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A SEMICLASSICAL BIRKHOFF NORMAL FORM FOR CONSTANT-RANK MAGNETIC FIELDS

LÉO MORIN

This paper deals with classical and semiclassical nonvanishing magnetic fields on a Riemannian manifold of arbitrary dimension. We assume that the magnetic field $B = dA$ has constant rank and admits a discrete well. On the classical part, we exhibit a harmonic oscillator for the Hamiltonian $H = |p - A(q)|^2$ near the zero-energy surface: the cyclotron motion. On the semiclassical part, we describe the semiexcited spectrum of the magnetic Laplacian $\mathcal{L}_\hbar = (i\hbar d + A)^*(i\hbar d + A)$. We construct a semiclassical Birkhoff normal form for $\mathcal{L}_\hbar$ and deduce new asymptotic expansions of the smallest eigenvalues in powers of $\hbar^{1/2}$ in the limit $\hbar \to 0$. In particular we see the influence of the kernel of B on the spectrum: it raises the energies at order $\hbar^{3/2}$.

1. Introduction

1A. *Context.* We consider the semiclassical magnetic Laplacian with Dirichlet boundary conditions

$$\mathcal{L}_\hbar = (i\hbar \mathrm{d} + A)^*(i\hbar \mathrm{d} + A)$$

on a d-dimensional oriented Riemannian manifold (M, g), which is either compact with boundary, or the Euclidean $\mathbb{R}^d$. A denotes a smooth 1-form on M, the magnetic potential. The magnetic field is the 2-form $B = \mathrm{d}A$.

The spectral theory of the magnetic Laplacian has given rise to many investigations, and appeared to have very various behaviors according to the variations of B and the geometry of M. We refer to the books and review [Helffer and Kordyukov 2014; Fournais and Helffer 2010; Raymond 2017] for a description of these works. Here we focus on the Dirichlet realization of $\mathcal{L}_\hbar$, and we give a description of semiexcited states, eigenvalues of order $\mathcal{O}(\hbar)$ in the semiclassical limit $\hbar \to 0$. As explained in the above references, the *magnetic intensity* has a great influence on these eigenvalues, and one can define it in the following way.

Using the isomorphism $T_qM \simeq T_qM^*$ given by the metric, one can define the following skew-symmetric operator $\boldsymbol{B}(q) : T_qM \to T_qM$ by

$$B_q(X, Y) = g_q(X, \boldsymbol{B}(q)Y) \quad \text{for all } X, Y \in T_qM, \text{ for all } q \in M. \tag{1-1}$$

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Since the operator $\boldsymbol{B}(q)$ is skew-symmetric with respect to the scalar product g_q, its eigenvalues are purely imaginary and symmetric with respect to the real axis. We denote these repeated eigenvalues by

$$\pm i\beta_1(q), \quad \dots, \quad \pm i\beta_s(q), \quad 0,$$

with $\beta_j(q) > 0$. In particular, the rank of $\boldsymbol{B}(q)$ is $2s$ and may depend on q. However, we will focus on the constant-rank case. We denote by k the dimension of the kernel of $\boldsymbol{B}(q)$, so that $d = 2s + k$. The magnetic intensity (or "trace+") is the scalar-valued function

$$b(q) = \sum_{j=1}^{s} \beta_j(q).$$

The function b is continuous on M, but nonsmooth in general. We are interested in discrete magnetic wells and nonvanishing magnetic fields.

Assumption 1. We assume that:

- The magnetic intensity is nonvanishing and admits a unique global minimum $b_0 > 0$ at $q_0 \in M \setminus \partial M$.
- The rank of $\boldsymbol{B}(q)$ is constant equal to $2s > 0$ on a neighborhood Ω of q_0.
- $\beta_i(q_0) \neq \beta_j(q_0)$ for every $1 \leq i < j \leq s$, and the minimum of b is nondegenerate.
- In the noncompact case $M = \mathbb{R}^d$,

$$b_\infty := \liminf_{|q| \to +\infty} b(q) > b_0$$

and there exists a $C > 0$ such that

$$|\partial_\ell \boldsymbol{B}_{ij}(q)| \leq C(1 + |\boldsymbol{B}(q)|) \quad \text{for all } \ell, i, j, \text{ for all } q \in \mathbb{R}^d.$$

Remark 1.1. Since the nonzero eigenvalues of $\boldsymbol{B}$ are simple at q_0, the function b is smooth on a neighborhood of q_0. In particular, it is meaningful to say that the minimum of b is nondegenerate.

Under Assumption 1, the following useful inequality was proven in [Helffer and Mohamed 1996]. There is a $C_0 > 0$ such that, for $\hbar$ small enough,

$$(1 + \hbar^{1/4} C_0)\langle \mathcal{L}_\hbar u, u\rangle \geq \int_M \hbar(b(q) - \hbar^{1/4} C_0)|u(q)|^2 \,\mathrm{d}q \quad \text{for all } u \in \mathrm{Dom}(\mathcal{L}_\hbar). \tag{1-2}$$

Remark 1.2. Actually, one has the better inequality obtained replacing $\hbar^{1/4}$ by $\hbar$. This was proved in [Guillemin and Uribe 1988] in the case of a nondegenerate B, in [Borthwick and Uribe 1996] in the constant rank case, and in [Ma and Marinescu 2002] in a more general setting.

Remark 1.3. Using this inequality, one can prove Agmon-like estimates for the eigenfunctions of $\mathcal{L}_\hbar$. Namely, the eigenfunctions associated to an eigenvalue $< b_1\hbar$ are exponentially small outside $K_{b_1} = \{q : b(q) \leq b_1\}$. We will use this result to localize our analysis to the neighborhood Ω of q_0. In particular, the greater b_1 is, the larger Ω must be.

Under Assumption 1, estimates on the ground states of $\mathcal{L}_\hbar$ in the semiclassical limit $\hbar \to 0$ were proven in several works, especially in dimensions $d = 2, 3$.

On $M=\mathbb{R}^2$, asymptotics for the j-th eigenvalue of $\mathcal{L}_\hbar$

$$\lambda_j(\mathcal{L}_\hbar)=b_0\hbar+(\alpha(2j-1)+c_1)\hbar^2+o(\hbar^2) \tag{1-3}$$

with explicit α, $c_1\in\mathbb{R}$ were proven in [Helffer and Morame 2001] (for $j=1$) and [Helffer and Kordyukov 2011] ($j\geq 1$). Actually, this second paper contains a description of some higher eigenvalues. They proved that, for any integers n, $j\in\mathbb{N}$, there exist $\hbar_{jn}>0$ and for $\hbar\in(0,\hbar_{jn})$ an eigenvalue $\lambda_{n,j}(\hbar)\in\operatorname{sp}(\mathcal{L}_\hbar)$ such that

$$\lambda_{n,j}(\hbar)=(2n-1)(b_0\hbar+((2j-1)\alpha+c_n)\hbar^2)+o(\hbar^2)$$

for another explicit constant c_n. In particular, it gives a description of *some* semiexcited states (of order $(2n-1)b_0\hbar$). Finally, [Raymond and Vũ Ngọc 2015] (and [Helffer and Kordyukov 2015]) gives a description of the whole spectrum below $b_1\hbar$, for any fixed $b_1\in(b_0,b_\infty)$. More precisely, they proved that this part of the spectrum is given by a family of effective operators $\mathcal{N}_\hbar^{[n]}$ ($n\in\mathbb{N}$) modulo $\mathcal{O}(\hbar^\infty)$. These effective operators are $\hbar$-pseudodifferential operators with principal symbol given by the function $\hbar(2n-1)b$. More interestingly, they explained why the two quantum oscillators

$$(2n-1)b_0\hbar \quad\text{and}\quad (2j-1)\alpha\hbar^2$$

appearing in the eigenvalue asymptotics correspond to two oscillatory motions in classical dynamics: the cyclotron motion and a rotation around the minimum point of b. The results of Raymond and Vũ Ngọc were generalized to an arbitrary d-dimensional Riemannian manifold in [Morin 2022b], under the assumption $k=0$ ($\boldsymbol{B}(q)$ has full rank), proving in particular similar estimates (1-3) in a general setting. Actually, these eigenvalue estimates were proven simultaneously in [Kordyukov 2019] in the context of the Bochner Laplacian.

We are interested on the influence of the kernel of $\boldsymbol{B}$ ($k>0$). Since the rank of $\boldsymbol{B}$ is even, this kernel always exists in odd dimensions: if $d=3$, the kernel directions correspond to the usual field lines. On $M=\mathbb{R}^3$, Helffer and Kordyukov [2013] proved the existence of $\lambda_{nmj}(\hbar)\in\operatorname{sp}(\mathcal{L}_\hbar)$ such that

$$\lambda_{nmj}(\hbar)=(2n-1)b_0\hbar+(2n-1)^{1/2}(2m-1)\nu_0\hbar^{3/2}+((2n-1)(2j-1)\alpha+c_{nm})\hbar^2+\mathcal{O}(\hbar^{9/4})$$

for some $\nu_0>0$ and α, $c_{nm}\in\mathbb{R}$. Motivated by this result and the 2-dimensional case, Helffer, Kordyukov, Raymond and Vũ Ngọc [Helffer et al. 2016] gave a description of the whole spectrum below $b_1\hbar$, proving in particular the eigenvalue estimates

$$\lambda_j(\mathcal{L}_\hbar)=b_0\hbar+\nu_0\hbar^{3/2}+\alpha(2j-1)\hbar^2+\mathcal{O}(\hbar^{5/2}). \tag{1-4}$$

Their results exhibit a new classical oscillatory motion in the directions of the field lines, corresponding to the quantum oscillator $(2m-1)\nu_0\hbar^{3/2}$.

The aim of this paper is to generalize the results of [Helffer et al. 2016] to an arbitrary Riemannian manifold M, under Assumption 1. In particular we describe the influence of the kernel of $\boldsymbol{B}$ in a general geometric and dimensional setting. Their approach, which we adapt, is based on a *semiclassical Birkhoff normal form*. The *classical* Birkhoff normal form has a long story in physics and goes back to [Delaunay 1860; Lindstedt 1883]. This formal normal form was the starting point of a lot of studies on stability near equilibrium, and KAM theory (after [Kolmogorov 1954; Arnold 1963; Moser 1962]). The name of this normal form comes from [Birkhoff 1927; Gustavson 1966]. We refer to the books [Moser 1968; Hofer and

Zehnder 1994] for precise statements. Our approach here relies on a quantization. Physicists and quantum chemists already noticed in the 1980s that a quantum analogue of the Birkhoff normal form could be used to compute energies of molecules [Delos et al. 1983; Jaffé and Reinhardt 1982; Marcus 1985; Shirts and Reinhardt 1982]. Joyeux and Sugny [2002] also used such techniques to describe the dynamics of excited states. Sjöstrand [1992] constructed a semiclassical Birkhoff normal form for a Schrödinger operator $-\hbar^2\Delta + V$ using the Weyl quantization, to make a mathematical study of semiexcited states. Raymond and Vũ Ngọc [2015] had the idea to adapt this method for $\mathcal{L}_\hbar$ on $\mathbb{R}^2$, and with Helffer and Kordyukov on $\mathbb{R}^3$ [Helffer et al. 2016]. This method is reminiscent of Ivrii's approach [2019].

1B. ***Main results.*** The first idea is to link the classical dynamics of a particle in the magnetic field B with the spectrum of $\mathcal{L}_\hbar$ using pseudodifferential calculus. Indeed, $\mathcal{L}_\hbar$ is an $\hbar$-pseudodifferential operator with principal symbol

$$H(q,p) = |p - A_q|^2 \quad \text{for all } p \in T_qM^*, \text{ for all } q \in M,$$

and H is the classical Hamiltonian associated to the magnetic field B. One can use this property to prove that, in the phase space T^*M, the eigenfunctions (with eigenvalue $< b_1\hbar$) are microlocalized on an arbitrarily small neighborhood of

$$\Sigma = H^{-1}(0) \cap T^*\Omega = \{(q,p) \in T^*\Omega : p = A_q\}.$$

Hence, the second main idea is to find a normal form for H on a neighborhood of Σ. Namely, we find canonical coordinates near Σ in which H has a "simple" form. The symplectic structure of Σ as a submanifold of T^*M is thus of great interest. One can see that the restriction of the canonical symplectic form $\mathrm{d}p \wedge \mathrm{d}q$ on T^*M to Σ is given by B (Lemma 2.1), and when B has constant rank, one can find Darboux coordinates $\varphi : \Omega' \subset \mathbb{R}^{2s+k}_{(y,\eta,t)} \to \Omega$ such that

$$\varphi^* B = \mathrm{d}\eta \wedge \mathrm{d}y,$$

up to shrinking Ω. We will start from these coordinates to get the following normal form for H.

Theorem 1.4. *Under Assumption 1, there exists a diffeomorphism*

$$\Phi_1 : U_1' \subset \mathbb{R}^{4s+2k} \to U_1 \subset T^*M$$

between neighborhoods U_1' of 0 *and U_1 of Σ such that*

$$\widehat{H}(x,\xi,y,\eta,t,\tau) := H \circ \Phi_1(x,\xi,y,\eta,t,\tau)$$

satisfies (with the notation $\hat{\beta}_j = \beta_j \circ \varphi$)

$$\widehat{H} = \langle M(y,\eta,t)\tau,\tau\rangle + \sum_{j=1}^{s} \hat{\beta}_j(y,\eta,t)(\xi_j^2 + x_j^2) + \mathcal{O}((x,\xi,\tau)^3)$$

uniformly with respect to (y,η,t) for some (y,η,t)-dependent positive definite matrix $M(y,\eta,t)$. Moreover,

$$\Phi_1^*(\mathrm{d}p \wedge \mathrm{d}q) = \mathrm{d}\xi \wedge \mathrm{d}x + \mathrm{d}\eta \wedge \mathrm{d}y + \mathrm{d}\tau \wedge \mathrm{d}t.$$

Remark 1.5. We will use the following notation for our canonical coordinates:

$$z=(x,\xi)\in\mathbb{R}^{2s},\quad w=(y,\eta)\in\mathbb{R}^{2s},\quad \tau=(t,\tau)\in\mathbb{R}^{2k}.$$

This theorem gives the Taylor expansion of H on a neighborhood of Σ. In particular $(x,\xi,\tau)\in\mathbb{R}^d$ measures the distance to Σ, whereas $(y,\eta,t)\in\mathbb{R}^d$ are canonical coordinates on Σ.

Remark 1.6. This theorem exhibits the harmonic oscillator $\xi_j^2+x_j^2$ in the expansion of H. This oscillator, which is due to the nonvanishing magnetic field, corresponds to the well-known cyclotron motion.

Actually, one can use the *Birkhoff normal form* algorithm to improve the remainder. Using this algorithm, we can change the $\mathcal{O}((x,\xi)^3)$ remainder into an explicit function of $\xi_j^2+x_j^2$, plus some smaller remainders $\mathcal{O}((x,\xi)^r)$. This remainder power r is restricted by resonances between the coefficients β_j. Thus, we take an integer $r_1\in\mathbb{N}$ such that,

$$\text{for all } \alpha\in\mathbb{Z}^s,\quad 0<|\alpha|<r_1 \quad\Longrightarrow\quad \sum_{j=1}^{s}\alpha_j\beta_j(q_0)\neq 0. \tag{1-5}$$

Here, $|\alpha|=\sum_j|\alpha_j|$. Moreover, we can use the pseudodifferential calculus to apply the Birkhoff algorithm to $\mathcal{L}_\hbar$, changing the classical oscillator $\xi_j^2+x_j^2$ into the quantum harmonic oscillator

$$\mathcal{I}_\hbar^{(j)}=-\hbar^2\partial_{x_j}^2+x_j^2,$$

whose spectrum consists of the simple eigenvalues $(2n-1)\hbar,\ n\in\mathbb{N}$. Following this idea we construct a normal form for $\mathcal{L}_\hbar$ in Theorem 3.4. We also deduce a description of its spectrum.

Theorem 1.7. *Let $\varepsilon>0$. Under Assumption 1, there exist $b_1\in(b_0,b_\infty)$, an integer $N_{\max}>0$ and a compactly supported function $f_1^\star\in\mathcal{C}^\infty(\mathbb{R}^{2s+2k}\times\mathbb{R}^s\times[0,1))$ such that*

$$|f_1^\star(y,\eta,t,\tau,I,\hbar)|\lesssim\big((|I|+\hbar)^2+|\tau|(|I|+\hbar)+|\tau|^3\big)$$

satisfying the following properties. For $n\in\mathbb{N}^s$, denote by $\mathcal{N}_\hbar^{[n]}$ the $\hbar$-pseudodifferential operator in (y,t) with symbol

$$N_\hbar^{[n]}=\langle M(y,\eta,t)\tau,\tau\rangle+\sum_{j=1}^{s}\hat\beta_j(y,\eta,t)(2n_j-1)\hbar+f_1^\star(y,\eta,t,\tau,(2n-1)\hbar,\hbar).$$

For $\hbar\ll 1$, there exists a bijection

$$\Lambda_\hbar:\operatorname{sp}(\mathcal{L}_\hbar)\cap(-\infty,b_1\hbar)\to\bigcup_{|n|\le N_{\max}}\operatorname{sp}(\mathcal{N}_\hbar^{[n]})\cap(-\infty,b_1\hbar)$$

such that $\Lambda_\hbar(\lambda)=\lambda+\mathcal{O}(\hbar^{r_1/2-\varepsilon})$ uniformly with respect to λ.

Remark 1.8. In this theorem $\operatorname{sp}(\mathcal{A})$ denotes the *repeated* eigenvalues of an operator $\mathcal{A}$, so that there might be some multiple eigenvalues, but $\Lambda_\hbar$ preserves this multiplicity. We only consider self-adjoint operators with discrete spectrum.

Remark 1.9. One should care of how large b_1 can be. As mentioned above, the eigenfunctions of energy $<b_1\hbar$ are exponentially small outside $K_{b_1}=\{q\in M : b(q)\le b_1\}$. Thus, we will chose b_1 such that $K_{b_1}\subset\Omega$, where Ω is some neighborhood of q_0. Hence the larger Ω is, the greater b_1 can be. However, there are three restrictions on the size of Ω:

- The rank of $\boldsymbol{B}(q)$ is constant on Ω.
- There exist canonical coordinates φ on Ω (i.e., such that $\varphi^* B = \mathrm{d}\eta \wedge \mathrm{d}y$).
- There is no resonance in Ω:

$$\text{for all } q \in \Omega,\ \text{for all } \alpha \in \mathbb{Z}^s, \quad 0 < |\alpha| < r_1 \quad \Longrightarrow \quad \sum_{j=1}^{s} \alpha_j \beta_j(q) \neq 0.$$

Of course the last condition is the most restrictive. However, if we forget the second condition, which is of global geometric nature, given a magnetic field and an r_1 one can estimate an associated b_1 satisfying the third condition. In particular we can construct simple examples on $\mathbb{R}^d$ such that the threshold $b_1\hbar$ includes several Landau levels.

Remark 1.10. If $k = 0$ we recover the result of [Morin 2022b]. Here we want to study the influence of a nonzero kernel $k > 0$. This result generalizes the result of [Helffer et al. 2016], which corresponds to $d = 3$, $s = k = 1$ on the Euclidean $\mathbb{R}^3$. However, this generalization is not straightforward since the magnetic geometry is much more complicated in higher dimensions, in particular if $k > 1$. Moreover, there is a new phenomena in higher dimensions: resonances between the functions β_j (as in [Morin 2022b]).

The spectrum of $\mathcal{L}_\hbar$ in $(-\infty, b_1\hbar)$ is given by the operators $\mathcal{N}_\hbar^{[n]}$. Actually if we choose b_1 small enough, it is only given by the first operator $\mathcal{N}_\hbar^{[1]}$ (here we denote the multi-integer $1 = (1, \dots, 1) \in \mathbb{N}^s$). Hence in the second part of this paper, we study the spectrum $\mathcal{N}_\hbar^{[1]}$ using a second Birkhoff normal form. Indeed, the symbol of $\mathcal{N}_\hbar^{[1]}$ is

$$N_\hbar^{[1]}(w, t, \tau) = \langle M(w, t)\tau, \tau \rangle + \hbar \hat{b}(w, t) + \mathcal{O}(\hbar^2) + \mathcal{O}(\tau\hbar) + \mathcal{O}(\tau^3),$$

so if we denote by $s(w)$ the minimum point of $t \mapsto \hat{b}(w, t)$ (which is unique on a neighborhood of 0), we get the expansion

$$N_\hbar^{[1]}(w, t, \tau) = \langle M(w, s(w))\tau, \tau \rangle + \frac{\hbar}{2} \Big\langle \frac{\partial^2 \hat{b}}{\partial t^2}(w, s(w)) \cdot (t - s(w)), t - s(w) \Big\rangle + \cdots, \tag{1-6}$$

where we will show that the remaining terms are only perturbations. As explained in Section 5, in (1-6) we can recognize a harmonic oscillator with frequencies $\sqrt{\hbar}\nu_j(w)$ $(1 \le j \le k)$, where $(\nu_j^2(w))_{1 \le j \le k}$ are the eigenvalues of the symmetric matrix

$$M(w, s(w))^{1/2} \cdot \tfrac{1}{2} \partial_t^2 \hat{b}(w, s(w)) \cdot M(w, s(w))^{1/2}.$$

These frequencies are smooth nonvanishing functions of w on a neighborhood of 0, as soon as we assume that they are simple.

Assumption 2. For indices $1 \le i < j \le k$, we have $\nu_i(0) \neq \nu_j(0)$.

We fix an integer $r_2 \in \mathbb{N}$ such that,

$$\text{for all } \alpha \in \mathbb{Z}^k, \quad 0 < |\alpha| < r_2 \quad \Longrightarrow \quad \sum_{j=1}^{k} \alpha_j \nu_j(0) \neq 0,$$

and we construct a normal form for $\mathcal{N}_\hbar^{[1]}$ in Theorem 5.4. Again, we deduce a description of its spectrum.

Theorem 1.11. *Let $c > 0$ and $\delta \in \left(0, \frac{1}{2}\right)$. Under Assumptions 1 and 2, with $k > 0$, there exists a compactly supported function $f_2^\star \in \mathcal{C}^\infty(\mathbb{R}^{2s} \times \mathbb{R}^k \times [0,1))$ such that*

$$|f_2^\star(y, \eta, J, \sqrt{\hbar})| \lesssim (|J| + \sqrt{\hbar})^2$$

satisfying the following properties. For $n \in \mathbb{N}^k$, denote by $\mathcal{M}_\hbar^{[n]}$ the $\hbar$-pseudodifferential operator in y with symbol

$$M_\hbar^{[n]}(y, \eta) = \hat{b}(y, \eta, s(y, \eta)) + \sqrt{\hbar} \sum_{j=1}^{k} \nu_j(y, \eta)(2n_j - 1) + f_2^\star(y, \eta, (2n-1)\sqrt{\hbar}, \sqrt{\hbar}).$$

For $\hbar \ll 1$, there exists a bijection

$$\Lambda_\hbar : \mathrm{sp}(\mathcal{N}_\hbar^{[1]}) \cap (-\infty, (b_0 + c\hbar^\delta)\hbar) \to \bigcup_{n \in \mathbb{N}^k} \mathrm{sp}(\hbar \mathcal{M}_\hbar^{[n]}) \cap (-\infty, (b_0 + c\hbar^\delta)\hbar)$$

such that $\Lambda_\hbar(\lambda) = \lambda + \mathcal{O}(\hbar^{1+\delta r_2/2})$ uniformly with respect to λ.

Remark 1.12. The threshold $b_0 + c\hbar^\delta$ is needed to get microlocalization of the eigenfunctions of $\mathcal{N}_\hbar^{[1]}$ in an arbitrarily small neighborhood of $\tau = 0$.

Remark 1.13. This second harmonic oscillator (in variables (t, τ)) corresponds to a classical oscillation in the directions of the field lines. We see that this new motion, due to the kernel of $\boldsymbol{B}$, induces powers of $\sqrt{\hbar}$ in the spectrum.

As a corollary, we get a description of the low-lying eigenvalues of $\mathcal{L}_\hbar$ by the effective operator $\hbar\mathcal{M}_\hbar^{[1]}$.

Corollary 1.14. *Let $\varepsilon > 0$ and $c \in (0, \min_j \nu_j(0))$. Define $\nu(0) = \sum_j \nu_j(0)$ and $r = \min(2r_1, r_2 + 4)$. Under Assumptions 1 and 2, with $k > 0$, there exists a bijection*

$$\Lambda_\hbar : \mathrm{sp}(\mathcal{L}_\hbar) \cap (-\infty, \hbar b_0 + \hbar^{3/2}(\nu(0) + 2c)) \to \mathrm{sp}(\hbar\mathcal{M}_\hbar^{[1]}) \cap (-\infty, \hbar b_0 + \hbar^{3/2}(\nu(0) + 2c))$$

such that $\Lambda_\hbar(\lambda) = \lambda + \mathcal{O}(\hbar^{r/4-\varepsilon})$ uniformly with respect to λ.

We deduce the following eigenvalue asymptotics.

Corollary 1.15. *Under the assumptions of Corollary 1.14, for $j \in \mathbb{N}$, the j-th eigenvalue of $\mathcal{L}_\hbar$ admits an expansion*

$$\lambda_j(\mathcal{L}_\hbar) = \hbar \sum_{\ell=0}^{\lfloor r/2 \rfloor - 2} \alpha_{j\ell} \hbar^{\ell/2} + \mathcal{O}(\hbar^{r/4-\varepsilon}),$$

with coefficients $\alpha_{j\ell} \in \mathbb{R}$ such that

$$\alpha_{j,0} = b_0, \quad \alpha_{j,1} = \sum_{j=1}^{k} \nu_j(0), \quad \alpha_{j,2} = E_j + c_0,$$

where $c_0 \in \mathbb{R}$ and $\hbar E_j$ is the j-th eigenvalue of an s-dimensional harmonic oscillator.

Remark 1.16. Note $\hbar E_j$ is the j-th eigenvalue of a harmonic oscillator whose symbol is given by the Hessian at $w = 0$ of $\hat{b}(w, s(w))$. Hence, it corresponds to a third classical oscillatory motion: a rotation in the space of field lines.

Remark 1.17. The asymptotics

$$\lambda_j(\mathcal{L}_\hbar) = b_0\hbar + \nu(0)\hbar^{3/2} + (E_j + c_0)\hbar^2 + o(\hbar^2)$$

were unknown before, except in the special 3-dimensional case $M = \mathbb{R}^3$ in [Helffer et al. 2016].

1C. ***Related questions and perspectives.*** In this paper, we are restricted to energies $\lambda < b_1\hbar$, and as mentioned in Remark 1.9, the threshold $b_1 > b_0$ is limited by three conditions, including the nonresonance one:

$$\text{for all } q \in \Omega, \text{ for all } \alpha \in \mathbb{Z}^s, \quad 0 < |\alpha| < r_1 \quad \Longrightarrow \quad \sum_{j=1}^{s} \alpha_j \beta_j(q) \neq 0.$$

It would be interesting to study the influence of resonances between the functions β_j on the spectrum of $\mathcal{L}_\hbar$. Maybe the Grushin techniques could help, as in [Helffer and Kordyukov 2015] for instance. A Birkhoff normal form was given in [Charles and Vũ Ngọc 2008] for a Schrödinger operator $-\hbar^2\Delta + V$ with resonances, but the situation is somehow simpler, since the analogues of $\beta_j(q)$ are independent of q in this context.

We are also restricted by the existence of Darboux coordinates φ on (Σ, B) such that $\varphi^* B = \mathrm{d}\eta \wedge \mathrm{d}y$. Indeed, the coordinates (y, η) on Σ are necessary to use the Weyl quantization. To study the influence of the global geometry of B, one should consider another quantization method for the presymplectic manifold (Σ, B). In the symplectic case, for instance in dimension $d = 2$, a Toeplitz quantization may be useful. This quantization is linked to the complex structure induced by B on Σ, and the operator $\mathcal{L}_\hbar$ can be linked with this structure in the following way:

$$\mathcal{L}_\hbar = 4\hbar^2\Big(\bar{\partial} + \frac{i}{2\hbar}A\Big)^*\Big(\bar{\partial} + \frac{i}{2\hbar}A\Big) + \hbar B = 4\hbar^2\bar{\partial}_A^*\bar{\partial}_A + \hbar B,$$

with

$$A = A_1 + iA_2, \quad B = \partial_1 A_2 - \partial_2 A_1, \quad 2\bar{\partial} = \partial_1 + i\partial_2.$$

In [Tejero Prieto 2006], this is used to compute the spectrum of $\mathcal{L}_\hbar$ on a bidimensional Riemann surface M with constant curvature and constant magnetic field. See also [Charles 2020; Kordyukov 2022], where semiexcited states for constant magnetic fields in higher dimensions are considered.

If the 2-form B is not exact, we usually consider a Bochner Laplacian on the p-th tensor product of a complex line bundle L over M, with curvature B. This Bochner Laplacian Δ_p depends on $p \in \mathbb{N}$, and the limit $p \to +\infty$ is interpreted as the semiclassical limit. The Bochner Laplacian Δ_p is a good generalization of the magnetic Laplacian because *locally* it can be written $(1/\hbar^2)(i\hbar\nabla + \boldsymbol{A})^2$, where the potential $\boldsymbol{A}$ is a local primitive of B, and $\hbar = p^{-1}$. For details, we refer to [Kordyukov 2019; 2020; Marinescu and Savale 2018]. Kordyukov [2019] constructed quasimodes for Δ_p in the case of a symplectic B and discrete wells. He proved expansions

$$\lambda_j(\Delta_p) \sim \sum_{\ell \geq 0} \alpha_{j\ell}\, p^{-\ell/2}.$$

Our work also gives such expansions for Δ_p as explained in [Morin 2022a].

In this paper, we only mention the study of the eigenvalues of $\mathcal{L}_\hbar$: what about the eigenfunctions? WKB expansions for the j-th eigenfunction were constructed on $\mathbb{R}^2$ in [Bonthonneau and Raymond 2020] and on a 2-dimensional Riemannian manifold in [Bonthonneau et al. 2021a]. We do not know how to construct magnetic WKB solutions in higher dimensions. This article suggests that the directions corresponding to the kernel of B could play a specific role.

Another related question is the decreasing of the real eigenfunctions. Agmon estimates only give a $\mathcal{O}(e^{-c/\sqrt{\hbar}})$ decay outside any neighborhood of q_0, but the 2-dimensional WKB suggests a $\mathcal{O}(e^{-c/\hbar})$ decay. Recently Bonthonneau, Raymond and Vũ Ngọc [Bonthonneau et al. 2021b] proved this on $\mathbb{R}^2$ using the FBI transform to work on the phase space $T^*\mathbb{R}^2$. This kind of question is motivated by the study of the tunneling effect: the exponentially small interaction between two magnetic wells for example.

Finally, we only have investigated the spectral theory of the stationary Schrödinger equation with a pure magnetic field; it would be interesting to describe the long-time dynamics of the full Schrödinger evolution, as was done in the Euclidean 2-dimensional case in [Boil and Vũ Ngọc 2021].

1D. ***Structure of the paper.*** In Section 2 we prove Theorem 1.4, describing the symbol H of $\mathcal{L}_\hbar$ on a neighborhood of $\Sigma = H^{-1}(0)$. In Section 3 we construct the normal form, first in a space of formal series (Section 3B) and then the quantized version $\mathcal{N}_\hbar$ (Section 3C). In Section 4 we prove Theorem 1.7. For this we describe the spectrum of $\mathcal{N}_\hbar$ (Section 4A), then we prove microlocalization properties on the eigenfunctions of $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar$ (Section 4B), and finally we compare the spectra of $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar$ (Section 4C).

In Section 5 we focus on Theorem 1.11 which describes the spectrum of the effective operator $\mathcal{N}_\hbar^{[1]}$. In Section 5A we study its symbol, in Section 5B we construct a second formal Birkhoff normal form, and in Section 5C the quantized version $\mathcal{M}_\hbar$. In Section 5D we compare the spectra of $\mathcal{N}_\hbar^{[1]}$ and $\mathcal{M}_\hbar$.

Finally, Sections 6 and 7 are dedicated to the proofs of Corollaries 1.14 and 1.15 respectively.

2. Geometry of the classical Hamiltonian

2A. ***Notation.*** $\mathcal{L}_\hbar$ is an $\hbar$-pseudodifferential operator on M with principal symbol H:

$$H(q,p) = |p - A_q|^2_{g_q^*}, \quad p \in T_q^*M, \ q \in M.$$

Here, T^*M denotes the cotangent bundle of M, and $p \in T_q^*M$ is a linear form on T_qM. The scalar product g_q on T_qM induces a scalar product g_q^* on T_q^*M, and $|\cdot|_{g_q^*}$ denotes the associated norm. In this section we prove Theorem 1.4, thus describing H on a neighborhood of its minimum:

$$\Sigma = \{(q,p) \in T^*M : q \in \Omega, \ p = A_q\}.$$

Recall that Ω is a small neighborhood of $q_0 \in M \setminus \partial M$. We will construct canonical coordinates $(z, w, v) \in \mathbb{R}^{2d}$ on Ω, with

$$z = (x, \xi) \in \mathbb{R}^{2s}, \quad w = (y, \eta) \in \mathbb{R}^{2s}, \quad v = (t, \tau) \in \mathbb{R}^{2k}.$$

$\mathbb{R}^{2d}$ is endowed with the canonical symplectic form

$$\omega_0 = \mathrm{d}\xi \wedge \mathrm{d}x + \mathrm{d}\eta \wedge \mathrm{d}y + \mathrm{d}\tau \wedge \mathrm{d}t.$$

We will identify Σ with

$$\Sigma' = \{(x, \xi, y, \eta, t, \tau) \in \mathbb{R}^{2d} : x = \xi = 0,\ \tau = 0\} = \mathbb{R}^{2s+k}_{(y,\eta,t)} \times \{0\}.$$

We will use several lemmas to prove Theorem 1.4. Before constructing the diffeomorphism Φ_1^{-1} on a neighborhood U_1 of Σ, we will first define it on Σ. Thus we need to understand the structure of Σ induced by the symplectic structure on T^*M (Section 2B). Then we will construct Φ_1 and finally prove Theorem 1.4 (Section 2C).

2B. ***Structure of Σ.*** Recall that on T^*M we have the Liouville 1-form α defined by

$$\alpha_{(q,p)}(\mathcal{V}) = p((\mathrm{d}\pi)_{(q,p)}\mathcal{V}) \quad \text{for all } (q, p) \in T^*M,\ \mathcal{V} \in T_{(q,p)}(T^*M),$$

where $\pi : T^*M \to M$ is the canonical projection: $\pi(q, p) = q$, and $\mathrm{d}\pi$ is its differential. T^*M is endowed with the symplectic form $\omega = \mathrm{d}\alpha$. Σ is a d-dimensional submanifold of T^*M which can be identified with Ω using

$$j : q \in \Omega \mapsto (q, A_q) \in \Sigma$$

and its inverse, which is π.

Lemma 2.1. *The restriction of ω to Σ is $\omega_\Sigma = \pi^* B$.*

Proof. Fix $q \in \Omega$ and $Q \in T_q M$. Then

$$(j^*\alpha)_q(Q) = \alpha_{j(q)}((\mathrm{d}j)Q) = A_q((\mathrm{d}\pi) \circ (\mathrm{d}j)Q) = A_q(Q),$$

because $\pi \circ j = \mathrm{Id}$. Thus $j^*\alpha = A$ and $\alpha_\Sigma = \pi^* j^* \alpha = \pi^* A$. Taking the exterior derivative we get

$$\omega_\Sigma = \mathrm{d}\alpha_\Sigma = \pi^*(\mathrm{d}A) = \pi^* B. \qquad \square$$

Since B is a closed 2-form with constant rank equal to $2s$, $(\Sigma, \pi^* B)$ is a presymplectic manifold. It is equivalent to (Ω, B), using j. We recall the Darboux lemma, which states that such a manifold is locally equivalent to $(\mathbb{R}^{2s+k}, \mathrm{d}\eta \wedge \mathrm{d}y)$.

Lemma 2.2. *Up to shrinking Ω, there exists an open subset Σ' of $\mathbb{R}^{2s+k}_{(y,\eta,t)}$ and a diffeomorphism $\varphi : \Sigma' \to \Omega$ such that $\varphi^* B = \mathrm{d}\eta \wedge \mathrm{d}y$.*

One can always take any coordinate system on Ω. Up to working in these coordinates, it is enough to consider the case $M = \mathbb{R}^d$ with

$$H(q, p) = \sum_{k,\ell=1}^{d} g^{k\ell}(q)(p_k - A_k(q))(p_\ell - A_\ell(q)), \quad (q, p) \in T^*\mathbb{R}^d \simeq \mathbb{R}^{2d},$$

to prove Theorem 1.4. This is what we will do. In coordinates, ω is given by

$$\omega = \mathrm{d}p \wedge \mathrm{d}q = \sum_{j=1}^{d} \mathrm{d}p_j \wedge \mathrm{d}q_j$$

and Σ is the submanifold

$$\Sigma = \{(q, \boldsymbol{A}(q)) : q \in \Omega\} \subset \mathbb{R}^{2d},$$

and $j \circ \varphi : \Sigma' \to \Sigma$.

In order to extend $j \circ \varphi$ to a neighborhood of Σ' in $\mathbb{R}^{2d}$ *in a symplectic way*, it is convenient to split the tangent space $T_{j(q)}(\mathbb{R}^{2d})$ according to tangent and normal directions to Σ. This is the purpose of the following two lemmas.

Lemma 2.3. *Fix $j(q) = (q, \boldsymbol{A}(q)) \in \Sigma$. Then the tangent space to Σ is*

$$T_{j(q)}\Sigma = \{(Q, P) \in \mathbb{R}^{2d} : P = \nabla_q \boldsymbol{A} \cdot Q\}.$$

Moreover, the ω-orthogonal $T_{j(q)}\Sigma^{\perp}$ is

$$T_{j(q)}\Sigma^{\perp} = \{(Q, P) \in \mathbb{R}^{2d} : P = (\nabla_q \boldsymbol{A})^T \cdot Q\}.$$

Finally,

$$T_{j(q)}\Sigma \cap T_{j(q)}\Sigma^{\perp} = \operatorname{Ker}(\pi^* B).$$

Proof. Since Σ is the graph of $q \mapsto \boldsymbol{A}(q)$, its tangent space is the graph of the differential $Q \mapsto (\nabla_q \boldsymbol{A}) \cdot Q$. In order to characterize $T\Sigma^{\perp}$, note that the symplectic form $\omega = \mathrm{d}p \wedge \mathrm{d}q$ is defined by

$$\omega_{(q,p)}((Q_1, P_1), (Q_2, P_2)) = \langle P_2, Q_1\rangle - \langle P_1, Q_2\rangle, \tag{2-1}$$

where $\langle\,\cdot\,,\cdot\,\rangle$ denotes the Euclidean scalar product on $\mathbb{R}^d$. Thus,

$$\begin{aligned}
(Q, P) \in T_{j(q)}\Sigma^{\perp} \quad &\Longleftrightarrow \quad \omega_{j(q)}((Q_0, \nabla_q \boldsymbol{A} \cdot Q_0), (Q, P)) = 0 \quad \text{for all } Q_0 \in \mathbb{R}^d \\
&\Longleftrightarrow \quad \langle P, Q_0\rangle - \langle(\nabla_q \boldsymbol{A}) \cdot Q_0, Q\rangle = 0 \quad \text{for all } Q_0 \in \mathbb{R}^d \\
&\Longleftrightarrow \quad \langle P - (\nabla_q \boldsymbol{A})^T \cdot Q, Q_0\rangle = 0 \quad \text{for all } Q_0 \in \mathbb{R}^d \\
&\Longleftrightarrow \quad P = (\nabla_q \boldsymbol{A})^T \cdot Q.
\end{aligned}$$

Finally, with Lemma 2.1 we know that the restriction of ω to $T\Sigma$ is given by $\pi^* B$. Hence, $T_{j(q)}\Sigma \cap T_{j(q)}\Sigma^{\perp}$ is the set of $(Q, P) \in T_{j(q)}\Sigma$ such that

$$\pi^* B((Q, P), (Q_0, P_0)) = 0 \quad \text{for all } (Q_0, P_0) \in T_{j(q)}\Sigma.$$

It is the kernel of $\pi^* B$. □

Now we define specific basis of $T_{j(q)}\Sigma$ and its orthogonal. Since $\boldsymbol{B}(q)$ is skew-symmetric with respect to g, there exist orthonormal vectors

$$\boldsymbol{u}_1(q),\ \boldsymbol{v}_1(q),\ \dots,\ \boldsymbol{u}_s(q),\ \boldsymbol{v}_s(q),\ \boldsymbol{w}_1(q),\ \dots,\ \boldsymbol{w}_k(q) \in \mathbb{R}^d$$

such that

$$\begin{cases} \boldsymbol{B}\boldsymbol{u}_j = -\beta_j \boldsymbol{v}_j, & 1 \le j \le s, \\ \boldsymbol{B}\boldsymbol{v}_j = \beta_j \boldsymbol{u}_j, & 1 \le j \le s, \\ \boldsymbol{B}\boldsymbol{w}_j = 0, & 1 \le j \le k. \end{cases} \tag{2-2}$$

These vectors are smooth functions of q because the nonzero eigenvalues $\pm i\beta_j(q)$ are simple. They define a basis of $\mathbb{R}^d$. Define the following ω-orthogonal vectors to $T\Sigma$:

$$\begin{cases} \boldsymbol{f}_j(q) := (1/\sqrt{\beta_j(q)})(\boldsymbol{u}_j(q), (\nabla_q \boldsymbol{A})^T \cdot \boldsymbol{u}_j(q)), & 1 \le j \le s, \\ \boldsymbol{f}_j'(q) := (1/\sqrt{\beta_j(q)})(\boldsymbol{v}_j(q), (\nabla_q \boldsymbol{A})^T \boldsymbol{v}_j(q)), & 1 \le j \le s. \end{cases} \tag{2-3}$$

These vectors are linearly independent and

$$T_{j(q)}\Sigma^{\perp} = K \oplus F,$$

with

$$K = \operatorname{Ker}(\pi^* B), \qquad F = \operatorname{span}(\boldsymbol{f}_1, \boldsymbol{f}_1', \dots, \boldsymbol{f}_s, \boldsymbol{f}_s').$$

Similarly, the tangent space $T_{j(q)}\Sigma$ admits a decomposition

$$T_{j(q)}\Sigma = E \oplus K$$

defined as follows. The map $j \circ \varphi : \Sigma' \to \Sigma$ from Lemma 2.2 satisfies $(j \circ \varphi)^*(\pi^* B) = \mathrm{d}\eta \wedge \mathrm{d}y$. Thus its differential $\mathrm{d}(j \circ \varphi)$ maps the kernel of $\mathrm{d}\eta \wedge \mathrm{d}y$ on the kernel of $\pi^* B$:

$$K = \{\mathrm{d}(j \circ \varphi)_q(0, T) : T \in \mathbb{R}^k\}. \tag{2-4}$$

A complementary space of K in $T\Sigma$ is given by

$$E := \{\mathrm{d}(j \circ \varphi)_q(W, 0) : W \in \mathbb{R}^{2s}\}. \tag{2-5}$$

From all these considerations we deduce:

Lemma 2.4. *Fix $j(q) = (q, \boldsymbol{A}(q)) \in \Sigma$. Then we have the decomposition*

$$T_{j(q)}(\mathbb{R}^{2d}) = \underbrace{E \oplus \overbrace{K}^{}}_{T\Sigma}\!\!\overbrace{\oplus F}^{T\Sigma^{\perp}} \oplus L,$$

where L is any Lagrangian complement of K in $(E \oplus F)^{\perp}$.

Proof. We have $T\Sigma + T\Sigma^{\perp} = E \oplus K \oplus F$, and the restriction of $\omega = \mathrm{d}p \wedge \mathrm{d}q$ to this space has kernel $K = T\Sigma \cap T\Sigma^{\perp}$. Hence, the restriction $\omega_{E\oplus F}$ of ω to $E \oplus F$ is nondegenerate and its orthogonal $(E \oplus F)^{\perp}$ as well. Moreover $(E \oplus F)^{\perp}$ has dimension $2d - 4s = 2k$, and we have

$$T_{j(q)}\mathbb{R}^{2d} = (E \oplus F) \oplus (E \oplus F)^{\perp}.$$

K is a Lagrangian subspace of $(E \oplus F)^{\perp}$. Therefore it admits a complementary Lagrangian: a subspace L of $(E \oplus F)^{\perp}$ with dimension k such that $\omega_L = 0$ and $(E \oplus F)^{\perp} = K \oplus L$. □

Remark 2.5. From now on, we fix any choice of Lagrangian complement L. With this choice, we define a basis $(\boldsymbol{\ell}_j)$ of L as follows. First note that the decomposition $(E \oplus F)^{\perp} = K \oplus L$ yields a bijection between L and the dual K^*, which is $\ell \mapsto \omega(\ell, \cdot)$. We emphasize that this bijection *depends on the choice of L*. Using this bijection, we define $\boldsymbol{\ell}_j$ to be the unique vector in L satisfying

$$\omega(\boldsymbol{\ell}_j, \mathrm{d}(j \circ \varphi)(0, T)) = T_j \quad \text{for all } T \in \mathbb{R}^k. \tag{2-6}$$

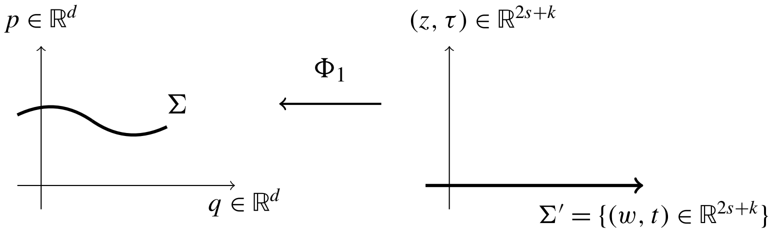

Figure 1. Using the canonical coordinates (w, t, τ, z), we identify Σ with Σ'.

2C. ***Construction of Φ_1 and proof of Theorem 1.4.*** We identified the "curved" manifold Σ with an open subset Σ' of $\mathbb{R}^{2s+k}$ using $j \circ \varphi$. Moreover, we did this in such a way that $(j \circ \varphi)^* \pi^* B = \mathrm{d}\eta \wedge \mathrm{d}y$. In this section we prove that we can identify a whole neighborhood of Σ in $\mathbb{R}^{2d}_{(q,p)}$ with a neighborhood of Σ' in $\mathbb{R}^{4s+2k}_{(z,w,v)}$, via a symplectomorphism Φ_1. See Figure 1.

Lemma 2.6. *There exists a diffeomorphism*

$$\Phi_1 : U_1' \subset \mathbb{R}^{2s+2k+2s}_{(w,t,\tau,z)} \to U_1 \subset \mathbb{R}^{2d}_{(q,p)}$$

between neighborhoods U_1 of Σ and U_1' of Σ' such that $\Phi_1^* \omega = \omega_0$ and $\Phi_1(w, t, 0, 0) = j \circ \varphi(w, t)$. Moreover its differential at $(w, t, \tau = 0, z = 0) \in \Sigma'$ is

$$\mathrm{d}\Phi_1(W, T, \mathcal{T}, Z) = \mathrm{d}_{(w,t)} j \circ \varphi(W, T) + \sum_{j=1}^{k} \mathcal{T}_j \hat{\boldsymbol{\ell}}_j(w, t) + \sum_{j=1}^{s} X_j \hat{\boldsymbol{f}}_j(w, t) + \Xi_j \hat{\boldsymbol{f}}_j'(w, t).$$

Remark 2.7. In this lemma we used the notation $Z = (X, \Xi)$ and $\hat{\boldsymbol{\ell}}_j = \boldsymbol{\ell}_j \circ \varphi$, $\hat{\boldsymbol{f}}_j = \boldsymbol{f}_j \circ \varphi$, and $\hat{\boldsymbol{f}}_j' = \boldsymbol{f}_j' \circ \varphi$.

Proof. We will first construct Φ such that $\Phi^* \omega_{|\Sigma'} = \omega_{0\ |\Sigma'}$ only on $\Sigma' = \Phi^{-1}(\Sigma)$. Then, we will use the Theorem B.2 to slightly change Φ into Φ_1 such that $\Phi_1^* \omega = \omega_0$ on a neighborhood of Σ'.

We define Φ by

$$\Phi(w, t, \tau, z) = j \circ \varphi(w, t) + \sum_{j=1}^{k} \tau_j \hat{\boldsymbol{\ell}}_j(w, t) + \sum_{j=1}^{s} x_j \hat{\boldsymbol{f}}_j(w, t) + \xi_j \hat{\boldsymbol{f}}_j'(w, t). \tag{2-7}$$

Its differential at $(w, t, 0, 0)$ has the desired form. Let us fix a point $(w, t, 0, 0) \in \Sigma'$ and compute $\Phi^* \omega$ at this point. By definition,

$$\Phi^* \omega_{(w,t,0,0)}(\,\cdot\,, \cdot\,) = \omega_{j(q)}((\mathrm{d}\Phi)\,\cdot\,, (\mathrm{d}\Phi)\,\cdot\,),$$

where $q = \varphi(w, t)$. Computing this 2-form in the canonical basis of $\mathbb{R}^{4s+2k}$ amounts to computing ω on the vectors $\boldsymbol{\ell}_j$, $\boldsymbol{f}_j$, $\boldsymbol{f}_j'$ and $\mathrm{d}(j \circ \varphi)(W, T)$. By (2-3) and (2-1) we have

$$\begin{aligned}
\omega(\boldsymbol{f}_i, \boldsymbol{f}_j) &= \frac{1}{\sqrt{\beta_i \beta_j}} \big(\langle (\nabla_q \boldsymbol{A})^\perp \cdot \boldsymbol{u}_j, \boldsymbol{u}_i \rangle - \langle (\nabla_q \boldsymbol{A})^\perp \cdot \boldsymbol{u}_i, \boldsymbol{u}_j \rangle \big) \\
&= \frac{1}{\sqrt{\beta_i \beta_j}} \langle ((\nabla_q \boldsymbol{A})^\perp - (\nabla_q \boldsymbol{A})) \cdot \boldsymbol{u}_j, \boldsymbol{u}_i \rangle \\
&= \frac{1}{\sqrt{\beta_i \beta_j}} B(\boldsymbol{u}_j, \boldsymbol{u}_i) = \frac{1}{\sqrt{\beta_i \beta_j}} g(\boldsymbol{u}_j, \boldsymbol{B}\boldsymbol{u}_i) = 0,
\end{aligned}$$

because $\boldsymbol{B}\boldsymbol{u}_i = -\beta_i \boldsymbol{v}_i$ is orthogonal to $\boldsymbol{u}_j$. Similarly we find

$$\omega(\boldsymbol{f}_i, \boldsymbol{f}_j') = \delta_{ij}, \quad \omega(\boldsymbol{f}_i', \boldsymbol{f}_j') = 0.$$

Moreover, $\boldsymbol{\ell}_i \in L \subset F^\perp$ so

$$\omega(\boldsymbol{\ell}_i, \boldsymbol{f}_j) = \omega(\boldsymbol{\ell}_i, \boldsymbol{f}_j') = 0.$$

Since L is Lagrangian we also have $\omega(\boldsymbol{\ell}_i, \boldsymbol{\ell}_j) = 0$. The vector $\mathrm{d}(j \circ \varphi)(W, T)$ is tangent to Σ and $\boldsymbol{f}_j, \boldsymbol{f}_j' \in T\Sigma^\perp$ so

$$\omega(\boldsymbol{f}_j, \mathrm{d}(j \circ \varphi)(W, T)) = \omega(\boldsymbol{f}_j', \mathrm{d}(j \circ \varphi)(W, T)) = 0.$$

Since $\boldsymbol{\ell}_i \in L \subset E^\perp$ and using (2-6), we have

$$\omega(\boldsymbol{\ell}_j, \mathrm{d}(j \circ \varphi)(W, T)) = \omega(\boldsymbol{\ell}_j, \mathrm{d}(j \circ \varphi)(0, T)) = T_j.$$

Finally, $(j \circ \varphi)^* \omega = \varphi^* B = \mathrm{d}\eta \wedge \mathrm{d}y$ so that

$$\omega(\mathrm{d}(j \circ \varphi)(W, T), \mathrm{d}(j \circ \varphi)(W', T')) = \mathrm{d}\eta \wedge \mathrm{d}y((W, T), (W', T')).$$

All these computations show that $(\Phi^* \omega)_{(w,t,0,0)}$ coincide with $\omega_0 = \mathrm{d}\xi \wedge \mathrm{d}x + \mathrm{d}\eta \wedge \mathrm{d}y + \mathrm{d}\tau \wedge \mathrm{d}t$. Thus $\Phi^* \omega = \omega_0$ on Σ. With Theorem B.2, we can change Φ into $\Phi_1(w, t, \tau, z) = \Phi(w, t, \tau, z) + \mathcal{O}((z, \tau)^2)$ such that $\Phi_1^* \omega = \omega_0$ on a neighborhood U_1' of Σ'. In particular, the differential of Φ_1 at $(w, t, 0, 0)$ coincides with the differential of Φ. □

Finally, the following lemma concludes the proof of Theorem 1.4.

Lemma 2.8. *The Hamiltonian $\widehat{H} = H \circ \Phi_1$ has the Taylor expansion*

$$\widehat{H}(w, t, \tau, x, \xi) = \tfrac{1}{2} \langle \partial_\tau^2 \widehat{H}(w, t, 0) \tau, \tau \rangle + \sum_{j=1}^{s} \hat{\beta}_j(w, t)(\xi_j^2 + x_j^2) + \mathcal{O}((\tau, x, \xi)^3).$$

Proof. Let us compute the differential and Hessian of

$$H(q, p) = \sum_{k,\ell=1}^{d} g^{k\ell}(q)(p_k - A_k(q))(p_\ell - A_\ell(q))$$

at a point $(q, \boldsymbol{A}(q)) \in \Sigma$. First,

$$\nabla_{(q,p)} H \cdot (Q, P) = \sum_{k,\ell=1}^{d} 2g^{k\ell}(q)(p_k - A_k(q))(P_\ell - \nabla_q A_\ell \cdot Q) + (p_k - A_k(q))(p_\ell - A_\ell(q)) \nabla_q g \cdot Q, \quad (2\text{-}8)$$

and at $p = \boldsymbol{A}(q)$ the Hessian is

$$\langle \nabla_{j(q)}^2 H \cdot (Q, P), (Q', P') \rangle = 2 \sum_{k,\ell=1}^{d} g^{k\ell}(q)(P_k - \nabla_q A_k \cdot Q)(P_\ell' - \nabla_q A_\ell \cdot Q'). \quad (2\text{-}9)$$

We can deduce a Taylor expansion of $\widehat{H}(w, t, \tau, z)$ with respect to (τ, z) (with fixed $q = \varphi(w, t)$). First,

$$\widehat{H}(w, t, 0, 0) = H(q, \boldsymbol{A}(q)) = 0.$$

Then we can compute the partial differential using Lemma 2.6,

$$\partial_{\tau,z}\widehat{H}(w,t,0,0)\cdot(W,T)=\nabla_{j(q)}H\cdot\partial_{\tau,z}\Phi_1(w,t,0,0)\cdot(W,T)=\nabla_{j(q)}H\cdot \mathrm{d}(j\circ\varphi)(W,T)=0,$$

because $\mathrm{d}(j\circ\varphi)(W,T)\in T_{j(q)}\Sigma$. The Taylor expansion of $\widehat{H}$ is thus

$$\widehat{H}(w,t,\tau,z)=\tfrac{1}{2}\langle\partial^2_{\tau,z}\widehat{H}(w,t,0)\cdot(\tau,z),(\tau,z)\rangle+\mathcal{O}((\tau,z)^3),$$

where $\partial^2_{\tau,z}\widehat{H}$ is the partial Hessian with respect to (τ,z). We have

$$\partial^2_{\tau,z}\widehat{H}=(\partial_{(\tau,z)}\Phi_1)^T\cdot\nabla^2_{j(q)}H\cdot(\partial_{(\tau,z)}\Phi_1),$$

and computing the Hessian matrix amounts to computing $\nabla^2_{j(q)}H$ on the vectors $\boldsymbol{g}_j$, $\boldsymbol{f}_j$, and $\boldsymbol{f}'_j$. If $(Q,P)\in T_{j(q)}\Sigma^\perp$, then $P=(\nabla_q A)^\perp\cdot Q$ so that, with (2-9),

$$\begin{aligned}\tfrac{1}{2}\nabla^2_{j(q)}H((Q,P),(Q',P'))&=\sum_{k,\ell,i,j=1}^{d}g^{k\ell}(q)(\partial_k A_j Q_j-\partial_j A_k Q_j)(\partial_\ell A_i Q'_i-\partial_i A_\ell Q'_i)\\&=\sum_{k,\ell,i,j}g^{k\ell}(q)B_{kj}Q_jB_{\ell i}Q'_i.\end{aligned}$$

But $\sum_k g^{k\ell}B_{kj}=\boldsymbol{B}_{\ell j}$ (by (1-1)) so

$$\tfrac{1}{2}\nabla^2_{j(q)}H((Q,P),(Q',P'))=\sum_{i,j,\ell}B_{\ell i}(\boldsymbol{B}_{\ell j}Q_j)Q'_i=B(\boldsymbol{B}\cdot Q,Q').$$

In the special case $(Q,P)=\boldsymbol{f}_j$ we have

$$\tfrac{1}{2}\nabla^2_{j(q)}H(\boldsymbol{f}_i,\boldsymbol{f}_j)=\frac{1}{\sqrt{\beta_i\beta_j}}B(\boldsymbol{B}\boldsymbol{u}_i,\boldsymbol{u}_j)=\frac{1}{\sqrt{\beta_i\beta_j}}g(\boldsymbol{B}\boldsymbol{u}_i,\boldsymbol{B}\boldsymbol{u}_j)=\sqrt{\beta_i\beta_j}\,g(\boldsymbol{v}_i,\boldsymbol{v}_j)=\sqrt{\beta_i\beta_j}\,\delta_{ij},$$

and similarly

$$\tfrac{1}{2}\nabla^2_{j(q)}H(\boldsymbol{f}'_i,\boldsymbol{f}'_j)=\sqrt{\beta_i\beta_j}\,\delta_{ij},\qquad \tfrac{1}{2}\nabla^2_{j(q)}H(\boldsymbol{f}_i,\boldsymbol{f}'_j)=0.$$

Finally, it remains to prove

$$\nabla^2_{j(q)}H(\boldsymbol{\ell}_i,\boldsymbol{f}_j)=\nabla^2_{j(q)}H(\boldsymbol{\ell}_i,\boldsymbol{f}'_j)=0 \tag{2-10}$$

to conclude that the Hessian of $\widehat{H}$ is

$$\tfrac{1}{2}\partial^2_{\tau,z}\widehat{H}(w,t,0,0)=\begin{pmatrix}\tfrac{1}{2}\partial^2_\tau\widehat{H}(w,t,0,0) & & & & & \\ & \beta_1 & & & & \\ & & \beta_1 & & & \\ & & & \ddots & & \\ & & & & \beta_s & \\ & & & & & \beta_s\end{pmatrix}.$$

Actually, (2-10) follows from the identity

$$L\subset F^\perp=(T\Sigma^\perp)^{\perp H}, \tag{2-11}$$

where $\perp H$ denotes the orthogonal with respect to the quadratic form $\nabla^2 H$ (which is different from the symplectic orthogonal $\perp$). Indeed, to prove (2-11) note that

$$\begin{aligned}
(Q,P)\in(T\Sigma^{\perp})^{\perp H} \quad &\Longrightarrow \quad \nabla^2 H((Q,P),(Q',(\nabla_q \boldsymbol{A})^T\cdot Q'))=0 \quad \text{for all } Q'\in\mathbb{R}^d\\
&\Longrightarrow \quad \sum_{k,\ell,j} g^{k\ell}(P_k-\nabla_q A_k\cdot Q)B_{\ell j}Q'_j=0 \quad \text{for all } Q'\in\mathbb{R}^d\\
&\Longrightarrow \quad \sum_{k,j}(P_k-\nabla_q A_k\cdot Q)\boldsymbol{B}_{kj}Q'_j=0 \quad \text{for all } Q'\in\mathbb{R}^d\\
&\Longrightarrow \quad \langle P-\nabla_q \boldsymbol{A}\cdot Q,\boldsymbol{B}Q'\rangle=0 \quad \text{for all } Q'\in\mathbb{R}^d\\
&\Longrightarrow \quad \langle P,\boldsymbol{B}Q'\rangle-\langle Q,(\nabla_q\boldsymbol{A})^T\cdot\boldsymbol{B}Q'\rangle=0 \quad \text{for all } Q'\in\mathbb{R}^d\\
&\Longrightarrow \quad \omega((Q,P),(\boldsymbol{B}Q',(\nabla_q\boldsymbol{A})^T\cdot\boldsymbol{B}Q'))=0 \quad \text{for all } Q'\in\mathbb{R}^d,
\end{aligned}$$

and we have

$$\begin{aligned}
F&=\{(V:(\nabla_q\boldsymbol{A})^T V),\ V\in\operatorname{span}(\boldsymbol{u}_1,\boldsymbol{v}_1,\dots,\boldsymbol{u}_s,\boldsymbol{v}_s)\}\\
&=\{(\boldsymbol{B}Q:(\nabla_q\boldsymbol{A})^T\boldsymbol{B}Q),\ Q\in\mathbb{R}^d\},
\end{aligned}$$

because the vectors $\boldsymbol{u}_j,\boldsymbol{v}_j$ span the range of $\boldsymbol{B}$. Hence we find

$$(Q,P)\in(T\Sigma^{\perp})^{\perp H} \quad\Longleftrightarrow\quad (Q,P)\in F^{\perp}. \qquad \square$$

3. Construction of the normal form $\mathcal{N}_\hbar$

3A. *Formal series.* Define $U=U'_1\cap\Sigma'\subset\mathbb{R}^{2s+k}_{(w,t)}\times\{0\}$. We construct the Birkhoff normal form in the space

$$\mathcal{E}_1=\mathcal{C}^\infty(U)[\![x,\xi,\tau,\hbar]\!].$$

It is a space of formal series in $(x,\xi,\tau,\hbar)$ with coefficients smoothly depending on (w,t). We see these formal series as Taylor series of symbols, which we quantize using the Weyl quantization. Given an $\hbar$-pseudodifferential operator $\mathcal{A}_\hbar=\operatorname{Op}_\hbar^w a_\hbar$ (with symbol $a_\hbar$ admitting an expansion in powers of $\hbar$ in some standard class), we denote by $[a_\hbar]$ or $\sigma^T(\mathcal{A}_\hbar)$ the Taylor series of $a_\hbar$ with respect to (x,ξ,τ) at $(x,\xi,\tau)=0$. Conversely, given a formal series $\rho\in\mathcal{E}_1$, we can find a bounded symbol $a_\hbar$ such that $[a_\hbar]=\rho$. This symbol is not uniquely defined, but any two such symbols differ by $\mathcal{O}((x,\xi,\hbar)^\infty)$, uniformly with respect to $(w,t)\in U$.

Remark 3.1. We prove below that the eigenfunctions of $\mathcal{L}_\hbar$ are microlocalized, where $(w,t)\in U$ and $|(x,\xi)|\lesssim\hbar^{1/2}$, so that the remainders $\mathcal{O}((x,\xi,\hbar)^\infty)$ are negligible.

• In order to make operations on Taylor series compatible with the Weyl quantization, we endow $\mathcal{E}_1$ with the Weyl–Moyal product $\star$, defined by $\operatorname{Op}_\hbar^w(a)\operatorname{Op}_\hbar^w(b)=\operatorname{Op}_\hbar^w(a\star b)$. This product satisfies

$$a_1\star a_2=\sum_{k=0}^{N}\frac{1}{k!}\Big(\frac{\hbar}{2i}\Box\Big)^k a_1(w,t,\tau,z)a_2(w',t',\tau',z')|_{w'=w,t'=t,\tau'=\tau,z'=z}+\mathcal{O}(\hbar^N),$$

where

$$\Box=\sum_{j=1}^{s}(\partial_{\eta_j}\partial_{y'_j}-\partial_{y_j}\partial_{\eta'_j})+\sum_{j=1}^{s}(\partial_{\xi_j}\partial_{x'_j}-\partial_{x_j}\partial_{\xi'_j})+\sum_{j=1}^{k}(\partial_{\tau_j}\partial_{t'_j}-\partial_{t_j}\partial_{\tau'_j}).$$

Note that to define such a product it is necessary to assume that our formal series depend smoothly on (w, t).

• The degree of a monomial is

$$\deg(x^\alpha \xi^{\alpha'} \tau^{\alpha''} \hbar^\ell) = |\alpha| + |\alpha'| + |\alpha''| + 2\ell. \tag{3-1}$$

We denote by $\mathcal{D}_N$ the $\mathcal{C}^\infty(U)$-module spanned by monomials of degree N, and

$$\mathcal{O}_N = \bigoplus_{n \geq N} \mathcal{D}_N, \tag{3-2}$$

which satisfies

$$\mathcal{O}_{N_1} \star \mathcal{O}_{N_2} \subset \mathcal{O}_{N_1+N_2}.$$

If $\rho_1, \rho_2 \in \mathcal{E}_1$, we denote their commutator by

$$[\rho_1, \rho_2] = \operatorname{ad}_{\rho_1} \rho_2 = \rho_1 \star \rho_2 - \rho_2 \star \rho_1,$$

and we have the formula

$$[\rho_1, \rho_2] = 2 \sinh\Big(\frac{\hbar}{2i} \square\Big) \rho_1 \rho_2. \tag{3-3}$$

In particular,

$$\text{for all } \rho_1 \in \mathcal{O}_{N_1}, \text{ for all } \rho_2 \in \mathcal{O}_{N_2}, \quad \frac{i}{\hbar}[\rho_1, \rho_2] \in \mathcal{O}_{N_1+N_2-2},$$

and $(i/\hbar)[\rho_1, \rho_2] = \{\rho_1, \rho_2\} + \mathcal{O}(\hbar^2)$. The Birkhoff normal form algorithm is based on the following lemma. We recall the definition (1-5) of r_1.

Lemma 3.2. *For $1 \leq j \leq s$, define $z_j = x_j + i\xi_j$ and $|z_j|^2 = x_j^2 + \xi_j^2$.*

(1) *Every series $\rho \in \mathcal{E}_1$ satisfies*

$$\frac{i}{\hbar} \operatorname{ad}_{|z_j|^2} \rho = \{|z_j|^2, \rho\}.$$

(2) *Let $0 \leq N < r_1$. For every $R_N \in \mathcal{D}_N$, there exist $\rho_N, K_N \in \mathcal{D}_N$ such that*

$$R_N = K_N + \sum_{j=1}^{s} \hat{\beta}_j(w, t) \frac{i}{\hbar} \operatorname{ad}_{|z_j|^2} \rho_N$$

and $[K_N, |z_j|^2] = 0$ for $1 \leq j \leq s$.

(3) *If $K \in \mathcal{E}_1$, then $[K, |z_j|^2] = 0$ for all $1 \leq j \leq s$ if and only if there exists a formal series $F \in \mathcal{C}^\infty(U)[\![I_1, \dots, I_s, \tau, \hbar]\!]$ such that*

$$K = F(|z_1|^2, \dots, |z_s|^2, \tau, \hbar).$$

Proof. The first statement is a simple computation. For the second and the third, it suffices to consider monomials $R_N = c(w, t) z^\alpha \bar{z}^{\alpha'} \tau^{\alpha''} \hbar^\ell$. Note that

$$\operatorname{ad}_{|z_j|^2}(c(w, t) z^\alpha \bar{z}^{\alpha'} \tau^{\alpha''} \hbar^\ell) = (\alpha'_j - \alpha_j) c(w, t) z^\alpha \bar{z}^{\alpha'} \tau^{\alpha''} \hbar^\ell,$$

so that R_N commutes with every $|z_j|^2$ $(1 \leq j \leq s)$ if and only if $\alpha = \alpha'$, which amounts to saying that R_N is a function of $|z_j|^2$ and proves (3). Moreover,

$$\sum_j \hat{\beta}_j \operatorname{ad}_{|z_j|^2}(z^\alpha \bar{z}^{\alpha'} \tau^{\alpha''} \hbar^\ell) = \langle \alpha' - \alpha, \hat{\beta} \rangle z^\alpha \bar{z}^{\alpha'} \tau^{\alpha''} \hbar^\ell,$$

where $\langle \gamma, \hat{\beta} \rangle = \sum_{j=1}^s \gamma_j \hat{\beta}_j(w, t)$. Under the assumption $|\alpha| + |\alpha'| + |\alpha''| + 2\ell < r_1$, we have $|\alpha - \alpha'| < r_1$ and by the definition of r_1 the function $\langle \alpha' - \alpha, \hat{\beta}(w, t) \rangle$ cannot vanish for $(w, t) \in U$, unless $\alpha = \alpha'$. If $\alpha = \alpha'$, we choose $\rho_N = 0$ and $R_N = K_N$ commutes with $|z_j|^2$. If $\alpha \neq \alpha'$, we choose $K_N = 0$ and

$$\rho_N = \frac{c(w, t)}{\langle \alpha' - \alpha, \hat{\beta}(w, t) \rangle} z^\alpha \bar{z}^{\alpha'} \tau^{\alpha''} \hbar^\ell,$$

and this proves (2). □

3B. *Formal Birkhoff normal form.* In this section we construct the Birkhoff normal form at a formal level. We will work with the Taylor series of the symbol H of $\mathcal{L}_\hbar$, in the new coordinates Φ_1. According to Theorem 1.4, $\widehat{H} = H \circ \Phi_1$ defines a formal series

$$[\widehat{H}] = H_2 + \sum_{k \geq 3} H_k,$$

where $H_k \in \mathcal{D}_k$ and

$$H_2 = \langle M(w, t)\tau, \tau \rangle + \sum_{j=1}^s \hat{\beta}_j(w, t)|z_j|^2. \tag{3-4}$$

At a formal level, the normal form can be stated as follows.

Theorem 3.3. *For every $\gamma \in \mathcal{O}_3$, there are $\kappa, \rho \in \mathcal{O}_3$ such that*

$$e^{(i/\hbar)\operatorname{ad}_\rho}(H_2 + \gamma) = H_2 + \kappa + \mathcal{O}_{r_1},$$

where κ is a function of harmonic oscillators:

$$\kappa = F(|z_1|^2, \dots, |z_s|^2, \tau, \hbar), \quad \textit{with some } F \in \mathcal{C}^\infty(U)[\![I_1, \dots, I_s, \tau, \hbar]\!].$$

Moreover, *if γ has real-valued coefficients*, *then so do ρ, κ and the remainder $\mathcal{O}_{r_1}$.*

Proof. We prove this by induction on an integer $N \geq 3$. Assume that we found $\rho_{N-1}, K_3, \dots, K_{N-1} \in \mathcal{O}_3$, with $[K_i, |z_j|^2] = 0$ for every (i, j) and $K_i \in \mathcal{D}_i$ such that

$$e^{(i/\hbar)\operatorname{ad}_{\rho_{N-1}}}(H_2 + \gamma) = H_2 + K_3 + \cdots + K_{N-1} + \mathcal{O}_N.$$

Rewriting the remainder as $R_N + \mathcal{O}_{N+1}$, with $R_N \in \mathcal{D}_N$, we have

$$e^{(i/\hbar)\operatorname{ad}_{\rho_{N-1}}}(H_2 + \gamma) = H_2 + K_3 + \cdots + K_{N-1} + R_N + \mathcal{O}_{N+1}.$$

We are looking for a $\rho' \in \mathcal{O}_N$. For such a ρ' we apply $e^{(i/\hbar)\operatorname{ad}_{\rho'}}$:

$$e^{(i/\hbar)\operatorname{ad}_{\rho_{N-1}+\rho'}}(H_2 + \gamma) = e^{(i/\hbar)\operatorname{ad}_{\rho'}}(H_2 + K_3 + \cdots + K_{N-1} + R_N + \mathcal{O}_{N+1}).$$

Since $(i/\hbar)\,\mathrm{ad}_{\rho'} : \mathcal{O}_k \to \mathcal{O}_{k+N-2}$ we have

$$e^{(i/\hbar)\,\mathrm{ad}_{\rho_{N-1}+\rho'}}(H_2+\gamma) = H_2 + K_3 + \cdots + K_{N-1} + R_N + \frac{i}{\hbar}\,\mathrm{ad}_{\rho'}(H_2) + \mathcal{O}_{N+1}. \tag{3-5}$$

The new term $(i/\hbar)\,\mathrm{ad}_{\rho'}(H_2) = -(i/\hbar)\,\mathrm{ad}_{H_2}(\rho')$ can still be simplified. Indeed by (3-4),

$$\frac{i}{\hbar}\,\mathrm{ad}_{H_2}(\rho') = \frac{i}{\hbar}[\langle M(w,t)\tau,\tau\rangle,\rho'] + \sum_{j=1}^{s}\Big(\hat{\beta}_j\frac{i}{\hbar}[|z_j|^2,\rho'] + |z_j|^2\frac{i}{\hbar}[\hat{\beta}_j,\rho']\Big), \tag{3-6}$$

with

$$\frac{i}{\hbar}[\hat{\beta}_j,\rho'] = \sum_{i=1}^{s}\bigg(\frac{\partial\hat{\beta}_j}{\partial y_i}\frac{\partial\rho'}{\partial\eta_i} - \frac{\partial\hat{\beta}_j}{\partial\eta_i}\frac{\partial\rho'}{\partial y_i}\bigg) + \sum_{i=1}^{k}\frac{\partial\hat{\beta}_j}{\partial t_i}\frac{\partial\rho'}{\partial\tau_i} + \mathcal{O}_{N-1} = \mathcal{O}_{N-1},$$

because a derivation with respect to (y,η,t) does not decrease the degree. Similarly,

$$\frac{i}{\hbar}[\langle M(w,t)\tau,\tau\rangle,\rho'] = \sum_{j=1}^{k}\bigg(\langle\partial_{t_j}M(w,t)\tau,\tau\rangle\frac{\partial\rho'}{\partial\tau_j} - \frac{\partial\langle M(w,t)\tau,\tau\rangle}{\partial\tau_j}\frac{\partial\rho'}{\partial t_j}\bigg) + \mathcal{O}_{N+1} = \mathcal{O}_{N+1},$$

and thus (3-6) becomes

$$\frac{i}{\hbar}\,\mathrm{ad}_{H_2}(\rho') = \sum_{j=1}^{s}\Big(\hat{\beta}_j\frac{i}{\hbar}\,\mathrm{ad}_{|z_j|^2}(\rho')\Big) + \mathcal{O}_{N+1}.$$

Using this formula in (3-5) we get

$$e^{(i/\hbar)\,\mathrm{ad}_{\rho_{N-1}+\rho'}}(H_2+\gamma) = H_2 + K_3 + \cdots + K_{N-1} + R_N - \sum_{j=1}^{s}\hat{\beta}_j\frac{i}{\hbar}\,\mathrm{ad}_{|z_j|^2}(\rho') + \mathcal{O}_{N+1}.$$

Thus, we are looking for $K_N, \rho' \in \mathcal{D}_N$ such that

$$R_N = K_N + \sum_{j=1}^{s}\hat{\beta}_j\frac{i}{\hbar}\,\mathrm{ad}_{|z_j|^2}(\rho'),$$

with $[K_N, |z_j|^2] = 0$. By Lemma 3.2, we can solve this equation provided $N < r_1$, and this concludes the proof. Moreover, $(i/\hbar)\,\mathrm{ad}_{|z_j|^2}$ is a real endomorphism, so we can solve this equation on $\mathbb{R}$. □

3C. *Quantizing the normal form.* We now construct the normal form $\mathcal{N}_\hbar$, quantizing Theorems 1.4 and 3.3. We denote by $\mathcal{I}_\hbar^{(j)}$ the harmonic oscillator with respect to x_j, defined by

$$\mathcal{I}_\hbar^{(j)} = \mathrm{Op}_\hbar^w(\xi_j^2 + x_j^2) = -\hbar^2\frac{\partial^2}{\partial x_j^2} + x_j^2.$$

We prove the following theorem.

Theorem 3.4. *There exist*

(1) *a microlocally unitary operator* $\mathrm{U}_\hbar : \mathrm{L}^2(\mathbb{R}^d_{x,y,t}) \to \mathrm{L}^2(M)$ *quantizing a symplectomorphism* $\tilde{\Phi}_1 = \Phi_1 + \mathcal{O}((x,\xi,\tau)^2)$, *microlocally on* $U_1' \times U_1$,

(2) *a function* $f_1^\star : \mathbb{R}^{2s+2k}_{y,\eta,t,\tau} \times \mathbb{R}^s_I \times [0,1]$ *which is* $\mathcal{C}^\infty$ *with compact support such that*

$$f_1^\star(y,\eta,t,\tau,I,\hbar) \le C\big((|I|+\hbar)^2 + |\tau|(|I|+\hbar) + |\tau|^3\big),$$

(3) *an* $\hbar$*-pseudodifferential operator* $\mathcal{R}_\hbar$, *whose symbol is* $\mathcal{O}((x,\xi,\tau,\hbar^{1/2})^{r_1})$ *on* U_1',

such that

$$\mathrm{U}_\hbar^* \mathcal{L}_\hbar \mathrm{U}_\hbar = \mathcal{N}_\hbar + \mathcal{R}_\hbar,$$

with

$$\mathcal{N}_\hbar = \mathrm{Op}_\hbar^w \langle M(w,t)\tau, \tau\rangle + \sum_{j=1}^{s} \mathcal{I}_\hbar^{(j)} \, \mathrm{Op}_\hbar^w \, \hat{\beta}_j(w,t) + \mathrm{Op}_\hbar^w \, f_1^\star(y,\eta,t,\tau,\mathcal{I}_\hbar^{(j)},\dots,\mathcal{I}_\hbar^{(s)},\hbar).$$

Remark 3.5. $\mathrm{U}_\hbar$ is a Fourier integral operator quantizing the symplectomorphism $\tilde{\Phi}_1$; see [Martinez 2002; Zworski 2012]. In particular, if $\mathcal{A}_\hbar$ is a pseudodifferential operator on M with symbol $a_\hbar = a_0 + \mathcal{O}(\hbar^2)$, then $\mathrm{U}_\hbar^* \mathcal{A}_\hbar \mathrm{U}_\hbar$ is a pseudodifferential operator on $\mathbb{R}^d$ with symbol

$$\sigma_\hbar = a_0 \circ \tilde{\Phi}_1 + \mathcal{O}(\hbar^2) \quad \text{on } U_1'.$$

Remark 3.6. Due to the parameters (y,η,t,τ) in the formal normal form, an additional quantization is needed, hence the $\mathrm{Op}_\hbar^w \, f_1^\star$-term. It is a quantization with respect to (y,η,t,τ) of an operator-valued symbol $f_1^\star(y,\eta,t,\tau,\mathcal{I}_\hbar^{(1)},\dots,\mathcal{I}_\hbar^{(s)})$. Actually, this operator symbol is simple since one can diagonalize it explicitly. Denoting by $h_{n_j}^j(x_j)$ the n_j-th eigenfunction of $\mathcal{I}_\hbar^{(j)}$, associated to the eigenvalue $(2n_j-1)\hbar$, we have for all $n \in \mathbb{N}^s$

$$f_1^\star(y,\eta,t,\tau,\mathcal{I}_\hbar^{(1)},\dots,\mathcal{I}_\hbar^{(s)},\hbar) h_n(x) = f_1^\star(y,\eta,\tau,(2n-1)\hbar,\hbar) h_n(x),$$

where $h_n(x) = h_{n_1}^1(x_1)\cdots h_{n_s}^s(x_s)$. Thus the operator $\mathrm{Op}_\hbar^w \, f_1^\star$ satisfies, for $u \in \mathrm{L}^2(\mathbb{R}^{s+k}_{(y,t)})$,

$$(\mathrm{Op}_\hbar^w \, f_1^\star) u \otimes h_n = \big(\mathrm{Op}_\hbar^w \, f_1^\star(y,\eta,t,\tau,(2n-1)\hbar,\hbar)u\big) \otimes h_n.$$

Proof. In order to prove Theorem 3.4, we first quantize Theorem 1.4. Using the Egorov theorem, there exists a microlocally unitary operator $V_\hbar : \mathrm{L}^2(\mathbb{R}^d) \to \mathrm{L}^2(M)$ quantizing the symplectomorphism $\Phi_1 : U_1' \to U_1$. Thus,

$$V_\hbar^* \mathcal{L}_\hbar V_\hbar = \mathrm{Op}_\hbar^w(\sigma_\hbar)$$

for some symbol $\sigma_\hbar$ such that

$$\sigma_\hbar = \widehat{H} + \mathcal{O}(\hbar^2) \quad \text{on } U_1'.$$

Then we use the following lemma to quantize the formal normal form and conclude. □

Lemma 3.7. *There exists a bounded pseudodifferential operator $\mathcal{Q}_\hbar$ with compactly supported symbol such that*

$$e^{(i/\hbar)\mathcal{Q}_\hbar} \, \mathrm{Op}_\hbar^w(\sigma_\hbar) e^{-(i/\hbar)\mathcal{Q}_\hbar} = \mathcal{N}_\hbar + \mathcal{R}_\hbar,$$

where $\mathcal{N}_\hbar$ and $\mathcal{R}_\hbar$ satisfy the properties stated in Theorem 3.4.

Remark 3.8. As explained below, the principal symbol Q of $\mathcal{Q}_\hbar$ is $\mathcal{O}((x,\xi,\tau)^3)$. Thus, the symplectic flow φ_t associated to the Hamiltonian Q is $\varphi_t(x,\xi,\tau) = (x,\xi,\tau) + \mathcal{O}((x,\xi,\tau)^2)$. Moreover, the Egorov theorem implies that $e^{-(i/\hbar)\mathcal{Q}_\hbar}$ quantizes the symplectomorphism φ_1. Hence, $V_\hbar e^{-(i/\hbar)\mathcal{Q}_\hbar}$ quantizes the symplectomorphism $\tilde{\Phi}_1 = \Phi_1 \circ \varphi_1 = \Phi_1 + \mathcal{O}((x,\xi,\tau)^2)$.

Proof. The proof of this lemma follows the exact same lines as in the case $k = 0$ [Morin 2022b, Theorem 4.1]. Let us recall the main arguments. The symbol $\sigma_\hbar$ is equal to $\widehat{H} + \mathcal{O}(\hbar^2)$ on U_1'. Thus, its associated formal series is $[\sigma_\hbar] = H_2 + \gamma$ for some $\gamma \in \mathcal{O}_3$. Using the Birkhoff normal form algorithm (Theorem 3.3), we get $\kappa, \rho \in \mathcal{O}_3$ such that

$$e^{(i/\hbar)\,\mathrm{ad}_\rho}(H_2 + \gamma) = H_2 + \kappa + \mathcal{O}_{r_1}.$$

If $Q_\hbar$ is a smooth compactly supported symbol with Taylor series $[Q_\hbar] = \rho$, then by the Egorov theorem the operator

$$e^{i\hbar^{-1}\mathrm{Op}_\hbar^w Q_\hbar}\,\mathrm{Op}_\hbar^w(\sigma_\hbar)e^{-i\hbar^{-1}\mathrm{Op}_\hbar^w Q_\hbar} \tag{3-7}$$

has a symbol with Taylor series $H_2 + \kappa + \mathcal{O}_{r_1}$. Since κ commutes with the oscillator $|z_j|^2$, it can be written as

$$\kappa = \sum_{2|\alpha|+|\alpha'|+2\ell\geq 3} c_{\alpha\alpha'\ell}(w,t)|z_1|^{2\alpha_1}\cdots|z_s|^{2\alpha_s}\tau_1^{\alpha_1'}\cdots\tau_k^{\alpha_k'}\hbar^\ell.$$

We can reorder this formal series using the monomials $(|z_j|^2)^{\star\alpha_j} = |z_j|^2 \star\cdots\star |z_j|^2$:

$$\kappa = \sum_{2|\alpha|+|\alpha'|+2\ell\geq 3} c^\star_{\alpha\alpha'\ell}(w,t)(|z_1|^2)^{\star\alpha_1}\cdots(|z_s|^2)^{\star\alpha_s}\tau_1^{\alpha_1'}\cdots\tau_k^{\alpha_k'}\hbar^\ell.$$

If $f_1^\star$ is a smooth compactly supported function with Taylor series

$$[f_1^\star] = \sum_{2|\alpha|+|\alpha'|+2\ell\geq 3} c^\star_{\alpha\alpha'\ell}(w,t)I_1^{\alpha_1}\cdots I_s^{\alpha_s}\tau_1^{\alpha_1'}\cdots\tau_k^{\alpha_k'}\hbar^\ell,$$

then the operator (3-7) is equal to

$$\mathcal{N}_\hbar = \mathrm{Op}_\hbar^w H_2 + \mathrm{Op}_\hbar^w f_1^\star(y,\eta,t,\tau,\mathcal{I}_\hbar^{(1)},\dots,\mathcal{I}_\hbar^{(s)},\hbar)$$

modulo $\mathcal{O}_{r_1}$. □

4. Comparing the spectra of $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar$

4A. *Spectrum of $\mathcal{N}_\hbar$.* In this section we describe the spectral properties of $\mathcal{N}_\hbar$. We can use the properties of harmonic oscillators to diagonalize it in the following way. For $1 \leq j \leq s$ and $n_j \geq 1$, we recall that the n_j-th Hermite function $h_{n_j}^j(x_j)$ is an eigenfunction of $\mathcal{I}_\hbar^{(j)}$,

$$\mathcal{I}_\hbar^{(j)}h_{n_j}^j = \hbar(2n_j - 1)h_{n_j}^j,$$

and the functions $(h_n)_{n\in\mathbb{N}^s}$ defined by

$$h_n(x) = h_{n_1}^1\otimes\cdots\otimes h_{n_s}^s(x) = h_{n_1}^1(x_1)\cdots h_{n_s}^s(x_s)$$

form a Hilbertian basis of $\mathrm{L}^2(\mathbb{R}_x^s)$. Thus, we can use this basis to decompose the space $\mathrm{L}^2(\mathbb{R}_{x,y,t}^{2s+k})$ on which $\mathcal{N}_\hbar$ acts:

$$\mathrm{L}^2(\mathbb{R}^{2s+k}) = \bigoplus_{n\in\mathbb{N}^s}(\mathrm{L}^2(\mathbb{R}_{y,t}^{s+k})\otimes h_n).$$

$\mathcal{N}_\hbar$ preserves this decomposition and

$$\mathcal{N}_\hbar = \bigoplus_{n\in\mathbb{N}^s} \mathcal{N}_\hbar^{[n]},$$

where $\mathcal{N}_\hbar^{[n]}$ is the pseudodifferential operator with symbol

$$N_\hbar^{[n]} = \langle M(w,t)\tau,\tau\rangle + \sum_{j=1}^{s} \hat{\beta}_j(w,t)(2n_j+1)\hbar + f_1^\star(w,t,\tau,(2n-1)\hbar,\hbar). \tag{4-1}$$

In particular, the spectrum of $\mathcal{N}_\hbar$ is given by

$$\operatorname{sp}(\mathcal{N}_\hbar) = \bigcup_{n\in\mathbb{N}^s} \operatorname{sp}(\mathcal{N}_\hbar^{[n]}).$$

Moreover, as in the $k=0$ case, for any $b_1>0$ there is an $N_{\max}>0$ (independent of $\hbar$) such that

$$\operatorname{sp}(\mathcal{N}_\hbar)\cap(-\infty,b_1\hbar) = \bigcup_{|n|\le N_{\max}} \operatorname{sp}(\mathcal{N}_\hbar^{[n]})\cap(-\infty,b_1\hbar).$$

The reason is that the symbol $N_\hbar^{[n]}$ is greater than $b_1\hbar$ for n large enough. Finally, to prove our main result, Theorem 1.7, it remains to compare the spectra of $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar$.

4B. ***Microlocalization of the eigenfunctions.*** Here we prove microlocalization results for the eigenfunctions of $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar$. These results are needed to show that the remainders $\mathcal{O}((x,\xi,\tau)^{r_1})$ we got are small. More precisely, for each operator we need to prove that the eigenfunctions are microlocalized

- inside Ω (space localization),
- where $|(x,\xi,\tau)|\lesssim\hbar^\delta$ for $\delta\in\left(0,\tfrac{1}{2}\right)$ (i.e., close to Σ).

Fix $\tilde{b}_1$ such that

$$K_{\tilde{b}_1} = \{q\in M : b(q)\le\tilde{b}_1\}\Subset\Omega.$$

Lemma 4.1 (space localization for $\mathcal{L}_\hbar$). *Let $b_1\in(b_0,\tilde{b}_1)$ and $\chi_0\in\mathcal{C}_0^\infty(M)$ be a cutoff function such that $\chi_0=1$ on $K_{\tilde{b}_1}$. Then every normalized eigenfunction $\psi_\hbar$ of $\mathcal{L}_\hbar$ associated with an eigenvalue $\lambda_\hbar\le b_1\hbar$ satisfies*

$$\psi_\hbar = \chi_0\psi_\hbar + \mathcal{O}(\hbar^\infty),$$

where the $\mathcal{O}(\hbar^\infty)$ is independent of $(\lambda_\hbar,\psi_\hbar)$.

Proof. This follows from the Agmon estimates,

$$\| e^{d(q,K_{\tilde{b}_1})\hbar^{-1/4}}\psi_\hbar\| \le C\|\psi_\hbar\|^2, \tag{4-2}$$

as in the $k=0$ case (in [Morin 2022b]). Indeed, from (4-2) we deduce

$$\|(1-\chi_0)\psi\| \le Ce^{-\varepsilon\hbar^{-1/4}}\|\psi_\hbar\|,$$

as soon as $\chi_0=1$ on an ε-neighborhood of $K_{\tilde{b}_1}$. □

Lemma 4.2 (microlocalization near Σ for $\mathcal{L}_\hbar$). *Let $\delta \in \left(0, \frac{1}{2}\right)$, $b_1 \in (b_0, \tilde{b}_1)$ and $\chi_1 \in \mathcal{C}^\infty(T^*M)$ be a cutoff function equal to* 1 *on a neighborhood of* Σ. *Then every eigenfunction $\psi_\hbar$ of $\mathcal{L}_\hbar$ associated with an eigenvalue $\lambda_\hbar \leq b_1\hbar$ satisfies*

$$\psi_\hbar = \mathrm{Op}_\hbar^w \chi_1(\hbar^{-\delta}(q, p))\psi_\hbar + \mathcal{O}(\hbar^\infty)\psi_\hbar,$$

where the $\mathcal{O}(\hbar^\infty)$ is in the space of bounded operators $\mathcal{L}(\mathrm{L}^2, \mathrm{L}^2)$ and independent of $(\lambda_\hbar, \psi_\hbar)$.

Proof. Let $g_\hbar \in \mathcal{C}_0^\infty(\mathbb{R})$ be such that

$$g_\hbar(\lambda) = \begin{cases} 1 & \text{if } \lambda \leq b_1\hbar, \\ 0 & \text{if } \lambda \geq \tilde{b}_1\hbar. \end{cases}$$

Then the eigenfunction $\psi_\hbar$ satisfies

$$\psi_\hbar = g_\hbar(\lambda_\hbar)\psi_\hbar = g_\hbar(\mathcal{L}_\hbar)\psi_\hbar.$$

With the notation $\chi = 1 - \chi_1$, we will prove that

$$\|\mathrm{Op}_\hbar^w \chi(\hbar^{-\delta}(q, p))g_\hbar(\mathcal{L}_\hbar)\|_{\mathcal{L}(\mathrm{L}^2,\mathrm{L}^2)} = \mathcal{O}(\hbar^\infty), \tag{4-3}$$

from which will follow $\psi_\hbar = \mathrm{Op}_\hbar^w \chi_1(\hbar^{-\delta}(q, p))\psi_\hbar + \mathcal{O}(\hbar^\infty)\psi_\hbar$, uniformly with respect to $(\lambda_\hbar, \psi_\hbar)$.

To lighten the notation, we define $\chi^w := \mathrm{Op}_\hbar^w \chi(\hbar^{-\delta}(q, p))$. For every $\psi \in \mathrm{L}^2(M)$ we define $\varphi = g_\hbar(\mathcal{L}_\hbar)\psi$. Then,

$$\langle \mathcal{L}_\hbar \chi^w \varphi, \chi^w \varphi\rangle = \langle \chi^w \mathcal{L}_\hbar \varphi, \chi^w \varphi\rangle + \langle [\mathcal{L}_\hbar, \chi^w]\varphi, \chi^w \varphi\rangle. \tag{4-4}$$

We will bound from above the right-hand side, and from below the left-hand side. First, since $g_\hbar(\lambda)$ is supported where $\lambda \leq \tilde{b}_1\hbar$, we have

$$\langle \chi^w \mathcal{L}_\hbar \varphi, \chi^w \varphi\rangle \leq \tilde{b}_1\hbar \|\chi^w \varphi\|^2. \tag{4-5}$$

Moreover, the commutator $[\mathcal{L}_\hbar, \chi^w]$ is a pseudodifferential operator of order $\hbar$, with symbol supported on $\operatorname{supp} \chi$. Hence, if $\underline{\chi}$ is a cutoff function having the same general properties of χ, such that $\underline{\chi} = 1$ on $\operatorname{supp} \chi$, we have

$$\langle [\mathcal{L}_\hbar, \chi^w]\varphi, \chi^w \varphi\rangle \leq C\hbar \|\underline{\chi}^w \varphi\| \, \|\chi^w \varphi\|. \tag{4-6}$$

Finally, the symbol of χ^w is equal to 0 on an $\hbar^\delta$-neighborhood of Σ, and thus the symbol $|p - A(q)|^2$ of $\mathcal{L}_\hbar$ is $\geq c\hbar^{2\delta}$ on the support of χ^w. Hence the Gårding inequality yields

$$\langle \mathcal{L}_\hbar \chi^w \varphi, \chi^w \varphi\rangle \geq c\hbar^{2\delta} \|\chi^w \varphi\|^2. \tag{4-7}$$

Using this last inequality in (4-4), and bounding the right-hand side with (4-5) and (4-6) we find

$$c\hbar^{2\delta} \|\chi^w \varphi\|^2 \leq \tilde{b}_1\hbar \|\chi^w \varphi\|^2 + C\hbar \|\underline{\chi}^w \varphi\| \, \|\chi^w \varphi\|,$$

and we deduce that

$$\|\chi^w \varphi\| \leq C\hbar^{1-2\delta} \|\underline{\chi}^w \varphi\|.$$

Iterating with $\underline{\chi}$ instead of χ, we finally get, for arbitrarily large $N > 0$,

$$\|\chi^w \varphi\| \leq C_N \hbar^N \|\varphi\|.$$

This is true for every ψ, with $\varphi = g_\hbar(\mathcal{L}_\hbar)\psi$, and thus $\|\chi^w g_\hbar(\mathcal{L}_\hbar)\| = \mathcal{O}(\hbar^\infty)$. ☐

Lemma 4.3 (microlocalization near Σ for $\mathcal{N}_\hbar$). *Let $\delta \in \left(0, \frac{1}{2}\right)$, $b_1 \in (b_0, \tilde{b}_1)$ and $\chi_1 \in \mathcal{C}_0^\infty(\mathbb{R}^{2s+k}_{x,\xi,\tau})$ be a cutoff function equal to* 1 *on a neighborhood of* 0. *Then every eigenfunction $\psi_\hbar$ of $\mathcal{N}_\hbar$ associated with an eigenvalue $\lambda_\hbar \le b_1\hbar$ satisfies*

$$\psi_\hbar = \mathrm{Op}_\hbar^w\, \chi_1(\hbar^{-\delta}(x, \xi, \tau)) + \mathcal{O}(\hbar^\infty)\psi_\hbar,$$

where the $\mathcal{O}(\hbar^\infty)$ is in $\mathcal{L}(\mathrm{L}^2, \mathrm{L}^2)$ and independent of $(\lambda_\hbar, \psi_\hbar)$.

Proof. Just as in the previous lemma, it is enough to show that

$$\|\chi^w g_\hbar(\mathcal{N}_\hbar)\| = \mathcal{O}(\hbar^\infty),$$

where $\chi^w = \mathrm{Op}_\hbar^w(1 - \chi_1(\hbar^{-\delta}(x, \xi, \tau)))$. We prove this using the same method. If $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ and $\varphi = g_\hbar(\mathcal{N}_\hbar)\psi$,

$$\langle \mathcal{N}_\hbar \chi^w \varphi, \chi^w \varphi\rangle = \langle \chi^w \mathcal{N}_\hbar \varphi, \chi^w \varphi\rangle + \langle [\mathcal{N}_\hbar, \chi^w]\varphi, \chi^w\varphi\rangle. \tag{4-8}$$

The right-hand side can be bounded from above as before. On the left-hand side we find $\varepsilon > 0$ such that

$$\langle \mathcal{N}_\hbar \chi^w \varphi, \chi^w \varphi\rangle \ge (1-\varepsilon)\langle \mathcal{H}_2 \chi^w \varphi, \chi^w\varphi\rangle, \tag{4-9}$$

with $\mathcal{H}_2 = \mathrm{Op}_\hbar^w\big(\langle M(w,t)\tau, \tau\rangle + \sum \hat{\beta}_j(w,t)|z_j|^2\big)$. The symbol of χ^w vanishes on an $\hbar^\delta$-neighborhood of $x = \xi = \tau = 0$. Thus we can bound from below the symbol of $\mathcal{H}_2$ and use the Gårding inequality:

$$\langle \mathcal{H}_2 \chi^w \varphi, \chi^w \varphi\rangle \ge c\hbar^{2\delta}\|\chi^w\varphi\|^2.$$

We conclude the proof as in Lemma 4.2. □

Lemma 4.4 (space localization for $\mathcal{N}_\hbar$). *Let $b_1 \in (b_0, \tilde{b}_1)$ and $\chi_0 \in \mathcal{C}_0^\infty(\mathbb{R}^{2s+k}_{y,\eta,t})$ be a cutoff function equal to* 1 *on a neighborhood of $\{\hat{b}(y,\eta,t) \le \tilde{b}_1\}$. Then every eigenfunction $\psi_\hbar$ of $\mathcal{N}_\hbar$ associated with an eigenvalue $\lambda_\hbar \le b_1\hbar$ satisfies*

$$\psi_\hbar = \mathrm{Op}_\hbar^w\, \chi_0(w,t)\psi_\hbar + \mathcal{O}(\hbar^\infty)\psi_\hbar,$$

where the $\mathcal{O}(\hbar^\infty)$ is in $\mathcal{L}(\mathrm{L}^2, \mathrm{L}^2)$ and independent of $(\lambda_\hbar, \psi_\hbar)$.

Proof. Every eigenfunction of $\mathcal{N}_\hbar$ is given by $\psi_\hbar(x,y,t) = u_\hbar(y,t)h_n(x)$ for some Hermite function h_n with $|n| \le N_{\max}$ and some eigenfunction $u_\hbar$ of $\mathcal{N}_\hbar^{[n]}$. Thus, it is enough to prove the lemma for the eigenfunctions of $\mathcal{N}_\hbar^{[n]}$. If $u_\hbar$ is such an eigenfunction, associated with an eigenvalue $\lambda_\hbar \le b_1\hbar$, then

$$u_\hbar = g_\hbar(\mathcal{N}_\hbar^{[n]})u_\hbar.$$

We will prove that $\|\chi^w g_\hbar(\mathcal{N}_\hbar^{[n]})\| = \mathcal{O}(\hbar^\infty)$, with $\chi^w = \mathrm{Op}_\hbar^w(1-\chi_0)$, which is enough to conclude. If $u \in \mathrm{L}^2(\mathbb{R}^{k+s}_{y,t})$ and $\varphi = g_\hbar(\mathcal{N}_\hbar^{[n]})u$, then

$$\langle \mathcal{N}_\hbar^{[n]} \chi^w \varphi, \chi^w \varphi\rangle = \langle \chi^w \mathcal{N}_\hbar^{[n]} \varphi, \chi^w \varphi\rangle + \langle [\mathcal{N}_\hbar^{[n]}, \chi^w]\varphi, \chi^w\varphi\rangle. \tag{4-10}$$

We first have the bound

$$\langle \chi^w \mathcal{N}_\hbar^{[n]} \varphi, \chi^w\varphi\rangle \le \tilde{b}_1\hbar\|\chi^w\varphi\|^2. \tag{4-11}$$

The commutator $[\mathcal{N}_\hbar^{[n]}, \chi^w]$ is a pseudodifferential operator of order $\hbar$ with symbol supported on $\operatorname{supp}\chi$. Moreover, its principal symbol is $\{N_\hbar^{[n]}, \chi\}$. From the definition of $N_\hbar^{[n]}$ we deduce

$$\langle[\mathcal{N}_\hbar^{[n]}, \chi^w]\varphi, \chi^w\varphi\rangle \le C\hbar\langle\underline{\chi}^w|\tau|^w\varphi, \chi^w\varphi\rangle,$$

where $\underline{\chi}$ has the same general properties as χ, and is equal to 1 on $\operatorname{supp}\chi$. By Lemma 4.3, we can find a cutoff where $|\tau| \lesssim \hbar^\delta$ and we get

$$\langle[\mathcal{N}_\hbar^{[n]}, \chi^w]\varphi, \chi^w\varphi\rangle \le C\hbar^{1+\delta}\|\underline{\chi}^w\varphi\|\,\|\chi^w\varphi\|. \tag{4-12}$$

Finally for $\varepsilon > 0$ small enough we have the lower bound

$$\langle\mathcal{N}_\hbar^{[n]}\chi^w\varphi, \chi^w\varphi\rangle \ge \hbar(\tilde{b}_1+\varepsilon)\|\chi^w\varphi\|^2,$$

because $N_\hbar^{[n]}(w,t) \ge \hbar\hat{b}(w,t)$ and χ vanishes on a neighborhood of $\{\hat{b}(w,t) \le \tilde{b}_1\}$. Using this lower bound in (4-10), and bounding the right-hand side with (4-11) and (4-12) we get

$$\hbar(\tilde{b}_1+\varepsilon)\|\chi^w\varphi\|^2 \le \hbar\tilde{b}_1\|\chi^w\varphi\|^2 + C\hbar^{1+\delta}\|\underline{\chi}^w\varphi\|\,\|\chi^w\varphi\|. \tag{4-13}$$

Thus

$$\varepsilon\|\chi^w\varphi\| \le C\hbar^\delta\|\underline{\chi}^w\varphi\|,$$

and we can iterate with $\underline{\chi}$ instead of χ to conclude. □

4C. *Proof of Theorem 1.7.* To conclude the proof of Theorem 1.7, it remains to show that

$$\lambda_n(\mathcal{L}_\hbar) = \lambda_n(\mathcal{N}_\hbar) + \mathcal{O}(\hbar^{r_1/2-\varepsilon})$$

uniformly with respect to $n \in [1, N_\hbar^{\max}]$ with

$$N_\hbar^{\max} = \max\{n \in \mathbb{N} : \lambda_n(\mathcal{L}_\hbar) \le b_1\hbar\}.$$

Here $\lambda_n(\mathcal{A})$ denotes the n-th eigenvalue of the self-adjoint operator $\mathcal{A}$, repeated with multiplicities.

Lemma 4.5. *One has*

$$\lambda_n(\mathcal{L}_\hbar) = \lambda_n(\mathcal{N}_\hbar) + \mathcal{O}(\hbar^{r_1/2-\varepsilon})$$

uniformly with respect to $n \in [1, N_\hbar^{\max}]$.

Proof. Let us focus on the "$\le$" inequality. For $n \in [1, N_\hbar^{\max}]$, denote by $\psi_n^\hbar$ the normalized eigenfunction of $\mathcal{N}_\hbar$ associated with $\lambda_n(\mathcal{N}_\hbar)$, and

$$\varphi_n^\hbar = \mathrm{U}_\hbar\psi_n^\hbar,$$

where $\mathrm{U}_\hbar$ is given by Theorem 3.4. We will use $\varphi_n^\hbar$ as quasimode for $\mathcal{L}_\hbar$. Let $N \in [1, N_\hbar^{\max}]$ and

$$V_N^\hbar = \operatorname{span}\{\varphi_n^\hbar : 1 \le n \le N\}.$$

For $\varphi \in V_N^\hbar$ we use the notation $\psi = \mathrm{U}_\hbar^{-1}\varphi$. By Theorem 3.4, we have

$$\langle\mathcal{L}_\hbar\varphi, \varphi\rangle = \langle\mathcal{N}_\hbar\psi, \psi\rangle + \langle\mathcal{R}_\hbar\psi, \psi\rangle \le \lambda_N(\mathcal{N}_\hbar)\|\psi\|^2 + \langle\mathcal{R}_\hbar\psi, \psi\rangle. \tag{4-14}$$

According to Lemmas 4.3 and 4.4, ψ is microlocalized, where $(w,t) \in \{\hat{b}(w,t) \le \tilde{b}_1\} \subset U$ and $|(x,\xi,\tau)| \le \hbar^{\delta}$. But the symbol of $\mathcal{R}_\hbar$ is such that $R_\hbar = \mathcal{O}((x,\xi,\tau,\hbar^{1/2})^{r_1})$ for $(w,t) \in U$, so

$$\langle \mathcal{R}_\hbar \psi, \psi\rangle = \mathcal{O}(\hbar^{\delta r_1}) = \mathcal{O}(\hbar^{r_1/2-\varepsilon}) \tag{4-15}$$

for suitable $\delta \in \left(0, \frac{1}{2}\right)$. By (4-14) and (4-15) we have

$$\langle \mathcal{L}_\hbar \varphi, \varphi\rangle \le (\lambda_N(\mathcal{N}_\hbar) + C\hbar^{r_1/2-\varepsilon})\|\varphi\|^2 \quad \text{for all } \varphi \in V_N^\hbar.$$

Since $V_N^\hbar$ is N-dimensional, the minimax principle implies that

$$\lambda_N(\mathcal{L}_\hbar) \le \lambda_N(\mathcal{N}_\hbar) + C\hbar^{r_1/2-\varepsilon}. \tag{4-16}$$

The reversed inequality is proved in the same way: we take the eigenfunctions of $\mathcal{L}_\hbar$ as quasimodes for $\mathcal{N}_\hbar$, and we use the microlocalization lemma, Lemma 4.2. □

5. A second normal form in the case $k > 0$

In the previous sections, we compared the spectrum of $\mathcal{L}_\hbar$ and the spectrum of the normal form $\mathcal{N}_\hbar$. Moreover, if $b_1 > b_0$ is sufficiently close to b_0 the spectrum of $\mathcal{N}_\hbar$ in $(-\infty, b_1\hbar)$ is given by the spectrum of $\mathcal{N}_\hbar^{[1]}$, an $\hbar$-pseudodifferential operator on $\mathbb{R}^{s+k}_{(y,t)}$ with symbol

$$N_\hbar^{[1]} = \langle M(y,\eta,t)\tau,\tau\rangle + \hbar\hat{b}(y,\eta,t) + f_1^\star(y,\eta,t,\tau,\hbar). \tag{5-1}$$

In this section, we will construct a Birkhoff normal form again, to describe the spectrum of $\mathcal{N}_\hbar^{[1]}$ by an effective operator $\mathcal{M}_\hbar$ on $\mathbb{R}^s_y$. For that purpose, in Section 5A we will find new canonical variables $(\hat{t},\hat{\tau})$ in which $N_\hbar^{[1]}$ is the perturbation of a harmonic oscillator. In Sections 5B and 5C we will construct the semiclassical Birkhoff normal form $\mathcal{M}_\hbar$. In Section 5D we will prove that the spectrum of $\mathcal{N}_\hbar^{[1]}$ is given by the spectrum of $\mathcal{M}_\hbar$.

Under Assumption 1 we know that $t \mapsto \hat{b}(w,t)$ admits a nondegenerate minimum at $s(w)$ for w in a neighborhood of 0, and we denote by $(\nu_1^2(w), \dots, \nu_k^2(w))$ the eigenvalues of the positive symmetric matrix

$$M(w,s(w))^{1/2} \cdot \tfrac{1}{2}\partial_t^2 \hat{b}(w,s(w)) \cdot M(w,s(w))^{1/2}.$$

The maps $\nu_1, \dots, \nu_k$ are smooth nonvanishing functions in a neighborhood of $w = 0$.

5A. *Geometry of the symbol $\mathcal{N}_\hbar^{[1]}$.*

We prove the following lemma.

Lemma 5.1. *There exists a canonical (symplectic) transformation $\Phi_2 : U_2 \to V_2$ between neighborhoods U_2, V_2 of $0 \in \mathbb{R}^{2s+2k}_{(y,\eta,t,\tau)}$ such that*

$$\widehat{N}_\hbar := N_\hbar^{[1]} \circ \Phi_2 = \hbar\hat{b}(w,s(w)) + \sum_{j=1}^{k} \nu_j(w)(\tau_j^2 + \hbar t_j^2) + \mathcal{O}(|t|^3|\tau|^2 + |t|^3\hbar + \hbar^2 + \hbar|\tau| + |\tau|^3 + |t||\tau|^2).$$

Proof. We want to expand $\mathcal{N}_\hbar^{[1]}$ near its minimum with respect to the variables $v = (t,\tau)$. First, from the Taylor expansion of $f_1^\star$ we deduce

$$N_\hbar^{[1]} = \langle M(w,t)\tau,\tau\rangle + \hbar\hat{b}(w,t) + \mathcal{O}(\hbar^2 + \tau\hbar + \tau^3).$$

We will Taylor-expand $t \mapsto \hat{b}(w,t)$ on a neighborhood of its minimum point $s(w)$. For that purpose, we define new variables $(\tilde{y},\tilde{\eta},\tilde{t},\tilde{\tau}) = \tilde{\varphi}(y,\eta,t,\tau)$ by

$$\begin{cases} \tilde{y} = y - \sum_{j=1}^{k} \tau_j \nabla_\eta s_j(y,\eta), \\ \tilde{\eta} = \eta + \sum_{j=1}^{k} \tau_j \nabla_y s_j(y,\eta), \\ \tilde{t} = t - s(y,\eta), \\ \tilde{\tau} = \tau. \end{cases}$$

Then $\tilde{\varphi}^*\omega_0 = \omega_0 + \mathcal{O}(\tau)$. Using Theorem B.2, we can make $\tilde{\varphi}$ symplectic on a neighborhood of 0, up to a change of order $\mathcal{O}(\tau^2)$. In these new variables, the symbol $\widetilde{N}_\hbar := N_\hbar^{[1]} \circ \tilde{\varphi}^{-1}$ is

$$\begin{aligned} \widetilde{N}_\hbar &= \langle M[\tilde{w} + \mathcal{O}(\tilde{\tau}), \tilde{t} + s(\tilde{w} + \mathcal{O}(\tilde{\tau}))]\tilde{\tau}, \tilde{\tau}\rangle + \hbar\hat{b}[\tilde{y} + \mathcal{O}(\tilde{\tau}), \tilde{\eta} + \mathcal{O}(\tilde{\tau}), s(\tilde{y},\tilde{\eta}) + \tilde{t} + \mathcal{O}(\tilde{\tau})] \\ &\qquad + \mathcal{O}(\hbar^2 + \hbar\tilde{\tau} + \tilde{\tau}^3) \\ &= \langle M(\tilde{w}, \tilde{t} + s(\tilde{w}))\tilde{\tau}, \tilde{\tau}\rangle + \hbar\hat{b}[\tilde{y}, \tilde{\eta}, s(\tilde{y},\tilde{\eta}) + \tilde{t}] + \mathcal{O}(\hbar^2 + \hbar\tilde{\tau} + \tilde{\tau}^3). \end{aligned}$$

Then we remove the tildes and expand this symbol in powers of $t, \tau, \hbar$. We find

$$\widetilde{N}_\hbar = \langle M(w, s(w))\tau, \tau\rangle + \hbar\hat{b}(w, s(w)) + \frac{\hbar}{2}\langle \partial_t^2 \hat{b}(w, s(w))t, t\rangle + \mathcal{O}(|t|^3\hbar + \hbar^2 + \hbar|\tau| + |\tau|^3 + |t|\,|\tau|^2).$$

Now, we want to diagonalize the positive quadratic forms $M(w, s(w))$ and $\frac{1}{2}\partial_t^2\hat{b}[w, s(w)]$. The diagonalization of quadratic forms in orthonormal coordinates implies that there exists a matrix $P(w)$ such that

$$^tP\ M^{-1} P = I \quad \text{and} \quad ^tP\ \tfrac{1}{2}\partial_t^2\hat{b}\ P = \operatorname{diag}(\nu_1^2, \dots, \nu_k^2).$$

We define the new coordinates $(\check{y},\check{\eta},\check{t},\check{\tau}) = \check{\varphi}(y,\eta,t,\tau)$ by

$$\begin{cases} \check{t} = P(w)^{-1} t, \\ \check{\tau} = {}^tP(w)\tau, \\ \check{y} = y + {}^t[\nabla_\eta(P^{-1}t)] \cdot {}^tP\tau, \\ \check{\eta} = \eta - {}^t[\nabla_y(P^{-1}t)] \cdot {}^tP\tau, \end{cases}$$

so that $\check{\varphi}^*\omega_0 - \omega_0 = \mathcal{O}(|t|^2 + |\tau|)$. Again, we can make it symplectic up to a change of order $\mathcal{O}(|t|^3 + |\tau|^2)$ by Theorem B.2. In these new variables, the symbol becomes (after removing the "checks")

$$\check{N}_\hbar = \hbar\hat{b}(w, s(w)) + \sum_{j=1}^{k} (\tau_j^2 + \hbar\nu_j(w)^2 t_j^2) + \mathcal{O}\big(|t|^3|\tau|^2 + |t|^3\hbar + \hbar^2 + \hbar|\tau| + |\tau|^3 + |t|\,|\tau|^2\big).$$

The last change of coordinates $(\hat{y},\hat{\eta},\hat{t},\hat{\tau}) = \hat{\varphi}(y,\eta,t,\tau)$, defined by

$$\begin{cases} \hat{t}_j = \nu_j(w)^{1/2} t_j, \\ \hat{\tau}_j = \nu_j(w)^{-1/2}\tau_j, \\ \hat{y}_j = y_j + \sum_{i=1}^{k} \nu_i^{-1/2}\tau_i \partial_{\eta_j}\nu_i^{1/2} t_i, \\ \hat{\eta} = \eta - \sum_{i=1}^{k} \nu_i^{-1/2}\tau_i \partial_{y_j}\nu_i^{1/2} t_i, \end{cases}$$

is such that $\hat{\varphi}^*\omega_0 = \omega_0 + \mathcal{O}(\tau)$, so it can be corrected modulo $\mathcal{O}(|\tau|^2)$ to be symplectic, and we get the new symbol

$$\widehat{N}_\hbar = \hbar\hat{b}(w, s(w)) + \sum_{j=1}^{k} \nu_j(w)(\tau_j^2 + \hbar t_j^2) + \mathcal{O}\big(|t|^3|\tau|^2 + |t|^3\hbar + \hbar^2 + \hbar|\tau| + |\tau|^3 + |t|\,|\tau|^2\big),$$

which concludes the proof. ∎

5B. ***Second formal normal form.*** The harmonic oscillators appearing in $\widehat{N}_\hbar$ are

$$\mathcal{J}_\hbar^{(j)} = \mathrm{Op}_\hbar^w(\hbar^{-1}\tau_j^2 + t_j^2), \quad 1 \le j \le k.$$

If we define

$$h = \sqrt{\hbar},$$

the symbol of $\mathcal{J}_\hbar^{(j)}$ for the h-quantization is $\tilde{\tau}_j^2 + t_j^2$. This is why we use the mixed quantization

$$\mathrm{Op}_\sharp^w(\mathrm{a})u(y_0, t_0) = \frac{1}{(2\pi\hbar)^{n-k}(2\pi\sqrt{\hbar})^k} \int e^{(i/\hbar)\langle y_0 - y, \eta\rangle} e^{(i/\sqrt{\hbar})\langle t_0 - t, \tilde{\tau}\rangle} \mathrm{a}(\sqrt{\hbar}, y, \eta, t, \tilde{\tau})\, \mathrm{d}y\, \mathrm{d}\eta\, \mathrm{d}t\, \mathrm{d}\tilde{\tau}. \quad (5\text{-}2)$$

It is related to the $\hbar$-quantization by the relation

$$\tau = h\tilde{\tau}, \quad h = \sqrt{\hbar}.$$

In other words, if a is a symbol in some standard class $S(m)$, and if we define

$$\mathrm{a}(h, y, \eta, t, \tilde{\tau}) = a(h^2, y, \eta, t, h\tilde{\tau}),$$

then we have

$$\mathrm{Op}_\sharp^w(\mathrm{a}) = \mathrm{Op}_\hbar^w(a).$$

However, if we take $\mathrm{a} \in S(m)$, then $\mathrm{Op}_\sharp^w(\mathrm{a})$ is not necessarily an $\hbar$-pseudodifferential operator, since the associated a may not be bounded with respect to $\hbar$, and thus it does not belong to any standard class. For instance, we have

$$\partial_\tau a = \frac{1}{\sqrt{\hbar}} \partial_{\tilde{\tau}} \mathrm{a}.$$

But still $\mathrm{Op}_\sharp^w(\mathrm{a})$ is an h-pseudodifferential operator, with symbol

$$\mathfrak{a}(h, y, \tilde{\eta}, t, \tilde{\tau}) = \mathrm{a}(h, y, h\tilde{\eta}, t, \tilde{\tau}).$$

With this notation

$$\mathrm{Op}_\sharp^w(\mathrm{a}) = \mathrm{Op}_h^w(\mathfrak{a}).$$

Thus, in this sense, we can use the properties of $\hbar$-pseudodifferential and h-pseudodifferential operators to deal with our mixed quantization.

Remark 5.2. Operators of the form (5-2) are just special cases of the usual h-pseudodifferential operators for which the reader can refer to [Martinez 2002; Zworski 2012]. Moreover, our mixed quantization could be interpreted as a $\sqrt{\hbar}$-quantization with operator-valued symbols for which we refer to [Keraval 2018; Martinez 2007]. Indeed we can write

$$\mathrm{Op}_\sharp^w(\mathrm{a}) = \mathrm{Op}_h^w(\mathrm{Op}_\hbar^w\, \mathrm{a}), \quad (5\text{-}3)$$

where we first quantize with respect to (y, η) so that $\mathrm{Op}_\hbar^w\, \mathrm{a}$ is an operator-valued symbol which depends on $(t, \tilde{\tau})$. In the following we could have used this formalism, thus dealing with operator-valued symbols in $(t, \tilde{\tau})$ instead of real-valued symbols and mixed quantization.

In our case, we have

$$\mathrm{Op}^w_\sharp(\mathrm{N}_h) = \mathrm{Op}^w_h(\widehat{N}_\hbar),$$

with

$$\mathrm{N}_h = h^2\hat{b}(w, s(w)) + h^2\sum_{j=1}^{k}\nu_j(w)(\tilde{\tau}_j^2 + t_j^2) + \mathcal{O}(h^2|t|^3 + h^4 + h^3|\tilde{\tau}| + h^2|t||\tilde{\tau}|^2).$$

Let us construct a semiclassical Birkhoff normal form with respect to this quantization. We will work in the space of formal series

$$\mathcal{E}_2 := \mathcal{C}^\infty(U)[\![t, \tilde{\tau}, h]\!], \tag{5-4}$$

where $U = U_2 \cap \mathbb{R}^{2s}_w \times \{0\}$. This space is endowed with the star product $\star$ adapted to our mixed quantization. In other words

$$\mathrm{Op}^w_\sharp(\mathrm{a}\star\mathrm{b}) = \mathrm{Op}^w_\sharp(\mathrm{a})\,\mathrm{Op}^w_\sharp(\mathrm{b}).$$

The change of variable $\tau = h\tilde{\tau}$ between the usual $\hbar$-quantization and our mixed quantization yields the following formula for the star product:

$$\mathrm{a}\star\mathrm{b} = \sum_{k\geq 0}\frac{1}{k!}\Big(\frac{h}{2i}\Big)^k A_h(\partial)^k(\mathrm{a}(h, y_1, \eta_1, t_1, \tilde{\tau}_1)\mathrm{b}(h, y_2, \eta_2, t_2, \tilde{\tau}_2))_{|(t_1,\tau_1,y_1,\eta_1)=(t_2,\tau_2,y_2,\eta_2)}, \tag{5-5}$$

with

$$A_h(\partial) = \sum_{j=1}^{k}\frac{\partial}{\partial t_{1j}}\frac{\partial}{\partial\tilde{\tau}_{2j}} - \frac{\partial}{\partial t_{2j}}\frac{\partial}{\partial\tilde{\tau}_{1j}} + h\sum_{j=1}^{s}\frac{\partial}{\partial y_{1j}}\frac{\partial}{\partial\eta_{2j}} - \frac{\partial}{\partial y_{2j}}\frac{\partial}{\partial\eta_{1j}}.$$

The degree function on $\mathcal{E}_2$ is defined by

$$\deg(t^{\alpha_1}\tilde{\tau}^{\alpha_2}h^\ell) = |\alpha_1| + |\alpha_2| + 2\ell.$$

We denote by $\mathcal{D}_N$ the $\mathcal{C}^\infty(U)$-module spanned by monomials of degree N, and

$$\mathcal{O}_N = \bigoplus_{n\geq N}\mathcal{D}_n.$$

For $\tau_1, \tau_2 \in \mathcal{E}_2$, we define

$$\mathrm{ad}_{\tau_1}(\tau_2) = [\tau_1, \tau_2] = \tau_1\star\tau_2 - \tau_2\star\tau_1,$$

and if $\tau_1 \in \mathcal{O}_{N_1}$ and $\tau_2 \in \mathcal{O}_{N_2}$,

$$\frac{i}{h}\,\mathrm{ad}_{\tau_1}(\tau_2) \in \mathcal{O}_{N_1+N_2-2}.$$

We define

$$N_0 = \hat{b}(w, s(w)) \in \mathcal{D}_0 \quad\text{and}\quad N_2 = \sum_{j=1}^{k}\nu_j(w)|\tilde{v}_j|^2 \in \mathcal{D}_2,$$

with the notation $\tilde{v}_j = t_j + i\tilde{\tau}_j$, so that

$$\frac{1}{h^2}\mathrm{N}_h = N_0 + N_2 + \mathcal{O}_3.$$

Now we construct the following normal form. Recall that r_2 is an integer chosen such that,

$$\text{for all } \alpha \in \mathbb{Z}^k, \quad 0 < |\alpha| < r_2, \quad \sum_{j=1}^{s} \alpha_j \nu_j(0) \neq 0.$$

Moreover, this nonresonance relation at $w = 0$ can be extended to a small neighborhood of 0.

Lemma 5.3. *For any $\gamma \in \mathcal{O}_3$, there exist $\kappa, \tau \in \mathcal{O}_3$ and $\rho \in \mathcal{O}_{r_2}$ such that*

$$e^{(i/h)\,\mathrm{ad}_\tau}(N_0 + N_2 + \gamma) = N_0 + N_2 + \kappa + \rho, \tag{5-6}$$

and $[\kappa, |\tilde{v}_j|^2] = 0$ for $1 \le j \le k$.

Proof. We prove this result by induction. Assume that we have, for some $N > 0$, a $\tau \in \mathcal{O}_3$ such that

$$e^{(i/h)\,\mathrm{ad}_\tau}(N_0 + N_2 + \gamma) = N_0 + N_2 + K_3 + \cdots + K_{N-1} + R_N + \mathcal{O}_{N+1},$$

with $R_N \in \mathcal{D}_N$ and $K_i \in \mathcal{D}_i$ such that $[K_i, |\tilde{v}_j|^2] = 0$. We are looking for a $\tau_N \in \mathcal{D}_N$. For such a τ_N, $(i/h)\,\mathrm{ad}_{\tau_N} : \mathcal{O}_j \to \mathcal{O}_{N+j-2}$ so

$$e^{(i/h)\,\mathrm{ad}_{\tau+\tau_N}}(N_0 + N_2 + \gamma) = N_0 + N_2 + K_3 + \cdots + K_{N-1} + R_N + \frac{i}{h}\,\mathrm{ad}_{\tau_N}(N_0 + N_2) + \mathcal{O}_{N+1}.$$

Moreover N_0 does not depend on (t, τ) so the expansion (5-5) yields

$$\frac{i}{h}\,\mathrm{ad}_{\tau_N}(N_0) = h \sum_{j=1}^{s} \left(\frac{\partial \tau_N}{\partial y_j} \frac{\partial N_0}{\partial \eta_j} - \frac{\partial \tau_N}{\partial \eta_j} \frac{\partial N_0}{\partial y_j} \right) + \mathcal{O}_{N+6} = \mathcal{O}_{N+2},$$

and thus

$$e^{(i/h)\,\mathrm{ad}_{\tau+\tau_N}}(N_0 + N_2 + \gamma) = N_0 + N_2 + K_3 + \cdots + K_{N-1} + R_N + \frac{i}{h}\,\mathrm{ad}_{\tau_N}(N_2) + \mathcal{O}_{N+1}.$$

So we are looking for $\tau_N, K_N \in \mathcal{D}_N$ solving the equation

$$R_N = K_N + \frac{i}{h}\,\mathrm{ad}_{N_2}\,\tau_N + \mathcal{O}_{N+1}. \tag{5-7}$$

To solve this equation, we study the operator $(i/h)\,\mathrm{ad}_{N_2} : \mathcal{O}_N \to \mathcal{O}_N$,

$$\frac{i}{h}\,\mathrm{ad}_{N_2}(\tau_N) = \sum_{j=1}^{k} \left(\nu_j(w) \frac{i}{h}\,\mathrm{ad}_{|\tilde{v}_j|^2}(\tau_N) + \frac{i}{h}\,\mathrm{ad}_{\nu_j}(\tau_N)|\tilde{v}_j|^2 \right),$$

and since ν only depends on w, expansion (5-5) yields

$$\frac{i}{h}\,\mathrm{ad}_{\nu_i}(\tau_N) = \sum_{j=1}^{s} h \left(\frac{\partial \nu_i}{\partial y_j} \frac{\partial \tau_N}{\partial \eta_j} - \frac{\partial \nu_i}{\partial \eta_j} \frac{\partial \tau_N}{\partial y_j} \right) + \mathcal{O}_{N+6} = \mathcal{O}_{N+2}.$$

Hence,

$$\frac{i}{h}\,\mathrm{ad}_{N_2}(\tau_N) = \sum_{j=1}^{k} \nu_j(w) \frac{i}{h}\,\mathrm{ad}_{|\tilde{v}_j|^2}(\tau_N) + \mathcal{O}_{N+2},$$

and (5-7) becomes

$$R_N = K_N + \sum_{j=1}^{k} \nu_j(w) \frac{i}{h}\,\mathrm{ad}_{|\tilde{v}_j|^2}(\tau_N) + \mathcal{O}_{N+1}. \tag{5-8}$$

Moreover, $(i/h)\,\mathrm{ad}_{|\tilde{v}_j|^2}$ acts as

$$\sum_{j=1}^{k} \nu_j(w)\frac{i}{h}\,\mathrm{ad}_{|\tilde{v}_j|^2}(v^{\alpha_1}\bar{v}^{\alpha_2}h^{\ell}) = \langle \nu(w), \alpha_2-\alpha_1\rangle v^{\alpha_1}\bar{v}^{\alpha_2}h^{\ell}.$$

The definition of r_2 ensures that $\langle \nu(w), \alpha_2-\alpha_1\rangle$ does not vanish on a neighborhood of $w=0$ if $N=|\alpha_1|+|\alpha_2|+2\ell<r_2$ and $\alpha_1\neq\alpha_2$. Hence we can decompose every R_N as in (5-8), where K_N contains the terms with $\alpha_1=\alpha_2$. These terms are exactly the ones commuting with $|\tilde{v}_j|^2$ for $1\le j\le k$. □

5C. *Second quantized normal form.* Now we can quantize Lemmas 5.1 and 5.3 to prove the following theorem.

Theorem 5.4. *There exist*

(1) *a unitary operator* $\mathrm{U}_{2,\hbar}:\mathrm{L}^2(\mathbb{R}^{s+k}_{(y,t)})\to\mathrm{L}^2(\mathbb{R}^{s+k}_{(y,t)})$ *quantizing a symplectomorphism* $\tilde{\Phi}_2=\Phi_2+\mathcal{O}((t,\tau)^2)$ *microlocally near* 0,

(2) *a function* $f_2^{\star}:\mathbb{R}^{2s}_w\times\mathbb{R}^k_J\times[0,1)\to\mathbb{R}$ *which is* $\mathcal{C}^\infty$ *with compact support such that*

$$|f_2^{\star}(w, J_1,\dots,J_k,\sqrt{\hbar})|\le C(|J|+\sqrt{\hbar})^2,$$

(3) *a* $\sqrt{\hbar}$*-pseudodifferential operator* $\mathcal{R}_{2,\hbar}$ *with symbol* $\mathcal{O}((t,\tilde{\tau},\hbar^{1/4})^{r_2})$ *on a neighborhood of* 0

such that

$$\mathrm{U}_{2,\hbar}^{*}\mathcal{N}_{\hbar}^{[1]}\mathrm{U}_{2,\hbar}=\hbar\mathcal{M}_{\hbar}+\hbar\mathcal{R}_{2,\hbar},$$

where $\mathcal{M}_\hbar$ *is the* $\hbar$*-pseudodifferential operator*

$$\mathcal{M}_{\hbar}=\mathrm{Op}_{\hbar}^{w}\,\hat{b}(w,s(w))+\sum_{j=1}^{k}\mathcal{J}_{\hbar}^{(j)}\,\mathrm{Op}_{\hbar}^{w}\,\nu_j+\mathrm{Op}_{\hbar}^{w}\,f_2^{\star}(w,\mathcal{J}_{\hbar}^{(1)},\dots,\mathcal{J}_{\hbar}^{(k)},\sqrt{\hbar}).$$

Proof. Lemma 5.1 provides us with a symplectomorphism Φ_2 such that

$$N_{\hbar}^{[1]}\circ\Phi_2=\hbar\hat{b}(w,s(w))+\sum_{j=1}^{k}\nu_j(w)(\tau_j^2+\hbar t_j^2)+\mathcal{O}(|t|^3|\tau|^2+|t|^3\hbar+\hbar^2+\hbar|\tau|+|\tau|^3+|t|\,|\tau|^2).$$

We can apply the Egorov theorem to get a Fourier integral operator $V_{2,\hbar}$ such that

$$V_{2,\hbar}^{*}\,\mathrm{Op}_{\hbar}^{w}(N_{\hbar}^{[1]})V_{2,\hbar}=\mathrm{Op}_{\hbar}^{w}(\widehat{N}_{\hbar}),$$

with $\widehat{N}_{\hbar}=N_{\hbar}^{[1]}\circ\Phi_2+\mathcal{O}(\hbar^2)$ on a neighborhood of $w=0$. We define

$$\mathrm{N}_h(y,\eta,t,\tilde{\tau})=\widehat{N}_{\hbar}(y,\eta,t,h\tilde{\tau}),$$

and following the notation of Section 5B, we have the associated formal series

$$\frac{1}{h^2}\mathrm{N}_h=N_0+N_2+\gamma,\quad \gamma\in\mathcal{O}_3.$$

We apply Lemma 5.3 and we get formal series κ, ρ such that

$$e^{(i/h)\,\mathrm{ad}_\rho}(N_0+N_2+\gamma)=N_0+N_2+\kappa+\mathcal{O}_{r_2}.$$

We take a compactly supported symbol $\mathrm{a}(h, w, t, \tilde{\tau})$ with Taylor series ρ. Then the operator

$$e^{ih^{-1}\operatorname{Op}_\sharp^w(\mathrm{a})}\operatorname{Op}_\sharp^w(h^{-2}\mathrm{N}_h)e^{-ih^{-1}\operatorname{Op}_\sharp^w(\mathrm{a})} \tag{5-9}$$

has a symbol with Taylor series $N_0 + N_2 + \kappa + \mathcal{O}_{r_2}$. Since $\kappa \in \mathcal{O}_3$ commutes with $|\tilde{v}_j|^2$, it can be written

$$\kappa = \sum_{2|\alpha|+2\ell\geq 3} c^\star_{\alpha\ell}(w)(|\tilde{v}_1|^2)^{\star\alpha_1}\cdots(|\tilde{v}_k|^2)^{\star\alpha_k}h^\ell.$$

If we take $f_2^\star(h, w, J_1, \dots, J_k)$ a smooth compactly supported function with Taylor series

$$[f_2^\star] = \sum_{2|\alpha|+2\ell\geq 3} c^\star_{\alpha\ell}(w)J_1^{\alpha_1}\cdots J_k^{\alpha_k}h^\ell,$$

then the operator (5-9) is equal to

$$\operatorname{Op}_\sharp^w N_0 + \operatorname{Op}_\sharp^w N_2 + \operatorname{Op}_\hbar^w f_2^\star(h, w, \mathcal{J}_\hbar^{(1)}, \dots, \mathcal{J}_\hbar^{(k)})$$

modulo $\mathcal{O}_{r_2}$. Multiplying by h^2, and getting back to the $\hbar$-quantization, we get

$$e^{ih^{-1}\operatorname{Op}_\sharp^w(\mathrm{a})}\operatorname{Op}_\hbar^w(\widehat{N}_\hbar)e^{-ih^{-1}\operatorname{Op}_\sharp^w(\mathrm{a})} = \hbar\mathcal{M}_\hbar + \hbar\mathcal{R}_\hbar,$$

with

$$\mathcal{M}_\hbar = \operatorname{Op}_\hbar^w \hat{b}(w, s(w)) + \sum_{j=1}^k \operatorname{Op}_\hbar^w \nu_j(w)\mathcal{J}_\hbar^{(j)} + \operatorname{Op}_\hbar^w f_2^\star(\sqrt{\hbar}, w, \mathcal{J}_\hbar^{(1)}, \dots, \mathcal{J}_\hbar^{(k)}),$$

and $\mathcal{R}_\hbar$ a $\sqrt{\hbar}$-pseudodifferential operator with symbol $\mathcal{O}_{r_2}$. Note that $\mathcal{M}_\hbar$ is an $\hbar$-pseudodifferential operator whose symbol admits an expansion in powers of $\sqrt{\hbar}$. □

5D. *Proof of Theorem 1.11.* In order to prove Theorem 1.11, we need the following microlocalization lemma.

Lemma 5.5. *Let $\delta \in \left(0, \frac{1}{2}\right)$ and $c > 0$. Let $\chi_0 \in \mathcal{C}_0^\infty(\mathbb{R}^{2s}_{(y,\eta)})$ and $\chi_1 \in \mathcal{C}_0^\infty(\mathbb{R}^{2k}_{(t,\tilde{\tau})})$ both equal to* 1 *on a neighborhood of* 0. *Then every eigenfunction $\psi_\hbar$ of $\mathcal{N}_\hbar$ or $\hbar\mathcal{M}_\hbar$ associated to an eigenvalue $\lambda_\hbar \leq \hbar(b_0 + c\hbar^\delta)$ satisfies*

$$\psi_\hbar = \operatorname{Op}_{\sqrt{\hbar}}^w \chi_0(\sqrt{\hbar}^{-\delta}(t, \tilde{\tau}))\operatorname{Op}_\hbar^w \chi_1(y, \eta)\psi_\hbar + \mathcal{O}(\hbar^\infty)\psi_\hbar.$$

Proof. Using the mixed quantization and $h = \sqrt{\hbar}$, we have $\mathcal{N}_\hbar^{[1]} = \operatorname{Op}_\sharp^w \mathrm{N}_h^{[1]}$, with

$$\mathrm{N}_h^{[1]}(y, \eta, t, \tilde{\tau}) = h^2\langle M(y, \eta, t)\tilde{\tau}, \tilde{\tau}\rangle + h^2\hat{b}(w, t) + f_1^\star(y, \eta, t, h\tilde{\tau}, h^2).$$

The principal part of $\mathrm{N}_h^{[1]}$ is of order h^2, and implies a microlocalization of the eigenfunctions, where

$$h^2\langle M(w, t)\tilde{\tau}, \tilde{\tau}\rangle + h^2\hat{b}(w, t) \leq \lambda_h \leq h^2(b_0 + ch^{2\delta}).$$

Since $\hat{b}$ admits a unique and nondegenerate minimum b_0 at 0, this implies that w lies in an arbitrarily small neighborhood of 0, and that

$$|t|^2 \leq Ch^{2\delta}, \quad |\tilde{\tau}|^2 \leq Ch^{2\delta}.$$

The technical details follow the same ideas of Lemmas 4.2, 4.3 and 4.4. Now we can focus on $\mathcal{M}_\hbar$, whose principal symbol with respect to the $\mathrm{Op}_\sharp^w$-quantization is

$$\mathrm{M}_0(y,\eta,t,\tilde{\tau}) = \hat{b}(y,\eta,s(y,\eta)) + \sum_{j=1}^{k} \nu_j(y,\eta)(\tilde{\tau}_j^2 + t_j^2).$$

Hence its eigenfunctions are microlocalized where

$$\hat{b}(y,\eta,s(y,\eta)) + \sum_{j=1}^{k} \nu_j(y,\eta)(\tilde{\tau}_j^2 + t_j^2) \le b_0 + ch^{2\delta},$$

which implies again that w lies in an arbitrarily small neighborhood of 0 and that

$$|t|^2 \le Ch^{2\delta}, \quad |\tilde{\tau}|^2 \le Ch^{2\delta}. \qquad \square$$

Using the same method as before, we deduce from Theorem 5.4 and Lemma 5.5 a comparison of the spectra of $\mathcal{N}_\hbar^{[1]}$ and $\mathcal{M}_\hbar$. With the notation

$$N_\hbar^{\max}(c,\delta) = \max\{n \in \mathbb{N} : \lambda_n(\mathcal{N}_\hbar^{[1]}) \le \hbar(b_0 + c\hbar^\delta)\},$$

the following lemma concludes the proof of Theorem 1.11.

Lemma 5.6. *Let $\delta \in \left(0, \frac{1}{2}\right)$ and $c > 0$. We have*

$$\lambda_n(\mathcal{N}_\hbar^{[1]}) = \hbar\lambda_n(\mathcal{M}_\hbar) + \mathcal{O}(\hbar^{1+\delta r_2/2}),$$

uniformly with respect to $n \in [1, N_\hbar^{\max}(c,\delta)]$.

Proof. We use the same method as before (see Lemma 4.5). The remainder $\mathcal{R}_{2,\hbar}$ is $\mathcal{O}((t,\tilde{\tau},\sqrt{\hbar})^{r_2})$ and the eigenfunctions are microlocalized where $|t| + |\tilde{\tau}| \le C\hbar^{\delta/2}$. Hence the $\hbar\mathcal{R}_{2,\hbar}$ term yields an error in $\hbar^{1+\delta r_2/2}$. $\square$

6. Proof of Corollary 1.14

In this section we prove that the spectrum of $\mathcal{L}_\hbar$ below $\hbar b_0 + \hbar^{3/2}(\nu(0) + 2c)$ is given by the spectrum of $\hbar\mathcal{M}_\hbar^{[1]}$, up to $\mathcal{O}(\hbar^{r/4-\varepsilon})$. We recall that $c \in (0, \min_j \nu_j(0))$ and $r = \min(2r_1, r_2 + 4)$.

We can apply Theorem 1.7 for $b_1 > b_0$ arbitrarily close to b_0. Thus the spectrum of $\mathcal{L}_\hbar$ in $(-\infty, b_1\hbar)$ is given by the spectrum of $\bigoplus_{n\in\mathbb{N}^s} \mathcal{N}_\hbar^{[n]}$ modulo $\mathcal{O}(\hbar^{r_1/2-\varepsilon}) = \mathcal{O}(\hbar^{r/4-\varepsilon})$. Moreover, the symbol of $\mathcal{N}_\hbar^{[n]}$ for $n \ne (1,\dots,1)$ satisfies

$$N_\hbar^{[n]}(y,\eta,t,\tau) \ge \hbar(b_0 + 2\min\beta_j - C\hbar),$$

and we deduce from the Gårding inequality that

$$\langle \mathcal{N}_\hbar^{[n]}\psi, \psi\rangle \ge \hbar b_1 \|\psi\|^2 \quad \text{for all } \psi \in \mathrm{L}^2(\mathbb{R}^{s+k}),$$

if b_1 is close enough to b_0. Hence the spectrum of $\mathcal{L}_\hbar$ below $b_1\hbar$ is given by the spectrum of $\mathcal{N}_\hbar^{[1]}$. Then, we apply Theorem 1.11 for δ close enough to $\frac{1}{2}$, and we see that the spectrum of $\mathcal{N}_\hbar^{[1]}$ below $(b_0 + \hbar^\delta)\hbar$

is given by the spectrum of $\bigoplus_{n\in\mathbb{N}^k} \hbar\mathcal{M}_\hbar^{[n]}$ modulo $\mathcal{O}(\hbar^{1+r_2/4-\varepsilon}) = \mathcal{O}(\hbar^{r/4-\varepsilon})$. The symbol of $\mathcal{M}_\hbar^{[n]}$ for $n \neq 1$ satisfies

$$\mathcal{M}_\hbar^{[n]}(y,\eta) \geq b_0 + \hbar^{1/2}\sum_{j=1}^{k} \nu_j(y,\eta)(2n_j - 1) - C\hbar,$$

and the eigenfunctions of $\mathcal{M}_\hbar^{[n]}$ are microlocalized in an arbitrarily small neighborhood of $(y,\eta) = 0$ (Lemma 5.5), and $M_\hbar^{[n]}$ satisfies in this neighborhood

$$\begin{aligned} M_\hbar^{[n]}(y,\eta) &\geq b_0 + \hbar^{1/2}\sum_{j=1}^{k} \nu_j(0)(2n_j - 1) - \hbar^{1/2}\varepsilon - C\hbar \\ &\geq b_0 + \hbar^{1/2}(\nu(0) + 2\min_j \nu_j(0) - \varepsilon) - C\hbar. \end{aligned}$$

Using the Gårding inequality, the spectrum of $\mathcal{M}_\hbar^{[n]}$ $(n \neq 1)$ is thus $\geq b_0 + \hbar^{1/2}(\nu(0) + 2c)$ for ε and $\hbar$ small enough. It follows that the spectrum of $\mathcal{N}_\hbar^{[1]}$ below $\hbar b_0 + \hbar^{3/2}(\nu(0) + 2c)$ is given by the spectrum of $\hbar\mathcal{M}_\hbar^{[1]}$.

7. Proof of Corollary 1.15

We explain here where the asymptotics for $\lambda_j(\mathcal{L}_\hbar)$ come from. First we use Corollary 1.14 so that the spectrum of $\mathcal{L}_\hbar$ below $\hbar b_0 + \hbar^{3/2}(\nu(0) + 2c)$ is given by $\mathcal{M}_\hbar^{[1]}$, modulo $\mathcal{O}(\hbar^{r/4-\varepsilon})$. The symbol of $\mathcal{M}_\hbar^{[1]}$ has the expansion

$$M_\hbar^{[1]}(w) = \hat{b}(w, s(w)) + \hbar^{1/2}\nu(0) + \hbar^{1/2}\nabla\nu(0)\cdot w + \hbar\tilde{c}_0 + \mathcal{O}(\hbar w + \hbar^{3/2} + \hbar^{1/2}w^2),$$

with $\nu(w) = \sum_{j=1}^k \nu_j(w)$. The principal part admits a unique minimum at 0, which is nondegenerate. The asymptotics of the first eigenvalues of such an operator are well known. First one can make a linear change of canonical coordinates diagonalizing the Hessian of $\hat{b}$ and get a symbol of the form

$$\widehat{M}_\hbar^{[1]}(w) = b_0 + \sum_{j=1}^{s} \mu_j(\eta_j^2 + y_j^2) + \hbar^{1/2}\nu(0) + \hbar^{1/2}\nabla\nu(0)\cdot w + \hbar\tilde{c}_0 + \mathcal{O}(w^3 + \hbar w + \hbar^{3/2} + \hbar^{1/2}w^2).$$

One can factor the $\nabla\nu(0)\cdot w$ term to get

$$\begin{aligned} \widehat{M}_\hbar^{[1]}(w) = b_0 + \sum_{j=1}^{s} \mu_j\left(\left(\eta_j + \frac{\partial_{\eta_j}\nu(0)}{2\mu_j}\hbar^{1/2}\right)^2 + \left(y_j + \frac{\partial_{y_j}\nu(0)}{2\mu_j}\hbar^{1/2}\right)^2\right) + \hbar^{1/2}\nu(0) + \hbar c_0 \\ + \mathcal{O}(w^3 + \hbar w + \hbar^{3/2} + \hbar^{1/2}w^2), \end{aligned}$$

with a new $c_0 \in \mathbb{R}$. Conjugating $\mathrm{Op}_\hbar^w\, \widehat{M}_\hbar^{[1]}$ by the unitary operator $U_\hbar$,

$$U_\hbar v(x) = \exp\left(\frac{i}{\sqrt{\hbar}}\sum_{j=1}^{s} \frac{\partial_{\eta_j}\nu(0)}{2\mu_j} y_j\right) v\left(x - \sum_{j=1}^{s} \frac{\partial_{y_j}\nu(0)}{2\mu_j}\hbar^{1/2}\right),$$

amounts to making a phase-space translation and changes the symbol into

$$\widetilde{M}_\hbar^{[1]}(w) = b_0 + \sum_{j=1}^{s} \mu_j(\eta_j^2 + y_j^2) + \hbar^{1/2}\nu(0) + \hbar c_0 + \mathcal{O}(w^3 + \hbar w + \hbar^{3/2} + \hbar^{1/2}w^2).$$

For an operator with such symbol (i.e., harmonic oscillator + remainders) one can apply the results of [Charles and Vũ Ngọc 2008, Theorem 4.7] or [Helffer and Sjöstrand 1984] and deduce that the j-th eigenvalue $\lambda_j(\mathcal{M}_\hbar^{[1]})$ admits an asymptotic expansion in powers of $\hbar^{1/2}$ such that

$$\lambda_j(\mathcal{M}_\hbar^{[1]}) = b_0 + \hbar^{1/2}\nu(0) + \hbar(c_0 + E_j) + \hbar^{3/2}\sum_{m=0}^{\infty}\alpha_{j,m}\hbar^{m/2},$$

where $\hbar E_j$ is the j-th repeated eigenvalue of the harmonic oscillator with symbol $\sum_{j=1}^{s}\mu_j(\eta_j^2 + y_j^2)$.

Appendix A: Local coordinates

If we choose local coordinates $q = (q_1, \dots, q_d)$ on M, we get the corresponding vector field basis $(\partial_{q_1}, \dots, \partial_{q_d})$ on $T_q M$, and the dual basis $(\mathrm{d}q_1, \dots, \mathrm{d}q_d)$ on $T_q M^*$. In these bases, g_q can be identified with a symmetric matrix $(g_{ij}(q))$ with determinant $|g|$, and g_q^* is associated with the inverse matrix $(g^{ij}(q))$. We can write the 1-form A and the 2-form B in the coordinates:

$$A \equiv A_1\mathrm{d}q_1 + \cdots + A_d\,\mathrm{d}q_d, \qquad B = \sum_{i<j} B_{ij}\mathrm{d}q_i \wedge \mathrm{d}q_j,$$

with $\boldsymbol{A} = (A_j)_{1\le j\le d} \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and

$$B_{ij} = \partial_i A_j - \partial_j A_i = ({}^t\mathrm{d}\boldsymbol{A} - \mathrm{d}\boldsymbol{A})_{ij}. \tag{A-1}$$

Let us denote by $(\boldsymbol{B}_{ij}(q))_{1\le i,j\le d}$ the matrix of the operator $\boldsymbol{B}(q) : T_q M \to T_q M$ in the basis $(\partial_{q_1}, \dots, \partial_{q_d})$. With this notation, (1-1) relating $\boldsymbol{B}$ to B can be rewritten,

$$\text{for all } Q, \widetilde{Q} \in \mathbb{R}^d, \quad \sum_{ijk} g_{kj}\boldsymbol{B}_{ki} Q_i \widetilde{Q}_j = \sum_{ij} B_{ij} Q_i \widetilde{Q}_j,$$

which means that,

$$\text{for all } i, j, \quad B_{ij} = \sum_k g_{kj}\boldsymbol{B}_{ki}. \tag{A-2}$$

Finally, in the coordinates, H is given by

$$H(q, p) = \sum_{i,j} g^{ij}(q)(p_i - A_i(q))(p_j - A_j(q)), \tag{A-3}$$

and $\mathcal{L}_\hbar$ acts as the differential operator:

$$\mathcal{L}_\hbar^{\text{coord}} = \sum_{k,l=1}^{d} |g|^{-1/2}(i\hbar\partial_k + A_k)g^{kl}|g|^{1/2}(i\hbar\partial_l + A_l). \tag{A-4}$$

Appendix B: Darboux-Weinstein lemmas

We used the following presymplectic Darboux lemma.

Theorem B.1. *Let M be a d-dimensional manifold endowed with a closed constant-rank-2 form ω. We denote by $2s$ the rank of ω and by k the dimension of its kernel. For every $q_0 \in M$, there exist a* *neighborhood V of q_0, a neighborhood U of $0 \in \mathbb{R}^{2s+k}_{(y,\eta,t)}$, and a diffeomorphism*

$$\varphi : U \to V$$

such that

$$\varphi^*\omega = \mathrm{d}\eta \wedge \mathrm{d}y.$$

We also used the following Weinstein result; see [Weinstein 1971]. We follow the proof given in [Raymond and Vũ Ngọc 2015].

Theorem B.2. *Let ω_0 and ω_1 be two 2-forms on $\mathbb{R}^d$ which are closed and nondegenerate. Let us split $\mathbb{R}^d$ into $\mathbb{R}^k_x \times \mathbb{R}^{d-k}_y$. We assume that $\omega_0 = \omega_1 + \mathcal{O}(|x|^\alpha)$ for some $\alpha \geq 1$. Then there exists a neighborhood of $0 \in \mathbb{R}^d$ and a change of coordinates ψ on this neighborhood such that*

$$\psi^*\omega_1 = \omega_0 \quad \textit{and} \quad \psi = \mathrm{Id} + \mathcal{O}(|x|^{\alpha+1}).$$

Proof. First we recall how to find a 1-form σ on a neighborhood of $x = 0$ such that

$$\tau := \omega_1 - \omega_0 = \mathrm{d}\sigma \quad \text{and} \quad \sigma = \mathcal{O}(|x|^{\alpha+1}).$$

We define the family $(\phi_t)_{0 \leq t \leq 1}$ by

$$\phi_t(x, y) = (tx, y).$$

We have

$$\phi_0^*\tau = 0 \quad \text{and} \quad \phi_1^*\tau = \tau. \tag{B-1}$$

Let us denote by X_t the vector field associated with ϕ_t,

$$X_t = \frac{\mathrm{d}\phi_t}{\mathrm{d}t} \circ \phi_t^{-1} = t^{-1}(x, 0).$$

The Lie derivative of τ along X_t is given by $\phi_t^*\mathcal{L}_{X_t}\tau = (\mathrm{d}/\mathrm{d}t)\phi_t^*\tau$. From the Cartan formula we have

$$\mathcal{L}_{X_t}\tau = \iota(X_t)\mathrm{d}\tau + \mathrm{d}(\iota(X_t)).$$

Since τ is closed, $\mathrm{d}\tau = 0$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^*\tau = \mathrm{d}(\phi_t^*\iota(X_t)\tau). \tag{B-2}$$

We choose the following 1-form (where (e_j) denotes the canonical basis of $\mathbb{R}^d$):

$$\sigma_t := \phi_t^*\iota(X_t)\tau = \sum_{j=1}^{k} x_j \tau_{\phi_t(x,y)}(e_j, \nabla\phi_t(.)) = \mathcal{O}(|x|^{\alpha+1}).$$

Equation (B-2) shows that $t \mapsto \phi_t^*\tau$ is smooth on $[0, 1]$. Thus, we can define $\sigma = \int_0^1 \sigma_t \, \mathrm{d}t$. From (B-2) and (B-1) we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^*\tau = \mathrm{d}\sigma_t \quad \text{and} \quad \tau = \mathrm{d}\sigma.$$

Then we use the Moser deformation argument. For $t \in [0, 1]$, we let $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$. The 2-form ω_t is closed and nondegenerate on a small neighborhood of $x = 0$. We look for ψ_t such that

$$\psi_t^*\omega_t = \omega_0.$$

For that purpose, let us determine the associated vector field Y_t,

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = Y_t(\psi_t).$$

The Cartan formula yields

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\psi_t^*\omega_t = \psi_t^*\Big(\frac{\mathrm{d}}{\mathrm{d}t}\omega_t + \iota(Y_t)\mathrm{d}\omega_t + \mathrm{d}(\iota(Y_t)\omega_t)\Big).$$

So

$$\omega_0 - \omega_1 = \mathrm{d}(\iota(Y_t)\omega_t),$$

and we are led to solve

$$\iota(Y_t)\omega_t = -\sigma.$$

By the nondegeneracy of ω_t, this determines Y_t. We know ψ_t exists until time $t = 1$ on a small enough neighborhood of $x = 0$, and $\psi_t^*\omega_t = \omega_0$. Thus $\psi = \psi_1$ is the desired diffeomorphism. Since $\sigma = \mathcal{O}(|x|^{\alpha+1})$, we get $\psi = \mathrm{Id} + \mathcal{O}(|x|^{\alpha+1})$. □

Appendix C: Pseudodifferential operators

We refer to [Zworski 2012; Martinez 2002] for the general theory of $\hbar$-pseudodifferential operators. If $m \in \mathbb{Z}$, we denote by

$$S^m(\mathbb{R}^{2d}) = \{a \in \mathcal{C}^\infty(\mathbb{R}^{2d}) : |\partial_x^\alpha \partial_\xi^\beta a| \le C_{\alpha\beta}\langle\xi\rangle^{m-|\beta|} \text{ for all } \alpha, \beta \in \mathbb{N}^d\}$$

the class of Kohn–Nirenberg symbols. If a depends on the semiclassical parameter $\hbar$, we require that the coefficients $C_{\alpha\beta}$ are uniform with respect to $\hbar \in (0, \hbar_0]$. For $a_\hbar \in S^m(\mathbb{R}^{2d})$, we define its associated Weyl quantization $\mathrm{Op}_\hbar^w(a_\hbar)$ by the oscillatory integral

$$\mathcal{A}_\hbar u(x) = \mathrm{Op}_\hbar^w(a_\hbar)u(x) = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^{2d}} e^{(i/\hbar)\langle x-y,\xi\rangle} a_\hbar\Big(\frac{x+y}{2}, \xi\Big)u(y)\,\mathrm{d}y\,\mathrm{d}\xi,$$

and we define

$$a_\hbar = \sigma_\hbar(\mathcal{A}_\hbar).$$

If M is a compact manifold, a pseudodifferential operator $\mathcal{A}_\hbar$ on $\mathrm{L}^2(M)$ is an operator acting as a pseudodifferential operator in coordinates. Then the principal symbol of $\mathcal{A}_\hbar$ (and its Kohn–Nirenberg class) does not depend on the coordinates, and we denote it by $\sigma_0(\mathcal{A}_\hbar)$. The subprincipal symbol $\sigma_1(\mathcal{A}_\hbar)$ is also well-defined, up to imposing that the charts be volume-preserving (in other words, if we see $\mathcal{A}_\hbar$ as acting on half-densities, its subprincipal symbol is well-defined). In the case where M is a compact manifold, $\mathcal{L}_\hbar$ is a pseudodifferential operator, and its principal and subprincipal symbols are

$$\sigma_0(\mathcal{L}_\hbar) = H, \quad \sigma_1(\mathcal{L}_\hbar) = 0.$$

If $M = \mathbb{R}^d$ and m is an order function on $\mathbb{R}^{2d}$, we denote by

$$S(m) = \{a \in \mathcal{C}^\infty(\mathbb{R}^{2d}) : |\partial_x^\alpha \partial_\xi^\beta a| \le C_{\alpha\beta} m(x, \xi) \text{ for all } \alpha, \beta \in \mathbb{N}^d\}$$

the class of standard symbols, and we similarly define the operator $\mathrm{Op}_\hbar^w(a)$ for such symbols. In this case, we assume that B belongs to some standard class. This is equivalent to assuming that H belongs to some (other) standard class. Then, $\mathcal{L}_\hbar$ is a pseudodifferential operator with total symbol H.

Appendix D: Egorov theorem

In this paper, we used several versions of the Egorov theorem. See for example [Robert 1987; Zworski 2012; Helffer et al. 2016].

Theorem D.1. *Let P and Q be $\hbar$-pseudodifferential operators on $\mathbb{R}^d$, with symbols $p \in S(m)$, $q \in S(m')$, where m and m' are order functions such that*

$$m' = \mathcal{O}(1), \quad mm' = \mathcal{O}'(1).$$

Then the operator $e^{(i/\hbar)Q} P e^{-(i/\hbar)Q}$ is a pseudodifferential operator whose symbol is in $S(m)$, and its symbol is

$$p \circ \kappa + \hbar S(1),$$

where the canonical transformation κ is the time-1 Hamiltonian flow associated with q.

We can use this result with the $\sqrt{\hbar}$-quantization to get an Egorov theorem for our mixed quantization $\mathrm{Op}_\sharp^w$.

Theorem D.2. *Let P be an h-pseudodifferential operator on $\mathbb{R}^d$, and $\mathrm{a} \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. Then*

$$e^{(i/\hbar)\,\mathrm{Op}_\sharp^w(\mathrm{a})} P e^{-(i/\hbar)\,\mathrm{Op}_\sharp^w(\mathrm{a})}$$

is an h-pseudodifferential operator on $\mathbb{R}^d$.

Proof. $\mathrm{Op}_\sharp^w(\mathrm{a})$ is an h-pseudodifferential operator. Thus, we can apply the Egorov theorem, and we deduce that $e^{(i/\hbar)\,\mathrm{Op}_\sharp^w(\mathrm{a})} P e^{-(i/\hbar)\,\mathrm{Op}_\sharp^w(\mathrm{a})}$ is an h-pseudodifferential operator on $\mathbb{R}^d$. □

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LÉO MORIN: `leo.morin@ens-rennes.fr`
Department of Mathematics, Aarhus University, Aarhus, Denmark

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