mathrm{d}}{\mathrm{d}t}\omega_t + \iota(Y_t)\mathrm{d}\omega_t + \mathrm{d}(\iota(Y_t)\omega_t)\Big).$$

So

$$\omega_0 - \omega_1 = \mathrm{d}(\iota(Y_t)\omega_t),$$

and we are led to solve

$$\iota(Y_t)\omega_t = -\sigma.$$

By the nondegeneracy of ω_t, this determines Y_t. We know ψ_t exists until time $t = 1$ on a small enough neighborhood of $x = 0$, and $\psi_t^*\omega_t = \omega_0$. Thus $\psi = \psi_1$ is the desired diffeomorphism. Since $\sigma = \mathcal{O}(|x|^{\alpha+1})$, we get $\psi = \mathrm{Id} + \mathcal{O}(|x|^{\alpha+1})$. □

Appendix C: Pseudodifferential operators

We refer to [Zworski 2012; Martinez 2002] for the general theory of $\hbar$-pseudodifferential operators. If $m \in \mathbb{Z}$, we denote by

$$S^m(\mathbb{R}^{2d}) = \{a \in \mathcal{C}^\infty(\mathbb{R}^{2d}) : |\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha\beta}\langle\xi\rangle^{m-|\beta|} \text{ for all } \alpha, \beta \in \mathbb{N}^d\}$$

the class of Kohn–Nirenberg symbols. If a depends on the semiclassical parameter $\hbar$, we require that the coefficients $C_{\alpha\beta}$ are uniform with respect to $\hbar \in (0, \hbar_0]$. For $a_\hbar \in S^m(\mathbb{R}^{2d})$, we define its associated Weyl quantization $\mathrm{Op}_\hbar^w(a_\hbar)$ by the oscillatory integral

$$\mathcal{A}_\hbar u(x) = \mathrm{Op}_\hbar^w(a_\hbar)u(x) = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^{2d}} e^{(i/\hbar)\langle x-y,\xi\rangle} a_\hbar\Big(\frac{x+y}{2}, \xi\Big)u(y)\,\mathrm{d}y\,\mathrm{d}\xi,$$

and we define

$$a_\hbar = \sigma_\hbar(\mathcal{A}_\hbar).$$

If M is a compact manifold, a pseudodifferential operator $\mathcal{A}_\hbar$ on $\mathrm{L}^2(M)$ is an operator acting as a pseudodifferential operator in coordinates. Then the principal symbol of $\mathcal{A}_\hbar$ (and its Kohn–Nirenberg class) does not depend on the coordinates, and we denote it by $\sigma_0(\mathcal{A}_\hbar)$. The subprincipal symbol $\sigma_1(\mathcal{A}_\hbar)$ is also well-defined, up to imposing that the charts be volume-preserving (in other words, if we see $\mathcal{A}_\hbar$ as acting on half-densities, its subprincipal symbol is well-defined). In the case where M is a compact manifold, $\mathcal{L}_\hbar$ is a pseudodifferential operator, and its principal and subprincipal symbols are

$$\sigma_0(\mathcal{L}_\hbar) = H, \quad \sigma_1(\mathcal{L}_\hbar) = 0.$$

If $M = \mathbb{R}^d$ and m is an order function on $\mathbb{R}^{2d}$, we denote by

$$S(m) = \{a \in \mathcal{C}^\infty(\mathbb{R}^{2d}) : |\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha\beta} m(x, \xi) \text{ for all } \alpha, \beta \in \mathbb{N}^d\}$$

the class of standard symbols, and we similarly define the operator $\mathrm{Op}_\hbar^w(a)$ for such symbols. In this case, we assume that B belongs to some standard class. This is equivalent to assuming that H belongs to some (other) standard class. Then, $\mathcal{L}_\hbar$ is a pseudodifferential operator with total symbol H.

Appendix D: Egorov theorem

In this paper, we used several versions of the Egorov theorem. See for example [Robert 1987; Zworski 2012; Helffer et al. 2016].

Theorem D.1. *Let P and Q be $\hbar$-pseudodifferential operators on $\mathbb{R}^d$, with symbols $p \in S(m)$, $q \in S(m')$, where m and m' are order functions such that*

$$m' = \mathcal{O}(1), \quad mm' = \mathcal{O}'(1).$$

Then the operator $e^{(i/\hbar)Q} P e^{-(i/\hbar)Q}$ is a pseudodifferential operator whose symbol is in $S(m)$, and its symbol is

$$p \circ \kappa + \hbar S(1),$$

where the canonical transformation κ is the time-1 Hamiltonian flow associated with q.

We can use this result with the $\sqrt{\hbar}$-quantization to get an Egorov theorem for our mixed quantization $\mathrm{Op}_\sharp^w$.

Theorem D.2. *Let P be an h-pseudodifferential operator on $\mathbb{R}^d$, and $\mathrm{a} \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. Then*

$$e^{(i/\hbar)\,\mathrm{Op}_\sharp^w(\mathrm{a})} P e^{-(i/\hbar)\,\mathrm{Op}_\sharp^w(\mathrm{a})}$$

is an h-pseudodifferential operator on $\mathbb{R}^d$.

Proof. $\mathrm{Op}_\sharp^w(\mathrm{a})$ is an h-pseudodifferential operator. Thus, we can apply the Egorov theorem, and we deduce that $e^{(i/\hbar)\,\mathrm{Op}_\sharp^w(\mathrm{a})} P e^{-(i/\hbar)\,\mathrm{Op}_\sharp^w(\mathrm{a})}$ is an h-pseudodifferential operator on $\mathbb{R}^d$. □

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BLOW-UP OF SOLUTIONS OF CRITICAL ELLIPTIC EQUATIONS IN THREE DIMENSIONS

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We describe the asymptotic behavior of positive solutions u_ε of the equation $-\Delta u + au = 3u^{5-\varepsilon}$ in $\Omega \subset \mathbb{R}^3$ with a homogeneous Dirichlet boundary condition. The function a is assumed to be critical in the sense of Hebey and Vaugon, and the functions u_ε are assumed to be an optimizing sequence for the Sobolev inequality. Under a natural nondegeneracy assumption we derive the exact rate of the blow-up and the location of the concentration point, thereby proving a conjecture of Brezis and Peletier (1989). Similar results are also obtained for solutions of the equation $-\Delta u + (a + \varepsilon V)u = 3u^5$ in Ω.

1. Introduction and main results

We are interested in the behavior of solutions to certain semilinear elliptic equations that are perturbations of the critical equation

$$-\Delta U = 3U^5 \quad \text{in } \mathbb{R}^3.$$

It is well known that all positive solutions to the latter equation are given by

$$U_{x,\lambda}(y) := \frac{\lambda^{1/2}}{(1+\lambda^2|y-x|^2)^{1/2}} \tag{1-1}$$

with parameters $x \in \mathbb{R}^3$ and $\lambda > 0$. This equation arises as the Euler–Lagrange equation of the optimization problem related to the Sobolev inequality

$$\int_{\mathbb{R}^3} |\nabla z|^2 \geq S \left(\int_{\mathbb{R}^3} z^6 \right)^{\frac{1}{3}}$$

with sharp constant [Aubin 1976; Rodemich 1966; Rosen 1971; Talenti 1976]

$$S := 3\left(\frac{\pi}{2}\right)^{\frac{4}{3}}.$$

The perturbed equations that we are interested in are posed in a bounded open set $\Omega \subset \mathbb{R}^3$ and involve a function a on Ω such that the operator $-\Delta + a$ with Dirichlet boundary conditions is coercive. (Later, we will be more precise concerning regularity assumptions on Ω and a.) One of the two families of equations also involves another rather arbitrary function V on Ω. The case where a and V are constants is also of interest.

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We consider solutions $u = u_\varepsilon$, parametrized by $\varepsilon > 0$, to the following two families of equations:

$$\begin{cases} -\Delta u + au = 3u^{5-\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-2}$$

and

$$\begin{cases} -\Delta u + (a + \varepsilon V)u = 3u^5 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1-3}$$

While there are certain differences between the problems (1-2) and (1-3), the methods used to study them are similar, and we will treat both in this paper. We are interested in the behavior of the solutions u_ε as $\varepsilon \to 0$, and we assume that in this limit the solutions form a minimizing sequence for the Sobolev inequality. More precisely, for (1-3) we assume

$$\lim_{\varepsilon \to 0} \frac{\int_\Omega |\nabla u_\varepsilon|^2}{\left(\int_\Omega u_\varepsilon^6\right)^{1/3}} = S, \tag{1-4}$$

and for (1-2) we assume

$$\lim_{\varepsilon \to 0} \frac{\int_\Omega |\nabla u_\varepsilon|^2}{\left(\int_\Omega u_\varepsilon^{6-\varepsilon}\right)^{2/(6-\varepsilon)}} = S. \tag{1-5}$$

For example, when Ω is the unit ball, $a = -\frac{1}{4}\pi^2$, and $V = -1$, then (1-3) has a solution if and only if $0 < \varepsilon < \frac{3}{4}\pi^2$; see [Brezis and Nirenberg 1983, Section 1.2]. Note that in this case π^2 is the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary conditions on Ω.

Returning to the general situation, the existence of solutions to (1-2) and (1-3) satisfying (1-4) and (1-5) can be proved via minimization under certain assumptions on a and V; see, e.g., [Frank et al. 2021] for (1-3). Moreover, it is not hard to prove, based on the characterization of optimizers in Sobolev's inequality, that these functions converge weakly to zero in $H_0^1(\Omega)$ and that u_ε^6 converges weakly in the sense of measures to a multiple of a delta function; see Proposition 2.2. In this sense, the functions u_ε blow up.

The problem of interest is to describe this blow-up behavior more precisely. This question was advertised in an influential paper by Brezis and Peletier [1989], who presented a detailed study of the case where Ω is a ball and a and V are constants. For earlier results on (1-2) with $a \equiv 0$, see [Atkinson and Peletier 1987; Budd 1987]. Concerning the case of general open sets $\Omega \subset \mathbb{R}^3$, the Brezis–Peletier paper contains three conjectures, the first two of which concern the blow-up behavior of solutions to the analogues of (1-2) and (1-3) in dimensions $N \geq 3$ ($N \geq 4$ for (1-3)) with $a \equiv 0$. These conjectures were proved independently in seminal works of Han [1991] and Rey [1989; 1990].

In the present paper, under a natural nondegeneracy condition, we prove the third Brezis–Peletier conjecture, which has remained open so far. It concerns the blow-up behavior of solutions of (1-2) for certain nonzero a in the three-dimensional case. We also prove the corresponding result for (1-3). This latter result is not stated explicitly as a conjecture in [Brezis and Peletier 1989], but it is contained there in spirit and could have been formulated using the same heuristics. Indeed, it is the version with $a \not\equiv 0$ of the second Brezis–Peletier conjecture in the same way as, concerning (1-2), the third conjecture is the $a \not\equiv 0$ version of the first one.

A characteristic feature of the three-dimensional case is the notion of criticality for the function a. To motivate this concept, let

$$S(a) := \inf_{0 \not\equiv z \in H_0^1(\Omega)} \frac{\int_\Omega (|\nabla z|^2 + a z^2)}{\left(\int_\Omega z^6\right)^{1/3}}.$$

One of the findings of [Brezis and Nirenberg 1983] is that if a is small (for instance, in $L^\infty(\Omega)$) but possibly nonzero, then $S(a) = S$. This is in stark contrast to the case of dimensions $N \geq 4$, where the corresponding analogue of $S(a)$ (with the exponent 6 replaced by $2N/(N-2)$) is always strictly below the corresponding Sobolev constant, whenever a is negative somewhere.

This phenomenon leads naturally to the following definition due to [Hebey and Vaugon 2001]. A continuous function a on $\overline{\Omega}$ is said to be *critical* in Ω if $S(a) = S$ and if for any continuous function $\tilde{a}$ on $\overline{\Omega}$ with $\tilde{a} \leq a$ and $\tilde{a} \not\equiv a$ one has $S(\tilde{a}) < S(a)$. Throughout this paper we assume that a is critical in Ω.

A key role in our analysis is played by the regular part of the Green's function and its zero set. To introduce these, we follow the sign and normalization convention of [Rey 1990]. Since the operator $-\Delta + a$ in Ω with Dirichlet boundary conditions is assumed to be coercive, it has a Green's function G_a satisfying, for each fixed $y \in \Omega$,

$$\begin{cases} -\Delta_x G_a(x, y) + a(x) G_a(x, y) = 4\pi \delta_y & \text{in } \Omega, \\ G_a(\cdot, y) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1-6}$$

The regular part H_a of G_a is defined by

$$H_a(x, y) := \frac{1}{|x-y|} - G_a(x, y). \tag{1-7}$$

It is well known that for each $y \in \Omega$ the function $H_a(\cdot, y)$, which is originally defined in $\Omega \setminus \{y\}$, extends to a continuous function in Ω, and we abbreviate

$$\phi_a(y) := H_a(y, y).$$

It was proved by Brezis [1986] that $\inf_{y \in \Omega} \phi_a(y) < 0$ implies $S(a) < S$. The reverse implication, which was stated in [Brezis 1986] as an open problem, was proved by Druet [2002]. Hence, as a consequence of criticality we have

$$\inf_{y \in \Omega} \phi_a(y) = 0; \tag{1-8}$$

see also [Esposito 2004] and [Frank et al. 2021, Proposition 5.1] for alternative proofs. Note that (1-8) implies, in particular, that each point x with $\phi_a(x) = 0$ is a critical point of ϕ_a.

Let us summarize the setting in this paper. In the sequel we set

$$\mathcal{N}_a := \{x \in \Omega : \phi_a(x) = 0\}.$$

Assumptions 1.1. (a) $\Omega \subset \mathbb{R}^3$ is a bounded, open set with C^2 boundary.

(b) $a \in C^{0,1}(\overline{\Omega}) \cap C^{2,\sigma}_{\mathrm{loc}}(\Omega)$ for some $\sigma > 0$.

(c) a is critical in Ω.

(d) Any point in $\mathcal{N}_a$ is a nondegenerate critical point of ϕ_a, that is, for any $x_0 \in \mathcal{N}_a$, the Hessian $D^2\phi_a(x_0)$ does not have a zero eigenvalue.

Let us briefly comment on these items. Assumptions (a) and (b) are modest regularity assumptions, which can probably be further relaxed with more effort. Concerning assumption (d) we first note that $\phi_a \in C^2(\Omega)$ by Lemma 4.1, and therefore any point in $\mathcal{N}_a$ is a critical point of ϕ_a; see (1-8). We believe that assumption (d) is "generically" true. (For results in this spirit, but in the noncritical case $a \equiv 0$, see [Micheletti and Pistoia 2014].) The corresponding assumption for $a \equiv 0$ appears frequently in the literature, for instance, in [Rey 1990; del Pino et al. 2004]. Assumption (d) holds, in particular, if Ω is a ball and a is a constant, as can be verified by explicit computation.

To leading order, the blow-up behavior of solutions of (1-3) will be given by the projection of a solution (1-1) of the unperturbed whole space equation to $H_0^1(\Omega)$. For parameters $x \in \mathbb{R}^3$ and $\lambda > 0$ we introduce $PU_{x,\lambda} \in H_0^1(\Omega)$ as the unique function satisfying

$$\Delta PU_{x,\lambda} = \Delta U_{x,\lambda} \quad \text{in } \Omega, \qquad PU_{x,\lambda} = 0 \quad \text{on } \partial\Omega. \tag{1-9}$$

Moreover, let

$$T_{x,\lambda} := \operatorname{span}\{PU_{x,\lambda},\ \partial_\lambda PU_{x,\lambda},\ \partial_{x_1} PU_{x,\lambda} \partial_{x_2} PU_{x,\lambda} \partial_{x_3} PU_{x,\lambda}\},$$

and let $T_{x,\lambda}^\perp$ be the orthogonal complement of $T_{x,\lambda}$ in $H_0^1(\Omega)$ with respect to the inner product $\int_\Omega \nabla u \cdot \nabla v$. By $\Pi_{x,\lambda}$ and $\Pi_{x,\lambda}^\perp$ we denote the orthogonal projections in $H_0^1(\Omega)$ onto $T_{x,\lambda}$ and $T_{x,\lambda}^\perp$, respectively.

Here are our main results. We begin with those pertaining to (1-2), and we first provide an asymptotic expansion of u_ε with a remainder in $H_0^1(\Omega)$.

Theorem 1.2 (asymptotic expansion of u_ε). *Let (u_ε) be a family of solutions to* (1-2) *satisfying* (1-5). *Then there are sequences $(x_\varepsilon) \subset \Omega$, $(\lambda_\varepsilon) \subset (0,\infty)$, $(\alpha_\varepsilon) \subset \mathbb{R}$ and $(r_\varepsilon) \subset T_{x_\varepsilon,\lambda_\varepsilon}^\perp$ such that*

$$u_\varepsilon = \alpha_\varepsilon (PU_{x_\varepsilon,\lambda_\varepsilon} - \lambda_\varepsilon^{-1/2} \Pi_{x_\varepsilon,\lambda_\varepsilon}^\perp (H_a(x_\varepsilon,\cdot) - H_0(x_\varepsilon,\cdot)) + r_\varepsilon) \tag{1-10}$$

and a point $x_0 \in \Omega$ with $\nabla\phi_a(x_0) = 0$ such that, *along a subsequence*,

$$|x_\varepsilon - x_0| = o(1), \tag{1-11}$$

$$\lim_{\varepsilon\to 0} \varepsilon\lambda_\varepsilon = \frac{32}{\pi}\phi_a(x_0), \tag{1-12}$$

$$\alpha_\varepsilon^{4-\varepsilon} = 1 + \frac{\varepsilon}{2}\log\lambda_\varepsilon + \begin{cases} \mathcal{O}(\lambda_\varepsilon^{-1}) & \text{if } \phi_a(x_0) \neq 0, \\ \frac{64}{3\pi}\phi_0(x_0)\lambda_\varepsilon^{-1} + o(\lambda_\varepsilon^{-1}) & \text{if } \phi_a(x_0) = 0, \end{cases} \tag{1-13}$$

$$\|\nabla r_\varepsilon\|_2 = \begin{cases} \mathcal{O}(\lambda_\varepsilon^{-1}) & \text{if } \phi_a(x_0) \neq 0, \\ \mathcal{O}(\lambda_\varepsilon^{-3/2}) & \text{if } \phi_a(x_0) = 0. \end{cases} \tag{1-14}$$

Moreover, *if $\phi_a(x_0) = 0$*, *then*

$$\lim_{\varepsilon\to 0} \varepsilon\lambda_\varepsilon^2 = -32a(x_0). \tag{1-15}$$

Our second main result concerns the pointwise blow-up behavior, both at the blow-up point and away from it, and, in the special case of constant a, verifies the conjecture from [Brezis and Peletier 1989] under the natural nondegeneracy assumption (d).

Theorem 1.3 (Brezis–Peletier conjecture). *Let* (u_ε) *be a family of solutions to* (1-2) *satisfying* (1-5).

(a) *The asymptotics close to the concentration point* x_0 *are given by*

$$\lim_{\varepsilon\to 0} \varepsilon\|u_\varepsilon\|_\infty^2 = \lim_{\varepsilon\to 0} \varepsilon|u_\varepsilon(x_\varepsilon)|^2 = \frac{32}{\pi}\phi_a(x_0).$$

If $\phi_a(x_0)=0$, *then*

$$\lim_{\varepsilon\to 0} \varepsilon\|u_\varepsilon\|_\infty^4 = \lim_{\varepsilon\to 0} \varepsilon|u_\varepsilon(x_\varepsilon)|^4 = -32a(x_0). \tag{1-16}$$

(b) *The asymptotics away from the concentration point* x_0 *are given by*

$$u_\varepsilon(x) = \lambda_\varepsilon^{-1/2} G_a(x,x_0) + o(\lambda_\varepsilon^{-1/2})$$

for every fixed $x \in \Omega\setminus\{x_0\}$. *The convergence is uniform for* x *away from* x_0.

Strictly speaking, the Brezis–Peletier conjecture [1989] is stated without the criticality assumption (c) on a, but rather under the assumption $\phi_a \geq 0$ on Ω. (Note that [Brezis and Peletier 1989] uses the opposite sign convention for the regular part of the Green's function. Also, their Green's function is normalized to be $\frac{1}{4\pi}$ times ours.) The remaining case, however, is much simpler and can be proved with existing methods. Indeed, by Druet's theorem [2002], the inequality $\phi_a \geq 0$ on Ω is equivalent to $S(a)=S$, and the assumption that a is critical is equivalent to $\min\phi_a = 0$. Thus, the case of the Brezis–Peletier conjecture that is not covered by our Theorem 1.3 is when $\min\phi_a > 0$. This case can be treated in the same way as the case $a\equiv 0$ in [Han 1991; Rey 1989] (or as we treat the case $\phi_a(x_0)>0$). Note that in this case the nondegeneracy assumption (d) is not needed. Whether this assumption can be removed in the case where $\phi_a(x_0)=0$ is an open problem.

We note that Theorems 1.2 and 1.3 and, in particular, the asymptotics (1-15) and (1-16) hold independently of whether $a(x_0)=0$ or not. We note that $a(x_0)\leq 0$ if $\phi_a(x_0)=0$, as shown in [Frank et al. 2021, Corollary 2.2]. We are grateful to H. Brezis (personal communication) for raising the question of whether $a(x_0)=0$ can happen and what the asymptotics of λ_ε and $\|u_\varepsilon\|_\infty$ would be in this case, or whether one can show that $\phi_a(x_0)=0$ implies $a(x_0)<0$. Deciding which alternative holds does not appear to be easy, in particular due to the nonlocal nature of $\phi_a(x_0)$. Here is a simple observation that may illustrate the expected level of difficulty: In the spirit of [Frank et al. 2021, Theorem 2.1 and Corollary 2.2], $a(x_0)<0$ would follow if one could exhibit a family of very refined test functions $\eta_{x_0,\lambda}$ such that when $\inf_\Omega \phi_a = \phi_a(x_0)=0$, the Sobolev quotient defining $S(a)$ satisfies $\mathcal{S}_a[\eta_{x_0,\lambda}] = S - c_1 a(x_0)\lambda^{-2} - c_2\lambda^{-\tau} + o(\lambda^{-\tau})$ for some $c_1, c_2 > 0$ and $\tau > 2$, say. However, extracting such an explicit term $c_2\lambda^{-\tau}$ is beyond the precision of both [Frank et al. 2021] and the present paper.

We also point out that the conjecture in [Brezis and Peletier 1989] is formulated with assumption (1-4) rather than (1-5). However, the latter assumption is typically used in the posterior literature dealing with problem (1-2), see, e.g., [Grossi and Pacella 2005; Han 1991], and we follow this convention.

We now turn our attention to the results for the second family of equations, namely (1-3). Whenever we deal with that problem, we impose the following additional assumptions:

Assumptions 1.4. (e) $a < 0$ in $\mathcal{N}_a$.

(f) $V \in C^{0,1}(\overline{\Omega})$.

Again, assumption (f) is a modest regularity assumption, which can probably be further relaxed with more effort. Assumption (e) is not severe, as we know from [Frank et al. 2021, Corollary 2.2] that any critical a satisfies $a \leq 0$ on $\mathcal{N}_a$; see also the above discussion of the question by Brezis of whether or not this assumption is automatically satisfied. In particular, it is fulfilled if a is a negative constant.

Let

$$Q_V(x) := \int_\Omega V(y) G_a(x,y)^2, \quad x \in \Omega. \tag{1-17}$$

Again, we first provide an asymptotic expansion of u_ε with a remainder in $H_0^1(\Omega)$.

Theorem 1.5 (asymptotic expansion of u_ε). *Let (u_ε) be a family of solutions to* (1-3) *satisfying* (1-4). *Then there are sequences $(x_\varepsilon) \subset \Omega$, $(\lambda_\varepsilon) \subset (0,\infty)$, $(\alpha_\varepsilon) \subset \mathbb{R}$ and $(r_\varepsilon) \subset T^\perp_{x_\varepsilon,\lambda_\varepsilon}$ such that*

$$u_\varepsilon = \alpha_\varepsilon (PU_{x_\varepsilon,\lambda_\varepsilon} - \lambda_\varepsilon^{-1/2} \Pi^\perp_{x_\varepsilon,\lambda_\varepsilon}(H_a(x_\varepsilon,\cdot) - H_0(x_\varepsilon,\cdot)) + r_\varepsilon) \tag{1-18}$$

and a point $x_0 \in \mathcal{N}_a$ with $Q_V(x_0) \leq 0$ such that, along a subsequence,

$$|x_\varepsilon - x_0| = o(\varepsilon^{1/2}), \tag{1-19}$$

$$\phi_a(x_\varepsilon) = o(\varepsilon), \tag{1-20}$$

$$\lim_{\varepsilon\to 0} \varepsilon\lambda_\varepsilon = 4\pi^2 \frac{|a(x_0)|}{|Q_V(x_0)|}, \tag{1-21}$$

$$\alpha_\varepsilon = 1 + \frac{4}{3\pi^3} \frac{\phi_0(x_0)|Q_V(x_0)|}{|a(x_0)|}\varepsilon + o(\varepsilon), \tag{1-22}$$

$$\|\nabla r_\varepsilon\|_2 = \mathcal{O}(\varepsilon^{3/2}). \tag{1-23}$$

If $Q_V(x_0) = 0$, the right side of (1-21) *is to be interpreted as ∞.*

The following result concerns the pointwise blow-up behavior.

Theorem 1.6. *Let (u_ε) be a family of solutions to* (1-3) *satisfying* (1-4).

(a) *The asymptotics close to the concentration point x_0 are given by*

$$\lim_{\varepsilon\to 0} \varepsilon \|u_\varepsilon\|_\infty^2 = \lim_{\varepsilon\to 0} \varepsilon |u_\varepsilon(x_\varepsilon)|^2 = 4\pi^2 \frac{|a(x_0)|}{|Q_V(x_0)|}.$$

If $Q_V(x_0) = 0$, the right side is to be interpreted as ∞.

(b) *The asymptotics away from the concentration point x_0 are given by*

$$u_\varepsilon(x) = \lambda_\varepsilon^{-1/2} G_a(x,x_0) + o(\lambda_\varepsilon^{-1/2})$$

for every fixed $x \in \Omega \setminus \{x_0\}$. The convergence is uniform for x away from x_0.

Theorems 1.2 and 1.5 state that, to leading order, the solution is given by a projected bubble $PU_{x_\varepsilon,\lambda_\varepsilon}$. One of the main points of these theorems, which enters crucially in the proof of Theorems 1.3 and 1.6, is the identification of the localization length λ_ε^{-1} of the projected bubble as an explicit constant times ε (for (1-2) if $\phi_a(x_0) \neq 0$ and for (1-3) if $Q_V(x_0) < 0$) or $\varepsilon^{1/2}$ (for (1-2) if $\phi_a(x_0) = 0$ and $a(x_0) \neq 0$).

The fact that the solutions are given to leading order by a projected bubble is a rather general phenomenon, which is shared, for instance, also by the higher-dimensional generalizations of (1-2) and (1-3). In contrast to the higher-dimensional case, however, in order to compute the asymptotics of the localization length λ_ε^{-1}, we need to extract the leading order correction to the bubble. Remarkably, for both problems (1-2) and (1-3) this correction is given by $\lambda_\varepsilon^{-1/2}\Pi^\perp_{x_\varepsilon,\lambda_\varepsilon}(H_a(x_\varepsilon,\cdot) - H_0(x_\varepsilon,\cdot))$.

In this relation it is natural to wonder whether the above projected bubble $PU_{x,\varepsilon}$ can be replaced by a different projected bubble $\widetilde{PU}_{x,\lambda}$, namely where the projection is defined with respect to the scalar product coming from the operator $-\Delta + a$, leading to

$$(-\Delta + a)\widetilde{PU}_{x,\lambda} = (-\Delta + a)U_{x,\lambda}, \quad \widetilde{PU}_{x,\lambda}|_{\partial\Omega} = 0.$$

Such a choice is probably possible and would even simplify some computations, but it would lead to additional difficulties elsewhere (for instance, in the proofs of Propositions 2.2 and 5.1 our choice allows us to apply the classical results by Bahri and Coron).

Moreover, for both problems the concentration point x_0 is shown to satisfy $\nabla\phi_a(x_0) = 0$. Here, however, we see an interesting difference between the two problems. Namely, for (1-3) we also know that $\phi_a(x_0) = 0$, whereas we know from [del Pino et al. 2004, Theorem 2(b)] that there are solutions of (1-2) concentrating at any critical point of ϕ_a which is not necessarily in $\mathcal{N}_a$. (These solutions also satisfy (1-4).)

An asymptotic expansion very similar to that in Theorem 1.5 is proved in [Frank et al. 2021] for energy-minimizing solutions of (1-3); see also [Frank et al. 2020] for the simpler higher-dimensional case. There, we did not assume the nondegeneracy of $D^2\phi_a(x_0)$, but we did assume that $Q_V < 0$ in $\mathcal{N}_a$. Moreover, in the energy minimizing setting we showed that x_0 satisfies

$$\frac{Q_V(x_0)^2}{|a(x_0)|} = \sup_{x\in\mathcal{N}_a,\, Q_V(x)<0} \frac{Q_V(x)^2}{|a(x)|},$$

but this cannot be expected in the more general setting of the present paper.

Before describing the technical challenges that we overcome in our proofs, let us put our work into perspective. In the past three decades there has been an enormous literature on blow-up phenomena of solutions to semilinear equations with critical exponent, which is impossible to summarize. We mention here only a few recent works from which, we hope, a more complete bibliography can be reconstructed. In some sense, the situation in the present paper is the simplest blow-up situation, as it concerns single bubble blow-up of positive solutions in the interior. Much more refined blow-up scenarios have been studied, including, for instance, multibubbling, sign-changing solutions or concentration on the boundary under Neumann boundary conditions. For an introduction we refer to [Druet et al. 2004; Hebey 2014]. In this paper we are interested in the description of the behavior of a given family of solutions. For the converse problem of constructing blow-up solutions in our setting, see [Musso and Salazar 2018; del Pino et al. 2004], and for a survey of related results, see [Pistoia 2013] and references therein. Obstructions to the existence of solutions in three dimensions were studied in [Druet and Laurain 2010]. The spectrum near zero of the linearization of solutions was studied in [Choi et al. 2016; Grossi and Pacella 2005]. There are also connections to the question of compactness of solutions; see [Brendle and Marques 2009; Khuri et al. 2009].

What makes the critical case in three dimensions significantly harder than the higher-dimensional analogues solved by Han [1991] and Rey [1989; 1990] is a certain cancellation, which is related to the fact that $\inf \phi_a = 0$. Thus, the term that in higher dimensions completely determines the blow-up vanishes in our case. Our way around this impasse is to iteratively improve our knowledge about the functions u_ε. The mechanism behind this iteration is a certain coercivity inequality, due to Esposito [2004], which we state in Lemma 2.3, and a crucial feature of our proof is to apply this inequality repeatedly, at different orders of precision. To arrive at the level of precision stated in Theorems 1.2 and 1.5, two iterations are necessary (plus a zeroth one, hidden in the proof of Proposition 2.2).

The first iteration, contained in Sections 2 and 5, is relatively standard and follows Rey's ideas [1990] with some adaptions due to Esposito [2004] to the critical case in three dimensions. The two main outcomes of the first iteration are that concentration occurs in the interior, and an order-sharp bound in H_0^1 on the remainder $\alpha_\varepsilon^{-1} u_\varepsilon - PU_{x_\varepsilon,\lambda_\varepsilon}$.

The second iteration, contained in Sections 3 and 6, is more specific to the problem at hand. Its main outcome is the extraction of the subleading correction

$$\lambda_\varepsilon^{-1/2} \Pi_{x_\varepsilon,\lambda_\varepsilon}^{\perp} (H_a(x_\varepsilon,\cdot) - H_0(x_\varepsilon,\cdot)).$$

Using the nondegeneracy of $D^2\phi_a(x_0)$ we will be able to show in the proofs of Theorems 1.2 and 1.5 that λ_ε is proportional to ε^{-1} (for (1-2) if $\phi_a(x_0) \neq 0$ and for (1-3) if $Q_V(x_0) < 0$) or $\varepsilon^{-1/2}$ (for (1-2) if $\phi_a(x_0) = 0$ and $a(x_0) \neq 0$).

The arguments described so far are, for the most part, carried out in H_0^1 norm. Once one has completed the two iterations, we apply in Sections 4C and 7B a Moser iteration argument in order to show that the remainder $\alpha_\varepsilon^{-1} u_\varepsilon - PU_{x_\varepsilon,\lambda_\varepsilon}$ is negligible also in L^∞ norm. This will then allow us to deduce Theorems 1.3 and 1.6.

As we mentioned before, Theorem 1.5 is the generalization of the corresponding theorem in [Frank et al. 2021] for energy-minimizing solutions. In that previous paper, we also used a similar iteration technique. Within each iteration step, however, minimality played an important role, and we used the iterative knowledge to further expand the energy functional evaluated at a minimizer. There is no analogue of this procedure in the current paper. Instead, as in most other works in this area, starting with [Brezis and Peletier 1989], Pohozaev identities now play an important role. These identities were not used in [Frank et al. 2021]. In fact, in that paper we did not use (1-3) at all and our results there are valid as well for a certain class of "almost minimizers".

There are five types of Pohozaev-type identities corresponding, in some sense, to the five linearly independent functions in the kernel of the Hessian at an optimizer of the Sobolev inequality on $\mathbb{R}^3$ (resulting from its invariance under multiplication by constants, by dilations and by translations). All five identities will be used to control the five parameters α_ε, λ_ε and x_ε in (1-10) and (1-18), which precisely correspond to the five asymptotic invariances. In fact, all five of these identities are used in the first iteration and then again in the second iteration. (To be more precise, in the first iteration in the proof of Theorem 1.5 it is more economical to only use four identities, since the information from the fifth identity is not particularly useful at this stage, due to the above mentioned cancellation $\phi_a(x_0) = 0$.)

Thinking of the five Pohozaev-type identities as coming from the asymptotic invariances is useful, but it is an oversimplification. Indeed, there are several possible choices for the multipliers in each category, for instance, u, $PU_{x,\lambda}$, $\psi_{x,\lambda}$ corresponding to multiplication by constants, $y \cdot \nabla u$, $\partial_\lambda PU_{x,\lambda}$, $\partial_\lambda \psi_{x,\lambda}$ corresponding to dilations, and $\partial_{x_j} u$, $\nabla_{x_j} PU_{x,\lambda}$, $\nabla_{x_j} PU_{x,\lambda}$ corresponding to translations. (Here $\psi_{x,\lambda}$ is a modified bubble defined below in (3-1).) The choice of the multiplier is subtle and depends on the available knowledge at the moment of applying the identity and the desired precision of the outcome. In any case, the upshot is that these identities can be brought together in such a way that they give the final result of Theorems 1.2 and 1.5 concerning the expansion in $H_0^1(\Omega)$. As mentioned before, the desired pointwise bounds in Theorems 1.3 and 1.6 then follow in a relatively straightforward way using a Moser iteration.

We believe that our techniques are robust enough to derive blow-up asymptotics for (1-2) and (1-3) in more general situations containing a nonzero weak limit and/or multiple concentration points. Since our main motivation was to solve the Brezis–Peletier conjecture stated for single blow-up [1989] and to limit the amount of calculations needed, we do not attempt to pursue this further here.

Let us also mention that a problem similar to, but different from, (1-2) has been studied in the recent article [Malchiodi and Mayer 2021] using a similar approach. While the analysis there, carried out on a Riemannian manifold M of dimension $n \geq 5$, is rather comprehensive and also treats the case of multiple blow-up points, it does not seem to contain an analogue of the vanishing phenomenon for $\phi_a(x_0)$ nor, as a consequence, of our refined iteration step described above.

The structure of this paper is as follows. The first part of the paper, consisting of Sections 2, 3 and 4, is devoted to problem (1-3), while the second part, consisting of Sections 5, 6 and 7, is devoted to (1-2). The two parts are presented in a parallel manner, but the emphasis in the second part is on the necessary changes compared to the first part. The preliminary Sections 2 and 5 contain an initial expansion, the subsequent Sections 3 and 6 contain its refinement and, finally, in Sections 4 and 7 the main theorems presented in this introduction are proved. Some technical results are deferred to two appendices.

2. Additive case: a first expansion

In this and the following section we will prepare for the proof of Theorems 1.5 and 1.6.

The main result from this section is the following preliminary asymptotic expansion of the family of solutions (u_ε).

Proposition 2.1. *Let* (u_ε) *be a family of solutions to* (1-3) *satisfying* (1-4). *Then, up to extraction of a subsequence, there are sequences* $(x_\varepsilon) \subset \Omega$, $(\lambda_\varepsilon) \subset (0, \infty)$, $(\alpha_\varepsilon) \subset \mathbb{R}$ *and* $(w_\varepsilon) \subset T^\perp_{x_\varepsilon,\lambda_\varepsilon}$ *such that*

$$u_\varepsilon = \alpha_\varepsilon (PU_{x_\varepsilon,\lambda_\varepsilon} + w_\varepsilon) \tag{2-1}$$

and a point $x_0 \in \Omega$ *such that*

$$|x_\varepsilon - x_0| = o(1), \quad \alpha_\varepsilon = 1 + o(1), \quad \lambda_\varepsilon \to \infty, \quad \|\nabla w_\varepsilon\|_2 = \mathcal{O}(\lambda^{-1/2}). \tag{2-2}$$

This proposition follows to a large extent by an adaptation of existing results in the literature. We include the proof since we have not found the precise statement and since related arguments will appear in the following section in a more complicated setting.

An initial qualitative expansion follows from works of Struwe [1984] and Bahri and Coron [1988]. In order to obtain the statement of Proposition 2.1, we then need to show two things, namely, the bound on $\|\nabla w\|$ and the fact that $x_0 \in \Omega$. The proof of the bound on $\|\nabla w\|$ that we give is rather close to that of Esposito [2004]. The setting in [Esposito 2004] is slightly different (there, V is equal to a negative constant and, more importantly, the solutions are assumed to be energy minimizing), but this part of the proof extends to our setting. On the other hand, the proof in [Esposito 2004] of the fact that $x_0 \in \Omega$ relies on the energy-minimizing property and does not work for us. Instead, we adapt some ideas from Rey [1990]. The proof in [Rey 1990] is only carried out in dimensions ≥ 4 and without the background a, but, as we will see, it extends with some effort to our situation.

We subdivide the proof of Proposition 2.1 into a sequence of subsections. The main result of each subsection is stated as a proposition at the beginning and summarizes the content of the corresponding subsection.

2A. ***A qualitative initial expansion.*** As a first important step, we derive the following expansion, which is already of the form of that in Proposition 2.1 except that all remainder bounds are nonquantitative and the limit point x_0 may a priori be on the boundary $\partial\Omega$.

Proposition 2.2. *Let (u_ε) be a family of solutions to* (1-3) *satisfying* (1-4). *Then, up to extraction of a subsequence, there are sequences $(x_\varepsilon) \subset \Omega$, $(\lambda_\varepsilon) \subset (0,\infty)$, $(\alpha_\varepsilon) \subset \mathbb{R}$ and $(w_\varepsilon) \subset T^{\perp}_{x_\varepsilon,\lambda_\varepsilon}$ such that* (2-1) *holds and a point $x_0 \in \overline{\Omega}$ such that*

$$|x_\varepsilon - x_0| = o(1), \quad \alpha_\varepsilon = 1 + o(1), \quad d_\varepsilon \lambda_\varepsilon \to \infty, \quad \|\nabla w_\varepsilon\|_2 = o(1), \tag{2-3}$$

where we write $d_\varepsilon := d(x_\varepsilon, \partial\Omega)$.

Proof. We shall only prove that $u_\varepsilon \rightharpoonup 0$ in $H^1_0(\Omega)$. Once this is shown, we can use standard arguments, due to Lions [1985], Struwe [1984] and Bahri and Coron [1988], to complete the proof of the proposition; see, for instance, [Rey 1990, Proof of Proposition 2].

<u>Step 1</u>: We begin by showing that (u_ε) is bounded in $H^1_0(\Omega)$ and that $\|u_\varepsilon\|_6 \gtrsim 1$. Integrating the equation for u_ε against u_ε, we obtain

$$\int_\Omega (|\nabla u_\varepsilon|^2 + (a - \varepsilon V) u_\varepsilon^2) = 3 \int_\Omega u_\varepsilon^6, \tag{2-4}$$

and therefore

$$3\left(\int_\Omega u_\varepsilon^6\right)^{\frac{2}{3}} = \frac{\int_\Omega |\nabla u_\varepsilon|^2}{\left(\int_\Omega u_\varepsilon^6\right)^{1/3}} + \frac{\int_\Omega (a + \varepsilon V) u_\varepsilon^2}{\left(\int_\Omega u_\varepsilon^6\right)^{1/3}}.$$

On the right side, the first quotient converges by (1-4) and the second quotient is bounded by Hölder's inequality. Thus, (u_ε) is bounded in $L^6(\Omega)$. By (1-4) we obtain boundedness in $H^1_0(\Omega)$. By coercivity of $-\Delta + a$ in $H_0^1(\Omega)$ and Sobolev's inequality, for all sufficiently small $\varepsilon > 0$, the left side in (2-4) is bounded from below by a constant times $\|u_\varepsilon\|_6^2$. This yields the lower bound on $\|u_\varepsilon\|_6 \gtrsim 1$.

Step 2: According to Step 1, (u_ε) has a weak limit point in $H_0^1(\Omega)$ and we denote by u_0 one of those. Our goal is to show that $u_0 \equiv 0$. Throughout this step, we restrict ourselves to a subsequence of ε's along which $u_\varepsilon \rightharpoonup u_0$ in $H_0^1(\Omega)$. By Rellich's lemma, after passing to a subsequence, we may also assume that $u_\varepsilon \to u_0$ almost everywhere. Moreover, passing to a further subsequence, we may also assume that $\|\nabla u_\varepsilon\|$ has a limit. Then, by (1-4), $\|u_\varepsilon\|_6$ has a limit as well and, by Step 1, none of these limits is zero.

We now argue as in the proof of [Frank et al. 2021, Proposition 3.1] and note that, by weak convergence,

$$\mathcal{T} = \lim_{\varepsilon\to 0} \int_\Omega |\nabla(u_\varepsilon - u_0)|^2 \ \text{ exists and satisfies } \ \lim_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega |\nabla u_0|^2 + \mathcal{T}$$

and, by the Brezis–Lieb lemma [Brezis and Lieb 1983],

$$\mathcal{M} = \lim_{\varepsilon\to 0} \int_\Omega (u_\varepsilon - u_0)^6 \ \text{ exists and satisfies } \ \lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon^6 = \int_\Omega u_0^6 + \mathcal{M}.$$

Thus, (1-4) gives

$$S\left(\int_\Omega u_0^6 + \mathcal{M}\right)^{\frac{1}{3}} = \int_\Omega |\nabla u_0|^2 + \mathcal{T}.$$

We bound the left side from above with the help of the elementary inequality

$$\left(\int_\Omega u_0^6 + \mathcal{M}\right)^{\frac{1}{3}} \le \left(\int_\Omega u_0^6\right)^{\frac{1}{3}} + \mathcal{M}^{1/3},$$

and, by the Sobolev inequality for $u_\varepsilon - u_0$, we bound the right side from below using

$$\mathcal{T} \ge S\mathcal{M}^{1/3}.$$

Thus,

$$S\left(\int_\Omega u_0^6\right)^{\frac{1}{3}} \ge \int_\Omega |\nabla u_0|^2.$$

Thus, either $u_0 \equiv 0$ or u_0 is an optimizer for the Sobolev inequality. Since u_0 has support in $\Omega \subsetneq \mathbb{R}^3$, the latter is impossible and we conclude that $u_0 \equiv 0$, as claimed. □

Convention. Throughout the rest of the paper, we assume that the sequence (u_ε) satisfies the assumptions and conclusions from Proposition 2.2. We will make no explicit mention of subsequences. Moreover, we typically drop the index ε from u_ε, α_ε, x_ε, λ_ε, d_ε and w_ε.

2B. *Coercivity.*

The following coercivity inequality from [Esposito 2004, Lemma 2.2] is a crucial tool for us in subsequently refining the expansion of u_ε. It states, roughly speaking, that the subleading error terms coming from the expansion of u_ε can be absorbed into the leading term, at least under some orthogonality condition.

Lemma 2.3. *There are constants $T_* < \infty$ and $\rho > 0$ such that, for all $x \in \Omega$, all $\lambda > 0$ with $d\lambda \geq T_*$ and all $v \in T_{x,\lambda}^{\perp}$,*

$$\int_\Omega (|\nabla v|^2 + av^2 - 15U_{x,\lambda}^4 v^2) \geq \rho \int_\Omega |\nabla v|^2. \tag{2-5}$$

The proof proceeds by compactness, using the inequality [Rey 1990, (D.1)]

$$\int_\Omega (|\nabla v|^2 - 15U_{x,\lambda}^4 v^2) \geq \frac{4}{7}\int_\Omega |\nabla v|^2 \quad \text{for all } v \in T_{x,\lambda}^{\perp}.$$

For details of the proof, we refer to [Esposito 2004].

In the following subsection, we use Lemma 2.3 to deduce a refined bound on $\|\nabla w\|_2$. We will use it again in Section 3B below to obtain improved bounds on the refined error term $\|\nabla r\|_2$, with $r \in T_{x,\lambda}^{\perp}$ defined in (3-4).

2C. ***The bound on $\|\nabla w\|_2$.*** The goal of this subsection is to prove:

Proposition 2.4. *As $\varepsilon \to 0$,*

$$\|\nabla w\|_2 = \mathcal{O}(\lambda^{-1/2}) + \mathcal{O}((\lambda d)^{-1}). \tag{2-6}$$

Using this bound, in Section 2D we prove that $d^{-1} = \mathcal{O}(1)$ and therefore the bound in Proposition 2.4 becomes $\|\nabla w\|_2 = \mathcal{O}(\lambda^{-1/2})$, as claimed in Proposition 2.1.

Proof. The starting point is the equation satisfied by w. Since

$$-\Delta PU_{x,\lambda} = -\Delta U_{x,\lambda} = 3U_{x,\lambda}^5,$$

from (2-1) and (1-3) we obtain

$$(-\Delta + a)w = -3U_{x,\lambda}^5 + 3\alpha^4(PU_{x,\lambda} + w)^5 - (a + \varepsilon V)PU_{x,\lambda} - \varepsilon V w. \tag{2-7}$$

Integrating this equation against w and using

$$\int_\Omega U_{x,\lambda}^5 w = \frac{1}{3}\int_\Omega \nabla PU_{x,\lambda} \cdot \nabla w = 0,$$

we get

$$\int_\Omega (|\nabla w|^2 + aw^2) = 3\alpha^4 \int_\Omega (PU_{x,\lambda} + w)^5 w - \int_\Omega (a + \varepsilon V)PU_{x,\lambda} w - \int_\Omega \varepsilon V w^2. \tag{2-8}$$

We estimate the three terms on the right-hand side separately.

The second and third terms are easy: We have by Lemma A.1

$$\left| \int_\Omega (a + \varepsilon V)PU_{x,\lambda} w \right| \lesssim \|w\|_6 \|U_{x,\lambda}\|_{6/5} \lesssim \lambda^{-1/2}\|\nabla w\|_2.$$

Moreover,

$$\left| \int_\Omega \varepsilon V w^2 \right| \lesssim \varepsilon \|w\|_6^2 = o(\|\nabla w\|_2^2).$$

The first term on the right side of (2-8) needs a bit more care. We write $PU_{x,\lambda} = U_{x,\lambda} - \varphi_{x,\lambda}$ as in Lemma A.2 and expand

$$\begin{aligned}\int_\Omega (PU_{x,\lambda}+w)^5 w \\ &= \int_\Omega U_{x,\lambda}^5 w + 5\int_\Omega U_{x,\lambda}^4 w^2 + \mathcal{O}\biggl(\int_\Omega (U_{x,\lambda}^4\varphi_{x,\lambda}|w| + U_{x,\lambda}^3(|w|^3+|w|\varphi_{x,\lambda}^2) + \varphi_{x,\lambda}^5|w| + w^6)\biggr) \\ &= 5\int_\Omega U_{x,\lambda}^4 w^2 + \mathcal{O}\biggl(\int_\Omega U_{x,\lambda}^4\varphi_{x,\lambda}|w| + \|\nabla w\|_2\|\varphi_{x,\lambda}\|_6^2 + \|\nabla w\|_2^3\biggr),\end{aligned}$$

where we again used $\int_\Omega U_{x,\lambda}^5 w = 0$. By Lemmas A.1 and A.2, we have $\|\varphi_{x,\lambda}\|_6^2 \lesssim (d\lambda)^{-1}$ and

$$\int_\Omega U_{x,\lambda}^4\varphi_{x,\lambda}|w| \lesssim \|w\|_6\|\varphi_{x,\lambda}\|_\infty\|U_{x,\lambda}\|_{24/5}^4 \lesssim \|\nabla w\|_2(d\lambda)^{-1}.$$

Putting all the estimates together, we deduce from (2-8) that

$$\int_\Omega (|\nabla w|^2 + aw^2 - 15\alpha^4U^4w^2) = \mathcal{O}((d\lambda)^{-1}\|\nabla w\|_2 + \lambda^{-1/2}\|\nabla w\|_2) + o(\|\nabla w\|_2^2).$$

Due to the coercivity inequality from Lemma 2.3, the left side is bounded from below by a positive constant times $\|\nabla w\|_2^2$. Thus, (2-6) follows. □

2D. ***Excluding boundary concentration.*** The goal of this subsection is to prove:

Proposition 2.5. $$d^{-1} = \mathcal{O}(1).$$

By integrating the equation for u against ∇u, one obtains the Pohozaev-type identity

$$-\int_\Omega (\nabla(a+\varepsilon V))u^2 = \int_{\partial\Omega} n\Bigl(\frac{\partial u}{\partial n}\Bigr)^2. \tag{2-9}$$

Inserting the decomposition $u = \alpha(PU + w)$, we get

$$\int_{\partial\Omega} n\Bigl(\frac{\partial PU_{x,\lambda}}{\partial n}\Bigr)^2 = -\int_{\partial\Omega} n\biggl(2\frac{\partial PU_{x,\lambda}}{\partial n}\frac{\partial w}{\partial n} + \Bigl(\frac{\partial w}{\partial n}\Bigr)^2\biggr) - \int_\Omega (\nabla(a+\varepsilon V))(PU_{x,\lambda}+w)^2. \tag{2-10}$$

Since $a, V \in C^1(\overline{\Omega})$, the volume integral is bounded by

$$\biggl|\int_\Omega (\nabla(a+\varepsilon V))(PU_{x,\lambda}+w)^2\biggr| \lesssim \|PU_{x,\lambda}\|_2^2 + \|w\|_2^2 \lesssim \lambda^{-1} + (\lambda d)^{-2}, \tag{2-11}$$

where we used (2-6) and Lemmas A.1 and A.2.

The function $\partial PU_{x,\lambda}/\partial n$ on the boundary is discussed in Lemma A.3. We now control the function $\partial w/\partial n$ on the boundary.

Lemma 2.6. $$\int_{\partial\Omega}\Bigl(\frac{\partial w}{\partial n}\Bigr)^2 = \mathcal{O}(\lambda^{-1}d^{-1}) + o(\lambda^{-1}d^{-2}).$$

Proof. The following proof is analogous to [Rey 1990, Appendix C]. It relies on the inequality

$$\Bigl\|\frac{\partial z}{\partial n}\Bigr\|_{L^2(\partial\Omega)}^2 \lesssim \|\Delta z\|_{L^{3/2}(\Omega)}^2 \quad \text{for all } z \in H^2(\Omega)\cap H_0^1(\Omega). \tag{2-12}$$

This inequality is well known and is contained in [Rey 1990, Appendix C]. A proof can be found, for instance, in [Hang et al. 2009].

We write (2-7) for w as $-\Delta w = F$ with

$$F := 3\alpha^4 (PU_{x,\lambda} + w)^5 - 3U_{x,\lambda}^5 - (a + \varepsilon V)(PU_{x,\lambda} + w). \tag{2-13}$$

We fix a smooth $0 \le \chi \le 1$ with $\chi \equiv 0$ on $\{|y| \le \frac{1}{2}\}$ and $\chi \equiv 1$ on $\{|y| \ge 1\}$ and define the cut-off function

$$\zeta(y) := \chi\left(\frac{y-x}{d}\right). \tag{2-14}$$

Then $\zeta w \in H^2(\Omega) \cap H_0^1(\Omega)$ and

$$-\Delta(\zeta w) = \zeta F - 2\nabla\zeta \cdot \nabla w - (\Delta\zeta) w.$$

The function F satisfies the simple pointwise bound

$$|F| \lesssim U_{x,\lambda}^5 + |w|^5 + U_{x,\lambda} + |w|, \tag{2-15}$$

which, when combined with inequality (2-12), yields

$$\begin{aligned}\left\|\frac{\partial w}{\partial n}\right\|_{L^2(\partial\Omega)}^2 = \left\|\frac{\partial(\zeta w)}{\partial n}\right\|_{L^2(\partial\Omega)}^2 &\lesssim \|\zeta F - 2\nabla\zeta\cdot\nabla w - (\Delta\zeta)w\|_{3/2}^2 \\ &\lesssim \|\zeta(U_{x,\lambda}^5 + |w|^5 + U_{x,\lambda} + |w|)\|_{3/2}^2 + \||\nabla\zeta||\nabla w|\|_{3/2}^2 + \|(\Delta\zeta)w\|_{3/2}^2.\end{aligned}$$

It remains to bound the norms on the right side. The term most difficult to estimate is $\|\zeta w^5\|_{3/2}$, because $5 \cdot \frac{3}{2} = \frac{15}{2} > 6$, and we shall come back to it later. The other terms can all be estimated using bounds on $\|U\|_{L^p(\Omega\setminus B_{d/2}(x))}$ from Lemma A.1, as well as the bound $\|w\|_6 \lesssim \lambda^{-1/2} + \lambda^{-1} d^{-1}$ from Proposition 2.4. Indeed, we have

$$\begin{aligned}\|\zeta U_{x,\lambda}^5\|_{3/2}^2 &\lesssim \|U_{x,\lambda}\|_{L^{15/2}(\Omega\setminus B_{d/2}(x))}^{10} \lesssim \lambda^{-5} d^{-6} = o(\lambda^{-1} d^{-2}), \\ \|\zeta U_{x,\lambda}\|_{3/2}^2 &\lesssim \|U_{x,\lambda}\|_{L^{3/2}(\Omega\setminus B_d)}^2 \lesssim \lambda^{-1} = \mathcal{O}(\lambda^{-1} d^{-1}), \\ \|\zeta w\|_{3/2}^2 &\lesssim \|w\|_6^2 \lesssim \lambda^{-1} + \lambda^{-2} d^{-2} = \mathcal{O}(\lambda^{-1} d^{-1}) + o(\lambda^{-1} d^{-2}), \\ \||\nabla\zeta||\nabla w|\|_{3/2}^2 &\lesssim \|\nabla w\|_2^2 \|\nabla\zeta\|_6^2 \lesssim (\lambda^{-1} + \lambda^{-2} d^{-2}) d^{-1} = \mathcal{O}(\lambda^{-1} d^{-1}) + o(\lambda^{-1} d^{-2})\end{aligned}$$

and

$$\|(\Delta\zeta)w\|_{3/2}^2 \lesssim \|w\|_6^2 \|\Delta\zeta\|_2^2 \lesssim (\lambda^{-1} + \lambda^{-2} d^{-2}) d^{-1} = \mathcal{O}(\lambda^{-1} d^{-1}) + o(\lambda^{-1} d^{-2}).$$

In order to estimate the difficult term $\|\zeta w^5\|_{3/2}$, we multiply the equation $-\Delta w = F$ by $\zeta^{1/2}|w|^{1/2} w$ and integrate over Ω to obtain

$$\int_\Omega \nabla(\zeta^{1/2}|w|^{1/2} w) \cdot \nabla w \le \int_\Omega |F| \zeta^{1/2} |w|^{3/2}. \tag{2-16}$$

We now note that there are universal constants $c > 0$ and $C < \infty$ such that, pointwise a.e.,

$$\nabla(\zeta^{1/2}|w|^{1/2} w) \cdot \nabla w \ge c|\nabla(\zeta^{1/4}|w|^{1/4} w)|^2 - C|w|^{5/2} |\nabla(\zeta^{1/4})|^2. \tag{2-17}$$

Indeed, by repeated use of the product rule and chain rule for Sobolev functions, one finds

$$\begin{aligned}\nabla(\zeta^{1/2}|w|^{1/2}w)\cdot\nabla w &= \tfrac{3}{2}\big(\tfrac{4}{5}\big)^2|\nabla(\zeta^{1/4}|w|^{1/4}w)|^2+\big(\tfrac{3}{2}\big(\tfrac{4}{5}\big)^2-\tfrac{4}{5}\cdot 2\big)|w|^{5/2}|\nabla(\zeta^{1/4})|^2\\ &\quad -\big(\tfrac{3}{2}\big(\tfrac{4}{5}\big)^2\cdot 2-\tfrac{4}{5}\cdot 2\big)|w|^{1/4}w\nabla(\zeta^{1/4})\cdot\nabla(\zeta^{1/4}|w|^{1/4}w).\end{aligned}$$

The claimed inequality (2-17) follows by applying Schwarz's inequality $v_1\cdot v_2\geq -\varepsilon|v_1|^2-|v_2|^2/(4\varepsilon)$ to the cross term on the right side with $\varepsilon>0$ small enough.

As a consequence of (2-17), we can bound the left side in (2-16) from below by

$$\int_\Omega \nabla(\zeta^{1/2}|w|^{1/2}w)\cdot\nabla w\geq c\int_\Omega|\nabla(\zeta^{1/4}|w|^{1/4}w)|^2-C\int_\Omega|w|^{5/2}|\nabla(\zeta^{1/4})|^2.$$

Thus, by the Sobolev inequality for the function $\zeta^{1/4}|w|^{1/4}w$ and (2-16), we get

$$\begin{aligned}\|\zeta w^5\|_{3/2}^2=\bigg(\int_\Omega\big|\zeta^{1/4}|w|^{1/4}w\big|^6\bigg)^{\frac{4}{3}} &\lesssim\bigg(\int_\Omega|\nabla(\zeta^{1/4}|w|^{1/4}w)|^2\bigg)^4\\ &\lesssim\bigg(\int_\Omega|w|^{5/2}|\nabla(\zeta^{1/4})|^2\bigg)^4+\bigg(\int_\Omega|F|\zeta^{1/2}|w|^{3/2}\bigg)^4.\end{aligned}\tag{2-18}$$

For the first term on the right side, we have

$$\begin{aligned}\bigg(\int_\Omega|w|^{5/2}|\nabla(\zeta^{1/4})|^2\bigg)^4\leq\|w\|_6^{10}\bigg(\int_\Omega|\nabla(\zeta^{1/4})|^{24/7}\bigg)^{\frac{7}{3}} &\lesssim(\lambda^{-5}+\lambda^{-10}d^{-10})d^{-1}\\ &=\mathcal{O}(\lambda^{-1}d^{-1})+o(\lambda^{-1}d^{-2}).\end{aligned}$$

To control the second term on the right side of (2-18), we use again the pointwise estimate (2-15). The contribution of the $|w|^5$ term to the second term on the right side of (2-18) is

$$\bigg(\int_\Omega|w|^{5+3/2}\zeta^{1/2}\bigg)^4=\bigg(\int_\Omega(\zeta^{1/2}w^{5/2})w^4\bigg)^4\leq\|\zeta w^5\|_{3/2}^2\|w\|_6^{16}=o(\|\zeta w^5\|_{3/2}^2),$$

which can be absorbed into the left side of (2-18).

For the remaining terms, we have

$$\begin{aligned}\bigg(\int_\Omega|w|^{3/2}U_{x,\lambda}^5\zeta^{1/2}\bigg)^4&\lesssim\|w\|_6^6\|U_{x,\lambda}\|_{L^{20/3}(\Omega\setminus B_{d/2}(x))}^{20}=(\lambda^{-3}+(d\lambda)^{-6})(\lambda^{-10}d^{-11}),\\ \bigg(\int_\Omega|w|^{3/2}U_{x,\lambda}\zeta^{1/2}\bigg)^4&\lesssim\|w\|_6^6\|U_{x,\lambda}\|_{L^{4/3}(\Omega)}^4=(\lambda^{-3}+(d\lambda)^{-6})\lambda^{-2},\\ \bigg(\int_\Omega|w|^{5/2}\zeta^{1/2}\bigg)^4&\lesssim\|w\|_6^{10}=\lambda^{-5}+(d\lambda)^{-10},\end{aligned}$$

all of which is $\mathcal{O}(\lambda^{-1}d^{-1})+o(\lambda^{-1}d^{-2})$. This concludes the proof of the bound

$$\|\zeta w^5\|_{3/2}^2=\mathcal{O}(\lambda^{-1}d^{-1})+o(\lambda^{-1}d^{-2})$$

and thus of Lemma 2.6. □

It is now easy to complete the proof of the main result of this section.

Proof of Proposition 2.5. The identity (2-10), together with the bound (2-11) and Lemma A.3(a), yields

$$C\lambda^{-1}\nabla\phi_0(x) = \mathcal{O}(\lambda^{-1}) + o(\lambda^{-1}d^{-2}) + \mathcal{O}\left(\left\|\frac{\partial PU_{x,\lambda}}{\partial n}\right\|_{L^2(\partial\Omega)}\left\|\frac{\partial w}{\partial n}\right\|_{L^2(\partial\Omega)} + \left\|\frac{\partial w}{\partial n}\right\|^2_{L^2(\partial\Omega)}\right)$$

for some $C > 0$. By Lemmas A.3(c) and 2.6, the last term on the right-hand side is bounded by $\lambda^{-1}d^{-3/2} + o(\lambda^{-1}d^{-2})$, so we get

$$\nabla\phi_0(x) = \mathcal{O}(d^{-3/2}) + o(d^{-2}).$$

On the other hand, according to [Rey 1990, (2.9)], we have $|\nabla\phi_0(x)| \gtrsim d^{-2}$. Hence

$$d^{-2} = \mathcal{O}(d^{-3/2}) + o(d^{-2}),$$

which yields $d^{-1} = \mathcal{O}(1)$, as claimed. □

2E. ***Proof of Proposition 2.1.*** Existence of the expansion follows from Proposition 2.2. Proposition 2.5 implies that $d^{-1} = \mathcal{O}(1)$, which implies that $x_0 \in \Omega$. Moreover, inserting the bound $d^{-1} = \mathcal{O}(1)$ into Proposition 2.4, we obtain $\|\nabla w\|_2 = \mathcal{O}(\lambda^{-1/2})$, as claimed in Proposition 2.1. This completes the proof of the proposition. □

3. Additive case: refining the expansion

Our goal in this section is to improve the decomposition given in Proposition 2.1. As in [Frank et al. 2021], our goal is to discover that a better approximation to u_ε is given by the function

$$\psi_{x,\lambda} := PU_{x,\lambda} - \lambda^{-1/2}(H_a(x,\cdot) - H_0(x,\cdot)). \tag{3-1}$$

Let us set

$$q_\varepsilon := w_\varepsilon + \lambda_\varepsilon^{-1/2}(H_a(x_\varepsilon,\cdot) - H_0(x_\varepsilon,\cdot)), \tag{3-2}$$

so that

$$u_\varepsilon = \alpha_\varepsilon(\psi_{x_\varepsilon,\lambda_\varepsilon} + q_\varepsilon).$$

As in [Frank et al. 2021], we further decompose

$$q_\varepsilon = s_\varepsilon + r_\varepsilon, \tag{3-3}$$

with $s_\varepsilon \in T_{x_\varepsilon,\lambda_\varepsilon}$ and $r_\varepsilon \in T^\perp_{x_\varepsilon,\lambda_\varepsilon}$ given by

$$r_\varepsilon := \Pi^\perp_{x_\varepsilon,\lambda_\varepsilon} q \quad \text{and} \quad s_\varepsilon := \Pi_{x_\varepsilon,\lambda_\varepsilon} q. \tag{3-4}$$

We note that the notation r_ε is consistent with that used in Theorem 1.5 since, using $w_\varepsilon \in T^\perp_{x_\varepsilon,\lambda_\varepsilon}$ where we write $w_\varepsilon = q_\varepsilon + \lambda_\varepsilon^{-1/2}(H_a(x_\varepsilon,\cdot) - H_0(x_\varepsilon,\cdot))$, we have

$$s_\varepsilon = \lambda_\varepsilon^{-1/2}\Pi_{x_\varepsilon,\lambda_\varepsilon}(H_a(x_\varepsilon,\cdot) - H_0(x_\varepsilon,\cdot)). \tag{3-5}$$

The following proposition summarizes the results of this section.

Proposition 3.1. *Let* (u_ε) *be a family of solutions to* (1-3) *satisfying* (1-4). *Then, up to extraction of a subsequence, there are sequences* $(x_\varepsilon) \subset \Omega,\ (\lambda_\varepsilon) \subset (0,\infty),\ (\alpha_\varepsilon) \subset \mathbb{R},\ (s_\varepsilon) \subset T_{x_\varepsilon,\lambda_\varepsilon}$ *and* $(r_\varepsilon) \subset T^\perp_{x_\varepsilon,\lambda_\varepsilon}$ *such that*

$$u_\varepsilon = \alpha_\varepsilon(\psi_{x_\varepsilon,\lambda_\varepsilon} + s_\varepsilon + r_\varepsilon) \tag{3-6}$$

and a point $x_0 \in \Omega$ *such that, in addition to Proposition 2.1,*

$$\begin{aligned} \|\nabla r_\varepsilon\|_2 &= \mathcal{O}(\varepsilon\lambda_\varepsilon^{-1/2}), \\ \phi_a(x_\varepsilon) &= a(x_\varepsilon)\pi\lambda_\varepsilon^{-1} - \frac{\varepsilon}{4\pi}Q_V(x_\varepsilon) + o(\lambda_\varepsilon^{-1}) + o(\varepsilon), \\ \nabla\phi_a(x_\varepsilon) &= \mathcal{O}(\varepsilon^\mu) \quad \textit{for any } \mu < 1, \\ \lambda_\varepsilon^{-1} &= \mathcal{O}(\varepsilon), \\ \alpha_\varepsilon^4 &= 1 + \frac{64}{3\pi}\phi_0(x_\varepsilon)\lambda_\varepsilon^{-1} + \mathcal{O}(\varepsilon\lambda_\varepsilon^{-1}). \end{aligned} \tag{3-7}$$

The expansion of $\phi_a(x)$ will be of great importance also in the final step of the proof of Theorem 1.5. Indeed, by using the bound on $|\nabla\phi_a(x)|$ we will show that in fact $\phi_a(x) = o(\lambda^{-1}) + o(\varepsilon)$. This allows us to determine $\lim_{\varepsilon\to 0}\varepsilon\lambda_\varepsilon$.

We prove Proposition 3.1 in the following subsections. Again the strategy is to expand suitable energy functionals.

3A. *Bounds on s.*

In this section we record bounds on the function s introduced in (3-4) and on the coefficients β, γ and δ_j defined by the decomposition

$$s = \Pi_{x,\lambda} q =: \lambda^{-1}\beta PU_{x,\lambda} + \gamma\partial_\lambda PU_{x,\lambda} + \lambda^{-3}\sum_{i=1}^{3}\delta_i\partial_{x_i}PU_{x,\lambda}. \tag{3-8}$$

Since $PU_{x,\lambda},\ \partial_\lambda PU_{x,\lambda}$ and $\partial_{x_i}PU_{x,\lambda},\ i = 1, 2, 3$, are linearly independent for sufficiently small ε, the numbers $\beta,\ \gamma$ and $\delta_i,\ i = 1, 2, 3$, (depending on ε, of course) are uniquely determined. The choice of the different powers of λ multiplying these coefficients is motivated by the following proposition.

Proposition 3.2. *The coefficients appearing in* (3-8) *satisfy*

$$\beta,\ \gamma,\ \delta_i = \mathcal{O}(1). \tag{3-9}$$

Moreover, we have the bounds

$$\|s\|_\infty = \mathcal{O}(\lambda^{-1/2}), \quad \|\nabla s\|_2 = \mathcal{O}(\lambda^{-1}) \quad \textit{and} \quad \|s\|_2 = \mathcal{O}(\lambda^{-3/2}), \tag{3-10}$$

as well as

$$\|\nabla s\|_{L^2(\Omega\setminus B_{d/2}(x))} = \mathcal{O}(\lambda^{-3/2}). \tag{3-11}$$

Proof. Because of (3-5), s_ε depends on u_ε only through the parameters λ and x. Since these parameters satisfy the same properties $\lambda \to \infty$ and $d^{-1} = \mathcal{O}(1)$ as in [Frank et al. 2021], the results on s_ε there are applicable. In particular, the bound (3-9) follows from [Frank et al. 2021, Lemma 6.1].

The bounds stated in (3-10) follow readily from (3-8) and (3-9), together with the corresponding bounds on the basis functions $PU_{x,\lambda}$, $\partial_\lambda PU_{x,\lambda}$ and $\partial_{x_i} PU_{x,\lambda}$, $i=1,2,3$, which come from

$$\|U_{x,\lambda}\|_\infty \lesssim \lambda^{1/2}, \quad \|\nabla U_{x,\lambda}\|_2 \lesssim 1, \quad \|U_{x,\lambda}\|_2 \lesssim \lambda^{-1/2},$$

and similar bounds on $\partial_\lambda U_{x,\lambda}$ and $\partial_{x_i} U_{x,\lambda}$, compare Lemma A.1, as well as

$$\|H_0(x,\cdot)\|_2 + \|\nabla_x H_0(x,\cdot)\|_2 + \|\nabla_x \nabla_y H_0(x,y)\|_2 \lesssim 1.$$

It remains to prove (3-11). Again by (3-8) and (3-9), it suffices to show that

$$\lambda^{-1}\|\nabla PU_{x,\lambda}\|_{L^2(\Omega\setminus B_{d/2}(x))} + \|\nabla\partial_\lambda PU_{x,\lambda}\|_{L^2(\Omega\setminus B_{d/2}(x))} + \lambda^{-3}\|\nabla\partial_{x_i} PU_{x,\lambda}\|_{L^2(\Omega\setminus B_{d/2}(x))} \lesssim \lambda^{-3/2}. \tag{3-12}$$

(In fact, there is a better bound on $\nabla\partial_{x_i} PU_{x,\lambda}$, but we do not need this.) Since the three bounds in (3-12) are all proved similarly, we only prove the second one.

By integration by parts, we have

$$\int_{\Omega\setminus B_{d/2}(x)} |\nabla\partial_\lambda PU_{x,\lambda}|^2 = 15\int_{\Omega\setminus B_{d/2}(x)} U_{x,\lambda}^4 \partial_\lambda U_{x,\lambda}\partial_\lambda PU_{x,\lambda} + \int_{\partial B_{d/2}(x)} \frac{\partial(\partial_\lambda PU_{x,\lambda})}{\partial n}\partial_\lambda PU_{x,\lambda}.$$

By the bounds from Lemmas A.1 and A.2, the volume integral is estimated by

$$\int_{\Omega\setminus B_{d/2}(x)} U_{x,\lambda}^4 \partial_\lambda U_{x,\lambda}\partial_\lambda PU_{x,\lambda} \le \int_{\mathbb{R}^3\setminus B_{d/2}(x)} U_{x,\lambda}^4 (\partial_\lambda U_{x,\lambda})^2 + \|\partial_\lambda \varphi_{x,\lambda}\|_\infty \int_{\mathbb{R}^3\setminus B_{d/2}(x)} U_{x,\lambda}^4 |\partial_\lambda U_{x,\lambda}| \lesssim \lambda^{-5}.$$

Since

$$\nabla\partial_\lambda U_{x,\lambda}(y) = \frac{\lambda^{3/2}}{2}\frac{(-5+3\lambda^2|y-x|^2)(y-x)}{(1+\lambda^2|y-x|^2)^{5/2}},$$

we find $|\nabla\partial_\lambda U_{x,\lambda}| \lesssim \lambda^{-3/2}$ on $\partial B_{d/2}(x)$. By the mean value formula for the harmonic function $\partial_\lambda\varphi_{x,\lambda}$ and the bound from Lemma A.2,

$$|\nabla\partial_\lambda\varphi_{x,\lambda}(y)| = \|\partial_\lambda\varphi_{x,\lambda}\|_\infty \lesssim \lambda^{-3/2} \quad \text{for all } y\in\partial B_{d/2}(x).$$

This implies that $|\nabla(\partial_\lambda PU_{x,\lambda})| \lesssim \lambda^{-3/2}$ on $\partial B_{d/2}(x)$. Thus, the boundary integral is estimated by

$$\int_{\partial B_{d/2}(x)} \frac{\partial(\partial_\lambda PU_{x,\lambda})}{\partial n}\partial_\lambda PU_{x,\lambda} = \|\nabla(\partial_\lambda PU_{x,\lambda})\|_{L^\infty(\partial B_{d/2}(x))}(\|\partial_\lambda U_{x,\lambda}\|_{L^\infty(\Omega\setminus B_{d/2}(x))} + \|\partial_\lambda\varphi_{x,\lambda}\|_\infty) \lesssim \lambda^{-3},$$

since $\|\partial_\lambda U_{x,\lambda}\|_{L^\infty(\Omega\setminus B_{d/2}(x))} \lesssim \lambda^{-3/2}$ by Lemma A.1. Collecting these estimates, we find that

$$\|\nabla\partial_\lambda PU_{x,\lambda}\|_{L^2(\Omega\setminus B_{d/2}(x))} \lesssim \lambda^{-3/2},$$

which is the second bound in (3-12). □

Later we will also need the leading order behavior of the zero-mode coefficients β and γ in (3-8).

Proposition 3.3. *As* $\varepsilon \to 0$,

$$\beta = \frac{16}{3\pi}(\phi_a(x) - \phi_0(x)) + \mathcal{O}(\lambda^{-1}), \quad \gamma = -\frac{8}{5}\beta + \mathcal{O}(\lambda^{-1}). \tag{3-13}$$

Proof. According to (3-5), we have

$$\int_\Omega \nabla s \cdot \nabla PU_{x,\lambda} = \lambda^{-1/2} \int_\Omega \nabla(H_a(x,\cdot) - H_0(x,\cdot)) \cdot \nabla PU_{x,\lambda}, \tag{3-14}$$

$$\int_\Omega \nabla s \cdot \nabla \partial_\lambda PU_{x,\lambda} = \lambda^{-1/2} \int_\Omega \nabla(H_a(x,\cdot) - H_0(x,\cdot)) \nabla \partial_\lambda PU_{x,\lambda}. \tag{3-15}$$

By (3-8), the left side of (3-14) is

$$\begin{aligned}\beta\lambda^{-1} \int_\Omega |\nabla PU_{x,\lambda}|^2 + \gamma \int_\Omega \nabla \partial_\lambda PU_{x,\lambda} \cdot \nabla PU_{x,\lambda} + \lambda^{-3} \sum_{i=1}^3 \delta_i \int_\Omega \nabla \partial_{x_i} PU_{x,\lambda} \cdot \nabla PU_{x,\lambda} \\ = 3\beta\lambda^{-1}\frac{\pi^2}{4} + \mathcal{O}(\lambda^{-2}),\end{aligned}$$

where we used the facts that, by [Rey 1990, Appendix B],

$$\begin{gathered}\int_\Omega |\nabla PU_{x,\lambda}|^2 = \frac{3\pi^2}{4} + \mathcal{O}(\lambda^{-1}), \quad \int_\Omega \nabla \partial_\lambda PU_{x,\lambda} \cdot \nabla PU_{x,\lambda} = \mathcal{O}(\lambda^{-2}), \\ \int_\Omega \nabla \partial_{x_i} PU_{x,\lambda} \cdot \nabla PU_{x,\lambda} = \mathcal{O}(\lambda^{-1}).\end{gathered} \tag{3-16}$$

On the other hand, the right side of (3-14) is

$$\begin{aligned}\lambda^{-1/2} \int_\Omega \nabla(H_a(x,\cdot) - H_0(x,\cdot)) \cdot \nabla PU &= 3\lambda^{-1/2} \int_\Omega (H_a(x,\cdot) - H_0(x,\cdot)) U_{x,\lambda}^5 \\ &= 4\pi(\phi_a(x) - \phi_0(x))\lambda^{-1} + \mathcal{O}(\lambda^{-2})\end{aligned} \tag{3-17}$$

by Lemma B.3. Comparing both sides yields the expansion of β stated in (3-13).

Similarly, by (3-8), the left side of (3-15) is

$$\begin{aligned}\frac{\beta}{\lambda^2} \int_\Omega \nabla PU_{x,\lambda} \cdot \nabla \partial_\lambda PU_{x,\lambda} + \gamma \int_\Omega |\nabla \partial_\lambda PU_{x,\lambda}|^2 + \lambda^{-3} \sum_{i=1}^3 \delta_i \int_\Omega \nabla \partial_{x_i} PU_{x,\lambda} \cdot \nabla \partial_\lambda PU_{x,\lambda} \\ = \frac{15\pi^2\gamma}{64\lambda^2} + \mathcal{O}(\lambda^{-3}),\end{aligned}$$

where, besides (3-16), we used $\int_\Omega \nabla \partial_{x_i} PU_{x,\lambda} \cdot \nabla \partial_\lambda PU_{x,\lambda} = \mathcal{O}(\lambda^{-2})$ by [Rey 1990, Appendix B] and

$$\int_\Omega |\nabla \partial_\lambda PU_{x,\lambda}|^2 = \int_\Omega |\nabla \partial_\lambda U_{x,\lambda}|^2 + \mathcal{O}(\lambda^{-3}) = \frac{15\pi^2}{64}\lambda^{-2} + \mathcal{O}(\lambda^{-3}).$$

(The numerical value comes from an explicit evaluation of the integral in terms of beta functions, which we omit.) On the other hand, the right side of (3-15) is

$$\begin{aligned}\lambda^{-1/2} \int_\Omega \nabla(H_a(x,\cdot) - H_0(x,\cdot)) \cdot \nabla \partial_\lambda PU_{x,\lambda} &= 15\lambda^{-1/2} \int_\Omega (H_a(x,\cdot) - H_0(x,\cdot)) U_{x,\lambda}^4 \partial_\lambda U_{x,\lambda} \\ &= -2\pi(\phi_a(x) - \phi_0(x))\lambda^{-2} + \mathcal{O}(\lambda^{-3})\end{aligned}$$

by Lemma B.3. Comparing both sides yields the expansion of γ stated in (3-13). $\square$

3B. ***The bound on $\|\nabla r\|_2$.*** The goal of this subsection is to prove:

Proposition 3.4. *As $\varepsilon \to 0$,*

$$\|\nabla r\|_2 = \mathcal{O}(\phi_a(x)\lambda^{-1}) + \mathcal{O}(\lambda^{-3/2}) + \mathcal{O}(\varepsilon\lambda^{-1/2}). \tag{3-18}$$

Using $\Delta(H_a(x,\cdot) - H_0(x,\cdot)) = -aG_a(x,\cdot)$ and introducing the function $g_{x,\lambda}$ from (A-4), we see that (2-7) for w implies

$$(-\Delta + a)r = -3U_{x,\lambda}^5 + 3\alpha^4(\psi_{x,\lambda} + s + r)^5 + a(f_{x,\lambda} + g_{x,\lambda}) - as - \varepsilon V(\psi_{x,\lambda} + s + r) + \Delta s. \tag{3-19}$$

Integrating against r and using the orthogonality conditions

$$\int_\Omega (\Delta s) r = -\int_\Omega \nabla s \cdot \nabla r = 0 \quad \text{and} \quad 3\int_\Omega U_{x,\lambda}^5 r = \int_\Omega \nabla PU_{x,\lambda} \cdot \nabla r = 0,$$

we obtain

$$\int_\Omega (|\nabla r|^2 + ar^2) = 3\alpha^4 \int_\Omega (\psi_{x,\lambda} + s + r)^5 r - \int_\Omega a(s - f_{x,\lambda} - g_{x,\lambda}) r - \int_\Omega \varepsilon V(\psi_{x,\lambda} + s + r) r. \tag{3-20}$$

The terms appearing in (3-20) satisfy the following bounds.

Lemma 3.5. *As $\varepsilon \to 0$, the following hold*:

(a) $\displaystyle \left| 3\alpha^4 \int_\Omega (\psi_{x,\lambda} + s + r)^5 r - 15\alpha^4 \int_\Omega U_{x,\lambda}^4 r^2 \right| \lesssim (\lambda^{-3/2} + \lambda^{-1}\phi_a(x) + \|r\|_6^2)\|r\|_6.$

(b) $\displaystyle \left| \int_\Omega (a(s - f_{x,\lambda} - g_{x,\lambda}) + \varepsilon V(\psi_{x,\lambda} + s + r)) r \right| \lesssim (\lambda^{-3/2} + \varepsilon\lambda^{-1/2})\|r\|_6.$

Proof. (a) We write $\psi_{x,\lambda} = U_{x,\lambda} - \lambda^{-1/2} H_a(x,\cdot) - f_{x,\lambda}$ and bound pointwise

$$\begin{aligned}(\psi_{x,\lambda} + s + r)^5 = U_{x,\lambda}^5 + 5U_{x,\lambda}^4(s + r) + \mathcal{O}(U_{x,\lambda}^4(\lambda^{-1/2}|H_a(x,\cdot)| + |f_{x,\lambda}|) + U_{x,\lambda}^3(r^2 + s^2)) \\ + \mathcal{O}(\lambda^{-5/2}|H_a(x,\cdot)|^5 + |f_{x,\lambda}|^5 + |r|^5 + |s|^5).\end{aligned} \tag{3-21}$$

When integrated against r, the first term vanishes by orthogonality. Let us bound the contribution coming from the second term, that is, from $5U_{x,\lambda}^4 s$. We write

$$s = \lambda^{-1}\beta U_{x,\lambda} + \gamma\partial_\lambda U_{x,\lambda} + \tilde{s},$$

so $\tilde{s}$ consists of the zero-mode contributions involving the δ_i, plus contributions from the difference between $PU_{x,\lambda}$ and $U_{x,\lambda}$ in the terms involving β and γ. By orthogonality, we have

$$\int_\Omega U_{x,\lambda}^4 s r = \int_\Omega U_{x,\lambda}^4 \tilde{s} r = \mathcal{O}(\|U_{x,\lambda}\|_6^4 \|\tilde{s}\|_6 \|r\|_6),$$

and, by Lemmas A.1 and A.2 as well as Proposition 3.2,

$$\|\tilde{s}\|_6 \le (|\beta| + |\gamma|)(\lambda^{-1}\|\varphi_{x,\lambda}\|_6 + \|\partial_\lambda \varphi_{x,\lambda}\|_6) + \lambda^{-3}\sum_{i=1}^3 |\delta_i| \|\partial_{x_i} PU_{x,\lambda}\|_6 \lesssim \lambda^{-3/2}.$$

This proves

$$\int_\Omega U_{x,\lambda}^4 s r = \mathcal{O}(\lambda^{-3/2}\|r\|_6). \tag{3-22}$$

It remains to bound the remainder terms in (3-21). We write $H_a(x,y) = \phi_a(x) + \mathcal{O}(|x-y|)$ and bound

$$\int_\Omega U_{x,\lambda}^{24/5} |H_a(x,\cdot)|^{6/5} \lesssim \phi_a(x)^{6/5} \int_\Omega U_{x,\lambda}^{24/5} + \int_\Omega U_{x,\lambda}^{24/5} |x-y|^{6/5} \lesssim \lambda^{-3/5}\phi_a(x)^{6/5} + \lambda^{-9/5}.$$

Hence

$$\begin{aligned}\left|\int_\Omega U_{x,\lambda}^4 (\lambda^{-1/2}|H_a(x,\cdot)| + |f_{x,\lambda}|)|r|\right| &\le (\lambda^{-1/2}\|U_{x,\lambda}^4 H_a(x,\cdot)\|_{6/5} + \|U_{x,\lambda}^4\|_{6/5}\|f_{x,\lambda}\|_\infty)\|r\|_6 \\ &\lesssim (\lambda^{-1}\phi_a(x) + \lambda^{-2})\|r\|_6. \qquad (3\text{-}23)\end{aligned}$$

Finally, using Proposition 3.2,

$$\begin{aligned}&\int_\Omega U_{x,\lambda}^3 (r^2+s^2)|r| + \int_\Omega (\lambda^{-5/2}|H_a(x,\cdot)|^5 + |f_{x,\lambda}|^5 + |r|^5 + |s|^5)|r| \\ &\qquad \lesssim (\|r\|_6^2 + \|s\|_6^2 + \lambda^{-5/2} + \|f_{x,\lambda}\|_\infty^5 + \|r\|_6^5 + \|s\|_6^5)\|r\|_6 \lesssim (\|r\|_6^2 + \lambda^{-2})\|r\|_6.\end{aligned}$$

(b) We have

$$\begin{aligned}&\left|\int_\Omega (a(s - f_{x,\lambda} - g_{x,\lambda}) + \varepsilon V(\psi_{x,\lambda} + s + r))r\right| \\ &\qquad \lesssim (\|s\|_{6/5} + \|f_{x,\lambda}\|_{6/5} + \|g_{x,\lambda}\|_{6/5} + \varepsilon\|\psi_{x,\lambda}\|_{6/5} + \varepsilon\|r\|_{6/5})\|r\|_6.\end{aligned}$$

By Proposition 3.2, $\|s\|_{6/5} \lesssim \|s\|_2 \lesssim \lambda^{-3/2}$. By Lemma A.2, $\|f_{x,\lambda}\|_{6/5} \lesssim \|f_{x,\lambda}\|_\infty \lesssim \lambda^{-5/2}$. By Lemma A.4, $\|g_{x,\lambda}\|_{6/5} \lesssim \lambda^{-2}$. By Lemmas A.1 and A.2, $\|\psi_{x,\lambda}\|_{6/5} \lesssim \lambda^{-1/2}$. Finally, $\|r\|_{6/5} \lesssim \|r\|_6$. This proves the claimed bound. ☐

Proof of Proposition 3.4. We deduce from identity (3-20) together with Lemma 3.5 that

$$\int_\Omega (|\nabla r|^2 + ar^2 - 15\alpha^4 U_{x,\lambda}^4 r^2) \lesssim (\lambda^{-1}\phi_a(x) + \lambda^{-3/2} + \varepsilon\lambda^{-1/2} + \|\nabla r\|_2^2 + \varepsilon\|\nabla r\|_2)\|\nabla r\|_2.$$

Since $\alpha^4 \to 1$ and $r \in T_{x,\lambda}^\perp$, the coercivity inequality (2-5) implies that for all sufficiently small $\varepsilon > 0$ the left side is bounded from below by $c\|\nabla r\|_2^2$ with a universal constant $c > 0$. Thus,

$$\|\nabla r\|_2 \lesssim \lambda^{-1}\phi_a(x) + \lambda^{-3/2} + \varepsilon\lambda^{-1/2} + \|\nabla r\|_2^2 + \varepsilon\|\nabla r\|_2.$$

For all sufficiently small $\varepsilon > 0$, the last two terms on the right side can be absorbed into the left side and we obtain the claimed inequality (3-18). ☐

Proposition 3.4 is a first step to prove the bound (3-7) in Proposition 3.1. In Section 3D we will show that $\phi_a(x) = \mathcal{O}(\lambda^{-1} + \varepsilon)$ and $\lambda^{-1} = \mathcal{O}(\varepsilon)$. Combining these with Proposition 3.4 we will obtain (3-7).

3C. *Expanding α^4*. In this subsection, we will prove:

Proposition 3.6. *As $\varepsilon \to 0$,*

$$\alpha^4 = 1 - 4\beta\lambda^{-1} + \mathcal{O}(\phi_a(x)\lambda^{-1} + \lambda^{-2} + \varepsilon\lambda^{-1}), \qquad (3\text{-}24)$$

where β is the zero-mode coefficient from (3-8).

To prove (3-24), we expand the energy identity obtained by integrating the equation for u against u. Writing $u=\alpha(\psi_{x,\lambda}+q)$, this yields

$$\int_\Omega |\nabla(\psi_{x,\lambda}+q)|^2+\int_\Omega (a+\varepsilon V)(\psi_{x,\lambda}+q)^2=3\alpha^4\int_\Omega(\psi_{x,\lambda}+q)^6,$$

which we write as

$$\int_\Omega(|\nabla\psi_{x,\lambda}|^2+(a+\varepsilon V)\psi_{x,\lambda}^2-3\alpha^4\psi_{x,\lambda}^6)+2\int_\Omega(\nabla q\cdot\nabla\psi_{x,\lambda}+(a+\varepsilon V)q\psi_{x,\lambda}-9\alpha^4q\psi_{x,\lambda}^5)=\mathcal{R}_0, \tag{3-25}$$

with

$$\mathcal{R}_0:=-\int_\Omega(|\nabla q|^2+(a+\varepsilon V)q^2)+3\alpha^4\sum_{k=2}^{6}\binom{6}{k}\int_\Omega\psi_{x,\lambda}^{6-k}q^k.$$

The following lemma provides the expansions of the terms in (3-25).

Lemma 3.7. *As $\varepsilon\to 0$, the following hold*:

(a) $\displaystyle\int_\Omega(|\nabla\psi_{x,\lambda}|^2+(a+\varepsilon V)\psi_{x,\lambda}^2-3\alpha^4\psi_{x,\lambda}^6)=(1-\alpha^4)\frac{3\pi^2}{4}+\mathcal{O}(\phi_a(x)\lambda^{-1}+\lambda^{-2}+\varepsilon\lambda^{-1}).$

(b) $\displaystyle\int_\Omega(\nabla q\cdot\nabla\psi_{x,\lambda}+(a+\varepsilon V)q\psi_{x,\lambda}-9\alpha^4q\psi_{x,\lambda}^5)=(1-3\alpha^4)\frac{3\pi^2}{4}\beta\lambda^{-1}+\mathcal{O}(\lambda^{-2}+\varepsilon^2\lambda^{-1}).$

(c) $\mathcal{R}_0=\mathcal{O}(\lambda^{-2}+\varepsilon^2\lambda^{-1})$.

Proof. (a) In [Frank et al. 2021, Theorem 2.1], we have shown the expansions

$$\int_\Omega(|\nabla\psi_{x,\lambda}|^2+(a+\varepsilon V)\psi_{x,\lambda}^2)=\frac{3\pi^2}{4}+\mathcal{O}(\phi_a(x)\lambda^{-1}+\lambda^{-2}+\varepsilon\lambda^{-1}),$$
$$3\int_\Omega\psi_{x,\lambda}^6=\frac{3\pi^2}{4}+\mathcal{O}(\phi_a(x)\lambda^{-1}+\lambda^{-2}),$$

which immediately imply the bound in (a).

(b) Since $\Delta(H_a(x,\cdot)-H_0(x,\cdot))=-aG_a(x,\cdot)$, we have $-\Delta\psi_{x,\lambda}=3U_{x,\lambda}^5-\lambda^{-1/2}aG_a(x,\cdot)$. Since $\psi_{x,\lambda}=\lambda^{-1/2}G_a(x,\cdot)-f_{x,\lambda}-g_{x,\lambda}$ with $g_{x,\lambda}$ from (A-4), we can rewrite this as

$$-\Delta\psi_{x,\lambda}+a\psi_{x,\lambda}=3U_{x,\lambda}^5-a(f_{x,\lambda}+g_{x,\lambda}). \tag{3-26}$$

Thus,

$$\begin{aligned}\int_\Omega(\nabla q\cdot\nabla\psi_{x,\lambda}&+(a+\varepsilon V)q\psi_{x,\lambda}-9\alpha^4q\psi_{x,\lambda}^5)\\&=3(1-3\alpha^4)\int_\Omega qU_{x,\lambda}^5-\int_\Omega q(9\alpha^4(\psi_{x,\lambda}^5-U_{x,\lambda}^5)+a(f_{x,\lambda}+g_{x,\lambda})+\varepsilon V\psi_{x,\lambda}).\end{aligned}$$

By orthogonality and the computations in the proof of Proposition 3.3,

$$3\int_\Omega qU_{x,\lambda}^5=\int_\Omega\nabla s\cdot\nabla PU_{x,\lambda}=\frac{3\pi^2}{4}\beta\lambda^{-1}+\mathcal{O}(\lambda^{-2}).$$

Moreover,

$$\left|\int_\Omega q(9\alpha^4(\psi_{x,\lambda}^5 - U_{x,\lambda}^5) + a(f_{x,\lambda}+g_{x,\lambda}) + \varepsilon V\psi_{x,\lambda})\right| \\ \lesssim \|q\|_6(\|\psi_{x,\lambda}^5 - U_{x,\lambda}^5\|_{6/5} + \|f_{x,\lambda}\|_{6/5} + \|g_{x,\lambda}\|_{6/5} + \varepsilon\|\psi_{x,\lambda}\|_{6/5}).$$

By Propositions 3.2 and 3.4 we have

$$\|q\|_6 \lesssim \|\nabla q\|_2 \lesssim \lambda^{-1} + \varepsilon\lambda^{-1/2}, \tag{3-27}$$

by Lemma A.2 we have $\|f_{x,\lambda}\|_\infty \lesssim \lambda^{-5/2}$ and, by Lemma A.4 we have $\|g_{x,\lambda}\|_{6/5} \lesssim \lambda^{-2}$. Moreover, writing $\psi_{x,\lambda} = U_{x,\lambda} - \lambda^{-1/2}H_a(x,\cdot) - f_{x,\lambda}$ and using Lemmas A.1 and A.2 and (B-1), we get $\|\psi_{x,\lambda}\|_{6/5} \lesssim \lambda^{-1/2}$. Also, bounding

$$|\psi_{x,\lambda}^5 - U_{x,\lambda}^5| \lesssim \psi_{x,\lambda}^4(\lambda^{-1/2}|H_a(x,\cdot)| + |f_{x,\lambda}|) + \lambda^{-5/2}|H_a(x,\cdot)|^5 + |f_{x,\lambda}|^5,$$

we obtain from Lemmas A.1 and A.2 and (B-1)

$$\|\psi_{x,\lambda}^5 - U_{x,\lambda}^5\|_{6/5} \lesssim \lambda^{-1/2}\|\psi_{x,\lambda}\|_{24/5}^4 + \lambda^{-5/2} \lesssim \lambda^{-1}.$$

Collecting all the terms, we obtain the claimed bound.

(c) Because of the second inequality in (3-27), the first integral in the definition of $\mathcal{R}_0$ is $\mathcal{O}(\lambda^{-2}+\varepsilon^2\lambda^{-1})$. The second integral is bounded, in absolute value, by a constant times

$$\int_\Omega (\psi_{x,\lambda}^4 q^2 + q^6) \le \|\psi_{x,\lambda}\|_6^4\|q\|_6^2 + \|q\|_6^6 \lesssim \lambda^{-2} + \varepsilon^2\lambda^{-1}.$$

This completes the proof. ☐

Proof of Proposition 3.6. The claim follows from (3-25) and Lemma 3.7. ☐

3D. *Expanding $\phi_a(x)$.* In this subsection we prove the following important expansion.

Proposition 3.8. *As $\varepsilon \to 0$,*

$$\phi_a(x) = \pi a(x)\lambda^{-1} - \frac{\varepsilon}{4\pi}Q_V(x) + o(\lambda^{-1}) + o(\varepsilon) \tag{3-28}$$

Before proving it, let us note the following consequence.

Corollary 3.9. *We have $\phi_a(x_0) = 0$, $Q_V(x_0) \le 0$ and*

$$\lambda^{-1} = \mathcal{O}(\varepsilon), \tag{3-29}$$

as $\varepsilon \to 0$. Moreover, $\|\nabla r\|_2 = \mathcal{O}(\varepsilon\lambda^{-1/2})$ and $\alpha^4 = 1 + \frac{64}{3\pi}\phi_0(x)\lambda^{-1} + \mathcal{O}(\varepsilon\lambda^{-1})$.

Proof. The fact that $\phi_a(x_0) = 0$ follows immediately from (3-28). Since $\phi_a(x) \ge 0$ by criticality and since $a(x_0) < 0$ by assumption, we deduce from (3-28) that $Q_V(x_0) \le 0$ and that

$$\lambda^{-1} \le \frac{|Q_V(x_0)| + o(1)}{4\pi^2|a(x_0)| + o(1)}\varepsilon = \mathcal{O}(\varepsilon).$$

Reinserting this into (3-28), we find $\phi_a(x) = \mathcal{O}(\varepsilon)$. Inserting this into Proposition 3.4, we obtain the claimed bound on $\|\nabla r\|_2$, and inserting it into (3-24) and (3-13), we obtain the claimed expansion of α^4. □

The proof of (3-28) is based on the Pohozaev identity obtained by integrating the equation for u against $\partial_\lambda \psi_{x,\lambda}$. We write the resulting equality in the form

$$\begin{aligned}\int_\Omega (\nabla \psi_{x,\lambda} \cdot \nabla \partial_\lambda \psi_{x,\lambda} + (a+\varepsilon V)\psi_{x,\lambda}\partial_\lambda \psi_{x,\lambda} - 3\alpha^4 \psi_{x,\lambda}^5 \partial_\lambda \psi_{x,\lambda}) \\ = -\int_\Omega (\nabla q \cdot \nabla \partial_\lambda \psi_{x,\lambda} + aq\partial_\lambda \psi_{x,\lambda} - 15\alpha^4 q \psi_{x,\lambda}^4 \partial_\lambda \psi_{x,\lambda}) + 30\alpha^4 \int_\Omega q^2 \psi_{x,\lambda}^3 \partial_\lambda \psi_{x,\lambda} + \mathcal{R}, \end{aligned} \tag{3-30}$$

with

$$\mathcal{R} = -\varepsilon \int_\Omega Vq\partial_\lambda \psi_{x,\lambda} + 3\alpha^4 \sum_{k=3}^{5} \binom{5}{k} \int_\Omega \psi_{x,\lambda}^{5-k} q^k \partial_\lambda \psi_{x,\lambda}.$$

The involved terms can be expanded as follows.

Lemma 3.10. *As $\varepsilon \to 0$, the following hold*:

(a) $$\begin{aligned}\int_\Omega (\nabla \psi_{x,\lambda} \cdot \nabla \partial_\lambda \psi_{x,\lambda} + (a+\varepsilon V)\psi_{x,\lambda}\partial_\lambda \psi_{x,\lambda} - 3\alpha^4 \psi_{x,\lambda}^5 \partial_\lambda \psi_{x,\lambda}) \\ = -2\pi\phi_a(x)\lambda^{-2} - \tfrac{1}{2} Q_V(x)\varepsilon\lambda^{-2} + (1-\alpha^4)4\pi\phi_a(x)\lambda^{-2} + (2\pi^2 a(x) + 15\pi^2 \phi_a(x)^2)\lambda^{-3} \\ + o(\lambda^{-3}) + o(\varepsilon\lambda^{-2}).\end{aligned}$$

(b) $$\begin{aligned}\int_\Omega (\nabla q \cdot \nabla \partial_\lambda \psi_{x,\lambda} + aq\partial_\lambda \psi_{x,\lambda} - 15\alpha^4 q \psi_{x,\lambda}^4 \partial_\lambda \psi_{x,\lambda}) \\ = -(1-\alpha^4)2\pi(\phi_a(x) - \phi_0(x))\lambda^{-2} + \mathcal{O}(\phi_a(x)\lambda^{-3}) + o(\varepsilon\lambda^{-2}) + o(\lambda^{-3}).\end{aligned}$$

(c) $$30\alpha^4 \int_\Omega q^2 \psi_{x,\lambda}^3 \partial_\lambda \psi_{x,\lambda} = \frac{15\pi^2}{16}\beta\gamma\lambda^{-3} + \mathcal{O}(\phi_a(x)\lambda^{-3}) + o(\varepsilon\lambda^{-2}) + o(\lambda^{-3}).$$

(d) $\mathcal{R} = \mathcal{O}(\phi_a(x)\lambda^{-3}) + o(\varepsilon\lambda^{-2}) + o(\lambda^{-3})$.

We emphasize that the proof of Lemma 3.10 is independent of the expansion of α^4 in (3-24). We only use the fact that $\alpha = 1 + o(1)$.

Proof. (a) Because of (3-26), the quantity of interest can be written as

$$\begin{aligned}\int_\Omega (\nabla \psi_{x,\lambda} \cdot \nabla \partial_\lambda \psi_{x,\lambda} + (a+\varepsilon V)\psi_{x,\lambda}\partial_\lambda \psi_{x,\lambda} - 3\alpha^4 \psi_{x,\lambda}^5 \partial_\lambda \psi_{x,\lambda}) \\ = 3\int_\Omega (U_{x,\lambda}^5 - \alpha^4 \psi_{x,\lambda}^5)\partial_\lambda \psi_{x,\lambda} - \int_\Omega a(f_{x,\lambda} + g_{x,\lambda})\partial_\lambda \psi_{x,\lambda} + \varepsilon \int_\Omega V\psi_{x,\lambda}\partial_\lambda \psi_{x,\lambda}.\end{aligned} \tag{3-31}$$

We discuss the three integrals on the right side separately. As a general rule, terms involving $f_{x,\lambda}$ will be negligible as a consequence of the bounds $\|f_{x,\lambda}\|_\infty = \mathcal{O}(\lambda^{-5/2})$ and $\|\partial_\lambda f_{x,\lambda}\|_\infty = \mathcal{O}(\lambda^{-7/2})$ in Lemma A.2. This will not always be carried out in detail.

We have

$$\int_\Omega (U_{x,\lambda}^5 - \alpha^4 \psi_{x,\lambda}^5)\partial_\lambda \psi_{x,\lambda} = (1-\alpha^4)\int_\Omega U_{x,\lambda}^5 \partial_\lambda \psi_{x,\lambda} + \alpha^4 \int_\Omega (U_{x,\lambda}^5 - \psi_{x,\lambda}^5)\partial_\lambda \psi_{x,\lambda}. \tag{3-32}$$

The first integral is, since $\psi_{x,\lambda} = U_{x,\lambda} - \lambda^{-1/2} H_a(x,\cdot) - f_{x,\lambda}$,

$$\int_\Omega U_{x,\lambda}^5 \partial_\lambda \psi_{x,\lambda} = \int_\Omega U_{x,\lambda}^5 \partial_\lambda U_{x,\lambda} + \frac{1}{2}\lambda^{-3/2}\int_\Omega U_{x,\lambda}^5 H_a(x,\cdot) + \mathcal{O}(\lambda^{-4}). \tag{3-33}$$

Since $\int_{\mathbb{R}^3} U_{x,\lambda}^5 \partial_\lambda U_{x,\lambda} = \frac{1}{6}\partial_\lambda \int_{\mathbb{R}^3} U_{x,\lambda}^6 = 0$, we have

$$\left|\int_\Omega U_{x,\lambda}^5 \partial_\lambda U_{x,\lambda}\right| = \left|\int_{\mathbb{R}^3\setminus\Omega} U_{x,\lambda}^5 \partial_\lambda U_{x,\lambda}\right| \lesssim \lambda^{-1}\int_{d\lambda}^\infty \left|\frac{r^2-r^4}{(1+r^2)^4}\right| dr = \mathcal{O}(\lambda^{-4}). \tag{3-34}$$

Next, by Lemma B.3,

$$\frac{1}{2}\lambda^{-3/2}\int_\Omega U_{x,\lambda}^5 H_a(x,\cdot) = \frac{2\pi}{3}\phi_a(x)\lambda^{-2} + \mathcal{O}(\lambda^{-3}).$$

This completes our discussion of the first term on the right side of (3-32). For the second term we have similarly,

$$\begin{aligned}
&\int_\Omega (U_{x,\lambda}^5 - \psi_{x,\lambda}^5)\partial_\lambda \psi_{x,\lambda} \\
&\quad= \int_\Omega (U_{x,\lambda}^5 - (U_{x,\lambda} - \lambda^{-1/2}H_a(x,\cdot))^5)\partial_\lambda (U_{x,\lambda} - \lambda^{-1/2}H_a(x,\cdot)) + o(\lambda^{-3}) \\
&\quad= 5\lambda^{-1/2}\int_\Omega U_{x,\lambda}^4 H_a(x,\cdot)\partial_\lambda U_{x,\lambda} + \frac{5}{2}\lambda^{-2}\int_\Omega U_{x,\lambda}^4 H_a(x,\cdot)^2 - 10\lambda^{-1}\int_\Omega U_{x,\lambda}^3 H_a(x,\cdot)^2 \partial_\lambda U_{x,\lambda} \\
&\qquad + \sum_{k=3}^5 \binom{5}{k}(-1)^k \lambda^{-k/2}\int_\Omega U_{x,\lambda}^{5-k} H_a(x,\cdot)^k \partial_\lambda U_{x,\lambda} \\
&\qquad\qquad - \frac{1}{2}\sum_{k=2}^5 \binom{5}{k}(-1)^k \lambda^{-(k+3)/2}\int_\Omega U_{x,\lambda}^{5-k} H_a(x,\cdot)^{k+1} + o(\lambda^{-3}).
\end{aligned} \tag{3-35}$$

Again, by Lemma B.3,

$$\begin{aligned}
&5\lambda^{-1/2}\int_\Omega U_{x,\lambda}^4 H_a(x,\cdot)\partial_\lambda U_{x,\lambda} + \frac{5}{2}\lambda^{-2}\int_\Omega U_{x,\lambda}^4 H_a(x,\cdot)^2 - 10\lambda^{-1}\int_\Omega U_{x,\lambda}^3 H_a(x,\cdot)^2 \partial_\lambda U_{x,\lambda} \\
&\qquad = -\frac{2\pi}{3}\phi_a(x)\lambda^{-2} + (2\pi a(x) + 5\pi^2 \phi_a(x)^2)\lambda^{-3} + o(\lambda^{-3}).
\end{aligned} \tag{3-36}$$

Finally, the two sums are bounded, in absolute value, by

$$\begin{aligned}
&\int_\Omega (U_{x,\lambda}^2 \lambda^{-3/2}|H_a(x,\cdot)|^3 + \lambda^{-5/2}|H_a(x,\cdot)|^5)|\partial_\lambda U_{x,\lambda}| + \int_\Omega (U_{x,\lambda}^3 \lambda^{-5/2}|H_a(x,\cdot)|^3 + \lambda^{-4}|H_a(x,\cdot)|^6) \\
&\qquad \lesssim \|\partial_\lambda U_{x,\lambda}\|_6 (\|U_{x,\lambda}\|_{12/5}^2 \lambda^{-3/2} + \lambda^{-5/2}) + \|U_{x,\lambda}\|_3^3 \lambda^{-5/2} + \lambda^{-4} = o(\lambda^{-3}).
\end{aligned}$$

This completes our discussion of the second term on the right side of (3-32) and therefore of the first term on the right side of (3-31).

For the second term on the right side of (3-31) we get, using $\psi_{x,\lambda} = U_{x,\lambda} - \lambda^{-1/2} H_a(x,\cdot) - f_{x,\lambda}$,

$$\int_\Omega a(f_{x,\lambda} + g_{x,\lambda})\partial_\lambda \psi_{x,\lambda} = \int_\Omega a g_{x,\lambda} \partial_\lambda U_{x,\lambda} + \frac{1}{2}\lambda^{-3/2} \int_\Omega a g_{x,\lambda} H_a(x,\cdot) + o(\lambda^{-3}).$$

The second integral is negligible since, by Lemma A.4,

$$\left| \frac{1}{2}\lambda^{-3/2} \int_\Omega a g_{x,\lambda} H_a(x,\cdot) \right| \lesssim \lambda^{-3/2} \int_\Omega g_{x,\lambda} \lesssim \lambda^{-4} \log \lambda.$$

Since a is differentiable, we can expand the first integral as

$$\int_\Omega a g_{x,\lambda} \partial_\lambda U_{x,\lambda} = a(x) \int_\Omega g_{x,\lambda} \partial_\lambda U_{x,\lambda} + \mathcal{O}\left(\int_\Omega |x-y| g_{x,\lambda} |\partial_\lambda U_{x,\lambda}| \right).$$

We have

$$\int_\Omega g_{x,\lambda} \partial_\lambda U_{x,\lambda} = \lambda^{-3} \int_{\lambda(\Omega - x)} g_{0,1} \partial_\lambda U_{0,1} = \lambda^{-3} \int_{\mathbb{R}^3} g_{0,1} \partial_\lambda U_{0,1} + o(\lambda^{-3})$$

and

$$\int_{\mathbb{R}^3} g_{0,1} \partial_\lambda U_{0,1} = 4\pi \int_0^\infty \left(\frac{1}{r} - \frac{1}{\sqrt{1+r^2}} \right) \frac{1-r^2}{2(1+r^2)^{3/2}} r^2 \, dr = 2\pi(3-\pi).$$

Using similar bounds one verifies that

$$\int_\Omega |x-y| g_{x,\lambda} |\partial_\lambda U_{x,\lambda}| \lesssim \lambda^{-4} \int_{\lambda(\Omega-x)} |z| g_{0,1} |\partial_\lambda U_{0,1}| \lesssim \lambda^{-4}.$$

This completes our discussion of the second term on the right side of (3-31).

For the third term on the right side of (3-31), we write

$$\psi_{x,\lambda} = \lambda^{-1/2} G_a(x,\cdot) - f_{x,\lambda} - g_{x,\lambda}$$

and get

$$\begin{aligned}
&\int_\Omega V \psi_{x,\lambda} \partial_\lambda \psi_{x,\lambda} \\
&\quad = \int_\Omega V(\lambda^{-1/2} G_a(x,\cdot) - g_{x,\lambda}) \partial_\lambda (\lambda^{-1/2} G_a(x,\cdot) - g_{x,\lambda}) + o(\lambda^2) \\
&\quad = -\tfrac{1}{2}\lambda^{-2} Q_V(x) + \mathcal{O}\left(\lambda^{-3/2} \int_\Omega G_a(x,\cdot) g_{x,\lambda} + \lambda^{-1/2} \int_\Omega G_a(x,\cdot) |\partial_\lambda g_{x,\lambda}| + \int_\Omega g_{x,\lambda} |\partial_\lambda g_{x,\lambda}| \right) + o(\lambda^2) \\
&\quad = -\tfrac{1}{2}\lambda^{-2} Q_V(x) + \mathcal{O}\big(\lambda^{-3/2} \|G_a(x,\cdot)\|_2 \|g_{x,\lambda}\|_2 \\
&\qquad\qquad + \lambda^{-1} \|G_a(x,\cdot)\|_2 \|\partial_\lambda g_{x,\lambda}\|_2 + \|g_{x,\lambda}\|_2 \|\partial_\lambda g_{x,\lambda}\|_2 \big) + o(\lambda^{-2}) \\
&\quad = -\tfrac{1}{2}\lambda^{-2} Q_V(x) + o(\lambda^{-2}).
\end{aligned}$$

In the last equality we used the bounds from Lemma A.4 and the fact that $G_a(x,\cdot) \in L^2(\Omega)$. This completes our discussion of the third term on the right side of (3-31) and concludes the proof of (a).

(b) We note that (3-26) yields

$$-\Delta\partial_\lambda\psi_{x,\lambda}+a\partial_\lambda\psi_{x,\lambda}=15U_{x,\lambda}^4\partial_\lambda U_{x,\lambda}-a(\partial_\lambda f_{x,\lambda}+\partial_\lambda g_{x,\lambda}).$$

Because of this equation, the quantity of interest can be written as

$$\begin{aligned}\int_\Omega(\nabla q\cdot\nabla\partial_\lambda\psi_{x,\lambda}&+aq\partial_\lambda\psi_{x,\lambda}-15\alpha^4q\psi_{x,\lambda}^4\partial_\lambda\psi_{x,\lambda})\\&=15\int_\Omega q(U_{x,\lambda}^4\partial_\lambda U_{x,\lambda}-\alpha^4\psi_{x,\lambda}^4\partial_\lambda\psi_{x,\lambda})-\int_\Omega aq(\partial_\lambda f_{x,\lambda}+\partial_\lambda g_{x,\lambda}).\end{aligned}\tag{3-37}$$

We discuss the two integrals on the right side separately.

We have

$$\begin{aligned}\int_\Omega q(U_{x,\lambda}^4&\partial_\lambda U_{x,\lambda}-\alpha^4\psi_{x,\lambda}^4\partial_\lambda\psi_{x,\lambda})\\&=(1-\alpha^4)\int_\Omega qU_{x,\lambda}^4\partial_\lambda U_{x,\lambda}+\alpha^4\int_\Omega q(U_{x,\lambda}^4\partial_\lambda U_{x,\lambda}-\psi_{x,\lambda}^4\partial_\lambda\psi_{x,\lambda}).\end{aligned}\tag{3-38}$$

The first integral is, by the orthogonality condition $0=\int_\Omega\nabla w\cdot\nabla\partial_\lambda PU_{x,\lambda}=15\int_\Omega wU_{x,\lambda}^4\partial_\lambda U_{x,\lambda}$,

$$\begin{aligned}\int_\Omega qU_{x,\lambda}^4\partial_\lambda U_{x,\lambda}&=\lambda^{-1/2}\int_\Omega(H_a(x,\cdot)-H_0(x,\cdot))U_{x,\lambda}^4\partial_\lambda U_{x,\lambda}\\&=-\frac{2\pi}{15}(\phi_a(x)-\phi_0(x))\lambda^{-2}+\mathcal{O}(\lambda^{-3}).\end{aligned}\tag{3-39}$$

For the second integral on the right side of (3-38), we have

$$\begin{aligned}\int_\Omega q(U_{x,\lambda}^4&\partial_\lambda U_{x,\lambda}-\psi_{x,\lambda}^4\partial_\lambda\psi_{x,\lambda})\\&=\int_\Omega q(U_{x,\lambda}^4\partial_\lambda U_{x,\lambda}-(U_{x,\lambda}-\lambda^{-1/2}H_a(x,\cdot))^4\partial_\lambda(U_{x,\lambda}-\lambda^{-1/2}H_a(x,\cdot)))+o(\lambda^{-3})\\&=\mathcal{O}(\phi_a(x)\lambda^{-3})+o(\varepsilon\lambda^{-2})+o(\lambda^{-3}).\end{aligned}\tag{3-40}$$

Let us justify the claimed bound here for a typical term. We write $H_a(x,y)=\phi_a(x)+\mathcal{O}(|x-y|)$ and get

$$\int_\Omega qU_{x,\lambda}^4\lambda^{-3/2}H_a(x,\cdot)=\lambda^{-3/2}\phi_a(x)\int_\Omega qU_{x,\lambda}^4+\mathcal{O}\Big(\lambda^{-3/2}\int_\Omega qU_{x,\lambda}^4|x-y|\Big).$$

Using the bound (3-27) on q and Lemma A.1 we get

$$\left|\int_\Omega qU_{x,\lambda}^4\right|\le\|q\|_6\|U_{x,\lambda}\|_{24/5}^4\lesssim\lambda^{-3/2}+\varepsilon\lambda^{-1}.$$

The remainder term is better because of the additional factor of $|x-y|$. We gain a factor of λ^{-1} since

$$\||x-\cdot|^{1/4}U_{x,\lambda}\|_{24/5}^4\lesssim\lambda^{-3/2}.$$

Another typical term,

$$\int_\Omega qU_{x,\lambda}^3\lambda^{-1/2}H_a(x,\cdot)\partial_\lambda U_{x,\lambda},$$

can be treated in the same way, since the bounds for $\partial_\lambda U_{x,\lambda}$ are the same as for $\lambda^{-1} U_{x,\lambda}$; see Lemma A.1. The remaining terms are easier. This completes our discussion of the first term on the right side of (3-37).

The second term on the right side of (3-37) is negligible. Indeed,

$$\int_\Omega aq(\partial_\lambda f_{x,\lambda} + \partial_\lambda g_{x,\lambda}) = \mathcal{O}(\|q\|_6 \|\partial_\lambda g_{x,\lambda}\|_{6/5}) + o(\lambda^{-3}) = o(\lambda^{-3}), \tag{3-41}$$

where we used Lemma A.4 and the same bound on q as before. This completes our discussion of the second term on the right side of (3-37) and concludes the proof of (b).

(c) We use the form (3-8) of the zero modes s, as well as the bounds on $\|\nabla s\|_2$ and $\|\nabla r\|_2$ from (3-10) and (3-18), to find

$$\begin{aligned}\int_\Omega q^2 \psi_{x,\lambda}^3 \partial_\lambda \psi_{x,\lambda} &= \int_\Omega s^2 \psi_{x,\lambda}^3 \partial_\lambda \psi_{x,\lambda} + \mathcal{O}(\phi_a(x)\lambda^{-3}) + o(\lambda^{-3}) + o(\varepsilon\lambda^{-2}) \\ &= \beta^2 \lambda^{-2} \int_\Omega U_{x,\lambda}^5 \partial_\lambda U_{x,\lambda} + 2\beta\gamma\lambda^{-1} \int_\Omega U_{x,\lambda}^4 (\partial_\lambda U_{x,\lambda})^2 + \gamma^2 \int_\Omega U_{x,\lambda}^3 (\partial_\lambda U_{x,\lambda})^3 \\ &\qquad + \mathcal{O}(\phi_a(x)\lambda^{-3}) + o(\lambda^{-3}) + o(\varepsilon\lambda^{-2}). \end{aligned} \tag{3-42}$$

A direct calculation using (B-15) gives

$$\lambda^{-2} \int_\Omega U_{x,\lambda}^5 \partial_\lambda U_{x,\lambda} = o(\lambda^{-3}), \quad \int_\Omega U_{x,\lambda}^3 (\partial_\lambda U_{x,\lambda})^3 = o(\lambda^{-3})$$

and

$$\begin{aligned}\int_\Omega U_{x,\lambda}^4 (\partial_\lambda U_{x,\lambda})^2 &= \frac{1}{4}\lambda^{-2} \int_\Omega U_{x,\lambda}^6 - \lambda^3 \int_\Omega \frac{|x-y|^2}{(1+\lambda^2|x-y|^2)^4} + \lambda^5 \int_\Omega \frac{|x-y|^4}{(1+\lambda^2|x-y|^2)^5} \\ &= \frac{\pi^2}{16}\lambda^{-2} - 4\pi\lambda^{-2} \int_0^\infty \frac{t^4\,dt}{(1+t^2)^4} + 4\pi\lambda^{-2} \int_0^\infty \frac{t^6\,dt}{(1+t^2)^5} + o(\lambda^{-2}) \\ &= \frac{\pi^2}{64}\lambda^{-2} + o(\lambda^{-2}). \end{aligned}$$

Inserting this into (3-42) gives the claimed expansion (c).

The proof of (d) uses similar bounds as in the rest of the proof and is omitted. □

Proof of Proposition 3.8. Combining (3-30) with Lemma 3.10 yields

$$0 = -4\pi\phi_a(x)\lambda^{-2} - Q_V(x)\varepsilon\lambda^{-2} + 4\pi^2 a(x)\lambda^{-3} + \lambda^{-3} R + \mathcal{O}(\phi_a(x)\lambda^{-3}) + o(\lambda^{-3}) + o(\varepsilon\lambda^{-2}), \tag{3-43}$$

with

$$R = \lambda(1-\alpha^4)4\pi(\phi_a(x) + \phi_0(x)) + 30\pi^2\phi_a(x)^2 - \tfrac{15}{8}\beta\gamma\pi^2.$$

We now make use of the expansion (3-24) of $\alpha^4 - 1$ and obtain

$$R = 16\beta\pi\phi_0(x) - \tfrac{15}{8}\beta\gamma\pi^2 + \mathcal{O}(\phi_a(x) + \lambda^{-1} + \varepsilon).$$

Inserting the expansions (3-13) of β and γ, we find the cancellation

$$R = \mathcal{O}(\phi_a(x) + \lambda^{-1} + \varepsilon). \tag{3-44}$$

In particular, $R = \mathcal{O}(1)$ and, inserting this into (3-43), we obtain

$$\phi_a(x) = \mathcal{O}(\lambda^{-1} + \varepsilon).$$

In particular, for the error term in (3-43), we have $\phi_a(x)\lambda^{-3} = o(\lambda^{-3})$ and, moreover, by (3-44), we have $R = \mathcal{O}(\lambda^{-1} + \varepsilon)$. Inserting this bound into (3-43), we obtain the claimed expansion (3-28). $\square$

3E. *Bounding $\nabla\phi_a(x)$.* In this subsection we prove the bound on $\nabla\phi_a(x)$ in Proposition 3.1.

Proposition 3.11. *For every $\mu < 1$, as $\varepsilon \to 0$,*

$$|\nabla\phi_a(x)| \lesssim \varepsilon^{\mu}. \tag{3-45}$$

The proof of this proposition is a refined version of the proof of Proposition 2.5. It is also based on expanding the Pohozaev identity (2-9). Abbreviating, for $v, z \in H^1(\Omega)$,

$$I[v, z] := \int_{\partial\Omega} \frac{\partial v}{\partial n}\frac{\partial z}{\partial n} n + \int_{\Omega} (\nabla a) v z \tag{3-46}$$

and writing $u = \alpha(\psi_{x,\lambda} + q)$, we can write identity (2-9) as

$$0 = I[\psi_{x,\lambda}] + 2I[\psi_{x,\lambda}, q] + I[q] + \varepsilon \int_{\Omega} (\nabla V)(\psi_{x,\lambda} + q)^2. \tag{3-47}$$

The following lemma extracts the leading contribution from the main term $I[\psi_{x,\lambda}]$.

Lemma 3.12. $I[\psi_{x,\lambda}] = 4\pi\nabla\phi_a(x)\lambda^{-1} + \mathcal{O}(\lambda^{-1-\mu})$ *for every $\mu < 1$.*

On the other hand, the next lemma allows us to control the error terms involving q.

Lemma 3.13.
$$\left\| \frac{\partial q}{\partial n} \right\|_{L^2(\partial\Omega)} \lesssim \varepsilon\lambda^{-1/2}.$$

Before proving these two lemmas, let us use them to give the proof of Proposition 3.11. In that proof, and later in this subsection, we will use the inequality

$$\|q\|_2 \lesssim \varepsilon\lambda^{-1/2}. \tag{3-48}$$

This follows from the bound (3-10) on s and the bounds in Corollary 3.9 on λ^{-1} and r. Note that (3-48) is better than the bound (3-27) in the L^6 norm.

Proof of Proposition 3.11. We shall make use of the bounds

$$\|\psi_{x,\lambda}\|_2 + \left\| \frac{\partial \psi_{x,\lambda}}{\partial n} \right\|_{L^2(\partial\Omega)} \lesssim \lambda^{-1/2}. \tag{3-49}$$

The first bound follows by writing $\psi_{x,\lambda} = U_{x,\lambda} - \lambda^{-1/2} H_a(x, \cdot) + f_{x,\lambda}$ and using the bounds in Lemmas A.1 and A.2 and in (B-1). We write $\psi_{x,\lambda} = PU_{x,\lambda} - \lambda^{-1/2}(H_a(x, \cdot) - H_0(x, \cdot))$ and use the bounds in Lemmas A.3 and B.1 for the second bound.

Combining the bounds (3-49) with the corresponding bounds for q from Lemma 3.13 and (3-48), we obtain

$$|I[\psi_{x,\lambda}, q]| \lesssim \varepsilon\lambda^{-1} \quad \text{and} \quad I[q] \lesssim \varepsilon^2\lambda^{-1}.$$

Moreover, by (3-48) and (3-49),

$$\varepsilon\left|\int_\Omega (\nabla V)(\psi_{x,\lambda}+q)^2\right| \lesssim \varepsilon\lambda^{-1}.$$

In view of these bounds, Lemma 3.12 and (3-47) imply $|\nabla\phi_a(x)| \lesssim \varepsilon+\lambda^{-\mu}$. Because of (3-29), this implies (3-45). □

It remains to prove Lemmas 3.12 and 3.13.

Proof of Lemma 3.12. We integrate (3-26) for $\psi_{x,\lambda}$ against $\nabla\psi_{x,\lambda}$ and obtain

$$-\tfrac{1}{2}I[\psi_{x,\lambda}] = 3\int_\Omega U_{x,\lambda}^5\nabla\psi_{x,\lambda} - \int_\Omega a(f_{x,\lambda}+g_{x,\lambda})\nabla\psi_{x,\lambda}. \tag{3-50}$$

For the first integral on the right side we write $\psi_{x,\lambda} = U_{x,\lambda} - \lambda^{-1/2}H_a(x,\cdot) + f_{x,\lambda}$ and integrate by parts to obtain

$$\begin{aligned}3\int_\Omega U_{x,\lambda}^5\nabla\psi_{x,\lambda} = 3\int_{\partial\Omega} U_{x,\lambda}^5\bigl(\tfrac{1}{6}U_{x,\lambda} - \lambda^{-1/2}H_a(x,\cdot) + f_{x,\lambda}\bigr)n \\ + 15\int_\Omega U_{x,\lambda}^4(\nabla U_{x,\lambda})(\lambda^{-1/2}H_a(x,\cdot) - f_{x,\lambda}).\end{aligned}$$

By Lemma B.3 (see also Remark B.4) we have

$$\int_\Omega U_{x,\lambda}^4(\nabla U_{x,\lambda})H_a(x,\cdot) = -\int_\Omega U_{x,\lambda}^4(\nabla_x U_{x,\lambda})H_a(x,\cdot) = -\frac{2\pi}{15}\nabla\phi_a(x)\lambda^{-1/2} + \mathcal{O}(\lambda^{-1/2-\mu}).$$

Finally, since $U_{x,\lambda} \lesssim \lambda^{-1/2}$ on $\partial\Omega$ and by the bounds on $U_{x,\lambda}$, $f_{x,\lambda}$ and $H_a(x,\cdot)$ from Lemmas A.1 and A.2 and from (B-1), we have

$$3\int_{\partial\Omega} U_{x,\lambda}^5\bigl(\tfrac{1}{6}U_{x,\lambda} - \lambda^{-1/2}H_a(x,\cdot) + f_{x,\lambda}\bigr)n + 15\int_\Omega U_{x,\lambda}^4(\nabla U_{x,\lambda})f_{x,\lambda} = \mathcal{O}(\lambda^{-2}).$$

This shows that the first term on the right side of (3-50) gives the claimed contribution.

On the other hand, for the second term on the right side of (3-50) we have

$$\begin{aligned}\int_\Omega a(f_{x,\lambda}+g_{x,\lambda})\nabla\psi_{x,\lambda} &= \int_\Omega a(f_{x,\lambda}+g_{x,\lambda})\nabla(U_{x,\lambda}-\lambda^{-1/2}H_a(x,\cdot)) - \frac{1}{2}\int_\Omega(\nabla a)f_{x,\lambda}^2 \\ &\qquad - \int_\Omega(a\nabla g_{x,\lambda}+g_{x,\lambda}\nabla a)f_{x,\lambda} + \frac{1}{2}\int_{\partial\Omega} af_{x,\lambda}^2 + \int_{\partial\Omega} af_{x,\lambda}g_{x,\lambda} \\ &= \int_\Omega ag_{x,\lambda}\nabla U_{x,\lambda} + \mathcal{O}(\lambda^{-3}).\end{aligned}$$

Here we used bounds from Lemmas A.2 and A.4 and from the proof of the latter. Finally, we write $a(y) = a(x) + \mathcal{O}(|x-y|)$ and use the oddness of $g_{x,\lambda}\nabla U_{x,\lambda}$ to obtain

$$\int_\Omega ag_{x,\lambda}\nabla U_{x,\lambda} = \mathcal{O}\biggl(\int_\Omega |x-y|g_{x,\lambda}|\nabla U_{x,\lambda}|\biggr) = \mathcal{O}(\lambda^{-2}).$$

This proves the claimed bound on the second term on the right side of (3-50). □

Proof of Lemma 3.13. The proof is analogous to that of Lemma 2.6. By combining (2-7) for w with $\Delta(H_a(x,\cdot)-H_0(x,\cdot)) = -aG_a(x,\cdot)$, we obtain $-\Delta q = F$ with

$$F := -3U_{x,\lambda}^5 + 3\alpha^4(\psi_{x,\lambda}+q)^5 - aq + a(f_{x,\lambda}+g_{x,\lambda}) - \varepsilon V(\psi_{x,\lambda}+q).$$

(We use the same notation as in the proof of Lemma 2.6 for analogous but different objects.)

We define the cut-off function ζ as before, but now in our bounds we do not make the dependence on d explicit, since we know already $d^{-1} = \mathcal{O}(1)$ by Proposition 2.5. Then $\zeta q \in H^2(\Omega)\cap H_0^1(\Omega)$ and

$$-\Delta(\zeta q) = \zeta F - 2\nabla\zeta\cdot\nabla q - (\Delta\zeta)q.$$

We claim that

$$\zeta|F| \lesssim \zeta|q|^5 + \varepsilon\zeta U_{x,\lambda} + |q| + \varepsilon\lambda^{-1/2}. \tag{3-51}$$

Indeed, on $\Omega\setminus B_{d/2}(x)$, we have $U_{x,\lambda}\lesssim\lambda^{-1/2}$ and $g_{x,\lambda}\lesssim\lambda^{-5/2}$. By Corollary 3.9, we have $\lambda^{-5/2} = \mathcal{O}(\varepsilon\lambda^{-1/2})$. Moreover, we write $\psi_{x,\lambda} = U_{x,\lambda} - \lambda^{-1/2}H_a(x,\cdot) + f_{x,\lambda}$ and use the bounds on $f_{x,\lambda}$ and $H_a(x,\cdot)$ from Lemma A.2 and (B-1).

Combining (3-51) with inequality (2-12), we obtain

$$\begin{aligned}\left\|\frac{\partial q}{\partial n}\right\|_{L^2(\partial\Omega)} = \left\|\frac{\partial(\zeta q)}{\partial n}\right\|_{L^2(\partial\Omega)} &\lesssim \|\Delta(\zeta q)\|_{3/2} = \|\zeta F - 2\nabla\zeta\cdot\nabla q - (\Delta\zeta)q\|_{3/2}\\ &\lesssim \|\zeta q^5\|_{3/2} + \varepsilon\|\zeta U_{x,\lambda}\|_{3/2} + \|q\|_{3/2} + \varepsilon\lambda^{-1/2} + \||\nabla\zeta||\nabla q|\|_{3/2} + \|(\Delta\zeta)q\|_{3/2}.\end{aligned}$$

It remains to bound the norms on the right side. All terms, except for the first one, are easily bounded. Indeed, by (3-48),

$$\|q\|_{3/2} + \|(\Delta\zeta)q\|_{3/2} \lesssim \|q\|_2 \lesssim \varepsilon\lambda^{-1/2}$$

and

$$\||\nabla\zeta||\nabla q|\|_{3/2} \lesssim \|\nabla q\|_{L^2(\Omega\setminus B_{d/2}(x))} \le \|\nabla s\|_{L^2(\Omega\setminus B_{d/2}(x))} + \|\nabla r\|_2 \lesssim \varepsilon\lambda^{-1/2},$$

where we used $\|\nabla s\|_{L^2(\Omega\setminus B_{d/2}(x))}\lesssim\lambda^{-3/2}$ by Equation (3-10) and $\|\nabla r\|_2\lesssim\varepsilon\lambda^{-1/2}$ by Corollary 3.9. (Notice that for the estimate on s it is crucial that the integral avoids $B_{d/2}(x)$.) Moreover, by Lemma A.1,

$$\|\zeta U_{x,\lambda}\|_{3/2} \lesssim \|U_{x,\lambda}\|_{L^{3/2}(\Omega\setminus B_{d/2}(x))} \lesssim \lambda^{-1/2}.$$

To bound the remaining term $\|\zeta q^5\|_{3/2}$ we argue as in Lemma 2.6 above and get

$$\begin{aligned}\|\zeta q^5\|_{3/2} &= \left(\int_\Omega |\zeta^{1/4}|q|^{1/4}q|^6\right)^{\frac{2}{3}} \lesssim \left(\int_\Omega |\nabla(\zeta^{1/4}|q|^{1/4}q)|^2\right)^2\\ &\lesssim \left(\int_\Omega |q|^{5/2}|\nabla(\zeta^{1/4})|^2\right)^2 + \left(\int_\Omega |F|\zeta^{1/2}|q|^{3/2}\right)^2 \lesssim \|q\|_6^5 + \left(\int_\Omega |F|\zeta^{1/2}|q|^{3/2}\right)^2.\end{aligned}$$

We use the pointwise estimate (3-51) on ζF, which is equally valid for $\zeta^{1/2}F$. The term coming from $|q|^5$ is bounded by

$$\left(\int_\Omega |q|^{5+3/2}\zeta^{1/2}\right)^2 = \left(\int_\Omega (\zeta|q|^5)^{1/2}q^4\right)^2 \le \|\zeta q^5\|_{3/2}\|q\|_6^8 = o(\|\zeta q^5\|_{3/2}),$$

which can be absorbed into the left side. The contributions from the remaining terms in the pointwise bound on $\zeta^{1/2}|F|$ can by easily controlled, and we obtain

$$\|\zeta q^5\|_{3/2} \lesssim \|q\|_6^5 + \lambda^{-5} + (\varepsilon\lambda^{-1/2})^5 \lesssim \varepsilon\lambda^{-1/2}.$$

Collecting all the estimates, we obtain the claimed bound. □

4. Proof of Theorems 1.5 and 1.6

4A. ***The behavior of ϕ_a near x_0.*** We are now in a position to complete the proof of Theorem 1.5. Our main remaining goal is to prove

$$\phi_a(x) = o(\varepsilon). \tag{4-1}$$

Once this is shown, we will be able to find a relation between λ and ε. The proof of (4-1) (and only this proof) relies on the nondegeneracy of critical points of ϕ_a.

We already know that $\phi_a(x_0) = 0$ and that $\phi_a(y) \geq 0$ for all $y \in \Omega$, hence x_0 is a critical point of ϕ_a. In this subsection we collect the necessary ingredients which exploit this fact.

Lemma 4.1. *The function ϕ_a is of class C^2 on Ω.*

Since we were unable to find a proof for this fact in the literature, we provide one in Section B2.

Thus, the following general lemma applies to ϕ_a.

Lemma 4.2. *Let u be C^2 near the origin and suppose that $u(0) = 0$, $\nabla u(0) = 0$ and that* Hess $u(0)$ *is invertible. Then, as $x \to 0$,*

$$u(x) = \tfrac{1}{2}\nabla u(x) \cdot (\operatorname{Hess} u(0))^{-1} \nabla u(x) + o(|x|^2). \tag{4-2}$$

Suppose additionally that Hess $u(0) \geq c$ *for some $c > 0$ in the sense of quadratic forms, i.e., the origin is a nondegenerate minimum of u. Then, as $x \to 0$,*

$$u(x) \lesssim |\nabla u(x)|^2. \tag{4-3}$$

Proof. We abbreviate $H(x) = \operatorname{Hess} u(x)$ and make a Taylor expansion around x to get

$$0 = u(0) = u(x) - \nabla u(x) \cdot x + \tfrac{1}{2} x \cdot H(x)x + o(|x|^2) \tag{4-4}$$

and

$$0 = \nabla u(0) = \nabla u(x) - H(x)x + o(|x|^2). \tag{4-5}$$

We infer from (4-5) and the invertibility of $H(0)$ that

$$x = H(x)^{-1}\nabla u(x) + o(|x|^2).$$

Inserting this into (4-4) gives

$$0 = u(x) - \tfrac{1}{2}\nabla u(x) \cdot H(x)^{-1}\nabla u(x) + o(|x|^2).$$

Since $H(x)^{-1} = H(0)^{-1} + o(|x|)$, this yields (4-2).

To prove (4-3), if zero is a nondegenerate minimum, then a Taylor expansion around zero shows

$$u(x) = \tfrac{1}{2} x \cdot H(0)x + o(|x|^2) \geq \tfrac{1}{4} c|x|^2 \tag{4-6}$$

for small enough $|x|$. Thus the $o(|x|^2)$ in (4-2) can be absorbed in the left side, and thus (4-3) holds. □

4B. ***Proof of Theorem 1.5.*** Equation (1-18) follows from Proposition 2.1, together with (3-2), (3-3) and (3-5). The facts that $x_0 \in \mathcal{N}_a$ and $Q_V(x_0) \leq 0$ follow from Corollary 3.9.

By Lemma 4.1 and the assumption that x_0 is a nondegenerate minimum of ϕ_a, we can apply Lemma 4.2 to the function $u(x) := \phi_a(x + x_0)$ to get

$$\phi_a(x) \lesssim |\nabla \phi_a(x)|^2.$$

Therefore, by the bound on $\nabla \phi_a(x)$ in Proposition 3.1 with some fixed $\mu \in \left(\frac{1}{2}, 1\right)$, we get

$$\phi_a(x) \lesssim |\nabla \phi_a(x)|^2 = o(\varepsilon). \tag{4-7}$$

This proves (1-20) and, by nondegeneracy of x_0, also (1-19). Moreover, inserting (4-7) into the expansion of $\phi_a(x)$ from Proposition 3.1, we find

$$0 = a(x)\pi\lambda^{-1} - \frac{\varepsilon}{4\pi} Q_V(x) + o(\lambda^{-1}) + o(\varepsilon),$$

that is,

$$\varepsilon\lambda = 4\pi^2 \frac{|a(x_0)| + o(1)}{|Q_V(x_0)| + o(1)}$$

with the understanding that this means $\varepsilon\lambda \to \infty$ if $Q_V(x_0) = 0$. This proves (1-21).

The remaining claims in Theorem 1.5 follow from Proposition 3.1.

4C. ***A bound on*** $\|w\|_\infty$**.** In this subsection, we prove a crude bound on the L^∞ norm of the first-order remainder w appearing in the decomposition $u = \alpha(PU_{x,\lambda} + w)$, and also on some of its L^p norms which cannot be controlled through Sobolev inequalities, i.e., $p > 6$. This bound was not needed in the proof of Theorem 1.5, but will be in that of Theorem 1.6.

Proposition 4.3. *As* $\varepsilon \to 0$,

$$\|w\|_p \lesssim \lambda^{-3/p} \quad \textit{for all } p \in (6, \infty). \tag{4-8}$$

Moreover, for every $\mu > 0$,

$$\|w\|_\infty = o(\lambda^\mu). \tag{4-9}$$

Our proof follows [Rey 1989, Proof of (25)], which concerns the case $N \geq 4$ and $a = 0$. Since some of the required modifications are rather complicated to state, we give details for the convenience of the reader.

Proof. We begin by proving the first bound in the proposition, which we write as

$$\|w\|_{3(r+1)}^{r+1} \lesssim \lambda^{-1} \quad \text{for all } r \in (1, \infty).$$

To prove this, we define F by (2-13), multiply (2-7) with $|w|^{r-1}w$ and integrate by parts to obtain

$$\frac{4r}{(r+1)^2}\int_\Omega \big|\nabla |w|^{\frac{r+1}{2}}\big|^2 = \int_\Omega F|w|^{r-1}w.$$

Thus, by Sobolev's inequality applied to $v=|w|^{(r+1)/2}$,

$$\|w\|_{3(r+1)}^{r+1} \lesssim \int_\Omega |F||w|^r. \tag{4-10}$$

In order to estimate the right side of (4-10), we make use of the bound

$$|F| \lesssim |\alpha^4-1|U_{x,\lambda}^5 + U_{x,\lambda}^4|w| + |w|^5 + U_{x,\lambda}^4\varphi_{x,\lambda} + U_{x,\lambda} + \varphi_{x,\lambda} + |w|. \tag{4-11}$$

This is a refinement of (3-51), which is obtained by writing $PU_{x,\lambda}=U_{x,\lambda}-\varphi_{x,\lambda}$ and using Lemma A.2 to bound $\varphi_{x,\lambda}^5 \lesssim \varphi_{x,\lambda}$.

We estimate the resulting terms separately. Using Hölder's inequality, Lemma A.1, Proposition 3.6 and the fact that for any $\eta, p, q>0$ with $p^{-1}+q^{-1}=1$ there is $C_\eta>0$ such that for any $a,b>0$ one has $ab\le \eta a^p + C_\eta b^q$, we obtain

$$\begin{aligned}
|\alpha^4-1|\int_\Omega U_{x,\lambda}^5|w|^r &\le \lambda^{-1}\|w\|_{3(r+1)}^r\|U\|_{5\cdot\frac{3r+3}{2r+3}}^5 \lesssim \lambda^{-1}\|w\|_{3(r+1)}^r\lambda^{\frac12\cdot\frac{r-1}{r+1}}\\
&= \|w\|_{3(r+1)}^r\lambda^{-\frac{r+3}{2(r+1)}} \le \eta\|w\|_{3(r+1)}^{r+1} + C_\eta\lambda^{-\frac{r+3}{2}};
\end{aligned}$$

$$\int_\Omega U_{x,\lambda}^4|w|^{r+1} \le \left(\int_\Omega U_{x,\lambda}^5|w|^r\right)^{\frac45}\left(\int_\Omega |w|^{r+5}\right)^{\frac15} \le \|w\|_{3(r+1)}^{r+\frac15}\lambda^{-\frac{4}{5(r+1)}} \le \eta\|w\|_{3(r+1)}^{r+1} + C_\eta\lambda^{-1};$$

$$\int_\Omega |w|^{5+r} \le \|w\|_{3(r+1)}^{r+1}\|w\|_6^4 \lesssim \|w\|_{3(r+1)}^{r+1}\lambda^{-2};$$

$$\begin{aligned}
\int_\Omega U_{x,\lambda}^4|w|^r\varphi_{x,\lambda} \le \lambda^{-\frac12}\|w\|_{3(r+1)}^r\|U_{x,\lambda}\|_{4\cdot\frac{3r+3}{2r+3}}^4 = \lambda^{-\frac12-\frac{1}{r+1}}\|w\|_{3(r+1)}^r &= \lambda^{-\frac{r+3}{2(r+1)}}\|w\|_{3(r+1)}^r\\
&\le \eta\|w\|_{3(r+1)}^{r+1} + C_\eta\lambda^{-\frac{r+3}{2}};
\end{aligned}$$

$$\int_\Omega U_{x,\lambda}|w|^r \le \|w\|_{3(r+1)}^r\|U_{x,\lambda}\|_{\frac{3r+3}{2r+3}} \lesssim \|w\|_{3(r+1)}^r\lambda^{-\frac12} \le \eta\|w\|_{3(r+1)}^{r+1} + C_\eta\lambda^{-\frac{r+1}{2}};$$

$$\int_\Omega \varphi_{x,\lambda}|w|^r \lesssim \lambda^{-\frac12}\|w\|_{3(r+1)}^r \le \eta\|w\|_{3(r+1)}^{r+1} + C_\eta\lambda^{-\frac{r+1}{2}};$$

$$\int_\Omega |w|^{r+1} \lesssim \left(\int_\Omega |w|^{5+r}\right)^{\frac{r+1}{r+5}} \lesssim \|w\|_{3(r+1)}^{\frac{(r+1)^2}{r+5}}\lambda^{-\frac{2(r+1)}{r+5}} \le \eta\|w\|_{3(r+1)}^{r+1} + C_\eta\lambda^{-\frac{r+1}{2}}.$$

By choosing η small enough (but independent of λ), we can absorb the term $\eta\|w\|_{3(r+1)}^{r+1}$, as well as the term $\lambda^{-2}\|w\|_{3(r+1)}^{r+1}$, into the left-hand side of inequality (4-10) to get

$$\|w\|_{3(r+1)}^{r+1} \lesssim \lambda^{-\frac{r+3}{2}} + \lambda^{-1} + \lambda^{-\frac{r+1}{2}} \lesssim \lambda^{-1}.$$

This is the claimed bound.

We now turn to the bound of the L^∞ norm of w. We write (2-7) for w as

$$w(x) = \frac{1}{4\pi}\int_\Omega G_0(x,y)F(y). \tag{4-12}$$

By Hölder's inequality and the fact that $0 \le G_0(x,y) \le |x-y|^{-1}$, we have for every $\delta \in (0,2)$

$$\|w\|_\infty \le \sup_{x\in\Omega} \|G_0(x,\cdot)\|_{3-\delta}\|F\|_{\frac{3-\delta}{2-\delta}} \lesssim \|F\|_{\frac{3-\delta}{2-\delta}}. \tag{4-13}$$

Hence it suffices to estimate $\|F\|_q$ with some $q := (3-\delta)/(2-\delta) > \frac{3}{2}$.

We use again the bound (4-11). The L^q norms of the resulting terms are easy to estimate. Indeed, since $|\alpha^4-1| \lesssim \lambda^{-1}$ by Proposition 3.6, we have by Lemma A.1

$$|\alpha^4-1|\|U_{x,\lambda}^5\|_q \lesssim \lambda^{-1}\|U\|_{5q}^5 \lesssim \lambda^{\frac{3}{2}-\frac{3}{q}}.$$

Next, by Lemma A.1 and A.2,

$$\|U_{x,\lambda}^4\varphi_{x,\lambda}\|_q \lesssim \lambda^{-\frac{1}{2}}\|U_{x,\lambda}\|_{4q}^4 =\lesssim \lambda^{\frac{3}{2}-\frac{3}{q}}.$$

Using additionally the bound on $\|\nabla w\|$ from Proposition 2.1, we can estimate, for every $q < 3$,

$$\|U_{x,\lambda}+\varphi_{x,\lambda}+|w|\|_q \le \|U_{x,\lambda}\|_q + \|\varphi_{x,\lambda}\|_\infty + \|\nabla w\|_6 \lesssim \lambda^{-\frac{1}{2}}.$$

Finally, using the bound (4-8),

$$\|U_{x,\lambda}^4 w\|_q \le \|U_{x,\lambda}\|_{5q}^4\|w\|_{5q} \lesssim \lambda^{2-\frac{12}{5q}}\|w\|_{5q} \lesssim \lambda^{2-\frac{3}{q}}$$

and

$$\|w^5\|_q = \|w\|_{5q}^5 \lesssim \lambda^{-\frac{3}{q}}.$$

Inserting these estimates into (4-13) yields

$$\|w\|_\infty \lesssim \lambda^{2-\frac{3}{q}} \quad \text{for every } q \in \big(\tfrac{3}{2}, 3\big).$$

As $\delta \searrow 0$ in (4-13), we have $q \searrow \frac{3}{2}$ and hence $2-\frac{3}{q} \searrow 0$. Thus (4-9) is proved. □

4D. *Proof of Theorem 1.6.* By Proposition 2.1, we have $u = \alpha(PU_{x,\lambda}+w)$ with $\alpha = 1+o(1)$. Moreover, by Proposition 4.3, $\|w\|_\infty = o(\lambda^{1/2})$. On the other hand, by Lemma A.2 we have

$$\|PU_{x,\lambda}\|_\infty = \|U_{x,\lambda}\|_\infty + \mathcal{O}(\|\varphi_{x,\lambda}\|_\infty) = \lambda^{\frac{1}{2}} + \mathcal{O}(\lambda^{-\frac{1}{2}}).$$

Putting these estimates together, we obtain

$$\varepsilon\|u_\varepsilon\|_\infty^2 = \varepsilon(\lambda^{\frac{1}{2}}+o(\lambda^{\frac{1}{2}}))^2 = \varepsilon\lambda(1+o(1)) = 4\pi^2\frac{|a(x_0)|}{|Q_V(x_0)|}(1+o(1))$$

by the relationship between ε and λ proved in Theorem 1.5. Moreover, $U_{x,\lambda}(x) = \lambda^{1/2} = \|U_{x,\lambda}\|_\infty$. This finishes the proof of part (a) in Theorem 1.6.

The proof of part (b) necessitates significantly fewer prerequisites. It only relies on the crude expansion of u given in Proposition 2.1 and the rough bounds on w from Proposition 4.3.

By applying $(-\Delta+a)^{-1}$, we write (1-3) as

$$u(z)=\frac{3}{4\pi}\int_\Omega G_a(z,y)u(y)^5-\frac{\varepsilon}{4\pi}\int_\Omega G_a(z,y)V(y)u(y). \tag{4-14}$$

We fix a sequence $\delta=\delta_\varepsilon=o(1)$ with $\lambda^{-1}=o(\delta_\varepsilon)$. This condition, together with the bounds from Proposition 2.1, easily implies $\frac{3}{4\pi}\int_{B_\delta(x)}u(y)^5=\lambda^{-1/2}+o(\lambda^{-1/2})$. Hence

$$\frac{3}{4\pi}\int_{B_\delta(x)}G_a(z,y)u(y)^5=\frac{3}{4\pi}\int_{B_\delta(x)}(G_a(z,x_0)+o(1))u(y)^5=\lambda^{-1/2}G_a(z,x_0)+o(\lambda^{-1/2}).$$

On the complement of $B_\delta(x)$, using Proposition 4.3 and Lemma A.1, we bound

$$\left|\int_{\Omega\setminus B_\delta(x)}G_a(z,y)u(y)^5\right|\lesssim\|G_a(z,\cdot)\|_2(\|U_{x,\lambda}\|^5_{L^{10}(\Omega\setminus B_\delta(x))}+\|w\|_{10}^5)\lesssim\lambda^{-5/2}\delta^{-7/2}+\lambda^{-3/2}.$$

Choosing, e.g., $\delta=\lambda^{-2/7}$, the last bound is $o(\lambda^{-1/2})$.

The second term on the right side of (4-14) is easily bounded by

$$\varepsilon\left|\int_\Omega G_a(z,y)V(y)u(y)\right|\lesssim\varepsilon\|G_a(z,\cdot)\|_2(\|U\|_2+\|w\|_2)\lesssim\varepsilon\lambda^{-1/2}$$

using the bounds from Proposition 2.1 and from Lemma A.1. Collecting the above estimates, part (b) of Theorem 1.6 follows.

5. Subcritical case: a first expansion

In the remainder of the paper we will deal with the proof of Theorems 1.2 and 1.3. The structure of our argument is very similar to that leading to Theorems 1.5 and 1.6. Namely, in the present section we derive a preliminary asymptotic expansion of u_ε and the involved parameters, which is refined subsequently in Section 6 below. Because of the similarities to the above argument, we will not always give full details.

The following proposition summarizes the results of this section.

Proposition 5.1. *Let (u_ε) be a family of solutions to* (1-2) *satisfying* (1-5). *Then, up to the extraction of a subsequence, there are sequences $(x_\varepsilon)\subset\Omega$, $(\lambda_\varepsilon)\subset(0,\infty)$, $(\alpha_\varepsilon)\subset\mathbb{R}$ and $(w_\varepsilon)\subset T^\perp_{x_\varepsilon,\lambda_\varepsilon}$ such that*

$$u_\varepsilon=\alpha_\varepsilon(PU_{x_\varepsilon,\lambda_\varepsilon}+w_\varepsilon) \tag{5-1}$$

and a point $x_0\in\Omega$ such that

$$|x_\varepsilon-x_0|=o(1),\quad \alpha_\varepsilon=1+o(1),\quad \lambda_\varepsilon\to\infty,\quad \|\nabla w_\varepsilon\|_2=\mathcal{O}(\lambda_\varepsilon^{-1/2}),\quad \varepsilon=\mathcal{O}(\lambda_\varepsilon^{-1}). \tag{5-2}$$

5A. ***A qualitative initial expansion.*** As a first step towards Proposition 5.1, we observe that the qualitative expansion from Proposition 2.2 still holds true, that is, there are sequences $(x_\varepsilon)\subset\Omega$, $(\lambda_\varepsilon)\subset(0,\infty)$, $(\alpha_\varepsilon)\subset\mathbb{R}$ and $(w_\varepsilon)\subset T^\perp_{x_\varepsilon,\lambda_\varepsilon}$ such that (5-1) holds and a point $x_0\in\overline{\Omega}$ such that, along a subsequence,

$$|x_\varepsilon-x_0|=o(1),\quad \alpha_\varepsilon=1+o(1),\quad d_\varepsilon\lambda_\varepsilon\to\infty,\quad \|\nabla w_\varepsilon\|_2=o(1),$$

where, as before, $d_\varepsilon:=d(x_\varepsilon,\partial\Omega)$.

Indeed, as explained in the proof of Proposition 2.2, it suffices to prove $u_\varepsilon \rightharpoonup 0$ in $H_0^1(\Omega)$ up to a subsequence. To achieve this, we first integrate (1-2) against u_ε to obtain

$$3\left(\int_\Omega u_\varepsilon^{6-\varepsilon}\right)^{\frac{4-\varepsilon}{6-\varepsilon}} = \frac{\int_\Omega |\nabla u_\varepsilon|^2}{\left(\int_\Omega u_\varepsilon^{6-\varepsilon}\right)^{2/(6-\varepsilon)}} + \frac{\int_\Omega a u_\varepsilon^2}{\left(\int_\Omega u_\varepsilon^{6-\varepsilon}\right)^{2/(6-\varepsilon)}}.$$

By (1-5) and Hölder's inequality, the right side is bounded, hence $\|u_\varepsilon\|_{6-\varepsilon} \lesssim 1$. By (1-5) again, $\|\nabla u_\varepsilon\|_2 \lesssim 1$. On the other hand, the right side is bounded from below by a positive constant by coercivity of $-\Delta + a$, which is a consequence of criticality, and by Hölder's inequality. This gives $\|u_\varepsilon\|_{6-\varepsilon} \gtrsim 1$, and hence $\|\nabla u_\varepsilon\|_2 \gtrsim 1$ by the inequalities of Sobolev and Hölder. This completes the analogue of Step 1 in the proof of Proposition 2.2.

Let us now turn to Step 2 in that proof. We denote by u_0 a weak limit point of u_ε in $H_0^1(\Omega)$, which exists by Step 1. Still by Step 1, we may assume that the quantities $\|u_\varepsilon\|_{6-\varepsilon}$ and $\|\nabla u_\varepsilon\|_2$ have nonzero limits. The only difference to Proposition 2.2 is now that we modify the definition of $\mathcal{M}$ to

$$\mathcal{M} = \lim_{\varepsilon\to 0} \int_\Omega (u_\varepsilon - u_0)^{6-\varepsilon},$$

where the exponent is $6-\varepsilon$ instead of 6. Thanks to the uniform bound $\|u_\varepsilon\|_{6-\varepsilon} \lesssim 1$ by Step 1, it can be easily checked that the proof of the Brezis–Lieb lemma (see, e.g., [Lieb and Loss 1997]) still yields

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon^{6-\varepsilon} = \lim_{\varepsilon\to 0} \int_\Omega u_0^{6-\varepsilon} + \mathcal{M} = \int_\Omega u_0^6 + \mathcal{M}.$$

Then the modified assumption (1-5) can be used to conclude

$$S\left(\int_\Omega u_0^6 + \mathcal{M}\right)^{\frac{1}{3}} = \int_\Omega |\nabla u_0|^2 + \mathcal{T}.$$

The rest of the proof is identical to Proposition 2.2.

We again adopt the convention in the remainder of the proof that we only consider the above subsequence and we will drop the subscript ε.

In order to prove Proposition 5.1, we will prove in the following subsections that $x_0 \in \Omega$, $\|\nabla w\|_2 = \mathcal{O}(\lambda^{-1/2})$ and $\varepsilon = \mathcal{O}(\lambda^{-1})$.

5B. ***The bound on*** $\|\nabla w\|_2$**.** The goal of this subsection is to prove:

Proposition 5.2. *As* $\varepsilon \to 0$,

$$\|\nabla w\|_2 = \mathcal{O}(\lambda^{-1/2}) + \mathcal{O}((\lambda d)^{-1}) + \mathcal{O}(\varepsilon). \tag{5-3}$$

Note that, in contrast to Proposition 2.4, there appears an additional error $\mathcal{O}(\varepsilon)$. We will prove in an extra step (Proposition 5.5) that $\varepsilon = \mathcal{O}((\lambda d)^{-1})$, so this extra term will disappear later.

The proof of Proposition 5.2 is somewhat lengthy, and we precede it by an auxiliary result, which is a simple consequence of the fact that $\alpha \to 1$.

Lemma 5.3. *As* $\varepsilon \to 0$,

$$\varepsilon \log \lambda = o(1).$$

A useful consequence of this lemma is that

$$U_{x,\lambda}^{-\varepsilon} \lesssim 1 \quad \text{in } \Omega. \tag{5-4}$$

Indeed, this follows from the lemma together with the fact that $U_{x,\lambda} \gtrsim \lambda^{-1/2}$ in Ω.

Proof. We integrate (1-2) against u and use the decomposition (5-1). This gives

$$\int_\Omega |\nabla(PU_{x,\lambda}+w)|^2 + \int_\Omega a(PU_{x,\lambda}+w)^2 = 3\alpha^{4-\varepsilon}\int_\Omega (PU_{x,\lambda}+w)^{6-\varepsilon}. \tag{5-5}$$

By orthogonality

$$\int_\Omega |\nabla(PU_{x,\lambda}+w)|^2 = \int_\Omega |\nabla PU_{x,\lambda}|^2 + \int_\Omega |\nabla w|^2 = \frac{3\pi^2}{4} + o(1).$$

Moreover, using Lemmas A.1 and A.2 we find $\int_\Omega a(PU_{x,\lambda}+w)^2 = o(1)$. On the other hand,

$$\int_\Omega (PU_{x,\lambda}+w)^{6-\varepsilon} = \int_\Omega U_{x,\lambda}^{6-\varepsilon} + o(1).$$

Hence (5-5) combined with the fact that $\alpha \to 1$ implies

$$\int_\Omega U_{x,\lambda}^{6-\varepsilon} = \frac{\pi^2}{4} + o(1). \tag{5-6}$$

Since

$$\int_\Omega U_{x,\lambda}^{6-\varepsilon} = \lambda^{-\varepsilon/2}\lambda^3 \int_\Omega (1+\lambda^2|x-y|^2)^{-3+\varepsilon/2} = \lambda^{-\varepsilon/2}\frac{\pi^2}{4}(1+o(1)),$$

we have $\lambda^{-\varepsilon/2} \to 1$ and hence the claim. □

The next result quantifies the difference between $\int_\Omega U_{x,\lambda}^{5-\varepsilon} v$ and $\int_\Omega U_{x,\lambda}^5 v = 0$ for $v \in T_{x,\lambda}^\perp$.

Lemma 5.4. *For every $v \in T_{x,\lambda}^\perp$,*

$$\left|\int_\Omega U_{x,\lambda}^{5-\varepsilon} v\right| \lesssim \varepsilon \|v\|_6. \tag{5-7}$$

Proof. By orthogonality,

$$\int_\Omega U_{x,\lambda}^{5-\varepsilon} v = \lambda^{-\varepsilon/2}\int_\Omega U_{x,\lambda}^5 e^{\varepsilon \log\sqrt{1+\lambda^2|x-y|^2}} v = \lambda^{-\varepsilon/2}\int_\Omega U_{x,\lambda}^5 (e^{\varepsilon \log\sqrt{1+\lambda^2|x-y|^2}} - 1) v.$$

By Lemma 5.3,

$$\varepsilon \log\sqrt{1+\lambda^2|x-y|^2} = o(1) \tag{5-8}$$

uniformly in x and y. Hence

$$0 < e^{\varepsilon \log\sqrt{1+\lambda^2|x-y|^2}} - 1 \lesssim \varepsilon \log\sqrt{1+\lambda^2|x-y|^2} \le \varepsilon\lambda|x-y|, \tag{5-9}$$

where we have used the inequality $\log\sqrt{1+t^2} \le |t|$. Since

$$\| |x-y| U_{x,\lambda}^5 \|_{6/5} = \mathcal{O}(\lambda^{-1}),$$

the result follows from the Hölder inequality. □

We are now in position to give the following:

Proof of Proposition 5.2. From (1-2) for u we obtain the following equation for w:

$$-\Delta w + aw = -3U_{x,\lambda}^5 - aPU_{x,\lambda} + 3\alpha^{4-\varepsilon}(PU_{x,\lambda}+w)^{5-\varepsilon}. \tag{5-10}$$

Integrating this equation against w gives

$$\int_\Omega (|\nabla w|^2 + aw^2) = -\int_\Omega aPU_{x,\lambda}w + 3\alpha^{4-\varepsilon}\int_\Omega w(PU_{x,\lambda}+w)^{5-\varepsilon}. \tag{5-11}$$

As before, the first term on the right-hand side is controlled easily by Hölder's inequality,

$$\left|\int_\Omega aPU_{x,\lambda}w\right| \lesssim \|PU_{x,\lambda}\|_2\|w\|_2 \lesssim \lambda^{-1/2}\|\nabla w\|_2.$$

In order to control the second term we use the fact that $PU_{x,\lambda} = U_{x,\lambda} - \varphi_{x,\lambda}$. Moreover, by a Taylor expansion and (5-4),

$$\begin{aligned}(PU_{x,\lambda}+w)^{5-\varepsilon} &= (U_{x,\lambda} - \varphi_{x,\lambda} + w)^{5-\varepsilon} \\ &= U_{x,\lambda}^{5-\varepsilon} + (5-\varepsilon)U_{x,\lambda}^{4-\varepsilon}w + \mathcal{O}(U_{x,\lambda}^4\varphi_{x,\lambda} + U_{x,\lambda}^3 w^2 + |w|^{5-\varepsilon} + \varphi_{x,\lambda}^{5-\varepsilon}).\end{aligned} \tag{5-12}$$

Hence,

$$\begin{aligned}&\left|\int_\Omega (PU_{x,\lambda}+w)^{5-\varepsilon}w - (5-\varepsilon)\alpha^{4-\varepsilon}\int_\Omega U_{x,\lambda}^{4-\varepsilon}w^2\right| \\ &\qquad\le \left|\int_\Omega U_{x,\lambda}^{5-\varepsilon}w\right| + \mathcal{O}\Big(\int_\Omega U_{x,\lambda}^4\varphi_{x,\lambda}|w|\Big) + \mathcal{O}(\|\nabla w\|_2^3 + \|\nabla w\|_2\|\varphi_{x,\lambda}\|_6^{5-\varepsilon}).\end{aligned}$$

We estimate the first term on the right side using Lemma 5.4. For the second term on the right side we argue as in the proof of Proposition 2.4 and obtain

$$\int_\Omega U_{x,\lambda}^4\varphi_{x,\lambda}|w| = \mathcal{O}((\lambda d)^{-1}\|\nabla w\|_2).$$

For the last term on the right side we use $\|\varphi_{x,\lambda}\|_6^2 = \mathcal{O}((\lambda d)^{-1})$. Moreover, in view of (5-9),

$$\begin{aligned}\int_\Omega U_{x,\lambda}^{4-\varepsilon}w^2 &\le \lambda^{-\varepsilon/2}\int_\Omega U_{x,\lambda}^4 w^2 + C\varepsilon\lambda\int_\Omega U_{x,\lambda}^4|x-y|w^2 \\ &\le (1+o(1))\int_\Omega U_{x,\lambda}^4 w^2 + \mathcal{O}(\varepsilon\lambda^{-1/2}\|\nabla w\|_2^2).\end{aligned} \tag{5-13}$$

Altogether we obtain, from (5-11),

$$\int_\Omega (|\nabla w|^2 + aw^2 - 15\alpha^{4-\varepsilon}U_{x,\lambda}^4 w^2) \lesssim ((\lambda d)^{-1} + \lambda^{-1/2} + \varepsilon)\|\nabla w\|_2 + o(\|\nabla w\|_2^2).$$

An application of the coercivity inequality of Lemma 2.3 now implies (5-3). □

5C. *The bound on ε.* The goal of this subsection is to prove:

Proposition 5.5. *As* $\varepsilon \to 0$,

$$\varepsilon = \mathcal{O}((\lambda d)^{-1}). \tag{5-14}$$

We note that the analogue of this proposition is not needed in Section 2 when studying (1-3).

The proof of Proposition 5.5 is based on the Pohozaev-type identity

$$\int_\Omega \nabla PU_{x,\lambda} \cdot \nabla \partial_\lambda PU_{x,\lambda} + \int_\Omega a(PU_{x,\lambda} + w)\partial_\lambda PU_{x,\lambda} = \alpha^{4-\varepsilon} 3 \int_\Omega (PU_{x,\lambda} + w)^{5-\varepsilon} \partial_\lambda PU_{x,\lambda}, \tag{5-15}$$

which arises from integrating (4-4) against $\partial_\lambda PU_{x,\lambda}$ and inserting the following bounds.

Lemma 5.6. *As* $\varepsilon \to 0$, *we have*

$$\int_\Omega \nabla PU_{x,\lambda} \cdot \nabla \partial_\lambda PU_{x,\lambda} + \int_\Omega a(PU_{x,\lambda} + w)\partial_\lambda PU_{x,\lambda} = \mathcal{O}(\lambda^{-2} d^{-1} + \lambda^{-1} \|\nabla w\|_2^2) \tag{5-16}$$

and

$$3 \int_\Omega (PU_{x,\lambda} + w)^{5-\varepsilon} \partial_\lambda PU_{x,\lambda} = -\tfrac{1}{16}(1 + o(1))\varepsilon \lambda^{-1} + \mathcal{O}(\lambda^{-2} d^{-1} + \lambda^{-1} \|\nabla w\|_2^2). \tag{5-17}$$

Before proving Lemma 5.6, let us use it to deduce the main result of this subsection.

Proof of Proposition 5.5. Inserting (5-16) and (5-17) into (5-15) and applying the bound (5-3) on $\|\nabla w\|$ we obtain

$$(1 + o(1))\varepsilon \lesssim (\lambda d)^{-1} + \|\nabla w\|_2^2 \lesssim (\lambda d)^{-1} + \varepsilon^2.$$

Since $\varepsilon = o(1)$, (5-14) follows. ☐

In the proof of Lemma 5.6 we need the following auxiliary bound.

Lemma 5.7. *For every* $v \in T_{x,\lambda}^\perp$,

$$\left| \int_\Omega U_{x,\lambda}^{4-\varepsilon} \partial_\lambda U_{x,\lambda} v \right| \lesssim \varepsilon \lambda^{-1} \|\nabla v\|_2. \tag{5-18}$$

The proof of this lemma is analogous to that of Lemma 5.4 and is omitted.

Proof of Lemma 5.6. We begin with proving (5-16). First, by [Rey 1990, (B.5)],

$$\int_\Omega \nabla PU_{x,\lambda} \cdot \nabla \partial_\lambda PU_{x,\lambda} = \mathcal{O}(\lambda^{-2} d^{-1}).$$

Writing $PU_{x,\lambda} = U_{x,\lambda} - \varphi_{x,\lambda}$, the second term in (5-16) is bounded by

$$\begin{aligned} \left| \int_\Omega a(PU_{x,\lambda} + w)\partial_\lambda PU_{x,\lambda} \right| &\lesssim (\|U_{x,\lambda}\|_2 + \|w\|_2)(\|\partial_\lambda U_{x,\lambda}\|_2 + \|\partial_\lambda \varphi_{x,\lambda}\|_2) \\ &\lesssim \lambda^{-2} d^{-1/2} + \lambda^{-3/2} d^{-1/2} \|\nabla w\|_2 \lesssim \lambda^{-2} d^{-1} + \lambda^{-1} \|\nabla w\|_2^2, \end{aligned}$$

by Lemma A.1 and (A-3), followed by Young's inequality.

Next, we prove (5-17). Using (5-12) and (5-4) we bound pointwise

$$\begin{aligned}(PU_{x,\lambda}+w)^{5-\varepsilon}\partial_\lambda PU_{x,\lambda} = U_{x,\lambda}^{5-\varepsilon}\partial_\lambda U_{x,\lambda} + (5-\varepsilon)U_{x,\lambda}^{4-\varepsilon}\partial_\lambda U_{x,\lambda} w \\ + \mathcal{O}((U_{x,\lambda}^4\varphi_{x,\lambda} + U_{x,\lambda}^3 w^2 + |w|^{5-\varepsilon} + \varphi_{x,\lambda}^{5-\varepsilon})|\partial_\lambda U_{x,\lambda}|) \\ + \mathcal{O}((U_{x,\lambda}^5 + |w|^{5-\varepsilon} + \varphi_{x,\lambda}^{5-\varepsilon})|\partial_\lambda \varphi_{x,\lambda}|).\end{aligned} \tag{5-19}$$

The integral over Ω of the two remainder terms is bounded by a constant times

$$\begin{aligned}\|\varphi_{x,\lambda}\|_\infty \|U_{x,\lambda}\|_5^4 \|\partial_\lambda U_{x,\lambda}\|_5 + (\|U_{x,\lambda}\|_6^3\|w\|_6^2 + \|w\|_6^{5-\varepsilon} + \|\varphi_{x,\lambda}\|_6^{5-\varepsilon})\|\partial_\lambda U_{x,\lambda}\|_6 \\ + \|U_{x,\lambda}\|_5^5 \|\partial_\lambda\varphi_{x,\lambda}\|_\infty + (\|w\|_6^{5-\varepsilon} + \|\varphi_{x,\lambda}\|_6^{5-\varepsilon})\|\partial_\lambda\varphi_{x,\lambda}\|_6 \lesssim \lambda^{-2}d^{-1} + \lambda^{-1}\|w\|_6^2,\end{aligned}$$

where in the last inequality we used the bounds from Lemmas A.1 and A.2.

By Lemma 5.7, the integral over Ω of the second term on the right side of (5-19) is bounded by a constant times $\varepsilon\lambda^{-1}\|\nabla w\|_2 = o(\varepsilon\lambda^{-1})$.

Finally, by an explicit calculation,

$$\begin{aligned}\int_\Omega U_{x,\lambda}^{5-\varepsilon}\partial_\lambda U_{x,\lambda} &= \int_\Omega U_{x,\lambda}^{5-\varepsilon}\left(\frac{U_{x,\lambda}}{2\lambda} - \frac{\lambda^{3/2}|x-y|^2}{(1+\lambda^2|x-y|^2)^{3/2}}\right) \\ &= \pi\lambda^{-1-\varepsilon/2}\left[\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3-\varepsilon}{2}\right)}{\Gamma\left(3-\frac{\varepsilon}{2}\right)} - \frac{2\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3-\varepsilon}{2}\right)}{\Gamma\left(4-\frac{\varepsilon}{2}\right)}\right] + \mathcal{O}(\lambda^{-4}d^{-3}) \\ &= -\frac{\pi^{3/2}}{4}\varepsilon\lambda^{-1-\varepsilon/2}\frac{\Gamma\left(\frac{3-\varepsilon}{2}\right)}{\Gamma\left(4-\frac{\varepsilon}{2}\right)} + \mathcal{O}(\lambda^{-4}d^{-3}) \\ &= -\frac{\pi^2}{48}\varepsilon\lambda^{-1}(1+o(1)) + \mathcal{O}(\lambda^{-4}d^{-3}),\end{aligned} \tag{5-20}$$

where, in the last step, we used Lemma 5.3. This completes the proof of (5-17). □

5D. ***Excluding boundary concentration.*** The goal of this subsection is to prove:

Proposition 5.8. $d^{-1} = \mathcal{O}(1)$.

Proof. The proof is very similar to that of Proposition 2.5 and we will be brief. Integrating the first equation in (1-2) against ∇u implies the Pohozaev-type identity

$$-\int_\Omega (\nabla a)u^2 = \int_{\partial\Omega} n\left(\frac{\partial u}{\partial n}\right)^2. \tag{5-21}$$

The volume integral on the left side can be estimated as before, since by Propositions 5.2 and 5.5 we have the same bound

$$\|\nabla w\|_2^2 \lesssim \lambda^{-1} + (\lambda d)^{-2}$$

as before. To bound the surface integral, we use the fact that

$$\int_{\partial\Omega}\left(\frac{\partial w}{\partial n}\right)^2 = \mathcal{O}(\lambda^{-1}d^{-1}) + o(^{-1}d^{-2}).$$

This is the analogue of Lemma 2.6. We only note that by (5-10) we have

$$F := -\Delta w = 3\alpha^{4-\varepsilon}(PU_{x,\lambda} + w)^{5-\varepsilon} - 3U_{x,\lambda}^5 - a(PU_{x,\lambda} + w) \tag{5-22}$$

and that this function satisfies (2-15). Therefore, using the above bound on $\|\nabla w\|_2$ we can proceed exactly in the same way as in the proof of Lemma 2.6.

Thus, as before, we obtain

$$C\lambda^{-1}\nabla\phi_0(x) = \mathcal{O}(\lambda^{-1}d^{-3/2}) + o(\lambda^{-1}d^{-2})$$

and then from $|\nabla\phi_0(x)| \gtrsim d^{-2}$ we conclude that $d^{-1} = \mathcal{O}(1)$, as claimed. □

5E. *Proof of Proposition 5.1.* The existence of the expansion is discussed in Section 5A. Proposition 5.8 implies that $d^{-1} = \mathcal{O}(1)$, which implies that $x_0 \in \Omega$. Moreover, inserting the bound $d^{-1} = \mathcal{O}(1)$ into Propositions 5.2 and 5.5, we obtain $\varepsilon = \mathcal{O}(\lambda^{-1})$ and $\|\nabla w\|_2 = \mathcal{O}(\lambda^{-1/2})$, as claimed in Proposition 5.1. This completes the proof of the proposition. □

6. Subcritical case: refining the expansion

As in the additive case, we refine the analysis of the remainder term w_ε in Proposition 5.1, which we write as $w_\varepsilon = \lambda_\varepsilon^{-1/2}(H_0(x_\varepsilon,\cdot) - H_a(x_\varepsilon,\cdot)) + s_\varepsilon + r_\varepsilon$ with s_ε and r_ε as in (3-4).

The following proposition summarizes the main results of this section.

Proposition 6.1. *Let* (u_ε) *be a family of solutions to* (1-2) *satisfying* (1-5). *Then, up to the extraction of a subsequence, there are sequences* $(x_\varepsilon) \subset \Omega$, $(\lambda_\varepsilon) \subset (0,\infty)$, $(\alpha_\varepsilon) \subset \mathbb{R}$, $(s_\varepsilon) \subset T_{x_\varepsilon,\lambda_\varepsilon}$ *and* $(r_\varepsilon) \subset T^\perp_{x_\varepsilon,\lambda_\varepsilon}$ *such that*

$$u_\varepsilon = \alpha_\varepsilon(\psi_{x_\varepsilon,\lambda_\varepsilon} + s_\varepsilon + r_\varepsilon) \tag{6-1}$$

and a point $x_0 \in \Omega$ *such that, in addition to Proposition 5.1,*

$$\|\nabla r_\varepsilon\|_2 = \mathcal{O}(\varepsilon + \lambda_\varepsilon^{-3/2} + \phi_a(x_\varepsilon)\lambda_\varepsilon^{-1}), \tag{6-2}$$

$$\phi_a(x_\varepsilon) = \pi a(x_\varepsilon)\lambda_\varepsilon^{-1} + \frac{\pi}{32}\varepsilon\lambda_\varepsilon(1 + o(1)) + o(\lambda_\varepsilon^{-1}), \tag{6-3}$$

$$\nabla\phi_a(x) = \mathcal{O}(\varepsilon\lambda_\varepsilon^{1/2} + \lambda_\varepsilon^{-\mu} + \phi_a(x_\varepsilon)\lambda_\varepsilon^{-1/2}) \quad \textit{for any } \mu < 1, \tag{6-4}$$

$$\alpha_\varepsilon^{4-\varepsilon} = 1 + \frac{\varepsilon}{2}\log\lambda_\varepsilon - 4\beta\lambda_\varepsilon^{-1} + \mathcal{O}(\varepsilon + \phi_a(x_\varepsilon)\lambda_\varepsilon^{-1}) + o(\lambda_\varepsilon^{-1}). \tag{6-5}$$

We will prove Proposition 6.1 through a series of propositions in the following subsections.

6A. *The bound on* $\|\nabla r\|_2$**.** The following proposition contains the bound on $\|\nabla r\|_2$ from Proposition 6.1.

Proposition 6.2. *As* $\varepsilon \to 0$,

$$\|\nabla r\|_2 = \mathcal{O}(\varepsilon + \lambda^{-3/2} + \phi_a(x)\lambda^{-1}). \tag{6-6}$$

Proof. Notice that

$$-\Delta r = -3U_{x,\lambda}^5 + 3\alpha^{4-\varepsilon}(\psi_{x,\lambda} + s + r)^{5-\varepsilon} + a(g_{x,\lambda} + f_{x,\lambda}) - a(s + r) + \Delta s,$$

with $g_{x,\lambda}$ as in (A-4). Hence

$$\int_\Omega (|\nabla r|^2 + ar^2) = 3\alpha^{4-\varepsilon} \int_\Omega (\psi_{x,\lambda} + s + r)^{5-\varepsilon} r - \int_\Omega a\Big(U_{x,\lambda} - \frac{\lambda^{-1/2}}{|x-y|} + s - f_{x,\lambda}\Big) r. \tag{6-7}$$

By Lemma 3.5(b)

$$\left|\int_\Omega a(g_{x,\lambda} + f_{x,\lambda} - s) r\right| \lesssim \lambda^{-3/2} \|r\|_6.$$

Now,

$$\begin{aligned}\int_\Omega (\psi_{x,\lambda} + s + r)^{5-\varepsilon} r = \int_\Omega U_{x,\lambda}^{5-\varepsilon} r + (5-\varepsilon) \int_\Omega U_{x,\lambda}^{4-\varepsilon} r^2 + (5-\varepsilon) \int_\Omega U_{x,\lambda}^{4-\varepsilon} rs \\ - (5-\varepsilon) \int_\Omega U_{x,\lambda}^{4-\varepsilon} (\lambda^{-1/2} H_a(x,\cdot) + f_{x,\lambda}) r + T_{3,\varepsilon},\end{aligned} \tag{6-8}$$

where similarly as in the proof Lemma 3.5 we find that

$$|T_{3,\varepsilon}| \lesssim \lambda^{-2} \|r\|_6 + \|r\|_6^3.$$

Moreover, similarly as in (5-13) we obtain

$$3\alpha^{4-\varepsilon}(5-\varepsilon) \int_\Omega U_{x,\lambda}^{4-\varepsilon} r^2 \le 15 \int_\Omega U_{x,\lambda}^4 r^2 + o(\|r\|_6^2).$$

Next, we write

$$\int_\Omega U_{x,\lambda}^{4-\varepsilon} rs = \lambda^{-\varepsilon/2} \Big(\int_\Omega U_{x,\lambda}^4 rs + \int_\Omega U_{x,\lambda}^4 (e^{\varepsilon \log \sqrt{1+\lambda^2 |x-y|^2}} - 1) rs \Big).$$

The prefactor $\lambda^{-\varepsilon/2}$ on the right side tends to 1 by Lemma 5.3. The first integral in the parentheses is bounded in (3-22). For the second integral we proceed again as in (5-13) and obtain

$$\left| \int_\Omega U_{x,\lambda}^4 (e^{\varepsilon \log \sqrt{1+\lambda^2|x-y|^2}} - 1) rs \right| \lesssim \lambda\varepsilon \|U^4 |x-y|\|_{3/2} \|r\|_6 \|s\|_6 \lesssim \varepsilon \lambda^{-1} \|r\|_6,$$

where we used (3-10) in the last inequality. Thus, recalling the bound on ε in (5-2),

$$\left| \int_\Omega U_{x,\lambda}^{4-\varepsilon} rs \right| \lesssim \lambda^{-3/2} \|r\|_6.$$

The fourth term on the right side of (6-8) is bounded, in absolute value, by a constant times

$$\int_\Omega U_{x,\lambda}^4 (\lambda^{-1/2} |H_a(x,\cdot)| + |f_{x,\lambda}|) |r| \lesssim (\lambda^{-1} \phi_a(x) + \lambda^{-2}) \|r\|_6,$$

where we used (3-23).

Using Lemma 5.4 to control the first term on the right-hand side of (6-8) and putting all the estimates into (6-7) we finally get

$$\int_\Omega (|\nabla r|^2 + ar^2 - 15 U_{x,\lambda}^4 r^2) \lesssim (\varepsilon + \lambda^{-1} \phi_a(x) + \lambda^{-3/2}) \|r\|_6 + o(\|r\|_6^2).$$

This, in combination with the coercivity inequality of Lemma 2.3, implies the claim. ☐

6B. ***Expanding*** $\boldsymbol{\alpha^{4-\varepsilon}}$**.** In this subsection, we prove the expansion of $\alpha^{4-\varepsilon}$ in Proposition 6.1.

Proposition 6.3. *As* $\varepsilon \to 0$,

$$\alpha^{4-\varepsilon} = 1 + \frac{\varepsilon}{2}\log\lambda - 4\beta\lambda^{-1} + \mathcal{O}(\varepsilon + \phi_a(x)\lambda^{-1}) + o(\lambda^{-1}). \tag{6-9}$$

Proof. As in the proof of Lemma 5.3 we will integrate (1-2) against u. However, this time we write $u = \alpha(\psi_{x,\lambda} + q)$ and obtain

$$\int_\Omega |\nabla(\psi_{x,\lambda} + q)|^2 + \int_\Omega a(\psi_{x,\lambda} + q)^2 = 3\alpha^{4-\varepsilon}\int_\Omega (\psi_{x,\lambda} + q)^{6-\varepsilon},$$

which we write as

$$\int_\Omega (|\nabla\psi_{x,\lambda}|^2 + a\psi_{x,\lambda}^2 - 3\alpha^{4-\varepsilon}|\psi_{x,\lambda}|^{6-\varepsilon}) \\ + 2\int_\Omega \left(\nabla q\cdot\nabla\psi_{x,\lambda} + aq\psi_{x,\lambda} - \frac{3(6-\varepsilon)}{2}\alpha^{4-\varepsilon} q|\psi_{x,\lambda}|^{4-\varepsilon}\psi_{x,\lambda}\right) = \mathcal{R}_0, \tag{6-10}$$

with

$$\mathcal{R}_0 := -\int_\Omega (|\nabla q|^2 + aq^2) + 3\alpha^{4-\varepsilon}\int_\Omega ((\psi_{x,\lambda} + q)^{6-\varepsilon} - |\psi_{x,\lambda}|^{6-\varepsilon} - (6-\varepsilon)|\psi_{x,\lambda}|^{4-\varepsilon}\psi_{x,\lambda} q).$$

We discuss separately the three terms that are involved in (6-10).

First, we claim that

$$\int_\Omega (|\nabla\psi_{x,\lambda}|^2 + a\psi_{x,\lambda}^2 - 3\alpha^{4-\varepsilon}|\psi_{x,\lambda}|^{6-\varepsilon}) = (1-\alpha^{4-\varepsilon})\frac{3\pi^2}{4} + \frac{3\pi^2}{8}\alpha^{4-\varepsilon}\varepsilon\log\lambda + \mathcal{O}(\varepsilon + \phi_a(x)\lambda^{-1} + \lambda^{-2}).$$

Indeed, this follows in the same way as in the proof of Lemma 3.7(a) together with the fact that

$$\int_\Omega (|\psi_{x,\lambda}|^{6-\varepsilon} - \psi_{x,\lambda}^6) = -\frac{\pi^2}{8}\varepsilon\log\lambda + \mathcal{O}(\varepsilon + \phi_a(x)\lambda^{-1} + \lambda^{-5/2}).$$

To prove the latter expansion, we write $\psi_{x,\lambda} = U_{x,\lambda} - \lambda^{-1/2}H_a(x,\cdot) - f_{x,\lambda}$ and expand, recalling (5-4),

$$|\psi_{x,\lambda}|^{6-\varepsilon} - \psi_{x,\lambda}^6 = U_{x,\lambda}^{6-\varepsilon} - U_{x,\lambda}^6 + \mathcal{O}(U_{x,\lambda}^5(\lambda^{-1/2}|H_a(x,\cdot)| + |f_{x,\lambda}|) + \lambda^{-5/2}|H_a(x,\cdot)|^5 + |f_{x,\lambda}|^5).$$

Using the bounds from Lemma A.2, (B-1) and proceeding as in the proof of Lemma B.3, we obtain

$$\int_\Omega (U_{x,\lambda}^5(\lambda^{-1/2}|H_a(x,\cdot)| + |f_{x,\lambda}|) + \lambda^{-5/2}|H_a(x,\cdot)|^5 + |f_{x,\lambda}|^5) = \mathcal{O}(\phi_a(x)\lambda^{-1} + \lambda^{-5/2}).$$

On the other hand, by an explicit computation,

$$\int_\Omega (U_{x,\lambda}^{6-\varepsilon} - U_{x,\lambda}^6) = \int_{\mathbb{R}^3} (U_{x,\lambda}^{6-\varepsilon} - U_{x,\lambda}^6) + \mathcal{O}(\lambda^{-3}) = \pi^{3/2}\left(\lambda^{-\varepsilon/2}\frac{\Gamma\left(\frac{3-\varepsilon}{2}\right)}{\Gamma\left(3-\frac{\varepsilon}{2}\right)} - \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}\right) + \mathcal{O}(\lambda^{-3}) \\ = -\frac{\pi^2}{8}\varepsilon\log\lambda + \mathcal{O}(\varepsilon + \lambda^{-3}),$$

proving the claimed expansion of the first term on the left side of (6-10).

We turn now to the second term on the left side of (6-10) and claim that

$$\int_\Omega \Big(\nabla q \cdot \nabla \psi_{x,\lambda} + a q \psi_{x,\lambda} - \frac{3(6-\varepsilon)}{2} \alpha^{4-\varepsilon} q |\psi_{x,\lambda}|^{4-\varepsilon} \psi_{x,\lambda} \Big) = (1 - 3\alpha^{4-\varepsilon}) \frac{3\pi^2}{4} \beta \lambda^{-1} + \mathcal{O}(\lambda^{-2}).$$

To show this, we proceed as in the proof of Lemma 3.7(b) and use the equation for $\psi_{x,\lambda}$ to write

$$\begin{aligned}
&\int_\Omega \Big(\nabla q \cdot \nabla \psi_{x,\lambda} + a q \psi_{x,\lambda} - \frac{3(6-\varepsilon)}{2} \alpha^{4-\varepsilon} q |\psi_{x,\lambda}|^{4-\varepsilon} \psi_{x,\lambda} \Big) \\
&\quad = 3\Big(1 - \frac{6-\varepsilon}{2}\alpha^{4-\varepsilon}\Big) \int_\Omega q U_{x,\lambda}^5 - \frac{3(6-\varepsilon)}{2} \int_\Omega q (U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5) \\
&\qquad\qquad - \int_\Omega q \Big(\frac{3(6-\varepsilon)}{2} (|\psi_{x,\lambda}|^{4-\varepsilon} \psi_{x,\lambda} - U_{x,\lambda}^{5-\varepsilon}) + a(f_{x,\lambda} + g_{x,\lambda}) \Big).
\end{aligned}$$

The first term on the right side was already computed in the proof of Lemma 3.7(b), and the last term on the right side can be bounded in the same way as there, except that now, instead of (3-27), we use the bound

$$\|\nabla q\|_2 \lesssim \lambda^{-1}, \tag{6-11}$$

which follows from the bounds on s and r in Propositions 3.2 and (6-6). For the second term on the right side we proceed as in the proof of Lemma 5.4 and obtain

$$\Big| \int_\Omega q (U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5) \Big| \lesssim \varepsilon \lambda^{1-\varepsilon/2} \int_\Omega |q| U_{x,\lambda}^5 |x-y| \le \varepsilon \lambda^{1-\varepsilon/2} \|U^5 |x-y|\|_{6/5} \|q\|_6 \lesssim \varepsilon \|q\|_6 \lesssim \varepsilon \lambda^{-1}.$$

By Proposition 5.5, this is $\mathcal{O}(\lambda^{-2})$.

Finally, we bound $\mathcal{R}_0$, the term on the right side of (6-10). Because of (6-11), the first integral in the definition of $\mathcal{R}_0$ is $\mathcal{O}(\lambda^{-2})$. The second integral is bounded, in absolute value, by a constant times

$$\int_\Omega (|\psi_{x,\lambda}|^{4-\varepsilon} q^2 + |q|^{6-\varepsilon}) \lesssim \|\psi_{x,\lambda}\|_6^{4-\varepsilon} \|q\|_6^2 + \|q\|_6^{6-\varepsilon} \lesssim \lambda^{-2}.$$

Inserting all the bounds in (6-10), we obtain the claimed bound. □

6C. *Expanding $\phi_a(x)$.* In this subsection we prove the following important expansion.

Proposition 6.4. *As $\varepsilon \to 0$,*

$$\phi_a(x) = \pi a(x) \lambda^{-1} + \frac{\pi}{32} \varepsilon \lambda (1 + o(1)) + o(\lambda^{-1}). \tag{6-12}$$

The proof of this proposition, which is the analogue of Proposition 3.8, is a refined version of the proof of Proposition 5.5. We integrate (1-2) for u against $\partial_\lambda \psi_{x,\lambda}$, and we write the resulting equality in the form

$$\begin{aligned}
&\int_\Omega (\nabla \psi_{x,\lambda} \cdot \nabla \partial_\lambda \psi_{x,\lambda} + a \psi_{x,\lambda} \partial_\lambda \psi_{x,\lambda} - 3\alpha^{4-\varepsilon} |\psi_{x,\lambda}|^{4-\varepsilon} \psi_{x,\lambda} \partial_\lambda \psi_{x,\lambda}) \\
&\quad = - \int_\Omega (\nabla q \cdot \nabla \partial_\lambda \psi_{x,\lambda} + a q \partial_\lambda \psi_{x,\lambda} - 3(5-\varepsilon) \alpha^{4-\varepsilon} q |\psi_{x,\lambda}|^{4-\varepsilon} \partial_\lambda \psi_{x,\lambda}) \\
&\qquad\qquad + \frac{3(5-\varepsilon)(4-\varepsilon)}{2} \alpha^{4-\varepsilon} \int_\Omega q^2 |\psi_{x,\lambda}|^{2-\varepsilon} \psi_{x,\lambda} \partial_\lambda \psi_{x,\lambda} + \mathcal{R},
\end{aligned} \tag{6-13}$$

with

$$\mathcal{R} = 3\alpha^{4-\varepsilon} \int_\Omega \Bigg((\psi_{x,\lambda} + q)^{5-\varepsilon} - |\psi_{x,\lambda}|^{4-\varepsilon}\psi_{x,\lambda} - (5-\varepsilon)|\psi_{x,\lambda}|^{4-\varepsilon} q \\ - \frac{(5-\varepsilon)(4-\varepsilon)}{2} |\psi_{x,\lambda}|^{2-\varepsilon}\psi_{x,\lambda} q^2 \Bigg) \partial_\lambda \psi_{x,\lambda}.$$

Lemma 6.5. *As* $\varepsilon \to 0$, *the following hold*:

(a) $$\int_\Omega (\nabla\psi_{x,\lambda} \cdot \nabla\partial_\lambda\psi_{x,\lambda} + a\psi_{x,\lambda}\partial_\lambda\psi_{x,\lambda} - 3\alpha^{4-\varepsilon}|\psi_{x,\lambda}|^{4-\varepsilon}\psi_{x,\lambda}\partial_\lambda\psi_{x,\lambda}) \\ = -2\pi\phi_a(x)\lambda^{-2}(1+o(1)) + \frac{\pi^2}{16}\varepsilon\lambda^{-1}(1+o(1)) + 2\pi^2 a(x)\lambda^{-3} + o(\lambda^{-3}).$$

(b) $$\int_\Omega (\nabla q \cdot \nabla\partial_\lambda\psi_{x,\lambda} + aq\partial_\lambda\psi_{x,\lambda} - 3(5-\varepsilon)\alpha^{4-\varepsilon} q|\psi_{x,\lambda}|^{4-\varepsilon}\partial_\lambda\psi_{x,\lambda}) \\ = -(1-\alpha^{4-\varepsilon})2\pi(\phi_a(x) - \phi_0(x))\lambda^{-2} + \mathcal{O}(\varepsilon\lambda^{-2}\log\lambda + \phi_a(x)\lambda^{-3}) + o(\lambda^{-3}).$$

(c) $$\int_\Omega q^2|\psi_{x,\lambda}|^{2-\varepsilon}\psi_{x,\lambda}\partial_\lambda\psi_{x,\lambda} = \frac{\pi^2}{32}\beta\gamma\lambda^{-3} + \mathcal{O}(\varepsilon\lambda^{-2} + \phi_a(x)\lambda^{-3}) + o(\lambda^{-3}).$$

(d) $\mathcal{R} = o(\lambda^{-3})$.

The proof of Lemma 6.5 is independent of the expansion of $\alpha^{4-\varepsilon}$ in Proposition 6.3. We only use the fact that $\alpha = 1 + o(1)$.

Proof. (a) As in the proof of Lemma 3.10(a), see (3-31), we have

$$\int_\Omega (\nabla\psi_{x,\lambda} \cdot \nabla\partial_\lambda\psi_{x,\lambda} + a\psi_{x,\lambda}\partial_\lambda\psi_{x,\lambda} - 3\alpha^{4-\varepsilon}|\psi_{x,\lambda}|^{4-\varepsilon}\psi_{x,\lambda}\partial_\lambda\psi_{x,\lambda}) \\ = 3\int_\Omega (U_{x,\lambda}^5 - \alpha^{4-\varepsilon}|\psi_{x,\lambda}|^{4-\varepsilon}\psi_{x,\lambda})\partial_\lambda\psi_{x,\lambda} - \int_\Omega a(f_{x,\lambda} + g_{x,\lambda})\partial_\lambda\psi_{x,\lambda}.$$

The second integral on the right side was shown in the proof of Lemma 3.10(a) to satisfy

$$\int_\Omega a(f_{x,\lambda} + g_{x,\lambda})\partial_\lambda\psi_{x,\lambda} = 2\pi(3-\pi)a(x)\lambda^{-3} + o(\lambda^{-3}).$$

We write the first integral on the right side as

$$\int_\Omega (U_{x,\lambda}^5 - \alpha^{4-\varepsilon}|\psi_{x,\lambda}|^{4-\varepsilon}\psi_{x,\lambda})\partial_\lambda\psi_{x,\lambda} = (1-\alpha^{4-\varepsilon})\int_\Omega U_{x,\lambda}^5\partial_\lambda\psi_{x,\lambda} - \alpha^{4-\varepsilon}\int_\Omega (U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5)\partial_\lambda\psi_{x,\lambda} \\ - \alpha^{4-\varepsilon}\int_\Omega (|\psi_{x,\lambda}|^{4-\varepsilon}\psi_{x,\lambda} - U_{x,\lambda}^{5-\varepsilon})\partial_\lambda\psi_{x,\lambda}. \quad (6\text{-}14)$$

As shown in the proof of Lemma 3.10(a),

$$\int_\Omega U_{x,\lambda}^5\partial_\lambda\psi_{x,\lambda} = \frac{2\pi}{3}\phi_a(x)\lambda^{-2} + \mathcal{O}(\lambda^{-3}).$$

Next, by Lemma A.2,

$$\int_\Omega (U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5)\partial_\lambda\psi_{x,\lambda} = \int_\Omega (U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5)\partial_\lambda U_{x,\lambda} + \tfrac{1}{2}\lambda^{-3/2}\int_\Omega (U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5)H_a(x,\cdot) + o(\lambda^{-3}).$$

For the first term, we use (5-20) and the bounds from the proof of Lemma 3.10(a) to get

$$\int_\Omega (U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5)\partial_\lambda U_{x,\lambda} = -\frac{\pi^2}{48}\varepsilon\lambda^{-1}(1+o(1)) + \mathcal{O}(\lambda^{-4}).$$

For the second term, we use the bound $\|U_{x,\lambda}^{-\varepsilon} - 1\|_\infty = \mathcal{O}(\varepsilon \log\lambda)$ and compute

$$\lambda^{-3/2}\left|\int_\Omega (U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5) H_a(x,\cdot)\right| \lesssim \varepsilon\lambda^{-3/2}\log\lambda \int_\Omega U_{x,\lambda}^5 H_a(x,\cdot) \lesssim \varepsilon\lambda^{-2}\log\lambda = o(\varepsilon\lambda^{-1}).$$

Concerning the last term on the right-hand side of (6-14), we will prove

$$\int_\Omega (|\psi_{x,\lambda}|^{4-\varepsilon}\psi_{x,\lambda} - U_{x,\lambda}^{5-\varepsilon})\partial_\lambda\psi_{x,\lambda} \\ = \frac{2\pi}{3}\phi_a(x)\lambda^{-2}(1+o(1)) - 2\pi a(x)\lambda^{-3} + \mathcal{O}(\phi_a(x)^2\lambda^{-3}) + o(\lambda^{-3}). \quad (6\text{-}15)$$

This will complete our discussion of the right-hand side of (6-14) and hence the proof of (a).

The proof of (6-15) is similar to the corresponding argument in the proof of Lemma 3.10(a), but we include some details. We bound pointwise

$$|\psi_{x,\lambda}|^{4-\varepsilon}\psi_{x,\lambda} - U_{x,\lambda}^{5-\varepsilon} = -(5-\varepsilon)\lambda^{-1/2}U_{x,\lambda}^{4-\varepsilon}H_a(x,\cdot) + \tfrac{1}{2}(5-\varepsilon)(4-\varepsilon)\lambda^{-1}U_{x,\lambda}^{3-\varepsilon}H_a(x,\cdot)^2 \\ + \mathcal{O}(\lambda^{-3/2}U_{x,\lambda}^2|H_a(x,\cdot)|^3 + \lambda^{-5/2}|H_a(x,\cdot)|^5 + U_{x,\lambda}^4|f_{x,\lambda}| + |f_{x,\lambda}|^5).$$

Using the bounds from Lemmas A.1 and A.2, we easily find that the remainder term, when integrated against $|\partial_\lambda\psi_{x,\lambda}|$, is $o(\lambda^{-3})$. Using expansion (B-5) we obtain, by an explicit calculation similar to (B-11) and (B-13),

$$\int_\Omega U_{x,\lambda}^{4-\varepsilon}H_a(x,\cdot)\partial_\lambda\psi_{x,\lambda} \\ = \int_\Omega U_{x,\lambda}^{4-\varepsilon}\partial_\lambda U_{x,\lambda}H_a(x,\cdot) + \mathcal{O}(\lambda^{-5/2}\phi_a(x)^2) + o(\lambda^{-5/2}) \\ = -\Big(\frac{2\pi}{15} + \mathcal{O}(\varepsilon)\Big)\phi_a(x)\lambda^{-(3+\varepsilon)/2} + \frac{2\pi}{5}a(x)\lambda^{-5/2} + \mathcal{O}(\lambda^{-5/2}\phi_a(x)^2) + o(\lambda^{-5/2}) \\ = -\frac{2\pi}{15}\phi_a(x)\lambda^{-3/2}(1+o(1)) + \frac{2\pi}{5}a(x)\lambda^{-5/2} + \mathcal{O}(\lambda^{-5/2}\phi_a(x)^2) + o(\lambda^{-5/2}),$$

where we used Lemma 5.3. In the same way, we get

$$\int_\Omega U_{x,\lambda}^{3-\varepsilon}H_a(x,\cdot)^2\partial_\lambda\psi_{x,\lambda} = \mathcal{O}(\lambda^{-2}\phi_a^2(x)) + o(\lambda^{-2}).$$

This proves (6-15).

(b) As in the proof of Lemma 3.10(b) we have

$$\int_\Omega (\nabla q\cdot\nabla\partial_\lambda\psi_{x,\lambda} + aq\partial_\lambda\psi_{x,\lambda} - 3(5-\varepsilon)\alpha^{4-\varepsilon}|\psi_{x,\lambda}|^{4-\varepsilon}q\partial_\lambda\psi_{x,\lambda}) \\ = 3\int_\Omega q(5U_{x,\lambda}^4\partial_\lambda U_{x,\lambda} - (5-\varepsilon)\alpha^{4-\varepsilon}|\psi_{x,\lambda}|^{4-\varepsilon}\partial_\lambda\psi_{x,\lambda}) - \int_\Omega aq(\partial_\lambda f_{x,\lambda} + \partial_\lambda g_{x,\lambda}).$$

According to (3-41), the second term on the right side is $o(\lambda^{-3})$. (Note that we now use the bound (6-11) instead of (3-27).) We write the first integral as

$$\begin{aligned}\int_\Omega q(5U_{x,\lambda}^4\partial_\lambda U_{x,\lambda}-(5-\varepsilon)\alpha^{4-\varepsilon}|\psi_{x,\lambda}|^{4-\varepsilon}\partial_\lambda\psi_{x,\lambda})\\ &= (5(1-\alpha^{4-\varepsilon})+\varepsilon\alpha^{4-\varepsilon})\int_\Omega qU_{x,\lambda}^4\partial_\lambda U_{x,\lambda}+(5-\varepsilon)\alpha^{4-\varepsilon}\int_\Omega q(U_{x,\lambda}^4\partial_\lambda U_{x,\lambda}-\psi_{x,\lambda}^4\partial_\lambda\psi_{x,\lambda})\\ &\quad + (5-\varepsilon)\alpha^{4-\varepsilon}\int_\Omega q(\psi_{x,\lambda}^4-|\psi_{x,\lambda}|^{4-\varepsilon})\partial_\lambda\psi_{x,\lambda}.\end{aligned}$$

According to (3-39),

$$\begin{aligned}(5(1-\alpha^{4-\varepsilon})+\varepsilon\alpha^{4-\varepsilon})\int_\Omega qU_{x,\lambda}^4\partial_\lambda U_{x,\lambda} &= (5(1-\alpha^{4-\varepsilon})+\varepsilon\alpha^{4-\varepsilon})\Big(-\frac{2\pi}{15}(\phi_a(x)-\phi_0(x))\lambda^{-2}+\mathcal{O}(\lambda^{-3})\Big)\\ &= -\frac{2\pi}{3}(1-\alpha^{4-\varepsilon})(\phi_a(x)-\phi_0(x))\lambda^{-2}+\mathcal{O}(\varepsilon\lambda^{-2})+o(\lambda^{-3}),\end{aligned}$$

and according to (3-40), using (6-11) instead of (3-27),

$$\int_\Omega q(U_{x,\lambda}^4\partial_\lambda U_{x,\lambda}-\psi_{x,\lambda}^4\partial_\lambda\psi_{x,\lambda})=\mathcal{O}(\phi_a(x)\lambda^{-3})+o(\lambda^{-3}).$$

Finally, for any fixed $\delta\in(0,d(x))$ and for any $p>1$ we have, by Lemma A.2,

$$\|\psi_{x,\lambda}^p\partial_\lambda\psi_{x,\lambda}\|_{L^\infty(B_\delta(x)^c\cap\Omega)}=\mathcal{O}(\lambda^{-(3+p)/2}). \tag{6-16}$$

On the other hand, taking δ sufficiently small (but independent of ε) we obtain $U_{x,\lambda}\lesssim\psi_{x,\lambda}\lesssim U_{x,\lambda}$ on $B_\delta(x)$. The latter implies $\psi_{x,\lambda}^{-\varepsilon}=U_{x,\lambda}^{-\varepsilon}(1+\mathcal{O}(\varepsilon))$ on $B_\delta(x)$, and therefore

$$\|1-\psi_{x,\lambda}^{-\varepsilon}\|_{L^\infty(B_\delta(x))}=\mathcal{O}(\varepsilon\log\lambda).$$

Consequently, using (6-11) and (6-16),

$$\left|\int_\Omega q(\psi_{x,\lambda}^4-|\psi_{x,\lambda}|^{4-\varepsilon})\partial_\lambda\psi_{x,\lambda}\right|\lesssim\|q\|_6(\varepsilon\log\lambda\|\psi_{x,\lambda}^4\partial_\lambda\psi_{x,\lambda}\|_{6/5}+\lambda^{-7/2})\lesssim\varepsilon\lambda^{-2}\log\lambda+\lambda^{-9/2}.$$

Collecting all the bounds, we arrive at the claimed expansion in (b).

(c) The relevant term with exponent $2-\varepsilon$ replaced by 2 was computed in Lemma 3.10(c). The same computation, but with Proposition 6.2 instead of Proposition 3.4, gives

$$\int_\Omega q^2\psi_{x,\lambda}^3\partial_\lambda\psi_{x,\lambda}=\frac{\pi^2}{32}\beta\gamma\lambda^{-3}+\mathcal{O}(\varepsilon\lambda^{-2}+\phi_a(x)\lambda^{-3})+o(\lambda^{-3}).$$

(The $\mathcal{O}(\varepsilon\lambda^{-2})$ term comes from bounding $\int_\Omega rs\psi_{x,\lambda}^3\partial_\lambda\psi_{x,\lambda}$.)

We bound the difference similarly as at the end of the previous part (b), namely,

$$\begin{aligned}\left|\int_\Omega q^2(|\psi_{x,\lambda}|^{2-\varepsilon}\psi_{x,\lambda}-\psi_{x,\lambda}^3)\partial_\lambda\psi_{x,\lambda}\right| &\lesssim\|q\|_6^2(\varepsilon\log\lambda\|\psi_{x,\lambda}^3\partial_\lambda\psi_{x,\lambda}\|_{3/2}+\lambda^{-3})\\ &\lesssim\varepsilon\lambda^{-3}\log\lambda+\lambda^{-5}=o(\lambda^{-3}).\end{aligned}$$

The proof of (d) uses similar bounds as in the rest of the proof and is omitted. ☐

Proof of Proposition 6.4. Inserting the bounds from Lemma 6.5 into (6-13), we obtain

$$\phi_a(x)(1+o(1)) - \frac{\pi}{32}\varepsilon\lambda(1+o(1)) - \pi a(x)\lambda^{-1} - (1-\alpha^{4-\varepsilon})\phi_0(x) + \frac{15\pi}{32}\beta\gamma\lambda^{-1} = o(\lambda^{-1}).$$

Inserting the expansion of $\alpha^{4-\varepsilon}$ from Proposition 6.3, this becomes

$$\phi_a(x)(1+o(1)) - \frac{\pi}{32}\varepsilon\lambda(1+o(1)) - \pi a(x)\lambda^{-1} - 4\beta\phi_0(x)\lambda^{-1} + \frac{15\pi}{32}\beta\gamma\lambda^{-1} = o(\lambda^{-1}).$$

Using the expansions (3-13) of β and γ, this can be simplified to

$$\phi_a(x)(1+o(1)) - \frac{\pi}{32}\varepsilon\lambda(1+o(1)) - \pi a(x)\lambda^{-1} = o(\lambda^{-1}),$$

which is the assertion. □

6D. *Bounding $\nabla\phi_a$.* In this subsection we prove the bound on $\nabla\phi_a(x)$ in Proposition 6.1.

Proposition 6.6. *For every $\mu < 1$, as $\varepsilon \to 0$,*

$$|\nabla\phi_a(x)| \lesssim \varepsilon\lambda^{1/2} + \lambda^{-\mu} + \phi_a(x)\lambda^{-1/2}. \tag{6-17}$$

Note that together with (5-2) it follows from Proposition 6.6 that x_0 is a critical point of ϕ_a.

The proof of Proposition 6.6 is a refined version of the proof of Proposition 5.8 and is again based on the Pohozaev identity (5-21). The latter reads, in the notation of (3-46),

$$0 = I[\psi_{x,\lambda}] + 2I[\psi_{x,\lambda}, q] + I[q]. \tag{6-18}$$

To control the boundary integrals involving q in this identity, we need the following lemma, which is the analogue of Lemma 3.13.

Lemma 6.7.
$$\left\|\frac{\partial q}{\partial n}\right\|_{L^2(\partial\Omega)} \lesssim \varepsilon + \lambda^{-3/2} + \phi_a(x)\lambda^{-1}.$$

Before proving this lemma, let us use it to complete the proof of Proposition 6.6. In that proof, and later in this subsection, we will use the inequality

$$\|q\|_2 \lesssim \varepsilon + \lambda^{-3/2} + \phi_a(x)\lambda^{-1}. \tag{6-19}$$

This follows from the bound (3-10) on s and the bound in Proposition 6.2 on r.

Proof of Proposition 6.6. It follows from Lemma 6.7 and the bounds (6-19) and (3-49) that

$$|I[\psi_{x,\lambda}, q]| \lesssim \varepsilon\lambda^{-1/2} + \lambda^{-2} + \phi_a(x)\lambda^{-3/2}, \quad |I[q]| \lesssim \varepsilon^2 + \lambda^{-3} + \phi_a(x)^2\lambda^{-2}.$$

The claim thus follows from Lemma 3.12 and (6-18). □

Proof of Lemma 6.7. Note that $-\Delta q = F$, with

$$F := -3U_{x,\lambda}^5 + 3\alpha^{4-\varepsilon}(\psi_{x,\lambda} + q)^{5-\varepsilon} - aq + a(f_{x,\lambda} + g_{x,\lambda}).$$

With the cut-off function ζ defined as in the proof of Lemma 2.6, we have

$$-\Delta(\zeta q) = \zeta F - 2\nabla\zeta\cdot\nabla q - (\Delta\zeta)q.$$

Arguing as in (3-51) we deduce that

$$\zeta|F| \lesssim \zeta|q|^{5-\varepsilon} + |q| + \lambda^{-5/2}. \tag{6-20}$$

Now we follow the line of arguments in the proof of Lemma 3.13. The only difference is that instead of (3-48) we have the bound

$$\|q\|_2 \lesssim \varepsilon + \lambda^{-3/2} + \phi_a(x)\lambda^{-1}, \tag{6-21}$$

which follows from (3-10) and Proposition 6.2. Using this estimate we find

$$\|\Delta(\zeta q)\|_{3/2} \lesssim \varepsilon + \lambda^{-3/2} + \phi_a(x)\lambda^{-1}.$$

In combination with (2-12), this proves the claim. □

7. Proof of Theorems 1.2 and 1.3

7A. ***Proof of Theorem 1.2.*** Equation (1-10) follows from Proposition 5.1, together with (3-2), (3-3) and (3-5). Proposition 5.1 gives also $|x_\varepsilon - x_0| = o(1)$. Moreover, the bound on λ in (5-2) together with (6-4) gives $\nabla\phi_a(x_0) = 0$, and (6-2) gives $\|\nabla r\|_2 = \mathcal{O}(\varepsilon + \lambda^{-3/2} + \phi_a(x)\lambda^{-1})$. By the bound on λ in (5-2), this proves the claimed bound on $\|\nabla r\|_2$ if $\phi_a(x_0) \neq 0$. In the case $\phi_a(x_0) = 0$, we will see below that $\phi_a(x) = o(\lambda^{-1})$ and $\varepsilon = \mathcal{O}(\lambda^{-2})$, so we again obtain the claimed bound.

Next, (6-3) shows that

$$\lim_{\varepsilon\to 0} \varepsilon\lambda = \frac{32}{\pi}\phi_a(x_0), \tag{7-1}$$

which is (1-12).

Equation (1-13) follows from (6-5). In the case $\phi_a(x_0) \neq 0$ this is immediate, and in the case $\phi_a(x_0) = 0$ we use, in addition, the expansion of β from Proposition 3.3 and the fact that $\varepsilon = o(\lambda^{-1})$ by (7-1).

Finally, let us assume $\phi_a(x_0) = 0$ and prove (1-15). We apply Lemma 4.2 to the function $u(x) := \phi_a(x + x_0)$ and get $\phi_a(x) \lesssim |\nabla\phi_a(x)|^2$. From (6-4), together with the fact that $\varepsilon = o(\lambda^{-1})$ by (7-1), we then get

$$\phi_a(x) = o(\lambda^{-1}). \tag{7-2}$$

Inserting this into (6-3), we obtain

$$\pi a(x)\lambda^{-1} + \frac{\pi}{32}\varepsilon\lambda(1 + o(1)) = o(\lambda^{-1}),$$

which is (1-15). This completes the proof of Theorem 1.2. □

7B. ***A bound on $\|w\|_\infty$.*** To complete the proof of Theorem 1.3 it remains to establish a suitable bound on $\|w\|_\infty$, as well as on $\|w\|_p$ for $p > 6$. This is provided by the following modification of Proposition 4.3.

Proposition 7.1. *As $\varepsilon \to 0$,*

$$\|w\|_p \lesssim \lambda^{-3/p} \quad \textit{for every } p \in (6, \infty). \tag{7-3}$$

Moreover, for every $\mu > 0$,

$$\|w\|_\infty = o(\lambda^\mu). \tag{7-4}$$

Proof. To prove the bound (7-3), let $r > 1$ and F be given by (5-22). As in the proof of Proposition 4.3, we obtain the same bound (4-10), where, similarly to (4-11), F satisfies

$$|F| \lesssim U_{x,\lambda}^{5-\varepsilon}|\alpha^{4-\varepsilon}-1| + |U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5| + U_{x,\lambda}^4(|w| + \varphi_{x,\lambda}) + |w|^5 + \varphi_{x,\lambda} + U_{x,\lambda} + |w|. \tag{7-5}$$

Using the bounds $\varepsilon \lesssim \lambda^{-1}$ from Proposition 5.1 and $|\alpha^{4-\varepsilon}-1| \lesssim \varepsilon \log\lambda$ by Proposition 6.3, we can estimate, for every $r > 1$,

$$\begin{aligned}\int_\Omega (U_{x,\lambda}^{5-\varepsilon}|\alpha^{4-\varepsilon}-1| + |U_{x,\lambda}^5 - U_{x,\lambda}^{5-\varepsilon}|)|w|^r \\ \lesssim \|w\|_{3(r+1)}^r (\|U_{x,\lambda}^{5-\varepsilon}\|_{\frac{3r+3}{2r+3}}|\alpha^{4-\varepsilon}-1| + \|U_{x,\lambda}^5 - U_{x,\lambda}^{5-\varepsilon}\|_{\frac{3r+3}{2r+3}}) \lesssim \|w\|_{3(r+1)}^r \varepsilon\log\lambda \|U_{x,\lambda}\|_{5\cdot\frac{3r+3}{2r+3}}^5 \\ \lesssim \|w\|_{3(r+1)}^r \varepsilon \log\lambda \lambda^{\frac{1}{2}\cdot\frac{r-1}{r+1}} \le \eta\|w\|_{3(r+1)}^{r+1} + C_\eta(\log\lambda)^{r+1}\lambda^{-\frac{r+3}{2}} \le \eta\|w\|_{3(r+1)}^{r+1} + C_\eta\lambda^{-1}.\end{aligned}$$

Hence the right side of (4-10) fulfills the same estimate as in the proof of Proposition 4.3, and we conclude (7-3) as we did there.

We now turn to the bound (7-4). From (5-10) we deduce that

$$w(x) = \frac{1}{4\pi}\int_\Omega G_0(x,y)F(y). \tag{7-6}$$

As in Proposition 4.3, we need to estimate $\|F\|_q$ for some $q > \frac{3}{2}$ using (7-5). We bound

$$\|U_{x,\lambda}^{5-\varepsilon}|\alpha^{4-\varepsilon}-1|\|_q \lesssim (\varepsilon\log\lambda + \lambda^{-1})\|U_{x,\lambda}\|_{5q}^5 \lesssim \lambda^{3/2-3/q}\log\lambda$$

for every $q > \frac{3}{2}$. Similarly,

$$\|U_{x,\lambda}^{5-\varepsilon} - U_{x,\lambda}^5\|_q \lesssim \varepsilon\log\lambda\|U_{x,\lambda}\|_{5q}^5 \lesssim \lambda^{3/2-3/q}\log\lambda$$

for every $q > \frac{3}{2}$. The other terms resulting from (7-5) are identical to those already estimated in Proposition 4.3. As there, we thus obtain $\|F\|_q \lesssim \lambda^{2-3/q}\log\lambda$. Letting $q \searrow \frac{3}{2}$ yields (7-4). □

7C. *Proof of Theorem 1.3.* At this point, the proof of Theorem 1.3 is almost identical to the proof of Theorem 1.6. We provide some details nevertheless.

By the bound $\|w\|_\infty = o(\lambda^{1/2})$ from Proposition 7.1 and Proposition 2.1, we have $\|u_\varepsilon\|_\infty = \lambda^{1/2} + o(\lambda^{1/2})$. Thus part (a) of Theorem 1.3 follows from (1-12) and (1-15), respectively.

To prove part (b), we rewrite (1-3) as

$$u(z) = \frac{3}{4\pi}\int_\Omega G_a(z,y)u(y)^{5-\varepsilon}.$$

Fix again $\delta = \delta_\varepsilon = o(1)$ with $\lambda^{-1} = o(\delta_\varepsilon)$, so that $\frac{3}{4\pi}\int_{B_{\delta_\varepsilon(x)}} u(y)^5 = 1 + o(1)$. Then

$$\frac{3}{4\pi}\int_{B_\delta(x)} G_a(z,y)u(y)^5 = \frac{3}{4\pi}\int_{B_\delta(x)}(G_a(z,x_0)+o(1))u(y)^5 = \lambda^{-1/2-\varepsilon/2}G_a(z,x_0)+o(\lambda^{-1/2-\varepsilon/2}).$$

On the other hand, by Lemmas 7.1 and A.1,

$$\left|\int_{\Omega\setminus B_\delta(x)} G_a(z,y)u(y)^{5-\varepsilon}\right| \lesssim \|G_a(z,\cdot)\|_2(\|U_{x,\lambda}\|^{5-\varepsilon}_{L^{10}(\Omega\setminus B_\delta(x))}+\|w\|_{10}^{5-\varepsilon}) \lesssim \lambda^{-5/2}\delta^{-7/2}+\lambda^{-3/2}.$$

Choosing $\delta=\lambda^{-c}$ with $c>0$ small enough and observing that $\lambda^{-\varepsilon/2}=1+o(1)$ by Lemma 5.3, the proof of part (b) of Theorem 1.3 is complete. □

Appendix A: Some useful bounds

In this section, we collect some bounds which will be of frequent use in our estimates.

Lemma A.1. *Let $x\in\Omega$ and let $1\le q<\infty$. As $\lambda\to\infty$, we have*

$$\|U_{x,\lambda}\|_{L^q(\Omega)} \lesssim \begin{cases}\lambda^{-1/2}, & 1\le q<3,\\ \lambda^{-1/2}(\log\lambda)^{1/3}, & q=3,\\ \lambda^{1/2-3/q}, & q>3.\end{cases} \tag{A-1}$$

Moreover, we have

$$\partial_{x_i}U_{x,\lambda}(y)=\lambda^{5/2}\frac{y_i-x_i}{(1+\lambda^2|x-y|^2)^{3/2}},$$

with

$$\|\partial_{x_i}U_{x,\lambda}\|_{L^q(\Omega)} \lesssim \begin{cases}\lambda^{-1/2}, & 1\le q<\frac32,\\ \lambda^{-1/2}(\log\lambda)^{2/3}, & q=\frac32,\\ \lambda^{3/2-3/q}, & q>\frac32,\end{cases}$$

and

$$\partial_\lambda U_{x,\lambda}(y)=\tfrac12\lambda^{-1/2}\frac{1-\lambda^2|x-y|^2}{(1+\lambda^2|x-y|^2)^{3/2}},$$

with

$$\|\partial_\lambda U\|_q\le\lambda^{-1}\|U\|_q \quad \textit{for any } 1\le q\le\infty.$$

Moreover, for any $\rho=\rho_\lambda$ with $\rho\lambda\to\infty$,

$$\|U\|_{L^q(\Omega\setminus B_\rho(x))} \lesssim \begin{cases}\lambda^{-1/2}, & 1\le q<3,\\ \lambda^{-1/2}(\log\lambda)^{1/3}, & q=3,\\ \lambda^{-1/2}\rho^{(3-q)/q}, & q>3,\end{cases}$$

and

$$\|\partial_\lambda U\|_{L^q(\Omega\setminus B_\rho(x))} \lesssim \begin{cases}\lambda^{-3/2}, & 1\le q<3,\\ \lambda^{-3/2}(\log\lambda)^{1/3}, & q=3,\\ \lambda^{-3/2}\rho^{(3-q)/q}, & q>3,\end{cases}$$

and

$$\|\partial_{x_i}U\|_{L^q(\Omega\setminus B_\rho(x))} \lesssim \begin{cases}\lambda^{-1/2}, & 1\le q<\frac32,\\ \lambda^{-1/2}(\log\lambda)^{2/3}, & q=\frac32,\\ \lambda^{-1/2}\rho^{(3-2q)/q}, & q>\frac32.\end{cases}$$

Proof. Taking $R > 0$ such that $\Omega \subset B_R(x)$, we have

$$\int_\Omega U_{x,\lambda}^q \lesssim \lambda^{-3+q/2} \int_0^{\lambda R} \frac{r^2}{(1+r^2)^{q/2}} \lesssim \lambda^{-3+q/2} \int_1^{\lambda R} r^{2-q} \lesssim \begin{cases} \lambda^{-q/2}, & 1 \leq q < 3, \\ \lambda^{-q/2}(\log\lambda)^{1/3}, & q = 3, \\ \lambda^{q/2-3}, & q > 3. \end{cases}$$

This proves (A-1). The remaining bounds follow by analogous explicit computations, which we omit. □

Lemma A.2. *We have*

$$U_{x,\lambda} = PU_{x,\lambda} + \lambda^{-1/2} H_0(x,\cdot) + f_{x,\lambda},$$

with

$$\|f_{x,\lambda}\|_\infty \lesssim \lambda^{-5/2} d^{-3}, \quad \|\partial_\lambda f_{x,\lambda}\|_\infty \lesssim \lambda^{-7/2} d^{-3}, \quad \|\partial_{x_i} f_{x,\lambda}\|_\infty \lesssim \lambda^{-5/2} d^{-4}. \tag{A-2}$$

The function $\varphi_{x,\lambda} := \lambda^{-1/2} H_0(x,\cdot) + f_{x,\lambda}$ *satisfies* $0 \leq \varphi_{x,\lambda} \leq U_{x,\lambda}$ *as well as*

$$\|\varphi_{x,\lambda}\|_6 \lesssim \lambda^{-1/2} d^{-1/2}, \quad \|\varphi_{x,\lambda}\|_\infty \lesssim \lambda^{-1/2} d^{-1}. \tag{A-3}$$

Moreover,

$$\|\partial_\lambda \varphi_{x,\lambda}\|_6 \lesssim \lambda^{-3/2} d^{-1/2}, \quad \|\partial_\lambda \varphi_{x,\lambda}\|_\infty \lesssim \lambda^{-3/2} d^{-1}$$

and

$$\|\partial_{x_i} \varphi_{x,\lambda}\|_6 \lesssim \lambda^{-1/2} d^{-1/2}, \quad \|\partial_{x_i} \varphi_{x,\lambda}\|_\infty \lesssim \lambda^{-1/2} d^{-2}.$$

Proof. Everything, except for the L^∞ bounds on $\varphi_{x,\lambda}$, $\partial_{x_i}\varphi_{x,\lambda}$ and $\partial_\lambda \varphi_{x,\lambda}$, is taken from [Rey 1990, Proposition 1]. Since these functions are harmonic, the remaining bounds follow from the maximum principle. □

Lemma A.3. *We have*

(a) $\displaystyle\int_{\partial\Omega} n \left(\frac{\partial PU_{x,\lambda}}{\partial n}\right)^2 = C\lambda^{-1} \nabla\phi_0(x) + o(\lambda^{-1} d^{-2})$ *for some constant* $C > 0$,

(b) $\displaystyle\int_{\partial\Omega} y \cdot n \left(\frac{\partial PU_{x,\lambda}}{\partial n}\right)^2 = \mathcal{O}(\lambda^{-1} d^{-2})$,

(c) $\displaystyle\int_{\partial\Omega} \left(\frac{\partial PU_{x,\lambda}}{\partial n}\right)^2 = \mathcal{O}(\lambda^{-1} d^{-2})$.

For the proof of Lemma A.3 we refer to [Rey 1990] Equations (2.7), (2.10), and (B.25), respectively.

We define the function

$$g_{x,\lambda}(y) := \frac{\lambda^{-1/2}}{|x-y|} - U_{x,\lambda}(y). \tag{A-4}$$

Lemma A.4. *As* $\lambda \to \infty$,

$$\|g_{x,\lambda}\|_p \lesssim \lambda^{1/2-3/p} \quad \textit{and} \quad \|\partial_\lambda g_{x,\lambda}\|_p \lesssim \lambda^{-1/2-3/p}$$

hold if $1 \leq p < 3$. *Moreover*, $\nabla g_{x,\lambda} \in L^p(\mathbb{R}^3)$ *for all* $1 \leq p < \frac{3}{2}$.

Proof. We have $g_{x,\lambda}(y) = \lambda^{1/2} g_{0,1}(\lambda(x-y))$ with $g_{0,1}(z) = |z|^{-1} - (1+|z|^2)^{-1/2}$. As $|z| \to \infty$,

$$g_{0,1}(z) = |z|^{-1}(1-(1+|z|^{-2})^{-1/2}) \lesssim |z|^{-3}.$$

Hence $g_{0,1} \in L^p(\mathbb{R}^3)$ for all $1 \le p < 3$, which yields $\|g_{x,\lambda}\|_p \le \lambda^{1/2-3/p} \|g_{0,1}\|_{L^p(\mathbb{R}^3)}$.

Next, by direct calculation,

$$\nabla g_{0,1}(z) = -\frac{z}{|z|^3} + \frac{z}{(1+|z|^2)^{3/2}} \lesssim |z|^{-4} \quad \text{as } |z| \to \infty.$$

Hence $\nabla g_{0,1} \in L^p(\mathbb{R}^3)$ for all $1 \le p < \frac{3}{2}$ and since $\nabla g_{x,\lambda}(x,y) = \lambda^{3/2}(\nabla g_{0,1})(\lambda(x-y))$, we conclude that $\nabla g_{x,\lambda} \in L^p(\mathbb{R}^3)$ for all $1 \le p < \frac{3}{2}$.

Finally, we observe

$$\partial_\lambda g_{x,\lambda}(y) = \lambda^{-1} g_{x,\lambda} + \lambda^{1/2}(x-y) \cdot (\nabla g_{0,1})(\lambda(x-y)).$$

By the above, we have $z \cdot \nabla g_{0,1} \in L^p(\mathbb{R}^3)$ for all $1 \le p < 3$ and thus

$$\|\partial_\lambda g_{x,\lambda}\|_p \le \lambda^{-1} \|g_{x,\lambda}\|_p + \lambda^{-1/2-3/p} \|z \cdot \nabla g_{0,1}\|_{L^p(\mathbb{R}^3)}$$

for all $1 \le p < 3$. □

Appendix B: Properties of the functions $H_a(x, y)$

In this appendix, we prove some properties of $H_a(x, y)$ needed in the proofs of the main results. Since these properties hold independently of the criticality of a, we state them for a generic function b which satisfies the same regularity conditions as a, namely,

$$b \in C(\overline{\Omega}) \cap C^{2,\sigma}_{\mathrm{loc}}(\Omega) \quad \text{for some } 0 < \sigma < 1.$$

(In fact, in Section B1 we only use $b \in C(\overline{\Omega}) \cap C^{1,\sigma}_{\mathrm{loc}}(\Omega)$ for some $0 < \sigma < 1$.) In addition, we assume that $-\Delta + b$ is coercive in Ω with Dirichlet boundary conditions. Note that the choice $b = 0$ is allowed.

B1. *Estimates on $H_b(x, \cdot)$.* We start by recalling the bound

$$\|H_b(x, \cdot)\|_\infty \lesssim d(x)^{-1} \quad \text{for all } x \in \Omega, \tag{B-1}$$

see [Frank et al. 2021, Equation (2.6)]. We next prove a similar bound for the derivatives of $H_b(x, \cdot)$.

Lemma B.1. *Let $x, y \in \Omega$ with $x \ne y$. Then $\nabla_x H_b(x, y)$ and $\nabla_y H_b(x, y)$ exist and satisfy*

$$\sup_{y \in \Omega \setminus \{x\}} |\nabla_x H_b(x, y)| \le C, \tag{B-2}$$

$$\sup_{y \in \Omega \setminus \{x\}} |\nabla_y H_b(x, y)| \le C, \tag{B-3}$$

with C uniform for x in compact subsets of Ω.

Proof. Step 1: We first prove the bounds for the special case $b = 0$, which we shall need as an ingredient for the general proof. Since $H_0(x, \cdot)$ is harmonic, we have $\Delta_y \nabla_y H_0(x, y) = 0$. Moreover, we have the bound $\nabla_y G_0(x,y) \lesssim |x-y|^{-2}$ uniformly for $x, y \in \Omega$ [Widman 1967, Theorem 2.3]. This implies that for x in a compact subset of Ω and for $y \in \partial\Omega$,

$$|\nabla_y H_0(x,y)| = |\nabla_y(|x-y|^{-1}) - \nabla_y G_0(x,y)| \le C.$$

We now conclude by the maximum principle.

The proof for the bound on $\nabla_x H_0(x,y)$ is analogous, but simpler, because $\nabla_x G_0(x,y) = 0$ for $y \in \partial\Omega$.

Step 2: For general b, we first prove the bounds for both x and y lying in a compact subset of Ω. By [Frank et al. 2021, Proof of Lemma 2.5] we have

$$H_b(x,y) = \phi_b(x) + \Psi_x(y) - \tfrac{1}{2} b(x)|y-x|,$$

with $\|\Psi_x\|_{C^{1,\mu}(K)} \le C$ for every $0 < \mu < 1$ and every compact subset K of Ω, and with C uniform for x in compact subsets. This shows that $|\nabla_y H_b(x,y)| \le C$ uniformly for x, y in compact subsets of Ω. By symmetry of H_b, this also implies $|\nabla_x H_b(x,y)| \le C$ uniformly for x, y in compact subsets of Ω.

Step 3: We complete the proof of the lemma by treating the case when x remains in a compact subset but y is close to the boundary. In particular, for what follows we may assume

$$|x-y|^{-1} \lesssim 1. \tag{B-4}$$

By the resolvent formula, we write

$$H_b(x,y) = H_0(x,y) + \frac{1}{4\pi} \int_\Omega G_0(x,z) b(z) G_b(z,y)\, dz.$$

By Step 1, the derivatives of $H_0(x,y)$ are uniformly bounded.

We thus only need to consider the integral term. Its ∂_{x_i}-derivative equals

$$\begin{aligned}\int_\Omega \partial_{x_i}\left(\frac{1}{|x-z|}\right) b(z) G_b(z,y)\, dz - \int_\Omega \partial_{x_i} H_0(x,z) b(z) G_b(z,y)\, dz \\ \lesssim \int_\Omega \frac{1}{|x-z|^2} \frac{1}{|z-y|}\, dz + 1 \lesssim \frac{1}{|x-y|^2} + 1 \lesssim 1,\end{aligned}$$

where we again used the fact that (B-2) holds for $b = 0$, together with (B-4). This completes the proof of (B-2).

The proof of (B-3) can be completed analogously. It suffices to write the resolvent formula as

$$H_b(x,y) = H_0(x,y) + \frac{1}{4\pi} \int_\Omega G_b(x,z) b(z) G_0(z,y)\, dz$$

in order to ensure that the ∂_{y_i}-derivative falls on G_0 and we can use (B-3) for $b = 0$. □

We now prove an expansion of $H_b(x,y)$ on the diagonal which improves upon [Frank et al. 2021, Lemma 2.5].

Lemma B.2. *Let $0 < \mu < 1$. If $y \to x$, then uniformly for x in compact subsets of Ω,*

$$H_b(x,y) = \phi_b(x) + \tfrac{1}{2}\nabla\phi_b(x)\cdot(y-x) - \tfrac{1}{2} b(x)|y-x| + \mathcal{O}(|y-x|^{1+\mu}). \tag{B-5}$$

Proof. In [Frank et al. 2021, Lemma 2.5], it is proved that

$$\Psi_x(y) := H_b(x,y) - \phi_b(x) + \tfrac{1}{2} b(x)|y-x| \tag{B-6}$$

is in $C^{1,\mu}_{\mathrm{loc}}(\Omega)$ (as a function of y) for any $\mu < 1$. Thus, by expanding $\Psi_x(y)$ near $y = x$,

$$H_b(x,y) = \phi_b(x) + \nabla\Psi_x(x)\cdot(y-x) - \tfrac{1}{2} b(x)|y-x| + \mathcal{O}(|y-x|^{1+\mu}). \tag{B-7}$$

This gives (B-5) provided we can show that, for each fixed $x \in \Omega$,

$$\nabla\Psi_x(x) = \tfrac{1}{2}\nabla\phi_b(x). \tag{B-8}$$

Indeed, by using (B-7) twice with the roles of x and y exchanged, subtracting and recalling $H_b(x,y) = H_b(y,x)$, we get

$$\begin{aligned} \phi_b(y) - \phi_b(x) &= (\nabla\Psi_y(y) + \nabla\Psi_x(x))(y-x) + \tfrac{1}{2}(b(y)-b(x))|x-y| + \mathcal{O}(|x-y|^{1+\mu}) \\ &= (\nabla\Psi_y(y) + \nabla\Psi_x(x))(y-x) + \mathcal{O}(|x-y|^{1+\mu}), \end{aligned} \tag{B-9}$$

because $b \in C^{0,\mu}_{\mathrm{loc}}(\Omega)$. We now argue that $\Psi_y \to \Psi_x$ in $C^1_{\mathrm{loc}}(\Omega)$, which implies $\nabla\Psi_y(y) \to \nabla\Psi_x(x)$. Together with this, (B-8) follows from (B-9).

To justify the convergence of Ψ_y we argue similarly as in [Frank et al. 2021, Lemma 2.5]. We note that $-\Delta_z\Psi_y = F_y(z)$, with

$$F_y(z) := \frac{b(z)-b(y)}{|z-y|} - b(z)H_b(y,z).$$

We claim that $F_y \to F_x$ in $L^p_{\mathrm{loc}}(\Omega)$ for any $p < \infty$. Indeed, the first term in the definition of F_y converges pointwise to F_x in $\Omega \setminus \{x\}$ and is locally bounded, independently of y, since $b \in C^{0,1}_{\mathrm{loc}}(\Omega)$. Thus, by dominated convergence it converges in $L^p_{\mathrm{loc}}(\Omega)$ for any $p < \infty$. Convergence in $L^\infty_{\mathrm{loc}}(\Omega)$ of the second term in the definition of F_y follows from the bound on the gradient of H_b in Lemma B.1. This proves the claim.

By elliptic regularity, the convergence $F_y \to F_x$ in $L^p_{\mathrm{loc}}(\Omega)$ implies the convergence $\Psi_y \to \Psi_x$ in $C^{1,1-3/p}_{\mathrm{loc}}(\Omega)$. This completes the proof. □

Lemma B.3. *For any $x \in \Omega$ we have, as $\lambda \to \infty$,*

$$\int_\Omega U_{x,\lambda}^5 H_b(x,\cdot) = \frac{4\pi}{3}\phi_b(x)\lambda^{-1/2} - \frac{4\pi}{3} b(x)\lambda^{-3/2} + o(\lambda^{-3/2}), \tag{B-10}$$

$$\int_\Omega U_{x,\lambda}^4 \partial_\lambda U_{x,\lambda} H_b(x,\cdot) = -\frac{2\pi}{15}\phi_b(x)\lambda^{-3/2} + \frac{2\pi}{5} b(x)\lambda^{-5/2} + o(\lambda^{-5/2}), \tag{B-11}$$

$$\int_\Omega U_{x,\lambda}^4 \partial_{x_i} U_{x,\lambda} H_b(x,\cdot) = \frac{2\pi}{15}\nabla\phi_b(x)\lambda^{-1/2} + o(\lambda^{-1/2}), \tag{B-12}$$

$$\int_\Omega U_{x,\lambda}^4 H_b(x,\cdot)^2 = \pi^2\phi_b(x)^2\lambda^{-1} + o(\lambda^{-1}), \tag{B-13}$$

$$\int_\Omega U_{x,\lambda}^3 \partial_\lambda U_{x,\lambda} H_b(x,\cdot)^2 = -\frac{\pi^2}{4}\phi_b(x)^2\lambda^{-2} + o(\lambda^{-2}). \tag{B-14}$$

The implied constants can be chosen uniformly for x in compact subsets of Ω.

Proof. Equalities (B-10) and (B-13) are proved in [Frank et al. 2021, Lemmas 2.5 and 2.6]. To prove (B-11), we write

$$\partial_\lambda U_{x,\lambda} = \frac{U_{x,\lambda}}{2\lambda} - \lambda^{3/2}\frac{|x-y|^2}{(1+\lambda^2|x-y|^2)^{3/2}}, \tag{B-15}$$

and therefore, using (B-10),

$$\int_\Omega H_b(x,y)U_{x,\lambda}^4\partial_\lambda U_{x,\lambda} = \frac{2\pi}{3}\phi_b(x)\lambda^{-3/2} - \frac{2\pi}{3}b(x)\lambda^{-5/2} - \lambda^{7/2}\int_\Omega H_b\frac{|x-y|^2}{(1+\lambda^2|x-y|^2)^{7/2}} + o(\lambda^{-5/2}).$$

With the help of (B-5) and the bound (B-1) we get

$$\begin{aligned}\int_\Omega H_b\frac{|x-y|^2}{(1+\lambda^2|x-y|^2)^{7/2}} &= 4\pi\phi_b(x)\lambda^{-5}\int_0^\infty\frac{t^4\,dt}{(1+t^2)^{7/2}} - 2\pi b(x)\lambda^{-6}\int_0^\infty\frac{t^5\,dt}{(1+t^2)^{7/2}} + o(\lambda^{-6})\\ &= \frac{4\pi}{5}\phi_b(x)\lambda^{-5} - \frac{16\pi}{15}b(x)\lambda^{-6} + o(\lambda^{-6}).\end{aligned}$$

Combining the last two equations gives (B-11).

For the proof of (B-14) we again use (B-15), but now we use (B-13) instead of (B-10). The constant comes from

$$\int_0^\infty\frac{t^4\,dt}{(1+t^2)^3} = \frac{3\pi}{16}.$$

We omit the details.

For the proof of (B-12) we use the explicit formula for $\partial_{x_i}U_{x,\lambda}$ in Lemma A.1. We split the integral into $B_d(x)$ and $\Omega\setminus B_d(x)$. In the first one, we used the bound (B-1) and the expansion (B-5). By oddness, the contribution coming from $\phi_a(x)$ cancels, as does the contribution from $\sum_{k\neq i}\partial_k\phi_b(x)(y_k-x_k)$. For the remaining term we use

$$\int_{B_d(x)}U_{x,\lambda}^4(y)\partial_{x_i}U_{x,\lambda}(y)(y_i-x_i) = \frac{4\pi}{3}\lambda^{-1/2}\int_0^{\lambda d}\frac{t^4\,dt}{(1+t^2)^{7/2}} = \frac{4\pi}{15}\lambda^{-1/2} + \mathcal{O}(\lambda^{-5/2}).$$

A similar computation shows that the contribution from the error $|x-y|^{1+\mu}$ on $B_d(x)$ is $\mathcal{O}(\lambda^{-1/2-\mu})$. Finally, the bounds from Lemma A.1 show that the contribution from $\Omega\setminus B_d(x)$ is $\mathcal{O}(\lambda^{-5/2})$. This completes the proof. □

Remark B.4. The proof just given shows that (B-12) holds with the error bound $\mathcal{O}(\lambda^{-1/2-\mu})$ for any $0<\mu<1$ instead of $o(\lambda^{-1/2})$.

B2. ***C^2 differentiability of ϕ_a.*** In this subsection, we prove Lemma 4.1. The argument is independent of the criticality of a, and we give the proof for a general function $b\in C^{0,1}(\overline{\Omega})\cap C^{2,\sigma}_{\mathrm{loc}}(\Omega)$ for some $0<\sigma<1$. The following argument is similar to [Frank et al. 2021, Lemma 2.5], where a first-order differentiability result is proved, and to [del Pino et al. 2004, Lemma A.1], where it is shown that $\phi_b\in C^\infty(\Omega)$ for constant b.

Let

$$\Psi(x,y) := H_b(x,y) + \tfrac{1}{4}(b(x)+b(y))|x-y|, \quad (x,y)\in\Omega\times\Omega. \tag{B-16}$$

Then $\phi_b(x)=\Psi(x,x)$, so it suffices to show that $\Psi\in C^2(\Omega\times\Omega)$.

Using $-\Delta_y |x-y| = -2|x-y|^{-1}$ and $-\Delta_y H_b(x,y) = b(y)G_b(x,y)$, we have

$$-\Delta_y \Psi(x,y) = -b(y)H_b(x,y) - \frac{1}{2}\frac{b(x)-b(y)-\nabla b(y)\cdot(x-y)}{|x-y|} - \frac{1}{4}\Delta b(y)|x-y|.$$

Since $b \in C^{2,\sigma}_{\mathrm{loc}}(\Omega)$ and since H_b is Lipschitz by Lemma B.1, the right side is in $C^{0,\sigma}_{\mathrm{loc}}(\Omega)$ as a function of y. By elliptic regularity, $\Psi(x,y)$ is in $C^{2,\sigma}_{\mathrm{loc}}(\Omega)$ as a function of y. Since $\Psi(x,y)$ is symmetric in x and y, we infer that $\Psi(x,y)$ is in $C^{2,\sigma}_{\mathrm{loc}}(\Omega)$ as a function of x.

It remains to justify the existence of mixed derivatives $\partial_{y_j}\partial_{x_i}\Psi(x,y)$. For this, we carry out a similar elliptic regularity argument for the function $\partial_{x_i}\Psi(x,y)$. We have

$$\begin{aligned}-\Delta_y \partial_{x_i}\Psi(x,y) = -b(y)\partial_{x_i}H_b(x,y) - \frac{1}{4}\Delta b(y)\frac{x_i-y_i}{|x-y|} - \frac{1}{2}\frac{\partial_i b(x)-\partial_i b(y)}{|x-y|}& \\ + \frac{1}{2}\frac{x_i-y_i}{|x-y|^3}(b(x)-b(y)-\nabla b(y)\cdot(x-y)).&\end{aligned}$$

Since $b \in C^{1,1}_{\mathrm{loc}}(\Omega)$ and since $\partial_{x_i}H_b$ is bounded by Lemma B.1, the right side is in $L^\infty_{\mathrm{loc}}(\Omega)$ as a function of y. By elliptic regularity, $\partial_{x_i}\Psi(x,y) \in C^{1,\mu}(\Omega)$ for every $\mu < 1$ as a function of y. In particular, the mixed derivative $\partial_{y_j}\partial_{x_i}\Psi(x,y)$ is in $C^{0,\mu}_{\mathrm{loc}}(\Omega)$ as a function of y. By symmetry, the same argument shows that the mixed derivative $\partial_{x_j}\partial_{y_i}\Psi(x,y)$ is in $C^{0,\mu}_{\mathrm{loc}}(\Omega)$ as a function of x.

The proof of Lemma 4.1 is therefore complete. □

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Added in proof

The topic of this paper has been further pursued in [König and Laurain 2022; König and Laurain 2023], where the case of several blow-up points is analyzed.

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A DETERMINATION OF THE BLOWUP SOLUTIONS TO THE FOCUSING, QUINTIC NLS WITH MASS EQUAL TO THE MASS OF THE SOLITON

BENJAMIN DODSON

We prove the only blowup solutions to the focusing, quintic nonlinear Schrödinger equation with mass equal to the mass of the soliton are rescaled solitons or the pseudoconformal transformation of those solitons.

1. Introduction

The one dimensional, focusing, mass-critical nonlinear Schrödinger equation is given by

$$iu_t + u_{xx} + |u|^4 u = 0, \quad u(0,x) = u_0(x) \in L^2(\mathbb{R}). \tag{1-1}$$

This equation is a special case of the Hamiltonian equation

$$iu_t + u_{xx} + |u|^{p-1} u = 0, \quad u(0,x) = u_0(x), \quad p > 1. \tag{1-2}$$

If $u(t,x)$ is a solution to (1-2), then

$$v(t,x) = \lambda^{2/(p-1)} u(\lambda^2 t, \lambda x) \tag{1-3}$$

is a solution to (1-2) with appropriately rescaled initial data. Furthermore,

$$\|\lambda^{2/(p-1)} u(0,\lambda x)\|_{\dot{H}^s(\mathbb{R})} = \lambda^{2/(p-1)+s-1/2} \|u_0\|_{\dot{H}^s(\mathbb{R})},$$

so, for $s_p = \frac{1}{2} - 2/(p-1)$, the $\dot{H}^{s_p}(\mathbb{R})$ norm of the initial data is invariant under the scaling symmetry (1-3).

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The scaling symmetry in (1-3) controls the local well-posedness theory of (1-1). In that case, $p = 5$ and $s_p = 0$.

Theorem 1. *The initial value problem* (1-1) *is locally well-posed for any* $u_0 \in L^2$.

(1) *For any* $u_0 \in L^2$, *there exists* $T(u_0) > 0$ *such that* (1-1) *is locally well-posed on the interval* $(-T, T)$.

(2) *If* $\|u_0\|_{L^2}$ *is small then* (1-1) *is globally well-posed, and the solution scatters both forward and backward in time. That is, there exist* $u_-, u_+ \in L^2(\mathbb{R})$ *such that*

$$\lim_{t \nearrow +\infty} \|u(t) - e^{it\Delta} u_+\|_{L^2} = 0 \quad \textit{and} \quad \lim_{t \searrow -\infty} \|u(t) - e^{it\Delta} u_-\|_{L^2} = 0.$$

(3) *If* I *is the maximal interval of existence for a solution to* (1-1) *with initial data* u_0, *we say* u *blows up forward in time if*

$$\lim_{T \nearrow \sup(I)} \|u\|_{L^6_{t,x}([0,T]\times\mathbb{R})} = +\infty.$$

If u *does not blow up forward in time, then* $\sup(I) = +\infty$ *and* u *scatters forward in time.*

(4) *If* $\sup(I) < \infty$ *then, for any* $s > 0$,

$$\lim_{t \nearrow \sup(I)} \|u(t)\|_{H^s} = +\infty.$$

(5) *Time reversal symmetry implies that the results corresponding to* (3) *and* (4) *also hold going backward in time.*

Remark. It is very important to emphasize that throughout this paper, blow up in positive time may be in finite time or infinite time, unless specified otherwise. The same is true for blow up in negative time.

Proof. Theorem 4 was proved in [Cazenave and Weissler 1990]. See also [Ginibre and Velo 1979a; 1979b; 1985; Kato 1987]. The proof uses the Strichartz estimates

$$\|u\|_{L_t^\infty L_x^2 \cap L_t^4 L_x^\infty (I\times\mathbb{R})} \lesssim \|u_0\|_{L^2(\mathbb{R})} + \|F\|_{L_t^1 L_x^2 + L_t^{4/3} L_x^1 (I\times\mathbb{R})},$$

where u is the solution to

$$iu_t + u_{xx} = F, \quad u(0,x) = u_0,$$

on the interval I, where $0 \in I$. The Strichartz estimates were proved in [Ginibre and Velo 1992; Strichartz 1977; Yajima 1987]. Theorem 4 was proved using Picard iteration, so u is a strong solution to (1-1). For all $t \in I$, where I is the open interval on which local well-posedness of (1-1) holds,

$$u(t) = e^{it\partial_{xx}} u_0 + i \int_0^t e^{i(t-\tau)\partial_{xx}} (|u(\tau)|^4 u(\tau))\, d\tau.$$

See [Tao 2006] for different notions of a solution. □

Furthermore, a solution to (1-1) has the conserved quantities mass,

$$M(u(t)) = \int |u(t,x)|^2\, dx = M(u(0)),$$

and energy,

$$E(u(t)) = \frac{1}{2}\int |u_x(t,x)|^2\,dx - \frac{1}{6}\int |u(t,x)|^6\,dx = E(u(0)).$$

For the more general equation (1-2), the Hamiltonian is given by

$$E(u(t)) = \frac{1}{2}\int |u_x(t,x)|^2\,dx - \frac{1}{p+1}\int |u(t,x)|^{p+1}\,dx = E(u(0)). \tag{1-4}$$

For $p < 5$, [Ginibre and Velo 1979a] proved global well-posedness of (1-2) with initial data $u_0 \in H^1(\mathbb{R})$. Indeed, by a straightforward application of the fundamental theorem of calculus and Hölder's inequality, if $u(t,x) \in L^2 \cap \dot{H}^1$,

$$|u(t,x)|^2 \le \int_x^\infty \big|\partial_y |u(t,y)|^2\big|\,dy \le 2\int_x^\infty |\partial_y u(t,y)|\,|u(t,y)|\,dy \le 2\|u\|_{\dot{H}^1(\mathbb{R})}\|u\|_{L^2(\mathbb{R})}. \tag{1-5}$$

Therefore,

$$\|u(t)\|_{L^{p+1}(\mathbb{R})}^{p+1} \lesssim \|u(t)\|_{\dot{H}^1(\mathbb{R})}^{(p-1)/2}\|u(t)\|_{L^2(\mathbb{R})}^{(p+3)/2},$$

so (1-4) implies the existence of a uniform upper bound on $\|u(t)\|_{\dot{H}^1}$ when $p < 5$.

For $p > 5$, there exist singular solutions of (1-2), that is, solutions on the finite interval $[0,T)$, $T < \infty$, for which

$$\lim_{t\to T}\|u(t)\|_{H^1(\mathbb{R})} = \infty.$$

See [Glassey 1977; Weinstein 1986].

When $p = 5$, (1-5) implies

$$\int |u(t,x)|^6\,dx \lesssim \|u(t)\|_{\dot{H}^1(\mathbb{R})}^2\|u(t)\|_{L^2(\mathbb{R})}^4, \tag{1-6}$$

which implies the existence of a threshold mass M_0 for which, if $\|u_0\|_{L^2} < M_0$,

$$E(u(t)) \gtrsim_{M_0} \|u(t)\|_{\dot{H}^1(\mathbb{R})}^2,$$

with implicit constant $\searrow 0$ as $\|u_0\|_{L^2} \nearrow M_0$.

From [Weinstein 1982], the optimal constant in (1-6) is given by the Gagliardo–Nirenberg inequality,

$$\|u\|_{L^6(\mathbb{R})}^6 \le 3\left(\frac{\|u\|_{L^2}^2}{\|Q\|_{L^2}^2}\right)^2\|u_x\|_{L^2}^2, \tag{1-7}$$

where

$$Q(x) = \left(\frac{3}{\cosh(2x)^2}\right)^{1/4}. \tag{1-8}$$

Therefore, if $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then (1-7) implies

$$E(u(t)) \gtrsim_{\|u_0\|_{L^2}} \|u(t)\|_{\dot{H}^1}^2, \tag{1-9}$$

which implies global well-posedness of (1-1) with initial data $u_0 \in H^1$ and $\|u_0\|_{L^2} < \|Q\|_{L^2}$. Furthermore, the identities

$$\frac{d}{dt}\int x^2|u(t,x)|^2\,dx = 4\,\mathrm{Im}\int x\overline{u(t,x)}u_x(t,x)\,dx$$

and

$$\frac{d^2}{dt^2}\int x^2|u(t,x)|^2\,dx = 16E(u(t))$$

imply scattering for (1-1) with initial data

$$u_0 \in H^1(\mathbb{R}) \cap \Sigma = \left\{u : \int x^2|u(x)|^2\,dx < \infty\right\}, \quad \|u_0\|_{L^2} < \|Q\|_{L^2}.$$

More recently, [Dodson 2015; 2016a] proved that (1-1) is globally well-posed and scattering for any initial data $u_0 \in L^2$, $\|u_0\|_{L^2} < \|Q\|_{L^2}$. The proof used the concentration compactness result of [Keraani 2006; Tao et al. 2008] which states that if $u(t)$ is a blowup solution to (1-1) of minimal mass, if t_n is a sequence of times approaching $\sup(I)$, and if u blows up forward in time on the maximal interval of existence I, then $u(t_n,x)$ has a subsequence that converges in L^2, up to the symmetries of (1-1). Using this fact, [Dodson 2015] proved that if u is a minimal mass blowup solution to (1-1), then there exists a sequence $t_n' \to \sup(I)$, for which $E(v_n) \searrow 0$, where v_n is a good approximation of $u(t_n',x)$, acted on by appropriate symmetries. Since (1-9) implies that the only u with mass less than $\|Q\|_{L^2}^2$ and zero energy is $u \equiv 0$, and the small data scattering result implies that the zero solution is stable under small perturbations, there cannot exist a minimal mass blowup solution to (1-1) with mass less than $\|Q\|_{L^2}^2$.

When $\|u\|_{L^2} = \|Q\|_{L^2}$, (1-7) only implies that $E(u) \geq 0$. The $Q(x)$ in (1-8) is the unique, positive solution to

$$Q_{xx} + Q^5 = Q. \tag{1-10}$$

See [Berestycki et al. 1981; Berestycki and Lions 1978; Kwong 1989; Strauss 1977] for existence and uniqueness of a ground state solution in general dimensions. Also observe that by the Pohozaev identity,

$$E(Q) = \frac{1}{2}\int (Q - Q_{xx} - Q^5)\left(\tfrac{1}{2}Q + xQ_x\right)dx = 0.$$

Up to the scaling (1-3), multiplication by a modulus one constant, and translation in space, Q is the unique minimizer of the energy functional with mass $\|Q\|_{L^2}$. See [Cazenave and Lions 1982; Weinstein 1986].

It is straightforward to verify that (1-8) solves (1-10), and that $e^{it}Q$ solves (1-1). Since $\|e^{it}Q\|_{L^6}$ is constant for all $t \in \mathbb{R}$, we have that $e^{it}Q$ blows up both forward and backward in time. Furthermore, the pseudoconformal transformation of $e^{it}Q(x)$,

$$u(t,x) = \frac{1}{t^{1/2}}\exp\left[\frac{-i}{t} + \frac{ix^2}{4t}\right]Q\left(\frac{x}{t}\right), \quad t > 0, \tag{1-11}$$

is a solution to (1-1) that blows up as $t \searrow 0$ and scatters as $t \to \infty$. Note that the mass is preserved under the pseudoconformal transformation of $e^{it}Q$.

It has long been conjecture that, up to symmetries of (1-1), the only nonscattering solutions to (1-1) are the soliton $e^{it}Q$ and the pseudoconformal transformation of the soliton, (1-11). Partial progress has been made in this direction.

Theorem 2. *If $u_0 \in H^1$, $\|u_0\|_{L^2} = \|Q\|_{L^2}$, and the solution $u(t)$ to* (1-1) *blows up in finite time $T > 0$, then $u(t,x)$ is equal to* (1-11), *up to symmetries of* (1-1).

Proof. This result was proved in [Merle 1992; 1993], and was proved for the focusing, mass-critical nonlinear Schrödinger equation in every dimension. □

For the mass-critical nonlinear Schrödinger equation in higher dimensions with radially symmetric initial data, [Killip et al. 2009] proved:

Theorem 3. *If $\|u_0\|_{L^2} = \|Q\|_{L^2}$ is radially symmetric, u is the solution to the focusing, mass-critical nonlinear Schrödinger equation with initial data u_0, and u blows up both forward and backward in time, then u is equal to the soliton, up to symmetries of the mass-critical nonlinear Schrödinger equation.*

In this paper we completely resolve this conjecture in one dimension, showing that the only blowup solutions to (1-1) with mass $\|u_0\|_{L^2}^2 = \|Q\|_{L^2}^2$ are the soliton and the pseudoconformal transformation of the soliton. This result should also hold in higher dimensions, which will be addressed in a forthcoming paper.

It is convenient to begin by considering solutions symmetric in x first.

Theorem 4. *The only symmetric solutions to* (1-1) *with mass $\|u_0\|_{L^2} = \|Q\|_{L^2}$ that blow up forward in time are the family of soliton solutions*

$$e^{-i\theta} e^{i\lambda^2 t} \lambda^{1/2} Q(\lambda x), \quad \lambda > 0, \ \ \theta \in \mathbb{R}, \tag{1-12}$$

and the pseudoconformal transformation of the soliton solution

$$\frac{1}{(T-t)^{1/2}} e^{i\theta} \exp\left[\frac{ix^2}{4(t-T)}\right] \exp\left[i\frac{\lambda^2}{t-T}\right] Q\left(\frac{\lambda x}{T-t}\right), \quad \lambda > 0, \ \ \theta \in \mathbb{R}, \ \ T \in \mathbb{R}, \ \ t < T. \tag{1-13}$$

The proof of Theorem 4 will occupy most of the paper. Once we have proved Theorem 4, we will remove the symmetry assumption on u_0, proving:

Theorem 5. *The only solutions to* (1-1) *with mass $\|u_0\|_{L^2} = \|Q\|_{L^2}$ that blow up forward in time are the family of soliton solutions*

$$e^{-i\theta - it\xi_0^2} e^{i\lambda^2 t} e^{ix\xi_0} \lambda^{1/2} Q(\lambda(x - 2t\xi_0) + x_0), \quad \lambda > 0, \ \ \theta \in \mathbb{R}, \ \ x_0 \in \mathbb{R}, \ \ \xi_0 \in \mathbb{R}, \tag{1-14}$$

and the pseudoconformal transformation of the family of solitons,

$$\frac{1}{(T-t)^{1/2}} e^{i\theta} \exp\left[\frac{i(x-\xi_0)^2}{4(t-T)}\right] \exp\left[i\frac{\lambda^2}{t-T}\right] Q\left(\frac{\lambda(x-\xi_0) - (T-t)x_0}{T-t}\right),$$
$$\text{where } \lambda > 0, \ \ \theta \in \mathbb{R}, \ \ x_0 \in \mathbb{R}, \ \ \xi_0 \in \mathbb{R}, \ \ T \in \mathbb{R}, \ \ t < T. \tag{1-15}$$

Applying time reversal symmetry to (1-1), *this theorem completely settles the question of qualitative behavior of solutions to* (1-1) *for initial data satisfying $\|u_0\|_{L^2} = \|Q\|_{L^2}$.*

The reader should see [Nakanishi and Schlag 2011] for this result for the Klein–Gordon equation.

Before proceeding to Section 2, the proof of Theorems 4 and 5 will be outlined. The first step (in Section 2) in the proof of Theorem 4 is to use the sequential convergence result of [Fan 2021] to show that Theorem 4 reduces to considering a symmetric solution to (1-1) that blows up forward in time with $\|u_0\|_{L^2} = \|Q\|_{L^2}$ and $\lambda(t)$ and $\gamma(t)$ continuous functions of time for which

$$\left\| \lambda(t)^{-1/2} e^{-i\gamma(t)} u\left(t, \frac{x}{\lambda(t)}\right) - Q(x) \right\|_{L^2} \leq \eta_*, \quad \eta_* \ll 1, \quad \text{for all } t > 0. \tag{1-16}$$

Then, in Section 3, the machinery in [Martel and Merle 2002] is used to choose $\lambda(t)$ and $\gamma(t)$ satisfying (1-16) for which

$$(\epsilon, Q_x) = (i\epsilon, Q_x) = (\epsilon, Q^3) = (i\epsilon, Q^3) = 0, \quad \text{where } \epsilon(t, x) = \lambda(t)^{-1/2} e^{-i\gamma(t)} u\left(t, \frac{x}{\lambda(t)}\right) - Q(x).$$

In Sections 4–6, the spectral theory of ϵ is combined with the long-time Strichartz estimates in [Dodson 2016a], proving

$$\int_a^b \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}\, dt \leq 3\big(\epsilon_2(a), \big(\tfrac{1}{2}Q + xQ_x\big)\big)_{L^2} - 3\big(\epsilon_2(b), \tfrac{1}{2}Q + xQ_x\big)_{L^2} + O\left(\frac{1}{T^9}\right),$$
$$\text{when } \int_a^b \lambda(t)^{-2}\, dt = T \text{ and } a > 0. \tag{1-17}$$

In Section 7, we use (1-17) to show that if $[0, T)$ is the maximal interval of existence of (1-1) in the forward time direction,

$$\int_0^T \|\epsilon(t)\|_{L^2}^p \lambda(t)^{-2} < \infty \quad \text{for any } p > 1. \tag{1-18}$$

Note that for the pseudoconformal solution (1-11), (1-18) holds, but fails when $p = 1$. Then in Section 8, we use the virial identity in [Merle and Raphael 2005] to show that $\lambda(t)$ is approximately monotone decreasing. In Section 9, the monotonicity of $\lambda(t)$ combined with long-time Strichartz estimates and conservation of energy implies that u is a soliton solution when $T = \infty$. When $T < \infty$, a pseudoconformal transformation of the solution must satisfy $T = \infty$, so therefore u must be a pseudoconformal transformation of a soliton. Finally, in Section 10, the above argument is generalized to the nonsymmetric case. We conclude with an Appendix describing U^p and V^p spaces, an important tool used in long-time Strichartz estimates.

2. Reductions of a symmetric blowup solution

Let u be a symmetric blowup solution to (1-1) with mass $\|u_0\|_{L^2} = \|Q\|_{L^2}$. Defining the distance to the two dimensional manifold of symmetries acting on the soliton (1-8) by

$$\inf_{\lambda > 0,\, \gamma \in \mathbb{R}} \|u_0(x) - e^{i\gamma} \lambda^{1/2} Q(\lambda x)\|_{L^2}, \tag{2-1}$$

there exist $\lambda_0 > 0$ and $\gamma_0 \in \mathbb{R}$ where this infimum is attained. Indeed:

Lemma 6. *There exist $\lambda_0 > 0$ and $\gamma_0 \in \mathbb{R}$ such that*

$$\|u_0(x) - e^{-i\gamma_0} \lambda_0^{-1/2} Q(\lambda_0^{-1} x)\|_{L^2(\mathbb{R})} = \inf_{\gamma, \lambda} \|u_0(x) - e^{-i\gamma} \lambda^{-1/2} Q(\lambda^{-1} x)\|_{L^2}.$$

Proof. Since Q, along with all its derivatives, is rapidly decreasing,

$$\|u_0(x) - e^{-i\gamma}\lambda^{-1/2}Q(\lambda^{-1}x)\|_{L^2}^2 \tag{2-2}$$

is differentiable and hence continuous as a function of λ and γ.

Next, by the dominated convergence theorem,

$$\begin{aligned}\lim_{\lambda\nearrow\infty}\inf_{\gamma\in[0,2\pi]} &\|u_0(x) - e^{-i\gamma}\lambda^{-1/2}Q(\lambda^{-1}x)\|_{L^2}^2 \\ &= \|u_0\|_{L^2}^2 + \|Q\|_{L^2}^2 - 2\lim_{\lambda\nearrow\infty}\sup_{\gamma\in[0,2\pi]} |(e^{-i\gamma}\lambda^{-1/2}Q(\lambda^{-1}x), u_0(x))_{L^2}| = 2\|Q\|_{L^2}^2. \end{aligned}\tag{2-3}$$

Here, $(f, g)_{L^2}$ denotes the L^2-inner product

$$(f, g)_{L^2} = \mathrm{Re}\int f(x)\overline{g(x)}\,dx.$$

Meanwhile, rescaling (2-3),

$$(e^{-i\gamma}\lambda^{-1/2}Q(\lambda^{-1}x), u_0(x))_{L^2} = (Q(x), e^{i\gamma}\lambda^{1/2}u_0(\lambda x))_{L^2},$$

and therefore,

$$\lim_{\lambda\searrow 0}\inf_{\gamma\in[0,2\pi]} \|u_0(x) - e^{-i\gamma}\lambda^{-1/2}Q(\lambda^{-1}x)\|_{L^2}^2 = 2\|Q\|_{L^2}^2. \tag{2-4}$$

Finally, the polarization identity

$$\|u_0(x) - \lambda^{-1/2}Q(\lambda^{-1}x)\|_{L^2}^2 + \|u_0(x) + \lambda^{-1/2}Q(\lambda^{-1}x)\|_{L^2}^2 = 4\|Q\|_{L^2}^2$$

implies that

$$\frac{1}{2\pi}\int_0^{2\pi} \|u_0(x) - e^{-i\gamma}\lambda^{-1/2}Q(\lambda^{-1}x)\|_{L^2}^2\,d\gamma = 2\|Q\|_{L^2}^2. \tag{2-5}$$

If, for all $\lambda > 0$,

$$\inf_{\gamma\in[0,2\pi]} \|u_0(x) - e^{-i\gamma}\lambda^{-1/2}Q(\lambda^{-1}x)\|_{L^2}^2 = 2\|Q\|_{L^2}^2,$$

then (2-5) implies

$$\|u_0(x) - e^{-i\gamma}\lambda^{-1/2}Q(\lambda^{-1}x)\|_{L^2}^2 = 2\|Q\|_{L^2}^2 \quad \text{for all } \lambda > 0,\ \ \gamma \in [0, 2\pi]. \tag{2-6}$$

In this case simply take $\lambda_0 = 1$ and $\gamma_0 = 0$.

Remark. Equation (2-6) is not possible, since (2-6) is equivalent to the statement that there exists $\|u_0\|_{L^2} = \|Q\|_{L^2}$ satisfying

$$(u_0(x), e^{-i\gamma}\lambda^{-1/2}Q(\lambda^{-1}x))_{L^2} = 0 \quad \text{for all } \gamma \in [0, 2\pi] \text{ and for all } \lambda > 0. \tag{2-7}$$

Since Q and u_0 are symmetric, let $R(y) = e^{y/2}Q(y)$ and $v(y) = e^{y/2}u_0(e^y)$. Then (2-7) implies that $v(y)$ is orthogonal to all translations of $R(y)$. Since $R(y)$ is exponentially decreasing, the Fourier transform of $R(y)$ is analytic in the strip, and therefore must have isolated zeros. Thus, its zeros are a set of measure zero, so the translates of R are dense in L^2. The author is grateful to an anonymous referee for pointing this fact out.

On the other hand, if

$$\inf_{\lambda>0}\inf_{\gamma\in\mathbb{R}}\|u_0(x)-e^{-i\gamma}\lambda^{-1/2}Q(\lambda^{-1}x)\|_{L^2}^2<2\|Q\|_{L^2}^2,$$

then (2-3) and (2-4) imply that there exist $0<\lambda_1<\lambda_2<\infty$ such that

$$\inf_{\lambda>0}\inf_{\gamma\in[0,2\pi]}\|u_0(x)-e^{-i\gamma}\lambda^{-1/2}Q(\lambda^{-1}x)\|_{L^2}^2=\inf_{\lambda\in[\lambda_1,\lambda_2]}\inf_{\gamma\in[0,2\pi]}\|u_0(x)-e^{-i\gamma}\lambda^{-1/2}Q(\lambda^{-1}x)\|_{L^2}^2.$$

Since (2-2) is continuous as a function of $\lambda>0$, $\gamma\in[0,2\pi]$, and $[\lambda_1,\lambda_2]\times[0,2\pi]$ is a compact set, there exist $\lambda_0>0$ and $\gamma_0\in[0,2\pi]$ such that

$$\|u_0(x)-e^{-i\gamma_0}\lambda_0^{-1/2}Q(\lambda_0^{-1}x)\|_{L^2(\mathbb{R})}=\inf_{\gamma\in[0,2\pi],\,\lambda>0}\|u_0(x)-e^{-i\gamma}\lambda^{-1/2}Q(\lambda^{-1}x)\|_{L^2}. \qquad \square$$

Using the weak sequential convergence result of [Fan 2021], Theorem 4 may be reduced to considering solutions that blow up in positive time for which (2-1) is small for all $t>0$.

Theorem 7. *Let $0<\eta_*\ll 1$ be a small, fixed constant to be defined later. If u is a symmetric solution to* (1-1) *on the maximal interval of existence $I\subset\mathbb{R}$, $\|u_0\|_{L^2}=\|Q\|_{L^2}$, u blows up forward in time, and*

$$\sup_{t\in[0,\sup(I))}\inf_{\lambda,\gamma}\|e^{i\gamma}\lambda^{1/2}u(t,\lambda x)-Q(x)\|_{L^2}\le\eta_*, \tag{2-8}$$

then u is a soliton solution of the form (1-12) *or a pseudoconformal transformation of a soliton of the form* (1-13).

Remark. Scaling symmetries imply that (2-1) and the left-hand side of (2-8) at a fixed time are equal.

Proof that Theorem 7 implies Theorem 4. Let $u(t)$ be the solution to (1-1) with symmetric initial data u_0 that satisfies $\|u_0\|_{L^2}=\|Q\|_{L^2}$. If

$$\lim_{t\nearrow\sup(I)}\inf_{\lambda>0,\,\gamma\in\mathbb{R}}\|\lambda^{1/2}e^{i\gamma}u(t,\lambda x)-Q\|_{L^2}=0, \tag{2-9}$$

then (2-8) holds for all $t>t_0$, for some $t_0\in I$. After translating in time so that $t_0=0$, Theorem 7 easily implies Theorem 4 in this case.

However, the convergence theorem of [Fan 2021] only implies $u(t)$ must converge to Q along a subsequence after rescaling and multiplying by a complex number of modulus one.

Theorem 8. *Let u be a symmetric solution to* (1-1) *that satisfies $\|u_0\|_{L^2}=\|Q\|_{L^2}$ and blows up forward in time. Let $(T_-(u),T_+(u))$ be the maximal lifespan of the solution u. Then there exists a sequence $t_n\to T_+(u)$ and a family of parameters $\lambda_n>0$, $\gamma_n\in\mathbb{R}$ such that*

$$e^{i\gamma_n}\lambda_n^{1/2}u(t_n,\lambda_n x)\to Q \quad \text{in } L^2. \tag{2-10}$$

If (2-9) does not hold but there exists some $t_0>0$ such that

$$\sup_{t\in[t_0,\sup(I))}\inf_{\lambda,\gamma}\|e^{i\gamma}\lambda^{1/2}u(t,\lambda x)-Q(x)\|_{L^2}\le\eta_*,$$

then after translating in time so that $t_0=0$, (2-8) holds.

Now suppose (2-9) does not hold and furthermore that there exists a sequence $t_n^- \nearrow \sup(I)$ such that

$$\inf_{\gamma\in\mathbb{R},\,\lambda>0} \|e^{i\gamma}\lambda^{1/2}u(t_n^-,\lambda x) - Q\|_{L^2} > \eta_* \tag{2-11}$$

for every n. After passing to a subsequence, suppose that, for every n, we have $t_n^- < t_n < t_{n+1}^-$, where t_n is the sequence in (2-10) and t_n^- is the sequence in (2-11). The fact that

$$\inf_{\gamma\in\mathbb{R},\,\lambda>0} \|e^{i\gamma}\lambda^{1/2}u(t,\lambda x) - Q\|_{L^2} \tag{2-12}$$

is upper semicontinuous as a function of t and is continuous for every t such that (2-12) is small guarantees that there exists a small, fixed $0 < \eta_* \ll 1$ such that the sequence t_n^+ defined by

$$t_n^+ = \inf\Bigl\{t \in I : \sup_{\tau\in[t,t_n]} \inf_{\lambda,\gamma} \|\lambda^{1/2}e^{i\gamma}u(\tau,\lambda x) - Q\|_{L^2} < \eta_*\Bigr\}$$

satisfies $t_n^+ \nearrow \sup(I)$ and

$$\inf_{\lambda>0,\,\gamma\in\mathbb{R}} \|e^{i\gamma}\lambda^{1/2}u(t_n^+,\lambda x) - Q(x)\|_{L^2} = \eta_*. \tag{2-13}$$

Indeed, the fact that (2-12) is upper semicontinuous as a function of t implies that

$$\Bigl\{0 \le t < t_n : \inf_{\lambda>0,\,\gamma\in\mathbb{R}} \|e^{i\gamma}\lambda^{1/2}u(t,\lambda x) - Q(x)\|_{L^2} \ge \eta_*\Bigr\}$$

is a closed set. Since this set is also contained in a bounded set, it has a maximal element t_n^+, and $t_n^+ \ge t_n^-$. The fact that (2-12) is upper semicontinuous in time also implies

$$\inf_{\lambda>0,\,\gamma\in\mathbb{R}} \|e^{i\gamma}\lambda^{1/2}u(t_n^+,\lambda x) - Q\|_{L^2} \ge \eta_*.$$

On the other hand, since

$$\inf_{\lambda>0,\,\gamma\in\mathbb{R}} \|e^{i\gamma}\lambda^{1/2}u(t,\lambda x) - Q\|_{L^2} < \eta_* \quad \text{for all } t_n^+ < t < t_n$$

and (2-12) is continuous at times $t \in I$ where (2-12) is small,

$$\inf_{\lambda>0,\,\gamma\in\mathbb{R}} \|e^{i\gamma}\lambda^{1/2}u(t_n^+,\lambda x) - Q\|_{L^2} = \eta_*. \tag{2-14}$$

Remark. The constant $0 < \eta_* \ll 1$ will be chosen to be a small fixed quantity that is sufficiently small to satisfy the hypotheses of Theorem 10, sufficiently small such that (2-12) is continuous in time when (2-12) is bounded by η_*, sufficiently small such that $\eta_* \le \eta_0$, where η_0 is the constant in the induction on frequency arguments in Theorem 13, and such that $T_* = 1/\eta_*$ is sufficiently large to satisfy the hypotheses of Theorem 18.

Theorem 9 (upper semicontinuity of the distance to a soliton). *The quantity*

$$\inf_{\lambda,\gamma} \|e^{i\gamma}\lambda^{1/2}u(t,\lambda x) - Q(x)\|_{L^2(\mathbb{R})}, \tag{2-15}$$

is upper semicontinuous as a function of time for any $t \in I$, where I is the maximal interval of existence for u. The quantity (2-15) *is also continuous in time when* (2-15) *is small.*

Proof. Choose some $t_0 \in I$ and suppose without loss of generality that

$$\|u(t_0,x)-Q(x)\|_{L^2}=\inf_{\lambda,\gamma}\|e^{i\gamma}\lambda^{1/2}u(t_0,\lambda x)-Q(x)\|_{L^2}.$$

For t close to t_0, let

$$\epsilon(t,x)=u(t,x)-e^{i(t-t_0)}Q(x). \tag{2-16}$$

Since $e^{i(t-t_0)}Q$ solves (1-1),

$$\begin{aligned} i\epsilon_t+\epsilon_{xx}+|u|^4u-e^{it}|Q|^4Q& \\ &=i\epsilon_t+\epsilon_{xx}+3|Q|^4\epsilon+2e^{2i(t-t_0)}|Q|^2Q^2\bar{\epsilon}+O\biggl(\sum_{j=2}^{5}\epsilon^{2+j}Q^{5-j}\biggr)=0. \end{aligned} \tag{2-17}$$

Equations (2-16), (2-17), and Strichartz estimates imply that, for $J\subset\mathbb{R},\ t_0\in J$,

$$\|\epsilon\|_{L_t^\infty L_x^2\cap L_t^4L_x^\infty(J\times\mathbb{R})}\lesssim\|\epsilon(t_0)\|_{L^2}+\|\epsilon\|_{L_t^\infty L_x^2(J\times\mathbb{R})}\|u\|^4_{L_t^4L_x^\infty(J\times\mathbb{R})}+\|\epsilon\|^5_{L^6_{t,x}(J\times\mathbb{R})}.$$

Local well-posedness of (1-1) combined with Strichartz estimates implies that $\|u\|_{L_t^4L_x^\infty(J\times\mathbb{R})}=1$ on some open neighborhood J of t_0. Therefore, for $\|\epsilon(t_0)\|_{L^2}$ small, partitioning J into finitely many pieces,

$$\sup_{t\in J}\|\epsilon(t)\|_{L^2}\lesssim\|\epsilon(t_0)\|_{L^2} \tag{2-18}$$

and

$$\lim_{t\to t_0}\|\epsilon(t)\|_{L^2}=\|\epsilon(t_0)\|_{L^2}. \tag{2-19}$$

Therefore,

$$\lim_{t\to t_0}\inf_{\lambda,\gamma}\|\lambda^{1/2}e^{i\gamma}u(t,\lambda x)-Q\|_{L^2}\le\|u(t_0,x)-Q\|_{L^2}=\inf_{\lambda>0,\gamma\in\mathbb{R}}\|\lambda^{1/2}e^{i\gamma}u(t_0,\lambda x)-Q\|_{L^2}.$$

Furthermore, if

$$\lim_{t\to t_0}\inf_{\lambda,\gamma}\|\lambda^{1/2}e^{i\gamma}u(t,\lambda x)-Q\|_{L^2}<\|u(t_0,x)-Q\|_{L^2},$$

then there exists a sequence $t_n'\to t_0,\ \lambda_n'>0,\ \gamma_n'\in\mathbb{R}$ such that

$$\lim_{n\to\infty}\|\lambda_n'^{1/2}e^{i\gamma_n'}u(t_n',\lambda_n'x)-Q\|_{L^2}<\inf_{\lambda,\gamma}\|\lambda^{1/2}e^{i\gamma}u(t_0,\lambda x)\|_{L^2}.$$

For t_n' sufficiently close to t_0, repeating the arguments giving (2-18) and (2-19) with t_n' as the initial data gives a contradiction.

When $\|\epsilon(t_0)\|_{L^2}$ is large, (2-17) implies

$$\frac{d}{dt}\|\epsilon(t)\|^2_{L^2}\lesssim\|Q\|^4_{L^\infty}\|\epsilon\|^2_{L^2}+\|u\|^4_{L^\infty}\|\epsilon\|^2_{L^2}.$$

Therefore, Gronwall's inequality and the fact that $u\in L^4_{t,\mathrm{loc}}L^\infty_x$ imply

$$\lim_{t\to t_0}\inf_{\lambda>0,\gamma\in\mathbb{R}}\|e^{i\gamma}\lambda^{1/2}u(t,\lambda x)-Q\|_{L^2}\le\inf_{\lambda>0,\gamma\in\mathbb{R}}\|e^{i\gamma}\lambda^{1/2}u(t_0,\lambda x)-Q\|_{L^2},$$

which implies upper semicontinuity. □

Making a profile decomposition of $u(t_n^+, x)$, the fact that u is a minimal mass blowup solution that blows up forward in time and $t_n^+ \nearrow \sup(I)$ implies that there exist $\lambda(t_n^+) > 0$ and $\gamma(t_n^+) \in \mathbb{R}$ such that

$$\lambda(t_n^+)^{1/2} e^{i\gamma(t_n^+)} u(t_n^+, \lambda(t_n^+)x) \to \tilde{u}_0$$

in L^2. Also, $t_n^+ \nearrow \sup(I)$ implies $\|\tilde{u}_0\|_{L^2} = \|Q\|_{L^2}$ is the initial data for a solution to (1-1) that blows up forward and backward in time, and by (2-14),

$$\inf_{\lambda>0,\, \gamma\in\mathbb{R}} \|\lambda^{1/2} e^{i\gamma} \tilde{u}_0(\lambda x) - Q\|_{L^2} = \eta_*. \tag{2-20}$$

Moreover, observe that (2-10), (2-13), and (2-18) directly imply that

$$\lim_{n\to\infty} \|u\|_{L^6_{t,x}([t_n^+, t_n]\times\mathbb{R})} = \infty \quad \text{and} \quad \lim_{n\to\infty} \|u\|_{L^6_{t,x}([0, t_n^+]\times\mathbb{R})} = \infty,$$

so if $\tilde{u}$ is the solution to (1-1) with initial data $\tilde{u}_0$,

$$\inf_{\lambda>0,\, \gamma\in\mathbb{R}} \|\lambda^{1/2} e^{i\gamma} \tilde{u}(t, \lambda x) - Q\|_{L^2} \le \eta_*$$

for all $t \in [0, \sup(\tilde{I}))$, where $\tilde{I}$ is the interval of existence of the solution $\tilde{u}$ to (1-1) with initial data $\tilde{u}_0$, and $\tilde{u}$ blows up both forward and backward in time. However, Theorem 7 and (2-20) imply that $\tilde{u}$ must be of the form (1-13). Such a solution scatters backward in time and is well approximated by a linear solution

$$e^{it\Delta} f = (4\pi t)^{-1/2} e^{-i\pi/4} \int \exp\left[i \frac{(x-y)^2}{4t}\right] f(y)\, dy,$$

which contradicts the fact that $\tilde{u}$ blows up both forward and backward in time.

Therefore, Theorem 7 implies that (2-11) cannot hold for any symmetric solution to (1-1) with mass $\|u_0\|_{L^2} = \|Q\|_{L^2}$, so by Theorem 7, any symmetric solution to (1-1) that blows up forward in time must be of the form (1-12) or (1-13). □

3. Decomposition of the solution near Q

Turning now to the proof of Theorem 7, make a decomposition of a symmetric solution close to Q, up to rescaling and multiplication by a modulus one constant. This result is classical; see, e.g., [Martel and Merle 2002], although here there is an additional technical complication due to the fact that u need not lie in H^1.

Theorem 10. *Take $u \in L^2$. There exists $\alpha > 0$ sufficiently small such that if there exist $\lambda_0 > 0$, $\gamma_0 \in \mathbb{R}$ that satisfy*

$$\|e^{i\gamma_0} \lambda_0^{1/2} u(\lambda_0 x) - Q\|_{L^2} \le \alpha,$$

then there exist unique $\lambda > 0$, $\gamma \in \mathbb{R}$ which satisfy

$$(\epsilon, Q^3)_{L^2} = (\epsilon, iQ^3)_{L^2} = 0, \tag{3-1}$$

where

$$\epsilon(x) = e^{i\gamma} \lambda^{1/2} u(\lambda x) - Q. \tag{3-2}$$

Furthermore,

$$\|\epsilon\|_{L^2} + \left|\frac{\lambda}{\lambda_0} - 1\right| + |\gamma - \gamma_0| \lesssim \|e^{i\gamma_0} \lambda_0^{1/2} u(\lambda_0 x) - Q\|_{L^2}. \tag{3-3}$$

Remark. Since $e^{i\gamma}$ is 2π-periodic, the γ in (3-2) is unique up to translations by $2\pi k$ for some integer k.

Proof. By Hölder's inequality,

$$|(e^{i\gamma_0}\lambda_0^{1/2}u(\lambda_0 x)-Q(x),Q^3)_{L^2}|\lesssim\|e^{i\gamma_0}\lambda_0^{1/2}u(\lambda_0 x)-Q\|_{L^2},$$
$$|(e^{i\gamma_0}\lambda_0^{1/2}u(\lambda_0 x)-Q(x),iQ^3)_{L^2}|\lesssim\|e^{i\gamma_0}\lambda_0^{1/2}u(\lambda_0 x)-Q\|_{L^2}.$$

First suppose that $\lambda_0=1$ and $\gamma_0=0$. The inner products

$$(e^{i\gamma}\lambda^{1/2}u(\lambda x)-Q(x),Q^3)_{L^2}\quad\text{and}\quad(e^{i\gamma}\lambda^{1/2}u(\lambda x)-Q(x),iQ^3)_{L^2}\tag{3-4}$$

are C^1 as functions of λ and γ. Indeed,

$$\frac{\partial}{\partial\gamma}(e^{i\gamma}\lambda^{1/2}u(\lambda x)-Q(x),Q^3)_{L^2}=(ie^{i\gamma}\lambda^{1/2}u(\lambda x),Q^3)_{L^2}\lesssim\|u\|_{L^2}\|Q\|_{L^6}^3,$$
$$\frac{\partial}{\partial\gamma}(e^{i\gamma}\lambda^{1/2}u(\lambda x)-Q(x),iQ^3)_{L^2}=(ie^{i\gamma}\lambda^{1/2}u(\lambda x),iQ^3)_{L^2}\lesssim\|u\|_{L^2}\|Q\|_{L^6}^3.$$

Next, integrating by parts,

$$\begin{aligned}\frac{\partial}{\partial\lambda}(e^{i\gamma}\lambda^{1/2}u(\lambda x)-Q(x),Q^3)_{L^2}&=\left(\frac{e^{i\gamma}}{2\lambda^{1/2}}u(\lambda x)+xe^{i\gamma}\lambda^{1/2}u_x(\lambda x),Q^3\right)_{L^2}\\&=\left(\frac{e^{i\gamma}}{2\lambda^{1/2}}u(\lambda x)-\frac{1}{\lambda^{1/2}}e^{i\gamma}u(\lambda x),Q^3\right)_{L^2}-\frac{3}{\lambda^{1/2}}(e^{i\gamma}u(\lambda x),Q^2Q_x)_{L^2}\\&\lesssim\frac{1}{\lambda}\|u\|_{L^2}\|Q\|_{L^6}^3+\frac{1}{\lambda}\|u\|_{L^2}\|xQ_x\|_{L^2}\|Q\|_{L^\infty}^2\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial\lambda}(e^{i\gamma}\lambda^{1/2}u(\lambda x)-Q(x),iQ^3)_{L^2}&=\left(\frac{e^{i\gamma}}{2\lambda^{1/2}}u(\lambda x)+xe^{i\gamma}\lambda^{1/2}u_x(\lambda x),iQ^3\right)_{L^2}\\&=\left(\frac{e^{i\gamma}}{2\lambda^{1/2}}u(\lambda x)-\frac{1}{\lambda^{1/2}}e^{i\gamma}u(\lambda x),iQ^3\right)_{L^2}-\frac{3}{\lambda^{1/2}}(e^{i\gamma}u(\lambda x),iQ^2Q_x)_{L^2}\\&\lesssim\frac{1}{\lambda}\|u\|_{L^2}\|Q\|_{L^6}^3+\frac{1}{\lambda}\|u\|_{L^2}\|xQ_x\|_{L^2}\|Q\|_{L^\infty}^2.\end{aligned}$$

Similar calculations prove uniform bounds on the Hessians of the inner products given in (3-4).

Next, compute

$$\frac{\partial}{\partial\gamma}(e^{i\gamma}\lambda^{1/2}u(\lambda x)-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,u=Q}=(iQ,Q^3)_{L^2}=0,$$
$$\frac{\partial}{\partial\gamma}(e^{i\gamma}\lambda^{1/2}u(\lambda x)-Q(x),iQ^3)_{L^2}\Big|_{\lambda=1,\gamma=0,u=Q}=(iQ,iQ^3)_{L^2}=\|Q\|_{L^4}^4,$$
$$\frac{\partial}{\partial\lambda}(e^{i\gamma}\lambda^{1/2}u(\lambda x)-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,u=Q}=\left(\tfrac12Q+xQ_x,Q^3\right)_{L^2}=\tfrac14\|Q\|_{L^4}^4,$$
$$\frac{\partial}{\partial\lambda}(e^{i\gamma}\lambda^{1/2}u(\lambda x),iQ)_{L^2}\Big|_{\lambda=1,\gamma=0,u=Q}=\left(\tfrac12Q+xQ_x,iQ\right)_{L^2}=0.$$

Therefore, by the inverse function theorem, if $\lambda_0 = 1$ and $\gamma_0 = 0$, there exist λ and γ satisfying

$$|\lambda - 1| + |\gamma| \lesssim \|e^{i\gamma_0} u(x) - Q(x)\|_{L^2}$$

such that

$$(e^{i\gamma}\lambda^{1/2} u(t, \lambda x) - Q(x), Q^3)_{L^2} = (e^{i\gamma}\lambda^{1/2} u(t, \lambda x) - Q(x), i Q^3)_{L^2} = 0.$$

The inverse function theorem also guarantees that λ and γ are unique for all $\lambda, \gamma \in [1-\delta, 1+\delta] \times [-\delta, \delta]$ for some $\delta > 0$, up to 2π-periodicity.

For λ outside $[1-\delta, 1+\delta]$, observe that

$$\begin{aligned} &\|e^{i\gamma}\lambda^{1/2} u(\lambda x) - Q\|_{L^2}^2 \\ &\quad = \|u\|_{L^2}^2 + \|Q\|_{L^2}^2 - 2(e^{i\gamma}\lambda^{1/2} Q(\lambda x), Q)_{L^2} - 2(e^{i\gamma}\lambda^{1/2}[u - Q](\lambda x), Q)_{L^2} \gtrsim \delta^2 - O(\alpha). \end{aligned} \tag{3-5}$$

Similarly, for γ outside $[-\delta, \delta]$, up to 2π-multiplicity,

$$\begin{aligned} &\|e^{i\gamma}\lambda^{1/2} u(\lambda x) - Q\|_{L^2}^2 \\ &\quad = \|u\|_{L^2}^2 + \|Q\|_{L^2}^2 - 2(e^{i\gamma}\lambda^{1/2} Q(\lambda x), Q)_{L^2} - 2(e^{i\gamma}\lambda^{1/2}[u - Q](\lambda x), Q)_{L^2} \gtrsim \delta^2 - O(\alpha), \end{aligned} \tag{3-6}$$

which implies uniqueness for $\alpha > 0$ sufficiently small.

For general λ_0 and γ_0, after rescaling,

$$\left|\frac{\lambda}{\lambda_0} - 1\right| + |\gamma - \gamma_0| \lesssim \|e^{i\gamma_0}\lambda_0^{1/2} u(t, \lambda_0 x) - Q(x)\|_{L^2}. \tag{3-7}$$

Finally, using scaling symmetries, the triangle inequality, and (3-7),

$$\begin{aligned} &\|e^{i\gamma}\lambda^{1/2} u(t, \lambda x) - Q(x)\|_{L^2} \\ &\quad = \left\| u(x) - e^{-i\gamma}\lambda^{-1/2} Q\left(\frac{x}{\lambda}\right)\right\|_{L^2} \\ &\quad \le \left\| u(x) - e^{-i\gamma_0}\lambda_0^{-1/2} Q\left(\frac{x}{\lambda_0}\right)\right\|_{L^2} + \left\| e^{-i\gamma_0}\lambda_0^{-1/2} Q\left(\frac{x}{\lambda_0}\right) - e^{-i\gamma}\lambda_0^{-1/2} Q\left(\frac{x}{\lambda_0}\right)\right\|_{L^2} \\ &\qquad + \left\| e^{-i\gamma}\lambda_0^{-1/2} Q\left(\frac{x}{\lambda_0}\right) - e^{-i\gamma}\lambda^{-1/2} Q\left(\frac{x}{\lambda}\right)\right\|_{L^2} \\ &\quad \lesssim \|e^{i\gamma_0} u(x) - Q(x)\|_{L^2}. \end{aligned}$$

This proves (3-3). □

Therefore, in Theorem 7, there exist functions

$$\lambda : I \to (0, \infty) \quad \text{and} \quad \gamma : I \to \mathbb{R}$$

such that (3-1) holds for all $t \in [0, \sup(I))$.

Theorem 11. *Under the hypotheses of Theorem 7, the functions $\lambda(t)$ and $\gamma(t)$ are continuous as functions of time on $[0, \sup(I))$. Additionally, $\lambda(t)$ and $\gamma(t)$ are differentiable in time almost everywhere on $[0, \sup(I))$.*

Proof. Suppose $J=[a,b]$ is an interval that satisfies

$$\|u\|_{L_t^4L_x^\infty(J\times\mathbb{R})}\le 1$$

and $J\subset[0,\sup(I))$. Suppose without loss of generality that $\lambda(a)=1$ and $\gamma(a)=0$. Also, suppose for now that $\|u(a)\|_{\dot H^1}<\infty$. Strichartz estimates and local well-posedness theory imply that

$$\|u\|_{L_t^\infty\dot H^1(J\times\mathbb{R})}\lesssim\|u(a)\|_{\dot H^1}. \tag{3-8}$$

Since $\lambda(a)=1$ and $\gamma(a)=0$,

$$(u(a,x)-Q(x),Q^3)_{L^2}=(u(a,x)-Q(x),iQ^3)_{L^2}=0.$$

Then, by direct calculation and the fact that Q is smooth and rapidly decreasing,

$$\begin{aligned}\frac{d}{dt}(u(t,x)-Q,Q^3)_{L^2}&=(iu_{xx},Q^3)_{L^2}+(i|u|^4u,Q^3)_{L^2}\\&=(iu,\partial_{xx}(Q^3))_{L^2}+i(|u|^4u,Q^3)_{L^2}\lesssim\|u\|_{L^2}+\|u\|_{L^\infty}^3\|u\|_{L^2}^2.\end{aligned}$$

Therefore, (3-8) implies that $(u(t,x)-Q(x),Q^3)_{L^2}$ is Lipschitz in time on J as is $(u(t,x)-Q(x),iQ^3)_{L^2}$ by an identical calculation. Then by the proof of Theorem 10, $\lambda(t)$ and $\gamma(t)$ are Lipschitz as a function of time for t close to a, and by the Lebesgue differentiation theorem, λ and γ are differentiable almost everywhere for t near a.

Recall from (3-1) that

$$\epsilon(t,x)=e^{i\gamma(t)}\lambda(t)^{1/2}u(t,\lambda(t)x)-Q(x).$$

By direct computation, for almost every t near a,

$$\begin{aligned}\epsilon_t&=i\dot\gamma(t)(Q+\epsilon)+\frac{\dot\lambda(t)}{\lambda(t)}\big(\tfrac12Q+xQ_x+\tfrac12\epsilon+x\epsilon_x\big)+i\lambda(t)^{-2}(Q_{xx}+\epsilon_{xx})\\&\qquad\qquad+i\lambda(t)^{1/2}e^{i\gamma(t)}|u(t,\lambda(t)x)|^4u(t,\lambda(x))\\&=i(\dot\gamma(t)+\lambda(t)^{-2})Q+\frac{\dot\lambda(t)}{\lambda(t)}\big(\tfrac12Q+xQ_x\big)+i\lambda(t)^{-2}(\epsilon_{xx}+5Q^4\operatorname{Re}(\epsilon)+iQ^4\operatorname{Im}(\epsilon)-\epsilon)\\&\qquad+i(\dot\gamma(t)+\lambda(t)^{-2})\epsilon+\frac{\dot\lambda(t)}{\lambda(t)}\big(\tfrac12\epsilon+x\epsilon_x\big)+\lambda(t)^{-2}O(|Q|^3|\epsilon|^2+|\epsilon|^5).\end{aligned}\tag{3-9}$$

Since a is arbitrary, λ and γ are differentiable at almost every $t\in[0,\sup(I))$.

Next, define the monotone function $s:[0,\sup(I))\to\mathbb{R}$,

$$s(t)=\int_0^t\lambda(\tau)^{-2}\,d\tau. \tag{3-10}$$

Making a change of variables $\epsilon_s=\lambda^2\epsilon_t$, by (3-9),

$$\begin{aligned}\epsilon_s&=i(\gamma_s+1)Q+\frac{\lambda_s}{\lambda}\big(\tfrac12Q+xQ_x\big)+i(\epsilon_{xx}+5Q^4\operatorname{Re}(\epsilon)+iQ^4\operatorname{Im}(\epsilon)-\epsilon)\\&\qquad+i(\gamma_s+1)\epsilon+\frac{\lambda_s}{\lambda}\big(\tfrac12\epsilon+x\epsilon_x\big)+O(|Q|^3|\epsilon|^2+|\epsilon|^2|u|^3).\end{aligned}\tag{3-11}$$

Plugging (3-11) into (3-2) and integrating by parts,

$$\frac{d}{ds}(\epsilon, Q^3) = (\epsilon_s, Q^3) = 0 = \frac{\lambda_s}{4\lambda}\|Q\|_{L^4}^4 - (\operatorname{Im}(\epsilon), \mathcal{L}_- Q^3)_{L^2} + O(|\gamma_s + 1|\|\epsilon\|_{L^2}) + O\left(\frac{\lambda_s}{\lambda}\|\epsilon\|_{L^2}\right) + O(\|\epsilon\|_{L^2}^2) + O(\|\epsilon\|_{L^2}^2\|u\|_{L^\infty}^3)$$

and

$$\frac{d}{ds}(\epsilon, iQ^3) = (\epsilon_s, iQ^3) = 0 = (\gamma_s + 1)\|Q\|_{L^4}^4 + (\epsilon, \mathcal{L}Q^3)_{L^2} + O(|\gamma_s + 1|\|\epsilon\|_{L^2}) + O\left(\frac{\lambda_s}{\lambda}\|\epsilon\|_{L^2}\right) + O(\|\epsilon\|_{L^2}^2) + O(\|\epsilon\|_{L^2}^2\|u\|_{L^\infty}^3),$$

where $\mathcal{L}_-$ and $\mathcal{L}$ are the linear operators

$$\mathcal{L}_- f = -f_{xx} - Q^4 f + f \quad \text{and} \quad \mathcal{L}f = -f_{xx} - 5Q^4 f + f. \tag{3-12}$$

Since $\mathcal{L}Q^3 = -8Q^3$ and $(\epsilon, Q^3)_{L^2} = 0$,

$$\frac{\|Q\|_{L^4}^4}{4}\frac{\lambda_s}{\lambda} = (\operatorname{Im}(\epsilon), \mathcal{L}_- Q^3)_{L^2} + O(|\gamma_s + 1|\|\epsilon\|_{L^2}) + O\left(\frac{\lambda_s}{\lambda}\|\epsilon\|_{L^2}\right) + O(\|\epsilon\|_{L^2}^2) + O(\|\epsilon\|_{L^2}^2\|u\|_{L^\infty}^3) \tag{3-13}$$

and

$$\|Q\|_{L^4}^4(\gamma_s + 1) = O(|\gamma_s + 1|\|\epsilon\|_{L^2}) + O\left(\frac{\lambda_s}{\lambda}\|\epsilon\|_{L^2}\right) + O(\|\epsilon\|_{L^2}^2) + O(\|\epsilon\|_{L^2}^2\|u\|_{L^\infty}^3). \tag{3-14}$$

Doing some algebra, (3-13), (3-14), and the computations proving (2-18) imply that, for any $a \in \mathbb{Z}_{\geq 0}$,

$$\int_a^{a+1}\left|\frac{\lambda_s}{\lambda}\right| ds \lesssim \int_a^{a+1}\|\epsilon\|_{L^2}\, ds \tag{3-15}$$

and

$$\int_a^{a+1}|\gamma_s + 1|\, ds \lesssim \int_a^{a+1}\|\epsilon\|_{L^2}\, ds. \tag{3-16}$$

Indeed, the computations proving (2-18) imply that

$$\sup_{s\in[a,a+1]}\|\epsilon(s)\|_{L^2} \lesssim \int_a^{a+1}\|\epsilon(s)\|_{L^2}\, ds, \tag{3-17}$$

so

$$\int_a^{a+1}\|\epsilon\|_{L^2}^2\|u\|_{L^\infty}^3\, ds \lesssim \int_a^{a+1}\|\epsilon(s)\|_{L^2}^2\, ds \cdot \int_a^{a+1}\|u\|_{L^\infty}^3\, ds. \tag{3-18}$$

Furthermore, Strichartz estimates and the computations proving (2-18) imply that $\int_a^{a+1}\|u\|_{L^\infty}^4\, ds \lesssim 1$, and crucially, the bound is independent of $\|u(a)\|_{\dot{H}^1}$.

For a general $u(a) \in L^2$, let $u^N(a) = P_{\leq N}u(a)$. Taking N large enough that

$$\|e^{i\gamma(a)}(\lambda(a))^{1/2}u^N(a, \lambda(a)x) - Q\|_{L^2} \leq 2\eta_*,$$

Theorem 10 implies that there exist $\gamma^N(s)$ and $\lambda^N(s)$ for any $s \in [a, a+1]$ such that (3-1) holds. Furthermore, $\lambda^N(s)$ and $\gamma^N(s)$ satisfy (3-13) and (3-14), and $\gamma^N(s)$ and $\lambda^N(s)$ converge to $\gamma(s)$

and $\lambda(s)$ uniformly on $[a, a+1]$, so $\gamma(s)$ and $\lambda(s)$ are continuous as functions of s. Furthermore, $\epsilon^N \to \epsilon$ in L^2 uniformly in s and $u^N \to u$ in $L_s^4 L_x^\infty$.

Therefore, plugging $\lambda^N(s)$, $\gamma^N(s)$, ϵ^N, and u^N into (3-13) and (3-14) and doing some algebra implies, by the dominated convergence theorem,

$$\tfrac{1}{4}\|Q\|_{L^4}^4[\ln\lambda(s) - \ln\lambda(a)] = \int_a^s O((\mathrm{Im}(\epsilon), \mathcal{L}_- Q^3)_{L^2}) + O(\|\epsilon\|_{L^2}^2) + O(\|\epsilon\|_{L^2}^2\|u\|_{L^\infty}^3)\,ds$$

and

$$\|Q\|_{L^4}^4[\gamma(s) - \gamma(a) + (s-a)] = \int_a^s O(\|\epsilon\|_{L^2}^2) + O(\|\epsilon\|_{L^2}^2\|u\|_{L^\infty}^3)\,ds.$$

Therefore, by the Lebesgue differentiation theorem, λ_s/λ and γ_s exist for almost every $s \in [a, a+1]$ and satisfy (3-13) and (3-14). □

Following [Merle 2001], the decomposition in Theorem 10 gives a positivity result.

Theorem 12. *If $\epsilon(t,x)$ is a symmetric function, $\epsilon \perp Q^3$, $\epsilon \perp iQ^3$, $\|\epsilon(t,x)\|_{L^2} \ll 1$, and $\|Q+\epsilon\|_{L^2} = \|Q\|_{L^2}$, then*

$$E(Q+\epsilon) \gtrsim \|\epsilon(t)\|_{H^1(\mathbb{R})}^2 = \int |\epsilon_x(t,x)|^2\,dx + \int |\epsilon(t,x)|^2\,dx.$$

Proof. Decomposing the energy and integrating by parts, since Q is a real-valued function,

$$\begin{aligned} E(Q+\epsilon) = \frac{1}{2}\int Q_x^2\,dx + \mathrm{Re}\int Q_x(x)\epsilon_x(t,x)\,dx + \frac{1}{2}\|\epsilon_x\|_{L^2}^2 - \frac{1}{6}\int Q(x)^6\,dx \\ - \mathrm{Re}\int Q(x)^5\epsilon(t,x)\,dx - \frac{3}{2}\int Q(x)^4|\epsilon(t,x)|^2\,dx \\ - \mathrm{Re}\int Q(x)^4\epsilon(t,x)^2\,dx - \int O(|\epsilon(t,x)|^3 Q^3 + |\epsilon(t,x)|^6)\,dx. \end{aligned}$$

First observe that, since $E(Q) = 0$,

$$\frac{1}{2}\int Q_x^2\,dx - \frac{1}{6}\int Q^6\,dx = 0.$$

Next, by (1-10) and integrating by parts,

$$\mathrm{Re}\int Q_x(x)\epsilon_x(t,x) - \mathrm{Re}\int Q(x)^5\epsilon(t,x) = -\mathrm{Re}\int (Q_{xx}(x) + Q(x)^5)\epsilon(t,x)\,dx = -\mathrm{Re}\int Q(x)\epsilon(t,x)\,dx.$$

Using the fact that $\|Q+\epsilon\|_{L^2} = \|Q\|_{L^2}$,

$$\tfrac{1}{2}\|Q\|_{L^2}^2 - \tfrac{1}{2}\|Q+\epsilon\|_{L^2}^2 + \tfrac{1}{2}\|\epsilon\|_{L^2}^2 = -(Q,\epsilon)_{L^2} = -\mathrm{Re}\int Q(x)\epsilon(t,x)\,dx = \tfrac{1}{2}\|\epsilon\|_{L^2}^2. \tag{3-19}$$

Therefore,

$$\begin{aligned} E(Q+\epsilon) = \frac{1}{2}\|\epsilon\|_{L^2}^2 + \frac{1}{2}\|\epsilon_x\|_{L^2}^2 - \frac{3}{2}\int Q(x)^4|\epsilon(t,x)|^2\,dx \\ - \mathrm{Re}\int Q(x)^4\epsilon(t,x)^2\,dx - \int O(|\epsilon(t,x)|^3 Q^3 + |\epsilon(t,x)|^6)\,dx. \end{aligned}$$

Decomposing the terms of order ϵ^2 into real and imaginary parts,

$$\begin{aligned}\frac{1}{2}\|\epsilon_x\|_{L^2}^2+\frac{1}{2}\|\epsilon\|_{L^2}^2-\frac{3}{2}\int Q(x)^4|\epsilon(t,x)|^2\,dx-\operatorname{Re}\int Q(x)^4\epsilon(t,x)^2\,dx \\ =\frac{1}{2}\int \operatorname{Re}(\epsilon)_x^2\,dx+\frac{1}{2}\int \operatorname{Re}(\epsilon)^2\,dx-\frac{5}{2}\int Q(x)^4\operatorname{Re}(\epsilon)^2\,dx+\frac{1}{2}\int \operatorname{Im}(\epsilon)_x^2\,dx+\frac{1}{2}\int \operatorname{Im}(\epsilon)^2\,dx \\ -\frac{1}{2}\int Q(x)^4\operatorname{Im}(\epsilon)^2\,dx.\end{aligned}$$

Recalling (3-12),

$$\frac{1}{2}\int \operatorname{Re}(\epsilon)_x^2\,dx+\frac{1}{2}\int \operatorname{Re}(\epsilon)^2\,dx-\frac{5}{2}\int Q(x)^4\operatorname{Re}(\epsilon)^2\,dx=\frac{1}{2}(\mathcal{L}\operatorname{Re}(\epsilon),\operatorname{Re}(\epsilon))_{L^2}.$$

It is well known, see, e.g., [Merle 2001], that $\mathcal{L}$ has one negative eigenvector, $\mathcal{L}(Q^3)=-8Q^3$, and one zero eigenvector, $\mathcal{L}(Q_x)=0$. Since $\operatorname{Re}(\epsilon)\perp Q^3$ and $\operatorname{Re}(\epsilon)$ symmetric guarantees that $\operatorname{Re}(\epsilon)\perp Q_x$,

$$\frac{1}{2}\int \operatorname{Re}(\epsilon)_x^2\,dx+\frac{1}{2}\int \operatorname{Re}(\epsilon)^2\,dx-\frac{5}{2}\int Q(x)^4\operatorname{Re}(\epsilon)^2\,dx\ge\frac{1}{2}\int \operatorname{Re}(\epsilon)^2\,dx.$$

Next, doing some algebra,

$$\frac{1}{2}\int \operatorname{Re}(\epsilon)_x^2\,dx=\frac{1}{2}(\mathcal{L}\operatorname{Re}(\epsilon),\operatorname{Re}(\epsilon))-\frac{1}{2}\int \operatorname{Re}(\epsilon)^2\,dx+\frac{5}{2}\int Q(x)^4\operatorname{Re}(\epsilon)^2\,dx\le C(\mathcal{L}\operatorname{Re}(\epsilon),\operatorname{Re}(\epsilon)).$$

By similar calculations, since $\operatorname{Im}(\epsilon)\perp Q^3$ and $\operatorname{Im}(\epsilon)\perp Q_x$,

$$\begin{aligned}\frac{1}{2}\int \operatorname{Im}(\epsilon)_x^2\,dx+\frac{1}{2}\int \operatorname{Im}(\epsilon)^2\,dx-\frac{1}{2}\int Q(x)^4\operatorname{Im}(\epsilon)^2\,dx \\ =\frac{1}{2}(\mathcal{L}\operatorname{Im}(\epsilon),\operatorname{Im}(\epsilon))+2\int Q(x)^4\operatorname{Im}(\epsilon)^2 \\ \ge\frac{1}{2}(\mathcal{L}\operatorname{Im}(\epsilon),\operatorname{Im}(\epsilon))\ge\frac{1}{2}\int \operatorname{Im}(\epsilon)^2\,dx+\frac{1}{2C}\int \operatorname{Im}(\epsilon)_x^2\,dx.\end{aligned}$$

Finally, by the Sobolev embedding theorem and $\|\epsilon\|_{L^2}\ll 1$,

$$\int|\epsilon|^6\,dx\lesssim\|\epsilon\|_{\dot H^1}^2\|\epsilon\|_{L^2}^4\ll\|\epsilon\|_{\dot H^1}^2$$

and

$$\int Q(x)^3|\epsilon(t,x)|^3\,dx\lesssim\|\epsilon\|_{L^2}^{3/2}\|\epsilon\|_{L^6}^{3/2}\lesssim\|\epsilon\|_{L^2}^{5/2}\|\epsilon\|_{\dot H^1}^{1/2}\lesssim\|\epsilon\|_{L^2}^{5/2}+\|\epsilon\|_{L^2}^{5/2}\|\epsilon\|_{\dot H^1}^2\ll\|\epsilon\|_{L^2}^2+\|\epsilon\|_{\dot H^1}^2,$$

which completes the proof of Theorem 12. □

4. A long-time Strichartz estimate

Having shown that it is enough to consider solutions to (1-1) that are close to the family of solitons and that there is a good decomposition of solutions that are close to the family of solitons, the next task is to obtain a good frequency-localized Morawetz estimate. The proof of the frequency-localized Morawetz estimate will occupy Sections 4–6.

The proof of scattering in [Dodson 2015] for (1-1) when $\|u_0\|_{L^2}<\|Q\|_{L^2}$ utilized a frequency-localized Morawetz estimate. There, the Morawetz estimate was used to show that $E(P_n u(t_n))\to 0$

along a subsequence, where P_n is a Fourier truncation operator that converges to the identity in the strong L^2-operator topology. Then the Gagliardo–Nirenberg inequality, (1-7), and the stability of the zero solution to (1-1) implies that $u \equiv 0$. In the case that $\|u_0\|_{L^2} = \|Q\|_{L^2}$, [Dodson 2021; Fan 2021] proved that $E(P_n u(t_n)) \to 0$ along a subsequence, so the almost periodicity of u implies that $u(t_n)$ converges to a rescaled version of Q.

In fact, [Dodson 2021; Fan 2021] proved more, that $E(Pu(t)) \to 0$ in an averaged sense on an interval $[0, T] \subset I$. The operator P is fixed on a fixed time interval, but P converges to the identity in the strong L^2-operator topology as $T \to \sup(I)$. The proof of Theorem 7 will argue that if $E(Pu(t))$ goes to zero in a time-averaged sense, then u must be equal to the soliton if the solution is global. If the solution blows up in finite time, then u must equal a pseudoconformal transformation of the soliton.

An essential ingredient in this proof is an improved version of the long-time Strichartz estimates in [Dodson 2016a]. The proof will make use of the bilinear estimates of [Planchon and Vega 2009], which were also used in the two dimensional problem [Dodson 2016b].

Eventually, the proof of Theorem 7 will make use of long-time Strichartz estimates on an interval $J = [a, b]$ for

$$1 \le \lambda(t) \le T^{1/100}, \tag{4-1}$$

where $T = s(b) - s(a)$ and $s(t) : [0, \sup(I)) \to [0, \infty)$ is the function given by (3-10). However, to avoid obscuring the main idea, it will be convenient to consider the case when $\lambda(t) = 1$ first, since the generalization to the case (4-1) is fairly straightforward.

Suppose without loss of generality that $a = 0$ and $b = T$. Choose

$$0 < \eta_1 \ll \eta_0 \ll 1$$

to be small constants, suppose

$$\|\epsilon(t, x)\|_{L^2} \le \eta_0 \tag{4-2}$$

for all $t \in J$, and choose $\eta_1 \ll \eta_0$ small enough that

$$\int_{|\xi| \ge \eta_1^{-1/2}} |\widehat{Q}(\xi)|^2 \, d\xi \le \eta_0^2, \tag{4-3}$$

and therefore

$$\sup_{t \in J} \int_{|\xi| \ge \eta_1^{-1/2}} |\hat{u}(t, \xi)|^2 \, d\xi \le 4\eta_0^2.$$

Then rescale from $\lambda(t) = 1$ to $\lambda(t) = 1/\eta_1$ and $[0, T] \mapsto [0, \eta_1^{-2} T]$.

When $i \in \mathbb{Z}$, $i > 0$, let P_i denote the standard Littlewood–Paley projection operator. When $i = 0$, let P_i denote the projection operator $P_{\le 0}$, and when $i < 0$, let P_i denote the zero operator.

Definition. Suppose $\eta_1^{-2} T = 2^{3k}$ for some $k \in \mathbb{Z}_{\ge 0}$. Then define the norm

$$\|u\|^2_{X([0, \eta_1^{-2} T] \times \mathbb{R})} = \sup_{0 \le i \le k} \sup_{1 \le a < 2^{3k-3i}} \|P_i u\|^2_{U^2_\Delta([(a-1)2^{3i}, a2^{3i}] \times \mathbb{R})} + 2^i \|(P_{\ge i} u)(P_{\le i-3} u)\|^2_{L^2_{t,x}([(a-1)2^{3i}, a2^{3i}] \times \mathbb{R})}. \tag{4-4}$$

Also, for any $0 \leq j \leq k$, let

$$\|u\|^2_{X_j([0,\eta_1^{-2}T]\times\mathbb{R})} = \sup_{0\leq i\leq j} \sup_{1\leq a<2^{3k-3i}} \|P_i u\|^2_{U^2_\Delta([(a-1)2^{3i},\,a2^{3i}]\times\mathbb{R})} + 2^i \|(P_{\geq i}u)(P_{\leq i-3}u)\|^2_{L^2_{t,x}([(a-1)2^{3i},\,a2^{3i}]\times\mathbb{R})}.$$

See [Koch and Tataru 2007] for a definition of the U^p_Δ and V^p_Δ norms. See also [Dodson 2016a; 2016b; 2019].

Theorem 13. *The long-time Strichartz estimate*

$$\|u\|_{X([0,\eta_1^{-2}T]\times\mathbb{R})} \lesssim 1$$

holds with implicit constant independent of T.

Proof. This estimate is proved by induction on j. Local well-posedness arguments combined with the fact that $\lambda(t) = \eta_1^{-1}$ for any $t \in [0, \eta_1^{-2}T]$ imply that

$$\|u\|_{U^2_\Delta([a,a+1]\times\mathbb{R})} \lesssim 1,$$

and when $i = 0$,

$$(P_{\geq i}u)(P_{\leq i-3}u) = 0.$$

Therefore,

$$\|u\|_{X_0([0,\eta_1^{-2}T]\times\mathbb{R})} \lesssim 1. \tag{4-5}$$

This is the base case.

Remark. The implicit constant in (4-5) does not depend on T or η_1.

To prove the inductive step, recall, by Duhamel's principle, that if $J = [(a-1)2^{3k-3i}, a2^{3k-3i}]$, then, for any $t_0 \in J$,

$$u(t) = e^{i(t-t_0)\Delta}u(t_0) + i\int_{t_0}^t e^{i(t-\tau)\Delta}(|u|^4u)\,d\tau$$

and

$$\|P_{\geq i}u\|_{U^2_\Delta(J\times\mathbb{R})} \lesssim \|P_{\geq i}(u(t_0))\|_{L^2} + \left\|\int_{t_0}^t e^{i(t-\tau)\Delta}P_{\geq i}(|u|^4u)\,d\tau\right\|_{U^2_\Delta(J\times\mathbb{R})}.$$

By (4-3) and the fact that $\lambda(t) = 1/\eta_1$ for all $t \in [0, \eta_1^{-2}T]$, if $i > 0$,

$$\sup_{t_0\in[0,\eta_1^{-2}T]} \|P_{\geq i}u(t_0)\|_{L^2} \lesssim \eta_0. \tag{4-6}$$

Next, choose $v \in V^2_\Delta(J\times\mathbb{R})$ such that $\|v\|_{V^2_\Delta(J\times\mathbb{R})} = 1$ and $\hat{v}(t,\xi)$ is supported on the Fourier support of P_i. It is a well-known fact that

$$\left\|\int_{t_0}^t e^{i(t-\tau)\Delta}P_{\geq i}(|u|^4u)\,d\tau\right\|_{U^2_\Delta(J\times\mathbb{R})} \lesssim \sup_v \|vP_{\geq i}(|u|^4u)\|_{L^1_{t,x}},$$

where $\sup_v$ is the supremum over all such v supported on P_i satisfying $\|v\|_{V^2_\Delta(J\times\mathbb{R})} = 1$. See [Hadac et al. 2009] for a proof.

By Hölder's inequality,

$$\begin{aligned}\|v(u_{\geq i-3})^2(u_{\leq i-3})^3\|_{L^1_{t,x}} + \|v(u_{\geq i-3})^3(u_{\leq i-3})^2\|_{L^1_{t,x}} + \|v(u_{\geq i-3})^4(u_{\leq i-3})\|_{L^1_{t,x}} + \|v(u_{\geq i-3})^5\|_{L^1_{t,x}} \\ \lesssim \|v\|_{L_t^\infty L_x^2}\|u_{\geq i-3}\|^5_{L_t^5 L_x^{10}} + \|v\|_{L_t^4 L_x^\infty}\|u_{\geq i-3}\|^4_{L_t^{16/3} L_x^8}\|u_{\leq i-3}\|_{L_t^\infty L_x^2} \\ + \|v\|_{L^6_{t,x}}\|(u_{\geq i-3})(u_{\leq i-6})\|_{L^2_{t,x}}\|u_{\leq i-6}\|_{L^\infty_{t,x}}\|u_{\geq i-3}\|^2_{L^6_{t,x}} \\ + \|v\|_{L^6_{t,x}}\|u_{\geq i-3}\|^3_{L^6_{t,x}}\|u_{\geq i-6}\|^2_{L^6_{t,x}} + \|v\|_{L^6_{t,x}}\|(u_{\geq i-3})(u_{\leq i-6})\|^{3/2}_{L^2_{t,x}}\|u_{\leq i-6}\|^{3/2}_{L^\infty_{t,x}}\|u_{\geq i-3}\|^{1/2}_{L^6_{t,x}} \\ + \|v\|_{L^6_{t,x}}\|u_{\geq i-3}\|^2_{L^6_{t,x}}\|u_{\geq i-6}\|^3_{L^6_{t,x}}.\end{aligned} \tag{4-7}$$

Since $V^2_\Delta \subset U^p_\Delta$ for any $p > 2$, again see [Hadac et al. 2009],

$$\|v\|_{L_t^\infty L_x^2} + \|v\|_{L^6_{t,x}} + \|v\|_{L_t^4 L_x^\infty} \lesssim \|v\|_{V^2_\Delta} \lesssim 1. \tag{4-8}$$

Next, when $i > 4$, since $U^2_\Delta \subset U^4_\Delta$, (4-3) and (4-6) imply

$$\|u_{\geq i-3}\|^5_{L_t^5 L_x^{10}} \lesssim \|u_{\geq i-3}\|^4_{L_t^4 L_x^\infty}\|u_{\geq i-3}\|_{L_t^\infty L_x^2} \lesssim \eta_0\|u_{\geq i-3}\|^4_{L_t^4 L_x^\infty} \lesssim \eta_0\|u\|^4_{X_{i-3}([0,T]\times\mathbb{R})}. \tag{4-9}$$

When $i \leq 4$, the fact that for any $a \in \mathbb{Z}$,

$$\|u\|_{L_t^4 L_x^\infty([a,a+1]\times\mathbb{R})} \lesssim 1, \tag{4-10}$$

the fact that the Fourier inversion formula and Hölder's inequality imply

$$\left\| \int_{|\xi|\leq\eta_1^{1/2}} e^{ix\cdot\xi}\hat{u}(t,\xi)\,d\xi \right\|_{L^\infty} \lesssim \eta_1^{1/4}\|u(t)\|_{L^2}, \tag{4-11}$$

and the fact that (4-3) implies, after rescaling $\lambda(t) = 1 \mapsto \lambda(t) = 1/\eta_1$,

$$\left(\int_{|\xi|\geq\eta_1^{1/2}} |\hat{u}(t,\xi)|^2\,d\xi\right)^{1/2} \lesssim \eta_0 \tag{4-12}$$

combine to imply that

$$\|u_{\geq i-3}\|^5_{L_t^5 L_x^{10}} \lesssim \eta_0. \tag{4-13}$$

Similar calculations can be made for the terms

$$\|u_{\geq i-3}\|^4_{L_t^{16/3} L_x^8}\|u_{\leq i-3}\|_{L_t^\infty L_x^2} + \|u_{\geq i-3}\|^3_{L^6_{t,x}}\|u_{\geq i-6}\|^2_{L^6_{t,x}} + \|u_{\geq i-3}\|^2_{L^6_{t,x}}\|u_{\geq i-6}\|^3_{L^6_{t,x}}. \tag{4-14}$$

Therefore,

$$\begin{aligned}\|u_{\geq i-3}\|^5_{L_t^5 L_x^{10}} + \|u_{\geq i-3}\|^4_{L_t^{16/3} L_x^8}\|u_{\leq i-3}\|_{L_t^\infty L_x^2} + \|u_{\geq i-3}\|^3_{L^6_{t,x}}\|u_{\geq i-6}\|^2_{L^6_{t,x}} + \|u_{\geq i-3}\|^2_{L^6_{t,x}}\|u_{\geq i-6}\|^3_{L^6_{t,x}} \\ \lesssim \eta_0\|u\|^4_{X_{i-3}([0,T]\times\mathbb{R})} + \eta_0\|u\|^3_{X_{i-3}([0,T]\times\mathbb{R})} + \eta_0.\end{aligned} \tag{4-15}$$

Next, by the definition on page 1710,

$$\|(u_{\geq i-3})(u_{\leq i-6})\|_{L^2_{t,x}}\|u_{\leq i-6}\|_{L^\infty_{t,x}}\|u_{\geq i-3}\|^2_{L^6_{t,x}} \lesssim 2^{-i/2}\|u\|^3_{X_{i-3}([0,T]\times\mathbb{R})}\|u_{\leq i-6}\|_{L^\infty_{t,x}}. \tag{4-16}$$

By (4-11), (4-12), and the Sobolev embedding theorem,

$$2^{-i/2}\|u_{\le i-6}\|_{L^\infty_{t,x}} \lesssim \eta_0, \tag{4-17}$$

so

$$2^{-i/2}\|u\|^3_{X_{i-3}([0,T]\times\mathbb{R})}\|u_{\le i-6}\|_{L^\infty_{t,x}} \lesssim \eta_0\|u\|^3_{X_{i-3}([0,T]\times\mathbb{R})}. \tag{4-18}$$

Making a similar calculation,

$$\|(u_{\ge i-3})(u_{\le i-6})\|^{3/2}_{L^2_{t,x}}\|u_{\le i-6}\|^{3/2}_{L^\infty_{t,x}}\|u_{\ge i-3}\|^{1/2}_{L^6_{t,x}} \lesssim \eta_0^{3/2}\|u\|^2_{X_{i-3}([0,T]\times\mathbb{R})}. \tag{4-19}$$

Since

$$P_{\ge i}(|u_{\le i-3}|^4 u_{\le i-3}) = 0,$$

it only remains to compute, using the definition on page 1710, (4-15), and (4-17),

$$\begin{aligned}\|v((P_{\ge i-3}u)(P_{\le i-3}u)^4)\|_{L^1_{t,x}} &\lesssim \|(P_{\ge i-3}u)(P_{\le i-6}u)\|_{L^2_{t,x}}\|P_{\le i-6}u\|_{L^\infty_{t,x}}\|v(P_{\le i-3}u)^2\|_{L^2_{t,x}}\\ &\qquad + \|P_{\ge i-3}u\|_{L^6_{t,x}}\|P_{\ge i-6}u\|^2_{L^6_{t,x}}\|v(P_{\le i-3}u)^2\|_{L^2_{t,x}}\\ &\lesssim \eta_0\|u\|^2_{X_{i-3}([0,T]\times\mathbb{R})}\|v(P_{\le i-3}u)^2\|_{L^2_{t,x}} + \eta_0\|u\|^4_{X_{i-3}([0,T]\times\mathbb{R})} + \eta_0.\end{aligned}$$

By the Sobolev embedding theorem,

$$\|P_{\le i-3}u\|_{L^{18}_{t,x}} \lesssim \sum_{0\le j\le i-3}\|P_j u\|_{L^{18}_{t,x}} \lesssim \sum_{0\le j\le i-3} 2^{(3i-3j)/18}2^{j/3}\|u\|_{X_{i-3}([0,T]\times\mathbb{R})} \lesssim 2^{i/3}\|u\|_{X_{i-3}([0,T]\times\mathbb{R})}.$$

Also, by $V^2_\Delta \subset U^{9/4}_\Delta$ and the Sobolev embedding theorem,

$$\|v(P_{\le i-3}u)\|_{L^{9/4}_{t,x}} \lesssim 2^{i/9}\|v\|_{V^{12/5}_\Delta(J\times\mathbb{R})}\cdot\sup_{v_0}\|(e^{it\Delta}v_0)(u_{\le i-3})\|^{8/9}_{L^2_{t,x}},$$

where $\sup_{v_0}$ is over all $\|v_0\|_{L^2} = 1$ supported in Fourier space on the support of P_i. Therefore, we have finally proved

$$\begin{aligned}\|P_{\ge i}u\|_{U^2_\Delta(J\times\mathbb{R})} &\lesssim \eta_0 + \eta_0\|u\|^2_{X_{i-3}([0,T]\times\mathbb{R})} + \eta_0\|u\|^4_{X_{i-3}([0,T]\times\mathbb{R})}\\ &\qquad + \eta_0\|u\|^3_{X_{i-3}([0,T]\times\mathbb{R})}\cdot 2^{4i/9}\sup_{v_0}\|(e^{it\Delta}v_0)(u_{\le i-3})\|^{8/9}_{L^2_{t,x}}.\end{aligned} \tag{4-20}$$

To complete the proof of Theorem 13, it only remains to prove

$$2^{i/2}\sup_{v_0}\|(e^{it\Delta}v_0)(u_{\le i-3})\|_{L^2_{t,x}} \lesssim 1 + \|u\|_{X_{i-3}([0,T]\times\mathbb{R})}. \tag{4-21}$$

Indeed, assuming that (4-21) is true, (4-20) becomes

$$\|P_{\ge i}u\|_{U^2_\Delta(J\times\mathbb{R})} \lesssim \eta_0 + \eta_0\|u\|^2_{X_{i-3}([0,T]\times\mathbb{R})} + \eta_0\|u\|^4_{X_{i-3}([0,T]\times\mathbb{R})}.$$

Equation (4-21) would also imply

$$\begin{aligned}2^{i/2}\|(P_{\ge i}u)(P_{\le i-3}u)\|_{L^2_{t,x}(J\times\mathbb{R})} &\lesssim \|P_{\ge i}u\|_{U^2_\Delta(J\times\mathbb{R})}(1 + \|u\|_{X_{i-3}([0,T]\times\mathbb{R})})\\ &\lesssim \eta_0 + \eta_0\|u\|_{X_{i-3}([0,T]\times\mathbb{R})} + \eta_0\|u\|^5_{X_{i-3}([0,T]\times\mathbb{R})}.\end{aligned}$$

Then taking a supremum over $0 \le i \le j$,

$$\|u\|_{X_j([0,T]\times\mathbb{R})} \lesssim 1 + \eta_0 \|u\|_{X_{j-3}([0,T]\times\mathbb{R})} + \eta_0 \|u\|^5_{X_{j-3}([0,T]\times\mathbb{R})},$$

which by induction on j, starting from the base case (4-5), proves Theorem 13.

The bilinear estimate (4-21) is proved using the interaction Morawetz estimate (see [Dodson 2016b; Planchon and Vega 2009]). To simplify notation, let

$$v(t,x) = e^{it\Delta} v_0,$$

where $\|v_0\|_{L^2} = 1$ and $\hat{v}_0$ is supported on the Fourier support of P_j for some $j \ge i$. Then take the Morawetz potential

$$M(t) = \int |v(t,y)|^2 \frac{(x-y)}{|x-y|} \operatorname{Im}[\bar{u}_{\le i-3} \partial_x u_{\le i-3}]\, dx\, dy + \int |u_{\le i-3}|^2 \frac{(x-y)}{|x-y|} \operatorname{Im}[\bar{v} v_x]\, dx\, dy.$$

Let $F(u) = |u|^4 u$. Then $u_{\le i-3}$ solves the equation

$$i\partial_t u_{\le i-3} + \Delta u_{\le i-3} + F(u_{\le i-3}) = F(u_{\le i-3}) - P_{\le i-3} F(u) = -\mathcal{N}_{i-3}. \tag{4-22}$$

Following [Planchon and Vega 2009],

$$\begin{aligned}\frac{d}{dt} M(t) = 8 \int |\partial_x(\overline{v(t,x)} u_{\le i-3})(t,x)|^2\, dx - \frac{8}{3} \int |v(t,x)|^2 |u_{\le i-3}(t,x)|^6\, dx \\ + \int |v(t,y)|^2 \frac{(x-y)}{|x-y|} \operatorname{Re}[\bar{u}_{\le i-3} \partial_x \mathcal{N}_{i-3}](t,x)\, dx \\ - \int |v(t,y)|^2 \frac{(x-y)}{|x-y|} \operatorname{Re}[\overline{\mathcal{N}}_{i-3} \partial_x u_{\le i-3}](t,x)\, dx \\ + 2 \int \operatorname{Im}[\bar{u}_{\le i-3} \mathcal{N}_{i-3}](t,y) \frac{(x-y)}{|x-y|} \operatorname{Im}[\bar{v} \partial_x v](t,x)\, dx\, dy.\end{aligned}$$

Then by the fundamental theorem of calculus, Bernstein's inequality, the Fourier support of $\bar{v} u_{\le i-3}$, $\|v_0\|_{L^2} = 1$, and the fact that $\|u\|_{L^2} = \|Q\|_{L^2}$,

$$\begin{aligned}2^{2j} \|\bar{v} u_{\le i-3}\|^2_{L^2_{t,x}(J\times\mathbb{R})} \lesssim 2^j + \| |v|^2 |u_{\le i-3}|^6 \|_{L^1_{t,x}} - \int |v(t,y)|^2 \frac{(x-y)}{|x-y|} \operatorname{Re}[\overline{\mathcal{N}}_{i-3} \partial_x u_{\le i-3}](t,x)\, dx \\ + \int |v(t,y)|^2 \frac{(x-y)}{|x-y|} \operatorname{Re}[\bar{u}_{\le i-3} \partial_x \mathcal{N}_{i-3}](t,x)\, dx \\ + 2 \int \operatorname{Im}[\bar{u}_{\le i-3} \mathcal{N}_{i-3}](t,y) \frac{(x-y)}{|x-y|} \operatorname{Im}[\bar{v} \partial_x v](t,x)\, dx\, dy.\end{aligned} \tag{4-23}$$

Also note that

$$\|\bar{v} u_{\le i-3}\|^2_{L^2_{t,x}} = \|\bar{v} v \bar{u}_{\le i-3} u_{\le i-3}\|_{L^1_{t,x}} = \|v u_{\le i-3}\|^2_{L^2_{t,x}}, \tag{4-24}$$

so it is not too important to pay attention to complex conjugates in the proceeding calculations.

First, by (4-17),

$$\| |v|^2 |u_{\le i-3}|^6 \|_{L^1_{t,x}(J\times\mathbb{R})} \lesssim \|v u_{\le i-3}\|^2_{L^2_{t,x}} \|u_{\le i-3}\|^4_{L^\infty_{t,x}} \lesssim \eta_0^4 2^{2i} \|v u_{\le i-3}\|^2_{L^2_{t,x}}. \tag{4-25}$$

Now consider the term

$$\mathcal{N}_{i-3} = P_{\leq i-3} F(u) - F(u_{\leq i-3}). \tag{4-26}$$

Since by Fourier support arguments

$$P_{\leq i-3} F(u_{\leq i-6}) - F(u_{\leq i-6}) = 0, \tag{4-27}$$

we have

$$\begin{aligned}\mathcal{N}_{i-3} &= P_{\leq i-3}(3|u_{\leq i-6}|^4 u_{\geq i-6} + 2|u_{\leq i-6}|^2 (u_{\leq i-6})^2 \bar{u}_{\geq i-6}) \\ &\qquad - (3|u_{\leq i-6}|^4 u_{i-6\leq \cdot \leq i-3} + 2|u_{\leq i-6}|^2 (u_{\leq i-6})^2 \bar{u}_{i-6\leq \cdot \leq i-3}) \\ &\qquad\qquad + P_{\leq i-3} O((u_{\geq i-6})^2 u^3) + O((u_{i-6\leq \cdot \leq i-3})^2 u^3) \\ &= \mathcal{N}^{(1)}_{i-3} + \mathcal{N}^{(2)}_{i-3}.\end{aligned} \tag{4-28}$$

Following (4-7)–(4-19),

$$\begin{aligned}\| |\mathcal{N}^{(2)}_{i-3}| \, |u_{\leq i-3}| \|_{L^1_{t,x}} &\lesssim \| |u_{\geq i-6}|^2 |u_{\leq i-9}|^4 \|_{L^1_{t,x}} + \| |u_{\geq i-6}|^2 |u_{\geq i-9}|^4 \|_{L^1_{t,x}} \\ &\lesssim \|(u_{\geq i-6})(u_{\leq i-9})\|^2_{L^2_{t,x}} \|u_{\leq i-9}\|^2_{L^\infty_{t,x}} + \|u_{\geq i-6}\|^2_{L^6_{t,x}} \|u_{\geq i-9}\|^4_{L^6_{t,x}} \\ &\lesssim \eta_0 (1 + \|u\|^6_{X_{i-3}([0,T]\times\mathbb{R})}).\end{aligned} \tag{4-29}$$

Therefore, since $\|v_0\|_{L^2} = 1$,

$$\begin{aligned}&- \iiint |v(t,y)|^2 \frac{(x-y)}{|x-y|} \operatorname{Re}[\overline{\mathcal{N}}^{(2)}_{i-3} \partial_x u_{\leq i-3}](t,x)\, dx\, dy\, dt \\ &\qquad + \iiint |v(t,y)|^2 \frac{(x-y)}{|x-y|} \operatorname{Re}[\bar{u}_{\leq i-3} \partial_x \mathcal{N}^{(2)}_{i-3}](t,x)\, dx\, dy\, dt \\ &\qquad\qquad + 2 \iiint \operatorname{Im}[\bar{u}_{\leq i-3} \mathcal{N}^{(2)}_{i-3}](t,y) \frac{(x-y)}{|x-y|} \operatorname{Im}[\bar{v} \partial_x v](t,x)\, dx\, dy\, dt \\ &\lesssim \eta_0 2^j (1 + \|u\|^6_{X_{i-3}([0,T]\times\mathbb{R})}).\end{aligned} \tag{4-30}$$

Next, observe that

$$\begin{aligned}&3P_{\leq i-3}(|u_{\leq i-6}|^4 u_{\geq i-6}) - 3(|u_{\leq i-6}|^4 u_{i-6\leq \cdot \leq i-3}) \\ &\qquad = 3P_{> i-3}(|u_{\leq i-6}|^4 u_{i-6\leq \cdot \leq i-3}) + 3P_{\leq i-3}(|u_{\leq i-6}|^4 u_{> i-3}).\end{aligned} \tag{4-31}$$

Again following (4-7)–(4-19),

$$\begin{aligned}&\|P_{\leq i-3}(|u_{\leq i-6}|^4 u_{> i-3})(u_{i-6\leq \cdot \leq i-3})\|_{L^1_{t,x}} + \|P_{> i-3}(|u_{\leq i-6}|^4 u_{i-6\leq \cdot \leq i-3})(u_{i-6\leq \cdot \leq i-3})\|_{L^1_{t,x}} \\ &\qquad \lesssim \eta_0 (1 + \|u\|^6_{X_{i-3}([0,T]\times\mathbb{R})}).\end{aligned} \tag{4-32}$$

Finally, observe that the Fourier support of

$$3P_{> i-3}(|u_{\leq i-6}|^4 u_{i-6\leq \cdot \leq i-3})(u_{\leq i-6}) + 3P_{\leq i-3}(|u_{\leq i-6}|^4 u_{> i-3})(u_{\leq i-6}) \tag{4-33}$$

is on frequencies $|\xi| \geq 2^{i-6}$. Therefore, integrating by parts,

$$\begin{aligned}
&\iiint \operatorname{Im}[\bar{u}_{\leq i-6} P_{>i-3}(|u_{\leq i-6}|^4 u_{i-6\leq \cdot \leq i-3})](t,y)\frac{(x-y)}{|x-y|}\operatorname{Im}[\bar{v}\partial_x v](t,x)\,dx\,dy\,dt \\
&\quad= \iiint \operatorname{Im}[\bar{v}\partial_x v](t,x)\cdot\frac{\partial_x}{\partial_x^2}\operatorname{Im}[\bar{u}_{\leq i-6} P_{>i-3}(|u_{\leq i-6}|^4 u_{i-6\leq \cdot \leq i-3})](t,x)\,dx\,dy\,dt \\
&\quad\lesssim 2^{-i}\|v_x\|_{L_t^4 L_x^\infty}\|v\|_{L_t^4 L_x^\infty}\|(u_{i-6\leq\cdot\leq i-3})(u_{\leq i-9})\|_{L^2_{t,x}}\|u_{\leq i-6}\|^4_{L_t^\infty L_x^8} \\
&\qquad\quad + 2^{-i}\|v_x\|_{L_t^4 L_x^\infty}\|v\|_{L_t^4 L_x^\infty}\|u_{i-6\leq\cdot\leq i-3}\|_{L_t^4 L_x^\infty}\|u_{i-9\leq\cdot\leq i-6}\|_{L_t^4 L_x^\infty}\|u_{\leq i-6}\|^4_{L_t^\infty L_x^4} \\
&\quad\lesssim \eta_0 2^j \|u\|^2_{X_{i-3}([0,T]\times\mathbb{R})}. \qquad (4\text{-}34)
\end{aligned}$$

A similar calculation gives the estimate

$$\begin{aligned}
&\iiint \operatorname{Im}[\bar{u}_{\leq i-6} P_{\leq i-3}(|u_{\leq i-6}|^4 u_{>i-3})](t,y)\frac{(x-y)}{|x-y|}\operatorname{Im}[\bar{v}\partial_x v](t,x)\,dx\,dy\,dt \\
&\qquad\qquad\lesssim \eta_0 2^j\|u\|^2_{X_{i-3}([0,T]\times\mathbb{R})}. \qquad (4\text{-}35)
\end{aligned}$$

The terms

$$\iiint |v(t,y)|^2\frac{(x-y)}{|x-y|}\operatorname{Re}[\overline{\mathcal{N}}^{(1)}_{i-3}\partial_x u_{\leq i-3}](t,x)\,dx\,dy\,dt \tag{4-36}$$

and

$$\iiint |v(t,y)|^2\frac{(x-y)}{|x-y|}\operatorname{Re}[\bar{u}_{\leq i-3}\partial_x \mathcal{N}^{(1)}_{i-3}](t,x)\,dx\,dy\,dt \tag{4-37}$$

may be analyzed in a similar manner.

Plugging (4-24)–(4-37) into (4-23) gives

$$2^{2j}\|\bar{v}u_{\leq i-3}\|^2_{L^2_{t,x}} + 2^{2j}\|vu_{\leq i-3}\|^2_{L^2_{t,x}} \lesssim 2^j + \eta_0 2^j(1+\|u\|^6_{X_{i-3}}).$$

Summing up over $j \geq i$ implies (4-21), which completes the proof of Theorem 13. □

Theorem 13 may be upgraded to take advantage of the fact that u is close to the soliton.

Theorem 14. *When $\lambda(t) = 1/\eta_1$ and $T = 2^{3k}$ for some positive integer k,*

$$\|P_{\geq k}u\|_{U^2_\Delta([0,T]\times\mathbb{R})} \lesssim \left(\frac{\eta_1^2}{T}\int_0^{\eta_1^{-2}T}\|\epsilon(t)\|^2_{L^2}\,dt\right)^{1/2} + \frac{1}{T^{10}}.$$

Proof. Make another induction on frequency argument starting at level $\frac{1}{2}k$. Observe that Theorem 13 implies that for any $a \in \mathbb{Z}$, $0 \leq a < \eta_1^{-1}T^{1/2}$,

$$\|P_{\geq k/2}u\|_{U^2_\Delta([a\eta_1^{-1}T^{1/2},(a+1)\eta_1^{-1}T^{1/2}]\times\mathbb{R})} \lesssim 1.$$

Next, following Theorem 13,

$$\begin{aligned}\|P_{\geq k/2+3}u\|_{U_\Delta^2([512a\eta_1^{-1}T^{1/2},512(a+1)\eta_1^{-1}T^{1/2}]\times\mathbb{R})} &\lesssim \inf_{t\in[512a\eta_1^{-1}T^{1/2},512(a+1)\eta_1^{-1}T^{1/2}]}\|P_{\geq k/2+3}u(t)\|_{L^2}\\ &\quad+\eta_0\|P_{\geq k/2}u\|_{U_\Delta^2([512a\eta_1^{-1}T^{1/2},512(a+1)\eta_1^{-1}T^{1/2}]\times\mathbb{R})}. \end{aligned}\tag{4-38}$$

Since Q is a smooth function, if $\gamma(t)$ and $\lambda(t)$ are given by Theorem 10 and $\lambda(t)=1/\eta_1$,

$$\begin{aligned}\|P_{\geq k/2+3}u(t)\|_{L^2} &\leq \|e^{i\gamma(t)}\lambda(t)^{1/2}u(t,\lambda(t)x)-Q(x)\|_{L^2}+\|P_{\geq k/2+3}Q(x)\|_{L^2}\\ &\lesssim \|\epsilon(t)\|_{L^2}+T^{-10}.\end{aligned}\tag{4-39}$$

Plugging (4-39) back into (4-38),

$$\begin{aligned}&\|P_{\geq k/2+3}u\|_{U_\Delta^2([512a\eta_1^{-1}T^{1/2},512(a+1)\eta_1^{-1}T^{1/2}]\times\mathbb{R})}\\ &\lesssim\left(\frac{\eta_1}{512T^{1/2}}\int_{512a\eta_1^{-1}T^{1/2}}^{512(a+1)\eta_1^{-1}T^{1/2}}\|\epsilon(t,x)\|_{L^2}^2\,dt\right)^{1/2}\\ &\qquad+T^{-10}+\eta_0\left(\sum_{j=1}^{512}\|P_{\geq k/2}u\|^2_{U_\Delta^2([(512a+(j-1))\eta_1^{-1}T^{1/2},(512a+j)\eta_1^{-1}T^{1/2}]\times\mathbb{R})}\right)^{1/2}.\end{aligned}\tag{4-40}$$

Arguing by induction in k, taking $\lfloor \frac{1}{6}k\rfloor$ steps in all, for η_0 sufficiently small,

$$\begin{aligned}\|P_{\geq k}u\|_{U_\Delta^2([0,\eta_1^{-2}T]\times\mathbb{R})} &\lesssim T^{-10}+2^{k/2}\eta_0^{-k/6}+\left(\frac{\eta_1^2}{T}\int_0^{\eta_1^{-2}T}\|\epsilon(t,x)\|_{L^2}^2\,dt\right)^{1/2}\\ &\lesssim T^{-10}+\left(\frac{\eta_1^2}{T}\int_0^{\eta_1^{-2}T}\|\epsilon(t,x)\|_{L^2}^2\,dt\right)^{1/2}.\end{aligned}$$

□

Remark. If C is the implicit constant in (4-40), then for $\eta_0\ll 1$ sufficiently small,

$$(C\eta_0)^{\lfloor k/6\rfloor}\leq T^{-10}.\tag{4-41}$$

The same argument can also be made when $\lambda(t)\geq 1/\eta_1$ for all $t\in J$.

Theorem 15. *When $\lambda(t)\geq 1/\eta_1$ on $J=[a,b]$,*

$$\int_J\lambda(t)^{-2}\,dt=T,$$

and $\eta_1^{-2}T=2^{3k}$, we have

$$\|P_{\geq k}u\|_{U_\Delta^2([a,b]\times\mathbb{R})}\lesssim T^{-10}+\left(\frac{1}{T}\int_a^b\|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}\,dt\right)^{1/2}.$$

The same argument could also be made for $\lambda(t)$ having a different lower bound, by rescaling $\lambda(t)$ to $\lambda(t)\geq 1/\eta_1$, computing long-time Strichartz estimates, and then rescaling back.

5. Almost conservation of energy

Since (3-11) implies that $\|\epsilon(t)\|_{L^2}$ is continuous as a function of time, the mean value theorem implies that under the conditions of Theorem 15, there exists $t_0 \in [a, b]$ such that

$$\|\epsilon(t_0)\|_{L^2}^2 = \frac{1}{T}\int_a^b \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}\, dt.$$

The next step in proving Theorem 7 is to control

$$\sup_{t\in[a,b]} \|\epsilon(t)\|_{L^2}$$

as a function of $\|\epsilon(t_0)\|_{L^2}$. Theorem 12 would be a very useful tool for doing so, except that while Q lies in $H^s(\mathbb{R})$ for any $s > 0$, it is not the case that ϵ must belong to $H^s(\mathbb{R})$ for any $s > 0$. Therefore, Theorem 12 will be used in conjunction with the Fourier truncation method of [Bourgain 1998]. See also the I-method, for example, in [Colliander et al. 2002].

Theorem 16. *Let $J = [a, b]$ be an interval such that*

$$\int_J \lambda(t)^{-2}\, dt = T,$$

$\eta_1^{-2} T = 2^{3k}$, *and $\lambda(t) \geq 1/\eta_1$ for all $t \in [a, b]$. Then,*

$$\sup_{t\in J} E(P_{\leq k+9} u(t)) \lesssim \frac{2^{2k}}{T}\int_J \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}\, dt + 2^{2k} T^{-10}.$$

Proof. By the mean value theorem, there exists $t_0 \in J$ such that

$$\|\epsilon(t_0)\|_{L^2}^2 \lesssim \frac{1}{T}\int_J \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}\, dt.$$

Next, decompose the energy. Let $\widetilde{Q}$ refer to a rescaled version of Q, that is, $\widetilde{Q} = \lambda(t_0)^{-1/2} Q(\lambda(t_0)^{-1} x)$, and let $\tilde{\epsilon}$ denote the rescaled ϵ given by $\tilde{\epsilon} = \lambda(t_0)^{-1/2}\tilde{\epsilon}(t_0, \lambda(t_0)^{-1} x)$. It is also convenient to split $\tilde{\epsilon}$ into real and imaginary parts:

$$\tilde{\epsilon} = \epsilon_1 + i\epsilon_2.$$

As in Theorem 12, by (3-2),

$$\begin{aligned}
E(P_{\leq k+9} u) &= E(P_{\leq k+9}\widetilde{Q} + P_{\leq k+9}\tilde{\epsilon}) \\
&= E(P_{\leq k+9}\widetilde{Q}) + \operatorname{Re}\int P_{\leq k+9}\widetilde{Q}_x \overline{P_{\leq k+9}\tilde{\epsilon}_x}\, dx - \operatorname{Re}\int (P_{\leq k+9}\widetilde{Q})^5 (\overline{P_{\leq k+9}\tilde{\epsilon}})\, dx \\
&\quad + \frac{1}{2}\|P_{\leq k+9}\tilde{\epsilon}\|_{\dot{H}^1}^2 - \frac{5}{2}\int (P_{\leq k+9}\widetilde{Q})^4 (P_{\leq k+9}\tilde{\epsilon}_1)^2\, dx - \frac{1}{2}\int (P_{\leq k+9}\widetilde{Q})^4 (P_{\leq k+9}\epsilon_2)^2 \\
&\quad - \int O(|P_{\leq k+9}\widetilde{Q}|^3 |P_{\leq k+9}\tilde{\epsilon}|^3) + O(|P_{\leq k+9}\tilde{\epsilon}|^6)\, dx.
\end{aligned} \tag{5-1}$$

Since $\widetilde{Q}$ is smooth and rapidly decreasing, $E(\widetilde{Q}) = 0$, and $\lambda(t_0) \geq 1/\eta_1$, Bernstein's inequality implies that

$$E(P_{\leq k+9}\widetilde{Q}) = \frac{1}{2}\int (P_{\leq k+9}\widetilde{Q}_x)^2 - \frac{1}{6}\int (P_{\leq k+9}\widetilde{Q})^6 \lesssim 2^{-30k}. \tag{5-2}$$

Next, integrating by parts and using (3-19), the smoothness of Q, and Bernstein's inequality,

$$\begin{aligned}\operatorname{Re}\int P_{\leq k+9}\widetilde{Q}_x \overline{P_{\leq k+9}\tilde{\epsilon}_x}\,dx - \operatorname{Re}\int (P_{\leq k+9}\widetilde{Q})^5\,\overline{P_{\leq k+9}\tilde{\epsilon}}\,dx \\ = -\operatorname{Re}\int (\overline{P_{\leq k+9}\epsilon})(P_{\leq k+9}\widetilde{Q}_{xx} + (P_{\leq k+9}\widetilde{Q})^5)\,dx = \frac{1}{2\lambda(t_0)^2}\|\epsilon\|_{L^2}^2 + O(2^{-30k}).\end{aligned} \tag{5-3}$$

Next, by Hölder's inequality, since $\lambda(t_0) \geq 1/\eta_1$,

$$\begin{aligned}\frac{1}{2}\|P_{\leq k+9}\tilde{\epsilon}\|_{\dot{H}^1}^2 - \frac{5}{2}\int (P_{\leq k+9}\widetilde{Q})^4 (P_{\leq k+9}\tilde{\epsilon}_1)^2\,dx - \frac{1}{2}\int (P_{\leq k+9}\widetilde{Q})^4 (P_{\leq k+9}\tilde{\epsilon}_2)^2\,dx \\ \lesssim \|P_{\leq k+9}\tilde{\epsilon}\|_{\dot{H}^1}^2 + \frac{1}{\lambda(t_0)^2}\|P_{\leq k+9}\tilde{\epsilon}\|_{L^2}^2 \lesssim 2^{2k}\|\epsilon(t_0)\|_{L^2}^2.\end{aligned} \tag{5-4}$$

By the Sobolev embedding theorem,

$$\begin{aligned}&\int |P_{\leq k+9}\tilde{\epsilon}|^3 |P_{\leq k+9}\widetilde{Q}|^3 + |P_{\leq k+9}\tilde{\epsilon}|^6\,dx \\ &\quad\lesssim \frac{1}{\lambda(t_0)^{3/2}}\|P_{\leq k+9}\tilde{\epsilon}(t_0)\|_{L^2}^{5/2}\|P_{\leq k+9}\tilde{\epsilon}(t_0)\|_{\dot{H}^1}^{1/2} + \|P_{\leq k+9}\tilde{\epsilon}(t_0)\|_{L^2}^4\|P_{\leq k+9}\tilde{\epsilon}(t_0)\|_{\dot{H}^1}^2 \\ &\quad\lesssim \frac{1}{\lambda(t_0)^2}\|P_{\leq k+9}\tilde{\epsilon}(t_0)\|_{L^2}^2 + \|P_{\leq k+9}\tilde{\epsilon}(t_0)\|_{L^2}^4\|P_{\leq k+9}\tilde{\epsilon}(t_0)\|_{\dot{H}^1}^2.\end{aligned} \tag{5-5}$$

Therefore, since $\lambda(t_0) \geq 1/\eta_1$,

$$E(P_{\leq k+9}\tilde{u}(t_0)) \lesssim 2^{2k}\|\epsilon(t_0)\|_{L^2}^2 + 2^{-30k}. \tag{5-6}$$

Next compute the change of energy

$$\begin{aligned}\frac{d}{dt}E(P_{\leq k+9}u) &= -(P_{\leq k+9}u_t, \Delta P_{\leq k+9}u)_{L^2} - (P_{\leq k+9}u_t, |P_{\leq k+9}u|^4 P_{\leq k+9}u)_{L^2} \\ &= -(P_{\leq k+9}u_t, P_{\leq k+9}F(u) - F(P_{\leq k+9}u))_{L^2} \\ &= (i\Delta P_{\leq k+9}u + iP_{\leq k+9}(|u|^4 u), P_{\leq k+9}F(u) - F(P_{\leq k+9}u))_{L^2}.\end{aligned}$$

First compute

$$\int_{t_0}^{t'} (i\Delta P_{\leq k+9}u, P_{\leq k+9}F(u) - F(P_{\leq k+9}u))_{L^2}\,dt$$

for some $t' \in J$. Making a Littlewood–Paley decomposition,

$$\begin{aligned}&\int_{t_0}^{t'} (i\Delta P_{\leq k+9}u, P_{\leq k+9}F(u) - F(P_{\leq k+9}u))_{L^2}\,dt \\ &\sim \sum_{0\leq k_5\leq k_4\leq k_3\leq k_2\leq k_1}\ \sum_{0\leq k_6\leq k+9}\int_{t_0}^{t'} \big(i\Delta P_{k_6}u, P_{\leq k+9}(P_{k_1}u\cdots P_{k_5}u) \\ &\qquad\qquad - (P_{\leq k+9}P_{k_1}u)\cdots(P_{\leq k+9}P_{k_5}u)\big)_{L^2}\,dt.\end{aligned}$$

Remark. For these computations, it is not so important to distinguish between u and $\bar{u}$.

Case 1: $k_1 \leq k+6$. In this case $P_{\leq k+9} P_{k_1} = P_{k_1}$ and $P_{\leq k+9}(P_{k_1} u \cdots P_{k_5} u) = P_{k_1} u \cdots P_{k_5} u$, so the contribution of these terms is zero. That is, for $k_1, \dots, k_5 \leq k+6$,

$$\int_{t_0}^{t'} (i\Delta P_{k_6} u, P_{\leq k+9}(P_{k_1} u \cdots P_{k_5} u) - (P_{\leq k+9} P_{k_1} u) \cdots (P_{\leq k+9} P_{k_5} u))_{L^2}\, dt = 0.$$

Case 2: $k_1 \geq k+6$ and $k_2 \leq k$. In this case, Fourier support properties imply that $k_6 \geq k+3$. Then by Theorem 15, Theorem 13, and (4-21),

$$\begin{aligned}\int_{t_0}^{t'} (i\Delta P_{k+3\leq \cdot \leq k+9} u, P_{\leq k+9}((P_{\leq k} u)^4 (P_{\geq k+6} u)) - (P_{\leq k} u)^4 (P_{k+6 \leq \cdot \leq k+9} u))_{L^2}\, dt \\ \lesssim 2^{2k} \|(P_{\geq k+3} u)(P_{\leq k} u)\|_{L^2_{t,x}} \|(P_{\geq k+6} u)(P_{\leq k} u)\|_{L^2_{t,x}} \|P_{\leq k} u\|^2_{L^\infty_{t,x}} \\ \lesssim \frac{2^{2k}}{T} \int_J \|\epsilon(t)\|^2_{L^2} \lambda(t)^{-2}\, dt + 2^{2k} T^{-10}.\end{aligned}$$

Case 3: $k_1 \geq k+6,\ k_2 \geq k,\ k_3 \leq k$. If $k_6 \leq k$, then by Fourier support properties, $k_2 \geq k+3$. Here,

$$\begin{aligned}&\int_{t_0}^{t'} \big(i\Delta P_{\leq k} u, P_{\leq k+9}((P_{\geq k+6} u)(P_{\geq k+3} u)(P_{\leq k} u)^3) \\ &\qquad\qquad - (P_{k+6\leq \cdot \leq k+9} u)(P_{k+3 \leq \cdot \leq k+9} u)(P_{\leq k} u)^3\big)_{L^2}\, dt \\ &\lesssim 2^{2k} \|(P_{\geq k+6} u)(P_{\leq k} u)\|_{L^2_{t,x}} \|(P_{\geq k+3} u)(P_{\leq k} u)\|_{L^2_{t,x}} \|P_{\leq k} u\|^2_{L^\infty_{t,x}} \\ &\lesssim \frac{2^{2k}}{T} \int_J \|\epsilon(t)\|^2_{L^2} \lambda(t)^{-2}\, dt + 2^{2k} T^{-10}.\end{aligned}$$

In the case when $k_6 \geq k$,

$$\begin{aligned}&\int_{t_0}^{t'} \big(i\Delta P_{k\leq \cdot \leq k+9} u, P_{\leq k+9}((P_{\geq k+6} u)(P_{\geq k} u)(P_{\leq k} u)^3) \\ &\qquad\qquad - (P_{k+6\leq \cdot \leq k+9} u)(P_{k \leq \cdot \leq k+9} u)(P_{\leq k} u)^3\big)_{L^2}\, dt \\ &\lesssim 2^{2k} \|(P_{\geq k+6} u)(P_{\leq k} u)\|_{L^2_{t,x}} \|P_{\geq k} u\|^2_{L^4_t L^\infty_x} \|P_{\leq k} u\|_{L^\infty_{t,x}} \|u\|_{L^\infty_t L^2_x} \\ &\lesssim \frac{2^{2k}}{T} \int_J \|\epsilon(t)\|^2_{L^2} \lambda(t)^{-2}\, dt + 2^{2k} T^{-10}.\end{aligned}$$

Case 4: $k_1 \geq 2^{k+6}$ and $k_2, k_3 \geq k$. In this case,

$$\begin{aligned}&\int_{t_0}^{t'} (i\Delta P_{\leq k+9} u, P_{\leq k+9}((P_{\geq k+6} u)(P_{\geq k} u)^2 u^2) - (P_{k+6\leq \cdot \leq k+9} u)(P_{k\leq \cdot \leq k+9} u)^2 (P_{\leq k+9} u)^2)_{L^2}\, dt \\ &\lesssim 2^{2k} \|(P_{\geq k+6} u)(P_{\leq k} u)\|_{L^2_{t,x}} \|P_{\geq k} u\|^2_{L^4_t L^\infty_x} \|P_{\leq k} u\|_{L^\infty_{t,x}} \|u\|_{L^\infty_t L^2_x} + 2^{2k} \|P_{\geq k} u\|^4_{L^4_t L^\infty_x} \|u\|^2_{L^\infty_t L^2_x} \\ &\lesssim \frac{2^{2k}}{T} \int_J \|\epsilon(t)\|^2_{L^2} \lambda(t)^{-2}\, dt + 2^{2k} T^{-10}.\end{aligned}$$

The contribution of the nonlinear terms is similar, using the fact that

$$(i P_{\leq k+9} F(u), P_{\leq k+9} F(u) - F(P_{\leq k+9} u))_{L^2} = (i P_{\leq k+9} F(u), F(P_{\leq k+9} u))_{L^2}.$$

Then make a Littlewood–Paley decomposition

$$\begin{aligned}(i P_{\leq k+9} F(u), &F(P_{\leq k+9} u))_{L^2} \\ &= \sum_{0 \leq k_5 \leq k_4 \leq k_3 \leq k_2 \leq k_1} \sum_{0 \leq k_5' \leq k_4' \leq k_3' \leq k_2' \leq k_1'} (i P_{\leq k+9}(u_{k_1} \cdots u_{k_5}), (P_{\leq k+9} P_{k_1} u) \cdots (P_{\leq k+9} P_{k_5} u))_{L^2}. \end{aligned} \tag{5-7}$$

Case 1: $k_1, k_1' \leq k+6$. Once again, if $k_1, k_1' \leq k+6$, then the right-hand side of (5-7) is zero.

Case 2: k_1 or $k_1' \geq k+6$, eight terms are $\leq k$. In the case that k_1 or $k_1' \geq k+6$ and eight of the terms in (5-7) are at frequency $\leq k$, then by Fourier support properties the final term should be at frequency $\geq k+3$. The contribution in this case is bounded by

$$\|(P_{\geq k+6} u)(P_{\leq k} u)\|_{L^2_{t,x}} \|(P_{\geq k+3} u)(P_{\leq k} u)\|_{L^2_{t,x}} \|P_{\leq k} u\|^6_{L^\infty_{t,x}} \lesssim 2^{2k} \frac{1}{T} \int_J \|\epsilon(t)\|^2_{L^2}\, dt + 2^{2k} T^{-10}.$$

Case 3: k_1 or $k_1' \geq k+6$, two terms are $\geq k$. The contribution of the case that k_1 or $k_1' \geq k+6$, two additional terms in (5-7) are at frequency $\geq k$, and the other seven terms are at frequency $\leq k$ is bounded by

$$\|(P_{\geq k+6} u)(P_{\leq k} u)\|_{L^2_{t,x}} \|P_{\geq k} u\|^2_{L^4_t L^\infty_x} \|P_{\leq k} u\|^5_{L^\infty_{t,x}} \|u\|_{L^\infty_t L^2_x} \lesssim 2^{2k} \frac{1}{T} \int_J \|\epsilon(t)\|^2_{L^2}\, dt + 2^{2k} T^{-10}.$$

Case 4: k_1 or $k_1' \geq k+6$ and at least three additional terms in (5-7) are at frequencies $\geq k$. This case may be reduced to a case where at least four terms in (5-7) are at frequency $\geq k$, and at least four terms are at frequency $\leq k+9$. To see why, notice that all five terms in $F(P_{\leq k+9} u)$ are at frequency $\leq k+9$, so if four or five of the terms in $P_{\leq k+9} F(u)$ are at frequency $\geq k$, then we are fine.

If exactly three terms in $P_{\leq k+9} F(u)$ are at frequency $\geq k$, then take the two terms in $P_{\leq k+9} F(u)$ that are at frequency $\leq k$ to be terms at frequency $\leq k+9$. Meanwhile, since at least four terms are at frequency $\geq k$,

$$F(P_{\leq k+9} u) \sim (P_{k \leq \cdot \leq k+9} u)(P_{\leq k+9} u)^4, \tag{5-8}$$

so in (5-8) there is one term at frequency $\geq k$ and two more terms at frequency $\leq k+9$.

If exactly two terms in $P_{\leq k+9} F(u)$ are at frequency $\geq k$, then there are three terms that are at frequency $\leq k$. In that case,

$$F(P_{\leq k+9} u) \sim (P_{k \leq \cdot \leq k+9} u)^2 (P_{\leq k+9} u)^3, \tag{5-9}$$

so in (5-9) there are two terms at frequency $\geq k$ and one term at frequency $\leq k+9$.

If one term in $P_{\leq k+9} F(u)$ is at frequency $\geq k$, then there are four terms in $P_{\leq k+9} F(u)$ at frequency $\leq k$. Then there must be at least three more in $F(P_{\leq k+9} u)$, so

$$F(P_{\leq k+9} u) \sim (P_{k \leq \cdot \leq k+9} u)^3 u^2. \tag{5-10}$$

If no terms in $P_{\leq k+9} F(u)$ are at frequency $\geq k$, then there must be four in $F(P_{\leq k+9} u)$, so

$$F(P_{\leq k+9} u) \sim (P_{k \leq \cdot \leq k+9} u)^4 u. \tag{5-11}$$

The contribution of all the different subcases of case four, (5-8)–(5-11), may be bounded by

$$\|P_{\geq k}u\|_{L_t^4 L_x^\infty}^4 \|u\|_{L_t^\infty L_x^2}^2 \|P_{\leq k+9}u\|_{L_{t,x}^\infty}^4 \lesssim 2^{2k}\frac{1}{T}\int_J \|\epsilon(t)\|_{L^2}^2\,dt + 2^{2k}T^{-10}.$$

This proves Theorem 16. □

Corollary 17. *If*

$$\frac{1}{\eta_1} \leq \lambda(t) \leq \frac{1}{\eta_1}T^{1/100}$$

and

$$\int_J \lambda(t)^{-2}\,dt = T,$$

then

$$\sup_{t\in J}\left\| P_{\leq k+9}\frac{1}{\lambda(t)^{1/2}}\epsilon\left(t, \frac{x}{\lambda(t)}\right)\right\|_{\dot H^1}^2 \lesssim \frac{2^{2k}}{T}\int_J \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}\,dt + 2^{2k}T^{-10}$$

and

$$\sup_{t\in J}\|\epsilon(t)\|_{L^2}^2 \lesssim \frac{T^{1/50}}{\eta_1^2}\frac{2^{2k}}{T}\int_J \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}\,dt + \frac{T^{1/50}}{\eta_1^2}2^{2k}T^{-10}.$$

Proof. The proof uses Theorem 16, Theorem 12, rescaling, and the fact that Q is smooth and all its derivatives are rapidly decreasing. □

6. A frequency-localized Morawetz estimate

The next step will be to combine long-time Strichartz estimates with almost conservation of energy to prove a frequency-localized Morawetz estimate adapted to the case when $\lambda(t)$ does not vary too much.

Theorem 18. *Let $J = [a, b]$ be an interval on which* (4-2) *holds for all $t \in J$, $1/\eta_1 \leq \lambda(t) \leq T^{1/100}/\eta_1$ for all $t \in J$, and*

$$\int_J \lambda(t)^{-2}\,dt = T. \tag{6-1}$$

Also suppose $2^{3k} = \eta_1^{-2}T$ and that $\epsilon = \epsilon_1 + i\epsilon_2$, where ϵ is given by Theorem 10. Then for T sufficiently large,

$$\int_a^b \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}\,dt \leq 3\big(\epsilon_2(a), \big(\tfrac{1}{2}Q + xQ_x\big)\big)_{L^2} - 3\big(\epsilon_2(b), \tfrac{1}{2}Q + xQ_x\big)_{L^2} + O\left(\frac{1}{T^9}\right). \tag{6-2}$$

Remark. The signs on the right-hand side of (6-2) are very important.

Proof. The proof uses a frequency-localized Morawetz estimate. The Morawetz potential is the same as the Morawetz potential used in [Dodson 2015]. See also [Dodson 2016a].

Let $\psi(x) \in C^\infty(\mathbb{R})$ be a smooth, even function, satisfying $\psi(x) = 1$ on $|x| \leq 1$ and supported on $|x| \leq 2$. Then for some large R ($R = T^{1/25}$ will do), let

$$\phi(x) = \int_0^x \chi\left(\frac{\eta_1 y}{R}\right)dy = \int_0^x \psi^2\left(\frac{\eta_1 y}{R}\right)dy, \tag{6-3}$$

and let

$$M(t) = \int \phi(x)\,\mathrm{Im}[\overline{P_{\leq k+9}u}\,\partial_x P_{\leq k+9}u](t,x)\,dx.$$

Doing some algebra using (3-2), as in (5-1),

$$u(t,x) = e^{-i\gamma(t)}\lambda(t)^{-1/2}Q\left(\frac{x}{\lambda(t)}\right) + e^{-i\gamma(t)}\lambda(t)^{-1/2}\epsilon\left(t,\frac{x}{\lambda(t)}\right) = e^{-i\gamma(t)}\widetilde{Q}(x) + e^{-i\gamma(t)}\tilde{\epsilon}(t,x). \quad (6\text{-}4)$$

Since $\mathrm{Im}[\overline{P_{\leq k+9}u}\,\partial_x(P_{\leq k+9}u)]$ is invariant under the multiplication operator $u \mapsto e^{-i\gamma(t)}u$,

$$M(t) = \int \phi(x)\,\mathrm{Im}[\overline{P_{\leq k+9}\widetilde{Q}(x) + P_{\leq k+9}\tilde{\epsilon}(t,x)}\,\partial_x(P_{\leq k+9}\widetilde{Q}(x) + P_{\leq k+9}\tilde{\epsilon}(t,x))]\,dx.$$

Since Q is real-valued,

$$\int \phi(x)\,\mathrm{Im}[\overline{P_{\leq k+9}\widetilde{Q}(x)}\,\partial_x(P_{\leq k+9}\widetilde{Q}(x))]\,dx = 0.$$

Next, by Corollary 17,

$$\int \phi(x)\,\mathrm{Im}[\overline{P_{\leq k+9}\tilde{\epsilon}(t,x)}\,\partial_x(P_{\leq k+9}\tilde{\epsilon}(t,x))]\,dx \lesssim \frac{R}{\eta_1^2}\frac{2^{2k}}{T^{99/100}}\int_J \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}\,dt + \frac{R}{\eta_1^2}2^{2k}T^{-9.99}.$$

Next, since $\lambda(t) \geq 1/\eta_1$, Q and $\partial_x Q$ are rapidly decreasing, $\phi(x) = x$ for $|x| \leq R/\eta_1$, and $|\phi_x(x)| \leq 1$,

$$\int \phi(x)\,\mathrm{Im}[\overline{P_{\leq k+9}\tilde{\epsilon}(t,x)}\,\partial_x(P_{\leq k+9}\widetilde{Q}(x))]\,dx = -(\epsilon_2, xQ_x)_{L^2} + O(T^{-10}). \quad (6\text{-}5)$$

Indeed, since Q is real, by rescaling,

$$\int x\,\mathrm{Im}[\overline{\tilde{\epsilon}(t,x)}\,\partial_x(\widetilde{Q}(x))]\,dx = -(\epsilon_2(t), xQ_x)_{L^2}. \quad (6\text{-}6)$$

Next, since $\lambda(t) \leq T^{1/100}/\eta_1$ and $R = T^{1/25}$,

$$\int x\,\mathrm{Im}[\overline{\tilde{\epsilon}(t,x)}\,\partial_x(\widetilde{Q}(x))]\,dx - \int \phi(x)\,\mathrm{Im}[\overline{\tilde{\epsilon}(t,x)}\,\partial_x(\widetilde{Q}(x))]\,dx$$
$$\leq \int_{|x|\geq R/\eta_1} \frac{x}{\lambda(t)^{3/2}}\left|Q_x\left(\frac{x}{\lambda(t)}\right)\right|\left|\frac{1}{\lambda(t)^{1/2}}\epsilon\left(t,\frac{x}{\lambda(t)}\right)\right|dx \lesssim T^{-10}. \quad (6\text{-}7)$$

Also, since Q and all its derivatives are rapidly decreasing, $\lambda(t) \geq 1/\eta_1$, $R = T^{1/25}$, and $2^{3k} = \eta_1^{-2}T$,

$$\int \phi(x)\,\mathrm{Im}[\overline{\tilde{\epsilon}(t,x)}\,\partial_x(\widetilde{Q}(x))]\,dx - \int \phi(x)\,\mathrm{Im}[\overline{\tilde{\epsilon}(t,x)}\,\partial_x(P_{\leq k+9}\widetilde{Q}(x))]\,dx$$
$$\lesssim R\|\epsilon\|_{L^2}\|P_{\geq k+9}\widetilde{Q}_x(x)\|_{L^2} \lesssim T^{-10}. \quad (6\text{-}8)$$

Next, (6-3) implies that $|\phi^{(j)}(x)| \lesssim 1$ for any $j \geq 1$, and since Q is smooth and all its derivatives are rapidly decreasing, integrating by parts, for j sufficiently large, yields

$$\int \phi(x)\,\mathrm{Im}[\overline{P_{\geq k+9}\tilde{\epsilon}(t,x)}\,\partial_x(P_{\leq k+9}\widetilde{Q}(x))]\,dx$$
$$= \int \phi(x)\,\mathrm{Im}\left[\overline{\frac{\Delta^j}{\Delta^j}P_{\geq k+9}\tilde{\epsilon}(t,x)}\,\partial_x(P_{\leq k+9}\widetilde{Q}(x))\right]dx \lesssim T^{-10}, \quad (6\text{-}9)$$

so (6-6)–(6-9) imply (6-5). Finally,

$$\int \phi(x)\,\mathrm{Im}[\overline{P_{\leq k+9}\widetilde{Q}(x)}\partial_x(P_{\leq k+9}\tilde{\epsilon}(t,x))]\,dx = \text{(6-5)} - \int \chi\Big(\frac{\eta_1 x}{R}\Big)\,\mathrm{Im}[\overline{P_{\leq k+9}\widetilde{Q}(x)}\cdot P_{\leq k+9}\tilde{\epsilon}(t,x)]\,dx.$$

Making an argument similar to (6-6)–(6-9),

$$-\int \chi\Big(\frac{\eta_1 x}{R}\Big)\,\mathrm{Im}[\overline{P_{\leq k+9}\widetilde{Q}(x)}\cdot P_{\leq k+9}\tilde{\epsilon}(t,x)]\,dx = -(\epsilon_2, Q)_{L^2} + O(T^{-10}). \tag{6-10}$$

Therefore,

$$M(b)-M(a) = 2\big(\epsilon_2(a), \tfrac{1}{2}Q + xQ_x\big)_{L^2} - 2\big(\epsilon_2(b), \tfrac{1}{2}Q + xQ_x\big)_{L^2} + O(T^{-10}) \\ + O\bigg(\frac{R}{\eta_1^2}\frac{2^{2k}}{T^{99/100}}\int_J \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}\,dt\bigg) + O\bigg(\frac{R}{\eta_1^2}2^{2k}T^{-9.99}\bigg).$$

Following (4-22),

$$i\partial_t P_{\leq k+9}u + \Delta P_{\leq k+9}u + F(P_{\leq k+9}u) = F(P_{\leq k+9}u) - P_{\leq k+9}F(u) = -\mathcal{N}. \tag{6-11}$$

Plugging in (6-11) and integrating by parts,

$$\begin{aligned}\frac{d}{dt}M(t) &= \int \phi(x)\,\mathrm{Re}[-\overline{P_{\leq k+9}u}_{xx}P_{\leq k+9}u_x + \overline{P_{\leq k+9}u}\,P_{\leq k+9}u_{xxx}] \\ &\quad+ \int \phi(x)\,\mathrm{Re}[-|P_{\leq k+9}u|^4\,\overline{P_{\leq k+9}u}(P_{\leq k+9}u_x) + \overline{P_{\leq k+9}u}\,\partial_x(|P_{\leq k+9}u|^4 P_{\leq k+9}u)] \\ &\quad+ \int \phi(x)\,\mathrm{Re}[\overline{P_{\leq k+9}u}\,\partial_x\mathcal{N}](t,x) - \int \phi(x)\,\mathrm{Re}[\overline{\mathcal{N}}\partial_x P_{\leq k+9}u](t,x) \\ &= 2\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|\partial_x P_{\leq k+9}u|^2\,dx - \frac{\eta_1^2}{2R^2}\int \chi''\Big(\frac{\eta_1 x}{R}\Big)|P_{\leq k+9}u|^2\,dx \\ &\quad- \frac{2}{3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\leq k+9}u|^6\,dx + \int \phi(x)\,\mathrm{Re}[\overline{P_{\leq k+9}u}\,\partial_x\mathcal{N}](t,x) \\ &\quad- \int \phi(x)\,\mathrm{Re}[\overline{\mathcal{N}}\partial_x P_{\leq k+9}u](t,x).\end{aligned} \tag{6-12}$$

Next, following (4-28),

$$\begin{aligned}\mathcal{N} &= P_{\leq k+9}(3|u_{\leq k}|^4 u_{\geq k+6} + 2|u_{\leq k}|^2(u_{\leq k})^2\bar{u}_{\geq k+6}) \\ &\quad- (3|u_{\leq k}|^4 u_{\geq k+6} + 2|u_{\leq k}|^2(u_{\leq k})^2\,\overline{P_{k+6\leq\cdot\leq k+9}u}) + P_{\leq k+9}O((u_{\geq k})(u_{\geq k+6})u^3) \\ &\quad+ O((P_{k+6\leq\cdot\leq k+9}u)(P_{k\leq\cdot\leq k+9}u)u^3) \\ &= \mathcal{N}^{(1)} + \mathcal{N}^{(2)}.\end{aligned}$$

As in (4-29), since $|\phi(x)| \lesssim \eta_1^{-1}R$, by Theorems 13 and 14,

$$\int_a^b\!\!\int \phi(x)\,\mathrm{Re}[\overline{P_{\leq k+9}u}\,\partial_x\mathcal{N}^{(2)}]\,dx\,dt - \int_a^b\!\!\int \phi(x)\,\mathrm{Re}[\overline{\mathcal{N}}^{(2)}\partial_x P_{\leq k+9}u]\,dx\,dt \\ \lesssim \frac{2^k R\eta_1^{-1}}{T}\int \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}\,dt + \frac{2^k R\eta_1^{-1}}{T^{10}}. \tag{6-13}$$

Making calculations identical to the estimate (6-13),

$$\begin{aligned}\int_a^b\int \phi(x)\,\mathrm{Re}[\overline{P_{k+3\le\cdot\le k+9}u}\,\partial_x\mathcal{N}^{(1)}]\,dx\,dt - \int_a^b\int \phi(x)\,\mathrm{Re}[\overline{\mathcal{N}}^{(1)}\partial_x P_{k+3\le\cdot\le k+9}u]\,dx\,dt \\ \lesssim 2^k R\|(u_{\ge k+6})(u_{\le k})\|_{L^2_{t,x}}\|(u_{\ge k+3})(u_{\le k})\|_{L^2_{t,x}}\|u_{\le k}\|^2_{L^\infty_{t,x}} \\ \lesssim \frac{2^k R\eta_1^{-1}}{T}\int_a^b \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}\,dt + \frac{2^k R\eta_1^{-1}}{T^{10}}.\end{aligned}$$

Finally, using Bernstein's inequality and the integration by parts argument in (4-34),

$$\begin{aligned}\int_0^T\int \phi(x)\,\mathrm{Re}[\overline{P_{\le k+3}u}\,\partial_x\mathcal{N}^{(1)}]\,dx\,dt - \int_0^T\int \phi(x)\,\mathrm{Re}[\overline{\mathcal{N}}^{(1)}\partial_x P_{\le k+3}u]\,dx\,dt \\ \lesssim \|P_{\ge k+6}\phi(x)\|_{L^\infty}\|(P_{\ge k+6}u)(P_{\le k}u)\|_{L^2_{t,x}}\|u_{\le k}\|^4_{L^8_{t,x}} \lesssim \frac{1}{T^{10}}.\end{aligned}$$

Remark. The last estimate follows from the fact that ϕ is smooth and $\int_J \lambda(t)^{-2}\,dt = T$, which by a local well-posedness argument implies

$$\|u\|_{L^6_{t,x}(J\times\mathbb{R})} \lesssim T^{1/6}.$$

Plugging this estimate of the error term back into (6-12),

$$\begin{aligned}2\int_a^b\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|\partial_x P_{\le k+9}u|^2\,dx\,dt - \frac{\eta_1^2}{2R^2}\int_a^b\int \chi''\Big(\frac{\eta_1 x}{R}\Big)|P_{\le k+9}u|^2\,dx\,dt \\ -\frac{2}{3}\int_a^b\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\le k+9}u|^6\,dx\,dt \\ = 2\big(\epsilon_2(a), \tfrac12 Q + xQ_x\big)_{L^2} - 2\big(\epsilon_2(b), \tfrac12 Q + xQ_x\big)_{L^2} \\ + O\bigg(\frac{2^{2k}T^{1/20}}{T}\int_a^b \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}\,dt\bigg) + O\Big(\frac{1}{T^9}\Big).\end{aligned} \tag{6-14}$$

Since Q is a real-valued function,

$$|P_{\le k+9}u|^2 = (P_{\le k+9}\widetilde{Q}(x))^2 + 2P_{\le k+9}\widetilde{Q}(x)\cdot P_{\le k+9}\tilde{\epsilon}_1(t,x) + |P_{\le k+9}\tilde{\epsilon}(t,x)|^2.$$

The support of $\psi''(x)$, the fact that $\lambda(t) \le T^{1/100}/\eta_1$, and (1-8) imply that

$$\frac{\eta_1^2}{R^2}\int \chi''\Big(\frac{\eta_1 x}{R}\Big)\widetilde{Q}(x)^2\,dx \lesssim \frac{\eta_1^2}{R^2}\frac{1}{T^{11}} \lesssim \frac{1}{\lambda(t)^2}\frac{1}{T^{11}}. \tag{6-15}$$

Also, since Q and all its derivatives are rapidly decreasing and $\lambda(t) \ge 1/\eta_1$,

$$\|P_{\ge k+9}\widetilde{Q}(x)\|^2_{L^2} \lesssim 2^{-30k}. \tag{6-16}$$

Therefore, since $R = T^{1/25}$ and $\lambda(t) \le T^{1/100}/\eta_1$, (6-15), (6-16), and the Cauchy–Schwarz inequality imply

$$\frac{\eta_1^2}{R^2}\int \chi''\Big(\frac{\eta_1 x}{R}\Big)|P_{\le k+9}u(t,x)|^2\,dx \lesssim \frac{1}{\lambda(t)^2}\frac{1}{T^{11}} + \frac{1}{\lambda(t)^2}\frac{1}{R}\|\epsilon\|^2_{L^2}.$$

Next, letting $\epsilon_{1x} + i\epsilon_{2x} = \partial_x \epsilon$, write the decomposition

$$\begin{aligned}
&2\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\leq k+9}u_x|^2\,dx - \frac{2}{3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\leq k+9}u|^6\,dx \\
&= \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{1}{2}P_{\leq k+9}Q_x\Big(\frac{x}{\lambda(t)}\Big)^2 - \frac{1}{6}P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^6\Big)\,dx \\
&\quad + \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(P_{\leq k+9}Q_x\Big(\frac{x}{\lambda(t)}\Big)P_{\leq k+9}\epsilon_{1x}\Big(t,\frac{x}{\lambda(t)}\Big) - P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^5 P_{\leq k+9}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\Big)\,dx \\
&\quad + \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{1}{2}\Big(P_{\leq k+9}\epsilon_{1x}\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2 - \frac{5}{2}P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^4\Big(P_{\leq k+9}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2\Big)\,dx \\
&\quad + \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{1}{2}\Big(P_{\leq k+9}\epsilon_{2x}\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2 - \frac{1}{2}P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^4\Big(P_{\leq k+9}\epsilon_2\Big(t,\frac{x}{\lambda(t)}\Big)\Big)\Big)^2\,dx \\
&\quad - \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{10}{3}\Big(P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)\Big)^3\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^3 \\
&\qquad\qquad + \frac{5}{2}\Big(P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)\Big)^2\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^4 \\
&\qquad\qquad + \Big(P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)\Big)\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^5 + \frac{1}{6}\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^6\Big)\,dx. \qquad (6\text{-}17)
\end{aligned}$$

Remark. Due to the presence of derivatives in

$$2\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\leq k+9}u_x|^2\,dx - \frac{2}{3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\leq k+9}u|^6\,dx,$$

it is convenient to dispense with the $\widetilde{Q}(x)$ and $\tilde{\epsilon}(t,x)$ notation and return to the Q and ϵ notation. We understand that $P_{\leq k+9}Q(x/\lambda(t))$ denotes the frequency projection after rescaling, not a rescaled projection. A rescaled projection appears in (6-18).

For terms of order ϵ^3 and higher, it is not too important to pay attention to complex conjugates, since these terms will be estimated using Hölder's inequality.

First, using the fact that

$$\tfrac{1}{2}Q_x^2 - \tfrac{1}{6}Q^6 = \tfrac{1}{2}Q^2 - \tfrac{1}{3}Q^6$$

combined with the fact that $1/\eta_1 \leq \lambda \leq T^{1/100}/\eta_1$, $R = T^{1/25}$, and Q is smooth and rapidly decreasing,

$$\begin{aligned}
&\frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{1}{2}Q_x\Big(\frac{x}{\lambda(t)}\Big)^2 - \frac{1}{6}Q\Big(\frac{x}{\lambda(t)}\Big)^6\Big)\,dx \\
&\qquad = \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{1}{2}Q\Big(\frac{x}{\lambda(t)}\Big)^2 - \frac{1}{3}Q\Big(\frac{x}{\lambda(t)}\Big)^6\Big) \lesssim \frac{1}{\lambda(t)^2}\frac{1}{T^{11}}.
\end{aligned}$$

Also, since $\eta_1^{-2}T = 2^{3k}$ and Q and its derivatives are smooth and rapidly decreasing, $\lambda(t) \geq 1/\eta_1$ and Bernstein's inequality implies that

$$\frac{2}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)Q_x\Big(\frac{x}{\lambda(t)}\Big)^2\,dx - \frac{2}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)P_{\leq k+9}Q_x\Big(\frac{x}{\lambda(t)}\Big)^2\,dx \lesssim \frac{1}{\lambda(t)^2}\frac{1}{T^{11}},$$

and

$$\frac{2}{3\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big) Q\Big(\frac{x}{\lambda(t)}\Big)^6 dx - \frac{2}{3\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big) P_{\leq k+9} Q\Big(\frac{x}{\lambda(t)}\Big)^6 dx \lesssim \frac{1}{\lambda(t)^2}\frac{1}{T^{11}}.$$

Therefore,

$$\begin{aligned}
&2\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\leq k+9}u_x|^2\,dx - \frac{2}{3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\leq k+9}u|^6\,dx \\
&= \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(P_{\leq k+9}Q_x\Big(\frac{x}{\lambda(t)}\Big)P_{\leq k+9}\epsilon_{1x}\Big(t,\frac{x}{\lambda(t)}\Big) - P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^5 P_{\leq k+9}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\Big)\,dx \\
&\quad + \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{1}{2}\Big(P_{\leq k+9}\epsilon_{1x}\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2 - \frac{5}{2}P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^4\Big(P_{\leq k+9}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2\Big)\,dx \\
&\quad + \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{1}{2}\Big(P_{\leq k+9}\epsilon_{2x}\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2 - \frac{1}{2}P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^4\Big(P_{\leq k+9}\epsilon_2\Big(t,\frac{x}{\lambda(t)}\Big)\Big)\Big)^2\,dx \\
&\quad - \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{10}{3}\Big(P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)\Big)^3\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^3 \\
&\qquad\qquad + \frac{5}{2}\Big(P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)\Big)^2\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^4 \\
&\qquad\qquad + \Big(P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)\Big)\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^5 + \frac{1}{6}\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^6\Big)\,dx \\
&\qquad\qquad\qquad + O\Big(\frac{1}{\lambda(t)^2}\frac{1}{T^{11}}\Big).
\end{aligned}$$

Integrating by parts,

$$\begin{aligned}
&\frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(P_{\leq k+5}Q_x\Big(\frac{x}{\lambda(t)}\Big)P_{\leq k+5}\epsilon_{1x}\Big(t,\frac{x}{\lambda(t)}\Big) - P_{\leq k+5}Q\Big(\frac{x}{\lambda(t)}\Big)^5 P_{\leq k+5}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)\,dx \\
&\quad = -\frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)(P_{\leq k+5}Q_{xx} + P_{\leq k+5}Q^5)\Big(\frac{x}{\lambda(t)}\Big)\cdot P_{\leq k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\,dx \\
&\qquad - \frac{8\eta_1}{R\lambda(t)^2}\int \psi\Big(\frac{\eta_1 x}{R}\Big)\psi'\Big(\frac{\eta_1 x}{R}\Big)P_{\leq k+5}Q_x\Big(\frac{x}{\lambda(t)}\Big)P_{\leq k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\,dx \\
&\qquad + \frac{4}{\lambda(t)^3}\int \psi\Big(\frac{\eta_1 x}{R}\Big)^2 P_{\leq k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)[(P_{\leq k+5}Q)^5 - P_{\leq k+5}Q^5]\Big(\frac{x}{\lambda(t)}\Big)\,dx.
\end{aligned}$$

Again using (1-8), $1/\eta_1 \leq \lambda(t) \leq T^{1/100}/\eta_1$, and the support of $\psi'(x)$,

$$\frac{8\eta_1}{R\lambda(t)^2}\int \psi\Big(\frac{\eta_1 x}{R}\Big)\psi'\Big(\frac{\eta_1 x}{R}\Big)Q_x\Big(\frac{x}{\lambda(t)}\Big)P_{\leq k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\,dx \lesssim \frac{1}{T^6}\frac{1}{\lambda(t)^2}\|\epsilon\|_{L^2}.$$

Also, since Q and all its derivatives are rapidly decreasing, by Bernstein's inequality,

$$\frac{8\eta_1}{R\lambda(t)^2}\int \psi\Big(\frac{\eta_1 x}{R}\Big)\psi'\Big(\frac{\eta_1 x}{R}\Big)P_{\geq k+5}Q_x\Big(\frac{x}{\lambda(t)}\Big)P_{\leq k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\,dx \lesssim \frac{1}{T^6\lambda(t)^2}\|\epsilon\|_{L^2},$$

and

$$\frac{4}{\lambda(t)^3}\int \psi\Big(\frac{\eta_1 x}{R}\Big)^2 P_{\leq k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)[(P_{\leq k+5}Q)^5 - P_{\leq k+5}Q^5]\Big(\frac{x}{\lambda(t)}\Big)\,dx \lesssim \frac{1}{T^6\lambda(t)^2}\|\epsilon\|_{L^2}.$$

Meanwhile, by conservation of mass, (1-8), (1-10), the upper and lower bounds of $\lambda(t)$, and the fact that Q and all its derivatives are rapidly decreasing,

$$\begin{aligned}
&-\frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big) P_{\leq k+5}(Q_{xx}+Q^5)\Big(\frac{x}{\lambda(t)}\Big)\cdot P_{\leq k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\,dx\\
&\quad= -\frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)(Q_{xx}+Q^5)\Big(\frac{x}{\lambda(t)}\Big)\cdot P_{\leq k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\,dx + O\Big(\frac{1}{\lambda(t)^2T^6}\|\epsilon\|_{L^2}\Big)\\
&\quad= -\frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)Q\Big(\frac{x}{\lambda(t)}\Big)\cdot P_{\leq k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\,dx + O\Big(\frac{1}{\lambda(t)^2T^6}\|\epsilon\|_{L^2}\Big)\\
&\quad= -\frac{4}{\lambda(t)^3}\int Q\Big(\frac{x}{\lambda(t)}\Big)\cdot P_{\leq k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\,dx + O\Big(\frac{1}{T^6\lambda(t)^2}\|\epsilon\|_{L^2}\Big)\\
&\quad= -\frac{4}{\lambda(t)^3}\int Q\Big(\frac{x}{\lambda(t)}\Big)\cdot \epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\,dx + O\Big(\frac{1}{T^6\lambda(t)^2}\|\epsilon\|_{L^2}\Big) = \frac{2}{\lambda(t)^2}\|\epsilon\|_{L^2}^2 + O\Big(\frac{1}{T^6\lambda(t)^2}\|\epsilon\|_{L^2}\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&2\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\leq k+9}u_x|^2\,dx - \frac{2}{3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\leq k+9}u|^6\,dx\\
&= \frac{2}{\lambda(t)^2}\|\epsilon\|_{L^2}^2\\
&\quad+\frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{1}{2}\Big(P_{\leq k+9}\epsilon_{1x}\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2 - \frac{5}{2}P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^4\Big(P_{\leq k+9}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2\Big)\,dx\\
&\quad+\frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{1}{2}\Big(P_{\leq k+9}\epsilon_{2x}\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2 - \frac{1}{2}P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^4\Big(P_{\leq k+9}\epsilon_2\Big(t,\frac{x}{\lambda(t)}\Big)\Big)\Big)^2\,dx\\
&\quad-\frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{10}{3}\Big(P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)\Big)^3\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^3\\
&\qquad\qquad+\frac{5}{2}\Big(P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)\Big)^2\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^4\\
&\qquad\qquad+\Big(P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)\Big)\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^5 + \frac{1}{6}\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^6\Big)\,dx\\
&\qquad\qquad+O\Big(\frac{1}{\lambda(t)^2}\frac{1}{T^{11}}\Big) + O\Big(\frac{1}{R\lambda(t)^2}\|\epsilon\|_{L^2}^2\Big).
\end{aligned}$$

Next, by Bernstein's inequality, since $1/\eta_1 \leq \lambda(t) \leq T^{1/100}/\eta_1$,

$$\begin{aligned}
&\frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{1}{2}\Big(P_{\leq k+9}\epsilon_{1x}\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2 - \frac{5}{2}P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^4\Big(P_{\leq k+9}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2\Big)\,dx\\
&\quad= \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{1}{2}\Big(P_{\leq k+9}\epsilon_{1x}\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2 - \frac{5}{2}Q\Big(\frac{x}{\lambda(t)}\Big)^4\Big(P_{\leq k+9}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2\Big)\,dx\\
&\qquad\qquad+O\Big(\frac{1}{\lambda(t)^2R}\|\epsilon\|_{L^2}^2\Big).
\end{aligned}$$

Taking $k(t) \in \mathbb{R}$ that satisfies $2^{k(t)} = \lambda(t)$ and rescaling,

$$\frac{4}{\lambda(t)^3}\int \psi^2\left(\frac{\eta_1 x}{R}\right)\left(\frac{1}{2}\left(P_{\leq k+9}\epsilon_{1x}\left(t,\frac{x}{\lambda(t)}\right)\right)^2 - \frac{5}{2}Q\left(\frac{x}{\lambda(t)}\right)^4\left(P_{\leq k+9}\epsilon_1\left(t,\frac{x}{\lambda(t)}\right)\right)^2\right)dx$$
$$= \frac{4}{\lambda(t)^2}\int \psi^2\left(\frac{\eta_1\lambda(t)x}{R}\right)\left(\frac{1}{2}(P_{\leq k+9+k(t)}\epsilon_{1x}(t,x))^2 - \frac{5}{2}Q(x)^4(P_{\leq k+9+k(t)}\epsilon_1(t,x))^2\right)dx.$$

Integrating by parts,

$$\frac{4}{\lambda(t)^2}\int \psi^2\left(\frac{\eta_1\lambda(t)x}{R}\right)\left(\frac{1}{2}(P_{\leq k+9+k(t)}\epsilon_1(t,x))_x^2 - \frac{5}{2}Q(x)^4(P_{\leq k+9+k(t)}\epsilon_1(t,x))^2\right)dx$$
$$= \frac{2}{\lambda(t)^2}\left\|\psi\left(\frac{\eta_1\lambda(t)x}{R}\right)(P_{\leq k+9+k(t)}\epsilon_1(t,x))\right\|_{\dot{H}^1}^2$$
$$- \frac{10}{\lambda(t)^2}\int \psi\left(\frac{\eta_1\lambda(t)x}{R}\right)^2 Q\left(\frac{x}{\lambda(t)}\right)^4(P_{\leq k+9+k(t)}\epsilon_1(t,x))^2\,dx + O\left(\frac{\eta_1^2}{R^2}\|\epsilon\|_{L^2}^2\right)$$
$$= \frac{2}{\lambda(t)^2}(\mathcal{L}\tilde{\epsilon},\tilde{\epsilon}) - \frac{2}{\lambda(t)^2}\|\tilde{\epsilon}\|_{L^2}^2 + O\left(\frac{1}{R\lambda(t)^2}\|\epsilon\|_{L^2}^2\right), \tag{6-18}$$

where $\mathcal{L}$ is given in (3-12) and

$$\tilde{\epsilon} = \psi\left(\frac{\eta_1\lambda(t)x}{R}\right)(P_{\leq k+9+k(t)}\epsilon_1(t,x)).$$

Remark. This $\tilde{\epsilon}$ is not the same as the $\tilde{\epsilon}$ in (6-4).

For a function $u \perp Q^3$ and $u \perp Q_x$, by the spectral properties of $\mathcal{L}$,

$$(\mathcal{L}u,u)_{L^2} - (u,u)_{L^2} \geq 0.$$

For a general $u \in L^2$,

$$u = a_1 Q^3 + a_2 Q_x + u^{\perp},$$

where $u^{\perp} \perp Q^3$ and $u^{\perp} \perp Q_x$, we have

$$(\mathcal{L}u,u)_{L^2} - (u,u)_{L^2} \geq -O(a_1^2) - O(a_2^2).$$

Since $\epsilon_1 \perp Q^3$ and $\epsilon_1 \perp Q_x$, by Bernstein's inequality and the fact that $1/\eta_1 \leq \lambda(t) \leq T^{1/100}/\eta_1$,

$$(\tilde{\epsilon},Q^3)_{L^2} = (\epsilon_1,Q^3)_{L^2} - \left(\left(1-\psi\left(\frac{x\eta_1\lambda(t)}{R}\right)\right)\epsilon_1, Q^3\right)_{L^2} - \left(\psi\left(\frac{\eta_1\lambda(t)x}{R}\right)P_{\geq k+9+k(t)}\epsilon_1, Q^3\right)_{L^2}$$
$$\lesssim \frac{1}{R}\|\epsilon\|_{L^2}$$

and

$$(\tilde{\epsilon},Q_x)_{L^2} = (\epsilon_1,Q_x)_{L^2} - \left(\left(1-\psi\left(\frac{x\eta_1\lambda(t)}{R}\right)\right)\epsilon_1, Q_x\right)_{L^2} - \left(\psi\left(\frac{\eta_1\lambda(t)x}{R}\right)P_{\geq k+9+k(t)}\epsilon_1, Q_x\right)_{L^2}$$
$$\lesssim \frac{1}{R}\|\epsilon\|_{L^2}.$$

Therefore, for some $0 \ll \delta < 1$ ($\delta = 1/100$ will do), since $|Q(x)|^3 \leq 3$,

$$\frac{2}{\lambda(t)^2}(\mathcal{L}\tilde{\epsilon}, \tilde{\epsilon}) - \frac{2}{\lambda(t)^2}\|\tilde{\epsilon}\|_{L^2}^2 + O\left(\frac{1}{R\lambda(t)^2}\|\epsilon\|_{L^2}^2\right) \geq \frac{\delta}{\lambda(t)^3}\left\|\psi\left(\frac{\eta_1 x}{R}\right)\left(P_{\leq k+9}\epsilon_{1x}\left(t, \frac{x}{\lambda(t)}\right)\right)\right\|_{L^2}^2 - O\left(\frac{1}{R}\frac{1}{\lambda(t)^2}\|\epsilon\|_{L^2}^2\right) - \frac{15\delta}{\lambda(t)^2}\|\epsilon_1\|_{L^2}^2.$$

Likewise, since $\epsilon \perp iQ^3$ and $\epsilon \perp iQ_x$,

$$\frac{4}{\lambda(t)^3}\int \psi^2\left(\frac{\eta_1 x}{R}\right)\left(\frac{1}{2}\left(P_{\leq k+9}\epsilon_{2x}\left(t, \frac{x}{\lambda(t)}\right)\right)^2 - \frac{1}{2}P_{\leq k+9}Q\left(\frac{x}{\lambda(t)}\right)^4\left(P_{\leq k+9}\epsilon_2\left(t, \frac{x}{\lambda(t)}\right)\right)^2\right)dx \geq \frac{\delta}{\lambda(t)^3}\left\|\psi\left(\frac{\eta_1 x}{R}\right)\left(P_{\leq k+9}\epsilon_{2x}\left(t, \frac{x}{\lambda(t)}\right)\right)\right\|_{L^2}^2 - O\left(\frac{1}{R}\frac{1}{\lambda(t)^2}\|\epsilon\|_{L^2}^2\right) - \frac{15\delta}{\lambda(t)^2}\|\epsilon_2\|_{L^2}^2.$$

Therefore,

$$\begin{aligned}
&2\int \psi^2\left(\frac{\eta_1 x}{R}\right)|P_{\leq k+9}u_x|^2\,dx - \frac{2}{3}\int \psi^2\left(\frac{\eta_1 x}{R}\right)|P_{\leq k+9}u|^6\,dx \\
&\geq \frac{3}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2 + \frac{\delta}{\lambda(t)^3}\left\|\psi\left(\frac{\eta_1 x}{R}\right)P_{\leq k+9}\epsilon_x\left(t, \frac{x}{\lambda(t)}\right)\right\|_{L^2}^2 \\
&\quad - \frac{4}{\lambda(t)^3}\int \psi^2\left(\frac{\eta_1 x}{R}\right)\left(\frac{10}{3}P_{\leq k+9}Q\left(\frac{x}{\lambda(t)}\right)^3 P_{\leq k+9}\epsilon\left(t, \frac{x}{\lambda(t)}\right)^3\right. \\
&\qquad\qquad\left. + \frac{5}{2}P_{\leq k+9}Q\left(\frac{x}{\lambda(t)}\right)^2 P_{\leq k+9}\epsilon\left(t, \frac{x}{\lambda(t)}\right)^4\right)dx \\
&\quad + \frac{1}{\lambda(t)^3}\int\left(P_{\leq k+9}Q\left(\frac{x}{\lambda(t)}\right)P_{\leq k+9}\epsilon\left(t, \frac{x}{\lambda(t)}\right)^5 + \frac{1}{6}P_{\leq k+9}\epsilon\left(t, \frac{x}{\lambda(t)}\right)^6\right)dx \\
&\qquad\qquad - O\left(\frac{1}{\lambda(t)^2 T^{11}}\right) - O\left(\frac{1}{R\lambda(t)^2}\|\epsilon\|_{L^2}^2\right).
\end{aligned}$$

Now, by the fundamental theorem of calculus and the product rule, for any $x \in \mathbb{R}$,

$$\begin{aligned}
\frac{1}{\lambda(t)}\psi\left(\frac{\eta_1 x}{R}\right)\left|P_{\leq k+9}\epsilon\left(t, \frac{x}{\lambda(t)}\right)\right|^2 &\lesssim \frac{1}{\lambda(t)}\int\left|\partial_x\left(\psi\left(\frac{\eta_1 x}{R}\right)\left|P_{\leq k+9}\epsilon\left(t, \frac{x}{\lambda(t)}\right)\right|^2\right)\right|dx \\
&\lesssim \frac{1}{\lambda(t)^{3/2}}\|\epsilon\|_{L^2(\mathbb{R})}\left\|\psi\left(\frac{\eta_1 x}{R}\right)P_{\leq k+9}\epsilon_x\left(t, \frac{x}{\lambda(t)}\right)\right\|_{L^2} + \frac{\eta_1}{R}\|\epsilon\|_{L^2}^2.
\end{aligned}$$

Therefore, by Hölder's inequality, the fact that $\|\epsilon\|_{L^2} \leq \eta_*$, the fact that $1/\eta_1 \leq \lambda(t) \leq T^{1/100}/\eta_1$, and $R = T^{1/25}$,

$$\begin{aligned}
\frac{1}{\lambda(t)^3}\int \psi\left(\frac{\eta_1 x}{R}\right)^2\left|P_{\leq k+9}\epsilon\left(t, \frac{x}{\lambda(t)}\right)\right|^6 dx &\lesssim \frac{1}{\lambda(t)^3}\left\|\psi\left(\frac{\eta_1 x}{R}\right)P_{\leq k+9}\epsilon_x\left(t, \frac{x}{\lambda(t)}\right)\right\|_{L^2}^2\|\epsilon\|_{L^2}^4 + \frac{\eta_1^2}{R^2}\|\epsilon\|_{L^2}^6 \\
&\lesssim \frac{\eta_*^4}{\lambda(t)^3}\left\|\psi\left(\frac{\eta_1 x}{R}\right)P_{\leq k+9}\epsilon_x\left(t, \frac{x}{\lambda(t)}\right)\right\|_{L^2}^2 + \frac{\eta_*^4}{R\lambda(t)^2}\|\epsilon\|_{L^2}^2.
\end{aligned}$$

Next, by Hölder's inequality and the Cauchy–Schwarz inequality, for $j = 3, 4, 5$,

$$\begin{aligned}\frac{1}{\lambda(t)^3}\int \psi\Big(\frac{\eta_1 x}{R}\Big)^2 \Big|P_{\le k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big|^j Q\Big(\frac{x}{\lambda(t)}\Big)^{6-j}\,dx \\ \lesssim \frac{1}{\lambda(t)^{(6-j)/2}}\bigg(\frac{1}{\lambda(t)^3}\int \psi\Big(\frac{\eta_1 x}{R}\Big)^2 \Big|P_{\le k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big|^6\,dx\bigg)^{(j-2)/4}\|\epsilon\|_{L^2}^{(6-j)/2} \\ \lesssim \frac{\eta_*}{\lambda(t)^2}\|\epsilon\|_{L^2}^2 + \frac{\eta_*}{\lambda(t)^3}\Big\|\psi\Big(\frac{\eta_1 x}{R}\Big)P_{\le k+9}\epsilon_x\Big(t,\frac{x}{\lambda(t)}\Big)\Big\|_{L^2}^2.\end{aligned}$$

Therefore, for $\eta_* \ll \delta$ sufficiently small and T sufficiently large,

$$2\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\le k+9}u_x|^2\,dx - \frac{2}{3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\le k+9}u|^6\,dx \ge \frac{1}{\lambda(t)^2}\|\epsilon\|_{L^2}^2 - O\Big(\frac{1}{\lambda(t)^2T^{11}}\Big). \tag{6-19}$$

Plugging (6-19) into (6-14), integrating in time, and using the fact that $2^{3k} = \eta_1^{-2}T$ for $T(\eta_1)$ sufficiently large, the term

$$O\bigg(\frac{2^{2k}T^{1/20}}{T}\int_a^b \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}\,dt\bigg)$$

can be absorbed into the integral of the first term on the right-hand side of (6-19). Since

$$\int_J \lambda(t)^{-2}\,dt = T,$$

the proof of Theorem 18 is complete. □

Since both the left and right-hand sides of (6-2) are scale invariant, the same argument also holds for an interval J where

$$A \le \lambda(t) \le AT^{1/100} \tag{6-20}$$

for any $A > 0$.

Corollary 19. *Let $J = [a, b]$ be an interval where* (6-1) *holds for some T sufficiently large and* (6-20) *also holds. Then* (6-2) *holds.*

7. An L_s^p bound on $\|\epsilon(s)\|_{L^2}$ when $p > 1$

Transitioning to s variables, under the change of variables (3-10), Theorem 18 and Corollary 19 imply that if $[a, a+T] \subset [0,\infty)$ is an interval on which

$$\frac{\sup_{s\in[a,a+T]}\lambda(s)}{\inf_{s\in[a,a+T]}\lambda(s)} \le T^{1/100},$$

then

$$\int_a^{a+T}\|\epsilon(s)\|_{L^2}^2\,ds \le 3\big(\epsilon(a), \tfrac{1}{2}Q + xQ_x\big)_{L^2} - 3\big(\epsilon(a+T), \tfrac{1}{2}Q + xQ_x\big)_{L^2} + O\Big(\frac{1}{T^9}\Big).$$

Theorem 18 implies good L_s^p integrability bounds on $\|\epsilon(s)\|_{L^2}$ under (2-8), which is equivalent to

$$\sup_{s\in[0,\infty)}\|\epsilon(s)\|_{L^2} \le \eta_*.$$

Theorem 20. *Let u be a symmetric solution to* (1-1) *that satisfies $\|u\|_{L^2} = \|Q\|_{L^2}$, and suppose*

$$\sup_{s \in [0,\infty)} \|\epsilon(s)\|_{L^2} \leq \eta_* \tag{7-1}$$

and $\|\epsilon(0)\|_{L^2} = \eta_*$. Then

$$\int_0^\infty \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim \eta_*, \tag{7-2}$$

with implicit constant independent of η_* when $\eta_* \ll 1$ is sufficiently small.

Furthermore, for any $j \in \mathbb{Z}_{\geq 0}$, let

$$s_j = \inf\{s \in [0,\infty) : \|\epsilon(s)\|_{L^2} = 2^{-j}\eta_*\}.$$

By definition, $s_0 = 0$, and the continuity of $\|\epsilon(s)\|_{L^2}$ combined with Theorem 8 implies that such an s_j exists for any $j > 0$. Then,

$$\int_{s_j}^\infty \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim 2^{-j}\eta_* \tag{7-3}$$

for each j, with implicit constant independent of η_* and $j \geq 0$.

Proof. Set $T_* = 1/\eta_*$ and suppose that T_* is large enough that Theorem 18 holds. Then by (3-15) and (7-1), for any $s' \geq 0$,

$$\Big| \sup_{s \in [s', s'+T_*]} \ln \lambda(s) - \inf_{s \in [s', s'+T_*]} \ln \lambda(s) \Big| \lesssim 1, \tag{7-4}$$

with implicit constant independent of $s' \geq 0$. Let J be the largest dyadic integer that satisfies

$$J = 2^{j_*} \leq -\ln \eta_*^{1/2}.$$

By (7-4) and the triangle inequality,

$$\Big| \sup_{s \in [s', s'+JT_*]} \ln \lambda(s) - \inf_{s \in [s', s'+JT_*]} \ln \lambda(s) \Big| \lesssim J,$$

and therefore

$$\frac{\sup_{s \in [s', s'+3JT^*]} \lambda(s)}{\inf_{s \in [s', s'+3JT^*]} \lambda(s)} \lesssim T_*^{1/100}. \tag{7-5}$$

Therefore, Theorem 18 may be utilized on $[s', s'+JT_*]$. In particular, for any $s' \geq 0$,

$$\int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim \|\epsilon(s')\|_{L^2} + \|\epsilon(s'+JT_*)\|_{L^2} + O\Big(\frac{1}{J^9 T_*^9}\Big). \tag{7-6}$$

In fact, if $s' > JT_*$, then by (7-5),

$$\int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim \inf_{s \in [s'-JT_*, s']} \|\epsilon(s)\|_{L^2} + \inf_{s \in [s'+JT_*, s'+2JT_*]} \|\epsilon(s)\|_{L^2} + O\Big(\frac{1}{J^9 T_*^9}\Big). \tag{7-7}$$

In particular, for a fixed $s' \geq 0$,

$$\sup_{a>0} \int_{s'+aJT_*}^{s'+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 \lesssim \frac{1}{J^{1/2}T_*^{1/2}} \left(\sup_{a\geq 0} \int_{s'+aJT_*}^{s'+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \right)^{1/2} + O\left(\frac{1}{J^9 T_*^9}\right). \quad (7\text{-}8)$$

Meanwhile, when $a = 0$,

$$\int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2}^2 \lesssim \|\epsilon(s')\|_{L^2} + \frac{1}{J^{1/2}T_*^{1/2}} \left(\sup_{a\geq 0} \int_{s'+aJT_*}^{s'+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \right)^{1/2} + O\left(\frac{1}{J^9 T_*^9}\right). \quad (7\text{-}9)$$

Therefore, taking $s' = s_{j_*}$,

$$\sup_{a\geq 0} \int_{s_{j_*}+aJT_*}^{s_{j_*}+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim 2^{-j_*}\eta_* + O(2^{-9j_*}\eta_*^9). \quad (7\text{-}10)$$

Then by the triangle inequality,

$$\sup_{s'\geq s_{j_*}} \int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim 2^{-j_*}\eta_*,$$

and by Hölder's inequality,

$$\sup_{s'\geq s_{j_*}} \int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2} \, ds \lesssim 1.$$

In fact, arguing by induction, there exists a constant $C < \infty$ such that

$$\sup_{s'\geq s_{nj_*}} \int_{s'}^{s'+J^nT_*} \|\epsilon(s)\|_{L^2} \, ds \leq C \quad (7\text{-}11)$$

for some $n > 0$ implies that

$$\sup_{s'\geq s_{(n+1)j_*}} \int_{s'}^{s'+J^{n+1}T_*} \|\epsilon(s)\|_{L^2}^2 \, ds \leq CJ^{-(n+1)}T_*^{-1}, \quad (7\text{-}12)$$

and by Hölder's inequality,

$$\sup_{s'\geq s_{(n+1)j_*}} \int_{s'}^{s'+J^{n+1}T_*} \|\epsilon(s)\|_{L^2} \, ds \leq C^{1/2}.$$

Therefore, (7-11) holds for any integer $n > 0$.

Now take any $j \in \mathbb{Z}$ and suppose $nj_* < j \leq (n+1)j_*$. Then by (7-11),

$$\sup_{a\geq 0} \int_{s_j+aJ^{n+1}T_*}^{s_j+(a+1)J^{n+1}T_*} \|\epsilon(s)\|_{L^2} \, ds \lesssim J.$$

Therefore, as in (7-10),

$$\sup_{a\geq 0} \int_{s_j+aJ^{n+1}T_*}^{s_j+(a+1)J^{n+1}T_*} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim 2^{-j}\eta_*,$$

and therefore by Hölder's inequality, for any $s' \geq s_j$,

$$\sup_{s' \geq s_j} \int_{s'}^{s'+2^j T_*} \|\epsilon(s)\|_{L^2} \, ds \lesssim 1,$$

with bound independent of j. Then by the triangle inequality, (7-5) holds for the interval $[s', s' + 3 \cdot 2^j J T_*]$, and by (7-6)–(7-9),

$$\int_{s_j}^{s_j + 2^j J T_*} \|\epsilon(s)\|_{L^2}^2 \lesssim 2^{-j} \eta_*, \tag{7-13}$$

and therefore, by the mean value theorem,

$$\inf_{s \in [s_j, s_j + 2^j J T_*]} \|\epsilon(s)\|_{L^2} \lesssim 2^{-j} \eta_* J^{-1/2},$$

which implies

$$s_{j+1} \in [s_j, s_j + 2^j J T_*].$$

Therefore, by (7-13) and Hölder's inequality,

$$\int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim 2^{-j} \eta_* \quad \text{and} \quad \int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2} \, ds \lesssim 1, \tag{7-14}$$

with constant independent of j. Summing in j gives (7-2) and (7-3). □

Now, by (2-18),

$$\|\epsilon(s')\|_{L^2} \sim \|\epsilon(s)\|_{L^2}$$

for any $s' \in [s, s+1]$, so (7-2) implies

$$\lim_{s \to \infty} \|\epsilon(s)\|_{L^2} = 0.$$

Next, by definition of s_j, (7-14) implies

$$\int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2} \, ds \lesssim 1,$$

and, for any $1 < p < \infty$,

$$\int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2}^p \, ds \lesssim \eta_*^{p-1} 2^{-j(p-1)}, \tag{7-15}$$

which implies that $\|\epsilon(s)\|_{L^2}$ belongs to L_s^p for any $p > 1$ but not to L_s^1.

Comparing (7-15) to the pseudoconformal transformation of the soliton, (1-11), for $0 < t < 1$,

$$\lambda(t) \sim t \quad \text{and} \quad \|\epsilon(t)\|_{L^2} \sim t,$$

so

$$\int_0^1 \|\epsilon(t)\|_{L^2} \lambda(t)^{-2} \, dt = \infty,$$

but for any $p > 1$,

$$\int_0^1 \|\epsilon(t)\|_{L^2}^p \lambda(t)^{-2} \, dt < \infty.$$

For the soliton, $\epsilon(s) \equiv 0$ for any $s \in \mathbb{R}$, so obviously $\epsilon \in L_s^p$ for $1 \leq p \leq \infty$.

8. Monotonicity of λ

Next, using a virial identity from [Merle and Raphael 2005], it is possible to show that $\lambda(s)$ is an approximately monotone decreasing function.

Theorem 21. *For any $s \geq 0$, let*

$$\tilde{\lambda}(s) = \inf_{\tau \in [0, s]} \lambda(\tau).$$

Then for any $s \geq 0$,

$$1 \leq \frac{\lambda(s)}{\tilde{\lambda}(s)} \leq 3. \tag{8-1}$$

Proof. Suppose there exist $0 \leq s_- \leq s_+ < \infty$ satisfying

$$\frac{\lambda(s_+)}{\lambda(s_-)} = e. \tag{8-2}$$

Then we can show that u is a soliton solution to (1-1), which is a contradiction since $\lambda(s)$ is constant in that case.

The proof that (8-2) implies that u is a soliton uses a virial identity from [Merle and Raphael 2005]. Using (3-11), compute

$$\begin{aligned} &\frac{d}{ds}(\epsilon, y^2 Q) + \frac{\lambda_s}{\lambda}\|yQ\|_{L^2}^2 + 4\big(\tfrac{1}{2}Q + yQ_y, \epsilon_2\big)_{L^2} \\ &\qquad = O(|\gamma_s + 1|\|\epsilon\|_{L^2}) + O\bigg(\bigg|\frac{\lambda_s}{\lambda}\bigg|\|\epsilon\|_{L^2}\bigg) + O(\|\epsilon\|_{L^2}^2) + O(\|\epsilon\|_{L^2}\|\epsilon\|_{L^8}^4). \end{aligned} \tag{8-3}$$

Indeed, by direct computation,

$$\partial_{xx}(x^2 Q) + Q^4(x^2 Q) - x^2 Q = 4\big(\tfrac{1}{2}Q + xQ_x\big).$$

Then by (3-15), (3-16), (7-2), and the fundamental theorem of calculus,

$$\|yQ\|_{L^2}^2 + 4\int_{s_-}^{s_+} \big(\epsilon_2, \tfrac{1}{2}Q + xQ_x\big)_{L^2} = O(\eta_*).$$

Therefore, there exists $s' \in [s_-, s_+]$ such that

$$\big(\epsilon_2, \tfrac{1}{2}Q + xQ_x\big)_{L^2} < 0. \tag{8-4}$$

Since $s' \geq 0$, there exists some $j \geq 0$ such that $s_j \leq s' + T_* < s_{j+1}$. Using the proof of Theorem 20, in particular (7-14),

$$\int_{s'}^{s_{j+1+J}} \bigg|\frac{\lambda_s}{\lambda}\bigg|\, ds \lesssim J. \tag{8-5}$$

Then by Theorem 18, (8-4) implies

$$\int_{s'}^{s_{j+1+J}} \|\epsilon(s)\|_{L^2}^2\, ds \lesssim 2^{-(j+1+J)}\eta_*,$$

and therefore by definition of s_{j+1+J},

$$\int_{s'}^{s_{j+1+J}} \|\epsilon(s)\|_{L^2}\,ds \lesssim 1. \tag{8-6}$$

Then, (8-6) implies that (8-5) holds on the interval $[s', s_{j+1+2J}]$, and arguing by induction, for any $k \geq 1$,

$$\int_{s'}^{s_{j+k}} \|\epsilon(s)\|_{L^2}^2\,ds \lesssim 2^{-j-k}\eta_*$$

and

$$\int_{s'}^{s_{j+k}} \|\epsilon(s)\|_{L^2}\,ds \lesssim 1,$$

with implicit constant independent of k. Taking $k \to \infty$,

$$\int_{s'}^{\infty} \|\epsilon(s)\|_{L^2}^2\,ds = 0,$$

which implies that $\epsilon(s) = 0$ for all $s \geq s'$. Therefore,

$$u_0 = \lambda^{1/2} Q(\lambda x) e^{i\gamma}$$

for some $\gamma \in \mathbb{R}$ and $\lambda > 0$, which proves that u is a soliton solution. □

9. Almost monotone $\lambda(t)$

The almost monotonicity of λ implies that when $\sup(I) = \infty$, u is equal to a soliton solution, and when $\sup(I) < \infty$, u is the pseudoconformal transformation of the soliton solution.

Theorem 22. *If u satisfies the conditions of Theorem 7, u blows up forward in time, and*

$$\sup(I) = \infty,$$

then u is equal to a soliton solution.

Proof. For any integer $k \geq 0$, let

$$I(k) = \{s \geq 0 : 2^{-k+2} \leq \tilde{\lambda}(s) \leq 2^{-k+3}\}. \tag{9-1}$$

Then by (8-1),

$$2^{-k} \leq \lambda(s) \leq 2^{-k+3} \tag{9-2}$$

for all $s \in I(k)$. By (3-10), the fact that $\sup(I) = \infty$ implies that

$$\sum 2^{-2k} |I(k)| = \infty.$$

Therefore, there exists a sequence $k_n \nearrow \infty$ such that

$$|I(k_n)| 2^{-2k_n} \geq \frac{1}{k_n^2}$$

and such that $|I(k_n)| \geq |I(k)|$ for all $k \leq k_n$.

Lemma 23. *For n sufficiently large, there exists $s_n \in I(k_n)$ such that*

$$\|\epsilon(s_n)\|_{L^2} \lesssim k_n^2 2^{-2k_n}.$$

Proof. Let $I(k_n) = [a_n, b_n]$. By Theorem 18, for n sufficiently large,

$$\int_{I(k_n)} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim \eta_* + 2^{-18k_n} k_n^{18} \lesssim \eta_*.$$

Then, using the virial identity in (8-3),

$$\int_{a_n}^{(3a_n+b_n)/4} \big(\epsilon_2, \tfrac{1}{2}Q + xQ_x\big)_{L^2} \, ds = O(\eta_*) + O(1).$$

Therefore, by the mean value theorem, there exists $s_n^- \in \big[a_n, \frac{1}{4}(3a_n + b_n)\big]$ such that

$$\big|\big(\epsilon_2(s_n^-), \tfrac{1}{2}Q + xQ_x\big)_{L^2}\big| \lesssim \frac{1}{|I(k_n)|}. \tag{9-3}$$

By a similar calculation, there exists $s_n^+ \in \big[\frac{1}{4}(a_n + 3b_n), b_n\big]$ such that

$$\big|\big(\epsilon_2(s_n^+), \tfrac{1}{2}Q + xQ_x\big)_{L^2}\big| \lesssim \frac{1}{|I(k_n)|}. \tag{9-4}$$

Plugging (9-3) and (9-4) into Theorem 18,

$$\int_{s_n^-}^{s_n^+} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim \frac{1}{|I(k_n)|}.$$

Then by the mean value theorem there exists $s_n \in [s_n^-, s_n^+]$ such that

$$\|\epsilon(s_n)\|_{L^2}^2 \lesssim \frac{1}{|I(k_n)|^2}.$$

Since $|I(k_n)| \geq 2^{2k_n} k_n^{-2}$, the proof of Lemma 23 is complete. □

Returning to the proof of Theorem 22, let m be the smallest integer such that

$$\frac{2^{2k_n}}{k_n^2} 2^m \geq |I(k_n)|. \tag{9-5}$$

Since $|I(k)| \leq |I(k_n)|$ for all $0 \leq k \leq k_n$, (9-5) implies that

$$|s_n| \leq 2^{2k_n+m+1}.$$

Let r_n be the smallest integer that satisfies

$$2^{(2k_n+m+1)/3} 2^{k_n} \frac{1}{\eta_1} \leq 2^{r_n}.$$

Since $\lambda(s) \geq 2^{-k_n}$ for all $s \in [0, s_n]$, setting $t_n = s^{-1}(s_n)$, rescaling so that $\lambda(t) \geq 1/\eta_1$ on $[0, 2^{2k_n}\eta_1^{-2}t_n]$, applying Theorem 18, then rescaling back,

$$\|P_{\geq r_n} u\|_{U^2_\Delta([0,t_n]\times\mathbb{R})} \lesssim \eta_*.$$

Arguing by induction on frequency and using (4-41) and the preceding computations,

$$\|P_{\geq r_n+k_n/4+m/4} u\|_{U^2_\Delta([0,t_n]\times\mathbb{R})} \lesssim k_n^2 2^{-2k_n} 2^{-m}. \tag{9-6}$$

Then using the computations in (5-1)–(5-6),

$$E(P_{\leq r_n+k_n/4+m/4} u(t_n)) \lesssim (k_n^2 2^{-2k_n} 2^{-m} 2^{r_n+k_n/4+m/4})^2 \sim (k_n^2 2^{-k_n/12-5m/12} \eta_1^{-1})^2.$$

Next, following the computations in the proof of Theorem 16 and using (9-6),

$$\sup_{t\in[0,t_n]} E(P_{\leq r_n+k_n/4+m/4} u(t)) \lesssim (k_n^2 2^{-k_n/12-5m/12} \eta_1^{-1})^2.$$

Since $m \geq 0$ for any n, taking $n \to \infty$ implies that $E(u_0) = 0$. Then by the Gagliardo–Nirenberg inequality, u_0 is a soliton. □

It only remains to show that when $\sup(I) < \infty$, u is a pseudoconformal transformation of the soliton. If one could show that the energy of u_0 is finite, then this fact would follow directly from the result of [Merle 1993]. Similarly, if one could generalize the result of that paper to data that need not have finite energy, then the proof would also be complete.

We do not quite prove this fact. Instead, suppose without loss of generality that $\sup(I) = 0$ and

$$\sup_{-1<t<0} \|\epsilon(t)\|_{L^2} \leq \eta_*.$$

Then write the decomposition

$$u(t,x) = \frac{e^{-i\gamma(t)}}{\lambda(t)^{1/2}} Q\left(\frac{x}{\lambda(t)}\right) + \frac{e^{-i\gamma(t)}}{\lambda(t)^{1/2}} \epsilon\left(t, \frac{x}{\lambda(t)}\right)$$

and apply the pseudoconformal transformation to $u(t,x)$. For $-\infty < t < -1$, let

$$\begin{aligned} v(t,x) &= \frac{1}{t^{1/2}} \overline{u\left(\frac{1}{t}, \frac{x}{t}\right)} e^{ix^2/4t} \\ &= \frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)}}{\lambda(1/t)^{1/2}} Q\left(\frac{x}{t\lambda(1/t)}\right) e^{ix^2/4t} + \frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)}}{\lambda(1/t)^{1/2}} \overline{\epsilon\left(\frac{1}{t}, \frac{x}{t\lambda(1/t)}\right)} e^{ix^2/4t}. \end{aligned}$$

Since the L^2 norm is preserved by the pseudoconformal transformation,

$$\lim_{t\searrow-\infty} \left\| \frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)}}{\lambda(1/t)^{1/2}} \overline{\epsilon\left(\frac{1}{t}, \frac{x}{t\lambda(1/t)}\right)} e^{ix^2/4t} \right\|_{L^2} = 0$$

and

$$\sup_{-\infty<t<-1} \left\| \frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)}}{\lambda(1/t)^{1/2}} \overline{\epsilon\left(\frac{1}{t}, \frac{x}{t\lambda(1/t)}\right)} e^{ix^2/4t} \right\|_{L^2} \leq \eta_*.$$

Since

$$\frac{1}{t^{1/2}}\frac{e^{i\gamma(1/t)}}{\lambda(1/t)^{1/2}}Q\biggl(\frac{x}{t\lambda(1/t)}\biggr)$$

is in the form

$$\frac{e^{i\tilde{\gamma}(t)}}{\tilde{\lambda}(t)^{1/2}}Q\biggl(\frac{x}{\tilde{\lambda}(t)}\biggr),$$

it only remains to estimate

$$\biggl\|\frac{1}{t^{1/2}}\frac{e^{i\gamma(1/t)}}{\lambda(1/t)^{1/2}}Q\biggl(\frac{x}{t\lambda(1/t)}\biggr)(e^{ix^2/4t}-1)\biggr\|_{L^2}.$$

Once again take (9-1). As in (9-2), for any $k \geq 0$, we have $\lambda(s) \sim 2^{-k}$ for all $s \in I(k)$. Furthermore, by (3-15), $\|\epsilon(t)\|_{L^2} \to 0$ as $t \nearrow 0$ implies that there exists a sequence $c_k \nearrow \infty$ such that

$$|I(k)| \geq c_k \quad \text{for all } k \geq 0.$$

Then by (3-10), there exists $r(t) \searrow 0$ as $t \nearrow 0$ such that

$$\lambda(t) \leq t^{1/2} r(t), \quad \text{so } \lambda(1/t) \leq t^{-1/2} r(1/t). \tag{9-7}$$

Therefore, since Q is rapidly decreasing,

$$\lim_{t \searrow -\infty}\biggl\|\frac{1}{t^{1/2}\lambda(1/t)^{1/2}}Q\biggl(\frac{x}{t\lambda(1/t)}\biggr)\frac{x^2}{4t}\biggr\|_{L^2} = 0 \tag{9-8}$$

as well as

$$\lim_{t \searrow -\infty}\biggl\|\frac{1}{t^{1/2}\lambda(1/t)^{1/2}}Q\biggl(\frac{x}{t\lambda(1/t)}\biggr)(e^{ix^2/4t}-1)\biggr\|_{L^2} = 0.$$

Therefore, by time reversal symmetry, v satisfies the conditions of Theorem 7, and v is a solution that blows up backward in time at $\inf(I) = -\infty$, so therefore, by Theorem 22, v must be a soliton. In particular,

$$v(t,x) = e^{i\lambda^2 t}e^{i\theta}\lambda^{1/2}Q(\lambda x) = \frac{1}{t^{1/2}}\overline{u\biggl(\frac{1}{t},\frac{x}{t}\biggr)}e^{ix^2/4t}.$$

Doing some algebra,

$$\overline{u\biggl(\frac{1}{t},\frac{x}{t}\biggr)} = e^{i\lambda^2 t}e^{i\theta}e^{-ix^2/4t}t^{1/2}\lambda^{1/2}Q(\lambda x),$$

so

$$u(t,x) = e^{-i\lambda^2/t}e^{-i\theta}e^{ix^2/4t}\frac{1}{t^{1/2}}\lambda^{1/2}Q\biggl(\frac{\lambda x}{t}\biggr).$$

This is clearly the pseudoconformal transformation of a soliton. This finally completes the proof of Theorem 7.

10. A nonsymmetric solution

When there is no symmetry assumption on u, there is no preferred origin, either in space or in frequency. As a result, two additional group actions on a solution u must be accounted for, translation in space:

$$u(t,x) \mapsto u(t,x-x_0), \quad x_0 \in \mathbb{R}, \tag{10-1}$$

and the Galilean symmetry:

$$e^{-it\xi_0^2} e^{ix\xi_0} u(t,x-2t\xi_0), \quad \xi_0 \in \mathbb{R}. \tag{10-2}$$

This gives a four parameter family of soliton solutions to (1-1), given by (1-14). Making the pseudoconformal transformation of (1-14) gives a solution in the form of (1-15).

In this section we prove Theorem 5, that the only nonsymmetric blowup solutions to (1-1) with mass $\|u_0\|_{L^2}^2 = \|Q\|_{L^2}^2$ belong to the family of solitons and pseudoconformal transformation of a soliton. To prove this, we will go through the proof of Theorem 4 in Sections 2–9, section by section, generalizing each step to the nonsymmetric case. There are several steps for which the argument in the symmetric case has an easy generalization to the nonsymmetric case, after accounting for the additional group actions (10-1) and (10-2). There are other steps for which the nonsymmetric case will require substantially more work.

10.1. *Reductions of a nonsymmetric blowup solution.* Using the same arguments showing that Theorem 4 may be reduced to Theorem 7, Theorem 5 may be reduced to:

Theorem 24. *Let $0 < \eta_* \ll 1$ be a small fixed constant to be defined later. If u is a solution to* (1-1) *on the maximal interval of existence $I \subset \mathbb{R}$, $\|u_0\|_{L^2} = \|Q\|_{L^2}$, u blows up forward in time, and*

$$\sup_{t \in [0,\sup(I))} \inf_{\lambda,\gamma,\xi_0,x_0} \|e^{i\gamma} e^{ix\xi_0} \lambda^{1/2} u(t,\lambda x + x_0) - Q(x)\|_{L^2} \le \eta_*, \tag{10-3}$$

then u is a soliton solution of the form (1-14) *or the pseudoconformal transformation of a soliton of the form* (1-15).

Reducing Theorem 5 to Theorem 24 requires the following generalization of Theorem 8, which was proved in [Dodson 2021, Theorem 2].

Theorem 25. *Assume that u is a solution to* (1-1) *with $\|u_0\|_{L^2} = \|Q\|_{L^2}$ that does not scatter forward in time. Let $(T^-(u), T^+(u))$ be its lifespan ($T^-(u)$ could be $-\infty$ and $T^+(u)$ could be $+\infty$). Then there exists a sequence $t_n \nearrow T^+(u)$ and a family of parameters $\lambda_n > 0$, $\xi_n \in \mathbb{R}$, $x_n \in \mathbb{R}$, and $\gamma_n \in \mathbb{R}$ such that*

$$\lambda_n^{1/2} e^{ix\xi_n} e^{i\gamma_n} u(t_n, \lambda_n x + x_n) \to Q \quad \text{in } L^2.$$

Lemma 6 can be generalized to the nonsymmetric case, proving that $\|e^{i\gamma} e^{ix\xi_0} \lambda^{1/2} u_0(\lambda x + x_0) - Q\|_{L^2}$ attains its infimum on $\gamma \in \mathbb{R}$, $\xi_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}$, and $\lambda > 0$. Theorem 9 is also easily generalized to the nonsymmetric case, showing that the left-hand side of (10-3) is upper semicontinuous in time and continuous in time when small. Therefore, Theorem 5 is easily reduced to Theorem 24 using the same argument that reduced Theorem 4 to Theorem 7.

10.2. *Decomposition of a nonsymmetric solution near Q.* When a nonsymmetric u is close to a soliton, it is possible to make a decomposition of u, generalizing Theorem 10 to account for the additional group actions in (10-1) and (10-2).

Theorem 26. *Take $u \in L^2$. There exists $\alpha > 0$ sufficiently small such that if there exist $\lambda_0 > 0$, $\gamma_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}$, and $\xi_0 \in \mathbb{R}$ that satisfy*

$$\| e^{i\gamma_0} e^{ix\xi_0} \lambda_0^{1/2} u(\lambda_0 x + x_0) - Q(x) \|_{L^2} \leq \alpha,$$

then there exist unique $\lambda > 0$, $\gamma \in \mathbb{R}$, $\tilde{x} \in \mathbb{R}$, and $\xi \in \mathbb{R}$ that satisfy

$$(\epsilon, Q^3)_{L^2} = (\epsilon, iQ^3)_{L^2} = (\epsilon, Q_x)_{L^2} = (\epsilon, iQ_x)_{L^2} = 0,$$

where

$$\epsilon(x) = e^{i\gamma} e^{ix\xi} \lambda^{1/2} u(\lambda x + \tilde{x}) - Q. \tag{10-4}$$

Furthermore,

$$\|\epsilon\|_{L^2} + \left| \frac{\lambda}{\lambda_0} - 1 \right| + |\gamma - \gamma_0 - \xi_0(\tilde{x} - x_0)| + \left| \xi - \frac{\lambda}{\lambda_0}\xi_0 \right| + \left| \frac{\tilde{x} - x_0}{\lambda_0} \right| \lesssim \| e^{i\gamma_0} e^{ix\xi_0} \lambda_0^{1/2} u(\lambda_0 x + x_0) - Q \|_{L^2}.$$

Remark. Once again, since $e^{i\gamma}$ is 2π-periodic, the γ in (10-4) is unique up to translations by $2\pi k$ for some integer k.

Proof. By Hölder's inequality, if $\epsilon = e^{i\gamma_0} e^{ix\xi_0} \lambda_0^{1/2} u(\lambda_0 x + x_0) - Q(x)$, then

$$|(\epsilon, Q^3)_{L^2}| + |(\epsilon, Q_x)_{L^2}| + |(\epsilon, iQ^3)_{L^2}| + |(\epsilon, iQ_x)_{L^2}| \lesssim \| e^{i\gamma_0} e^{ix\xi_0} \lambda_0^{1/2} u(\lambda_0 x + x_0) - Q(x) \|_{L^2}.$$

As in the proof of Theorem 10,

$$(e^{i\gamma} e^{ix\xi} \lambda^{1/2} u(\lambda x + \tilde{x}) - Q(x), f)_{L^2} \tag{10-5}$$

is C^1 as a function of γ, λ, $\tilde{x}$, and ξ, when

$$f \in \{Q^3, iQ^3, Q_x, iQ_x\}.$$

Indeed, by Hölder's inequality and the L^2-invariance of the scaling symmetry,

$$\frac{\partial}{\partial\gamma}(e^{i\gamma} e^{ix\xi} \lambda^{1/2} u(\lambda x + \tilde{x}) - Q(x), f)_{L^2} = (ie^{i\gamma} e^{ix\xi} \lambda^{1/2} u(\lambda x + \tilde{x}), f)_{L^2} \lesssim \|u\|_{L^2} \|f\|_{L^2}.$$

Next,

$$\frac{\partial}{\partial\xi}(e^{i\gamma} e^{ix\xi} \lambda^{1/2} u(\lambda x + \tilde{x}) - Q(x), f)_{L^2} = (ixe^{i\gamma} e^{ix\xi} \lambda^{1/2} u(\lambda x + \tilde{x}), f)_{L^2} \lesssim \|u\|_{L^2} \|xf\|_{L^2}. \tag{10-6}$$

Since Q and all its derivatives are rapidly decreasing, $xf \in L^2$ and (10-6) is well defined.

Next, integrating by parts,

$$
\begin{aligned}
&\frac{\partial}{\partial \lambda}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x), f)_{L^2}\\
&\quad=\left(\frac{1}{2\lambda}e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})+xe^{i\gamma}e^{ix\xi}\lambda^{1/2}u_x(\lambda x+\tilde{x}), f\right)_{L^2}\\
&\quad=\frac{1}{2\lambda}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x}), f)_{L^2}-\frac{1}{\lambda}\left(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x}), f\right)_{L^2}\\
&\qquad\qquad-\frac{\xi}{\lambda}(ie^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x}), xf)_{L^2}-\frac{1}{\lambda}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x}), xf_x)_{L^2}\\
&\quad\lesssim\frac{1}{\lambda}\|u\|_{L^2}\|f\|_{L^2}+\frac{1}{\lambda}\|u\|_{L^2}\|xf_x\|_{L^2}+\frac{|\xi|}{\lambda}\|u\|_{L^2}\|xf\|_{L^2}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\frac{\partial}{\partial \tilde{x}}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x), f)_{L^2}\\
&\qquad=(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u_x(\lambda x+\tilde{x}), f)_{L^2}\\
&\qquad=-\frac{1}{\lambda}(i\xi e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x}), f)_{L^2}-\frac{1}{\lambda}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x}), f_x)_{L^2}\\
&\qquad\lesssim\frac{1}{\lambda}\|u\|_{L^2}\|f\|_{L^2}+\frac{|\xi|}{\lambda}\|u\|_{L^2}\|f_x\|_{L^2}.
\end{aligned}
$$

Similar calculations also prove uniform bounds on the Hessians of (10-5).

Suppose $\lambda_0=1$, $\gamma_0=0$, $x_0=0$, and $\xi_0=0$. Compute

$$
\begin{aligned}
\frac{\partial}{\partial \lambda}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x), Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=\left(\tfrac{1}{2}Q+xQ_x, Q^3\right)_{L^2}=\tfrac{1}{4}\|Q\|_{L^4}^4,\\
\frac{\partial}{\partial \lambda}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x), iQ^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=\left(\tfrac{1}{2}Q+xQ_x, iQ^3\right)_{L^2}=0,\\
\frac{\partial}{\partial \lambda}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x), Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=\left(\tfrac{1}{2}Q+xQ_x, Q_x\right)_{L^2}=0,\\
\frac{\partial}{\partial \lambda}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x), Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=\left(\tfrac{1}{2}Q+xQ_x, iQ_x\right)_{L^2}=0;
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \gamma}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x), Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=(iQ, Q^3)_{L^2}=0,\\
\frac{\partial}{\partial \gamma}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x), iQ^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=(iQ, iQ^3)_{L^2}=\|Q\|_{L^4}^4,\\
\frac{\partial}{\partial \gamma}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x), Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=(iQ, Q_x)_{L^2}=0,\\
\frac{\partial}{\partial \gamma}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x), Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=(iQ, iQ_x)_{L^2}=0;
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \tilde{x}}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x), Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=(Q_x, Q^3)_{L^2}=0,\\
\frac{\partial}{\partial \tilde{x}}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x), iQ^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=(Q_x, iQ^3)_{L^2}=0,
\end{aligned}
$$

$$\frac{\partial}{\partial \tilde{x}}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q} = (Q_x,Q_x)_{L^2} = \|Q_x\|_{L^2}^2,$$
$$\frac{\partial}{\partial \tilde{x}}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q} = (Q_x,iQ_x)_{L^2} = 0;$$

$$\frac{\partial}{\partial \xi}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q} = (ixQ,Q^3)_{L^2} = 0,$$
$$\frac{\partial}{\partial \xi}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),iQ^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q} = (ixQ,iQ^3)_{L^2} = 0,$$
$$\frac{\partial}{\partial \xi}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),Q_x)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q} = (ixQ,Q_x)_{L^2} = 0,$$
$$\frac{\partial}{\partial \xi}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q} = (ixQ,iQ_x)_{L^2} = -\tfrac{1}{2}\|Q\|_{L^2}^2.$$

Therefore, by the inverse function theorem, if $\lambda_0=1,\ \gamma_0=0,\ \xi_0=0$, and $x_0=0$, there exists $\lambda>0$, $\gamma\in\mathbb{R},\ \xi\in\mathbb{R}$, and $\tilde{x}\in\mathbb{R}$ satisfying

$$\|\epsilon\|_{L^2}+|\lambda-1|+|\gamma|+|\xi|+|\tilde{x}| \lesssim \|e^{i\gamma_0}e^{ix\xi_0}\lambda_0^{1/2}u(\lambda_0 x+x_0)-Q\|_{L^2}. \tag{10-7}$$

As in (3-5) and (3-6), $\lambda>0,\ \gamma\in\mathbb{R}$, and $\tilde{x}\in\mathbb{R}$ are unique, and $\gamma\in\mathbb{R}$ is unique in $\mathbb{R}/2\pi n$.

For general $\lambda_0>0,\ x_0\in\mathbb{R},\ \xi_0\in\mathbb{R}$, and $\gamma_0\in\mathbb{R}$, combining (10-7) with symmetries of (1-1) yields

$$\|\epsilon\|_{L^2}+\left|\frac{\lambda}{\lambda_0}-1\right|+|\gamma-\gamma_0-\xi_0(\tilde{x}-x_0)|+\left|\xi-\frac{\lambda}{\lambda_0}\xi_0\right|+\left|\frac{\tilde{x}-x_0}{\lambda_0}\right| \lesssim \|e^{i\gamma_0}e^{ix\xi_0}\lambda_0^{1/2}u(\lambda_0 x+x_0)-Q\|_{L^2}. \qquad \square$$

As in Theorem 11, it is possible to show that $\lambda(t),\ \gamma(t),\ x(t)$, and $\xi(t)$ are continuous functions on $[0,\sup(I))$ and are differentiable almost everywhere on $[0,\sup(I))$. Let $s(t)$ be as in (3-10). Since $s:[0,\sup(I))\to[0,\infty)$ is monotone, the function is invertible: $t(s):[0,\infty)\to[0,\sup(I))$. Letting

$$\gamma(s)=\gamma(t(s)),\quad \lambda(s)=\lambda(t(s)),\quad x(s)=x(t(s)),\quad \xi(s)=\xi(t(s)),$$

and letting

$$\epsilon(s,x)=e^{i\gamma(s)}e^{ix\xi(s)}\lambda(s)^{1/2}u(t(s),\lambda(x)x+x(s))-Q(x), \tag{10-8}$$

we can compute

$$\begin{aligned}\epsilon_s = i\gamma_s(Q+\epsilon)+i\xi_s x(Q+\epsilon)+\frac{\lambda_s}{\lambda}\big(\tfrac{1}{2}(Q+\epsilon)+x(Q+\epsilon)_x\big)-i\frac{\lambda_s}{\lambda}\xi(s)x(Q+\epsilon)\\ +\frac{x_s}{\lambda}(Q+\epsilon)_x-i\frac{x_s}{\lambda}\xi(s)(Q+\epsilon)+i(Q+\epsilon)_{xx}\\ +2\xi(s)(Q+\epsilon)_x-i\xi(s)^2(Q+\epsilon)+i|Q+\epsilon|^4(Q+\epsilon).\end{aligned} \tag{10-9}$$

Taking $f\in\{Q^3,iQ^3,Q_x,iQ_x\}$,

$$\frac{d}{ds}(\epsilon,f)_{L^2}=(\epsilon_s,f)_{L^2}=0.$$

Using the fact that f belongs to the span of $\{Q^3, iQ^3, Q_x, iQ_x\}$ if and only if if belongs to the span of $\{Q^3, iQ^3, Q_x, iQ_x\}$ as a real vector space, compute the following:

$$\begin{aligned} &(i\gamma_s(Q+\epsilon), f)_{L^2} = (i\gamma_s Q, f)_{L^2} = 0 \quad \text{if } f = Q^3, Q_x, iQ_x, \\ &(i\gamma_s(Q+\epsilon), iQ^3) = \gamma_s \|Q\|_{L^4}^4; \end{aligned} \tag{10-10}$$

$$\Bigl(\xi_s - \frac{\lambda_s}{\lambda}\xi(s)\Bigr)(ix(Q+\epsilon), f)_{L^2} = \begin{cases} -\frac{1}{2}\bigl(\xi_s(s) - \frac{\lambda_s}{\lambda}\xi(s)\bigr)\|Q\|_{L^2}^2 \\ \quad + O\bigl(\bigl|\xi_s(s) - \frac{\lambda_s}{\lambda}\xi(s)\bigr|\|\epsilon\|_{L^2}\bigr) & \text{if } f = iQ_x, \\ O\bigl(\bigl|\xi_s(s) - \frac{\lambda_s}{\lambda}\xi(s)\bigr|\|\epsilon\|_{L^2}\bigr) & \text{if } f \in \{Q^3, Q_x, iQ^3\}; \end{cases} \tag{10-11}$$

$$\begin{aligned} &\Bigl(-i\frac{x_s}{\lambda}\xi(s)(Q+\epsilon), f\Bigr)_{L^2} = \Bigl(-i\frac{x_s}{\lambda}\xi(s)Q, f\Bigr)_{L^2} = 0 \quad \text{if } f = Q^3, Q_x, iQ_x, \\ &\Bigl(-i\frac{x_s}{\lambda}\xi(s), Q, iQ^3\Bigr)_{L^2} = -\frac{x_s}{\lambda}\xi(s)\|Q\|_{L^4}^4; \end{aligned} \tag{10-12}$$

$$\begin{aligned} &(i\xi(s)^2(Q+\epsilon), f)_{L^2} = (i\xi(s)^2 Q, f)_{L^2} = 0 \quad \text{if } f = Q^3, Q_x, iQ_x, \\ &(i\xi(s)^2 Q, iQ^3) = \xi(s)^2\|Q\|_{L^4}^4; \end{aligned} \tag{10-13}$$

$$\begin{aligned} &\frac{\lambda_s}{\lambda}\bigl(\tfrac{1}{2}(Q+\epsilon) + x(Q+\epsilon)_x, Q^3\bigr)_{L^2} = \frac{\lambda_s}{4\lambda}\|Q\|_{L^4}^4 + O\Bigl(\Bigl|\frac{\lambda_s}{\lambda}\Bigr|\|\epsilon\|_{L^2}\Bigr), \\ &\frac{\lambda_s}{\lambda}\bigl(\tfrac{1}{2}(Q+\epsilon) + x(Q+\epsilon)_x, f\bigr)_{L^2} = O\Bigl(\Bigl|\frac{\lambda_s}{\lambda}\Bigr|\|\epsilon\|_{L^2}\Bigr) \qquad \text{if } f = Q_x, iQ^3, iQ_x; \end{aligned} \tag{10-14}$$

$$\begin{aligned} &\Bigl(\frac{x_s}{\lambda} + 2\xi(s)\Bigr)((Q+\epsilon)_x, Q_x)_{L^2} = \Bigl(\frac{x_s}{\lambda} + 2\xi(s)\Bigr)\|Q_x\|_{L^2}^2 + O\Bigl(\Bigl|\frac{x_s}{\lambda} + 2\xi(s)\Bigr|\|\epsilon\|_{L^2}\Bigr), \\ &\Bigl(\frac{x_s}{\lambda} + 2\xi(s)\Bigr)((Q+\epsilon)_x, f)_{L^2} = O\Bigl(\Bigl|\frac{x_s}{\lambda} + 2\xi(s)\Bigr|\|\epsilon\|_{L^2}\Bigr) \qquad \text{if } f = Q^3, iQ^3, iQ_x. \end{aligned} \tag{10-15}$$

Finally, taking $\epsilon = \epsilon_1 + i\epsilon_2$,

$$\begin{aligned} &(i(Q+\epsilon)_{xx} + i|Q+\epsilon|^4(Q+\epsilon), f)_{L^2} \\ &\qquad = (iQ, f)_{L^2} + (i\mathcal{L}\epsilon_1 - \mathcal{L}_-\epsilon_2, f)_{L^2} + O((|\epsilon|^2(|\epsilon|^3 + |Q|^3), f)_{L^2}, \end{aligned} \tag{10-16}$$

where $\mathcal{L}$, $\mathcal{L}_-$ are given by (3-12). Since $\mathcal{L}$, $\mathcal{L}_-$ are self-adjoint operators, $(\epsilon_1, Q^3)_{L^2} = (\epsilon_2, Q^3)_{L^2} = 0$, $\mathcal{L}Q_x = 0$, and

$$\begin{aligned} &(i(Q+\epsilon)_{xx} + i|Q+\epsilon|^4(Q+\epsilon), f)_{L^2} \\ &\qquad = \begin{cases} \|Q\|_{L^4}^4 + O((|\epsilon|^2(|\epsilon|^3 + |Q|^3), f)_{L^2} & \text{if } f = iQ^3, \\ O((|\epsilon|^2(|\epsilon|^3 + |Q|^3), f)_{L^2} & \text{if } f = iQ_x, \\ -(\epsilon_2, \mathcal{L}_- Q^3)_{L^2} + O((|\epsilon|^2(|\epsilon|^3 + |Q|^3), f)_{L^2} & \text{if } f = Q^3, \\ -(\epsilon_2, \mathcal{L}_- Q_x)_{L^2} + O((|\epsilon|^2(|\epsilon|^3 + |Q|^3), f)_{L^2} & \text{if } f = Q_x. \end{cases} \end{aligned} \tag{10-17}$$

Combining (10-10)–(10-17), we have proved

$$\Big(\gamma_s + 1 - \frac{x_s}{\lambda}\xi(s) - \xi(s)^2\Big)\|Q\|_{L^4}^4 + O\Big(\Big|\xi_s - \frac{\lambda_s}{\lambda}\xi(s)\Big|\|\epsilon\|_{L^2}\Big) + O\Big(\Big|\frac{\lambda_s}{\lambda}\Big|\|\epsilon\|_{L^2}\Big) + O\Big(\Big|\frac{x_s}{\lambda} + 2\xi(s)\Big|\|\epsilon\|_{L^2}\Big) + O(\|\epsilon\|_{L^2}^2(\|Q\|_{L^\infty}^3 + \|\epsilon\|_{L^\infty}^3)) = 0, \tag{10-18}$$

$$-\frac{1}{2}\Big(\xi_s - \frac{\lambda_s}{\lambda}\xi(s)\Big)\|Q\|_{L^2}^2 + O\Big(\Big|\xi_s - \frac{\lambda_s}{\lambda}\xi(s)\Big|\|\epsilon\|_{L^2}\Big) + O\Big(\Big|\frac{\lambda_s}{\lambda}\Big|\|\epsilon\|_{L^2}\Big) + O\Big(\Big|\frac{x_s}{\lambda} + 2\xi(s)\Big|\|\epsilon\|_{L^2}\Big) + O(\|\epsilon\|_{L^2}^2(\|Q\|_{L^\infty}^3 + \|\epsilon\|_{L^\infty}^3)) = 0, \tag{10-19}$$

$$\frac{\lambda_s}{4\lambda}\|Q\|_{L^4}^4 - (\epsilon_2, \mathcal{L}_- Q^3)_{L^2} + O\Big(\Big|\xi_s - \frac{\lambda_s}{\lambda}\xi(s)\Big|\|\epsilon\|_{L^2}\Big) + O\Big(\Big|\frac{\lambda_s}{\lambda}\Big|\|\epsilon\|_{L^2}\Big) + O\Big(\Big|\frac{x_s}{\lambda} + 2\xi(s)\Big|\|\epsilon\|_{L^2}\Big) + O(\|\epsilon\|_{L^2}^2(\|Q\|_{L^\infty}^3 + \|\epsilon\|_{L^\infty}^3)) = 0, \tag{10-20}$$

$$\Big(\frac{x_s}{\lambda} + 2\xi\Big)\|Q_x\|_{L^2}^2 - (\epsilon_2, \mathcal{L}_- Q_x)_{L^2} + O\Big(\Big|\xi_s - \frac{\lambda_s}{\lambda}\xi(s)\Big|\|\epsilon\|_{L^2}\Big) + O\Big(\Big|\frac{\lambda_s}{\lambda}\Big|\|\epsilon\|_{L^2}\Big) + O\Big(\Big|\frac{x_s}{\lambda} + 2\xi(s)\Big|\|\epsilon\|_{L^2}\Big) + O(\|\epsilon\|_{L^2}^2(\|Q\|_{L^\infty}^3 + \|\epsilon\|_{L^\infty}^3)) = 0. \tag{10-21}$$

Using the same analysis as in (3-15)–(3-18), for any $a \in \mathbb{Z}_{\geq 0}$,

$$\int_a^{a+1} \Big|\gamma_s + 1 - \frac{x_s}{\lambda}\xi(s) - \xi(s)^2\Big|\,ds \lesssim \int_a^{a+1} \|\epsilon(s)\|_{L^2}^2\,ds, \tag{10-22}$$

$$\int_a^{a+1} \Big|\xi_s - \frac{\lambda_s}{\lambda}\xi(s)\Big|\,ds \lesssim \int_a^{a+1} \|\epsilon(s)\|_{L^2}^2\,ds, \tag{10-23}$$

$$\int_a^{a+1} \Big|\frac{\lambda_s}{\lambda}\Big|\,ds \lesssim \int_a^{a+1} \|\epsilon(s)\|_{L^2}\,ds, \tag{10-24}$$

$$\int_a^{a+1} \Big|\frac{x_s}{\lambda} + 2\xi\Big|\,ds \lesssim \int_a^{a+1} \|\epsilon(s)\|_{L^2}\,ds. \tag{10-25}$$

10.3. ***A long-time Strichartz estimate in the nonsymmetric case.*** The symmetry (10-1) does not impact the long-time Strichartz estimates in Theorems 13–15 at all. However, the Galilean symmetry (10-2) does, since it involves a translation in frequency, and therefore will impact estimates of u under frequency cutoffs. Nevertheless, it is possible to prove a modification of Theorem 15 using virtually the same arguments.

Theorem 27. *Suppose $\lambda(t)$, $x(t)$, $\xi(t)$, and $\gamma(t)$ are as in* (10-4). *Also suppose that on the interval $J = [a, b]$,*

$$\lambda(t) \geq \frac{1}{\eta_1}, \quad \int_J \lambda(t)^{-2}\,dt = T, \quad \text{and} \quad \eta_1^{-2} T = 2^{3k}.$$

Furthermore, suppose that

$$\frac{|\xi(t)|}{\lambda(t)} \leq \eta_0 \quad \text{for all } t \in [a, b].$$

Then

$$\|P_{\geq k}u\|_{U^2_\Delta([a,b]\times\mathbb{R})} \lesssim T^{-10} + \left(\frac{1}{T}\int_a^b \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}\,dt\right)^{1/2}.$$

Proof. Observe that by (10-4),

$$u(t,x) = e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}Q\left(\frac{x-x(t)}{\lambda(t)}\right) + e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}\epsilon\left(t,\frac{x-x(t)}{\lambda(t)}\right). \tag{10-26}$$

Then by (4-3), (4-4), and (10-26),

$$\|P_{>0}u\|^2_{L^\infty_t L^2_x([a,b]\times\mathbb{R})} \leq 4\eta_0^2.$$

Applying the induction on frequency arguments in Theorems 13–15 gives the same results. □

10.4. *Almost conservation of energy for a nonsymmetric solution.* It is possible to use the long-time Strichartz estimates in Theorem 27 to prove an almost conservation of energy for a nonsymmetric solution.

Theorem 28. *Let $J=[a,b]$ be an interval such that*

$$\lambda(t)\geq\frac{1}{\eta_1},\quad \frac{|\xi(t)|}{\lambda(t)}\leq\eta_0 \quad \textit{for all } t\in J \qquad \textit{and} \qquad \int_J \lambda(t)^{-2}\,dt = T,\quad \eta_1^{-2}T = 2^{3k}.$$

Then,

$$\sup_{t\in J} E(P_{\leq k+9}u(t)) \lesssim \frac{2^{2k}}{T}\int_J \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}\,dt + \left(\sup_{t\in J}\frac{\xi(t)}{\lambda(t)}\right)^2 + 2^{2k}T^{-10}. \tag{10-27}$$

Proof. Decompose the energy as in Theorem 12. Since $E(Q)=0$ and $(\epsilon_2, Q_x)=0$,

$$\begin{aligned}
E(u) &= E\left(e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}Q\left(\frac{x-x(t)}{\lambda(t)}\right) + e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}\epsilon\left(t,\frac{x-x(t)}{\lambda(t)}\right)\right)\\
&= \frac{1}{2\lambda(t)^2}\|Q_x\|^2_{L^2} + \frac{\xi(t)^2}{2\lambda(t)^2}\|Q\|^2_{L^2} - \frac{1}{6\lambda(t)^2}\|Q\|^6_{L^6} + \frac{1}{2\lambda(t)^2}\|\epsilon\|^2_{L^2} - \frac{2\xi(t)}{\lambda(t)^2}(Q_x,\epsilon_2)_{L^2}\\
&\quad - \frac{\xi(t)^2}{2\lambda(t)^2}\|\epsilon\|^2_{L^2} + \frac{1}{2\lambda(t)^2}\|\nabla\epsilon\|^2_{L^2} - \frac{\xi(t)}{\lambda(t)^2}(\nabla\epsilon_1,\epsilon_2)_{L^2} + \frac{\xi(t)}{\lambda(t)^2}(\nabla\epsilon_2,\epsilon_1)_{L^2} + \frac{\xi(t)^2}{2\lambda(t)^2}\|\epsilon\|^2_{L^2}\\
&\quad - \frac{5}{2\lambda(t)^2}\int Q(x)^4\epsilon_1(t,x)^2\,dx - \frac{1}{2\lambda(t)^2}\int Q(x)^4\epsilon_2(t,x)^2\,dx\\
&\quad + O\left(\frac{1}{\lambda(t)^2}\|\epsilon\|^3_{L^3} + \frac{1}{\lambda(t)^2}\|\epsilon\|^6_{L^6}\right)\\
&= \frac{\xi(t)^2}{2\lambda(t)^2}\|Q\|^2_{L^2} + \frac{1}{2\lambda(t)^2}\|\epsilon\|^2_{L^2} - \frac{\xi(t)}{\lambda(t)^2}(\nabla\epsilon_1,\epsilon_2)_{L^2} + \frac{\xi(t)}{\lambda(t)^2}(\nabla\epsilon_2,\epsilon_1)_{L^2} + \frac{1}{2\lambda(t)^2}\|\nabla\epsilon\|^2_{L^2}\\
&\quad - \frac{5}{2\lambda(t)^2}\int Q(x)^4\epsilon_1(t,x)^2\,dx - \frac{1}{2\lambda(t)^2}\int Q(x)^4\epsilon_2(t,x)^2\,dx\\
&\quad + O\left(\frac{1}{\lambda(t)^2}\|\epsilon\|^3_{L^3} + \frac{1}{\lambda(t)^2}\|\epsilon\|^6_{L^6}\right).
\end{aligned}\tag{10-28}$$

Using the bounds on $|\xi(t)|/\lambda(t)$, the fact that Q and all its derivatives are rapidly decreasing, Fourier truncation, and the mean value theorem implies that (10-27) holds for some $t_0 \in J$. Then, using the long-time Strichartz estimates in Theorem 27 and following the proof of Theorem 16 gives Theorem 28. □

It is also possible to generalize Corollary 17 to the nonsymmetric case.

Corollary 29. *If*

$$\frac{1}{\eta_1} \le \lambda(t) \le \frac{1}{\eta_1} T^{1/100}, \quad \frac{|\xi(t)|}{\lambda(t)} \le \eta_0 \quad \textit{for all } t \in J \qquad \textit{and} \qquad \int_J \lambda(t)^{-2}\,dt = T, \quad \eta_1^{-2} T = 2^{3k},$$

then

$$\sup_{t\in J} \left\| P_{\le k+9}\left(\frac{e^{-i\gamma(t)} e^{-ix\xi(t)/\lambda(t)}}{\lambda(t)^{1/2}} \epsilon\left(t, \frac{x-x(t)}{\lambda(t)}\right)\right)\right\|_{\dot H^1}^2 \lesssim \frac{2^{2k}}{T} \int_J \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}\,dt + \left(\sup_{t\in J} \frac{\xi(t)^2}{\lambda(t)^2}\right) + 2^{2k} T^{-10}$$

and

$$\sup_{t\in J} \|\epsilon(t)\|_{L^2}^2 \lesssim \frac{2^{2k} T^{1/50}}{\eta_1^2 T} \int_J \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}\,dt + \frac{T^{1/50}}{\eta_1^2}\left(\sup_{t\in J} \frac{\xi(t)^2}{\lambda(t)^2}\right) + 2^{2k} \frac{T^{1/50}}{\eta_1^2} T^{-10}.$$

Proof. As in the proof of Theorem 12, since $\epsilon \perp \{Q^3, Q_x, iQ^3, iQ_x\}$, there exists some $c > 0$ such that

$$\frac{1}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2 + \frac{1}{2\lambda(t)^2}\|\nabla\epsilon\|_{L^2}^2 - \frac{5}{2\lambda(t)^2}\int Q(x)^4 \epsilon_1(t,x)^2\,dx - \frac{1}{2\lambda(t)^2}\int Q(x)^4 \epsilon_2(t,x)^2\,dx \ge \frac{1}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2 + \frac{c}{\lambda(t)^2}\|\nabla\epsilon\|_{L^2}^2. \tag{10-29}$$

Next, for $\|\epsilon\|_{L^2} \le \eta_0$ sufficiently small, by the Cauchy–Schwarz inequality, taking $\delta = \|\epsilon\|_{L^2}/\|Q\|_{L^2}$ in the last step,

$$\begin{aligned}\frac{\xi(t)^2}{\lambda(t)^2}\|Q\|_{L^2}^2 - \frac{\xi(t)}{\lambda(t)^2}(\nabla\epsilon_1, \epsilon_2)_{L^2} + \frac{\xi(t)}{\lambda(t)^2}(\nabla\epsilon_2, \epsilon_1)_{L^2} &\ge \frac{\xi(t)^2}{\lambda(t)^2}\|Q\|_{L^2}^2 - \frac{1}{\delta}\frac{\xi(t)^2}{\lambda(t)^2}\|\epsilon\|_{L^2}^2 - \frac{\delta}{\lambda(t)^2}\|\nabla\epsilon\|_{L^2}^2 \\ &\ge -O\left(\frac{\eta_0}{\lambda(t)^2}\right)\|\nabla\epsilon\|_{L^2}^2. \end{aligned} \tag{10-30}$$

Finally, by Hölder's inequality and the Sobolev embedding theorem, since $\|\epsilon\|_{L^2} \ll 1$,

$$O\left(\frac{1}{\lambda(t)^2}\|\epsilon\|_{L^3}^3 + \frac{1}{\lambda(t)^2}\|\epsilon\|_{L^6}^6\right) \ll \frac{1}{\lambda(t)^2}\|\epsilon\|_{L^2}^2 + \frac{1}{\lambda(t)^2}\|\nabla\epsilon\|_{L^2}^2. \tag{10-31}$$

Plugging (10-29)–(10-31) into (10-28) proves the corollary. □

10.5. ***A frequency-localized Morawetz estimate for nonsymmetric u.*** As in Section 6, the long-time Strichartz estimates of Theorem 27 and the energy estimates of Theorem 28 and Corollary 29 give a theorem analogous to Theorem 18 in the nonsymmetric case.

Theorem 30. *Let $J = [a, b]$ be an interval on which*

$$\frac{|\xi(t)|}{\lambda(t)} \leq \eta_0, \quad \frac{1}{\eta_1} \leq \lambda(t) \leq \frac{1}{\eta_1} T^{1/100} \quad \textit{for all } t \in J \qquad \textit{and} \qquad \int_J \lambda(t)^{-2}\, dt = T, \quad \eta_1^{-2} T = 2^{3k}.$$

Also suppose $\epsilon = \epsilon_1 + i\epsilon_2$, where ϵ is given by Theorem 10. Finally, suppose there exists a uniform bound on $x(t)$,

$$\sup_{t \in J} |x(t)| \leq R = T^{1/25}. \tag{10-32}$$

Finally, suppose that $\xi(a) = 0$ and $x(b) = 0$. Then for T sufficiently large,

$$\int_a^b \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}\, dt \\ \leq 3\big(\epsilon_2(a), \big(\tfrac{1}{2} Q + x Q_x\big)\big)_{L^2} - 3\big(\epsilon_2(b), \tfrac{1}{2} Q + x Q_x\big)_{L^2} + \frac{T^{1/50}}{\eta_1^2} \sup_{t \in J} \frac{\xi(t)^2}{\lambda(t)^2} + O\Big(\frac{1}{T^9}\Big).$$

Proof. This time let

$$\phi(x) = \int_0^x \chi\Big(\frac{\eta_1 y}{2R}\Big)\, dy = \int_0^x \psi^2\Big(\frac{\eta_1 y}{2R}\Big)\, dy, \tag{10-33}$$

and let

$$M(t) = \int \phi(x) \operatorname{Im}[\overline{P_{\leq k+9} u}\, \partial_x P_{\leq k+9} u](t, x)\, dx.$$

Since

$$|\phi(x)| \lesssim \eta_1^{-1} R \quad \text{and} \quad |\xi(t)|/\lambda(t) \leq \eta_0,$$

Theorem 27 implies that the error terms arising from frequency truncation may be handled in exactly the same manner as in Theorem 18.

Next, observe that by (10-30) and (10-31), the additional terms in the left-hand side of (6-17) that arise from the fact that $\xi(t)$ need not be zero may be handled in exactly the same manner as the terms involving ϵ^3 and higher powers of ϵ.

Now decompose $M(b) - M(a)$. Since Q is real-valued, symmetric, and rapidly decreasing, (10-33), the bounds on $\lambda(t)$, and (10-32) imply

$$\int \phi(x) \operatorname{Im}\Big[\overline{e^{-i\gamma(t)} e^{-ix\xi(t)/\lambda(t)} \lambda(t)^{-1/2} P_{\leq k+9} Q\Big(\frac{x - x(t)}{\lambda(t)}\Big)} \\ \times \partial_x \Big(e^{-i\gamma(t)} e^{-ix\xi(t)/\lambda(t)} \lambda(t)^{-1/2} P_{\leq k+9} Q\Big(\frac{x - x(t)}{\lambda(t)}\Big)\Big)\Big]\, dx \\ = \frac{\xi(t)}{\lambda(t)^2} \int \phi(x) Q\Big(\frac{x - x(t)}{\lambda(t)}\Big)^2 dx + O(T^{-10}) = \frac{\xi(t)}{\lambda(t)} x(t) \|Q\|_{L^2}^2 + O(T^{-10}). \tag{10-34}$$

Since $\xi(a) = 0$ and $x(b) = 0$,

$$\frac{\xi(t)}{\lambda(t)} x(t) \|Q\|_{L^2}^2 \Big|_a^b = 0.$$

Next, by Corollary 29,

$$\begin{aligned}\int \phi(x)\,\mathrm{Im}\Big[&\overline{P_{\le k+9}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}\epsilon\Big(t,\frac{x-x(t)}{\lambda(t)}\Big)}\\ &\times \partial_x\Big(P_{\le k+9}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}\epsilon\Big(t,\frac{x-x(t)}{\lambda(t)}\Big)\Big)\Big]\,dx\\ \lesssim \frac{R}{\eta_1^2}\frac{2^{2k}}{T^{99/100}}\int_J \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}\,dt &+ \frac{R}{\eta_1^2}2^{2k}T^{-9.99}+\frac{T^{1/50}}{\eta_1^2}\sup_{t\in J}\frac{\xi(t)^2}{\lambda(t)^2}.\end{aligned} \tag{10-35}$$

Next, using the computations proving (6-5) combined with the fact that $(\epsilon_2, Q_x)=0$,

$$\begin{aligned}\int \phi(x)\,\mathrm{Im}\Big[&\overline{P_{\le k+9}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}\epsilon\Big(t,\frac{x-x(t)}{\lambda(t)}\Big)}\\ &\times \partial_x\Big(P_{\le k+9}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}Q\Big(\frac{x-x(t)}{\lambda(t)}\Big)\Big)\Big]\,dx\\ &= -(\epsilon_2, xQ_x)_{L^2}+\frac{x(t)}{\lambda(t)}(\epsilon_2,Q_x)_{L^2}-\frac{\xi(t)}{\lambda(t)}(\epsilon_1,Q)_{L^2}+O(T^{-10})\\ &= -(\epsilon_2, xQ_x)_{L^2}-\frac{\xi(t)}{2\lambda(t)}\|\epsilon\|_{L^2}^2+O(T^{-10}).\end{aligned} \tag{10-36}$$

Finally, integrating by parts,

$$\begin{aligned}\int \phi(x)\,\mathrm{Im}\Big[&\overline{P_{\le k+9}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}Q\Big(\frac{x-x(t)}{\lambda(t)}\Big)}\\ &\times \partial_x\Big(P_{\le k+9}\lambda(t)^{-1/2}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\epsilon\Big(t,\frac{x-x(t)}{\lambda(t)}\Big)\Big)\Big]\,dx\\ = \text{(10-36)} - \int \chi\Big(\frac{\eta_1 x}{2R}\Big)\,\mathrm{Im}\Big[&\overline{P_{\le k+9}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}Q\Big(\frac{x-x(t)}{\lambda(t)}\Big)}\\ &\times P_{\le k+9}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}\epsilon\Big(t,\frac{x-x(t)}{\lambda(t)}\Big)\Big]\,dx.\end{aligned} \tag{10-37}$$

As in (6-10),

$$\begin{aligned}-\int \chi\Big(\frac{\eta_1 x}{2R}\Big)\,\mathrm{Im}\Big[&\overline{P_{\le k+9}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}Q\Big(\frac{x-x(t)}{\lambda(t)}\Big)}\\ &\times P_{\le k+9}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}\epsilon\Big(t,\frac{x-x(t)}{\lambda(t)}\Big)\Big]\,dx\\ &= -(\epsilon_2, Q)_{L^2}+O(T^{-10}).\end{aligned} \tag{10-38}$$

Summing up (10-34)–(10-38) and using the fundamental theorem of calculus and the Morawetz estimate completes the proof of Theorem 30. □

10.6. ***An L_s^p bound on $\|\epsilon(s)\|_{L^2}$ when $p>1$ for nonsymmetric u.*** Combining Theorem 30 with (10-18)–(10-21), it is possible to prove Theorem 20 for nonsymmetric u.

Theorem 31. *Let u be a nonsymmetric solution to* (1-1) *that satisfies $\|u\|_{L^2} = \|Q\|_{L^2}$, and suppose*

$$\sup_{s \in [0,\infty)} \|\epsilon(s)\|_{L^2} \leq \eta_* \tag{10-39}$$

and $\|\epsilon(0)\|_{L^2} = \eta_*$. Then

$$\int_0^\infty \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim \eta_*, \tag{10-40}$$

with implicit constant independent of η_* when $\eta_* \ll 1$ is sufficiently small.

Furthermore, for any $j \in \mathbb{Z}_{\geq 0}$, let

$$s_j = \inf\{s \in [0,\infty) : \|\epsilon(s)\|_{L^2} = 2^{-j}\eta_*\}.$$

By definition, $s_0 = 0$, and, as in Theorem 20, such an s_j exists for any $j > 0$. Then,

$$\int_{s_j}^\infty \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim 2^{-j}\eta_* \tag{10-41}$$

for each j, with implicit constant independent of η_* and $j \geq 0$.

Proof. Set $T_* = 1/\eta_*$ and suppose that T_* is large enough that Theorem 30 holds. Then by (10-39) and (10-22),

$$\Big| \sup_{s \in [s', s'+T_*]} \ln \lambda(s) - \inf_{s \in [s', s'+T_*]} \ln \lambda(s) \Big| \lesssim 1.$$

Let J be the largest dyadic integer that satisfies

$$J = 2^{j_*} \leq -\ln \eta_*^{1/4}.$$

By (10-24) and the triangle inequality,

$$\Big| \sup_{s \in [s', s'+JT_*]} \ln \lambda(s) - \inf_{s \in [s', s'+JT_*]} \ln \lambda(s) \Big| \lesssim J,$$

and therefore,

$$\frac{\sup_{s \in [s', s'+3JT_*]} \lambda(s)}{\inf_{s \in [s', s'+3JT^*]} \lambda(s)} \lesssim T_*^{1/100}.$$

Rescale so that $\inf_{s \in [s', s'+3JT_*]} \lambda(s) = 1/\eta_1$. Then make a Galilean transformation so that $\xi(s') = 0$ and a translation in space so that $x(s'') = 0$ when $s'' \in [s', s'+3JT_*]$ is the other endpoint of the interval of integration. Then by (10-23) and (10-25),

$$\sup_{s \in [s', s'+3JT_*]} \frac{|\xi(s)|}{\lambda(s)} \lesssim \eta_* J \eta_1 \ll \eta_0 \quad \text{and} \quad \sup_{s \in [s', s'+3JT_*]} |x(s)| \lesssim J^2 T_*^{1/100} + \frac{1}{\eta_1} T_*^{1/100} J \ll T_*^{1/25}.$$

Therefore, by Theorem 30,

$$\sup_{a > 0} \int_{s'+aJT_*}^{s'+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 \lesssim \frac{1}{J^{1/2} T_*^{1/2}} \left(\sup_{a \geq 0} \int_{s'+aJT_*}^{s'+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \right)^{1/2} + T_*^{1/50} \eta_*^2 + O\left(\frac{1}{J^9 T_*^9} \right),$$

and when $a = 0$,

$$\int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2}^2 \lesssim \|\epsilon(s')\|_{L^2} + \frac{1}{J^{1/2}T_*^{1/2}} \left(\sup_{a \geq 0} \int_{s'+aJT_*}^{s'+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \right)^{1/2} + T_*^{1/50} \eta_*^2 + O\left(\frac{1}{J^9 T_*^9} \right).$$

Therefore, taking $s' = s_{j_*}$,

$$\sup_{a \geq 0} \int_{s_{j_*}+aJT_*}^{s_{j_*}+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim 2^{-j_*} \eta_* + O(2^{-9j_*} \eta_*^9).$$

By the triangle inequality,

$$\sup_{s' \geq s_{j_*}} \int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim 2^{-j_*} \eta_*,$$

and by Hölder's inequality,

$$\sup_{s' \geq s_{j_*}} \int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2} \, ds \lesssim 1.$$

It is therefore possible to prove Theorem 31 by induction. Indeed, suppose that for some $n > 0$,

$$\sup_{s' \geq s_{nj_*}} \int_{s'}^{s'+J^nT_*} \|\epsilon(s)\|_{L^2} \, ds \leq C \quad \text{and} \quad \sup_{s' \geq s_{nj_*}} \int_{s'}^{s'+J^nT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \leq C^2 J^{-n} \eta_*.$$

Then by (10-24),

$$\sup_{s' \geq s_{j_*}} \Big| \sup_{s \in [s', s'+J^{n+1}T_*]} \ln \lambda(s) - \inf_{s \in [s', s'+JT_*]} \ln \lambda(s) \Big| \lesssim CJ. \tag{10-42}$$

Next, rescaling so that $\inf_{s \in [s', s'+J^{n+1}T_*]} \lambda(s) = 1/\eta_1$ and setting $\xi(s') = 0$, (10-23) implies

$$\sup_{s \in [s', s'+J^{n+1}T_*]} \frac{|\xi(s)|}{\lambda(s)} \lesssim 2^{-j_*n} \eta_* \eta_1 C^2 J \ll \eta_0, \tag{10-43}$$

and by (10-25), if $x(s'') = 0$, where s'' is the other endpoint of the interval of integration,

$$\sup_{s \in [s'+J^{n+1}T_*]} |x(s)| \lesssim C^2 J^2 T_*^{1/100} + C \frac{1}{\eta_1} T_*^{1/100} J \ll T_*^{1/25}. \tag{10-44}$$

Then by Theorem 30, as in (7-12),

$$\sup_{s' \geq s_{(n+1)j_*}} \int_{s'}^{s'+J^{n+1}T_*} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim J^{-(n+1)} T_*^{-1} + T_*^{1/25} 2^{-2j_*n} \eta_*^2 C^4 J^2 \lesssim J^{-(n+1)} T_*^{-1}, \tag{10-45}$$

and by Hölder's inequality,

$$\sup_{s' \geq s_{(n+1)j_*}} \int_{s'}^{s'+J^{n+1}T_*} \|\epsilon(s)\|_{L^2} \, ds \lesssim 1. \tag{10-46}$$

It is important to observe that the implicit constants in (10-45) and (10-46) are independent of C so long as the final inequalities in (10-43) and (10-44) hold and $C \ll T_*^{1/2}$.

Now take any $j \in \mathbb{Z}$ and suppose $nj_* < j \leq (n+1)j_*$. Then by (10-45) and (10-46),

$$\sup_{a \geq 0} \int_{s_j + aJ^{n+1}T_*}^{s_j+(a+1)J^{n+1}T_*} \|\epsilon(s)\|_{L^2}\, ds \lesssim J \quad \text{and} \quad \sup_{a \geq 0} \int_{s_j + aJ^{n+1}T_*}^{s_j+(a+1)J^{n+1}T_*} \|\epsilon(s)\|_{L^2}^2\, ds \lesssim 2^{-j}\eta_*,$$

and therefore, after appropriate rescaling and Galilean and spatial translation, (10-42)–(10-44) hold. Therefore, by Theorem 30,

$$\sup_{s' \geq s_j} \int_{s'}^{s'+2^j T_*} \|\epsilon(s)\|_{L^2}\, ds \lesssim 1 \quad \text{and} \quad \int_{s'}^{s'+2^j T_*} \|\epsilon(s)\|_{L^2}^2\, ds \lesssim 2^{-j}\eta_*,$$

with implicit constant independent of j. Furthermore, as in (7-13),

$$\int_{s'}^{s'+2^j JT_*} \|\epsilon(s)\|_{L^2}^2\, ds \lesssim 2^{-j}\eta_*,$$

so by the mean value theorem,

$$\inf_{s \in [s_j, s_j + 2^j JT_*]} \|\epsilon(s)\|_{L^2} \lesssim 2^{-j}\eta_* J^{-1/2},$$

which implies

$$s_{j+1} \in [s_j, s_j + 2^j JT_*].$$

Therefore,

$$\int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2}^2\, ds \lesssim 2^{-j}\eta_* \quad \text{and} \quad \int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2}\, ds \lesssim 1,$$

with constant independent of j. Summing in j gives (10-40) and (10-41). □

Now then, as in Section 7,

$$\lim_{s \to \infty} \|\epsilon(s)\|_{L^2} = 0, \qquad \int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2}\, ds \lesssim 1,$$

and, for any $1 < p < \infty$

$$\int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2}^p\, ds \lesssim \eta_*^{p-1} 2^{-j(p-1)},$$

which implies that $\|\epsilon(s)\|_{L^2}$ belongs to L_s^p for any $p > 1$ but not to L_s^1.

10.7. ***Monotonicity of λ in the nonsymmetric case.*** It is possible to use the virial identity from [Merle and Raphael 2005] to show monotonicity in the nonsymmetric case as well.

Theorem 32. *For any $s \geq 0$, let*

$$\tilde{\lambda}(s) = \inf_{\tau \in [0,s]} \lambda(\tau).$$

Then for any $s \geq 0$,

$$1 \leq \frac{\lambda(s)}{\tilde{\lambda}(s)} \leq 3.$$

Proof. Suppose there exist $0 \leq s_- \leq s_+ < \infty$ satisfying

$$\frac{\lambda(s_+)}{\lambda(s_-)} = e.$$

Then using (10-9) and the computations in Theorem 21,

$$\begin{aligned}
&\frac{d}{ds}(\epsilon, y^2 Q)_{L^2} + \frac{\lambda_s}{\lambda}\|yQ\|_{L^2}^2 + 4\big(\epsilon_2, \tfrac{1}{2}Q + yQ_y\big)_{L^2} \\
&\quad = O\bigg(\bigg|\gamma_s + 1 - \frac{x_s}{\lambda}\xi(s) - \xi(s)^2\bigg|\|\epsilon\|_{L^2}\bigg) + O\bigg(\bigg|\frac{\lambda_s}{\lambda}\bigg|\|\epsilon\|_{L^2}\bigg) + O\bigg(\bigg|\frac{x_s}{\lambda} + 2\xi(s)\bigg|\|\epsilon\|_{L^2}\bigg) \\
&\qquad + O\bigg(\bigg|\xi_s - \frac{\lambda_s}{\lambda}\xi(s)\bigg|\|\epsilon\|_{L^2}\bigg) + O(\|\epsilon\|_{L^2}^2) + O(\|\epsilon\|_{L^2}^2\|\epsilon\|_{L^\infty}^3).
\end{aligned} \tag{10-47}$$

Then by Theorem 30 and the fundamental theorem of calculus,

$$\|yQ\|_{L^2}^2 + 4\int_{s_-}^{s_+}\big(\epsilon_2, \tfrac{1}{2}Q + xQ_x\big)_{L^2} = O(\eta_*).$$

Therefore, there exists $s' \in [s_-, s_+]$ such that

$$\big(\epsilon_2, \tfrac{1}{2}Q + xQ_x\big)_{L^2} < 0.$$

Make a Galilean transformation setting $\xi(s') = 0$ and a translation in space such that $x(s'') = 0$, where s'' is the other endpoint of the interval of integration. Also rescale so that $\lambda(s') = T_*^{1/200}/\eta_1$. Since $s' \geq 0$, there exists some $j \geq 0$ such that $s_j \leq s' + T_* < s_{j+1}$. By Theorem 31 and (10-23),

$$\begin{aligned}
&\int_{s'}^{s_{j+1+J}}\bigg|\frac{\lambda_s}{\lambda}\bigg|\,ds \lesssim J \quad \implies \quad \frac{1}{\eta_1} \leq \lambda(t) \leq \frac{1}{\eta_1}T_*^{1/100}, \\
&\sup_{s\in[s', s_{j+1+J}]}\frac{|\xi(s)|}{\lambda(s)} \ll \eta_0, \quad \text{and} \quad \sup_{s\in[s', s_{j+1+J}]}|x(s)| \ll T_*^{1/25}.
\end{aligned} \tag{10-48}$$

Then by Theorems 30 and 31,

$$\int_{s'}^{s_{j+1+J}}\|\epsilon(s)\|_{L^2}^2\,ds \lesssim 2^{-(j+1+J)}\eta_* + T_*^{1/50}\eta_*\int_{s'}^{s_{j+1+J}}\|\epsilon(s)\|_{L^2}^2\,ds \lesssim 2^{-(j+1+J)}\eta_*,$$

and therefore by definition of s_{j+1+J},

$$\int_{s'}^{s_{j+1+J}}\|\epsilon(s)\|_{L^2}\,ds \lesssim 1.$$

Then, (10-48) holds on the interval $[s', s_{j+1+2J}]$, and arguing by induction, for any $k \geq 1$,

$$\int_{s'}^{s_{j+k}}\|\epsilon(s)\|_{L^2}^2\,ds \lesssim 2^{-j-k}\eta_* \quad \text{and} \quad \int_{s'}^{s_{j+k}}\|\epsilon(s)\|_{L^2}\,ds \lesssim 1,$$

with implicit constant independent of k. Taking $k \to \infty$,

$$\int_{s'}^{\infty}\|\epsilon(s)\|_{L^2}^2\,ds = 0,$$

which implies that $\epsilon(s) = 0$ for all $s \geq s'$. Therefore, u is a soliton solution to (1-1). □

10.8. ***Almost monotone $\lambda(t)$.*** In the nonsymmetric case, when $\sup(I) = \infty$, u is equal to a soliton solution, and when $\sup(I) < \infty$, u is the pseudoconformal transformation of the soliton solution.

Theorem 33. *If u satisfies the conditions of Theorem 24, u blows up forward in time, and*

$$\sup(I) = \infty,$$

then u is equal to a soliton solution.

Proof. As in Theorem 22, for any integer $k \geq 0$, let

$$I(k) = \{s \geq 0 : 2^{-k+2} \leq \tilde{\lambda}(s) \leq 2^{-k+3}\}.$$

As in the proof of Theorem 22, there exists a sequence $k_n \nearrow \infty$ such that

$$|I(k_n)| 2^{-2k_n} \geq \frac{1}{k_n^2}$$

and such that $|I(k_n)| \geq |I(k)|$ for all $k \leq k_n$.

Lemma 34. *For n sufficiently large, there exists $s_n \in I(k_n)$ such that*

$$\|\epsilon(s_n)\|_{L^2} \lesssim k_n^2 2^{-2k_n}.$$

Proof. Let $I(k_n) = [a_n, b_n]$. By Theorem 31,

$$\int_{I(k_n)} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim \eta_*,$$

Then, using the virial identity in (10-47),

$$\int_{a_n}^{(3a_n+b_n)/4} \big(\epsilon_2, \tfrac{1}{2} Q + x Q_x\big)_{L^2} \, ds = O(\eta_*) + O(1).$$

Therefore, by the mean value theorem, there exists $s_n^- \in \big[a_n, \frac{1}{4}(3a_n + b_n)\big]$ such that

$$\big|\big(\epsilon_2(s_n^-), \tfrac{1}{2} Q + x Q_x\big)_{L^2}\big| \lesssim \frac{1}{|I(k_n)|}. \tag{10-49}$$

By a similar calculation, there exists $s_n^+ \in \big[\frac{1}{4}(a_n + 3b_n), b_n\big]$ such that

$$\big|\big(\epsilon_2(s_n^+), \tfrac{1}{2} Q + x Q_x\big)_{L^2}\big| \lesssim \frac{1}{|I(k_n)|}. \tag{10-50}$$

Therefore, by Theorem 30, (10-49) and (10-50) imply

$$\int_{s_n^-}^{s_n^+} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim \frac{1}{|I(k_n)|}. \tag{10-51}$$

Indeed, rescale so that $\lambda(s_n^-) = 1/\eta_1$. Then by Galilean transformation, suppose $\xi(s_n^-) = 0$ and by translation in space, suppose $x(s_n^+) = 0$. For all $s \in [s_n^-, s_n^+]$, by (10-23) and Theorem 31,

$$\frac{|\xi(s)|}{\lambda(s)} \lesssim \eta_1 \eta_* \quad \text{and} \quad |x(s)| \ll T_*^{1/25}. \tag{10-52}$$

Therefore, by Theorem 30 and (10-23),

$$\int_{s_n^-}^{s_n^+} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim \frac{1}{|I(k_n)|} + \eta_* T_*^{1/25} \int_{s_n^-}^{s_n^+} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim \frac{1}{|I(k_n)|}.$$

Remark. To make these computations completely rigorous, partition $[s_n^-, s_n^+]$ into a dyadic integer number of subintervals of length $\sim 1/\eta_*$, and then following the arguments proving Theorem 31, it is possible to prove that (10-52) holds on subintervals of length $\sim J/\eta_*$, and then by induction, (10-52) holds on $[s_n^-, s_n^+]$, which by Theorem 30 implies that (10-51) holds.

Then by the mean value theorem,

$$\|\epsilon(s_n)\|_{L^2}^2 \lesssim \frac{1}{|I(k_n)|^2}.$$

Since $|I(k_n)| \geq 2^{2k_n} k_n^{-2}$, the proof of Lemma 34 is complete. □

Make a Galilean transformation so that $\xi(s_n) = 0$. Then by (10-23), since $\lambda(s) \gtrsim 2^{-k_n}$ for all $s \in [0, s_n]$,

$$\frac{|\xi(s)|}{\lambda(s)} \lesssim 2^{k_n} \eta_*. \tag{10-53}$$

Now let m be the smallest integer such that

$$\frac{2^{2k_n}}{k_n^2} 2^m \geq |I(k_n)|. \tag{10-54}$$

Since $|I(k)| \leq |I(k_n)|$ for all $0 \leq k \leq k_n$, (10-54) implies that

$$|s_n| \leq 2^{2k_n+m+1}.$$

Let r_n be the smallest integer that satisfies

$$2^{(2k_n+m+1)/3} 2^{k_n} \frac{1}{\eta_1} \leq 2^{r_n}.$$

Then, as in the proof of Theorem 27, setting $t_n = s^{-1}(s_n)$, (10-53) and induction on frequency implies

$$\|P_{\geq r_n} u\|_{U_\Delta^2([0,t_n]\times\mathbb{R})} \lesssim \eta_*$$

and

$$\|P_{\geq r_n + k_n/4 + m/4} u\|_{U_\Delta^2([0,t_n]\times\mathbb{R})} \lesssim k_n^2 2^{-2k_n} 2^{-m}.$$

Furthermore,

$$E(P_{\leq r_n + k_n/4 + m/4} u(t_n)) \lesssim (k_n^2 2^{-2k_n} 2^{-m} 2^{r_n + k_n/4 + m/4})^2 \sim (k_n^2 2^{-k_n/12 - 5m/12} \eta_1^{-1})^2$$

and

$$\sup_{t \in [0,t_n]} E(P_{\leq r_n + k_n/4 + m/4} u(t)) \lesssim (k_n^2 2^{-k_n/12 - 5m/12} \eta_1^{-1})^2.$$

By (10-30), if $\xi_n(s)$ is the $\xi(s)$ in (10-8) for which $\xi_n(s_n) = 0$,

$$\sup_{0 \le s \le s_n} \frac{|\xi_n(s)|^2}{\lambda(s)^2} \lesssim (k_n^2 2^{-k_n/12 - 5m/12} \eta_1^{-1})^2,$$

which implies that $\xi(s)$ converges to some ξ_∞ as $s \to \infty$. Making a Galilean transformation that maps ξ_∞ to the origin and taking $n \to \infty$, since $m \ge 0$, (10-10) implies that $E(u_0) = 0$. Therefore, by the Gagliardo–Nirenberg inequality, u_0 is a soliton. ☐

When $\sup(I) < \infty$, suppose without loss of generality that $\sup(I) = 0$, and

$$\sup_{-1 < t < 0} \|\epsilon(t)\|_{L^2} \le \eta_*.$$

Then write the decomposition

$$u(t,x) = \frac{e^{-i\gamma(t)} \exp\left[-ix\frac{\xi(t)}{\lambda(t)}\right]}{\lambda(t)^{1/2}} Q\left(\frac{x - x(t)}{\lambda(t)}\right) + \frac{e^{-i\gamma(t)} \exp\left[-ix\frac{\xi(t)}{\lambda(t)}\right]}{\lambda(t)^{1/2}} \epsilon\left(t, \frac{x - x(t)}{\lambda(t)}\right),$$

and apply the pseudoconformal transformation to $u(t,x)$. For $-\infty < t < -1$,

$$\begin{aligned} v(t,x) = \frac{1}{t^{1/2}} \overline{u\left(\frac{1}{t}, \frac{x}{t}\right)} e^{ix^2/4t} = \frac{1}{t^{1/2}} &\frac{e^{i\gamma(1/t)} \exp\left[ix\frac{\xi(1/t)}{\lambda(1/t)}\right]}{\lambda(1/t)^{1/2}} Q\left(\frac{x - tx(1/t)}{t\lambda(1/t)}\right) e^{ix^2/4t} \\ &+ \frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)} \exp\left[ix\frac{\xi(1/t)}{\lambda(1/t)}\right]}{\lambda(1/t)^{1/2}} \overline{\epsilon\left(\frac{1}{t}, \frac{x - tx(1/t)}{t\lambda(1/t)}\right)} e^{ix^2/4t}. \end{aligned}$$

Since the L^2 norm is preserved by the pseudoconformal transformation,

$$\lim_{t \searrow -\infty} \left\| \frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)} \exp\left[ix\frac{\xi(1/t)}{\lambda(1/t)}\right]}{\lambda(1/t)^{1/2}} \overline{\epsilon\left(\frac{1}{t}, \frac{x - tx(1/t)}{t\lambda(1/t)}\right)} e^{ix^2/4t} \right\|_{L^2} = 0.$$

Next,

$$\frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)} \exp\left[ix\frac{\xi(1/t)}{\lambda(1/t)}\right]}{\lambda(1/t)^{1/2}} Q\left(\frac{x - tx(1/t)}{t\lambda(1/t)}\right) \exp\left[ix\frac{x(1/t)}{2}\right] \exp\left[-i\frac{t}{4}x\left(\frac{1}{t}\right)^2\right]$$

is of the form

$$e^{-i\tilde{\gamma}(t)} \exp\left[-ix\frac{\tilde{\xi}(t)}{\tilde{\lambda}(t)}\right] \tilde{\lambda}(t)^{-1/2} Q\left(\frac{x - \tilde{x}(t)}{\tilde{\lambda}(t)}\right),$$

where

$$\tilde{\gamma}(t) = \gamma\left(\frac{1}{t}\right) - \frac{1}{4}x\left(\frac{1}{t}\right)^2 t, \quad \tilde{\xi}(t) = \xi\left(\frac{1}{t}\right) + \frac{1}{2}x\left(\frac{1}{t}\right) t\lambda\left(\frac{1}{t}\right),$$
$$\tilde{\lambda}(t) = t\lambda\left(\frac{1}{t}\right), \quad \text{and} \quad \tilde{x}(t) = tx\left(\frac{1}{t}\right).$$

Also,

$$\begin{aligned}\Bigg\| &\frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)} \exp\big[ix\frac{\xi(1/t)}{\lambda(1/t)}\big]}{\lambda(1/t)^{1/2}} Q\Big(\frac{x - tx(1/t)}{t\lambda(1/t)}\Big) e^{ix^2/4t} \\ &\qquad - \frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)} \exp\big[ix\frac{\xi(1/t)}{\lambda(1/t)}\big]}{\lambda(1/t)^{1/2}} Q\Big(\frac{x - tx(1/t)}{t\lambda(1/t)}\Big) \exp\Big[ix\frac{x(1/t)}{2}\Big] e^{itx(1/t)^2} \Bigg\|_{L^2} \\ &= \Bigg\| \frac{1}{t^{1/2}\lambda(1/t)^{1/2}} Q\Big(\frac{x - tx(1/t)}{t\lambda(1/t)}\Big) \Big(\exp\Big[i\frac{(x - tx(1/t))^2}{4t}\Big] - 1\Big) \Bigg\|_{L^2}.\end{aligned}$$

As in (9-7) and (9-8),

$$\lim_{t \searrow -\infty} \Bigg\| \frac{1}{t^{1/2}\lambda(1/t)^{1/2}} Q\Big(\frac{x - tx(1/t)}{t\lambda(1/t)}\Big) \Big(\exp\Big[i\frac{(x - tx(1/t))^2}{4t}\Big] - 1\Big) \Bigg\|_{L^2} = 0.$$

Therefore, by time reversal symmetry, v satisfies the conditions of Theorem 24, and v is a solution that blows up backward in time at $\inf(I) = -\infty$, so therefore, by Theorem 33, v must be a soliton. Therefore, u is the pseudoconformal transformation of a soliton, which proves Theorem 5.

Appendix: U^p and V^p spaces

The description here of U^p and V^p spaces comes from Section 5.3 of [Dodson 2019]. See also [Koch et al. 2014].

Definition (U^p space). Suppose $u \in U^p$. We say that u is a U^p atom if there exists a sequence $\{t_k\} \nearrow \infty$ satisfying

$$u = \sum_k 1_{[t_k, t_{k+1})} u_k,$$

and

$$\sum \|u_k\|^p_{L^2(\mathbb{R}^d)} = 1.$$

Then define the norm

$$\|u(t)\|_{U^p(\mathbb{R}\times\mathbb{R}^d)} = \inf\Big\{\sum_\lambda |c_\lambda| : u(t) = \sum_\lambda c_\lambda u_\lambda \text{ for almost every } t \in \mathbb{R}, \text{ where } u_\lambda(t) \text{ is a } U^p \text{ atom}\Big\}.$$

Then set

$$\|u\|_{U^p_\Delta(\mathbb{R}\times\mathbb{R}^d)} = \|e^{-it\Delta}u\|_{U^p(\mathbb{R}\times\mathbb{R}^d)}.$$

Functions with finite U^p_Δ norm have finite Strichartz norms $L^{\tilde p}_t L^q_x$ when $p \le \tilde p \le \infty$ and (p, q) is an admissible pair. Bilinear Strichartz estimates also hold for $\tilde p$ in the appropriate range.

Theorem 35. *If I is an interval with $t_0 \in I$, for any $1 < p < \infty$, if $1/p + 1/p' = 1$,*

$$\Big\| \int_{t_0}^t e^{i(t-\tau)\Delta} F(\tau)\, d\tau \Big\|_{U^p_\Delta(I\times\mathbb{R}^d)} \lesssim \sup_{\|G\|_{V^{p'}_\Delta(I\times\mathbb{R}^d)}=1} \int_I \langle G, F\rangle \, d\tau.$$

The V_{Δ}^{p} space is defined as follows.

Definition (V_{Δ}^{p} spaces). Suppose $I = [0, T]$ is a compact interval. Define the partition

$$\mathcal{Z} = \{0 = t_0 < t_1 < \cdots < t_n = T\}.$$

Then for $1 < p < \infty$ define the norm

$$\|v\|_{V^p(\mathcal{Z}; I \times \mathbb{R}^d)}^{p} = \sum_{k=1}^{n} \|v(t_k) - v(t_{k-1})\|_{L^2(\mathbb{R}^d)}^{p}.$$

Then write

$$\|v\|_{V_{\Delta}^{p}(\mathcal{Z}; I \times \mathbb{R}^d)}^{p} = \|e^{-it\Delta} v(t)\|_{V_{\Delta}^{p}(\mathcal{Z}; I \times \mathbb{R}^d)}^{p},$$

and define the norm

$$\|v\|_{V_{\Delta}^{p}(I \times \mathbb{R}^d)} = \sup_{\mathcal{Z}} \|v\|_{V_{\Delta}^{p}(\mathcal{Z}; I \times \mathbb{R}^d)} + \|v\|_{L_t^{\infty} L_x^2 (I \times \mathbb{R}^d)}.$$

The V^p space embedding will be extremely useful.

Theorem 36. *If* $p < q$,

$$V^p \subset U^q.$$

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UNIFORM STABILITY IN THE EUCLIDEAN ISOPERIMETRIC PROBLEM FOR THE ALLEN–CAHN ENERGY

FRANCESCO MAGGI AND DANIEL RESTREPO

We consider the isoperimetric problem defined on the whole $\mathbb{R}^n$ by the Allen–Cahn energy functional. For nondegenerate double-well potentials, we prove sharp quantitative stability inequalities of quadratic type which are uniform in the length scale of the phase transitions. We also derive a rigidity theorem for critical points analogous to the classical Alexandrov theorem for constant mean curvature boundaries.

1. Introduction

1A. ***Overview.*** We study the family of "Euclidean isoperimetric problems" on $\mathbb{R}^n$, $n \geq 2$, given by

$$\Psi(\sigma, m) = \inf\left\{\mathcal{AC}_\sigma(u) : \int_{\mathbb{R}^n} V(u) = m,\ u \in H^1(\mathbb{R}^n; [0,1])\right\}, \quad \sigma, m > 0, \tag{1-1}$$

associated to the Allen–Cahn energy functionals of a nondegenerate double-well potential W (see (1-11) and (1-12) below)

$$\mathcal{AC}_\sigma(u) = \sigma \int_{\mathbb{R}^n} |\nabla u|^2 + \frac{1}{\sigma}\int_{\mathbb{R}^n} W(u), \quad \sigma > 0. \tag{1-2}$$

We analyze in particular the relation of these problems to the classical Euclidean isoperimetric problem

$$\Psi_{\mathrm{iso}}(m) = \inf\{P(E) : E \subset \mathbb{R}^n,\ |E| = m\} = n\omega_n^{1/n} m^{(n-1)/n}, \quad m > 0, \tag{1-3}$$

in the natural regime where the phase transition length scale σ and the volume constraint m satisfy

$$0 < \sigma < \varepsilon_0 m^{1/n} \tag{1-4}$$

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for some sufficiently small (dimensionless) constant $\varepsilon_0 = \varepsilon_0(n, W)$. The volume constraint in $\Psi(\sigma, m)$ is prescribed by means of the potential $V(t) = \big(\int_0^t \sqrt{W}\big)^{n/(n-1)}$. This specific choice is natural in light of the classical estimate obtained by combining Young's inequality with the BV-Sobolev inequality/Euclidean isoperimetry, and showing that, if $u \in H^1(\mathbb{R}^n; [0, 1])$, then, for $\Phi(t) = \int_0^t \sqrt{W}$,

$$\mathcal{AC}_\sigma(u) \geq 2\int_{\mathbb{R}^n} |\nabla u|\sqrt{W(u)} = 2\int_{\mathbb{R}^n} |\nabla \Phi(u)| > 2n\omega_n^{1/n}\bigg(\int_{\mathbb{R}^n} V(u)\bigg)^{(n-1)/n}. \tag{1-5}$$

In particular, by our choice of V, $\Psi(\sigma, m)$ is always nontrivial,[1] with

$$\Psi(\sigma, m) > 2\Psi_{\mathrm{iso}}(m) \quad \text{for all } \sigma, m > 0. \tag{1-6}$$

(The strict sign does not follow from (1-5) alone, but also requires the existence of minimizers in (1-5).) By combining (1-6) with a standard construction of competitors for $\Psi(\sigma, m)$, one sees immediately that

$$\lim_{\sigma\to 0^+} \Psi(\sigma, m) = 2\Psi_{\mathrm{iso}}(m) \quad \text{for all } m > 0. \tag{1-7}$$

The relation between the Allen–Cahn energy and the perimeter functional is of course a widely explored subject (without trying to be exhaustive, see, for example, [Modica and Mortola 1977; Modica 1987a; Sternberg 1988; Luckhaus and Modica 1989; Hutchinson and Tonegawa 2000; Röger and Tonegawa 2008; Le 2011; Tonegawa and Wickramasekera 2012; Dal Maso et al. 2015; Le 2015; Gaspar 2020]), and so is the relation between the "volume-constrained" minimization of $\mathcal{AC}_\sigma$ and relative isoperimetry/capillarity theory in bounded or periodic domains (see, e.g., [Modica 1987b; Sternberg and Zumbrun 1998; 1999; Pacard and Ritoré 2003; Carlen et al. 2006; Bellettini et al. 2006; Leoni and Murray 2016]). The goal of this paper is exploring in detail the proximity of $\Psi(\sigma, m)$ to the classical Euclidean isoperimetric problem $\Psi_{\mathrm{iso}}(m)$ in connection with two fundamental properties of the latter:

(i) The validity of the sharp quantitative Euclidean isoperimetric inequality [Fusco et al. 2008]: if $E \subset \mathbb{R}^n$ has finite perimeter $P(E)$ and positive and finite volume (Lebesgue measure) $\mathcal{L}^n(E)$, then

$$C(n)\sqrt{\frac{P(E)}{n\omega_n^{1/n}\mathcal{L}^n(E)^{(n-1)/n}} - 1} \geq \inf_{x_0\in\mathbb{R}^n} \frac{\mathcal{L}^n(E\Delta B_r(x_0))}{\mathcal{L}^n(E)}, \quad r = \bigg(\frac{\mathcal{L}^n(E)}{\omega_n}\bigg)^{1/n}, \tag{1-8}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.

(ii) Alexandrov's theorem [1962] (see [Delgadino and Maggi 2019] for a distributional version): a bounded open set whose boundary is smooth and has constant mean curvature is a ball; in other words, among bounded sets, the only volume-constrained critical points of the perimeter functional are its (global) volume-constrained minimizers.

[1] Obviously, this is not always true with others choices of V. For example, setting $V(t) = t$ in (1-1), which is the most common choice in addressing diffuse interface capillarity problems in bounded containers, one has $\Psi(\sigma, m) = 0$ by a simple scaling argument. Among the possible choices that make $\Psi(\sigma, m)$ nontrivial, ours has of course the advantage of appearing naturally in the lower bound (1-5). For this reason, and in the interest of definiteness and simplicity, we have not considered more general options here.

Concerning property (i), the natural question in relation to $\Psi(\sigma, m)$ is if a sharp stability estimate similar to (1-8) holds *uniformly* with respect to the ratio $\sigma/m^{1/n} \in (0, \varepsilon_0)$ for $\Psi(\sigma, m)$. Uniformity in $\sigma/m^{1/n}$ seems indeed a necessary feature for a stability estimate of this kind to be physically meaningful and interesting.

Concerning property (ii), we notice that the notion of smooth, volume-constrained critical point of $\Psi(\sigma, m)$ is that of a nonzero function $u \in C^2(\mathbb{R}^n; [0, 1])$ such that the semilinear PDE

$$-2\sigma^2 \Delta u = \sigma\lambda V'(u) - W'(u) \quad \text{on } \mathbb{R}^n \tag{1-9}$$

holds for a Lagrange multiplier $\lambda \in \mathbb{R}$. The boundedness assumption in Alexandrov's theorem is crucial to avoid examples of nonspherical constant mean curvature boundaries, like cylinders and unduloids. This is directly translated, for solutions of (1-9), into the requirement that $u(x) \to 0$ as $|x| \to \infty$, without which semilinear PDEs like (1-9) are known to possess nonradial solutions modeled on the aforementioned examples of unbounded constant mean curvature boundaries; see, e.g., [Pacard and Ritoré 2003].

Under the decay assumption $u(x) \to 0$ as $|x| \to \infty$, and without further constraints on σ and λ, every solution of (1-9) will be radial symmetric thanks to the moving-planes method [Gidas et al. 1981]. However, even in presence of symmetry, possible solutions to (1-9) will have a geometric meaning (and thus a chance of being exhausted by the family of global minimizers of $\Psi(\sigma, m)$) only if the parameters σ and λ are taken in the "geometric regime" where $\sigma\,\lambda$ is small. To explain why we consider such regime geometrically significant, we notice that the Lagrange multiplier λ in (1-9) has the dimension of an inverse length, which, geometrically, is the dimensionality of curvature. For σ to be the length of a phase transition around an interface of curvature λ, it must be that

$$0 < \sigma\,\lambda < \nu_0 \tag{1-10}$$

for some sufficiently small (dimensionless) constant $\nu_0 = \nu_0(n, W)$. Notice that since inverse length is volume$^{-1/n} = m^{-1/n}$, (1-10) is compatible with (1-4). We conclude that a natural generalization of Alexandrov's theorem to the Allen–Cahn setting is showing the existence of constants ε_0 and ν_0, depending on n and W only, such that, if $u \in C^2(\mathbb{R}^n; [0, 1])$ vanishes at infinity and solves (1-9) for σ and λ as in (1-10), then u is a minimizer of $\Psi(\sigma, m)$ for some value m such that (1-4) holds.

1B. ***Statement of the main theorem.*** We start by setting the following notation and conventions:

Assumptions on W. The double-well potential $W \in C^{2,1}[0, 1]$ satisfies the standard set of nondegeneracy assumptions

$$W(0) = W(1) = 0, \quad W > 0 \text{ on } (0, 1), \quad W''(0), W''(1) > 0, \tag{1-11}$$

as well as the normalization

$$\int_0^1 \sqrt{W} = 1. \tag{1-12}$$

Correspondingly to W, we introduce the potential V used in imposing the volume constraint in $\Psi(\sigma, m)$, by setting

$$V(t) = \Phi(t)^{n/(n-1)}, \quad \Phi(t) = \int_0^t \sqrt{W}, \quad t \in [0, 1]. \tag{1-13}$$

Notice that both V and Φ are strictly increasing on [0, 1], with $V(1)=\Phi(1)=1$ and $\Phi(t)\approx t^2$ and $V(t)\approx t^{2n/(n-1)}$ as $t\to 0^+$. All the relevant properties of W, Φ and V are collected in Section A3.

Classes of radial decreasing functions. We say that $u:\mathbb{R}^n\to\mathbb{R}$ is radial if $u(x)=\zeta(|x|)$ for some $\zeta:[0,\infty)\to\mathbb{R}$, and that u is radial decreasing if, in addition, ζ is decreasing. We denote by

$$\mathcal{R}_0,\quad \mathcal{R}_0^*,$$

the family of radial decreasing and radial strictly decreasing functions. For the sake of simplicity, when u is radial we shall simply write u in place of ζ, that is, we shall use interchangeably $u(x)$ and $u(r)$ to denote the value of u at x with $|x|=r$. Similarly, we shall write u', u'', etc. for the radial derivatives of u.

Universal constants and rates. We say that a real number is a *universal constant* it is positive and can be defined in terms of the dimension n and of the double-well potential W only. Following a widely used convention, we will use the latter C for a generically "large" universal constant, and $1/C$ for a generically "small" one. We will use ε_0, δ_0, ν_0, ℓ_0, etc. for small universal constants whose value will be typically "chosen" at the end of an argument to make products like $C\varepsilon_0$ "sufficiently small". Finally, given $k\in\mathbb{N}$, we will write "$f(\varepsilon)=\mathrm{O}(\varepsilon^k)$ as $\varepsilon\to 0^+$" if there exists a universal constant C such that $|f(\varepsilon)|\le C\varepsilon^k$ for every $\varepsilon\in(0,1/C)$; similar definitions are given for "$\mathrm{O}(t)$ as $t\to\infty$", etc.

Theorem 1.1 (main theorem). *If $n\ge 2$ and $W\in C^{2,1}[0,1]$ satisfies* (1-11) *and* (1-12), *then there exists a universal constant ε_0 such that setting*

$$\mathcal{X}(\varepsilon_0)=\{(\sigma,m):0<\sigma<\varepsilon_0 m^{1/n}\}$$

the following hold:

(i) *For every $(\sigma,m)\in\mathcal{X}(\varepsilon_0)$ there exists a minimizer $u_{\sigma,m}$ of $\Psi(\sigma,m)$ such that $u_{\sigma,m}\in\mathcal{R}_0^*\cap C^2(\mathbb{R}^n;(0,1))$, every other minimizer of $\Psi(\sigma,m)$ is obtained from $u_{\sigma,m}$ by translation, and the Euler–Lagrange equation*

$$-2\sigma^2\Delta u_{\sigma,m}=\sigma\Lambda(\sigma,m)V'(u_{\sigma,m})-W'(u_{\sigma,m}) \tag{1-14}$$

holds on $\mathbb{R}^n$ for some $\Lambda(\sigma,m)>0$.

(ii) Ψ *is continuous on $\mathcal{X}(\varepsilon_0)$ and*

$$\Psi(\sigma,\cdot)\ \text{is strictly concave, strictly increasing, and continuously differentiable on}\ ((\sigma/\varepsilon_0)^n,\infty), \tag{1-15}$$

$$\Lambda(\sigma,\cdot)=\frac{\partial\Psi}{\partial m}(\sigma,\cdot)\ \text{is strictly decreasing and continuous on}\ ((\sigma/\varepsilon_0)^n,\infty), \tag{1-16}$$

$$\Psi(\cdot,m)\ \text{is strictly increasing on}\ (0,\varepsilon_0 m^{1/n}). \tag{1-17}$$

Moreover, setting $\varepsilon=\sigma/m^{1/n}$, we have

$$\frac{\Psi(\sigma,m)}{m^{(n-1)/n}}=2n\omega_n^{1/n}+2n(n-1)\omega_n^{2/n}\kappa_0\varepsilon+\mathrm{O}(\varepsilon^2), \tag{1-18}$$

$$m^{1/n}\Lambda(\sigma,m)=2(n-1)\omega_n^{1/n}+\mathrm{O}(\varepsilon), \tag{1-19}$$

as $\varepsilon \to 0^+$ with $(\sigma, m) \in \mathcal{X}(\varepsilon_0)$. Here κ_0 is the universal constant defined by

$$\kappa_0 = \int_{\mathbb{R}} (V'(\eta)\eta' + W(\eta))s\,ds, \tag{1-20}$$

and η is the unique solution to $\eta' = -\sqrt{W(\eta)}$ on $\mathbb{R}$ with $\eta(0) = \frac{1}{2}$.

(iii) *Uniform stability*: *for every $(\sigma, m) \in \mathcal{X}(\varepsilon_0)$ and $u \in H^1(\mathbb{R}^n; [0, 1])$ with $\int_{\mathbb{R}^n} V(u) = m$ we have*, *for a universal constant C*,

$$C\sqrt{\frac{\mathcal{AC}_\sigma(u)}{\Psi(\sigma, m)} - 1} \geq \inf_{x_0 \in \mathbb{R}^n} \frac{1}{m} \int_{\mathbb{R}^n} |\Phi(u) - \Phi(T_{x_0} u_{\sigma,m})|^{n/(n-1)}, \tag{1-21}$$

where $T_{x_0} u_{\sigma,m}(x) = u_{\sigma,m}(x - x_0)$, $x \in \mathbb{R}^n$;

(iv) *Rigidity of critical points*: *there exists a universal constant ν_0 such that*, *if $\sigma > 0$, $u \in C^2(\mathbb{R}^n; [0, 1])$, $u(x) \to 0^+$ as $|x| \to \infty$, and u is a solution of*

$$-2\sigma^2 \Delta u = \sigma\lambda V'(u) - W'(u) \quad \text{on } \mathbb{R}^n \tag{1-22}$$

for a parameter λ such that

$$0 < \sigma\lambda < \nu_0, \tag{1-23}$$

then there exist $x_0 \in \mathbb{R}^n$ and $m > 0$ such that

$$\sigma < \varepsilon_0 m^{1/n}, \quad \lambda = \Lambda(\sigma, m), \quad u = T_{x_0} u_{\sigma,m}.$$

In particular, *u is a minimizer of $\Psi(\sigma, m)$.*

1C. *Relation of Theorem 1.1*(iii) *to Euclidean isoperimetric stability.* We start with some remarks connecting the (σ, m)-uniform stability estimate (1-21) to the sharp quantitative Euclidean isoperimetric inequality (1-8). To this end, it will be convenient to introduce the unit volume problem

$$\psi(\varepsilon) = \Psi(\varepsilon, 1) = \inf\left\{\mathcal{AC}_\varepsilon(u) : \int_{\mathbb{R}^n} V(u) = 1,\ u \in H^1(\mathbb{R}^n; [0, 1])\right\}, \quad \varepsilon > 0,$$

and correspondingly set

$$\lambda(\varepsilon) = \Lambda(\varepsilon, 1) = \frac{\partial \Psi}{\partial m}(\varepsilon, 1), \quad u_\varepsilon = u_{\varepsilon,1}, \quad \varepsilon > 0.$$

Notice that all the information about $\Psi(\sigma, m)$, $u_{\sigma,m}$, and $\Lambda(\sigma, m)$, is contained in $\psi(\varepsilon)$, u_ε and $\lambda(\varepsilon)$, thanks to the identities

$$\frac{\Psi(\sigma, m)}{m^{(n-1)/n}} = \psi\left(\frac{\sigma}{m^{1/n}}\right), \quad m^{1/n}\,\Lambda(\sigma, m) = \lambda\left(\frac{\sigma}{m^{1/n}}\right), \quad u_{\sigma,m}(x) = u_{\sigma/m^{1/n}}\left(\frac{x}{m^{1/n}}\right),$$

which are easily proved by a scaling argument (see (A-1) and (A-2)).

With this terminology at hand, we start by noticing that the right-hand side of (1-21) is bounded from above by $C(n)$ thanks to the volume constraint $\int_{\mathbb{R}^n} V(u) = m$. Therefore, in proving (1-21) with, say,

$(\sigma, m) = (\varepsilon, 1)$, one can directly assume that u is a "low-energy competitor for $\psi(\varepsilon)$" in the sense that, for a suitably small universal constant ℓ_0,

$$\mathcal{AC}_\varepsilon(u) \le \psi(\varepsilon) + \ell_0. \tag{1-24}$$

Now, if u is such a low-energy competitor u, then $f = \Phi(u)$ is $(\ell_0 + C\varepsilon)$-close to being an equality case for the BV-Sobolev inequality

$$|Df|(\mathbb{R}^n) \ge n\omega_n^{1/n} \quad \text{if } \textstyle\int_{\mathbb{R}^n} |f|^{n/(n-1)} = 1, \tag{1-25}$$

where $|Df|$ denotes the total variation measure of $f \in \mathrm{BV}(\mathbb{R}^n)$, and $|Df| = |\nabla f|\,dx$ if $f \in W^{1,1}(\mathbb{R}^n)$; see [Ambrosio et al. 2000]. Indeed, by an elementary comparison argument, we have

$$\psi(\varepsilon) \le 2n\omega_n^{1/n} + C\varepsilon \quad \text{for all } \varepsilon < \varepsilon_0, \tag{1-26}$$

while (1-5) gives

$$\mathcal{AC}_\varepsilon(u) - 2n\omega_n^{1/n} = \int_{\mathbb{R}^n} \left(\sqrt{\varepsilon}|\nabla u| - \sqrt{\frac{W(u)}{\varepsilon}}\right)^2 + 2\left\{\int_{\mathbb{R}^n} |\nabla[\Phi(u)]| - n\omega_n^{1/n}\right\}, \tag{1-27}$$

so that the combination of (1-24), (1-26) and (1-27) gives

$$\int_{\mathbb{R}^n} |\nabla[\Phi(u)]| - n\omega_n^{1/n} \le C(\ell_0 + \varepsilon),$$

while, clearly, $\int_{\mathbb{R}^n} f^{n/(n-1)} = \int_{\mathbb{R}^n} V(u) = 1$.

It is well known that (1-25) boils down to the Euclidean isoperimetric inequality if $f = 1_E$ is the characteristic function of $E \subset \mathbb{R}^n$, and that equality holds in (1-25) if and only if $f = a\,1_{B_r(x_0)}$ for some $r, a \ge 0$. A sharp quantitative version of (1-25) was proved in [Fusco et al. 2008] on sets, and then in [Fusco et al. 2007, Theorem 1.1] on functions, and takes the following form: if $n \ge 2$, $f \in \mathrm{BV}(\mathbb{R}^n)$, $f \ge 0$, and $\int_{\mathbb{R}^n} f^{n/(n-1)} = 1$, then there exist $x_0 \in \mathbb{R}^n$ and $r > 0$ such that

$$C(n)\sqrt{|Df|(\mathbb{R}^n) - n\omega_n^{1/n}} \ge \inf_{x_0 \in \mathbb{R}^n, r>0} \int_{\mathbb{R}^n} |f - a(r)1_{B_r(x_0)}|^{n/(n-1)}, \tag{1-28}$$

where $a(r)$ is defined by $\omega_n r^n a(r)^{n/(n-1)} = 1$. The uniform stability estimate (1-21) is thus modeled after (1-28), where of course one is working with a different "deficit", namely, $\mathcal{AC}_\varepsilon(u) - \psi(\varepsilon)$ rather than $|Df|(\mathbb{R}^n) - n\,\omega_n^{1/n}$ for $f = \Phi(u)$, and with a different "asymmetry", namely, the $n/(n-1)$-th power of the distance of $\Phi(u)$ from Φ composed with u_ε rather than with the multiple of the characteristic function of a ball.

The key result behind (1-21) is the following *Fuglede-type estimate* for $\psi(\varepsilon)$ (Theorem 4.1): there exist universal constants δ_0 and ε_0 such that if $\varepsilon < \varepsilon_0$, $u \in H^1(\mathbb{R}^n; [0, 1])$ is a radial (but not necessarily radial decreasing) function, $\int_{\mathbb{R}^n} V(u) = 1$ and

$$\int_{\mathbb{R}^n} |u - u_\varepsilon|^2 \le C\varepsilon, \quad \|u - u_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \le \delta_0, \tag{1-29}$$

then

$$C(\mathcal{AC}_\varepsilon(u) - \psi(\varepsilon)) \ge \int_{\mathbb{R}^n} \varepsilon|\nabla(u - u_\varepsilon)|^2 + \frac{(u - u_\varepsilon)^2}{\varepsilon}. \tag{1-30}$$

Note carefully the restriction here to *radial* functions. The right-hand side of (1-30) is the natural ε-dependent Hilbert norm associated to $\mathcal{AC}_\varepsilon$. By the usual trick based on Young's inequality, (1-30) implies

$$C(\mathcal{AC}_\varepsilon(u)-\psi(\varepsilon))\geq\int_{\mathbb{R}^n}|\nabla[(u-u_\varepsilon)^2]|\quad\text{for all } u \text{ radial, } \textstyle\int_{\mathbb{R}^n}V(u)=1, \tag{1-31}$$

and, then, thanks to the H^1-Sobolev inequality,

$$C(\mathcal{AC}_\varepsilon(u)-\psi(\varepsilon))\geq\left(\int_{\mathbb{R}^n}|u-u_\varepsilon|^{2n/(n-1)}\right)^{(n-1)/n}\quad\text{for all } u \text{ radial, } \textstyle\int_{\mathbb{R}^n}V(u)=1. \tag{1-32}$$

The ε-independent stability estimate (1-32) (and, a fortiori, the stronger estimate (1-31)) cannot hold on general $u\in H^1(\mathbb{R}^n;[0,1])$ with $\int_{\mathbb{R}^n}V(u)=1$: indeed, if this were the case, one could take in (1-32) $u=v_\varepsilon$ to be a family of smoothings of 1_E for any set $E\subset\mathbb{R}^n$, and then let $\varepsilon\to0^+$, to find a version of (1-8) with linear rather than quadratic rate. However, such linear estimate is well known to be false, since the rate in (1-8) is saturated, for example, by a family of ellipsoids converging to a ball.

We conclude that, on radial functions, one can get estimates, like (1-30), (1-31) and (1-32), that are stronger than what is available for generic functions. We notice in this regard that the validity of stronger stability estimates in presence of symmetries is well-known. For example, in the case of the BV-Sobolev inequality, it was proved in [Fusco et al. 2007, Theorem 3.1] that if $f\in\mathrm{BV}(\mathbb{R}^n)$ is radial decreasing, $f\geq0$, and $\int_{\mathbb{R}^n}f^{n/(n-1)}=1$, then (1-28) can be improved to

$$C(n)(|Df|(\mathbb{R}^n)-n\,\omega_n^{1/n})\geq\int_{\mathbb{R}^n}|f-a(r)1_{B_r}|^{n/(n-1)}; \tag{1-33}$$

i.e., the quadratic rate in (1-28) is refined into a linear rate.

We finally notice that (1-21) implies the sharp quantitative form of the Euclidean isoperimetric inequality (1-8) by a standard approximation argument. However, since our proof of (1-21) exploits (1-8), we are not really providing a new proof of (1-8). We approach the proof of (1-21) as follows. Adopting the general selection principle strategy of [Cicalese and Leonardi 2012] we start by deducing (1-21) on radial functions from the Fuglede-type inequality (1-30). Then we adapt to our setting the quantitative symmetrization method from the proof of (1-8) originally devised in [Fusco et al. 2008], and thus reduce the proof of (1-21) from the general case to the radial decreasing case. (It is in this reduction step, see in particular Theorem 5.4, that we exploit (1-8).) In principle, one could have tried to approach (1-21) by working on general functions in both the selection principle and in the Fuglede-type estimate steps. This approach does not seem convenient, however, since it would not save the work needed to implement the selection principle and the Fuglede-type estimates on radial functions, while, at the same time, it would still require the repetition of all the work done in [Cicalese and Leonardi 2012] to prove (1-8). In other words, an advantage of the approach followed here is that it separates neatly the two stability mechanisms at work in (1-21), the one related to the relation with the Euclidean isoperimetric problem, and the one specific to optimal transition profile problem (which is entirely captured by working with radial functions).

1D. ***Remarks on the Alexandrov-type result.*** We now make some comments on the proof of Theorem 1.1(iv) and explain why this result is closely related to the stability problem addressed in Theorem 1.1(iii).

We start by noticing that any $u \in C^2(\mathbb{R}^n; [0,1])$, with $u(x) \to 0$ as $|x| \to \infty$, and solving (1-22) for some $\sigma > 0$ and $\lambda \in \mathbb{R}$, will necessarily be a radial function by the moving planes method of [Gidas et al. 1981]; see Theorem 6.2(i) below.

However, as explained in the overview, there is no clear reason to expect these solutions to have a geometric meaning unless σ and λ are in a meaningful geometric relation, which, interpreting λ as a curvature and σ as a phase transition length, must take the form of $0 < \sigma\lambda < \nu_0$ for some sufficiently small ν_0; see (1-10). In Theorem 6.2(ii) we apply to (1-22) a classical result of [Peletier and Serrin 1983] about the uniqueness of radial solutions of semilinear PDEs on $\mathbb{R}^n$. Interestingly, the condition $0 < \sigma\lambda < \nu_0$, which was introduced because its natural geometric interpretation, plays a crucial role in checking the validity of one of the assumptions of the Peletier–Serrin uniqueness theorem.[2]

Once symmetry and uniqueness have been addressed by means of classical results like [Gidas et al. 1981; Peletier and Serrin 1983], proving Theorem 1.1(iv) essentially amounts to answering the following question: what is the range of values of λ in (1-22) corresponding to the minimizers $u_{\sigma,m}$ of $\Psi(\sigma, m)$ (with $0 < \sigma < \varepsilon_0 m^{1/n}$)? Can we show that every λ satisfying $0 < \sigma\,\lambda < \nu_0$ for a sufficiently small universal ν_0 falls in that range?

Looking back at (1-14) we are thus trying to identify the range of $m \mapsto \Lambda(\sigma, m) = (\partial\Psi/\partial m)(\sigma, m)$ for $m > (\sigma/\varepsilon_0)^n$, and to show that it contains an interval of the form $(0, \nu_0/\sigma)$. Such range is indeed proved to be an interval in Theorem 1.1(ii), where we show that $\Lambda(\sigma, \cdot)$ is decreasing and continuous. The fact that this interval contains a subinterval of the form $(0, \nu_0/\sigma)$ is also something that is established in Theorem 1.1(ii), specifically when we analyze the asymptotic behavior of $\Lambda(\sigma, m)$ as $\sigma/m^{1/n} \to 0$; see (1-19). Here we want to stress, however, the role of the *continuity* of $\Lambda(\sigma, \cdot)$, which is of course crucial in showing that $\{\Lambda(\sigma, m)\}_{m>(\sigma/\varepsilon_0)^n}$ covers the *interval* of values between the end-points $\Lambda(\sigma, +\infty) = 0$ and $\Lambda(\sigma, (\sigma/\varepsilon_0)^n)$. In turn, the Fuglede-type stability estimate (1-30) plays a crucial role in our proof of this continuity property: see Step 3 in the proof of Corollary 4.2.

The importance of the Fuglede-type estimate (1-30) in answering both questions of uniform stability and of Alexandrov-type rigidity is the main reason why both problems have been addressed in a same paper.

1E. ***Organization of the paper and proof of Theorem 1.1.*** The existence of minimizers of $\psi(\varepsilon)$ (for $\varepsilon < \varepsilon_0$) and the fact that such minimizers must be radial decreasing (although not necessarily unique up to translations) is established in Section 2 (see Theorem 2.1) through a careful concentration-compactness argument, which exploits both the quantitative stability for the BV-Sobolev inequality (in ruling out vanishing) and the specific properties of the Allen–Cahn energy (in ruling out dichotomy). After deducing the validity of the Euler–Lagrange equation (which, because of the range constraint $0 \le u \le 1$, holds initially only as a system of variational inequalities), the radial decreasing rearrangement of a minimizer is proved to be *strictly* decreasing, so that the Brothers–Ziemer theorem [1988] can be used to infer that generic minimizers belong to $\mathcal{R}_0^*$. This existence argument is then adapted to a more general family of perturbations of $\psi(\varepsilon)$, which later plays a crucial role in obtaining the main stability estimates (1-21) on

[2] In particular, it is not obvious to us if, outside of the "geometrically natural" regime defined by (1-10), we should expect uniqueness of radial solutions of (1-22) with decay at infinity.

radial decreasing functions; see Theorem 2.2. Here the notion of "critical sequence" for $\psi(\varepsilon_j)$, $\varepsilon_j \in (0, \varepsilon_0)$, which mixes the notion of "low-energy sequence" to that of "Palais–Smale sequence", is introduced.

In Section 3 we prove a resolution result for minimizers of $\psi(\varepsilon)$ (and, more generally, for the above-mentioned notion of critical sequence). In particular, in Theorem 3.1, we show, quantitatively in ε, that minimizers u_ε of $\psi(\varepsilon)$ in $\mathcal{R}_0$ are close to an ansatz which is well-known in the literature (see, e.g., [Niethammer 1995; Leoni and Murray 2016]) and is given by

$$u_\varepsilon(x) \approx \eta\left(\frac{|x| - R_0}{\varepsilon} - \tau_0\right), \quad R_0 = \frac{1}{\omega_n^{1/n}}, \quad \tau_0 = \int_{\mathbb{R}} \eta' V'(\eta) s \, ds,$$

where η is the unique solution of $\eta' = -\sqrt{W(\eta)}$ on $\mathbb{R}$ with $\eta(0) = \frac{1}{2}$. Exponential decay rates against this ansatz are then obtained in that same theorem. Our analysis is comparably simpler than that of [Leoni and Murray 2016] because our solutions are monotonic decreasing, and, in particular, cannot exhibit the oscillatory behavior at infinity also described, for positive solutions of general semilinear PDEs like (1-22), in [Ni 1983].

Section 4 is devoted to the proof of the Fuglede-type estimate (1-30). This is crucially based on the resolution theorem and on a careful contradiction argument based on the concentration-compactness principle. The Fuglede-type estimate is then shown to imply the uniqueness of radial minimizers (in particular, there is a unique minimizer u_ε of $\psi(\varepsilon)$ in $\mathcal{R}_0$, and every other minimizer of $\psi(\varepsilon)$ is obtained from u_ε by translation), the continuity of $\lambda(\varepsilon)$ on $\varepsilon < \varepsilon_0$, and the expansions as $\varepsilon \to 0^+$ for $\psi(\varepsilon)$ and $\lambda(\varepsilon)$ (which, by scaling, imply (1-18) and (1-19)).

In Section 5 we prove the uniform stability inequality (1-21). As explained in the remarks above, we first prove (1-21) on radial decreasing functions by means of the selection principle method of [Cicalese and Leonardi 2012] (this is where Theorem 2.2 and the above-mentioned notion of critical sequence are used), and then reduce the proof of (1-21) from the general case to the radial decreasing case by adapting to our setting the quantitative symmetrization method introduced in [Fusco et al. 2008] for proving (1-8).

In Section 6 we prove the Alexandrov-type result along the lines already illustrated in Section 1D.

Finally, in the Appendix we collect, for ease of reference, some basic facts and results which are frequently used throughout the paper. Readers are recommended to quickly familiarize themselves with the basic estimates for the potentials W, Φ and V contained therein before entering into the technical aspects of our proofs.

2. Existence and radial decreasing symmetry of minimizers

We begin by proving the following existence and symmetry result for minimizers of $\psi(\varepsilon)$.

Theorem 2.1. *If $n \geq 2$ and $W \in C^{2,1}[0, 1]$ satisfies* (1-11) *and* (1-12), *then there exists a universal constant ε_0 such that ψ is continuous on $(0, \varepsilon_0)$ and, for every $\varepsilon < \varepsilon_0$, there exist minimizers of $\psi(\varepsilon)$. Moreover, if u_ε is a minimizer of $\psi(\varepsilon)$ with $\varepsilon < \varepsilon_0$, then, up to a translation, $u_\varepsilon \in \mathcal{R}_0^* \cap C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ for every $\alpha \in (0, 1)$, $0 < u_\varepsilon < 1$ on $\mathbb{R}^n$, and, for some $\lambda \in \mathbb{R}$, u_ε solves*

$$-2\varepsilon^2 \Delta u_\varepsilon = \varepsilon \lambda V'(u_\varepsilon) - W'(u_\varepsilon) \quad \text{on } \mathbb{R}^n, \tag{2-1}$$

where λ *satisfies*

$$\lambda = \frac{(n-1)}{n}\psi(\varepsilon) + \frac{1}{n}\left\{\frac{1}{\varepsilon}\int_{\mathbb{R}^n} W(u_\varepsilon) - \varepsilon\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2\right\}. \tag{2-2}$$

Finally, λ *obeys the bound*

$$|\lambda - 2(n-1)\omega_n^{1/n}| \le C\sqrt{\varepsilon} \quad \textit{for all } \varepsilon < \varepsilon_0, \tag{2-3}$$

so that, *in particular*, $0 < 1/C \le \lambda \le C$ *for a universal constant* C.

Proof. Step 1: We show the existence of universal constants ℓ_0, M_0, and C such that if $\varepsilon < \varepsilon_0$ and $u \in H^1(\mathbb{R}^n; [0,1])$ satisfies

$$\mathcal{AC}_\varepsilon(u) \le 2n\omega_n^{1/n} + \ell, \qquad \int_{\mathbb{R}^n} V(u) = 1, \tag{2-4}$$

for some $\ell < \ell_0$, then, up to a translation,

$$\int_{B_{M_0}} V(u) \ge 1 - C\sqrt{\ell}. \tag{2-5}$$

Moreover, in the special case when $u \in \mathcal{R}_0$, the factor $\sqrt{\ell}$ in (2-5) can replaced by ℓ.

Indeed, by applying (1-28) to $f = \Phi(u)$ and exploiting the identity (1-27), we deduce that, up to a translation of u, we have

$$\int_{\mathbb{R}^n} |\Phi(u) - (\omega_n^{1/n} r)^{1-n} 1_{B_r}|^{n/(n-1)} \le C(n)\left(\frac{\mathcal{AC}_\varepsilon(u)}{2} - n\omega_n^{1/n}\right)^{1/2} \le C\sqrt{\ell} \tag{2-6}$$

for suitable $r > 0$, with ℓ in place of $\sqrt{\ell}$ if $u \in \mathcal{R}_0$ thanks to (1-33). Clearly, (2-6) implies

$$\int_{B_r^c} V(u) \le C\sqrt{\ell}. \tag{2-7}$$

Let us now define M_0 by setting

$$\Phi\left(\tfrac{1}{4}\right)[\omega_n^{1/n} M_0]^{n-1} = 1.$$

Clearly, if $r \le M_0$, then (2-7) gives

$$\int_{B_{M_0}^c} V(u) \le C\sqrt{\ell},$$

and (2-5) follows. Assuming by contradiction that $r > M_0$, by the definition of M_0 we find

$$[\omega_n^{1/n} r]^{1-n} < [\omega_n^{1/n} M_0]^{1-n} = \Phi\left(\tfrac{1}{4}\right) < \Phi\left(\tfrac{1}{2}\right),$$

so that

$$\int_{\{u \ge 1/2\} \cap B_r} \left|\Phi\left(\tfrac{1}{2}\right) - [\omega_n^{1/n} r]^{1-n}\right|^{n/(n-1)} \le \int_{\{u \ge 1/2\}} |\Phi(u) - [\omega_n^{1/n} r]^{1-n} 1_{B_r}|^{n/(n-1)}.$$

In particular, (2-6) and the fact that $\Phi\left(\tfrac{1}{2}\right) - \Phi\left(\tfrac{1}{4}\right)$ is a universal constant imply

$$\left|\left\{u \ge \tfrac{1}{2}\right\} \cap B_r\right| \le C\sqrt{\ell_0}. \tag{2-8}$$

At the same time (A-13) gives

$$\int_{\{u<1/2\}} V(u) \le C \int_{\{u<1/2\}} W(u) \le C\varepsilon \mathcal{AC}_\varepsilon(u) \le C\varepsilon. \tag{2-9}$$

By using, in the order, (2-9), the fact that V is increasing with $V(1)=1$, (2-8) and (2-7), we conclude

$$\begin{aligned} 1 = \int_{\mathbb{R}^n} V(u) &\le \int_{\{u\ge 1/2\}} V(u) + C\varepsilon \le \left|\left\{u \ge \tfrac{1}{2}\right\} \cap B_r\right| + \int_{B_r^c} V(u) + C\varepsilon \\ &\le C(\sqrt{\ell_0} + \varepsilon_0), \end{aligned}$$

which is a contradiction provided we take ℓ_0 and ε_0 small enough.

<u>Step 2</u>: We show the existence of a universal constant ℓ_0 such that, if $\varepsilon < \varepsilon_0$ and $\{u_j\}_j$ is a sequence in $H^1(\mathbb{R}^n; [0,1])$ with

$$\mathcal{AC}_\varepsilon(u_j) \le \psi(\varepsilon) + \ell_0, \qquad \int_{\mathbb{R}^n} V(u_j) = 1 \quad \text{for all } j, \tag{2-10}$$

then there exists $u \in H^1(\mathbb{R}^n; [0,1])$ such that, up to extracting subsequences and up to translations, $\Phi(u_j) \to \Phi(u)$ in $L^{n/(n-1)}(\mathbb{R}^n)$ and, in particular, $\int_{\mathbb{R}^n} V(u) = 1$.

We first notice that, by the elementary upper bound (1-26) and by (2-10), we have $\mathcal{AC}_\varepsilon(u_j) \le C$ for every j. Next, we apply the concentration-compactness principle (see Section A2) to $\{V(u_j)\,dx\}_j$. By (2-5) in Step 1, we find that

$$\int_{B_{M_0}} V(u_j) \ge 1 - C\sqrt{\ell_0} \quad \text{for all } j. \tag{2-11}$$

This rules out the vanishing case. We consider the case that the dichotomy case occurs. To that end, it will be convenient to notice the validity of the Lipschitz estimate

$$|\mathcal{AC}_\varepsilon(u) - \mathcal{AC}_{\nu\,\varepsilon}(u)| \le C|1-\nu|\mathcal{AC}_\varepsilon(u) \quad \text{for all } \nu \ge \frac{1}{C},\ u \in H^1(\mathbb{R}^n; [0,1]), \tag{2-12}$$

which is deduced immediately from

$$\mathcal{AC}_{\nu\varepsilon}(u) - \mathcal{AC}_\varepsilon(u) = (\nu-1)\varepsilon \int_{\mathbb{R}^n} |\nabla u|^2 + \left(\frac{1}{\nu} - 1\right)\frac{1}{\varepsilon}\int_{\mathbb{R}^n} W(u).$$

By (2-11), if we are in the dichotomy case, then there exists

$$\alpha \in (1 - C\sqrt{\ell_0}, 1) \tag{2-13}$$

such that for every $\tau \in (0, \alpha/2)$ we can find $S(\tau) > 0$ and $S_j(\tau) \to \infty$ as $j \to \infty$ such that

$$\left|\alpha - \int_{B_{S(\tau)}} V(u_j)\right| < \tau, \qquad \left|(1-\alpha) - \int_{B^c_{S_j(\tau)}} V(u_j)\right| < \tau \quad \text{for all } j. \tag{2-14}$$

We now pick a cut-off function[3] φ between $B_{S(\tau)}$ and $B_{S_j(\tau)}$, so that $\varphi \in C_c^\infty(B_{S_j(\tau)})$ with $0 \le \varphi \le 1$ and $|\nabla\varphi| \le (S_j(\tau) - S(\tau))^{-1} \le 2S_j(\tau)^{-1}$ on $\mathbb{R}^n$, and with $\varphi = 1$ on $B_{S(\tau)}$. We notice that (2-14) and the

[3]Notice that φ depends on both j and τ. We will not stress this dependency in the notation.

monotonicity of V give

$$\left|\alpha - \int_{\mathbb{R}^n} V(\varphi\, u_j)\right| < 2\tau, \quad \left|(1-\alpha) - \int_{\mathbb{R}^n} V((1-\varphi)u_j)\right| < 2\tau \quad \text{for all } j. \tag{2-15}$$

We compute that

$$\begin{aligned}
\mathcal{AC}_\varepsilon(u_j) &= \mathcal{AC}_\varepsilon(\varphi u_j) + \mathcal{AC}_\varepsilon((1-\varphi)u_j) + a_j + b_j,\\
a_j &= 2\varepsilon \int_{B_{S_j(\tau)}\setminus B_{S(\tau)}} \varphi(1-\varphi)|\nabla u_j|^2 - u_j^2|\nabla\varphi|^2 - (1-2\varphi)u_j\nabla u_j\cdot\nabla\varphi,\\
b_j &= \frac{1}{\varepsilon}\int_{B_{S_j(\tau)}\setminus B_{S(\tau)}} W(u_j) - W(\varphi u_j) - W((1-\varphi)u_j),
\end{aligned}$$

where we have taken into account that $\varphi\,(1-\varphi)$ and $\nabla\varphi$ are supported in $B_{S_j(\tau)}\setminus B_{S(\tau)}$, as well as that $W(0)=0$. Let us now set, for $\sigma\in(0,1)$,

$$\Gamma_j^+(\tau,\sigma) = (B_{S_j(\tau)}\setminus B_{S(\tau)})\cap\{u_j>\sigma\}, \quad \Gamma_j^-(\tau,\sigma) = (B_{S_j(\tau)}\setminus B_{S(\tau)})\cap\{u_j<\sigma\}.$$

By (2-14), we have

$$V(\sigma)\mathcal{L}^n(\Gamma_j^+(\tau,\sigma)) \le \int_{B_{S_j(\tau)}\setminus B_{S(\tau)}} V(u_j) \le C\tau \quad \text{for all } j.$$

Taking into account (A-11), if $\sigma<\delta_0$, then we have

$$\mathcal{L}^n(\Gamma_j^+(\tau,\sigma)) \le C\frac{\tau}{V(\sigma)} \le C\frac{\tau}{\sigma^{2n/(n-1)}} \quad \text{for all } j.$$

Provided $\tau\le\tau_*$ for a suitable small universal constant τ_* we can thus guarantee that

$$\sigma(\tau) := \tau^{1/[1+(2n/(n-1))]} = \tau^{(n-1)/(3n-1)} < \delta_0, \tag{2-16}$$

and, therefore, that, setting for brevity $\sigma=\sigma(\tau)$ as in (2-16),

$$\mathcal{L}^n(\Gamma_j^+(\tau,\sigma)) \le C\tau^{(n-1)/(3n-1)} = C\sigma \quad \text{for all } j.$$

At the same time, we can apply (A-5) with $b=u_j$ and $a=0$ to get

$$\left|W(u_j) - W''(0)\frac{u_j^2}{2}\right| \le Cu_j^3 \le C\sigma u_j^2 \quad \text{on } \Gamma_j^-(\tau,\sigma), \tag{2-17}$$

and identical inequalities with φu_j and $(1-\varphi)u_j$ in place of u_j, thus finding

$$\begin{aligned}
b_j &\ge \frac{W''(0)}{2\varepsilon}\int_{\Gamma_j^-(\tau,\sigma)} u_j^2 - (\varphi u_j)^2 - ((1-\varphi)u_j)^2 - \frac{C\sigma}{\varepsilon}\int_{\Gamma_j^-(\tau,\sigma)} u_j^2 - \frac{C}{\varepsilon}\mathcal{L}^n(\Gamma_j^+(\tau,\sigma))\\
&\ge \frac{W''(0)}{\varepsilon}\int_{\Gamma_j^-(\tau,\sigma)} \varphi(1-\varphi)u_j^2 - \frac{C\sigma}{\varepsilon}\int_{\Gamma_j^-(\tau,\sigma)} u_j^2 - C\frac{\sigma}{\varepsilon}\\
&\ge -\frac{C\sigma}{\varepsilon}\int_{\mathbb{R}^n} W(u_j) - C\frac{\sigma}{\varepsilon} \ge -C\frac{\sigma}{\varepsilon},
\end{aligned}$$

where, in the last line, we have used $W''(0) \geq 0$, $\varepsilon^{-1} \int_{\mathbb{R}^n} W(u_j) \leq \mathcal{AC}_\varepsilon(u_j) \leq C$, and the fact that (A-6) and $u_j \leq \sigma \leq \delta_0$ on $\Gamma_j^-(\tau, \sigma)$. This gives us

$$u_j^2 \leq C W(u_j) \quad \text{on } \Gamma_j^-(\tau, \sigma). \tag{2-18}$$

Similarly, if we discard the first term in the expression for a_j (which is, indeed, nonnegative), we find

$$\begin{aligned} a_j &\geq -2\varepsilon \int_{B_{S_j(\tau)} \setminus B_{S(\tau)}} u_j^2 |\nabla\varphi|^2 + u_j |\nabla u_j| |\nabla\varphi| \\ &\geq -C\varepsilon \|\nabla\varphi\|_{C^0(\mathbb{R}^n)} \int_{B_{S_j(\tau)} \setminus B_{S(\tau)}} \varepsilon |\nabla u_j|^2 + \frac{u_j^2}{\varepsilon} \geq -\frac{C}{S_j(\tau)}, \end{aligned}$$

where we have used $\|\nabla\varphi\|_{C^0(\mathbb{R}^n)} \leq 2S_j(\tau)^{-1}$ and that $S_j(\tau) \to \infty$ as $j \to \infty$, as well as noticed that

$$\varepsilon \int_{\mathbb{R}^n} |\nabla u_j|^2 \leq C \mathcal{AC}_\varepsilon(u_j) \leq C,$$

$$\int_{\mathbb{R}^n} u_j^2 \leq \mathcal{L}^n(\{u_j \geq \delta_0\}) + C \int_{\{u_j \leq \delta_0\}} W(u_j) \leq C \int_{\{u_j \geq \delta_0\}} V(u_j) + C\varepsilon \mathcal{AC}_\varepsilon(u_j) \leq C,$$

thanks to $V(t) \geq 1/C$ for $t \in (\delta_0, 1)$ and to $W(t) \geq t^2/C$ on for $t \in (0, \delta_0)$; see (A-6) and (A-14). Combining the lower bounds for a_j and b_j, we have thus proved

$$\mathcal{AC}_\varepsilon(u_j) \geq \mathcal{AC}_\varepsilon(\varphi u_j) + \mathcal{AC}_\varepsilon((1-\varphi)u_j) - C\left(\frac{\sigma}{\varepsilon} + \frac{1}{S_j(\tau)}\right). \tag{2-19}$$

If we set

$$m_j = \int_{\mathbb{R}^n} V(\varphi u_j), \quad n_j = \int_{\mathbb{R}^n} V((1-\varphi)u_j),$$

and define

$$v_j(x) = (\varphi u_j)(m_j^{1/n} x), \quad w_j(x) = ((1-\varphi)u_j)(n_j^{1/n} x), \quad x \in \mathbb{R}^n, \tag{2-20}$$

then by (A-1) and (A-2) we find

$$\int_{\mathbb{R}^n} V(v_j) = 1, \quad \mathcal{AC}_{\varepsilon/m_j^{1/n}}(v_j) = m_j^{(1-n)/n} \mathcal{AC}_\varepsilon(\varphi u_j), \tag{2-21}$$

with analogous identities for w_j. By (2-15) and (2-12), and keeping in mind (2-13), we find

$$\begin{aligned} \mathcal{AC}_\varepsilon(\varphi u_j) &= m_j^{(n-1)/n} \mathcal{AC}_{\varepsilon/m_j^{1/n}}(v_j) \\ &\geq (\alpha - C\tau)^{(n-1)/n} (1 - C|m_j^{-1/n} - 1|) \mathcal{AC}_\varepsilon(v_j) \\ &\geq (\alpha - C\tau)^{(n-1)/n} (1 - C|\alpha - 1| - C\tau) \psi(\varepsilon). \end{aligned} \tag{2-22}$$

Similarly, taking τ small enough with respect to $1-\alpha$, since $\int_{\mathbb{R}^n} V(w_j) = 1$ we have

$$\mathcal{AC}_\varepsilon((1-\varphi)u_j) = n_j^{(n-1)/n} \mathcal{AC}_{\varepsilon/n_j^{1/n}}(w_j) \geq ((1-\alpha) - C\tau)^{(n-1)/n} 2n\omega_n^{1/n}. \tag{2-23}$$

By combining (2-22) and (2-23) with (2-19) we get

$$\frac{\mathcal{AC}_\varepsilon(u_j)}{\psi(\varepsilon)} \geq (\alpha - C\tau)^{(n-1)/n} (1 - C|\alpha - 1| - C\tau) + \frac{c(n)}{\psi(\varepsilon)} ((1-\alpha) - C\tau)^{(n-1)/n} - \frac{C}{\psi(\varepsilon)} \left(\frac{\sigma}{\varepsilon} + \frac{1}{S_j(\tau)}\right).$$

Considering that $\psi(\varepsilon) \le C$ for $\varepsilon < \varepsilon_0$, we let first $j \to \infty$ and then $\tau \to 0^+$ (recall that $\sigma \to 0^+$ as $\tau \to 0^+$) to find

$$\begin{aligned} 1 &\ge (1 - C|\alpha - 1|)\alpha^{(n-1)/n} + c(n)(1-\alpha)^{(n-1)/n} \\ &\ge 1 - C|\alpha - 1| + c(n)(1-\alpha)^{(n-1)/n}. \end{aligned} \tag{2-24}$$

Since $1 > \alpha > 1 - C\sqrt{\ell_0}$, by taking ℓ_0 small enough we can make α arbitrarily close to 1 in terms of n and W, thus obtaining a contradiction with (2-24). This proves that $\{V(u_j)\,dx\}_j$ is in the compactness case of the concentration–compactness principle. Since (2-10) implies that $\{\Phi(u_j)\}_j$ has bounded total variation on $\mathbb{R}^n$ and since $V(u_j) = \Phi(u_j)^{n/(n-1)}$ does not concentrate mass at infinity, the compactness statement now follows by standard considerations.

<u>Step 3</u>: Let $\{u_j\}_j$ be a minimizing sequence of $\psi(\varepsilon)$ for some $\varepsilon < \varepsilon_0$. By (1-26) we can assume that for every j

$$\mathcal{AC}_\varepsilon(u_j) \le \psi(\varepsilon) + C\varepsilon \le 2n\omega_n^{1/n} + C\varepsilon.$$

We can then apply the compactness statement of Step 2 to deduce the existence of minimizers of $\psi(\varepsilon)$. To prove the continuity of ψ on $(0, \varepsilon_0)$, let $\varepsilon_j \to \varepsilon_* \in (0, \varepsilon_0)$ as $j \to \infty$, and, for each ε_j, let u_j be a minimizer of $\psi(\varepsilon_j)$. By (1-26) we can apply Step 2 to $\{u_j\}_j$ and deduce the existence, up to translations and up to extracting subsequences, of $u_* \in H^1(\mathbb{R}^n; [0, 1])$ such that $\Phi(u_j) \to \Phi(u_*)$ in $L^{n/(n-1)}(\mathbb{R}^n)$ as $j \to \infty$. If $v \in H^1(\mathbb{R}^n; [0, 1])$ with $\int_{\mathbb{R}^n} V(v) = 1$, then

$$\mathcal{AC}_{\varepsilon_j}(u_j) \le \mathcal{AC}_{\varepsilon_j}(v)$$

so that, letting $j \to \infty$ and using lower semicontinuity,

$$\mathcal{AC}_{\varepsilon_*}(u_*) \le \liminf_{j\to\infty} \mathcal{AC}_{\varepsilon_j}(u_j) \le \lim_{j\to\infty} \mathcal{AC}_{\varepsilon_j}(v) = \mathcal{AC}_{\varepsilon_*}(v).$$

Since $\int_{\mathbb{R}^n} V(u_*) = 1$, we conclude that u_* is a minimizer of $\psi(\varepsilon_*)$; and by plugging $v = u_*$ in the previous chain of inequalities, we find that $\psi(\varepsilon_j) \to \psi(\varepsilon_*)$ as $j \to \infty$.

<u>Step 4</u>: We now notice that, by the Pólya–Szegő inequality [Brothers and Ziemer 1988], once there is a minimizer of $\psi(\varepsilon)$, there is also a minimizer of $\psi(\varepsilon)$ which belongs to $\mathcal{R}_0$, or, in brief, a *radial decreasing minimizer* (more precisely, a radial decreasing minimizer with maximum at 0). In this step we prove that every radial decreasing minimizer u_ε of $\psi(\varepsilon)$ satisfies $0 < u_\varepsilon < 1$ on $\mathbb{R}^n$ and $u_\varepsilon \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$, and that in correspondence of u_ε one can find $\lambda \in \mathbb{R}$ such that

$$-2\varepsilon^2 \Delta u_\varepsilon = \varepsilon\lambda V'(u_\varepsilon) - W'(u_\varepsilon) \quad \text{on } \mathbb{R}^n. \tag{2-25}$$

To begin with, since u_ε is radial decreasing and has finite Dirichlet energy, u_ε is continuous on $\mathbb{R}^n$. In particular, there exist $0 \le a < b \le +\infty$ such that

$$\{u_\varepsilon > 0\} = B_b, \quad \{u_\varepsilon < 1\} = \mathbb{R}^n \setminus \bar{B}_a = \{x : |x| > a\}.$$

A standard first variation argument shows the existence of $\lambda \in \mathbb{R}$ such that

$$-2\varepsilon^2 \Delta u_\varepsilon = \varepsilon\lambda V'(u_\varepsilon) - W'(u_\varepsilon) \quad \text{in } \mathcal{D}'(\Omega),\ \Omega = B_b \setminus \bar{B}_a. \tag{2-26}$$

Since (2-26) implies that Δu_ε is bounded in Ω, by the Calderon–Zygmund theorem we find that $u_\varepsilon \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$. As a consequence, (2-26) gives that $-2\varepsilon^2\Delta u_\varepsilon = f(u_\varepsilon)$ for some $f \in C^1(0,1)$, and thus, by Schauder's theory, $u_\varepsilon \in C^{2,\alpha}_{\mathrm{loc}}(\Omega)$ for every $\alpha \in (0,1)$. We complete this step by showing that $\Omega = \mathbb{R}^n$.

Proof that $\Omega = \mathbb{R}^n$: Considering functions of the form $u + t\,\varphi$ with $t \ge 0$ and either $\varphi \in C^\infty_c(\mathbb{R}^n \setminus \bar{B}_a)$, $\varphi \ge 0$, or $\varphi \in C^\infty_c(B_b)$, $\varphi \le 0$, and then adjusting the volume constraint by a suitable variation localized in $B_b \setminus \bar{B}_a$, we also obtain the validity, in distributional sense, of the inequalities

$$-2\varepsilon^2\Delta u_\varepsilon \ge \varepsilon\lambda V'(u_\varepsilon) - W'(u_\varepsilon) \quad \text{in } \mathcal{D}'(\mathbb{R}^n \setminus \bar{B}_a), \tag{2-27}$$

$$-2\varepsilon^2\Delta u_\varepsilon \le \varepsilon\lambda V'(u_\varepsilon) - W'(u_\varepsilon) \quad \text{in } \mathcal{D}'(B_b). \tag{2-28}$$

We prove only (2-27) *in detail*: Pick any $\psi \in C^\infty_c(\{0 < u_\varepsilon < 1\})$ with $\psi \ge 0$ and $\int_{\mathbb{R}^n} V'(u_\varepsilon)\psi = 1$ (such choice is possible since $\{0 < u_\varepsilon < 1\}$ is nonempty and $\int_{\mathbb{R}^n} V'(u_\varepsilon) > 0$), and notice for future use that, thanks to (2-26),

$$\varepsilon\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla\psi + \frac{1}{\varepsilon}\int_{\mathbb{R}^n} W'(u_\varepsilon)\psi = \lambda\int_{\mathbb{R}^n} V'(u_\varepsilon)\psi = \lambda. \tag{2-29}$$

Given $\varphi \in C^\infty_c(\mathbb{R}^n \setminus \bar{B}_a)$ with $\varphi \ge 0$, since $\mathbb{R}^n \setminus \bar{B}_a = \{u_\varepsilon < 1\}$, we can find t_0, s_0 positive such that $u + t\varphi + s\psi$ takes values in $[0,1]$ whenever $(t,s) \in A_0 := [0,t_0] \times [-s_0, s_0]$. Setting $h(t,s) = \int_{\mathbb{R}^n} V(u_\varepsilon + t\varphi + s\psi)$, we see that $h \in C^2(A_0)$ with

$$h(0,0) = 1, \quad \frac{\partial h}{\partial t}(0,0) = \int_{\mathbb{R}^n} V'(u_\varepsilon)\varphi, \quad \frac{\partial h}{\partial s}(0,0) = \int_{\mathbb{R}^n} V'(u_\varepsilon)\psi = 1. \tag{2-30}$$

Moreover, by the strict monotonicity of V, we see that $h(0,s_0) = \int_{\mathbb{R}^n} V(u + s_0\psi) > h(0,0) = 1$, and similarly $h(0,-s_0) < 1$, so that, by continuity and up to decreasing t_0 and s_0,

$$h(t,s_0) > 1 > h(t,-s_0) \quad \text{for every } t \in [0,t_0], \qquad \frac{\partial h}{\partial s} \ge \frac{1}{2} \quad \text{on } A_0. \tag{2-31}$$

Therefore there is $s(t) : [0,t_0] \to (-s_0, s_0)$ such that $h(t, s(t)) = 1$. Differentiating and exploiting (2-30), we find $s'(0) = -\int_{\mathbb{R}^n} V'(u_\varepsilon)\varphi$, so that, by minimality of u_ε and by (2-29)

$$\begin{aligned}
0 &\le \frac{d}{dt}\Big|_{t=0^+} \mathcal{AC}_\varepsilon(u_\varepsilon + t\varphi + s(t)\psi) \\
&= \varepsilon\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla\varphi + \frac{1}{\varepsilon}\int_{\mathbb{R}^n} W'(u_\varepsilon)\varphi + s'(0)\varepsilon\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla\psi + \frac{1}{\varepsilon}\int_{\mathbb{R}^n} W'(u_\varepsilon)\psi \\
&= \varepsilon\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla\varphi + \frac{1}{\varepsilon}\int_{\mathbb{R}^n} W'(u_\varepsilon)\varphi - \lambda\int_{\mathbb{R}^n} V'(u_\varepsilon)\varphi.
\end{aligned}$$

By the arbitrariness of φ we thus find (2-27).

Having (2-27) and (2-28) at our disposal, we now prove $\Omega = \mathbb{R}^n$. We stress that, in the rest of the argument, the only property of

$$f(t) = \varepsilon\lambda V'(t) - W'(t), \quad t \in [0,1],$$

that will be used is the validity of the bound

$$|f(t)| \le C(1 + |\lambda|)t(1-t) \quad \text{for all } t \in [0,1]. \tag{2-32}$$

This remark will be useful to avoid repetitions when we come to Step 2 of the proof of Theorem 2.2. Notice that (2-32) indeed holds true thanks to (A-6) and (A-11), and that in (2-32) we cannot absorb $|\lambda|$ into C since we do not know yet that $|\lambda|$ admits a universal bound (this will actually be proved in Step 5 below).

By (2-32), (2-27) implies

$$-2\varepsilon^2\left\{u_\varepsilon''+(n-1)\frac{u_\varepsilon'}{t}\right\}\geq -C(1+|\lambda|)u_\varepsilon \quad \text{in } \mathcal{D}'(a,\infty). \tag{2-33}$$

Assuming by contradiction that $b<\infty$, let $r\in(a,b)$, s be such that $(r-s,r+s)\subset(a,b)$, and ζ_s be the Lipschitz function with $\zeta_s=0$ on $(0,r-s)$, $\zeta_s=1$ on $(r+s,\infty)$, and $\zeta_s'=1/(2s)$ on $(r-s,r+s)$. Testing (2-33) with $-u_\varepsilon'\zeta_s\geq 0$ (which is compactly supported in (a,∞)) we find that

$$\varepsilon^2\int_a^\infty [(u_\varepsilon')^2]'\zeta_s+2(n-1)\frac{(u_\varepsilon')^2}{t}\zeta_s\geq C(1+|\lambda|)\int_a^\infty u_\varepsilon u_\varepsilon'\zeta_s,$$

so that, after integration by parts, we obtain

$$2(n-1)\varepsilon^2\int_a^\infty\frac{(u_\varepsilon')^2}{t}\zeta_s+\frac{C(1+|\lambda|)}{2s}\int_{r-s}^{r+s}\frac{u_\varepsilon^2}{2}\geq\frac{\varepsilon^2}{2s}\int_{r-s}^{r+s}(u_\varepsilon')^2.$$

Letting $s\to 0^+$ we obtain

$$2(n-1)\varepsilon^2\int_r^b\frac{(u_\varepsilon')^2}{t}+C(1+|\lambda|)\frac{u_\varepsilon(r)^2}{2}\geq\varepsilon^2u_\varepsilon'(r)^2.$$

Finally letting $r\to b^-$ we conclude that $u_\varepsilon'(b^-)=0$. This fact, combined with $u_\varepsilon(b)=0$ and the uniqueness theorem for the second-order ODE (2-26), implies that $u_\varepsilon=0$ on (a,b), which is in contradiction with the continuity of u_ε if $a>0$, and with $\int_{\mathbb{R}^n}V(u_\varepsilon)=1$ if $a=0$. This proves that $b=+\infty$ (and thus that $u_\varepsilon>0$ on $\mathbb{R}^n$).

The proof of $a=0$ (that is, of $u_\varepsilon<1$ on $\mathbb{R}^n$) is analogous. After the change of variables $v=1-u_\varepsilon$, we have $v\geq 0$, $v'\geq 0$, $v=0$ on $(0,a)$, and, thanks to (2-28),

$$-2\varepsilon^2\left\{v''+(n-1)\frac{v'}{t}\right\}\geq -C(1+|\lambda|)v \quad \text{in } \mathcal{D}'(0,\infty). \tag{2-34}$$

Notice that (2-34) is identical to (2-33), and that an even reflection by $r=a$ maps the boundary conditions of v into those of u_ε: the same argument used for proving $u_\varepsilon'(b^-)=0$ will thus show that $v'(a^+)=0$. For the sake of clarity we give some details. We pick $r>a$, introduce a Lipschitz function $\bar\zeta_s$ with $\bar\zeta_s=1$ on $(0,r-s)$, $\bar\zeta_s=0$ on $(r+s,\infty)$, and $\bar\zeta_s'=-1/(2s)$ on $(r-s,r+s)$, and test (2-34) with $v'\bar\zeta_s\geq 0$, to get

$$-\varepsilon^2\int_0^\infty[(v')^2]'\bar\zeta_s+2(n-1)\frac{(v')^2}{t}\bar\zeta_s\geq -C(1+|\lambda|)\int_0^\infty vv'\bar\zeta_s.$$

Integration by parts now gives

$$-\frac{\varepsilon^2}{2s}\int_{r-s}^{r+s}(v')^2-2(n-1)\varepsilon^2\int_a^{r+s}\frac{(v')^2}{t}\bar\zeta_s\geq -\frac{C(1+|\lambda|)}{2s}\int_{r-s}^{r+s}\frac{v^2}{2},$$

so that in the limit $s \to 0^+$, and then $r \to a^+$, we find $v'(a^+) = 0$, that is to say, $u'_\varepsilon(a^+) = 0$. If $a > 0$ and thus $u_\varepsilon(a) = 1$, this, combined with (2-26), implies $u_\varepsilon = 1$ on $\mathbb{R}^n$, a contradiction.

Step 5: Given a radial decreasing minimizer u_ε of $\psi(\varepsilon)$, we prove that the corresponding $\lambda \in \mathbb{R}$ such that (2-25) holds satisfies

$$n\lambda = (n-1)\mathcal{AC}_\varepsilon(u_\varepsilon) + \frac{1}{\varepsilon}\int_{\mathbb{R}^n} W(u_\varepsilon) - \varepsilon \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2, \tag{2-35}$$

as well as

$$|\lambda - 2(n-1)\omega_n^{1/n}| \le C\sqrt{\varepsilon}. \tag{2-36}$$

In particular, up to decreasing the value of ε_0, we always have $1/C \le \lambda \le C$ for a universal constant C. To prove (2-35), following [Luckhaus and Modica 1989], we test the distributional form of (2-25) with $\varphi = X \cdot \nabla u_\varepsilon$ for some $X \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$, and get

$$\begin{aligned} 2\varepsilon \int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla X[\nabla u_\varepsilon] &= -\int_{\mathbb{R}^n} \left\{ 2\varepsilon \nabla^2 u_\varepsilon[\nabla u_\varepsilon] + \left(\frac{W'(u_\varepsilon)}{\varepsilon} - \lambda\, V'(u_\varepsilon) \right) \nabla u_\varepsilon \right\} \cdot X \\ &= \int_{\mathbb{R}^n} \left\{ \varepsilon |\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon} - \lambda V(u_\varepsilon) \right\} \operatorname{Div} X. \end{aligned} \tag{2-37}$$

We now pick $\eta \in C_c^\infty(B_2)$ with $0 \le \eta \le 1$ on B_2 and $\eta = 1$ in B_1. We set $\eta_R(x) = \eta(x/R)$ and test (2-37) with $X(x) = \eta_R(x)\, x$. We notice that $\operatorname{Div} X = n\, \eta_R + (x/R) \cdot (\nabla \eta)_R$, and that, by dominated convergence,

$$\begin{aligned} &\lim_{R\to\infty} \int_{\mathbb{R}^n} \left\{ \varepsilon |\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon} - \lambda V(u_\varepsilon) \right\} n\eta_R = n(\mathcal{AC}_\varepsilon(u_\varepsilon) - \lambda), \\ &\lim_{R\to\infty} \int_{\mathbb{R}^n} \left\{ \varepsilon |\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon} - \lambda V(u_\varepsilon) \right\} \frac{x}{R} \cdot (\nabla\eta)_R = 0, \\ &\lim_{R\to\infty} \int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \left(\eta_R \mathrm{Id} + \frac{x}{R} \otimes (\nabla \eta)_R \right) [\nabla u_\varepsilon] = \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2. \end{aligned} \tag{2-38}$$

In particular, (2-37) implies

$$n\lambda = n\mathcal{AC}_\varepsilon(u_\varepsilon) - 2\varepsilon \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2,$$

which can be easily rearranged into (2-35). At the same time, by (1-26) we find

$$\begin{aligned} \int_{\mathbb{R}^n} \left| \varepsilon |\nabla u_\varepsilon|^2 - \frac{W(u_\varepsilon)}{\varepsilon} \right| &\le \left(\int_{\mathbb{R}^n} \left| \sqrt{\varepsilon} |\nabla u_\varepsilon| - \sqrt{\frac{W(u_\varepsilon)}{\varepsilon}} \right|^2 \right)^{1/2} \left(\int_{\mathbb{R}^n} \left| \sqrt{\varepsilon} |\nabla u_\varepsilon| + \sqrt{\frac{W(u_\varepsilon)}{\varepsilon}} \right|^2 \right)^{1/2} \\ &= \left(\mathcal{AC}_\varepsilon(u_\varepsilon) - 2\int_{\mathbb{R}^n} |\nabla \Phi(u_\varepsilon)| \right)^{1/2} \left(\int_{\mathbb{R}^n} \left| \sqrt{\varepsilon} |\nabla u_\varepsilon| + \sqrt{\frac{W(u_\varepsilon)}{\varepsilon}} \right|^2 \right)^{1/2} \\ &\le C\sqrt{\varepsilon}\sqrt{\mathcal{AC}_\varepsilon(u_\varepsilon)} \le C\sqrt{\varepsilon}, \end{aligned}$$

which can be combined with (2-35) and with (1-26) to deduce (2-36).

Step 6: We are left to prove that every minimizer of $\psi(\varepsilon)$ is radial decreasing. Indeed, let u be a generic, possibly nonradial, minimizer of $\psi(\varepsilon)$, and let $v \in \mathcal{R}_0$ denote its radial decreasing rearrangement. By standard properties of rearrangements, $\int_{\mathbb{R}^n} V(u) = \int_{\mathbb{R}^n} V(v) = 1$, while by the Pólya–Szegő inequality

$\mathcal{AC}_\varepsilon(u) \ge \mathcal{AC}_\varepsilon(v)$, so that v is a minimizer of $\psi(\varepsilon)$ and equality holds in the Pólya–Szegő inequality for u, that is,

$$\int_{\mathbb{R}^n} |\nabla u|^2 = \int_{\mathbb{R}^n} |\nabla v|^2. \tag{2-39}$$

By Steps 4 and 5, v solves the ODE

$$2\varepsilon^2\left\{v'' + (n-1)\frac{v'}{r}\right\} = W'(v) - \lambda\varepsilon V'(v) \quad \text{on } (0,\infty), \tag{2-40}$$

with $0 < 1/C \le \lambda \le C$. Multiplying in (2-40) by v' and integrating over $(0,r)$ for some $r>0$, we obtain

$$\varepsilon^2 v'(r)^2 + 2(n-1)\int_0^r \frac{(v')^2}{t} = W(v(r)) - \lambda\varepsilon V(v(r)) + \lambda\varepsilon V(v(0)) \quad \text{for all } r>0, \tag{2-41}$$

where we have used $v'(0)=0$, $v(1)=1$, and $W(1)=0$. If r is such that $v(r)\le\delta_0$, then by (A-6), (A-11) and (2-41) we find

$$\varepsilon^2 v'(r)^2 \ge W(v) - C\varepsilon V(v) \ge \frac{v(r)^2}{C} - C\varepsilon\frac{v(r)^{2n/(n-1)}}{C} \ge \frac{v(r)^2}{C},$$

which gives, in particular, $v'(r)<0$; if r is such that $v(r)\in(\delta_0, 1-\delta_0)$, then, by the same method and thanks to $\inf_{(\delta_0,1-\delta_0)} W \ge 1/C$, we find that

$$\varepsilon^2 v'(r)^2 \ge W(v) - C\varepsilon V(v) \ge \frac{1}{C} - C\varepsilon \ge \frac{1}{C},$$

so that, once again, $v'(r)<0$; finally, if the interval $\{v \ge 1-\delta_0\}$ is nonempty, then it has the form $(0,a]$ for some $a>0$; multiplying (2-40) by r^{n-1}, integrating over $(0,r)$, and taking into account that $W'<0$ on $(1-\delta_0,1)$, $V'>0$ on $(0,1)$ and $\lambda>0$, we find

$$2\varepsilon^2 r^{n-1} v'(r) = \int_0^r [W'(v) - \lambda\varepsilon V'(v)] r^{n-1}\,dr < 0,$$

that is, once again $v'(r)<0$. We have thus proved that $v'<0$ on $(0,\infty)$. This information, combined with (2-39), allows us to exploit the Brothers–Ziemer theorem [1988] to conclude that u is a translation of v. This shows that every minimizer of $\psi(\varepsilon)$ is in $\mathcal{R}_0^*$, and concludes the proof of the theorem. □

The compactness argument used in the proof of Theorem 2.1 is relevant also in the implementation of the selection principle used in the proof of the stability estimate (1-21) in the radial decreasing case. Specifically, an adaptation of that argument is needed in showing the existence of minimizers in the variational problems used in the selection principle strategy. In the interest of clarity, it thus seems convenient to discuss this adaptation in this same section. We thus turn to the proof of Theorem 2.2 below. In the statement of this theorem we use for the first time the quantity

$$d_\Phi(u,v) = \int_{\mathbb{R}^n} |\Phi(u) - \Phi(v)|^{n/(n-1)}, \tag{2-42}$$

which is finite whenever $u, v \in H^1(\mathbb{R}^n; [0, 1])$ (indeed, $u \in H^1(\mathbb{R}^n; [0, 1])$ and $W(t) \le Ct^2$ for $t \in [0, 1]$ imply $\mathcal{AC}_\varepsilon(u) < \infty$, thus $|D(\Phi(u))|(\mathbb{R}^n) < \infty$, and hence $\Phi(u) \in L^{n/(n-1)}(\mathbb{R}^n)$ by the BV-Sobolev inequality).

Theorem 2.2. *If $n \ge 2$ and $W \in C^{2,1}[0, 1]$ satisfies* (1-11) *and* (1-12), *then there exist universal constants ε_0, a_0, ℓ_0 and C with the following properties*:

(i) *If $a \in (0, a_0)$, $\varepsilon < \varepsilon_0$, u_ε is a minimizer of $\psi(\varepsilon)$, and $v_\varepsilon \in H^1(\mathbb{R}^n; [0, 1])$ is such that*

$$\int_{\mathbb{R}^n} V(v_\varepsilon) = 1, \quad \mathcal{AC}_\varepsilon(v_\varepsilon) \le \psi(\varepsilon) + a\ell_0, \quad d_\Phi(v_\varepsilon, u_\varepsilon) \le \ell_0, \tag{2-43}$$

then the variational problem

$$\gamma(\varepsilon, a, v_\varepsilon) = \inf\left\{\mathcal{AC}_\varepsilon(w) + a d_\Phi(w, v_\varepsilon) : w \in H^1(\mathbb{R}^n; [0, 1]), \int_{\mathbb{R}^n} V(w) = 1\right\}$$

admits minimizers.

(ii) *If*, *in addition, $v_\varepsilon \in \mathcal{R}_0$, then $\gamma(\varepsilon, a, v_\varepsilon)$ admits a minimizer $w_\varepsilon \in \mathcal{R}_0$. Every such minimizer satisfies $w_\varepsilon \in \mathcal{R}_0^* \cap C^{2,1/(n-1)}_{\mathrm{loc}}(\mathbb{R}^n)$, $0 < w_\varepsilon < 1$ on $\mathbb{R}^n$, and solves*

$$-2\varepsilon^2 \Delta w_\varepsilon = \varepsilon w_\varepsilon(1 - w_\varepsilon)\mathrm{E}_\varepsilon - W'(w_\varepsilon) \quad \text{on } \mathbb{R}^n, \tag{2-44}$$

where E_ε is a continuous radial function on $\mathbb{R}^n$ with

$$\sup_{\mathbb{R}^n} |\mathrm{E}_\varepsilon| \le C. \tag{2-45}$$

Proof. Step 1: Set $\gamma = \gamma(\varepsilon, a, v_\varepsilon)$ for the sake of brevity, and let $\{u_j\}_j$ be a minimizing sequence for γ. Since $a > 0$, we can assume that

$$\mathcal{AC}_\varepsilon(u_j) + a d_\Phi(u_j, v_\varepsilon) \le \gamma + a\ell_0 \quad \text{for all } j. \tag{2-46}$$

In particular, comparing u_j by means of (2-46) with v_ε and u_ε respectively, we obtain the two basic bounds

$$\mathcal{AC}_\varepsilon(u_j) + a d_\Phi(u_j, v_\varepsilon) \le \mathcal{AC}_\varepsilon(v_\varepsilon) + a\ell_0 \le \psi(\varepsilon) + 2\ell_0, \tag{2-47}$$

$$\mathcal{AC}_\varepsilon(u_j) + a d_\Phi(u_j, v_\varepsilon) \le \psi(\varepsilon) + a d_\Phi(u_\varepsilon, v_\varepsilon) + a\ell_0. \tag{2-48}$$

Subtracting $\psi(\varepsilon)$ from (2-48), noticing that $\mathcal{AC}_\varepsilon(u_j) \ge \psi(\varepsilon)$, and using (2-43), we also find

$$d_\Phi(u_j, v_\varepsilon) \le d_\Phi(u_\varepsilon, v_\varepsilon) + \ell_0 \le 2\ell_0, \tag{2-49}$$

and hence, using again (2-43),

$$d_\Phi(u_j, u_\varepsilon) \le C\ell_0. \tag{2-50}$$

Finally, by (2-43), (2-47), and $\psi(\varepsilon) \le 2n\omega_n^{1/n} + C\varepsilon$, we can apply Step 1 of the proof of Theorem 2.1 to u_j, u_ε and v_ε, to find

$$\min\left\{\int_{B_{M_0}} V(u_j), \int_{B_{M_0}} V(u_\varepsilon), \int_{B_{M_0}} V(v_\varepsilon)\right\} \ge 1 - C\sqrt{\ell_0 + \varepsilon_0} \quad \text{for all } j, \tag{2-51}$$

where M_0 is a universal constant. Since (2-51) rules out the possibility of the vanishing case for $\{V(u_j)\,dx\}_j$, we can directly assume that the dichotomy case occurs, and in particular that there exists

$$\alpha \in (1 - C\sqrt{\ell_0 + \varepsilon_0}, 1) \tag{2-52}$$

such that for every $\tau \in (0, \min\{\alpha/2, \tau_*\})$ (here τ_* is as in (2-16)) we can find $S(\tau) > 0$, $S_j(\tau) \to \infty$ and a cut-off function φ between $B_{S(\tau)}$ and $B_{S_j(\tau)}$ such that $|\nabla\varphi| \le 2S_j(\tau)^{-1}$ on $\mathbb{R}^n$, and

$$\begin{aligned} \alpha - C\tau &\le \int_{B_{S(\tau)}} V(u_j), \int_{\mathbb{R}^n} V(\varphi u_j) \le \alpha + C\tau, \\ (1-\alpha) - C\tau &\le \int_{B^c_{S_j(\tau)}} V(u_j), \int_{\mathbb{R}^n} V((1-\varphi)u_j) \le (1-\alpha) + C\tau. \end{aligned} \tag{2-53}$$

We can now verbatim repeat the argument used in Step 2 of the proof of Theorem 2.1 to deduce (2-19) and find that, if $\sigma = \tau^{(n-1)/(3n-1)}$ as in (2-16), then

$$\mathcal{AC}_\varepsilon(u_j) \ge \mathcal{AC}_\varepsilon(\varphi u_j) + \mathcal{AC}_\varepsilon((1-\varphi)u_j) - C\left(\frac{\sigma}{\varepsilon} + \frac{1}{S_j(\tau)}\right); \tag{2-54}$$

in the same vein, by exactly the same argument used to deduce (2-23), we also have

$$\mathcal{AC}_\varepsilon((1-\varphi)u_j) \ge c(n)((1-\alpha) - C\tau)^{(n-1)/n}. \tag{2-55}$$

We now need to show that the $\mathcal{AC}_\varepsilon(\varphi\, u_j)$-term is larger than γ up to $\mathrm{O}(1-\alpha)$ and $\mathrm{O}(\tau)$ errors, but, for reasons that will become clearer in a moment, we cannot do this by just taking a rescaling of φu_j as done in Theorem 2.1. We will rather need to introduce the "localized" family of rescalings which we now describe.

We let $\zeta \in C^\infty_c(B_{2M_0}; [0,1]) \cap \mathcal{R}_0$ with $\zeta = 1$ on B_{M_0} and $|\zeta'| \le 2/M_0$. In particular,

$$|x|\,|\zeta'| \le 2 \quad \text{on } \mathbb{R}^n. \tag{2-56}$$

Next, we set $f_t(x) = x + t\,\zeta(|x|)\,x$ and $\hat{x} = x/|x|$ for $x \in \mathbb{R}^n$ and $t > 0$. By (2-56), if $|t| \le t_0 = t_0(n) < 1$, then $f_t : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism with

$$\begin{aligned} f_t(x) &= x \qquad \text{on } B^c_{2M_0}, \\ f_t(x) &= (1+t)x \quad \text{on } B_{M_0}, \\ \nabla f_t(x) &= (1+t\zeta)\mathrm{Id} + t|x|\zeta'\hat{x}\otimes\hat{x}, \\ Jf_t(x) &= (1+t\zeta)^{n-1}(1+t(\zeta+|x|\zeta')) = 1 + (n\zeta + |x|\zeta')t + \mathrm{O}(t^2). \end{aligned}$$

We set $v_j(t) = (\varphi\, u_j)\circ f_t$, so that $v_j(0) = \varphi\, u_j$, and consider the functions

$$b_j(t) = \int_{\mathbb{R}^n} V(v_j(t)) = \int_{\mathbb{R}^n} V(\varphi u_j) Jf_t, \qquad |t| \le t_0.$$

Clearly we have

$$b_j(0) = \int_{\mathbb{R}^n} V(\varphi u_j) \in [\alpha - C\tau, \alpha + C\tau], \tag{2-57}$$

$$|b_j''(t)| = \int_{\mathbb{R}^n} V(\varphi u_j)\left|\frac{d^2(Jf_t)}{dt^2}\right| \le C \quad \text{for all } |t| \le t_0; \tag{2-58}$$

more crucially, if we choose ε_0 and ℓ_0 small enough, then by (2-51) and (2-56) we find

$$b_j'(0)=\int_{\mathbb{R}^n} V(\varphi u_j)(n\zeta+|x|\zeta')\ge n\int_{B_{M_0}} V(u_j)-(n+2)\int_{B_{2M_0}\setminus B_{M_0}} V(u_j)\ge \frac{n}{2}.$$

As a consequence, by (2-58), we can find a universal constant t_1 such that

$$b_j'(t)\ge \frac{n}{3}\quad \text{for all } |t|\le t_1. \tag{2-59}$$

In particular, b_j is strictly increasing on $[-t_1,t_1]$, with

$$\begin{aligned} b_j(t_1)&\ge b_j(0)+\frac{n}{3}t_1\ge \alpha-C\tau+\frac{n}{3}t_1>1-C(\ell_0+\varepsilon_0+\tau)+\frac{n}{3}t_1>1,\\ b_j(-t_1)&\le b_j(0)-\frac{n}{3}t_1\le \alpha+C\tau-\frac{n}{3}t_1\le 1+C(\ell_0+\varepsilon_0+\tau)-\frac{n}{3}t_1<1-\frac{n}{4}t_1,\end{aligned}$$

so that, for every j, there exists $t_j\in(-t_1,t_1)$ such that $b_j(t_j)=1$: in other words,

$$\int_{\mathbb{R}^n} V(v_j(t_j))=1. \tag{2-60}$$

We now compare the energy of $v_j(t_j)=(\varphi\, u_j)\circ f_{t_j}$ to that of $\varphi\, u_j$. To this end, we first notice that, by comparing $b_j(0)=\int_{\mathbb{R}^n} V(\varphi\, u_j)=\alpha+\mathrm{O}(\tau)$ to $b_j(t_j)=1$, thanks to (2-59) we conclude that

$$|t_j|\le C((1-\alpha)+\tau)\quad \text{for all } j. \tag{2-61}$$

Denoting by $\|A\|$ the operator norm of a linear map A, we have

$$\|\nabla f_t(x)-\mathrm{Id}\|\le C|t|,\quad |Jf_t(x)-1|\le C|t|\quad \text{for all } x\in\mathbb{R}^n,$$

so that

$$\begin{aligned}\mathcal{AC}_\varepsilon(v_j(t))&=\int_{\mathbb{R}^n}\left\{\varepsilon|(\nabla f_t\circ f_t^{-1})[\nabla(\varphi u_j)]|^2+\frac{W(\varphi u_j)}{\varepsilon}\right\}Jf_t\\ &\le \int_{\mathbb{R}^n}\left\{\varepsilon(1+C|t|)^2|\nabla(\varphi u_j)|^2+\frac{W(\varphi u_j)}{\varepsilon}\right\}(1+C|t|)\le(1+C|t|)\mathcal{AC}_\varepsilon(\varphi u_j).\end{aligned}$$

Therefore if we combine (2-54), (2-55), and (2-61) with this last estimate, and take into account that $\mathcal{AC}_\varepsilon(u_j),\mathcal{AC}_\varepsilon(\varphi\, u_j)\le C$, then we obtain

$$\begin{aligned}\mathcal{AC}_\varepsilon(u_j)+a d_\Phi(u_j,v_\varepsilon)&\ge \mathcal{AC}_\varepsilon(v_j(t_j))+a d_\Phi(v_j(t_j),v_\varepsilon)+a(d_\Phi(u_j,v_\varepsilon)-d_\Phi(v_j(t_j),v_\varepsilon))\\ &\quad+c(n)((1-\alpha)-C\tau)^{(n-1)/n}-C\Big((1-\alpha)+\tau+\frac{1}{S_j(\tau)}+\frac{\sigma}{\varepsilon}\Big).\end{aligned} \tag{2-62}$$

We notice that for every $u,v\in H^1(\mathbb{R}^n;[0,1])$, thanks to the triangular inequality in $L^{n/(n-1)}$ and to $|b^{1/n'}-a^{1/n'}|\ge c(n)\,b^{-1/n}(b-a)$ for $0<a<b$, we have

$$c(n)\,\frac{|d_\Phi(u,v_\varepsilon)-d_\Phi(v,v_\varepsilon)|}{\max\{d_\Phi(u,v_\varepsilon),d_\Phi(v,v_\varepsilon)\}^{1/n}}\le d_\Phi(u,v)^{(n-1)/n}. \tag{2-63}$$

We apply (2-63) with $u = u_j$ and $v = u_j\,\varphi$ to find

$$\begin{aligned}|d_\Phi(u_j, v_\varepsilon) - d_\Phi(\varphi u_j, v_\varepsilon)| &\le C\int_{\mathbb{R}^n} |\Phi(u_j) - \Phi(\varphi u_j)|^{n/(n-1)}\\ &\le \int_{\mathbb{R}^n\setminus B_{S(\tau)}} V(u_j) \le C((1-\alpha)+\tau),\end{aligned}$$

where we have used (2-53). Similarly, noticing that

$$\begin{aligned}\frac{d}{ds}\Phi(v_j(s)) &= \sqrt{W(v_j(s))}[\nabla(\varphi\,u_j)\circ f_s]\cdot\frac{d}{ds}f_s\\ &= \sqrt{W(v_j(s))}[\nabla(\varphi u_j)\circ f_s]\cdot(\zeta(|x|)x),\end{aligned}$$

with $\zeta(x)\,|x| \le 2M_0$ for every $x\in\mathbb{R}^n$ by (2-56), we find[4]

$$\begin{aligned}|d_\Phi(v_j(t_j), v_\varepsilon) - d_\Phi(\varphi\,u_j, v_\varepsilon)| &\le C\int_{\mathbb{R}^n}|\Phi(v_j(t_j)) - \Phi(\varphi u_j)|^{n/(n-1)} \le C\int_{\mathbb{R}^n}|\Phi(v_j(t_j)) - \Phi(\varphi u_j)|\\ &\le C\left|\int_0^{t_j} ds\int_{\mathbb{R}^n}\sqrt{W(v_j(s))}[\nabla(\varphi\,u_j)\circ f_s]\cdot(\zeta(|x|)\,x)\right|\\ &\le C\left|\int_0^{t_j} ds\int_{\mathbb{R}^n}\sqrt{W(\varphi u_j)}\nabla(\varphi u_j)\cdot(\zeta(|f_s^{-1}|)\,f_s^{-1})Jf_s\right|\\ &\le CM_0|t_j|\int_{\mathbb{R}^n}\sqrt{W(\varphi u_j)}|\nabla(\varphi u_j)| \le C|t_j|\mathcal{AC}_\varepsilon(\varphi u_j).\end{aligned}\tag{2-64}$$

We finally combine (2-61), (2-62), (2-64), and the fact that $v_j(t_j)$ is a competitor for γ to conclude that

$$\mathcal{AC}_\varepsilon(u_j) + a d_\Phi(u_j, v_\varepsilon) \ge \gamma + c(n)((1-\alpha) - C\tau)^{(n-1)/n} - C\left((1-\alpha)+\tau+\frac{\sigma}{\varepsilon}+\frac{1}{S_j(\tau)}\right).$$

Letting $j\to\infty$ and then $\tau\to 0^+$ (so that $\sigma\to 0^+$ thanks to (2-16)), we finally conclude

$$0 \ge c(n)(1-\alpha)^{(n-1)/n} - C(1-\alpha),$$

which gives a contradiction with (2-52) if ε_0 and ℓ_0 are small enough. Having excluded vanishing and dichotomy, by a standard argument we deduce the existence of a minimizer of γ.

Step 2: We now assume that $v_\varepsilon\in\mathcal{R}_0$. Since Φ is an increasing function on [0, 1], if u^* denotes the radial decreasing rearrangement of $u:\mathbb{R}^n\to[0,\infty)$, then $\Phi(u^*) = \Phi(u)^*$. In particular, by a standard property of rearrangements,

$$d_\Phi(u, v) = \int_{\mathbb{R}^n}|\Phi(u) - \Phi(v)|^{n/(n-1)} \ge \int_{\mathbb{R}^n}|\Phi(u)^* - \Phi(v)^*|^{n/(n-1)} = d_\Phi(u^*, v^*).$$

This fact, combined with the Pólya–Szegő inequality and the fact that $v_\varepsilon^* = v_\varepsilon$, implies that the radial decreasing rearrangement of a minimizer of γ is also a minimizer of γ (in brief, a radial decreasing minimizer).

[4]This is the key step where using $f_t(x)$ rather than $(1+t)\,x$ (as done when proving Theorem 2.1) makes a substantial difference. Indeed, by using a global rescaling to fix the volume constraint of $\varphi\,u_j$, we end up having to control, in the analogous estimate to (2-64), the first moment of the energy density of $\varphi\,u_j$, i.e., $\int_{\mathbb{R}^n}|x|(\varepsilon|\nabla(\varphi u_j)|^2 + W(\varphi u_j)/\varepsilon)$, rather than the trivially bounded quantity $M_0\mathcal{AC}_\varepsilon(u_j)$.

We now show that every radial decreasing minimizer w_ε of γ satisfies $0 < w_\varepsilon < 1$ on $\mathbb{R}^n$, that $w_\varepsilon \in C^{2,1/(n-1)}_{\text{loc}}(\mathbb{R}^n)$, and that (2-44) holds for a radial continuous function E_ε bounded by a universal constant. Arguing as in Step 4 of the proof of Theorem 2.1, with $0 \le a < b \le +\infty$ and $\Omega = B_b \setminus \bar{B}_a = \{0 < w_\varepsilon < 1\}$, we see that w_ε solves

$$-2\varepsilon^2 \Delta w_\varepsilon = \varepsilon\lambda V'(w_\varepsilon) - W'(w_\varepsilon) - a\varepsilon Z_\varepsilon(x, w_\varepsilon) \quad \text{in } \mathcal{D}'(\Omega), \tag{2-65}$$

where, for $x \in \mathbb{R}^n$ and $t \in [0,1]$, we have set

$$Z_\varepsilon(x,t) = \frac{n}{n-1} |\Phi(t) - \Phi(v_\varepsilon)|^{(n/(n-1))-2} (\Phi(t) - \Phi(v_\varepsilon)) \sqrt{W(t)}.$$

By (2-65), Δw_ε is bounded in Ω, and thus, by the Calderon–Zygmund theorem, $w_\varepsilon \in \mathrm{Lip}_{\text{loc}}(\Omega)$. This implies that $Z_\varepsilon(x,t) \in C^{0,1/(n-1)}_{\text{loc}}(\Omega)$, and thus, by Schauder's theory, that $w_\varepsilon \in C^{2,1/(n-1)}_{\text{loc}}(\Omega)$. We now want to prove that $\Omega = \mathbb{R}^n$. By the same variational arguments used in deriving (2-27) and (2-28), we have

$$-2\varepsilon^2 \Delta w_\varepsilon \ge f(x,t) \quad \text{in } \mathcal{D}'(\mathbb{R}^n \setminus \bar{B}_a), \tag{2-66}$$

$$-2\varepsilon^2 \Delta w_\varepsilon \le f(x,t) \quad \text{in } \mathcal{D}'(B_b), \tag{2-67}$$

where $f(x,t)$ satisfies

$$|f(x,t)| \le Ct(1-t) \quad \text{for all } (x,t) \in \mathbb{R}^n \times [0,1], \tag{2-68}$$

thanks to (A-6) and (A-11) (which, in particular, give $|Z_\varepsilon(x,t)| \le Ct(1-t)$ for every $(x,t) \in \mathbb{R}^n \times [0,1]$). By repeating the same argument used in Step 4 of the proof of Theorem 2.1, we thus see that $\Omega = \mathbb{R}^n$. Finally, it is easily seen that (2-65), with $\Omega = \mathbb{R}^n$ and $w_\varepsilon \in C^2(\mathbb{R}^n)$, takes the form

$$-2\varepsilon^2 \Delta w_\varepsilon = \varepsilon w_\varepsilon(1 - w_\varepsilon)\mathrm{E}_\varepsilon - W'(w_\varepsilon) \quad \text{on } \mathbb{R}^n, \tag{2-69}$$

for a radial function E_ε bounded by a universal constant on $\mathbb{R}^n$, as claimed. □

3. Resolution of almost-minimizing sequences

In the main result of this section, Theorem 3.1 below, we provide a sharp description, up to first order as $\varepsilon \to 0^+$, of the minimizers of $\psi(\varepsilon)$. This resolution result is proved not only for minimizers of $\psi(\varepsilon)$, but also for a general notion of "critical sequence for $\psi(\varepsilon_j)$ as $\varepsilon_j \to 0^+$" modeled after the selection principle minimizers of Theorem 2.2.

In the following statement, η is the solution of $\eta' = -\sqrt{W(\eta)}$ on $\mathbb{R}$ with $\eta(0) = \frac{1}{2}$,

$$\tau_0 = \int_{\mathbb{R}} \eta' V'(\eta) s \, ds, \qquad \tau_1 = \int_{\mathbb{R}} W(\eta) s \, ds,$$

and $R_0 = \omega_n^{-1/n}$. Relevant properties of η are collected in Section A4.

Theorem 3.1. *If $n \ge 2$ and $W \in C^{2,1}[0,1]$ satisfies* (1-11) *and* (1-12), *then there exist universal constants ε_0, δ_0, and ℓ_0 with the following properties*:

Ansatz: *For every $\varepsilon < \varepsilon_0$ there exists a unique $\tau_\varepsilon \in \mathbb{R}$ such that if we set*

$$z_\varepsilon(x) = \eta\left(\frac{|x| - R_0}{\varepsilon} - \tau_\varepsilon\right), \tag{3-1}$$

then

$$\int_{\mathbb{R}^n} V(z_\varepsilon) = 1. \tag{3-2}$$

Moreover, we have $|\tau_\varepsilon - \tau_0| \le C\varepsilon$ and, in the limit as $\varepsilon \to 0^+$,

$$\mathcal{AC}_\varepsilon(z_\varepsilon) = 2n\omega_n^{1/n} + 2n(n-1)\omega_n^{2/n}(\tau_0 + \tau_1)\varepsilon + \mathrm{O}(\varepsilon^2). \tag{3-3}$$

Resolution of critical sequences: *If $\varepsilon_j \to 0^+$ as $j \to \infty$, $\{v_j\}_j$ is a sequence in $C^2(\mathbb{R}^n; [0, 1]) \cap \mathcal{R}_0$ such that*

$$\int_{\mathbb{R}^n} V(v_j) = 1, \tag{3-4}$$

$$\mathcal{AC}_{\varepsilon_j}(v_j) \le 2n\omega_n^{1/n} + \ell_0, \tag{3-5}$$

and $\{\mathrm{E}_j\}_j$ is a sequence of radial continuous functions on $\mathbb{R}^n$ with

$$-2\varepsilon_j^2 \Delta v_j = \varepsilon_j v_j(1 - v_j)\mathrm{E}_j - W'(v_j) \quad \text{on } \mathbb{R}^n, \tag{3-6}$$

$$\sup_j \|\mathrm{E}_j\|_{C^0(\mathbb{R}^n)} \le C, \tag{3-7}$$

then, for j large enough, we have

$$v_j(x) = z_{\varepsilon_j}(x) + f_j\left(\frac{|x| - R_0}{\varepsilon_j}\right), \quad x \in \mathbb{R}^n, \tag{3-8}$$

where $f_j \in C^2(-R_0/\varepsilon_j, \infty)$, and

$$|f_j(s)| \le C\varepsilon_j\, e^{-|s|/C} \quad \text{for all } s \ge -R_0/\varepsilon_j. \tag{3-9}$$

Moreover, for j large enough, there exist positive constants b_j and c_j such that

$$\begin{aligned} v_j(R_0 + c_j) &= \delta_0, \\ v_j(R_0 - b_j) &= 1 - \delta_0, \end{aligned} \tag{3-10}$$

and b_j and c_j satisfy

$$\frac{\varepsilon_j}{C} \le b_j, c_j \le C\varepsilon_j. \tag{3-11}$$

Finally, one has

$$\frac{C}{\varepsilon_j} \ge -v_j'(r) \ge \frac{1}{C\varepsilon_j} \quad \text{for all } r \in [R_0 - b_j, R_0 + c_j], \tag{3-12}$$

$$\begin{cases} v_j(r) \le Ce^{-(r-R_0)/(C\varepsilon_j)}, \\ |v_j^{(k)}(r)| \le \dfrac{C}{\varepsilon_j^k} e^{-(r-R_0)/(C\varepsilon_j)} \end{cases} \quad \text{for all } r \in [R_0 + c_j, \infty),\ k = 1, 2, \tag{3-13}$$

$$\begin{cases} 1 - v_j(r) \le C e^{-(R_0 - r)/(C\varepsilon_j)}, \\ |v_j'(r)| \le C \min\left\{\dfrac{r}{\varepsilon_j^2}, \dfrac{1}{\varepsilon_j}\right\} e^{-(R_0-r)/(C\varepsilon_j)}, \\ |v_j''(r)| \le \dfrac{C}{\varepsilon_j^2} e^{-(R_0-r)/(C\varepsilon_j)} \end{cases} \qquad \textit{for all } r \in (0, R_0 - b_j). \tag{3-14}$$

Proof. The first two steps of the proof take care of the ansatz-part of the statement, while starting from Step 3 we address the resolution result. We use the fact that, if we set $z_\tau(x) = \eta([(|x| - R_0)/\varepsilon] - \tau)$, then $f(\tau) = \int_{\mathbb{R}^n} V(z_\tau)$ is strictly increasing in τ with $f(-\infty) = 0$ and $f(+\infty) = +\infty$. For this reason, τ_ε is indeed uniquely defined by (3-2).

Step 1: We prove that if $\{w_\varepsilon\}_{\varepsilon>0}$ is defined by

$$w_\varepsilon(x) = \eta\left(\frac{|x| - R_0}{\varepsilon} - t_\varepsilon\right) + f_\varepsilon\left(\frac{|x| - R_0}{\varepsilon}\right), \quad x \in \mathbb{R}^n, \ \varepsilon > 0,$$

for some $t_\varepsilon \in \mathbb{R}$ and some functions $f_\varepsilon \in C^2(-R_0/\varepsilon, \infty)$ such that

$$\int_{\mathbb{R}^n} V(w_\varepsilon) = 1, \tag{3-15}$$

$$|f_\varepsilon(s)| \le C\varepsilon e^{-|s|/C} \quad \text{for all } s \ge -R_0/\varepsilon, \tag{3-16}$$

then

$$|t_\varepsilon - \tau_0| \le C\varepsilon \quad \text{for all } \varepsilon < \varepsilon_0. \tag{3-17}$$

Of course, in the particular case when $f_\varepsilon \equiv 0$, we have $w_\varepsilon = z_\varepsilon$ and $t_\varepsilon = \tau_\varepsilon$ thanks to (3-1) and (3-2).

Indeed, setting $z_0(x) = \eta([(|x| - R_0)/\varepsilon] - \tau_0)$ for $x \in \mathbb{R}^n$, and recalling (3-2) and (3-15), we consider the quantity

$$\kappa_\varepsilon = \int_{\mathbb{R}^n} V(1_{B_{R_0}}) - V(z_0) = \int_{\mathbb{R}^n} V(w_\varepsilon) - V(z_0). \tag{3-18}$$

We look at the first expression for κ_ε, passing first to the radial coordinate $r = |x|$ and then changing variables into $s = (r - R_0)/\varepsilon$. By taking into account the fact that τ_0 satisfies

$$\int_{\mathbb{R}} (1_{(-\infty,0)}(s) - V(\eta(s - \tau_0)))\, ds = 0,$$

see (A-19), we find

$$\begin{aligned} \frac{\kappa_\varepsilon}{n\,\omega_n} &= \varepsilon \int_{-R_0/\varepsilon}^{\infty} (1_{(-\infty,0)}(s) - V(\eta(s - \tau_0)))(R_0 + \varepsilon s)^{n-1}\, ds \\ &= \varepsilon R_0^{n-1} \int_{-R_0/\varepsilon}^{\infty} (1_{(-\infty,0)}(s) - V(\eta(s - \tau_0)))\, ds \\ &\qquad\qquad + \varepsilon \int_{-R_0/\varepsilon}^{\infty} (1_{(-\infty,0)}(s) - V(\eta(s - \tau_0)))[(R_0 + \varepsilon s)^{n-1} - R_0^{n-1}]\, ds \\ &= -\varepsilon R_0^{n-1} \int_{-\infty}^{-R_0/\varepsilon} (1_{(-\infty,0)}(s) - V(\eta(s - \tau_0)))\, ds \\ &\qquad\qquad + \varepsilon \sum_{k=0}^{n-2} a_k \int_{-R_0/\varepsilon}^{\infty} (1_{(-\infty,0)}(s) - V(\eta(s - \tau_0))) R_0^k (s\varepsilon)^{n-1-k}\, ds, \end{aligned}$$

with $a_k = \binom{n-1}{k}$. Since $\tau_0 = \tau_0(W)$, by the decay properties (A-16) of η, we have

$$|\,1_{(-\infty,0)}(s) - V(\eta(s-\tau_0))| \le Ce^{-|s|/C} \quad \text{for all } s \in \mathbb{R}, \tag{3-19}$$

so that

$$\left|\int_{-\infty}^{-R_0/\varepsilon} (1_{(-\infty,0)}(s) - V(\eta(s-\tau_0)))\,ds\right| \le C\int_{-\infty}^{-R_0/\varepsilon} e^{-|s|/C}\,ds \le Ce^{-R_0/(C\varepsilon)},$$

and, recalling that $\omega_n R_0^n = 1$,

$$|\kappa_\varepsilon| \le C\varepsilon e^{-R_0/(C\varepsilon)} + C\varepsilon^2 \sum_{j=1}^{n-1} \int_{-R_0/\varepsilon}^{\infty} |1_{(-\infty,0)}(s) - V(\eta(s-\tau_0))|\,|s|^j\,ds \le C\varepsilon^2,$$

where in the last inequality we have used (3-19) again. Taking into account the second formula for κ_ε in (3-18), we have thus proved

$$C\varepsilon^2 \ge \left|\int_{\mathbb{R}^n} V(w_\varepsilon) - V(z_0)\right|. \tag{3-20}$$

With the same change of variables used before we have

$$C\varepsilon \ge \left|\int_{-R_0/\varepsilon}^{\infty} \{V(\eta(s-t_\varepsilon) + f_\varepsilon(s)) - V(\eta(s-\tau_0))\}(R_0+\varepsilon s)^{n-1}\,ds\right|,$$

while the decay properties of f_ε assumed in (3-16) give

$$\begin{aligned}\left|\int_{-R_0/\varepsilon}^{\infty} \{V(\eta(s-t_\varepsilon) + f_\varepsilon(s)) - V(\eta(s-t_\varepsilon))\}(R_0+\varepsilon s)^{n-1}\,ds\right| \\ \le \int_{-R_0/\varepsilon}^{\infty} f_\varepsilon(s)(R_0+\varepsilon s)^{n-1}\,ds \int_0^1 V'(\eta(s-t_\varepsilon)+rf_\varepsilon(s))\,dr \le C\varepsilon;\end{aligned}$$

by combining the last two inequalities we thus find

$$\begin{aligned} C\varepsilon &\ge \left|\int_{-R_0/\varepsilon}^{\infty} \{V(\eta(s-t_\varepsilon)) - V(\eta(s-\tau_0))\}(R_0+\varepsilon s)^{n-1}\,ds\right| \\ &= \int_{-R_0/\varepsilon}^{\infty} |V(\eta(s-t_\varepsilon)) - V(\eta(s-\tau_0))|(R_0+\varepsilon s)^{n-1}\,ds, \end{aligned} \tag{3-21}$$

where in the last step we have used that $\tau \to V(\eta(\cdot - \tau))$ is strictly increasing in τ. Since (3-21) implies $t_\varepsilon \to \tau_0$ as $\varepsilon \to 0^+$, we can choose $\varepsilon_0 = \varepsilon_0(n, W)$ so that $|t_\varepsilon - \tau_0| \le 1$ and $R_0 + \varepsilon\,(\tau_0 - 1) \ge R_0/2$. Since $V \circ \eta$ is strictly decreasing on $\mathbb{R}$, we have $|(V\circ\eta)'| \ge 1/C$ on $[-2, 2]$, and noticing that if $|s-\tau_0| \le 1$, then $|s - t_\varepsilon| < 2$, we finally conclude

$$C\varepsilon \ge \int_{\tau_0-1}^{\tau_0+1} \frac{|(s-t_\varepsilon)-(s-\tau_0)|}{C}(R_0+\varepsilon s)^{n-1}\,ds \ge \frac{|\tau_0 - t_\varepsilon|}{C},$$

thus proving (3-17).

<u>Step 2</u>: We compute $\mathcal{AC}_\varepsilon(z_\varepsilon)$. Passing to the radial coordinate $r=|x|$, setting first $r=R_0+\varepsilon s$ and then $t=s-\tau_\varepsilon$, recalling that $\eta'=-\sqrt{W(\eta)}$, and exploiting the decay property (A-16) of η at $-\infty$, we find that, as $\varepsilon\to 0^+$,

$$\begin{aligned}\mathcal{AC}_\varepsilon(z_\varepsilon)&=n\omega_n\int_{-R_0/\varepsilon}^{\infty}(\eta'(s-\tau_\varepsilon)^2+W(\eta(s-\tau_\varepsilon)))(R_0+\varepsilon s)^{n-1}\,ds\\&=2n\omega_n\int_{-\tau_\varepsilon-R_0/\varepsilon}^{\infty}W(\eta(t))(R_0+\varepsilon(t+\tau_\varepsilon))^{n-1}\,dt\\&=2n\omega_n\int_{-\infty}^{\infty}W(\eta(t))(R_0+\varepsilon(t+\tau_\varepsilon))^{n-1}\,dt+\mathrm{O}(e^{-C/\varepsilon})\\&=2n\omega_n\int_{-\infty}^{\infty}W(\eta(t))(R_0+\varepsilon(t+\tau_0))^{n-1}\,dt+\mathrm{O}(\varepsilon^2),\end{aligned}\tag{3-22}$$

where in the last step we have used $\tau_\varepsilon=\tau_0+\mathrm{O}(\varepsilon)$. Recalling that, by (1-12),

$$\int_{\mathbb{R}}W(\eta)=-\int_{\mathbb{R}}\sqrt{W(\eta)}\,\eta'=-\int_{\mathbb{R}}\Phi'(\eta)\eta'=\Phi(\eta(-\infty))-\Phi(\eta(+\infty))=\Phi(1)=1,$$

as well as that $\omega_n\,R_0^n=1$, we find

$$\mathcal{AC}_\varepsilon(z_\varepsilon)=2n\omega_n^{1/n}+2n(n-1)\omega_n^{2/n}(\tau_0+\tau_1)\varepsilon+\mathrm{O}(\varepsilon^2)$$

as $\varepsilon\to 0^+$, that is (3-3). This proves the first part of the statement of the theorem.

<u>Step 3</u>: In preparation to the proof of the second part of the statement, we show that if $\varepsilon<\varepsilon_0$ and $u\in H^1(\mathbb{R}^n;[0,1])$ satisfies

$$\mathcal{AC}_\varepsilon(u)\le 2n\omega_n^{1/n}+\ell_0,\qquad \int_{\mathbb{R}^n}V(u)=1,\tag{3-23}$$

then

$$\int_{\mathbb{R}^n}|\Phi(u)-1_{B_{R_0}}|^{n/(n-1)}\le C((\sqrt{\ell_0})^{(n-1)/(2n)}+\varepsilon).\tag{3-24}$$

Moreover, if $u\in\mathcal{R}_0$, then $\sqrt{\ell_0}$ can be replaced by ℓ_0 in (3-24).

Indeed, by (3-23), as seen in Step 1 of the proof of Theorem 2.1, we have

$$\int_{\mathbb{R}^n}|\Phi(u)-(\omega_n^{1/n}r(u))^{1-n}1_{B_{r(u)}}|^{n/(n-1)}\le C\sqrt{\ell_0}\tag{3-25}$$

for some $r(u)\in(0,M_0]$, where M_0 is a universal constant. Setting $f(r)=(\omega_n^{1/n}\,r)^{1-n}$, and noticing that $f(R_0)=1$, it is enough to prove that

$$|r(u)-R_0|\le C((\sqrt{\ell_0})^{(n-1)/(2n)}+\varepsilon),\qquad |f(r(u))-1|\le C((\sqrt{\ell_0})^{(n-1)/(2n)}+\varepsilon).\tag{3-26}$$

Since $\mathrm{Lip}(f,[R_0/2,2R_0])\le C$ and $f(R_0)=1$, it is enough to prove the first estimate in (3-26). To this end, we start noticing that if $r(u)<R_0$, then $f(r(u))>f(R_0)=1\ge\Phi(u)$, and (3-25) gives

$$\begin{aligned}C\sqrt{\ell_0}\ge\int_{B_{r(u)}}|\Phi(u)-f(r(u))|^{n/(n-1)}&\ge\omega_n r(u)^n(f(r(u))-1)^{n/(n-1)}\\&=(u)^n(f(r(u))-f(R_0))^{n/(n-1)}=c(n)(1-(r(u)/R_0)^{n-1})^{n/(n-1)}\\&\ge c(n)(R_0-r(u))^{n/(n-1)},\end{aligned}$$

as desired. If, instead $r(u) > R_0$, then by $\int_{\mathbb{R}^n} W(u) \le \varepsilon \mathcal{AC}_\varepsilon(u) \le C$, $f(r(u)) \in (0,1)$ and (A-8) (that is, $\Phi(b) - \Phi(a) \ge (b-a)^2/C$ if $0 \le a \le b \le 1$), we deduce that

$$\begin{aligned} C\varepsilon &\ge \int_{B_{R_0}} W(u) \ge \int_{B_{R_0}} W(\Phi^{-1}(f(r(u)))) - C\int_{B_{R_0}} |u - \Phi^{-1}(f(r(u)))| \\ &\ge \int_{B_{R_0}} W(\Phi^{-1}(f(r(u)))) - C\int_{B_{R_0}} |\Phi(u) - f(r(u))|^{1/2} \\ &\ge \int_{B_{R_0}} W(\Phi^{-1}(f(r(u)))) - C\biggl(\int_{B_{R_0}} |\Phi(u) - f(r(u))|^{n/(n-1)}\biggr)^{(n-1)/(2n)}, \end{aligned}$$

where in the last inequality we have used the Hölder inequality with $p = (2n)/(n-1) > 1$ and the fact that $\mathcal{L}^n(B_{R_0})$ is a universal constant. Hence, by $B_{R_0} \subset B_{r(u)}$, (3-25) and $\omega_n R_0^n = 1$,

$$W(\Phi^{-1}(f(r(u)))) \le C((\sqrt{\ell_0})^{(n-1)/(2n)} + \varepsilon).$$

Now, $R_0 < r(u) \le M_0$ implies $1 > f(r(u)) \ge f(M_0) \ge \delta_0$ (provided we further decrease the value of δ_0). In particular, by $W(t) \ge (1-t)^2/C$ on $(\delta_0, 1)$ (which can be assumed as done with (A-13)), we have

$$C((\sqrt{\ell_0})^{(n-1)/(2n)} + \varepsilon) \ge (1 - \Phi^{-1}(f(r(u))))^2.$$

By (A-7), we have

$$1 - \Phi^{-1}(s) \ge \frac{\sqrt{1-s}}{C} \quad \text{for all } s \in (0,1),$$

thus concluding

$$\begin{aligned} C((\sqrt{\ell_0})^{(n-1)/(2n)} + \varepsilon) &\ge 1 - f(r(u)) = c(n)(R_0^{1-n} - r(u)^{1-n}) \\ &\ge \frac{c(n)}{r(u)^{n-1}}\biggl(\biggl(\frac{r(u)}{R_0}\biggr)^{n-1} - 1\biggr) \ge \frac{c(n)}{M_0^{n-1}}(r(u) - R_0). \end{aligned}$$

This completes the proof of (3-26), and thus of (3-24).

Step 4: We now consider $\{\varepsilon_j, v_j, \mathrm{E}_j\}_j$ as in the statement, and begin the proof of the resolution result. We introduce the radius $R_j(t)$ by setting $v_j(R_j(t)) = t$ for every t in the range of v_j. In this step we prove that both δ_0 (defined in Section A3) and $1 - \delta_0$ belong to the range of each v_j, that

$$3R_0 \ge R_j(\delta_0) \ge R_j(1-\delta_0) \ge \frac{R_0}{3}, \tag{3-27}$$

$$\frac{\varepsilon_j}{C} \le R_j(\delta_0) - R_j(1-\delta_0) \le C\varepsilon_j, \tag{3-28}$$

and that

$$-\frac{C}{\varepsilon_j} \le v_j' \le -\frac{1}{C\varepsilon_j} \quad \text{on } (R_j(1-\delta_0), R_j(\delta_0)). \tag{3-29}$$

In particular, the constants b_j and c_j introduced in (3-10) are well-defined, they satisfy

$$c_j = R_j(\delta_0) - R_0, \quad b_j = R_0 - R_j(1-\delta_0), \tag{3-30}$$

and property (3-12) in the statement boils down to (3-29).

By Step 3, for j large enough and considering that $v_j \in \mathcal{R}_0$, we have

$$\int_{\mathbb{R}^n} |1_{B_{R_0}} - \Phi(v_j)|^{n/(n-1)} \le C(\ell_0^{(n-1)/(2n)} + \varepsilon_0). \tag{3-31}$$

By (3-31), if ℓ_0 and ε_0 are small enough, then both δ_0 and $1-\delta_0$ must belong to the range of each v_j. Now, if $R_j(\delta_0) \le R_0$, then

$$\int_{B_{R_0}\setminus B_{R_j(\delta_0)}} |1_{B_{R_0}} - \Phi(v_j)|^{n/(n-1)} \ge \omega_n (R_0^n - R_j(\delta_0)^n)(1-\Phi(\delta_0))^{n/(n-1)} \ge \frac{R_0^n - R_j(\delta_0)^n}{C},$$

and $R_j(\delta_0) \ge R_0/2$ follows by (3-31) for ℓ_0 and ε_0 small enough; if, instead, $R_j(\delta_0) \ge R_0$, then

$$\int_{B_{R_j(\delta_0)}\setminus B_{R_0}} |1_{B_{R_0}} - \Phi(v_j)|^{n/(n-1)} \ge \omega_n (R_j(\delta_0)^n - R_0^n)\Phi(\delta_0)^{n/(n-1)} \ge \frac{R_j(\delta_0)^n - R_0^n}{C},$$

and $R_j(\delta_0) \le 2R_0$ follows, again, for ℓ_0 and ε_0 small enough; we have thus proved $R_0/2 \le R_j(\delta_0) \le 2R_0$. Since (3-5) implies $\mathcal{AC}_{\varepsilon_j}(v_j) \le C$ we also have

$$C\varepsilon_j \ge \int_{\mathbb{R}^n} W(v_j) \ge \frac{R_j(\delta_0)^n - R_j(1-\delta_0)^n}{C} \ge \frac{R_j(\delta_0) - R_j(1-\delta_0)}{C},$$

where in the last inequality we have used $R_j(\delta_0) \ge R_0/2$. Thus, we have so far proved (3-27) and the upper bound in (3-28). Before proving the lower bound in (3-28), we prove (3-29). To this end, we multiply (3-6) by v_j', and then integrate over an arbitrary interval $(0, r)$ to get

$$\varepsilon_j^2\left((v_j')^2 + 2(n-1)\int_0^r \frac{v_j'(t)^2}{t}\,dt\right) = W(v_j) - W(v_j(0)) - \varepsilon_j \int_0^r v_j(1-v_j)\mathrm{E}_j v_j'. \tag{3-32}$$

By (3-7), the right-hand side of (3-32) is bounded in terms of n and W, so that (3-32) implies $\varepsilon_j^2(v_j')^2 \le C$ on $(0,\infty)$; the lower bound in (3-29) then follows by $v_j' \le 0$. To obtain the upper bound in (3-29), we multiply again (3-6) by v_j', but this time we integrate over (r,∞) for $r \in (R_j(1-\delta_0), R_j(\delta_0))$, thus obtaining

$$\varepsilon_j^2\left(-v_j'(r)^2 + 2(n-1)\int_r^\infty \frac{v_j'(t)^2}{t}\,dt\right) = -W(v_j(r)) - \varepsilon_j \int_r^\infty v_j(1-v_j)\mathrm{E}_j v_j'. \tag{3-33}$$

By $W(v_j(r)) \ge \inf_{[\delta_0, 1-\delta_0]} W \ge 1/C$, (3-7), and the nonnegativity of the integral on the left-hand side of (3-33), we deduce that

$$2\varepsilon_j^2 v_j'(r)^2 \ge W(v_j(r)) - C\varepsilon_j \ge \frac{1}{C} \quad \text{for all } r \in (R_j(1-\delta_0), R_j(\delta_0)),$$

which, again by $v_j' \le 0$, implies the upper bound in (3-29). To finally prove the lower bound in (3-28), we notice that thanks to the lower bound in (3-29) we have

$$\frac{C}{\varepsilon_j}(R_j(\delta_0) - R_j(1-\delta_0)) \ge \int_{R_j(1-\delta_0)}^{R_j(\delta_0)} (-v_j') = 1 - 2\delta_0.$$

We have completed the proofs of (3-27), (3-28) and (3-29).

Step 5: We obtain sharp estimates for v_j as $r \to \infty$: precisely, we prove that for every $r \geq R_j(\delta_0)$ one has

$$v_j(r) \leq C e^{-(r-R_j(\delta_0))/(C\varepsilon_j)}, \tag{3-34}$$

$$|v_j^{(k)}(r)| \leq \frac{C}{\varepsilon_j^k} e^{-(r-R_j(\delta_0))/(C\varepsilon_j)}, \quad k = 1, 2. \tag{3-35}$$

We first transform (3-6) to get rid of the first-order term and capture the polynomial factor of the form $r^{(1-n)/2}$. To this end we consider the so-called Emden–Fowler change of variables. More precisely, we set $v_j = q\, w_j$ and notice that (3-6) gives

$$\varepsilon_j^2 \left\{ q w_j'' + w_j q'' + w_j' \left(2q' + \frac{(n-1)q}{r} \right) + \frac{(n-1)q' w_j}{r} \right\} = \frac{1}{2}(W'(v_j) - \varepsilon_j\, v_j\, (1-v_j)\, \mathrm{E}_j).$$

Thus setting $q(r) = r^{-a}$ with $a = (n-1)/2$ we find the following ODE for w_j:

$$\varepsilon_j^2 w_j'' = \frac{w_j}{2} \left(\varepsilon_j^2 \frac{2a(a-1)}{r^2} + \frac{W'(v_j) - \varepsilon_j v_j (1-v_j) \mathrm{E}_j}{v_j} \right). \tag{3-36}$$

Recasting (3-6) in spherical coordinates, exploiting (3-7) and (A-6), and taking j large enough to give $\varepsilon_j < \sigma_0$, we deduce that

$$\varepsilon_j^2 w_j'' \geq \frac{w_j}{2} \left(\varepsilon_j^2 \frac{2a(a-1)}{r^2} + \frac{1}{C} - C\varepsilon_j \right) \geq \frac{w_j}{2C_*} \tag{3-37}$$

for some C_* universal. We now notice that

$$w_*(r) = \delta_0 e^{-(r-R_j(\delta_0))/(\sqrt{2C_*}\varepsilon_j)}$$

satisfies $\varepsilon_j^2 w_*'' = w_*/2C_*$ and

$$w_*(R_j(\delta_0)) = \delta_0 = w_j(R_j(\delta_0)).$$

Therefore, if $r \geq R_j(\delta_0)$, then

$$w_j(r) \leq w_*(r) = \delta_0 e^{-(r-R_j(\delta_0))/(\sqrt{2C_*}\varepsilon_j)}, \tag{3-38}$$

from which we deduce

$$v_j(r) \leq \frac{\delta_0}{r^{(n-1)/2}} e^{-(r-R_j(\delta_0))/(\sqrt{2C_*}\varepsilon_j)} \quad \text{for all } r \geq R_j(\delta_0),$$

that is, (3-34). By combining (3-36) with (3-38) we first find

$$|w_j''(r)| \leq \frac{C}{\varepsilon_j^2} e^{-(r-R_j(\delta_0))/(\sqrt{2C_*}\varepsilon_j)} \quad \text{for all } r \geq R_j(\delta_0),$$

and then, by integration,

$$|w_j'(r)| \leq \int_r^\infty |w_j''(s)|\, ds \leq \frac{C}{\varepsilon_j} e^{-(r-R_j(\delta_0))/(\sqrt{2C_*}\varepsilon_j)} \quad \text{for all } r \geq R_j(\delta_0);$$

these last two estimates, combined with $v_j = r^{-(n-1)/2}\, w_j$ and the Leibniz rule, yield (3-35) for $k = 1, 2$.

Step 6: We obtain sharp estimates for $v_j(r)$ when $r \to 0^+$; precisely, we prove that for every $r \le R_j(1-\delta_0)$ one has

$$1 - v_j(r) \le C e^{-(R_j(1-\delta_0)-r)/(C\varepsilon_j)}, \tag{3-39}$$

$$|v_j'(r)| \le C \min\left\{\frac{r}{\varepsilon_j^2}, \frac{1}{\varepsilon_j}\right\} e^{-(R_j(1-\delta_0)-r)/(C\varepsilon_j)}, \tag{3-40}$$

$$|v_j''(r)| \le \frac{C}{\varepsilon_j^2} e^{-(R_j(1-\delta_0)-r)/(C\varepsilon_j)}. \tag{3-41}$$

To this end, it is convenient to recast (3-6) in terms of $w_j = 1 - v_j$, so that

$$2\varepsilon_j^2\left\{w_j'' + (n-1)\frac{w_j'}{r}\right\} = -W'(1-w_j) + \varepsilon_j w_j(1-w_j)\mathrm{E}_j. \tag{3-42}$$

By (A-6) and (3-7), if $r \le R_j(1-\delta_0)$, then

$$-W'(1-w_j) + \varepsilon_j w_j(1-w_j)\mathrm{E}_j \le C(1-w_j), \tag{3-43}$$

so that (3-42) implies in particular

$$2\varepsilon_j^2\left\{w_j'' + (n-1)\frac{w_j'}{r}\right\} \le C w_j \quad \text{on } (0, R_j(1-\delta_0)). \tag{3-44}$$

Multiplying by $w_j' \ge 0$ and integrating on $(0, r) \subset (0, R_j(1-\delta_0))$ we deduce

$$\varepsilon_j^2\left\{w_j'(r)^2 + \int_0^r \frac{(w_j')^2}{t}\right\} \le C(w_j(r)^2 - w_j(0)^2) \le C w_j(r)^2,$$

that is,

$$\varepsilon_j w_j' \le C w_j \quad \text{on } (0, R_j(1-\delta_0)). \tag{3-45}$$

Combining (3-45) with (3-42), (A-6) and (3-7), we find that

$$\begin{aligned} 2\varepsilon_j^2 w_j'' + C\varepsilon_j w_j &\ge 2\varepsilon_j^2\left\{w_j'' + \frac{n-1}{r} w_j'\right\} \\ &= -W'(1-w_j) + \varepsilon_j w_j(1-w_j)\mathrm{E}_j \ge \frac{w_j}{C} - C\varepsilon_j w_j \end{aligned}$$

on $[R_0/4, R_j(1-\delta_0))$, so that, for j large enough and for a constant C_* depending on n and W only, we have

$$\varepsilon_j^2 w_j'' \ge \frac{w_j}{C_*} \quad \text{on } [R_0/4, R_j(1-\delta_0)). \tag{3-46}$$

Correspondingly to C_*, we introduce the barrier

$$w_*(r) = \delta_0\{e^{((R_0/4)-r)/\sqrt{C_*\varepsilon_j^2}} + e^{(r-R_j(1-\delta_0))/\sqrt{C_*\varepsilon_j^2}}\}, \qquad r > 0.$$

By the monotonicity of w_j and by $R_j(1-\delta_0) \ge R_0/3$ (recall (3-27)),

$$w_*(R_0/4) \ge \delta_0 = w_j(R_j(1-\delta_0)) \ge w_j(R_0/4),$$

$$w_*(R_j(1-\delta_0)) \ge \delta_0 = w_j(R_j(1-\delta_0)),$$

$$\varepsilon_j^2 w_*'' = \frac{w_*}{C_*} \quad \text{on } [0, \infty).$$

We thus find $w_j \le w_*$ on $[R_0/4,\, R_j(1-\delta_0))$; that is, for every $R_0/4 \le r \le R_j(1-\delta_0)$,

$$1 - v_j(r) \le \delta_0\{e^{((R_0/4)-r)/\sqrt{C_*\varepsilon_j^2}} + e^{(r-R_j(1-\delta_0))/\sqrt{C_*\varepsilon_j^2}}\}. \tag{3-47}$$

By testing (3-47) with

$$r_* = \frac{R_0/4 + R_0/3}{2}$$

and exploiting the monotonicity of v_j, we find that for $r \in (0, r_*]$

$$1 - v_j(r) \le \delta_0 e^{-1/(C\varepsilon_j)} \quad \text{for all } r \in (0, r_*] \tag{3-48}$$

(thus obtaining the crucial information that, for j large enough and, for every $k \in \mathbb{N}$, $\|1-v_j\|_{C^0[0,r_*]} = \mathrm{o}(\varepsilon_j^k)$ as $j \to \infty$). At the same time, for $r_* \le r \le R_j(1-\delta_0)$, the second exponential in (3-47) is bounded from below in terms of a universal constant, while the first exponential is bounded from above by $e^{-1/C\varepsilon_j}$, so that (3-47) and (3-48) can be combined into

$$1 - v_j(r) \le C e^{-(R_j(1-\delta_0)-r)/(C\varepsilon_j)} \quad \text{for all } r \in (0, R_j(1-\delta_0)],$$

that is, (3-39). By combining (3-39) and (3-45) we also find

$$-v_j'(r) \le \frac{C}{\varepsilon_j} e^{-(R_j(1-\delta_0)-r)/(C\varepsilon_j)} \quad \text{for all } r \in (0, R_j(1-\delta_0)], \tag{3-49}$$

which is half of the estimate for $|v_j'|$ in (3-40). Multiplying (3-44) by r^{n-1} we find

$$2\varepsilon_j^2 (r^{n-1} w_j')' \le C r^{n-1} w_j \quad \text{for all } r \in (0, R_j(1-\delta_0)],$$

which we integrate over $(0, r) \subset (0, R_j(1-\delta_0))$ to conclude that

$$\varepsilon_j^2 r^{n-1}(-v_j'(r)) \le C \int_0^r w_j(t) t^{n-1}\, dt \le C(1 - v_j(r)) r^n \quad \text{for all } r \in (0, R_j(1-\delta_0)];$$

in particular, by combining this last inequality with (3-39) we find

$$-v_j'(r) \le C \frac{r}{\varepsilon_j^2} e^{-(R_j(1-\delta_0)-r)/(C\varepsilon_j)} \quad \text{for all } r \in (0, R_j(1-\delta_0)],$$

that is, the missing half of (3-40). Finally, by (3-42) with (3-43) we find

$$\varepsilon_j^2 |v_j''| \le C\left\{(1 - v_j) + \frac{|v_j'|}{r}\right\} \quad \text{on } (0, R_j(1-\delta_0)),$$

and then (3-41) follows from (3-39) and (3-40).

Step 7: We now improve the first set of inequalities in (3-27), and show that

$$R_0 - C\varepsilon_j \le R_j(1-\delta_0) < R_j(\delta_0) \le R_0 + C\varepsilon_j. \tag{3-50}$$

Let us set

$$\alpha_j = \int_{B_{R_j(1-\delta_0)}} V(v_j), \quad \beta_j = \int_{B_{R_j(\delta_0)} \setminus B_{R_j(1-\delta_0)}} V(v_j), \quad \gamma_j = \int_{B_{R_j(\delta_0)^c}} V(v_j).$$

By (A-11), (3-39) and (3-27) we have

$$\begin{aligned}|\alpha_j - \omega_n R_j(1-\delta_0)^n| &= \int_{B_{R_j(1-\delta_0)}} 1 - V(v_j) \le C \int_{B_{R_j(1-\delta_0)}} (1-v_j)^2 \\ &\le C \int_{B_{R_j(1-\delta_0)}} e^{-(R_j(1-\delta_0)-|x|)/(C\varepsilon_j)}\,dx \\ &= C \int_0^{R_j(1-\delta_0)} e^{-(R_j(1-\delta_0)-r)/(C\varepsilon_j)} r^{n-1}\,dr \\ &= C\varepsilon_j \int_{-R_j(1-\delta_0)/\varepsilon_j}^{0} e^{s/C}(R_j(1-\delta_0)+\varepsilon_j s)^{n-1}\,ds \le C\varepsilon_j.\end{aligned}$$

Similarly, by (A-11), (3-27) and (3-34) we find

$$\begin{aligned}|\gamma_j| = \int_{B^c_{R_j(\delta_0)}} V(v_j) \le C \int_{B_{R_j(\delta_0)^c}} v_j^{2n/(n-1)} \le C \int_{R_j(\delta_0)}^{\infty} e^{-(r-R_j(\delta_0))/(C\varepsilon_j)} r^{n-1}\,dr \\ = C\varepsilon_j \int_0^\infty e^{-s/C}(R_j(\delta_0)+\varepsilon_j s)^{n-1}\,ds \le C\varepsilon_j.\end{aligned}$$

Finally, thanks to (3-27),

$$|\beta_j| = \int_{B_{R_j(\delta_0)}\setminus B_{R_j(1-\delta_0)}} V(v_j) \le C(R_j(\delta_0) - R_j(1-\delta_0)) \le C\varepsilon_j.$$

Combining the estimates for α_j, β_j and γ_j with the fact that

$$\omega_n R_0^n = 1 = \int_{\mathbb{R}^n} V(v_j) = \alpha_j + \beta_j + \gamma_j,$$

we conclude that

$$C\varepsilon_j \ge \omega_n |R_0^n - R_j(1-\delta_0)^n| \le \frac{|R_0 - R_j(1-\delta_0)|}{C},$$

so that (3-50) follows by (3-27).

Step 8: We conclude the proof of the theorem: (3-29), (3-30) and (3-50) imply (3-10) and (3-12), as well as

$$|b_j|, |c_j| \le C\varepsilon_j, \tag{3-51}$$

which is a weaker form of (3-11); (3-34) and (3-35) imply (3-13), while (3-39), (3-40), and (3-41) imply (3-14). We are thus left to prove the full form of (3-11) (which includes a positive lower bound in the form ε_j/C for both b_j and c_j), as well as (3-8): that is, we want to show that if v_j satisfies (3-4), (3-5), (3-6) and (3-7), then, for every $x \in \mathbb{R}^n$ and j large enough, we have

$$v_j(x) = z_{\varepsilon_j}(x) + f_j\Big(\frac{|x|-R_0}{\varepsilon_j}\Big) = \eta\Big(\frac{|x|-R_0}{\varepsilon_j} - \tau_j\Big) + f_j\Big(\frac{|x|-R_0}{\varepsilon_j}\Big), \tag{3-52}$$

with functions $f_j \in C^2(I_j)$ such that

$$|f_j(s)| \le C\varepsilon_j e^{-|s|/C} \quad \text{for all } s \in I_j = (-R_0/\varepsilon_j, \infty), \tag{3-53}$$

and with $\tau_j = \tau_{\varepsilon_j}$ for τ_ε defined by (3-1) and (3-2). In fact, (3-52) and (3-53) imply the full form of (3-11): for example, combined with (3-12) and (3-17), they give

$$\begin{aligned} C\frac{b_j}{\varepsilon_j} &\geq \int_{R_0-b_j}^{R_0}(-v_j') = v_j(R_0-b_j) - v_j(R_0) = (1-\delta_0) - \eta(-\tau_j) - f_j(0) \\ &\geq 1 - \delta_0 - \eta(-\tau_0) - C\varepsilon_j, \end{aligned}$$

where the latter quantity is positive provided j is large enough and we further decrease the value of δ_0 to have $\delta_0 < 1 - \eta(-\tau_0)$.

We can thus focus on (3-52) and (3-53), which we recast by looking at the functions

$$\eta_j(s) = v_j(R_0 + \varepsilon_j s), \quad s \in I_j,$$

in terms of which $f_j(s) = \eta_j(s) - \eta(s-\tau_j)$. Thus, our goal becomes proving that

$$|\eta_j(s) - \eta(s-\tau_j)| \leq C\varepsilon_j e^{-|s|/C} \quad \text{for all } s \in I_j. \tag{3-54}$$

We start noticing that, by (3-12), (3-13) and (3-14), we have

$$C \geq -\eta_j'(s) \geq \frac{1}{C} \qquad \text{for all } s \in (-b_j/\varepsilon_j, c_j/\varepsilon_j), \tag{3-55}$$

$$\eta_j^{(k)}(s) \leq Ce^{-s/C} \qquad \text{for all } s \in (c_j/\varepsilon_j, \infty), k = 0, 1, 2, \tag{3-56}$$

$$\begin{cases} (1-\eta_j(s)) + |\eta_j''(s)| \leq Ce^{s/C}, \\ |\eta_j'| \leq C\min\left\{\dfrac{R_0 + \varepsilon_j\, s}{\varepsilon_j}, 1\right\} e^{s/C} \end{cases} \qquad \text{for all } s \in (-R_0/\varepsilon_j, -b_j/\varepsilon_j) \tag{3-57}$$

(while the analogous estimates for η are found in (A-16) and (A-18)). In order to estimate $f_j(s) = \eta_j(s) - \eta(s-\tau_j)$, we introduce

$$g_j(s) = \eta_j(s) - \eta(s-t_j)$$

for t_j defined by the identity

$$\eta(-(b_j/\varepsilon_j) - t_j) = 1 - \delta_0. \tag{3-58}$$

(Notice that the definition is well-posed by $\eta' < 0$ and $\eta(\mathbb{R}) = (0,1)$.) We claim that the proof of (3-53) can be reduced to that of

$$|g_j(s)| \leq C\varepsilon_j e^{-|s|/C} \quad \text{for all } s \in I_j. \tag{3-59}$$

Indeed, by (3-4), if (3-59) holds, then we are in the position to apply Step 1, and deduce from (3-17) that $|t_j - \tau_0| \leq C\varepsilon_j$. Having also (by the same argument) $|\tau_j - \tau_0| \leq C\varepsilon_j$, we deduce that

$$|\tau_j - t_j| \leq C\varepsilon_j,$$

which we exploit in combination with (3-56) and (3-57) to deduce

$$\begin{aligned} |f_j(s) - g_j(s)| = |\eta(s-t_j) - \eta(s-\tau_j)| &\leq C\int_0^1 |\eta'(s - \tau_j - t(t_j - \tau_j))|\,dt \\ &\leq C\varepsilon_j e^{-|s|/C} \quad \text{for all } s \in I_j. \end{aligned}$$

We are thus left to prove (3-59). To this end, we preliminarily notice that, since $\eta_j(-b_j/\varepsilon_j)=v_j(R_0-b_j)=1-\delta_0$, the definition of t_j is such that

$$g_j(-b_j/\varepsilon_j)=0. \tag{3-60}$$

Moreover, by the decay properties (A-16) of η and by $|b_j|\le C\varepsilon_j$, (3-58) implies

$$|t_j|\le C. \tag{3-61}$$

We now divide the proof of (3-59) in three separate arguments:

We prove (3-59) *for* $|s|\ge C\log(1/\varepsilon_j)$: This is trivial from the decay properties of η and η_j. Indeed, by (A-16), (3-61), (3-56) and (3-57) we find that

$$|g_j(s)|\le K_1e^{-|s|/K_1}\quad\text{for all } s\in I_j. \tag{3-62}$$

for a universal constant K_1. In particular, we trivially have

$$|g_j(s)|\le K_1\varepsilon_j e^{-|s|/(2K_1)}\quad\text{for all } s\in I_j,\ |s|\ge 2K_1\log\Big(\frac{1}{\varepsilon_j}\Big). \tag{3-63}$$

We will later increase the value of K_1 in (3-62) so that (3-74) below holds too.

We prove (3-59) *on arbitrary compact subsets of* I_j: More precisely, we show that for every $K>0$ we can find $C_K=C_K(n,W)$ (that is, a constant that depends on n, W and K only) such that

$$|g_j(s)|\le C_K\varepsilon_j\quad\text{for all } s\in I_j,\ |s|\le K. \tag{3-64}$$

To this end, setting $\mathrm{E}_j^*(s)=\mathrm{E}_j(R_0+\varepsilon_j s)$, we deduce from (3-6) that η_j satisfies the ODE

$$2\eta_j''+2\varepsilon_j\frac{n-1}{R_0+\varepsilon_j s}\eta_j'=W'(\eta_j)-\varepsilon_j\eta_j(1-\eta_j)\mathrm{E}_j^*\quad\text{on } I_j. \tag{3-65}$$

Multiplying (3-65) by $-\eta_j'$ and integrating over (s,∞) we find

$$\eta_j'(s)^2-2\varepsilon_j(n-1)\int_s^\infty\frac{\eta_j'(t)^2}{R_0+\varepsilon_j t}\,dt=W(\eta_j(s))+\varepsilon_j\int_s^\infty\eta_j(1-\eta_j)\eta_j'\mathrm{E}_j^*. \tag{3-66}$$

Since $\eta'(s-t_j)^2=W(\eta(s-t_j))$ for every $s\in\mathbb{R}$, we find that

$$\eta_j'(s)^2-\eta'(s-t_j)^2=W(\eta_j(s))-W(\eta(s-t_j))+\varepsilon_jL_j(s),$$
$$\text{where } L_j(s)=\int_s^\infty\bigg(2(n-1)\frac{\eta_j'(t)^2}{R_0+\varepsilon_j t}+\eta_j(1-\eta_j)\eta_j'\mathrm{E}_j^*\bigg)\,dt. \tag{3-67}$$

Setting

$$\ell_j(s)=\frac{W(\eta_j(s))-W(\eta(s-t_j))}{\eta_j(s)-\eta(s-t_j)},\quad d_j(s)=\eta_j'(s)+\eta'(s-t_j),\quad \Gamma_j(s)=\frac{\ell_j(s)}{d_j(s)},$$

and noticing that $d_j<0$ on I_j, (3-67) takes the form

$$g_j'(s)-\Gamma_j(s)g_j(s)=\frac{\varepsilon_jL_j(s)}{d_j(s)}\quad\text{for all } s\in I_j. \tag{3-68}$$

Multiplying (3-68) by $\exp(-\int_0^s \Gamma_j)$, integrating over an interval $(-b_j/\varepsilon_j, s)$, and taking into account (3-60), we find

$$g_j(s)e^{-\int_0^s \Gamma_j} = \varepsilon_j \int_{-b_j/\varepsilon_j}^{s} \frac{e^{-\int_0^t \Gamma_j}}{d_j(t)} L_j(t)\,dt \quad \text{for all } s \in I_j. \tag{3-69}$$

We now notice that by (3-7), (3-56) and (3-57),

$$|L_j(s)| \le C \min\{1, e^{-s/C}\} \quad \text{for all } s \in I_j. \tag{3-70}$$

Moreover, by $\operatorname{Lip} W \le C$ we have $|\ell_j| \le C$ on I_j, while $\eta_j' \le 0$ and (3-61) give

$$d_j(s) \le \eta'(s - t_j) \le -\frac{1}{C_K} \quad \text{for all } |s| \le K, \tag{3-71}$$

and, in particular, $|\Gamma_j(s)| \le C_K$ for $|s| \le K$. Now, assuming without loss of generality that K is large enough to give $K \ge |b_j|/\varepsilon_j$ (as we can do since $|b_j| \le C\varepsilon_j$ for a universal constant C), we can combine (3-69), (3-70), (3-71) and $|\Gamma_j| \le C_K$ on $[-K, K]$ to get (3-64).

Finally, *we prove* (3-59) *in the remaining case*: Having in mind (3-63) and (3-64), we are left to prove the existence of a sufficiently large universal constant K_2 such that (3-59) holds (provided j is large enough) for every $s \in I_j$ with $K_2 \le |s| \le 2K_1 \log(1/\varepsilon_j)$. To this end, we start by subtracting $2\eta'' = W(\eta)$ from (3-65), and obtain

$$2g_j'' - m_j g_j = \varepsilon_j \left\{ \eta_j(1-\eta_j)\mathrm{E}_j^* - 2(n-1)\frac{\eta_j'}{R_0 + \varepsilon_j s} \right\} \quad \text{for all } s \in I_j, \tag{3-72}$$

where

$$m_j(s) = \frac{W'(\eta_j(s)) - W'(\eta(s-t_j))}{\eta_j(s) - \eta(s-t_j)}, \quad s \in I_j.$$

The coefficient m_j is uniformly positive: indeed, the decay properties of η and η_j at infinity, combined with $|t_j| \le C$, imply the existence of a universal constant K_2 such that if $|s| \ge K_2$, then $\eta_j(s)$ and $\eta(s-t_j)$ are both at distance at most δ_0 from $\{0, 1\}$, and since $W'' \ge 1/C$ on $(0, \delta_0) \cup (1-\delta_0, 1)$ by (A-6), we conclude that, up to further increasing the value of K_2,

$$m_j(s) \ge \frac{1}{K_2} \quad \text{for all } s \in I_j,\ |s| \ge K_2. \tag{3-73}$$

At the same time, the right-hand side of (3-72) has exponential decay: indeed, by (3-7), (3-55), (3-56) and (3-57), if $|s| \le \log(1/\varepsilon_j)$, $s \in I_j$, then we get

$$\left| \eta_j(1-\eta_j)\mathrm{E}_j^* - 2(n-1)\frac{\eta_j'}{R_0 + \varepsilon_j s} \right| \le K_1 \varepsilon_j e^{-|s|/K_1}, \tag{3-74}$$

up to further increasing the value of the universal constant K_1 introduced in (3-63). Let us thus consider

$$g_*(s) = C_1 \varepsilon_j e^{-|s|/\sqrt{2C_2}}, \quad s \in \mathbb{R},$$

for C_1 and C_2 universal constants to be determined. By combining (3-72) with (3-73) and (3-74) we find that, if $s \in I_j$ with $K_2 \le |s| \le 2K_1 \log(1/\varepsilon_j)$, then

$$\begin{aligned} 2(g_j - g_*)'' - m_j(g_j - g_*) &\ge m_j g_* - 2g_*'' - K_1 \varepsilon_j e^{-|s|/K_1} \\ &\ge \left(\frac{1}{K_2} - \frac{1}{C_2}\right) g_* - K_1 \varepsilon_j e^{-|s|/K_1} \\ &= \varepsilon_j \left\{ \frac{C_1}{K_1}\left(\frac{1}{K_2} - \frac{1}{C_2}\right) e^{[(1/K_1)-(1/\sqrt{2C_2})]|s|} - 1 \right\} K_1 e^{-|s|/K_1}, \end{aligned}$$

where the latter quantity is nonnegative for every $|s| \ge K_2$ provided

$$C_1 \ge 3K_1K_2 e^{-K_0/(2K_1)}, \quad C_2 \ge \max\{2K_2, 2K_1^2\}. \tag{3-75}$$

At the same time, by (3-63),

$$|g_j(\pm 2K_1 \log(1/\varepsilon_j))| \le K_1 \varepsilon_j^2,$$

while $C_2 \ge 2K_1^2$ gives

$$g_*(\pm 2K_1 \log(1/\varepsilon_j)) = C_1 \varepsilon_j e^{-2K_1 \log(1/\varepsilon_j)/\sqrt{2C_2}} \ge C_1 \varepsilon_j^2.$$

Upon further requiring $C_1 \ge K_1$ we thus have

$$g_*(s) \ge |g_j(s)| \quad \text{at } s = \pm 2K_1 \log(1/\varepsilon_j). \tag{3-76}$$

Similarly, by (3-64),

$$|g_j(\pm K_2)| \le C_{K_2} \varepsilon_j,$$

while $C_{\ge} 2K_2$ gives

$$g_*(\pm K_2) = C_1 \varepsilon_j e^{-K_2/\sqrt{2C_2}} \ge C_1 \varepsilon_j e^{-\sqrt{K_2}/2}.$$

Upon requiring that $C_1 \ge C_{K_2} e^{\sqrt{K_2}/2}$, we find that

$$g_*(s) \ge |g_j(s)| \quad \text{at } s = \pm K_2. \tag{3-77}$$

In summary, we have proved that if K_1 satisfies (3-62) and (3-74), K_2 satisfies (3-73), and C_1 and C_2 are taken large enough in terms of K_1 and K_2, then (3-76) and (3-77) holds. In particular, $h_j = g_j - g_*$ is nonpositive on the boundary of the intervals $[-2K_1 \log(1/\varepsilon_j), -K_2]$ and $[K_2, 2K_1 \log(1/\varepsilon_j)]$, with $h_j'' - m_j h \ge 0,\ m_j \ge 0$, on those intervals thanks to (3-75) and (3-73); correspondingly, by the maximum principle, $h_j \le 0$ there, that is,

$$g_j(s) \le C_1 \varepsilon_j e^{-|s|/\sqrt{2C_2}} \quad \text{for all } s \in I_j,\ K_2 \le |s| \le 2K_1 \log(1/\varepsilon_j).$$

To get the matching lower bound we notice that, again by (3-74),

$$(-g_* - g_j)'' - m_j(-g_* - g_j) \ge m_j g_* - g_*'' - K_1 \varepsilon_j e^{-|s|/K_1}$$

so that, by the same considerations made before, the maximum principle can be applied to $k_j = -g_* - g_j$ on $[-2K_1 \log(1/\varepsilon_j), -K_2] \cup [K_2, 2K_1 \log(1/\varepsilon_j)]$ to deduce $g_j \ge -g_*$. This completes the proof of (3-59). □

4. Strict stability among radial functions

In this section we are going to exploit the resolution result in Theorem 3.1 to deduce a stability estimate for $\psi(\varepsilon)$ on radial (not necessarily decreasing) functions. More precisely, we shall prove the following statement.

Theorem 4.1 (Fuglede-type estimate). *If $n \geq 2$ and $W \in C^{2,1}[0,1]$ satisfies* (1-11) *and* (1-12), *then there exist universal constants δ_0 and ε_0 with the following property*: *if $\varepsilon < \varepsilon_0$, $u_\varepsilon \in \mathcal{R}_0$ is a minimizer of $\psi(\varepsilon)$, and $u \in H^1(\mathbb{R}^n; [0,1])$ is radial and such that*

$$\int_{\mathbb{R}^n} V(u) = 1, \tag{4-1}$$

$$\int_{\mathbb{R}^n} (u - u_\varepsilon)^2 \leq C\varepsilon, \tag{4-2}$$

$$\|u - u_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \leq \delta_0, \tag{4-3}$$

then, *setting $h = u - u_\varepsilon$*,

$$\mathcal{AC}_\varepsilon(u) - \psi(\varepsilon) \geq \frac{1}{C} \int_{\mathbb{R}^n} \varepsilon |\nabla h|^2 + \frac{h^2}{\varepsilon}. \tag{4-4}$$

Before entering into the proof of Theorem 4.1, we show how it can be used to improve on the conclusions of Theorem 2.1. In particular, it gives the uniqueness of minimizers in $\psi(\varepsilon)$ and, together with the resolution result in Theorem 3.1, allows us to compute the precise asymptotic behavior of $\psi(\varepsilon)$ and $\lambda(\varepsilon)$ up to second and first order in $\varepsilon \to 0^+$ respectively. Notice in particular that (4-7) sharply improves (2-3).

Corollary 4.2. *If $n \geq 2$ and $W \in C^{2,1}[0,1]$ satisfies* (1-11) *and* (1-12), *then there exists a universal constant ε_0 such that*, *if $\varepsilon < \varepsilon_0$, then $\psi(\varepsilon)$ admits a unique minimizer* (*modulo translations*). *In particular*, *for every $\varepsilon < \varepsilon_0$, $\lambda(\varepsilon)$ is unambiguously defined as the Lagrange multiplier of the unique minimizer $u_\varepsilon \in \mathcal{R}_0$ of $\psi(\varepsilon)$ by the identity* (2-2), *i.e.*,

$$\lambda(\varepsilon) = \left(1 - \frac{1}{n}\right)\psi(\varepsilon) + \frac{1}{n}\left\{\frac{1}{\varepsilon}\int_{\mathbb{R}^n} W(u_\varepsilon) - \varepsilon \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2\right\}. \tag{4-5}$$

Finally, *$\varepsilon \in (0, \varepsilon_0) \mapsto \lambda(\varepsilon)$ is continuous and the following expansions hold as $\varepsilon \to 0^+$*:

$$\psi(\varepsilon) = 2n\omega_n^{1/n} + 2n(n-1)\omega_n^{2/n}\kappa_0\varepsilon + \mathrm{O}(\varepsilon^2), \tag{4-6}$$

$$\lambda(\varepsilon) = 2(n-1)\omega_n^{1/n} + \mathrm{O}(\varepsilon), \tag{4-7}$$

where $\kappa_0 = \tau_0 + \tau_1 = \int_{\mathbb{R}} [\eta' V'(\eta) + W(\eta)] s\, ds$ and η is the unique solution to $\eta' = -\sqrt{W(\eta)}$ on $\mathbb{R}$ with $\eta(0) = \frac{1}{2}$.

Proof of Corollary 4.2. Step 1: Let $\varepsilon \in (0, \varepsilon_0)$ and let u_ε and v_ε be two minimizers of $\psi(\varepsilon)$, so that, up to translations, $u_\varepsilon, v_\varepsilon \in \mathcal{R}_0^*$ thanks to Theorem 2.1. By Theorem 3.1, if we set $h_\varepsilon = v_\varepsilon - u_\varepsilon$, then

$$h_\varepsilon(x) = f_\varepsilon\left(\frac{|x| - R_0}{\varepsilon}\right),$$

where $f_\varepsilon \in C^2(-R_0/\varepsilon, \infty)$, and

$$|f_\varepsilon(s)| \le C\varepsilon e^{-s/C} \quad \text{for all } s \ge -R_0/\varepsilon. \tag{4-8}$$

We thus see that $u = v_\varepsilon$ satisfies (4-1) and (4-3). Moreover, by (4-8),

$$\int_{\mathbb{R}^n} h_\varepsilon^2 = n\omega_n \int_{-R_0/\varepsilon}^{\infty} f_\varepsilon(s)^2 (R_0 + \varepsilon s)^{n-1} \varepsilon \, ds \le C\varepsilon^2,$$

so that (4-2) holds too. We can thus apply (4-4) with $u = v_\varepsilon$, and exploit the minimality of v_ε to deduce that

$$0 = \mathcal{AC}_\varepsilon(v_\varepsilon) - \psi(\varepsilon) \ge \frac{1}{C} \int_{\mathbb{R}^n} \varepsilon |\nabla h_\varepsilon|^2 + \frac{h_\varepsilon^2}{\varepsilon},$$

that is, $h_\varepsilon = 0$ on $\mathbb{R}^n$, as claimed.

Step 2: We prove (4-6) and (4-7). If u_ε is the minimizer of $\psi(\varepsilon)$ in $\mathcal{R}_0$, then by Theorem 3.1 we have $u_\varepsilon(x) = z_\varepsilon(x) + f_\varepsilon((|x| - R_0)/\varepsilon)$ for every $x \in \mathbb{R}^n$, and with f_ε satisfying (4-8). Moreover, as proved in (3-3), we have

$$\mathcal{AC}_\varepsilon(z_\varepsilon) = 2n\omega_n^{1/n} + 2n(n-1)\omega_n^{2/n}\kappa_0 + \mathrm{O}(\varepsilon^2).$$

Since $\mathcal{AC}_\varepsilon(u_\varepsilon) \le \mathcal{AC}_\varepsilon(z_\varepsilon)$, we are left to prove that $\mathcal{AC}_\varepsilon(u_\varepsilon) \ge \mathcal{AC}_\varepsilon(z_\varepsilon) - C\varepsilon^2$. Setting $|x| = R_0 + \varepsilon s$, we have

$$u_\varepsilon(x) = \eta(s - \tau_\varepsilon) + f_\varepsilon(s), \qquad \nabla u_\varepsilon(x) = \frac{\eta'(s - \tau_\varepsilon) + f_\varepsilon'(s)}{\varepsilon} \frac{x}{|x|},$$

while z_ε satisfies the same identities with $f_\varepsilon = 0$, so that

$$\begin{aligned}\mathcal{AC}_\varepsilon(u_\varepsilon) - \mathcal{AC}_\varepsilon(z_\varepsilon) = &\int_{-R_0/\varepsilon}^{\infty} \big(2\eta'(s - \tau_\varepsilon) f_\varepsilon'(s) + f_\varepsilon'(s)^2\big)(R_0 + \varepsilon s)^{n-1} \, ds \\ &+ \int_{-R_0/\varepsilon}^{\infty} \big(W(\eta(s - \tau_\varepsilon) + f_\varepsilon(s)) - W(\eta(s - \tau_\varepsilon))\big)(R_0 + \varepsilon s)^{n-1} \, ds.\end{aligned} \tag{4-9}$$

Integration by parts and $2\eta'' = W'(\eta)$ give

$$\begin{aligned}\int_{-R_0/\varepsilon}^{\infty} 2\eta'(s - \tau_\varepsilon) f_\varepsilon'(s)(R_0 + \varepsilon s)^{n-1} \, ds = &-\int_{-R_0/\varepsilon}^{\infty} W'(\eta(s - \tau_\varepsilon)) f_\varepsilon(s)(R_0 + \varepsilon s)^{n-1} \, ds \\ &- 2(n-1)\varepsilon \int_{-R_0/\varepsilon}^{\infty} \eta'(s - \tau_\varepsilon) f_\varepsilon(s)(R_0 + \varepsilon s)^{n-2} \, ds.\end{aligned}$$

Dropping the nonnegative term with $f_\varepsilon'(s)^2$ in (4-9), and noticing that, by (A-5) and (4-8), we have

$$|W(\eta(s - \tau_\varepsilon) + f_\varepsilon(s)) - W(\eta(s - \tau_\varepsilon)) - W'(\eta(s - \tau_\varepsilon)) f_\varepsilon(s)| \le C f_\varepsilon(s)^2$$

for every $s > -R_0/\varepsilon$, we thus find

$$\begin{aligned}&\mathcal{AC}_\varepsilon(u_\varepsilon) - \mathcal{AC}_\varepsilon(z_\varepsilon) \\ &\quad\ge -2(n-1)\varepsilon \int_{-R_0/\varepsilon}^{\infty} \eta'(s - \tau_\varepsilon) f_\varepsilon(s)(R_0 + \varepsilon s)^{n-2} \, ds - C \int_{-R_0/\varepsilon}^{\infty} f_\varepsilon(s)^2 (R_0 + \varepsilon s)^{n-1} \, ds \ge -C\varepsilon^2,\end{aligned}$$

where in the last inequality we have used (4-8), $|\tau_\varepsilon| \le C$ and the decay estimate for η' in (A-18). Coming to (4-7), rearranging terms in (4-5) we have

$$\lambda(\varepsilon) = \Big(1 - \frac{2}{n}\Big)\psi(\varepsilon) + \frac{2}{n}\frac{1}{\varepsilon}\int_{\mathbb{R}^n} W(u_\varepsilon). \tag{4-10}$$

By (4-8)

$$\frac{1}{\varepsilon}\int_{\mathbb{R}^n} W(u_\varepsilon) = \frac{1}{\varepsilon}\int_{\mathbb{R}^n} W(z_\varepsilon) + \mathrm{O}(\varepsilon) = \frac{\psi(\varepsilon)}{2} + \mathrm{O}(\varepsilon),$$

where in the second identity we have used (3-22). Hence $\lambda(\varepsilon) = (1-(1/n))\,\psi(\varepsilon) + \mathrm{O}(\varepsilon)$ and (4-7) follows from (4-6).

Step 3: We prove the continuity of λ on $(0, \varepsilon_0)$. Let $\varepsilon_j \to \varepsilon_* \in (0, \varepsilon_0)$ as $j \to \infty$ and set $h_j = u_{\varepsilon_j} - u_{\varepsilon_*}$. By the resolution formula (3-8) we have

$$\begin{aligned} |u_{\varepsilon_j}(x) - u_{\varepsilon_*}(x)| &\le \left|\eta\Big(\frac{|x|-R_0}{\varepsilon_j} - \tau_{\varepsilon_j}\Big) - \eta\Big(\frac{|x|-R_0}{\varepsilon_*} - \tau_{\varepsilon_*}\Big)\right| + \left|f_{\varepsilon_j}\Big(\frac{|x|-R_0}{\varepsilon_j}\Big) - f_{\varepsilon_*}\Big(\frac{|x|-R_0}{\varepsilon_*}\Big)\right| \\ &\le C\varepsilon_* e^{-(|x|-R_0)/C\varepsilon_*} + \left|\eta\Big(\frac{|x|-R_0}{\varepsilon_j} - \tau_0\Big) - \eta\Big(\frac{|x|-R_0}{\varepsilon_*} - \tau_0\Big)\right|, \end{aligned}$$

where we have used (3-17), (3-9) and (A-16). Similarly, since $\varepsilon_j \to \varepsilon_* > 0$, for j large enough we see that

$$\begin{aligned} \left|\eta\Big(\frac{|x|-R_0}{\varepsilon_j} - \tau_0\Big) - \eta\Big(\frac{|x|-R_0}{\varepsilon_*} - \tau_0\Big)\right| &\le \int_0^1 \left|\eta'\Big(\frac{|x|-R_0}{\varepsilon_* + t(\varepsilon_j - \varepsilon_*)} - \tau_0\Big)\right| \frac{||x|-R_0|}{(\varepsilon_* + t(\varepsilon_j-\varepsilon_*))^2}|\varepsilon_j - \varepsilon_*| \\ &\le C\frac{|\varepsilon_j - \varepsilon_*|}{\varepsilon_*^2} e^{-(|x|-R_0)/(C\varepsilon_*)}||x|-R_0| \le C\varepsilon_* e^{-(|x|-R_0)/(C\varepsilon_*)}. \end{aligned}$$

Setting $h_j = u_{\varepsilon_j} - u_{\varepsilon_*}$ we see that (4-1), (4-2) and (4-3) hold with $\varepsilon = \varepsilon_*$ and for j large enough, thus deducing that

$$\frac{1}{C}\int_{\mathbb{R}^n} \varepsilon_*|\nabla h_j|^2 + \frac{h_j^2}{\varepsilon_*} \le \mathcal{AC}_{\varepsilon_*}(u_{\varepsilon_j}) - \psi(\varepsilon_*) \le \max\Big\{\frac{\varepsilon_j}{\varepsilon_*}, \frac{\varepsilon_*}{\varepsilon_j}\Big\}\psi(\varepsilon_j) - \psi(\varepsilon_*).$$

From the continuity of ψ on $(0, \varepsilon_0)$ (Theorem 2.1) we conclude that

$$\lim_{j\to\infty}\int_{\mathbb{R}^n} |\nabla u_{\varepsilon_j} - \nabla u_{\varepsilon_*}|^2 = 0, \qquad \lim_{j\to\infty}\int_{\mathbb{R}^n} W(u_{\varepsilon_j}) = \int_{\mathbb{R}^n} W(u_{\varepsilon_*}),$$

and thus λ is continuous on $(0, \varepsilon_0)$ thanks to (4-10). □

We now turn to the proof of Theorem 4.1. This is based on a series of three lemmas, each containing a different stability estimate, coming increasingly closer to (4-4).

Lemma 4.3 (first stability lemma). *Let η be the unique solution to $\eta' = -\sqrt{W(\eta)}$ on $\mathbb{R}$ with $\eta(0) = \frac{1}{2}$. Let $n \ge 2$, let $W \in C^{2,1}[0, 1]$ satisfy* (1-11) *and* (1-12), *and let*

$$Q(u) = \int_{\mathbb{R}} 2(u')^2 + W''(\eta)u^2, \quad u \in H^1(\mathbb{R}).$$

Then $Q(u) \ge 0$ on $H^1(\mathbb{R})$, and $Q(u) = 0$ if and only if $u = t\eta'$ for some $t \in \mathbb{R}$.

Proof. Let us consider the variational problem

$$\gamma = \inf\left\{ Q(u) : \int_{\mathbb{R}} u^2 = 1 \right\}.$$

By (A-18) we have $\eta' \in H^1(\mathbb{R})$. Differentiating $2\eta'' = W'(\eta)$ we find $2(\eta')'' = W''(\eta)\eta'$, and then integration by parts gives $Q(\eta') = 0$. At the same time we clearly have $Q(u) \ge -\|W''\|_{C^0(0,1)} \int_{\mathbb{R}} u^2$ for every $u \in H^1(\mathbb{R})$, so that

$$-\|W''\|_{C^0(0,1)} \le \gamma \le 0.$$

We now prove that γ is attained. Let $\{w_j\}_j$ be a minimizing sequence for γ. By the concentration-compactness principle, $\{w_j^2\,dx\}_j$ is in the vanishing case if

$$\lim_{j\to\infty} \int_{I_R} w_j^2 = 0 \quad \text{for all } R > 0, \tag{4-11}$$

where we have set $I_R = (-R, R)$. By (A-16) and (A-6) there exists S_0 such that

$$W''(\eta) \ge \frac{1}{C} \quad \text{on } \mathbb{R} \setminus I_{S_0}. \tag{4-12}$$

Therefore by applying (4-11) twice with $R = S_0$ we find

$$\begin{aligned} \limsup_{j\to\infty} \int_{\mathbb{R}} w_j^2 &= \limsup_{j\to\infty} \int_{\mathbb{R}\setminus I_{S_0}} w_j^2 \le C \limsup_{j\to\infty} \int_{\mathbb{R}\setminus I_{S_0}} W''(\eta) w_j^2 \\ &= C \limsup_{j\to\infty} \int_{\mathbb{R}} W''(\eta) w_j^2 \le \lim_{j\to\infty} Q(w_j) = \gamma \le 0, \end{aligned}$$

a contradiction to $\int_{\mathbb{R}} w_j^2 = 1$. If, instead, $\{w_j^2\,dx\}_j$ is in the dichotomy case, then there is $\alpha \in (0,1)$ such that for every $\tau \in (0, \alpha/2)$ there exist $R > 0$ and $R_j \to \infty$ as $j \to \infty$ such that

$$\left| 1 - \alpha - \int_{I_R} w_j^2 \right| < \tau, \qquad \left| \alpha - \int_{\mathbb{R}\setminus I_{R_j}} w_j^2 \right| < \tau, \tag{4-13}$$

where, without loss of generality, we can assume $R \ge S_0$ for S_0 as in (4-12). In particular, if φ is a cut-off function between I_R and I_{R_j}, then we have

$$Q(w_j) = Q(\varphi\, w_j) + Q((1-\varphi) w_j) + E_j, \tag{4-14}$$

where, taking into account that φ' and $(1-\varphi)\,\varphi$ are supported in $I_{R_j} \setminus I_R$, we have

$$E_j = 2 \int_{I_{R_j}\setminus I_R} W''(\eta)(1-\varphi)\varphi w_j^2 + 4 \int_{I_{R_j}\setminus I_R} (\varphi w_j)'((1-\varphi) w_j)'. \tag{4-15}$$

The first integral in (4-15) is nonnegative by (4-12), while the second integral contains a nonnegative term of the form $\varphi(1-\varphi)(w_j')^2$; therefore, by (4-13),

$$\begin{aligned} E_j &\ge 4 \int_{I_{R_j}\setminus I_R} w_j w_j' (1-\varphi)\varphi' - w_j w_j' \varphi \varphi' - w_j^2 (\varphi')^2 \\ &\ge -C \int_{I_{R_j}\setminus I_R} w_j^2 - C \left(\int_{I_{R_j}\setminus I_R} w_j^2 \right)^{1/2} \left(\int_{\mathbb{R}} (w_j')^2 \right)^{1/2} \ge -C\sqrt{\tau}, \end{aligned} \tag{4-16}$$

where we have also used $Q(w_j) \to \gamma$ as $j \to \infty$ to infer

$$\int_{\mathbb{R}} (w_j')^2 \le Q(w_j) + \|W\|_{C^0[0,1]} \le C.$$

We can take φ supported in I_{R+1}. In this way, up to extracting a subsequence, we have that $\varphi\, w_j$ admits a weak limit w in $H^1(\mathbb{R})$. By lower semicontinuity, homogeneity of Q and (4-13) we have

$$\liminf_{j\to\infty} Q(\varphi w_j) \ge Q(w) \ge \gamma \int_{\mathbb{R}} w^2 \ge (1-\alpha)\gamma - C\tau. \tag{4-17}$$

Finally, since $(1-\varphi)$ is supported on $\mathbb{R} \setminus I_{S_0}$, by (4-12) we have

$$\int_{\mathbb{R}} Q((1-\varphi)v_j) \ge \frac{1}{C}\int_{\mathbb{R}} (1-\varphi)^2 w_j^2 \ge \frac{\alpha}{C} - C\tau,$$

so that, combining (4-14), (4-16), and (4-17) we find

$$\gamma \ge (1-\alpha)\gamma + \frac{\alpha}{C} - C\sqrt{\tau}.$$

Letting $\tau \to 0^+$ we find a contradiction with $\gamma \le 0$ and $\alpha > 0$. Having excluded vanishing and dichotomy, we have proved the existence of minimizers of γ.

Let now u be a minimizer of γ. Up to replacing u with $|u|$ we can assume $u \ge 0$. By a standard variational argument there exists $\lambda \in \mathbb{R}$ such that

$$\int_{\mathbb{R}} 2u'v' + W''(\eta)uv = \lambda \int_{\mathbb{R}} uv \quad \text{for all } v \in H^1(\mathbb{R}). \tag{4-18}$$

Testing with $v = \eta'$ and recalling that $2(\eta')'' = W''(\eta)\eta'$, we deduce that

$$\lambda \int_{\mathbb{R}} \eta' u = 0,$$

and, since $u \ge 0$, $\int_{\mathbb{R}} u^2 = 1$, and $\eta' < 0$, we find $\lambda = 0$. From here, if we test (4-18) with the same minimizer u, we conclude that $Q(u) = 0$ and, therefore, that $\gamma = 0$. We remark that this latter observation also implies that η' is a minimizer of γ.

We claim now that any minimizer of γ has to be either positive or negative on the whole line. Indeed, let v be any minimizer of γ. Therefore, $u = |v|$ is a nonnegative minimizer satisfying (4-18) with $\lambda = 0$. Thus, u is a C^2-solution of the ODE

$$2u'' = W''(\eta)u$$

on $\mathbb{R}$. If $0 = v(r_0) = u(r_0)$ for some $r_0 \in \mathbb{R}$, then $u'(r_0) \ne 0$ (otherwise we would have $u = 0$ on $\mathbb{R}$, against $\int_{\mathbb{R}} u^2 = 1$), and $u'(r_0) \ne 0$ contradicts $u \ge 0$ on $\mathbb{R}$. Hence, $u > 0$ on $\mathbb{R}$, and, therefore, v must have one sign too.

If u is also minimizer of γ, then, again by (4-18),

$$Q(u + s\eta') = Q(u) + s^2 Q(\eta') = 0 \quad \text{for all } s \in \mathbb{R}.$$

In particular, if $s\in\mathbb{R}$ is such that $u+s\eta'$ is not identically zero on $\mathbb{R}$, then $(u+s\eta')/\|u+s\eta'\|^2_{L^2(\mathbb{R})}$ is a minimizer of γ, and thus $u+s\eta'$ is either positive or negative on the whole $\mathbb{R}$. Let $s_0=\inf\{s:u+s\eta'<0 \text{ on } \mathbb{R}\}$. If, say, u is a negative minimizer (like η' is), then $s_0\le 0$; while, clearly, $s_0>-\infty$, since, for s negative enough, we must have $u+s\eta'>0$ at at least one point, and thus everywhere. Since $u+s_0\eta'\le 0$ on $\mathbb{R}$ with $u+s_0\eta'=0$ at at least one point, we deduce that $u+s_0\eta'=0$ on $\mathbb{R}$. □

Lemma 4.4 (second stability lemma). *If $n\ge 2$ and $W\in C^{2,1}[0,1]$ satisfies* (1-11) *and* (1-12), *then there exists a universal constant ε_0 with the following property. If $u_\varepsilon\in\mathcal{R}_0^*$ is a minimizer of $\psi(\varepsilon)$ for $\varepsilon<\varepsilon_0$ and $h\in H^1(\mathbb{R}^n)$ is a radial function such that*

$$\int_{\mathbb{R}^n} V'(u_\varepsilon)h=0, \tag{4-19}$$

then

$$\int_{\mathbb{R}^n} 2\varepsilon|\nabla h|^2+\Big(\frac{W''(u_\varepsilon)}{\varepsilon}-\lambda(\varepsilon)V''(u_\varepsilon)\Big)h^2\ge\frac{1}{C}\int_{\mathbb{R}^n}\varepsilon|\nabla h|^2+\frac{h^2}{\varepsilon}, \tag{4-20}$$

where $\lambda(\varepsilon)$ is the Lagrange multiplier of u_ε as in (4-5).

Proof. Step 1: We show that is enough to prove the lemma with

$$\int_{\mathbb{R}^n} 2\varepsilon|\nabla h|^2+\Big(\frac{W''(u_\varepsilon)}{\varepsilon}-\lambda(\varepsilon)V''(u_\varepsilon)\Big)h^2\ge\frac{1}{C}\int_{\mathbb{R}^n}\frac{h^2}{\varepsilon} \tag{4-21}$$

in place of (4-20). Indeed, if ε_0 is small enough, then $|\lambda(\varepsilon)|\le c(n)$ thanks to (2-3), and thus we can find a universal constant C_* such that

$$\int_{\mathbb{R}^n}\Big|\frac{1}{\varepsilon}W''(u_\varepsilon)-\lambda(\varepsilon)V''(u_\varepsilon)\Big|h^2\le C_*\int_{\mathbb{R}^n}\frac{h^2}{\varepsilon},$$

whenever u_ε is a minimizer of $\psi(\varepsilon)$, $\varepsilon<\varepsilon_0$, and $h\in H^1(\mathbb{R}^n)$. Let us now fix a radial function $h\in H^1(\mathbb{R}^n)$ satisfying (4-19). If $C_*\int_{\mathbb{R}^n}h^2/\varepsilon\le\int_{\mathbb{R}^n}\varepsilon|\nabla h|^2$, then we trivially have

$$\int_{\mathbb{R}^n} 2\varepsilon|\nabla h|^2+\Big(\frac{W''(u_\varepsilon)}{\varepsilon}-\lambda(\varepsilon)\,V''(\zeta_\varepsilon)\Big)h^2\ge\int_{\mathbb{R}^n}2\varepsilon|\nabla h|^2-C_*\int_{\mathbb{R}^n}\frac{h^2}{\varepsilon}\ge\int_{\mathbb{R}^n}\varepsilon|\nabla h|^2;$$

if, instead, $C_*\int_{\mathbb{R}^n}h^2/\varepsilon\ge\int_{\mathbb{R}^n}\varepsilon|\nabla h|^2$, then we deduce from (4-21)

$$\int_{\mathbb{R}^n} 2\varepsilon|\nabla h|^2+\Big(\frac{W''(u_\varepsilon)}{\varepsilon}-\lambda(\varepsilon)V''(u_\varepsilon)\Big)h^2\ge\frac{1}{C}\int_{\mathbb{R}^n}\frac{h^2}{\varepsilon}\ge\frac{1}{CC_*}\int_{\mathbb{R}^n}\varepsilon|\nabla h|^2.$$

In both cases, (4-20) is easily deduced.

Step 2: We prove (4-21). We argue by contradiction, and consider $\varepsilon_j\to 0^+$ as $j\to\infty$, $u_j\in\mathcal{R}_0^*$ minimizers of $\psi(\varepsilon_j)$, and radial functions $h_j\in H^1(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} V'(u_j)h_j=0, \tag{4-22}$$

$$\int_{\mathbb{R}^n} 2\varepsilon_j|\nabla h_j|^2+\Big(\frac{W''(u_j)}{\varepsilon_j}-\lambda_jV''(u_j)\Big)h_j^2<\frac{1}{j}\int_{\mathbb{R}^n}\frac{h_j^2}{\varepsilon_j}, \tag{4-23}$$

where λ_j are the Lagrange multipliers corresponding to u_j. By the homogeneity of (4-22) and (4-23) we can also assume that

$$\int_{\mathbb{R}^n} \frac{h_j^2}{\varepsilon_j} = 1. \tag{4-24}$$

Therefore, setting

$$\eta_j(s) = u_j(R_0 + \varepsilon_j s), \quad \beta_j(s) = h_j(R_0 + \varepsilon_j s), \quad s \geq -\frac{R_0}{\varepsilon_j},$$

we can recast (4-23) and (4-24) as

$$\int_{-R_0/\varepsilon_j}^{\infty} \big(2(\beta_j')^2 + (W''(\eta_j) - \varepsilon_j \lambda_j V''(\eta_j))\beta_j^2\big)(R_0 + \varepsilon_j s)^{n-1}\, ds \leq \frac{1}{j}, \tag{4-25}$$

$$\int_{-R_0/\varepsilon_j}^{\infty} \beta_j(s)^2 (R_0 + \varepsilon_j s)^{n-1}\, ds = 1. \tag{4-26}$$

By $\varepsilon_j \to 0^+$ and by (2-3) we know $\lambda_j \to c(n)$ as $j \to \infty$, which combined with $\|V''\|_{C^0[0,1]} \leq C$ and $\varepsilon_j \to 0^+$ shows that (4-25) and (4-26) imply

$$\limsup_{j\to\infty} \int_{-R_0/\varepsilon_j}^{\infty} \{2(\beta_j')^2 + W''(\eta_j)\beta_j^2\}(R_0 + \varepsilon_j s)^{n-1}\, ds \leq 0. \tag{4-27}$$

Since W'' is bounded on [0, 1], by (4-26) and (4-27) we deduce that $\{\beta_j\}_j$ is bounded in $H^1(-s_0, s_0)$ for every $s_0 > 0$. In particular there exists $\beta \in H^1_{\mathrm{loc}}(\mathbb{R})$ such that, up to extracting subsequences, β is the weak limit of $\{\beta_j\}_j$ in $H^1(-s_0, s_0)$ for every $s_0 > 0$. By $\beta_j' \rightharpoonup \beta'$ in $L^2(-s_0, s_0)$ for every $s_0 > 0$ we easily find

$$\liminf_{j\to\infty} \int_{-R_0/\varepsilon_j}^{\infty} 2\beta_j'(s)^2 (R_0 + \varepsilon_j s)^{n-1}\, ds \geq R_0^{n-1} \int_{\mathbb{R}} 2(\beta')^2. \tag{4-28}$$

We now apply the concentration-compactness principle to the sequence of measures

$$\mu_j = 1_{(-R_0/\varepsilon_j, \infty)}(s)\beta_j(s)^2 (R_0 + \varepsilon_j s)^{n-1}\, ds,$$

which satisfy $\mu_j(\mathbb{R}) = 1$ thanks to (4-24). We claim that, if the compactness case holds, and thus

$$\lim_{s_0\to+\infty} \sup_j \mu_j(\mathbb{R} \setminus [-s_0, s_0]) = 0, \tag{4-29}$$

then we can reach a contradiction, and complete the proof of the lemma. To prove this claim, let us set

$$\eta_0(s) = \eta(s - \tau_0)$$

for τ_0 as in (A-19), and let us notice that, for every $s_0 > 0$ we have

$$\begin{aligned} \limsup_{j\to\infty} &\left| \int_{-R_0/\varepsilon_j}^{\infty} W''(\eta_j)\beta_j(s)^2 (R_0 + \varepsilon_j s)^{n-1}\, ds - R_0^{n-1} \int_{\mathbb{R}} W''(\eta_0)\beta^2 \right| \\ &\leq \limsup_{j\to\infty} \int_{-s_0}^{s_0} |W''(\eta_j)\beta_j(s)^2 (R_0 + \varepsilon_j s)^{n-1} - R_0^{n-1} W''(\eta_0)\beta^2| \\ &\qquad + \|W''\|_{C^0[0,1]} \sup_{j\in\mathbb{N}} \mu_j(\mathbb{R} \setminus [-s_0, s_0]) + R_0^{n-1} \|W''\|_{C^0[0,1]} \int_{\mathbb{R}\setminus[-s_0,s_0]} \beta^2. \end{aligned} \tag{4-30}$$

Since $\beta_j \to \beta$ in $L^2_{\mathrm{loc}}(\mathbb{R})$ and $\eta_j \to \eta_0$ locally uniformly on $\mathbb{R}$ thanks to Theorem 3.1, the first term on the right-hand side of (4-30) is equal to zero. Letting now $s_0 \to \infty$, the second term goes to zero thanks to (4-29), while the third term goes to zero thanks to the fact that (4-29) implies in particular

$$R_0^{n-1} \int_{\mathbb{R}} \beta^2 = 1. \tag{4-31}$$

We can combine this information with (4-28) and finally deduce from (4-27) that

$$\int_{\mathbb{R}} 2(\beta')^2 + W''(\eta_0)\beta^2 \le 0. \tag{4-32}$$

By Lemma 4.3 we deduce that, if we set $\beta_0(s) = \beta(s+\tau_0)$, then $\beta_0 = t\,\eta'$ for some $t \neq 0$ ($t = 0$ being ruled out by (4-31)). In particular, $\beta = t\eta_0'$, and therefore

$$\int_{\mathbb{R}} V'(\eta_0)\beta = tV(\eta_0)\big|_{-\infty}^{+\infty} = tV(1) = t \neq 0.$$

However, by (4-22), we see that

$$0 = \int_{\mathbb{R}^n} V'(u_j)h_j = \int_{-R_0/\varepsilon_j}^{\infty} V'(\eta_j)\beta_j(s)(R_0 + s\varepsilon_j)^{n-1}\,ds \quad \text{for all } j,$$

and we can thus obtain a contradiction by showing that

$$\lim_{j\to\infty} \int_{-R_0/\varepsilon_j}^{\infty} V'(\eta_j)\beta_j(s)(R_0 + s\varepsilon_j)^{n-1}\,ds = R_0^{n-1} \int_{\mathbb{R}} V'(\eta_0)\beta. \tag{4-33}$$

This is proved by noticing that (A-11), (A-16), (3-56) and (3-57) give

$$0 \le \max\{V'(\eta_j),\, V'(\eta_0)\} \le Ce^{-|s|/C}$$

for every $s \in \mathbb{R}$ (or for every $s \ge -R_0/\varepsilon_j$, in the case of η_j). In particular,

$$\begin{aligned} &\lim_{s_0\to\infty} \limsup_{j\to\infty} \left[\int_{-R_0/\varepsilon_j}^{-s_0} + \int_{s_0}^{\infty}\right] V'(\eta_j)|\beta_j|(R_0 + s\varepsilon_j)^{n-1}\,ds \\ &\qquad \le C \lim_{s_0\to\infty} \limsup_{j\to\infty} \left(\int_{\{|s|>s_0\}} e^{-|s|/C}(R_0 + s\varepsilon_j)^{n-1}\,ds\right)^{1/2} \mu_j(\mathbb{R}\setminus[-s_0, s_0])^{1/2} = 0, \end{aligned}$$

so that a similar argument to the one used in (4-30) can be repeated to prove (4-33).

We are thus left to prove that the sequence of probability measures $\{\mu_j\}_j$ cannot be in the vanishing case nor in the dichotomy case of the concentration-compactness principle.

To exclude that $\{\mu_j\}_j$ *is in the vanishing case*: Since $\eta_j \to \eta$ locally uniformly on $\mathbb{R}$, up to take j large enough and for S_0 as in (4-12) we have $W''(\eta_j(s)) \ge 1/C$ for $|s| \ge S_0$, $s \ge -R_0/\varepsilon_j$. Since we are in the vanishing case, it holds

$$\lim_{j\to\infty} \int_{-S_0}^{S_0} \beta_j(s)^2(R_0 + \varepsilon_j s)^{n-1}\,ds = 0, \tag{4-34}$$

so that, by using first the lower bound on W'', and then (4-34), we get

$$\frac{1}{C}\limsup_{j\to\infty}\biggl[\int_{-R_0/\varepsilon_j}^{-S_0}+\int_{S_0}^{\infty}\biggr]\beta_j(s)^2(R_0+\varepsilon_j s)^{n-1}\,ds$$
$$\le\limsup_{j\to\infty}\biggl[\int_{-R_0/\varepsilon_j}^{-S_0}+\int_{S_0}^{\infty}\biggr]W''(\eta_j)\beta_j(s)^2(R_0+\varepsilon_j s)^{n-1}\,ds$$
$$=\limsup_{j\to\infty}\int_{-R_0/\varepsilon_j}^{\infty}W''(\eta_j)\beta_j(s)^2(R_0+\varepsilon_j s)^{n-1}\,ds\le 0,$$

where in the last inequality we have used (4-27). Combining this information with (4-34) we obtain a contradiction to (4-26), thus excluding the vanishing case.

To exclude that $\{\mu_j\}_j$ is in the dichotomy case: With S_0 as above, if we are in the dichotomy case, then there exists $\alpha\in(0,1)$ such that for every $\tau\in(0,\alpha/2)$ there exist $R>S_0$ and $R_j\to\infty$ such that

$$|\mu_j(I_R)-(1-\alpha)|<\tau,\quad |\mu_j(\mathbb{R}\setminus I_{R_j})-\alpha|<\tau\quad\text{for all } j. \tag{4-35}$$

Setting $A_j=\varphi\beta_j$, $B_j=(1-\varphi)\beta_j$, where φ is a cut-off function between B_R and B_{R+1}, and setting for the sake of brevity

$$Q_j(A,B)=\int_{-R_0/\varepsilon_j}^{\infty}\{2A'B'+W''(\eta_j)AB\}(R_0+\varepsilon_j s)^{n-1}\,ds,\qquad Q_j(A)=Q_j(A,A),$$

we can rewrite (4-27) as

$$\limsup_{j\to\infty} Q_j(A_j)+Q_j(B_j)+2Q_j(A_j,B_j)\le 0. \tag{4-36}$$

Now, since φ' and $(1-\varphi)\varphi$ are supported in $I_{R+1}\setminus I_R$, we see that

$$Q_j(A_j,B_j)\ge 2\int_{I_{R+1}\setminus I_R}(1-2\varphi)\varphi'\beta_j\beta_j'(R_0+\varepsilon_j s)^{n-1}\,ds+\int_{I_{R+1}\setminus I_R}\{W''(\eta_j)-(\varphi')^2\}\beta_j^2(R_0+\varepsilon_j s)^{n-1}\,ds,$$

where, thanks to (4-27) and the Hölder inequality,

$$\int_{I_{R+1}\setminus I_R}(1-2\varphi)\varphi'\beta_j\beta_j'(R_0+\varepsilon_j s)^{n-1}\,ds\le C\mu_j(I_{R+1}\setminus I_R)^{1/2}\le C\sqrt{\tau},$$
$$\int_{I_{R+1}\setminus I_R}\{W''(\eta_j)-(\varphi')^2\}\beta_j^2(R_0+\varepsilon_j s)^{n-1}\,ds\le C\mu_j(I_{R+1}\setminus I_R)\le C\tau.$$

We thus conclude that $Q_j(A_j,B_j)\ge -C\sqrt{\tau}$ for every j, and thus, by (4-36), that

$$\limsup_{j\to\infty} Q_j(A_j)+Q_j(B_j)\le C\sqrt{\tau}. \tag{4-37}$$

Now, since the supports of the A_j are uniformly bounded, we easily see that there exists $A\in H^1(\mathbb{R})$ such that $A_j\rightharpoonup A$ weakly in $H^1(\mathbb{R})$; in particular,

$$\liminf_{j\to\infty} Q_j(A_j)\ge\int_{\mathbb{R}}2(A')^2+W''(\eta_0)A^2\ge 0,$$

where in the last inequality we have used Lemma 4.3. By combining this last inequality with (4-37), $W''(\eta_j) \ge 1/C$ on $\mathbb{R} \setminus I_{S_0}$, and $R \ge S_0$, we conclude that

$$C\sqrt{\tau} \ge \limsup_{j\to\infty} Q_j(B_j) \ge \frac{1}{C} \limsup_{j\to\infty} \int_{-R_0/\varepsilon_j}^{\infty} (1-\varphi)^2 \beta_j^2 (R + s\varepsilon_j)^{n-1}\, ds$$

and thus, by (4-35), that $C\sqrt{\tau} \ge (\alpha/C) - C\tau$. Letting $\tau \to 0^+$ we obtain a contradiction with $\alpha > 0$. □

Lemma 4.5 (third stability lemma). *If $n \ge 2$ and $W \in C^{2,1}[0, 1]$ satisfies* (1-11) *and* (1-12), *then there exist universal constants δ_0 and ε_0 such that, if $u_\varepsilon \in \mathcal{R}_0^*$ is a minimizer of $\psi(\varepsilon)$ for $\varepsilon < \varepsilon_0$ and $u \in H^1(\mathbb{R}^n; [0, 1])$ is a radial function with*

$$\int_{\mathbb{R}^n} V(u) = 1, \tag{4-38}$$

$$\int_{\mathbb{R}^n} (u - u_\varepsilon)^2 \le C\varepsilon, \tag{4-39}$$

$$\|u - u_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \le \delta_0, \tag{4-40}$$

then, setting $h = u - u_\varepsilon$,

$$\int_{\mathbb{R}^n} 2\varepsilon|\nabla h|^2 + \left(\frac{W''(u_\varepsilon)}{\varepsilon} - \lambda(\varepsilon)V''(u_\varepsilon)\right)h^2 \ge \frac{1}{C}\int_{\mathbb{R}^n} \varepsilon|\nabla h|^2 + \frac{h^2}{\varepsilon}, \tag{4-41}$$

where $\lambda(\varepsilon)$ is the Lagrange multiplier of u_ε as in (4-5).

Proof. It will be convenient to set

$$P_\varepsilon(u, v) = \int_{\mathbb{R}^n} \varepsilon\nabla u \cdot \nabla v + \frac{uv}{\varepsilon},$$
$$Q_\varepsilon(u, v) = \int_{\mathbb{R}^n} \varepsilon\nabla u \cdot \nabla v + \left(\frac{W''(u_\varepsilon)}{\varepsilon} - \lambda(\varepsilon)V''(u_\varepsilon)\right)uv,$$

as well as $P_\varepsilon(u) = P_\varepsilon(u, u)$ and $Q_\varepsilon(u) = Q_\varepsilon(u, u)$. Let us start noticing that by Theorem 3.1 we have

$$\lim_{\sigma\to 0} \sup_{\varepsilon<\sigma} \sup_{v_\varepsilon} \left| \int_{\mathbb{R}^n} V'(v_\varepsilon)v_\varepsilon - R_0^{n-1} \int_{\mathbb{R}} V'(\eta)\eta \right| = 0,$$

where v_ε runs over all radial minimizers of $\psi(\varepsilon)$. Since $\int_{\mathbb{R}} V'(\eta)\eta$ is a positive constant depending on n and W only, this shows in particular that

$$\frac{1}{C} \le \int_{\mathbb{R}^n} V'(u_\varepsilon)u_\varepsilon \le C \quad \text{for all } \varepsilon < \varepsilon_0. \tag{4-42}$$

By (4-42), given $h = u - u_\varepsilon$ as in the statement, we can always find $t \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} V'(u_\varepsilon)(h + tu_\varepsilon) = 0, \quad \text{i.e.,} \quad t = -\frac{\int_{\mathbb{R}^n} V'(u_\varepsilon)h}{\int_{\mathbb{R}^n} V'(u_\varepsilon)u_\varepsilon}. \tag{4-43}$$

By (A-12), (4-40), and since $0 \le u_\varepsilon + h \le 1$, we have that, on $\mathbb{R}^n$,

$$\left| V(u_\varepsilon + h) - V(u_\varepsilon) - V'(u_\varepsilon)h - V''(u_\varepsilon)\frac{h^2}{2} \right| \le C\delta_0 h^2, \tag{4-44}$$

so that, by (4-38),

$$\left|\int_{\mathbb{R}^n} V'(u_\varepsilon)h + V''(u_\varepsilon)\frac{h^2}{2}\right| \le C\delta_0 \int_{\mathbb{R}^n} h^2, \tag{4-45}$$

and thus, thanks to $\|V''\|_{C^0[0,1]} \le C$, (4-42), (4-39), and (4-43),

$$|t| \le C\int_{\mathbb{R}^n} h^2 \le C\varepsilon \min\{P_\varepsilon(h), 1\}. \tag{4-46}$$

By (4-43) we can apply Lemma 4.4 to $u_\varepsilon + th$ and find that

$$Q_\varepsilon(h + tu_\varepsilon) \ge \frac{P_\varepsilon(h + tu_\varepsilon)}{C},$$

which can be more conveniently rewritten as

$$Q_\varepsilon(h) \ge \frac{P_\varepsilon(h)}{C} + 2t\left\{\frac{P_\varepsilon(h, u_\varepsilon)}{C} - Q_\varepsilon(h, u_\varepsilon)\right\} + t^2\left\{\frac{P_\varepsilon(u_\varepsilon)}{C} - Q_\varepsilon(u_\varepsilon)\right\}. \tag{4-47}$$

By Theorem 3.1, we see that $P_\varepsilon(u_\varepsilon) + |Q_\varepsilon(u_\varepsilon)| \le C$ (uniformly on $\varepsilon < \varepsilon_0$), so that (4-47) and (4-46) give

$$Q_\varepsilon(h) \ge \frac{P_\varepsilon(h)}{C} + 2t\left\{\frac{P_\varepsilon(h, u_\varepsilon)}{C} - Q_\varepsilon(h, u_\varepsilon)\right\}. \tag{4-48}$$

By the Hölder inequality, $ab \le (a^2 + b^2)/2$, $P_\varepsilon(u_\varepsilon) \le C$, and (4-46) we see that

$$|t|P_\varepsilon(h, u_\varepsilon) \le \frac{|t|}{2}(P_\varepsilon(h) + P_\varepsilon(u_\varepsilon)) \le C\varepsilon P_\varepsilon(h), \tag{4-49}$$

while by $|V'| + |W''| \le C$ and $|\lambda(\varepsilon)| \le C$ for $\varepsilon < \varepsilon_0$ we find, arguing as in (4-49),

$$|t|Q_\varepsilon(h, u_\varepsilon) \le |t|\left\{\varepsilon\int_{\mathbb{R}^n} |\nabla h||\nabla u_\varepsilon| + \frac{C}{\varepsilon}\int_{\mathbb{R}^n} |h|u_\varepsilon\right\} \le C\varepsilon P_\varepsilon(h). \tag{4-50}$$

By combining (4-48), (4-49), and (4-50) we conclude that $Q_\varepsilon(h) \ge P_\varepsilon(h)/C$, as desired. □

We are finally ready to prove Theorem 4.1.

Proof of Theorem 4.1. We are given u_ε and h as in Lemma 4.5, and now want to prove that

$$\mathcal{AC}_\varepsilon(u_\varepsilon + h) - \psi(\varepsilon) \ge \frac{1}{C}\int_{\mathbb{R}^n} \varepsilon|\nabla h|^2 + \frac{h^2}{\varepsilon} \tag{4-51}$$

holds. By (A-5) and (4-40) we have

$$\left|W(u_\varepsilon + h) - W(u_\varepsilon) - W'(u_\varepsilon)h - W''(u_\varepsilon)\frac{h^2}{2}\right| \le C\delta_0 h^2 \quad \text{on } \mathbb{R}^n;$$

therefore

$$\mathcal{AC}_\varepsilon(u_\varepsilon + h) - \mathcal{AC}_\varepsilon(u_\varepsilon) \ge \int_{\mathbb{R}^n} 2\varepsilon\nabla u_\varepsilon\cdot\nabla h + \frac{W'(u_\varepsilon)}{\varepsilon}h + \int_{\mathbb{R}^n} \varepsilon|\nabla h|^2 + \frac{W''(u_\varepsilon)}{2\varepsilon}h^2 - C\delta_0\int_{\mathbb{R}^n} h^2. \tag{4-52}$$

By the Euler–Lagrange equation for u_ε, see (2-1), we have

$$\int_{\mathbb{R}^n} 2\varepsilon\nabla u_\varepsilon\cdot\nabla h + \frac{W'(u_\varepsilon)}{\varepsilon}h = \lambda(\varepsilon)\int_{\mathbb{R}^n} V'(u_\varepsilon)h. \tag{4-53}$$

Moreover, by (4-45),

$$\left|\int_{\mathbb{R}^n} V'(u_\varepsilon)h + \int_{\mathbb{R}^n} V''(u_\varepsilon)\frac{h^2}{2}\right| \le C\delta_0 \int_{\mathbb{R}^n} h^2. \tag{4-54}$$

On combining (4-52), (4-53), and (4-54) with (4-41) we find that

$$\begin{aligned}\mathcal{AC}_\varepsilon(u_\varepsilon + h) - \psi(\varepsilon) &\ge \frac{1}{2}\int_{\mathbb{R}^n} 2\varepsilon|\nabla h|^2 + \left\{\frac{1}{\varepsilon}W''(u_\varepsilon) - \lambda(\varepsilon)V''(u_\varepsilon)\right\}h^2 - C\delta_0 \int_{\mathbb{R}^n} h^2 \\ &\ge \int_{\mathbb{R}^n} \varepsilon|\nabla h|^2 + \frac{h^2}{\varepsilon} - C\delta_0 \int_{\mathbb{R}^n} h^2,\end{aligned}$$

so that (4-51) follows by taking δ_0 small enough. □

5. Proof of the uniform stability theorem

In this section we prove Theorem 1.1(iii), i.e., we prove (1-21). We focus directly on the case $(\sigma, m) = (\varepsilon, 1)$, from which the general case follows immediately by scaling.

Theorem 5.1. *If $n \ge 2$ and $W \in C^{2,1}[0, 1]$ satisfies* (1-11) *and* (1-12), *then there exist universal constants $\varepsilon_0 > 0$ and C such that if $\varepsilon < \varepsilon_0$ and $u \in H^1(\mathbb{R}^n; [0, 1])$ with $\int_{\mathbb{R}^n} V(u) = 1$, then*

$$C\sqrt{\mathcal{AC}_\varepsilon(u) - \psi(\varepsilon)} \ge \inf_{x_0\in\mathbb{R}^n} \int_{\mathbb{R}^n} |\Phi(u) - \Phi(T_{x_0}u_\varepsilon)|^{n/(n-1)}, \tag{5-1}$$

where $T_{x_0}u_\varepsilon(x) = u_\varepsilon(x - x_0)$, $x \in \mathbb{R}^n$, and u_ε denotes the unique minimizer of $\psi(\varepsilon)$ in $\mathcal{R}_0$.

In order to streamline the exposition of the proof of Theorem 5.1, we introduce the isoperimetric deficit and asymmetry of $u \in H^1(\mathbb{R}^n; [0, 1])$ with $\int_{\mathbb{R}^n} V(u) = 1$, by setting

$$\begin{aligned}\delta_\varepsilon(u) &= \mathcal{AC}_\varepsilon(u) - \psi(\varepsilon),\\ \alpha_\varepsilon(u) &= \inf_{x_0\in\mathbb{R}^n} d_\Phi(u, T_{x_0}u_\varepsilon).\end{aligned}$$

Here, as in Theorem 2.2,

$$d_\Phi(u, v) = \int_{\mathbb{R}^n} |\Phi(u) - \Phi(v)|^{n/(n-1)} \quad \text{for all } u, v \in H^1(\mathbb{R}^n; [0, 1]).$$

With this notation, Theorem 5.1 states the existence of universal constants C and ε_0 such that if $\varepsilon < \varepsilon_0$, then

$$C\sqrt{\delta_\varepsilon(u)} \ge \alpha_\varepsilon(u) \quad \text{for all } u \in H^1(\mathbb{R}^n; [0, 1]), \int_{\mathbb{R}^n} V(u) = 1. \tag{5-2}$$

In the following subsections we discuss some key steps of the proof of Theorem 5.1, which is then presented at the end of this section.

5A. ***Reduction to the small asymmetry case.*** Thanks to the volume constraint $\int_{\mathbb{R}^n} V(u) = 1$ and to the triangular inequality in $L^{n/(n-1)}$, we always have $\alpha_\varepsilon(u) \le 2^{n/(n-1)}$. In particular, in proving (5-2), we can always assume that $\delta_\varepsilon(u) \le \delta_0$ for a universal constant δ_0. This is useful because, by the following lemma, by assuming $\delta_\varepsilon(u) \le \delta_0$ we can take $\alpha_\varepsilon(u)$ as small as needed independent of n and W.

Lemma 5.2 (ε-uniform qualitative stability). *If $n \ge 2$ and $W \in C^{2,1}[0,1]$ satisfies* (1-11) *and* (1-12), *then there exists a universal constant ε_0 with the following property: for every $\alpha > 0$ there exists $\delta > 0$ such that*

$$u \in H^1(\mathbb{R}^n;[0,1]), \qquad \int_{\mathbb{R}^n} V(u) = 1, \qquad \varepsilon < \varepsilon_0, \qquad \delta_\varepsilon(u) \le \delta$$

imply

$$\alpha_\varepsilon(u) \le \alpha.$$

Proof. We pick ε_0 such that Theorem 2.1 and Corollary 4.2 hold. If the lemma is false for such ε_0, then there exists $\alpha_* > 0$ and a sequence $\{u_j\}_j$ in $H^1(\mathbb{R}^n;[0,1])$ with $\int_{\mathbb{R}^n} V(u_j) = 1$ such that

$$\delta_{\varepsilon_j}(u_j) \to 0^+ \quad \text{as } j \to \infty, \tag{5-3}$$

for some $\varepsilon_j \to \varepsilon_* \in [0, \varepsilon_0]$ and with $\alpha_{\varepsilon_j}(u_j) \ge \alpha_*$. By (5-3), there is $\ell_j \to 0^+$ as $j \to \infty$ such that

$$\mathcal{AC}_{\varepsilon_j}(u_j) \le \psi(\varepsilon_j) + \ell_j \quad \text{for all } j, \tag{5-4}$$

We now distinguish two cases:

<u>Case 1</u>: $\varepsilon_* > 0$. In this case, by the continuity of ψ (see Theorem 2.1) and since

$$\mathcal{AC}_{\varepsilon_*}(u_j) - \psi(\varepsilon_*) \le b_j(\mathcal{AC}_{\varepsilon_j}(u_j) - \psi(\varepsilon_j)) + b_j\psi(\varepsilon_j) - \psi(\varepsilon_*), \qquad b_j = \max\left\{\frac{\varepsilon_j}{\varepsilon_*}, \frac{\varepsilon_*}{\varepsilon_j}\right\},$$

we can assume that $\mathcal{AC}_{\varepsilon_*}(u_j) - \psi(\varepsilon_*) \le \ell_0$ for ℓ_0 as in Step 2 of the proof of Theorem 2.1. We can thus apply that statement and conclude that, up to translations and up to subsequences, there is $u \in H^1(\mathbb{R}^n;[0,1])$ with $\int_{\mathbb{R}^n} V(u) = 1$ such that $d_\Phi(u_j, u) \to 0$ as $j \to \infty$. In particular, u is a minimizer of $\psi(\varepsilon_*)$, and therefore, up to a translation, we can assume that $u = u_{\varepsilon_*} \in \mathcal{R}_0$. Now, by repeating this same argument with the minimizers u_{ε_j} of $\psi(\varepsilon_j)$ in $\mathcal{R}_0$ in place of u_j, we see that

$$d_\Phi(u_{\varepsilon_j}, u_{\varepsilon_*}) \to 0 \quad \text{as } j \to \infty,$$

so that, thanks to (2-63), we find the contradiction

$$\alpha_* \le \alpha_{\varepsilon_j}(u_j) \le d_\Phi(u_j, u_{\varepsilon_j}) \le d_\Phi(u_j, u_{\varepsilon_*}) + C d_\Phi(u_{\varepsilon_j}, u_{\varepsilon_*})^{(n-1)/n} \to 0^+$$

as $j \to \infty$.

<u>Case 2</u>: $\varepsilon_* = 0$. In this case, thanks to (5-4),

$$2|D[\Phi(u_j)]|(\mathbb{R}^n) \le \mathcal{AC}_{\varepsilon_j}(u_j) \le \psi(\varepsilon_j) + \ell_j \le 2n\omega_n^{1/n} + C\varepsilon_j + \ell_j,$$

so that $\{\Phi(u_j)\}_j$ is asymptotically optimal for the sharp BV-Sobolev inequality. By the concentration-compactness principle (see, e.g., [Fusco et al. 2007, Theorem A.1]), up to subsequences and up to translations, $\Phi(u_j) \to a 1_{B_r}$ in $L^{n/(n-1)}(\mathbb{R}^n)$ as $j \to \infty$ for some a and r such that $a^{n/(n-1)}\omega_n r^n = 1$. The fact that $\mathcal{AC}_{\varepsilon_j}(v_j)$ is bounded implies that $v_j \to \{0,1\}$ a.e. on $\mathbb{R}^n$; therefore, by $\Phi(0) = 0$ and $\Phi(1) = 1$, it must be $a = 1$ and $R = R_0$ for $\omega_n R_0^n = 1$. By Theorem 3.1, if u_{ε_j} is a the minimizer of $\psi(\varepsilon_j)$ in $\mathcal{R}_0$, then

$$d_\Phi(u_{\varepsilon_j}, 1_{B_{R_0}}) \to 0 \quad \text{as } j \to \infty,$$

which gives the contradiction

$$\alpha_* \le \alpha_{\varepsilon_j}(u_j) \le d_\Phi(u_j, u_{\varepsilon_j}) \le d_\Phi(u_j, 1_{B_{R_0}}) + C d_\Phi(u_{\varepsilon_j}, 1_{B_{R_0}})^{(n-1)/n} \to 0^+$$

as $j \to \infty$. □

5B. ***Proof of Theorem 5.1 in the radial decreasing case.*** We start by noticing that, thanks to the results proved in the previous sections, we can quickly prove Theorem 5.1 for functions in $\mathcal{R}_0$.

Theorem 5.3. *If $n \ge 2$ and $W \in C^{2,1}[0,1]$ satisfies* (1-11) *and* (1-12), *then there exist universal constants C and ε_0 such that, for every $\varepsilon < \varepsilon_0$, denoting by u_ε the unique minimizer of $\psi(\varepsilon)$ in $\mathcal{R}_0$, one has*

$$C\sqrt{\delta_\varepsilon(u)} \ge d_\Phi(u, u_\varepsilon), \tag{5-5}$$

whenever $u \in H^1(\mathbb{R}^n;[0,1]) \cap \mathcal{R}_0$ with $\int_{\mathbb{R}^n} V(u) = 1$.

Proof. Arguing by contradiction, we can find $\varepsilon_j \to 0^+$ and $\{v_j\}_j$ in $H^1(\mathbb{R}^n;[0,1]) \cap \mathcal{R}_0$ with

$$\int_{\mathbb{R}^n} V(v_j) = 1, \qquad a_j = \frac{\mathcal{AC}_{\varepsilon_j}(v_j) - \psi(\varepsilon_j)}{d_\Phi(v_j, u_j)^2} \to 0 \quad \text{as } j \to \infty,$$

where $u_j = u_{\varepsilon_j}$ and, thanks to Lemma 5.2 and to $a_j \to 0^+$, we have

$$\lim_{j\to\infty} d_\Phi(v_j, u_j) = 0. \tag{5-6}$$

Correspondingly we consider the variational problems

$$\gamma_j = \gamma(\varepsilon_j, a_j, v_j) = \inf\left\{\mathcal{AC}_{\varepsilon_j}(w) + a_j d_\Phi(w, v_j) : w \in H^1(\mathbb{R}^n;[0,1]), \int_{\mathbb{R}^n} V(w) = 1\right\}.$$

With a_0, ℓ_0 and ε_0 as in Theorem 2.2, we notice that, for j large enough, we have $a_j \in (0, a_0)$, $\varepsilon_j < \varepsilon_0$, and

$$\mathcal{AC}_{\varepsilon_j}(v_j) \le \psi(\varepsilon_j) + a_j \ell_0, \quad d_\Phi(v_j, u_j) \le \ell_0. \tag{5-7}$$

In particular we can apply Theorem 2.2, and deduce the existence of minimizers w_j of γ_j. We claim that, as $j \to \infty$,

$$\lim_{j\to\infty} \frac{\mathcal{AC}_{\varepsilon_j}(w_j) - \psi(\varepsilon_j)}{d_\Phi(w_j, u_j)^2} = 0. \tag{5-8}$$

To show this, we first notice that, by comparing w_j to u_j we have

$$\mathcal{AC}_{\varepsilon_j}(w_j) + a_j\, d_\Phi(w_j, v_j) \le \psi(\varepsilon_j) + a_j\, d_\Phi(u_j, v_j),$$

so that (5-6) gives $\delta_{\varepsilon_j}(w_j) \to 0$, and then Lemma 5.2 implies

$$\lim_{j\to\infty} d_\Phi(w_j, u_j) = 0. \tag{5-9}$$

Next, comparing w_j to v_j we find that

$$\mathcal{AC}_{\varepsilon_j}(w_j) + a_j\, d_\Phi(w_j, v_j) \le \mathcal{AC}_{\varepsilon_j}(v_j),$$

so that $\psi(\varepsilon_j) \le \mathcal{AC}_{\varepsilon_j}(w_j)$ and the definition of a_j give

$$d_\Phi(w_j, v_j) \le \frac{\mathcal{AC}_{\varepsilon_j}(v_j) - \psi(\varepsilon_j)}{a_j} = d_\Phi(v_j, u_j)^2. \tag{5-10}$$

By (2-63), (5-6), (5-9), and (5-10) we find

$$
\begin{aligned}
|d_\Phi(w_j,u_j)-d_\Phi(v_j,u_j)| &\le C\max\{d_\Phi(w_j,u_j),d_\Phi(v_j,u_j)\}^{1/n}\,d_\Phi(w_j,v_j)^{(n-1)/n}\\
&= \mathrm{o}(d_\Phi(v_j,u_j)^{2(n-1)/n}),
\end{aligned}
$$

where $2(n-1)/n\ge 1$ thanks to $n\ge 2$. Thus, $d_\Phi(w_j,u_j)\ge d_\Phi(v_j,u_j)/C$ for j large enough, and $\mathcal{AC}_{\varepsilon_j}(w_j)\le \mathcal{AC}_{\varepsilon_j}(v_j)$ gives

$$
\frac{\mathcal{AC}_{\varepsilon_j}(w_j)-\psi(\varepsilon_j)}{d_\Phi(w_j,u_j)^2}\le C\,\frac{\mathcal{AC}_{\varepsilon_j}(v_j)-\psi(\varepsilon_j)}{d_\Phi(v_j,u_j)^2}\to 0^+,
$$

as claimed in (5-8).

We now derive a contradiction to (5-8). By Theorem 2.2, we know that $w_j\in\mathcal{R}_0^*\cap C^{2,1/(n-1)}_{\mathrm{loc}}(\mathbb{R}^n)$, $0<w_j<1$ on $\mathbb{R}^n$, and

$$
-2\varepsilon_j^2\Delta w_j=\varepsilon_j w_j(1-w_j)\mathrm{E}_j-W'(w_j)\quad\text{on }\mathbb{R}^n,\tag{5-11}
$$

where E_j is a continuous radial function on $\mathbb{R}^n$ with

$$
\sup_{\mathbb{R}^n}|\mathrm{E}_j|\le C.\tag{5-12}
$$

We can thus apply Theorem 3.1 to w_j. In particular, since both u_j and w_j obey the resolution formula (3-8), we have that $h_j=w_j-u_j$ satisfies

$$
|h_j(R_0+\varepsilon_j s)|\le C\varepsilon_j e^{-|s|/C}\quad\text{for all }s\ge-\frac{R_0}{\varepsilon_j}.\tag{5-13}
$$

In particular,

$$
\|h_j\|_{L^\infty(\mathbb{R}^n)}\le C\varepsilon_j,\qquad \int_{\mathbb{R}^n}h_j^2\le C\varepsilon_j,
$$

and we can thus apply Theorem 4.1 to deduce

$$
\begin{aligned}
\mathcal{AC}_{\varepsilon_j}(w_j)-\psi(\varepsilon_j) &\ge \frac{1}{C}\int_{\mathbb{R}^n}\varepsilon_j|\nabla h_j|^2+\frac{h_j^2}{\varepsilon_j}\\
&\ge \frac{1}{C}\int_{\mathbb{R}^n}|\nabla(h_j^2)|\ge\frac{1}{C}\biggl(\int_{\mathbb{R}^n}|h_j|^{2n/(n-1)}\biggr)^{(n-1)/n},
\end{aligned}\tag{5-14}
$$

where we have also used the BV-Sobolev inequality. By (5-13), and by applying (3-14) to u_j in combination with (A-6), we find that, if $A_j=B_{R_0+c_j}\setminus B_{R_0-b_j}$, then, for every $x\in\mathbb{R}^n\setminus A_j$, we have

$$
|\Phi(u_j(x))-\Phi(w_j(x))|\le|h_j(x)|\int_0^1\sqrt{W(u_j(x)+t h_j(x))}\,dt\le C|h_j(x)|e^{-||x|-R_0|/(C\varepsilon_j)},
$$

and, therefore,

$$
\int_{\mathbb{R}^n\setminus A_j}|\Phi(u_j)-\Phi(w_j)|^{n/(n-1)}\le C\int_{\mathbb{R}^n\setminus A_j}|h_j|^{n/(n-1)}e^{-||x|-R_0|/C\varepsilon_j}\le C\sqrt{\varepsilon_j}\biggl(\int_{\mathbb{R}^n}|h_j|^{2n/(n-1)}\biggr)^{1/2}.\tag{5-15}
$$

If, instead, $x\in A_j$, then by $|\Phi(u_j)-\Phi(w_j)|\le C|h_j|$ and $\mathcal{L}^n(A_j)\le C\varepsilon_j$ we find

$$
\int_{A_j}|\Phi(u_j)-\Phi(w_j)|^{n/(n-1)}\le C\sqrt{\varepsilon_j}\biggl(\int_{\mathbb{R}^n}|h_j|^{2n/(n-1)}\biggr)^{1/2}.\tag{5-16}
$$

By combining (5-14), (5-15) and (5-16), and thanks to $\varepsilon_j \le 1$, $n/(n-1) \ge 1$, and $\delta_{\varepsilon_j}(w_j) \le 1$, we conclude that

$$d_\Phi(u_j, w_j) \le C\sqrt{\varepsilon_j}\,\delta_{\varepsilon_j}(w_j)^{n/(2(n-1))} \le C\sqrt{\delta_{\varepsilon_j}(w_j)},$$

in contradiction to (5-8). □

Remark. The argument we have just presented provides further indication that (5-5) should not provide a sharp rate on radial decreasing functions. The sharp stability estimate on small radial perturbations of u_ε is clearly given in Theorem 4.1, but it is not clear what form the sharp stability estimate should take on $\mathcal{R}_0$ (or, more generally, on arbitrary radial functions).

5C. ***Reduction to radial decreasing functions.*** We now discuss the reduction of (5-2) to the case of radial decreasing functions. We do this by adapting to our setting the "quantitative symmetrization" strategy developed in [Fusco et al. 2007; 2008] in the study of Euclidean isoperimetry.

Given $n \ge 2$ and $k \in \{1, \dots, n\}$, we say that $u : \mathbb{R}^n \to \mathbb{R}$ is *k-symmetric* if there exist k mutually orthogonal hyperplanes such that u is symmetric by reflection through each of these hyperplanes. The class of n-symmetric functions is particularly convenient when it comes to quantifying sharp inequalities involving radial decreasing rearrangements. Consider for example the Pólya–Szegő inequality

$$\int_{\mathbb{R}^n} |\nabla u|^2 \ge \int_{\mathbb{R}^n} |\nabla u^*|^2, \tag{5-17}$$

where u^* is the radial decreasing rearrangement of u. A classical result of [Brothers and Ziemer 1988] shows that equality can hold in (5-17) without u being a translation of u^*; in general, the additional condition that $(u^*)' < 0$ a.e. must be assumed to deduce symmetry from equality in (5-17) (compare with Step 6 in the proof of Theorem 2.1). However, if u is n-symmetric, then equality in (5-17) automatically implies that u is radial decreasing. A quantitative version of this statement is proved in [Fusco et al. 2007, Theorem 2.2] in the BV-case of (5-17), and in [Cianchi et al. 2009, Theorem 3] in the Sobolev case. The following theorem is an adaptation of those results to our setting.

Theorem 5.4 (reduction from n-symmetric to radial decreasing functions). *If $n \ge 2$ and $W \in C^{2,1}[0,1]$ satisfies* (1-11) *and* (1-12), *then there exists a universal constant C with the following property. If $u \in H^1(\mathbb{R}^n; [0,1])$ is an n-symmetric function with $\int_{\mathbb{R}^n} V(u) = 1$ and u^* is its radial decreasing rearrangement, then*

$$d_\Phi(u, u^*) \le C\left(\int_{\mathbb{R}^n} W(u)\right)^{1/2}\left(\int_{\mathbb{R}^n} |\nabla u|^2 - \int_{\mathbb{R}^n} |\nabla u^*|^2\right)^{1/2}. \tag{5-18}$$

Moreover, for every $\varepsilon > 0$ we have

$$\alpha_\varepsilon(u) \le C\big(\alpha_\varepsilon(u^*) + (\mathcal{AC}_\varepsilon(u)\delta_\varepsilon(u))^{1/2}\big). \tag{5-19}$$

Proof. We first claim that

$$d_\Phi(u, u^*) \le \frac{n}{n-1}\int_0^1 \mathcal{L}^n(E_t)\Phi(t)^{1/(n-1)}\sqrt{W(t)}\,dt, \tag{5-20}$$

$$\int_{\mathbb{R}^n} |\nabla u|^2 - \int_{\mathbb{R}^n} |\nabla u^*|^2 \ge \frac{1}{C(n)}\int_0^1 \left(\frac{\mathcal{L}^n(E_t)}{\mu(t)}\right)^2 \frac{\mu(t)^{2(n-1)/n}}{-\mu'(t)}\,dt, \tag{5-21}$$

where $E_t = \{u > t\}\Delta\{u^* > t\}$, $\mu(t) = \mathcal{L}^n(\{u > t\})$, and $\mu'(t)$ denotes the absolutely continuous part of the distributional derivative of the decreasing function μ. To prove (5-20) we recall that, by [Cianchi et al. 2009, Lemma 5], we have

$$d_\Phi(u, u^*) \le \frac{n}{n-1}\int_0^1 \mathcal{L}^n(F_s)s^{1/(n-1)}\,ds,$$

provided $F_s = \{\Phi(u) > s\}\Delta\{\Phi(u^*) > s\}$. Since Φ is strictly increasing, we have $F_{\Phi(t)} = E_t$, so that the change of variables $s = \Phi(t)$ gives (5-20). To prove (5-21) we just notice that this is [Cianchi et al. 2009, equation (3.18)]. Now, by the Hölder inequality and (5-20), we find that

$$\begin{aligned}\int_0^1 \mathcal{L}^n(E_s)\Phi^{1/(n-1)}\sqrt{W} &= \int_0^1 \frac{\mathcal{L}^n(E_s)}{\mu}\frac{\mu^{(n-1)/n}}{(-\mu')^{1/2}}\frac{(-\mu')^{1/2}}{\mu^{-1/n}}\Phi^{1/(n-1)}\sqrt{W}\\ &\le \left(\int_0^1\left(\frac{\mathcal{L}^n(E_s)}{\mu}\right)^2\frac{\mu^{2(n-1)/n}}{-\mu'}\right)^{1/2}\left(\int_0^1 \frac{-\mu'}{\mu^{-2/n}}\Phi^{2/(n-1)}W\right)^{1/2}.\end{aligned}$$

By $1 = \int_{\mathbb{R}^n} V(u) \ge V(t)\mu(t)$ for every $t \in (0, 1)$, we have

$$\int_0^1 \frac{-\mu'}{\mu^{-2/n}}\Phi^{2/(n-1)}W \le \int_0^1 -\mu'(V\mu)^{2/n}W \le \int_0^1 -\mu' W \le \int_{\mathbb{R}^n} W(u),$$

where in the last inequality we have used $-\mu'\,d\mathcal{L}^1 \le -D\mu$, integration by parts and Fubini's theorem to deduce

$$-\int_0^1 W\,d[D\mu] = \int_0^1 W'(t)\mu(t)\,dt = \int_{\mathbb{R}^n} dx\int_0^{u(x)} W'(t)\,dt = \int_{\mathbb{R}^n} W(u).$$

By combining (5-20), (5-21) and these estimates we find (5-18). To prove (5-19), we notice that, by $\int_{\mathbb{R}^n} W(u) = \int_{\mathbb{R}^n} W(u^*)$ and $\int_{\mathbb{R}^n} V(u^*) = 1$, (5-18) gives

$$d_\Phi(u, u^*) \le C\mathcal{AC}_\varepsilon(u)^{1/2}(\mathcal{AC}_\varepsilon(u) - \mathcal{AC}_\varepsilon(u^*))^{1/2} \le C\mathcal{AC}_\varepsilon(u)^{1/2}\delta_\varepsilon(u)^{1/2} \tag{5-22}$$

and then (5-19) follows by the triangular inequality in $L^{n/(n-1)}(\mathbb{R}^n)$. □

Next we discuss the reduction from generic functions to n-symmetric ones.

Theorem 5.5 (reduction to n-symmetric functions). *If $n \ge 2$ and $W \in C^{2,1}[0, 1]$ satisfies* (1-11) *and* (1-12), *then there exist universal constants ε_0 and δ_0 with the following property. If $u \in H^1(\mathbb{R}^n; [0, 1])$, $\int_{\mathbb{R}^n} V(u) = 1$ and $\delta_\varepsilon(u) \le \delta_0$ for some $\varepsilon < \varepsilon_0$, then there exists $v \in H^1(\mathbb{R}^n; [0, 1])$ with $\int_{\mathbb{R}^n} V(v) = 1$ such that v is n-symmetric and*

$$\alpha_\varepsilon(u) \le C\alpha_\varepsilon(v), \quad \delta_\varepsilon(v) \le C\delta_\varepsilon(u). \tag{5-23}$$

Proof. Without loss of generality we can assume that $\delta_\varepsilon(u) \le \delta_0$ for a universal constant δ_0. By Lemma 5.2 we can choose δ_0 so that $\alpha_\varepsilon(u) \le \alpha_0$ for α_0 a universal constant of our choice. We divide the proof into a few steps.

Step 1: We prove that, if u is k-symmetric, $\{H_i\}_{i=1}^k$ are the mutually orthogonal hyperplanes of symmetry of u, and $J = \bigcap_{i=1}^k H_i$, then

$$\alpha_\varepsilon(u; J) = \inf_{x\in J} d_\Phi(u, T_x u_\varepsilon) \le C(n)\alpha_\varepsilon(u). \tag{5-24}$$

In other words, in computing the asymmetry of u in the proof of an estimate like (5-2), we can compare u with a translation of u_ε with maximum on J.

Indeed, let $x_0 \in \mathbb{R}^n$ be such that $\alpha_\varepsilon(u) = d_\Phi(u, T_{x_0}u_\varepsilon)$. Without loss of generality, we can assume $x_0 \notin J$. In particular, if y_0 denotes the reflection of x_0 with respect to J, then $y_0 \neq x_0$ and

$$d_\Phi(u, T_{y_0}u_\varepsilon) = d_\Phi(u, T_{x_0}u_\varepsilon) = \alpha_\varepsilon(u), \tag{5-25}$$

that is, also y_0 is an optimal center for computing $\alpha_\varepsilon(u)$. Let $z_0 = (x_0 + y_0)/2$, so that $z_0 \in J$, let $\nu = (x_0 - y_0)/|x_0 - y_0|$ (which is well-defined by $x_0 \neq y_0$), and let H be the open half-space orthogonal to ν, containing x_0, and such that $z_0 \in \partial H$. By $T_{z_0+t\nu}u_\varepsilon(x) = u_\varepsilon(x - z_0 - t\nu)$, we have

$$\frac{d}{dt}T_{z_0+t\nu}u_\varepsilon(x) = -\nu \cdot \frac{x - z_0 - t\nu}{|x - z_0 - t\nu|}u_\varepsilon'(|x - x_0 - t\nu|) > 0 \quad \text{for all } x \in H, t < 0,$$

since $u_\varepsilon' < 0$, and since the fact that ν points inside H gives

$$\begin{aligned} (z - z_0) \cdot \nu > 0 &\quad \text{for all } z \in H, \\ z = x - t\nu \in H &\quad \text{for all } x \in H,\ t < 0. \end{aligned}$$

We thus find that, if $t < 0$,

$$\begin{aligned} \frac{d}{dt}\int_H |\Phi(T_{x_0}u) - \Phi(T_{z_0+t\nu}u_\varepsilon)|^{n/(n-1)} \\ = \frac{n}{n-1}\int_H |\Phi(u) - \Phi(T_{z_0+t\nu}u_\varepsilon)|^{1/(n-1)}\sqrt{W(T_{z_0+t\nu}u_\varepsilon)}\frac{d}{dt}T_{z_0+t\nu}u_\varepsilon > 0, \end{aligned}$$

so that

$$\begin{aligned} \int_H |\Phi(T_{x_0}u) - \Phi(T_{y_0}u_\varepsilon)|^{n/(n-1)} &= \int_{H^-} |\Phi(T_{x_0}u) - \Phi(T_{z_0+t\,\nu}u_\varepsilon)|^{n/(n-1)}|_{t=-|x_0-y_0|/2} \\ &\le \int_H |\Phi(T_{x_0}u) - \Phi(T_{z_0+t\,\nu}u_\varepsilon)|^{n/(n-1)}|_{t=0} \\ &\le \int_H |\Phi(T_{x_0}u) - \Phi(T_{z_0}u_\varepsilon)|^{n/(n-1)}. \end{aligned} \tag{5-26}$$

Now, since both u and $T_{z_0}u_\varepsilon$ are symmetric by reflection with respect to ∂H, we have

$$\int_{\mathbb{R}^n} |\Phi(u) - \Phi(T_{z_0}u_\varepsilon)|^{n/(n-1)} = 2\int_H |\Phi(u) - \Phi(T_{z_0}u_\varepsilon)|^{n/(n-1)}; \tag{5-27}$$

therefore, by (5-25), (5-26) and (5-27) we conclude that

$$\begin{aligned} \alpha_\varepsilon(u; J) \le d_\Phi(u, T_{z_0}u_\varepsilon) &= 2\int_H |\Phi(u) - \Phi(T_{z_0}u_\varepsilon)|^{n/(n-1)} \\ &\le C(n)\biggl(\int_H |\Phi(u) - \Phi(T_{x_0}u_\varepsilon)|^{n/(n-1)} + \int_H |\Phi(T_{x_0}u_\varepsilon) - \Phi(T_{z_0}u_\varepsilon)|^{n/(n-1)}\biggr) \\ &\le C(n)\biggl(\alpha_\varepsilon(u) + \int_H |\Phi(T_{y_0}u_\varepsilon) - \Phi(T_{x_0}u_\varepsilon)|^{n/(n-1)}\biggr) \\ &\le C(n)(\alpha_\varepsilon(u) + d_\Phi(T_{y_0}u_\varepsilon, T_{x_0}u_\varepsilon)) \\ &\le C(n)(\alpha_\varepsilon(u) + d_\Phi(T_{y_0}u_\varepsilon, u) + d_\Phi(u, T_{x_0}u_\varepsilon)) = C(n)\alpha_\varepsilon(u), \end{aligned}$$

that is, (5-24).

Step 2: Let H_1 and H_2 be two orthogonal hyperplanes through the origin, let $H_i^{\pm}$ be the half-spaces defined by H_i, and let $x_i^{\pm} \in \partial H_i$. For $i = 1, 2$, consider the functions

$$U[u_\varepsilon, H_i, x_i^+, x_i^-] = 1_{H_i^+}\, T_{x_i^+} u_\varepsilon + 1_{H_i^-}\, T_{x_i^-} u_\varepsilon$$

obtained by "gluing" the restriction of u_ε to H_i^+ translated by x_1^+ to the restriction of u_ε to H_i^- translated by x_1^+ (notice that translating by $x_i^{\pm}$ brings H_i^+ and H_i^- into themselves). Setting for brevity

$$U_{\varepsilon,i} = U[u_\varepsilon, H_i, x_i^+, x_i^-],$$

we claim that, for every $a \in (0, 1)$ there is $\kappa = \kappa(a, n, W) > 0$ such that if

$$\max\{|x_1^+ - x_1^-|, |x_2^+ - x_2^-|, |x_1^+ - x_2^+|\} \le \kappa, \tag{5-28}$$

then, for every $\varepsilon < \varepsilon_0$,

$$\max\{d_\Phi(T_{x_1^+} u_\varepsilon, T_{x_1^-} u_\varepsilon), d_\Phi(T_{x_2^+} u_\varepsilon, T_{x_2^-} u_\varepsilon)\} \le \frac{8}{1-a}\, d_\Phi(U_{\varepsilon,1}, U_{\varepsilon,2}). \tag{5-29}$$

Indeed, since H_1 and H_2 are hyperplanes through the origin and $u_\varepsilon \in \mathcal{R}_0$, we have

$$\int_{H_1^{\pm}} V(T_{x_1^{\pm}} u_\varepsilon) = \frac{1}{2}, \qquad \int_{H_2^{\pm}} V(T_{x_2^{\pm}} u_\varepsilon) = \frac{1}{2}.$$

It is in general not true that, say, $H_1^+ \cap H_2^+$ has measure $\frac{1}{4}$ for either $V(T_{x_1^{\pm}} u_\varepsilon)\, dx$ or $V(T_{x_1^{\pm}} u_\varepsilon)\, dx$. However, provided we choose κ sufficiently small, thanks to Theorem 3.1, we can definitely ensure that, for every $\varepsilon < \varepsilon_0$ and $\beta, \gamma \in \{+, -\}$, we have

$$\int_{H_1^\beta \cap H_2^\gamma} |\Phi(T_{x_1^\beta} u_\varepsilon) - \Phi(T_{x_2^\gamma} u_\varepsilon)|^{n/(n-1)} \ge \frac{1-a}{4} d_\Phi(T_{x_1^\beta} u_\varepsilon, T_{x_2^\gamma} u_\varepsilon).$$

Correspondingly,

$$\begin{aligned} d_\Phi(U_{\varepsilon,1}, U_{\varepsilon,2}) &\ge \int_{H_1^\beta \cap H_2^\gamma} |\Phi(U_{\varepsilon,1}) - \Phi(U_{\varepsilon,2})|^{n/(n-1)} \\ &= \int_{H_1^\beta \cap H_2^\gamma} |\Phi(T_{x_1^\beta} u_\varepsilon) - \Phi(T_{x_2^\gamma} u_\varepsilon)|^{n/(n-1)} \ge \frac{1-a}{4} d_\Phi(T_{x_1^\beta} u_\varepsilon, T_{x_2^\gamma} u_\varepsilon), \end{aligned}$$

and thus

$$\begin{aligned} d_\Phi(T_{x_1^+} u_\varepsilon, T_{x_1^-} u_\varepsilon)^{(n-1)/n} &\le d_\Phi(T_{x_1^+} u_\varepsilon, T_{x_2^+} u_\varepsilon)^{(n-1)/n} + d_\Phi(T_{x_2^+} u_\varepsilon, T_{x_1^-} u_\varepsilon)^{(n-1)/n} \\ &\le \Big(\frac{8}{1-a}\, d_\Phi(U_{\varepsilon,1}, U_{\varepsilon,2})\Big)^{(n-1)/n}, \end{aligned}$$

as claimed.

Step 3: Given $u \in H^1(\mathbb{R}^n; [0, 1])$ with $\int_{\mathbb{R}^n} V(u) = 1$, we now consider a hyperplane H such that, if H^+ and H^- denote the two open half-spaces defined by H, then

$$\int_{H^+} V(u) = \int_{H^-} V(u) = \frac{1}{2}.$$

Denoting by ρ_H the reflection with respect to H, we let

$$u^+ = 1_{H^+} u + 1_{H^-}(u \circ \rho_H), \quad u^- = 1_{H^-} u + 1_{H^+}(u \circ \rho_H), \tag{5-30}$$

and notice that $u^{\pm} \in H^1(\mathbb{R}^n; [0, 1])$, with

$$2\mathcal{AC}_\varepsilon(u) = \mathcal{AC}_\varepsilon(u^+) + \mathcal{AC}_\varepsilon(u^-), \quad \int_{\mathbb{R}^n} V(u^+) = \int_{\mathbb{R}^n} V(u^-) = 1. \tag{5-31}$$

We claim that

$$\max\{\delta_\varepsilon(u^+), \delta_\varepsilon(u^-)\} \le 2\delta_\varepsilon(u), \quad \alpha_\varepsilon(u) \le C(n), \{\alpha_\varepsilon(u^+) + \alpha_\varepsilon(u^-) + d_\Phi(T_{x^+}u_\varepsilon, T_{x^-}u_\varepsilon)\}, \tag{5-32}$$

provided $T_{x^+}u_\varepsilon =$ and $T_{x^-}u_\varepsilon = T_{x^-}u_\varepsilon$ are such that $x^+, x^- \in H$, with

$$\alpha_\varepsilon(u^+; H) = d_\Phi(u^+, T_{x^+}u_\varepsilon), \quad \alpha_\varepsilon(u^-; H) = d_\Phi(u^-, T_{x^-}u_\varepsilon).$$

The first inequality in (5-32) is obvious from (5-31). To prove the second one we notice that

$$\begin{aligned}
\alpha_\varepsilon(u) &\le d_\Phi(u, T_{x^+}u_\varepsilon) \\
&= \int_{H^+} |\Phi(u) - \Phi(T_{x^+}u_\varepsilon)|^{n/(n-1)} + \int_{H^-} |\Phi(u) - \Phi(T_{x^+}u_\varepsilon)|^{n/(n-1)} \\
&= \int_{H^+} |\Phi(u^+) - \Phi(T_{x^+}u_\varepsilon)|^{n/(n-1)} + \int_{H^-} |\Phi(u^-) - \Phi(T_{x^+}u_\varepsilon)|^{n/(n-1)} \\
&\le C(n)\{d_\Phi(u^+, T_{x^+}u_\varepsilon) + d_\Phi(u^-, T_{x^-}u_\varepsilon) + d_\Phi(T_{x^-}u_\varepsilon, T_{x^+}u_\varepsilon)\},
\end{aligned}$$

that is, the second inequality in (5-32).

With these preliminary considerations in place, we now prove that if $u \in H^1(\mathbb{R}^n; [0, 1])$ with $\int_{\mathbb{R}^n} V(u) = 1$, if H_1 and H_2 are orthogonal hyperplanes such that the corresponding half-spaces $H_i^{\pm}$ satisfy

$$\int_{H_i^{\pm}} V(u) = \frac{1}{2},$$

and if $u_i^{\pm}$ as in (5-30) starting from H_i, then there is at least one $v \in \{u_1^+, u_1^-, u_2^+, u_2^-\}$ such that (5-23) holds. Given that $\delta_\varepsilon(v) \le 2\delta_\varepsilon(u)$ for every $v \in \{u_1^+, u_1^-, u_2^+, u_2^-\}$, we need to show that

$$\text{there exists } v \in \{u_1^+, u_1^-, u_2^+, u_2^-\} \text{ such that } \alpha_\varepsilon(u) \le C\alpha_\varepsilon(v). \tag{5-33}$$

Denoting by $x_i^{\pm}$ the points in H_i such that

$$\alpha_\varepsilon(u_i^{\pm}; H_i) = d_\Phi(u_i^{\pm}, T_{x_i^{\pm}}u_\varepsilon),$$

we notice that (5-33) follows if we can show that, provided α_0 is small enough,

$$\text{either} \quad d_\Phi(T_{x_1^+}u_\varepsilon, T_{x_1^-}u_\varepsilon) \le M\{\alpha_\varepsilon(u_1^+; H_1) + \alpha_\varepsilon(u_1^-; H_1)\} \tag{5-34}$$

$$\text{or} \quad d_\Phi(T_{x_2^+}u_\varepsilon, T_{x_2^-}u_\varepsilon) \le M\{\alpha_\varepsilon(u_2^+; H_2) + \alpha_\varepsilon(u_2^-; H_2)\} \tag{5-35}$$

for a constant M (as it turns out, any $M > 16$ works). Indeed, if, for example, (5-34) holds, then (5-24) and (5-32) with $H = H_1$ give

$$\alpha_\varepsilon(u) \le C\{\alpha_\varepsilon(u_1^+) + \alpha_\varepsilon(u_1^-) + \alpha_\varepsilon(u_1^+; H_1) + \alpha_\varepsilon(u_1^-; H_1)\} \le C\{\alpha_\varepsilon(u_1^+) + \alpha_\varepsilon(u_1^-)\},$$

and then either $C\alpha_\varepsilon(u_1^+) \ge \alpha_\varepsilon(u)$ or $C\alpha_\varepsilon(u_2^+) \ge \alpha_\varepsilon(u)$; in particular, (5-33) holds. We now want to prove that either (5-34) or (5-35) holds. We argue by contradiction. Recalling that $\alpha_\varepsilon(u_i^{\pm}; H_i) = d_\Phi(u_i^{\pm}, T_{x_i^{\pm}}u_\varepsilon)$, let us thus assume that both

$$d_\Phi(T_{x_1^+}u_\varepsilon, T_{x_1^-}u_\varepsilon) > M\{d_\Phi(u_1^+, T_{x_1^+}u_\varepsilon) + d_\Phi(u_1^-, T_{x_1^-}u_\varepsilon)\}, \tag{5-36}$$

$$d_\Phi(T_{x_2^+}u_\varepsilon, T_{x_2^-}u_\varepsilon) > M\{d_\Phi(u_2^+, T_{x_2^+}u_\varepsilon) + d_\Phi(u_2^-, T_{x_2^-}u_\varepsilon)\} \tag{5-37}$$

hold for M to be determined. In particular, if $U_{\varepsilon,i}$, $i = 1, 2$, are defined as in Step 2, and α_0 is small enough that (5-28) holds, then, by (5-29), we have

$$\begin{aligned}
&\max\{d_\Phi(T_{x_1^+}u_\varepsilon, T_{x_1^-}u_\varepsilon), d_\Phi(T_{x_2^+}u_\varepsilon, T_{x_2^-}u_\varepsilon)\}^{(n-1)/n} \\
&\qquad\le \left(\frac{8}{1-a} d_\Phi(U_{\varepsilon,1}, U_{\varepsilon,2})\right)^{(n-1)/n} \le \left(\frac{8}{1-a}\right)^{(n-1)/n} \sum_{i=1}^{2} d_\Phi(U_{\varepsilon,i}, u)^{(n-1)/n} \\
&\qquad= \left(\frac{8}{1-a}\right)^{(n-1)/n} \sum_{i=1}^{2} \left(\sum_{\beta=+,-} \int_{H_i^\beta} |\Phi(T_{x_i^\beta}u_\varepsilon) - \Phi(u_i^\beta)|^{n/(n-1)}\right)^{(n-1)/n} \\
&\qquad\le \left(\frac{8}{M(1-a)}\right)^{(n-1)/n} \sum_{i=1}^{2} (d_\Phi(T_{x_i^+}u_\varepsilon, T_{x_i^-}u_\varepsilon))^{(n-1)/n} \\
&\qquad\le \left(\frac{16}{M\,(1-a)}\right)^{(n-1)/n} \max\{d_\Phi(T_{x_1^+}u_\varepsilon, T_{x_1^-}u_\varepsilon), d_\Phi(T_{x_2^+}u_\varepsilon, T_{x_2^-}u_\varepsilon)\}^{(n-1)/n}.
\end{aligned}$$

We fix $M > 16$ and apply the above with $a \in (0, 1)$ such that $M\,(1-a) > 16$. We find that either $x_1^+ = x_1^-$ (a contradiction to (5-36)) or $x_2^+ = x_2^-$ (a contradiction to (5-37)).

Step 4: We now pick a family of n mutually orthogonal hyperplanes $\{H_i\}_{i=1}^n$ such that, denoting by $H_i^\pm$ the corresponding half-spaces, we have

$$\int_{H_i^\pm} V(u) = \frac{1}{2} \quad \text{for all } i = 1, \dots, n.$$

Considering the hyperplanes in pairs and arguing inductively on Step 3, up to a relabeling we reduce to a situation where there exists a function v, symmetric by reflection with respect to each H_i, $i = 1, \dots, n-1$, and such that

$$\alpha_\varepsilon(u) \le C\alpha_\varepsilon(v), \quad \delta_\varepsilon(v) \le 2^n\,\delta_\varepsilon(v), \quad \int_{H_n^\pm} V(v) = \frac{1}{2}.$$

We can thus consider the functions $v^\pm$ obtained by reflecting v with respect to H_n as in Step 3. By (5-32) we have

$$\max\{\delta_\varepsilon(v^+), \delta_\varepsilon(v^-)\} \le 2\delta_\varepsilon(v), \quad \alpha_\varepsilon(u) \le C(n)\{\alpha_\varepsilon(v^+) + \alpha_\varepsilon(v^-) + d_\Phi(T_{x^+}u_\varepsilon, T_{x^-}u_\varepsilon)\},$$

where x^+ and x^- are optimal centers for $\alpha_\varepsilon\big(v^+; \bigcap_{i=1}^n H_i\big)$ and $\alpha_\varepsilon\big(v^-; \bigcap_{i=1}^n H_i\big)$. However, $\bigcap_{i=1}^n H_i$ *is a point*; therefore $x^+ = x^-$ and we have actually proved

$$\alpha_\varepsilon(u) \le C(n)\{\alpha_\varepsilon(v^+) + \alpha_\varepsilon(v^-)\}.$$

Either v^+ or v^- is an n-symmetric function with the required properties. □

5D. ***Proof of Theorem 5.1.*** We finally prove Theorem 5.1. By Theorem 5.5 we can directly assume that u is n-symmetric. Hence, by Theorem 5.4, we can directly assume that $u \in \mathcal{R}_0$. For $u \in \mathcal{R}_0$, the conclusion follows from Theorem 5.3. Theorem 5.1 is proved.

6. Proof of the Alexandrov-type theorem

In this section we complete the proof of Theorem 1.1, including in particular proof of the Alexandrov-type result of part (iv) of the statement. We begin by proving some of the properties of $\Psi(\sigma, m)$ stated in Theorem 1.1(ii) and not yet discussed. We then review, in Section 6B, some classical uniqueness and symmetry results for semilinear PDEs in relation to our setting. Finally, in Section 6C we review how the various results of the paper combine into Theorem 1.1.

6A. ***Some properties of*** $\boldsymbol{\Psi(\sigma, m)}$**.** We prove here the properties of $\Psi(\sigma, m)$ stated in Theorem 1.1(ii). As explained in the Introduction, these properties will be crucial in proving Theorem 1.1(iv).

Theorem 6.1. *If* $n \ge 2$ *and* $W \in C^{2,1}[0, 1]$ *satisfies* (1-11) *and* (1-12), *then there exists a universal constant* ε_0 *such that*, *setting*

$$\mathcal{X}(\varepsilon_0) = \{(\sigma, m) : 0 < \sigma < \varepsilon_0 m^{1/n}\},$$

the following hold:

(1) *For every* $\sigma > 0$, $\Psi(\sigma, \cdot)$ *is concave on* $(0, \infty)$; *it is strictly concave on* $(0, \infty)$ *in* $n \ge 3$ *and on* $((\sigma/\varepsilon_0)^n, \infty)$ *if* $n = 2$.

(2) $\Lambda(\sigma, m)$ *is continuous on* $\mathcal{X}(\varepsilon_0)$ *and*

$$|m^{1/n}\Lambda(\sigma, m) - 2(n-1)\omega_n^{1/n}| \le C\frac{\sigma}{m^{1/n}} \quad \textit{for all } (\sigma, m) \in \mathcal{X}(\varepsilon_0). \tag{6-1}$$

(3) $\Psi(\sigma, \cdot)$ *is differentiable with*

$$\frac{\partial \Psi}{\partial m}(\sigma, m) = \Lambda(\sigma, m) \quad \textit{for all } (\sigma, m) \in \mathcal{X}(\varepsilon_0). \tag{6-2}$$

In particular, *for every* $\sigma > 0$

$$\Psi(\sigma, \cdot) \textit{ is strictly increasing on } ((\sigma/\varepsilon_0)^n, \infty),$$
$$\Lambda(\sigma, \cdot) \textit{ is strictly decreasing } ((\sigma/\varepsilon_0)^n, \infty).$$

(4) *For every* $m > 0$, $\Psi(\cdot, m)$ *is increasing on* $(0, \varepsilon_0 m^{1/n})$.

Proof. We recall for convenience the scaling formulas

$$\int_{\mathbb{R}^n} f(\rho_t u) = \frac{1}{t}\int_{\mathbb{R}^n} f(u), \tag{6-3}$$

$$\int_{\mathbb{R}^n} |\nabla(\rho_t u)|^2 = t^{(2/n)-1}\int_{\mathbb{R}^n} |\nabla u|^2,$$

$$\mathcal{AC}_\varepsilon(\rho_t u) = \varepsilon t^{(2/n)-1}\int_{\mathbb{R}^n} |\nabla u|^2 + \frac{1}{\varepsilon t}\int_{\mathbb{R}^n} W(u) = \frac{\mathcal{AC}_{\varepsilon t^{1/n}}(u)}{t^{(n-1)/n}}, \tag{6-4}$$

$$\Psi(\sigma, m) = m^{(n-1)/n}\psi\left(\frac{\sigma}{m^{1/n}}\right),$$

where $\rho_t u(x) = u(t^{1/n}x)$ for $x \in \mathbb{R}^n$ and $t > 0$, and the divide the argument in a few steps.

Step 1: We prove the concavity of $\Psi(\sigma,\cdot)$. Given $m_2>m_1>0$, $t\in(0,1)$, $\sigma>0$, and a minimizing sequence $\{w_j\}_j$ for $\Psi(\sigma,tm_1+(1-t)m_2)$, we set

$$\alpha_1=\frac{tm_1+(1-t)\,m_2}{m_1},\qquad \alpha_2=\frac{tm_1+(1-t)m_2}{m_2},$$

so that $t/\alpha_1+(1-t)/\alpha_2=1$. Since $\rho_{\alpha_1}w_j$ and $\rho_{\alpha_2}w_j$ are competitors for $\Psi(\sigma,m_1)$ and $\Psi(\sigma,m_2)$ respectively, by the concavity of $t\mapsto t^{(n-2)/n}$ (strict if $n\geq 3$), we see that

$$\begin{aligned} t\Psi(\sigma,m_1)+(1-t)\Psi(\sigma,m_2) &\leq t\mathcal{AC}_\sigma(\rho_{\alpha_1}w_j)+(1-t)\mathcal{AC}_\sigma(\rho_{\alpha_2}w_j) && \text{(6-5)}\\ &=\frac{t}{\alpha_1}\left(\int_{\mathbb{R}^n}\sigma\alpha_1^{2/n}|\nabla w_j|^2+\frac{W(w_j)}{\sigma}\right)+\frac{1-t}{\alpha_2}\left(\int_{\mathbb{R}^n}\sigma\alpha_2^{2/n}|\nabla w_j|^2+\frac{W(w_j)}{\sigma}\right)\\ &=\mathcal{AC}_\sigma(w_j)+\left(t\left(\frac{1}{\alpha_1}\right)^{(n-2)/n}+(1-t)\left(\frac{1}{\alpha_2}\right)^{(n-2)/n}-1\right)\sigma\int_{\mathbb{R}^n}|\nabla w_j|^2 && \text{(6-6)}\\ &\leq\mathcal{AC}_\sigma(w_j). && \text{(6-7)}\end{aligned}$$

Letting $j\to\infty$ we deduce the concavity of $\Psi(\sigma,\cdot\,)$ on $(0,\infty)$ (strict, if $n\geq 3$). If $n=2$ and $m_1\geq(\sigma/\varepsilon_0)^n$, then by Theorem 2.1 we can replace the minimizing sequence $\{w_j\}_j$ in the above argument with a minimizer w of $\Psi(\sigma,t\,m_1+(1-t)\,m_2)$. Since w solves the Euler–Lagrange equation (1-9), there cannot be a $t\neq 1$ such that $\rho_t w$ solves (1-9) with the same σ and some $t\in\mathbb{R}$. Thus, $\rho_{\alpha_i}w$ cannot be a minimizer of $\Psi(\sigma,m_i)$, and therefore we have a strict inequality in (6-5), and no need to take a limit in (6-7) (since $\mathcal{AC}_\sigma(w)=\Psi(\sigma,tm_1+(1-t)m_2)$).

Step 2: By Theorem 2.1 and Corollary 4.2 for every $m>0$ and $\sigma<\varepsilon_0 m^{1/n}$ there exists a unique $u_{\sigma,m}\in\mathcal{R}_0$ such that $u_{\sigma,m}$ is a minimizer of $\Psi(\sigma,m)$ and every other minimizer of $\Psi(\sigma,m)$ is a translation of $u_{\sigma,m}$. Moreover, there is $\Lambda(\sigma,m)>0$ such that

$$-2\sigma^2\Delta u_{\sigma,m}=\sigma\Lambda(\sigma,m)V'(u_{\sigma,m})-W'(u_{\sigma,m})\quad\text{on }\mathbb{R}^n.$$

Hence, if u_ε denotes as usual the unique minimizer of $\psi(\varepsilon)$ in $\mathcal{R}_0$, then by (6-3) and (6-4) we find

$$u_{\sigma,m}=\rho_{1/m}\,u_\varepsilon,\qquad \varepsilon=\frac{\sigma}{m^{1/n}},$$

and thus

$$\Lambda(\sigma,m)=\frac{\lambda(\varepsilon)}{m^{1/n}},\qquad \varepsilon=\frac{\sigma}{m^{1/n}}. \tag{6-8}$$

By combining (6-8) with Corollary 4.2 and with (4-7) we thus find that Λ is continuous on $\mathcal{X}(\varepsilon_0)$, with

$$\left|\Lambda(\sigma,m)-\frac{2(n-1)\omega_n^{1/n}}{m^{1/n}}\right|\leq C\frac{\sigma}{m^{2/n}}. \tag{6-9}$$

Step 3: We prove statement (iii). For $(\sigma,m)\in\mathcal{X}(\varepsilon_0)$, we set

$$a(t)=\mathcal{AC}_\sigma((1+t)u_{\sigma,m}),\qquad m(t)=\int_{\mathbb{R}^n}V((1+t)u_{\sigma,m}).$$

Then

$$m'(0)=\frac{n}{n-1}\int_{\mathbb{R}^n}\Phi(u_{\sigma,m})^{1/(n-1)}\sqrt{W(u_{\sigma,m})}u_{\sigma,m}>0$$

and thus there exist $t_* > 0$ and an open interval I of m such that m is strictly increasing from $(-t_*, t_*)$ to I with $m(0) = m$. From $\Psi(\sigma, m(t)) \le a(t)$ for every $|t| < t_*$ and from that fact that a is differentiable on $(-t_*, t_*)$ we deduce that, if m is such that $\Psi(\sigma, \cdot\,)$ is differentiable at m, then

$$\frac{\partial \Psi}{\partial m}(\sigma, m) = \frac{a'(0)}{m'(0)} = \frac{\int_{\mathbb{R}^n} 2\sigma\, \nabla u_{\sigma,m} \cdot \nabla u_{\sigma,m} + (1/\sigma) W'(u_{\sigma,m}) u_{\sigma,m}}{\int_{\mathbb{R}^n} V'(u_{\sigma,m}) u_{\sigma,m}} = \Lambda(\sigma, m).$$

Now, by statement (i), $\Psi(\sigma, \cdot\,)$ is differentiable a.e. on $((\sigma/\varepsilon_0)^n, \infty)$, as well as absolutely continuous, while $\Lambda(\sigma, \cdot\,)$ is continuous on $((\sigma/\varepsilon_0)^n, \infty)$: by the fundamental theorem of calculus we thus conclude that $(\partial\Psi/\partial m)(\sigma, \cdot\,)$ exists for every $m > (\sigma/\varepsilon_0)^n$ and agrees with $\Lambda(\sigma, m)$.

Step 4: We prove statement (iv). Recalling that

$$\Psi(\sigma, m) = m^{(n-1)/n} \psi\left(\frac{\sigma}{m^{1/n}}\right) \quad \text{for all } \sigma, m > 0, \tag{6-10}$$

we see that, since $\Psi(\sigma, \cdot\,)$ is differentiable on $((\sigma/\varepsilon_0)^n, \infty)$, we know ψ is differentiable on $(0, \varepsilon_0)$. Since ψ is differentiable on $(0, \varepsilon_0$, by (6-10) we see that $\Psi(\,\cdot, m)$ is differentiable on $(0, \varepsilon_0 m^{1/n})$ for every $m > 0$, with

$$\frac{\partial \Psi}{\partial \sigma} = m^{(n-2)/n} \psi'\left(\frac{\sigma}{m^{1/n}}\right).$$

Statement (iv) will thus follow by proving that $\psi' > 0$ on $(0, \varepsilon_0)$. To derive a useful formula for ψ we differentiate (6-10) in m and use (6-2) and $\lambda(\sigma/m^{1/n}) = m^{1/n}\Lambda(\sigma, m)$ to find that

$$\frac{n-1}{n} \frac{1}{m^{1/n}} \psi\left(\frac{\sigma}{m^{1/n}}\right) - \frac{1}{n} \frac{\sigma}{m^{2/n}} \psi'\left(\frac{\sigma}{m^{1/n}}\right) = \frac{\lambda(\sigma/m^{1/n})}{m^{1/n}}.$$

In particular, by (4-5),

$$\varepsilon \psi'(\varepsilon) = (n-1)\psi(\varepsilon) - n\lambda(\varepsilon) = \varepsilon \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon} \int_{\mathbb{R}^n} W(u_\varepsilon).$$

By (3-8), if we set $\eta_\varepsilon(s) = \eta(s - \tau_\varepsilon)$ and change variables according to $|x| = R_0 + \varepsilon s$ we find

$$\varepsilon \psi'(\varepsilon) = \int_{-R_0/\varepsilon}^{\infty} \{(\eta_\varepsilon' + f_\varepsilon')^2 - W(\eta_\varepsilon + f_\varepsilon)\}(R_0 + \varepsilon s)^{n-1}\, ds. \tag{6-11}$$

Multiplying by u_ε' and then integrating on (r, ∞) the Euler–Lagrange equation

$$-2\varepsilon^2 \left\{ u_\varepsilon'' + (n-1)\frac{u_\varepsilon'}{r} \right\} = \varepsilon\lambda(\varepsilon) V'(u_\varepsilon) - W'(u_\varepsilon),$$

we obtain as usual

$$\varepsilon^2 (u_\varepsilon')^2 - 2(n-1)\varepsilon^2 \int_r^\infty \frac{(u_\varepsilon')^2}{\rho}\, d\rho = W(u_\varepsilon) - \varepsilon\lambda(\varepsilon) V(u_\varepsilon)$$

for every $r > 0$; by the change of variables $r = R_0 + \varepsilon s$ we thus find

$$(\eta_\varepsilon' + f_\varepsilon')^2 - 2(n-1)\varepsilon \int_s^\infty \frac{(\eta_\varepsilon' + f_\varepsilon')^2}{R_0 + \varepsilon t}\, dt = W(\eta_\varepsilon + f_\varepsilon) - \lambda(\varepsilon)\varepsilon V(\eta_\varepsilon + f_\varepsilon)$$

for every $s \in (-R_0/\varepsilon, \infty)$. We combine this identity into (6-11) to find

$$\varepsilon\psi'(\varepsilon) = \int_{-R_0/\varepsilon}^{\infty} \left\{ 2(n-1)\varepsilon \int_s^\infty \frac{(\eta_\varepsilon' + f_\varepsilon')^2}{R_0 + \varepsilon\, t}\, dt - \lambda(\varepsilon)\varepsilon V(\eta_\varepsilon + f_\varepsilon) \right\} (R_0 + \varepsilon s)^{n-1}\, ds. \tag{6-12}$$

We now notice that, by (A-16), (A-18), and (3-9) (that is, by the exponential decay of η, η', η'' and by $|f_\varepsilon(s)| \le C\varepsilon e^{-|s|/C\varepsilon}$ for $s > -R_0/\varepsilon$), we have

$$\begin{aligned} \int_s^\infty \frac{(\eta_\varepsilon' + f_\varepsilon')^2 - (\eta_\varepsilon')^2}{R_0 + \varepsilon t}\, dt &\ge 2\int_s^\infty \frac{\eta_\varepsilon' f_\varepsilon'}{R_0 + \varepsilon t}\, dt \\ &= -2\frac{\eta_\varepsilon'(s) f_\varepsilon(s)}{R_0 + \varepsilon s} - 2\int_s^\infty f_\varepsilon(s) \left(\frac{\eta_\varepsilon'}{R_0 + \varepsilon\, t} \right)' dt \ge -C\varepsilon e^{-|s|/C} \end{aligned}$$

so that (6-12) gives

$$\varepsilon\psi'(\varepsilon) \ge \int_{-R_0/\varepsilon}^{\infty} \left\{ 2(n-1)\varepsilon \int_s^\infty \frac{(\eta_\varepsilon')^2\, dt}{R_0 + \varepsilon t} - \lambda(\varepsilon)\varepsilon V(\eta_\varepsilon + f_\varepsilon) \right\} (R_0 + \varepsilon s)^{n-1}\, ds - C\varepsilon^2. \tag{6-13}$$

By (4-7), (3-9), $R_0 = \omega_n^{-1/n}$ and (6-13), we have

$$\psi'(\varepsilon) \ge 2(n-1)\omega_n^{1/n} \int_{-R_0/\varepsilon}^{\infty} \left\{ \int_s^\infty (\eta_\varepsilon')^2\, dt - V(\eta_\varepsilon) \right\} (R_0 + \varepsilon s)^{n-1}\, ds - C\varepsilon. \tag{6-14}$$

Since $\int_s^\infty (\eta_\varepsilon')^2 = \Phi(\eta_\varepsilon(s))$ thanks to $\eta_\varepsilon' = -\sqrt{W(\eta_\varepsilon)} = -\Phi'(\eta_\varepsilon)$, by (6-14) we have

$$\begin{aligned} \psi'(\varepsilon) &\ge 2(n-1)\omega_n^{1/n} \int_{\mathbb{R}} (\Phi(\eta_\varepsilon) - V(\eta_\varepsilon))(R_0 + \varepsilon s)^{n-1}\, ds - C\varepsilon \\ &\ge 2(n-1)\omega_n^{1/n} R_0^{n-1} \int_{\mathbb{R}} (\Phi(\eta) - V(\eta))\, ds - C\varepsilon. \end{aligned}$$

Since Φ takes values in $(0, 1)$, $V = \Phi^{n/(n-1)} < \Phi$ on $(0, 1)$, and

$$\int_{\mathbb{R}} (\Phi(\eta) - V(\eta))\, ds$$

is a universal constant. In particular, $\psi'(\varepsilon) \ge 1/C$ for every $\varepsilon < \varepsilon_0$. □

6B. ***General criteria for radial symmetry and uniqueness.*** In this brief section we exploit two classical results from [Gidas et al. 1981; Peletier and Serrin 1983] to deduce a symmetry and uniqueness result for the kind of semilinear PDE arising as the Euler–Lagrange equation of $\Psi(\sigma, m)$.

Theorem 6.2. *Let $n \ge 2$, let $W \in C^{2,1}[0, 1]$ satisfy* (1-11) *and* (1-12), *and consider $\ell \in \mathbb{R}$ and $\sigma > 0$.*

(1) *If $u \in C^2(\mathbb{R}^n; [0, 1])$ is a nonzero solution to*

$$-2\sigma^2 \Delta u = \sigma\ell V'(u) - W'(u) \quad \text{on } \mathbb{R}^n, \tag{6-15}$$

with $u(x) \to 0$ as $|x| \to \infty$, then $0 < u < 1$ on $\mathbb{R}^n$ and $u \in \mathcal{R}_0^*$.

(2) *There exists a universal constant ν_0 such that, if $0 < \sigma\,\ell < \nu_0$, then, modulo translation,* (6-15) *has a unique solution among functions $u \in \mathcal{R}_0^*$, with $u(x) \to 0$ as $|x| \to \infty$ and $0 < u < 1$ on $\mathbb{R}^n$.*

Remark. Notice that the smallness of $\sigma\ell$ is required only for proving statement (ii).

Proof. Step 1: We prove statement (i). We intend to apply the following particular case of [Gidas et al. 1981, Theorem 2]: *if* $n \ge 2$, $u \in C^2(\mathbb{R}^n; [0,1])$, $u > 0$ *on* $\mathbb{R}^n$, $u(x) \to 0$ *as* $|x| \to \infty$, $-\Delta u + m\, u = g(u)$ *on* $\mathbb{R}^n$, *with* $m > 0$ *and* $g \in C^1[0,1]$ *with* $g(t) = \mathrm{O}(t^{1+\alpha})$ *as* $t \to 0^+$ *for some* $\alpha > 0$, *then, up to translations,* $u \in \mathcal{R}_0^*$.

To this end we reformulate (6-15) as

$$-\Delta u + mu = g(u) \quad \text{on } \mathbb{R}^n, \tag{6-16}$$

where $m = W''(0)/(2\sigma^2) > 0$ and

$$g(t) = \frac{\ell V'(t)}{2\sigma} + \frac{W''(0)t - W'(t)}{2\sigma^2}, \quad t \in [0,1].$$

As noticed in Section A3, $V \in C^{2,\gamma}[0,1]$ for some $\gamma \in (0,1]$, while $W \in C^{2,1}[0,1]$: in particular $g \in C^1[0,1]$. By $W \in C^{2,1}[0,1]$ with $W'(0) = 0$ we have $|W'(t) - W''(0)t| \le Ct^2$, while (A-11) states that $|V'(t)| \le Ct^{1+\alpha}$ for $t \in [0,1]$ for some $\alpha > 0$, so that

$$|g(t)| \le C(n, W, \ell, \sigma)t^{1+\alpha} \quad \text{for all } t \in [0,1]. \tag{6-17}$$

To check that $u > 0$ on $\mathbb{R}^n$, we notice that, by (6-17), for every $m' \in (0, m)$, we can find $t_0 > 0$ such that (6-16) implies that $-\Delta u + m'\, u \ge 0$ on the open set $\{u < t_0\}$. Since $u \ge 0$ and u is nonzero, we conclude by the strong maximum principle that $u > 0$ on $\{u < t_0\}$, and thus, on $\mathbb{R}^n$. We are thus in the position to apply the stated particular case of [Gidas et al. 1981, Theorem 2] and conclude that $u \in \mathcal{R}_0^*$.

We prove that $u < 1$ on $\mathbb{R}^n$. Let us set

$$f(t) = \frac{\ell V'(t)}{2\sigma} - \frac{W'(t)}{2\sigma^2}, \quad t \in [0,1], \tag{6-18}$$

and notice that (6-15) is equivalent to $-\Delta u = f(u)$ on $\mathbb{R}^n$. Since f is a Lipschitz function on $[0,1]$ with $f(1) = 0$, we can find $c > 0$ such that $f(t) + ct$ is increasing on $[0,1]$, and rewrite $-\Delta u = f(u)$ as

$$-\Delta(1-u) + c(1-u) = (f(t) + ct)\big|_{t=u}^{t=1} \ge 0.$$

We thus conclude that $v = 1 - u$ is nonnegative on $\mathbb{R}^n$ and such that $-\Delta v + cv \ge 0$. Since v is nonzero (thanks to $u(x) \to 0$ as $|x| \to \infty$), by the strong maximum principle we conclude that $v > 0$ on $\mathbb{R}^n$, i.e., $u < 1$ on $\mathbb{R}^n$.

Step 2: We prove statement (ii). We intend to use [Peletier and Serrin 1983, Theorem 2]: *if*

(a) f *locally Lipschitz on* $(0, \infty)$,

(b) $f(t)/t \to -m$ *as* $t \to 0^+$ *where* $m > 0$,

(c) *setting* $F(t) = \int_0^t f(s)\, ds$, *there exists* $\delta > 0$ *such that* $F(\delta) > 0$,

(d) *setting* $\beta = \inf\{t > 0 : F(t) > 0\}$ (*so that by* (b) *and* (c), $\beta \in (0, \delta)$), *the function* $t \mapsto f(t)/(t - \beta)$ *is decreasing on* $(\beta, \infty) \cap \{f > 0\}$,

then there is at most one $u \in C^2(\mathbb{R}^n) \cap \mathcal{R}_0$, *with* $u > 0$ *on* $\mathbb{R}^n$ *and* $u(x) \to 0$ *as* $|x| \to \infty$, *solving* $-\Delta u = f(u)$ *on* $\mathbb{R}^n$.

Since, by statement (i), solutions to (6-15) satisfy $0 < u < 1$ on $\mathbb{R}^n$, in checking that f as in (6-18) satisfies the above assumptions it is only the behavior of f on $(0, 1)$ (and not on $(0, \infty)$) that matters. Evidently (a) holds, since $f \in C^{1,\alpha}[0, 1]$ for some $\alpha \in (0, 1)$. Assumption (b) holds with $m = W''(0)/(2\sigma^2)$. Property (c) holds (with $\delta \in (0, 1)$) since

$$F(t) = \int_0^t f(s)\, ds = \frac{\ell V(t)}{2\sigma} - \frac{W(t)}{2\sigma^2}, \quad t \in [0, 1],$$

and $F(1) = (\ell V(1)/2\sigma) = \ell/2\sigma > 0$ by $\ell > 0$ and $W(1) = 0$. We finally prove (d). Notice that, clearly, $\beta \in (0, 1)$ and, by the continuity of F, $F(\beta) = 0$, so that, taking (A-3) and (A-6) into account, and using $\sigma\,\ell < \nu_0$ and $V(1) = 1$,

$$\frac{\min\{\beta^2, (1-\beta)^2\}}{C} \le W(\beta) = \sigma \ell V(\beta) \le \nu_0. \tag{6-19}$$

If $\nu_0 < 1$, then by (A-6) and (A-11) we find

$$2\sigma^2 F(t) = \sigma \ell V(t) - W(t) \le V(t) - W(t) \le C t^{2n/(n-1)} - \frac{t^2}{C} < 0 \quad \text{for all } t \in (0, \delta_0). \tag{6-20}$$

By (6-20) it must be $\beta \ge \delta_0$. Hence, by (6-19), if ν_0 is sufficiently small, then $(1-\beta)^2 \le C\nu_0$. Up to further decreasing the value of ν_0, we can finally get that $(\beta, 1) \subset (1-\delta_0, 1)$, with δ_0 as in Section A3.

We are now going to check property (d) by showing that

$$f'(t)(t-\beta) \le f(t) \quad \text{for all } t \in (\beta, 1) \tag{6-21}$$

(recall that $0 < u < 1$ on $\mathbb{R}^n$, so we can use a version of [Peletier and Serrin 1983, Theorem 2] localized to $(0, 1)$). Using the explicit formula for f, (6-21) is equivalent to

$$\sigma \ell V''(t)(t-\beta) \le \sigma \ell V'(t) - W'(t) + W''(t)(t-\beta) \quad \text{for all } t \in (\beta, 1). \tag{6-22}$$

By (A-6), we have $W''(t)(t-\beta) > 0$ on $(\beta, 1) \subset (1-\delta_0, 1)$, and since $V' \ge 0$ on $[0, 1]$, (6-22) is implied by checking that, for every $t \in (\beta, 1)$,

$$-W'(t) \ge \sigma \ell V''(t) = \sigma \ell \left\{ \frac{n}{(n-1)^2} \frac{W(t)}{\left(\int_0^t \sqrt{W}\right)^{(n-2)/(n-1)}} + \frac{n}{n-1} \left(\int_0^t \sqrt{W} \right)^{1/(n-1)} \frac{W'(t)}{2\sqrt{W(t)}} \right\}.$$

In turn, since $W' < 0$ on $(1-\delta_0, 1)$ and $\sigma\,\ell < \nu_0 < 1$, it is actually enough to check that

$$-W'(t) \ge \frac{n}{(n-1)^2} \frac{W(t)}{\left(\int_0^t \sqrt{W}\right)^{(n-2)/(n-1)}} \quad \text{for all } t \in (1-\delta_0, 1).$$

But, up to further decreasing the value of δ_0, this is obvious: indeed (A-6) gives $-W'(t) \ge (1-t)/C$ and $W(t) \le C(1-t)^2$ for every $t \in (1-\delta_0, 1)$. $\square$

6C. ***Proof of Theorem 1.1.*** Theorem 2.1, Corollary 4.2, Theorem 6.1 and a scaling argument show the validity of statements (i) and (ii), while statement (iii) follows similarly by scaling and by Theorem 5.1.

To prove the Alexandrov-type theorem, that is, statement (iv)[5] we consider $u \in C^2(\mathbb{R}^n; [0, 1])$, with $u(x) \to 0$ as $|x| \to \infty$, and solving

$$-2\sigma^2 \Delta u = \sigma \ell V'(u) - W'(u) \quad \text{on } \mathbb{R}^n, \tag{6-23}$$

for some σ and ℓ with $0 < \sigma\ell < \nu_0$. By Theorem 6.2(i), $u \in \mathcal{R}_0^*$, and by Theorem 6.2, provided ν_0 is small enough, we know that there is at most one radial solution to (6-23). Since we know that $u_{\sigma,m}$ is a radial solution of (6-23) with $\ell = \Lambda(\sigma, m)$, we are left to prove that for every $\ell \in (0, \nu_0/\sigma)$ there exists a unique $m \in ((\sigma/\varepsilon_0)^n, \infty)$ such that $\Lambda(\sigma, m) = \ell$.

To this end, we first notice that, by (4-7) and by scaling, for every $\sigma > 0$ we have

$$\Lambda(\sigma, m) = \frac{1}{m^{1/n}} \lambda\left(\frac{\sigma}{m^{1/n}}\right) \to 0^+ \quad \text{as } m \to +\infty.$$

In particular, since, by Theorem 6.1, $\Lambda(\sigma, \cdot)$ is continuous and strictly decreasing on $((\sigma/\varepsilon_0)^n, \infty)$, we have

$$\left\{\Lambda(\sigma, m) : m > \left(\frac{\sigma}{\varepsilon_0}\right)^n\right\} = \left(0, \Lambda\left(\sigma, \left(\frac{\sigma}{\varepsilon_0}\right)^n\right)\right).$$

Now, setting $m = (\sigma/\varepsilon_0)^n$ in (6-1), that is, in

$$|m^{1/n}\Lambda(\sigma, m) - 2(n-1)\omega_n^{1/n}| \le C\frac{\sigma}{m^{1/n}},$$

we find that

$$\left|\sigma\Lambda\left(\sigma, \left(\frac{\sigma}{\varepsilon_0}\right)^n\right) - 2(n-1)\omega_n^{1/n}\varepsilon_0\right| \le C\varepsilon_0^2,$$

which implies

$$\Lambda\left(\sigma, \left(\frac{\sigma}{\varepsilon_0}\right)^n\right) \ge \frac{(n-1)\omega_n^{1/n}\varepsilon_0}{\sigma} \quad \text{for all } \sigma > 0,$$

provided ε_0 is small enough. Up to further decreasing the value of ν_0 so to have $\nu_0 \le (n-1)\,\omega_n^{1/n}\,\varepsilon_0$, we have proved that

$$\left(0, \frac{\nu_0}{\sigma}\right) \subset \left\{\Lambda(\sigma, m) : m > \left(\frac{\sigma}{\varepsilon_0}\right)^n\right\},$$

and that for each $\ell \in (0, \nu_0/\sigma)$ there is a unique $m > (\sigma/\varepsilon_0)^n$ such that $\ell = \Lambda(\sigma, m)$, as claimed. This completes the proof of Theorem 1.1.

Appendix: Frequently used auxiliary facts

A1. *Scaling identities.* If $u \in H^1(\mathbb{R}^n; [0, \infty))$, $t > 0$, we set

$$\rho_t\, u(x) = u(t^{1/n}\, x), \qquad x \in \mathbb{R}^n,$$

and notice that

$$\int_{\mathbb{R}^n} f(\rho_t u) = \frac{1}{t}\int_{\mathbb{R}^n} f(u), \qquad \int_{\mathbb{R}^n} |\nabla(\rho_t u)|^2 = t^{(2/n)-1}\int_{\mathbb{R}^n} |\nabla u|^2, \tag{A-1}$$

[5]Notice that we are using ℓ in (6-23) rather than λ (as done in (1-22)) to denote the Lagrange multiplier of u. This is meant to avoid confusion with the function $\lambda(\varepsilon) = (\partial\Psi/\partial m)(\varepsilon, 1)$ appearing in the argument.

$$\mathcal{AC}_\varepsilon(\rho_t u) = \varepsilon t^{(2/n)-1} \int_{\mathbb{R}^n} |\nabla u|^2 + \frac{1}{\varepsilon t} \int_{\mathbb{R}^n} W(u) = \frac{\mathcal{AC}_{\varepsilon t^{1/n}}(u)}{t^{(n-1)/n}}. \tag{A-2}$$

whenever $f : \mathbb{R} \to \mathbb{R}$ is continuous.

A2. ***Concentration-compactness principle.*** Denoting by $B_r(x)$ the ball of center x and radius r in $\mathbb{R}^n$, and setting $B_r = B_r(0)$ when $x = 0$, we provide a reference statement for Lions' concentration-compactness criterion, which is repeatedly used in our arguments: *if $\{\mu_j\}_j$ is a sequence of probability measures in $\mathbb{R}^n$, then, up to extracting subsequences and composing each μ_j with a translation, one of the following mutually excluding possibilities holds*:

<u>Compactness case</u>: *for every $\tau > 0$ there exists $R > 0$ such that*

$$\inf_j \mu_j(B_R) \geq 1 - \tau.$$

<u>Vanishing case</u>: *for every $R > 0$,*

$$\lim_{j\to\infty} \sup_{x\in\mathbb{R}^n} \mu_j(B_R(x)) = 0.$$

<u>Dichotomy case</u>: *there exists $\alpha \in (0, 1)$ such that for every $\tau > 0$ one can find $S > 0$ with $S_j \to \infty$ such that*

$$\sup_j |\alpha - \mu_j(B_S)| < \tau, \qquad \sup_j |(1-\alpha) - \mu_j(\mathbb{R}^n \setminus B_{S_j})| < \tau.$$

Notice that the formulation of the dichotomy case used here is a bit more descriptive than the original one presented in [Lions 1984, Lemma I]. Its validity is inferred by a quick inspection of the proof presented in the cited reference.

A3. ***Estimates for W, Φ and V.*** Throughout the paper we work with a double-well potential $W \in C^{2,1}[0, 1]$ satisfying (1-11) and (1-12), that is,

$$W(0) = W(1) = 0, \quad W > 0 \text{ on } (0, 1), \quad W''(0), W''(1) > 0, \tag{A-3}$$

$$\int_0^1 \sqrt{W} = 1. \tag{A-4}$$

Frequently used properties of W are the validity, for a universal constant C, of the expansion

$$\left| W(b) - W(a) - W'(a)(b-a) - W''(a)\frac{(b-a)^2}{2} \right| \leq C|b-a|^3 \quad \text{for all } a, b \in [0, 1], \tag{A-5}$$

and the existence of a universal constant $\delta_0 < \frac{1}{2}$ such that

$$\begin{aligned} &\frac{1}{C} \leq \frac{W}{t^2}, \frac{W'}{t}, W'' \leq C && \text{on } (0, \delta_0], \\ &\frac{1}{C} \leq \frac{W}{(1-t)^2}, \frac{-W'}{1-t}, W'' \leq C && \text{on } [1-\delta_0, 1). \end{aligned} \tag{A-6}$$

We can use (A-6) to quantify the behaviors near the wells of Φ and, crucially, of V. We first notice that, by (A-3), $\Phi \in C^3_{\mathrm{loc}}(0, 1)$, with

$$\Phi' = \sqrt{W}, \quad \Phi'' = \frac{W'}{2\sqrt{W}}, \quad \Phi''' = \frac{W''}{2\sqrt{W}} - \frac{(W')^2}{4W^{3/2}} \quad \text{on } (0, 1).$$

By (A-6) and (A-4) we thus see that Φ satisfies

$$\begin{aligned} &\frac{1}{C} \le \frac{\Phi}{t^2}, \frac{\Phi'}{t}, \Phi'' \le C && \text{on } (0, \delta_0], \\ &\frac{1}{C} \le \frac{1-\Phi}{(1-t)^2}, \frac{\Phi'}{1-t}, -\Phi'' \le C && \text{on } [1-\delta_0, 1), \end{aligned} \tag{A-7}$$

from which we easily deduce

$$|\Phi(b) - \Phi(a)| \ge \frac{(b-a)^2}{C} \quad \text{for all } a, b \in [0, 1]. \tag{A-8}$$

Moreover, by exploiting (A-7) and setting for brevity $a = W''(0)$, we see that as $t \to 0^+$

$$\Phi''' = \frac{2W''W - (W')^2}{4W^{3/2}} = \frac{2(a + \mathrm{O}(t))(a(t^2/2) + \mathrm{O}(t^3)) - (at + \mathrm{O}(t^2))^2}{4(a(t^2/2) + \mathrm{O}(t^3))^{3/2}} = \frac{\mathrm{O}(t^3)}{4a^{3/2}t^3 + \mathrm{o}(t^3)},$$

and a similar computation holds for $t \to 1^-$, so that

$$|\Phi'''| \le C \quad \text{on } (0, \delta_0) \cup (1-\delta_0, 1). \tag{A-9}$$

By (A-7) and (A-9) we see that $\Phi \in C^{2,1}[0, 1]$ with a universal estimate on its $C^{2,1}[0, 1]$-norm: in particular,

$$\left|\Phi(b) - \Phi(a) - \Phi'(a)(b-a) - \Phi''(a)\frac{(b-a)^2}{2}\right| \le C|b-a|^3 \quad \text{for all } a, b \in (0, 1). \tag{A-10}$$

Since $V = \Phi^{1+\alpha}$ for $\alpha = 1/(n-1) \in (0, 1]$ (recall that $n \ge 2$) and $\Phi(t) = 0$ if and only if $t = 0$, we easily see that $V \in C^3_{\mathrm{loc}}(0, 1)$, with

$$V' = (1+\alpha)\Phi^\alpha \Phi', \quad V'' = (1+\alpha)\left\{\alpha \frac{(\Phi')^2}{\Phi^{1-\alpha}} + \Phi^\alpha \Phi''\right\}, \quad |V'''| \le C(\alpha)\left\{\frac{(\Phi')^3}{\Phi^{2-\alpha}} + \frac{\Phi'|\Phi''|}{\Phi^{1-\alpha}} + \Phi^\alpha |\Phi'''|\right\}.$$

By (A-10), and keeping track of the sign of Φ'' and of the fact that negative powers of $\Phi(t)$ are large only near $t = 0$, but are bounded near $t = 1$, we find that

$$\begin{aligned} &\frac{1}{C} \le \frac{V}{t^{2+2\alpha}}, \frac{V'}{t^{1+2\alpha}}, \frac{V''}{t^{2\alpha}} \le C, && |V'''| \le \frac{C}{t^{1-2\alpha}} && \text{on } (0, \delta_0], \\ &\frac{1}{C} \le \frac{1-V}{(1-t)^2}, \frac{V'}{1-t} \le C, && |V''|, |V'''| \le C && \text{on } [1-\delta_0, 1). \end{aligned} \tag{A-11}$$

In particular, $V \in C^{2,\gamma(n)}[0, 1]$, $\gamma(n) = \min\{1, 2/(n-1)\} \in (0, 1]$, with second-order Taylor expansions of the form

$$\left|V(b) - V(a) - V'(a)(b-a) - V''(a)\frac{(b-a)^2}{2}\right| \le C|b-a|^{2+\gamma(n)} \quad \text{for all } a, b \in (0, 1). \tag{A-12}$$

We finally notice that we can find a universal constant C such that

$$\frac{t^2}{C} \le W(t), \quad V(t) \le Ct^2, \quad V(t) \le CW(t) \quad \text{for all } t \in (0, 1-\delta_0) \tag{A-13}$$

(as it is easily deduced from the bounds on W and V in (A-6) and (A-11) and from the fact that $W > 0$ on $(0, 1)$), and that we can also find C so that

$$V(t) \geq \frac{1}{C} \quad \text{for all } t \in (\delta_0, 1). \tag{A-14}$$

A4. ***Estimates for the optimal transition profile η.*** A crucial object in the analysis of the Allen–Cahn energy is of course the optimal transition profile η, defined by the first-order ODE

$$\begin{cases} \eta' = -\sqrt{W(\eta)} & \text{on } \mathbb{R}, \\ \eta(0) = \frac{1}{2}, \end{cases} \tag{A-15}$$

which can be seen to satisfy (see, e.g., [Leoni and Murray 2016]) $\eta \in C^{2,1}(\mathbb{R})$, $\eta' < 0$ on $\mathbb{R}$ (and $-C \leq \eta' \leq -1/C$ for $|s| \leq 1$), $\eta(-\infty) = 1$, and $\eta(+\infty) = 0$, with the exponential decay properties

$$1 - \eta(s) \leq C e^{s/C} \quad \text{for all } s < 0, \qquad \eta(s) \leq C e^{-s/C} \quad \text{for all } s > 0, \tag{A-16}$$

for a universal constant C. Similarly, by combining (A-16) with (A-15), with the second-order ODE satisfied by η, namely,

$$2\eta'' = W'(\eta) \quad \text{on } \mathbb{R}, \tag{A-17}$$

and with (A-6) we see that also the first and second derivatives of η decay exponentially

$$|\eta'(s)|, |\eta''(s)| \leq C e^{-|s|/C} \quad \text{for all } s \in \mathbb{R}. \tag{A-18}$$

Combining again (A-16) and (A-6) we also see that

$$s \in \mathbb{R} \mapsto 1_{(-\infty,0)}(s) - V(\eta(s-\tau))$$

belongs to $L^1(\mathbb{R})$ for every $\tau \in \mathbb{R}$, with

$$\tau \in \mathbb{R} \mapsto \int_{-\infty}^{\infty} (1_{(-\infty,0)}(s) - V(\eta(s-\tau)))\, ds$$

increasing in τ and converging to $\mp\infty$ as $\tau \to \pm\infty$. In particular, there is a unique universal constant τ_0 such that

$$\int_{-\infty}^{\infty} (1_{(-\infty,0)}(s) - V(\eta(s-\tau_0)))\, ds = 0. \tag{A-19}$$

The constant τ_0 appears in the computation of the first-order expansion of $\psi(\varepsilon)$ as $\varepsilon \to 0^+$ and can be characterized, equivalently, to be

$$\tau_0 = \int_{\mathbb{R}} \eta' V'(\eta) s\, ds. \tag{A-20}$$

Indeed, (A-19) gives

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} (1_{(-\infty,0)}(s) - V(\eta(s-\tau_0)))\, ds \\ &= \int_{-\infty}^{0} (1 - V(\eta(s-\tau_0)))\, ds - \int_{0}^{\infty} V(\eta(s-\tau_0))\, ds \\ &= -\int_{-\infty}^{0} ds \int_{-\infty}^{s-\tau_0} \eta'(t) V'(\eta(t))\, dt + \int_{0}^{\infty} ds \int_{s-\tau_0}^{\infty} \eta'(t) V'(\eta(t))\, dt. \end{aligned}$$

Both integrands are nonnegative; therefore by Fubini's theorem

$$\begin{aligned}0&=-\int_{-\infty}^{-\tau_0}dt\int_{t+\tau_0}^{0}\eta'(t)V'(\eta(t))\,ds-\int_{-\tau_0}^{\infty}dt\int_{0}^{t+\tau_0}\eta'(t)V'(\eta(t))\,ds\\&=\int_{-\infty}^{-\tau_0}(t+\tau_0)\eta'(t)V'(\eta(t))\,dt+\int_{-\tau_0}^{\infty}(t+\tau_0)\eta'(t)V'(\eta(t))\,dt,\end{aligned}$$

that is,

$$\int_{\mathbb{R}}\eta'V'(\eta)t\,dt=-\tau_0\int_{\mathbb{R}}\eta'V'(\eta)=V(1)\tau_0=\tau_0,$$

as claimed.

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CONNECTIVITY CONDITIONS AND BOUNDARY POINCARÉ INEQUALITIES

OLLI TAPIOLA AND XAVIER TOLSA

Inspired by recent work of Mourgoglou and the second author, and earlier work of Hofmann, Mitrea and Taylor, we consider connections between the local John condition, the Harnack chain condition and weak boundary Poincaré inequalities in open sets $\Omega \subset \mathbb{R}^{n+1}$, with codimension-1 Ahlfors–David regular boundaries. First, we prove that if Ω satisfies both the local John condition and the exterior corkscrew condition, then Ω also satisfies the Harnack chain condition (and hence is a chord-arc domain). Second, we show that if Ω is a 2-sided chord-arc domain, then the boundary $\partial\Omega$ supports a Heinonen–Koskela-type weak 1-Poincaré inequality. We also construct an example of a set $\Omega \subset \mathbb{R}^{n+1}$ such that the boundary $\partial\Omega$ is Ahlfors–David regular and supports a weak boundary 1-Poincaré inequality but Ω is not a chord-arc domain. Our proofs utilize significant advances in particularly harmonic measure, uniform rectifiability and metric Poincaré theories.

1. Introduction

Extending the PDE theory from smooth or otherwise "nice" domains to spaces with rough geometries has been under intensive research over the previous decades. The new tools and techniques developed by numerous authors have helped to overcome many difficulties in this field but many questions remain open. See [Hofmann 2019; Mattila 2023] for recent surveys related to key developments in some parts of this and related research.

We consider connections between John-type conditions, the Harnack chain condition and Poincaré inequalities in spaces with rough boundaries, inspired by the recent work [Mourgoglou and Tolsa 2021].

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In their work, they solved a long-standing open problem [Kenig 1994, Problem 3.2.2] (see also [Toro 2010, Question 2.5]) about the solvability of the regularity problem for the Laplacian. More precisely, they proved the following theorem. We give the exact definitions of the objects and concepts in the theorem in Sections 2 and 3.

Theorem 1.1 [Mourgoglou and Tolsa 2021, part of Theorems 1.2 and 1.5]. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded open set satisfying the corkscrew condition* (*see Definition 2.7*), *with n-dimensional Ahlfors–David regular boundary $\partial\Omega$* (*see Definition 2.6*). *For $1 < p \le 2$ we have*:

(a) *The regularity problem for the Laplacian is solvable in L^p for Ω* (*see Definition 2.31*) *if and only if the Dirichlet problem is solvable in $L^{p'}$* (*see Definition 2.30*), *where p' satisfies $1/p + 1/p' = 1$.*

(b) *Suppose that either $\partial\Omega$ supports a weak p-Poincaré inequality* (*see Definition 2.29*) *or that Ω satisfies the 2-sided local John condition* (*see Definitions 2.1 and 3.2*). *If the regularity problem for the Laplacian is solvable in L^p for Ω, then the tangential regularity problem for the Laplacian is also solvable in L^p for Ω* (*see Definition 2.31*).

Remark that, in particular, from (a) in the theorem above it follows that the regularity problem is solvable in L^p for chord-arc domains (see Definition 2.10) for some $p > 1$; see, e.g., [David and Jerison 1990; Semmes 1990]. This extends previous results of [Jerison and Kenig 1982b] in the plane, and of [Verchota 1984] in Lipschitz domains.

Our goal is to revisit the assumptions of Theorem 1.1(b) by studying them from the point of view of 2-sided chord-arc domains: we show that a 2-sided local John domain with codimension 1 Ahlfors–David regular boundary is a 2-sided chord-arc domain, and the boundary of any 2-sided chord-arc domain supports weak Poincaré inequalities. To be more precise, we prove the following two results:

Theorem 1.2. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is an open set with n-dimensional Ahlfors–David regular boundary that satisfies the local John condition and the exterior corkscrew condition* (*see Definitions 2.1 and 2.7*). *Then Ω also satisfies the Harnack chain condition* (*see Definition 2.9*). *In particular*, *a 2-sided local John domain with codimension 1 Ahlfors–David regular boundary is a 2-sided chord-arc domain.*

Theorem 1.3. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a 2-sided chord-arc domain. Then the following weak 1-Poincaré inequality for Lipschitz functions on $\partial\Omega$ holds*: *there exist constants $C \ge 1$ and $\Lambda \ge 1$ such that for every Lipschitz function f on $\partial\Omega$ and every $\Delta = \Delta(y, r) = B(y, r) \cap \partial\Omega$ we have*

$$\fint_{\Delta} |f(x) - \langle f \rangle_{\Delta}| \, d\sigma(x) \le Cr \fint_{\Lambda\Delta} |\nabla_{\mathrm{t}} f(x)| \, d\sigma(x), \tag{1.4}$$

where $\nabla_{\mathrm{t}} f$ is the tangential gradient of f (*see Definition 2.27*), *$\sigma := \mathcal{H}^n|_{\partial\Omega}$ is the surface measure and*

$$\langle f \rangle_{\Delta} := \frac{1}{\sigma(\Delta)} \int_{\Delta} f(y) \, d\sigma(y)$$

is the integral average of f over Δ.

We note that the conclusion of Theorem 1.2 holds if the local John condition is "good enough" in the sense that a D_0-local John condition implies the Harnack chain condition but a (D_0, R_0)-local John condition for $R_0 = c \cdot \operatorname{diam}(\partial\Omega)$ and $c < 1$ small enough does not if $\operatorname{diam}(\partial\Omega) < \infty$. See the definitions and discussion in Section 3.

Theorem 1.3 and its consequence Corollary 1.6 improve some results in the literature. Earlier, Semmes proved that a weak 2-Poincaré inequality for the tangential gradient (that is, inequality (1.4) with the right-hand side replaced by $Cr(\fint_{\Lambda\Delta} |\nabla_t f(x)|^2 \, d\sigma(x))^{1/2}$) holds for smooth functions on any *chord-arc surface with small constant* [Semmes 1991, Lemma 1.1] (see the introduction of that work for the definition of these surfaces). Theorem 1.3 both provides a stronger inequality and generalizes the class of surfaces considered by Semmes. A key element in the proof of Theorem 1.3 is the machinery built by Hofmann, Mitrea and Taylor [Hofmann et al. 2010]. In their paper, they prove a weak (p, p)-Poincaré inequality with a tail for the Hofmann–Mitrea–Taylor Sobolev space $L^p_1(\partial\Omega)$ with respect to the Hofmann–Mitrea–Taylor gradient (see Definition 2.28) for any $1 < p < \infty$ on boundaries of 2-sided local John domains [Hofmann et al. 2010, Proposition 4.13]. Combining Theorem 1.3 with some density results in [Hofmann et al. 2010] and tools in [Mourgoglou and Tolsa 2021] shows us that the tail in their inequality can be removed, at least when Ω is a bounded 2-sided chord-arc domain (see Corollary 7.13).

Our results have some immediate consequences. First, we note that since chord-arc domains satisfy the local John condition, Theorem 1.2 gives us the following characterization result:

Corollary 1.5. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set satisfying a 2-sided corkscrew condition, with n-dimensional Ahlfors–David regular boundary $\partial\Omega$. Then the following conditions are equivalent*:

(a) *Ω satisfies the local John condition.*

(b) *Ω satisfies the Harnack chain condition.*

For recent related results for semiuniform domains and chord-arc domains, see [Azzam 2021b; Azzam et al. 2017].

Second, Theorem 1.3 combined with a Lipschitz characterization of Poincaré inequalities [Keith 2003, Theorem 2] gives us the following Heinonen–Koskela-type weak 1-Poincaré inequality:

Corollary 1.6. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 2-sided chord-arc domain. There exist constants $C \geq 1$ and $\Lambda \geq 1$ such that for every $\Delta = \Delta(y, r)$ we have*

$$\fint_{\Delta} |f(x) - \langle f \rangle_{\Delta}| \, d\sigma(x) \leq Cr \fint_{\Lambda\Delta} \rho(x) \, d\sigma(x) \tag{1.7}$$

for any $f \in L^1_{\mathrm{loc}}(\partial\Omega)$ and any upper gradient ρ of f (see Definition 2.25), where $\sigma := \mathcal{H}^n|_{\partial\Omega}$ is the surface measure.

We note that this weak 1-Poincaré inequality implies a weak p-Poincaré inequality (that is, inequality (1.7) with the right-hand side replaced by $Cr\big(\fint_{\Lambda\Delta} |\rho(x)|^p \, d\sigma(x)\big)^{1/p}$) for any $1 < p < \infty$ by Hölder's inequality. Furthermore, by [Heinonen et al. 2015, Corollary 9.14], these weak p-Poincaré inequalities imply weak (p, p)-Poincaré inequalities of the type

$$\left(\fint_{\Delta} |f(x) - \langle f \rangle_{\Delta}|^p \, d\sigma(x)\right)^{1/p} \leq \widetilde{C}r\left(\fint_{\Lambda\Delta} |\rho(x)|^p \, d\sigma(x)\right)^{1/p},$$

where we used the same notation as in Corollary 1.6. See also Corollary 7.13 for an inequality of this type for the Hofmann–Mitrea–Taylor Sobolev spaces in bounded 2-sided chord-arc domains. We also note that the conclusion of Corollary 1.6 is the inequality appearing in Theorem 1.1. It is natural to ask if Corollary 1.6 can be strengthened into a characterization but this is not possible: there exist non-chord-arc domain open sets $\Omega \subset \mathbb{R}^{n+1}$ with n-dimensional Ahlfors–David regular boundaries $\partial\Omega$ that support weak Poincaré inequalities. See Section 8 for an example and discussion. However, we point out that a result for the converse direction was recently proven by Azzam [2021a] who showed that weak Poincaré inequalities imply uniform rectifiability (see Definition 2.8) for Ahlfors–David regular sets $E \subset \mathbb{R}^{n+1}$.

Furthermore, Corollary 1.6, combined with [Heinonen and Koskela 1998, Theorem 5.7; Korte 2007, Theorem 3.3; Cheeger 1999, Theorem 17.1] (see also, e.g., [Merhej 2017; Heinonen et al. 2015]), immediately gives us the following two new results about the geometric structure of boundaries of 2-sided chord-arc domains:

Corollary 1.8. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 2-sided chord-arc domain. Then $\partial\Omega$ is a Loewner space* (see Definition 2.24).

Examples of Loewner spaces include the Euclidean space, Carnot groups and Riemannian manifolds of nonnegative Ricci curvature [Heinonen and Koskela 1998, Section 6]. For other examples, see [Heinonen et al. 2015, Section 14.2].

Corollary 1.9. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 2-sided chord-arc domain. Then*

- *if $n = 1$, then $\partial\Omega$ is quasiconvex (see Definition 2.22),*
- *if $n > 1$, then $\partial\Omega$ is annularly quasiconvex (see Definition 2.22).*

We note that the case $n = 1$ in Corollary 1.9 cannot be improved to annular quasiconvexity: $\Omega = B(0, 1)$ (the unit disc) is a 2-sided chord-arc domain but for any $z \in \partial\Omega$ and any $0 < r < \frac{1}{2}$ there exist points $x, y \in B(z, 2r) \setminus B(z, r)$ such that x and y can be joined in $\partial\Omega \setminus \{z\}$ only with paths γ such that $\ell(\gamma) \geq 1$.

The proofs of Theorems 1.2 and 1.3 and Corollary 1.6 utilize significant advances in geometric analysis over the past 25 years. For Theorem 1.2, we use harmonic measure theory (particularly the very recent results of Azzam, Hofmann, Martell, Mourgoglou and the second author [Azzam et al. 2020]) and uniform rectifiability techniques (particularly the bilateral weak geometric lemma of [David and Semmes 1993]). For Theorem 1.3 and Corollary 1.6, we combine layer potential techniques of [Hofmann et al. 2010] and pointwise and Lipschitz characterizations of Poincaré inequalities of [Heinonen and Koskela 1998; Keith 2003] (see also [Heinonen 2001]) with suitable localization and truncation arguments. One of the novelties in the proof of Theorem 1.3 is the use of a weak type-(1, 1) version of a weak Poincaré inequality with a tail, analogous to the strong type-(p, p) version proved previously in [Hofmann et al. 2010].

The paper is organized as follows. In Section 2 we fix the basic notation, review the numerous definitions needed in the paper and consider some auxiliary results from the literature. In Section 3, we define three different John-type conditions and compare them. In Section 4, we consider the bilateral weak geometric lemma of David and Semmes and prove some straightforward related results for the proof of Theorem 1.2, and in Section 5 we prove Theorem 1.2. In Section 6, we consider the Hofmann–Mitrea–Taylor-type weak p-Poincaré inequality with a tail for $1 < p < \infty$ and use techniques from its proof to prove a weak-type estimate for the case $p = 1$. In Section 7, we use this weak-type estimate together with some key results from the theory of Poincaré inequalities in metric spaces to prove Theorem 1.3 and Corollary 1.6. In Section 8, we end the paper by constructing an example that shows us that the assumptions in Corollary 1.6 are not optimal and consider some questions related to this work.

2. Notation, basic definitions and auxiliary results

Definition 2.1. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and consider a geometric condition; such as the local John condition (see Definition 3.2). If the interior of Ω^c satisfies the condition, we say that Ω satisfies the *exterior* version of the condition. If both Ω and the interior of Ω^c satisfy the same condition, we say that Ω satisfies the 2-*sided* version of the same condition.

We use the following basic notation and terminology:

- $\Omega \subset \mathbb{R}^{n+1}$ is an open set with n-dimensional boundary $\partial\Omega$. We denote the surface measure of $\partial\Omega$ by $\sigma := \mathcal{H}^n|_{\partial\Omega}$. Unless explicitly mentioned, we assume that $\partial\Omega$ is Ahlfors–David regular (see Definition 2.6) and that both Ω and $\operatorname{int}\Omega^c$ satisfy the corkscrew condition (that is, Ω satisfies the 2-sided corkscrew condition; see Definitions 2.1 and 2.7).

- Usually, we use capital letters X, Y, Z, and so on, to denote points in Ω, and lowercase letters x, y, z, and so on, to denote points in $\partial\Omega$.

- For every point $X \in \mathbb{R}^{n+1}$, we define $\delta(X) := \operatorname{dist}(X, \partial\Omega)$.

- We denote the open $(n+1)$-dimensional Euclidean ball with radius $r > 0$ by $B(X, r)$ or $B(x, r)$, depending on whether the center point lies in Ω or $\partial\Omega$. For any $x \in \partial\Omega$ and any $r > 0$, we denote the surface ball centered at x with radius r by $\Delta(x, r) := B(x, r) \cap \partial\Omega$.

- Given a Euclidean ball $B := B(X, r)$ or a surface ball $\Delta := \Delta(x, r)$ and a constant $\kappa > 0$, we define $\kappa B := B(X, \kappa r)$ and $\kappa\Delta := \Delta(x, \kappa r)$.

- For a metric measure space (X, d, μ), a function f and an open ball B, we denote the average of f over B by
$$\langle f \rangle_B := \fint_B f \, d\mu := \frac{1}{\mu(B)} \int_B f \, d\mu.$$

- A *path* is a continuous function $\gamma : [0, 1] \to X$, where X is a metric space. With slight abuse of terminology, we call a path $\gamma : [0, 1] \to \overline{\Omega}$ a *path in* Ω if $\gamma(t) \in \Omega$ for every $t \in (0, 1]$. With slight abuse of notation, we write $Z \in \gamma$ if there exists $t \in [0, 1]$ such that $\gamma(t) = Z$. We say that a path γ is *from* X_1 *to* X_2 if $\gamma(0) = X_1$ and $\gamma(1) = X_2$.

- The *length of a path* $\gamma : [0, 1] \to \overline{\Omega}$ is defined as
$$\ell(\gamma) := \sup\left\{ \sum_{i=0}^{k} |\gamma(t_i) - \gamma(t_{i+1})| \right\},$$
where the supremum is taken over all finite partitions $0 = t_0 < t_1 < \cdots < t_k = 1$ of the interval $[0, 1]$. We say that a path γ is *rectifiable* if the length of γ is finite.

• Given a rectifiable path γ and a function f, we denote the *arc-length parametrization of* γ (that is, the reparametrization of γ with respect to $\ell(\gamma)$) by $\gamma_\ell : [0, \ell(\gamma)] \to \mathbb{R}^{n+1}$ and the integral of f over γ by

$$\int_\gamma f := \int_0^{\ell(\gamma)} f \circ \gamma_\ell(t)\, dt.$$

• For a path γ and points $X_1, X_2 \in \gamma$ with $\gamma(t) = X_1$ and $\gamma(s) = X_2$ for $t, s \in [0, 1]$, $t < s$, we denote the piece of γ from X_1 to X_2 by $\gamma(X_1, X_2)$ and its length by $\ell(\gamma(X_1, X_2))$. Again, with slight abuse of notation, we write $Z \in \gamma(X_1, X_2)$ if there exists $u \in (s, t)$ such that $\gamma(u) = Z$.

• We denote harmonic measure with pole at $X \in \Omega$ by ω^X. Usually, we drop the pole from the notation if we consider properties that hold for every $X \in \Omega$.

• Let $A \subset \mathbb{R}^{n+1}$, $f : A \to \mathbb{R}$, $\alpha > 0$ and $\beta \geq 1$. We say that f is *α-Lipschitz* if

$$|f(x) - f(y)| \leq \alpha |x - y|$$

for all $x, y \in A$. We say that f is *locally α-Lipschitz* if

$$\limsup_{\substack{A \ni y \to x \\ y \neq x}} \frac{|f(x) - f(y)|}{|x - y|} \leq \alpha \tag{2.2}$$

for every $x \in A$. We say that f is *β-bi-Lipschitz* if

$$\frac{1}{\beta}|x - y| \leq |f(x) - f(y)| \leq \beta |x - y|$$

for all $x, y \in A$.

• We denote the *measure-theoretic boundary of* Ω by $\partial_*\Omega$: we have $x \in \partial_*\Omega$ if and only if $x \in \partial\Omega$, and

$$\liminf_{r \to 0^+} \frac{|B(x, r) \cap \Omega|}{r^{n+1}} > 0 \quad \text{and} \quad \liminf_{r \to 0^+} \frac{|B(x, r) \setminus \overline{\Omega}|}{r^{n+1}} > 0. \tag{2.3}$$

• For any $p > 1$, we denote the *Hölder conjugate of p* by p'. The numbers p and p' satisfy $1/p + 1/p' = 1$.

• We denote the *nontangential maximal operator* by N_*: for a function u in Ω, N_*u is a function $\partial\Omega$ defined as

$$N_*u(x) := \sup_{Y \in \Gamma_\alpha(x)} |u(Y)|, \tag{2.4}$$

where $\Gamma_\alpha(x)$ is the *cone at $x \in \partial\Omega$ with aperture α*,

$$\Gamma_\alpha(x) := \{Y \in \mathbb{R}^{n+1} : |x - Y| < \alpha\, \delta(Y)\}. \tag{2.5}$$

We say that a function u in Ω *converges nontangentially* to a function f on $\partial\Omega$ if $u(Y) \to f(x)$ as $Y \to x$ inside $\Gamma_\alpha(x)$.

• The letters c and C and their obvious variations denote constants that depend only on dimension, ADR constant (see Definition 2.6), UR constants (see Definition 2.8) and other similar parameters. The values of c and C may change from one occurrence to another. We do not track how our bounds depend on these constants and usually just write $\alpha_1 \lesssim \alpha_2$ if $\alpha_1 \leq c\, \alpha_2$ for a constant like this c and $\alpha_1 \approx \alpha_2$ if $\alpha_1 \lesssim \alpha_2 \lesssim \alpha_1$.

If the constant c_κ depends only on parameters of the previous type and some other parameter κ, we usually write $\alpha_1 \lesssim_\kappa \alpha_2$ instead of $\alpha_1 \le c_\kappa \alpha_2$.

2A. ***ADR, UR, NTA, CAD, and the corkscrew condition.***

Definition 2.6 (ADR). We say that a closed set $E \subset \mathbb{R}^{n+1}$ is a *d-ADR* (*Ahlfors–David regular*) set if there exists a constant $D \ge 1$ such that

$$\frac{1}{D} r^d \le \mathcal{H}^d(B(x,r)\cap E) \le D r^d$$

for every $x \in E$ and every $r \in (0, \operatorname{diam}(E))$, where $\operatorname{diam}(E)$ may be infinite.

Definition 2.7 (corkscrew condition). We say that Ω satisfies the *corkscrew condition* if there exists a constant $c \in (0,1)$ such that for every surface ball $\Delta := \Delta(x,r)$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, there exists a point $X_\Delta \in \Omega$ such that $B(X_\Delta, cr) \subset B(x,r)\cap\Omega$,

Definition 2.8 (UR). Following [David and Semmes 1991; 1993], we say that an n-ADR set $E \subset \mathbb{R}^{n+1}$ is *UR* (*uniformly rectifiable*) if it contains *big pieces of Lipschitz images* of $\mathbb{R}^n$, i.e., there exist constants $\theta, M > 0$ such that for every $x \in E$ and $r \in (0, \operatorname{diam}(E))$ there is a Lipschitz mapping $\rho = \rho_{x,r} : \mathbb{R}^n \to \mathbb{R}^{n+1}$, with Lipschitz norm no larger than M, such that

$$\mathcal{H}^n(E \cap B(x,r) \cap \rho(\{y \in \mathbb{R}^n : |y| < r\})) \ge \theta r^n.$$

Definition 2.9 (NTA). Following [Jerison and Kenig 1982a], we say that a domain $\Theta \subset \mathbb{R}^{n+1}$ is *NTA* (*nontangentially accessible*) if

- Θ satisfies the *Harnack chain condition*: there exists a uniform constant C such that for every $\rho > 0$, $\Lambda \ge 1$ and $X, X' \in \Theta$ with $\delta(X), \delta(X') \ge \rho$ and $|X - X'| < \Lambda\rho$ there exists a chain of open balls $B_1, \dots, B_N \subset \Theta$, $N \le C(\Lambda)$, with $X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \ne \varnothing$ and $C^{-1} \operatorname{diam}(B_k) \le \operatorname{dist}(B_k, \partial\Theta) \le C \operatorname{diam}(B_k)$,
- both Θ and $\mathbb{R}^{n+1} \setminus \Theta$ satisfy the corkscrew condition.

Definition 2.10 (CAD). An open set $\Omega \subset \mathbb{R}^{n+1}$ is a *CAD* (*chord-arc domain*) if it is NTA, and $\partial\Omega$ is n-ADR.

The following result originates from [David and Jerison 1990; Semmes 1990] (see also [Hofmann et al. 2010, Definition 3.7 and Corollary 3.9]):

Theorem 2.11. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is an open set satisfying the two-sided corkscrew condition and that $\partial\Omega$ is ADR. Then $\partial\Omega$ is UR and $\sigma(\partial\Omega \setminus \partial_*\Omega) = 0$.*

2A1. *Dyadic cubes.*

Theorem 2.12 (see, e.g., [Christ 1990; Sawyer and Wheeden 1992; Hytönen and Kairema 2012]). *Suppose that E is a d-ADR set. Then there exists a countable collection $\mathbb{D}$ (that we call a **dyadic system**),*

$$\mathbb{D} := \bigcup_{k\in\mathbb{Z}} \mathbb{D}_k, \qquad \mathbb{D}_k := \{Q_\alpha^k : \alpha \in \mathcal{A}_k\},$$

*of Borel sets Q_α^k (that we call **(dyadic) cubes**) such that*

(i) *the collection* $\mathbb{D}$ *is* ***nested***: *if* $Q, P \in \mathbb{D}$, *then* $Q \cap P \in \{\varnothing, Q, P\}$,

(ii) $E = \bigcup_{Q \in \mathbb{D}_k} Q$ *for every* $k \in \mathbb{Z}$ *and the union is disjoint*,

(iii) *there exist constants* $c_1 > 0$ *and* $C_1 \geq 1$ *such that*

$$\Delta(z_\alpha^k, c_1 2^{-k}) \subseteq Q_\alpha^k \subseteq \Delta(z_\alpha^k, C_1 2^{-k}), \tag{2.13}$$

(iv) *for every set* Q_α^k *there exist at most* N *cubes* $Q_{\beta_i}^{k+1}$ *(called the* ***children*** *of* Q_α^k*) such that* $Q_\alpha^k = \bigcup_i Q_{\beta_i}^{k+1}$, *where the constant* N *depends only on the ADR constant of* E.

Notation 2.14. We shall use the following notational conventions.

(1) For each k, and for every cube $Q_\alpha^k := Q \in \mathbb{D}_k$, we define $\ell(Q) := C_1 2^{-k}$ and $x_Q := z_\alpha^k$. We call $\ell(Q)$ the *side length* of Q, and x_Q the *center* of Q. If the set E is bounded or disconnected, the side length might not be well-defined, but we can fix this problem easily by, for example, considering the minimum of the numbers $C_1 2^{-k}$ such that $Q \subset \Delta(x_Q, C_1 2^{-k})$.

(2) For every $Q = Q_\alpha^k$ and $\kappa \geq 1$, we define

$$\kappa B_Q := B(z_\alpha^k, \kappa \ell(Q)).$$

For $\kappa = 1$, we simply let $\kappa B_Q = B_Q$.

Definition 2.15. We say that a collection $\mathcal{A} \subset \mathbb{D}$ satisfies a *Carleson packing condition* if there exists a constant $C \geq 1$ such that

$$\sum_{Q \in \mathcal{A}, Q \subset Q_0} \sigma(Q) \leq C \sigma(Q_0)$$

for every cube $Q_0 \in \mathbb{D}$. We call the smallest such constant C the *Carleson packing norm of* $\mathcal{A}$ and denote it by $\mathscr{C}_\mathcal{A}$.

We need the following straightforward lemma in the proof of Theorem 1.2:

Lemma 2.16. *Let* $\mathcal{A} \subset \mathbb{D}$ *be a collection satisfying a Carleson packing condition. Also, let* $Q_0 \in \mathbb{D}$ *be a cube and* $A \subset Q_0$ *a measurable subset such that* $\sigma(A) \geq c\, \sigma(Q_0)$ *for a constant* $c \in (0, 1)$. *Then there exists a cube* $Q \in \mathbb{D} \setminus \mathcal{A}$ *such that* $\sigma(Q \cap A) > 0$ *and* $\ell(Q) \approx_{\mathscr{C}_\mathcal{A}, c} \ell(Q_0)$.

Proof. Let us consider the first $K > \mathscr{C}_\mathcal{A}/c$ generations of subcubes of Q_0. Each of these generations covers the set A. For contradiction, suppose that $\sigma(Q \cap A) = 0$ for each of these subcubes such that $Q \notin \mathcal{A}$. Then, for every $m = 1, \ldots, K$, we can cover the set A (up to a set of measure 0) by cubes from the collection $\{Q \in \mathcal{A} : \ell(Q) = 2^{-m} \ell(Q_0)\}$. In particular, we get

$$\sum_{Q \in \mathcal{A}, Q \subset Q_0} \sigma(Q) = \sum_{m=0}^{\infty} \sum_{\substack{Q \in \mathcal{A} \\ \ell(Q) = 2^{-m} \ell(Q_0)}} \sigma(Q) \geq \sum_{m=1}^{K} \sum_{\substack{Q \in \mathcal{A} \\ \ell(Q) = 2^{-m} \ell(Q_0)}} \sigma(Q \cap A) = \sum_{m=1}^{K} \sigma(A) \geq K c\, \sigma(Q_0) > \mathscr{C}_\mathcal{A}\, \sigma(Q_0),$$

which contradicts the Carleson packing condition. Hence, there exists at least one cube Q from the first $\lceil \mathscr{C}_\mathcal{A}/c \rceil$ generations of subcubes of Q_0 such that $\sigma(Q \cap A) > 0$ and $Q \in \mathbb{D} \setminus \mathcal{A}$. $\square$

2B. ***Harmonic measure and the weak-A_∞ condition.***

Definition 2.17 (weak A_∞ for harmonic measure). For harmonic measure ω, we write $\omega \in \text{weak-}A_\infty(\sigma)$ if there exist constants $C \geq 1$ and $s > 0$ such that if $B := B(x, r)$ with $x \in \partial\Omega$ and $r \in \bigl(0, \frac{1}{2}\operatorname{diam}(\partial\Omega)\bigr)$ and $A \subset \Delta := B \cap \partial\Omega$ is a Borel set, then

$$\omega^Y(A) \leq C\left(\frac{\sigma(A)}{\sigma(\Delta)}\right)^s \omega^Y(2\Delta) \tag{2.18}$$

for every $Y \in \Omega \setminus 4B$.

We note that the constant 2 in (2.18) can be replaced with any constant $c > 1$ without changing the class weak-$A_\infty(\sigma)$ and that the weak-A_∞ property is equivalent with a weak reverse Hölder property for the Radon–Nikodym derivative; see, e.g., [Anderson et al. 2017, Section 8].

We use the following lemma from [Azzam et al. 2020] in the proof of Theorem 1.2. The lemma is a key ingredient for the proof of the geometric characterization of the weak-A_∞ property of harmonic measure:

Lemma 2.19 [Azzam et al. 2020, Section 10]. *Suppose that Ω has a uniformly rectifiable boundary $\partial\Omega$ and that $\omega \in$ weak-A_∞. Suppose also that $R_0 \in \mathbb{D}(\partial\Omega)$ is a dyadic cube and $Y \in \Omega \setminus 4B_{R_0}$ is a point such that*

$$c_1 \ell(R_0) \leq \delta(Y) \leq \operatorname{dist}(Y, R_0) \leq c_1^{-1} \ell(R_0)$$

and $\omega^Y(R_0) \geq c_2 > 0$. Then there exist a constant $c_3 > 0$ and a subset $A \subset R_0$ such that $\sigma(A) \geq c_3\, \sigma(R_0)$ and each point $x \in A$ can be joined to Y by a D-nontangential path (*see Definition 3.1*), *where c_3 and D depend only on c_1, c_2, n, the weak-A_∞ constants and the uniform rectifiability constants.*

We also need the following classical estimate (sometimes referred to as *Bourgain's estimate* [1987, Lemma 1]):

Lemma 2.20. *There exist uniform constants $c_0 \in (0, 1)$ and $C_0 > 1$, depending only on n and the ADR constant, such that the following holds*: *if $x \in \partial\Omega$, $r \in (0, \operatorname{diam}(\partial\Omega))$ and $Y \in B(x, c_0 r)$, then $\omega^Y(\Delta(x, r)) \geq 1/C_0 > 0$.*

2C. ***Quasiconvexity, annular quasiconvexity and Loewner spaces.***

Definition 2.21. Let (X, d) be a metric space. We say that a nonempty set $E \subset X$ is a *continuum* if it is compact and connected. We call a continuum *nondegenerate* if it contains more than one point. We say that points $x, y \in F \subset X$ can be *joined in F* if there exists a continuum $E \subset F$ such that $x, y \in E$.

Definition 2.22. Let (X, d) be a metric space. We say that X is

(i) *quasiconvex* if there exists a constant $C \geq 1$ such that for any pair of points $x, y \in X$ there exists a path $\gamma_{x,y} : [0, 1] \to X$ such that $\gamma_{x,y}(0) = x$, $\gamma_{x,y}(1) = y$ and $\ell(\gamma_{x,y}) \leq C\, d(x, y)$,

(ii) *annularly quasiconvex* if there exists a constant $C \geq 1$ such that if $x, y \in B(z, 2r) \setminus B(z, r)$ for $z \in X$ and $r < (1/C)\operatorname{diam}(X)$, then x and y can be joined by a path $\gamma = \gamma_{x,y}$ in $B(z, Cr) \setminus B(z, r/C)$ such that $\ell(\gamma) \leq C\, d(x, y)$.

(iii) *rectifiably connected* if for any pair of points $x, y \in X$ there exists a rectifiable path $\gamma_{x,y}$ from x to y.

Definition 2.23. Let (X, d, μ) be a rectifiably connected metric measure space with a locally finite Borel measure μ. Let $E, F \subset X$ be two disjoint nondegenerate continua, $(E, F) = (E, F; X)$ be the family of paths in X connecting E and F and $p \geq 1$. We define the *p-modulus of* (E, F) as

$$\operatorname{mod}_p(E, F) := \inf_{\varrho} \int_X \varrho^p \, d\mu,$$

where the infimum is taken over all nonnegative Borel functions $\varrho : X \to [0, \infty)$ satisfying

$$\int_\gamma \varrho \, ds \geq 1$$

for every γ in (E, F).

Definition 2.24. Let (X, d, μ) be a d-dimensional, rectifiably connected metric measure space with a locally finite Borel measure μ. We say that X is a *d-Loewner space* if there exists a function $\phi : (0, \infty) \to (0, \infty)$ such that if E and F are two disjoint, nondegenerate continua in X satisfying

$$\frac{\operatorname{dist}(E, F)}{\min\{\operatorname{diam}(E), \operatorname{diam}(F)\}} \leq t,$$

then $\phi(t) \leq \operatorname{mod}_d(E, F)$.

2D. *Upper, Hajłasz, tangential and Hofmann–Mitrea–Taylor gradients, Sobolev spaces, and weak Poincaré inequalities on $\partial\Omega$.*

Definition 2.25. Let $f : \partial\Omega \to \mathbb{R}$. We say that a Borel-measurable function $\rho : \partial\Omega \to [0, \infty]$ is an *upper gradient of f* if we have

$$|f(x) - f(y)| \leq \int_\gamma \rho = \int_0^{\ell(\gamma)} \rho \circ \gamma_\ell(t) \, dt$$

for all $x, y \in \partial\Omega$ and every rectifiable path γ from x to y, where $\gamma_\ell : [0, \ell(\gamma)] \to \mathbb{R}$ is the arc-length parametrization of γ.

The notion of upper gradients originates from [Heinonen and Koskela 1996; 1998], where they were called *very weak gradients*.

Definition 2.26. Let $f : \partial\Omega \to \mathbb{R}$. We say that a Borel-measurable function $g : \partial\Omega \to \mathbb{R}$ is a *Hajłasz gradient of f* if we have

$$|f(x) - f(y)| \leq (g(x) + g(y))|x - y|$$

for almost every $x, y \in \partial\Omega$. We denote the class of all Hajłasz gradients of f by $D(f)$. For $p \geq 1$, we denote the space of Borel functions with a Hajłasz gradient in $L^p(\partial\Omega)$ by $\dot{W}^{1,p}(\partial\Omega)$ and define a seminorm $\|\cdot\|_{\dot{W}^{1,p}(\partial\Omega)}$ for this space by setting

$$\|f\|_{\dot{W}^{1,p}(\partial\Omega)} := \inf_{g \in D(f)} \|g\|_{L^p(\partial\Omega)}.$$

The notion of Hajłasz gradients originates from [Hajłasz 1996]. By [Jiang et al. 2015], if $f, g \in L^1_{\mathrm{loc}}(\partial\Omega)$ and g is a Hajłasz gradient of f, then there exist functions $\tilde{f}, \tilde{g} \in L^1_{\mathrm{loc}}(\partial\Omega)$ such that $f = \tilde{f}$ and $g = \tilde{g}$ almost everywhere and $4\tilde{g}$ is an upper gradient of $\tilde{f}$.

Definition 2.27. Let $f : \partial\Omega \to \mathbb{R}$ be a Lipschitz function and let $x \in \partial\Omega$ be a point such that the approximate tangent plane $T_x\partial\Omega$ exists. Let $\tilde{f} : \mathbb{R}^{n+1} \to \mathbb{R}$ be any Lipschitz extension of f to $\mathbb{R}^{n+1}$. We say that f is *tangentially differentiable at x* if $\tilde{f}|_{x+T_x\partial\Omega}$ is differentiable at x. When it exists, we denote the corresponding *tangential gradient at x* by $\nabla_{\mathrm{t}} f(x)$.

A thorough reference for these types of differentiability results is [Maggi 2012]. In particular, since $\partial\Omega$ is uniformly rectifiable by Theorem 2.11 (and therefore $\partial\Omega$ is rectifiable; see [Hofmann et al. 2010, p. 2629] for an explicit proof), we know that the approximate tangent plane exists for almost every point $x \in \partial\Omega$ by [Maggi 2012, Theorem 10.2], the tangential gradient exists for σ-a.e. point $x \in \partial\Omega$ by [loc. cit., Theorem 11.4] and the definition of the tangential gradient is independent of the choice of the Lipschitz extension by [loc. cit., Theorem 10.1, Proposition 10.5, and Lemma 11.5].

By [Hofmann et al. 2010], the next definitions make sense if we know that — on top of the standing assumptions on Ω and $\partial\Omega$ (see the beginning of Section 2) — Ω is a set of locally finite perimeter and $\sigma(\partial\Omega \setminus \partial_*\Omega) = 0$ (recall (2.3)). Since we only use the following objects when Ω satisfies the 2-sided local John condition (or, equivalently by Corollary 1.5, when Ω is a 2-sided chord-arc domain), the assumptions are automatically satisfied by Theorem 2.11 and [Hofmann et al. 2010, Corollary 3.14].

Definition 2.28. Let $\nu(x) := (\nu_j(x))_{j=1}^{n+1}$ be the outer unit normal at $x \in \partial\Omega$. For a function $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$, $\varphi \in C^1_{\mathrm{c}}$, we define the (*Hofmann–Mitrea–Taylor*) *tangential derivatives of* φ as

$$\partial_{\mathrm{t},j,k}\varphi := \nu_j(\partial_k\varphi)|_{\partial\Omega} - \nu_k(\partial_j\varphi)|_{\partial\Omega}$$

for $1 \le j, k \le n+1$. The *Hofmann–Mitrea–Taylor Sobolev space* $L^p_1(\partial\Omega)$ is the space of the functions $f \in L^p(\partial\Omega)$ such that there exists a finite constant C_f such that

$$\sum_{1\le j,k\le n+1} \left| \int_{\partial\Omega} f\,\partial_{\mathrm{t},k,j}\varphi\,d\sigma \right| \le C_f \|\varphi\|_{L^{p'}(\sigma)}$$

for every $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^{n+1})$. By the Riesz representation theorem, for every $f \in L^p_1(\partial\Omega)$ and each $j, k = 1, 2, \ldots, n+1$ there exists a function $h_{j,k} \in L^p(\partial\Omega)$ satisfying

$$\int_{\partial\Omega} h_{j,k}\varphi\,d\sigma = \int_{\partial\Omega} f\,\partial_{\mathrm{t},k,j}\varphi\,d\sigma$$

for every $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^{n+1})$. We set $\partial_{\mathrm{t},j,k} f := h_{j,k}$ and define the *Hofmann–Mitrea–Taylor gradient* $\nabla_{\mathrm{HMT}} f$ by setting

$$\nabla_{\mathrm{HMT}} f := \Big(\sum_k \nu_k \partial_{\mathrm{t},j,k} f \Big)_{j=1}^{n+1}.$$

For the comprehensive theory of Hofmann–Mitrea–Taylor Sobolev spaces, see in particular Section 3 and 4 in [Hofmann et al. 2010].

Definition 2.29. Let $1 \le p < \infty$. We say that $\partial\Omega$ supports a *weak* (*Heinonen–Koskela-type*) *p-Poincaré inequality* if there exist constants $C = C_p \ge 1$ and $\Lambda \ge 1$ such that for every $\Delta = \Delta(y, r)$ we have

$$⨍_{\Delta} |f(x) - \langle f\rangle_\Delta|\,d\sigma(x) \le Cr\Big(⨍_{\Lambda\Delta} \rho(x)^p\,d\sigma(x)\Big)^{1/p}$$

for any $f \in L^1_{\mathrm{loc}}(\partial\Omega)$ and any upper gradient ρ of f.

2E. ***Solvability for the Laplacian.***

Definition 2.30. Let $1 \leq p < \infty$. We say that the Dirichlet problem (for the Laplacian) is *solvable in L^p for* Ω if there exists a constant C such that for any continuous function $f \in C(\partial\Omega)$ the solution $u = u_f$ to the Dirichlet problem with datum f converges nontangentially to f σ-a.e. and

$$\|N_* u\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)},$$

where N_* is the nontangential maximal operator (recall (2.4)).

Definition 2.31. Let $1 \leq p < \infty$. We say that the *regularity problem* (*for the Laplacian*) *is solvable in L^p for* Ω if there exists a constant C such that for any Lipschitz function $f : \partial\Omega \to \mathbb{R}$ the solution $u = u_f$ to the Dirichlet problem with datum f converges nontangentially to f σ-a.e. and

$$\|N_*(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{\dot{W}^{1,p}(\partial\Omega)}, \tag{2.32}$$

where $\|\cdot\|_{\dot{W}^{1,p}(\partial\Omega)}$ is the Hajłasz seminorm (see Definition 2.26). For $1 < p < \infty$, we say that the *tangential regularity problem* (*for the Laplacian*) *is solvable in L^p* if the previous holds after we replace (2.32) with the estimate

$$\|N_*(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|\nabla_t f\|_{L^p(\partial\Omega)}.$$

3. John, local John and weak local John conditions

In this section, we define different John-type conditions, compare them by considering some examples and make some remarks related to literature. We assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set, with n-dimensional Ahlfors–David regular boundary. We do not assume that the corkscrew conditions hold in general but we discuss their role below.

Definition 3.1 (nontangential paths). Let $\gamma : [0, 1] \to \overline{\Omega}$ be a path in Ω from X to Y. For $D \geq 1$, we say that γ is a *D-nontangential path* if we have

$$\ell(\gamma(X, Z)) \leq D\,\delta(Z)$$

for every $Z \in \gamma$.

Notice that Definition 3.1 is not symmetric with respect to X and Y. The general idea is that we use nontangential paths to measure how well we can connect boundary points to certain points inside the space. We use the name nontangential path to emphasize the connection between these paths and nontangential convergence we discussed in Section 2. Indeed, if there exists a D-nontangential path γ from $x \in \partial\Omega$ to $Y \in \Omega$, then, by definition, we have

$$|x - Z| \leq \ell(\gamma(x, Z)) \leq D\,\delta(Z)$$

for every $Z \in \gamma$ and therefore we have $Z \in \Gamma_D(x)$ for every $Z \in \gamma$. Here $\Gamma_D(x)$ is the cone at x of aperture D (see (2.5)).

Definition 3.2 (John, local John and weak local John conditions). Let $D_0 \geq 1$. We say that Ω satisfies

(i) the *D_0-John condition* if there exists a point $X_0 \in \Omega$ such that for every $Y \in \Omega$ there exists a D_0-nontangential path in Ω from Y to X_0,

(ii) the *local D_0-John condition* if for every $x \in \partial\Omega$ and every $r \in (0, \operatorname{diam}(\partial\Omega))$ there exists a point $Y_x \in B(x, r) \cap \Omega$ (that we call a *local John point*) such that $B(Y_x, r/D_0) \subset \Omega$ and for every $z \in \Delta(x, r)$ there exists a D_0-nontangential path γ_z in Ω from z to Y_x such that $\ell(\gamma_z) \leq D_0 r$,

(iii) the *weak local D_0-John condition* if there exist constants $\theta \in (0, 1]$ and $R \geq 2$ such that for every $X \in \Omega$ there exists a Borel set $F \subset \Delta_X := B(X, R\,\delta(X)) \cap \partial\Omega$ such that $\sigma(F) \geq \theta\sigma(\Delta_X)$ and for every $z \in F$ there exists a D_0-nontangential path γ_z in Ω from z to X such that $\ell(\gamma_z) \leq D_0 R\,\delta(X)$.

The John condition was first used in [John 1961] but the terminology originates from [Martio and Sarvas 1979]. The local John condition was first used in [Hofmann et al. 2010, Definition 3.12] and weak local John condition originates from [Azzam et al. 2020, Definition 2.11].

Generally, the John condition does not imply the local John condition, the local John condition does not imply the John condition, and there are domains that satisfy the weak local John condition but not the local John condition. We can see this by considering some straightforward examples:

Example 3.3. In $\mathbb{R}^2$ we have the following:

(1) $\Omega_1 := B(0, 1) \setminus \{(x, 0) : x \in [0, 1]\}$ satisfies the John condition (we can choose, e.g., $X_0 = \left(0, -\frac{1}{2}\right)$ as the "John point") but not the local John condition. We can see this by noticing that any ball $B((1, 0), r) \cap \Omega_1$ contains points $(y, t) \in \Omega_1$ and $(z, t) \in \partial\Omega_1$ for both $t < 0$ and $t > 0$ with arbitrarily small $|t|$. Hence, no matter how we choose the point $\widetilde{X}_0 \in B((1, 0), r) \cap \Omega_1$, there are points in $B((1, 0), r) \cap \partial\Omega_1$ that can be connected to $\widetilde{X}_0$ inside Ω_1 only with paths γ satisfying $\ell(\gamma) \geq 1$.

(2) $\Omega_2 := \mathbb{R}^2 \setminus \partial B(0, 1)$ satisfies the local John condition but not the John condition because we cannot connect a point in $B(0, 1)$ to a point in $\mathbb{R}^2 \setminus B(0, 1)$ with a path in Ω_2.

(3) $\Omega_3 := B(0, 1) \setminus \{(x, 0) : x \in (-1, 1)\}$ satisfies the weak local John condition but not the John condition or the local John condition. This is because Ω_3 is not connected (and hence cannot satisfy the John condition) and it cannot satisfy the local John condition for the same reason why Ω_1 in the part (1) of this example does not satisfy it.

To the best knowledge of the authors, it is not known if the local John condition alone implies the weak local John condition. We note that if there exists an open set $\Omega \subset \mathbb{R}^{n+1}$ with n-ADR boundary such that it satisfies the weak local John condition but not the local John condition, then Ω cannot satisfy the exterior corkscrew condition (see Lemma 3.6).

The John condition can be seen as a stronger form of connectivity of the space, but it does not imply connectivity for the boundary, not even if the exterior corkscrew condition holds. We can see this by considering the annulus $A := B(0, 2) \setminus B(0, 1)$. By the same example and the set Ω_2 in Example 3.3(2), we see that the local John condition does not generally imply connectivity for the space nor the boundary. However, by Theorem 1.2, we know that if the exterior corkscrew condition holds and the boundary of the space is Ahlfors–David regular, the local John condition implies connectivity for the space (but not the boundary, as we saw from the annulus A). By Corollaries 1.5 and 1.9, we know that the 2-sided local John condition combined with Ahlfors–David regularity of the boundary implies annular quasiconvexity (and therefore connectivity) for the boundary.

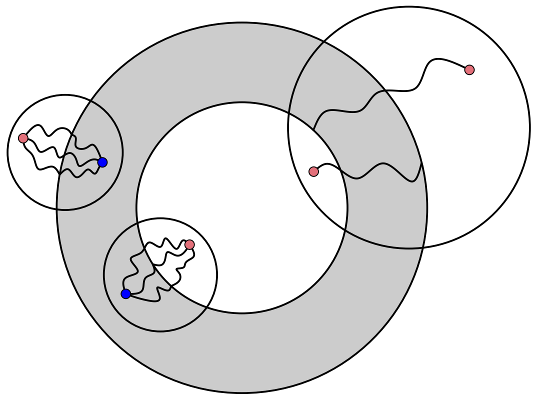

Figure 1. The 2-sided local (D_0, R_0)-John condition does not imply the Harnack chain condition or any kind of connectivity if R_0 is not large enough. In Example 3.5, for the complement of a closed annulus, we can find local John points for small balls centered at either the inner or the outer boundary. However, if the ball is large enough, then it contains both inner and outer boundary points, and we cannot find a point in Ω that connects to all the boundary points inside the ball without passing through the annulus itself.

We note that the local John condition in Definition 3.2 is the "extreme" case in the definition given in [Hofmann et al. 2010, Definition 3.12]. To be more precise, let us consider the following weaker version of the local John condition:

Definition 3.4. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, $D_0 \geq 1$ and $0 < R_0 \leq \operatorname{diam}(\partial\Omega)$. We say that Ω satisfies the *local* (D_0, R_0)*-John condition* if for every $x \in \partial\Omega$ and every $r \in (0, R_0)$ there exists a point $Y_x \in B(x, r) \cap \Omega$ (that we call a *local John point*) such that $B(Y_x, r/D_0) \subset \Omega$ and for every $z \in \Delta(x, r)$ there exists a D_0-nontangential path γ_z in Ω from z to Y_x such that $\ell(\gamma_z) \leq D_0 r$.

Thus, the local D_0-John condition and the local $(D_0, \operatorname{diam}(\partial\Omega))$-condition are the same thing. The reason why we consider only the case $(D_0, \operatorname{diam}(\partial\Omega))$ in Theorem 1.2 is simply because the result may fail if the local John condition is not good enough. We see this by the following simple example (see also Figure 1):

Example 3.5. Let $\Omega := B(0, 1) \cup \{X \in \mathbb{R}^2 : |X| > 2\} \subset \mathbb{R}^2$ be the interior of the complement of the annulus $B(0, 2) \setminus B(0, 1)$. Now $\partial\Omega$ is 1-ADR and Ω satisfies the 2-sided local $\left(D_0, \frac{1}{2}\right)$-John condition for suitable $D_0 > 1$, but Ω is not a chord-arc domain. In addition, the boundary $\partial\Omega$ is not connected.

In the proof of Theorem 1.2, we use harmonic measure theory and techniques that require that harmonic measure belongs to the class weak-$A_\infty(\sigma)$. By the main result of [Azzam et al. 2020], we know that, in our context, the weak-A_∞ property is equivalent to uniform rectifiability for $\partial\Omega$ and the weak local John condition for Ω. We note that although we do not assume the weak local John condition, it follows from the assumptions of Theorem 1.2. Indeed, by [David and Jerison 1990, p. 842] (see also [Semmes 1990]), we know that the 2-sided corkscrew condition implies the *interior big pieces of Lipschitz graphs condition* (see [Bennewitz and Lewis 2004, p. 572] for the definition). This condition then implies that harmonic measure is in weak-A_∞ by [Bennewitz and Lewis 2004, Theorem 1]. Combining this with [Azzam et al. 2020, Theorem 1.1] gives us the weak local John condition:

Lemma 3.6. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set satisfying the 2-sided corkscrew condition, with n-dimensional Ahlfors–David regular boundary. Then $\omega \in$ weak-$A_\infty(\sigma)$ and Ω satisfies the weak local John condition.*

4. The bilateral weak geometric lemma and nontangential approach

In this section, we consider some tools that we use in the proof of Theorem 1.2 to connect two pieces of different paths to each other without going too close to the boundary. Our approach is based on the use of β-numbers of [Jones 1990] and the bilateral weak geometric lemma of [David and Semmes 1993]. Throughout the section, we assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set satisfying the 2-sided corkscrew condition, with uniformly rectifiable boundary $\partial\Omega$, and $\mathbb{D}$ is a dyadic system on $\partial\Omega$. Recall that $\sigma := \mathcal{H}^n|_{\partial\Omega}$ is the surface measure and $\delta(X) := \operatorname{dist}(X, \partial\Omega)$ for $X \in \mathbb{R}^{n+1}$.

Given a ball $B = B(x, r) \subset \mathbb{R}^{n+1}$, a hyperplane $L \subset \mathbb{R}^{n+1}$ and a constant $c > 0$, we define

$$b\beta_{\partial\Omega}(B, L) := \sup_{y \in B \cap \partial\Omega} \frac{\operatorname{dist}(y, L)}{r} + \sup_{Y \in L \cap B} \frac{\delta(Y)}{r},$$
$$\mathcal{U}_c(L) := \{Y \in \mathbb{R}^{n+1} : \operatorname{dist}(Y, L) < c\}.$$

For any ball $B \subset \mathbb{R}^{n+1}$, we define the *bilateral β-number* as

$$b\beta_{\partial\Omega}(B) := \inf_L b\beta_{\partial\Omega}(B, L),$$

where the infimum is taken over all hyperplanes $L \subset \mathbb{R}^{n+1}$.

Recall from Notation 2.14 that for a cube $Q \in \mathbb{D}$ we write $B_Q = B(x_Q, \ell(Q))$, where x_Q is the center of Q and $\ell(Q)$ is its side length. By a straightforward reformulation of [David and Semmes 1993, Chapter I.2, Theorem 2.4], the following version of the *bilateral weak geometric lemma* (*BWGL*) holds:

Lemma 4.1. *For every $\varepsilon > 0$, there exists a constant $C_\varepsilon \geq 1$ such that*

$$\sum_{\substack{Q \in \mathbb{D}, Q \subset R, \\ b\beta_{\partial\Omega}(2B_Q) > \varepsilon}} \sigma(Q) \leq C_\varepsilon \sigma(R) \tag{4.2}$$

for any $R \in \mathbb{D}$, i.e., for any $\varepsilon > 0$ the collection $\{Q \in \mathbb{D} : b\beta_{\partial\Omega}(2B_Q) > \varepsilon\}$ satisfies a Carleson packing condition with Carleson packing norm depending only on ε, n and uniform rectifiability constants.

The BWGL actually characterizes the uniform rectifiability property but we only need the part written in Lemma 4.1.

In the proof of Theorem 1.2, we use the BWGL combined with the following lemmas that help us with technicalities related to constructing a path from a point to a nearby local John point. Alternatively, we could use the Whitney region constructions of Hofmann, Martell and Mayboroda [Hofmann et al. 2016, Section 3] (and their straightforward geometric applications in [Hofmann and Tapiola 2020; 2021]) for the same purpose, but this alternative approach is slightly less elementary than the one we present in this paper.

Lemma 4.3. *There exists $\varepsilon_0 > 0$, depending only on the 2-sided corkscrew condition, such that the following holds: if $B = B(x, r)$ is a ball with $x \in \partial\Omega$ and L_B is a hyperplane such that $b\beta_{\partial\Omega}(B, L_B) < \varepsilon \le \varepsilon_0$, then $B \setminus \mathcal{U}_{\varepsilon r}(L_B)$ consists of two convex components, B^+ and B^-, such that*

$$B^+ \subset \Omega \quad \textit{and} \quad B^- \subset \mathbb{R}^{n+1} \setminus \overline{\Omega}.$$

Proof. Let $B = B(x, r)$ be a ball with $x \in \partial\Omega$, $\varepsilon > 0$ and L_B be a hyperplane such that $b\beta_{\partial\Omega}(B, L_B) < \varepsilon$. By the definitions of $b\beta_{\partial\Omega}(B, L_B)$ and $\mathcal{U}_{\varepsilon r}(L_B)$, we know that

$$\partial\Omega \cap B \subset \mathcal{U}_{\varepsilon r}(L_B),$$

and $B \setminus \mathcal{U}_{\varepsilon r}(L_B)$ has exactly two components if ε is small enough, say $\varepsilon < \frac{1}{10}$. Thus, the two connected components V_1 and V_2 of $B \setminus \mathcal{U}_{\varepsilon r}(L_B)$ are contained in $\mathbb{R}^{n+1} \setminus \partial\Omega = \Omega \cup (\mathbb{R}^{n+1} \setminus \overline{\Omega})$. Furthermore, the components V_1 and V_2 are convex by the definition of $\mathcal{U}_{\varepsilon r}(L_B)$, and each V_i is either fully contained in Ω or fully contained in $\mathbb{R}^{n+1} \setminus \overline{\Omega}$. Indeed, if V_i intersects both Ω and $\mathbb{R}^{n+1} \setminus \overline{\Omega}$, then the line segment from any point $Z_1 \in V_i \cap \Omega$ to any point $Z_2 \in V_1 \cap \mathbb{R}^{n+1} \setminus \overline{\Omega}$ has to contain a point $z_0 \in \partial\Omega$ which then has to belong to V_i by convexity. This is impossible because $V_i \subset B \setminus \mathcal{U}_{\varepsilon r}(L_B)$ and therefore $V_i \cap \partial\Omega = \varnothing$.

Let us then show that if ε is small enough, one of the components V_i lies in Ω and the other lies in $\mathbb{R}^{n+1} \setminus \overline{\Omega}$. By the 2-sided corkscrew condition (applied for the surface ball $\Delta(x, r) = B(x, r) \cap \partial\Omega$), there exist balls

$$B_1 := B(Z^+, cr) \subset \Omega \cap B, \quad B_2 := B(Z^-, cr) \subset (\mathbb{R}^{n+1} \setminus \overline{\Omega}) \cap B,$$

where $c \in (0, 1)$ is independent of Δ. Let us assume that $\varepsilon \le \frac{c}{5}$. Now neither B_1 nor B_2 can be contained in $\mathcal{U}_{\varepsilon r}(L_B)$ and therefore both the balls intersect $V_1 \cup V_2$. Since each V_i is either fully contained in Ω or fully contained in $\mathbb{R}^{n+1} \setminus \overline{\Omega}$, we know that $V_i \cap B_1 \neq \varnothing$ implies $V_i \subset \Omega$ and $V_i \cap B_2 \neq \varnothing$ implies $V_i \subset \mathbb{R}^{n+1} \setminus \overline{\Omega}$. We also notice that if $V_1 \cap B_1 \neq \varnothing$, then $V_1 \cap B_2 = \varnothing$ since otherwise V_1 intersects both Ω and $\mathbb{R}^{n+1} \setminus \overline{\Omega}$. Thus, since both B_1 and B_2 intersect $V_1 \cup V_2$, we know that $V_1 \cap B_1 \neq \varnothing$ implies $V_2 \cap B_2 \neq \varnothing$, and similarly $V_1 \cap B_2 \neq \varnothing$ implies $V_2 \cap B_1 \neq \varnothing$. In particular, B_1 intersects exactly one of the components V_i, say V_1, which is then contained in Ω, and B_2 then intersects V_2, which is contained in $\mathbb{R}^{n+1} \setminus \overline{\Omega}$. Thus, we may set $B^+ = V_1$ and $B^- = V_2$ and choose $\varepsilon_0 = \min\{\frac{1}{10}, \frac{c}{5}\}$. □

Lemma 4.4. *Let $B = B(x_0, r)$ be a ball with $x_0 \in \partial\Omega$, $X_1 \in \Omega$ a point such that $|X_1 - x_0| \approx \delta(X_1) \ge \frac{r}{2}$ and γ a D-nontangential path from x_0 to X_1, where $D \ge 1$. Let $\varepsilon_0 > 0$ be as in Lemma 4.3 and suppose that $0 < \varepsilon < \min\{\varepsilon_0, \frac{1}{12D}\}$. Let L_B be a hyperplane such that $b\beta_{\partial\Omega}(B, L_B) < \varepsilon$, and let B^+ and B^- be the components of B as in Lemma 4.3. Now γ intersects $B^+ \setminus \mathcal{U}_{2\varepsilon r}(L_B)$.*

Proof. Since $|X_1 - x_0| \ge \frac{r}{2}$, we know that $X_0 \notin B(x_0, \frac{r}{4})$. Thus, there exists a point $Y_0 \in \gamma \cap \partial B(x_0, \frac{r}{4})$. We claim that $Y_0 \in B^+ \setminus \mathcal{U}_{2\varepsilon r}(L_B)$.

We notice that, by the definition of D-nontangential paths, we have

$$\delta(Y_0) \ge \frac{1}{D}\ell(\gamma(x_0, Y_0)) \ge \frac{1}{D}|x_0 - Y_0| = \frac{1}{4D}r. \tag{4.5}$$

For any point $X \in \mathbb{R}^{n+1}$, let pr_X be its orthogonal projection onto L_B. Let $Z \in \frac{1}{2}B \cap \mathcal{U}_{2\varepsilon r}(L_B)$. Now it holds that

$$|\,\mathrm{pr}_Z - x_0| \le |\,\mathrm{pr}_Z - Z| + |Z - x_0| < 2\varepsilon r + \tfrac{1}{2}r < r,$$

and thus, $\mathrm{pr}_Z \in B$. In particular, since $\mathrm{pr}_Z \in L_B$ and $b\beta_{\partial\Omega}(B, L_B) < \varepsilon$, we have

$$\delta(Z) \le |Z - \mathrm{pr}_Z| + \delta(\mathrm{pr}_Z) \le 2\varepsilon r + \varepsilon r = 3\varepsilon r < \frac{1}{4D} r \tag{4.6}$$

since $\varepsilon < \frac{1}{12D}$. In particular, by (4.5) and (4.6), we know that $Y_0 \notin \frac{1}{2}B \cap \mathcal{U}_{2\varepsilon r}(L_B)$. On the other hand, since $Y_0 \in \gamma \cap \partial B\big(x_0, \frac{r}{4}\big)$, we know that $Y_0 \in B \cap \Omega$ and hence $Y_0 \in B^+$ by Lemma 4.3. In particular, $Y_0 \in \gamma \cap B^+ \setminus \mathcal{U}_{2\varepsilon r}(L_B)$, which proves the claim. □

Lemma 4.7. *Let $\varepsilon_0 > 0$ be as in Lemma 4.3 and suppose that $0 < \varepsilon < \frac{1}{8}\varepsilon_0$. Let $B = B(x_0, r)$ be a ball with $x_0 \in \partial\Omega$, L_B be a hyperplane such that $b\beta_{\partial\Omega}(B, L_B) < \varepsilon$ and $y_0 \in \frac{1}{2}B \cap \partial\Omega$. Now we have*

- $b\beta_{\partial\Omega}\big(B\big(y_0, \frac{r}{4}\big), L_B\big) < 4\varepsilon < \frac{1}{2}\varepsilon_0$,
- *the set $\Omega \cap B\big(y_0, \frac{r}{4}\big) \setminus \mathcal{U}_{\varepsilon r}(L_B)$ is convex,*
- *for any point $Y \in \Omega \cap B\big(y_0, \frac{r}{4}\big) \setminus \mathcal{U}_{2\varepsilon r}(L_B)$ we have $\delta(Y) \ge \varepsilon r$.*

Proof. The second claim follows from the first claim combined with Lemma 4.3, and the first claim follows from the definition of β-numbers and the facts $B\big(y_0, \frac{r}{4}\big) \subset B$ and $b\beta_{\partial\Omega}(B, L_B) < \varepsilon < \frac{1}{8}\varepsilon_0$ in a straightforward way:

$$\begin{aligned} b\beta_{\partial\Omega}(B, L_B) &= \sup_{y \in B \cap \partial\Omega} \frac{\mathrm{dist}(y, L_B)}{r} + \sup_{Y \in L_B \cap B} \frac{\delta(Y)}{r} \\ &\ge \frac{1}{4} \sup_{y \in B(y_0, r/4) \cap \partial\Omega} \frac{\mathrm{dist}(y, L_B)}{\frac{r}{4}} + \sup_{Y \in L_B \cap B(y_0, r/4)} \frac{\delta(Y)}{\frac{r}{4}} = \frac{1}{4} b\beta_{\partial\Omega}\Big(B\Big(y_0, \frac{r}{4}\Big), L\Big). \end{aligned}$$

For the third claim, let $Y \in \Omega \cap B\big(y_0, \frac{r}{4}\big) \setminus \mathcal{U}_{2\varepsilon r}(L_B)$. Since $y_0 \in \partial\Omega$, we know that $\delta(Y) < \frac{r}{4}$. On the other hand, for any point $Z \in \mathbb{R}^{n+1} \setminus B$ we have

$$r \le |x_0 - Z| \le |x_0 - y_0| + |y_0 - Y| + |Y - Z| < \frac{r}{2} + \frac{r}{4} + |Y - Z|,$$

and thus, $|Y - Z| > \frac{r}{4}$. In particular, we have $\delta(Y) = \inf_{z \in \partial\Omega \cap B} |Y - z|$. Let $z_Y \in \partial\Omega \cap B$ be a point such that $\delta(Y) = |Y - z_Y|$. Since $Y \notin \mathcal{U}_{2\varepsilon r}(L_B)$, we know that $\mathrm{dist}(Y, L_B) \ge 2\varepsilon r$, and since $b\beta_{\partial\Omega}(B, L_B) < \varepsilon$, we know that $\mathrm{dist}(z_Y, L_B) < \varepsilon r$. In particular,

$$2\varepsilon r \le \mathrm{dist}(Y, L_B) \le |Y - z_Y| + \mathrm{dist}(z_Y, L_B) < |Y - z_Y| + \varepsilon r = \delta(Y) + \varepsilon r,$$

and therefore $\delta(Y) \ge \varepsilon r$, as claimed. □

5. Local John and exterior corkscrews imply Harnack chains

In this section, we prove Theorem 1.2, i.e., that the local John condition together with the exterior corkscrew condition implies the existence of Harnack chains. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with n-ADR boundary $\partial\Omega$, and suppose that Ω satisfies the local D_1-John condition and exterior corkscrew condition. Throughout the section, $\mathbb{D}$ is a dyadic system on $\partial\Omega$.

Proof of Theorem 1.2. Let $X, Y \in \Omega$ with $\delta(X), \delta(Y) \ge \rho > 0$ and $|X - Y| < \Lambda\rho$ for $\Lambda \ge 1$. We start by noticing that by Theorem 2.11 we know that $\partial\Omega$ is UR, and by Lemma 3.6 we know that harmonic measure belongs to the class weak-$A_\infty(\sigma)$.

We will construct a path between X and Y that stays far away from the boundary and use this path to construct the Harnack chain between X and Y. To ensure that we stay away from the boundary in a quantitative way, we have to be careful with the construction. This will make the construction quite technical and therefore we divide the proof into a few different parts.

The argument we present below works as it is in the case $\operatorname{diam}(\partial\Omega) = \infty$. We discuss the other cases in the end of the proof.

Part 1: choosing suitable cubes for X and Y for Lemma 2.19. Take a point $z_X \in \partial\Omega$ such that $\delta(X) = |X - z_X|$. Let c_0 and C_0 be the constants from Lemma 2.20. Now, by Lemma 2.20, since $X \in B(z_X, c_0 \cdot 2\delta(X)/c_0)$, we know that $\omega^X(\Delta(z_X, 2\delta(X)/c_0)) \geq 1/C_0$. Let us cover $\Delta(z_X, 2\delta(X)/c_0)$ by dyadic cubes Q_i of the same side length such that $\ell(Q_i) \approx 2\delta(X)/c_0$ and $X \notin 4B_{Q_i}$ for any i. There are at most c_n of these types of cubes Q_i. Since $\omega^X(\Delta(z_X, 2\delta(X)/c_0)) \geq 1/C_0$ and the cubes Q_i cover $\Delta(z_X, 2\delta(X)/c_0)$, we know that there exists a cube $Q_X \in \{Q_i\}_i$ such that $\omega^X(Q_X) \geq (c_n C_0)^{-1}$. In addition, the cube Q_X satisfies $\ell(Q_X) \approx \delta(X) \approx \operatorname{dist}(X, Q_X)$ with uniformly bounded implicit constants since

- one of the cubes Q_i contains the point z_X and $\delta(X) = |X - z_X|$,
- there is only a uniformly bounded number of the cubes Q_i, and
- $\ell(Q_i) \approx 2\delta(X)/c_0$ for every i.

Similarly, we can choose a cube Q_Y that has the same properties but with respect to Y instead of X.

Part 2: choosing a local John point. Let us consider the ball $B(z_X, D_1 r_0)$, where

$$r_0 := C_n \cdot \max\{\delta(X), \delta(Y), |X - Y|\} \tag{5.1}$$

for a large enough dimensional constant C_n such that $Q_X, Q_Y \subset B(z_X, D_1 r_0)$. Let $Z_0 \in \Omega$ be a local John point for the ball $B(z_X, D_1 r_0)$. By the local John condition, we know that $B(Z_0, D_1 r_0/D_1) = B(Z_0, r_0) \subset \Omega$ and there exist D_1-nontangential paths from the points of $\Delta(z_X, D_1 r_0)$ to Z_0. In particular, there exist D_1-nontangential paths from the points on Q_X and Q_Y to Z_0.

Part 3: choosing starting points for paths. Let us consider the cube Q_X. By the choice of Q_X and Lemma 2.19, we know that there exist constants $\alpha \in (0, 1]$ and $D_2 \geq 1$ (independent of X and Q_X) and a subset $A_X \subset Q_X$ such that

- $\sigma(A_X) \geq \alpha\, \sigma(Q_X)$, and
- there exist D_2-nontangential paths from the points on A_X to X.

Let $\varepsilon_0 > 0$ be as in Lemma 4.3 and set

$$\varepsilon := \frac{1}{8} \min\left\{\frac{1}{10}\varepsilon_0, \frac{1}{12D_1}, \frac{1}{12D_2}\right\}. \tag{5.2}$$

By the bilateral weak geometric lemma (Lemma 4.1) and Lemma 2.16, we know that there exists a cube R_X such that $b\beta_{\partial\Omega}(2B_{R_X}) < \varepsilon$, $\sigma(R_X \cap A_X) > 0$ and $\ell(R_X) \approx \ell(Q_X)$, where the implicit constant depends only on α and the Carleson packing norm of the collection of cubes Q such that $b\beta_{\partial\Omega}(2B_Q) > \varepsilon$. Let us choose any point $\tilde{z}_X \in R_X \cap A_X$. Similarly, we can choose sets A_Y and R_Y and a point $\tilde{z}_Y$ for the point Y.

Part 4: constructing paths between the local John point and X and Y. Let us recap:

- $R_X \subset \partial\Omega$ is a dyadic cube such that $\ell(R_X) \approx \delta(X)$.
- $\varepsilon > 0$ is a number defined in (5.2) and we have $b\beta_{\partial\Omega}(2B_{R_X}) < \varepsilon$.
- $\tilde{z}_X \in R_X$ is a point such that there exists a D_2-nontangential path from $\tilde{z}_X$ to X.
- $Z_0 \in \Omega$ is the local John point for the ball $B(z_X, D_1 r_0)$, where $|X - z_X| = \delta(X)$ and $r_0 > 0$ is the radius defined in (5.1), and $\tilde{z}_X \in B(z_X, D_1 r_0)$.

Let γ_1 be a D_1-nontangential path from $\tilde{z}_X$ to Z_0 and γ_2 be a D_2-nontangential path from $\tilde{z}_X$ to X. Let L_X be a hyperplane such that $b\beta_{\partial\Omega}(2B_{R_X}, L_X) < \varepsilon$ and consider the ball $B\big(\tilde{z}_X, \frac{1}{2}\ell(R_X)\big)$. Since we know that $\tilde{z}_X \in R_X \subset B_{R_X} \subset 2B_{R_X}$, Lemma 4.7 gives us

$$b\beta_{\partial\Omega}\Big(B\Big(\tilde{z}_X, \frac{1}{2}\ell(R_X)\Big), L_X\Big) < 4\varepsilon < \frac{1}{2}\min\Big\{\frac{1}{10}\varepsilon_0, \frac{1}{12D_1}, \frac{1}{12D_2}\Big\}.$$

By Lemma 4.4 (applied for $b\beta_{\partial\Omega}\big(B\big(\tilde{z}_X, \frac{1}{2}\ell(R_X)\big), L_X\big) < 4\varepsilon$), we know that both γ_1 and γ_2 intersect

$$B\big(\tilde{z}_X, \tfrac{1}{2}\ell(R_X)\big) \setminus \mathcal{U}_{2\cdot 4\varepsilon\cdot\ell(R_X)/2}(L_X) = B\big(\tilde{z}_X, \tfrac{1}{2}\ell(R_X)\big) \setminus \mathcal{U}_{2\varepsilon\cdot 2\ell(R_X)}(L_X).$$

Let $Z_1 \in \gamma_1$ and $X_1 \in \gamma_2$ be any points such that $Z_1, X_1 \in B\big(\tilde{z}_X, \frac{1}{2}\ell(R_X)\big) \setminus \mathcal{U}_{2\varepsilon\cdot 2\ell(R_X)}(L_X)$, and let γ_3 be the line segment connecting Z_1 to X_1. By Lemma 4.7, we know that γ_3 is fully contained in $B\big(\tilde{z}_X, \frac{1}{2}\ell(R_X)\big) \setminus \mathcal{U}_{2\varepsilon\cdot 2\ell(R_X)}(L_X)$ and we have

$$\delta(\widehat{X}) \geq \varepsilon \cdot 2\ell(R_X) \gtrsim \delta(X) \tag{5.3}$$

for every $\widehat{X} \in \gamma_3$, where the implicit constant depends on ε_0, D_1, D_2 and the structural constants appearing in the proof. Since γ_3 is line segment that is fully contained in $B\big(\tilde{z}_X, \frac{1}{2}\ell(R_X)\big) \setminus \mathcal{U}_{2\varepsilon\cdot 2\ell(R_X)}(L_X)$, we know

$$\ell(\gamma_3) \leq \ell(R_X) \lesssim \delta(X). \tag{5.4}$$

See Figure 2 for an illustration of the situation.

We now build a path γ_X from Z_0 to X by gluing together (after rescaling) the reversed part of γ_1 that travels from Z_1 to Z_0, the whole γ_3 (from Z_1 to X_1) and the part of γ_2 that travels from X_1 to X. Similarly, we can choose a hyperplane L_Y for R_Y and points $Z_2, Y_1 \in B\big(\tilde{z}_Y, \frac{1}{2}\ell(R_Y)\big) \setminus \mathcal{U}_{2\varepsilon\cdot 2\ell(R_Y)}(L_Y)$ and construct a path γ_Y from Z_0 to Y that passes through Z_2 and Y_1.

Part 5: constructing the Harnack chains. Let us consider the path γ_X. Since γ_2 is a nontangential path (from $\tilde{z}_X$ to X) and $\delta(X_1) \approx \delta(X)$, we know that $\ell(\gamma_2) \lesssim \delta(X)$ and $\delta(\widehat{X}) \approx \delta(X)$ for every $\widehat{X} \in \gamma_2(X_1, X)$. By (5.4), we know that $\ell(\gamma_3) \lesssim \delta(X)$, and by (5.3), $\delta(\widehat{X}) \approx \delta(X)$ for every $\widehat{X} \in \gamma_3$. Thus, for a suitable uniform implicit constant, we may cover $\gamma_X \setminus \gamma_1$ by a uniformly bounded number of balls B_i with radii $r_i \approx \delta(X)$ satisfying $\operatorname{dist}(B_i, \partial\Omega) \approx \operatorname{diam}(B_i)$. As for γ_1, we notice that since γ_1 is a nontangential path (from $\tilde{z}_X$ to Z_0) given by the local John condition and $\delta(Z_1) \approx \delta(X)$, we have

$$\ell(\gamma_1(\tilde{z}_X, Z_1)) \lesssim \delta(Z_1) \approx \delta(X) \quad \text{and} \quad \ell(\gamma_1) \lesssim r_0 = C_n \cdot \max\{\delta(X), \delta(Y), |X - Y|\},$$

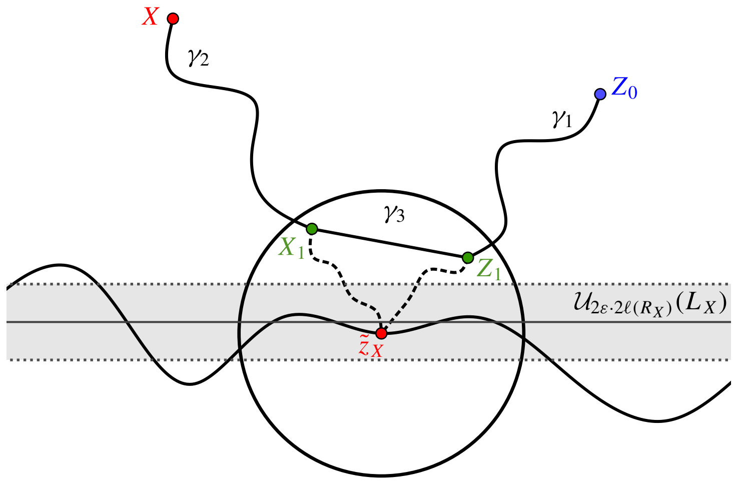

Figure 2. We construct a path between X and the local John point Z_0 that stays far away from the boundary the following way. By a careful choice of a point $\tilde{z}_X \in \partial\Omega$, we know that there exists a nontangential path γ_1 from $\tilde{z}_X$ to Z_0 (given by the local John condition) and a nontangential path γ_2 from $\tilde{z}_X$ to X (given by the weak-A_∞ property of harmonic measure through Lemma 2.19). By the BWGL, there exists a hyperplane L_X such that L_X approximates $B \cap \partial\Omega$ well for a suitable $\varepsilon > 0$ and a ball $B = B\big(\tilde{z}_X, \frac{1}{2}\ell(R_X)\big)$, where R_X is a dyadic cube containing $\tilde{z}_X$ such that $\ell(R_X) \approx \delta(X)$. By applications of the BWGL, we know that there exist points $Z_1 \in \gamma_1 \cap B \cap \Omega$ and $X_1 \in \gamma_2 \cap B \cap \Omega$ that do not lie on the strip $\mathcal{U}_{2\varepsilon\cdot 2\ell(R_X)}(L_X)$, i.e., they lie reasonably far away from the boundary. Because $\Omega \cap B \setminus \mathcal{U}_{2\varepsilon\cdot 2\ell(R_X)}(L_X)$ is convex, we can connect Z_1 and X_1 to each other with a line segment γ_3. We can now travel from Z_0 to X by using pieces of the paths γ_1, γ_3 and γ_2.

and therefore $\delta(\widehat{X}) \gtrsim \delta(X)$ for all $\widehat{X} \in \gamma_1(Z_1, Z_0)$. In particular, we can cover $\gamma_X \setminus (\gamma_2 \cup \gamma_3)$ by $N_0 \lesssim r_0/\delta(X)$ balls B_k of radii $r_k \approx \delta(X)$ such that $\operatorname{dist}(B_k, \partial\Omega) \approx \operatorname{diam}(B_k)$ for each k. Recall that $\delta(X), \delta(Y) \geq \rho > 0$ and $|X - Y| < \Lambda\rho$ for $\Lambda \geq 1$. Let us consider different cases:

- Suppose that $r_0 \lesssim |X - Y|$. Now we have
$$N_0 \lesssim \frac{|X - Y|}{\delta(X)} = \frac{\rho}{\delta(X)}\Lambda \leq \Lambda.$$
- Suppose that $r_0 = C_n\delta(X)$. Then
$$N_0 \lesssim \frac{C_n\delta(X)}{\delta(X)} = C_n \leq C_n\Lambda.$$
- Suppose that $r_0 = C_n\delta(Y)$ and $\delta(Y) \gg |X - Y|$. Then, by the triangle inequality, $\delta(X) \approx \delta(Y)$ and
$$N_0 \lesssim \frac{C_n\delta(Y)}{\delta(X)} \approx \frac{C_n\delta(X)}{\delta(X)} = C_n \leq C_n\Lambda.$$

By almost identical arguments, we know that same estimates hold for γ_Y. Thus, we can connect X to Y by taking the chain of balls B_i that covers γ_X and γ_Y. These balls satisfy $\operatorname{dist}(B_i, \partial\Omega) \approx \operatorname{diam}(B_i)$ for every i with possibly different implicit constants. We may choose a constant $\widetilde{C} \geq 1$ such that

$$\frac{1}{\widetilde{C}} \operatorname{diam}(B_i) \leq \operatorname{dist}(B_i, \partial\Omega) \leq \widetilde{C} \operatorname{diam}(B_i)$$

for every i since we used only a finite number of different constants in the construction (six, to be more precise, since we built γ_X and γ_Y using three pieces for each path with uniform implicit constants for each piece). We needed a uniformly bounded number of balls to cover the two out of three pieces of γ_X (same for γ_Y) and $N \lesssim \Lambda$ balls to cover the last piece of γ_X (same for γ_Y). Thus, the number of balls B_i is bounded by $\mathcal{C}\Lambda$, where $\mathcal{C}$ is a large enough constant depending only on n and the ADR, UR, local John, weak-A_∞ and corkscrew constants. Hence, $(B_i)_i$ is a Harnack chain between X and Y. This completes the proof for the case $\operatorname{diam}(\partial\Omega) = \infty$.

Let us then consider the remaining two cases. Suppose that $\operatorname{diam}(\partial\Omega) < \infty$ and $\operatorname{diam}(\Omega) < \infty$. Now, if $\max\{\delta(X), \delta(Y), |X - Y|\} \ll \operatorname{diam}(\partial\Omega)$, things work just as earlier. Thus, we may assume that $\max\{\delta(X), \delta(Y), |X - Y|\} \approx \operatorname{diam}(\partial\Omega)$. Since $\operatorname{diam}(\partial\Omega) < \infty$, there exists a point $Z_0 \in \Omega$ such that for any $z \in \partial\Omega$ there exists a D_1-nontangential path from z to Z_0. Now the previous proof works when we simply choose this "global" local John point instead of the point we chose in Part 2.

Finally, suppose that $\operatorname{diam}(\partial\Omega) < \infty$ and $\operatorname{diam}(\Omega) = \infty$. Let us consider the following three cases:

- If $\max\{\delta(X), \delta(Y), |X - Y|\} \ll \operatorname{diam}(\partial\Omega)$, we proceed as in the case "$\operatorname{diam}(\partial\Omega) = \infty$".
- If $\delta(X) \lesssim \operatorname{diam}(\partial\Omega)$, $\delta(Y) \lesssim \operatorname{diam}(\partial\Omega)$ and $|X - Y| \approx \operatorname{diam}(\partial\Omega)$, we proceed as in the case "$\operatorname{diam}(\partial\Omega) < \infty$ and $\operatorname{diam}(\Omega) < \infty$".
- If $\delta(X) \gg \operatorname{diam}(\partial\Omega)$ and $\delta(Y) \gg \operatorname{diam}(\partial\Omega)$, we can construct a Harnack chain from X to Y in the simple geometry of $\mathbb{R}^{n+1} \setminus B(Z, s)$ for $Z \in \Omega$ and $s \approx \operatorname{diam}(\partial\Omega)$.

Thus, we may assume that $\delta(X) \lesssim \operatorname{diam}(\partial\Omega)$ and $\delta(Y) \gg \operatorname{diam}(\partial\Omega)$. The previous procedure does not work directly in this case because we cannot apply Lemmas 2.20 and 2.19 for $\delta(Y) \gg \operatorname{diam}(\partial\Omega)$. Instead, we connect Y to a point that is close enough to the boundary and then connect this point to X.

Let $\hat{y} \in \partial\Omega$ be a point such that $\delta(Y) = |Y - \hat{y}|$. Let us consider the line segment L with endpoints $\hat{y}$ and Y. Since L is a line segment and $\delta(Y) = |Y - \hat{y}|$, we have $\delta(Z) = |Z - \hat{y}|$ for any $Z \in L$. Let us take a point $\widehat{Y} \in L$ such that $\delta(\widehat{Y}) = \delta(X)$. Now we can use the earlier procedure to construct a Harnack chain $(B_i)_i$ from X to $\widehat{Y}$, possibly using a "global" local John point as we did in the case "$\operatorname{diam}(\partial\Omega) < \infty$ and $\operatorname{diam}(\Omega) < \infty$". Since $\delta(X) = \delta(\widehat{Y}) \lesssim \operatorname{diam}(\partial\Omega)$ and $\delta(Y) \gg \operatorname{diam}(\partial\Omega)$, the length of the chain $(B_i)_i$ depends only on

$$\frac{|X - \widehat{Y}|}{\delta(X)} \lesssim \frac{|X - Y|}{\delta(X)} \leq \frac{\rho}{\delta(X)} \Lambda \leq \Lambda.$$

We then continue this chain from $\widehat{Y}$ to Y by covering the line segment from $\widehat{Y}$ to Y using balls $\widehat{B}_k$ with radii $r_k \approx \delta(\widehat{Y}) = \delta(X)$ such that $\operatorname{dist}(\widehat{B}_k, \partial\Omega) \approx \operatorname{diam}(\widehat{B}_k)$. The number of balls $\widehat{B}_k$ that we need is approximately

$$\frac{\ell(L)}{\delta(X)} \approx \frac{\delta(Y)}{\delta(X)} \approx \frac{|X - Y|}{\delta(X)} \leq \frac{\rho}{\delta(X)} \Lambda \leq \Lambda,$$

where we used that $\delta(X) \lesssim \operatorname{diam}(\partial\Omega)$ and $\delta(Y) \gg \operatorname{diam}(\partial\Omega)$ (and therefore we have $\delta(Y) \approx |X - Y|$). Hence, combining the chains $(B_i)_i$ and $(\widehat{B}_k)_k$ gives us a Harnack chain from X to Y. This completes the proof of the last case. □

6. Weak (1,1)-version of the Hofmann–Mitrea–Taylor Poincaré inequality and quasiconvexity

Let Ω be a 2-sided chord-arc domain. In this section, we consider some parts of the Hofmann–Mitrea–Taylor theory that we need for the proof of Theorem 1.3. Recall the definitions of the tangential gradient $\nabla_{\mathrm{t}} f$, the Hofmann–Mitrea–Taylor Sobolev space $L_1^p(\partial\Omega)$ and the Hofmann–Mitrea–Taylor tangential derivatives $\partial_{\mathrm{t},j,k} f$ and gradient $\nabla_{\mathrm{HMT}} f$ in Section 2D. It is straightforward to check that for a compactly supported Lipschitz function f on $\partial\Omega$ we have $f \in L_1^p(\partial\Omega)$ for every $1 < p < \infty$ and hence the Hofmann–Mitrea–Taylor gradient $\nabla_{\mathrm{HMT}} f$ exists.

Remark 6.1. In this section and in Section 7, we mostly consider compactly supported Lipschitz functions but this is enough for our purposes: by the results of [Keith 2003] (see Theorem 7.12 below) and [Mourgoglou and Tolsa 2021] (see Lemmas 6.2 and 6.4 below), verifying Theorem 1.3 for compactly supported Lipschitz functions implies Corollary 1.6, which then allows us to give up the assumption about compact support for Theorem 1.3. Assuming that our Lipschitz functions are compactly supported ensures that the Hofmann–Mitrea–Taylor gradients exist and we can use the machinery in [Hofmann et al. 2010; Mourgoglou and Tolsa 2021] without additional considerations.

Let us start by recalling two lemmas from [Mourgoglou and Tolsa 2021]. First, when considering compactly supported Lipschitz functions, the norm of the Hofmann–Mitrea–Taylor gradient agrees with the norm of the tangential gradient almost everywhere:

Lemma 6.2 [Mourgoglou and Tolsa 2021, Lemma 6.4]. *Let f be a compactly supported Lipschitz function on $\partial\Omega$. Then*

$$|\nabla_{\mathrm{t}} f| = |\nabla_{\mathrm{HMT}} f| \tag{6.3}$$

σ-almost everywhere.

In fact, [Mourgoglou and Tolsa 2021, Lemma 6.4] shows us that $\nabla_{\mathrm{t}} f = -\nabla_{\mathrm{HMT}} f$ almost everywhere, but we only need the comparability of the norms. We note that that lemma is formulated for bounded domains but a routine inspection of the proof shows us that it holds for compactly supported Lipschitz functions also in unbounded domains.

Furthermore, for Lipschitz functions, the norm of the tangential gradient (and hence the norm of the Hofmann–Mitrea–Taylor gradient) agrees almost everywhere with the "local Lipschitz constant" function:

Lemma 6.4 [Mourgoglou and Tolsa 2021, Lemma 2.2]. *Let f be a Lipschitz function on $\partial\Omega$. Then*

$$|\nabla_{\mathrm{t}} f(x)| = \limsup_{\partial\Omega \ni y \to x} \frac{|f(x) - f(y)|}{|x-y|} \approx \limsup_{r \to 0} \fint_{\Delta(x,r)} \frac{|f(x) - f(y)|}{|x-y|}\, d\sigma(y) \tag{6.5}$$

for σ-a.e. $x \in \partial\Omega$.

As a part of their extensive work, Hofmann, Mitrea and Taylor proved the following weak (p, p)-Poincaré inequality with a tail:

Theorem 6.6 [Hofmann et al. 2010, Proposition 4.13]. *Let $1 < p < \infty$. There exists a constant $C = C(\Omega, p)$ such that for every $f \in L^p_1(\partial\Omega)$, $x \in \partial\Omega$ and $r > 0$ we have*

$$\left(\fint_{\Delta} |f - \langle f\rangle_\Delta|^p \, d\sigma\right)^{1/p} \leq Cr\left(\fint_{5\Delta} |\nabla_{\mathrm{HMT}} f|^p \, d\sigma\right)^{1/p} + Cr\sum_{j=2}^{\infty} \frac{2^{-j}}{\sigma(2^j\Delta)} \int_{2^j\Delta\setminus 2^{j-1}\Delta} |\nabla_{\mathrm{HMT}} f| \, d\sigma, \tag{6.7}$$

where $\Delta := \Delta(x, r)$.

Some remarks related to the formulation of Theorem 6.6 are in order. In [Hofmann et al. 2010], Theorem 6.6 is formulated assuming only that $\partial\Omega$ is Ahlfors–David regular and Ω satisfies the 2-sided local (D_0, R_0)-John condition (recall Definition 3.4), and the result holds for all $r \in (0, R_0)$ instead of all $r \in (0, \operatorname{diam}(\partial\Omega))$. As we noted in Example 3.5, if R_0 is not large enough, the 2-sided local (D_0, R_0)-John condition is not strong enough to imply that $\partial\Omega$ is connected. Since Heinonen–Koskela-type weak Poincaré inequalities (recall Definition 2.29) imply connectivity (see, e.g., [Cheeger 1999, Theorem 17.1]), we know that the Hofmann–Mitrea–Taylor Poincaré inequalities may hold even when Heinonen–Koskela-type weak Poincaré inequalities fail.

This being said, if Ω is a 2-sided chord-arc domain (as we assumed in the beginning of this section), then the Hofmann–Mitrea–Taylor Poincaré inequalities hold as in Theorem 6.6 and they self-improve to Heinonen–Koskela-type Poincaré inequalities. The first claim is due to [Hofmann et al. 2010, Lemma 3.13]: any NTA domain satisfies the local (D_0, R_0)-John condition where R_0 is the upper bound for the scales of the corkscrew conditions. Since we assume that the corkscrew conditions hold for any $0 < r < \operatorname{diam}(\partial\Omega)$, we get the local D_0-John condition. The second claim follows from Corollary 1.6 which we prove using a Hofmann–Mitrea–Taylor-type Poincaré estimate (see Lemma 6.8 below) in the next section.

For the case $p > 1$, we can use directly the estimate (6.7) together with the arguments in the next section to obtain a weak p-Poincaré inequality. However, for the case $p = 1$, we need to revisit estimate (6.7) since an inspection of its proof shows us that a simple limiting argument does not work. Because of this, we prove the following weak $(1, 1)$-type version of the Hofmann–Mitrea–Taylor Poincaré inequality:

Lemma 6.8. *There exists a constant $C = C(\Omega)$ such that the following holds. For every compactly supported Lipschitz function f on $\partial\Omega$ and every $\xi \in \partial\Omega$, $r > 0$ and $\Delta := \Delta(\xi, r)$, there exists $a_\Delta \in \mathbb{R}$ such that*

$$\sup_{\lambda>0} \lambda\,\sigma(\{y \in \Delta : |f(y) - a_\Delta| > \lambda\}) \leq Cr\int_{5\Delta} |\nabla_{\mathrm{HMT}} f| \, d\sigma + Cr\sum_{j=2}^{\infty} 2^{-j(n+1)} \int_{2^j\Delta\setminus 2^{j-1}\Delta} |\nabla_{\mathrm{HMT}} f| \, d\sigma. \tag{6.9}$$

The proof of Lemma 6.8 is similar to the proof of estimate (6.7), which uses heavily the machinery built in [Hofmann et al. 2010]. For the convenience of the reader, we provide the key arguments below. For the proof, we recall the definitions of the Riesz transform $\mathcal{R}_\sigma$, the maximal truncation of the Riesz transform $\mathcal{R}_{\sigma,*}$ and the double layer potential $\mathcal{D}$ of a suitable function f on $\partial\Omega$: for $\varepsilon > 0$, $X \in \mathbb{R}^{n+1}$ and $Z \in \Omega$, we set

$$\mathcal{R}_{\sigma,\varepsilon} f(X) := \frac{1}{C_n} \int_{\partial\Omega\setminus B(X,\varepsilon)} \frac{X - y}{|X - y|^{n+1}} f(y) \, d\sigma(y), \qquad \mathcal{R}_\sigma f(X) := \lim_{\varepsilon\to 0} \mathcal{R}_{\sigma,\varepsilon} f(X),$$

$$\mathcal{D} f(Z) := \frac{1}{C_n} \int_{\partial\Omega} \frac{\nu(y)\cdot(y - Z)}{|Z - y|^{n+1}} f(y) \, d\sigma(y), \qquad \mathcal{R}_{\sigma,*} f(X) := \sup_{\varepsilon>0} |\mathcal{R}_{\sigma,\varepsilon} f(X)|,$$

where ν is the measure-theoretic outer unit normal of Ω (see (2.2.11) in [Hofmann et al. 2010]) and C_n is the surface area of the unit sphere in $\mathbb{R}^{n+1}$. We extend $\mathcal{D}f$ to the whole $\mathbb{R}^{n+1} \setminus \partial\Omega$ by changing the direction of the normal ν for points $X \in \operatorname{int}\Omega^{\mathrm{c}}$.

Proof. Let f be a compactly supported Lipschitz function on $\partial\Omega$, $\xi \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$. Let $\Delta := \Delta(\xi, r) = B(\xi, r) \cap \partial\Omega$. By the theory of layer potentials we know that for σ-a.e. $x \in \partial\Omega$ we have

$$f(x) = \mathcal{D}^+ f(x) - \mathcal{D}^- f(x),$$

where $\mathcal{D}^+ f(x)$ and $\mathcal{D}^- f(x)$ are the inner and outer nontangential limits of $\mathcal{D}f$ at x, respectively (see, e.g., [Hofmann et al. 2010, Section 3.3] and apply the results separately for Ω and $\operatorname{int}\Omega^{\mathrm{c}}$). These nontangential limits exist σ-almost everywhere. For a constant vector h_Δ to be fixed below, we consider the function

$$u(X) := \mathcal{D}f(X) - X \cdot h_\Delta, \quad X \in \mathbb{R}^{n+1} \setminus \partial\Omega.$$

Then, by the σ-a.e. existence of the limits $\mathcal{D}^\pm f$, the inner and outer nontangential limits $u^\pm(x)$ of u exist for σ-a.e. $x \in \partial\Omega$. Thus, we have

$$f(x) = u^+(x) - u^-(x) \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega.$$

To prove the estimate (6.9), we choose

$$a_\Delta = u(X_\Delta^+) - u(X_\Delta^-),$$

where X_Δ^+ and X_Δ^- are interior and exterior local John points inside $B(\xi, r)$, respectively. Since $B(X_\Delta^\pm, cr) \subset B(\xi, r) \setminus \partial\Omega$, we know that $\delta(X_\Delta^\pm) \approx r$. Now, for any $\lambda > 0$, we have

$$\begin{aligned}\sigma(\{x \in \Delta : |f(x) - a_\Delta| > \lambda\}) \\ \le \sigma\Big(\Big\{x \in \Delta : |u^+(x) - u(X_\Delta^+)| > \frac{\lambda}{2}\Big\}\Big) + \sigma\Big(\Big\{x \in \Delta : |u^-(x) - u(X_\Delta^-)| > \frac{\lambda}{2}\Big\}\Big).\end{aligned} \tag{6.10}$$

We only estimate the first term on the right-hand side of (6.10) since the second one is estimated similarly. For any $x \in \Delta$, let γ_x^+ be a nontangential path in Ω from x to X_Δ^+. Such a nontangential path with a uniform constant exists for every $x \in \Delta$ by the local John condition. Since γ_x^+ is a nontangential path with a uniform constant, we know that $\gamma_x^+ \subset B(\xi, Ar)$ for some fixed $A \ge 1$. Then, for any $Y \in \gamma_x^+ \cap \Omega$, the mean value theorem gives us

$$|u(Y) - u(X_\Delta^+)| \le \mathcal{H}^1(\gamma_x^+) \sup_{Z \in \gamma_x^+ \cap \Omega} |\nabla u(Z)| \lesssim r\, N_*(\chi_{B(\xi,Ar)} |\nabla u|)(x),$$

where N_* is the nontangential maximal operator for suitable aperture constant $\alpha > 1$ as defined in (2.4) (recall that a D-nontangential path from $x \in \partial\Omega$ to $X \in \Omega$ travels inside the cone $\Gamma_D(x)$, like we discussed in the beginning of Section 3). Letting $Y \to x$ then gives us

$$|u^+(x) - u(X_\Delta^+)| \lesssim r\, N_*(\chi_{B(\xi,Ar)} |\nabla u|)(x).$$

Thus,

$$\sigma\Big(\Big\{x \in \Delta : |u^+(x) - u(X_\Delta^+)| > \frac{\lambda}{2}\Big\}\Big) \lesssim \sigma(\{x \in \Delta : r\, N_*(\chi_{B(\xi,Ar)} |\nabla u|)(x) > c\lambda\}). \tag{6.11}$$

Let us then estimate the right-hand side of (6.11). We notice that for all $X \in \Gamma_\alpha(x) \cap B(\xi, Ar)$ it holds that $\nabla u(X) = \nabla \mathcal{D} f(X) - h_\Delta$. By (3.6.29) in [Hofmann et al. 2010], we know that for all $X \in \Omega$ it holds that

$$\nabla \mathcal{D} f(X) = \biggl(\sum_i \int_{\partial\Omega} \partial_i \mathcal{E}(X-y)\, \partial_{\mathrm{t},j,i} f(y)\, d\sigma(y)\biggr)_{1\le j\le n+1},$$

where $\mathcal{E}$ is the fundamental solution to the Laplacian in $\mathbb{R}^{n+1}$, that is,

$$\mathcal{E}(X) := \begin{cases} \dfrac{1}{C_n(1-n)} \dfrac{1}{|X|^{n-1}} & \text{if } n \ge 2, \\ \dfrac{1}{2\pi} \log |X| & \text{if } n = 1 \end{cases} \tag{6.12}$$

for $X \in \mathbb{R}^{n+1} \setminus \{0\}$, where C_n is the surface area of the unit sphere in $\mathbb{R}^{n+1}$. Thus, choosing

$$h_\Delta := \biggl(\sum_i \int_{\partial\Omega \setminus 2A\Delta} \partial_i \mathcal{E}(\xi - y)\, \partial_{\mathrm{t},j,i} f(y)\, d\sigma(y)\biggr)_{1\le j\le n+1}$$

gives us

$$\begin{aligned} \nabla u(X) &= \biggl(\sum_i \int_{2A\Delta} \partial_i \mathcal{E}(X-y)\, \partial_{\mathrm{t},j,i} f(y)\, d\sigma(y)\biggr)_{1\le j\le n+1} \\ &\qquad + \biggl(\sum_i \int_{\partial\Omega\setminus 2A\Delta} (\partial_i \mathcal{E}(X-y) - \partial_i \mathcal{E}(\xi - y))\, \partial_{\mathrm{t},j,i} f(y)\, d\sigma(y)\biggr)_{1\le j\le n+1} \\ &= \biggl(\sum_i \mathcal{R}_{i,\sigma}(\chi_{2A\Delta}\, \partial_{\mathrm{t},j,i} f)(X)\biggr)_{1\le j\le n+1} \\ &\qquad + \biggl(\sum_i \mathcal{R}_{i,\sigma}(\chi_{(2A\Delta)^{\mathrm{c}}}\, \partial_{\mathrm{t},j,i} f)(X) - \mathcal{R}_{i,\sigma}(\chi_{(2A\Delta)^{\mathrm{c}}}\, \partial_{\mathrm{t},j,i} f)(\xi)|\biggr)_{1\le j\le n+1} \\ &=: I(X) + II(X), \end{aligned} \tag{6.13}$$

where $\mathcal{R}_{i,\sigma}$ stands for the i-th component of the Riesz transform $\mathcal{R}_\sigma$. We then have

$$\begin{aligned} &\sigma(\{x \in \Delta : r\, N_*(\chi_{B(\xi,Ar)}|\nabla u|)(x) > c\lambda\}) \\ &\qquad \le \sigma\Bigl(\Bigl\{x \in \Delta : r\, N_*(\chi_{B(\xi,Ar)}|I|)(x) > \frac{c\lambda}{2}\Bigr\}\Bigr) + \sigma\Bigl(\Bigl\{x \in \Delta : r\, N_*(\chi_{B(\xi,Ar)}|II|)(x) > \frac{c\lambda}{2}\Bigr\}\Bigr). \end{aligned}$$

We first estimate the term $\sigma\bigl(\bigl\{x \in \Delta : r\, N_*(\chi_{B(\xi,Ar)}|I|)(x) > \frac{1}{2}c\lambda\bigr\}\bigr)$. Let $x \in \Delta$ and $Y \in \Gamma_\alpha(x) \cap B(\xi, Ar)$, where $\Gamma_\alpha(x)$ is the cone at x with aperture α (recall (2.5)). Let $\varepsilon = \varepsilon_{x,Y} := 2|x - Y|$ and $\Delta_\varepsilon := \Delta(x, \varepsilon)$. We then have

$$\begin{aligned} |I(Y)| &\le \sum_{i,j} |\mathcal{R}_{i,\sigma}(\chi_{2A\Delta}\, \partial_{\mathrm{t},j,i} f)(Y)| \\ &\le \sum_{i,j} |\mathcal{R}_{i,\sigma}(\chi_{2A\Delta\cap\Delta_\varepsilon}\, \partial_{\mathrm{t},j,i} f)(Y)| + \sum_{i,j} |\mathcal{R}_{i,\sigma}(\chi_{2A\Delta\setminus\Delta_\varepsilon}\, \partial_{\mathrm{t},j,i} f)(Y) - \mathcal{R}_{i,\sigma}(\chi_{2A\Delta\setminus\Delta_\varepsilon}\, \partial_{\mathrm{t},j,i} f)(x)| \\ &\qquad + \sum_{i,j} |\mathcal{R}_{i,\sigma}(\chi_{2A\Delta\setminus\Delta_\varepsilon}\, \partial_{\mathrm{t},j,i} f)(x)|. \end{aligned} \tag{6.14}$$

Since $\operatorname{dist}(Y, \partial\Omega) \approx \varepsilon$, the first term on the right-hand side of (6.14) satisfies

$$\begin{aligned} \sum_{i,j} |\mathcal{R}_{i,\sigma}(\chi_{2A\Delta\cap\Delta_\varepsilon}\, \partial_{\mathrm{t},j,i} f)(Y)| &\lesssim \sum_{i,j} \int_{\Delta_\varepsilon} \frac{|\chi_{2A\Delta}\, \partial_{\mathrm{t},j,i} f|}{\varepsilon^n}\, d\sigma \\ &\approx \sum_{i,j} \fint_{\Delta_\varepsilon} |\chi_{2A\Delta}\, \partial_{\mathrm{t},j,i} f|\, d\sigma \le M(\chi_{2A\Delta}|\nabla_{\mathrm{HMT}} f|)(x), \end{aligned}$$

where we used Ahlfors–David regularity for the estimate $\varepsilon^n \approx \sigma(\Delta(x,\varepsilon))$ and M stands for the centered maximal Hardy–Littlewood operator on $\partial\Omega$. For the second term on the right-hand side of (6.14), we get

$$\begin{aligned}\sum_{i,j} |\mathcal{R}_{i,\sigma}(\chi_{2A\Delta\setminus\Delta_\varepsilon}\, \partial_{\mathrm{t},j,i} f)(Y) - \mathcal{R}_{i,\sigma}(\chi_{2A\Delta\setminus\Delta_\varepsilon}\, \partial_{\mathrm{t},j,i} f)(x)| \\ &\overset{\text{(A)}}{\lesssim} \int_{2A\Delta\setminus\Delta_\varepsilon} \frac{|Y-x|}{|x-y|^{n+1}} |\nabla_{\mathrm{HMT}} f(y)|\, d\sigma(y) \\ &\lesssim \int_{\partial\Omega\setminus\Delta_\varepsilon} \frac{\varepsilon}{|x-y|^{n+1}} |\chi_{2A\Delta}(y)\nabla_{\mathrm{HMT}} f(y)|\, d\sigma(y) \\ &\overset{\text{(B)}}{\lesssim} \sum_{k=1}^{\infty} \int_{2^k\Delta_\varepsilon\setminus 2^{k-1}\Delta_\varepsilon} \frac{\varepsilon}{(2^k\varepsilon)^{n+1}} |\chi_{2A\Delta}(y)\nabla_{\mathrm{HMT}} f(y)|\, d\sigma(y) \\ &\overset{\text{(C)}}{\lesssim} \sum_{k=1}^{\infty} \frac{2^{-k}}{\sigma(2^k\Delta_\varepsilon)} \int_{2^k\Delta_\varepsilon\setminus 2^{k-1}\Delta_\varepsilon} |\chi_{2A\Delta}\nabla_{\mathrm{HMT}} f|\, d\sigma \lesssim M(\chi_{2A\Delta}\nabla_{\mathrm{HMT}} f)(x),\end{aligned}$$

where we used

(A) the mean value theorem for the Riesz kernel functions $X = (X_1, X_2, \dots, X_{n+1}) \mapsto X_i/|X|^{n+1}$ and the estimate $|\nabla X_i/|X|^{n+1}| \lesssim 1/|X|^{n+1}$,

(B) the fact that x is the center point of Δ_ε and hence, $|x-y| \approx 2^k\varepsilon$ for $y \in 2^k\Delta_\varepsilon \setminus 2^{k-1}\Delta_\varepsilon$,

(C) Ahlfors–David regularity.

In addition, the last term on the right-hand side of (6.14) is bounded above by the sum

$$\sum_{i,j} \mathcal{R}_{i,\sigma,*}(\chi_{2A\Delta}\, \partial_{\mathrm{t},j,i} f)(x),$$

where $\mathcal{R}_{i,\sigma,*}$ stands for the maximal truncation of the Riesz transform defined using only the i-th component of $\mathcal{R}_\sigma$. Thus,

$$|I(Y)| \lesssim M(\chi_{2A\Delta}\nabla_{\mathrm{HMT}} f)(x) + \sum_{i,j} \mathcal{R}_{i,\sigma,*}(\chi_{2A\Delta}\, \partial_{\mathrm{t},j,i} f)(x),$$

and so

$$\begin{aligned}\sigma\Big(\Big\{x \in \Delta : r\, N_*(\chi_{B(\xi,2r)}|I|)(x) > \frac{c\lambda}{2}\Big\}\Big) \\ \le \sigma\Big(\Big\{x \in \Delta : M(\chi_{2A\Delta}\nabla_{\mathrm{HMT}} f)(x) > \frac{c'\lambda}{r}\Big\}\Big) + \sum_{i,j} \sigma\Big(\Big\{x \in \Delta : \mathcal{R}_{i,\sigma,*}(\chi_{2A\Delta}\, \partial_{\mathrm{t},j,i} f)(x) > \frac{c'\lambda}{r}\Big\}\Big).\end{aligned}$$

Since $\partial\Omega$ is uniformly rectifiable, $\mathcal{R}_\sigma$ is bounded from $L^2(\sigma)$ to $L^2(\sigma)$ [David and Semmes 1991], and therefore $\mathcal{R}_{i,\sigma,*}$ is of weak type-(1, 1) with respect to σ by classical Calderón–Zygmund-type techniques; see, e.g., [Grafakos 2014, Section 5]. This and the weak type-(1, 1) of the Hardy–Littlewood maximal operator, combined with the previous estimates, then give us

$$\sigma\Big(\Big\{x \in \Delta : r\, N_*(\chi_{B(\xi,Ar)}|I|)(x) > \frac{c\lambda}{2}\Big\}\Big) \lesssim \frac{r}{\lambda} \int_{2A\Delta} |\nabla_{\mathrm{HMT}} f|\, d\sigma. \tag{6.15}$$

Let us then consider the term $II(X)$ in (6.13). For $x \in \Delta$ and $X \in \Gamma_\alpha(x) \cap B(\xi, Ar)$, the same arguments as with the middle sum in (6.14) give us

$$|II(X)| \lesssim \int_{\partial\Omega\setminus 2A\Delta} \frac{r}{|\xi - y|^{n+1}} |\nabla_{\mathrm{HMT}} f(y)|\, d\sigma(y) \lesssim \sum_{j=2}^{\infty} \frac{2^{-j}}{\sigma(2^j\Delta)} \int_{2^j\Delta\setminus 2^{j-1}\Delta} |\nabla_{\mathrm{HMT}} f|\, d\sigma.$$

Therefore,

$$\sigma\Bigl(\Bigl\{x \in \Delta : r\, N_*(\chi_{B(\xi,2r)}|II|)(x) > \frac{c\lambda}{2}\Bigr\}\Bigr) \lesssim \frac{r}{\lambda}\, \sigma(\Delta) \sum_{j=2}^{\infty} 2^{-j(n+1)} \int_{2^j\Delta\setminus 2^{j-1}\Delta} |\nabla_{\mathrm{HMT}} f|\, d\sigma. \tag{6.16}$$

Combining (6.15) and (6.16) with (6.11) and other previous estimates gives us

$$\sigma\Bigl(\Bigl\{x \in \Delta : |u^+(x) - u(X_\Delta^+)| > \frac{\lambda}{2}\Bigr\}\Bigr) \lesssim \frac{r}{\lambda} \int_{2A\Delta} |\nabla_{\mathrm{HMT}} f|\, d\sigma + \frac{r}{\lambda} \sum_{j=2}^{\infty} 2^{-j(n+1)} \int_{2^j\Delta\setminus 2^{j-1}\Delta} |\nabla_{\mathrm{HMT}} f|\, d\sigma.$$

By a straightforward covering argument, the right-hand side of the preceding inequality is comparable to the right-hand side of (6.9), with the comparability constant depending on A. By similar arguments, one can check that the same estimate holds replacing u^+ by u^- and X_Δ^+ by X_Δ^-. The claim follows then by applying the inequalities for u^+ and u^- to (6.10). □

For the proof of the case $\operatorname{diam}(\partial\Omega) < \infty$ in Theorem 1.3, we give a short proof of the fact that Theorem 6.6 implies quasiconvexity for bounded 2-sided chord-arc domains:

Lemma 6.17. *Suppose that Ω is a 2-sided chord-arc domain such that $\operatorname{diam}(\partial\Omega) < \infty$. Then the boundary $\partial\Omega$ is quasiconvex.*

Notice that Corollary 1.6 implies stronger connectivity properties than the conclusion of Lemma 6.17 (recall Corollary 1.9) but we need Lemma 6.17 to prove Corollary 1.6. Lemma 6.17 follows almost directly from the results reviewed in this section when we combine them with the following result of Durand-Cartagena, Jaramillo and Shanmugalingam:

Theorem 6.18 [Durand-Cartagena et al. 2013, Theorem 3.6]. *Let (X, d, μ) be a complete metric measure space with a doubling measure μ. Suppose that for every bounded Lipschitz function f that is locally 1-Lipschitz there exists a functional $a_f : \mathcal{B} \to [0, \infty)$ such that*

$$⨍_B |f - \langle f\rangle_B|\, d\mu \le a_f(B) \le C r_B,$$

where $\mathcal{B}$ is the collection of all open balls in (X, d), C is a uniform constant and r_B is the radius of the ball B. Then the space (X, d) is quasiconvex.

Proof of Lemma 6.17. Let f be a bounded Lipschitz function on $\partial\Omega$ such that f is locally 1-Lipschitz, and let $\mathcal{B} = \{\Delta(x, r) : x \in \partial\Omega, r < \operatorname{diam}(\partial\Omega)\}$. Since $\operatorname{diam}(\partial\Omega) < \infty$, we know that $\nabla_{\mathrm{HMT}} f$ exists (recall Remark 6.1). Let us define the functional $a_f : \mathcal{B} \to \mathbb{R}$ by setting

$$a_f(\Delta) = C_2 r \Bigl(⨍_\Delta |\nabla_{\mathrm{HMT}} f|^2\, d\sigma\Bigr)^{1/2} + C_2 r \sum_{j=2}^{\infty} \frac{2^{-j}}{\sigma(2^j\Delta)} \int_{2^j\Delta\setminus 2^{j-1}\Delta} |\nabla_{\mathrm{HMT}} f|\, d\sigma$$

for every $\Delta = \Delta(x, r) \in \mathcal{B}$, where C_2 is the constant given by Theorem 6.6 for the arbitrary choice $p = 2$. By part (1) of Lemma 6.2, the functional a_f is well-defined, and the fact that $a_f(\Delta) < \infty$ for each $\Delta \in \mathcal{B}$ follows from the argument below. Now, by Lemmas 6.2 and 6.4, and the fact that f is locally 1-Lipschitz, we know that $|\nabla_{\mathrm{HMT}} f| \leq 1$ almost everywhere. In particular, for $\Delta = \Delta(x, r) \in \mathcal{B}$ we have

$$a_f(\Delta) \leq C_2 r + C_2 r \sum_{j=2}^{\infty} \frac{2^{-j}}{\sigma(2^j \Delta)} \sigma(2^j \Delta \setminus 2^{j-1} \Delta) \leq C_2 r + \sum_{j=2}^{\infty} 2^{-j} \leq 2 C_2 r. \tag{6.19}$$

Thus, we now only need to notice that by Hölder's inequality and the estimate (6.19), we have

$$\fint_{\Delta} |f - \langle f \rangle_{\Delta}| \, d\sigma \leq \left(\fint_{\Delta} |f - \langle f \rangle_{\Delta}|^2 \, d\sigma \right)^{1/2} \leq a_f(\Delta) \leq 2 C_2 r$$

for any $\Delta = \Delta(x, r) \in \mathcal{B}$, and the claim follows from Theorem 6.18. □

7. Weak 1-Poincaré inequality for boundaries of 2-sided chord-arc domains

Let Ω be a 2-sided chord-arc domain. In this section, we prove Theorem 1.3 and Corollary 1.6 with the help of some tools from the literature. As a simple consequence of Theorem 1.3 and some results in the literature, we also show that the tail in the Hofmann–Mitrea–Taylor weak Poincaré inequality (Theorem 6.6) can be removed, at least when Ω is a bounded 2-sided chord-arc domain (see Corollary 7.13).

Instead of proving a Poincaré-type inequality directly, we use the following result to reduce the proof to a pointwise estimate:

Theorem 7.1 [Heinonen et al. 2015, part of Theorem 8.1.7]. *Let (X, d, μ) be a metric measure space with a doubling measure μ and V be a Banach space. Suppose that $1 \leq p < \infty$, $u : X \to V$ is integrable on balls and $g : X \to [0, \infty]$ is measurable. Then the following two conditions are equivalent*:

(a) *There exist constants $C, \lambda \geq 1$ such that*

$$\fint_B |u(x) - \langle u \rangle_B| \, d\mu(x) \leq C \operatorname{diam}(B) \left(\fint_{\lambda B} g(x)^p \, d\mu(x) \right)^{1/p}$$

for every open ball B in X.

(b) *There exist constants $C, \lambda \geq 1$ such that*

$$|u(x) - u(y)| \leq C d(x, y) \big(M_{\lambda d(x,y)}(g^p)(x) + M_{\lambda d(x,y)}(g^p)(y) \big)^{1/p}$$

for almost all $x, y \in X$, where M_R is the R-truncated centered Hardy–Littlewood maximal operator on $\partial\Omega$,

$$M_R f(z_0) := \sup_{r < R} \fint_{\Delta(z_0, r)} |f(z)| \, d\sigma(z).$$

These types of characterizations with respect to pointwise inequalities originate from [Heinonen and Koskela 1998].

Thus, to prove Theorem 1.3, it is enough for us to prove the following lemma (recall also Remark 6.1):

Lemma 7.2. *Suppose that u is a compactly supported Lipschitz function on $\partial\Omega$. There exists a universal constant $C \geq 1$ such that*

$$|u(x)-u(y)| \leq C|x-y|\big(M_{C|x-y|}(|\nabla_{\mathrm{t}} u|)(x)+M_{C|x-y|}(|\nabla_{\mathrm{t}} u|)(y)\big)$$

for all points x, $y \in \partial\Omega$ for which the tangential gradient $\nabla_{\mathrm{t}} u$ exists and (6.5) *holds.*

Let u be a compactly supported Lipschitz function on $\partial\Omega$. As we noted in Section 2D, the tangential gradient $\nabla_{\mathrm{t}} u$ exists for almost every point $x \in \partial\Omega$. By Lemma 6.4, we know that (6.5) holds for almost every point $x \in \partial\Omega$. Thus, the points in Lemma 7.2 are almost all points in $\partial\Omega$, as is required in Theorem 7.1.

In the proof of Lemma 7.2, we use a smooth cutoff function for balls. The construction of the function uses the usual mollifier technique. For the convenience of the reader — particularly because we need a quantitative bound for the norm of the gradient — we give the key details below.

Let us start by defining the standard mollifier. We set $\eta : \mathbb{R}^{n+1} \to \mathbb{R}$,

$$\eta(X)=\begin{cases} c_n e^{-1/(1-|X|^2)} & \text{if } |X|<1,\\ 0 & \text{if } |X| \geq 1,\end{cases}$$

where the constant c_n is chosen so that $\int_{\mathbb{R}^{n+1}} \eta(X)\,dX = 1$. For any $\kappa > 0$, we set

$$\eta_\kappa(X)=\frac{1}{\kappa^{n+1}}\eta\Big(\frac{X}{\kappa}\Big).$$

Notice that $\operatorname{supp}\eta_\kappa \subset B(0,\kappa)$. Using the standard mollifier, we define the smooth cutoff function φ_κ for the ball $B(0,\kappa)$ characteristic function using convolutions: we set $\varphi_\kappa : \mathbb{R}^{n+1} \to \mathbb{R}$,

$$\varphi_\kappa(X)=\eta_\kappa * \chi_{B(0,\kappa)}(X). \tag{7.3}$$

Thus, we have

$$\begin{aligned}\varphi_\kappa(X) &= \int_{\mathbb{R}^{n+1}} \eta_\kappa(X-Y)\,\chi_{B(0,\kappa)}(Y)\,dY\\ &= \int_{\mathbb{R}^{n+1}} \eta_\kappa(Y)\,\chi_{B(0,\kappa)}(X-Y)\,dY = \int_{B(0,\kappa)} \eta_\kappa(Y)\,\chi_{B(0,\kappa)}(X-Y)\,dY.\end{aligned}$$

From this representation, we see that $\varphi_\kappa \equiv 1$ on $B(0,\kappa)$ and $\varphi_\kappa \equiv 0$ on $\mathbb{R}^{n+1} \setminus B(0,2\kappa)$. By the standard theory of mollifiers (see, e.g., [Evans and Gariepy 1992, p. 123–124]), we know that φ_κ is smooth and it satisfies

$$\nabla\varphi_\kappa(X)=\int_{\mathbb{R}^{n+1}} \nabla_X\eta_\kappa(X-Y)\,\chi_{B(0,\kappa)}(Y)\,dY. \tag{7.4}$$

In particular, by the construction, the smoothness and compact support of η and (7.4), for $X=(X_1,\dots,X_{n+1})$ we have

$$\begin{aligned}\left|\frac{\partial_i}{\partial X_i}\varphi_\kappa(X)\right| &= \left|\int_{\mathbb{R}^{n+1}} \frac{\partial_i}{\partial X_i}\,\eta_\kappa(X-Y)\chi_{B(0,\kappa)}(Y)\,dY\right|\\ &\leq \int_{\mathbb{R}^{n+1}}\left|\frac{\partial_i}{\partial X_i}\eta_\kappa(X-Y)\right|dY = \frac{1}{\kappa}\int_{\mathbb{R}^{n+1}}\left|\frac{\partial_i}{\partial X_i}\eta(X-Y)\right|dY \leq \frac{C}{\kappa},\end{aligned}$$

where the constant C does not depend κ. Thus, we have $|\nabla\varphi_\kappa(X)| \lesssim \frac{1}{\kappa}$. We note that this implies that φ_κ is a compactly supported Lipschitz function. Therefore the tangential gradient of φ_κ exists for almost every point $x \in \partial\Omega$, and (7.5) and Lemma 6.4 imply that

$$|\nabla_t\varphi_\kappa(x)| \lesssim \frac{1}{\kappa}. \tag{7.5}$$

By translating and adjusting the constant κ, we can construct a smooth cutoff function for any ball $B(X, r)$ in $\mathbb{R}^{n+1}$.

Proof of Lemma 7.2. Let us fix points $x, y \in \partial\Omega$ such that the tangential gradient exists at x and y and (6.5) holds for x and y.

We prove the bound by using truncation and localization arguments which help us to control the values of the function inside balls $\Delta(x, c|x-y|)$ and $\Delta(y, c|x-y|)$ and allow us to deal with the tail in the right-hand side of inequality (6.9). We first assume that $\operatorname{diam}(\partial\Omega) = \infty$. Without loss of generality, we may assume that $u(x) = 0 < u(y)$. In particular, we have $|u(x) - u(y)| = u(y)$. We set

$$\tilde{u}(z) := \begin{cases} u(z) & \text{if } u(x) < u(z) < u(y), \\ 0 & \text{if } u(z) \le 0, \\ u(y) & \text{if } u(z) \ge u(y). \end{cases}$$

Notice that $\tilde{u}$ is a Lipschitz function since we get it by truncating the Lipschitz function u from above and below. Thus, by Lemma 6.4, we have

$$|\nabla_t u(z_0)| = \limsup_{\partial\Omega\ni z\to z_0} \frac{|u(z_0) - u(z)|}{|z_0 - z|} \quad \text{and} \quad |\nabla_t\tilde{u}(z_0)| = \limsup_{\partial\Omega\ni z\to z_0} \frac{|\tilde{u}(z_0) - \tilde{u}(z)|}{|z_0 - z|}$$

for σ-a.e. $z_0 \in \partial\Omega$. Let $z_0 \in \partial\Omega$ be a point such that the above holds. By continuity, we have the following:

(i) If $u(z_0) < u(x) = 0$ or $u(z_0) > u(y)$, then $|\nabla_t\tilde{u}(z_0)| = 0 \le |\nabla_t u(z_0)|$.

(ii) If $u(x) < u(z_0) < u(y)$, then $|\nabla_t\tilde{u}(z_0)| = |\nabla_t u(z_0)|$.

(iii) If $u(z_0) = u(x) = 0$, then there exists $r = r_{z_0} > 0$ such that if $|z_0 - z| < r$, then $u(z) < u(y)$. For such z,
- if $u(z) > u(x)$, then $|\tilde{u}(z_0) - \tilde{u}(z)| = |u(z_0) - u(z)|$, and
- if $u(z) \le u(x)$, then $|\tilde{u}(z_0) - \tilde{u}(z)| = 0 \le |u(z_0) - u(z)|$.

In particular, we have $|\nabla_t\tilde{u}(z_0)| \le |\nabla_t u(z_0)|$.

(iv) If $u(z_0) = u(y)$, then $|\nabla_t\tilde{u}(z_0)| \le |\nabla_t u(z_0)|$ by an argument similar to that in (iii).

Thus, for almost every $z_0 \in \partial\Omega$, the tangential gradients exist for u and $\tilde{u}$ and we have

$$|\nabla_t\tilde{u}(z_0)| \le |\nabla_t u(z_0)|. \tag{7.6}$$

Let us then start processing $|u(x) - u(y)|$. We define

$$R := |x - y|, \quad \Delta_0 := \Delta(x, R), \quad \Delta_0' := \Delta(y, R), \quad \Delta_j := 2^{-j}\Delta_0 \quad \text{and} \quad \Delta_j' := 2^{-j}\Delta_0'.$$

Let $\varphi = \varphi_{2^\alpha R}$ be a smooth cutoff function for the ball $B(x, 2^\alpha R)$ as in (7.3) (after translation, for the choice $\kappa = 2^\alpha R$) for large $\alpha \in \mathbb{N}$ to be fixed later, and let $v := \varphi\, \tilde{u}$. By Lemma 6.8, there exist numbers $a_{\Delta_j} \in \mathbb{R}$ such that

$$\sup_{\lambda>0} \lambda\, \sigma(\{z \in \Delta_j : |v(z) - a_{\Delta_j}| > \lambda\}) \le C r_j\, \sigma(\Delta_j)\, S_j, \tag{7.7}$$

where $r_j := 2^{-j} R$ and

$$S_j := \fint_{5\Delta_j} |\nabla_{\mathrm{HMT}} v|\, d\sigma + \sum_{k=2}^{\infty} \frac{2^{-k}}{\sigma(2^k \Delta_j)} \int_{2^k \Delta_j \setminus 2^{k-1} \Delta_j} |\nabla_{\mathrm{HMT}} v|\, d\sigma. \tag{7.8}$$

An analogous estimate holds when we replace Δ_j, a_{Δ_j} and S_j by Δ'_j, $a_{\Delta'_j}$ and S'_j, respectively, where S'_j is defined as S_j with Δ'_j in place of Δ_j. We claim now that

$$\lim_{j\to\infty} a_{\Delta_j} = v(x) = \tilde{u}(x) = u(x) = 0. \tag{7.9}$$

Indeed, for any $\lambda \in (0, |a_{\Delta_j} - \langle v\rangle_{\Delta_j}|)$ and any $z \in \Delta_j$, we have

$$\lambda < |a_{\Delta_j} - \langle v\rangle_{\Delta_j}| \le |v(z) - a_{\Delta_j}| + |v(z) - \langle v\rangle_{\Delta_j}|.$$

Hence, by (7.7) and Chebyshev's inequality, we get

$$\begin{aligned}\sigma(\Delta_j) &\le \sigma\Big(\Big\{z \in \Delta_j : |v(z) - a_{\Delta_j}| > \frac{\lambda}{2}\Big\}\Big) + \sigma\Big(\Big\{z \in \Delta_j : |v(z) - \langle v\rangle_{\Delta_j}| > \frac{\lambda}{2}\Big\}\Big)\\ &\lesssim \frac{1}{\lambda}\, r_j\, \sigma(\Delta_j)\, S_j + \frac{1}{\lambda} \int_{\Delta_j} |v - \langle v\rangle_{\Delta_j}|\, d\sigma.\end{aligned}$$

Therefore,

$$\lambda \lesssim r_j\, S_j + \fint_{\Delta_j} |v - \langle v\rangle_{\Delta_j}|\, d\sigma. \tag{7.10}$$

Since v is a compactly supported Lipschitz function, Lemmas 6.2 and 6.4 give $|S_j| \lesssim \|\nabla_{\mathrm{HMT}} v\|_{L^\infty(\sigma)} < \infty$, and thus $r_j\, S_j \to 0$ as $j \to \infty$. By the continuity of v, we also know that $\fint_{\Delta_j} |v(z) - \langle v\rangle_{\Delta_j}|\, d\sigma \to 0$ as $j \to \infty$. Then, choosing $\lambda = \frac{1}{2}|a_{\Delta_j} - \langle v\rangle_{\Delta_j}|$ gives us (7.9) by (7.10) and continuity of v.

Analogously to (7.9), we also have

$$\lim_{j\to\infty} a_{\Delta'_j} = v(y) = \tilde{u}(y) = u(y).$$

Thus,

$$\begin{aligned}u(y) = |u(x) - u(y)| &= |v(x) - v(y)|\\ &= \Big|\Big(\sum_{j=0}^{\infty}(a_{\Delta_{j+1}} - a_{\Delta_j}) + a_{\Delta_0}\Big) - \Big(\sum_{j=0}^{\infty}(a_{\Delta'_{j+1}} - a_{\Delta'_j}) + a_{\Delta'_0}\Big)\Big|\\ &\le \sum_{j=0}^{\infty} |a_{\Delta_{j+1}} - a_{\Delta_j}| + \sum_{j=0}^{\infty} |a_{\Delta'_{j+1}} - a_{\Delta'_j}| + |a_{\Delta_0} - a_{\Delta'_0}|\\ &=: I + II + III.\end{aligned}$$

Let us consider the sum I first. We analyze I by applying (7.7) again for $|a_{\Delta_j} - a_{\Delta_{j+1}}|$. For any $\lambda \in (0, |a_{\Delta_j} - a_{\Delta_{j+1}}|)$ and any $z \in \Delta_j$, we have

$$\lambda < |a_{\Delta_j} - a_{\Delta_{j+1}}| \le |v(z) - a_{\Delta_j}| + |v(z) - a_{\Delta_{j+1}}|,$$

and therefore (7.7) gives us

$$\begin{aligned}\sigma(\Delta_j) &\le \sigma\Big(\Big\{z \in \Delta_j : |v(z) - a_{\Delta_j}| > \frac{\lambda}{2}\Big\}\Big) + \sigma\Big(\Big\{z \in \Delta_{j+1} : |v(z) - a_{\Delta_{j+1}}| > \frac{\lambda}{2}\Big\}\Big) \\ &\lesssim \frac{1}{\lambda}\, r_j\, \sigma(\Delta_j)\, S_j + \frac{1}{\lambda}\, r_{j+1}\, \sigma(\Delta_{j+1})\, S_{j+1} \\ &\lesssim \frac{1}{\lambda}\, r_j\, \sigma(\Delta_j)\, S_j.\end{aligned}$$

Thus, $\lambda \lesssim r_j\, S_j$. Since this holds for all $\lambda \in (0, |a_{\Delta_j} - a_{\Delta_{j+1}}|)$, we have

$$|a_{\Delta_j} - a_{\Delta_{j+1}}| \lesssim r_j\, S_j. \tag{7.11}$$

This then gives us

$$\begin{aligned}I &\lesssim \sum_{j=0}^{\infty} r_j\, S_j \\ &\overset{(A)}{\lesssim} \sum_{j=0}^{\infty}\Big(2^{-j} R \fint_{5\Delta_j} |\nabla_{\mathrm{t}}(\tilde{u}\varphi)|\, d\sigma + 2^{-j} R \sum_{k=2}^{j+\alpha+1} 2^{-k} \fint_{2^k\Delta_j} |\nabla_{\mathrm{t}}(\tilde{u}\varphi)|\, d\sigma\Big) \\ &\overset{(B)}{\le} \sum_{j=0}^{\infty} 2^{-j} R \sum_{k=2}^{j+\alpha+1} 2^{-k} \fint_{2^k\Delta_j} |\nabla_{\mathrm{t}}\tilde{u}|\, d\sigma + \sum_{j=0}^{\infty} 2^{-j} R \cdot 2^{-j-\alpha-1} \frac{1}{\sigma(2^{\alpha+1}\Delta_0)} \int_{2^{\alpha+1}\Delta_0 \setminus 2^{\alpha}\Delta_0} |\tilde{u}(z) \nabla_{\mathrm{t}}\varphi(z)|\, d\sigma(z) \\ &\overset{(C)}{\lesssim} R \cdot M_{2^{\alpha+2}R}(\nabla_{\mathrm{t}}\tilde{u})(x) + \frac{\tilde{u}(y)}{2^{\alpha}},\end{aligned}$$

where we used

(A) the definition of S_j (see (7.8)), Lemma 6.2 and the fact that φ vanishes outside $2^{\alpha+1}\Delta_0$,

(B) Ahlfors–David regularity, the product rule for gradients and the properties of the cutoff function φ: $\varphi \le 1$ everywhere and φ is constant on $2^{\alpha}\Delta_0$ and outside $2^{\alpha+1}\Delta_0$,

(C) the fact that $2^k\Delta_j \subset 2^{\alpha+1}\Delta_0$ for all the relevant k and j, the definition of the truncated Hardy–Littlewood maximal operator, the fact that $\tilde{u}(z) \le \tilde{u}(y)$ for all z, and (7.5).

Using the same techniques gives us the bound

$$II \lesssim R \cdot M_{2^{\alpha+2}R}(\nabla_{\mathrm{t}}\tilde{u})(y) + \frac{\tilde{u}(y)}{2^{\alpha}}.$$

As for III, we notice that

$$III \le |a_{\Delta_0} - a_{2\Delta_0}| + |a_{2\Delta_0} - a_{\Delta_0'}| \lesssim r_0\, S_0.$$

Indeed, the fact that $|a_{\Delta_0} - a_{2\Delta_0}| \lesssim r_0\, S_0$ follows from (7.11), and an analogous estimate holds for the term $|a_{2\Delta_0} - a_{\Delta_0'}|$. Thus, the estimate obtained above for the term I is also valid for III:

$$III \lesssim \sum_{j=0}^{\infty} r_j\, S_j \lesssim R \cdot M_{2^{\alpha+2}R}(\nabla_{\mathrm{t}}\tilde{u})(x) + \frac{\tilde{u}(y)}{2^{\alpha}}.$$

Note that none of the implicit constants for the bounds for I, II and III depends on α. Thus, recalling that $u(x) = 0 < u(y) = \tilde{u}(y)$ and $R = |x - y|$, there exists a constant C such that

$$\begin{aligned} u(y) = |u(x) - u(y)| &\le I + II + III \\ &\le CR(M_{2^{\alpha+2}R}(\nabla_{\mathrm{t}}\tilde{u})(x) + M_{2^{\alpha+2}R}(\nabla_{\mathrm{t}}\tilde{u})(y)) + \frac{C}{2^\alpha}\tilde{u}(y) \\ &\overset{(7.6)}{\le} 2C|x - y|(M_{2^{\alpha+2}R}(\nabla_{\mathrm{t}} u)(x) + M_{2^{\alpha+2}R}(\nabla_{\mathrm{t}} u)(y)) + \frac{C}{2^\alpha} u(y). \end{aligned}$$

By choosing large enough α, we may absorb $(C/2^\alpha)u(y)$ to the left-hand side. This completes the proof for the case $\operatorname{diam}(\partial\Omega) = \infty$.

Let us then assume that $\operatorname{diam}(\partial\Omega) < \infty$. We have to consider this case separately because in the bounded case there might not exist cutoff functions with gentle enough gradient slope. However, the proof is still based on the previous case. The proof works as previously if $|x-y| \ll \operatorname{diam}(\partial\Omega)$ and thus, we may assume that $|x-y| \approx \operatorname{diam}(\partial\Omega)$. Now, by Lemma 6.17, we know that there exists a path $\gamma_{x,y}$ from x to y in $\partial\Omega$ such that $\ell(\gamma_{x,y}) \le C_0|x-y| \approx C_0 \cdot \operatorname{diam}(\partial\Omega)$. Using a covering argument for $\gamma_{x,y}$, we find a uniformly bounded number of points $z_0, z_1, \ldots, z_J \in \partial\Omega$ with $z_0 = x$, $z_J = y$ and $|z_j - z_{j+1}| \ll \operatorname{diam}(\partial\Omega)$. We get

$$\begin{aligned} |u(x) - u(y)| &\le \sum_{j=0}^{J-1} |u(z_j) - u(z_{j+1})| \\ &\lesssim \sum_{j=0}^{J-1} |z_j - z_{j+1}|\big(M_{C|z_j - z_{j+1}|}(|\nabla_{\mathrm{t}} u|)(z_j) + M_{C|z_j - z_{j+1}|}(|\nabla_{\mathrm{t}} u|)(z_{j+1})\big) \\ &\lesssim |x - y|\big(M_{\widetilde{C}|x-y|}(|\nabla_{\mathrm{t}} u|)(x) + M_{\widetilde{C}|x-y|}(|\nabla_{\mathrm{t}} u|)(y)\big), \end{aligned}$$

which is what we wanted. □

Corollary 1.6 follows now immediately when we combine Theorem 1.3 and Lemma 6.4 with the following key result of Keith (which is an improvement of an earlier result of [Heinonen and Koskela 1999, Theorem 1.1]):

Theorem 7.12 [Keith 2003, Theorem 2]. *Let (X, d, μ) be a complete metric measure space with a doubling measure μ and $1 \le p < \infty$. Then the following conditions are equivalent*:

(a) *The space (X, d, μ) supports a weak p-Poincaré inequality.*

(b) *There exist constants $C, \lambda \ge 1$ such that*

$$⨍_B |u(x) - \langle u\rangle_B|\, d\mu(x) \le C \operatorname{diam}(B)\bigg(⨍_{\lambda B} (\operatorname{Lip} u(x))^p\, d\mu(x)\bigg)^{1/p}$$

for every compactly supported Lipschitz function u and every ball B in X, where

$$\operatorname{Lip} u(x) := \limsup_{\substack{y\to x \\ y\neq x}} \frac{|f(x) - f(y)|}{d(x, y)}.$$

As a consequence of Theorem 1.3, we get the following improvement of Theorem 6.6 in bounded 2-sided chord-arc domains:

Corollary 7.13. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded 2-sided chord-arc domain, and let $1 < p < \infty$. There exists a constant $C = C(\Omega, p)$ such that for every $f \in L_1^p(\partial\Omega)$, $x \in \partial\Omega$ and $r > 0$ we have*

$$\left(\fint_{\Delta} |f - \langle f\rangle_\Delta|^p \, d\sigma\right)^{1/p} \le Cr\left(\fint_{\Lambda\Delta} |\nabla_{\mathrm{HMT}} f|^p \, d\sigma\right)^{1/p},$$

where $\Delta := \Delta(x, r)$ and Λ is the constant from Theorem 1.3.

It is likely that the boundedness assumption is not necessary for Corollary 7.13 but since the density results in [Hofmann et al. 2010, Section 4.3] are stated in the case where the boundary $\partial\Omega$ is compact, we only consider this case. We do not consider the case $p = 1$ since the theory of the Hofmann–Mitrea–Taylor Sobolev spaces $L_1^p(\partial\Omega)$ has been developed only for $1 < p < \infty$.

Proof of Corollary 7.13. Let $f \in L_1^p(\partial\Omega)$, $x \in \partial\Omega$, $r > 0$ and $\Delta := \Delta(x, r)$. By [Hofmann et al. 2010, Corollary 4.28], we know that Lipschitz functions form a dense subset of $L_1^p(\partial\Omega)$ when $L_1^p(\partial\Omega)$ is equipped with the natural Sobolev-type norm; see [Hofmann et al. 2010, Section 3.6] for details. In particular, since Ω is a bounded 2-sided chord-arc domain, there exists a sequence of compactly supported Lipschitz functions (f_k) such that $f_k \to f$ and $\nabla_{\mathrm{HMT}} f_k \to \nabla_{\mathrm{HMT}} f$ in $L^p(\partial\Omega)$. By Theorem 1.3, the functions f_k satisfy a weak 1-Poincaré inequality with respect to the tangential gradient, which implies a weak (p, p)-Poincaré inequality with respect to tangential gradient by Hölder's inequality and [Heinonen et al. 2015, Corollary 9.14]. These observations together with Lemma 6.2 give us

$$\left(\fint_{\Delta} |f_k - \langle f_k\rangle_\Delta|^p \, d\sigma\right)^{1/p} \le Cr\left(\fint_{\Lambda\Delta} |\nabla_{\mathrm{t}} f_k|^p \, d\sigma\right)^{1/p} = Cr\left(\fint_{\Lambda\Delta} |\nabla_{\mathrm{HMT}} f_k|^p \, d\sigma\right)^{1/p}.$$

Letting $k \to \infty$ then gives us

$$\left(\fint_{\Delta} |f - \langle f\rangle_\Delta|^p \, d\sigma\right)^{1/p} \le Cr\left(\fint_{\Lambda\Delta} |\nabla_{\mathrm{HMT}} f|^p \, d\sigma\right)^{1/p}$$

by standard L^p convergence arguments, which is what we wanted. □

8. A counterexample and questions

By Corollary 1.6, we know that the boundary of any 2-sided chord-arc domain supports weak Heinonen–Koskela-type Poincaré inequalities. It is natural to ask the following:

(1) Can Corollary 1.6 be reversed, i.e., if $\Omega \subset \mathbb{R}^{n+1}$ is an open set with n-dimensional Ahlfors–David regular boundary such that $\partial\Omega$ supports weak Poincaré inequalities, is Ω a 2-sided chord-arc domain?

(2) Can the assumptions of Corollary 1.6 be weakened in the obvious way, i.e., if $\Omega \subset \mathbb{R}^{n+1}$ is a (1-sided) chord-arc domain with connected boundary, does $\partial\Omega$ support weak Poincaré inequalities?

In the second question, the connectivity assumption for $\partial\Omega$ is necessary since weak Poincaré inequalities imply connectivity; see, e.g., [Cheeger 1999, Theorem 17.1]. However, the answer to both of these questions is no. For the first question, this follows from Example 8.4 below. For the second question, this follows from the example constructed in [Mourgoglou and Tolsa 2021, Section 10]. They construct a chord-arc domain with connected boundary such that the tangential regularity problem for the Laplacian is not solvable in L^p for any $1 \le p < \infty$ (recall Definition 2.31). The boundary of this chord-arc domain cannot support weak Poincaré inequalities by [Mourgoglou and Tolsa 2021, Theorem 1.2].

For our example, we need some results in the literature. In particular, we need the following result of Heinonen and Koskela about how weak Poincaré inequalities survive under unions. We formulate the result only in our context but we note that the result holds more generally for certain types of unions of Ahlfors–David regular metric spaces.

Theorem 8.1 [Heinonen and Koskela 1998, special case of Theorem 6.15]. *Let $E_1, E_2 \subset \mathbb{R}^{n+1}$ be two n-ADR sets such that both E_1 and E_2 support a weak p-Poincaré inequality for some $1 \le p < \infty$, and $\mathcal{H}^n(E_1 \cap E_2 \cap B(x,r)) \ge cr^n$ for all $x \in E_1 \cap E_2$ and all $0 < r < \min\{\operatorname{diam}(E_1), \operatorname{diam}(E_2)\}$. Then also the union $E_1 \cup E_2$ supports a weak p-Poincaré inequality when we equip $E_1 \cup E_2$ with the metric d that equals the usual Euclidean distance individually on E_1 and E_2 and*

$$d(x,y) := \inf_{Z \in E_1 \cap E_2} |X - Z| + |Z - Y| \tag{8.2}$$

for $X \in E_1 \setminus E_2$ and $Y \in E_2 \setminus E_1$.

We will also use repeatedly the fact that bi-Lipschitz mappings preserve weak Poincaré inequalities (see [Björn and Björn 2011, Proposition 4.16] for an explicit proof in the context of more general metric spaces and inequalities):

Proposition 8.3. *Let $E_1, E_2 \subset \mathbb{R}^{n+1}$ be two n-ADR sets equipped with metrics d_1 and d_2, respectively. Suppose that there exists a bi-Lipschitz mapping $\Phi : (E_1, d_1) \to (E_2, d_2)$ such that $\mathcal{H}^n(A) \approx \mathcal{H}^n(\Phi(A))$ for every measurable set $A \subset E_1$ and a uniform implicit constant. If E_1 supports a weak p-Poincaré inequality for some $1 \le p < \infty$, then also E_2 supports a weak p-Poincaré inequality.*

Let us then construct a disconnected non-chord-arc domain example of a set whose boundary still supports a weak 1-Poincaré inequality:

Example 8.4. Let us consider a "twice-pinched annulus" in $\mathbb{R}^2$ which we construct the following way. We start by considering the boundary of a usual annulus $A := B(0,4) \setminus \overline{B(0,3)}$. We remove all the points that lie on the strip $\{(x,y) \in \mathbb{R}^2 : -1 < y < 1\}$. This leaves us with four circular arcs: two inner arcs and two outer arcs. We then connect these arcs to each other with four line segments so that the inner arcs connect to outer arcs, and vice versa. This leaves us with a shape that looks like an annulus that has been pinched in two places so that the interior is no longer connected but the boundary is. We let Ω be the disconnected interior of this pinched annulus (see Figure 3).

The set Ω satisfies the 2-sided corkscrew condition and the boundary $\partial\Omega$ is 1-ADR but since Ω is not connected, it is not a chord-arc domain (however, we can easily modify the example to make it connected but still not a chord-arc domain; see Remark 8.6). The boundary of Ω still supports a weak 1-Poincaré inequality. We see this by noticing that we can express $\partial\Omega$ as a union of pieces that satisfy weak 1-Poincaré inequalities and that we can glue together with ample intersections to give back $\partial\Omega$ (that is, we can use Theorem 8.1 for these pieces). Indeed, $\partial\Omega$ consists of a slightly distorted inner circle and a slightly distorted outer circle. These distorted circles intersect only in two points (the two places

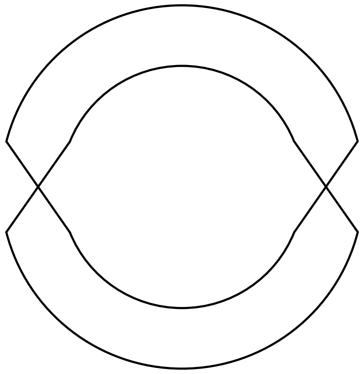

Figure 3. The set Ω in Example 8.4 is the disconnected interior of a twice-pinched annulus in $\mathbb{R}^2$. Unlike with a usual annulus, the boundary of Ω is connected (which is one of the minimum requirements Poincaré inequalities).

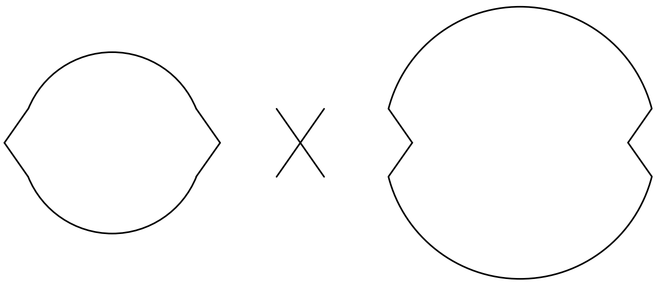

Figure 4. In Example 8.4, the boundary of Ω consists of a slightly distorted inner circle (on the left) and a slightly distorted outer circle (on the right) that intersect each other only at two points. However, the cross-like unions of two line segments (one copy in the middle) have ample intersections with both of these distorted circles.

where the line segments cross over each other) and therefore they alone are not enough for Theorem 8.1. Because of this, as two additional pieces, we take the cross-like unions of the pairs of line segments that cross over each other. These pieces have ample intersections with both of the distorted circles (see Figure 4).

As a 1-dimensional compact Riemannian manifold, a circle supports a weak 1-Poincaré; see [Heinonen and Koskela 1998, Section 6.1]. Since both of the distorted circles are bi-Lipschitz equivalent to a regular circle, both of them support a weak 1-Poincaré inequality by Proposition 8.3. As for the unions of two line segments, we first notice that a line segment supports a weak 1-Poincaré inequality because it is bi-Lipschitz equivalent to a piece of the real line (and a connected piece of the real line supports a weak 1-Poincaré inequality by definition and the classical result that the Euclidean space supports a 1-Poincaré inequality; see [Heinonen et al. 2015, Section 8.1]). The two line segments in the cross-like union meet only at one point and therefore we cannot use Theorem 8.1 directly for them. Because of this, we take three line segments and transform them (with bi-Lipschitz mappings) into three V-like shapes which we

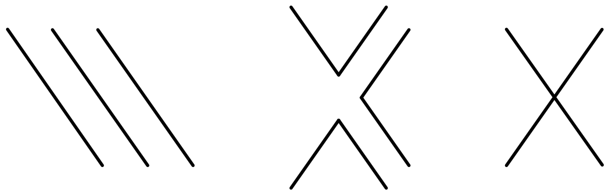

Figure 5. By Theorem 8.1, we can preserve weak Poincaré inequalities in intersections if the intersections are ample enough. In Example 8.4, the two intersecting line segments do not have ample intersection but we can create the cross-like shape by gluing three V-like shapes one by one into each other, and we get these V-like shapes by using three line segments and bi-Lipschitz mappings.

can then glue one by one to each other with ample intersections to create the original cross-like piece (see Figure 5).

We now get the boundary $\partial\Omega$ by gluing first two cross-like pieces to either one of the distorted circles and then gluing the remaining distorted circle to this shape. Gluing sets together using Theorem 8.1 preserves the weak Poincaré inequalities but with a different metric, the one in (8.2). However, for shapes as simple as the ones we use, it is straightforward to see that the new metrics we get are bi-Lipschitz equivalent with the Euclidean metric. Thus, by gluing together the two distorted circles with the help of two cross-like unions of two line segments, we see that $\partial\Omega$ supports a weak 1-Poincaré inequality. We note that we had to use two cross-like pieces for the gluing process because with just one cross-like piece the ample intersection requirement of Theorem 8.1 does not hold for the final step for one of the two intersection points of the distorted circles. By the original, more general form of Theorem 8.1 in [Heinonen and Koskela 1998], we can leave this type of a problematic isolated point out of the gluing process, but this would end up giving us a metric space with a different structure than $\partial\Omega$.

Remark 8.5. Example 8.4 gives us a 2-dimensional example but we can use the same techniques to construct higher dimensional examples. These examples are of the type $\Omega \times (0, 1)^n$, where Ω is the 2-dimensional set from Example 8.4. Thus, for example, the 3-dimensional example would be a hollow, twice-pinched cylinder with a thick boundary.

Remark 8.6. By using a once-pinched annulus instead of the twice-pinched one we used in Example 8.4, we get a connected set Ω such that it satisfies the 2-sided corkscrew condition, the boundary $\partial\Omega$ is 1-ADR and $\partial\Omega$ supports a weak 1-Poincaré inequality. However, despite connectivity, Ω is still not a chord-arc domain: there are points arbitrarily close to each other on different sides of the pinched part of the annulus such that they can be connected inside Ω only by circling around almost the entire annulus.

Thus, Corollary 1.6 cannot be reversed and we cannot weaken its assumptions in the obvious way. It is natural to formulate the following problem:

Problem 8.7. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with n-dimensional Ahlfors–David regular boundary. Give a geometric characterization for weak Heinonen–Koskela-type boundary p-Poincaré inequalities for $1 \leq p \leq n$.

By a geometric characterization in Problem 8.7, we mean a characterization of the type "$\omega \in$ weak-$A_\infty(\sigma)$ if and only if $\partial\Omega$ is UR and Ω satisfies the weak local John condition" (which is the main result of [Azzam et al. 2020]). By [Cheeger 1999, Theorem 17.1], we know that $\partial\Omega$ has to be quasiconvex, and by [Azzam 2021a], we know that $\partial\Omega$ has to be UR. By Corollary 1.6, Example 8.4 and [Mourgoglou and Tolsa 2021, Section 10], we know that 2-sided chord-arc domains are too strong for the characterization and (1-sided) chord-arc domains are not strong enough for a characterization. However, the answer does not lie somewhere between these two classes of domains: by Example 8.4, the connectivity properties of Ω itself do not play a big role in this problem.

Concerning John-type conditions, we recall the open problem we mentioned in Section 3:

Problem 8.8. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with n-UR (or just n-ADR) boundary. Suppose that Ω satisfies the local John condition. Does Ω also satisfy the weak local John condition?

Problem 8.8 is interesting only if Ω does not satisfy the exterior corkscrew condition: the local John condition implies the corkscrew condition and therefore the answer is trivially true in the presence exterior corkscrews by Lemma 3.6.

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