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A DETERMINATION OF THE BLOWUP SOLUTIONS TO THE FOCUSING, QUINTIC NLS WITH MASS EQUAL TO THE MASS OF THE SOLITON

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We prove the only blowup solutions to the focusing, quintic nonlinear Schrödinger equation with mass equal to the mass of the soliton are rescaled solitons or the pseudoconformal transformation of those solitons.

1. Introduction

The one dimensional, focusing, mass-critical nonlinear Schrödinger equation is given by

$$iu_t + u_{xx} + |u|^4 u = 0, \quad u(0,x) = u_0(x) \in L^2(\mathbb{R}). \tag{1-1}$$

This equation is a special case of the Hamiltonian equation

$$iu_t + u_{xx} + |u|^{p-1} u = 0, \quad u(0,x) = u_0(x), \quad p > 1. \tag{1-2}$$

If $u(t,x)$ is a solution to (1-2), then

$$v(t,x) = \lambda^{2/(p-1)} u(\lambda^2 t, \lambda x) \tag{1-3}$$

is a solution to (1-2) with appropriately rescaled initial data. Furthermore,

$$\|\lambda^{2/(p-1)} u(0,\lambda x)\|_{\dot{H}^s(\mathbb{R})} = \lambda^{2/(p-1)+s-1/2} \|u_0\|_{\dot{H}^s(\mathbb{R})},$$

so, for $s_p = \frac{1}{2} - 2/(p-1)$, the $\dot{H}^{s_p}(\mathbb{R})$ norm of the initial data is invariant under the scaling symmetry (1-3).

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The scaling symmetry in (1-3) controls the local well-posedness theory of (1-1). In that case, $p = 5$ and $s_p = 0$.

Theorem 1. *The initial value problem* (1-1) *is locally well-posed for any* $u_0 \in L^2$.

(1) *For any* $u_0 \in L^2$, *there exists* $T(u_0) > 0$ *such that* (1-1) *is locally well-posed on the interval* $(-T, T)$.

(2) *If* $\|u_0\|_{L^2}$ *is small then* (1-1) *is globally well-posed, and the solution scatters both forward and backward in time. That is, there exist* $u_-, u_+ \in L^2(\mathbb{R})$ *such that*

$$\lim_{t \nearrow +\infty} \|u(t) - e^{it\Delta} u_+\|_{L^2} = 0 \quad \text{and} \quad \lim_{t \searrow -\infty} \|u(t) - e^{it\Delta} u_-\|_{L^2} = 0.$$

(3) *If* I *is the maximal interval of existence for a solution to* (1-1) *with initial data* u_0, *we say* u *blows up forward in time if*

$$\lim_{T \nearrow \sup(I)} \|u\|_{L^6_{t,x}([0,T] \times \mathbb{R})} = +\infty.$$

If u *does not blow up forward in time, then* $\sup(I) = +\infty$ *and* u *scatters forward in time.*

(4) *If* $\sup(I) < \infty$ *then, for any* $s > 0$,

$$\lim_{t \nearrow \sup(I)} \|u(t)\|_{H^s} = +\infty.$$

(5) *Time reversal symmetry implies that the results corresponding to* (3) *and* (4) *also hold going backward in time.*

Remark. It is very important to emphasize that throughout this paper, blow up in positive time may be in finite time or infinite time, unless specified otherwise. The same is true for blow up in negative time.

Proof. Theorem 4 was proved in [Cazenave and Weissler 1990]. See also [Ginibre and Velo 1979a; 1979b; 1985; Kato 1987]. The proof uses the Strichartz estimates

$$\|u\|_{L^\infty_t L^2_x \cap L^4_t L^\infty_x (I \times \mathbb{R})} \lesssim \|u_0\|_{L^2(\mathbb{R})} + \|F\|_{L^1_t L^2_x + L^{4/3}_t L^1_x (I \times \mathbb{R})},$$

where u is the solution to

$$iu_t + u_{xx} = F, \quad u(0,x) = u_0,$$

on the interval I, where $0 \in I$. The Strichartz estimates were proved in [Ginibre and Velo 1992; Strichartz 1977; Yajima 1987]. Theorem 4 was proved using Picard iteration, so u is a strong solution to (1-1). For all $t \in I$, where I is the open interval on which local well-posedness of (1-1) holds,

$$u(t) = e^{it\partial_{xx}} u_0 + i \int_0^t e^{i(t-\tau)\partial_{xx}} (|u(\tau)|^4 u(\tau)) \, d\tau.$$

See [Tao 2006] for different notions of a solution. □

Furthermore, a solution to (1-1) has the conserved quantities mass,

$$M(u(t)) = \int |u(t,x)|^2 \, dx = M(u(0)),$$

and energy,

$$E(u(t)) = \frac{1}{2}\int |u_x(t,x)|^2\,dx - \frac{1}{6}\int |u(t,x)|^6\,dx = E(u(0)).$$

For the more general equation (1-2), the Hamiltonian is given by

$$E(u(t)) = \frac{1}{2}\int |u_x(t,x)|^2\,dx - \frac{1}{p+1}\int |u(t,x)|^{p+1}\,dx = E(u(0)). \tag{1-4}$$

For $p < 5$, [Ginibre and Velo 1979a] proved global well-posedness of (1-2) with initial data $u_0 \in H^1(\mathbb{R})$. Indeed, by a straightforward application of the fundamental theorem of calculus and Hölder's inequality, if $u(t,x) \in L^2 \cap \dot{H}^1$,

$$|u(t,x)|^2 \le \int_x^\infty \bigl|\partial_y |u(t,y)|^2\bigr|\,dy \le 2\int_x^\infty |\partial_y u(t,y)|\,|u(t,y)|\,dy \le 2\|u\|_{\dot{H}^1(\mathbb{R})}\|u\|_{L^2(\mathbb{R})}. \tag{1-5}$$

Therefore,

$$\|u(t)\|_{L^{p+1}(\mathbb{R})}^{p+1} \lesssim \|u(t)\|_{\dot{H}^1(\mathbb{R})}^{(p-1)/2}\|u(t)\|_{L^2(\mathbb{R})}^{(p+3)/2},$$

so (1-4) implies the existence of a uniform upper bound on $\|u(t)\|_{\dot{H}^1}$ when $p < 5$.

For $p > 5$, there exist singular solutions of (1-2), that is, solutions on the finite interval $[0, T)$, $T < \infty$, for which

$$\lim_{t\to T} \|u(t)\|_{H^1(\mathbb{R})} = \infty.$$

See [Glassey 1977; Weinstein 1986].

When $p = 5$, (1-5) implies

$$\int |u(t,x)|^6\,dx \lesssim \|u(t)\|_{\dot{H}^1(\mathbb{R})}^2\|u(t)\|_{L^2(\mathbb{R})}^4, \tag{1-6}$$

which implies the existence of a threshold mass M_0 for which, if $\|u_0\|_{L^2} < M_0$,

$$E(u(t)) \gtrsim_{M_0} \|u(t)\|_{\dot{H}^1(\mathbb{R})}^2,$$

with implicit constant $\searrow 0$ as $\|u_0\|_{L^2} \nearrow M_0$.

From [Weinstein 1982], the optimal constant in (1-6) is given by the Gagliardo–Nirenberg inequality,

$$\|u\|_{L^6(\mathbb{R})}^6 \le 3\left(\frac{\|u\|_{L^2}^2}{\|Q\|_{L^2}^2}\right)^2\|u_x\|_{L^2}^2, \tag{1-7}$$

where

$$Q(x) = \left(\frac{3}{\cosh(2x)^2}\right)^{1/4}. \tag{1-8}$$

Therefore, if $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then (1-7) implies

$$E(u(t)) \gtrsim_{\|u_0\|_{L^2}} \|u(t)\|_{\dot{H}^1}^2, \tag{1-9}$$

which implies global well-posedness of (1-1) with initial data $u_0 \in H^1$ and $\|u_0\|_{L^2} < \|Q\|_{L^2}$. Furthermore, the identities

$$\frac{d}{dt}\int x^2 |u(t,x)|^2\,dx = 4\,\mathrm{Im}\int x\overline{u(t,x)}u_x(t,x)\,dx$$

and

$$\frac{d^2}{dt^2}\int x^2 |u(t,x)|^2\,dx = 16E(u(t))$$

imply scattering for (1-1) with initial data

$$u_0 \in H^1(\mathbb{R}) \cap \Sigma = \left\{u : \int x^2 |u(x)|^2\,dx < \infty\right\}, \quad \|u_0\|_{L^2} < \|Q\|_{L^2}.$$

More recently, [Dodson 2015; 2016a] proved that (1-1) is globally well-posed and scattering for any initial data $u_0 \in L^2$, $\|u_0\|_{L^2} < \|Q\|_{L^2}$. The proof used the concentration compactness result of [Keraani 2006; Tao et al. 2008] which states that if $u(t)$ is a blowup solution to (1-1) of minimal mass, if t_n is a sequence of times approaching $\sup(I)$, and if u blows up forward in time on the maximal interval of existence I, then $u(t_n, x)$ has a subsequence that converges in L^2, up to the symmetries of (1-1). Using this fact, [Dodson 2015] proved that if u is a minimal mass blowup solution to (1-1), then there exists a sequence $t_n' \to \sup(I)$, for which $E(v_n) \searrow 0$, where v_n is a good approximation of $u(t_n', x)$, acted on by appropriate symmetries. Since (1-9) implies that the only u with mass less than $\|Q\|_{L^2}^2$ and zero energy is $u \equiv 0$, and the small data scattering result implies that the zero solution is stable under small perturbations, there cannot exist a minimal mass blowup solution to (1-1) with mass less than $\|Q\|_{L^2}^2$.

When $\|u\|_{L^2} = \|Q\|_{L^2}$, (1-7) only implies that $E(u) \geq 0$. The $Q(x)$ in (1-8) is the unique, positive solution to

$$Q_{xx} + Q^5 = Q. \tag{1-10}$$

See [Berestycki et al. 1981; Berestycki and Lions 1978; Kwong 1989; Strauss 1977] for existence and uniqueness of a ground state solution in general dimensions. Also observe that by the Pohozaev identity,

$$E(Q) = \frac{1}{2}\int (Q - Q_{xx} - Q^5)\left(\tfrac{1}{2}Q + xQ_x\right)dx = 0.$$

Up to the scaling (1-3), multiplication by a modulus one constant, and translation in space, Q is the unique minimizer of the energy functional with mass $\|Q\|_{L^2}$. See [Cazenave and Lions 1982; Weinstein 1986].

It is straightforward to verify that (1-8) solves (1-10), and that $e^{it}Q$ solves (1-1). Since $\|e^{it}Q\|_{L^6}$ is constant for all $t \in \mathbb{R}$, we have that $e^{it}Q$ blows up both forward and backward in time. Furthermore, the pseudoconformal transformation of $e^{it}Q(x)$,

$$u(t,x) = \frac{1}{t^{1/2}}\exp\left[\frac{-i}{t} + \frac{ix^2}{4t}\right]Q\left(\frac{x}{t}\right), \quad t > 0, \tag{1-11}$$

is a solution to (1-1) that blows up as $t \searrow 0$ and scatters as $t \to \infty$. Note that the mass is preserved under the pseudoconformal transformation of $e^{it}Q$.

It has long been conjecture that, up to symmetries of (1-1), the only nonscattering solutions to (1-1) are the soliton $e^{it}Q$ and the pseudoconformal transformation of the soliton, (1-11). Partial progress has been made in this direction.

Theorem 2. *If $u_0 \in H^1$, $\|u_0\|_{L^2} = \|Q\|_{L^2}$, and the solution $u(t)$ to* (1-1) *blows up in finite time $T > 0$, then $u(t, x)$ is equal to* (1-11), *up to symmetries of* (1-1).

Proof. This result was proved in [Merle 1992; 1993], and was proved for the focusing, mass-critical nonlinear Schrödinger equation in every dimension. □

For the mass-critical nonlinear Schrödinger equation in higher dimensions with radially symmetric initial data, [Killip et al. 2009] proved:

Theorem 3. *If $\|u_0\|_{L^2} = \|Q\|_{L^2}$ is radially symmetric, u is the solution to the focusing, mass-critical nonlinear Schrödinger equation with initial data u_0, and u blows up both forward and backward in time, then u is equal to the soliton, up to symmetries of the mass-critical nonlinear Schrödinger equation.*

In this paper we completely resolve this conjecture in one dimension, showing that the only blowup solutions to (1-1) with mass $\|u_0\|_{L^2}^2 = \|Q\|_{L^2}^2$ are the soliton and the pseudoconformal transformation of the soliton. This result should also hold in higher dimensions, which will be addressed in a forthcoming paper.

It is convenient to begin by considering solutions symmetric in x first.

Theorem 4. *The only symmetric solutions to* (1-1) *with mass $\|u_0\|_{L^2} = \|Q\|_{L^2}$ that blow up forward in time are the family of soliton solutions*

$$e^{-i\theta} e^{i\lambda^2 t} \lambda^{1/2} Q(\lambda x), \quad \lambda > 0, \ \theta \in \mathbb{R}, \tag{1-12}$$

and the pseudoconformal transformation of the soliton solution

$$\frac{1}{(T-t)^{1/2}} e^{i\theta} \exp\left[\frac{ix^2}{4(t-T)}\right] \exp\left[i\frac{\lambda^2}{t-T}\right] Q\left(\frac{\lambda x}{T-t}\right), \quad \lambda > 0, \ \theta \in \mathbb{R}, \ T \in \mathbb{R}, \ t < T. \tag{1-13}$$

The proof of Theorem 4 will occupy most of the paper. Once we have proved Theorem 4, we will remove the symmetry assumption on u_0, proving:

Theorem 5. *The only solutions to* (1-1) *with mass $\|u_0\|_{L^2} = \|Q\|_{L^2}$ that blow up forward in time are the family of soliton solutions*

$$e^{-i\theta - it\xi_0^2} e^{i\lambda^2 t} e^{ix\xi_0} \lambda^{1/2} Q(\lambda(x - 2t\xi_0) + x_0), \quad \lambda > 0, \ \theta \in \mathbb{R}, \ x_0 \in \mathbb{R}, \ \xi_0 \in \mathbb{R}, \tag{1-14}$$

and the pseudoconformal transformation of the family of solitons,

$$\frac{1}{(T-t)^{1/2}} e^{i\theta} \exp\left[\frac{i(x-\xi_0)^2}{4(t-T)}\right] \exp\left[i\frac{\lambda^2}{t-T}\right] Q\left(\frac{\lambda(x-\xi_0) - (T-t)x_0}{T-t}\right),$$
$$\text{where } \lambda > 0, \ \theta \in \mathbb{R}, \ x_0 \in \mathbb{R}, \ \xi_0 \in \mathbb{R}, \ T \in \mathbb{R}, \ t < T. \tag{1-15}$$

Applying time reversal symmetry to (1-1), *this theorem completely settles the question of qualitative behavior of solutions to* (1-1) *for initial data satisfying $\|u_0\|_{L^2} = \|Q\|_{L^2}$.*

The reader should see [Nakanishi and Schlag 2011] for this result for the Klein–Gordon equation.

Before proceeding to Section 2, the proof of Theorems 4 and 5 will be outlined. The first step (in Section 2) in the proof of Theorem 4 is to use the sequential convergence result of [Fan 2021] to show that Theorem 4 reduces to considering a symmetric solution to (1-1) that blows up forward in time with $\|u_0\|_{L^2} = \|Q\|_{L^2}$ and $\lambda(t)$ and $\gamma(t)$ continuous functions of time for which

$$\left\| \lambda(t)^{-1/2} e^{-i\gamma(t)} u\left(t, \frac{x}{\lambda(t)}\right) - Q(x) \right\|_{L^2} \leq \eta_*, \quad \eta_* \ll 1, \quad \text{for all } t > 0. \tag{1-16}$$

Then, in Section 3, the machinery in [Martel and Merle 2002] is used to choose $\lambda(t)$ and $\gamma(t)$ satisfying (1-16) for which

$$(\epsilon, Q_x) = (i\epsilon, Q_x) = (\epsilon, Q^3) = (i\epsilon, Q^3) = 0, \quad \text{where } \epsilon(t,x) = \lambda(t)^{-1/2} e^{-i\gamma(t)} u\left(t, \frac{x}{\lambda(t)}\right) - Q(x).$$

In Sections 4–6, the spectral theory of ϵ is combined with the long-time Strichartz estimates in [Dodson 2016a], proving

$$\int_a^b \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}\, dt \leq 3\big(\epsilon_2(a), \big(\tfrac{1}{2}Q + xQ_x\big)\big)_{L^2} - 3\big(\epsilon_2(b), \tfrac{1}{2}Q + xQ_x\big)_{L^2} + O\left(\frac{1}{T^9}\right),$$
$$\text{when } \int_a^b \lambda(t)^{-2}\, dt = T \text{ and } a > 0. \tag{1-17}$$

In Section 7, we use (1-17) to show that if $[0, T)$ is the maximal interval of existence of (1-1) in the forward time direction,

$$\int_0^T \|\epsilon(t)\|_{L^2}^p \lambda(t)^{-2} < \infty \quad \text{for any } p > 1. \tag{1-18}$$

Note that for the pseudoconformal solution (1-11), (1-18) holds, but fails when $p = 1$. Then in Section 8, we use the virial identity in [Merle and Raphael 2005] to show that $\lambda(t)$ is approximately monotone decreasing. In Section 9, the monotonicity of $\lambda(t)$ combined with long-time Strichartz estimates and conservation of energy implies that u is a soliton solution when $T = \infty$. When $T < \infty$, a pseudoconformal transformation of the solution must satisfy $T = \infty$, so therefore u must be a pseudoconformal transformation of a soliton. Finally, in Section 10, the above argument is generalized to the nonsymmetric case. We conclude with an Appendix describing U^p and V^p spaces, an important tool used in long-time Strichartz estimates.

2. Reductions of a symmetric blowup solution

Let u be a symmetric blowup solution to (1-1) with mass $\|u_0\|_{L^2} = \|Q\|_{L^2}$. Defining the distance to the two dimensional manifold of symmetries acting on the soliton (1-8) by

$$\inf_{\lambda > 0,\, \gamma \in \mathbb{R}} \|u_0(x) - e^{i\gamma} \lambda^{1/2} Q(\lambda x)\|_{L^2}, \tag{2-1}$$

there exist $\lambda_0 > 0$ and $\gamma_0 \in \mathbb{R}$ where this infimum is attained. Indeed:

Lemma 6. *There exist $\lambda_0 > 0$ and $\gamma_0 \in \mathbb{R}$ such that*

$$\|u_0(x) - e^{-i\gamma_0} \lambda_0^{-1/2} Q(\lambda_0^{-1} x)\|_{L^2(\mathbb{R})} = \inf_{\gamma, \lambda} \|u_0(x) - e^{-i\gamma} \lambda^{-1/2} Q(\lambda^{-1} x)\|_{L^2}.$$

Proof. Since Q, along with all its derivatives, is rapidly decreasing,

$$\|u_0(x) - e^{-i\gamma}\lambda^{-1/2} Q(\lambda^{-1}x)\|_{L^2}^2 \tag{2-2}$$

is differentiable and hence continuous as a function of λ and γ.

Next, by the dominated convergence theorem,

$$\begin{aligned}\lim_{\lambda \nearrow \infty} \inf_{\gamma \in [0,2\pi]} &\|u_0(x) - e^{-i\gamma}\lambda^{-1/2} Q(\lambda^{-1}x)\|_{L^2}^2 \\ &= \|u_0\|_{L^2}^2 + \|Q\|_{L^2}^2 - 2 \lim_{\lambda \nearrow \infty} \sup_{\gamma \in [0,2\pi]} |(e^{-i\gamma}\lambda^{-1/2} Q(\lambda^{-1}x), u_0(x))_{L^2}| = 2\|Q\|_{L^2}^2. \end{aligned} \tag{2-3}$$

Here, $(f, g)_{L^2}$ denotes the L^2-inner product

$$(f, g)_{L^2} = \mathrm{Re} \int f(x)\overline{g(x)}\, dx.$$

Meanwhile, rescaling (2-3),

$$(e^{-i\gamma}\lambda^{-1/2} Q(\lambda^{-1}x), u_0(x))_{L^2} = (Q(x), e^{i\gamma}\lambda^{1/2} u_0(\lambda x))_{L^2},$$

and therefore,

$$\lim_{\lambda \searrow 0} \inf_{\gamma \in [0,2\pi]} \|u_0(x) - e^{-i\gamma}\lambda^{-1/2} Q(\lambda^{-1}x)\|_{L^2}^2 = 2\|Q\|_{L^2}^2. \tag{2-4}$$

Finally, the polarization identity

$$\|u_0(x) - \lambda^{-1/2} Q(\lambda^{-1}x)\|_{L^2}^2 + \|u_0(x) + \lambda^{-1/2} Q(\lambda^{-1}x)\|_{L^2}^2 = 4\|Q\|_{L^2}^2$$

implies that

$$\frac{1}{2\pi} \int_0^{2\pi} \|u_0(x) - e^{-i\gamma}\lambda^{-1/2} Q(\lambda^{-1}x)\|_{L^2}^2\, d\gamma = 2\|Q\|_{L^2}^2. \tag{2-5}$$

If, for all $\lambda > 0$,

$$\inf_{\gamma \in [0,2\pi]} \|u_0(x) - e^{-i\gamma}\lambda^{-1/2} Q(\lambda^{-1}x)\|_{L^2}^2 = 2\|Q\|_{L^2}^2,$$

then (2-5) implies

$$\|u_0(x) - e^{-i\gamma}\lambda^{-1/2} Q(\lambda^{-1}x)\|_{L^2}^2 = 2\|Q\|_{L^2}^2 \quad \text{for all } \lambda > 0, \ \ \gamma \in [0, 2\pi]. \tag{2-6}$$

In this case simply take $\lambda_0 = 1$ and $\gamma_0 = 0$.

Remark. Equation (2-6) is not possible, since (2-6) is equivalent to the statement that there exists $\|u_0\|_{L^2} = \|Q\|_{L^2}$ satisfying

$$(u_0(x), e^{-i\gamma}\lambda^{-1/2} Q(\lambda^{-1}x))_{L^2} = 0 \quad \text{for all } \gamma \in [0, 2\pi] \text{ and for all } \lambda > 0. \tag{2-7}$$

Since Q and u_0 are symmetric, let $R(y) = e^{y/2} Q(y)$ and $v(y) = e^{y/2} u_0(e^y)$. Then (2-7) implies that $v(y)$ is orthogonal to all translations of $R(y)$. Since $R(y)$ is exponentially decreasing, the Fourier transform of $R(y)$ is analytic in the strip, and therefore must have isolated zeros. Thus, its zeros are a set of measure zero, so the translates of R are dense in L^2. The author is grateful to an anonymous referee for pointing this fact out.

On the other hand, if

$$\inf_{\lambda>0}\inf_{\gamma\in\mathbb{R}}\|u_0(x)-e^{-i\gamma}\lambda^{-1/2}Q(\lambda^{-1}x)\|_{L^2}^2<2\|Q\|_{L^2}^2,$$

then (2-3) and (2-4) imply that there exist $0<\lambda_1<\lambda_2<\infty$ such that

$$\inf_{\lambda>0}\inf_{\gamma\in[0,2\pi]}\|u_0(x)-e^{-i\gamma}\lambda^{-1/2}Q(\lambda^{-1}x)\|_{L^2}^2=\inf_{\lambda\in[\lambda_1,\lambda_2]}\inf_{\gamma\in[0,2\pi]}\|u_0(x)-e^{-i\gamma}\lambda^{-1/2}Q(\lambda^{-1}x)\|_{L^2}^2.$$

Since (2-2) is continuous as a function of $\lambda>0$, $\gamma\in[0,2\pi]$, and $[\lambda_1,\lambda_2]\times[0,2\pi]$ is a compact set, there exist $\lambda_0>0$ and $\gamma_0\in[0,2\pi]$ such that

$$\|u_0(x)-e^{-i\gamma_0}\lambda_0^{-1/2}Q(\lambda_0^{-1}x)\|_{L^2(\mathbb{R})}=\inf_{\gamma\in[0,2\pi],\,\lambda>0}\|u_0(x)-e^{-i\gamma}\lambda^{-1/2}Q(\lambda^{-1}x)\|_{L^2}.\qquad\square$$

Using the weak sequential convergence result of [Fan 2021], Theorem 4 may be reduced to considering solutions that blow up in positive time for which (2-1) is small for all $t>0$.

Theorem 7. *Let $0<\eta_*\ll 1$ be a small, fixed constant to be defined later. If u is a symmetric solution to* (1-1) *on the maximal interval of existence $I\subset\mathbb{R}$, $\|u_0\|_{L^2}=\|Q\|_{L^2}$, u blows up forward in time, and*

$$\sup_{t\in[0,\sup(I))}\inf_{\lambda,\gamma}\|e^{i\gamma}\lambda^{1/2}u(t,\lambda x)-Q(x)\|_{L^2}\le\eta_*,\tag{2-8}$$

then u is a soliton solution of the form (1-12) *or a pseudoconformal transformation of a soliton of the form* (1-13).

Remark. Scaling symmetries imply that (2-1) and the left-hand side of (2-8) at a fixed time are equal.

Proof that Theorem 7 implies Theorem 4. Let $u(t)$ be the solution to (1-1) with symmetric initial data u_0 that satisfies $\|u_0\|_{L^2}=\|Q\|_{L^2}$. If

$$\lim_{t\nearrow\sup(I)}\inf_{\lambda>0,\,\gamma\in\mathbb{R}}\|\lambda^{1/2}e^{i\gamma}u(t,\lambda x)-Q\|_{L^2}=0,\tag{2-9}$$

then (2-8) holds for all $t>t_0$, for some $t_0\in I$. After translating in time so that $t_0=0$, Theorem 7 easily implies Theorem 4 in this case.

However, the convergence theorem of [Fan 2021] only implies $u(t)$ must converge to Q along a subsequence after rescaling and multiplying by a complex number of modulus one.

Theorem 8. *Let u be a symmetric solution to* (1-1) *that satisfies $\|u_0\|_{L^2}=\|Q\|_{L^2}$ and blows up forward in time. Let $(T_-(u),T_+(u))$ be the maximal lifespan of the solution u. Then there exists a sequence $t_n\to T_+(u)$ and a family of parameters $\lambda_n>0$, $\gamma_n\in\mathbb{R}$ such that*

$$e^{i\gamma_n}\lambda_n^{1/2}u(t_n,\lambda_n x)\to Q\quad\text{in } L^2.\tag{2-10}$$

If (2-9) does not hold but there exists some $t_0>0$ such that

$$\sup_{t\in[t_0,\sup(I))}\inf_{\lambda,\gamma}\|e^{i\gamma}\lambda^{1/2}u(t,\lambda x)-Q(x)\|_{L^2}\le\eta_*,$$

then after translating in time so that $t_0=0$, (2-8) holds.

Now suppose (2-9) does not hold and furthermore that there exists a sequence $t_n^- \nearrow \sup(I)$ such that

$$\inf_{\gamma\in\mathbb{R},\,\lambda>0} \|e^{i\gamma}\lambda^{1/2}u(t_n^-,\lambda x)-Q\|_{L^2} > \eta_* \tag{2-11}$$

for every n. After passing to a subsequence, suppose that, for every n, we have $t_n^- < t_n < t_{n+1}^-$, where t_n is the sequence in (2-10) and t_n^- is the sequence in (2-11). The fact that

$$\inf_{\gamma\in\mathbb{R},\,\lambda>0} \|e^{i\gamma}\lambda^{1/2}u(t,\lambda x)-Q\|_{L^2} \tag{2-12}$$

is upper semicontinuous as a function of t and is continuous for every t such that (2-12) is small guarantees that there exists a small, fixed $0<\eta_*\ll 1$ such that the sequence t_n^+ defined by

$$t_n^+ = \inf\Bigl\{t\in I : \sup_{\tau\in[t,t_n]}\inf_{\lambda,\gamma}\|\lambda^{1/2}e^{i\gamma}u(\tau,\lambda x)-Q\|_{L^2}<\eta_*\Bigr\}$$

satisfies $t_n^+ \nearrow \sup(I)$ and

$$\inf_{\lambda>0,\,\gamma\in\mathbb{R}} \|e^{i\gamma}\lambda^{1/2}u(t_n^+,\lambda x)-Q(x)\|_{L^2} = \eta_*. \tag{2-13}$$

Indeed, the fact that (2-12) is upper semicontinuous as a function of t implies that

$$\Bigl\{0\le t<t_n : \inf_{\lambda>0,\,\gamma\in\mathbb{R}}\|e^{i\gamma}\lambda^{1/2}u(t,\lambda x)-Q(x)\|_{L^2}\ge\eta_*\Bigr\}$$

is a closed set. Since this set is also contained in a bounded set, it has a maximal element t_n^+, and $t_n^+\ge t_n^-$. The fact that (2-12) is upper semicontinuous in time also implies

$$\inf_{\lambda>0,\,\gamma\in\mathbb{R}} \|e^{i\gamma}\lambda^{1/2}u(t_n^+,\lambda x)-Q\|_{L^2} \ge \eta_*.$$

On the other hand, since

$$\inf_{\lambda>0,\,\gamma\in\mathbb{R}} \|e^{i\gamma}\lambda^{1/2}u(t,\lambda x)-Q\|_{L^2} < \eta_* \quad \text{for all } t_n^+<t<t_n$$

and (2-12) is continuous at times $t\in I$ where (2-12) is small,

$$\inf_{\lambda>0,\,\gamma\in\mathbb{R}} \|e^{i\gamma}\lambda^{1/2}u(t_n^+,\lambda x)-Q\|_{L^2} = \eta_*. \tag{2-14}$$

Remark. The constant $0<\eta_*\ll 1$ will be chosen to be a small fixed quantity that is sufficiently small to satisfy the hypotheses of Theorem 10, sufficiently small such that (2-12) is continuous in time when (2-12) is bounded by η_*, sufficiently small such that $\eta_*\le\eta_0$, where η_0 is the constant in the induction on frequency arguments in Theorem 13, and such that $T_* = 1/\eta_*$ is sufficiently large to satisfy the hypotheses of Theorem 18.

Theorem 9 (upper semicontinuity of the distance to a soliton). *The quantity*

$$\inf_{\lambda,\gamma} \|e^{i\gamma}\lambda^{1/2}u(t,\lambda x)-Q(x)\|_{L^2(\mathbb{R})}, \tag{2-15}$$

is upper semicontinuous as a function of time for any $t\in I$, where I is the maximal interval of existence for u. The quantity (2-15) *is also continuous in time when* (2-15) *is small.*

Proof. Choose some $t_0 \in I$ and suppose without loss of generality that

$$\|u(t_0,x)-Q(x)\|_{L^2} = \inf_{\lambda,\gamma} \|e^{i\gamma}\lambda^{1/2}u(t_0,\lambda x)-Q(x)\|_{L^2}.$$

For t close to t_0, let

$$\epsilon(t,x) = u(t,x) - e^{i(t-t_0)}Q(x). \tag{2-16}$$

Since $e^{i(t-t_0)}Q$ solves (1-1),

$$\begin{aligned} i\epsilon_t + \epsilon_{xx} + |u|^4u - e^{it}|Q|^4Q \\ = i\epsilon_t + \epsilon_{xx} + 3|Q|^4\epsilon + 2e^{2i(t-t_0)}|Q|^2Q^2\bar{\epsilon} + O\biggl(\sum_{j=2}^{5}\epsilon^{2+j}Q^{5-j}\biggr) = 0. \end{aligned} \tag{2-17}$$

Equations (2-16), (2-17), and Strichartz estimates imply that, for $J \subset \mathbb{R},\ t_0 \in J$,

$$\|\epsilon\|_{L_t^\infty L_x^2 \cap L_t^4 L_x^\infty(J\times\mathbb{R})} \lesssim \|\epsilon(t_0)\|_{L^2} + \|\epsilon\|_{L_t^\infty L_x^2(J\times\mathbb{R})}\|u\|^4_{L_t^4L_x^\infty(J\times\mathbb{R})} + \|\epsilon\|^5_{L^6_{t,x}(J\times\mathbb{R})}.$$

Local well-posedness of (1-1) combined with Strichartz estimates implies that $\|u\|_{L_t^4L_x^\infty(J\times\mathbb{R})} = 1$ on some open neighborhood J of t_0. Therefore, for $\|\epsilon(t_0)\|_{L^2}$ small, partitioning J into finitely many pieces,

$$\sup_{t\in J}\|\epsilon(t)\|_{L^2} \lesssim \|\epsilon(t_0)\|_{L^2} \tag{2-18}$$

and

$$\lim_{t\to t_0}\|\epsilon(t)\|_{L^2} = \|\epsilon(t_0)\|_{L^2}. \tag{2-19}$$

Therefore,

$$\lim_{t\to t_0}\inf_{\lambda,\gamma}\|\lambda^{1/2}e^{i\gamma}u(t,\lambda x)-Q\|_{L^2} \le \|u(t_0,x)-Q\|_{L^2} = \inf_{\lambda>0,\,\gamma\in\mathbb{R}}\|\lambda^{1/2}e^{i\gamma}u(t_0,\lambda x)-Q\|_{L^2}.$$

Furthermore, if

$$\lim_{t\to t_0}\inf_{\lambda,\gamma}\|\lambda^{1/2}e^{i\gamma}u(t,\lambda x)-Q\|_{L^2} < \|u(t_0,x)-Q\|_{L^2},$$

then there exists a sequence $t_n' \to t_0,\ \lambda_n' > 0,\ \gamma_n' \in \mathbb{R}$ such that

$$\lim_{n\to\infty}\|\lambda_n'^{1/2}e^{i\gamma_n'}u(t_n',\lambda_n'x)-Q\|_{L^2} < \inf_{\lambda,\gamma}\|\lambda^{1/2}e^{i\gamma}u(t_0,\lambda x)\|_{L^2}.$$

For t_n' sufficiently close to t_0, repeating the arguments giving (2-18) and (2-19) with t_n' as the initial data gives a contradiction.

When $\|\epsilon(t_0)\|_{L^2}$ is large, (2-17) implies

$$\frac{d}{dt}\|\epsilon(t)\|^2_{L^2} \lesssim \|Q\|^4_{L^\infty}\|\epsilon\|^2_{L^2} + \|u\|^4_{L^\infty}\|\epsilon\|^2_{L^2}.$$

Therefore, Gronwall's inequality and the fact that $u \in L^4_{t,\mathrm{loc}}L_x^\infty$ imply

$$\lim_{t\to t_0}\inf_{\lambda>0,\,\gamma\in\mathbb{R}}\|e^{i\gamma}\lambda^{1/2}u(t,\lambda x)-Q\|_{L^2} \le \inf_{\lambda>0,\,\gamma\in\mathbb{R}}\|e^{i\gamma}\lambda^{1/2}u(t_0,\lambda x)-Q\|_{L^2},$$

which implies upper semicontinuity. □

Making a profile decomposition of $u(t_n^+, x)$, the fact that u is a minimal mass blowup solution that blows up forward in time and $t_n^+ \nearrow \sup(I)$ implies that there exist $\lambda(t_n^+) > 0$ and $\gamma(t_n^+) \in \mathbb{R}$ such that

$$\lambda(t_n^+)^{1/2} e^{i\gamma(t_n^+)} u(t_n^+, \lambda(t_n^+)x) \to \tilde{u}_0$$

in L^2. Also, $t_n^+ \nearrow \sup(I)$ implies $\|\tilde{u}_0\|_{L^2} = \|Q\|_{L^2}$ is the initial data for a solution to (1-1) that blows up forward and backward in time, and by (2-14),

$$\inf_{\lambda>0,\, \gamma\in\mathbb{R}} \|\lambda^{1/2} e^{i\gamma} \tilde{u}_0(\lambda x) - Q\|_{L^2} = \eta_*. \tag{2-20}$$

Moreover, observe that (2-10), (2-13), and (2-18) directly imply that

$$\lim_{n\to\infty} \|u\|_{L^6_{t,x}([t_n^+, t_n]\times\mathbb{R})} = \infty \quad \text{and} \quad \lim_{n\to\infty} \|u\|_{L^6_{t,x}([0, t_n^+]\times\mathbb{R})} = \infty,$$

so if $\tilde{u}$ is the solution to (1-1) with initial data $\tilde{u}_0$,

$$\inf_{\lambda>0,\, \gamma\in\mathbb{R}} \|\lambda^{1/2} e^{i\gamma} \tilde{u}(t, \lambda x) - Q\|_{L^2} \le \eta_*$$

for all $t \in [0, \sup(\tilde{I}))$, where $\tilde{I}$ is the interval of existence of the solution $\tilde{u}$ to (1-1) with initial data $\tilde{u}_0$, and $\tilde{u}$ blows up both forward and backward in time. However, Theorem 7 and (2-20) imply that $\tilde{u}$ must be of the form (1-13). Such a solution scatters backward in time and is well approximated by a linear solution

$$e^{it\Delta} f = (4\pi t)^{-1/2} e^{-i\pi/4} \int \exp\left[i \frac{(x-y)^2}{4t}\right] f(y)\, dy,$$

which contradicts the fact that $\tilde{u}$ blows up both forward and backward in time.

Therefore, Theorem 7 implies that (2-11) cannot hold for any symmetric solution to (1-1) with mass $\|u_0\|_{L^2} = \|Q\|_{L^2}$, so by Theorem 7, any symmetric solution to (1-1) that blows up forward in time must be of the form (1-12) or (1-13). □

3. Decomposition of the solution near Q

Turning now to the proof of Theorem 7, make a decomposition of a symmetric solution close to Q, up to rescaling and multiplication by a modulus one constant. This result is classical; see, e.g., [Martel and Merle 2002], although here there is an additional technical complication due to the fact that u need not lie in H^1.

Theorem 10. *Take $u \in L^2$. There exists $\alpha > 0$ sufficiently small such that if there exist $\lambda_0 > 0$, $\gamma_0 \in \mathbb{R}$ that satisfy*

$$\|e^{i\gamma_0} \lambda_0^{1/2} u(\lambda_0 x) - Q\|_{L^2} \le \alpha,$$

then there exist unique $\lambda > 0$, $\gamma \in \mathbb{R}$ which satisfy

$$(\epsilon, Q^3)_{L^2} = (\epsilon, iQ^3)_{L^2} = 0, \tag{3-1}$$

where

$$\epsilon(x) = e^{i\gamma} \lambda^{1/2} u(\lambda x) - Q. \tag{3-2}$$

Furthermore,

$$\|\epsilon\|_{L^2} + \left|\frac{\lambda}{\lambda_0} - 1\right| + |\gamma - \gamma_0| \lesssim \|e^{i\gamma_0} \lambda_0^{1/2} u(\lambda_0 x) - Q\|_{L^2}. \tag{3-3}$$

Remark. Since $e^{i\gamma}$ is 2π-periodic, the γ in (3-2) is unique up to translations by $2\pi k$ for some integer k.

Proof. By Hölder's inequality,

$$|(e^{i\gamma_0}\lambda_0^{1/2}u(\lambda_0 x)-Q(x),Q^3)_{L^2}|\lesssim\|e^{i\gamma_0}\lambda_0^{1/2}u(\lambda_0 x)-Q\|_{L^2},$$
$$|(e^{i\gamma_0}\lambda_0^{1/2}u(\lambda_0 x)-Q(x),iQ^3)_{L^2}|\lesssim\|e^{i\gamma_0}\lambda_0^{1/2}u(\lambda_0 x)-Q\|_{L^2}.$$

First suppose that $\lambda_0=1$ and $\gamma_0=0$. The inner products

$$(e^{i\gamma}\lambda^{1/2}u(\lambda x)-Q(x),Q^3)_{L^2}\quad\text{and}\quad(e^{i\gamma}\lambda^{1/2}u(\lambda x)-Q(x),iQ^3)_{L^2}\tag{3-4}$$

are C^1 as functions of λ and γ. Indeed,

$$\frac{\partial}{\partial\gamma}(e^{i\gamma}\lambda^{1/2}u(\lambda x)-Q(x),Q^3)_{L^2}=(ie^{i\gamma}\lambda^{1/2}u(\lambda x),Q^3)_{L^2}\lesssim\|u\|_{L^2}\|Q\|_{L^6}^3,$$
$$\frac{\partial}{\partial\gamma}(e^{i\gamma}\lambda^{1/2}u(\lambda x)-Q(x),iQ^3)_{L^2}=(ie^{i\gamma}\lambda^{1/2}u(\lambda x),iQ^3)_{L^2}\lesssim\|u\|_{L^2}\|Q\|_{L^6}^3.$$

Next, integrating by parts,

$$\begin{aligned}\frac{\partial}{\partial\lambda}(e^{i\gamma}\lambda^{1/2}u(\lambda x)-Q(x),Q^3)_{L^2}&=\left(\frac{e^{i\gamma}}{2\lambda^{1/2}}u(\lambda x)+xe^{i\gamma}\lambda^{1/2}u_x(\lambda x),Q^3\right)_{L^2}\\&=\left(\frac{e^{i\gamma}}{2\lambda^{1/2}}u(\lambda x)-\frac{1}{\lambda^{1/2}}e^{i\gamma}u(\lambda x),Q^3\right)_{L^2}-\frac{3}{\lambda^{1/2}}(e^{i\gamma}u(\lambda x),Q^2Q_x)_{L^2}\\&\lesssim\frac{1}{\lambda}\|u\|_{L^2}\|Q\|_{L^6}^3+\frac{1}{\lambda}\|u\|_{L^2}\|xQ_x\|_{L^2}\|Q\|_{L^\infty}^2\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial\lambda}(e^{i\gamma}\lambda^{1/2}u(\lambda x)-Q(x),iQ^3)_{L^2}&=\left(\frac{e^{i\gamma}}{2\lambda^{1/2}}u(\lambda x)+xe^{i\gamma}\lambda^{1/2}u_x(\lambda x),iQ^3\right)_{L^2}\\&=\left(\frac{e^{i\gamma}}{2\lambda^{1/2}}u(\lambda x)-\frac{1}{\lambda^{1/2}}e^{i\gamma}u(\lambda x),iQ^3\right)_{L^2}-\frac{3}{\lambda^{1/2}}(e^{i\gamma}u(\lambda x),iQ^2Q_x)_{L^2}\\&\lesssim\frac{1}{\lambda}\|u\|_{L^2}\|Q\|_{L^6}^3+\frac{1}{\lambda}\|u\|_{L^2}\|xQ_x\|_{L^2}\|Q\|_{L^\infty}^2.\end{aligned}$$

Similar calculations prove uniform bounds on the Hessians of the inner products given in (3-4).

Next, compute

$$\frac{\partial}{\partial\gamma}(e^{i\gamma}\lambda^{1/2}u(\lambda x)-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,u=Q}=(iQ,Q^3)_{L^2}=0,$$
$$\frac{\partial}{\partial\gamma}(e^{i\gamma}\lambda^{1/2}u(\lambda x)-Q(x),iQ^3)_{L^2}\Big|_{\lambda=1,\gamma=0,u=Q}=(iQ,iQ^3)_{L^2}=\|Q\|_{L^4}^4,$$
$$\frac{\partial}{\partial\lambda}(e^{i\gamma}\lambda^{1/2}u(\lambda x)-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,u=Q}=\left(\tfrac{1}{2}Q+xQ_x,Q^3\right)_{L^2}=\tfrac{1}{4}\|Q\|_{L^4}^4,$$
$$\frac{\partial}{\partial\lambda}(e^{i\gamma}\lambda^{1/2}u(\lambda x),iQ)_{L^2}\Big|_{\lambda=1,\gamma=0,u=Q}=\left(\tfrac{1}{2}Q+xQ_x,iQ\right)_{L^2}=0.$$

Therefore, by the inverse function theorem, if $\lambda_0 = 1$ and $\gamma_0 = 0$, there exist λ and γ satisfying

$$|\lambda - 1| + |\gamma| \lesssim \|e^{i\gamma_0} u(x) - Q(x)\|_{L^2}$$

such that

$$(e^{i\gamma}\lambda^{1/2} u(t, \lambda x) - Q(x), Q^3)_{L^2} = (e^{i\gamma}\lambda^{1/2} u(t, \lambda x) - Q(x), iQ^3)_{L^2} = 0.$$

The inverse function theorem also guarantees that λ and γ are unique for all $\lambda, \gamma \in [1-\delta, 1+\delta] \times [-\delta, \delta]$ for some $\delta > 0$, up to 2π-periodicity.

For λ outside $[1-\delta, 1+\delta]$, observe that

$$\begin{aligned}\|e^{i\gamma}\lambda^{1/2} u(\lambda x) - Q\|_{L^2}^2 \\ = \|u\|_{L^2}^2 + \|Q\|_{L^2}^2 - 2(e^{i\gamma}\lambda^{1/2} Q(\lambda x), Q)_{L^2} - 2(e^{i\gamma}\lambda^{1/2}[u - Q](\lambda x), Q)_{L^2} \gtrsim \delta^2 - O(\alpha).\end{aligned} \tag{3-5}$$

Similarly, for γ outside $[-\delta, \delta]$, up to 2π-multiplicity,

$$\begin{aligned}\|e^{i\gamma}\lambda^{1/2} u(\lambda x) - Q\|_{L^2}^2 \\ = \|u\|_{L^2}^2 + \|Q\|_{L^2}^2 - 2(e^{i\gamma}\lambda^{1/2} Q(\lambda x), Q)_{L^2} - 2(e^{i\gamma}\lambda^{1/2}[u - Q](\lambda x), Q)_{L^2} \gtrsim \delta^2 - O(\alpha),\end{aligned} \tag{3-6}$$

which implies uniqueness for $\alpha > 0$ sufficiently small.

For general λ_0 and γ_0, after rescaling,

$$\left|\frac{\lambda}{\lambda_0} - 1\right| + |\gamma - \gamma_0| \lesssim \|e^{i\gamma_0}\lambda_0^{1/2} u(t, \lambda_0 x) - Q(x)\|_{L^2}. \tag{3-7}$$

Finally, using scaling symmetries, the triangle inequality, and (3-7),

$$\begin{aligned}&\|e^{i\gamma}\lambda^{1/2} u(t, \lambda x) - Q(x)\|_{L^2} \\ &= \left\|u(x) - e^{-i\gamma}\lambda^{-1/2} Q\left(\frac{x}{\lambda}\right)\right\|_{L^2} \\ &\le \left\|u(x) - e^{-i\gamma_0}\lambda_0^{-1/2} Q\left(\frac{x}{\lambda_0}\right)\right\|_{L^2} + \left\|e^{-i\gamma_0}\lambda_0^{-1/2} Q\left(\frac{x}{\lambda_0}\right) - e^{-i\gamma}\lambda_0^{-1/2} Q\left(\frac{x}{\lambda_0}\right)\right\|_{L^2} \\ &\qquad + \left\|e^{-i\gamma}\lambda_0^{-1/2} Q\left(\frac{x}{\lambda_0}\right) - e^{-i\gamma}\lambda^{-1/2} Q\left(\frac{x}{\lambda}\right)\right\|_{L^2} \\ &\lesssim \|e^{i\gamma_0} u(x) - Q(x)\|_{L^2}.\end{aligned}$$

This proves (3-3). □

Therefore, in Theorem 7, there exist functions

$$\lambda : I \to (0, \infty) \quad \text{and} \quad \gamma : I \to \mathbb{R}$$

such that (3-1) holds for all $t \in [0, \sup(I))$.

Theorem 11. *Under the hypotheses of Theorem 7, the functions* $\lambda(t)$ *and* $\gamma(t)$ *are continuous as functions of time on* $[0, \sup(I))$. *Additionally,* $\lambda(t)$ *and* $\gamma(t)$ *are differentiable in time almost everywhere on* $[0, \sup(I))$.

Proof. Suppose $J = [a, b]$ is an interval that satisfies

$$\|u\|_{L_t^4 L_x^\infty(J\times\mathbb{R})} \le 1$$

and $J \subset [0, \sup(I))$. Suppose without loss of generality that $\lambda(a) = 1$ and $\gamma(a) = 0$. Also, suppose for now that $\|u(a)\|_{\dot{H}^1} < \infty$. Strichartz estimates and local well-posedness theory imply that

$$\|u\|_{L_t^\infty \dot{H}^1(J\times\mathbb{R})} \lesssim \|u(a)\|_{\dot{H}^1}. \tag{3-8}$$

Since $\lambda(a) = 1$ and $\gamma(a) = 0$,

$$(u(a, x) - Q(x), Q^3)_{L^2} = (u(a, x) - Q(x), iQ^3)_{L^2} = 0.$$

Then, by direct calculation and the fact that Q is smooth and rapidly decreasing,

$$\begin{aligned}\frac{d}{dt}(u(t, x) - Q, Q^3)_{L^2} &= (iu_{xx}, Q^3)_{L^2} + (i|u|^4 u, Q^3)_{L^2}\\ &= (iu, \partial_{xx}(Q^3))_{L^2} + i(|u|^4 u, Q^3)_{L^2} \lesssim \|u\|_{L^2} + \|u\|_{L^\infty}^3 \|u\|_{L^2}^2.\end{aligned}$$

Therefore, (3-8) implies that $(u(t, x)-Q(x), Q^3)_{L^2}$ is Lipschitz in time on J as is $(u(t, x)-Q(x), iQ^3)_{L^2}$ by an identical calculation. Then by the proof of Theorem 10, $\lambda(t)$ and $\gamma(t)$ are Lipschitz as a function of time for t close to a, and by the Lebesgue differentiation theorem, λ and γ are differentiable almost everywhere for t near a.

Recall from (3-1) that

$$\epsilon(t, x) = e^{i\gamma(t)} \lambda(t)^{1/2} u(t, \lambda(t)x) - Q(x).$$

By direct computation, for almost every t near a,

$$\begin{aligned}\epsilon_t &= i\dot{\gamma}(t)(Q+\epsilon) + \frac{\dot{\lambda}(t)}{\lambda(t)}\big(\tfrac{1}{2}Q + xQ_x + \tfrac{1}{2}\epsilon + x\epsilon_x\big) + i\lambda(t)^{-2}(Q_{xx} + \epsilon_{xx})\\ &\qquad\qquad + i\lambda(t)^{1/2} e^{i\gamma(t)} |u(t, \lambda(t)x)|^4 u(t, \lambda(x))\\ &= i(\dot{\gamma}(t) + \lambda(t)^{-2})Q + \frac{\dot{\lambda}(t)}{\lambda(t)}\big(\tfrac{1}{2}Q + xQ_x\big) + i\lambda(t)^{-2}(\epsilon_{xx} + 5Q^4 \operatorname{Re}(\epsilon) + iQ^4 \operatorname{Im}(\epsilon) - \epsilon)\\ &\qquad\qquad + i(\dot{\gamma}(t) + \lambda(t)^{-2})\epsilon + \frac{\dot{\lambda}(t)}{\lambda(t)}\big(\tfrac{1}{2}\epsilon + x\epsilon_x\big) + \lambda(t)^{-2} O(|Q|^3|\epsilon|^2 + |\epsilon|^5).\end{aligned} \tag{3-9}$$

Since a is arbitrary, λ and γ are differentiable at almost every $t \in [0, \sup(I))$.

Next, define the monotone function $s : [0, \sup(I)) \to \mathbb{R}$,

$$s(t) = \int_0^t \lambda(\tau)^{-2}\, d\tau. \tag{3-10}$$

Making a change of variables $\epsilon_s = \lambda^2 \epsilon_t$, by (3-9),

$$\begin{aligned}\epsilon_s &= i(\gamma_s + 1)Q + \frac{\lambda_s}{\lambda}\big(\tfrac{1}{2}Q + xQ_x\big) + i(\epsilon_{xx} + 5Q^4 \operatorname{Re}(\epsilon) + iQ^4 \operatorname{Im}(\epsilon) - \epsilon)\\ &\qquad\qquad + i(\gamma_s + 1)\epsilon + \frac{\lambda_s}{\lambda}\big(\tfrac{1}{2}\epsilon + x\epsilon_x\big) + O(|Q|^3|\epsilon|^2 + |\epsilon|^2|u|^3).\end{aligned} \tag{3-11}$$

Plugging (3-11) into (3-2) and integrating by parts,

$$\frac{d}{ds}(\epsilon, Q^3) = (\epsilon_s, Q^3) = 0 = \frac{\lambda_s}{4\lambda}\|Q\|_{L^4}^4 - (\operatorname{Im}(\epsilon), \mathcal{L}_- Q^3)_{L^2} + O(|\gamma_s+1|\|\epsilon\|_{L^2}) + O\biggl(\frac{\lambda_s}{\lambda}\|\epsilon\|_{L^2}\biggr) + O(\|\epsilon\|_{L^2}^2) + O(\|\epsilon\|_{L^2}^2\|u\|_{L^\infty}^3)$$

and

$$\frac{d}{ds}(\epsilon, iQ^3) = (\epsilon_s, iQ^3) = 0 = (\gamma_s+1)\|Q\|_{L^4}^4 + (\epsilon, \mathcal{L}Q^3)_{L^2} + O(|\gamma_s+1|\|\epsilon\|_{L^2}) + O\biggl(\frac{\lambda_s}{\lambda}\|\epsilon\|_{L^2}\biggr) + O(\|\epsilon\|_{L^2}^2) + O(\|\epsilon\|_{L^2}^2\|u\|_{L^\infty}^3),$$

where $\mathcal{L}_-$ and $\mathcal{L}$ are the linear operators

$$\mathcal{L}_- f = -f_{xx} - Q^4 f + f \quad \text{and} \quad \mathcal{L} f = -f_{xx} - 5Q^4 f + f. \tag{3-12}$$

Since $\mathcal{L}Q^3 = -8Q^3$ and $(\epsilon, Q^3)_{L^2} = 0$,

$$\frac{\|Q\|_{L^4}^4}{4}\frac{\lambda_s}{\lambda} = (\operatorname{Im}(\epsilon), \mathcal{L}_- Q^3)_{L^2} + O(|\gamma_s+1|\|\epsilon\|_{L^2}) + O\biggl(\frac{\lambda_s}{\lambda}\|\epsilon\|_{L^2}\biggr) + O(\|\epsilon\|_{L^2}^2) + O(\|\epsilon\|_{L^2}^2\|u\|_{L^\infty}^3) \tag{3-13}$$

and

$$\|Q\|_{L^4}^4(\gamma_s+1) = O(|\gamma_s+1|\|\epsilon\|_{L^2}) + O\biggl(\frac{\lambda_s}{\lambda}\|\epsilon\|_{L^2}\biggr) + O(\|\epsilon\|_{L^2}^2) + O(\|\epsilon\|_{L^2}^2\|u\|_{L^\infty}^3). \tag{3-14}$$

Doing some algebra, (3-13), (3-14), and the computations proving (2-18) imply that, for any $a \in \mathbb{Z}_{\geq 0}$,

$$\int_a^{a+1}\biggl|\frac{\lambda_s}{\lambda}\biggr|\,ds \lesssim \int_a^{a+1}\|\epsilon\|_{L^2}\,ds \tag{3-15}$$

and

$$\int_a^{a+1}|\gamma_s+1|\,ds \lesssim \int_a^{a+1}\|\epsilon\|_{L^2}\,ds. \tag{3-16}$$

Indeed, the computations proving (2-18) imply that

$$\sup_{s\in[a,a+1]}\|\epsilon(s)\|_{L^2} \lesssim \int_a^{a+1}\|\epsilon(s)\|_{L^2}\,ds, \tag{3-17}$$

so

$$\int_a^{a+1}\|\epsilon\|_{L^2}^2\|u\|_{L^\infty}^3\,ds \lesssim \int_a^{a+1}\|\epsilon(s)\|_{L^2}^2\,ds\cdot\int_a^{a+1}\|u\|_{L^\infty}^3\,ds. \tag{3-18}$$

Furthermore, Strichartz estimates and the computations proving (2-18) imply that $\int_a^{a+1}\|u\|_{L^\infty}^4\,ds \lesssim 1$, and crucially, the bound is independent of $\|u(a)\|_{\dot{H}^1}$.

For a general $u(a) \in L^2$, let $u^N(a) = P_{\leq N}u(a)$. Taking N large enough that

$$\|e^{i\gamma(a)}(\lambda(a))^{1/2}u^N(a,\lambda(a)x) - Q\|_{L^2} \leq 2\eta_*,$$

Theorem 10 implies that there exist $\gamma^N(s)$ and $\lambda^N(s)$ for any $s \in [a, a+1]$ such that (3-1) holds. Furthermore, $\lambda^N(s)$ and $\gamma^N(s)$ satisfy (3-13) and (3-14), and $\gamma^N(s)$ and $\lambda^N(s)$ converge to $\gamma(s)$

and $\lambda(s)$ uniformly on $[a, a+1]$, so $\gamma(s)$ and $\lambda(s)$ are continuous as functions of s. Furthermore, $\epsilon^N \to \epsilon$ in L^2 uniformly in s and $u^N \to u$ in $L_s^4 L_x^\infty$.

Therefore, plugging $\lambda^N(s)$, $\gamma^N(s)$, ϵ^N, and u^N into (3-13) and (3-14) and doing some algebra implies, by the dominated convergence theorem,

$$\tfrac{1}{4}\|Q\|_{L^4}^4[\ln\lambda(s) - \ln\lambda(a)] = \int_a^s O((\mathrm{Im}(\epsilon), \mathcal{L}_- Q^3)_{L^2}) + O(\|\epsilon\|_{L^2}^2) + O(\|\epsilon\|_{L^2}^2 \|u\|_{L^\infty}^3)\, ds$$

and

$$\|Q\|_{L^4}^4[\gamma(s) - \gamma(a) + (s-a)] = \int_a^s O(\|\epsilon\|_{L^2}^2) + O(\|\epsilon\|_{L^2}^2 \|u\|_{L^\infty}^3)\, ds.$$

Therefore, by the Lebesgue differentiation theorem, λ_s/λ and γ_s exist for almost every $s \in [a, a+1]$ and satisfy (3-13) and (3-14). □

Following [Merle 2001], the decomposition in Theorem 10 gives a positivity result.

Theorem 12. *If $\epsilon(t,x)$ is a symmetric function, $\epsilon \perp Q^3$, $\epsilon \perp iQ^3$, $\|\epsilon(t,x)\|_{L^2} \ll 1$, and $\|Q+\epsilon\|_{L^2} = \|Q\|_{L^2}$, then*

$$E(Q+\epsilon) \gtrsim \|\epsilon(t)\|_{H^1(\mathbb{R})}^2 = \int |\epsilon_x(t,x)|^2\, dx + \int |\epsilon(t,x)|^2\, dx.$$

Proof. Decomposing the energy and integrating by parts, since Q is a real-valued function,

$$\begin{aligned} E(Q+\epsilon) = \frac{1}{2}\int Q_x^2\, dx + \mathrm{Re}\int Q_x(x)\epsilon_x(t,x)\, dx + \frac{1}{2}\|\epsilon_x\|_{L^2}^2 - \frac{1}{6}\int Q(x)^6\, dx \\ - \mathrm{Re}\int Q(x)^5 \epsilon(t,x)\, dx - \frac{3}{2}\int Q(x)^4 |\epsilon(t,x)|^2\, dx \\ - \mathrm{Re}\int Q(x)^4 \epsilon(t,x)^2\, dx - \int O(|\epsilon(t,x)|^3 Q^3 + |\epsilon(t,x)|^6)\, dx. \end{aligned}$$

First observe that, since $E(Q) = 0$,

$$\frac{1}{2}\int Q_x^2\, dx - \frac{1}{6}\int Q^6\, dx = 0.$$

Next, by (1-10) and integrating by parts,

$$\mathrm{Re}\int Q_x(x)\epsilon_x(t,x) - \mathrm{Re}\int Q(x)^5\epsilon(t,x) = -\mathrm{Re}\int (Q_{xx}(x) + Q(x)^5)\epsilon(t,x)\, dx = -\mathrm{Re}\int Q(x)\epsilon(t,x)\, dx.$$

Using the fact that $\|Q+\epsilon\|_{L^2} = \|Q\|_{L^2}$,

$$\tfrac{1}{2}\|Q\|_{L^2}^2 - \tfrac{1}{2}\|Q+\epsilon\|_{L^2}^2 + \tfrac{1}{2}\|\epsilon\|_{L^2}^2 = -(Q,\epsilon)_{L^2} = -\mathrm{Re}\int Q(x)\epsilon(t,x)\, dx = \tfrac{1}{2}\|\epsilon\|_{L^2}^2. \tag{3-19}$$

Therefore,

$$\begin{aligned} E(Q+\epsilon) = \frac{1}{2}\|\epsilon\|_{L^2}^2 + \frac{1}{2}\|\epsilon_x\|_{L^2}^2 - \frac{3}{2}\int Q(x)^4|\epsilon(t,x)|^2\, dx \\ - \mathrm{Re}\int Q(x)^4\epsilon(t,x)^2\, dx - \int O(|\epsilon(t,x)|^3 Q^3 + |\epsilon(t,x)|^6)\, dx. \end{aligned}$$

Decomposing the terms of order ϵ^2 into real and imaginary parts,

$$\begin{aligned}\frac{1}{2}\|\epsilon_x\|_{L^2}^2 + \frac{1}{2}\|\epsilon\|_{L^2}^2 - \frac{3}{2}\int Q(x)^4|\epsilon(t,x)|^2\,dx - \operatorname{Re}\int Q(x)^4\epsilon(t,x)^2\,dx \\ = \frac{1}{2}\int \operatorname{Re}(\epsilon)_x^2\,dx + \frac{1}{2}\int \operatorname{Re}(\epsilon)^2\,dx - \frac{5}{2}\int Q(x)^4\operatorname{Re}(\epsilon)^2\,dx + \frac{1}{2}\int \operatorname{Im}(\epsilon)_x^2\,dx + \frac{1}{2}\int \operatorname{Im}(\epsilon)^2\,dx \\ - \frac{1}{2}\int Q(x)^4\operatorname{Im}(\epsilon)^2\,dx.\end{aligned}$$

Recalling (3-12),

$$\frac{1}{2}\int \operatorname{Re}(\epsilon)_x^2\,dx + \frac{1}{2}\int \operatorname{Re}(\epsilon)^2\,dx - \frac{5}{2}\int Q(x)^4\operatorname{Re}(\epsilon)^2\,dx = \frac{1}{2}(\mathcal{L}\operatorname{Re}(\epsilon), \operatorname{Re}(\epsilon))_{L^2}.$$

It is well known, see, e.g., [Merle 2001], that $\mathcal{L}$ has one negative eigenvector, $\mathcal{L}(Q^3) = -8Q^3$, and one zero eigenvector, $\mathcal{L}(Q_x) = 0$. Since $\operatorname{Re}(\epsilon) \perp Q^3$ and $\operatorname{Re}(\epsilon)$ symmetric guarantees that $\operatorname{Re}(\epsilon) \perp Q_x$,

$$\frac{1}{2}\int \operatorname{Re}(\epsilon)_x^2\,dx + \frac{1}{2}\int \operatorname{Re}(\epsilon)^2\,dx - \frac{5}{2}\int Q(x)^4\operatorname{Re}(\epsilon)^2\,dx \geq \frac{1}{2}\int \operatorname{Re}(\epsilon)^2\,dx.$$

Next, doing some algebra,

$$\frac{1}{2}\int \operatorname{Re}(\epsilon)_x^2\,dx = \frac{1}{2}(\mathcal{L}\operatorname{Re}(\epsilon), \operatorname{Re}(\epsilon)) - \frac{1}{2}\int \operatorname{Re}(\epsilon)^2\,dx + \frac{5}{2}\int Q(x)^4\operatorname{Re}(\epsilon)^2\,dx \leq C(\mathcal{L}\operatorname{Re}(\epsilon), \operatorname{Re}(\epsilon)).$$

By similar calculations, since $\operatorname{Im}(\epsilon) \perp Q^3$ and $\operatorname{Im}(\epsilon) \perp Q_x$,

$$\begin{aligned}\frac{1}{2}\int \operatorname{Im}(\epsilon)_x^2\,dx + \frac{1}{2}\int \operatorname{Im}(\epsilon)^2\,dx - \frac{1}{2}\int Q(x)^4\operatorname{Im}(\epsilon)^2\,dx \\ = \frac{1}{2}(\mathcal{L}\operatorname{Im}(\epsilon), \operatorname{Im}(\epsilon)) + 2\int Q(x)^4\operatorname{Im}(\epsilon)^2 \\ \geq \frac{1}{2}(\mathcal{L}\operatorname{Im}(\epsilon), \operatorname{Im}(\epsilon)) \geq \frac{1}{2}\int \operatorname{Im}(\epsilon)^2\,dx + \frac{1}{2C}\int \operatorname{Im}(\epsilon)_x^2\,dx.\end{aligned}$$

Finally, by the Sobolev embedding theorem and $\|\epsilon\|_{L^2} \ll 1$,

$$\int |\epsilon|^6\,dx \lesssim \|\epsilon\|_{\dot{H}^1}^2\|\epsilon\|_{L^2}^4 \ll \|\epsilon\|_{\dot{H}^1}^2$$

and

$$\int Q(x)^3|\epsilon(t,x)|^3\,dx \lesssim \|\epsilon\|_{L^2}^{3/2}\|\epsilon\|_{L^6}^{3/2} \lesssim \|\epsilon\|_{L^2}^{5/2}\|\epsilon\|_{\dot{H}^1}^{1/2} \lesssim \|\epsilon\|_{L^2}^{5/2} + \|\epsilon\|_{L^2}^{5/2}\|\epsilon\|_{\dot{H}^1}^2 \ll \|\epsilon\|_{L^2}^2 + \|\epsilon\|_{\dot{H}^1}^2,$$

which completes the proof of Theorem 12. □

4. A long-time Strichartz estimate

Having shown that it is enough to consider solutions to (1-1) that are close to the family of solitons and that there is a good decomposition of solutions that are close to the family of solitons, the next task is to obtain a good frequency-localized Morawetz estimate. The proof of the frequency-localized Morawetz estimate will occupy Sections 4–6.

The proof of scattering in [Dodson 2015] for (1-1) when $\|u_0\|_{L^2} < \|Q\|_{L^2}$ utilized a frequency-localized Morawetz estimate. There, the Morawetz estimate was used to show that $E(P_n u(t_n)) \to 0$

along a subsequence, where P_n is a Fourier truncation operator that converges to the identity in the strong L^2-operator topology. Then the Gagliardo–Nirenberg inequality, (1-7), and the stability of the zero solution to (1-1) implies that $u \equiv 0$. In the case that $\|u_0\|_{L^2} = \|Q\|_{L^2}$, [Dodson 2021; Fan 2021] proved that $E(P_n u(t_n)) \to 0$ along a subsequence, so the almost periodicity of u implies that $u(t_n)$ converges to a rescaled version of Q.

In fact, [Dodson 2021; Fan 2021] proved more, that $E(Pu(t)) \to 0$ in an averaged sense on an interval $[0, T] \subset I$. The operator P is fixed on a fixed time interval, but P converges to the identity in the strong L^2-operator topology as $T \to \sup(I)$. The proof of Theorem 7 will argue that if $E(Pu(t))$ goes to zero in a time-averaged sense, then u must be equal to the soliton if the solution is global. If the solution blows up in finite time, then u must equal a pseudoconformal transformation of the soliton.

An essential ingredient in this proof is an improved version of the long-time Strichartz estimates in [Dodson 2016a]. The proof will make use of the bilinear estimates of [Planchon and Vega 2009], which were also used in the two dimensional problem [Dodson 2016b].

Eventually, the proof of Theorem 7 will make use of long-time Strichartz estimates on an interval $J = [a, b]$ for

$$1 \le \lambda(t) \le T^{1/100}, \tag{4-1}$$

where $T = s(b) - s(a)$ and $s(t) : [0, \sup(I)) \to [0, \infty)$ is the function given by (3-10). However, to avoid obscuring the main idea, it will be convenient to consider the case when $\lambda(t) = 1$ first, since the generalization to the case (4-1) is fairly straightforward.

Suppose without loss of generality that $a = 0$ and $b = T$. Choose

$$0 < \eta_1 \ll \eta_0 \ll 1$$

to be small constants, suppose

$$\|\epsilon(t, x)\|_{L^2} \le \eta_0 \tag{4-2}$$

for all $t \in J$, and choose $\eta_1 \ll \eta_0$ small enough that

$$\int_{|\xi| \ge \eta_1^{-1/2}} |\hat{Q}(\xi)|^2 \, d\xi \le \eta_0^2, \tag{4-3}$$

and therefore

$$\sup_{t \in J} \int_{|\xi| \ge \eta_1^{-1/2}} |\hat{u}(t, \xi)|^2 \, d\xi \le 4\eta_0^2.$$

Then rescale from $\lambda(t) = 1$ to $\lambda(t) = 1/\eta_1$ and $[0, T] \mapsto [0, \eta_1^{-2} T]$.

When $i \in \mathbb{Z}$, $i > 0$, let P_i denote the standard Littlewood–Paley projection operator. When $i = 0$, let P_i denote the projection operator $P_{\le 0}$, and when $i < 0$, let P_i denote the zero operator.

Definition. Suppose $\eta_1^{-2} T = 2^{3k}$ for some $k \in \mathbb{Z}_{\ge 0}$. Then define the norm

$$\|u\|^2_{X([0, \eta_1^{-2} T] \times \mathbb{R})} = \sup_{0 \le i \le k} \sup_{1 \le a < 2^{3k-3i}} \|P_i u\|^2_{U^2_\Delta([(a-1)2^{3i}, a2^{3i}] \times \mathbb{R})} + 2^i \|(P_{\ge i} u)(P_{\le i-3} u)\|^2_{L^2_{t,x}([(a-1)2^{3i}, a2^{3i}] \times \mathbb{R})}. \tag{4-4}$$

Also, for any $0 \leq j \leq k$, let

$$\|u\|^2_{X_j([0,\eta_1^{-2}T]\times\mathbb{R})} = \sup_{0\leq i\leq j}\ \sup_{1\leq a<2^{3k-3i}} \|P_i u\|^2_{U^2_\Delta([(a-1)2^{3i},\,a2^{3i}]\times\mathbb{R})} + 2^i \|(P_{\geq i}u)(P_{\leq i-3}u)\|^2_{L^2_{t,x}([(a-1)2^{3i},\,a2^{3i}]\times\mathbb{R})}.$$

See [Koch and Tataru 2007] for a definition of the U^p_Δ and V^p_Δ norms. See also [Dodson 2016a; 2016b; 2019].

Theorem 13. *The long-time Strichartz estimate*

$$\|u\|_{X([0,\eta_1^{-2}T]\times\mathbb{R})} \lesssim 1$$

holds with implicit constant independent of T.

Proof. This estimate is proved by induction on j. Local well-posedness arguments combined with the fact that $\lambda(t) = \eta_1^{-1}$ for any $t \in [0, \eta_1^{-2}T]$ imply that

$$\|u\|_{U^2_\Delta([a,a+1]\times\mathbb{R})} \lesssim 1,$$

and when $i = 0$,

$$(P_{\geq i}u)(P_{\leq i-3}u) = 0.$$

Therefore,

$$\|u\|_{X_0([0,\eta_1^{-2}T]\times\mathbb{R})} \lesssim 1. \tag{4-5}$$

This is the base case.

Remark. The implicit constant in (4-5) does not depend on T or η_1.

To prove the inductive step, recall, by Duhamel's principle, that if $J = [(a-1)2^{3k-3i}, a2^{3k-3i}]$, then, for any $t_0 \in J$,

$$u(t) = e^{i(t-t_0)\Delta}u(t_0) + i\int_{t_0}^t e^{i(t-\tau)\Delta}(|u|^4u)\,d\tau$$

and

$$\|P_{\geq i}u\|_{U^2_\Delta(J\times\mathbb{R})} \lesssim \|P_{\geq i}(u(t_0))\|_{L^2} + \left\|\int_{t_0}^t e^{i(t-\tau)\Delta}P_{\geq i}(|u|^4u)\,d\tau\right\|_{U^2_\Delta(J\times\mathbb{R})}.$$

By (4-3) and the fact that $\lambda(t) = 1/\eta_1$ for all $t \in [0, \eta_1^{-2}T]$, if $i > 0$,

$$\sup_{t_0\in[0,\eta_1^{-2}T]} \|P_{\geq i}u(t_0)\|_{L^2} \lesssim \eta_0. \tag{4-6}$$

Next, choose $v \in V^2_\Delta(J\times\mathbb{R})$ such that $\|v\|_{V^2_\Delta(J\times\mathbb{R})} = 1$ and $\hat{v}(t,\xi)$ is supported on the Fourier support of P_i. It is a well-known fact that

$$\left\|\int_{t_0}^t e^{i(t-\tau)\Delta}P_{\geq i}(|u|^4u)\,d\tau\right\|_{U^2_\Delta(J\times\mathbb{R})} \lesssim \sup_v \|vP_{\geq i}(|u|^4u)\|_{L^1_{t,x}},$$

where $\sup_v$ is the supremum over all such v supported on P_i satisfying $\|v\|_{V^2_\Delta(J\times\mathbb{R})} = 1$. See [Hadac et al. 2009] for a proof.

By Hölder's inequality,

$$\begin{aligned}\|v(u_{\geq i-3})^2(u_{\leq i-3})^3\|_{L^1_{t,x}} &+ \|v(u_{\geq i-3})^3(u_{\leq i-3})^2\|_{L^1_{t,x}} + \|v(u_{\geq i-3})^4(u_{\leq i-3})\|_{L^1_{t,x}} + \|v(u_{\geq i-3})^5\|_{L^1_{t,x}} \\ \lesssim \|v\|_{L^\infty_t L^2_x}\|u_{\geq i-3}\|^5_{L^5_t L^{10}_x} &+ \|v\|_{L^4_t L^\infty_x}\|u_{\geq i-3}\|^4_{L^{16/3}_t L^8_x}\|u_{\leq i-3}\|_{L^\infty_t L^2_x} \\ &+ \|v\|_{L^6_{t,x}}\|(u_{\geq i-3})(u_{\leq i-6})\|_{L^2_{t,x}}\|u_{\leq i-6}\|_{L^\infty_{t,x}}\|u_{\geq i-3}\|^2_{L^6_{t,x}} \\ &+ \|v\|_{L^6_{t,x}}\|u_{\geq i-3}\|^3_{L^6_{t,x}}\|u_{\geq i-6}\|^2_{L^6_{t,x}} + \|v\|_{L^6_{t,x}}\|(u_{\geq i-3})(u_{\leq i-6})\|^{3/2}_{L^2_{t,x}}\|u_{\leq i-6}\|^{3/2}_{L^\infty_{t,x}}\|u_{\geq i-3}\|^{1/2}_{L^6_{t,x}} \\ &+ \|v\|_{L^6_{t,x}}\|u_{\geq i-3}\|^2_{L^6_{t,x}}\|u_{\geq i-6}\|^3_{L^6_{t,x}}. \end{aligned} \tag{4-7}$$

Since $V^2_\Delta \subset U^p_\Delta$ for any $p > 2$, again see [Hadac et al. 2009],

$$\|v\|_{L^\infty_t L^2_x} + \|v\|_{L^6_{t,x}} + \|v\|_{L^4_t L^\infty_x} \lesssim \|v\|_{V^2_\Delta} \lesssim 1. \tag{4-8}$$

Next, when $i > 4$, since $U^2_\Delta \subset U^4_\Delta$, (4-3) and (4-6) imply

$$\|u_{\geq i-3}\|^5_{L^5_t L^{10}_x} \lesssim \|u_{\geq i-3}\|^4_{L^4_t L^\infty_x}\|u_{\geq i-3}\|_{L^\infty_t L^2_x} \lesssim \eta_0\|u_{\geq i-3}\|^4_{L^4_t L^\infty_x} \lesssim \eta_0\|u\|^4_{X_{i-3}([0,T]\times\mathbb{R})}. \tag{4-9}$$

When $i \leq 4$, the fact that for any $a \in \mathbb{Z}$,

$$\|u\|_{L^4_t L^\infty_x([a,a+1]\times\mathbb{R})} \lesssim 1, \tag{4-10}$$

the fact that the Fourier inversion formula and Hölder's inequality imply

$$\left\| \int_{|\xi|\leq \eta_1^{1/2}} e^{ix\cdot\xi}\hat{u}(t,\xi)\,d\xi \right\|_{L^\infty} \lesssim \eta_1^{1/4}\|u(t)\|_{L^2}, \tag{4-11}$$

and the fact that (4-3) implies, after rescaling $\lambda(t) = 1 \mapsto \lambda(t) = 1/\eta_1$,

$$\left(\int_{|\xi|\geq \eta_1^{1/2}} |\hat{u}(t,\xi)|^2\,d\xi \right)^{1/2} \lesssim \eta_0 \tag{4-12}$$

combine to imply that

$$\|u_{\geq i-3}\|^5_{L^5_t L^{10}_x} \lesssim \eta_0. \tag{4-13}$$

Similar calculations can be made for the terms

$$\|u_{\geq i-3}\|^4_{L^{16/3}_t L^8_x}\|u_{\leq i-3}\|_{L^\infty_t L^2_x} + \|u_{\geq i-3}\|^3_{L^6_{t,x}}\|u_{\geq i-6}\|^2_{L^6_{t,x}} + \|u_{\geq i-3}\|^2_{L^6_{t,x}}\|u_{\geq i-6}\|^3_{L^6_{t,x}}. \tag{4-14}$$

Therefore,

$$\begin{aligned}\|u_{\geq i-3}\|^5_{L^5_t L^{10}_x} + \|u_{\geq i-3}\|^4_{L^{16/3}_t L^8_x}\|u_{\leq i-3}\|_{L^\infty_t L^2_x} + \|u_{\geq i-3}\|^3_{L^6_{t,x}}\|u_{\geq i-6}\|^2_{L^6_{t,x}} + \|u_{\geq i-3}\|^2_{L^6_{t,x}}\|u_{\geq i-6}\|^3_{L^6_{t,x}} \\ \lesssim \eta_0\|u\|^4_{X_{i-3}([0,T]\times\mathbb{R})} + \eta_0\|u\|^3_{X_{i-3}([0,T]\times\mathbb{R})} + \eta_0. \end{aligned} \tag{4-15}$$

Next, by the definition on page 1710,

$$\|(u_{\geq i-3})(u_{\leq i-6})\|_{L^2_{t,x}}\|u_{\leq i-6}\|_{L^\infty_{t,x}}\|u_{\geq i-3}\|^2_{L^6_{t,x}} \lesssim 2^{-i/2}\|u\|^3_{X_{i-3}([0,T]\times\mathbb{R})}\|u_{\leq i-6}\|_{L^\infty_{t,x}}. \tag{4-16}$$

By (4-11), (4-12), and the Sobolev embedding theorem,

$$2^{-i/2}\|u_{\leq i-6}\|_{L^\infty_{t,x}} \lesssim \eta_0, \tag{4-17}$$

so

$$2^{-i/2}\|u\|^3_{X_{i-3}([0,T]\times\mathbb{R})}\|u_{\leq i-6}\|_{L^\infty_{t,x}} \lesssim \eta_0\|u\|^3_{X_{i-3}([0,T]\times\mathbb{R})}. \tag{4-18}$$

Making a similar calculation,

$$\|(u_{\geq i-3})(u_{\leq i-6})\|^{3/2}_{L^2_{t,x}}\|u_{\leq i-6}\|^{3/2}_{L^\infty_{t,x}}\|u_{\geq i-3}\|^{1/2}_{L^6_{t,x}} \lesssim \eta_0^{3/2}\|u\|^2_{X_{i-3}([0,T]\times\mathbb{R})}. \tag{4-19}$$

Since

$$P_{\geq i}(|u_{\leq i-3}|^4 u_{\leq i-3}) = 0,$$

it only remains to compute, using the definition on page 1710, (4-15), and (4-17),

$$\begin{aligned}\|v((P_{\geq i-3}u)(P_{\leq i-3}u)^4)\|_{L^1_{t,x}} &\lesssim \|(P_{\geq i-3}u)(P_{\leq i-6}u)\|_{L^2_{t,x}}\|P_{\leq i-6}u\|_{L^\infty_{t,x}}\|v(P_{\leq i-3}u)^2\|_{L^2_{t,x}} \\ &\qquad + \|P_{\geq i-3}u\|_{L^6_{t,x}}\|P_{\geq i-6}u\|^2_{L^6_{t,x}}\|v(P_{\leq i-3}u)^2\|_{L^2_{t,x}} \\ &\lesssim \eta_0\|u\|^2_{X_{i-3}([0,T]\times\mathbb{R})}\|v(P_{\leq i-3}u)^2\|_{L^2_{t,x}} + \eta_0\|u\|^4_{X_{i-3}([0,T]\times\mathbb{R})} + \eta_0.\end{aligned}$$

By the Sobolev embedding theorem,

$$\|P_{\leq i-3}u\|_{L^{18}_{t,x}} \lesssim \sum_{0\leq j\leq i-3}\|P_j u\|_{L^{18}_{t,x}} \lesssim \sum_{0\leq j\leq i-3} 2^{(3i-3j)/18}2^{j/3}\|u\|_{X_{i-3}([0,T]\times\mathbb{R})} \lesssim 2^{i/3}\|u\|_{X_{i-3}([0,T]\times\mathbb{R})}.$$

Also, by $V^2_\Delta \subset U^{9/4}_\Delta$ and the Sobolev embedding theorem,

$$\|v(P_{\leq i-3}u)\|_{L^{9/4}_{t,x}} \lesssim 2^{i/9}\|v\|_{V^{12/5}_\Delta(J\times\mathbb{R})}\cdot\sup_{v_0}\|(e^{it\Delta}v_0)(u_{\leq i-3})\|^{8/9}_{L^2_{t,x}},$$

where $\sup_{v_0}$ is over all $\|v_0\|_{L^2} = 1$ supported in Fourier space on the support of P_i. Therefore, we have finally proved

$$\begin{aligned}\|P_{\geq i}u\|_{U^2_\Delta(J\times\mathbb{R})} &\lesssim \eta_0 + \eta_0\|u\|^2_{X_{i-3}([0,T]\times\mathbb{R})} + \eta_0\|u\|^4_{X_{i-3}([0,T]\times\mathbb{R})} \\ &\qquad + \eta_0\|u\|^3_{X_{i-3}([0,T]\times\mathbb{R})}\cdot 2^{4i/9}\sup_{v_0}\|(e^{it\Delta}v_0)(u_{\leq i-3})\|^{8/9}_{L^2_{t,x}}.\end{aligned} \tag{4-20}$$

To complete the proof of Theorem 13, it only remains to prove

$$2^{i/2}\sup_{v_0}\|(e^{it\Delta}v_0)(u_{\leq i-3})\|_{L^2_{t,x}} \lesssim 1 + \|u\|_{X_{i-3}([0,T]\times\mathbb{R})}. \tag{4-21}$$

Indeed, assuming that (4-21) is true, (4-20) becomes

$$\|P_{\geq i}u\|_{U^2_\Delta(J\times\mathbb{R})} \lesssim \eta_0 + \eta_0\|u\|^2_{X_{i-3}([0,T]\times\mathbb{R})} + \eta_0\|u\|^4_{X_{i-3}([0,T]\times\mathbb{R})}.$$

Equation (4-21) would also imply

$$\begin{aligned}2^{i/2}\|(P_{\geq i}u)(P_{\leq i-3}u)\|_{L^2_{t,x}(J\times\mathbb{R})} &\lesssim \|P_{\geq i}u\|_{U^2_\Delta(J\times\mathbb{R})}(1 + \|u\|_{X_{i-3}([0,T]\times\mathbb{R})}) \\ &\lesssim \eta_0 + \eta_0\|u\|_{X_{i-3}([0,T]\times\mathbb{R})} + \eta_0\|u\|^5_{X_{i-3}([0,T]\times\mathbb{R})}.\end{aligned}$$

Then taking a supremum over $0 \leq i \leq j$,

$$\|u\|_{X_j([0,T]\times\mathbb{R})} \lesssim 1 + \eta_0 \|u\|_{X_{j-3}([0,T]\times\mathbb{R})} + \eta_0 \|u\|^5_{X_{j-3}([0,T]\times\mathbb{R})},$$

which by induction on j, starting from the base case (4-5), proves Theorem 13.

The bilinear estimate (4-21) is proved using the interaction Morawetz estimate (see [Dodson 2016b; Planchon and Vega 2009]). To simplify notation, let

$$v(t,x) = e^{it\Delta} v_0,$$

where $\|v_0\|_{L^2} = 1$ and $\hat{v}_0$ is supported on the Fourier support of P_j for some $j \geq i$. Then take the Morawetz potential

$$M(t) = \int |v(t,y)|^2 \frac{(x-y)}{|x-y|} \operatorname{Im}[\bar{u}_{\leq i-3} \partial_x u_{\leq i-3}]\, dx\, dy + \int |u_{\leq i-3}|^2 \frac{(x-y)}{|x-y|} \operatorname{Im}[\bar{v} v_x]\, dx\, dy.$$

Let $F(u) = |u|^4 u$. Then $u_{\leq i-3}$ solves the equation

$$i\partial_t u_{\leq i-3} + \Delta u_{\leq i-3} + F(u_{\leq i-3}) = F(u_{\leq i-3}) - P_{\leq i-3} F(u) = -\mathcal{N}_{i-3}. \tag{4-22}$$

Following [Planchon and Vega 2009],

$$\begin{aligned} \frac{d}{dt} M(t) = 8 \int |\partial_x (\overline{v(t,x)} u_{\leq i-3})(t,x)|^2\, dx - \frac{8}{3} \int |v(t,x)|^2 |u_{\leq i-3}(t,x)|^6\, dx \\ + \int |v(t,y)|^2 \frac{(x-y)}{|x-y|} \operatorname{Re}[\bar{u}_{\leq i-3} \partial_x \mathcal{N}_{i-3}](t,x)\, dx \\ - \int |v(t,y)|^2 \frac{(x-y)}{|x-y|} \operatorname{Re}[\overline{\mathcal{N}}_{i-3} \partial_x u_{\leq i-3}](t,x)\, dx \\ + 2 \int \operatorname{Im}[\bar{u}_{\leq i-3} \mathcal{N}_{i-3}](t,y) \frac{(x-y)}{|x-y|} \operatorname{Im}[\bar{v} \partial_x v](t,x)\, dx\, dy. \end{aligned}$$

Then by the fundamental theorem of calculus, Bernstein's inequality, the Fourier support of $\bar{v} u_{\leq i-3}$, $\|v_0\|_{L^2} = 1$, and the fact that $\|u\|_{L^2} = \|Q\|_{L^2}$,

$$\begin{aligned} 2^{2j} \|\bar{v} u_{\leq i-3}\|^2_{L^2_{t,x}(J\times\mathbb{R})} \lesssim 2^j + \| |v|^2 |u_{\leq i-3}|^6 \|_{L^1_{t,x}} - \int |v(t,y)|^2 \frac{(x-y)}{|x-y|} \operatorname{Re}[\overline{\mathcal{N}}_{i-3} \partial_x u_{\leq i-3}](t,x)\, dx \\ + \int |v(t,y)|^2 \frac{(x-y)}{|x-y|} \operatorname{Re}[\bar{u}_{\leq i-3} \partial_x \mathcal{N}_{i-3}](t,x)\, dx \\ + 2 \int \operatorname{Im}[\bar{u}_{\leq i-3} \mathcal{N}_{i-3}](t,y) \frac{(x-y)}{|x-y|} \operatorname{Im}[\bar{v} \partial_x v](t,x)\, dx\, dy. \end{aligned} \tag{4-23}$$

Also note that

$$\|\bar{v} u_{\leq i-3}\|^2_{L^2_{t,x}} = \|\bar{v} v \bar{u}_{\leq i-3} u_{\leq i-3}\|_{L^1_{t,x}} = \|v u_{\leq i-3}\|^2_{L^2_{t,x}}, \tag{4-24}$$

so it is not too important to pay attention to complex conjugates in the proceeding calculations.

First, by (4-17),

$$\| |v|^2 |u_{\leq i-3}|^6 \|_{L^1_{t,x}(J\times\mathbb{R})} \lesssim \|v u_{\leq i-3}\|^2_{L^2_{t,x}} \|u_{\leq i-3}\|^4_{L^\infty_{t,x}} \lesssim \eta_0^4 2^{2i} \|v u_{\leq i-3}\|^2_{L^2_{t,x}}. \tag{4-25}$$

Now consider the term

$$\mathcal{N}_{i-3} = P_{\leq i-3} F(u) - F(u_{\leq i-3}). \tag{4-26}$$

Since by Fourier support arguments

$$P_{\leq i-3} F(u_{\leq i-6}) - F(u_{\leq i-6}) = 0, \tag{4-27}$$

we have

$$\begin{aligned}\mathcal{N}_{i-3} = P_{\leq i-3}&(3|u_{\leq i-6}|^4 u_{\geq i-6} + 2|u_{\leq i-6}|^2 (u_{\leq i-6})^2 \bar{u}_{\geq i-6}) \\ &- (3|u_{\leq i-6}|^4 u_{i-6\leq \cdot \leq i-3} + 2|u_{\leq i-6}|^2 (u_{\leq i-6})^2 \bar{u}_{i-6\leq \cdot \leq i-3}) \\ &\qquad + P_{\leq i-3} O((u_{\geq i-6})^2 u^3) + O((u_{i-6\leq \cdot \leq i-3})^2 u^3) \\ = \mathcal{N}_{i-3}^{(1)} &+ \mathcal{N}_{i-3}^{(2)}.\end{aligned} \tag{4-28}$$

Following (4-7)–(4-19),

$$\begin{aligned}\| |\mathcal{N}_{i-3}^{(2)}| |u_{\leq i-3}| \|_{L^1_{t,x}} &\lesssim \| |u_{\geq i-6}|^2 |u_{\leq i-9}|^4 \|_{L^1_{t,x}} + \| |u_{\geq i-6}|^2 |u_{\geq i-9}|^4 \|_{L^1_{t,x}} \\ &\lesssim \|(u_{\geq i-6})(u_{\leq i-9})\|^2_{L^2_{t,x}} \|u_{\leq i-9}\|^2_{L^\infty_{t,x}} + \|u_{\geq i-6}\|^2_{L^6_{t,x}} \|u_{\geq i-9}\|^4_{L^6_{t,x}} \\ &\lesssim \eta_0 (1 + \|u\|^6_{X_{i-3}([0,T]\times\mathbb{R})}).\end{aligned} \tag{4-29}$$

Therefore, since $\|v_0\|_{L^2} = 1$,

$$\begin{aligned}-\iiint |v(t,y)|^2 \frac{(x-y)}{|x-y|} \operatorname{Re}[\overline{\mathcal{N}}_{i-3}^{(2)} \partial_x u_{\leq i-3}](t,x)\, dx\, dy\, dt \\ + \iiint |v(t,y)|^2 \frac{(x-y)}{|x-y|} \operatorname{Re}[\bar{u}_{\leq i-3} \partial_x \mathcal{N}_{i-3}^{(2)}](t,x)\, dx\, dy\, dt \\ + 2 \iiint \operatorname{Im}[\bar{u}_{\leq i-3} \mathcal{N}_{i-3}^{(2)}](t,y) \frac{(x-y)}{|x-y|} \operatorname{Im}[\bar{v} \partial_x v](t,x)\, dx\, dy\, dt \\ \lesssim \eta_0 2^j (1 + \|u\|^6_{X_{i-3}([0,T]\times\mathbb{R})}).\end{aligned} \tag{4-30}$$

Next, observe that

$$\begin{aligned}3P_{\leq i-3}(|u_{\leq i-6}|^4 u_{\geq i-6}) - 3(|u_{\leq i-6}|^4 u_{i-6\leq \cdot \leq i-3}) \\ = 3P_{>i-3}(|u_{\leq i-6}|^4 u_{i-6\leq \cdot \leq i-3}) + 3P_{\leq i-3}(|u_{\leq i-6}|^4 u_{>i-3}).\end{aligned} \tag{4-31}$$

Again following (4-7)–(4-19),

$$\begin{aligned}\|P_{\leq i-3}(|u_{\leq i-6}|^4 u_{>i-3})(u_{i-6\leq \cdot \leq i-3})\|_{L^1_{t,x}} + \|P_{>i-3}(|u_{\leq i-6}|^4 u_{i-6\leq \cdot \leq i-3})(u_{i-6\leq \cdot \leq i-3})\|_{L^1_{t,x}} \\ \lesssim \eta_0 (1 + \|u\|^6_{X_{i-3}([0,T]\times\mathbb{R})}).\end{aligned} \tag{4-32}$$

Finally, observe that the Fourier support of

$$3P_{>i-3}(|u_{\leq i-6}|^4 u_{i-6\leq \cdot \leq i-3})(u_{\leq i-6}) + 3P_{\leq i-3}(|u_{\leq i-6}|^4 u_{>i-3})(u_{\leq i-6}) \tag{4-33}$$

is on frequencies $|\xi| \geq 2^{i-6}$. Therefore, integrating by parts,

$$\begin{aligned}
&\iiint \operatorname{Im}[\bar{u}_{\leq i-6} P_{>i-3}(|u_{\leq i-6}|^4 u_{i-6\leq\cdot\leq i-3})](t,y)\frac{(x-y)}{|x-y|}\operatorname{Im}[\bar{v}\partial_x v](t,x)\,dx\,dy\,dt \\
&\quad= \iiint \operatorname{Im}[\bar{v}\partial_x v](t,x)\cdot\frac{\partial_x}{\partial_x^2}\operatorname{Im}[\bar{u}_{\leq i-6} P_{>i-3}(|u_{\leq i-6}|^4 u_{i-6\leq\cdot\leq i-3})](t,x)\,dx\,dy\,dt \\
&\quad\lesssim 2^{-i}\|v_x\|_{L_t^4 L_x^\infty}\|v\|_{L_t^4 L_x^\infty}\|(u_{i-6\leq\cdot\leq i-3})(u_{\leq i-9})\|_{L^2_{t,x}}\|u_{\leq i-6}\|^4_{L_t^\infty L_x^8} \\
&\qquad\quad + 2^{-i}\|v_x\|_{L_t^4 L_x^\infty}\|v\|_{L_t^4 L_x^\infty}\|u_{i-6\leq\cdot\leq i-3}\|_{L_t^4 L_x^\infty}\|u_{i-9\leq\cdot\leq i-6}\|_{L_t^4 L_x^\infty}\|u_{\leq i-6}\|^4_{L_t^\infty L_x^4} \\
&\quad\lesssim \eta_0 2^j \|u\|^2_{X_{i-3}([0,T]\times\mathbb{R})}.
\end{aligned} \tag{4-34}$$

A similar calculation gives the estimate

$$\iiint \operatorname{Im}[\bar{u}_{\leq i-6} P_{\leq i-3}(|u_{\leq i-6}|^4 u_{>i-3})](t,y)\frac{(x-y)}{|x-y|}\operatorname{Im}[\bar{v}\partial_x v](t,x)\,dx\,dy\,dt \lesssim \eta_0 2^j\|u\|^2_{X_{i-3}([0,T]\times\mathbb{R})}. \tag{4-35}$$

The terms

$$\iiint |v(t,y)|^2\frac{(x-y)}{|x-y|}\operatorname{Re}[\overline{\mathcal{N}}^{(1)}_{i-3}\partial_x u_{\leq i-3}](t,x)\,dx\,dy\,dt \tag{4-36}$$

and

$$\iiint |v(t,y)|^2\frac{(x-y)}{|x-y|}\operatorname{Re}[\bar{u}_{\leq i-3}\partial_x \mathcal{N}^{(1)}_{i-3}](t,x)\,dx\,dy\,dt \tag{4-37}$$

may be analyzed in a similar manner.

Plugging (4-24)–(4-37) into (4-23) gives

$$2^{2j}\|\bar{v}u_{\leq i-3}\|^2_{L^2_{t,x}} + 2^{2j}\|vu_{\leq i-3}\|^2_{L^2_{t,x}} \lesssim 2^j + \eta_0 2^j(1+\|u\|^6_{X_{i-3}}).$$

Summing up over $j \geq i$ implies (4-21), which completes the proof of Theorem 13. □

Theorem 13 may be upgraded to take advantage of the fact that u is close to the soliton.

Theorem 14. *When $\lambda(t) = 1/\eta_1$ and $T = 2^{3k}$ for some positive integer k,*

$$\|P_{\geq k}u\|_{U^2_\Delta([0,T]\times\mathbb{R})} \lesssim \left(\frac{\eta_1^2}{T}\int_0^{\eta_1^{-2}T}\|\epsilon(t)\|^2_{L^2}\,dt\right)^{1/2} + \frac{1}{T^{10}}.$$

Proof. Make another induction on frequency argument starting at level $\frac{1}{2}k$. Observe that Theorem 13 implies that for any $a \in \mathbb{Z}$, $0 \leq a < \eta_1^{-1}T^{1/2}$,

$$\|P_{\geq k/2}u\|_{U^2_\Delta([a\eta_1^{-1}T^{1/2},(a+1)\eta_1^{-1}T^{1/2}]\times\mathbb{R})} \lesssim 1.$$

Next, following Theorem 13,

$$\begin{aligned}\|P_{\geq k/2+3}u\|_{U^2_\Delta([512a\eta_1^{-1}T^{1/2},512(a+1)\eta_1^{-1}T^{1/2}]\times\mathbb{R})} \\ \lesssim \inf_{t\in[512a\eta_1^{-1}T^{1/2},512(a+1)\eta_1^{-1}T^{1/2}]} \|P_{\geq k/2+3}u(t)\|_{L^2} \\ + \eta_0\|P_{\geq k/2}u\|_{U^2_\Delta([512a\eta_1^{-1}T^{1/2},512(a+1)\eta_1^{-1}T^{1/2}]\times\mathbb{R})}. \quad (4\text{-}38)\end{aligned}$$

Since Q is a smooth function, if $\gamma(t)$ and $\lambda(t)$ are given by Theorem 10 and $\lambda(t)=1/\eta_1$,

$$\begin{aligned}\|P_{\geq k/2+3}u(t)\|_{L^2} &\leq \|e^{i\gamma(t)}\lambda(t)^{1/2}u(t,\lambda(t)x)-Q(x)\|_{L^2}+\|P_{\geq k/2+3}Q(x)\|_{L^2} \\ &\lesssim \|\epsilon(t)\|_{L^2}+T^{-10}. \quad (4\text{-}39)\end{aligned}$$

Plugging (4-39) back into (4-38),

$$\begin{aligned}\|P_{\geq k/2+3}u\|_{U^2_\Delta([512a\eta_1^{-1}T^{1/2},512(a+1)\eta_1^{-1}T^{1/2}]\times\mathbb{R})} \\ \lesssim \left(\frac{\eta_1}{512T^{1/2}}\int_{512a\eta_1^{-1}T^{1/2}}^{512(a+1)\eta_1^{-1}T^{1/2}}\|\epsilon(t,x)\|_{L^2}^2\,dt\right)^{1/2} \\ +T^{-10}+\eta_0\left(\sum_{j=1}^{512}\|P_{\geq k/2}u\|^2_{U^2_\Delta([(512a+(j-1))\eta_1^{-1}T^{1/2},(512a+j)\eta_1^{-1}T^{1/2}]\times\mathbb{R})}\right)^{1/2}. \quad (4\text{-}40)\end{aligned}$$

Arguing by induction in k, taking $\lfloor \frac{1}{6}k\rfloor$ steps in all, for η_0 sufficiently small,

$$\begin{aligned}\|P_{\geq k}u\|_{U^2_\Delta([0,\eta_1^{-2}T]\times\mathbb{R})} &\lesssim T^{-10}+2^{k/2}\eta_0^{-k/6}+\left(\frac{\eta_1^2}{T}\int_0^{\eta_1^{-2}T}\|\epsilon(t,x)\|_{L^2}^2\,dt\right)^{1/2} \\ &\lesssim T^{-10}+\left(\frac{\eta_1^2}{T}\int_0^{\eta_1^{-2}T}\|\epsilon(t,x)\|_{L^2}^2\,dt\right)^{1/2}. \qquad \square\end{aligned}$$

Remark. If C is the implicit constant in (4-40), then for $\eta_0 \ll 1$ sufficiently small,

$$(C\eta_0)^{\lfloor k/6\rfloor} \leq T^{-10}. \quad (4\text{-}41)$$

The same argument can also be made when $\lambda(t)\geq 1/\eta_1$ for all $t\in J$.

Theorem 15. *When $\lambda(t)\geq 1/\eta_1$ on $J=[a,b]$,*

$$\int_J \lambda(t)^{-2}\,dt = T,$$

and $\eta_1^{-2}T=2^{3k}$, we have

$$\|P_{\geq k}u\|_{U^2_\Delta([a,b]\times\mathbb{R})} \lesssim T^{-10}+\left(\frac{1}{T}\int_a^b\|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}\,dt\right)^{1/2}.$$

The same argument could also be made for $\lambda(t)$ having a different lower bound, by rescaling $\lambda(t)$ to $\lambda(t)\geq 1/\eta_1$, computing long-time Strichartz estimates, and then rescaling back.

5. Almost conservation of energy

Since (3-11) implies that $\|\epsilon(t)\|_{L^2}$ is continuous as a function of time, the mean value theorem implies that under the conditions of Theorem 15, there exists $t_0 \in [a,b]$ such that

$$\|\epsilon(t_0)\|_{L^2}^2 = \frac{1}{T}\int_a^b \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}\,dt.$$

The next step in proving Theorem 7 is to control

$$\sup_{t\in[a,b]} \|\epsilon(t)\|_{L^2}$$

as a function of $\|\epsilon(t_0)\|_{L^2}$. Theorem 12 would be a very useful tool for doing so, except that while Q lies in $H^s(\mathbb{R})$ for any $s>0$, it is not the case that ϵ must belong to $H^s(\mathbb{R})$ for any $s>0$. Therefore, Theorem 12 will be used in conjunction with the Fourier truncation method of [Bourgain 1998]. See also the I-method, for example, in [Colliander et al. 2002].

Theorem 16. *Let $J=[a,b]$ be an interval such that*

$$\int_J \lambda(t)^{-2}\,dt = T,$$

$\eta_1^{-2}T = 2^{3k}$, *and* $\lambda(t)\geq 1/\eta_1$ *for all* $t\in[a,b]$. *Then,*

$$\sup_{t\in J} E(P_{\leq k+9}u(t)) \lesssim \frac{2^{2k}}{T}\int_J \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}\,dt + 2^{2k}T^{-10}.$$

Proof. By the mean value theorem, there exists $t_0\in J$ such that

$$\|\epsilon(t_0)\|_{L^2}^2 \lesssim \frac{1}{T}\int_J \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}\,dt.$$

Next, decompose the energy. Let $\widetilde{Q}$ refer to a rescaled version of Q, that is, $\widetilde{Q} = \lambda(t_0)^{-1/2}Q(\lambda(t_0)^{-1}x)$, and let $\tilde{\epsilon}$ denote the rescaled ϵ given by $\tilde{\epsilon} = \lambda(t_0)^{-1/2}\tilde{\epsilon}(t_0, \lambda(t_0)^{-1}x)$. It is also convenient to split $\tilde{\epsilon}$ into real and imaginary parts:

$$\tilde{\epsilon} = \epsilon_1 + i\epsilon_2.$$

As in Theorem 12, by (3-2),

$$\begin{aligned} E(P_{\leq k+9}u) &= E(P_{\leq k+9}\widetilde{Q} + P_{\leq k+9}\tilde{\epsilon}) \\ &= E(P_{\leq k+9}\widetilde{Q}) + \operatorname{Re}\int P_{\leq k+9}\widetilde{Q}_x\,\overline{P_{\leq k+9}\tilde{\epsilon}_x}\,dx - \operatorname{Re}\int (P_{\leq k+9}\widetilde{Q})^5(\overline{P_{\leq k+9}\tilde{\epsilon}})\,dx \\ &\quad + \frac{1}{2}\|P_{\leq k+9}\tilde{\epsilon}\|_{\dot{H}^1}^2 - \frac{5}{2}\int (P_{\leq k+9}\widetilde{Q})^4(P_{\leq k+9}\tilde{\epsilon}_1)^2\,dx - \frac{1}{2}\int (P_{\leq k+9}\widetilde{Q})^4(P_{\leq k+9}\epsilon_2)^2 \\ &\quad - \int O(|P_{\leq k+9}\widetilde{Q}|^3|P_{\leq k+9}\tilde{\epsilon}|^3) + O(|P_{\leq k+9}\tilde{\epsilon}|^6)\,dx. \end{aligned} \tag{5-1}$$

Since $\widetilde{Q}$ is smooth and rapidly decreasing, $E(\widetilde{Q}) = 0$, and $\lambda(t_0) \geq 1/\eta_1$, Bernstein's inequality implies that

$$E(P_{\leq k+9}\widetilde{Q}) = \frac{1}{2}\int (P_{\leq k+9}\widetilde{Q}_x)^2 - \frac{1}{6}\int (P_{\leq k+9}\widetilde{Q})^6 \lesssim 2^{-30k}. \tag{5-2}$$

Next, integrating by parts and using (3-19), the smoothness of Q, and Bernstein's inequality,

$$\begin{aligned}\operatorname{Re}\int P_{\leq k+9}\widetilde{Q}_x \overline{P_{\leq k+9}\tilde{\epsilon}_x}\,dx - \operatorname{Re}\int (P_{\leq k+9}\widetilde{Q})^5 \overline{P_{\leq k+9}\tilde{\epsilon}}\,dx \\ = -\operatorname{Re}\int (\overline{P_{\leq k+9}\epsilon})(P_{\leq k+9}\widetilde{Q}_{xx} + (P_{\leq k+9}\widetilde{Q})^5)\,dx = \frac{1}{2\lambda(t_0)^2}\|\epsilon\|_{L^2}^2 + O(2^{-30k}).\end{aligned} \tag{5-3}$$

Next, by Hölder's inequality, since $\lambda(t_0) \geq 1/\eta_1$,

$$\begin{aligned}\frac{1}{2}\|P_{\leq k+9}\tilde{\epsilon}\|_{\dot{H}^1}^2 - \frac{5}{2}\int (P_{\leq k+9}\widetilde{Q})^4 (P_{\leq k+9}\tilde{\epsilon}_1)^2\,dx - \frac{1}{2}\int (P_{\leq k+9}\widetilde{Q})^4 (P_{\leq k+9}\tilde{\epsilon}_2)^2\,dx \\ \lesssim \|P_{\leq k+9}\tilde{\epsilon}\|_{\dot{H}^1}^2 + \frac{1}{\lambda(t_0)^2}\|P_{\leq k+9}\tilde{\epsilon}\|_{L^2}^2 \lesssim 2^{2k}\|\epsilon(t_0)\|_{L^2}^2.\end{aligned} \tag{5-4}$$

By the Sobolev embedding theorem,

$$\begin{aligned}&\int |P_{\leq k+9}\tilde{\epsilon}|^3 |P_{\leq k+9}\widetilde{Q}|^3 + |P_{\leq k+9}\tilde{\epsilon}|^6\,dx \\ &\quad\lesssim \frac{1}{\lambda(t_0)^{3/2}}\|P_{\leq k+9}\tilde{\epsilon}(t_0)\|_{L^2}^{5/2}\|P_{\leq k+9}\tilde{\epsilon}(t_0)\|_{\dot{H}^1}^{1/2} + \|P_{\leq k+9}\tilde{\epsilon}(t_0)\|_{L^2}^4\|P_{\leq k+9}\tilde{\epsilon}(t_0)\|_{\dot{H}^1}^2 \\ &\quad\lesssim \frac{1}{\lambda(t_0)^2}\|P_{\leq k+9}\tilde{\epsilon}(t_0)\|_{L^2}^2 + \|P_{\leq k+9}\tilde{\epsilon}(t_0)\|_{L^2}^4\|P_{\leq k+9}\tilde{\epsilon}(t_0)\|_{\dot{H}^1}^2.\end{aligned} \tag{5-5}$$

Therefore, since $\lambda(t_0) \geq 1/\eta_1$,

$$E(P_{\leq k+9}\tilde{u}(t_0)) \lesssim 2^{2k}\|\epsilon(t_0)\|_{L^2}^2 + 2^{-30k}. \tag{5-6}$$

Next compute the change of energy

$$\begin{aligned}\frac{d}{dt}E(P_{\leq k+9}u) &= -(P_{\leq k+9}u_t, \Delta P_{\leq k+9}u)_{L^2} - (P_{\leq k+9}u_t, |P_{\leq k+9}u|^4 P_{\leq k+9}u)_{L^2} \\ &= -(P_{\leq k+9}u_t, P_{\leq k+9}F(u) - F(P_{\leq k+9}u))_{L^2} \\ &= (i\Delta P_{\leq k+9}u + iP_{\leq k+9}(|u|^4 u), P_{\leq k+9}F(u) - F(P_{\leq k+9}u))_{L^2}.\end{aligned}$$

First compute

$$\int_{t_0}^{t'} (i\Delta P_{\leq k+9}u, P_{\leq k+9}F(u) - F(P_{\leq k+9}u))_{L^2}\,dt$$

for some $t' \in J$. Making a Littlewood–Paley decomposition,

$$\begin{aligned}&\int_{t_0}^{t'} (i\Delta P_{\leq k+9}u, P_{\leq k+9}F(u) - F(P_{\leq k+9}u))_{L^2}\,dt \\ &\sim \sum_{0\leq k_5\leq k_4\leq k_3\leq k_2\leq k_1}\ \sum_{0\leq k_6\leq k+9}\int_{t_0}^{t'} \big(i\Delta P_{k_6}u, P_{\leq k+9}(P_{k_1}u\cdots P_{k_5}u) \\ &\qquad\qquad - (P_{\leq k+9}P_{k_1}u)\cdots(P_{\leq k+9}P_{k_5}u)\big)_{L^2}\,dt.\end{aligned}$$

Remark. For these computations, it is not so important to distinguish between u and $\bar{u}$.

Case 1: $k_1 \leq k+6$. In this case $P_{\leq k+9}P_{k_1} = P_{k_1}$ and $P_{\leq k+9}(P_{k_1}u \cdots P_{k_5}u) = P_{k_1}u \cdots P_{k_5}u$, so the contribution of these terms is zero. That is, for $k_1, \dots, k_5 \leq k+6$,

$$\int_{t_0}^{t'} (i\Delta P_{k_6}u, P_{\leq k+9}(P_{k_1}u \cdots P_{k_5}u) - (P_{\leq k+9}P_{k_1}u)\cdots(P_{\leq k+9}P_{k_5}u))_{L^2}\,dt = 0.$$

Case 2: $k_1 \geq k+6$ and $k_2 \leq k$. In this case, Fourier support properties imply that $k_6 \geq k+3$. Then by Theorem 15, Theorem 13, and (4-21),

$$\begin{aligned}\int_{t_0}^{t'} (i\Delta P_{k+3\leq\cdot\leq k+9}u, P_{\leq k+9}((P_{\leq k}u)^4(P_{\geq k+6}u)) - (P_{\leq k}u)^4(P_{k+6\leq\cdot\leq k+9}u))_{L^2}\,dt \\ \lesssim 2^{2k}\|(P_{\geq k+3}u)(P_{\leq k}u)\|_{L^2_{t,x}}\|(P_{\geq k+6}u)(P_{\leq k}u)\|_{L^2_{t,x}}\|P_{\leq k}u\|^2_{L^\infty_{t,x}} \\ \lesssim \frac{2^{2k}}{T}\int_J \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}\,dt + 2^{2k}T^{-10}.\end{aligned}$$

Case 3: $k_1 \geq k+6,\ k_2 \geq k,\ k_3 \leq k$. If $k_6 \leq k$, then by Fourier support properties, $k_2 \geq k+3$. Here,

$$\begin{aligned}&\int_{t_0}^{t'} \big(i\Delta P_{\leq k}u, P_{\leq k+9}((P_{\geq k+6}u)(P_{\geq k+3}u)(P_{\leq k}u)^3) \\ &\qquad\qquad - (P_{k+6\leq\cdot\leq k+9}u)(P_{k+3\leq\cdot\leq k+9}u)(P_{\leq k}u)^3\big)_{L^2}\,dt \\ &\lesssim 2^{2k}\|(P_{\geq k+6}u)(P_{\leq k}u)\|_{L^2_{t,x}}\|(P_{\geq k+3}u)(P_{\leq k}u)\|_{L^2_{t,x}}\|P_{\leq k}u\|^2_{L^\infty_{t,x}} \\ &\lesssim \frac{2^{2k}}{T}\int_J \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}\,dt + 2^{2k}T^{-10}.\end{aligned}$$

In the case when $k_6 \geq k$,

$$\begin{aligned}&\int_{t_0}^{t'} \big(i\Delta P_{k\leq\cdot\leq k+9}u, P_{\leq k+9}((P_{\geq k+6}u)(P_{\geq k}u)(P_{\leq k}u)^3) \\ &\qquad\qquad - (P_{k+6\leq\cdot\leq k+9}u)(P_{k\leq\cdot\leq k+9}u)(P_{\leq k}u)^3\big)_{L^2}\,dt \\ &\lesssim 2^{2k}\|(P_{\geq k+6}u)(P_{\leq k}u)\|_{L^2_{t,x}}\|P_{\geq k}u\|^2_{L^4_tL^\infty_x}\|P_{\leq k}u\|_{L^\infty_{t,x}}\|u\|_{L^\infty_tL^2_x} \\ &\lesssim \frac{2^{2k}}{T}\int_J \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}\,dt + 2^{2k}T^{-10}.\end{aligned}$$

Case 4: $k_1 \geq 2^{k+6}$ and $k_2, k_3 \geq k$. In this case,

$$\begin{aligned}&\int_{t_0}^{t'} (i\Delta P_{\leq k+9}u, P_{\leq k+9}((P_{\geq k+6}u)(P_{\geq k}u)^2u^2) - (P_{k+6\leq\cdot\leq k+9}u)(P_{k\leq\cdot\leq k+9}u)^2(P_{\leq k+9}u)^2)_{L^2}\,dt \\ &\lesssim 2^{2k}\|(P_{\geq k+6}u)(P_{\leq k}u)\|_{L^2_{t,x}}\|P_{\geq k}u\|^2_{L^4_tL^\infty_x}\|P_{\leq k}u\|_{L^\infty_{t,x}}\|u\|_{L^\infty_tL^2_x} + 2^{2k}\|P_{\geq k}u\|^4_{L^4_tL^\infty_x}\|u\|^2_{L^\infty_tL^2_x} \\ &\lesssim \frac{2^{2k}}{T}\int_J \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}\,dt + 2^{2k}T^{-10}.\end{aligned}$$

The contribution of the nonlinear terms is similar, using the fact that

$$(iP_{\leq k+9}F(u), P_{\leq k+9}F(u) - F(P_{\leq k+9}u))_{L^2} = (iP_{\leq k+9}F(u), F(P_{\leq k+9}u))_{L^2}.$$

Then make a Littlewood–Paley decomposition

$$(iP_{\leq k+9}F(u), F(P_{\leq k+9}u))_{L^2} = \sum_{0\leq k_5\leq k_4\leq k_3\leq k_2\leq k_1} \sum_{0\leq k_5'\leq k_4'\leq k_3'\leq k_2'\leq k_1'} (iP_{\leq k+9}(u_{k_1}\cdots u_{k_5}), (P_{\leq k+9}P_{k_1}u)\cdots(P_{\leq k+9}P_{k_5}u))_{L^2}. \tag{5-7}$$

<u>Case 1</u>: $k_1, k_1' \leq k+6$. Once again, if $k_1, k_1' \leq k+6$, then the right-hand side of (5-7) is zero.

<u>Case 2</u>: k_1 or $k_1' \geq k+6$, eight terms are $\leq k$. In the case that k_1 or $k_1' \geq k+6$ and eight of the terms in (5-7) are at frequency $\leq k$, then by Fourier support properties the final term should be at frequency $\geq k+3$. The contribution in this case is bounded by

$$\|(P_{\geq k+6}u)(P_{\leq k}u)\|_{L^2_{t,x}} \|(P_{\geq k+3}u)(P_{\leq k}u)\|_{L^2_{t,x}} \|P_{\leq k}u\|^6_{L^\infty_{t,x}} \lesssim 2^{2k}\frac{1}{T}\int_J \|\epsilon(t)\|^2_{L^2}\,dt + 2^{2k}T^{-10}.$$

<u>Case 3</u>: k_1 or $k_1' \geq k+6$, two terms are $\geq k$. The contribution of the case that k_1 or $k_1' \geq k+6$, two additional terms in (5-7) are at frequency $\geq k$, and the other seven terms are at frequency $\leq k$ is bounded by

$$\|(P_{\geq k+6}u)(P_{\leq k}u)\|_{L^2_{t,x}} \|P_{\geq k}u\|^2_{L^4_t L^\infty_x} \|P_{\leq k}u\|^5_{L^\infty_{t,x}} \|u\|_{L^\infty_t L^2_x} \lesssim 2^{2k}\frac{1}{T}\int_J \|\epsilon(t)\|^2_{L^2}\,dt + 2^{2k}T^{-10}.$$

<u>Case 4</u>: k_1 or $k_1' \geq k+6$ and at least three additional terms in (5-7) are at frequencies $\geq k$. This case may be reduced to a case where at least four terms in (5-7) are at frequency $\geq k$, and at least four terms are at frequency $\leq k+9$. To see why, notice that all five terms in $F(P_{\leq k+9}u)$ are at frequency $\leq k+9$, so if four or five of the terms in $P_{\leq k+9}F(u)$ are at frequency $\geq k$, then we are fine.

If exactly three terms in $P_{\leq k+9}F(u)$ are at frequency $\geq k$, then take the two terms in $P_{\leq k+9}F(u)$ that are at frequency $\leq k$ to be terms at frequency $\leq k+9$. Meanwhile, since at least four terms are at frequency $\geq k$,

$$F(P_{\leq k+9}u) \sim (P_{k\leq\cdot\leq k+9}u)(P_{\leq k+9}u)^4, \tag{5-8}$$

so in (5-8) there is one term at frequency $\geq k$ and two more terms at frequency $\leq k+9$.

If exactly two terms in $P_{\leq k+9}F(u)$ are at frequency $\geq k$, then there are three terms that are at frequency $\leq k$. In that case,

$$F(P_{\leq k+9}u) \sim (P_{k\leq\cdot\leq k+9}u)^2(P_{\leq k+9}u)^3, \tag{5-9}$$

so in (5-9) there are two terms at frequency $\geq k$ and one term at frequency $\leq k+9$.

If one term in $P_{\leq k+9}F(u)$ is at frequency $\geq k$, then there are four terms in $P_{\leq k+9}F(u)$ at frequency $\leq k$. Then there must be at least three more in $F(P_{\leq k+9}u)$, so

$$F(P_{\leq k+9}u) \sim (P_{k\leq\cdot\leq k+9}u)^3u^2. \tag{5-10}$$

If no terms in $P_{\leq k+9}F(u)$ are at frequency $\geq k$, then there must be four in $F(P_{\leq k+9}u)$, so

$$F(P_{\leq k+9}u) \sim (P_{k\leq\cdot\leq k+9}u)^4u. \tag{5-11}$$

The contribution of all the different subcases of case four, (5-8)–(5-11), may be bounded by

$$\|P_{\geq k}u\|_{L_t^4L_x^\infty}^4\|u\|_{L_t^\infty L_x^2}^2\|P_{\leq k+9}u\|_{L_{t,x}^\infty}^4 \lesssim 2^{2k}\frac{1}{T}\int_J \|\epsilon(t)\|_{L^2}^2\,dt + 2^{2k}T^{-10}.$$

This proves Theorem 16. □

Corollary 17. *If*

$$\frac{1}{\eta_1} \leq \lambda(t) \leq \frac{1}{\eta_1}T^{1/100}$$

and

$$\int_J \lambda(t)^{-2}\,dt = T,$$

then

$$\sup_{t\in J}\left\| P_{\leq k+9}\frac{1}{\lambda(t)^{1/2}}\epsilon\left(t,\frac{x}{\lambda(t)}\right)\right\|_{\dot H^1}^2 \lesssim \frac{2^{2k}}{T}\int_J \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}\,dt + 2^{2k}T^{-10}$$

and

$$\sup_{t\in J}\|\epsilon(t)\|_{L^2}^2 \lesssim \frac{T^{1/50}}{\eta_1^2}\frac{2^{2k}}{T}\int_J \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}\,dt + \frac{T^{1/50}}{\eta_1^2}2^{2k}T^{-10}.$$

Proof. The proof uses Theorem 16, Theorem 12, rescaling, and the fact that Q is smooth and all its derivatives are rapidly decreasing. □

6. A frequency-localized Morawetz estimate

The next step will be to combine long-time Strichartz estimates with almost conservation of energy to prove a frequency-localized Morawetz estimate adapted to the case when $\lambda(t)$ does not vary too much.

Theorem 18. *Let $J = [a, b]$ be an interval on which* (4-2) *holds for all $t \in J$, $1/\eta_1 \leq \lambda(t) \leq T^{1/100}/\eta_1$ for all $t \in J$, and*

$$\int_J \lambda(t)^{-2}\,dt = T. \tag{6-1}$$

Also suppose $2^{3k} = \eta_1^{-2}T$ and that $\epsilon = \epsilon_1 + i\epsilon_2$, where ϵ is given by Theorem 10. Then for T sufficiently large,

$$\int_a^b \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}\,dt \leq 3\big(\epsilon_2(a),\big(\tfrac{1}{2}Q + xQ_x\big)\big)_{L^2} - 3\big(\epsilon_2(b),\tfrac{1}{2}Q + xQ_x\big)_{L^2} + O\left(\frac{1}{T^9}\right). \tag{6-2}$$

Remark. The signs on the right-hand side of (6-2) are very important.

Proof. The proof uses a frequency-localized Morawetz estimate. The Morawetz potential is the same as the Morawetz potential used in [Dodson 2015]. See also [Dodson 2016a].

Let $\psi(x) \in C^\infty(\mathbb{R})$ be a smooth, even function, satisfying $\psi(x) = 1$ on $|x| \leq 1$ and supported on $|x| \leq 2$. Then for some large R ($R = T^{1/25}$ will do), let

$$\phi(x) = \int_0^x \chi\left(\frac{\eta_1 y}{R}\right)dy = \int_0^x \psi^2\left(\frac{\eta_1 y}{R}\right)dy, \tag{6-3}$$

and let

$$M(t) = \int \phi(x)\,\mathrm{Im}[\overline{P_{\leq k+9}u}\,\partial_x P_{\leq k+9}u](t,x)\,dx.$$

Doing some algebra using (3-2), as in (5-1),

$$u(t,x) = e^{-i\gamma(t)}\lambda(t)^{-1/2}Q\left(\frac{x}{\lambda(t)}\right) + e^{-i\gamma(t)}\lambda(t)^{-1/2}\epsilon\left(t,\frac{x}{\lambda(t)}\right) = e^{-i\gamma(t)}\widetilde{Q}(x) + e^{-i\gamma(t)}\tilde{\epsilon}(t,x). \quad (6\text{-}4)$$

Since $\mathrm{Im}[\overline{P_{\leq k+9}u}\,\partial_x(P_{\leq k+9}u)]$ is invariant under the multiplication operator $u \mapsto e^{-i\gamma(t)}u$,

$$M(t) = \int \phi(x)\,\mathrm{Im}[\overline{P_{\leq k+9}\widetilde{Q}(x) + P_{\leq k+9}\tilde{\epsilon}(t,x)}\,\partial_x(P_{\leq k+9}\widetilde{Q}(x) + P_{\leq k+9}\tilde{\epsilon}(t,x))]\,dx.$$

Since Q is real-valued,

$$\int \phi(x)\,\mathrm{Im}[\overline{P_{\leq k+9}\widetilde{Q}(x)}\,\partial_x(P_{\leq k+9}\widetilde{Q}(x))]\,dx = 0.$$

Next, by Corollary 17,

$$\int \phi(x)\,\mathrm{Im}[\overline{P_{\leq k+9}\tilde{\epsilon}(t,x)}\,\partial_x(P_{\leq k+9}\tilde{\epsilon}(t,x))]\,dx \lesssim \frac{R}{\eta_1^2}\frac{2^{2k}}{T^{99/100}}\int_J \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}\,dt + \frac{R}{\eta_1^2}2^{2k}T^{-9.99}.$$

Next, since $\lambda(t) \geq 1/\eta_1$, Q and $\partial_x Q$ are rapidly decreasing, $\phi(x) = x$ for $|x| \leq R/\eta_1$, and $|\phi_x(x)| \leq 1$,

$$\int \phi(x)\,\mathrm{Im}[\overline{P_{\leq k+9}\tilde{\epsilon}(t,x)}\,\partial_x(P_{\leq k+9}\widetilde{Q}(x))]\,dx = -(\epsilon_2, xQ_x)_{L^2} + O(T^{-10}). \quad (6\text{-}5)$$

Indeed, since Q is real, by rescaling,

$$\int x\,\mathrm{Im}[\overline{\tilde{\epsilon}(t,x)}\,\partial_x(\widetilde{Q}(x))]\,dx = -(\epsilon_2(t), xQ_x)_{L^2}. \quad (6\text{-}6)$$

Next, since $\lambda(t) \leq T^{1/100}/\eta_1$ and $R = T^{1/25}$,

$$\int x\,\mathrm{Im}[\overline{\tilde{\epsilon}(t,x)}\,\partial_x(\widetilde{Q}(x))]\,dx - \int \phi(x)\,\mathrm{Im}[\overline{\tilde{\epsilon}(t,x)}\,\partial_x(\widetilde{Q}(x))]\,dx$$
$$\leq \int_{|x|\geq R/\eta_1} \frac{x}{\lambda(t)^{3/2}}\left|Q_x\left(\frac{x}{\lambda(t)}\right)\right|\left|\frac{1}{\lambda(t)^{1/2}}\epsilon\left(t,\frac{x}{\lambda(t)}\right)\right|\,dx \lesssim T^{-10}. \quad (6\text{-}7)$$

Also, since Q and all its derivatives are rapidly decreasing, $\lambda(t) \geq 1/\eta_1$, $R = T^{1/25}$, and $2^{3k} = \eta_1^{-2}T$,

$$\int \phi(x)\,\mathrm{Im}[\overline{\tilde{\epsilon}(t,x)}\,\partial_x(\widetilde{Q}(x))]\,dx - \int \phi(x)\,\mathrm{Im}[\overline{\tilde{\epsilon}(t,x)}\,\partial_x(P_{\leq k+9}\widetilde{Q}(x))]\,dx$$
$$\lesssim R\|\epsilon\|_{L^2}\|P_{\geq k+9}\widetilde{Q}_x(x)\|_{L^2} \lesssim T^{-10}. \quad (6\text{-}8)$$

Next, (6-3) implies that $|\phi^{(j)}(x)| \lesssim 1$ for any $j \geq 1$, and since Q is smooth and all its derivatives are rapidly decreasing, integrating by parts, for j sufficiently large, yields

$$\int \phi(x)\,\mathrm{Im}[\overline{P_{\geq k+9}\tilde{\epsilon}(t,x)}\,\partial_x(P_{\leq k+9}\widetilde{Q}(x))]\,dx$$
$$= \int \phi(x)\,\mathrm{Im}\left[\overline{\frac{\Delta^j}{\Delta^j}P_{\geq k+9}\tilde{\epsilon}(t,x)}\,\partial_x(P_{\leq k+9}\widetilde{Q}(x))\right]\,dx \lesssim T^{-10}, \quad (6\text{-}9)$$

so (6-6)–(6-9) imply (6-5). Finally,

$$\int \phi(x)\,\mathrm{Im}[\overline{P_{\leq k+9}\widetilde{Q}(x)}\partial_x(P_{\leq k+9}\tilde{\epsilon}(t,x))]\,dx = \text{(6-5)} - \int \chi\Big(\frac{\eta_1 x}{R}\Big)\,\mathrm{Im}[\overline{P_{\leq k+9}\widetilde{Q}(x)}\cdot P_{\leq k+9}\tilde{\epsilon}(t,x)]\,dx.$$

Making an argument similar to (6-6)–(6-9),

$$-\int \chi\Big(\frac{\eta_1 x}{R}\Big)\,\mathrm{Im}[\overline{P_{\leq k+9}\widetilde{Q}(x)}\cdot P_{\leq k+9}\tilde{\epsilon}(t,x)]\,dx = -(\epsilon_2, Q)_{L^2} + O(T^{-10}). \tag{6-10}$$

Therefore,

$$M(b) - M(a) = 2\big(\epsilon_2(a), \tfrac{1}{2}Q + xQ_x\big)_{L^2} - 2\big(\epsilon_2(b), \tfrac{1}{2}Q + xQ_x\big)_{L^2} + O(T^{-10}) + O\Big(\frac{R}{\eta_1^2}\frac{2^{2k}}{T^{99/100}}\int_J \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}\,dt\Big) + O\Big(\frac{R}{\eta_1^2}2^{2k}T^{-9.99}\Big).$$

Following (4-22),

$$i\partial_t P_{\leq k+9}u + \Delta P_{\leq k+9}u + F(P_{\leq k+9}u) = F(P_{\leq k+9}u) - P_{\leq k+9}F(u) = -\mathcal{N}. \tag{6-11}$$

Plugging in (6-11) and integrating by parts,

$$\begin{aligned}
\frac{d}{dt}M(t) &= \int \phi(x)\,\mathrm{Re}[-\overline{P_{\leq k+9}u}_{xx}P_{\leq k+9}u_x + \overline{P_{\leq k+9}u}\,P_{\leq k+9}u_{xxx}] \\
&\quad + \int \phi(x)\,\mathrm{Re}[-|P_{\leq k+9}u|^4\,\overline{P_{\leq k+9}u}(P_{\leq k+9}u_x) + \overline{P_{\leq k+9}u}\,\partial_x(|P_{\leq k+9}u|^4 P_{\leq k+9}u)] \\
&\quad + \int \phi(x)\,\mathrm{Re}[\overline{P_{\leq k+9}u}\,\partial_x\mathcal{N}](t,x) - \int \phi(x)\,\mathrm{Re}[\overline{\mathcal{N}}\partial_x P_{\leq k+9}u](t,x) \\
&= 2\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|\partial_x P_{\leq k+9}u|^2\,dx - \frac{\eta_1^2}{2R^2}\int \chi''\Big(\frac{\eta_1 x}{R}\Big)|P_{\leq k+9}u|^2\,dx \\
&\quad - \frac{2}{3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\leq k+9}u|^6\,dx + \int \phi(x)\,\mathrm{Re}[\overline{P_{\leq k+9}u}\,\partial_x\mathcal{N}](t,x) \\
&\quad - \int \phi(x)\,\mathrm{Re}[\overline{\mathcal{N}}\partial_x P_{\leq k+9}u](t,x).
\end{aligned} \tag{6-12}$$

Next, following (4-28),

$$\begin{aligned}
\mathcal{N} &= P_{\leq k+9}(3|u_{\leq k}|^4 u_{\geq k+6} + 2|u_{\leq k}|^2(u_{\leq k})^2\bar{u}_{\geq k+6}) \\
&\quad - (3|u_{\leq k}|^4 u_{\geq k+6} + 2|u_{\leq k}|^2(u_{\leq k})^2\,\overline{P_{k+6\leq\cdot\leq k+9}u}) + P_{\leq k+9}O((u_{\geq k})(u_{\geq k+6})u^3) \\
&\quad + O((P_{k+6\leq\cdot\leq k+9}u)(P_{k\leq\cdot\leq k+9}u)u^3) \\
&= \mathcal{N}^{(1)} + \mathcal{N}^{(2)}.
\end{aligned}$$

As in (4-29), since $|\phi(x)| \lesssim \eta_1^{-1}R$, by Theorems 13 and 14,

$$\int_a^b\!\!\int \phi(x)\,\mathrm{Re}[\overline{P_{\leq k+9}u}\,\partial_x\mathcal{N}^{(2)}]\,dx\,dt - \int_a^b\!\!\int \phi(x)\,\mathrm{Re}[\overline{\mathcal{N}}^{(2)}\partial_x P_{\leq k+9}u]\,dx\,dt \lesssim \frac{2^k R\eta_1^{-1}}{T}\int \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}\,dt + \frac{2^k R\eta_1^{-1}}{T^{10}}. \tag{6-13}$$

Making calculations identical to the estimate (6-13),

$$\begin{aligned}\int_a^b\!\!\int \phi(x)\,\mathrm{Re}[\overline{P_{k+3\le\cdot\le k+9}u}\,\partial_x\mathcal{N}^{(1)}]\,dx\,dt - \int_a^b\!\!\int \phi(x)\,\mathrm{Re}[\overline{\mathcal{N}}^{(1)}\partial_x P_{k+3\le\cdot\le k+9}u]\,dx\,dt \\ \lesssim 2^k R\|(u_{\ge k+6})(u_{\le k})\|_{L^2_{t,x}}\|(u_{\ge k+3})(u_{\le k})\|_{L^2_{t,x}}\|u_{\le k}\|^2_{L^\infty_{t,x}} \\ \lesssim \frac{2^k R\eta_1^{-1}}{T}\int_a^b \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}\,dt + \frac{2^k R\eta_1^{-1}}{T^{10}}.\end{aligned}$$

Finally, using Bernstein's inequality and the integration by parts argument in (4-34),

$$\begin{aligned}\int_0^T\!\!\int \phi(x)\,\mathrm{Re}[\overline{P_{\le k+3}u}\,\partial_x\mathcal{N}^{(1)}]\,dx\,dt - \int_0^T\!\!\int \phi(x)\,\mathrm{Re}[\overline{\mathcal{N}}^{(1)}\partial_x P_{\le k+3}u]\,dx\,dt \\ \lesssim \|P_{\ge k+6}\phi(x)\|_{L^\infty}\|(P_{\ge k+6}u)(P_{\le k}u)\|_{L^2_{t,x}}\|u_{\le k}\|^4_{L^8_{t,x}} \lesssim \frac{1}{T^{10}}.\end{aligned}$$

Remark. The last estimate follows from the fact that ϕ is smooth and $\int_J \lambda(t)^{-2}\,dt = T$, which by a local well-posedness argument implies

$$\|u\|_{L^6_{t,x}(J\times\mathbb{R})} \lesssim T^{1/6}.$$

Plugging this estimate of the error term back into (6-12),

$$\begin{aligned}2\int_a^b\!\!\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|\partial_x P_{\le k+9}u|^2\,dx\,dt - \frac{\eta_1^2}{2R^2}\int_a^b\!\!\int \chi''\Big(\frac{\eta_1 x}{R}\Big)|P_{\le k+9}u|^2\,dx\,dt \\ - \frac{2}{3}\int_a^b\!\!\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\le k+9}u|^6\,dx\,dt \\ = 2\big(\epsilon_2(a), \tfrac12 Q + xQ_x\big)_{L^2} - 2\big(\epsilon_2(b), \tfrac12 Q + xQ_x\big)_{L^2} \\ + O\bigg(\frac{2^{2k}T^{1/20}}{T}\int_a^b \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}\,dt\bigg) + O\Big(\frac{1}{T^9}\Big).\end{aligned} \tag{6-14}$$

Since Q is a real-valued function,

$$|P_{\le k+9}u|^2 = (P_{\le k+9}\widetilde{Q}(x))^2 + 2P_{\le k+9}\widetilde{Q}(x)\cdot P_{\le k+9}\tilde{\epsilon}_1(t,x) + |P_{\le k+9}\tilde{\epsilon}(t,x)|^2.$$

The support of $\psi''(x)$, the fact that $\lambda(t) \le T^{1/100}/\eta_1$, and (1-8) imply that

$$\frac{\eta_1^2}{R^2}\int \chi''\Big(\frac{\eta_1 x}{R}\Big)\widetilde{Q}(x)^2\,dx \lesssim \frac{\eta_1^2}{R^2}\frac{1}{T^{11}} \lesssim \frac{1}{\lambda(t)^2}\frac{1}{T^{11}}. \tag{6-15}$$

Also, since Q and all its derivatives are rapidly decreasing and $\lambda(t) \ge 1/\eta_1$,

$$\|P_{\ge k+9}\widetilde{Q}(x)\|^2_{L^2} \lesssim 2^{-30k}. \tag{6-16}$$

Therefore, since $R = T^{1/25}$ and $\lambda(t) \le T^{1/100}/\eta_1$, (6-15), (6-16), and the Cauchy–Schwarz inequality imply

$$\frac{\eta_1^2}{R^2}\int \chi''\Big(\frac{\eta_1 x}{R}\Big)|P_{\le k+9}u(t,x)|^2\,dx \lesssim \frac{1}{\lambda(t)^2}\frac{1}{T^{11}} + \frac{1}{\lambda(t)^2}\frac{1}{R}\|\epsilon\|^2_{L^2}.$$

Next, letting $\epsilon_{1x} + i\epsilon_{2x} = \partial_x \epsilon$, write the decomposition

$$\begin{aligned}
&2\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\leq k+9}u_x|^2\,dx - \frac{2}{3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\leq k+9}u|^6\,dx \\
&= \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\bigg(\frac{1}{2}P_{\leq k+9}Q_x\Big(\frac{x}{\lambda(t)}\Big)^2 - \frac{1}{6}P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^6\bigg)\,dx \\
&\quad + \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\bigg(P_{\leq k+9}Q_x\Big(\frac{x}{\lambda(t)}\Big)P_{\leq k+9}\epsilon_{1x}\Big(t,\frac{x}{\lambda(t)}\Big) - P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^5 P_{\leq k+9}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\bigg)\,dx \\
&\quad + \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\bigg(\frac{1}{2}\Big(P_{\leq k+9}\epsilon_{1x}\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2 - \frac{5}{2}P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^4\Big(P_{\leq k+9}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2\bigg)\,dx \\
&\quad + \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\bigg(\frac{1}{2}\Big(P_{\leq k+9}\epsilon_{2x}\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2 - \frac{1}{2}P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^4\Big(P_{\leq k+9}\epsilon_2\Big(t,\frac{x}{\lambda(t)}\Big)\Big)\bigg)^2\,dx \\
&\quad - \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\bigg(\frac{10}{3}\Big(P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)\Big)^3\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^3 \\
&\qquad\qquad + \frac{5}{2}\Big(P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)\Big)^2\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^4 \\
&\qquad\qquad + \Big(P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)\Big)\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^5 + \frac{1}{6}\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^6\bigg)\,dx. \qquad (6\text{-}17)
\end{aligned}$$

Remark. Due to the presence of derivatives in

$$2\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\leq k+9}u_x|^2\,dx - \frac{2}{3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\leq k+9}u|^6\,dx,$$

it is convenient to dispense with the $\widetilde{Q}(x)$ and $\tilde{\epsilon}(t,x)$ notation and return to the Q and ϵ notation. We understand that $P_{\leq k+9}Q(x/\lambda(t))$ denotes the frequency projection after rescaling, not a rescaled projection. A rescaled projection appears in (6-18).

For terms of order ϵ^3 and higher, it is not too important to pay attention to complex conjugates, since these terms will be estimated using Hölder's inequality.

First, using the fact that

$$\tfrac{1}{2}Q_x^2 - \tfrac{1}{6}Q^6 = \tfrac{1}{2}Q^2 - \tfrac{1}{3}Q^6$$

combined with the fact that $1/\eta_1 \leq \lambda \leq T^{1/100}/\eta_1$, $R = T^{1/25}$, and Q is smooth and rapidly decreasing,

$$\begin{aligned}
&\frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\bigg(\frac{1}{2}Q_x\Big(\frac{x}{\lambda(t)}\Big)^2 - \frac{1}{6}Q\Big(\frac{x}{\lambda(t)}\Big)^6\bigg)\,dx \\
&\qquad = \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\bigg(\frac{1}{2}Q\Big(\frac{x}{\lambda(t)}\Big)^2 - \frac{1}{3}Q\Big(\frac{x}{\lambda(t)}\Big)^6\bigg) \lesssim \frac{1}{\lambda(t)^2}\frac{1}{T^{11}}.
\end{aligned}$$

Also, since $\eta_1^{-2}T = 2^{3k}$ and Q and its derivatives are smooth and rapidly decreasing, $\lambda(t) \geq 1/\eta_1$ and Bernstein's inequality implies that

$$\frac{2}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)Q_x\Big(\frac{x}{\lambda(t)}\Big)^2\,dx - \frac{2}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)P_{\leq k+9}Q_x\Big(\frac{x}{\lambda(t)}\Big)^2\,dx \lesssim \frac{1}{\lambda(t)^2}\frac{1}{T^{11}},$$

and

$$\frac{2}{3\lambda(t)^3}\int\psi^2\Big(\frac{\eta_1 x}{R}\Big)Q\Big(\frac{x}{\lambda(t)}\Big)^6\,dx-\frac{2}{3\lambda(t)^3}\int\psi^2\Big(\frac{\eta_1 x}{R}\Big)P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^6\,dx\lesssim\frac{1}{\lambda(t)^2}\frac{1}{T^{11}}.$$

Therefore,

$$\begin{aligned}
&2\int\psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\leq k+9}u_x|^2\,dx-\frac{2}{3}\int\psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\leq k+9}u|^6\,dx\\
&=\frac{4}{\lambda(t)^3}\int\psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(P_{\leq k+9}Q_x\Big(\frac{x}{\lambda(t)}\Big)P_{\leq k+9}\epsilon_{1x}\Big(t,\frac{x}{\lambda(t)}\Big)-P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^5P_{\leq k+9}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\Big)\,dx\\
&\quad+\frac{4}{\lambda(t)^3}\int\psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{1}{2}\Big(P_{\leq k+9}\epsilon_{1x}\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2-\frac{5}{2}P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^4\Big(P_{\leq k+9}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2\Big)\,dx\\
&\quad+\frac{4}{\lambda(t)^3}\int\psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{1}{2}\Big(P_{\leq k+9}\epsilon_{2x}\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2-\frac{1}{2}P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^4\Big(P_{\leq k+9}\epsilon_2\Big(t,\frac{x}{\lambda(t)}\Big)\Big)\Big)^2\,dx\\
&\quad-\frac{4}{\lambda(t)^3}\int\psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{10}{3}\Big(P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)\Big)^3\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^3\\
&\qquad\qquad+\frac{5}{2}\Big(P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)\Big)^2\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^4\\
&\qquad\qquad+\Big(P_{\leq k+9}Q\Big(\frac{x}{\lambda(t)}\Big)\Big)\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^5+\frac{1}{6}\Big(P_{\leq k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^6\Big)\,dx\\
&\qquad\qquad+O\Big(\frac{1}{\lambda(t)^2}\frac{1}{T^{11}}\Big).
\end{aligned}$$

Integrating by parts,

$$\begin{aligned}
&\frac{4}{\lambda(t)^3}\int\psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(P_{\leq k+5}Q_x\Big(\frac{x}{\lambda(t)}\Big)P_{\leq k+5}\epsilon_{1x}\Big(t,\frac{x}{\lambda(t)}\Big)-P_{\leq k+5}Q\Big(\frac{x}{\lambda(t)}\Big)^5P_{\leq k+5}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)\,dx\\
&=-\frac{4}{\lambda(t)^3}\int\psi^2\Big(\frac{\eta_1 x}{R}\Big)(P_{\leq k+5}Q_{xx}+P_{\leq k+5}Q^5)\Big(\frac{x}{\lambda(t)}\Big)\cdot P_{\leq k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\,dx\\
&\quad-\frac{8\eta_1}{R\lambda(t)^2}\int\psi\Big(\frac{\eta_1 x}{R}\Big)\psi'\Big(\frac{\eta_1 x}{R}\Big)P_{\leq k+5}Q_x\Big(\frac{x}{\lambda(t)}\Big)P_{\leq k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\,dx\\
&\quad+\frac{4}{\lambda(t)^3}\int\psi\Big(\frac{\eta_1 x}{R}\Big)^2P_{\leq k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)[(P_{\leq k+5}Q)^5-P_{\leq k+5}Q^5]\Big(\frac{x}{\lambda(t)}\Big)\,dx.
\end{aligned}$$

Again using (1-8), $1/\eta_1\leq\lambda(t)\leq T^{1/100}/\eta_1$, and the support of $\psi'(x)$,

$$\frac{8\eta_1}{R\lambda(t)^2}\int\psi\Big(\frac{\eta_1 x}{R}\Big)\psi'\Big(\frac{\eta_1 x}{R}\Big)Q_x\Big(\frac{x}{\lambda(t)}\Big)P_{\leq k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\,dx\lesssim\frac{1}{T^6}\frac{1}{\lambda(t)^2}\|\epsilon\|_{L^2}.$$

Also, since Q and all its derivatives are rapidly decreasing, by Bernstein's inequality,

$$\frac{8\eta_1}{R\lambda(t)^2}\int\psi\Big(\frac{\eta_1 x}{R}\Big)\psi'\Big(\frac{\eta_1 x}{R}\Big)P_{\geq k+5}Q_x\Big(\frac{x}{\lambda(t)}\Big)P_{\leq k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\,dx\lesssim\frac{1}{T^6\lambda(t)^2}\|\epsilon\|_{L^2},$$

and

$$\frac{4}{\lambda(t)^3}\int\psi\Big(\frac{\eta_1 x}{R}\Big)^2P_{\leq k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)[(P_{\leq k+5}Q)^5-P_{\leq k+5}Q^5]\Big(\frac{x}{\lambda(t)}\Big)\,dx\lesssim\frac{1}{T^6\lambda(t)^2}\|\epsilon\|_{L^2}.$$

Meanwhile, by conservation of mass, (1-8), (1-10), the upper and lower bounds of $\lambda(t)$, and the fact that Q and all its derivatives are rapidly decreasing,

$$\begin{aligned}
&-\frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)P_{\le k+5}(Q_{xx}+Q^5)\Big(\frac{x}{\lambda(t)}\Big)\cdot P_{\le k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\,dx \\
&\quad= -\frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)(Q_{xx}+Q^5)\Big(\frac{x}{\lambda(t)}\Big)\cdot P_{\le k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\,dx + O\Big(\frac{1}{\lambda(t)^2T^6}\|\epsilon\|_{L^2}\Big) \\
&\quad= -\frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)Q\Big(\frac{x}{\lambda(t)}\Big)\cdot P_{\le k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\,dx + O\Big(\frac{1}{\lambda(t)^2T^6}\|\epsilon\|_{L^2}\Big) \\
&\quad= -\frac{4}{\lambda(t)^3}\int Q\Big(\frac{x}{\lambda(t)}\Big)\cdot P_{\le k+5}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\,dx + O\Big(\frac{1}{T^6\lambda(t)^2}\|\epsilon\|_{L^2}\Big) \\
&\quad= -\frac{4}{\lambda(t)^3}\int Q\Big(\frac{x}{\lambda(t)}\Big)\cdot \epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\,dx + O\Big(\frac{1}{T^6\lambda(t)^2}\|\epsilon\|_{L^2}\Big) = \frac{2}{\lambda(t)^2}\|\epsilon\|_{L^2}^2 + O\Big(\frac{1}{T^6\lambda(t)^2}\|\epsilon\|_{L^2}\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&2\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\le k+9}u_x|^2\,dx - \frac{2}{3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)|P_{\le k+9}u|^6\,dx \\
&= \frac{2}{\lambda(t)^2}\|\epsilon\|_{L^2}^2 \\
&\quad + \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{1}{2}\Big(P_{\le k+9}\epsilon_{1x}\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2 - \frac{5}{2}P_{\le k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^4\Big(P_{\le k+9}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2\Big)\,dx \\
&\quad + \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{1}{2}\Big(P_{\le k+9}\epsilon_{2x}\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2 - \frac{1}{2}P_{\le k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^4\Big(P_{\le k+9}\epsilon_2\Big(t,\frac{x}{\lambda(t)}\Big)\Big)\Big)^2\,dx \\
&\quad - \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{10}{3}\Big(P_{\le k+9}Q\Big(\frac{x}{\lambda(t)}\Big)\Big)^3\Big(P_{\le k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^3 \\
&\qquad\qquad + \frac{5}{2}\Big(P_{\le k+9}Q\Big(\frac{x}{\lambda(t)}\Big)\Big)^2\Big(P_{\le k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^4 \\
&\qquad\qquad + \Big(P_{\le k+9}Q\Big(\frac{x}{\lambda(t)}\Big)\Big)\Big(P_{\le k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^5 + \frac{1}{6}\Big(P_{\le k+9}\epsilon\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^6\Big)\,dx \\
&\qquad\qquad + O\Big(\frac{1}{\lambda(t)^2}\frac{1}{T^{11}}\Big) + O\Big(\frac{1}{R\lambda(t)^2}\|\epsilon\|_{L^2}^2\Big).
\end{aligned}$$

Next, by Bernstein's inequality, since $1/\eta_1 \le \lambda(t) \le T^{1/100}/\eta_1$,

$$\begin{aligned}
&\frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{1}{2}\Big(P_{\le k+9}\epsilon_{1x}\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2 - \frac{5}{2}P_{\le k+9}Q\Big(\frac{x}{\lambda(t)}\Big)^4\Big(P_{\le k+9}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2\Big)\,dx \\
&\quad= \frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{1}{2}\Big(P_{\le k+9}\epsilon_{1x}\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2 - \frac{5}{2}Q\Big(\frac{x}{\lambda(t)}\Big)^4\Big(P_{\le k+9}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2\Big)\,dx \\
&\qquad\qquad + O\Big(\frac{1}{\lambda(t)^2R}\|\epsilon\|_{L^2}^2\Big).
\end{aligned}$$

Taking $k(t) \in \mathbb{R}$ that satisfies $2^{k(t)} = \lambda(t)$ and rescaling,

$$\frac{4}{\lambda(t)^3}\int \psi^2\Big(\frac{\eta_1 x}{R}\Big)\Big(\frac{1}{2}\Big(P_{\le k+9}\epsilon_{1x}\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2-\frac{5}{2}Q\Big(\frac{x}{\lambda(t)}\Big)^4\Big(P_{\le k+9}\epsilon_1\Big(t,\frac{x}{\lambda(t)}\Big)\Big)^2\Big)\,dx$$
$$=\frac{4}{\lambda(t)^2}\int \psi^2\Big(\frac{\eta_1\lambda(t)x}{R}\Big)\Big(\frac{1}{2}(P_{\le k+9+k(t)}\epsilon_{1x}(t,x))^2-\frac{5}{2}Q(x)^4(P_{\le k+9+k(t)}\epsilon_1(t,x))^2\Big)\,dx.$$

Integrating by parts,

$$\frac{4}{\lambda(t)^2}\int \psi^2\Big(\frac{\eta_1\lambda(t)x}{R}\Big)\Big(\frac{1}{2}(P_{\le k+9+k(t)}\epsilon_1(t,x))_x^2-\frac{5}{2}Q(x)^4(P_{\le k+9+k(t)}\epsilon_1(t,x))^2\Big)\,dx$$
$$=\frac{2}{\lambda(t)^2}\Big\|\psi\Big(\frac{\eta_1\lambda(t)x}{R}\Big)(P_{\le k+9+k(t)}\epsilon_1(t,x))\Big\|_{\dot H^1}^2$$
$$-\frac{10}{\lambda(t)^2}\int \psi\Big(\frac{\eta_1\lambda(t)x}{R}\Big)^2 Q\Big(\frac{x}{\lambda(t)}\Big)^4(P_{\le k+9+k(t)}\epsilon_1(t,x))^2\,dx+O\Big(\frac{\eta_1^2}{R^2}\|\epsilon\|_{L^2}^2\Big)$$
$$=\frac{2}{\lambda(t)^2}(\mathcal{L}\tilde\epsilon,\tilde\epsilon)-\frac{2}{\lambda(t)^2}\|\tilde\epsilon\|_{L^2}^2+O\Big(\frac{1}{R\lambda(t)^2}\|\epsilon\|_{L^2}^2\Big), \tag{6-18}$$

where $\mathcal{L}$ is given in (3-12) and

$$\tilde\epsilon=\psi\Big(\frac{\eta_1\lambda(t)x}{R}\Big)(P_{\le k+9+k(t)}\epsilon_1(t,x)).$$

Remark. This $\tilde\epsilon$ is not the same as the $\tilde\epsilon$ in (6-4).

For a function $u \perp Q^3$ and $u \perp Q_x$, by the spectral properties of $\mathcal{L}$,

$$(\mathcal{L}u,u)_{L^2}-(u,u)_{L^2}\ge 0.$$

For a general $u \in L^2$,

$$u=a_1Q^3+a_2Q_x+u^\perp,$$

where $u^\perp \perp Q^3$ and $u^\perp \perp Q_x$, we have

$$(\mathcal{L}u,u)_{L^2}-(u,u)_{L^2}\ge -O(a_1^2)-O(a_2^2).$$

Since $\epsilon_1 \perp Q^3$ and $\epsilon_1 \perp Q_x$, by Bernstein's inequality and the fact that $1/\eta_1 \le \lambda(t) \le T^{1/100}/\eta_1$,

$$(\tilde\epsilon,Q^3)_{L^2}=(\epsilon_1,Q^3)_{L^2}-\Big(\Big(1-\psi\Big(\frac{x\eta_1\lambda(t)}{R}\Big)\Big)\epsilon_1,Q^3\Big)_{L^2}-\Big(\psi\Big(\frac{\eta_1\lambda(t)x}{R}\Big)P_{\ge k+9+k(t)}\epsilon_1,Q^3\Big)_{L^2}$$
$$\lesssim \frac{1}{R}\|\epsilon\|_{L^2}$$

and

$$(\tilde\epsilon,Q_x)_{L^2}=(\epsilon_1,Q_x)_{L^2}-\Big(\Big(1-\psi\Big(\frac{x\eta_1\lambda(t)}{R}\Big)\Big)\epsilon_1,Q_x\Big)_{L^2}-\Big(\psi\Big(\frac{\eta_1\lambda(t)x}{R}\Big)P_{\ge k+9+k(t)}\epsilon_1,Q_x\Big)_{L^2}$$
$$\lesssim \frac{1}{R}\|\epsilon\|_{L^2}.$$

Therefore, for some $0 \ll \delta < 1$ ($\delta = 1/100$ will do), since $|Q(x)|^3 \leq 3$,

$$\frac{2}{\lambda(t)^2}(\mathcal{L}\tilde{\epsilon}, \tilde{\epsilon}) - \frac{2}{\lambda(t)^2}\|\tilde{\epsilon}\|_{L^2}^2 + O\bigg(\frac{1}{R\lambda(t)^2}\|\epsilon\|_{L^2}^2\bigg) \geq \frac{\delta}{\lambda(t)^3}\bigg\|\psi\bigg(\frac{\eta_1 x}{R}\bigg)\bigg(P_{\leq k+9}\epsilon_{1x}\bigg(t, \frac{x}{\lambda(t)}\bigg)\bigg)\bigg\|_{L^2}^2 - O\bigg(\frac{1}{R}\frac{1}{\lambda(t)^2}\|\epsilon\|_{L^2}^2\bigg) - \frac{15\delta}{\lambda(t)^2}\|\epsilon_1\|_{L^2}^2.$$

Likewise, since $\epsilon \perp iQ^3$ and $\epsilon \perp iQ_x$,

$$\frac{4}{\lambda(t)^3}\int \psi^2\bigg(\frac{\eta_1 x}{R}\bigg)\bigg(\frac{1}{2}\bigg(P_{\leq k+9}\epsilon_{2x}\bigg(t, \frac{x}{\lambda(t)}\bigg)\bigg)^2 - \frac{1}{2}P_{\leq k+9}Q\bigg(\frac{x}{\lambda(t)}\bigg)^4\bigg(P_{\leq k+9}\epsilon_2\bigg(t, \frac{x}{\lambda(t)}\bigg)\bigg)^2\bigg)\,dx \geq \frac{\delta}{\lambda(t)^3}\bigg\|\psi\bigg(\frac{\eta_1 x}{R}\bigg)\bigg(P_{\leq k+9}\epsilon_{2x}\bigg(t, \frac{x}{\lambda(t)}\bigg)\bigg)\bigg\|_{L^2}^2 - O\bigg(\frac{1}{R}\frac{1}{\lambda(t)^2}\|\epsilon\|_{L^2}^2\bigg) - \frac{15\delta}{\lambda(t)^2}\|\epsilon_2\|_{L^2}^2.$$

Therefore,

$$\begin{aligned}
&2\int \psi^2\bigg(\frac{\eta_1 x}{R}\bigg)|P_{\leq k+9}u_x|^2\,dx - \frac{2}{3}\int \psi^2\bigg(\frac{\eta_1 x}{R}\bigg)|P_{\leq k+9}u|^6\,dx \\
&\geq \frac{3}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2 + \frac{\delta}{\lambda(t)^3}\bigg\|\psi\bigg(\frac{\eta_1 x}{R}\bigg)P_{\leq k+9}\epsilon_x\bigg(t, \frac{x}{\lambda(t)}\bigg)\bigg\|_{L^2}^2 \\
&\quad - \frac{4}{\lambda(t)^3}\int \psi^2\bigg(\frac{\eta_1 x}{R}\bigg)\bigg(\frac{10}{3}P_{\leq k+9}Q\bigg(\frac{x}{\lambda(t)}\bigg)^3 P_{\leq k+9}\epsilon\bigg(t, \frac{x}{\lambda(t)}\bigg)^3 \\
&\qquad\qquad + \frac{5}{2}P_{\leq k+9}Q\bigg(\frac{x}{\lambda(t)}\bigg)^2 P_{\leq k+9}\epsilon\bigg(t, \frac{x}{\lambda(t)}\bigg)^4\bigg)\,dx \\
&\quad + \frac{1}{\lambda(t)^3}\int\bigg(P_{\leq k+9}Q\bigg(\frac{x}{\lambda(t)}\bigg)P_{\leq k+9}\epsilon\bigg(t, \frac{x}{\lambda(t)}\bigg)^5 + \frac{1}{6}P_{\leq k+9}\epsilon\bigg(t, \frac{x}{\lambda(t)}\bigg)^6\bigg)\,dx \\
&\qquad\qquad - O\bigg(\frac{1}{\lambda(t)^2 T^{11}}\bigg) - O\bigg(\frac{1}{R\lambda(t)^2}\|\epsilon\|_{L^2}^2\bigg).
\end{aligned}$$

Now, by the fundamental theorem of calculus and the product rule, for any $x \in \mathbb{R}$,

$$\begin{aligned}
\frac{1}{\lambda(t)}\psi\bigg(\frac{\eta_1 x}{R}\bigg)\bigg|P_{\leq k+9}\epsilon\bigg(t, \frac{x}{\lambda(t)}\bigg)\bigg|^2 &\lesssim \frac{1}{\lambda(t)}\int\bigg|\partial_x\bigg(\psi\bigg(\frac{\eta_1 x}{R}\bigg)\bigg|P_{\leq k+9}\epsilon\bigg(t, \frac{x}{\lambda(t)}\bigg)\bigg|^2\bigg)\bigg|\,dx \\
&\lesssim \frac{1}{\lambda(t)^{3/2}}\|\epsilon\|_{L^2(\mathbb{R})}\bigg\|\psi\bigg(\frac{\eta_1 x}{R}\bigg)P_{\leq k+9}\epsilon_x\bigg(t, \frac{x}{\lambda(t)}\bigg)\bigg\|_{L^2} + \frac{\eta_1}{R}\|\epsilon\|_{L^2}^2.
\end{aligned}$$

Therefore, by Hölder's inequality, the fact that $\|\epsilon\|_{L^2} \leq \eta_*$, the fact that $1/\eta_1 \leq \lambda(t) \leq T^{1/100}/\eta_1$, and $R = T^{1/25}$,

$$\begin{aligned}
\frac{1}{\lambda(t)^3}\int \psi\bigg(\frac{\eta_1 x}{R}\bigg)^2\bigg|P_{\leq k+9}\epsilon\bigg(t, \frac{x}{\lambda(t)}\bigg)\bigg|^6\,dx &\lesssim \frac{1}{\lambda(t)^3}\bigg\|\psi\bigg(\frac{\eta_1 x}{R}\bigg)P_{\leq k+9}\epsilon_x\bigg(t, \frac{x}{\lambda(t)}\bigg)\bigg\|_{L^2}^2\|\epsilon\|_{L^2}^4 + \frac{\eta_1^2}{R^2}\|\epsilon\|_{L^2}^6 \\
&\lesssim \frac{\eta_*^4}{\lambda(t)^3}\bigg\|\psi\bigg(\frac{\eta_1 x}{R}\bigg)P_{\leq k+9}\epsilon_x\bigg(t, \frac{x}{\lambda(t)}\bigg)\bigg\|_{L^2}^2 + \frac{\eta_*^4}{R\lambda(t)^2}\|\epsilon\|_{L^2}^2.
\end{aligned}$$

Next, by Hölder's inequality and the Cauchy–Schwarz inequality, for $j = 3, 4, 5$,

$$\begin{aligned}\frac{1}{\lambda(t)^3}\int \psi\left(\frac{\eta_1 x}{R}\right)^2 \left|P_{\le k+9}\epsilon\left(t, \frac{x}{\lambda(t)}\right)\right|^j Q\left(\frac{x}{\lambda(t)}\right)^{6-j} dx \\ \lesssim \frac{1}{\lambda(t)^{(6-j)/2}}\left(\frac{1}{\lambda(t)^3}\int \psi\left(\frac{\eta_1 x}{R}\right)^2 \left|P_{\le k+9}\epsilon\left(t, \frac{x}{\lambda(t)}\right)\right|^6 dx\right)^{(j-2)/4} \|\epsilon\|_{L^2}^{(6-j)/2} \\ \lesssim \frac{\eta_*}{\lambda(t)^2}\|\epsilon\|_{L^2}^2 + \frac{\eta_*}{\lambda(t)^3}\left\|\psi\left(\frac{\eta_1 x}{R}\right) P_{\le k+9}\epsilon_x\left(t, \frac{x}{\lambda(t)}\right)\right\|_{L^2}^2.\end{aligned}$$

Therefore, for $\eta_* \ll \delta$ sufficiently small and T sufficiently large,

$$2\int \psi^2\left(\frac{\eta_1 x}{R}\right)|P_{\le k+9}u_x|^2\,dx - \frac{2}{3}\int \psi^2\left(\frac{\eta_1 x}{R}\right)|P_{\le k+9}u|^6\,dx \ge \frac{1}{\lambda(t)^2}\|\epsilon\|_{L^2}^2 - O\left(\frac{1}{\lambda(t)^2 T^{11}}\right). \tag{6-19}$$

Plugging (6-19) into (6-14), integrating in time, and using the fact that $2^{3k} = \eta_1^{-2} T$ for $T(\eta_1)$ sufficiently large, the term

$$O\left(\frac{2^{2k} T^{1/20}}{T}\int_a^b \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}\,dt\right)$$

can be absorbed into the integral of the first term on the right-hand side of (6-19). Since

$$\int_J \lambda(t)^{-2}\,dt = T,$$

the proof of Theorem 18 is complete. □

Since both the left and right-hand sides of (6-2) are scale invariant, the same argument also holds for an interval J where

$$A \le \lambda(t) \le A T^{1/100} \tag{6-20}$$

for any $A > 0$.

Corollary 19. *Let $J = [a, b]$ be an interval where* (6-1) *holds for some T sufficiently large and* (6-20) *also holds. Then* (6-2) *holds.*

7. An L_s^p bound on $\|\epsilon(s)\|_{L^2}$ when $p > 1$

Transitioning to s variables, under the change of variables (3-10), Theorem 18 and Corollary 19 imply that if $[a, a+T] \subset [0, \infty)$ is an interval on which

$$\frac{\sup_{s\in[a,a+T]}\lambda(s)}{\inf_{s\in[a,a+T]}\lambda(s)} \le T^{1/100},$$

then

$$\int_a^{a+T} \|\epsilon(s)\|_{L^2}^2\,ds \le 3\big(\epsilon(a), \tfrac{1}{2}Q + xQ_x\big)_{L^2} - 3\big(\epsilon(a+T), \tfrac{1}{2}Q + xQ_x\big)_{L^2} + O\left(\frac{1}{T^9}\right).$$

Theorem 18 implies good L_s^p integrability bounds on $\|\epsilon(s)\|_{L^2}$ under (2-8), which is equivalent to

$$\sup_{s\in[0,\infty)} \|\epsilon(s)\|_{L^2} \le \eta_*.$$

Theorem 20. *Let u be a symmetric solution to* (1-1) *that satisfies $\|u\|_{L^2} = \|Q\|_{L^2}$, and suppose*

$$\sup_{s \in [0,\infty)} \|\epsilon(s)\|_{L^2} \leq \eta_* \tag{7-1}$$

and $\|\epsilon(0)\|_{L^2} = \eta_*$. Then

$$\int_0^\infty \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim \eta_*, \tag{7-2}$$

with implicit constant independent of η_* when $\eta_* \ll 1$ is sufficiently small.

Furthermore, for any $j \in \mathbb{Z}_{\geq 0}$, let

$$s_j = \inf\{s \in [0,\infty) : \|\epsilon(s)\|_{L^2} = 2^{-j}\eta_*\}.$$

By definition, $s_0 = 0$, and the continuity of $\|\epsilon(s)\|_{L^2}$ combined with Theorem 8 implies that such an s_j exists for any $j > 0$. Then,

$$\int_{s_j}^\infty \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim 2^{-j}\eta_* \tag{7-3}$$

for each j, with implicit constant independent of η_* and $j \geq 0$.

Proof. Set $T_* = 1/\eta_*$ and suppose that T_* is large enough that Theorem 18 holds. Then by (3-15) and (7-1), for any $s' \geq 0$,

$$\Big| \sup_{s \in [s', s'+T_*]} \ln \lambda(s) - \inf_{s \in [s', s'+T_*]} \ln \lambda(s) \Big| \lesssim 1, \tag{7-4}$$

with implicit constant independent of $s' \geq 0$. Let J be the largest dyadic integer that satisfies

$$J = 2^{j_*} \leq -\ln \eta_*^{1/2}.$$

By (7-4) and the triangle inequality,

$$\Big| \sup_{s \in [s', s'+JT_*]} \ln \lambda(s) - \inf_{s \in [s', s'+JT_*]} \ln \lambda(s) \Big| \lesssim J,$$

and therefore

$$\frac{\sup_{s \in [s', s'+3JT^*]} \lambda(s)}{\inf_{s \in [s', s'+3JT^*]} \lambda(s)} \lesssim T_*^{1/100}. \tag{7-5}$$

Therefore, Theorem 18 may be utilized on $[s', s'+JT_*]$. In particular, for any $s' \geq 0$,

$$\int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim \|\epsilon(s')\|_{L^2} + \|\epsilon(s'+JT_*)\|_{L^2} + O\Big(\frac{1}{J^9 T_*^9}\Big). \tag{7-6}$$

In fact, if $s' > JT_*$, then by (7-5),

$$\int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim \inf_{s \in [s'-JT_*, s']} \|\epsilon(s)\|_{L^2} + \inf_{s \in [s'+JT_*, s'+2JT_*]} \|\epsilon(s)\|_{L^2} + O\Big(\frac{1}{J^9 T_*^9}\Big). \tag{7-7}$$

In particular, for a fixed $s' \geq 0$,

$$\sup_{a>0} \int_{s'+aJT_*}^{s'+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 \lesssim \frac{1}{J^{1/2}T_*^{1/2}} \left(\sup_{a\geq 0} \int_{s'+aJT_*}^{s'+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \right)^{1/2} + O\left(\frac{1}{J^9 T_*^9}\right). \quad (7\text{-}8)$$

Meanwhile, when $a = 0$,

$$\int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2}^2 \lesssim \|\epsilon(s')\|_{L^2} + \frac{1}{J^{1/2}T_*^{1/2}} \left(\sup_{a\geq 0} \int_{s'+aJT_*}^{s'+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \right)^{1/2} + O\left(\frac{1}{J^9 T_*^9}\right). \quad (7\text{-}9)$$

Therefore, taking $s' = s_{j_*}$,

$$\sup_{a\geq 0} \int_{s_{j_*}+aJT_*}^{s_{j_*}+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim 2^{-j_*}\eta_* + O(2^{-9j_*}\eta_*^9). \quad (7\text{-}10)$$

Then by the triangle inequality,

$$\sup_{s'\geq s_{j_*}} \int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim 2^{-j_*}\eta_*,$$

and by Hölder's inequality,

$$\sup_{s'\geq s_{j_*}} \int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2} \, ds \lesssim 1.$$

In fact, arguing by induction, there exists a constant $C < \infty$ such that

$$\sup_{s'\geq s_{nj_*}} \int_{s'}^{s'+J^nT_*} \|\epsilon(s)\|_{L^2} \, ds \leq C \quad (7\text{-}11)$$

for some $n > 0$ implies that

$$\sup_{s'\geq s_{(n+1)j_*}} \int_{s'}^{s'+J^{n+1}T_*} \|\epsilon(s)\|_{L^2}^2 \, ds \leq CJ^{-(n+1)}T_*^{-1}, \quad (7\text{-}12)$$

and by Hölder's inequality,

$$\sup_{s'\geq s_{(n+1)j_*}} \int_{s'}^{s'+J^{n+1}T_*} \|\epsilon(s)\|_{L^2} \, ds \leq C^{1/2}.$$

Therefore, (7-11) holds for any integer $n > 0$.

Now take any $j \in \mathbb{Z}$ and suppose $nj_* < j \leq (n+1)j_*$. Then by (7-11),

$$\sup_{a\geq 0} \int_{s_j+aJ^{n+1}T_*}^{s_j+(a+1)J^{n+1}T_*} \|\epsilon(s)\|_{L^2} \, ds \lesssim J.$$

Therefore, as in (7-10),

$$\sup_{a\geq 0} \int_{s_j+aJ^{n+1}T_*}^{s_j+(a+1)J^{n+1}T_*} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim 2^{-j}\eta_*,$$

and therefore by Hölder's inequality, for any $s' \geq s_j$,

$$\sup_{s' \geq s_j} \int_{s'}^{s' + 2^j T_*} \|\epsilon(s)\|_{L^2} \, ds \lesssim 1,$$

with bound independent of j. Then by the triangle inequality, (7-5) holds for the interval $[s', s' + 3 \cdot 2^j J T_*]$, and by (7-6)–(7-9),

$$\int_{s_j}^{s_j + 2^j J T_*} \|\epsilon(s)\|_{L^2}^2 \lesssim 2^{-j} \eta_*, \tag{7-13}$$

and therefore, by the mean value theorem,

$$\inf_{s \in [s_j, s_j + 2^j J T_*]} \|\epsilon(s)\|_{L^2} \lesssim 2^{-j} \eta_* J^{-1/2},$$

which implies

$$s_{j+1} \in [s_j, s_j + 2^j J T_*].$$

Therefore, by (7-13) and Hölder's inequality,

$$\int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim 2^{-j} \eta_* \quad \text{and} \quad \int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2} \, ds \lesssim 1, \tag{7-14}$$

with constant independent of j. Summing in j gives (7-2) and (7-3). □

Now, by (2-18),

$$\|\epsilon(s')\|_{L^2} \sim \|\epsilon(s)\|_{L^2}$$

for any $s' \in [s, s+1]$, so (7-2) implies

$$\lim_{s \to \infty} \|\epsilon(s)\|_{L^2} = 0.$$

Next, by definition of s_j, (7-14) implies

$$\int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2} \, ds \lesssim 1,$$

and, for any $1 < p < \infty$,

$$\int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2}^p \, ds \lesssim \eta_*^{p-1} 2^{-j(p-1)}, \tag{7-15}$$

which implies that $\|\epsilon(s)\|_{L^2}$ belongs to L_s^p for any $p > 1$ but not to L_s^1.

Comparing (7-15) to the pseudoconformal transformation of the soliton, (1-11), for $0 < t < 1$,

$$\lambda(t) \sim t \quad \text{and} \quad \|\epsilon(t)\|_{L^2} \sim t,$$

so

$$\int_0^1 \|\epsilon(t)\|_{L^2} \lambda(t)^{-2} \, dt = \infty,$$

but for any $p > 1$,

$$\int_0^1 \|\epsilon(t)\|_{L^2}^p \lambda(t)^{-2} \, dt < \infty.$$

For the soliton, $\epsilon(s) \equiv 0$ for any $s \in \mathbb{R}$, so obviously $\epsilon \in L_s^p$ for $1 \leq p \leq \infty$.

8. Monotonicity of λ

Next, using a virial identity from [Merle and Raphael 2005], it is possible to show that $\lambda(s)$ is an approximately monotone decreasing function.

Theorem 21. *For any $s \geq 0$, let*

$$\tilde{\lambda}(s) = \inf_{\tau \in [0,s]} \lambda(\tau).$$

Then for any $s \geq 0$,

$$1 \leq \frac{\lambda(s)}{\tilde{\lambda}(s)} \leq 3. \tag{8-1}$$

Proof. Suppose there exist $0 \leq s_- \leq s_+ < \infty$ satisfying

$$\frac{\lambda(s_+)}{\lambda(s_-)} = e. \tag{8-2}$$

Then we can show that u is a soliton solution to (1-1), which is a contradiction since $\lambda(s)$ is constant in that case.

The proof that (8-2) implies that u is a soliton uses a virial identity from [Merle and Raphael 2005]. Using (3-11), compute

$$\frac{d}{ds}(\epsilon, y^2 Q) + \frac{\lambda_s}{\lambda}\|yQ\|_{L^2}^2 + 4\big(\tfrac{1}{2}Q + yQ_y, \epsilon_2\big)_{L^2} \\ = O(|\gamma_s + 1|\|\epsilon\|_{L^2}) + O\bigg(\bigg|\frac{\lambda_s}{\lambda}\bigg|\|\epsilon\|_{L^2}\bigg) + O(\|\epsilon\|_{L^2}^2) + O(\|\epsilon\|_{L^2}\|\epsilon\|_{L^8}^4). \tag{8-3}$$

Indeed, by direct computation,

$$\partial_{xx}(x^2 Q) + Q^4(x^2 Q) - x^2 Q = 4\big(\tfrac{1}{2}Q + xQ_x\big).$$

Then by (3-15), (3-16), (7-2), and the fundamental theorem of calculus,

$$\|yQ\|_{L^2}^2 + 4\int_{s_-}^{s_+} \big(\epsilon_2, \tfrac{1}{2}Q + xQ_x\big)_{L^2} = O(\eta_*).$$

Therefore, there exists $s' \in [s_-, s_+]$ such that

$$\big(\epsilon_2, \tfrac{1}{2}Q + xQ_x\big)_{L^2} < 0. \tag{8-4}$$

Since $s' \geq 0$, there exists some $j \geq 0$ such that $s_j \leq s' + T_* < s_{j+1}$. Using the proof of Theorem 20, in particular (7-14),

$$\int_{s'}^{s_{j+1+J}} \bigg|\frac{\lambda_s}{\lambda}\bigg|\, ds \lesssim J. \tag{8-5}$$

Then by Theorem 18, (8-4) implies

$$\int_{s'}^{s_{j+1+J}} \|\epsilon(s)\|_{L^2}^2\, ds \lesssim 2^{-(j+1+J)}\eta_*,$$

and therefore by definition of s_{j+1+J},

$$\int_{s'}^{s_{j+1+J}} \|\epsilon(s)\|_{L^2}\,ds \lesssim 1. \tag{8-6}$$

Then, (8-6) implies that (8-5) holds on the interval $[s', s_{j+1+2J}]$, and arguing by induction, for any $k \geq 1$,

$$\int_{s'}^{s_{j+k}} \|\epsilon(s)\|_{L^2}^2\,ds \lesssim 2^{-j-k}\eta_*$$

and

$$\int_{s'}^{s_{j+k}} \|\epsilon(s)\|_{L^2}\,ds \lesssim 1,$$

with implicit constant independent of k. Taking $k \to \infty$,

$$\int_{s'}^{\infty} \|\epsilon(s)\|_{L^2}^2\,ds = 0,$$

which implies that $\epsilon(s) = 0$ for all $s \geq s'$. Therefore,

$$u_0 = \lambda^{1/2} Q(\lambda x) e^{i\gamma}$$

for some $\gamma \in \mathbb{R}$ and $\lambda > 0$, which proves that u is a soliton solution. □

9. Almost monotone $\lambda(t)$

The almost monotonicity of λ implies that when $\sup(I) = \infty$, u is equal to a soliton solution, and when $\sup(I) < \infty$, u is the pseudoconformal transformation of the soliton solution.

Theorem 22. *If u satisfies the conditions of Theorem 7, u blows up forward in time, and*

$$\sup(I) = \infty,$$

then u is equal to a soliton solution.

Proof. For any integer $k \geq 0$, let

$$I(k) = \{s \geq 0 : 2^{-k+2} \leq \tilde{\lambda}(s) \leq 2^{-k+3}\}. \tag{9-1}$$

Then by (8-1),

$$2^{-k} \leq \lambda(s) \leq 2^{-k+3} \tag{9-2}$$

for all $s \in I(k)$. By (3-10), the fact that $\sup(I) = \infty$ implies that

$$\sum 2^{-2k} |I(k)| = \infty.$$

Therefore, there exists a sequence $k_n \nearrow \infty$ such that

$$|I(k_n)| 2^{-2k_n} \geq \frac{1}{k_n^2}$$

and such that $|I(k_n)| \geq |I(k)|$ for all $k \leq k_n$.

Lemma 23. *For n sufficiently large, there exists $s_n \in I(k_n)$ such that*

$$\|\epsilon(s_n)\|_{L^2} \lesssim k_n^2 2^{-2k_n}.$$

Proof. Let $I(k_n) = [a_n, b_n]$. By Theorem 18, for n sufficiently large,

$$\int_{I(k_n)} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim \eta_* + 2^{-18k_n} k_n^{18} \lesssim \eta_*.$$

Then, using the virial identity in (8-3),

$$\int_{a_n}^{(3a_n+b_n)/4} \left(\epsilon_2, \tfrac{1}{2}Q + xQ_x\right)_{L^2} ds = O(\eta_*) + O(1).$$

Therefore, by the mean value theorem, there exists $s_n^- \in \left[a_n, \frac{1}{4}(3a_n + b_n)\right]$ such that

$$\left|\left(\epsilon_2(s_n^-), \tfrac{1}{2}Q + xQ_x\right)_{L^2}\right| \lesssim \frac{1}{|I(k_n)|}. \tag{9-3}$$

By a similar calculation, there exists $s_n^+ \in \left[\frac{1}{4}(a_n + 3b_n), b_n\right]$ such that

$$\left|\left(\epsilon_2(s_n^+), \tfrac{1}{2}Q + xQ_x\right)_{L^2}\right| \lesssim \frac{1}{|I(k_n)|}. \tag{9-4}$$

Plugging (9-3) and (9-4) into Theorem 18,

$$\int_{s_n^-}^{s_n^+} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim \frac{1}{|I(k_n)|}.$$

Then by the mean value theorem there exists $s_n \in [s_n^-, s_n^+]$ such that

$$\|\epsilon(s_n)\|_{L^2}^2 \lesssim \frac{1}{|I(k_n)|^2}.$$

Since $|I(k_n)| \geq 2^{2k_n} k_n^{-2}$, the proof of Lemma 23 is complete. □

Returning to the proof of Theorem 22, let m be the smallest integer such that

$$\frac{2^{2k_n}}{k_n^2} 2^m \geq |I(k_n)|. \tag{9-5}$$

Since $|I(k)| \leq |I(k_n)|$ for all $0 \leq k \leq k_n$, (9-5) implies that

$$|s_n| \leq 2^{2k_n+m+1}.$$

Let r_n be the smallest integer that satisfies

$$2^{(2k_n+m+1)/3} 2^{k_n} \frac{1}{\eta_1} \leq 2^{r_n}.$$

Since $\lambda(s) \geq 2^{-k_n}$ for all $s \in [0, s_n]$, setting $t_n = s^{-1}(s_n)$, rescaling so that $\lambda(t) \geq 1/\eta_1$ on $[0, 2^{2k_n}\eta_1^{-2} t_n]$, applying Theorem 18, then rescaling back,

$$\| P_{\geq r_n} u \|_{U_\Delta^2([0,t_n]\times\mathbb{R})} \lesssim \eta_*.$$

Arguing by induction on frequency and using (4-41) and the preceding computations,

$$\| P_{\geq r_n + k_n/4 + m/4} u \|_{U_\Delta^2([0,t_n]\times\mathbb{R})} \lesssim k_n^2 2^{-2k_n} 2^{-m}. \tag{9-6}$$

Then using the computations in (5-1)–(5-6),

$$E(P_{\leq r_n + k_n/4 + m/4} u(t_n)) \lesssim (k_n^2 2^{-2k_n} 2^{-m} 2^{r_n + k_n/4 + m/4})^2 \sim (k_n^2 2^{-k_n/12 - 5m/12} \eta_1^{-1})^2.$$

Next, following the computations in the proof of Theorem 16 and using (9-6),

$$\sup_{t\in[0,t_n]} E(P_{\leq r_n + k_n/4 + m/4} u(t)) \lesssim (k_n^2 2^{-k_n/12 - 5m/12} \eta_1^{-1})^2.$$

Since $m \geq 0$ for any n, taking $n \to \infty$ implies that $E(u_0) = 0$. Then by the Gagliardo–Nirenberg inequality, u_0 is a soliton. □

It only remains to show that when $\sup(I) < \infty$, u is a pseudoconformal transformation of the soliton. If one could show that the energy of u_0 is finite, then this fact would follow directly from the result of [Merle 1993]. Similarly, if one could generalize the result of that paper to data that need not have finite energy, then the proof would also be complete.

We do not quite prove this fact. Instead, suppose without loss of generality that $\sup(I) = 0$ and

$$\sup_{-1<t<0} \|\epsilon(t)\|_{L^2} \leq \eta_*.$$

Then write the decomposition

$$u(t,x) = \frac{e^{-i\gamma(t)}}{\lambda(t)^{1/2}} Q\left(\frac{x}{\lambda(t)}\right) + \frac{e^{-i\gamma(t)}}{\lambda(t)^{1/2}} \epsilon\left(t, \frac{x}{\lambda(t)}\right)$$

and apply the pseudoconformal transformation to $u(t,x)$. For $-\infty < t < -1$, let

$$\begin{aligned} v(t,x) &= \frac{1}{t^{1/2}} \overline{u\left(\frac{1}{t}, \frac{x}{t}\right)} e^{ix^2/4t} \\ &= \frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)}}{\lambda(1/t)^{1/2}} Q\left(\frac{x}{t\lambda(1/t)}\right) e^{ix^2/4t} + \frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)}}{\lambda(1/t)^{1/2}} \overline{\epsilon\left(\frac{1}{t}, \frac{x}{t\lambda(1/t)}\right)} e^{ix^2/4t}. \end{aligned}$$

Since the L^2 norm is preserved by the pseudoconformal transformation,

$$\lim_{t \searrow -\infty} \left\| \frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)}}{\lambda(1/t)^{1/2}} \overline{\epsilon\left(\frac{1}{t}, \frac{x}{t\lambda(1/t)}\right)} e^{ix^2/4t} \right\|_{L^2} = 0$$

and

$$\sup_{-\infty<t<-1} \left\| \frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)}}{\lambda(1/t)^{1/2}} \overline{\epsilon\left(\frac{1}{t}, \frac{x}{t\lambda(1/t)}\right)} e^{ix^2/4t} \right\|_{L^2} \leq \eta_*.$$

Since

$$\frac{1}{t^{1/2}}\frac{e^{i\gamma(1/t)}}{\lambda(1/t)^{1/2}}Q\biggl(\frac{x}{t\lambda(1/t)}\biggr)$$

is in the form

$$\frac{e^{i\tilde{\gamma}(t)}}{\tilde{\lambda}(t)^{1/2}}Q\biggl(\frac{x}{\tilde{\lambda}(t)}\biggr),$$

it only remains to estimate

$$\biggl\|\frac{1}{t^{1/2}}\frac{e^{i\gamma(1/t)}}{\lambda(1/t)^{1/2}}Q\biggl(\frac{x}{t\lambda(1/t)}\biggr)(e^{ix^2/4t}-1)\biggr\|_{L^2}.$$

Once again take (9-1). As in (9-2), for any $k \geq 0$, we have $\lambda(s) \sim 2^{-k}$ for all $s \in I(k)$. Furthermore, by (3-15), $\|\epsilon(t)\|_{L^2} \to 0$ as $t \nearrow 0$ implies that there exists a sequence $c_k \nearrow \infty$ such that

$$|I(k)| \geq c_k \quad \text{for all } k \geq 0.$$

Then by (3-10), there exists $r(t) \searrow 0$ as $t \nearrow 0$ such that

$$\lambda(t) \leq t^{1/2} r(t), \quad \text{so } \lambda(1/t) \leq t^{-1/2} r(1/t). \tag{9-7}$$

Therefore, since Q is rapidly decreasing,

$$\lim_{t \searrow -\infty}\biggl\|\frac{1}{t^{1/2}\lambda(1/t)^{1/2}}Q\biggl(\frac{x}{t\lambda(1/t)}\biggr)\frac{x^2}{4t}\biggr\|_{L^2} = 0 \tag{9-8}$$

as well as

$$\lim_{t \searrow -\infty}\biggl\|\frac{1}{t^{1/2}\lambda(1/t)^{1/2}}Q\biggl(\frac{x}{t\lambda(1/t)}\biggr)(e^{ix^2/4t}-1)\biggr\|_{L^2} = 0.$$

Therefore, by time reversal symmetry, v satisfies the conditions of Theorem 7, and v is a solution that blows up backward in time at $\inf(I) = -\infty$, so therefore, by Theorem 22, v must be a soliton. In particular,

$$v(t,x) = e^{i\lambda^2 t}e^{i\theta}\lambda^{1/2}Q(\lambda x) = \frac{1}{t^{1/2}}\overline{u\biggl(\frac{1}{t},\frac{x}{t}\biggr)}e^{ix^2/4t}.$$

Doing some algebra,

$$\overline{u\biggl(\frac{1}{t},\frac{x}{t}\biggr)} = e^{i\lambda^2 t}e^{i\theta}e^{-ix^2/4t}t^{1/2}\lambda^{1/2}Q(\lambda x),$$

so

$$u(t,x) = e^{-i\lambda^2/t}e^{-i\theta}e^{ix^2/4t}\frac{1}{t^{1/2}}\lambda^{1/2}Q\biggl(\frac{\lambda x}{t}\biggr).$$

This is clearly the pseudoconformal transformation of a soliton. This finally completes the proof of Theorem 7.

10. A nonsymmetric solution

When there is no symmetry assumption on u, there is no preferred origin, either in space or in frequency. As a result, two additional group actions on a solution u must be accounted for, translation in space:

$$u(t,x) \mapsto u(t,x-x_0), \quad x_0 \in \mathbb{R}, \tag{10-1}$$

and the Galilean symmetry:

$$e^{-it\xi_0^2} e^{ix\xi_0} u(t,x-2t\xi_0), \quad \xi_0 \in \mathbb{R}. \tag{10-2}$$

This gives a four parameter family of soliton solutions to (1-1), given by (1-14). Making the pseudoconformal transformation of (1-14) gives a solution in the form of (1-15).

In this section we prove Theorem 5, that the only nonsymmetric blowup solutions to (1-1) with mass $\|u_0\|_{L^2}^2 = \|Q\|_{L^2}^2$ belong to the family of solitons and pseudoconformal transformation of a soliton. To prove this, we will go through the proof of Theorem 4 in Sections 2–9, section by section, generalizing each step to the nonsymmetric case. There are several steps for which the argument in the symmetric case has an easy generalization to the nonsymmetric case, after accounting for the additional group actions (10-1) and (10-2). There are other steps for which the nonsymmetric case will require substantially more work.

10.1. ***Reductions of a nonsymmetric blowup solution.*** Using the same arguments showing that Theorem 4 may be reduced to Theorem 7, Theorem 5 may be reduced to:

Theorem 24. *Let $0 < \eta_* \ll 1$ be a small fixed constant to be defined later. If u is a solution to* (1-1) *on the maximal interval of existence $I \subset \mathbb{R}$, $\|u_0\|_{L^2} = \|Q\|_{L^2}$, u blows up forward in time, and*

$$\sup_{t \in [0, \sup(I))} \inf_{\lambda,\gamma,\xi_0,x_0} \|e^{i\gamma} e^{ix\xi_0} \lambda^{1/2} u(t, \lambda x + x_0) - Q(x)\|_{L^2} \le \eta_*, \tag{10-3}$$

then u is a soliton solution of the form (1-14) *or the pseudoconformal transformation of a soliton of the form* (1-15).

Reducing Theorem 5 to Theorem 24 requires the following generalization of Theorem 8, which was proved in [Dodson 2021, Theorem 2].

Theorem 25. *Assume that u is a solution to* (1-1) *with $\|u_0\|_{L^2} = \|Q\|_{L^2}$ that does not scatter forward in time. Let $(T^-(u), T^+(u))$ be its lifespan ($T^-(u)$ could be $-\infty$ and $T^+(u)$ could be $+\infty$). Then there exists a sequence $t_n \nearrow T^+(u)$ and a family of parameters $\lambda_n > 0$, $\xi_n \in \mathbb{R}$, $x_n \in \mathbb{R}$, and $\gamma_n \in \mathbb{R}$ such that*

$$\lambda_n^{1/2} e^{ix\xi_n} e^{i\gamma_n} u(t_n, \lambda_n x + x_n) \to Q \quad \text{in } L^2.$$

Lemma 6 can be generalized to the nonsymmetric case, proving that $\|e^{i\gamma} e^{ix\xi_0} \lambda^{1/2} u_0(\lambda x + x_0) - Q\|_{L^2}$ attains its infimum on $\gamma \in \mathbb{R}$, $\xi_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}$, and $\lambda > 0$. Theorem 9 is also easily generalized to the nonsymmetric case, showing that the left-hand side of (10-3) is upper semicontinuous in time and continuous in time when small. Therefore, Theorem 5 is easily reduced to Theorem 24 using the same argument that reduced Theorem 4 to Theorem 7.

10.2. ***Decomposition of a nonsymmetric solution near Q.*** When a nonsymmetric u is close to a soliton, it is possible to make a decomposition of u, generalizing Theorem 10 to account for the additional group actions in (10-1) and (10-2).

Theorem 26. *Take $u \in L^2$. There exists $\alpha > 0$ sufficiently small such that if there exist $\lambda_0 > 0$, $\gamma_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}$, and $\xi_0 \in \mathbb{R}$ that satisfy*

$$\|e^{i\gamma_0} e^{ix\xi_0} \lambda_0^{1/2} u(\lambda_0 x + x_0) - Q(x)\|_{L^2} \leq \alpha,$$

then there exist unique $\lambda > 0$, $\gamma \in \mathbb{R}$, $\tilde{x} \in \mathbb{R}$, and $\xi \in \mathbb{R}$ that satisfy

$$(\epsilon, Q^3)_{L^2} = (\epsilon, iQ^3)_{L^2} = (\epsilon, Q_x)_{L^2} = (\epsilon, iQ_x)_{L^2} = 0,$$

where

$$\epsilon(x) = e^{i\gamma} e^{ix\xi} \lambda^{1/2} u(\lambda x + \tilde{x}) - Q. \tag{10-4}$$

Furthermore,

$$\|\epsilon\|_{L^2} + \left|\frac{\lambda}{\lambda_0} - 1\right| + |\gamma - \gamma_0 - \xi_0(\tilde{x} - x_0)| + \left|\xi - \frac{\lambda}{\lambda_0}\xi_0\right| + \left|\frac{\tilde{x} - x_0}{\lambda_0}\right| \lesssim \|e^{i\gamma_0} e^{ix\xi_0} \lambda_0^{1/2} u(\lambda_0 x + x_0) - Q\|_{L^2}.$$

Remark. Once again, since $e^{i\gamma}$ is 2π-periodic, the γ in (10-4) is unique up to translations by $2\pi k$ for some integer k.

Proof. By Hölder's inequality, if $\epsilon = e^{i\gamma_0} e^{ix\xi_0} \lambda_0^{1/2} u(\lambda_0 x + x_0) - Q(x)$, then

$$|(\epsilon, Q^3)_{L^2}| + |(\epsilon, Q_x)_{L^2}| + |(\epsilon, iQ^3)_{L^2}| + |(\epsilon, iQ_x)_{L^2}| \lesssim \|e^{i\gamma_0} e^{ix\xi_0} \lambda_0^{1/2} u(\lambda_0 x + x_0) - Q(x)\|_{L^2}.$$

As in the proof of Theorem 10,

$$(e^{i\gamma} e^{ix\xi} \lambda^{1/2} u(\lambda x + \tilde{x}) - Q(x), f)_{L^2} \tag{10-5}$$

is C^1 as a function of γ, λ, $\tilde{x}$, and ξ, when

$$f \in \{Q^3, iQ^3, Q_x, iQ_x\}.$$

Indeed, by Hölder's inequality and the L^2-invariance of the scaling symmetry,

$$\frac{\partial}{\partial \gamma}(e^{i\gamma} e^{ix\xi} \lambda^{1/2} u(\lambda x + \tilde{x}) - Q(x), f)_{L^2} = (ie^{i\gamma} e^{ix\xi} \lambda^{1/2} u(\lambda x + \tilde{x}), f)_{L^2} \lesssim \|u\|_{L^2} \|f\|_{L^2}.$$

Next,

$$\frac{\partial}{\partial \xi}(e^{i\gamma} e^{ix\xi} \lambda^{1/2} u(\lambda x + \tilde{x}) - Q(x), f)_{L^2} = (ixe^{i\gamma} e^{ix\xi} \lambda^{1/2} u(\lambda x + \tilde{x}), f)_{L^2} \lesssim \|u\|_{L^2} \|xf\|_{L^2}. \tag{10-6}$$

Since Q and all its derivatives are rapidly decreasing, $xf \in L^2$ and (10-6) is well defined.

Next, integrating by parts,

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),f)_{L^2}\\
&=\left(\frac{1}{2\lambda}e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})+xe^{i\gamma}e^{ix\xi}\lambda^{1/2}u_x(\lambda x+\tilde{x}),f\right)_{L^2}\\
&=\frac{1}{2\lambda}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x}),f)_{L^2}-\frac{1}{\lambda}\left(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x}),f\right)_{L^2}\\
&\qquad-\frac{\xi}{\lambda}(ie^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x}),xf)_{L^2}-\frac{1}{\lambda}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x}),xf_x)_{L^2}\\
&\lesssim\frac{1}{\lambda}\|u\|_{L^2}\|f\|_{L^2}+\frac{1}{\lambda}\|u\|_{L^2}\|xf_x\|_{L^2}+\frac{|\xi|}{\lambda}\|u\|_{L^2}\|xf\|_{L^2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{\partial}{\partial\tilde{x}}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),f)_{L^2}\\
&\qquad=(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u_x(\lambda x+\tilde{x}),f)_{L^2}\\
&\qquad=-\frac{1}{\lambda}(i\xi e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x}),f)_{L^2}-\frac{1}{\lambda}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x}),f_x)_{L^2}\\
&\qquad\lesssim\frac{1}{\lambda}\|u\|_{L^2}\|f\|_{L^2}+\frac{|\xi|}{\lambda}\|u\|_{L^2}\|f_x\|_{L^2}.
\end{aligned}$$

Similar calculations also prove uniform bounds on the Hessians of (10-5).

Suppose $\lambda_0=1,\ \gamma_0=0,\ x_0=0$, and $\xi_0=0$. Compute

$$\begin{aligned}
\frac{\partial}{\partial\lambda}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=\left(\tfrac{1}{2}Q+xQ_x,Q^3\right)_{L^2}=\tfrac{1}{4}\|Q\|_{L^4}^4,\\
\frac{\partial}{\partial\lambda}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),iQ^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=\left(\tfrac{1}{2}Q+xQ_x,iQ^3\right)_{L^2}=0,\\
\frac{\partial}{\partial\lambda}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=\left(\tfrac{1}{2}Q+xQ_x,Q_x\right)_{L^2}=0,\\
\frac{\partial}{\partial\lambda}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=\left(\tfrac{1}{2}Q+xQ_x,iQ_x\right)_{L^2}=0;
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial\gamma}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=(iQ,Q^3)_{L^2}=0,\\
\frac{\partial}{\partial\gamma}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),iQ^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=(iQ,iQ^3)_{L^2}=\|Q\|_{L^4}^4,\\
\frac{\partial}{\partial\gamma}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=(iQ,Q_x)_{L^2}=0,\\
\frac{\partial}{\partial\gamma}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=(iQ,iQ_x)_{L^2}=0;
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial\tilde{x}}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=(Q_x,Q^3)_{L^2}=0,\\
\frac{\partial}{\partial\tilde{x}}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),iQ^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}&=(Q_x,iQ^3)_{L^2}=0,
\end{aligned}$$

$$\frac{\partial}{\partial \tilde{x}}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}=(Q_x,Q_x)_{L^2}=\|Q_x\|_{L^2}^2,$$
$$\frac{\partial}{\partial \tilde{x}}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}=(Q_x,iQ_x)_{L^2}=0;$$

$$\frac{\partial}{\partial \xi}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}=(ixQ,Q^3)_{L^2}=0,$$
$$\frac{\partial}{\partial \xi}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),iQ^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}=(ixQ,iQ^3)_{L^2}=0,$$
$$\frac{\partial}{\partial \xi}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),Q_x)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}=(ixQ,Q_x)_{L^2}=0,$$
$$\frac{\partial}{\partial \xi}(e^{i\gamma}e^{ix\xi}\lambda^{1/2}u(\lambda x+\tilde{x})-Q(x),Q^3)_{L^2}\Big|_{\lambda=1,\gamma=0,\tilde{x}=0,\xi=0,u=Q}=(ixQ,iQ_x)_{L^2}=-\tfrac{1}{2}\|Q\|_{L^2}^2.$$

Therefore, by the inverse function theorem, if $\lambda_0=1,\ \gamma_0=0,\ \xi_0=0$, and $x_0=0$, there exists $\lambda>0$, $\gamma\in\mathbb{R},\ \xi\in\mathbb{R}$, and $\tilde{x}\in\mathbb{R}$ satisfying

$$\|\epsilon\|_{L^2}+|\lambda-1|+|\gamma|+|\xi|+|\tilde{x}|\lesssim\|e^{i\gamma_0}e^{ix\xi_0}\lambda_0^{1/2}u(\lambda_0x+x_0)-Q\|_{L^2}. \tag{10-7}$$

As in (3-5) and (3-6), $\lambda>0,\ \gamma\in\mathbb{R}$, and $\tilde{x}\in\mathbb{R}$ are unique, and $\gamma\in\mathbb{R}$ is unique in $\mathbb{R}/2\pi n$.

For general $\lambda_0>0,\ x_0\in\mathbb{R},\ \xi_0\in\mathbb{R}$, and $\gamma_0\in\mathbb{R}$, combining (10-7) with symmetries of (1-1) yields

$$\|\epsilon\|_{L^2}+\left|\frac{\lambda}{\lambda_0}-1\right|+|\gamma-\gamma_0-\xi_0(\tilde{x}-x_0)|+\left|\xi-\frac{\lambda}{\lambda_0}\xi_0\right|+\left|\frac{\tilde{x}-x_0}{\lambda_0}\right|$$
$$\lesssim\|e^{i\gamma_0}e^{ix\xi_0}\lambda_0^{1/2}u(\lambda_0x+x_0)-Q\|_{L^2}.\quad\square$$

As in Theorem 11, it is possible to show that $\lambda(t),\ \gamma(t),\ x(t)$, and $\xi(t)$ are continuous functions on $[0,\sup(I))$ and are differentiable almost everywhere on $[0,\sup(I))$. Let $s(t)$ be as in (3-10). Since $s:[0,\sup(I))\to[0,\infty)$ is monotone, the function is invertible: $t(s):[0,\infty)\to[0,\sup(I))$. Letting

$$\gamma(s)=\gamma(t(s)),\quad \lambda(s)=\lambda(t(s)),\quad x(s)=x(t(s)),\quad \xi(s)=\xi(t(s)),$$

and letting

$$\epsilon(s,x)=e^{i\gamma(s)}e^{ix\xi(s)}\lambda(s)^{1/2}u(t(s),\lambda(x)x+x(s))-Q(x), \tag{10-8}$$

we can compute

$$\begin{aligned}\epsilon_s=i\gamma_s(Q+\epsilon)+i\xi_sx(Q+\epsilon)+\frac{\lambda_s}{\lambda}\big(\tfrac{1}{2}(Q+\epsilon)+x(Q+\epsilon)_x\big)-i\frac{\lambda_s}{\lambda}\xi(s)x(Q+\epsilon)\\ +\frac{x_s}{\lambda}(Q+\epsilon)_x-i\frac{x_s}{\lambda}\xi(s)(Q+\epsilon)+i(Q+\epsilon)_{xx}\\ +2\xi(s)(Q+\epsilon)_x-i\xi(s)^2(Q+\epsilon)+i|Q+\epsilon|^4(Q+\epsilon).\end{aligned} \tag{10-9}$$

Taking $f\in\{Q^3,iQ^3,Q_x,iQ_x\}$,

$$\frac{d}{ds}(\epsilon,f)_{L^2}=(\epsilon_s,f)_{L^2}=0.$$

Using the fact that f belongs to the span of $\{Q^3, iQ^3, Q_x, iQ_x\}$ if and only if if belongs to the span of $\{Q^3, iQ^3, Q_x, iQ_x\}$ as a real vector space, compute the following:

$$\begin{aligned} &(i\gamma_s(Q+\epsilon), f)_{L^2} = (i\gamma_s Q, f)_{L^2} = 0 \quad \text{if } f = Q^3, Q_x, iQ_x, \\ &(i\gamma_s(Q+\epsilon), iQ^3) = \gamma_s \|Q\|_{L^4}^4; \end{aligned} \tag{10-10}$$

$$\left(\xi_s - \frac{\lambda_s}{\lambda}\xi(s)\right)(ix(Q+\epsilon), f)_{L^2} = \begin{cases} -\frac{1}{2}\left(\xi_s(s) - \frac{\lambda_s}{\lambda}\xi(s)\right)\|Q\|_{L^2}^2 \\ \quad + O\left(\left|\xi_s(s) - \frac{\lambda_s}{\lambda}\xi(s)\right|\|\epsilon\|_{L^2}\right) & \text{if } f = iQ_x, \\ O\left(\left|\xi_s(s) - \frac{\lambda_s}{\lambda}\xi(s)\right|\|\epsilon\|_{L^2}\right) & \text{if } f \in \{Q^3, Q_x, iQ^3\}; \end{cases} \tag{10-11}$$

$$\begin{aligned} &\left(-i\frac{x_s}{\lambda}\xi(s)(Q+\epsilon), f\right)_{L^2} = \left(-i\frac{x_s}{\lambda}\xi(s)Q, f\right)_{L^2} = 0 \quad \text{if } f = Q^3, Q_x, iQ_x, \\ &\left(-i\frac{x_s}{\lambda}\xi(s), Q, iQ^3\right)_{L^2} = -\frac{x_s}{\lambda}\xi(s)\|Q\|_{L^4}^4; \end{aligned} \tag{10-12}$$

$$\begin{aligned} &(i\xi(s)^2(Q+\epsilon), f)_{L^2} = (i\xi(s)^2 Q, f)_{L^2} = 0 \quad \text{if } f = Q^3, Q_x, iQ_x, \\ &(i\xi(s)^2 Q, iQ^3) = \xi(s)^2\|Q\|_{L^4}^4; \end{aligned} \tag{10-13}$$

$$\begin{aligned} &\frac{\lambda_s}{\lambda}\left(\tfrac{1}{2}(Q+\epsilon) + x(Q+\epsilon)_x, Q^3\right)_{L^2} = \frac{\lambda_s}{4\lambda}\|Q\|_{L^4}^4 + O\left(\left|\frac{\lambda_s}{\lambda}\right|\|\epsilon\|_{L^2}\right), \\ &\frac{\lambda_s}{\lambda}\left(\tfrac{1}{2}(Q+\epsilon) + x(Q+\epsilon)_x, f\right)_{L^2} = O\left(\left|\frac{\lambda_s}{\lambda}\right|\|\epsilon\|_{L^2}\right) \qquad \text{if } f = Q_x, iQ^3, iQ_x; \end{aligned} \tag{10-14}$$

$$\begin{aligned} &\left(\frac{x_s}{\lambda} + 2\xi(s)\right)((Q+\epsilon)_x, Q_x)_{L^2} = \left(\frac{x_s}{\lambda} + 2\xi(s)\right)\|Q_x\|_{L^2}^2 + O\left(\left|\frac{x_s}{\lambda} + 2\xi(s)\right|\|\epsilon\|_{L^2}\right), \\ &\left(\frac{x_s}{\lambda} + 2\xi(s)\right)((Q+\epsilon)_x, f)_{L^2} = O\left(\left|\frac{x_s}{\lambda} + 2\xi(s)\right|\|\epsilon\|_{L^2}\right) \qquad \text{if } f = Q^3, iQ^3, iQ_x. \end{aligned} \tag{10-15}$$

Finally, taking $\epsilon = \epsilon_1 + i\epsilon_2$,

$$\begin{aligned} &(i(Q+\epsilon)_{xx} + i|Q+\epsilon|^4(Q+\epsilon), f)_{L^2} \\ &\qquad = (iQ, f)_{L^2} + (i\mathcal{L}\epsilon_1 - \mathcal{L}_-\epsilon_2, f)_{L^2} + O((|\epsilon|^2(|\epsilon|^3 + |Q|^3), f)_{L^2}, \end{aligned} \tag{10-16}$$

where $\mathcal{L}$, $\mathcal{L}_-$ are given by (3-12). Since $\mathcal{L}$, $\mathcal{L}_-$ are self-adjoint operators, $(\epsilon_1, Q^3)_{L^2} = (\epsilon_2, Q^3)_{L^2} = 0$, $\mathcal{L}Q_x = 0$, and

$$\begin{aligned} &(i(Q+\epsilon)_{xx} + i|Q+\epsilon|^4(Q+\epsilon), f)_{L^2} \\ &\qquad = \begin{cases} \|Q\|_{L^4}^4 + O((|\epsilon|^2(|\epsilon|^3 + |Q|^3), f)_{L^2} & \text{if } f = iQ^3, \\ O((|\epsilon|^2(|\epsilon|^3 + |Q|^3), f)_{L^2} & \text{if } f = iQ_x, \\ -(\epsilon_2, \mathcal{L}_- Q^3)_{L^2} + O((|\epsilon|^2(|\epsilon|^3 + |Q|^3), f)_{L^2} & \text{if } f = Q^3, \\ -(\epsilon_2, \mathcal{L}_- Q_x)_{L^2} + O((|\epsilon|^2(|\epsilon|^3 + |Q|^3), f)_{L^2} & \text{if } f = Q_x. \end{cases} \end{aligned} \tag{10-17}$$

Combining (10-10)–(10-17), we have proved

$$\Big(\gamma_s + 1 - \frac{x_s}{\lambda}\xi(s) - \xi(s)^2\Big)\|Q\|_{L^4}^4 + O\Big(\Big|\xi_s - \frac{\lambda_s}{\lambda}\xi(s)\Big|\|\epsilon\|_{L^2}\Big) + O\Big(\Big|\frac{\lambda_s}{\lambda}\Big|\|\epsilon\|_{L^2}\Big) + O\Big(\Big|\frac{x_s}{\lambda} + 2\xi(s)\Big|\|\epsilon\|_{L^2}\Big) + O(\|\epsilon\|_{L^2}^2(\|Q\|_{L^\infty}^3 + \|\epsilon\|_{L^\infty}^3)) = 0, \tag{10-18}$$

$$-\frac{1}{2}\Big(\xi_s - \frac{\lambda_s}{\lambda}\xi(s)\Big)\|Q\|_{L^2}^2 + O\Big(\Big|\xi_s - \frac{\lambda_s}{\lambda}\xi(s)\Big|\|\epsilon\|_{L^2}\Big) + O\Big(\Big|\frac{\lambda_s}{\lambda}\Big|\|\epsilon\|_{L^2}\Big) + O\Big(\Big|\frac{x_s}{\lambda} + 2\xi(s)\Big|\|\epsilon\|_{L^2}\Big) + O(\|\epsilon\|_{L^2}^2(\|Q\|_{L^\infty}^3 + \|\epsilon\|_{L^\infty}^3)) = 0, \tag{10-19}$$

$$\frac{\lambda_s}{4\lambda}\|Q\|_{L^4}^4 - (\epsilon_2, \mathcal{L}_- Q^3)_{L^2} + O\Big(\Big|\xi_s - \frac{\lambda_s}{\lambda}\xi(s)\Big|\|\epsilon\|_{L^2}\Big) + O\Big(\Big|\frac{\lambda_s}{\lambda}\Big|\|\epsilon\|_{L^2}\Big) + O\Big(\Big|\frac{x_s}{\lambda} + 2\xi(s)\Big|\|\epsilon\|_{L^2}\Big) + O(\|\epsilon\|_{L^2}^2(\|Q\|_{L^\infty}^3 + \|\epsilon\|_{L^\infty}^3)) = 0, \tag{10-20}$$

$$\Big(\frac{x_s}{\lambda} + 2\xi\Big)\|Q_x\|_{L^2}^2 - (\epsilon_2, \mathcal{L}_- Q_x)_{L^2} + O\Big(\Big|\xi_s - \frac{\lambda_s}{\lambda}\xi(s)\Big|\|\epsilon\|_{L^2}\Big) + O\Big(\Big|\frac{\lambda_s}{\lambda}\Big|\|\epsilon\|_{L^2}\Big) + O\Big(\Big|\frac{x_s}{\lambda} + 2\xi(s)\Big|\|\epsilon\|_{L^2}\Big) + O(\|\epsilon\|_{L^2}^2(\|Q\|_{L^\infty}^3 + \|\epsilon\|_{L^\infty}^3)) = 0. \tag{10-21}$$

Using the same analysis as in (3-15)–(3-18), for any $a \in \mathbb{Z}_{\geq 0}$,

$$\int_a^{a+1} \Big|\gamma_s + 1 - \frac{x_s}{\lambda}\xi(s) - \xi(s)^2\Big|\, ds \lesssim \int_a^{a+1} \|\epsilon(s)\|_{L^2}^2\, ds, \tag{10-22}$$

$$\int_a^{a+1} \Big|\xi_s - \frac{\lambda_s}{\lambda}\xi(s)\Big|\, ds \lesssim \int_a^{a+1} \|\epsilon(s)\|_{L^2}^2\, ds, \tag{10-23}$$

$$\int_a^{a+1} \Big|\frac{\lambda_s}{\lambda}\Big|\, ds \lesssim \int_a^{a+1} \|\epsilon(s)\|_{L^2}\, ds, \tag{10-24}$$

$$\int_a^{a+1} \Big|\frac{x_s}{\lambda} + 2\xi\Big|\, ds \lesssim \int_a^{a+1} \|\epsilon(s)\|_{L^2}\, ds. \tag{10-25}$$

10.3. ***A long-time Strichartz estimate in the nonsymmetric case.*** The symmetry (10-1) does not impact the long-time Strichartz estimates in Theorems 13–15 at all. However, the Galilean symmetry (10-2) does, since it involves a translation in frequency, and therefore will impact estimates of u under frequency cutoffs. Nevertheless, it is possible to prove a modification of Theorem 15 using virtually the same arguments.

Theorem 27. *Suppose $\lambda(t)$, $x(t)$, $\xi(t)$, and $\gamma(t)$ are as in* (10-4). *Also suppose that on the interval $J = [a, b]$,*

$$\lambda(t) \geq \frac{1}{\eta_1}, \quad \int_J \lambda(t)^{-2}\, dt = T, \quad \textit{and} \quad \eta_1^{-2} T = 2^{3k}.$$

Furthermore, suppose that

$$\frac{|\xi(t)|}{\lambda(t)} \leq \eta_0 \quad \textit{for all } t \in [a, b].$$

Then

$$\|P_{\geq k}u\|_{U^2_\Delta([a,b]\times\mathbb{R})} \lesssim T^{-10} + \left(\frac{1}{T}\int_a^b \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}\,dt\right)^{1/2}.$$

Proof. Observe that by (10-4),

$$u(t,x) = e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}Q\left(\frac{x-x(t)}{\lambda(t)}\right) + e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}\epsilon\left(t,\frac{x-x(t)}{\lambda(t)}\right). \quad (10\text{-}26)$$

Then by (4-3), (4-4), and (10-26),

$$\|P_{>0}u\|^2_{L_t^\infty L_x^2([a,b]\times\mathbb{R})} \leq 4\eta_0^2.$$

Applying the induction on frequency arguments in Theorems 13–15 gives the same results. □

10.4. ***Almost conservation of energy for a nonsymmetric solution.*** It is possible to use the long-time Strichartz estimates in Theorem 27 to prove an almost conservation of energy for a nonsymmetric solution.

Theorem 28. *Let $J=[a,b]$ be an interval such that*

$$\lambda(t) \geq \frac{1}{\eta_1}, \quad \frac{|\xi(t)|}{\lambda(t)} \leq \eta_0 \quad \textit{for all } t\in J \qquad \textit{and} \qquad \int_J \lambda(t)^{-2}\,dt = T, \quad \eta_1^{-2}T = 2^{3k}.$$

Then,

$$\sup_{t\in J} E(P_{\leq k+9}u(t)) \lesssim \frac{2^{2k}}{T}\int_J \|\epsilon(t)\|^2_{L^2}\lambda(t)^{-2}\,dt + \left(\sup_{t\in J}\frac{\xi(t)}{\lambda(t)}\right)^2 + 2^{2k}T^{-10}. \quad (10\text{-}27)$$

Proof. Decompose the energy as in Theorem 12. Since $E(Q)=0$ and $(\epsilon_2, Q_x)=0$,

$$\begin{aligned}
E(u) &= E\left(e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}Q\left(\frac{x-x(t)}{\lambda(t)}\right) + e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}\epsilon\left(t,\frac{x-x(t)}{\lambda(t)}\right)\right)\\
&= \frac{1}{2\lambda(t)^2}\|Q_x\|^2_{L^2} + \frac{\xi(t)^2}{2\lambda(t)^2}\|Q\|^2_{L^2} - \frac{1}{6\lambda(t)^2}\|Q\|^6_{L^6} + \frac{1}{2\lambda(t)^2}\|\epsilon\|^2_{L^2} - \frac{2\xi(t)}{\lambda(t)^2}(Q_x,\epsilon_2)_{L^2}\\
&\quad - \frac{\xi(t)^2}{2\lambda(t)^2}\|\epsilon\|^2_{L^2} + \frac{1}{2\lambda(t)^2}\|\nabla\epsilon\|^2_{L^2} - \frac{\xi(t)}{\lambda(t)^2}(\nabla\epsilon_1,\epsilon_2)_{L^2} + \frac{\xi(t)}{\lambda(t)^2}(\nabla\epsilon_2,\epsilon_1)_{L^2} + \frac{\xi(t)^2}{2\lambda(t)^2}\|\epsilon\|^2_{L^2}\\
&\quad - \frac{5}{2\lambda(t)^2}\int Q(x)^4\epsilon_1(t,x)^2\,dx - \frac{1}{2\lambda(t)^2}\int Q(x)^4\epsilon_2(t,x)^2\,dx\\
&\qquad + O\left(\frac{1}{\lambda(t)^2}\|\epsilon\|^3_{L^3} + \frac{1}{\lambda(t)^2}\|\epsilon\|^6_{L^6}\right)\\
&= \frac{\xi(t)^2}{2\lambda(t)^2}\|Q\|^2_{L^2} + \frac{1}{2\lambda(t)^2}\|\epsilon\|^2_{L^2} - \frac{\xi(t)}{\lambda(t)^2}(\nabla\epsilon_1,\epsilon_2)_{L^2} + \frac{\xi(t)}{\lambda(t)^2}(\nabla\epsilon_2,\epsilon_1)_{L^2} + \frac{1}{2\lambda(t)^2}\|\nabla\epsilon\|^2_{L^2}\\
&\quad - \frac{5}{2\lambda(t)^2}\int Q(x)^4\epsilon_1(t,x)^2\,dx - \frac{1}{2\lambda(t)^2}\int Q(x)^4\epsilon_2(t,x)^2\,dx\\
&\quad + O\left(\frac{1}{\lambda(t)^2}\|\epsilon\|^3_{L^3} + \frac{1}{\lambda(t)^2}\|\epsilon\|^6_{L^6}\right).
\end{aligned} \quad (10\text{-}28)$$

Using the bounds on $|\xi(t)|/\lambda(t)$, the fact that Q and all its derivatives are rapidly decreasing, Fourier truncation, and the mean value theorem implies that (10-27) holds for some $t_0 \in J$. Then, using the long-time Strichartz estimates in Theorem 27 and following the proof of Theorem 16 gives Theorem 28. □

It is also possible to generalize Corollary 17 to the nonsymmetric case.

Corollary 29. *If*

$$\frac{1}{\eta_1} \leq \lambda(t) \leq \frac{1}{\eta_1} T^{1/100}, \quad \frac{|\xi(t)|}{\lambda(t)} \leq \eta_0 \quad \textit{for all } t \in J \qquad \textit{and} \qquad \int_J \lambda(t)^{-2}\, dt = T, \quad \eta_1^{-2} T = 2^{3k},$$

then

$$\sup_{t \in J} \left\| P_{\leq k+9}\left(\frac{e^{-i\gamma(t)} e^{-ix\xi(t)/\lambda(t)}}{\lambda(t)^{1/2}} \epsilon\left(t, \frac{x - x(t)}{\lambda(t)}\right)\right)\right\|_{\dot{H}^1}^2 \lesssim \frac{2^{2k}}{T} \int_J \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}\, dt + \left(\sup_{t \in J} \frac{\xi(t)^2}{\lambda(t)^2}\right) + 2^{2k} T^{-10}$$

and

$$\sup_{t \in J} \|\epsilon(t)\|_{L^2}^2 \lesssim \frac{2^{2k} T^{1/50}}{\eta_1^2 T} \int_J \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}\, dt + \frac{T^{1/50}}{\eta_1^2}\left(\sup_{t \in J} \frac{\xi(t)^2}{\lambda(t)^2}\right) + 2^{2k} \frac{T^{1/50}}{\eta_1^2} T^{-10}.$$

Proof. As in the proof of Theorem 12, since $\epsilon \perp \{Q^3, Q_x, iQ^3, iQ_x\}$, there exists some $c > 0$ such that

$$\frac{1}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2 + \frac{1}{2\lambda(t)^2}\|\nabla\epsilon\|_{L^2}^2 - \frac{5}{2\lambda(t)^2}\int Q(x)^4 \epsilon_1(t,x)^2\, dx - \frac{1}{2\lambda(t)^2}\int Q(x)^4 \epsilon_2(t,x)^2\, dx \geq \frac{1}{2\lambda(t)^2}\|\epsilon\|_{L^2}^2 + \frac{c}{\lambda(t)^2}\|\nabla\epsilon\|_{L^2}^2. \tag{10-29}$$

Next, for $\|\epsilon\|_{L^2} \leq \eta_0$ sufficiently small, by the Cauchy–Schwarz inequality, taking $\delta = \|\epsilon\|_{L^2}/\|Q\|_{L^2}$ in the last step,

$$\frac{\xi(t)^2}{\lambda(t)^2}\|Q\|_{L^2}^2 - \frac{\xi(t)}{\lambda(t)^2}(\nabla\epsilon_1, \epsilon_2)_{L^2} + \frac{\xi(t)}{\lambda(t)^2}(\nabla\epsilon_2, \epsilon_1)_{L^2} \geq \frac{\xi(t)^2}{\lambda(t)^2}\|Q\|_{L^2}^2 - \frac{1}{\delta}\frac{\xi(t)^2}{\lambda(t)^2}\|\epsilon\|_{L^2}^2 - \frac{\delta}{\lambda(t)^2}\|\nabla\epsilon\|_{L^2}^2 \geq -O\left(\frac{\eta_0}{\lambda(t)^2}\right)\|\nabla\epsilon\|_{L^2}^2. \tag{10-30}$$

Finally, by Hölder's inequality and the Sobolev embedding theorem, since $\|\epsilon\|_{L^2} \ll 1$,

$$O\left(\frac{1}{\lambda(t)^2}\|\epsilon\|_{L^3}^3 + \frac{1}{\lambda(t)^2}\|\epsilon\|_{L^6}^6\right) \ll \frac{1}{\lambda(t)^2}\|\epsilon\|_{L^2}^2 + \frac{1}{\lambda(t)^2}\|\nabla\epsilon\|_{L^2}^2. \tag{10-31}$$

Plugging (10-29)–(10-31) into (10-28) proves the corollary. □

10.5. ***A frequency-localized Morawetz estimate for nonsymmetric u.*** As in Section 6, the long-time Strichartz estimates of Theorem 27 and the energy estimates of Theorem 28 and Corollary 29 give a theorem analogous to Theorem 18 in the nonsymmetric case.

Theorem 30. *Let $J = [a, b]$ be an interval on which*

$$\frac{|\xi(t)|}{\lambda(t)} \leq \eta_0, \quad \frac{1}{\eta_1} \leq \lambda(t) \leq \frac{1}{\eta_1} T^{1/100} \quad \textit{for all } t \in J \qquad \textit{and} \qquad \int_J \lambda(t)^{-2}\, dt = T, \quad \eta_1^{-2} T = 2^{3k}.$$

Also suppose $\epsilon = \epsilon_1 + i\epsilon_2$, where ϵ is given by Theorem 10. Finally, suppose there exists a uniform bound on $x(t)$,

$$\sup_{t \in J} |x(t)| \leq R = T^{1/25}. \tag{10-32}$$

Finally, suppose that $\xi(a) = 0$ and $x(b) = 0$. Then for T sufficiently large,

$$\int_a^b \|\epsilon(t)\|_{L^2}^2 \lambda(t)^{-2}\, dt \\ \leq 3\big(\epsilon_2(a), \big(\tfrac{1}{2} Q + x Q_x\big)\big)_{L^2} - 3\big(\epsilon_2(b), \tfrac{1}{2} Q + x Q_x\big)_{L^2} + \frac{T^{1/50}}{\eta_1^2} \sup_{t \in J} \frac{\xi(t)^2}{\lambda(t)^2} + O\Big(\frac{1}{T^9}\Big).$$

Proof. This time let

$$\phi(x) = \int_0^x \chi\Big(\frac{\eta_1 y}{2R}\Big)\, dy = \int_0^x \psi^2\Big(\frac{\eta_1 y}{2R}\Big)\, dy, \tag{10-33}$$

and let

$$M(t) = \int \phi(x) \operatorname{Im}[\overline{P_{\leq k+9} u}\, \partial_x P_{\leq k+9} u](t, x)\, dx.$$

Since

$$|\phi(x)| \lesssim \eta_1^{-1} R \quad \text{and} \quad |\xi(t)|/\lambda(t) \leq \eta_0,$$

Theorem 27 implies that the error terms arising from frequency truncation may be handled in exactly the same manner as in Theorem 18.

Next, observe that by (10-30) and (10-31), the additional terms in the left-hand side of (6-17) that arise from the fact that $\xi(t)$ need not be zero may be handled in exactly the same manner as the terms involving ϵ^3 and higher powers of ϵ.

Now decompose $M(b) - M(a)$. Since Q is real-valued, symmetric, and rapidly decreasing, (10-33), the bounds on $\lambda(t)$, and (10-32) imply

$$\int \phi(x) \operatorname{Im}\bigg[\overline{e^{-i\gamma(t)} e^{-ix\xi(t)/\lambda(t)} \lambda(t)^{-1/2} P_{\leq k+9} Q\Big(\frac{x - x(t)}{\lambda(t)}\Big)} \\ \times \partial_x \Big(e^{-i\gamma(t)} e^{-ix\xi(t)/\lambda(t)} \lambda(t)^{-1/2} P_{\leq k+9} Q\Big(\frac{x - x(t)}{\lambda(t)}\Big)\Big)\bigg]\, dx \\ = \frac{\xi(t)}{\lambda(t)^2} \int \phi(x) Q\Big(\frac{x - x(t)}{\lambda(t)}\Big)^2 dx + O(T^{-10}) = \frac{\xi(t)}{\lambda(t)} x(t) \|Q\|_{L^2}^2 + O(T^{-10}). \tag{10-34}$$

Since $\xi(a) = 0$ and $x(b) = 0$,

$$\frac{\xi(t)}{\lambda(t)} x(t) \|Q\|_{L^2}^2 \Big|_a^b = 0.$$

Next, by Corollary 29,

$$\int \phi(x)\,\mathrm{Im}\Big[\overline{P_{\le k+9}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}\epsilon\Big(t,\frac{x-x(t)}{\lambda(t)}\Big)} \times \partial_x\Big(P_{\le k+9}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}\epsilon\Big(t,\frac{x-x(t)}{\lambda(t)}\Big)\Big)\Big]\,dx$$
$$\lesssim \frac{R}{\eta_1^2}\frac{2^{2k}}{T^{99/100}}\int_J \|\epsilon(t)\|_{L^2}^2\lambda(t)^{-2}\,dt + \frac{R}{\eta_1^2}2^{2k}T^{-9.99} + \frac{T^{1/50}}{\eta_1^2}\sup_{t\in J}\frac{\xi(t)^2}{\lambda(t)^2}. \tag{10-35}$$

Next, using the computations proving (6-5) combined with the fact that $(\epsilon_2, Q_x) = 0$,

$$\int \phi(x)\,\mathrm{Im}\Big[\overline{P_{\le k+9}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}\epsilon\Big(t,\frac{x-x(t)}{\lambda(t)}\Big)} \times \partial_x\Big(P_{\le k+9}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}Q\Big(\frac{x-x(t)}{\lambda(t)}\Big)\Big)\Big]\,dx$$
$$= -(\epsilon_2, xQ_x)_{L^2} + \frac{x(t)}{\lambda(t)}(\epsilon_2, Q_x)_{L^2} - \frac{\xi(t)}{\lambda(t)}(\epsilon_1, Q)_{L^2} + O(T^{-10})$$
$$= -(\epsilon_2, xQ_x)_{L^2} - \frac{\xi(t)}{2\lambda(t)}\|\epsilon\|_{L^2}^2 + O(T^{-10}). \tag{10-36}$$

Finally, integrating by parts,

$$\int \phi(x)\,\mathrm{Im}\Big[\overline{P_{\le k+9}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}Q\Big(\frac{x-x(t)}{\lambda(t)}\Big)} \times \partial_x\Big(P_{\le k+9}\lambda(t)^{-1/2}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\epsilon\Big(t,\frac{x-x(t)}{\lambda(t)}\Big)\Big)\Big]\,dx$$
$$= \text{(10-36)} - \int \chi\Big(\frac{\eta_1 x}{2R}\Big)\,\mathrm{Im}\Big[\overline{P_{\le k+9}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}Q\Big(\frac{x-x(t)}{\lambda(t)}\Big)} \times P_{\le k+9}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}\epsilon\Big(t,\frac{x-x(t)}{\lambda(t)}\Big)\Big]\,dx. \tag{10-37}$$

As in (6-10),

$$-\int \chi\Big(\frac{\eta_1 x}{2R}\Big)\,\mathrm{Im}\Big[\overline{P_{\le k+9}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}Q\Big(\frac{x-x(t)}{\lambda(t)}\Big)} \times P_{\le k+9}e^{-i\gamma(t)}e^{-ix\xi(t)/\lambda(t)}\lambda(t)^{-1/2}\epsilon\Big(t,\frac{x-x(t)}{\lambda(t)}\Big)\Big]\,dx$$
$$= -(\epsilon_2, Q)_{L^2} + O(T^{-10}). \tag{10-38}$$

Summing up (10-34)–(10-38) and using the fundamental theorem of calculus and the Morawetz estimate completes the proof of Theorem 30. □

10.6. ***An L_s^p bound on $\|\epsilon(s)\|_{L^2}$ when $p > 1$ for nonsymmetric u.*** Combining Theorem 30 with (10-18)–(10-21), it is possible to prove Theorem 20 for nonsymmetric u.

Theorem 31. *Let u be a nonsymmetric solution to* (1-1) *that satisfies $\|u\|_{L^2} = \|Q\|_{L^2}$, and suppose*

$$\sup_{s \in [0,\infty)} \|\epsilon(s)\|_{L^2} \leq \eta_* \tag{10-39}$$

and $\|\epsilon(0)\|_{L^2} = \eta_*$. Then

$$\int_0^\infty \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim \eta_*, \tag{10-40}$$

with implicit constant independent of η_* when $\eta_* \ll 1$ is sufficiently small.

Furthermore, for any $j \in \mathbb{Z}_{\geq 0}$, let

$$s_j = \inf\{s \in [0,\infty) : \|\epsilon(s)\|_{L^2} = 2^{-j}\eta_*\}.$$

By definition, $s_0 = 0$, and, as in Theorem 20, such an s_j exists for any $j > 0$. Then,

$$\int_{s_j}^\infty \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim 2^{-j}\eta_* \tag{10-41}$$

for each j, with implicit constant independent of η_* and $j \geq 0$.

Proof. Set $T_* = 1/\eta_*$ and suppose that T_* is large enough that Theorem 30 holds. Then by (10-39) and (10-22),

$$\Big| \sup_{s \in [s', s'+T_*]} \ln \lambda(s) - \inf_{s \in [s', s'+T_*]} \ln \lambda(s) \Big| \lesssim 1.$$

Let J be the largest dyadic integer that satisfies

$$J = 2^{j_*} \leq -\ln \eta_*^{1/4}.$$

By (10-24) and the triangle inequality,

$$\Big| \sup_{s \in [s', s'+JT_*]} \ln \lambda(s) - \inf_{s \in [s', s'+JT_*]} \ln \lambda(s) \Big| \lesssim J,$$

and therefore,

$$\frac{\sup_{s \in [s', s'+3JT_*]} \lambda(s)}{\inf_{s \in [s', s'+3JT^*]} \lambda(s)} \lesssim T_*^{1/100}.$$

Rescale so that $\inf_{s \in [s', s'+3JT_*]} \lambda(s) = 1/\eta_1$. Then make a Galilean transformation so that $\xi(s') = 0$ and a translation in space so that $x(s'') = 0$ when $s'' \in [s', s'+3JT_*]$ is the other endpoint of the interval of integration. Then by (10-23) and (10-25),

$$\sup_{s \in [s', s'+3JT_*]} \frac{|\xi(s)|}{\lambda(s)} \lesssim \eta_* J \eta_1 \ll \eta_0 \quad \text{and} \quad \sup_{s \in [s', s'+3JT_*]} |x(s)| \lesssim J^2 T_*^{1/100} + \frac{1}{\eta_1} T_*^{1/100} J \ll T_*^{1/25}.$$

Therefore, by Theorem 30,

$$\sup_{a>0} \int_{s'+aJT_*}^{s'+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 \lesssim \frac{1}{J^{1/2} T_*^{1/2}} \bigg(\sup_{a \geq 0} \int_{s'+aJT_*}^{s'+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \bigg)^{1/2} + T_*^{1/50} \eta_*^2 + O\Big(\frac{1}{J^9 T_*^9}\Big),$$

and when $a = 0$,

$$\int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2}^2 \lesssim \|\epsilon(s')\|_{L^2} + \frac{1}{J^{1/2}T_*^{1/2}} \left(\sup_{a\geq 0} \int_{s'+aJT_*}^{s'+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \right)^{1/2} + T_*^{1/50}\eta_*^2 + O\left(\frac{1}{J^9 T_*^9}\right).$$

Therefore, taking $s' = s_{j_*}$,

$$\sup_{a\geq 0} \int_{s_{j_*}+aJT_*}^{s_{j_*}+(a+1)JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim 2^{-j_*}\eta_* + O(2^{-9j_*}\eta_*^9).$$

By the triangle inequality,

$$\sup_{s'\geq s_{j_*}} \int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim 2^{-j_*}\eta_*,$$

and by Hölder's inequality,

$$\sup_{s'\geq s_{j_*}} \int_{s'}^{s'+JT_*} \|\epsilon(s)\|_{L^2} \, ds \lesssim 1.$$

It is therefore possible to prove Theorem 31 by induction. Indeed, suppose that for some $n > 0$,

$$\sup_{s'\geq s_{nj_*}} \int_{s'}^{s'+J^nT_*} \|\epsilon(s)\|_{L^2} \, ds \leq C \quad \text{and} \quad \sup_{s'\geq s_{nj_*}} \int_{s'}^{s'+J^nT_*} \|\epsilon(s)\|_{L^2}^2 \, ds \leq C^2 J^{-n}\eta_*.$$

Then by (10-24),

$$\sup_{s'\geq s_{j_*}} \Big| \sup_{s\in[s',s'+J^{n+1}T_*]} \ln\lambda(s) - \inf_{s\in[s',s'+JT_*]} \ln\lambda(s) \Big| \lesssim CJ. \tag{10-42}$$

Next, rescaling so that $\inf_{s\in[s',s'+J^{n+1}T_*]} \lambda(s) = 1/\eta_1$ and setting $\xi(s') = 0$, (10-23) implies

$$\sup_{s\in[s',s'+J^{n+1}T_*]} \frac{|\xi(s)|}{\lambda(s)} \lesssim 2^{-j_*n}\eta_*\eta_1 C^2 J \ll \eta_0, \tag{10-43}$$

and by (10-25), if $x(s'') = 0$, where s'' is the other endpoint of the interval of integration,

$$\sup_{s\in[s'+J^{n+1}T_*]} |x(s)| \lesssim C^2 J^2 T_*^{1/100} + C\frac{1}{\eta_1} T_*^{1/100} J \ll T_*^{1/25}. \tag{10-44}$$

Then by Theorem 30, as in (7-12),

$$\sup_{s'\geq s_{(n+1)j_*}} \int_{s'}^{s'+J^{n+1}T_*} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim J^{-(n+1)}T_*^{-1} + T_*^{1/25} 2^{-2j_*n}\eta_*^2 C^4 J^2 \lesssim J^{-(n+1)}T_*^{-1}, \tag{10-45}$$

and by Hölder's inequality,

$$\sup_{s'\geq s_{(n+1)j_*}} \int_{s'}^{s'+J^{n+1}T_*} \|\epsilon(s)\|_{L^2} \, ds \lesssim 1. \tag{10-46}$$

It is important to observe that the implicit constants in (10-45) and (10-46) are independent of C so long as the final inequalities in (10-43) and (10-44) hold and $C \ll T_*^{1/2}$.

Now take any $j \in \mathbb{Z}$ and suppose $nj_* < j \leq (n+1)j_*$. Then by (10-45) and (10-46),

$$\sup_{a \geq 0} \int_{s_j + aJ^{n+1}T_*}^{s_j+(a+1)J^{n+1}T_*} \|\epsilon(s)\|_{L^2}\, ds \lesssim J \quad \text{and} \quad \sup_{a \geq 0} \int_{s_j + aJ^{n+1}T_*}^{s_j+(a+1)J^{n+1}T_*} \|\epsilon(s)\|_{L^2}^2\, ds \lesssim 2^{-j}\eta_*,$$

and therefore, after appropriate rescaling and Galilean and spatial translation, (10-42)–(10-44) hold. Therefore, by Theorem 30,

$$\sup_{s' \geq s_j} \int_{s'}^{s'+2^j T_*} \|\epsilon(s)\|_{L^2}\, ds \lesssim 1 \quad \text{and} \quad \int_{s'}^{s'+2^j T_*} \|\epsilon(s)\|_{L^2}^2\, ds \lesssim 2^{-j}\eta_*,$$

with implicit constant independent of j. Furthermore, as in (7-13),

$$\int_{s'}^{s'+2^j JT_*} \|\epsilon(s)\|_{L^2}^2\, ds \lesssim 2^{-j}\eta_*,$$

so by the mean value theorem,

$$\inf_{s \in [s_j, s_j+2^j JT_*]} \|\epsilon(s)\|_{L^2} \lesssim 2^{-j}\eta_* J^{-1/2},$$

which implies

$$s_{j+1} \in [s_j, s_j + 2^j JT_*].$$

Therefore,

$$\int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2}^2\, ds \lesssim 2^{-j}\eta_* \quad \text{and} \quad \int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2}\, ds \lesssim 1,$$

with constant independent of j. Summing in j gives (10-40) and (10-41). □

Now then, as in Section 7,

$$\lim_{s \to \infty} \|\epsilon(s)\|_{L^2} = 0, \qquad \int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2}\, ds \lesssim 1,$$

and, for any $1 < p < \infty$

$$\int_{s_j}^{s_{j+1}} \|\epsilon(s)\|_{L^2}^p\, ds \lesssim \eta_*^{p-1} 2^{-j(p-1)},$$

which implies that $\|\epsilon(s)\|_{L^2}$ belongs to L_s^p for any $p > 1$ but not to L_s^1.

10.7. ***Monotonicity of λ in the nonsymmetric case.*** It is possible to use the virial identity from [Merle and Raphael 2005] to show monotonicity in the nonsymmetric case as well.

Theorem 32. *For any $s \geq 0$, let*

$$\tilde{\lambda}(s) = \inf_{\tau \in [0,s]} \lambda(\tau).$$

Then for any $s \geq 0$,

$$1 \leq \frac{\lambda(s)}{\tilde{\lambda}(s)} \leq 3.$$

Proof. Suppose there exist $0 \leq s_- \leq s_+ < \infty$ satisfying

$$\frac{\lambda(s_+)}{\lambda(s_-)} = e.$$

Then using (10-9) and the computations in Theorem 21,

$$\begin{aligned}\frac{d}{ds}(\epsilon, y^2 Q)_{L^2} &+ \frac{\lambda_s}{\lambda}\|yQ\|_{L^2}^2 + 4\big(\epsilon_2, \tfrac{1}{2}Q + yQ_y\big)_{L^2} \\ &= O\bigg(\bigg|\gamma_s + 1 - \frac{x_s}{\lambda}\xi(s) - \xi(s)^2\bigg|\|\epsilon\|_{L^2}\bigg) + O\bigg(\bigg|\frac{\lambda_s}{\lambda}\bigg|\|\epsilon\|_{L^2}\bigg) + O\bigg(\bigg|\frac{x_s}{\lambda} + 2\xi(s)\bigg|\|\epsilon\|_{L^2}\bigg) \\ &\quad + O\bigg(\bigg|\xi_s - \frac{\lambda_s}{\lambda}\xi(s)\bigg|\|\epsilon\|_{L^2}\bigg) + O(\|\epsilon\|_{L^2}^2) + O(\|\epsilon\|_{L^2}^2\|\epsilon\|_{L^\infty}^3). \qquad (10\text{-}47)\end{aligned}$$

Then by Theorem 30 and the fundamental theorem of calculus,

$$\|yQ\|_{L^2}^2 + 4\int_{s_-}^{s_+}\big(\epsilon_2, \tfrac{1}{2}Q + xQ_x\big)_{L^2} = O(\eta_*).$$

Therefore, there exists $s' \in [s_-, s_+]$ such that

$$\big(\epsilon_2, \tfrac{1}{2}Q + xQ_x\big)_{L^2} < 0.$$

Make a Galilean transformation setting $\xi(s') = 0$ and a translation in space such that $x(s'') = 0$, where s'' is the other endpoint of the interval of integration. Also rescale so that $\lambda(s') = T_*^{1/200}/\eta_1$. Since $s' \geq 0$, there exists some $j \geq 0$ such that $s_j \leq s' + T_* < s_{j+1}$. By Theorem 31 and (10-23),

$$\begin{aligned}&\int_{s'}^{s_{j+1+J}}\bigg|\frac{\lambda_s}{\lambda}\bigg|\,ds \lesssim J \quad \Longrightarrow \quad \frac{1}{\eta_1} \leq \lambda(t) \leq \frac{1}{\eta_1}T_*^{1/100}, \\ &\sup_{s\in[s',s_{j+1+J}]}\frac{|\xi(s)|}{\lambda(s)} \ll \eta_0, \quad \text{and} \quad \sup_{s\in[s',s_{j+1+J}]}|x(s)| \ll T_*^{1/25}.\end{aligned} \qquad (10\text{-}48)$$

Then by Theorems 30 and 31,

$$\int_{s'}^{s_{j+1+J}}\|\epsilon(s)\|_{L^2}^2\,ds \lesssim 2^{-(j+1+J)}\eta_* + T_*^{1/50}\eta_*\int_{s'}^{s_{j+1+J}}\|\epsilon(s)\|_{L^2}^2\,ds \lesssim 2^{-(j+1+J)}\eta_*,$$

and therefore by definition of s_{j+1+J},

$$\int_{s'}^{s_{j+1+J}}\|\epsilon(s)\|_{L^2}\,ds \lesssim 1.$$

Then, (10-48) holds on the interval $[s', s_{j+1+2J}]$, and arguing by induction, for any $k \geq 1$,

$$\int_{s'}^{s_{j+k}}\|\epsilon(s)\|_{L^2}^2\,ds \lesssim 2^{-j-k}\eta_* \quad \text{and} \quad \int_{s'}^{s_{j+k}}\|\epsilon(s)\|_{L^2}\,ds \lesssim 1,$$

with implicit constant independent of k. Taking $k \to \infty$,

$$\int_{s'}^{\infty}\|\epsilon(s)\|_{L^2}^2\,ds = 0,$$

which implies that $\epsilon(s) = 0$ for all $s \geq s'$. Therefore, u is a soliton solution to (1-1). □

10.8. ***Almost monotone $\lambda(t)$.*** In the nonsymmetric case, when $\sup(I) = \infty$, u is equal to a soliton solution, and when $\sup(I) < \infty$, u is the pseudoconformal transformation of the soliton solution.

Theorem 33. *If u satisfies the conditions of Theorem 24, u blows up forward in time, and*

$$\sup(I) = \infty,$$

then u is equal to a soliton solution.

Proof. As in Theorem 22, for any integer $k \geq 0$, let

$$I(k) = \{s \geq 0 : 2^{-k+2} \leq \tilde{\lambda}(s) \leq 2^{-k+3}\}.$$

As in the proof of Theorem 22, there exists a sequence $k_n \nearrow \infty$ such that

$$|I(k_n)| 2^{-2k_n} \geq \frac{1}{k_n^2}$$

and such that $|I(k_n)| \geq |I(k)|$ for all $k \leq k_n$.

Lemma 34. *For n sufficiently large, there exists $s_n \in I(k_n)$ such that*

$$\|\epsilon(s_n)\|_{L^2} \lesssim k_n^2 2^{-2k_n}.$$

Proof. Let $I(k_n) = [a_n, b_n]$. By Theorem 31,

$$\int_{I(k_n)} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim \eta_*,$$

Then, using the virial identity in (10-47),

$$\int_{a_n}^{(3a_n+b_n)/4} \big(\epsilon_2, \tfrac{1}{2} Q + x Q_x\big)_{L^2} \, ds = O(\eta_*) + O(1).$$

Therefore, by the mean value theorem, there exists $s_n^- \in \big[a_n, \frac{1}{4}(3a_n + b_n)\big]$ such that

$$\big|\big(\epsilon_2(s_n^-), \tfrac{1}{2} Q + x Q_x\big)_{L^2}\big| \lesssim \frac{1}{|I(k_n)|}. \tag{10-49}$$

By a similar calculation, there exists $s_n^+ \in \big[\frac{1}{4}(a_n + 3b_n), b_n\big]$ such that

$$\big|\big(\epsilon_2(s_n^+), \tfrac{1}{2} Q + x Q_x\big)_{L^2}\big| \lesssim \frac{1}{|I(k_n)|}. \tag{10-50}$$

Therefore, by Theorem 30, (10-49) and (10-50) imply

$$\int_{s_n^-}^{s_n^+} \|\epsilon(s)\|_{L^2}^2 \, ds \lesssim \frac{1}{|I(k_n)|}. \tag{10-51}$$

Indeed, rescale so that $\lambda(s_n^-) = 1/\eta_1$. Then by Galilean transformation, suppose $\xi(s_n^-) = 0$ and by translation in space, suppose $x(s_n^+) = 0$. For all $s \in [s_n^-, s_n^+]$, by (10-23) and Theorem 31,

$$\frac{|\xi(s)|}{\lambda(s)} \lesssim \eta_1 \eta_* \quad \text{and} \quad |x(s)| \ll T_*^{1/25}. \tag{10-52}$$

Therefore, by Theorem 30 and (10-23),

$$\int_{s_n^-}^{s_n^+} \|\epsilon(s)\|_{L^2}^2\, ds \lesssim \frac{1}{|I(k_n)|} + \eta_* T_*^{1/25} \int_{s_n^-}^{s_n^+} \|\epsilon(s)\|_{L^2}^2\, ds \lesssim \frac{1}{|I(k_n)|}.$$

Remark. To make these computations completely rigorous, partition $[s_n^-, s_n^+]$ into a dyadic integer number of subintervals of length $\sim 1/\eta_*$, and then following the arguments proving Theorem 31, it is possible to prove that (10-52) holds on subintervals of length $\sim J/\eta_*$, and then by induction, (10-52) holds on $[s_n^-, s_n^+]$, which by Theorem 30 implies that (10-51) holds.

Then by the mean value theorem,

$$\|\epsilon(s_n)\|_{L^2}^2 \lesssim \frac{1}{|I(k_n)|^2}.$$

Since $|I(k_n)| \geq 2^{2k_n} k_n^{-2}$, the proof of Lemma 34 is complete. ☐

Make a Galilean transformation so that $\xi(s_n) = 0$. Then by (10-23), since $\lambda(s) \gtrsim 2^{-k_n}$ for all $s \in [0, s_n]$,

$$\frac{|\xi(s)|}{\lambda(s)} \lesssim 2^{k_n} \eta_*. \tag{10-53}$$

Now let m be the smallest integer such that

$$\frac{2^{2k_n}}{k_n^2} 2^m \geq |I(k_n)|. \tag{10-54}$$

Since $|I(k)| \leq |I(k_n)|$ for all $0 \leq k \leq k_n$, (10-54) implies that

$$|s_n| \leq 2^{2k_n+m+1}.$$

Let r_n be the smallest integer that satisfies

$$2^{(2k_n+m+1)/3} 2^{k_n} \frac{1}{\eta_1} \leq 2^{r_n}.$$

Then, as in the proof of Theorem 27, setting $t_n = s^{-1}(s_n)$, (10-53) and induction on frequency implies

$$\|P_{\geq r_n} u\|_{U_\Delta^2([0,t_n]\times\mathbb{R})} \lesssim \eta_*$$

and

$$\|P_{\geq r_n + k_n/4 + m/4} u\|_{U_\Delta^2([0,t_n]\times\mathbb{R})} \lesssim k_n^2 2^{-2k_n} 2^{-m}.$$

Furthermore,

$$E(P_{\leq r_n + k_n/4 + m/4} u(t_n)) \lesssim (k_n^2 2^{-2k_n} 2^{-m} 2^{r_n + k_n/4 + m/4})^2 \sim (k_n^2 2^{-k_n/12 - 5m/12} \eta_1^{-1})^2$$

and

$$\sup_{t\in[0,t_n]} E(P_{\leq r_n + k_n/4 + m/4} u(t)) \lesssim (k_n^2 2^{-k_n/12 - 5m/12} \eta_1^{-1})^2.$$

By (10-30), if $\xi_n(s)$ is the $\xi(s)$ in (10-8) for which $\xi_n(s_n) = 0$,

$$\sup_{0 \leq s \leq s_n} \frac{|\xi_n(s)|^2}{\lambda(s)^2} \lesssim (k_n^2 2^{-k_n/12 - 5m/12} \eta_1^{-1})^2,$$

which implies that $\xi(s)$ converges to some ξ_∞ as $s \to \infty$. Making a Galilean transformation that maps ξ_∞ to the origin and taking $n \to \infty$, since $m \geq 0$, (10-10) implies that $E(u_0) = 0$. Therefore, by the Gagliardo–Nirenberg inequality, u_0 is a soliton. □

When $\sup(I) < \infty$, suppose without loss of generality that $\sup(I) = 0$, and

$$\sup_{-1 < t < 0} \|\epsilon(t)\|_{L^2} \leq \eta_*.$$

Then write the decomposition

$$u(t,x) = \frac{e^{-i\gamma(t)} \exp\left[-ix\frac{\xi(t)}{\lambda(t)}\right]}{\lambda(t)^{1/2}} Q\left(\frac{x - x(t)}{\lambda(t)}\right) + \frac{e^{-i\gamma(t)} \exp\left[-ix\frac{\xi(t)}{\lambda(t)}\right]}{\lambda(t)^{1/2}} \epsilon\left(t, \frac{x - x(t)}{\lambda(t)}\right),$$

and apply the pseudoconformal transformation to $u(t,x)$. For $-\infty < t < -1$,

$$\begin{aligned} v(t,x) = \frac{1}{t^{1/2}} \overline{u\left(\frac{1}{t}, \frac{x}{t}\right)} e^{ix^2/4t} &= \frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)} \exp\left[ix\frac{\xi(1/t)}{\lambda(1/t)}\right]}{\lambda(1/t)^{1/2}} Q\left(\frac{x - tx(1/t)}{t\lambda(1/t)}\right) e^{ix^2/4t} \\ &\quad + \frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)} \exp\left[ix\frac{\xi(1/t)}{\lambda(1/t)}\right]}{\lambda(1/t)^{1/2}} \overline{\epsilon\left(\frac{1}{t}, \frac{x - tx(1/t)}{t\lambda(1/t)}\right)} e^{ix^2/4t}. \end{aligned}$$

Since the L^2 norm is preserved by the pseudoconformal transformation,

$$\lim_{t \searrow -\infty} \left\| \frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)} \exp\left[ix\frac{\xi(1/t)}{\lambda(1/t)}\right]}{\lambda(1/t)^{1/2}} \overline{\epsilon\left(\frac{1}{t}, \frac{x - tx(1/t)}{t\lambda(1/t)}\right)} e^{ix^2/4t} \right\|_{L^2} = 0.$$

Next,

$$\frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)} \exp\left[ix\frac{\xi(1/t)}{\lambda(1/t)}\right]}{\lambda(1/t)^{1/2}} Q\left(\frac{x - tx(1/t)}{t\lambda(1/t)}\right) \exp\left[ix\frac{x(1/t)}{2}\right] \exp\left[-i\frac{t}{4}x\left(\frac{1}{t}\right)^2\right]$$

is of the form

$$e^{-i\tilde{\gamma}(t)} \exp\left[-ix\frac{\tilde{\xi}(t)}{\tilde{\lambda}(t)}\right] \tilde{\lambda}(t)^{-1/2} Q\left(\frac{x - \tilde{x}(t)}{\tilde{\lambda}(t)}\right),$$

where

$$\tilde{\gamma}(t) = \gamma\left(\frac{1}{t}\right) - \frac{1}{4}x\left(\frac{1}{t}\right)^2 t, \quad \tilde{\xi}(t) = \xi\left(\frac{1}{t}\right) + \frac{1}{2}x\left(\frac{1}{t}\right)t\lambda\left(\frac{1}{t}\right),$$
$$\tilde{\lambda}(t) = t\lambda\left(\frac{1}{t}\right), \quad \text{and} \quad \tilde{x}(t) = tx\left(\frac{1}{t}\right).$$

Also,

$$\begin{aligned}\Bigg\| \frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)} \exp\left[ix\frac{\xi(1/t)}{\lambda(1/t)}\right]}{\lambda(1/t)^{1/2}} Q\Big(\frac{x - tx(1/t)}{t\lambda(1/t)}\Big) e^{ix^2/4t} \\ - \frac{1}{t^{1/2}} \frac{e^{i\gamma(1/t)} \exp\left[ix\frac{\xi(1/t)}{\lambda(1/t)}\right]}{\lambda(1/t)^{1/2}} Q\Big(\frac{x - tx(1/t)}{t\lambda(1/t)}\Big) \exp\Big[ix\frac{x(1/t)}{2}\Big] e^{itx(1/t)^2} \Bigg\|_{L^2} \\ = \Bigg\| \frac{1}{t^{1/2}\lambda(1/t)^{1/2}} Q\Big(\frac{x - tx(1/t)}{t\lambda(1/t)}\Big)\Big(\exp\Big[i\frac{(x - tx(1/t))^2}{4t}\Big] - 1\Big)\Bigg\|_{L^2}.\end{aligned}$$

As in (9-7) and (9-8),

$$\lim_{t \searrow -\infty} \Bigg\| \frac{1}{t^{1/2}\lambda(1/t)^{1/2}} Q\Big(\frac{x - tx(1/t)}{t\lambda(1/t)}\Big)\Big(\exp\Big[i\frac{(x - tx(1/t))^2}{4t}\Big] - 1\Big)\Bigg\|_{L^2} = 0.$$

Therefore, by time reversal symmetry, v satisfies the conditions of Theorem 24, and v is a solution that blows up backward in time at $\inf(I) = -\infty$, so therefore, by Theorem 33, v must be a soliton. Therefore, u is the pseudoconformal transformation of a soliton, which proves Theorem 5.

Appendix: U^p and V^p spaces

The description here of U^p and V^p spaces comes from Section 5.3 of [Dodson 2019]. See also [Koch et al. 2014].

Definition (U^p space). Suppose $u \in U^p$. We say that u is a U^p atom if there exists a sequence $\{t_k\} \nearrow \infty$ satisfying

$$u = \sum_k 1_{[t_k, t_{k+1})} u_k,$$

and

$$\sum \|u_k\|^p_{L^2(\mathbb{R}^d)} = 1.$$

Then define the norm

$$\|u(t)\|_{U^p(\mathbb{R}\times\mathbb{R}^d)} = \inf\Big\{\sum_\lambda |c_\lambda| : u(t) = \sum_\lambda c_\lambda u_\lambda \text{ for almost every } t \in \mathbb{R}, \text{ where } u_\lambda(t) \text{ is a } U^p \text{ atom}\Big\}.$$

Then set

$$\|u\|_{U^p_\Delta(\mathbb{R}\times\mathbb{R}^d)} = \|e^{-it\Delta}u\|_{U^p(\mathbb{R}\times\mathbb{R}^d)}.$$

Functions with finite U^p_Δ norm have finite Strichartz norms $L_t^{\tilde{p}} L_x^q$ when $p \le \tilde{p} \le \infty$ and (p, q) is an admissible pair. Bilinear Strichartz estimates also hold for $\tilde{p}$ in the appropriate range.

Theorem 35. *If I is an interval with $t_0 \in I$, for any $1 < p < \infty$, if $1/p + 1/p' = 1$,*

$$\Big\| \int_{t_0}^t e^{i(t-\tau)\Delta} F(\tau)\, d\tau \Big\|_{U^p_\Delta(I\times\mathbb{R}^d)} \lesssim \sup_{\|G\|_{V^{p'}_\Delta(I\times\mathbb{R}^d)}=1} \int_I \langle G, F\rangle\, d\tau.$$

The V_{Δ}^{p} space is defined as follows.

Definition (V_{Δ}^{p} spaces). Suppose $I = [0, T]$ is a compact interval. Define the partition

$$\mathcal{Z} = \{0 = t_0 < t_1 < \cdots < t_n = T\}.$$

Then for $1 < p < \infty$ define the norm

$$\|v\|_{V^p(\mathcal{Z}; I \times \mathbb{R}^d)}^p = \sum_{k=1}^{n} \|v(t_k) - v(t_{k-1})\|_{L^2(\mathbb{R}^d)}^p.$$

Then write

$$\|v\|_{V_{\Delta}^p(\mathcal{Z}; I \times \mathbb{R}^d)}^p = \|e^{-it\Delta} v(t)\|_{V_{\Delta}^p(\mathcal{Z}; I \times \mathbb{R}^d)}^p,$$

and define the norm

$$\|v\|_{V_{\Delta}^p(I \times \mathbb{R}^d)} = \sup_{\mathcal{Z}} \|v\|_{V_{\Delta}^p(\mathcal{Z}; I \times \mathbb{R}^d)} + \|v\|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)}.$$

The V^p space embedding will be extremely useful.

Theorem 36. *If* $p < q$,

$$V^p \subset U^q.$$

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