ANALYSIS & PDE

Volume 17 No. 5 2024

CHRISTOPH KEHLE AND MAXIME VAN DE MOORTEL

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ANALYSIS AND PDE
Vol. 17 (2024), No. 5, pp. 1501–1592
DOI: 10.2140/apde.2024.17.1501

STRONG COSMIC CENSORSHIP IN THE PRESENCE OF MATTER: THE DECISIVE EFFECT OF HORIZON OSCILLATIONS ON THE BLACK HOLE INTERIOR GEOMETRY

CHRISTOPH KEHLE AND MAXIME VAN DE MOORTEL

Motivated by the strong cosmic censorship conjecture in the presence of matter, we study the Einstein equations coupled with a charged/massive scalar field with spherically symmetric characteristic data relaxing to a Reissner–Nordström event horizon. Contrary to the vacuum case, the relaxation rate is conjectured to be *slow* (nonintegrable), opening the possibility that the matter fields and the metric coefficients *blow up in amplitude* at the Cauchy horizon, not just in energy. We show that whether this blow-up in amplitude occurs or not depends on a novel *oscillation condition* on the event horizon which determines whether or not a resonance is excited dynamically:

- If the oscillation condition is satisfied, then the resonance is not excited and we show boundedness and continuous extendibility of the matter fields and the metric across the Cauchy horizon.
- If the oscillation condition is violated, then by the *combined effect of slow decay and the resonance being excited*, we show that the massive uncharged scalar field blows up in amplitude.
  In a companion paper, we will show that in that case a novel *null contraction singularity* forms at the Cauchy horizon, across which the metric is not continuously extendible in the usual sense.

Heuristic arguments in the physics literature indicate that the oscillation condition should be satisfied generically on the event horizon. If these heuristics are true, then *our result falsifies the C^0-formulation of strong cosmic censorship by means of oscillation*.

1. Introduction

Is general relativity a deterministic theory? This fundamental question can only be addressed in the context of the initial value problem for the Einstein equations (see (1-1)), which govern the dynamics of spacetime in general relativity. Well-posedness for the initial value problem was established in [Choquet-Bruhat and Geroch 1969] (see also [Fourès-Bruhat 1952]), proving that any suitably regular Cauchy data admit a unique maximal future development, the so-called *maximal globally hyperbolic development* (MGHD). With this dynamical formulation at hand, general relativity can be considered deterministic if the MGHD of *generic* Cauchy data for the Einstein equations is inextendible. The genericity stipulation is clearly necessary because the MGHD of Kerr Cauchy data [1963] (rotating black holes) and of Reissner–Nordström Cauchy data [Reissner 1916; Nordström 1918] (their charged analogs) admit a future boundary, the Cauchy horizon, across which the metric is smoothly extendible. Heuristics of Penrose [1968] however suggest the instability of the Kerr/Reissner–Nordström Cauchy horizons and these led him to his famous

MSC2020: 35Q75, 35Q76, 83C05, 83C57, 83C75.
Keywords: strong cosmic censorship, black hole, Einstein equations, scattering, charged scalar field.

strong cosmic censorship conjecture [Penrose 1974] supporting the idea of determinism in general relativity. The most definitive and perhaps most desirable formulation of Penrose's strong cosmic censorship is the conjecture that the metric coefficients cannot be extended as continuous functions, namely:

Conjecture 1 (C^0-formulation of strong cosmic censorship). *The MGHD of generic asymptotically flat Cauchy data is inextendible as a continuous Lorentzian metric* (*we say the metric is C^0-inextendible*).

Conjecture 1 is related to the expectation that physical observers approaching the boundary of the MGHD of generic Cauchy data are destroyed. If Conjecture 1 is false, then one may still be able to prove a weaker version of inextendibility, but this would correspond to a weaker version of determinism.

Conjecture 1 is false in the absence of matter. In the celebrated work [Dafermos and Luk 2017], it is proved that, in vacuum, small perturbations of Kerr still admit a Cauchy horizon across which the spacetime is C^0-extendible — thus falsifying Conjecture 1 in the absence of matter. The key ingredient to their proof is an *integrable* inverse polynomial rate assumption for the decay of perturbations along the event horizon. Note, however, that a weaker H^1-formulation is still expected to hold [Christodoulou 2009; Dafermos and Luk 2017; Van de Moortel 2021]. If true, this would restore determinism at least in a weaker sense.

Can Conjecture 1 be salvaged in the presence of matter? In the present paper, we consider a nonvacuum model: the Einstein–Maxwell–Klein–Gordon (1-1)–(1-5) system in spherical symmetry governing the dynamics of gravitation coupled to a charged/massive scalar field. Arguments in the physics literature [Hod and Piran 1998; Koyama and Tomimatsu 2001; Konoplya and Zhidenko 2013; Burko and Khanna 2004; Oren and Piran 2003] suggest that perturbations of the exterior of Reissner–Nordström in this model settle down merely at a slow, *nonintegrable* rate (at least for massive and/or strongly charged perturbations), which is in stark contrast to the perturbations of Kerr in the vacuum case. As such, the methods of [Dafermos and Luk 2017] manifestly do not apply and the slow decay of perturbations may even raise hopes that for generic Cauchy data the metric is C^0-inextendible and thus, Conjecture 1 would be true after all for this matter model.

The question of C^0-extendibility across a future null boundary $\mathcal{CH}_{i^+}$. At first, it may appear that the slow decay in the above matter model in fact opens the possibility of a more drastic scenario where the singularity is everywhere spacelike inside the black hole. Notwithstanding, it was proven in [Van de Moortel 2018] that for this model black holes are bound to the future by a null boundary $\mathcal{CH}_{i^+} \neq \varnothing$ as depicted in Figure 1. We will continue using the term "Cauchy horizon" for $\mathcal{CH}_{i^+}$ by analogy with the Cauchy horizon of Reissner–Nordström, although the spacetime may or may not be C^0-extendible across the null boundary $\mathcal{CH}_{i^+}$. Therefore, although the future boundary is null and in particular not spacelike, the question of C^0-extendibility of the spacetime across $\mathcal{CH}_{i^+}$, i.e., Conjecture 1, remains open. This is the question that we shall now address.

Summary of our results. As we will show, the question of Conjecture 1 becomes unexpectedly subtle: In addition to the decay rates of perturbations on the exterior, it turns out that the validity of Conjecture 1 depends crucially on Fourier support properties of late-time perturbations due to a scattering resonance associated to the Cauchy horizon $\mathcal{CH}_{i^+}$. In our main Theorem I (i) we identify an *oscillation condition* on perturbations along the event horizon $\mathcal{H}^+$: *If the oscillation condition is satisfied by the perturbation*,

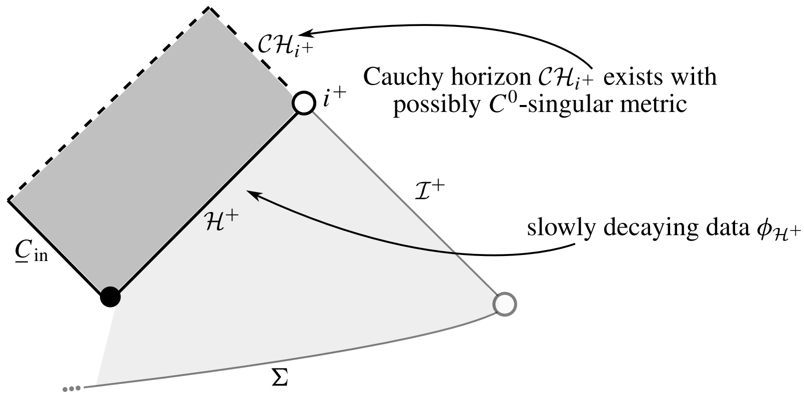

Figure 1. A Cauchy horizon $\mathcal{CH}_{i^+}$ exists for slowly decaying perturbations $\phi_{\mathcal{H}^+}$ as proven in [Van de Moortel 2018]; see Theorem A.

we show boundedness and continuous extendibility of the matter fields and the metric across the Cauchy horizon $\mathcal{CH}_{i^+}$ despite the obstruction created by slow decay. On the other hand, in Theorem I (ii) we show that *if the oscillation condition is violated on the event horizon* $\mathcal{H}^+$, *the resonance is excited and the uncharged scalar field blows up in amplitude*, namely $|\phi| \to +\infty$ at the Cauchy horizon $\mathcal{CH}_{i^+}$.

Heuristic and numerical arguments in the physics literature [Hod and Piran 1998; Koyama and Tomimatsu 2001; Konoplya and Zhidenko 2013; Burko and Khanna 2004; Oren and Piran 2003] suggest that the oscillation condition is indeed satisfied on $\mathcal{H}^+$ for generic perturbations of the black hole exterior. Assuming this, our result Theorem II *falsifies the* C^0*-formulation of strong cosmic censorship by means of oscillation*.

In Theorem III, we show that for both oscillating and nonoscillating perturbations,[1] *the scalar field blows up in the* $W^{1,1}_{\text{loc}}$*-norm at the Cauchy horizon* $\mathcal{CH}_{i^+}$, i.e., $\int |D_v\phi|\, \mathrm{d}v = +\infty$ schematically. This $W^{1,1}$ blow-up is in contrast to the vacuum case where the analogous statement is false [Dafermos and Luk 2017]. This shows that for both oscillating and nonoscillating perturbations, the Cauchy horizon $\mathcal{CH}_{i^+}$ is more singular in the presence of matter than in vacuum. Moreover, the blow-up of the scalar field in $W^{1,1}$ indicates that our result cannot be captured using only physical space techniques which have been used previously.

Finally, in our companion paper [Kehle and Van de Moortel $\geq$ 2024] we will prove Theorem IV, which shows that blow-up in amplitude of the scalar field indeed gives rise to a C^0-inextendibility statement on the metric within a spherically symmetric class. Theorem IV, in conjunction with Theorem I (ii), provides the first example of a dynamically formed singularity leading to a C^0-inextendibility statement of the metric across a null spacetime boundary (albeit within a restricted spherically symmetric class). Whether this statement can be upgraded to the full C^0-inextendibility of the spacetime remains open.[2]

Similarities with the $\Lambda < 0$ case. In the asymptotically AdS case ($\Lambda < 0$), solutions to the linear wave equation on AdS black holes also decay at a slow, nonintegrable rate [Holzegel and Smulevici 2014]. It turns out that in this context, oscillations also play a crucial role [Kehle 2020b; 2022] in addressing

[1] Up to a genericity condition in the charged scalar field case, which we can get rid of in the uncharged case; see Theorem III.

[2] Unrestricted C^0-inextendibility results (even for spacelike singularities) are known to be notoriously difficult to show; see, e.g., [Sbierski 2018] for the proof of C^0-inextendibility of the Schwarzschild solution across the spacelike singularity $\{r = 0\}$.

the question of the validity of the linear analog of Conjecture 1. The slow inverse logarithmic decay in the $\Lambda < 0$ case however arises from the superposition of infinitely many high ℓ angular modes. This is different from the present problem for $\Lambda = 0$, where the slow decay is inverse-polynomial (see Section 1A) and already occurs in spherical symmetry.

Outline of the Introduction. In Section 1A we introduce the Einstein–Maxwell–Klein–Gordon system and give a more detailed overview of our new results addressing the issue of strong cosmic censorship within this matter model in spherical symmetry. Further, we present a first version of our main theorems. In Section 1B we outline the important differences between the EMKG model and other models regarding the existence of a Cauchy horizon and the continuous extendibility of the metric. In Section 1C we mention previous results on the dynamical formation of weak null singularities at the Cauchy horizon, which we compare to the new singularities that dynamically form in our setting. In Section 1D we present previous results on scattering inside Reissner–Nordström black holes which are important for our proof. In Section 1E we elaborate on the interior of black holes with $\Lambda < 0$, in which oscillations turn out to play an important role as well. In Section 1F we briefly discuss the strategy of the proof.

1A. *Main results: first versions.*

1A1. *The EMKG system and existence of a Cauchy horizon for slowly decaying scalar fields.*

The EMKG model in spherical symmetry. We study the Einstein equations coupled to a charged massive scalar field: the Einstein–Maxwell–Klein–Gordon (EMKG) model in spherical symmetry

$$\mathrm{Ric}_{\mu\nu}(g) - \tfrac{1}{2}R(g)g_{\mu\nu} = \mathbb{T}^{EM}_{\mu\nu} + \mathbb{T}^{KG}_{\mu\nu}, \tag{1-1}$$

$$\mathbb{T}^{EM}_{\mu\nu} = 2\big(g^{\alpha\beta}F_{\alpha\nu}F_{\beta\mu} - \tfrac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu}\big), \tag{1-2}$$

$$\mathbb{T}^{KG}_{\mu\nu} = 2\big(\Re(D_\mu\phi\overline{D_\nu\phi}) - \tfrac{1}{2}(g^{\alpha\beta}D_\alpha\phi\overline{D_\beta\phi} + m^2|\phi|^2)g_{\mu\nu}\big), \tag{1-3}$$

$$\nabla^\mu F_{\mu\nu} = \frac{q_0}{2}i(\phi\overline{D_\nu\phi} - \bar{\phi}D_\nu\phi), \quad F = \mathrm{d}A, \tag{1-4}$$

$$g^{\mu\nu}D_\mu D_\nu\phi = m^2\phi, \quad D_\mu = \nabla_\mu + iq_0A_\mu \tag{1-5}$$

for a quintuplet $(\mathcal{M}, g, F, A, \phi)$, where $(\mathcal{M}, g)$ is a (3+1)-dimensional Lorentzian manifold, ϕ is a complex-valued scalar field, A is a real-valued 1-form, and F is a real-valued 2-form. Here $q_0 \in \mathbb{R}$ and $m \geq 0$ are fixed constants representing respectively the charge and the mass of the scalar field. The EMKG model describes self-gravitating matter and provides a setting for studying spherical gravitational collapse of charged and massive matter if $q_0 \neq 0$ and $m^2 \neq 0$ (see the discussion in Section 1C3). This model has attracted much attention in the literature [An and Lim 2022; Bizoń and Wasserman 2000; Dias et al. 2019; Kommemi 2013; Gajic and Luk 2019; Van de Moortel 2018; 2019; 2021; 2022]; see also [Yang and Yu 2019; Lindblad and Sterbenz 2006; Klainerman and Machedon 1994; Krieger et al. 2015; Oh and Tataru 2016; Rodnianski and Tao 2004] for work on the flat Minkowski background.

Setting of the problem. Consider the maximal globally hyperbolic development of suitably regular spherically symmetric Cauchy data prescribed on an asymptotically flat initial hypersurface Σ as depicted in Figure 1. General results for the EMKG model in spherical symmetry [Kommemi 2013] allow us to define null infinity $\mathcal{I}^+$ — a conformal boundary where idealized far away observers live, and the black hole interior region as the complement of the causal past of $\mathcal{I}^+$. If the black hole interior is nonempty, we also define the event horizon $\mathcal{H}^+$ as the past boundary of the black hole interior which separates the black hole interior from the black hole exterior.

In the current paper we will only be interested in the dynamics of the black hole interior. In particular, instead of studying the Cauchy problem with data on Σ, we will prescribe the scalar field ϕ and the metric on an ingoing cone $\underline{C}_{\text{in}}$ and on an outgoing cone $\mathcal{H}^+$ emulating the event horizon of an already-formed black hole. This setting corresponds to a characteristic initial value problem with data imposed on $\mathcal{H}^+ \cup \underline{C}_{\text{in}}$; see Figure 1. Our study of this characteristic initial value problem will be entirely self-contained. We will however continue to depict Σ on Figure 1 and subsequent figures for completeness. Our assumptions on the characteristic initial data on $\mathcal{H}^+ \cup \underline{C}_{\text{in}}$ will be made in accordance with the conjectured late-time tails on the event horizon $\mathcal{H}^+$ arising from generic Cauchy data on asymptotically flat Σ; see the discussion below.

Conjectured late-time asymptotics on the event horizon $\mathcal{H}^+$ and contrast with the vacuum case. Heuristic arguments regarding the black hole exterior in the physics literature (see [Hod and Piran 1998; Koyama and Tomimatsu 2001; Konoplya and Zhidenko 2013; Burko and Khanna 2004; Oren and Piran 2003]) indicate that (spherically symmetric) dynamical black holes arising from Cauchy data on Σ for the EMKG model relax to Reissner–Nordström along the event horizon $\mathcal{H}^+$ at a slow,[3] *nonintegrable* rate v^{-s}, $s \in \left(\frac{1}{2}, 1\right]$ for large v, in a standard Eddington–Finkelstein coordinate v. This is in contrast to the faster and integrable rate $s > 1$ proved in the uncharged massless case $m^2 = q_0 = 0$ [Dafermos and Rodnianski 2005], or assumed in vacuum in [Dafermos and Luk 2017] (see (1-21)). This fast, integrable rate v^{-s}, $s > 1$ in vacuum is indeed sufficient to prove the existence of a Cauchy horizon $\mathcal{CH}_{i^+}$, across which the spacetime is continuously extendible: this led to a *falsification of Conjecture 1 in vacuum without symmetry assumptions* [Dafermos and Luk 2017] (or for spherically symmetric models as in [Dafermos 2003; Van de Moortel 2018]); see Section 1B2.

Existence of a Cauchy horizon $\mathcal{CH}_{i^+}$ for slowly decaying scalar fields. Returning to the EMKG model, the first step in addressing Conjecture 1 is to understand whether, for slowly decaying characteristic data on the event horizon $\mathcal{H}^+$, the future boundary inside the black hole is null (a Cauchy horizon) or spacelike. In view of the slow decay on the event horizon $\mathcal{H}^+$, the spacelike singularity scenario is plausible and indeed desirable (if it were true, then Conjecture 1 would likely be valid). Despite the obstruction created by the slow decay of event horizon perturbations, it turns out however that the black hole future boundary has a nonempty null component $\mathcal{CH}_{i^+} \neq \emptyset$ emanating from i^+, see Figure 1, and is not everywhere spacelike as one might have hoped:

Theorem A ([Van de Moortel 2018], rough version; precise version recalled in Section 4A). *Consider spherically symmetric characteristic initial data for* (1-1)–(1-5) *on the event horizon* $\mathcal{H}^+$ (*and on an ingoing cone*). *Assume the following* ***slow decay*** *upper bound on the scalar field* $\phi_{\mathcal{H}^+}$ *on the event horizon* $\mathcal{H}^+ = [v_0, +\infty)$ *as*

$$|\phi_{\mathcal{H}^+}(v)| \le C_0 v^{-s}, \quad |D_v \phi_{\mathcal{H}^+}| \le C_0 v^{-s} \tag{1-6}$$

[3] Precisely, these slow rates hold conjecturally for a massive ($m^2 \neq 0$) scalar field and/or strongly charged $\left(|q_0 e| \ge \frac{1}{2}\right)$ one.

for all $v \geq v_0$ in a standard Eddington–Finkelstein-type v-coordinate on $\mathcal{H}^+ = [v_0, +\infty)$ for some $C_0 > 0$ and some decay rate $s > \frac{1}{2}$.

Then the spacetime is bound to the future by an ingoing null boundary $\mathcal{CH}_{i^+} \neq \varnothing$ (the Cauchy horizon) foliated by spheres of positive radius and emanating from i^+, and the Penrose diagram is given by the dark gray region in Figure 1.

Since by Theorem A the black hole future boundary is not everywhere spacelike and has a null component $\mathcal{CH}_{i^+} \neq \varnothing$, one may at first expect continuous extendibility across $\mathcal{CH}_{i^+}$. It turns out however that the spacetime of Theorem A may or may not be continuously extendible across $\mathcal{CH}_{i^+}$. This is perhaps unexpected, since all previous instances of black hole spacetimes with a null future boundary component are at least continuously extendible across that component [Dafermos and Luk 2017; Dafermos 2003; Luk and Oh 2019a]. Thus, Theorem A is not sufficient to fully address Conjecture 1 and the question of continuous extendibility across the null boundary $\mathcal{CH}_{i^+}$ has remained open.

The slow rate $s > \frac{1}{2}$ assumed in Theorem A is indeed *too slow* to prove the C^0-extendibility of spacetime across the Cauchy horizon $\mathcal{CH}_{i^+}$ using the same method as [Dafermos and Luk 2017] in vacuum. The method of [Dafermos and Luk 2017] requires the faster integrable decay assumption $s > 1$ and does not extend to the nonintegrable case $s \leq 1$, a failure that may even raise the attractive possibility that Conjecture 1 is true after all for the EMKG matter model. ***This could mean that determinism is in better shape in the presence of matter!***

1A2. *Theorem I: event horizon oscillations are decisive for the C^0 extendibility of the metric.* Our main result however shows that the situation is more subtle than one may first think: assuming that the scalar field ϕ oscillates sufficiently on the event horizon $\mathcal{H}^+$, we show in Theorem I (i) that *ϕ is uniformly bounded in the black hole interior and the metric is continuously extendible*. The event horizon oscillation assumption is sharp in the following sense: conversely assuming that the scalar field ϕ does not oscillate sufficiently on the event horizon $\mathcal{H}^+$, we show in Theorem I (ii) that *ϕ blows up in amplitude at the Cauchy horizon $\mathcal{CH}_{i^+}$*. It turns out that the oscillation condition on the event horizon $\mathcal{H}^+$, i.e., the main assumption of Theorem I (i), is conjecturally satisfied for generic Cauchy data on an asymptotically flat Σ, and thus, the *hope that determinism is in better shape in the presence of matter in the end does not come true*! (See Section 1A3.)

Theorems I (i) and I (ii) show that uniform boundedness or blow-up of the matter fields unexpectedly relies on fine properties of the scalar field ϕ on the event horizon $\mathcal{H}^+$ in both physical and Fourier space. At the heart of our novel oscillation condition lies the resonant frequency

$$\omega_{\text{res}}(M, e, q_0) := \omega_-(M, e, q_0) - \omega_+(M, e, q_0), \tag{1-7}$$

where

$$\omega_- = \omega_-(M, e, q_0) := \frac{q_0 e}{r_-(M, e)}, \quad \omega_+ = \omega_+(M, e, q_0) := \frac{q_0 e}{r_+(M, e)}$$

for asymptotic black hole parameters $0 < |e| < M$.

In what follows we will give rough versions of Theorems I (i) and I (ii). For the precise versions we refer the reader to Sections 4B and 4C.

Theorem I (i) (boundedness (rough version; precise version in Section 4B)). *Consider spherically symmetric characteristic initial data for* (1-1)–(1-5) *on the event horizon* $\mathcal{H}^+$ (*and on an ingoing cone*). *Assume the following* ***slow decay*** *upper bound on the scalar field* $\phi_{\mathcal{H}^+}$ *on the event horizon* $\mathcal{H}^+ = [v_0, +\infty)$ *as*

$$|\phi_{\mathcal{H}^+}(v)| \leq C v^{-s}, \quad |D_v \phi_{\mathcal{H}^+}| \leq C v^{-s} \tag{1-8}$$

for all $v \geq v_0$ *in a standard Eddington–Finkelstein-type* v*-coordinate on* $\mathcal{H}^+ = [v_0, +\infty)$, *for* $v_0 > 1$ *sufficiently large and for some* $C > 0$ *and some* (*nonintegrable*) *decay rate*

$$\tfrac{3}{4} < s \leq 1. \tag{1-9}$$

By Theorem A, *the spacetime*, *i.e.*, *the dark gray region in Figure 1*, *is bound to the future by a null boundary* $\mathcal{CH}_{i^+} \neq \varnothing$ (*the Cauchy horizon*). *Then*, *in the gauge* $A_v = 0$, *the following hold true*:

- *If* $\phi_{\mathcal{H}^+}$ *satisfies the* ***qualitative oscillation condition*** *on* $\mathcal{H}^+ = [v_0, +\infty)$, *i.e.*, *if for all* $O(v^{1-2s})$ *functions*

$$\limsup_{\tilde{v} \to +\infty} \left| \int_{v_0}^{\tilde{v}} \phi_{\mathcal{H}^+}(v) e^{i\omega_{\mathrm{res}} v (1 + O(v^{1-2s}))} \, \mathrm{d}v \right| < +\infty, \tag{1-10}$$

then the scalar field ϕ *is uniformly bounded in amplitude up to and including the Cauchy horizon* $\mathcal{CH}_{i^+}$.

- *If* $\phi_{\mathcal{H}^+}$ *satisfies the* ***strong qualitative oscillation condition*** *on* $\mathcal{H}^+ = [v_0, +\infty)$, *i.e.*, *if for all* $O(v^{1-2s})$ *functions*

$$\lim_{\tilde{v} \to +\infty} \left| \int_{v_0}^{\tilde{v}} \phi_{\mathcal{H}^+}(v) e^{i\omega_{\mathrm{res}} v (1 + O(v^{1-2s}))} \, \mathrm{d}v \right| \text{ exists and is finite}, \tag{1-11}$$

then additionally the metric g *and the scalar field* ϕ *are continuously extendible across the Cauchy horizon* $\mathcal{CH}_{i^+}$.

- *If* $\phi_{\mathcal{H}^+}$ *satisfies the* ***quantitative oscillation condition*** *on* $\mathcal{H}^+ = [v_0, +\infty)$, *i.e.*, *if there exist* $E > 0$, $\epsilon > 1 - s$ *such that for all* $O(v^{1-2s})$ *functions*

$$\lim_{\tilde{v} \to +\infty} \left| \int_{v_1}^{\tilde{v}} \phi_{\mathcal{H}^+}(v) e^{i\omega_{\mathrm{res}} v (1 + O(v^{1-2s}))} \, \mathrm{d}v \right| \leq E v_1^{-\epsilon} \quad \text{for all } v_1 \geq v_0, \tag{1-12}$$

then, *additionally the Maxwell field contraction* $F_{\mu\nu} F^{\mu\nu}$ *is uniformly bounded in amplitude and continuously extendible across the Cauchy horizon* $\mathcal{CH}_{i^+}$.

We refer to Figure 2 for an illustration of Theorem I (i).

In the uncharged case $q_0 = 0$, where $\omega_{\mathrm{res}} = 0$, we show that the qualitative oscillation condition (1-10) is sharp to obtain boundedness.

Theorem I (ii) (blow-up (rough version; precise version in Section 4C)). *Consider spherically symmetric characteristic initial data for* (1-1)–(1-5) *on the event horizon* $\mathcal{H}^+$ (*and on an ingoing cone*). *Assume the following* ***slow decay*** *upper bound on the scalar field on the event horizon* $\mathcal{H}^+$ (*i.e.*, $\phi_{\mathcal{H}^+}$ *satisfies* (1-8) *where* s *satisfies* (1-9)). *Assume additionally* $q_0 = 0$ *and let* $m^2 > 0$ *be generic.*

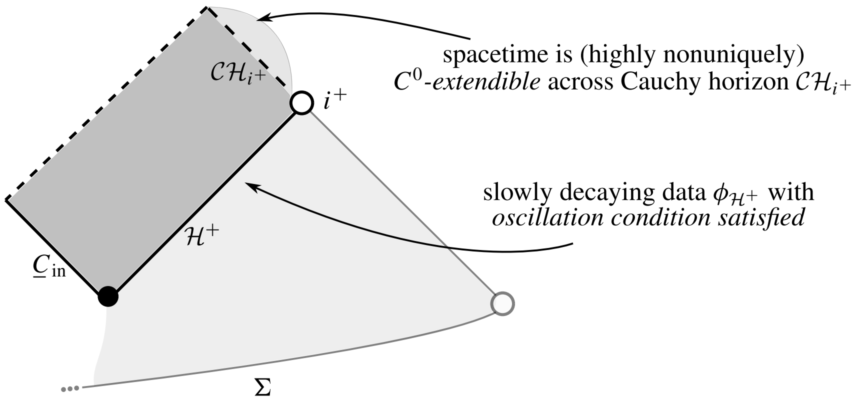

Figure 2. Theorem I (i): if the strong qualitative oscillation condition is satisfied, then the spacetime is C^0-extendible across the Cauchy horizon $\mathcal{CH}_{i^+}$.

Then, ϕ *blows up in amplitude at every point on the Cauchy horizon* $\mathcal{CH}_{i^+}$

$$\limsup_{(u,v)\to\mathcal{CH}_{i^+}} |\phi(u,v)| = +\infty \tag{1-13}$$

if and only if

$$\limsup_{\tilde{v}\to+\infty} \left| \int_{v_0}^{\tilde{v}} \phi_{\mathcal{H}^+}(v)\,\mathrm{d}v \right| = +\infty, \tag{1-14}$$

i.e., *if and only if* $\phi_{\mathcal{H}^+}$ *violates the qualitative oscillation condition* (1-10).

Further, *in the case where the scalar field* ϕ *blows up at the Cauchy horizon* $\mathcal{CH}_{i^+}$ *as in* (1-13), *a null contraction singularity forms at the Cauchy horizon* $\mathcal{CH}_{i^+}$ *as stated in Theorem IV and proved in* [Kehle and Van de Moortel ≥ 2024].

We refer to Figure 3 for an illustration of Theorem I (ii).

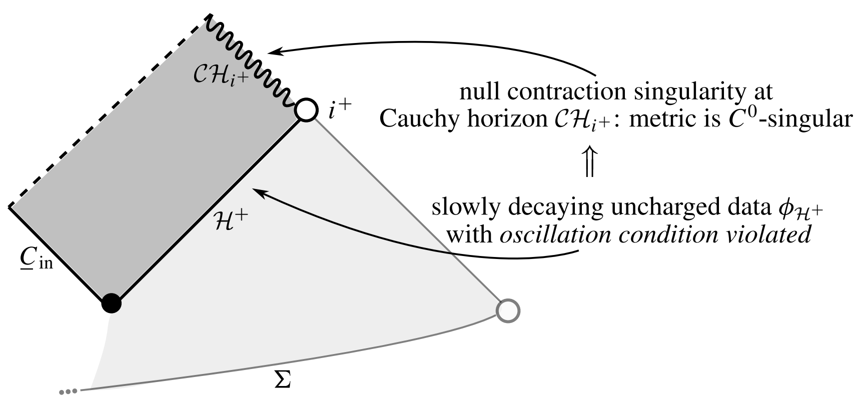

Figure 3. Theorem I (ii): If the oscillation condition is violated in the uncharged case, then a novel null contraction singularity forms at the Cauchy horizon $\mathcal{CH}_{i^+}$ and the metric is C^0-singular at $\mathcal{CH}_{i^+}$.

Theorem I (ii) also shows that it is impossible to prove boundedness of the scalar field ϕ only under the assumptions of Theorem A. This motivates a posteriori the introduction of the oscillation conditions (1-10), (1-11), (1-12), which are thus necessary to obtain boundedness and C^0 extendibility as claimed in Theorem I (i). Anticipating Section 1A5, we note that it is also impossible to prove the continuous extendibility of the metric in the usual sense only under the assumptions of Theorem A, by Theorem IV.

For concreteness, we will now give explicit examples of profiles $\phi_{\mathcal{H}^+}$ which satisfy (respectively violate) the oscillation condition (1-10), (1-11), (1-12) from above.

Example. For any fixed $\omega \neq \omega_{\text{res}}$ the profile $\phi_{\mathcal{H}^+} := e^{-i\omega v} v^{-s}$ satisfies the quantitative oscillation condition (1-12).

Nonexample. The profile $\phi_{\mathcal{H}^+} := e^{-i\omega_{\text{res}} v} v^{-s}$ violates the oscillation condition (1-10).

1A3. *Theorem II: the C^0-formulation of strong cosmic censorship is false.*

Slow decay on $\mathcal{H}^+$ for generic Cauchy data on Σ. We now return to Conjecture 1, which is formulated in terms of generic Cauchy data on an asymptotically flat Σ. First, the scalar field ϕ on the event horizon $\mathcal{H}^+$ is indeed expected to decay slowly for generic Cauchy data on Σ; i.e., $\phi_{\mathcal{H}^+}$ satisfies (1-8) only for $s \leq 1$, at least for almost every pair of parameters (m^2, q_0) (see Conjecture 2). This slow decay makes Theorems I (i) and I (ii) decisive to the study of Cauchy data on Σ as above, since the validity of Conjecture 1 now crucially depends on whether generic Cauchy data on Σ give rise to solutions for which the (slowly decaying) scalar field ϕ on the event horizon $\mathcal{H}^+$ satisfies or violates the oscillation condition (1-10) (or (1-11), its stronger analog).

Oscillations on $\mathcal{H}^+$ for generic Cauchy data on Σ. As it turns out, $\phi_{\mathcal{H}^+}$ is expected to satisfy the (even stronger) quantitative oscillation condition (1-12) for generic regular Cauchy data on Σ. This expectation is based on works in the physics literature relying on heuristic analysis [Hod and Piran 1998; Konoplya and Zhidenko 2013; Koyama and Tomimatsu 2001; 2002] or numerics [Burko and Khanna 2004; Oren and Piran 2003] giving precise asymptotic tails on the event horizon $\mathcal{H}^+$. We formulate this as the following conjecture, where $\phi_{\mathcal{H}^+}$ is the scalar field ϕ restricted to the event horizon $\mathcal{H}^+ = [v_0, +\infty)$, v is an Eddington–Finkelstein-type coordinate (see the gauge choice later defined in (3-6)), and electromagnetic gauge $A_v = 0$ (see (2-26)):

Conjecture 2. *Let $(\mathcal{M}, g, F, A, \phi)$ be a black hole solution of the system* (1-1)–(1-5) *arising from generic, spherically symmetric smooth Cauchy data on an asymptotically flat Σ. Then, the black hole exterior settles down to a Reissner–Nordström exterior with asymptotic mass M and asymptotic charge e satisfying $0 < |e| < M$. Moreover, the scalar field has the following late-time asymptotics on the event horizon $\mathcal{H}^+ = [v_0, +\infty)$:*

(1) *In the massive uncharged case, i.e., $m^2 > 0$, $q_0 = 0$,*

$$\phi_{\mathcal{H}^+}(v) = C(m \cdot M, D) \sin(mv + \omega_{\text{err}}(v)) \cdot v^{-5/6} + \phi_{\text{err}} \tag{1-15}$$

for fast decaying ϕ_{err} (*i.e.*, ϕ_{err} *satisfies* (1-8) *for* $s > 1$), *a constant* $C(m \cdot M, D) \neq 0$ *depending on* $m \cdot M$ *and the initial data* D, *and a sublinear growing phase*

$$\omega_{\text{err}}(v) = -\frac{3m}{2}(2\pi M)^{2/3} v^{1/3} + \omega(m \cdot M).$$

(2) *In the massless charged case*, *i.e.*, $m^2 = 0,\ q_0 \neq 0$,

$$\phi_{\mathcal{H}^+}(v) = C_H(q_0 e, D) \cdot e^{(iq_0 e/r_+)v} \cdot v^{-1-\delta} + \phi_{\text{err}}, \tag{1-16}$$

where $C_H(q_0 e, D) \neq 0$ *is a constant depending on* $q_0 e$ *and the initial data* D, $\delta(q_0 e) := \sqrt{1 - 4(q_0 e)^2} \in \mathbb{C}$, *and* ϕ_{err} *is fast decaying* (*i.e.*, ϕ_{err} *satisfies* (1-8) *for* $s > 1$).

(3) *In the massive charged case*, *i.e.*, $m^2 > 0,\ q_0 \neq 0$,

$$\phi_{\mathcal{H}^+}(v) = C(M \cdot m, D) \cdot e^{(iq_0 e/r_+)v} \cdot \sin(mv + \omega_{\text{err}}(v)) \cdot v^{-5/6} + \phi_{\text{err}}, \tag{1-17}$$

where all the quantities are as above and generically $|q_0 e| \neq r_- |m|$.

Falsification of Conjecture 1 assuming Conjecture 2. We will show that the conjectured profiles in (1-15), (1-16) and (1-17) indeed satisfy the quantitative oscillation (1-12). Thus, as a corollary of our main result Theorem I (i) we obtain a conditional, but otherwise definitive resolution of Conjecture 1:

Theorem II (rough version; precise version in Section 4D). *If* $\phi_{\mathcal{H}^+}$ *is as in Conjecture 2*, *then the metric* g *and the scalar field* ϕ *are continuously extendible across the Cauchy horizon* $\mathcal{CH}_{i^+}$.

In particular, *if Conjecture 2 is true*, *then Conjecture 1 is false for the Einstein–Maxwell–Klein–Gordon system in spherical symmetry.*

We refer to Section 4D for the precise statement of Theorem II.

The conjectured decay rates for $\phi_{\mathcal{H}^+}$ in Conjecture 2 are nonintegrable; i.e., $\phi_{\mathcal{H}^+}$ satisfies (1-8) with s in the range (1-9), except for the massless charged case with $|q_0 e| < \frac{1}{2}$. We also recall that nonintegrable decay of $\phi_{\mathcal{H}^+}$ is insufficient to prove continuous extendibility for g and ϕ by means of decay and indeed even leads to the blow-up of $|\phi|$ as shown in Theorem I (ii) in the case where the oscillation condition (1-10) is violated. In that sense, under the assumption of Conjecture 2, Theorem II shows that C^0-strong cosmic censorship for the EMKG model is false *only* by virtue of the oscillations of the scalar field ϕ on the event horizon $\mathcal{H}^+$.

Lack of oscillations for nongeneric Cauchy data on Σ. Having addressed the generic case in Conjecture 2, there remains still the possibility that there exist (nongeneric) Cauchy data for which the scalar field $\phi_{\mathcal{H}^+}$ on the event horizon $\mathcal{H}^+$ does not satisfy the (qualitative) oscillation condition (1-10). Indeed, on the basis of certain scattering arguments [Angelopoulos et al. 2020; Dafermos et al. 2018; Masaood 2022] we conjecture.[4]

Conjecture 3. *For any suitable finite-energy profile* $\phi_{\mathcal{H}^+}$ *there exist sufficiently regular Cauchy data on* Σ *for the EMKG system in spherical symmetry giving rise to a dynamical black hole for which the scalar field along the event horizon is given by* $\phi_{\mathcal{H}^+}$.

[4]We also note that Conjecture 3 is not specific to the EMKG system in spherical symmetry: similar conjectures can be made for a rather general class of models; see for instance [Angelopoulos et al. 2020; Dafermos et al. 2018].

In particular, if Conjecture 3 is true, this means that there exist Cauchy data on Σ for which the scalar field $\phi_{\mathcal{H}^+}$ on the event horizon $\mathcal{H}^+$ obeys (1-8) for $s > \frac{3}{4}$, but violates the oscillation condition (1-10); thus by Theorem I (ii), the scalar field ϕ blows up in amplitude at the Cauchy horizon $\mathcal{CH}_{i^+}$ (if $q_0 = 0$). Such (nongeneric) Cauchy data will be important in Section 1A5 as they will constitute examples of null contraction singularities at $\mathcal{CH}_{i^+}$; see Theorem IV. Finding the precise regularity (see [Dafermos and Shlapentokh-Rothman 2018; Dias et al. 2018b]) of such Cauchy data on Σ is also part of the resolution of Conjecture 3.

1A4. *Theorem III*: *$W^{1,1}$-blow-up along outgoing cones — a complete contrast with the vacuum case.* We remarked before that the falsification of the C^0-formulation of strong cosmic censorship in vacuum [Dafermos and Luk 2017] — the vacuum analog of Theorem II outside spherical symmetry — crucially relies on integrable decay along the event horizon $\mathcal{H}^+$ for perturbations and their derivatives (see (1-21)). Indeed, in their work, Dafermos and Luk propagate this integrable decay towards i^+ with suitable weighted energy estimates into the black hole interior. This integrable decay for outgoing derivatives is then used to show that the metric is actually $W^{1,1}$-extendible along outgoing null cones, i.e., with locally integrable Christoffel symbols. Note that this $W^{1,1}$-extendibility result of the metric is *strictly stronger* than the C^0-extendibility which subsequently follows by integrating. Mutatis mutandis, this robust physical space method of showing the stronger $W^{1,1}$-extendibility result as an intermediate step has been applied in various previous contexts to show C^0-extendibility; see, e.g., [Dafermos 2003; 2005a; Luk and Oh 2019a; Dafermos and Luk 2017], exploiting the null structure of the Einstein equations: in fact, this was the only known method to prove C^0-extendibility so far. For the EMKG model, however, only in the case $m^2 = 0$, $|q_0 e| < \frac{1}{2}$ do perturbations along the event horizon $\mathcal{H}^+$ decay at an *integrable* rate. For such integrable rates, the analog of Theorem II was shown already [Van de Moortel 2018] using the aforementioned physical space method and proving $W^{1,1}$-extendibility as an intermediate step (schematically $\int |\partial_v g| \, dv < \infty$):

Theorem [Van de Moortel 2018]. *Consider spherically symmetric characteristic initial data for* (1-1)–(1-5) *on the event horizon $\mathcal{H}^+$* (*and on an ingoing cone*). *Let the scalar field $\phi_{\mathcal{H}^+}$ decay fast on the event horizon $\mathcal{H}^+$* (*i.e., $\phi_{\mathcal{H}^+}$ satisfies* (1-8) *for $s > 1$*). *Then ϕ is uniformly bounded in amplitude and in $W^{1,1}$, i.e.,*

$$\sup_{(u,v)} |\phi|(u,v) < +\infty, \quad \sup_{u} \int_{v_0}^{+\infty} |D_v \phi|(u,v) \, dv < +\infty. \tag{1-18}$$

Moreover the metric g admits a $W^{1,1}$ extension $\tilde{g}$ across the Cauchy horizon $\mathcal{CH}_{i^+}$ and $\tilde{g}$ is C^0-admissible (*Definition 2.1*). *In particular, g is C^0-extendible.*

Note that the $W^{1,1}$-extendibility method provides a so-called C^0-admissible extension, which is a continuous extension also admitting null coordinates (a slightly stronger result than general C^0-extendibility).

Apart from the massless case $m^2 = 0$ with $|q_0 e| < \frac{1}{2}$, the scalar field ϕ on the event horizon $\mathcal{H}^+$ is expected to be *nonintegrable* along the event horizon $\mathcal{H}^+$ (Conjecture 2) and as such, the robust physical space methods of [Dafermos and Luk 2017; Luk and Oh 2019a; Dafermos 2003; Van de Moortel 2018] showing the intermediate and stronger $W^{1,1}_{\mathrm{loc}}$-extendibility fail.

We show in Theorem III below that indeed for a *generic nonintegrable* scalar field $\phi_{\mathcal{H}^+}$ on the event horizon $\mathcal{H}^+$, the scalar field ϕ *blows up* in $W^{1,1}$ (i.e., $\int |D_v \phi| \, dv = \infty$) at the Cauchy horizon $\mathcal{CH}_{i^+}$.

This is yet another manifestation of the fact that the C^0-extendibility result for the nonintegrable perturbations is unexpectedly subtle and crucially relies on the precise oscillations of the perturbation on the event horizon $\mathcal{H}^+$. In this sense, our result cannot be captured solely in physical space — making our mixed physical space-Fourier space approach seemingly necessary.

We now give a rough version of Theorem III and refer to Section 4E for the precise formulation.

Theorem III ($W^{1,1}$-blow-up along outgoing cones (rough version; precise version in Section 4E)). *Consider spherically symmetric characteristic initial data for* (1-1)–(1-5) *on the event horizon* $\mathcal{H}^+ = [v_0, +\infty)$ (*and on an ingoing cone*). *Then the following hold true*:

- *Consider arbitrary* $q_0 \in \mathbb{R},\ m^2 \geq 0$.

  Then, for generic $\phi_{\mathcal{H}^+}$ *satisfying* (1-8) *and* (1-9), *the scalar field* ϕ *blows up in* $W^{1,1}_{\mathrm{loc}}$ *at the Cauchy horizon* $\mathcal{CH}_{i^+}$; *i.e., for all u*
$$\int_{v_0}^{+\infty} |D_v\phi|(u, v)\, \mathrm{d}v = +\infty. \tag{1-19}$$

- *Consider either the small charge case* (*i.e.*, $0 < |q_0 e| < \epsilon(M, e, m^2)$ *for* $\epsilon(M, e, m^2) > 0$ *sufficiently small*, $m^2 \geq 0$) *or the uncharged case* $q_0 = 0$ *for almost every mass* $m^2 \in \mathbb{R}_{>0}$.

  Then, for all nonintegrable $\phi_{\mathcal{H}^+} \notin L^1$ *satisfying* (1-8) *and* (1-9), *the scalar field* ϕ *blows up in* $W^{1,1}_{\mathrm{loc}}$ *along outgoing cones at the Cauchy horizon* $\mathcal{CH}_{i^+}$: *i.e., for all u*
$$\int_{v_0}^{+\infty} |D_v\phi|(u, v)\, \mathrm{d}v = +\infty. \tag{1-20}$$

Theorem III shows that the Cauchy horizon $\mathcal{CH}_{i^+}$ is already more singular in the slowly decaying case (i.e., $\phi_{\mathcal{H}^+}$ obeys (1-8) for $s \leq 1$) than in the fast decaying case (i.e., $\phi_{\mathcal{H}^+}$ obeys (1-8) for $s > 1$) as the comparison with (1-18) illustrates.

Assuming that Conjecture 2 is true, as part of our novel Theorem III, we also show that the $W^{1,1}$ blow-up of ϕ given by (1-19) also occurs for generic and regular Cauchy data (for almost all parameters (q_0, m^2)).

Further Theorem III strongly suggests that generically the metric itself is also $W^{1,1}$-inextendible, i.e., does not admit locally integrable Christoffel symbols in any coordinate system. If true, this statement would be in dramatic contrast with the vacuum perturbations of Kerr considered in [Dafermos and Luk 2017] and the weak null singularities from [Luk 2018] (both enjoying the analog of fast decay on the event horizon $\mathcal{H}^+$; see Section 1B2) in which the metric is shown to be $W^{1,1}$-extendible across the Cauchy horizon $\mathcal{CH}_{i^+}$. Extending Theorem III to a full $W^{1,1}$-inextendibility result on the metric is however a difficult (albeit very interesting) open problem due to the geometric nature of such a statement; see [Dafermos and Luk 2017; Luk 2018; Sbierski 2018; 2022; Kehle and Van de Moortel $\geq$ 2024] for related discussions.

1A5. *Theorem IV*: *the null contraction singularity at the Cauchy horizon* $\mathcal{CH}_{i^+}$ *for perturbations violating the oscillation condition.* By Theorem I (ii), if $q_0 = 0$, then any scalar field $\phi_{\mathcal{H}^+}$ that violates on oscillation condition (1-10) on the event horizon $\mathcal{H}^+$ gives rise to ϕ that blows up in amplitude at the Cauchy horizon $\mathcal{CH}_{i^+}$. A natural question then emerges: how does this blow up of the matter field translate geometrically, i.e., does the metric admit a singularity?

This question is answered in the affirmative in our companion paper [Kehle and Van de Moortel $\geq$ 2024]: We show that the metric admits a novel type of C^0-singularity at the Cauchy horizon $\mathcal{CH}_{i^+}$ that we call a *null contraction singularity*. The main result of [Kehle and Van de Moortel $\geq$ 2024] is conditional: we show that the metric admits a null contraction singularity if $|\phi|$ blows up at the Cauchy horizon. Combining this result with Theorem I (ii) (if $q_0 = 0$) shows that a null contraction singularity is formed dynamically for a scalar field $\phi_{\mathcal{H}^+}$ violating the oscillation condition (1-10) on $\mathcal{H}^+$.

We emphasize that the null contraction singularity is a C^0-singularity and different (in particular stronger) from the usual blue-shift instability [Dafermos and Shlapentokh-Rothman 2018] for derivatives, which additionally occurs at the Cauchy horizon of dynamical EMKG black holes and triggers the blow up of curvature and of the Hawking mass (mass inflation); see [Van de Moortel 2018; 2021] and the discussion in Section 1C. Specifically, the null contraction singularity has the following novel characteristics.

Theorem IV [Kehle and Van de Moortel $\geq$ 2024]. *Consider spherically symmetric characteristic initial data for* (1-1)–(1-5) *on the event horizon* $\mathcal{H}^+$ (*and on an ingoing cone*). *Let the scalar field* $\phi_{\mathcal{H}^+}$ *decay slowly on the event horizon* $\mathcal{H}^+$ (*i.e.*, $\phi_{\mathcal{H}^+}$ *satisfies* (1-8), (1-9)). *Assume additionally that* ϕ *blows up in amplitude at the Cauchy horizon* $\mathcal{CH}_{i^+}$, *i.e., assume that* $\limsup_{(u,v)\to\mathcal{CH}_{i^+}} |\phi|(u, v) = +\infty$.

Then the metric g *admits a null contraction singularity in the following sense*:

(a) *The metric does not admit any* C^0-*admissible extension* (*as defined in Definition 2.1*) *across the Cauchy horizon* $\mathcal{CH}_{i^+}$.

(b) *The affine parameter time on ingoing null geodesics* (*with uniform but otherwise arbitrary normalization*) *between two radial causal curves with distinct endpoint at the Cauchy horizon* $\mathcal{CH}_{i^+}$ *tends to zero as the Cauchy horizon* $\mathcal{CH}_{i^+}$ *is approached.*

(c) *The angular tidal deformations of radial ingoing null geodesics* (*with uniform but otherwise arbitrary normalization*) *become arbitrarily large near the Cauchy horizon* $\mathcal{CH}_{i^+}$.

For the precise definitions of the terms employed in the statement of Theorem IV we refer the reader to [Kehle and Van de Moortel $\geq$ 2024]. Note that the null contraction singularity is named in reference to statement (b), the most emblematic: physically, it means that the (suitably renormalized) affine parameter time in the ingoing null direction between two observers tends to zero as both observers approach the Cauchy horizon $\mathcal{CH}_{i^+}$.

Theorem IV is the first instance of a *null contraction singularity*: statements (a)–(c) have only been shown to occur in the context of matter fields blowing up at the Cauchy horizon $\mathcal{CH}_{i^+}$, as we prove in [Kehle and Van de Moortel $\geq$ 2024]. In particular, statements (a)–(c) are all false on the exact Reissner–Nordström interior or on the spacetimes of Theorem I (i) for which ϕ is bounded.

In view of Theorem I (ii), we note that there exists a large class of characteristic data on $\mathcal{H}^+ \cup \underline{C}_{\text{in}}$ giving rise to a null contraction singularity at $\mathcal{CH}_{i^+}$; see Figure 3. Moreover, assuming Conjecture 3, we also note that there exist Cauchy data on asymptotically flat Σ which give rise to a null contraction singularity at $\mathcal{CH}_{i^+}$.

Finally, we note that statement (a) of Theorem IV is, to the best of the authors' knowledge, the first C^0-inextendibility result across a null boundary (in our case the Cauchy horizon $\mathcal{CH}_{i^+}$). The geometric statement (a) strongly suggests that the oscillation condition (1-10) is indeed crucial to falsify Conjecture 1. Note however that Theorem IV only proves the impossibility to extend the metric in a spherically symmetric C^0-class (also used in [Moschidis 2017]), where C^0 double null coordinates exist. It would be interesting to investigate whether statement (a) can be promoted to a full C^0-inextendibility statement. However such statements are notoriously difficult to obtain: even in the more singular case where the black hole boundary is spacelike,[5] the C^0-extendibility of the metric has only been proved for the Schwarzschild black hole [Sbierski 2018].

1B. *Cauchy horizons in other models: a comparison with our results.* Having introduced our main results on the EMKG model (1-1)–(1-5) in Section 1A, we will now mention selected results on the existence/regularity of Cauchy horizons and Conjecture 1 for different models, which will appear to be in dramatic contrast with the previous Theorem A and our new results given in Theorems I (i), I (ii), III and IV on the EMKG model in spherical symmetry.

1B1. *Spherically symmetric models with no Maxwell field: absence of a Cauchy horizon.* Before turning to models admitting Cauchy horizons emanating from i^+, it is useful to recall that there exist models for which such Cauchy horizons do not form. An example of such a model is given by the Einstein-scalar-field system (i.e., (1-1)–(1-5) with $F \equiv 0$, $m^2 = 0$) in spherical symmetry. This model was studied in the seminal series [Christodoulou 1991; 1993; 1999] where it is shown that the MGHD of generic spherically symmetric data is bound to the future by a spacelike boundary $\mathcal{S} = \{r=0\}$ (in particular, there exists no null component of the boundary) and observers approaching $\mathcal{S} = \{r=0\}$ experience infinite tidal deformations.

From [Christodoulou 1991], it follows that Conjecture 1 is *true* for the Einstein-scalar-field system in spherical symmetry in the sense that there exists no spherically symmetric C^0-extension of the metric.

1B2. *Stability of the Cauchy horizon and the downfall of Conjecture 1 for massless fields and in vacuum.*

The Einstein–Maxwell-uncharged-scalar-field in spherical symmetry. Christodoulou's spherically symmetric spacetimes however fail to capture the repulsive effect that angular momentum exerts on the geometry in nonspherical collapse. One way to model this repulsive effect while remaining in the realm of spherical symmetry is to add a Maxwell field to the Einstein-scalar-field equations: The electromagnetic force then plays the role of angular momentum in nonspherical collapse [Dafermos 2004]. The resulting Einstein–Maxwell-uncharged-scalar-field system, i.e., (1-1)–(1-5) with $m^2 = q_0 = 0$, admits a (spherically symmetric) stationary charged black hole, the Reissner–Nordström metric (for which $\phi \equiv 0$) whose MGHD is bound to the future by a smooth Cauchy horizon $\mathcal{CH}_{i^+}$; see Figure 4.

Falsification of Conjecture 1 for the Einstein–Maxwell-uncharged-scalar-field model in spherical symmetry. The interior dynamics[6] near i^+ for the Einstein–Maxwell-uncharged-scalar-field model were studied in the pioneering work [Dafermos 2003; 2005a], which proved that the interior of the black hole admits a Cauchy horizon $\mathcal{CH}_{i^+}$ across which the metric is continuously extendible, under the crucial

[5] A spacelike singularity is indeed widely associated to C^0-inextendibility, and viewed as a stronger singularity than a Cauchy horizon, notably because of the blow-up of tidal deformations experienced on timelike geodesics [Dafermos and Luk 2017; Sbierski 2018; 2022].

[6] For a discussion of the dynamics far away from i^+ in the context of gravitational collapse, see Section 1C3.

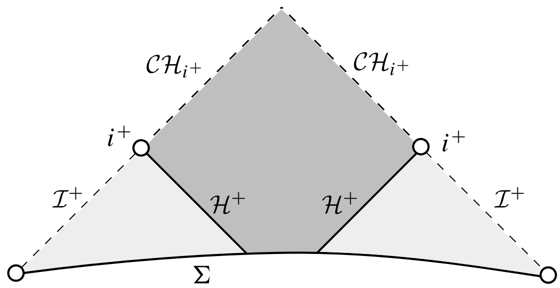

Figure 4. Penrose diagram of the subextremal Reissner–Nordström spacetime.

assumption of integrable decay of the scalar field on the event horizon $\mathcal{H}^+$. Integrable decay for the scalar field on the event horizon $\mathcal{H}^+$ (i.e., $\phi_{\mathcal{H}^+}$ satisfies (1-8) for $s > 1$) was later proved for sufficiently regular Cauchy data of [Dafermos and Rodnianski 2005]; therefore Conjecture 1 is *false* for the Einstein–Maxwell-uncharged-scalar-field model in spherical symmetry [Dafermos 2003; 2005a; Dafermos and Rodnianski 2005] *by means of fast decay* $s > 1$.

Moreover, for this spherically symmetric model, Dafermos [2014] characterized entirely the black hole future boundary for any small, two-ended perturbation of Reissner–Nordström. He indeed showed that the resulting dynamical black hole has no spacelike singularity: its maximal globally hyperbolic development is bound to the future by a null bifurcate Cauchy horizon $\mathcal{CH}_{i^+}$, and has the Penrose diagram of Figure 4.

Falsification of Conjecture 1 for the vacuum Einstein equations without symmetry. As we already mentioned in Section 1A, Conjecture 1 was also falsified in vacuum with no symmetry assumption in the celebrated work [Dafermos and Luk 2017]. In this case as well, the crucial assumption in [Dafermos and Luk 2017] is the fast decay of metric perturbations along the event horizon, i.e., schematically in a standard choice of v-coordinate

$$\|v^{s-1/2}(g - g_K)\|_{L^2(\mathcal{H}^+)} \leq \epsilon \quad \text{for some } s > 1, \tag{1-21}$$

where g_K is the Kerr metric and $\epsilon > 0$ is small. Note that (1-21) shows $|g - g_K|(v) \lesssim v^{-s}$ (at least along a sequence) and in that sense (1-21) is indeed the analog for $g - g_K$ of fast decay of the scalar field, i.e., (1-8) for $s > 1$.

The linear analog of (1-21) for the black hole exterior stability problem around Kerr has been established in [Dafermos et al. 2016; Shlapentokh-Rothman and Teixeira da Costa 2023]; see also the recent nonlinear work [Dafermos et al. 2021]. If (1-21) (and related estimates) are shown for the full Einstein equations in a neighborhood of Kerr, then the result of [Dafermos and Luk 2017] *unconditionally falsifies Conjecture 1 in vacuum*, by means of fast decay $s > 1$.

1C. ***Weak null singularities at the Cauchy horizon and a weaker formulation of strong cosmic censorship.*** In this section, we mention briefly other types of singularities at the Cauchy horizon $\mathcal{CH}_{i^+}$, and how they compare with the new singularities at the Cauchy horizon $\mathcal{CH}_{i^+}$ from Theorems III and IV.

1C1. *Weak null singularities and blue-shift instability.* As discussed earlier, our new results exhibit the first examples of Cauchy horizons $\mathcal{CH}_{i^+}$ singular at the C^0 level (for nonoscillating scalar fields at $\mathcal{H}^+$) and the $W^{1,1}$ level (for all slowly decaying scalar fields at $\mathcal{H}^+$). This new singularity at the Cauchy horizon $\mathcal{CH}_{i^+}$ is very different from the well-known weak null singularity at $\mathcal{CH}_{i^+}$ [Luk 2018; Van de Moortel 2023; Ori and Flanagan 1996; Brady et al. 1998; Burko and Ori 1998; Ori 1999], which corresponds to blow-up in the energy class (i.e., H^1 norm) at $\mathcal{CH}_{i^+}$ due to the celebrated blue-shift instability [Penrose 1968; McNamara 1978]. Blow-up in energy (i.e., H^1 norm in nondegenerate coordinate) at the Cauchy horizon of Kerr and Reissner–Nordström has indeed been proven to occur for the linear wave equation in [Dafermos and Shlapentokh-Rothman 2018; Luk and Sbierski 2016; Luk and Oh 2017]. Based on the blue-shift instability, Christodoulou suggested an alternative formulation of strong cosmic censorship that is weaker than Conjecture 1. Specifically, he conjectured in [Christodoulou 2009] that for generic asymptotically flat Cauchy data, the metric is H^1-inextendible i.e., admits no extension with square-integrable Christoffel symbols; see also [Chruściel 1991; Dafermos and Luk 2017].

More generally, we say that the Cauchy horizon $\mathcal{CH}_{i^+}$ is a weak null singularity if already the metric is C^2-inextendible across $\mathcal{CH}_{i^+}$, a property which is generally obtained from the blow-up of some curvature component in an appropriate frame [Van de Moortel 2021; Luk and Oh 2019a; Kommemi 2013].

1C2. *Dynamical formation of weak null singularities and known inextendibility results.* While examples of weak null singularities have been constructed in vacuum [Luk 2018], their dynamical formation from an "open set" of data with no symmetry assumption is still an open problem. Nevertheless, for the EMKG model in spherical symmetry, it was proven [Van de Moortel 2018; 2021] that *the Cauchy horizon $\mathcal{CH}_{i^+}$ of Theorem A is weakly singular*, i.e., the metric is C^2-inextendible across the Cauchy horizon $\mathcal{CH}_{i^+}$, under the assumptions of Theorem A and additional lower bounds on the scalar field consistent with Conjecture 2. In the uncharged massless model $q_0 = m^2 = 0$ of Section 1B2, the same result was previously proven unconditionally in [Luk and Oh 2019a; 2019b] for generic asymptotically flat two-ended Cauchy data. Both for the EMKG and the $q_0 = m^2 = 0$ model, the above C^2-inextendibility result was improved to a $C^{0,1}$-inextendibility statement in [Sbierski 2022].

1C3. *Weak null singularities in gravitational collapse.* We conclude this section by a brief discussion of the influence of a weak null singularity on the black hole geometry away from i^+. To study this question in the framework of gravitational collapse (i.e., one-ended spacetimes with a center Γ as in Figure 5), we cannot study the Einstein–Maxwell-uncharged-scalar-field model of Section 1B2 because of a well-known [Kommemi 2013; Van de Moortel 2023] topological obstruction caused by the scalar field being uncharged, i.e., $q_0 = 0$, forcing the initial data Σ to be two-ended [Dafermos 2014]. However in the EMKG model, where $q_0 \neq 0$, there is no such obstruction and one can study the one-ended global geometry of the black hole interior with a weak null singularity, even in spherical symmetry [Kommemi 2013]. The main known result in this context is that the *weak null singularity $\mathcal{CH}_{i^+}$ breaks down* [Van de Moortel 2023] before reaching the center: Consequently a so-called first singularity b_Γ is formed at the center Γ, as depicted in Figure 5. This is in complete contrast with the two-ended case where the future boundary is entirely null [Dafermos 2014] for a large class of spacetimes as we discussed in Section 1B2. In the conjecturally generic case where b_Γ is not a so-called *locally naked singularity* [Van de Moortel 2023; Kommemi 2013; Dafermos 2005b; Christodoulou 1999], then the breakdown of the weak null

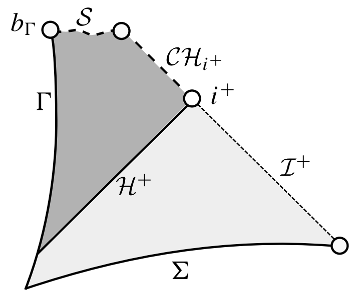

Figure 5. Conjectured Penrose diagram of a generic EMKG black hole with weakly singular $\mathcal{CH}_{i^+}$ [Van de Moortel 2023].

singularity $\mathcal{CH}_{i^+}$ proven [Van de Moortel 2023] implies that a stronger singularity $\mathcal{S} = \{r=0\}$ takes over and connects the weak null singularity $\mathcal{CH}_{i^+}$ to the center Γ as depicted in Figure 5.

1D. ***Scattering resonances associated to the Reissner–Nordström Cauchy horizon.*** We now turn to another result which is not directly concerned with the stability/instability of the Cauchy horizon but turns out to be important for the proofs of our main theorems: the finite-energy scattering theory for the linear wave equation on the interior of Reissner–Nordström developed in [Kehle and Shlapentokh-Rothman 2019]. A key insight to the result in that work was the absence of scattering resonances associated to the Killing generator of the Cauchy horizon, which is an exceptional feature of the massless and uncharged wave equation on exact Reissner–Nordström. Indeed, for the massive wave equation with generic masses $m^2 \in \mathbb{R}_{>0} - D(M, e)$ or for the charged equation, the scattering resonances are present and there does not exist an analogous scattering theory [Kehle and Shlapentokh-Rothman 2019]. As we will show, these scattering resonances are also the key sources of blow-up in amplitude of ϕ at the Cauchy horizon if the scalar field along the event horizon is nonoscillating and slowly decaying and thus, sufficiently resonant. In view of this, for the blow-up statement of Theorem I (ii) these exceptional masses for which the scattering resonances are absent have to be excluded. Refer also to [Mokdad 2022; Häfner et al. 2021] for a scattering theory of the Dirac equation on the interior of Reissner–Nordström and to [Bachelot 1994; Dimock and Kay 1987; Dafermos et al. 2018; Masaood 2022; Alford 2020] for scattering theories on the exterior.

1E. ***Connection to the linear analog of Conjecture 1 for negative cosmological constant $\Lambda < 0$.*** In the discussion above we have studied the Einstein equations with cosmological constant $\Lambda = 0$. Analogously, for $\Lambda \neq 0$, the Reissner–Nordström–(anti-)de Sitter and Kerr–(anti-)de Sitter spacetimes admit a smooth Cauchy horizon and the issue of strong cosmic censorship analogously arises in this setting. In particular, the case $\Lambda < 0$ has some similarities with our case in the sense that linear perturbations also only decay at a nonintegrable (inverse logarithmic for $\Lambda < 0$) rate due to a stable trapping phenomenon [Holzegel and Smulevici 2013; 2014; Holzegel et al. 2020]. A difference to our result is however that only perturbations consisting of a superposition of infinitely many high ℓ angular modes decay slowly and thus, the problem for $\Lambda < 0$ cannot be reasonably studied in spherical symmetry. Nevertheless, as in our case, this nonintegrable rate of decay might raise hopes that, in the case of negative cosmological constant $\Lambda < 0$, Conjecture 1 holds true.

On the one hand, for Reissner–Nordström–AdS, since stable trapping is a high-frequency phenomenon and uniform boundedness (on the linear level) is associated to zero-frequency scattering resonances of the Cauchy horizon, it was shown in [Kehle 2020b] that these difficulties decouple on Reissner–Nordström–AdS. (This decoupling can be seen as the analog of the fact that the oscillation condition of (1-10) is satisfied.) As a consequence of this frequency decoupling, it is shown in [Kehle 2020b] that, despite slow nonintegrable decay on the exterior, linear perturbations remain uniformly bounded and extend continuously across the Reissner–Nordström–AdS Cauchy horizon. This falsifies the linear analog of Conjecture 1 for Reissner–Nordström–AdS.

On the other hand, for Kerr–AdS, in view of the rotation of the black hole, frequency mixing occurs and trapped high-frequency perturbations on the exterior can at the same time be low-frequency when frequency is measured with respect to the Killing generator of the Cauchy horizon. In [Kehle 2020a; 2022] it is shown that this frequency mixing gives rise to a resonance phenomenon and an associated small divisors problem. In particular, for a set of Baire-generic Kerr–AdS black hole parameters, which are associated to a Diophantine condition, it is shown that linear perturbations ϕ blow up in amplitude at the Cauchy horizon. This shows that the linear analog of Conjecture 1 holds true for Baire-generic Kerr–AdS black holes.

There is yet another possible scenario in which the exteriors of AdS black holes are nonlinearly unstable (see [Moschidis 2017; 2020; 2023; Bizoń and Rostworowski 2011]) and the question of strong cosmic censorship would be thrown even more open.

Let us finally also briefly mention the case of positive cosmological constant $\Lambda > 0$, where perturbations on the exterior of Reissner–Nordström/Kerr–de Sitter decay at an exponential rate as proved in [Dyatlov 2011; Mavrogiannis 2023] for the linear wave equation and in [Hintz and Vasy 2018] for the vacuum Einstein equations. In view of this rapid decay, the theorem of [Dafermos and Luk 2017] manifestly also applies and thus, Conjecture 1 is false for $\Lambda > 0$. However, in view of this exponential decay, even weaker formulations such as the H^1-formulation of strong cosmic censorship mentioned in Section 1C may fail. We refer to [Dafermos 2014; Hintz and Vasy 2017; Dias et al. 2018a; 2018b; 2019; Dafermos and Shlapentokh-Rothman 2018; Costa et al. 2018; Costa and Franzen 2017; Mo et al. 2018; Hollands et al. 2020; Cardoso et al. 2018] for details.

1F. ***Summary of the strategy of the proof.*** We now turn to an outline of our proof and begin with the obstructions and difficulties encountered when attempting to prove boundedness of the scalar field at the Cauchy horizon $\mathcal{CH}_{i^+}$ and continuous extendibility of the metric.

- The physical space estimates used to show $\mathcal{CH}_{i^+} \neq \varnothing$ in the proof of Theorem A, under the assumption of a slowly decaying $\phi_{\mathcal{H}^+}$ on $\mathcal{H}^+$, i.e., obeying (1-8) and (1-9), are *consistent with the blow-up of the scalar field* ϕ at the Cauchy horizon $\mathcal{CH}_{i^+}$ and the failure of $\partial_v\phi$ to be integrable in v. As our new result shows, these estimates from [Van de Moortel 2018] are sharp by Theorem III and blow-up in amplitude indeed occurs for some perturbations by Theorem I (ii).

- The estimates of the proof of Theorem A however suggest that, *if* $\partial_v\phi$ *oscillates infinitely towards the Cauchy horizon* $\mathcal{CH}_{i^+}$ then ϕ *is bounded* (see Section 4F1): the hope would be that, although $\partial_v\phi$

is not Lebesgue-integrable (i.e., $\int_{v_0}^{+\infty} |\partial_v \phi| \, dv = +\infty$), it has a semiconvergent Riemann integral (i.e., $\lim_{\tilde{v}\to+\infty} \left|\int_{v_0}^{\tilde{v}} \partial_v \phi \, dv\right| < +\infty$ exists). A natural approach is then to attempt to propagate the event horizon oscillations (1-10) satisfied by $\phi_{\mathcal{H}^+}$ towards the Cauchy horizon $\mathcal{CH}_{i^+}$ in a suitable sense and deduce the boundedness of ϕ. However, this is not easy to show in physical space and prompts a Fourier space approach for the linearized equation.

- A complete understanding of the linearized problem is however insufficient in itself to prove the boundedness of ϕ since the *nonlinear terms cannot be treated purely perturbatively* in view of the slow decay. Consequently the precise structure of these nonlinear terms has to be understood and plays an important role in the argument (in contrast to the fast decay case $s > 1$) (see Section 4F3).

- Even once ϕ is proven to be bounded in amplitude, there is no clear mechanism yielding the continuous extendibility of the metric, contrary to the fast decay case $s > 1$ in which the mechanism is given by the integrability of the Christoffel symbols [Dafermos and Luk 2017; Luk and Oh 2019a] in a suitable sense (see Section 4F4 for a discussion).

Strategy. To address and overcome these difficulties in order to prove our main theorems as stated in Section 1A, we proceed as follows:

(1) We take advantage on the one hand of the previous result of Theorem A, the future black hole boundary is null, i.e., $\mathcal{CH}_{i^+} \neq \varnothing$ and the Penrose diagram is given by Figure 1, and on the other hand of the nonlinear estimates (see Section 4F1) that were already proven in [Van de Moortel 2018] for slowly decaying $\phi_{\mathcal{H}^+}$.

(2) We consider the massive/charged linear wave equation $g^{\mu\nu}_{\mathrm{RN}} D^{\mathrm{RN}}_{\mu} D^{\mathrm{RN}}_{\nu} \phi_{\mathcal{L}} = m^2 \phi_{\mathcal{L}}$ on a fixed Reissner–Nordström background g_{RN}, which we view as the linearization of the EMKG system (1-1)–(1-5). Using Fourier methods and a scattering approach, we prove uniform boundedness (respectively blow-up in amplitude) of $\phi_{\mathcal{L}}$ at the Cauchy horizon $\mathcal{CH}_{i^+}$ for an oscillating scalar field $\phi_{\mathcal{H}^+}$ obeying (1-10) at $\mathcal{H}^+$ (respectively nonoscillating $\phi_{\mathcal{H}^+}$, i.e., $\phi_{\mathcal{H}^+}$ violates (1-10) at $\mathcal{H}^+$); see Section 4F2.

(3) Independently of step (2), we prove nonlinear difference estimates on $g - g_{\mathrm{RN}}$. Although these estimates are, in a sense, weaker[7] than the nonlinear estimates of step (1), they are crucial in our proof that, for all slowly decaying $\phi_{\mathcal{H}^+}$, the linear solution $\phi_{\mathcal{L}}$ is bounded if and only if the nonlinear ϕ is bounded (at least in the $q_0 = 0$ case). In the charged $q_0 \neq 0$ case, we follow a similar logic but additional difficulties arise from the nonlinear backreaction of the Maxwell field. This step will be discussed in Section 4F3.

(4) With the boundedness of ϕ at hand from the previous step, we prove the continuous extendibility of the metric for oscillating perturbations $\phi_{\mathcal{H}^+}$ satisfying (1-11). For the proof, we introduce a crucial new quantity Υ (see (4-38)) exploiting the exact algebraic[8] structure of the nonlinear terms in the Einstein equations; see Section 4F4.

[7] In the sense that these estimates alone are insufficient to show that $\mathcal{CH}_{i^+} \neq \varnothing$ as proven in [Van de Moortel 2018] (see Theorem A).

[8] In contrast, when the decay is integrable as in vacuum, the null structure of the Einstein equations is sufficient [Dafermos and Luk 2017; Luk and Oh 2019a].

The proofs of Theorems I (i) and I (ii) are finally obtained by combining steps (1), (2), (3), and (4). Theorem II follows immediately. The proof of Theorem III is also derived from the strategy given by the same steps (1)–(4); see the last paragraphs in Section 4F4. We refer to Section 4F for a more detailed outline of the strategy of the proof.

1G. ***Outline of the paper.*** In Section 2, we set out notation, definitions and the geometric setting for the solutions of (1-1)–(1-5) under spherical symmetry. In Section 3, for any arbitrary slowly decaying scalar field $\phi_{\mathcal{H}^+}$, we construct and set up spherically symmetric characteristic data on the event horizon $\mathcal{H}^+$ and an ingoing cone such that the scalar field is given by $\phi_{\mathcal{H}^+}$ on $\mathcal{H}^+$. In Section 4, we give the precise formulations of our main results Theorems I (i), I (ii), II, III and their assumptions. We end this section with a detailed outline of our proof in Section 4F. In Section 5, we develop the linear theory and show our main linear results in Section 5D. In Section 6, we develop the nonlinear theory and show the boundedness of the scalar field for the coupled (1-1)–(1-5) and the continuous extendibility of the metric. We first outline in Section 6A the estimates proved in [Van de Moortel 2018], which will be useful for the nonlinear EMKG system. Then in Section 6B, we establish the main estimates necessary for the continuous extendibility of the metric. In Section 6C, we prove difference estimates which we combine in Section 6D with the linear estimates from Section 5 to prove our main results Theorems I (i), I (ii), II, and III.

2. Preliminaries

2A. ***The Reissner–Nordström interior.*** Reissner–Nordström black holes constitute a 2-parameter family of spherically symmetric spacetimes, indexed by charge and mass (e, M), which satisfy the Einstein–Maxwell system ((1-1)–(1-5) with $\phi \equiv 0$) in spherical symmetry. We are interested in the interiors of subextremal Reissner–Nordström black holes satisfying $0 < |e| < M$. To define these spacetimes, we first set

$$\Omega^2_{\mathrm{RN}}(r_{\mathrm{RN}}) := -\left(1 - \frac{2M}{r_{\mathrm{RN}}} + \frac{e^2}{r^2_{\mathrm{RN}}}\right), \tag{2-1}$$

which is nonnegative between the zeros given by

$$r_+(M, e) = M + \sqrt{M^2 - e^2} > 0,$$
$$r_-(M, e) = M - \sqrt{M^2 - e^2} > 0.$$

Now, we define the smooth manifold $\mathring{\mathcal{M}}_{\mathrm{RN}}$ as a 4-dimensional smooth manifold diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^2$. Up to the well-known degeneracy of the spherical coordinates on $\mathbb{S}^2$, let $(r_{\mathrm{RN}}, t, \theta, \varphi) \in (r_-, r_+) \times \mathbb{R} \times \mathbb{S}^2$ be a global chart. In that chart we define the smooth Lorentzian metric g_{RN} and Maxwell 2-form F_{RN}

$$g_{\mathrm{RN}} := -\Omega^{-2}_{\mathrm{RN}}\, \mathrm{d}r^2_{\mathrm{RN}} + \Omega^2_{\mathrm{RN}}\, \mathrm{d}t^2 + r^2_{\mathrm{RN}}(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2), \tag{2-2}$$

$$F_{\mathrm{RN}} = \mathrm{d}A^{\mathrm{RN}} = \frac{e}{r^2}\, \mathrm{d}t \wedge \mathrm{d}r. \tag{2-3}$$

We time-orient the Lorentzian manifold such that vector field $-\nabla r_{\mathrm{RN}}$ is future-directed. Further, we define the tortoise coordinate r^* by $\mathrm{d}r^* = -\Omega^{-2}_{\mathrm{RN}}\, \mathrm{d}r_{\mathrm{RN}}$ or more explicitly by

$$r^* = r^*(r_{\mathrm{RN}}) = r_{\mathrm{RN}} + \frac{1}{4K_+} \log(r_+ - r_{\mathrm{RN}}) + \frac{1}{4K_-} \log(r_{\mathrm{RN}} - r_-), \tag{2-4}$$

where $K_+(M,e)$, $K_-(M,e)$ are the surface gravities associated to the event/Cauchy horizon defined as

$$K_+(M,e)=\frac{1}{2r_+^2}\left(M-\frac{e^2}{r_+}\right)=\frac{r_+-r_-}{4r_+^2}>0,\quad K_-(M,e)=\frac{1}{2r_-^2}\left(M-\frac{e^2}{r_-}\right)=\frac{r_--r_+}{4r_-^2}<0. \tag{2-5}$$

We further introduce the null coordinates $(u,v,\theta,\varphi)\in\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2$ on $\mathring{\mathcal{M}}_{\mathrm{RN}}$ as

$$v=r^*(r)+t,\quad u=r^*(r)-t,\quad \theta=\theta,\quad \varphi=\varphi. \tag{2-6}$$

In this coordinate system the metric g_{RN} has the form

$$g_{\mathrm{RN}}=-\frac{\Omega_{\mathrm{RN}}^2}{2}(\mathrm{d}u\otimes\mathrm{d}v+\mathrm{d}v\otimes\mathrm{d}u)+r_{\mathrm{RN}}^2[\mathrm{d}\theta^2+\sin(\theta)^2\,\mathrm{d}\varphi^2]. \tag{2-7}$$

Now, we attach the (right) event horizon $\mathcal{H}^+$, the past/future bifurcation sphere $\mathcal{B}_-$, $\mathcal{B}_+$, the left event horizon $\mathcal{H}^{+,L}$, the (right) Cauchy horizon $\mathcal{CH}_{i^+}$, and the left Cauchy horizon $\mathcal{CH}_{i_L^+}$ to our manifold, formally defined as

$$\mathcal{H}^+=\{u=-\infty,\ v\in\mathbb{R}\},\quad \mathcal{CH}_{i_L^+}=\{u=+\infty,\ v\in\mathbb{R}\},\quad \mathcal{B}_-=\{u=-\infty,\ v=-\infty\},$$
$$\mathcal{H}^{+,L}=\{v=-\infty,\ u\in\mathbb{R}\},\quad \mathcal{CH}_{i^+}=\{v=+\infty,\ u\in\mathbb{R}\},\quad \mathcal{B}_+=\{u=+\infty,\ v=+\infty\}.$$

A word of caution. In the linear theory of Section 5 we will indeed denote by $\mathcal{H}^+$ the Reissner–Nordström event horizon $\{u=-\infty,\ v\in\mathbb{R}\}$. However, in the other parts of the paper we denote by $\mathcal{H}^+$ the dynamical event horizon $\{u=-\infty,\ v\ge v_0\}$ in the nonlinear part of Section 6 (see also the set-up of the characteristic data in Section 3 and the main theorems stated in Section 4). We do similarly for the Cauchy horizon $\mathcal{CH}_{i^+}$. We also note that the left event and the left Cauchy horizon only play a minor role in the linear part of Section 5 and we often omit "right" when referring to $\mathcal{H}^+$ and $\mathcal{CH}_{i^+}$.

The metric g_{RN} extends smoothly to the boundary and the resulting spacetime is a time-oriented Lorentzian manifold $(\mathcal{M}_{\mathrm{RN}},g_{\mathrm{RN}})$ with corners — the Reissner–Nordström interior. We remark that

$$\Omega_{\mathrm{RN}}^2\underset{r\to r_+}{\sim}C_{e,M}e^{4K_+r^*}=C_{e,M}e^{2K_+(u+v)},\quad \Omega_{\mathrm{RN}}^2\underset{r\to r_-}{\sim}C'_{e,M}e^{4K_-r^*}=C'_{e,M}e^{2K_-(u+v)} \tag{2-8}$$

for some $C_{e,M}>0$, $C'_{e,M}>0$. Further, we introduce regular coordinates (U,v) on $\mathring{\mathcal{M}}_{\mathrm{RN}}\cup\mathcal{H}^+$ as

$$\mathrm{d}U=\tfrac{1}{2}\Omega_{\mathrm{RN}}^2(u,v_0)\,\mathrm{d}u,\quad U(-\infty)=0,\ v=v \tag{2-9}$$

and note that $\mathcal{H}^+=\{U=0\}$. Here $v_0=v_0(M,e,D_1,s)$ will be determined in Proposition 3.2 later. In these coordinates we have obtained a different lapse function $(\Omega_{\mathrm{RN}}^2)_H=(\Omega_{\mathrm{RN}}^2)_H(U,v)=-2g_{\mathrm{RN}}(\partial_U,\partial_v)$ and the metric reads

$$g_{\mathrm{RN}}=-\frac{(\Omega_{\mathrm{RN}}^2)_H}{2}(\mathrm{d}U\otimes\mathrm{d}v+\mathrm{d}v\otimes\mathrm{d}U)+r_{\mathrm{RN}}^2[\mathrm{d}\theta^2+\sin(\theta)^2\,\mathrm{d}\varphi^2]. \tag{2-10}$$

Of course we can invert the coordinate change (2-9) and obtain

$$u=u(U),\quad v=v. \tag{2-11}$$

We also remark that $T:=\partial_t$ in $(r_{\mathrm{RN}},t,\theta,\phi)$-coordinates is a Killing vector field which extends smoothly to $(\mathcal{M}_{\mathrm{RN}},g_{\mathrm{RN}})$.

2B. ***Class of spacetimes, null coordinates, mass, charge.***

Spherically symmetric solution to the EMKG system. A smooth spherically symmetric solution of the EMGK system is described by a quintuplet $(\mathcal{M}, g, F, A, \phi)$, where $(\mathcal{M}, g)$ is a smooth (3+1)-dimensional Lorentzian manifold, ϕ is a smooth complex-valued scalar field, A is a smooth real-valued 1-form, and F is a smooth real-valued 2-form satisfying (1-1)–(1-5) and admitting a free SO(3) action on $(\mathcal{M}, g)$ which acts by isometry with spacelike 2-dimensional orbits (homeomorphic to $\mathbb{S}^2$) and which additionally leaves F, A and ϕ invariant.[9] In this case, the quotient $\mathcal{Q} = \mathcal{M}/\operatorname{SO}(3)$ is a 2-dimensional manifold with projection $\Pi : \mathcal{M} \to \mathcal{Q}$ taking a point of $\mathcal{M}$ into its spherical orbit. As SO(3) acts by isometry, $\mathcal{Q}$ inherits a natural metric, which we call $g_{\mathcal{Q}}$. The metric on $\mathcal{M}$ is then given by the warped product $g = g_{\mathcal{Q}} + r^2\, \mathrm{d}\sigma_{\mathbb{S}^2}$, where $r = \sqrt{\operatorname{Area}(\Pi^{-1}(p))/(4\pi)}$ for $p \in \mathcal{Q}$ is the area radius of the orbit and $\mathrm{d}\sigma_{\mathbb{S}^2}$ is the standard metric on the sphere. The Lorentzian metric $g_{\mathcal{Q}}$ over the smooth 2-dimensional manifold $\mathcal{Q}$ can be written in null coordinates (u, v) as a conformally flat metric

$$g_{\mathcal{Q}} := -\frac{\Omega^2}{2}(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u) \tag{2-12}$$

such that (in mild abuse of notation) we have upstairs

$$g = -\frac{\Omega^2}{2}(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u) + r^2\, \mathrm{d}\sigma_{\mathbb{S}^2}. \tag{2-13}$$

On $(\mathcal{Q}, g_{\mathcal{Q}})$, we now define the Hawking mass as

$$\rho := \frac{r}{2}(1 - g_{\mathcal{Q}}(\nabla r, \nabla r)), \tag{2-14}$$

as well as κ and ι as

$$\kappa := \frac{-\Omega^2}{2\partial_u r} \in \mathbb{R} \cup \{\pm\infty\}, \tag{2-15}$$

$$\iota := \frac{-\Omega^2}{2\partial_v r} \in \mathbb{R} \cup \{\pm\infty\}. \tag{2-16}$$

Electromagnetic fields on $\mathcal{Q}$. In what follows, we will abuse notation and denote by F the 2-form over $\mathcal{Q}$ that is the push-forward by Π of the electromagnetic 2-form originally on $\mathcal{M}$, and similarly for A and ϕ. In view of the SO(3) symmetry of the potential A we have (see [Kommemi 2013]) that F has the form

$$F = \frac{Q}{2r^2}\Omega^2 \mathrm{d}u \wedge \mathrm{d}v, \tag{2-17}$$

where Q is a scalar function called the electric charge. From $F = \mathrm{d}A$ we also obtain

$$[D_u, D_v] = iq_0 F_{uv} = \frac{iq_0 Q \Omega^2}{2r^2}.$$

Now we introduce the modified Hawking mass ϖ that involves the charge Q:

$$\varpi := \rho + \frac{Q^2}{2r}. \tag{2-18}$$

[9] Note that we assume that the SO(3) action is free, i.e., free of fixed points "$r = 0$" as we are interested in the region near i^+, i.e., away from $r = 0$.

An elementary computation relating geometric quantities (on the left) to coordinate-dependent ones (on the right) gives

$$1-\frac{2\rho}{r}=\frac{-4\partial_u r\,\partial_v r}{\Omega^2}=\frac{-\Omega^2}{\iota\kappa}=1-\frac{2\varpi}{r}+\frac{Q^2}{r^2}. \tag{2-19}$$

We also define the quantity

$$2K:=\frac{1}{r^2}\left(\varpi-\frac{Q^2}{r}\right), \tag{2-20}$$

and notice that, if $\varpi=M$ and $Q=e$, then $2K(r_\pm)=2K_\pm$. Further, we introduce the following notation, first used by Christodoulou:

$$\lambda=\partial_v r,\qquad \nu=\partial_u r.$$

Finally, note that (1-4)–(1-5) are invariant under electromagnetic gauge transformations (see Section 2C) and two solutions (ϕ, A) which differ by a gauge transformation represent the same physical behavior. An equivalent formulation to express this gauge freedom is to consider electromagnetism as a $U(1)$ gauge theory with principal $U(1)$-bundle $\pi\colon P\to M$: the charged scalar field is a global section of the associated complex line bundle $P\times_\rho\mathbb{C}$ through the representation ρ such that ϕ corresponds to an equivariant $\mathbb{C}$-valued map on P, i.e., $\phi(pg)=\rho(g)^{-1}\phi$. The representation $\rho: U(1)\to \mathrm{GL}(1,\mathbb{C})$ models the coupling of the scalar field and electromagnetic field. We refer to [Kommemi 2013, Section 1.1] and stick to our equivalent and more concrete formulation of the EMKG system.

C^0-admissible spacetimes and extensions. Lastly, we define the notion of a C^0-admissible extension of the metric (inspired from [Moschidis 2017, Definition A.3]). For the sake of brevity and concreteness we will give neither the most geometric nor the most general formulation and we refer to [Moschidis 2017; Kehle and Van de Moortel $\geq$ 2024] for further details.

Definition 2.1. We call $(\mathcal{M}, g)$ an admissible C^0 spherically symmetric spacetime if the following hold:

(1) $\mathcal{M}$ is a C^1-manifold diffeomorphic to $\mathcal{Q}\times\mathbb{S}^2$ for an open domain $\mathcal{Q}\subset\mathbb{R}^2$.

(2) g is an admissible C^0 spherically symmetric Lorentzian metric in the sense that for a diffeomorphism $\Phi\colon\mathcal{M}\to\mathcal{Q}\times\mathbb{S}^2$ there exist C^1-coordinates (u, v) on $\mathcal{Q}$ in which the metric $\Phi^*(g)$ on $\mathcal{Q}\times\mathbb{S}^2$ can be written as

$$\Phi^*(g)=-\frac{\Omega^2}{2}(\mathrm{d}u\otimes\mathrm{d}v+\mathrm{d}v\otimes\mathrm{d}u)+r^2 g_{\mathbb{S}^2}, \tag{2-21}$$

where $g_{\mathbb{S}^2}$ is the standard round metric on $\mathbb{S}^2$ and $\Omega^2, r^2\colon\mathcal{Q}\to(0,+\infty)$ are continuous.

(3) If $(\tilde u,\tilde v)$ is another C^1-coordinate system such that (2-21) holds with $\widetilde{\Omega}^2$ in place of Ω^2, then $\tilde u=U(u)$ and $\tilde v=V(v)$ for some unique and strictly monotonic C^1-functions U, V.

Remark 2.2. The pair (u, v) as above is called a null coordinate system. In the case where the metric g is locally Lipschitz such null coordinates always exist. Since we merely consider C^0 metrics, in our definition of admissible C^0 metric we additionally impose the existence and uniqueness (up to rescaling) of such null coordinates.

Definition 2.3. Let $(\mathcal{M}, g)$ and $(\widetilde{\mathcal{M}}, \tilde{g})$ be time-oriented admissible C^0 spherically symmetric spacetimes. We say that $(\widetilde{\mathcal{M}}, \tilde{g})$ is an admissible C^0 spherically symmetric future extension if

(1) there exists a C^1 embedding $i : \mathcal{M} \to \widetilde{\mathcal{M}}$ which is also a time-orientation-preserving isometry,

(2) there exists $p \in \widetilde{\mathcal{M}} - i(\mathcal{M})$ which is to the future of $i(\mathcal{M})$.

2C. ***Electromagnetic gauge choices.*** As remarked above, for a fixed metric g, the Maxwell–Klein–Gordon system of equations (1-4)–(1-5) is invariant under the *gauge* transform

$$\phi \to \tilde{\phi} = e^{-iq_0 f}\phi, \tag{2-22}$$

$$A \to \tilde{A} = A + \mathrm{d}f, \tag{2-23}$$

where f is a smooth real-valued function. Notice that for $\widetilde{D} := \nabla + \tilde{A}$ we have

$$\widetilde{D}\tilde{\phi} = e^{-iq_0 f} D\phi.$$

Therefore the quantities $|\phi|$ and $|D\phi|$ are gauge-invariant. In Section 6, we will use that these gauge-invariant quantities satisfy the following estimates which are an immediate consequence of the fundamental theorem of calculus, see, e.g., [Gajic and Luk 2019, Lemma 2.1]. In any (u, v)-coordinate system and for $u \geq u_1$ and $v \geq v_1$,

$$|f(u, v)| \leq |f(u_1, v)| + \int_{u_1}^{u} |D_u f|(u', v)\,\mathrm{d}u, \tag{2-24}$$

$$|f(u, v)| \leq |f(u, v_1)| + \int_{v_1}^{v} |D_v f|(u, v')\,\mathrm{d}v' \tag{2-25}$$

for any sufficiently regular function $f(u, v)$.

Although we will mainly estimate gauge-invariant quantities, to set up the characteristic data it is useful to fix an electromagnetic gauge. For the analysis of the nonlinear system in Section 6 in double null coordinates (u, v) we will impose

$$A_v \equiv 0. \tag{2-26}$$

In this gauge, the condition $F = \mathrm{d}A$ from (1-4) can be written (in any (u, v)-coordinate system) as

$$\partial_v A_u = -\frac{Q\Omega^2}{2r^2}. \tag{2-27}$$

To estimate the dynamics of $A = A_u\,\mathrm{d}u$ in the coupled system it is useful to define a background electromagnetic field A^{RN} which is governed by the fixed Maxwell form $F = F_{\mathrm{RN}}$ as in (2-3) on a fixed Reissner–Nordström background with mass and charge (M, e). Using coordinates (u, v) as defined in (2-6) we impose the gauge

$$A_v^{\mathrm{RN}} \equiv 0 \tag{2-28}$$

such that $F_{\mathrm{RN}} = \mathrm{d}A_{\mathrm{RN}}$ becomes

$$\partial_v A_u^{\mathrm{RN}} = -\frac{e\Omega_{\mathrm{RN}}^2(u, v)}{2r_{\mathrm{RN}}^2(u, v)}. \tag{2-29}$$

Moreover, we choose the normalization for A^{RN} to obtain

$$A^{\mathrm{RN}} = \left(-\frac{e}{r_{\mathrm{RN}}} + \frac{e}{r_+}\right) \mathrm{d}u \tag{2-30}$$

such that the 1-form A^{RN} extends smoothly to the right event horizon $\mathcal{H}^+$ on Reissner–Nordström.

For the linear theory in Section 5 we will work with the t-Fourier transform. In that context it is useful to use a gauge which is *different* from (2-30) and which is given (see (5-1)) by

$$A'_{\mathrm{RN}} = \left(\frac{e}{r_{\mathrm{RN}}} - \frac{e}{r_+}\right) \mathrm{d}t = \left(\frac{e}{r_+} - \frac{e}{r_{\mathrm{RN}}}\right) \frac{\mathrm{d}u - \mathrm{d}v}{2}. \tag{2-31}$$

2D. ***The Einstein–Maxwell–Klein–Gordon system in null coordinates.*** We now express the EMKG system (1-1)–(1-5) in a double-coordinate system (u, v) on $\mathcal{Q}$ using the electromagnetic gauge (2-26). The unknown functions $(r, \Omega^2, A_u, Q, \phi)$ on $\mathcal{Q}$ are subject to the system

$$\partial_u \partial_v r = \frac{-\Omega^2}{4r} - \frac{\partial_u r\, \partial_v r}{r} + \frac{\Omega^2}{4r^3} Q^2 + \frac{m^2 r}{4} \Omega^2 |\phi|^2 = -\frac{\Omega^2}{2} \cdot 2K + \frac{m^2 r}{4} \Omega^2 |\phi|^2, \tag{2-32}$$

$$\partial_u \partial_v \log(\Omega^2) = -2\Re(D_u \phi \partial_v \bar{\phi}) + \frac{\Omega^2}{2r^2} + \frac{2\partial_u r\, \partial_v r}{r^2} - \frac{\Omega^2}{r^4} Q^2, \tag{2-33}$$

the Raychaudhuri equations

$$\partial_u \left(\frac{\partial_u r}{\Omega^2}\right) = \frac{-r}{\Omega^2} |D_u \phi|^2, \tag{2-34}$$

$$\partial_v \left(\frac{\partial_v r}{\Omega^2}\right) = \frac{-r}{\Omega^2} |\partial_v \phi|^2, \tag{2-35}$$

the charged and massive Klein–Gordon equation

$$\partial_u \partial_v \phi = -\frac{\partial_u \phi\, \partial_v r}{r} - \frac{\partial_u r\, \partial_v \phi}{r} + \frac{q_0 i \Omega^2}{4r^2} Q\phi - \frac{m^2 \Omega^2}{4} \phi - i q_0 A_u \frac{\phi\, \partial_v r}{r} - i q_0 A_u\, \partial_v \phi, \tag{2-36}$$

and the Maxwell equations

$$\partial_u Q = -q_0 r^2 \Im(\phi \overline{D_u \phi}), \tag{2-37}$$

$$\partial_v Q = q_0 r^2 \Im(\phi \overline{\partial_v \phi}). \tag{2-38}$$

Finally, $F = \mathrm{d}A$ reads

$$\partial_v A_u = \frac{-Q\Omega^2}{2r^2}. \tag{2-39}$$

Note that (2-37) and (2-38) can be equivalently formulated introducing the quantity $\psi := r\phi$ as

$$\partial_u Q = -q_0 \Im(\psi \overline{D_u \psi}), \tag{2-40}$$

$$\partial_v Q = q_0 \Im(\psi \overline{\partial_v \psi}). \tag{2-41}$$

Further, (2-32) is equivalent to

$$\partial_u (r\, \partial_v r) = \frac{-\Omega^2}{4} + \frac{\Omega^2}{4r^2} Q^2 + \frac{m^2 r^2}{4} \Omega^2 |\phi|^2. \tag{2-42}$$

We can also rewrite (2-36) to control $|\partial_v \phi|$ more easily:

$$D_u\, \partial_v \phi = e^{-iq_0 \int_{u_0}^{u} A_u}\, \partial_u (e^{iq_0 \int_{u_0}^{u} A_u}\, \partial_v \phi) = -\frac{\partial_v r\, D_u \phi}{r} - \frac{\partial_u r\, \partial_v \phi}{r} + \frac{q_0 i \Omega^2}{4r^2} Q\phi - \frac{m^2 \Omega^2}{4} \phi. \tag{2-43}$$

We also have (recalling the notation $\psi = r\phi$)

$$e^{-iq_0 \int_{u_0}^{u} A_u} \partial_u (e^{iq_0 \int_{u_0}^{u} A_u} \partial_v \psi) = D_u(\partial_v \psi) = \frac{-\Omega^2 \phi}{4r} - \frac{\partial_u r\, \partial_v r \cdot \phi}{r} - \frac{\Omega^2 \phi}{4r^3} Q^2 + \frac{m^2 r}{4} \Omega^2 \phi |\phi|^2 - \frac{m^2 \Omega^2 r}{4} \phi - \frac{q_0 i \Omega^2}{4r} Q\phi \tag{2-44}$$

and

$$\partial_v (D_u \psi) = \frac{-\Omega^2 \phi}{4r} - \frac{\partial_u r\, \partial_v r \cdot \phi}{r} + \frac{\Omega^2 \phi}{4r^3} Q^2 + \frac{m^2 r}{4} \Omega^2 \phi |\phi|^2 - \frac{m^2 \Omega^2 r}{4} \phi - \frac{q_0 i \Omega^2}{4r} Q\phi. \tag{2-45}$$

3. Setup of the characteristic data and the oscillation condition

We first fix the arbitrary quantities

$$\text{subextremal charge and mass parameters } 0 < |e| < M, \tag{3-1}$$

$$\text{a decay rate } \tfrac{3}{4} < s \leq 1, \tag{3-2}$$

$$\text{constants } D_1, D_2 > 0. \tag{3-3}$$

These quantities will be kept fixed from now onward.

3A. ***Characteristic cones $\underline{C}_{\mathrm{in}}$, $\mathcal{H}^+$ and underlying manifold $\mathcal{Q}^+$.*** Our yet-to-be-constructed spacetime of study will be the future domain of dependence $\mathcal{Q}^+$ of the characteristic set $\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+ \subset \mathbb{R}^{1+1}$, where $\mathcal{H}^+ := \{U = 0,\ v_0 \leq v < +\infty\}$ and $\underline{C}_{\mathrm{in}} := \{0 \leq U \leq U_s,\ v = v_0\}$, which meet transversely at the common boundary point $p := \{U = 0,\ v = v_0\}$. Here, we use the convention that $f \in C^1(\mathcal{H}^+)$ means that $f \in C^1((v_0, \infty)) \cap C^0([v_0, \infty))$ with the property that $\partial_v f$ extends continuously to $v_0 = \partial\mathcal{H}^+$. Analogously, we define $C^1(\underline{C}_{\mathrm{in}})$. Moreover, we say that $f \in C^1(\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+)$ if f is continuous on $\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+$ and $f|_{\mathcal{H}^+} \in C^1(\mathcal{H}^+)$, $f|_{\underline{C}_{\mathrm{in}}} \in C^1(\underline{C}_{\mathrm{in}})$. In particular, note that if $f_1 \in C^1(\mathcal{H}^+)$ and $f_2 \in C^1(\underline{C}_{\mathrm{in}})$ satisfy $f_1(p) = f_2(p)$, then they define a function in $C^1(\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+)$. Analogously, we define C^k for $k \geq 2$. We define $\mathcal{Q}^+ := \{0 \leq U \leq U_s,\ v_0 \leq v < +\infty\}$. Here $v_0 = v_0(M, e, s, D_1) \geq 1$ only depends on M, e, s, D_1 and $U_s = U_s(M, e, s, D_2, D_1)$ only depends on M, e, s, D_2, D_1 — both of which will be determined in Proposition 3.2 below.

A new coordinate u. We will make use of other coordinates (u, v) on $\mathcal{Q}^+ - \mathcal{H}^+$ given by $u := u(U)$, $v = v$, where $u(U)$ is the function given through the condition (2-9) and (M, e) are as in (3-1). We also define $u_s := u(U_s)$.

An additional electromagnetic gauge freedom. At this point we recall our global electromagnetic gauge choice $A_v \equiv 0$ in Section 2C. An additional electromagnetic gauge freedom we have is the specification of A_U (or equivalently A_u) on $\underline{C}_{\mathrm{in}}$. We impose that A_U on $\underline{C}_{\mathrm{in}} = \{0 \leq U \leq U_s,\ v = v_0\}$ satisfies

$$A_U(U, v_0) = \left(-\frac{e}{r_{\mathrm{RN}}(U, v_0)} + \frac{e}{r_+(e, M)}\right)\frac{\mathrm{d}u}{\mathrm{d}U}(U) = 2\left(-\frac{e}{r_{\mathrm{RN}}(U, v_0)} + \frac{e}{r_+(e, M)}\right)\Omega_{\mathrm{RN}}^{-2}(U, v_0), \tag{3-4}$$

where we used (2-9) for the second identity and thus

$$A_u(u, v_0) = -\frac{e}{r_{\mathrm{RN}}(u, v_0)} + \frac{e}{r_+(e, M)}. \tag{3-5}$$

Here, r_{RN} is the r-value on Reissner–Nordström with parameters (M, e) as given in (3-1) and $r_+(M, e) = M^2 + \sqrt{M^2 - e^2}$.

3B. ***Coordinate gauge conditions on $\mathcal{H}^+$ and $\underline{C}_{\mathrm{in}}$.*** On $\mathcal{H}^+ = \{U = 0,\ v_0 \le v\}$ we will impose the gauge condition

$$\frac{\partial_U r(0, v)}{\Omega_H^2(0, v)} = -\frac{1}{2} \tag{3-6}$$

and on $\underline{C}_{\mathrm{in}} = \{0 \le U \le U_s,\ v = v_0\}$ we will impose

$$\partial_U r = -1. \tag{3-7}$$

3C. ***Free data $\phi \in C^1(\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+)$ with slow decay on $\mathcal{H}^+$ and construction of r, Q, Ω_H^2.*** Having set up the gauges we will now — in addition to the free prescription of $0 < |e| < M$ in (3-1) — freely prescribe data for ϕ on $\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+$. We recall (3-2) and (3-3) and define the class of slowly decaying data $\mathcal{SL}$ on the event horizon $\mathcal{H}^+$ in the following. In order to highlight that the definition does not depend on the gauge choice for the electromagnetic potential A, we formulate it in a gauge-invariant form (although we have already fixed the gauge $A_v \equiv 0$ in (2-26) and (3-5)).

Definition 3.1 (set of slowly decaying data $\mathcal{SL}$). We say that $\phi_{\mathcal{H}^+} \in C^1(\mathcal{H}^+, \mathbb{C})$ is *slowly decaying*, denoted by $\phi_{\mathcal{H}^+} \in \mathcal{SL}$, if

$$|\phi_{\mathcal{H}^+}|(v) + |D_v \phi_{\mathcal{H}^+}|(v) \le D_1 v^{-s} \tag{3-8}$$

for all $v \in \mathcal{H}^+$, where we recall $\frac{3}{4} < s \le 1$ was introduced and fixed in (3-2), and $D_1 > 0$ was introduced and fixed in (3-3).

Similarly, on $\underline{C}_{\mathrm{in}}$ we will also impose arbitrary (up to the corner condition) data $\phi_{\mathrm{in}} \in C^1(\underline{C}_{\mathrm{in}})$ satisfying

$$|D_U \phi_{\mathrm{in}}| \le D_2. \tag{3-9}$$

We will now finally conclude the setup of the initial data, where we recall that we freely prescribed subextremal e, M and the scalar field ϕ on $\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+$. In particular, using standard results about ODEs (recall that $s > \frac{3}{4}$; actually $s > \frac{1}{2}$ is sufficient to prove Proposition 3.2) we obtain:

Proposition 3.2. *There exist $v_0(M, e, s, D_1) \ge 1$ sufficiently large and $U_s(M, e, s, D_2, D_1) > 0$ sufficiently small such that the following holds true. Let $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ and $\phi_{\mathrm{in}} \in C^1(\underline{C}_{\mathrm{in}})$ satisfying* (3-9) *with $\phi_{\mathcal{H}^+}(p) = \phi_{\mathrm{in}}(p)$ be arbitrary. Then, there exist unique solutions $r \in C^2(\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+)$, $\Omega_H \in C^1(\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+)$ and $Q \in C^1(\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+)$ of the ODE system consisting of the Raychaudhuri equation* (2-35), *equation* (2-38), *the equation* (2-32) *using* (3-6) *on $\mathcal{H}^+$ and the ODE system consisting of* (3-7), (2-34) *and* (2-37) *on $\underline{C}_{\mathrm{in}}$ such that*

$$\lim_{v \to +\infty} r(0, v) = r_+(M, e) = M + \sqrt{M^2 + e^2}, \tag{3-10}$$

$$\lim_{v \to +\infty} Q(0, v) = e. \tag{3-11}$$

Moreover, $\mathcal{H}^+$ is affine complete, i.e., $\int_{v_0}^{+\infty} \Omega_H^2(0, v)\, \mathrm{d}v = +\infty$.

This shows that our free data (e, M, ϕ) and the gauge conditions give rise to a full set of data (r, Q, Ω_H^2, ϕ) on $\underline{C}_{\text{in}} \cup_p \mathcal{H}^+$ satisfying the constraint equations.

Further, note that (3-6) implies that

$$\kappa_{|\mathcal{H}^+} \equiv 1$$

in view of (2-15).

Remark 3.3. We also associate to (u, v) a lapse function Ω^2 through

$$\Omega^2 := \Omega_H^2 \frac{\mathrm{d}U}{\mathrm{d}u} \tag{3-12}$$

such that $\Omega^2 = -2g(\partial_u, \partial_v)$ and $\Omega_H^2 = -2g(\partial_U, \partial_v)$ once the spacetime is constructed.

Remark 3.4. In Theorem III we will introduce generic properties of functions in $\mathcal{SL}$. We remark that $\mathcal{SL}$ is the ball of size D_1 in the Banach space

$$\mathcal{SL}_0 := \Big\{ f \in C^1(\mathcal{H}^+; \mathbb{C}) : \sup_{v \geq v_0} (|v^s f| + |v^s D_v f|) < +\infty \Big\}. \tag{3-13}$$

In Theorem III (more precisely in Corollary 5.27) we identify a (exceptional) subspace $H_0 \subset \mathcal{SL}_0$ of infinite codimension. We then call functions $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ generic if $\phi_{\mathcal{H}^+} \in \mathcal{SL} - H$, where $H := H_0 \cap \mathcal{SL}$.

3D. ***Definitions of the oscillation spaces $\mathcal{O}$, $\mathcal{O}'$, $\mathcal{O}''$.*** We now define the subsets $\mathcal{O}, \mathcal{O}', \mathcal{O}'' \subset \mathcal{SL}$ of slowly decaying data on the event horizon describing the oscillation conditions. In order to highlight that the definitions do not depend on the gauge choice for the electromagnetic potential A we formulate them in a gauge-invariant form (although we have already fixed the gauge $A_v \equiv 0$ in (2-26) and (3-5)).

Definition 3.5 (qualitative oscillation condition $\mathcal{O}$). A function $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ is said to satisfy the *qualitative oscillation condition*, denoted by $\phi_{\mathcal{H}^+} \in \mathcal{O}$, if the qualitative condition

$$\limsup_{v \to +\infty} \left| \int_{v_0}^{v} \phi_{\mathcal{H}^+}(v') e^{i(\omega_{\text{res}} v' + q_0 \sigma_{\text{br}}(v'))} e^{i q_0 \int_{v_0}^{v'} (A_v)_{|\mathcal{H}^+}(v'')\, \mathrm{d}v''} \,\mathrm{d}v' \right| < +\infty \tag{3-14}$$

holds for all $D_{\text{br}} > 0$ and all functions $\sigma_{\text{br}} \in C^2([v_0, +\infty), \mathbb{R})$ satisfying

$$|\sigma_{\text{br}}(v)| \leq D_{\text{br}} \cdot (v^{2-2s} 1_{s<1} + \log(1+v) 1_{s=1}), \tag{3-15}$$

$$|\sigma'_{\text{br}}(v)| + |\sigma''_{\text{br}}(v)| \leq D_{\text{br}} v^{1-2s} \tag{3-16}$$

for all $v \geq v_0$, where we recall that $v_0(M, e, s, D_1) > 1$.

We will also denote by $\mathcal{NO} := \mathcal{SL} - \mathcal{O}$ the space of $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ violating (3-14).

Definition 3.6 (strong qualitative oscillation condition $\mathcal{O}'$). A function $\phi_{\mathcal{H}^+} \in \mathcal{O}$ is said to satisfy the *strong qualitative oscillation condition*, denoted by $\phi_{\mathcal{H}^+} \in \mathcal{O}'$, if the limit

$$\lim_{v \to +\infty} \left| \int_{v_0}^{v} \phi_{\mathcal{H}^+}(v') e^{i(\omega_{\text{res}} v' + q_0 \sigma_{\text{br}}(v'))} e^{i q_0 \int_{v_0}^{v'} (A_v)_{|\mathcal{H}^+}(v'')\, \mathrm{d}v''} \,\mathrm{d}v' \right| \tag{3-17}$$

exists (and is finite) for all $D_{\text{br}} > 0$ and all functions $\sigma_{\text{br}} \in C^2([v_0, +\infty), \mathbb{R})$ satisfying (3-15) and (3-16).

Definition 3.7 (quantitative oscillation condition $\mathcal{O}''$). A function $\phi_{\mathcal{H}^+} \in \mathcal{O}'$ is said to satisfy the *quantitative oscillation condition*, denoted by $\phi_{\mathcal{H}^+} \in \mathcal{O}''$, if for all $D_{\mathrm{br}} > 0$ there exist $E_{\mathcal{O}''}(D_{\mathrm{br}}) > 0$, $\eta_0(D_{\mathrm{br}}) > 0$ such that

$$\left| \int_v^{+\infty} e^{i(\omega_{\mathrm{res}} \cdot v' + q_0 \sigma_{\mathrm{br}}(v'))} e^{i q_0 \int_{v_0}^{v'} (A_v)_{|\mathcal{H}^+}(v'')\, dv''} \phi_{\mathcal{H}^+}(v')\, dv' \right| \leq E_{\mathcal{O}''} \cdot v^{s-1-\eta_0} \tag{3-18}$$

for all $v \geq v_0$ and all functions $\sigma_{\mathrm{br}} \in C^2([v_0, +\infty), \mathbb{R})$ satisfying (3-15) and (3-16).

Remark 3.8. Note that we have by definition the inclusions $\mathcal{O}'' \subset \mathcal{O}' \subset \mathcal{O} \subset \mathcal{SL}$. Moreover, note that $\mathcal{O}'' \not\subset L^1([v_0, +\infty))$; more generally, a generic function of $\mathcal{O}''$ is not in $L^1([v_0, +\infty))$.

Remark 3.9. The condition (3-14) and its stronger versions (3-17), (3-18) guarantee sufficiently robust nonresonant oscillations. These conditions are sufficient (our proof also suggests that they are necessary to some extent) to avoid that the backreaction of the Maxwell field (which, as we will show, creates unbounded but sublinear oscillations σ_{br} obeying (3-15), (3-16)) turns *linearly nonresonant* profiles into *nonlinearly resonant* profiles; see the last paragraph of Section 4F3 for a discussion.

Remark 3.10. In the uncharged case $q_0 = 0$, the backreaction of the electric field is absent. In this case note that (3-14) simplifies to a "finite average" condition.

4. Precise statements of the main theorems and outline of their proofs

4A. ***Existence of a Cauchy horizon $\mathcal{CH}_{i^+} \neq \varnothing$ and quantitative estimates in the black hole interior from*** [Van de Moortel 2018]**.** In [Van de Moortel 2018], the second author proved (among other results) that spherically symmetric EMKG black holes converging to a subextremal Reissner–Nordström admit a null boundary $\mathcal{CH}_{i^+} \neq \varnothing$ that we still call a Cauchy horizon. The proof of this main result in [Van de Moortel 2018] required many quantitative estimates that will be useful in the analysis of the current paper.

Theorem B [Van de Moortel 2018]. *Consider the characteristic data on $\underline{C}_{\mathrm{in}} \cup_p \mathcal{H}^+$ as described in Section 3 and fix the electromagnetic gauge* (2-26) *as in Section 2C. Let $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ be arbitrary, and let $\phi_{\mathrm{in}} \in C^1(\underline{C}_{\mathrm{in}})$ satisfying* (3-9) *with $\phi_{\mathrm{in}}(p) = \phi_{\mathcal{H}^+}(p)$ be arbitrary.*

Then, by choosing $U_s(M, e, s, D_2, D_1) > 0$ potentially smaller, the characteristic data give rise to the unique C^1 maximal globally hyperbolic development $(r, \Omega_H^2, A, Q, \phi)$ on $\mathcal{Q}^+$ solving the EMKG system of Section 2D. In addition, an (*ingoing*) *null boundary $\mathcal{CH}_{i^+} \neq \varnothing$* (*the Cauchy horizon*) *can be attached to $\mathcal{Q}^+$ on which r extends as a continuous function r_{CH} which remains bounded away from zero, depicted in the Penrose diagram in Figure 1. Note that $(r, \Omega_H^2, A, Q, \phi)$ on $\mathcal{Q}^+$ defines $(\mathcal{M}, g, A, F, \phi)$ which solves* (1-1)–(1-5).

Moreover, all the quantitative estimates stated in Propositions 6.1, 6.2, 6.3, 6.4 and 6.5 are satisfied.

If we additionally assume fast decay (*i.e., $\phi_{\mathcal{H}^+}$ satisfies* (3-8) *for $s > 1$*), *then ϕ is in $W^{1,1}_{\mathrm{loc}} \cap L^\infty$ at the Cauchy horizon $\mathcal{CH}_{i^+}$ and extends as a continuous function across the Cauchy horizon $\mathcal{CH}_{i^+}$. Moreover, in this case, the metric admits a C^0-admissible extension $\tilde{g}$ across the Cauchy horizon $\mathcal{CH}_{i^+}$ in the sense of Definition 2.1 and $\tilde{g}$ has locally integrable Christoffel symbols.*

Remark 4.1. We note that the above Theorem B showing $\mathcal{CH}_{i^+} \neq \varnothing$, together with all the quantitative estimates stated in Propositions 6.1, 6.2, 6.3, 6.4 and 6.5, actually holds under the weaker assumption of decay rate $s > \frac{1}{2}$ as opposed to $s > \frac{3}{4}$; see [Van de Moortel 2018]. For the purpose of extendibility across the Cauchy horizon $\mathcal{CH}_{i^+}$ for oscillating data as stated in our main result below, the decay assumption $s > \frac{3}{4}$ is needed and appears to be crucial; see the discussion in Section 4F.

4B. *Theorem I* (i): *scalar field boundedness and continuous extendibility for oscillating data.* In this section we give the precise version of Theorem I (i), which is proved as Corollary 6.18 in Section 6D1.

Theorem I (i) (boundedness). *Let the assumptions of Theorem B hold.*

(1) *If $\phi_{\mathcal{H}^+}$ satisfies the qualitative oscillation condition $\phi_{\mathcal{H}^+} \in \mathcal{O}$ (see Definition 3.5), then*

$$\sup_{(u,v)\in\mathcal{Q}^+} |\phi(u,v)| < +\infty. \tag{4-1}$$

(2) *If $\phi_{\mathcal{H}^+}$ satisfies the strong qualitative oscillation condition $\phi_{\mathcal{H}^+} \in \mathcal{O}'$ (see Definition 3.6), then* (4-1) *is true and moreover ϕ admits a continuous extension to $\mathcal{CH}_{i^+}$ and g admits a C^0-admissible extension to $\mathcal{CH}_{i^+}$ in the sense of Definition 2.1. In particular, g is continuously extendible.*

(3) *If $\phi_{\mathcal{H}^+}$ satisfies the quantitative oscillation condition $\phi_{\mathcal{H}^+} \in \mathcal{O}''$ (see Definition 3.7), then* (4-1) *is true, ϕ admits a continuous extension to $\mathcal{CH}_{i^+}$ and g admits a C^0-admissible extension to $\mathcal{CH}_{i^+}$. Moreover, Q is uniformly bounded on $\mathcal{Q}^+$ and admits a continuous extension to $\mathcal{CH}_{i^+}$. Further, there exists a constant $\widetilde{C} = \widetilde{C}(D_1, D_2, E_{\mathcal{O}''}, \eta_0, e, M, m^2, q_0, s) > 0$ such that for all $(u,v) \in \mathcal{LB} \subset \mathcal{Q}^+$*

$$|\phi|(u,v) \le \widetilde{C} \cdot |u|^{s-1-\eta_0}, \tag{4-2}$$

$$|Q - e|(u,v) \le \widetilde{C} \cdot |u|^{-\eta_0}, \tag{4-3}$$

where $E_{\mathcal{O}''} = E_{\mathcal{O}''}(D_{\mathrm{br}}) > 0$, $\eta_0 = \eta_0(D_{\mathrm{br}}) > 0$ are as in (3-18) *and $D_{\mathrm{br}} := D_{\mathrm{br}}(D_1, D_2, e, M, m^2, q_0, s) > 0$ is defined in the proof of Proposition 6.17. Here $\mathcal{LB}$ denotes the late blue-shift region (see Figure 7), a neighborhood of the Cauchy horizon which is defined in Section 6A.*

4C. *Theorem I* (ii): *blow-up in amplitude of the uncharged scalar field for nonoscillating data.* In this section we give the precise version of Theorem I (ii), which is proved as Corollary 6.20 in Section 6D2.

Theorem I (ii) (blow-up). *Let the assumptions of Theorem B hold and let $q_0 = 0$ and $m^2 \in \mathbb{R}_{>0} - D(M, e)$, where $D(M, e)$ is the discrete set of exceptional nonresonant masses as defined in* [Kehle and Shlapentokh-Rothman 2019, Theorem 7]. *In addition, assume that $\phi_{\mathcal{H}^+}$ violates the qualitative oscillation condition as in Definition 3.1, i.e., assume that $\phi_{\mathcal{H}^+} \in \mathcal{NO} := \mathcal{SL} - \mathcal{O}$.*

Then, for all $u \le u_s$, the scalar field blows up in amplitude at the Cauchy horizon $\mathcal{CH}_{i^+}$:

$$\limsup_{v\to+\infty} |\phi|(u,v) = +\infty. \tag{4-4}$$

4D. *Theorem II*: *falsification of C^0-formulation of strong cosmic censorship if Conjecture 2 is true.* We now give the precise version of Theorem II which is proved as Corollary 6.23 in Section 6D3.

Theorem II. *Let the assumptions of Theorem B hold. Additionally assume that Conjecture 2 is true, i.e., $\phi_{\mathcal{H}^+}$ is given by* (1-15) (*if* $q_0 = 0,\ m^2 > 0$), (1-16) (*if* $q_0 \neq 0,\ m^2 = 0$), *or* (1-17) (*if* $q_0 \neq 0,\ m^2 > 0$) *in the v-coordinate defined by* (3-6) *and that the generic condition $|q_0 e| \neq r_-(M, e)|m|$ holds.*

Then $|\phi|$, Q and the metric g admit a continuous extension to $\mathcal{CH}_{i^+}$ and the extension of g can be chosen to be C^0-admissible.

In the above sense, assuming that Conjecture 2 is true, then Conjecture 1 is false for the Einstein–Maxwell–Klein–Gordon system in spherical symmetry.

4E. ***Theorem III*: $W^{1,1}$ *blow-up of the scalar field for nonintegrable data.*** In this section we give the precise version of Theorem III, which is proved in Section 6D4. To state the theorem we first define the set

$$\mathcal{Z}_{\mathfrak{t}}(M, e, q_0, m^2) := \{\omega \in \mathbb{R} \colon \mathfrak{t}(\omega, M, e, q_0, m^2) = 0\} \subset \mathbb{R}, \tag{4-5}$$

which is the zero set of the renormalized transmission coefficient $\mathfrak{t}(\omega)$ defined in (5-23). At this point we note that $\mathcal{Z}_{\mathfrak{t}}(M, e, q_0, m^2)$ is discrete and, depending on the parameters (M, e, q_0, m^2), possibly empty. For small $\delta > 0$ we also define the smeared out set $\mathcal{Z}^{\delta}_{\mathfrak{t}}(M, e, q_0, m^2) \subset \mathbb{R}$ as the set of all $\omega \in \mathbb{R}$ with $\operatorname{dist}(\omega, \mathcal{Z}_{\mathfrak{t}}(M, e, q_0, m^2)) < \delta$. We remark that $\mathcal{Z}^{\delta}_{\mathfrak{t}}(M, e, q_0, m^2) = \varnothing$ if $\mathcal{Z}_{\mathfrak{t}}(M, e, q_0, m^2) = \varnothing$.

Associated to $\mathcal{Z}^{\delta}_{\mathfrak{t}}(M, e, q_0, m^2)$ we now define a family (parametrized by $\delta > 0$) of Fourier projection operators $P_\delta \colon f \in L^2([v_0, +\infty)) \mapsto \mathcal{F}^{-1}[\chi_\delta \mathcal{F}[\tilde{f}]] \in L^2(\mathbb{R})$, where $\tilde{f} \in L^2(\mathbb{R})$ is the extension of f by the zero function for $v < v_0$. Here, $\chi_\delta(\omega)$ is a family (parametrized by $\delta > 0$) of smooth functions which are positive on $\mathcal{Z}^{\delta}_{\mathfrak{t}}(M, e, q_0, m^2)$ and vanish otherwise. In the case where $\mathcal{Z}^{\delta}_{\mathfrak{t}}(M, e, q_0, m^2) = \varnothing$, also $\chi_\delta \equiv 0$. Further, for the Fourier transform, we use the convention

$$\mathcal{F}[\tilde{f}](\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{f}(v) e^{i\omega v}\, \mathrm{d}v.$$

Finally, we are in the position to state Theorem III which is proved in Section 6D4. The first part is shown as Corollary 6.25; the second part is shown as Corollary 6.26.

Theorem III. *Let the assumptions of Theorem B hold.*

***Part* 1.** *Let $\phi_{\mathcal{H}^+} \in \mathcal{SL} - L^1([v_0, +\infty))$ and let at least one of the following assumptions hold*:

(a) *$P_\delta \phi_{\mathcal{H}^+} \in L^1(\mathbb{R})$ for some $\delta > 0$,*

(b) *or $0 < |q_0 e| \leq \epsilon(M, e, m^2)$ for some $\epsilon(M, e, m^2) > 0$ sufficiently small or $q_0 = 0,\ m^2 \notin D(M, e)$.*

Then, the scalar field ϕ blows up in $W^{1,1}$ along outgoing cones at the Cauchy horizon $\mathcal{CH}_{i^+}$ in the sense that for all $u \leq u_s$

$$\int_{v_0}^{+\infty} |D_v \phi|(u, v)\, \mathrm{d}v = +\infty. \tag{4-6}$$

In particular, for any $q_0 \in \mathbb{R}$ and $m^2 \geq 0$, the set H of data $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ for which (4-6) *is not satisfied for all $u \leq u_s$ is exceptional in the sense that $H = H_0 \cap \mathcal{SL}$, where $H_0 \subset \mathcal{SL}_0$ is a subspace of infinite codimension within $\mathcal{SL}_0$* (*recall the definition of $\mathcal{SL}_0$ from* (3-13)). *In the above sense, $\mathcal{SL} - H$ is a generic set and thus $W^{1,1}$-blow-up of the scalar field at the Cauchy horizon $\mathcal{CH}_{i^+}$ is a generic property of the data $\phi_{\mathcal{H}^+} \in \mathcal{SL}$.*

Part 2. *Assume that* $\phi_{\mathcal{H}^+}$ *is given by* (1-15) (*if* $q_0 = 0,\ m^2 > 0$), (1-16) (*if* $q_0 \neq 0,\ m^2 = 0$), *or* (1-17) (*if* $q_0 \neq 0,\ m^2 > 0$) *in the* v*-coordinate defined by* (3-6). *Assume the conditions*

$$\mathcal{Z}_{\mathrm{t}} \cap \Theta = \varnothing, \tag{4-7}$$

$$(m^2, |q_0 e|) \notin \{0\} \times \left[0, \tfrac{1}{2}\right), \tag{4-8}$$

where $\mathcal{Z}_{\mathrm{t}}(M, e, q_0, m^2)$ *is defined in* (4-5) *and where*

$$\Theta(M, e, q_0, m^2) := \begin{cases} \{-m, +m\} & \text{if } q_0 = 0, m^2 \neq 0, \\ \{-q_0 e/r_+\} & \text{if } |q_0 e| \geq \frac{1}{2},\ m^2 = 0, \\ \{-m - q_0 e/r_+, m - q_0 e/r_+\} & \text{if } q_0 \neq 0,\ m^2 \neq 0. \end{cases}$$

Then, the scalar field ϕ *blows up in* $W^{1,1}$ *along outgoing cones at the Cauchy horizon* $\mathcal{CH}_{i^+}$, *i.e.*, (4-6) *holds for all* $u \leq u_s$.

Moreover, (4-7) *is satisfied generically in the sense that for given parameters* $m^2 \geq 0,\ q_0 \in \mathbb{R}$, *with* $m^2 \neq q_0^2$, *the condition* (4-7) *is satisfied for*

$$(M, e) \in \{(M, e) \in \mathbb{R}^2 : 0 < |e| < M\} - E_{m^2, q_0},$$

where $E_{m^2,q_0} \subset \mathbb{R}^2$ *is the zero set of an analytic function.*

In particular, *for fixed* $m^2 \geq 0,\ q_0 \in \mathbb{R}$ *with* $m^2 \neq q_0^2$ *and* $(m^2, |q_0 e|) \notin \{0\} \times \left[0, \tfrac{1}{2}\right)$ *and for almost all parameters*

$$(M, e) \in \{(M, e) \in \mathbb{R}^2 : 0 < |e| < M\},$$

assuming $\phi_{\mathcal{H}^+}$ *is as above*, *then* (4-6) *holds for all* $u \leq u_s$.

Remark 4.2. Note that (4-6) also implies the blow-up of the spacetime $W^{1,1}$ norm in (u, v)-coordinates, i.e., for all $u_1 < u_2 \leq u_s$

$$\int_{u_1}^{u_2} \int_{v_0}^{+\infty} |D_v \phi|(u, v)\, \mathrm{d}v\, \mathrm{d}u = +\infty.$$

The precise formulation and the proof of Theorem IV will be given in our companion paper [Kehle and Van de Moortel ≥ 2024].

4F. ***Outline of the proofs.*** In this section, we elaborate on steps (1)–(4) originally presented in Section 1F. The reader may wish to come back to the current section while consulting the proofs given in Sections 5 and 6. For convenience, we will conclude this section with a guide for the reader; see Section 4F5.

4F1. *A first approach in physical space and the difficulties associated to slow decay* (*step* (1)).

Physical space estimates for the nonlinear problem. Theorem B proving $\mathcal{CH}_{i^+} \neq \varnothing$ also comes with many quantitative stability estimates (see Section 6A) for the nonlinear problem (1-1)–(1-5) under the assumption of slowly decaying $\phi_{\mathcal{H}^+}$ satisfying (1-8) on $\mathcal{H}^+$ $\left(\text{not only for } s > \frac{3}{4} \text{ but also } s > \frac{1}{2}\right)$. These estimates already proven in [Van de Moortel 2018] will be our starting point in Section 6. Although these estimates are sharp, they are however not sufficient to prove the boundedness of ϕ in amplitude, in view of the slow decay obstruction if $s \le 1$ as we shall explain below. To illustrate our point, we start with one of the main estimates[10] obtained by physical space methods in [Van de Moortel 2018]:

$$|D_v\phi|(u,v) \lesssim v^{-s}. \tag{4-9}$$

Boundedness/continuous extendibility in the integrable case. In the integrable case $s > 1$, integrating (4-9) gives immediately boundedness

$$\|\phi\|_{L^\infty} \lesssim \text{data} + \sup_u \|D_v\phi(u,\cdot)\|_{L^1_v} \lesssim \text{data} + \|\langle v\rangle^{-s}\|_{L^1_v} < +\infty, \tag{4-10}$$

and also gives the $W^{1,1}$-extendibility of the metric (i.e., locally integrable Christoffel symbols). From the estimates giving the $W^{1,1}$-extendibility of the metric, one can immediately deduce the continuous extendibility of the metric (see the discussion in Section 4F4). All the known previous proofs of continuous extendibility of the metric indeed proceed via this method [Luk and Oh 2019a; Dafermos 2003; Dafermos and Luk 2017].

Slow decay obstruction in the nonintegrable case. In present paper we however have to deal with the nonintegrable case $s \le 1$, where we note that the above method fails as $\|\langle v\rangle^{-s}\|_{L^1_v}$ (the right-hand side of (4-10)) is infinite, even suggesting that the left-hand side $\|D_v\phi(u,\cdot)\|_{L^1_v}$ could be infinite as well. Indeed, we prove *blow-up* of $\|D_v\phi(u,\cdot)\|_{L^1_v}$ (the so-called $W^{1,1}$ norm on outgoing cones) for generic data $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ (Theorem III, see Section 4F4 for a description of its proof), which illustrates the obstruction to proving boundedness by the standard method previously used in the $s > 1$ case.

Summary of the rate numerology. To summarize, square-integrable decay (i.e., (3-8) with $s > \frac{1}{2}$) is sufficient to show that the black hole boundary admits a null component $\mathcal{CH}_{i^+}$ (the Cauchy horizon) by Theorem B, but is in general insufficient for $W^{1,1}$ extendibility and boundedness of the matter fields and metric coefficients (for which integrable decay, i.e., (3-8) with $s > 1$, is sufficient). In the rest of the section, we explain how to deal with the broader range $\frac{3}{4} < s \le 1$ ($s > \frac{3}{4}$ is important for the new nonlinear estimates; see Section 4F4 and Remark 4.3).

An ingoing derivative estimate. Yet another particularity of the nonintegrable case $s \le 1$ is that $|D_u\phi|$ *may potentially* blow up in amplitude at the Cauchy horizon [Van de Moortel 2018] (there are indeed known examples for which $|D_u\phi|$ blows up; see [Van de Moortel 2021]). Nevertheless, assuming $s > \frac{3}{4}$, we show that $D_u(r\phi)$ is *uniformly bounded* (Proposition 6.6), although not integrable, i.e., we prove that for all $\phi_{\mathcal{H}^+}$ satisfying (3-8)

$$|D_u(r\phi)|(u,v) \lesssim |u|^{-s}. \tag{4-11}$$

Note that, consistently with our result that $|\phi|$ blows up for some data, (4-11) cannot be integrated in u.

Compensate the failure of integrability with oscillations. Slow decay of the data, as we explained, leads to a lack of integrability of the metric and fields derivatives which are roughly of the form, for $\frac{3}{4} < s \le 1$,

$$|D_v\phi| \approx v^{-s}, \tag{4-12}$$

[10]The main difficulty in obtaining (4-9) is nonlinear in nature: its proof in [Van de Moortel 2018] exploits the structure of the Einstein equations to address the delicate issue of controlling the metric for a slow rate $s \le 1$. In contrast, the null condition suffices if $s > 1$.

which is not integrable as $v \to +\infty$ (i.e., towards the Cauchy horizon $\mathcal{CH}_{i^+}$). Nevertheless, boundedness of ϕ could be obtained *by means of the oscillations*, i.e., if we could propagate an estimate of the form

$$D_v\phi \approx e^{i\omega v} \cdot v^{-s} \tag{4-13}$$

for some $\omega \in \mathbb{R} - \{0\}$. However, the propagation of such oscillations, if present on the event horizon characteristic data $\phi_{\mathcal{H}^+}$, requires further estimates in Fourier space that we introduce in the following section.

4F2. *The linear problem* (*step* (2)). In this section, we discuss how to prove boundedness or blow-up of $\phi_{\mathcal{L}}$ solving the linearized equation. This step corresponds to the proof of our main linear result Theorem V in Section 5.

Representation formula using the Fourier transform. For the linear (charged massive) wave equation $g^{\mu\nu}_{\mathrm{RN}} D^{\mathrm{RN}}_\mu D^{\mathrm{RN}}_\nu \phi_{\mathcal{L}} = m^2\phi_{\mathcal{L}}$ on a fixed subextremal Reissner–Nordström interior metric (2-7), the physical space estimates of Section 4F1 also apply, but a Fourier approach is also possible, taking advantage of the Killing vector field ∂_t. Taking the Fourier transform in t, the wave equation then reduces to the so-called radial ODE (see (5-13)). Using this, we will view aspects of the interior propagation from the event horizon to the Cauchy horizon as a scattering problem mapping data on the event horizon to their evolution restricted to the Cauchy horizon; see [Kehle and Shlapentokh-Rothman 2019; Kehle 2022]. Formally, we have, in a suitable regular electromagnetic gauge at the Cauchy horizon:

$$\begin{aligned}\phi_{\mathcal{L}}\restriction_{\mathcal{CH}_{i^+}}(u) = \frac{r_+}{\sqrt{2\pi}r_-}\,\mathrm{p.v.}\int_{\mathbb{R}} \frac{\mathfrak{r}(\omega)}{\omega-\omega_{\mathrm{res}}}\mathcal{F}[\phi_{\mathcal{H}^+}](\omega)e^{i(\omega-\omega_{\mathrm{res}})u}\,\mathrm{d}\omega \\ + \lim_{v\to\infty}\frac{r_+}{\sqrt{2\pi}r_-}\,\mathrm{p.v.}\int_{\mathbb{R}} \frac{\mathfrak{t}(\omega)}{\omega-\omega_{\mathrm{res}}}\mathcal{F}[\phi_{\mathcal{H}^+}](\omega)e^{-i(\omega-\omega_{\mathrm{res}})v}\,\mathrm{d}\omega + \mathrm{Error},\end{aligned} \tag{4-14}$$

where Error is uniformly bounded by the energy of $\phi_{\mathcal{H}^+}$ along the event horizon $\mathcal{H}^+$ and $\omega_{\mathrm{res}}(M,e,q_0)$ is as in (1-7). Here, $\mathfrak{r}(\omega)$ and $\mathfrak{t}(\omega)$ are the (renormalized) scattering coefficients (see Definition 5.2).

Using that $\mathcal{F}[\mathrm{p.v}(1/x)] = i\pi\,\mathrm{sgn}$ and $\mathfrak{t}(\omega_{\mathrm{res}}) = -\mathfrak{r}(\omega_{\mathrm{res}})$ (see (5-25)) we formally obtain

$$\phi_{\mathcal{L}}\restriction_{\mathcal{CH}_{i^+}}(u) = \frac{\sqrt{2\pi}ir_+}{r_-}\mathfrak{r}(\omega_{\mathrm{res}})\lim_{v\to\infty}\int_{-u}^{v}\phi_{\mathcal{H}^+}(\tilde{v})e^{i\omega_{\mathrm{res}}\tilde{v}}\,\mathrm{d}\tilde{v} + \mathrm{Error}. \tag{4-15}$$

Note that $\mathfrak{r}(\omega)$ is real-analytic and in the charged case when $\omega_{\mathrm{res}} \neq 0$, then always $\mathfrak{r}(\omega=\omega_{\mathrm{res}}) \neq 0$. In this charged case, the formal scattering operator (4-14) has a resonance at $\omega = \omega_{\mathrm{res}}$. However, in the uncharged case $q_0 = \omega_{\mathrm{res}} = 0$, there exists a discrete set of nonresonant masses $m^2 \in D(M,e)$ (particularly $0 \in D(M,e)$) such that $\mathfrak{r}(\omega=\omega_{\mathrm{res}}=0)=0$ for $m^2 \in D(M,e)$ as shown in [Kehle and Shlapentokh-Rothman 2019]. In that case, the scattering pole is absent and this can be seen as a key observation towards the T-energy scattering theory on the interior of Reissner–Nordström for the uncharged massless wave equation developed in [Kehle and Shlapentokh-Rothman 2019]. However, it is shown in that work that, for generic masses $m^2 \in \mathbb{R}_{>0} - D(M,e)$, the resonance is present and scattering fails.

A sharp condition for boundedness or blow-up at the Cauchy horizon. Restricting to parameters $q_0 \neq 0$ or $q_0 = 0,\ m^2 \in \mathbb{R}_{>0} - D(M,e)$, the resonance is present and from the formal computation and (4-15) we read off that $|\phi_{\mathcal{L}}| \le C$ if the data $\phi_{\mathcal{H}^+}$ satisfy $\phi_{\mathcal{H}^+} \in L^1_v$ (in addition to having finite energy to control the error terms). Thus, in particular for fast decaying data (i.e., $\phi_{\mathcal{H}^+}$ satisfies (3-8) for $s > 1$), we formally obtain uniform boundedness of $\phi_{\mathcal{L}}$ at the Cauchy horizon.

For general $\phi_{\mathcal{H}^+} \in \mathcal{SL} - L^1$, the above reasoning does not hold, and blow-up in amplitude is possible. For concreteness, first consider the uncharged and massive case $q_0 = 0,\ m^2 \notin D(M, e)$. Then, $\omega_{\text{res}} = 0$ and, as we will show, $\phi_{\mathcal{L}}$ is uniformly bounded at the Cauchy horizon if and only if $\phi_{\mathcal{H}^+}$ satisfies

$$\sup_{v \in [v_0, +\infty)} \left| \int_{v_0}^{v} \phi_{\mathcal{H}^+}(v')\, \mathrm{d}v' \right| < +\infty. \tag{4-16}$$

For instance, (4-16) gives boundedness of $\phi_{\mathcal{L}}$ for data $\phi_{\mathcal{H}^+}$ of the form

$$\phi_{\mathcal{H}^+}(v) \approx e^{-i\omega v} \cdot v^{-s}, \tag{4-17}$$

where we recall $\frac{3}{4} < s \leq 1$, provided $\omega \in \mathbb{R} - \{0\}$: in this case, $\phi_{\mathcal{H}^+}$ obeys the quantitative oscillation condition $\phi_{\mathcal{H}^+} \in \mathcal{O}''$ as defined in Definition 3.7. If, however, $\omega = 0$ then $\phi_{\mathcal{H}^+}$ violates the oscillation condition, i.e., $\phi_{\mathcal{H}^+} \in \mathcal{NO} = \mathcal{SL} - \mathcal{O}$, and thus, $|\phi_{\mathcal{L}}|$ blows up at the Cauchy horizon $\mathcal{CH}_{i^+}$ in view of (4-16) (still assuming $q_0 = 0$).

In the charged case $q_0 \neq 0$, the resonance is always present and uniform boundedness of $\phi_{\mathcal{L}}$ at the Cauchy horizon is true for profiles satisfying the oscillation condition $\phi_{\mathcal{H}^+} \in \mathcal{O}$, e.g., profiles of the form

$$\phi_{\mathcal{H}^+} \approx e^{-i(\omega + \omega_{\text{res}}) \cdot v} \cdot v^{-s}, \tag{4-18}$$

where $\frac{3}{4} < s \leq 1$, provided $\omega \in \mathbb{R} - \{0\}$. If however $\omega = 0$ then $|\phi_{\mathcal{L}}|$ blows up at the Cauchy horizon $\mathcal{CH}_{i^+}$. We refer to Corollary 5.25 for a precise statement of the results of this paragraph.

Improved decay for $\phi_{\mathcal{H}^+} \in \mathcal{O}''$ to obtain the boundedness of the Maxwell field. Note that for the nonlinear EMKG system (1-1)–(1-5), the charge $Q(u, v)$ from (2-17) is a dynamical quantity (assuming $q_0 \neq 0$) that is nonlinearly coupled to ϕ and g, and hence the boundedness of Q is not guaranteed. Proving the boundedness of Q in amplitude indeed requires establishing further decay estimates proved in Corollary 5.25(3), whose proof we now outline. In the case where $\phi_{\mathcal{H}^+}$ satisfies the quantitative oscillation condition, i.e., $\phi_{\mathcal{H}^+} \in \mathcal{O}''$, the main term in (4-15) enjoys decay in $|u|$ as $u \to -\infty$ (corresponding to i^+ in Figure 1). In particular, for $\phi_{\mathcal{H}^+} \in \mathcal{O}''$ we will show (see Theorem V(B)) the quantitative control

$$|\phi_{\mathcal{L}}|(u, v) \lesssim |u|^{-1+s-\eta_0} \tag{4-19}$$

for some $\eta_0 > 0$. This (linear) quantitative estimate will be later useful to the boundedness proof of Q in the coupled case (see Section 4F4).

Towards the $W^{1,1}$-inextendibility. To illustrate the obstruction caused by slow decay explained in Section 4F1, we show in Theorem III that ϕ does not have locally outgoing integrable derivatives near the Cauchy horizon, i.e., $\int |D_v \phi|(u, v)\, \mathrm{d}v = +\infty$ for all u, consistently with the expectation given by (4-12). This blow-up in $W^{1,1}$ norm on outgoing cones justifies that, in the case where ϕ remains bounded, the reason is oscillation and not decay.

To show the $W^{1,1}$ blow-up in linear theory (see Corollary 5.27), we prove a representation formula for $\partial_v \phi_{\mathcal{L}}(u_0, v)$ (see (5-115)) and show that $\partial_v \phi_{\mathcal{L}}(u_0, v) \notin L^1_v$ for fixed u_0. Expressed in a regular gauge on the Cauchy horizon and neglecting error terms, we formally have

$$\partial_v \phi_{\mathcal{L}}(u_0, v) \approx -i \frac{r_+ e^{i\omega_{\mathrm{res}} u_0}}{\sqrt{2\pi} r_-} \int_{\mathbb{R}} \mathcal{F}[\phi_{\mathcal{H}^+}](\omega)\, \mathfrak{t}(\omega) e^{-i(\omega-\omega_{\mathrm{res}})v}\, \mathrm{d}\omega \tag{4-20}$$

close to the Cauchy horizon. We interpret (4-20) as a formal Fourier multiplication operator with multiplier $\mathfrak{t}(\omega)$, i.e., $T_{\mathfrak{t}}\colon \phi_{\mathcal{H}^+}(v) \mapsto \partial_v \phi_{\mathcal{L}}(u_0, v)$. Since our data $\phi_{\mathcal{H}^+}(v)$ are not integrable ($\phi_{\mathcal{H}^+} \notin L^1$) along the event horizon $\mathcal{H}^+$ and we aim to show that $\partial_v \phi_{\mathcal{L}}(u_0, v)$ is not in L^1_v, it is natural to consider to inverse operator $T_{\mathfrak{t}}^{-1} = T_{1/\mathfrak{t}}$ with Fourier multiplier $1/\mathfrak{t}(\omega)$. Formally, by Young's convolution inequality we have

$$\|\phi_{\mathcal{H}^+}\|_{L^1} = \|T_{\mathfrak{t}}^{-1}[\partial_v \phi_{\mathcal{L}}]\|_{L^1} = \|\mathcal{F}[\mathfrak{t}^{-1}] * \partial_v \phi_{\mathcal{L}}\|_{L^1} \leq \|\mathcal{F}[\mathfrak{t}^{-1}]\|_{L^1} \|\partial_v \phi_{\mathcal{L}}\|_{L^1}. \tag{4-21}$$

Since our data $\phi_{\mathcal{H}^+}$ are assumed to be nonintegrable (i.e., $\phi_{\mathcal{H}^+} \notin L^1$), the above formal argument shows $W^{1,1}$ blow-up for $\phi_{\mathcal{L}}(u_0, \cdot\,)$ if $\mathcal{F}[\mathfrak{t}^{-1}] \in L^1$. The above formal computation is made rigorous in the proof of Theorem V(E). Further, we will prove that the only obstruction to $\mathcal{F}[\mathfrak{t}^{-1}] \in L^1$ is potential zeros of $\mathfrak{t}(\omega)$. In the uncharged case $q_0 = 0$, however, the ODE analog of the T-energy identity yields that

$$|\mathfrak{t}(\omega)|^2 = |\mathfrak{r}(\omega)|^2 + |\omega|^2. \tag{4-22}$$

Moreover, since we exclude nonresonant masses (i.e., $m^2 \in \mathbb{R}_{>0} - D(M, e)$), we have $\mathfrak{t}(0) \neq 0$ and as such, $\mathfrak{t}(\omega)$ is nowhere zero. As a result, we show $\mathcal{F}[\mathfrak{t}^{-1}] \in L^1$. For the uncharged case with resonant masses, this shows that all characteristic data $\phi_{\mathcal{H}^+}$ on the event horizon $\mathcal{H}^+$ that are not integrable give rise to solutions which blow up in $W^{1,1}$ along outgoing cones at the Cauchy horizon $\mathcal{CH}_{i^+}$.

In the charged case, however, the analog of (4-22) becomes

$$|\mathfrak{t}(\omega)|^2 = |\mathfrak{r}(\omega)|^2 + \omega(\omega - \omega_{\mathrm{res}}) \tag{4-23}$$

such that $\mathfrak{t}(\omega)$ may have zeros for $\omega \in (0, \omega_{\mathrm{res}})$ or $\omega \in (\omega_{\mathrm{res}}, 0)$. For small charges, a perturbation argument shows that $\mathfrak{t}(\omega)$ does not have zeros but for general charges the set of zeros $\mathcal{Z}_t(M, e, q_0, m^2) = \{\omega \in \mathbb{R}\colon \mathfrak{t}(\omega, M, e, q_0, m^2) = 0\} \subset \{0 < |\omega| < |\omega_{\mathrm{res}}|\}$ could be (and in general will be) nonempty. In view of this, for nonintegrable data (i.e., $\phi_{\mathcal{H}^+} \notin L^1$) which satisfy $P_\delta \phi_{\mathcal{H}^+} \in L^1$ (recall the definition of P_δ from Section 4E), we show that the arising solution blows up in $W^{1,1}$ along outgoing cones. It follows $\phi_{\mathcal{L}}$ blows up in $W^{1,1}$ along outgoing cones for all $\phi_{\mathcal{H}^+} \in \mathcal{SL} - H$, where $H \subset \mathcal{SL}$ is an exceptional subset first introduced in the statement of Theorem III.

4F3. *The nonlinear problem, I*: *physical space estimates of the difference* (*step* (3)). As we explained, the physical space method does not capture the oscillations of the field which are crucial to our proof. On the other hand, the (global) frequency analysis used for the linear equation Klein–Gordon equation on Reissner–Nordström (see (5-3) and as explained above) relies on two key properties: the existence of the Killing vector field ∂_t and the linearity of the equation — none of which extends to the coupled system (1-1)–(1-5).

In the present paper we overcome these limitations by controlling the difference between the nonlinear evolution and its linear counterpart in physical space (i.e., $g - g_{\mathrm{RN}}$ and $\phi - \phi_{\mathcal{L}}$, see below). In the uncharged case $q_0 = 0$, this is exactly the strategy we adopt; see the first paragraph below. In the case $q_0 \neq 0$, unbounded backreaction oscillations of the Maxwell field however require a more sophisticated

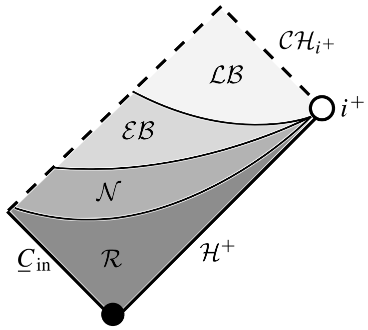

Figure 6. Division of a rectangular neighborhood of i^+ into four spacetime regions.

nonlinear scheme; see Section 4F4 and the second paragraph below. These unbounded backreaction oscillations motivate the precise definition of the oscillations spaces $\mathcal{O}$, $\mathcal{O}'$ and $\mathcal{O}''$ from Section 3D; see the third paragraph below.

The proof of the nonlinear differences estimates will be carried out in Section 6C and follows the splitting of spacetime into four different regions depicted in Figure 7 used already in [Van de Moortel 2018]; see Figure 6 (a similar splitting was first introduced in [Dafermos 2003] and subsequently used in [Franzen 2016; Dafermos and Luk 2017; Luk and Oh 2019a]). More specifically we refer the reader to Propositions 6.13–6.16.

It is important to note that the difference estimates described in this section (and proved in Section 6C) are completely independent of the estimates of Section 5 (whose description was outlined in Section 4F2), with the notable exception of the final formula (4-32) that uses the linear formula (4-15) “as a black box”.

Difference estimates near i^+ for $q_0 = 0$. Near the Cauchy horizon $\mathcal{CH}_{i^+}$ and close to i^+ as in Figure 1 (i.e., for u close to $-\infty$) we obtain difference estimates of the schematic form

$$|\phi - \phi_{\mathcal{L}}|(u, v) + |u|^{-s} \cdot (|g - g_{\mathrm{RN}}| + |\partial_u(g - g_{\mathrm{RN}})|)(u, v) \lesssim |u|^{1-3s}, \tag{4-24}$$

$$\left|\partial_v(\phi - \phi_{\mathcal{L}})\right|(u, v) + v^{-s} \cdot |\partial_v(g - g_{\mathrm{RN}})|(u, v) \lesssim v^{1-3s}, \tag{4-25}$$

where (g, F, A, ϕ) solve (1-1)–(1-5) with data $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ and $\phi_{\mathcal{L}}$ solves (1-5) with same data $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ on a fixed Reissner–Nordström background (2-7) (corresponding to the one g is converging towards i^+). The key point is that $\phi - \phi_{\mathcal{L}}$, unlike ϕ, will turn out to be $\dot{W}^{1,1}$ along outgoing cones at $\mathcal{CH}_{i^+}$, namely (4-25) gives

$$\sup_{u,v} |\phi - \phi_{\mathcal{L}}|(u, v) \lesssim \sup_{u} \int_{v_0}^{+\infty} |\partial_v(\phi - \phi_{\mathcal{L}})|(u, v)\, \mathrm{d}v \lesssim \int_{v_0}^{+\infty} v^{1-3s} \lesssim v_0^{2-3s} < \infty$$

as $s > \frac{3}{4} > \frac{2}{3}$. Therefore $\phi - \phi_{\mathcal{L}}$ *is bounded.* In particular, in the uncharged case $q_0 = 0$, uniform boundedness of ϕ in the region of Figure 1 is equivalent to that of $\phi_{\mathcal{L}}$. As we will see below, this is no longer true if $q_0 \neq 0$.

Difference estimates near i^+ for $q_0 \neq 0$. If $q_0 \neq 0$, the metric differences are similar, but the scalar field difference is now impacted by the Maxwell backreaction. In particular, the first term of (4-25) is replaced by an estimate of the schematic form (in the gauge (2-26) where $A_v = A_v^{\mathrm{RN}} = 0$)

$$\left| e^{-i\sigma_{\mathrm{br}}(u,v)}\, \partial_v \phi - \partial_v \phi_{\mathcal{L}} \right|(u,v) \lesssim v^{1-3s}, \tag{4-26}$$

$$\sigma_{\mathrm{br}}(u,v) := \int_{u_\gamma(v)}^{u} \big((A_u)^{\mathrm{CH}}(u') - (A_u^{\mathrm{RN}})^{\mathrm{CH}}(u') \big)\, \mathrm{d}u', \tag{4-27}$$

where $u_\gamma(v) \sim -v$ and $(A_u)^{\mathrm{CH}}(u')$, $(A_u^{\mathrm{RN}})^{\mathrm{CH}}(u')$ are defined as the extensions of $A_u(u,v)$, $A_u^{\mathrm{RN}}(u,v)$ to $\mathcal{CH}_{i^+}$; see Proposition 6.16 for a precise statement. The difficulty is that σ_{br} is unbounded in general; nevertheless, we prove sublinear growth estimates (in Proposition 6.16 again)

$$|\sigma_{\mathrm{br}}(u,v)| \lesssim v^{2-2s} 1_{s<1} + (1+\log(v)) 1_{s=1}, \tag{4-28}$$

$$|\partial_v \sigma_{\mathrm{br}}(u,v)| + |\partial_v^2 \sigma_{\mathrm{br}}(u,v)| \lesssim v^{1-2s}. \tag{4-29}$$

Note that this is not a gauge issue: in fact, σ_{br} is a gauge-independent quantity obtained by the expression

$$\sigma_{\mathrm{br}}(u,v) := \iint_{[u_\gamma(v),u]\times[v_0,+\infty)} \left(\frac{\Omega^2 Q}{r^2} - \frac{\Omega^2_{\mathrm{RN}} e}{r^2_{\mathrm{RN}}} \right) \mathrm{d}u\, \mathrm{d}v, \tag{4-30}$$

assuming (3-5). As a consequence, *it is no longer true that $\phi - \phi_{\mathcal{L}}$ is uniformly bounded.* Instead, the consequence of (4-26) is that the following quantity is in $\dot{W}^{1,1}$ along outgoing cones and hence bounded:

$$\left| \phi(u,v) - \int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')}\, \partial_v \phi_{\mathcal{L}}(u,v')\, \mathrm{d}v' \right| \lesssim |u|^{2-3s}, \tag{4-31}$$

where $v_\gamma(u) \sim -u$. Therefore, boundedness of ϕ is now down to the boundedness of

$$\int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')}\, \partial_v \phi_{\mathcal{L}}(u,v')\, \mathrm{d}v'.$$

By our representation formula (4-15), this expression becomes, up to error, *an explicit integral of the data*

$$\phi(u,v) = \int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v') + i\omega_{\mathrm{res}} v'} \phi_{\mathcal{H}^+}(v')\, \mathrm{d}v' + O(|u|^{2-3s}). \tag{4-32}$$

Thus, the nonlinear representation formula (4-32) gives boundedness of ϕ up to and including the Cauchy horizon $\mathcal{CH}_{i^+}$ for characteristic event horizon data $\phi_{\mathcal{H}^+} \in \mathcal{O}$, one of the main goals of Theorem I (i) (see Section 6D1).

Further, (4-32) will also show blow-up of ϕ in amplitude at the Cauchy horizon $\mathcal{CH}_{i^+}$ for event horizon characteristic data $\phi_{\mathcal{H}^+} \notin \mathcal{O}$. We postpone the related discussion to the last paragraph of Section 4F4.

The motivation to introduce σ_{br} in the definition of the spaces $\mathcal{O}$, $\mathcal{O}'$, $\mathcal{O}''$. As explained above, the Maxwell field exerts a nontrivial backreaction with in general *unbounded oscillation* σ_{br} (recall (4-28)). Recalling that $\phi_{\mathcal{L}}$ is bounded if and only if the right-hand side of (4-15) is finite (where σ_{br} is as in (4-30)), and that ϕ is bounded if and only if the right-hand side of (4-32) is finite, it becomes clear that *the Maxwell backreaction may turn some linearly nonresonant profiles into nonlinearly resonant ones and vice versa* (a phenomenon which is absent in the uncharged case $q_0 = 0$ where the nonlinear estimates show that ϕ is bounded if and only if $\phi_{\mathcal{L}}$ is bounded).

Therefore, to ensure that our class of oscillating data $\phi_{\mathcal{H}^+} \in \mathcal{O}$ (and analogously $\mathcal{O}'$, $\mathcal{O}''$) gives rise to a bounded ϕ (and not only bounded $\phi_{\mathcal{L}}$), we must define $\phi_{\mathcal{H}^+} \in \mathcal{O}$ (and analogously $\mathcal{O}'$, $\mathcal{O}''$) as a stronger condition than the right-hand side of (4-15) being finite. This stronger condition is to impose *sufficiently robust oscillations* that yield finiteness of the right-hand side of (4-32) for all functions σ_{br} satisfying (3-15), (3-16). In particular, for σ_{br} given by the formula (4-30) (which obeys (3-15), (3-16), as we show, see (4-28), (4-29)), the condition $\phi_{\mathcal{H}^+} \in \mathcal{O}$ (and analogously $\mathcal{O}'$, $\mathcal{O}''$) shows that the oscillations in the initial data are sufficiently robust to not be over-powered by the nonlinear backreaction of the Maxwell field in evolution.

4F4. *The nonlinear problem II*: *boundedness/blow-up of matter fields and metric extendibility* (*step* (4)). Earlier we explained how the nonlinear difference estimates, culminating with (4-32), show that qualitatively oscillating $\phi_{\mathcal{H}^+} \in \mathcal{O}$ on the event horizon $\mathcal{H}^+$ give rise to uniformly bounded scalar field ϕ up to and including $\mathcal{CH}_{i^+}$. In this section, we outline the proof of the following results that conclude the proof of our main theorems:

- C^0-extendibility of the metric (within a certain spherically symmetric class) is *equivalent* to boundedness of $|\phi|$ in amplitude (first paragraph below; see also statements (A) and (B)). From the above equivalence given by (A) and (B), we deduce the main statement of Theorem I (i): the C^0-extendibility of the metric across $\mathcal{CH}_{i^+}$ holds under the strong qualitative oscillation condition $\phi_{\mathcal{H}^+} \in \mathcal{O}'$ on the event horizon $\mathcal{H}^+$ (see the proof in Section 6B). In our companion paper [Kehle and Van de Moortel $\geq$ 2024], the implication (B) that "blow-up of ϕ implies C^0-inextendibility" will be used to prove Theorem IV.
- The charge $Q(u, v)$ of the Maxwell field is bounded for quantitatively oscillating $\phi_{\mathcal{H}^+} \in \mathcal{O}''$ on the event horizon $\mathcal{H}^+$ (second paragraph below, proved in Section 6D1): one of the statements of Theorem I (i).
- The scalar field ϕ blows up in $W^{1,1}$, i.e., $\int |D_v\phi|(u, v)\, \mathrm{d}v = \infty$ for generic slowly decaying $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ on the event horizon $\mathcal{H}^+$ (third paragraph below, proved in Section 6D4): this is Theorem III.
- The scalar field ϕ blows up in L^∞, i.e., $\sup_{(u,v)}|\phi|(u, v) = \infty$ for nonoscillating $\phi_{\mathcal{H}^+} \in \mathcal{NO} = \mathcal{SL} - \mathcal{O}$ on the event horizon, assuming $q_0 = 0$ (fourth paragraph below, proved in Section 6D2): this is Theorem I (ii).

Continuous extendibility of the metric as a consequence of scalar field boundedness. We explained above how to prove boundedness/blow-up of the scalar field depending on the data $\phi_{\mathcal{H}^+}$. Now we explain how to prove that C^0-extendibility on the metric is in a sense equivalent to the boundedness of ϕ up to and including $\mathcal{CH}_{i^+}$, as it turns out! Combining this novel conditional result with the previously discussed boundedness theorem for ϕ will give the main result of Theorem I (i), i.e., the C^0-extendibility of the metric for any characteristic data $\phi_{\mathcal{H}^+} \in \mathcal{O}'$. The proof relies on a nonlinear scheme adapted to the slow decay of the solutions and taking advantage of the algebraic structure of the Einstein equations as explained below.

We begin by recalling from [Van de Moortel 2018] that the following estimates for $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ hold true near the Cauchy horizon $\mathcal{CH}_{i^+}$ and for some $\alpha > 0$ (see Section 6A for details)

$$\Omega^2(u, v) \lesssim e^{-\alpha v}, \tag{4-33}$$

$$|\partial_u \log(\Omega^2)| \lesssim |u|^{1-2s}, \tag{4-34}$$

$$|\partial_v \log(\Omega^2)| \lesssim v^{1-2s}, \tag{4-35}$$

$$|\partial_u r| \lesssim |u|^{-2s}, \tag{4-36}$$

$$|\partial_v r| \lesssim v^{-2s}. \tag{4-37}$$

Since the r estimates (4-36) (4-37) are integrable, it can be shown that $r(u_n, v_n)$ is a Cauchy sequence for any $u_n \to u,\ v_n \to +\infty$: Therefore, r extends to a continuous function. In contrast, the (conjecturally sharp) decay for $\log(\Omega^2)$ is too weak to adopt the same reasoning since $s \leq 1$ ((4-34), (4-35) are nonintegrable).

Nevertheless, $\partial_u\partial_v \log(\Omega^2) + 2\Re(\overline{D_u\phi} D_v\phi)$ enjoys a better decay (see (6-54)), i.e., the weak decay from (4-34), (4-35) comes from a $\Re(\overline{D_u\phi} D_v\phi)$ term in the Einstein equations. It was first noticed by the second author in [Van de Moortel 2019] that it is useful to write the weakly decaying term as a *total derivative*, up to error

$$2\Re(\overline{D_u\phi} D_v\phi) = \partial_u\partial_v(|\phi|^2) + \cdots .$$

Exploiting the ideas of [Van de Moortel 2019], we introduce the following new quantity Υ, which is nonlinear and nonlocal:

$$\Upsilon(u, V) := \log(\Omega^2)(u, V) + |\phi|^2(u, V) + \int_u^{u_s} \frac{|\partial_u r|(u', V)}{r(u', V)} |\phi|^2(u', V)\, \mathrm{d}u', \tag{4-38}$$

where $\Omega^2 := -2g(\partial_u, \partial_V)$ for a suitably renormalized (u, V) coordinate system. We then prove that Υ is bounded and admits a continuous extension (see Section 6B2 for the proof).

Remark 4.3. To show that the right-hand side of (4-38) is bounded, we need the assumption $s > \frac{3}{4}$, which among other things, explains the numerology in the definition of $\mathcal{SL}$ (Definition 3.1); compare with Theorem A.

It turns out that the boundedness of Υ ultimately makes C^0-extendibility *equivalent* to the boundedness of ϕ in the following sense (see [Van de Moortel 2019]).

(A) If $|\phi|$ is bounded, then there exists a coordinate system (u, V) such that $\log(\Omega^2)$ is bounded.

(B) Conversely, if $|\phi|$ blows up, there *exists no coordinate system* (u, V) such that $\log(\Omega^2)$ is bounded.

Part (A) follows from the definition (4-38) and the (unconditional) boundedness of Υ (since $\partial_u r/r$ is also bounded). Moreover, because Υ is continuously extendible, if $|\phi|$ is continuously extendible, then $\log(\Omega^2)$ is also continuously extendible (hence so is Ω^2). In particular for data $\phi_{\mathcal{H}^+} \in \mathcal{O}'$, since we previously showed that $|\phi|$ is continuously extendible across $\mathcal{CH}_{i^+}$, we then obtain the continuous extendibility of g (see Section 6B3 for the proof), and a slightly improved statement: the existence of a C^0-admissible extension (Definition 2.1), i.e., a continuous extension admitting regular double null coordinates (u, V) given by the above pair (r, Ω^2).

Part (B) is more delicate and is proven in [Van de Moortel 2019, Theorem 2.3.5] (and used in [Kehle and Van de Moortel $\geq$ 2024] to prove Theorem IV): it implies that if $|\phi|$ blows up, then g does not admit any C^0-admissible extension.

Boundedness of the Maxwell field Q. We now outline the proof of the boundedness of the charge $Q(u, v)$ for $\phi_{\mathcal{H}^+} \in \mathcal{O}''$ given in Section 6D1. To prove boundedness of Q, we will actually need *decay as $u \to -\infty$ for ϕ* (in addition to its uniform boundedness already obtained assuming $\phi_{\mathcal{H}^+} \in \mathcal{O}$): this motivates the introduction of the space $\mathcal{O}'' \subset \mathcal{O}$ from Section 3D. We start taking advantage of the structure of the Maxwell equation:

$$\partial_u Q = r^2 \Im(\bar{\phi} D_u \phi) = \Im(\overline{r\phi} D_u(r\phi)).$$

Moreover, we use (4-11) to obtain the estimate (using also the boundedness of r):

$$|\partial_u Q| \lesssim |\phi| \cdot |D_u(r\phi)| \lesssim |\phi| \cdot |u|^{-s}.$$

To obtain boundedness, we integrate in u. For this, we take advantage of the quantitative $|u|$ decay of $|\phi|$ which is true if $\phi_{\mathcal{H}^+}$ satisfies the quantitative oscillation condition $\phi_{\mathcal{H}^+} \in \mathcal{O}''$. Combining both the linear estimate (4-19) on $\phi_{\mathcal{L}}$ and the nonlinear estimate (4-24) on $\phi - \phi_{\mathcal{L}}$, we obtain $|\phi| \lesssim |u|^{s-1-\eta_0}$ and thus

$$|\partial_u Q| \lesssim |u|^{-1-\eta_0},$$

which is integrable and thus sufficient to conclude the boundedness and continuous extendibility of Q.

$W^{1,1}$ blow-up of the scalar field. We now turn to the proof of $W^{1,1}$ blow-up on outgoing cones of ϕ for generic $\phi_{\mathcal{H}^+} \in \mathcal{SL} - L^1$(proof in Section 6D4). One of our nonlinear difference estimates gives near the Cauchy horizon $\mathcal{CH}_{i^+}$ and uniformly in u

$$\big||D_v\phi|(u, v) - |D_v^{\mathrm{RN}}\phi_{\mathcal{L}}|(u, v)\big| \lesssim v^{1-3s},$$

which is integrable, since $s > \frac{3}{4} > \frac{2}{3}$. Therefore, $\|D_v\phi(u, \cdot)\|_{L^1} = +\infty$ if and only if $\|D_v^{\mathrm{RN}}\phi_{\mathcal{L}}(u, \cdot)\|_{L^1} = +\infty$. For $|q_0 e|$ small enough, (4-21) gives blow up of $\|D_v\phi(u, \cdot)\|_{L^1}$ for any $\phi_{\mathcal{H}^+} \in \mathcal{SL} - L^1$ (and for any $\phi_{\mathcal{H}^+} \in \mathcal{SL} - H$ in the case $q_0 \neq 0$, what we call the generic case, recalling the discussion at the end of Section 4F2).

Blow-up in amplitude of the scalar field ϕ if $\phi_{\mathcal{H}^+} \notin \mathcal{O}$. We now explain how the nonlinear representation formula (4-32) can be used to prove the blow-up in amplitude of $\phi_{\mathcal{H}^+}$ for $\phi_{\mathcal{H}^+} \in \mathcal{NO} = \mathcal{SL} - \mathcal{O}$ (see Section 6D2 for the proof). Recall indeed that (4-32) formally states that the uniform boundedness of ϕ up to and including the Cauchy horizon $\mathcal{CH}_{i^+}$ is equivalent to the finiteness of the characteristic data integral on the event horizon $\mathcal{H}^+$, i.e., for all $|u| \geq v_0$

$$\sup_v |\phi|(u, v) = \infty \quad \Longleftrightarrow \quad \sup_v \left| \int_{-u}^{v} e^{i\sigma_{\mathrm{br}}(u,v') + i\omega_{\mathrm{res}} v'} \phi_{\mathcal{H}^+}(v')\, \mathrm{d}v' \right| = \infty \tag{4-39}$$

for σ_{br} defined by (4-27) and in the gauge (2-26). If for given characteristic data $\phi_{\mathcal{H}^+} \in \mathcal{SL} - \mathcal{O}$ on the event horizon $\mathcal{H}^+$, the upper bounds (4-28), (4-29) also hold as *lower bounds* up to the Cauchy horizon $\mathcal{CH}_{i^+}$, (4-39) shows that *ϕ blows up at the Cauchy horizon $\mathcal{CH}_{i^+}$*: for instance, one can check that for $\frac{2}{3} < s < 1$,

$$\text{for the choice } \phi_{\mathcal{H}^+}(v) = e^{-iq_0\omega_{\mathrm{res}} v} v^{-s}, \quad \limsup_{v \to +\infty} \left| \int_{v_0}^{v} e^{iq_0 (v')^{2-2s}} (v')^{-s}\, \mathrm{d}v' \right| = +\infty.$$

Unfortunately, while we conjecture that such lower bounds are true[11] for *most solutions*, it seems that fine-tuned ones could violate them. When these lower bounds are violated and σ'_{br} or σ''_{br} decay faster, we

[11] The identity (4-30) indeed suggests that σ_{br} is comparable schematically to $|g - g_{\mathrm{RN}}|$ which is formally of order $\alpha\ v^{1-2s} + o(v^{1-2s})$ for some $\alpha \in \mathbb{R}$. The case $\alpha = 0$ is presumably nongeneric but leads to faster decay for σ'_{br} and σ''_{br} notably.

have a linearly resonant profile ($\phi_{\mathcal{H}^+} \notin \mathcal{O}$) become nonlinearly nonresonant (meaning ϕ is bounded at the Cauchy horizon) (for instance: if σ''_{br} decays faster, say $\sigma''_{\mathrm{br}}(v) = O(v^{-5s+3})$, then the right-hand side of (4-39) is finite for the choice $\phi_{\mathcal{H}^+}(v) = e^{-iq_0\omega_{\mathrm{res}}v} v^{-s}$). To sum up: the difficulty to control precisely these backreaction oscillations explains the absence of blowing-up examples for $q_0 \neq 0$ in the present paper, but not their plausibility!

In the case $q_0 = 0$, and for $m^2 \notin D(M, e)$, we obtain blow-up for all data $\phi_{\mathcal{H}^+} \in \mathcal{SL} - \mathcal{O}$. As mentioned before, the restriction of the mass parameter m^2 is due to "exceptional" so-called *nonresonant masses* (see [Kehle and Shlapentokh-Rothman 2019]) for which boundedness of the linearized $\phi_{\mathcal{L}}$ (hence of the EMKG-coupled scalar field ϕ, by our result) is true, even though $\phi_{\mathcal{H}^+} \notin \mathcal{O}$. Nevertheless, the set of nonresonant masses $D(M, e)$ is the zero set of a nontrivial analytic function as proved in [Kehle and Shlapentokh-Rothman 2019], and as such, it is discrete and of zero Lebesgue measure.

4F5. *Guide to the reader.* We conclude this section with a short guide to help the reader read through the proofs of Sections 5 and 6. While the above outline of the proof was organized thematically to highlight the resolution of various difficulties, for technical reasons the rest of the paper is organized slightly differently as follows:

(1) In Section 5 we study the solution $\phi_{\mathcal{L}}$ of the linear charged and massive Klein–Gordon equation $g^{\mu\nu}_{\mathrm{RN}} D^{\mathrm{RN}}_{\mu} D^{\mathrm{RN}}_{\nu} \phi_{\mathcal{L}} = m^2 \phi_{\mathcal{L}}$ on a fixed Reissner–Nordström metric with slowly decaying characteristic data $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ on the event horizon $\mathcal{H}^+$. The approach is mostly focused on Fourier analysis, capturing the oscillations of $\phi_{\mathcal{L}}$ towards the Cauchy horizon $\mathcal{CH}_{i^+}$.

- (a) In Section 5A, we set up the radial ODE satisfied by the Fourier transform of $\phi_{\mathcal{L}}$ associated to the timelike Killing vector field ∂_t on (2-7).
- (b) In Section 5B, we first show the existence of a scattering resonance (i.e., a pole at the resonant frequency $\omega = \omega_{\mathrm{res}}$). Moreover, we show suitable resolvent estimates associated to the radial ODE. This allows us to prove properties of the (renormalized) scattering coefficients $\mathfrak{r}(\omega)$, $\mathfrak{t}(\omega)$.
- (c) In Section 5C, we show a first representation formula involving $\mathfrak{r}(\omega)$ and $\mathfrak{t}(\omega)$ for $\phi_{\mathcal{L}}$ in terms of the event horizon data $\phi_{\mathcal{H}^+}$.
- (d) In Section 5D, we take the limit of the representation formula to the Cauchy horizon of Reissner–Nordström which eventually yields our main linear result Theorem V.

(2) In Section 6 we estimate the solution (g, F, A, ϕ) of the nonlinear Einstein–Maxwell–Klein–Gordon system (1-1)–(1-5) with slowly decaying characteristic data $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ on the event horizon $\mathcal{H}^+$. The approach is mostly focused on physical space estimates, capturing the effect of ϕ on the metric g.

- (a) In Section 6A we recall the nonlinear estimates from [Van de Moortel 2018]. They are essential to the analysis, both to show the continuous extendibility of g and for the nonlinear difference estimates; see below.
- (b) In Section 6B, we show that, assuming ϕ is uniformly bounded, the metric g is continuously extendible. The proof exploits the special structure of the nonlinearity in the Einstein equations.

(c) In Section 6C, we estimate together the differences $g - g_{\mathrm{RN}}$ and $\phi - \phi_{\mathcal{L}}$. If $q_0 = 0$ this shows that boundedness of ϕ is equivalent to boundedness of $\phi_{\mathcal{L}}$. If $q_0 \neq 0$, we have (4-31) as a substitute.

(d) In Section 6D, we combine the results of Sections 5 and 6C to obtain the nonlinear representation formula (4-32). From (4-32) we can read off boundedness/blow-up of ϕ from the event horizon data $\phi_{\mathcal{H}^+}$. Combining with Section 6B gives the C^0-extendibility of g for oscillating event horizon data $\phi_{\mathcal{H}^+} \in \mathcal{O}'$ (Theorem I (i)). The other results follow from similar considerations.

5. Linear theory: the charged/massive Klein–Gordon equation on the Reissner–Nordström interior

We begin by studying the charged and massive scalar fields on the *fixed* subextremal Reissner–Nordström interior (2-7) with the subextremal parameters $0 < |e| < M$ from (3-1). In this section, the connection ∇ and the metric g_{RN} are the Reissner–Nordström connection and metric, respectively. As mentioned in Section 2C, we also use the electromagnetic gauge condition

$$A'_{\mathrm{RN}} = \left(\frac{e}{r} - \frac{e}{r_+}\right) \mathrm{d}t = \frac{1}{2}\left(\frac{e}{r} - \frac{e}{r_+}\right) \mathrm{d}v - \frac{1}{2}\left(\frac{e}{r} - \frac{e}{r_+}\right) \mathrm{d}u, \tag{5-1}$$

which satisfies $F_{\mathrm{RN}} = \mathrm{d}A'_{\mathrm{RN}}$ for

$$F_{\mathrm{RN}} = \frac{e}{2r^2}\Omega^2_{\mathrm{RN}}\, \mathrm{d}u \wedge \mathrm{d}v. \tag{5-2}$$

Note that F_{RN} satisfies the homogeneous Maxwell equations $\mathrm{d} * F_{\mathrm{RN}} = 0$, $\mathrm{d}F_{\mathrm{RN}} = 0$ and that (5-2) is the corresponding linear version of (2-17).

We now consider solutions $\phi'_{\mathcal{L}}$ of the charged Klein–Gordon equation (1-5), which reads

$$(\nabla_\mu + iq_0(A'_{\mathrm{RN}})_\mu)(\nabla^\mu + iq_0(A'_{\mathrm{RN}})^\mu)\phi'_{\mathcal{L}} - m^2\phi'_{\mathcal{L}} = 0, \tag{5-3}$$

where $q_0 \in \mathbb{R}$, $m^2 \geq 0$, are the charge and mass parameters of the field. We also recall

$$\omega_r = \frac{q_0 e}{r}, \quad \omega_+ = \frac{q_0 e}{r_+}, \quad \omega_- = \frac{q_0 e}{r_-}, \quad \omega_{\mathrm{res}} = \omega_- - \omega_+. \tag{5-4}$$

Note that in the gauge (5-1), we have

$$D^{\mathrm{RN}}_v = \partial_v + iq_0(A'_{\mathrm{RN}})_v = \partial_v + \frac{i}{2}(\omega_r - \omega_+), \tag{5-5}$$

$$D^{\mathrm{RN}}_u = \partial_u + iq_0(A'_{\mathrm{RN}})_u = \partial_u - \frac{i}{2}(\omega_r - \omega_+) \tag{5-6}$$

such that for any C^1 function we have

$$e^{-i\omega_{\mathrm{res}} r^*}\, \partial_v(e^{i\omega_{\mathrm{res}} r^*} f) = \partial_v f + i(\omega_- - \omega_+)(\partial_v r^*) f = D^{\mathrm{RN}}_v f + \frac{i}{2}(\omega_- - \omega_r) f \tag{5-7}$$

and similarly for D^{RN}_u. For $q_0 = m^2 = 0$, the field is uncharged and massless, and (5-3) reduces to the well-known wave equation

$$\Box_{g_{\mathrm{RN}}} \phi'_{\mathcal{L}} = 0. \tag{5-8}$$

For $q_0 \neq 0,\ m^2 = 0$, the field is charged and massless and is governed by

$$(\nabla_\mu + iq_0(A'_{\mathrm{RN}})_\mu)(\nabla^\mu + iq_0(A'_{\mathrm{RN}})^\mu)\phi'_{\mathcal{L}} = 0. \tag{5-9}$$

Finally, for $q_0 = 0,\ m^2 \neq 0$, the field is uncharged and massive and governed by the Klein–Gordon equation

$$\Box_{g_{\mathrm{RN}}}\phi'_{\mathcal{L}} - m^2\phi'_{\mathcal{L}} = 0. \tag{5-10}$$

Notation. Throughout Section 5 we will use the following notation. If X and Y are two (typically nonnegative) quantities, we use $X \lesssim Y$ or $Y \lesssim X$ to denote that $X \leq C(M, e, m^2, q_0, s)Y$ for some constant $C(M, e, m^2, q_0, s)$ depending on the parameters (M, e, m^2, q_0, s). If C depends on an additional parameter p, we also use the notation $\lesssim_p$, $\gtrsim_p$. We also use $X = O(Y)$ for $|X| \lesssim Y$. We use $X \sim Y$ for $X \lesssim Y \lesssim X$. We also recall that throughout Section 5 we use the convention that $\mathcal{H}^+ = \mathcal{H}^+_R = \{u = -\infty,\ v \in \mathbb{R}\}$ as stated in Section 2A.

5A. *Separation of variables and radial ODE.* Since $T = \partial_t$ is a Killing field of the Reissner–Nordström spacetime and in view of the specific choice of electromagnetic gauge A'_{RN}, (5-3) admits a separation of variables. Formally, let $\phi'_{\mathcal{L}} = \phi'_{\mathcal{L}}(t, r)$ be a solution to (5-3). Then, we define the t-Fourier transform

$$\mathcal{F}[\phi'_{\mathcal{L}}](r, \omega) = \hat{\phi}'_{\mathcal{L}} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi'_{\mathcal{L}}(r, t) e^{i\omega t}\, \mathrm{d}t. \tag{5-11}$$

Formally, since $\phi'_{\mathcal{L}}$ solves (5-3), we have that

$$u(r^*) = u(\omega, r^*) := r(r^*)\mathcal{F}[\phi'_{\mathcal{L}}](r(r^*), \omega) \tag{5-12}$$

solves

$$-u'' - (\omega - (\omega_r - \omega_+))^2 u + Vu = 0, \tag{5-13}$$

where

$$V = -\Omega^2_{\mathrm{RN}}(r_*)\left(\frac{2M}{r^3} - \frac{2e^2}{r^4} + m^2\right). \tag{5-14}$$

The radial ODE (5-13) admits the following fundamental pairs of solution associated to the event horizon ($r^* \to -\infty$) and the Cauchy horizon ($r^* \to +\infty$).

Definition 5.1. Let $u_{\mathcal{H}_R}$, $u_{\mathcal{H}_L}$, $u_{\mathcal{CH}_R}$ and $u_{\mathcal{CH}_L}$ be the unique smooth solutions to (5-13) satisfying

$$u_{\mathcal{H}_R}(r^*) = e^{-i\omega r^*} + O(\Omega^2_{\mathrm{RN}}) \qquad \text{as } r^* \to -\infty, \tag{5-15}$$

$$u_{\mathcal{H}_L}(r^*) = e^{i\omega r^*} + O(\Omega^2_{\mathrm{RN}}) \qquad \text{as } r^* \to -\infty, \tag{5-16}$$

$$u_{\mathcal{CH}_R}(r^*) = e^{i(\omega - \omega_{\mathrm{res}})r^*} + O(\Omega^2_{\mathrm{RN}}) \qquad \text{as } r^* \to +\infty, \tag{5-17}$$

$$u_{\mathcal{CH}_L}(r^*) = e^{-i(\omega - \omega_{\mathrm{res}})r^*} + O(\Omega^2_{\mathrm{RN}}) \qquad \text{as } r^* \to +\infty \tag{5-18}$$

for $\omega \in \mathbb{R}$. The pairs $(u_{\mathcal{H}_R}, u_{\mathcal{H}_L})$ and $(u_{\mathcal{CH}_R}, u_{\mathcal{CH}_L})$ span the solution space of (5-13) for $\omega \in \mathbb{R} - \{0\}$ and $\omega \in \mathbb{R} - \{\omega_{\mathrm{res}}\}$, respectively.

Using the fact that the Wronskian

$$\mathfrak{W}(f,g) := fg' - f'g \tag{5-19}$$

of two solution of (5-13) is independent of r^*, we define transmission and reflection coefficients $\mathfrak{T}(\omega)$ and $\mathfrak{R}(\omega)$ as follows.

Definition 5.2. For $\omega \in \mathbb{R} - \{\omega_{\text{res}}\}$, we define the transmission and reflection coefficients $\mathfrak{T}$ and $\mathfrak{R}$ as

$$\mathfrak{T}(\omega) := \frac{\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_R})}{\mathfrak{W}(u_{\mathcal{CH}_L}, u_{\mathcal{CH}_R})} = \frac{\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_R})}{2i(\omega - \omega_{\text{res}})}, \tag{5-20}$$

$$\mathfrak{R}(\omega) := \frac{\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_L})}{\mathfrak{W}(u_{\mathcal{CH}_R}, u_{\mathcal{CH}_L})} = \frac{\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_L})}{-2i(\omega - \omega_{\text{res}})}, \tag{5-21}$$

where $u_{\mathcal{H}_R}$, $u_{\mathcal{H}_L}$, $u_{\mathcal{CH}_R}$ and $u_{\mathcal{CH}_L}$ are defined in Definition 5.1. Indeed, this allows us to write

$$u_{\mathcal{H}_R} = \mathfrak{T} u_{\mathcal{CH}_L} + \mathfrak{R} u_{\mathcal{CH}_R} \tag{5-22}$$

for $\omega \in \mathbb{R} - \{\omega_{\text{res}}\}$. Moreover, we define the normalized transmission and reflection coefficients as

$$\mathfrak{t}(\omega) = (\omega - \omega_{\text{res}})\mathfrak{T}(\omega) = \frac{\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_R})}{2i}, \tag{5-23}$$

$$\mathfrak{r}(\omega) = (\omega - \omega_{\text{res}})\mathfrak{R}(\omega) = \frac{\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_L})}{-2i}, \tag{5-24}$$

which manifestly satisfy

$$\mathfrak{t}(\omega_{\text{res}}) = -\mathfrak{r}(\omega_{\text{res}}). \tag{5-25}$$

Remark 5.3. Note that the radial ODE (5-13) depends analytically on ω. Thus, $u_{\mathcal{H}_R}$, $u_{\mathcal{H}_L}$, $u_{\mathcal{CH}_R}$ and $u_{\mathcal{CH}_L}$ are real-analytic functions for ω for fixed r^*. In particular, this means that the Wronskians $\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_R})$, $\mathfrak{W}(u_{\mathcal{CH}_R}, u_{\mathcal{CH}_L})$ etc. are real-analytic functions for $\omega \in \mathbb{R}$ which can be extended holomorphically to a neighborhood of the real line.

We will also define the renormalized functions.

Definition 5.4. We define

$$\widetilde{u_{\mathcal{H}_R}}(r^*, \omega) := e^{i\omega r^*} u_{\mathcal{H}_R}(r^*, \omega), \tag{5-26}$$

$$\widetilde{u_{\mathcal{H}_L}}(r^*, \omega) := e^{-i\omega r^*} u_{\mathcal{H}_L}(r^*, \omega), \tag{5-27}$$

$$\widetilde{u_{\mathcal{CH}_R}}(r^*, \omega) := e^{-i(\omega - \omega_{\text{res}})r^*} u_{\mathcal{CH}_R}(r^*, \omega), \tag{5-28}$$

$$\widetilde{u_{\mathcal{CH}_L}}(r^*, \omega) := e^{i(\omega - \omega_{\text{res}})r^*} u_{\mathcal{CH}_L}(r^*, \omega). \tag{5-29}$$

5B. *Analysis for the radial ODE.*

Proposition 5.5. *Let either of the following two assumptions hold true.*

- $q_0 \neq 0$.
- $q_0 = 0$ *but* $m^2 \notin D(M, e)$, *where* $D(M, e)$ *is the discrete set of* [Kehle and Shlapentokh-Rothman 2019, Theorem 7].

Then, the transition and reflection coefficients $\mathfrak{T}(\omega)$ *and* $\mathfrak{R}(\omega)$, *as defined in Definition 5.2, have (nonremovable) poles of first order at* $\omega = \omega_{\text{res}}$.

Proof. First, note that $(\mathrm{Im}(u'\bar{u}))' = 0$ holds true for any C^1 solution of (5-13). Applying this to $u_{\mathcal{H}_R}$ and expanding $u_{\mathcal{H}_R}$ as $u_{\mathcal{H}_R} = \mathfrak{T} u_{\mathcal{CH}_L} + \mathfrak{R} u_{\mathcal{CH}_R}$, we conclude the *ODE energy identity*

$$|\mathfrak{T}|^2 - |\mathfrak{R}|^2 = \frac{\omega}{\omega - \omega_{\mathrm{res}}}. \tag{5-30}$$

If $q_0 \neq 0$ and thus, $\omega_{\mathrm{res}} \neq 0$, we have $|\mathfrak{T}|^2 \geq \omega/(\omega - \omega_{\mathrm{res}})$ for $|\omega| > \omega_{\mathrm{res}}$. Sending $\omega \to \omega_{\mathrm{res}}$, we conclude that $\mathfrak{T}$ blows up and since $\mathfrak{T}$ is meromorphic in a complex neighborhood of ω_{res}, the claim follows. In particular, we have that $\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_R})(\omega = \omega_{\mathrm{res}}) \neq 0$ and $\mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_L})(\omega = \omega_{\mathrm{res}}) \neq 0$. For $q_0 = 0$ and $m^2 \notin D(M, e)$, the claim follows from [Kehle and Shlapentokh-Rothman 2019, Theorem 7]. □

Proposition 5.6. *The solutions $u_{\mathcal{H}_R}$, $u_{\mathcal{CH}_L}$, $u_{\mathcal{CH}_R}$ and the renormalized functions $\widetilde{u_{\mathcal{H}_R}}$, $\widetilde{u_{\mathcal{CH}_L}}$, $\widetilde{u_{\mathcal{CH}_R}}$ as defined in Definitions 5.1 and 5.4, respectively, satisfy for $\omega \in \mathbb{R}$*

$$\sup_{r^* \in (-\infty, r_0^*]} |u_{\mathcal{H}_R}(\omega, r^*)| \lesssim_{r_0^*} 1, \tag{5-31}$$

$$\sup_{r^* \in (-\infty, r_0^*]} |u'_{\mathcal{H}_R}(\omega, r^*)| \lesssim_{r_0^*} |\omega| \tag{5-32}$$

for any fixed $r_0^* \in \mathbb{R}$ and

$$|\widetilde{u_{\mathcal{H}_R}}(\omega, r^*) - 1| \lesssim_{r_0^*} |\Omega^2_{\mathrm{RN}}(r^*)|, \tag{5-33}$$

$$|\widetilde{u'_{\mathcal{H}_R}}(\omega, r^*)| \lesssim_{r_0^*} |\Omega^2_{\mathrm{RN}}(r^*)| \tag{5-34}$$

uniformly for $r^* \leq r_0^*$. Moreover, for $\omega \in \mathbb{R}$ and any fixed $r_0^* \in \mathbb{R}$

$$\sup_{r^* \in [r_0^*, +\infty)} |u_{\mathcal{CH}_L}(\omega, r^*)| \lesssim_{r_0^*} 1, \tag{5-35}$$

$$\sup_{r^* \in [r_0^*, +\infty)} |u'_{\mathcal{CH}_L}(\omega, r^*)| \lesssim_{r_0^*} |\omega|, \tag{5-36}$$

$$\sup_{r^* \in [r_0^*, +\infty)} |u_{\mathcal{CH}_R}(\omega, r^*)| \lesssim_{r_0^*} 1, \tag{5-37}$$

$$\sup_{r^* \in [r_0^*, +\infty)} |u'_{\mathcal{CH}_R}(\omega, r^*)| \lesssim_{r_0^*} |\omega|, \tag{5-38}$$

and uniformly for $r^* \geq r_0^*$

$$|\widetilde{u_{\mathcal{CH}_L}}(\omega, r^*) - 1| \lesssim_{r_0^*} |\Omega^2_{\mathrm{RN}}(r^*)|, \tag{5-39}$$

$$|\widetilde{u'_{\mathcal{CH}_L}}(\omega, r^*)| \lesssim_{r_0^*} |\Omega^2_{\mathrm{RN}}|, \tag{5-40}$$

$$|\widetilde{u_{\mathcal{CH}_R}}(\omega, r^*) - 1| \lesssim_{r_0^*} |\Omega^2_{\mathrm{RN}}(r^*)|, \tag{5-41}$$

$$|\widetilde{u'_{\mathcal{CH}_R}}(\omega, r^*)| \lesssim_{r_0^*} |\Omega^2_{\mathrm{RN}}(r^*)|. \tag{5-42}$$

The transition and reflection coefficients as defined in Definition 5.2 satisfy

$$\sup_{|\omega - \omega_{\mathrm{res}}| \geq 1} (|\mathfrak{T}(\omega)| + |\mathfrak{R}(\omega)|) \lesssim 1. \tag{5-43}$$

Proof. It suffices to show the results for $u_{\mathcal{H}_R}$ and $\widetilde{u_{\mathcal{H}_R}}$ as the other cases follow completely analogously. We will consider the cases $|\omega| \leq \omega_0 := |\omega_{\mathrm{res}}| + 1$ and $|\omega| > \omega_0$ independently. First, for $|\omega| \leq \omega_0$, we note that $u_{\mathcal{H}_R}$ is the unique solution to the Volterra equation

$$u_{\mathcal{H}_R}(r^*,\omega) = e^{-i\omega r^*} + \int_{-\infty}^{r^*} \frac{\sin(\omega(r^*-y))}{\omega}\big(2\omega(\omega_r-\omega_+) - (\omega_r-\omega_+)^2 + V(y)\big)u_{\mathcal{H}_R}(y,\omega)\,\mathrm{d}y. \tag{5-44}$$

For $\omega = 0$, we mean $\sin(\omega(r^*-y))/\omega = r^*-y$. Now, since

$$\int_{-\infty}^{r_0^*} \sup_{y\le r^*<r_0^*} |K(r^*,y)|\,\mathrm{d}y \lesssim \Omega^2_{\mathrm{RN}}(r_0^*), \tag{5-45}$$

where

$$K(r^*,y) = \frac{\sin(\omega(r^*-y))}{\omega}\big(2\omega(\omega_r-\omega_+) - (\omega_r-\omega_+)^2 + V(y)\big), \tag{5-46}$$

we have by standard estimates on Volterra equations (e.g., [Kehle and Shlapentokh-Rothman 2019, Proposition 2.3] or [Olver 1974, §10]) that, for $|\omega| \le \omega_0$,

$$\|u_{\mathcal{H}_R}\|_{L^\infty(-\infty,r_0^*)} \lesssim_{r_0^*} 1, \tag{5-47}$$

as well as

$$|u_{\mathcal{H}_R} - e^{-i\omega r^*}| \lesssim |\Omega^2_{\mathrm{RN}}(r^*)| \tag{5-48}$$

uniformly for $r^* \le 0$. Similarly, we obtain

$$\|u'_{\mathcal{H}_R}\|_{L^\infty(-\infty,r_0^*)} \lesssim_{r_0^*} 1 + |\omega| \lesssim_{r_0^*} 1. \tag{5-49}$$

Note that this also shows that, for $|\omega| \le \omega_0$, we have

$$\|\widetilde{u_{\mathcal{H}_R}}'\|_{L^\infty(-\infty,r_0^*)} \lesssim_{r_0^*} 1, \tag{5-50}$$

$$|\widetilde{u_{\mathcal{H}_R}} - 1| \lesssim |\Omega^2_{\mathrm{RN}}(r^*)| \tag{5-51}$$

uniformly for $r^* \le 0$.

Now, we consider the case $|\omega| \ge \omega_0$. Note that in this frequency regime, the frequency-dependent potential

$$W := -(\omega - (\omega_r - \omega_+))^2 \tag{5-52}$$

satisfies

$$-W \gtrsim \omega^2, \tag{5-53}$$

$$|W'/W| \lesssim \Omega^2_{\mathrm{RN}}/|\omega|, \tag{5-54}$$

$$|W''/W| \lesssim \Omega^2_{\mathrm{RN}}/|\omega|, \tag{5-55}$$

and the radial potential V satisfies

$$|V|, |V'|, |V''| \lesssim \Omega^2_{\mathrm{RN}} \tag{5-56}$$

uniformly on $r^* \in \mathbb{R}$.

Now we will use a WKB approximation for $u_{\mathcal{H}_R}$. First, we will estimate the total variation $\mathcal{V}_{-\infty,+\infty}$ associated to the error-control function

$$F_{u_{\mathcal{H}_R}}(r^*,\omega) := \int_{-\infty}^{r^*} \frac{1}{|W|^{1/4}}\frac{\mathrm{d}^2}{\mathrm{d}x^2}|W|^{-1/4} - \frac{V}{|W|^{1/2}}\,\mathrm{d}y. \tag{5-57}$$

In view of (5-53)–(5-55), we estimate

$$\mathcal{V}_{-\infty,+\infty}(F_{u_{\mathcal{H}_R}}) = \int_{-\infty}^{+\infty} \left|\frac{1}{|W|^{1/4}}\frac{\mathrm{d}^2}{\mathrm{d}x^2}|W|^{-1/4} - \frac{V}{|W|^{1/2}}\right|\mathrm{d}y \lesssim \frac{1}{|\omega|}. \tag{5-58}$$

Thus, applying [Olver 1974, Theorem 2.2, p. 196] we obtain

$$u_{\mathcal{H}_R}(r^*,\omega)=\frac{|\omega|^{1/2}}{|W(r^*,\omega)|^{1/4}}e^{-i\omega r^*+i\int_{-\infty}^{r^*}\omega_r-\omega_+\mathrm{d}y}(1+\eta_{u_{\mathcal{H}_R}}), \tag{5-59}$$

where the error function $\eta_{u_{\mathcal{H}_R}}$ satisfies

$$|\eta_{u_{\mathcal{H}_R}}(r^*,\omega)|\lesssim\frac{1}{|\omega|}, \tag{5-60}$$

$$|\eta'_{u_{\mathcal{H}_R}}(r^*,\omega)|\lesssim|W(r^*,\omega)|^{1/2}\frac{1}{|\omega|}\lesssim 1 \tag{5-61}$$

uniformly for $r^*\in\mathbb{R}$ and $|\omega|\geq\omega_0$ as well as

$$|\eta_{u_{\mathcal{H}_R}}(r^*,\omega)|\lesssim\frac{\Omega^2_{\mathrm{RN}}}{|\omega|}, \tag{5-62}$$

$$|\eta'_{u_{\mathcal{H}_R}}(r^*,\omega)|\lesssim\Omega^2_{\mathrm{RN}} \tag{5-63}$$

uniformly for $r^*<0$ and $|\omega|\geq\omega_0$. This shows that for $|\omega|\geq\omega_0$ we have

$$\|u_{\mathcal{H}_R}\|_{L^\infty(\mathbb{R})}\lesssim 1, \tag{5-64}$$

$$\|u'_{\mathcal{H}_R}\|_{L^\infty(\mathbb{R})}\lesssim|\omega|. \tag{5-65}$$

Note also that $\widetilde{u_{\mathcal{H}_R}}=e^{i\omega r^*}u_{\mathcal{H}_R}$ similarly satisfies

$$\|\widetilde{u_{\mathcal{H}_R}}\|_{L^\infty(\mathbb{R})}\lesssim 1, \tag{5-66}$$

$$\|\widetilde{u_{\mathcal{H}_R}}'\|_{L^\infty(\mathbb{R})}\lesssim 1 \tag{5-67}$$

and

$$|\widetilde{u_{\mathcal{H}_R}}(r^*,\omega)-1|\lesssim_{r_0^*}\Omega^2_{\mathrm{RN}}, \tag{5-68}$$

$$|\widetilde{u_{\mathcal{H}_R}}'(r^*,\omega)|\lesssim_{r_0^*}\Omega^2_{\mathrm{RN}} \tag{5-69}$$

uniformly for $r^*\leq r_0^*$ and $\omega\in\mathbb{R}$. The other results for $u_{\mathcal{CH}_L}$ and $u_{\mathcal{CH}_R}$ are shown completely analogously.

Now, we will show the bounds on the transmission and reflection coefficients $\mathfrak{T}$ and $\mathfrak{R}$. The bound (5-43) follows from the fact that for $|\omega|$ sufficiently large, $|\mathfrak{W}(u_{\mathcal{H}_R},u_{\mathcal{CH}_R})|$, $|\mathfrak{W}(u_{\mathcal{H}_R},u_{\mathcal{CH}_L})|\lesssim|\omega|$ in view of (5-64), (5-65) and computing the Wronskian as $r^*\to+\infty$. For $|\omega|$ small, the bound follows from the continuity of $|\mathfrak{W}(u_{\mathcal{H}_R},u_{\mathcal{CH}_R})|$ and $|\mathfrak{W}(u_{\mathcal{H}_R},u_{\mathcal{CH}_L})|$. □

Lemma 5.7. *The bounds*

$$|\partial_\omega\widetilde{u_{\mathcal{CH}_R}}(\omega,r^*)|\lesssim\Omega^2_{\mathrm{RN}}, \tag{5-70}$$

$$|\partial_\omega\widetilde{u_{\mathcal{CH}_L}}(\omega,r^*)|\lesssim\Omega^2_{\mathrm{RN}} \tag{5-71}$$

and

$$|\partial_{r^*}\partial_\omega\widetilde{u_{\mathcal{CH}_R}}(\omega,r^*)|\lesssim\Omega^2_{\mathrm{RN}}\langle\omega\rangle, \tag{5-72}$$

$$|\partial_{r^*}\partial_\omega\widetilde{u_{\mathcal{CH}_L}}(\omega,r^*)|\lesssim\Omega^2_{\mathrm{RN}}\langle\omega\rangle \tag{5-73}$$

hold uniformly for $r^*\geq 0$ and $\omega\in\mathbb{R}$. (We recall that $\langle\omega\rangle:=\sqrt{1+\omega^2}$).

Moreover,

$$|\partial_\omega \widetilde{u_{\mathcal{H}_R}}(\omega, r^*)| \lesssim \Omega^2_{\mathrm{RN}}, \tag{5-74}$$

$$|\partial_{r^*}\partial_\omega \widetilde{u_{\mathcal{H}_R}}(\omega, r^*)| \lesssim \Omega^2_{\mathrm{RN}}\langle\omega\rangle \tag{5-75}$$

hold uniformly for $r^* \le 0$ *and* $\omega \in \mathbb{R}$.

Proof. First, we consider the range $|\omega - \omega_{\mathrm{res}}| \le 1$. First, note that $\widetilde{u_{\mathcal{CH}_R}}$ solves the Volterra integral equation

$$\begin{aligned}\widetilde{u_{\mathcal{CH}_R}}(r^*, \omega) = 1 + \int_{r^*}^{+\infty} &\frac{\sin[(\omega-\omega_{\mathrm{res}})(r^*-y)]}{\omega-\omega_{\mathrm{res}}} e^{-i(\omega-\omega_{\mathrm{res}})(r^*-y)} \\ &\times [V(y) - (\omega_- - \omega_{r(y)})(2\omega + 2\omega_+ - \omega_- - \omega_{r(y)})]\widetilde{u_{\mathcal{CH}_R}}(\omega, y)\, \mathrm{d}y.\end{aligned} \tag{5-76}$$

Thus, $\partial_\omega \widetilde{u_{\mathcal{CH}_R}}$ solves

$$\begin{aligned}\partial_\omega \widetilde{u_{\mathcal{CH}_R}}(r^*, \omega) = &\int_{r^*}^{+\infty} \frac{\sin[(\omega-\omega_{\mathrm{res}})(r^*-y)]}{\omega-\omega_{\mathrm{res}}} e^{-i(\omega-\omega_{\mathrm{res}})(r^*-y)} \\ &\qquad \times [V(y) - (\omega_- - \omega_{r(y)})(2\omega + 2\omega_+ - \omega_- - \omega_{r(y)})]\, \partial_\omega \widetilde{u_{\mathcal{CH}_R}}(\omega, y)\, \mathrm{d}y \\ &+ \int_{r^*}^{+\infty} \frac{\partial_\omega(\mathrm{sinc}[(\omega-\omega_{\mathrm{res}})(r^*-y)]e^{-i(\omega-\omega_{\mathrm{res}})(r^*-y)})}{r^*-y}(r^*-y)^2 \\ &\qquad \times [V(y) - (\omega_- - \omega_{r(y)})(2\omega + 2\omega_+ - \omega_- - \omega_{r(y)})]\widetilde{u_{\mathcal{CH}_R}}(\omega, y)\, \mathrm{d}y \\ &+ \int_{r^*}^{+\infty} \frac{\sin[(\omega-\omega_{\mathrm{res}})(r^*-y)]}{\omega-\omega_{\mathrm{res}}} e^{-i(\omega-\omega_{\mathrm{res}})(r^*-y)} \\ &\qquad \times 2[V(y) - (\omega_- - \omega_{r(y)}))]\widetilde{u_{\mathcal{CH}_R}}(\omega, y)\, \mathrm{d}y.\end{aligned} \tag{5-77}$$

Now, we have the following bounds uniformly for $r^* \ge 0$:

$$\left| \frac{\sin[(\omega-\omega_{\mathrm{res}})(r^*-y)]}{\omega-\omega_{\mathrm{res}}} e^{-i(\omega-\omega_{\mathrm{res}})(r^*-y)} \right| \lesssim (r^*-y), \tag{5-78}$$

$$\left| \frac{\partial_\omega(\mathrm{sinc}[(\omega-\omega_{\mathrm{res}})(r^*-y)]e^{-i(\omega-\omega_{\mathrm{res}})(r^*-y)})}{r^*-y} \right| \lesssim 1, \tag{5-79}$$

$$|V(y) - (\omega_- - \omega_{r(y)})(2\omega + 2\omega_+ - \omega_- - \omega_{r(y)})| \lesssim \Omega^2_{\mathrm{RN}}, \tag{5-80}$$

$$|V(y) - (\omega_- - \omega_{r(y)}))| \lesssim \Omega^2_{\mathrm{RN}}. \tag{5-81}$$

With these bounds, standard results (e.g., [Olver 1974, §10]) on estimates of solutions of Volterra integral equations show that

$$|\partial_\omega \widetilde{u_{\mathcal{CH}_R}}(r^*, \omega)| \lesssim \Omega^2_{\mathrm{RN}} \tag{5-82}$$

uniformly for $r^* \ge 0$. Similarly, we have

$$|\partial_\omega \widetilde{u_{\mathcal{CH}_L}}(r^*, \omega)| \lesssim \Omega^2_{\mathrm{RN}} \tag{5-83}$$

uniformly for $r^* \ge 0$.

Differentiation of (5-77) with respect to r^* also gives

$$|\partial_{r^*}\partial_\omega \widetilde{u_{\mathcal{CH}_R}}| \lesssim \Omega^2_{\mathrm{RN}} \tag{5-84}$$

and analogously we obtain

$$|\partial_{r^*}\partial_\omega \widetilde{u_{\mathcal{CH}_L}}| \lesssim \Omega^2_{\mathrm{RN}}. \tag{5-85}$$

Now, we consider the range $|\omega-\omega_{\mathrm{res}}| \geq 1$. Then, for $r^* \geq 0$, we have the bounds

$$\left|\frac{\sin[(\omega-\omega_{\mathrm{res}})(r^*-y)]}{\omega-\omega_{\mathrm{res}}} e^{-i(\omega-\omega_{\mathrm{res}})(r^*-y)}\right| \lesssim \langle\omega\rangle^{-1}, \tag{5-86}$$

$$|\partial_\omega(\mathrm{sinc}[(\omega-\omega_{\mathrm{res}})(r^*-y)]e^{-i(\omega-\omega_{\mathrm{res}})(r^*-y)})| \lesssim \langle\omega\rangle^{-1}\frac{1+|r^*-y|}{|r^*-y|}, \tag{5-87}$$

$$|V(y)-(\omega_- - \omega_{r(y)})(2\omega+2\omega_+ - \omega_- - \omega_{r(y)})| \lesssim \Omega^2_{\mathrm{RN}}\langle\omega\rangle, \tag{5-88}$$

$$|V(y)-(\omega_- - \omega_{r(y)}))| \lesssim \Omega^2_{\mathrm{RN}}. \tag{5-89}$$

Thus, analogously to the above, this gives uniformly for $r^* \geq 0$

$$|\partial_\omega \widetilde{u_{\mathcal{CH}_R}}(r^*,\omega)| \lesssim \Omega^2_{\mathrm{RN}}, \tag{5-90}$$

$$|\partial_\omega \widetilde{u_{\mathcal{CH}_L}}(r^*,\omega)| \lesssim \Omega^2_{\mathrm{RN}}, \tag{5-91}$$

as well as

$$|\partial_{r^*}\partial_\omega \widetilde{u_{\mathcal{CH}_R}}| \lesssim \Omega^2_{\mathrm{RN}}\langle\omega\rangle, \tag{5-92}$$

$$|\partial_{r^*}\partial_\omega \widetilde{u_{\mathcal{CH}_L}}| \lesssim \Omega^2_{\mathrm{RN}}\langle\omega\rangle. \tag{5-93}$$

The result on $u_{\mathcal{H}_R}$ follows completely analogously. □

Corollary 5.8. *The normalized transmission and reflection coefficients satisfy*

$$|\mathfrak{t}(\omega)| + |\mathfrak{r}(\omega)| \lesssim 1 + |\omega|. \tag{5-94}$$

Proof. This is a consequence of Propositions 5.5 and 5.6. □

Lemma 5.9. *We have*

$$|\partial_\omega \mathfrak{r}(\omega)| \lesssim \langle\omega\rangle, \tag{5-95}$$

$$|\partial_\omega \mathfrak{t}(\omega)| \lesssim \langle\omega\rangle. \tag{5-96}$$

Proof. We estimate

$$|\partial_\omega \mathfrak{r}| \lesssim |\partial_\omega \mathfrak{W}(u_{\mathcal{H}_R}, u_{\mathcal{CH}_R})| \lesssim |\mathfrak{W}(\partial_\omega u_{\mathcal{H}_R}, u_{\mathcal{CH}_R})(r^*=0)| + |\mathfrak{W}(u_{\mathcal{H}_R}, \partial_\omega u_{\mathcal{CH}_R})(r^*=0)| \lesssim \langle\omega\rangle \tag{5-97}$$

in view of Lemma 5.7 and Proposition 5.6. Analogously the same holds for $\mathfrak{t}$. □

Towards the $W^{1,1}$ inextendibility at the Cauchy horizon we need to analyze the zeros of the transmission coefficient $\mathfrak{t}$. To do so, we recall the definition of $\mathcal{Z}_{\mathfrak{t}}(M,e,q_0,m^2)$ from (4-5).

Lemma 5.10. (1) *Let $q_0 e \neq 0$. Then, $\mathcal{Z}_{\mathfrak{t}} \subset (0,\omega_{\mathrm{res}})$ if $q_0 e > 0$ or $\mathcal{Z}_{\mathfrak{t}} \subset (\omega_{\mathrm{res}}, 0)$ if $q_0 e < 0$.*

(2) *Let $0 < |q_0 e| < \epsilon(M,e,m^2)$ for some $\epsilon(M,e,m^2)$ sufficiently small. Then $\mathfrak{t}$ does not have any zeros, i.e., $\mathcal{Z}_{\mathfrak{t}} = \varnothing$.*

(3) *Let $q_0 = 0$ and let $m^2 \notin D(M,e)$, where $D(M,e)$ is the discrete set as in* [Kehle and Shlapentokh-Rothman 2019, Theorem 7]. *Then, $\mathfrak{t}(\omega)$ does not have any zeros, i.e., $\mathcal{Z}_{\mathfrak{t}}(M,e,0,m^2) = \varnothing$ if $m^2 \notin D(M,e)$.*

Proof. The first statement follows from the fact that $|\mathfrak{t}|^2 = |\mathfrak{r}|^2 + \omega(\omega - \omega_{\mathrm{res}}) \geq \omega(\omega - \omega_{\mathrm{res}})$, Proposition 5.5 and the fact that $\mathfrak{t}(\omega = 0) \neq 0$. Indeed, if $\mathfrak{t}(\omega = 0) = 0$, then $\mathfrak{r}(\omega = 0) = 0$ and thus $\mathfrak{T}(\omega = 0) = \mathfrak{R}(\omega = 0) = 0$. But this cannot be true, since otherwise $u_{\mathcal{H}_R} = \mathfrak{R} u_{\mathcal{CH}_R} + \mathfrak{T} u_{\mathcal{CH}_L}$ would be trivial. The second statement just follows from continuity of $\mathfrak{t}$ as a function of the parameters $q_0 e$. The third statement is shown in [Kehle and Shlapentokh-Rothman 2019, Theorem 7]. □

Remark 5.11. Note that for $q_0 = 0$ and $m^2 = 0$, we have that $\mathfrak{t}(\omega = 0) = 0$. This is a crucial observation for the existence of a T-energy scattering theory as established in [Kehle and Shlapentokh-Rothman 2019].

5C. *Representation formula.*

We recall that throughout Section 5 we consider the event horizon $\mathcal{H}^+$ as the set $\{u = -\infty\} \times \{v \in \mathbb{R}\}$ as in Section 2A.

Definition 5.12. For $f \in L^2(\mathcal{H}^+)$ we define the Fourier transform along the event horizon as

$$\mathcal{F}_{\mathcal{H}^+}[f](\omega) := r_+ \mathcal{F}[f](\omega) = \frac{r_+}{\sqrt{2\pi}} \int_{\mathbb{R}} f(\tilde{v}) e^{i\omega\tilde{v}} \, \mathrm{d}\tilde{v} \tag{5-98}$$

in mild abuse of notation.

Lemma 5.13. *Let $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \in C^\infty(\mathcal{H}^+)$ be spherically symmetric smooth data on the event horizon and assume that $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}$ is supported away from the past bifurcation sphere. Assume vanishing data on the left event horizon and let $\phi'_{\mathcal{L}}$ be the arising smooth solution to* (5-3) *attaining that data. Then, for any fixed v_1 and any $u \in \mathbb{R}$, $v \leq v_1$, we have*

$$\phi'_{\mathcal{L}}(u, v) = \frac{1}{\sqrt{2\pi} r} \int \mathcal{F}_{\mathcal{H}^+}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \chi_{\leq v_1}](\omega) \widetilde{u_{\mathcal{H}_R}}(r^*(u, v), \omega) e^{-i\omega v} \, \mathrm{d}\omega \tag{5-99}$$

and

$$\partial_v(r\phi'_{\mathcal{L}}(u, v)) = \frac{1}{\sqrt{2\pi}} \int \mathcal{F}_{\mathcal{H}^+}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \chi_{\leq v_1}](\omega) \, \partial_v(\widetilde{u_{\mathcal{H}_R}}(r^*(u, v), \omega) e^{-i\omega v}) \, \mathrm{d}\omega, \tag{5-100}$$

$$\partial_u(r\phi'_{\mathcal{L}}(u, v)) = \frac{1}{\sqrt{2\pi}} \int \mathcal{F}_{\mathcal{H}^+}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \chi_{\leq v_1}](\omega) \, \partial_u(\widetilde{u_{\mathcal{H}_R}}(r^*(u, v), \omega) e^{-i\omega v}) \, \mathrm{d}\omega, \tag{5-101}$$

where $\chi_{\leq v_1}(v) = \chi_0(v - v_1)$ and $\chi_0 \colon \mathbb{R} \to [0, 1]$ is a smooth cut-off which satisfies $\chi_0(x) = 1$ for $x \leq 0$ and $\chi_0(x) = 0$ for $x \geq 1$.

By a standard density argument, (5-99), (5-100) *and* (5-101) *hold also for spherically symmetric data $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \in C^1(\mathcal{H}^+)$ with $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}$ supported away from the past bifurcation sphere.*

Proof. Fix any v_1 and let (u, v) with $v \leq v_1$ be arbitrary. By the domain of dependence property, we have that $\phi'_{\mathcal{L}}$ satisfies $\phi'_{\mathcal{L}} = \phi'_{\mathcal{L}_{\leq v_1}}$ on (u, v) with $v \leq v_1$, where $\phi'_{\mathcal{L}_{\leq v_1}}$ is the unique solution arising from data $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \chi_{\leq v_1} \in C_c^\infty(\mathcal{H}^+)$ on the right event horizon $\mathcal{H}^+$ together with vanishing data on the left event horizon. Now, since $\mathcal{F}_{\mathcal{H}^+}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \chi_{\leq v_1}]$ is Schwartz, $u_{\mathcal{H}_R}$ satisfies (5-13), and $u_{\mathcal{H}_R}$ obeys the bounds as in Proposition 5.6, we can differentiate under the integral sign on the right-hand side of (5-99) and conclude that indeed the right-hand side of (5-99) solves (5-3). Finally, to show that $\phi'_{\mathcal{L}} = \phi'_{\mathcal{L}_{\leq v_1}}$ it suffices to show that the right-hand side assumes the data from which $\phi'_{\mathcal{L}_{\leq v_1}}$ arises. But again, since $\mathcal{F}_{\mathcal{H}^+}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \chi_{\leq v_1}]$ is Schwartz, we immediately obtain that the right-hand side of (5-99) converges to $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \chi_{\leq v_1}$ towards the right event horizon, and — after an application of the Riemann–Lebesgue lemma — to 0 towards the left event horizon. Now, (5-99) follows from uniqueness of the characteristic initial value problem. The formulae (5-100) and (5-101) now follow from differentiating under the integral sign, which can be applied as $\mathcal{F}_{\mathcal{H}^+}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v_1}]$ is a Schwartz function. □

Note that the above proposition immediately implies:

Corollary 5.14. *Let $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}$ be as in Lemma 5.13 and assume vanishing data on the left horizon. Let $\phi'_{\mathcal{L}}$ be the arising smooth solution attaining that data. Then,*

$$\phi'_{\mathcal{L}}(u,v) = \frac{1}{\sqrt{2\pi}r}\int_{\mathbb{R}} \mathcal{F}_{\mathcal{H}^+}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}](\omega)\widetilde{u_{\mathcal{H}_R}}(r(u,v),\omega)e^{-i\omega v}\,\mathrm{d}\omega \tag{5-102}$$

and

$$\partial_v(r\phi'_{\mathcal{L}}(u,v)) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{F}_{\mathcal{H}^+}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}](\omega)\,\partial_v(\widetilde{u_{\mathcal{H}_R}}(r(u,v),\omega)e^{-i\omega v})\,\mathrm{d}\omega, \tag{5-103}$$

$$\partial_u(r\phi'_{\mathcal{L}}(u,v)) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{F}_{\mathcal{H}^+}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}](\omega)\,\partial_u(\widetilde{u_{\mathcal{H}_R}}(r(u,v),\omega)e^{-i\omega v})\,\mathrm{d}\omega \tag{5-104}$$

for $u, v\in\mathbb{R}$, where $\chi_{\le v}$ is as in Lemma 5.13.

Proof. Choosing $v = v_1$ in Lemma 5.13 yields the result. □

5D. *Main results from the linear theory.* Before we state the main proposition about the linear theory, we define the following norms for sufficiently regular functions:

$$E_1[f] := \left(\int_{\mathbb{R}} |f(v)|^2 + |\partial_v f(v)|^2\,\mathrm{d}v\right)^{1/2}, \tag{5-105}$$

$$E_1^{\beta}[f] := \left(\int_{\mathbb{R}} (|f(v)|^2 + |\partial_v f(v)|^2)\langle v\rangle^{2\beta}\,\mathrm{d}v\right)^{1/2}, \tag{5-106}$$

$$F^{\beta}[f] := \sup_{v\ge 0}\langle v\rangle^{\beta}\left|\int_v^{+\infty} f(\tilde v)e^{i\omega_{\mathrm{res}}\tilde v}\,\mathrm{d}\tilde v\right|. \tag{5-107}$$

Further, for part (E) of the following proposition, we will use the Fourier projection operator P_δ defined in Section 4E. We will further state estimates in the so-called late blue-shift region $\mathcal{LB}$. This region is defined as

$$\mathcal{LB} = \left\{\Delta' + \frac{2s}{2|K_-|}\log(v) \le u+v\right\}$$

for some $\Delta'\ge 0$ chosen in Section 6A. (Note that the estimate below involving $\mathcal{LB}$ actually holds true uniformly for all $\Delta'\ge 0$.) For given u, we also define $v_\gamma(u)$ to satisfy

$$\Delta' + \frac{2s}{2|K_-|}\log(v_\gamma(u)) = u + v_\gamma(u).$$

Note that the estimate $\Omega^2_{\mathrm{RN}}(u,v)\lesssim v^{-2s}$ is satisfied in $\mathcal{LB}$. We refer to Figure 6 for a visualization of the region $\mathcal{LB}$ near i^+. In fact, in the region $\mathcal{LB}$ all the following estimates apply and $\mathcal{LB}$ is also the region in which we will make use of the linear theory for the nonlinear theory.

Theorem V. *Let $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\in C^1(\mathcal{H}^+)$ be spherically symmetric and assume that $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}$ is supported away from the past bifurcation sphere. Assume further that $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}$ has finite energy along the event horizon, i.e., that*

$$E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] < +\infty. \tag{5-108}$$

Let $\phi'_{\mathcal{L}}$ be the arising solution on the black hole interior with no incoming radiation from the left event horizon.

(A) *Then, for $v \geq 0$ and $u \in \mathbb{R}$ with $r^* = \frac{1}{2}(u+v) \geq 0$, we have*

$$e^{i\omega_{\mathrm{res}} r^*}\phi'_{\mathcal{L}}(u,v) = \frac{\sqrt{2\pi}\, i r_+}{r}\mathfrak{r}_{\omega_{\mathrm{res}}}(0) e^{i\omega_{\mathrm{res}} u}\left(\int_{-u}^{v} (\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(\tilde{v}) e^{i\omega_{\mathrm{res}}\tilde{v}}\, \mathrm{d}\tilde{v}\right) + \phi_{\mathrm{r}}(u,v) + \phi_{\mathrm{err}}(u,v), \tag{5-109}$$

where $\phi_{\mathrm{r}}(u,v)$ and $\phi_{\mathrm{err}}(u,v)$ satisfy the quantitative bounds

$$|\phi_{\mathrm{r}}(u,v)| \lesssim E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}], \tag{5-110}$$

$$|\phi_{\mathrm{err}}(u,v)| \lesssim_\alpha E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]\Omega_{\mathrm{RN}}^{2-\alpha}(u,v) \tag{5-111}$$

uniformly for $v \geq 0$, $u \in \mathbb{R}$, $2r^* = v+u \geq 2$ and any fixed $0<\alpha<2$. Further, $\phi_{\mathrm{r}}(u,v)$ and $\phi_{\mathrm{err}}(u,v)$ extend continuously to the right Cauchy horizon. In particular, $\lim_{n\to+\infty}\phi_{\mathrm{r}}(u_n,v_n)$ exists for any sequence $(u_n,v_n) \to (u,+\infty)$.

(B) *If additionally $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}$ satisfies*

$$E_1^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] < +\infty, \tag{5-112}$$

$$F^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] < +\infty \tag{5-113}$$

for some $0 < \beta \leq 1$, then

$$\langle u\rangle^\beta |\phi'_{\mathcal{L}}|(u,v) \lesssim \langle u\rangle^\beta \left|\mathfrak{r}_{\omega_{\mathrm{res}}}(0)\int_{-u}^{v} (\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(\tilde{v}) e^{i\omega_{\mathrm{res}}\tilde{v}}\, \mathrm{d}\tilde{v}\right| + E_1^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] + F^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \tag{5-114}$$

uniformly for all $v \geq 2$, $u \in \mathbb{R}$ such that $v \geq v_\gamma(u)$.

(C) *Moreover,*

$$\partial_v(r e^{i\omega_{\mathrm{res}} r^*}\phi'_{\mathcal{L}}(u,v)) = -i\frac{r_+ e^{i\omega_{\mathrm{res}} u}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega) e^{-i\omega v}\, \mathrm{d}\omega + \Phi_{\mathrm{error}}, \tag{5-115}$$

where Φ_{error} satisfy the quantitative bounds

$$|\Phi_{\mathrm{error}}|(u,v) \lesssim_\alpha E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]\Omega_{\mathrm{RN}}^{2-\alpha}(u,v) \tag{5-116}$$

for any fixed $0<\alpha<2$ and every (u,v) such that $r^*(u,v) \geq 1$.

(D) *Additionally to the assumptions in parts* (A) *and* (B), *let $\sigma_{\mathrm{br}} = \sigma_{\mathrm{br}}(u,v) \in C^1_{u,v}$ with $|\partial_v \sigma_{\mathrm{br}}| \lesssim \langle v\rangle^{1-2s}$ be arbitrary. Assume further that*

$$G^s[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] := \|\langle v\rangle^s (\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty} + \|\langle v\rangle^s\, \partial_v (\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty} < +\infty. \tag{5-117}$$

Then, for all $v \geq v_\gamma(u)$

$$\left|\int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')}\,\partial_{v'}(e^{i\omega_{\mathrm{res}} r^*} r\phi'_{\mathcal{L}}(u,v'))\,\mathrm{d}v'\right|$$
$$\lesssim \left|\int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')} e^{i\omega_{\mathrm{res}} v'} (\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(v')\,\mathrm{d}v'\right| + \langle u\rangle^{2-3s}(G^s[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] + E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]). \tag{5-118}$$

(E) *Let $u \in \mathbb{R}$ be arbitrary and assume that $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}$ is such that $\|\partial_v(re^{i\omega_{\mathrm{res}}r^*}\phi'_{\mathcal{L}}(u,v))\|_{L^1_v} < +\infty$.*

- *Assume in addition that $P_\delta(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} \in L^1_v(\mathbb{R})$ for some $\delta > 0$. Then,*
$$\|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v} \lesssim_\delta \|\partial_v(re^{i\omega_{\mathrm{res}}r^*}\phi'_{\mathcal{L}}(u,v))\|_{L^1_v} + E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] + \|P_\delta(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v}.$$
- *If $0 < |q_0 e| < \epsilon(M,e,m^2)$ or $(q_0,m^2) \in \{0\} \times \mathbb{R} - D(M,e)$ as in Lemma 5.10, then*
$$\|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v} \lesssim \|\partial_v(re^{i\omega_{\mathrm{res}}r^*}\phi'_{\mathcal{L}}(u,v))\|_{L^1_v} + E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}].$$

Proof of Theorem V. (A) We use the representation formula (5-99) in Lemma 5.13 and have

$$\phi'_{\mathcal{L}}(u,v) = \frac{r_+}{\sqrt{2\pi}r}\,\mathrm{p.v.}\int_{\mathbb{R}}\bigg[\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}](\omega) \times \frac{\mathfrak{r}(\omega)\widetilde{u_{\mathcal{CH}_R}}(\omega,r^*)e^{i(\omega-\omega_{\mathrm{res}})r^*}+\mathfrak{t}(\omega)\widetilde{u_{\mathcal{CH}_L}}(\omega,r^*)e^{-i(\omega-\omega_{\mathrm{res}})r^*}}{\omega-\omega_{\mathrm{res}}}e^{-i\omega t}\bigg]\,\mathrm{d}\omega. \tag{5-119}$$

After a change of variables $\omega \mapsto \omega + \omega_{\mathrm{res}}$, we obtain

$$\phi'_{\mathcal{L}}(u,v) = \frac{r_+e^{-i\omega_{\mathrm{res}}r^*}e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}r}\,\mathrm{p.v.}\int_{\mathbb{R}}\bigg[\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega) \times \frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)\widetilde{u_{\mathcal{CH}_R}}(\omega+\omega_{\mathrm{res}},r^*)e^{i\omega u}+\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)\widetilde{u_{\mathcal{CH}_L}}(\omega+\omega_{\mathrm{res}},r^*)e^{-i\omega v}}{\omega}\bigg]\,\mathrm{d}\omega, \tag{5-120}$$

where $\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega) = \mathfrak{r}(\omega+\omega_{\mathrm{res}})$ and $\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega) = \mathfrak{t}(\omega+\omega_{\mathrm{res}})$.

We now expand the numerator and obtain

$$\begin{aligned}
\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)\widetilde{u_{\mathcal{CH}_R}}(\omega+\omega_{\mathrm{res}},r^*) &= \mathfrak{r}_{\omega_{\mathrm{res}}}(0) + (\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{r}_{\omega_{\mathrm{res}}}(0)) + \mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}}+\omega,r^*)-1) && \text{(5-121)}\\
&= \mathfrak{r}_{\omega_{\mathrm{res}}}(0) && \text{(5-122)}\\
&\quad + (\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{r}_{\omega_{\mathrm{res}}}(0)) && \text{(5-123)}\\
&\quad + \mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}}+\omega,r^*)-\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)) && \text{(5-124)}\\
&\quad + \mathfrak{r}_{\omega_{\mathrm{res}}}(0)(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)-1) && \text{(5-125)}\\
&\quad + (\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{r}_{\omega_{\mathrm{res}}}(0))(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)-1), && \text{(5-126)}
\end{aligned}$$

as well as

$$\begin{aligned}
\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)\widetilde{u_{\mathcal{CH}_L}}(\omega+\omega_{\mathrm{res}},r^*) &= \mathfrak{t}_{\omega_{\mathrm{res}}}(0) + (\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{t}_{\omega_{\mathrm{res}}}(0)) + \mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)(\widetilde{u_{\mathcal{CH}_L}}(\omega_{\mathrm{res}}+\omega,r^*)-1) && \text{(5-127)}\\
&= \mathfrak{t}_{\omega_{\mathrm{res}}}(0) && \text{(5-128)}\\
&\quad + (\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{t}_{\omega_{\mathrm{res}}}(0)) && \text{(5-129)}\\
&\quad + \mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)(\widetilde{u_{\mathcal{CH}_L}}(\omega_{\mathrm{res}}+\omega,r^*)-\widetilde{u_{\mathcal{CH}_L}}(\omega_{\mathrm{res}},r^*)) && \text{(5-130)}\\
&\quad + \mathfrak{t}_{\omega_{\mathrm{res}}}(0)(\widetilde{u_{\mathcal{CH}_L}}(\omega_{\mathrm{res}},r^*)-1) && \text{(5-131)}\\
&\quad + (\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{t}_{\omega_{\mathrm{res}}}(0))(\widetilde{u_{\mathcal{CH}_L}}(\omega_{\mathrm{res}},r^*)-1). && \text{(5-132)}
\end{aligned}$$

We write

$$\frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)}{\omega} = \frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(0)}{\omega} + \mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega), \qquad \frac{\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)}{\omega} = \frac{\mathfrak{t}_{\omega_{\mathrm{res}}}(0)}{\omega} + \mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega), \tag{5-133}$$

where

$$\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega) := \frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{r}_{\omega_{\mathrm{res}}}(0)}{\omega}, \qquad \mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega) := \frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{t}_{\omega_{\mathrm{res}}}(0)}{\omega} \tag{5-134}$$

are real-analytic.

In the following we will estimate each term from (5-122)–(5-132) independently. We start with the main term coming from (5-122).

Lemma 5.15. *We have*

$$e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{mainR}}(u,v) := \frac{r_+e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}r}\,\mathrm{p.v.}\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(0)}{\omega}e^{i\omega u}\,\mathrm{d}\omega \tag{5-135}$$

satisfies

$$e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{mainR}}(u,v) = i\pi\frac{r_+e^{i\omega_{\mathrm{res}}u}\mathfrak{r}_{\omega_{\mathrm{res}}}(0)}{\sqrt{2\pi}r}\int_{\mathbb{R}}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(\tilde v)\chi_{\le v}(\tilde v)e^{i\omega_{\mathrm{res}}u}\,\mathrm{sgn}(\tilde v+u)\,\mathrm{d}\tilde v. \tag{5-136}$$

Proof. This follows directly from the fact that $\mathcal{F}[\mathrm{p.v.}(1/x)] = i\pi\,\mathrm{sgn}$. □

Lemma 5.16. *We have that*

$$\begin{aligned} e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{errorR1}}(u,v) :&= \frac{r_+e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}r}\,\mathrm{p.v.}\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{r}_{\omega_{\mathrm{res}}}(0)}{\omega}e^{i\omega u}\,\mathrm{d}\omega && (5\text{-}137)\\ &= \frac{r_+e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}r}\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{i\omega u}\,\mathrm{d}\omega && (5\text{-}138)\end{aligned}$$

extends continuously to the Cauchy horizon and satisfies

$$|\phi_{\mathrm{errorR1}}(u,v)| \lesssim E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]. \tag{5-139}$$

If additionally, $E_1^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] < +\infty$ for some $0<\beta\le 1$, we further have

$$|\langle u\rangle^\beta\phi_{\mathrm{errorR1}}(u,v)| \lesssim E_1^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \tag{5-140}$$

for all $r^*\ge 0$.

Proof. It suffices to show both claims for $\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{i\omega u}\,\mathrm{d}\omega$. We begin by showing (5-140) under the assumption $E_1^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]<\infty$. We will use the notation $\langle\partial_\omega\rangle^\beta$ to denote the Fourier multiplier with $(1+|u|^2)^{\beta/2}$, where u is the dual variable to ω. Using this, we estimate

$$\begin{aligned}&\left|\langle u\rangle^\beta\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{i\omega u}\,\mathrm{d}\omega\right|\\ &\quad= \left|\int_{\mathbb{R}}\langle\partial_\omega\rangle^\beta\big(\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)\big)e^{i\omega u}\,\mathrm{d}\omega\right|\\ &\quad\le \|\langle\partial_\omega\rangle^\beta(\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}]\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}})\|_{L^1_\omega}\\ &\quad\le \|\langle\partial_\omega\rangle^\beta(\langle\omega\rangle\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}])\|_{L^2_\omega}\|\langle\omega\rangle^{-1}\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}\|_{L^2_\omega}\\ &\qquad\qquad + \|\langle\omega\rangle\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}]\|_{L^2_\omega}\|\langle\partial_\omega\rangle^\beta(\langle\omega\rangle^{-1}\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}})\|_{L^2_\omega}\\ &\quad\lesssim \|\langle v\rangle^\beta(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}\|_{L^2_v} + \|\langle v\rangle^\beta\,\partial_v((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot})\|_{L^2_v}\\ &\quad\lesssim E_1^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \end{aligned}\tag{5-141}$$

in view of a Kato–Ponce inequality (see, e.g., [Grafakos and Oh 2014, Theorem 1]) and

$$\|\langle\omega\rangle^{-1}\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}\|_{L^2(\mathbb{R}_\omega)} \lesssim 1, \tag{5-142}$$

$$\|\langle\partial_\omega\rangle^{\beta}(\langle\omega\rangle^{-1}\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}})\|_{L^2(\mathbb{R}_\omega)} \lesssim 1, \tag{5-143}$$

which follow from the definition of $\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}$, $\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}$ as well as Lemma 5.9. Now, note that the previous estimates for $\beta = 0$ give (5-139).

For the continuous extendibility across the Cauchy horizon we need to show that for $(u_n, v_n) \to (u_0, +\infty)$, the limit

$$\lim_{n\to\infty}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v_n}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{i\omega u_n}\,\mathrm{d}\omega \tag{5-144}$$

exists and that the limiting function is continuous. In view of the triangle inequality we have

$$\begin{aligned}\biggl|\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v_n}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{i\omega u_n}\,\mathrm{d}\omega - \int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{i\omega u_0}\,\mathrm{d}\omega\biggr| \\ \lesssim \int_{\mathbb{R}} |\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)||e^{i\omega u_n}-e^{i\omega u_0}|\,\mathrm{d}\omega \\ + \int_{\mathbb{R}} |\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(1-\chi_{\le v_n})e^{i\omega_{\mathrm{res}}\cdot}](\omega)||\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)|\,\mathrm{d}\omega.\end{aligned} \tag{5-145}$$

In the first term of (5-145) we apply dominated convergence to interchange the limit with the integral which is justified as

$$\begin{aligned}\int_{\mathbb{R}} |\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)||e^{i\omega u_n}-e^{i\omega u_0}|\,\mathrm{d}\omega \\ \lesssim \int_{\mathbb{R}} |\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)|\,\mathrm{d}\omega \lesssim E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]\end{aligned} \tag{5-146}$$

in view of (5-142). For the second term in (5-145) we have that

$$\begin{aligned}\int_{\mathbb{R}} |\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(1-\chi_{\le v_n})e^{i\omega_{\mathrm{res}}\cdot}](\omega)||\mathfrak{r}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)|\,\mathrm{d}\omega \\ \lesssim \biggl(\int_{\mathbb{R}} |\partial_{\tilde{v}}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(\tilde{v})(1-\chi_{\le v_n}(\tilde{v})))|^2\,\mathrm{d}\tilde{v}\biggr)^{1/2} \to 0\end{aligned} \tag{5-147}$$

as $n \to \infty$ since $E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] < +\infty$. That the limit is continuous also follows from (5-146). □

Lemma 5.17. *We have that*

$$\begin{aligned}e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{errorR2}}(u,v) := \frac{r_+e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}r}\,\mathrm{p.v.}\!\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega) \\ \cdot\frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}}+\omega,r^*)-\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*))}{\omega}e^{i\omega u}\,\mathrm{d}\omega\end{aligned} \tag{5-148}$$

converges to zero towards the Cauchy horizon and satisfies the quantitative bound

$$|\phi_{\mathrm{errorR2}}(u,v)| \lesssim \Omega^2_{\mathrm{RN}}(u,v)E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \tag{5-149}$$

for $r^* \ge 1$.

Proof. We estimate

$$\left|\frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}}+\omega,r^*)-\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*))}{\omega}\right| \tag{5-150}$$

$$\lesssim \sup_{|\omega|\le 1}|\partial_\omega \widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}}+\omega,r^*)| + \sup_{|\omega|\ge 1}|\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}}+\omega,r^*)-\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)| \tag{5-151}$$

$$\lesssim \Omega^2_{\mathrm{RN}} \tag{5-152}$$

in view of Lemma 5.7 and Proposition 5.6. Now, (5-149) follows from a direct application of the Cauchy–Schwarz inequality. □

Lemma 5.18. *We have that*

$$e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{errorR3}}(u,v)$$

$$:= \frac{r_+ e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}\,r}\,\mathrm{p.v.}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(0)(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)-1)}{\omega}e^{i\omega u}\,\mathrm{d}\omega \tag{5-153}$$

$$= \mathfrak{r}_{\omega_{\mathrm{res}}}(0)(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)-1)\frac{r_+ e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}\,r}\,\mathrm{p.v.}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{1}{\omega}e^{i\omega u}\,\mathrm{d}\omega \tag{5-154}$$

converges to zero towards the Cauchy horizon and satisfies the quantitative bound

$$\begin{aligned}|\phi_{\mathrm{errorR3}}(u,v)| &\lesssim \Omega^2_{\mathrm{RN}}(u,v)\|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v+u}\|_{L^1_v}\\ &\lesssim \Omega^2_{\mathrm{RN}}(u,v)E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]\langle r^*\rangle^{1/2} \lesssim_\alpha \Omega^{2-\alpha}_{\mathrm{RN}}(u,v)E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]\end{aligned} \tag{5-155}$$

for $r^*\ge 1$ and any $\alpha>0$.

Proof. It suffices to control the principal value integral. A direct computation using that $\mathcal{F}[\mathrm{p.v.}(1/x)] = i\pi\,\mathrm{sgn}$ yields

$$\left|\mathrm{p.v.}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{1}{\omega}e^{i\omega u}\,\mathrm{d}\omega\right|$$

$$\lesssim \int_{\mathbb{R}} |(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(\tilde v-u)\chi_{\le v}(\tilde v-u)|\,\mathrm{d}\tilde v \le \|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v+u}\|_{L^1(\mathbb{R})}. \tag{5-156}$$

The second inequality in (5-155) is now a consequence of the Cauchy–Schwarz inequality. □

Now, we are in the position to control the last term as follows.

Lemma 5.19. *We have that*

$$e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{errorR4}}(u,v)$$

$$:= \frac{r_+ e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}\,r}\,\mathrm{p.v.}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{(\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{r}_{\omega_{\mathrm{res}}}(0))(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)-1)}{\omega}e^{i\omega u}\,\mathrm{d}\omega \tag{5-157}$$

$$= (\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)-1)e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{errorR1}} \tag{5-158}$$

converges to zero towards the Cauchy horizon and satisfies the quantitative bound

$$|\phi_{\mathrm{errorR4}}(u,v)| \lesssim \Omega^2_{\mathrm{RN}}(u,v)E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \tag{5-159}$$

for $r^*\ge 0$.

Proof. This follows immediately from Lemma 5.16. □

Now, we turn to the terms arising from the transmission coefficient. Completely analogous to Lemma 5.15 we obtain:

Lemma 5.20. *We have that*

$$e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{mainT}}(u,v) := \frac{r_+ e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}r}\,\mathrm{p.v.}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{\mathfrak{t}_{\omega_{\mathrm{res}}}(0)}{\omega}e^{-i\omega v}\,\mathrm{d}\omega \tag{5-160}$$

satisfies

$$e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{mainT}}(u,v) = i\pi\frac{r_+ e^{i\omega_{\mathrm{res}}u}\mathfrak{t}_{\omega_{\mathrm{res}}}(0)}{\sqrt{2\pi}r}\int_{\mathbb{R}}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(\tilde v)\chi_{\le v}(\tilde v)e^{i\omega_{\mathrm{res}}\tilde v}\,\mathrm{sgn}(\tilde v - v)\,\mathrm{d}\tilde v. \tag{5-161}$$

Lemma 5.21. *We have that*

$$\begin{aligned} e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{errorT1}}(u,v) &:= \frac{r_+ e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}r}\,\mathrm{p.v.}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{t}_{\omega_{\mathrm{res}}}(0)}{\omega}e^{-i\omega v}\,\mathrm{d}\omega &\text{(5-162)}\\ &= \frac{r_+ e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}r}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega &\text{(5-163)}\end{aligned}$$

extends continuously to zero at the right Cauchy horizon, i.e., for $v\to+\infty$ and $u\to u_0$. If in addition $E_1^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]<\infty$, then we have the quantitative decay

$$|\phi_{\mathrm{errorT1}}(u,v)| \lesssim \langle v\rangle^{-\beta}E_1^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]. \tag{5-164}$$

Proof. We first show the first claim without assuming that $E_1^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]<+\infty$. Doing the analogous estimate as in (5-147) it suffices to show that

$$\left|\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v}](\omega)\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega\right| \tag{5-165}$$

tends to zero as $v\to+\infty$. Thus, it suffices to show that $v\mapsto\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v}](\omega)\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega$ is an H^1 function. This again follows from

$$\int_{\mathbb{R}}(1+\omega^2)|\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v}](\omega)|^2|\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)|^2\,\mathrm{d}\omega \lesssim E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]\sup_{\omega\in\mathbb{R}}|\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)| \lesssim E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]. \tag{5-166}$$

We will now proceed to show the quantitative decay assuming $E^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]<\infty$. In this case we have

$$\begin{aligned}&\left|\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\tilde v}](\omega)\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega\right|\\ &\quad= \left|\frac{1}{v}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\tilde v}](\omega)\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)\,\partial_\omega e^{-i\omega v}\,\mathrm{d}\omega\right|\\ &\quad\lesssim \left|\frac{1}{v}\int_{\mathbb{R}} \partial_\omega\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\tilde v}](\omega)\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega\right|\\ &\qquad\qquad + \left|\frac{1}{v}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\tilde v}](\omega)(\partial_\omega\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega))e^{-i\omega v}\,\mathrm{d}\omega\right|\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left|\frac{1}{v}\int_{|\omega|\le 1}\mathcal{F}[\tilde{v}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\tilde{v}}](\omega)\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega\right| \\
&\qquad + \left|\frac{1}{v}\int_{|\omega|\le 1}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\tilde{v}}](\omega)(\partial_\omega\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega))e^{-i\omega v}\,\mathrm{d}\omega\right| \\
&\qquad\qquad + \left|\frac{1}{v}\int_{|\omega|\ge 1}\mathcal{F}[\tilde{v}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\tilde{v}}](\omega)\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega\right| \\
&\qquad\qquad\qquad + \left|\frac{1}{v}\int_{|\omega|\ge 1}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\tilde{v}}](\omega)(\partial_\omega\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}(\omega))e^{-i\omega v}\,\mathrm{d}\omega\right| \\
&\lesssim \frac{1}{v}\|\tilde{v}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\tilde{v}}\|_{L^2_{\tilde{v}}}\|\partial_\omega\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}\|_{L^2_\omega[-1,1]} + \frac{1}{v}\|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\tilde{v}}\|_{L^2_{\tilde{v}}}\|\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}\|_{L^2_\omega[-1,1]} \\
&\qquad + \frac{1}{v}\|\partial_{\tilde{v}}(\tilde{v}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\tilde{v}})\|_{L^2_{\tilde{v}}}\|\omega^{-1}\,\partial_\omega\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}\|_{L^2_\omega(\mathbb{R}-[-1,1])} \\
&\qquad\qquad + \frac{1}{v}\|\partial_{\tilde{v}}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\tilde{v}})\|_{L^2_{\tilde{v}}}\|\omega^{-1}\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}\|_{L^2_\omega(\mathbb{R}-[-1,1])} \\
&\lesssim \frac{1}{v^\beta}E_1^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]
\end{aligned}
$$

since

$$\|\partial_\omega\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}\|_{L^2_\omega[-1,1]},\ \|\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}\|_{L^2_\omega[-1,1]},\ \|\omega^{-1}\,\partial_\omega\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}\|_{L^2_\omega(\mathbb{R}-[-1,1])},\ \|\omega^{-1}\mathfrak{t}^{\mathrm{re}}_{\omega_{\mathrm{res}}}\|_{L^2_\omega(\mathbb{R}-[-1,1])}\lesssim 1. \qquad \square$$

Analogously to Lemma 5.17 we have:

Lemma 5.22. *We have that*

$$
\begin{aligned}
&e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{errorT2}}(u,v) \\
&:= \frac{r_+e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}r}\,\mathrm{p.v.}\!\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)(\widetilde{u_{\mathcal{CH}_L}}(\omega_{\mathrm{res}}+\omega,r^*)-\widetilde{u_{\mathcal{CH}_L}}(\omega_{\mathrm{res}},r^*))}{\omega}e^{-i\omega v}\,\mathrm{d}\omega \qquad (5\text{-}167)
\end{aligned}
$$

converges to zero towards the Cauchy horizon and satisfies the quantitative bound

$$|\phi_{\mathrm{errorT2}}(u,v)|\lesssim\Omega^2_{\mathrm{RN}}(u,v)E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \qquad (5\text{-}168)$$

for $r^*\ge 1$.

Analogously to Lemma 5.18 we further obtain:

Lemma 5.23. *We have that*

$$
\begin{aligned}
&e^{i\omega_{\mathrm{res}}r^*}\phi_{\mathrm{errorT3}}(u,v) \\
&:= \frac{r_+e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}r}\,\mathrm{p.v.}\!\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{\mathfrak{t}_{\omega_{\mathrm{res}}}(0)(\widetilde{u_{\mathcal{CH}_L}}(\omega_{\mathrm{res}},r^*)-1)}{\omega}e^{-i\omega v}\,\mathrm{d}\omega \qquad (5\text{-}169) \\
&= \mathfrak{t}_{\omega_{\mathrm{res}}}(0)(\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)-1)\frac{r_+e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}r}\,\mathrm{p.v.}\!\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\frac{1}{\omega}e^{-i\omega v}\,\mathrm{d}\omega \qquad (5\text{-}170)
\end{aligned}
$$

converges to zero towards the Cauchy horizon and satisfies the quantitative bound

$$|\phi_{\mathrm{errorT3}}(u,v)|\lesssim_\alpha\Omega^{2-\alpha}_{\mathrm{RN}}(u,v)E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \qquad (5\text{-}171)$$

for $r^*\ge 1$.

Finally, completely analogous to Lemma 5.19 we have:

Lemma 5.24. *We have that*

$$e^{i\omega_{\mathrm{res}} r^*}\phi_{\mathrm{errorT4}}(u,v) := \frac{r_+ e^{i\omega_{\mathrm{res}} u}}{\sqrt{2\pi}\, r}\, \mathrm{p.v.}\int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v} e^{i\omega_{\mathrm{res}}\cdot}](\omega) \cdot \frac{(\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{t}_{\omega_{\mathrm{res}}}(0))(\widetilde{u_{\mathcal{CH}_L}}(\omega_{\mathrm{res}},r^*)-1)}{\omega} e^{-i\omega v}\,\mathrm{d}\omega$$
$$= (\widetilde{u_{\mathcal{CH}_R}}(\omega_{\mathrm{res}},r^*)-1)e^{i\omega_{\mathrm{res}} r^*}\phi_{\mathrm{errorT1}} \tag{5-172}$$

converges to zero towards the Cauchy horizon and satisfies the quantitative bound

$$|\phi_{\mathrm{errorR4}}(u,v)| \lesssim \Omega^2_{\mathrm{RN}}(u,v) E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \tag{5-173}$$

for $r^* \geq 1$.

Having estimated each term independently in the integral appearing in (5-120) and noting that

$$e^{i\omega_{\mathrm{res}} r^*}(\phi_{\mathrm{mainR}}+\phi_{\mathrm{mainT}})(u,v) = \frac{\sqrt{2\pi}\, i r_+}{r}\mathfrak{r}_{\omega_{\mathrm{res}}}(0) e^{i\omega_{\mathrm{res}} u}\left(\int_{-u}^{v} (\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(\tilde{v}) e^{i\omega_{\mathrm{res}}\tilde{v}}\,\mathrm{d}\tilde{v}\right) \tag{5-174}$$

in view of $\mathfrak{r}_{\omega_{\mathrm{res}}}(0) = -\mathfrak{t}_{\omega_{\mathrm{res}}}(0)$, we finally obtain (5-109) with

$$\phi_{\mathrm{r}} = e^{i\omega_{\mathrm{res}} r^*}\phi_{\mathrm{errorR1}} \tag{5-175}$$

and

$$\phi_{\mathrm{error}} = e^{i\omega_{\mathrm{res}} r^*}(\phi_{\mathrm{errorR2}}+\phi_{\mathrm{errorR3}}+\phi_{\mathrm{errorR4}}+\phi_{\mathrm{errorT1}}+\phi_{\mathrm{errorT2}}+\phi_{\mathrm{errorT3}}+\phi_{\mathrm{errorT4}}). \tag{5-176}$$

The bounds and continuity statement for ϕ_{r} and ϕ_{error} now follow from Lemma 5.16 and (5-172).

(B) In view of part (A) and the fact that Ω_{RN} decays exponentially in $r^* = \frac{1}{2}(u+v)$ towards the Cauchy horizon, it suffices to show that

$$\langle u\rangle^\beta \left| \int_{-u}^{v+1} \chi_{\leq v}(\tilde{v})(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(\tilde{v}) e^{i\omega_{\mathrm{res}}\tilde{v}}\,\mathrm{d}\tilde{v}\right| + \langle u\rangle^\beta |\phi_{\mathrm{r}}(u,v)| \lesssim F^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] + E_1^\beta[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \tag{5-177}$$

as we consider the region $v \geq |u| + \log(v)/(2|K_-|)$ in which $\Omega^2_{\mathrm{RN}}(u,v) \lesssim \langle v\rangle^{-1}$. Now, the claim is a direct consequence of the second parts of Lemmas 5.16 and 5.21 together with the assumptions (5-113) and (5-112).

(C) We will now consider $\partial_v(r e^{i\omega_{\mathrm{res}} r^*}\phi'_{\mathcal{L}})$. We use the second part of Lemma 5.13 and end up with

$$\partial_v(e^{i\omega_{\mathrm{res}} r^*} r\phi'_{\mathcal{L}}(u,v))$$
$$= \frac{r_+ e^{i\omega_{\mathrm{res}} u}}{\sqrt{2\pi}}\,\mathrm{p.v.}\int_{\mathbb{R}} \Bigg[\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\leq v_1} e^{i\omega_{\mathrm{res}}\cdot}](\omega) \cdot \frac{\mathfrak{r}_{\omega_{\mathrm{res}}}(\omega)\,\partial_v \widetilde{u_{\mathcal{CH}_R}}(\omega+\omega_{\mathrm{res}},r^*) e^{i\omega u} + \mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)\,\partial_v(\widetilde{u_{\mathcal{CH}_L}}(\omega+\omega_{\mathrm{res}},r^*)e^{-i\omega v})}{\omega}\Bigg]\mathrm{d}\omega \tag{5-178}$$

for $v_1 > v$. Since $\partial_v \widetilde{u_{\mathcal{CH}_R}}$ and $\partial_v \widetilde{u_{\mathcal{CH}_L}}$ are bounded uniformly in absolute value by Ω^2_{RN} in view of Proposition 5.6, the terms of (5-178) which arise thereof are bounded by $\Omega^{2-\alpha}_{\mathrm{RN}} E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]$ for any $\alpha > 0$

as in part (A). Similarly, $\widetilde{u_{\mathcal{CH}_L}}-1$ is bounded by Ω^2_{RN} and thus, the main term arises from $\partial_v(e^{-i\omega v})$ and we obtain

$$\partial_v(e^{i\omega_{\mathrm{res}}r^*}r\phi'_{\mathcal{L}}(u,v)) = -i\frac{r_+e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\chi_{\le v_1}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega+\Phi^{v_1}_{\mathrm{error}}, \tag{5-179}$$

where $|\Phi^{v_1}_{\mathrm{error}}|\lesssim_\alpha \Omega^{2-\alpha}_{\mathrm{RN}}E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]$. Note that $\Phi^{v_1}_{\mathrm{error}}$ depends on v_1 but the upper bound is uniform in v_1. Since $\langle\omega\rangle\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\cdot}]\in L^2_\omega$ and $\langle\omega\rangle^{-1}\mathfrak{t}_{\omega_{\mathrm{res}}}\in L^\infty_\omega$, we can take the limit $v_1\to\infty$ and obtain

$$\partial_v(e^{i\omega_{\mathrm{res}}r^*}r\phi'_{\mathcal{L}}(u,v)) = -i\frac{r_+e^{i\omega_{\mathrm{res}}u}}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega+\Phi_{\mathrm{error}}, \tag{5-180}$$

where $|\Phi_{\mathrm{error}}(u,v)|\lesssim_\alpha \Omega^{2-\alpha}_{\mathrm{RN}}(u,v)E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]$.

(D) Note that Φ_{error} as in part (C) decays proportional to $\Omega^{2-\alpha}_{\mathrm{RN}}$ for any $\alpha>0$ and thus

$$\begin{aligned}\int_{v_\gamma(u)}^v|\Phi_{\mathrm{error}}|\,\mathrm{d}v' &\lesssim_\alpha (\Omega_{\mathrm{RN}})^{2-\alpha}(u,v_\gamma(u))E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]\lesssim_\alpha \langle u\rangle^{-s(2-\alpha)}E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]\\ &\lesssim \langle u\rangle^{2-3s}E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]\end{aligned} \tag{5-181}$$

choosing $\alpha>0$ sufficiently small (recall that $s\le 1$ therefore $2s>3s-2$). Thus, it suffices to show the result for the main part in (5-115). We further write

$$\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)=\mathfrak{t}^0_{\omega_{\mathrm{res}}}+\omega\mathfrak{t}^1_{\omega_{\mathrm{res}}}+\omega^2\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}(\omega), \tag{5-182}$$

where we note that

$$|\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}|=\left|\frac{\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega)-\mathfrak{t}^0_{\omega_{\mathrm{res}}}-\omega\mathfrak{t}^1_{\omega_{\mathrm{res}}}}{\omega^2}\right|\lesssim\langle\omega\rangle^{-1}$$

and $|\partial_\omega\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}|\lesssim\langle\omega\rangle^{-1}$ in view of Corollary 5.8 and Lemma 5.9. Hence,

$$\langle v\rangle\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})(v)\in L^2(\mathbb{R}_v) \tag{5-183}$$

and thus, $\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})\in L^1(\mathbb{R})$ by the Cauchy–Schwarz inequality.

Now, using (5-115) we obtain

$$\begin{aligned}&\left|\int_{v_\gamma(u)}^v e^{i\sigma_{\mathrm{br}}(u,v')}\,\partial_{v'}(e^{i\omega_{\mathrm{res}}r^*}r\phi'_{\mathcal{L}})\,\mathrm{d}v'\right|\\ &\quad\lesssim\left|\int_{v_\gamma(u)}^v e^{i\sigma_{\mathrm{br}}(u,v')}\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{t}^0_{\omega_{\mathrm{res}}}e^{-i\omega v'}\,\mathrm{d}\omega\,\mathrm{d}v'\right|\\ &\qquad+\left|\int_{v_\gamma(u)}^v e^{i\sigma_{\mathrm{br}}(u,v')}\int\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\omega\mathfrak{t}^1_{\omega_{\mathrm{res}}}e^{-i\omega v'}\,\mathrm{d}\omega\,\mathrm{d}v'\right|\\ &\qquad\quad+\left|\int_{v_\gamma(u)}^v e^{i\sigma_{\mathrm{br}}(u,v')}\int\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\omega^2\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v'}\,\mathrm{d}\omega\,\mathrm{d}v'\right|.\end{aligned} \tag{5-184}$$

For the first term we directly take the inverse Fourier transform and estimate

$$\begin{aligned}&\left|\int_{v_\gamma(u)}^v e^{i\sigma_{\mathrm{br}}(u,v')}\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\mathfrak{t}^0_{\omega_{\mathrm{res}}}e^{-i\omega v'}\,\mathrm{d}\omega\,\mathrm{d}v'\right|\\ &\qquad\lesssim\left|\int_{v_\gamma(u)}^v e^{i\sigma_{\mathrm{br}}(u,v')}e^{i\omega_{\mathrm{res}}v'}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(v')\,\mathrm{d}v'\right|.\end{aligned} \tag{5-185}$$

Similarly, for the second term we integrate by parts and obtain

$$\begin{aligned}\left|\int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')}\int_{\mathbb{R}}\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\omega\mathfrak{t}^1_{\omega_{\mathrm{res}}}e^{-i\omega v'}\,\mathrm{d}\omega\,\mathrm{d}v'\right|&\\ \lesssim\left|\int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')}\,\partial_{v'}(e^{i\omega_{\mathrm{res}}v'}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(v'))\,\mathrm{d}v'\right|&\\ \lesssim\langle u\rangle^{-s}\|\langle v\rangle^s(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty}+\left|\int_{v_\gamma(u)}^{v}|\partial_{v'}\sigma_{\mathrm{br}}(u,v')|\,|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(v')|\,\mathrm{d}v'\right|&\\ \lesssim\langle u\rangle^{2-3s}\|v^s(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty}.&\qquad(5\text{-}186)\end{aligned}$$

Using the same method as above, the third term satisfies

$$\left|\int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')}\int\mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\cdot}](\omega)\omega^2\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v'}\,\mathrm{d}\omega\,\mathrm{d}v'\right|$$

$$\lesssim\left|\int_{v_\gamma(u)}^{v}\partial_{v'}(e^{i\sigma_{\mathrm{br}}(u,v')})\int\mathcal{F}[\partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})](\omega)\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v'}\,\mathrm{d}\omega\,\mathrm{d}v'\right| \tag{5-187}$$

$$+\left|\int\mathcal{F}[\partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})](\omega)\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v}\,\mathrm{d}\omega\right| \tag{5-188}$$

$$+\left|\int\mathcal{F}[\partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})](\omega)\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}(\omega)e^{i\omega v_\gamma(u)}\,\mathrm{d}\omega\right|. \tag{5-189}$$

We will now estimate the three terms individually.

We start with integrand of (5-187) and note that the other terms (5-188) and (5-189) are treated analogously. We write

$$\begin{aligned}\left|\int\mathcal{F}[\partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})](\omega)\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}}(\omega)e^{-i\omega v'}\,\mathrm{d}\omega\right|&\lesssim\left|[\partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})]*\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})\right|(v')\\ &=\left|\int_{\mathbb{R}}\partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})(\tilde v)\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})(v'-\tilde v)\,\mathrm{d}\tilde v\right|.\end{aligned} \tag{5-190}$$

To estimate the convolution, we note that for $v'\geq 2R$, either $|\tilde v|\geq R$ or $|\tilde v-v'|\geq R$. Thus,

$$\begin{aligned}&\left|\int_{\mathbb{R}}\partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})(\tilde v)\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})(v'-\tilde v)\,\mathrm{d}\tilde v\right|\\ &\lesssim\left|\int_{|\tilde v|\geq R}\partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})(\tilde v)\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})(v'-\tilde v)\,\mathrm{d}\tilde v\right|+\left|\int_{|\tilde v-v'|\geq R}\partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})(\tilde v)\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})(v'-\tilde v)\,\mathrm{d}\tilde v\right|\\ &\lesssim R^{-s}\left|\int_{|\tilde v|\geq R}|\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})(v'-\tilde v)|\,\mathrm{d}\tilde v\right|(\|v^s(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty}+\|v^s\,\partial_v(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty})\\ &\qquad+R^{-1}\left|\int_{|\tilde v-v'|\geq R}|\partial_{\tilde v}((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}e^{i\omega_{\mathrm{res}}\tilde v})(\tilde v)|\,|v'-\tilde v|\,|\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})(v'-\tilde v)|\,\mathrm{d}\tilde v\right|\\ &\lesssim\langle v'\rangle^{-s}\|(\|v^s(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty}+\|v^s\,\partial_v(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty})\|\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})\|_{L^1}+\langle\tilde v\rangle^{-1}E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]\|\langle v\rangle\mathcal{F}(\tilde{\mathfrak{t}}_{\omega_{\mathrm{res}}})\|_{L^2}\\ &\lesssim\langle v'\rangle^{-s}((\|v^s(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty}+\|v^s\,\partial_v(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty})+E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]),\end{aligned} \tag{5-191}$$

where we used (5-183). Now, plugging these estimates in (5-187), (5-188) and (5-189) and using that $|\partial_v \sigma_{\mathrm{br}}| \lesssim \langle v\rangle^{1-2s}$, we obtain, since $\frac{3}{4} < s \le 1$

$$\left| \int_{v_\gamma(u)}^{v} e^{i\sigma_{\mathrm{br}}(u,v')}\, \partial_{v'}(e^{i\omega_{\mathrm{res}} r^*} r\phi'_{\mathcal{L}}) \right| \mathrm{d}v' \lesssim \langle u\rangle^{2-3s} (\|v^s (\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty} + \|v^s\, \partial_v (\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^\infty} + E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]).$$

This shows (D).

(E) Assume that $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}$ is such that the arising solution $\phi'_{\mathcal{L}}$ satisfies $\partial_v(e^{i\omega_{\mathrm{res}} r^*} r\phi'_{\mathcal{L}})(u,\cdot) \in L^1_v$ on some constant u surface. Then, in view of (5-180), we have that

$$\begin{aligned} \left\| \int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega) \mathfrak{t}_{\omega_{\mathrm{res}}}(\omega) e^{-i\omega v}\, \mathrm{d}\omega \right\|_{L^1_v} &\lesssim \|\partial_v (r e^{i\omega_{\mathrm{res}} r^*} \phi'_{\mathcal{L}}(u,v))\|_{L^1_v} + \|\Phi_{\mathrm{error}}\|_{L^1_v} \\ &\lesssim \|\partial_v (r e^{i\omega_{\mathrm{res}} r^*} \phi'_{\mathcal{L}}(u,v))\|_{L^1_v} + E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]. \end{aligned} \tag{5-192}$$

We will first consider the cases for which $\mathfrak{t}_{\omega_{\mathrm{res}}}$ does not have any zeros (i.e., $\mathcal{Z}_{\mathfrak{t}} = \varnothing$); see Lemma 5.10. Then $1/\mathfrak{t}_{\omega_{\mathrm{res}}} \lesssim \langle \omega\rangle^{-1}$ since $|\mathfrak{t}|^2 = |\mathfrak{r}|^2 + \omega(\omega - \omega_{\mathrm{res}})$. For that, also recall $\mathfrak{t}_{\omega_{\mathrm{res}}}(\omega) = \mathfrak{t}(\omega + \omega_{\mathrm{res}})$. Moreover, in this case, $\mathcal{F}^{-1}[1/\mathfrak{t}_{\omega_{\mathrm{res}}}] \in L^1_v$ since $1/\mathfrak{t}_{\omega_{\mathrm{res}}} \in L^2_\omega$, $\partial_\omega(1/\mathfrak{t}_{\omega_{\mathrm{res}}}) \in L^2_\omega$. Thus, $1/\mathfrak{t}_{\omega_{\mathrm{res}}}$ is a L^1 bounded Fourier multiplier. Hence, using that $1 = \mathfrak{t}_{\omega_{\mathrm{res}}}(1/\mathfrak{t}_{\omega_{\mathrm{res}}})$ and (5-192), we obtain

$$\begin{aligned} \|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v} &\lesssim \left\| \int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega) \mathfrak{t}_{\omega_{\mathrm{res}}}(\omega) e^{-i\omega v}\, \mathrm{d}\omega \right\|_{L^1_v} \\ &\lesssim \|\partial_v (r e^{i\omega_{\mathrm{res}} r^*} \phi'_{\mathcal{L}}(u,v))\|_{L^1_v} + E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}]. \end{aligned} \tag{5-193}$$

Now, we consider the case, where $\mathfrak{t}$ potentially has zeros, all of which have to lie in $\mathcal{Z}^\delta_{\mathfrak{t}}$. Then, by the inverse triangle inequality applied to (5-192) we obtain

$$\begin{aligned} &\|\partial_v (r e^{i\omega_{\mathrm{res}} r^*} \phi'_{\mathcal{L}}(u,v))\|_{L^1_v} + E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] \\ &\quad\gtrsim \left\| \int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega) \mathfrak{t}_{\omega_{\mathrm{res}}}(\omega) e^{-i\omega v}\, \mathrm{d}\omega \right\|_{L^1_v} \\ &\quad\ge \left\| \int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega) (1-\chi_\delta(\omega+\omega_{\mathrm{res}})) \mathfrak{t}_{\omega_{\mathrm{res}}}(\omega) e^{-i\omega v}\, \mathrm{d}\omega \right\|_{L^1_v} \\ &\qquad\qquad - \left\| \int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega) \chi_\delta(\omega+\omega_{\mathrm{res}}) \mathfrak{t}_{\omega_{\mathrm{res}}}(\omega) e^{-i\omega v}\, \mathrm{d}\omega \right\|_{L^1_v}, \end{aligned} \tag{5-194}$$

where we recall that χ_δ is supported in $\mathcal{Z}^\delta_{\mathfrak{t}}$. For the first term we use $|1/\mathfrak{t}| \lesssim_\delta \langle\omega\rangle^{-1}$ on $\mathbb{R} - \mathcal{Z}^\delta_{\mathfrak{t}}$ and obtain

$$\begin{aligned} \left\| \int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega) (1-\chi_\delta(\omega+\omega_{\mathrm{res}})) \mathfrak{t}_{\omega_{\mathrm{res}}}(\omega) e^{-i\omega v}\, \mathrm{d}\omega \right\|_{L^1_v} &\gtrsim_\delta \|(1-P_\delta)(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v} \\ &\ge \|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v} - \|P_\delta (\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v}. \end{aligned} \tag{5-195}$$

For the second term we use $\mathfrak{t} \cdot \chi_\delta \in C^\infty_c$ and obtain

$$\left\| \int_{\mathbb{R}} \mathcal{F}[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+} e^{i\omega_{\mathrm{res}}\cdot}](\omega) \chi_\delta(\omega+\omega_{\mathrm{res}}) \mathfrak{t}_{\omega_{\mathrm{res}}}(\omega) e^{-i\omega v}\, \mathrm{d}\omega \right\|_{L^1_v} \lesssim \|P_\delta (\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v}. \tag{5-196}$$

Putting everything together yields

$$\|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v} \lesssim_\delta \|\partial_v(re^{i\omega_{\mathrm{res}}r^*}\phi'_{\mathcal{L}}(u,v))\|_{L^1_v} + E_1[(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}] + \|P_\delta(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}\|_{L^1_v}. \tag{5-197}$$

This shows part (E) and concludes the proof of Theorem V. □

To connect with the nonlinear theory and the various oscillation spaces from Section 3D we state the following corollaries from Theorem V. We will also introduce a smooth positive cut-off supported only on $v \geq v_0+2$ and such that $\chi_{\geq v_0+3} = 1$ for $v \geq v_0+3$. We assume that $|\partial_v \chi_{\geq v_0+3}| \leq 2$. We also recall the notation $\psi'_{\mathcal{L}} = r\phi'_{\mathcal{L}}$.

Corollary 5.25. *Let $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ be arbitrary and define $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(v) := \chi_{\geq v_0+3}(v)\phi_{\mathcal{H}^+}(v)$, which we trivially extend for $v \leq v_0$. Let $\phi'_{\mathcal{L}}$ be the unique solution of* (5-3) *with data $(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}$ on $\mathcal{H}^+$ and no incoming data from the left event horizon. Note that by the definition of $\mathcal{SL}$ $\big($recalling $s \in \big(\frac{3}{4}, 1\big]\big)$ we have that for all $v \geq v_0$*

$$v^s(|(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}|(v) + |\partial_v(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}|(v)) \leq 4D_1. \tag{5-198}$$

(1) *If $\phi_{\mathcal{H}^+} \in \mathcal{O}$, then*

$$\sup_{v\geq v_0, u_0 \leq u_s} \left| \int_{v_0}^{v} e^{iq_0\sigma_{\mathrm{br}}(v')} e^{iq_0 \int_{v_0}^{v}(A'_{\mathrm{RN}})_v(u_0,v')\,dv'} D_v^{\mathrm{RN}} \psi'_{\mathcal{L}}(u_0,v')\,dv' \right| < +\infty \tag{5-199}$$

for all σ_{br} satisfying (3-15), (3-16).

(2) *If $\phi_{\mathcal{H}^+} \in \mathcal{O}'$, then additionally for all $u_0 \leq u_s$*

$$\lim_{v\to+\infty} \left| \int_{v_0}^{v} e^{iq_0\sigma_{\mathrm{br}}(v')} e^{iq_0 \int_{v_0}^{v}(A'_{\mathrm{RN}})_v(u_0,v')\,dv'} D_v^{\mathrm{RN}} \psi'_{\mathcal{L}}(u_0,v')\,dv' \right| \tag{5-200}$$

exists and is finite for all σ_{br} satisfying (3-15), (3-16).

(3) *If $\phi_{\mathcal{H}^+} \in \mathcal{O}''$, then additionally for all $D_{\mathrm{br}} > 0$ there exists $D' = D'(e, M, D_1, s, q_0, m^2, D_{\mathrm{br}}) > 0$ and $\tilde{\eta}_0(e, M, D_1, s, q_0, m^2, D_{\mathrm{br}}) > 0$ such that for all σ_{br} satisfying* (3-15), (3-16) *and for all $(u,v) \in \mathcal{LB}$*

$$\left| \int_{v_\gamma(u)}^{v} e^{iq_0\sigma_{\mathrm{br}}(v')} e^{iq_0 \int_{v_0}^{v}(A'_{\mathrm{RN}})_v(u,v')\,dv'} D_v^{\mathrm{RN}} \psi'_{\mathcal{L}}(u,v')\,dv' \right| \lesssim D' \cdot |u|^{s-1-\tilde{\eta}_0}. \tag{5-201}$$

(4) *Assume that $q_0 = 0$, $m^2 \notin D(M,e)$ and that $\phi_{\mathcal{H}^+} \in \mathcal{NO} = \mathcal{SL} - \mathcal{O}$. Then for all $u \in \mathbb{R}$*

$$\limsup_{v\to+\infty} |\phi'_{\mathcal{L}}|(u,v) = +\infty. \tag{5-202}$$

Remark 5.26. It should be noted that for the nonlinear problem we will impose nonzero data on $\underline{C}_{\mathrm{in}}$. For the difference estimates it however suffices if the linear data and the nonlinear data agree eventually on $\mathcal{H}^+$.

Proof. We begin by noting that $\phi_{\mathcal{H}^+} \in \mathcal{O}, \mathcal{O}', \mathcal{O}''$ if and only if $\frac{1}{4}(\phi'_{\mathcal{L}})_{|\mathcal{H}^+}(v) = \frac{1}{4}\chi_{\geq v_0+3}(v)\phi_{\mathcal{H}^+}(v) \in \mathcal{O}, \mathcal{O}', \mathcal{O}''$, respectively.[12]

[12] The factor $\frac{1}{4}$ is just to make sure that $\frac{1}{4}\chi_{\geq v_0+3}(v)\phi_{\mathcal{H}^+}(v) \in \mathcal{SL}$ if $\phi_{\mathcal{H}^+} \in \mathcal{SL}$.

Now, the first statement is a consequence of part (D) of Theorem V, the expression for the gauge derivative in (5-7) and the fact that for some bounded function $f(u)$

$$\begin{aligned} q_0 \int_{v_0}^{v} (A'_{\mathrm{RN}})_{v'}(u, v')\,\mathrm{d}v' &= -\frac{1}{2}\int_{v_0}^{v} (\omega_- - \omega_r)\,\mathrm{d}v' + \frac{1}{2}\omega_{\mathrm{res}} \cdot (v - v_0) \\ &= -\frac{1}{2}\int_{v_0}^{+\infty} (\omega_- - \omega_r)\,\mathrm{d}v' + \frac{1}{2}\int_{v}^{+\infty} (\omega_- - \omega_r)\,\mathrm{d}v' + \frac{1}{2}\omega_{\mathrm{res}} \cdot (2r^* - u - v_0) \\ &= \omega_{\mathrm{res}} r^* + f(u) + O(\Omega^2_{\mathrm{RN}}(r^*)). \end{aligned} \tag{5-203}$$

The second statement follows completely analogously. For the third statement, we use part (D) of Theorem V, and that, defining $0 < \tilde{\eta}\eta_0 = \min\{\eta_0, \frac{1}{10}(3s-4)\}$ (where η_0 is as in the definition of $\mathcal{O}''$), we have $\min(1 - s + \tilde{\eta}_0, 2s - 3) = 1 - s + \tilde{\eta}_0$ for some $\tilde{\eta}_0 > 0$ as $s > \frac{3}{4}$.

Now, we proceed to the last statement. Indeed, under the assumption $q_0 = 0$ and $m^2 \notin D(M, e)$, we have that $\mathfrak{r}(\omega = 0) \neq 0$. Thus, from Theorem V(A), and the assumption $\phi_{\mathcal{H}^+} \in \mathcal{NO}$, the claim follows. □

Moreover, we also deduce a result of $\dot{W}^{1,1}$ blow-up along outgoing cones for the linearized solution in the following sense. To state the following corollary we recall the definition of P_δ as in Section 4E.

Corollary 5.27. *Let the assumptions of Corollary 5.25 hold.*

(1) *Assume that $P_\delta(\phi_{\mathcal{H}^+}) \in L^1$ for some $\delta > 0$. Then, for all $u \le u_s$, we have*

$$\int_{v_0}^{+\infty} |\phi_{\mathcal{H}^+}|(v')\,\mathrm{d}v' \lesssim_\delta \int_{v_0}^{+\infty} |D_v^{\mathrm{RN}} \psi'_{\mathcal{L}}|(u, v)\,\mathrm{d}v + \|P_\delta(\phi_{\mathcal{H}^+})\|_{L^1_v} + D_1, \tag{5-204}$$

recalling the definition $\psi'_{\mathcal{L}} = r_{\mathrm{RN}}\phi'_{\mathcal{L}}$. In particular, if

$$\phi_{\mathcal{H}^+} \in \mathcal{SL} - L^1(\mathcal{H}^+) \quad \textit{with } P_\delta(\phi_{\mathcal{H}^+}) \in L^1(\mathbb{R}) \textit{ for some } \delta > 0, \tag{5-205}$$

then for all $u \le u_s$,

$$\int_{v_0}^{+\infty} |D_v^{\mathrm{RN}} \psi'_{\mathcal{L}}|(u, v')\,\mathrm{d}v' = +\infty. \tag{5-206}$$

Thus, the set of data $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ leading to blow-up for each $u \le u_s$ as in (5-206) *is generic in the sense that its complement H is the set $H = H_0 \cap \mathcal{SL}$ for some vector space $H_0 \subset \mathcal{SL}_0$ of infinite codimension in $\mathcal{SL}_0$, where we recall* (3-13) *for the definition of $\mathcal{SL}_0$.*

(2) *Assume $0 < |q_0 e| < \epsilon(M, e, m^2)$ or $q_0 = 0$ and $m^2 \notin D(M, e)$. Then, for all $u \le u_s$, we have*

$$\int_{v_0}^{+\infty} |\phi_{\mathcal{H}^+}|(v')\,\mathrm{d}v' \lesssim \int_{v_0}^{+\infty} |D_v^{\mathrm{RN}} \psi'_{\mathcal{L}}|(u, v')\,\mathrm{d}v' + D_0. \tag{5-207}$$

In particular, if $\phi_{\mathcal{H}^+} \in \mathcal{SL} - L^1(\mathcal{H}^+)$, then

$$\int_{v_0}^{+\infty} |D_v^{\mathrm{RN}} \psi'_{\mathcal{L}}|(u, v')\,\mathrm{d}v' = +\infty.$$

Proof. The statements follow from Theorem V(E). The genericity of $\mathcal{SL} - H$ in the first statement is a direct consequence of (5-205). We have also used that $P_\delta((\phi'_{\mathcal{L}})_{|\mathcal{H}^+}) \in L^1$ if and only if $P_\delta(\phi_{\mathcal{H}^+}) \in L^1$. □

6. Nonlinear estimates for the EMKG system and extendibility properties of the metric

We give a brief outline of Section 6:

(1) In Section 6A we recall the time-decay estimates that were established in the nonlinear setting by the second author in [Van de Moortel 2018] (see Theorem B). These estimates play a crucial role in the proof of the Cauchy horizon (in-)stability and will also be essential to the analysis of the present paper. Recall that the various gauges were defined in Sections 3 and 2C.

(2) In Sections 6B and 6B3, we provide some useful nonlinear estimates, and show how to deduce the continuous extendibility of the metric from the boundedness of the scalar field. To do so, we will in particular exploit the algebraic structure of the nonlinear terms in the Einstein equations.

(3) In Section 6C, we estimate the difference of the dynamical metric g with the Reissner–Nordström metric g_{RN} and the difference of the scalar field ϕ and its linear counterpart $\phi_{\mathcal{L}}$ ($\phi_{\mathcal{L}}$ differs from $\phi'_{\mathcal{L}}$ of Section 5 by a gauge change; see Section 6C). If $q_0 = 0$, we show that these differences are bounded, thus showing the coupled ϕ is bounded if and only if its linear counterpart $\phi_{\mathcal{L}}$ is bounded. If $q_0 \neq 0$, the estimates are more involved and include a backreaction contribution from the Maxwell field; see Section 6C4.

(4) In Section 6D, we combine the results from the linear theory (Section 5) with the results above to prove Theorems I (i) (Section 6D1), I (ii) (Section 6D2), II (Section 6D3) and III (Section 6D4).

Throughout Section 6 we will work under the assumptions of Theorem B.

6A. ***The existence of a Cauchy horizon for the EMKG system and previously proven nonlinear estimates.*** We use five different regions which partition the domain $[-\infty, u_s] \times [v_0, +\infty]$; see Figure 7. To this effect, we first introduce the function $h(v)$ as in [Van de Moortel 2018, Proposition 4.4]; namely we define $h(v)$ by the relation

$$\Omega_H^2(U=0, v) = e^{2K_+\cdot(v+h(v)-v_0)}. \tag{6-1}$$

Note that $h(v_0) = 0$ by gauges (3-7), (3-6). It is proven in [Van de Moortel 2018] that as $v \to +\infty$

$$h(v) = O(v^{2-2s})1_{s<1} + O(\log(v))1_{s=1}, \qquad h'(v) = O(v^{1-2s}), \qquad h''(v) = O(v^{-2s}). \tag{6-2}$$

Now we can introduce the five regions partitioning our spacetime $\{0 \le U \le U_s,\ v \ge v_0\}$:

(1) The event horizon $\mathcal{H}^+ = \{u = -\infty\} = \{U = 0\}$.

(2) The red-shift region $\mathcal{R} = \{u + v + h(v) \le -\Delta\}$.

(3) The no-shift region $\mathcal{N} := \{-\Delta \le u + v + h(v) \le \Delta_N\}$.

(4) The early blue-shift region

$$\mathcal{EB} := \left\{\Delta_N \le u + v + h(v) \le -\Delta' + \frac{2s}{2|K_-|}\log(v)\right\},$$

assuming that $|u_s|$is sufficiently large so that $\Delta_N + \Delta' < (2s/(2|K_-|))\log(v)$ in $\mathcal{EB}$.

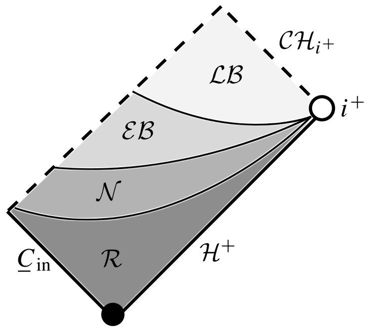

Figure 7. Division of a rectangular neighborhood of i^+ into five spacetime regions.

(5) The late blue-shift[13] region

$$\mathcal{LB} := \left\{ -\Delta' + \frac{2s}{2|K_-|} \log(v + h(v)) \le u + v + h(v) \right\}.$$

In the proof of Theorem B, it was shown that there exists a large constant $\Delta_0(M, e, q_0, m^2, s, D_1, D_2) > 0$ such that, if Δ, Δ_N, $\Delta' > \Delta_0$, the following estimates (as enumerated below) are true. In the course of the proof of the new result, we will implicitly always assume that Δ, Δ_N, $\Delta' > \Delta_0$ and choose when necessary Δ, Δ_N, $\Delta' > \Delta_1$ for some $\Delta_1(M, e, q_0, m^2, s, D_1, D_2) > \Delta_0$ that will be defined later.

Proposition 6.1 (nonlinear estimates on the event horizon $\mathcal{H}^+$ [Van de Moortel 2018]). *There exists a constant $D_H = D_H(M, e, q_0, m^2, s, D_1, D_2) > 0$ such that the following estimates hold true on $\mathcal{H}^+ = \{U = 0,\ v \ge v_0\}$:*

$$|Q(0, v) - e| \le D_H \cdot v^{1-2s}, \tag{6-3}$$

$$|\varpi(0, v) - M| \le D_H \cdot v^{1-2s}, \tag{6-4}$$

$$0 \le \lambda(0, v) \le D_H \cdot v^{-2s}, \tag{6-5}$$

$$0 \le r_+ - r(0, v) \le D_H \cdot v^{1-2s}, \tag{6-6}$$

$$|\partial_v \log(\Omega_H^2)(0, v) - 2K(0, v)| \le D_H \cdot v^{-2s}, \tag{6-7}$$

$$|2K_+ h'(v) + [2K_+ - 2K(0, v)]| \le D_H \cdot v^{-2s}, \tag{6-8}$$

$$|\partial_U \log(\Omega_H^2)|(0, v) \le D_H \cdot \Omega_H^2(0, v), \tag{6-9}$$

$$|\partial_U \phi|(0, v) \le D_H \cdot \Omega_H^2(0, v) \cdot v^{-s}, \tag{6-10}$$

$$|A_U|(0, v) \le D_H \cdot \Omega_H^2(0, v). \tag{6-11}$$

Proposition 6.2 (nonlinear estimates in the red-shift region $\mathcal{R}$ [Van de Moortel 2018]). *There exists a constant $D_R = D_R(M, e, q_0, m^2, s, D_1, D_2) > 0$ such that the following estimates hold true for all $(u, v) \in \mathcal{R}$:*

[13] Note that the late blue-shift differs slightly from [Van de Moortel 2018] where it was defined to be $\mathcal{LB} := \{-\Delta' + (2s/(2|K_-|)) \log(v) \le u + v + h(v)\}$.

$$|\phi|(u,v)+|D_v\phi|(u,v) \le D_R \cdot v^{-s}, \tag{6-12}$$

$$|D_u\phi|(u,v) \le D_R \cdot e^{2K_+\cdot(u+v+h(v))} \cdot v^{-s}, \tag{6-13}$$

$$|\log(\Omega^2(u,v)) - 2K_+\cdot(u+v+h(v))| \le D_R \cdot \Omega^2(u,v), \tag{6-14}$$

$$0 \le 1-\kappa(u,v) \le D_R \cdot \Omega^2(u,v) \cdot v^{-2s}, \tag{6-15}$$

$$|\partial_u \log \Omega^2(u,v)| \le D_R \cdot \Omega^2(u,v), \tag{6-16}$$

$$|\partial_v \log(\Omega^2)(u,v) - 2K(u,v)| \le D_R \cdot v^{-2s}, \tag{6-17}$$

$$0 \le r_+ - r(u,v) \le D_R \cdot \Omega^2(u,v) + v^{1-2s}, \tag{6-18}$$

$$|Q(u,v)-e| \le D_R \cdot v^{1-2s}, \tag{6-19}$$

$$|\varpi(u,v)-M| \le D_R \cdot v^{1-2s}, \tag{6-20}$$

$$|2K(u,v)-2K_+| \le D_R \cdot \Omega^2(u,v) + v^{1-2s}. \tag{6-21}$$

Proposition 6.3 (nonlinear estimates in the no-shift region $\mathcal{N}$ [Van de Moortel 2018]). *There exists a constant $D_N = D_N(M, e, q_0, m^2, s, D_1, D_2) > 0$ such that the following estimates hold true for all $(u,v) \in \mathcal{N}$:*

$$|\phi(u,v)| + |D_v\phi(u,v)| \le D_N \cdot v^{-s}, \tag{6-22}$$

$$|D_u\phi(u,v)| \le D_N \cdot v^{-s}, \tag{6-23}$$

$$\left|\log \Omega^2(u,v) - \log\left(-\left(1 - \frac{2M}{r(u,v)} + \frac{e^2}{r^2(u,v)}\right)\right)\right| \le D_N \cdot v^{1-2s}, \tag{6-24}$$

$$0 \le 1-\kappa(u,v) \le D_N \cdot v^{-2s}, \tag{6-25}$$

$$|1-\iota(u,v)| \le D_N \cdot v^{1-2s}, \tag{6-26}$$

$$|\partial_u \log(\Omega^2)(u,v) - 2K(u,v)| \le D_N \cdot v^{1-2s}, \tag{6-27}$$

$$|\partial_v \log(\Omega^2)(u,v) - 2K(u,v)| \le D_N \cdot v^{-2s}, \tag{6-28}$$

$$|Q(u,v)-e| \le D_N \cdot v^{1-2s}. \tag{6-29}$$

$$|\varpi(u,v)-M| \le D_N \cdot v^{1-2s}. \tag{6-30}$$

$$|\log(\Omega^2)|(u,v) + |\log(r)|(u,v) \le D_N. \tag{6-31}$$

Moreover, denoting by $\gamma_N := \{u+v+h(v) = \Delta_N\}$ the future boundary of $\mathcal{N}$, we have on γ_N

$$\Omega^2(u_\gamma(u), v) \le D_N \cdot e^{2K_-\cdot\Delta_N}. \tag{6-32}$$

Proposition 6.4 (nonlinear estimates in the early blue-shift region $\mathcal{EB}$ [Van de Moortel 2018]). *There exists a constant $D_E = D_E(M, e, q_0, m^2, s, D_1, D_2) > 0$ such that the following estimates hold true for all $(u,v) \in \mathcal{EB}$:*

$$|\phi(u,v)| \le D_E \cdot v^{-s}\log(v), \tag{6-33}$$

$$|D_v\phi(u,v)| \le D_E \cdot v^{-s}, \tag{6-34}$$

$$|D_u\phi(u,v)| \le D_E \cdot v^{-s}, \tag{6-35}$$

$$|\log \Omega^2(u,v) - 2K_- \cdot (u+v+h(v))| \le D_E \cdot \Delta \cdot e^{-2K_+\Delta} < 1, \tag{6-36}$$

$$0 \le 1-\kappa(u,v) \le \tfrac{1}{3}, \tag{6-37}$$

$$|1-\iota(u,v)| \le \tfrac{1}{3}, \tag{6-38}$$

$$|\partial_u \log(\Omega^2)(u,v) - 2K(u,v)| \le D_E \cdot v^{1-2s}\log(v)^3, \tag{6-39}$$

$$|\partial_v \log(\Omega^2)(u,v) - 2K(u,v)| \le D_E \cdot v^{-2s}\log(v)^3, \tag{6-40}$$

$$|2K(u,v) - 2K_-| \le \tfrac{1}{1000}|K_-|, \tag{6-41}$$

$$|Q(u,v)-e| \le D_E \cdot v^{1-2s}, \tag{6-42}$$

$$|\varpi(u,v)-M| \le D_E \cdot v^{1-2s}, \tag{6-43}$$

$$|r(u,v) - r_-(M,e)| \le D_E \cdot (v^{1-2s} + \Omega^2(u,v)). \tag{6-44}$$

Moreover, denoting by $\gamma := \{u+v+h(v) = -\Delta' + (s/(2|K_-|))\log(v)\}$ *the future boundary of* $\mathcal{EB}$, *we have on* γ

$$\Omega^2(u_\gamma(v), v) \le D_E \cdot v^{-2s}. \tag{6-45}$$

Proposition 6.5 (nonlinear estimates in the late blue-shift region $\mathcal{LB}$ [Van de Moortel 2018]). *There exists a constant* $D_L = D_L(M, e, q_0, m^2, s, D_1, D_2) > 0$ *such that the following estimates hold true*: *for all* $\eta > 0$, *there exists* $C_\eta > 0$ *such that for all* $(u,v) \in \mathcal{LB}$

$$\Omega^{2\eta}(u,v)|\phi|(u,v) \le C_\eta \cdot v^{-s}, \tag{6-46}$$

$$\Omega^{2\eta}(u,v)|Q-e|(u,v) \le C_\eta \cdot v^{1-2s}, \tag{6-47}$$

$$|\phi|^2(u,v) + Q^2(u,v) \le D_L \cdot v^{2-2s} 1_{\{s<1\}} + D_L \cdot [\log(v)]^2 1_{\{s=1\}}, \tag{6-48}$$

$$|D_v\phi|(u,v) \le D_L \cdot v^{-s}, \tag{6-49}$$

$$|\partial_v \log(\Omega^2_{\mathrm{CH}})|(u,v) \le D_L \cdot v^{1-2s} 1_{\{s<1\}} + D_L \cdot \log(v) \cdot v^{-1} 1_{\{s=1\}}, \tag{6-50}$$

$$0 < \Omega^2(u,v) \le -\lambda(u,v) \le D_L \cdot v^{-2s}, \tag{6-51}$$

$$0 < -\nu(u,v) \le D_L \cdot |u|^{-2s}. \tag{6-52}$$

6B. ***Nonlinear estimates exploiting the algebraic structure.*** We emphasize that we *do not necessarily assume that* $\phi_{\mathcal{H}^+} \in \mathcal{O}$ in this section. The specific assumptions of this type are made in Section 6D only. In fact, we use many of these estimates in our companion paper [Kehle and Van de Moortel ≥ 2024] as well (where it is assumed that $\phi_{\mathcal{H}^+} \notin \mathcal{O}$). Throughout Sections 6B–6D we use the notation $|f(u,v)| \lesssim |g(u,v)|$ if there exists a constant $\Gamma(M, e, m^2, q_0, D_1, D_2, s) > 0$ such that $|f(u,v)| \le \Gamma \cdot |g(u,v)|$ for all (u,v) in the spacetime region of interest.

6B1. *Boundedness and continuous extendibility of* $D_u\psi$. To reach the goals of this section, we must first prove preliminary estimates on $D_u\psi$, where $\psi := r\phi$ is (what is called in the black hole exterior) the radiation field. Since r is upper and lower bounded in our region of interest, it may be very surprising to consider this quantity in the black hole *interior*. However, as it turns out, $D_u\psi$ is always bounded, while $D_u\phi$ is bounded if and only if ϕ is (providing $\liminf_{v\to+\infty} |\nu|(u,v) > 0$, which is conjecturally a generic condition; see [Van de Moortel 2021] for a discussion and proof of this result).

Proposition 6.6. *We have the following (gauge-independent) estimate for all* $(u, v) \in \mathcal{LB}$:

$$|D_u\psi|(u, v) \lesssim |u|^{-s}. \tag{6-53}$$

Moreover, in the gauge (2-26), *both* $D_u\psi$ *and* A_u *admit a bounded extension to the Cauchy horizon, denoted by* $(D_u\psi)_{\mathrm{CH}}$ *and* $(A_u)^{\mathrm{CH}}$, *respectively.*

Proof. Using (2-45) with the estimates of Proposition 6.5, we have

$$|\partial_v(D_u\psi)| \lesssim |\lambda| \cdot |\nu| \cdot |\phi| + \Omega^{1.99} \cdot v^{-s}.$$

Finally with (6-51) and (6-48) we get

$$|\partial_v(D_u\psi)| \lesssim v^{1-3s} \cdot |u|^{-2s} + \Omega^{1.99} \cdot v^{-s}.$$

Now the left-hand side is integrable in v since $s > \frac{2}{3}$ so $D_u\psi$ admits a bounded extension by integrability and integrating from γ we obtain the estimate, in view of the estimate on γ from Proposition 6.4. To conclude, the extendibility of A_u follows from (2-39) and the estimates of Proposition 6.5 that show that $|\partial_v A_u|$ is integrable in v. □

6B2. *Key estimates for a candidate coordinate system* (u, V) *for a continuous extension.* In this section, we construct an adequate coordinate system (u, V), in which the boundedness of the metric coefficient $\log(\Omega^2_{\mathrm{CH}})$ related to (u, V) by $\Omega^2_{\mathrm{CH}} = -2g(\partial_u, \partial_V)$ follows from the boundedness of the scalar field ϕ.

Proposition 6.7. *There exists a coordinate system* (u, V) *for which* $V(v) < 1$, *and* $\lim_{v\to+\infty} V(v) = 1$ *and for which, defining the metric coefficient* $\Omega^2_{\mathrm{CH}}\, \mathrm{d}u\, \mathrm{d}V = \Omega^2\, \mathrm{d}u\, \mathrm{d}v$, *we have for all* $(u, v) \in \mathcal{LB}$:

$$\left|\partial_v\left(\log(\Omega^2_{\mathrm{CH}})(u, v) + |\phi|^2(u, v) + \int_u^{u_s} \frac{|\nu|}{r}|\phi|^2(u', v)\, \mathrm{d}u'\right)\right| \lesssim v^{2-4s} + v^{-2s}|\log(v)|^3, \tag{6-54}$$

and

$$\left|\partial_v\partial_u\left(\log(\Omega^2_{\mathrm{CH}})(u, v) + |\phi|^2(u, v) + \int_u^{u_s} \frac{|\nu|}{r}|\phi|^2(u', v)\, \mathrm{d}u'\right)\right| \lesssim |u|^{-2s} \cdot (v^{2-4s} + v^{-2s}|\log(v)|^3) + |u|^{-s} \cdot v^{1-3s}. \tag{6-55}$$

As a consequence, the quantity Υ *defined as*

$$\Upsilon(u, v) := \log(\Omega^2_{\mathrm{CH}}) + |\phi|^2 + \int_u^{u_s} \frac{|\nu|}{r}|\phi|^2\, \mathrm{d}u' \tag{6-56}$$

admits a continuous extension $\Upsilon_{\mathrm{CH}}(u)$ *across* $\mathcal{CH}_{i^+}$ *and*

$$\partial_u\Upsilon = \partial_u\left(\log(\Omega^2_{\mathrm{CH}}) + |\phi|^2 + \int_u^{u_s} \frac{|\nu|}{r}|\phi|^2\, \mathrm{d}u'\right) \tag{6-57}$$

admits a bounded extension across $\mathcal{CH}_{i^+}$.

Proof. We first use (2-43) to establish the two formulae

$$\frac{\partial_u\partial_v(r|\phi|^2)}{r} = \partial_u\partial_v(|\phi|^2) + \frac{\nu}{r}\,\partial_v(|\phi|^2) + \frac{1}{r}\partial_u(\lambda|\phi|^2),$$

$$-2\Re(D_u\phi\overline{\partial_v\phi}) = \frac{-\partial_u\partial_v(r|\phi|^2)}{r} + \left(\frac{\partial_u\partial_v r}{r} - \frac{m^2\Omega^2}{2}\right)|\phi|^2.$$

Now we define $2K_\gamma(v) := 2K(u_{\gamma(v)}, v)$ and we rewrite (2-33) using the two last formulae

$$\left|\partial_u(\partial_v \log(\Omega^2) - 2K_\gamma(v) + \partial_v(|\phi|^2)) + \frac{\nu}{r}\,\partial_v(|\phi|^2) + \frac{1}{r}\,\partial_u(\lambda|\phi|^2)\right| \lesssim |\lambda\nu|(1+|\phi|^2) + \Omega^2(1+Q^2+m^2|\phi|^2).$$

First note that the right-hand side is $O(|u|^{-2s}\cdot v^{2-4s} + |u|^{-2s}\cdot v^{-2s})$, using the estimates of Proposition 6.5. Using (2-32), (6-53) and the other estimates of Proposition 6.5 we get

$$|\partial_u(\lambda|\phi|^2)| = |\partial_u(r^{-2}\lambda|\psi|^2)| \lesssim |u|^{-2s}v^{2-4s} + |u|^{-s}\cdot v^{1-3s}.$$

This gives

$$\left|\partial_u(\partial_v \log(\Omega^2) - 2K_\gamma(v) + \partial_v(|\phi|^2)) + \frac{\nu}{r}\,\partial_v(|\phi|^2)\right| \lesssim |u|^{-2s}\cdot v^{-2s} + |u|^{-2s}v^{2-4s} + |u|^{-s}\cdot v^{1-3s}. \tag{6-58}$$

Now we want to integrate both sides on $[u_\gamma(v), u]$. Recall that on γ, $|\partial_v \log(\Omega^2)(u_\gamma(v), v) - 2K_\gamma(v)| \lesssim v^{-2s}|\log(v)|^3$ and $|\partial_v(\phi^2)| \lesssim v^{-2s}|\log(v)|$, as established in Proposition 6.4. Thus, we obtain

$$\left|\partial_v \log(\Omega^2) - 2K_\gamma(v) + \partial_v(|\phi|^2) + \int_{u_{\gamma(v)}}^{u} \frac{\nu}{r}\,\partial_v(|\phi|^2)\,du'\right| \lesssim v^{2-4s} + v^{-2s}|\log(v)|^3. \tag{6-59}$$

Now we write

$$\int_{u_{\gamma(v)}}^{u} \frac{\nu}{r}\,\partial_v(|\phi|^2)\,du' = \int_{u_{\gamma(v)}}^{u_s} \frac{\nu}{r}\,\partial_v(|\phi|^2)\,du' - \partial_v\left(\int_u^{u_s} \frac{\nu}{r}|\phi|^2\,du'\right) + \int_u^{u_s} \partial_v\left(\frac{\nu}{r}\right)|\phi|^2\,du'.$$

Using (2-32) and the estimates of Proposition 6.5 again, we see that

$$\left|\int_u^{u_s} \partial_v\left(\frac{\nu}{r}\right)|\phi|^2\,du'\right| \lesssim \int_u^{u_s} (|\nu|\,|\lambda| + \Omega^2(1+Q^2+|\phi|^2))|\phi|^2\,du' \lesssim v^{2-4s}.$$

Therefore we actually showed that

$$\left|\partial_v \log(\Omega^2) - 2K_\gamma(v) + \int_{u_\gamma(v)}^{u_s} \frac{\nu}{r}\,\partial_v(|\phi|^2)\,du' + \partial_v(|\phi|^2) - \partial_v\left(\int_u^{u_s} \frac{\nu}{r}|\phi|^2\,du'\right)\right| \lesssim v^{2-4s} + v^{-2s}|\log(v)|^3. \tag{6-60}$$

Note that the second and the third terms of the left-hand-side only depend on v and not on u.

We define a new coordinate system (u, V) with the equations

$$\frac{dV}{dv} = e^{f(v)}, \tag{6-61}$$

$$f'(v) = 2K_\gamma(v) + \int_{u_{\gamma(v)}}^{u_s} \frac{|\nu|}{r}\,\partial_v(|\phi|^2)(u', v)\,du'. \tag{6-62}$$

By the estimates of Proposition 6.5, note that $|f'(v) - 2K_-| \lesssim v^{1-2s}$ and we recall that $K_- < 0$; thus $V'(v)$ is integrable as $v \to +\infty$, and $V(v)$ increases towards a limit V_∞ which we can choose to be 1 without loss of generality. Therefore, we also have upon integration, as $v \to +\infty$:

$$1 - V(v) \approx e^{f(v)}.$$

We also denote by Ω^2_{CH} the metric coefficient in this system defined by $\Omega^2_{\mathrm{CH}} = -2g(\partial_u, \partial_V)$, i.e.,

$$\Omega^2_{\mathrm{CH}}\,\mathrm{d}u\,\mathrm{d}V = \Omega^2\,\mathrm{d}u\,\mathrm{d}v, \quad \text{hence} \quad \Omega^2_{\mathrm{CH}}(u,v) = \Omega^2(u,v)e^{-f(v)}.$$

We then have the claimed estimate (6-54)

$$\left|\partial_v\left(\log(\Omega^2_{\mathrm{CH}}) + |\phi|^2 + \int_u^{u_s} \frac{|\nu|}{r}|\phi|^2\,\mathrm{d}u'\right)\right| \lesssim v^{2-4s} + v^{-2s}|\log(v)|^3.$$

Clearly, (6-58) is a reformulation of (6-55). Since the right-hand sides of (6-54) and (6-55) are integrable in v for $s > \frac{3}{4}$, a standard Cauchy sequence argument shows that $\Upsilon(u,v)$ admits a continuous extension, and $\partial_u\Upsilon(u,v)$ has a (locally) bounded extension. □

6B3. *Metric extendibility conditional on the boundedness of the scalar field.* Now that we have built the quantity Υ and proven its extendibility, we will prove that the continuous extendibility of $|\phi|$ implies the continuous extendibility of the metric (conversely, the blow-up of $|\phi|$ implies that there exists no coordinate system (u,v) in which $\log(\Omega^2)$ is even bounded; see [Kehle and Van de Moortel ≥ 2024; Van de Moortel 2019]).

Lemma 6.8. *Assume that the function $(u,v) \in \mathcal{LB} \to |\phi|(u,v)$ extends continuously to $\mathcal{CH}_{i^+} \cap \{u \le u_s\}$ as a continuous function $|\phi|_{\mathrm{CH}}(u)$. Then $\int_u^{u_s} (\nu/r)|\phi|^2(u',V)\,\mathrm{d}u'$ extends continuously to $\mathcal{CH}_{i^+} \cap \{u \le u_s\}$ as a continuous function. Moreover, $\nu(u,v)$ extends to $\mathcal{CH}_{i^+} \cap \{u \le u_s\}$ as a bounded function $\nu_{\mathrm{CH}}(u)$.*

Remark 6.9. In fact, we do not prove directly that ν extends as continuous function across the Cauchy horizon, as we do not control $\partial_u\nu$. However, even though ν_{CH} might not be continuous in u, it is clearly in L^1_{loc} (and even in $L^1(\mathcal{CH}_{i^+} \cap \{u \le u_s\})$, as $|\nu_{\mathrm{CH}}| \lesssim |u|^{-2s}$) which is sufficient for our purpose.

Proof. Using the estimates of Proposition 6.5, we see that for $(u,v) \in \mathcal{LB}$

$$|\partial_v\nu|(u,v) \lesssim v^{-2s},$$

which shows, by integrability, that for all $u \le u_s$ there exists $\nu_{\mathrm{CH}}(u)$ such that $\lim_{v\to+\infty} \nu(u,v) = \nu_{\mathrm{CH}}(u)$. Now take again $u_\infty < u_s$ and two sequences $u_i \to u_\infty$, $V_i \to 1$, $V_i < 1$ and write

$$\begin{aligned}
&\left|\int_{u_i}^{u_s} \frac{\nu}{r}|\phi|^2(u',V_i)\,\mathrm{d}u' - \int_{u_i}^{u_s} \frac{\nu_{\mathrm{CH}}(u')}{r_{\mathrm{CH}}(u')}|\phi|^2_{\mathrm{CH}}(u')\,\mathrm{d}u'\right| \\
&\quad \le \left|\int_{u_i}^{u_\infty} \frac{\nu}{r}|\phi|^2(u',V_i)\,\mathrm{d}u'\right| + \left|\int_{u_i}^{u_\infty} \frac{\nu_{\mathrm{CH}}(u')}{r_{\mathrm{CH}}(u')}|\phi|^2_{\mathrm{CH}}(u')\,\mathrm{d}u'\right| \\
&\qquad\qquad + \left|\int_{u_\infty}^{u_s} \left(\frac{\nu}{r}|\phi|^2(u',V_i) - \frac{\nu_{\mathrm{CH}}(u')}{r_{\mathrm{CH}}(u')}|\phi|^2_{\mathrm{CH}}(u')\right)\mathrm{d}u'\right|.
\end{aligned}$$

Now both functions $(\nu/r)|\phi|^2(u,V)$ and $(\nu_{\mathrm{CH}}(u')/r_{\mathrm{CH}}(u'))|\phi|^2_{\mathrm{CH}}(u)$ are uniformly bounded in u and v on a set of the form $(u,V) \in [u_\infty - \epsilon, u_s] \times [1-\epsilon, 1]$ and

$$\lim_{i\to+\infty} \frac{\nu}{r}|\phi|^2(u',V_i) = \frac{\nu_{\mathrm{CH}}(u')}{r_{\mathrm{CH}}(u')}|\phi|^2_{\mathrm{CH}}(u'),$$

so by the dominated convergence theorem, the last term tends to 0 as i tends to $+\infty$.

Moreover, the integrands of the first two terms are uniformly bounded, and thus these two terms tend to 0 as i tends to $+\infty$. This concludes the proof of the lemma. □

Corollary 6.10. *Assume that the function* $(u, v) \in \mathcal{LB} \to |\phi|(u, v)$ *extends continuously to* $\mathcal{CH}_{i^+} \cap \{u \le u_s\}$ *as a continuous function* $|\phi|_{\mathrm{CH}}(u)$. *Then the metric* g *admits a continuous extension* $\tilde{g}$, *which can be chosen to be* C^0*-admissible* (*Definition 2.1*).

Proof. It follows from Proposition 6.7 and Lemma 6.8 that Ω^2_{CH} extends continuously to $\mathcal{CH}_{i^+} \cap \{u \le u_s\} = \{u \le u_s\} \times \{V = 1\}$. We know already that r extends continuously to $\mathcal{CH}_{i^+} \cap \{u \le u_s\} = \{u \le u_s\} \times \{V = 1\}$; therefore, in view of the form of the metric (2-13), the corollary is proved. □

6C. ***Difference-type estimates on the scalar field and metric difference estimates.*** In this section, we carry out the nonlinear difference estimates. To do this, we have to introduce a new coordinate involving $h(v)$ defined in (6-1) (see the difference estimate (6-64), to compare with (6-7)):

$$\tilde{v}(v) := v + h(v). \tag{6-63}$$

Recalling (6-2), it is clear that $\tilde{v} = v \cdot (1 + O(v^{1-2s}))$ and $\partial_{\tilde{v}} f = \partial_v f \cdot (1 + O(v^{1-2s}))$ for all f. Note also

$$\widetilde{\Omega}^2(u, \tilde{v}(v)) = \frac{\Omega^2(u, v)}{1 + h'(v)} = (1 + O(v^{1-2s})) \cdot \Omega^2(u, v),$$

$$\partial_{\tilde{v}} \log(\widetilde{\Omega}^2)(u, \tilde{v}(v)) = \frac{\partial_v \log(\Omega^2)(u, v)}{1 + h'(v)} - \frac{h''(v)}{[1 + h'(v)]^2} = (1 + O(v^{1-2s})) \cdot \partial_v \log(\Omega^2)(u, v) + O(v^{-2s}),$$

where $\widetilde{\Omega}^2 := -2g(\partial_u, \partial_{\tilde{v}})$. Estimates from Section 6A can be easily translated into $(u, \tilde{v})$-coordinates:

Lemma 6.11. *Defining* $\widetilde{\Omega}^2_H := -2g(\partial_U, \partial_{\tilde{v}})$, *the estimate* (6-7) *on* $\mathcal{H}^+$ *is replaced by*

$$\left|\log\left(\frac{\widetilde{\Omega}^2(0, \tilde{v})}{\Omega^2_{\mathrm{RN}}(0, \tilde{v})}\right)\right| = \left|\log\left(\frac{\widetilde{\Omega}^2_H(0, \tilde{v})}{(\Omega^2_{\mathrm{RN}})_H(0, \tilde{v})}\right)\right| \lesssim \tilde{v}^{1-2s}, \quad |\partial_{\tilde{v}} \log(\widetilde{\Omega}^2_H)(0, \tilde{v}) - 2K_+| \lesssim \tilde{v}^{-2s}. \tag{6-64}$$

Moreover, (6-14), (6-17) *are replaced by the following estimates valid in the spacetime region* $\mathcal{R}$:

$$e^{2K_+(u+\tilde{v})} \lesssim \widetilde{\Omega}^2(U, \tilde{v}) \lesssim e^{2K_+(u+\tilde{v})}, \quad |\partial_{\tilde{v}} \log(\widetilde{\Omega}^2)(U, \tilde{v}) - 2K_+| \lesssim \tilde{v}^{-2s}. \tag{6-65}$$

Finally, (6-28) *and* (6-40) *are replaced by the following* (*weaker*) *estimates in the regions* $\mathcal{N} \cup \mathcal{EB}$:

$$|\partial_{\tilde{v}} \log(\widetilde{\Omega}^2)(U, \tilde{v}) - 2K(U, \tilde{v})| \lesssim \tilde{v}^{1-2s}. \tag{6-66}$$

All the others estimates of Section 6A are still valid replacing v *by* $\tilde{v}$, Ω^2 *by* $\widetilde{\Omega}^2$ *and so on* (*adjusting the constants with no loss of generality*, *i.e.*, *replacing* D_H *by* $2D_H$, D_R *by* $2D_R$, $\frac{1}{3}$ *by* $\frac{2}{3}$ *etc.*).

Proof. This follows from the equation $\widetilde{\Omega}^2_H(0, \tilde{v}) = e^{2K_+ \cdot (\tilde{v} - v_0)}/(1 + h'(v))$ (using the identity (6-1)) and (6-8), (6-2). □

Notation. In view of Lemma 6.11, from now on and until the end of the paper, we make a mild abuse of notation and redefine v to be this new $\tilde{v}$ given by (6-63) with the necessary adjustments, i.e., λ becomes the notation for $\partial_{\tilde{v}} r$, Ω^2 the notation for $-2g(\partial_u, \partial_{\tilde{v}})$, etc. We will not use the old definition of v any longer in what follows.

The goal of this section is to take the difference between $\phi(u, v)$ and $\phi_{\mathcal{L}}(u, v)$ and estimate the quantity

$$\delta\phi(u, v) := \phi(u, v) - \phi_{\mathcal{L}}(u, v), \tag{6-67}$$

where $\phi_{\mathcal{L}}$ solves the linear equation

$$(\nabla_\mu + iq_0(A^{\mathrm{RN}})_\mu)(\nabla^\mu + iq_0(A^{\mathrm{RN}})^\mu)\phi_{\mathcal{L}} - m^2\phi_{\mathcal{L}} = 0 \tag{6-68}$$

on the fixed Reissner–Nordström background (2-7) in the gauge A^{RN} as in (2-30). More precisely, we will define data for $\phi_{\mathcal{L}}$ on $\mathcal{H}^+$ (data on $\underline{C}_{\mathrm{in}}$ is irrelevant) so as to match the data $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ for ϕ on $\mathcal{H}^+$ (see the paragraph immediately below): Our goal is then to prove that $\delta\phi$ is bounded and continuously extendible (for $q_0 = 0$), and similar estimates featuring nonlinear backreaction if $q_0 \neq 0$.

We now define $\phi_{\mathcal{L}}$ on $\mathcal{Q}^+$ as the unique solution of (6-68) on the fixed Reissner–Nordström metric (2-7) with parameters (M, e) and with data

$$\begin{aligned} \phi_{\mathcal{L}}(u, v_0) &\equiv 0 && \text{for all } u \in (-\infty, u_s], \\ (\phi_{\mathcal{L}})_{|\mathcal{H}^+}(v) &\equiv \chi_{\geq v_0+3}(v)\phi_{\mathcal{H}^+}(v) && \text{for all } v \in [v_0, +\infty), \end{aligned}$$

where $\chi_{v_0+\geq 3}$ is the smooth cut-off supported on $v \geq v_0 + 2$ and $\chi_{\geq v_0+3} = 1$ for $v \geq v_0 + 3$ as defined in Corollary 5.25.

Remark 6.12. Note that the unique solution $\phi'_{\mathcal{L}}$ arising from the above data in the gauge (2-31), which is used in Section 5, agrees with $\phi_{\mathcal{L}}$ up to a gauge transformation as the gauges agree for the initial data, in particular, $A_v^{\mathrm{RN}} = (A'_{\mathrm{RN}})_v = 0$ on the event horizon by construction.

Recall that $\phi_{\mathcal{L}}$ is also a solution of (2-36), (2-43), (2-45), (2-44) where $(r, \Omega^2, A, D, \phi)$ are all replaced by their Reissner–Nordström analogs $(r_{\mathrm{RN}}, \Omega^2_{\mathrm{RN}}, A^{\mathrm{RN}}, D^{\mathrm{RN}}, \phi_{\mathcal{L}})$. Similarly, $r_{\mathrm{RN}}, \Omega^2_{\mathrm{RN}}, A^{\mathrm{RN}}$ also satisfy the equations of Section 2D with $\phi \equiv 0$ (i.e., (2-7) satisfies the Einstein–Maxwell equations in spherical symmetry), a fact we will repetitively use.

The estimates of [Van de Moortel 2018], that are recalled in Section 6A and stated in Lemma 6.11 in our new coordinate system, are key to our new difference estimates. We will use these estimates throughout the argument, without necessarily referring to them explicitly.

6C1. *Difference estimates in the red-shift region.*

Proposition 6.13. *There exists $D'_H(M, e, q_0, m^2, s, D_1, D_2) > 0$ such that for all $(u, v) \in \mathcal{R}$*

$$\begin{aligned} |r(u, v) - r_{\mathrm{RN}}(u, v)| + |\lambda(u, v) - \lambda_{\mathrm{RN}}(u, v)| \\ + |Q(u, v) - e| + |\log(\Omega^2)(u, v) - \log(\Omega^2_{\mathrm{RN}})(u, v)| \leq D'_H \cdot v^{1-2s}, \end{aligned} \tag{6-69}$$

$$\begin{aligned} |\partial_u \log(\Omega^2)(u, v) - \partial_u \log(\Omega^2_{\mathrm{RN}})(u, v)| + |\nu(u, v) - \nu_{\mathrm{RN}}(u, v)| \\ + |A_u(u, v) - A_u^{\mathrm{RN}}(u, v)| \leq D'_H \cdot e^{2K_+(u+v)} \cdot v^{1-2s}, \end{aligned} \tag{6-70}$$

$$|\partial_u \delta\phi| \leq D'_H \cdot e^{2K_+(u+v)} \cdot v^{1-3s}, \tag{6-71}$$

$$|\delta\phi| + |\partial_v \delta\phi| \leq D'_H \cdot v^{1-3s}. \tag{6-72}$$

Proof. First, recall that $r_{\mathrm{RN}} \equiv r_+(M, e)$, $Q_{\mathrm{RN}} \equiv e$, $\varpi_{\mathrm{RN}} \equiv M$, and $\lambda_{\mathrm{RN}} \equiv 0$ on the event horizon $\mathcal{H}^+$, by definition. Lastly, recall that $A_u = A_u^{\mathrm{RN}}$ on $\underline{C}_{\mathrm{in}}$ by the gauge choice (3-5). Recalling that $D_H > 0$ is defined in Proposition 6.1, we bootstrap the estimates

$$|r(u, v) - r_{\mathrm{RN}}(u, v)| \leq 4D_H \cdot v^{1-2s}, \tag{6-73}$$

$$|\log(\Omega^2)(u, v) - \log(\Omega^2_{\mathrm{RN}})(u, v)| \leq 4B_H \cdot v^{1-2s} \tag{6-74}$$

for $B_H(M, e, q_0, m^2, D_1, D_2) > 0$ defined as the constant in (6-64) such that

$$\left|\log\left(\frac{\Omega^2(0, v)}{\Omega^2_{\mathrm{RN}}(0, v)}\right)\right| \leq B_H \cdot v^{1-2s}$$

in the new coordinate v. Plugging these bootstraps into (2-42) and using (6-10), (6-65), we find that

$$\begin{aligned}|\partial_u(r\,\partial_v r - r_{\mathrm{RN}}\,\partial_v r_{\mathrm{RN}})| = |\partial_u(r\lambda - r_{\mathrm{RN}}\lambda_{\mathrm{RN}})| &\lesssim |\Omega^2 - \Omega^2_{\mathrm{RN}}| + \Omega^2 \cdot (|\phi|^2 + |r - r_{\mathrm{RN}}| + |Q - e|)\\ &\lesssim e^{2K_+(u+v)} \cdot v^{1-2s},\end{aligned}$$

where we used

$$|\Omega^2 - \Omega^2_{\mathrm{RN}}| \lesssim \Omega^2 \cdot \left|\log\left(\frac{\Omega^2}{\Omega^2_{\mathrm{RN}}}\right)\right| \lesssim \Omega^2 \cdot v^{1-2s}.$$

This is also equivalent (recalling (3-12)) to

$$|\partial_U(r\lambda - r_{\mathrm{RN}}\lambda_{\mathrm{RN}})| \lesssim e^{2K_+ v} \cdot v^{1-2s}.$$

Integrating the above using (6-5) we get

$$|r\lambda - r\lambda_{\mathrm{RN}}| \lesssim v^{-2s} + \Omega^2 \cdot v^{1-2s}. \tag{6-75}$$

Writing now the difference for (2-32), taking advantage of (6-75) and the bootstraps gives

$$|\partial_v \partial_U(r - r_{\mathrm{RN}})| \lesssim |\lambda_{\mathrm{RN}}| \cdot |\partial_U r - \partial_U r_{\mathrm{RN}}| + e^{2K_+ v} v^{-2s} + e^{2K_+ v} v^{1-2s}.$$

Integrating in v using a Gronwall estimate and the boundedness of $\partial_U r$ on $\underline{C}_{\mathrm{in}}$ we get

$$|\partial_U r - \partial_U r_{\mathrm{RN}}| \lesssim 1 + e^{2K_+ v} \cdot v^{1-2s},$$

which, upon integrating in U this time and using (6-6) gives

$$|r - r_{\mathrm{RN}}| \leq D_H \cdot v^{1-2s} + D \cdot e^{2K_+(u+v)} e^{-2K_+ v} + D \cdot e^{2K_+(u+v)} \cdot v^{1-2s},$$

where $D(M, e, q_0, m^2, D_1, D_2) > 0$. Choosing Δ sufficiently large such that

$$e^{2K_+(u+v)} \leq e^{-2K_+\Delta} < D^{-1} \cdot D_H$$

allows us to retrieve bootstrap (6-73).

Similarly plugging (6-12), (6-13), (6-74) and the previously proven estimates into (2-33) we get

$$|\partial_u \partial_v(\log(\Omega^2) - \log(\Omega^2_{\mathrm{RN}}))| \lesssim |D_u\phi| \cdot |\partial_v\phi| + |\Omega^2 - \Omega^2_{\mathrm{RN}}| + \Omega^2 \cdot v^{1-2s} \lesssim \Omega^2 \cdot v^{1-2s},$$

or equivalently using (6-65)

$$|\partial_v \partial_U (\log(\Omega^2) - \log(\Omega^2_{\mathrm{RN}}))| \lesssim e^{2K_+ v} \cdot v^{1-2s}.$$

Integrating in v using the boundedness of $\partial_U \log(\Omega^2)$ and $\partial_U \log(\Omega^2_{\mathrm{RN}})$ on $\underline{C}_{\mathrm{in}}$ we get

$$|\partial_U (\log(\Omega^2) - \log(\Omega^2_{\mathrm{RN}}))| \lesssim 1 + e^{2K_+ v} \cdot v^{1-2s} \lesssim e^{2K_+ v} \cdot v^{1-2s},$$

from which we retrieve bootstrap (6-74), using the smallness of $e^{-2K_+\Delta}$ as we did above.

Now all bootstraps are closed and we continue with the proof of the claimed difference estimates. Taking the difference between (2-39) and its Reissner–Nordström version, and integrating in v using $A_u(u, v_0) - A_u^{\mathrm{RN}}(u, v_0) = 0$, we obtain

$$|A_u(u, v) - A_u^{\mathrm{RN}}(u, v)| \lesssim e^{2K_+ \cdot (u+v)} \cdot v^{1-2s}.$$

For $\delta\phi$, we introduce a new bootstrap assumption (completely independently from the other bootstrap assumptions that have already been retrieved), which is true on $\underline{C}_{\mathrm{in}}$ by assumption:

$$|\partial_u \delta\phi|(u, v) \le B_1 \cdot e^{2K_+(u+v)} \cdot v^{1-3s} \tag{6-76}$$

for some $B_1 > 0$ large enough to be chosen later. Integrating in u and using $|\delta\phi| \lesssim v^{1-3s}$ on the event horizon $\mathcal{H}^+$ (since $\delta\phi_{|\mathcal{H}^+} \equiv 0$ for $v \ge 3$) gives

$$|\delta\phi|(u, v) \lesssim (1 + B_1) \cdot e^{2K_+(u+v)} \cdot v^{1-3s} \lesssim (1 + B_1) \cdot e^{-2K_+\Delta} \cdot v^{1-3s}. \tag{6-77}$$

Now we take the difference of (2-36) obeyed by ϕ and the corresponding equation obeyed by $\phi_{\mathcal{L}}$, namely

$$\begin{aligned}\partial_u\partial_v(\delta\phi) = &-\frac{\partial_u\delta\phi\,\partial_v r}{r} - \frac{\partial_v\delta\phi\,\partial_u r}{r} + \frac{q_0 i\Omega^2}{4r^2} Q\,\delta\phi - \frac{m^2\Omega^2}{4}\delta\phi - iq_0 A_u \frac{\delta\phi\,\partial_v r}{r} - iq_0 A_u\,\partial_v\delta\phi\\ &- \partial_u\phi_{\mathcal{L}}\left[\frac{\partial_v r}{r} - \frac{\partial_v r_{\mathrm{RN}}}{r_{\mathrm{RN}}}\right] - \partial_v\phi_{\mathcal{L}}\left[\frac{\partial_u r}{r} - \frac{\partial_u r_{\mathrm{RN}}}{r_{\mathrm{RN}}}\right] + \left[\frac{q_0 i\Omega^2}{4r^2} Q - \frac{q_0 i\Omega^2_{\mathrm{RN}}}{4r^2_{\mathrm{RN}}} e\right]\phi_{\mathcal{L}}\\ &- \frac{m^2[\Omega^2 - \Omega^2_{\mathrm{RN}}]}{4}\phi_{\mathcal{L}} - iq_0\left[A_u\frac{\partial_v r}{r} - A_u^{\mathrm{RN}}\frac{\partial_v r_{\mathrm{RN}}}{r_{\mathrm{RN}}}\right]\phi_{\mathcal{L}} - iq_0[A_u - A_u^{\mathrm{RN}}]\partial_v\phi_{\mathcal{L}}.\end{aligned}$$

We get, using also (6-76), (6-77) and (6-65) (note that one can write $\phi_{\mathcal{L}} = \phi - \delta\phi$ and use (6-12), (6-13) to bound ϕ and (6-76), (6-77) to bound $\phi_{\mathcal{L}}$)

$$|\partial_u\partial_v\delta\phi| \lesssim e^{2K_+(u+v)} \cdot (1 + B_1) \cdot v^{1-3s} + v^{-s} \cdot e^{2K_+(u+v)} \cdot |\partial_v\delta\phi|. \tag{6-78}$$

Integrating in u and using Gronwall's estimate we get (recalling that $|\partial_v\delta\phi| \lesssim v^{1-3s}$ on $\mathcal{H}^+$) we get

$$|\partial_v\delta\phi| \lesssim (1 + B_1 \cdot e^{-2K_+\Delta}) \cdot v^{1-3s},$$

and using this in (6-78) we get

$$|\partial_v\partial_u\delta\phi| \lesssim e^{2K_+(u+v)} \cdot (1 + B_1) \cdot v^{1-3s} + B_1 \cdot e^{-2K_+\Delta} \cdot v^{1-4s} \cdot e^{2K_+(u+v)}.$$

Integrating in v this time, choosing B_1 appropriately and using the smallness of $e^{-2K_+\Delta}$, retrieves, together with another integration in u, bootstrap (6-76), gives the claimed estimates on $\delta\phi$ and concludes the proof. $\square$

6C2. *Difference estimates in the no-shift region.*

Proposition 6.14. *There exists $C_N = C_N(M, e, q_0, m^2, s, D_1, D_2) > 0$ such that the following estimates are satisfied for all $(u, v) \in \mathcal{N}$:*

$$|r(u,v) - r_{\mathrm{RN}}(u,v)| + |Q(u,v) - e| + |\log(\Omega^2)(u,v) - \log(\Omega^2_{\mathrm{RN}})(u,v)| \le C_N \cdot v^{1-2s},$$

$$\begin{aligned}|\partial_v \log(\Omega^2)(u,v) - \partial_v \log(\Omega^2_{\mathrm{RN}})(u,v)| + |\partial_u \log(\Omega^2)(u,v) - \partial_u \log(\Omega^2_{\mathrm{RN}})(u,v)| \\ + |\lambda(u,v) - \lambda_{\mathrm{RN}}(u,v)| + |\nu(u,v) - \nu_{\mathrm{RN}}(u,v)| + |A_u(u,v) - A_u^{\mathrm{RN}}(u,v)| \le C_N \cdot v^{1-2s},\end{aligned}$$

$$|\delta\phi| + |\partial_u \delta\phi| + |\partial_v \delta\phi| \le C_N \cdot v^{1-3s}.$$

Proof. The proof consists of combination of the proof of Proposition 6.13 with that of in [Van de Moortel 2018, Proposition 4.7]: we partition $\mathcal{N}$ into smaller regions $\mathcal{N}_k := \{-\Delta + (k-1)\epsilon \le u + v \le -\Delta + k\epsilon\}$ for $k \in [\![1, N]\!]$ and $N \cdot \epsilon = \Delta'$. We will prove the result by finite induction on k. The induction hypothesis is that the following estimates hold in $\mathcal{N}_k$:

$$|r(u,v) - r_{\mathrm{RN}}(u,v)| + |\lambda(u,v) - \lambda_{\mathrm{RN}}(u,v)| + |\log(\Omega^2)(u,v) - \log(\Omega^2_{\mathrm{RN}})(u,v)| \le C_k \cdot v^{1-2s}, \quad (6\text{-}79)$$

$$|\nu(u,v) - \nu_{\mathrm{RN}}(u,v)| + |A_u(u,v) - A_u^{\mathrm{RN}}(u,v)| \le C_k \cdot v^{1-2s}, \quad (6\text{-}80)$$

where $C_k = 2^k \cdot B_N$ for a large enough constant $B_N > 0$ to be determined later. The estimates of Proposition 6.13 render the initialization of the induction true for B_N large enough. So we assume that (6-79), (6-80) hold for $\mathcal{N}_k$ and we prove them in $\mathcal{N}_{k+1}$. As before we bootstrap

$$|r(u,v) - r_{\mathrm{RN}}(u,v)| + |\lambda(u,v) - \lambda_{\mathrm{RN}}(u,v)| + |\log(\Omega^2)(u,v) - \log(\Omega^2_{\mathrm{RN}})(u,v)| \le 4C_k \cdot v^{1-2s}, \quad (6\text{-}81)$$

$$|\nu(u,v) - \nu_{\mathrm{RN}}(u,v)| + |A_u(u,v) - A_u^{\mathrm{RN}}(u,v)| \le 4C_k \cdot v^{1-2s}. \quad (6\text{-}82)$$

We treat one typical term, to show the specificity of the no-shift region $\mathcal{N}$ compared to $\mathcal{R}$: under the bootstraps and (6-31), (6-29) we have

$$|\partial_u \partial_v r| \lesssim (1 + C_k) \cdot v^{1-2s} \sim C_k \cdot |u|^{1-2s}.$$

Upon integration in the u direction, it gives, using (6-79) in the past, for some $E(M, e, q_0, m^2, D_1, D_2) > 0$

$$|\lambda - \lambda_{\mathrm{RN}}| \le C_k \cdot v^{1-2s} + E \cdot \epsilon \cdot C_k \cdot v^{1-2s};$$

thus for $\epsilon > 0$ sufficiently small, so that $E \cdot \epsilon < 1$, we close the part of bootstrap (6-81) relative to $\lambda - \lambda_{\mathrm{RN}}$. The other terms are addressed similarly, we omit the details. Such estimates allow us to retrieve bootstraps (6-81), (6-82) and prove the induction hypothesis. Once this is done, we can prove difference estimates for $\delta\phi$ exactly as in Proposition 6.13. $\square$

6C3. *Difference estimates in the early blue-shift region.*

Proposition 6.15. *There exists a constant $C_E = C_E(M, e, q_0, m^2, s, D_1, D_2) > 0$ such that the following estimates are satisfied for all $(u, v) \in \mathcal{EB}$:*

$$|r(u,v)-r_{\mathrm{RN}}(u,v)| \le C_E \cdot v^{1-2s}, \tag{6-83}$$

$$|\nu(u,v)-\nu_{\mathrm{RN}}(u,v)|+|A_u(u,v)-A_u^{\mathrm{RN}}(u,v)|+|\lambda(u,v)-\lambda_{\mathrm{RN}}(u,v)| \le C_E \cdot v^{1-2s}, \tag{6-84}$$

$$|\partial_u \log(\Omega^2)(u,v)-\partial_u \log(\Omega^2_{\mathrm{RN}})(u,v)|+|\partial_v \log(\Omega^2)(u,v)-\partial_v \log(\Omega^2_{\mathrm{RN}})(u,v)| \le C_E \cdot v^{1-2s}, \tag{6-85}$$

$$|\partial_u \delta\phi|+|\partial_v \delta\phi| \le C_E \cdot v^{1-3s}, \tag{6-86}$$

$$|\delta\phi| \le C_E \cdot v^{1-3s} \cdot \log(v), \tag{6-87}$$

$$|\log(\Omega^2)(u,v)-\log(\Omega^2_{\mathrm{RN}})(u,v)| \le C_E \cdot v^{1-2s} \cdot \log(v). \tag{6-88}$$

Proof. Note that in $\mathcal{EB}$, as in $\mathcal{N}$, we have $v \sim |u|$ and that the size of the region is logarithmic, i.e., $u - u_{\gamma_N}(v) \lesssim \log(v) \sim \log(|u|)$ and $v - v_{\gamma_N}(u) \lesssim \log(|u|) \sim \log(v)$. As before, we start with bootstraps:

$$|\lambda(u,v) - \lambda_{\mathrm{RN}}(u,v)| + |\nu(u,v) - \nu_{\mathrm{RN}}(u,v)| \le 4C_N \cdot v^{1-2s}, \tag{6-89}$$

$$\Omega^2(u,v) \cdot |\log(\Omega^2)(u,v) - \log(\Omega^2_{\mathrm{RN}})(u,v)| \le 4C_N \cdot v^{1-3s}, \tag{6-90}$$

$$|r(u,v) - r_{\mathrm{RN}}(u,v)| \le B_N \cdot v^{1-2s} \tag{6-91}$$

for some $B_N > C_N$ to be determined later. The set of (u, v) for which these bootstraps are satisfied is nonempty by the estimates of Proposition 6.14.

Retrieving the bootstrap on $r - r_{\mathrm{RN}}$ is the most delicate. We use (2-19) and write the difference of the two identities below:

$$\lambda \cdot \kappa^{-1} = \nu \cdot \iota^{-1} = \frac{-2\lambda\nu}{\Omega^2} = \frac{1}{2} - \frac{\varpi}{r} + \frac{Q^2}{2r^2},$$

$$\nu_{\mathrm{RN}} = \lambda_{\mathrm{RN}} = -\frac{\Omega^2_{\mathrm{RN}}}{2} = \frac{1}{2} - \frac{M}{r_{\mathrm{RN}}} + \frac{e^2}{2r^2_{\mathrm{RN}}}.$$

Thus, we have

$$\begin{aligned}
&(\lambda - \lambda_{\mathrm{RN}}) \cdot \kappa^{-1} + \lambda_{\mathrm{RN}} \cdot (\kappa^{-1} - 1) \\
&\quad = (\lambda - \lambda_{\mathrm{RN}}) \cdot \kappa^{-1} + (\nu_{\mathrm{RN}} - \nu) + \nu \cdot (1 - e^{-\log(\Omega^2)+\log(\Omega^2_{\mathrm{RN}})})) \\
&\quad = -\frac{\varpi}{r} + \frac{Q^2}{2r^2} + \frac{M}{r_{\mathrm{RN}}} - \frac{e^2}{2r^2_{\mathrm{RN}}} \frac{M-\varpi}{r} + \frac{M}{r \cdot r_{\mathrm{RN}}} \cdot (r - r_{\mathrm{RN}}) + \frac{Q^2 - e^2}{2r^2} - \frac{e^2 \cdot (r + r_{\mathrm{RN}})}{2r^2 \cdot r^2_{\mathrm{RN}}} \cdot (r - r_{\mathrm{RN}});
\end{aligned}$$

hence, combined with the $(r - r_{\mathrm{RN}})$ terms, we have

$$\begin{aligned}
&\left(\frac{M}{r \cdot r_{\mathrm{RN}}} - \frac{e^2 \cdot (r + r_{\mathrm{RN}})}{2r^2 \cdot r^2_{\mathrm{RN}}}\right) \cdot (r - r_{\mathrm{RN}}) \\
&\qquad = \frac{2M \cdot r \cdot r_{\mathrm{RN}} - e^2 \cdot (r + r_{\mathrm{RN}})}{2r^2 \cdot r^2_{\mathrm{RN}}} \cdot (r - r_{\mathrm{RN}}) \\
&\qquad = (\lambda - \lambda_{\mathrm{RN}}) \cdot \kappa^{-1} + (\nu_{\mathrm{RN}} - \nu) + \nu \cdot (1 - e^{-\log(\Omega^2)(u,v)+\log(\Omega^2_{\mathrm{RN}})}) + \frac{-M + \varpi}{r} + \frac{-Q^2 + e^2}{2r^2}.
\end{aligned}$$

To conclude, we have to prove that the prefactor of the left-hand side $(2M\cdot r\cdot r_{\mathrm{RN}}-e^2\cdot(r+r_{\mathrm{RN}}))/(r^2\cdot r_{\mathrm{RN}}^2)$ is bounded away from zero: for this, notice that, since $0<|e|<M$, we have

$$r_-(M,e)=M-\sqrt{M^2-e^2}<e^2/M,$$

which is equivalent to

$$2M\cdot r_-^2<2e^2\cdot r_-.$$

By (6-44) and choosing Δ_N sufficiently large, there exists a small constant $\alpha(M,e)>0$ such that in $\mathcal{EB}$

$$\left|\frac{2M\cdot r\cdot r_{\mathrm{RN}}-e^2\cdot(r+r_{\mathrm{RN}})}{2r^2\cdot r_{\mathrm{RN}}^2}\right|>\alpha.$$

Thus, as a consequence of bootstrap (6-89) and (6-43), (6-42) and (6-37), there exists

$$C_N'(M,e,q_0,m^2,s,D_1,D_2)>0$$

such that

$$|r-r_{\mathrm{RN}}|\le C_N'\cdot v^{1-2s}+C_N'\cdot|\nu|\cdot|\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2)|\le C_N'\cdot v^{1-2s}+2C_N'\cdot 4C_N\cdot v^{1-2s}<B_N\cdot v^{1-2s},$$

where we chose $B_N=2C_N'+4C_N'\cdot 4C_N$ for the last inequality to be true. Therefore, bootstrap (6-91) is retrieved.

Now we turn to bootstrap (6-90), which is equally delicate (because we want to avoid a logarithmic loss). As in Proposition 6.13, we write the difference between (2-33) satisfied by Ω^2 and the analogous equation satisfied by Ω_{RN}^2. Using also (6-42) and bootstrap (6-89), (6-90), (6-91) we obtain

$$\begin{aligned}
&|\partial_v\partial_u(\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2))|\\
&\qquad\lesssim|D_u\phi|\cdot|\partial_v\phi|+\Omega^2\cdot(|Q-e|+|r-r_{\mathrm{RN}}|)+|\Omega^2-\Omega_{\mathrm{RN}}^2|+|\lambda|\cdot|\nu-\nu_{\mathrm{RN}}|+|\lambda-\lambda_{\mathrm{RN}}|\cdot|\nu|\\
&\qquad\lesssim v^{-2s}+\Omega^2\cdot v^{1-2s}+\Omega^2\cdot\left|\log\left(\frac{\Omega^2}{\Omega_{\mathrm{RN}}^2}\right)\right|+\Omega^2\cdot|\nu-\nu_{\mathrm{RN}}|+\Omega^2\cdot|\lambda-\lambda_{\mathrm{RN}}|\lesssim v^{-2s}+\Omega^2\cdot v^{1-2s},
\end{aligned}$$

where in the last line we have used (6-38), (6-37) as $|\lambda|,\ |\nu|\lesssim\Omega^2$ and the usual inequality

$$|\Omega^2-\Omega_{\mathrm{RN}}^2|\lesssim\Omega^2\cdot\left|\log\left(\frac{\Omega^2}{\Omega_{\mathrm{RN}}^2}\right)\right|$$

(which is true because $\Omega_{\mathrm{RN}}^2/\Omega^2\ge\frac{1}{10}$, an estimate which follows directly from (6-36)). Integrating in v (recall the v-difference is of size $\log(v)$), we get, using Proposition 6.14,

$$|\partial_u\log(\Omega^2)-\partial_u\log(\Omega_{\mathrm{RN}}^2)|(u,v)\lesssim v^{1-2s}. \tag{6-92}$$

Instead of integrating (6-92) directly (and incurring a logarithmic loss), we write an identity: for any $\eta>0$,

$$\partial_u[\Omega^\eta\cdot(\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2))]=\Omega^\eta\cdot\frac{\eta}{2}\cdot\partial_u\log(\Omega^2)\cdot(\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2))+\Omega^\eta\cdot\partial_u[\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2)],$$

from which we deduce, using also $\partial_u\log(\Omega^2)<0$ (see Proposition 6.4)

$$\begin{aligned}
&\partial_u[\Omega^{2\eta}\cdot(\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2))^2]\\
&\quad=2\eta\cdot\Omega^{2\eta}\cdot\partial_u\log(\Omega^2)\cdot(\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2))^2+2\Omega^\eta\cdot\partial_u[\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2)]\cdot(\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2))\\
&\quad\le 2\Omega^{2\eta}\cdot|\partial_u[\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2)]|\cdot|\log(\Omega^2)-\log(\Omega_{\mathrm{RN}}^2)|,
\end{aligned}$$

which in turn implies, using (6-92)

$$\partial_u[\Omega^\eta \cdot |\log(\Omega^2) - \log(\Omega^2_{\mathrm{RN}})|] = \Omega^\eta \cdot |\partial_u[\log(\Omega^2) - \log(\Omega^2_{\mathrm{RN}})]| \lesssim \Omega^\eta \cdot v^{1-2s}.$$

Integrating the above in u using $\partial_u \log(\Omega^2) \in (3K_-, K_-)$ (see Proposition 6.4) and the bounds from Proposition 6.14, we get for some $E'(M, e, q_0, m^2, s, D_1, D_2) > 0$

$$\Omega^\eta(u, v) \cdot |\log(\Omega^2)(u, v) - \log(\Omega^2_{\mathrm{RN}})(u, v)| \le C_N \cdot v^{1-2s} + E' \cdot \eta^{-1} \cdot \Omega^\eta(u_{\gamma_{\mathcal{N}}}(v), v) \cdot v^{1-2s}. \tag{6-93}$$

Applying (6-93) for $\eta = 2$, choosing Δ_N large enough so that $\Omega^\eta(u_{\gamma_{\mathcal{N}}}(v), v) \approx e^{2K_- \Delta_N} < C_N/(10E')$ retrieves bootstrap (6-90).

Retrieving bootstrap (6-89) is done similarly: we integrate the difference between (2-32) satisfied by r and the analog satisfied by r_{RN}, using Proposition 6.14, and we prove

$$|\lambda(u, v) - \lambda_{\mathrm{RN}}(u, v)| + |\nu(u, v) - \nu_{\mathrm{RN}}(u, v)| \le 3C_N \cdot v^{1-2s},$$

which closes all the bootstrap assumptions.

Now we turn to the rest of the differences estimates claimed in the statement of the proposition. Integrating the differences into (2-39), (2-33) as we did in Proposition 6.13 gives straightforwardly

$$|A_u(u, v) - A_u^{\mathrm{RN}}(u, v)| \lesssim v^{1-2s},$$
$$|\partial_v \log(\Omega^2)(u, v) - \partial_v \log(\Omega^2_{\mathrm{RN}})(u, v)| \lesssim v^{1-2s},$$

where we also used that the size of the region of integration is logarithmic, i.e., $\int_{u_\gamma(v)}^u v^{-2s}\, \mathrm{d}u \lesssim v^{-2s} \log(v)$.

For $\delta\phi$, we proceed as in Proposition 6.13 and make the following bootstrap assumptions for some $B' > 0$:

$$\Omega(u, v) \cdot |\delta\phi|(u, v) \le B' \cdot v^{1-3s}, \tag{6-94}$$
$$|\partial_v \delta\phi|(u, v) \le B' \cdot v^{1-3s}. \tag{6-95}$$

Plugging differences into (2-36) satisfied by ϕ and the analogous equation satisfied by $\delta\phi$, we get, using (6-94), (6-95) and the previously proven difference estimates,

$$|\partial_u \partial_v \delta\phi| \lesssim B' \cdot \Omega \cdot v^{1-3s} + \Omega^2 \cdot |\partial_u \delta\phi|, \tag{6-96}$$

from which we deduce, upon integrating in v and using a Gronwall estimate,

$$|\partial_u \delta\phi|(u, v) \lesssim (1 + B' \cdot \Omega(u, v_{\gamma_{\mathcal{N}}}(u))) \cdot v^{1-3s}, \tag{6-97}$$

and plugging (6-97) into (6-96) and integrating in u this time we get

$$|\partial_v \delta\phi|(u, v) \lesssim (1 + B' \cdot [\Omega(u, v_{\gamma_{\mathcal{N}}}(u)) + \Omega(v_{\gamma_N}(v), v)]) \cdot v^{1-3s}, \tag{6-98}$$

which is sufficient to retrieve bootstrap (6-95) after an appropriate choice of B' and choosing also Δ_N large enough (to obtain a small constant from $\Omega(u_{\gamma_{\mathcal{N}}}(v), v)$ as we did above).

To retrieve bootstrap (6-94), we proceed as with $\partial_u \log(\Omega^2)$ earlier, with the identity

$$\partial_u(\Omega^\eta \delta\phi) = \frac{\eta}{2} \cdot \partial_u \log(\Omega^2) \cdot \Omega^\eta \cdot \delta\phi + \Omega^\eta \cdot \partial_u \delta\phi,$$

which also implies, using (6-97) and by the same reasoning as for $\partial_u \log(\Omega^2)$ above

$$\partial_u(\Omega^\eta|\delta\phi|) \leq \Omega^\eta \cdot |\partial_u(\delta\phi)| \lesssim \Omega^\eta \cdot (1+B'\cdot\Omega(u, v_{\gamma_{\mathcal{N}}}(u)))\cdot v^{1-3s}.$$

Integrating this inequality in u for $\eta=1$, after an appropriate choice of B' and choosing also Δ_N large enough as we did above allows to retrieve bootstrap (6-94) and concludes the proof. □

6C4. *Difference estimates in the late blue-shift region.* In this section, we will not need to estimate metric differences anymore (although we will use the difference estimates from past sections); therefore, we do not require a bootstrap method and proceed directly.

Proposition 6.16. *There exists a constant $C_L = C_L(M, e, q_0, m^2, s, D_1, D_2) > 0$ such that the following are satisfied for all $(u, v) \in \mathcal{LB}$:*

$$|A_u(u,v) - A_u^{\mathrm{CH}}(u)| + |A_u^{\mathrm{RN}}(u,v) - (A_u^{\mathrm{RN}})^{\mathrm{CH}}(u)| \leq C_L\cdot\Omega^2(u,v) \leq C_L^2\cdot v^{-2s}, \tag{6-99}$$

$$|A_u(u,v) - A_u^{\mathrm{RN}}(u,v)| + |A_u^{\mathrm{CH}}(u) - (A_u^{\mathrm{RN}})^{\mathrm{CH}}(u)| \leq C_L\cdot|u|^{1-2s}, \tag{6-100}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}u}(A_u^{\mathrm{CH}} - (A_u^{\mathrm{RN}})^{\mathrm{CH}})\right|(u) \leq C_L\cdot|u|^{1-2s}, \tag{6-101}$$

$$\begin{aligned}\big||D_v\psi|(u,v) - |D_v\psi_{\mathcal{L}}|(u,v)\big| \\ \leq \left|e^{iq_0\int_{u_\gamma(v)}^u A_u^{\mathrm{CH}}(u')\,\mathrm{d}u'}\partial_v\psi(u,v) - e^{iq_0\int_{u_\gamma(v)}^u (A_u^{\mathrm{RN}})^{\mathrm{CH}}(u')\,\mathrm{d}u'}\partial_v\psi_{\mathcal{L}}(u,v)\right| \leq C_L\cdot v^{1-3s},\end{aligned} \tag{6-102}$$

$$\left|\psi(u,v) - \int_{v_\gamma(u)}^{v} e^{iq_0\int_{u_\gamma(v')}^u[(A_u^{\mathrm{RN}})^{\mathrm{CH}} - A_u^{\mathrm{CH}}](u')\,\mathrm{d}u'}\,\partial_v\psi_{\mathcal{L}}(u,v')\,\mathrm{d}v'\right| \leq C_L\cdot|u|^{2-3s}, \tag{6-103}$$

$$\big||D_u\psi|(u,v) - |D_u^{\mathrm{RN}}\psi_{\mathcal{L}}|(u,v)\big| \leq \big|D_u\psi(u,v) - D_u^{\mathrm{RN}}\psi_{\mathcal{L}}(u,v)\big| \leq C_L\cdot|u|^{1-3s}\cdot\log|u|. \tag{6-104}$$

Moreover, for every fixed $u < u_s$, there exists $f(u) \in \mathbb{C}$ such that

$$\lim_{v\to+\infty}\psi(u,v) - \int_{v_\gamma(u)}^{v} e^{iq_0\int_{u_\gamma(v')}^u[(A_u^{\mathrm{RN}})^{\mathrm{CH}} - A_u^{\mathrm{CH}}](u')\,\mathrm{d}u'}\,\partial_v\psi_{\mathcal{L}}(u,v')\,\mathrm{d}v' = f(u). \tag{6-105}$$

Proof. We start with estimates on the potentials: By (2-39) and (6-47) we have for $\eta = 0.01$

$$|\partial_v(A_u - A_u^{\mathrm{RN}})| \leq |\partial_v A_u| + |\partial_v A_u^{\mathrm{RN}}| \lesssim \Omega^{2-\eta} + \Omega_{\mathrm{RN}}^{2-2\eta},$$

which we can integrate from the curve γ; using (6-45) and (6-50) using [Van de Moortel 2018, Lemma 4.1] as before, we obtain, using also Proposition 6.15, the bound

$$|A_u - A_u^{\mathrm{RN}}| \lesssim |u|^{1-2s}. \tag{6-106}$$

Moreover, recall that we proved in Proposition 6.6 that $A_u(u,v)$ and $A_u^{\mathrm{RN}}(u,v)$ extend to $\mathcal{CH}_{i^+}$ as bounded functions $(A_u)^{\mathrm{CH}}(u)$ and $(A_u^{\mathrm{RN}})^{\mathrm{CH}}(u)$, respectively. Integrating (2-39) towards the past from the Cauchy horizon $\mathcal{CH}_{i^+}$ we also obtain the following estimates for all $(u,v)\in\mathcal{LB}$:

$$|A_u(u,v) - A_u^{\mathrm{CH}}(u)| + |A_u^{\mathrm{RN}}(u,v) - (A_u^{\mathrm{RN}})^{\mathrm{CH}}(u)| \lesssim \Omega^2(u,v) \lesssim v^{-2s}, \tag{6-107}$$

$$\int_{u_\gamma(v)}^u |A_u(u',v) - A_u^{\mathrm{CH}}(u')|\,\mathrm{d}u' + \int_{u_\gamma(v)}^u |A_u^{\mathrm{RN}}(u',v) - (A_u^{\mathrm{RN}})^{\mathrm{CH}}(u')|\,\mathrm{d}u' \lesssim v^{-2s}. \tag{6-108}$$

To obtain (6-101), note the following identity obtained using (2-39) with (3-5) (note that $A_u(u, v_0) = A_u^{\mathrm{RN}}(u, v_0)$):

$$A_u^{\mathrm{CH}}(u) - (A_u^{\mathrm{RN}})^{\mathrm{CH}}(u) = \int_{v_0}^{+\infty} -\frac{\Omega^2(u, v')Q(u, v')}{r^2(u, v')} + \frac{\Omega_{\mathrm{RN}}^2(u, v')e}{r_{\mathrm{RN}}^2(u, v')}\, \mathrm{d}v'. \tag{6-109}$$

We now commute (2-39) with ∂_u to estimate $(d/du)(A_u^{\mathrm{CH}}(u)-(A_u^{\mathrm{RN}})^{\mathrm{CH}}(u))$ and we obtain a formula analogous to (6-109). Using the fact that $\partial_u \log(\Omega^2)\Omega^{0.1}$ is bounded (by Proposition 6.5) to estimate the parts of the integral lying in $\mathcal{LB}$, and we obtain an estimate only involving the regions strictly to the past of $\mathcal{LB}$:

$$\left|\frac{\mathrm{d}}{\mathrm{d}u}(A_u^{\mathrm{CH}}(u)-(A_u^{\mathrm{RN}})^{\mathrm{CH}}(u))\right| \le \left|\int_{v_0}^{v_\gamma(u)} \partial_u\left(-\frac{\Omega^2(u, v')Q(u, v')}{r^2(u, v')} + \frac{\Omega_{\mathrm{RN}}^2(u, v')e}{r_{\mathrm{RN}}^2(u, v')}\right) \mathrm{d}v'\right| + |u|^{-2s}. \tag{6-110}$$

Therefore, it is sufficient to control the above integral in $\mathcal{R}\cup\mathcal{N}\cup\mathcal{EB}$. Note that the differences $\Omega^2-\Omega_{\mathrm{RN}}^2$, $\partial_u\Omega^2-\partial_u\Omega_{\mathrm{RN}}^2$, $Q-e$, $\nu-\nu_{\mathrm{RN}}$ and $r-r_{\mathrm{RN}}$ have been controlled with $|u|^{1-2s}$ weights in Propositions 6.13, 6.14 and 6.15; this gives (6-101).

Now we turn to the ϕ estimates. We write (2-44) for $u_0 = u_\gamma(v)$ and using the estimates from Proposition 6.5 (notably (6-47) and (6-46) with $\eta = 0.1$) we obtain

$$\begin{aligned}
&|\partial_u(e^{iq_0\int_{u_\gamma(v)}^u A_u(u',v)\,\mathrm{d}u'}\partial_v\psi - e^{iq_0\int_{u_\gamma(v)}^u A_u^{\mathrm{RN}}(u',v)\,\mathrm{d}u'}\partial_v\psi_{\mathcal{L}})| \\
&\qquad\lesssim |\partial_u(e^{iq_0\int_{u_\gamma(v)}^u A_u(u',v)\,\mathrm{d}u'}\partial_v\psi)| + |\partial_u(e^{iq_0\int_{u_\gamma(v)}^u A_u^{\mathrm{RN}}(u,v)\,\mathrm{d}u'}\partial_v\psi_{\mathcal{L}})| \\
&\qquad\lesssim |u|^{-2s}\cdot v^{1-3s} + (\Omega^{1.9}+\Omega_{\mathrm{RN}}^{1.9})\cdot v^{-s}.
\end{aligned}$$

Integrating in u and using (6-45) with the usual integration rules (i.e., [Van de Moortel 2018, Lemma 4.1]) we obtain

$$\begin{aligned}
&|e^{iq_0\int_{u_\gamma(v)}^u A_u(u',v)\,\mathrm{d}u'}\partial_v\psi(u, v) - e^{iq_0\int_{u_\gamma(v)}^u A_u^{\mathrm{RN}}(u',v)\,\mathrm{d}u'}\partial_v\psi_{\mathcal{L}}(u, v)| \\
&\qquad\lesssim |\partial_v\psi(u_\gamma(v), v) - \partial_v\psi_{\mathcal{L}}(u_\gamma(v), v)| + |u|^{1-2s}\cdot v^{1-3s} + v^{-2.8s} \lesssim v^{1-3s},
\end{aligned} \tag{6-111}$$

where we also used (6-86). Then by (6-111), (6-108), we obtain

$$\begin{aligned}
&|\partial_v\psi(u, v) - e^{iq_0\int_{u_\gamma(v)}^u[(A_u^{\mathrm{RN}})^{\mathrm{CH}}(u')-A_u^{\mathrm{CH}}(u')]\,\mathrm{d}u'}\partial_v\psi_{\mathcal{L}}(u, v)| \\
&\quad\le |e^{iq_0\int_{u_\gamma(v)}^u A_u(u',v)\,\mathrm{d}u'}\partial_v\psi(u, v) - e^{iq_0\int_{u_\gamma(v)}^u A_u^{\mathrm{RN}}(u',v)\,\mathrm{d}u'}\partial_v\psi_{\mathcal{L}}(u, v)| \\
&\qquad\qquad + |e^{iq_0\int_{u_\gamma(v)}^u[A_u^{\mathrm{RN}}(u',v)-(A_u^{\mathrm{RN}})^{\mathrm{CH}}(u')-A_u(u',v)+A_u^{\mathrm{CH}}(u')]\,\mathrm{d}u'} - 1|\cdot|\partial_v\psi_{\mathcal{L}}|(u, v) \\
&\quad\lesssim v^{1-3s} + |e^{iq_0\int_{u_\gamma(v)}^u[A_u^{\mathrm{RN}}(u',v)-(A_u^{\mathrm{RN}})^{\mathrm{CH}}(u')-A_u(u',v)+A_u^{\mathrm{CH}}(u')]\,\mathrm{d}u'} - 1|\cdot|\partial_v\psi_{\mathcal{L}}|(u, v) \\
&\quad\lesssim v^{1-3s} + v^{-2s}\cdot v^{-s} \lesssim v^{1-3s},
\end{aligned} \tag{6-112}$$

where in the first line we multiplied by the phase $e^{iq_0\int_{u_\gamma(v)}^u A_u(u',v)\,\mathrm{d}u'}$ inside the absolute value and we used (6-49) (applied to $\phi_{\mathcal{L}}$) in the last line. This implies (6-102) (the first inequality is obtained by the reverse triangular inequality). Integrating in v from γ then gives (6-103) and (6-105), using also (6-34) to control the boundary term $|\psi(u, v_\gamma(u))| \lesssim |u|^{-s} \lesssim |u|^{2-3s}$ (recall that $s \le 1$).

For (6-104) we estimate (2-45) using the estimate of Proposition 6.5 (naively, without taking advantage of a difference structure) and $A_v = A_v^{\mathrm{RN}} = 0$, and we get

$$|\partial_v(D_u\psi - D_u^{\mathrm{RN}}\psi_{\mathcal{L}})| \lesssim |u|^{-2s} \cdot v^{1-3s} + (\Omega^{1.9} + \Omega_{\mathrm{RN}}^{1.9}) \cdot v^{-s}.$$

Integrating in v, using the bounds of Proposition 6.15 and (6-106) (to control the difference on γ, similarly to what was done earlier in the proof) allows us to prove (6-104) thus concluding the proof. □

6D. *Combining the linear and the nonlinear estimates.* In this section, we combine the nonlinear difference estimates of Section 6C with the linear estimates on a fixed Reissner–Nordström background obtained in Section 5. This allows us to conclude the proof of the boundedness of ϕ if $\phi_{\mathcal{H}^+} \in \mathcal{O}$ and if $q_0 = 0$, blow up if $\phi_{\mathcal{H}^+} \notin \mathcal{O}$.

6D1. *Boundedness and extendibility of the matter fields for oscillating data and proof of Theorem I*(i).

Proposition 6.17. *Assume the following gauge-invariant condition*: *there exists $u_0 \leq u_s$ such that*

$$\lim_{v\to+\infty} \int_{v_0}^{v} e^{iq_0\sigma_{\mathrm{br}}(v')} e^{iq_0 \int_{v_0}^{v'} A_v(u_0,v'')\,\mathrm{d}v''} D_v\psi_{\mathcal{L}}(u_0, v')\,\mathrm{d}v' \tag{6-113}$$

exists and is finite for all σ_{br} satisfying (3-15), (3-16). *Then ϕ in the gauge* (2-26), (3-5) *admits a continuous extension to $\mathcal{CH}_{i^+}$. Moreover the gauge-independent quantities $|\phi|$ and the metric g also admit a continuous extension to $\mathcal{CH}_{i^+}$ and the extension of g can be chosen to be C^0-admissible as in Definition 2.1.*

If we additionally assume the following gauge-invariant condition: *for all $D_{\mathrm{br}} > 0$, there exists $\eta_0(D_{\mathrm{br}}) > 0$ such that for all σ_{br} satisfying* (3-15), (3-16) *and for all $(u, v) \in \mathcal{LB}$,*

$$\left| \int_{v_\gamma(u)}^{v} e^{iq_0\sigma_{\mathrm{br}}(v')} e^{iq_0 \int_{v_0}^{v'} A_v(u_0,v'')\,\mathrm{d}v''} D_v\psi_{\mathcal{L}}(u_0, v')\,\mathrm{d}v' \right| \lesssim D' \cdot |u|^{s-1-\eta_0}, \tag{6-114}$$

then Q and ϕ are bounded and the following estimates are true for all $(u, v) \in \mathcal{LB}$:

$$|\phi|(u, v) \lesssim |u|^{-1+s-\eta_0}, \tag{6-115}$$

$$|Q - e|(u, v) \lesssim |u|^{-\eta_0}, \tag{6-116}$$

where the implicit constants are allowed to depend on $\eta_0 > 0$. Moreover, Q extends to a continuous function $Q_{\mathrm{CH}}(u)$ on $\mathcal{CH}_{i^+}$.

Proof. Applying the assumption to $\sigma_{\mathrm{br}}(v) = \int_{u_\gamma(v)}^{u_0} [(A_u^{\mathrm{RN}})^{\mathrm{CH}} - A_u^{\mathrm{CH}}](u')\,\mathrm{d}u'$ (which satisfies (3-15) and (3-16) by Proposition 6.16) we get by Proposition 6.16 that for ψ in the gauge (2-26) (note that $A_v \equiv 0$),

$$\lim_{v\to+\infty} \psi(u_0, v) := \psi_{\mathrm{CH}}(u_0)$$

exists and is finite. Recall also from Proposition 6.6 that $D_u\psi$ and A_u admit (in the gauge (2-26), (3-5)) a bounded extension to $\mathcal{CH}_{i^+}$ which we denoted respectively by $(D_u\psi)_{\mathrm{CH}}$ and $(A_u)^{\mathrm{CH}}$. Recall also that one can write for any $u_0 \in \mathbb{R}$ the identity

$$\partial_u(e^{iq_0 \int_{u_0}^{u} A_u(u',v)\,\mathrm{d}u'} \psi(u, v)) = e^{iq_0 \int_{u_0}^{u} A_u(u',v)\,\mathrm{d}u'} D_u\psi(u, v),$$

which upon integration gives

$$\psi(u,v)=e^{-iq_0\int_{u_0}^{u}A_u(u',v)\,\mathrm{d}u'}\psi(u_0,v)+e^{-iq_0\int_{u_0}^{u}A_u(u',v)\,\mathrm{d}u'}\int_{u_0}^{u}e^{iq_0\int_{u_0}^{u'}A_u(u'',v)\,\mathrm{d}u''}D_u\psi(u',v)\,\mathrm{d}u'.$$

Now note by Proposition 6.16, $A_u\in L^\infty_{\mathrm{loc}}$; therefore by dominated convergence, the function $(u,v)\mapsto\int_{u_0}^{u}A_u(u',v)\,\mathrm{d}u'$ extends continuously to $\int_{u_0}^{u}(A_u)^{\mathrm{CH}}(u')\,\mathrm{d}u'$ at $\mathcal{CH}_{i^+}$. Since $D_u\psi\in L^\infty_{\mathrm{loc}}$ as well (by (6-53)), an other use of dominated convergence, together with the existence of the limit $\lim_{v\to+\infty}\psi(u_0,v)$ shows that $\psi(u,v)$ admits a continuous extension to $\mathcal{CH}_{i^+}$ denoted by $\psi_{\mathcal{CH}_{i^+}}(u)$. By Theorem B, r admits a continuous extension $r_{\mathcal{CH}_{i^+}}(u)$ to $\mathcal{CH}_{i^+}$ which is bounded away from zero. Therefore, $\phi(u,v)$ also admits a continuous extension to $\mathcal{CH}_{i^+}$ denoted by $\phi_{\mathcal{CH}_{i^+}}(u)$. The continuous extendibility of the metric g (and the C^0-admissible character of the extension) follows immediately as a consequence of Corollary 6.10.

Now we make the additional assumption (6-114). We define $(\sigma_{\mathrm{br}})_u(v):=\int_{u_\gamma(v)}^{u}[(A_u^{\mathrm{RN}})^{\mathrm{CH}}-A_u^{\mathrm{CH}}](u')\,\mathrm{d}u'$ for each $u\le u_s$. It follows from (6-100) and (6-101) that $(\sigma_{\mathrm{br}})_u$ satisfies (3-15), (3-16) with a constant $D_{\mathrm{br}}(M,e,q_0,m^2,s,D_1,D_2)>0$ *that is independent of* u. In view of this, (6-115) follows from (6-114) combined with (6-103) and the fact that $s>\frac{3}{4}$. Now we plug (6-115), the boundedness of r, and (6-53) into (2-40) to obtain the estimate in $\mathcal{LB}$:

$$|\partial_u Q|\lesssim|u|^{-1-\eta_0}.$$

Integrating this estimate from γ we obtain (6-116), in view of the estimate on γ from Proposition 6.4.

For the continuous extendibility of Q, we start integrating (2-40) to get for all $(u,v)\in\mathcal{LB}$

$$Q(u,v)=Q(u_\gamma(v),v)+q_0\int_{u_\gamma(v)}^{u}\Im(\bar\psi D_u\psi)(u',v)\,\mathrm{d}u'.$$

Note that the function $u\to\Im(\bar\psi D_u\psi)(u,v)$ is dominated by the integrable function $|u|^{-1-\eta_0}$ therefore by the dominated convergence theorem, $\int_{u_\gamma(v)}^{u}\Im(\bar\psi D_u\psi)(u',v)\,\mathrm{d}u'$ extends continuously to the function $\int_{-\infty}^{u}\Im(\bar\psi_{\mathrm{CH}}(D_u\psi)_{\mathrm{CH}})(u')\,\mathrm{d}u'$. Therefore, Q admits a continuous extension to $\mathcal{CH}_{i^+}$, which concludes the proof. □

Corollary 6.18. (1) *Assume that $\phi_{\mathcal{H}^+}\in\mathcal{O}$. Then ϕ is uniformly bounded on $\mathcal{LB}$ and thus* (4-1) *holds true.*

(2) *Assume additionally that $\phi_{\mathcal{H}^+}\in\mathcal{O}'$. Then $|\phi|$ and g are continuously extendible, and the extension of g can be chosen to be C^0-admissible.*

(3) *Assume additionally that $\phi_{\mathcal{H}^+}\in\mathcal{O}''$. Then* (6-115) *and* (6-116) *are true for all $(u,v)\in\mathcal{LB}$ and moreover Q admits a continuous extension to $\mathcal{CH}_{i^+}$.*

Proof. The first statement follows from (6-103) of Proposition 6.16 and Corollary 5.25. The others are direct applications of Proposition 6.17 and Corollary 5.25 (using that (6-113) and (6-114) are gauge-invariant conditions). □

In particular, Corollary 6.18 shows Theorem I (i).

6D2. *Blow-up of the scalar field for $\phi_{\mathcal{H}^+}\notin\mathcal{O}$ (nonoscillating data) if $q_0=0$ and proof of Theorem I*(ii).

Lemma 6.19. *Assume that there exists $u_0\le u_s$ such that*

$$\limsup_{v\to+\infty}|\phi|(u_0,v)=+\infty.$$

Then for all $u \leq u_s$ we have

$$\limsup_{v\to+\infty} |\phi|(u,v) = \limsup_{v\to+\infty} |\psi|(u,v) = +\infty.$$

Moreover we have the following bounds: for all $u \leq u_s$, there exists $f(u) > 0$ for all $v > v_\gamma(u)$ such that

$$\left| |\phi|(u,v) - \frac{r(u_0,v)}{r(u,v)} |\phi|(u_0,v) \right| \leq f(u),$$
$$\liminf_{v\to+\infty} \frac{|\phi|(u,v)}{|\phi|(u_0,v)} = \frac{r_{CH}(u_0)}{r_{CH}(u)} > 0. \tag{6-117}$$

Proof. This is an immediate consequence of the integrating of (6-53) and the continuous extendibility of r to a function which is bounded away from zero (by the definition of $\mathcal{CH}_{i^+}$). □

We will not use (6-117) in the present work, but it is an important estimate for our companion paper [Kehle and Van de Moortel $\geq$ 2024].

Corollary 6.20. *Assume that $q_0 = 0$ and that $\phi_{\mathcal{H}^+} \in \mathcal{SL} - \mathcal{O}$. Then for all $u \leq u_s$ we have the blow-up*

$$\limsup_{v\to+\infty} |\phi|(u,v) = \limsup_{v\to+\infty} |\psi|(u,v) = +\infty,$$

and moreover the asymptotics (6-117) *are satisfied.*

Proof. The result follows from a combined application of Corollary 5.25 (using that $\phi'_{\mathcal{L}}$ and $\phi_{\mathcal{L}}$ relate by a gauge transformation; hence $|\phi'_{\mathcal{L}}| = |\phi_{\mathcal{L}}|$), (6-105) in Proposition 6.16 and Lemma 6.19. □

In particular, Corollary 6.20 shows Theorem I (ii).

6D3. *Proof of Theorem II.* Before turning to the proof of Theorem II, we prove the following.

Lemma 6.21. *Let $s > \frac{3}{4}$ and $\omega_1 \in \mathbb{R} - \{\omega_{\text{res}}\}$ and $\phi_{\mathcal{H}^+}$ be given by*

$$\phi_{\mathcal{H}^+}(v) = e^{-i\omega_1 v + \omega_{\text{err}}(v)} v^{-s} + \phi_{\text{err}} \tag{6-118}$$

for any $\phi_{\text{err}} \in C^1([v_0,+\infty),\mathbb{R})$ satisfying (1-8) *with $s > 1$ and any $\omega_{\text{err}} \in C^2([v_0,+\infty),\mathbb{R})$ such that $\omega'_{\text{err}}(v) \to 0$ as $v \to +\infty$ and such that $|\omega''_{\text{err}}|(v) \leq D \cdot v^{-2+2s-\eta_0}$ for $v \geq v_0$ and some constants $D > 0$ and $\eta_0 > 0$. Then $\phi_{\mathcal{H}^+} \in \mathcal{O}''$, where we assume without loss of generality that $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ (by choosing $D_1 > 0$ possibly larger).*

Proof. Since $\phi_{\mathcal{H}^+} \in \mathcal{SL}$ (by possibly choosing $D_1 > 0$ larger) it suffices to check (3-18) independently for $e^{-i\omega_1 v + \omega_{\text{err}}(v)} v^{-s}$ and ϕ_{err}. First note that ϕ_{err} satisfies (3-18) since it satisfies (1-8) with $s > 1$.

For $e^{-i\omega_1 v + \omega_{\text{err}}(v)} v^{-s}$ we can assume with no loss of generality that $\frac{3}{4} < s \leq 1$ (since the case $s > 1$ follows immediately from integrability). It suffices to prove that there exists $\eta > 0$, $E > 0$ such that for all large enough $\tilde{v}$, v with $\tilde{v} < v$

$$\left| \int_{\tilde{v}}^{v} e^{i\omega_{\text{res}} v' - i\omega_1 v' + i\sigma_{\text{br}}(v') + i\omega_{\text{err}}(v')} (v')^{-s} \, dv' \right| \leq E \cdot \tilde{v}^{-1+s-\eta} \tag{6-119}$$

for all σ_{br} satisfying (3-15) and (3-16). For conciseness, we will introduce the notation $\omega = \omega_{\text{res}} - \omega_1 \neq 0$. We make use of integration by parts:

$$\begin{aligned}\int_{\tilde v}^{v} e^{i\omega v'+i\sigma_{\mathrm{br}}(v')+i\omega_{\mathrm{err}}(v')}(v')^{-s}\,\mathrm{d}v' &= -i\int_{\tilde v}^{v}\frac{\mathrm{d}}{\mathrm{d}v'}\big(e^{i\omega v'+i\sigma_{\mathrm{br}}(v')+i\omega_{\mathrm{err}}(v')}\big)\frac{(v')^{-s}}{\omega+\sigma'_{\mathrm{br}}(v')+\omega'_{\mathrm{err}}(v')}\,\mathrm{d}v' \\ &= -i\,\frac{v^{-s}e^{i\omega v+i\sigma_{\mathrm{br}}(v)+i\omega_{\mathrm{err}}(v)}}{\omega+\sigma'_{\mathrm{br}}(v)+\omega'_{\mathrm{err}}(v)}+i\,\frac{\tilde v^{-s}e^{i\omega\tilde v+i\sigma_{\mathrm{br}}(\tilde v)+i\omega_{\mathrm{err}}(\tilde v)}}{\omega+\sigma'_{\mathrm{br}}(\tilde v)+\omega'_{\mathrm{err}}(\tilde v)} \\ &\quad -is\int_{\tilde v}^{v} e^{i\omega v'+i\sigma_{\mathrm{br}}(v')+i\omega_{\mathrm{err}}(v')}\frac{(v')^{-s-1}}{\omega+\sigma'_{\mathrm{br}}(v')+\omega'_{\mathrm{err}}(v')}\,\mathrm{d}v' \\ &\quad -i\int_{\tilde v}^{v} e^{i\omega v'+i\sigma_{\mathrm{br}}(v')+i\omega_{\mathrm{err}}(v')}\frac{(v')^{-s}\cdot(\sigma''_{\mathrm{br}}(v')+\omega''_{\mathrm{err}}(v'))}{(\omega+\sigma'_{\mathrm{br}}(v')+\omega'_{\mathrm{err}}(v'))^2}\,\mathrm{d}v'.\end{aligned}$$

Note that, using (3-16) and the decay assumption on ω'_{err}, we have $\omega+\sigma'_{\mathrm{br}}(v')+\omega'_{\mathrm{err}}(v')$ is bounded away from zero for $\tilde v$ large enough (since $\omega\neq 0$). The first two terms obviously obey (6-119) since $s>\frac{1}{2}$. Similarly, the third term can be integrated to show

$$\left|\int_{\tilde v}^{v} e^{i\omega v'+i\sigma_{\mathrm{br}}(v')+i\omega_{\mathrm{err}}(v')}\frac{(v')^{-s-1}}{\omega+\sigma'_{\mathrm{br}}(v')+\omega'_{\mathrm{err}}(v')}\,\mathrm{d}v'\right|\lesssim \tilde v^{-s}.$$

For the last term, we write using $|\omega''_{\mathrm{err}}(v)|\lesssim v^{-2+2s-\eta_0}$ and (3-16)

$$\begin{aligned}\left|\int_{\tilde v}^{v} e^{i\omega v'+i\sigma_{\mathrm{br}}(v')+i\omega_{\mathrm{err}}(v')}\frac{(v')^{-s}\cdot(\sigma''_{\mathrm{br}}(v')+\omega''_{\mathrm{err}}(v'))}{(\omega+\sigma'_{\mathrm{br}}(v')+\omega'_{\mathrm{err}}(v'))^2}\,\mathrm{d}v'\right| &\lesssim \int_{\tilde v}^{v}(v')^{-s}\cdot(v^{-2+2s-\eta_0}+v^{1-2s})\,\mathrm{d}v' \\ &\lesssim \tilde v^{-1+s-\eta_0}+\tilde v^{2-3s}\lesssim \tilde v^{-1+s-\eta_0}\end{aligned}$$

for some $\eta_0>0$, where to obtain this estimate, we used the fact that $s<1+\eta_0$ for some $\eta_0>0$ and also $2-3s<-1+s-\eta_0$ $\big($since we assumed $s>\frac{3}{4}\big)$. $\square$

Proposition 6.22. *Assume that the parameters* (M,e,q_0,m^2) *are such that*

$$|q_0e|\neq r_-(M,e)|m|.$$

Let $\phi_{\mathcal{H}^+}$ *be given by either the profile of* (1-15) (*if* $m^2>0,\ q_0=0$) *or* (1-16) (*if* $m^2=0,\ q_0\neq 0$) *or* (1-17) (*if* $m^2>0,\ q_0\neq 0$). *Then* $\phi_{\mathcal{H}^+}\in\mathcal{O}''$, *where we again assume without loss of generality that* $\phi_{\mathcal{H}^+}\in\mathcal{SL}$ (*by choosing* $D_1>0$ *possibly larger*).

Proof. If $m^2=0$, $|q_0e|<\frac{1}{2}$, then $\phi_{\mathcal{H}^+}$ satisfies (3-8) for $s>1$ and thus $\phi_{\mathcal{H}^+}\in\mathcal{O}''$. Otherwise, we have three different cases:

(1) $q_0=0,\ m^2\neq 0$: It suffices to prove that $e^{\pm i(mv+\omega_{\mathrm{err}}(v))}\cdot v^{-5/6}\in\mathcal{O}''$, where $\omega_{\mathrm{err}}(v)=-\frac{3}{2}m(2\pi M)^{2/3}\cdot v^{1/3}+\omega(m\cdot M)$. Note that $\omega'_{\mathrm{err}}(v)\to 0$ as $v\to+\infty$ and such that $|\omega''_{\mathrm{err}}|(v)\lesssim v^{-5/3}\lesssim v^{-2+2\cdot(5/6)-\eta_0}$ for any $0<\eta_0<\frac{4}{3}$. Therefore by Lemma 6.21, $e^{\pm i(mv+\omega_{\mathrm{err}}(v))}\cdot v^{-5/6}\in\mathcal{O}''$.

(2) $|q_0e|\geq\frac{1}{2}$, $m^2=0$. Then $\delta=\pm i\sqrt{4(q_0e)^2-1}$ and $\phi_{\mathcal{H}^+}$ is of the form (6-118) with $\omega_1=-q_0e/r_+\neq\omega_{\mathrm{res}}$, $\omega_{\mathrm{err}}=-(\sqrt{4(q_0e)^2-1})\log(v)$ and $s=1$. Indeed we have $\omega'_{\mathrm{err}}(v)=o(1)$ and $|\omega''_{\mathrm{err}}|(v)\lesssim v^{-2}\lesssim v^{-2+2s-\eta_0}$ for $\eta_0>0$ since $2s-2=0$. Therefore, $\phi_{\mathcal{H}^+}\in\mathcal{O}''$ by Lemma 6.21.

(3) $q_0\neq 0,\ m^2\neq 0$: As in the case $q_0=0,\ m^2\neq 0$, we know $\phi_{\mathcal{H}^+}$ is a linear combination of two profiles of the form (6-118) with $\omega_1=\pm m-q_0e/r_+$. Since the parameters (M,e,q_0,m^2) do not satisfy $|q_0e|\neq r_-(M,e)|m|$, we know that $\omega_1\neq\omega_{\mathrm{res}}$. The rest of the argument follows as above. $\square$

Corollary 6.23. *Assume that the parameters* (M, e, q_0, m^2) *are such that*

$$|q_0 e| \neq r_-(M, e)|m|.$$

Let $\phi_{\mathcal{H}^+}$ *by either the profile of* (1-15) (*if* $m^2 > 0,\ q_0 = 0$) *or* (1-16) (*if* $m^2 = 0,\ q_0 \neq 0$) *or* (1-17) (*if* $m^2 > 0,\ q_0 \neq 0$). *Then,* (6-115) *and* (6-116) *are true for all* $(u, v) \in \mathcal{LB}$. *Moreover,* $|\phi|$, Q *and the metric* g *admit a continuous extension to* $\mathcal{CH}_{i^+}$ *and the extension of* g *can be chosen to be* C^0*-admissible.*

Proof. This is an immediate application of Proposition 6.22 and Corollary 6.18 (using that $|\phi'_{\mathcal{L}}| = |\phi_{\mathcal{L}}|$ since $\phi'_{\mathcal{L}}$ and $\phi_{\mathcal{L}}$ only differ by gauge transformation). □

In particular, Corollary 6.23 shows Theorem II.

6D4. $\dot{W}^{1,1}_{\mathrm{loc}}$ *blow-up of the scalar field on outgoing cones: proof of Theorem III.*

Proposition 6.24. *Assume that for all* $u \leq u_s$ *we have the blow up*

$$\int_{v_0}^{+\infty} |D_v^{\mathrm{RN}} \phi_{\mathcal{L}}|(u, v')\, \mathrm{d}v' = +\infty. \tag{6-120}$$

Then, for all $u \leq u_s$,

$$\int_{v_0}^{+\infty} |D_v \phi|(u, v')\, \mathrm{d}v' = \int_{v_0}^{+\infty} |D_v \psi|(u, v')\, \mathrm{d}v' = +\infty. \tag{6-121}$$

Conversely, (6-121) *implies* (6-120).

Proof. Note that $D_v^{\mathrm{RN}} \psi_{\mathcal{L}} = r D_v^{\mathrm{RN}} \phi_{\mathcal{L}} - (\Omega^2_{\mathrm{RN}}/2)\phi_{\mathcal{L}}$. Since r is lower-bounded on $\mathcal{LB}$ and in view of (6-46) (which also applies to $\phi_{\mathcal{L}}$), for all $u \leq u_s$

$$\int_{v_0}^{+\infty} |D_v^{\mathrm{RN}} \psi_{\mathcal{L}}|(u, v)\, \mathrm{d}v = +\infty.$$

Therefore, integrating (6-102) $\left(\text{since } s > \frac{3}{4} > \frac{2}{3}\right)$ we also obtain, for all $u \leq u_s$,

$$\int_{v_0}^{+\infty} |D_v \psi|(u, v)\, \mathrm{d}v = +\infty.$$

Since $D_v \psi = r D_v \phi + \lambda \phi$ and by (6-48), (6-51),

$$|\lambda \phi| \lesssim v^{1-3s}$$

is integrable; therefore, for all $u \leq u_s$,

$$\int_{v_0}^{+\infty} |D_v \phi|(u, v)\, \mathrm{d}v = +\infty.$$

The above also shows that (6-121) implies (6-120). □

Corollary 6.25. *Assume* $\phi_{\mathcal{H}^+} \in \mathcal{SL} - H$ (*defined in the proof of Corollary 5.27*). *Then* (6-121) *holds true.*

In the particular case $|q_0 e| \leq \epsilon(M, e, m^2)$ (*in particular if* $q_0 = 0$), *where* $\epsilon > 0$ *is defined in the proof of Corollary 5.27, for all* $\phi_{\mathcal{H}^+} \in \mathcal{SL} - L^1$ (6-121) *is satisfied.*

Proof. This follows from Corollary 5.27 (using that $\phi'_{\mathcal{L}}$ are $\phi_{\mathcal{L}}$ relate by a gauge transformation; hence $|\phi'_{\mathcal{L}}| = |\phi_{\mathcal{L}}|$ and $|D_v \phi'_{\mathcal{L}}| = |D_v \phi_{\mathcal{L}}|$) and Proposition 6.24. □

Corollary 6.25 thus concludes the proof of Part 1 of Theorem III. Now we turn to the proof of Part 2 of Theorem III.

Corollary 6.26. *Let $\phi_{\mathcal{H}^+}$ be given by either the profile of* (1-15) (*if* $m^2 > 0,\ q_0 = 0$) *or* (1-16) (*if* $m^2 = 0,\ q_0 \neq 0$) *or* (1-17) (*if* $m^2 > 0,\ q_0 \neq 0$). *Assume the condition $\mathcal{Z}_{\mathfrak{t}} \cap \Theta = \varnothing$. Then, there exists a $\delta(M, e, q_0, m^2) > 0$ sufficiently small such that $P_\delta \phi_{\mathcal{H}^+} \in L^1(\mathbb{R})$.*

Moreover, the condition $\mathcal{Z}_{\mathfrak{t}}(M, e, q_0, m^2) \cap \Theta(M, e, q_0, m^2) = \varnothing$ is generic in the sense that for given $m^2 \geq 0$, $q_0 \in \mathbb{R}$ with $m^2 \neq q_0^2$, the set of parameters (M, e) satisfying the conditions is the zero set of a nontrivial real-analytic function on $\{0 < |e| < M\}$. In particular, in view of Part 1 of Theorem III, we obtain Part 2 of Theorem III.

Proof. We start with the second claim. Fix $m^2 \geq 0,\ q_0 \in \mathbb{R}$ with $q_0^2 \neq m^2$. We define $f_{\pm,m^2,q_0}(M, e) := \mathfrak{t}(\pm m - q_0 e/r_+, M, e, q_0, m^2)$. By analyticity of $\mathfrak{t}$ (note that $\mathfrak{t}$ is the Wronskian of solutions to an ODE with analytic coefficients depending analytically on (ω, M, e)), we have that both $f_{\pm,m^2,q_0} \colon \{(M, e) \in \mathbb{R}^2 : 0 < |e| < M\} \to \mathbb{R}$ are analytic. It suffices to show that both $f_\pm$ are nontrivial. From the ODE energy identity, $|\mathfrak{t}|^2 = |\mathfrak{r}|^2 + \omega(\omega - \omega_{\mathrm{res}}) \geq \omega(\omega - \omega_{\mathrm{res}})$ we conclude

$$|f_\pm|^2 \geq \left(\pm m - \frac{q_0 e}{r_+}\right)\left(\pm m - \frac{q_0 e}{r_-}\right) \to \left(\pm m - \frac{q_0 e}{|e|}\right)^2 > 0$$

as $|e| \to M$. We used here that $m^2 \neq q_0^2$.

Now, fix $0 < \delta < \operatorname{dist}(\mathcal{Z}_{\mathfrak{t}}, \Theta)$. By Plancherel's theorem and the Cauchy–Schwarz inequality, it suffices to show that $\chi_\delta(\omega)\mathcal{F}(\phi_{\mathcal{H}^+})$ is in $H^{1/2+\tau}$ for some $\tau > 0$ (recalling the definition of $\chi_\delta(\omega)$ from Section 4E). Further, since χ_δ is smooth and has compact support ($\mathcal{Z}_{\mathfrak{t}}^\delta \subset [-|\omega_{\mathrm{res}}| - \delta, |\omega_{\mathrm{res}}| + \delta]$), and $\mathcal{F}(\phi_{\mathcal{H}^+}) \in L^2$, it suffices (e.g., by the Kato–Ponce inequality) to show that $\chi_\delta(\omega)\langle\partial_\omega\rangle^{1/2+\tau}\mathcal{F}(\phi_{\mathcal{H}^+})$ is in L^2. Thus, we need to show that $\mathcal{F}(\langle v\rangle^{1/2+\tau}\phi_{\mathcal{H}^+}) \in L^2(\mathcal{Z}_{\mathfrak{t}}^\delta)$ for some $\tau > 0$. We now fix $0 < \tau < s - \frac{1}{2}$. A direct adaption of the proofs of Lemma 6.21 and Proposition 6.22 then shows $\mathcal{F}(\langle v\rangle^{1/2+\tau}\phi_{\mathcal{H}^+}) \in L^\infty(\mathcal{Z}_{\mathfrak{t}}^\delta)$ from which the claim follows. □

Acknowledgements

The authors would like to express their gratitude to Mihalis Dafermos and Jonathan Luk for their support and many valuable comments on the manuscript. They also thank Jan Sbierski for insightful conversations. Van de Moortel is grateful to András Vasy for interesting discussions in the early stage of this project. Van de Moortel also wants to express gratitude to Amos Ori for very enlightening conversations related to this project. The authors are also grateful to an anonymous referee for several helpful comments and remarks to improve the manuscript. Kehle is supported by Dr. Max Rössler, the Walter Haefner Foundation and the ETH Zürich Foundation.

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Received 31 May 2021. Revised 7 Apr 2022. Accepted 21 Dec 2022.

CHRISTOPH KEHLE: christoph.kehle@eth-its.ethz.ch
kehle@mit.edu
Institute for Theoretical Studies, ETH Zürich, Zürich, Switzerland

and

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, United States

MAXIME VAN DE MOORTEL: mmoortel@princeton.edu
Department of Mathematics, Princeton University, Princeton, NJ, United States

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

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ANALYSIS & PDE

Volume 17 No. 5 2024