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**CONNECTIVITY CONDITIONS AND BOUNDARY POINCARÉ
INEQUALITIES**



CONNECTIVITY CONDITIONS AND BOUNDARY POINCARÉ INEQUALITIES

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Inspired by recent work of Mourgoglou and the second author, and earlier work of Hofmann, Mitrea and Taylor, we consider connections between the local John condition, the Harnack chain condition and weak boundary Poincaré inequalities in open sets $\Omega \subset \mathbb{R}^{n+1}$, with codimension-1 Ahlfors–David regular boundaries. First, we prove that if Ω satisfies both the local John condition and the exterior corkscrew condition, then Ω also satisfies the Harnack chain condition (and hence is a chord-arc domain). Second, we show that if Ω is a 2-sided chord-arc domain, then the boundary $\partial\Omega$ supports a Heinonen–Koskela-type weak 1-Poincaré inequality. We also construct an example of a set $\Omega \subset \mathbb{R}^{n+1}$ such that the boundary $\partial\Omega$ is Ahlfors–David regular and supports a weak boundary 1-Poincaré inequality but Ω is not a chord-arc domain. Our proofs utilize significant advances in particularly harmonic measure, uniform rectifiability and metric Poincaré theories.

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1. Introduction

Extending the PDE theory from smooth or otherwise “nice” domains to spaces with rough geometries has been under intensive research over the previous decades. The new tools and techniques developed by numerous authors have helped to overcome many difficulties in this field but many questions remain open. See [Hofmann 2019; Mattila 2023] for recent surveys related to key developments in some parts of this and related research.

We consider connections between John-type conditions, the Harnack chain condition and Poincaré inequalities in spaces with rough boundaries, inspired by the recent work [Mourgoglou and Tolsa 2021].

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In their work, they solved a long-standing open problem [Kenig 1994, Problem 3.2.2] (see also [Toro 2010, Question 2.5]) about the solvability of the regularity problem for the Laplacian. More precisely, they proved the following theorem. We give the exact definitions of the objects and concepts in the theorem in Sections 2 and 3.

Theorem 1.1 [Mourgoglou and Tolosa 2021, part of Theorems 1.2 and 1.5]. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded open set satisfying the corkscrew condition (see Definition 2.7), with n -dimensional Ahlfors–David regular boundary $\partial\Omega$ (see Definition 2.6). For $1 < p \leq 2$ we have:*

- (a) *The regularity problem for the Laplacian is solvable in L^p for Ω (see Definition 2.31) if and only if the Dirichlet problem is solvable in $L^{p'}$ (see Definition 2.30), where p' satisfies $1/p + 1/p' = 1$.*
- (b) *Suppose that either $\partial\Omega$ supports a weak p -Poincaré inequality (see Definition 2.29) or that Ω satisfies the 2-sided local John condition (see Definitions 2.1 and 3.2). If the regularity problem for the Laplacian is solvable in L^p for Ω , then the tangential regularity problem for the Laplacian is also solvable in L^p for Ω (see Definition 2.31).*

Remark that, in particular, from (a) in the theorem above it follows that the regularity problem is solvable in L^p for chord-arc domains (see Definition 2.10) for some $p > 1$; see, e.g., [David and Jerison 1990; Semmes 1990]. This extends previous results of [Jerison and Kenig 1982b] in the plane, and of [Verchota 1984] in Lipschitz domains.

Our goal is to revisit the assumptions of Theorem 1.1(b) by studying them from the point of view of 2-sided chord-arc domains: we show that a 2-sided local John domain with codimension 1 Ahlfors–David regular boundary is a 2-sided chord-arc domain, and the boundary of any 2-sided chord-arc domain supports weak Poincaré inequalities. To be more precise, we prove the following two results:

Theorem 1.2. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is an open set with n -dimensional Ahlfors–David regular boundary that satisfies the local John condition and the exterior corkscrew condition (see Definitions 2.1 and 2.7). Then Ω also satisfies the Harnack chain condition (see Definition 2.9). In particular, a 2-sided local John domain with codimension 1 Ahlfors–David regular boundary is a 2-sided chord-arc domain.*

Theorem 1.3. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a 2-sided chord-arc domain. Then the following weak 1-Poincaré inequality for Lipschitz functions on $\partial\Omega$ holds: there exist constants $C \geq 1$ and $\Lambda \geq 1$ such that for every Lipschitz function f on $\partial\Omega$ and every $\Delta = \Delta(y, r) = B(y, r) \cap \partial\Omega$ we have*

$$\int_{\Delta} |f(x) - \langle f \rangle_{\Delta}| d\sigma(x) \leq Cr \int_{\Lambda\Delta} |\nabla_{\mathfrak{t}} f(x)| d\sigma(x), \quad (1.4)$$

where $\nabla_{\mathfrak{t}} f$ is the tangential gradient of f (see Definition 2.27), $\sigma := \mathcal{H}^n|_{\partial\Omega}$ is the surface measure and

$$\langle f \rangle_{\Delta} := \frac{1}{\sigma(\Delta)} \int_{\Delta} f(y) d\sigma(y)$$

is the integral average of f over Δ .

We note that the conclusion of Theorem 1.2 holds if the local John condition is “good enough” in the sense that a D_0 -local John condition implies the Harnack chain condition but a (D_0, R_0) -local John

condition for $R_0 = c \cdot \text{diam}(\partial\Omega)$ and $c < 1$ small enough does not if $\text{diam}(\partial\Omega) < \infty$. See the definitions and discussion in [Section 3](#).

[Theorem 1.3](#) and its consequence [Corollary 1.6](#) improve some results in the literature. Earlier, Semmes proved that a weak 2-Poincaré inequality for the tangential gradient (that is, inequality (1.4) with the right-hand side replaced by $Cr(\int_{\Lambda\Delta} |\nabla_t f(x)|^2 d\sigma(x))^{1/2}$) holds for smooth functions on any chord-arc surface with small constant [[Semmes 1991](#), Lemma 1.1] (see the introduction of that work for the definition of these surfaces). [Theorem 1.3](#) both provides a stronger inequality and generalizes the class of surfaces considered by Semmes. A key element in the proof of [Theorem 1.3](#) is the machinery built by Hofmann, Mitrea and Taylor [[Hofmann et al. 2010](#)]. In their paper, they prove a weak (p, p) -Poincaré inequality with a tail for the Hofmann–Mitrea–Taylor Sobolev space $L_1^p(\partial\Omega)$ with respect to the Hofmann–Mitrea–Taylor gradient (see [Definition 2.28](#)) for any $1 < p < \infty$ on boundaries of 2-sided local John domains [[Hofmann et al. 2010](#), Proposition 4.13]. Combining [Theorem 1.3](#) with some density results in [[Hofmann et al. 2010](#)] and tools in [[Mourgoglou and Tolsa 2021](#)] shows us that the tail in their inequality can be removed, at least when Ω is a bounded 2-sided chord-arc domain (see [Corollary 7.13](#)).

Our results have some immediate consequences. First, we note that since chord-arc domains satisfy the local John condition, [Theorem 1.2](#) gives us the following characterization result:

Corollary 1.5. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set satisfying a 2-sided corkscrew condition, with n -dimensional Ahlfors–David regular boundary $\partial\Omega$. Then the following conditions are equivalent:*

- (a) Ω satisfies the local John condition.
- (b) Ω satisfies the Harnack chain condition.

For recent related results for semiuniform domains and chord-arc domains, see [[Azzam 2021b](#); [Azzam et al. 2017](#)].

Second, [Theorem 1.3](#) combined with a Lipschitz characterization of Poincaré inequalities [[Keith 2003](#), Theorem 2] gives us the following Heinonen–Koskela-type weak 1-Poincaré inequality:

Corollary 1.6. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 2-sided chord-arc domain. There exist constants $C \geq 1$ and $\Lambda \geq 1$ such that for every $\Delta = \Delta(y, r)$ we have*

$$\int_{\Delta} |f(x) - \langle f \rangle_{\Delta}| d\sigma(x) \leq Cr \int_{\Lambda\Delta} \rho(x) d\sigma(x) \tag{1.7}$$

for any $f \in L_{\text{loc}}^1(\partial\Omega)$ and any upper gradient ρ of f (see [Definition 2.25](#)), where $\sigma := \mathcal{H}^n|_{\partial\Omega}$ is the surface measure.

We note that this weak 1-Poincaré inequality implies a weak p -Poincaré inequality (that is, inequality (1.7) with the right-hand side replaced by $Cr(\int_{\Lambda\Delta} |\rho(x)|^p d\sigma(x))^{1/p}$) for any $1 < p < \infty$ by Hölder’s inequality. Furthermore, by [[Heinonen et al. 2015](#), Corollary 9.14], these weak p -Poincaré inequalities imply weak (p, p) -Poincaré inequalities of the type

$$\left(\int_{\Delta} |f(x) - \langle f \rangle_{\Delta}|^p d\sigma(x) \right)^{1/p} \leq \tilde{C}r \left(\int_{\Lambda\Delta} |\rho(x)|^p d\sigma(x) \right)^{1/p},$$

where we used the same notation as in [Corollary 1.6](#). See also [Corollary 7.13](#) for an inequality of this type for the Hofmann–Mitrea–Taylor Sobolev spaces in bounded 2-sided chord-arc domains. We also note that the conclusion of [Corollary 1.6](#) is the inequality appearing in [Theorem 1.1](#). It is natural to ask if [Corollary 1.6](#) can be strengthened into a characterization but this is not possible: there exist non-chord-arc domain open sets $\Omega \subset \mathbb{R}^{n+1}$ with n -dimensional Ahlfors–David regular boundaries $\partial\Omega$ that support weak Poincaré inequalities. See [Section 8](#) for an example and discussion. However, we point out that a result for the converse direction was recently proven by Azzam [[2021a](#)] who showed that weak Poincaré inequalities imply uniform rectifiability (see [Definition 2.8](#)) for Ahlfors–David regular sets $E \subset \mathbb{R}^{n+1}$.

Furthermore, [Corollary 1.6](#), combined with [[Heinonen and Koskela 1998](#), Theorem 5.7; [Korte 2007](#), Theorem 3.3; [Cheeger 1999](#), Theorem 17.1] (see also, e.g., [[Merhej 2017](#); [Heinonen et al. 2015](#)]), immediately gives us the following two new results about the geometric structure of boundaries of 2-sided chord-arc domains:

Corollary 1.8. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 2-sided chord-arc domain. Then $\partial\Omega$ is a Loewner space (see [Definition 2.24](#)).*

Examples of Loewner spaces include the Euclidean space, Carnot groups and Riemannian manifolds of nonnegative Ricci curvature [[Heinonen and Koskela 1998](#), Section 6]. For other examples, see [[Heinonen et al. 2015](#), Section 14.2].

Corollary 1.9. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 2-sided chord-arc domain. Then*

- *if $n = 1$, then $\partial\Omega$ is quasiconvex (see [Definition 2.22](#)),*
- *if $n > 1$, then $\partial\Omega$ is annularly quasiconvex (see [Definition 2.22](#)).*

We note that the case $n = 1$ in [Corollary 1.9](#) cannot be improved to annular quasiconvexity: $\Omega = B(0, 1)$ (the unit disc) is a 2-sided chord-arc domain but for any $z \in \partial\Omega$ and any $0 < r < \frac{1}{2}$ there exist points $x, y \in B(z, 2r) \setminus B(z, r)$ such that x and y can be joined in $\partial\Omega \setminus \{z\}$ only with paths γ such that $\ell(\gamma) \geq 1$.

The proofs of [Theorems 1.2](#) and [1.3](#) and [Corollary 1.6](#) utilize significant advances in geometric analysis over the past 25 years. For [Theorem 1.2](#), we use harmonic measure theory (particularly the very recent results of Azzam, Hofmann, Martell, Mouroglou and the second author [[Azzam et al. 2020](#)]) and uniform rectifiability techniques (particularly the bilateral weak geometric lemma of [[David and Semmes 1993](#)]). For [Theorem 1.3](#) and [Corollary 1.6](#), we combine layer potential techniques of [[Hofmann et al. 2010](#)] and pointwise and Lipschitz characterizations of Poincaré inequalities of [[Heinonen and Koskela 1998](#); [Keith 2003](#)] (see also [[Heinonen 2001](#)]) with suitable localization and truncation arguments. One of the novelties in the proof of [Theorem 1.3](#) is the use of a weak type-(1, 1) version of a weak Poincaré inequality with a tail, analogous to the strong type-(p, p) version proved previously in [[Hofmann et al. 2010](#)].

The paper is organized as follows. In [Section 2](#) we fix the basic notation, review the numerous definitions needed in the paper and consider some auxiliary results from the literature. In [Section 3](#), we define three different John-type conditions and compare them. In [Section 4](#), we consider the bilateral weak geometric lemma of David and Semmes and prove some straightforward related results for the proof of [Theorem 1.2](#), and in [Section 5](#) we prove [Theorem 1.2](#). In [Section 6](#), we consider the Hofmann–Mitrea–Taylor-type weak p -Poincaré inequality with a tail for $1 < p < \infty$ and use techniques from its proof to

prove a weak-type estimate for the case $p = 1$. In [Section 7](#), we use this weak-type estimate together with some key results from the theory of Poincaré inequalities in metric spaces to prove [Theorem 1.3](#) and [Corollary 1.6](#). In [Section 8](#), we end the paper by constructing an example that shows us that the assumptions in [Corollary 1.6](#) are not optimal and consider some questions related to this work.

2. Notation, basic definitions and auxiliary results

Definition 2.1. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and consider a geometric condition; such as the local John condition (see [Definition 3.2](#)). If the interior of Ω^c satisfies the condition, we say that Ω satisfies the *exterior* version of the condition. If both Ω and the interior of Ω^c satisfy the same condition, we say that Ω satisfies the *2-sided* version of the same condition.

We use the following basic notation and terminology:

- $\Omega \subset \mathbb{R}^{n+1}$ is an open set with n -dimensional boundary $\partial\Omega$. We denote the surface measure of $\partial\Omega$ by $\sigma := \mathcal{H}^n|_{\partial\Omega}$. Unless explicitly mentioned, we assume that $\partial\Omega$ is Ahlfors–David regular (see [Definition 2.6](#)) and that both Ω and $\text{int } \Omega^c$ satisfy the corkscrew condition (that is, Ω satisfies the 2-sided corkscrew condition; see [Definitions 2.1](#) and [2.7](#)).
- Usually, we use capital letters X, Y, Z , and so on, to denote points in Ω , and lowercase letters x, y, z , and so on, to denote points in $\partial\Omega$.
- For every point $X \in \mathbb{R}^{n+1}$, we define $\delta(X) := \text{dist}(X, \partial\Omega)$.
- We denote the open $(n+1)$ -dimensional Euclidean ball with radius $r > 0$ by $B(X, r)$ or $B(x, r)$, depending on whether the center point lies in Ω or $\partial\Omega$. For any $x \in \partial\Omega$ and any $r > 0$, we denote the surface ball centered at x with radius r by $\Delta(x, r) := B(x, r) \cap \partial\Omega$.
- Given a Euclidean ball $B := B(X, r)$ or a surface ball $\Delta := \Delta(x, r)$ and a constant $\kappa > 0$, we define $\kappa B := B(X, \kappa r)$ and $\kappa \Delta := \Delta(x, \kappa r)$.
- For a metric measure space (X, d, μ) , a function f and an open ball B , we denote the average of f over B by

$$\langle f \rangle_B := \int_B f \, d\mu := \frac{1}{\mu(B)} \int_B f \, d\mu.$$

- A *path* is a continuous function $\gamma : [0, 1] \rightarrow X$, where X is a metric space. With slight abuse of terminology, we call a path $\gamma : [0, 1] \rightarrow \overline{\Omega}$ a *path in Ω* if $\gamma(t) \in \Omega$ for every $t \in (0, 1]$. With slight abuse of notation, we write $Z \in \gamma$ if there exists $t \in [0, 1]$ such that $\gamma(t) = Z$. We say that a path γ is *from X_1 to X_2* if $\gamma(0) = X_1$ and $\gamma(1) = X_2$.
- The *length of a path* $\gamma : [0, 1] \rightarrow \overline{\Omega}$ is defined as

$$\ell(\gamma) := \sup \left\{ \sum_{i=0}^k |\gamma(t_i) - \gamma(t_{i+1})| \right\},$$

where the supremum is taken over all finite partitions $0 = t_0 < t_1 < \dots < t_k = 1$ of the interval $[0, 1]$. We say that a path γ is *rectifiable* if the length of γ is finite.

- Given a rectifiable path γ and a function f , we denote the *arc-length parametrization* of γ (that is, the reparametrization of γ with respect to $\ell(\gamma)$) by $\gamma_\ell : [0, \ell(\gamma)] \rightarrow \mathbb{R}^{n+1}$ and the integral of f over γ by

$$\int_\gamma f := \int_0^{\ell(\gamma)} f \circ \gamma_\ell(t) dt.$$

- For a path γ and points $X_1, X_2 \in \gamma$ with $\gamma(t) = X_1$ and $\gamma(s) = X_2$ for $t, s \in [0, 1], t < s$, we denote the piece of γ from X_1 to X_2 by $\gamma(X_1, X_2)$ and its length by $\ell(\gamma(X_1, X_2))$. Again, with slight abuse of notation, we write $Z \in \gamma(X_1, X_2)$ if there exists $u \in (s, t)$ such that $\gamma(u) = Z$.
- We denote harmonic measure with pole at $X \in \Omega$ by ω^X . Usually, we drop the pole from the notation if we consider properties that hold for every $X \in \Omega$.
- Let $A \subset \mathbb{R}^{n+1}, f : A \rightarrow \mathbb{R}, \alpha > 0$ and $\beta \geq 1$. We say that f is α -Lipschitz if

$$|f(x) - f(y)| \leq \alpha|x - y|$$

for all $x, y \in A$. We say that f is *locally α -Lipschitz* if

$$\limsup_{\substack{A \ni y \rightarrow x \\ y \neq x}} \frac{|f(x) - f(y)|}{|x - y|} \leq \alpha \tag{2.2}$$

for every $x \in A$. We say that f is β -*bi-Lipschitz* if

$$\frac{1}{\beta}|x - y| \leq |f(x) - f(y)| \leq \beta|x - y|$$

for all $x, y \in A$.

- We denote the *measure-theoretic boundary* of Ω by $\partial_*\Omega$: we have $x \in \partial_*\Omega$ if and only if $x \in \partial\Omega$, and

$$\liminf_{r \rightarrow 0^+} \frac{|B(x, r) \cap \Omega|}{r^{n+1}} > 0 \quad \text{and} \quad \liminf_{r \rightarrow 0^+} \frac{|B(x, r) \setminus \bar{\Omega}|}{r^{n+1}} > 0. \tag{2.3}$$

- For any $p > 1$, we denote the *Hölder conjugate* of p by p' . The numbers p and p' satisfy $1/p + 1/p' = 1$.
- We denote the *nontangential maximal operator* by N_* : for a function u in Ω , N_*u is a function $\partial\Omega$ defined as

$$N_*u(x) := \sup_{Y \in \Gamma_\alpha(x)} |u(Y)|, \tag{2.4}$$

where $\Gamma_\alpha(x)$ is the *cone at $x \in \partial\Omega$ with aperture α* ,

$$\Gamma_\alpha(x) := \{Y \in \mathbb{R}^{n+1} : |x - Y| < \alpha \delta(Y)\}. \tag{2.5}$$

We say that a function u in Ω *converges nontangentially* to a function f on $\partial\Omega$ if $u(Y) \rightarrow f(x)$ as $Y \rightarrow x$ inside $\Gamma_\alpha(x)$.

- The letters c and C and their obvious variations denote constants that depend only on dimension, ADR constant (see [Definition 2.6](#)), UR constants (see [Definition 2.8](#)) and other similar parameters. The values of c and C may change from one occurrence to another. We do not track how our bounds depend on these constants and usually just write $\alpha_1 \lesssim \alpha_2$ if $\alpha_1 \leq c \alpha_2$ for a constant like this c and $\alpha_1 \approx \alpha_2$ if $\alpha_1 \lesssim \alpha_2 \lesssim \alpha_1$.

If the constant c_κ depends only on parameters of the previous type and some other parameter κ , we usually write $\alpha_1 \lesssim_\kappa \alpha_2$ instead of $\alpha_1 \leq c_\kappa \alpha_2$.

2A. ADR, UR, NTA, CAD, and the corkscrew condition.

Definition 2.6 (ADR). We say that a closed set $E \subset \mathbb{R}^{n+1}$ is a d -ADR (Ahlfors–David regular) set if there exists a constant $D \geq 1$ such that

$$\frac{1}{D}r^d \leq \mathcal{H}^d(B(x, r) \cap E) \leq Dr^d$$

for every $x \in E$ and every $r \in (0, \text{diam}(E))$, where $\text{diam}(E)$ may be infinite.

Definition 2.7 (corkscrew condition). We say that Ω satisfies the *corkscrew condition* if there exists a constant $c \in (0, 1)$ such that for every surface ball $\Delta := \Delta(x, r)$, with $x \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$, there exists a point $X_\Delta \in \Omega$ such that $B(X_\Delta, cr) \subset B(x, r) \cap \Omega$,

Definition 2.8 (UR). Following [David and Semmes 1991; 1993], we say that an n -ADR set $E \subset \mathbb{R}^{n+1}$ is UR (uniformly rectifiable) if it contains *big pieces of Lipschitz images* of \mathbb{R}^n , i.e., there exist constants $\theta, M > 0$ such that for every $x \in E$ and $r \in (0, \text{diam}(E))$ there is a Lipschitz mapping $\rho = \rho_{x,r} : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$, with Lipschitz norm no larger than M , such that

$$\mathcal{H}^n(E \cap B(x, r) \cap \rho(\{y \in \mathbb{R}^n : |y| < r\})) \geq \theta r^n.$$

Definition 2.9 (NTA). Following [Jerison and Kenig 1982a], we say that a domain $\Theta \subset \mathbb{R}^{n+1}$ is NTA (nontangentially accessible) if

- Θ satisfies the *Harnack chain condition*: there exists a uniform constant C such that for every $\rho > 0$, $\Lambda \geq 1$ and $X, X' \in \Theta$ with $\delta(X), \delta(X') \geq \rho$ and $|X - X'| < \Lambda\rho$ there exists a chain of open balls $B_1, \dots, B_N \subset \Theta$, $N \leq C(\Lambda)$, with $X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \neq \emptyset$ and $C^{-1} \text{diam}(B_k) \leq \text{dist}(B_k, \partial\Theta) \leq C \text{diam}(B_k)$,
- both Θ and $\mathbb{R}^{n+1} \setminus \Theta$ satisfy the corkscrew condition.

Definition 2.10 (CAD). An open set $\Omega \subset \mathbb{R}^{n+1}$ is a CAD (*chord-arc domain*) if it is NTA, and $\partial\Omega$ is n -ADR.

The following result originates from [David and Jerison 1990; Semmes 1990] (see also [Hofmann et al. 2010, Definition 3.7 and Corollary 3.9]):

Theorem 2.11. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is an open set satisfying the two-sided corkscrew condition and that $\partial\Omega$ is ADR. Then $\partial\Omega$ is UR and $\sigma(\partial\Omega \setminus \partial_*\Omega) = 0$.*

2A1. Dyadic cubes.

Theorem 2.12 (see, e.g., [Christ 1990; Sawyer and Wheeden 1992; Hytönen and Kairema 2012]). *Suppose that E is a d -ADR set. Then there exists a countable collection \mathbb{D} (that we call a **dyadic system**),*

$$\mathbb{D} := \bigcup_{k \in \mathbb{Z}} \mathbb{D}_k, \quad \mathbb{D}_k := \{Q_\alpha^k : \alpha \in \mathcal{A}_k\},$$

*of Borel sets Q_α^k (that we call **(dyadic) cubes**) such that*

- (i) the collection \mathbb{D} is **nested**: if $Q, P \in \mathbb{D}$, then $Q \cap P \in \{\emptyset, Q, P\}$,
- (ii) $E = \bigcup_{Q \in \mathbb{D}_k} Q$ for every $k \in \mathbb{Z}$ and the union is disjoint,
- (iii) there exist constants $c_1 > 0$ and $C_1 \geq 1$ such that

$$\Delta(z_\alpha^k, c_1 2^{-k}) \subseteq Q_\alpha^k \subseteq \Delta(z_\alpha^k, C_1 2^{-k}), \tag{2.13}$$

- (iv) for every set Q_α^k there exist at most N cubes $Q_{\beta_i}^{k+1}$ (called the **children** of Q_α^k) such that $Q_\alpha^k = \bigcup_i Q_{\beta_i}^{k+1}$, where the constant N depends only on the ADR constant of E .

Notation 2.14. We shall use the following notational conventions.

- (1) For each k , and for every cube $Q_\alpha^k := Q \in \mathbb{D}_k$, we define $\ell(Q) := C_1 2^{-k}$ and $x_Q := z_\alpha^k$. We call $\ell(Q)$ the *side length* of Q , and x_Q the *center* of Q . If the set E is bounded or disconnected, the side length might not be well-defined, but we can fix this problem easily by, for example, considering the minimum of the numbers $C_1 2^{-k}$ such that $Q \subset \Delta(x_Q, C_1 2^{-k})$.
- (2) For every $Q = Q_\alpha^k$ and $\kappa \geq 1$, we define

$$\kappa B_Q := B(z_\alpha^k, \kappa \ell(Q)).$$

For $\kappa = 1$, we simply let $\kappa B_Q = B_Q$.

Definition 2.15. We say that a collection $\mathcal{A} \subset \mathbb{D}$ satisfies a *Carleson packing condition* if there exists a constant $C \geq 1$ such that

$$\sum_{Q \in \mathcal{A}, Q \subset Q_0} \sigma(Q) \leq C \sigma(Q_0)$$

for every cube $Q_0 \in \mathbb{D}$. We call the smallest such constant C the *Carleson packing norm* of \mathcal{A} and denote it by $\mathcal{C}_\mathcal{A}$.

We need the following straightforward lemma in the proof of [Theorem 1.2](#):

Lemma 2.16. *Let $\mathcal{A} \subset \mathbb{D}$ be a collection satisfying a Carleson packing condition. Also, let $Q_0 \in \mathbb{D}$ be a cube and $A \subset Q_0$ a measurable subset such that $\sigma(A) \geq c \sigma(Q_0)$ for a constant $c \in (0, 1)$. Then there exists a cube $Q \in \mathbb{D} \setminus \mathcal{A}$ such that $\sigma(Q \cap A) > 0$ and $\ell(Q) \approx_{\mathcal{C}_\mathcal{A}, c} \ell(Q_0)$.*

Proof. Let us consider the first $K > \mathcal{C}_\mathcal{A}/c$ generations of subcubes of Q_0 . Each of these generations covers the set A . For contradiction, suppose that $\sigma(Q \cap A) = 0$ for each of these subcubes such that $Q \notin \mathcal{A}$. Then, for every $m = 1, \dots, K$, we can cover the set A (up to a set of measure 0) by cubes from the collection $\{Q \in \mathcal{A} : \ell(Q) = 2^{-m} \ell(Q_0)\}$. In particular, we get

$$\sum_{Q \in \mathcal{A}, Q \subset Q_0} \sigma(Q) = \sum_{m=0}^{\infty} \sum_{\substack{Q \in \mathcal{A} \\ \ell(Q) = 2^{-m} \ell(Q_0)}} \sigma(Q) \geq \sum_{m=1}^K \sum_{\substack{Q \in \mathcal{A} \\ \ell(Q) = 2^{-m} \ell(Q_0)}} \sigma(Q \cap A) = \sum_{m=1}^K \sigma(A) \geq K c \sigma(Q_0) > \mathcal{C}_\mathcal{A} \sigma(Q_0),$$

which contradicts the Carleson packing condition. Hence, there exists at least one cube Q from the first $\lceil \mathcal{C}_\mathcal{A}/c \rceil$ generations of subcubes of Q_0 such that $\sigma(Q \cap A) > 0$ and $Q \in \mathbb{D} \setminus \mathcal{A}$. □

2B. Harmonic measure and the weak- A_∞ condition.

Definition 2.17 (weak A_∞ for harmonic measure). For harmonic measure ω , we write $\omega \in \text{weak-}A_\infty(\sigma)$ if there exist constants $C \geq 1$ and $s > 0$ such that if $B := B(x, r)$ with $x \in \partial\Omega$ and $r \in (0, \frac{1}{2} \text{diam}(\partial\Omega))$ and $A \subset \Delta := B \cap \partial\Omega$ is a Borel set, then

$$\omega^Y(A) \leq C \left(\frac{\sigma(A)}{\sigma(\Delta)} \right)^s \omega^Y(2\Delta) \quad (2.18)$$

for every $Y \in \Omega \setminus 4B$.

We note that the constant 2 in (2.18) can be replaced with any constant $c > 1$ without changing the class weak- $A_\infty(\sigma)$ and that the weak- A_∞ property is equivalent with a weak reverse Hölder property for the Radon–Nikodym derivative; see, e.g., [Anderson et al. 2017, Section 8].

We use the following lemma from [Azzam et al. 2020] in the proof of Theorem 1.2. The lemma is a key ingredient for the proof of the geometric characterization of the weak- A_∞ property of harmonic measure:

Lemma 2.19 [Azzam et al. 2020, Section 10]. *Suppose that Ω has a uniformly rectifiable boundary $\partial\Omega$ and that $\omega \in \text{weak-}A_\infty$. Suppose also that $R_0 \in \mathbb{D}(\partial\Omega)$ is a dyadic cube and $Y \in \Omega \setminus 4B_{R_0}$ is a point such that*

$$c_1 \ell(R_0) \leq \delta(Y) \leq \text{dist}(Y, R_0) \leq c_1^{-1} \ell(R_0)$$

and $\omega^Y(R_0) \geq c_2 > 0$. Then there exist a constant $c_3 > 0$ and a subset $A \subset R_0$ such that $\sigma(A) \geq c_3 \sigma(R_0)$ and each point $x \in A$ can be joined to Y by a D -nontangential path (see Definition 3.1), where c_3 and D depend only on c_1, c_2, n , the weak- A_∞ constants and the uniform rectifiability constants.

We also need the following classical estimate (sometimes referred to as Bourgain’s estimate [1987, Lemma 1]):

Lemma 2.20. *There exist uniform constants $c_0 \in (0, 1)$ and $C_0 > 1$, depending only on n and the ADR constant, such that the following holds: if $x \in \partial\Omega$, $r \in (0, \text{diam}(\partial\Omega))$ and $Y \in B(x, c_0 r)$, then $\omega^Y(\Delta(x, r)) \geq 1/C_0 > 0$.*

2C. Quasiconvexity, annular quasiconvexity and Loewner spaces.

Definition 2.21. Let (X, d) be a metric space. We say that a nonempty set $E \subset X$ is a *continuum* if it is compact and connected. We call a continuum *nondegenerate* if it contains more than one point. We say that points $x, y \in F \subset X$ can be *joined in F* if there exists a continuum $E \subset F$ such that $x, y \in E$.

Definition 2.22. Let (X, d) be a metric space. We say that X is

- (i) *quasiconvex* if there exists a constant $C \geq 1$ such that for any pair of points $x, y \in X$ there exists a path $\gamma_{x,y} : [0, 1] \rightarrow X$ such that $\gamma_{x,y}(0) = x$, $\gamma_{x,y}(1) = y$ and $\ell(\gamma_{x,y}) \leq C d(x, y)$,
- (ii) *annularly quasiconvex* if there exists a constant $C \geq 1$ such that if $x, y \in B(z, 2r) \setminus B(z, r)$ for $z \in X$ and $r < (1/C) \text{diam}(X)$, then x and y can be joined by a path $\gamma = \gamma_{x,y}$ in $B(z, Cr) \setminus B(z, r/C)$ such that $\ell(\gamma) \leq C d(x, y)$.
- (iii) *rectifiably connected* if for any pair of points $x, y \in X$ there exists a rectifiable path $\gamma_{x,y}$ from x to y .

Definition 2.23. Let (X, d, μ) be a rectifiably connected metric measure space with a locally finite Borel measure μ . Let $E, F \subset X$ be two disjoint nondegenerate continua, $(E, F) = (E, F; X)$ be the family of paths in X connecting E and F and $p \geq 1$. We define the p -modulus of (E, F) as

$$\text{mod}_p(E, F) := \inf_{\varrho} \int_X \varrho^p d\mu,$$

where the infimum is taken over all nonnegative Borel functions $\varrho : X \rightarrow [0, \infty)$ satisfying

$$\int_{\gamma} \varrho ds \geq 1$$

for every γ in (E, F) .

Definition 2.24. Let (X, d, μ) be a d -dimensional, rectifiably connected metric measure space with a locally finite Borel measure μ . We say that X is a d -Loewner space if there exists a function $\phi : (0, \infty) \rightarrow (0, \infty)$ such that if E and F are two disjoint, nondegenerate continua in X satisfying

$$\frac{\text{dist}(E, F)}{\min\{\text{diam}(E), \text{diam}(F)\}} \leq t,$$

then $\phi(t) \leq \text{mod}_d(E, F)$.

2D. Upper, Hajlasz, tangential and Hofmann–Mitrea–Taylor gradients, Sobolev spaces, and weak Poincaré inequalities on $\partial\Omega$.

Definition 2.25. Let $f : \partial\Omega \rightarrow \mathbb{R}$. We say that a Borel-measurable function $\rho : \partial\Omega \rightarrow [0, \infty]$ is an *upper gradient of f* if we have

$$|f(x) - f(y)| \leq \int_{\gamma} \rho = \int_0^{\ell(\gamma)} \rho \circ \gamma_{\ell}(t) dt$$

for all $x, y \in \partial\Omega$ and every rectifiable path γ from x to y , where $\gamma_{\ell} : [0, \ell(\gamma)] \rightarrow \mathbb{R}$ is the arc-length parametrization of γ .

The notion of upper gradients originates from [Heinonen and Koskela 1996; 1998], where they were called *very weak gradients*.

Definition 2.26. Let $f : \partial\Omega \rightarrow \mathbb{R}$. We say that a Borel-measurable function $g : \partial\Omega \rightarrow \mathbb{R}$ is a *Hajlasz gradient of f* if we have

$$|f(x) - f(y)| \leq (g(x) + g(y))|x - y|$$

for almost every $x, y \in \partial\Omega$. We denote the class of all Hajlasz gradients of f by $D(f)$. For $p \geq 1$, we denote the space of Borel functions with a Hajlasz gradient in $L^p(\partial\Omega)$ by $\dot{W}^{1,p}(\partial\Omega)$ and define a seminorm $\|\cdot\|_{\dot{W}^{1,p}(\partial\Omega)}$ for this space by setting

$$\|f\|_{\dot{W}^{1,p}(\partial\Omega)} := \inf_{g \in D(f)} \|g\|_{L^p(\partial\Omega)}.$$

The notion of Hajlasz gradients originates from [Hajlasz 1996]. By [Jiang et al. 2015], if $f, g \in L^1_{\text{loc}}(\partial\Omega)$ and g is a Hajlasz gradient of f , then there exist functions $\tilde{f}, \tilde{g} \in L^1_{\text{loc}}(\partial\Omega)$ such that $f = \tilde{f}$ and $g = \tilde{g}$ almost everywhere and $4\tilde{g}$ is an upper gradient of \tilde{f} .

Definition 2.27. Let $f : \partial\Omega \rightarrow \mathbb{R}$ be a Lipschitz function and let $x \in \partial\Omega$ be a point such that the approximate tangent plane $T_x \partial\Omega$ exists. Let $\tilde{f} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be any Lipschitz extension of f to \mathbb{R}^{n+1} . We say that f is *tangentially differentiable at x* if $\tilde{f}|_{x+T_x \partial\Omega}$ is differentiable at x . When it exists, we denote the corresponding *tangential gradient at x* by $\nabla_t f(x)$.

A thorough reference for these types of differentiability results is [Maggi 2012]. In particular, since $\partial\Omega$ is uniformly rectifiable by Theorem 2.11 (and therefore $\partial\Omega$ is rectifiable; see [Hofmann et al. 2010, p. 2629] for an explicit proof), we know that the approximate tangent plane exists for almost every point $x \in \partial\Omega$ by [Maggi 2012, Theorem 10.2], the tangential gradient exists for σ -a.e. point $x \in \partial\Omega$ by [loc. cit., Theorem 11.4] and the definition of the tangential gradient is independent of the choice of the Lipschitz extension by [loc. cit., Theorem 10.1, Proposition 10.5, and Lemma 11.5].

By [Hofmann et al. 2010], the next definitions make sense if we know that — on top of the standing assumptions on Ω and $\partial\Omega$ (see the beginning of Section 2) — Ω is a set of locally finite perimeter and $\sigma(\partial\Omega \setminus \partial_* \Omega) = 0$ (recall (2.3)). Since we only use the following objects when Ω satisfies the 2-sided local John condition (or, equivalently by Corollary 1.5, when Ω is a 2-sided chord-arc domain), the assumptions are automatically satisfied by Theorem 2.11 and [Hofmann et al. 2010, Corollary 3.14].

Definition 2.28. Let $\nu(x) := (\nu_j(x))_{j=1}^{n+1}$ be the outer unit normal at $x \in \partial\Omega$. For a function $\varphi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$, $\varphi \in C_c^1$, we define the (Hofmann–Mitrea–Taylor) *tangential derivatives of φ* as

$$\partial_{t,j,k} \varphi := \nu_j(\partial_k \varphi)|_{\partial\Omega} - \nu_k(\partial_j \varphi)|_{\partial\Omega}$$

for $1 \leq j, k \leq n + 1$. The Hofmann–Mitrea–Taylor Sobolev space $L^p_1(\partial\Omega)$ is the space of the functions $f \in L^p(\partial\Omega)$ such that there exists a finite constant C_f such that

$$\sum_{1 \leq j,k \leq n+1} \left| \int_{\partial\Omega} f \partial_{t,k,j} \varphi \, d\sigma \right| \leq C_f \|\varphi\|_{L^{p'}(\sigma)}$$

for every $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$. By the Riesz representation theorem, for every $f \in L^p_1(\partial\Omega)$ and each $j, k = 1, 2, \dots, n + 1$ there exists a function $h_{j,k} \in L^p(\partial\Omega)$ satisfying

$$\int_{\partial\Omega} h_{j,k} \varphi \, d\sigma = \int_{\partial\Omega} f \partial_{t,k,j} \varphi \, d\sigma$$

for every $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$. We set $\partial_{t,j,k} f := h_{j,k}$ and define the Hofmann–Mitrea–Taylor gradient $\nabla_{\text{HMT}} f$ by setting

$$\nabla_{\text{HMT}} f := \left(\sum_k \nu_k \partial_{t,j,k} f \right)_{j=1}^{n+1}.$$

For the comprehensive theory of Hofmann–Mitrea–Taylor Sobolev spaces, see in particular Section 3 and 4 in [Hofmann et al. 2010].

Definition 2.29. Let $1 \leq p < \infty$. We say that $\partial\Omega$ supports a *weak (Heinonen–Koskela-type) p -Poincaré inequality* if there exist constants $C = C_p \geq 1$ and $\Lambda \geq 1$ such that for every $\Delta = \Delta(y, r)$ we have

$$\int_{\Delta} |f(x) - \langle f \rangle_{\Delta}| \, d\sigma(x) \leq Cr \left(\int_{\Lambda \Delta} \rho(x)^p \, d\sigma(x) \right)^{1/p}$$

for any $f \in L^1_{\text{loc}}(\partial\Omega)$ and any upper gradient ρ of f .

2E. Solvability for the Laplacian.

Definition 2.30. Let $1 \leq p < \infty$. We say that the Dirichlet problem (for the Laplacian) is *solvable in L^p* for Ω if there exists a constant C such that for any continuous function $f \in C(\partial\Omega)$ the solution $u = u_f$ to the Dirichlet problem with datum f converges nontangentially to f σ -a.e. and

$$\|N_*u\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)},$$

where N_* is the nontangential maximal operator (recall (2.4)).

Definition 2.31. Let $1 \leq p < \infty$. We say that the *regularity problem (for the Laplacian) is solvable in L^p* for Ω if there exists a constant C such that for any Lipschitz function $f : \partial\Omega \rightarrow \mathbb{R}$ the solution $u = u_f$ to the Dirichlet problem with datum f converges nontangentially to f σ -a.e. and

$$\|N_*(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{\dot{W}^{1,p}(\partial\Omega)}, \quad (2.32)$$

where $\|\cdot\|_{\dot{W}^{1,p}(\partial\Omega)}$ is the Hajlasz seminorm (see Definition 2.26). For $1 < p < \infty$, we say that the *tangential regularity problem (for the Laplacian) is solvable in L^p* if the previous holds after we replace (2.32) with the estimate

$$\|N_*(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|\nabla_t f\|_{L^p(\partial\Omega)}.$$

3. John, local John and weak local John conditions

In this section, we define different John-type conditions, compare them by considering some examples and make some remarks related to literature. We assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set, with n -dimensional Ahlfors–David regular boundary. We do not assume that the corkscrew conditions hold in general but we discuss their role below.

Definition 3.1 (nontangential paths). Let $\gamma : [0, 1] \rightarrow \bar{\Omega}$ be a path in Ω from X to Y . For $D \geq 1$, we say that γ is a *D -nontangential path* if we have

$$\ell(\gamma(X, Z)) \leq D\delta(Z)$$

for every $Z \in \gamma$.

Notice that Definition 3.1 is not symmetric with respect to X and Y . The general idea is that we use nontangential paths to measure how well we can connect boundary points to certain points inside the space. We use the name nontangential path to emphasize the connection between these paths and nontangential convergence we discussed in Section 2. Indeed, if there exists a D -nontangential path γ from $x \in \partial\Omega$ to $Y \in \Omega$, then, by definition, we have

$$|x - Z| \leq \ell(\gamma(x, Z)) \leq D\delta(Z)$$

for every $Z \in \gamma$ and therefore we have $Z \in \Gamma_D(x)$ for every $Z \in \gamma$. Here $\Gamma_D(x)$ is the cone at x of aperture D (see (2.5)).

Definition 3.2 (John, local John and weak local John conditions). Let $D_0 \geq 1$. We say that Ω satisfies

- (i) the *D_0 -John condition* if there exists a point $X_0 \in \Omega$ such that for every $Y \in \Omega$ there exists a D_0 -nontangential path in Ω from Y to X_0 ,

- (ii) the *local D_0 -John condition* if for every $x \in \partial\Omega$ and every $r \in (0, \text{diam}(\partial\Omega))$ there exists a point $Y_x \in B(x, r) \cap \Omega$ (that we call a *local John point*) such that $B(Y_x, r/D_0) \subset \Omega$ and for every $z \in \Delta(x, r)$ there exists a D_0 -nontangential path γ_z in Ω from z to Y_x such that $\ell(\gamma_z) \leq D_0 r$,
- (iii) the *weak local D_0 -John condition* if there exist constants $\theta \in (0, 1]$ and $R \geq 2$ such that for every $X \in \Omega$ there exists a Borel set $F \subset \Delta_X := B(X, R\delta(X)) \cap \partial\Omega$ such that $\sigma(F) \geq \theta\sigma(\Delta_X)$ and for every $z \in F$ there exists a D_0 -nontangential path γ_z in Ω from z to X such that $\ell(\gamma_z) \leq D_0 R \delta(X)$.

The John condition was first used in [John 1961] but the terminology originates from [Martio and Sarvas 1979]. The local John condition was first used in [Hofmann et al. 2010, Definition 3.12] and weak local John condition originates from [Azzam et al. 2020, Definition 2.11].

Generally, the John condition does not imply the local John condition, the local John condition does not imply the John condition, and there are domains that satisfy the weak local John condition but not the local John condition. We can see this by considering some straightforward examples:

Example 3.3. In \mathbb{R}^2 we have the following:

- (1) $\Omega_1 := B(0, 1) \setminus \{(x, 0) : x \in [0, 1]\}$ satisfies the John condition (we can choose, e.g., $X_0 = (0, -\frac{1}{2})$ as the “John point”) but not the local John condition. We can see this by noticing that any ball $B((1, 0), r) \cap \Omega_1$ contains points $(y, t) \in \Omega_1$ and $(z, t) \in \partial\Omega_1$ for both $t < 0$ and $t > 0$ with arbitrarily small $|t|$. Hence, no matter how we choose the point $\tilde{X}_0 \in B((1, 0), r) \cap \Omega_1$, there are points in $B((1, 0), r) \cap \partial\Omega_1$ that can be connected to \tilde{X}_0 inside Ω_1 only with paths γ satisfying $\ell(\gamma) \geq 1$.
- (2) $\Omega_2 := \mathbb{R}^2 \setminus \partial B(0, 1)$ satisfies the local John condition but not the John condition because we cannot connect a point in $B(0, 1)$ to a point in $\mathbb{R}^2 \setminus B(0, 1)$ with a path in Ω_2 .
- (3) $\Omega_3 := B(0, 1) \setminus \{(x, 0) : x \in (-1, 1)\}$ satisfies the weak local John condition but not the John condition or the local John condition. This is because Ω_3 is not connected (and hence cannot satisfy the John condition) and it cannot satisfy the local John condition for the same reason why Ω_1 in the part (1) of this example does not satisfy it.

To the best knowledge of the authors, it is not known if the local John condition alone implies the weak local John condition. We note that if there exists an open set $\Omega \subset \mathbb{R}^{n+1}$ with n -ADR boundary such that it satisfies the weak local John condition but not the local John condition, then Ω cannot satisfy the exterior corkscrew condition (see Lemma 3.6).

The John condition can be seen as a stronger form of connectivity of the space, but it does not imply connectivity for the boundary, not even if the exterior corkscrew condition holds. We can see this by considering the annulus $A := B(0, 2) \setminus B(0, 1)$. By the same example and the set Ω_2 in Example 3.3(2), we see that the local John condition does not generally imply connectivity for the space nor the boundary. However, by Theorem 1.2, we know that if the exterior corkscrew condition holds and the boundary of the space is Ahlfors–David regular, the local John condition implies connectivity for the space (but not the boundary, as we saw from the annulus A). By Corollaries 1.5 and 1.9, we know that the 2-sided local John condition combined with Ahlfors–David regularity of the boundary implies annular quasiconvexity (and therefore connectivity) for the boundary.

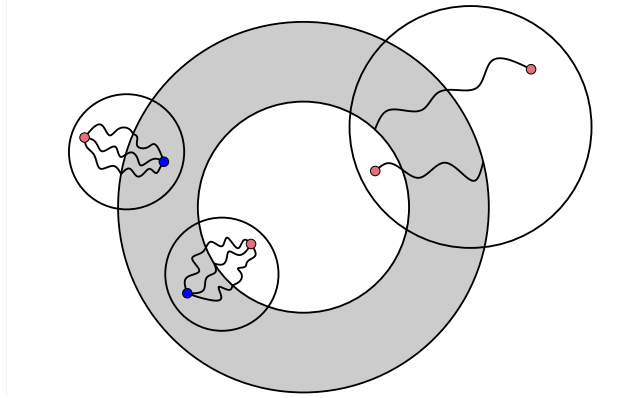


Figure 1. The 2-sided local (D_0, R_0) -John condition does not imply the Harnack chain condition or any kind of connectivity if R_0 is not large enough. In [Example 3.5](#), for the complement of a closed annulus, we can find local John points for small balls centered at either the inner or the outer boundary. However, if the ball is large enough, then it contains both inner and outer boundary points, and we cannot find a point in Ω that connects to all the boundary points inside the ball without passing through the annulus itself.

We note that the local John condition in [Definition 3.2](#) is the “extreme” case in the definition given in [\[Hofmann et al. 2010, Definition 3.12\]](#). To be more precise, let us consider the following weaker version of the local John condition:

Definition 3.4. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, $D_0 \geq 1$ and $0 < R_0 \leq \text{diam}(\partial\Omega)$. We say that Ω satisfies the *local (D_0, R_0) -John condition* if for every $x \in \partial\Omega$ and every $r \in (0, R_0)$ there exists a point $Y_x \in B(x, r) \cap \Omega$ (that we call a *local John point*) such that $B(Y_x, r/D_0) \subset \Omega$ and for every $z \in \Delta(x, r)$ there exists a D_0 -nontangential path γ_z in Ω from z to Y_x such that $\ell(\gamma_z) \leq D_0 r$.

Thus, the local D_0 -John condition and the local $(D_0, \text{diam}(\partial\Omega))$ -condition are the same thing. The reason why we consider only the case $(D_0, \text{diam}(\partial\Omega))$ in [Theorem 1.2](#) is simply because the result may fail if the local John condition is not good enough. We see this by the following simple example (see also [Figure 1](#)):

Example 3.5. Let $\Omega := B(0, 1) \cup \{X \in \mathbb{R}^2 : |X| > 2\} \subset \mathbb{R}^2$ be the interior of the complement of the annulus $B(0, 2) \setminus B(0, 1)$. Now $\partial\Omega$ is 1-ADR and Ω satisfies the 2-sided local $(D_0, \frac{1}{2})$ -John condition for suitable $D_0 > 1$, but Ω is not a chord-arc domain. In addition, the boundary $\partial\Omega$ is not connected.

In the proof of [Theorem 1.2](#), we use harmonic measure theory and techniques that require that harmonic measure belongs to the class weak- $A_\infty(\sigma)$. By the main result of [\[Azzam et al. 2020\]](#), we know that, in our context, the weak- A_∞ property is equivalent to uniform rectifiability for $\partial\Omega$ and the weak local John condition for Ω . We note that although we do not assume the weak local John condition, it follows from the assumptions of [Theorem 1.2](#). Indeed, by [\[David and Jerison 1990, p. 842\]](#) (see also [\[Semmes 1990\]](#)), we know that the 2-sided corkscrew condition implies the *interior big pieces of Lipschitz graphs condition* (see [\[Bennewitz and Lewis 2004, p. 572\]](#) for the definition). This condition then implies that

harmonic measure is in weak- A_∞ by [Bennewitz and Lewis 2004, Theorem 1]. Combining this with [Azzam et al. 2020, Theorem 1.1] gives us the weak local John condition:

Lemma 3.6. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set satisfying the 2-sided corkscrew condition, with n -dimensional Ahlfors–David regular boundary. Then $\omega \in \text{weak-}A_\infty(\sigma)$ and Ω satisfies the weak local John condition.*

4. The bilateral weak geometric lemma and nontangential approach

In this section, we consider some tools that we use in the proof of Theorem 1.2 to connect two pieces of different paths to each other without going too close to the boundary. Our approach is based on the use of β -numbers of [Jones 1990] and the bilateral weak geometric lemma of [David and Semmes 1993]. Throughout the section, we assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set satisfying the 2-sided corkscrew condition, with uniformly rectifiable boundary $\partial\Omega$, and \mathbb{D} is a dyadic system on $\partial\Omega$. Recall that $\sigma := \mathcal{H}^n|_{\partial\Omega}$ is the surface measure and $\delta(X) := \text{dist}(X, \partial\Omega)$ for $X \in \mathbb{R}^{n+1}$.

Given a ball $B = B(x, r) \subset \mathbb{R}^{n+1}$, a hyperplane $L \subset \mathbb{R}^{n+1}$ and a constant $c > 0$, we define

$$b\beta_{\partial\Omega}(B, L) := \sup_{y \in B \cap \partial\Omega} \frac{\text{dist}(y, L)}{r} + \sup_{Y \in L \cap B} \frac{\delta(Y)}{r},$$

$$\mathcal{U}_c(L) := \{Y \in \mathbb{R}^{n+1} : \text{dist}(Y, L) < c\}.$$

For any ball $B \subset \mathbb{R}^{n+1}$, we define the *bilateral β -number* as

$$b\beta_{\partial\Omega}(B) := \inf_L b\beta_{\partial\Omega}(B, L),$$

where the infimum is taken over all hyperplanes $L \subset \mathbb{R}^{n+1}$.

Recall from Notation 2.14 that for a cube $Q \in \mathbb{D}$ we write $B_Q = B(x_Q, \ell(Q))$, where x_Q is the center of Q and $\ell(Q)$ is its side length. By a straightforward reformulation of [David and Semmes 1993, Chapter I.2, Theorem 2.4], the following version of the *bilateral weak geometric lemma (BWGL)* holds:

Lemma 4.1. *For every $\varepsilon > 0$, there exists a constant $C_\varepsilon \geq 1$ such that*

$$\sum_{\substack{Q \in \mathbb{D}, Q \subset R, \\ b\beta_{\partial\Omega}(2B_Q) > \varepsilon}} \sigma(Q) \leq C_\varepsilon \sigma(R) \tag{4.2}$$

for any $R \in \mathbb{D}$, i.e., for any $\varepsilon > 0$ the collection $\{Q \in \mathbb{D} : b\beta_{\partial\Omega}(2B_Q) > \varepsilon\}$ satisfies a Carleson packing condition with Carleson packing norm depending only on ε , n and uniform rectifiability constants.

The BWGL actually characterizes the uniform rectifiability property but we only need the part written in Lemma 4.1.

In the proof of Theorem 1.2, we use the BWGL combined with the following lemmas that help us with technicalities related to constructing a path from a point to a nearby local John point. Alternatively, we could use the Whitney region constructions of Hofmann, Martell and Mayboroda [Hofmann et al. 2016, Section 3] (and their straightforward geometric applications in [Hofmann and Tapiola 2020; 2021]) for the same purpose, but this alternative approach is slightly less elementary than the one we present in this paper.

Lemma 4.3. *There exists $\varepsilon_0 > 0$, depending only on the 2-sided corkscrew condition, such that the following holds: if $B = B(x, r)$ is a ball with $x \in \partial\Omega$ and L_B is a hyperplane such that $b\beta_{\partial\Omega}(B, L_B) < \varepsilon \leq \varepsilon_0$, then $B \setminus \mathcal{U}_{\varepsilon r}(L_B)$ consists of two convex components, B^+ and B^- , such that*

$$B^+ \subset \Omega \quad \text{and} \quad B^- \subset \mathbb{R}^{n+1} \setminus \bar{\Omega}.$$

Proof. Let $B = B(x, r)$ be a ball with $x \in \partial\Omega$, $\varepsilon > 0$ and L_B be a hyperplane such that $b\beta_{\partial\Omega}(B, L_B) < \varepsilon$. By the definitions of $b\beta_{\partial\Omega}(B, L_B)$ and $\mathcal{U}_{\varepsilon r}(L_B)$, we know that

$$\partial\Omega \cap B \subset \mathcal{U}_{\varepsilon r}(L_B),$$

and $B \setminus \mathcal{U}_{\varepsilon r}(L_B)$ has exactly two components if ε is small enough, say $\varepsilon < \frac{1}{10}$. Thus, the two connected components V_1 and V_2 of $B \setminus \mathcal{U}_{\varepsilon r}(L_B)$ are contained in $\mathbb{R}^{n+1} \setminus \partial\Omega = \Omega \cup (\mathbb{R}^{n+1} \setminus \bar{\Omega})$. Furthermore, the components V_1 and V_2 are convex by the definition of $\mathcal{U}_{\varepsilon r}(L_B)$, and each V_i is either fully contained in Ω or fully contained in $\mathbb{R}^{n+1} \setminus \bar{\Omega}$. Indeed, if V_i intersects both Ω and $\mathbb{R}^{n+1} \setminus \bar{\Omega}$, then the line segment from any point $Z_1 \in V_i \cap \Omega$ to any point $Z_2 \in V_i \cap \mathbb{R}^{n+1} \setminus \bar{\Omega}$ has to contain a point $z_0 \in \partial\Omega$ which then has to belong to V_i by convexity. This is impossible because $V_i \subset B \setminus \mathcal{U}_{\varepsilon r}(L_B)$ and therefore $V_i \cap \partial\Omega = \emptyset$.

Let us then show that if ε is small enough, one of the components V_i lies in Ω and the other lies in $\mathbb{R}^{n+1} \setminus \bar{\Omega}$. By the 2-sided corkscrew condition (applied for the surface ball $\Delta(x, r) = B(x, r) \cap \partial\Omega$), there exist balls

$$B_1 := B(Z^+, cr) \subset \Omega \cap B, \quad B_2 := B(Z^-, cr) \subset (\mathbb{R}^{n+1} \setminus \bar{\Omega}) \cap B,$$

where $c \in (0, 1)$ is independent of Δ . Let us assume that $\varepsilon \leq \frac{c}{5}$. Now neither B_1 nor B_2 can be contained in $\mathcal{U}_{\varepsilon r}(L_B)$ and therefore both the balls intersect $V_1 \cup V_2$. Since each V_i is either fully contained in Ω or fully contained in $\mathbb{R}^{n+1} \setminus \bar{\Omega}$, we know that $V_i \cap B_1 \neq \emptyset$ implies $V_i \subset \Omega$ and $V_i \cap B_2 \neq \emptyset$ implies $V_i \subset \mathbb{R}^{n+1} \setminus \bar{\Omega}$. We also notice that if $V_1 \cap B_1 \neq \emptyset$, then $V_1 \cap B_2 = \emptyset$ since otherwise V_1 intersects both Ω and $\mathbb{R}^{n+1} \setminus \bar{\Omega}$. Thus, since both B_1 and B_2 intersect $V_1 \cup V_2$, we know that $V_1 \cap B_1 \neq \emptyset$ implies $V_2 \cap B_2 \neq \emptyset$, and similarly $V_1 \cap B_2 \neq \emptyset$ implies $V_2 \cap B_1 \neq \emptyset$. In particular, B_1 intersects exactly one of the components V_i , say V_1 , which is then contained in Ω , and B_2 then intersects V_2 , which is contained in $\mathbb{R}^{n+1} \setminus \bar{\Omega}$. Thus, we may set $B^+ = V_1$ and $B^- = V_2$ and choose $\varepsilon_0 = \min\{\frac{1}{10}, \frac{c}{5}\}$. □

Lemma 4.4. *Let $B = B(x_0, r)$ be a ball with $x_0 \in \partial\Omega$, $X_1 \in \Omega$ a point such that $|X_1 - x_0| \approx \delta(X_1) \geq \frac{r}{2}$ and γ a D -nontangential path from x_0 to X_1 , where $D \geq 1$. Let $\varepsilon_0 > 0$ be as in Lemma 4.3 and suppose that $0 < \varepsilon < \min\{\varepsilon_0, \frac{1}{12D}\}$. Let L_B be a hyperplane such that $b\beta_{\partial\Omega}(B, L_B) < \varepsilon$, and let B^+ and B^- be the components of B as in Lemma 4.3. Now γ intersects $B^+ \setminus \mathcal{U}_{2\varepsilon r}(L_B)$.*

Proof. Since $|X_1 - x_0| \geq \frac{r}{2}$, we know that $X_0 \notin B(x_0, \frac{r}{4})$. Thus, there exists a point $Y_0 \in \gamma \cap \partial B(x_0, \frac{r}{4})$. We claim that $Y_0 \in B^+ \setminus \mathcal{U}_{2\varepsilon r}(L_B)$.

We notice that, by the definition of D -nontangential paths, we have

$$\delta(Y_0) \geq \frac{1}{D} \ell(\gamma(x_0, Y_0)) \geq \frac{1}{D} |x_0 - Y_0| = \frac{1}{4D} r. \tag{4.5}$$

For any point $X \in \mathbb{R}^{n+1}$, let pr_X be its orthogonal projection onto L_B . Let $Z \in \frac{1}{2}B \cap \mathcal{U}_{2\varepsilon r}(L_B)$. Now it holds that

$$|\text{pr}_Z - x_0| \leq |\text{pr}_Z - Z| + |Z - x_0| < 2\varepsilon r + \frac{1}{2}r < r,$$

and thus, $\text{pr}_Z \in B$. In particular, since $\text{pr}_Z \in L_B$ and $b\beta_{\partial\Omega}(B, L_B) < \varepsilon$, we have

$$\delta(Z) \leq |Z - \text{pr}_Z| + \delta(\text{pr}_Z) \leq 2\varepsilon r + \varepsilon r = 3\varepsilon r < \frac{1}{4D}r \tag{4.6}$$

since $\varepsilon < \frac{1}{12D}$. In particular, by (4.5) and (4.6), we know that $Y_0 \notin \frac{1}{2}B \cap \mathcal{U}_{2\varepsilon r}(L_B)$. On the other hand, since $Y_0 \in \gamma \cap \partial B(x_0, \frac{r}{4})$, we know that $Y_0 \in B \cap \Omega$ and hence $Y_0 \in B^+$ by Lemma 4.3. In particular, $Y_0 \in \gamma \cap B^+ \setminus \mathcal{U}_{2\varepsilon r}(L_B)$, which proves the claim. \square

Lemma 4.7. *Let $\varepsilon_0 > 0$ be as in Lemma 4.3 and suppose that $0 < \varepsilon < \frac{1}{8}\varepsilon_0$. Let $B = B(x_0, r)$ be a ball with $x_0 \in \partial\Omega$, L_B be a hyperplane such that $b\beta_{\partial\Omega}(B, L_B) < \varepsilon$ and $y_0 \in \frac{1}{2}B \cap \partial\Omega$. Now we have*

- $b\beta_{\partial\Omega}(B(y_0, \frac{r}{4}), L_B) < 4\varepsilon < \frac{1}{2}\varepsilon_0$,
- the set $\Omega \cap B(y_0, \frac{r}{4}) \setminus \mathcal{U}_{\varepsilon r}(L_B)$ is convex,
- for any point $Y \in \Omega \cap B(y_0, \frac{r}{4}) \setminus \mathcal{U}_{2\varepsilon r}(L_B)$ we have $\delta(Y) \geq \varepsilon r$.

Proof. The second claim follows from the first claim combined with Lemma 4.3, and the first claim follows from the definition of β -numbers and the facts $B(y_0, \frac{r}{4}) \subset B$ and $b\beta_{\partial\Omega}(B, L_B) < \varepsilon < \frac{1}{8}\varepsilon_0$ in a straightforward way:

$$\begin{aligned} b\beta_{\partial\Omega}(B, L_B) &= \sup_{y \in B \cap \partial\Omega} \frac{\text{dist}(y, L_B)}{r} + \sup_{Y \in L_B \cap B} \frac{\delta(Y)}{r} \\ &\geq \frac{1}{4} \sup_{y \in B(y_0, r/4) \cap \partial\Omega} \frac{\text{dist}(y, L_B)}{\frac{r}{4}} + \sup_{Y \in L_B \cap B(y_0, r/4)} \frac{\delta(Y)}{\frac{r}{4}} = \frac{1}{4} b\beta_{\partial\Omega}\left(B\left(y_0, \frac{r}{4}\right), L\right). \end{aligned}$$

For the third claim, let $Y \in \Omega \cap B(y_0, \frac{r}{4}) \setminus \mathcal{U}_{2\varepsilon r}(L_B)$. Since $y_0 \in \partial\Omega$, we know that $\delta(Y) < \frac{r}{4}$. On the other hand, for any point $Z \in \mathbb{R}^{n+1} \setminus B$ we have

$$r \leq |x_0 - Z| \leq |x_0 - y_0| + |y_0 - Y| + |Y - Z| < \frac{r}{2} + \frac{r}{4} + |Y - Z|,$$

and thus, $|Y - Z| > \frac{r}{4}$. In particular, we have $\delta(Y) = \inf_{z \in \partial\Omega \cap B} |Y - z|$. Let $z_Y \in \partial\Omega \cap B$ be a point such that $\delta(Y) = |Y - z_Y|$. Since $Y \notin \mathcal{U}_{2\varepsilon r}(L_B)$, we know that $\text{dist}(Y, L_B) \geq 2\varepsilon r$, and since $b\beta_{\partial\Omega}(B, L_B) < \varepsilon$, we know that $\text{dist}(z_Y, L_B) < \varepsilon r$. In particular,

$$2\varepsilon r \leq \text{dist}(Y, L_B) \leq |Y - z_Y| + \text{dist}(z_Y, L_B) < |Y - z_Y| + \varepsilon r = \delta(Y) + \varepsilon r,$$

and therefore $\delta(Y) \geq \varepsilon r$, as claimed. \square

5. Local John and exterior corkscrews imply Harnack chains

In this section, we prove Theorem 1.2, i.e., that the local John condition together with the exterior corkscrew condition implies the existence of Harnack chains. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with n -ADR boundary $\partial\Omega$, and suppose that Ω satisfies the local D_1 -John condition and exterior corkscrew condition. Throughout the section, \mathbb{D} is a dyadic system on $\partial\Omega$.

Proof of Theorem 1.2. Let $X, Y \in \Omega$ with $\delta(X), \delta(Y) \geq \rho > 0$ and $|X - Y| < \Lambda\rho$ for $\Lambda \geq 1$. We start by noticing that by Theorem 2.11 we know that $\partial\Omega$ is UR, and by Lemma 3.6 we know that harmonic measure belongs to the class weak- $A_\infty(\sigma)$.

We will construct a path between X and Y that stays far away from the boundary and use this path to construct the Harnack chain between X and Y . To ensure that we stay away from the boundary in a quantitative way, we have to be careful with the construction. This will make the construction quite technical and therefore we divide the proof into a few different parts.

The argument we present below works as it is in the case $\text{diam}(\partial\Omega) = \infty$. We discuss the other cases in the end of the proof.

Part 1: choosing suitable cubes for X and Y for [Lemma 2.19](#). Take a point $z_X \in \partial\Omega$ such that $\delta(X) = |X - z_X|$. Let c_0 and C_0 be the constants from [Lemma 2.20](#). Now, by [Lemma 2.20](#), since $X \in B(z_X, c_0 \cdot 2\delta(X)/c_0)$, we know that $\omega^X(\Delta(z_X, 2\delta(X)/c_0)) \geq 1/C_0$. Let us cover $\Delta(z_X, 2\delta(X)/c_0)$ by dyadic cubes Q_i of the same side length such that $\ell(Q_i) \approx 2\delta(X)/c_0$ and $X \notin 4B_{Q_i}$ for any i . There are at most c_n of these types of cubes Q_i . Since $\omega^X(\Delta(z_X, 2\delta(X)/c_0)) \geq 1/C_0$ and the cubes Q_i cover $\Delta(z_X, 2\delta(X)/c_0)$, we know that there exists a cube $Q_X \in \{Q_i\}_i$ such that $\omega^X(Q_X) \geq (c_n C_0)^{-1}$. In addition, the cube Q_X satisfies $\ell(Q_X) \approx \delta(X) \approx \text{dist}(X, Q_X)$ with uniformly bounded implicit constants since

- one of the cubes Q_i contains the point z_X and $\delta(X) = |X - z_X|$,
- there is only a uniformly bounded number of the cubes Q_i , and
- $\ell(Q_i) \approx 2\delta(X)/c_0$ for every i .

Similarly, we can choose a cube Q_Y that has the same properties but with respect to Y instead of X .

Part 2: choosing a local John point. Let us consider the ball $B(z_X, D_1 r_0)$, where

$$r_0 := C_n \cdot \max\{\delta(X), \delta(Y), |X - Y|\} \tag{5.1}$$

for a large enough dimensional constant C_n such that $Q_X, Q_Y \subset B(z_X, D_1 r_0)$. Let $Z_0 \in \Omega$ be a local John point for the ball $B(z_X, D_1 r_0)$. By the local John condition, we know that $B(Z_0, D_1 r_0/D_1) = B(Z_0, r_0) \subset \Omega$ and there exist D_1 -nontangential paths from the points of $\Delta(z_X, D_1 r_0)$ to Z_0 . In particular, there exist D_1 -nontangential paths from the points on Q_X and Q_Y to Z_0 .

Part 3: choosing starting points for paths. Let us consider the cube Q_X . By the choice of Q_X and [Lemma 2.19](#), we know that there exist constants $\alpha \in (0, 1]$ and $D_2 \geq 1$ (independent of X and Q_X) and a subset $A_X \subset Q_X$ such that

- $\sigma(A_X) \geq \alpha \sigma(Q_X)$, and
- there exist D_2 -nontangential paths from the points on A_X to X .

Let $\varepsilon_0 > 0$ be as in [Lemma 4.3](#) and set

$$\varepsilon := \frac{1}{8} \min\left\{\frac{1}{10}\varepsilon_0, \frac{1}{12D_1}, \frac{1}{12D_2}\right\}. \tag{5.2}$$

By the bilateral weak geometric lemma ([Lemma 4.1](#)) and [Lemma 2.16](#), we know that there exists a cube R_X such that $b\beta_{\partial\Omega}(2B_{R_X}) < \varepsilon$, $\sigma(R_X \cap A_X) > 0$ and $\ell(R_X) \approx \ell(Q_X)$, where the implicit constant depends only on α and the Carleson packing norm of the collection of cubes Q such that $b\beta_{\partial\Omega}(2B_Q) > \varepsilon$. Let us choose any point $\tilde{z}_X \in R_X \cap A_X$. Similarly, we can choose sets A_Y and R_Y and a point \tilde{z}_Y for the point Y .

Part 4: constructing paths between the local John point and X and Y . Let us recap:

- $R_X \subset \partial\Omega$ is a dyadic cube such that $\ell(R_X) \approx \delta(X)$.
- $\varepsilon > 0$ is a number defined in (5.2) and we have $b\beta_{\partial\Omega}(2B_{R_X}) < \varepsilon$.
- $\tilde{z}_X \in R_X$ is a point such that there exists a D_2 -nontangential path from \tilde{z}_X to X .
- $Z_0 \in \Omega$ is the local John point for the ball $B(z_X, D_1r_0)$, where $|X - z_X| = \delta(X)$ and $r_0 > 0$ is the radius defined in (5.1), and $\tilde{z}_X \in B(z_X, D_1r_0)$.

Let γ_1 be a D_1 -nontangential path from \tilde{z}_X to Z_0 and γ_2 be a D_2 -nontangential path from \tilde{z}_X to X . Let L_X be a hyperplane such that $b\beta_{\partial\Omega}(2B_{R_X}, L_X) < \varepsilon$ and consider the ball $B(\tilde{z}_X, \frac{1}{2}\ell(R_X))$. Since we know that $\tilde{z}_X \in R_X \subset B_{R_X} \subset 2B_{R_X}$, Lemma 4.7 gives us

$$b\beta_{\partial\Omega}\left(B\left(\tilde{z}_X, \frac{1}{2}\ell(R_X)\right), L_X\right) < 4\varepsilon < \frac{1}{2} \min\left\{\frac{1}{10}\varepsilon_0, \frac{1}{12D_1}, \frac{1}{12D_2}\right\}.$$

By Lemma 4.4 (applied for $b\beta_{\partial\Omega}(B(\tilde{z}_X, \frac{1}{2}\ell(R_X)), L_X) < 4\varepsilon$), we know that both γ_1 and γ_2 intersect

$$B\left(\tilde{z}_X, \frac{1}{2}\ell(R_X)\right) \setminus \mathcal{U}_{2.4\varepsilon\ell(R_X)/2}(L_X) = B\left(\tilde{z}_X, \frac{1}{2}\ell(R_X)\right) \setminus \mathcal{U}_{2\varepsilon\cdot 2\ell(R_X)}(L_X).$$

Let $Z_1 \in \gamma_1$ and $X_1 \in \gamma_2$ be any points such that $Z_1, X_1 \in B(\tilde{z}_X, \frac{1}{2}\ell(R_X)) \setminus \mathcal{U}_{2\varepsilon\cdot 2\ell(R_X)}(L_X)$, and let γ_3 be the line segment connecting Z_1 to X_1 . By Lemma 4.7, we know that γ_3 is fully contained in $B(\tilde{z}_X, \frac{1}{2}\ell(R_X)) \setminus \mathcal{U}_{2\varepsilon\cdot 2\ell(R_X)}(L_X)$ and we have

$$\delta(\widehat{X}) \geq \varepsilon \cdot 2\ell(R_X) \gtrsim \delta(X) \tag{5.3}$$

for every $\widehat{X} \in \gamma_3$, where the implicit constant depends on ε_0, D_1, D_2 and the structural constants appearing in the proof. Since γ_3 is line segment that is fully contained in $B(\tilde{z}_X, \frac{1}{2}\ell(R_X)) \setminus \mathcal{U}_{2\varepsilon\cdot 2\ell(R_X)}(L_X)$, we know

$$\ell(\gamma_3) \leq \ell(R_X) \lesssim \delta(X). \tag{5.4}$$

See Figure 2 for an illustration of the situation.

We now build a path γ_X from Z_0 to X by gluing together (after rescaling) the reversed part of γ_1 that travels from Z_1 to Z_0 , the whole γ_3 (from Z_1 to X_1) and the part of γ_2 that travels from X_1 to X . Similarly, we can choose a hyperplane L_Y for R_Y and points $Z_2, Y_1 \in B(\tilde{z}_Y, \frac{1}{2}\ell(R_Y)) \setminus \mathcal{U}_{2\varepsilon\cdot 2\ell(R_Y)}(L_Y)$ and construct a path γ_Y from Z_0 to Y that passes through Z_2 and Y_1 .

Part 5: constructing the Harnack chains. Let us consider the path γ_X . Since γ_2 is a nontangential path (from \tilde{z}_X to X) and $\delta(X_1) \approx \delta(X)$, we know that $\ell(\gamma_2) \lesssim \delta(X)$ and $\delta(\widehat{X}) \approx \delta(X)$ for every $\widehat{X} \in \gamma_2(X_1, X)$. By (5.4), we know that $\ell(\gamma_3) \lesssim \delta(X)$, and by (5.3), $\delta(\widehat{X}) \approx \delta(X)$ for every $\widehat{X} \in \gamma_3$. Thus, for a suitable uniform implicit constant, we may cover $\gamma_X \setminus \gamma_1$ by a uniformly bounded number of balls B_i with radii $r_i \approx \delta(X)$ satisfying $\text{dist}(B_i, \partial\Omega) \approx \text{diam}(B_i)$. As for γ_1 , we notice that since γ_1 is a nontangential path (from \tilde{z}_X to Z_0) given by the local John condition and $\delta(Z_1) \approx \delta(X)$, we have

$$\ell(\gamma_1(\tilde{z}_X, Z_1)) \lesssim \delta(Z_1) \approx \delta(X) \quad \text{and} \quad \ell(\gamma_1) \lesssim r_0 = C_n \cdot \max\{\delta(X), \delta(Y), |X - Y|\},$$

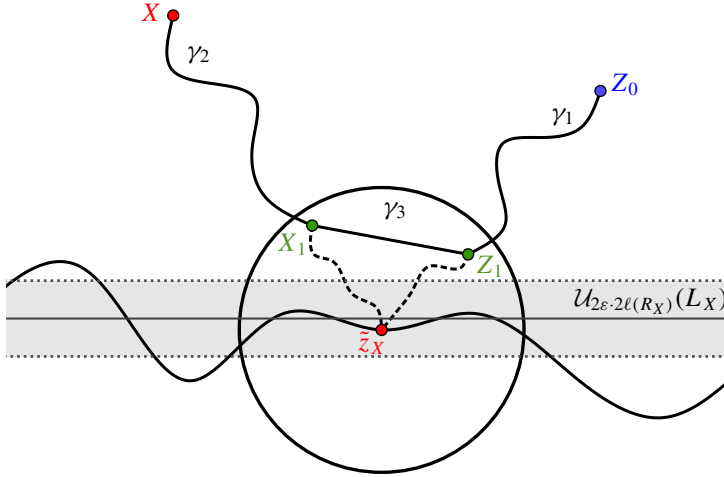


Figure 2. We construct a path between X and the local John point Z_0 that stays far away from the boundary the following way. By a careful choice of a point $\tilde{z}_X \in \partial\Omega$, we know that there exists a nontangential path γ_1 from \tilde{z}_X to Z_0 (given by the local John condition) and a nontangential path γ_2 from \tilde{z}_X to X (given by the weak- A_∞ property of harmonic measure through Lemma 2.19). By the BWGL, there exists a hyperplane L_X such that L_X approximates $B \cap \partial\Omega$ well for a suitable $\varepsilon > 0$ and a ball $B = B(\tilde{z}_X, \frac{1}{2}\ell(R_X))$, where R_X is a dyadic cube containing \tilde{z}_X such that $\ell(R_X) \approx \delta(X)$. By applications of the BWGL, we know that there exist points $Z_1 \in \gamma_1 \cap B \cap \Omega$ and $X_1 \in \gamma_2 \cap B \cap \Omega$ that do not lie on the strip $\mathcal{U}_{2^\varepsilon \cdot 2\ell(R_X)}(L_X)$, i.e., they lie reasonably far away from the boundary. Because $\Omega \cap B \setminus \mathcal{U}_{2^\varepsilon \cdot 2\ell(R_X)}(L_X)$ is convex, we can connect Z_1 and X_1 to each other with a line segment γ_3 . We can now travel from Z_0 to X by using pieces of the paths γ_1 , γ_3 and γ_2 .

and therefore $\delta(\widehat{X}) \gtrsim \delta(X)$ for all $\widehat{X} \in \gamma_1(Z_1, Z_0)$. In particular, we can cover $\gamma_X \setminus (\gamma_2 \cup \gamma_3)$ by $N_0 \lesssim r_0/\delta(X)$ balls B_k of radii $r_k \approx \delta(X)$ such that $\text{dist}(B_k, \partial\Omega) \approx \text{diam}(B_k)$ for each k . Recall that $\delta(X), \delta(Y) \geq \rho > 0$ and $|X - Y| < \Lambda\rho$ for $\Lambda \geq 1$. Let us consider different cases:

- Suppose that $r_0 \lesssim |X - Y|$. Now we have

$$N_0 \lesssim \frac{|X - Y|}{\delta(X)} = \frac{\rho}{\delta(X)} \Lambda \leq \Lambda.$$

- Suppose that $r_0 = C_n\delta(X)$. Then

$$N_0 \lesssim \frac{C_n\delta(X)}{\delta(X)} = C_n \leq C_n\Lambda.$$

- Suppose that $r_0 = C_n\delta(Y)$ and $\delta(Y) \gg |X - Y|$. Then, by the triangle inequality, $\delta(X) \approx \delta(Y)$ and

$$N_0 \lesssim \frac{C_n\delta(Y)}{\delta(X)} \approx \frac{C_n\delta(X)}{\delta(X)} = C_n \leq C_n\Lambda.$$

By almost identical arguments, we know that same estimates hold for γ_Y . Thus, we can connect X to Y by taking the chain of balls B_i that covers γ_X and γ_Y . These balls satisfy $\text{dist}(B_i, \partial\Omega) \approx \text{diam}(B_i)$ for

every i with possibly different implicit constants. We may choose a constant $\tilde{C} \geq 1$ such that

$$\frac{1}{\tilde{C}} \text{diam}(B_i) \leq \text{dist}(B_i, \partial\Omega) \leq \tilde{C} \text{diam}(B_i)$$

for every i since we used only a finite number of different constants in the construction (six, to be more precise, since we built γ_X and γ_Y using three pieces for each path with uniform implicit constants for each piece). We needed a uniformly bounded number of balls to cover the two out of three pieces of γ_X (same for γ_Y) and $N \lesssim \Lambda$ balls to cover the last piece of γ_X (same for γ_Y). Thus, the number of balls B_i is bounded by $C\Lambda$, where C is a large enough constant depending only on n and the ADR, UR, local John, weak- A_∞ and corkscrew constants. Hence, $(B_i)_i$ is a Harnack chain between X and Y . This completes the proof for the case $\text{diam}(\partial\Omega) = \infty$.

Let us then consider the remaining two cases. Suppose that $\text{diam}(\partial\Omega) < \infty$ and $\text{diam}(\Omega) < \infty$. Now, if $\max\{\delta(X), \delta(Y), |X - Y|\} \ll \text{diam}(\partial\Omega)$, things work just as earlier. Thus, we may assume that $\max\{\delta(X), \delta(Y), |X - Y|\} \approx \text{diam}(\partial\Omega)$. Since $\text{diam}(\partial\Omega) < \infty$, there exists a point $Z_0 \in \Omega$ such that for any $z \in \partial\Omega$ there exists a D_1 -nontangential path from z to Z_0 . Now the previous proof works when we simply choose this “global” local John point instead of the point we chose in Part 2.

Finally, suppose that $\text{diam}(\partial\Omega) < \infty$ and $\text{diam}(\Omega) = \infty$. Let us consider the following three cases:

- If $\max\{\delta(X), \delta(Y), |X - Y|\} \ll \text{diam}(\partial\Omega)$, we proceed as in the case “ $\text{diam}(\partial\Omega) = \infty$ ”.
- If $\delta(X) \lesssim \text{diam}(\partial\Omega)$, $\delta(Y) \lesssim \text{diam}(\partial\Omega)$ and $|X - Y| \approx \text{diam}(\partial\Omega)$, we proceed as in the case “ $\text{diam}(\partial\Omega) < \infty$ and $\text{diam}(\Omega) < \infty$ ”.
- If $\delta(X) \gg \text{diam}(\partial\Omega)$ and $\delta(Y) \gg \text{diam}(\partial\Omega)$, we can construct a Harnack chain from X to Y in the simple geometry of $\mathbb{R}^{n+1} \setminus B(Z, s)$ for $Z \in \Omega$ and $s \approx \text{diam}(\partial\Omega)$.

Thus, we may assume that $\delta(X) \lesssim \text{diam}(\partial\Omega)$ and $\delta(Y) \gg \text{diam}(\partial\Omega)$. The previous procedure does not work directly in this case because we cannot apply Lemmas 2.20 and 2.19 for $\delta(Y) \gg \text{diam}(\partial\Omega)$. Instead, we connect Y to a point that is close enough to the boundary and then connect this point to X .

Let $\hat{y} \in \partial\Omega$ be a point such that $\delta(Y) = |Y - \hat{y}|$. Let us consider the line segment L with endpoints \hat{y} and Y . Since L is a line segment and $\delta(Y) = |Y - \hat{y}|$, we have $\delta(Z) = |Z - \hat{y}|$ for any $Z \in L$. Let us take a point $\hat{Y} \in L$ such that $\delta(\hat{Y}) = \delta(X)$. Now we can use the earlier procedure to construct a Harnack chain $(B_i)_i$ from X to \hat{Y} , possibly using a “global” local John point as we did in the case “ $\text{diam}(\partial\Omega) < \infty$ and $\text{diam}(\Omega) < \infty$ ”. Since $\delta(X) = \delta(\hat{Y}) \lesssim \text{diam}(\partial\Omega)$ and $\delta(Y) \gg \text{diam}(\partial\Omega)$, the length of the chain $(B_i)_i$ depends only on

$$\frac{|X - \hat{Y}|}{\delta(X)} \lesssim \frac{|X - Y|}{\delta(X)} \leq \frac{\rho}{\delta(X)} \Lambda \leq \Lambda.$$

We then continue this chain from \hat{Y} to Y by covering the line segment from \hat{Y} to Y using balls \hat{B}_k with radii $r_k \approx \delta(\hat{Y}) = \delta(X)$ such that $\text{dist}(\hat{B}_k, \partial\Omega) \approx \text{diam}(\hat{B}_k)$. The number of balls \hat{B}_k that we need is approximately

$$\frac{\ell(L)}{\delta(X)} \approx \frac{\delta(Y)}{\delta(X)} \approx \frac{|X - Y|}{\delta(X)} \leq \frac{\rho}{\delta(X)} \Lambda \leq \Lambda,$$

where we used that $\delta(X) \lesssim \text{diam}(\partial\Omega)$ and $\delta(Y) \gg \text{diam}(\partial\Omega)$ (and therefore we have $\delta(Y) \approx |X - Y|$). Hence, combining the chains $(B_i)_i$ and $(\widehat{B}_k)_k$ gives us a Harnack chain from X to Y . This completes the proof of the last case. \square

6. Weak (1,1)-version of the Hofmann–Mitrea–Taylor Poincaré inequality and quasiconvexity

Let Ω be a 2-sided chord-arc domain. In this section, we consider some parts of the Hofmann–Mitrea–Taylor theory that we need for the proof of [Theorem 1.3](#). Recall the definitions of the tangential gradient $\nabla_t f$, the Hofmann–Mitrea–Taylor Sobolev space $L^p_1(\partial\Omega)$ and the Hofmann–Mitrea–Taylor tangential derivatives $\partial_{t,j,k} f$ and gradient $\nabla_{\text{HMT}} f$ in [Section 2D](#). It is straightforward to check that for a compactly supported Lipschitz function f on $\partial\Omega$ we have $f \in L^p_1(\partial\Omega)$ for every $1 < p < \infty$ and hence the Hofmann–Mitrea–Taylor gradient $\nabla_{\text{HMT}} f$ exists.

Remark 6.1. In this section and in [Section 7](#), we mostly consider compactly supported Lipschitz functions but this is enough for our purposes: by the results of [\[Keith 2003\]](#) (see [Theorem 7.12](#) below) and [\[Mourgoglou and Tolsa 2021\]](#) (see [Lemmas 6.2](#) and [6.4](#) below), verifying [Theorem 1.3](#) for compactly supported Lipschitz functions implies [Corollary 1.6](#), which then allows us to give up the assumption about compact support for [Theorem 1.3](#). Assuming that our Lipschitz functions are compactly supported ensures that the Hofmann–Mitrea–Taylor gradients exist and we can use the machinery in [\[Hofmann et al. 2010; Mourgoglou and Tolsa 2021\]](#) without additional considerations.

Let us start by recalling two lemmas from [\[Mourgoglou and Tolsa 2021\]](#). First, when considering compactly supported Lipschitz functions, the norm of the Hofmann–Mitrea–Taylor gradient agrees with the norm of the tangential gradient almost everywhere:

Lemma 6.2 [\[Mourgoglou and Tolsa 2021, Lemma 6.4\]](#). *Let f be a compactly supported Lipschitz function on $\partial\Omega$. Then*

$$|\nabla_t f| = |\nabla_{\text{HMT}} f| \tag{6.3}$$

σ -almost everywhere.

In fact, [\[Mourgoglou and Tolsa 2021, Lemma 6.4\]](#) shows us that $\nabla_t f = -\nabla_{\text{HMT}} f$ almost everywhere, but we only need the comparability of the norms. We note that that lemma is formulated for bounded domains but a routine inspection of the proof shows us that it holds for compactly supported Lipschitz functions also in unbounded domains.

Furthermore, for Lipschitz functions, the norm of the tangential gradient (and hence the norm of the Hofmann–Mitrea–Taylor gradient) agrees almost everywhere with the “local Lipschitz constant” function:

Lemma 6.4 [\[Mourgoglou and Tolsa 2021, Lemma 2.2\]](#). *Let f be a Lipschitz function on $\partial\Omega$. Then*

$$|\nabla_t f(x)| = \limsup_{\partial\Omega \ni y \rightarrow x} \frac{|f(x) - f(y)|}{|x - y|} \approx \limsup_{r \rightarrow 0} \int_{\Delta(x,r)} \frac{|f(x) - f(y)|}{|x - y|} d\sigma(y) \tag{6.5}$$

for σ -a.e. $x \in \partial\Omega$.

As a part of their extensive work, Hofmann, Mitrea and Taylor proved the following weak (p, p) -Poincaré inequality with a tail:

Theorem 6.6 [Hofmann et al. 2010, Proposition 4.13]. *Let $1 < p < \infty$. There exists a constant $C = C(\Omega, p)$ such that for every $f \in L^p_1(\partial\Omega)$, $x \in \partial\Omega$ and $r > 0$ we have*

$$\left(\int_{\Delta} |f - \langle f \rangle_{\Delta}|^p d\sigma \right)^{1/p} \leq Cr \left(\int_{5\Delta} |\nabla_{\text{HMT}} f|^p d\sigma \right)^{1/p} + Cr \sum_{j=2}^{\infty} \frac{2^{-j}}{\sigma(2^j \Delta)} \int_{2^j \Delta \setminus 2^{j-1} \Delta} |\nabla_{\text{HMT}} f| d\sigma, \tag{6.7}$$

where $\Delta := \Delta(x, r)$.

Some remarks related to the formulation of Theorem 6.6 are in order. In [Hofmann et al. 2010], Theorem 6.6 is formulated assuming only that $\partial\Omega$ is Ahlfors–David regular and Ω satisfies the 2-sided local (D_0, R_0) -John condition (recall Definition 3.4), and the result holds for all $r \in (0, R_0)$ instead of all $r \in (0, \text{diam}(\partial\Omega))$. As we noted in Example 3.5, if R_0 is not large enough, the 2-sided local (D_0, R_0) -John condition is not strong enough to imply that $\partial\Omega$ is connected. Since Heinonen–Koskela-type weak Poincaré inequalities (recall Definition 2.29) imply σ connectivity (see, e.g., [Cheeger 1999, Theorem 17.1]), we know that the Hofmann–Mitrea–Taylor Poincaré inequalities may hold even when Heinonen–Koskela-type weak Poincaré inequalities fail.

This being said, if Ω is a 2-sided chord-arc domain (as we assumed in the beginning of this section), then the Hofmann–Mitrea–Taylor Poincaré inequalities hold as in Theorem 6.6 and they self-improve to Heinonen–Koskela-type Poincaré inequalities. The first claim is due to [Hofmann et al. 2010, Lemma 3.13]: any NTA domain satisfies the local (D_0, R_0) -John condition where R_0 is the upper bound for the scales of the corkscrew conditions. Since we assume that the corkscrew conditions hold for any $0 < r < \text{diam}(\partial\Omega)$, we get the local D_0 -John condition. The second claim follows from Corollary 1.6 which we prove using a Hofmann–Mitrea–Taylor-type Poincaré estimate (see Lemma 6.8 below) in the next section.

For the case $p > 1$, we can use directly the estimate (6.7) together with the arguments in the next section to obtain a weak p -Poincaré inequality. However, for the case $p = 1$, we need to revisit estimate (6.7) since an inspection of its proof shows us that a simple limiting argument does not work. Because of this, we prove the following weak $(1, 1)$ -type version of the Hofmann–Mitrea–Taylor Poincaré inequality:

Lemma 6.8. *There exists a constant $C = C(\Omega)$ such that the following holds. For every compactly supported Lipschitz function f on $\partial\Omega$ and every $\xi \in \partial\Omega$, $r > 0$ and $\Delta := \Delta(\xi, r)$, there exists $a_{\Delta} \in \mathbb{R}$ such that*

$$\sup_{\lambda > 0} \lambda \sigma(\{y \in \Delta : |f(y) - a_{\Delta}| > \lambda\}) \leq Cr \int_{5\Delta} |\nabla_{\text{HMT}} f| d\sigma + Cr \sum_{j=2}^{\infty} 2^{-j(n+1)} \int_{2^j \Delta \setminus 2^{j-1} \Delta} |\nabla_{\text{HMT}} f| d\sigma. \tag{6.9}$$

The proof of Lemma 6.8 is similar to the proof of estimate (6.7), which uses heavily the machinery built in [Hofmann et al. 2010]. For the convenience of the reader, we provide the key arguments below. For the proof, we recall the definitions of the Riesz transform \mathcal{R}_{σ} , the maximal truncation of the Riesz transform $\mathcal{R}_{\sigma,*}$ and the double layer potential \mathcal{D} of a suitable function f on $\partial\Omega$: for $\varepsilon > 0$, $X \in \mathbb{R}^{n+1}$ and $Z \in \Omega$, we set

$$\begin{aligned} \mathcal{R}_{\sigma,\varepsilon} f(X) &:= \frac{1}{C_n} \int_{\partial\Omega \setminus B(X,\varepsilon)} \frac{X - y}{|X - y|^{n+1}} f(y) d\sigma(y), & \mathcal{R}_{\sigma} f(X) &:= \lim_{\varepsilon \rightarrow 0} \mathcal{R}_{\sigma,\varepsilon} f(X), \\ \mathcal{D}f(Z) &:= \frac{1}{C_n} \int_{\partial\Omega} \frac{\nu(y) \cdot (y - Z)}{|Z - y|^{n+1}} f(y) d\sigma(y), & \mathcal{R}_{\sigma,*} f(X) &:= \sup_{\varepsilon > 0} |\mathcal{R}_{\sigma,\varepsilon} f(X)|, \end{aligned}$$

where ν is the measure-theoretic outer unit normal of Ω (see (2.2.11) in [Hofmann et al. 2010]) and C_n is the surface area of the unit sphere in \mathbb{R}^{n+1} . We extend $\mathcal{D}f$ to the whole $\mathbb{R}^{n+1} \setminus \partial\Omega$ by changing the direction of the normal ν for points $X \in \text{int } \Omega^c$.

Proof. Let f be a compactly supported Lipschitz function on $\partial\Omega$, $\xi \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$. Let $\Delta := \Delta(\xi, r) = B(\xi, r) \cap \partial\Omega$. By the theory of layer potentials we know that for σ -a.e. $x \in \partial\Omega$ we have

$$f(x) = \mathcal{D}^+ f(x) - \mathcal{D}^- f(x),$$

where $\mathcal{D}^+ f(x)$ and $\mathcal{D}^- f(x)$ are the inner and outer nontangential limits of $\mathcal{D}f$ at x , respectively (see, e.g., [Hofmann et al. 2010, Section 3.3] and apply the results separately for Ω and $\text{int } \Omega^c$). These nontangential limits exist σ -almost everywhere. For a constant vector h_Δ to be fixed below, we consider the function

$$u(X) := \mathcal{D}f(X) - X \cdot h_\Delta, \quad X \in \mathbb{R}^{n+1} \setminus \partial\Omega.$$

Then, by the σ -a.e. existence of the limits $\mathcal{D}^\pm f$, the inner and outer nontangential limits $u^\pm(x)$ of u exist for σ -a.e. $x \in \partial\Omega$. Thus, we have

$$f(x) = u^+(x) - u^-(x) \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega.$$

To prove the estimate (6.9), we choose

$$a_\Delta = u(X_\Delta^+) - u(X_\Delta^-),$$

where X_Δ^+ and X_Δ^- are interior and exterior local John points inside $B(\xi, r)$, respectively. Since $B(X_\Delta^\pm, cr) \subset B(\xi, r) \setminus \partial\Omega$, we know that $\delta(X_\Delta^\pm) \approx r$. Now, for any $\lambda > 0$, we have

$$\begin{aligned} &\sigma(\{x \in \Delta : |f(x) - a_\Delta| > \lambda\}) \\ &\leq \sigma\left(\left\{x \in \Delta : |u^+(x) - u(X_\Delta^+)| > \frac{\lambda}{2}\right\}\right) + \sigma\left(\left\{x \in \Delta : |u^-(x) - u(X_\Delta^-)| > \frac{\lambda}{2}\right\}\right). \end{aligned} \tag{6.10}$$

We only estimate the first term on the right-hand side of (6.10) since the second one is estimated similarly. For any $x \in \Delta$, let γ_x^+ be a nontangential path in Ω from x to X_Δ^+ . Such a nontangential path with a uniform constant exists for every $x \in \Delta$ by the local John condition. Since γ_x^+ is a nontangential path with a uniform constant, we know that $\gamma_x^+ \subset B(\xi, Ar)$ for some fixed $A \geq 1$. Then, for any $Y \in \gamma_x^+ \cap \Omega$, the mean value theorem gives us

$$|u(Y) - u(X_\Delta^+)| \leq \mathcal{H}^1(\gamma_x^+) \sup_{Z \in \gamma_x^+ \cap \Omega} |\nabla u(Z)| \lesssim r N_*(\chi_{B(\xi, Ar)} |\nabla u|)(x),$$

where N_* is the nontangential maximal operator for suitable aperture constant $\alpha > 1$ as defined in (2.4) (recall that a D -nontangential path from $x \in \partial\Omega$ to $X \in \Omega$ travels inside the cone $\Gamma_D(x)$, like we discussed in the beginning of Section 3). Letting $Y \rightarrow x$ then gives us

$$|u^+(x) - u(X_\Delta^+)| \lesssim r N_*(\chi_{B(\xi, Ar)} |\nabla u|)(x).$$

Thus,

$$\sigma\left(\left\{x \in \Delta : |u^+(x) - u(X_\Delta^+)| > \frac{\lambda}{2}\right\}\right) \lesssim \sigma(\{x \in \Delta : r N_*(\chi_{B(\xi, Ar)} |\nabla u|)(x) > c\lambda\}). \tag{6.11}$$

Let us then estimate the right-hand side of (6.11). We notice that for all $X \in \Gamma_\alpha(x) \cap B(\xi, Ar)$ it holds that $\nabla u(X) = \nabla \mathcal{D}f(X) - h_\Delta$. By (3.6.29) in [Hofmann et al. 2010], we know that for all $X \in \Omega$ it holds that

$$\nabla \mathcal{D}f(X) = \left(\sum_i \int_{\partial\Omega} \partial_i \mathcal{E}(X - y) \partial_{t,j,i} f(y) d\sigma(y) \right)_{1 \leq j \leq n+1},$$

where \mathcal{E} is the fundamental solution to the Laplacian in \mathbb{R}^{n+1} , that is,

$$\mathcal{E}(X) := \begin{cases} \frac{1}{C_n(1-n)} \frac{1}{|X|^{n-1}} & \text{if } n \geq 2, \\ \frac{1}{2\pi} \log |X| & \text{if } n = 1 \end{cases} \tag{6.12}$$

for $X \in \mathbb{R}^{n+1} \setminus \{0\}$, where C_n is the surface area of the unit sphere in \mathbb{R}^{n+1} . Thus, choosing

$$h_\Delta := \left(\sum_i \int_{\partial\Omega \setminus 2A\Delta} \partial_i \mathcal{E}(\xi - y) \partial_{t,j,i} f(y) d\sigma(y) \right)_{1 \leq j \leq n+1}$$

gives us

$$\begin{aligned} \nabla u(X) &= \left(\sum_i \int_{2A\Delta} \partial_i \mathcal{E}(X - y) \partial_{t,j,i} f(y) d\sigma(y) \right)_{1 \leq j \leq n+1} \\ &\quad + \left(\sum_i \int_{\partial\Omega \setminus 2A\Delta} (\partial_i \mathcal{E}(X - y) - \partial_i \mathcal{E}(\xi - y)) \partial_{t,j,i} f(y) d\sigma(y) \right)_{1 \leq j \leq n+1} \\ &= \left(\sum_i \mathcal{R}_{i,\sigma}(\chi_{2A\Delta} \partial_{t,j,i} f)(X) \right)_{1 \leq j \leq n+1} \\ &\quad + \left(\sum_i \mathcal{R}_{i,\sigma}(\chi_{(2A\Delta)^c} \partial_{t,j,i} f)(X) - \mathcal{R}_{i,\sigma}(\chi_{(2A\Delta)^c} \partial_{t,j,i} f)(\xi) \right)_{1 \leq j \leq n+1} \\ &=: I(X) + II(X), \end{aligned} \tag{6.13}$$

where $\mathcal{R}_{i,\sigma}$ stands for the i -th component of the Riesz transform \mathcal{R}_σ . We then have

$$\begin{aligned} &\sigma(\{x \in \Delta : r N_*(\chi_{B(\xi, Ar)} |\nabla u|)(x) > c\lambda\}) \\ &\leq \sigma\left(\left\{x \in \Delta : r N_*(\chi_{B(\xi, Ar)} |I|)(x) > \frac{c\lambda}{2}\right\}\right) + \sigma\left(\left\{x \in \Delta : r N_*(\chi_{B(\xi, Ar)} |II|)(x) > \frac{c\lambda}{2}\right\}\right). \end{aligned}$$

We first estimate the term $\sigma(\{x \in \Delta : r N_*(\chi_{B(\xi, Ar)} |I|)(x) > \frac{1}{2}c\lambda\})$. Let $x \in \Delta$ and $Y \in \Gamma_\alpha(x) \cap B(\xi, Ar)$, where $\Gamma_\alpha(x)$ is the cone at x with aperture α (recall (2.5)). Let $\varepsilon = \varepsilon_{x,Y} := 2|x - Y|$ and $\Delta_\varepsilon := \Delta(x, \varepsilon)$.

We then have

$$\begin{aligned} |I(Y)| &\leq \sum_{i,j} |\mathcal{R}_{i,\sigma}(\chi_{2A\Delta} \partial_{t,j,i} f)(Y)| \\ &\leq \sum_{i,j} |\mathcal{R}_{i,\sigma}(\chi_{2A\Delta \cap \Delta_\varepsilon} \partial_{t,j,i} f)(Y)| + \sum_{i,j} |\mathcal{R}_{i,\sigma}(\chi_{2A\Delta \setminus \Delta_\varepsilon} \partial_{t,j,i} f)(Y) - \mathcal{R}_{i,\sigma}(\chi_{2A\Delta \setminus \Delta_\varepsilon} \partial_{t,j,i} f)(x)| \\ &\quad + \sum_{i,j} |\mathcal{R}_{i,\sigma}(\chi_{2A\Delta \setminus \Delta_\varepsilon} \partial_{t,j,i} f)(x)|. \end{aligned} \tag{6.14}$$

Since $\text{dist}(Y, \partial\Omega) \approx \varepsilon$, the first term on the right-hand side of (6.14) satisfies

$$\begin{aligned} \sum_{i,j} |\mathcal{R}_{i,\sigma}(\chi_{2A\Delta \cap \Delta_\varepsilon} \partial_{t,j,i} f)(Y)| &\lesssim \sum_{i,j} \int_{\Delta_\varepsilon} \frac{|\chi_{2A\Delta} \partial_{t,j,i} f|}{\varepsilon^n} d\sigma \\ &\approx \sum_{i,j} \int_{\Delta_\varepsilon} |\chi_{2A\Delta} \partial_{t,j,i} f| d\sigma \leq M(\chi_{2A\Delta} |\nabla_{\text{HMT}} f|)(x), \end{aligned}$$

where we used Ahlfors–David regularity for the estimate $\varepsilon^n \approx \sigma(\Delta(x, \varepsilon))$ and M stands for the centered maximal Hardy–Littlewood operator on $\partial\Omega$. For the second term on the right-hand side of (6.14), we get

$$\begin{aligned} \sum_{i,j} |\mathcal{R}_{i,\sigma}(\chi_{2A\Delta \setminus \Delta_\varepsilon} \partial_{t,j,i} f)(Y) - \mathcal{R}_{i,\sigma}(\chi_{2A\Delta \setminus \Delta_\varepsilon} \partial_{t,j,i} f)(x)| \\ \stackrel{(A)}{\lesssim} \int_{2A\Delta \setminus \Delta_\varepsilon} \frac{|Y-x|}{|x-y|^{n+1}} |\nabla_{\text{HMT}} f(y)| d\sigma(y) \\ \lesssim \int_{\partial\Omega \setminus \Delta_\varepsilon} \frac{\varepsilon}{|x-y|^{n+1}} |\chi_{2A\Delta}(y) \nabla_{\text{HMT}} f(y)| d\sigma(y) \\ \stackrel{(B)}{\lesssim} \sum_{k=1}^\infty \int_{2^k \Delta_\varepsilon \setminus 2^{k-1} \Delta_\varepsilon} \frac{\varepsilon}{(2^k \varepsilon)^{n+1}} |\chi_{2A\Delta}(y) \nabla_{\text{HMT}} f(y)| d\sigma(y) \\ \stackrel{(C)}{\lesssim} \sum_{k=1}^\infty \frac{2^{-k}}{\sigma(2^k \Delta_\varepsilon)} \int_{2^k \Delta_\varepsilon \setminus 2^{k-1} \Delta_\varepsilon} |\chi_{2A\Delta} \nabla_{\text{HMT}} f| d\sigma \lesssim M(\chi_{2A\Delta} \nabla_{\text{HMT}} f)(x), \end{aligned}$$

where we used

- (A) the mean value theorem for the Riesz kernel functions $X = (X_1, X_2, \dots, X_{n+1}) \mapsto X_i/|X|^{n+1}$ and the estimate $|\nabla X_i/|X|^{n+1}| \lesssim 1/|X|^{n+1}$,
- (B) the fact that x is the center point of Δ_ε and hence, $|x-y| \approx 2^k \varepsilon$ for $y \in 2^k \Delta_\varepsilon \setminus 2^{k-1} \Delta_\varepsilon$,
- (C) Ahlfors–David regularity.

In addition, the last term on the right-hand side of (6.14) is bounded above by the sum

$$\sum_{i,j} \mathcal{R}_{i,\sigma,*}(\chi_{2A\Delta} \partial_{t,j,i} f)(x),$$

where $\mathcal{R}_{i,\sigma,*}$ stands for the maximal truncation of the Riesz transform defined using only the i -th component of \mathcal{R}_σ . Thus,

$$|I(Y)| \lesssim M(\chi_{2A\Delta} \nabla_{\text{HMT}} f)(x) + \sum_{i,j} \mathcal{R}_{i,\sigma,*}(\chi_{2A\Delta} \partial_{t,j,i} f)(x),$$

and so

$$\begin{aligned} \sigma\left(\left\{x \in \Delta : r N_*(\chi_{B(\xi,2r)}|I|)(x) > \frac{c\lambda}{2}\right\}\right) \\ \leq \sigma\left(\left\{x \in \Delta : M(\chi_{2A\Delta} \nabla_{\text{HMT}} f)(x) > \frac{c'\lambda}{r}\right\}\right) + \sum_{i,j} \sigma\left(\left\{x \in \Delta : \mathcal{R}_{i,\sigma,*}(\chi_{2A\Delta} \partial_{t,j,i} f)(x) > \frac{c'\lambda}{r}\right\}\right). \end{aligned}$$

Since $\partial\Omega$ is uniformly rectifiable, \mathcal{R}_σ is bounded from $L^2(\sigma)$ to $L^2(\sigma)$ [David and Semmes 1991], and therefore $\mathcal{R}_{i,\sigma,*}$ is of weak type-(1, 1) with respect to σ by classical Calderón–Zygmund-type techniques; see, e.g., [Grafakos 2014, Section 5]. This and the weak type-(1, 1) of the Hardy–Littlewood maximal operator, combined with the previous estimates, then give us

$$\sigma\left(\left\{x \in \Delta : r N_*(\chi_{B(\xi,Ar)}|I|)(x) > \frac{c\lambda}{2}\right\}\right) \lesssim \frac{r}{\lambda} \int_{2A\Delta} |\nabla_{\text{HMT}} f| d\sigma. \tag{6.15}$$

Let us then consider the term $II(X)$ in (6.13). For $x \in \Delta$ and $X \in \Gamma_\alpha(x) \cap B(\xi, Ar)$, the same arguments as with the middle sum in (6.14) give us

$$|II(X)| \lesssim \int_{\partial\Omega \setminus 2A\Delta} \frac{r}{|\xi - y|^{n+1}} |\nabla_{\text{HMT}} f(y)| d\sigma(y) \lesssim \sum_{j=2}^\infty \frac{2^{-j}}{\sigma(2^j \Delta)} \int_{2^j \Delta \setminus 2^{j-1} \Delta} |\nabla_{\text{HMT}} f| d\sigma.$$

Therefore,

$$\sigma\left(\left\{x \in \Delta : r N_*(\chi_{B(\xi, 2r)} |II|)(x) > \frac{c\lambda}{2}\right\}\right) \lesssim \frac{r}{\lambda} \sigma(\Delta) \sum_{j=2}^\infty 2^{-j(n+1)} \int_{2^j \Delta \setminus 2^{j-1} \Delta} |\nabla_{\text{HMT}} f| d\sigma. \tag{6.16}$$

Combining (6.15) and (6.16) with (6.11) and other previous estimates gives us

$$\sigma\left(\left\{x \in \Delta : |u^+(x) - u(X_\Delta^+)| > \frac{\lambda}{2}\right\}\right) \lesssim \frac{r}{\lambda} \int_{2A\Delta} |\nabla_{\text{HMT}} f| d\sigma + \frac{r}{\lambda} \sum_{j=2}^\infty 2^{-j(n+1)} \int_{2^j \Delta \setminus 2^{j-1} \Delta} |\nabla_{\text{HMT}} f| d\sigma.$$

By a straightforward covering argument, the right-hand side of the preceding inequality is comparable to the right-hand side of (6.9), with the comparability constant depending on A . By similar arguments, one can check that the same estimate holds replacing u^+ by u^- and X_Δ^+ by X_Δ^- . The claim follows then by applying the inequalities for u^+ and u^- to (6.10). \square

For the proof of the case $\text{diam}(\partial\Omega) < \infty$ in Theorem 1.3, we give a short proof of the fact that Theorem 6.6 implies quasiconvexity for bounded 2-sided chord-arc domains:

Lemma 6.17. *Suppose that Ω is a 2-sided chord-arc domain such that $\text{diam}(\partial\Omega) < \infty$. Then the boundary $\partial\Omega$ is quasiconvex.*

Notice that Corollary 1.6 implies stronger connectivity properties than the conclusion of Lemma 6.17 (recall Corollary 1.9) but we need Lemma 6.17 to prove Corollary 1.6. Lemma 6.17 follows almost directly from the results reviewed in this section when we combine them with the following result of Durand-Cartagena, Jaramillo and Shanmugalingam:

Theorem 6.18 [Durand-Cartagena et al. 2013, Theorem 3.6]. *Let (X, d, μ) be a complete metric measure space with a doubling measure μ . Suppose that for every bounded Lipschitz function f that is locally 1-Lipschitz there exists a functional $a_f : \mathcal{B} \rightarrow [0, \infty)$ such that*

$$\int_B |f - \langle f \rangle_B| d\mu \leq a_f(B) \leq Cr_B,$$

where \mathcal{B} is the collection of all open balls in (X, d) , C is a uniform constant and r_B is the radius of the ball B . Then the space (X, d) is quasiconvex.

Proof of Lemma 6.17. Let f be a bounded Lipschitz function on $\partial\Omega$ such that f is locally 1-Lipschitz, and let $\mathcal{B} = \{\Delta(x, r) : x \in \partial\Omega, r < \text{diam}(\partial\Omega)\}$. Since $\text{diam}(\partial\Omega) < \infty$, we know that $\nabla_{\text{HMT}} f$ exists (recall Remark 6.1). Let us define the functional $a_f : \mathcal{B} \rightarrow \mathbb{R}$ by setting

$$a_f(\Delta) = C_2 r \left(\int_\Delta |\nabla_{\text{HMT}} f|^2 d\sigma \right)^{1/2} + C_2 r \sum_{j=2}^\infty \frac{2^{-j}}{\sigma(2^j \Delta)} \int_{2^j \Delta \setminus 2^{j-1} \Delta} |\nabla_{\text{HMT}} f| d\sigma$$

for every $\Delta = \Delta(x, r) \in \mathcal{B}$, where C_2 is the constant given by [Theorem 6.6](#) for the arbitrary choice $p = 2$. By part (1) of [Lemma 6.2](#), the functional a_f is well-defined, and the fact that $a_f(\Delta) < \infty$ for each $\Delta \in \mathcal{B}$ follows from the argument below. Now, by [Lemmas 6.2](#) and [6.4](#), and the fact that f is locally 1-Lipschitz, we know that $|\nabla_{\text{HMT}} f| \leq 1$ almost everywhere. In particular, for $\Delta = \Delta(x, r) \in \mathcal{B}$ we have

$$a_f(\Delta) \leq C_2 r + C_2 r \sum_{j=2}^{\infty} \frac{2^{-j}}{\sigma(2^j \Delta)} \sigma(2^j \Delta \setminus 2^{j-1} \Delta) \leq C_2 r + \sum_{j=2}^{\infty} 2^{-j} \leq 2C_2 r. \tag{6.19}$$

Thus, we now only need to notice that by Hölder’s inequality and the estimate [\(6.19\)](#), we have

$$\int_{\Delta} |f - \langle f \rangle_{\Delta}| d\sigma \leq \left(\int_{\Delta} |f - \langle f \rangle_{\Delta}|^2 d\sigma \right)^{1/2} \leq a_f(\Delta) \leq 2C_2 r$$

for any $\Delta = \Delta(x, r) \in \mathcal{B}$, and the claim follows from [Theorem 6.18](#). □

7. Weak 1-Poincaré inequality for boundaries of 2-sided chord-arc domains

Let Ω be a 2-sided chord-arc domain. In this section, we prove [Theorem 1.3](#) and [Corollary 1.6](#) with the help of some tools from the literature. As a simple consequence of [Theorem 1.3](#) and some results in the literature, we also show that the tail in the Hofmann–Mitrea–Taylor weak Poincaré inequality ([Theorem 6.6](#)) can be removed, at least when Ω is a bounded 2-sided chord-arc domain (see [Corollary 7.13](#)).

Instead of proving a Poincaré-type inequality directly, we use the following result to reduce the proof to a pointwise estimate:

Theorem 7.1 [[Heinonen et al. 2015](#), part of [Theorem 8.1.7](#)]. *Let (X, d, μ) be a metric measure space with a doubling measure μ and V be a Banach space. Suppose that $1 \leq p < \infty$, $u : X \rightarrow V$ is integrable on balls and $g : X \rightarrow [0, \infty]$ is measurable. Then the following two conditions are equivalent:*

(a) *There exist constants $C, \lambda \geq 1$ such that*

$$\int_B |u(x) - \langle u \rangle_B| d\mu(x) \leq C \text{diam}(B) \left(\int_{\lambda B} g(x)^p d\mu(x) \right)^{1/p}$$

for every open ball B in X .

(b) *There exist constants $C, \lambda \geq 1$ such that*

$$|u(x) - u(y)| \leq C d(x, y) (M_{\lambda d(x,y)}(g^p)(x) + M_{\lambda d(x,y)}(g^p)(y))^{1/p}$$

for almost all $x, y \in X$, where M_R is the R -truncated centered Hardy–Littlewood maximal operator on $\partial\Omega$,

$$M_R f(z_0) := \sup_{r < R} \int_{\Delta(z_0, r)} |f(z)| d\sigma(z).$$

These types of characterizations with respect to pointwise inequalities originate from [[Heinonen and Koskela 1998](#)].

Thus, to prove [Theorem 1.3](#), it is enough for us to prove the following lemma (recall also [Remark 6.1](#)):

Lemma 7.2. *Suppose that u is a compactly supported Lipschitz function on $\partial\Omega$. There exists a universal constant $C \geq 1$ such that*

$$|u(x) - u(y)| \leq C|x - y| (M_{C|x-y|}(|\nabla_{\mathbf{t}}u|)(x) + M_{C|x-y|}(|\nabla_{\mathbf{t}}u|)(y))$$

for all points $x, y \in \partial\Omega$ for which the tangential gradient $\nabla_{\mathbf{t}}u$ exists and (6.5) holds.

Let u be a compactly supported Lipschitz function on $\partial\Omega$. As we noted in Section 2D, the tangential gradient $\nabla_{\mathbf{t}}u$ exists for almost every point $x \in \partial\Omega$. By Lemma 6.4, we know that (6.5) holds for almost every point $x \in \partial\Omega$. Thus, the points in Lemma 7.2 are almost all points in $\partial\Omega$, as is required in Theorem 7.1.

In the proof of Lemma 7.2, we use a smooth cutoff function for balls. The construction of the function uses the usual mollifier technique. For the convenience of the reader — particularly because we need a quantitative bound for the norm of the gradient — we give the key details below.

Let us start by defining the standard mollifier. We set $\eta : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$,

$$\eta(X) = \begin{cases} c_n e^{-1/(1-|X|^2)} & \text{if } |X| < 1, \\ 0 & \text{if } |X| \geq 1, \end{cases}$$

where the constant c_n is chosen so that $\int_{\mathbb{R}^{n+1}} \eta(X) dX = 1$. For any $\kappa > 0$, we set

$$\eta_\kappa(X) = \frac{1}{\kappa^{n+1}} \eta\left(\frac{X}{\kappa}\right).$$

Notice that $\text{supp } \eta_\kappa \subset B(0, \kappa)$. Using the standard mollifier, we define the smooth cutoff function φ_κ for the ball $B(0, \kappa)$ characteristic function using convolutions: we set $\varphi_\kappa : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$,

$$\varphi_\kappa(X) = \eta_\kappa * \chi_{B(0,\kappa)}(X). \tag{7.3}$$

Thus, we have

$$\begin{aligned} \varphi_\kappa(X) &= \int_{\mathbb{R}^{n+1}} \eta_\kappa(X - Y) \chi_{B(0,\kappa)}(Y) dY \\ &= \int_{\mathbb{R}^{n+1}} \eta_\kappa(Y) \chi_{B(0,\kappa)}(X - Y) dY = \int_{B(0,\kappa)} \eta_\kappa(Y) \chi_{B(0,\kappa)}(X - Y) dY. \end{aligned}$$

From this representation, we see that $\varphi_\kappa \equiv 1$ on $B(0, \kappa)$ and $\varphi_\kappa \equiv 0$ on $\mathbb{R}^{n+1} \setminus B(0, 2\kappa)$. By the standard theory of mollifiers (see, e.g., [Evans and Gariepy 1992, p. 123–124]), we know that φ_κ is smooth and it satisfies

$$\nabla \varphi_\kappa(X) = \int_{\mathbb{R}^{n+1}} \nabla_X \eta_\kappa(X - Y) \chi_{B(0,\kappa)}(Y) dY. \tag{7.4}$$

In particular, by the construction, the smoothness and compact support of η and (7.4), for $X = (X_1, \dots, X_{n+1})$ we have

$$\begin{aligned} \left| \frac{\partial_i}{\partial X_i} \varphi_\kappa(X) \right| &= \left| \int_{\mathbb{R}^{n+1}} \frac{\partial_i}{\partial X_i} \eta_\kappa(X - Y) \chi_{B(0,\kappa)}(Y) dY \right| \\ &\leq \int_{\mathbb{R}^{n+1}} \left| \frac{\partial_i}{\partial X_i} \eta_\kappa(X - Y) \right| dY = \frac{1}{\kappa} \int_{\mathbb{R}^{n+1}} \left| \frac{\partial_i}{\partial X_i} \eta(X - Y) \right| dY \leq \frac{C}{\kappa}, \end{aligned}$$

where the constant C does not depend κ . Thus, we have $|\nabla\varphi_\kappa(X)| \lesssim \frac{1}{\kappa}$. We note that this implies that φ_κ is a compactly supported Lipschitz function. Therefore the tangential gradient of φ_κ exists for almost every point $x \in \partial\Omega$, and (7.5) and Lemma 6.4 imply that

$$|\nabla_t\varphi_\kappa(x)| \lesssim \frac{1}{\kappa}. \tag{7.5}$$

By translating and adjusting the constant κ , we can construct a smooth cutoff function for any ball $B(X, r)$ in \mathbb{R}^{n+1} .

Proof of Lemma 7.2. Let us fix points $x, y \in \partial\Omega$ such that the tangential gradient exists at x and y and (6.5) holds for x and y .

We prove the bound by using truncation and localization arguments which help us to control the values of the function inside balls $\Delta(x, c|x - y|)$ and $\Delta(y, c|x - y|)$ and allow us to deal with the tail in the right-hand side of inequality (6.9). We first assume that $\text{diam}(\partial\Omega) = \infty$. Without loss of generality, we may assume that $u(x) = 0 < u(y)$. In particular, we have $|u(x) - u(y)| = u(y)$. We set

$$\tilde{u}(z) := \begin{cases} u(z) & \text{if } u(x) < u(z) < u(y), \\ 0 & \text{if } u(z) \leq 0, \\ u(y) & \text{if } u(z) \geq u(y). \end{cases}$$

Notice that \tilde{u} is a Lipschitz function since we get it by truncating the Lipschitz function u from above and below. Thus, by Lemma 6.4, we have

$$|\nabla_t u(z_0)| = \limsup_{\partial\Omega \ni z \rightarrow z_0} \frac{|u(z_0) - u(z)|}{|z_0 - z|} \quad \text{and} \quad |\nabla_t \tilde{u}(z_0)| = \limsup_{\partial\Omega \ni z \rightarrow z_0} \frac{|\tilde{u}(z_0) - \tilde{u}(z)|}{|z_0 - z|}$$

for σ -a.e. $z_0 \in \partial\Omega$. Let $z_0 \in \partial\Omega$ be a point such that the above holds. By continuity, we have the following:

- (i) If $u(z_0) < u(x) = 0$ or $u(z_0) > u(y)$, then $|\nabla_t \tilde{u}(z_0)| = 0 \leq |\nabla_t u(z_0)|$.
- (ii) If $u(x) < u(z_0) < u(y)$, then $|\nabla_t \tilde{u}(z_0)| = |\nabla_t u(z_0)|$.
- (iii) If $u(z_0) = u(x) = 0$, then there exists $r = r_{z_0} > 0$ such that if $|z_0 - z| < r$, then $u(z) < u(y)$. For such z ,
 - if $u(z) > u(x)$, then $|\tilde{u}(z_0) - \tilde{u}(z)| = |u(z_0) - u(z)|$, and
 - if $u(z) \leq u(x)$, then $|\tilde{u}(z_0) - \tilde{u}(z)| = 0 \leq |u(z_0) - u(z)|$.

In particular, we have $|\nabla_t \tilde{u}(z_0)| \leq |\nabla_t u(z_0)|$.

- (iv) If $u(z_0) = u(y)$, then $|\nabla_t \tilde{u}(z_0)| \leq |\nabla_t u(z_0)|$ by an argument similar to that in (iii).

Thus, for almost every $z_0 \in \partial\Omega$, the tangential gradients exist for u and \tilde{u} and we have

$$|\nabla_t \tilde{u}(z_0)| \leq |\nabla_t u(z_0)|. \tag{7.6}$$

Let us then start processing $|u(x) - u(y)|$. We define

$$R := |x - y|, \quad \Delta_0 := \Delta(x, R), \quad \Delta'_0 := \Delta(y, R), \quad \Delta_j := 2^{-j} \Delta_0 \quad \text{and} \quad \Delta'_j := 2^{-j} \Delta'_0.$$

Let $\varphi = \varphi_{2^\alpha R}$ be a smooth cutoff function for the ball $B(x, 2^\alpha R)$ as in (7.3) (after translation, for the choice $\kappa = 2^\alpha R$) for large $\alpha \in \mathbb{N}$ to be fixed later, and let $v := \varphi \tilde{u}$. By Lemma 6.8, there exist numbers $a_{\Delta_j} \in \mathbb{R}$ such that

$$\sup_{\lambda > 0} \lambda \sigma(\{z \in \Delta_j : |v(z) - a_{\Delta_j}| > \lambda\}) \leq Cr_j \sigma(\Delta_j) S_j, \tag{7.7}$$

where $r_j := 2^{-j} R$ and

$$S_j := \int_{5\Delta_j} |\nabla_{\text{HMT}} v| d\sigma + \sum_{k=2}^{\infty} \frac{2^{-k}}{\sigma(2^k \Delta_j)} \int_{2^k \Delta_j \setminus 2^{k-1} \Delta_j} |\nabla_{\text{HMT}} v| d\sigma. \tag{7.8}$$

An analogous estimate holds when we replace Δ_j, a_{Δ_j} and S_j by $\Delta'_j, a_{\Delta'_j}$ and S'_j , respectively, where S'_j is defined as S_j with Δ'_j in place of Δ_j . We claim now that

$$\lim_{j \rightarrow \infty} a_{\Delta_j} = v(x) = \tilde{u}(x) = u(x) = 0. \tag{7.9}$$

Indeed, for any $\lambda \in (0, |a_{\Delta_j} - \langle v \rangle_{\Delta_j}|)$ and any $z \in \Delta_j$, we have

$$\lambda < |a_{\Delta_j} - \langle v \rangle_{\Delta_j}| \leq |v(z) - a_{\Delta_j}| + |v(z) - \langle v \rangle_{\Delta_j}|.$$

Hence, by (7.7) and Chebyshev's inequality, we get

$$\begin{aligned} \sigma(\Delta_j) &\leq \sigma\left(\left\{z \in \Delta_j : |v(z) - a_{\Delta_j}| > \frac{\lambda}{2}\right\}\right) + \sigma\left(\left\{z \in \Delta_j : |v(z) - \langle v \rangle_{\Delta_j}| > \frac{\lambda}{2}\right\}\right) \\ &\lesssim \frac{1}{\lambda} r_j \sigma(\Delta_j) S_j + \frac{1}{\lambda} \int_{\Delta_j} |v - \langle v \rangle_{\Delta_j}| d\sigma. \end{aligned}$$

Therefore,

$$\lambda \lesssim r_j S_j + \int_{\Delta_j} |v - \langle v \rangle_{\Delta_j}| d\sigma. \tag{7.10}$$

Since v is a compactly supported Lipschitz function, Lemmas 6.2 and 6.4 give $|S_j| \lesssim \|\nabla_{\text{HMT}} v\|_{L^\infty(\sigma)} < \infty$, and thus $r_j S_j \rightarrow 0$ as $j \rightarrow \infty$. By the continuity of v , we also know that $\int_{\Delta_j} |v(z) - \langle v \rangle_{\Delta_j}| d\sigma \rightarrow 0$ as $j \rightarrow \infty$. Then, choosing $\lambda = \frac{1}{2}|a_{\Delta_j} - \langle v \rangle_{\Delta_j}|$ gives us (7.9) by (7.10) and continuity of v .

Analogously to (7.9), we also have

$$\lim_{j \rightarrow \infty} a_{\Delta'_j} = v(y) = \tilde{u}(y) = u(y).$$

Thus,

$$\begin{aligned} u(y) - u(x) &= |u(x) - u(y)| = |v(x) - v(y)| \\ &= \left| \left(\sum_{j=0}^{\infty} (a_{\Delta_{j+1}} - a_{\Delta_j}) + a_{\Delta_0} \right) - \left(\sum_{j=0}^{\infty} (a_{\Delta'_{j+1}} - a_{\Delta'_j}) + a_{\Delta'_0} \right) \right| \\ &\leq \sum_{j=0}^{\infty} |a_{\Delta_{j+1}} - a_{\Delta_j}| + \sum_{j=0}^{\infty} |a_{\Delta'_{j+1}} - a_{\Delta'_j}| + |a_{\Delta_0} - a_{\Delta'_0}| \\ &=: I + II + III. \end{aligned}$$

Let us consider the sum I first. We analyze I by applying (7.7) again for $|a_{\Delta_j} - a_{\Delta_{j+1}}|$. For any $\lambda \in (0, |a_{\Delta_j} - a_{\Delta_{j+1}}|)$ and any $z \in \Delta_j$, we have

$$\lambda < |a_{\Delta_j} - a_{\Delta_{j+1}}| \leq |v(z) - a_{\Delta_j}| + |v(z) - a_{\Delta_{j+1}}|,$$

and therefore (7.7) gives us

$$\begin{aligned} \sigma(\Delta_j) &\leq \sigma\left(\left\{z \in \Delta_j : |v(z) - a_{\Delta_j}| > \frac{\lambda}{2}\right\}\right) + \sigma\left(\left\{z \in \Delta_{j+1} : |v(z) - a_{\Delta_{j+1}}| > \frac{\lambda}{2}\right\}\right) \\ &\lesssim \frac{1}{\lambda} r_j \sigma(\Delta_j) S_j + \frac{1}{\lambda} r_{j+1} \sigma(\Delta_{j+1}) S_{j+1} \\ &\lesssim \frac{1}{\lambda} r_j \sigma(\Delta_j) S_j. \end{aligned}$$

Thus, $\lambda \lesssim r_j S_j$. Since this holds for all $\lambda \in (0, |a_{\Delta_j} - a_{\Delta_{j+1}}|)$, we have

$$|a_{\Delta_j} - a_{\Delta_{j+1}}| \lesssim r_j S_j. \tag{7.11}$$

This then gives us

$$\begin{aligned} I &\lesssim \sum_{j=0}^{\infty} r_j S_j \\ &\stackrel{(A)}{\lesssim} \sum_{j=0}^{\infty} \left(2^{-j} R \int_{5\Delta_j} |\nabla_t(\tilde{u}\varphi)| d\sigma + 2^{-j} R \sum_{k=2}^{j+\alpha+1} 2^{-k} \int_{2^k \Delta_j} |\nabla_t(\tilde{u}\varphi)| d\sigma\right) \\ &\stackrel{(B)}{\leq} \sum_{j=0}^{\infty} 2^{-j} R \sum_{k=2}^{j+\alpha+1} 2^{-k} \int_{2^k \Delta_j} |\nabla_t \tilde{u}| d\sigma + \sum_{j=0}^{\infty} 2^{-j} R \cdot 2^{-j-\alpha-1} \frac{1}{\sigma(2^{\alpha+1} \Delta_0)} \int_{2^{\alpha+1} \Delta_0 \setminus 2^\alpha \Delta_0} |\tilde{u}(z) \nabla_t \varphi(z)| d\sigma(z) \\ &\stackrel{(C)}{\lesssim} R \cdot M_{2^{\alpha+2} R}(\nabla_t \tilde{u})(x) + \frac{\tilde{u}(y)}{2^\alpha}, \end{aligned}$$

where we used

- (A) the definition of S_j (see (7.8)), Lemma 6.2 and the fact that φ vanishes outside $2^{\alpha+1} \Delta_0$,
- (B) Ahlfors–David regularity, the product rule for gradients and the properties of the cutoff function φ : $\varphi \leq 1$ everywhere and φ is constant on $2^\alpha \Delta_0$ and outside $2^{\alpha+1} \Delta_0$,
- (C) the fact that $2^k \Delta_j \subset 2^{\alpha+1} \Delta_0$ for all the relevant k and j , the definition of the truncated Hardy–Littlewood maximal operator, the fact that $\tilde{u}(z) \leq \tilde{u}(y)$ for all z , and (7.5).

Using the same techniques gives us the bound

$$II \lesssim R \cdot M_{2^{\alpha+2} R}(\nabla_t \tilde{u})(y) + \frac{\tilde{u}(y)}{2^\alpha}.$$

As for III, we notice that

$$III \leq |a_{\Delta_0} - a_{2\Delta_0}| + |a_{2\Delta_0} - a_{\Delta'_0}| \lesssim r_0 S_0.$$

Indeed, the fact that $|a_{\Delta_0} - a_{2\Delta_0}| \lesssim r_0 S_0$ follows from (7.11), and an analogous estimate holds for the term $|a_{2\Delta_0} - a_{\Delta'_0}|$. Thus, the estimate obtained above for the term I is also valid for III:

$$III \lesssim \sum_{j=0}^{\infty} r_j S_j \lesssim R \cdot M_{2^{\alpha+2} R}(\nabla_t \tilde{u})(x) + \frac{\tilde{u}(y)}{2^\alpha}.$$

Note that none of the implicit constants for the bounds for *I*, *II* and *III* depends on α . Thus, recalling that $u(x) = 0 < u(y) = \tilde{u}(y)$ and $R = |x - y|$, there exists a constant C such that

$$\begin{aligned} u(y) = |u(x) - u(y)| &\leq I + II + III \\ &\leq CR(M_{2^{\alpha+2}R}(\nabla_t \tilde{u})(x) + M_{2^{\alpha+2}R}(\nabla_t \tilde{u})(y)) + \frac{C}{2^\alpha} \tilde{u}(y) \\ &\stackrel{(7.6)}{\leq} 2C|x - y|(M_{2^{\alpha+2}R}(\nabla_t u)(x) + M_{2^{\alpha+2}R}(\nabla_t u)(y)) + \frac{C}{2^\alpha} u(y). \end{aligned}$$

By choosing large enough α , we may absorb $(C/2^\alpha)u(y)$ to the left-hand side. This completes the proof for the case $\text{diam}(\partial\Omega) = \infty$.

Let us then assume that $\text{diam}(\partial\Omega) < \infty$. We have to consider this case separately because in the bounded case there might not exist cutoff functions with gentle enough gradient slope. However, the proof is still based on the previous case. The proof works as previously if $|x - y| \ll \text{diam}(\partial\Omega)$ and thus, we may assume that $|x - y| \approx \text{diam}(\partial\Omega)$. Now, by Lemma 6.17, we know that there exists a path $\gamma_{x,y}$ from x to y in $\partial\Omega$ such that $\ell(\gamma_{x,y}) \leq C_0|x - y| \approx C_0 \cdot \text{diam}(\partial\Omega)$. Using a covering argument for $\gamma_{x,y}$, we find a uniformly bounded number of points $z_0, z_1, \dots, z_J \in \partial\Omega$ with $z_0 = x, z_J = y$ and $|z_j - z_{j+1}| \ll \text{diam}(\partial\Omega)$. We get

$$\begin{aligned} |u(x) - u(y)| &\leq \sum_{j=0}^{J-1} |u(z_j) - u(z_{j+1})| \\ &\lesssim \sum_{j=0}^{J-1} |z_j - z_{j+1}| (M_{C|z_j - z_{j+1}|}(|\nabla_t u|)(z_j) + M_{C|z_j - z_{j+1}|}(|\nabla_t u|)(z_{j+1})) \\ &\lesssim |x - y| (M_{\tilde{C}|x - y|}(|\nabla_t u|)(x) + M_{\tilde{C}|x - y|}(|\nabla_t u|)(y)), \end{aligned}$$

which is what we wanted. □

Corollary 1.6 follows now immediately when we combine Theorem 1.3 and Lemma 6.4 with the following key result of Keith (which is an improvement of an earlier result of [Heinonen and Koskela 1999, Theorem 1.1]):

Theorem 7.12 [Keith 2003, Theorem 2]. *Let (X, d, μ) be a complete metric measure space with a doubling measure μ and $1 \leq p < \infty$. Then the following conditions are equivalent:*

- (a) *The space (X, d, μ) supports a weak p -Poincaré inequality.*
- (b) *There exist constants $C, \lambda \geq 1$ such that*

$$\int_B |u(x) - \langle u \rangle_B| d\mu(x) \leq C \text{diam}(B) \left(\int_{\lambda B} (\text{Lip } u(x))^p d\mu(x) \right)^{1/p}$$

for every compactly supported Lipschitz function u and every ball B in X , where

$$\text{Lip } u(x) := \limsup_{\substack{y \rightarrow x \\ y \neq x}} \frac{|f(x) - f(y)|}{d(x, y)}.$$

As a consequence of Theorem 1.3, we get the following improvement of Theorem 6.6 in bounded 2-sided chord-arc domains:

Corollary 7.13. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded 2-sided chord-arc domain, and let $1 < p < \infty$. There exists a constant $C = C(\Omega, p)$ such that for every $f \in L^p_1(\partial\Omega)$, $x \in \partial\Omega$ and $r > 0$ we have*

$$\left(\int_{\Delta} |f - \langle f \rangle_{\Delta}|^p d\sigma \right)^{1/p} \leq Cr \left(\int_{\Lambda_{\Delta}} |\nabla_{\text{HMT}} f|^p d\sigma \right)^{1/p},$$

where $\Delta := \Delta(x, r)$ and Λ is the constant from [Theorem 1.3](#).

It is likely that the boundedness assumption is not necessary for [Corollary 7.13](#) but since the density results in [[Hofmann et al. 2010](#), Section 4.3] are stated in the case where the boundary $\partial\Omega$ is compact, we only consider this case. We do not consider the case $p = 1$ since the theory of the Hofmann–Mitrea–Taylor Sobolev spaces $L^p_1(\partial\Omega)$ has been developed only for $1 < p < \infty$.

Proof of Corollary 7.13. Let $f \in L^p_1(\partial\Omega)$, $x \in \partial\Omega$, $r > 0$ and $\Delta := \Delta(x, r)$. By [[Hofmann et al. 2010](#), Corollary 4.28], we know that Lipschitz functions form a dense subset of $L^p_1(\partial\Omega)$ when $L^p_1(\partial\Omega)$ is equipped with the natural Sobolev-type norm; see [[Hofmann et al. 2010](#), Section 3.6] for details. In particular, since Ω is a bounded 2-sided chord-arc domain, there exists a sequence of compactly supported Lipschitz functions (f_k) such that $f_k \rightarrow f$ and $\nabla_{\text{HMT}} f_k \rightarrow \nabla_{\text{HMT}} f$ in $L^p(\partial\Omega)$. By [Theorem 1.3](#), the functions f_k satisfy a weak 1-Poincaré inequality with respect to the tangential gradient, which implies a weak (p, p) -Poincaré inequality with respect to tangential gradient by Hölder’s inequality and [[Heinonen et al. 2015](#), Corollary 9.14]. These observations together with [Lemma 6.2](#) give us

$$\left(\int_{\Delta} |f_k - \langle f_k \rangle_{\Delta}|^p d\sigma \right)^{1/p} \leq Cr \left(\int_{\Lambda_{\Delta}} |\nabla_{\text{t}} f_k|^p d\sigma \right)^{1/p} = Cr \left(\int_{\Lambda_{\Delta}} |\nabla_{\text{HMT}} f_k|^p d\sigma \right)^{1/p}.$$

Letting $k \rightarrow \infty$ then gives us

$$\left(\int_{\Delta} |f - \langle f \rangle_{\Delta}|^p d\sigma \right)^{1/p} \leq Cr \left(\int_{\Lambda_{\Delta}} |\nabla_{\text{HMT}} f|^p d\sigma \right)^{1/p}$$

by standard L^p convergence arguments, which is what we wanted. □

8. A counterexample and questions

By [Corollary 1.6](#), we know that the boundary of any 2-sided chord-arc domain supports weak Heinonen–Koskela-type Poincaré inequalities. It is natural to ask the following:

- (1) Can [Corollary 1.6](#) be reversed, i.e., if $\Omega \subset \mathbb{R}^{n+1}$ is an open set with n -dimensional Ahlfors–David regular boundary such that $\partial\Omega$ supports weak Poincaré inequalities, is Ω a 2-sided chord-arc domain?
- (2) Can the assumptions of [Corollary 1.6](#) be weakened in the obvious way, i.e., if $\Omega \subset \mathbb{R}^{n+1}$ is a (1-sided) chord-arc domain with connected boundary, does $\partial\Omega$ support weak Poincaré inequalities?

In the second question, the connectivity assumption for $\partial\Omega$ is necessary since weak Poincaré inequalities imply connectivity; see, e.g., [[Cheeger 1999](#), Theorem 17.1]. However, the answer to both of these questions is no. For the first question, this follows from [Example 8.4](#) below. For the second question, this follows from the example constructed in [[Mourgoglou and Tolsa 2021](#), Section 10]. They construct a

chord-arc domain with connected boundary such that the tangential regularity problem for the Laplacian is not solvable in L^p for any $1 \leq p < \infty$ (recall [Definition 2.31](#)). The boundary of this chord-arc domain cannot support weak Poincaré inequalities by [[Mourgoglou and Tolsa 2021](#), Theorem 1.2].

For our example, we need some results in the literature. In particular, we need the following result of Heinonen and Koskela about how weak Poincaré inequalities survive under unions. We formulate the result only in our context but we note that the result holds more generally for certain types of unions of Ahlfors–David regular metric spaces.

Theorem 8.1 [[Heinonen and Koskela 1998](#), special case of Theorem 6.15]. *Let $E_1, E_2 \subset \mathbb{R}^{n+1}$ be two n -ADR sets such that both E_1 and E_2 support a weak p -Poincaré inequality for some $1 \leq p < \infty$, and $\mathcal{H}^n(E_1 \cap E_2 \cap B(x, r)) \geq cr^n$ for all $x \in E_1 \cap E_2$ and all $0 < r < \min\{\text{diam}(E_1), \text{diam}(E_2)\}$. Then also the union $E_1 \cup E_2$ supports a weak p -Poincaré inequality when we equip $E_1 \cup E_2$ with the metric d that equals the usual Euclidean distance individually on E_1 and E_2 and*

$$d(x, y) := \inf_{Z \in E_1 \cap E_2} |X - Z| + |Z - Y| \quad (8.2)$$

for $X \in E_1 \setminus E_2$ and $Y \in E_2 \setminus E_1$.

We will also use repeatedly the fact that bi-Lipschitz mappings preserve weak Poincaré inequalities (see [[Björn and Björn 2011](#), Proposition 4.16] for an explicit proof in the context of more general metric spaces and inequalities):

Proposition 8.3. *Let $E_1, E_2 \subset \mathbb{R}^{n+1}$ be two n -ADR sets equipped with metrics d_1 and d_2 , respectively. Suppose that there exists a bi-Lipschitz mapping $\Phi : (E_1, d_1) \rightarrow (E_2, d_2)$ such that $\mathcal{H}^n(A) \approx \mathcal{H}^n(\Phi(A))$ for every measurable set $A \subset E_1$ and a uniform implicit constant. If E_1 supports a weak p -Poincaré inequality for some $1 \leq p < \infty$, then also E_2 supports a weak p -Poincaré inequality.*

Let us then construct a disconnected non-chord-arc domain example of a set whose boundary still supports a weak 1-Poincaré inequality:

Example 8.4. Let us consider a “twice-pinched annulus” in \mathbb{R}^2 which we construct the following way. We start by considering the boundary of a usual annulus $A := B(0, 4) \setminus \overline{B(0, 3)}$. We remove all the points that lie on the strip $\{(x, y) \in \mathbb{R}^2 : -1 < y < 1\}$. This leaves us with four circular arcs: two inner arcs and two outer arcs. We then connect these arcs to each other with four line segments so that the inner arcs connect to outer arcs, and vice versa. This leaves us with a shape that looks like an annulus that has been pinched in two places so that the interior is no longer connected but the boundary is. We let Ω be the disconnected interior of this pinched annulus (see [Figure 3](#)).

The set Ω satisfies the 2-sided corkscrew condition and the boundary $\partial\Omega$ is 1-ADR but since Ω is not connected, it is not a chord-arc domain (however, we can easily modify the example to make it connected but still not a chord-arc domain; see [Remark 8.6](#)). The boundary of Ω still supports a weak 1-Poincaré inequality. We see this by noticing that we can express $\partial\Omega$ as a union of pieces that satisfy weak 1-Poincaré inequalities and that we can glue together with ample intersections to give back $\partial\Omega$ (that is, we can use [Theorem 8.1](#) for these pieces). Indeed, $\partial\Omega$ consists of a slightly distorted inner circle and a slightly distorted outer circle. These distorted circles intersect only in two points (the two places

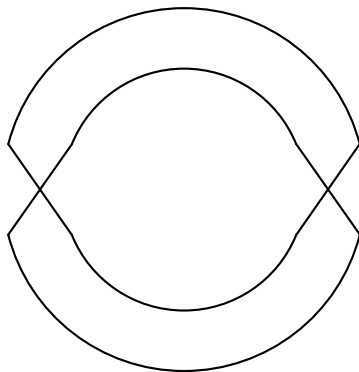


Figure 3. The set Ω in [Example 8.4](#) is the disconnected interior of a twice-pinned annulus in \mathbb{R}^2 . Unlike with a usual annulus, the boundary of Ω is connected (which is one of the minimum requirements Poincaré inequalities).

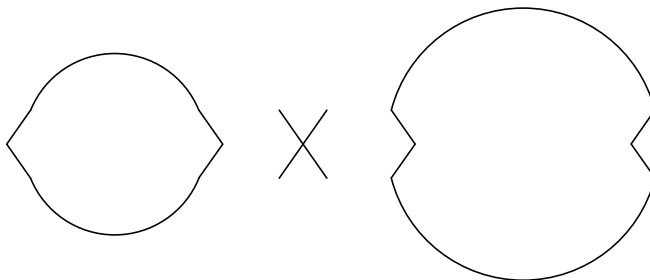


Figure 4. In [Example 8.4](#), the boundary of Ω consists of a slightly distorted inner circle (on the left) and a slightly distorted outer circle (on the right) that intersect each other only at two points. However, the cross-like unions of two line segments (one copy in the middle) have ample intersections with both of these distorted circles.

where the line segments cross over each other) and therefore they alone are not enough for [Theorem 8.1](#). Because of this, as two additional pieces, we take the cross-like unions of the pairs of line segments that cross over each other. These pieces have ample intersections with both of the distorted circles (see [Figure 4](#)).

As a 1-dimensional compact Riemannian manifold, a circle supports a weak 1-Poincaré; see [[Heinonen and Koskela 1998](#), Section 6.1]. Since both of the distorted circles are bi-Lipschitz equivalent to a regular circle, both of them support a weak 1-Poincaré inequality by [Proposition 8.3](#). As for the unions of two line segments, we first notice that a line segment supports a weak 1-Poincaré inequality because it is bi-Lipschitz equivalent to a piece of the real line (and a connected piece of the real line supports a weak 1-Poincaré inequality by definition and the classical result that the Euclidean space supports a 1-Poincaré inequality; see [[Heinonen et al. 2015](#), Section 8.1]). The two line segments in the cross-like union meet only at one point and therefore we cannot use [Theorem 8.1](#) directly for them. Because of this, we take three line segments and transform them (with bi-Lipschitz mappings) into three V-like shapes which we

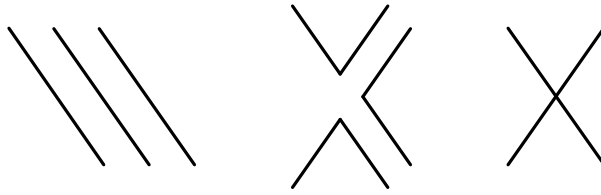


Figure 5. By [Theorem 8.1](#), we can preserve weak Poincaré inequalities in intersections if the intersections are ample enough. In [Example 8.4](#), the two intersecting line segments do not have ample intersection but we can create the cross-like shape by gluing three V-like shapes one by one into each other, and we get these V-like shapes by using three line segments and bi-Lipschitz mappings.

can then glue one by one to each other with ample intersections to create the original cross-like piece (see [Figure 5](#)).

We now get the boundary $\partial\Omega$ by gluing first two cross-like pieces to either one of the distorted circles and then gluing the remaining distorted circle to this shape. Gluing sets together using [Theorem 8.1](#) preserves the weak Poincaré inequalities but with a different metric, the one in [\(8.2\)](#). However, for shapes as simple as the ones we use, it is straightforward to see that the new metrics we get are bi-Lipschitz equivalent with the Euclidean metric. Thus, by gluing together the two distorted circles with the help of two cross-like unions of two line segments, we see that $\partial\Omega$ supports a weak 1-Poincaré inequality. We note that we had to use two cross-like pieces for the gluing process because with just one cross-like piece the ample intersection requirement of [Theorem 8.1](#) does not hold for the final step for one of the two intersection points of the distorted circles. By the original, more general form of [Theorem 8.1](#) in [[Heinonen and Koskela 1998](#)], we can leave this type of a problematic isolated point out of the gluing process, but this would end up giving us a metric space with a different structure than $\partial\Omega$.

Remark 8.5. [Example 8.4](#) gives us a 2-dimensional example but we can use the same techniques to construct higher dimensional examples. These examples are of the type $\Omega \times (0, 1)^n$, where Ω is the 2-dimensional set from [Example 8.4](#). Thus, for example, the 3-dimensional example would be a hollow, twice-pinched cylinder with a thick boundary.

Remark 8.6. By using a once-pinched annulus instead of the twice-pinched one we used in [Example 8.4](#), we get a connected set Ω such that it satisfies the 2-sided corkscrew condition, the boundary $\partial\Omega$ is 1-ADR and $\partial\Omega$ supports a weak 1-Poincaré inequality. However, despite connectivity, Ω is still not a chord-arc domain: there are points arbitrarily close to each other on different sides of the pinched part of the annulus such that they can be connected inside Ω only by circling around almost the entire annulus.

Thus, [Corollary 1.6](#) cannot be reversed and we cannot weaken its assumptions in the obvious way. It is natural to formulate the following problem:

Problem 8.7. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with n -dimensional Ahlfors–David regular boundary. Give a geometric characterization for weak Heinonen–Koskela-type boundary p -Poincaré inequalities for $1 \leq p \leq n$.

By a geometric characterization in [Problem 8.7](#), we mean a characterization of the type “ $\omega \in \text{weak-}A_\infty(\sigma)$ if and only if $\partial\Omega$ is UR and Ω satisfies the weak local John condition” (which is the main result of [\[Azzam et al. 2020\]](#)). By [\[Cheeger 1999, Theorem 17.1\]](#), we know that $\partial\Omega$ has to be quasiconvex, and by [\[Azzam 2021a\]](#), we know that $\partial\Omega$ has to be UR. By [Corollary 1.6, Example 8.4](#) and [\[Mourgoglou and Tolsa 2021, Section 10\]](#), we know that 2-sided chord-arc domains are too strong for the characterization and (1-sided) chord-arc domains are not strong enough for a characterization. However, the answer does not lie somewhere between these two classes of domains: by [Example 8.4](#), the connectivity properties of Ω itself do not play a big role in this problem.

Concerning John-type conditions, we recall the open problem we mentioned in [Section 3](#):

Problem 8.8. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with n -UR (or just n -ADR) boundary. Suppose that Ω satisfies the local John condition. Does Ω also satisfy the weak local John condition?

[Problem 8.8](#) is interesting only if Ω does not satisfy the exterior corkscrew condition: the local John condition implies the corkscrew condition and therefore the answer is trivially true in the presence exterior corkscrews by [Lemma 3.6](#).

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