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PROJECTIVE EMBEDDING OF STABLY DEGENERATING SEQUENCES OF HYPERBOLIC RIEMANN SURFACES

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Given a sequence of genus $g \geq 2$ curves converging to a punctured Riemann surface with complete metric of constant Gaussian curvature -1, we prove that the Kodaira embedding using an orthonormal basis of the Bergman space of sections of a pluricanonical bundle also converges to the embedding of the limit space together with extra complex projective lines.

1. Introduction

Let $\mathcal{M}_g$ be the moduli of a smooth compact Riemann surfaces of genus g. When $g \geq 2$, the compactification $\overline{\mathcal{M}}_g$, due to [Deligne and Mumford 1969], is the moduli of stable curves. Each smooth curve of genus g carries a unique Poincaré metric with constant Gaussian curvature -1. If $C \in \overline{\mathcal{M}}_g$ is a singular stable curve, then by removing the nodes, the smooth part carries a unique complete hyperbolic metric with constant Gaussian curvature -1. And if a holomorphic family $\pi : \mathcal{C} \to \boldsymbol{D}$ of compact smooth curves C_t degenerates to $C = C_0$, then the hyperbolic metrics are continuous on the vertical line bundle [Wolpert 1990].

In this article, from the point view of the quantization framework in [Donaldson 2001; Donaldson and Sun 2014], we are interested in the convergence of the pluricanonical Bergman embeddings of the family of hyperbolic surfaces in the complex projective spaces. More precisely, let (C_j, g_j) be a sequence of genus $g \geq 2$ Riemann surfaces with Riemannian metric g_j of constant Gaussian curvature -1 that converges, in the pointed Gromov–Hausdorff topology, to a punctured Riemann surface (C_0, g_0) — not necessarily connected — with a complete Riemannian metric g_0 of constant Gaussian curvature -1. Let K_{C_j} denote the canonical bundle of C_j; then K_{C_j} is endowed with a Hermitian metric h_j defined by the Kähler form ω_j associated to g_j. We consider the Bergman space $\mathcal{H}_{j,k}$ consisting of L^2-integrable holomorphic sections of $K^k_{C_j}$. Then $\mathcal{H}_{j,k}$ is a finite-dimensional Hermitian space with the Hermitian product defined by

$$\langle s, t \rangle = \int_{C_j} (s, t)_{h_j} \omega_j,$$

where, by abuse of notation, we still use h_j to denote the induced Hermitian metric on $K^k_{C_j}$. For k large enough, a basis of $\mathcal{H}_{j,k}$ will induce a Kodaira embedding of C_j to $\mathbb{CP}^{N_k}$, where $N_k = \dim \mathcal{H}_{j,k} - 1$ is

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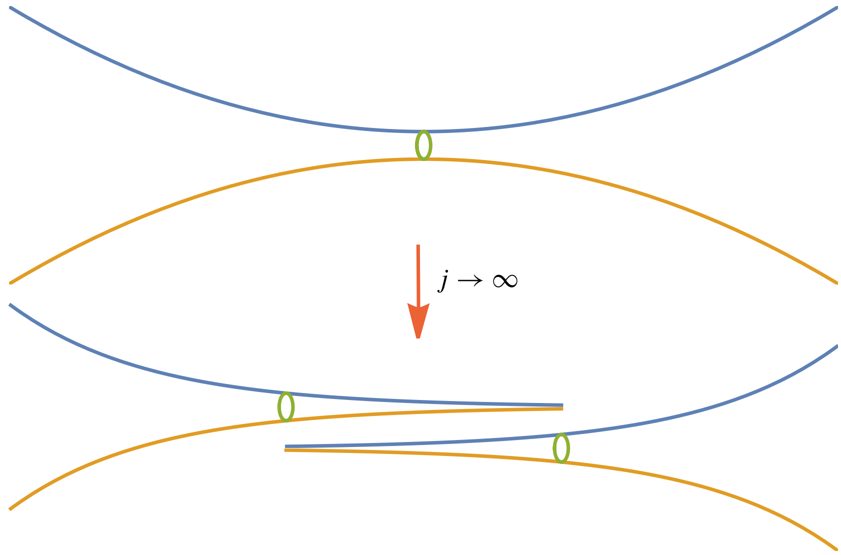

Figure 1. Degeneration of hyperbolic metrics.

independent of $j \geq 1$. For $j = 0$, the dimension of $\mathcal{H}_{j,k}$ is smaller than that of $j > 0$. It is natural to consider the embedding induced by an orthonormal basis for $\mathcal{H}_{j,k}$, which can be considered as a bridge from Kähler geometry to algebraic geometry [Donaldson and Sun 2017; Sun and Sun 2021]. It is worth mentioning that after this article was finished, the author learned that Dong [2023] recently proved that if a smooth family of hyperelliptic curves degenerate to a nodal curve, then their Bergman kernels also converge to the Bergman kernel of the nodal curve.

As the Gaussian curvature is -1, the degeneration of metrics can only be "pinching a nontrivial loop", namely a sequence of surfaces with increasingly thinner and longer handles, with the central loops degenerating to points. So C_0 has d pairs of punctures, which will be called ends. And for k large enough, the dimension of $\mathcal{H}_{0,k}$ equals $N_k + 1 - d$. Now we can state our main theorem.

Theorem 1.1. *For k large enough, we can choose an orthonormal basis for $\mathcal{H}_{j,k}$ for all $j \geq 0$, so that, as $j \to \infty$, the image of the embedding*

$$\Phi_{j,k} : C_j \to \mathbb{CP}^{N_k}$$

induced by the orthonormal basis converges to the image of C_0 under the embedding

$$\Phi_{0,k} : C_0 \to \mathbb{CP}^{N_k - d} \subset \mathbb{CP}^{N_k},$$

attached with d pairs of complex projective lines. To each pair of the ends $(p_\alpha, p_{\alpha+d})$ a pair of complex projective lines is associated, forming a connected chain between the images of these two points.

It is interesting to mention that during the process of taking the limit, the pair of complex projective lines are developed as a pair of bubbles. We illustrate this process in Figures 1 and 2. Also, we should mention that k depends only on the geometry of C_0 and does not need to be too big by the results in [Sun 2017].

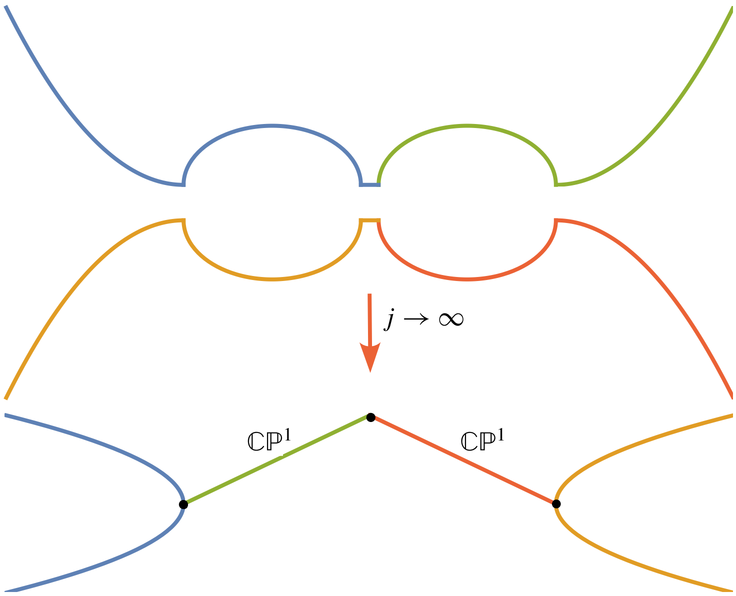

Figure 2. Degeneration in $\mathbb{CP}^{N_k}$.

The proof of this theorem makes heavy use of the methods we developed from [Sun and Sun 2021] to [Sun 2017]. Just as in [Donaldson and Sun 2014], the main point is basically proving the convergence of the Bergman kernels. We hope this result may shine a light on the study of the degeneration of higher-dimensional projective manifolds [Honda et al. 2019; Song 2017; Sun 2019].

The structure of this article is as follows. We will first quickly recall the necessary background for this article. Then we will calculate in the model for the thin handles, or "the collar", of the Riemann surfaces close to the limit. And in the end, we will finish the proof of the convergence of the pluricanonical Bergman embeddings.

2. Punctured model

On the punctured disk $\boldsymbol{D}^*$, the Poincaré metric is

$$\omega_P = \frac{2i\, dz \wedge d\bar{z}}{|z|^2(\log|z|^2)^2}.$$

Taking the local section of the canonical bundle $e = dz/z$, the local potential is

$$\varphi_P = -\log|e|^2 = -\log\frac{(\log 1/|z|^2)^2}{2}.$$

We use the notation $\tau = -\log|z|$, which yields $\varphi_P = -\log(2\tau^2)$. We are interested in the L^2-norm of the sections $z^a e^{k+1}$ of $K_{\boldsymbol{D}^*}^{k+1}$, $a \in \mathbb{Z}^+$. So we have the integrals

$$Y_a = \int (2\tau^2)^{k+1}|z|^{2a}\omega_P.$$

Further, we have

$$Y_a = 2^{k+2}\pi \int_0^\infty e^{-2a\tau+2k\log\tau}\, d\tau.$$

Writing $g_a(\tau) = -2a\tau + 2k\log\tau$, we have $g_a''(t) = -2k/\tau^2$. So $g_a(\tau)$ is a concave function which attains its only maximum at $\tau_a = k/a$. We will use the following basic result from [Sun 2017, Lemma 2.6].

Lemma 2.1. *Let $f(x)$ be a concave function. Suppose $f'(x_0) < 0$, then we have*

$$\int_{x_0}^\infty e^{f(x)}\, dx \le \frac{e^{f(x_0)}}{-f'(x_0)}.$$

We can use Laplace's method and the lemma above to estimate

$$Y_a \approx 2^{k+2}\pi e^{-2k+2k\log(k/a)}\sqrt{\frac{k\pi}{2a^2}}.$$

Of course, we can directly calculate the integral to get

$$Y_a = 2^{k+2}\pi \frac{(2k)!}{(2a)^{2k+1}},$$

but the idea of mass concentration is key to our arguments. The Bergman kernel of $\boldsymbol{D}^*$ is then

$$\rho_{0,k+1} = \frac{2^{2k}\tau^{2k+2}}{\pi(2k)!}\sum (a)^{2k+1}|z|^{2a}.$$

Let C_0 be a punctured Riemann surface obtained by removing $2d$ points $\{p_\alpha\}_{1\le\alpha\le 2d}$ from a compact Riemann surface. C_0 is endowed with a complete Poincaré metric ω with constant Gaussian curvature -1. The metric ω defines a Hermitian metric h on the canonical bundle K_{C_0}. Then, for any positive integer k, we denote by $\mathcal{H}_k$ the space of holomorphic sections of $K_{C_0}^k$ that are L^2-integrable, namely

$$\|s\|^2 = \int_{C_0} |s|_h^2\omega < \infty.$$

For each p_α, there is a neighborhood U_α with local coordinate z such that $\omega = \omega_P$ on $U_\alpha \backslash p_\alpha$. We can assume that U_α contains the points satisfying $|z| \le R_\alpha$. We note that the injective radius at the points $|z| = R_\alpha$ is about $\pi/(4(\log R_\alpha)^2)$. For simplicity, we let R be the minimum of the R_α, $1 \le \alpha \le 2d$. Clearly, for the complement of $\bigcup_{1\le\alpha\le 2d} U_\alpha$ in C_0, there is a positive lower bound λ_0 for the injective radius.

Let ρ_{k+1} denote the Bergman kernel of C_0 for the Bergman space $\mathcal{H}_{k+1}$. The basic conclusion of [Sun 2017] (which is also an implication of [Sun 2013], although not explicitly stated there) is that, for k large enough, in the "inside" of U_α, where $\tau = -\log|z| > \sqrt{k+1}$, the Bergman kernel ρ_{k+1} is very much like $\rho_{0,k+1}$, meaning that $|\rho_{k+1}/\rho_{0,k+1} - 1|$ is $o(1/k^N)$ for all N. We have shown in [Sun 2013; 2017] that $\rho_{0,k+1}$ is dominated by the terms $c_a|z|^{2a}$ for $a < k^{3/4}$, meaning that

$$\frac{\rho_{0,k+1}}{\sum_{a=1}^{k^{3/4}} c_a|z|^{2a}} - 1 = o\left(\frac{1}{k^N}\right)$$

for all N. In particular, $\rho_{0,k+1}$ is dominated by $c_1|z|^2$ when $\tau \ge k$. The sections for C_0 corresponding to z^a in the model $\boldsymbol{D}^*$ is constructed as follows. We let z_α denote the local coordinate z on U_α. Let $e_\alpha = dz_\alpha/z_\alpha$ be the local frame of K_{C_0}. Then $z_\alpha^a e_\alpha^{k+1}$, $a \ge 1$, are local sections of $K_{C_0}^{k+1}$. We choose and fix a cut-off function $\chi(r)$ that equals 1 for $r < \frac{1}{2}R$ and 0 for $r > \frac{2}{3}R$. Then we denote by χ_α the function $\chi(|z_\alpha|)$ defined on C_0. Then $\chi_\alpha z_\alpha^a e_\alpha^{k+1}$ is a global smooth L^2-integrable section of $K_{C_0}^{k+1}$. We then take the orthogonal projection of this section into the space $\mathcal{H}_{k+1}$, and then normalize the holomorphic section to be of norm 1, obtaining a section $s_{\alpha,a} \in \mathcal{H}_{k+1}$. We write

$$V_0 = \{s_{\alpha,a} : 1 \le \alpha \le 2d,\ 1 \le a < k^{3/4}\}.$$

For k large enough, within $r < \frac{1}{4}R$, the sections $s_{\alpha,a}$ are approximately equal to $\sqrt{c_a}z^a$ with relative error less than $1/k^2$.

We choose and fix an orthonormal basis $W_0 = \{s_j\}$ for the orthogonal complement $V_0^\perp \subset \mathcal{H}_{k+1}$.

To obtain global sections of L^k from local ones, we will need to use Hörmander's L^2 estimate. The following lemma is well known; see for example [Tian 1990].

Lemma 2.2. *Suppose (M, g) is a complete Kähler manifold of complex dimension n and $\mathcal{L}$ is a line bundle on M with hermitian metric h. If*

$$\langle -2\pi i\Theta_h + \mathrm{Ric}(g), v \wedge \bar{v}\rangle_g \ge C|v|_g^2$$

for any tangent vector v of type $(1, 0)$ at any point of M, where $C > 0$ is a constant and Θ_h is the curvature form of h, then, for any smooth $\mathcal{L}$-valued $(0, 1)$-form α on M with $\bar\partial\alpha = 0$ and $\int_M |\alpha|^2\, dV_g$ finite, there exists a smooth $\mathcal{L}$-valued function β on M such that $\bar\partial\beta = \alpha$ and

$$\int_M |\beta|^2\, dV_g \le \frac{1}{C}\int |\alpha|^2\, dV_g,$$

where dV_g is the volume form of g and the norms are induced by h and g.

In our setting, for a curve C_j, $j \ge 0$, with line bundle $K_{C_j}^{k+1}$, the constant k is independent of j.

3. The collar model

Let f_ε be a function depending only on $|z|$ satisfying the conditions

- $f_\varepsilon > 0$,
- $f_\varepsilon(1) = \varepsilon^2$,
- $f'_\varepsilon(1) = 0$,
- $\Delta_z \log f_\varepsilon(|z|) = 2f_\varepsilon/|z|^2$.

Clearly, such an f_ε exists and is unique in a neighborhood U of $|z| = 1$. Let

$$\omega_\varepsilon = \frac{f_\varepsilon i\, dz \wedge d\bar{z}}{2|z|^2}$$

be the Kähler metric defined on U. Then our choice of f_ε makes the metric have constant Gaussian curvature -1. Writing $t = \log|z|$ and abusing notation, we consider f_ε as a function of t. Then we have

$$\Delta_z \log f_\varepsilon = \frac{d^2 \log f_\varepsilon}{dt^2} \frac{1}{|z|^2}.$$

For simplicity, we will use $f(t)$ to denote f_ε. Therefore, we have

$$(\log f(t))'' = 2f(t).$$

The first fundamental form of the metric is

$$I = f(t)\,dt^2 + f(t)\,d\theta^2.$$

We use the arc-length parameter u for the t-curves. Then by the curvature condition, we have

$$f(t) = \varepsilon^2 \cosh^2 u, \quad u(0) = 0, \quad \text{and} \quad \frac{dt}{du} = \frac{1}{\varepsilon \cosh u}.$$

Therefore, we have

$$t = \frac{1}{\varepsilon} \tan^{-1}[\sinh u].$$

One can see that f can be extended to the annulus $\{z : -\pi/(2\varepsilon) < t < \pi/(2\varepsilon)\}$. It is worth noticing that $|u| \to \infty$ when t goes to the boundary $|t| = \pi/(2\varepsilon)$, meaning $f_\varepsilon \to \infty$. It is natural to use the notation

$$\tau_\varepsilon = \frac{\pi}{2\varepsilon} - t.$$

From now on, we will always assume that ε is very small compared to k^{-k}. So the region where f is defined is a large annulus. We will write

$$\mathbb{C}^*_\varepsilon = \left\{ z : -\frac{\pi}{2\varepsilon} < t < \frac{\pi}{2\varepsilon} \right\},$$

and we have our model $\mathbb{C}^*_\varepsilon = (\mathbb{C}^*_\varepsilon, \omega_\varepsilon)$.

We also use the frame $e = dz/z$ for the canonical bundle $K_{\mathbb{C}^*_\varepsilon}$, so we have

$$|e|^2 = \frac{2}{f}.$$

So for a section of $s = ge^{\otimes(k+1)}$ of $K^{k+1}_{\mathbb{C}^*_\varepsilon}$, the L^2-norm squared is

$$\|s\|^2 = \int_{\mathbb{C}^*_\varepsilon} \left(\frac{2}{f}\right)^{k+1} |g|^2 \omega_\varepsilon.$$

We are interested in the L^2-norm of the sections $z^a e^{k+1}$ of $K^{k+1}_{\mathbb{C}^*_\varepsilon}$, $a \in \mathbb{Z}$. So we have the integrals

$$I_{\varepsilon,a} = \int \left(\frac{2}{f}\right)^{k+1} |z|^{2a} \omega_\varepsilon.$$

Further, we have

$$I_{\varepsilon,a} = 2^{k+2}\pi \int_{-\pi/(2\varepsilon)}^{\pi/(2\varepsilon)} e^{2at-k\log f}\, dt.$$

Writing $g_a(t) = 2at - k\log f$, we have $g_a''(t) = -2kf(t)$. So $g_a(t)$ is a concave function which attains its only maximum at t_a satisfying $f'(t_a)/f(t_a) = 2a/k$. Write $u_a = u(t_a)$. Since $f'(t)/f(t) = 2\varepsilon \sinh u$, we have

$$\varepsilon \sinh u_a = \frac{a}{k}.$$

So $\sinh u_a = a/(k\varepsilon)$ is very large when $a \geq 1$, and $f(t_a) = a^2/k^2 + \varepsilon^2 > a^2/k^2$. When a is large, say $a \geq k$, we have that $g_a''(t_a)$ is also large. The third derivative is

$$g^{(3)}(t) = -4k\varepsilon \sinh u.$$

So $g^{(3)}(t_a) = -8a$, and, for $|t - t_a| < (\sqrt{k}\log k)/a$, we have that $g^{(3)}(t)$ is also $O(a)$. Therefore, for $|t - t_a| < (\sqrt{k}\log k)/a$,

$$\frac{g''(t)}{g''(t_a)} = 1 + O\left(\frac{\log k}{\sqrt{k}}\right).$$

We can use Lemma 2.1 to estimate

$$I_{\varepsilon,a} = \left(1 + O\left(\frac{\log k}{\sqrt{k}}\right)\right) 2^{k+2}\pi e^{2at_a - k\log f(t_a)} \sqrt{\frac{\pi}{2kf(t_a)}} = \sqrt{\frac{\pi}{2k}} \frac{2^{k+2}\pi}{(\varepsilon\cosh u_a)^{2k+1}} e^{2at_a}.$$

We have that the mass of $I_{\varepsilon,a}$ is concentrated within the neighborhood $\{|t - t_a| < (\sqrt{k}\log k)/a\}$ with relative error less than $k^{-\log k + 3/2}$, namely

$$I_{\varepsilon,a} = (1 + O(k^{-\log k+3/2}))2^{k+2}\pi \int_{|t-t_a|<(\sqrt{k}\log k)/a} e^{2at-k\log f}\, dt.$$

When $a < k$, we use the variable u to estimate the integral

$$\int_{-\pi/(2\varepsilon)}^{\pi/(2\varepsilon)} e^{2at-k\log f}\, dt = \int_{-\infty}^{\infty} e^{2at-\log f} \frac{du}{\varepsilon\cosh u}.$$

Let $-G(u) = k\log f - 2at - \log\cosh u$ be the exponent function. Then

$$G'(u) = 2k\frac{\sinh u}{\cosh u} - \frac{2a}{\varepsilon\cosh u} + \frac{\sinh u}{\cosh u},$$

$$G''(u) = \frac{2k}{\cosh^2 u} + \frac{2a\sinh u}{\varepsilon\cosh^2 u} + \frac{1}{\cosh^2 u},$$

$$G^{(3)}(u) = -\frac{(4k+2)\sinh u}{\cosh^3 u} + \frac{2a(1-\sinh^2 u)}{\varepsilon\cosh^3 u}.$$

So $G(u)$ is a convex function of u which attains its only minimum at u_a' satisfying

$$\sinh u_a' = \frac{2a}{\varepsilon(2k+1)},$$

and

$$G''(u'_a) = \frac{(2k+1)^3\varepsilon^2 + 4a^2(2k+1)}{4a^2 + (2k+1)^2\varepsilon^2} = 2k+1+O(\varepsilon^2),$$

which is large. Further,

$$G^{(3)}(u'_a) = -\frac{2k+1}{[(2k+1)^2/(4a^2)+1]^{3/2}} + O(\varepsilon^2).$$

So again, we can estimate

$$\begin{aligned} I_{\varepsilon,a} &= \left(1+O\left(\frac{\log k}{\sqrt{k}}\right)\right)\frac{2^{k+2}\pi}{\varepsilon\cosh u'_a} e^{2at'_a - k\log f(t'_a)}\sqrt{\frac{\pi}{2k+1}} \\ &= \left(1+O\left(\frac{\log k}{\sqrt{k}}\right)\right)\sqrt{\frac{\pi}{2k+1}}\frac{2^{k+2}\pi}{(\varepsilon\cosh u'_a)^{2k+1}} e^{2at'_a}, \end{aligned}$$

where $t'_a = t(u'_a)$.

So for $a \geq k$,

$$\begin{aligned} \frac{I_{\varepsilon,a+1}}{I_{\varepsilon,a}} &= \left(1+O\left(\frac{\log k}{\sqrt{k}}\right)\right) e^{2at_{a+1}-t_a}\left(\frac{\cosh u_a}{\cosh u_{a+1}}\right)^{2k+1} \\ &= \left(1+O\left(\frac{\log k}{\sqrt{k}}\right)\right) e^{2\varepsilon k/a}\frac{a^{2k+1}}{(a+1)^{2k+1}} = \left(1+O\left(\frac{\log k}{\sqrt{k}}\right)\right)\frac{a^{2k+1}}{(a+1)^{2k+1}}. \end{aligned}$$

Since

$$\frac{\varepsilon\sinh u'_a}{\varepsilon\sinh u_a} = 1 + \frac{1}{2k},$$

this approximation for $I_{\varepsilon,a+1}/I_{\varepsilon,a}$ works for all $a > 0$.

Since $\sinh u_a$ is very large, we can use the Taylor expansion of $\tan^{-1}$ around infinity to estimate

$$\frac{\pi}{2} - \varepsilon t_a = \frac{\varepsilon k}{a} + O\left(\left(\frac{\varepsilon k}{a}\right)^2\right)$$

and also

$$\frac{\pi}{2} - \varepsilon t'_a = \frac{\varepsilon(2k+1)}{2a} + O\left(\left(\frac{\varepsilon k}{a}\right)^2\right).$$

Therefore, for $k > a \geq 1$,

$$\frac{|z|^{2a}}{I_{\varepsilon,a}} = \left(1+O\left(\frac{\log k}{\sqrt{k}}\right)\right)\frac{\sqrt{2k}2^{k-1}}{\pi^{3/2}}\left(\frac{e}{2k+1}\right)^{2k+1} a^{2k+1}e^{-2a\tau_\varepsilon}.$$

Notice that the term

$$\frac{\sqrt{2k}2^{k-1}}{\pi^{3/2}}\left(\frac{e}{2k+1}\right)^{2k+1}$$

is independent of a.

Recall that the power series in the expression of $\rho_{0,k+1}$ is also

$$\sum a^{2k+1}|z|^{2a} = \sum a^{2k+1}e^{-2a\tau}$$

By the same argument as in [Sun 2017, Theorem 1.1] and [Sun 2013, p. 5535, Case II], for $t \in [0, t_1]$, the Bergman kernel is dominated by

$$\left[\frac{|z|^2}{I_{\varepsilon,1}} + \frac{1}{I_{\varepsilon,0}}\right]\left(\frac{2}{f}\right)^{k+1}.$$

The idea of the argument is very simple: by the mass concentration property of the integral $I_{\varepsilon,a}$, the contribution of $|z|^{2a}/I_{\varepsilon,a}$ to the Bergman kernel gets smaller and smaller when t moves further away from t_a. When $a < \sqrt{k}/\log k$, we have that $(a/(a+1))^{2k+1}$ is very small, meaning that t_a is already far enough from t_{a+1} for the integral $I_{\varepsilon,a+1}$, so that the contribution of $|z|^{2(a+1)}/I_{\varepsilon,a+1}$ to the Bergman kernel at t_a is negligible.

By symmetry, for $t \in [t_{-1}, 0]$, the Bergman kernel is dominated by

$$\left[\frac{|z|^{-2}}{I_{\varepsilon,-1}} + \frac{1}{I_{\varepsilon,0}}\right]\left(\frac{2}{f}\right)^{k+1}.$$

In particular, we have the following:

Lemma 3.1. *For any holomorphic section s of $K_{\mathbb{C}^*_\varepsilon}^{k+1}$ satisfying $\|s\| = 1$, we have*

$$|s|^2 < \varepsilon^2\left(\frac{\sqrt{e}\log\varepsilon}{k}\right)^{2k}$$

when

$$\cosh u \in \left(\frac{-1}{2\varepsilon\log\varepsilon}, \frac{-1}{\varepsilon\log\varepsilon}\right).$$

Proof. By symmetry, we can assume $t > 0$. For the right end of the interval, we only need to estimate the norms of

$$\frac{z}{I_{\varepsilon,1}}e^{k+1} \quad\text{and}\quad \frac{1}{I_{\varepsilon,0}}e^{k+1}$$

at t, where $\cosh u = -1/(\varepsilon\log\varepsilon)$. For the first one, we have

$$\left|\frac{z}{I_{\varepsilon,1}}e^{k+1}\right|^2 = \left(1 + O\left(\frac{\log k}{\sqrt{k}}\right)\right)\frac{\sqrt{2k}\varepsilon^2}{2k\pi^{3/2}}\left(\frac{\log\varepsilon}{k}\right)^{2k}e^{k+1/2}.$$

For the second one, notice that $u'_a = 0 = t'_a$ and

$$I_{\varepsilon,0} = \left(1 + O\left(\frac{\log k}{\sqrt{k}}\right)\right)\sqrt{\frac{\pi}{2k+1}}\frac{2^{k+2}\pi}{\varepsilon^{2k+1}}.$$

So we have

$$\left|\frac{1}{I_{\varepsilon,0}}e^{k+1}\right|^2 = \left(1 + O\left(\frac{\log k}{\sqrt{k}}\right)\right)\frac{\sqrt{2k}}{2\pi^{3/2}}\varepsilon^{2k+1}(\log\varepsilon)^{2k},$$

which is much smaller than the first one. For the left end of the interval, we have a smaller norm for the section $ze^{k+1}/I_{\varepsilon,1}$, and a still very small norm for the section $e^{k+1}/I_{\varepsilon,0}$. Combining these estimates, we have proved the lemma. □

Assume C_j converges to C_0 in the pointed Gromov–Hausdorff topology. For j big enough, C_j has exactly d closed geodesics whose arc length is less than $\frac{1}{4}\lambda_0$. We denote these circles by $\gamma_{j,\alpha}$, $1 \le \alpha \le d$, and the arc length of $\gamma_{j,\alpha}$ by $\varepsilon_{j,\alpha}$. Rearranging the points p_α, we can assume that $2\pi\varepsilon_{j,\alpha}$ converges to the pair $(p_\alpha, p_{\alpha+d})$ as $j \to \infty$. Also, for j large enough, there is a neighborhood $U_{j,\alpha}$, usually referred to as a collar, of each $\gamma_{j,\alpha}$ which is homeomorphic to an annulus. We define a map

$$h_{j,\alpha} : U_{j,\alpha} \to \mathbb{C}^*_\varepsilon,$$

with $\varepsilon = \varepsilon_{j,\alpha}$, as follows. Fix an isometry λ of $\gamma_{j,\alpha}$ to the circle $|z| = 1$ in $\mathbb{C}^*_\varepsilon$. Then, passing through each point q on $\gamma_{j,\alpha}$, there is an unique geodesic l_q orthogonal to $\gamma_{j,\alpha}$. We define $h_{j,\alpha}$ to be the map that sends each such geodesic l_q to the geodesic passing through $\lambda(q)$, orthogonal to the unit circle, and preserving λ and the orientation. Since both surfaces have constant Gaussian curvature -1, $h_{j,\alpha}$ is an isometry so long as the geodesics l_q do not intersect each other. But since the curvature is negative, by the Gauss–Bonnet theorem, these geodesics cannot intersect within $U_{j,\alpha}$. Therefore, $h_{j,\alpha}$ is also holomorphic, and we can use the coordinate z from $\mathbb{C}^*_\varepsilon$ as the holomorphic coordinate of $U_{j,\alpha}$. By switching p_α and $p_{\alpha+d}$ if necessary, we can assume that the part $|z| > 1$ of $U_{j,\alpha}$ converges to a neighborhood of p_α and the part $|z| < 1$ converges to that of $p_{\alpha+d}$. We can assume that $U_{j,\alpha} = \{1/M \le |z| \le M\}$, and, for j large enough, we can assume that the injective radius at $|z| = M$ is larger than

$$\frac{\pi}{4(\log 3R/4)^2}.$$

We denote by $U^+_{j,\alpha}$ the part of $U_{j,\alpha}$ with $|z| > 1$ and similarly $U^-_{j,\alpha}$ the part of $U_{j,\alpha}$ with $|z| < 1$. We then define a map

$$\varphi_{j,\alpha} : U^+_{j,\alpha} \to U_\alpha$$

by sending $\varepsilon \cosh u$ to $1/(2\tau)$ while preserving the circles $\{u = \text{constant}\}$. Clearly, we are only preserving the length of the circles. By symmetry, we also have

$$\varphi_{j,\alpha+d} : U^-_{j,\alpha} \to U_{\alpha+d}.$$

By our assumption on the injective radius, the image of $\varphi_{j,\alpha}$ contains the circle $|z_\alpha| = \frac{3}{4}R$. On U_α, the first fundamental form is

$$I_0 = \frac{1}{\tau^2}(d\tau^2 + d\theta^2).$$

The pullback

$$\varphi^*_{j,\alpha} I_0 = \tanh^2 u \, du^2 + (\varepsilon \cosh u)^2 \, d\theta^2$$

is almost isometric to the metric

$$I_j = du^2 + (\varepsilon \cosh u)^2 \, d\theta^2$$

when u is large. In particular, for the part where $\varepsilon \cosh u \ge -1/\log \varepsilon$, we have that $\varphi_{j,\alpha}$ converges to an isometry when $j \to \infty$.

Let $U_\alpha(r)$ denote the subset of U_α consisting of the points $|z_\alpha| < r$. Let $F = C_0 \setminus \bigcup_{1\le\alpha\le 2d} U_\alpha\big(\frac{2}{3}R\big)$, and let $\psi_j : F \to C_j$ be the diffeomorphism with its image. Since ψ_j converges to an isometry as $j\to\infty$, we can glue ψ_j^{-1} with the $\varphi_{j,\alpha}$ — rotating $\varphi_{j,\alpha}$ if necessary — for j large enough, to get a map

$$G_j : C_j \setminus \bigcup \gamma_{j,\alpha} \to C_0$$

with the following properties:

- G_j is a diffeomorphism of $C_j \setminus \bigcup \gamma_{j,\alpha}$ with its image.
- $G_j = \varphi_{j,\alpha}$ for $p \in \varphi_{j,\alpha}^{-1} U_\alpha\big(\frac{2}{3}R\big),\ 1 \le \alpha \le 2d$.
- G_j is almost an isometry on $C_j \setminus \bigcup_{1\le\alpha\le 2d} \varphi_{j,\alpha}^{-1} U_\alpha\big(\frac{2}{3}R\big)$ and converges to an isometry when $j \to \infty$.

For any conformal metric, the compatible complex structure J is just a counterclockwise rotation by $\frac{\pi}{2}$. We see that almost isometric implies almost holomorphic. Therefore $G_j^{-1*} K_{C_j}$ converges to K_{C_0} as subbundles of $T_{C_0} \otimes \mathbb{C}$. More precisely, let J_j be the complex structure compatible with the Riemannian metric g_j. If the pointwise norm

$$\sup_{v\in T_p, |v|_g = 1} |g_j(v, v) - g(v, v)| < \delta,$$

then we have

$$\sup_{v\in T_p, |v|_g = 1} |J_j(v) - J(v)|_g < \lambda\delta$$

for some constant λ independent of p and g. We call the supremum above the pointwise distance from J_j to J. Moreover, if g_j converges to g in C^2-norm, then J_j converges to J in C^2-norm also. If we denote by T_J the holomorphic tangent space with respect to J, then the orthogonal projection of T_{J_j} to T_J is close to an isometry if J_j is close to J. We identify T_{J_j} with T_j via this orthogonal projection, and similarly $K_j = T^*_{J_j}$ with $K = T^*_J$, which we will also call an orthogonal projection for simplicity. Since the metric on the canonical bundle is defined by the Kähler form ω and ω_j converges to ω, we have that the Chern connection ∇_j on K_j converges to the Chern connection ∇ on K.

4. Convergence of projective embedding

By assigning value 1 on $\gamma_{j,\alpha}$, we glue together the pullbacks $G_j^*\chi_\alpha$ and $G_j^*\chi_{\alpha+d}$ to get a function denoted by $\tilde\chi_\alpha$ for $1 \le \alpha \le d$. On each $\varphi_{j,\alpha}$, we also consider the $\tilde\chi_\alpha z^a$ as global smooth sections of $K_{C_j}^{k+1}$. Then we repeat the construction of V_0 by normalizing the orthogonal projection of $\tilde\chi_\alpha z^a$ onto $\mathcal{H}_{j,k+1}$ and denote the resulting section by $s_{j,\alpha,a},\ |a| < k^{3/4}$. Then we define

$$V_j = \{s_{j,\alpha,a} : 1 \le \alpha \le d,\ |a| < k^{3/4}\}.$$

We should remark here that the choice of the upper bound $k^{3/4}$ is not necessary, it is purely a habit from [Sun and Sun 2021]. Notice that the number of sections of V_j is larger than that of V_0 by the number d. Those extra sections are $\{s_{j,\alpha,0}\}_{1\le\alpha\le d}$. We consider $s_{j,\alpha,a}$ as a smooth section of $G_j^{-1*} K_{C_j}^{k+1}$ on image(G_j). We then define a piecewise smooth section $\tilde s_{j,\alpha,a}$ of $K_{C_0}^{k+1}$ which equals the orthogonal projection of $s_{j,\alpha,a}$ to $K_{C_0}^{k+1}$ on image(G_j), and equals 0 in the complement. For simplicity, we will say that $s_{j,\alpha,a}$ converges in some topology if $\tilde s_{j,\alpha,a}$ converges in that topology.

Proposition 4.1. *The smooth section $s_{j,\alpha,a}$ converges to $s_{\alpha,a}$ for $a > 0$ and to $s_{\alpha+d,-a}$ for $a < 0$, in L^2-norm, as $j \to \infty$.*

Proof. By symmetry, we only have to prove the result for $a > 0$. By taking j large enough, we can assume that $p \in C_0 \backslash \bigcup U_\alpha(\varepsilon(j,\alpha))$. When $\varepsilon \cosh u \geq -1/\log \varepsilon$, we observe that $\tanh^2 u = 1 - (\varepsilon \log \varepsilon)^2$ is very close to 1. So $\varphi_{j,\alpha}$ is very close to an isometry. For simplicity, we still use the notation ε for $\varepsilon(j,\alpha)$. Then we look at the integral

$$I_{j,\alpha,a} = 2^{k+2}\pi \int \tilde{\chi}_\alpha^2 \frac{1}{(\varepsilon \cosh u)^{2k}} e^{2at}\, dt = 2^{k+2}\pi e^{\pi a/\varepsilon} \int \tilde{\chi}_\alpha^2 \frac{1}{(\varepsilon \cosh u)^{2k}} e^{-2a\tau_\varepsilon}\, d\tau_\varepsilon.$$

On C_0, we have

$$J_{\alpha,a} = 2^{k+2}\pi \int \chi_\alpha^2 \tau^{2k} e^{-2a\tau}\, d\tau.$$

For $I_{j,\alpha,a}$, by Lemma 2.1, we can truncate the part $\tau_\varepsilon > -\log \varepsilon$ by introducing a relative error $< \varepsilon$. Also for $J_{\alpha,a}$, we can truncate the part $\tau > -\log \varepsilon$ by introducing a relative error $< \varepsilon$. Then for the part $\tau_\varepsilon \leq -\log \varepsilon$,

$$\tilde{\chi}_\alpha^2 \frac{1}{(\varepsilon \cosh u)^{2k}} e^{-2a\tau_\varepsilon}$$

converges to $\chi_\alpha^2 \tau^{2k} e^{-2a\tau}$ uniformly. Therefore $I_{j,\alpha,a} e^{-\pi a/\varepsilon}$ converges to $J_{\alpha,a}$ as $j \to \infty$, and therefore $I_{j,\alpha,a}^{-1/2} z_\alpha^a$ converges to $J_{\alpha,a}^{-1/2} z^a$ as $j \to \infty$. Also for this part, the 1-form dz_α/z_α converges uniformly to dz/z. The way to get orthogonal projection onto holomorphic sections is to find the solutions of the equations

$$\bar{\partial} v = \bar{\partial} J_{\alpha,a}^{-1/2} z^a \left(\frac{dz}{z}\right)^{k+1} \quad \text{and} \quad \bar{\partial}_j v_j = \bar{\partial}_j I_{j,\alpha,a}^{-1/2} z_\alpha^a \left(\frac{dz_\alpha}{z_\alpha}\right)^{k+1},$$

where we denote by $\bar{\partial}_j$ the $\bar{\partial}$ operator on C_j, with minimal L^2-norms. Here the L^2-norms of v_j and v are defined with v_j and v considered as sections of $K_{C_j}^{k+1}$ and $K_{C_0}^{k+1}$, respectively. In order to prove the conclusion of the lemma, it suffices to prove that v_j converges to v. Notice that $\bar{\partial} v$ is supported within the annulus $\frac{2}{3}R \leq |z| \leq \frac{3}{4}R \subset U_\alpha$. By the mass concentration property of z^a, the mass satisfies

$$\|\bar{\partial} v\| < \frac{1}{k^2}.$$

Therefore

$$\int |v|^2 \omega < \frac{1}{k^3},$$

and v is holomorphic outside the support of $\bar{\partial} v$. Since the Bergman kernel is dominated by $|\sqrt{Y_1^{-1}} z|^2$ and

$$\int_{|z|<\varepsilon} Y_1^{-1} |z|^2 \omega < \varepsilon,$$

we have

$$\int_{|z|<\varepsilon} |v|^2 \omega < \varepsilon$$

in every U_α. We pull back the restriction v to $C_0\backslash \bigcup U_\alpha(\varepsilon(j,\alpha))$ by G_j, and use cut-off functions κ_ε near the edges to get a global smooth section of $G_j^*(K_{C_0}^{k+1})$, which is then projected to a smooth section of $K_{C_j}^{k+1}$, called $\tilde{v}_j$. More explicitly, we define the κ_ε as a smooth function of τ_ε that equals 1 for $\tau_\varepsilon < -\log\varepsilon$, and equals 0 for $\tau_\varepsilon > -2\log\varepsilon$. We can also assume that $|\kappa_\varepsilon'| < 2/(-\log\varepsilon)$. We have

$$\nabla^*\nabla v = 2\bar{\partial}^*\bar{\partial} v + (k+1)v.$$

So

$$\int |\nabla v|^2\omega = \int 2|\bar{\partial} v|^2\omega + (k+1)\int |v|^2\omega < \frac{3}{k^2}.$$

Therefore,

$$\int |\nabla_j \tilde{v}_j|^2\omega_j < \frac{4}{k^2} \quad \text{and} \quad \int_{C_j} |\bar{\partial}_j \tilde{v}_j - \bar{\partial}_j v_j|^2\omega_j = \delta_j,$$

where $\delta_j \to 0$ as $j \to \infty$. Then we can solve the equation

$$\bar{\partial}_j u = \bar{\partial}_j \tilde{v}_j - \bar{\partial}_j v_j$$

with minimal L^2-norm, so that

$$\int_{C_j} |u|^2\omega_j \leq \frac{1}{k}\delta_j.$$

Since v_j is a minimal solution,

$$\int |\tilde{v}_j - u|^2\omega_j \geq \int |v_j|^2\omega_j.$$

So

$$\int |\tilde{v}_j|^2\omega_j \geq \int |v_j|^2\omega_j - \sqrt{\frac{\delta_j}{k^5}}.$$

Conversely, for each v_j, we first use the cut-off functions κ_ε to make it vanish near the edges, then we pull it back by G_j^{-1} to C_0. By orthogonal projection and extension by 0, we obtain a smooth section $\tilde{v}^j$ of $K_{C_0}^{k+1}$. Similar to $\tilde{v}_j$, by Lemma 3.1, we have

$$\int_{C_0} |\bar{\partial}\tilde{v}^j - \bar{\partial} v|^2\omega_j = \delta_j',$$

where $\delta_j' \to 0$ as $j \to \infty$. Then by solving a $\bar{\partial}$-equation again, we get

$$\int |\tilde{v}^j|^2\omega \geq \int |v|^2\omega - \sqrt{\frac{\delta_j'}{k^5}}.$$

Since the L^2-norm of $\tilde{v}_j$ is close to that of v and

$$\int |\tilde{v}^j|^2\omega \leq \int |v_j|^2\omega_j + \delta_j'',$$

where $\delta_j' \to 0$ as $j \to \infty$, we get

$$\int |\tilde{v}_j|^2\omega_j - \int |v_j|^2\omega_j \to 0$$

as $j \to \infty$. Then by the uniqueness of the minimal solution of the $\bar\partial$-equation, we get

$$\int |\tilde{v}_j - v_j|^2 \omega_j \to 0$$

as $j \to \infty$. Finally, since the L^2-norm of $s_{j,\alpha,a}$ on the area where

$$\frac{1}{\varepsilon \cosh u} < \frac{-1}{2 \log \varepsilon}$$

also goes to 0 as $j \to \infty$, we have proved the theorem. □

The same ideas can be used for the sections in W_0. For each $s \in W_0$, we have

$$\int_{|z|<\varepsilon} |s|^2 \omega < \varepsilon$$

in every U_α. We pull back the restriction s to $C_0 \setminus \bigcup U_\alpha(\varepsilon(j,\alpha))$ by G_j, and use cut-off functions κ_ε near the edges to get a global smooth section of $G_j^*(K_{C_0}^{k+1})$, which is then projected to a smooth section of $K_{C_j}^{k+1}$, called $\tilde{s}_j$. Then we have

$$\int_{C_j} |\bar\partial_j \tilde{s}_j|^2 \omega_j \to 0$$

as $j \to \infty$. So we can solve the equation

$$\bar\partial_j u_j = \bar\partial_j \tilde{s}_j$$

with minimal L^2-norm to get a holomorphic section $s_{,j} = u_j + \tilde{s}_j \in \mathcal{H}_{j,k+1}$ satisfying

$$\int_{C_j} |s_{,j}|^2 \omega_j - 1 \to 0 \quad \text{and} \quad s_{,j} \to s$$

in L^2-norm as $j \to \infty$. We have proved the following:

Lemma 4.2. *For each $s \in W_0$, we can find $s_{,j} \in \mathcal{H}_{j,k+1}$ such that $s_{,j}$ converges to s in the L^2-norm and $\|s_{,j}\| \to 1$ as $j \to \infty$.*

We will denote by W_j the set of sections $\{s_{,j} : s \in W_0\}$. Notice that W_j is not an orthonormal set, but gets closer as j gets larger. We fix an order on W_0, then we order each W_j accordingly. We give each V_j the dictionary order. Recall that $s_{j,\alpha,a}$ corresponds to $s_{\alpha,a}$ for $a > 0$ and $s_{\alpha+d,-a}$ for $a < 0$. Then we add 0 d times to V_0, and order the obtained set $\widetilde{V}_0$ according to the correspondence to V_j, where each 0 corresponds to a section $s_{j,\alpha,0}$, $1 \le \alpha \le d$. Then we define the embedding

$$\Phi_j : C_j \to \mathbb{CP}^{N_k},$$

where $N_k = \dim \mathcal{H}_{j,k+1} - 1$, by $\Phi_j = [V_j, W_j]$, where $[\cdots]$ means the homogeneous coordinates. Similarly we define the embedding

$$\Phi_0 : C_0 \to \mathbb{CP}^{N_k}$$

by $\Phi_0 = [\widetilde{V}_0, W_0]$.

Let $[Z_0, \dots, Z_N]$ be the homogeneous coordinates of $\mathbb{CP}^N$. $U_0 = \{[1, w], w \in \mathbb{C}^N\}$ is a coordinate patch with $w_i = Z_i/Z_0$. The Z_i can be identified as generating sections in $H^0(\mathbb{CP}^N, \mathcal{O}(1))$. In particular, Z_0 is a local frame in U_0. Then on U_0, the Fubini–Study form $\omega = \frac{1}{2} i\partial\bar{\partial}\log(1+|w|^2)$ has the explicit form

$$\omega = \frac{i}{2}\frac{(1+|w|^2)\sum dw^i \wedge d\bar{w}_i - \left(\sum \bar{w}_i\, dw_i\right)\left(\sum w_i\, d\bar{w}_i\right)}{(1+|w|^2)^2}.$$

On each $U_{j,\alpha}$, within the area $0 \le t \le t_0 = \pi/(2\varepsilon) + \log\varepsilon$, the image under Φ_j is dominated by the two sections $s_{j,\alpha,0}$ and $s_{j,\alpha,1}$, since the contribution of other sections is of relative size $< k^2\varepsilon$. We can estimate the map $[s_{j,\alpha,0}, s_{j,\alpha,1}, 0, \dots]$ by the local sections $[b_0, b_1 z, 0, \dots]$, with relative error $< 1/k^2$, where

$$b_0^{-2} = \frac{2^{k+2}\pi^{3/2}}{\sqrt{2k}}\varepsilon^{-2k-1} \quad \text{and} \quad b_1^{-2} = \frac{2^{k+2}\pi^{3/2}}{\sqrt{2k}}e^{\pi/\varepsilon}k^{2k+1}.$$

So the map is simplified to

$$[1, e^{-\pi/(2\varepsilon)}(\varepsilon k)^{-k-1/2}z, 0, \dots],$$

and we can estimate the length of the image of $\gamma_{j,\alpha}$ — which is approximately $2\pi e^{-\pi/(2\varepsilon)}(\varepsilon k)^{-k-1/2}$ — which goes to 0 as $j \to \infty$. So the image of $\gamma_{j,\alpha}$ converges to $[1, 0, \dots]$ in the current ordering of coordinates. Similarly, we can estimate the length of the image of the circle $t_0 = \pi/(2\varepsilon) + \log\varepsilon$ — which is approximately $(2\pi/\varepsilon)(\varepsilon k)^{k+1/2}$ — which goes to 0 as $j \to \infty$. Therefore, the image of the circle $t_0 = \pi/(2\varepsilon) + \log\varepsilon$ converges to $[0, 1, \dots]$ in the current ordering of coordinates. Notice that $e^{-\pi/(2\varepsilon)}(\varepsilon k)^{-k-1/2}|z|$ goes to ∞ at t_0, so the image of the area $0 \le t \le t_0 = \pi/(2\varepsilon) + \log\varepsilon$ converges to the complex projective line connecting the points $[1, 0, 0, \dots]$ and $[0, 1, 0, \dots]$ in the current ordering of coordinates. This area is the bubble mentioned in the introduction. By symmetry, the image of the area $0 \ge t \ge -t_0$ also converges to a complex projective line.

For the part $G_j^{-1}\left(C_0 \backslash \bigcup U_\alpha(\varepsilon(j, \alpha))\right)$, the sections $\{s_{j,\alpha,0}\}_{1\le\alpha\le d}$ are negligible. So the convergence of the remaining sections in V_j and W_j in the L^2 sense implies that the image of $G_j^{-1}\left(C_0 \backslash \bigcup U_\alpha(\varepsilon(j, \alpha))\right)$ under Φ_j converges to the image of Φ_0. To conclude the proof of the main theorem, we only have to notice that although $V_j \cup W_j$ is not orthonormal, we can modify them. The sections in W_j are almost orthonormal, so we can transform them to be orthonormal with a matrix A_j whose difference from the identity matrix goes to 0 as $j \to \infty$ in L^∞-norm. And the modified sections still converge to sections in W_0 in L^2. We first apply a Gram–Schmidt process to the set V_0, then we apply a Gram–Schmidt process following the order of V_0 to the set $V_j \backslash \{s_{j,\alpha,0}\}_{1\le\alpha\le d}$. Finally, since the sections $\{s_{j,\alpha,0}\}_{1\le\alpha\le d}$ are almost orthonormal and almost orthogonal to the other section in V_j, we can modify the new V_j again with a matrix B_j whose difference from the identity matrix goes to 0 as $j \to \infty$ in L^∞-norm to get an orthonormal set. And we have proved the main theorem.

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UNIQUENESS OF EXCITED STATES TO $-\Delta u + u - u^3 = 0$ IN THREE DIMENSIONS

ALEX COHEN, ZHENHAO LI AND WILHELM SCHLAG

We prove the uniqueness of several excited states to the ODE $\ddot{y}(t) + (2/t)\dot{y}(t) + f(y(t)) = 0$, $y(0) = b$, and $\dot{y}(0) = 0$, for the model nonlinearity $f(y) = y^3 - y$. The n-th excited state is a solution with exactly n zeros and which tends to 0 as $t \to \infty$. These represent all smooth radial nonzero solutions to the PDE $\Delta u + f(u) = 0$ in H^1. We interpret the ODE as a damped oscillator governed by a double-well potential, and the result is proved via rigorous numerical analysis of the energy and variation of the solutions. More specifically, the problem of uniqueness can be formulated entirely in terms of inequalities on the solutions and their variation, and these inequalities can be verified numerically.

1. Introduction

Consider the ODE

$$\ddot{y}(t) + \frac{2}{t}\dot{y}(t) + f(y(t)) = 0, \tag{1-1}$$

$$y(0) = b, \quad \dot{y}(0) = 0. \tag{1-2}$$

In this paper, we will only consider the model case $f(y) = y^3 - y$, but it will be convenient to use the more general notation for the nonlinearity. Smooth solutions to this ODE exist for all $t \geq 0$ and any $b \in \mathbb{R}$, and they are unique. We denote them by y_b, or simply y. The singular coefficient at $t = 0$ can be dealt with by a power series ansatz, or by Picard iteration. Solutions to this ODE correspond to radial smooth solutions of the PDE

$$\Delta u + f(u) = 0 \tag{1-3}$$

in three dimensions, under the identification $t = r$, the radial variable. Dynamics of (1-1) resemble particle motion in a double well as in Figure 1, with time varying friction. The qualitative behavior exhibits a trichotomy: we either have

- $(y_b(t), \dot{y}_b(t)) \to (1, 0)$,
- $(y_b(t), \dot{y}_b(t)) \to (-1, 0)$,
- $(y_b(t), \dot{y}_b(t)) \to (0, 0)$;

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Keywords: Klein–Gordon, interval arithmetic, soliton, excited state.

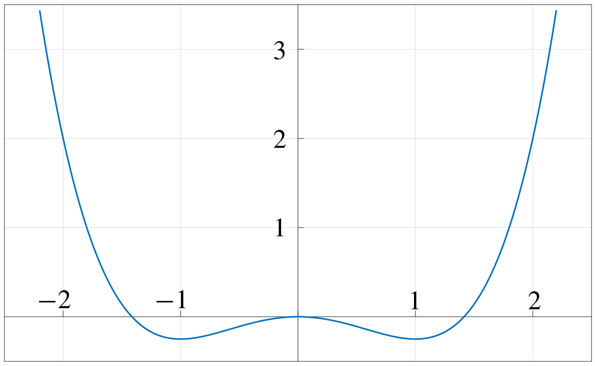

Figure 1. Potential function $V(y) = \frac{1}{4}y^4 - \frac{1}{2}y^2$.

see Section 3 below. A *bound state* is a nonzero solution with $y(t) \to 0$. Only these solutions give rise to nontrivial solutions $u \in H^1(\mathbb{R}^3)$ of the elliptic PDE (1-3). A *ground state* is a positive bound state, and an *n-th excited state* is a bound state with precisely n zero crossings. The 0-th bound state is the ground state. Existence and uniqueness of these bound states have been investigated since the 1960s. Using a variational characterization, [Ryder 1967, Theorem II] showed the existence of both ground and excited states with any finite number of zero crossings. Coffman [1972, Section 6] related Ryder's *characteristic values* to degree theory in infinite dimensions and Lyusternik–Schnirelmann techniques. Most importantly, [Coffman 1972] also established uniqueness of the ground state for the cubic case. For more general nonlinearities, ground state uniqueness was then shown by [Kwong 1989; McLeod and Serrin 1987; Peletier and Serrin 1983; 1986; Zhang 1991], and finally in greatest generality by McLeod [1993]. Clemons and Jones [1993] gave a different proof of McLeod's theorem based on the Emden–Folwer transformation and unstable manifold theory. Berestycki and Lions [1983a; 1983b] solved the existence problem of radial bound states for (1-3) for all H^1 subcritical nonlinearities $f(u)$ in all dimensions, see also the earlier work by Strauss [1977].

However, uniqueness of excited states in the radial class, i.e., for the ODE (1-1), remained open for most nonlinearities. In fact, [Hastings and McLeod 2012, Chapter 19] list this problem as one of three major open problems in nonlinear ODEs. We note that there has been some uniqueness results for specific nonlinearities; [Troy 2005] proved the uniqueness of the first excited state for a piecewise linear nonlinearity by analyzing the explicit solutions, and Cortázar, García-Huidobro, and Yarur [Cortázar et al. 2009] proved uniqueness of the first excited state with restrictions on $yf'(y)/f(y)$. However, neither cover the cubic nonlinearity. In this paper, we provide a rigorous computer-assisted proof of the uniqueness of the first excited state for the cubic nonlinearity. The proof technique combines analytical dynamics with the rigorous ODE solver `VNODE-LP`, see Section 2.3 and [Nedialkov 2011]. The latter works with interval arithmetic and therefore does not compute precise solutions (which is impossible), but rather intervals containing the solution at any given time. These inclusions accommodate all errors incurred through floating point arithmetic, and are therefore themselves free of errors.

Theorem 1. *The first twenty excited states of ODE* (1-1)–(1-2) *are unique for* $f(y) = -y + y^3$.

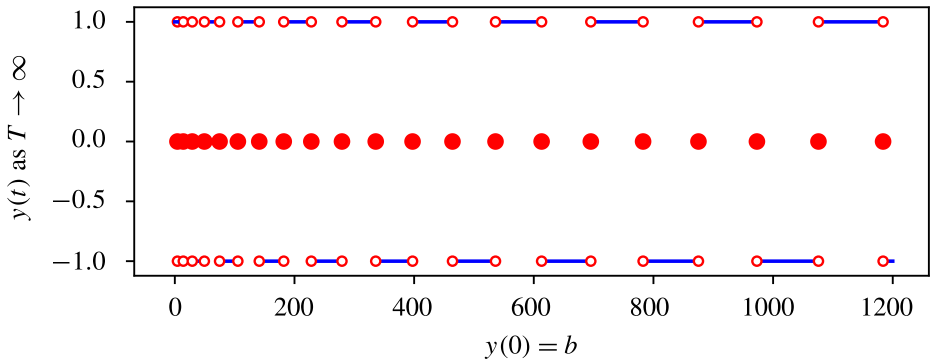

Figure 2. Limiting position $y_b(T)$ as $T \to \infty$, plotted as a function of the initial condition b up to $b = 1200$. The solid (red) dots represent bound states, and this graph holds due to Theorem 1 and the rigorous numerical work done in this paper.

The method is robust, and extends to both more general nonlinearities as well as other dimensions. But we will leave the verification of this claim for another paper. The code involved in the proof is publicly available, see the GitHub repository https://github.com/alexander-cohen/NLKG-Uniqueness-Prover. The readers can verify uniqueness of higher excited states beyond the 20-th using the arguments of this paper. Computation time is the main obstacle to going further than the twentieth state, to which the authors chose to limit themselves. See Figure 2 for a graph of the limiting position of $y_b(T)$ as a function of b, up to the twentieth excited state. The rigorous numerical work done in this paper proves that this graph holds.

The uniqueness property of the ground state soliton is of fundamental importance to the classification of its long-term evolution under the nonlinear cubic Schrödinger or Klein–Gordon flows. See for example [Cazenave 2003; Nakanishi and Schlag 2011]. The uniqueness property of the excited states should therefore also be seen as a bridge to dynamical results. As a first step, one needs to determine the spectrum of the linearized operator

$$H = -\Delta + 1 - 3\phi^2$$

in the radial subspace of $L^2(\mathbb{R}^3)$. Here ϕ is any radial bound state solution of the PDE (1-3). If ϕ is the ground state, then it is known that the spectrum over the radial functions contains a unique negative eigenvalue, and no other discrete spectrum up to and including zero energy (nonradially, due to the translation symmetry, 0 is an eigenvalue of multiplicity 3); see [Nakanishi and Schlag 2011]. A particularly delicate question pertains to the shape of the spectrum in the interval (0, 1], including the threshold 1 of the (absolutely) continuous spectrum. This was settled in [Costin et al. 2012] for the ground state of the cubic power in three dimensions. It turns out that (0, 1] is a spectral gap, including the threshold, which is not a resonance. Due to the absence of an explicit expression of the ground state soliton, the method of [Costin et al. 2012] depended on an approximation of this special solution. Note that without uniqueness such an approximation has no meaning. The authors intend to investigate the spectral problem of excited states in another publication using the methods of [Creek et al. 2017], which essentially require the uniqueness property of the special solution ϕ (in the case of [Creek et al. 2017], ϕ is the so-called Skyrmion).

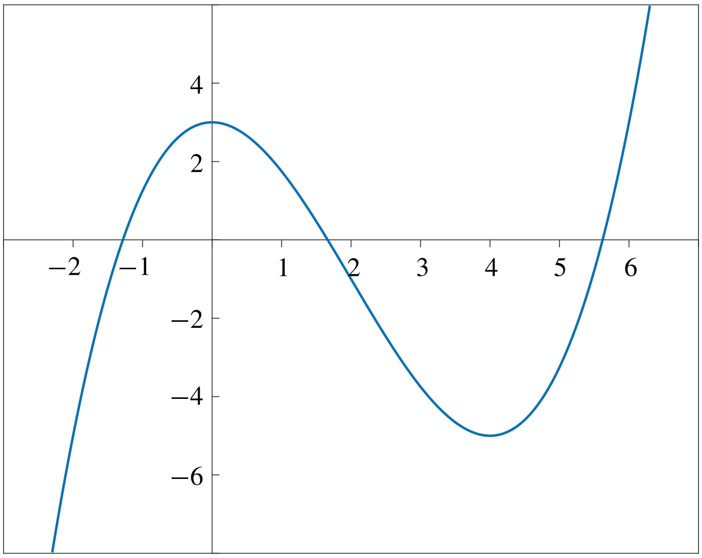

Figure 3. Toy example function.

2. Overview of approach

2.1. ***Toy example*: *finding zeros of a function.*** Suppose we wish to find the number of zeros of the function f as shown in Figure 3. Numerical computations make it clear that f has exactly 3 zeros — how can we use a computer to prove this rigorously?

A first approach might be to find the approximate location of those zeros with reasonably high precision, using floating point arithmetic. Say they lie at approximately $y_1,\ y_2,\ y_3$. Then we can use interval arithmetic to show rigorously that f is bounded away from zero everywhere but the three small intervals $\big(y_1 - \frac{1}{100},\ y_1 + \frac{1}{100}\big)$, $\big(y_2 - \frac{1}{100}, y_2 + \frac{1}{100}\big)$, $\big(y_3 - \frac{1}{100}, y_3 + \frac{1}{100}\big)$. Then, by interval arithmetic combined with the intermediate value theorem, f has at least one zero in each of these intervals. Finally, to show that each of them contains *at most* one zero, we can apply the mean value theorem. If we prove rigorously, using interval arithmetic, that f' is bounded away from zero in those intervals, then uniqueness follows. Notice that if f has infinitely many zeros with a limit point, then f' must be zero at that limit point. As expected, this method would break down in such a scenario.

2.2. ***Finding and isolating excited states.*** We apply a similar idea to the ODE (1-1). We now outline the approach by means of the ground state. Suppose we find numerically that the unique ground state should be at height $b_0 \approx 4.3373$. Using a rigorous ODE solver, we can prove that, for all $b \in (1, b_0 - 0.001)$, $y_b(t) > 0$ up to some time T and $E(T) < 0$. This will imply by analytical arguments, see the next section, that $y_b(t) \to 1$ and $y_b(t)$ is positive, so it is not a ground state. Similarly, we can show that, for $b \in (b_0 + 0.001, 50)$, the solution passes over $y = 0$ and thus is not a ground state. It follows from a connectedness argument that there is some ground state in the interval $(b_0 - 0.001, b_0 + 0.001)$. To prove that there is *exactly* one ground state in that interval, we find some large time T such that $\delta_b(T), \dot{\delta}_b(T) < 0$ for all $b \in (b_0 - 0.001, b_0 + 0.001)$, where $\delta_b = \partial_b y_b$. This means that if b_0^* is the actual ground state (rather than an approximation) for any $b > b_0^*$ in our interval, then $y_b(T) < y_{b_0^*}(T)$ and $\dot{y}_b(T) < \dot{y}_{b_0^*}(T)$ by the mean value theorem. One can then prove, assuming this condition, that $y_b(T)$ crosses over zero and lands in the second well, see Lemma 8. This will show that there is at most one ground state in the interval $(b_0 - 0.001, b_0 + 0.001)$. All that remains is showing that there is no ground state in the range $(50, \infty)$. To this end, we rescale the ODE (1-1) so that it takes the form $\ddot{w} + (2/t)\dot{w} + w^3 - b^{-2}w = 0,\ w(0) = 1$, $\dot{w}(0) = 0$. Again using `VNODE-LP`, we then show that the solution of this equation exhibits more than any given number of zeros provided b is sufficiently large. This then implies the same for y_b.

The same approach works just as well for excited states as it does for ground states.

2.3. ***Approximating solutions via interval arithmetic.*** We now outline our computational approach. Our main tool is the `VNODE-LP` package for rigorous ODE solving. The supporting website is at [Nedialkov 2010], and the documentation is available at [Nedialkov 2006].

`VNODE-LP` uses exact interval arithmetic, a toolset which allows for rigorous numerical computations. Rather than computing with floating point numbers as usual, interval arithmetic treats all values as *intervals* of real numbers, of the form $\boldsymbol{a} = [a_1, a_2]$, where a_1, a_2 are machine representable floating point numbers. All mathematical operations are rounded properly so that any input within the original interval ends up within the output interval. The `VNODE-LP` package combines interval arithmetic with ODE solving: given an initial value problem $\dot{y} = f(y, t)$ with initial values in an interval $\boldsymbol{b}$, a starting time interval $\boldsymbol{t}_1$, and an ending time interval $\boldsymbol{t}_2$, the package outputs an interval $\boldsymbol{y}$ such that, for any $b \in \boldsymbol{b}$, we have $t_1 \in \boldsymbol{t}_1,\ t_2 \in \boldsymbol{t}_2,\ y_{b,t_1}(t_2) \in \boldsymbol{y}$.

A difficulty in applying `VNODE-LP` to our problem is that ODE (1-1) is singular at $t = 0$. To deal with this, we approximate $y_b(t)$ near $t = 0$ by Picard iteration. We explicitly bound the error terms in this approximation so that we can rigorously obtain an interval containing $y_b(t_0),\ \dot{y}_b(t_0)$ for t_0 small. Then `VNODE-LP` can be applied to this desingularized initial value problem, and we will have rigorous bounds on our solutions and quantities defined in terms of the solutions.

Section 5 explains in detail how we use this software, and provides links to websites containing the code and all supporting data needed in the proof of our theorem. This will hopefully allow the reader to implement the methods of this paper in other related settings.

3. Analytical description of the damped oscillator dynamics

3.1. ***Basic properties of the ODE.*** It is an elementary property that smooth solutions of (1-1), (1-2) exist for all times $t \geq 0$; in fact, we will reestablish this fact below in passing. Taking it for granted, we note that the energy

$$E(t) := \tfrac{1}{2}\dot{y}^2(t) + V(y(t)) = \tfrac{1}{2}\dot{y}^2(t) + \tfrac{1}{4}y(t)^4 - \tfrac{1}{2}y(t)^2$$

satisfies

$$\dot{E}(t) = -\frac{2}{t}\dot{y}^2(t)$$

and thus $E(t) \leq E(0) = V(b)$ for all times. In fact, $E(t)$ is strictly decreasing unless it is a constant, and that can only happen for the unique stationary solutions $(y, \dot{y})$ equal to $(0, 0)$ or $(\pm 1, 0)$. In particular, if $V(b) \le 0$, then $E(t) < 0$ for all $t > 0$ unless $y(t) = 0$ is a constant. We will see below that this implies that $(y(t), \dot{y}(t)) \to (1, 0)$ as $t \to \infty$ (recall that we are assuming $b > 0$). In other words, $(y, \dot{y})(t)$ approaches the minimum of the potential well on the right of Figure 1, and so $\inf_{t>0} y_b(t) > 0$. The range of b here is $0 < b \le \sqrt{2}$.

On the other hand, if $b > \sqrt{2}$, then $V(y(t)) \le E(t) < E(0) = V(b)$ for all $t > 0$, whence

$$y(t)^2(y(t)^2 - 2) \le b^2(b^2 - 2)$$

and thus $|y(t)| \le b$ for all $t \ge 0$. We will assume from now on that $b > \sqrt{2}$. Rewriting the initial value problem (1-1), (1-2) in the form

$$\frac{d}{dt}(t^2\dot{y}(t)) + t^2 f(y(t)) = 0,$$

where $f(y) = y^3 - y$ throughout, we arrive at the integral equations

$$\begin{aligned} y(t) &= b + \int_0^t \dot{y}(s)\,ds, \\ \dot{y}(t) &= -\int_0^t \frac{s^2}{t^2} f(y(s))\,ds = \int_0^t \frac{s^2}{t^2} y(s)(1 - y(s)^2)\,ds. \end{aligned} \tag{3-1}$$

For short times, we obtain a unique solution by the contraction mapping principle which is smooth near $t = 0$. Picard iteration gives better constants, which is important for starting VNODE at some positive time. We shall determine the quantitative bounds in Section 3.3. But first we recall the equation of variation of (1-1) relative to the initial height b.

3.2. *The equation of variation.* We let

$$\delta_b(t) := \frac{\partial}{\partial b} y_b(t).$$

Then differentiating (1-1), $\delta_b(t)$ satisfies the ODE

$$\ddot{\delta} + \frac{2}{t}\dot{\delta} + f'(y)\delta = 0,$$

with initial conditions $\delta(0) = 1$ and $\dot{\delta}(0) = 0$. Notice that the ODE for δ depends on the solution $y_b(t)$. Altogether, we can make one ODE in four variables that includes y and δ:

$$\frac{d}{dt}\begin{pmatrix} y \\ v_y \\ \delta \\ v_\delta \end{pmatrix} = \begin{pmatrix} v_y \\ -\frac{2}{t}v_y - f(y) \\ v_\delta \\ -\frac{2}{t}v_\delta - f'(y)\delta \end{pmatrix}$$

with initial vector

$$\begin{pmatrix} y \\ v_y \\ \delta \\ v_\delta \end{pmatrix}(0) = \begin{pmatrix} b \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Switching again to the variables $t^2y(t)$ and $t^2\delta(t)$, respectively, and writing the resulting ODE in integral form, this is equivalent to

$$Z(t) := \begin{pmatrix} y \\ v_y \\ \delta \\ v_\delta \end{pmatrix}(t) = \begin{pmatrix} b \\ 0 \\ 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} v_y(s) \\ -t^{-2}s^2 f(y(s)) \\ v_\delta(s) \\ -t^{-2}s^2 f'(y(s))\delta(s) \end{pmatrix} ds$$
$$= \begin{pmatrix} b \\ 0 \\ 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} v_y(s) \\ t^{-2}s^2 y(s)(1-y(s)^2) \\ v_\delta(s) \\ t^{-2}s^2(1-3y(s)^2)\delta(s) \end{pmatrix} ds. \tag{3-2}$$

The first three Picard iterates of this system are

$$Z_0(t) = \begin{pmatrix} b \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad Z_1(t) = \begin{pmatrix} b \\ -\frac{1}{3}tf(b) \\ 1 \\ -\frac{1}{3}tf'(b) \end{pmatrix}, \quad Z_2(t) = \begin{pmatrix} b-\frac{1}{6}t^2 f(b) \\ -\frac{1}{3}tf(b) \\ 1-\frac{1}{6}t^2 f'(b) \\ -\frac{1}{3}tf'(b) \end{pmatrix}. \tag{3-3}$$

3.3. *Picard approximation.* The purpose of this section is to compare the actual solution $Z(t)$ in (3-2) to the second Picard iterate $Z_2(t)$ in (3-3), which we denote in the form

$$Z_2(t) =: \begin{pmatrix} \tilde{y}(t) \\ \dot{\tilde{y}}(t) \\ \tilde{\delta}(t) \\ \dot{\tilde{\delta}}(t) \end{pmatrix} = \begin{pmatrix} \tilde{y}(t) \\ \tilde{v}_y(t) \\ \tilde{\delta}(t) \\ \tilde{v}_\delta(t) \end{pmatrix}. \tag{3-4}$$

In fact, we will prove the following inequalities on each of the four entries of this vector.

Lemma 2. *Suppose $b \geq \sqrt{2}$. Then, for all times $t \geq 0$,*

$$\tilde{y}(t) \leq y(t) \leq \tilde{y}(t) + \frac{f(b)f'(b)}{120}t^4 \leq \tilde{y}(t) + \frac{b^5}{40}t^4,$$
$$\dot{\tilde{y}}(t) \leq \dot{y}(t) \leq \tilde{y}(t) + \frac{f(b)f'(b)}{30}t^3 \leq \dot{\tilde{y}}(t) + \frac{b^5}{10}t^3. \tag{3-5}$$

For all $0 \leq t \leq t_*$ with

$$t_* := \min\left(\sqrt{\frac{6(\sqrt{3}b-1)}{\sqrt{3}b(b^2-1)}}, \frac{\log 4}{\sqrt{3}b}\right), \tag{3-6}$$

one has

$$\tilde{\delta}(t) \leq \delta(t) \leq \tilde{\delta}(t) + \tfrac{1}{8}b^4t^4 \quad \textit{and} \quad \dot{\tilde{\delta}}(t) \leq \dot{\delta}(t) \leq \dot{\tilde{\delta}}(t) + \tfrac{1}{2}b^4t^3. \tag{3-7}$$

Proof. Since energy is decreasing and $b \geq \sqrt{2}$, we have $|y(t)| \leq b$ for all $t \geq 0$. Note that

$$f(b) \geq f(\sqrt{2}) > \frac{2}{3\sqrt{3}},$$

which is the absolute value of the local minima and maxima, so $|f(y)| \le f(b)$ for all $|y| \le b$. Therefore $|f(y(t))| \le f(b)$ for all $t \ge 0$. Substituting this bound into (3-1) yields

$$|\dot{y}(t)| \le \tfrac{1}{3} t f(b), \tag{3-8}$$

$$0 \le b - y(t) \le \tfrac{1}{6} t^2 f(b), \tag{3-9}$$

for all times $t \ge 0$. Leveraging these bounds, we now compare the actual solution to its second Picard iterates as in (3-3). In view of (3-4),

$$\tilde{y}(t) = b - \tfrac{1}{6} t^2 f(b) \quad \text{and} \quad \dot{\tilde{y}}(t) = -\tfrac{1}{3} t f(b),$$

and we obtain via the mean value theorem that

$$0 \le \dot{y}(t) - \dot{\tilde{y}}(t) = \int_0^t \frac{s^2}{t^2} [f(b) - f(y(s))]\, ds \le \frac{f(b) f'(b)}{30} t^3 \quad \text{and} \quad 0 \le y(t) - \tilde{y}(t) \le \frac{f(b) f'(b)}{120} t^4,$$

where we used that $|f'(y)| \le f'(b) = 3b^2 - 1$ for all $|y| \le b$.

The last two rows of (3-2) imply that

$$\begin{aligned} |\delta(t) - 1| &\le \int_0^t |\dot{\delta}(s)|\, ds, \\ |\dot{\delta}(t)| &\le t^{-2} f'(b) \int_0^t s^2 |\delta(s)|\, ds \le \frac{t}{3} f'(b) + t^{-2} f'(b) \int_0^t s^2 |\delta(s) - 1|\, ds \\ &\le t b^2 + 3b^2 \int_0^t |\delta(s) - 1|\, ds, \end{aligned}$$

whence $h(t) := |\dot{\delta}(t)| + \mu |\delta(t) - 1|$ with $\mu := \sqrt{3} b$ satisfies

$$h(t) \le t b^2 + \mu \int_0^t h(s)\, ds \quad \text{and} \quad h(t) \le \frac{b^2}{\mu} (e^{\mu t} - 1). \tag{3-10}$$

We infer from the last two rows of (3-2) and (3-4) that

$$\begin{aligned} \delta(t) - \tilde{\delta}(t) &= \int_0^t (v_\delta(s) - \tilde{v}_\delta(s))\, ds, \\ v_\delta(t) - \tilde{v}_\delta(t) &= t^{-2} \int_0^t s^2 (f'(b) - f'(y(s)) \delta(s))\, ds \\ &= t^{-2} \int_0^t s^2 [f'(b) - f'(y(s)) + f'(y(s))(1 - \delta(s))]\, ds \end{aligned} \tag{3-11}$$

as well as

$$1 - \delta(t) = -\int_0^t v_\delta(s)\, ds, \quad -v_\delta(t) = t^{-2} \int_0^t s^2 f'(y(s)) \delta(s)\, ds. \tag{3-12}$$

Let $t_* > 0$ be such that $f'(y(t)) \ge 0$ and $\delta(t) \ge 0$ for all $0 \le t \le t_*$. Then by (3-12), $\delta(t) \le 1$ for those times and thus by (3-11)

$$\delta(t) - \tilde{\delta}(t) \ge 0, \quad v_\delta(t) - \tilde{v}_\delta(t) \ge 0$$

for all $0 \le t \le t_*$. By (3-10), we have $\delta(t) \ge 0$ as long as

$$e^{\mu t} \le 4, \quad t \le \frac{\log 4}{\sqrt{3}b}.$$

Moreover, $f'(y(t)) \ge 0$ as long as $\sqrt{3}y(t) \ge 1$ which by (3-5) holds provided

$$\sqrt{3}\tilde{y}(t) \ge 1, \quad t \le \sqrt{\frac{6(\sqrt{3}b-1)}{\sqrt{3}b(b^2-1)}},$$

whence in summary we get (3-6) and the lower bounds in (3-7). For the upper bound, note that (3-12) and (3-5), respectively, imply that

$$-v_\delta(t) \le tb^2, \quad 1-\delta(t) \le \tfrac{1}{2}t^2b^2, \quad b-y(t) \le \tfrac{1}{6}t^2b^3.$$

Inserting these bounds into (3-11) yields, by the mean value theorem,

$$v_\delta(t)-\tilde{v}_\delta(t) \le t^{-2}\int_0^t s^2\left(\frac{b^4s^2+3b^4s^2}{2}\right)ds \le \frac{1}{2}b^4t^3,$$
$$\delta(t)-\tilde{\delta}(t) = \int_0^t (v_\delta(s)-\tilde{v}_\delta(s))\,ds \le \frac{1}{8}b^4t^4,$$

as claimed. □

3.4. ***The equation at infinity.*** As explained in Section 2.2, to prove uniqueness of the first excited state we will need to show that all y_b have at least two crossings for all sufficiently large b. For the second excited state, we need to do the same with three crossings, and so on. This will be accomplished by means of the following lemma.

Lemma 3. *Let $y(t)$ be a solution to ODE* (1-1)–(1-2). *Let $w(s)=b^{-1}y(s/b)$. Then w satisfies*

$$\ddot{w}+\frac{2}{s}\dot{w}+w^3-\beta^2 w=0, \tag{3-13}$$
$$w(0)=1, \quad \dot{w}(0)=0, \tag{3-14}$$

where $\beta := b^{-1}$.

Proof. The proof is immediate by scaling. □

We will analyse this initial value problem with VNODE-LP, but as before we can only start at positive times rather than at $t=0$. The analogue of Lemma 2 is the following. We only need to approximate the ODE in (3-13). Indeed, since the initial condition is fixed, the equation of variations does not arise.

Lemma 4. *Suppose $0<\beta\le\frac{1}{10}$, and let $\tilde{w}(t) := 1-\frac{1}{6}(1-\beta^2)t^2$ and $\dot{\tilde{w}}(t) := -\frac{1}{3}(1-\beta^2)t$. Then, for all times $t \ge 0$,*

$$\begin{aligned}\tilde{w}(t) &\le w(t) \le \tilde{w}(t)+\tfrac{1}{40}t^4,\\ \dot{\tilde{w}}(t) &\le \dot{w}(t) \le \dot{\tilde{w}}(t)+\tfrac{1}{10}t^3.\end{aligned} \tag{3-15}$$

Proof. We write (3-13) in the form

$$\frac{d}{dt}(t^2\dot{w}(t)) = -t^2 f_\beta(w(t)),$$

with

$$f_\beta(w) := w^3 - \beta^2 w = w(w^2-\beta^2).$$

Solutions are global, and the energy takes the form

$$E_\beta(t) = \tfrac{1}{2}\dot{w}^2(t) + V_\beta(w(t)), \quad V_\beta(w) = \tfrac{1}{4}w^4 - \tfrac{1}{2}\beta^2 w^2,$$

which is nonincreasing as before. Thus, $V_\beta(w(t)) \le E_\beta(t) \le V_\beta(1) = \frac{1}{4} - \frac{1}{2}\beta^2$, whence $|w(t)| \le 1$ for all times. The integral formulation of the initial value problem for w is of the form

$$\begin{aligned} w(t) &= 1 + \int_0^t \dot{w}(s)\,ds, \\ \dot{w}(t) &= -\int_0^t \frac{s^2}{t^2} f_\beta(w(s))\,ds = \int_0^t \frac{s^2}{t^2} w(s)(\beta^2 - w(s)^2)\,ds. \end{aligned} \tag{3-16}$$

Inserting $w = 1$ into the right-hand side of the second equation of (3-16) gives $\dot{\tilde{w}}(t) := -\frac{1}{3} f_\beta(1)t$, and $\tilde{w}$ is obtained by inserting this expression into the right-hand side of the first equation of (3-16). These are precisely the approximate solutions appearing in the formulation of the lemma. The stated bounds are now obtained as in Lemma 2, and we leave the details to the reader. □

3.5. ***Limit sets and convergence theorems.*** As we have already noted, an important quantity associated with (1-1) is the energy $E(y, \dot{y}) = \frac{1}{2}\dot{y}^2 + V(y)$, where $V(y) = \int_0^y f(y)$ is the potential energy. Explicitly, $V(y) = \frac{1}{4}y^4 - \frac{1}{2}y^2$ resembles a double well as in Figure 1. Were we to modify our ODE to $\ddot{y} + f(y) = 0$, then the energy would be preserved. The term $(2/t)\dot{y}$ adds a time dependent frictional force, so energy decreases monotonically:

$$\dot{E}(t) = \dot{y}\ddot{y}(t) + f(y(t))\dot{y}(t) = -\frac{2\dot{y}^2(t)}{t}.$$

The interpretation of the radial form of the PDE (1-3) as a damped oscillator with the role of time being played by the radial variable is of essential importance in this section. Tao [2006] emphasized this already in his exposition of ground state uniqueness, but here we will rely on this interpretation even more heavily. In particular, the proof of the long-term trichotomy given by the solution vector of the main ODE approaching one of the three critical points of the potential follows the dynamical argument in the damped oscillator paper [Cabot et al. 2009].

The following lemma determines the ω-limit set of every trajectory in phase space. The lemma combined with the monotonicity of the energy will help us determine the desired long-term trichotomy.

Lemma 5. *If $y(t) = y_b(t)$ is the global solution to the initial value problem* (1-1), (1-2), *then there exists an increasing unbounded sequence $\{t_j\}$ such that $(y(t_j), \dot{y}(t_j)) \to (0,0)$ or $(\pm 1, 0)$ as $j \to \infty$.*

Proof. From boundedness of the energy, we see that

$$\sum_{n=1}^{\infty} \frac{1}{n} I_n \le \int_0^\infty \frac{\dot{y}(t)^2}{t}\,dt < \infty, \quad I_n := \int_{n-1}^{n} \dot{y}(t)^2\,dt.$$

Therefore, $I_{n_j} \to 0$ as $j \to \infty$ for some subsequence. We can pick $t_j \in (n_j - 1, n_j)$, so that $\dot{y}(t_j) \to 0$. Since $E(t)$ and $y(t)$ are bounded, $\ddot{y}$ is bounded, and differentiating (1-1), we then see that $\dddot{y}$ is also bounded. Therefore $I_{n_j} \to 0$ implies that $\ddot{y}(t_j) \to 0$. This implies that

$$|f(y(t_j))| \le |\ddot{y}(t_j)| + \frac{2}{t}|\dot{y}(t_j)| \to 0,$$

so there must be a subsequence of $y(t_{n_j})$ that converges either to 0 or 1. □

Next, we establish that each trajectory must converge to the point in its limit set, cf. the convergence theorems in [Cabot et al. 2009].

Lemma 6. *Either $y_b(t) \to -1$, or $y_b(t) \to 0$, or $y_b(t) \to 1$ as $t \to \infty$. In all cases $\dot{y}(t) \to 0$.*

Proof. Since $E(t)$ is monotonically decreasing, the limit $\lim_{t\to\infty} E(t)$ exists as a real number, which is either negative or nonnegative. In the former case, there must be a sequence t_j such that $(y(t_j), \dot{y}(t_j)) \to (\pm 1, 0)$ by Lemma 5. Monotonicity of the energy then implies that $E(t) = E(y_b(t), \dot{y}_b(t))$ tends toward the global minimum value of the potential energy, which means that $(y_b(t), \dot{y}_b(t)) \to (\pm 1, 0)$.

If the limit is nonnegative, then Lemma 5 implies that $E(t) \to 0$ as $t \to \infty$. Suppose $y(t)$ does not converge to 0. Let τ_j denote the j-th time at which $\dot{y}(\tau_j) = 0$, and if $y(t)$ does not tend to 0, then $\{\tau_j\}$ is an infinite sequence. We will show that this leads to a contradiction since too much energy will be lost in each oscillation. To do so, we first bound $\tau_{j+1} - \tau_j$ from above. Assume without loss of generality that $\dot{y} > 0$ between τ_j and τ_{j+1}. We have $V(y) \le -\frac{1}{4}y^2$ for $y \in (-1, 1)$. Let $\tau_j < t_1 < t_2 < \tau_{j+1}$, so that $y(t_1) = -1$ and $y(t_2) = 1$. In particular, the portion of the trajectory between t_1 and t_2 is the part of the trajectory going over the hill in the potential, which should be the most time-consuming part of the trajectory, and indeed,

$$\begin{aligned}\int_{t_1}^{t_2} 1\,dt = \int_{-1}^{1} \frac{1}{y'}\,dy = \int_{-1}^{1} \frac{1}{\sqrt{2(E(y) - V(y))}}\,dy \le 2\int_0^1 \frac{1}{\sqrt{2E(t_2) + y^2/2}}\,dy \\ \le 2\int_0^{\sqrt{2E(t_2)}} \frac{1}{\sqrt{2E(t_2)}}\,dy + 2\sqrt{2}\int_{\sqrt{2E(t_2)}}^{1} \frac{1}{y}\,dy \lesssim -\log E(t_2),\end{aligned} \tag{3-17}$$

assuming that τ_j is large enough that $0 < E(t_2) \ll 1$. Note that from the energy, $\dot{y}(t)$ can only reverse sign if $|y(t)| > \sqrt{2}$. Since the energy is always positive,

$$E(t_2) \ge \int_{t_2}^{\tau_{j+1}} \frac{2\dot{y}(t)^2}{t}\,dt \ge \frac{1}{\tau_{j+1}} \int_1^{\sqrt{2}} 2\sqrt{2(E(y) - V(y))}\,dy \gtrsim \frac{1}{\tau_{j+1}}.$$

Substituting this into (3-17), we find that

$$t_2 - t_1 \lesssim \log \tau_{j+1}. \tag{3-18}$$

Finally, we show that for small energy, the time spent by $y(t)$ in one oscillation outside the interval $(-1, 1)$ is uniformly bounded by some constant. Fix some $0 < \varepsilon \ll \sqrt{2} - 1$ such that $|f(y)| \ge \alpha > 0$ for all $y \in B_\varepsilon(\pm\sqrt{2})$, the ε-neighborhoods of $\pm\sqrt{2}$. Then there exists a sufficiently large T such that $\sqrt{2} < |y(\tau_j)| < \sqrt{2} + \varepsilon$ for all $\tau_j > T$ and such that $|2\dot{y}(t)|/t < \frac{1}{2}\alpha$ for all $t > T$. This means that if $\tau_j > T$ and $\dot{y}(t) > 0$ for $t \in (\tau_j, \tau_{j+1})$, then

$$\begin{aligned} \ddot{y}(t) &= -f(y(t)) - \frac{2}{t}\dot{y}(t) \ge \frac{\alpha}{2} && \text{when } y(t) \in B_\varepsilon(-\sqrt{2}), \\ \ddot{y}(t) &= -f(y(t)) - \frac{2}{t}\dot{y}(t) \le -\alpha && \text{when } y(t) \in B_\varepsilon(+\sqrt{2}). \end{aligned}$$

In other words, between τ_j and τ_{j+1}, the initial acceleration and final deceleration are both uniformly bounded from below. Then there is a uniform constant bounding the time spent by the part of the trajectory in $B_\varepsilon(\pm\sqrt{2})$. Outside both $B_\varepsilon(\pm\sqrt{2})$ and the interval $(-1, 1)$, the velocity is uniformly bounded from below, so there is a uniform constant bounding the time in that region as well.

Therefore, (3-18) can in fact be improved to $\tau_{j+1} - \tau_j \lesssim \log \tau_{j+1}$, so $\tau_j \lesssim j \log \tau_j$. One first reads off $\tau_j \lesssim j^2$, and then applies this inequality once more to conclude

$$\tau_j \lesssim j \log j.$$

The cumulative loss in energy starting from some sufficiently large time τ_N is therefore

$$\int_{\tau_N}^{\infty} \frac{\dot{y}^2(t)}{t}\,dt \ge \sum_{j=N}^{\infty} \frac{1}{\tau_{j+1}} \int_{\tau_j}^{\tau_{j+1}} \dot{y}^2(t)\,dt \gtrsim \sum_{j=N}^{\infty} \frac{1}{j \log j},$$

which is not finite, a contradiction. □

3.6. *Passing over the saddle.* We now turn to a lemma which establishes the following natural property: consider the value $0 < y(T) = \varepsilon \ll 1$ of a bound state solution with T large enough that $y(t) > 0$ for all $t > T$. Then any other y_b with $y_b(T) \in (0, \varepsilon)$ and $\dot{y}_b(T) < \dot{y}(T)$ needs to cross 0 after time T. For simplicity, we prove the lemma for $f(y) = y^3 - y$, but it is easy to see that it works for many nonlinearities via the same argument.

Lemma 7. *Suppose $b^* \in (0, \infty)$ is a bound state, and assume $y_{b^*}(t)$ approaches* 0 *from the right without loss of generality. That is, $\dot{y}_{b^*}(t) < 0$ for all $t \ge T$, for some T. Then y_{b^*} has no more zero crossings after time T, and, increasing T if necessary, we may assume $0 < y_{b^*}(T) \le 1/\sqrt{3}$. If $y_b(t)$ is another solution with $0 < y_b(T) < y_{b^*}(T)$ and $\dot{y}_b(T) < \dot{y}_{b^*}(T)$, then $y_b(t)$ has a zero crossing after time T.*

Proof. Let $s(t) = y_{b^*}(t) - y_b(t)$. Then

$$\ddot{s}(t) + \frac{2}{t}\dot{s}(t) = f(y_b(t)) - f(y_{b^*}(t)). \tag{3-19}$$

At $t = T$, we have $s(T) > 0$ and $\dot{s}(T) > 0$. If $y_b(t)$ does not cross zero for any $t > T$, then $y_b(t) \to 0$ or 1. This means that $s(t) \to 0$ or -1. In either case, $s(t)$ must reach a maximum after $t = T$, so there exists a $t_* > T$ such that $\dot{s}(t_*) = 0$, $s(t_*) > 0$, and $\ddot{s}(t_*) \le 0$. Then by (3-19),

$$f(y_b(t_*)) - f(y_{b^*}(t_*)) \le 0. \tag{3-20}$$

It is clear that $f(y)$ is strictly decreasing for $y\in(0,1/\sqrt{3})$, so when $s(t_*)>0$, (3-20) leads to a contradiction with the assumption $y_{b^*}(T)\le 1/\sqrt{3}$. □

Note that the only property of f we used is that $f'(0)<0$, which holds for all nonlinearities associated with a double-well potential.

The next lemma provides a sufficient condition under which the trajectory will pass over the hill and be trapped in the following well. The underlying mechanism is the consumption of energy due to a necessary oscillation around the left well. If this amount exceeds the energy present at the pass over the saddle at $y=0$, then the remaining energy is negative, ensuring trapping. The lemma will ensure that if $y_{b^*}(t)$ is a bound state, then, for initial values $b\in(b^*,b^*+\varepsilon)$ for some small ϵ, we have that $y_b(t)$ will necessarily fall into the following potential well.

Lemma 8. *Suppose $y(t)$ is a solution of* (1-1), (1-2) *such that*, *for some $T>0$*,

$$0\le y(T)<\tfrac{1}{2},\quad \dot y(T)<0,\quad 0<E(T)<\tfrac{1}{4},\quad E(T)\big(T-2\ln E(T)+\tfrac{3}{2}\big)<\tfrac{3}{8}.$$

Then if $y(t)$ has a zero after (*or at*) *time T, it must proceed to fall into the left well. That is*, $y(t)\to -1$, *and $y(t)$ has no further zero crossings.*

Proof. Suppose $y(t)$ has another zero crossing, say the minimal time $t\ge T$ with this property is $t_0\ge T$. Then $\dot y(t_0)<0$ and there can be no reversal in the sign of $\dot y(t)$ until after $y(t)$ has passed -1. So we can define $T_1>T$ to be the first time after T at which $y(T_1)=-\frac{1}{2}$.

Suppose $y(t)$ does not fall into the left well; in this situation, $E(T_1)=\alpha>0$. Then we must have $E(T)>\alpha$. Let $t(s)$, $-\frac{1}{2}<s<\frac{1}{2}$, be the nearest time after/before T such that $y(t(s))=s$. Then we have $\big($recall $0<\alpha<\frac{1}{4}\big)$

$$\begin{aligned}T_1-T&<\int_{-1/2}^{1/2}\frac{1}{|\dot y(t(s))|}\,ds=\int_{-1/2}^{1/2}\frac{1}{\sqrt{2(E(t(s))-V(s))}}\,ds<\int_{-1/2}^{1/2}\frac{1}{\sqrt{2\alpha+s^2-s^4/2}}\,ds\\&<\int_{-1/2}^{1/2}\frac{1}{\sqrt{2\alpha+s^2/2}}\,ds<2\int_0^{\sqrt{\alpha}}\frac{1}{\sqrt{2\alpha}}\,ds+2\sqrt{2}\int_{\sqrt{\alpha}}^{1/2}\frac{1}{s}\,ds\\&=\sqrt{2}-4\ln(2)-2\ln(\alpha)<-2\ln(\alpha).\end{aligned}$$

Thus $T_1<T-2\ln(\alpha)$. Next, we observe that if the conditions of the lemma are satisfied, then more than α energy is lost in going from $y=-\frac{1}{2}$ to $y=-1$. Letting $t(s)$ be as before and assuming $y(t)$ does not fall into the left well, $E(t(s))>0$ for $-1<s<-\frac{1}{2}$. Using

$$\frac{dE}{ds}=\frac{1}{\dot y}\frac{dE}{dt}=\frac{2|\dot y|}{t},$$

we have

$$\begin{aligned}\Delta E&=2\int_{-1}^{-1/2}\frac{|\dot y(t(s))|}{t(s)}\,ds=2\int_{-1}^{-1/2}\frac{\sqrt{2(E(t(s))-V(s))}}{t(s)}\,ds\\&>2\int_{-1}^{-1/2}\frac{\sqrt{s^2-s^4/2}}{t(s)}\,ds>\frac{1}{T_2}\int_{-1}^{-1/2}|s|\,ds=\frac{3}{8T_2},\end{aligned}$$

where $T_2 > T$ is the first time at which $y = -1$. Now we have

$$T_2 - T_1 = \int_{-1}^{-1/2} \frac{1}{|\dot{y}(t(s))|}\,ds = \int_{-1}^{-1/2} \frac{1}{\sqrt{2(E(t(s)) - V(s))}}\,ds$$
$$< \int_{-1}^{-1/2} \frac{1}{\sqrt{y^2 - y^4/2}}\,dy < 2\int_{-1}^{-1/2} \frac{1}{|y|}\,dy < \frac{3}{2}.$$

Altogether, we have

$$\Delta E > \frac{3}{8}\frac{1}{T - 2\ln(\alpha) + \frac{3}{2}},$$

and if $\Delta E > \alpha$, then $E(T_2) < 0$, and the particle falls into the left well. This occurs when

$$\alpha\big(T - 2\ln(\alpha) + \tfrac{3}{2}\big) < \tfrac{3}{8}.$$

Because $\alpha < E(T)$ and $\alpha \ln(\alpha)$ is monotone decreasing for $0 < \alpha < \frac{1}{4}$, in the situation of the lemma, if

$$T E(T) - 2E(T) \ln E(T) + \tfrac{3}{2} E(T) < \tfrac{3}{8},$$

then the particle must fall into the left well if it crosses zero after time T. □

4. Proof of Theorem 1

4.1. ***Outline of proof.*** Theorem 1 is proved by running a `C++` computer program which combines the rigorous numerics of `VNODE-LP` with the analytical lemmas of the preceding section. This code is divided into two parts: a *planning* section, and a *proving* section. The planning section of the code creates a plan for proving the first several bound states are unique, and the proving section executes this plan and outputs a rigorous proof of uniqueness. Separating these two sections is advantageous because only the proving section must be mathematically rigorous, so only that part of the code needs to be checked for correctness. The planning section can be modified without fear of compromising the rigor of the code.

In what follows we treat `VNODE-LP` as a black box that takes in an input interval $\boldsymbol{b} = (b_1, b_2)$ and a time interval $\boldsymbol{t} = (t_1, t_2)$, and outputs an interval $\boldsymbol{y_b}(\boldsymbol{t}) = (y_1, y_2) \times (\dot{y}_1, \dot{y}_2) \times (\delta_1, \delta_2) \times (\dot{\delta}_1, \dot{\delta}_2)$ which contains $y_b(t)$ for any $b \in (b_1, b_2)$ and $t \in (t_1, t_2)$. We can also integrate the equation at infinity (3-13) rigorously. To implement this functionality, we use the explicit error bounds given in Lemma 2 to move past the singularity at $t = 0$. For instance, we may pick $\boldsymbol{t}_0 = (0.1, 0.101)$ and then use those error bounds to find $\boldsymbol{y}_0$ a vector of four intervals which contain any $y_b(t)$, $b \in (b_1, b_2)$, and $t \in (0.1, 0.101)$. At this point we may input the starting intervals $\boldsymbol{y}_0$, $\boldsymbol{t}_0$ directly into `VNODE-LP`, which rigorously integrates to the desired ending time. See Figure 4 for a depiction of `VNODE-LP` integration with solution intervals.

We now describe our procedure for the ground state and first excited state, before describing the planning and proving sections in detail. Bound states can only occur in the range $b \in (\sqrt{2}, \infty)$. To prove the ground state is unique, we split this range into four intersecting intervals: I_1, I_2, I_3, I_4. For instance, we can take

$$I_1 = (1.4, 4.26), \quad I_2 = (4.25, 4.43), \quad I_3 = (4.42, 6.32), \quad I_4 = (6.31, \infty).$$

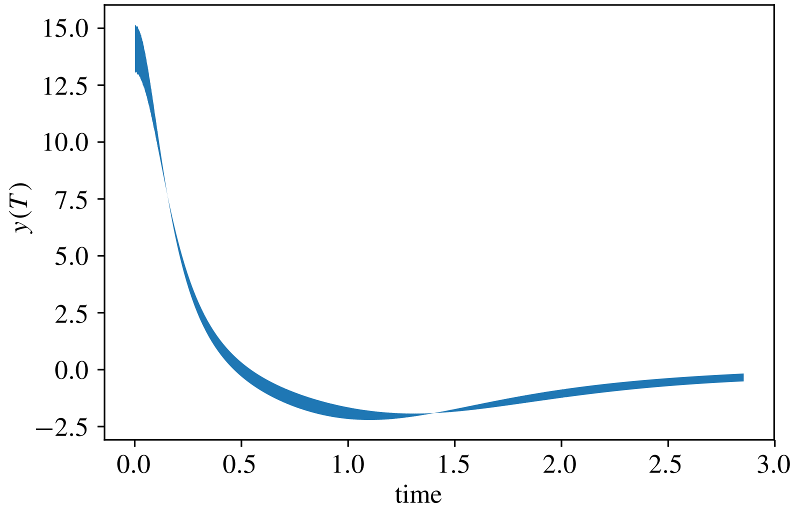

Figure 4. VNODE-LP numerical integration with solution intervals scaled up $\times 100$. The "pinch points" of near zero y-uncertainty occur when $\delta = y_b(t) \sim 0$, and although it is not shown here, the $\dot{y}$-uncertainty is larger at these points.

Now, numerical evidence shows that the ground state occurs in the range I_2. So, in the range I_1, the solution will eventually fall into the right energy well. We use VNODE-LP to prove this by splitting I_1 into smaller chunks, and verifying that in each of these chunks the energy of the solution eventually falls below zero. We deal with the range I_3 in the same way, by showing, for all $b \in I_3$, that $y_b(t)$ eventually has negative energy. In the interval I_4, the solution should always have at least one zero crossing. We prove this using the equation at infinity (3-16) as discussed in Section 3.4. The infinite range $b \in I_4$ corresponds to the finite range $\beta \in (0, 0.16)$, so by splitting this range into small chunks and verifying that in these chunks $w(t)$ is eventually negative, we prove that $y_b(t)$ eventually crosses zero for all $b \in I_4$. Notice that we could have replaced the interval I_4 by $I_3 \cup I_4$, and it would still be true that in this range there is at least one zero crossing. We treat I_3 separately because the numerics of ODE (3-16) are delicate near bound states, so I_3 acts as a *buffer interval*.

The only range left is I_2, which actually contains the ground state. We must show that I_2 contains at most one ground state, and that it contains no first or higher excited states. To this end, we use Lemmas 7 and 8, respectively. At some time $T > 0$, say $T = 6$, any $y_b(T)$ for $b \in I_2$ will be positive, moving in the negative direction, and small in magnitude. We use VNODE-LP to prove that $\delta_b(T), \dot{\delta}_b(T) < 0$ for $b \in I_2$. Let $b_0 \in I_2$ be a ground state. Then for any $b > b_0$, the mean value theorem implies that $y_b(T) < y_{b_0}(T)$ and $\dot{y}_b(T) < \dot{y}_{b_0}(T)$. By Lemma 7, $y_b(t)$ must cross zero. It follows that there is at most one ground state in I_2. Next, we check that the conditions of Lemma 8 are satisfied for all $y_b(T)$, $b \in I_2$. This implies that if any solution $y_b(t)$ does cross zero, it must fall into the left energy well, and cannot be a higher excited state. Altogether this shows that there is at most one ground state for the ODE (3-1), as desired. It also follows from this analysis that the ground state exists, so we have successfully shown the ground state exists and is unique.

We note a subtlety in this argument. A pathological issue would be that, by time $T = 6$, $y_b(T)$ crossed all the way to the left well, came back to the right well, and then started to approach $y = 0$ from the right. This would kill our later attempt to prove that the second excited state is unique, because we would miss a second excited state in I_2. To deal with this, we use energy considerations to bound $|\dot{y}_b(t)|$, and we make small enough time steps with VNODE-LP so that the solution cannot cross zero twice in between time steps. Then, we can be sure that the number of zero crossings observed by VNODE-LP up to some time T is the actual number of zero crossings for all solutions in our initial interval $\mathbf{b}$, up to time T.

To prove the ground state is unique, we split the range $(\sqrt{2}, \infty)$ into subintervals to which we applied three different proof methods. The method for I_1 and I_3 was FALL: we proved that the solution eventually has negative energy and thus cannot be a bound state. The method for I_4 was INFTY_CROSSES_MANY: we used the equation at infinity to show that there are sufficiently many zero crossings and thus no ground states. The method for I_2 was BOUND_STATE_GOOD: we used the analytical Lemmas 7 and 8 to show that there was at most one ground state and no other bound states. These are the same methods we use to deal with higher excited states. We have used the same notation here as is used in the code, for ease of verifying that the code follows the mathematical argument.

We extend our procedure to prove the first excited state is unique. We split up $(\sqrt{2}, \infty)$ into six pieces:

$$I_1 = (1.40, 4.26), \qquad I_2 = (4.25, 4.43), \qquad I_3 = (4.42, 14.10),$$
$$I_4 = (14.09, 14.12), \quad I_5 = (14.11, 16.11), \quad I_6 = (16.10, \infty).$$

We apply the FALL method to I_1, I_3, I_5, we apply the BOUND_STATE_GOOD method to I_2, I_4, and we apply the INFTY_CROSSES_MANY method to I_6. For the interval I_4, our careful stepping procedure as described above lets us find a time $T > 0$, for example $T = 8$, such that $y_b(T)$ crosses zero exactly once by time T for all $b \in I_4$. We can also verify that $\dot{y}_b(T) > 0$, $\delta_b(T) > 0$, and $\dot{\delta}_b(T) > 0$, so that the conditions of Lemma 7 are satisfied and there is at most one first excited state in the interval I_4. Next we verify that the conditions of Lemma 8 are satisfied uniformly for $b \in I_4$ at time T, so that there are no second or higher excited states in I_4. Altogether this shows that the first excited state exists and is unique, and sets us up to prove subsequent excited states are unique as well.

4.2. ***Planning section.*** We now describe the planning section of the code. Given a value $N \geq 0$, this section outputs a list of intervals $I_1, I_2, \ldots, I_k$, along with which method is to be used in each interval. The proving section will use this plan to verify that all bound states up to the N-th (that is, all bound states with $\leq N$ zero crossings, or in other words the first N excited states) are unique.

Let $b_0, b_1, \ldots, b_N$ denote the locations of the first N excited states (assuming for now that they are unique). We find their locations numerically with a binary search. To find b_k, we keep track of a lower bound $l < b_k$ and an upper bound $u > b_k$, and at each iteration, check how many times $y_m(t)$ crosses zero, $m = \frac{1}{2}(l + u)$. If $y_m(t)$ crosses zero more than k times, we set $u := m$, and if not we set $l := m$. We iterate until we have a small enough interval (l, m) containing b_k.

Next, we find small enough intervals around each bound state so that the BOUND_STATE_GOOD method can run successfully for each bound state. We start with a large interval around b_k, width 0.5, and then keep on dividing the width by two until BOUND_STATE_GOOD succeeds.

Third, we fill in the space between the bound states with FALL intervals. We include a buffer interval above the last bound state so as to make INFTY_CROSSES_MANY run faster.

Finally, we create an interval $\boldsymbol{\beta} = (0, \beta)$ corresponding to the infinite interval $(1/\beta, \infty)$, where we will show the ODE crosses zero at least $N + 1$ times.

4.3. ***Proving section.*** The proving section receives a list of intervals and methods from the planning section, and outputs a rigorous proof that the first N excited states are unique. The first step is to verify that subsequent intervals intersect each other, so that every real number in the range $(\sqrt{2}, \infty)$ is covered by some interval. Next, the different methods are implemented as follows.

The FALL method receives an interval, e.g., $(1.4, 4.2)$, and must prove that, for all b in that interval, $y_b(t)$ eventually has negative energy. It begins by attempting to integrate with that potentially very large input interval for b. Of course, VNODE-LP will likely fail to integrate with such a large input interval. If this happens, we bisect the interval into two halves, an upper and lower half, and recursively apply the FALL method to each half. Once the starting intervals are small enough, VNODE-LP will successfully integrate and prove that the energy is eventually negative. This bisection method allows us to use larger intervals away from the bound states and smaller intervals closer to the bound states, where the computations are more delicate.

The BOUND_STATE_GOOD method receives an interval I which supposedly contains an n-th bound state. It must prove that there is at most one n-th bound state, no lower bound states, and no higher bound states in I. We use a careful stepping procedure to find some time $T > 0$ such that, for all $b \in I$, $y_b(t)$ crosses zero exactly n times by time T. This already shows that I doesn't contain any lower bound states. Increasing T if necessary, we also verify that $\dot{y}_b(T)$, $\delta_b(T)$, and $\dot{\delta}_b(T)$ all have the opposite sign as $y_b(T)$ uniformly in I. As discussed earlier, Lemma 7 then implies that there is at most one n-th bound state in I. Finally, we verify that the conditions in Lemma 8 apply at time T, so there are no higher bound states in I. Throughout we use interval arithmetic, never floating point arithmetic.

The INFTY_CROSSES_MANY method works similarly to the FALL method. We bisect the interval $(0, \beta)$ into smaller pieces, and in each of these small pieces we prove that $w(t)$ has at least $N + 1$ crossings.

Altogether, these methods show that, if it exists, the n-th bound state (counting from $n = 0$) must be unique and lie in the n-th BOUND_STATE_GOOD interval. This proves that all bound states up to the N-th are unique, as desired. In fact, the code may also be used to show these bound states exist by counting crossing numbers, but this is already known by synthetic methods [Hastings and McLeod 2012].

5. Using VNODE-LP and the data

The code and full output logs from the proof procedure can be found at https://github.com/alexander-cohen/NLKG-Uniqueness-Prover, with 9cf63c06ca1838e64dd35fe11ca4fdfd45591714 the most recent commit at the time of writing. The code is contained in the single C++ file "nlkg_uniqueness_prover.cc", and output logs are titled "uniqueness_output_N=*.txt". The code proved the first 20 excited states are unique in ~ 4h running on a MacBook Pro 2017, 2.5 GHz. Time is the main limiting factor to proving uniqueness of more excited states. We summarize the output of the proof for $N = 3$ excited states in Table 1. Intervals are rounded for space; in actuality they have a nonempty intersection. See Figure 5 for a visual representation of the same information.

interval	method	details
[1.414, 4.266]	FALL	—
[4.266, 4.433]	BOUND_STATE_GOOD	bound state 0, used $T = 1.921$, $y(T) \in [0.127, 0.277]$, $\dot{y}(T) \in [-0.342, -0.283]$, $\delta \in [-0.374, -0.269]$, $\dot{\delta} \in [-0.139, -0.049]$
[4.433, 14.095]	FALL	—
[14.085, 14.115]	BOUND_STATE_GOOD	bound state 1, used $T = 2.855$, $y(T) \in -0.3[40, 43]$, $\dot{y}(T) \in 0.452[46, 55]$, $\delta \in 0.15[59, 63]$, $\dot{\delta} \in 0.00[05, 12]$
[14.115, 29.090]	FALL	—
[29.090, 29.174]	BOUND_STATE_GOOD	bound state 2, used $T = 4.970$, $y(T) \in 0.1[05, 29]$, $\dot{y}(T) \in -0.1[34, 46]$, $\delta \in -0.1[18, 26]$, $\dot{\delta} \in -0.0[57, 65]$
[29.174, 49.339]	FALL	—
[49.339, 49.381]	BOUND_STATE_GOOD	bound state 3, used $T = 5.908$, $y(T) \in -0.1[68, 76]$, $\dot{y}(T) \in [0.198, 0.202]$, $\delta \in 0.07[31, 55]$, $\dot{\delta} \in 0.01[72, 91]$
[49.381, 51.381]	FALL	—
[0.000, 0.019]	INFTY_CROSSES_MANY	—

Table 1. Output of the proof summarized for $N = 3$ excited states.

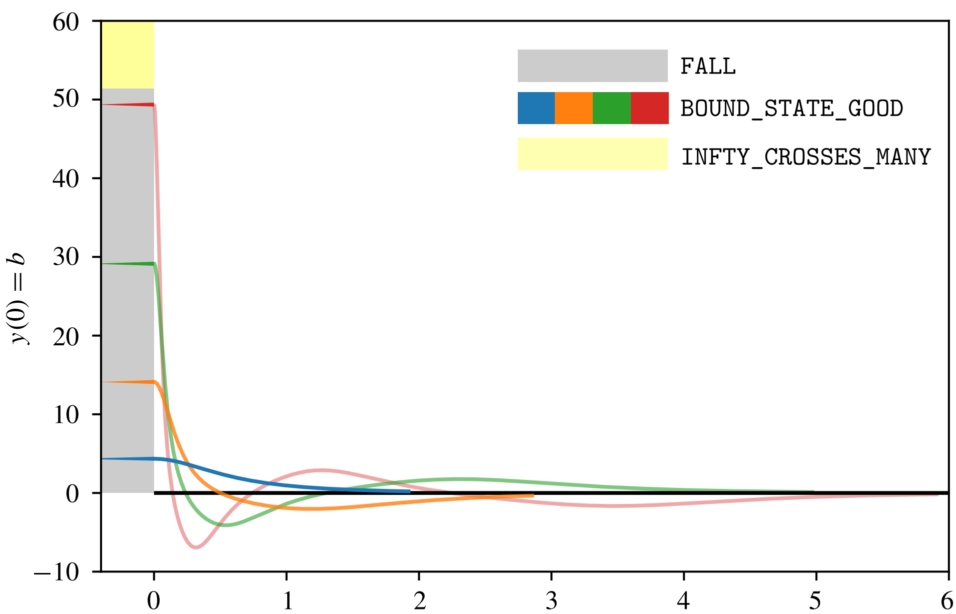

Figure 5. Graph showing the first three excited states and how the b-axis is partitioned by different proof methods.

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ON THE SPECTRUM OF NONDEGENERATE MAGNETIC LAPLACIANS

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We consider a compact Riemannian manifold with a Hermitian line bundle whose curvature is nondegenerate. Under a general condition, the Laplacian acting on high tensor powers of the bundle exhibits gaps and clusters of eigenvalues. We prove that for each cluster the number of eigenvalues that it contains is given by a Riemann–Roch number. We also give a pointwise description of the Schwartz kernel of the spectral projectors onto the eigenstates of each cluster, similar to the Bergman kernel asymptotics of positive line bundles. Another result is that gaps and clusters also appear in local Weyl laws.

1. Introduction

Consider a Hermitian line bundle L on a compact Riemannian manifold with a connection ∇ whose curvature is nondegenerate. We will be concerned with the eigenvalues and eigenstates of the Bochner Laplacians $\Delta_k = \frac{1}{2}\nabla^*\nabla + kV$ acting on positive tensor powers L^k of the bundle, V being a real function, in the limit where k tends to infinity. Physically, $k^{-2}\Delta_k$ is a magnetic Schrödinger operator with k the inverse of the Planck's constant, ∇ the magnetic potential and $k^{-1}V$ the electric potential.

A very particular case is the $\bar{\partial}$-Laplacian of high powers of a positive line bundle on a complex manifold. Its ground states are the holomorphic sections which play obviously a central role in algebraic/complex geometry, but also in mathematical physics: in Kähler quantization, the space of holomorphic sections is the quantum space and the large k limit is the semiclassical limit. Starting from [Guillemin and Uribe 1988], it has been understood that for a manifold that is not necessarily complex, the holomorphic sections can be replaced by the bounded states of the Bochner Laplacian Δ_k, where the potential V is suitably defined; bounded here means that the eigenvalues are bounded independently of k. These "almost" holomorphic sections have been used with success in various problems on symplectic manifolds from their projective embeddings to their quantizations [Borthwick and Uribe 1996; 2000; Ma and Marinescu 2007].

In the larger regime where we consider all the eigenvalues smaller than $k\Lambda$, with Λ arbitrary large but independent of k, few results are known: a general Weyl law was established in [Demailly 1985], which we will recall later, and for a specific class of connection ∇, [Faure and Tsujii 2015] showed that the spectrum of Δ_k exhibits some gaps and clusters, the first cluster consisting of the bounded states of [Guillemin and Uribe 1988].

A natural question is to determine the number of eigenvalues in each cluster. For the first cluster, in the case of holomorphic sections of a positive line bundle, the answer is provided by the Riemann–Roch–Hirzebruch theorem and the Kodaira vanishing theorem. More generally, when k is sufficiently large,

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the number of bounded states of the Bochner Laplacian of [Guillemin and Uribe 1988] is still given by the Riemann–Roch number of L^k. One of our main results is that the number of eigenvalues in each higher cluster is given as well by a Riemann–Roch number, associated to L^k tensored with a convenient auxiliary bundle F defined in terms of the cluster.

We are also concerned with results of local nature: we show that gaps and clusters appear as well in the local Weyl laws of Δ_k; local here means that each eigenvalue is counted with a weight given by the square of the pointwise norm of the corresponding eigensection. Furthermore we give a pointwise description of the Schwartz kernel of the spectral projectors associated to each cluster, generalizing the Bergman kernel asymptotics for positive line bundles.

The picture emerging from these results is that the restriction of the Bochner Laplacian Δ_k to each cluster is essentially a Berezin–Toeplitz operator with principal symbol an endomorphism of the auxiliary bundle F.

1A. ***The magnetic Laplacian.*** Let us turn to precise statements. Let M^{2n} be a closed manifold equipped with a Riemannian metric g, a volume form μ, a Hermitian line bundle L with a connection compatible with the metric, a Hermitian vector bundle A over M having an arbitrary rank r with a connection, and a section $V \in \mathcal{C}^\infty(M, \operatorname{End} A)$ such that $V(x)$ is Hermitian for any $x \in M$. Define the Laplacian

$$\Delta_k = \tfrac{1}{2}\nabla^*\nabla + kV : \mathcal{C}^\infty(L^k \otimes A) \to \mathcal{C}^\infty(L^k \otimes A). \tag{1}$$

Here $k \in \mathbb{N}$, ∇ is the covariant derivative of $L^k \otimes A$, ∇^* is its adjoint, the scalar products of sections of $L^k \otimes A$ or $L^k \otimes A \otimes T^*M$ are defined by integrating the pointwise scalar products against the volume form μ. The metric of T^*M is induced by the Riemannian metric.

We have introduced the bundle A with the endomorphism-valued section V to include some important Laplacians as the $\bar\partial$-Laplacian acting on p-forms or the square of some Dirac operators. Furthermore our results hold for a slightly more general class of operators than (1), which are defined in Section 3 and are locally of the form (B).

Since Δ_k is a formally self-adjoint elliptic operator on a compact manifold, it is essentially self-adjoint, its spectrum $\operatorname{sp}(\Delta_k)$ is a discrete subset of $[k \inf V_1, +\infty[$ and consists only of eigenvalues with finite multiplicities, and the eigenfunctions are smooth sections of $L^k \otimes A$. Here $V_1(x)$ is the lowest eigenvalue of $V(x)$.

The curvature of L has the form ω/i, with $\omega \in \Omega^2(M, \mathbb{R})$ a closed form. Let us assume that

$$\omega \text{ is nondegenerate at each point of } M. \tag{A}$$

Thus ω is a symplectic form. Associated to ω is the Liouville volume form $\mu_{\mathrm{L}} = \omega^n/n!$. We will assume that $\mu = \mu_{\mathrm{L}}$. This is not a restrictive assumption because if we multiply μ by a positive function ρ and the metric of A by ρ^{-1}, we do not change the scalar products of $\mathcal{C}^\infty(L^k \otimes A)$ and $\Omega^1(L^k \otimes A)$. Working with μ_{L} will simplify several statements.

1B. ***Pointwise data.*** We now introduce several pointwise data that will enter in our asymptotic description of the spectrum of Δ_k. Denote by j_B the section of $\operatorname{End}(TM)$ such that $\omega(\xi, \eta) = g(j_B\xi, \eta)$. Then M

has an almost-complex structure j compatible with ω defined by

$$j_y := |j_{B,y}|^{-1} j_{B,y} \quad \text{for all } y \in M.$$

So the vector bundle $T^{1,0}M = \operatorname{Ker}(j - i\operatorname{id}_{TM\otimes\mathbb{C}}) \subset TM \otimes \mathbb{C}$ has a Hermitian metric h given by $h(\xi, \eta) = \omega(\xi, \bar{\eta})/i$.

Moreover, the complexification of $j_{B,y}/i$ restricts to a positive endomorphism of $(T^{1,0}_y M, h_y)$. Denote its eigenvalues by $0 < B_1(y) \leqslant \cdots \leqslant B_n(y)$. We introduce an orthonormal basis (u_i) of $(T^{1,0}_y M, h_y)$ such that $j_{B,y} u_i = i B_i(y) u_i$.

Consider the space $\mathcal{D}(T_y M) = \mathbb{C}[T^{0,1}_y M]$ of antiholomorphic polynomials of $T_y M$. If (z_i) are the linear complex coordinates of $T_y M$ dual to the u_i, then $\mathcal{D}(T_y M) = \mathbb{C}[\bar{z}_1, \dots, \bar{z}_n]$. Define the endomorphism

$$\square_y = \sum_i B_i(y)\big(\mathfrak{a}_i^\dagger \mathfrak{a}_i + \tfrac{1}{2}\big) + V(y) : \mathcal{D}(T_y M) \otimes A_y \to \mathcal{D}(T_y M) \otimes A_y, \tag{2}$$

where $\mathfrak{a}_i$ and $\mathfrak{a}_i^\dagger$ are the endomorphisms of $\mathcal{D}(T_y M)$ acting by derivation with respect to $\bar{z}_i$ and multiplication by $\bar{z}_i$ respectively.

We introduce an eigenbasis (ζ_j) of $V(y)$: $V(y)\zeta_i = V_i(y)\zeta_i$, with $V_1(y) \leqslant \cdots \leqslant V_r(y)$. Then $\square_y$ is diagonalizable, with eigenbasis $(\bar{z}^\alpha \otimes \zeta_j,\ (\alpha, j) \in \mathbb{N}^n \times \{1, \dots, r\})$,

$$\square_y(\bar{z}^\alpha \otimes \zeta_j) = \bigg(\sum_i B_i(y)\big(\alpha(i) + \tfrac{1}{2}\big) + V_j(y)\bigg)\bar{z}^\alpha \otimes \zeta_j.$$

Let $\lambda_1(y) \leqslant \lambda_2(y) \leqslant \cdots$ be the eigenvalues of $\square_y$ ordered and repeated according to their multiplicities.

The operators $\square_y$ depend smoothly on y even if it is not obvious from (2), because in general there is no local smooth frame (u_i) of $T^{1,0}M$ which is an eigenbasis of $j_{B,y}$ at each y. The various eigenvalues $B_i(y)$, $V_j(y)$ and $\lambda_\ell(y)$ depend continuously on y.

1C. *Weyl laws.* Demailly [1985] proved a Weyl law for the operators $k^{-1}\Delta_k$. It says roughly that in the semiclassical limit $k \to \infty$, the spectrum of $k^{-1}\Delta_k$ is an aggregate of the spectra of the $\square_y$. More precisely, we introduce the counting functions $N_y(\lambda) = \sharp\{\ell : \lambda_\ell(y) \leqslant \lambda\}$ of the $\square_y$ and the one of $k^{-1}\Delta_k$

$$N(\lambda, k) = \sharp\, \operatorname{sp}(k^{-1}\Delta_k) \cap]-\infty, \lambda].$$

Here and in the sequel, an eigenvalue with multiplicity m is counted m times. Let $v : \mathbb{R} \to \mathbb{R}$ be the nondecreasing function $v(\lambda) := \int_M N_y(\lambda)\, d\mu_{\mathrm{L}}(y)$. Let D be the set of discontinuity points of v. Then for any $\lambda \in \mathbb{R} \setminus D$, we have

$$N(\lambda, k) = \left(\frac{k}{2\pi}\right)^n v(\lambda) + \mathrm{o}(k^n) \tag{3}$$

as k tends to infinity. We have slightly reformulated the original result [Demailly 1985, Theorem 0.6], which holds more generally for ω not necessarily nondegenerate and M not necessarily compact.

The subset D is in general nonempty. As an easy example, if $j_B = j$ and $V = 0$, then $D = \frac{1}{2}n + \mathbb{N}$, v is locally constant on $\mathbb{R} \setminus D$ and, for any $\ell \in \mathbb{N}$,

$$v\Big(\frac{n}{2} + \ell + 0\Big) = v\Big(\frac{n}{2} + \ell - 0\Big) + r\mu_{\mathrm{L}}(M)\binom{n+\ell-1}{\ell-1}.$$

Our goal is to understand the corrections to the Weyl law (3), in other words what is hidden in the remainder $o(k^n)$. For instance, if the function v is constant on a compact interval J, then (3) implies that $\sharp \operatorname{sp}(k^{-1}\Delta_k) \cap J = o(k^n)$. Actually, as we will see, in this situation, when k is sufficiently large, J contains no eigenvalue of $k^{-1}\Delta_k$. Furthermore, the numbers of eigenvalues between such intervals is given by Riemann–Roch numbers.

To state our results, we introduce the set $\Sigma = \bigcup_j \lambda_j(M)$. Σ is a locally finite union of closed disjoint intervals. The function v is locally constant on $\mathbb{R} \setminus \Sigma$ and Σ is the support of the Lebesgue–Stieltjes measure dv; see Section 2D for a proof of these statements.

If B is complex vector bundle of M, we denote by $\operatorname{RR}(B)$ the Riemann–Roch number of B, that is, the integral of the product of the Chern character of B by the Todd form of (M, j).

Theorem 1.1. *Let $a, b \in \mathbb{R} \setminus \Sigma$, with $a < b$. Then when k is sufficiently large,*

$$\sharp \operatorname{sp}(k^{-1}\Delta_k) \cap [a, b] = \begin{cases} \operatorname{RR}(L^k \otimes F) & \text{if } [a, b] \cap \Sigma \neq \varnothing, \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

where F is the vector bundle with fibers $F_y = \operatorname{Im} 1_{[a,b]}(\square_y)$, $y \in M$.

The assumption that $a, b \in \mathbb{R} \setminus \Sigma$ guarantees that the number of eigenvalues of $\square_y$ in $[a, b]$ is constant, so that F is a genuine smooth vector bundle. $\operatorname{RR}(L^k \otimes F)$ depends polynomially on k, with leading term

$$\operatorname{RR}(L^k \otimes F) = (\operatorname{rank} F)\left(\frac{k}{2\pi}\right)^n \mu_{\mathrm{L}}(M) + \mathcal{O}(k^{n-1}).$$

The result is consistent with the Weyl law (3) because when $a, b \in \mathbb{R} \setminus \Sigma$ we have $N_y(b) = N_y(a) + \operatorname{rank} F$ for any $y \in M$.

Theorem 1.1 holds not only for the magnetic Laplacians (1), but also for other remarkable geometric operators, as for instance the holomorphic Laplacians or the square of spin-c Dirac operators. The corresponding results are stated in Theorems 3.4 and 3.6. In these cases, $\Sigma = \mathbb{N}$, so the spectrum of $k^{-1}\Delta_k$ consists of clusters at nonnegative integers, the dimension of each cluster being given by the Riemann–Roch number $\operatorname{RR}(L^k \otimes F)$, where F is a sum of tensor products of symmetric and exterior powers of $T^{1,0}M$; see part (3) of Theorem 3.6.

Theorem 1.1 is relevant only when Σ has several components. Note that the set of (ω, g, V) such that Σ is not connected is open in $\mathcal{C}^0$-topology. Let us discuss some examples where the fiber bundle F can be made explicit.

First if $j = j_B$ and $V = 0$, the set Σ is $\frac{1}{2}n + \mathbb{N}$, and for $a = \frac{1}{2}(n-1) + \ell$, with $\ell \in \mathbb{N}$ and $b = a + 1$, the bundle F in Theorem 1.1 is $\operatorname{Sym}^\ell(T^{1,0}M) \otimes A$. More generally, suppose that $B_1 = \cdots = B_n$, that is, $j_B = Bj$, with $B \in \mathcal{C}^\infty(M, \mathbb{R}_{>0})$. Then $\Sigma \subset \bigcup_{\ell \in \mathbb{N}} [\sigma_\ell^-, \sigma_\ell^+]$, where

$$\sigma_\ell^- = \inf\Big(B\Big(\ell + \frac{n}{2}\Big) + V_1\Big), \quad \sigma_\ell^+ = \sup\Big(B\Big(\ell + \frac{n}{2}\Big) + V_r\Big).$$

Assume there exist $a, b \in \mathbb{R}$, such that $\sigma_{\ell-1}^+ < a < \sigma_\ell^-$ and $\sigma_\ell^+ < b < \sigma_{\ell+1}^-$ for some $\ell \in \mathbb{N}$. Then Theorem 1.1 holds and $F = \operatorname{Sym}^\ell(T^{1,0}M) \otimes A$.

Since Σ is the support of the Lebesgue–Stieltjes measure $d\upsilon$, the Weyl law (3) implies that, for any $\lambda \in \Sigma$, the distance $d(\lambda, \operatorname{sp}(k^{-1}\Delta_k))$ tends to 0 as $k \to \infty$. To the contrary, if $\lambda \notin \Sigma$, by the second case of (4), there exists $\epsilon > 0$ such that $d(\lambda, \operatorname{sp}(k^{-1}\Delta_k)) \geqslant \epsilon$ when k is sufficiently large.

The following theorem gives more precise estimates.

Theorem 1.2. *For any $\Lambda > 0$, there exists $C > 0$ such that, for any $\lambda \leqslant \Lambda$,*

$$\lambda \in \Sigma \quad \Longrightarrow \quad \operatorname{dist}(\lambda, \operatorname{sp}(k^{-1}\Delta_k)) \leqslant Ck^{-1/2}, \tag{5}$$

$$\lambda \in \operatorname{sp}(k^{-1}\Delta_k) \quad \Longrightarrow \quad \operatorname{dist}(\lambda, \Sigma) \leqslant Ck^{-1/2}. \tag{6}$$

When the bundle F of Theorem 1.1 has a definite parity, see Remark 7.3, (6) can be slightly improved. For instance, if as above $j_B = Bj$ and there exist a, b such that $\sigma^+_{\ell-1} < a < \sigma^-_\ell$ and $\sigma^+_\ell < b < \sigma^-_{\ell+1}$, then $\operatorname{sp}(k^{-1}\Delta_k) \cap [a, b] \subset [\sigma^-_\ell, \sigma^+_\ell] + \mathcal{O}(k^{-1})$.

Interestingly, some local Weyl laws hold with a similar gapped structure. Instead of Σ, the local law at $y \in M$ involves the spectrum $\Sigma_y = \{\lambda_i(y) : i \in \mathbb{N}\}$ of $\square_y$, which is a discrete subset of $\mathbb{R}$. Clearly, $\Sigma = \bigcup_y \Sigma_y$.

For any $k \in \mathbb{N}$, choose an orthonormal eigenbasis $(\Psi_{k,i})_{i\in\mathbb{N}}$ of $k^{-1}\Delta_k$ such that $k^{-1}\Delta_k\Psi_{k,i} = \lambda_{k,i}\Psi_{k,i}$, with $\lambda_{0,k} \leqslant \lambda_{1,k} \leqslant \cdots$. For any $y \in M$ and real numbers $a < b$, define

$$N(y, a, b, k) = \sum_{i:\lambda_{k,i}\in[a,b]} |\Psi_{k,i}(y)|^2,$$

so we count the eigenvalues in $[a, b]$ with weights given by the square of the pointwise norm at y of the corresponding eigenvectors.

Theorem 1.3. *For any $\Lambda \in \mathbb{R} \setminus \Sigma$, $y \in M$ and $a, b \in \,]-\infty, \Lambda] \setminus \Sigma_y$ such that $a < b$, the following holds: If $[a, b] \cap \Sigma_y$ is empty, then $N(y, a, b, k) = \mathcal{O}(k^{-\infty})$. Otherwise we have an asymptotic expansion*

$$N(y, a, b, k) = \left(\frac{k}{2\pi}\right)^n \sum_{\lambda\in\Sigma_y\cap[a,b]} \sum_{\ell=0}^{\infty} m_{\ell,\lambda} k^{-\ell} + \mathcal{O}(k^{-\infty}), \tag{7}$$

where the coefficients $m_{\ell,\lambda}$ do not depend on a, b, k. In particular, $m_{0,\lambda}$ is the multiplicity of the eigenvalue λ of $\square_y$.

We believe that the same result holds without the assumption that a, b are smaller than $\Lambda \in \mathbb{R} \setminus \Sigma$. Observe that the first-order term $\sum_{\lambda\in[a,b]} m_{0,\lambda}$ in (7) is merely the number of eigenvalues of $\square_y$ in $[a, b]$. In particular we recover the same structure as in the counting law (4) of Theorem 1.2: when the leading-order term is zero, $N(y, a, b, k) = \mathcal{O}(k^{-\infty})$. We interpret this as a gap in the local Weyl law.

Besides these gaps and clusters, another notable aspect in Theorems 1.1 and 1.3 is that we have full asymptotic expansions. For the Laplace–Beltrami operators or the Schrödinger operator without magnetic fields, the remainders in Weyl laws have a completely different behavior; see for instance the survey [Zelditch 2008, Section 8]. Another situation where clusters and gaps occur is for the pseudodifferential operators with a principal symbol having a periodic Hamiltonian flow. This has been studied in many papers; see for instance [Weinstein 1977; Colin de Verdière 1979], [Dozias 1997] for a semiclassical result and [Boutet de Monvel 1980; Boutet de Monvel and Guillemin 1981, Section 1] for earlier results, with

Riemann–Roch numbers already. For our magnetic Laplacians, the gaps are also connected to periodic Hamiltonians: the quantum harmonic oscillators $\mathfrak{a}_i^\dagger \mathfrak{a}_i$ of (2). In dimension 2, this lies at the origin of the cyclotron motion or resonance of a charged particle in a magnetic field.

1D. ***Schwartz kernels of spectral projectors.*** Another result we would like to emphasize in this introduction is the asymptotic description of the Schwartz kernel of $g(k^{-1}\Delta_k)$, where $g : \mathbb{R} \to \mathbb{C}$ is a bounded function with compact support satisfying some assumptions. These Schwartz kernels are by definition given at $(x, y) \in M^2$ by

$$g(k^{-1}\Delta_k)(x, y) = \sum_i g(\lambda_{k,i})\Psi_{k,i}(x) \otimes \overline{\Psi_{k,i}(y)} \in L_x^k \otimes A_x \otimes \bar{L}_y^k \otimes \bar{A}_y.$$

We will prove that $g(k^{-1}\Delta_k)$ belongs to the operator algebra $\mathcal{L}(A)$ introduced in [Charles 2024]. Let us recall the main characteristics of $\mathcal{L}(A)$; the complete definition will be given in Section 5.

$\mathcal{L}(A)$ consists of families $(P_k)_{k\in\mathbb{N}}$ such that, for any k, P_k is an endomorphism of $\mathcal{C}^\infty(M, L^k \otimes A)$ having a smooth Schwartz kernel in $\mathcal{C}^\infty(M^2, (L^k \otimes A) \boxtimes (\bar{L}^k \otimes \bar{A}))$ satisfying the following conditions. First, for any compact subset K of M^2 not intersecting the diagonal, for any N, we have $P_k(x, y) = \mathcal{O}(k^{-N})$ uniformly on K. Second, for any open set U of M identified with a convex open set of $\mathbb{R}^{2n}$ through a diffeomorphism, let $F \in \mathcal{C}^\infty(U^2, L \boxtimes \bar{L})$ be the unitary frame such that $F(x, y) = u \otimes \bar{v}$, where v is any vector in L_y with norm 1 and $u \in L_x$ is the parallel transport of v along the path $t \in [0, 1] \to y + t(x - y)$. We introduce a unitary trivialization of A on U and identify accordingly the sections of $A \boxtimes \bar{A}$ over U^2 with the functions of $\mathcal{C}^\infty(U^2, \mathbb{C}^r \otimes \bar{\mathbb{C}}^r)$. Then the Schwartz kernel of P_k has the following asymptotic expansion on U^2: for any $N \in \mathbb{N}$, for any $x \in U$ and $\xi \in T_xU$ such that $x + \xi \in U$,

$$P_k(x + \xi, x) = \left(\frac{k}{2\pi}\right)^n F^k(x + \xi, x)e^{-k|\xi|_x^2/4} \sum_{\ell=0}^{N} k^{-\ell} a_\ell(x, k^{1/2}\xi) + \mathcal{O}(k^{n-(N+1)/2}), \tag{8}$$

where $|\xi|_x^2 = \omega_x(\xi, j_x\xi)$, the coefficients $a_\ell(x, \cdot)$ are polynomial maps $T_xM \to \mathbb{C}^r \otimes \bar{\mathbb{C}}^r$ depending smoothly on x, and the $\mathcal{O}$ is uniform when $(x + \xi, x)$ runs over any compact set of U^2.

Such an operator $P = (P_k)$ has a symbol $\sigma_0(P)$, which at $y \in M$ is the endomorphism of $\mathcal{D}(T_yM) \otimes A_y$ defined by

$$(\sigma_0(P)(y))(f)(u) = (2\pi)^{-n} \int_{T_yM} e^{(u-v)\cdot\bar{v}} a_0(y, u - v) f(v)\, d\mu_y(v).$$

Here, the scalar product $u \cdot \bar{v}$ and the measure μ_y are defined in terms of linear complex coordinates $z_i : T_yM \to \mathbb{C}$ associated to an orthonormal frame of $(T_y^{1,0}M, h_y)$ by $u \cdot \bar{v} = \sum z_i(u)\overline{z_i(v)}$ and $\mu_y = |dz_1 \cdots dz_n d\bar{z}_1 \cdots d\bar{z}_n|$.

As a result, for any $(P_k) \in \mathcal{L}(A)$, we have $\|P_k\| = \mathcal{O}(1)$ and

$$\|P_k\| = \mathcal{O}(k^{-1/2}) \iff \sigma_0(P)(y) = 0 \text{ for all } y \in M \iff a_0(y, \cdot) = 0 \text{ for all } y \in M.$$

Furthermore $\mathcal{L}(A)$ is closed under composition and the map σ_0 is an algebra morphism. Here the product of the symbols at y is the composition of endomorphisms of $\mathcal{D}(T_yM) \otimes A_y$, which is not commutative.

Theorem 1.4. (1) *For any $a, b \in \mathbb{R} \setminus \Sigma$, the spectral projector $\Pi_k := 1_{[a.b]}(k^{-1}\Delta_k)$ and $k^{-1}\Delta_k \Pi_k$ belong to $\mathcal{L}(A)$ and their symbols at y are equal to $1_{[a,b]}(\square_y)$ and $\square_y 1_{[a,b]}(\square_y)$ respectively.*

(2) *For any $\Lambda \in \mathbb{R} \setminus \Sigma$, for any $g \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$ such that* $\operatorname{supp} g \subset]-\infty, \Lambda]$, $(g(k^{-1}\Delta_k))_k$ *belongs to $\mathcal{L}(A)$ and its symbol at y is $g(\square_y)$.*

The second assertion is actually a generalization of the first one because choosing $\Lambda > b$ such that $[b, \Lambda] \cap \Sigma = \varnothing$, one has $1_{[a,b]} = g$ on an open neighborhood of Σ with $g \in \mathcal{C}^\infty(\mathbb{R})$ supported in $]-\infty, \Lambda]$, and by Theorem 1.1, $1_{[a,b]}(\lambda) = g(\lambda)$ for any $\lambda \in \operatorname{sp}(k^{-1}\Delta_k)$ when k is sufficiently large.

1E. ***Comparison with earlier results.*** This work started as a collaboration with Yuri Kordyukov and some of the results presented here appeared also in [Kordyukov 2022]: the existence of spectrum gaps, that is, (4) when $[a, b] \cap \Sigma = \varnothing$, and a weak version of (6) with a $\mathcal{O}(k^{-1/4})$ instead of the $\mathcal{O}(k^{-1/2})$ are proved in [loc. cit., Theorem 1.2]. Moreover, under the assumption of Theorem 1.4, the Schwartz kernel of the spectral projector $\Pi_k = 1_{[a,b]}(k^{-1}\Delta_k)$ is described in [loc. cit., Theorem 1.6] in a way similar to our result.

In the case where $j_B = j$ and V is constant, the existence of spectrum gaps, that is, (4) when $[a, b] \cap \Sigma = \varnothing$, was proved in [Faure and Tsujii 2015, Theorem 10.2.2]. Our proof will follow the same line as in that work and is similar to the proof in [Kordyukov 2022].

In the case again where $j_B = j$ and $V = 0$, the first gap and the asymptotic description of the first cluster has a long history. When j is integrable so that M is a complex manifold and ω is Kähler, the gap follows from Kodaira vanishing theorem, the first cluster consists of the holomorphic sections of L^k, its dimension is given by the Riemann–Roch–Hirzebruch theorem, and the Schwartz kernel of the corresponding spectral projector is the Bergman kernel, whose asymptotic can be deduced from [Boutet de Monvel and Sjöstrand 1976] and which has been used in many papers starting from [Zelditch 1998]. The extension to almost-complex structure was done in [Guillemin and Uribe 1988; Borthwick and Uribe 2007; Ma and Marinescu 2008]. Parallel results for spin-c Dirac operators were proved in [Borthwick and Uribe 1996; Ma and Marinescu 2002; 2007].

The main tool we use in this paper is the algebra $\mathcal{L}(A)$ introduced in [Charles 2024]; a first weaker version was proposed in [Charles 2016]. The asymptotic expansions (8) or similar versions have been used before by several authors to describe the spectral projector on the first cluster and corresponding Toeplitz operators, [Shiffman and Zelditch 2002; Charles 2003; Ma and Marinescu 2007] for instance. In [Charles 2024], besides establishing the main properties of $\mathcal{L}(A)$, we considered some projectors (Π_k) in $\mathcal{L}(A)$ whose symbol at $y \in M$ is the projector onto the m-th level of a Landau Hamiltonian $\sum \mathfrak{a}_i^\dagger \mathfrak{a}_i$. In particular we computed the rank of Π_k as a Riemann–Roch number and we studied the corresponding Toeplitz algebra. By the results of the current paper, particular instances of such projectors are the spectral projectors on the m-cluster of a magnetic Laplacian with $j_B = j$ and $V = 0$.

In a different context, many works have been devoted to the magnetic Schrödinger operator in $\mathbb{R}^n$; see [Raymond 2017] for a general overview. The most significant result is a semiclassical description of the bottom of the spectrum in terms of effective operators whose principal symbols are the functions we denoted by λ_i; see for instance [Ivrii 1998, Theorem 6.2.7], [Raymond and Vũ Ngọc 2015, Theorem 1.6] or [Morin 2020, Theorem 2] for a statement in the manifold setting. These works differ in at least two ways from the current paper: The global gap assumption is generally replaced by a confinement hypothesis; typically the function we denote by λ_0 is assumed to have a nondegenerate minimum. Moreover, the general strategy is to put the Schrödinger operator on a normal form by conjugating it with a convenient Fourier integral operator.

1F. ***Outline of the paper.*** The main idea in the first part of the paper is to approximate the Laplacian Δ_k locally by a family of Laplacians $\Delta_{y,k},\ y \in M$, obtained from Δ_k by "freezing" the coordinates at y. In Section 2 we introduce these operators, recall the basic results regarding their spectrum and explain the relationship with the operators $\Box_y$ of Section 1B. In Section 3, we introduce a class of Laplacians slightly more general than the magnetic Laplacians Δ_k and which are well-approximated by the $\Delta_{y,k}$. This class contains the holomorphic Laplacians and some of their generalizations without integrable complex structure. In Section 4, we prove a weak version of Theorem 1.2 which says that $\mathrm{sp}(k^{-1}\Delta_k) \to \Sigma$ in the limit $k \to \infty$, by constructing on one hand some peaked sections which are approximate eigenmodes of Δ_k, and on the other hand, by inverting $\lambda - k^{-1}\Delta_k$ up to a $\mathcal{O}(k^{-1/4})$ when $\lambda \notin \Sigma$.

In the second part of the paper, Sections 5 and 6, we introduce the algebra $\mathcal{L}(A)$ and prove that the spectral projector $1_{[a,b]}(k^{-1}\Delta_k)$ belongs to $\mathcal{L}(A)$ when $a, b \in \mathbb{R} \setminus \Sigma$. The proof is divided into three steps: From the resolvent estimate of Section 4, we deduce that any operator of $\mathcal{L}(A)$ having symbol $1_{[a,b]}(\Box)$ is an approximation of $1_{[a,b]}(k^{-1}\Delta_k)$ up to a $\mathcal{O}(k^{-1/4})$. We then prove that $\mathcal{L}(A)/\mathcal{O}(k^{-\infty})$ has a unique self-adjoint projector having symbol $1_{[a,b]}(\Box)$ and commuting with Δ_k. Finally we prove that this operator is indeed the spectral projector.

In the last part, Section 7, we establish some spectral properties for the Toeplitz operators associated to the projectors of $\mathcal{L}(A)$, including a sharp Gårding inequality and the functional calculus. Then we deduce Theorems 1.1 and 1.3 and the second part of Theorem 1.4.

2. The linear pointwise data

In this section we consider a compact manifold M^{2n} equipped with a symplectic form ω and a Riemannian metric g. Let $A \to M$ be a Hermitian vector bundle with a section V of $\mathcal{C}^\infty(M, \mathrm{End}\, A)$ such that $V(x)$ is Hermitian for any $x \in M$. We choose a point $y \in M$.

2A. ***The complex structure.*** Let $j_{B,y}$ be the endomorphism of T_yM such that $\omega_y(\xi, \eta) = g_y(j_{B,y}\xi, \eta)$. It will be useful to work with the following normal form.

Lemma 2.1. *There exists* $0 < B_1(y) \leqslant \cdots \leqslant B_n(y)$ *such that* T_yM *has a basis* (e_i, f_i) *satisfying*

$$\omega_y(e_i, e_j) = \omega_y(f_i, f_j) = 0, \quad \omega_y(e_i, f_j) = \delta_{ij},$$
$$j_{B,y}e_i = B_i(y) f_i, \qquad j_{B,y} f_i = -B_i(y)e_i.$$

The vectors $u_i = \frac{1}{\sqrt{2}}(e_i - i f_i)$, $\bar{u}_i = \frac{1}{\sqrt{2}}(e_i + i f_i)$ *are a basis of* $T_yM \otimes \mathbb{C}$ *and*

$$\frac{1}{i}\omega_y(u_i, u_j) = \frac{1}{i}\omega(\bar{u}_i, \bar{u}_j) = 0, \quad \frac{1}{i}\omega(u_i, \bar{u}_j) = \delta_{ij},$$
$$j_{B,y}u_i = i B_i(y)u_i, \qquad j_{B,y}\bar{u}_i = -i B_i(y)\bar{u}_i.$$

Proof. Since $j_{B,y}$ is a g_y-antisymmetric invertible endomorphism of T_yM, there exists a g_y-orthonormal basis $(\tilde{e}_i, \tilde{f}_i)$ such that $j_{B,y}\tilde{e}_i = B_i(y)\tilde{f}_i$ and $j_{B,y}\tilde{f}_i = -B_i(y)\tilde{e}_i$, where the $B_i(y)$ are positive. We set $e_i = (B_i(y))^{-1/2}\tilde{e}_i$ and $f_i = (B_i(y))^{-1/2}\tilde{f}_i$, and the result follows by direct computations. □

We can interpret this result as follows: first, $j_{B,y}/i$ is $\mathbb{C}$-diagonalizable with only nonzero real eigenvalues, denoted by $\pm B_i(y)$. Second, the subspace W of $T_yM \otimes \mathbb{C}$ spanned by the u_i is the sum of the eigenspaces of $j_{B,y}/i$ with a positive eigenvalue. W is Lagrangian and the sesquilinear form h_y of $T_yM \otimes \mathbb{C}$ given by $h_y(u, v) = \omega_y(u, \bar{v})/i$ is positive on W. Equivalently the endomorphism j_y of T_yM such that $j_y = i$ on W is a complex structure of T_yM compatible with ω_y. So from now on, we will denote $W = \operatorname{Ker}(j_y - i)$ by $T_y^{1,0}M$, and by the definition of j_y, the restriction of $j_{B,y}/i$ to $T_y^{1,0}M$ is a positive endomorphism of $(T_y^{1,0}M, h_y)$ with eigenvalues the $B_i(y)$. Hence the vectors (u_i) in Lemma 2.1 are nothing else than a h_y-orthonormal eigenbasis of $T_y^{1,0}M$.

An important remark is that j_y depends smoothly on y, so it defines an almost complex structure of M. Indeed, the space $T_y^{1,0}M$ depends smoothly on y because $j_{B,y}/i$ being invertible, no eigenvalue can cross 0. Another reason is that $j_y = |j_{B,y}|^{-1} j_{B,y}$, where $|j_{B,y}|$ is the positive square root of the g_y-positive endomorphism $-j_{B,y}^2$. Actually, the construction of j is the classical proof of the fact that any symplectic manifold admits a compatible almost-complex structure; see [McDuff and Salamon 2017, Proposition 2.5.6].

To the contrary, in general, we cannot choose a local continuous symplectic frame (e_i, f_i) of TM such that $j_B e_i = B_i f_i$, $j_B f_i = -B_i e_i$, even if we renumber the eigenvalues $B_i(y)$ in a way depending on y. Indeed, as is well known, it is not possible in general to diagonalize smoothly a symmetric matrix, the symmetric matrix being $-(j_{B,y})^2$ in our case. More specifically, consider on $\mathbb{R}^2 \otimes \mathbb{R}^2$ with its usual Euclidean structure the endomorphism $j_B(s, t) = M(s, t) \otimes j_2$, where

$$M(s,t) = \begin{pmatrix} 1+s & t \\ t & 1-s \end{pmatrix}, \quad j_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with s and t parameters in a neighborhood of 0. Then j_B is nondegenerate and antisymmetric, and we can choose for each (s, t) a basis (e_i, f_i) satisfying the previous conditions, but not continuously with respect to (s, t). Indeed, $-j_B^2(s, t) = M^2(s, t) \otimes \mathrm{id}$ and for $s = 0$, t small nonzero, the eigenspaces of $M(s, t)$ are $(1, 1)\mathbb{R}$ and $(1, -1)\mathbb{R}$, whereas for $t = 0$ and s small nonzero, they are $(1, 0)\mathbb{R}$ and $(0, 1)\mathbb{R}$.

This example appears on $\mathbb{R}^4$ equipped with its usual Euclidean metric and the closed form

$$\omega = (1 + p_1)\, dp_1 \wedge dq_1 + (1 - p_2)\, dp_2 \wedge dq_2 + q_1\, dq_1 \wedge dp_2 - q_2\, dp_1 \wedge dq_2,$$

which is symplectic on a neighborhood of the origin. On the plane $\{p_1 = p_2 : q_1 = q_2\}$, the matrix of j_B is $M(p_1, q_1) \otimes j_2$.

We have also to be careful that the metric $\tilde{g}$ determined by (ω, j),

$$\tilde{g}_y(\xi, \eta) := \omega_y(\xi, j_y\eta) = g_y(|j_{B,y}|\xi, \eta), \tag{9}$$

is equal to g_y only when $B_1(y) = \cdots = B_n(y) = 1$, that is, when $j_{B,y}$ is itself a complex structure.

2B. ***The scalar Laplacian of T_yM.*** Consider now the covariant derivative

$$\nabla = d + \frac{1}{i}\alpha : \mathcal{C}^\infty(T_yM) \to \Omega^1(T_yM), \tag{10}$$

where $\alpha \in \Omega^1(T_yM, \mathbb{R})$ is given by $\alpha_\xi(\eta) = \frac{1}{2}\omega_y(\xi, \eta)$. Since $d\alpha = \omega_y$, the curvature of ∇ is ω_y/i. We then define the scalar Laplacian of T_yM by

$$\Delta_y^{\mathrm{scal}} := \tfrac{1}{2}\nabla^*\nabla : \mathcal{C}^\infty(T_yM) \to \mathcal{C}^\infty(T_yM). \tag{11}$$

Here the scalar products of $\mathcal{C}^\infty(T_yM)$ and $\Omega^1(T_yM)$ are defined by integrating the pointwise scalar products against a fixed constant volume form, the pointwise scalar product of $\Omega^1(T_yM)$ is defined from the metric g_y.

We can explicitly compute the spectrum and eigenfunctions of Δ_y^{scal} as follows. We introduce a basis (e_i, f_i) of T_yM as in Lemma 2.1. This basis is g_y-orthogonal and $g_y(e_i, e_i) = g_y(f_i, f_i) = B_i(y)^{-1}$, so we have

$$\Delta_y^{\mathrm{scal}} = -\frac{1}{2}\sum_{i=1}^n B_i(y)(\nabla_{e_i}^2 + \nabla_{f_i}^2) = \sum_{i=1}^n B_i(y)\big(-\nabla_{u_i}\nabla_{\bar{u}_i} + \tfrac{1}{2}\big),$$

where $u_i = \frac{1}{\sqrt{2}}(e_i - if_i)$, $\bar{u}_i = \frac{1}{\sqrt{2}}(e_i + if_i)$. Denote by z_i the linear complex coordinates dual to the u_i. If (p_i, q_i) are the real linear coordinates of T_yM in the basis (e_i, f_i), then $z_i = \frac{1}{\sqrt{2}}(p_i + iq_i)$. Since $\omega_y = i\sum dz_i \wedge d\bar{z}_i$, we have

$$\nabla = d + \frac{1}{2}\sum_{i=1}^n (z_i\, d\bar{z}_i - \bar{z}_i\, dz_i).$$

We introduce the function $s(\xi) := \exp(-|\xi|_y^2/4)$, $\xi \in T_yM$, where

$$|\xi|_y^2 = \sum_i (p_i^2 + q_i^2) = 2\sum_i |z_i|^2 = \tilde{g}_y(\xi, \xi).$$

Since $s = \exp(-|z|^2/2)$, we have $\nabla_{\bar{u}_i} s = 0$, so s is ∇-holomorphic.

Let us consider $\mathcal{C}^\infty(T_yM)$ as the space of sections of the trivial line bundle over T_yM and let us use s as a global frame. We introduce the operators

$$\mathfrak{a}_i = \partial_{\bar{z}_i}, \quad \mathfrak{a}_i^\dagger = \bar{z}_i - \partial_{z_i}. \tag{12}$$

Then $\nabla_{\bar{u}_i}(fs) = (\mathfrak{a}_i f)s$ and $\nabla_{u_i}(fs) = -(\mathfrak{a}_i^\dagger f)s$, so that

$$\Delta_y^{\mathrm{scal}}(fs) = (\widetilde{\square}_y^{\mathrm{scal}} f)s, \quad \text{with } \widetilde{\square}_y^{\mathrm{scal}} := \sum_{i=1}^n B_i(y)\big(\mathfrak{a}_i^\dagger \mathfrak{a}_i + \tfrac{1}{2}\big). \tag{13}$$

Let $\mathcal{P}(T_yM)$ be the space of polynomial functions $T_yM \to \mathbb{C}$, not necessarily holomorphic or antiholomorphic. With the coordinates (z_i), $\mathcal{P}(T_yM) = \mathbb{C}[z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n]$. Observe that $\mathfrak{a}_i$ and $\mathfrak{a}_i^\dagger$ preserve $\mathcal{P}(T_yM)$ and the same holds for $\widetilde{\square}_y^{\mathrm{scal}}$.

Since the $\mathfrak{a}_i$, $\mathfrak{a}_i^\dagger$ satisfy the so-called canonical commutation relations

$$[\mathfrak{a}_i, \mathfrak{a}_j] = [\mathfrak{a}_i^\dagger, \mathfrak{a}_j^\dagger] = 0, \quad [\mathfrak{a}_i, \mathfrak{a}_j^\dagger] = \delta_{ij},$$

we deduce by a classical argument that the endomorphisms $\mathfrak{a}_i^\dagger \mathfrak{a}_i$ of $\mathcal{P}(T_yM)$ are mutually commuting endomorphisms, each of them diagonalizable with spectrum $\mathbb{N}$; see for instance [Charles 2024, Proposition 4.1]. So we have a decomposition into joint eigenspaces

$$\mathcal{P}(T_yM) = \bigoplus_{\alpha\in\mathbb{N}^n} \mathfrak{L}_\alpha, \quad \text{with } \mathfrak{L}_\alpha = \bigcap_{i=1}^{n} \operatorname{Ker}(\mathfrak{a}_i^\dagger \mathfrak{a}_i - \alpha(i)). \tag{14}$$

Furthermore, $\mathfrak{L}_0 = \mathbb{C}[z_1, \dots, z_n]$ and $\mathfrak{L}_\alpha = (\mathfrak{a}^\dagger)^\alpha \mathfrak{L}_0$ for all $\alpha \in \mathbb{N}^n$ where $(\mathfrak{a}^\dagger)^\alpha = (\mathfrak{a}_1^\dagger)^{\alpha(1)} \cdots (\mathfrak{a}_n^\dagger)^{\alpha(n)}$. Consequently $\widetilde{\square}_y^{\text{scal}}$ is a diagonalizable endomorphism of $\mathcal{P}(T_yM)$ with spectrum Σ_y^{scal} given by

$$\Sigma_y^{\text{scal}} = \left\{ \sum_{j=1}^{n} B_j(y)\big(\alpha(j) + \tfrac{1}{2}\big) : \alpha \in \mathbb{N}^n \right\}. \tag{15}$$

Moreover the eigenspace $\mathcal{E}(\lambda)$ of the eigenvalue $\lambda \in \Sigma_y$ is the sum of the $\mathfrak{L}_\alpha$, where α runs over the multi-indices of $\mathbb{N}^n$ such that $\sum B_i(y)\big(\alpha(i) + \frac{1}{2}\big) = \lambda$.

We can deduce from these algebraic facts the L^2-spectral theory of Δ_y^{scal}. First of all, the space $\exp(-|\xi|_y^2/4)\mathcal{P}(T_yM)$ is dense in $L^2(T_yM)$ by the same proof that Hermite functions are dense. So we deduce from (14) a decomposition of $L^2(T_yM)$ in a Hilbert sum of orthogonal subspaces,

$$L^2(T_yM) = \bigoplus_{\alpha\in\mathbb{N}^n} \mathcal{K}_\alpha, \qquad \mathcal{K}_\alpha = \overline{e^{-|\xi|_y^2/4}\mathfrak{L}_\alpha}^{L^2(T_yM)} \quad \text{for all } \alpha \in \mathbb{N}^n \tag{16}$$

Let $\mathcal{G}$ be the subspace of $L^2(T_yM)$ consisting of the ψ having a decomposition $\sum \psi_\alpha$ in (16) such that $\sum |\alpha|^2 \|\psi_\alpha\|^2$ is finite. As a differential operator, Δ_y^{scal} acts on the distribution space $\mathcal{C}^{-\infty}(T_yM)$ and in particular on $L^2(T_yM)$. It is not difficult to see that $\mathcal{G}$ consists of the $\psi \in L^2(T_yM)$ such that $\Delta_y^{\text{scal}}\psi \in L^2(T_yM)$.

Lemma 2.2. $(\Delta_y^{\text{scal}}, \mathcal{G})$ *is a self-adjoint unbounded operator of* $L^2(T_yM)$, *which is the closure of* $(\Delta_y^{\text{scal}}, e^{-|\xi|_y^2/4}\mathcal{P}(T_yM))$. *Its spectrum is* Σ_y *and consists only of eigenvalues, the eigenspace of* $\lambda \in \Sigma_y$ *being the closure of* $\exp(-|\xi|_y^2/4)\mathcal{E}(\lambda)$.

This follows from (16), (14) and (13) by elementary standard arguments; see for instance [Davies 1995, Lemma 1.2.2]. Even if we won't need this in the sequel, it can be useful to note that

- the eigenspace $\mathcal{K}_0$ is the Bargmann space: $\psi \in \mathcal{K}_0$ if and only if $\psi \in L^2(T_yM)$ and $\psi = e^{-|\xi|_y/4} f(z_1, \dots, z_n)$ with f holomorphic.
- $\mathcal{G}$ is different from the Sobolev space $H^2 = \big\{\psi \in L^2(T_yM) : \sum(\partial_{p_i}^2 + \partial_{q_i}^2)\psi \in L^2\big\}$. Actually, $H^2 \cap \mathcal{G} = \mathcal{G} \cap \overline{\mathcal{G}} = H^2_{\text{iso}}(T_yM)$, the isotropic Sobolev space defined as $\big\{\psi \in H^2(T_yM) : \sum({p_i}^2 + {q_i}^2)\psi \in L^2\big\}$.

2C. ***The A_y-valued Laplacian Δ_y.*** We now consider the full Laplacian

$$\Delta_y := \Delta_y^{\text{scal}} + V(y) : \mathcal{C}^\infty(T_yM, A_y) \to \mathcal{C}^\infty(T_yM, A_y). \tag{17}$$

We deduce from the properties of Δ_y^{scal} that $(\Delta_y, \mathcal{G} \otimes A_y)$ is a selfadjoint unbounded operator of $L^2(T_yM) \otimes A_y$ with discrete spectrum

$$\Sigma_y = \left\{ \sum_{j=1}^{n} B_j(y)\big(\alpha(j) + \tfrac{1}{2}\big) + V_\ell(y) : \alpha \in \mathbb{N}^n,\ \ell = 1, \dots, r \right\}, \tag{18}$$

where $V_1(y) \leqslant \cdots \leqslant V_r(y)$ are the eigenvalues of $V(y)$. Let (ζ_ℓ) be an eigenbasis of $V(y)$, $V(y)\zeta_\ell = V_\ell(y)\zeta_\ell$. Then any $\lambda \in \Sigma_y$ is an eigenvalue of Δ_y with eigenspace the closure of the sum of the $\exp(-|\xi|_y^2/4)\mathcal{E}(\lambda') \otimes \mathbb{C}\zeta_\ell$ such that $\lambda' + V_\ell(y) = \lambda$.

In the sequel, we will mainly work with

$$\widetilde{\Box}_y = e^{|\xi|_y^2/4} \Delta_y e^{-|\xi|_y^2/4} = \widetilde{\Box}_y^{\mathrm{scal}} + V(y) \tag{19}$$

acting on $\mathcal{P}(T_yM) \otimes A_y$.

2D. *The set Σ and the function v.* Denote by $\lambda_1(y) \leqslant \lambda_2(y) \leqslant \cdots$ the eigenvalues of $\Box_y$ ordered and repeated according to their multiplicities. Let

$$\Sigma = \bigcup_{y \in M} \Sigma_y = \bigcup_{\ell} \lambda_\ell(M).$$

We introduce for any $y \in M$ the counting function $N_y(\lambda) = \sharp\{\ell : \lambda_\ell(y) \leqslant \lambda\}$ of $\Box_y$. Let $v : \mathbb{R} \to \mathbb{R}$ be the nondecreasing function $v(\lambda) := \int_M N_y(\lambda)\, d\mu_{\mathrm{L}}(y)$.

Lemma 2.3. *The functions λ_ℓ are continuous. Σ is locally a finite union of closed bounded intervals*; *it is the support of the Lebesgue–Stieltjes measure dv.*

Proof. First the functions B_i and V_j are continuous, so, for any $\alpha \in \mathbb{N}^n$ and j, $f_{\alpha,j} := \sum_i B_i\big(\alpha(i) + \frac{1}{2}\big) + V_j$ is continuous as well. Since M is compact, $c := \inf_{y \in M} B_1(y)$ is positive. Then $f_{\alpha,j} \geqslant c|\alpha| + \inf V_1$, with $|\alpha| = \alpha(1) + \cdots + \alpha(n)$. Thus, for any $\Lambda \in \mathbb{R}$, $f_{\alpha,j} \geqslant \Lambda$ except for a finite number of (α, j). Since $\Sigma = \bigcup_{\alpha,j} f_{\alpha,j}(M)$ and $f_{\alpha,j}(M)$ is compact, this proves that $\Sigma \cap\,]-\infty, \Lambda]$ is a finite union of closed bounded intervals. By the same reason, for any $y \in M$ and $\Lambda \in \mathbb{R}$, $f_{\alpha,j}(y) = \Lambda$ only for a finite number of (α, j). From this we deduce readily that the functions λ_ℓ are continuous.

For any λ, the function $y \to N_y(\lambda)$ takes only integral values. It is measurable because, for any ℓ, $\{y : N_y(\lambda) = \ell\} = \{\lambda_\ell \leqslant \lambda\} \cap \{\lambda_{\ell+1} > \lambda\}$ is the intersection of an open set with a closed set. So $v(\lambda)$ is well-defined. Then v is clearly nondecreasing, and the associated Lebesgue–Stieltjes measure $\nu = dv$ is defined by $\nu([a,b]) = v(b^+) - v(a^-)$.

Now $\lambda \notin \operatorname{supp}(\nu)$ if and only if v is constant on a neighborhood of λ. If $\lambda \notin \Sigma$, then there exists ℓ and $\epsilon > 0$ such that $\lambda_\ell \leqslant \lambda - \epsilon$ and $\lambda + \epsilon \leqslant \lambda_{\ell+1}$; thus, for any y, $N_y(\lambda - \epsilon) = N_y(\lambda + \epsilon)$ and so $v(\lambda - \epsilon) = v(\lambda + \epsilon)$, giving $\lambda \notin \operatorname{supp} \nu$. Conversely, if, for some $\epsilon > 0$, $v(\lambda - \epsilon) = v(\lambda + \epsilon)$, then $N_y(\lambda - \epsilon) = N_y(\lambda + \epsilon)$ for any $y \in A$, where $M \setminus A$ has measure zero. Since A is dense, this implies that $\lambda_\ell \leqslant \lambda - \epsilon$ and $\lambda_{\ell+1} \geqslant \lambda + \epsilon$, with $\ell = N_y(\lambda)$, so $\lambda \notin \Sigma$. ☐

2E. *The restriction $\Box_y$ of $\widetilde{\Box}_y$ to antiholomorphic polynomials.* Since the spaces $\mathcal{L}_\alpha$ in (14) are infinite-dimensional, the eigenvalues of $\widetilde{\Box}_y$ are infinitely degenerate. We can avoid this degeneracy by replacing $\mathcal{P}(T_yM)$ by the subspace $\mathcal{D}(T_yM) \subset \mathcal{P}(T_yM)$ of antiholomorphic polynomials. With the coordinates (z_i)

introduced previously, $\mathcal{D}(T_yM) = \mathbb{C}[\bar{z}_1, \dots, \bar{z}_n]$. This point is central in our treatment since it will lead us to the definition of the fiber bundle F of Theorem 1.1.

First, the annihilation and creation operators $\mathfrak{a}_i$, $\mathfrak{a}_i^\dagger$ preserve the subspace $\mathcal{D}(T_yM)$ in which they act respectively by $\partial_{\bar{z}_i}$ and $\bar{z}_i$. Moreover the joint eigenspaces $\mathfrak{L}_\alpha$ of the $\mathfrak{a}_i^\dagger \mathfrak{a}_i$ satisfy $\mathfrak{L}_\alpha \cap \mathcal{D}(T_xM) = \mathbb{C}\,\bar{z}^\alpha$. So $\widetilde{\square}_y$ preserves $\mathcal{D}(T_yM) \otimes A_y$ and its restriction $\square_y \in \operatorname{End}(\mathcal{D}(T_yM) \otimes A_y)$ has the same spectrum as $\widetilde{\square}_y$. For any eigenvalue λ, the corresponding eigenspaces of $\widetilde{\square}_y$ and $\square_y$ are $\bigoplus \mathfrak{L}_\alpha \otimes \mathbb{C}\zeta_\ell$ and $\bigoplus \mathbb{C}\bar{z}^\alpha \otimes \mathbb{C}\zeta_\ell$ respectively, where in both cases we sum over the (α, ℓ) such that $\sum B_i(y)\big(\alpha(i) + \frac{1}{2}\big) + V_\ell(y) = \lambda$.

For any $p \in \mathbb{N}$, the endomorphism $\square_y$ preserves the subspace $\mathcal{D}_{\leqslant p}(T_yM)$ of $\mathcal{D}(T_yM)$ of polynomials with degree smaller than p. These spaces are obviously finite-dimensional and their union $\mathcal{D}_{\leqslant p}(TM) = \bigcup_y \mathcal{D}_{\leqslant p}(T_yM)$ is a genuine vector bundle over M. Moreover $y \mapsto \square_y|_{\mathcal{D}_{\leqslant p}(T_yM)}$ is a smooth section of $\operatorname{End}(\mathcal{D}_{\leqslant p}(TM))$.

Lemma 2.4. (1) *For any $\Lambda > 0$, there exists $p \in \mathbb{N}$ such that, for any $y \in M$ and $\lambda \in \operatorname{sp}(\square_y) \cap]-\infty, \Lambda]$, the eigenspace $\operatorname{Ker}(\square_y - \lambda)$ is contained in $\mathcal{D}_{\leqslant p}(T_yM) \otimes A_y$.*

(2) *For any compact interval I whose endpoints do not belong to Σ, the spaces*

$$F_y = \bigoplus_{\lambda \in \operatorname{sp}(\square_y) \cap I} \operatorname{Ker}(\lambda - \square_y), \qquad y \in M,$$

are the fibers of a subbundle F of $\mathcal{D}_{\leqslant p}(TM) \otimes A$, with p a sufficiently large integer.

Proof. As in the beginning of the proof of Lemma 2.3, $\sum B_i(y)\big(\alpha(i) + \frac{1}{2}\big) + V_\ell(y) \leqslant \Lambda$ implies $c|\alpha| \leqslant \Lambda - \inf V_1$, with $c = \inf B_1 > 0$, which proves the first assertion with p any integer larger than $c^{-1}(\Lambda - \inf V_1)$.

Since I is bounded, by the first part, $F_y \subset \mathcal{D}_{\leqslant p}(T_yM) \otimes A_y$ for any y, when p is sufficiently large. The projector of $\operatorname{End}(\mathcal{D}_{\leqslant p}(T_yM) \otimes A_y)$ onto F_y is given by the Cauchy integral formula

$$(2\pi i)^{-1} \int_\gamma (\lambda - \square_{y,p})^{-1} \, d\lambda, \tag{20}$$

where $\square_{y,p}$ is the restriction of $\square_y$ to $\mathcal{D}_{\leqslant p}(T_yM) \otimes A_y$ and γ is a loop of $\mathbb{C} \setminus \Sigma_y$ which encircles I. By the assumption that the endpoints of I do not belong to Σ, we can choose γ independent of y. Hence (20) depends smoothly on y and its image F_y as well. $\square$

3. A class of magnetic Laplacians

Consider a compact Riemannian manifold (M, g) equipped with a Hermitian line bundle L with a connection ∇ of curvature ω/i, and a Hermitian vector bundle A with a section $V \in \mathcal{C}^\infty(M, \operatorname{End}(A))$ such that $V(x)$ is Hermitian for any $x \in M$.

The results we will prove later hold for families of differential operators

$$(\Delta_k : \mathcal{C}^\infty(M, L^k \otimes A) \to \mathcal{C}^\infty(M, L^k \otimes A), \ k \in \mathbb{N})$$

having the following local form: for any coordinate chart (U, x_i) of M and trivialization $A|_U \simeq U \times \mathbb{A}$, we have on U by identifying $\mathcal{C}^\infty(U, L^k \otimes A)$ with $\mathcal{C}^\infty(U, L^k \otimes \mathbb{A})$ that

$$\Delta_k = -\frac{1}{2} \sum g^{ij} \nabla_{i,k} \nabla_{j,k} + kV + \sum a_i \nabla_{i,k} + b, \tag{B}$$

where $g^{ij} = g(dx_i, dx_j)$, $\nabla_{i,k}$ is the covariant derivative of L^k with respect to ∂_{x^i}, and a_i, b are in $\mathcal{C}^\infty(U, \operatorname{End}(\mathbb{A}))$ and do not depend on k.

In this section, we will prove that various operators have the form (B): the magnetic Laplacians (1) defined in the Introduction, the holomorphic Laplacians and also some generalized Laplacians associated to semiclassical Dirac operators.

3A. *About Assumption* (B). The proof that some operators satisfy Assumption (B) consists in each case of establishing a Weitzenböck-type formula. Since we don't need to give a geometric definition of the coefficients a_i and b in (B), the computations will be rather simple once we know which terms to neglect. To give a systematic treatment and to have a better understanding of the approximations we do, we will introduce noncommutative symbols for the differential operator algebra generated by the $\nabla_{i,k}$ and $\mathcal{C}^\infty(U, \operatorname{End} \mathbb{A})$. Instead of the full algebra, we will only work with second-order operators. Everything in this section works without assuming that ω is degenerate, the dimension of M could be odd as well, but we will not insist on that.

Let (e_i) be a frame of TM on an open set U of M and $\mathbb{A}$ be a Hermitian vector space. Let $\nabla_{i,k}$ be the covariant derivation of $\mathcal{C}^\infty(U, L^k)$ with respect to e_i. For any $y \in M$, let $\nabla_{y,i}$ be the covariant derivative of $\mathcal{C}^\infty(T_y M)$ for the connection (10) with respect to $e_i(y)$.

We say that a family $P = (P_k : \mathcal{C}^\infty(U, L^k \otimes \mathbb{A}) \to \mathcal{C}^\infty(U, L^k \otimes \mathbb{A}),\ k \in \mathbb{N})$ of differential operators belongs to $\mathcal{G}_2$ if it has the form

$$P_k = \sum_{i \leqslant j} d_{ij} \nabla_{i,k} \nabla_{j,k} + kc + \sum b_i \nabla_{i,k} + a \tag{21}$$

for some coefficients $d_{ij}, c, b_i, a \in \mathcal{C}^\infty(U, \operatorname{End} \mathbb{A})$ independent of k. For such a family, we define

$$\sigma_2(P)(y) = \sum_{i \leqslant j} d_{ij}(y) \nabla_{y,i} \nabla_{y,j} + c(y) : \mathcal{C}^\infty(T_y M, \mathbb{A}) \to \mathcal{C}^\infty(T_y M, \mathbb{A}).$$

Similarly we define the subspaces $\mathcal{G}_0$ and $\mathcal{G}_1$ of $\mathcal{G}_2$ and the corresponding symbols as follows. Assume that P satisfies (21). Then

$$P \in \mathcal{G}_1 \iff d_{ij} = c = 0, \qquad \sigma_1(P)(y) = \sum b_i(y) \nabla_{y,i},$$
$$P \in \mathcal{G}_0 \iff d_{ij} = c = b_i = 0, \qquad \sigma_0(P)(y) = a(y).$$

The basic property we need is the following.

Lemma 3.1. *Let $P \in \mathcal{G}_N$, $P' \in \mathcal{G}_{N'}$, with $N + N' \leqslant 2$. Then*

- *$P^* := (P_k^*)$ belongs to $\mathcal{G}_N$ and $\sigma_N(P^*)(y) = (\sigma_N(P)(y))^*$.*
- *$PP' := (P_k P_k') \in \mathcal{G}_{N+N'}$ and $\sigma_{N+N'}(PP')(y) = \sigma_N(P)(y) \circ \sigma_{N'}(P')(y)$.*

Here the formal adjoints P_k^* are defined with respect to any volume form μ of U which is independent of k, whereas the adjoint of $\sigma_N(P)(y)$ is defined with respect to any constant volume form of T_yM.

Proof. This is easily proved, let us emphasize the main points. First $\nabla_{i,k}^* = -\nabla_{i,k} + \operatorname{div}_\mu(e_i)$, so $(\nabla_{i,k}^*)$ belongs to $\mathcal{G}_1$ and $\sigma_1(\nabla_{i,k}^*)(y) = -\nabla_{y,i} = \nabla_{y,i}^*$. Second $\nabla_{i,k} a = a\nabla_{i,k} + \mathcal{L}_{e_i} a$, so $(\nabla_{i,k} a)$ belongs to $\mathcal{G}_1$ and has symbol $\sigma_1(\nabla_{i,k} a)(y) = a(y)\nabla_{y,i} = \nabla_{y,i} a(y)$. Third

$$\nabla_{i,k}\nabla_{j,k} = \nabla_{j,k}\nabla_{i,k} + \frac{k}{i}\omega(e_i, e_j)$$

so when $i > j$, $(\nabla_{i,k}\nabla_{j,k})$ belongs to $\mathcal{G}_2$ and

$$\sigma_2(\nabla_{i,k}\nabla_{j,k})(y) = \nabla_{y,j}\nabla_{y,i} + \frac{1}{i}\omega(e_i, e_j)(y) = \nabla_{y,i}\nabla_{y,j}. \qquad \square$$

Remark 3.2. Viewing k^{-1} as a semiclassical parameter, we can consider the algebra generated by the $\nabla_{i,k}$ and $\mathcal{C}^\infty(U)$ as a semiclassical algebra. But the order and the symbol that we use here are different from the semiclassical ones. A first reason is that the product of the $\sigma_N(P)$ is not abelian. Let us compare the order of the generators.

If we define the order of $(P_k) \in \mathcal{G}_N$ as N, the covariant derivatives $\nabla_{i,k}$ have order 1, multiplication by k has order 2 and multiplication by a function f has order 0. In particular k has twice the order of $\nabla_{i,k}$.

In contrast, let us trivialize L over an open set U so that $\mathcal{C}^\infty(U, L^k) \simeq \mathcal{C}^\infty(U)$ and $\nabla^{L^k} = d + k\alpha/i$, with $\alpha \in \Omega^1(U, \mathbb{R})$ the connection 1-form. We introduce the semiclassical parameter $\hbar = k^{-1}$. Then the operators $(ik)^{-1}\nabla_{i,k} = \hbar\partial_{e_i}/i - \alpha(e_i)$ and multiplication by f are semiclassical differential operators of order 0. So $\nabla_{i,k}$ and k have the same order as semiclassical differential operators. $\square$

Notice that for any vector field X of U, $(\nabla_X^{L^k})$ belongs to $\mathcal{G}_1$ with symbol at y given by the covariant derivative of $\mathcal{C}^\infty(T_yM)$ with respect to $X(y)$. Using this and Lemma 3.1, we deduce that $\mathcal{G}_N$ and σ_N do not depend on the choice of the frame (e_i). Let us make the dependence with respect to $(U, \mathbb{A})$ explicit, so we write $\mathcal{G}_N(U, \mathbb{A})$ instead of $\mathcal{G}_N$.

Using again Lemma 3.1, we see that if $u \in \mathcal{C}^\infty(U, \operatorname{End}\mathbb{A})$ is invertible at each point, then, for any $P \in \mathcal{G}_N(U, \mathbb{A})$, uPu^{-1} belongs to $\mathcal{G}_N(U, \mathbb{A})$ and $\sigma_N(uPu^{-1})(y) = u(y)\sigma_N(P)(y)u(y)^{-1}$. So we can define $\mathcal{G}_N(A)$ as the space of differential operator families (P_k) such that for any k, P_k acts on $\mathcal{C}^\infty(M, L^k \otimes A)$ and for any trivialization $A|_U \simeq U \times \mathbb{A}$, the local representative of (P_k) belongs to $\mathcal{G}_N(U, \mathbb{A})$. The corresponding symbol $\sigma_N(P)(y)$ is invariantly defined as a differential operator of $\mathcal{C}^\infty(T_yM, A_y)$.

It is also useful to consider differential operators from $\mathcal{C}^\infty(M, L^k \otimes A)$ to $\mathcal{C}^\infty(M, L^k \otimes B)$, where B is a second auxiliary Hermitian vector bundle. To handle these operators, we define the subspace $\mathcal{G}_N(A, B)$ of $\mathcal{G}_N(A \oplus B)$ consisting of the (P_k) such that, for any k, $\operatorname{Im} P_k \subset \mathcal{C}^\infty(M, L^k \otimes B) \subset \operatorname{Ker} P_k$. The symbol at y of an element of $\mathcal{G}_N(A, B)$ is a differential operator $\mathcal{C}^\infty(T_yM, A_y) \to \mathcal{C}^\infty(T_yM, B_y)$.

Observe now that assumption (B) has the reformulation

$$(\Delta_k) \in \mathcal{G}_2(A) \qquad \text{and} \qquad \sigma_2(\Delta_k)(y) = \Delta_y^{\text{scal}} + V(y) \quad \text{for all } y \in M. \tag{B'}$$

3B. ***Magnetic Laplacian.*** The simplest example of an operator satisfying condition (B) is the magnetic Laplacian defined in Section 1A. So besides the line bundle L with its connection, the Riemannian metric g and the section $V \in \mathcal{C}^\infty(M, \operatorname{End} A)$, we introduce a connection on A not necessarily preserving the Hermitian structure and a volume form μ on M. Set

$$\Delta_k = \tfrac{1}{2}(\nabla^{L^k\otimes A})^*\nabla^{L^k\otimes A} + kV : \mathcal{C}^\infty(M, L^k \otimes A) \to \mathcal{C}^\infty(M, L^k \otimes A),$$

where the formal adjoint of $\nabla^{L^k\otimes A}$ is defined from the scalar product obtained by integrating pointwise scalar products against μ.

Proposition 3.3. (Δ_k) *satisfies assumption* (B).

Proof. This follows from Lemma 3.1 and the fact that $(\nabla^{L^k\otimes A})$ belongs to $\mathcal{G}_1(A, A \otimes T^*M)$ with symbol at y equal to the covariant derivative ∇ of $\mathcal{C}^\infty(T_yM)$ tensored with the identity of A_y. To see this, write locally

$$\nabla^{L^k\otimes A} = \sum_i \epsilon(e_i^*)\nabla^{L^k\otimes A}_{e_i} = \sum_i \epsilon(e_i^*)(\nabla_{i,k} + \gamma_i),$$

where (e_i^*) is the dual frame of (e_i), $\epsilon(e_i^*)$ is the exterior product by e_i^* and the $\gamma_i \in \mathcal{C}^\infty(U, \operatorname{End}\mathbb{A})$ are the coefficients of the connection 1-form of ∇^A in a trivialization $A|_U \simeq U \times \mathbb{A}$. □

3C. *Holomorphic Laplacian.* Assume that M is a complex manifold and L, A are holomorphic Hermitian bundles, L being positive in the sense that the curvature of its Chern connection if ω/i, where $\omega \in \Omega^{1,1}(M)$, is a Kähler form. Equip $T^{0,1}M$ with the metric $|u|^2 = \omega(\bar{u}, u)/i$, $u \in T^{0,1}M$, and let $\mu = \omega^n/n!$ be the Liouville volume form. Define the holomorphic Laplacian

$$\Delta''_k = (\bar\partial_{L^k\otimes A})^*\bar\partial_{L^k\otimes A} : \mathcal{C}^\infty(M, L^k \otimes A) \to \mathcal{C}^\infty(M, L^k \otimes A).$$

By Hodge theory, $\operatorname{Ker}\Delta''_k$ is isomorphic with the Dolbeault cohomology space $H^0(L^k \otimes A)$. When k is sufficiently large, the dimension of $H^0(L^k \otimes A)$ is the Riemann–Roch number $\operatorname{RR}(L^k \otimes A)$ defined as the evaluation of the product of the Chern character of $L^k \otimes A$ by the Todd class of M. Additionally, Δ''_k satisfies assumption (B), which leads to the following description of its spectrum.

Theorem 3.4. *For any* $\Lambda > 0$, *there exists* $C > 0$ *such that* $\operatorname{sp}(k^{-1}\Delta''_k) \cap [0, \Lambda]$ *is contained in* $\mathbb{N} + Ck^{-1}[-1, 1]$. *For any* $m \in \mathbb{N}$,

$$\sharp\operatorname{sp}(k^{-1}\Delta''_k) \cap \left[m - \tfrac{1}{2}, m + \tfrac{1}{2}\right] = \operatorname{RR}(L^k \otimes A \otimes \operatorname{Sym}^m(T^{1,0}M)),$$

when k is sufficiently large.

Notice that the first eigenvalue cluster is degenerate in the sense that $\operatorname{sp}(\Delta''_k) \cap \left[0, \tfrac{1}{2}\right] \subset \{0\}$ when k is sufficiently large.

Proof. Now $\bar\partial_{L^k\otimes A}$ belongs to $\mathcal{G}_1(A, A \otimes (T^*M)^{0,1})$ and its symbol at y is the $(0, 1)$-component of the connection ∇ defined in (10). Using the same notation (u_i) and (z_i) as in Section 2B, $\nabla^{0,1} = \sum \epsilon(d\bar{z}_i) \otimes \nabla_{\bar{u}_i}$. Since the adjoint of $\epsilon(d\bar{z}_i)$ is the interior product by $\bar{u}_i$, we have $\epsilon(d\bar{z}_i)^*\epsilon(d\bar{z}_i) = 1$ so that

$$\sigma_2(\Delta''_k)(y) = -\sum_i \nabla_{u_i}\nabla_{\bar{u}_i}.$$

Thus Δ''_k satisfies assumption (B) with $V(y) = -\tfrac{1}{2}n$ and $\Sigma_y = \mathbb{N}$. The result follows now from Corollary 7.2 with k^{-1} instead of $k^{-1/2}$ by Remark 7.3. □

Similarly we can consider the Laplacian acting on $(0, q)$-forms and prove the same result where $\mathbb{N}$ is replaced by $q+\mathbb{N}$ and the number of eigenvalues in $q+m+\left[-\frac{1}{2}, \frac{1}{2}\right]$ is the Riemann–Roch number of $L^k \otimes A \otimes \wedge^{0,q}(T^*M) \otimes \mathrm{Sym}^m(T^{1,0}M)$.

We can also generalize this to the case where the complex structure is not integrable. So assume that (M, ω) is a symplectic manifold with a compatible almost-complex structure j, that $L \to M$ is a Hermitian line bundle with a connection of curvature ω/i and that A a Hermitian vector bundle with a connection. Then Theorem 3.4 holds with the operator

$$\Delta_k'' = ((\nabla^{L^k\otimes A})^{(0,1)})^*(\nabla^{L^k\otimes A})^{(0,1)} : \mathcal{C}^\infty(L^k \otimes A) \to \mathcal{C}^\infty(L^k \otimes A)$$

and the proof is exactly the same. However, it is no longer true that the first eigenvalue cluster is nondegenerate. Using Dirac operators, one can generalize the previous result and still have the degeneracy of the first cluster, as explained in the next section.

3D. ***Semiclassical Dirac operators.*** In this section, (M, ω, j) is a symplectic manifold with an almost complex structure, (L, ∇) is a Hermitian line bundle on M with a connection having curvature ω/i and A is an auxiliary Hermitian vector bundle.

Let $S = \wedge^{0,\bullet}T^*M$ be the spinor bundle and S^+, S^- be the subbundles of even and odd forms respectively. For any $y \in M$, extend the covariant derivative ∇ defined in (10) to $\Omega^\bullet(T_yM)$ in the usual way and denote by $\nabla^{0,1}$ the restriction of its (0, 1)-component to $\Omega^{0,\bullet}(T_yM) = \mathcal{C}^\infty(T_yM \otimes S_y)$.

Definition 3.5. A semiclassical Dirac operator is a family $(D_k) \in \mathcal{G}_1(A \otimes S)$ with symbol

$$\sigma_1(D_k)(y) = \nabla^{0,1} + (\nabla^{0,1})^* : \Omega^{0,\bullet}(T_yM) \to \Omega^{0,\bullet}(T_yM) \quad \text{for all } y \in M$$

such that for any k, D_k is formally self-adjoint and odd.

Such an operator can be constructed as follows: we introduce a connection on S preserving S^+ and S^- and a connection on A and set

$$D_k = \sum_i \epsilon(\bar{\theta}_i)\nabla^{L^k\otimes A\otimes S}_{\bar{u}_i} + (\epsilon(\bar{\theta}_i)\nabla^{L^k\otimes A\otimes S}_{\bar{u}_i})^*,$$

where (u_i) is any orthonormal frame of $T^{1,0}M$, (θ_i) is the dual frame of $(T^*M)^{1,0}$ and the exterior product $\epsilon(\bar{\theta}_i)$ acts on S. Another example is provided by spin-c Dirac operators; see [Duistermaat 1996: Ma and Marinescu 2007, Section 1.3]. Observe as well that the semiclassical Dirac operator is unique up to a self-adjoint odd operator of $\mathcal{G}_0(A \otimes S)$. We denote by

$$D_k^\pm : \mathcal{C}^\infty(L^k \otimes A \otimes S^\pm) \to \mathcal{C}^\infty(L^k \otimes A \otimes S^\mp)$$

the restrictions of D_k and observe that D_k^- is the formal adjoint of D_k^+.

Theorem 3.6. *Let (D_k) be a semiclassical Dirac operator. Then the operator $\Delta_k = D_k^- D_k^+$ satisfies*:

(1) *For any $\Lambda > 0$, there exists $C > 0$ such that* $\mathrm{sp}(k^{-1}\Delta_k) \cap [0, \Lambda]$ *is contained in* $\mathbb{N} + Ck^{-1/2}[-1, 1]$.

(2) $\mathrm{sp}(k^{-1}\Delta_k) \cap \left[0, \frac{1}{2}\right] \subset \{0\}$ *and* $\mathrm{Ker}\,\Delta_k$ *has dimension* $\mathrm{RR}(L^k \otimes A)$ *when k is sufficiently large.*

(3) *For any $m \in \mathbb{N}$, when k is sufficiently large,*

$$\sharp\, \mathrm{sp}(k^{-1}\Delta_k) \cap \left[m - \tfrac{1}{2}, m + \tfrac{1}{2}\right] = \mathrm{RR}(L^k \otimes A_m),$$

where $A_m = \bigoplus_{(\ell,p)} A \otimes \mathrm{Sym}^\ell(T^{1,0}M) \otimes \wedge^{2p}(T^{1,0}M)$, the sum being over the $(\ell, p) \in \mathbb{N}^2$ such that $\ell + 2p = m$ and $p \leqslant n$.

Proof. As in Section 2B, let (u_i) be an orthonormal basis of $T_y^{1,0}M$ and (z_i) be the associated linear complex coordinates. We have $\nabla^*_{\bar{u}_i} = -\nabla_{u_i}$, $\epsilon(d\bar{z}_i)^* = \iota(\bar{u}_i)$ so that

$$\sigma_1(D_k)(y) = \sum \epsilon(d\bar{z}_i)\nabla_{\bar{u}_i} - \iota(\bar{u}_i) \otimes \nabla_{u_i}.$$

A standard computation using that ∇_{u_i}, $\nabla_{\bar{u}_i}$ commute with $\epsilon(d\bar{z}_j)$, $\iota(\bar{u}_j)$ and $[\nabla_{u_i}, \nabla_{u_j}] = [\nabla_{\bar{u}_i}, \nabla_{\bar{u}_j}] = 0$, $[\nabla_{u_i}, \nabla_{\bar{u}_j}] = \delta_{ij}$ leads to

$$\sigma_2(D_k^2)(y) = \sum(-\nabla_{u_i}\nabla_{\bar{u}_i} + \epsilon(d\bar{z}_i)\iota(\bar{u}_i)) = \Delta_y^{\mathrm{scal}} - \frac{n}{2} + \mathrm{N}_y,$$

where N_y is the number operator of S_y, that is, $\mathrm{N}_y\alpha = (\deg\alpha)\alpha$. Restricting to S^+, we deduce that (Δ_k) satisfies assumption (B) with $V(y) = -\frac{1}{2}n + \mathrm{N}_y$. So $\Sigma_y = \mathbb{N}$ and the first assertion of the theorem follows from the second part of Corollary 7.2.

In the same way, $(D_k^+ D_k^-)$ has the form (B) with $V(y) = -\frac{1}{2}n + \mathrm{N}_y$ as well, but the number operator takes odd value on S^-. Thus

$$\mathrm{sp}(k^{-1}D_k^+D_k^-) \subset [1 - Ck^{-1/2}, \infty[$$

for some positive C. Since, for any $\lambda \neq 0$, D_k^+ is an isomorphism between $\mathrm{Ker}(D_k^- D_k^+ - \lambda)$ and $\mathrm{Ker}(D_k^+ D_k^- - \lambda)$, this proves that $\mathrm{sp}(k^{-1}\Delta_k) \cap \left]0, \frac{1}{2}\right]$ is empty when k is sufficiently large and the first part of the second assertion follows.

The second part of the second assertion and the third assertion follow from Corollary 7.2. Indeed, for $V(y) = -\frac{1}{2}n + \mathrm{N}_y$ acting on $A_y \otimes S_y^+$, the bundle F with fiber $F_y = \ker(\square_y - m)$ is isomorphic to A_m as a complex vector bundle. □

4. Spectral estimates

Let $(\Delta_k : \mathcal{C}^\infty(M, L^k \otimes A) \to \mathcal{C}^\infty(M, L^k \otimes A))$ be a differential operator family satisfying (B). We assume that the curvature ω/i is nondegenerate. We assume as well that Δ_k is formally self-adjoint, where the scalar product of $\mathcal{C}^\infty(M, L^k \otimes A)$ is defined from the measure $\mu = \omega^n/n!$.

For any $y \in M$, by the Darboux lemma, there exists a coordinate system (U, x_i) of M centered at y such that ω is constant in these coordinates, that is, $\omega = \frac{1}{2}\sum \omega_{ij}\, dx_i \wedge dx_j$, with $\omega_{ij} = \omega(\partial_{x_i}, \partial_{x_j})$ constant functions. We identify U with a neighborhood of the origin of T_yM through these coordinates. We assume that this neighborhood is convex.

We introduce a unitary section F_y of $L \to U$ such that, for any $\xi \in U$, F_y is flat on the segment $[0, \xi]$. Then

$$\nabla F_y = \frac{1}{2i}\sum_{i,j} \omega_{i,j} x_i\, dx_j \otimes F_y. \tag{22}$$

Indeed $\nabla F_y = (\alpha/i) \otimes F_y$ with α satisfying $d\alpha = \omega$ and $\int_{[0,\xi]} \alpha = 0$ for any $\xi \in U$. We easily see that these conditions determine a unique α and that they are satisfied by $\alpha = \frac{1}{2} \sum \omega_{ij} x_i \, dx_j$.

So trivializing L on U by using this frame F_y, $L|_U \simeq U \times \mathbb{C}$ and ∇ becomes the linear connection defined in (10). Moreover trivializing L^k on U with F_y^k, the covariant derivative $\nabla_{j,k}$ of L^k with respect to ∂_{x_j} is

$$\nabla_{j,k} = \partial_{x_j} + \frac{ik}{2} \sum_i \omega_{i,j} x_i. \tag{23}$$

Now we introduce the Laplacian $\Delta_{y,k}$ of $\mathcal{C}^\infty(T_yM, A_y)$ associated to this covariant derivative, the constant metric g_y of T_yM and the constant potential $kV(y)$, that is,

$$\Delta_{y,k} = -\frac{1}{2} \sum g_y^{ij} \nabla_{i,k} \nabla_{j,k} + kV(y). \tag{24}$$

For $k = 1$, we recover the Laplacian Δ_y defined in (17).

We introduce a trivialization of the auxiliary vector bundle $A|_U = U \times A_y$ so that $\mathcal{C}^\infty(U, L^k \otimes A) \simeq \mathcal{C}^\infty(U, A_y)$. Then assumption (B) tells us that

$$\Delta_k - \Delta_{y,k} = \sum_{i,j} a_{ij} \nabla_{i,k} \nabla_{j,k} + \sum_i a_i \nabla_{i,k} + kc + b, \tag{25}$$

where $a_{ij} = -\frac{1}{2} g^{ij} + \frac{1}{2} g_y^{ij}$ and $c = V - V(y)$ are both equal to zero at the origin y. The identity (25) will be used later to compare the spectra of Δ_k and $\Delta_{y,k}$; see the proofs of Proposition 4.1 and Lemma 4.4.

Before that, let us compute the spectrum of $\Delta_{y,k}$. The Laplacian $k^{-1}\Delta_{y,k}$ is unitarily conjugated to Δ_y. Indeed, we introduce the rescaling map

$$S_k : \mathcal{C}^\infty(T_yM, A_y) \to \mathcal{C}^\infty(T_yM, A_y), \quad S_k(f)(x) = k^{n/2} f(k^{1/2}x). \tag{26}$$

Then, from the formula (23), we easily check that

$$k^{1/2} S_k \nabla_i = \nabla_{i,k} S_k, \quad k^{-1} \Delta_{y,k} S_k = S_k \Delta_y. \tag{27}$$

Consequently, the spectrum of $k^{-1}\Delta_{y,k}$ is Σ_y for any k.

4A. ***Peaked sections.*** As above, we identify a neighborhood U of y with a neighborhood of the origin in T_yM through Darboux coordinates, we introduce the frame F_y of L on U with covariant derivative given by (22), and we work with a trivialization $A|_U \simeq U \times A_y$. Choose a function $\psi \in \mathcal{C}_0^\infty(U, \mathbb{R})$ such that $\psi = 1$ on a neighborhood of y. Then to any polynomial $f \in \mathcal{P}(T_yM) \otimes A_y$, we associate the smooth section $\Phi_k(f)$ of $L^k \otimes A$ defined on U by

$$\Phi_k(f)(\xi) = k^{n/2} F_y^k(\xi) e^{-k|\xi|_y^2/4} f(k^{1/2}\xi) \psi(\xi) \tag{28}$$

and equal to 0 on $M \setminus U$.

Proposition 4.1. *We have*

(1) $\|\Phi_k(f)\|^2 = \int_{T_yM} e^{-|\xi|_y^2/2} |f(\xi)|^2 \, d\mu_y(\xi) + \mathcal{O}(e^{-C/k})$, *with* $\mu_y = \omega_y^n/n!$ *the Liouville form of* T_yM,

(2) $k^{-1}\Delta_k \Phi_k(f) = \Phi_k(g) + \mathcal{O}(k^{-1/2})$, *with* $g = \widetilde{\square}_y(f)$.

The peaked sections of [Charles 2024] are defined without using the Darboux coordinates, and for this reason the $\mathcal{O}(e^{-k/C})$ in the norm estimate is replaced by a $\mathcal{O}(k^{-1/2})$. Actually, the Darboux coordinates are not essential in this subsection, they only simplify slightly some estimates, whereas in Sections 4B and 4C it will be necessary to use them.

Proof. Since $\Phi_k(f)$ is supported in U, we can view it as a function of T_yM, so

$$\Phi_k(f) = \psi S_k(sf),$$

where $s(\xi) = e^{-|\xi|_y^2/4}$ as in Section 2B and S_k is the rescaling map (26). Since we work with Darboux coordinates, the volume form μ of M coincide on U with μ_y. So

$$\|\Phi_k(f)\|^2 = \int_{T_yM} |S_k(sf)|^2 \psi^2 \, d\mu_y.$$

We will need several times to estimate an integral having the form

$$I_k(\widetilde{\psi}) = \int_{T_yM} |S_k(sf)|^2 \widetilde{\psi} \, d\mu_y = k^n \int_{T_yM} e^{-k/2|\xi|_y^2} |f(k^{1/2}\xi)|^2 \, \tilde{\psi}(\xi) \, d\mu_y(\xi),$$

with $\widetilde{\psi} \in \mathcal{C}^\infty(T_yM)$ satisfying $\widetilde{\psi}(\xi) = \mathcal{O}(|\xi|^m)$ on T_yM for $m \geqslant 0$. We claim that $I_k(\widetilde{\psi}) = \mathcal{O}(k^{-m/2})$ and in the case where $\widetilde{\psi} = 0$ on a neighborhood of the origin, $I_k(\widetilde{\psi}) = \mathcal{O}(e^{-k/C})$ for some $C > 0$.

The first claim follows from the change of variable $\sqrt{k}\xi = \xi'$. For the second one, we use that $e^{-k|\xi|_y^2/2}\widetilde{\psi}(\xi) = \mathcal{O}(e^{-k/C}|\xi|^m e^{-k|\xi|_y^2/4})$ and do the same change of variable.

The first assertion of the proposition is an immediate consequence of the second claim with $\widetilde{\psi} = 1 - \psi^2$. For the second assertion, we start from (25) and using that $[\nabla_{i,k}, \psi] = \partial_{x_i}\psi$ repetitively, we obtain

$$\Delta_k \psi = \psi\big(\Delta_{y,k} + a_{ij}\nabla_i^k \nabla_j^k + \tilde{b}_i \nabla_i + kc + \tilde{c}\big), \tag{29}$$

where a_{ij}, c are the same functions as in (25), and $\tilde{b}_i$ and $\tilde{c}$ do not depend on k.

Now, by (29), $\Delta_k(\psi S_k(sf))$ is a sum of five terms, the first one being

$$\psi \Delta_{y,k} S_k(fs) = k\psi S_k(\Delta_y(sf)) = k\Phi_k(g), \quad \text{with } sg = \Delta_y(sf),$$

by (27). We will prove that the four other terms are in $\mathcal{O}(k^{1/2})$, which will conclude the proof.

Each time, we will apply the preliminary integral estimate with the convenient function $\widetilde{\psi}$. First since $|\psi\tilde{c}|$ is bounded, $\psi\tilde{c}\, S_k(sf) = \mathcal{O}(1)$. Second, c vanishes at the origin, $|\psi(\xi)c(\xi)|^2 = \mathcal{O}(|\xi|^2)$ so that $\psi c\, S_k(sf) = \mathcal{O}(k^{-1/2})$. Third, by (27),

$$\nabla_{i,k} S_k(sf) = k^{1/2} S_k(\nabla_i(sf)) = k^{1/2} S_k(sf_i),$$

with a new polynomial f_i, and since $\psi\tilde{b}_i$ is bounded, we get

$$\psi\tilde{b}_i \, \nabla_{i,k} S_k(sf) = \mathcal{O}(k^{1/2}).$$

Similarly, $\nabla_{i,k}\nabla_{j,k} S_k(sf) = kS_k(sf_{ij})$ with new polynomials f_{ij}, and a_{ij} vanishing at the origin, so we obtain

$$\psi a_{ij} \nabla_{i,k}\nabla_{j,k} S_k(sf) = \mathcal{O}(k^{1/2})$$

as was to be proved. ☐

Theorem 4.2. *Let* (Δ_k) *be a family of formally self-adjoint differential operators of the form* (B). *Then, if* $\lambda \in \Sigma_y$, *there exists* $C(y, \lambda)$ *such that*

$$\operatorname{dist}(\lambda, \operatorname{sp}(k^{-1}\Delta_k)) \leqslant C(y, \lambda) k^{-1/2} \quad \textit{for all } k.$$

Furthermore, for any $\Lambda > 0$, $C(y, \lambda)$ *stays bounded when* (y, λ) *runs over* $M \times]-\infty, \Lambda]$.

This proves the first assertion of Theorem 1.2.

Proof. By Section 2B, any eigenvalue λ of $\widetilde{\Box}_y$ has an eigenfunction $f \in \mathcal{P}(T_yM) \otimes A_y$. Normalizing conveniently f, we get by Proposition 4.1,

$$\|\Phi_k(f)\| = 1 + \mathcal{O}(e^{-k/C}), \quad k^{-1}\Delta_k \Phi_k(f) = \lambda \Phi_k(f) + \mathcal{O}, (k^{-1/2}),$$

which proves that $\operatorname{dist}(\lambda, \operatorname{sp}(k^{-1}\Delta_k)) = \mathcal{O}(k^{-1/2})$. To get a uniform $\mathcal{O}$ when $\lambda \leqslant \Lambda$, remember that by the first assertion of Lemma 2.4, we can choose $f \in \mathcal{D}_{\leqslant p}(T_yM) \otimes A_y$, where p is sufficiently large and independent of $y \in M$. Furthermore, for any $p \in \mathbb{N}$, the $\mathcal{O}$'s in Proposition 4.1 are uniform with respect to f describing the compact set $\{f \in \mathcal{D}_{\leqslant p}(TM) \otimes A : \|f\| = 1\}$. Here we can use any metric of $\mathcal{D}_{\leqslant p}(TM)$, the natural one in our situation being $\|f\|^2 = \int_{T_yM} e^{-|\xi|_y^2/2} |f(\xi)|^2 \, d\mu_y(\xi)$ for $f \in \mathcal{D}(T_yM)$. □

4B. ***A local approximate resolvent.*** Recall that $k^{-1}\Delta_{y,k} = S_k \Delta_y S_k^*$ so that $k^{-1}\Delta_{y,k}$ has the same spectrum Σ_y as Δ_y. For any $\lambda \in \mathbb{C} \setminus \Sigma_y$, we denote by

$$R_{y,k}(\lambda) := (\lambda - k^{-1}\Delta_{y,k})^{-1} : L^2(T_yM) \otimes A_y \to L^2(T_yM) \otimes A_y$$

the resolvent. We will need the following basic elliptic estimates.

Proposition 4.3. *For any* $\lambda \in \mathbb{C} \setminus \Sigma_y$, *the resolvent* $R_{y,k}(\lambda)$ *sends* $\mathcal{C}_0^\infty$ *to* $\mathcal{C}^\infty$ *and satisfies*

$$\|k^{-1/2}\nabla_{i,k} R_{y,k}(\lambda)\| \leqslant C_\Lambda d^{-1}, \quad \|k^{-1}\nabla_{i,k}\nabla_{j,k} R_{y,k}(\lambda)\| \leqslant C_\Lambda d^{-1} \tag{30}$$

if $|\lambda| \leqslant \Lambda$ *with* $d = \operatorname{dist}(\lambda, \Sigma_y)$ *and the constant* C_Λ *independent of* k.

Here and in the sequel, the norm $\|\cdot\|$ is the operator norm associated to the L^2-norm.

Proof. The first assertion follows from elliptic regularity: for any distribution ψ of T_yM, if $(\lambda - k^{-1}\Delta_{y,k})\psi$ is smooth then ψ is smooth.

Since $R_{y,k}(\lambda) = S_k R_{y,1}(\lambda) S_k^*$ and $k^{-1/2}\nabla_{i,k} = S_k \nabla_i S_k^*$, it suffices to prove the inequalities (30) for $k = 1$. We can assume that the frame $(\partial/\partial x_i)$ is g-orthonormal at y, so $g_y^{ij} = \delta_{ij}$, so $\Delta_y = -\frac{1}{2}\sum_i \nabla_i^2 + V(y)$. Since $\langle \Delta_y u, u\rangle = \frac{1}{2}\sum \|\nabla_i u\|^2 + \langle V(y)u, u\rangle$, we have by the Cauchy–Schwarz inequality

$$\|\nabla_i u\|^2 \leqslant C\|u\|(\|\Delta_y u\| + \|u\|). \tag{31}$$

Since $[\nabla_i, \nabla_j] = \omega_{i,j}/i$, we have

$$\begin{aligned}\|\nabla_i \nabla_j u\|^2 &= \langle \nabla_j \nabla_i^2 \nabla_j u, u\rangle = \langle \nabla_i^2 \nabla_j^2 u, u\rangle + \frac{2}{i}\omega_{ji}\langle \nabla_i \nabla_j u, u\rangle \\ &= \langle \nabla_j^2 u, \nabla_i^2 u\rangle + \frac{2}{i}\omega_{ij}\langle \nabla_j u, \nabla_i u\rangle.\end{aligned} \tag{32}$$

Moreover,

$$\frac{1}{4}\sum_{i,j}\langle \nabla_i^2 u, \nabla_j^2 u\rangle = \|\Delta_y u - V(y)u\|^2 \leqslant C(\|\Delta_y u\| + \|u\|)^2. \tag{33}$$

Estimating the first term of (32) with (33) and the second one with (31), it comes that

$$\|\nabla_i \nabla_j u\|^2 \leqslant C(\|\Delta_y u\| + \|u\|)^2. \tag{34}$$

To conclude the proof, we use that the norm of $R_y(\lambda) = (\lambda - \Delta_y)^{-1}$ is d^{-1} and $\Delta_y R_y(\lambda) = \lambda R_y(\lambda) - \mathrm{id}$ so when $|\lambda| \leqslant \Lambda$,

$$\|\Delta_y R_y(\lambda)\| \leqslant \Lambda d^{-1} + 1 \leqslant C_\Lambda d^{-1}$$

because d stays bounded when λ is. Hence it follows from (31) and (34) that

$$\|\nabla_i R_y(\lambda) v\| \leqslant C_\Lambda d^{-1}\|v\|, \quad \|\nabla_j \nabla_i R_y(\lambda) v\| \leqslant C_\Lambda d^{-1}\|v\|,$$

which corresponds to (30) for $k = 1$. □

Recall that we identified a neighborhood of $y \in M$ with a neighborhood U of the origin of $T_y M$ through Darboux coordinates. We introduce a smooth function $\chi : T_y M \to [0, 1]$ such that $\chi(\xi) = 1$ when $|\xi| \leqslant 1$ and $\chi(\xi) = 0$ when $|\xi| \geqslant 2$. Define $\chi_r(\xi) := \chi(\xi/r)$. In the sequel we assume that r is sufficiently small so that χ_r is supported in U. Then for any differential operator P acting on $\mathcal{C}^\infty(U)$, $\chi_r P$ and $P\chi_r$ are differential operators with coefficients supported in U, so we can view them as operators acting on $\mathcal{C}^\infty(T_y M)$.

In the following lemma, we prove that the resolvent $R_{y,k}(\lambda)$ of $k^{-1}\Delta_{y,k}$ is a local right-inverse of $(\lambda - k^{-1}\Delta_k)$ up to some error.

Lemma 4.4. *For any $\lambda \in \mathbb{C} \setminus \Sigma_y$ such that $|\lambda| \leqslant \Lambda$, we have with $d = d(\lambda, \Sigma_y)$*

$$\|(\lambda - k^{-1}\Delta_k)\chi_r R_{y,k}(\lambda) - \chi_r\| \leqslant C_\Lambda F(r, k^{-1}, d), \tag{35}$$

where $F(r, \hbar, d) = (r + \hbar^{1/2} + \hbar r^{-2} + \hbar^{1/2} r^{-1})d^{-1}$.

Proof. We compute

$$\begin{aligned}(\lambda - k^{-1}\Delta_k)\chi_r R_{y,k}(\lambda) - \chi_r &= -k^{-1}[\Delta_k, \chi_r]R_{y,k}(\lambda) + \chi_r(\lambda - k^{-1}\Delta_k)R_{y,k}(\lambda) - \chi_r \\ &= -k^{-1}[\Delta_k, \chi_r]R_{y,k}(\lambda) + \chi_r k^{-1}(\Delta_{y,k} - \Delta_k)R_{y,k}(\lambda). \end{aligned}\tag{36}$$

To estimate the first term, we start from assumption (B), which gives us

$$\begin{aligned}[\Delta_k, \chi_r] &= -\tfrac{1}{2} g^{ij}[\nabla_{i,k}\nabla_{j,k}, \chi_r] + a_j[\nabla_{j,k}, \chi_r] \\ &= -\tfrac{1}{2} g^{ij}\big((\partial_j\partial_i\chi_r) + (\partial_i\chi_r)\nabla_{j,k} + (\partial_j\chi_r)\nabla_{i,k}\big) + a_j(\partial_j\chi_r).\end{aligned}$$

Applying the estimates (30), we deduce that

$$\begin{aligned}\|k^{-1}[\Delta_k, \chi_r]R_{y,k}(\lambda)\| &\leqslant C(k^{-1}r^{-2}d^{-1} + k^{-1/2}r^{-1}d^{-1} + k^{-1}r^{-1}d^{-1}) \\ &\leqslant C(k^{-1}r^{-2} + k^{-1/2}r^{-1})d^{-1}.\end{aligned}$$

To estimate the second term of (36), we use the expression (25) and the fact that the a_{ij} and c vanish at the origin so that $|\chi_r a_{ij}| \leqslant Cr$ and $|\chi_r c| \leqslant Cr$. By (30) it follows that

$$\|\chi_r k^{-1}(\Delta_k - \Delta_{y,k})R_{y,k}(\lambda)\| \leqslant C(r + k^{-1/2} + k^{-1})d^{-1} \leqslant C(r + k^{-1/2})d^{-1},$$

which concludes the proof. □

4C. ***Globalization.*** The local approximation of the resolvent at y in the previous section was based on a choice of Darboux coordinates. To globalize this, we will first choose such coordinate charts depending smoothly on y. All the constructions to come depend on an auxiliary Riemannian metric. For any $y \in M$ and $r > 0$ let $B_y(r)$ be the open ball $\{\xi \in T_yM : \|\xi\| < r\}$.

Lemma 4.5. *There exist $r_0 > 0$ and a smooth family of embeddings $(\Psi_y : B_y(r_0) \to M,\ y \in M)$ such that, for any $y \in M$, $\Psi_y(0) = y$, $T_0\Psi_y = \mathrm{id}_{T_yM}$ and $\Psi_y^*\omega$ is constant on $B_y(r_0)$.*

The family $(\Psi_y,\ y \in M)$ is smooth in the sense that the map $\Psi(\xi) = \psi_y(\xi),\ \xi \in B_y(r_0)$, from the open set $\bigcup_{y\in M} B_y(r_0)$ of TM to M, is smooth.

Lemma 4.6. *There exist $N \in \mathbb{N}$, $r_1 > 0$ and for any $0 < r < r_1$ a finite subset $I(r)$ of M such that the open sets $\Psi_y(B_y(r)),\ y \in I(r)$, form a covering of M with multiplicity bounded by N.*

The multiplicity of a covering $\bigcup_{i\in I} U_i \supset M$ is the maximal number of U_i with nonempty intersection. The proofs of Lemmas 4.5 and 4.6 are standard and postponed to Section 8.

Recall that $\Sigma = \bigcup \Sigma_y$. So, for any $\lambda \in \mathbb{C} \setminus \Sigma$, the resolvents $R_{y,k}(\lambda) : \mathcal{C}_0^\infty(T_yM, A_y) \to \mathcal{C}^\infty(T_yM, A_y)$ are well-defined. As previously, we introduce a section F_y of $L \to \Psi_y(B_y(r))$ satisfying (22) and a trivialization of A on $\Psi_y(B_y(r))$, from which we identify $\mathcal{C}^\infty(\Psi_y(B_y(r)), L^k \otimes A) \simeq \mathcal{C}^\infty(B_y(r), A_y)$. Let

$$\widetilde{R}_{y,k}(\lambda) : \mathcal{C}_0^\infty(\Psi_y(B_y(r)), L^k \otimes A) \to \mathcal{C}^\infty(\Psi_y(B_y(r)), L^k \otimes A)$$

be the map corresponding to $R_{y,k}(\lambda)$ under these identifications.

For r sufficiently small, define the function $\chi_{y,r}$ supported in $\Psi_y(B_y(r_0))$ and such that $\chi_{y,r}(\Psi_y(\xi)) = \chi(\xi/r)$. We introduce a partition of unity $(\psi_{r,y},\ y \in I(r))$, subordinated to the cover $(\Psi_y(B_y(r)),\ y \in I(r))$. Then define the operator $R_k^r(\lambda)$ acting on $\mathcal{C}^\infty(M, L^k \otimes A)$ by

$$R_k^r(\lambda) := \sum_{y\in I(r)} \chi_{y,r} \widetilde{R}_{y,k}(\lambda)\psi_{r,y}. \tag{37}$$

Theorem 4.7. *Let (Δ_k) be a family of formally self-adjoint differential operators of the form* (B). *Then, for any $|\lambda| \leqslant \Lambda$,*

$$\|(\lambda - k^{-1}\Delta_k)R_k^r(\lambda) - 1\| \leqslant C_\Lambda F(r, k^{-1}, d), \tag{38}$$

with $d = \mathrm{dist}(\lambda, \Sigma)$ and F the same function as in Lemma 4.4.

Proof. Let (U_i) be a covering of M with multiplicity $N = \sup_x |\{i/x \in U_i\}|$. Then:

(1) If v_i is a family of sections such that $\operatorname{supp} v_i \subset U_i$ for any i, then $\left\|\sum v_i\right\|^2 \leqslant N \sum \|v_i\|^2$.

(2) For any section u, $\sum \|u\|_{U_i}^2 \leqslant N\|u\|^2$.

To prove the first claim, $\left\|\sum v_i\right\|^2=\sum_{i,j} M_{ij}\langle v_i, v_j\rangle \leqslant \sum M_{ij}\|v_i\|\|v_j\|$, where $M_{i,j}=1$ when $U_i\cap U_j\neq\varnothing$ and 0 otherwise. By Schur test applied to the matrix M, $\langle Ma, a\rangle \leqslant N\|a\|^2$ and the result follows. To prove the second claim, set $m(x)=\sum 1_{U_i}(x)$, which is bounded by N by assumption. Then $\sum \|u\|^2_{U_i} = \int_M |u(x)|^2 m(x)\,d\mu(x) \leqslant N\|u\|^2$.

We now apply this to the covering $\Psi_y(B_y(r))$, $y\in I(r)$. By Lemma 4.4, for any $u\in\mathcal{C}^\infty(M, L^k)$, we have $\|S^r_{y,k}\psi_{y,r}u\| \leqslant CF\|\psi_{y,r}u\|$, where

$$S^r_{y,k}=(\lambda-k^{-1}\Delta_k)\chi_{y,r}\widetilde{R}_{y,k}(\lambda)-\chi_{y,r},$$

$F=F(r,k^{-1},d)$ and the constant C can be chosen independently of y because everything depends continuously on y and M is compact. Since $R^r_k(\lambda)-1=\sum_{y\in I(r)} S_{y,k}\psi_{y,r}$, we have

$$\begin{aligned}\|R^r_k(\lambda)u-u\|^2 &\leqslant N\sum_{y\in I(r)}\|S^r_{y,k}\psi_{y,r}u\|^2 \leqslant N(CF)^2\sum_{y\in I(r)}\|\psi_{y,r}u\|^2\\ &\leqslant N(CF)^2\sum_{y\in I(r)}\|u\|^2_{\Psi_y(B_y(r))}\leqslant (NCF)^2\|u\|^2,\end{aligned}$$

which proves (38). □

Recall basic facts pertaining to the spectral theory of Δ_k; see for instance [Shubin 1987, Section 8.3]. As an elliptic formally self-adjoint differential operator of order 2 on a compact manifold, Δ_k is a self-adjoint unbounded operator with domain the Sobolev space $H^2(M, L^k\otimes A)$. Its spectrum $\mathrm{sp}(\Delta_k)$ is a discrete subset of $\mathbb{R}$ bounded from below and consists only of eigenvalues with finite multiplicities.

Corollary 4.8. *For any $\Lambda>0$, there exists $C>0$ such that for any k we have*

$$\mathrm{sp}(k^{-1}\Delta_k)\cap\,]-\infty,\Lambda]\subset\Sigma+Ck^{-1/4}[-1,1]. \tag{39}$$

So any $\lambda\in\mathbb{C}$ satisfying $|\lambda|\leqslant\Lambda$ and $d(\lambda,\Sigma)\geqslant Ck^{-1/4}$ does not belong to $\mathrm{sp}(k^{-1}\Delta_k)$. Moreover, for any such λ,

$$\|R^{r_k}_k(\lambda)-(\lambda-k^{-1}\Delta_k)^{-1}\|\leqslant Cd(\lambda,\Sigma)^{-2}k^{-1/4}, \tag{40}$$

with $r_k=k^{-1/4}$.

Equation (39) shows the second assertion of Theorem 1.2 with $k^{-1/4}$ instead of $k^{-1/2}$. The improvement with $k^{-1/2}$ will be proved in Corollary 7.2.

Proof. First, since $\|\widetilde{R}_{y,k}(\lambda)\|\leqslant d(\lambda,\Sigma_y)^{-1}\leqslant d^{-1}$ with $d=d(\lambda,\Sigma)$, we deduce from the first part of the proof of Theorem 4.7 that

$$\|R^r_k(\lambda)\|\leqslant Cd^{-1}, \tag{41}$$

where C does not depend on r, λ and k. From now on assume that $r=k^{-1/4}$. So $F(r,k^{-1},d)\leqslant C'k^{-1/4}d^{-1}$. By Theorem 4.7, as soon as $C_\Lambda C'k^{-1/4}d^{-1}\leqslant\frac{1}{2}$, we have $(\lambda-k^{-1}\Delta_k)R^r_k(\lambda)$ is invertible, so $\widetilde{R}_k:=R^r_k(\lambda)((\lambda-k^{-1}\Delta_k)R^r_k(\lambda))^{-1}$ is a bounded operator of L^2 satisfying

$$(\lambda-k^{-1}\Delta_k)\widetilde{R}_k=\mathrm{id} \tag{42}$$

and by (41),

$$\|\widetilde{R}_k-R^r_k(\lambda)\|\leqslant 2\|R^r_k(\lambda)\|\,\|(\lambda-k^{-1}\Delta_k)R^r_k(\lambda)-1\|\leqslant C''d^{-2}k^{-1/4}.$$

We claim that $\widetilde{R}_k$ is actually continuous $L^2 \to H^2$. Indeed, by classical result on elliptic operators [Shubin 1987, Theorem 5.1], there exists a pseudodifferential operator P_k of order -2 which is a parametrix of $\lambda - k^{-1}\Delta_k$; that is, $P_k(\lambda - k^{-1}\Delta_k) = \mathrm{id} + S_k$, where S_k is a smoothing operator. Then multiplying by $\widetilde{R}_k$, we obtain $P_k = \widetilde{R}_k + S_k\widetilde{R}_k$, so $\widetilde{R}_k = P_k - S_k\widetilde{R}_k$. Now, since P_k is of order -2 and S_k is smoothing, they are both continuous $L^2 \to H^2$, so the same holds for $\widetilde{R}_k$.

To finish the proof, we assume that λ is real. Then $k^{-1}\Delta_k - \lambda$ is a Fredholm operator from H^2 to L^2 with index 0, because it is formally self-adjoint; see [Shubin 1987, Theorem 8.1]. By (42), $\lambda - k^{-1}\Delta_k$ sends H^2 onto L^2, so its kernel is trivial, and thus λ is not an eigenvalue. □

5. The operator class $\mathcal{L}(A)$

5A. ***Symbol spaces.*** Let $\boldsymbol{E}$ be an n-dimensional Hermitian space. As we did in Section 2B for $\boldsymbol{E} = T_yM$, consider the spaces $\mathcal{P}(\boldsymbol{E})$, $\mathcal{D}(\boldsymbol{E})$ consisting respectively of polynomial maps and antiholomorphic polynomial maps from $\boldsymbol{E}$ to $\mathbb{C}$. We will introduce two subalgebras $\mathcal{S}(\boldsymbol{E})$ and $\widetilde{\mathcal{S}}(\boldsymbol{E})$ of $\mathrm{End}(\mathcal{D}(\boldsymbol{E}))$ and $\mathrm{End}(\mathcal{P}(\boldsymbol{E}))$ respectively. These algebras will be used later to define the symbols of the operators in the class $\mathcal{L}$.

First we equip $\mathcal{P}(\boldsymbol{E})$ with the scalar product

$$\langle f, g\rangle = (2\pi)^{-n} \int_{\boldsymbol{E}} e^{-|z|^2} f(z)\, \overline{g(z)}\, d\mu_{\boldsymbol{E}}(z), \tag{43}$$

where $\mu_{\boldsymbol{E}}$ is the measure $\prod dz_i\, d\bar{z}_i$ if (z_i) are linear complex coordinates associated to an orthonormal basis of $\boldsymbol{E}$. The Gaussian weight $e^{-|z|^2}$ appeared already in Section 2B through the pointwise norm of the frame $s = \exp(-|z|^2/2)$.

Choose linear complex coordinates (z_i) as above. Then the family $|\alpha\rangle := (\alpha!)^{-1/2}\bar{z}^{\alpha}$, $\alpha \in \mathbb{N}^n$, is an orthonormal basis of $\mathcal{D}(\boldsymbol{E})$. For any $\alpha, \beta \in \mathbb{N}^n$, we introduce the endomorphism $\rho_{\alpha\beta} := |\alpha\rangle\langle\beta|$ of $\mathcal{D}(\boldsymbol{E})$. Here we use the physicist notation, so $\rho_{\alpha\beta}(\bar{z}^{\gamma}) = 0$ when $\gamma \neq \beta$ and $\rho_{\alpha\beta}(|\beta\rangle) = |\alpha\rangle$.

Consider the creation and annihilation operators $\mathfrak{a}_i$, $\mathfrak{a}_i^{\dagger}$ defined in (12) as endomorphisms of $\mathcal{P}(\boldsymbol{E})$. Note that with the scalar product (43), $\mathfrak{a}_i^{\dagger}$ is the formal adjoint of $\mathfrak{a}_i$. We introduce the endomorphism $\tilde{\rho}_{\alpha\beta}$ of $\mathcal{P}(\boldsymbol{E})$

$$\tilde{\rho}_{\alpha\beta} := (\alpha!\beta!)^{-1/2}(\mathfrak{a}^{\dagger})^{\alpha}\tilde{\rho}_{00}\mathfrak{a}^{\beta},$$

where $\mathfrak{a}^{\beta} = \mathfrak{a}_1^{\beta(1)}\cdots\mathfrak{a}_n^{\beta(n)}$, $(\mathfrak{a}^{\dagger})^{\alpha} = (\mathfrak{a}_1^{\dagger})^{\alpha(1)}\cdots(\mathfrak{a}_n^{\dagger})^{\alpha(n)}$ and $\tilde{\rho}_{00}$ is the orthogonal projector onto the subspace $\mathfrak{L}_0$ of $\mathcal{P}(\boldsymbol{E})$ consisting of holomorphic polynomials.

Observe that the restriction of $\tilde{\rho}_{\alpha\beta}$ to $\mathcal{D}(\boldsymbol{E})$ is $\rho_{\alpha\beta}$. Furthermore, in the decomposition into orthogonal subspaces $\mathcal{P}(\boldsymbol{E}) = \bigoplus_{\alpha}\mathfrak{L}_{\alpha}$ considered in (14), $\tilde{\rho}_{\alpha\beta}$ is zero on $\mathfrak{L}_{\gamma}$ with $\gamma \neq \beta$ and restricts to an isomorphism from $\mathfrak{L}_{\beta}$ to $\mathfrak{L}_{\alpha}$. Also $\tilde{\rho}_{\alpha\alpha}$ is the orthogonal projector onto $\mathfrak{L}_{\alpha}$.

The algebras $\mathcal{S}(\boldsymbol{E})$ and $\widetilde{\mathcal{S}}(\boldsymbol{E})$ are defined as the subalgebras of $\mathrm{End}(\mathcal{D}(\boldsymbol{E}))$ and $\mathrm{End}(\mathcal{P}(\boldsymbol{E}))$ with basis the families $(\rho_{\alpha,\beta},\ \alpha, \beta \in \mathbb{N}^n)$ and $(\tilde{\rho}_{\alpha\beta},\ \alpha, \beta \in \mathbb{N}^n)$ respectively. As the notation suggests, these algebras do not depend on the coordinate choice. This follows from the following Schwartz kernel description.

Let $\mathrm{Op} : \mathcal{P}(\boldsymbol{E}) \to \mathrm{End}(\mathcal{P}(\boldsymbol{E}))$ be the linear map defined by

$$\mathrm{Op}(q)(f)(u) = (2\pi)^{-n} \int_{\boldsymbol{E}} e^{u\cdot\bar{v} - |v|^2} q(u - v) f(v)\, d\mu_{\boldsymbol{E}}(v), \tag{44}$$

where $u \cdot \bar{v}$ is the scalar product of u and v. By [Charles 2024, Lemma 4.3], $\tilde{\rho}_{\alpha,\beta} = \operatorname{Op}(p_{\alpha,\beta})$, where $p_{\alpha\beta}$ is the polynomial

$$p_{\alpha,\beta} := (\alpha!\,\beta!)^{-1/2}(\bar{z} - \partial_z)^{\alpha}(-z)^{\beta}, \quad \alpha, \beta \in \mathbb{N}^n. \tag{45}$$

Since these polynomials form a basis of $\mathcal{P}(\boldsymbol{E})$, Op is an isomorphism from $\mathcal{P}(\boldsymbol{E})$ to $\widetilde{\mathcal{S}}(\boldsymbol{E})$. Furthermore, the map sending $q \in \mathcal{P}(\boldsymbol{E})$ to $\operatorname{Op}(q)|_{\mathcal{D}(\boldsymbol{E})}$ is an isomorphism from $\mathcal{P}(\boldsymbol{E})$ to $\mathcal{S}(\boldsymbol{E})$.

In the sequel we will tensor the space $\mathcal{P}(\boldsymbol{E})$ with an auxiliary vector space $\mathbb{A}$ and extend the map Op from $\mathcal{P}(\boldsymbol{E}) \otimes \operatorname{End} \mathbb{A}$ to $\widetilde{\mathcal{S}}(\boldsymbol{E}) \otimes \operatorname{End} \mathbb{A}$.

5B. ***Eigenprojectors of Landau Hamiltonian.*** Choose now $\boldsymbol{E} = T_y M$ and recall that, for a convenient choice of complex coordinate (z_i), the associated Landau Hamiltonian $\widetilde{\square}_y$ is given by

$$\widetilde{\square}_y = e^{|\xi|_y^2/4} \Delta_y e^{-|\xi|_y^2/4} = \sum B_i(y)\big(\mathfrak{a}_i^{\dagger}\mathfrak{a}_i + \tfrac{1}{2}\big) + V(y) \tag{46}$$

acting on $\mathcal{P}(T_y M) \otimes A_y$. Its spectrum Σ_y and its eigenspaces were described in Section 2C in terms of the $\mathfrak{L}_\alpha$ and an eigenbasis (ζ_ℓ) of $V(y)$, $V(y)\zeta_\ell = V_\ell(y)\zeta_\ell$. Consequently if I is any bounded subset of $\mathbb{R}$, the spectral projector of $\widetilde{\square}_y$ for the eigenvalues in I is $\operatorname{Op}(\sigma^I(y))$, where

$$\sigma^I(y) = \sum_{(\alpha,\ell) \in \mathcal{I}_y} p_{\alpha\alpha} \otimes |\zeta_\ell\rangle\langle\zeta_\ell|,$$

and $\mathcal{I}_y = \big\{(\alpha, \ell) \in \mathbb{N}^n \times \{1, \dots, r\} / \sum_i B_i(y)\big(\alpha(i) + \tfrac{1}{2}\big) + V_\ell(y) \in I\big\}$.

The map $y \mapsto \sigma^I(y)$ is a section of the infinite-rank vector bundle $\mathcal{P}(TM)$, not smooth in general, not even continuous. In the sequel we will assume that

$$I \text{ is a compact interval with endpoints not belonging to } \Sigma. \tag{C}$$

Let $\mathcal{P}_{\leqslant p}(\boldsymbol{E})$ be the subspace of $\mathcal{P}(\boldsymbol{E})$ of polynomials with degrees in z and in $\bar{z}$ smaller than p. Let $\mathcal{P}_{\leqslant p}(TM)$ be the vector bundle over M with fiber at y equal to $\mathcal{P}_{\leqslant p}(T_y M)$.

Lemma 5.1. *If I satisfies* (C) *and p is sufficiently large, then $y \mapsto \sigma^I(y)$ is a smooth section of $\mathcal{P}_{\leqslant p}(TM) \otimes \operatorname{End} A$.*

Proof. Recall from Section 2E that $\square_y$ is the restriction of $\widetilde{\square}_y$ to $\mathcal{D}(T_y M)$. By Lemma 2.4, the spaces

$$F_y := \operatorname{Im} 1_I(\square_y) = \operatorname{Span}(\bar{z}^{\alpha} \otimes \zeta_\ell, (\alpha, \ell) \in \mathcal{I}_y) \tag{47}$$

are the fibers of a subbundle of $\mathcal{D}_{\leqslant p}(TM) \otimes A$ if p is sufficiently large. So the projector onto F_y depends smoothly on y; in other words, the map $y \to \operatorname{Op}(\sigma^I(y))|_{\mathcal{D}(T_y M) \otimes A_y}$ is a smooth section of $\operatorname{End}(\mathcal{D}_{\leqslant p}(TM) \otimes A)$.

Now we have an isomorphism

$$\mathcal{P}_{\leqslant p}(\boldsymbol{E}) \xrightarrow{\operatorname{Op}_p} \operatorname{End}(\mathcal{D}_{\leqslant p}(\boldsymbol{E})), \quad q \mapsto \text{the restriction of } \operatorname{Op}(q) \text{ to } \mathcal{D}_{\leqslant p}(\boldsymbol{E}).$$

Indeed, on one hand $(p_{\alpha\beta}, |\alpha|, |\beta| \leqslant p)$ is a basis of $\mathcal{P}_{\leqslant p}(\mathbb{C}^n)$ and on the other hand $(\rho_{\alpha\beta}, |\alpha|, |\beta| \leqslant p)$ is a basis of $\operatorname{End} \mathcal{D}_{\leqslant p}(\mathbb{C}^n)$. This gives a vector bundle isomorphism $\mathcal{P}_{\leqslant p}(TM) \otimes \operatorname{End} A \simeq \operatorname{End}(\mathcal{D}_{\leqslant p}(TM) \otimes A)$, and concludes the proof. ☐

Let $\mathcal{S}(TM)$ be the infinite-rank vector bundle over M with fibers $\mathcal{S}(T_yM)$ defined as in Section 5A. A section U of $\mathcal{S}(TM) \otimes \operatorname{End} A$ is *smooth* if it has the form

$$U(y) = \operatorname{Op}(q(y))|_{\mathcal{D}(T_yM)\otimes A_y}, \tag{48}$$

where $y \to q(y)$ is a smooth section of $\mathcal{P}_{\leqslant p}(TM) \otimes \operatorname{End} A$ for some p. By Lemma 5.1, for any interval I satisfying (C), we have a symbol $\pi^I \in \mathcal{C}^\infty(M, \mathcal{S}(TM) \otimes \operatorname{End} A)$ defined at y by

$$\pi^I(y) = 1_I(\Box_y) = \operatorname{Op}(\sigma^I(y))|_{\mathcal{D}(T_yM)\otimes A_y}, \tag{49}$$

which is the projector of $\mathcal{D}(T_yM) \otimes A_y$ onto the subspace F_y defined in Lemma 2.4.

5C. *Operators.* The operator class $\mathcal{L}(A)$ was introduced in [Charles 2024]. It depends on (M, ω, j), the prequantum bundle L, that is, L with its metric and connection, and the auxiliary Hermitian bundle A.

$\mathcal{L}(A)$ consists of families of operators $(P_k : \mathcal{C}^\infty(M, L^k \otimes A) \to \mathcal{C}^\infty(M, L^k \otimes A),\ k \in \mathbb{N})$ having smooth Schwartz kernels satisfying the following conditions. First, $P_k(x, y)$ is in $\mathcal{O}(k^{-\infty})$ outside the diagonal. More precisely, for any compact subset K of $M^2 \setminus \operatorname{diag} M$ and for any N, there exists $C > 0$ such that

$$|P_k(x, y)| \leqslant Ck^{-N} \quad \text{for all } k \in \mathbb{N}, \text{ for all } (x, y) \in K.$$

Second, for any open set U of M identified through a diffeomorphism with a convex open set of $\mathbb{R}^{2n}$ and any unitary trivialization $A|_U \simeq U \times \mathbb{C}^r$, we have on U^2 for any positive integers N, k

$$P_k(x+\xi, x) = \left(\frac{k}{2\pi}\right)^n F^k(x+\xi, x) e^{-k|\xi|_x^2/4} \sum_{\ell=0}^{N} k^{-\ell} a_\ell(x, k^{1/2}\xi) + r_{N,k}(x+\xi, x), \tag{50}$$

where the section $F : U^2 \to L \boxtimes \bar{L}$ is defined as in Section 1D, the coefficients $a_\ell(x, \xi) \in \mathbb{C}^r \otimes \bar{\mathbb{C}}^r$ depend smoothly on x and polynomialy on ξ, with degree bounded independently of x, and the remainder $r_{N,k}$ is in $\mathcal{O}(k^{n-(N+1)/2})$ uniformly on any compact subset of U^2.

The subspace $\mathcal{L}^+(A)$ of $\mathcal{L}(A)$ consists of the operator families (P_k) where the coefficients a_ℓ in the local expansions (50) satisfy $a_\ell(x, -\xi) = (-1)^\ell a_\ell(x, \xi)$. The symbol map is the application $\sigma_0 : \mathcal{L} \to \mathcal{C}^\infty(M, \mathcal{S}(TM) \otimes \operatorname{End} A)$ given locally by

$$\sigma_0(P)(x) = \operatorname{Op}(a_0(x, \cdot))|_{\mathcal{D}(T_xM)} \in \mathcal{S}(T_xM) \otimes \operatorname{End} A_x, \tag{51}$$

where we view $a_0(x, \xi)$ in $\mathbb{C}^r \otimes \bar{\mathbb{C}}^r \simeq \operatorname{End} \mathbb{C}^r \simeq \operatorname{End} A_x$.

Recall that for any compact interval I of $\mathbb{R}$, we denote by Π_k^I the corresponding spectral projector of $k^{-1}\Delta_k$. The central result of this paper is the following theorem.

Theorem 5.2. *Let* (Π_k^I) *be the spectral projector of a formally self-adjoint operator family* (Δ_k) *of the form* (B) *with* I *satisfying* (C). *Then* (Π_k^I) *belongs to* $\mathcal{L}^+(A)$ *and has symbol* π^I.

The proof is given in Section 6. We will actually prove a stronger result where we describe the Schwartz kernel derivatives as well.

5D. *The class $\mathcal{L}^\infty(A)$.* We need first a few definitions. Consider a real number N. We say that a sequence (f_k) of $\mathcal{C}^\infty(U)$ with U an open set of M is in $\mathcal{O}_\infty(k^{-N})$ if, for any $m \in \mathbb{N}$, for any vector fields $X_1, \dots, X_m$ of U, for any compact subset K of U, there exists $C > 0$ such that

$$|X_1 \cdots X_m f_k(x)| \leqslant C k^{-N+m} \quad \text{for all } x \in K,\ k \in \mathbb{N}.$$

Let $s = (s_k \in \mathcal{C}^\infty(M, L^k \otimes A),\ k \in \mathbb{N})$. We say that $s \in \mathcal{O}_\infty(k^{-N})$ if, for any unitary frames u and $(v_j)_{j=1}^r$ of L and A defined over the same open set U of M, the local representative sequences $(f_{k,j})$ such that $s_k = \sum f_{j,k} u^k \otimes v_j$, are in $\mathcal{O}_\infty(k^{-N})$. We say that s belongs to $\mathcal{O}_\infty(k^\infty)$ (resp. $\mathcal{O}_\infty(k^{-\infty})$) if $s \in \mathcal{O}_\infty(k^{-N})$ for some N (resp. for any N). So

$$\mathcal{O}_\infty(k^{-\infty}) \subset \mathcal{O}_\infty(k^{-N}) \subset \mathcal{O}_\infty(k^{-N'}) \subset \mathcal{O}_\infty(k^\infty) \quad \text{if } N \geqslant N'.$$

Replacing M, L and A by M^2, $L \boxtimes \bar{L}$ and $A \boxtimes \bar{A}$, we can apply these definitions to Schwartz kernels of operator families $(P_k : \mathcal{C}^\infty(M, L^k \otimes A) \to \mathcal{C}^\infty(M, L^k \otimes A),\ k \in \mathbb{N})$.

By definition, $\mathcal{L}^\infty(A)$ and $\mathcal{L}^\infty_\infty(A)$ are the subspaces of $\mathcal{L}(A)$ consisting of operator families with a Schwartz kernel in $\mathcal{O}_\infty(k^\infty)$ and $\mathcal{O}_\infty(k^{-\infty})$ respectively. By [Charles 2024, Proposition 6.3], the difference between $\mathcal{L}^\infty(A)$ and $\mathcal{L}(A)$ is rather small because for any $P \in \mathcal{L}(A)$, there exists $P' \in \mathcal{L}^\infty(A)$ such that the Schwartz kernel of $P - P'$ is in $\mathcal{O}(k^{-\infty})$, that is, $P_k(x, x') = P'_k(x, x') + \mathcal{O}(k^{-N})$ for any N, with $\mathcal{O}$ uniform on M^2. Furthermore P' is unique modulo $\mathcal{L}^\infty_\infty(A)$.

By [Charles 2024, Proposition 6.3], for any $(P_k) \in \mathcal{L}^\infty(A)$ the asymptotic expansion (50) holds with a remainder $r_{N,k}$ in $\mathcal{O}_\infty(k^{n-(N+1)/2})$.

Theorem 5.3. *Under the same assumptions as in Theorem 5.2, (Π^I_k) belongs to $\mathcal{L}^\infty(A)$.*

The proof will be given in Section 6. To end this section, let us state the following corollary of Theorems 5.2, 5.3 and Lemma 6.3.

Corollary 5.4. *Under the same assumptions as in Theorem 5.2, $(k^{-1}\Delta_k \Pi^I_k)$ belongs to $\mathcal{L}^+(A) \cap \mathcal{L}^\infty(A)$ and has symbol $\sigma_0(k^{-1}\Delta_k \Pi_k) = \square \circ \pi^I$.*

So the first part of Theorem 1.4 follows from Theorem 5.2 and Corollary 5.4.

6. Proof of Theorems 5.2 and 5.3

The first step, Lemma 6.1, is to show that any operator in $\mathcal{L}(A)$ with symbol π^I is an approximation of Π^I_k up to $\mathcal{O}(k^{-1/4})$. This will follow from the resolvent estimate given in Corollary 4.8 and the Cauchy–Riesz formula. The second step, Lemma 6.2, is the construction of a formal projector $(P_k) \in \mathcal{L}^+(A)$ with symbol π^I which almost commutes with Δ_k. The third step, Section 6C, is to show that this formal projector (P_k) is equal to Π^I_k up to $\mathcal{O}(k^{-\infty})$ and even up to $\mathcal{O}_\infty(k^{-\infty})$ when $(P_k) \in \mathcal{L}^\infty(A)$.

6A. *A first approximation.*

Lemma 6.1. *Under the same assumptions as in Theorem 5.2, $\Pi^I_k = P_k + \mathcal{O}(k^{-1/4})$ for any (P_k) in $\mathcal{L}(A)$ with symbol π^I.*

Proof. Step 1: The proof starts from the resolvent approximation given in Corollary 4.8. Choose a loop γ of $\mathbb{C} \setminus \Sigma$ which encircles I. When k is sufficiently large, by Corollary 4.8, γ does not meet the spectrum of $k^{-1}\Delta_k$. So by Riesz projection formula and (40),

$$\Pi_k^I = \frac{1}{2i\pi}\int_\gamma (\lambda - k^{-1}\Delta_k)^{-1}\,d\lambda = \frac{1}{2i\pi}\int_\gamma R_k^{r_k}(\lambda)\,d\lambda + \mathcal{O}(k^{-1/4}), \tag{52}$$

with $r_k = k^{-1/4}$. Since $R_k^r(\lambda) := \sum_{y\in I(r)} \chi_{y,r}\widetilde{R}_{y,k}(\lambda)\psi_{r,y}$, we get

$$\Pi_k^I = \sum_{y\in I(r_k)} \chi_{y,r_k}\widetilde{P}_{y,k}^I\psi_{r_k,y} + \mathcal{O}(k^{-1/4}), \tag{53}$$

where for any y

$$\widetilde{P}_{y,k}^I = \frac{1}{2i\pi}\int_\gamma \widetilde{R}_{y,k}(\lambda)\,d\lambda.$$

Recall that $\widetilde{R}_{y,k}(\lambda)$ is the restriction of the resolvent $(\lambda - k^{-1}\Delta_{y,k})^{-1}$ to $\mathcal{C}_0^\infty(B_y(r), \mathbb{C}^r)$ identified with $\mathcal{C}_0^\infty(\Psi_y(B_y(r)), L^k \otimes A)$. So by Riesz projection formula again, $\widetilde{P}_{y,k}^I$ is the restriction of the spectral projection

$$P_{y,k}^I = \frac{1}{2i\pi}\int_\gamma (\lambda - k^{-1}\Delta_{y,k})^{-1}\,d\lambda.$$

Step 2: Let $d : M^2 \to \mathbb{R}_{\geqslant 0}$ be a distance locally equivalent to the Euclidean distance in each chart and set $m_k(x', x) := k^n \exp(-kcd(x', x)^2)$ with $c > 0$. Then by Schur test, any operator family $(Q_k : \mathcal{C}^\infty(M, L^k \otimes A) \to \mathcal{C}^\infty(M, L^k \otimes A), k \in \mathbb{N})$ having a continuous Schwartz kernel satisfying $|Q_k(x', x)| = \mathcal{O}(m_k(x', x))$ uniformly with respect to x, x' and k, has a bounded operator norm; see [Charles 2024, proof of Lemma 5.1] for more details. Given this and (53), it suffices now to prove that

$$P_k(x', x) = \sum_{y\in I(r)} \chi_{y,r_k}(x')\widetilde{P}_{y,k}^I(x', x)\psi_{r_k,y}(x) + (m_k(x', x) + 1)\mathcal{O}(k^{-1/4}). \tag{54}$$

In the sequel, we will allow the constant c entering in the definition of m_k to decrease from one line to another. With this convention, for any $p > 0$, we can replace any $\mathcal{O}(d^p(x', x)m_k(x', x))$ by $\mathcal{O}(k^{p/2}m_k(x', x))$.

Step 3: Equation (54) follows from

$$P_k(x', x) = \widetilde{P}_{y,k}^I(x', x) + (m_k(x', x) + 1)\mathcal{O}(k^{-1/4}) \tag{55}$$

for all $(x', x) \in \Psi_y(B_y(2r)) \times \Psi_y(B_y(2r))$, with $\mathcal{O}$ uniform with respect to all the variables, y included. Indeed, since $\operatorname{supp}\psi_{r,y} \subset \Psi_y(B_y(r)) \subset \{\chi_{y,r} = 1\}$, we have

$$\chi_{y,r}(x')\psi_{r,y}(x) = \psi_{r,y}(x) + \mathcal{O}(d(x', x)r^{-1}) \quad \text{for all } x, x' \in \Psi_y(B_y(2r)).$$

Recall that by [Charles 2024, Lemma 5.1], $P_k(x', x) = \mathcal{O}(m_k(x', x)) + \mathcal{O}(k^{-N})$ for any N. Applying this to $N = \frac{1}{4}$ and using that $m_k d = \mathcal{O}(k^{-1/2}m_k)$ as explained above, we obtain

$$\chi_{y,r}(x')P_k(x', x)\psi_{r,y}(x) = P_k(x', x)\psi_{r,y}(x) + \mathcal{O}(k^{-1/2}m_k(x', x)r^{-1}) + \mathcal{O}(k^{-1/4}).$$

Assume now that (55) holds. Multiplying (55) by $\chi_{y,r}(x')\psi_{r,y}(x)$ and using the last equality, we obtain

$$P_k(x',x)\psi_{r,y}(x) = \chi_{y,r}(x')\widetilde{P}^I_{y,k}(x',x)\psi_{r,y}(x) + \mathcal{O}(k^{-1/2}m_k(x',x)r^{-1}) + \mathcal{O}(k^{-1/4}),$$

which holds for all $x', x \in M$. Recall that the covering $\bigcup \Psi_y(B_y(r))$, $y \in I(r)$, has a multiplicity bounded independently on r. So we can sum these estimates without multiplying the remainder by the number of summands and we obtain

$$P_k(x',x) = \sum_{y\in I(r)} \chi_{y,r}(x')\widetilde{P}^I_{y,k}(x',x)\psi_{r,y}(x) + \mathcal{O}(k^{-1/2}m_k(x',x)r^{-1}) + \mathcal{O}(k^{-1/4}).$$

This proves (54) because $r_k = k^{-1/4}$.

Step 4: We give a formula for the Schwartz kernel of the spectral projector $P^I_{y,k}$. First, by the rescaling (26), (27), we have

$$P^I_{y,k}(\xi,\eta) = k^n P^I_y(k^{1/2}\xi, k^{1/2}\eta), \tag{56}$$

with $P^I_y := P^I_{y,1}$. Second, the Schwartz kernel of P^I_y is given by

$$P^I_y(\eta+\xi,\eta) = (2\pi)^{-n} e^{(i/2)\omega_y(\eta,\xi) - |\xi|^2_y/4}\pi^I(y,\xi). \tag{57}$$

Indeed, by (46), $P^I_y = e^{-|\xi|_y/4}\operatorname{Op}(\sigma^I(y))e^{|\xi|_y/4}$ and it follows from (44) that

$$\begin{aligned} P^I_y(\xi,\eta) &= (2\pi)^{-n} e^{-|u|^2/2 + u\cdot\bar{v} - |v|^2/2}\sigma^I(y, u-v) \\ &= (2\pi)^{-n} e^{(u\cdot\bar{v} - \bar{u}\cdot v)/2 - |u-v|^2/2}\sigma^I(y,u-v), \end{aligned} \tag{58}$$

with (u_i), (v_i) the complex coordinates of ξ and η defined as in Section 2B, in particular $|\xi|^2_y = \frac{1}{2}|u|^2$ and $|\eta|^2_y = \frac{1}{2}|v|^2$. Since $\omega_y = i\sum_i du_i \wedge d\bar{u}_i$, (57) follows from (58). Inserting (57) into (56), we get

$$P^I_{y,k}(\eta+\xi,\eta) = \left(\frac{k}{2\pi}\right)^n F^k_y(\eta+\xi,\eta)e^{-k|\xi|^2_y/4}\sigma^I(y,k^{1/2}\xi), \tag{59}$$

with $F_y(\eta+\xi,\eta) = e^{(i/2)\omega_y(\eta,\xi)}$. F_y has the same characterization as the section F entering in the expansion (50), that is, $F_y(\eta,\eta) = 1$ and $\mathbb{R} \ni t \to F_y(\eta + t\xi, \eta)$ is flat for any ξ, η.

Step 5: The Schwartz kernel of P_k has the local expansion (50). By [Charles 2024, Lemma 5.1], the remainder $r_{N,k}$ is in $\mathcal{O}(k^{-N/2}m_k) + \mathcal{O}(k^{-N'})$ for any N'. So in particular,

$$P_k(x+\xi,x) = \left(\frac{k}{2\pi}\right)^n F^k(x+\xi,x)e^{-k|\xi|^2_x/4}\sigma^I(x,k^{1/2}\xi) + (m_k+1)\mathcal{O}(k^{-1/4}). \tag{60}$$

Step 6: We now prove (55) by comparing (59) and (60). So let $x, x' \in \Psi_y(B_y(2r))$ and $\xi = x' - x$. We will use several times that

$$d(x,y) \leqslant Cr, \quad C^{-1}d \leqslant |\xi| \leqslant Cd, \quad \text{where } d := d(x',x).$$

Let $\Phi_y : \Psi_y(B_y(r_0)) \to T_yM$ be the inverse of Ψ_y. We have to compare $P_k(x+\xi, x)$ with $\widetilde{P}^I_{y,k}(x+\xi, x) = P^I_{y,k}(\eta+\tilde{\xi}, \eta)$, where

$$\eta = \Phi_y(x), \quad \eta+\tilde{\xi} = \Phi_y(x+\xi).$$

We claim that

$$\tilde{\xi} = \xi + \mathcal{O}(rd+d^2). \tag{61}$$

To see this, write $\tilde{\xi} = \Phi_y(x+\xi) - \Phi_y(x) = L_y(x,\xi)\xi$, where $L_y(x,0) = T_x\Phi_y$. Since $L_y(y,0) = \mathrm{id}_{T_yM}$, we have

$$L_y(x,\xi) = L_y(x,0) + \mathcal{O}(|\xi|) = \mathrm{id}_{T_yM} + \mathcal{O}(d(x,y)+|\xi|).$$

So $\tilde{\xi} = \xi + \mathcal{O}(|\xi|(d(x,y)+|\xi|)) = \xi + \mathcal{O}(d(r+d))$.

Consider now a smooth function $(x,\xi) \to q(x,\xi)$ which is polynomial homogeneous in ξ with degree ℓ. Then

$$q(x,\xi) = q(y,\xi) + \mathcal{O}(d(x,y)|\xi|^\ell) = q(y,\xi) + \mathcal{O}(rd^\ell)$$

and by (61), $q(y,\xi) = q(y,\tilde{\xi}) + \mathcal{O}(d^\ell(r+d))$. So

$$q(x,k^{1/2}\xi) = q(y,k^{1/2}\tilde{\xi}) + \mathcal{O}((k^{1/2}d)^\ell(r+d)). \tag{62}$$

Consequently

$$\sigma^I(x,k^{1/2}\xi) = \sigma^I(y,k^{1/2}\tilde{\xi}) + \mathcal{O}(r+d)\sum(k^{1/2}d)^\ell, \tag{63}$$

where the sum on the right is over ℓ and finite.

By [Charles 2016, Section 2.6], the section $E(x+\xi,x) := F(x+\xi,x)e^{-|\xi|_x^2/4}$ depends on the coordinate choice up to a section vanishing to third order along the diagonal. So

$$E(x+\xi,x) = F_y(\eta+\tilde{\xi},\eta)e^{-|\tilde{\xi}|_x/4}e^{\mathcal{O}(d^3)} = E_y(\eta+\tilde{\xi},\eta)e^{\mathcal{O}(d^3+d^2r)},$$

with $E_y(\eta+\tilde{\xi},\eta) := F_y(\eta+\tilde{\xi},\eta)e^{-|\tilde{\xi}|_y/4}$ because $|\tilde{\xi}|_y^2 = |\xi|_x^2 + \mathcal{O}(d^2(r+d))$ by (62). So using that $|e^z-1| \leqslant |z|e^{|\operatorname{Re} z|}$ and that $k^nE^k(x+\xi,x) = \mathcal{O}(m_k)$, we have

$$\begin{aligned} k^n(E^k(x+\xi,x) - E^k_y(\eta+\tilde{\xi},\eta)) &= \mathcal{O}(d^2(d+r)m_k)e^{kCd^2(d+r)} = \mathcal{O}(d^2(d+r)m_k)e^{kCd^2(d+r)} \\ &= \mathcal{O}(k^{-5/4}m_k)e^{kCd^2(d+r)} = \mathcal{O}(k^{-5/4}m_k), \end{aligned} \tag{64}$$

where we have used that d and r are both in $\mathcal{O}(k^{-1/4})$, and always the same convention that the constant c in m_k can change from one line to another so that $d^pm_k = \mathcal{O}(k^{-p/2}m_k)$. Using again that $k^nE^k(x+\xi,x) = \mathcal{O}(m_k)$, it follows from (63),

$$\begin{aligned} k^nE^k(x+\xi,x)\sigma^I(x,k^{1/2}\xi) &= k^nE^k(x+\xi,x)\sigma^I(y,k^{1/2}\tilde{\xi}) + \mathcal{O}(k^{-1/4}m_k) \\ &= k^nE^k_y(\eta+\tilde{\xi},\eta)\sigma^I(y,k^{1/2}\tilde{\xi}) + \mathcal{O}(k^{-1/4}m_k) \end{aligned}$$

by (64), which ends the proof of (55). □

6B. ***A formal projector.*** This section is devoted to the proof of the following lemma.

Lemma 6.2. *Under the same assumptions as in Theorem 5.2, there exists $(P_k) \in \mathcal{L}^\infty(A) \cap \mathcal{L}^+(A)$ unique modulo $\mathcal{L}^\infty_\infty(A)$ such that $\sigma_0(P_k) = \pi^I$, $P_k = P_k^*$ for any k, $P_k \equiv P_k^2$ modulo $\mathcal{L}^\infty_\infty(A)$ and $[\Delta_k, P_k] \equiv 0$ modulo $\mathcal{L}^\infty_\infty(A)$.*

To show this, we will construct (P_k) by successive approximations. We introduce the filtration

$$\mathcal{L}^\infty_p(A) := \mathcal{L}^\infty(A) \cap \mathcal{O}_\infty(k^{-p/2}),$$

$p \in \mathbb{N}$. For any $p \in \mathbb{N}$, we have a symbol map

$$\sigma_p : \mathcal{L}^\infty_p(A) \to \mathcal{C}^\infty(M, \mathcal{S}(TM) \otimes \operatorname{End} A)$$

such that $\sigma_p(P) = \sigma_0(k^{p/2}P)$, where σ_0 was defined in (51). By [Charles 2024, Proposition 2.1 and Theorem 2.2], σ_p is onto, $\operatorname{Ker} \sigma_p = \mathcal{L}^\infty_{p+1}(A)$ and for any sequence (Q_p) of $\mathcal{L}^\infty(A)$ such that $Q_p \in \mathcal{L}^\infty_p(A)$ for any p, there exists $Q \in \mathcal{L}^\infty(A)$ such that $Q = Q_0 + \cdots + Q_p$ modulo $\mathcal{L}^\infty_{p+1}(A)$ for any p. Moreover:

(1) If Q and Q' belong to $\mathcal{L}^\infty_p(A)$ and $\mathcal{L}^\infty_{p'}(A)$ respectively, then their product belongs to $\mathcal{L}^\infty_{p+p'}(A)$. Furthermore, at any $x \in M$, $\sigma_{p+p'}(QQ')(x)$ is the product of $\sigma_p(Q)(x)$ and $\sigma_{p'}(Q')(x)$.

(2) If Q belongs to $\mathcal{L}^\infty_p(A)$, then its adjoint Q^* belongs to $\mathcal{L}^\infty_p(A)$ with symbol $\sigma_p(Q^*)(x) = \sigma_p(Q)(x)^*$.

By [Charles 2024, Theorem 2.5], $\mathcal{L}^+(A)$ is a subalgebra of $\mathcal{L}(A)$.

Lemma 6.3. *For any Q in $\mathcal{L}^\infty_p(A)$, $(k^{-1}\Delta_k Q_k)$ and $(k^{-1}Q_k\Delta_k)$ both belong to $\mathcal{L}^\infty_p(A)$ and their symbols at x are $\Box_x \circ \sigma_p(Q)(x)$ and $\sigma_p(Q)(x) \circ \Box_x$. If $Q \in \mathcal{L}^+(A)$ then the same holds for $(k^{-1}\Delta_k Q_k)$ and $(k^{-1}Q_k\Delta_k)$.*

Proof. By [Charles 2024, Proposition 6.3, Assertion 3c and 3d], $(k^{-1}\Delta_k Q_k)$ and $(k^{-1}Q_k\Delta_k)$ both belong to $\mathcal{L}^\infty_p(A)$. To compute the symbol, we can use the peaked sections of Section 4A. Indeed, if $\Phi_k(f)$ is defined by (28), with $f \in \mathcal{D}(T_xM) \otimes A_x$ and $(P_k) \in \mathcal{L}_0(A)$ then by [Charles 2024, Proposition 2.4], $P_k\Phi_k(f) = \Phi_k(g) + \mathcal{O}(k^{-1/2})$ with $g = \sigma_0(P_k)(x)f$. So the symbol of any operator of $\mathcal{L}_p(A)$ is characterized by its action on the peaked sections. Proposition 4.1 tells us how $k^{-1}\Delta_k$ acts on the peaked section and the first part of the result follows. To show that the composition with $k^{-1}\Delta_k$ preserves the subspace $\mathcal{L}^+(A)$ of even operators, one uses instead of the asymptotic expansion (50) the alternative expansion

$$P_k(x, y) = \left(\frac{k}{2\pi}\right)^n E^k(x, y) \sum k^{-\ell/2} b_\ell(x, y) + \mathcal{O}(k^{-\infty});$$

see [Charles 2024, equation (45) and Proposition 5.6]. The fact that (P_k) is even means that $b_\ell = 0$ when ℓ is odd. When $(P_k) \in \mathcal{L}^\infty(A)$, this expansion holds for the $\mathcal{C}^\infty$ topology, so we can compute the Schwartz kernel of $k^{-1}\Delta_k P_k$ by letting $k^{-1}\Delta_k$ act on each term of the expansion. Doing this with the expression (B), no half power of k appears so $k^{-1}\Delta_k P_k$ is even. The same argument works for $k^{-1}P_k\Delta_k$. ☐

In the sequel, to lighten the notation, we write π instead of π^I. Let L_1 and L_2 be the endomorphisms of $\mathcal{C}^\infty(M, \mathcal{S}(TM) \otimes \operatorname{End} A)$ defined by

$$L_1(f)(x) = \pi(x) \circ f(x) + f(x) \circ \pi(x) - f(x),$$
$$L_2(f)(x) = [\Box_x, f(x)].$$

Assuming that I satisfies (C), $\pi \in \mathcal{C}^\infty(M, \operatorname{End}(\mathcal{D}_{\leqslant p_0}(TM) \otimes \operatorname{End} A)$ for some p_0, so that L_1 is well-defined, meaning that $L_1(f)$ is a smooth section of $\mathcal{S}(TM) \otimes \operatorname{End} A$ when f is.

Lemma 6.4. *The following sequence is exact*:

$$0 \to \operatorname{Symb} \xrightarrow{L} \operatorname{Symb} \oplus \operatorname{Symb} \xrightarrow{L'} \operatorname{Symb} \to 0, \tag{65}$$

where $\operatorname{Symb} = \mathcal{C}^\infty(M, \mathcal{S}(TM) \otimes \operatorname{End} A)$, $L(f) = (L_1(f), L_2(f))$ *and* $L'(f_1, f_2) = L_2(f_1) - L_1(f_2)$.

Proof. $L' \circ L = 0$ is equivalent to $L_1 \circ L_2 = L_2 \circ L_1$, which follows from $[\Box, \pi] = 0$. Indeed $[\Box, \pi] = 0$ implies that $[\Box, f\pi] = [\Box, f]\pi$ and $[\Box, \pi f] = \pi[\Box, f]$ so that

$$\begin{aligned} L_2(L_1(f)) &= [\Box, f\pi + \pi f - f] \\ &= [\Box, f]\pi + \pi[\Box, f] - [\Box, f] = L_1(L_2(f)). \end{aligned}$$

Recall that $\operatorname{Symb} = \bigcup_{p \in \mathbb{N}} \operatorname{Symb}_p$, with $\operatorname{Symb}_p = \mathcal{C}^\infty(M, \operatorname{End}(\mathcal{D}_{\leqslant p}(TM) \otimes A))$. L_2 preserves each Symb_p and the same holds for L_1 when p is larger than p_0. So we have to prove that, for any $p \geqslant p_0$, the sequence (65) with Symb replaced by Symb_p is exact.

By Lemma 2.4, the image of π is a subbundle F of $\mathcal{D}_{\leqslant p}(TM) \otimes A$. Let $F^\perp$ be the orthogonal subbundle, so that $\mathcal{D}_{\leqslant p}(TM) \otimes A = F \oplus F^\perp$. Write the elements of Symb_p as block matrices according to this decomposition. The restrictions of π and $\Box$ to Symb_p have the particular forms

$$\pi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Box = \begin{pmatrix} \Box_{\text{in}} & 0 \\ 0 & \Box_{\text{out}} \end{pmatrix}$$

Writing

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have

$$L_1(f) = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}, \quad L_2(f) = \begin{pmatrix} [\Box_{\text{in}}, a] & E_1(b) \\ E_2(c) & [\Box_{\text{out}}, d] \end{pmatrix},$$

with

$$E_1(b) = \Box_{\text{in}} b - b\Box_{\text{out}}, \quad E_2(c) = \Box_{\text{out}} c - c\Box_{\text{in}} = -E_1(c^*)^*.$$

Let us prove that E_1 and E_2 are invertible endomorphisms of the spaces $\mathcal{C}^\infty(M, \operatorname{Hom}(F^\perp, F))$ and $\mathcal{C}^\infty(M, \operatorname{Hom}(F, F^\perp))$ respectively. For any $y \in M$, we introduce an orthonormal eigenbasis (e_i) of the restriction of $\Box_y$ to $\mathcal{D}_{\leqslant p}(T_y M) \otimes A_y$. So $\Box_y e_i = \lambda_i e_i$ and F_y (resp. $F_y^\perp$) is spanned by the e_i such that $\lambda_i \in I$ (resp. $\lambda_i \notin I$). Now the endomorphism

$$\operatorname{Hom}(F_y^\perp, F_y) \to \operatorname{Hom}(F_y^\perp, F_y), \quad b(y) \mapsto \Box_{\text{in}}(y) b(y) - b(y) \Box_{\text{out}}(y), \tag{66}$$

is diagonalizable with eigenvectors $|e_i\rangle\langle e_j|$ and eigenvalues $\lambda_i - \lambda_j$, where $\lambda_i \in I$ and $\lambda_j \notin I$. Since $\lambda_i - \lambda_j \neq 0$, (66) is invertible for any y, so the same holds for E_1. The proof for E_2 is similar.

From this, we deduce easily that the sequence is exact. In particular if $L'(f_1, f_2) = 0$ with

$$f_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \quad \text{for } i = 1 \text{ or } 2,$$

then $(f_1, f_2) = L(f)$ with

$$f = \begin{pmatrix} a_1 & E_1^{-1}(b_2) \\ E_2^{-1}(c_2) & -d_1 \end{pmatrix}.$$

Observe as well that $f_1 = f_1^*$ and $f_2 = -f_2^*$ imply that $f = f^*$. □

Proof of Lemma 6.2. Let $P \in \mathcal{L}^\infty(A)$ be self-adjoint with symbol $\sigma_0(P) = \pi$. Then $R_1 := P^2 - P$ and $R_2 := k^{-1}[\Delta_k, P]$ both belong to $\mathcal{L}_1^\infty(A)$. Indeed their σ_0-symbols are respectively $\pi^2 - \pi$ and $[\Box, \pi]$, and both of them vanish.

Let us prove by induction on $m \geqslant 1$ that there exists P as above such that R_1 and R_2 are in $\mathcal{L}_m^\infty(A)$. Define $P' = P + S$ with $S \in \mathcal{L}_m^\infty(A)$. Assume that R_1 and R_2 are in $\mathcal{L}_m^\infty(A)$. Then

$$(P')^2 - P' = R_1 + SP + PS - S \mod \mathcal{L}_{m+1}^\infty(A),$$
$$[k^{-1}\Delta_k, P'] = R_2 + [k^{-1}\Delta_k, S].$$

So $(P')^2 - P'$ and $[k^{-1}\Delta_k, P']$ belong to $\mathcal{L}_m(A)$ and their σ_m-symbols are respectively $f_1 + L_1(f)$ and $f_2 + L_2(f)$ with $f = \sigma_m(S)$, $f_1 = \sigma_m(R_1)$ and $f_2 = \sigma_m(R_2)$. Let us prove that we can choose f so that $f_1 + L_1(f) = 0$ and $f_2 + L_2(f) = 0$. By Lemma 6.4, it suffices to check that $L_1(f_2) = L_2(f_1)$. But $L_2(f_1)$ is the σ_m-symbol of $[k^{-1}\Delta_k, R_1]$, $L_1(f_2)$ is the σ_m-symbol of $PR_2 + R_2P - R_2$, and these operators are equal as shows a direct computation. So f exists. Furthermore $f = f^*$ by the remark at the end of the proof of Lemma 6.4. So we can choose S self-adjoint.

We conclude the proof with the convergence property with respect to the filtration $\mathcal{L}_m(A)$ recalled above. Observe also that if we start with $P \in \mathcal{L}^+(A)$, then we end with a formal projector in $\mathcal{L}^+(A)$. □

6C. ***Operator norm and pointwise estimates.*** Let us choose an operator (P_k) satisfying the conditions of Lemma 6.2. Recall that for any operator $Q \in \mathcal{L}_m(A)$, $Q_k = \mathcal{O}(k^{-m/2})$ in the sense that the operator norm of Q_k is in $\mathcal{O}(k^{-m/2})$. So P_k is self-adjoint, it is an almost projector $P_k^2 = P_k + \mathcal{O}(k^{-\infty})$ and it almost commutes with Δ_k in the sense that $[\Delta_k, P_k] = \mathcal{O}(k^{-\infty})$. Furthermore, by Lemma 6.1, $P_k = \Pi_k^I + \mathcal{O}(k^{-1/4})$.

Lemma 6.5. $$P_k = \Pi_k^I + \mathcal{O}(k^{-\infty}).$$

Proof. We omit the index k to simplify the notation. Let $\mathcal{H}_+ = \operatorname{Ran} \Pi^I$ and $\mathcal{H}_-$ be its orthogonal in $L^2(M, L^k \otimes A)$. We introduce the corresponding block decomposition of P

$$P = \begin{pmatrix} P_{++} & P_{+-} \\ P_{-+} & P_{--} \end{pmatrix}.$$

We first prove that P_{-+} and P_{+-} are in $\mathcal{O}(k^{-\infty})$.

By Corollary 4.8 and assumption (C), there exists ϵ such that when k is sufficiently large

$$\operatorname{dist}(I, \operatorname{sp}(k^{-1}\Delta_k) \setminus I) \geqslant \epsilon.$$

Let ξ_λ and ξ_μ be two eigenfunctions of $k^{-1}\Delta_k$ with eigenvalues λ and μ respectively. Then

$$\begin{aligned}(\lambda-\mu)\langle P\xi_\lambda, \xi_\mu\rangle &= k^{-1}(\langle P\Delta_k\xi_\lambda, \xi_\mu\rangle - \langle P\xi_\lambda, \Delta_k\xi_\mu\rangle)\\ &= k^{-1}\langle [P, \Delta_k]\xi_\lambda, \xi_\mu\rangle\\ &= \mathcal{O}(k^{-\infty})\|\xi_\lambda\|\,\|\xi_\mu\| \end{aligned}\tag{67}$$

because $[P, \Delta_k] = \mathcal{O}(k^{-\infty})$. Now for any $\xi_+ \in \mathcal{H}_+$ and $\xi_- \in \mathcal{H}_-$, write their decompositions into eigenvectors $\xi_+ = \sum \xi_\lambda$ and $\xi_- = \sum \xi_\mu$. So $\|\xi_+\|^2 = \sum \|\xi_\lambda\|^2$, $\|\xi_-\|^2 = \sum \|\xi_\mu\|^2$ and $\langle P\xi_-, \xi_+\rangle = \sum \langle P\xi_\lambda, \xi_\mu\rangle$. So by (67),

$$|\langle P\xi_-, \xi_+\rangle| \leqslant \epsilon^{-1}\mathcal{O}(k^{-\infty})\sum \|\xi_\lambda\|\,\|\xi_\mu\| \leqslant \epsilon^{-1}\mathcal{O}(k^{-\infty})\|\xi_-\|\,\|\xi_+\|$$

by the Cauchy–Schwarz inequality. This proves that $P_{+-} = \mathcal{O}(k^{-\infty})$. The same holds for its adjoint P_{-+}.

Now the fact that $P^2 = P + \mathcal{O}(k^{-\infty})$ implies $P_{++}^2 = P_{++} + \mathcal{O}(k^{-\infty})$ and the same for P_{--}. Indeed,

$$\begin{aligned}(\Pi^I P\Pi^I)^2 &= \Pi^I P\Pi^I P\Pi^I\\ &= \Pi^I P^2\Pi^I + \mathcal{O}(k^{-\infty}) \quad (\text{because } P_{-+} = \mathcal{O}(k^{-\infty}))\\ &= \Pi^I P\Pi^I + \mathcal{O}(k^{-\infty}) \quad (\text{because } P^2 = P + \mathcal{O}(k^{-\infty})).\end{aligned}$$

By Lemma 6.1, $P = \Pi^I + \mathcal{O}(k^{-1/4})$, so $P_{--} = \mathcal{O}(k^{-1/4})$. Then $P_{--}^2 = P_{--} + \mathcal{O}(k^{-\infty})$ implies

$$P_{--} = \mathcal{O}(k^{-\infty}).$$

In the same way, $(\mathrm{id}_{\mathcal{H}_+} - P_{++})^2 = \mathrm{id}_{\mathcal{H}_+} - P_{++} + \mathcal{O}(k^{-\infty})$ and $\mathrm{id}_{\mathcal{H}_+} - P_{++} = \mathcal{O}(k^{-1/4})$ imply $\mathrm{id}_{\mathcal{H}_+} - P_{++} = \mathcal{O}(k^{-\infty})$. So

$$P_{++} = \mathrm{id}_{\mathcal{H}_+} + \mathcal{O}(k^{-\infty}),$$

which concludes the proof. ☐

Lemma 6.6. *For any* $\ell, m \in \mathbb{N}$, *we have* $\Delta_k^\ell(P_k - \Pi_k^I)\Delta_k^m = \mathcal{O}(k^{-\infty})$.

Proof. On one hand, we have

$$\Delta_k^\ell P_k = \mathcal{O}(k^\ell), \qquad \Delta_k^\ell \Pi_k^I = \mathcal{O}(k^\ell), \tag{68}$$

where the first estimate is a consequence of $((k^{-1}\Delta_k)^\ell P_k) \in \mathcal{L}^\infty(A)$, and the second one is merely that Π_k^I is the spectral projector of $k^{-1}\Delta_k$ for the bounded interval I.

On the other hand, since for any $Q \in \mathcal{L}^\infty_\infty(A)$, $\Delta_k^\ell Q\Delta_k^m$ belongs to $\mathcal{L}^\infty_\infty(A)$ as well, we have

$$\begin{aligned}\Delta_k^\ell P_k^2\Delta_k^m &= \Delta_k^\ell P_k\Delta_k^m + \mathcal{O}(k^{-\infty}),\\ \Delta_k^\ell[k^{-1}\Delta_k, P_k]\Delta_k^m &= \mathcal{O}(k^{-\infty}).\end{aligned}\tag{69}$$

By the first equality, $\Delta_k^\ell P_k \Delta_k^m = \Delta_k^{\ell+m} P_k + \mathcal{O}(k^{-\infty})$. Since $[\Delta_k, \Pi_k^I] = 0$, it suffices to prove the final result for $m = 0$, that is, $\Delta_k^\ell (P_k - \Pi_k^I) = \mathcal{O}(k^{-\infty})$. We have

$$\begin{aligned}
\Delta_k^\ell (P_k - \Pi_k^I) &\overset{(69)}{=} \Delta_k^\ell (P_k^2 - \Pi_k^I) + \mathcal{O}(k^{-\infty}) \\
&= \Delta_k^\ell P_k (P_k - \Pi_k^I) + \Delta_k^\ell (P_k \Pi_k^I) - \Delta_k^\ell \Pi_k^I + \mathcal{O}(k^{-\infty}) \\
&\overset{(69)}{=} \Delta_k^\ell P_k (P_k - \Pi_k^I) + P_k \Delta_k^\ell \Pi_k^I - \Delta_k^\ell \Pi_k^I + \mathcal{O}(k^{-\infty}) \\
&= \Delta_k^\ell P_k (P_k - \Pi_k^I) + (P_k - \Pi_k^I) \Delta_k^\ell \Pi_k^I + \mathcal{O}(k^{-\infty}) \;=\; \mathcal{O}(k^\ell) \mathcal{O}(k^{-\infty})
\end{aligned}$$

by (68) and Lemma 6.5. □

We are now ready to conclude the proof of Theorems 5.2 and 5.3: we will show that the Schwartz kernel of $P_k - \Pi_k^I$ is in $\mathcal{O}_\infty(k^{-\infty})$, in the sense of Section 5D.

Choose two open sets U and U' of M equipped both with a set of coordinates and unitary trivializations of L and A, so that we can identify the sections of $L^k \otimes A$ on U with functions. Let $\varphi \in \mathcal{C}_0^\infty(U)$, $\varphi' \in \mathcal{C}_0^\infty(U')$. Then $\varphi(P_k - \Pi_k^I)\varphi'$ can be viewed as an operator of $\mathbb{R}^{2n}$. We introduce the differential operator

$$\Lambda_k = 1 - k^{-2} \sum_{i=1}^{2n} \partial_{x_i}^2$$

acting on $\mathcal{C}^\infty(\mathbb{R}^{2n})$.

Lemma 6.7. *For any $\ell \in \mathbb{N}$,*

$$\Lambda_k^\ell \, \varphi(P_k - \Pi_k^I)\varphi' \, \Lambda_k^\ell = \mathcal{O}(k^{-\infty}). \tag{70}$$

Consequently, the Schwartz kernel of $\varphi(P_k - \Pi_k^I)\varphi'$ is in $\mathcal{O}_\infty(k^{-\infty})$.

Proof. We will use basic results on semiclassical pseudodifferential operators of $\mathbb{R}^{2n}$, with the semiclassical parameter usually denoted by h equal here to k^{-1}. Choose $\psi_1, \psi_2 \in \mathcal{C}_0^\infty(U)$ such that $\operatorname{supp} \varphi \subset \{\psi_1 = 1\}$ and $\operatorname{supp} \psi_1 \subset \{\psi_2 = 1\}$. The operator $\psi_1(1 + (k^{-2}\Delta_k)^\ell)$, viewed as an operator of $\mathbb{R}^{2n}$, is a semiclassical differential operator with principal symbol $\psi_1(H^\ell + 1)$, where H is the symbol of Δ_k, so

$$H(x, \xi) = \sum g^{ij}(x)(\xi_i + \alpha_i(x))(\xi_j + \alpha_j(x)),$$

with $-i \sum \alpha_i \, dx_i$ the connection 1-form of L in the trivialization used to identify sections with functions. The operator $\varphi \Lambda_k^\ell$ is also a semiclassical differential operator with symbol $\varphi(x)\langle \xi \rangle^{2\ell}$. Since the symbol $\psi_1(H^\ell + 1)$ is elliptic on $\operatorname{supp} \varphi \times \overline{\mathbb{R}}^n$, we can factorize

$$\Lambda_k^\ell \varphi = Q_k \psi_1 (1 + (k^{-2}\Delta_k)^\ell) + S_k,$$

with Q_k a zero-order semiclassical pseudodifferential operator and S_k in the residual class. To do this, we only need the pseudodifferential calculus in the usual class $S_{1,0}^k(T^*\mathbb{R}^{2n})$ of symbols; see for instance [Dyatlov and Zworski 2019, Section E.1.5]. Composing with ψ_2,

$$\Lambda_k^\ell \varphi = Q_k \psi_1 (1 + (k^{-2}\Delta_k)^\ell) + S_k \psi_2. \tag{71}$$

Similarly, we have

$$\varphi' \Lambda_k^\ell = \psi_1'(1+(k^{-2}\Delta_k)^\ell) Q_k' + \psi_2' S_k'. \tag{72}$$

Now by Lemma 6.6,

$$(1+(k^{-2}\Delta_k)^\ell)(P_k - \Pi_k^I)(1+(k^{-2}\Delta_k)^m) = \mathcal{O}(k^{-\infty}), \tag{73}$$

and by the usual result on boundedness of pseudodifferential operators, see [Dyatlov and Zworski 2019, Proposition E.19], $Q_k, Q_k' = \mathcal{O}(1)$ and $S_k, S_k' = \mathcal{O}(k^{-\infty})$. We deduce (70) easily with (71), (72) and (73).

Now let H_k^m be the Sobolev space $H^m(\mathbb{R}^{2n})$ with the k-dependent norm $\|u\|_{H_k^m} = \|\langle k^{-1}\xi\rangle \hat{u}(\xi)\|_{L^2(\mathbb{R}^{2n})}$. Then Λ_k^ℓ is an isometry $H_k^m \to H_k^{m-2\ell}$. So (70) tells us that the operator norm $H_k^{-2\ell} \to H_k^{2\ell}$ of $R_k = \varphi(P_k - \Pi_k^I)\varphi'$ is in $\mathcal{O}(k^{-\infty})$. Since the Schwartz kernel of R_k at (x, y) is equal to $\delta_x(R_k \delta_y)$ and the Dirac δ_x belongs to H_k^{-m} with a norm in $\mathcal{O}(k^{2n})$ for any $m > n$, we have $R_k(x, y) = \mathcal{O}(k^{-\infty})$. Similarly, $\partial_x^\alpha \partial_y^\beta R_k(x, y) = \mathcal{O}(k^{-\infty})$ for any $\alpha, \beta \in \mathbb{N}^{2n}$ because the H_k^{-m}-norm of $\partial^\alpha \delta_x$ is a $\mathcal{O}(k^{2n})$ as soon as $m \geqslant n + |\alpha|$. □

7. Toeplitz operators

Let F be a vector subbundle of $\mathcal{D}_{\leqslant p}(TM) \otimes A$ for some p. Let $(\Pi_k) \in \mathcal{L}(A)$ such that, for each k, Π_k is a self-adjoint projector of $\mathcal{C}^\infty(M, L^k \otimes A)$ and, for any $x \in M$, the symbol $\pi(x) = \sigma_0(\Pi_k)(x)$ is the orthogonal projector onto F_x. Let $\mathcal{H}_k$ be the image of Π_k.

The corresponding Toeplitz operators are the $(P_k) \in \mathcal{L}(A)$ such that $\Pi_k P_k \Pi_k = P_k$. The symbol $\sigma_0(P)(x)$ of such an operator satisfies

$$\pi(x)\sigma_0(P)(x)\pi(x) = \sigma_0(P)(x).$$

So $\sigma_0(P)(x) = f(x)\pi(x)$, with $f(x) \in \operatorname{End} F_x$. This section f of $\operatorname{End} F$ can be considered as the Toeplitz symbol of (P_k).

We will establish several spectral results for these Toeplitz operators. Applied to the spectral projector $\Pi_k = 1_{[a,b]}(k^{-1}\Delta_k)$ and $P_k = k^{-1}\Delta_k \Pi_k$, this will complete the proofs of Theorems 1.1, 1.2, 1.3 and 1.4 stated in the Introduction.

7A. *Global spectral estimates.*

Theorem 7.1. (1) *When k is sufficiently large*, $\dim \mathcal{H}_k = \mathrm{RR}(L^k \otimes F)$.

(2) *For any $(P_k) \in \mathcal{L}(A)$ such that $P_k^* = P_k$ and $\Pi_k P_k \Pi_k = P_k$ for any k, we have for any $\Psi \in \mathcal{H}_k$ with $\|\Psi\| = 1$ that*

$$\inf_M f_- + \mathcal{O}(k^{-1/2}) \leqslant \langle P_k \Psi, \Psi\rangle \leqslant \sup_M f_+ + \mathcal{O}(k^{-1/2}), \tag{74}$$

where the $\mathcal{O}$'s are uniform with respect to Ψ and, for any $x \in M$, $f_-(x)$ and $f_+(x)$ are the smallest and largest eigenvalues of the restriction of $\sigma_0(P)(x)$ to F_x.

The proof is based on the generalized ladder operators introduced in [Charles 2024]: if $(A', F', \Pi_k', \mathcal{H}_k')$ is a second set of data satisfying the same assumption as $(A, F, \Pi_k, \mathcal{H}_k)$ and F, F' are isomorphic vector bundles, then there exist isomorphisms $U_k : \mathcal{H}_k \to \mathcal{H}_k'$ when k is sufficiently large. Then defining $\mathcal{H}_k'$

as the kernel of a well-chosen spin-c Dirac operator, $\dim \mathcal{H}'_k$ is given by the Atiyah–Singer theorem, which will prove the first statement. For the second one, choose $\mathcal{H}'_k$ so that $F' = A'$, $U_k P_k U_k^*$ is equal to a Toeplitz operator $\Pi'_k f \Pi'_k$ up to a $\mathcal{O}(k^{-1/2})$. The inspiration here comes from the proof of the sharp Gårding inequality for semiclassical pseudodifferential operator.

Proof. Consider a second self-adjoint projector $\Pi' \in \mathcal{L}(A')$ with $\sigma_0(\Pi')$ the orthogonal projector onto a vector bundle F' of $\mathcal{D}_{\leqslant p}(TM) \otimes A'$. Assume that F and F' are isomorphic vector bundles. Then there exists $u \in \mathcal{C}^\infty(M, \operatorname{Hom}(F, F'))$ such that, for any $x \in M$, $u(x)$ is a unitary isomorphism from F_x to F'_x. Extending $u(x)$ to a map $\mathcal{D}(T_x M) \otimes A_x \to \mathcal{D}(T_x M) \otimes A'_x$ which is zero on the orthogonal of F_x, we have

$$u^*(x)u(x) = \sigma_0(\Pi)(x), \quad u(x)u^*(x) = \sigma_0(\Pi')(x).$$

So if $(U_k) \in \mathcal{L}(A, A')$ has symbol u, then

$$U_k^* U_k = \Pi_k + \mathcal{O}(k^{-1/2}), \quad U_k U_k^* = \Pi'_k + \mathcal{O}(k^{-1/2}). \tag{75}$$

Furthermore replacing U_k by $\Pi'_k U_k \Pi_k$ does not modify the symbol of U_k so the same property holds and moreover $\Pi'_k U_k \Pi_k = U_k$. Consequently U_k restricts to an isomorphism from $\mathcal{H}_k$ to the image $\mathcal{H}'_k$ of Π'_k, when k is sufficiently large.

Hence for large k, the dimension of $\mathcal{H}_k$ only depends on the isomorphism class of F. To compute it, we introduce a spin-c Dirac operators D_k acting on $L^k \otimes A'$ with $A' = F \otimes \bigwedge^{0,\bullet} T^*M$ and define $\mathcal{H}'_k$ as the kernel of D_k. Then by a vanishing theorem [Borthwick and Uribe 1996; Ma and Marinescu 2002], $\dim \mathcal{H}'_k$ is equal to the index of D_k^+ when k is sufficiently large. By Atiyah–Singer index theorem, $\dim \mathcal{H}'_k = \operatorname{RR}(L^k \otimes F)$. Furthermore, it follows from [Ma and Marinescu 2007] that the projector (Π'_k) belongs to $\mathcal{L}(A')$, and $\sigma_0(\Pi'_k)$ is the projector onto $\mathbb{C} \otimes F \otimes \mathbb{C}$. Alternatively the vanishing theorem and the fact that $(\Pi'_k) \in \mathcal{L}(A')$ follows also from Corollary 4.8 and Theorem 5.2 applied to $D_k^- D_k^+$ as in the proof of Theorem 3.6.

To prove the second part, we choose $A' = F' = F$, that is, (Π'_k) belongs to $\mathcal{L}(F)$ and its symbol is the projection onto $\mathcal{D}_0(TM) \otimes F$. For instance, we can choose $\Pi'_k = 1_I(k^{-1}\Delta_k)$ with $I = \frac{1}{2}n + \left[-\frac{1}{2}, \frac{1}{2}\right]$ and Δ_k the magnetic Laplacian acting on $\mathcal{C}^\infty(M, L^k \otimes F)$ defined from any connection of F and the metric $\omega(\cdot, j\,\cdot)$ so that $\Sigma = \frac{1}{2}n + \mathbb{N}$.

Now let $P \in \mathcal{L}(A)$ be selfadjoint and such that $\Pi_k P_k \Pi_k = P_k$. Then the symbol $\sigma_0(P)(x)$ is self-adjoint and has the form $\sigma_0(P)(x) = f(x)\pi(x)$ with $f(x) \in \operatorname{End} F_x$. So $\sigma_0(P)(x) = u^*(x) f(x) u(x)$, and thus

$$P_k = U_k^* f U_k + \mathcal{O}(k^{-1/2}), \tag{76}$$

where f acts on $\mathcal{C}^\infty(M, L^k \otimes F)$ by pointwise multiplication. For any $\Psi' \in \mathcal{C}^\infty(M, L^k \otimes F)$,

$$(\inf_M f_-)\|\Psi'\|^2 \leqslant \langle f\Psi', \Psi'\rangle \leqslant (\sup_M f_+)\|\Psi'\|^2,$$

where $f_-(x)$ and $f_+(x)$ are the smallest and largest eigenvalues of $f(x)$ for any x. We conclude the proof by setting $\Psi' = U_k \Psi$ and using (75) and (76). □

Corollary 7.2. *Let (Δ_k) be a family of formally self-adjoint differential operators of the form* (B). *Let $a, b \in \mathbb{R} \setminus \Sigma$, with $a < b$. Then when k is sufficiently large*

$$\sharp\, \operatorname{sp}(k^{-1}\Delta_k) \cap [a, b] = \begin{cases} \operatorname{RR}(L^k \otimes F) & \text{if } [a, b] \cap \Sigma \neq \varnothing, \\ 0 & \text{otherwise}, \end{cases} \tag{77}$$

with F the bundle with fibers $F_x = 1_{[a,b]}(\square_x)$. Furthermore

$$\operatorname{sp}(k^{-1}\Delta_k) \cap [a, b] \subset [a, b] \cap \Sigma + \mathcal{O}(k^{-1/2}). \tag{78}$$

Proof. When $[a, b] \cap \Sigma$ is empty, we already know by Corollary 4.8 that $\operatorname{sp}(k^{-1}\Delta_k) \cap [a, b]$ is empty when k is sufficiently large. When $[a, b] \cap \Sigma \neq \varnothing$, by Theorem 5.2, the spectral projector $\Pi_k = 1_{[a,b]}(k^{-1}\Delta_k)$ belongs to $\mathcal{L}(A)$ with symbol $\pi = 1_{[a,b]}(\square)$. So the dimension of $\operatorname{Im} \Pi_k$ is given in the first assertion of Theorem 7.1.

Moreover, by Corollary 5.4, $(k^{-1}\Delta_k)\Pi_k$ belongs to $\mathcal{L}(A)$ and its symbol is $\square 1_{[a,b]}(\square)$. By the second assertion of Theorem 7.1,

$$\operatorname{sp}(k^{-1}\Delta_k) \cap [a, b] = \operatorname{sp}(k^{-1}\Delta_k \Pi_k) \subset [\inf f_-, \sup f_+] + \mathcal{O}(k^{-1/2})$$

where f is the restriction of $\square$ to $F = \operatorname{Im} \pi$.

This proves the inclusion (78) when $[a, b] \cap \Sigma$ is connected. Indeed,

$$[a, b] \cap \Sigma_y = [f_-(y), f_+(y)] \cap \Sigma_y.$$

So on one hand, M being compact, $\inf f_- = f(y_-)$ and $\sup f_+ = f(y_+)$ belongs to $[a, b] \cap \Sigma$. On the other hand $[a, b] \cap \Sigma \subset [\inf f_-, \sup f_+]$. Consequently $[a, b] \cap \Sigma = [\inf f_-, \sup f_+]$.

To treat the general case, we use that $[a, b] \cap \Sigma$ is a finite union of mutually disjoint compact intervals $I_1, \dots, I_\ell$. So there exists $a_1 = a < a_2 < \cdots < a_{\ell+1} = b$ in $\mathbb{R} \setminus \Sigma$ such that $I_i = [a_i, a_{i+1}] \cap \Sigma$ and by what we have proved, $\operatorname{sp}(k^{-1}\Delta_k) \cap [a_i, a_{i+1}] \subset I_i + \mathcal{O}(k^{-1/2})$. ☐

Remark 7.3. Decompose $\mathcal{D}(TM)$ into even and odd subspaces

$$\mathcal{D}^+(TM) = \bigoplus_{p \in \mathbb{N}} \mathcal{D}_{2p}(TM), \quad \mathcal{D}^-(TM) = \bigoplus_{p \in \mathbb{N}} \mathcal{D}_{2p+1}(TM).$$

Let us assume that (Π_k) is even and that F has a definite parity in the sense that F is a subbundle of $D^\epsilon(TM) \otimes A$ for $\epsilon = +$ or $-$. Then (74) and (78) hold with k^{-1} instead of $k^{-1/2}$.

Indeed, by [Charles 2024, Theorem 2.5], the σ_p-symbol of $P_k \in \mathcal{L}_p^+(A)$ has the same parity of p, meaning that $\sigma_p(P_k)$ sends $\mathcal{D}^\epsilon(TM) \otimes A$ into $\mathcal{D}^{\epsilon'}(TM) \otimes A$ with $\epsilon' = (-1)^p \epsilon$. So if an operator $(P_k) \in \mathcal{L}_1^+(A)$ is such that $\Pi_k P_k \Pi_k = P_k$, then its symbol $\sigma_1(P_k)$ is odd and has the form $g\pi$ for some $g \in \operatorname{End} F$. So g is odd, but F has a definite parity, so $g = 0$. Consequently $(P_k) \in \mathcal{L}_2^+(A)$. Moreover, by [Charles 2024, Theorem 3.4], we can construct $(U_k) \in \mathcal{L}(A, F)$ such that $U_k U_k^* = \mathrm{id}$ when k is sufficiently large and (U_k) has the same parity as F. So if $(P_k) \in \mathcal{L}^+(A)$, then $(U_k P_k U_k^*) \in \mathcal{L}^+(F)$. Then in the proof of Theorem 7.1, we can replace the $\mathcal{O}(k^{-1/2})$ in (75) and (76) by a $\mathcal{O}(k^{-1})$. ☐

7B. ***Local spectral estimates.***

Theorem 7.4. *Let $(P_k) \in \mathcal{L}(A)$ be such that $\Pi_k P_k \Pi_k = P_k$ and $P_k^* = P_k$. Let $f \in \mathcal{C}^\infty(M, \operatorname{End} F)$ be the restriction of $\sigma_0(P_k)$ to F.*

(1) *For any compact subsets C of M and I of $\mathbb{R}$ such that $I \cap \operatorname{sp}(f(x)) = \varnothing$ for any $x \in C$, we have for any N*

$$(\Pi_k 1_I(P_k)\Pi_k)(x,x) = \mathcal{O}(k^{-N}) \quad \text{for all } x \in C,$$

with a $\mathcal{O}$ uniform with respect to x.

(2) *For any $g \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$, $(\Pi_k g(P_k)\Pi_k)$ belongs to $\mathcal{L}(A)$ and its σ_0-symbol is $(g \circ f)\pi$. Moreover, if (Π_k) and (P_k) are in $\mathcal{L}^+(A)$, then the same holds for $(\Pi_k g(P_k)\Pi_k)$.*

Proof. Let U be the open set $\{x \in M : \operatorname{sp}(f(x)) \cap I = \varnothing\}$. Let $\varphi \in \mathcal{C}_0^\infty(U)$ and $\lambda \in I$. Observe that $\varphi(f-\lambda)^{-1} \in \mathcal{C}^\infty(M, \operatorname{End} F)$. So if $(Q_k) \in \mathcal{L}(A)$ has symbol $\varphi(f-\lambda)^{-1}\pi$, we have

$$\Pi_k Q_k \Pi_k (P_k - \lambda \Pi_k) = \Pi_k \varphi \Pi_k - R_k, \tag{79}$$

with $(R_k) \in \mathcal{L}_1(A)$. Let us improve this to obtain $(R_k) \in \mathcal{L}_\infty(A)$.

We need the following notion of support: for any $S \in \mathcal{L}(A)$, $\operatorname{supp} S$ is the closed set of M such that $x \notin \operatorname{supp} S$ if and only if $S_k(y,z) = \mathcal{O}(k^{-\infty})$ on a neighborhood of (x,x). Using that the Schwartz kernel of $S \in \mathcal{L}(A)$ is in $\mathcal{O}(k^{-\infty})$ on compact subsets of $M^2 \setminus \operatorname{diag} M$ and in $\mathcal{O}(k^n)$ on M^2, we prove that for any S, $S' \in \mathcal{L}(A)$ we have $\operatorname{supp}(SS') \subset (\operatorname{supp} S) \cap (\operatorname{supp} S')$.

Assume now that $(Q_k) \in \mathcal{L}(A)$ has the symbol $\varphi(f-\lambda)^{-1}\pi$ as above and is supported in U. Then $(R_k) \in \mathcal{L}_p(A)$ with $p \geqslant 1$, $\Pi_k R_k \Pi_k = R_k$ so that the symbol $r = \sigma_p(R_k)$ satisfies $\pi r \pi = r$. Furthermore, (R_k) is supported in U, so the same holds for r, so that $r(f-\lambda)^{-1} \in \mathcal{C}^\infty(M, \operatorname{End} F)$. Let $(Q'_k) \in \mathcal{L}_p(A)$ be supported in U and have symbol $\sigma_p(Q'_k) = r(f-\lambda)^{-1}\pi$. Then if we replace Q_k in (79) by $Q_k + Q'_k$, we have now $(R_k) \in \mathcal{L}_{p+1}(A)$. We deduce the existence of (Q_k) such that (79) holds with $(R_k) \in \mathcal{L}_\infty(A)$, so the operator norm of R_k is in $\mathcal{O}(k^{-\infty})$.

We claim that this construction can be realized so that we obtain an $\mathcal{O}(k^{-\infty})$ uniform with respect to $\lambda \in I$. To do this, we consider families

$$(S_k(\lambda)) \in \mathcal{L}(A), \quad \lambda \in I, \tag{80}$$

such that in the kernel expansion (50), the coefficients a_ℓ depend continuously on λ and the remainders $r_{N,k}$ are in $\mathcal{O}(k^{n-(N+1)/2})$ on compact subsets of U^2 with an $\mathcal{O}$ independent of λ. Then if $(S'_k(\lambda))$ is another family depending continuously on λ in the same sense, the same holds for the product $(S'_k(\lambda)S_k(\lambda))$. Furthermore, if $(S_k(\lambda)) \in \mathcal{L}_p(A)$ for any $\lambda \in I$, the operator norm of $S_k(\lambda)$ is in $\mathcal{O}(k^{-p/2})$ with an $\mathcal{O}$ independent of λ. The proof of these claims is the same as the proof of the same facts without λ. Later in (85), we will use these results again with the parameter λ describing a compact subset of $\mathbb{C}$.

Now we deduce from (79) with $\|R_k\| = \mathcal{O}(k^{-\infty})$ that, for any k, any normalized $\Psi \in \mathcal{H}_k$ such that $P_k\Psi = \lambda\Psi$ with $\lambda \in I$ satisfies $\langle \varphi\Psi, \Psi \rangle = \mathcal{O}(k^{-\infty})$ with an $\mathcal{O}$ independent of λ and Ψ. For any $x \in U$, we can choose φ equal to 1 on a neighborhood of x and we deduce the existence of a compact neighborhood V

of x, such that any Ψ as above satisfies

$$\int_V |\Psi(x)|^2 \, d\mu(x) = \mathcal{O}(k^{-\infty}).$$

Writing $\Psi = \Pi_k \Psi$ and using that the Schwartz kernel of Π_k is in $\mathcal{O}(k^n)$ on M^2 and in $\mathcal{O}(k^{-\infty})$ on compact subsets of M^2 not intersecting the diagonal, we get that on a neighborhood of x the pointwise norm of Ψ is in $\mathcal{O}(k^{-\infty})$. Since $(\Pi_k 1_I(P_k)\Pi_k)(x, x)$ is the sum of the $|\Psi_\ell(x)|^2$, where (Ψ_ℓ) is an orthonormal basis of $\mathcal{H}_k \cap \operatorname{Im} 1_I(P_k)$ consisting of eigenvectors of P_k, and $\dim \mathcal{H}_k = \mathcal{O}(k^n)$, we deduce that

$$(\Pi_k 1_I(P_k)\Pi_k)(x, x) = \mathcal{O}(k^{-\infty}) \quad \text{for all } x \in U,$$

with an $\mathcal{O}$ uniform on compact subsets of U. This ends the proof of the first assertion.

For the second assertion, since the operator norm of P_k is bounded independently of k, we can assume that $g \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{C})$. We will apply the Helffer–Sjöstrand formula, which we already used in a similar context for the functional calculus of Toeplitz operators [Charles 2003, Proposition 12]. So for $\widetilde{P}_k$ the restriction of P_k to $\mathcal{H}_k$, we have

$$g(\widetilde{P}_k) = \frac{1}{2\pi} \int_{\mathbb{C}} (\partial_{\bar z} \tilde g)(z)(z - \widetilde{P}_k)^{-1} \, |dz\, d\bar z|, \tag{81}$$

where $\tilde g \in \mathcal{C}_0^\infty(\mathbb{C}, \mathbb{C})$ is an extension of g such that $\partial_{\bar z} \tilde g$ vanishes to infinite order along the real axis [Zworski 2012, Theorem 14.8].

In the same way we proved (79), we can construct, for any $z \in \mathbb{C} \setminus \mathbb{R}$, $(Q_k(z)) \in \mathcal{L}(A)$ such that $\Pi_k Q_k(z) \Pi_k = Q_k(z)$ and

$$Q_k(z)(z - P_k) = \Pi_k - R_k(z), \tag{82}$$

with $(R_k(z)) \in \mathcal{L}_\infty(A)$. At the first step we set $Q_k(z) = \Pi_k \widetilde{Q}_k \Pi_k$, with $\widetilde{Q}_k(z)$ in $\mathcal{L}(A)$ having symbol $(z - f)^{-1}\pi$. We obtain (82) with $(R_k(z)) \in \mathcal{L}_1(A)$. Then if $(R_k(z)) \in \mathcal{L}_p(A)$ and has symbol $\sigma_p(R_k(z)) = r(z)$, we add to Q_k the operator $\Pi_k Q'_k(z) \Pi_k$, where $(Q'_k(z))$ is an operator of $\mathcal{L}_p(A)$ with symbol $\sigma(Q'_k(z)) = r(z)(z - f)^{-1}\pi$.

To apply this in (81), we need to control carefully the dependence with respect to z. For U an open set of M, we introduce the space $\mathcal{F}\mathcal{C}^\infty(U)$ consisting of family $(f(z, \cdot),\ z \in \mathbb{C} \setminus \mathbb{R})$ of $\mathcal{C}^\infty(U)$ having the form

$$g(z, x) = \frac{\sum_m a_m(x) z^m}{\sum_m b_m(x) z^m}$$

where the sums are finite, the coefficients a_m and b_m belong to $\mathcal{C}^\infty(U)$, and for any x the poles of $g(\,\cdot\,, x)$ lie on the real axis. Since $\mathcal{F}\mathcal{C}^\infty(U)$ is a $\mathcal{C}^\infty(U)$-module, we can define $\mathcal{F}\mathcal{C}^\infty(U, B)$ for any auxiliary bundle B as the space of z-dependent section of B on U with local representatives in $\mathcal{F}\mathcal{C}^\infty(U)$ for any z-independent frame of B on U.

Having in mind the construction of $Q_k(z)$ in (82), observe that $(z - f)^{-1}$ belongs to $\mathcal{F}\mathcal{C}^\infty(M, \operatorname{End} F)$. Moreover, $\mathcal{F}\mathcal{C}^\infty(U)$ being closed under product, for any $r(z) \in \mathcal{F}\mathcal{C}^\infty(M, \operatorname{End} F)$, we have $r(z)(z - f)^{-1} \in \mathcal{F}\mathcal{C}^\infty(M, \operatorname{End} F)$.

Now we introduce the space $\mathcal{FL}(A)$ consisting of families $(P_k(z), z \in \mathbb{C} \setminus \mathbb{R})$ of $\mathcal{L}(A)$ such that in the asymptotic expansion (50) satisfied by the Schwartz kernel of $P_k(z)$, the coefficients have the form

$$a_\ell(z, x, \xi) = \sum a_{\ell,\alpha}(z, x)\xi^\alpha, \tag{83}$$

with $a_{\ell,\alpha} \in \mathcal{FC}^\infty(U, \operatorname{End}\mathbb{C}^r)$, and each remainder $r_{N,k}$ is in $\mathcal{O}(k^{n-(N+1)/2})$ uniformly on $K \cap ((\mathbb{C}\setminus\mathbb{R}) \times U^2)$, where K is any compact subset of $\mathbb{C} \times U^2$. We claim that we can choose $Q_k(z) \in \mathcal{FL}(A)$ in (82). To see this, it suffices to prove that

$$S(z) \in \mathcal{FL}(A) \quad \Longrightarrow \quad \Pi_k S_k(z) \Pi_k (z - P_k) \in \mathcal{FL}(A), \tag{84}$$

and then to use what we said before on $r(z) \circ (z - f)^{-1}$. To prove (84), it suffices to show that, for any $S(z) \in \mathcal{FL}(A)$ and $T \in \mathcal{L}(A)$ independent of z, $TS(z)$ and $S(z)T$ belong to $\mathcal{FL}(A)$. To prove this, we can assume that the Schwartz kernel of $T(z)$ is contained in a compact subset of U^2 independent of k and z, where we have the expansion (50), and we can treat each term of the expansion independently of the others. Suppose we only have $a_\ell(z, x, \xi)$. Then by (83), $S(z) = \sum S_\alpha a_{\ell,\alpha}(z, \cdot)$, where the sum is finite, $S_\alpha \in \mathcal{L}_\ell(A)$ and does not depend on z. Since $TS(z) = \sum (TS_\alpha) a_{\ell,\alpha}(z, \cdot)$ and $TS_\alpha \in \mathcal{L}_\ell(A)$, for any α, $TS(z)$ belongs to $\mathcal{FL}_\ell(A)$. The product $S(z)T$ is more delicate to handle. By the same proof as [Charles 2016, Lemma 5.11], for any compact set K of U, there exists a family $(T_\beta, \beta \in \mathbb{N}^{2n})$ such that $T_\beta \in \mathcal{L}_{|\beta|}(A)$, and for any $f \in \mathcal{C}^\infty_K(U)$ we have $fT = \sum_{|\beta| \leqslant N} T_\beta(\partial^\beta f)$ modulo $\mathcal{L}_{N+1}(A)$. Consequently

$$S(z)T = \sum_\alpha S_\alpha (a_{\ell,\alpha}(z, \cdot)T) = \sum_{\alpha, |\beta| \leqslant N} S_\alpha T_\beta (\partial^\beta a_{\ell,\alpha}(z, \cdot))|vtw$$

modulo $\mathcal{L}_{N+1}(A)$. To conclude we use that $S_\alpha T_\beta \in \mathcal{L}_{\ell+|\beta|}(A)$ and $\partial^\beta a_{\ell,\alpha} \in \mathcal{FC}^\infty(U, \operatorname{End}\mathbb{C}^r)$.

Now the function $\xi(z) = (\operatorname{Im} z)^{-1} \partial_{\bar z} \tilde g(z)$ vanishes to infinite order along the real axis and its support is contained in the compact set $K = \operatorname{supp} \tilde g$. For any $f \in \mathcal{FC}^\infty(U)$, the product $\xi(z) f(z, \cdot)$ extends smoothly to $\mathbb{C}$. We deduce that there exists a family $(S_k(z))$ of $\mathcal{L}(A)$ depending continuously of $z \in K$ in the same sense as (80), and such that

$$\Pi_k S_k(z) \Pi_k = S_k(z), \quad S_k(z)(z - P_k) = \xi(z)\Pi_k + \mathcal{O}(k^{-\infty}), \tag{85}$$

with an $\mathcal{O}$ uniform with respect to z. Since $\|(z - \widetilde{P}_k)^{-1}\| = \mathcal{O}(|\operatorname{Im} z|^{-1})$, multiplying the last equality by $(\operatorname{Im} z)(z - \widetilde{P}_k)^{-1}$, we obtain

$$\partial_{\bar z} \tilde g(z)(z - \widetilde{P}_k)^{-1} \Pi_k = (\operatorname{Im} z) S_k(z) + R_k(z), \tag{86}$$

with $R_k(z) = \mathcal{O}(k^{-\infty})$. Since $\Pi_k R_k(z) \Pi_k = R_k(z)$ and the Schwartz kernel of Π_k is in $\mathcal{O}(k^n)$, this implies that the Schwartz kernel of $R_k(z)$ is in $\mathcal{O}(k^{-\infty})$ uniformly with respect to z. Inserting (86) in (81), it comes that $(g(\widetilde{P}_k)\Pi_k)$ belongs to $\mathcal{L}(A)$. To see this, we simply have to integrate with respect to z the coefficients $a_\ell(z, x, \xi)$ in the expansion (50) of the Schwartz kernel of $(\operatorname{Im} z) S_k(z)$. Since $\sigma_0((\operatorname{Im} z) S_k(z)) = \partial_{\bar z} \tilde g(z)(z - f)^{-1}\pi$, we deduce also that

$$\sigma_0(g(\widetilde{P}_k)\Pi_k) = \frac{1}{2\pi} \int_{\mathbb{C}} \partial_{\bar z} \tilde g(z)(z - f)^{-1} \pi \, |dz \, d\bar z| = g(f)\pi,$$

which concludes the proof. ☐

Corollary 7.5. *Let* (Δ_k) *be a family of formally self-adjoint differential operators of the form* (B). *Let* $\Lambda \in \mathbb{R} \setminus \Sigma$. *Then for any* $g \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$ *supported in* $]-\infty, \Lambda]$, $(g(k^{-1}\Delta_k))$ *belongs to* $\mathcal{L}^+(A)$ *and has symbol* $g(\Box)$.

Proof. Since g is supported in $]-\infty, \Lambda]$, we have $g(k^{-1}\Delta_k) = \Pi_k g(k^{-1}\Delta_k \Pi_k)\Pi_k$ where $\Pi_k = 1_{]-\infty,\Lambda]}(k^{-1}\Delta_k)$. By Theorem 5.2 and Corollary 5.4, (Π_k) and $k^{-1}\Delta_k\Pi_k$ belong to $\mathcal{L}^+(A)$ with symbols $\pi = 1_{]-\infty,\Lambda]}(\Box)$ and $f = g(\Box)$. So the result follows from the second assertion of Theorem 7.4. □

This proves the second part of Theorem 1.4. We end this section with the proof of the local Weyl laws, Theorem 1.3. The proof works for any (Δ_k) of the form (B).

Proof of Theorem 1.3. We use the same notation as in Corollary 7.5 and its proof. Let $a, b \in]-\infty, \Lambda] \setminus \Sigma_y$. We have $\operatorname{sp}(f(y)) = \Sigma_y \cap]-\infty, \Lambda]$. When $[a, b] \cap \Sigma_y$ is empty, the first part of Theorem 7.4 implies that $N(y, a, b, k) = \mathcal{O}(k^{-\infty})$. To the contrary, assume that $[a, b] \cap \Sigma_y = \{\lambda\}$. Then choose a function $g \in \mathcal{C}_0^\infty(]a, b[, \mathbb{R})$ which is equal to 1 on $]\lambda-\epsilon, \lambda+\epsilon[$ for some $\epsilon > 0$. Since $N(y, a, \lambda-\epsilon, k) = \mathcal{O}(k^{-\infty})$ and $N(y, \lambda+\epsilon, b, k) = \mathcal{O}(k^{-\infty})$ by the first part of the proof,

$$N(y, a, b, k) = g(k^{-1}\Delta_k)(y, y) + \mathcal{O}(k^{-\infty}).$$

Since $g(k^{-1}\Delta_k)$ is in $\mathcal{L}^+(A)$ and has symbol $g(\Box)$, we have by [Charles 2024, Theorem 2.2, Assertion 5 and Proposition 5.6]

$$g(k^{-1}\Delta_k)(y, y) = \left(\frac{k}{2\pi}\right)^n \sum_{\ell=0}^{\infty} m_{\ell,\lambda} k^{-\ell} + \mathcal{O}(k^{-\infty}),$$

with $m_{0,\lambda} = \operatorname{tr} g(\Box)(y)$, so $m_{0,\lambda}$ is the multiplicity of λ as an eigenvalue of $\Box_y$. □

8. Miscellaneous proofs

Proof of Lemma 4.5. This is essentially Darboux lemma with parameters. We can adapt the proof presented in [McDuff and Salamon 2017, Section 3.2]. A more efficient approach based on [Bursztyn et al. 2019] is as follows. First, if r is sufficiently small, for any y, the exponential map $\exp_y : T_yM \to M$ restricts to an embedding from $B_y(r)$ into M. Identify $U = \exp_y(B_y(r))$ with an open set of T_yM. We are looking for a diffeomorphism φ defined on a neighborhood of the origin of T_yM such that $\varphi(0) = 0$, $T_0\varphi = \mathrm{id}$ and $\varphi^*\omega$ is constant. The important point is to define φ in such a way that it depends smoothly on y.

Let α be the primitive of ω on U obtained by radial homotopy. So

$$\alpha_x(v) = \int_0^1 \omega_{tx}(tx, v)\, dt, \qquad x \in U,\ v \in T_yM, \tag{87}$$

and $d\alpha = \omega$. Let X be the vector field of U such that $\iota_X\omega = 2\alpha$. By the Poincaré lemma, $\mathcal{L}_X\omega = 2\omega$. Furthermore, linearizing α at the origin, we see that $X = E + \mathcal{O}(2)$, with E the Euler vector field of T_yM. Since $Z = X - E$ vanishes to second order at the origin, the family $Z_t(x) := Z(tx)/t^2$ extends smoothly at $t = 0$. Let φ_t be the flow of the time-dependent vector field Z_t of U, that is, $\varphi_0(x) = x$ and $\dot{\varphi}_t(x) = Z_t(\varphi_t(x))$. Since Z_t is zero at the origin, φ_1 is a germ of a diffeomorphism of $(T_yM, 0)$.

By the proof of Lemma 2.4 in [Bursztyn et al. 2019], $\varphi_1^* X = E$, where the pull-back is defined by $\varphi_1^* X = (\varphi_1^{-1})_* X$. So $\mathcal{L}_X \omega = 2\omega$ implies that $\mathcal{L}_E \varphi_1^* \omega = 2\varphi_1^* \omega$. So $\varphi_1^* \omega$ is constant.

To conclude, observe that φ_1 depends smoothly on y because α given in (87) depends smoothly on y, so the same holds for X and Z_t, and the solution of a first-order differential equation depending smoothly on a parameter, is smooth with respect to the parameter. Finally the radius r_0 is chosen so that φ_1 is defined on $B_y(r_0)$. Since M is compact, we can choose $r_0 > 0$ independent of y. □

Proof of Lemma 4.6. Let d be the geodesic distance of M associated to our Riemannian metric. Starting from $d(y, \exp_y(\xi)) = \|\xi\|$ when ξ is sufficiently close to the origin, we get

$$C^{-1} \|\xi\| \leqslant d(y, \Psi_y(\xi)) \leqslant C\|\xi\| \tag{88}$$

for any $\xi \in B_y(r_1)$ with r_1 sufficiently small. So if $B(y, r)$ is the open ball of the metric space (M, d), then $\Psi_y(B_y(r)) \subset B(y, rC))$ and $B(y, r) \subset \Psi_y(B_y(rC))$. Define

$$v_-(\epsilon) = \inf\{\mathrm{vol}(B(y, \epsilon)) : y \in M\}, \quad v_+(\epsilon) = \sup\{\mathrm{vol}(B(y, \epsilon)) : y \in M\}.$$

Then, replacing C by a larger constant if necessary, when ϵ is sufficiently small, $C^{-1}\epsilon^{2n} \leqslant v_-(\epsilon)$ and $v_+(\epsilon) \leqslant C\epsilon^{2n}$.

For any $\epsilon > 0$, choose a maximal subset $J(\epsilon)$ of M such that the balls $B(y, \epsilon/2)$, $y \in J(\epsilon)$, are mutually disjoint. From the maximality, $M \subset \bigcup_{y \in J(\epsilon)} B(y, \epsilon)$ so that the sets $U_y(\epsilon) := \Psi_y(B_y(\epsilon C))$, $y \in J(\epsilon)$, cover M. For any $x \in M$, let $N(x, \epsilon)$ be the number of $y \in J(\epsilon)$ such that $x \in U_y(\epsilon)$. If $x \in U_y(\epsilon)$, by triangle inequality, $B(y, \epsilon/2) \subset B(x, \epsilon(1+C^2))$. Since the balls $B(y, \epsilon/2)$, $y \in J(\epsilon)$ are mutually disjoint, we have

$$N(x, \epsilon) v_-(\epsilon/2) \leqslant \mathrm{vol}(B(x, \epsilon(1 + C^2))) \leqslant v_+(\epsilon(1 + C^2))$$

So $N(x, \epsilon) \leqslant C^2(2(1 + C^2))^{2n}$. Thus the multiplicity of the cover $U_y(\epsilon)$, $y \in J(\epsilon)$, is bounded independently of ϵ. □

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VARIATIONAL METHODS FOR THE KINETIC FOKKER–PLANCK EQUATION

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We develop a functional-analytic approach to the study of the Kramers and kinetic Fokker–Planck equations which parallels the classical H^1 theory of uniformly elliptic equations. In particular, we identify a function space analogous to H^1 and develop a well-posedness theory for weak solutions in this space. In the case of a conservative force, we identify the weak solution as the minimizer of a uniformly convex functional. We prove new functional inequalities of Poincaré- and Hörmander-type and combine them with basic energy estimates (analogous to the Caccioppoli inequality) in an iteration procedure to obtain the C^∞ regularity of weak solutions. We also use the Poincaré-type inequality to give an elementary proof of the exponential convergence to equilibrium for solutions of the kinetic Fokker–Planck equation which mirrors the classic dissipative estimate for the heat equation. Finally, we prove enhanced dissipation in a weakly collisional limit.

1. Introduction

1A. ***Motivation and informal summary of results.*** We develop a well-posedness and regularity theory for weak solutions of the hypoelliptic equation

$$-\Delta_v f + v \cdot \nabla_v f + v \cdot \nabla_x f + \boldsymbol{b} \cdot \nabla_v f = f^* \quad \text{in } \mathbb{T}^d \times \mathbb{R}^d. \tag{1-1}$$

The unknown function $f(x, v)$ is a function of the position variable $x \in \mathbb{T}^d$ and the velocity variable $v \in \mathbb{R}^d$. The PDE (1-1) is sometimes called the *Kramers equation*. We also consider the time-dependent version of this equation, namely

$$\partial_t f - \Delta_v f + v \cdot \nabla_v f + v \cdot \nabla_x f + \boldsymbol{b} \cdot \nabla_v f = f^* \quad \text{in } (0, \infty) \times \mathbb{T}^d \times \mathbb{R}^d, \tag{1-2}$$

which is often called the *kinetic Fokker–Planck equation*.

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These equations were first studied in [Kolmogorov 1934] and were the main motivating examples for the general theory of [Hörmander 1967] of hypoelliptic equations. They are of physical interest due to their relation with the *Langevin diffusion process* formally defined by

$$\ddot{X} = \boldsymbol{b}(X) - \dot{X} + \dot{B}, \tag{1-3}$$

where $\dot{X}$, $\ddot{X}$ stand respectively for the first and second time derivatives of X, a stochastic process taking values in $\mathbb{R}^d$, and $\dot{B}$ denotes a white noise process. Equation (1-3) can be interpreted as Newton's law of motion for a particle subject to the force field $\boldsymbol{b}(X)$, friction and thermal noise. This process can be recast as a Markovian evolution for the pair (X, V) evolving according to

$$\begin{cases} \dot{X} = -V, \\ \dot{V} = -\boldsymbol{b}(X) - V - \dot{B}. \end{cases}$$

The infinitesimal generator of this Markov process is the differential operator appearing on the left side of (1-1).

Kolmogorov [1934] gave an explicit formula for the fundamental solution of (1-2) in the case $\boldsymbol{b} = 0$ and $U = \mathbb{R}^d$, which gives the existence of smooth solutions of (1-1) and (1-2) and implies that the operators on the left sides of (1-1) and (1-2) are *hypoelliptic*—that is, if f is a distributional solution of either of these equations and f^* is smooth, then f is also smooth. This result is extended to more general equations in the celebrated paper [Hörmander 1967], where he gave an essentially complete classification of hypoelliptic operators. In the case of the particular equations (1-1) and (1-2), his arguments yield a more systematic proof of Kolmogorov's results and, in particular, interior regularity estimates.

The study of hypoelliptic equations often falls back on the theory of pseudodifferential operators; see for example Kohn's proof [1973] of Hörmander's classical result [1967], which is included in the monograph [Hörmander 1985]. The purpose of this paper is rather to present a functional-analytic and variational theory for (1-1) and (1-2) which has strong analogies to the familiar theory of uniformly elliptic equations. In particular, in this paper we

- identify a function space H^1_{hyp} based on the natural energy estimates and develop a notion of weak solutions in this space;
- prove functional inequalities for H^1_{hyp}, for instance a Poincaré-type inequality, which implies uniform coercivity of our equations and holds not just on the spatial domain $\mathbb{T}^d$ but on any C^1 domain;
- develop a well-posedness theory of weak solutions based on the minimization of a uniformly convex functional;
- develop a regularity theory for weak solutions, based on an iteration of energy estimates, which implies that weak solutions are smooth;
- prove dissipative estimates for solutions of (1-2), using the coercivity of the variational structure, which imply an exponential decay to equilibrium.

Such a theory has until now remained undeveloped, despite the attention these equations have received in the last half century. The definition of the space H^1_{hyp} is not new: it and variants of it have been studied previously in [Baouendi and Grisvard 1968; Papanicolaou and Varadhan 1985; Carrillo 1998]. However, the functional inequalities and other key properties which are required to work with this space are established here. A robust notion of weak solutions and corresponding well-posedness theory—besides allowing one to prove classical results for (1-1) and (1-2) in a different way—is important because it provides a natural framework for studying the stability of solutions (i.e., proving that a sequence of approximate solutions converges to a solution). In fact, it is just such an application—namely, developing a theory of homogenization for (1-2)—which motivated the present work. Furthermore, we expect that the theory developed here will provide a closer link between the hypoelliptic equations (1-1) and (1-2) and the classical theory of uniformly elliptic and parabolic equations, allowing, for example, for a more systematic development of regularity estimates for solutions of the former by analogy to the latter. For instance, it would be interesting to investigate a possible connection between the functional-analytic framework proposed in this paper and the recent works [Wang and Zhang 2009; 2011; Golse et al. 2019; Mouhot 2018], which develop De Giorgi–Nash-type Hölder estimates for generalizations of the kinetic Fokker–Planck equations with measurable coefficients.[1]

In the first part of the paper, we address the well-posedness of (1-1) under a weak formulation based on the Sobolev-type space $H^1_{\mathrm{hyp}}(\mathbb{T}^d)$, defined below in (1-10). In the case in which $\boldsymbol{b}$ is a potential field, we provide two proofs of well-posedness. The first relies on the abstract Lax–Milgram theorem, while the second identifies a *uniformly convex* functional that has the sought-after weak solution as its unique minimizer. The identification of the correct convex functional is inspired by [Brezis and Ekeland 1976a; 1976b] on variational formulations of parabolic equations (see also the more recent [Ghoussoub 2009; Armstrong et al. 2018]). The proof that our functional is coercive relies on a new Poincaré-type inequality for H^1_{hyp}; see Theorem 1.3 below. The Poincaré inequality in fact holds in a much more general setting than the periodic setting in which we consider (1-1). Our convex-analytic arguments for well-posedness can be immediately adapted to cover nonlinear equations such as those obtained by replacing $\Delta_v f$ in (1-1) with $\nabla_v \cdot (\boldsymbol{a}(x, v, \nabla_v f))$ for $p \mapsto \boldsymbol{a}(x, v, p)$ a Lipschitz and uniformly maximal monotone operator (uniformly over $x \in \mathbb{T}^d$ and $v \in \mathbb{R}^d$).

Roughly speaking, the norm $\|\cdot\|_{H^1_{\mathrm{hyp}}(U)}$ is a measure of the size of the vector fields $\nabla_v f$ and $v \cdot \nabla_x f$, but crucially, the former is measured in a strong $L^2_x L^2_v$-type norm and the latter in a weaker $L^2_x H^{-1}_v$-type norm (see (1-10) below). The importance of measuring the vector fields $\nabla_v f$ and $v \cdot \nabla_x f$ using different norms also features prominently in other works including [Bouchut 2002], but only spaces of positive regularity are considered there. Measuring the term $v \cdot \nabla_x f$ in a space of negative regularity in the v-variable is related to the idea of *velocity averaging*, the idea that one should expect better control of the spatial regularity of a solution of (1-1) or (1-2) after averaging in the velocity variable. This concept is therefore wired into the definition of the H^1_{hyp} norm, allowing us to perform velocity averaging in a systematic way. Once we have proved the existence of weak solutions to (1-1) in H^1_{hyp}, we are interested in showing that these solutions are in fact smooth. It is elementary to verify that the differential operators ∇_v and $v \cdot \nabla_x$ satisfy Hörmander's bracket condition, and therefore, as exposed in [Hörmander 1967], a control

[1]We refer to [Guerand and Imbert 2022; Anceschi and Rebucci 2022], which appeared after the first version of the present paper.

of both $\nabla_v f$ and $v \cdot \nabla_x f$ in $L^2_x L^2_v$ would yield control of the seminorm of the function f in a fractional Sobolev space of positive regularity, namely $H^{1/2}_x L^2_v$. However, since the natural definition of the function space $H^1_{\mathrm{hyp}}(U)$ provides us only with control of $v \cdot \nabla_x f$ in a space of *negative* regularity in v, we are forced to revisit the arguments of [Hörmander 1967]. A key step there is an interpolation-type inequality which converts the $L^2_x H^{-1}_v$ control on $v \cdot \nabla_x f$ (i.e., "velocity averaged" regularity) and $L^2_x H^1_v$ regularity on f into $L^2_x L^2_v$ regularity for a type of "fractional derivative" $(v \cdot \nabla_x)^{1/2} f$.[2] With this interpolation in hand, we then prove a functional inequality (see Theorem 1.4 below) which asserts that the $H^1_{\mathrm{hyp}}(U)$ norm controls exactly one-third of a derivative in arbitrary x-directions in the space $L^2_x L^2_v$ in a weaker (Besov) sense, and almost one-third of a derivative in a stronger (Sobolev) sense. The one-third exponent is identical to that in Hörmander's paper and is sharp.[3]

Once we have proved that an arbitrary H^1_{hyp} function possesses at least a fractional derivative in the x-variable, we are in a position to iterate the estimate by repeatedly differentiating the equation a fractional number of times to obtain higher regularity (and eventually smoothness, under appropriate assumptions on $\boldsymbol{b}$ and f^*) of weak solutions. In order to perform this iteration, we again depart from the original arguments of [Hörmander 1967] and subsequent treatments and rely on an appropriate version of the Caccioppoli inequality (i.e., the basic L^2 energy estimate) for (1-1). This avoids any recourse to sophisticated pseudodifferential operators and once again mimics the classical functional-analytic arguments in the uniformly elliptic setting.

The developments described above and even the variational structure identified for (1-1) are not restricted to the time-independent setting. Indeed, we show that they can be adapted in a very straightforward way to the kinetic Fokker–Planck equation (1-2), the main difference being that the first-order part in a "sum-of-squares" representation of the differential operator is now $\partial_t + v \cdot \nabla_x$ instead of just $v \cdot \nabla_x$. The adaptation thus consists in replacing the latter by the former throughout; the natural function space associated with (1-2), denoted by H^1_{kin}, is defined in (6-2)–(6-3). We also prove a Poincaré inequality for functions in H^1_{kin} which implies the uniform coercivity of the variational structure with respect to the H^1_{kin} norm. This allows us to give a rather direct and natural proof of exponential long-time decay to equilibrium for solutions of (1-2) with constant-in-time right-hand sides. This result (stated in Theorem 1.6 below) can be compared with the celebrated results of exponential convergence to equilibrium for kinetic Fokker–Planck equations on $\mathbb{R}^d$ with confining potentials; see in particular [Desvillettes and Villani 2001; Hérau and Nier 2004; Helffer and Nier 2005; Eckmann and Hairer 2003; Desvillettes and Villani 2005; Villani 2009; Baudoin 2017; Dolbeault et al. 2015]; see also [Camrud et al. 2022; Talay 1999; 2002;

[2]The analogous estimate for the heat equation is $f \in H^{1/2}_t L^2_x$.

[3]When translating [Hörmander 1967] into the present setting, the vector field is $X_0 = \partial_t + v \cdot \nabla_x$, and for simplicity we consider the "flat case" in which $X_1 = \nabla_v$. The regularity along X_0 is of index $\frac{1}{2}$, while the regularity along X_1 is of index 1. Then Theorem 4.3 of [Hörmander 1967] gives regularity along the commutator $\nabla_x = [X_1, X_0]$ of index $\frac{1}{3}$, since $1/\left(\frac{1}{3}\right) = 1/1 + 1/\left(\frac{1}{2}\right)$. In addition, the exponent $\frac{1}{3}$ arises naturally in the following way: consider $\partial_t f + v \cdot \nabla_x f - \varepsilon \Delta_v f = 0$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$. Dimensionally speaking, $[f] = M,\ [x] = L,\ [v] = L/T$, and $[\varepsilon] = L^2/T^3$. The above PDE has a two-parameter scaling symmetry which keeps ε fixed, namely, $f \to \rho f(\lambda^{2/3} t, \lambda x, \lambda^{1/3} v),\ \lambda, \rho > 0$. Here, ε is considered "dimensionless": $[\varepsilon] = 1$, that is, we identify $L^2 \sim T^3$. In this convention, the unique exponent α for which $\|(-\Delta)^{\alpha/2}_x f\|_{L^2_{t,x,v}}$ has the same dimensions as $\|\nabla_v f\|_{L^2_{t,x,v}}$ is $\alpha = \frac{1}{3}$. Furthermore, the "flat case" is the formal limit of (1-2) upon "zooming in."

Grothaus and Stilgenbauer 2015] for a probabilistic approach. Compared to previous approaches, our proof of exponential convergence is once again closer to the classical dissipative argument for the heat equation based on differentiating the square of the spatial L^2 norm of the solution. Informally, our method is based on the idea that hypocoercivity is simply coercivity with respect to the correct norm.

1B. *Statements of the main results.* We begin by introducing the Sobolev-type function space H^1_{hyp} associated with (1-1). We let $U \subseteq \mathbb{R}^d$ either be a bounded C^1 domain with boundary, or we consider the boundary-less settings of $\mathbb{R}^d$ itself or the torus $\mathbb{T}^d$ with periodic boundary conditions. While we do not prove unique solvability in H^1_{hyp} of the Dirichlet problem in bounded C^1 domains, we nonetheless can prove the Poincaré inequality, so we study the two settings (with and without boundary) in tandem. We denote by γ the standard Gaussian measure on $\mathbb{R}^d$, defined by

$$d\gamma(v) := (2\pi)^{-d/2} \exp\bigl(-\tfrac{1}{2}|v|^2\bigr)\, dv. \tag{1-4}$$

For each $p \in [1, \infty)$, we denote by $L^p_\gamma := L^p(\mathbb{R}^d, d\gamma)$ the Lebesgue space with norm

$$\|f\|_{L^p_\gamma} := \left(\int_{\mathbb{R}^d} |f(v)|^p \, d\gamma(v)\right)^{1/p},$$

and by H^1_γ the Banach space with norm

$$\|f\|_{H^1_\gamma} := (\|f\|^2_{L^2_\gamma} + \|\nabla f\|^2_{L^2_\gamma})^{1/2}.$$

The dual space of H^1_γ is denoted by H^{-1}_γ. By abuse of notation, we typically denote the canonical pairing $\langle \cdot , \cdot \rangle_{H^1_\gamma, H^{-1}_\gamma}$ between $f \in H^1_\gamma$ and $f^* \in H^{-1}_\gamma$ by

$$\int_{\mathbb{R}^d} f f^* \, d\gamma := \langle f, f^* \rangle_{H^1_\gamma, H^{-1}_\gamma}. \tag{1-5}$$

Concerning the vector field $\boldsymbol{b}$, we shall often make the following assumption. Throughout the rest of the paper, we shall remind the reader when this assumption is in effect, or when we take more general vector fields $\boldsymbol{b}$.

Assumption 1.1. There exists $W \in C^{0,1}(U; \mathbb{R})$ such that $\boldsymbol{b}(x) = -\nabla W(x)$ for almost every $x \in U$.

Under the above assumption, we denote by $d\sigma$ the measure on U defined by

$$d\sigma(x) := \exp(-W(x))\, dx \tag{1-6}$$

and by dm the measure on $U \times \mathbb{R}^d$ defined by

$$dm(x, v) := d\sigma(x)\, d\gamma(v) = \exp\bigl(-W(x) - \tfrac{1}{2}|v|^2\bigr)\, dx\, dv. \tag{1-7}$$

A consequence of this definition and integration by parts is the equality

$$\iint_{\mathbb{T}^d \times \mathbb{R}^d} (v \cdot \nabla_x f(x, v) + \boldsymbol{b}(x) \cdot \nabla_v f(x, v))\, dm = 0 \tag{1-8}$$

for all smooth $\mathbb{T}^d$-periodic functions f.

Given $p \in [1, \infty)$, $U \subseteq \mathbb{R}^d$ and an arbitrary Banach space X, we denote by $L^p(U; X)$ the Banach space consisting of measurable functions $f : U \to X$ with norm

$$\|f\|_{L^p(U;X)} := \left(\int_U \|f(x, \cdot)\|_X^p \, dx \right)^{1/p}.$$

It will occasionally be convenient to consider the space $L^p_\sigma(U; X)$, which contains functions for which the norm

$$\|f\|_{L^p_\sigma(U;X)} := \left(\int_U \|f(x, \cdot)\|_X^p \, d\sigma \right)^{1/p}$$

is finite. Notice that, on bounded domains, the above norms induced by dx and $d\sigma$ are equivalent under Assumption 1.1.

We define the space $H^1_{\mathrm{hyp}}(U)$ by

$$H^1_{\mathrm{hyp}}(U) := \{ f \in L^2(U; H^1_\gamma) : v \cdot \nabla_x f \in L^2(U; H^{-1}_\gamma) \} \tag{1-9}$$

and equip it with the norm

$$\|f\|_{H^1_{\mathrm{hyp}}(U)} := (\|f\|^2_{L^2(U;H^1_\gamma)} + \|v \cdot \nabla_x f\|^2_{L^2(U;H^{-1}_\gamma)})^{1/2}. \tag{1-10}$$

When $\boldsymbol{b}$ satisfies Assumption 1.1, it is natural to define the H^1_{hyp} norm with $\|v \cdot \nabla_x f + \boldsymbol{b} \cdot \nabla_v f\|_{L^2_\sigma(U;H^{-1}_\gamma)}$ replacing $\|v \cdot \nabla_x f\|_{L^2(U;H^{-1}_\gamma)}$ in (1-9). The two norms are evidently equivalent on a bounded domain.

Given a bounded domain $U \subseteq \mathbb{R}^d$ and a vector field $\boldsymbol{b} \in L^\infty(U \times \mathbb{R}^d)^d$, we say that a function $f \in H^1_{\mathrm{hyp}}(U)$ is a *weak solution of* (1-1) *in* $U \times \mathbb{R}^d$ if,

$$\text{for all } h \in L^2(U; H^1_\gamma), \quad \int_{U \times \mathbb{R}^d} \nabla_v h \cdot \nabla_v f \, dx \, d\gamma = \int_{U \times \mathbb{R}^d} h(f^* - v \cdot \nabla_x f - \boldsymbol{b} \cdot \nabla_v f) \, dx \, d\gamma.$$

As in (1-5), the precise interpretation of the right side is

$$\int_U \langle h(x, \cdot), (f^* - v \cdot \nabla_x f - \boldsymbol{b} \cdot \nabla_v f)(x, \cdot) \rangle_{H^1_\gamma, H^{-1}_\gamma} \, dx. \tag{1-11}$$

As mentioned previously, we assume throughout that the domain $U \subseteq \mathbb{R}^d$ is bounded and has a C^1 boundary, or that $U = \mathbb{T}^d$ with periodic boundary conditions or $U = \mathbb{R}^d$. In the case $U \neq \mathbb{T}^d, \mathbb{R}^d$, we denote by $\boldsymbol{n}_U$ the outward-pointing unit normal to ∂U and define the *hypoelliptic boundary* of U by

$$\partial_{\mathrm{hyp}} U := \{ (x, v) \in \partial U \times \mathbb{R}^d : v \cdot \boldsymbol{n}_U(x) < 0 \}.$$

We denote by $H^1_{\mathrm{hyp},0}(U)$ the closure in $H^1_{\mathrm{hyp}}(U)$ of the set of smooth functions with compact support in $\bar{U} \times \mathbb{R}^d$ which vanish on $\partial_{\mathrm{hyp}} U$.

We give a first demonstration that $H^1_{\mathrm{hyp}}(U)$ is indeed the natural function space on which to build a theory of weak solutions of (1-1) by presenting a well-posedness result for the Kramers equation.

Theorem 1.2 (well-posedness of the Kramers equation). *Let* $\boldsymbol{b}$ *satisfy Assumption 1.1, and let* $f^* \in L^2(\mathbb{T}^d; H^{-1}_\gamma)$ *be such that* $\iint_{\mathbb{T}^d \times \mathbb{R}^d} f^*(x, v) \, dm = 0$. *Then there exists a unique weak solution* $f \in H^1_{\mathrm{hyp}}(\mathbb{T}^d)$

to the Kramers equation

$$-\Delta_v f + v\cdot\nabla_v f + v\cdot\nabla_x f + \boldsymbol{b}\cdot\nabla_v f = f^* \quad \text{in } \mathbb{T}^d\times\mathbb{R}^d, \tag{1-12}$$

with $\iint_{\mathbb{T}^d\times\mathbb{R}^d} f(x,v)\,dm = 0$*. Furthermore, there exists a constant* $C(\boldsymbol{b},d)<\infty$ *such that* f *satisfies the estimate*

$$\|f\|_{H^1_{\text{hyp}}(\mathbb{T}^d)} \leqslant C\|f^*\|_{L^2(\mathbb{T}^d;H^{-1}_\gamma)}. \tag{1-13}$$

We next give an informal discussion regarding how one could naively guess that H^1_{hyp} is the "correct" space for solving (1-1), and how our proof of Theorem 1.2 will work. We take the simpler case of matrix inversion in finite dimensions as a starting point. Given two matrices A and B with B skew-symmetric and a vector f^*, consider the problem of finding f such that

$$(A^*A+B)f = f^*, \tag{1-14}$$

where A^* denotes the transpose of A. We propose to approach this problem by looking for a minimizer of the functional

$$f\mapsto \inf\bigl\{\tfrac12(Af-\boldsymbol{g},Af-\boldsymbol{g}) : A^*\boldsymbol{g} = f^*-Bf\bigr\},$$

where $(\cdot,\cdot)$ denotes the underlying scalar product. It is clear that the infimum is nonnegative, and if f is a solution to (1-14), then choosing $\boldsymbol{g}=Af$ shows that this infimum is actually zero (null). Moreover, since B is skew-symmetric, whenever $(f,\boldsymbol{g})$ satisfy the constraint in the infimum above, we have

$$\tfrac12(Af-\boldsymbol{g},Af-\boldsymbol{g}) = \tfrac12(Af,Af)+\tfrac12(\boldsymbol{g},\boldsymbol{g})-(f,f^*). \tag{1-15}$$

The latter quantity is clearly a convex function of the pair $(f,\boldsymbol{g})$. The point is that under very mild assumptions on A and B, it will in fact be *uniformly* convex on the set of pairs $(f,\boldsymbol{g})$ satisfying the (linear) constraint $A^*\boldsymbol{g}=f^*-Bf$. Informally, the functional in (1-15) is coercive with respect to the seminorm $(f,\boldsymbol{g})\mapsto |Af|+|\boldsymbol{g}|+|A(A^*A)^{-1}Bf|$.

With this analogy in mind, and assuming that $\boldsymbol{b}$ vanishes for simplicity, we rewrite the problem of finding a solution to (1-1) (with $\boldsymbol{b}\equiv 0$) as that of finding a null minimizer of the functional

$$f\mapsto \inf\biggl\{\int_{\mathbb{T}^d\times\mathbb{R}^d} \tfrac12|\nabla_v f-\boldsymbol{g}|^2\,dx\,d\gamma : \nabla_v^*\boldsymbol{g} = f^*-v\cdot\nabla_x f\biggr\}, \tag{1-16}$$

where $\nabla_v^* F := -\nabla_v\cdot F + v\cdot F$ is the formal adjoint of ∇_v in L^2_γ. It is clear that the infimum above is nonnegative, and if we are provided with a solution f to (1-1) (with $\boldsymbol{b}\equiv 0$), then choosing $\boldsymbol{g}=\nabla_v f$ reveals that this infimum vanishes at f. This functional gives strong credence to the definition of the space $H^1_{\text{hyp}}(U)$ given in (1-9). Using convex-analytic arguments, we show that the mapping in (1-16) is uniformly convex, and that its infimum is null. This implies the well-posedness of the problem (1-1) with $\boldsymbol{b}\equiv 0$. The proof of coercivity relies on the following Poincaré-type inequality for $H^1_{\text{hyp}}(U)$.

For every $f\in L^1(U;L^1_\gamma)$, we define $(f)_U := |U|^{-1}\int_{U\times\mathbb{R}^d} f(x,v)\,d\sigma(x)\,d\gamma(v)$. For the purposes of the Poincaré inequality, we may set $U=\mathbb{T}^d$ or $U\subseteq\mathbb{R}^d$ a general C^1 domain. See Proposition 3.3 and [Cao et al. 2023] for an extension to the case $U=\mathbb{R}^d$ with a confining potential.

Theorem 1.3 (Poincaré inequality for H^1_{hyp}). *For $U = \mathbb{T}^d$ or $U \subseteq \mathbb{R}^d$ a general bounded C^1 domain, there exists a constant $C(U, d) < \infty$ such that, for every $f \in H^1_{\mathrm{hyp}}(U)$, we have*

$$\|f - (f)_U\|_{L^2(U;L^2_\gamma)} \leqslant C(\|\nabla_v f\|_{L^2(U;L^2_\gamma)} + \|v \cdot \nabla_x f\|_{L^2(U;H^{-1}_\gamma)}). \tag{1-17}$$

Moreover, if in addition $f \in H^1_{\mathrm{hyp},0}(U)$, then we have

$$\|f\|_{L^2(U;L^2_\gamma)} \leqslant C(\|\nabla_v f\|_{L^2(U;L^2_\gamma)} + \|v \cdot \nabla_x f\|_{L^2(U;H^{-1}_\gamma)}). \tag{1-18}$$

The inequality (1-17) asserts that, up to an additive constant, the full $H^1_{\mathrm{hyp}}(U)$ norm of a function f is controlled by the seminorm

$$[\![f]\!]_{H^1_{\mathrm{hyp}}(U)} := \|\nabla_v f\|_{L^2(U;L^2_\gamma)} + \|v \cdot \nabla_x f\|_{L^2(U;H^{-1}_\gamma)}.$$

In particular, any distribution f with $[\![f]\!]_{H^1_{\mathrm{hyp}}(U)} < \infty$ is actually a function, which moreover belongs to $L^2_x L^2_\gamma$. The inequality (1-18) is a then simple extension which shows that for functions which vanish on the hypoelliptic boundary, the full H^1_{hyp} norm is controlled by the seminorm.

The proof of Theorem 1.3 thus necessarily uses the Hörmander bracket condition, although in this case the way it is used is rather implicit. If we follow Hörmander's ideas more explicitly, then we obtain more information, namely some positive (fractional) regularity in the x-variable. This is encoded in the following functional inequality, which we call the *Hörmander inequality*. The definitions of the fractional Sobolev spaces H^α used in the statement are given in Section 3B; see (3-30). The Besov space $Q^{1/3}_{\nabla_x}(U)$ is defined in (2-13) in Section 2C and measures difference quotients in the spatial variable x of fractional order $\frac{1}{3}$.

Theorem 1.4 (Hörmander inequality for H^1_{hyp}). *Let $\alpha \in \left[0, \frac{1}{3}\right)$, and let $U = \mathbb{T}^d$ or $U = \mathbb{R}^d$. There exists a constant $C(\alpha, d) < \infty$ such that, for every $f \in H^1_{\mathrm{hyp}}(U)$, we have the estimate*

$$\|f\|_{H^\alpha(U;L^2_\gamma)} \leqslant C\|f\|_{H^1_{\mathrm{hyp}}(U)}. \tag{1-19}$$

For $\alpha = \frac{1}{3}$, we have the estimate

$$\|f\|_{Q^{1/3}_{\nabla_x}(U)} \leqslant C\|f\|_{H^1_{\mathrm{hyp}}(U)}. \tag{1-20}$$

The inequality (1-19) gives control over a norm with nonnegative regularity in x and v. The estimate should be considered as an interior estimate in x; in other words, for U a general domain and any $f \in H^1_{\mathrm{hyp}}(U)$, we can apply the inequality (1-19) after multiplying f by a smooth cutoff function which vanishes for x near ∂U.

Our next main result asserts that weak solutions of (1-1) are actually smooth. This is accomplished by an argument which closely parallels the one for obtaining H^k regularity for solutions of uniformly elliptic equations. We first obtain a version of the Caccioppoli inequality, that is, a reverse Poincaré inequality, which states that the H^1_{hyp} seminorm of a solution of (1-1) can be controlled by its L^2 oscillation (see Lemma 5.1 for the precise statement). Combined with Theorem 1.4, this tells us that a fractional spatial derivative of a solution of (1-1) can be controlled by the L^2 oscillation of the function itself. This estimate can then be iterated: we repeatedly differentiate the equation a fractional amount to obtain estimates of the higher derivatives of the solution in the x-variable; we then obtain estimates for derivatives in the v-variable relatively easily.

Notice that the following statement implies that solutions of (1-1) are C^∞ in both variables (x, v) provided that the vector field $\boldsymbol{b}$ is assumed to be smooth. For convenience, in the statement below we use the convention $C^{-1,1} = L^\infty$.

Theorem 1.5 (interior Sobolev regularity for (1-1)). *Let $k \in \mathbb{N}$, $r \in (0, \infty)$ and $\boldsymbol{b} \in C^{k-1,1}(B_r \times \mathbb{R}^d; \mathbb{R}^d)$. There exists a constant $C < \infty$ depending on*

$$(d, k, r, \|\boldsymbol{b}\|_{C^{k-1,1}(B_r \times \mathbb{R}^d; \mathbb{R}^d)})$$

such that, for every $f \in H^1_{\mathrm{hyp}}(B_r)$ and $f^* \in L^2(B_r; H^{-1}_\gamma)$ satisfying

$$-\Delta_v f + v \cdot \nabla_v f + v \cdot \nabla_x f + \boldsymbol{b} \cdot \nabla_v f = f^* \quad \text{in } B_r \times \mathbb{R}^d, \tag{1-21}$$

the following holds: if $\partial^\alpha f^* \in L^2(B_r; H^{-1}_\gamma)$ for all multi-indices $\alpha \in \mathbb{N}^d \times \mathbb{N}^d$ with $|\alpha| \leqslant k$, then we have $\partial^\alpha f \in H^1_{\mathrm{hyp}}(B_{r/2})$ and the estimate

$$\|\partial^\alpha f\|_{H^1_{\mathrm{hyp}}(B_{r/2})} \leqslant C\Big(\|f - (f)_{B_r}\|_{L^2(B_r; L^2_\gamma)} + \sum_{|\beta| \leqslant k} \|\partial^\beta \tilde{f}^*\|_{L^2(B_r; H^{-1}_\gamma)}\Big)$$

for all multi-indices $\alpha \in \mathbb{N}^d \times \mathbb{N}^d$ with $|\alpha| \leqslant k$.

The results stated above are for the time-independent Kramers equation (1-1). In Section 6, we develop an analogous theory for the time-dependent kinetic Fokker–Planck equation (1-2) with an associated function space H^1_{kin} (defined in (6-2)–(6-3)) in place of H^1_{hyp}. In particular, we obtain analogues of the results above for (1-2) which are stated in Section 6.

The long-time behavior of solutions of (1-2) has been studied by many authors in the last two decades: see [Desvillettes and Villani 2001; Hérau and Nier 2004; Helffer and Nier 2005; Eckmann and Hairer 2003; Desvillettes and Villani 2005; Villani 2009]. Most of these papers consider the case in which $\boldsymbol{b}(x) = -\nabla W(x)$ for a potential W which has sufficient growth at infinity, in which case dm is an explicit invariant measure, and solutions of (1-2) can be expected to converge exponentially fast to the constant which is the integral of the initial data with respect to the invariant measure. This setting is in a certain sense easier than the Dirichlet problem, since one does not have to worry about the boundary. While our methods could also handle this setting, we formulate a result for the exponential convergence of a solution of the Cauchy–Dirichlet problem with constant-in-time right-hand side to the solution of the time-independent problem.

Theorem 1.6 (convergence to equilibrium). *Let $U \subseteq \mathbb{R}^d$ be a C^1 domain and $\boldsymbol{b} \in L^\infty(U; C^{0,1}(\mathbb{R}^d))^d$. There exists $\lambda(\|\boldsymbol{b}\|_{L^\infty(U \times \mathbb{R}^d)}, U, d) > 0$ satisfying the following property. Let $f^* \in L^2(U; H^{-1}_\gamma)$. Suppose that $f_\infty \in H^1_{\mathrm{hyp},0}(U)$ solves (1-12), and that, for every $T \in (0, \infty)$, $f \in H^1_{\mathrm{kin}}((0, T) \times U)$ solves*

$$\begin{cases} \partial_t f - \Delta_v f + v \cdot \nabla_v f + v \cdot \nabla_x f + \boldsymbol{b} \cdot \nabla_v f = f^* & \text{in } (0, T) \times U \times \mathbb{R}^d, \\ f = 0 & \text{on } (0, T) \times \partial_{\mathrm{hyp}} U, \end{cases} \tag{1-22}$$

where the boundary condition is satisfied in the sense that $f \in H^1_{\mathrm{kin},||}((0,T)\times U)$.[4] *Then, for every* $t \geqslant 0$, *we have*

$$\|f(t,\cdot) - f_\infty\|_{L^2(U;L^2_\gamma)} \leqslant 2\exp(-\lambda t)\|f(0,\cdot) - f_\infty\|_{L^2(U;L^2_\gamma)}. \tag{1-23}$$

Notice that interior regularity estimates immediately upgrade the L^2 convergence in (1-23) to convergence in spaces of higher regularity (at least in the interior) with the same exponential rate.

Unlike previous arguments establishing the exponential decay to equilibrium of solutions of (1-2) which are based on differentiation of perhaps nontransparent quantities involving the solution and several (possibly mixed) derivatives in both x and v, the proof of Theorem 1.6 we give here is elementary and close to the classical dissipative estimate for uniformly parabolic equations. The essential idea is to differentiate the square of the L^2 norm of the solution and then apply the Poincaré inequality. We cannot quite perform the computation exactly like this, and so we use a finite difference instead of the time derivative and apply a version of the Poincaré inequality adapted to the kinetic equation in a thin cylinder (see Proposition 6.2). Unlike previous approaches, our method therefore relates the positive constant λ in (1-23) to the optimal constant in a Poincaré-type inequality. One caveat of Theorem 1.6 is that, while we have a hypoelliptic Poincaré inequality in the above setting, we do not yet have a well-posedness theory in H^1_{kin} except when $U = \mathbb{T}^d$.

Finally, we prove an enhanced dissipation estimate for solutions to the kinetic Fokker–Planck equation on the torus $\mathbb{T}^d$ with no right-hand side and $\boldsymbol{b} \equiv 0$ in a weakly collisional limit $\varepsilon \to 0^+$. The PDE satisfied by f when initial data f_{in} is given then becomes

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = \varepsilon(\Delta_v f - v\cdot\nabla_v f) & \text{in } (0,\infty)\times\mathbb{T}^d\times\mathbb{R}^d, \\ f|_{t=0} = f_{\mathrm{in}}. \end{cases} \tag{1-24}$$

The spatial averages $f_{\mathrm{avg}}(t,v) := \fint_{\mathbb{T}^d} f(t,x,v)\,dx$ satisfy

$$\partial_t f_{\mathrm{avg}} = \varepsilon(\Delta_v f_{\mathrm{avg}} - v\cdot\nabla_v f_{\mathrm{avg}}) \tag{1-25}$$

and decay only on the dissipative timescale $T_{\mathrm{d}} \sim \varepsilon^{-1}$, as can be seen by rescaling t in (1-25). In the setting of (1-24), enhanced dissipation is the observation that $f - f_{\mathrm{avg}}$ decays on the faster timescale $T_{\mathrm{e}} \sim \varepsilon^{-1/3}$:

Theorem 1.7 (enhanced dissipation). *There exist constants* $C(d) < \infty$ *and* $c(d) > 0$ *such that, for every* $\varepsilon \in (0,1]$, *initial data* $f_{\mathrm{in}} \in L^2(\mathbb{T}^d; L^2_\gamma)$ *satisfying*

$$\int_{\mathbb{T}^d} f_{\mathrm{in}}(x,v)\,dx = 0 \quad \textit{for all } v \in \mathbb{R}^d, \tag{1-26}$$

and for f *the unique solution of* (1-24) *constructed in Proposition 6.10, we have*

$$\|f(t,\cdot,\cdot)\|_{L^2(\mathbb{T}^d;L^2_\gamma)} \leqslant C\|f_{\mathrm{in}}\|_{L^2(\mathbb{T}^d;L^2_\gamma)}\exp(-c\varepsilon^{-1/3}t). \tag{1-27}$$

[4] $H^1_{\mathrm{kin},||}((0,T)\times U)$ is defined to be the closure of test functions $C^\infty([0,T];U)$ vanishing on the lateral part of the hypoelliptic boundary; see Section 6E.

When enhancement cannot be extracted directly from an explicit solution formula, it is often approached by hypocoercivity techniques, which were developed in [Villani 2009] in the context of kinetic theory; see also [Guo 2002]. These methods were adapted to the context of fluid dynamics in work of Gallagher, Gallay, and Nier [Gallagher et al. 2009], Beck and Wayne [2013], and Bedrossian and Coti Zelati [2017]. In joint work of the first and last authors with Beekie [Albritton et al. 2022], we demonstrated enhancement for solutions of certain advection-diffusion equations (passive scalars in shear flows) by methods which adhered more closely to Hörmander's original paper [1967]. In particular, the H^1_{hyp} framework presented here was readily extended to problems requiring more brackets to span the tangent space. Theorem 1.7, which is inspired by [Albritton et al. 2022], follows from an appropriate time- and ε-dependent version of the Hörmander inequality from Theorem 1.4.

In principle, one may also prove (1-27) with $\boldsymbol{b}$ satisfying Assumption 1.1; see Remark 6.14. It would be interesting to understand this method in the context of the Boltzmann and Landau equations.

1C. ***On unique solvability of the Dirichlet problem.*** There is a subtle point in the analysis of the Dirichlet problem for (1-1) on general domains U which is due to the fact that we should prescribe the boundary condition only on part of the boundary, namely $\partial_{\mathrm{hyp}} U := \{(x, v) \in \partial U \times \mathbb{R}^d : v \cdot \boldsymbol{n}_U(x) < 0\}$, where $\boldsymbol{n}_U$ denotes the outer normal to U. There is a difficulty coming from the possibly wild behavior of the trace of an H^1_{hyp} function near the *singular set* $\{(x, v) \in \partial U \times \mathbb{R}^d : v \cdot \boldsymbol{n}_U(x) = 0\}$, sometimes called the *grazing set*. The following question remains open:[5]

Question 1.8. Does there exist $C(U, d) < \infty$ such that, for every $f \in C^\infty_c(\overline{U} \times \mathbb{R}^d)$,

$$\int_{\partial U \times \mathbb{R}^d} f^2 \, |v \cdot \boldsymbol{n}_U| \, dx \, d\gamma \leqslant C \|f\|^2_{H^1_{\mathrm{hyp}}(U)} \, ?$$

In the case of one spatial dimension ($d = 1$), this difficulty has been previously overcome and the well-posedness result was already proved in [Baouendi and Grisvard 1968]. A generalization to higher dimensions was announced in [Carrillo 1998], but we think that the argument given there is incomplete because the difficulty concerning the boundary behavior was not satisfactorily treated. This is explained in more detail in Appendix A of the original version [Armstrong and Mourrat 2019] of the present work. A different way to phrase the main difficulty is discussed in Remark 4.3.

The original version [Armstrong and Mourrat 2019] of this paper contained an error in the treatment of the Dirichlet and Cauchy–Dirichlet problems for the Kramers and kinetic Fokker–Planck equations, respectively.[6] We were unable to repair the proof; see Remark 4.3 below. In this version, we only prove unique solvability on the torus. *It remains an interesting open question whether unique solvability holds with boundary in the natural H^1_{hyp} class.*

In the intervening years, we succeeded in improving the results in other ways. Foremost, we sharpened the Hörmander-type inequality from $\alpha = \frac{1}{6}-$ to $\alpha = \frac{1}{3}-$ *without* cutoffs in the velocity variable. The second and third authors view this as a significant strengthening of the paper, essentially due to the first

[5] It is not difficult to define a pointwise a.e. trace away from the singular set, see Lemma 4.3 in the original version [Armstrong and Mourrat 2019] of this paper on arXiv, but apparently this has limited usefulness.

[6] See two equations below (4.20) in the original version on arXiv ("Arguing as in for the last term in (4.19). . . ").

and fourth authors. This allows us to prove enhanced relaxation to equilibrium, which was not contained in the first version of the paper. There have also been many works revisiting [Hörmander 1967] and at least partially inspired by the first version; see [Bedrossian et al. 2022; Bedrossian and Liss 2021; Armstrong et al. 2018; Guerand and Imbert 2022; Anceschi and Rebucci 2022; Brigati 2023; Cao et al. 2023; Lu and Wang 2022].

1D. ***Outline of the paper.*** In the next section we present the function space $H^1_{\mathrm{hyp}}(U)$ and its important properties, as well as the Besov spaces used in the Hörmander inequality. In Section 3 we prove the functional inequalities stated in Theorems 1.3 and 1.4 and establish the compactness of the embedding of $H^1_{\mathrm{hyp}}(U)$ into $L^2(U; L^2_\gamma)$. In Section 4 we give two proofs of Theorem 1.2 on the well-posedness of the Dirichlet problem for the Kramers equation. The interior regularity of solutions, and in particular Theorem 1.5, is obtained in Section 5. Finally, in Section 6 we prove the analogous results for the kinetic Fokker–Planck equation (1-2) as well as the exponential decay to equilibrium (Theorem 1.6) and the enhancement estimate (Theorem 1.7).

2. Function space basics

In this section, we establish some basic properties of the function space $H^1_{\mathrm{hyp}}(U)$ defined in (1-9)–(1-10) and introduce several Besov-type spaces which will be necessary for the proof of the Hörmander inequality.

2A. ***Properties of H^1_γ and H^{-1}_γ.*** We start by setting up some notation that will be used throughout the paper. We denote the formal adjoint of the operator ∇_v by ∇_v^*; that is, for every $F \in (H^1_\gamma)^d$, we define

$$\nabla_v^* F := -\nabla_v \cdot F + v \cdot F. \tag{2-1}$$

This definition can be extended to any $F \in (L^2_\gamma)^d$, in which case $\nabla_v^* F \in H^{-1}_\gamma$ and we have, for every $f \in H^1_\gamma$,

$$\int_{\mathbb{R}^d} f\, \nabla_v^* F \, d\gamma = \int_{\mathbb{R}^d} \nabla_v f \cdot F \, d\gamma.$$

Recall that the left side above is shorthand notation for the duality pairing between H^1_γ and H^{-1}_γ. We denote the average of a function $f \in L^1_\gamma$ by

$$\langle f \rangle_\gamma := \int_{\mathbb{R}^d} f \, d\gamma. \tag{2-2}$$

Since $1 \in H^1_\gamma$, the definition of $\langle f \rangle_\gamma$ can be extended to arbitrary $f \in H^{-1}_\gamma$. The Gaussian Poincaré inequality states that, for every $f \in H^1_\gamma$,

$$\| f - \langle f \rangle_\gamma \|_{L^2_\gamma} \leqslant \| \nabla_v f \|_{L^2_\gamma}.$$

We can thus replace $\| f \|_{L^2_\gamma}$ by $|\langle f \rangle_\gamma|$ in the definition of H^1_γ and have an equivalent norm:

$$|\langle f \rangle_\gamma|^2 + \| \nabla f \|^2_{L^2_\gamma} \leqslant \| f \|^2_{H^1_\gamma} \leqslant 2|\langle f \rangle_\gamma|^2 + 3\| \nabla f \|^2_{L^2_\gamma}.$$

This comparison of norms has the following counterpart for the dual space H^{-1}_γ.

Lemma 2.1 (identification of H_γ^{-1}). *There exists a universal constant $C<\infty$ such that, for every $f^*\in H_\gamma^{-1}$,*

$$C^{-1}\|f^*\|_{H_\gamma^{-1}} \leqslant |\langle f^*\rangle_\gamma| + \inf\{\|\boldsymbol{h}\|_{L^2_\gamma} : \nabla_v^*\boldsymbol{h} = f^* - \langle f^*\rangle_\gamma\} \leqslant C\|f^*\|_{H_\gamma^{-1}}. \tag{2-3}$$

Proof. The bilinear form

$$(f,g)\mapsto \langle f\rangle_\gamma\langle g\rangle_\gamma + \int_{\mathbb{R}^d}\nabla_v f\cdot\nabla_v g\,d\gamma$$

is a scalar product for the Hilbert space H^1_γ. By the Riesz representation theorem, for every $f^*\in H_\gamma^{-1}$, there exists $g\in H^1_\gamma$ such that,

$$\text{for all } f\in H^1_\gamma,\quad \int_{\mathbb{R}^d} ff^*\,d\gamma = \langle f\rangle_\gamma\,\langle g\rangle_\gamma + \int_{\mathbb{R}^d}\nabla_v f\cdot\nabla_v g\,d\gamma.$$

(Recall that the integral on the left side is convenient notation for the canonical pairing between H^1_γ and H^{-1}_γ.) We clearly have $\langle g\rangle_\gamma = \langle f^*\rangle_\gamma$, and thus

$$|\langle g\rangle_\gamma|^2 + \int_{\mathbb{R}^d}|\nabla_v g|^2\,d\gamma \leqslant \|g\|_{H^1_\gamma}\,\|f^*\|_{H^{-1}_\gamma}.$$

This implies that $\|\nabla_v g\|_{L^2_\gamma}\leqslant C\|f^*\|_{H^{-1}_\gamma}$, and since $\nabla_v^*\nabla_v g = f^*-\langle f^*\rangle_\gamma$, this proves the rightmost inequality in (2-3). Conversely, for any $\boldsymbol{h}\in L^2_\gamma$, if

$$f^* = \langle f^*\rangle_\gamma + \nabla_v^*\boldsymbol{h},$$

then, for every $f\in H^1_\gamma$,

$$\left|\int_{\mathbb{R}^d} ff^*\,d\gamma\right| \leqslant |\langle f\rangle_\gamma|\,|\langle f^*\rangle_\gamma| + \|\nabla f\|_{L^2_\gamma}\,\|\boldsymbol{h}\|_{L^2_\gamma},$$

and thus the leftmost inequality in (2-3) holds. □

We often work with the dual pair of Banach spaces $L^2(U;H^1_\gamma)$ and $L^2(U;H^{-1}_\gamma)$. With the identification given by Lemma 2.1, we have

$$\|f^*\|_{L^2(U;H^{-1}_\gamma)} \simeq \|\langle f^*\rangle_\gamma\|_{L^2(U)} + \inf\{\|\boldsymbol{g}\|_{L^2(U;L^2_\gamma)} : \nabla_v^*\boldsymbol{g} = f^*-\langle f^*\rangle_\gamma\}, \tag{2-4}$$

in the sense that the norms on each side are equivalent.

For convenience, for every $f\in L^1(U;L^1_\gamma)$, we use the shorthand notation

$$(f)_U := |U|^{-1}\int_{U\times\mathbb{R}^d} f(x,v)\,d\sigma(x)\,d\gamma(v). \tag{2-5}$$

We will occasionally also use this notation in the case when f depends only on the space variable x, in which case we simply have $(f)_U = |U|^{-1}\int_U f\,d\sigma(x)$.

In the proof of the Hörmander inequality, it will be beneficial to understand which type of finite differences are controlled by $\|f\|_{H^1_\gamma}$. Recall that

$$d\gamma(v) := (2\pi)^{-d/2}\exp\left(-\tfrac{1}{2}|v|^2\right)dv.$$

The fundamental issue is that $\gamma(\cdot + h)$ is not comparable to γ, above and below, uniformly in v. For instance, while the translation of the measure γ by a fixed vector $y \in \mathbb{R}^d$ is absolutely continuous with respect to γ, the associated Radon–Nikodym derivative is unbounded (unless $y = 0$). This distinguishes Gaussians from $e^{-\langle x\rangle}$, for example, and changes the finite difference characterization of the space

$$\|\nabla_v u\|_{L^2(U;L^2_\gamma)},$$

since its finite difference characterization is not in the seminorm

$$\sup_{h>0} h^{-1}\|u(x, v+h) - u(x, v)\|_{L^2(U;L^2_\gamma)}.$$

Towards an appropriate characterization, we first note that a consequence of the logarithmic Sobolev inequality and the Gaussian Poincaré inequality is the estimate

$$\| |v| u\|_{L^2(U;L^2_\gamma)} \lesssim \|\nabla_v u\|_{L^2(U;L^2_\gamma)} \tag{2-6}$$

for functions u satisfying $\langle u\rangle_\gamma = 0$; the reader may consult (3-35) and the ensuing discussion for details. The inequality (2-6), together with the product rule, gives

$$\|\nabla_v(u\gamma^{1/2})\|_{L^2(U;L^2(\mathbb{R}^d))} \lesssim \|\nabla_v u\|_{L^2(U;L^2_\gamma(\mathbb{R}^d))},$$

and since the left-hand side has a finite difference characterization, we have

$$\sup_{h\in\mathbb{R}^d\setminus\{0\}} |h|^{-1}\|u(x, v+h)\gamma^{1/2}(v+h) - u(x, v)\gamma^{1/2}(v)\|_{L^2(U;L^2(\mathbb{R}^d))} \lesssim \|\nabla_v u\|_{L^2(U;L^2_\gamma)}. \tag{2-7}$$

We refer to [Lunardi 2018] for further discussion.

2B. *Density of smooth functions in H^1_{hyp}.*

We show that the set of smooth functions is dense in H^1_{hyp}.

Proposition 2.2. *The set $C_c^\infty(\overline{U} \times \mathbb{R}^d)$ of smooth functions with compact support in $\overline{U} \times \mathbb{R}^d$ is dense in $H^1_{\mathrm{hyp}}(U)$.*

Proof. We focus on the case when $U \subseteq \mathbb{R}^d$ is a bounded C^1 domain. When $U = \mathbb{T}^d$, the proof can be done more simply by cutting off in v and mollifying.

We decompose the proof into three steps.

Step 1: In this step, we show that it suffices to consider the case when U satisfies a convenient quantitative form of the star-shape property. For every $z \in \partial U$, there exist a radius $r > 0$ and a C^1 function $\Psi \in C^1(\mathbb{R}^{d-1}; \mathbb{R})$ such that, up to a relabelling of the axes, we have

$$U \cap B(z, r) = \{x = (x_1, \ldots, x_d) \in B(z, r) : x_d > \Psi(x_1, \ldots, x_{d-1})\}.$$

Since Ψ is a C^1 function, there exists $\delta > 0$ such that for every $x \in U \cap B(z, r)$, we have the cone containment property

$$\left\{x + y : \frac{y_d}{|y|} \geqslant 1 - \delta\right\} \cap B(z, r) \subseteq U. \tag{2-8}$$

Setting

$$z' = z + \left(0, \dots, 0, \frac{r}{2}\right) \in \mathbb{R}^d,$$

and reducing $\delta > 0$ if necessary, we claim that, for every $x \in U \cap B(z, \delta^2)$ and $\varepsilon \in (0, 1]$, we have

$$B(x - \varepsilon(x - z'), \delta^2 \varepsilon) \subseteq U. \tag{2-9}$$

Assuming the contrary, let $y \in \mathbb{R}^d$ be such that

$$x + y \in B(x - \varepsilon(x - z'), \delta^2 \varepsilon) \setminus U.$$

Then

$$|y + \varepsilon(x - z')| \leqslant \delta^2 \varepsilon,$$

and therefore

$$\begin{aligned} \left| y - \varepsilon\left(0, \dots, 0, \frac{r}{2}\right)\right| &\leqslant \left| y + \varepsilon(x - z) - \varepsilon\left(0, \dots, 0, \frac{r}{2}\right)\right| + \varepsilon|x - z| \\ &\leqslant |y + \varepsilon(x - z')| + \varepsilon|x - z| \leqslant 2\delta^2 \varepsilon. \end{aligned}$$

Taking $\delta > 0$ sufficiently small, we arrive at a contradiction with the cone property (2-8). Now that (2-9) is proved for every x in a relative neighborhood of z, and up to a further reduction of the value of $\delta > 0$ if necessary, it is not difficult to show that one can find an open set U' containing z and z' and such that (2-9) holds for every $x \in U \cap U'$.

Summarizing, and using the fact that U is a bounded set, we have shown that there exist families of bounded open sets $U_1, \dots, U_M \subseteq \mathbb{R}^d$, points $x_1, \dots, x_M \in \mathbb{R}^d$ and a parameter $r > 0$ such that

$$U = \bigcup_{k=1}^{M} U_i$$

and for every $k \in \{1, \dots, M\}$, $x \in U_k$ and $\varepsilon \in (0, 1]$,

$$B(x - \varepsilon(x - x_k), r\varepsilon) \subseteq U_k.$$

By using a partition of unity, we can reduce our study to the case when this property is satisfied for the domain U itself (in place of each of the U_k's). By translation, we may assume that the reference point x_k is at the origin, and by scaling, we may also assume that this property holds with $r = 1$. That is, from now on, we assume that, for every $x \in U$ and $\varepsilon \in (0, 1]$, we have

$$B((1 - \varepsilon)x, \varepsilon) \subseteq U. \tag{2-10}$$

Step 2: Let $f \in H^1_{\mathrm{hyp}}(U)$. We aim to show that f belongs to the closure of the set $C^\infty_c(\overline{U} \times \mathbb{R}^d)$ in $H^1_{\mathrm{hyp}}(U)$. Without loss of generality, we may assume that f is compactly supported in $\overline{U} \times \mathbb{R}^d$. Indeed, if $\chi \in C^\infty_c(\mathbb{R}^d; \mathbb{R})$ is a smooth function with compact support and such that $\chi \equiv 1$ in a neighborhood of the origin, then the function $(x, v) \mapsto f(x, v)\chi(v/M)$ belongs to $H^1_{\mathrm{hyp}}(U)$ and converges to f in $H^1_{\mathrm{hyp}}(U)$ as M tends to infinity.

Let $\zeta \in C^\infty_c(\mathbb{R}^d; \mathbb{R})$ be a smooth function with compact support in $B(0, 1)$ and such that $\int_{\mathbb{R}^d} \zeta = 1$. For each $\varepsilon > 0$ and $x \in \mathbb{R}^d$, we write

$$\zeta_\varepsilon(x) := \varepsilon^{-d} \zeta(\varepsilon^{-1} x), \tag{2-11}$$

and we define, for each $\varepsilon \in \left(0, \frac{1}{2}\right]$, $x \in U$ and $v \in \mathbb{R}^d$,

$$f_\varepsilon(x, v) := \int_{\mathbb{R}^d} f((1-\varepsilon)x + y, v)\zeta_\varepsilon(y)\,dy.$$

Note that this definition makes sense by the assumption of (2-10). The goal of this step is to show that f belongs to the closure in $H^1_{\mathrm{hyp}}(U)$ of the convex hull of the set $\left\{f_\varepsilon : \varepsilon \in \left(0, \frac{1}{2}\right]\right\}$. By Mazur's lemma (see [Ekeland and Temam 1976, page 6]), it suffices to show that f_ε converges weakly to f in $H^1_{\mathrm{hyp}}(U)$. Since it is elementary to show that f_ε converges to f in the sense of distributions, this boils down to checking that f_ε is bounded in $H^1_{\mathrm{hyp}}(U)$. By Jensen's inequality,

$$\begin{aligned}\|\nabla_v f_\varepsilon\|^2_{L^2(U;L^2_\gamma)} &\leqslant \int_{U\times\mathbb{R}^d}\int_{\mathbb{R}^d} |\nabla_v f|^2((1-\varepsilon)x+y, v)\zeta_\varepsilon(y)\,dy\,dx\,d\gamma(v)\\ &\leqslant (1-\varepsilon)^{-d}\|\nabla_v f\|^2_{L^2(U;L^2_\gamma)}.\end{aligned}$$

In order to evaluate $\|v\cdot\nabla_x f_\varepsilon\|_{L^2(U;H^{-1}_\gamma)}$, we compute, for every $\varphi \in L^2(U; H^1_\gamma)$,

$$\begin{aligned}\int_{U\times\mathbb{R}^d} v\cdot\nabla_x f_\varepsilon\,\varphi\,dx\,d\gamma &= (1-\varepsilon)\int_{U\times\mathbb{R}^d}\int_{\mathbb{R}^d} v\cdot\nabla_x f((1-\varepsilon)x+y, v)\zeta_\varepsilon(y)\varphi(x,v)\,dy\,dx\,d\gamma(v)\\ &= (1-\varepsilon)^{1-d}\int_{U\times\mathbb{R}^d}\int_{\mathbb{R}^d} v\cdot\nabla_x f(x+y, v)\zeta_\varepsilon(y)\,\varphi\left(\frac{x}{1-\varepsilon}, v\right)dy\,dx\,d\gamma(v)\\ &= \int_{U\times\mathbb{R}^d}\int_{\mathbb{R}^d} v\cdot\nabla_x f(y, v)\zeta_\varepsilon(y-x)\,\varphi\left(\frac{x}{1-\varepsilon}, v\right)dy\,dx\,d\gamma(v).\end{aligned}$$

Since, by Jensen's inequality,

$$\int_{U\times\mathbb{R}^d}\left|\int_U \zeta_\varepsilon(y-x)\varphi\left(\frac{x}{1-\varepsilon}, v\right)dx\right|^2 dy\,d\gamma(v) \leqslant (1-\varepsilon)^{-d}\|\varphi\|^2_{L^2(U;L^2_\gamma)},$$

as well as

$$\int_{U\times\mathbb{R}^d}\left|\int_U \zeta_\varepsilon(y-x)\nabla_v\varphi\left(\frac{x}{1-\varepsilon}, v\right)dx\right|^2 dy\,d\gamma(v) \leqslant (1-\varepsilon)^{-d}\|\nabla_v\varphi\|^2_{L^2(U;L^2_\gamma)},$$

we deduce that

$$\int_{U\times\mathbb{R}^d} v\cdot\nabla_x f_\varepsilon\,\varphi\,dx\,d\gamma \leqslant (1-\varepsilon)^{1-3d/2}\|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}\|\varphi\|_{L^2(U;H^1_\gamma)},$$

and therefore

$$\|v\cdot\nabla_x f_\varepsilon\|_{L^2(U;H^{-1}_\gamma)} \leqslant (1-\varepsilon)^{1-3d/2}\|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}.$$

This completes the proof that the set $\left\{f_\varepsilon : \varepsilon \in \left(0, \frac{1}{2}\right]\right\}$ is bounded in $H^1_{\mathrm{hyp}}(U)$, and thus that f belongs to the closed convex hull of this set.

Step 3: It remains to be shown that for each fixed $\varepsilon \in \left(0, \frac{1}{2}\right]$, the function f_ε belongs to the closure in $H^1_{\mathrm{hyp}}(U)$ of the set $C^\infty_c(\overline{U}\times\mathbb{R}^d)$. For every $\eta \in (0, 1]$, we define

$$\begin{aligned}f_{\varepsilon,\eta}(x, v) &:= \int_{\mathbb{R}^d} f_\varepsilon(x, w)\zeta_\eta(v-w)\,dw\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y, w)\zeta_\varepsilon(y-(1-\varepsilon)x)\zeta_\eta(v-w)\,dy\,dw.\end{aligned}$$

From the last expression, we see that $f_{\varepsilon,\eta}$ belongs to $C_c^\infty(\overline{U}\times\mathbb{R}^d)$ (recall that f itself has compact support in $\overline{U}\times\mathbb{R}^d$). Moreover, since $\nabla_v f_\varepsilon \in L^2(U;L^2_\gamma)$ and

$$\nabla_v f_{\varepsilon,\eta}(x,v)=\int_{\mathbb{R}^d}\nabla_v f_\varepsilon(x,v-w)\zeta_\eta(w)\,dw,$$

it is classical to verify that $\nabla_v f_{\varepsilon,\eta}$ converges to $\nabla_v f_\varepsilon$ in $L^2(U;L^2_\gamma)$ as η tends to 0. By the definition of f_ε and the fact that f_ε is compactly supported, we have that $v\cdot\nabla_x f_\varepsilon\in L^2(U;L^2_\gamma)$. The same reasoning as above thus gives that $v\cdot\nabla_x f_{\varepsilon,\eta}$ converges to $v\cdot\nabla_x f_\varepsilon$ in $L^2(U;L^2_\gamma)$, and thus a fortiori in $L^2(U;H^{-1}_\gamma)$ as η tends to 0. This shows that

$$\lim_{\eta\to 0}\|f_{\varepsilon,\eta}-f_\varepsilon\|_{H^1_{\mathrm{hyp}}(U)}=0$$

and thus completes the proof of the proposition. □

2C. *Besov spaces.* We shall use the following Besov-type spaces in the proof of the Hörmander inequality. The first of these spaces measures fractional regularity along the vector field $v\cdot\nabla_x$, while the second measures fractional regularity along ∇_x. As the Hörmander inequality is an interior estimate, we only consider these spaces in the cases that $U=\mathbb{R}^d$ or $U=\mathbb{T}^d$. To lighten the notation, we may frequently write $\|\cdot\|_{Q^{1/2}_{v\cdot\nabla_x}}$ rather than $\|\cdot\|_{Q^{1/2}_{v\cdot\nabla_x}(U)}$, as the choice of $U=\mathbb{R}^d,\mathbb{T}^d$ plays no role in the argument. The Q stands for "quotient".

Definition 2.3. For measurable $f:U\times\mathbb{R}^d\to\mathbb{R}$, we define

$$\|f\|^2_{Q^{1/2}_{v\cdot\nabla_x}(U)}:=\sup_{0<\eta<\infty}\frac{1}{\eta^2}\iint_{U\times\mathbb{R}^d}(f(x+\eta^2 v,v)-f(x,v))^2\,d\gamma(v)\,dx. \tag{2-12}$$

Definition 2.4. For measurable $f:U\times\mathbb{R}^d\to\mathbb{R}$, we define

$$\|f\|^2_{Q^{1/3}_{\nabla_x}(U)}:=\sup_{\substack{0<\eta<\infty\\ x'\in\mathbb{S}^{d-1}}}\frac{1}{\eta^2}\iint_{U\times\mathbb{R}^d}(f(x+\eta^3 x',v)-f(x,v))^2\,d\gamma(v)\,dx. \tag{2-13}$$

3. Functional inequalities for H^1_{hyp}

In this section we present the proofs of Theorems 1.3 and 1.4.

3A. *The Poincaré inequality for H^1_{hyp}.* We begin with the proof of Theorem 1.3, the Poincaré-type inequality for the space $H^1_{\mathrm{hyp}}(U)$. The proof requires the following fact regarding the equivalence (up to additive constants) of the norms $\|h\|_{L^2(U)}$ and $\|\nabla h\|_{H^{-1}(U)}$.

Lemma 3.1. *Let U be a Lipschitz domain or $U=\mathbb{T}^d$. Then there exists $C(U,d)<\infty$ such that, for every $h\in L^2(U)$,*

$$\|h-(h)_U\|_{L^2(U)}\leqslant C\|\nabla h\|_{H^{-1}(U)}.$$

Proof. We begin by considering the case U is a Lipschitz domain. Without loss of generality, we assume $(h)_U=0$. We consider the problem

$$\begin{cases}\nabla\cdot \boldsymbol{f}=h & \text{in } U,\\ \boldsymbol{f}=0 & \text{on } \partial U.\end{cases} \tag{3-1}$$

Bogovskii's operator [1980] (see also [Galdi 2011, Section III.3]) guarantees the existence of a solution $\boldsymbol{f}$ with components in $H_0^1(U)$ satisfying the estimate

$$\|\boldsymbol{f}\|_{H^1(U)} \leqslant C\|h\|_{L^2(U)}. \tag{3-2}$$

Then we have

$$\|h\|_{L^2(U)}^2 = \int_U h\,\nabla\cdot\boldsymbol{f} = -\int_U \nabla h\cdot\boldsymbol{f} \leqslant \|\nabla h\|_{H^{-1}(U)}\,\|\boldsymbol{f}\|_{H^1(U)}.$$

The conclusion then follows by (3-2). In the case $U=\mathbb{T}^d$, the estimate follows from classical Littlewood–Paley estimates, and we omit the details. □

Proof of Theorem 1.3. Let $f\in H^1_{\mathrm{hyp}}(U)$. In view of Proposition 2.2, we can without loss of generality assume that f is a smooth function. We decompose the proof into five steps.

Step 1: We show that

$$\|f-\langle f\rangle_\gamma\|_{L^2(U;L^2_\gamma)} \leqslant \|\nabla_v f\|_{L^2(U;L^2_\gamma)}. \tag{3-3}$$

By the Gaussian Poincaré inequality, we have for every $x\in U$ that

$$\|f(x,\cdot\,)-\langle f\rangle_\gamma(x)\|_{L^2_\gamma} \leqslant \|\nabla_v f(x,\cdot\,)\|_{L^2_\gamma}.$$

This yields (3-3) after integration over $x\in U$.

Step 2: We show that

$$\|\nabla\langle f\rangle_\gamma\|_{H^{-1}(U)} \leqslant C(\|\nabla_v f\|_{L^2(U;L^2_\gamma)} + \|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}). \tag{3-4}$$

We select $\xi_1,\dots,\xi_d\in C_c^\infty(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d} v\xi_i(v)\,d\gamma(v) = e_i, \tag{3-5}$$

and, for each test function $\phi\in H_0^1(U)$ and $i\in\{1,\dots,d\}$, we compute

$$\begin{aligned}
&\int_U \partial_{x_i}\phi(x)\langle f\rangle_\gamma(x)\,dx\\
&\quad= \int_{U\times\mathbb{R}^d} v\cdot\nabla_x\phi(x)\langle f\rangle_\gamma(x)\xi_i(v)\,dx\,d\gamma(v)\\
&\quad= \int_{U\times\mathbb{R}^d} v\cdot\nabla_x\phi(x)f(x,v)\xi_i(v)\,dx\,d\gamma(v) + \int_{U\times\mathbb{R}^d} v\cdot\nabla_x\phi(x)(f(x,v)-\langle f\rangle_\gamma(x))\xi_i(v)\,dx\,d\gamma(v).
\end{aligned}$$

To control the first term on the right side, we perform an integration by parts to obtain

$$\begin{aligned}
\left|\int_{U\times\mathbb{R}^d} v\cdot\nabla_x\phi(x)f(x,v)\xi_i(v)\,dx\,d\gamma(v)\right| &= \left|\int_{U\times\mathbb{R}^d} \phi(x)\xi_i(v)\,v\cdot\nabla_x f(x,v)\,dx\,d\gamma(v)\right|\\
&\leqslant C\|\phi\xi_i\|_{L^2(U;H^1_\gamma)}\|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}\\
&\leqslant C\|\phi\|_{L^2(U)}\|\xi_i\|_{H^1_\gamma}\|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}\\
&\leqslant C\|\phi\|_{L^2(U)}\|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}.
\end{aligned}$$

To control the second term, we use (3-3) and the fact that ξ_i has compact support:

$$\begin{aligned}\left|\int_{U\times\mathbb{R}^d} v\cdot\nabla_x\phi(x)(f(x,v)-\langle f\rangle_\gamma(x))\xi_i(v)\,dx\,d\gamma(v)\right| \\ \leqslant C\int_{U\times\mathbb{R}^d}|v|\,|\xi_i(v)|\,|\nabla_x\phi(x)|\,|f(x,v)-\langle f\rangle_\gamma(x)|\,dx\,d\gamma(v) \\ \leqslant C\|\phi\|_{H^1(U)}\|\nabla_v f\|_{L^2(U;L^2_\gamma)}.\end{aligned}$$

Combining the above displays and taking the supremum over $\phi\in H^1_0(U)$ with $\|\phi\|_{H^1(U)}\leqslant 1$ yields (3-4).

Step 3: We deduce from Lemma 3.1, (3-3) and (3-4) that

$$\begin{aligned}\|f-(f)_U\|_{L^2(U;L^2_\gamma)} &\leqslant \|f-\langle f\rangle_\gamma\|_{L^2(U;L^2_\gamma)}+\|\langle f\rangle_\gamma-(f)_U\|_{L^2(U)}\\ &\leqslant \|f-\langle f\rangle_\gamma\|_{L^2(U;L^2_\gamma)}+C\|\nabla\langle f\rangle_\gamma\|_{H^{-1}(U)}\\ &\leqslant C(\|\nabla_v f\|_{L^2(U;L^2_\gamma)}+\|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}).\end{aligned}$$

This completes the proof of (1-17).

Step 4: The remaining steps are specific to the case with boundary. To complete the proof of (1-18), we must show that, under the additional assumption that $U\neq\mathbb{T}^d$ and $f\in H^1_{\mathrm{hyp},0}(U)$, we have

$$|(f)_U|\leqslant C(\|\nabla_v f\|_{L^2(U;L^2_\gamma)}+\|v\cdot\nabla_x f\|_{L^2(U;H^{-1}_\gamma)}). \tag{3-6}$$

Let f_1 be a test function belonging to $C^\infty_c(\overline{U}\times\mathbb{R}^d)$, to be constructed below, which satisfies

$$f_1=0 \quad \text{on } (\partial U\times\mathbb{R}^d)\setminus\partial_{\mathrm{hyp}}(U), \tag{3-7}$$

$$\fint_U\int_{\mathbb{R}^d} v\cdot\nabla_x f_1\,d\gamma\,dx=1 \tag{3-8}$$

and, for some constant $C(U,d)<\infty$,

$$\|v\cdot\nabla_x f_1\|_{L^2(U;L^2_\gamma)}\leqslant C. \tag{3-9}$$

The test function f_1 is constructed in Step 5 below. We first use it to obtain (3-6). We proceed by using (3-8) to split the mean of f as

$$(f)_U=\fint_U\int_{\mathbb{R}^d} f\,v\cdot\nabla_x f_1\,d\gamma\,dx-\fint_U\int_{\mathbb{R}^d}(f-(f)_U)v\cdot\nabla_x f_1\,d\gamma\,dx$$

and estimate the two terms on the right side separately. For the first term, we have

$$\begin{aligned}\left|\fint_U\int_{\mathbb{R}^d} f\,v\cdot\nabla_x f_1\,d\gamma\,dx\right| &= \left|-\fint_U\int_{\mathbb{R}^d} f_1\,v\cdot\nabla_x f\,d\gamma\,dx+\frac{1}{|U|}\int_{\partial U}\int_{\mathbb{R}^d}(v\cdot\boldsymbol{n}_U)ff_1\,d\gamma\,dx\right|\\ &=\left|\fint_U\int_{\mathbb{R}^d} f_1 v\cdot\nabla_x f\,d\gamma\,dx\right|,\end{aligned}$$

where we used that $(v\cdot\boldsymbol{n}_U)ff_1$ vanishes on $\partial U\times\mathbb{R}^d$ to remove the boundary integral. (Recall that by the definition of $H^1_{\mathrm{hyp},0}(U)$, we can assume without loss of generality that the function f is smooth, so the justification of the integration by parts above is classical.) We thus obtain

$$\left| \fint_U \int_{\mathbb{R}^d} f_1 v \cdot \nabla_x f \, d\gamma \, dx \right| \leqslant \frac{1}{|U|} \| f_1 \|_{L^2(U;H^1_\gamma)} \| v \cdot \nabla_x f \|_{L^2(U;H^{-1}_\gamma)}.$$

This completes the estimate for the first term. For the second term, we use (3-9) to get

$$\begin{aligned} \left| \fint_U \int_{\mathbb{R}^d} (f - (f)_U) \, v \cdot \nabla_x f_1 \, d\gamma \, dx \right| &\leqslant \| f - (f)_U \|_{L^2(U;L^2_\gamma)} \| v \cdot \nabla_x f_1 \|_{L^2(U;L^2_\gamma)} \\ &\leqslant C \| f - (f)_U \|_{L^2(U;L^2_\gamma)}, \end{aligned}$$

which is estimated using the result of Step 3. Putting these together yields (3-6).

Step 5: We construct the test function $f_1 \in C^\infty_c(\overline{U} \times \mathbb{R}^d)$ satisfying (3-7), (3-8) and (3-9). Fix $x_0 \in \partial U$, where $\boldsymbol{n}_U(x_0)$ is well-defined. Since the unit normal $\boldsymbol{n}_U$ is continuous at x_0, there exist $v_0 \in \mathbb{R}^d$ and $r > 0$ such that for every $x, v \in \mathbb{R}^d$ satisfying $(x, v) \in (B_r(x_0) \cap \partial U) \times B_r(v_0)$, we have $v \cdot \boldsymbol{n}_U(x) > 0$. In other words, every $(x, v) \in (B_r(x_0) \cap \partial U) \times B_r(v_0)$ is such that $(x, v) \in \partial_{\mathrm{hyp}} U$. Observe that, for every $f_1 \in C^\infty_c(\mathbb{R}^d \times \mathbb{R}^d)$, we have

$$\fint_U \int_{\mathbb{R}^d} v \cdot \nabla_x f_1 \, d\gamma \, dx = \frac{1}{|U|} \int_{\partial U} \int_{\mathbb{R}^d} (v \cdot \boldsymbol{n}_U) f_1 \, d\gamma \, dx.$$

We select a function $f_1 \in C^\infty_c(\mathbb{R}^d \times \mathbb{R}^d)$ with compact support in $B_r(x_0) \times B_r(v_0)$ and such that $f_1 \geqslant 0$ and $f_1(x_0, v_0) = 1$. In this case, the integral on the right side above is nonnegative, since f_1 vanishes whenever $v \cdot \boldsymbol{n}_U \leqslant 0$. In fact, since f_1 is positive on a set of positive measure on $\partial U \times \mathbb{R}^d$ (in the sense of the product of the $(d-1)$-dimensional Hausdorff and Lebesgue measures), the integral above is positive. Up to multiplying f_1 by a positive scalar if necessary, we can thus ensure that (3-8) holds. It is clear that this construction also ensures that (3-7) and (3-9) hold. □

Remark 3.2. As the argument above reveals, for the inequality (1-18) to hold, the assumption of $f \in H^1_{\mathrm{hyp},0}(U)$ can be weakened: it suffices that f vanishes on a relatively open piece of the boundary $\partial U \times \mathbb{R}^d$. The constant C in (1-18) then depends additionally on the identity of this piece of the boundary where f is assumed to vanish.

3A1. *Poincaré inequality with confining potential.* It is also interesting to understand Theorem 1.3 in the global setting with confining potential.[7]

Only in this subsection, we redefine $H^1_{\mathrm{hyp}}(\mathbb{R}^d)$ according to the norm

$$\| f \|_{H^1_{\mathrm{hyp}}(\mathbb{R}^d)} = \| f \|_{L^2_\sigma(\mathbb{R}^d;H^1_\gamma)} + \| v \cdot \nabla_x f + \boldsymbol{b} \cdot \nabla_v f \|_{L^2_\sigma(\mathbb{R}^d;H^{-1}_\gamma)}, \tag{3-10}$$

and when $\boldsymbol{b}$ satisfies Assumption 1.1 with $U = \mathbb{R}^d$, and $f \in L^1_\sigma(\mathbb{R}^d; L^1_\gamma)$, we use the notation

$$(f)_{\mathbb{R}^d} := \int f \, dm.$$

[7] A proof is also contained in [Cao et al. 2023] following the methods in the original version of this paper, which only discussed bounded domains.

Proposition 3.3 (Poincaré with confining potential). *Suppose that $\boldsymbol{b}$ satisfies Assumption 1.1 with $U = \mathbb{R}^d$, the potential W satisfies $W \in C^{1,1}(\mathbb{R}^d)$, and there exists a constant $C_W < \infty$ such that the following weighted Poincaré inequality holds for all $h \in H^1_\sigma(\mathbb{R}^d)$ with $(h)_{\mathbb{R}^d} = 0$:*

$$\int_U |\nabla_x W|^2 |h|^2 \, d\sigma \leqslant C_W \int_U |\nabla_x h|^2 \, d\sigma. \tag{3-11}$$

Then there exists a constant $C(W,d) < \infty$ such that, for all $f \in H^1_{\mathrm{hyp}}(\mathbb{R}^d)$, defined according to (3-10), *with $(f)_{\mathbb{R}^d} = 0$*

$$\|f\|_{L^2_\sigma(\mathbb{R}^d; L^2_\gamma)} \leqslant C(\|\nabla_v f\|_{L^2_\sigma(\mathbb{R}^d; L^2_\gamma)} + \|v \cdot \nabla_x f + \boldsymbol{b} \cdot \nabla_v f\|_{L^2_\sigma(\mathbb{R}^d; H^{-1}_\gamma)}).$$

First, we require an analogue of Lemma 3.1.

Lemma 3.4 (auxiliary lemma). *Under the assumptions of Proposition 3.3, there exists $C(W,d) < \infty$ such that, for every $h \in L^2_\sigma$,*

$$\|h - (h)_{\mathbb{R}^d}\|_{L^2_\sigma} \leqslant C \|\nabla_x h\|_{H^{-1}_\sigma}.$$

Proof. Without loss of generality, we assume that $(h)_{\mathbb{R}^d} = 0$. Consider the operators

$$\tilde{A} = \nabla_x, \quad \tilde{A}^* = -\operatorname{div}_x - \boldsymbol{b} \,\cdot\, .$$

We consider the problem

$$\tilde{A}^* \boldsymbol{g} = h \quad \text{in } \mathbb{R}^d, \tag{3-12}$$

where we seek $\boldsymbol{g} \in H^1_\sigma$. The problem can be solved by defining $\boldsymbol{g} = \tilde{A} f$ and solving

$$\tilde{A}^* \tilde{A} f = h \quad \text{in } \mathbb{R}^d, \tag{3-13}$$

with $(f)_{\mathbb{R}^d} = 0$. By the Lax–Milgram lemma, there exists a solution $f \in H^1_\sigma$ with $(f)_{\mathbb{R}^d} = 0$ and $\|f\|_{H^1_\sigma} \leqslant C \|h\|_{H^{-1}_\sigma} \leqslant C \|h\|_{L^2}$. To demonstrate that $\boldsymbol{g} \in H^1_\sigma$, we commute a derivative ∂_i through (3-13):

$$\tilde{A}^* \tilde{A} \partial_i f = -\Delta_x \partial_i f - \boldsymbol{b} \cdot \nabla_x \partial_i f = \partial_i h + \partial_i \boldsymbol{b} \cdot \nabla_x f =: F, \tag{3-14}$$

where F is a forcing term in H^{-1}_σ. Clearly, $\|\partial_i h\|_{H^{-1}_\sigma} \leqslant C \|h\|_{L^2_\sigma}$.[8] For the commutator term, we have

$$\|\partial_i \boldsymbol{b} \cdot \nabla_x f\|_{L^2_\sigma} \leqslant \|\partial_i \boldsymbol{b}\|_{L^\infty} \|\nabla_x f\|_{L^2_\sigma} \leqslant C \|h\|_{H^{-1}_\sigma},$$

where C depends on the $C^{1,1}$ regularity of W. By the Lax–Milgram lemma (or energy estimates) applied to (3-14) for each i, we have

$$\|\nabla_x \boldsymbol{g}\|_{L^2_\sigma} \leqslant C \|\nabla^2_x f\|_{L^2_\sigma} \leqslant C \|F\|_{H^{-1}_\sigma} \leqslant C \|h\|_{L^2_\sigma}. \tag{3-15}$$

While $\boldsymbol{g}$ may not have zero average, it was already controlled in L^2_σ. Finally, we have

$$\|h\|^2_{L^2_\sigma} = \int_{\mathbb{R}^d} h \, \nabla \cdot \boldsymbol{g} \, d\sigma = -\int_{\mathbb{R}^d} \nabla h \cdot \boldsymbol{g} \, d\sigma \leqslant \|\nabla h\|_{H^{-1}_\sigma} \|\boldsymbol{g}\|_{H^1_\sigma}.$$

The conclusion then follows by (3-15). □

[8] This follows from integration by parts against a test function $g \in H^1_\sigma$ and the Poincaré inequality in (3-11), which controls the term $\int \partial_i W g h \, d\sigma$ appearing when ∂_i hits the weight.

Proof of Proposition 3.3. Let $f \in H^1_{\mathrm{hyp}}(\mathbb{R}^d)$; see (3-10). By applying an approximation procedure with smooth cut-off in x and v and mollifying, we can without loss of generality assume that f is a compactly supported, smooth function. Again, we decompose the proof into three steps. Step 1 is identical, so we skip to the next step.

Step 2: We show that

$$\|\nabla \langle f\rangle_\gamma\|_{H^{-1}_\sigma(\mathbb{R}^d)} \leqslant C(\|\nabla_v f\|_{L^2_\sigma(\mathbb{R}^d;L^2_\gamma)} + \|(v\cdot\nabla_x + \boldsymbol{b}\cdot\nabla_v)f\|_{L^2_\sigma(\mathbb{R}^d;H^{-1}_\gamma)}). \tag{3-16}$$

We select $\xi_1,\dots,\xi_d \in C^\infty_c(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d} v\xi_i(v)\,d\gamma(v) = e_i,$$

and, for each test function $\phi \in H^1_\sigma(\mathbb{R}^d)$ and $i \in \{1,\dots,d\}$, we compute

$$\int \phi\partial_{x_i}\langle f\rangle_\gamma\,dm = \int \phi\xi_i(v)v\cdot\nabla_x\langle f\rangle_\gamma\,dm = -\int \phi\xi_i v\cdot\nabla_x(f-\langle f\rangle_\gamma)\,dm + \int \phi\xi_i v\cdot\nabla_x f\,dm. \tag{3-17}$$

We expand the second term on the right-hand side as

$$\int \phi\xi_i v\cdot\nabla_x f\,dm = \int \phi\xi_i(v\cdot\nabla_x + \boldsymbol{b}\cdot\nabla_v)f\,dm - \int \phi\xi_i\boldsymbol{b}\cdot\nabla_v(f-\langle f\rangle_\gamma)\,dm, \tag{3-18}$$

where we use that $\boldsymbol{b}\cdot\nabla_v\langle f\rangle_\gamma = 0$. Combining (3-17) and (3-18), we have

$$\int \phi\partial_{x_i}\langle f\rangle_\gamma\,dm = \int \phi\xi_i(v\cdot\nabla_x + \boldsymbol{b}\cdot\nabla_v)f\,dm - \int \phi\xi_i(v\cdot\nabla_x + \boldsymbol{b}\cdot\nabla_v)(f-\langle f\rangle_\gamma)\,dm = \mathrm{I} + \mathrm{II}.$$

For I, we have

$$\left|\int \phi\xi_i(v\cdot\nabla_x + \boldsymbol{b}\cdot\nabla_v)f\,dm\right| \leqslant C\|\phi\xi_i\|_{L^2_\sigma(\mathbb{R}^d;H^1_\gamma)}\|(v\cdot\nabla_x + \boldsymbol{b}\cdot\nabla_v)f\|_{L^2_\sigma(\mathbb{R}^d;H^{-1}_\gamma)}.$$

For II, we integrate by parts across the measure dm:

$$-\int \phi\xi_i(v\cdot\nabla_x + \boldsymbol{b}\cdot\nabla_v)(f-\langle f\rangle_\gamma)\,dm = \int v\cdot\nabla_x\phi\xi_i(f-\langle f\rangle_\gamma)\,dm + \int \phi\boldsymbol{b}\cdot\nabla_v\xi_i(f-\langle f\rangle_\gamma)\,dm = \mathrm{II}_a + \mathrm{II}_b.$$

For II_a, we use

$$\left|\int v\cdot\nabla_x\phi\xi_i(f-\langle f\rangle_\gamma)\,dm\right| \leqslant C\|v\xi_i\|_{L^\infty_\gamma}\|\phi\|_{L^2_\sigma}\|f-\langle f\rangle_\gamma\|_{L^2_\sigma(\mathbb{R}^d;L^2_\gamma)}.$$

For II_b, we use

$$\left|\int \phi\boldsymbol{b}\cdot\nabla_v\xi_i(f-\langle f\rangle_\gamma)\,dm\right| \leqslant \||\nabla W||\phi|\|_{L^2_\sigma}\|\nabla_v\xi_i\|_{L^\infty_\gamma}\|f-\langle f\rangle_\gamma\|_{L^2_\sigma(\mathbb{R}^d;L^2_\gamma)}.$$

We use the assumed Poincaré inequality (3-11) to control $\||\nabla W||\phi|\|_{L^2_\sigma}$ by $\|\phi\|_{H^1_\sigma}$. Then using (3-3) concludes the proof of (3-16).

Step 3: We deduce from Lemma 3.4, (3-3) and (3-16) that

$$\begin{aligned}\|f-(f)_U\|_{L^2_\sigma(\mathbb{R}^d;L^2_\gamma)} &\leqslant \|f-\langle f\rangle_\gamma\|_{L^2_\sigma(\mathbb{R}^d;L^2_\gamma)}+\|\langle f\rangle_\gamma-(f)_{\mathbb{R}^d}\|_{L^2_\sigma}\\ &\leqslant \|f-\langle f\rangle_\gamma\|_{L^2_\sigma(\mathbb{R}^d;L^2_\gamma)}+C\|\nabla\langle f\rangle_\gamma\|_{H^{-1}_\sigma}\\ &\leqslant C(\|\nabla_v f\|_{L^2_\sigma(\mathbb{R}^d;L^2_\gamma)}+\|(v\cdot\nabla_x+\boldsymbol{b}\cdot\nabla_v)f\|_{L^2_\sigma(\mathbb{R}^d;H^{-1}_\gamma)}).\end{aligned}$$ □

3B. ***Interpolation and Hörmander inequalities for H^1_{hyp}.*** In this subsection, we use the Hörmander bracket condition to obtain a functional inequality which provides some interior spatial regularity for general H^1_{hyp} functions. Both the statement and proof of the inequality follow closely the ideas of [Hörmander 1967]. Other variants of Hörmander's inequality have been previously obtained; see in particular [Bouchut 2002; Albritton et al. 2022]. We remind the reader that our initial estimates are phrased in terms of the Besov-type norms defined in Section 2C and are thus valid for $U=\mathbb{T}^d,\mathbb{R}^d$.

Proposition 3.5 (interpolation). *For every $\delta>0$, there exists $C(d,\delta)<\infty$ such that for $U=\mathbb{T}^d,\mathbb{R}^d$ and any smooth function $u:U\times\mathbb{R}^d\to\mathbb{R}$, we have*

$$\|u\|_{Q^{1/2}_{v\cdot\nabla_x}(U)}\leqslant C(\|u\|_{L^2(U;H^1_\gamma)}+\|v\cdot\nabla_x u\|_{L^2(U;H^{-1}_\gamma)})+\delta\|u\|_{Q^{1/3}_{\nabla_x}(U)}. \tag{3-19}$$

Proof. Step 1: Let $\phi\in C_0^\infty((-1,1)^d)$ be a smooth, positive, radial function with unit L^1 norm. For $t\in(0,\infty)$, we define $\phi_t u(x,v)$ by

$$\phi_t u(x,v)=\int_{\mathbb{R}^d}u(x+t^3x',v)\phi(x')\,dx',$$

where in the case $U=\mathbb{T}^d$ we have periodically extended u to a function defined on all of $\mathbb{R}^d$. Using Jensen's inequality, we calculate that

$$\begin{aligned}\|\phi_t u(x,v)-u(x,v)\|^2_{L^2(U;L^2_\gamma)} &= \iint_{\mathbb{R}^d\times U}\biggl(\int_{\mathbb{R}^d}\phi(x')(u(x+t^3x',v)-u(x,v))\,dx'\biggr)^2 dx\,d\gamma(v)\\ &\leqslant \iiint_{\mathbb{R}^d\times U\times\mathbb{R}^d}\phi(x')(u(x+t^3x',v)-u(x,v))^2\,dx'\,dx\,d\gamma(v)\\ &= \iiint_{\mathbb{R}^d\times U\times\mathbb{R}^d}\phi(x')t^2\frac{1}{t^2}(u(x+t^3x',v)-u(x,v))^2\,dx'\,dx\,d\gamma(v)\\ &\leqslant \int_{\mathbb{R}^d}\phi(x')t^2\|u\|^2_{Q^{1/3}_{\nabla_x}}\,dx',\end{aligned}$$

and thus we see that

$$\|\phi_t u(x,v)-u(x,v)\|^2_{L^2(U;L^2_\gamma)}\leqslant t^2\|u\|^2_{Q^{1/3}_{\nabla_x}(U)}. \tag{3-20}$$

Step 2: Let

$$f(t)=\|u(x+t^2v,v)-u(x,v)\|^2_{L^2(U;L^2_\gamma)}.$$

For $t\in(0,\infty)$, it will suffice to show that

$$f(t)\leqslant t^2\bigl(C(\|u\|_{L^2(U;H^1_\gamma)}+\|v\cdot\nabla_x u\|_{L^2(U;H^{-1}_\gamma)})+\delta\|u\|_{Q^{1/3}_{\nabla_x}(U)}\bigr)^2. \tag{3-21}$$

Moreover, for $t \geqslant 1$, we have the obvious estimate $f(t) \leqslant 4\|u\|^2_{L^2(U;L^2_\gamma)}$, so we consider only $t \in (0,1)$. We may write that

$$f(t) \leqslant \|\phi_{\delta t} u(x+t^2 v, v) - u(x+t^2 v, v)\|^2_{L^2(U;L^2_\gamma)} + \|\phi_{\delta t} u(x+t^2 v, v) - \phi_{\delta t} u(x,v)\|^2_{L^2(U;L^2_\gamma)} + \|\phi_{\delta t} u(x,v) - u(x,v)\|^2_{L^2(U;L^2_\gamma)}. \tag{3-22}$$

By Step 1, the first and third terms of (3-22) are bounded by

$$\delta^2 t^2 \|u\|^2_{Q^{1/3}_{\nabla_x}}.$$

Step 3: It remains to estimate the second term in (3-22). For $t \in (0,1)$ and $0 \leqslant \tau \leqslant t^2$, consider

$$F(\tau) = \|\phi_{\delta t} u(x+\tau v, v) - \phi_{\delta t} u(x,v)\|^2_{L^2(U;L^2_\gamma)}, \tag{3-23}$$

where $F(t^2)$ is precisely the second term in (3-22). Since $F(0)=0$, it will suffice to show that there exists $C(d,\delta) < \infty$ such that

$$F'(\tau) \leqslant C^2(\|u\|^2_{L^2(U;H^1_\gamma)} + \|v\cdot\nabla_x u\|^2_{L^2(U;H^{-1}_\gamma)}) + \delta^2\|u\|^2_{Q^{1/3}_{\nabla_x}}.$$

We have

$$\begin{aligned} F'(\tau) &= 2\iint_{\mathbb{R}^d\times U} (\phi_{\delta t} u(x+\tau v, v) - \phi_{\delta t} u(x,v))v\cdot\nabla_x(\phi_{\delta t} u)(x+\tau v, v)\,dx\,d\gamma(v) \\ &= 2\iint_{\mathbb{R}^d\times U} (\phi_{\delta t} u(x,v) - \phi_{\delta t} u(x-\tau v, v))v\cdot\nabla_x(\phi_{\delta t} u)(x,v)\,dx\,d\gamma(v). \end{aligned}$$

Since $[v\cdot\nabla_x, \phi_{\delta t}]u = [\nabla_v, \phi_{\delta t}]u = 0$ and we have a bound on $\|v\cdot\nabla_x u\|_{L^2(U;H^{-1}_\gamma)}$, we will achieve the desired estimate for $F'(\tau)$ if we can bound

$$(\phi_{\delta t} u(x,v) - \phi_{\delta t} u(x-\tau v, v)) \tag{3-24}$$

in $L^2(U;H^1_\gamma)$. The only nontrivial estimate comes when the ∇_v lands on the x-coordinate of the second term in (3-24), which we may write out as

$$\begin{aligned} \int_{\mathbb{R}^d} -\tau\nabla_x u(x+(\delta t)^3 x' - \tau v, v)\phi(x')\,dx' &= -\int_{\mathbb{R}^d} \frac{\tau}{(\delta t)^3}\nabla_{x'} u(x+(\delta t)^3 x' - \tau v, v)\phi(x')\,dx' \\ &= \int_{\mathbb{R}^d} \frac{\tau}{(\delta t)^3} u(x+(\delta t)^3 x' - \tau v, v)\nabla_{x'}\phi(x')\,dx' \\ &= \int_{\mathbb{R}^d} \frac{\tau}{(\delta t)^3} (u(x+(\delta t)^3 x' - \tau v, v) - u(x-\tau v, v))\nabla_{x'}\phi(x')\,dx'. \end{aligned}$$

But by Step 1, this is bounded in $L^2(U;L^2_\gamma)$ by a constant multiple of

$$\frac{\tau}{(\delta t)^3}|t|\,\|u\|_{Q^{1/3}_{\nabla_x}} \leqslant \frac{1}{\delta^3}\|u\|_{Q^{1/3}_{\nabla_x}},$$

where we have used the assumption that $\tau \leqslant t^2$. Note that in order to absorb the $1/\delta^3$ in the denominator, we may appeal to the Cauchy–Schwarz and Young inequalities in front of $\|v\cdot\nabla_x u\|_{L^2_x(U;H^{-1}_\gamma)}$, which leads to the estimate (3-19) after modifying δ to absorb any implicit constants. □

With Proposition 3.5 in hand, we can now prove a Hörmander inequality which provides regularity in the x-variable, measured in the $Q^{1/3}_{\nabla_x}$ space. The H^α estimate in Theorem 1.4 for $\alpha < \frac{1}{3}$ will be an immediate corollary, and essentially amounts to converting $B^{1/3}_{2,\infty}$-type regularity to $B^{\alpha}_{2,2}$-type regularity. Following [Hörmander 1967], the proof of Theorem 1.4 is based on the splitting of a first-order finite difference in the x-variable into finite differences which are either in the v-variable, or in the x-variable in the direction of v. Explicitly, we have

$$\begin{aligned} f(x+t^3y, v) - f(x, v) = f(x+t^3y, v) &- f(x+t^3y, v-ty) \\ &+ f(x+t^3y, v-ty) - f(x+t^3y+t^2(v-ty), v-ty) \\ &+ f(x+t^2v, v-ty) - f(x+tv, v) \\ &+ f(x+t^2v, v) - f(x, v). \end{aligned} \tag{3-25}$$

Notice that the right side consists of four finite differences, two for each of the derivatives ∇_v and $v \cdot \nabla_x$ which we can expect to control by the $L_2(U; H^1_\gamma)$ and $Q^{1/2}_{v\cdot\nabla_x}$ norms, respectively. The fact that the increment on the left is of size t^3 and those on the right side are of sizes t and t^2 suggests that we may expect to have one-third derivative in the statement of Theorem 1.4, which we are able to obtain in a Besov sense with the $Q^{1/3}_{\nabla_x}$ norm. The exponent $\frac{1}{3}$ is optimal, although it may be possible to improve the endpoint regularity from $B^{1/3}_{2,\infty}$-type to $B^{1/3}_{2,2}$ using more advanced microlocal techniques.

The relation (3-25) is a special case of Hörmander's bracket condition introduced in [Hörmander 1967], which for the particular equation we consider here is quite simple to check. Indeed, let $X_1, \dots, X_d$, $V_1, \dots, V_d$ denote the canonical vector fields and X_0 be the vector field $(x, v) \mapsto (v, 0)$. Then the Hörmander bracket condition is implied by the identity

$$[V_i, X_0] = X_i. \tag{3-26}$$

This is a local version of the identity (3-25). More precisely, for every vector field Z, if we denote by $t \mapsto \exp(tZ)$ the flow induced by the vector field Z on $\mathbb{R}^d \times \mathbb{R}^d$, then

$$\exp(-tV_i)\exp(-tX_0)\exp(tV_i)\exp(tX_0)(x, v) = (x, v) + t^2[V_i, X_0](x, v) + o(t^2), \quad t \to 0. \tag{3-27}$$

For the vector fields of interest, $Z \in \{X_0, X_1, \dots, X_d, V_1, \dots, V_d\}$, the flows take the very simple form

$$\exp(tZ)(x, v) = (x, v) + tZ(x, v),$$

the relation (3-27) becomes an identity (that is, the term $o(t^2)$ is actually zero), and loosely, this identity can be rephrased in the form of (3-25). The only difference is that, to exploit that our functions have only $\frac{1}{2}$ derivatives in the $v \cdot \nabla_x$ direction, it is advantageous to flow in the direction $v \cdot \nabla_x$ with speed t rather than unit speed.

Proposition 3.6 (Besov–type Hörmander inequality). *There exists a dimensional constant $C(d) < \infty$ such that, for $U = \mathbb{T}^d, \mathbb{R}^d$ and any smooth function $u : U \times \mathbb{R}^d \to \mathbb{R}$, we have the estimate*

$$\|u\|_{Q^{1/3}_{\nabla_x}(U)} \leqslant C(\|u\|_{Q^{1/2}_{v\cdot\nabla_x}(U)} + \|u\|_{L^2_x(U; H^1_\gamma)}). \tag{3-28}$$

Proof of Proposition 3.6. Let $f(x, v) = u(x, v)\gamma^{1/2}(v)$, and choose $\eta \in (0, \infty)$ and $x' \in \mathbb{S}^{d-1}$. Then we may write

$$\|u(x+\eta^3 x', v) - u(x, v)\|_{L^2(U;L^2_\gamma)} = \|f(x+\eta^3 x', v) - f(x, v)\|_{L^2(U;L^2)}$$

and

$$\begin{aligned} f(x+\eta^3 x', v) - f(x, v) = f(x+\eta^3 x', v) &- f(x+\eta^3 x', v-\eta x') \\ &+ f(x+\eta^3 x', v-\eta x') - f(x+\eta^3 x' + \eta^2(v-\eta x'), v-\eta x') \\ &+ f(x+\eta^2 v, v-\eta x') - f(x+\eta^2 v, v) \\ &+ f(x+\eta^2 v, v) - f(x, v). \end{aligned} \tag{3-29}$$

Dividing by η, integrating in $L^2(U; L^2(\mathbb{R}^d))$, and appealing to (2-7) bounds the first term:

$$\frac{1}{\eta^2}\iint_{\mathbb{R}^d\times U} (f(x+\eta^3 x', v) - f(x+\eta^3 x', v-\eta x'))^2\,dx\,dv \leqslant C\|\nabla_v u\|^2_{L^2(U;L^2_\gamma)},$$

with a similar bound holding for the third term. Dividing again by η and integrating in $L^2(U; L^2(\mathbb{R}^d))$ yields the bound

$$\frac{1}{\eta^2}\iint_{\mathbb{R}^d\times U} \bigl(f(x+\eta^3 x', v-\eta x') - f(x+\eta^3 x' + \eta^2(v-\eta x'), v-\eta x')\bigr)^2\,dx\,dv \leqslant \|u\|^2_{Q^{1/2}_{v\cdot\nabla_x}(U)},$$

with a similar bound holding for the fourth term. Appealing to (3-19) with a suitably small choice of δ concludes the proof. □

To obtain the statements in Theorem 1.4 for $\alpha < \frac{1}{3}$, we must work in H^α_x rather than $(B^\alpha_{2,\infty})_x$ spaces of fractional differentiability, and so we introduce the Banach space-valued fractional Sobolev spaces, defined as follows: for every domain $U \subseteq \mathbb{R}^d$, $\alpha \in (0, 1)$, Banach space X with norm $\|\cdot\|_X$ and $u \in L^2(U; X)$, we define the seminorm

$$[\![u]\!]_{H^\alpha(U;X)} := \left(\int_U\int_U \frac{\|u(x)-u(y)\|^2_X}{|x-y|^{d+2\alpha}}\,dx\,dy\right)^{1/2} \tag{3-30}$$

and the norm

$$\|u\|_{H^\alpha(U;X)} := (\|u\|^2_{L^2(U;X)} + [\![u]\!]^2_{H^\alpha(U;X)})^{1/2}.$$

We then define the fractional Sobolev space

$$H^\alpha(U; X) := \{u \in L^2(U; X) : \|u\|_{H^\alpha(U;X)} < \infty\}. \tag{3-31}$$

The space $H^\alpha(U; X)$ is a Banach space under the norm $\|\cdot\|_{H^\alpha(U;X)}$. We understand that $H^0(U; X) = L^2(U; X)$. We also set

$$\|u\|_{H^{1+\alpha}(U;X)} := (\|u\|^2_{L^2(U;X)} + \|\nabla u\|^2_{H^\alpha(U;X)})^{1/2},$$

and define the Banach space $H^{1+\alpha}(U; X)$ as in (3-31). We may now use Proposition 3.6 to prove the non-endpoint estimates from Theorem 1.4.

Proof of Theorem 1.4. Recall that we consider the domain $U = \mathbb{R}^d$ or $U = \mathbb{T}^d$. We have that, for $\alpha < \frac{1}{3}$,

$$\begin{aligned}
[\![u]\!]^2_{H^\alpha(U;L^2_\gamma)} &= \iint_{U\times U} \frac{\|u(x,\cdot)-u(y,\cdot)\|^2_{L^2_\gamma}}{|x-y|^{d+2\alpha}}\,dx\,dy \\
&= \iint_{U\times U} \frac{\|u(x'+y,\cdot)-u(y,\cdot)\|^2_{L^2_\gamma}}{|x'|^{d+2\alpha}}\,dx'\,dy \\
&= \iint_{\{|x'|<1\}\times U} \frac{\|u(x'+y,\cdot)-u(y,\cdot)\|^2_{L^2_\gamma}}{|x'|^{d+2\alpha}}\,dy\,dx' + \iint_{\{|x'|\geqslant 1\}\times U} \frac{\|u(x'+y,\cdot)-u(y,\cdot)\|^2_{L^2_\gamma}}{|x'|^{d+2\alpha}}\,dy\,dx' \\
&\leqslant \int_{\{|x'|<1\}} \frac{|x'|^{2/3}\|u\|^2_{Q^{1/3}_{\nabla_x}}}{|x'|^{d+2\alpha}}\,dx' \quad + \quad C(\alpha)\|u\|^2_{L^2(U;L^2_\gamma)} \\
&\leqslant C(\alpha)(\|u\|^2_{L^2(U;H^1_\gamma)} + \|v\cdot\nabla_x u\|^2_{L^2(U;H^{-1}_\gamma)}) \leqslant C(\alpha)\|u\|^2_{H^1_{\mathrm{hyp}}(U)},
\end{aligned}$$

concluding the proof. □

For the purposes of interpolation, we also need to consider fractional Sobolev spaces in the velocity variable. As discussed in the arguments leading to (2-7), the relevant spaces are weighted by the measure γ, which is strongly inhomogeneous. Because of this difficulty, we use the following definition. For each $f \in L^2_\gamma$ and $t > 0$, we set

$$K(t, f) := \inf\{\|f_0\|_{L^2_\gamma} + t\|f_1\|_{H^1_\gamma} : f = f_0 + f_1,\ f_0 \in L^2_\gamma,\ f_1 \in H^1_\gamma\},$$

and, for every $\alpha \in (0, 1)$, we define

$$\|f\|_{H^\alpha_\gamma} := \left(\int_0^\infty (t^{-\alpha}K(f,t))^2\,\frac{dt}{t}\right)^{1/2}. \tag{3-32}$$

We also define $H^{-\alpha}_\gamma$ to be the space dual to H^α_γ.

We may utilize interpolation to obtain embeddings into other similar spaces of positive regularity in both variables. In particular, appealing to Theorem 1.4 and the interpolation inequality

$$\|f\|_{H^{\theta\beta}(U;H^{1-\theta}_\gamma)} \leqslant \|f\|^\theta_{H^\beta(U;L^2_\gamma)}\|f\|^{1-\theta}_{L^2(U;H^1_\gamma)}, \quad \theta \in [0,1], \quad U = \mathbb{T}^d, \mathbb{R}^d,$$

immediately implies the following estimate.

Corollary 3.7 (Hörmander inequality for H^1_{hyp}). *Let $\alpha \in \left[0, \frac{1}{3}\right)$ and $U = \mathbb{T}^d, \mathbb{R}^d$. There exists a constant $C(\alpha, d) < \infty$ such that, for every $\theta \in [0, 1]$ and every $f \in H^1_{\mathrm{hyp}}(U)$, we have the estimate*

$$\|f\|_{H^{\theta\alpha}(U;H^{1-\theta}_\gamma)} \leqslant C\|f\|_{H^1_{\mathrm{hyp}}(U)}.$$

Observe that, by introducing a cutoff function in the spatial variable, we also obtain analogous embeddings for bounded domains $U \subseteq \mathbb{R}^d$, such as

$$H^1_{\mathrm{hyp}}(U) \hookrightarrow H^\alpha(U_\delta; L^2_\gamma),$$

valid for every $\alpha < \frac{1}{3}$ and $\delta > 0$, where $U_\delta := \{x \in U : \mathrm{dist}(x, \partial U) > \delta\}$.

3C. ***Compact embedding of H^1_{hyp} into $L^2(U; L^2_\gamma)$.*** Using the results of the previous subsection, we show that the embedding $H^1_{\mathrm{hyp}}(U) \hookrightarrow L^2(U; L^2_\gamma)$ is compact. In this section, we assume that $U \subseteq \mathbb{R}^d$ is a bounded C^1 domain or $\mathbb{T}^d$.

Proposition 3.8 (compact embedding of $H^1_{\mathrm{hyp}}(U)$ into $L^2(U; L^2_\gamma)$). *The inclusion map $H^1_{\mathrm{hyp}}(U) \hookrightarrow L^2(U; L^2_\gamma)$ is compact.*

The proof is straightforward on $\mathbb{T}^d$. First, approximate by functions in $C^\infty_0(\mathbb{T}^d \times \mathbb{R}^d)$. Next, we use the embedding $H^1_{\mathrm{hyp}}(\mathbb{T}^d) \subseteq H^\alpha(\mathbb{T}^d \times B_{v_0})$ for all $v_0 \in [1, +\infty)$. Finally, we apply the standard Rellich compactness theorem. Hence, we focus only on bounded C^1 domains $U \subseteq \mathbb{R}^d$ below.

Before we give the proof of Proposition 3.8, we need to review some basic facts concerning the logarithmic Sobolev inequality and a generalized Hölder inequality for Orlicz norms. The logarithmic Sobolev inequality states that, for some $C < \infty$,

$$\int_{\mathbb{R}^d} f^2(v)\log(1+f^2(v))\,d\gamma(v) \leqslant C\int_{\mathbb{R}^d} |\nabla f|^2\,d\gamma(v) \quad \text{for all } f \in H^1_\gamma \text{ with } \|f\|_{L^2_\gamma} = 1. \tag{3-33}$$

Let $F : \mathbb{R} \to [0, \infty)$ denote the (strictly) convex function

$$F(t) := |t|\log(1+|t|).$$

Let F^* denote its dual convex conjugate function, defined by

$$F^*(s) := \sup_{t\in\mathbb{R}}(st - F(t)).$$

Then (F, F^*) is a *Young pair* (see [Rao and Ren 1991]), that is, both F and F^* are nonnegative, even, convex, and satisfy $F(0) = F^*(0) = 0$, as well as

$$\lim_{|t|\to\infty} |t|^{-1}F(t) = \lim_{|s|\to\infty} |s|^{-1}F^*(s) = \infty.$$

Moreover, both F and F^* are strictly increasing on $[0, \infty)$ and in particular vanish only at $t = 0$. Given any measure space (X, ω), the *Orcliz spaces* $L_F(X, \omega)$ and $L_{F^*}(X, \omega)$, which are defined by the norms

$$\|g\|_{L_F(X,\omega)} := \inf\left\{t>0 : \int_X F(t^{-1}g)\,d\omega \leqslant F(1)\right\}, \quad \|g\|_{L_{F^*}(X,\omega)} := \inf\left\{t>0 : \int_X F^*(t^{-1}g)\,d\omega \leqslant F^*(1)\right\},$$

are dual Banach spaces and the following generalized version of the Hölder inequality is valid (see [Rao and Ren 1991, Proposition 3.3.1]):

$$\int_X |gg^*|\,d\omega \leqslant \|g\|_{L_F(X,\omega)}\|g^*\|_{L_{F^*}(X,\omega)} \quad \text{for all } g \in L_F(X, \omega),\ g^* \in L_{F^*}(X, \omega).$$

The logarithmic Sobolev inequality (3-33) may be written in terms of the Orcliz norm as

$$\|f^2\|_{L_F(\mathbb{R}^d,\gamma)} \leqslant C(|\langle f\rangle_\gamma|^2 + \|\nabla f\|^2_{L^2_\gamma}) \quad \text{for all } f \in H^1_\gamma.$$

The previous two displays imply

$$\left(\int_{U\times\mathbb{R}^d} g|f|^2\,dx\,d\gamma(v)\right)^{1/2} \leqslant C\|g\|^{1/2}_{L_{F^*}(U\times\mathbb{R}^d,dx\,d\gamma)}\|f\|_{L^2(U;H^1_\gamma)}. \tag{3-34}$$

We do not identify F^* with an explicit formula, although we notice that the inequality

$$s(t+1) \leqslant \exp(s) + t\log(1+t) \quad \text{for all } s, t \in (0,\infty)$$

implies

$$F^*(s) \leqslant \exp(s) - s.$$

This allows us in particular to obtain from (3-34) that

$$\left(\int_{U\times\mathbb{R}^d} |v|^2 |f|^2 \, dx \, d\gamma(v)\right)^{1/2} \leqslant C\|f\|_{L^2(U;H^1_\gamma)}. \tag{3-35}$$

We also point out that (3-35) also implies the existence of $C(d,U)<\infty$ such that, for every $f \in L^2(U; L^2_\gamma)$,

$$\|\nabla_v f\|_{L^2(U;H^{-1}_\gamma)} \leqslant C\|f\|_{L^2(U;L^2_\gamma)}. \tag{3-36}$$

We now turn to the proof of Proposition 3.8.

Proof of Proposition 3.8. For each $\theta > 0$, we define

$$U_\theta := \{x : \operatorname{dist}(x, \partial U) < \theta\}. \tag{3-37}$$

Since U is a C^1 domain, we can extend the outer normal $\boldsymbol{n}_U$ to a globally C^0 function on $\overline{U}$. We can moreover assume that, for some $\theta_0(U) > 0$, this extension $\boldsymbol{n}_U$ coincides with the gradient of the mapping $x \mapsto -\operatorname{dist}(x, \partial U)$ in U_{θ_0}.

By Proposition 2.2, we may work under the qualitative assumption that all of our $H^1_{\mathrm{hyp}}(U)$ functions belong to $C^\infty_c(\overline{U}\times\mathbb{R}^d)$. Select $\varepsilon > 0$ and a sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq H^1_{\mathrm{hyp}}(U)$ satisfying

$$\sup_{n\in\mathbb{N}} \|f_n\|_{H^1_{\mathrm{hyp}}(U)} \leqslant 1.$$

We will argue that there exists a subsequence $\{f_{n_k}\}$ such that

$$\limsup_{k\to\infty} \sup_{i,j\geqslant k} \|f_{n_i} - f_{n_j}\|_{L^2(U;L^2_\gamma)} \leqslant \varepsilon. \tag{3-38}$$

The proposition may then be obtained by a diagonalization argument.

Step 1: We claim that there exists $v_0 \in [1,\infty)$ such that, for every $f \in H^1_{\mathrm{hyp}}(U)$,

$$\left(\int_U \int_{\mathbb{R}^d\setminus B_{v_0}} |f(x,v)|^2 \, dx \, d\gamma(v)\right)^{1/2} \leqslant \frac{\varepsilon}{3}\|f\|_{H^1_{\mathrm{hyp}}(U)}.$$

Indeed, applying (3-34), we find that

$$\left(\int_U \int_{\mathbb{R}^d\setminus B_{v_0}} |f(x,v)|^2 \, dx \, d\gamma(v)\right)^{1/2} \leqslant C\|\mathbb{1}_{U\times(\mathbb{R}^d\setminus v_0)}\|^{1/2}_{L_{F^*}(U\times\mathbb{R}^d, dx d\gamma)} \|f\|_{H^1_{\mathrm{hyp}}(U)}.$$

Taking v_0 sufficiently large, depending on ε, ensures that

$$C\|\mathbb{1}_{U\times(\mathbb{R}^d\setminus v_0)}\|^{1/2}_{L_{F^*}(U\times\mathbb{R}^d, dx d\gamma)} \leqslant \frac{\varepsilon}{3}.$$

Step 2: We next claim that there exists $\delta \in \left(0, \tfrac{1}{2}\right]$ such that, for every $f \in H^1_{\mathrm{hyp}}(U)$,

$$\left(\int_U \int_{\mathbb{R}^d} |f(x,v)|^2 \mathbb{1}_{\{|\boldsymbol{n}_U\cdot v|<\delta\}} \, dx \, d\gamma(v)\right)^{1/2} \leqslant \frac{\varepsilon}{3}\|f\|_{H^1_{\mathrm{hyp}}(U)}.$$

The argument here is similar to the estimate in Step 1, above. We simply apply (3-34) after choosing δ small enough that

$$C\|\mathbb{1}_{\{|\boldsymbol{n}_U\cdot v|<\delta\}}\|^{1/2}_{L_{F^*}(U\times\mathbb{R}^d,dx\,d\gamma)} \leqslant \frac{\varepsilon}{3}.$$

Step 3: We next show that, for every $\delta>0$, there exists $\theta>0$ such that, for every function $f\in H^1_{\mathrm{hyp}}(U)$,

$$\left(\int_U\int_{\mathbb{R}^d}|f(x,v)|^2\mathbb{1}_{\{|\boldsymbol{n}_U\cdot v|\geqslant\delta\}}\mathbb{1}_{\{\mathrm{dist}(x,\partial U)<\theta\}}\,dx\,d\gamma(v)\right)^{1/2} \leqslant \frac{\varepsilon}{3}\|f\|_{H^1_{\mathrm{hyp}}(U)}. \tag{3-39}$$

For $\theta\in\left(0,\frac12\theta_0\right]$ to be taken sufficiently small in terms of $\delta>0$ in the course of the argument, we let $\varphi\in C^{1,1}(\overline{U})$ be defined by

$$\varphi(x) := -\eta(\mathrm{dist}(x,\partial U)),$$

where $\eta\in C^\infty_c([0,\infty))$ satisfies

$$0\leqslant\eta\leqslant 2\theta,\qquad 0\leqslant\eta'\leqslant 1,\qquad \eta(x)=x\quad\text{on }[0,\theta],\qquad \eta'=0\quad\text{on }[2\theta,\infty).$$

We have $-2\theta\leqslant\varphi\leqslant 0$. Moreover, by the definition of θ_0 below (3-37), its gradient $\nabla\varphi$ is proportional to $\boldsymbol{n}_U$ in U, it vanishes outside of $U_{2\theta}$, and $\nabla\varphi=\boldsymbol{n}_U$ in U_θ. We next select another test function $\chi\in C^\infty_c([0,\infty))$ satisfying

$$0\leqslant\chi\leqslant 1,\qquad \chi\equiv 0\quad\text{on }\left[0,\tfrac12\delta\right],\qquad \chi\equiv 1\quad\text{on }[\delta,\infty),\qquad |\chi'|\leqslant\delta^{-1},$$

and define

$$\psi_\pm(x,v) := \chi((v\cdot\boldsymbol{n}_U(x))_\pm),$$

where for $r\in\mathbb{R}$, we use the notation $r_- := \max(0,-r)$ and $r_+ := \max(0,r)$. Observe that

$$|\nabla_v\psi_\pm(x,v)| = |\chi'((v\cdot\boldsymbol{n}_U(x))_\pm)|\,|\boldsymbol{n}_U(x)| \leqslant C\delta^{-1}.$$

Therefore

$$\begin{aligned}\|\varphi f\psi_\pm\|_{L^2(U;H^1_\gamma)} &\leqslant C(\|\varphi f\psi_\pm\|_{L^2(U;L^2_\gamma)} + \|\varphi\nabla_v(f\psi_\pm)\|_{L^2(U;L^2_\gamma)})\\ &\leqslant C\theta(\|f\|_{L^2(U;L^2_\gamma)} + \|\nabla_v f\|_{L^2(U;L^2_\gamma)} + \|f\nabla_v\psi_\pm\|_{L^2(U;L^2_\gamma)})\\ &\leqslant C\theta\delta^{-1}\|f\|_{L^2(U;H^1_\gamma)},\end{aligned}$$

and hence

$$\left|\int_{U\times\mathbb{R}^d}\varphi f\psi_\pm v\cdot\nabla_x f\,dx\,d\gamma(v)\right| \leqslant C\theta\delta^{-1}\|f\|^2_{H^1_{\mathrm{hyp}}(U)}.$$

On the other hand,

$$\begin{aligned}\int_{U\times\mathbb{R}^d}\varphi f\psi_\pm v\cdot\nabla_x f\,dx\,d\gamma(v) &= -\frac12\int_{U\times\mathbb{R}^d} f^2 v\cdot\nabla_x(\varphi\psi_\pm)\,dx\,d\gamma(v)\\ &= -\frac12\int_{U\times\mathbb{R}^d}\varphi f^2 v\cdot\nabla_x\psi_\pm\,dx\,d\gamma(v) - \frac12\int_{U\times\mathbb{R}^d}\psi_\pm f^2 v\cdot\nabla\varphi\,dx\,d\gamma(v).\end{aligned}$$

Since $|v\cdot\nabla_x\psi_\pm(x,v)|\leqslant C\delta^{-1}|v|^2$, we have, by (3-35),

$$\left|\int_{U\times\mathbb{R}^d}\varphi f^2 v\cdot\nabla_x\psi_\pm\,dx\,d\gamma(v)\right| \leqslant C\theta\delta^{-1}\int_{U\times\mathbb{R}^d}|v|^2 f^2\,dx\,d\gamma(v) \leqslant C\theta\delta^{-1}\|f\|^2_{H^1_{\mathrm{hyp}}(U)}.$$

We deduce that

$$\left| \int_{U \times \mathbb{R}^d} \psi_\pm f^2 v \cdot \nabla \varphi \, dx \, d\gamma(v) \right| \leqslant C\theta\delta^{-1} \|f\|^2_{H^1_{\mathrm{hyp}}(U)}.$$

Finally, we observe from the properties of φ and $\psi_\pm$ that

$$\begin{aligned} \int_U \int_{\mathbb{R}^d} |f(x,v)|^2 \mathbb{1}_{\{|\boldsymbol{n}_U \cdot v| \geqslant \delta\}} \mathbb{1}_{\{\mathrm{dist}(x,\partial U) < \theta\}} \, dx \, d\gamma(v) \\ \leqslant \delta^{-1} \left(\left| \int_{U \times \mathbb{R}^d} \psi_+ f^2 v \cdot \nabla \varphi \, dx \, d\gamma(v) \right| + \left| \int_{U \times \mathbb{R}^d} \psi_- f^2 v \cdot \nabla \varphi \, dx \, d\gamma(v) \right| \right) \\ \leqslant C\theta\delta^{-2} \|f\|^2_{H^1_{\mathrm{hyp}}(U)}. \end{aligned}$$

Taking $\theta = c\varepsilon^2\delta^2$ for a sufficiently small constant $c > 0$ yields the claimed inequality (3-39).

Step 4: By the results of the previous three steps, to obtain (3-38) it suffices to exhibit a subsequence $\{f_{n_k}\}$ satisfying

$$\limsup_{k \to \infty} \sup_{i,j \geqslant k} \int_{U_\theta \times B_{v_0}} |f_{n_i} - f_{n_j}|^2 \, dx \, d\gamma(v) = 0.$$

This is an immediate consequence of Corollary 3.7 and the compactness of the embedding

$$H^{1/10}(U_\theta; H^{1/3}_\gamma) \hookrightarrow L^2(U_\theta; L^2_\gamma(B_{v_0}))$$

(see for instance [Adams and Fournier 2003, Theorem 2.32]). □

4. The Kramers equation

In this section, we present two proofs of the existence of weak solutions in $H^1_{\mathrm{hyp}}(\mathbb{T}^d)$ to the Kramers equation

$$-\Delta_v f + v \cdot \nabla_v f + v \cdot \nabla_x f + \boldsymbol{b} \cdot \nabla_v f = g^*, \tag{4-1}$$

where $g^* \in L^2(\mathbb{T}^d; H^{-1}_\gamma)$ satisfies $\iint_{\mathbb{T}^d \times \mathbb{R}^d} g^* \, dm = 0$ (recall that the weighted mean of g^* is well-defined by duality since the function 1 belongs to $L^2(\mathbb{T}^d; H^1_\gamma)$). The first proof uses the abstract Lions–Lax–Milgram theorem and a modification of (4-1) with a penalization term νf. The hypoelliptic energy estimates are used in sending the parameter ν to zero. This approach is partly inspired by [Carrillo 1998]. The second proof uses a dual variational approach which characterizes the weak solutions of (4-1) as the minimizers of a natural energy under an appropriate constraint, in analogy with the discussion following the statement of Theorem 1.2. In both cases, the Poincaré inequality from Theorem 1.3 provides the necessary coercivity.

Throughout this section, the force field $\boldsymbol{b}(x) = -\nabla W(x)$ is as in Assumption 1.1. In particular, $\boldsymbol{b}$ depends only on x and is conservative. Let dm be as defined in (1-7).

4A. ***The Lions–Lax–Milgram approach.*** We recall the abstract version of Lions' representation theorem from [Showalter 1997, Theorem 3.1, p. 109].

Lemma 4.1 (Lions' representation theorem). *Let H be a Hilbert space and Φ a pre-Hilbert space. Let $E : H \times \Phi \to \mathbb{R}$ be a bilinear form satisfying the continuity criterion*

$$E(\,\cdot\,, \phi) \in H^* \quad \textit{for all } \phi \in \Phi. \tag{4-2}$$

Then the following two properties are equivalent:

- (*coercivity*) *We have*

$$\inf_{\|\phi\|_\Phi=1} \sup_{\|h\|_H\leqslant 1} |E(h,\phi)| \geqslant c > 0. \tag{4-3}$$

- (*solvability*) *For each* $L \in \Phi^*$, *there exists* $f \in H$ *such that*

$$E(f,\phi) = L(\phi) \quad \textit{for all } \phi \in \Phi. \tag{4-4}$$

Notice that uniqueness and stability estimates are not guaranteed by Lemma 4.1 itself; they are concluded a posteriori.

Proof of Theorem 1.2. We split the argument into steps; in the first step, we solve a penalized problem, and in the second, we send the penalization parameter ν to zero.

Step 1: Consider the penalized problem

$$(v\cdot\nabla_x + \boldsymbol{b}\cdot\nabla_v) f + \nu f = g^* + \Delta f - v\cdot\nabla_v f \tag{4-5}$$

posed on the torus $\mathbb{T}^d$, where $\nu \in (0,1]$. We define the following objects:

(1) the *test function space*

$$\Phi = C_0^\infty(\mathbb{T}^d\times\mathbb{R}^d)$$

with inner product

$$(\phi,\psi) = \iint_{\mathbb{T}^d\times\mathbb{R}^d} \nabla_v\phi\cdot\nabla_v\psi\, dm + \iint_{\mathbb{T}^d\times\mathbb{R}^d} \phi\psi\, dm, \tag{4-6}$$

(2) the *solution space*

$$H = \{h \in L^2_\sigma(\mathbb{T}^d; H^1_\gamma) : (h)_{\mathbb{T}^d} = 0\},$$

with inner product (4-6),

(3) the penalized *bilinear form*

$$E(h,\phi) = \iint_{\mathbb{T}^d\times\mathbb{R}^d} \nabla_v h\cdot\nabla_v\phi\, dm + \nu\iint_{\mathbb{T}^d\times\mathbb{R}^d} h\phi\, dm - \iint_{\mathbb{T}^d\times\mathbb{R}^d} h(v\cdot\nabla_x + \boldsymbol{b}\cdot\nabla_v)\phi\, dm,$$

(4) and the *linear functional*

$$L = g^* \in L^2_\sigma(\mathbb{T}^d; H^{-1}_\gamma), \quad \text{with } (g^*)_{\mathbb{T}^d} = 0.$$

It is not difficult to verify that E is continuous (4-2) and coercive (4-3). Indeed, the key features are that the antisymmetric operator $v\cdot\nabla_x + \boldsymbol{b}\cdot\nabla_v$ hits the test function ϕ, and the penalization term $\nu\iint_{\mathbb{T}^d\times\mathbb{R}^d} |\phi|^2\, dm$ controls the "lower part" $(L^2(\mathbb{T}^d; L^2_\gamma))$ of the norm after testing with ϕ. Hence, Lemma 4.1 guarantees the existence of a solution $f \in H$ to (4-4), which is the distributional formulation of the penalized equation (4-5).

From the equation itself, we recover that $(v\cdot\nabla_x+\boldsymbol{b}\cdot\nabla_v)f\in L^2_\sigma(\mathbb{T}^d;\, H^{-1}_\gamma)$, and therefore, $f\in H^1_{\mathrm{hyp}}(\mathbb{T}^d)$ qualitatively. By the density of smooth functions in $H^1_{\mathrm{hyp}}(\mathbb{T}^d)$, this is enough regularity[9] to multiply (4-5) by f and integrate by parts to demonstrate the basic energy estimate:

$$\iint_{\mathbb{T}^d\times\mathbb{R}^d}|\nabla_v f|^2\,dm+\nu\iint_{\mathbb{T}^d\times\mathbb{R}^d}|f|^2\,dm\leqslant C\nu^{-1}\|g^*\|^2_{L^2_\sigma(\mathbb{T}^d;H^{-1}_\gamma)},\tag{4-7}$$

which guarantees that the solution is unique.[10] From the equation itself, we have

$$\begin{aligned}\|(v\cdot\nabla_x+\boldsymbol{b}\cdot\nabla_v)f\|_{L^2_\sigma(\mathbb{T}^d;H^{-1}_\gamma)}&\leqslant C\|A^*Af\|_{L^2_\sigma(\mathbb{T}^d;H^{-1}_\gamma)}+\|g^*\|_{L^2(\mathbb{T}^d;H^{-1}_\gamma)}+C\nu\|f\|_{L^2_\sigma(\mathbb{T}^d;H^{-1}_\gamma)}\\&\overset{(4\text{-}7)}{\leqslant} C\|g^*\|_{L^2_\sigma(\mathbb{T}^d;H^{-1}_\gamma)},\end{aligned}\tag{4-8}$$

where the constant C changes from line to line. Then (4-7), (4-8), and the hypoelliptic Poincaré inequality for mean-zero functions imply

$$\|f\|_{H^1_{\mathrm{hyp}}(\mathbb{T}^d)}\leqslant C\|g^*\|_{L^2_\sigma(\mathbb{T}^d;H^{-1}_\gamma)}.$$

Step 2: Next, we consider $\nu\to 0^+$. Let f^ν denote the unique solution of the penalized problem (4-5). Subtracting two solutions f^{ν_1} and f^{ν_2}, we have that the difference $\tilde f^{\nu_1,\nu_2}$ solves the equation

$$(v\cdot\nabla_x+\boldsymbol{b}\cdot\nabla_v)\tilde f^{\nu_1,\nu_2}+(\nu_1 f^{\nu_1}-\nu_2 f^{\nu_2})=(\Delta-v\cdot\nabla_v)\tilde f^{\nu_1,\nu_2}.\tag{4-9}$$

We may regard $\nu_1 f^{\nu_1}-\nu_2 f^{\nu_2}$ as a forcing term which is $O(\nu_1+\nu_2)$ in $L^2_\sigma(\mathbb{T}^d;\, H^{-1}_\gamma)$. By the hypoelliptic energy estimates for (4-9), we have

$$\|\tilde f^{\nu_1,\nu_2}\|_{H^1_{\mathrm{hyp}}(\mathbb{T}^d)}=O(\nu_1+\nu_2).$$

Choosing $\nu=2^{-k}$, the sequence (f_k) of solutions to (4-5) with penalization $\nu=2^{-k}$ is Cauchy in $H^1_{\mathrm{hyp}}(\mathbb{T}^d)$ and therefore converges to a solution f in $H^1_{\mathrm{hyp}}(\mathbb{T}^d)$ with $(f)_{\mathbb{T}^d}=0$. By passing to the distributional limit in each term in (4-5), we find that f solves (4-1) in the sense of distributions. □

Remark 4.2 (role of the penalization). The above proof requires a *coercive* bilinear form E which, in particular, controls the L^2 norm. The a priori estimates for solutions of (4-1) do indeed control the L^2 part of the norm through the hypoelliptic Poincaré inequality, but the control of $\|(v\cdot\nabla_x+\boldsymbol{b}\cdot\nabla_v)f\|_{L^2(\mathbb{T}^d;H^{-1}_\gamma)}$ is encoded by the PDE itself rather than the bilinear form E, which only encodes the energy estimate. This is why we include the penalization νf. In some sense, control of $\|(v\cdot\nabla_x+\boldsymbol{b}\cdot\nabla_v)f\|_{L^2(\mathbb{T}^d;H^{-1}_\gamma)}$ is concluded a posteriori.

In the time-dependent case, one can skip the penalization by instead considering the equation satisfied by $e^t f$; see Proposition 6.10.

Remark 4.3 (difficulty with boundary). Consider (4-1) in a bounded C^1 domain U with force f^* and zero Dirichlet condition on $\partial_{\mathrm{hyp}}U$. What goes wrong with the proof? One can demonstrate that there exists a solution $f^\nu\in H^1_{\mathrm{hyp}}(U)$ of the penalized equations which satisfies $f^\nu|_{\partial_{\mathrm{hyp}}U}=0$ *away from the singular set*. However, we do not know how to justify that $f^\nu\in H^1_{\mathrm{hyp},0}(U)$. That is, we cannot characterize $H^1_{\mathrm{hyp},0}(U)$

[9]To justify this, one may use the density of test functions demonstrated in Proposition 2.2.

[10]The estimate (4-7) can be made more convenient, without the factor ν^{-1}, if $\langle g^*\rangle_\gamma\equiv 0$.

as consisting of $H^1_{\mathrm{hyp}}(U)$ functions which vanish on $\partial_{\mathrm{hyp}} U$ away from the singular set. Consequently, we cannot justify the integration by parts that would generate the energy estimates that would imply uniqueness of f^ν and allow us to send $\nu \to 0^+$.

4B. ***The dual variational approach.*** Define

$$Bf := v \cdot \nabla_x f + \boldsymbol{b} \cdot \nabla_v f. \tag{4-10}$$

Consider the functional

$$\mathcal{J}[f, \boldsymbol{j}] = \iint_{\mathbb{T}^d \times \mathbb{R}^d} \tfrac{1}{2} |\nabla_v f - \boldsymbol{j}|^2 \, d\sigma(x) \, d\gamma(v) \tag{4-11}$$

evaluated at pairs $(f, \boldsymbol{j}) \in H^1_{\mathrm{hyp}}(\mathbb{T}^d) \times (L^2(\mathbb{T}^d; L^2_\gamma))^d$ satisfying

$$\nabla_v^* \boldsymbol{j} = g^* - Bf = g^* - (v \cdot \nabla_x f + \boldsymbol{b} \cdot \nabla_v f), \quad (f)_{\mathbb{T}^d} = 0. \tag{4-12}$$

In the remainder of this section, we *always* consider $f \in H^1_{\mathrm{hyp}}(\mathbb{T}^d)$ satisfying the second condition. We seek a null minimizer of $\mathcal{J}$ restricted to such pairs, which, if it exists, will satisfy the implication

$$\nabla_v f = \boldsymbol{j} \quad \Longrightarrow \quad \nabla_v^* \boldsymbol{j} = \nabla_v^* \nabla_v f = g^* - Bf,$$

which is precisely (4-1).

Proposition 4.4 (solvability of the Kramers equation). *Under Assumption 1.1 and the assumption that*

$$\iint_{\mathbb{T}^d \times \mathbb{R}^d} g^* \, d\gamma(v) \, d\sigma(x) = 0,$$

there exists a unique solution f to (4-1) *such that $(f)_{\mathbb{T}^d} = 0$, and f is given as the null minimizer of the functional $\mathcal{J}[f, \boldsymbol{j}]$ over pairs $(f, \boldsymbol{j})$ satisfying the constraint* (4-12).

Before proving Proposition 4.4, we argue that one may assume that $\langle g^* \rangle_\gamma = 0$ as a function of x. For this, we require:

Lemma 4.5. *Let $h \in L^2(\mathbb{T}^d)$ be given with $(h)_{\mathbb{T}^d} := \int_{\mathbb{T}^d} h(x) \, d\sigma(x) = 0$. Then there exists $g \in H^1_{\mathrm{hyp}}(\mathbb{T}^d)$ with $(g)_{\mathbb{T}^d} = 0$ such that*

$$\langle v \cdot \nabla_x g + \boldsymbol{b}(x) \cdot \nabla_v g \rangle_\gamma(x) = h(x), \quad \|g\|_{H^1_{\mathrm{hyp}}(\mathbb{T}^d)} \leqslant C \|h\|_{L^2(\mathbb{T}^d)}. \tag{4-13}$$

Suppose that we can solve (4-1) under the simplification $\langle g^* \rangle_\gamma = 0$. By Lemma 4.5 with $h = \langle g^* \rangle_\gamma$, we can find $g \in H^1_{\mathrm{hyp}}(\mathbb{T}^d)$ such that $\langle v \cdot \nabla_x g + \boldsymbol{b} \cdot \nabla_v g \rangle_\gamma = h$. Then, since $\langle -\Delta_v g + v \cdot \nabla_v g \rangle_\gamma = 0$, we can solve

$$-\Delta_v f + v \cdot \nabla_v f + v \cdot \nabla_x f + \boldsymbol{b} \cdot \nabla_v f = g^* - (-\Delta_v g + v \cdot \nabla_v g + v \cdot \nabla_x g + \boldsymbol{b} \cdot \nabla_v g),$$

so $f + g$ solves (4-1). We now show that such a g exists, and in the argument below we always work under the assumption that $\langle g^* \rangle_\gamma = 0$. We shall occasionally use the notation $g^* \in L^2(\mathbb{T}^d; \dot{H}^{-1}_\gamma)$ to signify that $\langle g^* \rangle_\gamma = 0$.

Proof of Lemma 4.5. Let $\boldsymbol{f} \in H^1(\mathbb{T}^d; \mathbb{R}^d)$ be a solution to the problem[11]

$$\nabla_x \cdot \boldsymbol{f}(x) + \boldsymbol{b}(x) \cdot \boldsymbol{f}(x) = h(x).$$

Let $\xi(s) : \mathbb{R} \to \mathbb{R}$ be a compactly supported, smooth, odd function of a single variable such that $\int_{\mathbb{R}} \xi(s) s \, ds \neq 0$. Define $\xi_i : \mathbb{R}^d \to \mathbb{R}$ by

$$\xi_i(v) = \xi(v_i) \prod_{i' \neq i} \xi'(v_{i'}),$$

so that ξ_i is odd in v_i and even in all other $v_{i'}$ for $i' \neq i$. Under an appropriate normalization, we find that

$$\int_{\mathbb{R}^d} \partial_{v_j} \xi_i(v) \, d\gamma(v) = \int_{\mathbb{R}^d} v_j \xi_i(v) \, d\gamma(v) = \delta_{ij},$$

since $v_j \xi_i(v) d\gamma(v)$ is odd in v_i unless $i = j$, in which case it is even in all components of v. Define

$$g(x, v) = \boldsymbol{f}_i(x) \xi_i(v),$$

where we have used the summation convention over repeated indices. By the smoothness of the ξ_i's and the $H^1(\mathbb{T}^d)$ regularity of $\boldsymbol{f}$, it is clear that $g \in H^1_{\mathrm{hyp}}(\mathbb{T}^d)$ with norm controlled by the sum of the respective H^1 norms of $\boldsymbol{f}$ and ξ. Furthermore, $(g)_{\mathbb{T}^d} = 0$ since, for $1 \leqslant i \leqslant d$, ξ_i is odd in v_i. Now we may compute

$$\begin{aligned} \langle Bg \rangle_\gamma(x) &= \int_{\mathbb{R}^d} (v_j \partial_{x_j} g(x, v) + \boldsymbol{b}_j(x) \partial_{v_j} g(x, v)) \, d\gamma(v) \\ &= \int_{\mathbb{R}^d} (v_j \partial_{x_j} \boldsymbol{f}_i(x) \xi_i(v) + \boldsymbol{b}_j \boldsymbol{f}_i(x) \partial_{v_j} \xi_i(v)) \, d\gamma(v) \\ &= \partial_i \boldsymbol{f}_i(x) + \boldsymbol{b}_i(x) \boldsymbol{f}_i(x) = h(x). \end{aligned}$$

□

Proof of Proposition 4.4. We split the argument into five steps.

<u>Step 1</u>: In this step, we show that the functional $\mathcal{J}$ is not uniformly equal to $+\infty$ and is uniformly convex on pairs $(f, \boldsymbol{j})$ satisfying the constraint (4-12). Let us denote the set of pairs satisfying the constraint by

$$\mathcal{A}(g^*) := \{(f, \boldsymbol{j}) \in H^1_{\mathrm{hyp}}(\mathbb{T}^d) \times (L^2(\mathbb{T}^d; L^2_\gamma))^d : \nabla_v^* \boldsymbol{j} = g^* - Bf, \ (f)_{\mathbb{T}^d} = 0\}.$$

First, since $g^* \in L^2(\mathbb{T}^d; \dot{H}^{-1}_\gamma)$, there exists $\boldsymbol{j} \in L^2(\mathbb{T}^d; L^2_\gamma)$ such that $g^* = A^* \boldsymbol{j}$. The pair $(0, \boldsymbol{j})$ belongs to $\mathcal{A}(g^*)$, and $\mathcal{J}(0, \boldsymbol{j}) < +\infty$.

We now demonstrate uniform convexity. Since, for every $(f', \boldsymbol{j}') \in \mathcal{A}(g^*)$ and $(f, \boldsymbol{j}) \in \mathcal{A}(0)$,

$$\tfrac{1}{2}\mathcal{J}[f' + f, \boldsymbol{j}' + \boldsymbol{j}] + \tfrac{1}{2}\mathcal{J}[f' - f, \boldsymbol{j}' - \boldsymbol{j}] - \mathcal{J}[f', \boldsymbol{j}'] = \mathcal{J}[f, \boldsymbol{j}], \tag{4-14}$$

it suffices to show that there exists $C(d) < \infty$ such that, for every $(f, \boldsymbol{j}) \in \mathcal{A}(0)$,

$$\mathcal{J}[f, \boldsymbol{j}] \geqslant C^{-1}(\|f\|^2_{H^1_{\mathrm{hyp}}(\mathbb{T}^d)} + \|\boldsymbol{j}\|^2_{L^2(\mathbb{T}^d; L^2_\gamma)}). \tag{4-15}$$

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[11]For example, one could argue as in the proof of Lemma 3.4 to produce $\boldsymbol{f}$ via the Lax–Milgram theorem satisfying the bound $\|\boldsymbol{f}\|_{H^1(\mathbb{T}^d)} \leqslant C\|h\|_{L^2(\mathbb{T}^d)}$.

Expanding the square and using that $\nabla_v^* \boldsymbol{j} = -Bf$, we find

$$\mathcal{J}[f, \boldsymbol{j}] = \iint_{\mathbb{T}^d \times \mathbb{R}^d} \left(\tfrac{1}{2}|\nabla_v f|^2 + \tfrac{1}{2}|\boldsymbol{j}|^2 + fBf\right) dm.$$

Moreover, by (1-8), the term $\iint_{\mathbb{T}^d \times \mathbb{R}^d} fBf\, dm$ vanishes. Finally, from $-Bf = \nabla_v^* \boldsymbol{j}$, we have $\langle Bf \rangle_\gamma = 0$, and thus

$$\begin{aligned} \|v \cdot \nabla_x f\|_{L^2(\mathbb{T}^d; H_\gamma^{-1})} &\leqslant \|Bf\|_{L^2(\mathbb{T}^d; H_\gamma^{-1})} + \|\boldsymbol{b}(x) \cdot \nabla_v f\|_{L^2(\mathbb{T}^d; H_\gamma^{-1})} \\ &\leqslant C\|\boldsymbol{j}\|_{L^2(\mathbb{T}^d; L^2_\gamma)} + C\|\nabla_v f\|_{L^2(\mathbb{T}^d; L^2_\gamma)}. \end{aligned}$$

Combining the last displays and Theorem 1.3 yields (4-15), and thus also the uniform convexity of the functional in (4-11).

Step 2: In this step, we rephrase the problem in terms of a perturbed convex minimization problem. Denote by $(f_1, \boldsymbol{j}_1)$ the unique minimizing pair of the functional $\mathcal{J}$ over $\mathcal{A}(g^*)$. We obviously have

$$\mathcal{J}[f_1, \boldsymbol{j}_1] \geqslant 0.$$

We now show that there is a one-to-one correspondence between solutions f of the Kramers equation and null minimizers $(f, \boldsymbol{j})$ of $\mathcal{J}$ satisfying the constraint (4-12): for every $f \in H^1_{\mathrm{hyp}}(\mathbb{T}^d)$ with $(f)_{\mathbb{T}^d} = 0$, we have

$$f \text{ solves (4-1)} \quad \Longleftrightarrow \quad \mathcal{J}[f, \boldsymbol{j}_1] = 0.$$

Indeed, the implication $\Longrightarrow$ is clear, since if f solves (4-1), then

$$(f, \nabla_v f) \in \mathcal{A}(g^*) \quad \text{and} \quad \mathcal{J}[f, \nabla_v f] = 0.$$

Conversely, if $\mathcal{J}[f_1, \boldsymbol{j}_1] = 0$, then by convexity we have $f = f_1$ (assuming the mean-zero constraint from (4-12)), and

$$\nabla_v f_1 = \boldsymbol{j}_1 \quad \text{a.e. in } \mathbb{T}^d \times \mathbb{R}^d.$$

Then since $\nabla_v^* \boldsymbol{j}_1 = g^* - Bf_1$, we recover that $f = f_1$ is indeed a solution of (4-1). In particular, the fact that there is at most one solution to (4-1) is clear.

To complete the proof, it thus remains to show that given the unique minimizing pair $(f_1, \boldsymbol{j}_1)$, we have

$$\mathcal{J}[f_1, \boldsymbol{j}_1] \leqslant 0. \tag{4-16}$$

We phrase this as a perturbed convex minimization problem for the functional G, which is defined for every $f^* \in L^2(\mathbb{T}^d; H_\gamma^{-1})$ with $(f^*)_{\mathbb{T}^d} = 0$ by

$$G(f^*) := \inf_{\substack{f \in H^1_{\mathrm{hyp}}(\mathbb{T}^d) \\ (f)_{\mathbb{T}^d} = 0}} \left(\iint_{\mathbb{T}^d \times \mathbb{R}^d} ff^*\, dm + \inf_{\substack{\boldsymbol{j} \in L^2(\mathbb{T}^d) \\ (f, \boldsymbol{j}) \in \mathcal{A}(f^* + g^*)}} \mathcal{J}[f, \boldsymbol{j}] \right).$$

To complete the proof, we must show that $G(0) \leqslant 0$. We decompose the argument into the next three steps.

Step 3: In this step, we show that G is convex and reduce the problem to showing that the convex dual of G is nonnegative. For every pair $(f, \boldsymbol{j})$ satisfying $(f, \boldsymbol{j}) \in \mathcal{A}(f^* + g^*)$, we have

$$\nabla_v^* \boldsymbol{j} = f^* + g^* - Bf, \quad (f)_{\mathbb{T}^d} = 0, \tag{4-17}$$

and so utilizing (1-8) we find that

$$\begin{aligned}\iint_{\mathbb{T}^d\times\mathbb{R}^d} ff^*\,dm+\mathcal{J}[f,\boldsymbol{j}] &= \iint_{\mathbb{T}^d\times\mathbb{R}^d} ff^*\,dm+\iint_{\mathbb{T}^d\times\mathbb{R}^d} \tfrac{1}{2}|\nabla_v f-\boldsymbol{j}|^2\,dm\\ &= \iint_{\mathbb{T}^d\times\mathbb{R}^d} ff^*\,dm+\iint_{\mathbb{T}^d\times\mathbb{R}^d} \tfrac{1}{2}|\nabla_v f|^2+\tfrac{1}{2}|\boldsymbol{j}|^2-f\nabla_v^*\boldsymbol{j}\,dm\\ &= \iint_{\mathbb{T}^d\times\mathbb{R}^d} ff^*\,dm+\iint_{\mathbb{T}^d\times\mathbb{R}^d} \tfrac{1}{2}|\nabla_v f|^2+\tfrac{1}{2}|\boldsymbol{j}|^2-f(f^*+g^*-Bf)\,dm\\ &= \iint_{\mathbb{T}^d\times\mathbb{R}^d} \tfrac{1}{2}|\nabla_v f|^2+\tfrac{1}{2}|\boldsymbol{j}|^2-g^*f\,dm.\end{aligned}$$

Taking the infimum over all $(f,\boldsymbol{j})$ satisfying the affine constraint $(f,\boldsymbol{j})\in\mathcal{A}(f^*+g^*)$, we obtain the quantity $G(f^*)$. We thus infer that G is convex in the variable f^*. By Lemma 4.5, given $f^*\in L^2(\mathbb{T}^d;H_\gamma^{-1})$ with vanishing mean, we may find $f_0\in H^1_{\mathrm{hyp}}(\mathbb{T}^d)$ such that $\langle Bf_0\rangle_\gamma=\langle f^*+g^*\rangle_\gamma=\langle f^*\rangle_\gamma$. Then since $\langle f^*+g^*-Bf_0\rangle_\gamma=0$, we may find $\boldsymbol{j}\in(L^2(\mathbb{T}^d;L^2_\gamma))^d$ such that $\nabla_v^*\boldsymbol{j}=f^*+g^*-Bf_0$, and we see that the function G is also locally bounded above. These two properties imply that G is lower semicontinuous; see [Ekeland and Temam 1976, Lemma I.2.1 and Corollary I.2.2]. We denote by G^* the convex dual of G, defined for every $h\in L^2(\mathbb{T}^d;H^1_\gamma)$ with $(h)_{\mathbb{T}^d}=0$ by

$$G^*(h):=\sup_{\substack{f^*\in L^2(\mathbb{T}^d;H_\gamma^{-1})\\ (f^*)_{\mathbb{T}^d}=0}}\left(-G(f^*)+\iint_{\mathbb{T}^d\times\mathbb{R}^d} hf^*\,dm\right),$$

and by G^{**} the bidual of G. Since G is lower semicontinuous, we have $G^{**}=G$ (see [Ekeland and Temam 1976, Proposition I.4.1]), and, in particular,

$$G(0)=G^{**}(0)=\sup_{\substack{h\in L^2(\mathbb{T}^d;H^1_\gamma)\\ (h)_{\mathbb{T}^d}=0}}(-G^*(h)).$$

In order to prove that $G(0)\leqslant 0$, it therefore suffices to show that,

$$\text{for all } h\in L^2(\mathbb{T}^d;H^1_\gamma) \text{ with } (h)_{\mathbb{T}^d}=0,\quad G^*(h)\geqslant 0. \tag{4-18}$$

Step 4: In this step we show that

$$G^*(h)<+\infty \quad\Longrightarrow\quad h\in H^1_{\mathrm{hyp}}(\mathbb{T}^d). \tag{4-19}$$

We rewrite $G^*(h)$ in the form

$$G^*(h)=\sup\left\{\iint_{\mathbb{T}^d\times\mathbb{R}^d}\left(-\tfrac{1}{2}|\nabla_v f-\boldsymbol{j}|^2-ff^*+hf^*\right)dm\right\}, \tag{4-20}$$

where the supremum is over every $f\in H^1_{\mathrm{hyp}}(\mathbb{T}^d)$, $\boldsymbol{j}\in L^2(\mathbb{T}^d;L^2_\gamma)^d$ and $f^*\in L^2(\mathbb{T}^d;H_\gamma^{-1})$ satisfying the constraint (4-17). Given f with $(f)_{\mathbb{T}^d}=0$, we choose to restrict the supremum above to $f^*:=Bf$ and $\boldsymbol{j}=\boldsymbol{j}_0$ the solution of $\nabla_v^*\boldsymbol{j}_0=g^*$. Recall that such a $\boldsymbol{j}_0\in L^2(\mathbb{T}^d;L^2_\gamma)^d$ exists since $\langle g^*\rangle_\gamma=0$. With such choices of f^* and $\boldsymbol{j}$, the constraint (4-17) is satisfied, and we obtain

$$G^*(h) \geqslant \sup\left\{ \iint_{\mathbb{T}^d\times\mathbb{R}^d} \big(-\tfrac{1}{2}|\nabla_v f - \boldsymbol{j}_0|^2 - fBf + hBf\big)\, dm : f \in H^1_{\mathrm{hyp}}(\mathbb{T}^d),\ (f)_{\mathbb{T}^d} = 0 \right\}.$$

Recalling that $\iint fBf\, dm = 0$, and using that $C_0^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ is dense in $H^1_{\mathrm{hyp}}(\mathbb{T}^d)$, we deduce

$$G^*(h) \geqslant \sup\left\{ \iint_{\mathbb{T}^d\times\mathbb{R}^d} \big(-\tfrac{1}{2}|\nabla_v f - \boldsymbol{j}_0|^2 + hBf\big)\, dm : f \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}^d),\ (f)_{\mathbb{T}^d} = 0 \right\}.$$

Then the assumption of $G^*(h) < \infty$ implies

$$\sup\left\{ \iint_{\mathbb{T}^d\times\mathbb{R}^d} hBf\, dm : f \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}^d),\ (f)_{\mathbb{T}^d} = 0,\ \|f\|_{L^2(\mathbb{T}^d; H^1_\gamma)} \leqslant 1 \right\} < \infty.$$

This then shows that the distribution Bh belongs to the dual of $L^2(\mathbb{T}^d; H^1_\gamma)$, which is $L^2(\mathbb{T}^d; H^{-1}_\gamma)$. Since

$$v \cdot \nabla_x h = Bh - \boldsymbol{b} \cdot \nabla_v h,$$

the proof of (4-19) is complete.

Step 5: In place of (4-18), we have left to show that,

$$\text{for all } h \in H^1_{\mathrm{hyp}}(\mathbb{T}^d) \text{ with } (h)_{\mathbb{T}^d} = 0, \quad G^*(h) \geqslant 0. \tag{4-21}$$

Since $Bf \in L^2(\mathbb{T}^d; H^{-1}_\gamma)$, we may replace f^* by $f^* + Bf$ in the variational formula (4-20) for G^* to get

$$G^*(h) = \sup\left\{ \iint_{\mathbb{T}^d\times\mathbb{R}^d} \big(-\tfrac{1}{2}|\nabla_v f - \boldsymbol{j}|^2 + (h - f)(f^* + Bf)\big)\, dm \right\}, \tag{4-22}$$

where the supremum is now over every $f \in H^1_{\mathrm{hyp}}(\mathbb{T}^d)$, $\boldsymbol{j} \in L^2(\mathbb{T}^d; L^2_\gamma)^d$ and $f^* \in L^2(\mathbb{T}^d; H^{-1}_\gamma)$ satisfying the constraint

$$\nabla_v^* \boldsymbol{j} = f^* + g^*, \quad (f)_{\mathbb{T}^d} = 0. \tag{4-23}$$

Setting $f = h$ in (4-22), we find that

$$G^*(h) \geqslant \sup\left\{ \iint_{\mathbb{T}^d\times\mathbb{R}^d} -\tfrac{1}{2}|\nabla_v h - \boldsymbol{j}|^2\, dm \right\},$$

with the supremum ranging over all $f^* \in L^2(\mathbb{T}^d; H^{-1}_\gamma)$ and $\boldsymbol{j} \in L^2(\mathbb{T}^d; L^2_\gamma)^d$ satisfying the constraint (4-23). We now simply select $\boldsymbol{j} = \nabla_v h \in L^2(\mathbb{T}^d; L^2_\gamma)^d$ and

$$f^* = \nabla_v^* \boldsymbol{j} - g^* \in L^2(\mathbb{T}^d; H^{-1}_\gamma),$$

at which point we conclude that $G^*(h) \geqslant 0$. □

5. Interior regularity of solutions

In this subsection, we use energy methods to obtain interior regularity estimates for solutions of the equation

$$-\Delta_v f + v \cdot \nabla_v f + v \cdot \nabla_x f + \boldsymbol{b} \cdot \nabla_v f + cf = f^*. \tag{5-1}$$

In analogy to the classical theory for uniformly elliptic equations (such as the Laplace or Poisson equations), we obtain an appropriate version of the Caccioppoli inequality, apply it iteratively to obtain H^1_{hyp} estimates on all spatial derivatives of the solution, and then apply the Hörmander and Sobolev inequalities to obtain pointwise estimates. In particular, we obtain higher regularity estimates — strong enough to imply that our weak solutions are C^∞ — without resorting to sophisticated theory for pseudodifferential operators.

We begin with a version of the Caccioppoli inequality for (5-1).

Lemma 5.1 (Caccioppoli inequality). *Suppose* $r > 0$, $\boldsymbol{b} \in L^\infty(B_r; L^\infty(\mathbb{R}^d; \mathbb{R}^d))$, $c \in L^\infty(B_r; L^\infty(\mathbb{R}^d))$, *and the pair* $(f, f^*) \in L^2(B_r; H^1_\gamma) \times L^2(B_r; H^{-1}_\gamma)$ *satisfies the equation*

$$-\Delta_v f + v \cdot \nabla_v f + v \cdot \nabla_x f + \boldsymbol{b} \cdot \nabla_v f + cf = f^* \quad \text{in } B_r \times \mathbb{R}^d. \tag{5-2}$$

Then $f \in H^1_{\mathrm{hyp}}(B_r)$, *and there exists* $C(d, r, \|\boldsymbol{b}\|_{L^\infty(B_r;L^\infty(\mathbb{R}^d))}, \|c|_{L^\infty(B_r;L^\infty(\mathbb{R}^d))}) < \infty$ *such that*

$$\|\nabla_v f\|_{L^2(B_{r/2};L^2_\gamma)} + \|v \cdot \nabla_x f\|_{L^2(B_{r/2};H^{-1}_\gamma)} \leqslant C\|f\|_{L^2(B_r;L^2_\gamma)} + C\|f^*\|_{L^2(B_r;H^{-1}_\gamma)}. \tag{5-3}$$

Proof. The PDE (5-2) guarantees that $f \in L^2(B_r; H^1_\gamma)$ belongs qualitatively to $H^1_{\mathrm{hyp}}(B_r)$.

Step 1: We show that there exists $C(d) < \infty$ such that

$$\|\nabla_v f\|_{L^2(B_{r/2};L^2_\gamma)} \leqslant C\Big(\frac{1}{r} + \|\boldsymbol{b}\|_{L^\infty(B_r\times\mathbb{R}^d)} + \|c\|^{1/2}_{L^\infty(B_r\times\mathbb{R}^d)}\Big)\|f\|_{L^2(B_r;L^2_\gamma)} + C(1+r)\|f^*\|_{L^2(B_r;H^{-1}_\gamma)}. \tag{5-4}$$

Select a smooth cutoff function $\phi \in C^\infty_c(B_r)$ which is compactly supported in B_r and satisfies $0 \leqslant \phi \leqslant 1$ in B_r, $\phi \equiv 1$ on $B_{r/2}$ and $\|\nabla\phi\|_{L^\infty(B_r)} \leqslant 8r^{-1}$. Testing (5-2) with $(x, v) \mapsto \phi^2(x) f(x, v)$ yields

$$\begin{aligned}\int_{B_r\times\mathbb{R}^d} \phi^2 |\nabla_v f|^2 \, dx\, d\gamma = \int_{B_r\times\mathbb{R}^d} \phi^2 f\, f^* \, dx\, d\gamma - \int_{B_r\times\mathbb{R}^d} \phi^2 f v \cdot \nabla_x f \, dx\, d\gamma \\ - \int_{B_r\times\mathbb{R}^d} \phi^2 f \boldsymbol{b} \cdot \nabla_v f \, dx\, d\gamma - \int_{B_r\times\mathbb{R}^d} \phi^2 c f^2 \, dx\, d\gamma.\end{aligned} \tag{5-5}$$

We estimate each of the terms on the right-hand side of (5-5) separately.

For the first term on the right side of (5-5), we use

$$\begin{aligned}\left|\int_{B_r\times\mathbb{R}^d} \phi^2 f\, f^* \, dx\, d\gamma\right| &\leqslant \|\phi^2 f\|_{L^2(B_r;H^1_\gamma)} \|f^*\|_{L^2(B_r;H^{-1}_\gamma)} \\ &\leqslant \big(\|\phi^2 \nabla_v f\|_{L^2(B_r;L^2_\gamma)} + \|f\|_{L^2(B_r;L^2_\gamma)}\big)\|f^*\|_{L^2(B_r;H^{-1}_\gamma)}\end{aligned} \tag{5-6}$$

and then apply Young's inequality to obtain

$$\begin{aligned}\left|\int_{B_r\times\mathbb{R}^d} \phi^2 f\, f^* \, dx\, d\gamma\right| \leqslant \frac{1}{6}\int_{B_r\times\mathbb{R}^d} \phi^2 |\nabla_v f|^2 \, dx\, d\gamma \\ + \frac{C}{r^2}\int_{B_r\times\mathbb{R}^d} f^2 \, dx\, d\gamma + C(1+r^2)\|f^*\|^2_{L^2(B_r;H^{-1}_\gamma)}.\end{aligned} \tag{5-7}$$

For the second term on the right side of (5-5), we integrate by parts to find

$$\begin{aligned}-\int_{B_r\times\mathbb{R}^d}\phi^2 f v\cdot\nabla_x f\,dx\,d\gamma &= -\int_{B_r\times\mathbb{R}^d}\phi^2 v\cdot\nabla_x\big(\tfrac12 f^2\big)\,dx\,d\gamma\\ &= \int_{B_r\times\mathbb{R}^d}\phi\nabla_x\phi\cdot vf^2\,dx\,d\gamma\\ &= \int_{B_r\times\mathbb{R}^d}\phi(x)\nabla_x\phi(x)\cdot v\exp\big(-\tfrac12|v|^2\big)f^2(x,v)\,dx\,dv\\ &= -\int_{B_r\times\mathbb{R}^d}2f\phi\nabla_x\phi\cdot\nabla_v f\,dx\,d\gamma.\end{aligned}$$

Thus, by Young's inequality,

$$\begin{aligned}\left|\int_{B_r\times\mathbb{R}^d}\phi^2 f v\cdot\nabla_x f\,dx\,d\gamma\right| &\leqslant \frac16\int_{B_r\times\mathbb{R}^d}\phi^2|\nabla_v f|^2\,dx\,d\gamma + C\int_{B_r\times\mathbb{R}^d}f^2|\nabla_x\phi|^2\,dx\,d\gamma\\ &\leqslant \frac16\int_{B_r\times\mathbb{R}^d}\phi^2|\nabla_v f|^2\,dx\,d\gamma + \frac{C}{r^2}\int_{B_r\times\mathbb{R}^d}f^2\,dx\,d\gamma.\end{aligned}\tag{5-8}$$

For the third term on the right side of (5-5), we use Young's inequality to obtain

$$\begin{aligned}\left|\int_{B_r\times\mathbb{R}^d}\phi^2 f\boldsymbol{b}\cdot\nabla_v f\right|dx\,d\gamma &\leqslant \frac16\int_{B_r\times\mathbb{R}^d}\phi^2|\nabla_v f|^2\,dx\,d\gamma + C\int_{B_r\times\mathbb{R}^d}\phi^2 f^2|\boldsymbol{b}|^2\,dx\,d\gamma\\ &\leqslant \frac16\int_{B_r\times\mathbb{R}^d}\phi^2|\nabla_v f|^2\,dx\,d\gamma + C\|\boldsymbol{b}\|^2_{L^\infty(B_r\times\mathbb{R}^d)}\int_{B_r\times\mathbb{R}^d}f^2\,dx\,d\gamma.\end{aligned}\tag{5-9}$$

To conclude, we combine (5-5)–(5-9) and the obvious estimate on the final term to obtain

$$\begin{aligned}\int_{B_r\times\mathbb{R}^d}\phi^2|\nabla_v f|^2\,dx\,d\gamma \leqslant \frac23\int_{B_r\times\mathbb{R}^d}\phi^2|\nabla_v f|^2\,dx\,d\gamma + \frac{C}{r^2}\int_{B_r\times\mathbb{R}^d}f^2\,dx\,d\gamma + C(1+r^2)\|f^*\|^2_{L^2(B_r;H^{-1}_\gamma)}\\ + C(\|\boldsymbol{b}\|^2_{L^\infty(B_r\times\mathbb{R}^d)} + \|c\|_{L^\infty(B_r\times\mathbb{R}^d)})\int_{B_r\times\mathbb{R}^d}f^2\,dx\,d\gamma.\end{aligned}$$

The first term on the right may now be reabsorbed on the left. Using that $\phi = 1$ on $B_{r/2}$, we thus obtain (5-4). The analysis in Step 1 is enough to conclude that $f\in H^1_{\mathrm{hyp}}(B_{r/2})$ and the gradient bound in (5-3).

Step 2: We show that there exists $C(d) < \infty$ such that

$$\begin{aligned}\|v\cdot\nabla_x f\|_{L^2(B_{r/2};H^{-1}_\gamma)} \leqslant C(1+\|\boldsymbol{b}\|_{L^\infty(B_{r/2}\times\mathbb{R}^d)})\|\nabla_v f\|_{L^2(B_{r/2};L^2_\gamma)}\\ + C\|c\|_{L^\infty(B_{r/2}\times\mathbb{R}^d)}\|f\|_{L^2(B_{r/2};L^2_\gamma)} + C\|f^*\|_{L^2(B_{r/2};H^{-1}_\gamma)}.\end{aligned}\tag{5-10}$$

This estimate may be combined with (5-4) to obtain the bound for the second term in (5-3), which completes the proof of the lemma.

To obtain (5-10), we test (5-2) with $w\in L^2(B_{r/2};H^1_\gamma)$ to find that

$$\int_{B_r\times\mathbb{R}^d}w\,(v\cdot\nabla_x f)\,dx\,d\gamma = -\int_{B_r\times\mathbb{R}^d}\nabla_v f\cdot(\nabla_v w + w\boldsymbol{b})\,dx\,d\gamma + \int_{B_r\times\mathbb{R}^d}wf^*\,dx\,d\gamma - \int_{B_r\times\mathbb{R}^d}cwf\,dx\,d\gamma.$$

We deduce that

$$\left|\int_{B_r\times\mathbb{R}^d} w\,(v\cdot\nabla_x f)\,dx\,d\gamma\right| \leqslant \|\nabla_v f\|_{L^2(B_{r/2};L^2_\gamma)}(\|\nabla_v w\|_{L^2(B_{r/2};L^2_\gamma)}+\|\boldsymbol{b}\|_{L^\infty(B_{r/2}\times\mathbb{R}^d)}\|w\|_{L^2(B_{r/2};L^2_\gamma)}) \\ +\|w\|_{L^2(B_{r/2};H^1_\gamma)}\|f^*\|_{L^2(B_{r/2};H^{-1}_\gamma)}+\|c\|_{L^\infty(B_r\times\mathbb{R}^d)}\|f\|_{L^2(B_r;L^2_\gamma)}\|w\|_{L^2(B_r;L^2_\gamma)}.$$

Taking the supremum over $w\in L^2(B_{r/2};H^1_\gamma)$ with $\|w\|_{L^2(B_{r/2};H^1_\gamma)}\leqslant 1$ yields (5-10).

The combination of (5-4) and (5-10) yields (5-3). □

In the next lemma, under appropriate regularity conditions on the coefficients, we differentiate (5-1) with respect to x_i to obtain an equation for $\partial_{x_i} f$, and then apply the previous lemma to obtain an interior H^1_{hyp} estimate for $\partial_{x_i} f$. We need to essentially differentiate the equation a fractional number of times (see [Mingione 2007; 2011]).

Lemma 5.2 (differentiating in x). *Fix $r\in(0,\infty)$ and coefficients $\boldsymbol{b}\in C^{0,1}(B_r\times\mathbb{R}^d;\mathbb{R}^d)$, $c\in C^{0,1}(B_r\times\mathbb{R}^d;\mathbb{R})$. Suppose that $f^*\in H^1(B_r;H^{-1}_\gamma)$ and $f\in H^1_{\mathrm{hyp}}(B_r)$ satisfy*

$$-\Delta_v f+v\cdot\nabla_v f+v\cdot\nabla_x f+\boldsymbol{b}\cdot\nabla_v f+cf=f^* \quad \text{in } B_r\times\mathbb{R}^d. \tag{5-11}$$

Then, for each $i\in\{1,\dots,d\}$, the function $h:=\partial_{x_i} f$ belongs to $H^1_{\mathrm{hyp}}(B_{r'})$ for all $r'\in(0,r)$ and satisfies

$$-\Delta_v h+v\cdot\nabla_v h+v\cdot\nabla_x h+\boldsymbol{b}\cdot\nabla_v h+ch=\partial_{x_i} f^*-\partial_{x_i}\boldsymbol{b}\cdot\nabla_v f-\partial_{x_i} c\,f \quad \text{in } B_{r'}\times\mathbb{R}^d. \tag{5-12}$$

Moreover, there exists $C(d,r,\|\boldsymbol{b}\|_{C^{0,1}(B_r\times\mathbb{R}^d)},\|c\|_{C^{0,1}(B_r\times\mathbb{R}^d)})<\infty$ such that

$$\|\partial_{x_i} f\|_{H^1_{\mathrm{hyp}}(B_{r/2})}\leqslant C\|f\|_{L^2(B_r;L^2_\gamma)}+C\|f^*\|_{H^1(B_r;H^{-1}_\gamma)}. \tag{5-13}$$

Proof. The argument is by induction on the fractional exponent of differentiability of f in the spatial variable x. Essentially, we want to differentiate the equation a fractional amount (almost $\frac{1}{3}$ times), apply the Caccioppoli inequality to the fractional derivative, and then iterate until we have one full spatial derivative.

Step 1: We first prove that, for every $(f,f^*)\in H^1_{\mathrm{hyp}}(B_r)\times H^1(B_r,H^{-1}_\gamma)$ satisfying (5-11), there exists $C(d,r,\|\boldsymbol{b}\|_{C^{0,1}(B_r\times\mathbb{R}^d)},\|c\|_{C^{0,1}(B_r\times\mathbb{R}^d)})<\infty$ such that f belongs to $H^1(B_{r/2};H^1_\gamma)$ and satisfies the estimate

$$\|\nabla_x f\|_{L^2(B_{r/2};H^1_\gamma)}\leqslant C\|f\|_{L^2(B_r;L^2_\gamma)}+C\|f^*\|_{H^1(B_r;H^{-1}_\gamma)}. \tag{5-14}$$

Suppose that $\alpha_0\in[0,1)$ is such that the following statement is valid: For every $\alpha\in[0,\alpha_0]$, $r>0$, and pair $(f,f^*)\in H^1_{\mathrm{hyp}}(B_r)\times H^\alpha(B_r,H^{-1}_\gamma)$ satisfying (5-11), we have $f\in H^\alpha(B_{r/2};H^1_\gamma)$ and, for $C(d,r,\|\boldsymbol{b}\|_{C^{0,1}(B_r\times\mathbb{R}^d)},\|c\|_{C^{0,1}(B_r\times\mathbb{R}^d)},\alpha)<\infty$, the estimate

$$\|f\|_{H^\alpha(B_{r/2};H^1_\gamma)}\leqslant C\|f\|_{L^2(B_r;L^2_\gamma)}+C\|f^*\|_{H^\alpha(B_r;H^{-1}_\gamma)}. \tag{5-15}$$

We argue that the statement is also valid for $\min\bigl(\alpha_0+\frac{1}{3}-\delta,1\bigr)$ in place of α_0 for all $\delta\in\bigl(0,\frac{1}{3}\bigr)$. Note that this statement is clearly valid for $\alpha_0=0$ by the Caccioppoli inequality (Lemma 5.1).

Fix $\alpha\in[0,\alpha_0]$ and a pair

$$(f,f^*)\in H^1_{\mathrm{hyp}}(B_r)\times H^\alpha(B_r,H^{-1}_\gamma)$$

satisfying (5-11), an index $i \in \{1, \dots, d\}$, and a cutoff function $\phi \in C_c^\infty(B_{r/2})$ with $0 \leqslant \phi \leqslant 1$ and $\phi \equiv 1$ on $B_{r/4}$. Define the functions

$$\tilde{f} := \phi^2 f,$$
$$\tilde{f}^* := \phi^2 f^* + 2 f \phi \, v \cdot \nabla_x \phi.$$

Observe that $\tilde{f} \in H^1_{\mathrm{hyp}}(\mathbb{R}^d)$ and $\tilde{f}^* \in H^\alpha(\mathbb{R}^d; H^{-1}_\gamma)$ are compactly supported in B_r and satisfy

$$\|\tilde{f}\|_{H^\alpha(\mathbb{R}^d; L^2_\gamma)} \leqslant C \|f\|_{H^\alpha(B_r; L^2_\gamma)},$$
$$\|\tilde{f}^*\|_{H^\alpha(\mathbb{R}^d; H^{-1}_\gamma)} \leqslant C(\|f^*\|_{H^\alpha(B_r; H^{-1}_\gamma)} + \|f\|_{H^\alpha(B_r; L^2_\gamma)}),$$

and the PDE (5-1) in $\mathbb{R}^d \times \mathbb{R}^d$.

Next, we mollify. This step ensures that the function qualitatively belongs to good enough spaces to justify the computations (the analogous step in Nirenberg's method is finite differences). Define

$$\bar{f} = \tilde{f} *_x \psi^\varepsilon,$$
$$\bar{f}^* = \tilde{f}^* *_x \psi^\varepsilon - [\psi^\varepsilon *_x, \boldsymbol{b} \cdot] \nabla_v \tilde{f} - [\psi^\varepsilon *_x, c] \tilde{f},$$

where ψ^ε is an appropriate mollification at scale ε. Then $(\bar{f}, \bar{f}^*)$ satisfies the PDE (5-1) in $\mathbb{R}^d \times \mathbb{R}^d$. We have

$$\|(1-\Delta_x)^{\alpha/2} \bar{f}\|_{L^2(\mathbb{R}^d; L^2_\gamma)} \leqslant C \|\tilde{f}\|_{H^\alpha(\mathbb{R}^d; L^2_\gamma)}, \tag{5-16}$$
$$\|(1-\Delta_x)^{\alpha/2} \bar{f}^*\|_{L^2(\mathbb{R}^d; H^{-1}_\gamma)} \leqslant C \|\tilde{f}^*\|_{H^\alpha(\mathbb{R}^d; H^{-1}_\gamma)} + C \|\tilde{f}\|_{H^\alpha(\mathbb{R}^d; L^2_\gamma)}, \tag{5-17}$$

since $[\psi^\varepsilon *_x, \boldsymbol{b} \cdot]$ and $[\psi^\varepsilon *_x, c]$ are $H^\alpha(\mathbb{R}^d; H^{-1}_\gamma)$-bounded for all $\alpha \in [0, 1]$, while $\boldsymbol{b}$ and c are Lipschitz. We apply $(1-\Delta_x)^{\alpha/2}$ to the PDE (5-1) satisfied by $(\bar{f}, \bar{f}^*)$ and define $f_\alpha = (1-\Delta_x)^{\alpha/2} \bar{f}$. We have that f_α satisfies the equation

$$-\Delta_v f_\alpha + v \cdot \nabla_v f_\alpha + v \cdot \nabla_x f_\alpha + \boldsymbol{b} \cdot \nabla_v f_\alpha + c f_\alpha = (1-\Delta)^{\alpha/2} \bar{f}^* - [(1-\Delta)^{\alpha/2}, \boldsymbol{b} \cdot] \nabla_v \bar{f} - [(1-\Delta)^{\alpha/2}, c] \bar{f}$$

in $\mathbb{R}^d \times \mathbb{R}^d$. The Cacciopoli inequality for $f_\alpha \in L^2(\mathbb{R}^d; H^1_\gamma)$, the Hörmander inequality, and (5-16)–(5-17) give

$$\|f_\alpha\|_{H^{1/3-\delta}(\mathbb{R}^d; L^2_\gamma)} + \|f_\alpha\|_{L^2(\mathbb{R}^d; H^1_\gamma)} \leqslant C \|\tilde{f}\|_{H^\alpha(\mathbb{R}^d; L^2_\gamma)} + C \|\tilde{f}^*\|_{H^\alpha(\mathbb{R}^d; H^{-1}_\gamma)} \tag{5-18}$$

for all $\delta \in \left(0, \frac{1}{3}\right)$, where C depends on δ. Sending the mollification parameter ε to 0^+ completes the induction and the proof. We emphasize that this induction demonstrates that $\partial_{x_i} f \in L^2(B_{r'}; H^1_\gamma)$ for all $r' < r$, where f is a function satisfying the hypotheses of Lemma 5.2. Once this is known, one may plainly differentiate the equation in ∂_{x_i} and apply Caccioppoli's inequality to conclude. □

Lemma 5.3 (differentiating in v). *Fix $r \in (0, \infty)$ and coefficients $\boldsymbol{b} \in C^{0,1}(B_r \times \mathbb{R}^d; \mathbb{R}^d)$, $c \in C^{0,1}(B_r \times \mathbb{R}^d; \mathbb{R})$. Suppose that $f^* \in H^1(B_r; L^2_\gamma)$ and $f \in H^1_{\mathrm{hyp}}(B_r)$ satisfy*

$$-\Delta_v f + v \cdot \nabla_v f + v \cdot \nabla_x f + \boldsymbol{b} \cdot \nabla_v f + c f = f^* \quad \text{in } B_r \times \mathbb{R}^d. \tag{5-19}$$

Then, for each $i \in \{1, \dots, d\}$, the function $h := \partial_{v_i} f$ belongs to $H^1_{\mathrm{hyp}}(B_{r'})$ for all $r' \in (0, r)$ and satisfies

$$-\Delta h + v \cdot \nabla_v h + v \cdot \nabla_x h + \boldsymbol{b} \cdot \nabla_v h + (c+1) h = h^* \quad \text{in } B_{r'} \times \mathbb{R}^d, \tag{5-20}$$

where

$$h^* := \partial_{v_i} f^* - \partial_{x_i} f - (\partial_{v_i} \boldsymbol{b}) \cdot \nabla_v f - (\partial_{v_i} c) f. \tag{5-21}$$

Moreover, there exists $C(d, r, \|\boldsymbol{b}\|_{C^{0,1}(B_r \times \mathbb{R}^d)}, \|c\|_{C^{0,1}(B_r \times \mathbb{R}^d)}) < \infty$ *such that*

$$\|\partial_{v_i} f\|_{H^1_{\mathrm{hyp}}(B_{r/2})} \leqslant C\|f\|_{L^2(B_r; L^2_\gamma)} + C\|f^*\|_{H^1(B_r; L^2_\gamma)}. \tag{5-22}$$

Proof. The standard procedure is to differentiate the equation and apply Caccioppoli's inequality. This introduces a forcing term h^*, defined in (5-21), which contains $\partial_{x_i} f$, and this is why we improve the spatial regularity beforehand in Lemma 5.2. That is, we already know

$$\|f\|_{H^1_{\mathrm{hyp}}(B_{r'})} + \|\partial_{x_i} f\|_{L^2(B_{r'}; H^1_\gamma)} \leqslant C(\|f\|_{L^2(B_r; L^2_\gamma)} + \|f^*\|_{H^1(B_r; H^{-1}_\gamma)}),$$

as in Lemma 5.2, where $r' = 7r/8$. In addition to this observation, we require a cut-off and mollification procedure to compensate for the fact that we did not assume *qualitatively* that $\partial_{v_i} f \in L^2(B_r; H^1_\gamma)$, which would be enough to make the energy estimate rigorous.

For $\ell \geqslant 1$, consider a standard cut-off function φ^ℓ in v at scale ℓ. Define

$$\begin{aligned} \tilde{f} &= \varphi^\ell f, \\ \tilde{f}^* &= \varphi^\ell f^* - 2\nabla_v f \cdot \nabla_v \varphi^\ell - f\Delta_v \varphi^\ell + f(v \cdot \nabla_v \varphi^\ell + \boldsymbol{b} \cdot \nabla_v \varphi^\ell), \end{aligned}$$

where we suppress the dependence on ℓ in the notation. Then $(\tilde{f}, \tilde{f}^*)$ solves (5-1) in $B_{r'} \times \mathbb{R}^d$, and it is not difficult to verify that

$$\begin{aligned} \|\tilde{f}\|_{L^2(B_{r'}; H^1_\gamma)} &\leqslant C\|f\|_{L^2(B_{r'}; H^1_\gamma)}, \\ \|\partial_{x_i} \tilde{f}\|_{L^2(B_{r'}; L^2_\gamma)} &\leqslant \|\partial_{x_i} f\|_{L^2(B_{r'}; L^2_\gamma)}, \\ \|\tilde{f}^*\|_{L^2(B_{r'}; L^2_\gamma)} &\leqslant C(\|f\|_{L^2(B_{r'}; H^1_\gamma)} + \|f^*\|_{L^2(B_{r'}; L^2_\gamma)}). \end{aligned}$$

Next, we mollify. Let ψ^ε be a standard mollification function in v at scale $0 < \varepsilon \ll 1$. Define

$$\begin{aligned} \bar{f} &= \psi^\varepsilon *_v \tilde{f}, \\ \bar{f}^* &= \psi^\varepsilon *_v \tilde{f}^* - [\psi^\varepsilon_v *_v, v\cdot](\nabla_v \tilde{f} + \nabla_x \tilde{f}) - ([\psi^\varepsilon *_v, \boldsymbol{b}\cdot]\nabla_v \tilde{f}) - [\psi^\varepsilon *_v, c]\tilde{f}, \end{aligned} \tag{5-23}$$

where again we suppress the dependence on ℓ, ε in the notation. Then $(\bar{f}, \bar{f}^*)$ is well-defined in $B_{r'} \times \mathbb{R}^d$ and solves (5-1) there.

We highlight a few features of the cut-off and mollification procedure. Translations of L^2_γ functions may not belong to L^2_γ, due to the superexponential nature of the weight (compare with exponential weights $e^{-c\langle v \rangle}$). Hence, mollification is not well-behaved on L^2_γ. The velocity cut-off φ^ℓ tames this issue. This cut-off has the additional benefit of taming commutators with v which occur naturally in the force term $\bar{f}^*$.

We claim

$$\begin{aligned} \limsup_{\varepsilon \to 0^+} \|\bar{f}\|_{L^2(B_{r'}; H^1_\gamma)} &\leqslant \|\tilde{f}\|_{L^2(B_{r'}; H^1_\gamma)}, \\ \limsup_{\varepsilon \to 0^+} \|\partial_{x_i} \bar{f}\|_{L^2(B_{r'}; L^2_\gamma)} &\leqslant \|\partial_{x_i} \tilde{f}\|_{L^2(B_{r'}; L^2_\gamma)}, \end{aligned} \tag{5-24}$$

and, more subtly,

$$\limsup_{\varepsilon\to 0^+} \|\bar{f}^*\|_{L^2(B_{r'};L^2_\gamma)} \leqslant C(\|\tilde{f}\|_{L^2(B_{r'};H^1_\gamma)} + \|\tilde{f}^*\|_{L^2(B_{r'};L^2_\gamma)}), \tag{5-25}$$

where (5-24) and (5-25) are for fixed ℓ. Both estimates in (5-24) are evident due to the support properties of $\tilde{f}$, so we focus on (5-25). For each fixed ℓ, we have

$$\|([\psi^\varepsilon *_v, \boldsymbol{b}\cdot]\nabla_v \tilde{f}) + [\psi^\varepsilon *_v, c]\tilde{f}\|_{L^2(B_{r'};L^2_\gamma)} \to 0$$

as $\varepsilon \to 0^+$.[12] Here, we use that the coefficients are Lipschitz and $\tilde{f}$ is compactly supported. It remains to analyze the second term in (5-23). From the compact support, we may replace v by $\varphi^{2\ell} v$. Then

$$\|[\psi^\varepsilon_v *_v, (\varphi^{2\ell} v)\cdot](\nabla_v \tilde{f})\|_{L^2(B_{r'};L^2_\gamma)} \to 0, \tag{5-26}$$

$$\|[\psi^\varepsilon_v *_v, (\varphi^{2\ell} v)\cdot](\nabla_x \tilde{f})\|_{L^2(B_{r'};L^2_\gamma)} \to 0 \tag{5-27}$$

as $\varepsilon \to 0^+$ for fixed ℓ.

Finally, we define $\bar{h} = \partial_{v_i} \bar{f}$ and

$$\bar{h}^* := \partial_{v_i} \bar{f}^* - \partial_{x_i} \bar{f} - (\partial_{v_i} \boldsymbol{b}) \cdot \nabla_v \bar{f} - (\partial_{v_i} c)\bar{f}, \tag{5-28}$$

which solve (5-20) in $B_{r'} \times \mathbb{R}^d$ and satisfy

$$\|\bar{h}\|_{L^2(B_{r'};L^2_\gamma)} \leqslant \|\bar{f}\|_{L^2(B_{r'};H^1_\gamma)},$$

$$\|\bar{h}^*\|_{L^2(B_{r'};H^{-1}_\gamma)} \leqslant C(\|\bar{f}^*\|_{L^2(B_{r'};L^2_\gamma)} + \|\partial_{x_i} \bar{f}\|_{L^2(B_{r'};L^2_\gamma)} + \|\bar{f}\|_{L^2(B_{r'};H^1_\gamma)}).$$

These, in turn, are estimated by the aforementioned inequalities for $\bar{f}$, $\tilde{f}$, and f. Applying Caccioppoli's inequality and sending $\varepsilon \to 0^+$ and $\ell \to +\infty$ completes the proof. □

Theorem 1.5 concerning the interior regularity, jointly in the variables x and v, is obtained by differentiating the equation and repeatedly applying Lemmas 5.2 and 5.3, and we omit the details.

6. The kinetic Fokker–Planck equation

In this last section, we study the time-dependent kinetic Fokker–Planck equation

$$\partial_t f - \varepsilon(\Delta_v f - v \cdot \nabla_v f) + v \cdot \nabla_x f + \boldsymbol{b} \cdot \nabla_v f = f^*. \tag{6-1}$$

The parameter ε is only relevant for the enhancement estimate, and one may imagine that $\varepsilon = 1$ until the final subsection. As with the Kramers equation, we prove a Poincaré inequality for bounded domains $V \subseteq \mathbb{R} \times \mathbb{R}^d$ which are either C^1 or cylindrical products $I \times U$ where $I \subseteq \mathbb{R}$ is a bounded interval and U is a bounded C^1 domain, but we consider the initial value problem only for $U = \mathbb{T}^d$.

[12] One may verify this by writing out the commutator explicitly and using the fundamental theorem of calculus for the difference terms that arise, such as $c(x, v - v') - c(x, v)$ if the mollification variable is v'.

6A. ***Function spaces.*** We define the function space

$$H^1_{\mathrm{kin}}(V) := \{f \in L^2(V; H^1_\gamma) : \partial_t f + v \cdot \nabla_x f \in L^2(V; H^{-1}_\gamma)\}, \tag{6-2}$$

equipped with the norm

$$\|f\|_{H^1_{\mathrm{kin}}(V)} := \|f\|_{L^2(V;H^1_\gamma)} + \|\partial_t f + v \cdot \nabla_x f\|_{L^2(V;H^{-1}_\gamma)}. \tag{6-3}$$

We denote the unit exterior normal to V by $\boldsymbol{n}_V \in L^\infty(\partial V; \mathbb{R}^{d+1})$. If V is a C^1 domain, then $\boldsymbol{n}_V(t,x)$ is well-defined for every $(t,x) \in \partial V$; if V is of the form $I \times U$, then $\boldsymbol{n}_V(t,x)$ is well-defined unless $(t,x) \in \partial I \times \partial U$, in which case we take the convention that $\boldsymbol{n}_V(t,x) = 0$. We define the hypoelliptic boundary of $V \subseteq \mathbb{R} \times \mathbb{R}^d$ as

$$\partial_{\mathrm{kin}}(V) := \left\{((t,x),v) \in \partial V \times \mathbb{R}^d : \begin{pmatrix} 1 \\ v \end{pmatrix} \cdot \boldsymbol{n}_V(t,x) < 0\right\}.$$

We denote by $H^1_{\mathrm{kin},0}(V)$ the closure in $H^1_{\mathrm{kin}}(V)$ of the set of smooth functions which vanish on $\partial_{\mathrm{kin}} V$.

Proposition 6.1 (density of smooth functions). *Let $V \subseteq \mathbb{R} \times \mathbb{R}^d$ be a bounded C^1 domain or cylindrical product $I \times U$, where U is a bounded C^1 domain. The set $C^\infty_c(\overline{V} \times \mathbb{R}^d)$ of smooth functions with compact support in $\overline{V} \times \mathbb{R}^d$ is dense in $H^1_{\mathrm{kin}}(V)$.*

Proof. Mimicking the first step of the proof of Proposition 2.2, which only uses that the domain is Lipschitz, we see that we can assume without loss of generality that, for every $z \in V$ and $\varepsilon \in (0,1]$, we have

$$B((1-\varepsilon)z, \varepsilon) \subseteq V.$$

Here we use z to denote a generic variable in $\mathbb{R} \times \mathbb{R}^d$; in standard notation, $z = (t,x)$. Let ζ_ε be a $(1+d)$-dimensional version of the mollifier defined in (2-11), and let $f \in H^1_{\mathrm{kin}}(V)$. We define, for every $\varepsilon \in \left(0, \frac{1}{2}\right]$, $z \in V$ and $v \in \mathbb{R}^d$,

$$f_\varepsilon(z,v) := \int_{\mathbb{R}^{1+d}} f((1-\varepsilon)z + z', v)\zeta_\varepsilon(z')\,dz'.$$

We then show as in Step 2 of the proof of Proposition 2.2 that f belongs to the closed convex hull of the set $\left\{f_\varepsilon : \varepsilon \in \left(0, \frac{1}{2}\right]\right\}$, and then, as in Step 3 of this proof, that for each $\varepsilon > 0$, we have that f_ε belongs to the closure of the set $C^\infty_c(\overline{V} \times \mathbb{R}^d)$. □

6B. ***Functional inequalities for H^1_{kin}.*** We next show a Poincaré inequality for $H^1_{\mathrm{kin}}(V)$. For the sake of generality, we allow for more flexible boundary conditions than in Theorem 1.3, in the spirit of Remark 3.2.

Proposition 6.2 (Poincaré inequality). *Let $V \subseteq \mathbb{R} \times \mathbb{R}^d$ be a bounded C^1 domain or a cylindrical product $I \times U$, where U is a bounded C^1 domain.*

(1) *There exists a constant $C(V,d) < \infty$ such that, for every $f \in H^1_{\mathrm{kin}}(V)$, we have*

$$\|f - (f)_V\|_{L^2(V;L^2_\gamma)} \leqslant C(\|\nabla_v f\|_{L^2(V;L^2_\gamma)} + \|v \cdot \nabla_x f + \partial_t f\|_{L^2(V;H^{-1}_\gamma)}).$$

(2) *Let W be a relatively open subset of $\partial V \times \mathbb{R}^d$. There exists a constant $C(V, W, d) < \infty$ such that for every $f \in C_c^\infty(\overline{V} \times \mathbb{R}^d)$ that vanishes on W, we have*

$$\|f\|_{L^2(V;L^2_\gamma)} \leqslant C(\|\nabla_v f\|_{L^2(V;L^2_\gamma)} + \|v \cdot \nabla_x f + \partial_t f\|_{L^2(V;H^{-1}_\gamma)}).$$

Proof of Proposition 6.2. The proof is similar to that of Theorem 1.3. By Proposition 6.1, we can assume that $f \in C_c^\infty(\overline{W} \times \mathbb{R}^d)$. We start by using the Gaussian Poincaré inequality to assert that

$$\|f - \langle f \rangle_\gamma\|_{L^2(V;L^2_\gamma)} \leqslant \|\nabla_v f\|_{L^2(V;L^2_\gamma)}.$$

Paralleling the second step of the proof of Theorem 1.3, we then aim to gain control on a negative Sobolev norm of the derivatives of $\langle f \rangle_\gamma$. Here we treat the time and space variables on an equal footing, and thus are interested in controlling $\partial_t \langle f \rangle_\gamma$ and $\nabla \langle f \rangle_\gamma$ in the $H^{-1}(V)$ norm. The precise claim is that there exists $C(d, V) < \infty$ such that for every test function $\phi \in C_c^\infty(V)$ satisfying

$$\|\phi\|_{L^2(V)} + \|\nabla\phi\|_{L^2(V)} + \|\partial_t\phi\|_{L^2(V)} \leqslant 1, \tag{6-4}$$

we have

$$\left|\int_V \phi\, \partial_t \langle f \rangle_\gamma\right| + \sum_{i=1}^d \left|\int_V \phi\, \partial_{x_i} \langle f \rangle_\gamma\right| \leqslant C(\|\nabla_v f\|_{L^2(V;L^2_\gamma)} + \|v \cdot \nabla_x f + \partial_t f\|_{L^2(V;H^{-1}_\gamma)}). \tag{6-5}$$

We start by showing that the first term on the left side of (6-5), which refers to the time derivative of $\langle f \rangle_\gamma$, is estimated by the right side of (6-5). We select a smooth function $\xi_0 \in C_c^\infty(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \xi_0(v)\, d\gamma(v) = 1 \quad \text{and} \quad \int_{\mathbb{R}^d} v\xi_0(v)\, d\gamma(v) = 0, \tag{6-6}$$

and observe that, using these properties of ξ_0, we can write

$$\begin{aligned}\int_V \partial_t\phi(t,x)\, \langle f \rangle_\gamma(t,x)\, dt\, dx &= \int_{V\times\mathbb{R}^d} \xi_0(v)(\partial_t\phi(t,x) + v\cdot\nabla_x\phi(t,x))\langle f \rangle_\gamma(t,x)\, dt\, dx\, d\gamma(v)\\ &= \int_{V\times\mathbb{R}^d} \xi_0(v)(\partial_t + v\cdot\nabla_x)\phi(t,x)\, f(t,x,v)\, dt\, dx\, d\gamma(v)\\ &\quad + \int_{V\times\mathbb{R}^d} \xi_0(v)(\partial_t + v\cdot\nabla_x)\phi(t,x)(\langle f \rangle_\gamma(t,x) - f(t,x,v))\, dt\, dx\, d\gamma(v).\end{aligned}$$

Using (6-4) and the fact that ξ_0 has compact support, we can bound the second integral above by

$$C\|f - \langle f \rangle_\gamma\|_{L^2(V;L^2_\gamma)} \leqslant C\|\nabla_v f\|_{L^2(V;L^2_\gamma)}.$$

By integration by parts, the absolute value of the first integral is equal to

$$\left|\int_{V\times\mathbb{R}^d} \xi_0(v)\phi(t,x)(v\cdot\nabla_x + \partial_t)f(t,x,v)\, dt\, dx\, d\gamma(v)\right| \leqslant C\|v\cdot\nabla_x f + \partial_t f\|_{L^2(V;H^{-1}_\gamma)}.$$

This completes the proof of the estimate in (6-5) involving the time derivative. To estimate the terms involving the space derivatives, we fix $i \in \{1, \dots, d\}$ and use a smooth function $\xi_i \in C_c^\infty(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d} \xi_i(v)\, d\gamma(v) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} v\xi_i(v)\, d\gamma(v) = e_i$$

to get

$$\int_V \partial_{x_i}\phi(t,x)\,\langle f\rangle_\gamma(t,x)\,dt\,dx = \int_{V\times\mathbb{R}^d} \xi_i(v)(v\cdot\nabla_x\phi(t,x)+\partial_t\phi(t,x))\langle f\rangle_\gamma(t,x)\,dt\,dx\,d\gamma(v).$$

The rest of the argument is then identical to the estimate involving the time derivative, and thus (6-5) is proved. The remainder of the proof is then identical to that for Theorem 1.3. Note that we need to invoke Lemma 3.1, which allows Lipschitz regularity, for the domain V. □

6C. *The Hörmander inequality for H^1_{kin}.* For the Hörmander inequality, we recall the parameter ε from (6-1) and assume that the spatial/temporal domain is $V=[0,\varepsilon^{-1/3}]\times\mathbb{T}^d$, although a similar estimate would hold for $V=[0,\varepsilon^{-1/3}]\times\mathbb{R}^d$. We emphasize that we have included this particular factor of ε due to the fact that the a priori estimates for (6-1) control only $\varepsilon^{1/2}\nabla_v f$, and also due to the scaling between the regularity exponent we shall be able to obtain for $\nabla_x f$ and the a priori estimate. This inequality for $H^1_{\mathrm{kin}}(V)$ is proved in an almost identical way to the one for $H^1_{\mathrm{hyp}}(\mathbb{T}^d)$; the only difference is that the time variable is *not* periodic as is the space variable. So a bit of care must be taken with the finite differences corresponding to the vector field $\partial_t+v\cdot\nabla_x$. We track the parameter ε throughout the proof for the purposes of the enhancement estimate later on. The version of (3-25) we use here is

$$\begin{aligned} &f(t,x+\eta^3\varepsilon^{1/2}x',v)-f(t,x,v)\\ &\quad= f(t,x+\eta^3\varepsilon^{1/2}x',v)-f(t,x+\eta^3\varepsilon^{1/2}x',v-\eta\varepsilon^{1/2}x')\\ &\qquad+f(t,x+\eta^3\varepsilon^{1/2}x',v-\eta\varepsilon^{1/2}x')-f(t+\eta^2,x+\eta^3\varepsilon^{1/2}x'+\eta^2(v-\varepsilon^{1/2}\eta x'),v-\eta\varepsilon^{1/2}x')\\ &\qquad\quad+f(t+\eta^2,x+\eta^2v,v-\eta\varepsilon^{1/2}x')-f(t+\eta^2,x+\eta^2v,v)\\ &\qquad\qquad+f(t+\eta^2,x+\eta^2v,v)-f(t,x,v). \end{aligned}\tag{6-7}$$

As before, we must define the following Besov spaces based on finite differences in the ∇_x and $D_t=\partial_t+v\cdot\nabla_x$ directions. The Besov space measuring fractional regularity in the x variable now depends *fundamentally* on ε and t, and so we denote this space $Q^{1/3,\varepsilon}_{\nabla_x}$. To lighten the notation, in the context of proofs in which ε is always fixed, we sometimes shall substitute the notation $Q^{1/3}_{\nabla_x}$ instead of the more cumbersome $Q^{1/3,\varepsilon}_{\nabla_x}$, and similarly for $Q^{1/2,\varepsilon}_{D_t}$.

Definition 6.3. For measurable $u:(0,\varepsilon^{-1/3})\times\mathbb{T}^d\times\mathbb{R}^d\to\mathbb{R}$, we define

$$\begin{aligned} &\|u\|^2_{Q^{1/2,\varepsilon}_{D_t}}\\ &:=\sup_{0<\eta\leqslant\sqrt{\varepsilon^{-1/3}/2}}\frac{1}{\eta^2}\biggl(\iiint_{(0,\varepsilon^{-1/3}/2)\times\mathbb{R}^d\times\mathbb{T}^d}(u(t+\eta^2,x+\eta^2v,v)-u(t,x,v))^2\,dx\,d\gamma(v)\,dt\\ &\qquad\qquad+\iiint_{(\varepsilon^{-1/3}/2,\varepsilon^{-1/3})\times\mathbb{R}^d\times\mathbb{T}^d}(u(t-\eta^2,x-\eta^2v,v)-u(t,x,v))^2\,dx\,d\gamma(v)\,dt\biggr). \end{aligned}\tag{6-8}$$

We define

$$\|u\|^2_{Q^{1/3,\varepsilon}_{\nabla_x}}:=\sup_{\substack{0<\eta\leqslant\sqrt{\varepsilon^{-1/3}/2}\\ x'\in\mathbb{S}^{d-1}}}\frac{1}{\eta^2}\iiint_{(0,\varepsilon^{-1/3})\times\mathbb{R}^d\times\mathbb{T}^d}(u(t,x+\varepsilon^{1/2}\eta^3x',v)-u(t,x,v))^2\,dx\,d\gamma(v)\,dt. \tag{6-9}$$

Notice that the quantity $\varepsilon^{1/2}\eta^3$ is of order 1 if η^2 takes its maximum value of $\varepsilon^{-1/3}/2$. Then by iterating the finite differences, the norm in (6-9) is equivalent to one in which the supremum is taken over values of η at least as large as the diameter of $\mathbb{T}^d$, at which point the norm is equivalent to one including all positive values of η.

To streamline the proof of the enhancement estimate later, we assume in the following proposition that $\langle \partial_t u + v \cdot \nabla_x u\rangle_\gamma \equiv 0$ (a condition which will be satisfied in the enhancement context). Then from Lemma 2.1, the $L^2_{t,x}H^{-1}_\gamma$ norm of $\partial_t u + v\cdot\nabla_x u$ may be obtained via duality against the gradients (in v) of $L^2_{t,x}H^1_\gamma$ functions which have vanishing means $\langle\,\cdot\,\rangle_\gamma$. Thus the inequality (6-10) does not require the $L^2_{t,x}L^2_\gamma$ norm of u on the right-hand side; one could easily adjust the statement in the case that $\langle \partial_t u + v\cdot\nabla_x u\rangle_\gamma \neq 0$ by including the necessary term.

Lemma 6.4 (interpolation). *For every $\delta > 0$, there exists a constant $C(\delta, d) < \infty$ (not depending on ε) such that, for any smooth function u satisfying $\langle \partial_t u + v\cdot\nabla_x u\rangle_\gamma \equiv 0$,*

$$\|u\|^2_{Q^{1/2,\varepsilon}_{D_t}} \leqslant \delta\|u\|^2_{Q^{1/3,\varepsilon}_{\nabla_x}} + C(\delta)\big(\varepsilon\|\nabla_v u\|^2_{L^2((0,\varepsilon^{-1/3})\times\mathbb{T}^d;L^2_\gamma)} + \varepsilon^{-1}\|\partial_t u + v\cdot\nabla_x u\|^2_{L^2((0,\varepsilon^{-1/3})\times\mathbb{T}^d;H^{-1}_\gamma)}\big). \tag{6-10}$$

Remark 6.5. The factors of ε ensure that the right-hand side remains of order 1 as $\varepsilon \to 0$ and arise naturally when deriving the a priori estimates for solutions to (6-1); see Section 6F for more details.

Proof. The proof is similar for both halves of (6-8), i.e., the forward and backward differences, and so we focus on the case of the forward difference.

Step 1: Let $\phi \in C_0^\infty((-1,1)^d)$ be a smooth, positive, radial function with unit L^1 norm. For $\zeta > 0$, we define $\phi_\zeta u(t,x,v)$ by

$$\phi_\zeta u(t,x,v) = \int_{\mathbb{R}^d} u(t, x + \zeta^3\varepsilon^{1/2}x', v)\phi(x')\,dx'.$$

Analogously to Step 1 from the proof of Theorem 1.4, we have

$$\|\phi_\zeta u(t,x,v) - u(t,x,v)\|^2_{L^2((0,\varepsilon^{-1/3})\times\mathbb{T}^d;L^2_\gamma)} \leqslant \zeta^2\|u\|^2_{Q^{1/3}_{\nabla_x}}. \tag{6-11}$$

Step 2: Let

$$f(\eta) = \|u(t+\eta^2, x+\eta^2 v, v) - u(t,x,v)\|^2_{L^2((0,\varepsilon^{-1/3}/2)\times\mathbb{T}^d;L^2_\gamma)}.$$

We may write

$$\begin{aligned} f(\eta) \lesssim{}& \|\phi_{\delta\eta}u(t+\eta^2, x+\eta^2 v, v) - u(t+\eta^2, x+\eta^2 v, v)\|^2_{L^2((0,\varepsilon^{-1/3}/2)\times\mathbb{T}^d;L^2_\gamma)} \\ &+ \|\phi_{\delta\eta}u(t+\eta^2, x+\eta^2 v, v) - \phi_{\delta\eta}u(t,x,v)\|^2_{L^2((0,\varepsilon^{-1/3}/2)\times\mathbb{T}^d;L^2_\gamma)} \\ &+ \|\phi_{\delta\eta}u(t,x,v) - u(t,x,v)\|^2_{L^2((0,\varepsilon^{-1/3}/2)\times\mathbb{T}^d;L^2_\gamma)}, \end{aligned} \tag{6-12}$$

where the implicit constant is independent of η, δ, and u. By Step 1 with $\zeta = \delta\eta$, the first and third terms are bounded by

$$\delta^2\eta^2\|u\|^2_{Q^{1/3}_{\nabla_x}}.$$

Step 3: It remains to estimate the second term in (6-12). For $\eta \in (0, \sqrt{\varepsilon^{-1/3}/2})$ and $0 \leqslant \tau \leqslant \eta^2$, consider

$$F(\tau) = \|\phi_{\delta\eta}u(t+\tau, x+\tau v, v) - \phi_{\delta\eta}u(t,x,v)\|^2_{L^2((0,\varepsilon^{-1/3}/2)\times\mathbb{T}^d;L^2_\gamma)}. \tag{6-13}$$

The term in question is $F(\eta^2)$. Since $F(0)=0$, it suffices to estimate $F'(\tau)$. We have

$$\begin{aligned}
&F'(\tau)\\
&=2\iiint_{(0,\varepsilon^{-1/3}/2)\times\mathbb{R}^d\times\mathbb{T}^d}(\phi_{\delta\eta}u(t+\tau,x+\tau v,v)-\phi_{\delta\eta}u(t,x,v))\cdot D_t(\phi_{\delta\eta}u)(t+\tau,x+\tau v,v)\,dx\,d\gamma(v)\,dt\\
&=2\iiint_{(\tau,\varepsilon^{-1/3}/2+\tau)\times\mathbb{R}^d\times\mathbb{T}^d}(\phi_{\delta\eta}u(t,x,v)-\phi_{\delta\eta}u(t-\tau,x-\tau v,v))\cdot D_t(\phi_{\delta\eta}u)(t,x,v)\,dx\,d\gamma(v)\,dt.
\end{aligned}\tag{6-14}$$

From $[D_t,\phi_{\delta\eta}]u=[\nabla_v,\phi_{\delta\eta}]u=0$, the assumption $\langle\partial_t u+v\cdot\nabla_x u\rangle_\gamma\equiv 0$, and our control of

$$\|\partial_t u+v\cdot\nabla_x u\|_{L^2((0,\varepsilon^{-1/3})\times\mathbb{T}^d;H_\gamma^{-1})},$$

we will achieve the desired estimate for $F'(\tau)$ if we can bound

$$\nabla_v(\phi_{\delta\eta}u(t,x,v)-\phi_{\delta\eta}u(t-\tau,x-\tau v,v))$$

in $L^2((\tau,\varepsilon^{-1/3}/2+\tau)\times\mathbb{T}^d;L^2_\gamma)$. Notice that after obtaining these bounds, we apply the Cauchy–Schwarz inequality with a prefactor of ε in front of one term and ε^{-1} in front of the other in order to obtain (6-10). The only nontrivial estimate comes when the ∇_v lands on the x-coordinate of the second term, which we may write out as

$$\begin{aligned}
&\int_{\mathbb{T}^d}-\tau\nabla_x u(t-\tau,x+(\delta\eta)^3\varepsilon^{1/2}x'-\tau v,v)\phi(x')\,dx'\\
&\qquad=-\int_{\mathbb{T}^d}\frac{\tau}{(\delta\eta)^3\varepsilon^{1/2}}\nabla_{x'}u(t-\tau,x+(\delta\eta)^3\varepsilon^{1/2}x'-\tau v,v)\phi(x')\,dx'\\
&\qquad=\int_{\mathbb{T}^d}\frac{\tau}{(\delta\eta)^3\varepsilon^{1/2}}u(t-\tau,x+(\delta\eta)^3\varepsilon^{1/2}x'-\tau v,v)\nabla_{x'}\phi(x')\,dx'\\
&\qquad=\int_{\mathbb{T}^d}\frac{\tau}{(\delta\eta)^3\varepsilon^{1/2}}(u(t-\tau,x+(\delta\eta)^3\varepsilon^{1/2}x'-\tau v,v)-u(t-\tau,x-\tau v,v))\nabla_{x'}\phi(x')\,dx'.
\end{aligned}$$

But slight adjustments to the argument from Step 1 show that this is bounded in $L^2((\tau,\varepsilon^{1/2}/2+\tau)\times\mathbb{T}^d;L^2_\gamma)$ by a constant independent of δ times

$$\frac{\tau}{(\delta\eta)^3\varepsilon^{1/2}}\delta\eta\|u\|_{Q^{1/3}_{\nabla_x}}\leqslant\frac{1}{\delta^2\varepsilon^{1/2}}\|u\|_{Q^{1/3}_{\nabla_x}},$$

where here we have used the assumption that $\tau\leqslant\eta^2$. Using the Cauchy–Schwarz and Young inequalities to absorb the negative powers of ε and δ with the $L^2_{t,x}H^{-1}_\gamma$ norm concludes the proof. □

We may now state and prove the following proposition. As with the interpolation, in the case that $\langle\partial_t u+v\cdot\nabla_x u\rangle_\gamma\neq 0$, one could adjust the statement of the second inequality to include the necessary $L^2_{t,x}L^2_\gamma$ norm of u.

Proposition 6.6 (Hörmander inequality). *There exists $C(d)<\infty$ (not depending on ε) such that for every smooth function u satisfying $\langle\partial_t u+v\cdot\nabla_x u\rangle_\gamma\equiv 0$, we have*

$$\begin{aligned}
\|u\|_{Q^{1/3,\varepsilon}_{\nabla_x}}&\leqslant C(\varepsilon^{1/2}\|\nabla_v u\|_{L^2((0,\varepsilon^{-1/3})\times\mathbb{T}^d;L^2_\gamma)}+\|u\|_{Q^{1/2,\varepsilon}_{D_t}})\\
&\leqslant C(\varepsilon^{1/2}\|\nabla_v u\|_{L^2((0,\varepsilon^{-1/3})\times\mathbb{T}^d;L^2_\gamma)}+\varepsilon^{-1/2}\|\partial_t u+v\cdot\nabla_x u\|_{L^2((0,\varepsilon^{-1/3})\times\mathbb{T}^d;H^{-1}_\gamma)}).
\end{aligned}\tag{6-15}$$

Proof of Proposition 6.6. Set

$$g(t,x,v)=f(t,x,v)\gamma^{1/2}(v),$$

and choose $\eta^2\in(0,\varepsilon^{-1/3}]$ and $x'\in\mathbb{S}^{d-1}$. Then we may write

$$\|f(t,x+\varepsilon^{1/2}\eta^3x',v)-f(t,x,v)\|_{L^2((0,\varepsilon^{-1/3}/2)\times\mathbb{T}^d;L^2_\gamma)} = \|g(t,x+\varepsilon^{1/2}\eta^3x',v)-g(t,x,v)\|_{L^2((0,\varepsilon^{-1/3}/2)\times\mathbb{T}^d;L^2(\mathbb{R}^d))}$$

and

$$\begin{aligned} g(t,x+\eta^3\varepsilon^{1/2}x',v)-g(t,x,v) &= g(t,x+\eta^3\varepsilon^{1/2}x',v)-g(t,x+\eta^3\varepsilon^{1/2}x',v-\eta\varepsilon^{1/2}x')\\ &\quad+g(t,x+\eta^3\varepsilon^{1/2}x',v-\eta\varepsilon^{1/2}x')\\ &\quad-g(t+\eta^2,x+\eta^3\varepsilon^{1/2}x'+\eta^2(v-\varepsilon^{1/2}\eta x'),v-\eta\varepsilon^{1/2}x')\\ &\quad+g(t+\eta^2,x+\eta^2v,v-\eta\varepsilon^{1/2}x')-g(t+\eta^2,x+\eta^2v,v)\\ &\quad+g(t+\eta^2,x+\eta^2v,v)-g(t,x,v). \end{aligned} \tag{6-16}$$

Dividing by η, integrating in $L^2((0,\varepsilon^{-1/3}/2)\times\mathbb{T}^d;L^2(\mathbb{R}^d))$, and appealing to (2-7) as in the time-independent case yields

$$\frac{1}{\eta}\|f(t,x+\varepsilon^{1/2}\eta^3x',v)-f(t,x,v)\|_{L^2((0,\varepsilon^{-1/3})\times\mathbb{T}^d;L^2_\gamma)}\lesssim\varepsilon^{1/2}\|\nabla_v f\|_{L^2((0,\varepsilon^{-1/3})\times\mathbb{T}^d;L^2_\gamma)}+\|f\|_{Q^{1/2}_{D_t}}.$$

For the other half of the time interval, it is easy to rewrite (6-16) with a backwards difference in the $\partial_t+v\cdot\nabla_x$ direction by first adding $\eta\varepsilon^{1/2}x'$ in the v-variable and then subtracting η^2 in the t-variable and $\eta^2(v+\varepsilon^{1/2}\eta x')$ in the x-variable. Arguing as for the forward differences produces an identical estimate. Then using Lemma 6.4 and absorbing the $\|f\|^2_{Q^{1/3}_{\nabla_x}}$ factor required to bound $\|f\|_{Q^{1/2}_{D_t}}$ from the right-hand side onto the left-hand side gives the result. □

Remark 6.7. From the embedding $Q^{1/3}_{\nabla_x}\hookrightarrow L^2((0,\varepsilon^{-1/3})\times\mathbb{T}^d;L^2_\gamma)$ for functions with vanishing x-mean $\langle u\rangle(t,v)=\fint_{\mathbb{T}^d}u(t,x,v)\,dx$ (see, for example, [Albritton et al. 2022]), we obtain the ε-dependent Poincaré inequality

$$\|u\|_{L^2((0,\varepsilon^{-1/3})\times\mathbb{T}^d;L^2_\gamma)}\leqslant C\varepsilon^{-1/6}\|u\|_{Q^{1/3}_{\nabla_x}}. \tag{6-17}$$

Note that to obtain this inequality, we have rescaled out the factors of ε used in the finite differences of the $Q^{1/3}_{\nabla_x}$ norm and then appealed to an ε-independent function space embedding.

Remark 6.8 (regularity in time). By an interpolation argument, the result of Proposition 6.6 implies some time regularity for a function $f\in H^1_{\mathrm{kin}}(V)$ for $V=(0,\varepsilon^{-1/3})\times\mathbb{T}^d$. Indeed, by the definition of the norm $\|\cdot\|_{H^1_{\mathrm{kin}}}$, we have

$$\|f\|_{L^2((0,\varepsilon^{-1/3})\times\mathbb{T}^d;H^1_\gamma)}\leqslant\|f\|_{H^1_{\mathrm{kin}}((0,\varepsilon^{-1/3})\times\mathbb{T}^d)}.$$

By interpolation and (6-15), for every $\theta\in[0,1]$ and $\alpha\in\left[0,\tfrac{1}{3}\right)$,

$$\|f\|_{L^2((0,\varepsilon^{-1/3});H^{\theta\alpha}(\mathbb{T}^d;H^{1-2\theta}_\gamma))}\leqslant C\|f\|_{H^1_{\mathrm{kin}}((0,\varepsilon^{-1/3})\times\mathbb{T}^d)}.$$

We also have, by (6-15), for any $\alpha \in \left[0, \frac{1}{3}\right)$,

$$\begin{aligned}\|f\|_{H^1((0,\varepsilon^{-1/3});H^{\alpha-1}(\mathbb{T}^d;H_\gamma^{-1}))} &\leqslant \|f\|_{L^2((0,\varepsilon^{-1/3});H^{\alpha-1}(\mathbb{T}^d;H_\gamma^{-1}))} + \|\partial_t f\|_{L^2((0,\varepsilon^{-1/3});H^{\alpha-1}(\mathbb{T}^d;H_\gamma^{-1}))} \\ &\leqslant \|f\|_{L^2((0,\varepsilon^{-1/3});L^2(\mathbb{T}^d;H_\gamma^{-1}))} + \|\partial_t f - v\cdot\nabla_x f\|_{L^2((0,\varepsilon^{-1/3});L^2(\mathbb{T}^d;H_\gamma^{-1}))} \\ &\qquad\qquad + \|v\cdot\nabla_x f\|_{L^2((0,\varepsilon^{-1/3});H^{\alpha-1}(\mathbb{T}^d;H_\gamma^{-1}))} \\ &\leqslant C\|f\|_{H^1_{\mathrm{kin}}((0,\varepsilon^{-1/3})\times\mathbb{T}^d)}.\end{aligned}$$

By interpolation of the previous two displays, we obtain, for any $\theta, \sigma \in [0, 1]$ and $\alpha \in \left[0, \frac{1}{3}\right)$,

$$\|f\|_{H^\sigma((0,\varepsilon^{-1/3});H^{\theta\alpha-\sigma(1-\alpha+\theta\alpha)}(\mathbb{T}^d;H_\gamma^{1-2(\theta+\sigma-\theta\sigma)}))} \leqslant C\|f\|_{H^1_{\mathrm{kin}}((0,\varepsilon^{-1/3})\times\mathbb{T}^d)}. \tag{6-18}$$

Each of the constants C above depends only on (α, d). Note that all three exponents can be made simultaneously positive, for example taking $\alpha = \theta = \frac{1}{4}$ and $\sigma = \frac{1}{32}$ yields

$$\|f\|_{H^{1/32}((0,\varepsilon^{-1/3});H^{1/32}(\mathbb{T}^d;H_\gamma^{7/16}))} \leqslant C\|f\|_{H^1_{\mathrm{kin}}((0,\varepsilon^{-1/3})\times\mathbb{T}^d)}. \tag{6-19}$$

By (6-19) and an argument very similar to the proof of Proposition 3.8, which we omit, we obtain the following compact embedding statement.

Proposition 6.9 (compact embedding of H^1_{kin} into L^2). *For any bounded C^1 domain $V \subseteq \mathbb{R}\times\mathbb{R}^d$ or cylindrical product $I \times U$ where U is a bounded C^1 domain, the inclusion map $H^1_{\mathrm{kin}}(V) \hookrightarrow L^2(V; L^2_\gamma)$ is compact.*

6D. *Well-posedness of the Cauchy problem.*

Proposition 6.10 (solvability of the kinetic Fokker–Planck equation). *Let $T \in (0, +\infty]$, $f_{\mathrm{in}} \in L^2_m$, and $g^* \in L^2(\mathbb{T}^d \times (0, T); H_\gamma^{-1})$. Under Assumption 1.1, there exists a unique solution*

$$f \in C([0, T]; L^2_m(\mathbb{T}^d \times \mathbb{R}^d)) \cap H^1_{\mathrm{kin}}((0, T) \times \mathbb{T}^d) \tag{6-20}$$

to the kinetic Fokker–Planck equation (6-1) *with initial data f_{in} and forcing term g^*.*

Proof. Let $T \in (0, +\infty]$. Let $f_{\mathrm{in}} \in L^2_m$ and $g^* \in L^2((0, T); L^2_\sigma(\mathbb{T}^d; H_\gamma^{-1})))$. A function g solves the kinetic Fokker–Planck equation if and only if $f(t, x, v) = g(t, x, v)e^t$ solves

$$\partial_t f + (v\cdot\nabla_x + \boldsymbol{b}\cdot\nabla_v)f + f = f^* + \varepsilon(\Delta f - v\cdot\nabla_v f), \tag{6-21}$$

where $f^* = e^t g^*$. We solve (6-21) on $(0, T) \times \mathbb{T}^d \times \mathbb{R}^d$ by applying Lemma 4.1 with an appropriate functional setup:

(1) the *test function space*

$$\Phi = C_0^\infty(\mathbb{T}^d \times \mathbb{R}^d \times [0, T)) \tag{6-22}$$

with inner product

$$(\phi, \psi) = \int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d} \nabla_v\phi\cdot\nabla_v\psi \, dm\, dt + \int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d} \phi\psi \, dm\, dt, \tag{6-23}$$

(2) the *solution space*

$$H = L^2(0,T; L^2_\sigma(\mathbb{T}^d; H^1_\gamma))$$

with inner product (6-23),

(3) the *bilinear form*

$$E(h,\phi) = \varepsilon \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d} \nabla_v h\cdot\nabla_v\phi\,dm\,dt + \int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d} h\phi\,dm\,dt - \int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d} h(\partial_t + v\cdot\nabla_x + \boldsymbol{b}\cdot\nabla_v)\phi\,dm\,dt,$$

(4) and the *linear functional*

$$L\phi = \int_{\mathbb{T}^d\times\mathbb{R}^d} f_{\text{in}}\phi(x,v,0)\,dm + g^*(\phi).$$

As before, in the Kramers equation, one may verify that E is continuous (4-2) on H for each fixed $\phi\in\Phi$. We now verify coercivity (4-3) and mention two essential new features: (i) the initial data f_{in} is built into the linear function L, and (ii) test functions $\phi\in\Phi$ vanish at $t=T$ but are not required to vanish at $t=0$ (which is necessary for them to "detect" the initial data). After integrating by parts in all variables, we have

$$E(\phi,\phi) = \varepsilon\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d} |\nabla_v\phi|^2\,dm\,dt + \int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d}|\phi|^2\,dm\,dt + \frac{1}{2}\int_{\mathbb{T}^d\times\mathbb{R}^d}|\phi(x,v,0)|^2\,dm \geqslant \varepsilon(\psi,\psi)_H.$$

Lemma 4.1 generates a weak solution $f\in H$ to $E(f,\phi)=L\phi$ for all $\phi\in\Phi$. In particular, choosing $\phi\in\Phi$ that additionally vanish near $t=0$ guarantees that the PDE (6-21) is satisfied in the sense of distributions. From the PDE itself, we recover that $f\in H^1_{\text{kin}}(\mathbb{T}^d\times(0,T))$ and, in particular, $f\in C([0,T]; L^2_\sigma(\mathbb{T}^d; L^2_\gamma))$; see Lemma 6.12. This is enough regularity to justify that the initial data is f_{in} and the basic energy estimate which guarantees uniqueness. □

We do not include a proof of the following statement in this paper, since the argument is a close adaptation of the one of Theorem 1.5. We define $V_r := (-r,r)\times B_r$ and denote by $\nabla_{t,x}$ the full gradient in t and x, that is, $\nabla_{t,x} = (\partial_t, \nabla_x)$.

Proposition 6.11 (interior regularity, kinetic Fokker–Planck). *Let $\boldsymbol{b}\in C^{k-1,1}(V_r\times\mathbb{R}^d;\mathbb{R}^d)$, $k\in\mathbb{N}$ and $r\in(0,\infty)$. There exists a constant $C<\infty$ depending on*

$$(d,k,r,\|\boldsymbol{b}\|_{C^{k-1,1}(V_r\times\mathbb{R}^d;\mathbb{R}^d)})$$

such that, for every $f\in H^1_{\text{kin}}(V_r)$ and $f^*\in L^2(V_r; H^{-1}_\gamma)$ satisfying

$$\partial_t f - \Delta_v f + v\cdot\nabla_v f + v\cdot\nabla_x f + \boldsymbol{b}\cdot\nabla_v f = f^* \quad \text{in } V_r\times\mathbb{R}^d, \tag{6-24}$$

the following holds: if $\partial^\alpha f^*\in L^2(B_r; H^{-1}_\gamma)$ for all multi-indices $\alpha\in\mathbb{N}\times\mathbb{N}^d\times\mathbb{N}^d$ satisfying $|\alpha|\leqslant k$, then we have $\partial^\alpha f\in H^1_{\text{kin}}(V_{r/2})$ and the estimate

$$\|\partial^\alpha f\|_{H^1_{\text{kin}}(V_{r/2})} \leqslant C\Bigg(\|f-(f)_{V_r}\|_{L^2(V_r;L^2_\gamma)} + \sum_{|\beta|\leqslant k}\|\partial^\beta\tilde{f}^*\|_{L^2(V_r;H^{-1}_\gamma)}\Bigg)$$

for all multi-indices $\alpha\in\mathbb{N}\times\mathbb{N}^d\times\mathbb{N}^d$ satisfying $|\alpha|\leqslant k$.

6E. *Exponential decay in time.* For each bounded interval $I = (I_-, I_+) \subseteq \mathbb{R}$ and bounded C^1 domain U, we denote by $H^1_{\mathrm{kin},||}(I \times U)$ the closure in $H^1_{\mathrm{kin}}(I \times U)$ of the set of smooth functions which vanish on $I \times \partial_{\mathrm{hyp}} U$. Note that in particular, we allow the trace of $f \in H^1_{\mathrm{kin},||}(I \times U)$ on the initial time slice $\{I_-\} \times U$ to be nonzero. In this section, we show that a solution to the kinetic Fokker–Planck equation with zero right-hand side and belonging to $H^1_{\mathrm{kin},||}(I \times U)$ decays to zero exponentially fast in time. We start with a preliminary classical lemma.

Lemma 6.12 (continuity in L^2). *Every function in $H^1_{\mathrm{kin},||}(I \times U)$ can be identified* (*up to a set of null measure*) *with an element of $C(\bar{I}; L^2(U; L^2_\gamma))$.*

Proof. If f is a smooth function which vanishes on $I \times \partial_{\mathrm{hyp}} U$, then, for every $t \in I$, we have

$$\partial_t \|f(t,\cdot)\|^2_{L^2(U;L^2_\gamma)} + \int_{\partial U \times \mathbb{R}^d} f^2(t,x,v)(v \cdot \boldsymbol{n}_U(x))_+ \, dx \, d\gamma(v) = 2 \int_{U \times \mathbb{R}^d} (f(\partial_t f + v \cdot \nabla_x f))(t,x,v) \, dx \, d\gamma(v),$$

where we recall that $(r)_+ := \max(0, r)$. Since the second integral on the left side is nonnegative, we deduce that, for every $s, t \in I$,

$$\left| \|f(t,\cdot)\|^2_{L^2(U;L^2_\gamma)} - \|f(s,\cdot)\|^2_{L^2(U;L^2_\gamma)} \right| \leqslant 2 \|f\|_{L^2((s,t)\times U; H^1_\gamma)} \|\partial_t f + v \cdot \nabla_x f\|_{L^2((s,t)\times U; H^{-1}_\gamma)},$$

and thus, for a constant $C(I) < \infty$,

$$\sup_{t \in \bar{I}} \|f(t,\cdot)\|_{L^2(U;L^2_\gamma)} \leqslant C \|f\|_{H^1_{\mathrm{kin}}(I \times U)}.$$

For a general $f \in H^1_{\mathrm{kin},||}(I \times U)$, there exists a sequence (f_n) of smooth functions which vanish on $I \times \partial_{\mathrm{hyp}} U$ and such that f_n converges to f in $H^1_{\mathrm{kin}}(I \times U)$. It follows from the inequality above that f_n converges to f with respect to the $L^\infty(I; L^2(U; L^2_\gamma))$ norm; in particular, $f \in C(\bar{I}; L^2(U; L^2_\gamma))$. □

We finally turn to the proof of Theorem 1.6, which is restated in the following proposition. Notice that, by linearity, it suffices to prove the theorem in the case $f^* = 0$ and $f_\infty = 0$.

Proposition 6.13 (exponential decay to equilibrium). *Let $U \subseteq \mathbb{R}^d$ be a bounded C^1 domain and $\boldsymbol{b} \in L^\infty(U \times \mathbb{R}^d)^d$. There exists $\lambda(\|\boldsymbol{b}\|_{L^\infty(U\times\mathbb{R}^d)}, U, d) > 0$ such that*, *for every $T \in (0, \infty)$ and $f \in H^1_{\mathrm{kin},||}((0,T) \times U)$ satisfying*

$$\partial_t f - \Delta_v f + v \cdot \nabla_v f + v \cdot \nabla_x f + \boldsymbol{b} \cdot \nabla_v f = 0 \quad \text{in } (0,T) \times U \times \mathbb{R}^d,$$

we have, *for every $t \in (0, T)$*,

$$\|f(t,\cdot)\|_{L^2(U;L^2_\gamma)} \leqslant 2 \exp(-\lambda t) \|f(0,\cdot)\|_{L^2(U;L^2_\gamma)}.$$

Proof. For every $0 \leqslant s < t$, we compute

$$\tfrac{1}{2}(\|f(t,\cdot)\|^2_{L^2(U;L^2_\gamma)} - \|f(s,\cdot)\|^2_{L^2(U;L^2_\gamma)}) \leqslant -\|\nabla_v f\|^2_{L^2((s,t)\times U; L^2_\gamma)}.$$

In particular,

$$\text{the mapping } t \mapsto \|f(t,\cdot)\|_{L^2(U;L^2_\gamma)} \text{ is nonincreasing.} \tag{6-25}$$

Since

$$-\nabla_v^* \nabla_v f = \partial_t f + v \cdot \nabla_x f + \boldsymbol{b} \cdot \nabla_v f,$$

we have

$$\begin{aligned} \|\partial_t f + v \cdot \nabla_x f\|_{L^2((s,t)\times U; H_\gamma^{-1})} &\leqslant \|\partial_t f + v \cdot \nabla_x f + \boldsymbol{b} \cdot \nabla_v f\|_{L^2((s,t)\times U; H_\gamma^{-1})} + \|\boldsymbol{b} \cdot \nabla_v f\|_{L^2((s,t)\times U; H_\gamma^{-1})} \\ &\leqslant C \|\nabla_v f\|_{L^2((s,t)\times U; L^2_\gamma)}, \end{aligned}$$

and thus

$$\begin{aligned} -(\|f(t,\cdot)\|^2_{L^2(U;L^2_\gamma)} - \|f(s,\cdot)\|^2_{L^2(U;L^2_\gamma)}) \\ \geqslant \frac{1}{C}(\|\nabla_v f\|^2_{L^2((s,t)\times U; L^2_\gamma)} + \|\partial_t f + v \cdot \nabla_x f\|^2_{L^2((s,t)\times U; H_\gamma^{-1})}). \end{aligned} \tag{6-26}$$

We aim to appeal to Proposition 6.2 to conclude. We define

$$V := [0, 1] \times U. \tag{6-27}$$

For every $t \geqslant 0$, we write

$$V_t := (t, 0) + V = \{(t+s, x) \in \mathbb{R} \times \mathbb{R}^d : (s, x) \in V\}.$$

Inequality (6-26) implies that, for every $t \geqslant 0$,

$$-(\|f(t+1,\cdot)\|^2_{L^2(U;L^2_\gamma)} - \|f(t,\cdot)\|^2_{L^2(U;L^2_\gamma)}) \geqslant \frac{1}{C}(\|\nabla_v f\|^2_{L^2(V_t;L^2_\gamma)} + \|\partial_t f - v \cdot \nabla_x f\|^2_{L^2(V_t;H_\gamma^{-1})}).$$

Proposition 6.2 yields

$$-(\|f(t+1,\cdot)\|^2_{L^2(U;L^2_\gamma)} - \|f(t,\cdot)\|^2_{L^2(U;L^2_\gamma)}) \geqslant \frac{1}{C}\|f\|^2_{L^2(V_t;L^2_\gamma)}.$$

Using (6-25) and (6-27), we deduce

$$-(\|f(t+1,\cdot)\|^2_{L^2(U;L^2_\gamma)} - \|f(t,\cdot)\|^2_{L^2(U;L^2_\gamma)}) \geqslant \frac{1}{C}\|f(t+1,\cdot)\|^2_{L^2(U;L^2_\gamma)}.$$

This implies exponential decay of the mapping $t \mapsto \|f(t,\cdot)\|_{L^2(U;L^2_\gamma)}$ along integer values of t, and we then obtain the conclusion of the proposition by using (6-25) once more. □

6F. *Enhancement.* Finally, we prove Theorem 1.7. Recall that f is assumed to be a solution to

$$\partial_t f + v \cdot \nabla_x f = \varepsilon(\Delta_v f - v \cdot \nabla_v f) \quad \text{in } (0,\infty) \times \mathbb{T}^d \times \mathbb{R}^d. \tag{6-28}$$

Proof of Theorem 1.7. After multiplying (6-28) by f and integrating over $(0, \varepsilon^{-1/3}) \times \mathbb{T}^d \times \mathbb{R}^d$, we obtain the a priori estimates

$$\begin{aligned} \varepsilon\|\nabla_v f\|^2_{L^2((0,\varepsilon^{-1/3})\times\mathbb{T}^d\times\mathbb{R}^d)} &\leqslant \|f_{\text{in}}\|^2_{L^2(\mathbb{T}^d;L^2_\gamma)} - \|f(\varepsilon^{-1/3},\cdot,\cdot)\|^2_{L^2(\mathbb{T}^d;L^2_\gamma)}, \\ \varepsilon^{-1}\|\partial_t f + v \cdot \nabla_x f\|^2_{L^2((0,\varepsilon^{-1/3})\times\mathbb{T}^d(H_\gamma^{-1}))} &\lesssim \|f_{\text{in}}\|^2_{L^2(\mathbb{T}^d;L^2_\gamma)} - \|f(\varepsilon^{-1/3},\cdot,\cdot)\|^2_{L^2(\mathbb{T}^d;L^2_\gamma)}. \end{aligned}$$

Applying the inequality in (6-15) from Proposition 6.6, which is justified since $\langle \partial_t f + v \cdot \nabla_x f \rangle_\gamma = \varepsilon \langle \Delta_v f - v \cdot \nabla_v f \rangle_\gamma \equiv 0$, we obtain

$$\|f\|^2_{Q^{1/3}_{\nabla_x}} \lesssim \varepsilon \|\nabla_v f\|^2_{L^2((0,\varepsilon^{-1/3})\times \mathbb{T}^d; L^2_\gamma)} \lesssim \|f_{\mathrm{in}}\|^2_{L^2(\mathbb{T}^d; L^2_\gamma)} - \|f(\varepsilon^{-1/3}, \cdot, \cdot)\|^2_{L^2(\mathbb{T}^d; L^2_\gamma)}.$$

From (6-17) and the observation that the mean-zero in x condition from (1-26) is propagated forward in time, we then obtain

$$\begin{aligned} \|f\|^2_{L^2((0,\varepsilon^{-1/3})\times \mathbb{T}^d; L^2_\gamma)} &\lesssim \varepsilon^{-1/3} \|f\|^2_{Q^{1/3}_{\nabla_x}} \\ &\lesssim \varepsilon^{2/3} \|\nabla_v f\|^2_{L^2((0,\varepsilon^{-1/3})\times \mathbb{T}^d; L^2_\gamma)} \\ &\lesssim \varepsilon^{-1/3} (\|f_{\mathrm{in}}\|^2_{L^2(\mathbb{T}^d; L^2_\gamma)} - \|f(\varepsilon^{-1/3}, \cdot, \cdot)\|^2_{L^2(\mathbb{T}^d; L^2_\gamma)}). \end{aligned}$$

Translating in time and iterating this procedure yields exponential decay with rate $\exp(-c\varepsilon^{-1/3} t)$ along integer multiples of $\varepsilon^{-1/3}$, similarly to the proof of Proposition 6.13. Applying (6-25), which holds as well for solutions to (6-28), we obtain (1-27). □

Remark 6.14. In principle, one can also incorporate a conservative $\boldsymbol{b}$ satisfying Assumption 1.1 into the enhancement estimate, since $[\boldsymbol{b}(x) \cdot \nabla_v, \partial_{v_i}] = 0$ for all $i = 1, \dots, d$.

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IMPROVED ENDPOINT BOUNDS FOR THE LACUNARY SPHERICAL MAXIMAL OPERATOR

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We prove new endpoint bounds for the lacunary spherical maximal operator and as a consequence obtain almost everywhere pointwise convergence of lacunary spherical means for functions locally in $L\log\log\log L(\log\log\log\log L)^{1+\epsilon}$ for any $\epsilon > 0$.

1. Introduction

Let $d \geq 2$ be a fixed dimension; all constants in this paper are allowed to depend on d. We use the asymptotic notation $X \lesssim Y$, $Y \gtrsim X$, or $X = O(Y)$ to denote the estimate $|X| \leq CY$ for a constant C that can depend on d, and $X \approx Y$ for $X \lesssim Y \lesssim X$.

Define the lacunary spherical maximal operator $\mathcal{M}$ by

$$\mathcal{M} f(x) := \sup_{k\in\mathbb{Z}} |f * \sigma_k(x)|,$$

where σ_k denotes the (L^1-normalized) surface measure on the $(d-1)$-sphere of radius 2^k centered at the origin.

Throughout this paper, $\log = \log_2$ denotes the logarithm to base 2, and we define the iterated logarithms

$$\operatorname{Log}(t) := \log(100+t), \quad \operatorname{Log}_3(t) := \operatorname{Log}\operatorname{Log}\operatorname{Log} t,$$
$$\operatorname{Log}_2(t) := \operatorname{Log}\operatorname{Log} t, \qquad \operatorname{Log}_4(t) := \operatorname{Log}\operatorname{Log}\operatorname{Log}\operatorname{Log} t.$$

It was shown by C. Calderón [1979] and Coifman and Weiss [1978] that $\mathcal{M}$ extends to a bounded operator on $L^p(\mathbb{R}^d)$ for $p > 1$, which implies almost everywhere pointwise convergence of lacunary spherical means for functions in $L^p(\mathbb{R}^d)$ for $p > 1$. An alternate proof of this result was later given in [Duoandikoetxea and Rubio de Francia 1986]. It has remained open, however, as to whether $\mathcal{M}$ is weak-type (1, 1), or equivalently, whether almost everywhere pointwise convergence of lacunary spherical means holds for functions in $L^1(\mathbb{R}^d)$.

Christ and Stein [1987] showed using an extrapolation argument that $\mathcal{M} f \in L^{1,\infty}(\mathbb{R}^d)$ for functions f on $\mathbb{R}^d$ supported in a cube Q satisfying $f \in L\operatorname{Log} L(Q)$. Christ [1988] also proved that $\mathcal{M}$ maps the Hardy space $H^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$. More recently, Seeger, Tao, and Wright [Seeger et al. 2003; 2004] showed that $\mathcal{M}$ maps the space $L \operatorname{Log}_2 L(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$. In this paper we prove that $\mathcal{M}$ maps all characteristic functions in $L \operatorname{Log}_3 L(\mathbb{R}^d)$ boundedly to $L^{1,\infty}(\mathbb{R}^d)$, and more generally maps the entire

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space $L\ \mathrm{Log}_3\, L\, \mathrm{Log}_4^{1+\epsilon}\, L(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ for every $\epsilon > 0$, thus obtaining almost everywhere pointwise convergence of lacunary spherical means for functions locally in $L\ \mathrm{Log}_3\, L\, \mathrm{Log}_4^{1+\epsilon}\, L(\mathbb{R}^d)$.

Proposition 1.1 (lacunary spherical maximal inequality for indicator functions). *For all measurable indicator functions $f = \chi_E$ and all $\alpha > 0$ we have*

$$|\{\mathcal{M} f > \alpha\}| \lesssim \frac{1}{\alpha} \int |f(x)|\, \mathrm{Log}_3 \frac{|f(x)|}{\alpha}\, dx. \tag{1-1}$$

Here and in the sequel we use $|E|$ to denote the Lebesgue measure of a subset E of $\mathbb{R}^d$.

Proposition 1.2 (lacunary spherical maximal inequality for arbitrary functions). *For every $\epsilon > 0$, all measurable f and all $\alpha > 0$ we have*

$$|\{\mathcal{M} f > \alpha\}| \le C_\epsilon \frac{1}{\alpha} \int |f(x)|\, \mathrm{Log}_3 \frac{|f(x)|}{\alpha}\, \mathrm{Log}_4^{1+\epsilon} \frac{|f(x)|}{\alpha}\, dx \tag{1-2}$$

for some constant C_ϵ depending only on ϵ.

By the usual limiting and truncation arguments we obtain the following corollary.

Theorem 1.3 (almost everywhere convergence of lacunary spherical means). *Let $\epsilon > 0$, and let f be locally in $L\ \mathrm{Log}_3\, L\ \mathrm{Log}_4^{1+\epsilon}\, L(\mathbb{R}^d)$. Then*

$$f * \sigma_k(x) \to f(x)$$

for almost every $x \in \mathbb{R}^d$.

Before we proceed with the proofs, we briefly outline the argument. In [Seeger et al. 2003], the restricted version of the argument relied crucially on a decomposition of the function $f = \chi_E$ on Whitney cubes into characteristic functions of sets called “generalized boxes”, which had properties called “length” and “thickness”. As the name suggests, in two dimensions such sets are a generalization of rectangular boxes, for which the length and thickness correspond to the long and short sides respectively of the rectangle. In the case of two dimensions, convolution of a rectangular box with the measure σ_k has measure equal to 2^k times the length of the box. Similarly, the length of a generalized box determines for how many scales k one may throw away the support of σ_k convolved with the characteristic function of the generalized box. Conversely, the thickness of the box determines what L^2 estimates one may obtain for σ_k convolved with the characteristic function of the generalized box.

The argument of [Seeger et al. 2003] proceeded by combining standard Calderón–Zygmund techniques along with this decomposition of E into generalized boxes on Whitney cubes, and by leveraging L^2 and exceptional set size estimates via the properties of length and thickness for each generalized box. Our argument will also make use of a similar decomposition, but there will be many new ingredients involved, and in general our argument will more closely use the geometry of the sphere.

For example, we exploit the geometry of caps on spheres to introduce a fairly involved algorithm for defining exceptional sets, and we throw away more exceptional sets than in [Seeger et al. 2003]. These exceptional sets are defined by covering $\mathbb{R}^d$ with collections of rotated grids of rectangles,[1] where the

[1] In higher dimensions $d > 2$, by a “rectangle” we refer to a rotated box of some dimensions $c \times \cdots \times c \times c'$, with one side c' shorter than the other sides c.

dimensions of the rectangles are determined by an iterative relationship between the dimensions of a given generalized box and the cap structure of the spherical measure. On fixing a particular direction in S^{d-1} which determines the orientation of the rectangular grids to be considered, we then subdivide the generalized box into rectangular pieces where the generalized box has sufficiently high "mass", and throw away as an exceptional set the sumset of this rectangular box and a piece of the cap on the sphere with normals pointing in similar directions as the short side of the rectangular box, so that such a set is contained in a translation of the fattening of the spherical cap by an amount comparable to the short side of the rectangular box.

We then decompose the kernel $\sigma_k * \sigma_k$ into linear combinations of characteristic functions of rectangles with dimensions corresponding to the caps that appear in our algorithm for defining exceptional sets. The L^2 estimates for each such piece of the kernel convolved with a given rectangular piece in a grid with similar orientation is determined by the mass of the generalized box on that rectangular piece. There are essentially double-logarithmically (in the relevant parameter) many such different sizes of caps that appear, which alone would lead to the desired L^2 estimates with an additional double-logarithmic factor. However, we are able to throw away triple-logarithmically many "intermediate scales"; that is, we may sum in L^1 the convolutions of characteristic functions of parts of the generalized boxes with intermediate masses with the associated cap measures. After doing so, we improve the L^2 estimates for the remaining "light scales" by the needed double-logarithmic factor, and also improve the support size estimates for the remaining "heavy scales" by a double-logarithmic factor.

2. Preliminary reductions

In Calderón–Zygmund theory, weak-type estimates are often established by a combination of L^1 and L^2 estimates outside of an exceptional set, and our arguments will be no exception to this strategy. It is convenient to introduce some notation to abstract this strategy.

Definition 2.1 (Calderón–Zygmund control). Let $\alpha, V > 0$. A *Calderón–Zygmund term* of threshold α and measure V is a measurable function $F : \mathbb{R}^d \to \mathbb{R}$ of one of the following types:

(type L^0) F is a function supported on a set of measure $O(V)$.

(type L^1) F is an L^1 function with $\|F\|_1 \lesssim \alpha V$.

(type L^2) F is an L^2 function with $\|F\|_2^2 \lesssim \alpha^2 V$.

Here and in the sequel we use $\|\cdot\|_p$ to denote the usual $L^p(\mathbb{R}^d)$ norms. A function F is *Calderón–Zygmund controlled* with threshold α and measure V if $|F|$ can be pointwise dominated by a sum of boundedly many Calderón–Zygmund terms $F_1, \ldots, F_n$, with $n = O(1)$, and each F_i a Calderón–Zygmund term (of type L^0, L^1, or L^2) of threshold α and measure V.

A model example of a Calderón–Zygmund controlled term to keep in mind is a simple function $\alpha\chi_E$, where E is of measure $O(V)$. The type-L^1 terms are in fact redundant as they can be easily split into the sum of a type-L^0 and a type-L^2 term, but we find it conceptually convenient to retain this intermediate term for our arguments.

We record some convenient properties of Calderón–Zygmund controlled functions:

Lemma 2.2 (basic properties of Calderón–Zygmund controlled functions). *Let* $\alpha > 0$.

(a) (*Chebyshev inequality*) *If* F *is* ***Calderón–Zygmund controlled*** *with threshold* α *and some measure* $V > 0$, *then* $|\{|F| > \alpha\}| \lesssim V$.

(b) (*triangle inequality for bounded sums*) *If* F, F_1, F_2 *obey the bounds* $|F| \lesssim |F_1| + |F_2|$ *and* F_1, F_2 *are Calderón–Zygmund controlled with threshold* α *and some measure* $V_1, V_2 > 0$ *respectively, then* F *is Calderón–Zygmund controlled with threshold* α *and measure* $V_1 + V_2$.

(c) (*triangle inequality for square functions and unbounded sums*) *If* $(F_q)_{q\in\mathfrak{Q}}$ *is a collection of functions, with each* F_q ***Calderón–Zygmund controlled*** *with threshold* α *and some measure* $V_q > 0$, *then the square function* $\big(\sum_{q\in\mathfrak{Q}} |F_q|^2\big)^{1/2}$ *is Calderón–Zygmund controlled with threshold* α *and measure* $\sum_{q\in\mathfrak{Q}} V_q$. *If the* F_q *are Calderón–Zygmund terms of type* L^0 *or* L^1 *of the threshold* α *and measure* V_q, *then* $\sum_{q\in\mathfrak{Q}} F_q$ *is also Calderón–Zygmund controlled at threshold* α *and measure* $\sum_{q\in\mathfrak{Q}} V_q$; *but if the* F_q *were instead* L^2 *terms of threshold* α *and measure* V_q, *then* $\sum_{q\in\mathfrak{Q}} F_q$ *can only be said to be an* L^2 *term of threshold* α *and measure* $\big(\sum_{q\in\mathfrak{Q}} V_q^{1/2}\big)^2$.

Proof. For (a), we bound $|F| \le F_1 + \cdots + F_n$ by the sum of Calderón–Zygmund terms F_i of threshold α and measure V. By Chebyshev's inequality (in the type-L^1 and type-L^2 cases) we have

$$\{F_i > \alpha/n\} \lesssim V$$

for all $i = 1, \dots, n$; summing, we obtain the claim.

The claim (b) is immediate from the triangle inequality, as is (c), after using the trivial bound $\big(\sum_{q\in\mathfrak{Q}} |F_q|^2\big)^{1/2} \le \sum_{q\in\mathfrak{Q}} |F_q|$ to handle type-L^1 terms arising from the square function. □

Most of this paper will be devoted to the proof of the following variant of Propositions 1.1 and 1.2. Call a function $f : \mathbb{R}^d \to \mathbb{R}$ *granular* if it is a finite linear combination of indicator functions of dyadic cubes. As in [Seeger et al. 2003], it is convenient for minor technical reasons to restrict attention to granular functions.

Proposition 2.3 (bounding the lacunary spherical maximal function). *Let* $0 \le \alpha \le 1$, *and let* f *be a granular function taking values in* $[0, 1]$. *Then* $\mathcal{M} f$ *is Calderón–Zygmund controlled with threshold* α *and measure* $(\mathrm{Log}_3(1/\alpha)/\alpha)\|f\|_1$.

By Lemma 2.2(a) we see that the conclusion of Proposition 2.3 implies the bound

$$|\{x \in \mathbb{R}^d : \mathcal{M} f(x) > \alpha\}| \lesssim \frac{\mathrm{Log}_3(1/\alpha)}{\alpha} \|f\|_1 \tag{2-1}$$

for granular f taking values in $[0, 1]$; the granularity hypothesis can then be removed by a standard limiting argument. It is then clear that Proposition 2.3 implies Proposition 1.1 as a special case. Let us now also see why it implies Proposition 1.2:

Proof of Proposition 1.2. Without loss of generality we may assume that f is nonnegative. Since we have the pointwise estimate $\mathcal{M}f \le \mathcal{M}(f\chi_{f\ge\alpha/2})+\alpha/2$, we may assume without loss of generality (after replacing α with $\alpha/2$) that $f(x)\ge\alpha$ for all x in the support of f. We then have the pointwise bound

$$\mathcal{M}f(x) \le \sum_{k=1}^{\infty} \mathcal{M}(f\chi_{\mathrm{Log}_3(f/\alpha)\approx 2^k})$$

and hence

$$|\{\mathcal{M}f>\alpha\}| \le \sum_{k=1}^{\infty} \left|\left\{\mathcal{M}(f\chi_{\mathrm{Log}_3(f/\alpha)\approx 2^k})(x) > \frac{\alpha}{C_\epsilon k^{1+\epsilon}}\right\}\right|$$

for a sufficiently large constant C_ϵ. By (2-1) and a simple rescaling we have

$$\left|\left\{\mathcal{M}(f\chi_{\mathrm{Log}_3(f/\alpha)\approx 2^k}) > \frac{\alpha}{C_\epsilon k^{1+\epsilon}}\right\}\right| \le C'_\epsilon (k^{1+\epsilon}2^k/\alpha)\int_{\mathbb{R}^d} f(x)\chi_{\mathrm{Log}_3(f(x)/\alpha)\approx 2^k}\,dx$$

for some quantity C'_ϵ depending only on ϵ. Summing in k, we obtain the claim. □

It remains to establish Proposition 2.3.

Reductions using Calderón–Zygmund theory. Similarly to [Seeger et al. 2003], we first make some standard reductions using Calderón–Zygmund theory. By the L^2 boundedness of the lacunary spherical maximal function, any expression of the form Mg with $\|g\|_2^2 \lesssim \alpha\|f\|_1$ will be a Calderón–Zygmund term of type L^2, threshold α, and measure $(1/\alpha)\|f\|_1$, and can thus be neglected. In particular, if we define the standard Calderón–Zygmund exceptional set

$$\Omega := \{M_{HL}(f)\ge\alpha\},$$

where M_{HL} is the Hardy–Littlewood maximal operator, then f is bounded almost everywhere outside of Ω by α, so in particular

$$\|f\chi_{\mathbb{R}^d\setminus\Omega}\|_2^2 \le \alpha\|f\|_1.$$

Thus the $\mathcal{M}(f\chi_{\mathbb{R}^d\setminus\Omega})$ gives a negligible contribution, and it suffices (by Lemma 2.2(b)) to control the contribution $\mathcal{M}(f\chi_\Omega)$ arising from the set Ω.

By the Whitney decomposition, we may partition Ω (up to null sets) by a family $\mathfrak{Q}$ of essentially disjoint cubes q on which $\int_q f \lesssim \alpha|q|$; setting $f_q := f\chi_q$, each f_q is granular, and we conclude that $f\chi_\Omega = \sum_{q\in\mathfrak{Q}} f_q$ almost everywhere, and by the Hardy–Littlewood maximal inequality one has $\sum_q |q| \lesssim (1/\alpha)\|f\|_1$. By arguing as in the start of [Seeger et al. 2003, §3] we may partition $\mathfrak{Q}$ into a bounded number of families $\mathfrak{Q}_i$ such that the cubes q in each $\mathfrak{Q}_i$ have their doubles $2q$ pairwise disjoint. By the triangle inequality (Lemma 2.2(b)), it suffices to show that for each i, the expression $\mathcal{M}\sum_{q\in\mathfrak{Q}_i} f_q$ is Calderón–Zygmund controlled at threshold α and measure $\mathrm{Log}_3(1/\alpha)\cdot\sum_{q\in\mathfrak{Q}_i}|q|$.

Henceforth we fix i and omit the constraint $q\in\mathfrak{Q}_i$ from summations for sake of brevity. Following [Seeger et al. 2003], we now introduce some cancellation, by defining the projection operator Π_q to be the projection operator onto a certain space of polynomials. That is, let $\{P_j\}_{j=1}^{L}$ be an orthonormal basis for the space of polynomials of degree $\leq 100d$ on the unit cube $\left[-\frac{1}{2}, \frac{1}{2}\right]^d$. If q is a cube with center x_q and sidelength $l(q)$, define

$$\Pi_q[h](x) := \chi_q(x) \sum_{j=1}^{L} P_j\left(\frac{x - x_q}{l(q)}\right) \int_q h(y) P_j\left(\frac{y - x_q}{l(q)}\right) \frac{dy}{l(q)^d}.$$

Introduce the "bad functions"

$$b_q := f_q - \Pi_q[f_q].$$

Since

$$|\Pi_q[f_q](x)| \lesssim \alpha \chi_q,$$

we have

$$\left\| \sum_q \Pi_q[f_q] \right\|_2^2 \lesssim \alpha^2 \sum_q |q|,$$

and so by the L^2-boundedness of the lacunary spherical maximal operator, the contribution of $\mathcal{M} \sum_q \Pi_q[f_q]$ is a type-L^2 Calderón–Zygmund term of threshold α and measure $\sum_q |q|$. Thus it remains to obtain Calderón–Zygmund control on the contribution

$$\sup_k \left| \sum_q b_q * \sigma_k \right|$$

of the bad functions b_q.

The contributions of those k with $2^k \leq l(q)$ are contained in $\bigcup_q 3q$, and are thus acceptable Calderón–Zygmund terms of type L^0. For the remaining k, we will replace the sup by an ℓ^2 norm; thus we will show that

$$\left(\sum_k \left| \sum_{q:\, 2^k > l(q)} b_q * \sigma_k \right|^2 \right)^{1/2}$$

is Calderón–Zygmund controlled at threshold α and measure $\sum_{q \in \mathfrak{Q}_i} |q|$. Expanding out the square and using the triangle inequality, it suffices to establish this claim for the diagonal contribution

$$\left(\sum_q \sum_{k:\, 2^k > l(q)} |b_q * \sigma_k|^2 \right)^{1/2} \tag{2-2}$$

and for the off-diagonal contribution

$$\left(\sum_k \sum_{q \neq q':\, 2^k > l(q), l(q')} (b_q * \sigma_k)(\overline{b_{q'} * \sigma_k}) \right)^{1/2}. \tag{2-3}$$

The off-diagonal expression (2-3) can be handled by existing arguments. Indeed, since

$$\left\| \left(\sum_k \sum_{q \neq q':\, 2^k > l(q), l(q')} (b_q * \sigma_k)(\overline{b_{q'} * \sigma_k}) \right)^{1/2} \right\|_2^2 \leq \sum_k \sum_{q \neq q'} |\langle b_q * \sigma_k, b_{q'} * \sigma_k \rangle|,$$

it would suffice (by the definition of a type-L^2 term) to show that

$$\sum_k \sum_{q \neq q'} |\langle b_q * \sigma_k, b_{q'} * \sigma_k \rangle| \lesssim \alpha^2 \sum_q |q|,$$

which can be proven as in [Seeger et al. 2003, (4.17)] by exploiting the smoothness of the kernel $\sigma_k * \sigma_k$ and the cancellation of b_q (and the hypothesis that the doubles $2q$ of $q \in \mathfrak{Q}_i$ are disjoint). Since the proof is nearly identical to that given in [Seeger et al. 2003], we omit it here.

It remains to control the diagonal contribution (2-2). It is easy to see that

$$\sum_q \sum_{k:\,2^k > l(q)} \|\Pi_q[f_q] * \sigma_k\|_2^2 \lesssim \sum_q \sum_{k:\,2^k > l(q)} \|\alpha \chi_q * \sigma_k\|_2^2 \lesssim \sum_q \sum_{k:\,2^k > l(q)} \alpha^2 2^{-k(d-1)} \ell(q)^{2d-1} \lesssim \alpha^2 \sum_q |q|.$$

Thus $\bigl(\sum_q \sum_{k:2^k > l(q)} |\Pi_q[f_q] * \sigma_k|^2\bigr)^{1/2}$ is a type-L^2 term of the required threshold and measure, and so by the triangle inequality we may replace (2-2) with

$$\Bigl(\sum_q \sum_{k:2^k > l(q)} |f_q * \sigma_k|^2\Bigr)^{1/2}.$$

To show that this expression is Calderón–Zygmund controlled at threshold α and measure $\sum_q |q|$, it suffices by Lemma 2.2 to show that the inner square functions $\bigl(\sum_{k:2^k > l(q)} |f_q * \sigma_k|^2\bigr)^{1/2}$ are Calderón–Zygmund controlled at threshold α and measure $|q|$ for each cube q. A simple scaling argument shows that we may then normalize q to be a unit cube. We have thus reduced matters to establishing:

Proposition 2.4 (bounding the lacunary spherical maximal function of f_q). *Let* $0 < \alpha < 1$, *and let* f_q *be a granular function supported on a unit cube q taking values in* $[0, 1]$ *with* $\int_q f_q \lesssim \alpha$. *Then the expression*

$$\Bigl(\sum_{k>0} |f_q * \sigma_k|^2\Bigr)^{1/2} \tag{2-4}$$

is Calderón–Zygmund controlled at threshold α *and measure* $\mathrm{Log}_3(1/\alpha)$.

The remainder of this paper will be devoted to the proof of this proposition.

Structural decomposition of f_q. Let q, f_q, α be as in Proposition 2.4. In [Seeger et al. 2003], the support of f_q was decomposed into structures referred to as "generalized boxes", which behaved in a certain way like 1-dimensional sets and which had associated quantities referred to as "length" and "thickness", the former which governed support size estimates and the latter which controlled L^2 bounds. We describe a decomposition of f_q that is in a similar spirit.

Lemma 2.5 (structural decomposition lemma). *Let* q, f_q, α *be as in Proposition 2.4. List the dyadic numbers between* α^2 *and* 1 *in increasing order as*

$$\alpha^2 < \gamma_0 < \gamma_1 < \cdots < \gamma_J = 1;$$

thus $J \approx \mathrm{Log}(1/\alpha)$. *Then we can take the decomposition*

$$f_q = \sum_{j=0}^{J} f_q^{\gamma_j} \tag{2-5}$$

such that, for each j, $f_q^{\gamma_j}$ *is a granular function taking values in* $[0, 1]$ *that is supported on a finite union of cubes* Q *in* q *whose total "length"* $\lambda(f_q^{\gamma_j}) := \sum_Q l(Q)$ *obeys the estimates*

$$\int f_q^{\gamma_j} \approx \gamma_j \cdot \lambda(f_q^{\gamma_j}) \tag{2-6}$$

if $j > 0$, *with just the upper bound*

$$\int f_q^{\gamma_j} \lesssim \gamma_j \cdot \lambda(f_q^{\gamma_j}) \tag{2-7}$$

for $j = 0$. *Furthermore, one has*

$$\int_Q f_q^{\gamma_j} \lesssim \gamma_j l(Q) \tag{2-8}$$

for every cube Q. *We refer to* γ_j *as the critical density of* $f_q^{\gamma_j}$.

We will use the decomposition (2-5) in an essential way throughout the rest of the paper, as well as the key properties (2-6), (2-7) and (2-8). In [Seeger et al. 2003], the analog of (2-7) is that for every generalized box B of thickness γ and length λ, we have $|B| \lesssim \gamma \cdot \lambda$.

Proof. We perform a greedy algorithm, extracting the "heaviest" cubes first. Given a (nonnegative) function f and a cube Q, we define the *weight*

$$\mathrm{wt}_Q[f] := \frac{1}{l(Q)} \int_Q f.$$

The symbol Q will always be understood to be a dyadic cube. We then inductively define

$$E_q^{\gamma_J} := \bigcup_{Q \subset q:\, \mathrm{wt}_Q[f_q] \geq \gamma_J} Q;$$

note that from the trivial bound $\mathrm{wt}_Q[f_q] \leq \mathrm{wt}_Q[\chi_q]$ there are only finitely many cubes Q that can contribute here. For $1 < j < J$, we define

$$E_q^{\gamma_j} := \bigcup_{Q \subset q:\, \mathrm{wt}_Q[f_q \chi_{q \setminus \bigcup_{l>j} E_q^{\gamma_l}}] \geq \gamma_j} Q; \tag{2-9}$$

again, this is a finite union of dyadic cubes. Set

$$E_q^{\gamma_0} := q \setminus \Bigl(\bigcup_{1 \leq j \leq N} E_q^{\gamma_j} \Bigr).$$

If we then set

$$f_q^{\gamma_j} := f_q \chi_{E_q^{\gamma_j} \setminus \bigcup_{j<l\leq J} E_q^{\gamma_l}},$$

we obtain (2-5), and the f_q are clearly granular. For $j > 0$, let $\mathcal{Q}_j$ be a maximal cover of $E_q^{\gamma_j}$ by dyadic cubes $Q \subset q$ obeying the stated condition

$$\mathrm{wt}_Q[f_q \chi_{q \setminus \bigcup_{l>j} E_q^{\gamma_l}}] \geq \gamma_j,$$

and set

$$\lambda(f_q^{\gamma_j}) := \sum_{Q \in \mathcal{Q}_j} l(Q). \tag{2-10}$$

Then $f_q^{\gamma_j}$ is supported on $\bigcup_{Q\in\mathcal{Q}_j} Q$, and the required claims (2-6), (2-8) follow from the construction of the $f_q^{\gamma_j}$ and (2-9) (and the upper bound $f_q \le \chi_q$, in the $j = J$ case). For $j = 0$, we still have (2-8), and the claim (2-7) follows by taking $\mathcal{Q}_0$ to consist just of the unit cube q. □

It remains to show that the expression

$$\Bigl(\sum_{k>0}\Bigl|\sum_{\gamma} f_q^{\gamma} * \sigma_k\Bigr|^2\Bigr)^{1/2} \tag{2-11}$$

is Calderón–Zygmund controlled of threshold α and measure $\operatorname{Log}_3(1/\alpha)$, where γ is implicitly restricted to $\gamma_0, \dots, \gamma_J$.

Further reductions. We record the basic L^0, L^1, L^2 estimates on $f_q^{\gamma} * \sigma_k$ (which were already implicit in [Seeger et al. 2003]):

Lemma 2.6 (L^0, L^1, L^2 estimates). *Let $k > 0$ and $\gamma \ge \gamma_0$.*

(L^0) *$f_q^{\gamma} * \sigma_k$ is a type-L^0 Calderón–Zygmund term of threshold α and measure $2^{k(d-1)}\lambda(f_q^{\gamma})$.*

(L^1) *$f_q^{\gamma} * \sigma_k$ is a type-L^1 Calderón–Zygmund term of threshold α and measure $(1/\alpha)\|f_q^{\gamma}\|_1$.*

(L^2) *$f_q^{\gamma} * \sigma_k$ is a type-L^2 Calderón–Zygmund term of threshold α and measure*

$$\frac{2^{-k(d-1)}\gamma}{\alpha^2}\operatorname{Log}\frac{2^{k(d-1)}}{\gamma}\cdot\|f_q^{\gamma}\|_1.$$

Proof. For the L^0 estimate, we decompose f_q^{γ} into functions $f_q^{\gamma}\chi_Q$ supported on cubes Q with $\sum_Q l(Q) = \lambda(f_q^{\gamma})$. A geometric calculation shows that $f_q^{\gamma}\chi_Q * \sigma_k$ is supported on an annular region of measure $O(2^{k(d-1)}l(Q))$, and the claim follows by summing in Q.

The L^1 estimate is immediate from Young's inequality, so we turn to the L^2 estimate. Using the well-known pointwise estimate

$$\sigma_k * \sigma_k(x) \lesssim \frac{2^{-k(d-1)}}{|x|}\chi_{|x|\le 2^{k+1}}, \tag{2-12}$$

we may expand

$$\begin{aligned}\|f_q^{\gamma} * \sigma_k\|_2^2 &= \langle f_q^{\gamma}, \sigma_k * \sigma_k * f_q^{\gamma}\rangle\\ &\lesssim 2^{-k(d-1)}\int f_q^{\gamma}(x)\Bigl(\sum_{l\le k+1} 2^{-l}\int_{y:\,|x-y|\approx 2^l} f_q^{\gamma}(y)\,dy\Bigr)\,dx\\ &\lesssim 2^{-k(d-1)}\|f_q^{\gamma}\|_1 \sup_x \sum_{l\le k+1} 2^{-l}\int_{y:\,|x-y|\approx 2^l} f_q^{\gamma}(y)\,dy.\end{aligned}$$

From (2-8) and the pointwise bound $f_q^{\gamma} \le f \le 1$ we have

$$\int_{y:|x-y|\approx 2^l} f_q^{\gamma}(y)\,dy \lesssim \min(\gamma 2^l, 2^{dl}) \tag{2-13}$$

and thus

$$\|f_q^\gamma * \sigma_k\|_2^2 \lesssim 2^{-k(d-1)} \|f_q^\gamma\|_1 \sum_{l\le k+1} \min(\gamma, 2^{(d-1)l}).$$

The summand is equal to γ for $O(\mathrm{Log}(2^{k(d-1)}/\gamma))$ terms, and decays geometrically otherwise, giving the claim. □

From the L^2 case of this lemma we see that $f_q^{\gamma_0} * \sigma_k$ is a type-L^2 Calderón–Zygmund term of threshold α and measure

$$\alpha^{-2} 2^{-k(d-1)} \gamma_0 \,\mathrm{Log}\, \frac{2^{k(d-1)}}{\gamma_0} \cdot \|f_q^{\gamma_0}\|_1 \lesssim 2^{-k(d-1)} \alpha \,\mathrm{Log}\, \frac{2^{k(d-1)}}{\alpha^2}$$

since $\gamma_0 \approx \alpha^2$ and $\|f_q^{\gamma_0}\|_1 \le \|f_q\|_1 \lesssim \alpha$. Summing over all positive k using Lemma 2.2(c), we conclude that

$$\Big(\sum_{k>0} |f_q^{\gamma_0} * \sigma_k|^2\Big)^{1/2}$$

is Calderón–Zygmund controlled of threshold α and measure $O(\alpha\,\mathrm{Log}(1/\alpha))$, which is acceptable. Thus we may delete the γ_0 term from (2-11) and focus attention on

$$\Big(\sum_{k>0} \Big| \sum_{\gamma>\gamma_0} f_q^\gamma * \sigma_k \Big|^2\Big)^{1/2}. \tag{2-14}$$

From the L^0 case of this lemma and Lemma 2.2(c), followed by (2-6), we see that

$$\Big(\sum_{k>0} \Big| \sum_{\gamma>\gamma_0:\, k(d-1)<\log(\gamma/\alpha)} f_q^\gamma * \sigma_k \Big|^2\Big)^{1/2}$$

is a type-L^0 Calderón–Zygmund term of threshold α and measure

$$\sum_{k>0} \sum_{\gamma>\gamma_0:\, k(d-1)<\log(\gamma/\alpha)} 2^{k(d-1)} \lambda(f_q^\gamma) \lesssim \sum_{\gamma>\gamma_0} \frac{\gamma}{\alpha} \lambda(f_q^\gamma) \approx \frac{1}{\alpha} \sum_{\gamma>\gamma_0} \|f_q^\gamma\|_1 \le \frac{\|f_q\|_1}{\alpha} \lesssim 1,$$

giving the claim. Thus the contribution of the "small scales" with $k(d-1) < \log(\gamma/\alpha)$ is acceptable.

Next, we claim that the contribution of the "nearly small scale" case

$$\log \frac{\gamma}{\alpha} \le k(d-1) < \log \frac{\gamma}{\alpha} + 100\,\mathrm{Log}_3 \frac{1}{\alpha} \tag{2-15}$$

is also acceptable. Indeed, from the L^1 case of Lemma 2.6 and Lemma 2.2(c), we see that

$$\Big(\sum_{k>0} \Big| \sum_{\gamma>\gamma_0:\text{(2-15)}} f_q^\gamma * \sigma_k \Big|^2\Big)^{1/2}$$

is a type-L^1 Calderón–Zygmund term of threshold α and measure

$$\sum_{k>0} \sum_{\gamma>\gamma_0:\,\text{(2-15)}} \frac{1}{\alpha} \|f_q^\gamma\|_1 \lesssim \frac{\mathrm{Log}_3(1/\alpha)}{\alpha} \sum_{\gamma>\gamma_0} \|f_q^\gamma\|_1 \le \frac{\mathrm{Log}_3(1/\alpha)}{\alpha} \|f_q\|_1 \lesssim \frac{\mathrm{Log}_3(1/\alpha)}{\alpha}$$

giving the claim.

Furthermore, as in [Seeger et al. 2003], we claim that the contribution of terms $\sigma_k * f_q^\gamma$ in the "large scale" case

$$k(d-1) \le \log \frac{\gamma}{\alpha} + 100 \operatorname{Log}_2 \frac{1}{\alpha} \tag{2-16}$$

is also acceptable (with some room to spare). Indeed, from Cauchy–Schwarz one has

$$\sum_{\gamma > \gamma_0 :\, (2\text{-}16)} f_q^\gamma * \sigma_k \lesssim \Biggl(\sum_{\gamma > \gamma_0 :\, (2\text{-}16)} \Bigl(k(d-1) - \log \frac{\gamma}{\alpha} \Bigr)^2 |f_q^\gamma * \sigma_k|^2 \Biggr)^{1/2}$$

and from this, the L^2 case of Lemma 2.6 and Lemma 2.2(c) we see that

$$\Biggl(\sum_{k>0} \Bigl| \sum_{\gamma > \gamma_0 :\, (2\text{-}16)} f_q^\gamma * \sigma_k \Bigr|^2 \Biggr)^{1/2}$$

is a type-L^2 Calderón–Zygmund term of threshold α and measure

$$\begin{aligned} \sum_{k>0} \sum_{\gamma > \gamma_0 :\, (2\text{-}16)} & \frac{(k(d-1) - \log(\gamma/\alpha))^2}{\alpha^2} 2^{-k(d-1)} \gamma \operatorname{Log} \frac{2^{k(d-1)}}{\gamma} \cdot \|f_q^\gamma\|_1 \\ & \lesssim \sum_{\gamma > \gamma_0} \frac{\operatorname{Log}_2^2(1/\alpha)}{\alpha^2} \Bigl(\frac{\gamma}{\alpha} \operatorname{Log}^{100} \frac{1}{\alpha} \Bigr)^{-1} \gamma \operatorname{Log}_2 \frac{1}{\alpha} \cdot \|f_q^\gamma\|_1 \lesssim \frac{1}{\alpha} \sum_{\gamma > \gamma_0} \|f_q^\gamma\|_1 \le \frac{\|f_q\|_1}{\alpha} \lesssim 1 \end{aligned}$$

as required.

We have now treated all scales k except for those in the "medium-scale" range $\mathcal{K}_\gamma$ defined by

$$\mathcal{K}_\gamma := \Bigl\{ k > 0 : \log \frac{\gamma}{\alpha} + 100 \operatorname{Log}_3 \frac{1}{\alpha} \le k(d-1) \le \log \frac{\gamma}{\alpha} + 100 \operatorname{Log}_2 \frac{1}{\alpha} \Bigr\}. \tag{2-17}$$

We have thus reduced Proposition 2.4 to the following.

Proposition 2.7. *Let* $0 < \alpha < 1$, *and let* f_q *be a granular function on a unit cube* q *taking values in* $[0, 1]$ *with*

$$\int_q f_q \lesssim \alpha. \tag{2-18}$$

Let f_q^γ *be as in Lemma 2.5, and* $\mathcal{K}_\gamma$ *be given by* (2-17). *Then the expression*

$$\Biggl(\sum_{k>0} \Bigl| \sum_{\gamma > \gamma_0 :\, k \in \mathcal{K}_\gamma} f_q^\gamma * \sigma_k \Bigr|^2 \Biggr)^{1/2} \tag{2-19}$$

is Calderón–Zygmund controlled at threshold α *and measure* $\operatorname{Log}_3(1/\alpha)$.

Remark 2.8. If we were willing to replace $\operatorname{Log}_3(1/\alpha)$ by of $\operatorname{Log}_2(1/\alpha)$ in the measure parameter of the conclusion then we could use the previous "nearly small scale" argument to express (2-19) as a type-L^1 term, recovering the results of [Seeger et al. 2003] (with essentially the same proof). The main innovation of this paper is to treat these medium-scale contributions by a more sophisticated argument than this simple L^1 argument, in particular constructing some additional exceptional sets outside of which one can establish good L^2 estimates at "light" scales.

3. Proof of Proposition 2.7

Let α, q, f_q, f_q^γ be as in the above proposition. To prove Proposition 2.7, we will identify an exceptional set to hold the "heavy" terms of type L^0, then split the remaining portions of (2-19) into "intermediate" terms that will be of type L^1, and finally "light" terms that will be of type L^2 outside of the previously identified exceptional set. To construct these terms we need to introduce some additional scales, and identify certain rectangles on which the f_q^γ are unusually "heavy".

Defining double-logarithmically many scales. Let us temporarily fix a critical density γ with $\alpha^2 \le \gamma \le 1$. For each such density we associate a key radius

$$r = r_\gamma := \max(1, (\gamma/\alpha)^{1/(d-1)}) \tag{3-1}$$

and note that the constraint (2-17) ensures that 2^k is a little bit larger than r:

$$2^k \ge r \operatorname{Log}_2^{100/(d-1)} \frac{1}{\alpha} \quad \text{for all } k \in \mathcal{K}_\gamma. \tag{3-2}$$

With the density γ fixed, we identify $O(\operatorname{Log}_2(1/\alpha))$ many natural scales

$$\gamma^{1/(d-1)} = c_0 \le c_1 \le \cdots \le c_N \le r$$

between $\gamma^{1/(d-1)}$ and r in our problem that will lead us to our $L \operatorname{Log}_3 L$ result. They will be defined recursively by initializing

$$c_0 := \gamma^{1/(d-1)} \le 1 \le r$$

and then taking iterated geometric means with r; thus

$$c_i := \sqrt{c_{i-1} r} \tag{3-3}$$

for all $i \ge 1$. More explicitly, we have

$$c_i = (\gamma^{1/(d-1)}/r)^{2^{-i}} r = \max(\gamma^{2^{-i}/(d-1)}, \gamma^{1/(d-1)} \alpha^{-(1-2^{-i})/(d-1)}) \tag{3-4}$$

for all $i \ge 0$. Geometrically, each c_i for $i \ge 1$ arises (up to constants) as the diameter of a spherical cap of thickness c_{i-1} on a sphere of radius r; see Figure 1. These scales are motivated by a decomposition of the kernel $\sigma_k * \sigma_k$ into linear combinations of characteristic functions of rectangles, which will appear later in the paper.

We terminate the sequence of scales c_i at the first $N = N_\gamma$ for which

$$c_N \ge 2^{-10} r.$$

Since $\alpha^2 < \gamma \le 1$, we have $1 \le r/\gamma \le \alpha^{-O(1)}$, and hence $N \lesssim \operatorname{Log}_2(1/\alpha)$.

Throwing away exceptional sets at each scale. We can now define some exceptional sets.

Definition 3.1. Let the parameters k, γ be fixed as above, and define the scales $c_0 \le \cdots \le c_N$ as in the previous section. Let $1 \le i \le N$. Let $M > 0$ be a dyadic number.

(1) Define $\{\Phi_{j,i}\}_{j \in \mathcal{J}_i}$ to be a maximal set of (c_{i-1}/c_i)-separated directions $\Phi_{j,i}$ in S^{d-1}, so that the number of directions is $\approx (c_i/c_{i-1})^{d-1}$.

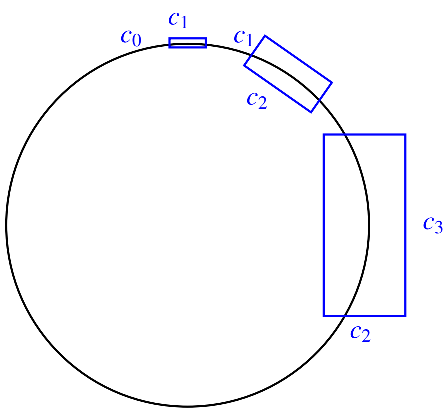

Figure 1. The scales c_i. Here $c_0 = \gamma^{1/(d-1)}$ and the circle has radius 2^k for some k with 2^k somewhat larger than r.

(2) For each direction $\Phi_{j,i}$ in the above set, partition $\mathbb{R}^d$ into $c_i \times c_i \times \cdots \times c_i \times c_{i-1}$ parallel rectangles that belong to some fixed grid, with the short direction parallel to the direction $\Phi_{j,i}$. Define $\mathcal{R}_{j,i,M}$ to be the collection of all such rectangles R such that

$$\int_R |f_q^\gamma| \geq c_{i-1} M \gamma. \tag{3-5}$$

(3) We then define our exceptional set as

$$S_{M,k,q,\gamma} := \bigcup_{i=1}^{N} \bigcup_{j \in \mathcal{J}_i} \operatorname{supp}(\sigma_{k,j,i} * \chi_{\bigcup_{R \in \mathcal{R}_{j,i,M}} 100R}),$$

where $\sigma_{k,j,i}$ is the restriction of σ_k to a spherical cap on the sphere of radius 2^k of angular width $100c_{i-1}/c_i$ centered at $2^k \Phi_{j,i}$.

Observe that the sets $S_{M,k,q,\gamma}$ decrease as M increases. We claim the crucial upper bound on the size of the exceptional sets defined above.

Lemma 3.2 (maximal inequality). *With the notation of Definition 3.1, we have the bound*

$$|S_{M,k,q,\gamma}| \lesssim \frac{2^{k(d-1)}}{M} \lambda(f_q^\gamma).$$

A key point here is that we do not lose a factor of N (which can be as large as $\operatorname{Log}_2(1/\alpha)$) on the right-hand side, by taking advantage of how the rectangles associated to different scales c_i nest within (dilates of) each other. This inequality can be viewed as a complicated variant of the Hardy–Littlewood maximal inequality.

Before we proceed with the proof of Lemma 3.2, we say a few words to motivate the previous definitions. We note that in some sense the support of $\sigma_{k,j,i}$ is "adapted" to translates of rectangles in $\mathcal{R}_{j,i,M}$, in the sense that convolution with characteristic functions of rectangles effectively fattens it by c_{i-1} and translates it. Thus we note that for each $R \in \mathcal{R}_{j,i}$, the set $\operatorname{supp}(\sigma_{k,j,i} * \chi_{100R})$ is contained in a

$1000c_{i-1}$-neighborhood of a translate of (a slightly wider version of) the cap $\mathrm{supp}(\sigma_{k,j,i})$. The rectangles $R \in \mathcal{R}_{j,i,M}$ that are sufficiently "heavy" in the sense of (3-5) correspond to (more or less) poor L^2 estimates for $\sigma_{k,j,i} * \chi_R$, and so we would like to remove the supports of $\sigma_{k,j,i} * \chi_R$. Since the support of this is essentially contained in a c_{i-1}-fattening of the cap $\mathrm{supp}(\sigma_{k,j,i})$, the heavier the rectangles we consider (the larger M is) the fewer number of such rectangles there can be, so the smaller the total size of exceptional sets thrown away. Thus using a pigeonholing argument, we can obtain the bound from Lemma 3.2.

Proof of Lemma 3.2. We need to impose a partial order relation on the directions $\Phi_{j,i}$. For any i, i' with $i > i'$, we will say that a direction $\Phi_{j,i}$ is an *ancestor* of $\Phi_{j',i'}$ and write $(j', i') \prec (j, i)$ if the ball of radius $10c_{i'-1}/c_{i'}$ centered at $\Phi_{j',i'}$ is contained in the ball of radius $10c_{i-1}/c_i$ centered at $\Phi_{j,i}$. This is easily seen to be a partial order, and every $\Phi_{j',i'}$ has at least one ancestor $\Phi_{j,i}$ at generation i.

By definition, $S_{M,k,q,\gamma}$ is contained in a set of the form

$$\bigcup_{1\le i\le N} \bigcup_{j\in\mathcal{J}_i} \bigcup_{R\in\mathcal{R}_{j,i,M}} \mathrm{supp}(\sigma_{k,j,i} * \chi_{1000R}),$$

where each $R \in \mathcal{R}_{j,i,M}$ is a $c_i \times c_i \times \cdots \times c_i \times c_{i-1}$ rectangle with short side pointing in direction $\Phi_{j,i}$ satisfying

$$\int_R f_q^\gamma \ge c_{i-1} M\gamma. \tag{3-6}$$

Moreover, if $(j, i) \prec (j', i')$ then for any $R \in \mathcal{R}_{j,i,M}$ and any $R' \in \mathcal{R}_{j',i',M}$, if $R \cap R' \ne \varnothing$ then $R \subset 100R'$. It follows that we can choose subcollections $\widetilde{\mathcal{R}}_{j,i,M} \subset \mathcal{R}_{j,i,M}$ satisfying

$$S_{M,k,q,\gamma} \subset \bigcup_{1\le i\le N} \bigcup_{j\in\mathcal{J}_i} \bigcup_{R\in\widetilde{\mathcal{R}}_{j,i,M}} \mathrm{supp}(\sigma_{k,j,i} * \chi_{10000R}),$$

so that for given any direction $\Phi_{j_1,1}$, a chain of ancestors

$$\Phi_{j_1,1} \prec \Phi_{j_2,2} \prec \cdots \prec \Phi_{j_N,N} \tag{3-7}$$

satisfies the property that the rectangles in the collections $\widetilde{\mathcal{R}}_{j_i,i,M}$, $1 \le i \le N$, are all pairwise disjoint. Indeed, we can choose $\widetilde{\mathcal{R}}_{j,i,M}$ to be the collection of rectangles $R \in \mathcal{R}_{j,i,M}$ which are maximal in the sense that they do not intersect any $R' \in \mathcal{R}_{j',i',M}$ for some ancestor $(j', i') \succ (j, i)$.

Since $\sigma_{k,j,i} * \chi_{10000R}$ is essentially supported in a c_{i-1}-fattening of (a slightly wider version of) the cap $\mathrm{supp}(\sigma_{k,j,i} * \chi_{10000R})$, the measure of its support is $\lesssim 2^{k(d-1)} c_{i-1}^d / c_i^{d-1}$. We can thus bound

$$|S_{M,k,q,\gamma}| \lesssim \Bigl| \bigcup_{1\le i\le N} \bigcup_{j\in\mathcal{J}_i} \bigcup_{R\in\widetilde{\mathcal{R}}_{j,i,M}} \mathrm{supp}(\sigma_{k,j,i} * \chi_{10000R}) \Bigr| \lesssim \sum_{i=1}^N \sum_{j\in\mathcal{J}_i} 2^{k(d-1)} \frac{c_{i-1}^d}{c_i^{d-1}} \cdot \mathrm{card}(\widetilde{\mathcal{R}}_{j,i,M}).$$

By the disjointness property mentioned above and (3-6), for a chain of ancestors as in (3-7), we have the bound

$$\int f_q^\gamma \ge \sum_{1\le i\le N} \sum_{R\in\widetilde{\mathcal{R}}_{j_i,i,M}} \int_R |f_q^\gamma| \ge \sum_{1\le i\le N} c_{i-1} M\gamma \cdot \mathrm{card}(\widetilde{\mathcal{R}}_{j_i,i,M}) \tag{3-8}$$

and hence by (2-6)

$$\sum_{1\le i\le N} c_{i-1}\,\mathrm{card}(\widetilde{\mathcal{R}}_{j_i,i,M}) \lesssim \frac{\lambda(f_q^\gamma)}{M}.$$

Since each direction $\Phi_{j,i}$ is the ancestor of $\lesssim ((c_{i-1}/c_i)\cdot(c_1/c_0))^{d-1}$ many directions $\Phi_{j',i'}$ with $i'=1$, it follows that

$$\begin{aligned}
|S_{M,k,q,\gamma}| &\lesssim \sum_i\sum_j 2^{k(d-1)}\frac{c_{i-1}^d}{c_i^{d-1}}\cdot\mathrm{card}(\widetilde{\mathcal{R}}_{j,i,M})\\
&\lesssim 2^{k(d-1)}\sum_{j_1}\sum_{j,i:\,(j,i)\succ(j_1,1)} 2^{k(d-1)}\frac{c_{i-1}^d}{c_i^{d-1}}\cdot\left(\frac{c_{i-1}}{c_i}\cdot\frac{c_1}{c_0}\right)^{1-d}\cdot\mathrm{card}(\widetilde{\mathcal{R}}_{j,i,M})\\
&\lesssim 2^{k(d-1)}\sum_{j_1}\sum_{j,i:\,(j,i)\succ(j_1,1)} c_{i-1}\left(\frac{c_0}{c_1}\right)^{d-1}\cdot\mathrm{card}(\widetilde{\mathcal{R}}_{j,i,M})\\
&\lesssim 2^{k(d-1)}\left(\frac{c_0}{c_1}\right)^{d-1}\sum_{j_1}\frac{\lambda(f_q^\gamma)}{M}\lesssim\frac{2^{k(d-1)}}{M}\cdot\lambda(f_q^\gamma).
\end{aligned}$$

□

For a given choice of k and γ, we define the upper height $M_+(k,\gamma)$ by the formula

$$\log_2 M_+(k,\gamma) := \left\lfloor k(d-1)+\log\frac{\alpha}{\gamma}+\mathrm{Log}_3\frac{1}{\alpha}\right\rfloor \tag{3-9}$$

and the lower height $M_-(k,\gamma)$ by the formula

$$\log_2 M_-(k,\gamma) := \left\lfloor k(d-1)+\log\frac{\alpha}{\gamma}-100\,\mathrm{Log}_3\frac{1}{\alpha}\right\rfloor. \tag{3-10}$$

The exceptional set associated to the upper height $M_+(k,\gamma)$ is of acceptable size:

Lemma 3.3. *We have the bound*

$$\left|\bigcup_{k,\gamma} S_{M_+(k,\gamma),k,q,\gamma}\right|\lesssim 1.$$

In particular, any component of (2-19) *that is supported in* $\bigcup_{k,\gamma} S_{M_+(k,\gamma),k,q,\gamma}$ *is a type-*L^0 *Calderón–Zygmund term of threshold* α *and measure* 1.

Proof. By Lemma 3.2, (2-6), and (3-9), we have

$$|S_{M_+(k,\gamma),k,q,\gamma}|\lesssim\frac{2^{k(d-1)}}{M_+(k,\gamma)}\lambda(f_q^\gamma)\lesssim\frac{1}{\alpha\,\mathrm{Log}_2(1/\alpha)}\|f_q^\gamma\|_1.$$

Summing over $k\in\mathcal{K}_\gamma$ using the fact that $|\mathcal{K}_\gamma|\lesssim\mathrm{Log}_2(1/\alpha)$, and then summing over γ, we obtain the desired bound thanks to (2-18). □

A decomposition of $f_q^\gamma * \sigma_k$. Recall that f is granular, and hence $f=\sum_l c_l\chi_{\omega_l}$, where each ω_l is a δ-grain, i.e., a dyadic cube of small sidelength $\delta>0$, which we can take to be smaller than (say) α^{100}. We now associate a natural spherical measure to each δ-grain ω_l, defined so that it is supported on those caps where there exists a "heavy" rectangle containing ω_l with short side essentially pointing in the direction normal to the corresponding cap.

Definition 3.4. For each δ-grain ω_l and for a given height M, define $\sigma_{k,\omega_l}^{M,\gamma}$ to be the restriction of σ_k to

$$\bigcup_{i=1}^{N} \bigcup_{j\in\mathcal{J}_i:\,\exists R\in\mathcal{R}_{j,i,M}(\omega_l\cap R\neq\varnothing)} \operatorname{supp}(\sigma_{k,j,i}),$$

where n ranges over $1\le n\le N$. Observe that these measures are decreasing as M increases.

Recall that the parameter i corresponds to the "height", in a sense, of the spherical measure $\sigma_{k,\omega_l}^{i,\gamma}$. We now decompose the function $f_q^\gamma * \sigma_k$ into different "heights" as follows. For a given height M, define the "projection of $f_q^\gamma * \sigma_k$ onto height M" as

$$g_k^{M,\gamma} := \sum_{\delta-\text{grains } \omega_l} \sigma_{k,\omega_l}^{M,\gamma} * (f_q^\gamma \chi_{\omega_l}). \tag{3-11}$$

Then we have the telescoping decomposition

$$f_q^\gamma * \sigma_k * = f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),\gamma} + \sum_{M\ge M_-(k,\gamma)} (g_k^{M,\gamma} - g_k^{2M,\gamma}). \tag{3-12}$$

As previously mentioned, we will see that we have efficient (even when summing over γ and over the relevant range of k) L^2 estimates for the term $f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),\gamma}$. This term represents the "projection of $f_q^\gamma * \sigma_k$ onto low heights".

Discarding the heavy terms via exceptional sets. We can easily dispose of the "heavy" terms in which $M \ge M_+(k,\gamma)$.

Proposition 3.5. *The terms $g_k^{M,\gamma} - g_k^{2M,\gamma}$ for $M \ge M_+(k,\gamma)$ are supported in $S_{M_+(k,\gamma),k,q,\gamma}$, and thus collectively contribute an acceptable L^0 Calderón–Zygmund term thanks to Lemma 3.3.*

Proof. For all δ-grains ω_l which appear in the expression defining $g_k^{M,\gamma}$ for some $M \ge M_+(k,\gamma)$, there is a "heavy" rectangle R containing ω_l such that $\operatorname{supp}(\sigma_{k,j,n} * \chi_R)$ is contained in $S_{M,k,q,\gamma}$ and hence in $S_{M_+(k,\gamma),k,q,\gamma}$. □

Handling the intermediate terms via L^1 estimates. Now we dispose of the "intermediate" terms in which $M_-(k,\gamma) \le M < M_+(k,\gamma)$.

Proposition 3.6. *The contribution of the terms $g_k^{M,\gamma} - g_k^{2M,\gamma}$ with $M_-(k,\gamma) \le M < M_+(k,\gamma)$ to* (2-19) *is an acceptable L^1 Calderón–Zygmund term.*

Proof. We need to establish the bound

$$\Big\| \Big(\sum_{k\ge 0} \Big| \sum_{\gamma>\gamma_0:\,k\in\mathcal{K}_\gamma} \sum_{M_-(k,\gamma)\le M<M_+(k,\gamma)} g_k^{M,\gamma} - g_k^{2M,\gamma} \Big|^2 \Big)^{1/2} \Big\|_1 \lesssim \alpha \operatorname{Log}_3 \frac{1}{\alpha}.$$

Bounding the ℓ^2 norm by the ℓ^1 norm, we can bound the left-hand side by

$$\sum_{k\ge 0} \sum_{\gamma>\gamma_0:\,k\in\mathcal{K}_\gamma} \sum_{M_-(k,\gamma)\le M<M_+(k,\gamma)} \|g_k^{M,\gamma} - g_k^{2M,\gamma}\|_1.$$

Writing $M = 2^j M_-(k,\gamma)$ for some $0 \le j \lesssim \mathrm{Log}_3(1/\alpha)$, it suffices to show that

$$\sum_{k\ge 0} \sum_{\gamma>\gamma_0:\, k\in\mathcal{K}_\gamma} \|g_k^{2^j M_-(k,\gamma),\gamma} - g_k^{2^{k+1} M_-(k,\gamma),\gamma}\|_1 \lesssim \alpha$$

for each such j.

Fix j. Since

$$\sum_\gamma \|f_q^\gamma\|_1 \le \|f\|_1 \lesssim \alpha,$$

it will suffice to show that

$$\sum_{k\in\mathcal{K}_\gamma} \|g_k^{2^j M_-(k,\gamma),\gamma} - g_k^{2^{k+1} M_-(k,\gamma),\gamma}\|_1 \lesssim \|f_q^\gamma\|_1.$$

By (3-11) and Young's inequality, it suffices to show that

$$\sum_{k\in\mathcal{K}_\gamma} \|\sigma_{k,\omega_l}^{2^j M_-(k,\gamma),\gamma} - \sigma_{k,\omega_l}^{2^{j+1} M_-(k,\gamma),\gamma}\| \lesssim 1$$

for each grain ω_l, where $\|\cdot\|$ denotes the total variation norm.

Fix ω_l. By rescaling all the spheres supporting σ_k to a common sphere, it suffices to show that the angles subtended by the spherical cap supporting each of the measures

$$\sigma_{k,\omega_l}^{2^j M_-(k,\gamma),\gamma} - \sigma_{k,\omega_l}^{2^{j+1} M_-(k,\gamma),\gamma}$$

are disjoint as k varies. But this follows directly from the definition of these measures, the telescoping nature of the decomposition, and the fact that $d-1 \ge 1$ ensures that for different values of k, the differences of these measures live at different "heights", and the differences of measures at consecutive heights isolate the height at which a certain angular piece first occurs. □

Estimating the L^2 norm of the light term. In view of the preceding calculations and Lemma 2.2(b), it will suffice to show that

$$\Bigg(\sum_{k\ge 0}\Bigg| \sum_{\gamma>\gamma_0:\, k\in\mathcal{K}_\gamma} f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),\gamma}\Bigg|^2\Bigg)^{1/2}$$

is a type-L^2 Calderón–Zygmund term of threshold α and measure 1 (we will no longer need to lose the additional factor of $\mathrm{Log}_3(1/\alpha)$). Because each k is associated to $O(\mathrm{Log}_2(1/\alpha))$ values of γ, it suffices by Cauchy–Schwarz to show that

$$\Bigg(\sum_{k\ge 0} \sum_{\gamma>\gamma_0:\, k\in\mathcal{K}_\gamma} \Big| f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),\gamma}\Big|^2\Bigg)^{1/2}$$

is a type-L^2 Calderón–Zygmund term of threshold α and measure $\mathrm{Log}_2^{-1}(1/\alpha)$. We rearrange this expression as

$$\Bigg(\sum_{\gamma>\gamma_0} \sum_{k\in\mathcal{K}_\gamma} |f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),\gamma}|^2\Bigg)^{1/2}.$$

Since

$$\sum_{\gamma>\gamma_0} \|f_q^\gamma\|_1 \le \|f_q\|_1 \lesssim \alpha,$$

it then suffices by Lemma 2.2(c) to show that, for each $\gamma > \gamma_0$ and $k \in \mathcal{K}_\gamma$, the quantity

$$f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),\gamma}$$

is a type-L^2 Calderón–Zygmund term of threshold α and measure

$$\frac{\mathrm{Log}_2^{-2}(1/\alpha)}{\alpha}\|f_q^\gamma\|_1.$$

In other words, it remains to establish the bound

$$\|f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),\gamma}\|_2^2 \lesssim \alpha\,\mathrm{Log}_2^{-2}\frac{1}{\alpha}\cdot\|f_q^\gamma\|_1. \tag{3-13}$$

The first step is to write

$$\begin{aligned}\|f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),\gamma}\|_2^2 &= \Bigl\|\sum_{\delta\text{-grains }\omega_l}(\sigma_k-\sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma}) * f_q^\gamma\chi_{\omega_l}\Bigr\|_{L^2}^2\\ &\lesssim \sum_{\delta\text{-grains }\omega_l}\langle(\sigma_k-\sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma}) * f_q^\gamma\chi_{\omega_l},\ \sigma_k * f_q^\gamma\rangle\\ &= \sum_{\delta\text{-grains }\omega_l}\langle f_q^\gamma\chi_{\omega_l},\ \sigma_k*(\sigma_k-\sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma}) * f_q^\gamma\rangle.\end{aligned} \tag{3-14}$$

Domination of the kernel $\sigma_k*\sigma_k$ by linear combinations of characteristic functions of rectangles. Recall from (2-12) that we have the pointwise estimate

$$\sigma_k*\sigma_k(x) \lesssim 2^{-k(d-1)}|x|^{-1}\chi_{B_k}(x),$$

where $B_k := \{|x| \le 2^{k+1}\}$ is the ball of radius 2^{k+1} around the origin. Inside this ball, we isolate the annulus

$$A_\gamma := \{x : \gamma^{1/(d-1)} \le |x| \le 2^{-100}r_\gamma\},$$

where we recall that the radius r_γ was defined in (3-1).

Thus the kernel $\sigma_k*\sigma_k$ can essentially be decomposed as follows. Fix q and γ, and let $\{c_i\}_{i=0}^N$ be the enumeration of the scales described earlier in (3-4). For $1 \le i \le N$, let $\mathcal{R}_i$ be a collection of $(c_i/c_{i-1})^{d-1}$ many rectangles of dimensions $c_i \times c_i \times \cdots \times c_i \times c_{i-1}$ centered at the origin, with short sides pointing in equally spaced directions, where $\{c_i\}_{i=1}^N$ are the scales described earlier. We may dominate

$$2^{-k(d-1)}|x|^{-1}\chi_{A_\gamma}(x) \lesssim \sum_{i=1}^N 2^{-k(d-1)}c_i^{-(d-1)}c_{i-1}^{d-2}\sum_{R\in\mathcal{R}_i}\chi_R. \tag{3-15}$$

Indeed, for each i, $c_i^{-(d-1)}c_{i-1}^{d-2}\sum_{r\in\mathcal{R}_i}\chi_R$ is essentially of size c_i^{-1} for $|x|\approx c_i$, since for $|x|\approx c_i$ the rectangles are essentially disjoint in the case that $d=2$, and for general d there are

$$\approx c_i^{-d}\times(c_i/c_{i-1})^{d-1}\times c_i^{d-1}c_{i-1} \approx (c_i/c_{i-1})^{d-2}$$

many rectangles that intersect a given x. By similar reasoning, one sees that for $c_{i-1}\le|x|\le c_i$, we also have $c_i^{-(d-1)}c_{i-1}^{d-2}\sum_{r\in\mathcal{R}_i}\chi_R$ is essentially of size $|x|^{-1}$.

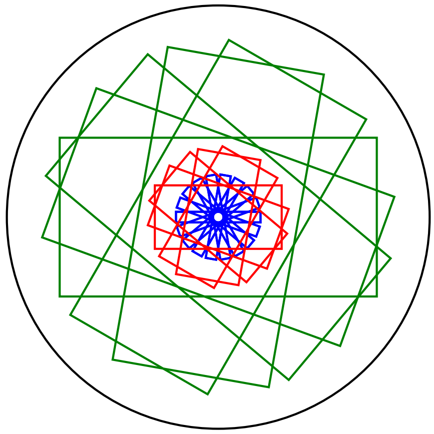

Figure 2. Domination of the kernel $\sigma_k * \sigma_k$ in the sphere of radius $2^{-100} r_\gamma$ centered at the origin.

From (3-15) and (2-12) we have the bound

$$\sigma_k * \sigma_k(x) \lesssim \sum_{i=1}^{N} 2^{-k(d-1)} c_i^{-(d-1)} c_{i-1}^{d-2} \sum_{R \in \mathcal{R}_i} \chi_R + 2^{-k(d-1)} |x|^{-1} \chi_{B_k \setminus A_\gamma}(x); \tag{3-16}$$

see Figure 2.

Eliminating bad rectangles. Now fix some i with $1 \le i \le N$, and suppose that R is a rectangle in $\mathcal{R}_i$ such that

$$\int_{\omega_l + R} f_q^\gamma > c_{i-1} \gamma M_-(k, \gamma).$$

Then by definition, the support of $\sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma}$ contains a spherical cap of angular width $50 c_{i-1}/c_i$ with some normal parallel to the short side of R. This implies that $\sigma_k * (\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma})$ is supported outside of the set

$$(R)_1 := \{x \in R : x \ge \tfrac{1}{10} c_i\}.$$

Indeed, for any $x \in R$ in the support of $\sigma_k * (\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma})$ with $|x| \ge \frac{1}{10} c_i$, we require there to exist y on the sphere of radius 2^k centered at the origin such that $x - y$ is also on the sphere of radius 2^k centered at the origin, but outside the cap of angular width $50 c_{i-1}/c_i$ with some normal parallel to the short side of R. Suppose toward a contradiction that $x \in R \cap \{z : \frac{1}{10} c_i \le |z| \le 10 c_i\}$. But for any such $x - y$, we have that $(x - y) + (R)_1$ lies outside the sphere of radius 2^k, since R will be transverse to the boundary of the sphere at $x - y$ (see Figure 3). Thus we have verified our claim that $\sigma_k * (\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma})$ is supported outside of the set $(R)_1$.

Repeating this process, if

$$\int_{\omega_l + (R \setminus (R)_1)} f_q^\gamma > c_{i-1} \gamma M_-(k, \gamma),$$

then by definition, the support of $\sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma}$ contains a spherical cap of angular width $50 \cdot 2 c_{i-1}/c_i$ with some normal parallel to the short side of R. As before, this implies that $\sigma_k * (\sigma_k - \sigma_{k,\omega_l}^{m(k,q,\gamma),\gamma})$ is supported

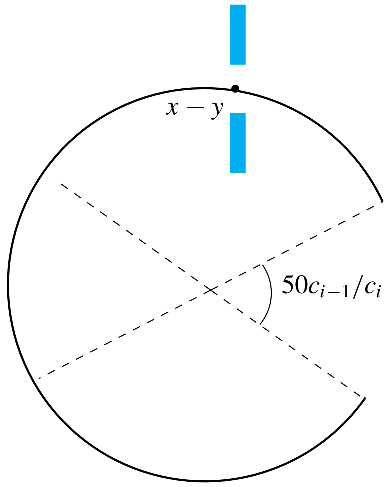

Figure 3. The two blue rectangles represent the set $x - y + (R)_1$.

outside of the set

$$(R)_2 := \left\{x \in R : |x| \geq \tfrac{1}{20} c_i\right\}.$$

Repeating again, if

$$\int_{\omega_l + (R \setminus (R)_2)} f_q^\gamma > c_{i-1} \gamma M_-(k, \gamma),$$

then $\sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma}$ contains a spherical cap of angular width $50 \cdot 4c_{i-1}/c_i$ with some tangent parallel to the long side of R. This implies that $\sigma_k * (\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma})$ is supported outside of the set $(R)_3$, where we define

$$(R)_3 := \left\{x \in \mathbb{R} : |x| \geq \tfrac{1}{40} c_i\right\}.$$

We continue this process until some stage L when

$$\int_{(\omega_l + (R \setminus (R)_L))} f_q^\gamma > c_{i-1} \gamma M_-(k, \gamma),$$

and $\sigma_k * (\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma})$ is supported outside of the set

$$(R)_L := \left\{x \in \mathbb{R} : |x| \geq \frac{1}{2^{L-1} \times 10} c_i\right\}.$$

(Note that this must eventually happen if $\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma}$ is not identically 0, since the set $(R)_L$ can potentially increase by continuing this process up to $\{x \in R : |x| \geq 10c_{i-1}\}$, which would imply that $\sigma_k - \sigma_{k,\omega_l}^{m(k,q,\gamma),\gamma}$ is identically 0.)

For convenience, we summarize the above argument in the following lemma.

Lemma 3.7. *Fix a δ-grain ω_l. For any rectangle $R \in \mathcal{R}_i$, there is a subset $(R)_L \subset R$ such that*

$$\int_{\omega_l + (R \setminus (R)_L)} |f_q^\gamma| \lesssim c_{i-1} \gamma M_-(k, \gamma)$$

and $\sigma_k * (\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma})$ is supported outside of the set $(R)_L$.

Finishing up the proof. Lemma 3.7 and (3-16) implies that, for each δ-grain ω_l and each rectangle $R \in \mathcal{R}_i$, there is a function h_R with $\int |h_R| \le c_{i-1} M_-(k,\gamma)\gamma$ so that by (3-14) we may dominate

$$\begin{aligned}\|f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),\gamma}\|_2^2 &\lesssim \sum_{\delta-\text{grains } \omega_l} \langle f_q^\gamma \chi_{\omega_l},\, \sigma_k * (\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma}) * f_q^\gamma\rangle \\ &\lesssim \sum_{\delta-\text{grains } \omega_l} \langle f_q^\gamma \chi_{\omega_l}, 2^{-k(d-1)} |x|^{-1} \chi_{B_k \setminus A_\gamma} * f_q^\gamma\rangle \\ &\qquad + \sum_{\delta-\text{grains } \omega_l} \sum_{i=1}^{N} 2^{-k(d-1)} \frac{c_{i-1}^{d-2}}{c_i^{d-1}} \sum_{R \in \mathcal{R}_i} \langle f_q^\gamma \chi_{\omega_l}, \chi_R * h_R\rangle. \end{aligned} \tag{3-17}$$

It is not difficult to show that

$$\langle f_q^\gamma \chi_{\omega_l}, 2^{-k(d-1)} |x|^{-1} \chi_{B_k \setminus A_\gamma} * f_q^\gamma\rangle \lesssim \alpha \operatorname{Log}_2^{-2} \frac{1}{\alpha} \|f_q^\gamma \chi_{\omega_l}\|_1. \tag{3-18}$$

Indeed, by Young's inequality it would suffice to show that

$$2^{-k(d-1)} \int_{|x-y| \in B_k \setminus A_\gamma} \frac{f_q^\gamma(y)}{|x-y|}\, dy \lesssim \alpha \operatorname{Log}_2^{-2} \frac{1}{\alpha} \tag{3-19}$$

for any x. From (2-13) we have

$$\int_{|x-y| \approx 2^l} \frac{f_q^\gamma(y)}{|x-y|}\, dy \lesssim \min(\gamma, 2^{l(d-1)})$$

so by dyadic decomposition we may bound the left-hand side of (3-19) by

$$2^{-k(d-1)} \Biggl(\sum_{2^l \lesssim \gamma^{1/(d-1)}} 2^{l(d-1)} + \sum_{r_\gamma \lesssim 2^l \lesssim 2^k} \gamma \Biggr),$$

which we can sum to

$$\lesssim 2^{-k(d-1)} \gamma \operatorname{Log} \frac{2^k}{r_\gamma} \lesssim 2^{-k(d-1)} \gamma \operatorname{Log} \frac{2^{k(d-1)}}{\gamma/\alpha}$$

thanks to (3-1). By (3-2), we have

$$\frac{2^{k(d-1)}}{\gamma/\alpha} \ge \operatorname{Log}_2^{100} \frac{1}{\alpha},$$

giving (3-18) as claimed.

This gives a satisfactory bound for the first term occurring in the right-hand side of (3-17). To bound the second term, we observe that since $\int |h_R| \lesssim c_{i-1} M_-(k,\gamma)\gamma$, we have

$$\langle f_q^\gamma \chi_{\omega_l}, \chi_R * h_R\rangle \lesssim c_{i-1} M_-(k,\gamma)\gamma \|f_q^\gamma \chi_{\omega_l}\|_1. \tag{3-20}$$

Combining (3-17), (3-18), and (3-20) and summing over all i and all δ-grains ω_l, using the fact that the cardinality of $\mathcal{R}_i$ is $\lesssim (c_i/c_{i-1})^{d-1}$ and $N \lesssim \operatorname{Log}_2(1/\alpha)$, and recalling the definition (3-10) of $M_-(k,\gamma)$, we obtain

$$\|f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),k,\gamma}\|_2^2 \lesssim \alpha \operatorname{Log}_2^{-2} \frac{1}{\alpha} \|f_q^\gamma\|_1,$$

which is the desired L^2 bound. This completes the proof of (3-13), and hence Propositions 1.1, 1.2, and Theorem 1.3.

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GLOBAL WELL-POSEDNESS FOR A SYSTEM OF QUASILINEAR WAVE EQUATIONS ON A PRODUCT SPACE

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We consider a system of quasilinear wave equations on the product space $\mathbb{R}^{1+3}\times\mathbb{S}^1$, which we want to see as a toy model for the Einstein equations with additional compact dimensions. We show global existence of solutions for small and regular initial data with polynomial decay at infinity. The method combines energy estimates on hyperboloids inside the light cone and weighted energy estimates outside the light cone.

1. Introduction

We address the problem of global existence of small solutions to a certain class of quasilinear systems of wave equations on the product space $\mathbb{R}^{1+3}\times\mathbb{S}^1$. Let $\Box_{x,y}=-\partial_t^2+\Delta_x+\partial_y^2$ denote the d'Alembertian operator in the (t,x,y)-variables, where $t\in\mathbb{R}$ is the time coordinate, $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ are the Cartesian coordinates and $y\in\mathbb{S}^1$ is the periodic coordinate. The system we consider has the form

$$\begin{cases}\Box_{x,y}u+u\partial_y^2u=\sum_{1\le i,j\le 2}\boldsymbol{N}_1(w_i,w_j),\\ \Box_{x,y}v+u\partial_y^2v=\sum_{1\le i,j\le 2}\boldsymbol{N}_2(w_i,w_j),\end{cases}\qquad (t,x,y)\in\mathbb{R}^{1+3}\times\mathbb{S}^1, \tag{1-1}$$

with initial conditions set at time $t_0=2$

$$(u,v)(2,x,y)=(\phi_0,\psi_0)(x,y),\quad (\partial_tu,\partial_tv)(2,x,y)=(\phi_1,\psi_1)(x,y). \tag{1-2}$$

The nonlinearities $\boldsymbol{N}_1(\cdot,\cdot)$, $\boldsymbol{N}_2(\cdot,\cdot)$ are linear combinations of the quadratic null forms

$$\begin{aligned} Q_0(\phi,\psi)&=\partial_t\phi\,\partial_t\psi-\nabla_x\phi\cdot\nabla_x\psi,\\ Q_{ij}(\phi,\psi)&=\partial_{x_i}\phi\,\partial_{x_j}\psi-\partial_{x_j}\phi\,\partial_{x_i}\psi,\quad 1\le i<j\le 3,\\ Q_{0i}(\phi,\psi)&=\partial_t\phi\,\partial_{x_i}\psi-\partial_{x_i}\phi\,\partial_t\psi,\qquad 1\le i\le 3,\end{aligned} \tag{1-3}$$

and $w_1,w_2=\{u,v\}$ denote the two-component solutions.

The main result we present in this paper asserts the global existence of solutions to (1-1) when the initial data are small and localized real functions. Our result also extends to semilinear interactions of the form $\partial w_i\cdot\partial_y w_j$ with ∂ being any of the derivatives in the (t,x,y)-variables.

1A. ***Notation.*** Below is a summary of some notation we will use throughout:

- $r=|x|$ is the radial coordinate in the x-variables.
- We use the Einstein summation convention and take the sum over repeated indexes.

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- ∂_j denotes the derivative ∂_{x_j} in the j-th direction for $j = \overline{1,3}$. Sometimes we may use the notation ∂_0 for ∂_t and ∂_4 for ∂_y.
- $\nabla_x = (\partial_1, \partial_2, \partial_3)$ denotes the gradient in the spatial variable x, $\nabla_{xy} = (\partial_1, \partial_2, \partial_3, \partial_4)$ denotes the gradient in the full set of spatial variables (x, y).
- ∂_x denotes any of the derivatives in $\{\partial_j : j = \overline{1,3}\}$.
- ∂_{xy} denotes any of the derivatives in $\{\partial_j : j = \overline{1,4}\}$. Analogously for ∂_{tx} and ∂_{txy}. We will use ∂ and ∂_{txy} interchangeably.
- $\square_{x,y} = -\partial_t^2 + \Delta_x + \partial_y^2$ and $\square_x = -\partial_t^2 + \Delta_x$.
- Given a multi-index $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_4) \in \mathbb{N}^5$, its length is computed classically as $|\alpha| = \sum_{i=0}^4 \alpha_i$ and $\partial^\alpha = \partial_0^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}\partial_4^{\alpha_4}$. We use the notation ∂_x^α for $\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ and it is then clear what ∂_{xy}^α or ∂_{tx}^α stand for.
- More generally, given a family of vector fields $\{\Gamma_1, \dots, \Gamma_n\}$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\Gamma^\alpha = \Gamma_1^{\alpha_1} \cdots \Gamma_n^{\alpha_n}$. Sometimes we make an abuse of notation and write Γ^k (resp. $\Gamma^{\le k}$) instead of $\sum_{\alpha:|\alpha|=k} \Gamma^\alpha$ (resp. $\sum_{\alpha:|\alpha|\le k} \Gamma^\alpha$).

1B. ***Motivation and a brief history.*** Our interest in studying nonlinear wave equations on product spaces comes from the theory of supergravity (SUGRA) in physics and more precisely from the Kaluza–Klein theory, which represents the classical approach to the unification of general relativity with electromagnetism and more generally with gauge fields; see the original works [Kaluza 1921; Klein 1926]. The Kaluza–Klein approach considers general relativity in $3+1+d$ dimensions with space-time factorizing as

$$\mathscr{M}^{(3+1+d)} = \mathbb{R}^{1+3} \times K,$$

where K is a compact d-manifold referred to as *internal space*. In the simplest case $d = 1$-dimensional gravity is compactified on a circle ($K = \mathbb{S}^1$) to obtain at low energies a coupled Einstein–Maxwell scalar system in 3+1 space-time dimensions. Kaluza–Klein space-times $\mathbb{R}^{1+3} \times \mathbb{S}^1$ have been studied the influential work [Witten 1982], where he proves instability at the semiclassical level but provides heuristic arguments for classical stability. The first result proving the classical stability of the Kaluza–Klein theory is obtained in [Wyatt 2018], where only perturbations depending on the noncompact coordinates are considered, using tools developed in [Lindblad and Rodnianski 2010]. More general space-times with supersymmetric compactifications $\mathscr{M} = \mathbb{R}^{1+n} \times K$ have recently been studied by Andersson, Blue, Wyatt, and Yau [Andersson et al. 2023]. The space-times $\mathscr{M}$ are equipped with the metric

$$\hat{g} = \eta_{\mathbb{R}^{1+n}} + k,$$

where $\eta_{\mathbb{R}^{1+n}}$ is the Minkowski metric in $\mathbb{R}^{1+n}$ and k is such that (K, k) is a compact Ricci-flat Riemannian manifold having a cover that admits a spin structure and a nonzero parallel spinor. A global stability result is proved in [Andersson et al. 2023] under the assumption $n \ge 9$ and for Cauchy data that are Schwarzschild near infinity, but it is conjectured that these conditions can be relaxed and that space-times with a supersymmetric compactification and $n = 3$ are nonlinearly stable. We also briefly mention a result on the stability of cosmological Kaluza–Klein space-times (where the Minkowski space-time is replaced by the four-dimensional Milne universe) under zero-modes perturbations by Branding, Fajman and Kröncke [Branding et al. 2019] and a stability result for semilinear wave equations on the cosmological Kaluza–Klein background in [Wang 2021].

In both the aforementioned works [Wyatt 2018; Andersson et al. 2023], as well as in many other works concerning the global stability problem for Einstein equations, the use of the so-called *wave-coordinates* allows one to write the Einstein equations as a system of quasilinear wave equations on the metric $g = (g^{\alpha\beta})_{\alpha\beta}$

$$g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} = P_{\mu\nu}(\partial g, \partial g) + G_{\mu\nu}(g)(\partial g, \partial g), \tag{1-4}$$

where the sum is taken over repeated indexes, $P_{\mu\nu}$ are quadratic forms and $G_{\mu\nu}(g)(\partial g, \partial g)$ contain cubic terms. In the present paper we focus on a toy model for the Einstein equations on $\mathbb{R}^{1+3}\times\mathbb{S}^1$ that only keeps a selection of terms from (1-4), precisely the semilinear terms with null structure and the quasilinear term $g^{yy}\partial_y^2$ where y is the periodic coordinate on $\mathbb{S}^1$. Our goal here is in fact to study the global well-posedness for quasilinear wave equations on the product space $\mathbb{R}^{1+3}\times\mathbb{S}^1$ without having to make use of the full structure of the Einstein equations. The unknown u in system (1-1) plays the role of the coefficient g_{yy} and the unknown v encodes any other metric coefficient $g_{\mu\nu}$.

We mention here that wave equations on product spaces also appear in other contexts, for instance when studying the propagation of waves along infinite homogeneous waveguides; see [Lesky and Racke 2003; Metcalfe et al. 2005; Metcalfe and Stewart 2008; Ettinger 2015].

One key observation when studying the small data global well-posedness problem for wave equations on product spaces of the form $\mathbb{R}^{1+3}\times\mathbb{T}^d$ is that they are closely related to infinite systems coupling wave and Klein–Gordon equations on the flat space $\mathbb{R}^{1+3}$. In fact, if $W = W(t, x, y)$ is solution to

$$\Box_{x,y} W = F, \quad (t, x, y)\in\mathbb{R}^{1+3}\times\mathbb{T}^d,$$

for some source term F, its Fourier modes in the periodic direction $\{W_k(t, x)\}_{k\in\mathbb{Z}}$ solve the following equations on the flat space $\mathbb{R}^{1+3}$:

$$(-\partial_t^2 + \Delta_x)W_k - |k|^2 W_k = \frac{1}{\mathrm{Vol}(\mathbb{T}^d)}\int_{\mathbb{T}^d} e^{-iy\cdot k} F\, dy, \quad (t, x)\in\mathbb{R}^{1+3}.$$

In particular, the zero-mode W_0 is a solution to a wave equation and any other nonzero mode W_k solves a Klein–Gordon equation of mass $|k|$.

The global well-posedness for systems coupling a finite number of wave and Klein–Gordon equations on the flat 3+1 space-time with small data have largely been studied. We cite the initial results by Georgiev [1990] and Katayama [2012], followed by LeFloch and Ma [2014], Wang [2015; 2020] and Ionescu and Pausader [2019], who study such systems as a model for the full Einstein–Klein–Gordon equations; the global stability of the full problem is successively proved in [LeFloch and Ma 2016; Ionescu and Pausader 2022]. In [LeFloch and Ma 2014; Wang 2015] global well-posedness is proved for compactly supported initial data and quadratic quasilinear nonlinearities that satisfy some suitable conditions, including the *null condition* of [Klainerman 1986] for self-interactions between the wave components of the solution. An idea used in these works is that of employing hyperbolic coordinates in the forward light cone; this was first introduced in [Klainerman 1985] for Klein–Gordon equations and in [Tataru 2002] in the wave context, and later reintroduced in [LeFloch and Ma 2014] under the name of *hyperboloidal foliation method*. In [Ionescu and Pausader 2019] global regularity and scattering is proved in the case of small smooth initial data that decay at a suitable rate at infinity and nonlinearities that do not verify the null condition but present a particular resonant structure. We also cite [Dong and Wyatt 2020a], which proves global well-posedness for a quadratic semilinear interaction in which there are no derivatives on the massless wave component. Other related results are [Bachelot 1988; Ozawa et al. 1995; Tsutaya 1996; Tsutsumi 2003a; 2003b; Klainerman et al. 2020; Dong et al. 2021] and see [Ma 2017a; 2017b; 2017c; 2020; 2021; Stingo 2023; Ifrim and Stingo 2019; Dong and Wyatt 2020b] for results about wave-Klein–Gordon systems in lower dimensions.

Our goal in this paper is to prove the global stability for (1-1) in the case where the initial data are not compactly supported but only have a mild polynomial decay at infinity. Our approach makes use of the vector field method in [Christodoulou and Klainerman 1993] and follows [LeFloch and Ma 2016] in that a big portion of the estimates recovered for the solution in the interior of the cone $\{t = r+1\} \times \mathbb{S}^1$ are estimates on hyperboloids. The main difference with [LeFloch and Ma 2014] is that the interior estimates need to be coupled with exterior estimates in the region outside the cone. Those are weighted energy estimates, also used in [Lindblad and Rodnianski 2010], that we have to propagate in time.

1C. ***The main result.*** In order to describe the initial data we consider for our problem we introduce the energy space $\mathscr{H}^0$ endowed with the norm

$$\|(u[t], v[t])\|^2_{\mathscr{H}^0} := \|u\|^2_{H^1_{xy}} + \|u_t\|^2_{L^2} + \|v\|^2_{H^1_{xy}} + \|v_t\|^2_{L^2}$$

and the higher-order energy spaces $\mathscr{H}^n$ for $n \geq 1$ endowed with the norm

$$\|(u[t], v[t])\|^2_{\mathscr{H}^n} := \sum_{|\alpha|\leq n} \|(\partial^\alpha_{xy} u[t], \partial^\alpha_{xy} v[t])\|^2_{\mathscr{H}^0}.$$

In the above definition, ∂^α_{xy} is a short notation for $\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\partial^{\alpha_4}_{y}$ given $\alpha = (\alpha_1, \dots, \alpha_4)$ and we use the following notation for the Cauchy data in (1-2) at time t:

$$(u[t], v[t]) := (u(t), u_t(t), v(t), v_t(t)).$$

The global well-posedness result that is the object of this paper is proved under some decay assumptions on the initial data. A preliminary version of our main theorem states the following:

Theorem 1. *Assume that the initial data* $(u[2], v[2])$ *for* (1-1) *satisfy*

$$\sum_{|\alpha|=0}^{5} \|\langle x\rangle^{\leq|\alpha|+\kappa/2}\partial^\alpha_{xy}(u[2], v[2])\|_{\mathscr{H}^0} \leq \epsilon \ll 1$$

for some positive fixed κ *and* $\langle x\rangle = \sqrt{1+|x|^2}$*. Then the system* (1-1) *is globally well-posed in the space* $\mathscr{H}^5$.

We remark here that our choice to set the initial data at time $t_0 = 2$ over the conventional $t_0 = 0$ is more convenient for our computations and comes at no expense as the system (1-1) is invariant under time translations.

1D. *The wave-Klein–Gordon structure.* The Cauchy problem (1-1)–(1-2) can be written in a more compact form as a vector equation for the unknown $W = (u, v)^T$

$$\Box_{x,y} W + u\partial_y^2 W = \boldsymbol{N}(W, W), \tag{1-5}$$

with data

$$W|_{t=2} = \Phi_0, \quad \partial_t W|_{t=2} = \Phi_1, \tag{1-6}$$

where $\Phi_0 = (\phi_0, \psi_0)^T$ and $\Phi_1 = (\phi_1, \psi_1)^T$ and

$$\boldsymbol{N}(W, W) = \sum_{1\le i,j\le 2} \begin{pmatrix} \boldsymbol{N}_1(w_i, w_j) \\ \boldsymbol{N}_2(w_i, w_j) \end{pmatrix}.$$

The projection of W onto the periodic direction y reveals the nature of (1-5) as a system coupling one (vector) wave equation with an infinite sequence of (vector) Klein–Gordon equations of variable mass $|k|\sqrt{1+u_0}$ with $k \in \mathbb{Z}^*$. If we denote by $W_k = (u_k, v_k)^T$ the projection of W onto the k-th frequency

$$W_k(t, x) = \int_{\mathbb{S}^1} e^{-iky} W(t, x, y)\,\frac{dy}{2\pi}, \quad k \in \mathbb{Z},$$

we see that the functions $\{W_k\}_k$ satisfy the coupled system

$$\begin{cases} (-\partial_t^2 + \Delta_x)W_k - |k|^2(1+u_0)W_k = \displaystyle\int_{\mathbb{S}^1} e^{-iky}\boldsymbol{N}(W, W)\,\frac{dy}{2\pi} - \int_{\mathbb{S}^1} e^{-iky}(u-u_0)\partial_y^2 W\,\frac{dy}{2\pi}, \\ k \in \mathbb{Z}. \end{cases}$$

The zero mode W_0 is solution to a wave equation, while any other nonzero mode W_k is solution to a Klein–Gordon equation of variable mass $|k|\sqrt{1+u_0}$. This distinction will be fundamental for our analysis and we will often work throughout the paper with the decomposition of W

$$W = W_0 + \mathtt{W}, \quad \mathtt{W}(t, x, y) = \sum_{k\neq 0} e^{iky} W_k(t, x), \tag{1-7}$$

so that (1-5) is equivalent to the system

$$\begin{cases} (-\partial_t^2 + \Delta_x)W_0 = \displaystyle\int_{\mathbb{S}^1} \boldsymbol{N}(W, W)\,\frac{dy}{2\pi} + \int_{\mathbb{S}^1} \partial_y u\,\partial_y \mathtt{W}\,\frac{dy}{2\pi}, \\ (-\partial_t^2 + \Delta_x)\mathtt{W} + (1+u)\,\partial_y^2\mathtt{W} = \boldsymbol{N}(W, W) - \displaystyle\int_{\mathbb{S}^1} \boldsymbol{N}(W, W)\,\frac{dy}{2\pi} - \int_{\mathbb{S}^1} \partial_y u\,\partial_y\mathtt{W}\,\frac{dy}{2\pi}. \end{cases} \tag{1-8}$$

Observe that the source term F_0 in the equation of W_0 does not contain mixed interactions:

$$F_0 = \boldsymbol{N}(W_0, W_0) + \int_{\mathbb{S}^1} \boldsymbol{N}(\mathtt{W}, \mathtt{W})\,\frac{dy}{2\pi} + \int_{\mathbb{S}^1} \partial_y u \cdot \partial_y \mathtt{W}\,\frac{dy}{2\pi}. \tag{1-9}$$

We are now able to state a more precise version of the main theorem.

Theorem 2. *Assume that for some positive fixed κ the initial data for* (1-5) *satisfy*

$$\sum_{|\alpha|=0}^{5} \|x^{\le|\alpha|+\kappa/2}\partial_{xy}^\alpha W[2]\|_{\mathcal{H}^0} \le \epsilon \ll 1.$$

Then the solution W to (1-5)–(1-6) *exists globally in time in $\mathscr{H}^5$ and the two components of the solution*, $W_0 = \fint_{\mathbb{S}^1} W\,(dy/2\pi)$ *and* $\mathtt{W} = W - W_0$, *satisfy the pointwise bounds*

$$
\begin{aligned}
|\partial^\alpha_{tx} W_0(t,x)| &\lesssim \epsilon \langle t+|x|\rangle^{-1}\langle t-|x|\rangle^{-1/2}, && |\alpha| = \overline{1,3},\\
\|\partial^j_y \partial^\alpha_{tx} \mathtt{W}(t,x,\cdot)\|_{L^2_y(\mathbb{S}^1)} &\lesssim \epsilon \langle t+|x|\rangle^{-3/2}, && j=\overline{0,1},\ |\alpha| = \overline{0,1},\\
\|\partial^j_y \partial^\alpha_{tx} \mathtt{W}(t,x,\cdot)\|_{L^2_y(\mathbb{S}^1)} &\lesssim \epsilon \langle t+|x|\rangle^{-1}\langle t-|x|\rangle^{-1/2}, && j=\overline{0,1}, |\alpha| = 2.
\end{aligned}
$$

1E. *Vector fields.* In order to describe the global bounds and decay properties of the solution $W = (u, v)^T$ to (1-5)–(1-6) we need to introduce the family of Killing vector fields associated to our problem. Those are the vector fields that exactly commute with $\Box_{x,y}$:

$$\partial_0, \partial_1, \partial_2, \partial_3, \partial_4, \tag{1-10}$$

$$\Omega_{ij} = x_j\partial_i - x_i\partial_j, \quad 1 \le i < j \le 3, \tag{1-11}$$

$$\Omega_{0i} = t\partial_i + x_i\partial_t, \qquad i = \overline{1,3}. \tag{1-12}$$

The expressions in (1-10) correspond to the translations in the coordinate directions; (1-11) correspond to the Euclidean rotations in the x-coordinates; (1-12) are the hyperbolic rotations, also called boosts. We also introduce the conformal scaling vector field

$$\mathscr{S} = t\partial_t + x\cdot\nabla_x, \tag{1-13}$$

which is not Killing for (1-5) but will appear later in the analysis of the problem. We refer to (1-11) and (1-12) as *Klainerman vector fields* and generally denote them by Z:

$$Z := \{\Omega_{ij}, \Omega_{0i}\}.$$

We denote the full set of admissible vector fields for $\Box_{x,y}$ as

$$\mathscr{Z} := \{\partial_0, \partial_1, \partial_2, \partial_3, \partial_4, \Omega_{ij}, \Omega_{0i}\} \tag{1-14}$$

and for any multi-index $\gamma = (\alpha, \beta)$ we define

$$\mathscr{Z}^\gamma = \partial^\alpha Z^\beta.$$

For any two nonnegative integers k, n with $k \le n$ we say that the multi-index $\gamma = (\alpha, \beta)$ is of type (n, k) if $|\gamma| = |\alpha| + |\beta| \le n$ and $|\beta| \le k$, in other words if there are at most k Klainerman vector fields among the $|\gamma|$ admissible vector fields in the product $\mathscr{Z}^\gamma$.

1F. *The null structure.* The nonlinearities we consider in this work are linear combinations of the classical quadratic null forms (1-3). An important feature of the null forms is that they are combinations of three types of products and can be expressed schematically as

$$\boldsymbol{N}(\phi,\psi) = \bar\partial\phi\cdot\partial\psi + \partial\phi\cdot\bar\partial\psi + \frac{t-r}{t}\partial\phi\cdot\partial\psi, \tag{1-15}$$

where $\bar\partial_j = t^{-1}\Omega_{0j}$ are the rescaled hyperbolic rotations, or also as

$$\boldsymbol{N}(\phi,\psi) = \mathscr{T}\phi\cdot\partial\psi + \partial\phi\cdot\mathscr{T}\psi, \tag{1-16}$$

where $\mathcal{T}_j = \partial_j + (x_j/r)\partial_t$ for $j = \overline{1,3}$ are the vector fields tangent to the cones $\{t - r = \text{const}\}$. The $\mathcal{T}$ vector fields are related to the boosts in general via the relation

$$\mathcal{T}_j = \frac{1}{t}\Omega_{0j} + \frac{(t-r)}{t}\frac{x_j}{r}\partial_t.$$

We will often make use of the two representations (1-15) and (1-16) to recover suitable pointwise and energy estimates for the solution, as they allow us to recover additional decay for the $W_0 \times W_0$ interactions.

1G. ***The interior and exterior region.*** The proof of our main theorem is based on the combination of a classical local existence result with a bootstrap argument. We will perform such argument separately in the two regions in which we decompose our space-time:

$$\begin{aligned} &\text{interior region} \quad \mathcal{D}^{\text{in}} := \{(t,x) : t \geq 2 \text{ and } |x| < t-1\} \times \mathbb{S}^1, \\ &\text{exterior region} \quad \mathcal{D}^{\text{ex}} := \{(t,x) : t \geq 2 \text{ and } |x| \geq t-1\} \times \mathbb{S}^1. \end{aligned}$$

In order to describe our bootstrap assumptions we first introduce some notation. Given any hyperboloid $\mathcal{H}_s$ in $\mathbb{R}^{1+3} \times \mathbb{S}^1$ we denote by $\mathcal{H}_s^{\text{in}}$ (resp. $\mathcal{H}_s^{\text{ex}}$) the branch of $\mathcal{H}_s$ contained in the interior region $\mathcal{D}^{\text{in}}$ (resp. in the exterior region $\mathcal{D}^{\text{ex}}$):

$$\begin{aligned} \mathcal{H}_s &= \{(t,x) : s^2 = t^2 - |x|^2\} \times \mathbb{S}^1, \\ \mathcal{H}_s^{\text{in}} &:= \big\{(t,x,y) \in \mathcal{H}_s : t \geq 2 \text{ and } |x| < \tfrac{1}{2}(s^2-1)\big\}, \\ \mathcal{H}_s^{\text{ex}} &:= \big\{(t,x,y) \in \mathcal{H}_s : t \geq 2 \text{ and } |x| \geq \tfrac{1}{2}(s^2-1)\big\}. \end{aligned}$$

Moreover we denote by $\mathcal{H}_{[2,s]}^{\text{in}}$ the hyperbolic interior region above $\mathcal{H}_2^{\text{in}}$ and below $\mathcal{H}_s^{\text{in}}$, and by $\mathcal{H}_{[2,s]}^{\text{ex}}$ the portion of the exterior region below $\mathcal{H}_s^{\text{ex}}$ for any $s \geq 2$ (see Figure 1 and Figure 2, which is displayed in Section 3):

$$\begin{aligned} \mathcal{H}_{[2,s]}^{\text{in}} &:= \{(t,x,y) \in \mathcal{D}^{\text{in}} : 2 \leq t^2 - |x|^2 \leq s^2\}, \\ \mathcal{H}_{[2,s]}^{\text{ex}} &:= \{(t,x,y) \in \mathcal{D}^{\text{ex}} : t^2 - |x|^2 \leq s^2\}. \end{aligned}$$

In the interior region the bootstrap assumptions will be energy bounds on the truncated hyperboloids $\mathcal{H}_s^{\text{in}}$ for $s \geq 2$ and pointwise bounds on the Z-derivative of the zero mode of the solution. The local wellposedness theory for this problem ensures the existence and smallness of the solution $W = (u,v)^T$ to (1-5)–(1-6) up to the interior hyperboloid $\mathcal{H}_2^{\text{in}}$; hence our goal will be to propagate the bootstrap assumptions in the hyperbolic interior region above $\mathcal{H}_2^{\text{in}}$

$$\mathcal{H}_{[2,\infty)}^{\text{in}} := \{(t,x,y) \in \mathcal{D}^{\text{in}} : 2 \leq t^2 - |x|^2\}.$$

In the exterior region the bootstrap assumptions will instead be weighted energy bounds on the constant time slices Σ_t^{ex} which foliate $\mathcal{D}^{\text{ex}}$ for $t \geq 2$,

$$\Sigma_t^{\text{ex}} := \{x \in \mathbb{R}^3 : |x| \geq t-1\} \times \mathbb{S}^1.$$

We warn the reader that throughout the paper we will work with functions defined on the product space $\mathbb{R}^{1+3} \times \mathbb{S}^1$ as well as with functions not depending on the y-variable and defined on the flat space $\mathbb{R}^{1+3}$.

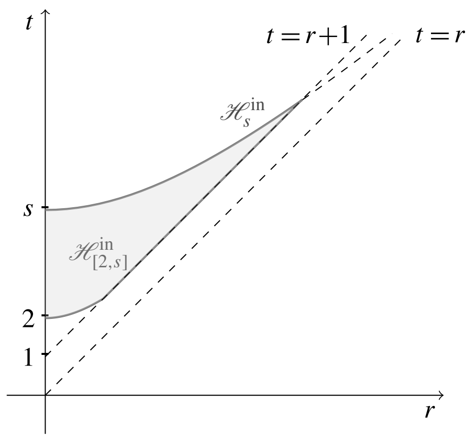

Figure 1. Vertical section of the region $\mathscr{H}_{[2,s]}^{\text{in}}$ projected onto $\mathbb{R}^{1+3}$.

With the purpose of keeping notation as light as possible, a region in $\mathbb{R}^{1+3}\times\mathbb{S}^1$ and its projection onto $\mathbb{R}^{1+3}$ will have the same name.

1H. ***The energy functionals.*** In the interior region we aim to propagate a priori energy bounds on truncated hyperboloids. The interior energy functional on $\mathscr{H}_s^{\text{in}}$ associated to the linear counterpart of (1-5) is

$$\begin{aligned} E^{\text{in}}(s,W) &:= \frac{1}{2\pi}\iint_{\mathscr{H}_s^{\text{in}}}\Big(\frac{s}{t}\Big)^2|\partial_t W|^2+|\bar\partial W|^2+|\partial_y W|^2\,dx\,dy \\ &= \frac{1}{2\pi}\iint_{\mathscr{H}_s^{\text{in}}}\Big(\frac{s}{t}\Big)^2|\partial_x W|^2+t^{-2}|\mathscr{S}W|^2+t^{-2}\sum_{1\le i<j\le 3}|\Omega_{ij}W|^2+|\partial_y W|^2\,dx\,dy, \end{aligned} \tag{1-17}$$

where $|\bar\partial W|^2$ stands for $\sum_{i=1}^3|\bar\partial_i W|^2$. The above functional hence controls the square of the norms

$$\|(s/t)\partial W\|_{L^2(\mathscr{H}_s^{\text{in}})},\quad \|\bar\partial W\|_{L^2(\mathscr{H}_s^{\text{in}})},\quad \|\partial_y W\|_{L^2(\mathscr{H}_s^{\text{in}})},\quad t^{-1}\|\mathscr{S}W\|_{L^2(\mathscr{H}_s^{\text{in}})},\quad t^{-1}\|ZW\|_{L^2(\mathscr{H}_s^{\text{in}})}.$$

From Parseval's identity we have the decomposition

$$E^{\text{in}}(t,W)=E^{\text{in}}(t,W_0)+E^{\text{in}}(t,\mathtt{W}),$$

where $E^{\text{in}}(t,\mathtt{W})$ is defined by replacing W with $\mathtt{W}$ in (1-17) and

$$\begin{aligned} E^{\text{in}}(s,W_0) &:= \int_{\mathscr{H}_s^{\text{in}}}\Big(\frac{s}{t}\Big)^2|\partial_t W_0|^2+|\bar\partial W_0|^2\,dx \\ &= \int_{\mathscr{H}_s^{\text{in}}}\Big(\frac{s}{t}\Big)^2|\partial_x W_0|^2+t^{-2}|\mathscr{S}W_0|^2+t^{-2}\sum_{1\le i<j\le 3}|\Omega_{ij}W_0|^2\,dx. \end{aligned}$$

We also introduce and work with the conformal energy functional on truncated hyperboloids associated to the linear flat wave equation on $\mathbb{R}^{1+3}$. This is the functional defined as

$$E^{\text{c,in}}(s,W_0):=\int_{\mathscr{H}_s^{\text{in}}}\frac{1}{t^2}|KW_0+2tW_0|^2+\frac{s^2}{t^2}\sum_{i=1}^3|\Omega_{0i}W_0|^2\,dx, \tag{1-18}$$

where $K=(t^2+r^2)\partial_t+2rt\partial_r$ is the Morawetz multiplier. Since $\Omega_{ij}=(x_i/t)\Omega_{0j}-(x_j/t)\Omega_{0i}$, this functional controls the square of the norms

$$t^{-1}\|KW_0+2W_0\|_{L^2(\mathscr{H}_s^{\text{in}})} \quad \text{and} \quad \|(s/t)ZW_0\|_{L^2(\mathscr{H}_s^{\text{in}})}.$$

We point out that conformal energies on hyperboloids have also been used in other works; see for instance [Ma and Huang 2017; Wong 2017].

In the exterior region we will describe the evolution of (1-5) by means of the weighted energy functional

$$E^{\text{ex},\kappa}(t,W)=\frac{1}{2\pi}\iint_{\Sigma_t^{\text{ex}}}(2+r-t)^{\kappa+1}[|\partial_t W|^2+|\nabla_x W|^2+|\partial_y W|^2]\,dx\,dy \tag{1-19}$$

and of stronger norm $X^{\text{ex},\kappa}_{T_0}$

$$\|W\|^2_{X^{\text{ex},\kappa}_{T_0}}:=\sup_{t\in[2,T_0]}E^{\text{ex},\kappa}(t,W)+\frac{(1+\kappa)}{2\pi}\int_2^{T_0}\iint_{\Sigma_t^{\text{ex}}}(2+r-t)^{\kappa}(|\mathscr{T}W|^2+|\partial_y W|^2)\,dx\,dy\,dt, \tag{1-20}$$

where $|\mathscr{T}W|^2$ stands for $\sum_{i=1}^3|\mathscr{T}_iW|^2$. The bootstrap assumptions in this region will be energy bounds on the $X^{\text{ex},\kappa}_{T_0}$ norm of the solution for any arbitrarily fixed $T_0>2$ and $\kappa>0$. This norm not only controls the weighted energy of the solution but also the weighted L^2 space-time norm of its *good derivatives*: the tangential derivatives $\mathscr{T}$ to the cones $\{t-r=\text{const}\}$ and the derivative along the periodic direction ∂_y.

As a result of the global energy bounds that will be proved to hold in the exterior region (see Section 4) we also obtain a control on the energy of W on the exterior hyperboloids

$$\begin{aligned}E^{\text{ex},h}(s,W)&=\frac{1}{2\pi}\iint_{\mathscr{H}_s^{\text{ex}}}\Big(\frac{s}{t}\Big)^2|\partial_t W|^2+|\bar\partial W|^2+|\partial_y W|^2\,dx\,dy\\&=\frac{1}{2\pi}\iint_{\mathscr{H}_s^{\text{ex}}}\Big(\frac{s}{t}\Big)^2|\partial_x W|^2+t^{-2}|\mathscr{S}W|^2+t^{-2}\sum_{1\le i<j\le 3}|\Omega_{ij}W|^2+|\partial_y W|^2\,dx\,dy,\end{aligned} \tag{1-21}$$

as well as on the exterior conformal energy of W_0 on constant time slices Σ_t^{ex}

$$E^{\text{c,ex}}(t,W_0):=\int_{\Sigma_t^{\text{ex}}}|\mathscr{S}W_0+2W_0|^2+\sum_{i=1}^3|\Omega_{0i}W_0|^2\,dx. \tag{1-22}$$

1I. *The quasilinear energies.* Equation (1-5) is quasilinear and in order to propagate both the aforementioned interior and exterior a priori energy bounds we need to consider a cubic modification of the energies introduced in the previous subsection. Such *quasilinear energies* are defined as

$$\begin{aligned}E^{\text{in}}_{\text{quasi}}(s,W)&:=E^{\text{in}}(s,W)+\frac{1}{2\pi}\iint_{\mathscr{H}_s^{\text{in}}}u|\partial_y W|^2\,dx\,dy,\\E^{\text{ex},\kappa}_{\text{quasi}}(t,W)&:=E^{\text{ex},\kappa}(t,W)+\frac{1}{2\pi}\iint_{\Sigma_t^{\text{ex}}}(2+r-t)^{\kappa+1}u|\partial_y W|^2\,dx\,dy,\\E^{\text{ex},h}_{\text{quasi}}(s,W)&:=E^{\text{ex},h}(s,W)+\frac{1}{2\pi}\iint_{\mathscr{H}_s^{\text{ex}}}u|\partial_y W|^2\,dx\,dy.\end{aligned}$$

We also introduce the quasilinear modification of the stronger norm $X^{\mathrm{ex},\kappa}_{T_0}$

$$\|W\|^2_{X^{\mathrm{ex},\kappa}_{\mathrm{quasi},T_0}} := \sup_{t\in[2,T_0]} E^{\mathrm{ex},\kappa}_{\mathrm{quasi}}(t,W) + \frac{(1+\kappa)}{2\pi}\int_2^{T_0}\iint_{\Sigma^{\mathrm{ex}}_t}(2+r-t)^\kappa\big(|\mathscr{T}W|^2+(1+u)|\partial_y W|^2\big)\,dx\,dy\,dt.$$

We immediately observe that under smallness assumptions on u, e.g., $|u|\le\frac{1}{10}$, our starting energies are equivalent to their corresponding quasilinear counterparts; i.e., for any $E=\{E^{\mathrm{in}},E^{\mathrm{ex},\kappa},E^{\mathrm{ex},h}\}$

$$\tfrac{9}{10}E(t,W)\le E_{\mathrm{quasi}}(t,W)\le\tfrac{11}{10}E(t,W).$$

The same holds true for the stronger norms $X^{\mathrm{ex},\kappa}_{T_0}$ and $X^{\mathrm{ex},\kappa}_{\mathrm{quasi},T_0}$.

1J. ***Higher-order norms.*** We use the vector fields introduced before to define the higher-order counterparts of the energy functionals and of the stronger exterior norm $X^{\mathrm{ex},\kappa}_{T_0}$

$$E_n(s,W):=\sum_{|\gamma|\le n}E(s,\mathscr{Z}^\gamma W),\qquad E=\{E^{\mathrm{in}},E^{\mathrm{ex},\kappa},E^{\mathrm{ex},h}\},$$
$$\|W\|_{X^{n,\kappa}_{T_0}}:=\sum_{|\gamma|\le n}\|\mathscr{Z}^\gamma W\|_{X^{\mathrm{ex},\kappa}_{T_0}}.$$

The higher-order energies of W_0 and $\mathtt{W}$ are defined analogously. We observe that the above higher-order energies control the high Sobolev regularity of the solution in the interior and exterior regions respectively and also keep track of the Z vector fields applied to the solution in addition to usual derivatives. In the interior region it will be important to keep track of the precise number of Klainerman vector fields acting on the $\mathtt{W}$-component of the solution and to that purpose we also introduce the energy

$$E^{\mathrm{in}}_{n,k}(s,\mathtt{W}):=\sum_{\mathscr{I}_{n,k}}E^{\mathrm{in}}(s,\mathscr{Z}^\gamma\mathtt{W}),$$

where $\mathscr{I}_{n,k}$ denotes the set of indexes of type (n,k). We finally introduce the higher-order counterparts of the conformal energy functionals (1-18) and (1-22) in order to control the conformal energies of pure products of Klainerman vector fields acting on W_0

$$E^{\mathrm{c,in}}_n(s,W_0):=\sum_{|\beta|\le n}E^{\mathrm{c,in}}(s,Z^\beta W_0),\qquad E^{\mathrm{c,ex}}_n(t,W_0):=\sum_{|\beta|\le n}E^{\mathrm{c,ex}}(s,Z^\beta W_0).$$

2. Overview of the proof

The proof of our main theorem is based on the combination of a classical local well-posedness result for (1-5) with a bootstrap argument. We will perform this argument separately in the interior region $\mathscr{D}^{\mathrm{in}}$ and the exterior region $\mathscr{D}^{\mathrm{ex}}$ in which we divide the space-time $\mathbb{R}^{1+3}\times\mathbb{S}^1$.

The bootstrap assumptions in the exterior region $\mathscr{D}^{\mathrm{ex}}$ are uniform-in-time energy bounds on the higher-order stronger norm $X^{5,\kappa}_{T_0}$ of the solution W for any arbitrarily fixed $T_0>2$ and $\kappa>0$:

$$\|W\|^2_{X^{5,\kappa}_{T_0}}\le 2C_0^2\epsilon^2. \tag{2-1}$$

The result we want to prove in the exterior region is the following:

Proposition 2.1. *There exists a constant $C_0 > 0$ sufficiently large and a constant $\epsilon_0 > 0$ sufficiently small such that for every $0 < \epsilon < \epsilon_0$ if $W = (u, v)^T$ is a solution to* (1-5)–(1-6) *in an interval* $[2, T_0]$ *and satisfies the energy bounds* (2-1) *then actually*

$$\|W\|^2_{X^{5,\kappa}_{T_0}} \le C_0^2 \epsilon^2.$$

In the above proposition the time T_0 is arbitrary; therefore the solution W exists globally in $\mathscr{D}^{\text{ex}}$ and satisfies the energy bound (2-1) for all times $T_0 > 2$. In particular, we have

$$\|\mathscr{Z}^{\le 5} W\|_{X^{\text{ex},\kappa}_\infty} := \lim_{T_0 \to \infty} \|\mathscr{Z}^{\le 5} W\|_{X^{\text{ex},\kappa}_{T_0}} \le 2C_0^2 \epsilon^2. \tag{2-2}$$

We also observe that, as a consequence of (2-1), there exists an integrable function $l \in L^1([2, T_0])$ and the following bounds hold true for all $t \in [2, T_0]$:

$$\|(2+r-t)^{(\kappa+1)/2} \partial \mathscr{Z}^{\le 5} W(t)\|_{L^2(\Sigma_t^{\text{ex}})} \le \sqrt{2} C_0 \epsilon, \tag{2-3}$$

$$\|(2+r-t)^{\kappa/2} \mathscr{T} \mathscr{Z}^{\le 5} W\|_{L^2(\Sigma_t^{\text{ex}})} + \|(2+r-t)^{\kappa/2} \partial_y \mathscr{Z}^{\le 5} W\|_{L^2(\Sigma_t^{\text{ex}})} \le C_0 \epsilon \sqrt{l(t)}. \tag{2-4}$$

The bootstrap assumptions in the interior region $\mathscr{D}^{\text{in}}$ are higher-order energy bounds on hyperboloids for the W_0- and $\mathtt{W}$-components of the solution and pointwise bounds on the Z derivative of W_0. Given an arbitrarily fixed $s_0 > 2$, these are

$$E_5^{\text{in}}(s, W_0) \le 2A^2 \epsilon^2, \tag{2-5}$$

$$E_{5,k}^{\text{in}}(s, \mathtt{W}) \le 2A^2 \epsilon^2 s^{2\delta_k}, \quad k = \overline{0, 5}, \tag{2-6}$$

for all $s \in [2, s_0]$ and

$$|Z W_0(t, x)| \le 2B\epsilon t^{-1} s^\sigma, \quad (t, x) \in \mathscr{H}^{\text{in}}_{[2, s_0]}. \tag{2-7}$$

In the above inequalities the parameters σ, δ_k are fixed small universal constants satisfying $0 < \sigma \ll \delta_k \ll \delta_{k+1}$ for $k = \overline{1, 4}$, $\delta_0 = 0$ and A and B are large universal constants which we will improve as a part of the conclusion of the proof. The result we want to prove in this region requires the global exterior energy bounds (2-2) and can be stated as follows:

Proposition 2.2. *There exist two constants $A, B > 0$ sufficiently large, $0 < \epsilon_0, \sigma, \delta_k \ll 1$ sufficiently small with $\delta_0 = 0$ and $\sigma \ll \delta_k \ll \delta_{k+1}$ for $k = \overline{1, 4}$ such that for every $0 < \epsilon < \epsilon_0$ if $W = (u, v)^T$ is a solution to* (1-5)–(1-6) *in the region $\mathscr{H}^{\text{in}}_{[2, s_0]} \cup \mathscr{D}^{\text{ex}}$ and satisfies the global exterior energy bounds* (2-2) *as well as the interior bounds* (2-5)–(2-7) *for all $s \in [2, s_0]$, then it actually satisfies the enhanced interior bounds*

$$E_5^{\text{in}}(s, W_0) \le A^2 \epsilon^2,$$

$$E_{5,k}^{\text{in}}(s, \mathtt{W}) \le A^2 \epsilon^2 s^{2\delta_k}, \quad k = \overline{0, 5},$$

$$|Z W_0(t, x)| \le B\epsilon t^{-1} s^\sigma$$

for all $s \in [2, s_0]$ and all $(t, x) \in \mathscr{H}^{\text{in}}_{[2, s_0]}$.

In the above proposition the hyperbolic time s_0 is arbitrary, which implies the global existence of the solution in the interior region $\mathscr{D}^{\text{in}}$.

The proof of Propositions 2.1 and 2.2 are classical in that they are built around two main steps: (i) pointwise bounds derived from the a priori energy estimates and (ii) vector field energy estimates. Some important remarks:

(a) As for wave-Klein–Gordon systems, the fact that (1-1) is not scaling-invariant prevents us from using Klainerman–Sobolev inequalities on constant time slices, which foliate the entire space-time and would yield pointwise bounds for the solution without distinguishing between interior and exterior regions. Such inequalities

$$(1+t+|x|)^2(1+|t-|x||)|W(t,x,y)|^2 \le \sum_{|\alpha|\le 3} \|\Gamma^\alpha W(t,\cdot)\|^2_{L^2(\mathbb{R}^3\times\mathbb{S}^1)}, \quad \Gamma\in\{\Omega_{ij},\Omega_{0j},\mathscr{S}\},$$

require, in fact, a good control on the L^2 norm of $\mathscr{S}W$ and its higher-order derivatives, which we do not have since $\mathscr{S}$ does not commute with the linear part of our system. For solutions arising from compactly supported data and hence supported in the interior of the cone $t=r+1$ this problem is overcome by the fact that Klainerman–Sobolev inequalities on hyperboloids — which entirely foliate this region — do not involve the scaling vector field and only require a good control of the L^2 norm of the hyperbolic derivatives of the solution (see Lemma 5.1). In the case of data that are not compactly supported and only have a mild decay at infinity, however, one also needs to treat the exterior region. The use of the scaling vector field in this region is avoided by recurring to weighted Sobolev inequalities in which the weight is a function of the distance to the cone (see Section 4A). This motivates our decomposition of the space-time into interior and exterior regions and the fact that the proof is done separately in these two regions.

(b) The second step of our proof, i.e., the propagation of the a priori vector field energy estimates, consists in writing a higher-order (interior and exterior respectively) energy inequality for the solution and in using the pointwise bounds previously obtained to perturbatively estimate the source terms (commutator terms and null terms) appearing in the equation of $\mathscr{Z}^\gamma W$ for $|\gamma|\le 5$. The quadratic null interactions satisfy good (i.e., integrable in time) L^2 estimates thanks to their representation via (1-15) or (1-16). Some commutator terms, on the contrary, only show a slow decay in time that is at the limit of integrability. Let us look for instance at the product $Z^\beta u_0\cdot\partial_y^2\mathtt{W}$, which appears in the equation of $\mathscr{Z}^\gamma\mathtt{W}$ from the commutator $[\mathscr{Z}^\gamma, u\partial_y^2]\mathtt{W}$ when γ is a multi-index of type (k,k), i.e., when $\mathscr{Z}^\gamma = Z^\beta$ is a pure product of Klainerman vector fields, after decomposing u according to (1-7). In the interior region, the L^2 norm of such a term can only be controlled in terms of the (square root of the) conformal energy $E^{\mathrm{c,in}}_{k-1}(s,W_0)$, but even assuming the sharp bounds

$$E^{\mathrm{c,in}}_{k-1}(s,W_0)\lesssim \epsilon^2 s \quad\text{and}\quad \sup_{\mathscr{H}^{\mathrm{in}}_s}|\partial_y^2\mathtt{W}|\lesssim \epsilon^2 s^{-3/2},$$

we are not able to recover a better L^2 bound than

$$\|Z^\beta u_0\cdot\partial_y^2\mathtt{W}\|_{L^2_{xy}(\mathscr{H}^{\mathrm{in}}_s)} \le E^{\mathrm{c,in}}_{k-1}(s,W_0)^{1/2}\|\partial_y^2\mathtt{W}\|_{L^\infty(\mathscr{H}^{\mathrm{in}}_s)}\lesssim \epsilon^2 s^{-1}.$$

These problematic commutator terms — which are absent in the equation of $\mathscr{Z}^\gamma W_0$ — prevent us from obtaining uniform-in-time energy bounds for $\mathtt{W}$, which explains why we distinguish between the energies of W_0 and of $\mathtt{W}$ in Proposition 2.2 and do not propagate uniform-in-time energy bounds for the latter. In the exterior region the commutator terms are treated using a weighted Hardy inequality (see Lemma 4.5) and do not lead to the same type of issue discussed above thanks to the weights; see step (2b) in the proof of Proposition 2.1.

(c) As observed in the previous point, it is important to have a sharp decay $\|\mathtt{W}\|_{L^\infty} \lesssim s^{-3/2}$ when estimating some of the commutator terms arising in the equation of $\mathscr{Z}^\gamma \mathtt{W}$. However, the pointwise bounds that we obtain from the energy in the interior region by means of Klainerman–Sobolev inequalities are not optimal due to the slow growth in time assumed in (2-6); see estimates (5-3). Therefore, we need to study the equation satisfied by $\mathtt{W}$ in order to recover more suitable pointwise bounds; see Proposition 5.4. We also point out that, since Klainerman–Sobolev inequalities only yield pointwise bounds for the usual derivatives of W_0, we also need to recover L^∞-L^∞ estimates for ZW_0 using the equation it satisfies in order to propagate (2-7).

(d) The interior and exterior energy inequalities will be obtained from the integration of the relation (3-6) over the interior and exterior regions respectively, which share a boundary (let us call it here $\mathscr{C}$) along the cone $t = r + 1$. From Stokes' theorem, these inequalities will both involve a boundary term (an integral over the region $\mathscr{C}$ that is controlled given the behavior of the solution in the exterior region) which feeds information from the exterior to the interior region.

The paper is structured as follows. In Section 3 we derive the energy inequalities for the linearized equation, both in the exterior and the interior regions. We also derive the conformal energy inequalities. In Section 4 we prove the global existence of the solution in the exterior region. In Section 5 we recover the pointwise estimates for $\mathtt{W}$ and W_0 in the interior region. Finally, in Section 6 we improve the energy estimates in the interior region, concluding the proof of Theorem 1.

3. The linearized equation

The purpose of this section is to write the energy inequality and the conformal energy inequality for the linearized equation associated to (1-5) in both the interior and exterior regions $\mathscr{D}^{\text{in}}$ and $\mathscr{D}^{\text{ex}}$. This set of inequalities will repeatedly be used in the following sections when propagating the higher-order energy assumptions, as the equations satisfied by the differentiated unknown will be cast in the form (3-1).

We will look at the linear inhomogeneous equation

$$(-\partial_t^2 + \Delta_x)\boldsymbol{W} + (1+u)\partial_y^2 \boldsymbol{W} = \boldsymbol{F}, \quad (t, x, y) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{S}^1, \tag{3-1}$$

where u is assumed to be a sufficiently small function, e.g. $|u| \le \frac{1}{10}$. We start by proving a weighted energy inequality on the exterior constant time slices Σ_t^{ex} for general linear inhomogeneous equations of the above form. In the following proposition the lifespan $T_0 > 2$ is arbitrary, $\mathscr{D}^{\text{ex}}_{T_0}$ denotes the portion of exterior region in the time strip $[2, T_0]$ and $\mathscr{C}_{[2,T_0]}$ is its null boundary

$$\mathscr{D}^{\text{ex}}_{T_0} = \{(t, x, y) \in \mathscr{D}^{\text{ex}} : 2 \le t \le T_0\},$$
$$\mathscr{C}_{[2,T_0]} = \{(t, x, y) \in \mathscr{D}^{\text{ex}} : 2 \le t \le T_0,\ r = t - 1\}.$$

Proposition 3.1. *Let $\boldsymbol{W}$ be a solution to* (3-1) *and $l \in L^1([2, T_0])$. Suppose that u is a function satisfying the following pointwise bounds in the exterior region $\mathscr{D}^{\mathrm{ex}}_{T_0}$*:

$$\|u\|_{L^\infty(\mathscr{D}^{\mathrm{ex}}_{T_0})} \le \tfrac{1}{10}, \tag{3-2}$$

$$\|(2+r-t)^{1/2}\partial_y u(t,x,\cdot)\|_{L^\infty(\mathbb{S}^1)} \lesssim \epsilon\sqrt{l(t)}, \tag{3-3}$$

$$\|(2+r-t)\partial u(t,x,\cdot)\|_{L^\infty(\mathbb{S}^1)} \lesssim \epsilon. \tag{3-4}$$

For any fixed $\kappa > 0$ the following inequality holds true:

$$\|\boldsymbol{W}\|^2_{X^{\mathrm{ex},\kappa}_{T_0}} + \iint_{\mathscr{C}_{[2,T_0]}} (2+r-t)^{\kappa+1}(|\mathscr{T}\boldsymbol{W}|^2 + |\partial_y \boldsymbol{W}|^2)\,dS \lesssim E^{\mathrm{ex},\kappa}(2,\boldsymbol{W}) + \|(2+r-t)^{(\kappa+1)/2}\boldsymbol{F}\|_{L^1_t L^2_{xy}(\mathscr{D}^{\mathrm{ex}}_{T_0})}\|\boldsymbol{W}\|_{X^{\mathrm{ex},\kappa}_{T_0}}, \tag{3-5}$$

where dS is the surface element of $\mathscr{C}_{[2,T_0]}$.

We remark that the result of Proposition 3.1 can be proved for any positive and increasing weight $\omega = \omega(r-t)$ only depending on the distance from the cone $\{t = r\}$ if the hypotheses on the function u are changed appropriately.

Proof. From the smallness assumption on u it will be enough to prove the inequality in the statement with $E^{\mathrm{ex},\kappa}(t,\boldsymbol{W})$ and $\|\boldsymbol{W}\|_{X^{\mathrm{ex},\kappa}_{T_0}}$ replaced by $E^{\mathrm{ex},\kappa}_{\mathrm{quasi}}(t,\boldsymbol{W})$ and $\|\boldsymbol{W}\|_{X^{\mathrm{ex},\kappa}_{\mathrm{quasi},T_0}}$ respectively. A simple computation shows that for any positive weight $\omega = \omega(r-t)$ one has

$$\begin{aligned}\omega(r-t)\partial_t \boldsymbol{W}[\Box_{x,y}\boldsymbol{W} + u\,\partial_y^2 \boldsymbol{W}] &= -\tfrac{1}{2}\partial_t[\omega(r-t)((\partial_t\boldsymbol{W})^2 + |\nabla_x \boldsymbol{W}|^2 + (1+u)(\partial_y\boldsymbol{W})^2)] \\ &\quad + \mathrm{div}_x(\omega(r-t)\partial_t\boldsymbol{W}\,\nabla_x\boldsymbol{W}) + \partial_y(\omega(r-t)(1+u)\partial_t\boldsymbol{W}\partial_y\boldsymbol{W}) \\ &\quad - \tfrac{1}{2}\omega'(r-t)(|\mathscr{T}\boldsymbol{W}|^2 + (1+u)(\partial_y\boldsymbol{W})^2) - \omega(r-t)\big(\partial_y u\,\partial_t\boldsymbol{W}\partial_y\boldsymbol{W} - \tfrac{1}{2}\partial_t u(\partial_y\boldsymbol{W})^2\big).\end{aligned} \tag{3-6}$$

We consider the case $\omega(z) = (2+z)^{\kappa+1}$ and integrate the above equality over the exterior region $\mathscr{D}^{\mathrm{ex}}_{T_0}$. We use Stokes' theorem when integrating in the (t,x)-variables, with normal vectors

$$\boldsymbol{n}_{\Sigma^{\mathrm{ex}}_{T_0}} = (1,0,0,0), \quad \boldsymbol{n}_{\Sigma^{\mathrm{ex}}_2} = -\boldsymbol{n}_{\Sigma^{\mathrm{ex}}_{T_0}}, \quad \boldsymbol{n}_{\mathscr{C}_{[2,T_0]}} = \Big(1, -\frac{x}{r}\Big). \tag{3-7}$$

We observe that

$$(\partial_t\boldsymbol{W})^2 + |\nabla_x\boldsymbol{W}|^2 + 2\frac{x}{r}\cdot\nabla_x\boldsymbol{W}\partial_t\boldsymbol{W} = |\mathscr{T}\boldsymbol{W}|^2$$

and hence recover

$$\begin{aligned}&\|\boldsymbol{W}\|^2_{X^{\mathrm{ex},\kappa}_{\mathrm{quasi},T_0}} + \iint_{\mathscr{C}_{[2,T_0]}} (2+r-t)^{\kappa+1}[|\mathscr{T}\boldsymbol{W}|^2 + (1+u)|\partial_y\boldsymbol{W}|^2]\,dS \\ &\le E^{\mathrm{ex},\kappa}_{\mathrm{quasi}}(2,\boldsymbol{W}) + \frac{1}{\pi}\iint_{\mathscr{D}^{\mathrm{ex}}_{T_0}} (2+r-t)^{\kappa+1}\big(|\boldsymbol{F}\,\partial_t\boldsymbol{W}| + |\partial_y u\,\partial_t\boldsymbol{W}\partial_y\boldsymbol{W}| + \tfrac{1}{2}|\partial_t u|\,(\partial_y\boldsymbol{W})^2\big)\,dx\,dy\,dt.\end{aligned} \tag{3-8}$$

We remark that the above inequality holds actually true for any index $\kappa > -1$. The smallness of u ensures that the integral in the above left-hand side is equivalent to the one in the left-hand side of inequality (3-5).

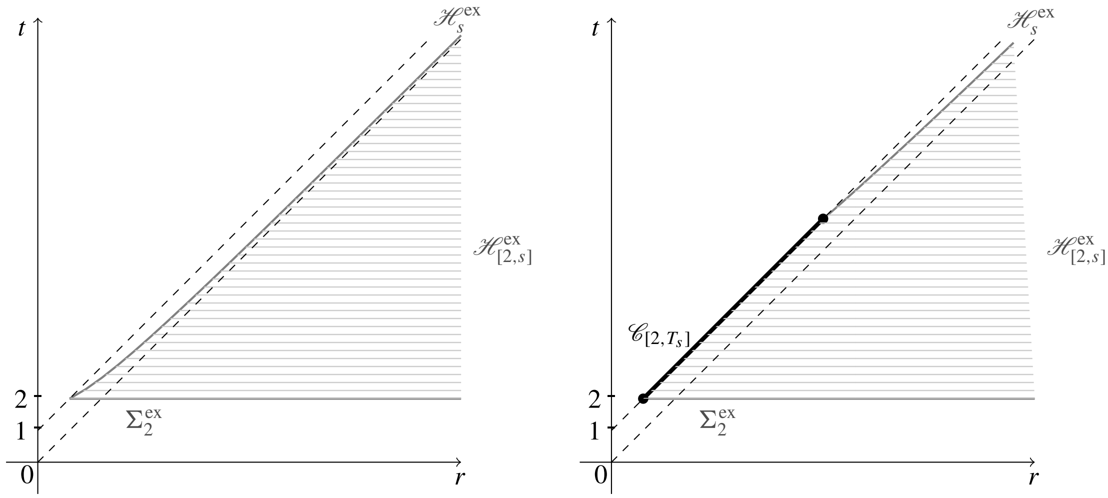

Figure 2. Vertical section of the region $\mathscr{H}_{[2,s]}^{\mathrm{ex}}$ and its foliation projected onto $\mathbb{R}^{1+3}$. In the left picture, $s=\sqrt{3}$; in the right one, $s>\sqrt{3}$ and the two bullets delimit the later boundary $\mathscr{C}_{[2,T_s]}$. The dashed lines represent the cones $t=r$ and $t=r+1$.

We also see, from the assumption (3-3) and the Cauchy–Schwarz inequality, that

$$\begin{aligned}\iint_{\mathscr{D}_{T_0}^{\mathrm{ex}}}(2+r-t)^{\kappa+1}|\partial_y u\partial_t \boldsymbol{W}\partial_y \boldsymbol{W}|\,dx\,dy\,dt &\lesssim \|(2+r-t)^{(\kappa+1)/2}\partial_y u\,\partial_y \boldsymbol{W}\|_{L^1_tL^2_{xy}(\mathscr{D}_{T_0}^{\mathrm{ex}})}\|\boldsymbol{W}\|_{X_{T_0}^{\mathrm{ex},\kappa}}\\ &\lesssim \|(2+r-t)^{1/2}\partial_y u\|_{L^2_tL^\infty_{xy}(\mathscr{D}_{T_0}^{\mathrm{ex}})}\|(2+r-t)^{\kappa/2}\partial_y \boldsymbol{W}\|_{L^2_{txy}(\mathscr{D}_{T_0}^{\mathrm{ex}})}\|\boldsymbol{W}\|_{X_{T_0}^{\mathrm{ex},\kappa}}\\ &\lesssim \epsilon\|\sqrt{l(\cdot)}\|_{L^2_t}\|\boldsymbol{W}\|^2_{X_{T_0}^{\mathrm{ex},\kappa}}\lesssim \epsilon\|\boldsymbol{W}\|^2_{X_{T_0}^{\mathrm{ex},\kappa}}\end{aligned}$$

and from (3-4)

$$\iint_{\mathscr{D}_{T_0}^{\mathrm{ex}}}(2+r-t)^{\kappa+1}|\partial_t u|(\partial_y \boldsymbol{W})^2\,dx\,dy\,dt \lesssim \epsilon\iint_{\mathscr{D}_{T_0}^{\mathrm{ex}}}(2+r-t)^{\kappa}(\partial_y \boldsymbol{W})^2\,dx\,dy\,dt\lesssim \epsilon\|\boldsymbol{W}\|^2_{X_{T_0}^{\mathrm{ex},\kappa}};$$

hence the above two integrals can be absorbed in the left-hand side of (3-8) if $\epsilon\ll 1$ is sufficiently small. Finally, from the Cauchy–Schwarz inequality we also have

$$\iint_{\mathscr{D}_{T_0}^{\mathrm{ex}}}(2+r-t)^{\kappa+1}|\boldsymbol{F}\partial_t \boldsymbol{W}|\,dx\,dy\,dt\le \|(2+r-t)^{(\kappa+1)/2}\boldsymbol{F}\|_{L^1_tL^2_{xy}(\mathscr{D}_{T_0}^{\mathrm{ex}})}\|\boldsymbol{W}\|_{X_{T_0}^{\mathrm{ex},\kappa}}. \qquad \square$$

If we assume that the function u satisfies the pointwise bounds (3-2)–(3-4) in the whole exterior region $\mathscr{D}^{\mathrm{ex}}$ we can also recover an inequality for the energy on exterior truncated hyperboloids $\mathscr{H}_s^{\mathrm{ex}}$ for any $s>0$, that is, on any branch of hyperboloid contained in the exterior region. We observe that $\mathscr{H}_{[2,s]}^{\mathrm{ex}}$ corresponds to the portion of exterior region delimited by $\mathscr{H}_s$ and Σ_2^{ex} whenever $0<s\le\sqrt{3}$ and by $\mathscr{H}_s^{\mathrm{ex}}$, $\mathscr{C}_{[2,T_s]}$ and Σ_2^{ex} whenever $s>\sqrt{3}$, where $T_s=\frac{1}{2}(s^2+1)$. We remind the definition the stronger norm $\|\cdot\|_{X_\infty^{\mathrm{ex},\kappa}}$ over the interval $[2,\infty)$

$$\|\boldsymbol{W}\|_{X_\infty^{\mathrm{ex},\kappa}}:=\lim_{T\to\infty}\|\boldsymbol{W}\|_{X_T^{\mathrm{ex},\kappa}}.$$

Proposition 3.2. *Assume that $\boldsymbol{W}$ is solution to the linear inhomogeneous equation* (3-1) *with u satisfying the decay properties* (3-2)–(3-4) *in the whole region $\mathscr{D}^{\mathrm{ex}}$. For any $s>0$*

$$E^{\mathrm{ex},h}(s,\boldsymbol{W})+\delta_{s>\sqrt{3}}\iint_{\mathscr{C}_{[2,T_s]}}|\mathscr{T}\boldsymbol{W}|^2+|\partial_y\boldsymbol{W}|^2\,dS\lesssim E^{\mathrm{ex},0}(2,\boldsymbol{W})+\epsilon\|\boldsymbol{W}\|^2_{X^{\mathrm{ex},0}_\infty}+\|\boldsymbol{F}\|_{L^1_tL^2_{xy}(\mathscr{H}^{\mathrm{ex}}_{[2,s]})}\|\boldsymbol{W}\|_{X^{\mathrm{ex},0}_\infty},$$

where $\delta_{s>\sqrt{3}}=0$ if $s\le\sqrt{3}$ and is 1 *otherwise.*

Proof. From the smallness assumption on u it will be enough to prove the statement with $E^{\mathrm{ex},h}(t,\boldsymbol{W})$ and $E^{\mathrm{ex},0}(2,\boldsymbol{W})$ replaced by $E^{\mathrm{ex},h}_{\mathrm{quasi}}(t,\boldsymbol{W})$ and $E^{\mathrm{ex},0}_{\mathrm{quasi}}(2,\boldsymbol{W})$ respectively. We integrate the relation (3-6) in the case where $\omega\equiv1$ over the region $\mathscr{H}^{\mathrm{ex}}_{[2,s]}$ and use Stokes's theorem when integrating in the (t,x)-variables, with normal vectors given in (3-7) and $\boldsymbol{n}_{\mathscr{H}^{\mathrm{ex}}_s}=(1,-x/t)$. We obtain

$$\begin{aligned}E^{\mathrm{ex},h}_{\mathrm{quasi}}(s,\boldsymbol{W})&+\delta_{s>\sqrt{3}}\iint_{\mathscr{C}_{[2,T_s]}}|\mathscr{T}\boldsymbol{W}|^2+(1+u)|\partial_y\boldsymbol{W}|^2\,dS\\&\le E^{\mathrm{ex},0}_{\mathrm{quasi}}(2,\boldsymbol{W})+\frac{1}{\pi}\iint_{\mathscr{H}^{\mathrm{ex}}_{[2,s]}}|\partial_yu\,\partial_t\boldsymbol{W}\partial_y\boldsymbol{W}|+\frac12|\partial_tu|\,(\partial_y\boldsymbol{W})^2+|\boldsymbol{F}\partial_t\boldsymbol{W}|\,dx\,dy\,dt.\end{aligned}$$

We foliate $\mathscr{H}^{\mathrm{ex}}_{[2,s]}$ by the constant time slices Σ^s_t for $t\ge2$ (see Figure 2), where

$$\Sigma^s_t=\bigl\{x\in\mathbb{R}^3:r\ge\max(t-1,\sqrt{t^2-s^2})\bigr\}\times\mathbb{S}^1,$$

and from the Cauchy–Schwarz inequality, the assumptions (3-3), (3-4) and the definition of the norm $X^{\mathrm{ex},0}_\infty$ we immediately see

$$\iint_{\mathscr{H}^{\mathrm{ex}}_{[2,s]}}|\partial_yu\,\partial_y\boldsymbol{W}\,\partial_y\boldsymbol{W}|+\tfrac12|\partial_tu|(\partial_y\boldsymbol{W})^2\,dx\,dy\,dt\lesssim\epsilon\|\boldsymbol{W}\|^2_{X^{\mathrm{ex},0}_\infty}.$$

Furthermore

$$\iint_{\mathscr{H}^{\mathrm{ex}}_{[2,s]}}|\boldsymbol{F}\partial_t\boldsymbol{W}|\,dx\,dy\,dt\le\|\boldsymbol{F}\|_{L^1_tL^2_{xy}(\mathscr{H}^{\mathrm{ex}}_{[2,s]})}\|\boldsymbol{W}\|_{X^{\mathrm{ex},0}_\infty}.$$ □

We now prove the interior energy inequality for (3-1). In the following proposition the hyperbolic lifespan $s_0>2$ is arbitrary and $\mathscr{C}_{[2,s]}$ will denote the later boundary of the hyperbolic region $\mathscr{H}^{\mathrm{in}}_{[2,s]}$, which is included in $\mathscr{C}_{[2,T_s]}$ for $T_s=\frac12(s^2+1)$:

$$\begin{aligned}\mathscr{C}_{[2,s]}&=\bigl\{(r+1,x):\tfrac12\le r\le\tfrac12(s^2-1)\bigr\}\times\mathbb{S}^1\\&=\bigl\{(t,t-1):\tfrac32\le t\le\tfrac12(s^2+1)\bigr\}\times\mathbb{S}^2\times\mathbb{S}^1.\end{aligned}$$

Proposition 3.3. *Let $\boldsymbol{W}$ be a solution to* (3-1) *and suppose that u is a function that satisfies the following bounds in the hyperbolic region $\mathscr{H}^{\mathrm{in}}_{[2,s_0]}$*

$$\|u\|_{L^\infty(\mathscr{H}^{\mathrm{in}}_{[2,s_0]})}\le\tfrac{1}{10},\tag{3-9}$$

$$|\partial_tu_0(t,x)|\lesssim\epsilon t^{-1/2}s^{-1},\tag{3-10}$$

$$\|\partial\mathtt{u}(t,x,\cdot)\|_{L^\infty(\mathbb{S}^1)}\lesssim\epsilon t^{-3/2+\delta}\tag{3-11}$$

for some small $\delta > 0$, *where* $u_0 = \fint_{\mathbb{S}^1} u(t,x,y)\,dy$ *and* $\mathtt{u} = u - u_0$. *Then*

$$E^{\text{in}}(s, \boldsymbol{W}) \lesssim E^{\text{in}}(2, \boldsymbol{W}) + \iint_{\mathscr{C}_{[2,s]}} |\mathscr{T}\boldsymbol{W}|^2 + |\partial_y \boldsymbol{W}|^2\,dS + \int_2^s \|\boldsymbol{F}\|_{L^2_{xy}(\mathscr{H}^{\text{in}}_\tau)} E^{\text{in}}(\tau, \boldsymbol{W})^{1/2}\,d\tau \tag{3-12}$$

for all $s \in [2, s_0]$.

Proof. For any fixed $s \in [2, s_0]$, we integrate the equality (3-6) with $\omega \equiv 1$ over the region $\mathscr{H}^{\text{in}}_{[2,s]}$ (see Figure 1), which we foliate for hyperboloids $\mathscr{H}^{\text{in}}_\tau$ with $\tau \in [2, s]$. We use Stokes' theorem when integrating in the variables (t, x), with normal vectors given by

$$\boldsymbol{n}_{\mathscr{H}^{\text{in}}_s} = \Big(1, -\frac{x}{t}\Big), \quad \boldsymbol{n}_{\mathscr{H}^{\text{in}}_2} = -\boldsymbol{n}_{\mathscr{H}^{\text{in}}_s}, \quad \boldsymbol{n}_{\mathscr{C}_{[2,s]}} = \Big(-1, \frac{x}{r}\Big) \tag{3-13}$$

and obtain

$$E^{\text{in}}_{\text{quasi}}(s, \boldsymbol{W}) \le E^{\text{in}}_{\text{quasi}}(2, \boldsymbol{W}) + \iint_{\mathscr{C}_{[2,s]}} |\mathscr{T}\boldsymbol{W}|^2 + (1+u)|\partial_y \boldsymbol{W}|^2\,dS$$
$$+ \frac{1}{\pi}\int_2^s \int_{\mathscr{H}^{\text{in}}_\tau} \Big(\frac{\tau}{t}\Big)|\partial_y u\,\partial_t \boldsymbol{W}\partial_y \boldsymbol{W}| + \frac{1}{2}\Big(\frac{\tau}{t}\Big)|\partial_t u|(\partial_y \boldsymbol{W})^2 + \Big(\frac{\tau}{t}\Big)|\boldsymbol{F}\partial_t \boldsymbol{W}|\,dx\,dy\,d\tau. \tag{3-14}$$

The integral on the null boundary $\mathscr{C}_{[2,s]}$ in the above right-hand side is bounded by its counterpart in the right-hand side of (3-12) thanks to the smallness assumption (3-9). From the assumption (3-11) and the fact that $\tau \le t$ we derive

$$\int_2^s \int_{\mathscr{H}^{\text{in}}_\tau} \Big(\frac{\tau}{t}\Big)|\partial_y u\,\partial_t \boldsymbol{W}\partial_y \boldsymbol{W}| + \frac{1}{2}\Big(\frac{\tau}{t}\Big)|\partial_t \mathtt{u}|\,(\partial_y \boldsymbol{W})^2\,dx\,dy\,d\tau$$
$$\lesssim \int_2^s \|\partial \mathtt{u}\|_{L^\infty(\mathscr{H}^{\text{in}}_\tau)} E^{\text{in}}(\tau, \boldsymbol{W})\,d\tau \lesssim \epsilon \int_2^s \tau^{-3/2+\delta} E^{\text{in}}(\tau, \boldsymbol{W})\,d\tau,$$

while from (3-10) we have

$$\int_2^s \int_{\mathscr{H}^{\text{in}}_\tau} \frac{1}{2}\Big(\frac{\tau}{t}\Big)|\partial_t u_0|\,(\partial_y \boldsymbol{W})^2\,dx\,dy\,d\tau \lesssim \epsilon \int_2^s \tau^{-3/2} E^{\text{in}}(\tau, \boldsymbol{W})\,d\tau.$$

The Cauchy–Schwarz inequality yields

$$\int_2^s \int_{\mathscr{H}^{\text{in}}_\tau} \Big(\frac{\tau}{t}\Big)|\boldsymbol{F}\partial_t \boldsymbol{W}|\,dx\,dy\,d\tau \le \int_2^{s_0} \|\boldsymbol{F}\|_{L^2_{xy}(\mathscr{H}^{\text{in}}_\tau)} E^{\text{in}}(\tau, \boldsymbol{W})^{1/2}\,d\tau,$$

and finally the use of the Gronwall inequality concludes the proof of the statement. □

As detailed in Section 6, it will be important to distinguish between the two components W_0 and $\mathtt{W}$ of the solution W to (1-5), in particular to show that the energies associated to the zero mode W_0 are uniformly bounded in time. We will make use of the following classical result about the energy on interior truncated hyperboloids of solutions of linear inhomogeneous wave equations on the flat space $\mathbb{R}^{1+3}$.

Proposition 3.4. *Let* $\boldsymbol{W}_0$ *be a solution of the linear inhomogeneous wave equation*

$$(-\partial_t^2 + \Delta_x)\boldsymbol{W}_0 = \boldsymbol{F}_0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3. \tag{3-15}$$

For all $s \in [2, s_0]$ *we have the energy inequality*

$$E^{\text{in}}(s, \boldsymbol{W}_0) \le E^{\text{in}}(2, \boldsymbol{W}_0) + \int_{\mathscr{C}_{[2,s]}} |\mathscr{T}\boldsymbol{W}_0|^2\,dS + 2\int_2^s \|\boldsymbol{F}_0\|_{L^2_x(\mathscr{H}^{\text{in}}_\tau)} E^{\text{in}}(\tau, \boldsymbol{W}_0)^{1/2}\,d\tau.$$

We also prove below the interior and exterior conformal energy inequalities for linear inhomogeneous wave equations on $\mathbb{R}^{1+3}$. We will, in fact, need to control the higher-order conformal energies of the zero-mode W_0 of our solution W in order to recover pointwise bounds for W_0 and ZW_0 later in the paper.

Proposition 3.5. *Let $\boldsymbol{W}_0$ be a solution to* (3-15). *Then*

$$E^{\mathrm{c,in}}(s,\boldsymbol{W}_0)\le E^{\mathrm{c,in}}(2,\boldsymbol{W}_0)+\int_2^s\|\tau\boldsymbol{F}_0\|_{L^2(\mathscr{H}_\tau^{\mathrm{in}})}E^{\mathrm{c,in}}(\tau,\boldsymbol{W}_0)^{1/2}\,d\tau \\ +\int_{\mathscr{C}_{[2,s]}}|(t+r)(\partial_t\boldsymbol{W}_0+\partial_r\boldsymbol{W}_0)+2\boldsymbol{W}_0|^2+(t-r)^2(|\nabla_x\boldsymbol{W}_0|^2-(\partial_r\boldsymbol{W}_0)^2)\,dS.$$

Proof. Let $K=(t^2+r^2)\partial_t+2rt\partial_r$ denote the Morawetz multiplier. We have the equality

$$\begin{aligned}-(K\boldsymbol{W}_0+2t\boldsymbol{W}_0)\Box_x\boldsymbol{W}_0 &= \partial_t\big[\tfrac12(t^2+r^2)(|\partial_t\boldsymbol{W}_0|^2+|\nabla_x\boldsymbol{W}_0|^2)+2rt\partial_t\boldsymbol{W}_0\partial_r\boldsymbol{W}_0+2t\boldsymbol{W}_0\partial_t\boldsymbol{W}_0-\boldsymbol{W}_0^2\big]\\ &\quad-\mathrm{div}_x[(t^2+r^2)\partial_t\boldsymbol{W}_0\nabla_x\boldsymbol{W}_0+2rt\partial_r\boldsymbol{W}_0\nabla_x\boldsymbol{W}_0+tx(|\partial_t\boldsymbol{W}_0|^2-|\nabla_x\boldsymbol{W}_0|^2)+2t\boldsymbol{W}_0\nabla_x\boldsymbol{W}_0],\end{aligned} \tag{3-16}$$

which we integrate over the interior hyperbolic region $\mathscr{H}^{\mathrm{in}}_{[2,s]}$. We apply Stokes' theorem (recall the normal vectors listed (3-13)). First, we compute the contribution to the integral on $\mathscr{H}_s^{\mathrm{in}}$. Algebraic computations show that

$$\begin{aligned}&\tfrac12(t^2+r^2)(|\partial_t\boldsymbol{W}_0|^2+|\nabla_x\boldsymbol{W}_0|^2)+2rt\partial_t\boldsymbol{W}_0\partial_r\boldsymbol{W}_0\\ &\qquad+[(t^2+r^2)\partial_t\boldsymbol{W}_0\nabla_x\boldsymbol{W}_0+2rt\partial_r\boldsymbol{W}_0\nabla_x\boldsymbol{W}_0+tx(|\partial_t\boldsymbol{W}_0|^2-|\nabla_x\boldsymbol{W}_0|^2)]\cdot\frac{x}{t}\\ &=\frac{1}{2t^2}|K\boldsymbol{W}_0|^2+\frac{s^2}{2t^2}\sum_{i=1}^3|\Omega_{0i}\boldsymbol{W}_0|^2\end{aligned}$$

and

$$2t\boldsymbol{W}_0\partial_t\boldsymbol{W}_0-\boldsymbol{W}_0^2+2t\boldsymbol{W}_0\nabla_x\boldsymbol{W}_0\cdot\frac{x}{t}=\frac{2}{t}\boldsymbol{W}_0\,K\boldsymbol{W}_0-\frac{r}{t}\Omega_{0r}(\boldsymbol{W}_0^2)-\boldsymbol{W}_0^2,$$

where $\Omega_{0r}=r\partial_t+t\partial_r$. The parametrization of $\mathscr{H}_s^{\mathrm{in}}$ by the hyperbolic angle θ, i.e.,

$$\mathscr{H}_s^{\mathrm{in}}=\{(s\cosh\theta,\,s\sinh\theta):\theta\in[\theta_1,\theta_2]\}$$

for some suitable θ_1,θ_2, and the change of variables $r=s\sinh\theta$ allows us to see that

$$\begin{aligned}\int_{\mathscr{H}_s^{\mathrm{in}}}\frac{r}{t}\Omega_{0r}(\boldsymbol{W}_0)^2\,dx &= \int_0^{(s^2-1)/2}\int_{\mathbb{S}^2}\frac{r}{t}\Omega_{0r}(\boldsymbol{W}_0)^2r^2\,dr\,d\sigma\\ &=\int_{\theta_1}^{\theta_2}\int_{\mathbb{S}^2}\frac{\sinh\theta}{\cosh\theta}\partial_\theta(\boldsymbol{W}_0)^2s^3\sinh^2\theta\cosh\theta\,d\theta\,d\sigma\\ &=-3\int_{\theta_1}^{\theta_2}\int_{\mathbb{S}^2}\boldsymbol{W}_0^2s^3\sinh^2\theta\cosh\theta\,d\theta\,d\sigma+\left[\int_{\mathbb{S}^2}\boldsymbol{W}_0^2s^3\sinh^3\theta\,d\sigma\right]_{\theta=\theta_1}^{\theta=\theta_2}\\ &=-3\int_{\mathscr{H}_s^{\mathrm{in}}}\boldsymbol{W}_0^2\,dx+\left[\int_{\mathbb{S}^2}\boldsymbol{W}_0^2r^3\,d\sigma\right]_{r=r_1}^{r=r_2},\end{aligned}$$

where $r_1 = s\sinh\theta_1 = \frac{1}{2}$ and $r_2 = s\sinh\theta_2 = \frac{1}{2}(s^2-1)$. Therefore the integral on the boundary $\mathscr{H}_s^{\text{in}}$ equals

$$\int_{\mathscr{H}_s^{\text{in}}} \frac{1}{2t^2}|K\boldsymbol{W}_0|^2 + \frac{s^2}{2t^2}\sum_{i=1}^3 |\Omega_{0i}\boldsymbol{W}_0|^2 + \frac{2}{t}\boldsymbol{W}_0\, K\boldsymbol{W}_0 + 2\boldsymbol{W}_0^2\, dx - \Big[\int_{\mathbb{S}^2} \boldsymbol{W}_0^2 s^3\sinh^3\theta\, d\sigma\Big]_{\theta=\theta_1}^{\theta=\theta_2}$$
$$= \int_{\mathscr{H}_s^{\text{in}}} \frac{1}{2t^2}|K\boldsymbol{W}_0 + 2\boldsymbol{W}_0|^2 + \frac{s^2}{2t^2}\sum_{i=1}^3 |\Omega_{0i}\boldsymbol{W}_0|^2\, dx - \Big[\int_{\mathbb{S}^2} \boldsymbol{W}_0^2 s^3\sinh^3\theta\, d\sigma\Big]_{\theta=\theta_1}^{\theta=\theta_2}.$$

We now compute the contributions to the integral on $\mathscr{C}_{[2,s]}$. Algebraic computations show that

$$\begin{aligned}
&\tfrac{1}{2}(t^2+r^2)(|\partial_t\boldsymbol{W}_0|^2 + |\nabla_x\boldsymbol{W}_0|^2) + 2rt\,\partial_t\boldsymbol{W}_0\partial_r\boldsymbol{W}_0\\
&\qquad + [(t^2+r^2)\partial_t\boldsymbol{W}_0\nabla_x\boldsymbol{W}_0 + 2rt\,\partial_r\boldsymbol{W}_0\nabla_x\boldsymbol{W}_0 + tx(|\partial_t\boldsymbol{W}_0|^2 - |\nabla_x\boldsymbol{W}_0|^2)]\cdot\frac{x}{r}\\
&= \tfrac{1}{2}(t+r)^2(\partial_t\boldsymbol{W}_0 + \partial_r\boldsymbol{W}_0)^2 + \tfrac{1}{2}(|\nabla_x\boldsymbol{W}_0|^2 - (\partial_r\boldsymbol{W}_0)^2)
\end{aligned}$$

and

$$\begin{aligned}
2t\boldsymbol{W}_0\partial_t\boldsymbol{W}_0 - \boldsymbol{W}_0^2 + 2t\boldsymbol{W}_0\nabla_x\boldsymbol{W}_0\cdot\frac{x}{r} &= 2t\boldsymbol{W}_0(\partial_t\boldsymbol{W}_0 + \partial_r\boldsymbol{W}_0) - \boldsymbol{W}_0^2\\
&= 2(t+r)\boldsymbol{W}_0(\partial_t\boldsymbol{W}_0 + \partial_r\boldsymbol{W}_0) - 2r\boldsymbol{W}_0(\partial_t\boldsymbol{W}_0 + \partial_r\boldsymbol{W}_0) - \boldsymbol{W}_0^2.
\end{aligned}$$

In the coordinates $(u, v) = (t-r, t+r)$ we have that $\partial_t + \partial_r = \partial_v$ and moreover the lateral boundary can be expressed as

$$\mathscr{C}_{[2,s]} = \Big\{\Big(\frac{v+1}{2}, \frac{v-1}{2}, \sigma\Big) : 2 \le v \le s^2,\ \sigma\in\mathbb{S}^2\Big\}.$$

Using polar coordinates and then the change of coordinates $r = \frac{1}{2}(v-1)$, we see that

$$\begin{aligned}
\int_{\mathscr{C}_{[2,s]}} 2r\boldsymbol{W}_0(\partial_t\boldsymbol{W}_0 + \partial_r\boldsymbol{W}_0)\, d\Sigma_s &= \int_{r_1}^{r_2}\int_{\mathbb{S}^2} (\partial_t+\partial_r)(\boldsymbol{W}_0^2)r^3\, dr\, d\sigma = \int_2^{s^2}\int_{\mathbb{S}^2} \partial_v(\boldsymbol{W}_0)^2\Big(\frac{v-1}{2}\Big)^3\frac{dv}{2}\, d\sigma\\
&= -3\int_2^{s^2}\int_{\mathbb{S}^2} \boldsymbol{W}_0^2\Big(\frac{v-1}{2}\Big)^2\frac{dv}{2}\, d\sigma + \Big[\int_{\mathbb{S}^2}\boldsymbol{W}_0^2\Big(\frac{v-1}{2}\Big)^3 d\sigma\Big]_{v=2}^{v=s^2}\\
&= -3\int_{\mathscr{C}_{[2,s]}} \boldsymbol{W}_0^2\, d\Sigma_s + \Big[\int_{\mathbb{S}^2}\boldsymbol{W}_0^2 r^3\, d\sigma\Big]_{r=r_1}^{r=r_2}.
\end{aligned}$$

Therefore, the integral on the lateral boundary $\mathscr{C}_{[2,s]}$ equals to the opposite of

$$\begin{aligned}
&\int_{\mathscr{C}_{[2,s]}} \frac{1}{2}(t+r)^2(\partial_t\boldsymbol{W}_0 + \partial_r\boldsymbol{W}_0)^2 + 2(t+r)\boldsymbol{W}_0(\partial_t\boldsymbol{W}_0 + \partial_r\boldsymbol{W}_0) + 2\boldsymbol{W}_0^2\, d\Sigma_s\\
&\qquad + \frac{1}{2}\int_{\mathscr{C}_{[2,s]}} (|\nabla_x\boldsymbol{W}_0|^2 - (\partial_r\boldsymbol{W}_0)^2)\, d\Sigma_s + \Big[\int_{\mathbb{S}^2}\boldsymbol{W}_0^2 r^3\, d\sigma\Big]_{r=r_1}^{r=r_2}\\
&= \frac{1}{2}\int_{\mathbb{S}^2} |(t+r)(\partial_t\boldsymbol{W}_0 + \partial_r\boldsymbol{W}_0) + 2\boldsymbol{W}_0|^2 + (|\nabla_x\boldsymbol{W}_0|^2 - (\partial_r\boldsymbol{W}_0)^2)\, d\Sigma_s + \Big[\int_{\mathbb{S}^2}\boldsymbol{W}_0^2 r^3\, d\sigma\Big]_{r=r_1}^{r=r_2}.
\end{aligned}$$

Summing everything up proves the result of the statement. □

Proposition 3.6. *Let $\boldsymbol{W}_0$ be a solution to* (3-15). *Then*

$$E^{c,ex}(T,\boldsymbol{W}_0)+\int_{\mathscr{C}_{[2,T]}}|(t+r)(\partial_t\boldsymbol{W}_0+\partial_r\boldsymbol{W}_0)+2\boldsymbol{W}_0|^2+(t-r)^2(|\nabla\boldsymbol{W}_0|^2-(\partial_r\boldsymbol{W}_0)^2)\,dS \\ \le E^{c,ex}(2,\boldsymbol{W}_0)+\int_2^T\|(t+r)\boldsymbol{F}_0\|_{L^2_x(\Sigma^{ex}_t)}E^{c,ex}(t,\boldsymbol{W}_0)^{1/2}\,dt.$$

Proof. The result of the proposition follows from the integration of equality (3-16) over the region $\mathscr{D}^{ex}_T$ combined with Stokes' theorem and the equality

$$K\boldsymbol{W}_0+2t\boldsymbol{W}_0=t(\mathscr{S}\boldsymbol{W}_0+2\boldsymbol{W}_0)+r\Omega_{0r}\boldsymbol{W}_0. \qquad \square$$

In the interior region we will recover pointwise bounds on ZW_0 from the higher-order conformal energies of W_0 via Klainerman–Sobolev inequalities on hyperboloids (see Lemma 5.1). This will require a control on the conformal energy of the solution on a portion of the hyperboloid $\mathscr{H}_s$ in the exterior region $\mathscr{D}^{ex}$ that in turn will be obtained from a control on the conformal energy on the flat hypersurfaces Σ^s_t defined below. We hence state the following modification of the conformal energy inequality.

Proposition 3.7. *For any s, T_1, T_2 with $2\le s<T_1<T_2$ and any $t\in[T_1,T_2]$, let*

$$\Sigma^s_t:=\{x\in\mathbb{R}^3:|x|\ge\sqrt{t^2-s^2}\}$$

and

$$E^{c,ex}_s(t,\boldsymbol{W}_0):=\int_{\Sigma^s_t}|\mathscr{S}W_0+2W_0|^2+\sum_{i=1}^3|\Omega_{0i}W_0|^2\,dx.$$

Then

$$E^{c,ex}_s(T_2,\boldsymbol{W}_0)+\int_{\mathscr{H}_s\cap[T_1,T_2]}\frac{1}{t^2}|K\boldsymbol{W}_0+2t\boldsymbol{W}_0|^2+\frac{s^2}{t^2}\sum_{i=1}^3|\Omega_{0i}\boldsymbol{W}_0|^2\,dx \\ \le E^{c,ex}_s(T_1,\boldsymbol{W}_0)+\int_{T_1}^{T_2}\|(t+r)\boldsymbol{F}_0\|_{L^2_x(\Sigma^s_t)}\,E^{c,ex}_s(t,\boldsymbol{W}_0)^{1/2}\,dt.$$

Proof. The result of the statement follows by integrating (3-16) over the region bounded by $\Sigma^s_{T_2}$, $\Sigma^s_{T_1}$ and $\mathscr{H}_s\cap[T_1,T_2]$, which can be foliated by the hypersurfaces Σ^s_t for $t\in[T_1,T_2]$. $\square$

4. Global existence in the exterior region

The main goal of this section is to prove Proposition 2.1, that is, the propagation of the a priori energy bounds on the weighted higher-order exterior energies of the solution. This in turn will imply the global existence of the solution to the Cauchy problem (1-1)–(1-2) under the assumptions of Theorem 1.

The proof of Proposition 2.1 unfolds in two main steps:

(1) We recover sharp pointwise bounds from the energy assumptions (2-1).

(2) We compute the equation satisfied by the differentiated variable $\mathscr{Z}^\gamma W$ and compare it to the inhomogeneous linear equation (3-1) in order to use the energy inequality of Proposition 3.1. We use the energy assumptions and the pointwise bounds obtained in step (1) to perturbatively estimate the source terms.

The main tools used in steps (1) and (2) are weighted Sobolev and Hardy inequalities, with weights depending on the distance from the light cone. We mention here that these inequalities already played an important role in the proof of [Lindblad and Rodnianski 2010] of the global stability of the Minkowski space-time for the vacuum Einstein equations. We also observe that, as a result of proving global energy bounds in the exterior region, we obtain bounds for the higher-order conformal energy $E^{\mathrm{c,ex}}_5(t, W)$ for all $t \geq 2$ and a uniform-in-time control of the higher-order energies $E^{\mathrm{ex},h}_5(s, W)$ on exterior hyperboloids, for all $s \geq 2$.

This section is organized as follows: in Sections 4A and 4B we prove weighted Sobolev and Hardy inequalities; in Section 4C we recover the aforementioned pointwise bounds; in Section 4D we finally propagate the bounds (2-1).

4A. *Weighted Sobolev inequalities.*

Lemma 4.1. *Let $\beta \in \mathbb{R}$. For any sufficiently smooth function w we have*

$$\sup_{\Sigma^{\mathrm{ex}}_t} (2+r-t)^\beta r^2 |w(t,x,y)|^2 \lesssim \iint_{\Sigma^{\mathrm{ex}}_t} (2+r-t)^{\beta+1}(\partial_r \mathscr{Z}^{\leq 2} w)^2 + (2+r-t)^{\beta-1}(\mathscr{Z}^{\leq 2} w)^2 \, dx\, dy. \tag{4-1}$$

Proof. Let (r, σ) be the spherical coordinates in $\mathbb{R}^3$, $r = |x|$ and $\sigma = x/|x| \in \mathbb{S}^2$. We begin by observing that the Sobolev embedding $H^2(\mathbb{S}^2 \times \mathbb{S}^1) \subset L^\infty(\mathbb{S}^2 \times \mathbb{S}^1)$ implies

$$\sup_{\mathbb{S}^2\times\mathbb{S}^1} |w(t,r,\sigma,y)|^2 \leq \sum_{0\leq l+k\leq 2} \int |\nabla^l_\sigma \partial^k_y w(t,r,\sigma,y)|^2 \, d\sigma\, dy.$$

We then remark that for any function v and $(t,x,y) \in \Sigma^{\mathrm{ex}}_t$

$$\begin{aligned}\partial_r[(2+r-t)^\beta r^2\, v(t,x,y)^2] &= 2(2+r-t)^\beta r^2\, v\partial_r v + \beta(2+r-t)^{\beta-1} r^2 v^2 + 2(2+r-t)^\beta r v^2 \\ &\geq 2(2+r-t)^\beta r^2 v\partial_r v + \beta(2+r-t)^{\beta-1} r^2 v^2,\end{aligned}$$

so if v is compactly supported in x we can write

$$\begin{aligned}(2+r-t)^\beta r^2\, v(t,x,y)^2 &= -\int_r^\infty \partial_\rho[(2+\rho-t)^\beta \rho^2\, v(t,x,y)^2]\, d\rho \\ &\lesssim_\beta \int_r^\infty (2+\rho-t)^\beta |v\partial_\rho v|\rho^2\, d\rho + \int_r^\infty (2+\rho-t)^{\beta-1} v^2 \rho^2\, d\rho \\ &\lesssim_\beta \int_r^\infty (2+\rho-t)^{\beta+1}(\partial_\rho v)^2\, \rho^2\, d\rho + \int_r^\infty (2+\rho-t)^{\beta-1} v^2\rho^2\, d\rho.\end{aligned} \tag{4-2}$$

By replacing v with $\nabla^l_\sigma \partial^k_y w(t,r,\sigma,y)$ for $k+l \leq 2$ we obtain (4-1) in the case where w is compactly supported. In the general case where w is not compactly supported we consider a cut-off function $\chi \in C^\infty_0(\mathbb{R})$ and apply the inequality (4-1) to $\chi(\epsilon r)w$ for any $\epsilon > 0$

$$\begin{aligned}\sup_{\Sigma^{\mathrm{ex}}_t} (2+r-t)^\beta r^2 |\chi(\epsilon r) w|^2 \lesssim \sum_{k+l\leq 2} \iint_{\Sigma^{\mathrm{ex}}_t} &(2+r-t)^{\beta+1}(\partial_r \nabla^k_\sigma \partial^l_y w)^2 + (2+r-t)^{\beta-1}(\nabla^k_\sigma \partial^l_y w)^2\, dx\, dy \\ &+ \sum_{k+l\leq 2} \iint_{\Sigma^{\mathrm{ex}}_t} (2+r-t)^{\beta+1}\epsilon^2 |\chi'(\epsilon r)|^2 (\nabla^k_\sigma \partial^l_y w)^2\, dx\, dy.\end{aligned}$$

On the intersection of Σ_t^{ex} with the support of $\chi'(\epsilon r)$ we have that $(2+r-t)^2\epsilon^2 \lesssim 1$ so

$$\iint_{\Sigma_t^{\mathrm{ex}}} (2+r-t)^{\beta+1}\epsilon^2|\chi'(\epsilon r)|^2(\nabla_\sigma^k \partial_y^l w)^2\,dx\,dy \lesssim \iint_{\Sigma_t^{\mathrm{ex}}} (2+r-t)^{\beta-1}(\nabla_\sigma^k \partial_y^l w)^2\,dx\,dy.$$

By letting $\epsilon \to 0$ we derive (4-1) also in the case of noncompactly supported w. □

Slight modifications of the above proof yield the following three results.

Lemma 4.2. *Let $\beta \in \mathbb{R}$. For a sufficiently regular function w we have*

$$\sup_{\Sigma_t^{\mathrm{ex}}} (2+r-t)^\beta r^2 |w(t,x,y)|^2 \lesssim \iint_{\Sigma_t^{\mathrm{ex}}} (2+r-t)^\beta((\partial_r \mathscr{Z}^{\le 2} w)^2 + (\mathscr{Z}^{\le 2} w)^2)\,dx\,dy. \tag{4-3}$$

Proof. It follows by estimating v and $\partial_\rho v$ in (4-2) with the same weight and using the fact that $2+r-t \ge 1$ on Σ_t^{ex}. □

Lemma 4.3. *Let $\beta \in \mathbb{R}$. For any sufficiently regular function w we have*

$$\sup_{\Sigma_t^{\mathrm{ex}}} (2+r-t)^\beta r^2 \|w(t,r,\cdot)\|^2_{L^2(\mathbb{S}^2\times\mathbb{S}^1)} \lesssim \iint_{\Sigma_t^{\mathrm{ex}}} (2+r-t)^{\beta+1}(\partial_r w)^2 + (2+r-t)^{\beta-1} w^2\,dx\,dy. \tag{4-4}$$

Proof. The inequality (4-4) follows by replacing v with the $L^2(\mathbb{S}^2\times\mathbb{S}^1)$ norm of w in the left- and right-hand sides of inequality (4-2). □

Lemma 4.4. *Let $\beta \in \mathbb{R}$. For any sufficiently regular function w we have*

$$\sup_{\Sigma_t^{\mathrm{ex}}} (2+r-t)^\beta r^2 \|w(t,r,\cdot)\|^2_{L^4(\mathbb{S}^2\times\mathbb{S}^1)} \lesssim \iint_{\Sigma_t^{\mathrm{ex}}} (2+r-t)^\beta (\partial_r^{\le 1} \mathscr{Z}^{\le 1} w)^2\,dx\,dy. \tag{4-5}$$

Proof. The inequality (4-5) follows by estimating v and $\partial_\rho v$ in (4-2) with the same weight, then applying the inequality with v replaced by the $L^4(\mathbb{S}^2\times\mathbb{S}^1)$ norm of w and finally using the Sobolev injection $H^1(\mathbb{S}^2\times\mathbb{S}^1) \subset L^4(\mathbb{S}^2\times\mathbb{S}^1)$. □

4B. *Weighted Hardy inequality.*

Lemma 4.5. *Let $\beta > -1$. For any sufficiently regular function w for which the integral in the following left-hand side is finite we have*

$$\iint_{\Sigma_t^{\mathrm{ex}}} (2+r-t)^\beta w^2\,dx\,dy \lesssim \iint_{\Sigma_t^{\mathrm{ex}}} (2+r-t)^{\beta+2}(\partial_r w)^2\,dx\,dy.$$

Proof. A simple computation shows that for any $\beta \in \mathbb{R}$ and $(t,x,y) \in \Sigma_t^{\mathrm{ex}}$

$$\partial_r[r^2(2+r-t)^{\beta+1}] = 2r(2+r-t)^{\beta+1} + (\beta+1)r^2(2+r-t)^\beta \ge (\beta+1)r^2(2+r-t)^\beta.$$

Consequently, if $\beta > -1$ and w is a compactly supported function

$$\begin{aligned}\int_{r\geq t-1}(2+r-t)^{\beta}w^2\,dx &= \int_{t-1}^{\infty}\int_{\mathbb{S}^2}(2+r-t)^{\beta}r^2w^2\,dr\,d\sigma \leq \frac{1}{\beta+1}\int_{t-1}^{\infty}\int_{\mathbb{S}^2}\partial_r(r^2(2+r-t)^{\beta+1})w^2\,dr\,d\sigma\\ &= -\frac{2}{\beta+1}\int_{r\geq t-1}(2+r-t)^{\beta+1}w\partial_r w\, r^2\,dr\,d\sigma - \frac{1}{\beta+1}\left[\int_{\mathbb{S}^2\times\mathbb{S}^1} w^2r^2\,d\sigma\right]_{r=t-1}\\ &\lesssim \left(\int_{r\geq t-1}(2+r-t)^{\beta}w^2\,dx\right)^{1/2}\left(\int_{r\geq t-1}(2+r-t)^{\beta+2}(\partial_r w)^2\,dx\right)^{1/2},\end{aligned}$$

and the inequality of the statement follows after further integration on $\mathbb{S}^1$.

Let now w be any sufficiently regular function, not necessarily compactly supported, and χ be any fixed cut-off function. For any $\epsilon > 0$, we apply the inequality of the statement to the compactly supported function $\chi(\epsilon r)w$ and obtain

$$\iint_{\Sigma_t^{\mathrm{ex}}}(2+r-t)^{\beta}\chi(\epsilon r)^2w^2\,dx\,dy \lesssim \iint_{\Sigma_t^{\mathrm{ex}}}(2+r-t)^{\beta+2}[\chi(\epsilon r)^2(\partial w)^2+\epsilon^2\chi'(\epsilon r)^2w^2]\,dx\,dy.$$

On the support of $\chi'(\epsilon r)$ we have $\epsilon^2(2+r-t)^2\lesssim 1$. Using that

$$\lim_{\epsilon\to 0}\int_{r\geq 1/\epsilon}(2+r-t)^{\beta}w^2\,dx = 0,$$

we obtain the result of the statement by passing to the limit $\epsilon\to 0$. □

4C. ***Pointwise bounds.***

Proposition 4.6. *Assume that the solution $W = (u, v)^T$ to* (1-5)–(1-6) *satisfies the a priori energy bounds* (2-1) *for some fixed $T_0 > 2$ and $\kappa > 0$. There exists an integrable function $l \in L^1([2, T_0])$ such that the following pointwise estimates hold true in $\mathscr{D}^{\mathrm{ex}}_{T_0}$:*

$$\sup_{\mathbb{S}^1}|W|\lesssim\epsilon, \tag{4-6}$$

$$\sup_{\mathbb{S}^1}|\partial\mathscr{Z}^{\leq 2}W|\lesssim C_0\epsilon r^{-1}(2+r-t)^{-(\kappa+1)/2}, \tag{4-7}$$

$$\sup_{\mathbb{S}^1}|\mathscr{T}\mathscr{Z}^{\leq 2}W|+|\partial_y\mathscr{Z}^{\leq 2}W|+|\mathscr{Z}^{\leq 2}\mathtt{W}|\lesssim C_0\epsilon r^{-1}\sqrt{l(t)}(2+r-t)^{-\kappa/2}, \tag{4-8}$$

$$\sup_{\mathbb{S}^1}|Z\mathscr{Z}^{\leq 2}W|\lesssim C_0\epsilon r^{-1}(2+r-t)^{-\kappa/2}. \tag{4-9}$$

Proof. The bounds (4-7) and (4-8) follow immediately from Lemma 4.2 with $\beta=\kappa+1$ and $\beta=\kappa$ respectively, from (2-4), the Poincaré inequality and the energy bounds (2-3), (2-4). The pointwise bound (4-9) follows instead applying Lemma 4.1 with $\beta=\kappa$ and Lemma 4.5 with $\beta=\kappa-1$ to write that

$$\iint_{\Sigma_t^{\mathrm{ex}}}(2+r-t)^{\kappa-1}(Z\mathscr{Z}^{\leq 4}W)^2\,dx\,dy \lesssim \iint_{\Sigma_t^{\mathrm{ex}}}(2+r-t)^{\kappa+1}(\partial\mathscr{Z}^{\leq 5}W)^2\,dx\,dy.$$

Finally, the bound (4-6) on W is obtained from the integration of (4-7) along the direction $\partial_q=\partial_r-\partial_t$ until the initial time slice $t_0=2$ and from the smallness assumption on the initial data. □

A trivial consequence of the decomposition (1-7) and the bounds (4-8)–(4-9), that will be useful later in Section 5, is the following estimate for the zero-mode W_0 of the solution:

$$\sup_{\mathbb{S}^1} |Z\mathscr{Z}^{\le 1} W_0| \lesssim C_0\epsilon r^{-1}(2+r-t)^{-\kappa/2}. \tag{4-10}$$

4D. ***Propagation of the exterior energy bounds.*** We start by considering any multi-index γ with $|\gamma| = n \le 5$ and compare the system satisfied by the differentiated function $W^\gamma = (u^\gamma, v^\gamma)^T$ — which is a shorthand notation for $\mathscr{Z}^\gamma W = (\mathscr{Z}^\gamma u, \mathscr{Z}^\gamma v)$ — to the inhomogeneous linear equation (3-1). Here the variable that plays the role of the linear variable $\boldsymbol{W}$ is W^γ. We remind the reader that all vector fields $\mathscr{Z}$ in the family (1-14) are related to the geometry of the problem and are the generators of the Lorentz transformations of the Minkowski space $\mathbb{R}^{1+3}$. In particular, they preserve the structure of system (1-1) and equivalently of (1-5).

The equation satisfied by W^γ is obtained by commuting $\mathscr{Z}^\gamma$ to (1-5)

$$(-\partial_t^2 + \Delta_x)W^\gamma + (1+u)\partial_y^2 W^\gamma = F^\gamma, \tag{4-11}$$

where the inhomogeneous term F^γ is given by

$$\begin{aligned} F^\gamma &= -[\mathscr{Z}^\gamma, u\partial_y^2]W + \sum_{|\gamma_1|+|\gamma_2|\le|\gamma|} \boldsymbol{N}(W^{\gamma_1}, W^{\gamma_2}), \\ [\mathscr{Z}^\gamma, u\partial_y^2]W &= \delta_\gamma \sum_{\substack{|\gamma_1|+|\gamma_2|=|\gamma| \\ |\gamma_2|<|\gamma|}} u^{\gamma_1}\partial_y^2 W^{\gamma_2}, \end{aligned} \tag{4-12}$$

where $\delta_\gamma = 0$ if $|\gamma| = 0$, and $\delta_\gamma = 1$ otherwise. The nonlinear term $\boldsymbol{N}(\cdot\,,\cdot)$ in the right-hand side of (4-12) is a two-vector of new linear combinations of the quadratic null forms introduced in (1-3), which arise from the commutation of the Klainerman vector fields with $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$.

We have seen in Proposition 4.6 that under the a priori energy assumption (2-1) the solution W satisfies the pointwise bounds (4-6) and (4-7). The hypotheses of Proposition 3.1 are then fulfilled and we have

$$\begin{aligned} \|W^\gamma\|^2_{X^{\mathrm{ex},\kappa}_{T_0}} &+ \iint_{\mathscr{C}_{[2,T_0]}} (2+r-t)^{\kappa+1}(|\mathscr{T}W^\gamma|^2 + |\partial_y W^\gamma|^2)\,dS \\ &\lesssim E^{\mathrm{ex},\kappa}(2, W^\gamma) + \|(2+r-t)^{(\kappa+1)/2}F^\gamma\|_{L^1_t L^2_{xy}(\mathscr{D}^{\mathrm{ex}}_{T_0})}\|W^\gamma\|_{X^{\mathrm{ex},\kappa}_{T_0}}. \end{aligned} \tag{4-13}$$

Proof of Proposition 2.1. It is enough for our purpose to estimate the weighted norm of the source term F^γ and prove that for every $|\gamma| \le 5$

$$\|(2+r-t)^{(\kappa+1)/2}F^\gamma\|_{L^2_{xy}(\Sigma^{\mathrm{ex}}_t)} \lesssim C_0^2\epsilon^2(t-1)^{-(\kappa+1)/2}\sqrt{l(t)}, \tag{4-14}$$

where $l \in L^1([2, T_0])$. In fact, if we plug (4-14) and a priori energy bound (2-1) into (4-13) we obtain that there exists some universal positive constant C so that

$$\|W\|^2_{X^{5,\kappa}_{T_0}} + \sum_{|\gamma|\le 5} \iint_{\mathscr{C}_{[2,T_0]}} (2+r-t)^{\kappa+1}(|\mathscr{T}W^\gamma|^2 + |\partial_y W^\gamma|^2)\,dS \le CE^{\mathrm{ex},\kappa}_5(2, W) + 2CC_0^3\epsilon^3.$$

For any fixed constant $K > 1$ (e.g. $K = 2$) we then choose $C_0 > 0$ sufficiently large so that

$$E^{\mathrm{ex},\kappa}_5(2, W) \le \frac{C_0^2\epsilon^2}{CK}$$

and $\epsilon_0 > 0$ sufficiently small so that $2CC_0^2\epsilon < 1/K$ for $\epsilon \le \epsilon_0$ to finally obtain

$$\|W\|^2_{X^{5,\kappa}_{T_0}} + \sum_{|\gamma|\le 5} \iint_{\mathscr{C}_{[2,T_0]}} (2+r-t)^{\kappa+1}(|\mathscr{T} W^\gamma|^2 + |\partial_y W^\gamma|^2)\, dS \le \frac{2C_0\epsilon^2}{K}.$$

We estimate the different contributions to F^γ separately. In all the estimates that follow we will use the a priori energy bound (2-1) and the fact that $r > t-1$ in the exterior region.

(1) *The null terms*: We use here the null form representation via the formula (1-16)

$$N(W^{\gamma_1}, W^{\gamma_2}) \sim \mathscr{T} W^{\gamma_1}\cdot \partial W^{\gamma_2} + \partial W^{\gamma_1}\cdot \mathscr{T} W^{\gamma_2}.$$

The products in the above right-hand side are equivalent given the range of γ_1 and γ_2 and we only focus on the analysis of the first one. We distinguish between the different values that γ_1 and γ_2 can take and remind the reader that $|\gamma_1| + |\gamma_2| \le |\gamma| = n \le 5$.

(a) *The case* $|\gamma_1| = 0$: Here $\gamma_2 = \gamma$ and we immediately obtain from (4-8) that

$$\begin{aligned}\|(2+r-t)^{(\kappa+1)/2}\mathscr{T} W\cdot\partial W^\gamma\|_{L^2_{xy}} &\le \|\mathscr{T} W\|_{L^\infty_{xy}}\|W^\gamma\|_{X^{\mathrm{ex},\kappa}_{T_0}}\\ &\lesssim C_0^2\epsilon^2(t-1)^{-1}\sqrt{l(t)}.\end{aligned}$$

(b) *The case* $|\gamma_2| = 0$: Here $\gamma_1 = \gamma$ and we obtain from (4-7) that

$$\begin{aligned}\|(2+r-t)^{(\kappa+1)/2}\mathscr{T} W^\gamma\cdot\partial W\|_{L^2_{xy}} &\le \|(2+r-t)^{1/2}\partial W\|_{L^\infty_{xy}}\|(2+r-t)^{\kappa/2}\mathscr{T} W^\gamma\|_{L^2_{xy}}\\ &\lesssim C_0^2\epsilon^2(t-1)^{-1}\sqrt{l(t)}.\end{aligned}$$

(c) *The case* $|\gamma_1|, |\gamma_2| > 0$: We use spherical polar coordinates and the Cauchy–Schwarz inequality to bound the weighted $L^2_{xy}(\Sigma^{\mathrm{ex}}_t)$ norm of $\mathscr{T} W^{\gamma_1}\cdot\partial W^{\gamma_2}$ as follows:

$$\begin{aligned}\|(2+r-t)^{(\kappa+1)/2}\mathscr{T} W^{\gamma_1}\cdot\partial W^{\gamma_2}\|^2_{L^2_{xy}} &= \int_{t-1}^\infty \iint_{\mathbb{S}^2\times\mathbb{S}^1} (2+r-t)^{\kappa+1}|\mathscr{T} W^{\gamma_1}|^2\,|\partial W^{\gamma_2}|^2\, r^2\, dr\, d\sigma\, dy\\ &\lesssim \int_{t-1}^\infty (2+r-t)^{\kappa+1}\|\mathscr{T} W^{\gamma_1}\|^2_{L^4(\mathbb{S}^2\times\mathbb{S}^1)}\|\partial W^{\gamma_2}\|^2_{L^4(\mathbb{S}^2\times\mathbb{S}^1)}\, r^2\, dr.\end{aligned}$$

In the case where $|\gamma_1| \le n-2$ we apply the inequality (4-5) to $\mathscr{T} W^{\gamma_1}$ with $\beta = \kappa$ and Sobolev's injection $H^1(\mathbb{S}^2\times\mathbb{S}^1) \subset L^4(\mathbb{S}^2\times\mathbb{S}^1)$ to ∂W^{γ_2}. We derive that

$$\begin{aligned}&\int_{t-1}^\infty (2+r-t)^{\kappa+1}\|\mathscr{T} W^{\gamma_1}\|^2_{L^4(\mathbb{S}^2\times\mathbb{S}^1)}\|\partial W^{\gamma_2}\|^2_{L^4(\mathbb{S}^2\times\mathbb{S}^1)}\, r^2\, dr\\ &\qquad\lesssim \|(2+r-t)^{\kappa/2}\mathscr{T}\mathscr{Z}^{\le n} W\|^2_{L^2_{xy}(\Sigma^{\mathrm{ex}}_t)} \int_{t-1}^\infty (2+r-t) r^{-2}\|\partial\mathscr{Z}^{\le n} W\|^2_{L^2(\mathbb{S}^2\times\mathbb{S}^1)} r^2\, dr\\ &\qquad\lesssim (t-1)^{-2}\|(2+r-t)^{\kappa/2}\mathscr{T}\mathscr{Z}^{\le n} W\|^2_{L^2_{xy}(\Sigma^{\mathrm{ex}}_t)}\|(2+r-t)^{(\kappa+1)/2}\partial\mathscr{Z}^{\le n} W\|^2_{L^2_{xy}(\Sigma^{\mathrm{ex}}_t)}.\end{aligned}$$

In the remaining case where $|\gamma_1| = n-1$ and $|\gamma_2| = 1$ we apply the inequality (4-5) to ∂W^{γ_2} with $\beta = \kappa+1$ and the injection $H^1(\mathbb{S}^2\times\mathbb{S}^1) \subset L^4(\mathbb{S}^2\times\mathbb{S}^1)$ to $\mathscr{T} W^{\gamma_1}$. We get

$$\int_{t-1}^{\infty} (2+r-t)^{\kappa+1} \|\mathscr{T} W^{\gamma_1}\|^2_{L^4(\mathbb{S}^2\times\mathbb{S}^1)} \|\partial W^{\gamma_2}\|^2_{L^4(\mathbb{S}^2\times\mathbb{S}^1)}\, r^2\, dr \\ \lesssim \|(2+r-t)^{(\kappa+1)/2} \partial \mathscr{Z}^{\le n} W\|^2_{L^2_{xy}(\Sigma^{\mathrm{ex}}_t)} \iint_{\Sigma^{\mathrm{ex}}_t} r^{-2} |\mathscr{T}\mathscr{Z}^{\le n} W|^2\, dx\, dy \\ \lesssim (t-1)^{-2} \|(2+r-t)^{\kappa/2} \mathscr{T}\mathscr{Z}^{\le n} W\|^2_{L^2_{xy}(\Sigma^{\mathrm{ex}}_t)} \|W\|^2_{X^{n,\kappa}_{T_0}}.$$

In both scenarios we obtain

$$\|(2+r-t)^{(\kappa+1)/2} \mathscr{T} W^{\gamma_1} \cdot \partial W^{\gamma_2}\|_{L^2_{xy}} \lesssim C_0^2 \epsilon^2 (t-1)^{-1} \sqrt{l(t)}.$$

(2) *The commutator terms*: Since $|\gamma_1| \ge 1$, we can write $u^{\gamma_1} = \mathscr{Z} u^{\tilde{\gamma}_1}$ for some $|\tilde{\gamma}_1| = |\gamma_1| - 1$. We can also write $\partial_y^2 W^{\gamma_2} = \partial_y W^{\tilde{\gamma}_2}$ for some other $\tilde{\gamma}_2$ such that $|\tilde{\gamma}_2| = |\gamma_2| + 1$ and observe that then $|\tilde{\gamma}_1| + |\tilde{\gamma}_2| = n$. Depending on $\gamma_1 = (\alpha_1, \beta_1)$ we can distinguish two cases:

(a) *The case* $|\alpha_1| > 0$: Here we choose $\tilde{\gamma}_1$ so that $\mathscr{Z} u^{\tilde{\gamma}_1} = \partial u^{\tilde{\gamma}_1}$. The products $\partial u^{\tilde{\gamma}_1} \cdot \partial_y W^{\tilde{\gamma}_2}$ have the same behavior of the null terms treated in case 1.

(b) *The case* $|\alpha_1| = 0$: Here $\mathscr{Z}^{\gamma_1} = Z^{\beta_1}$ is a pure product of Klainerman vector fields and $\mathscr{Z} u^{\tilde{\gamma}_1} = Z u^{\tilde{\gamma}_1}$. We choose the exponents (p_1, p_2) using

$$(p_1, p_2) = \begin{cases} (2, \infty) & \text{if } |\tilde{\gamma}_1| = n-1, \\ (\infty, 2) & \text{if } |\tilde{\gamma}_2| = n, \\ (4, 4) & \text{otherwise} \end{cases}$$

and will place the two factors in $L^{p_1}(\mathbb{S}^2\times\mathbb{S}^1)$ and $L^{p_2}(\mathbb{S}^2\times\mathbb{S}^1)$ respectively. We use the Sobolev injections $H^2(\mathbb{S}^2\times\mathbb{S}^1) \subset L^\infty(\mathbb{S}^2\times\mathbb{S}^1)$ and $H^1(\mathbb{S}^2\times\mathbb{S}^1) \subset L^4(\mathbb{S}^2\times\mathbb{S}^1)$ to derive

$$\|(2+r-t)^{(\kappa+1)/2} Z u^{\tilde{\gamma}_1} \partial_y W^{\tilde{\gamma}_2}\|^2_{L^2_{xy}} \lesssim \int_{t-1}^{\infty} (2+r-t)^{\kappa+1} \|Z u^{\tilde{\gamma}_1}\|^2_{L^{p_1}(\mathbb{S}^2\times\mathbb{S}^1)} \|\partial_y W^{\tilde{\gamma}_2}\|^2_{L^{p_2}(\mathbb{S}^2\times\mathbb{S}^1)}\, r^2\, dr \\ \lesssim \int_{t-1}^{\infty} (2+r-t)^{\kappa+1} \|Z \mathscr{Z}^{\le 4} u\|^2_{L^2(\mathbb{S}^2\times\mathbb{S}^1)} \|\partial_y \mathscr{Z}^{\le 5} W\|^2_{L^2(\mathbb{S}^2\times\mathbb{S}^1)}\, r^2\, dr.$$

Applying the inequality (4-4) to $Z\mathscr{Z}^{\le 4} u$ with $\beta = \kappa$ and successively the weighted Hardy inequality proved in Lemma 4.5 with $\beta = \kappa - 1$ we find

$$\sup_{r \ge t-1} (2+r-t)^{\kappa} r^2 \|Z\mathscr{Z}^{\le 4} u\|^2_{L^2(\mathbb{S}^2\times\mathbb{S}^1)} \\ \lesssim \|(2+r-t)^{(\kappa+1)/2} \partial_r \mathscr{Z}^{\le 5} u\|^2_{L^2_{xy}(\Sigma^{\mathrm{ex}}_t)} + \|(2+r-t)^{(\kappa-1)/2} Z\mathscr{Z}^{\le 4} u\|^2_{L^2_{xy}(\Sigma^{\mathrm{ex}}_t)} \\ \lesssim \|(2+r-t)^{(\kappa+1)/2} \partial \mathscr{Z}^{\le 5} u\|^2_{L^2_{xy}(\Sigma^{\mathrm{ex}}_t)}.$$

We can therefore continue the previous chain of inequalities

$$\lesssim \|(2+r-t)^{(\kappa+1)/2} \partial_r \mathscr{Z}^{\le 5} u\|^2_{L^2_{xy}(\Sigma^{\mathrm{ex}}_t)} \int_{t-1}^{\infty} (2+r-t) r^{-2} \|\partial_y \mathscr{Z}^{\le 5} W\|^2_{L^2(\mathbb{S}^2\times\mathbb{S}^1)}\, r^2\, dr \\ \lesssim (t-1)^{-1-\kappa} \|W\|^2_{X^{5,\kappa}_{T_0}} \|(2+r-t)^{\kappa/2} \partial_y \mathscr{Z}^{\le 5} W\|^2_{L^2_{xy}(\Sigma^{\mathrm{ex}}_t)}$$

and finally conclude that

$$\sum_{\substack{|\gamma_1|+|\gamma_2|=n\\ |\gamma_1|\geq 1}} \|(2+r-t)^{(\kappa+1)/2}u^{\gamma_1}\cdot\partial_y^2 W^{\gamma_2}\|_{L^2_{xy}} \lesssim C_0^2\epsilon^2(t-1)^{-(\kappa+1)/2}\sqrt{l(t)}. \qquad \square$$

As a byproduct of the above proof we have also obtained that, for any fixed $K>1$, there exist $C_0>0$ sufficiently large and $\epsilon_0>0$ sufficiently small such that

$$\sum_{|\gamma|\leq 5}\iint_{\mathscr{C}_{[2,T_0]}}(2+r-t)^{\kappa+1}(|\mathscr{T}\mathscr{Z}^\gamma W|^2+|\partial_y\mathscr{Z}^\gamma W|^2)\,dS \leq \frac{2C_0\epsilon^2}{K}. \tag{4-15}$$

An immediate consequence of the global energy bounds (2-2) obtained from Proposition 2.1 is the following estimate on the higher-order energies of the solution W on the truncated exterior hyperboloids $\mathscr{H}_s^{\mathrm{ex}}$ for any $s>0$.

Proposition 4.7. *Let $W=(u,v)^T$ be the global solution of the Cauchy problem* (1-5)–(1-6) *in the exterior region $\mathscr{D}^{\mathrm{ex}}$. There exists a constant $C>0$ such that for any $s>0$*

$$E_5^{\mathrm{ex},h}(s,W)+\delta_{s>\sqrt{3}}\sum_{|\gamma|\leq 5}\iint_{\mathscr{C}_{[2,T_s]}}|\mathscr{T}\mathscr{Z}^\gamma W|^2+|\partial_y\mathscr{Z}^\gamma W|^2\,dS \leq CC_0^2\epsilon^2, \tag{4-16}$$

where $\delta_{s>\sqrt{3}}=0$ if $s\leq\sqrt{3}$, 1 otherwise and $T_s=\frac{1}{2}(s^2+1)$.

Proof. The result follows by applying Proposition 3.2 with $\boldsymbol{W}=W^\gamma$ and $\boldsymbol{F}=F^\gamma$ for any multi-index γ such that $|\gamma|\leq 5$ and then using the global energy bound (2-2), the estimate (4-14) of the source term F^γ and the fact that

$$\|F^\gamma\|_{L^1_t L^2_{xy}(\mathscr{H}^{\mathrm{ex}}_{[2,s]})} \lesssim \int_2^\infty \|F^\gamma\|_{L^2_{xy}(\Sigma_t^{\mathrm{ex}})}\,dt \lesssim C_0^2\epsilon^2. \qquad \square$$

We conclude this section with the derivation of a bound for the higher-order exterior conformal energy of W_0 as well as for the higher-order conformal energy on portions of the hyperboloid $\mathscr{H}_s$ in the exterior region $\mathscr{D}^{\mathrm{ex}}$.

Proposition 4.8. *Assume the solution $W=(u,v)^T$ of the Cauchy problem* (1-5)–(1-6) *satisfies the a priori exterior energy bounds* (2-1). *Then*

$$\sum_{|\gamma|\leq 4}\int_{\mathscr{C}_{[2,T_0]}}|(t+r)(\partial_t W_0^\gamma+\partial_r W_0^\gamma)+2W_0^\gamma|^2+(t-r)^2(|\nabla_x W_0^\gamma|^2-(\partial_r W_0^\gamma)^2)\,dS + \sup_{[2,T_0]} E_4^{\mathrm{c,ex}}(t,W_0) \lesssim C_0^2\epsilon^2\ln T_0, \tag{4-17}$$

where the implicit constant only depends on C_0.

Proof. Let us fix $|\gamma|\leq 4$. By integrating (4-11) over the sphere $\mathbb{S}^1$ we obtain that W_0^γ is solution of the linear inhomogeneous wave equation

$$(-\partial_t^2+\Delta_x)W_0^\gamma = F_0^\gamma, \tag{4-18}$$

with source term

$$
\begin{aligned}
F_0^\gamma &= \int_{\mathbb{S}^1} F^\gamma \, \frac{dy}{2\pi} - \int_{\mathbb{S}^1} [\mathscr{Z}^\gamma, u\partial_y^2] W^\gamma \, \frac{dy}{2\pi} \\
&= \sum_{|\gamma_1|+|\gamma_2|\le|\gamma|} \int_{\mathbb{S}^1} \boldsymbol{N}(W^{\gamma_1}, W^{\gamma_2}) \, \frac{dy}{2\pi} + \sum_{|\gamma_1|+|\gamma_2|=|\gamma|} \int_{\mathbb{S}^1} \partial_y u^{\gamma_1} \cdot \partial_y W^{\gamma_2} \, \frac{dy}{2\pi}.
\end{aligned} \tag{4-19}
$$

When applying Proposition 3.6 with $\boldsymbol{W}_0 = W_0^\gamma$ and $\boldsymbol{F}_0 = F_0^\gamma$, we derive that for all $T \in [2, T_0]$

$$
\begin{aligned}
E^{\mathrm{c,ex}}(T, W_0^\gamma) + \int_{\mathscr{C}_{[2,T]}} |(t+r)(\partial_t W_0^\gamma + \partial_r W_0^\gamma) + 2W_0^\gamma|^2 + (t-r)^2(|\nabla_x W_0^\gamma|^2 - (\partial_r W_0^\gamma)^2) \, dS \\
\le E^{\mathrm{c,ex}}(2, W_0^\gamma) + \int_2^T \|(t+r)F_0^\gamma\|_{L^2_x(\Sigma_t^{\mathrm{ex}})} E^{\mathrm{c,ex}}(t, W_0^\gamma)^{1/2} \, dt,
\end{aligned} \tag{4-20}
$$

where

$$
\|(t+r)F_0^\gamma\|_{L^2_x(\Sigma_t^{\mathrm{ex}})} \le \sum_{|\gamma_1|+|\gamma_2|\le|\gamma|} \|(t+r)\boldsymbol{N}(W^{\gamma_1}, W^{\gamma_2})\|_{L^2_{xy}} + \sum_{|\gamma_1|+|\gamma_2|=|\gamma|} \|(t+r)\partial_y u^{\gamma_1} \cdot \partial_y W^{\gamma_2}\|_{L^2_{xy}}.
$$

We estimate the different contributions to F_0^γ separately. We start by observing that since $|\gamma_1| + |\gamma_2| \le 4$, at least one of the two multi-indexes has length less than or equal to 2. We call this index γ_j and will place the factor carrying it in L^∞, and the other one in L^2.

(1) *The commutator terms*: We use the pointwise bound (4-8) and the energy bound (2-4)

$$
\|(t+r)\partial_y u^{\gamma_1} \cdot \partial_y W^{\gamma_2}\|_{L^2_{xy}(\Sigma_t^{\mathrm{ex}})} \lesssim C_0\epsilon\sqrt{l(t)} \|\partial_y \mathscr{Z}^{\le 4} W\|_{L^2_{xy}(\Sigma_t^{\mathrm{ex}})} \lesssim C_0^2\epsilon^2 l(t).
$$

(2) *The null terms*: Here we use the null form representation via the formula (1-15) and the relation $\bar{\partial} = t^{-1}Z$ to write

$$
\boldsymbol{N}(W^{\gamma_1}, W^{\gamma_2}) = \frac{1}{t} Z W^{\gamma_1} \cdot \partial W^{\gamma_2} + \frac{1}{t} \partial W^{\gamma_1} \cdot Z W^{\gamma_2} + \frac{t-r}{t} \partial W^{\gamma_1} \cdot \partial W^{\gamma_2}. \tag{4-21}
$$

The first two products in the above right-hand side are equivalent given the range of γ_1 and γ_2 so we will just analyze the first one. In the case where $|\gamma_1| \le 2$ we deduce from the pointwise bound (4-9) that

$$
\|(t+r)t^{-1} Z W^{\gamma_1} \cdot \partial W^{\gamma_2}\|_{L^2_{xy}(\Sigma_t^{\mathrm{ex}})} \lesssim C_0\epsilon \|(t+r)r^{-1}t^{-1} \partial W^{\gamma_2}\|_{L^2_{xy}(\Sigma_t^{\mathrm{ex}})} \lesssim C_0\epsilon t^{-1} E_4^{\mathrm{ex},0}(t, W)^{1/2}.
$$

In the case where $|\gamma_1| \ge 3$, we bound ∂W^{γ_2} using (4-7) and decompose W^{γ_1} according to (1-7). We apply the Poincaré inequality to obtain

$$
\|(t+r)t^{-1} Z \mathtt{W}^{\gamma_1} \cdot \partial W^{\gamma_2}\|_{L^2_{xy}(\Sigma_t^{\mathrm{ex}})} \lesssim C_0\epsilon \|(t+r)r^{-1}t^{-1} \partial_y Z \mathtt{W}^{\gamma_1}\|_{L^2_{xy}(\Sigma_t^{\mathrm{ex}})} \lesssim C_0\epsilon t^{-1} E_5^{\mathrm{ex},0}(t, W)^{1/2}
$$

and use Lemma 4.5 with $\beta = \kappa - 1$ to get

$$
\begin{aligned}
\|(t+r)t^{-1} Z W_0^{\gamma_1} \cdot \partial W^{\gamma_2}\|_{L^2_{xy}(\Sigma_t^{\mathrm{ex}})} &\lesssim C_0\epsilon \|(t+r)t^{-1}r^{-1}(2+r-t)^{-\kappa+(\kappa-1)/2} Z W_0^{\gamma_1}\|_{L^2_{xy}(\Sigma_t^{\mathrm{ex}})} \\
&\lesssim C_0\epsilon t^{-1} E_5^{\mathrm{ex},\kappa}(t, W)^{1/2}.
\end{aligned}
$$

The last quadratic term in the right-hand side of (4-21) is estimated using again (4-7),

$$\begin{aligned}\|(t+r)(t-r)/t\,\partial W^{\gamma_1}\cdot\partial W^{\gamma_2}\|_{L^2_{xy}(\Sigma^{\mathrm{ex}}_t)} &\lesssim C_0\epsilon t^{-1}\|(2+r-t)^{(1-\kappa)/2}\partial\mathscr{Z}^{\le 4}W\|_{L^2_{xy}(\Sigma^{\mathrm{ex}}_t)}\\ &\lesssim C_0\epsilon t^{-1}E^{\mathrm{ex},0}_4(t,W)^{1/2},\end{aligned}$$

which gives

$$\|(t+r)\boldsymbol{N}(W^{\gamma_1},W^{\gamma_2})\|_{L^2_{xy}(\Sigma^{\mathrm{ex}}_t)}\lesssim C_0\epsilon t^{-1}E^{\mathrm{ex},\kappa}_5(t,W)^{1/2}.$$

The combination of steps (1) and (2) with the energy bound (2-1) yields

$$\|(t+r)F^{\gamma}_0\|_{L^2_{xy}(\Sigma^{\mathrm{ex}}_t)}\lesssim C_0^2\epsilon^2 l(t)+C_0^2\epsilon^2t^{-1}, \tag{4-22}$$

which plugged into (4-20) for all $|\gamma|\le 4$ gives

$$\begin{aligned}E^{\mathrm{c,ex}}_4(T,W_0)+\sum_{|\gamma|\le 4}\int_{\mathscr{C}_{[2,T]}}|(t+r)(\partial_tW^{\gamma}_0+\partial_rW^{\gamma}_0)+2W^{\gamma}_0|^2+(t-r)^2(|\nabla_xW^{\gamma}_0|^2-(\partial_rW^{\gamma}_0)^2)\,dS&\\ \lesssim E^{\mathrm{c,ex}}_4(2,W_0)+\int_2^T(C_0^2\epsilon^2l(t)+C_0^2\epsilon^2t^{-1})E^{\mathrm{c,ex}}_4(t,W_0)^{1/2}\,dt&\\ \lesssim E^{\mathrm{c,ex}}_4(2,W_0)+C_0^2\epsilon^2\sup_{[2,T_0]}E^{\mathrm{c,ex}}_4(t,W_0)+C_0^2\epsilon^2\ln T.&\end{aligned}$$

If $\epsilon\ll 1$ is sufficiently small we get

$$\begin{aligned}\sup_{[2,T_0]}E^{\mathrm{c,ex}}_4(t,W_0)+\sum_{|\gamma|\le 4}\int_{\mathscr{C}_{[2,T_0]}}|(t+r)(\partial_tW^{\gamma}_0+\partial_rW^{\gamma}_0)+2W^{\gamma}_0|^2+(t-r)^2(|\nabla_xW^{\gamma}_0|^2-(\partial_rW^{\gamma}_0)^2)\,dS&\\ \lesssim E^{\mathrm{c,ex}}_4(2,W_0)+C_0^2\epsilon^2\ln T_0,&\end{aligned}$$

so the result of the proposition follows from the smallness of the conformal energy at the initial time, which in turn follows from the assumptions on the initial data. □

Lemma 4.9. *Let $s\ge 2$ and $2<T_1<T_2$ be such that the portion of hyperboloid $\mathscr{H}_s$ in the time strip $[T_1,T_2]$ is entirely contained in the exterior region $\mathscr{D}^{\mathrm{ex}}$. Assume $W=(u,v)^T$ is the solution to the Cauchy problem* (1-5)–(1-6) *in $\mathscr{D}^{\mathrm{ex}}$ satisfying the global energy bounds* (2-1). *Then there exists a constant $C>0$ such that for all $|\gamma|\le 4$*

$$\Big\|\frac{1}{t}KW^{\gamma}_0+2W^{\gamma}_0\Big\|^2_{L^2(\mathscr{H}_s\cap[T_1,T_2])}+\sum_{i=1}^3\Big\|\frac{s}{t}\,\Omega_{0i}W^{\gamma}_0\Big\|^2_{L^2(\mathscr{H}_s\cap[T_1,T_2])}\le CC_0^2\epsilon^2\ln T_2.$$

Proof. We apply Proposition 3.7 with $\boldsymbol{W}_0=W^{\gamma}$ and $|\gamma|\le 4$. It follows from the hypotheses that $\Sigma^s_t\subset\Sigma^{\mathrm{ex}}_t$ and hence that $E^{\mathrm{c,ex}}_s(t,\boldsymbol{W}_0)\le E^{\mathrm{c,ex}}(t,\boldsymbol{W}_0)$ for all $t\in[T_1,T_2]$. Therefore, the result is obtained using estimates (4-17) and (4-22). □

5. Pointwise estimates in the interior region

The goal of this section is to recover pointwise estimates for solutions $W=(u,v)^T$ to (1-5) in the interior hyperbolic region $\mathscr{H}^{\mathrm{in}}_{[2,s_0]}$ under the a priori assumptions (2-5)–(2-7) and to propagate the a priori pointwise estimate (2-7) on ZW_0.

5A. ***Pointwise estimates from Klainerman–Sobolev inequalities.*** A first subset of pointwise estimates for W_0 and $\mathtt{W}$ is immediately obtained from (2-5) and (2-6) via the following Sobolev inequality on hyperboloids, whose proof can be found in [LeFloch and Ma 2018].

Lemma 5.1. *Let $\boldsymbol{W} = \boldsymbol{W}(t,x)$ be a sufficiently regular function in the cone $\mathscr{C} = \{t > r\}$. For all $(t,x) \in \mathscr{C}$, let $s = \sqrt{t^2 - r^2}$ and $B\big(x, \frac{1}{3}t\big)$ be the ball centered at x with radius $\frac{1}{3}t$. Then*

$$|\boldsymbol{W}(t,x)|^2 \le Ct^{-3} \sum_{|\gamma|\le 2} \int_{B(x,t/3)} |Z^\gamma \boldsymbol{W}(\sqrt{s^2+|\xi|^2}, \xi)|^2 \, d\xi,$$

where C is a positive universal constant and $Z_j = x_j \partial_t + t\partial_j$, $j = \overline{1,3}$.

We remind the reader that we proved uniform-in-time energy bounds (4-16) for the solution on exterior truncated hyperboloids $\mathscr{H}^{\mathrm{ex}}_s$ as well as the exterior pointwise bound (4-10) on ZW_0. We can then think of the a priori energy bounds (2-5), (2-6) as being valid not only on $\mathscr{H}^{\mathrm{in}}_s$ for $s \ge 2$ but on all (branches of) hyperboloids contained in the upper half plane $t \ge 2$. It is analogous for the pointwise bound (2-7), which can be thought to hold true for every (t,x,y) such that $t \ge 2$ and $t^2 - r^2 \le s_0^2$. Therefore, the following lemma provides us with pointwise estimates for the solution on the portion of the interior light cone below the hyperboloid $\mathscr{H}_{s_0}$, which we denote by $\mathscr{H}_{[2,s_0]}$,

$$\mathscr{H}_{[2,s_0]} := \{(t,x) : t \ge 2 \text{ and } t^2 - r^2 \le s_0^2\} \times \mathbb{S}^1.$$

Lemma 5.2. *Let $\mathscr{I}_{n,k}$ denote the set of multi-indexes of type (n,k). Under the a priori energy bounds* (2-5) *and* (2-6) *we have the following pointwise estimates in $\mathscr{H}_{[2,s_0]}$:*

$$|\partial \mathscr{Z}^{\le 3} W_0(t,x)| \lesssim \epsilon t^{-1/2} s^{-1}, \tag{5-1}$$

$$|\bar{\partial} \mathscr{Z}^{\le 3} W_0(t,x)| \lesssim \epsilon t^{-3/2}, \tag{5-2}$$

$$\sum_{\mathscr{I}_{3,k}} \|\mathscr{Z}^\gamma \mathtt{W}(t,x,\cdot)\|_{L^\infty(\mathbb{S}^1)} + \|\partial_y^{\le 1} \mathscr{Z}^\gamma \mathtt{W}(t,x,\cdot)\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon t^{-3/2} s^{\delta_{k+2}}, \quad k = \overline{0,3}, \tag{5-3}$$

$$\sum_{|\gamma|=3} \|\partial_{tx} \mathscr{Z}^\gamma \mathtt{W}(t,x,\cdot)\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon t^{-1/2} s^{-1+\delta_5}, \tag{5-4}$$

and

$$\sum_{\mathscr{I}_{2,k}} \|\bar{\partial} \partial_y^{\le 1} \mathscr{Z}^\gamma \mathtt{W}(t,x,\cdot)\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon t^{-5/2} s^{\delta_{k+3}}, \quad k = \overline{0,2}, \tag{5-5}$$

$$\sum_{\mathscr{I}_{1,k}} \|\bar{\partial}^2 \partial_y^{\le 1} \mathscr{Z}^\gamma \mathtt{W}(t,x,\cdot)\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon t^{-7/2} s^{\delta_{k+4}}, \quad k = \overline{0,1}, \tag{5-6}$$

$$\|\bar{\partial}^2 \partial_{tx} \mathscr{Z} \mathtt{W}(t,x,\cdot)\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon t^{-5/2} s^{-1+\delta_5}. \tag{5-7}$$

Moreover

$$\sup_{\mathscr{H}_{[2,s_0]}} |W| \lesssim \epsilon. \tag{5-8}$$

Proof. The estimates (5-1), (5-2) and (5-4) are immediate consequence of Lemma 5.1, the energy assumptions (2-5), (2-6) and the fact that $|[\mathscr{Z}, (s/t)]\boldsymbol{W}| \lesssim |\boldsymbol{W}|$ for any $\boldsymbol{W}$. The same is for the estimate of $\partial_y \mathscr{Z}^\gamma \mathtt{W}$ in (5-3), while the remaining norms of $\mathscr{Z}^\gamma \mathtt{W}$ there are obtained using also the Sobolev injection on $\mathbb{S}^1$ (for the L^∞ norm) and the Poincaré inequality (for the L^2 norm). The estimates on the $\bar\partial$ derivatives of $\mathtt{W}$ are deduced from (5-3) and (5-4) using the fact that $\bar\partial = t^{-1}Z$. Finally, if we set $\widetilde{W}_0(s,x) = W_0(\sqrt{s^2+r^2}, x)$, we see that

$$|\widetilde{W}_0(s,x) - \widetilde{W}_0(2,x)| \le \int_2^s |\partial_\tau \widetilde{W}_0(\tau,x)|\,d\tau = \int_2^s \frac{\tau}{t}|\partial_t W_0(\sqrt{\tau^2+x^2},x)|\,d\tau \lesssim \epsilon$$

and hence derive from the smallness of the initial data that

$$|W_0(t,x)| \lesssim |\widetilde{W}_0(s,x) - \widetilde{W}_0(2,x)| + |\widetilde{W}_0(2,x)| \lesssim \epsilon.$$

The combination of the above estimate with (5-3) yields (5-8). □

5B. ***Improved pointwise estimates on the nonzero modes.*** The bounds for the nonzero mode $\mathtt{W}$ of the solution obtained in Lemma 5.2 via Sobolev embeddings are affected by the small growth in s of the energies and are not sharp. However, they can be improved if one studies more closely the equation satisfied by $\mathtt{W}$. Enhancing such bounds and particularly (5-3) will be fundamental to propagate the a priori pointwise bound (2-7) later in Proposition 5.6. We make use of the following result, which is motivated by [Klainerman 1985] and whose proof is an adaptation of a similar estimate for Klein–Gordon equations initially proved in [LeFloch and Ma 2016], later revisited in [Dong and Wyatt 2020a] in the case of Klein–Gordon equations with variable mass.

Proposition 5.3. *Assume $\boldsymbol{W}$ is a solution of the equation*

$$\Box_{x,y}\boldsymbol{W} + u\Delta_y \boldsymbol{W} = \boldsymbol{F}, \quad (t,x,y)\in\mathbb{R}^{1+3}\times\mathbb{S}^1, \tag{5-9}$$

such that $\int_{\mathbb{S}^1}\boldsymbol{W}\,dy = 0$. For every fixed (t,x) in the region $\mathscr{C}=\{t>r\}$, let $s=\sqrt{t^2-r^2}$ and $Y_{tx}, A_{tx}, B_{tx}: \mathbb{R}^+\setminus\{0\}\to\mathbb{R}^+$ be the functions defined as

$$Y^2_{tx}(\lambda) := \int_{\mathbb{S}^1} \lambda\big|\tfrac{3}{2}\boldsymbol{W}_\lambda + (\mathscr{S}\boldsymbol{W})_\lambda\big|^2 + \lambda^3(1+u_\lambda)|\partial_y \boldsymbol{W}_\lambda|^2\,dy, \tag{5-10}$$

$$A_{tx}(\lambda) := \sup_{\mathbb{S}^1}\frac{1}{2\lambda}|(\mathscr{S}u)_\lambda| + \sup_{\mathbb{S}^1}|\partial_y u_\lambda|, \tag{5-11}$$

$$B^2_{tx}(\lambda) = \int_{\mathbb{S}^1}\lambda^{-1}|(R\boldsymbol{W})_\lambda|^2\,dy, \tag{5-12}$$

where $f_\lambda(t,x,y) = f(\lambda t/s, \lambda r/s, y)$ and

$$R\boldsymbol{W}(t,x,y) = s^2\bar\partial^i\bar\partial_i\boldsymbol{W} + x^ix^j\bar\partial_i\bar\partial_j\boldsymbol{W} + \tfrac{3}{4}\boldsymbol{W} + 3x^i\bar\partial_i\boldsymbol{W} - s^2\boldsymbol{F}.$$

Then $\boldsymbol{W}$ satisfies the following inequality in the hyperbolic region $\mathscr{H}_{[2,\infty)}$:

$$s^{3/2}(\|\boldsymbol{W}\|_{L^2(\mathbb{S}^1)} + \|\partial_y\boldsymbol{W}\|_{L^2(\mathbb{S}^1)}) + s^{1/2}\|\mathscr{S}\boldsymbol{W}\|_{L^2(\mathbb{S}^1)} \lesssim \bigg(Y_{tx}(2) + \int_2^s B_{tx}(\lambda)\,d\lambda\bigg)e^{\int_2^s A_{tx}(\lambda)\,d\lambda}.$$

Proof. For every fixed $(t,x,y)\in\mathscr{H}_{[2,\infty)}$ we define $\omega_{txy}(\lambda) := \lambda^{3/2}\boldsymbol{W}(\lambda t/s, \lambda x/s, y)$ to be the evaluation of $\boldsymbol{W}$ on the hyperboloid $\mathscr{H}_\lambda$ dilated by factor $\lambda^{3/2}$. We have

$$\dot\omega_{txy}(\lambda) = \lambda^{1/2}\big(\tfrac{3}{2}\boldsymbol{W}_\lambda + (\mathscr{S}\boldsymbol{W})_\lambda\big)$$

and

$$\ddot{\omega}_{txy}(\lambda)=\lambda^{-1/2}(P\boldsymbol{W})_\lambda,$$

where

$$P\boldsymbol{W}=\tfrac{3}{4}\boldsymbol{W}+3(t\partial_t\boldsymbol{W}+x^i\partial_i\boldsymbol{W})+(t^2\partial_t^2\boldsymbol{W}+2tx^i\partial_i\partial_t\boldsymbol{W}+x^ix^j\partial_i\partial_j\boldsymbol{W}).$$

Using (5-9) we derive that ω_{txy} satisfies the equation

$$\begin{aligned}\ddot{\omega}_{txy}-(1+u_\lambda)\partial_y^2\omega_{txy}&=-\lambda^{3/2}\boldsymbol{F}_\lambda+\lambda^{-1/2}\big(s^2\bar\partial^i\bar\partial_i\boldsymbol{W}+x^ix^j\bar\partial_i\bar\partial_j\boldsymbol{W}+\tfrac{3}{4}\boldsymbol{W}+3x^i\bar\partial_i\boldsymbol{W}\big)_\lambda\\&=\lambda^{-1/2}(R\boldsymbol{W})_\lambda.\end{aligned}$$

We drop the lower indexes in $\omega_{txy}(\lambda)$ in order to have lighter notation and simply denote it by $\omega(\lambda)$ in what follows. We multiply the above equation by $\partial_\lambda\omega$ and integrate over $\mathbb{S}^1$:

$$\begin{aligned}\int_{\mathbb{S}^1}&\partial_\lambda\omega(\partial_\lambda^2\omega-(1+u_\lambda)\partial_y^2\omega)\,dy\\&=\frac{d}{d\lambda}\bigg(\frac12\int_{\mathbb{S}^1}|\partial_\lambda\omega|^2\,dy\bigg)+\int_{\mathbb{S}^1}(1+u_\lambda)\partial_y\omega\,\partial_y\partial_\lambda\omega\,dy+\int_{\mathbb{S}^1}\partial_\lambda\omega\,\partial_yu_\lambda\,\partial_y\omega\,dy\\&=\frac{d}{d\lambda}\bigg(\frac12\int_{\mathbb{S}^1}|\partial_\lambda\omega|^2+(1+u_\lambda)|\partial_y\omega|^2\,dy\bigg)-\frac12\int_{\mathbb{S}^1}\partial_\lambda u_\lambda\,|\partial_y\omega|^2\,dy+\int_{\mathbb{S}^1}\partial_\lambda\omega\,\partial_yu_\lambda\,\partial_y\omega\,dy.\end{aligned}$$

We obtain

$$\frac{d}{d\lambda}Y_{tx}^2(\lambda)\lesssim A_{tx}(\lambda)Y_{tx}^2(\lambda)+B_{tx}(\lambda)Y_{tx}(\lambda),$$

with A_{tx}, B_{tx}, Y_{tx} as in the statement and from the Gronwall lemma

$$Y_{tx}(s)\lesssim\bigg(Y_{tx}(2)+\int_2^sB_{tx}(\lambda)\,d\lambda\bigg)e^{\int_2^sA_{tx}(\lambda)\,d\lambda}.$$

Finally, from the definition of Y_{tx}, the Poincaré inequality and the fact that $s\geq 2$ we get

$$s^{3/2}(\|\boldsymbol{W}\|_{L^2(\mathbb{S}^1)}+\|\partial_y\boldsymbol{W}\|_{L^2(\mathbb{S}^1)})+s^{1/2}\|\mathscr{S}\boldsymbol{W}\|_{L^2(\mathbb{S}^1)}\lesssim Y_{tx}(s).\qquad\square$$

Proposition 5.4. *Under the a priori assumptions* (2-5)–(2-7) *we have*

$$\sup_{\mathscr{H}_{[2,s_0]}}t^{3/2}(\|\partial^j\mathtt{W}\|_{L^2(\mathbb{S}^1)}+\|\partial_y\partial^j\mathtt{W}\|_{L^2(\mathbb{S}^1)})+\sup_{\mathscr{H}_{[2,s_0]}}t^{3/2}s^{-1}\|\mathscr{S}\partial^j\mathtt{W}\|_{L^2(\mathbb{S}^1)}\lesssim\epsilon,\quad j=\overline{0,1},\tag{5-13}$$

$$\sup_{\mathscr{H}_{[2,s_0]}}t^{3/2}(\|Z\mathtt{W}\|_{L^2(\mathbb{S}^1)}+\|\partial_yZ\mathtt{W}\|_{L^2(\mathbb{S}^1)})+\sup_{\mathscr{H}_{[2,s_0]}}t^{3/2}s^{-1}\|\mathscr{S}Z\mathtt{W}\|_{L^2(\mathbb{S}^1)}\lesssim\epsilon s^\sigma,\tag{5-14}$$

and

$$\sup_{\mathscr{H}_{[2,s_0]}}st^{1/2}(\|\partial^2\mathtt{W}\|_{L^2(\mathbb{S}^1)}+\|\partial_y\partial^2\mathtt{W}\|_{L^2(\mathbb{S}^1)})+\sup_{\mathscr{H}_{[2,s_0]}}t^{1/2}\|\mathscr{S}\partial^2\mathtt{W}\|_{L^2(\mathbb{S}^1)}\lesssim\epsilon,\tag{5-15}$$

$$\sup_{\mathscr{H}_{[2,s_0]}}st^{1/2}(\|\partial Z\mathtt{W}\|_{L^2(\mathbb{S}^1)}+\|\partial_y\partial Z\mathtt{W}\|_{L^2(\mathbb{S}^1)})+t^{1/2}\|\mathscr{S}\partial Z\mathtt{W}\|_{L^2(\mathbb{S}^1)}\lesssim\epsilon s^\sigma.\tag{5-16}$$

Proof. For any fixed $j=\overline{0,2}$ and $k=\overline{0,1}$ we compare the equation satisfied by the differentiated functions $\partial^j\mathtt{W}$ and $\partial^kZ\mathtt{W}$ respectively with (5-9) and apply the result of Proposition 5.3. A simple computation shows that

$$\Box_{x,y}\partial^j \mathtt{W} + u\partial_y^2\partial^j \mathtt{W} = F_1^j, \quad j = \overline{0,2},$$
$$\Box_{x,y}\partial^k Z\mathtt{W} + u\partial_y^2\partial^k Z\mathtt{W} = F_2^k, \quad k = \overline{0,1},$$

with source terms given by

$$F_1^0 = \boldsymbol{N}(W, W) - \int_{\mathbb{S}^1} \boldsymbol{N}(W, W)\,\frac{dy}{2\pi} + \int_{\mathbb{S}^1} \partial_y u \cdot \partial_y \mathtt{W}\,\frac{dy}{2\pi},$$
$$F_1^j = \partial^j F_1^0 - \sum_{1\le h\le j} \partial^h u \cdot \partial_y^2 \partial^{j-h}\mathtt{W}, \quad j = \overline{1,2},$$

and

$$F_2^0 = ZF_1^0 - Zu \cdot \partial_y^2 \mathtt{W},$$
$$F_2^1 = \partial Z F_1^0 - \partial Z u \cdot \partial_y^2 \mathtt{W} - \partial u \cdot \partial_y^2 Z\mathtt{W} - Zu \cdot \partial_y^2 \partial \mathtt{W}.$$

From Proposition 5.3 we have

$$s^{3/2}(\|\boldsymbol{W}\|_{L^2(\mathbb{S}^1)} + \|\partial_y \boldsymbol{W}\|_{L^2(\mathbb{S}^1)}) + s^{1/2}\|\mathscr{S}\boldsymbol{W}\|_{L^2(\mathbb{S}^1)} \lesssim \left(Y_{tx}(2) + \int_2^s B_{tx}(\lambda)\,d\lambda\right)e^{\int_2^s A_{tx}(\lambda)\,d\lambda},$$

with $\boldsymbol{W} = \{\partial^j \mathtt{W}, \partial^k Z\mathtt{W} : j = \overline{0,2},\ k = \overline{0,1}\}$ and corresponding source term $\boldsymbol{F} = \{F_1^j, F_2^k : j = \overline{0,2},\ k = \overline{0,1}\}$. In order to obtain the bounds in the statement we need to estimate the quantities $Y_{tx}(2)$, $A_{tx}(\lambda)$ and $B_{tx}(\lambda)$ defined in (5-10), (5-11), (5-12) for all the different values of $\boldsymbol{W}$ and $\boldsymbol{F}$.

(1) *The $A_{tx}(\lambda)$ term*: This is the same for all values of $\boldsymbol{W}$. Here we take the decomposition $u = u_0 + \mathtt{u}$ and rewrite the scaling vector field as

$$\mathscr{S} = (t-r)\partial_t + (r-t)\partial_r + (t\partial_r + r\partial_t). \tag{5-17}$$

Using the pointwise bounds (5-1)–(5-3) as well as the assumption (2-7) and the fact that $s/t \le 1$ in the interior of the light cone, we derive that

$$A_{tx}(\lambda) \lesssim \sup_{y\in\mathbb{S}^1} \frac{s}{t+r}\,|(\partial u)_\lambda| + \lambda^{-1}|(Zu)_\lambda| + |(\partial_y u)_\lambda| \lesssim \epsilon\lambda^{-3/2+\delta_2}$$

and consequently

$$\int_1^s A_{tx}(\lambda)d\lambda \lesssim \epsilon. \tag{5-18}$$

(2) *The $Y_{tx}(2)$ term*: The functions appearing here are evaluated on the hyperboloid $\mathscr{H}_2$. From the bound (5-3) on $\mathtt{W}$, the smallness of u given by (5-8) and the decomposition (5-17) it follows that for all values of $\boldsymbol{W}$ under consideration

$$|Y_{tx}(2)| \lesssim \|\boldsymbol{W}_2\|_{L^2(\mathbb{S}^1)} + \|(\mathscr{S}\boldsymbol{W})_2\|_{L^2(\mathbb{S}^1)} + \|1+u_2\|_{L^\infty(\mathbb{S}^1)}\|(\partial_y \boldsymbol{W})_2\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon\,(s/t)^{3/2}. \tag{5-19}$$

(3) *The $B_{tx}(\lambda)$ term*: These are the only ones for which we need to distinguish between the different values of $\boldsymbol{W}$ and hence of $\boldsymbol{F}$. Let us remark here that if G is any linear combination of the products

$$\begin{aligned}\partial\mathscr{Z}^{\gamma_1}W \cdot \partial\mathscr{Z}^{\gamma_2}\mathtt{W}, \quad \int_{\mathbb{S}^1} \partial\mathscr{Z}^{\gamma_1}\mathtt{W}\cdot\partial\mathscr{Z}^{\gamma_2}\mathtt{W}\,dy, &\quad |\gamma_1|+|\gamma_2| \le 2,\\ \partial\mathscr{Z}^{\gamma_1}u_0 \cdot \partial_y^2\mathscr{Z}^{\gamma_2}\mathtt{W}, &\quad |\gamma_1|+|\gamma_2| \le 1,\\ \mathscr{Z}^{\gamma_1}\mathtt{u}\cdot\partial_y^2\mathscr{Z}^{\gamma_2}\mathtt{W}, &\quad |\gamma_1|+|\gamma_2| \le 2, |\gamma_2| < 2,\end{aligned} \tag{5-20}$$

then from (5-1) and (5-3)

$$\|\lambda^{-1/2}(s^2G)_\lambda\|_{L^2(\mathbb{S}^1)} = \lambda^{3/2}\|G_\lambda\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon^2\lambda^{-3/2+2\delta_4}(s/t)^2. \tag{5-21}$$

(a) *The case* $\boldsymbol{W} = \mathscr{Z}^{\le 1}\mathtt{W}$: From the pointwise bounds (5-3), (5-5) and (5-6) we immediately obtain the estimate

$$\begin{aligned}\big\|\lambda^{-1/2}\big(s^2\bar\partial^i\bar\partial_i\boldsymbol{W} + x^ix^j\bar\partial_i\bar\partial_j\boldsymbol{W} + \tfrac{3}{4}\boldsymbol{W} + 3x^i\bar\partial_i\boldsymbol{W}\big)_\lambda\big\|_{L^2(\mathbb{S}^1)}&\\ \lesssim \lambda^{3/2}(1+r^2/s^2)\|(\bar\partial^2\boldsymbol{W})_\lambda\|_{L^2(\mathbb{S}^1)} + \lambda^{1/2}rs^{-1}\|(\bar\partial\boldsymbol{W})_\lambda\|_{L^2(\mathbb{S}^1)} + \lambda^{-1/2}\|\boldsymbol{W}_\lambda\|_{L^2(\mathbb{S}^1)}&\\ \lesssim \epsilon\lambda^{-2+\delta_5}(s/t)^{3/2}.&\end{aligned} \tag{5-22}$$

In the case where $\boldsymbol{W} = \partial^{\le 1}\mathtt{W}$, the corresponding $\boldsymbol{F}$ satisfies (5-21), yielding

$$\int_2^s B_{tx}(\lambda)\,d\lambda \lesssim \epsilon(s/t)^{3/2}, \tag{5-23}$$

and the combination of (5-18), (5-19), (5-23) gives (5-13). In the case where $\boldsymbol{W} = Z\mathtt{W}$, the only quadratic term in $\boldsymbol{F} = ZF_1^0$ that is not in (5-20) is $Zu_0\cdot\partial_y^2\mathtt{W}$. For this we apply the a priori bound (2-7) and the enhanced bound (5-13) and get

$$\|Zu_0\cdot\partial_y^2\mathtt{W}\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon^2t^{-5/2}s^\sigma.$$

Therefore

$$\lambda^{3/2}\|(F_2^0)_\lambda\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon^2\lambda^{-3/2+2\delta_4}(s/t)^2 + \epsilon^2\lambda^{-1+\sigma}(s/t)^{5/2} \lesssim \epsilon^2\lambda^{-1+\sigma}(s/t)^2,$$

which combined with (5-22) implies

$$\int_2^s B_{tx}(\lambda)\,d\lambda \lesssim \epsilon s^\sigma(s/t)^{3/2}.$$

The above estimate, together with (5-18) and (5-19), gives (5-14).

(b) *The case* $\boldsymbol{W} = \{\partial^2\mathtt{W}, \partial Z\mathtt{W}\}$: Here the bounds (5-3), (5-5) and (5-7) give

$$\begin{aligned}\big\|\lambda^{-1/2}(s^2\bar\partial^i\bar\partial_i\boldsymbol{W} + \tfrac{3}{4}\boldsymbol{W} + 3x^i\bar\partial_i\boldsymbol{W})_\lambda\big\|_{L^2(\mathbb{S}^1)}&\\ \lesssim \lambda^{3/2}\|(\bar\partial^2\boldsymbol{W})_\lambda\|_{L^2(\mathbb{S}^1)} + \lambda^{1/2}rs^{-1}\|(\bar\partial\boldsymbol{W})_\lambda\|_{L^2(\mathbb{S}^1)} + \lambda^{-1/2}\|\boldsymbol{W}_\lambda\|_{L^2(\mathbb{S}^1)}&\\ \lesssim \epsilon\lambda^{-2+\delta_5}(s/t)^{3/2}&\end{aligned} \tag{5-24}$$

and

$$\|\lambda^{-1/2}(x^ix^j\bar\partial_i\bar\partial_j\boldsymbol{W})_\lambda\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon\lambda^{-2+\delta_5}(s/t)^{1/2}. \tag{5-25}$$

In the case $\boldsymbol{W} = \partial^2\mathtt{W}$, the corresponding $\boldsymbol{F} = F_1^2$ satisfies (5-21), which summed up with the above estimates yields

$$\int_2^s B_{tx}(\lambda)\,d\lambda \lesssim \epsilon(s/t)^{1/2}$$

and therefore (5-15). In the case $\boldsymbol{W} = \partial Z\mathtt{W}$, the only term in $\boldsymbol{F} = F_2^1$ that is not in (5-20) is $Zu_0\cdot\partial_y^2\partial\mathtt{W}$. For this we apply the a priori bound (2-7) and the enhanced bound (5-15)

$$\|Zu_0\cdot\partial_y^2\partial\mathtt{W}\|_{L^2(\mathbb{S}^1)} \le \|Zu_0\|_{L^\infty(\mathbb{S}^1)}\|\partial_y^2\partial\mathtt{W}\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon^2t^{-3/2}s^{-1+\sigma}.$$

Therefore

$$\lambda^{3/2}\|(F_2^1)_\lambda\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon^2\lambda^{-3/2+2\delta_4}(s/t)^2 + \epsilon^2\lambda^{-1+\sigma}(s/t)^{3/2} \lesssim \epsilon^2\lambda^{-1+\sigma}(s/t)^{3/2},$$

which combined with (5-24), (5-25) implies

$$\int_2^s B_{txy}(\lambda)\,d\lambda \lesssim \epsilon s^\sigma (s/t)^{1/2}. \qquad \square$$

5C. ***The propagation of the a priori pointwise bound.*** In order to propagate the a priori bound (2-7) we use L^∞-L^∞-type estimates. For this we give a closer look to the wave equation satisfied by ZW_0 and use the enhanced pointwise bounds recovered in the previous subsection to estimate the nonlinear terms. We will make use of the following lemma, due to [Alinhac 2006].

Lemma 5.5. *Let $\boldsymbol{W}_0$ be the solution to $\Box_{tx}\boldsymbol{W}_0 = \boldsymbol{F}_0$ with zero initial data and suppose that $\boldsymbol{F}_0$ is spatially compactly supported satisfying the pointwise bound*

$$|\boldsymbol{F}_0(t,x)| \le Ct^{-2-\nu}(t-|x|)^{-1+\mu}$$

for some fixed $\mu, \nu > 0$ and some positive constant C. Then

$$|\boldsymbol{W}_0(t,x)| \lesssim \frac{C}{\mu\nu}(t-|x|)^{\mu-\nu}t^{-1}.$$

Proposition 5.6. *There exists a constant $B > 0$ sufficiently large and ϵ_0 sufficiently small such that for any $0 < \epsilon < \epsilon_0$ if $W = (u, v)^T$ is solution of the Cauchy problem* (1-5)–(1-6) *and satisfies the a priori bounds* (2-5)–(2-7) *in the interior region $\mathcal{H}_{[2,s_0]}$, together with the global energy bounds* (2-1) *in the exterior region $\mathcal{D}^{\text{ex}}$, then in $\mathcal{H}_{[2,s_0]}$ it actually satisfies the enhanced pointwise bound*

$$|ZW_0(t,x)| \le B\epsilon t^{-1}s^\sigma.$$

Proof. We consider a cut-off function $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi(z) = 1$ for $|z| \le \frac{1}{2}$ and $\chi(z) = 0$ for $|z| > 1$, and decompose W_0 into the sum $W_0^a + W_0^b$, where W_0^a and W_0^b solve to the Cauchy problems

$$\begin{aligned} &\Box_x W_0^a = \chi\big(\big(r+\tfrac{1}{2}\big)/t\big)F_0, && (W_0^a, \partial_t W_0^a)|_{t=2} = (0,0),\\ &\Box_x W_0^b = \big(1-\chi\big(\big(r+\tfrac{1}{2}\big)/t\big)\big)F_0, && (W_0^b, \partial_t W_0^b)|_{t=2} = (W_0, \partial_t W_0)|_{t=2}, \end{aligned}$$

with F_0 given by

$$F_0 = \int_{\mathbb{S}^1} \boldsymbol{N}(W,W)\,\frac{dy}{2\pi} + \int_{\mathbb{S}^1} \partial_y u \cdot \partial_y \mathtt{W}\,\frac{dy}{2\pi}.$$

The scope of such decomposition is to estimate ZW_0^a and ZW_0^b separately. We aim to apply Lemma 5.5 to ZW_0^a, since it is solution to a wave equation with zero data and source term supported in the interior of the cone $\big\{t = r + \frac{1}{2}\big\}$,

$$\Box_x ZW_0^a = \chi\big(\big(r+\tfrac{1}{2}\big)/t\big)ZF_0 - \big[Z, \chi\big(\big(r+\tfrac{1}{2}\big)/t\big)\big]F_0, \quad (ZW_0^a, \partial_t ZW_0^a)|_{t=2} = (0,0),$$

and Klainerman–Sobolev embeddings to estimate ZW_0^b.

We start by estimating F_0 and ZF_0 on the support of $\chi\big((r+\frac{1}{2})/t\big)$, after observing the uniform boundedness of the commutator term $\big[Z, \chi\big((r+\frac{1}{2})/t\big)\big]$ and the fact that F_0 does not contain mixed interactions, i.e.,

$$F_0 = \boldsymbol{N}(W_0, W_0) + \int_{\mathbb{S}^1} \boldsymbol{N}(\mathtt{W}, \mathtt{W})\,\frac{dy}{2\pi} + \int_{\mathbb{S}^1} \partial_y u\cdot\partial_y \mathtt{W}\,\frac{dy}{2\pi}.$$

On the one hand, we use the null structure representation (1-15) of $\boldsymbol{N}$ for the quadratic interactions $W_0\times W_0$ and apply (5-1), (5-2) to deduce that

$$|Z^{\le 1}\boldsymbol{N}(W_0, W_0)| \lesssim |\bar\partial Z^{\le 1} W_0|\,|\partial W_0| + |\partial Z^{\le 1}W_0|\,|\bar\partial W_0| + \frac{s^2}{t^2}|\partial Z^{\le 1}W_0|\,|\partial W_0| \lesssim \epsilon^2 t^{-2}s^{-1}.$$

On the other hand, from the improved bounds (5-13), (5-16) we get

$$\int_{\mathbb{S}^1} |Z^{\le 1}\boldsymbol{N}(\mathtt{W},\mathtt{W})|\,dy + \int_{\mathbb{S}^1} |Z^{\le 1}(\partial_y u\cdot\partial_y\mathtt{W})|\,dy \lesssim \|\partial Z^{\le 1}\mathtt{W}\|_{L^2(\mathbb{S}^1)}\|\partial \mathtt{W}\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon^2 t^{-2}s^{-1+\sigma}.$$

Therefore, on the support of $\chi(r/t)$ and for $t\ge 2$

$$|\boldsymbol{F}_0(t,x)| \lesssim \epsilon^2 t^{-5/2}\langle t-r\rangle^{-1/2} \quad\text{and}\quad |Z\boldsymbol{F}_0(t,x)| \lesssim \epsilon^2 t^{-5/2+\sigma/2}\langle t-r\rangle^{-1/2+\sigma/2}.$$

Lemma 5.5 yields then

$$|ZW_0^a(t,x)| \le C\epsilon^2 t^{-1}(t-r)^{\sigma}$$

for some constant $C = C(A,B)$ that depends quadratically on A and B.

The remaining term ZW_0^b to analyze is estimated via the Klainerman–Sobolev inequality of Lemma 5.1. Observe that for points $(t,x)\in\mathscr{H}_s^{\text{in}}$ close to the boundary of $\mathscr{H}_s^{\text{in}}$, the ball $B\big(x,\frac{1}{3}t\big)$ also intersects the exterior region and we get

$$t^{1/2}s|ZW_0^b(t,x)| \lesssim E_2^{\text{c,in}}(s, W_0^b)^{1/2} + \sum_{|\gamma|\le 2}\|(s/t)Z\mathscr{Z}^\gamma W_0^b\|_{L^2_x(\mathscr{H}_s\cap[T_1^s,T_2^s])},$$

where $s^2=t^2-r^2$, $T_1^s=\frac{1}{2}(s^2+1)$ is the time the hyperboloid $\mathscr{H}_s$ intersects the cone $t=r+1$ and $T_2^s=\sqrt{s^2+|\xi|^2}$ with $|\xi - x| = \frac{1}{3}t$ (the second term in the above right-hand side should be omitted when $T_2^s\le T_1^s$). The support of the source term in the equation satisfied by W_0^b is contained in the exterior region and hence it is that in the equation of $\mathscr{Z}^\gamma W_0^b$ for any $|\gamma|\le 2$. From Proposition 3.5 applied to $\boldsymbol{W}_0 = \mathscr{Z}^\gamma W_0^b$ with $|\gamma|\le 2$, the inequality (4-17) (which is valid also for W_0^b) with $T_0=\frac{1}{2}(s^2+1)$, and the smallness assumption on the initial data, we deduce that

$$E_2^{\text{c,in}}(s, W_0^b) \le E_2^{\text{c,in}}(2, W_0^b) + C_0^2\epsilon^2\ln s \lesssim C_0^2\epsilon^2\ln s.$$

Moreover, from Lemma 4.9 with $T_j = T_j^s$ for $j=1,2$ and the observation that $\ln T_2^s\lesssim \ln s$, we also derive that

$$\sum_{|\gamma|\le 2}\|(s/t)Z\mathscr{Z}^\gamma W_0^b\|_{L^2_x(\mathscr{H}_s\cap[T_1^s,T_2^s])} \lesssim C_0^2\epsilon^2\ln s.$$

This gives

$$|ZW_0^b(t,x)| \le \widetilde{C}C_0\epsilon t^{-1/2}s^{-1+\sigma}$$

for a universal constant $\widetilde{C}$ and therefore

$$|ZW_0(t,x)| \le |ZW_0^a(t,x)| + |ZW_0^b(t,x)| \le \widetilde{C}C_0\epsilon t^{-1/2}s^{-1+\sigma} + C\epsilon^2 t^{-1}(t-r)^\sigma \le B\epsilon t^{-1}s^\sigma$$

if we choose B sufficiently large so that $B \ge 2\widetilde{C}C_0$ and ϵ_0 sufficiently small so that $2C\epsilon \le B$. □

6. Energy estimates in the interior region

The goal of this section is to propagate the interior energy bounds (2-5) and (2-6) on the two components W_0 and $\mathtt{W}$ of the solution W to the Cauchy problem (1-5)–(1-6). We remind the reader that for any multi-index γ the differentiated function $W^\gamma = (u^\gamma, v^\gamma)$ is a solution to (4-11) with source term (4-12), while its zero-mode W_0^γ solves the inhomogeneous wave equation (4-18) with source term (4-19).

We first start by recovering an energy bound for the higher-order conformal energies of W_0. Such a bound follows from (2-5) and (2-6) as well as from the pointwise estimates obtained in Section 5. It will be necessary for the propagation of (2-6) and the computations that lead to it will be useful in the proof of Proposition 6.2.

Proposition 6.1. *Assume the solution $W = (u, v)^T$ to* (1-5)–(1-6) *satisfies the a priori estimates* (2-5)–(2-7) *in the region $\mathscr{H}_{[2,s_0]}$ as well as the global exterior energy bounds* (2-1) *in the exterior region $\mathscr{D}^{\mathrm{ex}}$. Then*

$$\sup_{[2,s]} E_k^{\mathrm{c,in}}(s, W_0) \le C\epsilon s^{1+2\mu_k}, \quad s\in[2,s_0],\ k=\overline{0,4}, \tag{6-1}$$

where $\mu_k = \delta_k$ if $k \le 3$ and $\mu_4 = \delta_2+\delta_4$.

Proof. We consider here only multi-indices γ of type (k,k), i.e., $\gamma = (0,\beta)$ with $|\beta| = k$ and $\mathscr{Z}^\gamma = Z^\beta$, and apply Proposition 3.5 with $\boldsymbol{W}_0 = W_0^\gamma$ and $\boldsymbol{F}_0 = F_0^\gamma$. We derive

$$\begin{aligned}\sup_{\tau\in[2,s]} E^{\mathrm{c,in}}(\tau, W_0^\gamma) \le E^{\mathrm{c,in}}(2, W_0^\gamma) + \int_2^s \|\tau \boldsymbol{F}_0^\gamma\|_{L^2(\mathscr{H}_\tau^{\mathrm{in}})} E^{\mathrm{c,in}}(\tau, W_0^\gamma)^{1/2}\,d\tau \\ + \int_{\mathscr{C}_{[2,s]}} |(t+r)(\partial_t W_0^\gamma + \partial_r W_0^\gamma) + 2W_0^\gamma|^2 + (t-r)^2(|\nabla W_0^\gamma|^2 - (\partial_r W_0^\gamma)^2)\,dS. \end{aligned}\tag{6-2}$$

The integral over the boundary $\mathscr{C}_{[2,s]}$ equals the integral in the left-hand side of (4-17) when $T_0 = \frac{1}{2}(s^2+1)$; hence

$$\int_{\mathscr{C}_{[2,s]}} |(t+r)(\partial_t W_0^\gamma + \partial_r W_0^\gamma) + 2W_0^\gamma|^2 + (t-r)^2(|\nabla W_0^\gamma|^2 - (\partial_r W_0^\gamma)^2)\,dS \lesssim \epsilon^2 \ln s. \tag{6-3}$$

The different contributions to the inhomogeneous term F_0^γ, which we remind are only pure interactions $W_0 \times W_0$ and $\mathtt{W}\times\mathtt{W}$ (see (1-9)), are estimated separately below:

(1) *The pure nonzero modes interactions*: After a Cauchy–Schwarz inequality in the y-variable, we will place the factor whose index has length smaller than $\frac{1}{2}k$ in $L_y^2 L_x^\infty$ and the remaining one in L_{xy}^2. We use the pointwise bound (5-3) in the case where $|\gamma_1| = |\gamma_2| = 2$ (which appears only if $k = 4$)

$$\|\partial\mathtt{W}^{\gamma_1}\cdot\partial\mathtt{W}^{\gamma_2}\|_{L_y^1 L_{tx}^2(\mathscr{H}_\tau^{\mathrm{in}})} \lesssim \epsilon\tau^{-3/2+\delta_4} E_2^{\mathrm{in}}(\tau, W)^{1/2},$$

the enhanced pointwise bound (5-13) whenever one of the two multi-indexes has length 0

$$\|\partial \mathtt{W}\cdot\partial \mathtt{W}^{\gamma_2}\|_{L^1_y L^2_{tx}(\mathscr{H}^{\rm in}_\tau)}\lesssim \epsilon\tau^{-3/2}E^{\rm in}_k(\tau,W)^{1/2},$$

and (5-16) otherwise

$$\|\partial \mathtt{W}\cdot\partial \mathtt{W}^{\gamma_2}\|_{L^1_y L^2_{tx}(\mathscr{H}^{\rm in}_\tau)}\lesssim \epsilon\tau^{-3/2+\sigma}E^{\rm in}_{k-1}(\tau,W)^{1/2}.$$

We get

$$\sum_{|\gamma_1|+|\gamma_2|\le k}\|\partial \mathtt{W}^{\gamma_1}\,\partial \mathtt{W}^{\gamma_2}\|_{L^1_y L^2_x(\mathscr{H}^{\rm in}_\tau)}\lesssim \epsilon\tau^{-3/2}[E^{\rm in}_k(\tau,W)^{1/2}+\tau^\sigma E^{\rm in}_{k-1}(\tau,W)^{1/2}+\nu_4\tau^{\delta_4}E^{\rm in}_2(\tau,W)^{1/2}],$$

where $\nu_4=1$ if $k=4$ and 0 otherwise.

(2) *The pure zero modes interactions*: We use the representation formula (1-15)

$$N(W_0^{\gamma_1},W_0^{\gamma_2})=\bar\partial W_0^{\gamma_1}\cdot\partial W_0^{\gamma_2}+\partial W_0^{\gamma_1}\cdot\bar\partial W_0^{\gamma_2}+\frac{t-r}{t}\partial W_0^{\gamma_1}\cdot\partial W_0^{\gamma_2}$$

and estimate all terms in the above right-hand side using (5-1) and (5-2) as follows:

$$\begin{aligned}&\|N(W_0^{\gamma_1},W_0^{\gamma_2})\|_{L^2_x(\mathscr{H}^{\rm in}_\tau)}\\&\quad\lesssim \|(t/\tau)\bar\partial\mathscr{Z}^{\le 3}W_0\|_{L^\infty_x(\mathscr{H}^{\rm in}_\tau)}\|(\tau/t)\partial\mathscr{Z}^{\le k}W_0\|_{L^2_x(\mathscr{H}^{\rm in}_\tau)}+\|\bar\partial\mathscr{Z}^{\le k}W_0\|_{L^2_x(\mathscr{H}^{\rm in}_\tau)}\|\partial\mathscr{Z}^{\le 3}W_0\|_{L^\infty_x(\mathscr{H}^{\rm in}_\tau)}\\&\quad\lesssim \epsilon\tau^{-3/2}E^{\rm in}_k(\tau,W)^{1/2}.\end{aligned}$$

We combine the estimates in step (1) and (2) with the a priori energy bounds (2-5), (2-6) and choose σ, δ_{k-1} small so that $\sigma+\delta_{k-1}\le\delta_k$ to get

$$\|F_0^\gamma\|_{L^2_x(\mathscr{H}^{\rm in}_\tau)}\lesssim\epsilon^2\tau^{-3/2+\mu_k},\quad \text{with } \mu_k=\begin{cases}\delta_k & \text{if } k\le 3,\\ \delta_4+\delta_2 & \text{if } k=4.\end{cases}\tag{6-4}$$

We finally plug (6-3) and (6-4) into (6-2) and use the smallness assumptions on the initial data to derive

$$\begin{aligned}\sup_{[2,s]}E^{\rm c,in}_k(\tau,W_0)&\lesssim E^{\rm c,in}_k(2,W_0)+\epsilon^2 s^{2C\epsilon}\ln s+\int_2^s\epsilon^2\tau^{-1/2+\mu_k}E^{\rm c,in}_k(\tau,W_0)^{1/2}\,d\tau\\&\lesssim\epsilon^2 s^{1+2\mu_k}+\epsilon^2\sup_{[2,s]}E^{\rm c,in}_k(\tau,W_0)\end{aligned}$$

and obtain the result of the proposition by choosing ϵ sufficiently small. □

As a result of the exterior bound (4-9) and the combination of Lemma 5.1 with the conformal energy bound (6-1) and Lemma 4.9 with $T_1=\frac12(s^2+1)$ and $T_2=\sqrt{s^2+|y|^2}$ for $|y-x|=\frac13 t$, we have the following additional pointwise bound for the Z derivatives of W_0 in the whole region $\mathscr{H}_{[2,s_0]}$:

$$|ZZ^jW_0(t,x)|\lesssim\epsilon t^{-1/2}s^{-1/2+\mu_{j+2}}\quad\text{for } j=\overline{0,2},\quad\text{where } \mu_k=\begin{cases}\delta_k & \text{if } k\le 3,\\ \delta_4+\delta_2 & \text{if } k=4.\end{cases}\tag{6-5}$$

We now have all the ingredients to propagate the a priori energy bounds (2-5) and (2-6).

Proposition 6.2. *There exists a constant $A>0$ sufficiently large, some small parameters $\sigma\ll\delta_k\ll\delta_{k+1}$ for $k=\overline{1,4}$ and $\epsilon_0>0$ sufficiently small such that, if for any $0<\epsilon<\epsilon_0$ the solution $W=(u,v)^T$ to the Cauchy problem* (1-5)–(1-6) *satisfies the a priori bounds* (2-5)–(2-7) *in the interior region $\mathscr{H}^{\rm in}_{[2,s_0]}$ and the* *global exterior energy bounds* (2-1) *in the exterior region* $\mathscr{D}^{\mathrm{ex}}$, *then it also satisfies the enhanced energy bound*

$$E_5^{\mathrm{in}}(s, W_0) \le A^2\epsilon^2, \quad s \in [2, s_0]. \tag{6-6}$$

Proof. For any $|\gamma| \le 5$, equation (4-18) satisfied by W_0^γ has the same structure as the inhomogeneous wave equation (3-15); therefore Proposition 3.4 with $\boldsymbol{W}_0 = W_0^\gamma$ and $\boldsymbol{F}_0 = F_0^\gamma$ implies

$$E^{\mathrm{in}}(s, W_0^\gamma) \lesssim E^{\mathrm{in}}(2, W_0^\gamma) + \int_{\mathscr{C}_{[2,s]}} |\mathscr{T} W_0^\gamma|^2 \, dS + \int_2^s \|F_0^\gamma\|_{L^2_x(\mathscr{H}_\tau^{\mathrm{in}})} E^{\mathrm{in}}(\tau, W_0^\gamma)^{1/2} \, d\tau$$

for all $s \in [2, s_0]$. The implicit constant in the above right-hand side is independent of s_0.

The fact that the integral over the boundary $\mathscr{C}_{[2,s]}$ is finite and small is a consequence of (4-16) and an estimate for the source term F_0^γ when $|\gamma| \le 4$ has already been obtained in (6-4). When $|\gamma| = 5$ we simply use the estimate (5-3) on the one hand, and (5-1), (5-2) and the null structure on the other hand, to deduce

$$\sum_{|\gamma_1|+|\gamma_2|\le 5} \|\partial \mathtt{W}^{\gamma_1} \cdot \partial \mathtt{W}^{\gamma_2}\|_{L^1_y L^2_x(\mathscr{H}_\tau^{\mathrm{in}})} \lesssim \epsilon \tau^{-3/2+\delta_4} E_5^{\mathrm{in}}(t, W)^{1/2},$$
$$\sum_{|\gamma_1|+|\gamma_2|\le 5} \|\boldsymbol{N}(W_0^{\gamma_1}, W_0^{\gamma_2})\|_{L^1_y L^2_x(\mathscr{H}_\tau^{\mathrm{in}})} \lesssim \epsilon \tau^{-3/2} E_5^{\mathrm{in}}(s, W)^{1/2}$$

and therefore get

$$\sum_{|\gamma|\le 5} \|F_0^\gamma\|_{L^2_x(\mathscr{H}_\tau^{\mathrm{in}})} \lesssim \epsilon \tau^{-3/2+\delta_4} E_5^{\mathrm{in}}(\tau, W)^{1/2}. \tag{6-7}$$

From the a priori energy bound (2-6) and the exterior energy bound (4-15) we obtain that, for some fixed $K > 1$ and some universal constant $C > 0$,

$$\begin{aligned} E_5^{\mathrm{in}}(s, W_0) &\le E_5^{\mathrm{in}}(2, W_0) + \frac{C_0^2\epsilon^2}{K} + \int_2^s A^2 C \epsilon^3 \tau^{-3/2+\delta_4+\delta_5} \, d\tau \\ &\le E_5^{\mathrm{in}}(2, W_0) + \frac{C_0^2\epsilon^2}{K} + A^2 C \epsilon^3. \end{aligned}$$

The desired improved energy bound then follows choosing $K = 3$, ϵ_0 small such that $3C\epsilon_0^2 < 1$ and $A \ge C_0$ sufficiently large so that

$$E_5^{\mathrm{in}}(2, W_0) \le \frac{A^2\epsilon_0^2}{3}. \qquad \square$$

Proposition 6.3. *There exists a constant* $A > 0$ *sufficiently large, some small parameters* $\sigma \ll \delta_k \ll \delta_{k+1}$ *for* $k = \overline{1, 4}$ *and* $\epsilon_0 > 0$ *sufficiently small such that, if for any* $0 < \epsilon < \epsilon_0$ *the solution* $W = (u, v)^T$ *to the Cauchy problem* (1-5)–(1-6) *satisfies the a priori bounds* (2-5)–(2-7) *in the hyperbolic region* $\mathscr{H}^{\mathrm{in}}_{[2,s_0]}$ *and the global exterior energy bounds* (2-1) *in the exterior region* $\mathscr{D}^{\mathrm{ex}}$, *then it also satisfies the enhanced energy bound*

$$E_{5,k}^{\mathrm{in}}(s, \mathtt{W}) \le A^2\epsilon^2 s^{2\delta_k}, \quad s \in [2, s_0]. \tag{6-8}$$

Proof. We start by considering a multi-index γ of type (n,k) with $k\le n\le 5$ and compare the equation satisfied by $\mathtt{W}^\gamma=(\mathrm{u}^\gamma,\mathrm{v}^\gamma)^T$ with the linear inhomogeneous equation (3-1)

$$\Box_{x,y}\mathtt{W}^\gamma+u\,\partial_y^2\mathtt{W}^\gamma=F^\gamma-F_0^\gamma,\quad (t,x,y)\in\mathbb{R}^{1+3}\times\mathbb{S}^1,$$

so applying Proposition 3.3 with $\boldsymbol{W}=\mathtt{W}^\gamma$ and $\boldsymbol{F}=F^\gamma-F_0^\gamma$ we derive the inequality

$$E^{\mathrm{in}}(s,\mathtt{W}^\gamma)\lesssim E^{\mathrm{in}}(2,\mathtt{W}^\gamma)+\iint_{\mathscr{C}_{[2,s]}}|\mathscr{T}\mathtt{W}^\gamma|^2+|\partial_y\mathtt{W}^\gamma|^2\,d\Sigma_s+\int_2^s\|F^\gamma-F_0^\gamma\|_{L^2_{xy}(\mathscr{H}_\tau^{\mathrm{in}})}E^{\mathrm{in}}(\tau,\mathtt{W}^\gamma)^{1/2}\,d\tau.\tag{6-9}$$

The integral over the boundary $\mathscr{C}_{[2,s]}$ is finite and small as a consequence of the exterior energy estimate (4-15). The pure zero-mode interactions with null structure and the pure nonzero modes interactions have been already examined; see estimate (6-7). The contributions to the source term $F^\gamma-F_0^\gamma$ that have not been estimated yet are the following types of quadratic terms:

$$\begin{aligned}&\partial W_0^{\gamma_1}\cdot\partial\mathtt{W}^{\gamma_2}\quad \text{for } |\gamma_1|+|\gamma_2|\le|\gamma|,\\ &u_0^{\gamma_1}\cdot\partial_y^2W^{\gamma_2}\quad \text{for } |\gamma_1|+|\gamma_2|=|\gamma|,\ |\gamma_2|<|\gamma| \text{ and } \gamma_1=(0,\beta_1).\end{aligned}$$

From the pointwise bounds (5-1) and (5-3) we immediately deduce

$$\sum_{|\gamma_1|+|\gamma_2|\le|\gamma|}\|\partial W_0^{\gamma_1}\cdot\partial\mathtt{W}^{\gamma_2}\|_{L^2_{xy}(\mathscr{H}_\tau^{\mathrm{in}})}\lesssim C_0\epsilon\tau^{-3/2+\delta_5}E_5^{\mathrm{in}}(\tau,W)^{1/2}.$$

The products $u_0^{\gamma_1}\cdot\partial_y^2W^{\gamma_2}$ with $\gamma_1=(0,\beta_1)$ and such that $\mathscr{Z}^{\gamma_1}=Z^{\beta_1}$ is a pure product of Klainerman vector fields are estimated separately depending on the values of γ_1 and γ_2. Let $k_1:=|\gamma_1|$. Then $k_1+|\gamma_2|=n$ and $k_1\le k$ and we distinguish the following cases:

- When $k_1=n$ and $|\gamma_2|=0$ we have $k=n$ and use the pointwise bound (5-13) and the conformal energy bound (6-1) to derive

$$\|u_0^{\gamma_1}\cdot\partial_y^2\mathtt{W}\|_{L^2_{xy}(\mathscr{H}_\tau^{\mathrm{in}})}\lesssim\|(t/\tau)\partial_y^2\mathtt{W}\|_{L^2_yL^\infty_x(\mathscr{H}_\tau^{\mathrm{in}})}E_{n-1}^{\mathrm{c,in}}(\tau,W_0)^{1/2}\lesssim\epsilon^2\tau^{-1+\mu_{k-1}}.$$

- When $k_1=n-1$ and $|\gamma_2|=1$ we have $n-1\le k\le n$ and use (5-15), (5-16), (6-1)

$$\|u_0^{\gamma_1}\cdot\partial_y^2\mathtt{W}^{\gamma_2}\|_{L^2_{xy}(\mathscr{H}_\tau^{\mathrm{in}})}\lesssim\|(t/\tau)\partial_y^2\mathtt{W}^{\gamma_2}\|_{L^2_yL^\infty_x(\mathscr{H}_\tau^{\mathrm{in}})}E_{n-2}^{\mathrm{c,in}}(\tau,W_0)^{1/2}\lesssim\epsilon^2\tau^{-1+\mu_{k-1}+\sigma}.$$

- When $k_1=n-2$ and $|\gamma_2|=2$ we have $n-1\le k\le n$. This case only appears when $3\le n\le 5$. If $\mathscr{Z}^{\gamma_2}=\{\partial^2,\partial Z\}$ or if $\mathscr{Z}^{\gamma_2}=\partial Z$ and $n=\overline{4,5}$ we apply (5-3) and (6-1):

$$\|u_0^{\gamma_1}\cdot\partial_y^2\mathtt{W}^{\gamma_2}\|_{L^2_{xy}(\mathscr{H}_\tau^{\mathrm{in}})}\lesssim\|(t/\tau)\partial_y^2\mathtt{W}^{\gamma_2}\|_{L^2_yL^\infty_x(\mathscr{H}_\tau^{\mathrm{in}})}E_{n-3}^{\mathrm{c,in}}(\tau,W_0)^{1/2}\lesssim\epsilon^2\tau^{-1+\mu_{k-2}+\delta_3},$$

while if $\mathscr{Z}^{\gamma_2}=\partial Z$ and $n=3$ (in which case $k=2$) we use (2-6) and (2-7):

$$\|u_0^{\gamma_1}\cdot\partial_y^2\mathtt{W}^{\gamma_2}\|_{L^2_{xy}(\mathscr{H}_\tau^{\mathrm{in}})}\lesssim\|Zu_0\|_{L^\infty(\mathscr{H}_\tau^{\mathrm{in}})}E_{3,1}^{\mathrm{in}}(\tau,W)^{1/2}\lesssim\epsilon^2\tau^{-1+\sigma+\delta_{k-1}}.$$

- When $k_1=1$ and $|\gamma_2|=n-1$ for $n=\overline{4,5}$, the a priori bounds (2-6) and (2-7) give

$$\|u_0^{\gamma_1}\cdot\partial_y^2\mathtt{W}^{\gamma_2}\|_{L^2_{xy}(\mathscr{H}_\tau^{\mathrm{in}})}\lesssim\|Zu_0\|_{L^\infty(\mathscr{H}_\tau^{\mathrm{in}})}E_{n,k-1}^{\mathrm{in}}(\tau,W)^{1/2}\lesssim\epsilon^2\tau^{-1+\sigma+\delta_{k-1}},$$

and in the last case $k_1 = 2$, $|\gamma_2| = 3$ — which only appears when $n = 5$ — the bounds (2-6) and (6-5) yield

$$\|u_0^{\gamma_1} \cdot \partial_y^2 \mathtt{W}^{\gamma_2}\|_{L^2_{xy}(\mathscr{H}_\tau^{\text{in}})} \lesssim \|Z^2 u_0\|_{L^\infty(\mathscr{H}_\tau^{\text{in}})} E^{\text{in}}_{4,k-2}(\tau, W)^{1/2} \lesssim \epsilon^2 \tau^{-1+2\delta_{k-2}}.$$

Choosing appropriately $\sigma \ll \delta_k \ll \delta_{k+1}$ for $k = \overline{1,4}$ so that δ_{k+1} is bigger than some linear combination of σ and δ_j with $j \le k$, we then obtain

$$\|u_0^{\gamma_1} \cdot \partial_y^2 \mathtt{W}^{\gamma_2}\|_{L^2_{xy}(\mathscr{H}_\tau^{\text{in}})} \lesssim \epsilon^2 \tau^{-1+\delta_k},$$

with an implicit constant that depends on A and B. The same bound holds then true for $F^\gamma - F_0^\gamma$ so plugging it into (6-9) together with the exterior energy estimate (4-15), and the a priori energy bounds (2-6), gives us

$$E^{\text{in}}_{5,k}(s, \mathtt{W}) \le C E^{\text{in}}_{5,k}(2, \mathtt{W}) + \frac{C C_0^2 \epsilon^2}{2} + \int_2^s A^2 C \epsilon^3 \tau^{-1+2\delta_k}\, d\tau.$$

The end of the proof follows finally by choosing appropriately the constants A and ϵ_0. □

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EXISTENCE OF RESONANCES FOR SCHRÖDINGER OPERATORS ON HYPERBOLIC SPACE

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We prove existence results and lower bounds for the resonances of Schrödinger operators associated to smooth, compactly support potentials on hyperbolic space. The results are derived from a combination of heat and wave trace expansions and asymptotics of the scattering phase.

1. Introduction

This article is devoted to establishing lower bounds on the resonance count for Schrödinger operators on the hyperbolic space $\mathbb{H}^{n+1}$. Although such results are well known in Euclidean scattering theory, the literature for Schrödinger operators on hyperbolic space is comparatively sparse. Upper bounds on resonances for such operators were considered in [Borthwick 2010; Borthwick and Crompton 2014]. Most other recent papers dealing with Schrödinger operators on hyperbolic space have focused on applications to nonlinear Schrödinger equations [Anker and Pierfelice 2009; Banica 2007; Banica et al. 2008; 2009; Borthwick and Marzuola 2015; Ionescu et al. 2012; Ionescu and Staffilani 2009]. As far as we are aware, the literature contains no general existence results for resonances in this context.

Let Δ denote the positive Laplacian operator on $\mathbb{H}^{n+1}$. For $V \in C_0^\infty(\mathbb{H}^{n+1}, \mathbb{R})$, the Schrödinger operator $\Delta + V$ has continuous spectrum $\left[\frac{1}{4}n^2, \infty\right)$. The resolvent of $\Delta + V$ is defined for $\operatorname{Re} s > \frac{1}{2}n$ by

$$R_V(s) := (\Delta + V - s(n-s))^{-1}. \tag{1-1}$$

As an operator on weighted L^2 spaces, $R_V(s)$ admits a meromorphic extension to $s \in \mathbb{C}$, with poles of finite rank, as described in Section 2. The resonance set $\mathcal{R}_V$ associated to V consists of the poles of $R_V(s)$, repeated according to the multiplicity given by

$$m_V(\zeta) := \operatorname{rank} \operatorname{Res}_\zeta R_V(s). \tag{1-2}$$

There are no eigenvalues embedded in the continuous spectrum, and no resonances on the line $\operatorname{Re} s = \frac{1}{2}n$ except possibly at $\frac{1}{2}n$. For a proof, see, for example, [Borthwick and Marzuola 2015, Theorem 1]. The discrete spectrum of $\Delta + V$ is therefore finite and lies below $\frac{1}{4}n^2$. An eigenvalue λ corresponds to a resonance $\zeta \in \left(\frac{1}{2}n, n\right)$ such that $\lambda = \zeta(n-\zeta)$.

The resonance counting function,

$$N_V(r) := \#\left\{\zeta \in \mathcal{R}_V : \left|\zeta - \tfrac{1}{2}n\right| \le r\right\}, \tag{1-3}$$

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satisfies a polynomial bound as $r \to \infty$:

$$N_V(r) = O(r^{n+1}). \tag{1-4}$$

This estimate is covered by the more general result of [Borthwick 2008, Theorem 1.1], but with $\mathbb{H}^{n+1}$ as the background metric one could also give a simpler proof following the approach that Guillopé and Zworski [1995] used for $n = 1$. A sharp constant for the bound (1-4), depending on the support of the potential, was obtained in [Borthwick 2010]. The corresponding lower bound was shown to hold in a generic sense in [Borthwick and Crompton 2014], but the existence question was not resolved.

The existence problem for resonances looks quite different in even and odd dimensions. In even dimensions, $\mathcal{R}_0$ contains resonances at negative integers, with multiplicities such that the polynomial bound (1-4) is already saturated for $V = 0$. Therefore our goal in even dimensions is to distinguish $\mathcal{R}_V$ from $\mathcal{R}_0$. On the other hand, $\mathcal{R}_0$ is empty for odd dimensional hyperbolic space. In that case we seek lower bounds on $\mathcal{R}_V$ itself.

In the present paper, we will prove the following:

Theorem 1.1. *Let $\mathcal{R}_V$ denote the set of resonances of $\Delta + V$ for $V \in C_0^\infty(\mathbb{H}^{n+1}, \mathbb{R})$, with $N_V(r)$ the corresponding counting function.*

(1) *If the dimension is even or equal to* 3, *then $\mathcal{R}_V = \mathcal{R}_0$ only if $V = 0$.*

(2) *For even dimension ≥ 6, if $V \neq 0$ then $\mathcal{R}_V$ and $\mathcal{R}_0$ differ by infinitely many points. This conclusion holds also if $\int V\,dg \geq 0$ for* $\dim = 2$, *and if $\int V\,dg \neq 0$ for* $\dim = 4$.

(3) *In all odd dimensions, if $\mathcal{R}_V$ is not empty then the resonance set is infinite and $N_V(r) \neq O(r)$.*

In even dimensions, we also show that the resonance set determines the scattering phase and wave trace completely, and in particular fixes all of the wave invariants. See Section 9 for the full set of inverse scattering results.

Theorem 1.1 is derived from the asymptotic expansions of the scattering phase and heat and wave traces. In the context of potential scattering in hyperbolic space, these expansions do not seem to have not been studied in the literature, so we give a full account of their adaptation to this setting. The explicit formulas for the wave invariants are stated in Proposition 6.4.

The organization of the paper is as follows. After reviewing some facts on the resolvent and its kernel in Section 2, we use the spectral resolution to define distributional traces in Section 3. In Section 4 we establish the Birman–Krein formula relating these traces to the scattering phase. The Poisson formula expressing the wave trace as a sum over resonances is proven in Section 5. In Section 6 the asymptotic expansion of the wave trace at $t = 0$ is established. The corresponding heat trace expansion is worked out in Section 7, which is then used to study asymptotics of the scattering phase in Section 8. Finally, in Section 9 these tools are applied to derive the existence results.

2. The resolvent

The resolvent of the free Laplacian on $\mathbb{H}^{n+1}$ is traditionally written with spectral parameter $s(n-s)$ as in (1-1). Derived by Patterson [1989, Proposition 2.2], the resolvent kernel is given by the well-known hypergeometric formula

$$R_0(s;z,w) = \frac{\pi^{-n/2}2^{-2s-1}\Gamma(s)}{\Gamma\big(s-\frac{1}{2}n+1\big)} \cosh^{-2s}\big(\tfrac{1}{2}d(z,w)\big) F\big(s, s-\tfrac{1}{2}(n-1), 2s-n+1; \cosh^{-2}\big(\tfrac{1}{2}d(z,w)\big)\big),$$

where $d(z,w)$ is the hyperbolic distance. Using hypergeometric identities [NIST 2010, §14.3(iii)], we can rewrite this formula as

$$R_0(s;z,w) = (2\pi)^{-(n+1)/2} \frac{\Gamma(s)}{\sinh^{\mu} d(z,w)} \boldsymbol{Q}^{\mu}_{\nu}(\cosh d(z,w)), \tag{2-1}$$

where $\nu := s-\frac{1}{2}(n+1)$, $\mu := \frac{1}{2}(n-1)$, and $\boldsymbol{Q}^{\mu}_{\nu}$ denotes the normalized Legendre function:

$$\boldsymbol{Q}^{\mu}_{\nu}(x) := \frac{e^{-i\pi\mu}}{\Gamma(\mu+\nu+1)} Q^{\mu}_{\nu}(x).$$

(Under this convention $\boldsymbol{Q}^{\mu}_{\nu}(x)$ is entire as a function of both indices.) The factor $\Gamma(s)$ in (2-1) has poles at negative integers, but these yield resonances only for $n+1$ even. In odd dimensions the poles are canceled by zeroes of $\boldsymbol{Q}^{\mu}_{\nu}$.

Let $(r,\omega) \in [0,\infty) \times \mathbb{S}^n$ denote geodesic polar coordinates on $\mathbb{H}^{n+1}$. We will take

$$\rho := \frac{1}{\cosh r}$$

as a boundary defining function for the radial compactification of $\mathbb{H}^{n+1}$ into a ball. The hypergeometric formula for $\boldsymbol{Q}^{\mu}_{\nu}(x)$ [Erdélyi et al. 1953, §3.2(5)] yields an expansion of the resolvent kernel:

$$R_0(s;z,w) = \pi^{-n/2}2^{-s-1} \frac{\Gamma(s)}{\Gamma\big(s-\frac{1}{2}n+1\big)} \sum_{k=0}^{\infty} \frac{a_k(s)}{(\cosh d(z,w))^{s+2k}}, \tag{2-2}$$

with $a_0(s) = 1$. In particular,

$$R_0(s;z,w) = O(e^{-sd(z,w)}) \quad \text{as } d(z,w) \to \infty, \tag{2-3}$$

which shows that $R_0(s)$ extends meromorphically to $s \in \mathbb{C}$ as a bounded operator $\rho^N L^2(\mathbb{H}^{n+1}) \to \rho^{-N} L^2(\mathbb{H}^{n+1})$ for $\operatorname{Re} s > -N+\frac{1}{2}n$.

For $V \in C_0^{\infty}(\mathbb{H}^{n+1}, \mathbb{R})$, the resolvent $R_V(s)$ defined in (1-1) is related to $R_0(s)$ by the identity

$$R_0(s) = R_V(s)(1+VR_0(s)). \tag{2-4}$$

The operator $1+VR_0(s)$ is invertible by Neumann series for $\operatorname{Re} s$ sufficiently large, and it follows from (2-3) that the operator $VR_0(s)$ is compact on $\rho^N L^2(\mathbb{H}^{n+1})$ for $\operatorname{Re} s > -N+\frac{1}{2}n$. Hence the analytic Fredholm theorem yields a meromorphic inverse $(1+VR_0(s))^{-1}$, with poles of finite rank, which is bounded on $\rho^N L^2(\mathbb{H}^{n+1})$ for $\operatorname{Re} s > -N+\frac{1}{2}n$. We thus obtain a meromorphic extension of $R_V(s)$ by setting

$$R_V(s) = R_0(s)(1+VR_0(s))^{-1},$$

which is bounded as an operator $\rho^N L^2(\mathbb{H}^{n+1}) \to \rho^{-N} L^2(\mathbb{H}^{n+1})$ for $\operatorname{Re} s > -N+\frac{1}{2}n$.

We can see from (2-2) that the free resolvent kernel $R_0(s; z, z')$ is polyhomogeneous as a function of $\rho(z')$ as $\rho(z') \to 0$, with leading term of order $\rho(z')^s$. It follows from (2-4) that the kernel of $R_V(s)$ has the same property. The Poisson kernel is defined as the leading coefficient in the expansion as $\rho(z') \to 0$:

$$E_V(s; z, \omega') := \lim_{r' \to \infty} \rho(z')^{-s} R_V(s; z, z'). \tag{2-5}$$

Interpreting this function as an integral kernel, with respect to the standard sphere metric, defines the Poisson operator

$$E_V(s) : C^\infty(\mathbb{S}^n) \to L^2(\mathbb{H}^{n+1}).$$

The Poisson operator maps boundary data to solutions of the generalized eigenfunction equation

$$(\Delta + V - s(n-s))u = 0.$$

By Stone's formula, the continuous part of the spectral resolution of $\Delta + V$ is given by the restriction of the operator $R_V(s) - R_V(n-s)$ to the line $\operatorname{Re} s = \frac{1}{2}n$. This is related to the Poisson operator by the identity

$$R_V(s) - R_V(n-s) = (n-2s) E_V(s) E_V(n-s)^t \tag{2-6}$$

as operators $C_0^\infty(\mathbb{H}^{n+1}) \to L^2(\mathbb{H}^{n+1})$, meromorphically for $s \in \mathbb{C}$. The proof of (2-6) is essentially the same as in the case $n = 1$ presented in [Borthwick 2016, Proposition 4.6].

3. Traces

Given $V \in C_0^\infty(\mathbb{H}^{n+1}, \mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$, the operator $f(\Delta + V) - f(\Delta)$ is of trace class. In fact, the map

$$f \mapsto \operatorname{tr}[f(\Delta + V) - f(\Delta)] \tag{3-1}$$

defines a tempered distribution. For the proof, see [Dyatlov and Zworski 2019, Theorem 3.50], which applies to the hyperbolic setting with only minor modifications.

The spectral theorem gives the representation

$$f(\Delta + V) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} [(\Delta + V - \lambda - i\varepsilon)^{-1} - (\Delta + V - \lambda + i\varepsilon)^{-1}] f(\lambda)\, d\lambda,$$

with the limit taken in the operator-norm topology. We can separate the contributions from the discrete and continuous spectrum, and write the continuous part in terms of $R_V(s)$ by setting $s(n-s) = \lambda \pm i\varepsilon$. The result is

$$f(\Delta + V) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Big[R_V\Big(\frac{1}{2}n - i\xi + \varepsilon\Big) - R_V\Big(\frac{1}{2}n + i\xi + \varepsilon\Big) \Big] f\Big(\frac{1}{4}n^2 + \xi^2\Big) \xi\, d\xi + \sum_{j=1}^{d} f(\lambda_j) \phi_j \otimes \bar{\phi}_j, \tag{3-2}$$

where the $\lambda_1, \dots, \lambda_d$ are the eigenvalues of $\Delta + V$, with corresponding normalized eigenvectors ϕ_j.

The self-adjointness of $\Delta + V$ implies an estimate

$$\|(\Delta + V - s(n-s))u\| \geq |\operatorname{Im} s(n-s)| \|u\|^2.$$

This shows that a pole of $R_V(s)$ at $s=\frac{1}{2}n$ could have at most second-order. (This argument is analogous to the Euclidean case; see [Dyatlov and Zworski 2019, Lemma 3.16].) A pole of order 2 can occur only if $\frac{1}{4}n^2$ is an eigenvalue, which is ruled out by [Bouclet 2013, Corollary 1.2]. Therefore $R_V(s)$ has at most a first-order pole at $\frac{1}{2}n$. The integrand in (3-2) is thus continuous at $\varepsilon=0$ in the operator topology because a pole would be canceled by the extra factor of ξ. Taking the limit $\varepsilon\to 0$ gives

$$f(\Delta+V)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\Big[R_V\Big(\frac{1}{2}n-i\xi\Big)-R_V\Big(\frac{1}{2}n+i\xi\Big)\Big]f\Big(\frac{1}{4}n^2+\xi^2\Big)\xi\,d\xi+\sum_{j=1}^{d}f(\lambda_j)\phi_j\otimes\bar{\phi}_j.$$

Let us define the integral kernel of the spectral resolution as

$$K_V(\xi;z,w):=\frac{\xi}{2\pi i}\Big[R_V\Big(\frac{1}{2}n-i\xi;z,w\Big)-R_V\Big(\frac{1}{2}n+i\xi;z,w\Big)\Big]. \tag{3-3}$$

For $V=0$ this kernel can be written explicitly using (2-1) and the Legendre connection formula [NIST 2010, §14.9(iii)]:

$$\frac{\boldsymbol{Q}^{\mu}_{-\nu-1}(x)}{\Gamma(\mu+\nu+1)}-\frac{\boldsymbol{Q}^{\mu}_{\nu}(x)}{\Gamma(\mu-\nu)}=\cos(\pi\nu)P^{-\mu}_{\nu}(x).$$

The result is

$$K_0(\xi;z,w):=c_n(\xi)(\sinh r)^{-\mu}P^{-\mu}_{-1/2+i\xi}(\cosh r), \tag{3-4}$$

where $\mu=\frac{1}{2}(n-1),\ r=d(z,w)$, and

$$c_n(\xi):=(2\pi)^{-\mu}\xi\sinh(\pi\xi)\Gamma\big(\tfrac{1}{2}n+i\xi\big)\Gamma\big(\tfrac{1}{2}n-i\xi\big).$$

The hypergeometric expansion [NIST 2010, (14.3.9)] of $P^{-\mu}_{\nu}(x)$ near $x=1$ shows that $K_0(\xi;z,w)$ is smooth for all $z,w\in\mathbb{H}^{n+1}\times\mathbb{H}^{n+1}$.

For the Schrödinger operator case, we note that (2-4) yields the identity

$$R_V(s)-R_V(n-s)=(1-R_V(s)V)(R_0(s)-R_0(n-s))(1-VR_V(n-s)).$$

Since $R_0(s)-R_0(n-s)$ has a smooth kernel for $\operatorname{Re}s=\frac{1}{2}n,\ V\in C_0^\infty(\mathbb{H}^{n+1},\mathbb{R})$, and $R_V(s)$ is a pseudodifferential operator of order -2, the identity implies that $R_V(s)-R_V(n-s)$ also has a smooth kernel for $\operatorname{Re}s=\frac{1}{2}n,\ s\neq\frac{1}{2}n$. The kernel $K_V(\xi;\cdot,\cdot)$ is thus continuous for $\xi\in\mathbb{R}$ and smooth as a function on $\mathbb{H}^{n+1}\times\mathbb{H}^{n+1}$.

In [Borthwick and Marzuola 2015, Proposition 6.1], it was shown that $\operatorname{Im}R_V\big(\frac{1}{2}n+i\xi;z,w\big)$ satisfies a polynomial bound as a function of $\xi\in\mathbb{R}$, uniformly in $\mathbb{H}^{n+1}\times\mathbb{H}^{n+1}$, provided there is no resonance at $s=\frac{1}{2}n$. This restriction can be removed for the K_V estimate because of the extra factor of ξ in (3-3), since, as noted above, $R_V(s)$ has at most a first-order pole at $s=\frac{1}{2}n$. We can thus use the spectral resolution formula to write the kernel of $f(\Delta+V)$ as

$$f(\Delta+V)(z,w)=\int_{-\infty}^{\infty}K_V(\xi;z,w)f\big(\tfrac{1}{4}n^2+\xi^2\big)\,d\xi+\sum_{j=1}^{d}f(\lambda_j)\phi_j(z)\bar{\phi}_j(w). \tag{3-5}$$

Since $f(\Delta+V)-f(\Delta)$ is trace-class and has a continuous kernel, the trace can be computed as an integral over the kernel by Duflo's theorem [1972, Theorem V.3.1.1]. This proves the following:

Proposition 3.1. *For* $V \in C_0^\infty(\mathbb{H}^{n+1}, \mathbb{R})$,

$$\operatorname{tr}[f(\Delta+V) - f(\Delta)] = \int_{\mathbb{H}^{n+1}} \int_{-\infty}^{\infty} [K_V(\xi; z, z) - K_0(\xi, z, z)] f\big(\tfrac{1}{4}n^2 + \xi^2\big)\, d\xi\, dg(z) + \sum_{j=1}^{d} f(\lambda_j).$$

4. Birman–Krein formula

The Birman–Krein formula relates the spectral resolution of $\Delta + V$ to the scattering matrix. This formula provides the crucial link between the traces discussed in Section 3 and the resonance set. The formula for the hyperbolic case is analogous to the Euclidean version [Dyatlov and Zworski 2019, Theorem 3.51].

The scattering matrix associated to V is defined as follows. The Poisson operator maps a function $f \in C^\infty(\mathbb{S}^n)$ to a generalized eigenfunction $E_V(s) f$, which admits an asymptotic expansion with leading terms

$$(2s - n) E_V(s) f \sim \rho^{n-s} f + \rho^s f', \tag{4-1}$$

where $f' \in C^\infty(\mathbb{S}^n)$ for $\operatorname{Re} s = \frac{1}{2}n,\ s \neq \frac{1}{2}n$. The structure of this expansion is well known and can be deduced from the resolvent identity (2-4).

The scattering matrix $S_V(s)$ is a family of pseudodifferential operators $S_V(s)$ on $\mathbb{S}^n$ that intertwines the leading coefficients of (4-1):

$$S_V(s) : f \mapsto f'.$$

For appropriate choices of s, we can interpret f as incoming boundary data, and f' as the corresponding outgoing data. By the meromorphic continuation of the resolvent, $S_V(s)$ extends meromorphically to $s \in \mathbb{C}$. The identities

$$S_V(s)^{-1} = S_V(n - s) \tag{4-2}$$

and

$$E_V(n - s) S_V(s) = -E_V(s), \tag{4-3}$$

which follow from (4-1), hold meromorphically in s.

The integral kernel of the scattering matrix (with respect to the standard sphere metric) can be derived from the resolvent by a boundary limit analogous to (2-5):

$$S_V(s; \omega, \omega') := (2s - n) \lim_{\rho, \rho' \to 0} (\rho\rho')^{-s} R_V(s; z, z')$$

for $\omega \neq \omega'$. We can thus see from (2-4) that

$$S_V(s) = S_0(s) - (2s - n) E_V(s) V E_0(s).$$

This gives a formula for the relative scattering matrix

$$S_V(s) S_0(s)^{-1} = I + (2s - n) E_V(s) V E_0(n - s). \tag{4-4}$$

Since $(2s - n) E_V(s) V E_0(n - s)$ is a smoothing operator, $S_V(s) S_0(s)^{-1}$ is of determinant class. We can thus define the relative scattering determinant

$$\tau(s) := \det[S_V(s) S_0(n - s)].$$

By (4-2), the scattering determinant satisfies

$$\tau(s)\tau(n-s)=1. \tag{4-5}$$

Also, since $S_V(s)$ is unitary on the critical line, $|\tau(s)|=1$ for $\operatorname{Re} s=\frac{1}{2}n$.

We can evaluate $\tau\left(\frac{1}{2}n\right)$ by noting that

$$S_V\left(\tfrac{1}{2}n\right)=-I+2P, \tag{4-6}$$

where P is an orthogonal projection of rank $m_V\left(\frac{1}{2}n\right)$. (See [Borthwick 2016, Lemma 8.9] for the argument.) This implies that

$$\tau\left(\tfrac{1}{2}n\right)=(-1)^{m_V(n/2)}.$$

The scattering phase $\sigma(\xi)$ for $\xi\in\mathbb{R}$ is defined as

$$\sigma(\xi):=\frac{i}{2\pi}\log\frac{\tau\left(\frac{1}{2}n+i\xi\right)}{\tau\left(\frac{1}{2}n\right)},$$

with the branch of logarithm chosen continuously from $\sigma(0):=0$. The reflection formula (4-5) implies

$$\sigma(-\xi)=-\sigma(\xi).$$

We will be particularly interested in the derivative of the scattering phase. By [Gohberg and Kreĭn 1969, §IV.1],

$$\frac{\tau'}{\tau}(s)=\operatorname{tr}\left[(S_V(s)S_0(n-s))^{-1}\frac{d}{ds}(S_V(s)S_0(n-s))\right]=\operatorname{tr}[S_V(n-s)S_V'(s)-S_0(s)S_0'(n-s)],$$

where $S_V'(s):=\partial_s S_V(s)$. For the scattering phase this gives

$$\sigma'(\xi)=-\frac{1}{2\pi}\operatorname{tr}\left[S_V\left(\frac{1}{2}n-i\xi\right)S_V'\left(\frac{1}{2}n+i\xi\right)-S_0\left(\frac{1}{2}n+i\xi\right)S_0'\left(\frac{1}{2}n-i\xi\right)\right]. \tag{4-7}$$

Theorem 4.1 (Birman–Krein formula). *For $V\in L^\infty_{\mathrm{cpt}}(\mathbb{H}^{n+1},\mathbb{R})$ and $f\in\mathcal{S}(\mathbb{R})$,*

$$\operatorname{tr}[f(\Delta+V)-f(\Delta)]=\int_0^\infty\sigma'(\xi)f\left(\tfrac{1}{4}n^2+\xi^2\right)d\xi+\sum_{j=1}^d f(\lambda_j)+\tfrac{1}{2}m_V\left(\tfrac{1}{2}n\right)f\left(\tfrac{1}{4}n^2\right),$$

where $\lambda_1,\dots,\lambda_d$ are the eigenvalues of $\Delta+V$ and $m_V\left(\frac{1}{2}n\right)$ is the multiplicity of $\frac{1}{2}n$ as a resonance of $\Delta+V$.

Proof. For convenience, let us assume that the discrete spectrum of $\Delta+V$ is empty, since the contribution to the trace from $\lambda_1,\dots,\lambda_d$ is easily dealt with. Under this assumption, Proposition 3.1 gives

$$\operatorname{tr}[f(\Delta+V)-f(\Delta)]=\int_{\mathbb{H}^{n+1}}\int_{-\infty}^\infty[K_V(\xi;z,z)-K_0(\xi,z,z)]f\left(\tfrac{1}{4}n^2+\xi^2\right)d\xi\,dg(z).$$

If the integral over z is restricted to the set $\{\rho(z) \geq \varepsilon\}$, then switching the order of integration is justified by the uniform polynomial bounds on K_V. We can thus write

$$\operatorname{tr}[f(\Delta+V)-f(\Delta)] = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} I_\varepsilon(\xi) f\left(\tfrac{1}{4}n^2+\xi^2\right) d\xi, \tag{4-8}$$

where

$$I_\varepsilon(\xi) := \int_{\{\rho\geq\varepsilon\}} [K_V(\xi; z, z) - K_0(\xi, z, z)]\, d\omega'\, dg(z).$$

To compute $I_\varepsilon(\xi)$, we first use (2-6) to write K_V in terms of E_V:

$$I_\varepsilon(\xi) := -\frac{(2s-n)^2}{4\pi}\int_{\{\rho\geq\varepsilon\}}\int_{\mathbb{S}^n}\big[E_V(s; z, \omega')E_V(n-s; z, \omega') \\ - E_0(s; z, \omega')E_0(n-s; z, \omega')\big]\, d\omega'\, dg(z), \tag{4-9}$$

where $d\omega'$ is the standard sphere measure. We are using the identification $s = \frac{1}{2}n + i\xi$ freely here, to simplify notation where possible. The next step is to apply a Maass–Selberg identity as described in the proof of [Borthwick 2016, Proposition 10.4]. Because $E_V(s)$ satisfies the eigenvalue equation, we can write

$$E_V(s')E(n-s)^t = \frac{1}{s(n-s)-s'(n-s')}[E_V(s')\Delta E(n-s)^t - \Delta E_V(s')E(n-s)^t].$$

Applying this to (4-9) yields

$$I_\varepsilon(\xi) := -\frac{1}{4\pi}\lim_{s'\to s}\frac{2s-n}{s'-s}\int_{\{\rho\geq\varepsilon\}}\int_{\mathbb{S}^n}\big[E_V(s'; z, \omega')\Delta E_V(n-s; z, \omega') \\ - \Delta E_V(s'; z, \omega')E_V(n-s; z, \omega') - E_0(s'; z, \omega')\Delta E_0(n-s; z, \omega') \\ - E_0(s'; z, \omega')\Delta E_0(n-s; z, \omega')\big]\, d\omega'\, dg(z),$$

with Δ acting on the z variable. By Green's formula applied to the region $\{\rho = \varepsilon\}$,

$$I_\varepsilon(\xi) := \frac{1}{4\pi}\lim_{s'\to s}\frac{2s-n}{s'-s}\int_{\{\rho=\varepsilon\}}\int_{\mathbb{S}^n}\big[E_V(s'; z, \omega')\partial_r E_V(n-s; z, \omega') \\ - \partial_r E_V(s'; z, \omega')E_V(n-s; z, \omega') - E_0(s'; z, \omega')\partial_r E_0(n-s; z, \omega') \\ - E_0(s'; z, \omega')\partial_r E_0(n-s; z, \omega')\big] \sinh^n r\, d\omega'\, d\omega,$$

where $z = (r, \omega)$ in geodesic polar coordinates. The same calculation with $s = s'$ yields zero, so we can evaluate the limit $s' \to s$ as a derivative:

$$I_\varepsilon(\xi) = \frac{2s-n}{4\pi}\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}\big[E'_V(s; z, \omega')\partial_r E_V(n-s; z, \omega') \\ - \partial_r E'_V(s; z, \omega')E_V(n-s; z, \omega') - E'_0(s; z, \omega')\partial_r E_0(n-s; z, \omega') \\ + \partial_r E'_0(s; z, \omega')E_0(n-s; z, \omega')\big] \sinh^n r\, d\omega'\, d\omega\Big|_{r=\cosh^{-1}(1/\varepsilon)},$$

where $E'_V = \partial_s E_V$. The integrand can be simplified using the identity (4-2) and the distributional asymptotic

$$(2s-n)E_V(s; z, \omega') \sim \rho^{n-s}\delta_\omega(\omega') + \rho^s S_V(s; \omega, \omega'),$$

which follows from (4-1). After cancelling terms between $E_V(s)$ and $E_0(s)$, we find

$$I_\varepsilon(\xi) = -\frac{1}{4\pi}\operatorname{tr}\Big[S_V\Big(\frac{1}{2}n-i\xi\Big)S_V'\Big(\frac{1}{2}n+i\xi\Big)-S_0\Big(\frac{1}{2}n-i\xi\Big)S_0'\Big(\frac{1}{2}n+i\xi\Big)\Big] \\ +\frac{\varepsilon^{-2i\xi}}{8\pi i\xi}\operatorname{tr}\Big[S_V\Big(\frac{1}{2}n-i\xi\Big)-S_0\Big(\frac{1}{2}n-i\xi\Big)\Big]-\frac{\varepsilon^{2i\xi}}{8\pi i\xi}\operatorname{tr}\Big[S_V\Big(\frac{1}{2}n+i\xi\Big)-S_0\Big(\frac{1}{2}n+i\xi\Big)\Big]+o(\varepsilon). \quad (4\text{-}10)$$

The first trace in (4-10) reduces to $\frac{1}{2}\sigma'(\xi)$ by (4-7). Thus, applying (4-10) in (4-8) gives

$$\operatorname{tr}[f(\Delta+V)-f(\Delta)]=\frac{1}{2}\int_{-\infty}^{\infty}\sigma'(\xi)f\Big(\frac{1}{4}n^2+\xi^2\Big)\,d\xi+\lim_{a\to\infty}\int_{-\infty}^{\infty}\Big[\frac{e^{i\xi a}}{8\pi i\xi}\varphi(\xi)-\frac{e^{-i\xi a}}{8\pi i\xi}\varphi(-\xi)\Big]\,d\xi, \quad (4\text{-}11)$$

where we have substituted $\varepsilon = e^{-a/2}$ and

$$\varphi(\xi) := \operatorname{tr}\big[S_V\big(\tfrac{1}{2}n-i\xi\big)-S_0\big(\tfrac{1}{2}n-i\xi\big)\big]f\big(\tfrac{1}{4}n^2+\xi^2\big). \quad (4\text{-}12)$$

To evaluate the limit $a\to\infty$ in (4-11), we need to control the growth of $\varphi(\xi)$. We can argue as in [Borthwick and Crompton 2014, Lemma 3.3] that, for $\chi \in C_0^\infty(\mathbb{H})$ equal to 1 on the support of V,

$$S_V(s)-S_0(s) = -(2s-n)E_0(s)^t\chi(1+VR_0(s)\chi)^{-1}VE_0(s).$$

The Hilbert–Schmidt norms of the cutoff factors $\chi E_0\big(\frac{1}{2}n\pm i\xi\big)$ are $O(|\xi|^{-1})$ by [Borthwick and Crompton 2014, Lemma 3.3]. The operator norm of

$$\big(1+VR_0\big(\tfrac{1}{2}n\pm i\xi\big)\chi\big)^{-1}$$

is $O(1)$ by the cutoff resolvent bound from [Guillarmou 2005, Proposition 3.2]. Therefore the trace in (4-12) has at most polynomial growth, and φ is integrable over $\xi\in\mathbb{R}$.

The Riemann–Lebesgue lemma gives

$$\lim_{a\to\infty}\int_{|\xi|>1}\Big[\frac{e^{i\xi a}}{8\pi i\xi}\varphi(\xi)-\frac{e^{-i\xi a}}{8\pi i\xi}\varphi(-\xi)\Big]\,d\xi=0$$

as well as

$$\lim_{a\to\infty}\int_{-1}^{1}\frac{e^{\pm i\xi a}}{8\pi i\xi}[\varphi(\pm\xi)-\varphi(0)]\,d\xi=0.$$

We can thus drop the portion of the integral with $|\xi|>1$ and replace $\varphi(\pm\xi)$ by $\varphi(0)$ for $|\xi|\le 1$ before taking the limit. This reduces the final term in (4-11) to

$$\lim_{a\to\infty}\int_{-\infty}^{\infty}\Big[\frac{e^{i\xi a}}{8\pi i\xi}\varphi(\xi)-\frac{e^{-i\xi a}}{8\pi i\xi}\varphi(-\xi)\Big]\,d\xi=\lim_{a\to\infty}\int_{-1}^{1}\frac{\sin(\xi a)}{4\pi\xi}\varphi(0)\,d\xi \\ =\varphi(0)\int_{-\infty}^{\infty}\frac{\sin(\xi)}{4\pi\xi}\,d\xi=\frac{1}{4}\varphi(0).$$

To complete the argument, we note that (4-6) implies

$$\operatorname{tr}\big[S_V\big(\tfrac{1}{2}n\big)-S_0\big(\tfrac{1}{2}n\big)\big]=2m_V\big(\tfrac{1}{2}n\big). \qquad \square$$

5. Poisson formula

The Poisson formula expresses the trace of the wave group as a sum over the resonance set. The relative wave trace,

$$\Theta_V(t) := \operatorname{tr}\Big[\cos\Big(t\sqrt{\Delta + V - \tfrac{1}{4}n^2}\Big) - \cos\Big(t\sqrt{\Delta - \tfrac{1}{4}n^2}\Big)\Big], \tag{5-1}$$

is defined distributionally as in Section 3. That is, for $\psi \in C_0^\infty(\mathbb{R})$,

$$(\Theta_V, \psi) := \operatorname{tr}[f(\Delta + V) - f(\Delta)],$$

where

$$f(x) := \chi(x) \int_{-\infty}^{\infty} \cos\Big(t\sqrt{x - \tfrac{1}{4}n^2}\Big)\psi(t)\, dt, \tag{5-2}$$

with χ a smooth cutoff which equals 1 on the spectrum of $\Delta + V - \frac{1}{4}n^2$ and vanishes on $(-\infty, c]$ for some $c < 0$. The cutoff is a technicality, included so that $f \in \mathcal{S}(\mathbb{R})$.

Theorem 5.1 (Poisson formula). *For a potential* $V \in C_0^\infty(\mathbb{H}^{n+1}, \mathbb{R})$,

$$t^{n+1}\Theta_V = t^{n+1}\Bigg[\frac{1}{2}\sum_{\zeta \in \mathcal{R}_V} e^{(\zeta - n/2)|t|} - u_0(t)\Bigg] \tag{5-3}$$

as a distribution on $\mathbb{R}$, *where*

$$u_0(t) := \begin{cases} \dfrac{\cosh(t/2)}{(2\sinh(t/2))^{n+1}} & \textit{for } n+1 \textit{ even},\\ 0 & \textit{for } n+1 \textit{ odd}. \end{cases}$$

A more general version of the Poisson formula for resonances for compactly supported black-box perturbations of $\mathbb{H}^{n+1}$ was stated in [Borthwick 2010, Theorem 3.4], with the proof omitted because of its similarity to the argument of [Guillopé and Zworski 1997]. Zworski has recently noted that the proof in that paper glossed over certain technical details concerning the computation of the distributional Fourier transform of the spectral resolution. Furthermore, the optimal factor of t^{n+1} was not obtained in these previous versions.

The technicalities of this proof are now worked out in [Dyatlov and Zworski 2019, Chapter 3], including the t prefactor. The proof of [Dyatlov and Zworski 2019, Theorem 3.53] relies only on a global upper bound on the counting function, as in (1-4), and a factorization formula for the scattering determinant, which we state as Proposition 5.2 below. It therefore essentially applies to Theorem 5.1.

However, there are some structural differences in the hyperbolic case, due to the shifted spectral parameter $z = s(1-s)$ and the nontrivial background contribution of $\mathbb{H}^{n+1}$ in even dimensions. For the sake of completeness, we will include a hyperbolic version of the proof.

The starting point is to apply the Birman–Krein formula (Theorem 4.1) to the relative wave trace. The relation (5-2) implies that

$$f\big(\tfrac{1}{4}n^2 + \xi^2\big) = \tfrac{1}{2}[\hat{\psi}(\xi) + \hat{\psi}(-\xi)].$$

Using this, and the fact that $\sigma'(\xi)$ is even, reduces the Birman–Krein formula to

$$(\Theta_V, \psi) = \frac{1}{2}\int_{-\infty}^{\infty} \sigma'(\xi)\hat{\psi}(\xi)\,d\xi + \sum_{j=1}^{d} f(\lambda_j) + \frac{1}{2}m_V\Big(\frac{1}{2}n\Big)\hat{\psi}(0). \tag{5-4}$$

To evaluate the integral in (5-4), we need some additional facts about the scattering determinant. Given the polynomial bound on the resonance counting function (1-4), we can define the Hadamard product

$$H_V(s) := s^{m_V(0)} \prod_{\zeta\in\mathcal{R}_V\setminus\{0\}} E\Big(\frac{s}{\zeta}, n+1\Big),$$

where

$$E(z, p) := (1-z)e^{z+z^2/2+\cdots+z^p/p}.$$

This yields an entire function with zeros located at the resonances.

The following factorization formula provides the connection between the Birman–Krein formula and the resonance set.

Proposition 5.2. *The relative scattering determinant admits a factorization*

$$\tau(s) = (-1)^{m_V(n/2)} e^{q(s)} \frac{H_V(n-s)}{H_V(s)} \frac{H_0(s)}{H_0(n-s)},$$

where q is a polynomial of degree at most $n+1$ satisfying $q(n-s) = -q(s)$.

Proposition 5.2 is a special case of [Borthwick 2010, Proposition 3.1], which applies to black-box perturbations of $\mathbb{H}^{n+1}$. That statement did not include the symmetry condition on $q(s)$, which follows from (4-5) once the parity of $m_V\big(\frac{1}{2}n\big)$ has been factored out. An analogous result for metric perturbations was given in [Borthwick 2008, Proposition 7.2], without the estimate on the degree of $q(s)$. These previous versions contained a typo in the Hadamard product, in that the $\zeta = 0$ term should always be treated as a separate factor $s^{m(0)}$.

In view of (3-1), Theorem 4.1 implies that the derivative σ' defines a tempered distribution. We will need the following estimate of its rate of growth.

Proposition 5.3. *For $V \in C_0^\infty(\mathbb{H}^{n+1}, \mathbb{R})$, the derivative of the scattering phase satisfies*

$$|\sigma'(\xi)| \le C_V(1+|\xi|)^{n-1}$$

for $\xi \in \mathbb{R}$.

The fact that σ' has at most polynomial growth follows from Proposition 5.2 by a general argument given in [Guillopé and Zworski 1997, Lemma 4.7], which in turn is based on a method introduced by Melrose [1988]. The explicit growth rate of Proposition 5.3 was proven in [Borthwick and Crompton 2014, Proposition 3.1].

With these ingredients in place, the strategy for the proof of the Poisson formula is essentially to compute the Fourier transform of σ'.

Proof of Theorem 5.1. Let us first show that the right-hand side of (5-3) defines a distribution. Indeed, if we exclude the finite number of terms with $\operatorname{Re}\zeta > \frac{1}{2}n$, which have exponential growth, the remaining sum gives a tempered distribution. To see this, consider a test function $\psi \in \mathcal{S}(\mathbb{R})$. Repeated integration by parts can be used to estimate, for $\operatorname{Re}\zeta > \frac{1}{2}n$,

$$\left|\int_{-\infty}^{\infty} t^{n+1} e^{(\zeta-n/2)t}\psi(t)\,dt\right| \leq \frac{C}{\left(1+\left|\zeta-\frac{1}{2}n\right|\right)^{n+2}} \sum_{k=0}^{n+1} \sup_{t\in\mathbb{R}} |\langle t\rangle^{n+3}\psi^{(k)}(t)|.$$

It then follows from the polynomial bound (1-4) that the sum

$$t^{n+1} \sum_{\operatorname{Re}\zeta<n/2} e^{(\zeta-n/2)|t|}$$

defines a tempered distribution on $\mathbb{R}$. The right-hand side of (5-3) is thus well defined as a distribution, since there are only finitely many terms with $\operatorname{Re}\zeta \geq \frac{1}{2}n$.

Let Θ_{sc} denote the tempered distribution defined by

$$(\Theta_{\mathrm{sc}},\psi) := \frac{1}{2}\int_{-\infty}^{\infty} \sigma'(\xi)\hat{\psi}(\xi)\,d\xi \tag{5-5}$$

for $\psi \in \mathcal{S}(\mathbb{R})$. This distribution accounts for the contributions to the Birman–Krein formula (5-4) from the continuous spectrum. The sum over the discrete spectrum can be rewritten as a sum over the resonances with $\operatorname{Re} s > \frac{1}{2}n$, using the fact that

$$\cos\left(t\sqrt{\lambda-\tfrac{1}{4}n^2}\right) = \cosh\left(t\left(\zeta-\tfrac{1}{2}n\right)\right)$$

for $\zeta \in \left(\frac{1}{2}n, \infty\right)$ and $\lambda = \zeta(n-\zeta)$. The Birman–Krein formula then becomes

$$\Theta_V(t) = \Theta_{\mathrm{sc}}(t) + \sum_{\operatorname{Re}\zeta>n/2} \cosh\left(t\left(\zeta-\tfrac{1}{2}n\right)\right) + \tfrac{1}{2}m_V\left(\tfrac{1}{2}n\right). \tag{5-6}$$

Since Θ_{sc} is tempered, it suffices to evaluate (5-5) under the assumption that $\hat{\psi} \in C_0^\infty(\mathbb{R})$. From Proposition 5.2 we calculate

$$\frac{\tau'}{\tau}(s) = q'(s) - \frac{H_V'}{H_V}(n-s) - \frac{H_V'}{H_V}(s) + \frac{H_0'}{H_0}(s) + \frac{H_0'}{H_0}(n-s).$$

The Hadamard product derivatives are given by

$$\frac{H_V'}{H_V}(s) = \frac{m_V(0)}{s} + \sum_{\zeta\in\mathcal{R}_V\setminus\{0\}} \left[\frac{1}{s-\zeta} + \frac{1}{\zeta} + \cdots + \frac{s^n}{\zeta^{n+1}}\right].$$

Hence we can write

$$\frac{H_V'}{H_V}(n-s) + \frac{H_V'}{H_V}(s) = \sum_{\zeta\in\mathcal{R}_V} \left[\frac{n-2\zeta}{(n-s-\zeta)(s-\zeta)} + p_\zeta(s)\right],$$

where $p_\zeta(s)$ is a polynomial of degree n for $\zeta \neq 0$, and $p_0(s) := 0$.

Switching to a ξ derivative for σ gives

$$\sigma'(\xi) = -\frac{1}{2\pi}\frac{\tau'}{\tau}\Big(\frac{1}{2}n + i\xi\Big),$$

which evaluates to

$$\sigma'(\xi) = -\frac{1}{2\pi}q'\Big(\frac{1}{2}n + i\xi\Big) + \frac{1}{2\pi}\sum_{\zeta\in\mathcal{R}_V}\Bigg[\frac{n-2\zeta}{\xi^2 + \big(\zeta - \frac{1}{2}n\big)^2} + p_\zeta\Big(\frac{1}{2}n + i\xi\Big)\Bigg] - \frac{1}{2\pi}\sum_{\zeta\in\mathcal{R}_0}\Bigg[\frac{n-2\zeta}{\xi^2 + \big(\zeta - \frac{1}{2}n\big)^2} + p_\zeta\Big(\frac{1}{2}n + i\xi\Big)\Bigg], \tag{5-7}$$

where p_ζ is a polynomial of degree at most n. The convergence is uniform on compact intervals. Note that there is no pole corresponding to the possible resonance at $\zeta = \frac{1}{2}n$, because a zero at this point would cancel out of $H_V(s)/H_V(n-s)$.

Assuming that $\hat{\psi}$ is compactly supported, the contributions of (5-7) to $(t^{n+1}\Theta_{\text{sc}}, \psi)$ can be evaluated term by term. Under the Fourier transform, the factor t^{n+1} becomes $(-i\partial_\xi)^{n+1}$, which knocks out all of the polynomial terms. Hence, after integrating by parts,

$$(t^{n+1}\Theta_{\text{sc}}, \psi) = \frac{1}{4\pi}\sum_{\zeta\in\mathcal{R}_V}\int_{-\infty}^{\infty}\frac{n-2\zeta}{\xi^2 + \big(\zeta - \frac{1}{2}n\big)^2}(i\partial_\xi)^{n+1}\hat{\psi}(\xi)\,d\xi - \frac{1}{4\pi}\sum_{\zeta\in\mathcal{R}_0}\int_{-\infty}^{\infty}\frac{n-2\zeta}{\xi^2 + \big(\zeta - \frac{1}{2}n\big)^2}(i\partial_\xi)^{n+1}\hat{\psi}(\xi)\,d\xi.$$

By a straightforward contour integration,

$$\int_{-\infty}^{\infty} e^{-i\xi t}\frac{n-2\zeta}{\xi^2 + \big(\zeta - \frac{1}{2}n\big)^2}\,d\xi = \begin{cases} -2\pi e^{-(\zeta - n/2)|t|}, & \operatorname{Re}\zeta > \frac{1}{2}n, \\ 2\pi e^{(\zeta - n/2)|t|}, & \operatorname{Re}\zeta < \frac{1}{2}n. \end{cases}$$

Using this calculation in the formula for $(t^{n+1}\Theta_{\text{sc}}, \psi)$ gives

$$(t^{n+1}\Theta_{\text{sc}}, \psi) = \frac{1}{2}\int_{-\infty}^{\infty} t^{n+1}\Bigg(\sum_{\substack{\zeta\in\mathcal{R}_V\\ \operatorname{Re}\zeta < n/2}} e^{(\zeta - n/2)|t|} - \sum_{\substack{\zeta\in\mathcal{R}_V\\ \operatorname{Re}\zeta > n/2}} e^{-(\zeta - n/2)|t|} - \sum_{\zeta\in\mathcal{R}_0} e^{(\zeta - n/2)|t|}\Bigg)\psi(t)\,dt. \tag{5-8}$$

This calculation contains no contribution from a resonance at $\zeta = \frac{1}{2}n$, because a zero at this point cancels out of the formula for $\tau(s)$.

To remove the restriction of compact support for $\hat{\psi}$, we note that the right-hand side of (5-8) defines a tempered distribution by the remarks at the beginning of the proof. Since Θ_{sc} is also tempered and $C_0^\infty(\mathbb{R})$ is dense in $\mathcal{S}(\mathbb{R})$, it follows that (5-8) holds for all $\psi \in \mathcal{S}(\mathbb{R})$.

Combining this computation of $t^{n+1}\Theta_{\text{sc}}$ with the formula (5-6) now yields the formula

$$t^{n+1}\Theta_V = \frac{1}{2}t^{n+1}\Bigg[\sum_{\zeta\in\mathcal{R}_V} e^{(\zeta - n/2)|t|} - \sum_{\zeta\in\mathcal{R}_0} e^{(\zeta - n/2)|t|}\Bigg].$$

Note that the constant term $\frac{1}{2}m_V\big(\frac{1}{2}n\big)$ from (5-6) is now incorporated into the sum over $\mathcal{R}_V$. This completes the proof for $n+1$ odd, because $\mathcal{R}_0$ is empty. If $n+1$ is even, then $\mathcal{R}_0$ is equal to $-\mathbb{N}_0$ as a set, with multiplicities given by the dimensions of spaces of spherical harmonics of degree k,

$$m_0(-k) = (2k+n)\frac{(k+1)\cdots(n+k-1)}{n!}.$$

The resulting sum over $\mathcal{R}_0$ was computed in [Guillarmou and Naud 2006, Lemma 2.4],

$$\frac{1}{2}\sum_{k=0}^{\infty} m_0(-k)e^{-(k+n/2)|t|} = \frac{\cosh(t/2)}{(2\sinh(t/2))^{n+1}}. \qquad \square$$

6. Wave trace expansion

In this section, we compute the expansion at $t = 0$ of the relative wave trace distribution Θ_V, as defined in (5-1), and determine the first two wave invariants explicitly. Although the existence of the wave-trace expansion is considered to be well known, we are not aware of any direct proof for Schrödinger operators in the literature. For the odd-dimensional Euclidean case, [Melrose 1995, §4.1] is generally cited, but this source does not include a proof. Because the hyperbolic setting leads to differences from the familiar Euclidean formulas, we will include the argument here.

To set up the expansion formula, we recall that $|t|^\beta$ is well defined as a meromorphic family of distributions on $\mathbb{R}$, with poles at negative odd integers. The residues at these poles are given by delta distributions. Dividing by $\Gamma\big(\frac{1}{2}(\beta+1)\big)$ cancels the poles and defines a holomorphic family

$$\vartheta^\beta(t) := \frac{|t|^\beta}{\Gamma\big(\frac{1}{2}(\beta+1)\big)}, \tag{6-1}$$

where

$$\vartheta^{-1-2j}(t) = (-1)^j \frac{j!}{(2j)!}\delta^{(2j)}(t)$$

for $j \in \mathbb{N}_0$ (see, e.g., [Kanwal 2004, §4.4, (52)]).

Theorem 6.1. *Let $V \in C_0^\infty(\mathbb{H}^{n+1})$ with $n \geq 1$. For each integer $N > \big[\frac{1}{2}(n+1)\big]$, there exist constants $a_k(V)$ (the wave invariants) such that*

$$\Theta_V(t) = \sum_{k=1}^{N} a_k(V)\vartheta^{-n+2k-1}(t) + F_N(t),$$

with $F_N \in C^{2N-n-1}(\mathbb{R})$ and $F_N(t) = O(|t|^{2N-n})$ as $t \to 0$.

The proof is adapted from [Bérard 1977] and relies on the Hadamard–Riesz construction of a parametrix for the wave kernel [Hadamard 1923; Riesz 1949]. For $V \in C_0^\infty(\mathbb{H}^{n+1})$, let

$$P_V := \Delta + V - \tfrac{1}{4}n^2.$$

We denote by e_V the fundamental solution of the Cauchy problem for the wave equation

$$\begin{aligned} (\partial_t^2 + P_V)e_V(t; z, w) &= 0, \\ e_V(0; z, w) &= \delta(z-w), \\ \partial_t e_V(0; z, w) &= 0, \end{aligned} \tag{6-2}$$

for $t \in \mathbb{R}$ and $z, w \in \mathbb{H}^{n+1}$. In other words, $e_V(t; \cdot, \cdot)$ is the integral kernel of the wave operator $\cos(t\sqrt{P_V})$.

For $\alpha \in \mathbb{C}$, we define the holomorphic family of distributions

$$\chi_+^\alpha := \frac{x_+^\alpha}{\Gamma(\alpha+1)}$$

using the notation of [Hörmander 1983, §3.2]. This family satisfies the derivative identity

$$\frac{d}{dx}\chi_+^\alpha = \chi_+^{\alpha-1}.$$

Since $\chi_+^0 = x_+$, it follows that χ_+^α is a point distribution at negative integers:

$$\chi_+^{-m} = \delta^{(m-1)}(x).$$

For $z, w \in \mathbb{H}^{n+1}$, we set $r := d(z, w)$ and denote by $\chi_+^\alpha(t^2 - r^2)$ the pullback of χ_+^α by the smooth map $\mathbb{H}^{n+1} \times \mathbb{H}^{n+1} \times \mathbb{R} \to \mathbb{R}$ given by $(z, w, t) \mapsto t^2 - d(z, w)^2$. Since χ_+^α is classically differentiable for $\operatorname{Re}\alpha > -1$, derivatives of $\chi_+^\alpha(t^2 - r^2)$ can be computed directly in this region, and then extended by analytic continuation. Hence the formulas

$$\begin{aligned} \partial_t[\chi_+^\alpha(t^2 - r^2)] &= 2t\chi_+^{\alpha-1}(t^2 - r^2), \\ \partial_r[\chi_+^\alpha(t^2 - r^2)] &= -2r\chi_+^{\alpha-1}(t^2 - r^2) \end{aligned} \tag{6-3}$$

are valid for all α.

Following [Bérard 1977, §D], we seek to construct the parametrix as a sum of the distributions $|t|\chi_+^\alpha(t^2 - r^2)$ with increasing values of α. The starting point for the expansion is dictated by the initial conditions in (6-2), so we need to understand the distributional limit of $|t|\chi_+^\alpha(t^2 - r^2)$ as $t \to 0$.

Lemma 6.2. *For $\psi \in C_0^\infty(\mathbb{H}^{n+1})$,*

$$\lim_{t\to 0}(|t|\chi_+^\alpha(t^2 - d(z,\cdot)^2), \psi) = \begin{cases} \pi^{n/2}\psi(z), & \alpha = -\frac{1}{2}n - 1, \\ 0, & \alpha > -\frac{1}{2}n - 1 \end{cases} \tag{6-4}$$

and

$$\lim_{t\to 0}(\partial_t[|t|\chi_+^\alpha(t^2 - d(z,\cdot)^2)], \psi) = 0 \tag{6-5}$$

for $\alpha = -\frac{1}{2}n - 1$ and $\alpha > -\frac{1}{2}(n+1)$.

Proof. The distribution is even in the variable t, so it suffices to consider $t > 0$. The first formula of (6-3) gives, for $k \in \mathbb{N}$,

$$\chi_+^\alpha(t^2 - r^2) = \left(\frac{1}{2t}\partial_t\right)^k \chi_+^{\alpha+k}(t^2 - r^2),$$

which can be used to shift the computation to the integrable range. For $t > 0$ and $\operatorname{Re}\alpha + k > -1$, we have

$$(\chi_+^\alpha(t^2 - d(z,\cdot)^2), \psi) = \frac{1}{\Gamma(\alpha+k+1)}\left(\frac{1}{2t}\partial_t\right)^k \int_0^t (t^2 - r^2)^{\alpha+k}\tilde{\psi}(r)r^n\,dr,$$

where in geodesic polar coordinates (r, ω) centered at z,

$$\tilde{\psi}(r) := \frac{\sinh^n r}{r^n}\int_{\mathbb{S}^n} \psi(r, \omega)\,d\omega.$$

Rescaling $r \to tr$ in the integral gives

$$(\chi_+^\alpha(t^2 - d(z,\cdot)^2), \psi) = \frac{1}{\Gamma(\alpha+k+1)} \left(\frac{1}{2t}\partial_t\right)^k \left[t^{2(\alpha+k)+n+1} \int_0^1 (1-r^2)^{\alpha+k} \tilde{\psi}(tr) r^n \, dr \right]. \tag{6-6}$$

Since $\tilde{\psi}$ is the spherical average of ψ centered at z, and the linear term in a Taylor approximation of ψ at z cancels out in this average,

$$\tilde{\psi}(r) = \mathrm{Vol}(\mathbb{S}^n)\psi(z) + O(r^2) = \frac{2\pi^{(n+1)/2}}{\Gamma\big(\frac{1}{2}(n+1)\big)} \psi(z) + O(r^2).$$

For the same reason, $\partial_r \tilde{\psi}(r) = O(r)$. Higher radial derivatives are bounded on $\{r > 0\}$. Hence, in the leading term from (6-6), all of the t derivatives are applied to the factor preceding the integral, which gives

$$\left(\frac{1}{2t}\partial_t\right)^k [t^{2(\alpha+k)+n+1}] = \frac{\Gamma\big(\alpha+k+\frac{1}{2}(n+3)\big)}{\Gamma\big(\alpha+\frac{1}{2}(n+3)\big)} t^{2\alpha+n+1}.$$

The leading contribution from the r integration can be calculated from Euler's beta function formula [NIST 2010, (5.12.1)]:

$$\int_0^1 (1-r^2)^{\alpha+k} r^n \, dr = \frac{\Gamma\big(\alpha+k+1\big)\Gamma\big(\frac{1}{2}(n+1)\big)}{2\Gamma\big(\alpha+k+\frac{1}{2}(n+3)\big)}.$$

Combining these results in (6-6) gives, for $t > 0$,

$$(\chi_+^\alpha(t^2 - d(z,\cdot)^2), \psi) = \frac{\pi^{(n+1)/2}}{\Gamma\big(\alpha+\frac{1}{2}(n+3)\big)} t^{2\alpha+n+1} \psi(z) + O(t^{2\alpha+n+3}). \tag{6-7}$$

This proves (6-4) once the extra factor of t has been inserted.

To establish (6-5), we note that

$$\partial_t [t\chi_+^\alpha(t^2 - r^2)] = \chi_+^\alpha(t^2 - r^2) + 2t^2 \chi_+^{\alpha-1}(t^2 - r^2)$$

by (6-3). We thus obtain from (6-7),

$$(\partial_t [t\chi_+^\alpha(t^2 - d(z,\cdot)^2)], \psi) = \pi^{(n+1)/2} \frac{2\alpha+n+2}{\Gamma\big(\alpha+\frac{1}{2}(n+3)\big)} t^{2\alpha+n+1} \psi(z) + O(t^{2\alpha+n+3}),$$

and (6-5) follows. □

We take the following ansatz for the parametrix:

$$e_{V,N}(t,z,w) := \pi^{-n/2} \sum_{k=0}^{N} u_{V,k}(z,w) |t| \chi_+^{-n/2+k-1}(t^2 - r^2), \tag{6-8}$$

where $u_{V,0}(z,z) = 1$ and the higher coefficients $u_{V,k}$ are to be chosen such that, in the expression for $(\partial_t^2 + P_V) e_{V,N}(t,z,\cdot)$, the coefficients of $|t|\chi_+^{-n/2+k-1}(t^2 - r^2)$ cancel for $k \le N$. Lemma 6.2 implies that the initial conditions are satisfied:

$$\begin{aligned} e_{V,N}(0; z, w) &= \delta(z-w), \\ \partial_t e_{V,N}(0; z, w) &= 0. \end{aligned} \tag{6-9}$$

To work out the equations for the coefficients, we will compute the action of $(\partial_t^2 + P_V)$ on each term. As above, it suffices to compute for $t > 0$ by evenness, and we will use the temporary abbreviations

$$u_{V,k}(z, \cdot) \rightsquigarrow u_k, \quad \chi_+^\alpha(t^2 - r^2) \to \chi_+^\alpha$$

to simplify the formulas. The time derivatives are calculated from (6-3):

$$\partial_t^2[t\chi_+^\alpha] = 6t\chi_+^{\alpha-1} + 4t^3\chi_+^{\alpha-2}.$$

Using the geodesic polar coordinate form of the Laplacian

$$\Delta = -\partial_r^2 - n\coth r\,\partial_r + \sinh^{-2} r\,\Delta_{\mathbb{S}^n}, \tag{6-10}$$

we also compute

$$P_V\chi_+^\alpha = (2 + 2nr\coth r)\chi_+^{\alpha-1} - 4r^2\chi_+^{\alpha-2}.$$

Putting these together gives

$$(\partial_t^2 + P_V)[u_k t\chi_+^\alpha] = (P_V u_k)t\chi_+^\alpha + 4r(\partial_r u_k)t\chi_+^{\alpha-1} + (8 + 2nr\coth r)u_k t\chi_+^{\alpha-1} + 4u_k t(t^2 - r^2)\chi_+^{\alpha-2}.$$

The final term simplifies:

$$(t^2 - r^2)\chi_+^{\alpha-2} = (\alpha - 1)\chi_+^{\alpha-1},$$

which reduces the formula to

$$(\partial_t^2 + P_V)[u_k t\chi_+^\alpha] = (P_V u_k)t\chi_+^\alpha + [4r\partial_r u_k + (4(\alpha+1) + 2nr\coth r)u_k]t\chi_+^{\alpha-1}.$$

After setting $\alpha = -\frac{1}{2}n + k - 1$ as in (6-8), we obtain, for $t > 0$,

$$(\partial_t^2 + P_V)[u_k t\chi_+^{-n/2+k-1}] = [4r\partial_r u_k + (2n(r\coth r - 1) + 4k)u_k]t\chi_+^{-n/2+k} + (P_V u_k)t\chi_+^{-n/2+k-1}. \tag{6-11}$$

The calculation (6-11) shows the cancelling of terms in $(\partial_t^2 + P_V)e_{V,N}(t, z, \cdot)$ is ensured by the transport equations

$$\begin{aligned} [4r\partial_r + 2n(r\coth r - 1)]u_{V,0}(z, \cdot) &= 0, \\ [4r\partial_r + 2n(r\coth r - 1) + 4k]u_{V,k}(z, \cdot) &= -P_V u_{V,k-1}(z, \cdot). \end{aligned} \tag{6-12}$$

To solve (6-12) we define

$$\phi(r) := \left(\frac{\sinh r}{r}\right)^{-n/2}, \tag{6-13}$$

and then set

$$\begin{aligned} u_{V,0}(z, w) &= \phi(r), \\ u_{V,k+1}(z, w) &= -\frac{1}{4}\phi(r)\int_0^1 \frac{s^k}{\phi(sr)} P_V u_{V,k}(z, \gamma(s))\,ds, \end{aligned} \tag{6-14}$$

where γ is the geodesic from z to w, parametrized by $s \in [0, 1]$, and P_V acts on the second variable of $u_{V,k}$. The coefficients $u_{V,k}$ are smooth for all k.

Proposition 6.3. *With $e_{V,N}$ defined as above, set*

$$q_{V,N}(t, z, w) := e_V(t, z, w) - e_{V,N}(t, z, w).$$

For $m \in \mathbb{N}$, we have $q_{V,N} \in C^m$ for N sufficiently large and

$$|q_{V,N}(t, z, w)| = O(|t|^{-n+N-1})$$

as $t \to 0$, uniformly in z, w.

Proof. From (6-8), (6-11), and the transport equations (6-12), we observe that

$$(\partial_t^2 + P_V)e_{V,N}(t, z, \cdot) = f_N(t, z, \cdot),$$

where

$$f_N(t, z, \cdot) = \pi^{-n/2} P_V u_{V,N}(z, \cdot)|t|\, \chi_+^{-n/2+N-1}(t^2 - r^2), \tag{6-15}$$

with P_V acting on the second variable. Since $e_{V,N}$ satisfies the same initial conditions (6-9) as e_V, this gives

$$\begin{aligned}(\partial_t^2 + P_V)q_{V,N}(t; z, w) &= f_N(t, z, w),\\ q_{V,N}(0; z, w) &= 0,\\ \partial_t q_{V,N}(0; z, w) &= 0.\end{aligned}$$

The coefficients $u_{V,k}$ are smooth, by (6-14), and $|t|\chi_+^{\alpha}(t^2 - r^2)$ is C^l for $\alpha > l + 1$. Hence, by (6-15), $f_N \in C^l$ for $l < N - \frac{1}{2}n - 2$ and has support in $\{r \le t\}$. It follows that $q_V \in C^m$ for N sufficiently large.

For any $b > 0$, the Sobolev norms of $f_N(t, z, \cdot)$ can be estimated by $O(|t|^b)$ for N sufficiently large. These estimates are uniform in z, since ϕ depends only on r and V has compact support. Standard regularity estimates for hyperbolic PDEs (see, for example, [Trèves 1975, Chapter 47]) then show that, for N_1 sufficiently large,

$$|q_{V,N_1}(t, z, w)| = O(|t|^{2N-n-1}),$$

uniformly in z, w. The estimate of $q_{V,N}$ as $t \to 0$ is then derived from

$$q_{V,N}(t, z, w) = \pi^{-n/2} \sum_{k=N+1}^{N_1} u_{V,k}(z, w)|t|\chi_+^{-n/2+k-1}(t^2 - r^2) + q_{V,N_1}(t, z, w). \qquad \square$$

With this estimate on the parametrix, we are now prepared to establish the wave trace expansion.

Proof of Theorem 6.1. For $\psi \in C_0^\infty(\mathbb{R})$, we write the integral kernel of the trace-class operator

$$\int_{-\infty}^{\infty} [\cos(t\sqrt{P_V}) - \cos(t\sqrt{P_0})]\psi(t)\, dt$$

as

$$\int_{-\infty}^{\infty} [e_V(t, z, w) - e_0(t, z, w)]\psi(t)\, dt,$$

which is smooth and compactly supported. Taking the trace gives

$$(\Theta_V, \psi) = \int_{\mathbb{H}^{n+1}} \left(\int_{-\infty}^{\infty} [e_V(t, z, w) - e_0(t, z, w)]\psi(t)\, dt \right)\Big|_{z=w} dg(z). \tag{6-16}$$

The wave kernel parametrices can be substituted into (6-16) and the contributions from the terms $k = 0, \dots, N$ evaluated separately.

For $\operatorname{Re}\alpha$ sufficiently large, we can verify directly that

$$|t|\, \chi_+^\alpha(t^2 - r^2)\big|_{r=0} = \vartheta^{2\alpha+1}(t),$$

and this formula extends to all $\alpha \in \mathbb{C}$ by analytic continuation. It follows from (6-16) and Proposition 6.3 that

$$\Theta_V(t) = \pi^{-n/2} \sum_{k=1}^{N} \left(\int_{\mathbb{H}^{n+1}} [u_{V,k}(z, z) - u_{0,k}(z, z)]\, dg(z) \right) \vartheta^{-n+2k-1}(t) + F_N(t),$$

where

$$F_N(t) := \int_{\mathbb{H}^{n+1}} [q_{V,N}(t, z, z) - q_{0,N}(t, z, z)]\, dg(z). \qquad \square$$

The proof of Theorem 6.1 yields a formula for the wave invariants:

$$a_k(V) = \pi^{-n/2} \int_{\mathbb{H}^{n+1}} [u_{V,k}(z, z) - u_{0,k}(z, z)]\, dg(z). \tag{6-17}$$

This formula can be simplified somewhat using the transport equations. By (6-12), we have

$$(L+4) \cdots (L+4k) u_{V,k}(z, \cdot) = (-1)^k P_V u_{V,0}(z, \cdot), \tag{6-18}$$

where, in geodesic polar coordinates centered at z, L is the differential operator

$$L := 4r\,\partial_r + 2n(r \coth r - 1).$$

Note that, for any smooth function f, Lf vanishes at $r = 0$. Therefore, evaluating (6-18) at the point z yields

$$u_{V,k}(z, z) = \frac{(-1)^k}{4^k k!} P_V^k u_{V,0}(z, z), \tag{6-19}$$

where $u_{V,0}(z, w) = \phi(d(z, w))$ and P_V^k acts on the second variable. In principle, (6-19) can be used to derive explicit formulas for all of the wave invariants. The first two are relatively simple.

Proposition 6.4. *For $V \in C_0^\infty(\mathbb{H}^{n+1}, \mathbb{R})$,*

$$a_1(V) = -\frac{1}{4}\pi^{-n/2} \int_{\mathbb{H}^{n+1}} V(z)\, dg(z)$$

and

$$a_2(V) = \frac{1}{32}\pi^{-n/2} \int_{\mathbb{H}^{n+1}} \left[\frac{2n - n^2}{6} V(z) + V(z)^2 \right] dg(z).$$

Proof. Since $u_{V,0}(z,z)=1$ and $P_V-P_0=V$, we see immediately from (6-19) that

$$u_{V,1}(z,z)-u_{0,1}(z,z)=-\tfrac{1}{4}V(z).$$

This gives the formula for $a_1(V)$.

For the second invariant, we use (6-19) to write

$$u_{V,2}(z,z)-u_{0,2}(z,z)=\tfrac{1}{32}(P_V^2\phi-P_0^2\phi)|_{r=0}=\tfrac{1}{32}[2V(z)P_0\phi(0)+\Delta V(z)+V(z)^2],$$

where r is the radius for geodesic polar coordinates centered at z, and we have used the facts that $\phi(0)=1$ and $\partial_r\phi(0)=0$. From (6-10) and (6-13) we compute

$$P_0\phi(0)=\frac{n(n+1)}{6}-\frac{n^2}{4}=\frac{2n-n^2}{12}.$$

This gives

$$u_{V,2}(z,z)-u_{0,2}(z,z)=\frac{1}{32}\left[\frac{2n-n^2}{6}V(z)+V(z)^2+\Delta V(z)\right].$$

When substituted into the formula for $a_2(V)$, the term $\Delta V(z)$ integrates to zero because V has compact support. □

7. Heat trace

The relative heat trace associated to a potential $V\in C_0^\infty(\mathbb{H}^{n+1},\mathbb{R})$ is defined by applying the distribution (3-1) to the function $f(x)=\chi(x)e^{-t(x-n^2/4)}$ for $t>0$, where χ is a smooth cutoff which equals 1 on the spectrum of $P_V:=\Delta-\frac{1}{4}n^2+V$ and vanishes on $(-\infty,c]$ for some $c<0$. The Birman–Krein formula (Theorem 4.1) gives

$$\operatorname{tr}[e^{-tP_V}-e^{-tP_0}]=\int_0^\infty\sigma'(\xi)e^{-\xi^2t}\,d\xi+\sum_{j=1}^{d}e^{t(n^2/4-\lambda_j)}+\tfrac{1}{2}m_V\big(\tfrac{1}{2}n\big). \tag{7-1}$$

We have no analog of the Poisson formula of Theorem 5.1 for the heat trace. This is because the values of $\big(\zeta-\frac{1}{2}n\big)^2$ are spread over the full complex plane, so there is no apparent regularization of the heat trace as a sum over the resonance set.

The asymptotic expansion of the heat trace at $t=0$ can be derived by a variety of methods. The simplest route for us is via the wave trace expansion.

Theorem 7.1. *As $t\to 0$, the relative heat trace admits an asymptotic expansion*

$$\operatorname{tr}[e^{-tP_V}-e^{-tP_0}]\sim\pi^{-1/2}\sum_{k=1}^{\infty}a_k(V)(4t)^{-(n+1)/2+k}, \tag{7-2}$$

where $a_k(V)$ are the wave invariants from Theorem 6.1.

Proof. Since $\Theta_{\mathrm{sc}}(s)$ is tempered, the definition (5-5) gives

$$\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\Theta_{\mathrm{sc}}(s)e^{-s^2/4t}\,ds=\int_0^\infty\sigma'(\xi)e^{-\xi^2t}\,d\xi$$

for $t > 0$. It then follows from (5-6) and (7-1) that

$$\operatorname{tr}[e^{-tP_V} - e^{-tP_0}] = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \Theta_V(s) e^{-s^2/4t}\, ds \tag{7-3}$$

even when $\Theta_V(s)$ is not tempered.

For $\operatorname{Re}\beta$ sufficiently large, we compute

$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \vartheta^\beta(s) e^{-s^2/4t}\, ds = \pi^{-1/2}(4t)^{\beta/2},$$

and this formula extends to all $\beta \in \mathbb{C}$ by analytic continuation. Theorem 6.1 thus yields the expansion

$$\operatorname{tr}[e^{-tP_V} - e^{-tP_0}] = \pi^{-1/2} \sum_{k=1}^{N} a_k(V)(4t)^{-(n+1)/2+k} + \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} F_N(s) e^{-s^2/4t}\, ds$$

for $N > \left[\frac{1}{2}(n+2)\right]$, where $F_N(s) = O(|s|^{2N-n})$ as $s \to 0$. From (5-6) we also see that $F_N(s) = O(e^{n|s|/2})$ as $|s| \to \infty$. It follows that

$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} F_N(s) e^{-s^2/4t}\, ds = O(t^{-n/2+N}). \qquad \square$$

Note that coefficients in (7-2) are not quite the usual heat invariants because of the shift $-n^2/4$ in the definition of P_V. This shift gives an extra factor of $e^{-n^2t/4}$ in (7-2), so the traditional heat invariants could be computed as finite linear combinations of the $a_k(V)$.

The behavior of the heat kernel as $t \to \infty$ is also of interest to us. According to (7-1), this behavior is dominated by exponential terms corresponding to eigenvalues and a constant term from the possible resonance at $s = \frac{1}{2}n$. If these contributions are absent, then the heat kernel decays at a rate independent of the dimension.

Proposition 7.2. *For the potential $V \in C^\infty(\mathbb{H}^{n+1}, \mathbb{R})$, suppose that P_V has no eigenvalues and no resonance at $\frac{1}{2}n$. Then the following bound holds uniformly for $t \in (0, \infty)$ and z, w in $\mathbb{H}$:*

$$e^{-tP_V}(t; z, w) \asymp t^{-3/2} e^{-r^2/(4t)-(nr)/2} \left(1 + \frac{r+1}{t}\right)^{n/2-1} (1+r), \tag{7-4}$$

where $r = d(z, w)$ and $\asymp$ means that the ratio of the two sides is bounded above and below by positive constants.

In the case $V = 0$, (7-4) was proven in [Davies and Mandouvalos 1988, Theorem 3.1]. There is no factor $e^{-n^2t/4}$ in (7-4) because of the shift $-\frac{1}{4}n^2$ in the definition of P_V. These estimates were generalized in [Chen and Hassell 2020, Theorem 5] to asymptotically hyperbolic Cartan–Hadamard manifolds with no eigenvalues and no resonance at $s = \frac{1}{2}n$ by methods that allow for the inclusion of a C_0^∞ potential. The power $t^{-3/2}$, independent of dimension, corresponds to the vanishing of the spectral resolution $K_V(\xi; \cdot, \cdot)$ to order ξ^2 at $\xi = 0$ under the assumption of no resonance at $\frac{1}{2}n$.

Proposition 7.2 implies a bound on the heat trace by the argument from [Sá Barreto and Zworski 1996, Proposition 3.1].

Corollary 7.3. *For a potential V satisfying the hypotheses of Proposition 7.2*,

$$\operatorname{tr}[e^{-tP_V} - e^{-tP_0}] = O(t^{-1/2})$$

as $t \to \infty$.

Proof. Duhamel's principle gives the trace estimate

$$|\operatorname{tr}[e^{-tP_V} - e^{-tP_0}]| \le \int_0^t \int_{\mathbb{H}^{n+1}} \int_{\mathbb{H}^{n+1}} e^{-(t-s)P_0}(w, z) e^{-sP_V}(z, w) |V(z)| \, dg(z) \, dg(w) \, ds. \tag{7-5}$$

The uniform bound (7-4) implies in particular that

$$e^{-sP_V}(z, w) \le C_V e^{-sP_0}(z, w),$$

so we can use the semigroup property to estimate

$$\int_{\mathbb{H}^{n+1}} e^{-(t-s)P_0}(w, z) e^{-sP_V}(z, w) \, dg(w) \le C_V e^{-tP_0}(z, z) \le C_V t^{-3/2}$$

as $t \to \infty$, uniformly in z. Applying this to (7-5) gives

$$|\operatorname{tr}[e^{-tP_V} - e^{-tP_0}]| \le C_V t^{-1/2} \|V\|_{L^1}.$$ □

8. Scattering phase asymptotics

The Birman–Krein formula allows us to connect the wave-trace invariants to corresponding asymptotic expansions for the scattering phase and its derivative. For Schrödinger operators in the odd-dimensional Euclidean setting, the asymptotic expansion of the scattering phase was established by Colin de Verdière [1981], Guillopé [1981], and Popov [1982], via formulas relating the scattering determinant to regularized determinants of the cutoff resolvent. An argument based on expansion of the scattering matrix is given in [Yafaev 2010, Theorem 9.2.12], and a semiclassical version is given in [Dyatlov and Zworski 2019, Theorem 3.62].

For hyperbolic space we have the following version of these results:

Theorem 8.1. *For $V \in C_0^\infty(\mathbb{H}^{n+1}, \mathbb{R})$ the function $\sigma'(\xi)$ admits a full asymptotic expansion as $\xi \to +\infty$. If the dimension $n+1$ is odd, then*

$$\sigma'(\xi) \sim \sum_{k=1}^{\infty} c_k(V) \xi^{n-2k}.$$

For $n+1$ even, the expansion is truncated:

$$\sigma'(\xi) = \sum_{k=1}^{[n/2]} c_k(V) \xi^{n-2k} + O(\xi^{-\infty}).$$

The coefficients are related to the wave invariants by

$$c_k(V) = \frac{2^{-n+2k}}{\pi^{1/2} \Gamma\big(\tfrac{1}{2}(n+1) - k\big)} a_k(V).$$

Before proving the theorem, we start by establishing the existence of the scattering phase expansion. The coefficients are relatively easy to calculate once this is known.

Proposition 8.2. *For* $V \in C_0^\infty(\mathbb{H}^{n+1}, \mathbb{R})$ *the function* $\sigma'(\xi)$ *admits an asymptotic expansion of the form*

$$\sigma'(\xi) \sim \sum_{j=0}^{\infty} b_j \xi^{n-j-1} \tag{8-1}$$

as $\xi \to \infty$.

Proof. Of the approaches mentioned above, the ray expansion method from [Yafaev 2010, §8.4] is the most easily adapted to the hyperbolic setting. In our context, the idea is to expand $E_V(s; z, \omega')$ in powers of s and then apply this expansion to the scattering phase.

To develop the approximation formula, we first consider $z \in \mathbb{H}^{n+1}$ with $\omega' = \infty$. In standard hyperbolic coordinates $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}_+$,

$$\Delta = -y^2 \partial_y^2 + (n-1) y \partial_y + y^2 \Delta_x, \tag{8-2}$$

and the unperturbed generalized eigenfunction has the form (see, e.g., [Borthwick 2010, §4])

$$E_0(s; z, \infty) = 2^{-2s-1} \pi^{-1/2} \frac{\Gamma(s)}{\Gamma\big(s - \frac{1}{2}n + 1\big)} y^s. \tag{8-3}$$

In geodesic coordinates $y = e^{-r}$, so this is the analog of a Euclidean plane wave with frequency $\xi = \operatorname{Im} s$.

Following the construction in [Yafaev 2010, §8.1], we define an approximate plane wave using the ansatz

$$\psi_N(s; z) = \sum_{j=0}^{N} s^{-j} y^s w_j(z), \tag{8-4}$$

with $w_0(z) = 1$. From (8-2), we have

$$[\Delta + V - s(n-s)](y^s w_j) = y^s (\Delta + V) w_j - 2 s y^{s-1} \partial_y w_j.$$

We can thus cancel coefficients up to order s^N by imposing the transport equation

$$2 y \partial_y w_{j+1} = (\Delta + V) w_j.$$

The solutions are given recursively by

$$w_{j+1}(z) := \frac{1}{2} \int_{-\infty}^{0} (\Delta + V) w_j(x, e^t y)\, dt \tag{8-5}$$

for $j \geq 1$. With these coefficients, the function (8-4) satisfies

$$[\Delta + V - s(n-s)] \psi_N(s; z) = s^{-N} y^s w_N(z). \tag{8-6}$$

In (8-5), the point (x, e^t) can be interpreted geometrically as the translation of $z = (x, y)$ by distance t along the vertical geodesic through z. Returning to the geodesic polar coordinates $z = (r, \omega) \in \mathbb{R}_+ \times \mathbb{S}^n$

used to define $E_V(s)$, we let $\phi_{z,\omega'}(t)$ denote the unique geodesic through z with limit point $\omega' \in \mathbb{S}^n$ as $t \to 0$. Let $w_0(z,\omega') = 1$ and define $w_j(z,\omega')$ for $j \geq 1$ by

$$w_{j+1}(z,\omega') := \frac{1}{2}\int_{-\infty}^{0} (\Delta + V) w_j(\phi_{z,\omega'}(t))\, dt.$$

For the approximate Poisson kernel

$$E_{V,N}(s; z, \omega') := \sum_{j=0}^{N} s^{-j} w_j(z,\omega') E_0(s, z, \omega'), \tag{8-7}$$

the calculation of (8-6) shows that

$$[\Delta + V - s(n-s)] E_{V,N}(s; z, \omega') := s^{-N} E_0(s, z, \omega')(\Delta + V) w_N(z, \omega').$$

The coefficients of (8-7) have support properties analogous to the approximate plane waves in the Euclidean case. That is, for $j \geq 1$, $w_j(z,\omega')$ vanishes unless z lies on a geodesic connecting a point in supp V to the limit point ω'. One can thus repeat the argument from [Yafaev 2010, Theorem 8.4.3] using the cutoff resolvent bound from [Guillarmou 2005, Proposition 3.2] in place of its Euclidean counterpart. The result is that

$$E_V(s; z, \omega') = E_{V,N}(s; z, \omega') + q_n(s; z, \omega'), \tag{8-8}$$

where, for $\operatorname{Re} s = \frac{1}{2}n$,

$$\|q_n(s; \cdot\,, \omega')\|_{L^2(B)} = O(s^{n/2-N}),$$

with B a ball in $\mathbb{H}^{n+1}$ containing supp V. The shift in the power in the error estimate comes from the Γ factors in the normalization of (8-3). The same error estimate applies when (8-8) is differentiated with respect to s.

The approximation (8-8) can be applied to the scattering phase through the formula (4-4), which gives

$$\tau(s) = \det(1 + T(s)),$$

where

$$T(s) := (2s - n) E_V(s) V E_0(n-s).$$

By the definition of the scattering phase and the fact that $1 + T\left(\frac{1}{2}n + i\xi\right)$ is unitary for $\xi \in \mathbb{R}$,

$$\sigma'(\xi) = -\frac{1}{2\pi} \operatorname{tr}\Big[\Big(1 + T\Big(\frac{1}{2}n + i\xi\Big)^*\Big) T'\Big(\frac{1}{2}n + i\xi\Big)\Big]. \tag{8-9}$$

The kernels of $T(s)$ and $T'(s)$ are smooth, and (8-8) gives uniform asymptotic expansions of their kernels for $\operatorname{Re} s = \frac{1}{2}n$, with leading term of order at most ξ^{n-1}. We can thus deduce the expansion of $\sigma'(\xi)$ from (8-9). □

Although the leading term in (8-1) matches the growth estimate of Proposition 5.3, this coefficient vanishes and the leading order is actually ξ^{n-2}. Computing coefficients through the construction of Proposition 8.2 is rather cumbersome, however. Comparison to the heat trace expansion via the Berman–Krein formula yields a much easier method.

Proof of Theorem 8.1. By a straightforward calculus argument (see [Dyatlov and Zworski 2019, Lemma 3.65]), the expansion (8-1) yields the corresponding expansion

$$\int_0^\infty \sigma'(\xi)e^{-\xi^2 t}d\xi \sim \frac{1}{2}\sum_{j=0}^{n-1}\Gamma\Big(\frac{1}{2}(n-j)\Big)b_j\, t^{-(n-j)/2} - \frac{1}{2}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}b_{n+2l}\, t^l \log t + \frac{1}{2}\sum_{l=0}^{\infty}\Gamma\Big(-l-\frac{1}{2}\Big)b_{n+2l+1}t^{l+1/2} + g(t)$$

as $t \to 0^+$, where $g \in C^\infty[0,\infty)$. The function g is not determined by the coefficients b_j. On the other hand, by (7-1) and Theorem 7.1 we have

$$\int_0^\infty \sigma'(\xi)e^{-\xi^2 t}\, d\xi \sim \pi^{-1/2}\sum_{k=1}^{\infty} a_k(V)(4t)^{-(n+1)/2+k} + h(t), \tag{8-10}$$

where $h \in C^\infty[0,\infty)$ is given by

$$h(t) := \sum_{j=1}^{d} e^{t(n^2/4-\lambda_j)} + \tfrac{1}{2}m_V\big(\tfrac{1}{2}n\big).$$

If $n+1$ is odd, then comparing these expansions shows that $b_j = 0$ if j is even and

$$b_{2k-1} = \frac{2^{-n+2k}}{\pi^{1/2}\Gamma\big(\frac{1}{2}(n+1)-k\big)}a_k(V) \tag{8-11}$$

for $k \in \mathbb{N}$. For $n+1$ even the heat trace expansion contains only integral powers of t. This implies that $b_j = 0$ for all $j \geq n$ and also for even values of $j < n$. For odd values of $j < n$, the coefficients are given by (8-11). □

Integrating the asymptotic expansion from Theorem 8.1 yields the following:

Corollary 8.3. *The scattering phase admits a full asymptotic expansion as $\xi \to 0$. If the dimension $n+1$ is odd, then*

$$\sigma(\xi) \sim \sum_{k=1}^{[n/2]}\frac{c_k(V)}{n-2k+1}\xi^{n-2k+1} + d + \tfrac{1}{2}m_V\big(\tfrac{1}{2}n\big) + \sum_{k>[n/2]}\frac{c_k(V)}{n-2k+1}\xi^{n-2k+1},$$

where d is the number of eigenvalues. For $n+1$ even,

$$\sigma(\xi) = \sum_{k=1}^{[n/2]}\frac{c_k(V)}{n-2k+1}\xi^{n-2k+1} + d + \tfrac{1}{2}m_V\big(\tfrac{1}{2}n\big) + O(\xi^{-\infty}).$$

Proof. By Theorem 8.1, the function

$$t \mapsto \int_0^\infty \Big[\sigma'(x) - \sum_{k=1}^{[n/2]} c_k(V)x^{n-2k}\Big]e^{-x^2 t}\, dx$$

is continuous for $t \in [0, \infty)$. By (8-10), taking the limit as $t \to 0^+$ yields

$$\int_0^\infty \left[\sigma'(x) - \sum_{k=1}^{[n/2]} c_k(V) x^{n-2k}\right] dx = d + \tfrac{1}{2} m_V\left(\tfrac{1}{2}n\right).$$

Splitting the integral at $x = \xi$ then gives, since $\sigma(0) = 0$,

$$\sigma(\xi) = \sum_{k=1}^{[n/2]} \frac{c_k(V)}{n-2k+1} \xi^{n-2k+1} + d + \tfrac{1}{2} m_V\left(\tfrac{1}{2}n\right) - \int_\xi^\infty \left[\sigma'(x) - \sum_{k=1}^{[n/2]} c_k(V) x^{n-2k}\right] dx.$$

By Theorem 8.1, for $n+1$ odd the final integral on the right can be integrated to produce an asymptotic expansion in ξ. For $n+1$ even, this integral gives an error term $O(\xi^{-\infty})$. □

9. Existence of resonances

The asymptotic expansions of the wave trace and scattering have significantly different behavior in odd and even dimensions, so we will consider the two cases separately.

Even dimensions. For $n+1$ even, all of the singularities in the wave trace expansion of Theorem 6.1 are detectable for $t \neq 0$. It thus follows immediately from Theorem 5.1 that, for $V \in C_0^\infty(\mathbb{H}^{n+1}, \mathbb{R})$, the resonance set $\mathcal{R}_V$ determines all of the wave invariants $a_k(V)$. In particular, since the vanishing of the first two wave invariants implies $V = 0$ by the formulas of Proposition 6.4, we obtain the following:

Theorem 9.1. *For $V \in C_0^\infty(\mathbb{H}^{n+1}, \mathbb{R})$ with $n+1$ even, if $\mathcal{R}_V = \mathcal{R}_0$ then $V = 0$.*

We can also deduce a lower bound on the resonance counting function from the wave trace in even dimensions. Note that $\Theta_V(t) = O(t^{-n+1})$ by Theorem 6.1, whereas the $\mathcal{R}_0$ contribution in (5-3) satisfies

$$u_0(t) \sim \frac{1}{t^{n+1}}$$

as $t \to 0$. It thus follows from (5-3) that

$$\sum_{\zeta \in \mathcal{R}_V} e^{(\zeta - n/2)t} \sim \frac{2}{t^{n+1}}. \tag{9-1}$$

The lower-bound argument from [Guillopé and Zworski 1997, Theorem 1.3] (see also [Borthwick 2016, §12.2]) can be applied to (9-1), yielding the following:

Theorem 9.2. *For $n+1$ even, the counting function for $\mathcal{R}_V$ satisfies*

$$N_V(r) \geq c r^{n+1}$$

for some constant $c > 0$ that depends only on n and the radius of supp V.

Proof. Choose $\phi \in C_0^\infty(\mathbb{R}_+)$ with $\phi \geq 0$ and $\phi(1) > 0$, and set

$$\phi_\lambda(t) := \lambda \phi(\lambda t).$$

By (9-1) we have

$$\int_0^\infty \Bigl(\sum_{\zeta\in\mathcal{R}_V} e^{(\zeta-n/2)t}\Bigr)\phi_\lambda(t)\,dt \geq c_n\lambda^{n+1},$$

where c_n does not depend on V. Using the Fourier transform to evaluate the right-hand side gives

$$\sum_{\zeta\in\mathcal{R}_V} \hat{\phi}\Bigl(\frac{i(\zeta-\frac{1}{2}n)}{\lambda}\Bigr) \geq c_n\lambda^{n+1}. \tag{9-2}$$

Since $\hat{\phi}(\xi)$ is rapidly decreasing, we can estimate $\hat{\phi}(\xi) = O(|\xi|^{-n-2})$ in particular. In terms of the counting function, (9-2) then implies

$$c_n\lambda^{n+1} \leq \int_0^\infty \Bigl(\frac{1+r}{\lambda}\Bigr)^{-n-2} dN_V(r) = (n+2)\int_0^\infty (1+r)^{-n-3}N_V(\lambda r)\,dr.$$

Splitting the integral at $r=a$ and adjusting the constant gives

$$c_n\lambda^{n+1} \leq N_V(\lambda a) + \int_a^\infty (1+r)^{-n-3}N_V(\lambda r)\,dr. \tag{9-3}$$

If V has support in a ball of radius R, then [Borthwick 2010, Theorem 1.1] gives an upper bound

$$N_V(r) \leq C_R r^{n+1}.$$

Applying this estimate to (9-3) gives

$$N_V(\lambda a) \geq c_n\lambda^{n+1} - C_R\lambda^{n+1}a^{-1}.$$

We can then set $a = 2C_R/c_n$ and rescale λ to obtain

$$N_V(\lambda) \geq \frac{1}{2}c_n\Bigl(\frac{c_n}{2C_R}\Bigr)^{n+1}\lambda^{n+1}. \qquad \square$$

The existence of a lower bound in even dimensions is not surprising, since the optimal order of growth is already attained for $V=0$. It is more interesting to examine the difference between $\mathcal{R}_V$ and the background resonance set. Note that when $n+1$ is even, the expansion of Theorem 8.1 contains only odd powers of ξ. Since $\sigma'(\xi)$ is an even function, this creates a discrepancy that we can exploit.

Theorem 9.3. *For $n+1$ even, suppose that $V_1, V_2 \in C_0^\infty(\mathbb{H}^{n+1}, \mathbb{R})$. If the resonance sets $\mathcal{R}_{V_1}$ and $\mathcal{R}_{V_2}$ differ by only finitely many points (counting multiplicities), then:*

(1) *The corresponding scattering phases σ_{V_1} and σ_{V_2} differ by a constant.*

(2) *The sets $\mathcal{R}_{V_1}\backslash\mathcal{R}_{V_2}$ and $\mathcal{R}_{V_2}\backslash\mathcal{R}_{V_1}$ are contained in $(0,n)$ and invariant under the reflection $s\mapsto n-s$.*

(3) *The wave invariants satisfy $a_k(V_1)=a_k(V_2)$ for $k=1,\dots,\frac{1}{2}(n-1)$.*

Furthermore, if $\mathcal{R}_{V_1}=\mathcal{R}_{V_2}$ (with multiplicities), then $\Theta_{V_1}=\Theta_{V_2}$, and hence all of the wave invariants match.

Proof. Under the assumption that $\mathcal{R}_{V_1}$ and $\mathcal{R}_{V_2}$ differ by only finitely many points, the factorization of Proposition 5.2 implies that

$$\frac{\tau_{V_1}(s)}{\tau_{V_2}(s)} = (-1)^{m_{V_1}(n/2)-m_{V_2}(n/2)} e^{p(s)} \prod_{\zeta\in\mathcal{R}_{V_1}\setminus\mathcal{R}_{V_2}} \frac{n-s-\zeta}{s-\zeta} \prod_{\zeta\in\mathcal{R}_{V_2}\setminus\mathcal{R}_{V_1}} \frac{s-\zeta}{n-s-\zeta}, \tag{9-4}$$

where p is a polynomial with degree at most $n+1$ satisfying $p(s) = p(n-s)$. It follows that $\sigma'_{V_1}(\xi)-\sigma'_{V_2}(\xi)$ is an even, rational function of ξ. Since the expansion formula from Theorem 8.1 contains only odd powers of ξ plus an $O(\xi^{-\infty})$ remainder, this implies that $\sigma'_{V_1}(\xi) = \sigma'_{V_2}(\xi)$. The equality of the wave invariants for $k = 1, \dots, \left[\frac{1}{2}n\right]$ then follows from the matching of expansion coefficients. Since $\sigma'_{V_1} = \sigma'_{V_2}$ also implies that $\tau_{V_1}(s)/\tau_{V_2}(s)$ is constant, the characterization of $\mathcal{R}_{V_1}\setminus\mathcal{R}_{V_2}$ and $\mathcal{R}_{V_2}\setminus\mathcal{R}_{V_1}$ follows from (9-4).

If $\mathcal{R}_{V_1} = \mathcal{R}_{V_2}$, then the same argument shows that the scattering phases are equal. It then follows from (5-6) that $\Theta_{V_1} = \Theta_{V_1}$. ☐

Let us apply Theorem 9.3 to compare $\mathcal{R}_V$ to $\mathcal{R}_0$. The hypothesis that $\mathcal{R}_V$ and $\mathcal{R}_0$ differ by finitely many points implies that $\sigma'(\xi) = 0$ and $a_k(V) = 0$ for $k \le \frac{1}{2}(n-1)$. Since $\mathcal{R}_0 \cap (0, n) = \varnothing$, it also implies that $\mathcal{R}_V$ is the union of $\mathcal{R}_0$ with a possible resonance at $\zeta = \frac{1}{2}n$ plus a finite set of pairs of the form

$$\zeta = \tfrac{1}{2}n \pm \sqrt{\tfrac{1}{4}n^2 - \lambda_j},$$

where λ_j is an eigenvalue.

As noted at the start of this section, the vanishing of the first two wave invariants implies that $V = 0$. From Theorem 9.3 we thus immediately obtain the following:

Corollary 9.4. *Let $V \in C_0^\infty(\mathbb{H}^{n+1}, \mathbb{R})$ with $n+1$ even and $n \ge 5$. If $V \ne 0$ then $\mathcal{R}_V$ differs from $\mathcal{R}_0$ by infinitely many points (counting multiplicities).*

For $n \le 3$, we cannot fully control the first two wave invariants. However, we can derive some extra information from the heat trace. If $\sigma'(\xi) = 0$, we see from (7-1) and (7-2) that

$$\sum_{j=1}^{d} e^{t(n^2/4-\lambda_j)} + \tfrac{1}{2}m_V\left(\tfrac{1}{2}n\right) \sim \sum_{k=(n+1)/2}^{\infty} 2^{-n+2k-1}\pi^{-1/2} a_k(V) t^{-(n+1)/2+k} \tag{9-5}$$

as $t \to 0$. Matching the coefficients in the expansion leads to a set of relationships between the discrete eigenvalues λ_j, the multiplicity $m_V\left(\frac{1}{2}n\right)$, and the wave invariants.

Corollary 9.5. *For $V \in C_0^\infty(\mathbb{H}^2, \mathbb{R})$, if $V \ne 0$ and $\int V\,dg \ge 0$, then $\mathcal{R}_V$ differs from $\mathcal{R}_0$ by infinitely many points. The same conclusion holds for $V \in C_0^\infty(\mathbb{H}^4, \mathbb{R})$, provided $\int V\,dg \ne 0$.*

Proof. Assume that $\mathcal{R}_V$ differs from $\mathcal{R}_0$ by finitely many points. For $n = 1$, the t^0 term in (9-5) gives

$$d + \frac{1}{2}m_V\left(\frac{1}{2}n\right) = -\frac{1}{4\pi}\int_{\mathbb{H}^2} V(z)\,dg(z).$$

Hence $\int V\,dg \ge 0$ implies $\mathcal{R}_V = \varnothing$, which gives $V = 0$ by Theorem 9.1.

For $n = 3$, the assumption that $\mathcal{R}_V$ differs from $\mathcal{R}_0$ by finitely many points gives $a_1(V) = 0$ by Theorem 9.3. This means $\int V\,dg = 0$. ☐

Assuming a finite discrepancy between $\mathcal{R}_V$ and $\mathcal{R}_0$, the expansion (9-5) also implies a set of relations between eigenvalues and wave invariants. For $n = 1$, we have

$$\frac{1}{(k-1)!}\sum_{j=1}^{d}\Big(\frac{1}{4}-\lambda_j\Big)^{k-1} = 4^{k-1}\pi^{-1/2}a_k(V)$$

for $k \geq 2$. If $n = 3$,

$$d + \tfrac{1}{2}m_V\big(\tfrac{3}{2}\big) = \pi^{-1/2}a_2(V)$$

and

$$\frac{1}{(k-2)!}\sum_{j=1}^{d}\Big(\frac{9}{4}-\lambda_j\Big)^{k-2} = 4^{k-2}\pi^{-1/2}a_k(V)$$

for $k \geq 3$. Although these relations seem rather delicate, they do not lead to any obvious contradictions.

Odd dimensions. In odd dimensions, the primary limitation to drawing implications from the wave trace is the fact that the terms in the expansion of Theorem 6.1 with $k \leq \frac{1}{2}n$ are distributions supported only at $t = 0$. Hence the trace formula of Theorem 5.1 yields no information about the first $\frac{1}{2}n$ wave invariants.

In the Euclidean case, [Sá Barreto and Zworski 1996] exploited the decay of the heat trace as $t \to \infty$ to prove an existence result. In the hyperbolic case, the corresponding decay rate from Corollary 7.3 is merely $O(t^{-1/2})$, independent of the dimension. Hence this approach fails and we obtain an existence result only for dimension three.

Theorem 9.6. *For $V \in C_0^\infty(\mathbb{H}^3, \mathbb{R})$, if $V \neq 0$ then $\mathcal{R}_V$ is not empty.*

Proof. For $n = 2$, if $\mathcal{R}_V = \varnothing$ then Theorems 5.1 and 6.1 show that $a_k(V) = 0$ for $k \geq 2$. By the formula from Proposition 6.4,

$$a_2(V) = \frac{1}{32\pi}\int_{\mathbb{H}^3} V(z)^2\,dg(z),$$

so $a_2(V) = 0$ implies $V = 0$ when $n = 2$. □

As long as at least one resonance exists, we can use the Poisson formula to show that there are infinitely many. The arguments from [Christiansen 1999, Theorem 1] and [Sá Barreto 2001, Theorem 1.3] can then be applied to produce a lower bound on the count.

Theorem 9.7. *For $V \in C_0^\infty(\mathbb{H}^{n+1}, \mathbb{R})$ with $n+1$ odd, either $\mathcal{R}_V = \varnothing$ or $\mathcal{R}_V$ is infinite and the counting function satisfies*

$$\limsup_{r\to\infty}\frac{N_V(r)}{r} > 0.$$

Proof. Suppose that $\mathcal{R}_V$ is finite. By Theorem 5.1 the wave trace is given by a finite sum:

$$\Theta_V(t) = \frac{1}{2}\sum_{\zeta\in\mathcal{R}_V} e^{(\zeta-n/2)|t|}$$

for $t \neq 0$. Hence

$$\lim_{t\to 0}\Theta_V(t) = \tfrac{1}{2}\#\mathcal{R}_V.$$

Since the wave trace expansion of Theorem 6.1 has no term of order t^0 for $n+1$ odd, this shows that $\mathcal{R}_V = \varnothing$.

Now assume that $\mathcal{R}_V$ is infinite. Since $\mathcal{R}_0 = \varnothing$ in odd dimensions, the factorization formula of Proposition 5.2 reduces to

$$\tau(s) = (-1)^{m_V(n/2)} e^{q(s)} \frac{H_V(n-s)}{H_V(s)}.$$

This is completely analogous to the factorization in the Euclidean case, once we shift the spectral parameter by setting $s = \frac{1}{2}n + i\xi$.

Suppose that $N_V(r) = O(r)$. Then, the scattering phase expansion of Corollary 8.3 allows us to apply [Sá Barreto 2001, Theorem 1.2] to deduce that

$$\left| \sum_{|\xi_j|<r} \frac{1}{\xi_j} \right| \le C$$

for all $r > 0$. The argument from the proof of [Sá Barreto 2001, Theorem 1.3] then yields a contradiction to the fact the asymptotic expansion from Corollary 8.3 has only integral powers of ξ. ☐

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CHARACTERIZATION OF RECTIFIABILITY VIA LUSIN-TYPE APPROXIMATION

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We prove that a Radon measure μ on $\mathbb{R}^n$ can be written as $\mu = \sum_{i=0}^n \mu_i$, where each of the μ_i is an i-dimensional rectifiable measure if and only if, for every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ and every $\varepsilon > 0$, there exists a function g of class C^1 such that $\mu(\{x \in \mathbb{R}^n : g(x) \neq f(x)\}) < \varepsilon$.

1. Introduction

A fundamental yet simple consequence of Rademacher's theorem and Whitney's theorem is the fact that Lipschitz functions on the Euclidean space admit a Lusin-type approximation with C^1-functions, namely, for every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ and every $\varepsilon > 0$, there exists a function $g : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that

$$\mathscr{L}^n(\{x \in \mathbb{R}^n : g(x) \neq f(x)\}) < \varepsilon,$$

where $\mathscr{L}^n$ denotes the Lebesgue measure; see [Simon 1983, Theorem 5.3]. This fact has a central role in many pivotal results in geometric measure theory, including the existence of the approximate tangent space to a rectifiable set [Simon 1983, Lemma 11.1] and the validity of area and coarea formulas [Simon 1983, § 12].

On the one hand, this approximation property does not only hold for the Lebesgue measure: for instance it holds trivially for a Dirac delta. It is not difficult to see that the same property holds for any rectifiable measure, and clearly the class of Radon measures for which the property holds is closed under finite sums.

On the other hand, it is known that there are measures μ for which Lipschitz functions do not admit a Lusin-type approximation with respect to μ with functions of class C^1; see [Marchese 2017]. In this note we completely classify those measures, proving that the validity of such an approximation property characterizes rectifiable measures, in the following sense.

Theorem 1.1. *Let μ be a positive Radon measure on $\mathbb{R}^n$. The measure μ can be written as $\mu = \sum_{i=0}^n \mu_i$, where each of the μ_i is an i-dimensional rectifiable measure if and only if, for every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ and every $\varepsilon > 0$, there exists a function g of class C^1 such that*

$$\mu(\{x \in \mathbb{R}^n : g(x) \neq f(x)\}) < \varepsilon. \tag{1}$$

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The proof of the "only if" part of Theorem 1.1 is a simple application of Whitney's theorem. The proof of the "if" part exploits some tools introduced in [Alberti and Marchese 2016], including the notion of the *decomposability bundle* of a measure μ: a map $x \mapsto V(\mu, x)$ which detects the maximal subspaces along which Lipschitz functions are differentiable μ-almost everywhere [Alberti and Marchese 2016, § 2.6]. For the purposes of this paper, we need to refine the result [Alberti and Marchese 2016, Theorem 1.1(ii)] on the existence of Lipschitz functions which are nondifferentiable along directions which do not belong to the decomposability bundle. In that paper, such nondifferentiability is proved by finding a Lipschitz function f and for μ-almost every point x a sequence of points $y_i := x + t_i v \in \mathbb{R}^n$ converging to x along a direction $v \notin V(\mu, x)$, such that the corresponding incremental ratios $(f(y_i) - f(x))/t_i$ do not converge. Here we need to find a function f such that there exist points y_i as above, with the additional requirement that $y_i \in \operatorname{supp}(\mu)$; see Proposition 3.1. For a nonrectifiable measure μ, the existence of a μ-positive set of points x for which there are points $y_i \in \operatorname{supp}(\mu)$ approaching x along a direction $v \notin V(\mu, x)$ is guaranteed by Lemma 2.1.

We plan to investigate similar questions in Carnot groups, exploiting tools and techniques introduced in [De Philippis et al. 2022]. In this setting, similar questions have already attracted some interest. For instance, in [Julia et al. 2023] the authors proved a suitable extension of Lusin's approximation-type theorem for the surface measure of 1-codimensional $C^1_{\mathbb{H}}$-rectifiable surfaces in the Heisenberg groups $\mathbb{H}^n$, $n \geq 2$, and where the regular approximation of Lipschitz functions are found in the class of $C^1_{\mathbb{H}}$-regular functions. The authors also prove that in $\mathbb{H}^1$ there is a regular surface and a Lipschitz function that cannot be approximated by $C^1_{\mathbb{H}}$-regular functions. This different behavior is connected to the algebraic structure of the tangents to 1-codimensional regular surfaces in the Heisenberg groups $\mathbb{H}^n$ when $n = 1$ or $n \geq 2$.

2. Notation and preliminaries

We denote by $U(x, r)$ the open ball in $\mathbb{R}^n$ with center x and radius r and by $B(x, r)$ the closed ball. In addition, for a Borel set E and a $\delta > 0$, we define $B(E, \delta) := \bigcup_{y \in E} B(y, \delta)$. The unit sphere is denoted by $\mathbb{S}^{n-1}$.

Given a Radon measure μ and a (possibly vector-valued) function f, we denote by $f\mu$ the measure

$$f\mu(A) := \int_A f \, d\mu \quad \text{for every Borel set } A.$$

For a measure μ and a Borel set E we denote by $\mu \llcorner E$ the restriction of μ to E, namely the measure defined by

$$\mu \llcorner E(A) := \mu(A \cap E) \quad \text{for every Borel set } A.$$

The support of a positive Radon measure μ, denoted $\operatorname{supp}(\mu)$, is the intersection of all closed sets C such that $\mu(\mathbb{R}^n \setminus C) = 0$. For $0 \leq k \leq n$, the symbol $\mathscr{H}^k$ denotes the k-dimensional Hausdorff measure on $\mathbb{R}^n$.

Definition (rectifiable sets and measures). For $0 \leq k \leq n$, a set $E \subset \mathbb{R}^n$ is *k-rectifiable* if there are sets E_i $(i = 1, 2, \ldots)$ such that

(i) E_i is a Lipschitz image of $\mathbb{R}^k$ for every i;

(ii) $\mathscr{H}^k\big(E \setminus \bigcup_{i \geq 1} E_i\big) = 0$.

A Radon measure is said to be *k-rectifiable* if it is absolutely continuous with respect to $\mathscr{H}^k \llcorner E$ for some k-rectifiable set E.

As usual, the symbol $\mathrm{Gr}(k, n)$ denotes the Grassmannian of k-planes in $\mathbb{R}^n$, and we define $\mathrm{Gr} := \bigcup_{0 \le k \le n} \mathrm{Gr}(k, n)$. We endow Gr with the topology induced by the distance

$$d(V, W) := d_{\mathscr{H}}(V \cap U(0, 1), W \cap U(0, 1)),$$

where $d_{\mathscr{H}}$ is the Hausdorff distance. We recall the following definition; see [Alberti and Marchese 2016, §2.6, §6.1 and Theorem 6.4].

Definition (decomposability bundle). Given a positive Radon measure μ on $\mathbb{R}^n$, its *decomposability bundle* is a map $V(\mu, \cdot)$ taking values in the set Gr defined as follows. A vector $v \in \mathbb{R}^n$ belongs to $V(\mu, x)$ if and only if there exists a vector-valued measure T with $\operatorname{div} T = 0$ such that

$$\lim_{r \to 0} \frac{\mathbb{M}((T - v\mu) \llcorner B(x, r))}{\mu(B(x, r))} = 0,$$

where $\mathbb{M}((T - v\mu) \llcorner B(x, r))$ denotes the total variation of the vector-valued measure $(T - v\mu) \llcorner B(x, r)$.

Definition (tangent measures). We define the map $T_{x,r}(y) = (y - x)/r$, and we denote by $T_{x,r}\mu$ the pushforward of μ under $T_{x,r}$, namely $T_{x,r}\mu(A) := \mu(x + rA)$ for every Borel set A. Given a measure μ and a point x, the family of *tangent measures* $\mathrm{Tan}(\mu, x)$, introduced in [Preiss 1987], consists of all the possible nonzero limits (with respect to the weak* convergence of measures) of $c_i T_{x,r_i}\mu$ for some sequence of positive real numbers c_i and some sequence of radii $r_i \to 0$. We know thanks to [Preiss 1987, Theorem 2.5] that $\mathrm{Tan}(\mu, x)$ is nonempty μ-almost everywhere.

Definition (cone over a k-plane). For any $k \in \{1, \dots, n-1\}$, $0 < \vartheta < 1$, $x \in \mathbb{R}^n$ and $V \in \mathrm{Gr}(k, n)$, we let

$$X(x, V, \vartheta) := x + \{v \in \mathbb{R}^n : |p_V(v)| \ge \vartheta |v|\},$$

where p_V denotes the orthogonal projection onto V. For notational convenience, for $k = 0$ and for every $0 < \vartheta < 1$, we define $X(x, 0, \vartheta) := \{x\}$.

Definition (F_K distance between measures). Given ϕ and ψ two Radon measures on $\mathbb{R}^n$, and given $K \subseteq \mathbb{R}^n$ a compact set, we define

$$F_K(\phi, \psi) := \sup\left\{\left|\int f\, d\phi - \int f\, d\psi\right| : f \in \mathrm{Lip}_1^+(K)\right\}, \tag{2}$$

where $\mathrm{Lip}_1^+(K)$ denotes the class of 1-Lipschitz nonnegative functions with support contained in K. We also write $F_{x,r}$ for $F_{B(x,r)}$.

Lemma 2.1. *Let μ be a Radon measure on $\mathbb{R}^n$ with $\dim(V(\mu, x)) = k < n$ for μ-almost every x. Assume that $\mu(R) = 0$ for every k-rectifiable set R. Then, for every $0 < \vartheta < 1$ and for every $\varepsilon > 0$,*

$$\mathrm{supp}(\mu) \cap B(x, \varepsilon) \setminus X(x, V(\mu, x), \vartheta) \ne \varnothing \tag{3}$$

for μ-almost every x.

Proof. Assume by contradiction that there exists a Borel set E with $\mu(E) > 0$ such that, for every $x \in E$, there exists $\varepsilon > 0$ such that (3) fails. We claim that this implies that, for μ-almost every $x \in E$, every tangent measure $\nu \in \mathrm{Tan}(\mu, x)$ satisfies

$$\mathrm{supp}(\nu) \subset X(0, V(\mu, x), \vartheta). \tag{4}$$

In order to prove (4), fix $x \in E$ such that $\mathrm{Tan}(\mu, x)$ is nonempty and consider any open ball $U(y, \rho) \subset \mathbb{R}^n \setminus X(0, V(\mu, x), \vartheta)$. Notice that since (3) fails, we have $T_{x,r}\mu(U(y, \rho)) = \mu(U(x + ry, r\rho)) = 0$ for every $r < \varepsilon/(|y|+\rho)$, which we conclude in view of [De Lellis 2008, Proposition 2.7]. Thanks to [Del Nin and Merlo 2022, Proposition 2.9] we infer that $\mathrm{supp}(\nu) \subset V(\mu, x)$ and in particular $\nu = c\mathscr{H}^k \llcorner V(\mu, x)$ for some $c > 0$. For every $W \in \mathrm{Gr}(k, n)$, define

$$E_W := \{x \in \mathbb{R}^n : (k+1)F_{0,1}(\mathscr{H}^k \llcorner V(\mu, x), \mathscr{H}^k \llcorner W) < 20^{-k-4}\}.$$

By the compactness of the Grassmannian, there exists $W \in \mathrm{Gr}(k, n)$ such that $\mu(E_W) > 0$. On the other hand, by [Preiss 1987, §4.4(5)] and by the locality of tangent measures, see [Preiss 1987, §2.3(4)], we conclude that $\mu \llcorner E_W$ is supported on a k-rectifiable set. This however contradicts the assumption that $\mu(R) = 0$ for every k-rectifiable set R. □

Definition (cone-null sets). For any $e \in \mathbb{S}^{n-1}$ and $\theta \in (0, 1)$, we let the *one-sided cone of axis e and amplitude* θ be the set

$$C(e, \theta) := \{v \in \mathbb{R}^n : \langle v, e\rangle \geq \theta|v|\}.$$

In the following we denote by $\Gamma(e, \theta)$ the family of Lipschitz curves $\gamma : E \subseteq \mathbb{R} \to \mathbb{R}^n$ such that $\gamma'(t) \in C(e, \theta)$ for $\mathcal{L}^1$-almost every $t \in E$. Finally, a Borel set B is said to be $C(e, \theta)$-null if $\mathscr{H}^1(\mathrm{im}(\gamma) \cap B) = 0$ for any $\gamma \in \Gamma(e, \theta)$.

Proposition 2.2. *Let E be a compact set in $\mathbb{R}^n$. Let $W \in \mathrm{Gr}(k, n)$, with $k < n$, and suppose that there exists $\theta_0 \in (0, 1)$ such that, for any $e \in W^\perp$, the set E is $C(e, \theta_0)$-null. Then, for any $\theta_0 \leq \theta < 1$ and $\varepsilon > 0$, there exists $\delta_0 > 0$ such that*

$$\mathscr{H}^1(\mathrm{im}(\gamma) \cap B(E, \delta_0)) \leq \varepsilon$$

for any $\gamma \in \Gamma(e, \theta)$. For any $\theta_0 \leq \theta < 1$, $0 < \delta < \delta_0$ and any $e \in W^\perp$, consider the function

$$\omega_{e,\theta,\delta}(x) := \sup_{\substack{\gamma\in\Gamma(e,\theta)\\ \gamma(b)=x+\lambda e}} \mathscr{H}^1(B(E, \delta) \cap \mathrm{im}(\gamma)) - \lambda|e|. \tag{5}$$

Then the following properties hold:

(i) $0 \leq \omega_{e,\theta,\delta}(x) \leq \varepsilon$ *for any* $x \in \mathbb{R}^n$,

(ii) $\omega_{e,\theta,\delta}(x) \leq \omega_{e,\theta,\delta}(x+se) \leq \omega_{e,\theta,\delta}(x)+s|e|$ *for every $s > 0$ and any $x \in \mathbb{R}^n$. Moreover, if the segment $[x, x + se]$ is contained in $B(E, \delta)$, then* $\omega_{e,\theta,\delta}(x + se) = \omega_{e,\theta,\delta}(x) + s|e|$,

(iii) $|\omega_{e,\theta,\delta}(x + v) - \omega_{e,\theta,\delta}(x)| \leq \theta(1 - \theta^2)^{-1/2}|v|$ *for every* $v \in V := e^\perp$,

(iv) $\omega_{e,\theta,\delta}$ *is* $(1 + (n - 1)\theta(1 - \theta^2)^{-1/2})$*-Lipschitz.*

Proof. The first part of the proposition is an immediate consequence of Step 1 in the proof of [Alberti and Marchese 2016, Lemma 4.12]. On the other hand, the construction of the function $\omega_{e,\theta,\delta}$ was performed in the second step of that proof. □

3. Construction of nondifferentiable functions

In this section we prove the existence of some suitable Lipschitz functions which are nondifferentiable along directions that are quantitatively far away from the decomposability bundle. Given a measure μ as in Lemma 2.1, we prove that there are *many* functions which are nondifferentiable on a set of positive μ-measure with the additional property that the nondifferentiability is "detected" by the points in the support of μ; see Proposition 3.1.

In this section we fix $k \in \{0, \dots, n-1\}$ and let μ be a Radon measure such that $\dim(V(\mu, x)) = k$ for μ-almost every $x \in \mathbb{R}^n$ and $\mu(R) = 0$ for any k-rectifiable set R. Thanks to the strong locality principle, see [Alberti and Marchese 2016, Proposition 2.9(i)], and Lusin's theorem, we can assume, up to restriction to a compact subset $\widetilde{K} \subset \operatorname{supp}(\mu)$ of positive μ-measure, that $V(\mu, x)$ is uniformly continuous on $\widetilde{K}$. Up to restricting to a subset where the oscillation of V is small, we can assume that there are n continuous vector fields $e_1, \dots, e_n : \mathbb{R}^n \to \mathbb{S}^{n-1}$ such that

$$V(\mu, x) = \operatorname{span}\{e_1(x), \dots, e_k(x)\} \quad \text{and} \quad V(\mu, x)^\perp = \operatorname{span}\{e_{k+1}(x), \dots, e_n(x)\} \quad \text{for every } x \in \widetilde{K}.$$

The aim of this section is to prove the following.

Proposition 3.1. *Let μ and $\widetilde{K}$ be as above. There exists a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ and a Borel set $E \subseteq \widetilde{K}$ of positive μ-measure such that, for μ-almost every $x \in E$, there exists a direction $v \notin V(\mu, x)$ and a sequence of points $y_i = y_i(x) \in \widetilde{K}$ such that*

$$\frac{y_i - x}{|y_i - x|} \to v \quad \textit{and} \quad \limsup_{i\to\infty} \frac{f(y_i) - f(x)}{|y_i - x|} - \liminf_{i\to\infty} \frac{f(y_i) - f(x)}{|y_i - x|} > 0.$$

Writing $\alpha = 1/\sqrt{n}$, we apply Lemma 2.1 with the choice $\vartheta = \sqrt{1-\alpha^2}$ to find a compact subset K_α of $\widetilde{K}$ with positive measure, where

$$\operatorname{supp}(\mu) \cap B(x, r) \setminus X(x, V(\mu, x), \sqrt{1-\alpha^2}) \neq \varnothing \quad \text{for any } r > 0 \text{ and every } x \in K_\alpha. \tag{6}$$

Lemma 3.2. *Let μ and K_α be as above. Then, we can find a compact set $K \subseteq K_\alpha$ of positive μ-measure and a continuous vector field $e : \mathbb{R}^n \to \mathbb{S}^{n-1}$ such that $e(x)$ is orthogonal to $V(\mu, x)$ at μ-almost every $x \in \mathbb{R}^n$ and such that*

$$\operatorname{supp}(\mu) \cap B(x, r) \cap C(e(x), (n-k)^{-1}\alpha) \setminus X(x, V(\mu, x), \sqrt{1-\alpha^2}) \neq \varnothing$$
$$\textit{for any } r > 0 \textit{ and for every } x \in K. \tag{7}$$

Proof. By the choice of α, the cones

$$C(e_{k+1}(x), (n-k)^{-1}\alpha), \dots, C(e_n(x), (n-k)^{-1}\alpha), C(-e_{k+1}(x), (n-k)^{-1}\alpha), \dots, C(-e_n(x), (n-k)^{-1}\alpha)$$

cover $\mathbb{R}^n \setminus X(0, V(\mu, x), \sqrt{1-\alpha^2})$ for every $x \in K_\alpha$. Hence there exists one vector field, which we denote by e, among the $e_{k+1}, \dots, e_n, -e_{k+1}, \dots, -e_n$ for which the set of those $x \in K_\alpha$ where (7) holds has positive μ-measure. □

Definition. Throughout the rest of this section we will let α_0 be as in (6) and we fix $0 < \alpha < \alpha_0$. We also fix the compact set K and the continuous vector field $e : \mathbb{R}^n \to \mathbb{S}^{n-1}$ yielded by Lemma 3.2. We let $e_1, \dots, e_k : \mathbb{R}^n \to \mathbb{S}^{n-1}$ be continuous orthonormal vector fields spanning $V(\mu, x)$ at every $x \in K$ and we complete $\{e_1, \dots, e_k, e\}$ to a basis of $\mathbb{R}^n$ of orthonormal continuous vector fields that we denote by $\{e_1, \dots, e_k, e, e_{k+1}, \dots, e_{n-1}\}$.

Fix a ball $B(0, r)$ such that $K \subset B(0, r-1)$. For any $\beta \in (0, 1)$, we denote by X_β the family of Lipschitz functions $f : B(0, r) \to \mathbb{R}$ such that

$$|D_e f(x)| \le 1 \qquad \text{and} \qquad |D_{e_j} f(x)| \le \beta \quad \text{for any } j = 1, \dots, n-1, \tag{8}$$

for $\mathscr{L}^n$-almost every $x \in \mathbb{R}^n$. We metrize X_β with the supremum norm and note that this make X_β a complete and separable metric space. Note also that X_β is nontrivial as it contains all the β-Lipschitz functions.

In the following definition we introduce some quantities which measure the incremental ratios "detected" by points in the support of μ, at fixed scales and along directions which are outside a cone whose axis is the decomposability bundle.

Definition. For any $\beta > 0$ and any $0 \le \sigma' < \sigma < 1$, we can define on X_β the maps

$$T^+_{\sigma',\sigma} f : x \mapsto \max\left\{\sup\left\{\frac{f(x+v) - f(x)}{|v|} : \sigma' < |v| \le \sigma \text{ and } x + v \in \operatorname{supp}(\mu) \setminus X(x, V(\mu, x), \sqrt{1-\alpha^2})\right\}, -n\right\},$$

$$T^-_{\sigma',\sigma} f : x \mapsto \min\left\{\inf\left\{\frac{f(x+v) - f(x)}{|v|} : \sigma' < |v| \le \sigma \text{ and } x + v \in \operatorname{supp}(\mu) \setminus X(x, V(\mu, x), \sqrt{1-\alpha^2})\right\}, n\right\}.$$

Proposition 3.3. *For any* $0 \le \sigma' < \sigma < 1$, *the functionals*

$$U^\pm_{\sigma',\sigma} f := \int_K T^\pm_{\sigma',\sigma} f(z)\, d\mu(z)$$

are Baire class 1 *on* X_β.

Proof. As a first step we show that the $T^+_{\sigma',\sigma} : X_\beta \to L^1(\mu \llcorner K)$ are continuous whenever $0 < \sigma' < \sigma < 1$. The functions $T^+_{\sigma',\sigma} f$ belong to $L^1(\mu \llcorner K)$ since K has finite measure and

$$|T^+_{\sigma',\sigma} f| \le \operatorname{Lip}(f) + n.$$

In addition, it is immediate to see that

$$|T^+_{\sigma',\sigma} f(x) - T^+_{\sigma',\sigma} g(x)| \le \frac{2\|f - g\|_\infty}{\sigma'} \quad \text{for } \mu\text{-almost every } x \in \mathbb{R}^n,$$

thanks to the fact that if at some $x \in \mathbb{R}^n$ we have

$$(B(x,\sigma) \setminus B(x,\sigma')) \cap (\mathrm{supp}(\mu) \setminus X(x, V(\mu,x), \sqrt{1-\alpha^2})) = \varnothing,$$

then $T_{\sigma',\sigma} f(x) = -n$ for any $f \in X_\beta$. Integrating with respect to μ, we infer that

$$\|T^+_{\sigma',\sigma} f(x) - T^+_{\sigma',\sigma} g(x)\|_{L^1(\mu \llcorner K)} \le \frac{2\mu(K)}{\sigma'} \|f-g\|_\infty.$$

This implies in particular that $U^+_{\sigma',\sigma}$ is a continuous functional on X_β. Following verbatim the argument above, one can also prove the continuity of the functionals $T^-_{\sigma',\sigma}$ and $U^-_{\sigma',\sigma}$.

In order to prove that $U^\pm_{0,\sigma}$ is of Baire class 1, thanks to [Kechris 1995, §24.B] we just need to show that, for any $f \in X_\beta$, we have

$$\lim_{j\to\infty} U^\pm_{j^{-1},\sigma} f = U^\pm_{0,\sigma} f. \tag{9}$$

This is an immediate consequence of the dominated convergence theorem since the sequence $(T^\pm_{j^{-1},\sigma} f)_j$ converges pointwise to $T^\pm_{0,\sigma} f$ and is dominated by the function constantly equal to n. □

We are now ready to prove the main result of the section, namely the fact that X_β contains *plenty* of Lipschitz functions whose nondifferentiability at some points of K is "detected" by points in the support of μ.

Proposition 3.4. *Let $\beta < (8n^2)^{-1}\alpha$. Then, for every $\sigma > 0$, the continuity points of $U^\pm_{0,\sigma}$ are contained in the set*

$$\mathcal{L}_\pm(\sigma) := \left\{ f \in X_\beta : \pm U^\pm_{0,\sigma} f \ge \frac{\alpha}{16n}\mu(K) \right\}.$$

In particular both $\mathcal{L}_+(\sigma)$ and $\mathcal{L}_-(\sigma)$ are residual in X_β.

Let us briefly explain here the idea of the proof. In our reduction, for every point $x \in K$ at any small scale, there is a point $y \in \mathrm{supp}(\mu)$ such that $y - x$ is far away from $V(\mu, x)$; see Lemma 3.2. Hence the point y is not reached by Lipschitz curves passing through x and lying inside $\mathrm{supp}(\mu)$. By Proposition 2.2, we can find a Lipschitz function ω with small supremum norm which "jumps" with high derivative along the segment $[x, y]$ for any such point y. Assuming by contradiction that at a continuity point $g \in X_\beta$ the value of $U^+_{0,\sigma}$ is below a certain threshold, we reach a contradiction perturbing g by adding ω, so that the value of $U^+_{0,\sigma}$ increases significantly.

Proof. We prove the result just for $U^+_{0,\sigma}$. The argument to prove the analogous statement for $U^-_{0,\sigma}$ can be obtained following verbatim that for $U^+_{0,\sigma}$ while making suitable changes of sign.

Assume for contradiction that g is a continuity point for $U^+_{0,\sigma}$ contained in $X_\beta \setminus \mathcal{L}_+(\sigma)$. It is easy to see by convolution that smooth functions are dense in X_β. Since g is a continuity point for $U^+_{0,\sigma}$, for any $\ell \in \mathbb{N}$, we can find a smooth function $h_\ell \in X_\beta$ such that

$$\|g - h_\ell\|_\infty \le 2^{-\ell} \quad \text{and} \quad U^+_{0,\sigma} h_\ell \le \alpha\mu(K)/(8n),$$

and, for *any* $x \in \mathbb{R}^n$, we have

$$|D_e h_\ell(x)| \le 1 \qquad \text{and} \qquad |D_{e_j} h_\ell(x)| \le \beta \quad \text{for any } j = 1, \dots, n-1.$$

Let

$$A := \{y \in K : T^+_{0,\sigma} h_\ell(y) \le \alpha/(8n)\}.$$

Thanks to Besicovitch's covering theorem and [Alberti and Marchese 2016, Lemma 7.5], we can cover μ-almost all A with countably many closed and disjoint balls $\{B(y_j, r_j)\}_{j\in\mathbb{N}}$ such that, for $0 < \eta, \chi < (n2^{10\ell})^{-1}\beta^2$,

(i) $r_j \le 2^{-\ell}$, $\mu(A \cap B(y_j, r_j)) \ge (1-\eta)\mu(B(y_j, r_j))$ and $\mu(\partial B(y_j, r_j)) = 0$,

(ii) for any $z \in B(y_j, r_j)$,

$$|e(z) - e(y_j)| + |\nabla h_\ell(y_j) - \nabla h_\ell(z)| + \left|\frac{h_\ell(z) - h_\ell(y_j)}{|z - y_j|} - \nabla h_\ell(z)\left[\frac{z - y_j}{|z - y_j|}\right]\right| \le \chi^4,$$

(iii) for any $j \in \mathbb{N}$, we can find $0 < \rho_j < (n2^\ell)^{-1}\beta^2$ and a compact subset $\tilde{A}_j$ of $A \cap B(y_j, (1-2\rho_j)r_j)$ such that $\mu(\tilde{A}_j) \ge (1-2\eta)\mu(B(y_j, r_j))$ and $\tilde{A}_j$ is $C(e(y_j), 2^{-10\ell}\chi^2)$-null.

For any $j \in \mathbb{N}$, we let ϕ_j be a smooth $2(\rho_j r_j)^{-1}$-Lipschitz function such that $0 \le \phi_j \le 1$, $\phi_j = 1$ on $B(y_j, (1-\rho_j)r_j)$ and it is supported on $B(y_j, r_j)$. Now fix $0 < \varepsilon < \beta\chi^2$. Thanks to Proposition 2.2 we can find $\delta_j \le 2^{-j}\rho_j r_j$ and a function ω_j such that:

(1) $0 \le \omega_j(x) \le \varepsilon\beta\rho_j r_j$ for any $x \in \mathbb{R}^n$.

(2) $\omega_j(x) \le \omega_j(x + se(y_j)) \le \omega_j(x) + s$ for every $s > 0$ and any $x \in \mathbb{R}^n$. Moreover, if the segment $[x, x + se(y_j)]$ is contained in $B(\tilde{A}_j, \delta_j)$, then $\omega_j(x + se(y_j)) = \omega_j(x) + s$.

(3) $|\omega_j(x + v) - \omega_j(x)| \le 2^{-9\ell}\chi^2|v|$ for every $v \in e(y_j)^\perp$.

(4) ω_j is $1 + 2^{-9\ell}\chi^2$-Lipschitz.

We thus define the function g_ℓ as

$$g_\ell := (1-2\chi)\left(h_\ell + \sum_{j\in\mathbb{N}}[-\langle\nabla h_\ell(y_j), e(y_j)\rangle + 1]\phi_j\omega_j\right). \tag{10}$$

First we estimate the supremum distance

$$\begin{aligned}
\|g - g_\ell\|_\infty &\le \|g - h_\ell\|_\infty + 2\chi\|h_\ell\|_\infty + (1-2\chi)\|h_\ell - (1-2\chi)^{-1}g_\ell\|_\infty \\
&\le 2^{-\ell} + \chi(\|g\|_\infty + 2^{-\ell}) + (1-2\chi)\left\|\sum_{j\in\mathbb{N}}(1 - \langle\nabla h_\ell(y_j), e(y_j)\rangle)\right\|_\infty \\
&\le 2^{-\ell}(2 + \|g\|_\infty + (1 + (n-1)\beta^2)^{1/2}) \le 2^{-\ell}(4 + \|g\|_\infty),
\end{aligned} \tag{11}$$

where the last inequality follows from the choice of β. The above computation shows that the sequence g_ℓ converges in the supremum distance.

Let us now prove that $g_\ell \in X_\beta$. If $z \notin \bigcup_j B(y_j, r_j)$, then the functions h_ℓ and g_ℓ and their gradients coincide at z and hence g_ℓ satisfies (8) on $\left(\bigcup_j B(y_j, r_j)\right)^c$. If on the other hand $z \in \bigcup_j B(y_j, r_j)$, there exists a unique $j \in \mathbb{N}$ such that $z \in B(y_j, r_j)$. In particular, differentiating (10) we get

$$\nabla g_\ell(z) = (1-2\chi)\big[\nabla h_\ell(z) + [-\langle \nabla h_\ell(y_j), e(y_j)\rangle + 1]\nabla\phi_j(z)\omega_j(z) \\ + [-\langle \nabla h_\ell(y_j), e(y_j)\rangle + 1]\phi_j(z)\nabla\omega_j(z)\big],$$

so that, for $\mathscr{L}^n$-almost every $x \in \mathbb{R}^n$, we have

$$|\langle \nabla g_\ell(z), e(z)\rangle| \le (1-2\chi)|\langle \nabla h_\ell(z), e(z)\rangle + [-\langle \nabla h_\ell(y_j), e(y_j)\rangle + 1]\phi_j(z)\langle \nabla\omega_j(z), e(z)\rangle| + 2\varepsilon\beta,$$

where in the estimate above we have used the facts that

$$|-\langle \nabla h_\ell(y_j), e(y_j)\rangle + 1| \le 2, \quad \|\nabla\phi\|_{L^\infty(\mathscr{L}^n)} \le 2(\rho_j r_j)^{-1} \quad \text{and} \quad \|\omega_j\|_\infty \le \varepsilon\beta\rho_j r_j.$$

Now we replace z with y_j in the first addendum, by means of the estimate (ii), obtaining

$$|\langle \nabla g_\ell(z), e(z)\rangle| \le 3(1-2\chi)\chi^2 + (1-2\chi)\big|\langle \nabla h_\ell(y_j), e(y_j)\rangle(1-\phi_j(z)\langle \nabla\omega_j(z), e(z)\rangle) \\ + \phi_j(z)\langle \nabla\omega_j(z), e(z)\rangle\big| + 2\varepsilon\beta.$$

Finally, substituting z with y_j in the argument of the vector field e, we deduce thanks to (ii) that

$$\begin{aligned}|\langle \nabla g_\ell(z), e(z)\rangle| &\le 3(1-2\chi)\chi^2 + 2\varepsilon\beta + 6(1-2\chi)(1+2^{-9\ell}\chi)\chi^2 \\ &\quad + (1-2\chi)|\langle \nabla h_\ell(y_j), e(y_j)\rangle(1-\phi_j(z)\langle \nabla\omega_j(z), e(y_j)\rangle) + \phi_j(z)\langle \nabla\omega_j(z), e(y_j)\rangle| \\ &\le 3(1-2\chi)\chi^2 + 2\varepsilon\beta + 6(1-2\chi)(1+2^{-9\ell}\chi)\chi^2 + (1-2\chi) \le 1,\end{aligned}$$

where the last inequality follows from the choice of $\chi,\ \beta,\ \varepsilon$. Furthermore, for any $q = 1, \dots, n-1$, we infer similarly that

$$\begin{aligned}&|g_\ell(z+te_q(z)) - g_\ell(z)| \\ &\quad\le (1-2\chi)|h_\ell(z+te_q(z)) - h_\ell(z)| \\ &\qquad + (1-2\chi)|[1-\langle \nabla h_\ell(y_j), e(y_j)\rangle](\phi_j(z+te_q(z)) - \phi_j(z))\omega_j(z)| \\ &\qquad + (1-2\chi)|[1-\langle \nabla h_\ell(y_j), e(y_j)\rangle]\phi_j(z)(\omega_j(z+te_q(y_j)) - \omega_j(z))| \\ &\qquad + (1-2\chi)|[1-\langle \nabla h_\ell(y_j), e(y_j)\rangle]\phi_j(z)(\omega_j(z+te_q(z)) - \omega_j(z+te_q(y_j)))| + o(|t|) \\ &\quad\le (1-2\chi)\beta|t| + 4(1-2\chi)(\beta\varepsilon\rho_j r_j)(\rho_j r_j)^{-1}|t| + 3\cdot 2^{-9\ell}(1-2\chi)\chi^2|t| \\ &\qquad + 3(1-2\chi)(1+2^{-9\ell}\chi)\chi^4|t| + o(|t|) \\ &\quad\le (1-2\chi)(\beta + 4\beta\varepsilon + 4\cdot 2^{-9\ell}\chi^2 + 4(1+2^{-9\ell}\chi)\chi^4)|t| \\ &\quad\le (1-2\chi)(1+10\chi^2)\beta|t| + o(|t|) < \beta|t|,\end{aligned}$$

provided $|t|$ is chosen sufficiently small (depending on z) and where the second to last inequality holds thanks to the choice of χ, ε and for ℓ large enough that $2^{-\ell} \le \beta$. The above bound implies that, in particular,

$$|\langle \nabla g_\ell(z), e_q(z)\rangle| \le \beta \quad \text{for } \mathscr{L}^n\text{-almost every } x \in \mathbb{R}^n. \tag{12}$$

This concludes the proof that, for ℓ sufficiently large, we have $g_\ell \in X_\beta$.

The next step in the proof is to show that the functions g_ℓ satisfy the inequality $U^+_{0,\sigma} g_\ell \geq \alpha\mu(K)/(8n)$ for ℓ sufficiently large, which contradicts the continuity of $U^+_{0,\sigma}$ at g (recall that we supposed $U^+_{0,\sigma} g \geq \alpha\mu(K)/(16n)$). In order to see this, we first estimate from below the partial derivative of g_ℓ along e on the points of $\tilde{A}_j$ for any j. So, let us fix for any $j \in \mathbb{N}$ a point $z \in \tilde{A}_j$. Then, let $0 < \lambda_0 < \delta_j$ be small enough that $\phi_j(z + \lambda e(z)) = 1$ for any $0 < \lambda < \lambda_0$, and note that

$$
\begin{aligned}
&\langle g_\ell(z+\lambda e(z)) - g_\ell(z), e(z)\rangle \\
&\qquad \geq (1-2\chi)[(h_\ell(z+\lambda e(z)) - h_\ell(z)) + [1 - \langle \nabla h_\ell(y_j), e(y_j)\rangle](\omega_j(z+\lambda e(z)) - \omega_j(z))] \\
&\qquad \geq (1-2\chi)[-\chi^2\lambda + \lambda\langle \nabla h_\ell(z), e(z)\rangle + [1 - \langle \nabla h_\ell(y_j), e(y_j)\rangle]\lambda] \\
&\qquad \geq \lambda(1-2\chi)(1-4\chi^2) \geq (1-6\chi)\lambda.
\end{aligned}
$$

This implies in particular that, for any unit vector $v \in C(e(z), (n-k)^{-1}\alpha)$ and for any $\lambda > 0$, we have

$$
\begin{aligned}
g_\ell(z+\lambda v) - g_\ell(z) &\geq g_\ell(z+\lambda v) - g_\ell(z+\lambda\langle e(z), v\rangle e(z)) + g_\ell(z+\lambda\langle e(z), v\rangle e(z)) - g_\ell(z) \\
&\geq \alpha(n-k)^{-1}(1-6\chi)\lambda - \beta\sqrt{n-1}\lambda \geq \frac{\alpha}{2(n-k)}\lambda - \beta n\lambda > \alpha\frac{\lambda}{4(n-k)}, \qquad (13)
\end{aligned}
$$

where the last inequality follows from the choice of β. However, thanks to the choice of K, see (7), we infer that

$$
T^+_{0,\sigma} g_\ell(z) \geq \frac{\alpha}{4(n-k)} \quad \text{for any } z \in \bigcup_j \tilde{A}_j.
$$

This allows us to infer that

$$
\begin{aligned}
U^+_{0,\sigma} g_\ell &= \int_A T^+_{0,\sigma} g_\ell \, d\mu + \int_{K\setminus A} T^+_{0,\sigma} g_\ell \, d\mu \geq \int_A T^+_{0,\sigma} g_\ell \, d\mu + \alpha\mu(K\setminus A) \\
&= \int_{A\setminus\bigcup_j \tilde{A}_j} T^+_{0,\sigma} g_\ell \, d\mu + \sum_{j\in\mathbb{N}} \int_{A_j} T^+_{0,\sigma} g_\ell \, d\mu + \alpha\mu(K\setminus A) \\
&\geq -\mu\Bigl(A \setminus \bigcup_{j\in\mathbb{N}} A_j\Bigr)\operatorname{Lip}(g_\ell) + \frac{\alpha}{4(n-k)}\mu\Bigl(\bigcup_{j\in\mathbb{N}} A_j\Bigr) + \alpha\mu(K\setminus A) \\
&\geq -2\mu\Bigl(A \setminus \bigcup_{j\in\mathbb{N}} A_j\Bigr) + \frac{\alpha}{4(n-k)}\mu\Bigl((K\setminus A) \cup \bigcup_{j\in\mathbb{N}} A_j\Bigr) \\
&\geq -4\eta\mu(K) + \frac{\alpha}{4(n-k)}(1-2\eta)\mu(K) \geq \frac{\alpha}{8n}\mu(K)
\end{aligned}
$$

for ℓ sufficiently large.

Since the functional $U^+_{0,\sigma}$ is of Baire class 1, thanks to [Oxtoby 1971, Chapter 7] we know that the set of the continuity points of $U^+_{0,\sigma}$ is residual. However, since, thanks to the above argument, $\mathcal{L}_+(\sigma)$ contains the continuity points of $U^+_{0,\sigma}$, we conclude that $\mathcal{L}_+(\sigma)$ is residual in X_β. □

Proof of Proposition 3.1. Let $\beta := (16n^2)^{-1}\alpha$ and let $\mathfrak{c}(\alpha) := \alpha/(16n)$. Note that since the countable intersection of residual sets is residual, we can find a Lipschitz function f in X_β such that $f \in \bigcap_{\sigma\in\mathbb{Q}\cap(0,1)}(\mathcal{L}_+(\sigma) \cap \mathcal{L}_-(\sigma))$. In particular, for any $\sigma > 0$, we have

$$
U^-_{0,\sigma} f \leq -\mathfrak{c}(\alpha)\mu(K) < \mathfrak{c}(\alpha)\mu(K) \leq U^+_{0,\sigma} f.
$$

Letting $\Delta T_\sigma f(z) := T^+_{0,\sigma} f(z) - T^-_{0,\sigma} f(z)$ and $C_\sigma := \{z \in K : \Delta T_\sigma(z) > \mathfrak{c}(\alpha)\}$, we have

$$2\mathfrak{c}(\alpha)\mu(K) \le \int_K \Delta T_\sigma(z)\, d\mu \llcorner K(z) \le \mu(K \setminus C_\sigma)\mathfrak{c}(\alpha) + 2\,\mathrm{Lip}(f)\mu(C_\sigma).$$

Thanks to the above computation we infer in particular that $\mu(C_\sigma) \ge \mathfrak{c}(\alpha)\mu(K)/(2\,\mathrm{Lip}(f))$ for any $\sigma > 0$. Thus, defining $E := \bigcap_{j\in\mathbb{N}} \bigcup_{l\ge j} C_{1/l}$, Fatou's lemma implies that

$$\frac{\mathfrak{c}(\alpha)\mu(K)}{2\,\mathrm{Lip}(f)} \le \limsup_{p\to\infty} \mu(C_{1/p}) \le \int \limsup_{p\to\infty} \mathbb{1}_{C_{1/p}}\, d\mu = \mu(E),$$

where $\mathbb{1}_{C_{1/p}}$ denotes the indicator function of the set $C_{1/p}$. Therefore, E is a Borel set of positive μ-measure such that, for μ-almost every $z \in E$, there exists a sequence of natural numbers (depending on z) such that $p \to \infty$ and $\Delta T_{1/p} > \mathfrak{c}(\alpha)$. In particular, for μ-almost every $z \in E$, we have

$$\mathfrak{c}(\alpha) < \liminf_{p\to\infty} (T^+_{0,1/p} f(z) - T^-_{0,1/p} f(z)) = \lim_{p\to\infty} (T^+_{0,1/p} f(z) - T^-_{0,1/p} f(z)), \tag{14}$$

where the last identity comes from the fact that $p \mapsto T^+_{0,1/p} f(z)$ is decreasing and $p \mapsto T^-_{0,1/p} f(z)$ is increasing for any z. However, thanks to the definitions of $T^+_{0,1/p} f$ and $T^-_{0,1/p} f$, it is immediate to see that, for μ-almost every $z \in E$, we can find a sequence

$$y_i = y_i(z) \in \mathrm{supp}(\mu) \cap B(z, i^{-1}) \setminus X(0, V(\mu, x), \sqrt{1-\alpha^2})$$

such that

$$\frac{y_i - z}{|y_i - z|} \to v \quad \text{and} \quad \limsup_{i\to\infty} \frac{f(y_i) - f(z)}{|y_i - z|} - \liminf_{i\to\infty} \frac{f(y_i) - f(z)}{|y_i - z|} > \frac{\mathfrak{c}(\alpha)}{2}. \qquad \square$$

4. Proof of Theorem 1.1

Without loss of generality we can restrict our attention to finite measures. Assume that μ is a finite sum of rectifiable measures. For every $\varepsilon > 0$, there exist finitely many disjoint, compact submanifolds S_j for $j = 1, \dots, N$ of class C^1 (of any dimension between 0 and n) such that, defining $K := \bigcup_{j=1}^N S_j$, we have $\mu(\mathbb{R}^n \setminus K) < \frac{1}{2}\varepsilon$. Consider now any Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$. By [Alberti and Marchese 2016, Theorem 1.1(i)] and Lusin's theorem, we can find a closed subset $C \subset K$ such that $\mu(K \setminus C) < \frac{1}{2}\varepsilon$ and, for every $x \in C$, the differential $d_{V(\mu,x)} f(x)$, see [Alberti and Marchese 2016, §2.1], exists and is continuous. Let $d : C \to \mathbb{R}^n$ be obtained by extending $d_{V(\mu,\cdot)} f$ to be zero in the directions orthogonal to $V(\mu, \cdot)$. By [Alberti and Marchese 2016, Proposition 2.9(iii)] and since the S_j's have positive mutual distances, we can apply Whitney's extension theorem, see [Evans and Gariepy 1992, Theorem 6.10], deducing that there exists a function $g : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $g = f$ and $dg = d$ on C. Hence Lipschitz functions admit a Lusin-type approximation with respect to μ with functions of class C^1.

Assume now that μ is not a finite sum of rectifiable measures, and write $\mu = \sum_{k=0}^n \mu \llcorner E_k$, where

$$E_k := \{x \in \mathbb{R}^n : \dim(V(\mu, x)) = k\}.$$

Then there exists $k \in \{0, \dots, n-1\}$ such that $\mu \llcorner E_k$ is not a k-rectifiable measure: the case $k = n$ can be excluded by combining [Alberti and Marchese 2016, Theorem 1.1(i)] and [De Philippis and Rindler 2016, Theorem 1.14] so as to ensure that a measure on $\mathbb{R}^n$ whose decomposability bundle has dimension n is absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^n$. Let ν be the supremum of all k-rectifiable measures $\sigma \le \mu \llcorner E_k$, and let E be any Borel set such that $\nu = \mu \llcorner (\mathbb{R}^n \setminus E)$. We claim that $\mu \llcorner E$ satisfies the assumptions of Lemma 2.1.

To prove the claim, consider a k-dimensional surface S that is the graph of some function $h : W \to W^\perp$ of class C^1, where $W \in \mathrm{Gr}(k, n)$. Assume for contradiction that $\eta := \mu \llcorner (E \cap S)$ is nonzero. If

$$G = \{\mu_t := \mathscr{H}^1 \llcorner E_t\}_{t\in I} \in \mathscr{F}_\eta$$

is a family as in [Alberti and Marchese 2016, Proposition 2.8(ii)], then $\mathrm{supp}(\mu_t) \subset S$ for almost every $t \in I$. Since both $V(\eta, x)$ and $\mathrm{Tan}(S, x)$ are k-dimensional, this implies that $V(\eta, x) = \mathrm{Tan}(S, x)$ for η-almost every x. Fix now a point $y \in \mathrm{supp}(\eta)$, and observe that the family $\{\mathscr{H}^1 \llcorner p_W(E_t)\}_{t\in I}$ belongs to $\mathscr{F}_{(p_W)_\sharp\eta}$ (as $(p_W)_\sharp\mu_t$ is absolutely continuous with respect to $\mathscr{H}^1 \llcorner p_W(E_t)$ for any t) and that $V((p_W)_\sharp\eta, \cdot)$ is k-dimensional $(p_W)_\sharp\eta$-almost everywhere. By [De Philippis and Rindler 2016, Corollary 1.12], we infer that $(p_W)_\sharp\eta$ is absolutely continuous with respect to $\mathscr{H}^k \llcorner W$. Finally, since p_W is locally bi-Lipschitz from S to W, this implies that η is absolutely continuous with respect to $\mathscr{H}^k \llcorner S$, which contradicts the maximality of σ. Hence $\mu \llcorner E$ satisfies the assumptions of Lemma 2.1.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be the Lipschitz function obtained from Proposition 3.1. Clearly there exists no function $g : \mathbb{R}^n \to \mathbb{R}$ of class C^1 which coincides with f on a set of positive $\mu \llcorner E$ measure, hence Lipschitz functions do not admit a Lusin-type approximation with respect to μ with functions of class C^1.

Remarks. (1) It is evident from the last lines in the proof of Theorem 1.1 that the condition that g is of class C^1 can be replaced by the condition that g is differentiable everywhere.

(2) In Theorem 1.1 the condition (1) can be strengthened to

$$\mu(\{x \in \mathbb{R}^n : g(x) \ne f(x) \text{ or } d_V g(x) \ne d_V f(x)\}) < \varepsilon, \tag{15}$$

where d_V denotes the "tangential differential" defined in [Alberti and Marchese 2016, Theorem 1.1]. This follows immediately from [De Philippis et al. 2022, Proposition 6.2]; see also [Julia et al. 2023, Theorem B]. On the other hand one cannot replace (1) with the condition

$$\mu(\{x \in \mathbb{R}^n : d_V g(x) \ne d_V f(x)\}) < \varepsilon, \tag{16}$$

since the latter does not force any geometric structure on μ. More precisely, for every Radon measure μ and every Lipschitz function f, for every $\varepsilon > 0$, one can find a function g of class C^1 such that (16) holds; see [Marchese and Schioppa 2019, Theorem 2.1].

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ON THE ENDPOINT REGULARITY IN ONSAGER'S CONJECTURE

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Onsager's conjecture states that the conservation of energy may fail for three-dimensional incompressible Euler flows with Hölder regularity below $\frac{1}{3}$. This conjecture was recently solved by the author, yet the endpoint case remains an interesting open question with further connections to turbulence theory. In this work, we construct energy nonconserving solutions to the three-dimensional incompressible Euler equations with space-time Hölder regularity converging to the critical exponent at small spatial scales and containing the entire range of exponents $\left[0, \frac{1}{3}\right)$.

Our construction improves the author's previous result towards the endpoint case. To obtain this improvement, we introduce a new method for optimizing the regularity that can be achieved by a convex integration scheme. A crucial point is to avoid loss of powers in frequency in the estimates of the iteration. This goal is achieved using localization techniques of Isett and Oh (*Arch. Ration. Mech. Anal.* **221**:2 (2016), 725–804) to modify the convex integration scheme.

We also prove results on general solutions at the critical regularity that may not conserve energy. These include a theorem on intermittency stating roughly that energy dissipating solutions cannot have absolute structure functions satisfying the Kolmogorov–Obukhov scaling for any $p > 3$ if their singular supports have space-time Lebesgue measure zero.

1. Introduction

We consider the endpoint regularity in Onsager's conjecture for the incompressible Euler equations on $\mathbb{R} \times \mathbb{T}^3$, which we write in conservation form as

$$\begin{aligned} \partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p &= 0, \\ \nabla_j v^j &= 0, \end{aligned} \tag{E}$$

using the summation convention for summing repeated indices. We are concerned mainly with weak solutions to the incompressible Euler equations, which are defined most generally as a locally square-integrable vector field v (called the velocity field) and scalar function p (called the pressure) that together satisfy (E) in the sense of distributions.

Onsager's conjecture states that for any Hölder exponent $\alpha < \frac{1}{3}$ there exist periodic weak solutions to the three-dimensional incompressible Euler equations that belong to the Hölder class $v \in L_t^\infty C_x^\alpha$ and fail to conserve the total kinetic energy $\frac{1}{2} \int_{\mathbb{T}^3} |v(t, x)|^2 \, dx$. The endpoint case of the conjecture is that the same statement should hold for $\alpha = \frac{1}{3}$. The above statements originate from [Onsager 1949] on the statistical theory of hydrodynamic turbulence, where he postulated that dissipation of energy may occur

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in the absence of viscosity[1] through the mechanism of an energy cascade modeled by the incompressible Euler equations.

Onsager's argument predicts that such energy dissipation should be possible for incompressible Euler flows with regularity exactly $\frac{1}{3}$. Specifically, Onsager argued that the energy cascade occurring in a turbulent flow will result in an energy spectrum with a statistical power law consistent with exactly the (Besov or Hölder) regularity $\frac{1}{3}$ in the inertial range of frequencies, which agrees with the scaling laws of turbulence predicted by Kolmogorov's theory [1941]. (See also [De Lellis and Székelyhidi 2013a; Eyink and Sreenivasan 2006] for more detailed reviews of these statements and computations.) On the other hand, Onsager asserted that conservation of energy must hold for every incompressible Euler flow $v \in L_t^\infty C_x^\alpha(I \times \mathbb{T}^3)$ with Hölder regularity α strictly above $\frac{1}{3}$. A strengthening of this latter assertion was proved in [Constantin et al. 1994] after initial work in [Eyink 1994], with the sharpest known result being that conservation of energy holds for weak solutions in the Besov class $v \in L_t^3 B^{1/3}_{3,c_0(\mathbb{N})}$ [Cheskidov et al. 2008]. These results leave open the possibility that energy dissipation as considered by Onsager may be possible for solutions to incompressible Euler with exactly the critical regularity $\frac{1}{3}$ (e.g., for weak solutions in the class $v \in C_t C_x^{1/3}$), while the construction in [Eyink 1994] of initial data with critical regularity and nonzero energy flux provides further evidence that dissipation of energy for weak solutions at the critical regularity should indeed exist.

The existence of weak solutions to incompressible Euler equations in the class $v \in L_t^\infty C_x^\alpha(\mathbb{R} \times \mathbb{T}^3)$ that fail to conserve energy has been established by the author for all $\alpha < \frac{1}{3}$ in [Isett 2018]. The solutions are constructed using the method of convex integration, which was first introduced to the incompressible Euler equations by De Lellis and Székelyhidi [2009; 2013b; 2014] and was further developed towards improved partial results towards Onsager's conjecture in [Buckmaster et al. 2015; 2016; Isett 2017a]. The proof in [Isett 2018] relies also on the use of Mikado flows introduced in [Daneri and Székelyhidi 2017] to implement convex integration in combination with a new "gluing approximation" technique.

In the present work, we improve upon the result in [Isett 2018] to construct solutions with borderline regularity that approaches the endpoint case at small length scales while failing to conserve energy. Our main result is the following.

Theorem 1.1. *There exists a weak solution (v, p) to the incompressible Euler equations that has nonempty, compact support in time on $\mathbb{R} \times \mathbb{T}^3$ and belongs to the class $v \in \bigcap_{\alpha<1/3} C^\alpha_{t,x}$. Moreover, one may arrange that v also satisfies an estimate of the form*

$$|v(t, x + \Delta x) - v(t, x)| \le C|\Delta x|^{1/3 - B\sqrt{(\log\log|\Delta x|^{-1})/(\log|\Delta x|^{-1})}} \tag{1}$$

for some constants C and B and for all $(t, x) \in \mathbb{R} \times \mathbb{T}^3$ and all $|\Delta x| \le 10^{-2}$.

The theorem is significant for the following reasons:

- Theorem 1.1 demonstrates how close the method of convex integration can come to achieving the self-similar $L_t^\infty C_x^{1/3}$ regularity that corresponds to the Kolmogorov theory.

[1] A related and important open question is whether such energy dissipating solutions arise as zero viscosity limits of solutions to the Navier–Stokes equations.

- The theorem is the first result proved by convex integration that approaches the endpoint regularity and avoids the strictly positive gap in regularity from the endpoint faced by previous results. In particular, we have that $v \in \bigcap_{\alpha<1/3} C^{\alpha}_{t,x}$ rather than having regularity bounded strictly below the limiting exponent (i.e., $v \in C^{1/3-\epsilon}_{t,x}$ for some $\epsilon > 0$).
- The proof of Theorem 1.1 is based on a new algorithm that optimizes the regularity coming from a convex integration construction, which may be useful for future numerical simulations of convex integration solutions. This algorithm also apparently identifies an evident barrier towards achieving the endpoint regularity exactly using the convex integration method.
- The proof of Theorem 1.1 clarifies which techniques in the literature yield the sharpest regularity.

The constant B, which determines[2] the rate at which the regularity $\frac{1}{3}$ is approached at small scales, can be taken to be $B = 2\sqrt{\frac{2}{3}} + o(1)$, and this bound can be improved to $B = \frac{4}{3} + o(1)$ by combining our methods with the approach to the gluing approximation taken in [Buckmaster et al. 2019a] (see Sections 11–12 below). For comparison, note that inequality (1) with

$$O\biggl(\sqrt{\frac{\log\log|\Delta x|^{-1}}{\log|\Delta x|^{-1}}}\biggr) \quad \text{replaced by} \quad O\biggl(\frac{1}{\log|\Delta x|^{-1}}\biggr)$$

would correspond to exactly the endpoint regularity $L^{\infty}_t C^{1/3}_x$.

The algorithm we develop to prove Theorem 1.1, presented in Sections 11–12, is the main novelty of our paper relating to the construction of solutions. Later on we will discuss theorems that elaborate a general theory of endpoint solutions. We expect that our algorithm can be adapted to give similar borderline regularity results in any known convex integration construction of Hölder-continuous solutions in which loss of powers in the frequency in the estimates can be avoided. In particular, the method is likely to generalize to isometric embeddings as in [Conti et al. 2012] (but not [De Lellis et al. 2018]), to nondegenerate active scalar equations [Isett and Vicol 2015], to the two-dimensional Monge–Ampère equation [Lewicka and Pakzad 2017], and to the surface quasigeostrophic (SQG) equation [Buckmaster et al. 2019b]. In these cases, there is no logarithmic loss in the main lemma and the $\log\log|\Delta x|^{-1}$ term appearing in (1) should be replaced by a large constant. (In the present case, the gluing technique gives rise to a logarithmic loss.) It is hopeful that our algorithm for optimizing the regularity may also be useful for potential applications to simulating convex integration solutions.

To achieve solutions with borderline regularity, it is necessary that the proof avoids losses of powers of the frequency in the estimates of the iteration scheme. An important point in this regard is that the approach to the gluing construction taken in [Isett 2018] obtains estimates that lose only a power of the logarithm of the frequency. These estimates require extending the timescale of the gluing beyond the standard timescale in the local existence theory for incompressible Euler, which would be inversely proportional to some C^{α} norm of the initial velocity gradient. (We note in contrast that the approach taken in [Buckmaster et al. 2019a] leads to loss of powers in the frequency at several points in the proof. These occur both in the gluing and convex integration in parts of the proof where local well-posedness theory,

[2] Note that changing the value of B in (1) corresponds to an inequivalent norm.

Schauder estimates and Calderón–Zygmund commutator estimates are employed.) Still there is one point in the proof in [Isett 2018], which occurs during the convex integration step, where one encounters a loss of powers in the frequency, and it is necessary to modify the convex integration part of the proof to obtain our borderline result. This loss of powers occurs specifically when solving the divergence equation $\nabla_j R^{j\ell} = U^\ell$ for a symmetric tensor $R^{j\ell}$.

To avoid this power loss, we adapt the strategy of [Isett and Oh 2016b] for localizing the convex integration method, which relies on two main modifications to the construction to gain the necessary estimate. The first point is to modify the construction using waves that are localized to small length scales and are each forced to obey the conservation of angular momentum in addition to the conservation of linear momentum. The second point is to make use of a family of operators developed in [Isett and Oh 2016b] that give compactly supported, symmetric solutions to the divergence equation when the necessary conditions for solving the symmetric divergence equation are satisfied. In combination, these modifications allow one to avoid the loss of powers in the frequency that had been present in [Isett 2017a] while enabling the authors to extend previous work of [Isett 2017a] on $\left(\frac{1}{5} - \epsilon\right)$-Hölder Euler flows to the nonperiodic setting of $\mathbb{R} \times \mathbb{R}^3$. Here we adapt these ideas to the present scheme to achieve an analogous improvement in our bounds. We note that it is important for this gain that we rely on the approach to the nonstationary phase estimate based on a parametrix and nonlinear phase functions introduced in [Isett 2017a].

Obtaining the endpoint case of Onsager's conjecture will require further new ideas, and it is of interest to study the behavior of potential energy nonconserving solutions with endpoint regularity and possible approaches to constructing them. A convex integration approach to the endpoint regularity would be possible if something sufficiently close to an "ideal" main lemma can be proven where one has neither logarithmic nor loss of powers in the frequency and the constant in the frequency growth is equal to $\widehat{C} = 1$ (as in a remark of [Isett and Oh 2016b]) or approaches $\widehat{C} = 1$ asymptotically at a rate such that $\sum_k \log \widehat{C}_{(k)}$ converges.

Such a construction appears to be presently out of reach; however, it may be considered favorable that convex integration constructions are able in general to yield solutions whose singularities occupy regions of space with positive volume. As the following theorem demonstrates, singularities with positive Lebesgue measure are necessary for any energy nonconserving solution with critical regularity to exist provided the integrability exponent for this regularity is greater than 3.

Theorem 1.2 (intermittency theorem). *A weak solution (v, p) to incompressible Euler on $I \times \mathbb{T}^d$ or $I \times \mathbb{R}^d$ that dissipates or otherwise fails to conserve energy cannot belong to an endpoint class $v \in L^r_t B^{1/3}_{r,\infty} \cap L^2_{t,x}$ with an integrability exponent $r > 3$ if its singular support has space-time Lebesgue measure zero.*

Here singular support is in the sense of distributions — the closed set whose complement is the largest set on which v is locally C^∞. In fact, the precise theorem we obtain is a sharper result where singular support is improved to singular support relative to the conservative Onsager critical space $L^3_t \dot{B}^{1/3}_{3,c(\mathbb{N})}$ — that is, the closed set whose complement is the largest open set on which v is locally represented by an $L^3_t \dot{B}^{1/3}_{3,c(\mathbb{N})}$ function (see Section 3).

Theorem 1.2 has a special significance in terms of intermittent scaling exponents in turbulence. The K41 theory [Kolmogorov 1941] predicts a scaling law of the form $\langle |v(x+\Delta x) - v(x)|^p \rangle^{1/p} \sim |\Delta x|^{1/3}$ for absolute structure functions (the Kolmogorov–Obukhov law), which mathematically corresponds to $B^{1/3}_{p,\infty}$ control of the velocity field. The idea that "intermittency" (deviations from self-similarity and homogeneity) in the energy dissipation and singular structure of turbulence can lead to the failure of this scaling law for $p \neq 3$ was first attributed to Landau by Kolmogorov in the 1940's (see [Frisch 1991, Section 5]). Moreover, experimental studies have found evidence of such intermittency in the energy dissipation of turbulent flows accompanied by deviations from the Kolmogorov–Obukhov law arising from a multifractal structure [Meneveau and Sreenivasan 1987; 1991; Meneveau et al. 1990]. Theorem 1.2 and its proof provide a rigorous sense in which lower dimensional singularities or energy dissipation in fact logically imply deviations from the Kolmogorov–Obukhov law, thus reinforcing the experimental findings.

Theorem 1.2 is a consequence of two facts that are also new remarks in the literature, which are a local version of the sharp energy conservation criterion in [Cheskidov et al. 2008] and a result on integrability of the energy dissipation measure (see Theorems 3.1 and 3.2 below). One would most likely expect that energy nonconserving solutions exist for the entire spectrum of endpoint spaces above, including the endpoint case of $L^\infty_t C^{1/3}_x$. For a more precise formulation of Theorem 1.2, we refer to Section 3. We also note the works [Buckmaster et al. 2021; Cheskidov and Shvydkoy 2014; 2023; De Rosa and Haffter 2023; Luo and Shvydkoy 2015; Novack and Vicol 2023; Shvydkoy 2018] for further mathematical results related to intermittency.

In addition to having the endpoint regularity, Onsager's paper [1949] describes Euler flows that furthermore have decreasing kinetic energy. Related to this point, we state the following Theorem.

Theorem 1.3. *If* (v, p) *are a weak solution to* (E) *on* $I \times \mathbb{T}^d$, $d \geq 2$, *with* $v \in C_t C^{1/3}_x$ (*or more generally with* $v \in C_t B^{1/3}_{3,\infty}$) *then the total kinetic energy* $e(t) = \int_{\mathbb{T}^d} \frac{1}{2} |v(t,x)|^2 \, dx$ *is* C^1 *in time.*

Theorem 1.3 implies that the task of finding an energy dissipating solution in the class $v \in C_t C^{1/3}_x$ can be reduced to finding any example of a solution in this class that fails to satisfy energy conservation. Such a solution would have total kinetic energy that is either strictly increasing or strictly decreasing on some open interval of time. After possibly reversing time one obtains a solution with a decreasing energy profile on an open interval. For $\alpha < \frac{1}{3}$, the existence of energy-dissipating solutions in $C_t C^\alpha_x$ was proven in [Buckmaster et al. 2019a] by introducing an additional idea in the convex integration part of the proof to prescribe the energy profile of the solutions. We expect that this technique[3] should be possible to extend to the class described by (1) for example by modifying the statement of our main lemma in a way similar to the analysis in [Isett and Oh 2016b; 2017].

The proof of Theorem 1.3, presented in Section 2 below, suggests that the failure of energy conservation for solutions in the critical space $v \in C_t C^{1/3}_x$ should be very common. The proof reduces the existence of an energy-dissipating solution to solving the Euler equations with appropriate initial data in the desired critical space for a short time. However, one must be cautious that the Euler equations are ill-posed

[3] A related technical point is that the approach to prescribing the energy profile in [Buckmaster et al. 2019a] involves requiring the stress tensor $R^{j\ell}$ to be trace-free in addition to being symmetric. It is also possible to prescribe the energy profile without imposing the trace-free requirement on $R^{j\ell}$; see [Isett and Oh 2016b; 2017].

in $C_t C_x^\alpha$ or in $C_t B^\alpha_{3,\infty}$ for all $\alpha < 1$, as has been shown in [Bardos and Titi 2010; Cheskidov and Shvydkoy 2010], which presents a significant difficulty for constructing solutions in these spaces. Complementing these negative results, our proof of Theorem 1.3 yields as a byproduct a necessary condition for a given divergence-free vector field to be the initial datum of a solution in the class $v \in C_t B^{1/3}_{3,\infty}$.

With regard to energy dissipation and solving the Cauchy problem, a natural question is whether there is a simple, Onsager-critical function space in which *local* dissipation of energy is guaranteed along with total kinetic energy dissipation if one can solve the Cauchy problem in that space with the appropriate initial data. A simple criterion of this type is provided in Theorem 3.3 of Section 3A. The regularity condition imposed to maintain local dissipation in this criterion is notably stronger than that assumed to control total kinetic energy in Theorem 1.3, as our Theorem 3.3 involves solutions in the function space $C_t C_x^{1/3}$ rather than assuming only Besov regularity in space.

We now summarize the organization of the paper and the proof of our borderline result, Theorem 1.1. The general theory of endpoint solutions, including Theorems 1.2–3.3, is contained in Sections 2–3A. We then summarize notation for the main body of the paper in Section 4. Sections 5–10 contain the main lemma of the paper and our modification of the convex integration construction of [Isett 2018]. These sections assume familiarity with the convex integration construction in that paper. Section 11 explains the proof of Theorem 1.1 using the main lemma, and presents our new method for optimizing the regularity in a general convex integration scheme. Section 12 outlines how to combine our methods with the approach to the gluing approximation taken in [Buckmaster et al. 2019a] to improve the rate of convergence to the critical exponent in the estimate (1).

2. Regularity of kinetic energy at the critical exponent

We start with a proof of Theorem 1.3 on the C^1 regularity of the kinetic energy profile for solutions of class $C_t B^{1/3}_{3,\infty}$. In the next section we prove Theorem 1.2. We will use the summation convention for summing repeated upper and lower spatial indices, so that $v^\ell v_\ell = |v|^2$ and $\nabla_\ell v^\ell = \operatorname{div} v$.

The proof of Theorem 1.3 is an extension of the argument of [Constantin et al. 1994] for proving energy conservation for weak solutions in the class $v \in L^3_t B^{1/3+\epsilon}_{3,\infty}$ and of a remark in [Isett 2023] on the endpoint case. Namely, suppose that (v, p) is a weak solution to (E) with velocity of class $v \in C_t B^{1/3}_{3,\infty}(I \times \mathbb{T}^d)$, $d \geq 2$, with I an open interval. Let η_ϵ be a standard mollifier in $\mathbb{R}^d$ at length scale ϵ, and let $v^\ell_\epsilon = \eta_\epsilon * v^\ell$ denote the mollification of v in the spatial variables. Then, as in [Constantin et al. 1994], one has (using $v \in C_t L^2_x$) that

$$\frac{d}{dt}\int_{\mathbb{T}^d} \frac{|v|^2(t,x)}{2}\,dx = \lim_{\epsilon\to 0}\frac{d}{dt}\int_{\mathbb{T}^d}\frac{|v_\epsilon|^2(t,x)}{2}\,dx = -\lim_{\epsilon\to 0}\int_{\mathbb{T}^d}\nabla_j(v_\epsilon)_\ell R^{j\ell}_\epsilon(t,x)\,dx, \tag{2}$$
$$R^{j\ell}_\epsilon(t,x) := v^j_\epsilon(t,x)v^\ell_\epsilon(t,x) - \eta_\epsilon * (v^j v^\ell)(t,x),$$

where the convergence in (2) holds in $\mathcal{D}'(I)$. (See [Isett and Oh 2016a, Proof of Theorem 2.2] for a detailed presentation of this point.) The rightmost term in (2) gives rise to the family of trilinear forms $T_\epsilon[v,v,v](t) := \int_{\mathbb{T}^d}\nabla_j(v_\epsilon)_\ell R^{j\ell}_\epsilon(t,x)\,dx$ that satisfy, uniformly in ϵ, the bound

$$|T_\epsilon[u,v,w]|(t) \lesssim \|u(t,\cdot)\|_{B^{1/3}_{3,\infty}}\|v(t,\cdot)\|_{B^{1/3}_{3,\infty}}\|w(t,\cdot)\|_{B^{1/3}_{3,\infty}}, \tag{3}$$

by the commutator estimate of [Constantin et al. 1994]. Using (3), we have that the family of functions $T_\epsilon[v, v, v](t)$ are both uniformly bounded and equicontinuous on every compact subinterval of I, as they satisfy

$$|T_\epsilon[v, v, v](t) - T_\epsilon[v, v, v](t_0)| \lesssim \|v(t,\cdot) - v(t_0,\cdot)\|_{B^{1/3}_{3,\infty}} \|v\|^2_{C_t B^{1/3}_{3,\infty}},$$

and their moduli of continuity can therefore be bounded uniformly in ϵ in terms of the modulus of continuity of $v(t,\cdot)$ into $B^{1/3}_{3,\infty}(\mathbb{T}^d)$ and local bounds for $\|v(t,\cdot)\|_{B^{1/3}_{3,\infty}}$. Consequently, the convergence in (2) is actually uniform-in-t on every open interval J with compact closure in I, as the weak limit in $\mathcal{D}'(J)$, which is unique, must also be achieved uniformly along subsequences by Arzelà–Ascoli. (If the convergence were not uniform, there would exist a subsequence converging uniformly to a continuous function different from (2), which contradicts the weak convergence.) The energy flux in (2), a priori in $\mathcal{D}'(I)$, is thus continuous in t on I, and the kinetic energy profile is therefore C^1 in t on I.

Note that one would typically expect the energy flux given by the right-hand side of (2) to be nonzero at any given time t_0 for a vector field with $v(t_0,\cdot) \in C^{1/3}_x$, as examples of divergence-free initial data $v_0(x) \in C^{1/3}$ for which this limit can be positive are given in [Cheskidov et al. 2008; Eyink 1994].

We note also that our argument provides a necessary condition for a vector field $v_0(x) \in B^{1/3}_{3,\infty}$ to be realized as the initial datum of an Euler flow in the class $v \in C_t B^{1/3}_{3,\infty}$, which is that the limit $\lim_{\epsilon\to 0} T_\epsilon[v_0, v_0, v_0]$ on the right-hand side of (2) must exist and must also be independent of the chosen mollifying kernel η_ϵ, so that the instantaneous rate of energy dissipation is well defined at time 0.

We now turn to the proof of Theorem 1.2.

3. Singularities of dissipative solutions with critical regularity

We now establish Theorem 1.2 on the necessity of positive measure singularities of Onsager critical solutions with integrability exponent $p > 3$ that do not conserve energy, which is an immediate consequence of Theorems 3.1 and 3.2 below. Both theorems are stated in terms of Besov spaces whose basic properties we recall within the proofs. We state the first Theorem 3.1 in a sharp, critical space to make clear the severity of the singularity that is implicitly discussed in Theorem 1.2.

Theorem 3.1. *Let (v, p) be a weak solution to the incompressible Euler equations of class $v \in L^3_{t,x}$ on $I \times \mathbb{T}^d$ or $I \times \mathbb{R}^d$, with I an open interval. Then the distribution*

$$-D[v, p] := \partial_t\big(\tfrac{1}{2}|v|^2\big) + \nabla_j\big(\big(\tfrac{1}{2}|v|^2 + p\big)v^j\big)$$

has support contained in the singular support of v relative to the critical space $L^3_t B^{1/3}_{3,c_0(\mathbb{N})}$.

Here we define the singular support of v relative to the space $L^3_t B^{1/3}_{3,c_0(\mathbb{N})}$ to be the complement of those points $q = (t, x)$ for which there exists an open neighborhood O_q of q on which v is represented by a distribution of class $L^3_t B^{1/3}_{3,c_0(\mathbb{N})}$. We recall the standard characterization of the $B^{1/3}_{r,\infty}$ norm of a vector field on an open set Ω in $\mathbb{R}^d$, which is given by $\|v\|_{L^r(\Omega)} + \sup_{h\in\mathbb{R}^d\setminus\{0\}} |h|^{-1/3}\|v(\cdot - h) - v(\cdot)\|_{L^r_x(\Omega\cap(\Omega+h))}$, and we also recall that $C^\infty(\Omega)$ is dense in $B^{1/3}_{r,c_0(\mathbb{N})}(\Omega)$ with respect to the $B^{1/3}_{r,\infty}$ norm. It is clear that the singular support of v relative to $L^3_t B^{1/3}_{3,c_0(\mathbb{N})}$ is a subset of the usual singular support of v as a distribution.

Related restrictions on the support of $D[v, p]$ under different hypotheses and with different proofs are given in [Cheskidov and Shvydkoy 2014, Theorem 4.3], [Drivas and Nguyen 2018, Theorem 1] and [Bardos et al. 2019, Theorem 3.1].

Our second theorem asserts that weak solutions of class $v \in L^r_t B^{1/3}_{r,\infty}$ for integrability exponents $r > 3$ possess integrability for their corresponding energy dissipation measure $D[v, p]$. The assumptions are given in a way that is sufficient for our application to proving Theorem 1.2.

Theorem 3.2. *Let (v, p) be a weak solution to incompressible Euler of class $v \in L^r_t B^{1/3}_{r,\infty}$ for some $r \geq 3$ on $I \times \mathbb{T}^d$ or $I \times \mathbb{R}^d$, with I an open interval. Then the distribution $D[v, p]$ above is a (signed) measure. If furthermore $r > 3$, this measure is absolutely continuous with respect to the Lebesgue measure, and its Radon–Nikodym derivative is of class $D[v, p] \in L^{r/3}_{t,x}$.*

It will be clear that the proof of Theorem 3.2 does not give absolute continuity in the case $r = 3$. For example, the proof would apply to many other equations such as Burgers', where shock solutions give examples of $L^\infty_t B^{1/3}_{3,\infty}$ solutions for which the corresponding energy dissipation measure is not absolutely continuous. There also exist time-independent divergence-free vector fields demonstrating that our approach would not yield absolute continuity in the $r = 3$ case.[4]

Proof of Theorem 1.2. Let us observe now that Theorem 1.2 follows from Theorems 3.1 and 3.2, focusing on the case of $I \times \mathbb{R}^d$. Namely, if a weak solution (v, p) is of class $v \in L^3_{t,x} \cap L^2_{t,x}$ and does not conserve kinetic energy (meaning that the distribution $e(t) := \frac{1}{2} \int_{\mathbb{R}^d} |v|^2(t, x)\, dx$ is not a constant), then the distribution $D[v, p]$ is well defined and cannot be the 0 distribution. This statement can be checked by verifying that, for any test function $\psi \in C^\infty_c(I)$, by dominated convergence one has

$$\begin{aligned}\langle \psi(t), e'(t)\rangle_{\mathcal{D}'(I)} &= \lim_{R\to\infty} \langle \psi(t)\chi_R(x), -D[v, p]\rangle_{\mathcal{D}'(I\times\mathbb{R}^d)} \\ &:= -\int_I \psi'(t) e(t)\, dt \\ &= -\lim_{R\to\infty} \int_{I\times\mathbb{R}^d} \left[\psi'(t)\chi_R(x)\frac{|v|^2}{2} + \psi(t)\nabla_j\chi_R(x)\left(\frac{|v|^2}{2} + p\right)v^j\right] dt\, dx,\end{aligned}$$

where $\chi_R(x) = \chi(x/R)$ is a rescaled bump function that is equal to 1 in a growing neighborhood of the origin that encompasses the whole space as $R \to \infty$. We use here that $\left(\frac{1}{2}|v|^2 + p\right)v^j$ and $\frac{1}{2}|v|^2$ are both in $L^1_{t,x}(I \times \mathbb{R}^d)$ as $v \in L^2_{t,x} \cap L^3_{t,x}$ and $p = \Delta^{-1}\nabla_j\nabla_\ell(v^j v^\ell) \in L^{3/2}_{t,x}$ by Calderón–Zygmund theory,[5] which implies that $\Delta^{-1}\nabla_j\nabla_\ell$ acts as a bounded operator on $L^{3/2}_{t,x}$ mapping two-tensors to scalars. In fact the weaker condition $(1 + |x|)^{-1}\left(\frac{1}{2}|v|^2 + p\right)v^j \in L^1_{t,x}$ suffices for this proof.

For a solution of class $v \in L^r_t B^{1/3}_{r,\infty}$ with $r > 3$, we have by Theorem 3.2 that $D[v, p]$ is of class $L^{r/3}_{t,x}$. For $D[v, p]$ to be nonzero, the support of $D[v, p]$ as a distribution must then occupy a closed set with positive Lebesgue measure. From Theorem 3.1, the nontrivial support of $D[v, p]$ gives a lower bound for the singular support of v as a distribution, which implies Theorem 1.2. □

[4]R. Shvydkoy, personal communication.

[5]The case of $\mathbb{T}^d$ appears to be less standard than the $\mathbb{R}^d$ case but can be deduced from the $\mathbb{R}^d$ case using the local Calderón–Zygmund theory in $\mathbb{R}^d$ as in [Wang 2003]. See, e.g., [Isett 2017b, Proof of Theorem 6.2].

We now prove Theorem 3.1 along with Theorem 3.2. The proof is a local version of the energy conservation criteria of [Cheskidov et al. 2008; Constantin et al. 1994]. The observation that the proof of energy conservation in [Constantin et al. 1994] can be localized is originally due to [Duchon and Robert 2000] and has recently been of use to several authors in the context of bounded domains [Bardos and Titi 2018; Bardos et al. 2019; Drivas and Nguyen 2018]. Some issues that are not central to our goals here have been avoided as our hypotheses suffice to guarantee $p = \Delta^{-1}\nabla_j\nabla_\ell(v^j v^\ell) \in L^{3/2}_{t,x}$. The norms and function space in what follows refer to the entire space $I \times \mathbb{T}^d$ or $I \times \mathbb{R}^d$ unless otherwise stated. We will focus on the $\mathbb{R}^d$ cases in what follows as the results for $\mathbb{T}^d$ follow from the same proofs.

Proof of Theorems 3.2 and 3.1. Let (v, p) be a weak solution of class $v \in L^r_t B^{1/3}_{r,\infty} \cap L^2_{t,x}$ for some $r \geq 3$. Then

$$v \in L^r_{t,x} \cap L^2_{t,x} \quad \text{and} \quad p = \Delta^{-1}\nabla_j\nabla_\ell(v^j v^\ell) \in L^{r/2}_{t,x}$$

by Calderón–Zygmund theory as before. The key formula we use is the analogue of the formula from [Duchon and Robert 2000] involving the commutator of [Constantin et al. 1994]:

$$\begin{aligned} -D[v, p] = \partial_t\big(\tfrac{1}{2}|v|^2\big) + \nabla_j\big[\big(\tfrac{1}{2}|v|^2 + p\big)v^j\big] &= \lim_{\epsilon\to 0} \nabla_j v_{\epsilon\ell} R^{j\ell}_\epsilon, \\ R^{j\ell}_\epsilon &= \eta_\epsilon * (v^j v^\ell) - v^j_\epsilon v^\ell_\epsilon, \end{aligned} \tag{4}$$

where $v^\ell_\epsilon = \eta_\epsilon * v^\ell$ is a standard mollification of v^ℓ in the spatial variables at length scale ϵ, and the limit (4) holds for any fixed test function on $I \times \mathbb{R}^d$ or $I \times \mathbb{T}^d$.

We first prove Theorems 3.2 and 3.1 assuming (4). By Hölder's inequality with $3/r = 1/r + 2/r$ and the commutator estimates of [Constantin et al. 1994], one has the following bound uniformly in ϵ:

$$\begin{aligned} \|\nabla_j v_{\epsilon\ell} R^{j\ell}_\epsilon\|_{L^{r/3}_{t,x}} \leq \|\nabla_j v_{\epsilon\ell}\|_{L^r_{t,x}} \|R^{j\ell}_\epsilon\|_{L^{r/2}_{t,x}} &\lesssim (\epsilon^{-1+1/3}\|v\|_{L^r_t B^{1/3}_{r,\infty}}) \|R^{j\ell}_\epsilon\|_{L^{r/2}_{t,x}} \\ &\lesssim (\epsilon^{-1+1/3}\|v\|_{L^r_t B^{1/3}_{r,\infty}}) \epsilon^{2/3} \|v\|^2_{L^r_t B^{1/3}_{r,\infty}} \lesssim \|v\|^3_{L^r_t B^{1/3}_{r,\infty}}. \end{aligned} \tag{5}$$

The sequence $\nabla_j v_{\epsilon\ell} R^{j\ell}_\epsilon$ is therefore uniformly bounded in $L^{r/3}_{t,x}$ independent of $\epsilon > 0$.

As a consequence, using $r \geq 3$, the weak limit $D[v, p] = \lim_{\epsilon\to 0} \nabla_j v_{\epsilon\ell} R^{j\ell}_\epsilon$ is a Radon measure. That is, by (5) and Hölder's inequality (with the characteristic function of K as one of the factors), for any compact set K and any test function $\phi(t, x)$ supported in K, one has

$$|\langle \phi, D[v, p]\rangle_{\mathcal{D}'(I\times\mathbb{R}^d)}| \leq C_K \|\phi\|_{C^0} \|v\|^3_{L^r_t B^{1/3}_{r,\infty}}.$$

Moreover, for $r > 3$, the measure $D[v, p]$ is absolutely continuous with density function in $L^{r/3}_{t,x}$ by the duality characterization of the latter space, thus confirming Theorem 3.2. Namely, if $s \in (1, \infty)$ is the dual exponent with $1/s + 3/r = 1$, we have

$$|\langle \phi, D[v, p]\rangle_{\mathcal{D}'(I\times\mathbb{R}^d)}| \leq C \|\phi\|_{L^s_{t,x}} \|v\|^3_{L^r_t B^{1/3}_{r,\infty}}.$$

From the density of test functions in $L^s_{t,x}$, we have that $D[v, p]$ is in the dual of $L^s_{t,x}$, which is the space $L^{r/3}_{t,x}$.

The proof of Theorem 3.1 is more subtle as the statement concerns the function space $L^3_t B^{1/3}_{3,c_0(\mathbb{N})}$ and is more local in nature. In particular, our approach is local as compared to the Fourier-analytic approach of [Cheskidov et al. 2008]; the details in the presentation below are similar to those of [Isett and Oh 2016a].

Let $v \in L^3_{t,x}$ be a weak solution, so that $p \in L^{3/2}_{t,x}$, and let q be a point in the complement of the singular support of v relative to $L^3_t B^{1/3}_{3,c_0(\mathbb{N})}$. That is, there is an open neighborhood of q that can be taken to have the form $J \times B_q$ with J a finite open subinterval of I and B_q a spatial ball such that $v \in L^3_t B^{1/3}_{3,c_0(\mathbb{N})}(J \times B_q)$. Let $\phi \in C^\infty_c(J \times B_q)$ be a fixed test function and $B'_q \subseteq B_q$ be a smaller spatial ball such that $\operatorname{supp} \phi \subseteq J \times B'_q$. From (4), we have

$$\langle \phi, -D[v, p]\rangle = \lim_{\epsilon\to 0} \int_J \int_{B'_q} \phi(t, x) \nabla_j v_{\epsilon\ell} R^{j\ell}_\epsilon \, dx\, dt,$$

where by assumption $v \in L^3_t B^{1/3}_{3,c_0(\mathbb{N})}(J \times B_q)$. Then as in the proof of (5) one has that

$$|\langle \phi, -D[v, p]\rangle| \le \limsup_{\epsilon\to 0} \|\phi\|_{C^0} \int_J \|\nabla v_\epsilon(t, \cdot)\|_{L^3(B'_q)} \|R^{j\ell}_\epsilon(t, \cdot)\|_{L^{3/2}(B'_q)}\, dt, \tag{6}$$

and that the dt integrand is bounded uniformly in ϵ by $C\|v(t, \cdot)\|^3_{B^{1/3}_{3,\infty}(B_q)}$, which is integrable over J. Moreover, for almost every $t \in J$, one has that $v(t, \cdot) \in B^{1/3}_{3,c_0(\mathbb{N})}$ belongs to the closure of $C^\infty(B_q)$ in the $B^{1/3}_{3,\infty}$ norm. For each such t, the improved bound

$$\limsup_{\epsilon\to 0} \epsilon^{1-1/3} \|\nabla v_\epsilon(t, \cdot)\|_{L^3(B'_q)} = 0$$

holds, as can be seen by a smooth approximation argument. Combined with $\|R^{j\ell}_\epsilon(t, \cdot)\|_{L^{3/2}(B'_q)} \le C_t \epsilon^{2/3}$ on the same set of t, we have the convergence to zero for almost every t in (6), which implies the limit in (6) is zero by the Lebesgue dominated convergence theorem.

The last remaining point is to justify the limit in (4) for any fixed test function, which we prove using the definition of a weak solution following details similar to [Isett and Oh 2016a]. Let (v, p) be a weak solution of class $v \in L^3_{t,x}$, so that $p \in L^{3/2}_{t,x}$ on $I \times \mathbb{R}^d$ as before. Let $\phi \in C^\infty_c$ be a test function on $I \times \mathbb{R}^d$ and V_ϕ be an open set with compact closure in $I \times \mathbb{R}^d$ that contains $\operatorname{supp} \phi$. Let $\eta_\epsilon(h) = \epsilon^{-d}\eta(h/\epsilon)$ and $\zeta_\delta(\tau) = \delta^{-1}\zeta(\tau/\delta)$ be even mollifying kernels in the space and time variables, respectively, with respective supports $\operatorname{supp} \eta_\epsilon \subseteq B_\epsilon(0)$ in $\mathbb{R}^d$ and $\operatorname{supp} \zeta_\delta \subseteq B_\delta(0)$ in $\mathbb{R}$. Define $\eta_{\epsilon\delta}(\tau, h) = \zeta_\delta(\tau)\eta_\epsilon(h)$ and the vector field $\omega^\ell_{\epsilon\delta} = \eta_{\epsilon\delta} * (\phi\eta_{\epsilon\delta} * v^\ell)$, where the convolution is in both space and time. We will write $*_x$ or $*_t$ to mean convolution in only the space or time variables. Taking $\omega^\ell_{\epsilon\delta}$ as our test function in the weak formulation of Euler (i.e., multiplying the equation and integrating by parts) gives

$$-\int_{I\times\mathbb{R}^d} [v^\ell \partial_t \eta_{\epsilon\delta} * (\phi\eta_{\epsilon\delta} * v_\ell) + v^j v^\ell \nabla_j \eta_{\epsilon\delta} * (\phi\eta_{\epsilon\delta} * v_\ell) + p\nabla^\ell \eta_{\epsilon\delta} * (\phi\eta_{\epsilon\delta} * v_\ell)]\, dx\, dt = 0.$$

Using the self-adjointness of $\eta_{\epsilon\delta}*$ and the divergence-free property of $\eta_{\epsilon\delta} * v^\ell$, one obtains

$$-\int_{I\times\mathbb{R}^d} \left[\partial_t \phi(t, x) \frac{|\eta_{\epsilon\delta} * v^\ell|^2}{2} + (v^j v^\ell)\eta_{\epsilon\delta} * \nabla_j[\phi\eta_{\epsilon\delta} * v_\ell] + p\eta_{\epsilon\delta} * (\nabla^\ell \phi\eta_{\epsilon\delta} * v_\ell)\right] dx\, dt = 0.$$

As $v \in L^3_{t,x} \cap L^2_{t,x}(V_\phi)$ and $p \in L^{3/2}_{t,x}(V_\phi)$, we may safely let $\delta \to 0$ at this point with $\epsilon > 0$ fixed using uniform-in-δ boundedness of the convolution operators in the formula (including the operators $\nabla_j \eta_{\epsilon\delta} *$ that appear from the product rule) and the strong convergence of $\eta_{\epsilon\delta} * v^\ell \to v^\ell_\epsilon := \eta_\epsilon *_x v^\ell$ in $L^2_{t,x} \cap L^3_{t,x}(\operatorname{supp}\phi)$ for each fixed $\epsilon > 0$. Taking the $\delta \to 0$ limit, we may replace each appearance of $\eta_{\epsilon\delta} * = \eta_\epsilon *_x [\zeta_\delta *_t \cdot\,]$ in the formula with $\eta_\epsilon *_x$, which we now write more simply as $\eta_\epsilon * := \eta_\epsilon *_x$.

Using the self-adjointness of $\eta_\epsilon *$ and the divergence-free property of v^ℓ_ϵ, which are justified by the same limiting argument, one then obtains

$$-\int_{I\times\mathbb{R}^d}\left[\partial_t\phi(t,x)\frac{|\eta_\epsilon * v^\ell|^2}{2} + \nabla_j\phi(t,x)\left(\frac{|v_\epsilon|^2}{2}v^j_\epsilon + \eta_\epsilon * p v^j_\epsilon\right)\right]dx\,dt = \int_{I\times\mathbb{R}^d}\phi(t,x)\nabla_j v_{\epsilon\ell} R^{j\ell}_\epsilon\,dx\,dt + Z_\epsilon,$$

$$Z_\epsilon := \int_{I\times\mathbb{R}^d}\nabla_j\phi R^{j\ell}_\epsilon v_{\epsilon\ell}\,dx\,dt.$$

Note that the left-hand side of the first equation tends to exactly $\langle\phi, -D[v,p]\rangle_{\mathcal{D}'(I\times\mathbb{R}^d)}$ as $\epsilon \to 0$, using that $v^\ell_\epsilon = \eta_\epsilon * v^\ell \to v^\ell$ in $L^3_{t,x} \cap L^2_{t,x}(V_\phi)$ and that $p \in L^{3/2}_{t,x}$ again. Thus formula (4) will be proven once it is shown that $\lim_{\epsilon\to 0} Z_\epsilon = 0$.

To this end, write $R^{j\ell}_\epsilon$ in terms of bilinear operators $R^{j\ell}_\epsilon = B_\epsilon[v^j, v^\ell]$, where the operators B_ϵ are defined for smooth u^j and w^ℓ by $B_\epsilon[u^j, w^\ell] := \eta_\epsilon * (u^j w^\ell) - \eta_\epsilon * u^j \eta_\epsilon * w^\ell$. One has then that

$$\|B_\epsilon[u,w]\|_{L^{3/2}_{t,x}(V_\phi)} \to 0 \quad \text{as } \epsilon \to 0$$

whenever u^j and w^ℓ are smooth vector fields on $I \times \mathbb{R}^d$, and that

$$\|B_\epsilon[u,w]\|_{L^{3/2}_{t,x}(V_\phi)} \le C\|u\|_{L^3_{t,x}}\|w\|_{L^3_{t,x}(I\times\mathbb{R}^d)}$$

uniformly in $\epsilon > 0$. Combining these properties and using the density of smooth vector fields in $L^3_{t,x}(I\times\mathbb{R}^d)$, we obtain that $\|R^{j\ell}_\epsilon\|_{L^{3/2}_{t,x}(V_\phi)} \to 0$ as $\epsilon \to 0$, and $Z_\epsilon \to 0$ as well by applying Hölder's inequality with v_ϵ bounded in $L^3_{t,x}(V_\phi)$. □

3A. *Stability of local energy dissipation in a critical class.* In this section we prove Theorem 3.3, which provides a simple function space criterion from which one can deduce local dissipation on an open interval of time from local dissipation at time 0.

Theorem 3.3. *Let $\bar{v}$ be a divergence-free vector field of class $\bar{v} \in C^{1/3}(\mathbb{T}^d)$ for which the local energy dissipation is everywhere bounded by a strictly negative constant. Then any weak solution (v, p) of class $v \in C_t C^{1/3}(I \times \mathbb{T}^d)$ that obtains the initial data $\bar{v}$ must satisfy the local energy inequality $D[v,p] < 0$ on some open time interval containing $t = 0$.*

The precise condition on the initial data $\bar{v}$ will be specified in line (8) of the proof below.

Proof. Let $\bar{v}$ be as above and let (v, p) be a weak solution to the Euler equations of class $v \in C_t C^{1/3}(I\times\mathbb{T}^d)$ on an open interval of time containing $t = 0$ with initial data $\bar{v}(x)$. Let $\tilde{I}$ be an open subinterval of I containing $t = 0$, and let $\phi \in C^\infty_c(\tilde{I}\times\mathbb{T}^d)$ be a nonnegative test function supported in $t \in \tilde{I}$. As in the previous sections, we have

$$\langle\phi, D[v,p]\rangle = \lim_{\delta\to 0}\int_{\tilde{I}}\int_{\mathbb{T}^d}\phi(t,x)T_\delta[v](t,x)\,dx\,dt, \qquad T_\delta[v](t,x) = \nabla_j v_{\delta\ell} R^{j\ell}_\delta(t,x),$$

where T_δ is the trilinear form from Section 2 and $D[v, p]$ is as in the previous sections. We write

$$\langle \phi, D[v,p]\rangle = \lim_{\delta\to 0}\int_{\tilde{I}}\int_{\mathbb{T}^d} \phi(t,x)(T_\delta[v](t,x) - T_\delta[v](0,x))\,dx\,dt \\ + \lim_{\delta\to 0}\int_{\tilde{I}}\int_{\mathbb{T}^d} \phi(t,x) T_\delta[v](0,x)\,dx\,dt. \tag{7}$$

The precise assumption placed on the initial condition $\bar{v}$ is that

$$\lim_{\delta\to 0} T_\delta[v](0,x) = \lim_{\delta\to 0} \nabla_j \bar{v}_{\delta\ell} R_\delta^{j\ell}(0,x) \le -\varepsilon < 0 \tag{8}$$

in the sense of distributions on $\mathbb{T}^d$ for some constant $\varepsilon > 0$. Integrating (8) against the nonnegative test function

$$\tilde{\phi}(x) = \int_{\tilde{I}} \phi(t,x)\,dt \in C^\infty(\mathbb{T}^d),$$

we have that

$$\text{second term of (7)} \le -\varepsilon \int_{\mathbb{T}^d}\left[\int_{\tilde{I}} \phi(t,x)\,dt\right] dx.$$

For a sufficiently small time interval $\tilde{I}$, we can obtain the bound

$$\sup_{\delta>0} \|T_\delta[v](t,x) - T_\delta[v](0,x)\|_{L^\infty(\tilde{I}\times\mathbb{T}^d)} \le \tfrac{1}{2}\varepsilon$$

using that the T_δ are uniformly bounded trilinear forms mapping $C^{1/3}$ to L^∞, the assumption that $v \in C_t C_x^{1/3}$ is continuous in time with values in $C^{1/3}$, and the commutator estimate of [Constantin et al. 1994] to control the bilinear term. Combining these estimates with the sign condition on ϕ gives

$$\langle \phi, D[v,p]\rangle \le \|\phi\|_{L^1(\tilde{I}\times\mathbb{T}^d)}\frac{\varepsilon}{2} - \varepsilon\int_{\tilde{I}}\int_{\mathbb{T}^d} \phi(t,x)\,dx\,dt \\ \le -\frac{\varepsilon}{2}\int_{\tilde{I}}\int_{\mathbb{T}^d} \phi(t,x)\,dx\,dt$$

for all nonnegative $\phi \in C_c^\infty(\tilde{I}\times\mathbb{T}^d)$. This bound shows that $D[v,p] \le -\frac{1}{2}\varepsilon < 0$ as a distribution when restricted to $\tilde{I}\times\mathbb{T}^d$, which concludes the proof of Theorem 3.3. □

With Theorems 1.2–3.3 now proven, we turn to the notation that will be used for the remainder of the paper and the proof of Theorem 1.1.

4. Notation

We will follow the same notational conventions as introduced in [Isett 2018, Section 2]. In particular, multi-indices will be represented in vector notation. For example, if $\vec{a} = (a_1, a_2, a_3)$ is a multi-index of order $|\vec{a}| = 3$, each $a_i \in \{1,2,3\}$, then $\nabla_{\vec{a}} = \nabla_{a_1}\nabla_{a_2}\nabla_{a_3}$ denotes the corresponding third-order partial derivative operator. We use $\mathrm{supp}_t\, f$ to indicate the time support of a function f with domain in $\mathbb{R}\times\mathbb{T}^3$ (i.e., the closed set of times for which $\{t\}\times\mathbb{T}^3$ intersects the usual support).

We recall the definitions of an Euler–Reynolds flow and frequency-energy levels.

Definition 4.1. A vector field $v^\ell : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}^3$, function $p : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$ and symmetric tensor field $R^{j\ell} : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}^3 \otimes \mathbb{R}^3$ satisfy the *Euler–Reynolds equations* if the equations

$$\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p = \nabla_j R^{j\ell},$$
$$\nabla_j v^j = 0$$

hold on $\mathbb{R} \times \mathbb{T}^3$. Any solution to the Euler–Reynolds equations (v, p, R) is called an *Euler–Reynolds flow*. The symmetric tensor field $R^{j\ell}$ is called the *stress* tensor.

Definition 4.2. Let (v, p, R) be a solution of the Euler–Reynolds equation, $\Xi \geq 3$ and $e_v \geq e_R > 0$ be positive numbers. We say that (v, p, R) have *frequency-energy levels* bounded by (Ξ, e_v, e_R) to order L in C^0 if v and R are of class $C_t C_x^L$ and the following estimates hold:

$$\|\nabla_{\vec{a}} v\|_{C^0} \leq \Xi^{|\vec{a}|} e_v^{1/2} \quad \text{for all } 1 \leq |\vec{a}| \leq L,$$
$$\|\nabla_{\vec{a}} R\|_{C^0} \leq \Xi^{|\vec{a}|} e_R \quad \text{for all } 0 \leq |\vec{a}| \leq L.$$

Here ∇ refers only to derivatives in the spatial variables.

5. The main lemma

The first goal of the paper will be to improve on the main lemma in [Isett 2018], so that we remove the need for a double exponential growth of frequencies. The main lemma of our paper states the following:

Lemma 5.1 (main lemma). *Let $L = 3$. There exists constants $\widehat{C}$ and C_L such that the following holds. Let (v, p, R) be any solution of the Euler–Reynolds equation with frequency-energy levels bounded by (Ξ, e_v, e_R) to order L in C^0, and let J be an open subinterval of $\mathbb{R}$ such that*

$$\operatorname{supp}_t v \cup \operatorname{supp}_t R \subseteq J.$$

Define the parameter $\widehat{\Xi} = \Xi (e_v/e_R)^{1/2}$. Let N be any positive number obeying the condition

$$N \geq (e_v/e_R)^{1/2}. \tag{9}$$

Then there exists a solution (v_1, p_1, R_1) of Euler–Reynolds with frequency-energy levels bounded by

$$(\Xi', e_v', e_R') = \left(\widehat{C} N \Xi, (\log \widehat{\Xi}) e_R, (\log \widehat{\Xi})^{5/2} \frac{e_v^{1/2} e_R^{1/2}}{N} \right) \tag{10}$$

to order L in C^0 such that

$$\operatorname{supp}_t v_1 \cup \operatorname{supp}_t R_1 \subseteq N(J; \Xi^{-1} e_v^{-1/2})$$

and such that the correction $V = v_1 - v$ obeys the estimate

$$\|V\|_{C^0} \leq C_L (\log \widehat{\Xi})^{1/2} e_R^{1/2}. \tag{11}$$

The crucial difference between the main lemma above as compared to [Isett 2018, Lemma 2.1] is that we do not require any lower bound of the form $N \geq \Xi^\eta$ for the frequency growth parameter N in inequality (9). This difference enables us to avoid double exponential growth of frequencies in constructing solutions as in [Isett and Oh 2016b]. Likewise, the constants $\widehat{C}$ and C_L in the estimates do not depend on such a parameter η.

We establish Lemma 5.1 by modifying the proof of the convex integration lemma, [Isett 2018, Lemma 3.3], as the proof of this lemma contains the only step in which the assumption $N \geq \Xi^\eta$ is used.

6. The improved convex integration lemma

As in [Isett 2018], we will establish Lemma 5.1 by combining a gluing approximation lemma and a convex integration lemma. In Lemma 6.1 below, we summarize the result of combining the regularization lemma and the gluing approximation lemma from [Isett 2018, Section 3]. (Here we have renamed the Euler–Reynolds flow that were $(\tilde{v}, \tilde{p}, \widetilde{R})$ to be (v, p, R).)

Lemma 6.1 (gluing approximation lemma). *There are absolute constants $C_1 \geq 2$ and $\delta_0 \in \left(0, \frac{1}{25}\right)$ such that the following holds. Let (v_0, p_0, R_0) be an Euler–Reynolds flow with frequency-energy levels bounded by (Ξ, e_v, e_R) to order 3 in C^0 such that $\operatorname{supp}_t v_0 \cup \operatorname{supp}_t R_0 \subseteq J$. Define the parameters*

$$\widehat{N} := (e_v/e_R)^{1/2}, \quad \widehat{\Xi} := \widehat{N}\,\Xi = (e_v/e_R)^{1/2}\,\Xi.$$

Then, for any $0 < \delta \leq \delta_0$, there exist a constant $C_\delta \geq 1$, a constant $\theta > 0$, a sequence of times $\{t(I)\}_{I \in \mathbb{Z}} \subseteq \mathbb{R}$ and an Euler–Reynolds flow (v, p, R), $R = \sum_{I \in \mathbb{Z}} R_I$, that satisfy the support restrictions

$$\operatorname{supp}_t v \cup \operatorname{supp}_t R \subseteq N(J; 3^{-1}\Xi^{-1}e_v^{-1/2}), \tag{12}$$

$$2^{-1}\delta(\log\widehat{\Xi})^{-2}\Xi^{-1}e_v^{-1/2} \leq \theta \leq \delta(\log\widehat{\Xi})^{-2}\Xi^{-1}e_v^{-1/2}, \tag{13}$$

$$\operatorname{supp}_t R_I \subseteq \left[t(I) - \tfrac{1}{2}\theta, t(I) + \tfrac{1}{2}\theta\right], \tag{14}$$

$$\bigcup_I \bigcup_{I' \neq I} [t(I) - \theta, t(I) + \theta] \cap [t(I') - \theta, t(I') + \theta] = \varnothing, \tag{15}$$

and the estimates

$$\begin{aligned} \|v - v_0\|_{C^0} &\leq C_1 e_R^{1/2}, \\ \|\nabla_{\vec{a}} v\|_{C^0} &\leq C_1 \Xi^{|\vec{a}|} e_v^{1/2}, \quad |\vec{a}| = 1, 2, 3, \qquad (16) \\ \sup_I \|\nabla_{\vec{a}} R_I\|_{C^0} &\leq C_\delta \widehat{N}^{(|\vec{a}|-2)_+} \Xi^{|\vec{a}|} \log\widehat{\Xi} e_R, \quad |\vec{a}| = 0, 1, 2, 3, \qquad (17) \\ \sup_I \|\nabla_{\vec{a}}(\partial_t + v \cdot \nabla) R_I\|_{C^0} &\leq C_\delta (\log\widehat{\Xi})^3 \Xi e_v^{1/2} \Xi^{|\vec{a}|} e_R, \quad |\vec{a}| = 0, 1, 2. \end{aligned}$$

Our improved convex integration lemma may then be stated as follows.

Lemma 6.2 (convex integration lemma). *There exists an absolute constant b_0 such that, for any $C_1, C_\delta \geq 1$ and $\delta > 0$, there is a constant $\widetilde{C} = \widetilde{C}_{\delta, C_1, C_\delta}$ for which the following holds. Suppose J is a subinterval of $\mathbb{R}$ and (v, p, R) is an Euler–Reynolds flow, $R = \sum_I R_I$, that satisfy the conclusions* (12)–(15) *and* (16)–(17) *of Lemma 6.1 for some (Ξ, e_v, e_R), some $\theta > 0$ and some sequence of times $\{t(I)\}_{I \in \mathbb{Z}} \subseteq \mathbb{R}$. Also suppose*

$$|\theta|\,\|\nabla v\|_{C^0} \leq b_0. \tag{18}$$

Let $N \geq (e_v/e_R)^{1/2}$. Then there is an Euler–Reynolds flow (v_1, p_1, R_1) with frequency-energy levels in the sense of Definition 4.2 bounded by

$$(\Xi', e_v', e_R') = \left(\widetilde{C} N \Xi, (\log\widehat{\Xi}) e_R, (\log\widehat{\Xi})^{5/2} \frac{e_v^{1/2} e_R^{1/2}}{N}\right) \tag{19}$$

such that

$$\operatorname{supp}_t v_1 \cup \operatorname{supp}_t R_1 \subseteq N(J;\, \Xi^{-1} e_v^{-1/2}),$$
$$\|v_1 - v\|_{C^0} \le \widetilde{C} (\log \widehat{\Xi})^{1/2} e_R^{1/2}.$$

Lemma 5.1 now follows by combining Lemmas 6.1 and 6.2 as explained in [Isett 2018, Section 3]. (Here Lemma 6.1 is applied with (v_0, p_0, R_0) taken to be the (v, p, R) given in the assumptions of Lemma 5.1.) The only important difference in the present case is that we have removed the assumption $N \ge \Xi^\eta$ and the constants $\widehat{C}$ and C_L (which can be set equal if desired) do not depend on η.

We now explain how to prove Lemma 6.2 by modifying the proof of [Isett 2018, Lemma 3.3].

7. Modifying the convex integration

We now proceed with the proof of Lemma 6.2. The construction will be based on the proof of [Isett 2018, Lemma 3.3] implementing convex integration with the Mikado flows of [Daneri and Székelyhidi 2017], but modified to adapt the localization strategy of [Isett and Oh 2016b] to our setting.

Let (v, p, R), $R = \sum_I R_I$ be given as in the assumptions of Lemma 6.2, which are the conclusions of Lemma 6.1. We will use the symbol $\lesssim$ to denote inequalities involving explicit constants that are allowed to depend on the parameters C_1, δ and C_δ, but never on (Ξ, e_v, e_R), N, θ, $\widehat{\Xi}$, etc.

We obtain the new Euler–Reynolds flow (v_1, p_1, R_1) of Lemma 6.2 by adding carefully designed corrections $v_1^\ell = v^\ell + V^\ell$ and $p_1 = p + P$ to the velocity and pressure, respectively, and using the resulting equation for (v_1, p_1) to construct the appropriate R_1. The correction V^ℓ will be a sum of divergence-free, high-frequency vector fields indexed by a set $\mathcal{J}$:

$$V^\ell = \sum_{J \in \mathcal{J}} V_J^\ell, \quad \nabla_\ell V_J^\ell = 0 \quad \text{for all } J \in \mathcal{J}.$$

The index $J \in \mathcal{J}$ will have several components, $J = (I, J_1, J_2, J_3, f)$, that together specify the time interval and spatial location in which V_J will be supported as well as the direction in which V_J takes values. Specifically, we choose an even integer $\Pi \in [3\Xi, 6\Xi] \cap 2\mathbb{Z}$ of size comparable to Ξ and define

$$\mathcal{J} := \mathbb{Z} \times (\mathbb{Z}/\Pi\mathbb{Z})^3 \times \mathbb{F}, \quad \mathbb{F} := \{e_i \pm e_j : 1 \le i < j \le 3\}.$$

Each V_J, $J = (I, J_1, J_2, J_3, f)$, will be supported in a time interval of length $\sim \theta$ around time $t(I)$, and initially at time $t(I)$ will be supported in a ball of size $\sim \Xi^{-1}$ around the point

$$x_0(J) := \Pi^{-1}(J_1, J_2, J_3) \in (\mathbb{R}/\mathbb{Z})^3.$$

The component $f \in \mathbb{F}$ specifies which of the $\#\mathbb{F} = 6$ directions in $\mathbb{R}^3$ in which V_J^ℓ approximately takes values.

As in [Isett 2017a, Section 12], let $v_\epsilon = \eta_\epsilon * v$ be the coarse scale velocity field obtained by mollification in space at scale ϵ. Let $\Phi_s : \mathbb{R} \times \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R} \times \mathbb{T}^3$ be the *coarse scale flow* (the flow map of v_ϵ)

$$\Phi_s(t, x) = (t + s, \Phi_s^i(t, x)), \quad \frac{d}{ds}\Phi_s^i(t, x) = v_\epsilon^i(\Phi_s(t, x)), \quad \Phi_0(t, x) = (t, x), \tag{20}$$

and let $\Gamma_I : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{T}^3$ be the *back-to-labels map* associated to v_ϵ from the initial time $t(I)$:

$$\begin{aligned}(\partial_t + v_\epsilon^i \nabla_i)\Gamma_I(t,x) &= 0,\\ \Gamma_I(t(I),x) &= x.\end{aligned} \tag{21}$$

We also define the coarse scale advective derivative $\bar{D}_t := (\partial_t + v_\epsilon \cdot \nabla)$.

To localize the waves V_J, we construct a smooth, quadratic partition of unity initiating from each time $t(I)$ that follows the flow of v_ϵ and has length scale $\sim \Xi^{-1}$. The elements of this partition of unity are functions $\chi_{(I,[k])} : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$ that are indexed by $(I,[k]) \in \mathbb{Z} \times (\mathbb{Z}/\Pi\mathbb{Z})^3$, and they satisfy

$$\sum_{[k]\in(\mathbb{Z}/\Pi\mathbb{Z})^3} \chi^2_{(I,[k])}(t,x) = 1 \quad \text{for all } I \in \mathbb{Z}, \;\; (t,x) \in \mathbb{R}\times\mathbb{T}^3, \tag{22}$$

$$\bar{D}_t \chi_{(I,[k])}(t,x) = 0 \quad \text{for all } (I,[k]) \in \mathbb{Z}\times(\mathbb{Z}/\Pi\mathbb{Z})^3, \;\; (t,x)\in\mathbb{R}\times\mathbb{T}^3. \tag{23}$$

To construct the initial data for the partition of unity, choose a smooth $\bar{\chi} : \mathbb{R}^3 \to \mathbb{R}$ with support in $\left[-\frac{3}{4}, \frac{3}{4}\right]^3$ such that $\sum_{m\in\mathbb{Z}^3} \bar{\chi}^2(h-m) = 1$ for all $h \in \mathbb{R}^3$, then periodize and rescale to define

$$\chi_{(I,[k])}(t(I),x) := \sum_{m\in\mathbb{Z}^3} \bar{\chi}(\Pi x - [k] - \Pi m). \tag{24}$$

Observe that $\chi_{(I,[k])}(t(I),x)$ does not depend on how we represent the equivalence classes of $x \in (\mathbb{R}/\mathbb{Z})^3$ or $[k] \in (\mathbb{Z}/\Pi\mathbb{Z})^3$, and that (22) holds at time $t(I)$. The same identity holds for all time $t \in \mathbb{R}$ by (23) and uniqueness of solutions to the transport equation. Observe also, since $3\Xi \le \Pi \le 6\Xi$, that the initial data for $\chi_{(I,[k])}(t(I),\cdot)$ is supported in a ball of radius Ξ^{-1} around $\Pi^{-1}[k]$ in $(\mathbb{R}/\mathbb{Z})^3$, and satisfies estimates of the form $\|\nabla_{\vec{a}} \chi_{(I,[k])}(t(I),\cdot)\|_{C^0} \lesssim_{|\vec{a}|} \Xi^{|\vec{a}|}$.

7A. ***Localizing the convex integration construction.*** Unlike the scheme in [Isett 2018], our scheme will involve many Mikado flow based waves at any given time that are supported within overlapping regions. In general, interference between overlapping Mikado flows would produce error terms that cannot be controlled for the iteration. We avoid this interference by "threading" the Mikado flows together, so that, at the initial time, the main terms of the waves V_J will have disjoint support. The support then remains disjoint as the Mikado flows are advected along the coarse scale flow.

To accomplish this construction, let $f \in \mathbb{F}$ and let $[k] \in (\mathbb{Z}/2\mathbb{Z})^3$. Choose an $r_0 > 0$ and choose disjoint, periodic lines $\ell_{(f,[k])} = \{p_{(f,[k])} + tf : t \in \mathbb{R}\}$ that are separated from each other by a distance greater than $6r_0$ in the torus $(\mathbb{R}/\mathbb{Z})^3$. Choose smooth functions $\psi_{(f,[k])} : \mathbb{T}^3 \to \mathbb{R}$ of the form $\psi_{(f,[k])}(X) = g(\mathrm{dist}(X, \ell_{(f,[k])}))$, $\operatorname{supp} g(\cdot) \subseteq \left[\frac{1}{2}r_0, r_0\right]$, such that

$$\int_{\mathbb{T}^3} \psi_{(f,[k])}(X)\, dX = 0, \qquad \int_{\mathbb{T}^3} \psi^2_{(f,[k])}(X)\, dX = 1. \tag{25}$$

With these choices, the functions $\psi_{(f,[k])}$ have disjoint support and have gradients orthogonal to f:

$$\begin{gathered}\nabla_\ell \psi_{(f,[k])}(X) f^\ell = 0,\\ \operatorname{supp}\psi_{(f,[k])} \cap \operatorname{supp}\psi_{(\tilde{f},[\tilde{k}])} = \varnothing \quad \text{if } f \ne \tilde{f} \text{ or } [k] \ne [\tilde{k}] \text{ in } (\mathbb{Z}/2\mathbb{Z})^3.\end{gathered} \tag{26}$$

Conditions (26) and (25) imply that $\psi_{(f,[k])}(X) f^\ell$ is divergence-free with mean zero, which implies that there is[6] a smooth tensor field

$$\Omega^{\alpha\beta}_{(f,[k])} : \mathbb{T}^3 \to \mathbb{R}^3 \otimes \mathbb{R}^3$$

that is antisymmetric in $\alpha\beta$ and satisfies

$$\nabla_\alpha \Omega^{\alpha\beta}_{(f,[k])}(X) = \psi_{(f,[k])}(X) f^\beta, \quad \int_{\mathbb{T}^3} \Omega^{\alpha\beta}_{(f,[k])}(X)\, dX = 0 \qquad \text{for all } 1 \le \alpha, \beta \le 3.$$

Since all components of the $\Omega^{\alpha\beta}_{(f,[k])}$ have mean zero on the torus, we can further construct tensor fields

$$\widetilde{\Omega}^{\alpha\beta\gamma}_{(f,[k])} : \mathbb{T}^3 \to \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3,$$

also antisymmetric in $\alpha\beta$, such that

$$\nabla_\gamma \widetilde{\Omega}^{\alpha\beta\gamma}_{(f,[k])}(X) = \Omega^{\alpha\beta}_{(f,[k])}(X), \quad \int_{\mathbb{T}^3} \widetilde{\Omega}^{\alpha\beta\gamma}_{(f,[k])}(X)\, dX = 0 \qquad \text{for all } 1 \le \alpha, \beta, \gamma \le 3.$$

For example, we can take

$$\widetilde{\Omega}^{\alpha\beta\gamma}_{(f,[k])} := \nabla^\gamma \Delta^{-1} \Omega^{\alpha\beta}_{(f,[k])}.$$

These second-order potentials will be used to impose local conservation of angular momentum similar to the use of double-curl form waves in [Isett and Oh 2016b].

For $J = (I, J_1, J_2, J_3, f)$, let $[J] := [(J_1, J_2, J_3)]$. We define the corrections V^ℓ_J to have the form

$$V^\ell_J = \mathring{V}^\ell_J + \delta V^\ell_J, \quad \mathring{V}^\ell_J = v^\ell_J \psi_J(t,x), \quad \psi_J(t,x) := \psi_{(f,[J])}(\lambda \Gamma_I(t,x)). \tag{27}$$

The amplitudes v^ℓ_J have the same form as in [Isett 2018, Section 13] except they incorporate the partition of unity χ_J. In particular, they take values orthogonal to the gradient of the oscillatory functions ψ_J:

$$v^\ell_J = \chi_J [e_I^{1/2}(t) \gamma_{(I,f)}(t,x) (\nabla \Gamma_I^{-1})^\ell_a f^a], \tag{28}$$

$$\operatorname{supp}_t e_I^{1/2}(t) \subseteq [t(I) - \theta, t(I) + \theta], \tag{29}$$

$$\chi_J(t,x) = \chi_{(I,[J_1,J_2,J_3])}(t,x), \quad J = (I, J_1, J_2, J_3, f),$$

$$v^\ell_J \nabla_\ell \psi_J = 0. \tag{30}$$

Note in particular that by construction the main terms of each wave have disjoint supports

$$\operatorname{supp} \mathring{V}_J \cap \operatorname{supp} \mathring{V}_K = \varnothing \quad \text{if } J \neq K. \tag{31}$$

Indeed, if $J = (J_0, J_1, J_2, J_3, f)$ and $K = (K_0, K_1, K_2, K_3, f')$ are not equal and $J_0 \neq K_0$, then V^ℓ_J and V^ℓ_K live on different time intervals. If $J_0 = K_0 = I$, one has either $f \neq f'$ or $(J_1, J_2, J_3) \neq (K_1, K_2, K_3)$ mod 2, either case implying $\operatorname{supp} \psi_J \cap \operatorname{supp} \psi_K = \varnothing$, or $f = f'$ and $(J_1, J_2, J_3) = (K_1, K_2, K_3)$ mod 2. In the last case, one has $\operatorname{supp} \chi_J \cap \operatorname{supp} \chi_K = \varnothing$ unless $J = K$.

[6] We can take for instance $\Omega^{\alpha\beta}_{(f,[k])} = \nabla^\alpha \Delta^{-1}[\psi_{(f,[k])} f^\beta] - \nabla^\beta \Delta^{-1}[\psi_{(f,[k])} f^\alpha]$.

The correction V_J^ℓ is made to be divergence-free and to have the form (27) by making V_J^ℓ the divergence of an antisymmetric tensor built from the Lie transport of the potentials $\widetilde{\Omega}^{\alpha\beta\gamma}_{(f,[k])}$ above:

$$V_J^\ell = \lambda^{-2}\nabla_a\nabla_c[\chi_J(\nabla\Gamma_I^{-1})^a_\alpha(\nabla\Gamma_I^{-1})^\ell_\beta(\nabla\Gamma_I^{-1})^c_\gamma e_I^{1/2}(t)\gamma_{(I,f)}\widetilde{\Omega}^{\alpha\beta\gamma}_J], \tag{32}$$

$$\begin{aligned}
\delta V_J^\ell &= \delta v^\ell_{J,\alpha\beta}\Omega_J^{\alpha\beta} + \delta v^\ell_{J,\alpha\beta\gamma}\widetilde{\Omega}_J^{\alpha\beta\gamma},\\
\Omega_J^{\alpha\beta}(t,x) &:= \Omega^{\alpha\beta}_{(f,[J_1,J_2,J_3])}(\lambda\Gamma_I),\\
\widetilde{\Omega}_J^{\alpha\beta\gamma}(t,x) &:= \widetilde{\Omega}^{\alpha\beta\gamma}_{(f,[J_1,J_2,J_3])}(\lambda\Gamma_I),\\
\delta v^\ell_{J,\alpha\beta} &:= \lambda^{-1}\nabla_a[\chi_J(\nabla\Gamma_I^{-1})^a_\alpha(\nabla\Gamma_I^{-1})^\ell_\beta e_I^{1/2}(t)\gamma_{(I,f)}],\\
\delta v^\ell_{J,\alpha\beta\gamma} &:= \lambda^{-2}\nabla_a\nabla_c[\chi_J(\nabla\Gamma_I^{-1})^a_\alpha(\nabla\Gamma_I^{-1})^\ell_\beta(\nabla\Gamma_I^{-1})^c_\gamma e_I^{1/2}(t)\gamma_{(I,f)}].
\end{aligned} \tag{33}$$

Note that the main term $\mathring{V}_J^\ell$ in (27)–(28) appears when the derivatives ∇_a and ∇_c both fall on $\widetilde{\Omega}_J^{\alpha\beta\gamma}$. Since V_J^ℓ has the form $V_J^\ell = \nabla_a W_J^{a\ell}$, where $W_J^{a\ell}$ is antisymmetric in $a\ell$, we have that V_J^ℓ is divergence-free.

The amplitudes constructed here are related to those constructed in [Isett 2018, Section 13] (which are indexed by $(I,f)\in\mathbb{Z}\times\mathbb{F}$ and do not involve spatial cutoffs) by the formula

$$v_J^\ell = \chi_J v^\ell_{(I,f)}, \quad J = (I, J_1, J_2, J_3, f). \tag{34}$$

This comparison allows us to see that the parameter $\epsilon = \epsilon_v$ in the mollification of $v \mapsto v_\epsilon$ can be chosen to have the same value $\epsilon_v = c_v N^{-1/2}\Xi^{-1}$ as in [Isett 2018, Section 16], which is based on the requirement

$$\|v - v_\epsilon\|_{C^0}\max_J \||v_J||\psi_J|\|_{C^0} \le (\log\widehat{\Xi})^{1/2}\frac{e_v^{1/2}e_R^{1/2}}{500N}.$$

Since we have chosen the same parameter in the mollification $v\mapsto\epsilon_v$ as that chosen in [Isett 2018], we obtain the same estimates for v_ϵ:

$$\|\nabla_{\vec a}v_\epsilon\|_{C^0} \lesssim_{|\vec a|} N^{(|\vec a|-2)_+/2}\Xi^{|\vec a|}e_v^{1/2} \quad \text{if } |\vec a|\ge 1, \tag{35}$$

where the implicit constant is equal to 1 for $|\vec a| = 1$. From this fact we will see in the following Section 8 that all the remaining estimates for the components of the construction coincide with those in the proof of [Isett 2018, Lemma 3.3].

8. Estimates for components of the construction

Here we summarize the estimates for the components of the construction, which coincide with those of [Isett 2018]. The following elementary lemma will be convenient:

Lemma 8.1. *For $u\ge 0$, an integer $M\ge 0$ and for $g:\mathbb{T}^3\to\mathbb{R}$, define (for $N\ge 1,\ \Xi>0$)*

$$H_{M,u}[g] := \max_{0\le|\vec a|\le M}\frac{\|\nabla_{\vec a}g\|_{C^0}}{N^{(|\vec a|-u)_+/2}\Xi^{|\vec a|}}. \tag{36}$$

Then, for $\lambda\ge N^{1/2}\Xi$, we have for any first-order partial derivative ∇_a

$$\begin{aligned}
H_{M,u}[\lambda^{-1}\nabla_a g] &\le H_{M+1,u}[g],\\
H_{M,u}[\Xi^{-1}\nabla_a g] &\le H_{M+1,u+1}[g].
\end{aligned}$$

We also have the triangle inequality $H_{M,u}[g_{(1)}+g_{(2)}] \le H_{M,u}[g_{(1)}]+H_{M,u}[g_{(2)}]$ *and product estimate*

$$H_{M,u}[g_{(1)}g_{(2)}] \lesssim_M H_{M,u}[g_{(1)}]H_{M,u}[g_{(2)}]. \tag{37}$$

All the properties follow quickly from the definition (36). Inequality (37) follows from the expansion

$$\nabla_{\vec{a}}(g_{(1)}g_{(2)}) = \sum_{|\vec{a}_1|+|\vec{a}_2|=|\vec{a}|} c_{\vec{a}_1,\vec{a}_2}\nabla_{\vec{a}_1}g_{(1)}\nabla_{\vec{a}_2}g_{(2)},$$

the bound

$$\|\nabla_{\vec{a}_i}g_{(i)}\|_{C^0} \le N^{(|\vec{a}_i|-u)_+/2}\,\Xi H_{M,u}[g_{(i)}]$$

and the inequality $(|\vec{a}_1|-u)_+ + (|\vec{a}_2|-u)_+ \le (|\vec{a}|-u)_+$.

The estimates for the construction may now be summarized as follows. Here we use the fact that the frequency $\lambda := B_\lambda N\,\Xi$ is larger than $N^{1/2}\,\Xi$ to conclude that the lower-order terms $\delta v^\ell_{J,\alpha\beta\gamma}$ obey the same bounds as the $\delta v^\ell_{J,\alpha\beta}$.

Proposition 8.2. *The following bounds hold with constants depending only on* $|\vec{a}|$:

$$\|\nabla_{\vec{a}}\gamma_{(I,f)}\|_{C^0} + \|\nabla_{\vec{a}}(\nabla\Gamma_I^{-1})\|_{C^0} \lesssim N^{(|\vec{a}|-1)_+/2}\,\Xi^{|\vec{a}|}, \tag{38}$$

$$\|\nabla_{\vec{a}}\overline{D}_t\gamma_{(I,f)}\|_{C^0} + \|\nabla_{\vec{a}}\overline{D}_t(\nabla\Gamma_I^{-1})\|_{C^0} \lesssim (\log\widehat{\Xi})^2\,\Xi e_v^{1/2}N^{(|\vec{a}|-1)_+/2}\,\Xi^{|\vec{a}|}, \tag{39}$$

$$\sup_{t\in\mathbb{R}}(e_I^{1/2}(t)+\theta|\partial_t e_I^{1/2}(t)|) \lesssim (\log\widehat{\Xi})^{1/2}e_R^{1/2}, \tag{40}$$

$$\|\nabla_{\vec{a}}\chi_J\|_{C^0} \lesssim N^{(|\vec{a}|-1)_+/2}\,\Xi^{|\vec{a}|}, \tag{41}$$

$$\|\nabla_{\vec{a}}v_J^\ell\|_{C^0} \lesssim (\log\widehat{\Xi})^{1/2}N^{(|\vec{a}|-1)_+/2}\,\Xi^{|\vec{a}|}e_R^{1/2}, \tag{42}$$

$$\|\nabla_{\vec{a}}\overline{D}_t v_J^\ell\|_{C^0} \lesssim (\log\widehat{\Xi})^{5/2}N^{(|\vec{a}|-1)_+/2}\,\Xi^{|\vec{a}|}e_R^{1/2}, \tag{43}$$

$$\|\nabla_{\vec{a}}\delta v^\ell_{J,\alpha\beta}\|_{C^0} + \|\nabla_{\vec{a}}\delta v^\ell_{J,\alpha\beta\gamma}\|_{C^0} \lesssim \lambda^{-1}(\log\widehat{\Xi})^{1/2}N^{|\vec{a}|/2}\,\Xi^{1+|\vec{a}|}e_R^{1/2}, \tag{44}$$

$$\|\nabla_{\vec{a}}\overline{D}_t\delta v^\ell_{J,\alpha\beta}\|_{C^0} + \|\nabla_{\vec{a}}\overline{D}_t\delta v^\ell_{J,\alpha\beta\gamma}\|_{C^0} \lesssim \lambda^{-1}(\log\widehat{\Xi})^{5/2}N^{|\vec{a}|/2}\,\Xi^{|\vec{a}|+2}e_v^{1/2}e_R^{1/2}. \tag{45}$$

Proof. Inequalities (38)–(40) follow from the bounds in [Isett 2018, Section 17.1]. Inequality (41) for $|\vec{a}|=0$ follows from the maximum principle for $\overline{D}_t\chi_J=0$. To obtain (41), we apply [Isett 2017a, Proposition 17.4] in the case of order $L=2$ frequency-energy levels to obtain

$$E_M[\chi_J](\Phi_s(t,x)) \le e^{C_M\Xi e_v^{1/2}|s|}E_M[\chi_J](t(I),x), \tag{46}$$

$$E_M[\chi_J](t,x) := \sum_{0\le|\vec{a}|\le M}\Xi^{-2|\vec{a}|}N^{-(|\vec{a}|-1)_+}|\nabla_{\vec{a}}\nabla\chi_J(t,x)|^2,$$

and we use the fact that, by the construction in (24),

$$E_M[\chi_J](t(I),x) \lesssim_M \sum_{0\le|\vec{a}|\le M}\Xi^{-2|\vec{a}|}N^{-(|\vec{a}|-1)_+}(\Xi^{|\vec{a}|+1})^2 \lesssim_M \Xi^2.$$

We have $\Xi e_v^{1/2}|s| \le \Xi e_v^{1/2}\theta \le 1$ on the support of the time cutoff $e_I^{1/2}$ from (29), so (46) yields

$$\|\nabla_{\vec{a}}\chi_J\|_{C^0} \lesssim N^{(|\vec{a}|-2)_+/2}\,\Xi^{|\vec{a}|},$$

which implies (41).

The proofs of estimates (42)–(45) for v_J^ℓ and for $\delta v_{J,\alpha\beta\gamma}^\ell$ are exactly as in [Isett 2018, Section 17.1] with the addition of the cutoff function χ_J. For instance, note that

$$\chi_J(\nabla\Gamma_I^{-1})_\alpha^a \quad \text{and} \quad \bar{D}_t[\chi_J(\nabla\Gamma_I^{-1})_\alpha^a] = \chi_J\bar{D}_t(\nabla\Gamma_I^{-1})_\alpha^a$$

obey the same bounds as $(\nabla\Gamma_I^{-1})_\alpha^a$ and $\bar{D}_t(\nabla\Gamma_I^{-1})_\alpha^a$, respectively, up to constants, so we may absorb the cutoff χ_J into the first factor of $(\nabla\Gamma^{-1})$ in estimating formulas (28) and (33) while repeating the proofs in [Isett 2018, Section 17.1].

It remains to check (42)–(45) for the lower-order term $\delta v_{J,\alpha\beta\gamma}^\ell$. Applying Lemma 8.1, we obtain

$$\begin{aligned}
\lambda\Xi^{-1}\delta v_{J,\alpha\beta\gamma}^\ell &= \Xi^{-1}\lambda^{-1}\nabla_a\nabla_c[\chi_J(\nabla\Gamma_I^{-1})_\alpha^a(\nabla\Gamma_I^{-1})_\beta^\ell(\nabla\Gamma_I^{-1})_\gamma^c e_I^{1/2}(t)\gamma_{(I,f)}],\\
\lambda\Xi^{-1}H_{M,0}[\delta v_{J,\alpha\beta\gamma}^\ell] &\lesssim_M H_{M+1,1}[\lambda^{-1}\nabla_c[\chi_J(\nabla\Gamma_I^{-1})_\alpha^a(\nabla\Gamma_I^{-1})_\beta^\ell(\nabla\Gamma_I^{-1})_\gamma^c e_I^{1/2}(t)\gamma_{(I,f)}]\\
&\lesssim_M H_{M+2,1}[\chi_J(\nabla\Gamma_I^{-1})_\alpha^a(\nabla\Gamma_I^{-1})_\beta^\ell(\nabla\Gamma_I^{-1})_\gamma^c e_I^{1/2}(t)\gamma_{(I,f)}]\\
&\lesssim_M e_I^{1/2}(t)H_{M+2,1}[\chi_J]H_{M+2,1}[(\nabla\Gamma_I^{-1})]^3H_{M+2,1}[\gamma_{(I,f)}], \qquad (47)\\
H_{M,0}[\delta v_{J,\alpha\beta\gamma}^\ell] &\lesssim_M \lambda^{-1}(\log\widehat{\Xi})^{1/2}\Xi e_R^{1/2}. \qquad (48)
\end{aligned}$$

Here every term in (47) is bounded by $\lesssim_M 1$ except $e_I^{1/2}(t)$. Note that (48) is equivalent to (44).

To prove (45), we proceed similarly by commuting in the advective derivative weighted by the parameter $\theta\sim(\log\widehat{\Xi})^{-2}\Xi^{-1}e_v^{-1/2}$:

$$\begin{aligned}
(\lambda\Xi^{-1}\theta)\bar{D}_t\delta v_{J,\alpha\beta\gamma}^\ell &= \Xi^{-1}\lambda^{-1}\nabla_a\nabla_c[\theta\bar{D}_t[\chi_J(\nabla\Gamma_I^{-1})_\alpha^a(\nabla\Gamma_I^{-1})_\beta^\ell(\nabla\Gamma_I^{-1})_\gamma^c e_I^{1/2}(t)\gamma_{(I,f)}]] \qquad (49)\\
&\quad -\theta(\nabla_a v_\epsilon^i)\Xi^{-1}\lambda^{-1}\nabla_i\nabla_c[\chi_J(\nabla\Gamma_I^{-1})_\alpha^a(\nabla\Gamma_I^{-1})_\beta^\ell(\nabla\Gamma_I^{-1})_\gamma^c e_I^{1/2}(t)\gamma_{(I,f)}] \qquad (50)\\
&\quad -\Xi^{-1}\nabla_a[\nabla_c v_\epsilon^i\lambda^{-1}\nabla_i[\chi_J(\nabla\Gamma_I^{-1})_\alpha^a(\nabla\Gamma_I^{-1})_\beta^\ell(\nabla\Gamma_I^{-1})_\gamma^c e_I^{1/2}(t)\gamma_{(I,f)}]]. \qquad (51)
\end{aligned}$$

The terms (50) and (51) may be estimated using Lemma 8.1 as in the proof of (47)–(48) to obtain

$$\begin{aligned}
H_{M,0}[(50)]+H_{M,0}[(51)] &\lesssim_M e_I^{1/2}(t)H_{M+1,1}[\theta\nabla v_\epsilon]H_{M+2,1}[\chi_J]H_{M+2,1}[(\nabla\Gamma_I^{-1})]^3H_{M+2,1}[\gamma_{(I,f)}]\\
&\overset{(35)-(41)}{\lesssim_M} e_I^{1/2}(t)\lesssim(\log\widehat{\Xi})^{1/2}e_R^{1/2}.
\end{aligned}$$

For (49), apply the product rule for $\theta\bar{D}_t$ and apply Lemma 8.1 repeatedly to obtain

$$\begin{aligned}
H_{M,0}[(49)] \lesssim_M (e_I^{1/2}(t)+\theta|\partial_t e_I^{1/2}(t)|)H_{M+2,1}[\chi_J]\cdot(H_{M+2,1}[(\nabla\Gamma_I^{-1})]+\theta H_{M+2,1}[\bar{D}_t(\nabla\Gamma_I^{-1})])^3&\\
\cdot(H_{M+2,1}[\gamma_{(I,f)}]+\theta H_{M+2,1}[\bar{D}_t\gamma_{(I,f)}]).&
\end{aligned}$$

Since

$$\theta H_{M+2,1}[\bar{D}_t\gamma_{(I,f)}] \quad \text{and} \quad \theta H_{M+2,1}[\bar{D}_t(\nabla\Gamma_I^{-1})]$$

are bounded by $\lesssim_M 1$ from (38)–(39), we have

$$\begin{aligned}
H_{M,0}[\delta v_{J,\alpha\beta\gamma}] &\le \theta^{-1}\lambda^{-1}\Xi(H_{M,0}[(49)]+H_{M,0}[(50)]+H_{M,0}[(51)])\\
&\lesssim_M \theta^{-1}\lambda^{-1}\Xi(e_I^{1/2}(t)+\theta|\partial_t e_I^{1/2}(t)|)\\
&\lesssim \theta^{-1}\lambda^{-1}\Xi(\log\widehat{\Xi})^{1/2}e_R^{1/2}.
\end{aligned}$$

This bound is equivalent to the desired bound (45) for $\delta v_{J,\alpha\beta\gamma}$. □

As (42)–(45) are the same bounds for the components of the correction as those proven for $v^\ell_{(I,f)}$ and $\delta v^\ell_{(I,f),\alpha\beta}$ in [Isett 2018, Section 17], we have the following bounds from [Isett 2018, Proposition 17.3].

Proposition 8.3 (correction estimates). *For* $0 \le |\vec{a}| \le 3$, *we have*

$$\begin{aligned} \sup_J \|\nabla_{\vec{a}} \mathring{V}_J\|_{C^0} &\lesssim (B_\lambda N \Xi)^{|\vec{a}|} (\log \widehat{\Xi})^{1/2} e_R^{1/2}, \\ \sup_J \|\nabla_{\vec{a}} \delta V_J\|_{C^0} &\lesssim (B_\lambda N \Xi)^{|\vec{a}|-1} \Xi (\log \widehat{\Xi})^{1/2} e_R^{1/2}, \\ \|V\|_{C^0} &\lesssim (B_\lambda N \Xi)^{|\vec{a}|} (\log \widehat{\Xi})^{1/2} e_R^{1/2}, \qquad (52) \\ \operatorname{supp}_t V &\subseteq \bigcup_I \operatorname{supp}_t e_I \subseteq \bigcup_I [t(I) - \theta, t(I) + \theta]. \end{aligned}$$

For the estimate (52), we use that at most a bounded number (say 2^3) distinct V^ℓ_J are supported at any given point (t, x). This detail will be explained following (76) below. We now consider the error terms and their estimates.

9. The error terms

Given the Euler–Reynolds flow (v, p, R), the new velocity field $v^\ell_1 = v^\ell + V^\ell$, with

$$V^\ell = \sum_J V^\ell_J = \sum_J \mathring{V}^\ell_J + \delta V^\ell_J,$$

and pressure $p_1 = p + P$ will solve the Euler–Reynolds equations when coupled to a new Reynolds stress tensor $R_1^{j\ell}$. The new stress tensor $R_1^{j\ell}$ will be composed of terms that solve

$$R_1^{j\ell} = R_M^{j\ell} + R_T^{j\ell} + R_S^{j\ell} + R_H^{j\ell}, \qquad (53)$$

$$R_M^{j\ell} = (v^j - v^j_\epsilon) V^\ell + V^j (v^\ell - v^\ell_\epsilon) + (R^{j\ell} - R^{j\ell}_\epsilon),$$

$$\nabla_j R_T^{j\ell} = \partial_t V^\ell + \nabla_j (v^j_\epsilon V^\ell + V^j v^\ell_\epsilon), \qquad (54)$$

$$R_S^{j\ell} = \sum_{J,K \in \mathcal{J}} \delta V^j_J \mathring{V}^\ell_K + \mathring{V}^j_J \delta V^\ell_K + \delta V^j_J \delta V^\ell_K,$$

$$\nabla_j R_H^{j\ell} = \nabla_j \Bigg[\sum_{J \in \mathcal{J}} \mathring{V}^j_J \mathring{V}^\ell_J + P \delta^{j\ell} + R^{j\ell}_\epsilon \Bigg]. \qquad (55)$$

In writing (55), we have made the crucial observation that all of the off-diagonal terms in the summation $\sum_{J,K \in \mathcal{J}} \mathring{V}^j_J \mathring{V}^\ell_K$ vanish due to the disjointness of support stated in (31).

Our construction has been designed in such a way that

$$\sum_{J \in \mathcal{J}} v^j_J v^\ell_J + P \delta^{j\ell} + R^{j\ell}_\epsilon = 0. \qquad (56)$$

From (27) and (56), equation (55) reduces to

$$\nabla_j R_H^{j\ell} = \nabla_j \Bigg[\sum_{J \in \mathcal{J}} v^j_J v^\ell_J (\psi_J^2 - 1) \Bigg]. \qquad (57)$$

To verify (56), note that, for each $I \in \mathbb{Z}$ and $\mathcal{J}(I) := \{I\} \times (\mathbb{Z}/\Pi\mathbb{Z})^3 \times \mathbb{F}$, we have from (22) and (34) that

$$\sum_{J \in \mathcal{J}(I)} v_J^j v_J^\ell = \sum_{[k] \in (\mathbb{Z}/\Pi\mathbb{Z})^3} \sum_{f \in \mathbb{F}} \chi^2_{(I,[k])} v^j_{(I,f)} v^\ell_{(I,f)} = \sum_{f \in \mathbb{F}} v^j_{(I,f)} v^\ell_{(I,f)}, \tag{58}$$

where $v_{(I,f)}$ are the amplitudes from the construction in [Isett 2018]. The equality

$$\sum_{I \in \mathbb{Z}} \sum_{f \in \mathbb{F}} v^j_{(I,f)} v^\ell_{(I,f)} + P\delta^{j\ell} + R^{j\ell}_\epsilon = 0$$

proved in [Isett 2018, Sections 14–15] now implies the equality (56) in the present construction using (58).

It now remains to show that, when $R_T^{j\ell}$ and $R_H^{j\ell}$ are chosen appropriately, the tensor $R_1^{j\ell}$ defined by (53) satisfies the bounds required by Lemma 6.2.

10. Solving the symmetric divergence equation

To estimate the error tensor R_1 defined in (53), the only terms that require a different treatment from [Isett 2018] are the terms R_T and R_H. Namely, since our choice of v_ϵ and R_ϵ and our estimates for $\mathring{V}_J$ and δV_J also coincide with those of that paper, Proposition 17.4 there shows that

$$\begin{aligned}
\|R_M\|_{C^0} + \|R_S\|_{C^0} &\le (\log \widehat{\Xi}) \frac{e_v^{1/2} e_R^{1/2}}{10N}, \\
\|\nabla_{\vec{a}} R_M\|_{C^0} + \|\nabla_{\vec{a}} R_S\|_{C^0} &\lesssim (B_\lambda N \Xi)^{|\vec{a}|} (\log \widehat{\Xi}) \frac{e_v^{1/2} e_R^{1/2}}{N}, \quad 1 \le |\vec{a}| \le 3, \\
\operatorname{supp}_t R_M \cup \operatorname{supp}_t R_S &\subseteq \bigcup_I [t(I) - \theta, t(I) + \theta],
\end{aligned} \tag{59}$$

provided we choose the constant B_λ in the definition of $\lambda = B_\lambda N \Xi$ to be larger than a certain, absolute constant $\bar{B}_\lambda$.

The tensors R_T and R_H are defined as summations of the form

$$R_T^{j\ell} = \sum_{J \in \mathcal{J}} R^{j\ell}_{T,J}, \quad R_H^{j\ell} = \sum_{J \in \mathcal{J}} R^{j\ell}_{H,J}, \tag{60}$$

where each term is symmetric and is localized both in space and in time around the support of V_J^ℓ.

We expand the terms (54) and (57) (using the orthogonality $v_J^j \nabla_j \psi_J = 0$ stated in (30) in the case of R_H, and using $\nabla_j v_\epsilon^j = \nabla_j V_J^j = 0$ in the case of R_T) to obtain the equations

$$\nabla_j R^{j\ell}_{T,J} = \partial_t V_J^\ell + \nabla_j (v_\epsilon^j V_J^\ell + V_J^j v_\epsilon^\ell), \tag{61}$$

$$\nabla_j R^{j\ell}_{T,J} = u^\ell_{TJ} \psi_J + u^\ell_{TJ,\alpha\beta} \Omega_J^{\alpha\beta} + u^\ell_{TJ,\alpha\beta\gamma} \widetilde{\Omega}_J^{\alpha\beta\gamma}, \tag{62}$$

$$\nabla_j R^{j\ell}_{H,J} = u^\ell_{HJ} (\psi_J^2 - 1), \tag{63}$$

$$\begin{aligned}
u^\ell_{HJ} &= \nabla_j [v_J^j v_J^\ell], \\
u^\ell_{TJ} &:= \bar{D}_t v_J^\ell + v_J^j \nabla_j v_\epsilon^\ell, \\
u^\ell_{TJ,\alpha\beta} &:= \bar{D}_t \delta v^\ell_{J,\alpha\beta} + \delta v^j_{J,\alpha\beta} \nabla_j v_\epsilon^\ell, \\
u^\ell_{TJ,\alpha\beta\gamma} &:= \bar{D}_t \delta v^\ell_{J,\alpha\beta\gamma} + \delta v^j_{J,\alpha\beta\gamma} \nabla_j v_\epsilon^\ell.
\end{aligned}$$

By the construction in Section 7A, each of the functions ψ_J, $(\psi_J^2 - 1)$, $\Omega_J^{\alpha\beta}$ and $\widetilde{\Omega}_J^{\alpha\beta\gamma}$ have the form $\omega(\lambda\Gamma_I(t,x))$, where $\omega : \mathbb{T}^3 \to \mathbb{R}$ belongs to a finite set of smooth functions of mean zero on $\mathbb{T}^3$. We may therefore apply the following proposition, which is similar to [Isett 2018, Proposition 17.6] and is proven in Section 10A below using the same parametrix expansion technique.

Proposition 10.1 (nonstationary phase). *If $U^\ell : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}^3$ is a smooth vector field of the form $U^\ell = u^\ell \omega(\lambda\Gamma_I)$, where $\omega : \mathbb{T}^3 \to \mathbb{R}$ is a smooth function of mean zero, then, for any $D \geq 1$, there exist a smooth, symmetric tensor field $Q_{(D)}^{j\ell} : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}^3 \otimes \mathbb{R}^3$ and a vector field $U_{(D)}^\ell$ satisfying*

$$U^\ell = \nabla_j Q_{(D)}^{j\ell} + U_{(D)}^\ell,$$

$$\sup_{0\leq|\vec{a}|\leq 3} \lambda^{-|\vec{a}|} \|\nabla_{\vec{a}} Q_{(D)}^{j\ell}\|_{C^0} \lesssim \lambda^{-1} \sup_{0\leq|\vec{a}|\leq D+3} \frac{\|\nabla_{\vec{a}} u^\ell\|_{C^0}}{N^{|\vec{a}|/2}\,\Xi^{|\vec{a}|}},$$

$$\sup_{0\leq|\vec{a}|\leq 3} \lambda^{-|\vec{a}|} \|\nabla_{\vec{a}} U^\ell\|_{C^0} \lesssim B_\lambda^{-1} N^{-D/2} \sup_{0\leq|\vec{a}|\leq D+3} \frac{\|\nabla_{\vec{a}} u^\ell\|_{C^0}}{N^{|\vec{a}|/2}\,\Xi^{|\vec{a}|}},$$

$$\operatorname{supp} Q_{(D)}^{j\ell} \cup \operatorname{supp} U_{(D)}^\ell \subseteq \operatorname{supp} U^\ell,$$

where the implicit constant depends only on ω and D.

We apply Proposition 10.1 to each of the terms in (62) and (63) and use the estimates

$$H_{D+3,0}[u_{TJ}^\ell] + H_{D+3,0}[u_{TJ,\alpha\beta}^\ell] + H_{D+3,0}[u_{TJ,\alpha\beta\gamma}^\ell] + H_{D+3,0}[u_{HJ}^\ell] \lesssim (\log \widehat{\Xi})^{5/2}\,\Xi e_v^{1/2} e_R^{1/2},$$

$$H_{D+3,0}[u] := \sup_{0\leq|\vec{a}|\leq D+3} \frac{\|\nabla_{\vec{a}} u^\ell\|_{C^0}}{N^{|\vec{a}|/2}\,\Xi^{|\vec{a}|}},$$

which follow from (42)–(45) and Lemma 8.1 (and are saturated only by u_{TJ}^ℓ), to obtain the decompositions

$$(62) = \nabla_j Q_{TJ,(D)}^{j\ell} + U_{TJ,(D)}^\ell, \quad (63) = \nabla_j Q_{HJ,(D)}^{j\ell} + U_{HJ,(D)}^\ell, \tag{64}$$

where the symmetric tensors $Q_{TJ,(D)}$ and $Q_{HJ,(D)}$ and remainder terms $U_{TJ,(D)}$ and $U_{HJ,(D)}$ satisfy

$$\begin{aligned}\sup_{0\leq|\vec{a}|\leq 3} \lambda^{-|\vec{a}|} (\|\nabla_{\vec{a}} Q_{TJ,(D)}^{j\ell}\|_{C^0} + \|\nabla_{\vec{a}} Q_{HJ,(D)}^{j\ell}\|_{C^0}) &\lesssim_D \lambda^{-1} (\log \widehat{\Xi})^{5/2}\,\Xi e_v^{1/2} e_R^{1/2} \\ &\lesssim_D B_\lambda^{-1} (\log \widehat{\Xi})^{5/2} \frac{e_v^{1/2} e_R^{1/2}}{N},\end{aligned} \tag{65}$$

$$\sup_{0\leq|\vec{a}|\leq 3} \lambda^{-|\vec{a}|} (\|\nabla_{\vec{a}} U_{TJ,(D)}^{j\ell}\|_{C^0} + \|\nabla_{\vec{a}} U_{HJ,(D)}^{j\ell}\|_{C^0}) \lesssim_D B_\lambda^{-1} N^{-D/2} (\log \widehat{\Xi})^{5/2}\,\Xi e_v^{1/2} e_R^{1/2}, \tag{66}$$

$$\operatorname{supp} U_{TJ,(D)} \cup \operatorname{supp} U_{HJ,(D)} \cup \operatorname{supp} Q_{TJ,(D)} \cup \operatorname{supp} Q_{HJ,(D)} \subseteq \operatorname{supp} \chi_J \cdot e_I^{1/2}(t). \tag{67}$$

To complete the construction of $R_{T,J}^{j\ell}$ and $R_{HJ}^{j\ell}$ to (62)–(63), we construct solutions to the equations

$$\nabla_j R_{TJ,(D)}^{j\ell} = U_{TJ,(D)}^\ell, \quad \nabla_j R_{HJ,(D)}^{j\ell} = U_{HJ,(D)}^\ell \tag{68}$$

that are localized around space-time cylinders containing the supports of v_J by using the inverses for the symmetric divergence equation that were constructed in [Isett and Oh 2016b]. We first recall the notions of Lagrangian and Eulerian cylinders from that paper.

Definition 10.2. Let Φ_s be the flow map associated to v_ϵ as defined in (20). Given a point in space-time $(t_0, x_0) \in \mathbb{R} \times \mathbb{T}^3$ and positive numbers $\tau, \rho > 0$, we define the *v_ϵ-adapted Eulerian cylinder* $\widehat{C}(\tau, \rho; t_0, x_0)$ with duration 2τ and base radius ρ as well as the *v_ϵ-adapted Lagrangian cylinder* $\widehat{\Gamma}(\tau, \rho; t_0, x_0)$ with duration 2τ and base radius ρ to be

$$\widehat{C}(\tau, \rho; t_0, x_0) := \{\Phi_s(t_0, x_0) + (0, h) : 0 \le |s| \le \tau,\ 0 \le |h| \le \rho\},$$
$$\widehat{\Gamma}(\tau, \rho; t_0, x_0) := \{\Phi_s(t_0, x_0 + h) : 0 \le |s| \le \tau,\ 0 \le |h| \le \rho\}.$$

The two notions are related (see [Isett and Oh 2016b, Lemma 5.2]) by

$$(t', x') \in \widehat{C}(\tau, \rho; t_0, x_0) \iff (t, x) \in \widehat{\Gamma}_v(\tau, \rho; t', x'), \tag{69}$$
$$\widehat{\Gamma}(\tau, e^{-\tau\|\nabla v_\epsilon\|_{C^0}}\rho; t_0, x_0) \subseteq \widehat{C}(\tau, \rho; t_0, x_0) \subseteq \widehat{\Gamma}(\tau, e^{\tau\|\nabla v_\epsilon\|_{C^0}}\rho; t_0, x_0). \tag{70}$$

It follows that the amplitudes constructed in Section 7A are supported in an Eulerian cylinder

$$\begin{aligned} \operatorname{supp} \chi_J \cdot e_I^{1/2}(t) \subseteq \widehat{\Gamma}(\theta, \Pi^{-1}; t(I), x_0(J)) &\subseteq \widehat{C}(\theta, e^{\theta\|\nabla v_\epsilon\|_{C^0}}\Pi^{-1}; t(I), x_0(J)) \\ &\subseteq \widehat{C}(\theta, \Xi^{-1}; t(I), x_0(J)), \end{aligned} \tag{71}$$

and the remainder terms $U^\ell_{TJ,(D)}$ and $U^\ell_{HJ,(D)}$ are supported in the same Eulerian cylinder by (67).

Before we can obtain symmetric tensors that solve the equations in (68), we must check that the necessary orthogonality conditions

$$\int_{\mathbb{R}^3} U^\ell(t, x)\, dx = 0, \qquad \int_{\mathbb{R}^3} (x^j U^\ell - x^\ell U^j)(t, x)\, dx = 0, \qquad 1 \le j, \ell \le 3 \tag{72}$$

are satisfied, where U^ℓ is the (nonperiodic restriction of) $U^\ell_{TJ,(D)}$ or $U^\ell_{HJ,(D)}$. To check condition (72), note that $U^\ell_{HJ,(D)}$ is by construction in (57) and (64) the divergence of a smooth *symmetric* tensor with compact support, and that $U^\ell_{TJ,(D)}$ has the form $\nabla_a \nabla_c [T_J^{ac\ell}] + \nabla_j U_J^{j\ell}$ (using (32), (61), (64)), where $U_J^{j\ell}$ is symmetric and both $T_J^{ac\ell}$ and $U_J^{j\ell}$ have compact support in the cylinder (71). Integrating by parts, one obtains the conditions (72) for the nonperiodic restrictions of both $U^\ell_{TJ,(D)}$ and $U^\ell_{HJ,(D)}$.

We now have the necessary inputs to solve the symmetric divergence equation with good control over the support and boundedness properties of the solution map. We recall[7] the following result of [Isett and Oh 2016b, Section 10] (in particular Lemmas 10.3 and 10.4).

Lemma 10.3. *Suppose U is a smooth vector field on $\mathbb{R} \times \mathbb{R}^d$ with support in an Eulerian cylinder $\widehat{C}(\theta, \rho; t_0, x_0)$ relative to a smooth vector field $\bar{v}$. If U is orthogonal at all times to the rotation and translation vector fields on $\mathbb{R}^d$ in the sense of* (72), *then there is a symmetric tensor field $R_U^{j\ell}$ that is also supported in the same Eulerian cylinder and that solves $\nabla_j R_U^{j\ell} = U^\ell$. The solution can be taken to depend linearly on U and to satisfy the bounds*

$$\|\nabla_{\vec{a}} R_U\|_{C^0} \lesssim \rho \sum_{\vec{a}_1 + \vec{a}_2 = \vec{a}} \rho^{-|\vec{a}_1|} \|\nabla_{\vec{a}_2} U\|_{C^0}.$$

[7] Here we do not need to use the additional advective derivative estimates that were used in [Isett and Oh 2016b] since we only need to bound spatial derivatives.

Applying this lemma, we obtain symmetric tensors solving (68) such that

$$\operatorname{supp} R^{j\ell}_{TJ,(D)} \cup \operatorname{supp} R^{j\ell}_{HJ,(D)} \subseteq \widehat{C}(\theta,\, \Xi^{-1};\, t(I), x_0(J)), \tag{73}$$

$$\begin{aligned}\|R^{j\ell}_{TJ,(D)}\|_{C^0} + \|R^{j\ell}_{HJ,(D)}\|_{C^0} &\lesssim \Xi^{-1}(\|U^{\ell}_{TJ,(D)}\|_{C^0} + \|U^{\ell}_{HJ,(D)}\|_{C^0}) \\ &\overset{(66)}{\lesssim} B_\lambda^{-1} N^{-D/2} (\log \widehat{\Xi})^{5/2} e_v^{1/2} e_R^{1/2},\end{aligned} \tag{74}$$

$$\|\nabla_{\vec{a}} R^{j\ell}_{TJ,(D)}\|_{C^0} + \|\nabla_{\vec{a}} R^{j\ell}_{HJ,(D)}\|_{C^0} \lesssim_{|\vec{a}|} \Xi^{-1} \sum_{|\vec{b}|\le|\vec{a}|} \Xi^{|\vec{a}|-|\vec{b}|}(\|\nabla_{\vec{b}} U^{\ell}_{TJ,(D)}\|_{C^0} + \|\nabla_{\vec{b}} U^{\ell}_{HJ,(D)}\|_{C^0}). \tag{75}$$

We now set $D = 2$ and define

$$R^{j\ell}_{TJ} = Q^{j\ell}_{TJ,(D)} + R^{j\ell}_{TJ,(D)} \quad \text{and} \quad R^{j\ell}_{HJ} = Q^{j\ell}_{HJ,(D)} + R^{j\ell}_{HJ,(D)}.$$

Combining (65), (67), (73), and (74) into (60), we obtain the estimate

$$\|R_T\|_{C^0} + \|R_H\|_{C^0} \lesssim B_\lambda^{-1} (\log \widehat{\Xi})^{5/2} \frac{e_v^{1/2} e_R^{1/2}}{N}. \tag{76}$$

To sum the estimates we have also used the fact that the number of distinct cylinders of the form (73) that can intersect at a given point in space-time (t, x) is bounded by an absolute constant. To check this fact, note that if two cylinders indexed by J and J' intersect at a point $(t^*, x^*) \in \mathbb{R} \times \mathbb{T}^3$, then

$$\begin{aligned}&(t^*, x^*) \in \widehat{C}(\theta,\, \Xi^{-1};\, t(I), x_0(J)) \cap \widehat{C}(\theta,\, \Xi^{-1};\, t(I'), x_0(J')) \quad \Rightarrow \\ &\qquad I = I' \quad \text{and} \quad (t(I), x_0(J)), (t(I), x_0(J')) \overset{(69)}{\in} \widehat{\Gamma}(\theta,\, \Xi^{-1};\, t^*, x^*) \\ &\qquad\qquad \overset{(70)}{\in} \widehat{C}(\theta,\, e^{\theta\|\nabla v_\epsilon\|_{C^0}}\Xi^{-1};\, t^*, x^*) \subseteq \widehat{C}(\theta,\, 3\Xi^{-1};\, t^*, x^*).\end{aligned}$$

The number of indices $J = (I, f)$ for which $(t(I), x_0(J))$ can belong to a given ball of radius $3\Xi^{-1} \lesssim \Pi^{-1}$ is bounded by an absolute constant by the construction of the cutoff functions.

We can now take B_λ to be a sufficiently large number such that the right-hand side of (76) is bounded by $(\log \widehat{\Xi})^{5/2} e_v^{1/2} e_R^{1/2}/(20N)$ (and so that $\lambda = B_\lambda N \Xi \in \mathbb{Z}$ is an integer). This choice achieves our desired bound for $\|R_1\|_{C^0}$ when combined with (59). The desired bounds for higher derivatives

$$\|\nabla_{\vec{a}} R_T\|_{C^0} + \|\nabla_{\vec{a}} R_H\|_{C^0} \lesssim (N\Xi)^{|\vec{a}|} (\log \widehat{\Xi})^{5/2} \frac{e_v^{1/2} e_R^{1/2}}{N}, \quad 1 \le |\vec{a}| \le 3,$$

now follow from (65), (66), (75) and the observations concerning the overlaps of the cylinders (73). The assertions about the desired support of $R_1^{j\ell}$ asserted in Lemma 6.2 are clear from construction.

The proof of Lemma 6.2 will now be complete after explaining the proof of Proposition 10.1.

10A. ***The parametrix expansion.*** We now prove Proposition 10.1 using the argument in the proof of [Isett 2018, Proposition 17.6]. Let $U^\ell = u^\ell \omega(\lambda \Gamma_I)$ be given as in the assumptions of Proposition 10.1. By Fourier-expanding $\omega(X)$ as a function on $\mathbb{T}^3$, we have

$$U^\ell = \sum_{m \ne 0} \hat{\omega}(m) e^{i\lambda \xi_m(t,x)} u^\ell(t, x), \tag{77}$$

where $m \in \mathbb{Z}^3$ and $\xi_m(t,x) := m \cdot \Gamma_I(t,x)$. Following the proof of [Isett 2018, Proposition 17.6], we set

$$Q^{j\ell}_{(D)} = \sum_{m\neq 0} \hat{\omega}(m) Q^{j\ell}_{(D),m}, \quad Q^{j\ell}_{(D),m} := \lambda^{-1} \sum_{k=1}^{D} e^{i\lambda\xi_m} q^{j\ell}_{(k),m}. \tag{78}$$

The amplitudes $q^{j\ell}_{(k),m}$ are constructed inductively with a sequence of amplitudes $u^{\ell}_{(k),m}$ such that

$$\begin{aligned} i\nabla_j \xi_m q^{j\ell}_{(k),m} &= u^{\ell}_{(k-1),m}, \\ u^{\ell}_{(k),m} &= -\lambda^{-1}\nabla_j q^{j\ell}_{(k),m} \end{aligned} \tag{79}$$

and $u^{\ell}_{(0),m} = u^{\ell}$. By (77), (79) and induction on D, we then obtain

$$\begin{aligned} U^{\ell} &= \nabla_j Q^{j\ell}_{(D)} + U^{\ell}_{(D)}, \\ U^{\ell}_{(D)} &= \sum_{m\neq 0} \hat{\omega}(m) e^{i\lambda\xi_m} u^{\ell}_{(D),m}. \end{aligned} \tag{80}$$

More specifically, to solve (79) we first choose smooth functions $\bar{q}^{j\ell}_a(p)$ of a variable $p \in \mathbb{R}^3 \setminus \{0\}$, symmetric in $j\ell$, such that each $\bar{q}^{j\ell}_a(p)$ is degree -1 homogeneous ($\bar{q}^{j\ell}_a(\alpha p) = \alpha^{-1}\bar{q}^{j\ell}_a(p)$ if $\alpha \in \mathbb{R}\setminus\{0\}$) and such that $ip_j \bar{q}^{j\ell}_a(p) = \delta^{\ell}_a$ for all $p \neq 0$. See [Isett 2018, Proposition 17.6] for an explicit example. We then set $q^{j\ell}_{(k),m} := \bar{q}^{j\ell}_a(\nabla\xi_m) u^a_{(k-1),m}$, so that (79) is satisfied.

From this construction we see that both $Q^{\ell}_{(D)}$ and $U^{\ell}_{(D)}$ have support contained in $\operatorname{supp} u^{\ell}$. We obtain the desired estimates for $Q^{\ell}_{(D)}$ and $U^{\ell}_{(D)}$ stated in Proposition 10.1 from the formulas (78) and (80) by using the bounds

$$\begin{aligned} \|\nabla_{\vec{a}} q^{j\ell}_{(k),m}\|_{C^0} &\lesssim N^{-(k-1)/2} N^{|\vec{a}|/2} \Xi^{|\vec{a}|} H_{D+3,0}[u] \quad \text{for all } 0 \le |\vec{a}| \le D-k+4, \quad 1 \le k \le D, \\ \|\nabla_{\vec{a}} u^{\ell}_{(k),m}\|_{C^0} &\lesssim B_\lambda^{-1} N^{-k/2} N^{|\vec{a}|/2} \Xi^{|\vec{a}|} H_{D+3,0}[u] \quad \text{for all } 0 \le |\vec{a}| \le D-k+3, \quad 1 \le k \le D, \end{aligned}$$

from the proof of [Isett 2018, Proposition 17.6] (where $H_{D+3,0}[u]$ is written simply as H), and by using the rapid decay of $|\hat{\omega}(m)| \lesssim (1+|m|)^{-40}$ to ensure convergence in the summation over $m \in \mathbb{Z}^3$. (The main point in the estimate is that each spatial derivative of the sum costs at most a factor of λ.)

11. Iterating the main lemma

We now explain the proof of Theorem 1.1. Similar to other convex integration constructions, the theorem will be proven by repeatedly applying Lemma 5.1 to obtain a sequence of Euler–Reynolds flows $(v_{(k)}, p_{(k)}, R_{(k)})$ indexed by k (with frequency-energy levels bounded by $(\Xi_{(k)}, e_{v,(k)}, e_{R,(k)})$) that will converge uniformly to the solution v stated in Theorem 1.1. Unlike previous works, we introduce here a new and sharper approach to estimating the regularity and to optimizing the choice of parameters governing the growth of frequencies.

To initialize the construction, we construct a smooth Euler–Reynolds flow $(v_{(1)}, p_{(1)}, R_{(1)})$ with compact support in time that satisfies

$$\sup_{x\in\mathbb{T}^3} v_{(1)}(0,x) \ge 10 \tag{81}$$

and has frequency-energy levels (to order 3 in C^0) bounded by $(\Xi_{(1)}, e_{R,(1)}, e_{R,(1)})$, where $\Xi_{(1)} = \widehat{\Xi}_{(1)}$ and $e_{R,(1)}$ are large and small parameters, respectively, that remain to be chosen. One way to produce such an Euler–Reynolds flow is to apply the main lemma in the convex integration scheme of [Isett 2017a] (as was done in [Isett 2018]). This approach has some added benefits such as the ability to obtain arbitrarily large increases in energy within an arbitrarily small time interval [Isett 2017a]. For the present purpose it will suffice to take a simpler approach.

We take $v_{(1)}$ to have the form $v^\ell_{(1)} = \psi(B^{-1}t)U^\ell$, where ψ is a smooth cutoff with $\psi(0) = 1$ and $0 \le \psi(t) \le 1$ for all t, B is a large parameter, and $U^\ell : \mathbb{T}^3 \to \mathbb{R}^3$ is a smooth vector field that satisfies

$$\int_{\mathbb{T}^3} U^\ell(x)\,dx = 0, \quad \nabla_\ell U^\ell = 0, \quad \nabla_j(U^j U^\ell) = 0, \quad \sup_{x\in\mathbb{T}^3} U^\ell(x) \ge 10.$$

For example, one can take a sufficiently large Mikado flow for $U^\ell(x)$. We then take $p_{(1)} = 0$ and $R_{(1)}$ to be a symmetric tensor that solves

$$\nabla_j R^{j\ell}_{(1)} = \partial_t v^\ell_{(1)} = B^{-1}\psi'(B^{-1}t)U^\ell(X) \tag{82}$$

by applying an appropriate, degree -1 Fourier-multiplier to the right-hand side of (82). The Euler–Reynolds flow $(v_{(1)}, p_{(1)}, R_{(1)})$ obtained in this way has frequency-energy levels (to order 3 in C^0) bounded by $(\overline{\Xi}, 1, e_{R,(1)})$, where $\overline{\Xi}$ depends only on U^ℓ, and where $e_{R,(1)} \lesssim B^{-1}$ can be made arbitrarily small by taking B large depending on U^ℓ. It follows from Definition 4.2 that $(v_{(1)}, p_{(1)}, R_{(1)})$ also have frequency-energy levels bounded by

$$(\Xi_{(1)}, e_{v,(1)}, e_{R,(1)}) := (\overline{\Xi} e^{-1/2}_{R,(1)}, e_{R,(1)}, e_{R,(1)}),$$

where we have now fixed our choice of $\Xi_{(1)} := \overline{\Xi} e^{-1/2}_{R,(1)}$ in terms of the small parameter $e_{R,(1)}$ that remains to be chosen.

11A. ***Heuristics and deriving the optimization problem for the parameters.*** The sequence of frequency-energy levels $(\Xi, e_v, e_R)_{(k)}$ and Euler–Reynolds flows will now be determined by repeatedly applying Lemma 5.1, so that the following rules hold. (Here $\widehat{C}$ and C_L denote the two constants of Lemma 5.1 and $\widehat{\Xi}_{(k)} := (e_v/e_R)^{1/2}_{(k)} \Xi_{(k)}$.)

$$\Xi_{(k+1)} = \widehat{C} N_{(k)} \Xi_{(k)}, \tag{83}$$

$$e_{v,(k+1)} = (\log \widehat{\Xi}_{(k)}) e_{R,(k)}, \tag{84}$$

$$e_{R,(k+1)} = \frac{e_{R,(k)}}{g_{(k)}}, \tag{85}$$

$$N_{(k)} = (\log \widehat{\Xi}_{(k)})^A \left(\frac{e_v}{e_R}\right)^{1/2}_{(k)} g_{(k)}, \quad A := \tfrac{5}{2}. \tag{86}$$

The sequence $g_{(k)} > 1$ describes the "gain" in the size of the error after stage k, and the sequence of frequency growth parameters $N_{(k)}$ is determined by inequality (10) in Lemma 5.1, so that this choice of $N_{(k)}$ achieves the desired gain. To work with the estimate (11), it will also be useful to impose that

$$(\log \widehat{\Xi}_{(k+1)})^{1/2} e^{1/2}_{R,(k+1)} \le \tfrac{1}{2} (\log \widehat{\Xi}_{(k)})^{1/2} e^{1/2}_{R,(k)} \quad \text{for all } k \ge 1. \tag{87}$$

The Euler–Reynolds flows constructed by repeatedly applying Lemma 5.1 using the above choice of parameters $N_{(k)}$ will converge uniformly to the velocity field $v^\ell = v^\ell_{(1)} + \sum_{k=1}^\infty V^\ell_{(k)}$. Assuming (87), which is verified in Proposition 11.1 below, this solution will be nontrivial and continuous for $e_{R,(1)}$ chosen small enough (depending on $\overline{\Xi}$, $\widehat{C}$ and C_L) thanks to (81) and

$$\sum_{k=1}^\infty \|V^\ell_{(k)}\|_{C^0} \overset{(11),(87)}{\leq} \sum_{k=0}^\infty C_L (\log \widehat{\Xi}_{(1)})^{1/2} e_{R,(1)} 2^{-k} \leq 5. \tag{88}$$

As $R_{(k)}$ converges uniformly to 0, one has from the Euler–Reynolds system that the associated sequence of pressures $p_{(k)} = \Delta^{-1}\nabla_j\nabla_\ell(R^{j\ell}_{(k)} - v^j_{(k)}v^\ell_{(k)})$ converge weakly in $\mathcal{D}'(\mathbb{R}\times\mathbb{T}^3)$ to $p = -\Delta^{-1}\nabla_j\nabla_\ell(v^j v^\ell)$, and that the pair (v, p) form a weak solution to the Euler equations.

Our goal is now to choose $g_{(k)}$ that optimize the regularity of the solution v. The key evolution rule that isolates $\frac{1}{3}$ as the limiting regularity and plays a key role in our analysis will be the following:

$$\delta_{(k)}\big(\tfrac{1}{3}\log\widehat{\Xi}_{(k)} + \tfrac{1}{2}\log e_{R,(k)}\big) = \big(\tfrac{1}{3}A + \tfrac{1}{6}\big)\log\log\widehat{\Xi}_{(k)} + \log\widehat{C}. \tag{89}$$

Here $\delta_{(k)}[f_{(k)}] = f_{(k+1)} - f_{(k)}$ is the discrete differencing operator and $A = \frac{5}{2}$. A crucial point is that (89) holds for *all* possible choices of $g_{(k)}$.

With the goal of computing regularity in mind, suppose $\Delta x \in \mathbb{R}^3$ with, say, $0 < |\Delta x| \leq 10^{-2}$. Writing

$$v = v_{(\bar{k})} + \sum_{k\geq\bar{k}} V_{(k)} \quad \text{and} \quad L_k := \log\widehat{\Xi}_{(k)},$$

we can bound $|v(t, x+\Delta x) - v(t,x)|$ using (87) by

$$\begin{aligned}|v(t,x+\Delta x) - v(t,x)| &\leq \|\nabla v_{(\bar{k})}\|_{C^0}|\Delta x| + \sum_{k\geq\bar{k}} 2\|V_{(k)}\|_{C^0} \leq \Xi_{(\bar{k})} e^{1/2}_{v,(\bar{k})}|\Delta x| + 4C_L(\log\widehat{\Xi}_{(\bar{k})})^{1/2} e^{1/2}_{R,(\bar{k})} \\ &\leq 4C_L L_{\bar{k}}(\widehat{\Xi}_{(\bar{k})}|\Delta x| + 1)e^{1/2}_{R,(\bar{k})}.\end{aligned} \tag{90}$$

The estimate is optimized by choosing $\bar{k}$ to be the largest value k for which $\widehat{\Xi}_{(k)}|\Delta x| \leq 1$. Now assuming $\bar{k}$ has been chosen as this value, the estimate (90) leads to

$$\begin{aligned}|v(t,x+\Delta x) - v(t,x)| &\leq 8C_L L_{\bar{k}} e^{1/2}_{R,(\bar{k})} = 8C_L L_{\bar{k}} \widehat{\Xi}^{-1/3}_{(\bar{k})} \exp\big(\tfrac{1}{3}\log\widehat{\Xi}_{(\bar{k})} + \tfrac{1}{2}\log e_{R,(\bar{k})}\big) \\ &\lesssim L_{\bar{k}}\widehat{\Xi}^{-1/3}_{(\bar{k}+1)} \exp\big(\tfrac{1}{3}\delta_{(k)}\log\widehat{\Xi}_{(k)}\big|_{k=\bar{k}} + \tfrac{1}{3}\log\widehat{\Xi}_{(k)} + \tfrac{1}{2}\log e_{R,(k)}\big) \\ &\lesssim |\Delta x|^{1/3} L_{\bar{k}} \exp\big(\tfrac{1}{3}\delta_{(k)}\log\widehat{\Xi}_{(k)}\big|_{k=\bar{k}} + \tfrac{1}{3}\log\widehat{\Xi}_{(k)} + \tfrac{1}{2}\log e_{R,(k)}\big).\end{aligned} \tag{91}$$

Using (89) to expand $\frac{1}{3}\log\widehat{\Xi}_{(k)} + \frac{1}{2}\log e_{R,(k)}$, we minimize the right-hand side of (91) if we minimize

$$H_{\bar{k}} := \left(\frac{1}{3}(\log\widehat{\Xi}_{(\bar{k}+1)} - \log\widehat{\Xi}_{(\bar{k})}) + \sum_{k=1}^{\bar{k}-1}(\log\log\widehat{\Xi}_{(k)} + \log\widehat{C})\right). \tag{92}$$

The expression (92) now reveals the optimization problem for choosing $g_{(k)}$. Namely, to control the term $\delta_{(k)}\log\widehat{\Xi}_{(k)}$, the frequencies should not grow too quickly. However, a slow growth of frequencies produces a long summation and a poor estimate for the sum as the construction is iterated many times before achieving a given length scale. Intuitively, the best estimate should be achieved if the two terms are balanced, which suggests the parameters $L_k = \log \widehat{\Xi}_{(k)}$ should satisfy the discrete version of the equation

$$\frac{dL}{dk} = 3 \int_1^k (\log L(\kappa) + c)\, d\kappa,$$

whose solutions grow like $L_k = (3 + o(1))k^2 \log k$ at infinity.

We will see that the regularity is optimal precisely when L_k are chosen to have this growth.

11B. *Parameter asymptotics and optimization.* With this motivation, we take $g_{(k)} = e^{\gamma k \log k}$, where $\gamma > 0$ is a parameter that will be chosen to optimize the regularity. To simplify the algebra we can restrict to $k \geq 2$ by assuming that the Euler–Reynolds flows $(v_{(1)}, p_{(1)}, R_{(1)}) = (v_{(2)}, p_{(2)}, R_{(2)})$ and their frequency-energy levels are equal.

Before estimating the regularity, we wish to fix our choice of the parameter $e_{R,(1)}$ that dictates the initial frequency-energy levels. We therefore verify the assumption (87) (restricting now to $\gamma \geq 2$).

Proposition 11.1. *If $\gamma \geq 2$ and $e_{R,(1)}$ is small enough depending on $\widehat{C}$, then* (87) *holds for all $k \geq 2$.*

Proof. Taking logs of (87), it suffices to bound the quantity

$$\tfrac{1}{2}\delta_{(k)} \log\log \widehat{\Xi}_{(k)} + \tfrac{1}{2}\delta_{(k)} \log e_{R,(k)} = \tfrac{1}{2}\delta_{(k)} \log\log \widehat{\Xi}_{(k)} - \tfrac{1}{2} \log g_{(k)} \tag{93}$$

by $-\log 2$ uniformly in k.

Towards this goal, we set $Z_k := \widehat{C}(\log \widehat{\Xi}_{(k)})^{A+1/2}$ to be the lower-order factor from (83) and (86). Linearizing $\log(\,\cdot\,)$ around $L_k := \log \widehat{\Xi}_{(k)}$ and using (83)–(86) and concavity, we have

$$\delta_{(k)} \log\log \widehat{\Xi}_{(k)} = \log(\log \widehat{\Xi}_{(k)} + \log(Z_k g_{(k)}^{3/2})) - \log\log \widehat{\Xi}_{(k)} \leq \frac{\log(Z_k g_{(k)}^{3/2})}{\log \widehat{\Xi}_{(k)}}. \tag{94}$$

We now substitute (94) into (93) and take $e_{R,(1)}$ small enough to ensure that $\widehat{\Xi}_{(k)} \geq \Xi_{(k)} \geq \Xi_{(1)} = \overline{\Xi} e_{R,(1)}^{-1/2}$ is large enough so that the following bound holds for all $k \geq 2$:

$$(93) \leq \frac{\log Z_k}{\log \widehat{\Xi}_{(k)}} - \frac{1}{3} \log g_{(k)}. \tag{95}$$

Taking $e_{R,(1)}$ smaller and hence $\Xi_{(1)}$ larger, we can ensure that the function

$$f(\Xi) := \frac{\log(\widehat{C}(\log \Xi)^{A+1/2})}{\log \Xi}$$

is decreasing in Ξ on the interval $\Xi \in [\Xi_{(1)}, \infty)$. From $\widehat{\Xi}_{(k)} \geq \Xi_{(1)}$ and (95) we obtain

$$(95) \leq \frac{\log(\widehat{C}(\log \Xi_{(1)})^{A+1/2})}{\log \Xi_{(1)}} - \frac{1}{3} \log g_{(2)} \quad \text{for all } k \geq 2. \tag{96}$$

We have that $-\frac{1}{3} \log g_{(2)} = -\frac{2}{3}\gamma \log 2 \leq -\frac{4}{3} \log 2$. Taking $e_{R,(1)}$ small and thus $\Xi_{(1)}$ large, we can bound (96) and therefore (93) by $-\log 2$, which establishes Proposition 11.1. □

At this point, we choose $e_{R,(1)}$ sufficiently small (depending on $\widehat{C}$ and C_L) to satisfy the assumptions of Proposition 11.1 and such that (88) holds.

With the initial frequency-energy levels determined, we now turn to the asymptotics of the frequency-energy levels for large k. These asymptotics are summarized as follows.

Proposition 11.2. *For all $k \geq 3$ and the above choice of $g_{(k)}$, we have the asymptotics*

$$-\log e_{R,(k)} = \frac{\gamma k^2}{2} \log k + O(k \log k), \tag{97}$$

$$\frac{1}{2} \log\left(\frac{e_v}{e_R}\right)_{(k)} = \frac{1}{2}\gamma k \log k + O(\log k), \tag{98}$$

$$\delta_{(k)} \log \widehat{\Xi}_{(k)} = \frac{3}{2}\gamma k \log k + O(\log k), \tag{99}$$

$$\log \widehat{\Xi}_{(k)} = \frac{3}{2}\frac{\gamma k^2}{2} \log k + O(k \log k), \tag{100}$$

$$\log\log \widehat{\Xi}_{(k)} = 2 \log k + O(1), \tag{101}$$

$$\frac{1}{3} \log \widehat{\Xi}_{(k)} + \frac{1}{2} \log e_{R,(k)} = 2\left(\frac{A}{3} + \frac{1}{6}\right) k \log k + O(k), \tag{102}$$

$$\Xi_{(k)} = \exp\left(\frac{3\gamma k^2}{4} \log k + O(k \log k)\right), \tag{103}$$

together with the bounds

$$(\log \widehat{\Xi}_{(k)})^{-1} = O(k^{-2}(\log k)^{-1}), \tag{104}$$

$$\log\log \widehat{\Xi}_{(k)} = O(\log k). \tag{105}$$

Here the implicit constants in the $O(\,\cdot\,)$ notation depend only on $\widehat{C}$, γ, $\Xi_{(1)}$, $e_{R,(1)}$ and $A = \frac{5}{2}$.

The proof will proceed by induction on $k \geq 3$ and will use some extra notation for the induction. We write $C_{(97)}, \ldots, C_{(105)}$ to refer to the implicit constants in the $O(\,\cdot\,)$ notation in the proposition. For example the term in (105) is bounded by $|O(\log k)| \leq C_{(105)} \log k$. We assume at the onset that all the constants $C_{(97)}, \ldots, C_{(105)}$ are sufficiently large depending on $\Xi_{(1)}$ and $e_{R,(1)} = e_{v,(1)}$ such that the bounds (97)–(105) hold for $k = 3$. The proof will make use of the Taylor expansion formula

$$f(X+Y) = f(X) + Y \int_0^1 f'(X + \sigma Y)\, d\sigma = f(X) + f'(X)Y + Y^2 \int_0^1 (1-\sigma) f''(X + \sigma Y)\, d\sigma. \tag{106}$$

Proof of (97). The equality follows from the evolution rule $\log e_{R,(k+1)} = -\log g_{(k)} + \log e_{R,(k)}$ and

$$\sum_{1 \leq I \leq k} \log g_{(I)} = \sum_{1 \leq I \leq k} \gamma I \log I = \frac{\gamma k^2}{2} \log k + O(k \log k), \quad k \geq 3$$

(where the constant above depends on γ). □

Proof of (104). From $\log \widehat{\Xi}_{(k+1)} \geq \log g_{(k)} + \log \widehat{\Xi}_{(k)}$, we have

$$k^2 \log k \lesssim \sum_{3 \leq I \leq k} \log g_{(I)} \leq \log \widehat{\Xi}_{(k)}.$$ □

Proof of (105). Let $L_k := \log \widehat{\Xi}_{(k)}$ and $Z_k = \widehat{C}(\log \widehat{\Xi}_{(k)})^{A+1/2}$. Then for some $A_0 \geq 1$ and all $k \geq 3$,

$$\begin{aligned}\delta_{(k)} \log\log \widehat{\Xi}_{(k)} &= \log(L_k + \log(Z_k g_{(k)}^{3/2})) - \log L_k \\ &\leq L_k^{-1}(\log Z_k + \log g_{(k)}^{3/2}) \leq A_0 C_{(104)}(k^{-2}(\log k)^{-1} \log\log \widehat{\Xi}_{(k)} + k^{-1}).\end{aligned}$$

Choose $k^* = k^*(C_{(104)})$ large enough that $A_0 C_{(104)} k^{-2} \leq 10^{-1}\delta_{(k)} \log k$ for all $k \geq k^*$, and assume that $C_{(105)}$ is large enough that (105) holds for $k \leq k^*$.

We now proceed by induction on k to obtain (105) for $k > k^*$. Assuming (105) for k, we have

$$\delta_{(k)} \log\log \widehat{\Xi}_{(k)} \leq 10^{-1} C_{(105)} \delta_{(k)} \log k + A_0 C_{(104)} k^{-1} \leq C_{(105)} \delta_{(k)} \log k \quad \text{for } k \geq k^*$$

if $C_{(105)}$ is sufficiently large, which implies (105) for $k+1$, and thus for all $k \geq k^*$ by induction. □

Proof of (98). The equality follows from (105) and

$$\frac{1}{2}\log\left(\frac{e_v}{e_R}\right)_{(k+1)} = \frac{1}{2}(\log g_{(k)} + \log\log \widehat{\Xi}_{(k)}).$$ □

Proof of (99)–(100). For $k \geq 3$, we have by (98) and (105) $\big(\text{for } A = \frac{5}{2}\big)$

$$\begin{aligned}\delta_{(k)} \log \widehat{\Xi}_{(k)} &= \frac{1}{2}\log\left(\frac{e_v}{e_R}\right)_{(k+1)} + \log g_{(k)} + A \log\log \widehat{\Xi}_{(k)} \\ &= \frac{3\gamma}{2} k \log k + O(\log k) = \frac{3\gamma}{2}\delta_{(k)}\left[\frac{k^2}{2}\log k\right] + O(\delta_{(k)}[k \log k]),\end{aligned}$$

which implies both (99) and (100) after summing over k. □

Proof of (101). Again writing $L_k = \log \widehat{\Xi}_{(k)}$ and $Z_k = \widehat{C}(\log \widehat{\Xi}_{(k)})^{A+1/2}$, we have by Taylor expansion

$$\begin{aligned}\delta_{(k)} \log\log \widehat{\Xi}_{(k)} &= \log(L_k + \log(Z_k g_{(k)}^{3/2})) - \log L_k \\ &= (\log \widehat{\Xi}_{(k)})^{-1} \log(Z_k g_{(k)}^{3/2}) - \int_0^1 d\sigma \frac{(\log(Z_k g_{(k)}^{3/2}))^2(1-\sigma)}{(L_k + \sigma \log(Z_k g_{(k)}^{3/2}))^2}.\end{aligned}$$

The main term is

$$(\log \widehat{\Xi}_{(k)})^{-1} \log g_{(k)}^{3/2} = 2k^{-1} + O(k^{-2}) = 2\delta_{(k)} \log k + O(k^{-2})$$

by (100). The remaining terms are of size $O(k^{-2})$ by (105) and (100) again. Summing over k gives the desired result (101). □

Proof of (102). Equation (102) follows from (89), (101) and summation over k. □

Proof of (103). Equation (103) follows from

$$\log \Xi_{(k)} = \log \widehat{\Xi}_{(k)} - \frac{1}{2}\log\left(\frac{e_v}{e_R}\right)_{(k)},$$

equation (100) and (98). □

We now return to analyzing the regularity estimate (91). From (100), (99), (102), and by the definitions of $\bar{k}$ and $\widehat{\Xi}_{(\bar{k})}$, we obtain (using (106) with $f(X) = X^{-1}$ or $\log X$) that, for all $|\Delta x| \le 10^{-2}$,

$$\bar{k}^2 \log \bar{k} \lesssim \log \widehat{\Xi}_{(\bar{k})} \le \log |\Delta x|^{-1} \le \log \widehat{\Xi}_{(\bar{k}+1)} \lesssim \bar{k}^2 \log \bar{k},$$

$$\frac{3\gamma}{4}\bar{k}^2 \log \bar{k} = \log |\Delta x|^{-1} + O(\bar{k} \log \bar{k}), \tag{107}$$

$$(\log |\Delta x|^{-1})^{-1} = \left(\frac{4}{3\gamma} + O(\bar{k}^{-1})\right)\bar{k}^{-2}(\log \bar{k})^{-1},$$

$$\log(\bar{k}^2) = \log\log |\Delta x|^{-1} + O(\log\log \bar{k}). \tag{108}$$

To bound (91) purely in terms of $|\Delta x|$, we first estimate the logarithm of the term $L_{\bar{k}} \exp(H_{\bar{k}})$ appearing in (91)–(92) (using $A = \frac{5}{2}$ and $\frac{1}{3}A + \frac{1}{6} = 1$) by

$$\begin{aligned}(\log |\Delta x|^{-1})^{-1} \cdot (H_{\bar{k}} + \log L_{\bar{k}}) &= \left(\frac{4}{3\gamma}(1 + O(\bar{k}^{-1}))(\bar{k}^2 \log \bar{k})^{-1}\right) \cdot \left(\left(\frac{\gamma}{2} + 2\right)\bar{k} \log \bar{k} + O(\bar{k})\right) \\ &= \frac{4}{3\gamma}\left(\frac{\gamma}{2} + 2\right)\bar{k}^{-1} + O(\bar{k}^{-1}(\log \bar{k})^{-1}) \\ &= \frac{4}{3\gamma}\left(\frac{\gamma}{2} + 2\right)(\bar{k}^2 \log \bar{k})^{-1/2}(\log \bar{k})^{1/2} + O(\bar{k}^{-1}(\log \bar{k})^{-1}) \\ &= 2^{-1/2}\left(\frac{4}{3\gamma}\right)\left(\frac{\gamma}{2} + 2\right)(\bar{k}^2 \log \bar{k})^{-1/2}(\log\log |\Delta x|^{-1})^{1/2} \\ &\qquad + O\left(\frac{\log\log \bar{k}}{(\bar{k}^2 \log \bar{k})^{1/2}(\log\log |\Delta x|^{-1})^{1/2}}\right).\end{aligned}$$

In the last line we used (108) and (106) with $f(X) = X^{1/2}$. From (107) and (106) we then have

$$\begin{aligned}(\log |\Delta x|^{-1})^{-1} \cdot (H_{\bar{k}} + \log L_{\bar{k}}) &= 2^{-1/2}\left(\frac{4}{3\gamma}\right)^{1/2}\left(\frac{\gamma}{2} + 2\right)(\log |\Delta x|^{-1})^{-1/2}(\log\log |\Delta x|^{-1})^{1/2} \\ &\qquad + O\left(\frac{\log\log\log |\Delta x|^{-1}}{(\log |\Delta x|^{-1})^{1/2}(\log\log |\Delta x|^{-1})^{1/2}}\right).\end{aligned} \tag{109}$$

The bound (109) is optimized by taking $\gamma = 4$, which is precisely the value that leads to the asymptotic $\log \widehat{\Xi}_{(k)} = (3 + o(1))k^2 \log k$ predicted by the heuristics at the conclusion of Section 11A. Substituting into (91), we finally obtain

$$|v(t, x + \Delta x) - v(t, x)| \lesssim |\Delta x|^{1/3 - B\sqrt{(\log\log |\Delta x|^{-1})/(\log |\Delta x|^{-1})}}, \tag{110}$$

where one can take the constant $B = 2\sqrt{\frac{2}{3}}$ at the expense of introducing the additional lower-order term[8] from (109). In particular, v belongs to $\bigcap_{\alpha<1/3} L_t^\infty C_x^\alpha$, and therefore belongs to $\bigcap_{\alpha<1/3} C_{t,x}^\alpha$ by the results in [Isett 2023]. To check that v has compact support in time, note that the time support in each iteration grows by at most a factor

$$\Xi_{(k)}^{-1} e_{v,(k)}^{-1/2} = \widehat{\Xi}_{(k)}^{-1} e_{R,(k)}^{-1/2} = \widehat{\Xi}_{(k)}^{-2/3} \exp\left(-\tfrac{1}{3} \log \widehat{\Xi}_{(k)} - \tfrac{1}{2} \log e_{R,(k)}\right).$$

[8]The derivation of (92) suggests that taking $g_{(k)} = \left(\sum_{I=1}^{k} (\log\log \widehat{\Xi}_{(I)} + \log \widehat{C})\right) + (\log\log \widehat{\Xi}_k/2)$ would optimize the lower-order terms as well, although this alternative choice would not affect the leading-order terms.

Using (100) and (102), we conclude that the series $\sum_k \Xi_{(k)}^{-1} e_{v,(k)}^{-1/2}$ converges, and hence the limiting solution is supported on a finite time interval. This calculation concludes the proof of Theorem 1.1.

12. Improving the borderline estimate

In this section, we sketch roughly how the value of the B appearing in the regularity estimate (110) can be improved by combining with the approach to the gluing lemma introduced in [Buckmaster et al. 2019a].

Recall that, in the notation of [Isett 2018], the gluing lemma is proved by introducing, for a given Euler–Reynolds flow (v, p, R), corrections

$$y^\ell = \sum_I \eta_I y_I^\ell \quad \text{and} \quad \bar{p} = \sum_I \eta_I \bar{p}_I$$

to the velocity and pressure such that the new velocity field $\tilde{v}^\ell = v^\ell + y^\ell$ and pressure $\tilde{p} = p + \bar{p}$ solve the Euler–Reynolds system with a new Reynolds stress $\widetilde{R}$ that is supported in disjoint time intervals of width $\theta \sim (\log \widehat{\Xi})^{-2} \Xi^{-1} e_v^{-1/2}$. The new stress $\widetilde{R}$ is constructed in terms of symmetric tensors $r_I^{j\ell}$ that solve $\nabla_j r_I^{j\ell} = y_I^\ell$, which are obtained by solving the following initial value problem:[9]

$$\begin{aligned}(\partial_t + v^i \nabla_i) r_I^{j\ell} &= \mathcal{R}^{j\ell}[\nabla_i[\nabla_a v^i r_I^{ab}] - y_I^i \nabla_i v^b] - y_I^j y_I^\ell - \bar{p}_I \delta^{j\ell} - R^{j\ell},\\ r_I^{j\ell}(t(I), x) &= 0.\end{aligned} \tag{111}$$

Here $\mathcal{R}^{j\ell}$ is an order -1 operator that inverts the divergence equation in symmetric tensors, and the identity $\nabla_j r_I^{j\ell} = y_I^\ell$ can be checked using the equation

$$\begin{aligned}\partial_t y_I^\ell + v^i \nabla_i y_I^\ell + y_I^i \nabla_i v^\ell + \nabla_j (y_I^j y_I^\ell) + \nabla^\ell \bar{p}_I &= -\nabla_j R^{j\ell},\\ \partial_t y_I^\ell + v^i \nabla_i y_I^\ell + y_I^i \nabla_i u_I^\ell + \nabla^\ell \bar{p}_I &= -\nabla_j R^{j\ell},\end{aligned} \tag{112}$$

where $u_I^\ell = v^\ell + y_I^\ell$ is the classical solution to incompressible Euler equations with initial data $v^\ell(t_0(I), x)$.

In [Buckmaster et al. 2019a], a different approach to solving and estimating solutions of the equation $\nabla_j r_I^{j\ell} = y_I^\ell$ is taken. There, one first considers the potential $\tilde{z}_I = \Delta^{-1} \nabla \times y_I$, which solves $\nabla \times \tilde{z}_I = y_I$, $\operatorname{div} \tilde{z}_I = 0$, and turns out to satisfy an evolution equation that (like (111)) has a good structure. From $\tilde{z}_I$, one then obtains a symmetric antidivergence for y_I by applying a zeroth-order operator (e.g., $r_I^{j\ell} = \mathcal{R}^{j\ell}[\nabla \times \tilde{z}_I]$), which is estimated using Schauder and commutator estimates for Calderón–Zygmund operators (CZOs). (We note that, conversely, estimates for $\tilde{z}_I$ can be deduced from those of $r_I^{j\ell}$ above by similar zeroth-order commutator estimates.) The key simplification comes in treating the term $\Delta^{-1} \nabla \times [y_I \cdot \nabla v]$ that is analogous to the term $\mathcal{R}^{j\ell}[y_I \cdot \nabla v]$ in (111), the latter of which had been treated by a decomposition into frequency increments in [Isett 2018]. For the present applications, the estimates employed in [Buckmaster et al. 2019a], which apply the classical local well-posedness theory for Euler and Schauder and commutator estimates for CZOs, are not strong enough as they lose small powers of the frequency Ξ, which restricts the regularity to $\frac{1}{3} - \epsilon$ for some $\epsilon > 0$. However, as we now explain, combining the techniques in

[9] Here we have simplified the equations by combining the equations for the $\rho_I^{j\ell}$ and $z_I^{j\ell}$ from [Isett 2018] into one equation.

[Buckmaster et al. 2019a] and [Isett 2018] leads to a logarithmic improvement in the timescale of the gluing and hence a logarithmic improvement in the main estimate of the iteration.

The approach of [Buckmaster et al. 2019a] can be extended to any dimension using the antisymmetric potential[10] defined by

$$\psi_I^{ab} = \mathcal{B}^{ab}[y_I] := \Delta^{-1}(\nabla^a y_I^b - \nabla^b y_I^a),$$

which solves the Hodge system[11]

$$\nabla_a \psi^{ab} = y_I^b, \quad (\nabla \wedge \psi)^{abc} := \nabla^a \psi^{bc} - \nabla^b \psi^{ac} + \nabla^c \psi^{ab} = 0, \quad \int_{\mathbb{T}^3} \psi(x)\, dx = 0. \tag{113}$$

Using the antisymmetry of ψ_I^{ab}, one obtains the identity

$$y_I^i \nabla_i v^\ell = \nabla_a[\psi_I^{ai} \nabla_i v^\ell]. \tag{114}$$

Using (114) and $\psi_I^{ab} = \mathcal{B}^{ab}[y_I^j] = \mathcal{B}^{ab}\nabla_i[r_I^{ij}]$ can provide an alternative approach to treating the low-frequency part of the term $\mathcal{R}^{j\ell}[y_I^i \nabla_i v^b]$ in (111) and the analogous term in the pressure.

Towards improving the timescale of the gluing, apply (112) along with the calculus identity (which we express in both index and invariant notation)

$$\begin{aligned} \Delta\psi^{ab} &= (\nabla \wedge [\nabla \neg \psi])^{ab} + (\nabla \neg [\nabla \wedge \psi])^{ab}, \\ \nabla_i[\nabla^i \psi^{ab}] &= (\nabla^a[\nabla_j \psi^{jb}] - \nabla^b[\nabla_j \psi^{ja}]) + \nabla_i[\nabla^i \psi^{ab} + \nabla^b \psi^{ia} + \nabla^a \psi^{bi}], \end{aligned}$$

to derive the following equation for the potential ψ_I^{ab}, generalizing [Buckmaster et al. 2019a, Section 3.3]:[12]

$$\begin{aligned} \Delta[(\partial_t + v^i \nabla_i)\psi_I^{jk}] = \nabla_a \nabla_i[(\psi_I^{jk} \wedge \nabla^a) v^i] - \nabla^j \wedge [\nabla_a(\psi_I^{ai} \nabla_i v^k) + \nabla_a[y_I^a y_I^k] + \nabla_i R^{ik}] \\ + \nabla^j \wedge [\nabla_i[\nabla_a v^i \psi_I^{ak}]], \\ \psi_I^{jk} \wedge \nabla^a v^i := \psi_I^{jk} \nabla^a v^i - \psi_I^{ak} \nabla^j v^i + \psi_I^{aj} \nabla^k v^i. \end{aligned} \tag{115}$$

This derivation relies on (113) and $\nabla^j \nabla^k \bar{p}_I - \nabla^k \nabla^j \bar{p}_I = 0$, and uses that $\nabla_i v^i = 0$ to maintain the divergence form. The convention above for $\nabla^j \wedge$ applied to a vector field is $\nabla^j \wedge u^k := \nabla^j u^k - \nabla^k u^j$, while $(\psi_I^{jk} \wedge \nabla^a) v^i$ indicates a sum over cyclic permutations of jka in $\psi_I^{jk} \nabla^a v^i$.

One may now couple (115) to (111) while writing

$$\mathcal{R}^{j\ell}[y_I^i \nabla_i v^\ell] = \mathcal{R}^{j\ell} \nabla_a[\psi_I^{ai} \nabla_i v^\ell]$$

and similarly for the analogous term $\Delta^{-1}\nabla_\ell[y_I^i \nabla_i v^\ell]$ appearing in the pressure $\bar{p}_I$. By considering a weighted norm $\hbar(t) = \hbar_I(t)$ such that (setting $\widehat{N} := (e_v/e_R)^{1/2}$ and, for instance, $\alpha = \frac{1}{7}$)

$$\begin{aligned} \|\nabla_{\vec{a}} r_I\|_{C^0} + \|\nabla_{\vec{a}} \psi_I\|_{C^0} + \widehat{\Xi}^{-\alpha}(\|\nabla_{\vec{a}} r_I\|_{\dot{C}^\alpha} + \|\nabla_{\vec{a}} \psi_I\|_{\dot{C}^\alpha}) \le \widehat{N}^{(|\vec{a}|-2)_+} \Xi^{|\vec{a}|} (\Xi e_v^{1/2})^{-1} e_R \hbar(t), \\ \|\nabla_{\vec{a}} y_I\|_{C^0} + \widehat{\Xi}^{-\alpha}\|y_I\|_{\dot{C}^\alpha} \le \widehat{N}^{(|\vec{a}|-2)_+} \Xi^{|\vec{a}|} e_R^{1/2} \hbar(t) \qquad \text{for } 0 \le |\vec{a}| \le 3, \end{aligned}$$

[10] We write ψ^{ab} to agree with the usual stream function ψ in dimension 2, which is related by $\psi^{ab} = \psi\epsilon^{ab}$, where the two-dimensional volume element ϵ^{ab} is the unique antisymmetric tensor with $\epsilon^{12} = 1$.

[11] We caution the reader that our normalizations for wedge products are taken to elucidate the present calculations, but do not agree with all standard normalizations, which can differ up to multiplication by constants.

[12] A slight departure from [Buckmaster et al. 2019a] is the isolation of quadratic terms of the form $y_I^j y_I^\ell$, which would be estimated jointly in $y_I^i \nabla_i v^\ell + \nabla_i(y_I^i y_I^\ell) = y_I^i \nabla_i u_I^\ell$ in the approach of that paper. The $y_I^j y_I^\ell$ terms are kept separate here in order to avoid a resulting additional derivative loss in the estimates.

and following the Littlewood–Paley approach to the gluing estimates in [Isett 2018], we obtain the bound

$$\hbar(t) \lesssim (\log \widehat{\Xi})\,\Xi e_v^{1/2} \int_0^t (1+\hbar(\tau))^2\, d\tau. \tag{116}$$

The prefactor in (116) improves the analogous prefactor in [Isett 2018, Proposition 10.1] by a factor of $(\log \widehat{\Xi})^{-1}$, which thus improves the timescale θ by a logarithmic factor to $\theta \sim (\log \widehat{\Xi})^{-1}(\Xi e_v^{1/2})^{-1}$. What this improvement in timescale yields is that the time cutoff factors of η_I' in the terms of the form $\sim \eta_I' r_I^{j\ell}$ that compose the new stress error $\widetilde{R}$ have become smaller by a factor $(\log \widehat{\Xi})^{-1}$ in size, while the antidivergence terms $r_I^{j\ell}$ have increased in size by a factor of $(\log \widehat{\Xi})$ over the elongated time scale.

Although the estimate $\|\widetilde{R}\|_{C^0} \lesssim (\log \widehat{\Xi}) e_R$ on the stress does not improve, the estimate on the advective derivative improves logarithmically to

$$\|D_t \widetilde{R}\|_{C^0} \lesssim (\log \widehat{\Xi})^2\, \Xi e_v^{1/2} e_R.$$

The bound (19) for the new frequency-energy levels in the main lemma similarly improves by one power of $\log \widehat{\Xi}$ to become

$$(\Xi', e_v', e_R') = \left(\widetilde{C} N \Xi,\, (\log \widehat{\Xi}) e_R,\, (\log \widehat{\Xi})^A \frac{e_v^{1/2} e_R^{1/2}}{N}\right), \quad A = \frac{3}{2}. \tag{117}$$

(One can alternatively pursue an approach closer to [Buckmaster et al. 2019a] wherein the equation for ψ_I^{ab} is coupled to the evolution equation[13] for a different, symmetric antidivergence such as

$$\tilde{r}_I^{j\ell} := \Delta^{-1}(\nabla^j y_I^\ell + \nabla^\ell y_I^j).$$

Implementing this alternative approach requires additional, sharper commutator estimates.)

The improvement in the power $A = \frac{3}{2}$ of $\log \widehat{\Xi}$ in (117) then leads to an improvement in the constant B in the leading-order term of the regularity estimate (110). Namely, repeating the analysis of Section 11 but with $A = \frac{3}{2}$ instead of $\frac{5}{2}$ improves the leading-order term in (102), which leads to a factor of $\left(\frac{1}{2}\gamma + \frac{4}{3}\right)$ in (109) in place of $\left(\frac{1}{2}\gamma + 2\right)$. After choosing $\gamma = \frac{8}{3}$ to optimize (109), one obtains a leading-order constant of $B = \frac{4}{3} = 2\left(\frac{2}{3}\right)$ instead of $B = 2\sqrt{\frac{2}{3}}$. Note that, with the improved constant, the function space implicitly defined by the estimate (110) is strictly contained in the one with the larger value of B, and the corresponding norms are not comparable to each other.

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[13] Using the evolution equation for the symmetric antidivergence is important to avoid an additional logarithmic loss that would be incurred from attempting to deduce estimates for $\tilde{r}_I$ directly from those for ψ_I.

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EXTREME TEMPORAL INTERMITTENCY IN THE LINEAR SOBOLEV TRANSPORT ALMOST SMOOTH NONUNIQUE SOLUTIONS

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We revisit the notion of temporal intermittency to obtain sharp nonuniqueness results for linear transport equations. We construct divergence-free vector fields with sharp Sobolev regularity $L^1_t W^{1,p}$ for all $p<\infty$ in space dimensions $d \geq 2$ whose transport equations admit nonunique weak solutions belonging to $L^p_t C^k$ for all $p<\infty$ and $k \in \mathbb{N}$. In particular, our result shows that the time-integrability assumption in the uniqueness of the DiPerna–Lions theory is essential. The same result also holds for transport-diffusion equations with diffusion operators of arbitrarily large order in any dimensions $d \geq 2$.

1. Introduction

We consider the linear transport equation on the torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ with $d \geq 2$:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho|_{t=0} = \rho_0, \end{cases} \tag{1-1}$$

where $\rho : \mathbb{T}^d \times [0,T] \to \mathbb{R}$ is a scalar density function and $u : \mathbb{T}^d \times [0,T] \to \mathbb{R}^d$ is a given incompressible vector field, i.e., $\operatorname{div} u = 0$ and $\rho_0 : \mathbb{T}^d \to \mathbb{R}$ is a given initial datum. The linearity of the equation allows us to prove the existence of weak solutions — even for very rough vector fields — that satisfy the equation in the sense of distributions

$$\int_{\mathbb{T}^d} \rho_0 \varphi(\,\cdot\,,0)\,dx = \int_0^T \int_{\mathbb{T}^d} \rho(\partial_t \varphi + u \cdot \nabla \varphi)\,dx\,dt \quad \text{for all } \varphi \in C_c^\infty(\mathbb{T}^d \times [0,T)). \tag{1-2}$$

In this paper, we focus on the issue of the uniqueness/nonuniqueness of weak solutions satisfying (1-2) with $\rho \in L^1_{t,x}$ and $\rho u \in L^1_{t,x}$, for vector fields with Sobolev regularity. The celebrated DiPerna–Lions theory provides natural criteria for the uniqueness of the weak solutions for Sobolev vector fields:

Theorem 1.1 [DiPerna and Lions 1989]. *Let $p, q \in [1,\infty]$, and let $u \in L^1(0,T; W^{1,q}(\mathbb{T}^d))$ be a divergence-free vector field. For any $\rho_0 \in L^p(\mathbb{T}^d)$, there exists a unique renormalized solution $\rho \in C([0,T]; L^p(\mathbb{T}^d))$ to* (1-1). *Moreover, if*

$$\frac{1}{p} + \frac{1}{q} \leq 1, \tag{1-3}$$

then this solution ρ is unique among all weak solutions in the class $L^\infty(0,T; L^p(\mathbb{T}^d))$.

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In recent years, there has been a growing interest [Brué et al. 2021; Cheskidov and Luo 2021; Modena and Sattig 2020; Modena and Székelyhidi 2018] in showing the (possible) sharpness of the DiPerna–Lions condition (1-3), but so far the nonuniqueness constructions have not reached the full complement of (1-3) in the class of $L^\infty_t L^p$ solutions. In this paper, we show that the time-integrability assumption in the DiPerna–Lions uniqueness theorem is essential. More precisely, we show the following.

Theorem 1.2. *For any dimension $d \geq 2$, there exists $u : \mathbb{T}^d \times [0, T] \to \mathbb{R}^d$, a divergence-free velocity vector field, satisfying $u \in L^1(0, T; W^{1,p}(\mathbb{T}^d))$ for all $p < \infty$ such that the uniqueness of* (1-1) *fails in the class*

$$\rho \in \bigcap_{\substack{p<\infty \\ k\in\mathbb{N}}} L^p(0, T; C^k(\mathbb{T}^d)) \quad \textit{and} \quad \rho u \in L^1(\mathbb{T}^d \times [0, T]).$$

This result is proved by the convex integration technique, which was brought to fluid dynamics by the pioneering work [De Lellis and Székelyhidi 2009] and has seen applications to the transport equation in [Brué et al. 2021; Cheskidov and Luo 2021; Modena and Sattig 2020; Modena and Székelyhidi 2018]. More details on the background and historical development will be discussed shortly. The key ingredient in the proof of Theorem 1.2 is the use of temporal intermittency, introduced in our previous works [Cheskidov and Luo 2021; 2022; 2023]. In particular, it improves our previous result [Cheskidov and Luo 2021] in terms of the integrability in time of the solution ρ and the spatial regularity of u and ρ. Moreover, Theorem 1.2 is sharp in the following two ways:

(1) The vector field cannot be $L^1_t W^{1,\infty}$ for which any $L^1_{t,x}$ solution of (1-1) with $\rho u \in L^1_{t,x}$ must coincide[1] a.e. with the Lagrangian solution.

(2) The density class cannot have any $L^\infty_t C^k$ regularity for $k \in \mathbb{N}$ due to the DiPerna–Lions condition (1-3).

Background and comparison. While the classical method of characteristics implies the well-posedness of (1-1) for Lipschitz vector fields, for non-Lipschitz vector fields, the method of characteristics no longer applies, and the well-posedness of (1-1) becomes challenging. The renormalization theory of [DiPerna and Lions 1989] provides powerful well-posedness of (1-1) under suitable Sobolev regularity assumptions on the vector field, and the renormalized solutions are shown to be unique in the regime (1-3).

Since Aizenman's example [1978], there have been examples of nonuniqueness at the Lagrangian level [Alberti et al. 2019; Colombini et al. 2003; Depauw 2003; Drivas et al. 2022; Yao and Zlatoš 2017], that is, constructions of vector fields whose flow maps exhibit degeneration. However, for a long time, the existence of nonunique (Eulerian) weak solutions of (1-1) for divergence-free Sobolev vector fields $u \in L^1_t W^{1,p}$ was unknown. To our knowledge, the first Eulerian construction of nonuniqueness was obtained in [Crippa et al. 2015] using the framework of [De Lellis and Székelyhidi 2009] for bounded vector fields.

Inspired by the spatially intermittent construction in [Buckmaster and Vicol 2019], the breakthrough result [Modena and Székelyhidi 2018] gave the first example of a Sobolev vector field with nonunique weak solutions to (1-1) and led to a lot of interest in improving nonuniqueness constructions to larger functional classes. Below we list the regimes where the nonuniqueness has been achieved:

[1] For instance, by a duality argument using estimates of the flow as in [Ambrosio et al. 2005, Proposition 8.1.7].

(1) [Modena and Székelyhidi 2018; 2019]: $\rho \in C_t L^p$ when $u \in C_t W^{1,q}$ for $1/p + 1/q > 1 + 1/(d-1)$ and $d \geq 3$.

(2) [Modena and Sattig 2020]: $\rho \in C_t L^p$ when $u \in C_t W^{1,q}$ for $1/p + 1/q > 1 + 1/d$.

(3) Bruè, Colombo, and De Lellis [Brué et al. 2021]: positive[2] $\rho \in C_t L^p$ when $u \in C_t W^{1,q}$ for $1/p + 1/q > 1 + 1/d$.

(4) [Cheskidov and Luo 2021]: $\rho \in L^1_t L^p$ when $u \in L^1_t W^{1,q}$ for $1/p + 1/q > 1$ and $d \geq 3$.

In summary, in the class of $L^\infty_t L^p$ densities, the nonuniqueness has been achieved in the regime $1/p + 1/q > 1 + 1/d$, while nonuniqueness in the regime $1/p + 1/q > 1$ is possible if one settles for $L^1_t L^p$ densities. However, it was not known whether $1/p + 1/q = 1$ is still the critical threshold for $L^1_t L^p$ densities.

Our main goal here is to show that the DiPerna–Lions scaling $1/p + 1/q = 1$ becomes irrelevant once the time integrability of ρ is slightly weakened. In particular, Theorem 1.2 follows from the following convex integration construction.

Theorem 1.3. *Let $d \geq 2$, $\varepsilon > 0$, and $N \in \mathbb{N}$. Let $\tilde{\rho} \in C^\infty(\mathbb{T}^d \times \mathbb{R})$ be such that $\operatorname{supp}_t \tilde{\rho} \subset (0, T)$ and $\fint_{\mathbb{T}^d} \rho(x, t)\, dx = 0$ for all $t \in \mathbb{R}$.*

Then there exist a divergence-free vector field $u : \mathbb{T}^d \times [0, T] \to \mathbb{R}^d$ and a density $\rho : \mathbb{T}^d \times [0, T] \to \mathbb{R}$ such that all of the following hold:

(1) *$u \in L^1(0, T; W^{1,p}(\mathbb{T}^d))$ and $\rho \in L^p(0, T; C^k(\mathbb{T}^d))$ for all $1 \leq p < \infty$ and $k \in \mathbb{N}$.*

(2) *$\rho u \in L^1(\mathbb{T}^d \times [0, T])$ and (ρ, u) is a weak solution to (1-1) in the sense of (1-2).*

(3) *The deviation of ρ in $C^N(\mathbb{T}^d)$ norm is small: $\|\rho - \tilde{\rho}\|_{L^N_t C^N} \leq \varepsilon$.*

(4) *ρ has a compact temporal support: $\operatorname{supp}_t \rho \subset \operatorname{supp}_t \tilde{\rho}$.*

Remarks. (1) Here our initial data is always zero and attained in the classical sense. It is also easy to show that the obtained solution ρ is continuous in time in the sense of distributions (see Lemma 7.7 in [Cheskidov and Luo 2021] for details).

(2) Theorem 1.3 continues to hold for the transport-diffusion equation with a parabolic regularization $\Delta^m \rho$ of arbitrary order in the same regularity classes $(\rho, u) \in L^p_t C^k \times L^1_t W^{1,p}$, see Theorem 6.1. To our knowledge, this is the first example of a PDE where parabolic regularization does not provide any additional rigidity for the uniqueness of a class of weak solutions.

(3) The nonunique solutions ρ must change their signs — it is known by [Caravenna and Crippa 2021, Corollary 5.4] that any sign-definite solution $\rho \in L^1_{t,x}$ of $L^1_t W^{1,d+}$ vector fields is Lagrangian, see also [Brué et al. 2021, Section 8.2].

(4) By the linearity of (1-1), for any initial data $\rho_0 \in L^p(\mathbb{T}^d)$, the constructed vector field gives nonunique solutions in the class $\rho \in L^q_t L^p$ for any $q < \infty$. Indeed, one can add the constructed solution on top of the renormalized solution associated to ρ_0.

[2]Well-posedness for positive ρ can go beyond the DiPerna–Lions range, see [Brué et al. 2021, Theorem 1.5].

Strategy of the proof. We conclude with some final remarks on the proof. As said, we used the convex integration technique brought to fluid dynamics by the pioneering work of [De Lellis and Székelyhidi 2009]. The groundbreaking technique of that work resulted in breakthroughs in the fluids community over the last decade, and we refer readers to [Buckmaster et al. 2019; Buckmaster and Vicol 2019; De Lellis and Székelyhidi 2009; 2013; Isett 2018; Modena and Székelyhidi 2018] for a complete account.

The construction follows the same framework of temporal intermittency in our previous work [Cheskidov and Luo 2021]. A key difference is the regularity $L^p_t C^k$ for the density, which requires extreme intermittency in time when progressing to high frequencies. Since the density does not enjoy any "reasonable" L^∞_t regularity, from the duality $\rho u \in L^1_{t,x}$ we can gain a surprising regularity of almost L^1_t Lipschitz of the vector field. As in that previous work, this extreme temporal intermittency necessitates the use of stationary building blocks, as otherwise, the error produced by the large acceleration of the density becomes insurmountable with the non-Lipschitzness of the vector field, see Lemma 4.1 below. Once extreme intermittency in time is achieved, a little deduction of the time regularity of the density from L^∞_t to L^p_t allows us to gain essentially infinitely many derivatives in space for the density.

Finally, since the density enjoys essentially infinite many derivatives in space, the same construction also holds for transport-diffusion equations with diffusion operators of arbitrarily large order in any dimension $d \geq 2$. Surprisingly, even in dimension $d = 2$ a diffusion of an arbitrarily high order is not able to provide uniqueness for this class of weak solutions.

Organization. The rest of the paper is organized as follows.

- We prove the main theorem stated in the introduction in Section 2 by assuming Proposition 2.1, whose proof is the main content of this paper.
- In Section 3, we first introduce temporal intermittency into the construction, which is essential for our scheme. Next, we recall Mikado densities and Mikado flows as spatial building blocks. Finally, we use these temporal and spatial building blocks to define the density and velocity perturbations.
- In Section 4, we first specify the oscillation and concentration parameters and obtain estimates on the velocity and density perturbations claimed in Proposition 2.1.
- Section 5 is devoted to deriving the new defect field and its estimates, finishing the proof of Proposition 2.1.
- In Section 6, we show that the same nonuniqueness holds for transport-diffusion equations with arbitrarily high order of diffusion as well.
- In the Appendix, we recall some (now standard) technical tools in convex integration, namely the improved Hölder inequalities and antidivergence operators.

2. The main proposition and proof of Theorem 1.3

Notations. Throughout the paper, we fix the spatial domain $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, identified with a periodic box $[0, 1]^d$. Average over $\mathbb{T}^d$ is denoted by $\fint f = \int_{\mathbb{T}^d} f$. Functions on $\mathbb{T}^d$ are identified as periodic ones in $\mathbb{R}^d$, and we say f is $\sigma^{-1}\mathbb{T}^d$-periodic if

$$f(x+\sigma^{-1}k) = f(x) \quad \text{for any } k \in \mathbb{Z}^d.$$

Spatial Lebesgue norms are denoted by $\|\cdot\|_{L^p} = \|\cdot\|_{L^p(\mathbb{T}^d)}$, while we write $\|\cdot\|_{L^p_{t,x}}$ for Lebesgue norms taken in the space-time domain $\mathbb{T}^d \times [0,T]$. If a function f is time-dependent, we write $\|f(t)\|_{L^p}$ to indicate that the spatial norm is taken at a time slice $t \in [0,T]$. For a Banach space X, we use the notation $\|\cdot\|_{L^p_t X}$ to denote the norm on Bochner spaces $L^p([0,T]; X)$, such as $\|\cdot\|_{L^1_t W^{k,p}}$ and $\|\cdot\|_{L^p_t C^k}$.

The differentiation operations such as ∇, Δ, and div are meant for differentiation in space only.

We use the notation $X \lesssim Y$, which means $X \le CY$ for some constant $C > 0$. The notation $X \sim Y$ means both $X \lesssim Y$ and $Y \lesssim X$ at the same time.

Continuity-defect equation. As in [Modena and Székelyhidi 2018], we consider the continuity-defect equation to obtain approximate solutions to the transport equation

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = \operatorname{div} R, \\ \operatorname{div} u = 0, \end{cases} \tag{2-1}$$

where $R : \mathbb{T}^d \times [0,T] \to \mathbb{R}^d$ is called the defect field. In what follows, a triple (ρ, u, R) will denote a smooth solution to (2-1). Recall that for a function $f \in L^1_{t,x}$, its temporal support $\operatorname{supp}_t f$ is the closure of the set

$$\{t \in [0,T] : \|f(\cdot,t)\|_{L^1(\mathbb{T}^d)} > 0\}.$$

We now state the main proposition of the paper and use it to prove Theorem 1.3.

Proposition 2.1. *Let $d \ge 2$. There exists a universal constant $M > 0$ such that the following holds.*

Suppose (ρ, u, R) is a smooth solution of (2-1) on $[0,1]$ such that $\operatorname{supp}_t R \subset (0,1)$. Then, for any $1 \le p \in \mathbb{N}$ and any $0 < \delta < \frac{1}{2}$, there exists another smooth solution (ρ_1, u_1, R_1) of (2-1) on $[0,1]$ such that the density perturbation $\theta := \rho_1 - \rho$ and the vector field perturbation $w = u_1 - u$ satisfy the following:

(1) *Both θ and w have zero spacial mean and*

$$\operatorname{supp}_t \theta \subset \operatorname{supp}_t R. \tag{2-2}$$

(2) *θ and w satisfy the estimates*

$$\|\theta\|_{L^p_t C^p} \le \delta, \tag{2-3}$$

$$\|w\|_{L^1_t W^{1,p}} \le \delta, \tag{2-4}$$

$$\|\theta w + \theta u + \rho w\|_{L^1_{t,x}} \le M \|R\|_{L^1_{t,x}}. \tag{2-5}$$

(3) *The new defect field R_1 satisfies*

$$\operatorname{supp}_t R_1 \subset \operatorname{supp}_t R \tag{2-6}$$

and the estimate

$$\|R_1\|_{L^1_{t,x}} \le \delta. \tag{2-7}$$

Proof of Theorem 1.3. We assume $T = 1$ without loss of generality. We will construct a sequence (ρ_n, u_n, R_n), $n = 1, 2, \dots$ of solutions to (2-1) as follows. For $n = 1$, we set

$$\begin{aligned} \rho_1(t) &:= \tilde{\rho}, \\ u_1(t) &:= 0, \\ R_1(t) &:= \mathcal{R}(\partial_t \tilde{\rho}), \end{aligned}$$

where $\mathcal{R} = \Delta^{-1}\nabla$ is the inverse divergence in the Appendix. Then (ρ_1, u_1, R_1) solves (2-1) trivially by the constant mean assumption on $\tilde{\rho}$.

Next, we apply Proposition 2.1 inductively to obtain (ρ_n, u_n, R_n) for $n = 2, 3, \dots$ as follows. Given (ρ_n, u_n, R_n), we apply Proposition 2.1 with parameters

$$p_n = N2^n, \quad \delta_n = \varepsilon 2^{-n},$$

to obtain a new triple $(\rho_{n+1}, u_{n+1}, R_{n+1})$. Then the perturbations $\theta_n := \rho_{n+1} - \rho_n$ and $w_n := u_{n+1} - u_n$ and the defect field R_n satisfy

$$\|\theta_n\|_{L^{p_n}_t C^{p_n}} \le \delta_n, \quad \|w_n\|_{L^1_t W^{1,p_n}} \le \delta_n, \tag{2-8a}$$

$$\|R_{n+1}\|_{L^1_{t,x}} \le \delta_n, \tag{2-8b}$$

$$\|\theta_n w_n + \theta_n u_n + \rho_n w_n\|_{L^1_{t,x}} \le M \|R_n\|_{L^1_{t,x}}, \tag{2-8c}$$

for all $n = 1, 2, \dots$. In addition, due to (2-6) and (2-2), we have

$$\operatorname{supp}_t \theta_n \subset \operatorname{supp}_t \tilde{\rho} \quad \text{for all } n \in \mathbb{N}. \tag{2-9}$$

Hence by (2-8a) there exists $(\rho, u) \in L^p_t C^p \times L^1_t W^{1,p}$ for all $p \in \mathbb{N}$ such that

$$\rho_n \to \rho \quad \text{in } L^p_t C^p \quad \text{and} \quad u_n \to u \quad \text{in } L^1_t W^{1,p} \quad \text{for all } p \in \mathbb{N}. \tag{2-10}$$

Moreover, $\operatorname{supp}_t \rho \subset \operatorname{supp}_t \tilde{\rho}$ due to (2-9). Since $p_n \ge N$ and the time interval is of length 1,

$$\|\rho - \tilde{\rho}\|_{L^N_t C^N} \le \sum_{n \ge 1} \|\theta_n\|_{L^N_t C^N} \le \sum_{n \ge 1} \|\theta_n\|_{L^{p_n}_t C^{p_n}} \le \varepsilon.$$

It remains to show (ρ, u) is a weak solution. We first prove that $\rho u \in L^1_{t,x}$ and $\rho_n u_n \to \rho u$ in $L^1_{t,x}$. Using (2-8c),

$$\|\rho_{n+1} u_{n+1} - \rho_n u_n\|_{L^1_{t,x}} \le M \delta_{n-1} \quad \text{for } n \ge 2. \tag{2-11}$$

Thus the sequence $\rho_n u_n$ is Cauchy in $L^1_{t,x}$, and consequently there is $G \in L^1_{t,x}$ such that $\rho_n u_n \to G$ in $L^1_{t,x}$. Now we claim that $G = \rho u$. Thanks to (2-11), passing to subsequences and dropping subindices, we get $\rho_n \to \rho$ and $u_n \to u$ a.e. in $\mathbb{T}^d \times [0, 1]$. So $\rho_n u_n \to G$ a.e. in $\mathbb{T}^d \times [0, 1]$, and hence $\rho u = G$ and $\rho_n u_n \to \rho u$ in $L^1_{t,x}$. Since in addition $R_n \to 0$ in $L^1_{t,x}$ by (2-8b), it is standard to show that (ρ, u) is a weak solution to (1-1). □

3. Temporal intermittency, building blocks, and perturbations

The rest of the paper is devoted to the proof of Proposition 2.1. In this section, we introduce the temporal and spatial building blocks and use them to define the density and velocity perturbations.

Summary of parameters. Given arbitrarily large $p \in \mathbb{N}$ as in the statement of Proposition 2.1, we will fix three exponents in Lemma 4.1 below: $r > 1$ very close to 1, $0 < \alpha \ll 1$, and $0 < \gamma \ll 1$. These exponents are used to define three large parameters: the concentrations $\kappa, \mu \geq 1$ and oscillation $\sigma \in \mathbb{N}$. These three large parameters satisfy the hierarchy $1 \ll \sigma \ll \mu \ll \kappa$ — whose meaning will be made precise in Section 4 — but their exact values will be fixed at the end depending on the given solution (ρ, u, R).

Temporal functions $\tilde{g}_k$ and g_k. We start with defining the intermittent oscillatory functions $\tilde{g}_k$ and g_k that lie at the heart of our scheme. First, we fix a profile function $\widetilde{G} \in C_c^\infty((0,1))$ such that

$$\int_{[0,1]} \widetilde{G}^2 \, dt = 1, \quad \int_{[0,1]} \widetilde{G} \, dt = 0, \quad \|\widetilde{G}\|_{L^\infty} \leq 2, \tag{3-1}$$

and, for $k = 1, \dots, d$, define G_k to be the 1-periodic extension of $\widetilde{G}(\kappa(t - t_k))$, where $t_k \in [0,1]$ are chosen such that G_k have disjoint supports for different k. In other words, $G_k(t) = \sum_{n \in \mathbb{Z}} \widetilde{G}(n + \kappa(t - t_k))$. We will refer to $\kappa \geq 1$ as the temporal concentration parameter.

Next, for a large oscillation parameter $\sigma \in \mathbb{N}$ and a small exponent $0 < \alpha < 1$ to be fixed later, we define σ^{-1}-periodic functions

$$\tilde{g}_k(t) = \kappa^\alpha G_k(\sigma t), \quad g_k(t) = \kappa^{1-\alpha} G_k(\sigma t). \tag{3-2}$$

We will use $\tilde{g}_\kappa$ to oscillate the densities Φ_k, and g_κ to oscillate the vectors $\boldsymbol{W}_k$, defined in the following section. Note that, by (3-1),

$$\int_{[0,1]} \tilde{g}_k g_k \, dt = 1, \tag{3-3}$$

and by definitions of $\tilde{g}_k$ and g_k,

$$\|\tilde{g}_k\|_{L^q([0,1])} \sim \kappa^{\alpha - 1/q}, \quad \|\tilde{g}_k'\|_{L^q([0,1])} \sim (\kappa\sigma)\kappa^{\alpha - 1/q}, \quad \|g_k\|_{L^q([0,1])} \sim \kappa^{1-\alpha-1/q}. \tag{3-4}$$

Temporal correction function h_k. Now we define a σ^{-1}-periodic function $h_k : \mathbb{R} \to \mathbb{R}$ by

$$h_k(t) := \sigma \int_0^t (\tilde{g}_k g_k - 1) \, d\tau, \tag{3-5}$$

so that

$$\sigma^{-1} \partial_t h_k = \tilde{g}_k g_k - 1. \tag{3-6}$$

Thanks to (3-3), we have $\int_{[0,\sigma^{-1}]} \tilde{g}_k g_k \, dt = \sigma^{-1}$. Since $\tilde{g}_k g_k \geq 0$ by their definitions, it follows that h_k is well-defined and satisfies the estimate

$$\|h_k\|_{L^\infty[0,1]} \leq 1. \tag{3-7}$$

The function h_k will be used to design the temporal corrector θ_o in (3-17).

Mikado densities and flows. Here we recall the spatial building blocks for our convex integration construction; the Mikado densities and Mikado flows introduced in [Daneri and Székelyhidi 2017] and [Modena and Székelyhidi 2018]. These are periodic objects supported on pipes with a small radius. Note that we do not require them to have disjoint supports in space — each Mikado object will be coupled with a temporal function $\tilde{g}_k$ or g_k to achieve disjoint supports in space-time.

For $k = 1, \dots, d$, we denote each standard Euclidean basis vector by $\boldsymbol{e}_k = (0, \dots, 1, \dots, 0)$. For any $x \in \mathbb{R}^d$ and $k = 1, \dots, d$, we write $x'_k \in \mathbb{R}^{d-1}$ for the vector $x'_k = (x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_d)$.

Let $d \geq 2$ be the spatial dimension. We fix a vector field $\Omega \in C_c^\infty(\mathbb{R}^{d-1})$ and a scalar density $\phi \in C_c^\infty(\mathbb{R}^{d-1})$ such that

$$\operatorname{supp} \Omega \subset (0,1)^{d-1}, \quad \operatorname{div} \Omega = \phi, \quad \int_{\mathbb{R}^{d-1}} \phi^2 = 1. \tag{3-8}$$

For each $k = 1, \dots, d$, we define the nonperiodic Mikado objects

$$\begin{aligned} \widetilde{\Phi}_k(x) &= \phi(\mu x'_k), \\ \widetilde{\Omega}_k(x) &= \mu^{-1} \Omega(\mu x'_k), \\ \widetilde{W}_k(x) &= \mu^{d-1} \phi(\mu x'_k) \boldsymbol{e}_k, \end{aligned} \tag{3-9}$$

define the 1-periodic objects $\Omega_k : \mathbb{T}^d \to \mathbb{R}^d$, $\Phi_k : \mathbb{T}^d \to \mathbb{R}$, and $W_k : \mathbb{T}^d \to \mathbb{R}^d$ as the 1-periodic extensions of (3-9), and then rescale them by a large oscillation factor $\sigma \in \mathbb{N}$:

$$\Phi_k(x) = \Phi_k(\sigma x), \quad \Omega_k(x) = \Omega_k(\sigma x), \quad \boldsymbol{W}_k(x) = W_k(\sigma x). \tag{3-10}$$

We now summarize the properties of the constructed building blocks Ω_k, Φ_k, and $\boldsymbol{W}_k$ in the following theorem.

Theorem 3.1. *For all $\sigma \in \mathbb{N}$ and $\mu \geq 1$, the density Φ_k, potential Ω_k, and vector field $\boldsymbol{W}_k$ defined by* (3-10) *satisfy the following for every $k = 1, \dots, d$*:

(1) $\boldsymbol{W}_k : \mathbb{T}^d \to \mathbb{R}^d$, $\Phi_k : \mathbb{T}^d \to \mathbb{R}$, *and* $\Omega_k : \mathbb{T}^d \to \mathbb{R}^d$ *are smooth $\sigma^{-1}\mathbb{T}^d$-periodic functions, and $\boldsymbol{W}_k$, Φ_k have zero mean on $\mathbb{T}^d$.*

(2) $\operatorname{div} \boldsymbol{W}_k = \operatorname{div}(\Phi_k \boldsymbol{W}_k) = 0$, *and the density Φ_k is the divergence of the potential $\sigma^{-1}\Omega_k$*:

$$\operatorname{div} \Omega_k = \sigma \Phi_k. \tag{3-11}$$

(3) *For any $1 \leq p \leq \infty$ and $s \geq 0$,*

$$\|\Omega_k\|_{L^p} \lesssim \mu^{-1-(d-1)/p}, \tag{3-12a}$$

$$\|\Phi_k\|_{W^{s,p}} \lesssim (\sigma\mu)^s \mu^{-d-1/p}, \tag{3-12b}$$

$$\|\boldsymbol{W}_k\|_{W^{s,p}} \lesssim (\sigma\mu)^s \mu^{(d-1)(1-1/p)}. \tag{3-12c}$$

(4) *The following identity holds*:

$$\fint_{\mathbb{T}^d} \Phi_k(x) \boldsymbol{W}_k(x)\, dx = \boldsymbol{e}_k. \tag{3-13}$$

Proof. The first two points are direct consequences of the definitions while the last point follows from (3-8).

When $s = 0$, the bounds (3-12a)–(3-12c) follow from the small supports of the nonperiodic objects $\widetilde{\Phi}_k$, $\widetilde{\Omega}_k$, $\widetilde{W}_k$: the support set is a cylinder of radius $\sim \mu^{-1}$ and length 1. The general case $s > 0$ can be obtained by interpolation between the cases $s \in \mathbb{N}$. □

Density and velocity perturbations. Here we define perturbations (θ, w) given a defect field R as in Proposition 2.1.

Recall that the concentration parameters $\mu, \kappa \geq 1$ and the oscillation parameter $\sigma \in \mathbb{N}$ introduced so far will be specified in Lemma 4.1 below. The velocity perturbation is defined by

$$w := \sum_{1 \leq k \leq d} g_k \boldsymbol{W}_k. \tag{3-14}$$

For the density perturbation, first we decompose the defect field

$$R(x,t) = \sum_{1 \leq k \leq d} R_k(x,t) \boldsymbol{e}_k, \tag{3-15}$$

where the $\boldsymbol{e}_k$ are the standard Euclidean basis as before. We define the density perturbation as the sum of the zero-mean projection of the principal part and a small oscillation correction:

$$\theta = \mathbb{P}_{\neq 0} \theta_p + \theta_o,$$

where $\mathbb{P}_{\neq 0} f = f - \fint f$ is the projection removing the spatial mean, and

$$\theta_p := - \sum_{1 \leq k \leq d} \tilde{g}_k R_k \Phi_k, \tag{3-16}$$

$$\theta_o = \sigma^{-1} \operatorname{div} \sum_{1 \leq k \leq d} h_k R_k \boldsymbol{e}_k. \tag{3-17}$$

Note that $\operatorname{div} w = 0$ for all t since $\boldsymbol{W}_k$ is divergence-free, which also implies that

$$\operatorname{div}([\mathbb{P}_{\neq 0} \theta_p] w) = \operatorname{div}(\theta_p w).$$

By definitions, $\operatorname{supp}_t \theta \subset \operatorname{supp}_t R$ as required in (2-2) of Proposition 2.1.

4. Estimates of the density and velocity perturbations

The goal of this section is to obtain estimates (2-3), (2-4), and (2-5) on θ and w claimed in Proposition 2.1.

Choice of parameters. Now we specify all the oscillation and concentration parameters in the perturbation as explicit powers of a large frequency number $\lambda > 0$ that will be fixed in the end.

(1) Oscillation $\sigma \in \mathbb{N}$:

$$\sigma = \lceil \lambda^{2\gamma} \rceil.$$

Without loss of generality, we only consider values of λ such that $\sigma = \lambda^{2\gamma} \in \mathbb{N}$ in what follows.

(2) Concentration $\kappa, \mu \geq 1$:

$$\mu = \lambda, \quad \kappa = \lambda^{(d-2\gamma)/\alpha}.$$

Lemma 4.1. *For any $p \in \mathbb{N}$, there exist constants $\alpha > 0$, $0 < \gamma < \frac{1}{4}$, and $r > 1$ such that the following holds*:

$$(\sigma\mu)^p \kappa^{\alpha - 1/p} \le \lambda^{-\gamma} \quad (\theta_p \in L^p_t C^p), \tag{4-1}$$

$$\kappa^{-\alpha} (\sigma\mu)^1 \mu^{(d-1)(1-1/p)} \le \lambda^{-\gamma} \quad (w \in L^1_t W^{1,p}), \tag{4-2}$$

$$\kappa^{\alpha} \mu^{-1-(d-1)/r} \le \lambda^{-\gamma} \quad (\textit{acceleration error}). \tag{4-3}$$

Proof. We first fix $\gamma > 0$. Condition (4-2) in terms of power of λ reads

$$\frac{d-1}{p} \ge 5\gamma.$$

Since $p < \infty$, this condition is satisfied for $0 < \gamma < \frac{1}{4}$ sufficiently small. Expressing (4-1) in terms of power of λ gives

$$\alpha \le \frac{1}{p} \frac{d - 2\gamma}{(2p\gamma + p + d - \gamma)}.$$

Since $0 < \gamma < \frac{1}{4}$, this condition on α is automatically satisfied when

$$\alpha < \frac{d - \frac{1}{2}}{2p^2 + 2dp}.$$

We then fix $\alpha > 0$ according to this condition.

For condition (4-3), taking $r = 1$, the left-hand side becomes

$$\lambda^{d-2\gamma-1-d+1} = \lambda^{-2\gamma}.$$

Therefore, by continuity, (4-3) holds for $r > 1$ close enough to 1. □

We remark that Lemma 4.1 cannot hold for $p = \infty$ from its proof — the conditions (4-2) and (4-3) become incompatible when $p = \infty$. This is consistent with the $L^1_{t,x}$ unconditional uniqueness of $L^1_t W^{1,\infty}$ vector fields as in [Ambrosio et al. 2005, Proposition 8.1.7].

Estimates for the perturbations. In what follows, C_R will stand for a large constant that only depends on the triple (ρ, u, R) provided as the input by Proposition 2.1. It is important that C_R can *never* depend on the free parameters σ, μ, and κ in the building blocks that we used to define θ and w.

Lemma 4.2 (estimate on θ). *The density perturbation θ satisfies*

$$\|\theta\|_{L^p_t C^p} \le C_R \lambda^{-\gamma}.$$

Proof. For the principle part θ_p, since the space $C^p(\mathbb{T}^d)$ is an algebra, using Hölder's inequality, (3-4), and (3-12b), we obtain

$$\|\theta_p\|_{L^p_t C^p} \le \sum_{1 \le k \le d} \|\tilde{g}_k\|_{L^p} \|R_k\|_{L^\infty_t C^p} \|\Phi_k\|_{L^\infty_t C^p} \le C_R (\sigma\mu)^p \kappa^{\alpha - 1/p} \le C_R \lambda^{-\gamma},$$

where the last inequality holds due to condition (4-1).

For the temporal corrector θ_0 defined in (3-17), by Hölder's inequality and (3-7), we have

$$\|\theta_o\|_{L^\infty_t C^p} \le \sigma^{-1} \sum_{1\le k\le d} \|h_k\|_{L^\infty([0,1])} \|\operatorname{div}(R_k \boldsymbol{e}_k)\|_{L^\infty_t C^p} \le C_R \sigma^{-1}, \tag{4-4}$$

and the final bound holds by the definition of σ. □

Lemma 4.3 (estimate on w). *The velocity perturbation w satisfies*

$$\|w\|_{L^1_t W^{1,p}} \lesssim \lambda^{-\gamma}.$$

Proof. Using Hölder's inequality, (3-4), and (3-12c), we obtain

$$\|w\|_{L^1_t W^{1,p}} \le \sum_{1\le k\le d} \|g_k\|_{L^1} \|\boldsymbol{W}_k\|_{W^{1,p}} \lesssim \kappa^{-\alpha}(\sigma\mu)\mu^{(d-1)(1-1/p)}.$$

The conclusion holds thanks to (4-2). □

Lemma 4.4 (estimate on θw). *The following estimate holds*:

$$\|\theta w + \theta u + \rho w\|_{L^1_{t,x}} \lesssim \|R\|_{L^1_{t,x}} + C_R \lambda^{-\gamma}.$$

Proof. Taking the L^1 norm in space and using Lemma A.1 and the fact that $\Phi_k \boldsymbol{W}_k$ is $\sigma^{-1}\mathbb{T}^d$-periodic in space, we obtain

$$\begin{aligned}
\|\theta(t)w(t)\|_{L^1} &\le \sum_{1\le k\le d} |\tilde{g}_k(t) g_k(t)| \|R_k(t)\Phi_k \boldsymbol{W}_k\|_{L^1} \\
&\lesssim \sum_{1\le k\le d} |\tilde{g}_k(t) g_k(t)| \|\Phi_k \boldsymbol{W}_k\|_{L^1} (\|R_k(t)\|_{L^1} + \sigma^{-1}\|R_k(t)\|_{C^1}) \\
&\lesssim \sum_{1\le k\le d} |\tilde{g}_k(t) g_k(t)| (\|R_k(t)\|_{L^1} + \sigma^{-1}\|R_k\|_{C^1_{t,x}}),
\end{aligned}$$

where we used $\|\Phi_k \boldsymbol{W}_k\|_{L^1_x} = 1$ by (3-12b) and (3-12c) in the last step. Now taking the L^1 norm in time, using Lemma A.1 together with σ-periodicity of $g_k(t) g_k(t)$ and the smoothness of $t \mapsto \|R_k(t)\|_{L^1}$, and recalling that $\|g_k g_k\|_{L^1} = 1$, we arrive at

$$\|\theta w\|_{L^1_{t,x}} \lesssim \sum_{1\le k\le d} \|\tilde{g}_k g_k\|_{L^1} (\|R_k\|_{L^1_{t,x}} + \sigma^{-1} C_R) \lesssim \sum_{1\le k\le d} (\|R_k\|_{L^1_{t,x}} + \sigma^{-1} C_R) \lesssim \|R\|_{L^1_{t,x}} + C_R \sigma^{-1},$$

where the implicit constant does not depend on the parameter λ or the given solution (ρ, u, R).

The estimates for the other two terms θu and ρw follow from Lemmas 4.2 and 4.3. Indeed, Hölder's inequality gives

$$\|\theta u\|_{L^1_{t,x}} \le \|\theta\|_{L^1_{t,x}} \|u\|_{L^\infty_{t,x}} \le C_R \lambda^{-\gamma}$$

and

$$\|\rho w\|_{L^1_{t,x}} \le \|w\|_{L^1_{t,x}} \|\rho\|_{L^\infty_{t,x}} \le C_R \lambda^{-\gamma}. \qquad \square$$

5. The new defect field R_1 and its estimates

We continue with the proof of Proposition 2.1. Our next goal is to define a suitable defect field R_1 such that the new density ρ_1 and vector field u_1,

$$\rho_1 := \rho + \theta, \quad u_1 := u + w,$$

solve the continuity-defect equation

$$\partial_t \rho_1 + u_1 \cdot \nabla \rho_1 = \operatorname{div} R_1. \tag{5-1}$$

The defect field R_1 will consist of three parts,

$$R_1 = R_{\mathrm{osc}} + R_{\mathrm{lin}} + R_{\mathrm{cor}},$$

each solving the corresponding divergence equation

$$\begin{aligned} \operatorname{div} R_{\mathrm{osc}} &= \partial_t \theta + \operatorname{div}(\theta_p w + R), \\ \operatorname{div} R_{\mathrm{lin}} &= \operatorname{div}(\theta u + \rho w), \\ \operatorname{div} R_{\mathrm{cor}} &= \operatorname{div}(\theta_o w). \end{aligned}$$

So we define the linear error $R_{\mathrm{lin}} = \theta u + \rho w$ and the correction error $R_{\mathrm{cor}} = \theta_o w$ in the usual way and the oscillation error R_{osc} in the following important lemma. Recall that $\mathcal{R}$ and $\mathcal{B}$ are the antidivergence operators defined in the Appendix.

Definition of the oscillation error.

Lemma 5.1 (space-time oscillations). *The following identity holds*:

$$\partial_t \theta + \operatorname{div}(\theta_p w + R) = \operatorname{div}(R_{\mathrm{osc,x}} + R_{\mathrm{osc,t}} + R_{\mathrm{acc}}),$$

where $R_{\mathrm{osc,x}}$ *is the spatial oscillation error*

$$R_{\mathrm{osc,x}} = -\sum_{1 \le k \le d} \tilde{g}_k g_k \mathcal{B}\Big(\nabla R_k, \Big(\Phi_k \boldsymbol{W}_k - \fint_{\mathbb{T}^d} \Phi_k \boldsymbol{W}_k\Big)\Big),$$

$R_{\mathrm{osc,t}}$ *is the temporal oscillation error*

$$R_{\mathrm{osc,t}} = \sigma^{-1} \sum_{1 \le k \le d} h_k \partial_t R_k \boldsymbol{e}_k,$$

and R_{acc} *is the acceleration error*

$$R_{\mathrm{acc}} = -\sum_{1 \le k \le d} \mathcal{B}(\partial_t(\tilde{g}_k R_k), \Phi_k).$$

Proof. By definition of θ_p and w, using the disjointedness of supports of $\tilde{g}_k$ and $g_{k'}$ for $k \neq k'$, we obtain

$$\operatorname{div}(\theta_p w) = -\sum_{1 \le k \le d} \tilde{g}_k g_k \operatorname{div}(R_k \Phi_k \boldsymbol{W}_k). \tag{5-2}$$

Thanks to $\operatorname{div}(\Phi_k \boldsymbol{W}_k) = 0$, for each k,

$$\operatorname{div}(R_k \Phi_k \boldsymbol{W}_k) = \nabla R_k \cdot \mathbb{P}_{\neq 0}(\Phi_k \boldsymbol{W}_k) + \operatorname{div}(R_k \boldsymbol{e}_k)$$

such that from (5-2) we have the decomposition

$$\partial_t \theta + \operatorname{div}(\theta_p w + R) = O_1 + O_2 + O_3, \tag{5-3}$$

with

$$\begin{aligned} O_1 &:= \partial_t \mathbb{P}_{\neq 0} \theta_p, \\ O_2 &:= -\sum_{1 \le k \le d} \tilde{g}_k g_k \nabla R_k \cdot \mathbb{P}_{\neq 0}(\Phi_k \boldsymbol{W}_k), \\ O_3 &:= \partial_t \theta_o - \sum_{1 \le k \le d} \tilde{g}_k g_k \operatorname{div}(R_k \boldsymbol{e}_k) + \operatorname{div} R. \end{aligned}$$

By the definitions of R_{acc} and $\mathcal{R}$, the first term $O_1 = \operatorname{div} R_{\text{acc}}$ since Φ_k has zero mean.

For the second term O_2, by definition of $\mathcal{B}$ and (A-2), we observe that

$$\fint_{\mathbb{T}^d} R_k \operatorname{div}(\Phi_k \boldsymbol{W}_k) + \nabla R_k \cdot \left(\Phi_k \boldsymbol{W}_k - \fint_{\mathbb{T}^d} \Phi_k \boldsymbol{W}_k \right) = \operatorname{div} \mathcal{B}\left(\nabla R_k, \left(\Phi_k \boldsymbol{W}_k - \fint_{\mathbb{T}^d} \Phi_k \boldsymbol{W}_k \right) \right), \tag{5-4}$$

where the meaning of the vector-valued argument is that $\mathcal{B}$ is applied to each of its components. So (5-4) implies $O_2 = \operatorname{div} R_{\text{osc,x}}$.

Finally, for the last term O_3, by the definition of θ_o (3-17), (3-6), and (3-3),

$$\begin{aligned} \partial_t \theta_o &= \sigma^{-1} \sum_{1 \le k \le d} h_k' \operatorname{div}(R_k \boldsymbol{e}_k) + \sigma^{-1} \sum_{1 \le k \le d} h_k \operatorname{div}(\partial_t R_k \boldsymbol{e}_k) \\ &= (\tilde{g}_\kappa g_\kappa - 1) \sum_{1 \le k \le d} \operatorname{div}(R_k \boldsymbol{e}_k) + \sigma^{-1} \sum_{1 \le k \le d} h_k \operatorname{div}(\partial_t R_k \boldsymbol{e}_k), \end{aligned}$$

which implies $O_3 = \operatorname{div} R_{\text{osc,t}}$. □

Estimates of the new defect error. In the remainder of this section, we finish the proof of Proposition 2.1. Given $\delta > 0$, we will show that the sum of $L^1_{t,x}$ norms of each error is less than $C_R \lambda^{-\gamma}$. This concludes the proof provided λ is chosen large enough.

R_{acc} estimate. Taking advantage of the potential Ω_k as in (3-11), we obtain

$$\begin{aligned} \|R_{\text{acc}}\|_{L^1_{t,x}} &= \sigma^{-1} \|\mathcal{B}(\partial_t(\tilde{g}_k R_k), \operatorname{div} \Omega_k)\|_{L^1_{t,x}} \\ &\lesssim C_R \sigma^{-1} \|\tilde{g}_k\|_{W^{1,1}} \|\mathcal{R} \operatorname{div} \Omega_k\|_{L^1} && \text{(by Lemma A.2)} \\ &\lesssim C_R \sigma^{-1} \|\tilde{g}_k\|_{W^{1,1}} \|\Omega_k\|_{L^r} && \text{(by boundedness of } \mathcal{R} \text{ in } L^r) \\ &\lesssim C_R \kappa^\alpha \mu^{-1-(d-1)/r} && \text{(by (3-4) and (3-12a))} \\ &\lesssim C_R \lambda^{-\gamma} && \text{(by (4-3)).} \end{aligned} \tag{5-5}$$

$R_{\mathrm{osc,x}}$ *estimate.* By Hölder's inequality, Lemma A.2, and the bounds $\|\tilde{g}_k g_k\|_{L^1_t} = 1$ and $\|\Phi_k \boldsymbol{W}_k\|_{L^1} = 1$, we obtain

$$\|R_{\mathrm{osc,x}}\|_{L^1_{t,x}} \leq \sum_{1\leq k\leq d} \|\tilde{g}_k g_k\|_{L^1} \|\mathcal{B}(\nabla R_k, \mathbb{P}_{\neq 0}(\Phi_k \boldsymbol{W}_k))\|_{L^\infty_t L^1} \lesssim C_R \sum_{1\leq k\leq d} \|\mathcal{R}\mathbb{P}_{\neq 0}(\Phi_k \boldsymbol{W}_k)\|_{L^1} \leq C_R \lambda^{-\gamma}.$$

$R_{\mathrm{osc,t}}$ *estimate.* By (3-7),

$$\|R_{\mathrm{osc},t}\|_{L^1_{t,x}} = \Big\|\sigma^{-1} \sum_{1\leq k\leq d} h_k \partial_t R_k \boldsymbol{e}_k\Big\|_{L^1_{t,x}} \leq C_R \sigma^{-1} \sum_{1\leq k\leq d} \|h_k\|_{L^1} \leq C_R \lambda^{-\gamma}.$$

R_{lin} *estimate.* We start with Hölder's inequality

$$\|R_{\mathrm{lin}}\|_{L^1_{t,x}} \leq \|\theta\|_{L^1_{t,x}} \|u\|_{L^\infty_{t,x}} + \|\rho\|_{L^\infty_{t,x}} \|w\|_{L^1_{t,x}}.$$

It suffices to show $\|\theta\|_{L^1_{t,x}} \leq C_R \lambda^{-\gamma}$ and $\|w\|_{L^1_{t,x}} \leq C_R \lambda^{-\gamma}$. These follow from Lemmas 4.2 and 4.3 since $p \geq 1$.

R_{cor} *estimate.* By Hölder's inequality,

$$\|R_{\mathrm{cor}}\|_{L^1_{t,x}} \leq \|\theta_o\|_{L^\infty_{t,x}} \|w\|_{L^1_{t,x}}.$$

Since $\|\theta_o\|_{L^\infty_{t,x}} \leq C_R \lambda^{-\gamma}$ from its definition (or by (4-4) from Lemma 4.2), by Lemma 4.3 we also have $\|R_{\mathrm{cor}}\|_{L^1_{t,x}} \leq C_R \lambda^{-\gamma}$.

Conclusion of the proof of Proposition 2.1. The first point is proved in Section 3 while the second point is proved in Section 4 provided λ is sufficiently large. For the last point, (2-6) follows from the definition of the new defect error R_1, and the estimate follows from the ones in the subsection above by choosing λ sufficiently large once again. Hence Proposition 2.1 is proved.

6. Extension to transport-diffusion equations

In this section, we extend the main results to general transport-diffusion equations

$$\begin{cases} \partial_t \rho - L\rho + u \cdot \nabla \rho = 0, \\ \rho|_{t=0} = \rho_0, \end{cases} \tag{6-1}$$

where L is a given constant coefficient differential operator of order $k \in \mathbb{N}$. Weak solutions to (6-1) can be defined analogously to (1-2) by the adjoint of L, and we impose the minimum regularity $\rho \in L^1_{t,x}$ and $\rho u \in L^1_{t,x}$ as before.

The following nonuniqueness result holds for (6-1).

Theorem 6.1. *Let $d \geq 2$ and L be any constant coefficient differential operator of order $k \geq 1$. There exists a divergence-free velocity vector field $u \in L^1(0, T; W^{1,p}(\mathbb{T}^d))$ for all $p < \infty$ such that the uniqueness of* (6-1) *fails in the class*

$$\rho \in \bigcap_{\substack{p<\infty \\ k\in\mathbb{N}}} L^p(0, T; C^k(\mathbb{T}^d)) \quad \textit{and} \quad \rho u \in L^1(\mathbb{T}^d \times [0, T]).$$

Proof. We only need to check that Proposition 2.1 holds for (6-1). It suffices to check that the linear term $L\rho$ results in a small error, which is defined as

$$R_L := \mathcal{R} L \sum_{1\le k\le d} \tilde{g}_k R_k \Phi_k.$$

Indeed, by L^1 boundedness of $\mathcal{R}$,

$$\|R_L\|_{L^1_{t,x}} \lesssim C_R \sum_{1\le k\le d} \|\tilde{g}_k\|_{L^1} \|\Phi_k\|_{W^{k,1}},$$

where $k \ge 1$ is the order of the linear operator L. Since we only need to prove the results for p large, we can assume $k \le p$, so that, as in the proof of Lemma 4.2,

$$\|R_L\|_{L^1_{t,x}} \lesssim C_R \kappa^{\alpha-1} (\sigma\mu)^p \le C_R \lambda^{-\gamma}.$$

Hence there is no additional constraint coming from the diffusion. □

Appendix: Standard tools in convex integration

In this section, we recall several technical results that are now standard in convex integration.

Improved Hölder's inequality on $\mathbb{T}^d$. We recall the following result due to [Modena and Székelyhidi 2018, Lemma 2.1], which was inspired by [Buckmaster and Vicol 2019, Lemma 3.7].

Lemma A.1. *Let $d \ge 2$, $r \in [1,\infty]$, and $a, f : \mathbb{T}^d \to \mathbb{R}$ be smooth functions. Then, for every $\sigma \in \mathbb{N}$,*

$$\Big| \|af(\sigma\,\cdot)\|_{L^r(\mathbb{T}^d)} - \|a\|_{L^r(\mathbb{T}^d)} \|f\|_{L^r(\mathbb{T}^d)} \Big| \lesssim \sigma^{-1/r} \|a\|_{C^1(\mathbb{T}^d)} \|f\|_{L^r(\mathbb{T}^d)}. \tag{A-1}$$

Note that the error term on the right-hand side can be made arbitrarily small by increasing the oscillation factor σ.

Antidivergence operators $\mathcal{R}$ and $\mathcal{B}$. We will use the standard antidivergence operator $\Delta^{-1}\nabla$ on $\mathbb{T}^d$, which will be denoted by $\mathcal{R}$. We write $C_0^\infty(\mathbb{T}^d)$ for the space of smooth functions with zero mean on $\mathbb{T}^d$.

It is well known that, for any $f \in C^\infty(\mathbb{T}^d)$, there exists a unique $u \in C_0^\infty(\mathbb{T}^d)$ such that

$$\Delta u = f - \fint f.$$

For any smooth scalar function $f \in C^\infty(\mathbb{T}^d)$, the standard antidivergence operator $\mathcal{R} : C^\infty(\mathbb{T}^d) \to C_0^\infty(\mathbb{T}^d, \mathbb{R}^d)$ can be defined as

$$\mathcal{R} f := \Delta^{-1} \nabla f,$$

which satisfies

$$\operatorname{div}(\mathcal{R} f) = f - \fint_{\mathbb{T}^d} f \quad \text{for all } f \in C^\infty(\mathbb{T}^d).$$

It is well known (see for instance [Modena and Székelyhidi 2018, Lemma 2.2]) that $\mathcal{R}$ is bounded on Sobolev spaces $W^{k,p}(\mathbb{T}^d)$ for all $k \in \mathbb{N}$ and that $\mathcal{R}\operatorname{div}$ is a Calderón–Zygmund operator:

$$\|\mathcal{R}(\operatorname{div} u)\|_{L^r(\mathbb{T}^d)} \lesssim \|u\|_{L^r(\mathbb{T}^d)} \quad \text{for all } u \in C^\infty(\mathbb{T}^d, \mathbb{R}^d) \text{ and } 1 < r < \infty.$$

Recall the following useful fact about $\mathcal{R}$:

$$\mathcal{R}f(\sigma\,\cdot) = \sigma^{-1}\mathcal{R}f \quad \text{for any } f \in C_0^\infty(\mathbb{T}^d) \text{ and any positive } \sigma \in \mathbb{N}.$$

We will also use its bilinear counterpart $\mathcal{B} : C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d, \mathbb{R}^d)$, defined by

$$\mathcal{B}(a, f) := a\mathcal{R}f - \mathcal{R}(\nabla a \cdot \mathcal{R}f).$$

It is easy to see that $\mathcal{B}$ is a left-inverse of the divergence:

$$\operatorname{div}(\mathcal{B}(a, f)) = af - \fint_{\mathbb{T}^d} af\,dx \quad \text{provided that } f \in C_0^\infty(\mathbb{T}^d), \tag{A-2}$$

which can be proved easily using integration by parts. The following estimate is a direct consequence of the boundedness of $\mathcal{R}$ on Sobolev spaces $W^{k,p}(\mathbb{T}^d)$.

Lemma A.2. *Let $d \geq 2$ and $1 \leq r \leq \infty$. Then, for any $a, f \in C^\infty(\mathbb{T}^d)$,*

$$\|\mathcal{B}(a, f)\|_{L^r(\mathbb{T}^d)} \lesssim \|a\|_{C^1(\mathbb{T}^d)} \|\mathcal{R}f\|_{L^r(\mathbb{T}^d)}.$$

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L^p-POLARITY, MAHLER VOLUMES, AND THE ISOTROPIC CONSTANT

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This article introduces L^p versions of the support function of a convex body K and associates to these canonical L^p-polar bodies $K^{\circ,p}$ and Mahler volumes $\mathcal{M}_p(K)$. Classical polarity is then seen as L^∞-polarity. This one-parameter generalization of polarity leads to a generalization of the Mahler conjectures, with a subtle advantage over the original conjecture: conjectural uniqueness of extremizers for each $p \in (0, \infty)$. We settle the upper bound by demonstrating the existence and uniqueness of an L^p-Santaló point and an L^p-Santaló inequality for symmetric convex bodies. The proof uses Ball's Brunn–Minkowski inequality for harmonic means, the classical Brunn–Minkowski inequality, symmetrization, and a systematic study of the $\mathcal{M}_p$ functionals. Using our results on the L^p-Santaló point and a new observation motivated by complex geometry, we show how Bourgain's slicing conjecture can be reduced to lower bounds on the L^p-Mahler volume coupled with a certain conjectural convexity property of the logarithm of the Monge–Ampère measure of the L^p-support function. We derive a suboptimal version of this convexity using Kobayashi's theorem on the Ricci curvature of Bergman metrics to illustrate this approach to slicing. Finally, we explain how Nazarov's complex-analytic approach to the classical Mahler conjecture is instead precisely an approach to the L^1-Mahler conjecture.

1. Introduction

The polar K° and the support function h_K of a convex body K are fundamental objects in functional and convex analysis. The Mahler and Bourgain conjectures have motivated an enormous amount of research in those fields over the past 85 years. One of the goals of this article is to point out that K° and h_K are L^∞-versions of a more general one-parameter family of objects

$$K^{\circ,p} \quad \text{and} \quad h_{p,K},$$

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introduce the associated one-parameter generalization of the Mahler volume $\mathcal{M}_p$ and conjectures, and establish some of their fundamental properties. As we explain in detail and back up with explicit computations, minimizers should be unique (see Figure 3 and the discussion surrounding it). This is a subtle, but perhaps crucial, advantage, as compared to Mahler's original conjecture. To quote [Tao 2007] (see also [Błocki 2015, p. 90]),

> *In my opinion, the main reason why this conjecture is so difficult is that unlike the upper bound, in which there is essentially only one extremiser up to affine transformations (namely the ball), there are many distinct extremisers for the lower bound…*

As an application of the theory of L^p-polarity, we develop a connection between these new objects (L^p-support functions and L^p-Mahler volumes) and Bourgain's slicing conjecture, e.g., making contact with Kobayashi's theorem on the Ricci curvature of Bergman metrics. Finally, we explain how Nazarov's and Błocki's work on a complex-analytic approach to the classical Mahler conjecture fits in, being precisely an approach to the L^1-Mahler conjecture.

Our approach is loosely motivated by complex geometry, but the article in its entirety can be read with no knowledge of complex methods. As is probably clear from the text, the authors are novices in the study of the Mahler and Bourgain conjectures and are sorry for any omission in accrediting results properly. The motivation for this article lies not so much in the particular results as in showing the link between complex geometry and this beautiful area. It should also be stressed that the list of references is far from complete. We have tried to make the text accessible to both convex and complex analysts and so perhaps included a bit more background than usual.

1A. *Motivation from Bergman kernels.* Denote by

$$K^\circ := \{y \in \mathbb{R}^n : \langle x, y\rangle \le 1 \text{ for all } x \in K\} \tag{1-1}$$

the polar body associated to a convex body (compact and convex with nonempty interior) $K \subset \mathbb{R}^n$. A key step in Nazarov's complex-analytic approach to the Bourgain–Milman inequality [1987, Theorem 1] is a bound on the *Mahler volume*

$$\mathcal{M}(K) := n!\,|K||K^\circ| \tag{1-2}$$

of a symmetric (i.e., $-K = K$) convex body K from below by a multiple of the Bergman kernel $\mathcal{K}_{T_K}(z, w)$ of the tube domain $T_K := \mathbb{R}^n + \sqrt{-1}K$ over K, evaluated on the diagonal at the origin [Nazarov 2012, p. 338]. This was generalized by Hultgren [2013, Lemma 11] and two of us [Mastrantonis and Rubinstein 2022, Proposition 6] to any convex body K:

$$\pi^n |K|^2 \mathcal{K}_{T_K}(\sqrt{-1}b(K), \sqrt{-1}b(K)) \le \mathcal{M}(K - b(K)), \tag{1-3}$$

where

$$b(K) := \int_K x \,\frac{\mathrm{d}x}{|K|}$$

is the barycenter of K.

This article, however, is not about Bergman kernels (though we come back to Bergman kernels in Sections 1F and 6E). Nonetheless, the L^p-*Mahler volumes* introduced below are partly motivated by (1-3).

In order to prove (1-3) one uses Jensen's inequality together with an explicit formula for the Bergman kernels of tube domains evaluated on the diagonal, due to [Rothaus 1960, Theorem 2.6; Korányi 1962, Theorem 2; Hsin 2005, (1.2)], that as observed recently can be expressed as [Mastrantonis and Rubinstein 2022, Remark 36]

$$\mathcal{K}_{T_K}(0,0) = \frac{1}{(4\pi)^n} \int_{\mathbb{R}^n} e^{-h_{1,K}(y)} \frac{\mathrm{d}y}{|K|}, \tag{1-4}$$

where, following [Mastrantonis and Rubinstein 2022, Definition 13], we denote by

$$h_{1,K}(y) := \log \int_K e^{\langle x,y\rangle} \frac{\mathrm{d}x}{|K|} \tag{1-5}$$

the logarithmic Laplace transform of the convex indicator function $\mathbf{1}_K^\infty$ ($\mathbf{1}_K^\infty$ is 0 on K and ∞ otherwise). Therefore, the left-hand side of (1-3) becomes $\pi^n|K| \int_{\mathbb{R}^n} e^{-h_{1,K-b(K)}(y)}\,\mathrm{d}y$, bearing a curious resemblance to the standard formula for the Mahler volume (1-2),

$$\mathcal{M}(K) = |K| \int_{\mathbb{R}^n} e^{-h_K(y)}\,\mathrm{d}y \tag{1-6}$$

(see (4-2) below), where

$$h_K(y) := \sup_{x\in K} \langle x, y\rangle \tag{1-7}$$

is the (classical) support function of K.

1B. ***L^p-support function, -polarity, and -Mahler volume.*** Motivated by the preceding discussion and [Mastrantonis and Rubinstein 2022, Remark 36], we introduce the L^p-*support function* of a compact body (compact with nonempty interior) $K \subset \mathbb{R}^n$ for all $p > 0$,

$$h_{p,K}(y) := \log\biggl(\int_K e^{p\langle x,y\rangle} \frac{\mathrm{d}x}{|K|}\biggr)^{\frac{1}{p}}, \quad y \in \mathbb{R}^n, \tag{1-8}$$

unifying and interpolating between (1-5) and (1-7) (notice that $h_{\infty,K} := \lim_{p\to\infty} h_{p,K} = h_K$ by Corollary 2.7). These are convex functions in y, monotone increasing in p, and take the Cartesian product of bodies to the sum of the respective L^p-support functions (Lemma 2.2). Less obviously, they also enjoy a convexity property in p (Lemma 2.4), and a "concavity" property in K (Lemma 2.5).

Generalizing (1-6), we introduce the L^p-*Mahler volume*,

$$\mathcal{M}_p(K) := |K| \int_{\mathbb{R}^n} e^{-h_{p,K}(y)}\,\mathrm{d}y. \tag{1-9}$$

The functional $\mathcal{M}_p$ shares many (but not all) of the properties of $\mathcal{M} = \mathcal{M}_\infty$ (by Corollary 2.7), e.g., invariance under the action of $\mathrm{GL}(n,\mathbb{R})$ (Lemma 4.7), tensoriality (Remark 2.3), existence and uniqueness of a Santaló point (Proposition 1.5), and a Santaló inequality for symmetric bodies (Theorem 1.6).

It is natural to ask whether there is an analogue of (1-2) for $\mathcal{M}_p$, i.e., is there a canonically associated body to K for which $\mathcal{M}_p$ can be expressed as the volume of a product body in $\mathbb{R}^{2n}$? We answer this affirmatively. To that end, we introduce the following:

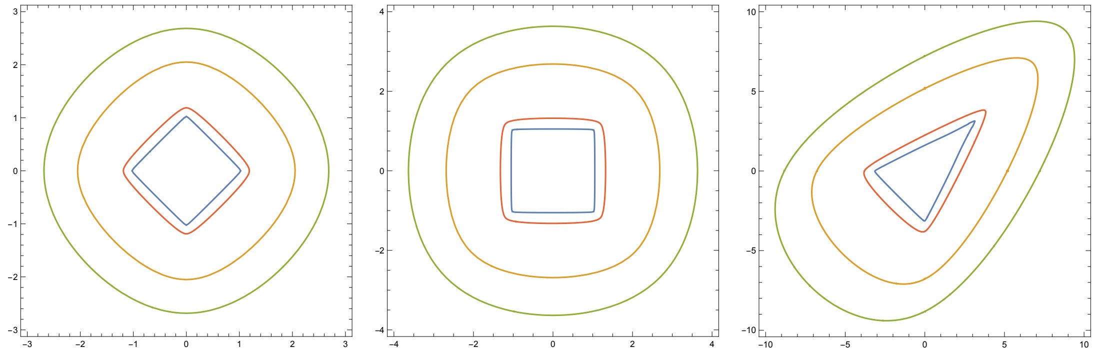

Figure 1. The L^p-polars of the square $B^2_\infty := [-1,1]^2$ (left), the diamond $B^2_1 := (B^2_\infty)^\circ$ (middle), the 2-simplex centered at the origin (right) for $p = \frac{1}{2}$ (green), $p = 1$ (orange), $p = 10$ (red) and $p = 100$ (blue).

Definition 1.1. Let $K \subset \mathbb{R}^n$. Define the L^p-polar body of K by

$$K^{\circ,p} := \left\{ y \in \mathbb{R}^n : \int_0^\infty r^{n-1} e^{-h_{p,K}(ry)}\, \mathrm{d}r \geq (n-1)! \right\}. \tag{1-10}$$

Our first result answers the aforementioned question.

Theorem 1.2. *Let $p \in (0,\infty]$. For a convex body $K \subset \mathbb{R}^n$, $K^{\circ,p}$ is convex, closed, has nonempty interior, and*

$$\mathcal{M}_p(K) = n!\,|K|\,|K^{\circ,p}|. \tag{1-11}$$

It is compact (bounded) if and only if $0 \in \operatorname{int} K$. For K symmetric, $K^{\circ,p}$ is symmetric.

Theorem 1.2 justifies the notation

$$\|y\|_{K^{\circ,p}} := \left(\frac{1}{(n-1)!} \int_0^\infty r^{n-1} e^{-h_{p,K}(ry)}\, \mathrm{d}r \right)^{-\frac{1}{n}} \tag{1-12}$$

(the power serves to homogenize), and $K^{\circ,p} = \{y \in \mathbb{R}^n : \|y\|_{K^{\circ,p}} \leq 1\}$. For $p = \infty$ one recovers the usual polar body, i.e., $K^{\circ,\infty} = K^\circ$ (Lemma 3.6). The case $p = 0$ is treated in Section 3B1. Figure 1 illustrates some explicit examples.

As p approaches 0, the L^p-polars of all three of the bodies pictured in Figure 1 increase to $\mathbb{R}^2$. In fact, for any convex body $K \subset \mathbb{R}^n$,, $K^{\circ,p}$ increases to $\{y : \langle y, b(K)\rangle \leq 1\}$ as $p \to 0$ (Proposition 3.7), so we define $K^{\circ,0}$ to be exactly that (Definition 3.10). In particular, $K^{\circ,0}$ is either $\mathbb{R}^2$ or a half-space depending on whether or not $b(K)$ vanishes. By Example 3.11, we may plot a few of the L^p-polars of the standard simplex on the plane (1-14); see Figure 2. Note that $\Delta_2^{\circ,0}$ is a half-space since $b(\Delta_2) \neq 0$.

The proof of Theorem 1.2 has several parts. To obtain (1-11) we rely on a result of Ball (Theorem 5.20) that implies that (1-12) has all the properties of a norm, except that it is, in general, only *positively* 1-homogeneous, i.e., $\|\lambda y\|_{K^{\circ,p}} = \lambda \|y\|_{K^{\circ,p}}$ for $\lambda > 0$. If K is symmetric then $\|\cdot\|_{K^{\circ,p}}$ is fully 1-homogeneous, i.e., a norm (then $K^{\circ,p}$ is also symmetric). For completeness, we include a detailed and self-contained proof of Ball's result in the Appendix. In particular, $\|\cdot\|_{K^{\circ,p}}$ is convex and so is $K^{\circ,p}$.

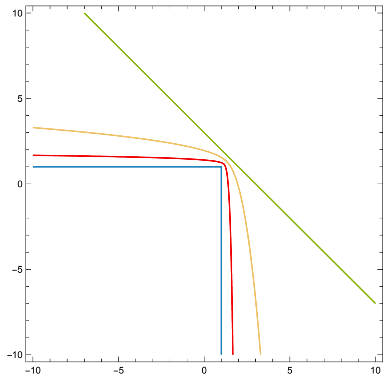

Figure 2. The boundary of $(\Delta_2)^{\circ,p}$ for $p=0$ (green), $p=2$ (orange), $p=10$ (red), and $p=\infty$ (blue).

Equality (1-11) follows from a standard formula relating the volume of a convex body to the surface integral of $\|\cdot\|^{-n}_{K^{\circ,p}}$ over the unit sphere (see (3-2)). Nonemptiness of the interior follows from $K^\circ \subset K^{\circ,p}$ (Lemma 3.6). This inclusion also implies that $K^{\circ,p}$ is unbounded when $0 \notin \operatorname{int} K$. The converse is slightly more subtle: when $0 \in \operatorname{int} K$ one has a small cube $[-\varepsilon,\varepsilon]^n \subset K$. For classical polarity this would be the end of the argument; yet unlike classical polarity, L^p-polarity does not invert inclusions, so we cannot simply argue that $K^{\circ,p} \subset ([-\varepsilon,\varepsilon]^n)^{\circ,p}$. Instead, we use the existence of a small cube inside of K to obtain a lower bound on $h_{p,K}$ in terms of $h_{p,[-\varepsilon,\varepsilon]^n}$ (see (3-8)), which then induces an upper bound on $K^{\circ,p}$ by a multiple of $([-\varepsilon,\varepsilon]^n)^{\circ,p}$. The latter can be shown to be bounded (Claim 3.4), from which the boundedness of $K^{\circ,p}$ follows by using yet another key estimate (Lemma 2.6).

1C. ***L^p-Mahler conjectures and uniqueness of minimizers.*** For $q>0$, denote by

$$B^n_q := \{x \in \mathbb{R}^n : |x_1|^q + \cdots + |x_n|^q \le 1\} \tag{1-13}$$

the (closed) n-dimensional q-ball, and denote by

$$\Delta_n := \{x \in [0,\infty)^n : x_1 + \cdots + x_n \le 1\} \tag{1-14}$$

the standard simplex in $\mathbb{R}^n$. We propose a 1-parameter generalization of Mahler's conjectures. Mahler's original conjectures [1939a; 1939b, p. 96] amount to setting $p=\infty$ in the following statements.

Conjecture 1.3. *Let $p \in (0,\infty]$. For a symmetric convex body $K \subset \mathbb{R}^n$,*

$$\mathcal{M}_p([-1,1]^n) \le \mathcal{M}_p(K) \le \mathcal{M}_p(B^n_2).$$

Conjecture 1.4. *Let $p \in (0,\infty]$. For a convex body $K \subset \mathbb{R}^n$,*

$$\inf_{x\in\Delta_n} \mathcal{M}_p(\Delta_n - x) \le \mathcal{M}_p(K).$$

By Proposition 1.5 below, the infimum in Conjecture 1.4 is attained by a unique point.

By the Bourgain–Milman inequality [1987, Corollary 6.1], there is $c>0$ independent of dimension so that $\mathcal{M}(K) \ge c^n$ for all convex bodies $K \subset \mathbb{R}^n$. By Lemma 3.12 below, this induces a lower bound on $\mathcal{M}_p$ for all p with the constant only depending on p. The best known constant for $\mathcal{M}$ in dimensions

$n \geq 4$ with K symmetric is $c = \pi$ [Kuperberg 2008, Corollary 1.6; Berndtsson 2021, Theorem 2.1]. The sharp bound $c = 4$ is due to [Mahler 1939a, (2)] in dimension $n = 2$ and [Iriyeh and Shibata 2020, Theorem 1.1] in dimension $n = 3$ (see also [Fradelizi et al. 2022]). For general K, the best known constant is $c = 2$ for $n = 3$ and $c = \frac{\pi}{2}$ for $n \geq 4$ by the symmetric bound and a symmetrization trick (see, for example, [Mastrantonis and Rubinstein 2022, Corollary 55]). In dimension $n = 2$ the sharp bound is due to [Mahler 1939a, (1)]. One may also formulate other versions of Mahler's original conjecture, e.g., to zonoids [Reisner 1986] or unconditional bodies [Saint-Raymond 1981, §4] and generalize these to all p, but in this article we focus on Conjectures 1.3 and 1.4. In the special case $p = 1$, using (1-4) one can show that the lower bound of Conjecture 1.3 is equivalent to a conjecture of Błocki [2014, p. 56], while Conjecture 1.4 reduces to a conjecture of [Mastrantonis and Rubinstein 2022, Conjecture 10], both stated in terms of Bergman kernels of tube domains.

Conjectures 1.3 and 1.4 for all $p \in (0, \infty)$ imply Mahler's conjectures, as we show in Lemma 3.12. On the surface, the former look harder to deal with. However, there is a subtle, perhaps crucial, advantage in the "regularized" version of the symmetric Mahler conjecture (Conjecture 1.3 for $p \in (0, \infty)$) compared to the classical version ($p = \infty$) of that conjecture. This has to do with the nonuniqueness of minimizers in the classical symmetric Mahler conjecture which has been pointed out by experts [Tao 2008, §1.3; 2007] (see, in particular, the comments in the latter) as one of the main obstacles to tackling it (see also the quote by Tao in the Introduction). Let us elaborate on that.

Indeed, tensoriality of $\mathcal{M} = \mathcal{M}_\infty$ together with its invariance under classical polarity leads to the conjectured nonuniqueness of symmetric minimizers, referred to as Hanner polytopes (nonuniqueness here is in the strong sense: after taking the quotient by $\mathrm{GL}(n, \mathbb{R})$, i.e., there are minimizing bodies that are in different $\mathrm{GL}(n, \mathbb{R})$-orbits). Hanner polytopes are symmetric convex polytopes that are defined inductively: $[-1, 1]$ is the unique Hanner polytope in dimension $n = 1$. In higher dimensions, a Hanner polytope is given either as the Cartesian product of two lower-dimensional Hanner polytopes, or as the polar of such [Hanner 1956, Theorems 3.1–3.2, 7.1; Hansen and Lima 1981, Corollary 7.4]. For example, in dimension $n = 3$ there are precisely two non-$\mathrm{GL}(n, \mathbb{R})$ equivalent Hanner polytopes: the cube $[-1, 1]^3$, as the product of lower-dimensional Hanner polytopes, and its polar B_1^3 [Hanner 1956, pp. 86–87].

By contrast, our L^p-polarity operation (1-10) is no longer a duality, i.e., $(K^{\circ,p})^{\circ,p} \neq K$ in general. In fact, the L^p-polar always has a smooth boundary for $p \in (0, \infty)$, and hence L^p-polarity is never a duality operation among polytopes. By (1-11) this means $\mathcal{M}_p$ is not invariant under L^p-polarity. We conjecture that for all $p \in (0, \infty)$, up to the action of $\mathrm{GL}(n, \mathbb{R})$, $\mathcal{M}_p$ is uniquely minimized by the cube among symmetric convex bodies, and by the simplex, appropriately repositioned, among general convex bodies. If true, this would give some motivation for studying $\mathcal{M}_p$ and show that the original Mahler conjecture has (for better and for worse) additional invariance absent from our L^p-Mahler conjectures. Figure 3 illustrates this symmetry-breaking property of $\mathcal{M}_p$ in $n = 3$:

We emphasize that the above discussion pertains to the symmetric case, since in the nonsymmetric case, the simplex, appropriately repositioned, is already conjectured to be the unique (up to $\mathrm{GL}(n, \mathbb{R})$) minimizer for the classical nonsymmetric Mahler conjecture [Tao 2007]. That is, $\mathcal{M}$ should be minimized by $\Delta_n - b(\Delta_n)$, where $b(\Delta_n)$ coincides with the Santaló point of Δ_n. Note that $(\Delta_n - b(\Delta_n))^\circ$ is

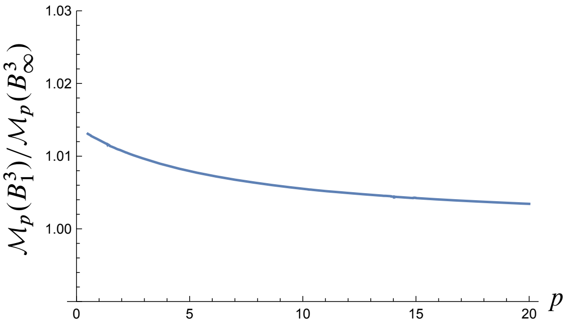

Figure 3. $\mathcal{M}_p(B_1^3)/\mathcal{M}_p(B_\infty^3)$ for $p \in [\frac{1}{2}, 20]$.

a $\mathrm{GL}(n,\mathbb{R})$ image of $\Delta_n - b(\Delta_n)$, so polarity does not produce a non-$\mathrm{GL}(n,\mathbb{R})$ equivalent minimizer in this case. The conjectured uniqueness of the minimizer in the nonsymmetric case (regardless of p) is perhaps related to the fact that Δ_n cannot be expressed as a product of polytopes of lower dimension.

1D. ***L^p-Santaló theorem.*** For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, denote by

$$V(f) := \int_{\mathbb{R}^n} e^{-f(x)}\,\mathrm{d}x \quad \text{and} \quad b(f) := \frac{1}{V(f)} \int_{\mathbb{R}^n} x e^{-f(x)}\,\mathrm{d}x$$

its volume and barycenter respectively. This terminology is motivated by $V(h_K) = n!\,|K^\circ|$ (see (4-2)), and $b(h_K) = (n+1)b(K^\circ)$ (see (4-1)). By Theorem 1.2, $V(h_{p,K}) = n!\,|K^{\circ,p}|$. However, lacking homogeneity, it is not clear how $b(h_{p,K})$ can be directly related to $b(K^{\circ,p})$ (Section 4). Our next result generalizes the Santaló point.

Proposition 1.5. *Let $p \in (0,\infty]$. For a convex body $K \subset \mathbb{R}^n$ there exists a unique $x_{p,K} \in \mathbb{R}^n$ with*

$$\mathcal{M}_p(K - x_{p,K}) = \inf_{x \in \mathbb{R}^n} \mathcal{M}_p(K - x),$$

which is also the unique point such that $b(h_{p,K-x_{p,K}}) = 0$. Moreover, $x_{p,K} \in \mathrm{int}\,K$.

Part of the proof of Proposition 1.5 is almost identical to Santaló's proof [1949, §2] of the existence and uniqueness of Santaló points. The idea is to show that the function $x \mapsto \mathcal{M}_p(K - x)$ is ∞ for $x \notin \mathrm{int}\,K$ (Lemma 4.2), and smooth and strictly convex for $x \in \mathrm{int}\,K$ (Lemma 4.4). This forces the existence of a unique minimum. The main difference is that we study $\int_{\mathbb{R}^n} e^{-h_{p,K}(y)}\,\mathrm{d}y$ under translations of K, while Santaló [1949, (1.1)] studied the surface integral $\int_{\partial B_2^n} h_K(u)^{-n}\,\mathrm{d}u$.

One of our main results is a generalization of Santaló's theorem, verifying the upper bound in Conjecture 1.3:

Theorem 1.6. *Let $p \in (0,\infty]$. For a symmetric convex body $K \subset \mathbb{R}^n$,*

$$\mathcal{M}_p(K) \le \mathcal{M}_p(B_2^n).$$

In particular, by taking $p \to \infty$, one recovers Santaló's inequality [1949, (1.3)] $\mathcal{M}(K) \le \mathcal{M}(B_2^n)$ (though, of course, for this purpose alone there are direct, easier, proofs, e.g., [Saint-Raymond 1981, Theorem 14]).

The L^p-polar $K^{\circ,p}$ (1-10) is central to the proof of Theorem 1.6. One idea behind the proof is standard: for $u \in \partial B_2^n$, the Steiner symmetrization with respect to a hyperplane through the origin

$$u^\perp := \{x \in \mathbb{R}^n : \langle x, u\rangle = 0\},$$

increases the volume of the L^p-polar $|(\sigma_u K)^{\circ,p}| \geq |K^{\circ,p}|$ (Proposition 5.1). Yet proving this seems nonstandard and rather nontrivial. We achieve it by proving the estimate

$$\tfrac{1}{2}(K^{\circ,p} \cap (u^\perp + tu)) + \tfrac{1}{2}(K^{\circ,p} \cap (u^\perp - tu)) \subset (\sigma_u K)^{\circ,p} \cap (u^\perp + tu) \quad \text{for all } t \in \mathbb{R}, \tag{1-15}$$

which compares the slices of K and those of $\sigma_u K$ over $u^\perp$, and then using the (classical) Brunn–Minkowski inequality. To obtain (1-15) we use Ball's Brunn–Minkowski inequality for harmonic means (Theorem 5.20), together with the convexity of $x \mapsto \log\bigl(\frac{1}{t}\sinh(t)\bigr)$ (Claim 5.19).

Remark 1.7. Theorem 1.6 is different from the L^p Santaló inequalities of Lutwak and Zhang, who introduced the symmetrized L^p-centroid body $\Gamma_p K$ with support function given by

$$h_{\Gamma_p K}(y) := \left(\frac{1}{c_{n,p}} \int_K |\langle x, y\rangle|^p \frac{\mathrm{d}x}{|K|}\right)^{\frac{1}{p}}$$

(where $c_{n,p}$ is a constant that depends on n and p determined by $\Gamma_p B_2^n = B_2^n$) for which they proved $|K||(\Gamma_p K)^\circ| \leq |B_2^n|^2$ [Lutwak and Zhang 1997]. Their construction is restricted to symmetric bodies since $\Gamma_p K$ is always symmetric (regardless of whether K is), and the large p limit does not recover the polar body but rather the reflection body: $\lim_{p\to\infty} \Gamma_p K = K \cup (-K)$ (since $\lim_{p\to\infty} h_{\Gamma_p K}(y) = \sup_{x\in K} |\langle x, y\rangle|$). Subsequently, Ludwig and Haberl–Schuster extended this to nonsymmetric bodies [Ludwig 2005, p. 4195; Haberl and Schuster 2009, §3] introducing the L^p-centroid body $M_p K^+$ whose support function is

$$h_{M_p K^+}(y) := \left(C_{n,p}(n+p) \int_K \max\{\langle x, y\rangle, 0\}^p \,\mathrm{d}x\right)^{\frac{1}{p}}.$$

Note that as $p \to \infty$, we have $K^{\circ,p} \to K^\circ$ (Lemma 3.6), while $M_p K^+ \to K$ [Haberl and Schuster 2009, p. 9]. Yet for fixed p, it is not apparent to us if there is a precise relation between $M_p K^+$ and our $K^{\circ,p}$ (though the polar of former are "isomorphic" to the latter — see Remark 3.14). They seem to be distinct. For example, $\Gamma_2 K$ is the Legendre ellipsoid of the convex body; thus bounding $|K||(\Gamma_2 K)^\circ|$ from below by a bound of the form c^n, where c is a constant independent of dimension, would imply Bourgain's conjecture (Conjecture 1.8) [Lutwak and Zhang 1997, p. 14]. On the contrary, by Lemma 3.12 below, the Bourgain–Milman inequality implies bounds of this type for $\mathcal{M}_p$ for all $p > 0$. It would be interesting to investigate relations between these constructions and ours, as well as relations to the level-sets of the logarithmic Laplace transform (see Remark 3.14), e.g., as in [Klartag and Milman 2012; Latała and Wojtaszczyk 2008].

1E. *Relation to the isotropic constant and Bourgain's slicing conjecture.* The L^p-support function $h_{p,K}$ is related to the covariance matrix of a convex body (Lemma 6.3),

$$\mathrm{Cov}_{ij}(K) := \int_K x_i x_j \frac{\mathrm{d}x}{|K|} - \int_K x_i \frac{\mathrm{d}x}{|K|} \int_K x_j \frac{\mathrm{d}x}{|K|} \tag{1-16}$$

via the identity

$$\nabla^2 h_{p,K}(0) = p\,\mathrm{Cov}(K). \tag{1-17}$$

This turns out to have an interesting connection to the slicing problem. Set

$$\mathcal{C}(K) := \frac{|K|^2}{\det \mathrm{Cov}(K)}. \tag{1-18}$$

Note

$$\mathcal{C}(K) = \frac{1}{L_K^{2n}}, \tag{1-19}$$

where L_K is the isotropic constant [Brazitikos et al. 2014, Definition 2.3.11]. Bourgain [1986, Remark, p. 1470; 1991, (1.9)] conjectured the following.

Conjecture 1.8. *There exists a constant $c > 0$ independent of dimension such that $\mathcal{C}(K) \geq c^n$ for all $n \in \mathbb{N}$ and all convex bodies $K \subset \mathbb{R}^n$.*

Let $B > 0$. We introduce the following convexity hypothesis:

$$u_{B,K} := \log\det \nabla^2 h_{1,K} + B h_{1,K} \quad \text{is convex.} \tag{$*_B$}$$

Note here that $h_{1,K}$ is twice differentiable (Lemma 4.4). We restrict to $p = 1$ since property $(*_B)$ is equivalent to a similar convexity property on $h_{p,K}$ (see Remark 6.15).

Theorem 1.9. *Let $K \subset \mathbb{R}^n$ be a convex body for which $(*_B)$ holds for some $B > 0$. Then*:

(i) *There is $x_K \in \operatorname{int} K$ with*

$$\mathcal{C}(K) \geq \frac{\mathcal{M}_{\frac{1}{B}}(K - x_K)^2}{\mathcal{M}_{\frac{1}{2B}}(K - x_K)} \left(\frac{2}{eB}\right)^n.$$

(ii) *There is $x_K \in \operatorname{int} K$ with*

$$\mathcal{C}(K) \geq \frac{\mathcal{M}(K - x_K)}{e^{2n} B^n} \geq \left(\frac{\pi}{2e^2 B}\right)^n.$$

(iii) *If K is symmetric*,

$$\mathcal{C}(K) \geq \frac{\mathcal{M}(K)}{e^n B^n} \geq \left(\frac{\pi}{eB}\right)^n.$$

Theorem 1.9 has the following consequence for Bourgain's slicing conjecture.

Corollary 1.10. *If there is a constant $B > 0$ independent of dimension such that $(*_B)$ holds for all convex bodies in all dimensions*, *then Conjecture 1.8 holds.*

In this direction, we have the following partial progress:

Theorem 1.11. *Property $(*_{n+1})$ holds for all convex bodies $K \subset \mathbb{R}^n$.*

As an immediate corollary of Theorems 1.9 and 1.11 we recover the so-called "folklore" bound on the isotropic constant due to [Milman and Pajor 1989, p. 96].

Corollary 1.12. *For a convex body $K \subset \mathbb{R}^n$*,

$$\mathcal{C}(K) \geq \left(\frac{\pi}{2e^2 n}\right)^n.$$

Corollary 1.12 is equivalent to an upper bound on the isotropic constant,

$$L_K \le C\sqrt{n} \tag{1-20}$$

for $C = e\sqrt{2/\pi}$, and hence is far from optimal: (1-20) holds with $C = 2\pi e$ by Milman and Pajor, $L_K \le Cn^{1/4}\log n$ by [Bourgain 1991, Theorem 1.6], and $L_K \le Cn^{1/4}$ by [Klartag 2006, Corollary 1.2], while very recently Chen [2021] obtained $L_K \le C_1 e^{C_2\sqrt{\log(n)}\sqrt{\log\log(3n)}}$ (in particular, $L_K \le Cn^{\varepsilon}$ for all $\varepsilon > 0$); see also [Klartag and Lehec 2022, (1)]. On these foundations several authors improved this to $L_K \le C(\log(n))^q$ for various values of q [Klartag and Lehec 2022; Jambulapati et al. 2022; Klartag 2023]; Conjecture 1.8 remains open.

The proof of Theorem 1.9 starts with the observation (1-17). The convexity assumption $(*_B)$ allows for the application of Jensen's inequality with respect to any probability measure μ. Because of (1-17) this will only be useful if μ is centered at the origin, i.e.,

$$b(\mu) := \int_{\mathbb{R}^n} y \, \mathrm{d}\mu(y) = 0 \in \mathbb{R}^n.$$

We use the family of log-concave measures given by the $\frac{1}{p}$-support functions,

$$\nu_{p,K} := \frac{e^{-h_{1/p,K}(y)}\mathrm{d}y}{\int_{\mathbb{R}^n} e^{-h_{1/p,K}(y)}\,\mathrm{d}y} = \frac{e^{-ph_{1,K}(y)}\mathrm{d}y}{\int_{\mathbb{R}^n} e^{-ph_{1,K}(y)}\,\mathrm{d}y}, \tag{1-21}$$

and optimize over p (the equality in (1-21) follows from Lemma 2.2(i) below). Proposition 1.5 is crucial here, since it ensures that K may be translated to a position for which $b(\nu_{p,K}) = 0$ (Corollary 4.5). After applying Jensen's inequality for the measures $\nu_{p,K}$, it remains to bound $\int_{\mathbb{R}^n} \log\det \nabla^2 h_{1,K}\,\mathrm{d}\nu_{p,K}(y)$ and $\int_{\mathbb{R}^n} h_{1,K}(y)\,\mathrm{d}\nu_{p,K}(y)$; this is done in Lemmas 6.10 and 6.13 respectively. The L^p-Mahler volumes $\mathcal{M}_p$ figure quite prominently throughout the proofs.

The proof of Theorem 1.11 is based upon an explicit computation

$$\log\det \nabla^2 h_{p,K}(y) = -p(n+1)h_{p,K}(y) + \log\psi(y), \tag{1-22}$$

ψ being the determinant of a positive-definite matrix. This relies on writing $\det \nabla^2 h_{p,K}$ as the determinant of the $(n+1)\times(n+1)$ Gram matrix M of the first moments of the measure

$$\frac{e^{p\langle x,y\rangle}}{e^{ph_{p,K}(y)}}\frac{\mathbf{1}_K^\infty(x)\,\mathrm{d}x}{|K|}.$$

Each entry of M then involves an $e^{-ph_{p,K}(y)}$ term; thus $\det\nabla^2 h_{p,K} = \det M = e^{-(n+1)ph_{p,K}}\psi$ for a positive $\psi > 0$. Taking the logarithm gives (1-22). For the remaining terms, ψ, being the sum of products of $n+1$ integrals over K, can be written as an integral over K^{n+1},

$$\psi(y) = C\int_{K^{n+1}} |\Delta(z)|^2 e^{p\langle z,(y,\dots,y)\rangle}\,\mathrm{d}z,$$

from which the convexity of $\log\psi$ can be deduced (Lemma 6.16), and hence the claim of Theorem 1.11.

Finally, we generalize Theorem 1.11 to the setting of a general probability measure — this is formulated in Theorem 6.19. In this generality, we show that the constant $B = n+1$ is actually optimal. In Section 6E we give a completely different proof of both theorems using, surprisingly, Kobayashi's theorem on the Ricci curvature of Bergman metrics, coming back full circle to the point of departure of this article in Section 1A: Bergman kernels.

1F. ***Perspective on the work on Nazarov and Błocki.*** Having presented L^p-polarity, it is perhaps worthwhile to revisit our original motivation for developing this theory: [Nazarov 2012; Blocki 2014; 2015].

Nazarov applied the theory of Bergman kernels of tube domains to tackle the symmetric Mahler conjecture. The constant he obtained $c = \frac{\pi^3}{16}$ in the inequality $\mathcal{M}(K) \geq c^n$ for symmetric convex bodies $K \subset \mathbb{R}^n$ was suboptimal compared to the conjectured value of $c = 4$ (see Section 1C) but the possibility remained open that perhaps a better choice of holomorphic L^2 function and weight function in Hörmander's $\bar{\partial}$-technique would allow to tackle the Mahler conjectures, or that perhaps, as Nazarov [2012, p. 337] suggested

> *. . . in order to get the Mahler conjecture itself on this way, one would have to work directly with the Paley–Wiener space by either finding a good analogue of the Hörmander theorem allowing to control the Paley–Wiener norm of the solution, or by finding some novel way to construct decaying analytic functions of several variables.*

Nazarov's approach was subsequently revisited by Błocki [2014; 2015], Hultgren [2013], and ourselves [Berndtsson 2022; Mastrantonis and Rubinstein 2022]. It became plausible after [Błocki 2015, p. 96] that Nazarov's approach might not yield Mahler's conjectures. In view of the results in the present article (e.g., Lemma 3.12) it is now clear why this is so, and exactly how Nazarov's approach fits in our story: it is an approach to the case $p=1$ of Conjectures 1.3–1.4. It is a beautiful coincidence that L^1-Mahler volumes can be expressed in terms of Bergman kernels (see Section 1A and [Mastrantonis and Rubinstein 2022, (42)]),

$$\mathcal{M}_1(K - b(K)) = (4\pi)^n |K|^2 \mathcal{K}_{T_K}(\sqrt{-1}b(K), \sqrt{-1}b(K)); \tag{1-23}$$

but even if one had a complete understanding of the variation of such kernels among tube domains, solving the classical Mahler conjectures would still require bridging the gap between L^1 and L^∞.

Finally, we touch upon an observation encountered in [Błocki 2015, p. 96]:

> *This shows (although only numerically) that the Bergman kernel for tube domains does not behave well under taking duals.*

Indeed, the theory of Bergman kernels of tube domains corresponds to $\mathcal{M}_1$ and L^1-polarity and the lack of homogeneity of $h_{1,K}$ leads to incompatibility with L^∞-polarity, i.e., with classical polarity/duality.

Organization. In Section 2A basic properties of $h_{p,K}$ are laid out, namely the convexity of $h_{p,K}$ (Lemma 2.1), its behavior under affine transformations of K, Cartesian products, and its monotonicity with respect to p (Lemma 2.2). Convexity properties of $h_{p,K}$ with respect to p or K are studied in Section 2B. In Section 2C, an upper bound to the support function h_p in terms of $h_{p,K}$ for bodies with barycenter at the origin $b(K) = 0$ is given. Section 2D is dedicated to the explicit computation of $h_{p,[-1,1]^n}$ for the cube. In Section 3A the L^p-polar $K^{\circ,p}$ is introduced, for which $\mathcal{M}_p(K) = n!\,|K||K^{\circ,p}|$, $K^\circ \subset K^{\circ,p}$ and $\bigcap_{p>0} K^{\circ,p} = K^\circ$. Inequalities relating $\mathcal{M}$ to $\mathcal{M}_p$ are established in Section 3C, and Section 3D is dedicated to computing $\mathcal{M}_p([-1,1]^n)$. In Section 3E, the L^p-support of the diamond B_1^n is explicitly computed in all dimensions and for all p (Lemma 3.17). Section 4 establishes the existence and uniqueness of Santaló points for $\mathcal{M}_p$ (Proposition 1.5), and in Section 5 we prove a Santaló inequality for $\mathcal{M}_p$ for symmetric convex bodies, showing that the 2-ball B_2^n is the maximizer (Theorem 1.6). In Section 6, we study the isotropic constant and the relations between $h_{p,K}$, $\mathcal{M}_p$, and Bourgain's conjecture. In particular, we prove Theorem 1.9, Theorem 1.11, and its generalization, Theorem 6.19. We conclude by giving an alternative proof of the latter using Bergman kernel methods and Kobayashi's theorem. In the Appendix, we verify that $K^{\circ,p}$ is a convex body by proving Proposition A.1, and provide a detailed proof of Ball's Brunn–Minkowski inequality for the harmonic mean (Theorem 5.20).

2. L^p support functions

In this section we lay out basic properties for $h_{p,K}$. In Section 2A we show convexity of $y \mapsto h_{p,K}(y)$ (Lemma 2.1) and list several properties in Lemma 2.2, e.g., how $h_{p,K}$ transforms under affine transformations of K or with respect to Cartesian products. In Section 2B we study convexity properties of $h_{p,K}$ in terms of convex combinations of p (Lemma 2.4) or K (Lemma 2.5). An upper bound for the support function h_K by $h_{p,K}$ for bodies with barycenter at the origin $b(K)=0$ is given in Section 2C. Finally, in Section 2D we carry out explicit computations for the cube.

2A. ***Basic properties of $h_{p,K}$.*** The functions $h_{p,K}$ defined by (1-8) are convex, even if the underlying body K is only compact.

Lemma 2.1. *Let $p \in (0,\infty)$. For a compact body $K \subset \mathbb{R}^n$, $h_{p,K}(y)$ is a convex function of y.*

Proof. Let $y, z \in \mathbb{R}^n$ and $\lambda \in (0,1)$. By Hölder's inequality,

$$\begin{aligned} h_{p,K}((1-\lambda)y+\lambda z) &= \frac{1}{p}\log\biggl(\int_K e^{p\langle x,(1-\lambda)y+\lambda z\rangle}\,\frac{\mathrm{d}x}{|K|}\biggr) \\ &= \frac{1}{p}\log\biggl(\int_K (e^{p\langle x,y\rangle})^{1-\lambda}(e^{p\langle x,z\rangle})^{\lambda}\,\frac{\mathrm{d}x}{|K|}\biggr) \\ &\le \frac{1}{p}\log\biggl[\biggl(\int_K e^{p\langle x,y\rangle}\,\frac{\mathrm{d}x}{|K|}\biggr)^{1-\lambda}\biggl(\int_K e^{p\langle x,z\rangle}\,\frac{\mathrm{d}x}{|K|}\biggr)^{\lambda}\biggr] \\ &= \frac{1-\lambda}{p}\log\biggl(\int_K e^{p\langle x,y\rangle}\,\frac{\mathrm{d}x}{|K|}\biggr)+\frac{\lambda}{p}\log\biggl(\int_K e^{p\langle x,z\rangle}\,\frac{\mathrm{d}x}{|K|}\biggr) \\ &= (1-\lambda)h_{p,K}(y)+\lambda h_{p,K}(z). \end{aligned}$$

□

Next, we list some properties of L^p-support functions that will be useful throughout.

Lemma 2.2. *Let $0<p<q<\infty$. For compact bodies $K \subset \mathbb{R}^n$, $L \subset \mathbb{R}^m$, and $A \in \mathrm{GL}(n,\mathbb{R})$, $a \in \mathbb{R}^n$:*

(i) $h_{p,K}(y) = \frac{1}{p}h_{1,K}(py)$.

(ii) $h_{p,K-a}(y) = h_{p,K}(y) - \langle a, y\rangle$.

(iii) $h_{p,AK}(y) = h_{p,K}(A^T y)$.

(iv) $h_{p,K\times L}(y,z)=h_{p,K}(y)+h_{p,L}(z),\ y\in\mathbb{R}^n,\ z\in\mathbb{R}^m$.

(v) $h_{p,K}\le h_{q,K}\le h_K$.

Proof. (i) By definition,

$$h_{p,K}(y)=\frac{1}{p}\log\int_K e^{p\langle x,y\rangle}\,\frac{\mathrm{d}x}{|K|}=\frac{1}{p}\log\int_K e^{\langle x,py\rangle}\,\frac{\mathrm{d}x}{|K|}=\frac{1}{p}h_{1,K}(py).$$

(ii) Changing variables $x=u-a$ for $x\in K-a,\ u\in K$, and $\mathrm{d}x=\mathrm{d}u$,

$$\begin{aligned}h_{p,K-a}(y)&=\frac{1}{p}\log\biggl(\int_{K-a}e^{p\langle x,y\rangle}\,\frac{\mathrm{d}x}{|K-a|}\biggr)=\frac{1}{p}\log\biggl(\int_K e^{p\langle u-a,y\rangle}\,\frac{\mathrm{d}u}{|K|}\biggr)\\&=\frac{1}{p}\log\biggl(\int_K e^{p\langle u,y\rangle}\,\frac{\mathrm{d}u}{|K|}e^{-p\langle a,y\rangle}\biggr)=h_{p,K}(y)-\langle a,\,y\rangle.\end{aligned}$$

(iii) For $x=Au,\ \mathrm{d}x=|\det A|\mathrm{d}u$,

$$\begin{aligned}h_{p,AK}(y)&=\frac{1}{p}\log\biggl(\int_{AK}e^{p\langle x,y\rangle}\,\frac{\mathrm{d}x}{|AK|}\biggr)=\frac{1}{p}\log\biggl(\int_K e^{p\langle Au,y\rangle}\,\frac{|\det A|\mathrm{d}u}{|\det A||K|}\biggr)\\&=\frac{1}{p}\log\biggl(\int_K e^{p\langle u,A^Ty\rangle}\,\frac{\mathrm{d}u}{|K|}\biggr)=h_{p,K}(A^Tu).\end{aligned}$$

(iv) By Tonelli's theorem [Folland 1999, §2.37; Mastrantonis and Rubinstein 2022, Claim 22],

$$\begin{aligned}h_{p,K\times L}(y,z)&=\frac{1}{p}\log\biggl(\int_{K\times L}e^{p\langle (x,u),(y,z)\rangle}\,\frac{\mathrm{d}x\mathrm{d}u}{|K\times L|}\biggr)\\&=\frac{1}{p}\log\biggl(\int_{K\times L}e^{p\langle x,y\rangle}e^{p\langle z,u\rangle}\,\frac{\mathrm{d}x\mathrm{d}u}{|K||L|}\biggr)\\&=\frac{1}{p}\log\biggl[\biggl(\int_K e^{p\langle x,y\rangle}\,\frac{\mathrm{d}x}{|K|}\biggr)\biggl(\int_L e^{p\langle z,u\rangle}\,\frac{\mathrm{d}u}{|L|}\biggr)\biggr]\\&=\frac{1}{p}\log\biggl(\int_K e^{p\langle x,y\rangle}\,\frac{\mathrm{d}x}{|K|}\biggr)+\frac{1}{p}\log\biggl(\int_L e^{p\langle z,u\rangle}\,\frac{\mathrm{d}u}{|L|}\biggr)\\&=h_{p,K}(y)+h_{p,L}(z).\end{aligned}$$

(v) By (1-7),

$$h_{q,K}(y):=\frac{1}{q}\log\biggl(\int_K e^{q\langle x,y\rangle}\,\frac{\mathrm{d}x}{|K|}\biggr)\le\frac{1}{q}\log\biggl(\int_K e^{qh_K(y)}\,\frac{\mathrm{d}x}{|K|}\biggr)=\frac{1}{q}\log e^{qh_K(y)}=h_K(y).$$

By Hölder's inequality (note $\frac{q}{p}>1$),

$$\begin{aligned}h_{p,K}(y)&=\frac{1}{p}\log\biggl(\int_K e^{p\langle x,y\rangle}\,\frac{\mathrm{d}x}{|K|}\biggr)\le\frac{1}{p}\log\biggl[\biggl(\int_K e^{\frac{q}{p}p\langle x,y\rangle}\,\frac{\mathrm{d}x}{|K|}\biggr)^{\frac{p}{q}}\biggl(\int_K\frac{\mathrm{d}x}{|K|}\biggr)^{1-\frac{p}{q}}\biggr]\\&=\frac{1}{p}\frac{p}{q}\log\biggl(\int_K e^{q\langle x,y\rangle}\,\frac{\mathrm{d}x}{|K|}\biggr)=h_{q,K}(y).\end{aligned}$$

□

Remark 2.3. One may wonder why we have a factor of $n!$ in (1-2) and (1-11). The first reason is that then one has (1-6) and (1-9). The second, more important, reason is that then $\mathcal{M}_p$ is *tensorial*. Indeed, by Lemma 2.2(iv) and (1-9),

$$\begin{aligned}\mathcal{M}_p(K\times L) &:= |K\times L|\int_{\mathbb{R}^n\times\mathbb{R}^m} e^{-h_{p,K\times L}(y,z)}\,\mathrm{d}y\,\mathrm{d}z\\ &= |K||L|\int_{\mathbb{R}^n\times\mathbb{R}^m} e^{-h_{p,K}(y)}e^{-h_{p,L}(z)}\,\mathrm{d}y\,\mathrm{d}z = \mathcal{M}_p(K)\mathcal{M}_p(L).\end{aligned}$$

2B. ***Additional convexity properties.*** Lemma 2.1 states that $y\mapsto h_{p,K}(y)$ is convex regardless of the convexity of K. Regarding p and K as the variables, we show two more properties: Lemma 2.4 describes convexity in p, and Lemma 2.5 shows an asymptotic (in p) concavity in K. These two lemmas are not used elsewhere in the article and we state them for their independent interest.

Lemma 2.4. *Let $p,q\in(0,\infty)$. For a convex body $K\subset\mathbb{R}^n$ and $\lambda\in(0,1)$,*

$$h_{(1-\lambda)p+\lambda q,K}\le \frac{(1-\lambda)p}{(1-\lambda)p+\lambda q}h_{p,K}+\frac{\lambda q}{(1-\lambda)p+\lambda q}h_{q,K}.$$

Proof. By Hölder's inequality,

$$\begin{aligned}h_{(1-\lambda)p+\lambda q,K}(y) &= \frac{1}{(1-\lambda)p+\lambda q}\log\biggl(\int_K e^{((1-\lambda)p+\lambda q)\langle x,y\rangle}\frac{\mathrm{d}x}{|K|}\biggr)\\ &= \frac{1}{(1-\lambda)p+\lambda q}\log\biggl(\int_K e^{(1-\lambda)p\langle x,y\rangle}e^{\lambda q\langle x,y\rangle}\frac{\mathrm{d}x}{|K|}\biggr)\\ &\le \frac{1}{(1-\lambda)p+\lambda q}\log\biggl[\biggl(\int_K e^{p\langle x,y\rangle}\frac{\mathrm{d}x}{|K|}\biggr)^{1-\lambda}\biggl(\int_K e^{q\langle x,y\rangle}\frac{\mathrm{d}x}{|K|}\biggr)^{\lambda}\biggr]\\ &= \frac{(1-\lambda)p}{(1-\lambda)p+\lambda q}\frac{1}{p}\log\biggl(\int_K e^{p\langle x,y\rangle}\frac{\mathrm{d}x}{|K|}\biggr)+\frac{\lambda q}{(1-\lambda)p+\lambda q}\frac{1}{q}\log\biggl(\int_K e^{q\langle x,y\rangle}\frac{\mathrm{d}x}{|K|}\biggr)\\ &= \frac{(1-\lambda)p}{(1-\lambda)p+\lambda q}h_{p,K}(y)+\frac{\lambda q}{(1-\lambda)p+\lambda q}h_{q,K}(y).\end{aligned}$$ ☐

Lemma 2.5. *Let $p\in(0,\infty)$. For convex bodies $K,L\subset\mathbb{R}^n$ and $\lambda\in(0,1)$,*

$$h_{p,(1-\lambda)K+\lambda L}\ge(1-\lambda)h_{p,K}+\lambda h_{p,L}-\frac{1}{p}\log\biggl(\frac{|(1-\lambda)K+\lambda L|}{|K|^{1-\lambda}|L|^{\lambda}}\biggr).$$

Proof. Fix $y\in\mathbb{R}^n$. Note that

$$\mathbf{1}_{(1-\lambda)K+\lambda L}((1-\lambda)x+\lambda z)e^{p\langle(1-\lambda)x+\lambda z,y\rangle}\ge(\mathbf{1}_K(x)e^{p\langle x,y\rangle})^{1-\lambda}(\mathbf{1}_L(z)e^{p\langle z,y\rangle})^{\lambda}$$

for all $x,z\in\mathbb{R}^n$. Therefore, by the Prékopa–Leindler inequality [Prékopa 1973, Theorem 3],

$$\int_{(1-\lambda)K+\lambda L}e^{p\langle x,y\rangle}\,\mathrm{d}x\ge\biggl(\int_K e^{p\langle x,y\rangle}\,\mathrm{d}x\biggr)^{1-\lambda}\biggl(\int_L e^{p\langle x,y\rangle}\,\mathrm{d}x\biggr)^{\lambda}.$$

As a result,

$$\begin{aligned} h_{p,(1-\lambda)K+\lambda L}(y) &= \frac{1}{p}\log\biggl(\int_{(1-\lambda)K+\lambda L} e^{p\langle x,y\rangle}\,\frac{\mathrm{d}x}{|(1-\lambda)K+\lambda L|}\biggr) \\ &\geq \frac{1}{p}\log\biggl[\biggl(\int_K e^{p\langle x,y\rangle}\,\mathrm{d}x\biggr)^{1-\lambda}\biggl(\int_L e^{p\langle x,y\rangle}\,\mathrm{d}x\biggr)^{\lambda}\frac{1}{|(1-\lambda)K+\lambda L|}\biggr] \\ &= \frac{1}{p}\log\biggl[\biggl(\int_K e^{p\langle x,y\rangle}\,\frac{\mathrm{d}x}{|K|}\biggr)^{1-\lambda}\biggl(\int_L e^{p\langle x,y\rangle}\,\frac{\mathrm{d}x}{|L|}\biggr)^{\lambda}\frac{|K|^{1-\lambda}|L|^{\lambda}}{|(1-\lambda)K+\lambda L|}\biggr] \\ &= (1-\lambda)h_{p,K}(y)+\lambda h_{p,K}(y)-\frac{1}{p}\log\biggl(\frac{|(1-\lambda)K+\lambda L|}{|K|^{1-\lambda}|L|^{\lambda}}\biggr), \end{aligned}$$

as claimed. □

2C. ***A reverse inequality.*** By Lemma 2.2(v),

$$h_{p,K} \leq h_K$$

regardless of the position of K. A reverse inequality holds when the barycenter is at the origin:

Lemma 2.6. *Let $p \in (0,\infty)$. For a convex body $K \subset \mathbb{R}^n$ with $b(K)=0$, and $\lambda \in (0,1)$,*

$$h_K(y) \leq h_{p,K}\biggl(\frac{y}{\lambda}\biggr) - \frac{n}{p}\log(1-\lambda).$$

Proof. Let $x \in K$, $y \in \mathbb{R}^n$ and $\lambda \in (0,1)$. The aim is to use Jensen's inequality to get an upper bound on $\langle x,y\rangle$. Since $b(K)=0$,

$$\begin{aligned} \langle x,y\rangle &= \biggl\langle \lambda x, \frac{y}{\lambda}\biggr\rangle = \biggl\langle \lambda x + (1-\lambda)b(K), \frac{y}{\lambda}\biggr\rangle \\ &= \biggl\langle \lambda x + (1-\lambda)\int_K u\,\frac{\mathrm{d}u}{|K|}, \frac{y}{\lambda}\biggr\rangle = \int_K \biggl\langle \lambda x + (1-\lambda)u, \frac{y}{\lambda}\biggr\rangle \frac{\mathrm{d}u}{|K|}. \end{aligned} \tag{2-1}$$

By convexity, $(1-\lambda)x+\lambda u$ lies in K as $x,u \in K$. Therefore, by (2-1), Jensen's inequality, and the change of variables $v = \lambda x + (1-\lambda)u$,

$$\begin{aligned} \langle x,y\rangle = \log e^{\langle x,y\rangle} &\leq \log\biggl(\int_K e^{\langle \lambda x+(1-\lambda)u, \frac{y}{\lambda}\rangle}\,\frac{\mathrm{d}u}{|K|}\biggr) \\ &= \log\biggl(\int_{\lambda x+(1-\lambda)K} e^{\langle v, \frac{y}{\lambda}\rangle}\,\frac{(1-\lambda)^{-n}\mathrm{d}v}{|K|}\biggr) \\ &= \log\biggl(\frac{1}{(1-\lambda)^n}\int_{\lambda x+(1-\lambda)K} e^{p\langle v, \frac{y}{p\lambda}\rangle}\,\frac{\mathrm{d}v}{|K|}\biggr) \\ &\leq \log\biggl(\frac{1}{(1-\lambda)^n}\int_K e^{p\langle v, \frac{y}{p\lambda}\rangle}\,\frac{\mathrm{d}v}{|K|}\biggr) = ph_{p,K}\biggl(\frac{y}{p\lambda}\biggr) - n\log(1-\lambda). \end{aligned}$$

A supremum over $x \in K$ gives $h_K(y) \leq ph_{K,p}\bigl(\frac{y}{p\lambda}\bigr) - n\log(1-\lambda)$. By a change of variable, $h_K(py) \leq ph_{K,p}\bigl(\frac{y}{\lambda}\bigr) - n\log(1-\lambda)$. The lemma now follows from homogeneity of h_K. □

Corollary 2.7. *Let $q \in (0,\infty]$. For a convex body $K \subset \mathbb{R}^n$,*

$$\lim_{p\to q} h_{p,K}(y) = h_{q,K}(y).$$

Proof. First, let $q \in (0,\infty)$. Since K is bounded, there exists $M > 0$ with $|x| \le M$ for all $x \in K$. In particular, $e^{p\langle x,y\rangle} \le e^{2qM|y|}$ for all $x \in K$ and $p \le 2q$. By dominated convergence [Folland 1999, §2.24],

$$\lim_{p\to q} \int_K e^{p\langle x,y\rangle} \frac{\mathrm{d}x}{|K|} = \int_K e^{q\langle x,y\rangle} \frac{\mathrm{d}x}{|K|}.$$

Therefore,

$$\lim_{p\to q} h_{p,K}(y) = \lim_{p\to q}\left(\frac{1}{p}\log \int_K e^{p\langle x,y\rangle} \frac{\mathrm{d}x}{|K|}\right) = \frac{1}{q}\log \int_K e^{q\langle x,y\rangle} \frac{\mathrm{d}x}{|K|} = h_{q,K}(y).$$

Next, consider $q = \infty$. By Lemma 2.2(v), $h_{p,K}(y)$ is monotone increasing in p, with $h_{p,K}(y) \le h_K(y)$. Thus the limit exists with $\lim_{p\to\infty} h_{p,K}(y) \le h_K(y)$; equivalently, $\lim_{p\to\infty}[h_{p,K}(y) - \langle y, b(K)\rangle] \le h_K(y) - \langle y, b(K)\rangle$. By Lemma 2.2(ii), this is

$$\lim_{p\to\infty} h_{p,K-b(K)}(y) \le h_{K-b(K)}(y).$$

On the other hand, as $b(K - b(K)) = 0$, Lemma 2.6 applies:

$$h_{K-b(K)}(y) = \frac{h_{K-b(K)}(\lambda y)}{\lambda} \le \frac{h_{p,K-b(K)}(y)}{\lambda} - \frac{n}{\lambda p}\log(1-\lambda),$$

where we used the homogeneity of h_K (here λ can be taken as any fixed value in $(0, 1)$). Letting first $p \to \infty$ and then $\lambda \to 1$,

$$h_{K-b(K)}(y) \le \lim_{p\to\infty} h_{p,K-b(K)}(y).$$

In conclusion, $h_{K-b(K)}(y) = \lim_{p\to\infty} h_{p,K-b(K)}(y)$ and using Lemma 2.2(ii) again we obtain $h_K(y) = \lim_{p\to\infty} h_{p,K}(y)$. □

2D. *The cube.* We explicitly compute the L^p-support functions and L^p-Mahler volumes of the cube $[-1, 1]^n$. This will be useful in proving Lemma 4.2 later.

Lemma 2.8. *For $p \in (0,\infty)$,*

$$h_{p,[-1,1]^n}(y) = \frac{1}{p}\sum_{i=1}^n \log\left(\frac{\sinh(py_i)}{py_i}\right), \quad y \in \mathbb{R}^n.$$

Proof. By Claim 2.9 below,

$$\begin{aligned} h_{p,[-1,1]^n}(y) &= \frac{1}{p}\log\left(\int_{[-1,1]^n} e^{p\langle x,y\rangle} \frac{\mathrm{d}x}{|[-1,1]^n|}\right) \\ &= \frac{1}{p}\log\left(2^n \prod_{i=1}^n \frac{\sinh(py_i)}{py_i}\frac{1}{2^n}\right) = \frac{1}{p}\sum_{i=1}^n \log\left(\frac{\sinh(py_i)}{py_i}\right). \end{aligned}$$ □

Claim 2.9. *For $y \in \mathbb{R}^n$,*

$$\int_{[-1,1]^n} e^{\langle x,y\rangle}\,\mathrm{d}x = 2^n \prod_{i=1}^n \frac{\sinh(y_i)}{y_i}.$$

Proof. This may be expressed as the product of integrals

$$\int_{[-1,1]^n} e^{\langle x,y\rangle}\,\mathrm{d}x = \prod_{i=1}^{n}\int_{-1}^{1} e^{x_i y_i}\,\mathrm{d}x_i,$$

because $e^{\langle x,y\rangle} = e^{x_1 y_1}\cdots e^{x_n y_n}$ and $[-1,1]^n$ is the product of n copies of $[-1,1]$. It is therefore enough to take $n=1$ and $y\in\mathbb{R}$. Suppose first that $y\neq 0$. Then

$$\int_{-1}^{1} e^{xy}\,\mathrm{d}x = \left[\frac{e^{xy}}{y}\right]_{x=-1}^{1} = \frac{e^{y}-e^{-y}}{y} = \frac{2\sinh(y)}{y}.$$

For $y=0$, we have $\int_{-1}^{1} e^{x\cdot 0}\,\mathrm{d}x = 2$. By L'Hôpital's rule also

$$\lim_{y\to 0}\frac{2\sinh(y)}{y} = \lim_{y\to 0}\frac{e^{y}-e^{-y}}{y} = \lim_{y\to 0}\frac{e^{y}+e^{-y}}{1} = 2,$$

verifying the formula for all y. □

3. L^p-polarity and L^p-Mahler volumes

In Section 3A, we motivate the definition of the L^p-polar body $K^{\circ,p}$ (Definition 1.1) and prove Theorem 1.2. In Section 3B, we establish the continuity of $K^{\circ,p}$ in p (Lemma 3.6) and show that, for p converging to 0, $K^{\circ,p}$ converges either to $\mathbb{R}^n$ or a half-space (Proposition 3.7). In Section 3C we generalize (1-3) to a lower bound of $\mathcal{M}$ in terms of $\mathcal{M}_p$, for all $p>0$, for bodies with $b(K)=0$ (see (3-10)). In Sections 3D and 3E calculations for $\mathcal{M}_p([-1,1]^n)$ and h_{p,B_1^n} are carried out and used to numerically approximate $\mathcal{M}_p(B_1^3)$, providing evidence that $\mathcal{M}_p([-1,1]^3) < \mathcal{M}_p(B_1^3)$ when $p<\infty$ (Figure 3).

3A. *The L^p-polar body.*

3A1. *Motivating the definition.* The support function of a convex body is convex and 1-homogeneous and hence its sublevel set

$$K^\circ := \{y\in\mathbb{R}^n : h_K(y)\le 1\}$$

defines a convex body such that $\mathcal{M}(K) = \mathcal{M}_\infty(K) = |K|\int_{\mathbb{R}^n} e^{-h_K(y)}\,\mathrm{d}y = n!\,|K||K^\circ|$. This is special for the case $p=\infty$. To see why, first recall the definition (1-9), $\mathcal{M}_p(K) := |K|\int_{\mathbb{R}^n} e^{-h_{p,K}}$. Yet despite the suggestive notation, for $p\in(0,\infty)$, $h_{p,K}$ is not the support function of a convex body since it is not 1-homogeneous. On the other hand, by Lemma 2.1, $h_{p,K}$ is convex and hence the sublevel set $\{h_{p,K}\le 1\} := \{y\in\mathbb{R}^n : h_{p,K}(y)\le 1\}$ is a convex body. Nonetheless, the volume of $\{h_{p,K}\le 1\}$ is not related to $\mathcal{M}_p(K)$ since despite having

$$\int_{\mathbb{R}^n} e^{-h_{p,K}(x)}\,\mathrm{d}x = \int_{-\infty}^{\infty} e^{-t}\,|\{h_{p,K}\le t\}|\,\mathrm{d}t;$$

without 1-homogeneity it is not clear how $\{h_{p,K}\le t\}$ relates to $\{h_{p,K}\le 1\}$ for all t.

In order to properly define the "L^p-polar" body we replace $h_{p,K}$ by a 1-homogeneous cousin. An equivalent way of defining a convex body L is via its "norm":

$$\|x\|_L := \inf\{t>0 : x\in tL\}. \tag{3-1}$$

This is a norm only when L is symmetric, but it is always positively 1-homogeneous and subadditive with $L = \{x \in \mathbb{R}^n : \|x\|_L \le 1\}$ [Gruber 2007, Theorem 4.3]. Given such a "norm", the volume of L can be expressed as an integral over the sphere:

$$|L| = \int_{\{x\in\mathbb{R}^n:\|x\|_L\le 1\}} \mathrm{d}x = \int_{\{(r,u)\in[0,\infty)\times\partial B_2^n:\|ru\|_L\le 1\}} r^{n-1}\,\mathrm{d}r\,\mathrm{d}u$$
$$= \int_{\partial B_2^n}\int_{r=0}^{1/\|u\|_L} r^{n-1}\,\mathrm{d}r\,\mathrm{d}u = \frac{1}{n}\int_{\partial B_2^n}\frac{\mathrm{d}u}{\|u\|_L^n}. \tag{3-2}$$

Looking at (3-2) one may be able to recover the "norm" of a convex body by writing its volume as an integral over ∂B_2^n. Our aim is to define a convex body $K^{\circ,p}$ with volume $|K^{\circ,p}| = \frac{1}{n!}\int e^{-h_{p,K}}$. Starting from the volume we guess its norm: we need to write $\int e^{-h_{p,K}}$ as an integral on the sphere matching (3-2),

$$|K^{\circ,p}| = \frac{1}{n!}\int_{\mathbb{R}^n} e^{-h_{p,K}(y)}\,\mathrm{d}y = \frac{1}{n!}\int_{\partial B_2^n}\int_0^\infty e^{-h_{p,K}(ru)}r^{n-1}\,\mathrm{d}r\,\mathrm{d}u$$
$$= \frac{1}{n}\int_{\partial B_2^n}\frac{\mathrm{d}u}{\big[\big(\frac{1}{(n-1)!}\int_0^\infty r^{n-1}e^{-h_{p,K}(ru)}\,\mathrm{d}r\big)^{-\frac{1}{n}}\big]^n}. \tag{3-3}$$

This justifies the definition of $\|\cdot\|_{K^{\circ,p}}$ via (1-12) and $K^{\circ,p}$ as the convex body associated to that "norm" (Definition 1.1).

3A2. *Proof of Theorem 1.2.* In this subsection we conclude the proof of Theorem 1.2. We start with two lemmas.

Lemma 3.1. *Let* $0 < p < q$ *and recall* (1-12). *For a compact body* K, $\|\cdot\|_{K^{\circ,p}} \le \|\cdot\|_{K^{\circ,q}} \le h_K(\cdot)$. *In particular*, $K^\circ \subset K^{\circ,q} \subset K^{\circ,p}$.

Note the support function of a compact body coincides with the "norm" of the polar body (3-1),

$$h_K(\cdot) = \|\cdot\|_{K^\circ}, \tag{3-4}$$

since $y \in K^\circ$ if and only if $h_K(y) \le 1$ [Gruber 2007, p. 56]. Also, for convex bodies [Rockafellar 1970, Corollary 13.1.1],

$$L \subset K \quad \text{if and only if} \quad \|\cdot\|_K \le \|\cdot\|_L \quad \text{if and only if} \quad h_{K^\circ}(\cdot) \le h_{L^\circ}(\cdot). \tag{3-5}$$

Lemma 3.2. *Let* $p \in (0,\infty]$. *For a convex body* $K \subset \mathbb{R}^n$, $K^{\circ,p}$ *is bounded* (*compact*) *if and only if* $0 \in \operatorname{int} K$.

In particular, since K° has nonempty interior [Rockafellar 1970, Corollary 14.6.1], Lemma 3.1 shows that $K^{\circ,p}$ is nonempty and has nonempty interior.

Before proving Lemmas 3.1 and 3.2, let us recall an integral formula regarding 1-homogeneous functions that will be useful throughout.

Claim 3.3. *Let* $k \in \mathbb{N}$. *For a* 1*-homogeneous function* $f : \mathbb{R}^n \to \mathbb{R}$ *and* $x \in \mathbb{R}^n$ *with* $f(x) \ne 0$,

$$\int_0^\infty r^{k-1}e^{-f(rx)}\,\mathrm{d}r = \frac{(k-1)!}{f(x)^k}.$$

Proof. By homogeneity of f, $f(rx) = rf(x)$ for all $r > 0$. Setting $\rho = rf(x)$,

$$\int_0^\infty r^{k-1} e^{-f(rx)}\,\mathrm{d}r = \int_0^\infty r^{k-1} e^{-rf(x)}\,\mathrm{d}r = \int_0^\infty \frac{\rho^{k-1}}{f(x)^{k-1}} e^{-\rho}\,\frac{\mathrm{d}\rho}{f(x)}$$
$$= \frac{1}{f(x)^k}\int_0^\infty \rho^{k-1} e^{-\rho}\,\mathrm{d}\rho = \frac{(k-1)!}{f(x)^k},$$

as claimed. □

Proof of Lemma 3.1. Let $p \le q$. By Lemma 2.2(v), $h_{p,K} \le h_{q,K}$. Thus by (1-12),

$$\|x\|_{K^{\circ,p}} = \left(\frac{1}{(n-1)!}\int_0^\infty r^{n-1} e^{-h_{p,K}(rx)}\,\mathrm{d}r\right)^{-\frac{1}{n}}$$
$$\le \left(\frac{1}{(n-1)!}\int_0^\infty r^{n-1} e^{-h_{q,K}(rx)}\,\mathrm{d}r\right)^{-\frac{1}{n}} = \|x\|_{K^{\circ,q}}. \tag{3-6}$$

So, for $x \in K^{\circ,q}$, $\|x\|_{K^{\circ,p}} \le \|x\|_{K^{\circ,q}} \le 1$; thus $x \in K^{\circ,p}$. In addition, by homogeneity of h_K, Claim 3.3 gives

$$\frac{1}{(n-1)!}\int_0^\infty r^{n-1} e^{-h_K(rx)}\,\mathrm{d}r = \frac{1}{h_K(x)^n}. \tag{3-7}$$

Since by Lemma 2.2(v) $h_{p,K} \le h_K$, and by (3-4), (3-7) and a computation similar to (3-6) $\|x\|_{K^{\circ,p}} \le h_K(x) = \|x\|_{K^\circ}$, it follows that $K^\circ \subset K^{\circ,p}$ by (3-5). □

For the proof of Lemma 3.2, it is useful to know that the L^p-polars of $[-1,1]^n$ are bounded.

Claim 3.4. *For $p \in (0,\infty]$, we have $([-1,1]^n)^{\circ,p}$ is bounded.*

Proof. Since $b([-1,1]^n) = 0$, by Lemma 2.6, with $\lambda = \frac{1}{2}$,

$$h_{[-1,1]^n}\left(\frac{ry}{2}\right) \le h_{p,[-1,1]^n}(ry) + \frac{n}{p}\log 2$$

for all $y \in \mathbb{R}^n$ and $r > 0$. Thus by (1-12),

$$\|y\|_{([-1,1]^n)^{\circ,p}} = \left(\frac{1}{(n-1)!}\int_0^\infty r^{n-1} e^{-h_{p,[-1,1]^n}(ry)}\,\mathrm{d}r\right)^{-\frac{1}{n}}$$
$$\ge \left(\frac{1}{(n-1)!}\int_0^\infty r^{n-1} e^{-h_{[-1,1]^n}\left(r\frac{y}{2}\right)} e^{\frac{n}{p}\log 2}\,\mathrm{d}r\right)^{-\frac{1}{n}}$$
$$= e^{-\frac{\log 2}{p}}\left(\frac{1}{h_{[-1,1]^n}\left(\frac{y}{2}\right)^n}\right)^{-\frac{1}{n}} = \frac{e^{-\frac{\log 2}{p}}}{2} h_{[-1,1]^n}(y) = \frac{e^{-\frac{\log 2}{p}}}{2}\|y\|_{([-1,1]^n)^\circ},$$

by Claim 3.3, the homogeneity of $h_{[-1,1]^n}$, and (3-4). By (3-5),

$$([-1,1]^n)^{\circ,p} \subset 2e^{\frac{\log 2}{p}}([-1,1]^n)^\circ = 2e^{\frac{\log 2}{p}} B_1^n,$$

which is bounded. □

Proof of Lemma 3.2. Assume $0 \in \operatorname{int} K$ and let $r > 0$ be such that $[-r, r]^n \subset K$. Then,

$$\begin{aligned} h_{p,K}(y) &= \frac{1}{p}\log\left(\int_K e^{p\langle x,y\rangle}\,\frac{\mathrm{d}x}{|K|}\right) \geq \frac{1}{p}\log\left(\int_{[-r,r]^n} e^{p\langle x,y\rangle}\,\frac{\mathrm{d}x}{|K|}\right) \\ &= \frac{1}{p}\log\left(\int_{[-r,r]^n} e^{p\langle x,y\rangle}\,\frac{\mathrm{d}x}{|[-r,r]^n|}\frac{|[-r,r]^n|}{|K|}\right) \\ &= h_{p,[-r,r]^n}(y) + \frac{1}{p}\log\frac{(2r)^n}{|K|}. \end{aligned} \tag{3-8}$$

Using this and (1-12),

$$\|y\|_{K^{\circ,p}} \geq \left(\frac{1}{(n-1)!}\int_0^\infty r^{n-1}e^{-h_{p,[-r,r]^n}(\rho y)-\frac{1}{p}\log\frac{(2r)^n}{|K|}}\,\mathrm{d}\rho\right)^{-\frac{1}{n}} = \frac{(2r)^{\frac{1}{p}}}{|K|^{\frac{1}{np}}}\|y\|_{([-r,r]^n)^{\circ,p}}.$$

Thus, by (3-5),

$$K^{\circ,p} \subset \frac{|K|^{\frac{1}{np}}}{(2r)^{\frac{1}{p}}}([-r,r]^n)^{\circ,p},$$

which is bounded by Claim 3.4.

For the converse, we claim that if $0 \notin \operatorname{int} K$ then $K^{\circ,p}$ is unbounded. By Lemma 3.1, $K^\circ \subset K^{\circ,p}$ so it is enough to show K° is unbounded. This is classical [Rockafellar 1970, Corollary 14.5.1]. □

Proof of Theorem 1.2. By Proposition A.1, $\|\cdot\|_{K^{\circ,p}}$ is positively 1-homogeneous and subadditive. The nonemptiness of the interior of $K^{\circ,p}$ follows from Lemma 3.1 since K° has nonempty interior. It is also closed and convex as the sublevel set of a continuous, convex function. Convexity of $\|\cdot\|_{K^{\circ,p}}$ follows from its 1-homogeneity and subadditivity: for $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$,

$$\|(1-\lambda)x+\lambda y\|_{K^{\circ,p}} \leq \|(1-\lambda)x\|_{K^{\circ,p}} + \|\lambda y\|_{K^{\circ,p}} = (1-\lambda)\|x\|_{K^{\circ,p}} + \lambda\|y\|_{K^{\circ,p}}.$$

If K is symmetric, i.e., $-K = K$, then

$$h_{p,K}(-y) = \frac{1}{p}\log\int_K e^{p\langle x,-y\rangle}\,\frac{\mathrm{d}x}{|K|} = \frac{1}{p}\log\int_{-K} e^{p\langle -z,-y\rangle}\,\frac{\mathrm{d}z}{|K|} = \frac{1}{p}\log\int_K e^{p\langle z,y\rangle}\,\frac{\mathrm{d}z}{|K|} = h_{p,K}(y).$$

Therefore,

$$\|-x\|_{K^{\circ,p}} = \left(\int_0^\infty r^{n-1}e^{-h_{p,K}(-rx)}\,\mathrm{d}r\right)^{-\frac{1}{n}} = \left(\int_0^\infty r^{n-1}e^{-h_{p,K}(rx)}\,\mathrm{d}r\right)^{-\frac{1}{n}} = \|x\|_{K^{\circ,p}},$$

making $\|\cdot\|_{K^{\circ,p}}$ a norm, and $K^{\circ,p}$ symmetric. Finally, (1-11) follows from (3-3) and the definition of $\|\cdot\|_{K^{\circ,p}}$. □

Remark 3.5. Ball showed that for a convex function $\phi : \mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$ and $q \geq 1$, setting

$$\|y\|_{\phi,q} := \left(\int_0^\infty r^{q-1}e^{-\phi(ry)}\,\mathrm{d}r\right)^{-\frac{1}{q}}$$

defines a positively 1-homogeneous, subadditive function that is also a norm when ϕ is even [Ball 1988, Theorem 5]. Then, $\{y \in \mathbb{R}^n : \|y\|_{\phi,q} \leq 1\}$ defines a convex body (for even ϕ [Ball 1988, Theorem 5], for general ϕ [Klartag 2006, Theorem 2.2]). In this notation, (1-12) reads $\|y\|^n_{K^{\circ,p}} = (n-1)!\,\|y\|^n_{h_{p,K},n}$. For a statement and proof of Ball's theorem, see Proposition A.1 below.

3B. ***Continuity of $\mathcal{M}_p$ and limiting cases.*** First, we translate (pointwise) convergence of L^p-support functions to convergence of the norms of the L^p-polars.

Lemma 3.6. *Let $0 < p < q \le \infty$. For a compact body $K \subset \mathbb{R}^n$, $K^{\circ,p} \subset K^{\circ,q}$ and*

$$\lim_{p \to q} \|x\|_{K^{\circ,p}} = \|x\|_{K^{\circ,q}}.$$

In particular, $\bigcap_{0<p<q} K^{\circ,p} = K^{\circ,q}$.

Proof. By Corollary 2.7, $h_{p,K}$ increases to $h_{q,K}$ as p increases to q. Therefore, one may use the monotone convergence theorem [Folland 1999, §2.14] to take the limit under the integral in the definition of $h_{p,K}$,

$$\begin{aligned}
\lim_{p\to\infty} \|x\|_{K^{\circ,p}} &= \lim_{p\to\infty} \left(\frac{1}{(n-1)!} \int_{r=0}^{\infty} r^{n-1} e^{-h_{p,K}(rx)}\, \mathrm{d}r \right)^{-\frac{1}{n}} \\
&= \left(\frac{1}{(n-1)!} \int_{r=0}^{\infty} \lim_{p\to q} (r^{n-1} e^{-h_{p,K}(rx)})\, \mathrm{d}r \right)^{-\frac{1}{n}} \\
&= \left(\frac{1}{(n-1)!} \int_{r=0}^{\infty} r^{n-1} e^{-h_{q,K}(rx)}\, \mathrm{d}r \right)^{-\frac{1}{n}} = \|x\|_{K^{\circ,q}}.
\end{aligned}$$

□

3B1. *The cases $p = 0$ and $p = \infty$.* By Lemma 3.6, as $p \to \infty$, $K^{\circ,p}$ converges to the polar body K° in (1-1). In this subsection we focus on the other extreme case $p = 0$ and show that in the limit $p \to 0$, $K^{\circ,p}$ converges either to $\mathbb{R}^n$ or to a half-space, depending on whether $b(K) = 0$ or not.

Proposition 3.7. *For a compact body $K \subset \mathbb{R}^n$,*

$$\lim_{p\to 0} \|y\|_{K^{\circ,p}} = \langle y, b(K)\rangle$$

and

$$\overline{\bigcup_{p>0} K^{\circ,p}} = \{y \in \mathbb{R}^n : \langle y, b(K)\rangle \le 1\}.$$

Proposition 3.7 and Lemma 3.6 imply the following inclusion for all $K^{\circ,p}$.

Corollary 3.8. *For a compact body $K \subset \mathbb{R}^n$, $K^{\circ,p} \subset \{y \in \mathbb{R}^n : \langle y, b(K)\rangle \le 1\}$ for all $p \in (0, \infty]$.*

The statement of Corollary 3.8 is trivial when $p = \infty$ because $b(K) \in K$; thus, by the definition of the polar, $\langle y, b(K)\rangle \le 1$ for all $y \in K^\circ$. The proof of Proposition 3.7 follows from the fact that the L^p-support functions converge to a linear function as $p \to 0$.

Lemma 3.9. *For a compact body $K \subset \mathbb{R}^n$ and $y \in \mathbb{R}^n$,*

$$\lim_{p\to 0} h_{p,K}(y) = \langle y, b(K)\rangle.$$

Proof. Expanding the exponential,

$$\begin{aligned}
h_{p,K}(y) &= \frac{1}{p} \log \int_K e^{p\langle x,y\rangle} \frac{\mathrm{d}x}{|K|} \\
&= \frac{1}{p} \log\left(\int_K 1 + p\langle x, y\rangle + O(p^2) \frac{\mathrm{d}x}{|K|} \right) \\
&= \frac{1}{p} \log(1 + p\langle y, b(K)\rangle + O(p^2)).
\end{aligned}$$

By L'Hôpital's rule,

$$\lim_{p\to 0} h_{p,K}(y) = \lim_{p\to 0} \frac{\log(1+p\langle y, b(K)\rangle + O(p^2))}{p} = \lim_{p\to 0} \frac{\langle y, b(K)\rangle + O(p)}{1+p\langle y, b(K)\rangle + O(p^2)} = \langle y, b(K)\rangle.$$

Alternative proof:

$$\begin{aligned}\lim_{p\to 0} h_{p,K}(y) &= \lim_{p\to 0} \frac{1}{p} \log \int_K e^{p\langle x,y\rangle} \frac{\mathrm{d}x}{|K|} \\ &= \lim_{p\to 0} \frac{\int_K \langle x, y\rangle e^{p\langle x,y\rangle} \frac{\mathrm{d}x}{|K|}}{\int_K e^{p\langle x,y\rangle} \frac{\mathrm{d}x}{|K|}} = \int_K \langle x, y\rangle \frac{\mathrm{d}x}{|K|} = \langle y, b(K)\rangle,\end{aligned}$$

again by L'Hôpital's rule. □

Proof of Proposition 3.7. For $y \in \mathbb{R}^n$ with $\langle y, b(K)\rangle \neq 0$, by the monotone convergence theorem [Folland 1999, §2.14] and Lemma 3.9,

$$\begin{aligned}\lim_{p\to 0} \|y\|_{K^{\circ,p}} &= \lim_{p\to 0} \left(\frac{1}{(n-1)!} \int_0^\infty r^{n-1} e^{-h_{p,K}(ry)} \,\mathrm{d}r\right)^{-\frac{1}{n}} \\ &= \left(\frac{1}{(n-1)!} \int_0^\infty r^{n-1} e^{-r\langle y, b(K)\rangle} \,\mathrm{d}r\right)^{-\frac{1}{n}} = \langle y, b(K)\rangle,\end{aligned}$$

where Claim 3.3 was used on the 1-homogeneous $y \mapsto \langle y, b(K)\rangle$. If $\langle y, b(K)\rangle = 0$, similarly,

$$\lim_{p\to 0} \|y\|_{K^{\circ,p}} = \left(\frac{1}{(n-1)!} \int_0^\infty r^{n-1} \,\mathrm{d}r\right)^{-\frac{1}{n}} = 0 = \langle y, b(K)\rangle. \qquad \square$$

Proposition 3.7 motivates the following definition.

Definition 3.10. For a compact body $K \subset \mathbb{R}^n$, let

$$K^{\circ,0} := \{y \in \mathbb{R}^n : \langle y, b(K)\rangle \leq 1\}.$$

For a set $A \subset \mathbb{R}^n$, denote by

$$\operatorname{co} A$$

its *convex hull* defined as the smallest convex set in $\mathbb{R}^n$ containing A.

Example 3.11. The polar body of the standard 2-dimensional simplex Δ_2 is given by the intersection of two half-spaces

$$\Delta_2^\circ = \{(x, y) \in \mathbb{R}^2 : x \leq 1 \text{ and } y \leq 1\}.$$

That is because $\Delta_2 = \operatorname{co}\{(0,0), (1,0), (0,1)\}$; thus $(x, y) \in \Delta^\circ$ if and only if $x = \langle (x, y), (1,0)\rangle \leq 1$ and $y = \langle (x, y), (0,1)\rangle \leq 1$. In addition, $|\Delta_2| = \frac{1}{2}$; thus the x-coordinate of the barycenter of Δ_2 is

$$\frac{1}{|\Delta_2|} \int_{\Delta_2} x \,\mathrm{d}x \,\mathrm{d}y = 2 \int_{x=0}^1 \int_{y=0}^{1-x} x \,\mathrm{d}y \,\mathrm{d}x = 2 \int_0^1 x(1-x) \,\mathrm{d}x = \frac{1}{3}.$$

Similarly, $\frac{1}{|\Delta_2|} \int_{\Delta_2} y \,\mathrm{d}y = \frac{1}{3}$, and hence $b(\Delta_2) = \left(\frac{1}{3}, \frac{1}{3}\right)$. As a result,

$$\Delta^{\circ,0} = \{(x, y) \in \mathbb{R}^2 : x + y \leq 3\}.$$

By Lemma 3.6, $\{x \le 1\} \cap \{y \le 1\} \subset (\Delta_2)^{\circ,p} \subset \{x + y \le 3\}$ for all $p \ge 0$. By direct calculation,

$$h_{p,\Delta_2}(x, y) = \frac{1}{p} \log\left(\frac{\frac{e^{px}-1}{px} - \frac{e^{py}-1}{py}}{p\frac{x-y}{2}} \right),$$

from which we get Figure 2 in the Introduction. By Lemma 2.2(ii),

$$h_{p,\Delta_2 - b(\Delta_2)}(y) = h_{p,\Delta_2} - \langle (x, y), b(\Delta_2) \rangle = \frac{1}{p} \log\left(\frac{\frac{e^{px}-1}{px} - \frac{e^{py}-1}{py}}{p\frac{x-y}{2}} \right) - \frac{x}{3} - \frac{y}{3},$$

leading to Figure 1, right, in the Introduction.

3C. *Inequalities between $\mathcal{M}_p$ and $\mathcal{M}$.* By Lemma 2.2(v), $h_{p,K} \le h_K$ for all p; thus

$$\mathcal{M}(K) \le \mathcal{M}_p(K). \tag{3-9}$$

In view of Lemma 2.6, a reverse inequality holds under the extra assumption of $b(K) = 0$.

Lemma 3.12. *Let $p \in (0, \infty)$. For a convex body $K \subset \mathbb{R}^n$ with $b(K) = 0$,*

$$\left(\frac{p}{(1+p)^{1+\frac{1}{p}}} \right)^n \mathcal{M}_p(K) \le \mathcal{M}(K).$$

Hence, $\lim_{p\to\infty} \mathcal{M}_p(K) = \mathcal{M}(K)$.

Remark 3.13. Lemma 3.12 generalizes the Bergman kernel inequality (1-3) (recall (1-23)).

Proof. Assume $b(K) = 0$. Lemma 2.6 applies to give

$$\begin{aligned} \mathcal{M}(K) = |K| \int_{\mathbb{R}^n} e^{-h_K(y)}\, \mathrm{d}y &\ge (1-\lambda)^{\frac{n}{p}} |K| \int_{\mathbb{R}^n} e^{-h_{p,K}(\frac{y}{\lambda})}\, \mathrm{d}y \\ &= (1-\lambda)^{\frac{n}{p}} \lambda^n |K| \int_{\mathbb{R}^n} e^{-h_{p,K}(y)}\, \mathrm{d}y = ((1-\lambda)^{\frac{1}{p}} \lambda)^n \mathcal{M}_p(K). \end{aligned} \tag{3-10}$$

It remains to maximize $f(\lambda) := (1-\lambda)^{1/p}\lambda$. The derivative

$$f'(\lambda) = -\frac{1}{p}(1-\lambda)^{\frac{1}{p}-1}\lambda + (1-\lambda)^{\frac{1}{p}} = (1-\lambda)^{\frac{1}{p}-1}\left(-\frac{\lambda}{p} + 1 - \lambda \right) \tag{3-11}$$

is positive for $\lambda \in \left(0, \frac{p}{p+1}\right)$ and nonpositive for $\lambda \in \left(\frac{p}{p+1}, 1\right)$, so plugging $\lambda = \frac{p}{p+1}$ in (3-10) proves the claim.

Finally, note

$$\lim_{p\to\infty} \frac{p}{(1+p)^{1+\frac{1}{p}}} = \lim_{p\to\infty} \left(\frac{1}{(1+p)^{\frac{1}{p}}} - \frac{1}{(1+p)^{1+\frac{1}{p}}} \right) = 1;$$

thus $\lim_{p\to\infty} \mathcal{M}_p(K) = \mathcal{M}(K)$. □

Remark 3.14. For convex $K \subset \mathbb{R}^n$ with $b(K) = 0$, and any $\lambda \in (0, 1)$,

$$\begin{aligned} h_K(y) &= \left(\frac{1}{(n-1)!} \int_0^\infty r^{n-1} e^{-h_K(ry)}\, \mathrm{d}r \right)^{-\frac{1}{n}} \\ &\le \left(\frac{1}{(n-1)!} \int_0^\infty r^{n-1} e^{-h_{p,K}(\frac{ry}{\lambda}) + \frac{n}{p}\log(1-\lambda)}\, \mathrm{d}r \right)^{-\frac{1}{n}} = \frac{\|y\|_{K^{\circ,p}}}{(1-\lambda)^{\frac{1}{p}}\lambda}, \end{aligned}$$

where we used Lemma 2.6 and Claim 3.3. So,

$$K^\circ \subset K^{\circ,p} \subset \frac{1}{(1-\lambda)^{\frac{1}{p}}\lambda} K^\circ \subset \frac{(1+p)^{1+\frac{1}{p}}}{p} K^\circ$$

(optimizing over λ as in the proof of Lemma 3.12). This yields inclusions independent of K or the dimension. Thus for convex bodies with $b(K)=0$, all the L^p-polars $K^{\circ,p}$ are "isomorphic" to (each other and to) the classical polar body K°. They are also "isomorphic" to the sublevel sets of $h_{1,K}$; see [Klartag and Milman 2005, Lemma 2.2; 2012, p. 16]. Furthermore, the latter (at least in the symmetric case) are "isomorphic" to the Lutwak–Zhang centroid bodies from Remark 1.7 [Klartag and Milman 2012, Lemma 2.3]. Nonetheless, "isomorphic" in this context means that inclusions in both directions exist by dilations independent of dimension. Consequently, such equivalences are not typically helpful when one is concerned with sharp lower bounds as in the Mahler conjectures. Given Lemma 3.12 and the remarks in the Introduction, we believe that our L^p-polars could be helpful in the pursuit of sharp bounds, e.g., as in the Mahler conjectures.

3D. ***The cube.*** The next lemma computes the L^p-Mahler volume of the cube.

Lemma 3.15. *For $p \in (0,\infty)$,*

$$\mathcal{M}_p([-1,1]^n) = 4^n \left(\frac{1}{p} \int_0^\infty \left(\frac{y}{\sinh(y)} \right)^{\frac{1}{p}} \mathrm{d}y \right)^n.$$

Note that

$$\mathcal{M}_p([-1,1]^n) = (\mathcal{M}_p([-1,1]))^n,$$

in agreement with Remark 2.3.

Proof. By Lemma 2.8,

$$\begin{aligned}
\mathcal{M}_p([-1,1]^n) &= |[-1,1]^n| \int_{\mathbb{R}^n} e^{-h_{p,[-1,1]^n}(y)} \,\mathrm{d}y = 2^n \int_{\mathbb{R}^n} \prod_{i=1}^n \left(\frac{py_i}{\sinh(py_i)} \right)^{\frac{1}{p}} \mathrm{d}y \\
&= 2^n \prod_{i=1}^n \int_{\mathbb{R}} \left(\frac{py_i}{\sinh(py_i)} \right)^{\frac{1}{p}} \mathrm{d}y = 2^n \left(\int_{\mathbb{R}} \left(\frac{py}{\sinh(py)} \right)^{\frac{1}{p}} \mathrm{d}y \right)^n.
\end{aligned}$$

The claim follows from the evenness of $py/\sinh(py)$ and the change of variables $z = py$. □

In the notation of Section 1A, Błocki [2014, (7)]. obtained

$$|[-1,1]^n|^2 \mathcal{K}_{T_{[-1,1]^n}}(0,0) = \left(\tfrac{\pi}{4}\right)^n.$$

This agrees with our next corollary as $\mathcal{M}_1(K) = (4\pi)^n |K|^2 \mathcal{K}_{T_K}(0,0)$ by (1-4).

Corollary 3.16. $$\mathcal{M}_1([-1,1]^n) = \pi^{2n}.$$

Proof. Setting $p=1$ in Lemma 3.15,

$$\mathcal{M}_1([-1,1]^n) = \left(2 \int_{\mathbb{R}} \frac{y}{\sinh(y)} \,\mathrm{d}y \right)^n = \left(4 \int_0^\infty \frac{y}{\sinh(y)} \,\mathrm{d}y \right)^n,$$

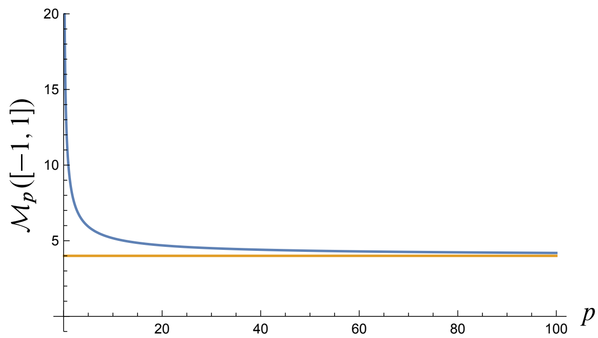

Figure 4. $\mathcal{M}_p([-1,1])$ for $p \in (0,100)$ compared to $\mathcal{M}([-1,1]) = 4$.

because $y/\sinh(y)$ is even. Using $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k$ for $0 < x < 1$, expand the integrand

$$\frac{y}{\sinh(y)} = \frac{2y}{e^y - e^{-y}} = \frac{2ye^{-y}}{1-e^{-2y}} = 2ye^{-y}\sum_{k=0}^{\infty} e^{-2ky} = \sum_{k=0}^{\infty} 2ye^{-(2k+1)y}.$$

Therefore, by integration by parts

$$\begin{aligned}\int_0^\infty \frac{y}{\sinh(y)}\,\mathrm{d}y &= \sum_{k=0}^{\infty}\int_0^\infty 2ye^{-(2k+1)y}\,\mathrm{d}y = \sum_{k=0}^{\infty}\frac{2}{2k+1}\int_0^\infty e^{-(2k+1)y}\,\mathrm{d}y \\ &= 2\sum_{k=0}\frac{1}{(2k+1)^2} = 2\biggl(\sum_{k=1}^{\infty}\frac{1}{k^2} - \sum_{k=1}^{\infty}\frac{1}{(2k)^2}\biggr) \\ &= 2\biggl(\sum_{k=1}^{\infty}\frac{1}{k^2} - \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}\biggr) = \frac{3}{2}\sum_{k=0}^{\infty}\frac{1}{k^2} = \frac{3}{2}\frac{\pi^2}{6} = \frac{\pi^2}{4},\end{aligned}$$

and hence

$$\mathcal{M}_1([-1,1]^n) = \biggl(4\int_0^\infty \frac{y}{\sinh(y)}\,\mathrm{d}y\biggr)^n = \pi^{2n}. \qquad \square$$

A numerical approximation of $\mathcal{M}_p([-1,1])$ gives Figure 4.

3E. ***Cube, diamond, and uniqueness of minimizers.*** Let $B_\infty^n = [-1,1]^n$ and $B_1^n = (B_\infty^n)^\circ$ be the cube and diamond (recall (1-13)). The L^p-support function of the cube was computed in Lemma 2.8 and its L^p-Mahler volume is given by Lemma 3.15. Lemma 3.17 below is the considerably harder computation of the L^p-support function of the diamond.

Lemmas 3.15 and 3.17 allow the comparison of the L^p-Mahler volumes of the cube and the diamond. We carried this out numerically for $n = 3$ and those computations lead to Figure 3 from the Introduction. As discussed in Section 1C, this provides evidence that the cube is the unique minimizer for Conjecture 1.3.

Lemma 3.17. *For $p \in (0,\infty)$,*

$$h_{p,B_1^n}(y) = \frac{1}{p}\log\biggl(\frac{n!}{p^n}\sum_{j=1}^{n}\frac{y_j^{n-2}(e^{py_j} + (-1)^n e^{-py_j})}{(y_j^2 - y_1^2)\cdots(y_j^2 - y_{j-1}^2)(y_j^2 - y_{j+1}^2)\cdots(y_j^2 - y_n^2)}\biggr). \tag{3-12}$$

The special case $p = 1$ of (3-12) was stated by Błocki [2015, pp. 96–97] in terms of Bergman kernels without proof.

For the proof of Lemma 3.17 we require the following claim.

Claim 3.18. *For $n \geq 2$, and distinct $y_1, \dots, y_n \in \mathbb{R}$,*

$$\sum_{j=1}^{n} \frac{y_j^k}{(y_j - y_1)\cdots(y_j - y_{j-1})(y_j - y_{j+1})\cdots(y_j - y_n)} = \begin{cases} 0, & 0 \leq k < n-1, \\ 1, & k = n-1. \end{cases}$$

Proof. Consider the rational function

$$f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}, \quad z \mapsto \frac{z^k}{(z - y_2)\cdots(z - y_n)} + \sum_{j=2}^{n} \frac{y_j^k}{(y_j - z)(y_j - y_2)\cdots(y_j - y_n)},$$

i.e., think of y_1 as a complex variable.

The claim is that f is a polynomial. It is enough to show that its poles at $y_2, \dots, y_n$ are removable singularities. By symmetry, it is enough to do it for y_2. There are only two terms involving $(z - y_2)$ in the denominator. Write their sum as

$$\frac{z^k}{(z-y_2)\cdots(z-y_n)} + \frac{y_2^k}{(y_2-z)\cdots(y_2-y_n)} = \left(\frac{z^k}{(z-y_3)\cdots(z-y_n)} - \frac{y_2^k}{(y_2-y_3)\cdots(y_2-y_n)}\right)^{-1} \frac{1}{z-y_2}.$$

We claim the numerator can be written in the form $(z - y_2)p(z)$ for some polynomial p. Indeed,

$$\begin{aligned}
&\frac{z^k}{(z - y_3)\cdots(z - y_n)} - \frac{y_2^k}{(y_2 - y_3)\cdots(y_2 - y_n)} \\
&\quad = \frac{z^k}{(z - y_3)\cdots(z - y_n)} - \frac{y_2^k}{(z - y_3)\cdots(z - y_n)} + \frac{y_2^k}{(z - y_3)\cdots(z - y_n)} - \frac{y_2^k}{(y_2 - y_3)\cdots(y_2 - y_n)} \\
&\quad = \frac{(z - y_2)(z^{k-1} + \cdots + y_2^{k-1})}{(z - y_3)\cdots(z - y_n)} - y_2^k \frac{(z - y_3)\cdots(z - y_n) - (y_2 - y_3)\cdots(y_2 - y_n)}{(z - y_3)\cdots(z - y_n)(y_2 - y_3)\cdots(y_2 - y_n)} \\
&\quad = \frac{(z - y_2)(z^{k-1} + \cdots + y_2^{k-1})}{(z - y_3)\cdots(z - y_n)} - y_2^k \frac{(z - y_2)p(z)}{(z - y_3)\cdots(z - y_n)(y_2 - y_3)\cdots(y_2 - y_n)} \\
&\quad = \frac{z^{k-1} + \cdots + y_2^{k-1}}{(z - y_3)\cdots(z - y_n)} - y_2^k \frac{p(z)}{(z - y_3)\cdots(z - y_n)(y_2 - y_3)\cdots(y_2 - y_n)},
\end{aligned}$$

where $p(z)$ is a polynomial such that

$$(z - y_3)\cdots(z - y_n) - (y_2 - y_3)\cdots(y_2 - y_n) = (z - y_2)p(z),$$

since the left-hand side is a polynomial that vanishes at y_2. In sum, f is a polynomial. In addition,

$$\lim_{z\to\infty} f(z) = \begin{cases} 0, & k < n-1, \\ 1, & k = n-1, \end{cases}$$

proving, by Liouville's theorem [Ahlfors 1978, p. 122], that f is constant (as a bounded, entire function) and equal to 0 when $0 < k < n - 1$, or 1 when $k = n - 1$. □

Proof of Lemma 3.17. Since B_1^n is the union of 2^n simplices of volume $1/(n!)$, $|B_1^n| = 2^n/(n!)$. In addition, by splitting the integral into 2^n integrals over the simplex,

$$e^{ph_{p,B_1^n}(\frac{y}{p})} = \int_{B_1^n} e^{\langle x,y\rangle} \frac{\mathrm{d}x}{|B_1^n|} = n! \int_{\Delta_n} \cosh(x_1 y_1)\cdots\cosh(x_n y_n)\,\mathrm{d}x. \tag{3-13}$$

The rest of the proof is by induction on n.

For $n = 2$, by (3-13),

$$\begin{aligned} e^{ph_{p,B_1^2}(\frac{y}{p})} &= 2\int_{\Delta_2} \cosh(x_1 y_1)\cosh(x_2 y_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &= 2\int_0^1 \cosh(x_1 y_1)\int_0^{1-x_1} \cosh(x_2 y_2)\,\mathrm{d}x_2\,\mathrm{d}x_1 = \int_0^1 \cosh(x_1 y_1)\frac{\sinh((1-x_1)y_2)}{y_2}\,\mathrm{d}x_1 \\ &= \frac{1}{y_2}\left[\frac{y_1\sinh(y_1 x_1)\sinh((1-x_1)y_2) + y_2\cosh(x_1 y_1)\cosh((1-x_1)y_2)}{y_1^2 - y_2^2}\right]_{x_1=0}^1 \\ &= \frac{\cosh(y_1)-\cosh(y_2)}{y_1^2-y_2^2} = \frac{\cosh(y_1)}{y_1^2-y_2^2} + \frac{\cosh(y_2)}{y_2^2-y_1^2}, \end{aligned}$$

where

$$\int \cosh(ax+c)\sinh(bx+d)\,\mathrm{d}x = \frac{a\sinh(ax+c)\sinh(bx+d) - b\cosh(ax+c)\cosh(bx+d)}{a^2-b^2} + C$$

was used.

For $n \geq 2$, by (3-13),

$$\begin{aligned} \frac{e^{ph_{p,B_1^{n+1}}(\frac{y}{p})}}{(n+1)!} &= \int_{\Delta_{n+1}} \cosh(x_1 y_1)\cdots\cosh(x_{n+1} y_{n+1})\,\mathrm{d}x \\ &= \int_{x_{n+1}=0}^1 \cosh(x_{n+1} y_{n+1})\int_{(1-x_{n+1})\Delta_n} \cosh(x_1 y_1)\cdots\cosh(x_n y_n)\,\mathrm{d}x \\ &= \int_{x_{n+1}=0}^1 \cosh(x_{n+1} y_{n+1})\frac{e^{ph_{p,B_1^n}(\frac{(1-x_{n+1})y}{p})}}{n!}(1-x_{n+1})^n\,\mathrm{d}x_{n+1} \end{aligned} \tag{3-14}$$

because by (3-13) and changing variables,

$$\begin{aligned} &\int_{(1-x_{n+1})\Delta_n} \cosh(x_1 y_1)\cdots\cosh(x_n y_n)\,\mathrm{d}x \\ &\qquad = \int_{\Delta_n} \cos((1-x_{n+1})z_1 y_1)\cdots\cosh((1-x_{n+1})z_n y_n)(1-x_{n+1})^n\,\mathrm{d}z \\ &\qquad = \frac{e^{ph_{p,B_1^n}(\frac{(1-x_{n+1})y}{p})}}{n!}(1-x_{n+1})^n. \end{aligned}$$

By induction,

$$\begin{aligned} e^{ph_{p,B_1^n}(\frac{(1-x_{n+1})y}{p})} &= \frac{n!}{p^n}\sum_{j=1}^n \frac{\left(\frac{(1-x_{n+1})y_j}{p}\right)^{n-2}(e^{(1-x_{n+1})y_j} + (-1)^n e^{-(1-x_{n+1})y_j})}{\left(\frac{1-x_{n+1}}{p}\right)^{2(n-1)}(y_j^2-y_1^2)\cdots(y_j^2-y_{j-1}^2)(y_j^2-y_{j+1}^2)\cdots(y_j^2-y_n^2)} \\ &= \frac{n!}{(1-x_{n+1})^n}\sum_{j=1}^n \frac{y_j^{n-2}(e^{(1-x_{n+1})y_j} + (-1)^n e^{-(1-x_{n+1})y_j})}{(y_j^2-y_1^2)\cdots(y_j^2-y_{j-1}^2)(y_j^2-y_{j+1}^2)\cdots(y_j^2-y_n^2)}. \end{aligned} \tag{3-15}$$

Therefore, by (3-14) and (3-15),

$$\frac{e^{ph_{p,B_1^{n+1}}(\frac{y}{p})}}{(n+1)!} = \sum_{j=1}^{n} \frac{y_j^{n-2} \int_0^1 \cosh(x_{n+1}y_{n+1})(e^{(1-x_{n+1})y_j} + (-1)^n e^{-(1-x_{n+1})y_j})\, \mathrm{d}x_{n+1}}{(y_j^2 - y_1^2)\cdots(y_j^2 - y_{j-1}^2)(y_j^2 - y_{j+1}^2)\cdots(y_j^2 - y_n^2)}. \tag{3-16}$$

To complete the proof, compute

$$\begin{aligned}\int_0^1 \cosh(x_{n+1}y_{n+1}) e^{(1-x_{n+1})y_j}\, \mathrm{d}x_{n+1} &= e^{y_j} \int_0^1 \cosh(x_{n+1}y_{n+1}) e^{-x_{n+1}y_j}\, \mathrm{d}x_{n+1} \\ &= \frac{1}{2} e^{y_j} \int_0^1 e^{x_{n+1}(y_{n+1}-y_j)} + e^{-x_{n+1}(y_{n+1}+y_j)}\, \mathrm{d}x_{n+1} \\ &= \frac{1}{2} e^{y_j} \left(\frac{e^{y_{n+1}-y_j} - 1}{y_{n+1} - y_j} - \frac{e^{-(y_{n+1}+y_j)} - 1}{y_{n+1} + y_j} \right) \\ &= \frac{1}{2} \left(\frac{e^{y_{n+1}} - e^{y_j}}{y_{n+1} - y_j} - \frac{e^{-y_{n+1}} - e^{y_j}}{y_{n+1} + y_j} \right) \\ &= \frac{y_j e^{y_j} - y_j \cosh(y_{n+1}) - y_{n+1} \sinh(y_{n+1})}{y_j^2 - y_{n+1}^2},\end{aligned} \tag{3-17}$$

and hence, replacing y_j by $-y_j$ in (3-17),

$$\int_0^1 \cosh(x_{n+1}y_{n+1}) e^{-(1-x_{n+1})y_j}\, \mathrm{d}x_{n+1} = \frac{-y_j e^{-y_j} + y_j \cosh(y_{n+1}) - y_{n+1} \sinh(y_{n+1})}{y_j^2 - y_{n+1}^2}. \tag{3-18}$$

Therefore, by (3-17) and (3-18),

$$\begin{aligned}&\int_0^1 \cosh(x_{n+1})(e^{(1-x_{n+1}y_j)} + (-1)^n e^{-(1-x_{n+1})y_j})\, \mathrm{d}x_{n+1} \\ &\quad = \frac{y_j e^{y_j} + (-1)^{n+1} y_j e^{y_j}}{y_j^2 - y_{n+1}^2} - \frac{(1-(-1)^n) y_j \cosh(y_{n+1})}{y_j^2 - y_{n+1}^2} - \frac{(1+(-1)^n) y_{n+1} \sinh(y_{n+1})}{y_j^2 - y_{n+1}^2}.\end{aligned} \tag{3-19}$$

By (3-16), (3-19) and Claim 3.18,

$$\begin{aligned}\frac{1}{(n+1)!} e^{ph_{p,B_1^{n+1}}(\frac{y}{p})} &= \sum_{j=1}^{n} \frac{y_j^{n-2} \int_0^1 \cosh(x_{n+1}y_{n+1})(e^{(1-x_{n+1})y_j} + (-1)^n e^{-(1-x_{n+1})y_j})\, \mathrm{d}x_{n+1}}{(y_j^2-y_1^2)\cdots(y_j^2-y_{j-1}^2)(y_j^2-y_{j+1}^2)\cdots(y_j^2-y_n^2)} \\ &= \sum_{j=1}^{n} \frac{y_j^{n-1}(e^{y_j} + (-1)^{n+1} e^{-y_j})}{(y_j^2-y_1^2)\cdots(y_j^2-y_{j-1}^2)(y_j^2-y_{j+1}^2)\cdots(y_j^2-y_n^2)(y_j^2-y_{n+1}^2)} \\ &\quad -(1-(-1)^n) \sum_{j=1}^{n} \frac{y_j^{n-1} \cosh(y_{n+1})}{(y_j^2-y_1^2)\cdots(y_j^2-y_{j-1}^2)(y_j^2-y_{j+1}^2)\cdots(y_j^2-y_n^2)(y_j^2-y_{n+1}^2)} \\ &\quad -(1+(-1)^n) \sum_{j=1}^{n} \frac{y_j^{n-2} y_{n+1} \sinh(y_{n+1})}{(y_j^2-y_1^2)\cdots(y_j^2-y_{j-1}^2)(y_j^2-y_{j+1}^2)\cdots(y_j^2-y_n^2)(y_j^2-y_{n+1}^2)}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \frac{y_j^{n-1}(e^{y_j} + (-1)^{n+1} e^{-y_j})}{(y_j^2 - y_1^2)\cdots(y_j^2 - y_{j-1}^2)(y_j^2 - y_{j+1}^2)\cdots(y_j^2 - y_n^2)(y_j^2 - y_{n+1}^2)} \\
&\quad + (1 - (-1)^n)\cosh(y_{n+1}) \frac{y_{n+1}^{n-1}}{(y_{n+1}^2 - y_1^2)\cdots(y_{n+1}^2 - y_n^2)} \\
&\qquad + (1 + (-1)^n)\sinh(y_{n+1}) \frac{y_{n+1}^{n-1}}{(y_{n+1}^2 - y_1^2)\cdots(y_{n+1}^2 - y_n^2)} \\
&= \sum_{j=1}^{n} \frac{y_j^{n-1}(e^{y_j} + (-1)^{n+1} e^{-y_j})}{(y_j^2 - y_1^2)\cdots(y_j^2 - y_{j-1}^2)(y_j^2 - y_{j+1}^2)\cdots(y_j^2 - y_n^2)(y_j^2 - y_{n+1}^2)} \\
&\quad + (e^{y} + (-1)^{n+1} e^{-y}) \frac{y_{n+1}^{n-1}}{(y_{n+1}^2 - y_1^2)\cdots(y_{n+1}^2 - y_n^2)},
\end{aligned}
$$

as desired. □

Therefore, in dimension $n = 3$, for distinct values of x, y and z,

$$
h_{p,B_1^3}(x,y,z) = \frac{1}{p}\log\left[\frac{6}{p^3}\left(\frac{x\sinh(px)}{(x^2-y^2)(x^2-z^2)} + \frac{y\sinh(py)}{(y^2-x^2)(y^2-z^2)} + \frac{z\sinh(pz)}{(z^2-x^2)(z^2-y^2)}\right)\right],
$$

which smoothly extends to $\mathbb{R}^3$. In particular,

$$
h_{p,B_1^3}(x,y,z) = \begin{cases}
\frac{1}{p}\log\left[\frac{6}{p^3}\left(\frac{p\cosh(px)}{2(x^2-z^2)} - \frac{x^2+z^2}{(x^2-z^2)^2}\frac{\sinh(px)}{2x} + \frac{z\sinh(pz)}{(x^2-z^2)^2}\right)\right], & x = y \neq z, \\
\frac{1}{p}\log\left[\frac{6}{p^3}\left(\frac{p\cosh(px)}{2(x^2-y^2)} - \frac{x^2+y^2}{(x^2-y^2)^2}\frac{\sinh(px)}{2x} + \frac{y\sinh(py)}{(x^2-y^2)^2}\right)\right], & x = z \neq y, \\
\frac{1}{p}\log\left[\frac{6}{p^3}\left(\frac{p\cosh(py)}{2(y^2-x^2)} - \frac{y^2+x^2}{(y^2-x^2)^2}\frac{\sinh(py)}{2y} + \frac{x\sinh(px)}{(y^2-x^2)^2}\right)\right], & y = z \neq x, \\
\frac{1}{p}\log\left[\frac{6}{p^3}\left(\frac{xp\cosh(px) - \sinh(px) + x^2p^2\sinh(px)}{8x^3}\right)\right], & x = y = z \neq 0, \\
0, & x = y = z = 0.
\end{cases}
$$

4. The L^p-Santaló point

In this section, we prove Proposition 1.5.

First, let us elucidate the similarities and differences from the case $p = \infty$. The Santaló point [1949, (2.3)] of K is the unique point $x_{\infty,K} \in \operatorname{int} K$ for which $b((K - x_{\infty,K})^\circ) = 0$. This is equivalent to $b(h_{K - x_{\infty,K}}) = 0$ since

$$
b(h_K) = (n+1)b(K^\circ). \tag{4-1}
$$

However, since $h_{p,K}$ is not 1-homogeneous for $p < \infty$, it is not in general true that $b(K^{\circ,p})$ vanishes when $b(h_{p,K})$ does. To verify (4-1), first compute $V(h_K)$. Since h_K is 1-homogeneous and $h_K = \|\cdot\|_{K^\circ}$, by Claim 3.3 and (3-2),

$$V(h_K)=\int_{\mathbb{R}^n} e^{-h_K(y)}\,\mathrm{d}y=\int_{\partial B_2^n}\int_0^\infty r^{n-1}e^{-h_K(ru)}\,\mathrm{d}r\,\mathrm{d}u=(n-1)!\int_{\partial B_2^n}\frac{\mathrm{d}u}{h_K(u)^n}=n!\,|K^\circ|. \tag{4-2}$$

Another way to see (4-2) is to start with (1-9) and (1-11), i.e.,

$$V(h_{p,K})=n!\,|K^{\circ,p}|, \tag{4-3}$$

and take $p\to\infty$.

For the barycenters, compute in polar coordinates,

$$\begin{aligned}
b(K^{\circ,p})&=\frac{1}{|K^{\circ,p}|}\int_{\{\|y\|_{K^{\circ,p}}\le 1\}} y\,\mathrm{d}y\\
&=\frac{1}{|K^{\circ,p}|}\int_{\{(r,u)\in(0,\infty)\times\partial B_2^n:\|ru\|_{K^{\circ,p}}\le 1\}} ru r^{n-1}\,\mathrm{d}r\,\mathrm{d}u\\
&=\frac{1}{|K^{\circ,p}|}\int_{\partial B_2^n}\int_{r=0}^{1/\|u\|_{K^{\circ,p}}} r^n u\,\mathrm{d}r\,\mathrm{d}u\\
&=\frac{1}{|K^{\circ,p}|}\frac{1}{n+1}\int_{\partial B_2^n}\frac{u}{\|u\|_{K^{\circ,p}}^{n+1}}\,\mathrm{d}u\\
&=\frac{1}{n+1}\frac{1}{|K^{\circ,p}|}\int_{\partial B_2^n} u\biggl(\frac{1}{(n-1)!}\int_0^\infty r^{n-1}e^{-h_{p,K}(ru)}\,\mathrm{d}r\biggr)^{\frac{n+1}{n}}\mathrm{d}u.
\end{aligned} \tag{4-4}$$

In addition, by (4-3),

$$b(h_{p,K})=\frac{1}{V(h_{p,K})}\int_{\mathbb{R}^n} ye^{-h_{p,K}(y)}\,\mathrm{d}y=\frac{1}{|K^{\circ,p}|}\frac{1}{n!}\int_{\partial B_2^n} u\int_0^\infty r^n e^{-h_{p,K}(ru)}\,\mathrm{d}r\,\mathrm{d}u. \tag{4-5}$$

For $p=\infty$, since $h_{\infty,K}=h_K$ is homogeneous. Claim 3.3 gives

$$\biggl(\frac{1}{(n-1)!}\int_0^\infty r^{n-1}e^{-h_K(ru)}\,\mathrm{d}r\biggr)^{\frac{n+1}{n}}=\biggl(\frac{1}{h_K(u)^n}\biggr)^{\frac{n+1}{n}}=\frac{1}{h_K(u)^{n+1}}=\frac{1}{n!}\int_0^\infty r^n e^{-h_{p,K}(ru)}\,\mathrm{d}r, \tag{4-6}$$

so (4-1) follows from (4-4)–(4-6), but without homogeneity such a relation does not hold.

Remark 4.1. While (4-1) does not hold for all p, one can show a weaker inequality of the form

$$\biggl(\frac{1}{(n-1)!}\int_0^\infty r^{n-1}e^{-h_{p,K}(ru)}\,\mathrm{d}r\biggr)^{\frac{n+1}{n}}\le(n+1)\frac{\|e^{-h_{p,K}}\|_\infty^{\frac{1}{n}}}{(n!)^{\frac{1}{n}}}\frac{1}{n!}\int_0^\infty r^n e^{-h_{p,K}(ru)}\,\mathrm{d}r,$$

by using [Brazitikos et al. 2014, Lemma 2.2.4].

The proof of Proposition 1.5 is based on three key lemmas, proved in Sections 4A and 4B.

Lemma 4.2. *Let* $p\in(0,\infty]$. *For a convex body* $K\subset\mathbb{R}^n$, $\mathcal{M}_p(K-x)<\infty$ *if and only if* $x\in\operatorname{int}K$.

Lemma 4.3. *Let* $p\in(0,\infty]$. *For a convex body* $K\subset\mathbb{R}^n$ *and* $x_0\in\partial K$,

$$\lim_{x\to x_0}\mathcal{M}_p(K-x)=\infty.$$

Lemma 4.4. *Let $p \in (0, \infty]$. For a convex body $K \subset \mathbb{R}^n$, $x \mapsto \mathcal{M}_p(K - x)$, $x \in \operatorname{int} K$, is twice differentiable and strictly convex with $\nabla_x \mathcal{M}_p(K - x) = \mathcal{M}_p(K - x) b(h_{p,K-x})$.*

Proof of Proposition 1.5. Since, by Lemmas 4.2 and 4.4, $x \mapsto \mathcal{M}_p(K - x)$ is strictly convex in int K and blows up on $\mathbb{R}^n \setminus \operatorname{int} K$, it must have a unique minimum at some $x_{p,K} \in \operatorname{int} K$. This is a critical point and therefore, by Lemma 4.4,

$$0 = \nabla_x \mathcal{M}_p(K - x_{p,K}) = \mathcal{M}_p(K - x_{p,K}) b(h_{p,K-x_{p,K}}).$$

Thus $b(h_{p,K-x_{p,K}}) = 0$. □

We call $x_{p,K}$ the L^p-Santaló point of K. For future reference we record its characterization:

Corollary 4.5. *Let $p \in (0, \infty]$. For a convex body $K \subset \mathbb{R}^n$, there exists a unique $x_{p,K} \in \mathbb{R}^n$ such that $b(h_{p,K-x_{p,K}}) = 0$.*

It is not clear to us how to directly prove Corollary 4.5 if not by Proposition 1.5. In general, for a convex function $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ with $b(\phi) \in \mathbb{R}^n$, it is not hard to see that there is an $x \in \mathbb{R}^n$ such that under the translation

$$T_x : \mathbb{R}^n \to \mathbb{R}^n, \quad y \mapsto y - x,$$

the pull-back of ϕ

$$T_x^* \phi(y) := \phi(y - x)$$

has its barycenter at the origin, $b(T_x^* \phi) = 0$. This is because

$$b(T_x^* \phi) = \int_{\mathbb{R}^n} y e^{-T_x^* \phi(y)} \frac{\mathrm{d}y}{V(\phi)} = \int_{\mathbb{R}^n} y e^{-\phi(y-x)} \frac{\mathrm{d}y}{V(\phi)} = \int_{\mathbb{R}^n} (y + x) e^{-\phi(y)} \frac{\mathrm{d}y}{V(\phi)} = b(\phi) + x,$$

so it is enough to choose $x = -b(\phi)$. However, functional translation of $h_{p,K}$ does not correspond to the translation of the body. That is, in general, $T_x^* h_{p,K} \neq h_{p,K-x}$. In fact, by Lemma 2.2(ii), $h_{p,K-x}(y) = h_{p,K}(y) - \langle y, x \rangle$, and hence

$$b(h_{p,K-x}) = \int_{\mathbb{R}^n} y e^{-h_{p,K-x}(y)} \frac{\mathrm{d}y}{V(h_{p,K-x})} = \int_{\mathbb{R}^n} y e^{-h_{p,K}(y)} e^{\langle y,x \rangle} \frac{\mathrm{d}y}{V(h_{p,K-x})},$$

from which is not clear what x should be so that $b(h_{p,K-x}) = 0$.

Remark 4.6. While we discuss lack of translation-invariance of some quantities, it will be helpful to note how $\mathcal{M}_p$ transforms under the $\mathrm{GL}(n, \mathbb{R})$-action. For $p > 0$, a convex body $K \subset \mathbb{R}^n$, and $A \in \mathrm{GL}(n, \mathbb{R})$, by Lemma 2.2(iii),

$$\|x\|_{(AK)^{\circ,p}} := \left(\int_0^\infty r^{n-1} e^{-h_{p,AK}(rx)} \, \mathrm{d}r \right)^{-\frac{1}{n}} = \left(\int_0^\infty r^{n-1} e^{-h_{p,K}(rA^T x)} \, \mathrm{d}r \right)^{-\frac{1}{n}} = \|A^T x\|_{K^{\circ,p}};$$

hence

$$(AK)^{\circ,p} = (A^{-1})^T K^{\circ,p}. \tag{4-7}$$

In sum:

Lemma 4.7. *Let $p \in (0, \infty]$. For a compact body $K \subset \mathbb{R}^n$ and $A \in \mathrm{GL}(n, \mathbb{R})$, $\mathcal{M}_p(AK) = \mathcal{M}_p(K)$.*

This $\mathrm{GL}(n,\mathbb{R})$-invariance will be useful in several places, e.g., in the proof of Claim 4.8 below and in proving Theorem 1.6 when we deal with Steiner symmetrization.

4A. *Finiteness of $\mathcal{M}_p$.* Lemma 4.2 follows from the following two claims.

Claim 4.8. *Let $p\in(0,\infty]$. For a convex body $K\subset\mathbb{R}^n$ with $0\in\operatorname{int}K$, and $r>0$ such that $[-r,r]^n\subset K$,*

$$\mathcal{M}_p(K)\le\frac{|K|^{1+\frac{1}{p}}}{(2r)^{n+\frac{n}{p}}}\mathcal{M}_p([-1,1]^n).$$

In particular, $\mathcal{M}_p(K)<\infty$.

Proof. Since $0\in\operatorname{int}K$, there is $r>0$ such that $[-r,r]^n\subset\operatorname{int}K$. By (3-8),

$$\begin{aligned}\mathcal{M}_p(K)&:=|K|\int_{\mathbb{R}^n}e^{-h_{p,K}(y)}\,\mathrm{d}y\\&\le|K|\int_{\mathbb{R}^n}e^{-h_{p,[-r,r]^n}(y)}\frac{|K|^{\frac{1}{p}}}{(2r)^{\frac{n}{p}}}\,\mathrm{d}y=\frac{|K|^{1+\frac{1}{p}}}{(2r)^{n+\frac{n}{p}}}(2r)^n\int_{\mathbb{R}^n}e^{-h_{p,[-r,r]^n}(y)}\,\mathrm{d}y\\&=\frac{|K|^{1+\frac{1}{p}}}{(2r)^{n+\frac{n}{p}}}\mathcal{M}_p([-r,r]^n)=\frac{|K|^{1+\frac{1}{p}}}{(2r)^{n+\frac{n}{p}}}\mathcal{M}_p([-1,1]^n),\end{aligned}$$

where we used Lemma 4.7. By Lemma 3.12, since $b([-1,1]^n)=0$,

$$\mathcal{M}_p([-1,1]^n)\le\left(\frac{(1+p)^{1+\frac{1}{p}}}{p}\right)^n\mathcal{M}([-1,1]^n)=\left(\frac{(1+p)^{1+\frac{1}{p}}}{p}\right)^n4^n,$$

concluding the proof. □

Claim 4.9. *Let $p\in(0,\infty]$. For a convex body $K\subset\mathbb{R}^n$ with $0\notin\operatorname{int}K$, $\mathcal{M}_p(K)=\infty$.*

Proof. By convexity of K, since $0\notin\operatorname{int}K$, there is a hyperplane through the origin

$$u^\perp:=\{x\in\mathbb{R}^n:\langle x,u\rangle=0\}$$

such that $K\subset\{x\in\mathbb{R}^n:\langle x,u\rangle\ge0\}$. In particular, $\langle x,-u\rangle\le0$ for all $x\in K$, and hence

$$c:=\int_K e^{p\langle x,-u\rangle}\frac{\mathrm{d}x}{|K|}<1.$$

If it was exactly equal to 1, then $\langle x,u\rangle=0$ for all $x\in K$, that is, $K\subset u^\perp$, which is a contradiction because K has nonempty interior. Let $U\subset\partial B_2^n$ be an open neighborhood of $-u$ such that

$$\int_K e^{p\langle x,v\rangle}\frac{\mathrm{d}x}{|K|}\le\frac{1+c}{2}<1\quad\text{for all } v\in U.$$

For $r\ge1$ and $v\in U$, $x\in K$, since $p\langle x,v\rangle<0$, we have $rp\langle x,v\rangle\le p\langle x,v\rangle$. Thus

$$\int_K e^{rp\langle x,v\rangle}\frac{\mathrm{d}x}{|K|}\le\int_K e^{p\langle x,v\rangle}\frac{\mathrm{d}x}{|K|}\le\frac{1+c}{2}<1,\quad v\in U,\ r\ge1. \tag{4-8}$$

In polar coordinates, by (4-8),

$$\begin{aligned}\mathcal{M}_p(K) &= |K|\int_{\mathbb{R}^n} e^{-h_{p,K}(y)}\,\mathrm{d}y = |K|\int_{\mathbb{R}^n}\frac{\mathrm{d}y}{\left(\int_K e^{p\langle x,y\rangle}\frac{1}{|K|}\,\mathrm{d}x\right)^{\frac{1}{p}}}\\ &= |K|\int_{\partial B_2^n}\int_0^\infty \frac{r^{n-1}\,\mathrm{d}r\,\mathrm{d}v}{\left(\int_K e^{rp\langle x,v\rangle}\frac{1}{|K|}\,\mathrm{d}x\right)^{\frac{1}{p}}}\\ &\geq |K|\int_U\int_1^\infty \frac{r^{n-1}\,\mathrm{d}r\,\mathrm{d}v}{\left(\int_K e^{rp\langle x,v\rangle}\frac{1}{|K|}\,\mathrm{d}x\right)^{\frac{1}{p}}}\\ &\geq |K|\left(\frac{1+c}{2}\right)^{-\frac{1}{p}}\int_U\int_1^\infty r^{n-1}\,\mathrm{d}r = \infty. \qquad \square\end{aligned}$$

Proof of Lemma 4.3. As $h_{p,K}\leq h_K$ (Lemma 2.2(v)), $|(K-x)^{\circ,p}|\geq|(K-x)^\circ|$ for all $x\in\mathbb{R}^n$ and $p\in(0,\infty]$. It is therefore enough to prove the claim for $p=\infty$. By Lemma 4.2, $\mathcal{M}(K-x)=\infty$ for $x\notin \operatorname{int} K$. Hence, we may further restrict our attention to $x\in\operatorname{int} K$.

By rotating K we may take $-e_n$ as the outward-pointing unit normal of K at x_0. For $x\in\operatorname{int}K$, let $\varepsilon=\varepsilon(x)>0$ such that $K-x\subset\{x_n\geq-\varepsilon\}$. Since K is bounded, there exists $M>0$ such that $K\subset MB_2^n$. Now, $(K-x)^\circ$ contains the cone

$$C := \left\{(\eta,y_n)\in\mathbb{R}^{n-1}\times\mathbb{R} : |\eta|\leq\frac{1+\varepsilon y_n}{M},\ y_n\in[0,-\varepsilon^{-1}]\right\}\subset(K-x)^\circ.$$

The volume of the cone is given by

$$|C| = \int_{-\frac{1}{\varepsilon}}^0\int_{\left(\frac{1+\varepsilon y_n}{M}\right)B_2^{n-1}}\mathrm{d}\eta\,\mathrm{d}y_n = \frac{|B_2^{n-1}|}{M^{n-1}}\int_{-\frac{1}{\varepsilon}}^0(1+\varepsilon y_n)^{n-1}\,\mathrm{d}y_n = \frac{|B_2^{n-1}|}{nM^{n-1}\varepsilon}.$$

As $x\to x_0$, $\varepsilon=\varepsilon(x_0)\to0^+$; hence $|(K-x)^\circ|\to\infty$. $\square$

4B. *Smoothness and convexity of $\mathcal{M}_p$.*

Proof of Lemma 4.4. Denote by $e_1,\dots,e_n$ the standard basis of $\mathbb{R}^n$. For $x\in\operatorname{int}K$ there is $r>0$ such that $x+2rB_2^n\subset\operatorname{int}K$. Using Lemma 2.2(ii), for $0<\varepsilon<r$,

$$\begin{aligned}\frac{n!\,|(K-x-\varepsilon e_i)^{\circ,p}|-n!\,|(K-x)^{\circ,p}|}{\varepsilon} &= \frac{1}{\varepsilon}\int_{\mathbb{R}^n} e^{-h_{p,K-x-\varepsilon e_i}(y)}-e^{-h_{p,K-x}(y)}\,\mathrm{d}y\\ &= \frac{1}{\varepsilon}\int_{\mathbb{R}^n} e^{-h_{p,K-x}(y)}e^{\langle\varepsilon e_i,y\rangle}-e^{-h_{p,K-x}(y)}\,\mathrm{d}y\\ &= \int_{\mathbb{R}^n}\frac{e^{\varepsilon y_i}-1}{\varepsilon}e^{-h_{p,K-x}(y)}\,\mathrm{d}y. \end{aligned}\tag{4-9}$$

For $0<\varepsilon<r$,

$$\left|\frac{e^{\varepsilon y_i}-1}{\varepsilon}\right|\leq\sum_{m=1}^\infty\frac{\varepsilon^{m-1}|y_i|^m}{m!}\leq\sum_{m=1}^\infty\frac{r^{m-1}|y_i|^m}{m!}=\frac{1}{r}\sum_{m=1}^\infty\frac{r^m|y_i|^m}{m!}\leq\frac{1}{r}e^{r|y_i|};$$

hence

$$\left|\frac{e^{\varepsilon y_i}-1}{\varepsilon}e^{-h_{p,K-x}(y)}\right| \le \frac{1}{r}e^{r|y_i|}e^{-h_{p,K-x}(y)},$$

and

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{1}{r}e^{r|y_i|}e^{-h_{p,K-x}(y)} &= \frac{1}{r}\int_{\mathbb{R}^{n-1}}\left(\int_0^\infty e^{ry_i}+\int_{-\infty}^0 e^{-ry_i}\right)e^{-h_{p,K-x}(y)}\,\mathrm{d}y\\ &\le \frac{1}{r}\int_{\mathbb{R}^n} e^{ry_i}e^{-h_{p,K-x}(y)}\,\mathrm{d}y+\frac{1}{r}\int_{\mathbb{R}^n} e^{-ry_i}e^{-h_{p,K-x}(y)}\,\mathrm{d}y\\ &= \frac{1}{r}\big(n!\,|(K-x-re_i)^{\circ,p}|+n!\,|(K-x+re_i)^{\circ,p}|\big)\end{aligned}$$

is finite by Lemma 4.2 as $x+re_i$ and $x-re_i$ are both in the interior of K. Therefore, dominated convergence applies to (4-9):

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{n!\,|(K-x-\varepsilon e_i)^{\circ,p}|-n!\,|(K-x)^{\circ,p}|}{\varepsilon} &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\frac{e^{\varepsilon y_i}-1}{\varepsilon}e^{-h_{p,K-x}(y)}\,\mathrm{d}y\\ &= \int_{\mathbb{R}^n}\lim_{\varepsilon\to 0}\frac{e^{\varepsilon y_i}-1}{\varepsilon}e^{-h_{p,K-x}(y)}\,\mathrm{d}y\\ &= \int_{\mathbb{R}^n} y_i e^{-h_{p,K-x}(y)}\,\mathrm{d}y.\end{aligned}$$

That is, $x\mapsto|(K-x)^\circ|$, or equivalently $x\mapsto\mathcal{M}_p(K-x)$, is differentiable in $\operatorname{int}K$ with gradient

$$\nabla_x\mathcal{M}_p(K-x)=|K|\int_{\mathbb{R}^n} ye^{-h_{p,K-x}(y)}\,\mathrm{d}y=\mathcal{M}_p(K-x)b(h_{p,K_x}),$$

as

$$b(h_{p,K-x})=\int_{\mathbb{R}^n} ye^{-h_{p,K-x}(y)}\,\frac{\mathrm{d}y}{V(h_{p,K-x})}=\frac{1}{n!\,|(K-x)^{\circ,p}|}\int_{\mathbb{R}^n} ye^{-h_{p,K-x}(y)}\,\mathrm{d}y.$$

Similarly, one can show that the second-order derivatives exist and are continuous. Differentiating under the integral sign,

$$\frac{\partial^2}{\partial x_i\,\partial x_j}\mathcal{M}_p(K-x)=|K|\int_{\mathbb{R}^n} y_i y_j e^{-h_{p,K-x}(y)}\,\mathrm{d}y.$$

Therefore, for $v\in\mathbb{R}^n$,

$$v^T\nabla_x^2\mathcal{M}_p(K-x)v=|K|\sum_{i,j=1}^n\int_{\mathbb{R}^n} v_i v_j y_i y_j e^{-h_{p,K-x}(y)}\,\mathrm{d}y=|K|\int_{\mathbb{R}^n}\langle v,y\rangle^2 e^{-h_{p,K-x}(y)}\,\mathrm{d}y\ge 0,$$

with equality if and only if $\langle y,v\rangle=0$ for almost all y, or equivalently $v=0$, proving strict convexity. □

5. The upper bound on $\mathcal{M}_p$

This section is dedicated to proving the L^p-Santaló theorem, Theorem 1.6. As expected, we use symmetrization. However, there are a number of intricate details that need to be carefully dealt with, since L^p-polarity is a highly nonlocal operation compared to classical polarity. On the surface of it though, as in the case $p=\infty$, the key estimate we need to prove is the monotonicity of volume under Steiner symmetrization:

Proposition 5.1. *Let $p\in(0,\infty]$. For a symmetric convex body $K\subset\mathbb{R}^n$ and $u\in\partial B_2^n$, let $\sigma_u K$ be the Steiner symmetral of K (Definition 5.5). Then, $|(\sigma_u K)^{\circ,p}|\geq|K^{\circ,p}|$.*

5A. ***Outline of the proof of Proposition 5.1.*** Proposition 5.1 is proved in Section 5G. For $n=1$, $\sigma_u K=K$ if $K=-K$. Thus, take $n>1$ for the rest of the section. We follow a classical proof for the case $p=\infty$ [Gruber 2007, Proposition 9.2; Artstein-Avidan et al. 2015, Proposition 1.1.15] and make the appropriate modifications to $p\in(0,\infty)$. This involves comparing the volume of the "slices" of the polar body perpendicular to the vector used for Steiner symmetrization. For a convex body $K\subset\mathbb{R}^n$, and $x_n\in\mathbb{R}$, denote by

$$K(x_n):=\{\xi\in\mathbb{R}^{n-1}:(\xi,x_n)\in K\}\tag{5-1}$$

the slice of K at height x_n. By Tonelli's theorem [Folland 1999, §2.37], the volume of a convex body may be expressed as an integral of the volume of its slices,

$$|K|=\int_{\{(\xi,x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}:\xi\in K(x_n)\}}\mathrm{d}\xi\,\mathrm{d}x_n=\int_{-\infty}^{\infty}|K(x_n)|\,\mathrm{d}x_n.\tag{5-2}$$

In view of (5-2), Proposition 5.1 follows from the next lemma. Denote by $e_1,\dots,e_n$ the standard basis of $\mathbb{R}^n$.

Lemma 5.2. *Let $p\in(0,\infty]$. For a symmetric convex body $K\subset\mathbb{R}^n$, $|(\sigma_{e_n}K)^{\circ,p}(x_n)|\geq|K^{\circ,p}(x_n)|$ for all $x_n\in\mathbb{R}$.*

Lemma 5.2, in turn, follows from the Brunn–Minkowski inequality and the following monotonicity property of the average of antipodal slices under Steiner symmetrization.

Lemma 5.3. *Let $p\in(0,\infty]$. For a convex body $K\subset\mathbb{R}^n$,*

$$\frac{K^{\circ,p}(x_n)+K^{\circ,p}(-x_n)}{2}\subset\frac{(\sigma_{e_n}K)^{\circ,p}(x_n)+(\sigma_{e_n}K)^{\circ,p}(-x_n)}{2}=(\sigma_{e_n}K)^{\circ,p}(x_n).\tag{5-3}$$

The equality on the right-hand side holds because $\sigma_{e_n}K$, and hence $(\sigma_{e_n}K)^{\circ,p}$ (Lemma 5.17), are by construction symmetric with respect to $e_n^{\perp}$. Nonetheless, note that no symmetry on K is assumed for Lemma 5.3, in contrast to Lemma 5.2. Applying the Brunn–Minkowski inequality on Lemma 5.3 gives

$$|(\sigma_{e_n}K)^{\circ,p}(x_n)|^{\frac{1}{n-1}}\geq\tfrac{1}{2}|K^{\circ,p}(x_n)|^{\frac{1}{n-1}}+\tfrac{1}{2}|K^{\circ,p}(-x_n)|^{\frac{1}{n-1}}.$$

Without any symmetry assumption on K, $|K^{\circ,p}(x_n)|$ and $|K^{\circ,p}(-x_n)|$ may be unrelated. For symmetric convex bodies, $K^{\circ,p}(-x_n)=-K^{\circ,p}(x_n)$ (Claim 5.14) and hence $|K^{\circ,p}(-x_n)|=|K^{\circ,p}(x_n)|$, justifying the symmetry assumption in Lemma 5.2. See Figure 5.

In order to obtain the inclusion of Lemma 5.3, we first obtain an inequality relating the norms before and after symmetrization:

$$\left\|\left(\frac{\xi+\xi'}{2},x_n\right)\right\|_{(\sigma_{e_n}K)^{\circ,p}}\leq\frac{\|(\xi,x_n)\|_{K^{\circ,p}}+\|(\xi',-x_n)\|_{K^{\circ,p}}}{2}.\tag{5-4}$$

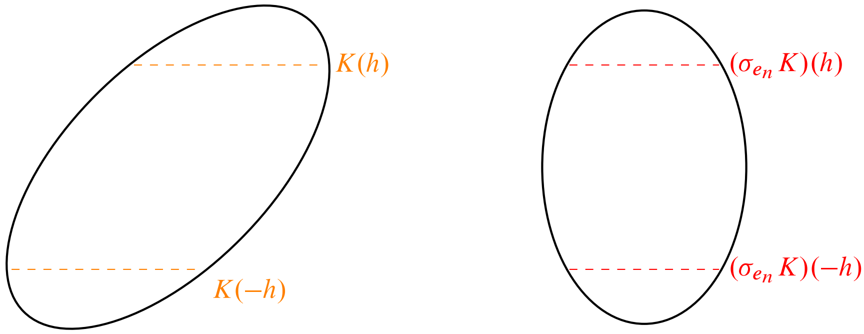

Figure 5. Comparing the slices.

For $p=\infty$, by (3-4), (5-4) reads

$$h_{\sigma_{e_n} K}\left(\frac{\xi+\xi'}{2}, x_n\right) \leq \frac{h_K(\xi, x_n)+h_K(\xi', -x_n)}{2}, \tag{5-5}$$

which is classical and simple to prove: any element of $\sigma_{e_n} K$ is of the form $\left(z, \frac{t-s}{2}\right)$ for $(z, t), (z, s) \in K$, so

$$\begin{aligned}
\left\langle\left(\frac{\xi+\xi'}{2}, x_n\right),\left(z, \frac{t-s}{2}\right)\right\rangle &= \left\langle\frac{\xi+\xi'}{2}, z\right\rangle + x_n \frac{t-s}{2} \\
&= \frac{\langle\xi, z\rangle + x_n t}{2} + \frac{\langle\xi', z\rangle - x_n s}{2} \\
&= \frac{\langle(\xi, x_n),(z, t)\rangle}{2} + \frac{\langle(\xi', -x_n),(z, -s)\rangle}{2} \\
&\leq \frac{h_K(\xi, x_n)+h_K(\xi', -x_n)}{2},
\end{aligned}$$

and (5-5) follows. One of our key estimates in this section is a 3-parameter (p, s, t) family generalization of (5-5):

Lemma 5.4. *Let $p \in (0, \infty]$, and $K \subset \mathbb{R}^n$ a convex body. For $\xi, \xi' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$ and $r, t, s > 0$ with $\frac{2}{r} = \frac{1}{t} + \frac{1}{s}$,*

$$h_{p, \sigma_{e_n} K}\left(r \frac{\xi+\xi'}{2}, r x_n\right) \leq \frac{s}{t+s} h_{p, K}(t\xi, t x_n) + \frac{t}{t+s} h_{p, K}(s\xi', -s x_n).$$

For $p=\infty$, Lemma 5.4 is equivalent to (5-4). Lacking homogeneity, for $p \in (0, \infty)$ this is no longer the case. Notwithstanding, Lemma 5.4 is exactly the condition necessary to apply Ball's Brunn–Minkowski inequality for harmonic means (Theorem 5.20, proven in the Appendix) from which we deduce (5-4). The next step in the proof of Proposition 5.1 is to use (5-4) to obtain Lemma 5.3. Finally, Lemma 5.3 and a symmetry property for antipodal slices of symmetric bodies (Corollary 5.16) give Lemma 5.2 from which Proposition 5.1 follows by (5-2).

The proof of Proposition 5.1 is organized as follows. Sections 5B and 5C are preparatory. In Section 5B we recall a few basics of Steiner symmetrization. In Section 5C, Lemma 5.11 establishes the continuity of $\mathcal{M}_p$ in the Hausdorff topology (Definition 5.9). Section 5D establishes several symmetries between antipodal slices for symmetric convex bodies. Section 5E is dedicated to proving Lemma 5.4, and Section 5F to proving Lemmas 5.3 and 5.2. In Section 5G, we complete the proofs of Proposition 5.1 and Theorem 1.6.

5B. ***Steiner symmetrization.*** For a vector $u \in \partial B_2^n$ denote by

$$u^\perp := \{x \in \mathbb{R}^n : \langle x, u \rangle = 0\}$$

the hyperplane through the origin that is normal to u. Let, also,

$$\pi_{u^\perp} : \mathbb{R}^n \to u^\perp, \quad x \mapsto x - \langle x, u \rangle u,$$

be the projection onto $u^\perp$. Given $u \in \partial B_2^n$, one may foliate any convex body K by a family of straight line segments parametrized by a hyperplane $u^\perp$. The Steiner symmetral $\sigma_u K$ is the unique such foliation for which the line segments have their midpoints in $u^\perp$ [Steiner 1838, pp. 286–287] (see also [Gruber 2007, §9; Artstein-Avidan et al. 2015, Definition 1.1.13]):

Definition 5.5. For $K \subset \mathbb{R}^n$ a convex body and $u \in \partial B_2^n$, the Steiner symmetral in the u direction is given by

$$\sigma_u(K) := \left\{x + tu : x \in \pi_{u^\perp}(K) \text{ and } |t| \le \tfrac{1}{2}|K \cap (x + \mathbb{R}u)|\right\}.$$

Steiner symmetrization produces a convex body that is symmetric with respect to $u^\perp$.

Definition 5.6. A convex body $K \subset \mathbb{R}^n$ is symmetric with respect to a hyperplane $u^\perp$ if for all $x \in K$

$$x - 2\langle x, u \rangle u \in K.$$

Equivalently, K remains invariant under reflection with respect to $u^\perp$. Steiner symmetrization also preserves volume and convexity [Gruber 2007, Proposition 9.1]:

Lemma 5.7. *For a convex body $K \subset \mathbb{R}^n$ and $u \in \partial B_2^n$, $\sigma_u(K)$ is a convex body, symmetric with respect to $u^\perp$, with $|\sigma_u(K)| = |K|$.*

Orthogonal transformations preserve volume and, by (4-7), commute with L^p-polarity. The following lemma then justifies working with $u = e_n$ throughout.

Lemma 5.8. *For a convex body $K \subset \mathbb{R}^n$, $u \in \partial B_2^n$, and $A \in O(n)$,*

$$\sigma_u(K) = A^{-1}\sigma_{Au}(AK).$$

In particular, $|\sigma_u(K)| = |\sigma_{Au}(AK)|$.

Proof. Since $A \in O(n)$ is invertible, it is enough to show $A^{-1}\sigma_{Au}(AK) \subset \sigma_u(K)$. Let $x + tAu \in \sigma_{Au}(AK)$ with

$$x \in \pi_{(Au)^\perp}(AK) \quad \text{and} \quad |t| \le \tfrac{1}{2}|(AK) \cap (x + \mathbb{R}Au)|.$$

First,

$$\pi_{(Au)^\perp}(AK) = A\pi_{u^\perp}(K). \tag{5-6}$$

Indeed, for $z \in \mathbb{R}^n$,

$$\pi_{(Au)^\perp}(Az) = Az - \langle Az, Au\rangle Au = Az - \langle z, u\rangle Au = A(z - \langle z, u\rangle u) = A\pi_{u^\perp}(z),$$

because, since $A \in O(n)$, we have $\langle Az, Au\rangle = \langle z, A^T Au\rangle = \langle z, u\rangle$.

Second,

$$(AK) \cap (x + \mathbb{R}Au) = A(K \cap (A^{-1}x + \mathbb{R}u)). \tag{5-7}$$

That is because, $y \in (AK) \cap (x + \mathbb{R}Au)$ if and only if $y \in AK$ and $y = x + sAu$, $x \in K$, $s \in \mathbb{R}$. Equivalently, $A^{-1}y \in K$ and $A^{-1}y = A^{-1}x + su \in A^{-1}x + \mathbb{R}u$, i.e., $A^{-1}y \in K \cap (A^{-1}x + \mathbb{R}u)$.

Using (5-7) and as $A \in O(n)$ preserves volume, $|K \cap (A^{-1}x + \mathbb{R}u)| = |(AK) \cap (x + \mathbb{R}Au)|$. Thus $A^{-1}(x + tAu) = A^{-1}x + tu$ is such that $A^{-1}x \in A^{-1}\pi_{(Au)^\perp}(AK) = A^{-1}(A\pi_{u^\perp}(K)) = \pi_{u^\perp}(K)$ (using (5-6)), and $|t| \le \frac{1}{2}|(AK) \cap (x + \mathbb{R}Au)| = \frac{1}{2}|K \cap (A^{-1}x + \mathbb{R}u)|$, that is, $A^{-1}(x + tAu) = A^{-1}x + tu \in \sigma_u(K)$. □

Recall the definition of the Hausdorff metric.

Definition 5.9. For $K, L \subset \mathbb{R}^n$ two compact bodies, let

$$d_H(K, L) := \inf\{\varepsilon > 0 : K \subset L + \varepsilon B_2^n \text{ and } L \subset K + \varepsilon B_2^n\}$$

be the Hausdorff distance between K and L.

Repeated Steiner symmetrizations Hausdorff converge to a 2-ball [Gross 1917; Gruber 2007, Theorem 9.1].

Lemma 5.10. *For a convex body $K \subset \mathbb{R}^n$, there is $\rho > 0$ and a sequence of vectors $u_j \in \partial B_2^n$ such that if $K_j := \sigma_{u_j}(K_{j-1})$, where $K_0 := K$, then $K_j \to \rho B_2^n$ in the Hausdorff metric.*

5C. *Hausdorff continuity of $\mathcal{M}_p$.* The aim of this subsection is to verify that $\mathcal{M}_p$ is continuous under Hausdorff convergence (Lemma 5.11).

By Lemma 5.10, iterated applications of Steiner symmetrization d_H-converge to a 2-ball. Therefore, in order to obtain Theorem 1.6, it is necessary to show that $\mathcal{M}_p$ is d_H-continuous.

Lemma 5.11. *Let $p \in (0, \infty]$ and $\{K_j\}_{j\ge1} \subset \mathbb{R}^n$ be a sequence of convex bodies d_H-converging to a convex body $K \subset \mathbb{R}^n$ with $|K^{\circ,p}| < \infty$. Then, $\mathcal{M}_p(K_j) \to \mathcal{M}_p(K)$.*

Lemma 5.11 follows from the next two claims. First, the volume of convex bodies is continuous under the Hausdorff metric. Note this is not true without the convexity assumption, e.g., for space-filling curves. Denote by

$$\mathbf{1}_K(x) := \begin{cases} 1, & x \in K, \\ 0, & x \notin K \end{cases}$$

the indicator function of K.

Claim 5.12. *Let $\{K_j\}_{j\ge1} \subset \mathbb{R}^n$ be a sequence of convex bodies d_H-converging to $K \subset \mathbb{R}^n$. Then, $|K_j| \to |K|$.*

Proof. Since $d_H(K_j, K) \to 0$, there are $\varepsilon_j > 0$ such that $K_j \subset K + \varepsilon_j B_2^n$ and $K + \varepsilon_j B_2^n \subset K_j$, with $\varepsilon_j \to 0$. In particular, $\{\varepsilon_j\}_{j\geq 1}$ is bounded. For simplicity, take $\varepsilon_j \leq 1$. In particular, $K_j \subset K + \varepsilon_j B_2^n \subset K + B_2^n$; thus $\mathbf{1}_{K_j} \leq \mathbf{1}_{K+B_2^n}$ for all j. This allows for the use of dominated convergence. It is therefore enough to show

$$\lim_{j\to\infty} \mathbf{1}_{K_j}(x) = \mathbf{1}_K(x), \quad x \in (\operatorname{int} K) \cup (\mathbb{R}^n \setminus K). \tag{5-8}$$

Then, by dominated convergence,

$$\lim_{j\to\infty} |K_j| = \lim_{j\to\infty} \int_{\mathbb{R}^n} \mathbf{1}_{K_j} = \int_{\mathbb{R}^n} \lim_{j\to\infty} \mathbf{1}_{K_j} = \int_{\mathbb{R}^n} \mathbf{1}_K = |K|.$$

For (5-8), let $x \in \operatorname{int} K$. There is $\varepsilon > 0$ such that $x + \varepsilon B_2^n \subset K$. Since $\varepsilon_j \to 0$, there is $j_0 \geq 1$ such that $\varepsilon_j < \varepsilon$ for all $j \geq j_0$. Therefore, $x + \varepsilon B_2^n \subset K \subset K_j + \varepsilon_j B_2^n \subset K_j + \varepsilon B_2^n$. By the cancellation law for the Minkowski sum of convex bodies [Gruber 2007, Theorem 6.1(i)], $\{x\} \subset K_j$, i.e., $x \in K_j$. Therefore, $\mathbf{1}_{K_j}(x) = 1 = \mathbf{1}_K(x)$ for all $j \geq j_0$.

For $x \in \mathbb{R}^n \setminus K$, since K is closed, $\mathbb{R}^n \setminus K$ is open. Thus there is $\varepsilon > 0$ such that $x + 2\varepsilon B_2^n \subset \mathbb{R}^n \setminus K$, i.e., $(x + 2\varepsilon B_2^n) \cap K = \varnothing$. Let $j_0 \geq 1$ with $\varepsilon_j < \varepsilon$ for all $j \geq j_0$. Then, $K_j \subset K + \varepsilon B_2^n$ and hence

$$(x + \varepsilon B_2^n) \cap K_j \subset (x + \varepsilon B_2^n) \cap (K + \varepsilon B_2^n) = \varnothing,$$

because, for $y \in (x + \varepsilon B_2^n) \cap (K + \varepsilon B_2^n)$, we have $y = x + \varepsilon u = z + \varepsilon v$ for $u, v \in B_2^n$ and $z \in K$. That is, $z = x + \varepsilon(u - v) \in x + 2\varepsilon B_2^n$; thus $z \in K \cap (x + 2\varepsilon B_2^n) = \varnothing$, a contradiction. Therefore, $x \notin K_j$ for all $j \geq j_0$, i.e., $\mathbf{1}_{K_j}(x) = 0 = \mathbf{1}_K(x)$ for all $j \geq j_0$, proving (5-8). □

Second, the volume of the L^p-polars is also continuous under Hausdorff convergence given that the limit is a convex body with finite $\mathcal{M}_p$ volume.

Claim 5.13. *Let $p \in (0, \infty]$ and $\{K_j\}_{j\geq 1} \subset \mathbb{R}^n$ be a sequence of convex bodies d_H-converging to a convex body K with $|K^{\circ,p}| < \infty$. Then, $|K_j^{\circ,p}| \to |K^{\circ,p}|$.*

Proof. Since $d_H(K_j, K) \to 0$, there are $\varepsilon_j > 0$ such that $K_j \subset K + \varepsilon_j B_2^n$ and $K \subset K_j + \varepsilon_j B_2^n$ with $\varepsilon_j \to 0$. In particular, $\{\varepsilon_j\}_{j\geq 1}$ is bounded. For simplicity, take $\varepsilon_j \leq \frac{1}{2}$. In particular,

$$K_j \subset K + \varepsilon_j B_2^n \subset K + B_2^n,$$

so K_j are uniformly bounded. Let $M > 0$ such that $|x| \leq M$ for all $x \in K_j$ and all j. For $y \in \mathbb{R}^n$,

$$\begin{aligned} \big||K_j| e^{p h_{p,K_j}(y)} - |K| e^{p h_{p,K}(y)}\big| &= \left| \int_{K_j} e^{p\langle x,y\rangle}\, dx - \int_K e^{p\langle x,y\rangle}\, dx \right| \\ &\leq \int_{(K_j\setminus K)\cup(K\setminus K_j)} e^{p\langle x,y\rangle}\, dx \leq |(K_j \setminus K) \cup (K \setminus K_j)| e^{pM|y|}. \end{aligned} \tag{5-9}$$

Note that

$$\mathbf{1}_{(K_j\setminus K)\cup(K\setminus K)}(y) = |\mathbf{1}_{K_j}(y) - \mathbf{1}_K(y)|,$$

which converges to 0 almost everywhere by (5-8). By dominated convergence, $|(K_j \setminus K) \cup (K \setminus K_j)| \to 0$. Taking $j \to \infty$ in (5-9), $|K_j| e^{p h_{p,K_j}(y)} \to |K| e^{p h_{p,K}(y)}$. By Claim 5.12, $|K_j| \to |K|$; thus

$$\lim_{j\to\infty} h_{p,K_j}(y) = h_{p,K}(y), \quad y \in \mathbb{R}^n, \tag{5-10}$$

establishing the pointwise convergence.

The aim is to use dominated convergence on $e^{-h_{p,K_j}}$, for which a uniform (independent of j) and integrable upper bound is necessary. By assumption $|K^{\circ,p}| < \infty$, or equivalently, by Lemma 4.2, $0 \in \operatorname{int} K$. That is, there is $r > 0$ such that $[-2r, 2r]^n \subset K$. Therefore, for large enough $j_0 > 0$, we have $[-r, r]^n \subset K_j$ for all $j \geq j_0$. In addition, by Claim 5.12, $|K_j| \to |K| > 0$; thus there is $M' > 0$ with $|K_j| \leq M'$ for all j. As a result,

$$h_{p,K_j}(y) = \frac{1}{p} \log \int_{K_j} e^{p\langle x,y\rangle} \frac{\mathrm{d}x}{|K_j|} \geq \frac{1}{p} \log \int_{[-r,r]^n} e^{p\langle x,y\rangle} \frac{\mathrm{d}x}{M'} = h_{p,[-r,r]^n}(y) + \log \frac{(2r)^n}{M'},$$

and hence

$$e^{-h_{p,K_j}(y)} \leq \frac{M'}{(2r)^n} e^{-h_{p,[-r,r]^n}(y)}.$$

The right-hand side is integrable since by (4-7)

$$\int_{\mathbb{R}^n} e^{-h_{p,[-r,r]^n}(y)}\, \mathrm{d}y = \frac{\mathcal{M}_p([-r,r]^n)}{|[-r,r]^n|} = \frac{1}{(2r)^n} \mathcal{M}_p([-1,1]^n),$$

which is finite by Lemma 3.12. The claim now follows from (5-10) and the dominated convergence theorem. □

Proof of Lemma 5.11. By Claims 5.12–5.13, $|K_j| \to |K|$ and $|K_j^{\circ,p}| \to |K^{\circ,p}|$; thus by (1-11), $\lim_{j\to\infty} \mathcal{M}_p(K_j) = \lim_{j\to\infty} n!\,|K_j|\,|K_j^{\circ,p}| = n!\,|K|\,|K^{\circ,p}| = \mathcal{M}_p(K)$. □

5D. *Slice analysis of symmetric convex bodies.*

5D1. *Symmetry with respect to a hyperplane.* Antipodal slices are related when $-K = K$: $\xi \in K(-x_n)$ if and only if $(\xi, -x_n) \in K$ or $-(\xi, -x_n) = (-\xi, x_n) \in K$, i.e., if and only if $-\xi \in K(x_n)$. In sum:

Claim 5.14. *For a symmetric convex body $K \subset \mathbb{R}^n$, $K(-x_n) = -K(x_n)$ for all $x_n \in \mathbb{R}$.*

If, instead, one assumes K to be symmetric with respect to the hyperplane $e_n^{\perp}$, then antipodal slices are exactly equal: note that $\xi \in K(x_n)$ if and only if $(\xi, x_n) \in K$, which by the symmetry of K with respect to $e_n^{\perp}$ is equivalent to $(\xi, -x_n) \in K$ or $\xi \in K(-x_n)$. Thus:

Claim 5.15. *For a convex body $K \subset \mathbb{R}^n$ symmetric with respect to $e_n^{\perp}$, $K(-x_n) = K(x_n)$ for all $x_n \in \mathbb{R}$.*

5D2. *L^p-polarity preserves symmetries.*

Corollary 5.16. *Let $p \in (0, \infty]$. For a symmetric convex body K, $K^{\circ,p}(-x_n) = -K^{\circ,p}(x_n)$ for all $x_n \in \mathbb{R}$.*

Proof. By Theorem 1.2, $K^{\circ,p}$ is symmetric. Thus, by Claim 5.14, $K^{\circ,p}(-x_n) = -K^{\circ,p}(x_n)$. □

In addition, $K^{\circ,p}$ inherits symmetries with respect to hyperplanes from K.

Lemma 5.17. *Let $p \in (0, \infty]$, $u \in \partial B_2^n$ and K a convex body symmetric with respect to $u^{\perp}$. Then, $K^{\circ,p}$ is symmetric with respect to $u^{\perp}$.*

Proof. By symmetry with respect to $u^\perp$, $\pi_{u^\perp}(K) = K \cap u^\perp$. There is concave $f : K \cap u^\perp \to [0, \infty)$ such that

$$K = \{x + tu : x \in K \cap u^\perp \text{ and } |t| \le f(x)\}.$$

For $y \in K \cap u^\perp,\ s \in \mathbb{R}$,

$$\begin{aligned}
h_{p,K}(y+su) &= \frac{1}{p}\log\biggl(\int_K e^{p\langle z, y+su\rangle}\,\frac{\mathrm{d}z}{|K|}\biggr)\\
&= \frac{1}{p}\log\biggl(\int_{x\in K\cap u^\perp}\int_{t=-f(x)}^{f(x)} e^{p\langle x+tu, y+su\rangle}\,\frac{\mathrm{d}t\,\mathrm{d}x}{|K|}\biggr)\\
&= \frac{1}{p}\log\biggl(\int_{x\in K\cap u^\perp} e^{p\langle x,y\rangle}\int_{t=-f(x)}^{f(x)} e^{pts}\,\frac{\mathrm{d}t\,\mathrm{d}x}{|K|}\biggr)\\
&= \frac{1}{p}\log\biggl(\int_{x\in K\cap u^\perp} e^{p\langle x,y\rangle}\int_{\tau=-f(x)}^{f(x)} e^{-p\tau s}\,\frac{\mathrm{d}\tau\,\mathrm{d}x}{|K|}\biggr) = h_{p,K}(y-su),
\end{aligned}$$

by the change of variables $\tau = -t$. As a result, $\|y+su\|_{K^{\circ,p}} = \|y-su\|_{K^{\circ,p}}$, and hence $y+su \in K^{\circ,p}$ if and only if $y-su \in K^{\circ,p}$ as desired. □

By Lemma 5.7, $\sigma_{e_n}K$ is symmetric with respect to $e_n^\perp$; thus, by Lemma 5.17, $(\sigma_{e_n}K)^{\circ,p}$ also is. Therefore, by Claim 5.15 its antipodal slices are equal.

Corollary 5.18. *Let $p \in (0,\infty]$. For a convex body $K \subset \mathbb{R}^n$, $(\sigma_{e_n}K)^{\circ,p}(x_n) = (\sigma_{e_n}K)^{\circ,p}(-x_n)$ for all $x_n \in \mathbb{R}$.*

5E. *Proof of Lemma 5.4.* The only two ingredients required for the proof of Lemma 5.4 are Hölder's inequality and the log-convexity of $\sinh(t)/t$ (Claim 5.19 below).

Proof of Lemma 5.4. Let $f, g : \pi_{e_n^\perp}(K) \to \mathbb{R}, g \le f$, so that

$$K = \{(\xi, x_n) \in \pi_{e_n^\perp}(K) \times \mathbb{R} : g(\xi) \le x_n \le f(\xi)\}.$$

Then,

$$\sigma_{e_n}K = \bigl\{(\xi, x_n) \in \pi_{e_n^\perp}(K) \times \mathbb{R} : |x_n| \le \tfrac{1}{2}(f(\xi) - g(\xi))\bigr\}.$$

In the integrals below it will be convenient to use slice-coordinates

$$(\eta, y_n) \in \sigma_{e_n K}, \quad \text{with } \eta \in (\sigma_{e_n}K) \cap e_n^\perp, \quad y_n \in \mathbb{R}.$$

Since $|\sigma_{e_n}K| = |K|$ and $(\sigma_{e_n}K) \cap e_n^\perp = \pi_{e_n}^\perp(K)$,

$$\begin{aligned}
h_{p,\sigma_{e_n}K}\Bigl(r\frac{\xi+\xi'}{2}, rx_n\Bigr) &= \frac{1}{p}\log\biggl(\int_{\sigma_{e_n}K} e^{p\langle r\frac{\xi+\xi'}{2}, \eta\rangle} e^{prx_ny_n}\,\frac{\mathrm{d}\eta\,\mathrm{d}y_n}{|\sigma_{e_n}K|}\biggr)\\
&= \frac{1}{p}\log\biggl(\int_{\eta\in(\sigma_{e_n}K)\cap e_n^\perp}\int_{y_n=-\frac{f(\eta)-g(\eta)}{2}}^{\frac{f(\eta)-g(\eta)}{2}} e^{pr\langle\frac{\xi+\xi'}{2},\eta\rangle} e^{prx_ny_n}\,\frac{\mathrm{d}y_n\,\mathrm{d}\eta}{|K|}\biggr)\\
&= \frac{1}{p}\log\biggl(\int_{\pi_{e_n^\perp}(K)} e^{pr\langle\frac{\xi+\xi'}{2},\eta\rangle}\,\frac{e^{prx_n\frac{f(\eta)-g(\eta)}{2}} - e^{-prx_n\frac{f(\eta)-g(\eta)}{2}}}{prx_n}\,\frac{\mathrm{d}\eta}{|K|}\biggr)\\
&= \frac{1}{p}\log\biggl(\int_{\pi_{e_n^\perp}(K)} e^{pr\langle\frac{\xi+\xi'}{2},\eta\rangle}\,\frac{2}{prx_n}\sinh\Bigl(prx_n\frac{f(\eta)-g(\eta)}{2}\Bigr)\frac{\mathrm{d}\eta}{|K|}\biggr).
\end{aligned} \tag{5-11}$$

Also,

$$\begin{aligned} h_{p,K}(t\xi, tx_n) &= \frac{1}{p}\log\biggl(\int_K e^{p\langle t\xi,\eta\rangle} e^{ptx_n y_n}\,\frac{\mathrm{d}\eta\,\mathrm{d}y_n}{|K|}\biggr) \\ &= \frac{1}{p}\log\biggl(\int_{\pi_{e_n^\perp}(K)}\int_{y_n=g(\eta)}^{f(\eta)} e^{pt\langle\xi,\eta\rangle} e^{ptx_n y_n}\,\frac{\mathrm{d}y_n\,\mathrm{d}\eta}{|K|}\biggr) \\ &= \frac{1}{p}\log\biggl(\int_{\pi_{e_n^\perp}(K)} e^{pt\langle\xi,\eta\rangle}\,\frac{1}{px_n t}\bigl(e^{px_n tf(\eta)} - e^{px_n tg(\eta)}\bigr)\,\frac{\mathrm{d}\eta}{|K|}\biggr) \\ &= \frac{1}{p}\log\biggl(\int_{\pi_{e_n^\perp}(K)} e^{pt\langle\xi,\eta\rangle}\,\frac{2}{px_n t} e^{px_n t\frac{f(\eta)+g(\eta)}{2}}\sinh\Bigl(px_n t\,\frac{f(\eta)-g(\eta)}{2}\Bigr)\,\frac{\mathrm{d}\eta}{|K|}\biggr), \end{aligned} \tag{5-12}$$

because

$$\begin{aligned} e^{px_n tf(\eta)} - e^{px_n tg(\eta)} &= e^{px_n t\frac{f(\eta)+g(\eta)}{2}}\bigl(e^{px_n t\frac{f(\eta)-g(\eta)}{2}} - e^{-px_n t\frac{f(\eta)-g(\eta)}{2}}\bigr) \\ &= 2e^{px_n t\frac{f(\eta)+g(\eta)}{2}}\sinh\Bigl(px_n t\,\frac{f(\eta)-g(\eta)}{2}\Bigr). \end{aligned}$$

Similarly,

$$\begin{aligned} &h_{p,K}(s\xi', -sx_n) \\ &\quad= \frac{1}{p}\log\biggl(\int_{\pi_{e_n^\perp}(K)} e^{p\langle s\xi',\eta\rangle}\,\frac{2}{p(-sx_n)} e^{-px_n s\frac{f(\eta)+g(\eta)}{2}}\sinh\Bigl(p(-sx_n)\frac{f(\eta)-g(\eta)}{2}\Bigr)\,\frac{\mathrm{d}\eta}{|K|}\biggr) \\ &\quad= \frac{1}{p}\log\biggl(\int_{\pi_{e_n^\perp}(K)} e^{ps\langle\xi',\eta\rangle}\,\frac{2}{px_n s} e^{-px_n s\frac{f(\eta)+g(\eta)}{2}}\sinh\Bigl(px_n s\frac{f(\eta)-g(\eta)}{2}\Bigr)\,\frac{\mathrm{d}\eta}{|K|}\biggr). \end{aligned} \tag{5-13}$$

By (5-12)–(5-13) and Hölder's inequality,

$$\begin{aligned} &\frac{s}{t+s}h_{p,K}(t\xi, tx_n) + \frac{t}{t+s}h_{p,K}(s\xi', -sx_n) \\ &\quad= \frac{1}{p}\log\biggl[\biggl(\int_{\pi_{e_n^\perp}(K)} e^{pt\langle\xi,\eta\rangle}\,\frac{2}{px_n t} e^{px_n t\frac{f(\eta)+g(\eta)}{2}}\sinh\Bigl(px_n t\,\frac{f(\eta)-g(\eta)}{2}\Bigr)\,\frac{\mathrm{d}\eta}{|K|}\biggr)^{\frac{s}{t+s}} \\ &\qquad\qquad\times\biggl(\int_{\pi_{e_n^\perp}(K)} e^{ps\langle\xi',\eta\rangle}\,\frac{2}{px_n s} e^{-px_n s\frac{f(\eta)+g(\eta)}{2}}\sinh\Bigl(px_n s\frac{f(\eta)-g(\eta)}{2}\Bigr)\,\frac{\mathrm{d}\eta}{|K|}\biggr)^{\frac{t}{t+s}}\biggr] \\ &\quad\geq \frac{1}{p}\log\biggl(\int_{\pi_{e_n^\perp}(K)} e^{p\frac{ts}{t+s}\langle\xi,\eta\rangle}\Bigl(\frac{2}{px_n t}\Bigr)^{\frac{s}{t+s}} e^{px_n\frac{ts}{t+s}\frac{f(\eta)+g(\eta)}{2}}\Bigl(\sinh\Bigl(px_n t\,\frac{f(\eta)-g(\eta)}{2}\Bigr)^{\frac{s}{t+s}}\Bigr) \\ &\qquad\qquad\times e^{p\frac{ts}{t+s}\langle\xi',\eta\rangle}\Bigl(\frac{2}{px_n s}\Bigr)^{\frac{t}{t+s}} e^{-px_n\frac{ts}{t+s}\frac{f(\eta)+g(\eta)}{2}}\Bigl(\sinh\Bigl(px_n s\frac{f(\eta)-g(\eta)}{2}\Bigr)^{\frac{t}{t+s}}\Bigr)\,\frac{\mathrm{d}\eta}{|K|}\biggr) \\ &\quad= \frac{1}{p}\log\biggl(\int_{\pi_{e_n^\perp}(K)} e^{p\frac{ts}{t+s}\langle\xi+\xi',\eta\rangle} J(\eta,t)^{\frac{s}{t+s}} J(\eta,s)^{\frac{t}{t+s}}\,\frac{\mathrm{d}\eta}{|K|}\biggr) \\ &\quad= \frac{1}{p}\log\biggl(\int_{\pi_{e_n^\perp}(K)} e^{pr\langle\frac{\xi+\xi'}{2},\eta\rangle} J(\eta,t)^{\frac{s}{t+s}} J(\eta,s)^{\frac{t}{t+s}}\,\frac{\mathrm{d}\eta}{|K|}\biggr), \end{aligned} \tag{5-14}$$

where

$$J(\eta,t) := \frac{2}{px_n t}\sinh\biggl(px_n t\,\frac{f(\eta)-g(\eta)}{2}\biggr).$$

By Claim 5.19 below, $\log J$ is convex in t, and therefore

$$J(\eta,t)^{\frac{s}{t+s}} J(\eta,s)^{\frac{t}{t+s}} \ge J\biggl(\eta, \frac{s}{t+s}t+\frac{t}{t+s}s\biggr) = J\biggl(\eta,\frac{2ts}{t+s}\biggr) = J(\eta,r), \tag{5-15}$$

because $\frac{2ts}{t+s}=r$. Therefore, by (5-11), (5-14) and (5-15),

$$\begin{aligned}\frac{s}{t+s}h_{p,K}(t\xi,tx_n)+\frac{t}{t+s}h_{p,K}(s\xi',-sx_n)&\\ &\ge \frac{1}{p}\log\biggl(\int_{K\cap e_n^\perp} e^{pr\langle\frac{\xi+\xi'}{2},\eta\rangle}\frac{2}{px_n r}\sinh\biggl(px_n r\frac{f(\eta)-g(\eta)}{2}\biggr)\frac{d\eta}{|K|}\biggr)\\ &= h_{p,\sigma_{e_n}K}\biggl(r\frac{\xi+\xi'}{2},rx_n\biggr),\end{aligned}$$

as desired. □

Claim 5.19. *For any $x>0$, $t\mapsto \log\bigl(\frac{1}{t}\sinh(tx)\bigr)$, $t>0$, is convex.*

Proof. Write

$$f(t) := \log\biggl(\frac{1}{t}\sinh(tx)\biggr) = \log(\sinh(tx)) - \log t.$$

Compute the derivatives

$$f'(t) = x\frac{\cosh(tx)}{\sinh(tx)} - \frac{1}{t}$$

and

$$\begin{aligned}f''(t) &= x^2\frac{\sinh(tx)}{\sinh(tx)} - x^2\frac{(\cosh(tx))^2}{(\sinh(tx))^2}+\frac{1}{t^2} = x^2\biggl(1-\frac{(\cosh(tx))^2}{(\sinh(tx))^2}+\frac{1}{(tx)^2}\biggr)\\ &= x^2\biggl(1-\frac{1+(\sinh(tx))^2}{(\sinh(tx))^2}+\frac{1}{(tx)^2}\biggr) = x^2\biggl(\frac{1}{(tx)^2}-\frac{1}{(\sinh(tx))^2}\biggr)\ge 0,\end{aligned}$$

because $\sinh(y)\ge y$ for all $y\ge 0$. □

5F. *Slice analysis of $K^{\circ,p}$ under Steiner symmetrization.*

5F1. *A monotonicity property for the average of antipodal slices.* For the proof of Lemma 5.3, we first prove (5-4). The aim is to apply the following theorem due to [Ball 1986, Theorem 4.10] for F, G, H appropriate exponentials of the L^p-support functions.

Theorem 5.20. *Let $F, G, H : (0,\infty)\to[0,\infty)$ be measurable functions, not almost everywhere* 0, *with*

$$H(r) \ge F(t)^{\frac{s}{t+s}} G(s)^{\frac{t}{t+s}} \quad \text{for all} \quad \frac{2}{r} = \frac{1}{t}+\frac{1}{s}. \tag{5-16}$$

Then, for $q\ge 1$,

$$2\biggl(\int_0^\infty r^{q-1}H(r)\,dr\biggr)^{-\frac{1}{q}} \le \biggl(\int_0^\infty t^{q-1}F(t)\,dt\biggr)^{-\frac{1}{q}} + \biggl(\int_0^\infty s^{q-1}G(s)\,ds\biggr)^{-\frac{1}{q}}.$$

For the reader's convenience, we give a proof in the Appendix. Applying Theorem 5.20 to prove (5-4) becomes possible by Lemma 5.4.

Proof of Lemma 5.3. Set

$$F(t) := e^{-h_{p,K}(t\xi, tx_n)}, \quad G(s) := e^{-h_{p,K}(s\xi', -sx_n)}, \quad H(r) := e^{-h_{p,\sigma_{e_n}K}\left(r\frac{\xi+\xi'}{2}, rx_n\right)}.$$

By Lemma 5.4, for any $t, s > 0$ with $\frac{2}{r} = \frac{1}{t} + \frac{1}{s}$,

$$H(r) \geq F(t)^{\frac{s}{t+s}} G(s)^{\frac{t}{t+s}};$$

thus, by Theorem 5.20 for $q = n$,

$$\begin{aligned}
\left\| \left(\frac{\xi+\xi'}{2}, x_n \right) \right\|_{(\sigma_{e_n}K)^{\circ,p}} &= \left(\frac{1}{(n-1)!} \int_0^\infty r^{n-1} e^{-h_{p,\sigma_{e_n}K}\left(r\frac{\xi+\xi'}{2}, rx_n\right)} \, \mathrm{d}r \right)^{-\frac{1}{n}} \\
&\leq \frac{1}{2} \left(\frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-h_{p,K}(t\xi, tx_n)} \, \mathrm{d}t \right)^{-\frac{1}{n}} \\
&\qquad + \frac{1}{2} \left(\frac{1}{(n-1)!} \int_0^\infty s^{n-1} e^{-h_{p,K}(s\xi', -sx_n)} \, \mathrm{d}s \right)^{-\frac{1}{n}} \\
&= \frac{1}{2} \|(\xi, x_n)\|_{K^{\circ,p}} + \frac{1}{2} \|(\xi', -x_n)\|_{K^{\circ,p}}.
\end{aligned}$$

verifying (5-4).

For $\xi \in (K^{\circ,p})(x_n)$ and $\xi' \in (K^{\circ,p})(-x_n)$, by definition (5-1), $(\xi, x_n) \in K^{\circ,p}$ and $(\xi', -x_n) \in K^{\circ,p}$, i.e., $\|(\xi, x_n)\|_{K^{\circ,p}} \leq 1$ and $\|(\xi', -x_n)\|_{K^{\circ,p}} \leq 1$. By (5-4),

$$\left\| \left(\frac{\xi+\xi'}{2}, x_n \right) \right\|_{(\sigma_{e_n}K)^{\circ,p}} \leq \frac{\|(\xi, x_n)\|_{K^{\circ,p}} + \|(\xi', -x_n)\|_{K^{\circ,p}}}{2} \leq 1,$$

i.e., $\left(\frac{\xi+\xi'}{2}, x_n\right) \in (\sigma_{e_n}K)^{\circ,p}$ or $\frac{\xi+\xi'}{2} \in (\sigma_{e_n}K)^{\circ,p}(x_n)$. Finally, by Corollary 5.18, $(\sigma_{e_n^\perp}K)^{\circ,p}(x_n) = (\sigma_{e_n^\perp}K)^{\circ,p}(-x_n)$; hence we have the equality in the right-hand side of (5-3). □

5F2. *Monotonicity of the volume of slices under Steiner symmetrization.*

Proof of Lemma 5.2. By the Brunn–Minkowski inequality and Lemma 5.3,

$$\begin{aligned}
|(\sigma_{e_n}K)^{\circ,p}(x_n)|^{\frac{1}{n-1}} &\geq \frac{|K^{\circ,p}(x_n) + K^{\circ,p}(-x_n)|^{\frac{1}{n-1}}}{2} \\
&\geq \frac{|K^{\circ,p}(x_n)|^{\frac{1}{n-1}} + |K^{\circ,p}(-x_n)|^{\frac{1}{n-1}}}{2} = |K^{\circ,p}(x_n)|^{\frac{1}{n-1}},
\end{aligned}$$

because K is symmetric thus, by Corollary 5.16, $K^{\circ,p}(-x_n) = -K^{\circ,p}(x_n)$, and hence their volumes are equal $|K^{\circ,p}(-x_n)| = |K^{\circ,p}(x_n)|$. □

5G. *Proof of Theorem 1.6.*

We now complete the proofs of Proposition 5.1 and Theorem 1.6.

Proof of Proposition 5.1. Take for a moment $u = e_n$. By (5-2) and Lemma 5.2,

$$|(\sigma_{e_n}K)^{\circ,p}| = \int_{-\infty}^{\infty} |(\sigma_{e_n}K)^{\circ,p}(x_n)| \, \mathrm{d}x_n \geq \int_{-\infty}^{\infty} |(K^{\circ,p})(x_n)| \, \mathrm{d}x_n = |K^{\circ,p}|. \tag{5-17}$$

In general, for $u \in \partial B_2^n$, there is $A \in O(n)$ such that $Au = e_n$. By Lemma 5.8, $\sigma_u K = A^{-1}(\sigma_{Au}(AK)) = A^{-1}(\sigma_{e_n}(AK))$. By (4-7), $(\sigma_u K)^{\circ,p} = (A^{-1}\sigma_{e_n}(AK))^{\circ,p} = A^T(\sigma_{e_n}(AK))^{\circ,p}$. Thus by (5-17),

$$\begin{aligned}|(\sigma_u K)^{\circ,p}| &= |\det A^T||(\sigma_{e_n}(AK))^{\circ,p}| \\ &\geq |\det A^T||(AK)^{\circ,p}| = |A^T(AK)^{\circ,p}| = |K^{\circ,p}|,\end{aligned}$$

because, again by (4-7), $A^T(AK)^{\circ,p} = (A^{-1}AK)^{\circ,p} = K^{\circ,p}$. □

Theorem 1.6 follows from Proposition 5.1 and the fact that repeated Steiner symmetrizations converge to a dilated 2-ball (Lemma 5.10).

Proof of Theorem 1.6. There is $\rho > 0$ and a sequence $\{u_j\}_{j\geq 1} \subset \partial B_2^n$ such that for

$$K_0 := K, \quad K_j := \sigma_{u_j} K_{j-1},$$

$K_j \to \rho B_2^n$ in the Hausdorff metric [Artstein-Avidan et al. 2015, Theorem 1.1.16]. By Proposition 5.1,

$$\begin{aligned}\mathcal{M}_p(K_j) &= n!\,|K_j||K_j^{\circ,p}| = n!\,|K||K_j^{\circ,p}| \\ &\leq n!\,|K||(\sigma_{u_{j+1}} K_j)^{\circ,p}| = n!\,|K||K_{j+1}^{\circ,p}| = \mathcal{M}_p(K_{j+1}).\end{aligned}$$

In particular, $\mathcal{M}_p(K) \leq \mathcal{M}_p(K_j)$ for all j. Sending $j \to \infty$, $K_j \to \rho B_2^n$ in the Hausdorff metric, and hence, by Lemmas 4.7 and 5.11, $\mathcal{M}_p(K_j) \to \mathcal{M}_p(\rho B_2^n) = \mathcal{M}_p(B_2^n)$; thus $\mathcal{M}_p(K) \leq \mathcal{M}_p(B_2^n)$. □

6. A connection to Bourgain's slicing problem

In this section we explore the relationship between the L^p support functions $h_{p,K}$ (1-8) and the slicing problem (Conjecture 1.8). The aim is to prove Theorem 1.9 and then illustrate how it implies a suboptimal upper bound on the isotropic constant (Corollary 1.12) originally due to Milman and Pajor. We also explain some interesting connections to and motivations from complex geometry.

In Section 6A2 we recall the definitions of the covariance matrix and the isotropic constant, and relate these to $h_{p,K}$ (Lemma 6.3). In Section 6A3 we recall the definition of the Monge–Ampère measure and its basic properties. Theorem 1.9 is proved in Section 6B. The proof consists of two parts: using Jensen's inequality to bound $\int \log\det \nabla^2 h_{1,K}$ (Lemma 6.10), and then bounding $\int_{\mathbb{R}^n} h_{1,K}\, d\nu_{p,K}$ (Lemma 6.13). In Section 6C, we show $\log\det \nabla^2 h_{p,K} + p(n+1)h_{p,K}$ is convex, proving Theorem 1.11. From Theorems 1.9 and 1.11 we then obtain an upper bound on the isotropic constant of order $O(\sqrt{n})$ (Corollary 1.12). In Section 6D, we define the L^p support functions of compactly supported probability measures and show that Theorem 1.11 cannot be improved in that setting (Example 6.20). Finally, in Section 6E we explain some novel connections of our work to complex geometry, in particular to Ricci curvature, Fubini–Study metrics, Bergman metrics, Kobayashi's theorem, and holomorphic line bundles.

6A. *Preliminaries.*

6A1. *Affine-invariance of $\mathcal{C}$.* The isotropic constant is an affine invariant (e.g., [Brazitikos et al. 2014, p. 77]); hence so is $\mathcal{C}$. As we could not find precisely the following lemma in the literature, we include its proof for completeness.

Lemma 6.1. *For $K \subset \mathbb{R}^n$ and $A \in \mathrm{GL}(n,\mathbb{R})$, $b \in \mathbb{R}^n$,*

$$\mathrm{Cov}(AK+b) = A\,\mathrm{Cov}(K)A^T$$

where $\mathrm{Cov}(K)$ *is defined in* (1-16).

Proof. Write $A = [A_i^j]_{i,j=1}^n$, $b = (b_1,\dots,b_n)$ and $T(x) = Ax+b$. The Einstein summation convention of summing over repeated indices is used. Changing variables $y = T^{-1}x = A^{-1}x - A^{-1}b$, $\mathrm{d}y = |\det A^{-1}|\mathrm{d}x = |\det A|^{-1}\mathrm{d}x$,

$$\begin{aligned}
\mathrm{Cov}_{ij}(AK+b) &= \int_{T(K)} x_i x_j \frac{\mathrm{d}x}{|T(K)|} - \int_{T(K)} x_i \frac{\mathrm{d}x}{|T(K)|} \int_{T(K)} x_j \frac{\mathrm{d}x}{|T(K)|} \\
&= \int_K (Ay+b)_i (Ay+b)_j \frac{|\det A|\,\mathrm{d}y}{|AK+b|} - \int_K (Ay+b)_i \frac{|\det A|\,\mathrm{d}y}{|AK+b|} \int_K (Ay+b)_j \frac{|\det A|\,\mathrm{d}y}{|AK+b|} \\
&= \int_K (A_i^k y_k + b_i)(A_j^l y_l + b_j) \frac{\mathrm{d}y}{|K|} - \int_K (A_i^k y_k + b_i) \frac{\mathrm{d}y}{|K|} \int_K (A_j^l y_l + b_j), \frac{\mathrm{d}y}{|K|} \\
&= A_i^k A_j^l \int_K y_k y_l \frac{\mathrm{d}y}{|K|} + b_j A_i^k \int_K y_k \frac{\mathrm{d}y}{|K|} + b_i A_j^l \int_K y_j \frac{\mathrm{d}y}{|K|} + b_i b_j \\
&\qquad - A_i^k A_j^l \int_K y_k y_l \frac{\mathrm{d}y}{|K|} - b_j A_i^k \int_K y_k \frac{\mathrm{d}y}{|K|} - b_i A_j^l \int_K y_l \frac{\mathrm{d}y}{|K|} - b_i b_j \\
&= A_i^k A_j^l \left(\int_K y_k y_l \frac{\mathrm{d}y}{|K|} - \int_K y_k \frac{\mathrm{d}y}{|K|} \int_K y_l \frac{\mathrm{d}y}{|K|} \right) \\
&= A_i^k A_j^l \,\mathrm{Cov}_{kl}(K),
\end{aligned}$$

proving the claim. □

Let $A \in \mathrm{GL}(n,\mathbb{R})$ and $b \in \mathbb{R}^n$. By Lemma 6.1,

$$\mathcal{C}(AK+b) = \frac{|AK+b|^2}{\det \mathrm{Cov}(AK+b)} = \frac{(\det A)^2 |K|^2}{\det(A\,\mathrm{Cov}(K)A^T)} = \frac{(\det A)^2 |K|^2}{(\det A)^2 \det \mathrm{Cov}(K)} = \mathcal{C}(K),$$

proving:

Corollary 6.2. *$\mathcal{C}$ is an affine invariant.*

6A2. *L^p-support functions and the isotropic constant.* Next, we relate the functional $\mathcal{C}$ (1-18) to $h_{p,K}$ (1-8) (for $p = 1$ see [Klartag 2006, Lemma 3.1]).

Lemma 6.3. *Let $p > 0$. For a convex body $K \subset \mathbb{R}^n$, we have $\nabla^2 h_{p,K}(0) = p\,\mathrm{Cov}(K)$ and*

$$\mathcal{C}(K) = \frac{p^n |K|^2}{\det \nabla^2 h_{p,K}(0)}.$$

Proof. By direct calculation,

$$\frac{\partial}{\partial y_i} h_{p,K}(y) = \frac{\int_K x_i e^{p\langle x,y\rangle}\,\mathrm{d}x}{\int_K e^{p\langle x,y\rangle}\,\mathrm{d}x}$$

and

$$\frac{\partial^2}{\partial y_i \partial y_j} h_{p,K}(y) = \frac{p\int_K x_i x_j e^{p\langle x,y\rangle}\,\mathrm{d}x \int_K e^{p\langle x,y\rangle}\mathrm{d}x - p\int_K x_i e^{p\langle x,y\rangle}\,\mathrm{d}x \int_K x_j e^{p\langle x,y\rangle}\,\mathrm{d}x}{\left(\int_K e^{p\langle x,y\rangle}\,\mathrm{d}x\right)^2}.$$

Since for $y=0$, $\int_K e^{p\langle x,0\rangle}\,\mathrm{d}x = |K|$,

$$\frac{\partial^2 h_{p,K}}{\partial y_i \partial y_j}(0) = p\int_K x_i x_j \frac{\mathrm{d}x}{|K|} - p\int_K x_i \frac{\mathrm{d}x}{|K|}\int_K x_j \frac{\mathrm{d}x}{|K|} = p\,\mathrm{Cov}_{i,j}(K)$$

and

$$\det \nabla^2 h_{p,K}(0) = \det(p\,\mathrm{Cov}(K)) = p^n \det \mathrm{Cov}(K) = p^n \frac{|K|^2}{\mathcal{C}(K)},$$

as claimed. □

6A3. *The Monge–Ampère measure.* We review some basic details concerning the Monge–Ampère measure, following [Rauch and Taylor 1977]. Legendre duality is defined by $f^*(y) := \sup_{x\in\mathbb{R}^n}[\langle y,x\rangle - f(x)]$.

Definition 6.4 [Rockafellar 1970, p. 215]. For a convex function $\phi : \mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$ and $x \in \mathbb{R}^n$, the subdifferential of ϕ at x is

$$\partial\phi(x) := \{y \in \mathbb{R}^n : \phi(z) \ge \phi(x) + \langle y, z-x\rangle \text{ for all } z \in \mathbb{R}^n\}.$$

Lemma 6.5 [Rockafellar 1970, Theorem 23.5]. *For $\phi : \mathbb{R}^n \to \mathbb{R}$ convex, $\partial\phi(\mathbb{R}^n) \subset \{\phi^* < \infty\}$.*

Proof. By definition of the subgradient, for $y \in \partial\phi(x)$, we have $\phi(z) \ge \phi(x) + \langle y, z-x\rangle$ for all $z \in \mathbb{R}^n$, i.e., $\langle y,x\rangle - \phi(x) \ge \langle y,z\rangle - \phi(z)$. Taking supremum over all $z \in \mathbb{R}^n$,

$$\phi^*(y) \le \langle y,x\rangle - \phi(x) < \infty,$$

as claimed. □

Corollary 6.6. *For all $p \in (0,\infty)$,*

$$\partial h_K(\mathbb{R}^n) \subset K \quad \textit{and} \quad \partial h_{p,K}(\mathbb{R}^n) \subset K.$$

Proof. Since $h_K^* = \mathbf{1}_K^\infty$, by Lemma 6.5, $\partial h_K(\mathbb{R}^n) \subset \{\mathbf{1}_K^\infty < \infty\} = K$. Similarly, since $h_{p,K} \le h_K$, the Legendre transform satisfies $\mathbf{1}_K^\infty = h_K^* \le h_{p,K}^*$; thus, by Lemma 6.5,

$$\partial h_{p,K}(\mathbb{R}^n) \subset \{h_{p,K}^* < \infty\} \subset \{\mathbf{1}_K^\infty < \infty\} = K. \qquad \square$$

Definition 6.7 [Rauch and Taylor 1977, Definition 2.6]. For a convex function ϕ, let

$$(\mathrm{MA}\phi)(U) := |\partial\phi(U)|,$$

where the right-hand side denotes the Lebesgue measure of $\partial\phi(U)$ in $\mathbb{R}^n$.

Lemma 6.8. $$\mathrm{MA}h_{p,K}(\mathbb{R}^n) \le |K|.$$

Proof. By definition, $\mathrm{MA}h_{p,K}(\mathbb{R}^n) = |\partial h_{p,K}(\mathbb{R}^n)| \le |K|$ because $\partial h_{p,K}(\mathbb{R}^n) \subset K$ by Corollary 6.6. □

Remark 6.9. In fact, equality holds in Lemma 6.8. In particular, $h_{p,K}$ is a smooth, strictly convex function with $\nabla h_{p,K}(\mathbb{R}^n) = \mathrm{int}\,K$ (see [Klartag 2006, Lemma 3.1] for the case $p=1$). By the smoothness of $h_{p,K}$ we also know the density of $\mathrm{MA}h_{p,K}$ equals $\det \nabla^2 h_{p,K}$.

6B. ***Conditional lower bounds on the isotropic constant.*** The proof of Theorem 1.9 relies on the following observation. Assume that K satisfies $(*_B)$ for some $B > 0$, i.e.,

$$u_{B,K}(y) := \log\det \nabla^2 h_{1,K}(y) + B h_{1,K}(y)$$

is convex. Note that $h_{1,K}(0) = 0$; thus $u_{B,K}(0) = \log\det \nabla^2 h_{1,K}(0)$. By Lemma 6.3,

$$\mathcal{C}(K) = \frac{|K|^2}{\det \mathrm{Cov}(K)} = \frac{|K|^2}{\det \nabla^2 h_{1,K}(0)} = |K|^2 e^{-u_{B,K}(0)}. \tag{6-1}$$

Since $u_{B,K}$ is convex by assumption, for a probability measure μ with $b(\mu) = 0$, by Jensen's inequality,

$$\begin{aligned} u_{B,K}(0) = u_{B,K}\biggl(\int_{\mathbb{R}^n} y \,\mathrm{d}\mu(y)\biggr) &\le \int_{\mathbb{R}^n} u_{B,K}(y)\,\mathrm{d}\mu(y) \\ &= \int_{\mathbb{R}^n} \log\det \nabla^2 h_{1,K}(y)\,\mathrm{d}\mu(y) + B\int_{\mathbb{R}^n} h_{1,K}(y)\,\mathrm{d}\mu(y). \end{aligned} \tag{6-2}$$

By (6-1) and (6-2), in order to get bounds on $\mathcal{C}(K)$ it is enough to bound $\int \log \mathrm{MA} h_{1,K}\,\mathrm{d}\mu$ and $\int_{\mathbb{R}^n} h_{1,K}\,\mathrm{d}\mu$, for a suitable probability measure μ.

Here, we consider the probability measures (1-21) for which we obtain the desired bounds (Lemmas 6.10 and 6.13). By Corollary 4.5, we may translate K to a suitable position in order to obtain estimates on $\int_{\mathbb{R}^n} \log \mathrm{MA} h_{1,K}(y)\,\mathrm{d}\nu_{p,K}(y)$ (Lemma 6.10(ii) and (iii)).

6B1. *A bound on $\int \log\det \nabla^2 h_{1,K}$ in terms of L^p-Mahler volumes.*

Lemma 6.10. *Let $p > 0$. For a convex body $K \subset \mathbb{R}^n$, and $\nu_{p,K}$ as in* (1-21):

(i) *We have*

$$\int_{\mathbb{R}^n} \log\det \nabla^2 h_{1,K}(y)\,\mathrm{d}\nu_{p,K}(y) \le \log\biggl(|K|^2 \frac{\mathcal{M}_{\frac{1}{2p}}(K)}{\mathcal{M}_{\frac{1}{p}}(K)^2}\frac{p^n}{2^n}\biggr).$$

(ii) *If $b(\nu_{p,K}) = 0$, then*

$$\int_{\mathbb{R}^n} \log\det \nabla^2 h_{1,K}(y)\,\mathrm{d}\nu_{p,K}(y) \le \log\biggl(\frac{|K|e^n}{\int_{\mathbb{R}^n} e^{-ph_{1,K}(y)}\,\mathrm{d}y}\biggr).$$

(iii) *If $b(K) = 0$, then*

$$\int_{\mathbb{R}^n} \log\det \nabla^2 h_{1,K}(y)\,\mathrm{d}\nu_{p,K}(y) \le \log\biggl(\frac{|K|}{\int_{\mathbb{R}^n} e^{-ph_{1,K}(y)}\,\mathrm{d}y}\biggr).$$

For the proof of Lemma 6.10 we need the following.

Claim 6.11. *Let $p > 0$. For a convex body $K \subset \mathbb{R}^n$,*

$$\int_{\mathbb{R}^n} \log\det \nabla^2 h_{1,K}(y)\,\mathrm{d}\nu_{p,K}(y) \le \log\biggl(|K| \frac{\int_{\mathbb{R}^n} e^{-2ph_{1,K}(y)}\,\mathrm{d}y}{\bigl(\int_{\mathbb{R}^n} e^{-ph_{1,K}(y)}\,\mathrm{d}y\bigr)^2}\biggr).$$

Proof. By Jensen's inequality and Cauchy–Schwarz,

$$\begin{aligned}\int_{\mathbb{R}^n} \log\det\nabla^2 h_{1,K}(y)\,\mathrm{d}\nu_{p,K}(y) &= 2\int_{\mathbb{R}^n} \log(\det\nabla^2 h_{1,K}(y))^{\frac12}\,\mathrm{d}\nu_{p,K}(y)\\ &\le 2\log\int_{\mathbb{R}^n}(\det\nabla^2 h_{1,K}(y))^{\frac12}\,\mathrm{d}\nu_{p,K}(y)\\ &= 2\log\int_{\mathbb{R}^n}(\det\nabla^2 h_{1,K}(y))^{\frac12}\,\frac{e^{-ph_{1,K}(y)}}{\int_{\mathbb{R}^n} e^{-ph_{1,K(y)}}\,\mathrm{d}y}\,\mathrm{d}y\\ &\le 2\log\biggl(\int_{\mathbb{R}^n}\det\nabla^2 h_{1,K}(y)\,\mathrm{d}y\,\frac{\int_{\mathbb{R}^n} e^{-2ph_{1,K}(y)}\,\mathrm{d}y}{\bigl(\int_{\mathbb{R}^n} e^{-ph_{1,K}(y)}\,\mathrm{d}y\bigr)^2}\biggr)^{\frac12}\\ &\le \log\biggl(|K|\,\frac{\int_{\mathbb{R}^n} e^{-2ph_{1,K}(y)}\,\mathrm{d}y}{\bigl(\int_{\mathbb{R}^n} e^{-ph_{1,K}(y)}\,\mathrm{d}y\bigr)^2}\biggr),\end{aligned}$$

because by Lemma 6.8 and Remark 6.9, $\int_{\mathbb{R}^n}\det\nabla^2 h_{1,K}(y)\,\mathrm{d}y = \mathrm{MA}h_{1,K}(\mathbb{R}^n)\le|K|$. □

Proof of Lemma 6.10. (i) In view of Claim 6.11, it is enough to compute the following two integrals,

$$\int_{\mathbb{R}^n} e^{-2ph_{1,K}(y)}\,\mathrm{d}y = \frac{1}{(2p)^n}\int_{\mathbb{R}^n} e^{-2ph_{1,K}(\frac{y}{2p})}\,\mathrm{d}y = \frac{1}{(2p)^n}\int_{\mathbb{R}^n} e^{-h_{1/(2p),K}(y)}\,\mathrm{d}y = \frac{1}{(2p)^n}\frac{\mathcal{M}_{\frac{1}{2p}}(K)}{|K|^{1+2p}},$$

because by Lemma 2.2(i), $2ph_{1,K}\bigl(\frac{y}{2p}\bigr) = h_{1/(2p),K}(y)$. Similarly,

$$\int_{\mathbb{R}^n} e^{-ph_{1,K}(y)}\,\mathrm{d}y = \frac{1}{p^n}\frac{\mathcal{M}_{\frac1p}(K)}{|K|^{1+p}}.$$

Therefore,

$$|K|\frac{\int_{\mathbb{R}^n} e^{-2ph_{1,K}(y)}\,\mathrm{d}y}{\bigl(\int_{\mathbb{R}^n} e^{-ph_{1,K}(y)}\,\mathrm{d}y\bigr)^2} = |K|\frac{\mathcal{M}_{\frac{1}{2p}}(K)}{(2p)^n|K|^{1+2p}}\,\frac{p^{2n}|K|^{2+2p}}{\mathcal{M}_{\frac1p}(K)^2} = |K|^2\frac{p^n}{2^n}\frac{\mathcal{M}_{\frac{1}{2p}}(K)}{\mathcal{M}_{\frac1p}(K)^2}. \tag{6-3}$$

The claim follows from (6-3) and Claim 6.11.

(ii) Since $b(\nu_{p,K}) = b(ph_{1,K}) = 0$, by Lemma 6.12 below, $ph_{1,K}(y)\ge ph_{1,K}(0)-n = -n$. Therefore,

$$\int_{\mathbb{R}^n} e^{-2ph_{1,K}(y)}\,\mathrm{d}y = \int_{\mathbb{R}^n} e^{-ph_{1,K}(y)}e^{-ph_{1,K}(y)}\,\mathrm{d}y \le e^n\int_{\mathbb{R}^n} e^{-ph_{1,K}(y)}\,\mathrm{d}y. \tag{6-4}$$

The claim follows directly from (6-4) and Claim 6.11.

(iii) Since $b(K)=0$, $\int_K\langle x,y\rangle\,\mathrm{d}x = 0$ for all $y\in\mathbb{R}^n$. As a result, by Jensen's inequality,

$$h_{1,K}(y) = \log\int_K e^{\langle x,y\rangle}\,\frac{\mathrm{d}x}{|K|} \ge \int_K \log e^{\langle x,y\rangle}\,\frac{\mathrm{d}x}{|K|} = \int_K\langle x,y\rangle\,\frac{\mathrm{d}x}{|K|} = 0,$$

i.e., $h_{1,K}(y)\ge 0$. Therefore, for $p>0$, $2ph_{1,K}(y)\ge ph_{1,K}(y)$ and hence

$$\int_{\mathbb{R}^n} e^{-2ph_{1,K}(y)}\,\mathrm{d}y \le \int_{\mathbb{R}^n} e^{-ph_{1,K}(y)}\,\mathrm{d}y. \tag{6-5}$$

The claim follows directly from (6-5) and Claim 6.11. □

In the previous proof we made use of the following estimate of [Fradelizi 1997, Theorem 4], stated without proof. We include a proof for the reader's convenience (see also [Brazitikos et al. 2014, Theorem 2.2.2]).

Lemma 6.12. *For a convex function $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$,*

$$\inf_{x\in\mathbb{R}^n} \phi(x) \geq \phi(b(\phi)) - n.$$

Proof. To begin with, it is enough to consider ϕ to be smooth, strictly convex, and bounded from below by $C|x|^2$ for large $|x|$. That is because for a smooth, nonnegative, compactly supported mollifier $\chi : \mathbb{R}^n \to [0, \infty)$ we know that

$$\phi_\varepsilon(x) := \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \phi(x - y)\chi\left(\frac{y}{\varepsilon}\right) \mathrm{d}y$$

is smooth, convex and decreases to ϕ as $\varepsilon \to 0$. Let

$$\phi_{j,\varepsilon}(x) := \phi_\varepsilon(x) + \frac{1}{j}\frac{|x|^2}{2},$$

smooth, convex functions that decrease to ϕ as $\varepsilon \to 0^+$ and $j \to \infty$ [Klimek 1991, Theorem 2.5.5]. In addition, $\phi_{j,\varepsilon}(x) \geq C|x|^2$ for large enough $|x|$, since ϕ_ε can be estimated by a linear term due to convexity, that is, $\phi_\varepsilon(x) \geq \phi_\varepsilon(0) + \langle \nabla\phi_\varepsilon(0), x\rangle$. By monotone convergence, $b(\phi_{j,\varepsilon}) \to b(\phi)$ as $\varepsilon \to 0$ and $j \to \infty$. By convexity, $\phi(x) \geq \phi(b(\phi)) + \langle \partial\phi(b(\phi)), x - b(\phi)\rangle$ for all x, so if the claim holds for $\phi_{j,\varepsilon}$, then

$$\begin{aligned}
\phi_{j,\varepsilon}(y) &\geq \phi_{j,\varepsilon}(b(\phi_{j,\varepsilon})) - n \\
&\geq \phi(b(\phi_{j,\varepsilon})) - n \\
&\geq \phi(b(\phi)) + \langle \partial\phi(b(\phi)), b(\phi_{j,\varepsilon}) - b(\phi)\rangle - n,
\end{aligned}$$

because $\phi_{j,\varepsilon} \geq \phi$. Taking $j \to \infty$ and $\varepsilon \to 0$ yields $\phi(y) \geq \phi(b(\phi)) - n$.

For ϕ smooth, strictly convex with $\phi(x) \geq C|x|^2$ for large $|x|$, by Jensen's inequality,

$$\phi(b(\phi)) = \phi\left(\int_{\mathbb{R}^n} xe^{-\phi(x)} \frac{\mathrm{d}x}{V(\phi)}\right) \leq \int_{\mathbb{R}^n} \phi(x)e^{-\phi(x)} \frac{\mathrm{d}x}{V(\phi)}. \tag{6-6}$$

By convexity, for all $x, y \in \mathbb{R}^n$, $\phi(y) \geq \phi(x) + \langle \nabla\phi(x), y - x\rangle$; thus, integrating with respect $e^{-\phi(x)} \frac{\mathrm{d}x}{V(\phi)}$,

$$\begin{aligned}
\phi(y) &\geq \int_{\mathbb{R}^n} \phi(x)e^{-\phi(x)} \frac{\mathrm{d}x}{V(\phi)} + \int_{\mathbb{R}^n} \langle \nabla\phi(x), y - x\rangle e^{-\phi(x)} \frac{\mathrm{d}x}{V(\phi)} \\
&= \int_{\mathbb{R}^n} \phi(x)e^{-\phi(x)} \frac{\mathrm{d}x}{V(\phi)} + \sum_{i=1}^n \int_{\mathbb{R}^n} \frac{\partial\phi}{\partial x_i}(x)(y_i - x_i)e^{-\phi(x)} \frac{\mathrm{d}x}{V(\phi)} \\
&= \int_{\mathbb{R}^n} \phi(x)e^{-\phi(x)} \frac{\mathrm{d}x}{V(\phi)} - \sum_{i=1}^n \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i}(e^{-\phi(x)})(y_i - x_i) \frac{\mathrm{d}x}{V(\phi)} \\
&= \int_{\mathbb{R}^n} \phi(x)e^{-\phi(x)} \frac{\mathrm{d}x}{V(\phi)} - \sum_{i=1}^n \int_{\mathbb{R}^n} e^{-\phi(x)} \frac{\mathrm{d}x}{V(\phi)} \\
&= \int_{\mathbb{R}^n} \phi(x)e^{-\phi(x)} \frac{\mathrm{d}x}{V(\phi)} - n,
\end{aligned} \tag{6-7}$$

because, by integration by parts,

$$\int_{\mathbb{R}} \frac{\partial}{\partial x_i}(e^{-\phi(x)})(y_i - x_i)\,\mathrm{d}x = 0 - \int_{\mathbb{R}} e^{-\phi(x)}\,\mathrm{d}x,$$

since

$$\lim_{|x_i|\to\infty} e^{-\phi(x)}|y_i - x_i| \le \lim_{x_i\to\infty} e^{-C|x|^2}|y-x| = 0.$$

By (6-6) and (6-7), $\phi(b(\phi)) \le \phi(y) + n$ for all $y \in \mathbb{R}^n$, from which the claim follows. □

6B2. *A bound on* $\int_{\mathbb{R}^n} h_{1,K}\,\mathrm{d}\nu_{p,K}$.

Lemma 6.13. *Let* $p > 0$. *For a convex body* $K \subset \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} h_{1,K}(y)\,\mathrm{d}\nu_{p,K}(y) \le \frac{n}{p}.$$

Proof of Lemma 6.13. By Lemma 2.2(v), $h_{p,K}$ increases to h_K with p. Therefore, by Lemma 2.2(i),

$$\begin{aligned} F(p) := \log \int_{\mathbb{R}^n} e^{-h_{p,K}(y)}\,\mathrm{d}y &= \log \int_{\mathbb{R}^n} e^{-\frac{1}{p}h_{1,K}(py)}\,\mathrm{d}y \\ = \log \int_{\mathbb{R}^n} e^{-\frac{1}{p}h_{1,K}(y)}\,\frac{\mathrm{d}y}{p^n} &= \log \int_{\mathbb{R}^n} e^{-\frac{1}{p}h_{1,K}(y)}\,\mathrm{d}y - n\log p \end{aligned}$$

is decreasing with p, and hence, its derivative must be nonpositive,

$$0 \ge \frac{\mathrm{d}F}{\mathrm{d}p} = \frac{\int_{\mathbb{R}^n} e^{-\frac{1}{p}h_{1,K}(y)} h_{1,K}(y)\,\mathrm{d}y}{p^2 \int_{\mathbb{R}^n} e^{-\frac{1}{p}h_{1,K}(y)}\,\mathrm{d}y} - \frac{n}{p} = \frac{1}{p^2}\int_{\mathbb{R}^n} h_{1,K}(y)\,\mathrm{d}\nu_{\frac{1}{p}}(y) - \frac{n}{p},$$

and the lemma follows. □

6B3. *Proof of Theorem 1.9.*

Claim 6.14. *Let* $p > 0$. *For a convex body* $K \subset \mathbb{R}^n$ *with* $0 \in \operatorname{int} K$ *and* $|K| = 1$,

$$\int_{\mathbb{R}^n} e^{-ph_{1,K}(y)}\,\mathrm{d}y \ge \frac{\mathcal{M}(K)}{p^n}.$$

Proof. Since $h_{p,K} \le h_K$, by homogeneity of h_K,

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-ph_{1,K}(y)}\,\mathrm{d}y &\ge \int_{\mathbb{R}^n} e^{-ph_K(y)}\,\mathrm{d}y = \int_{\mathbb{R}^n} e^{-h_K(py)}\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} e^{-h_K(v)}\,\frac{\mathrm{d}v}{p^n} = \frac{n!\,|K^\circ|}{p^n} = \frac{\mathcal{M}(K)}{p^n}, \end{aligned}$$

because $|K| = 1$; thus $\mathcal{M}(K) := n!\,|K||K^\circ| = n!\,|K^\circ|$. □

Proof of Theorem 1.9. By assumption, $(*_B)$ holds. Thus (6-2) applies for probability measures with barycenter at the origin.

(i) In order to apply the estimate (6-2), it is necessary to have a measure with barycenter at the origin. By Corollary 4.5, we may translate K so that $b(\nu_{p,K}) = b(h_{1/p,K}) = 0$. By Corollary 6.2, this does not affect $\mathcal{C}(K)$. By (6-2) and Lemmas 6.13 and 6.10(i),

$$u_{B,K}(0) \le \log\left(|K|^2 \frac{\mathcal{M}_{\frac{1}{2p}}(K)}{\mathcal{M}_{\frac{1}{p}}(K)^2}\frac{p^n}{2^n}\right) + \frac{Bn}{p}.$$

As a result, by (6-1),

$$\mathcal{C}(K) \geq \frac{\mathcal{M}_{\frac{1}{p}}(K)^2}{\mathcal{M}_{\frac{1}{2p}}(K)} \frac{2^n}{p^n} e^{-\frac{Bn}{p}}.$$

Choosing $p = B$,

$$\mathcal{C}(K) \geq \frac{\mathcal{M}_{\frac{1}{B}}(K)^2}{\mathcal{M}_{\frac{1}{2B}}(K)} \frac{2^n}{B^n} e^{-n}.$$

(ii) Similarly, to apply (6-2) we need a measure with barycenter at the origin. By Corollary 4.5, we may translate K so that $b(\nu_{p,K}) = b(h_{1/p,K}) = 0$. Also, rescale so that $|K| = 1$. By Corollary 6.2 this does not affect $\mathcal{C}(K)$. By (6-2), Lemmas 6.13 and 6.10(ii),

$$u_{B,K}(0) \leq \log\left(\frac{e^n}{\int_{\mathbb{R}^n} e^{-ph_{1,K}(y)}\,\mathrm{d}y}\right) + \frac{Bn}{p} \leq \log\left(\frac{e^n p^n}{\mathcal{M}(K)^n}\right) + \frac{Bn}{p},$$

where we used Claim 6.14 for the last inequality. As a result, by (6-1), since $|K| = 1$,

$$\mathcal{C}(K) = e^{-u_{B,K}(0)} \geq \frac{\mathcal{M}(K)}{e^n p^n} e^{-\frac{Bn}{p}}. \tag{6-8}$$

We can now optimize over all p on the right-hand side. Setting

$$f(p) := p^n e^{(1+\frac{B}{p})n}$$

gives

$$f'(p) = e^{n+\frac{nB}{p}} \cdot \left[np^{n-1} - p^n \cdot \frac{nB}{p^2}\right] = ne^{n+n\frac{B}{p}} p^{n-2}(p - B),$$

and the second derivative gives

$$\begin{aligned} f''(p) &= ne^{n+\frac{nB}{p}}[-nBp^{-2}p^{n-2}(p-B) + (n-2)p^{n-3}(p-B) + p^{n-2}] \\ &= ne^{n+\frac{nB}{p}} p^{n-4}[-nB(p-B) + (n-2)p(p-B) + p^2], \end{aligned}$$

so $f''(B) = ne^{2n}B^{n-2} > 0$ as long as $B > 0$. This confirms $p = B$ is a minimum. Thus, choosing $p = B$ in (6-8),

$$\mathcal{C}(K) \geq \frac{\mathcal{M}(K)}{e^{2n}B^n}.$$

(iii) Since K is symmetric, $b(K) = b(\nu_{p,K}) = 0$. Rescale K so that $|K| = 1$. $\mathcal{C}(K)$ remains invariant under rescaling by Corollary 6.2. By (6-2), Lemmas 6.13 and 6.10(iii), and Claim 6.14,

$$u_{B,K}(0) \leq \log\left(\frac{1}{\int_{\mathbb{R}^n} e^{-ph_{1,K}(y)}\,\mathrm{d}y}\right) + \frac{Bn}{p} \leq \log\left(\frac{p^n}{\mathcal{M}(K)}\right) + \frac{Bn}{p}.$$

As a result, by (6-1), since $|K| = 1$,

$$\mathcal{C}(K) = e^{-u_{B,K}(0)} \geq \frac{\mathcal{M}(K)}{p^n} e^{-\frac{Bn}{p}}.$$

Thus, choosing $p = B$,

$$\mathcal{C}(K) \geq \frac{\mathcal{M}(K)}{e^n B^n},$$

concluding the proof of Theorem 1.9 □

Remark 6.15. It is enough to formulate Theorem 1.9 in terms of $h_{1,K}$: By Lemma 2.2(i),

$$h_{p,K}(y) = \frac{1}{p} h_{1,K}(py).$$

Therefore $\nabla^2 h_{p,K}(y) = p\nabla^2 h_{1,K}(py)$. As a result,

$$\log\det\nabla^2 h_{p,K}(y) + pBh_{p,K}(y) = \log\det\nabla^2 h_{1,K}(py) + Bh_{1,K}(py) + n\log p.$$

Thus, $\log\det\nabla^2 h_{p,K}(y) + pBh_{p,K}(y)$ is convex if and only if $\log\det\nabla^2 h_{1,K}(y) + Bh_{1,K}(y)$ is.

6C. ***A suboptimal bound.*** We prove Theorem 1.11, i.e., we show that $\log\det\nabla^2 h_{p,K} + p(n+1)h_{p,K}$ is convex. Corollary 1.12 then follows from Theorem 1.9(ii).

Proof of Corollary 1.12. By Theorems 1.9(ii) and 1.11,

$$\mathcal{C}(K) \geq \frac{\mathcal{M}(K)}{e^{2n}(n+1)^n} \geq \left(\frac{\pi}{2e^2}\right)^n \frac{1}{(n+1)^n}.$$

By tensorization (replacing $(n+1)^{-n}$ by n^{-n} [Mastrantonis and Rubinstein 2022, Appendix A]),

$$\mathcal{C}(K) \geq \left(\frac{\pi}{2e^2 n}\right)^n. \qquad \square$$

Proof of Theorem 1.11. Recall by the proof of Lemma 6.3,

$$\nabla^2 h_{p,K}(y) = p\left(\int_K x_i x_j \, \mathrm{d}\nu^y(x) - \int_K x_i \, \mathrm{d}\nu^y(x) \int_K x_j \, \mathrm{d}\nu^y(x)\right)_{i,j}, \tag{6-9}$$

where

$$\mathrm{d}\nu^y(x) := \frac{e^{p\langle x,y\rangle}}{\int_K e^{p\langle x,y\rangle} \frac{\mathrm{d}x}{|K|}} \frac{\mathbf{1}_K^\infty(x)\mathrm{d}x}{|K|},$$

a probability measure that depends on y. Consider the $(n+1)\times(n+1)$ matrix

$$M := \begin{bmatrix} 1 & \int_K x_1 \, \mathrm{d}\nu^y(x) \ \cdots \ \int_K x_n \, \mathrm{d}\nu^y(x) \\ \int_K x_1 \, \mathrm{d}\nu^y(x) & \\ \vdots & \left[\int_K x_i x_j \, \mathrm{d}\nu^y(x)\right]_{i,j=1}^n \\ \int_K x_n \, \mathrm{d}\nu^y(x) & \end{bmatrix}.$$

By row reduction and (6-9),

$$\det M = p^{-n} \det\nabla^2 h_{p,K}.$$

Note that for $i, j \in \{0, 1, \ldots, n\}$, we have $M_{ij} = \langle x_i, x_j\rangle_{L^2(\mathrm{d}\nu^y)}$, where $x_0 = 1$. For

$$\Delta(x^{(0)}, \ldots, x^{(n)}) := \det \begin{bmatrix} 1 & x_1^{(0)} & \cdots & x_n^{(0)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_1^{(n)} & \cdots & x_n^{(n)} \end{bmatrix},$$

by Andréief's formula [Forrester 2019, (1.7)],

$$\begin{aligned}\det M &= \frac{1}{(n+1)!}\int_{K^{n+1}}\left(\det\begin{bmatrix}1 & x_1^{(0)} & \cdots & x_n^{(0)}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & x_1^{(n)} & \cdots & x_n^{(n)}\end{bmatrix}\right)^2 \mathrm{d}\nu^y(x^{(0)})\cdots \mathrm{d}\nu^y(x^{(n)})\\ &= \frac{1}{(n+1)!}\int_{K^{n+1}}|\Delta|^2\frac{e^{p\langle x^{(0)},y\rangle}}{\int_K e^{p\langle x^{(0)},y\rangle}\,\mathrm{d}x^{(0)}}\,\mathrm{d}\lambda(x^{(0)})\cdots\frac{e^{p\langle x^{(n)},y\rangle}}{\int_K e^{p\langle x^{(n)},y\rangle}\,\mathrm{d}x^{(n)}}\,\mathrm{d}\lambda(x^{(n)})\\ &= \frac{1}{(n+1)!}\frac{1}{\left(\int_K e^{p\langle x,y\rangle}\,\mathrm{d}x\right)^{n+1}}\int_{K^{n+1}}|\Delta|^2 e^{p\langle\sum_{j=0}^n x^{(j)},y\rangle}\,\mathrm{d}x^{(0)}\cdots\mathrm{d}x^{(n)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\log\det\nabla^2 h_{p,K}(y) &= n\log p+\log\det M\\ &= n\log p-\log(n+1)!-(n+1)\log\int_K e^{p\langle x,y\rangle}\,\mathrm{d}x+\log\phi(y)\\ &= n\log p-\log(n+1)!-p(n+1)h_{p,K}(y)+\log\phi(y),\end{aligned}$$

where

$$\phi(y):=\int_{K^{n+1}}|\Delta|^2 e^{p\langle\sum_{j=0}^n x^{(j)},y\rangle}\,\mathrm{d}x^{(0)}\cdots\mathrm{d}x^{(n)}.$$

Since $\log\phi$ is convex (Lemma 6.16 below), and

$$\log\det\nabla^2 h_{p,K}(y)+p(n+1)h_{p,K}(y)=n\log p-\log(n+1)!+\log\phi(y),$$

the claim follows. □

Lemma 6.16. *Let $K\subset\mathbb{R}^n$ be a convex body, $m\in\mathbb{N}$, and $f:\mathbb{R}^{nm}\to[0,+\infty)$ a measurable function. Then,*

$$\phi(y):=\log\int_{K^m} f(x_1,\dots,x_m)e^{\langle x_1+\cdots+x_m,y\rangle}\,\mathrm{d}x_1\cdots dx_m,\quad y\in\mathbb{R}^n,$$

is convex.

Proof. Write $x=(x_1,\dots,x_m)\in\mathbb{R}^{nm}$ and let $\lambda\in(0,1)$, $y_1,y_2\in\mathbb{R}^n$. Since

$$f(x)e^{\langle x_1+\cdots+x_m,(1-\lambda)y_1+\lambda y_2\rangle}=(f(x)e^{\langle x_1+\cdots+x_m,y_1\rangle})^{1-\lambda}(f(x)e^{\langle x_1+\cdots+x_m,y_2\rangle})^{\lambda},$$

by Hölder's inequality for $p=\frac{1}{1-\lambda}$ and $q=\frac{1}{\lambda}$,

$$\begin{aligned}\int_{K^m} f(x)e^{\langle x_1+\cdots+x_m,(1-\lambda)y_1+\lambda y_2\rangle}\,\mathrm{d}x \le &\left(\int_{K^m} f(x)e^{\langle x_1+\cdots+x_m,y_1\rangle}\,\mathrm{d}x\right)^{1-\lambda}\\ &\left(\int_{K^m} f(x)e^{\langle x_1+\cdots+x_m,y_2\rangle}\,\mathrm{d}x\right)^{\lambda}.\end{aligned}$$

Taking logarithms yields $\phi((1-\lambda)y_1+\lambda y_2)\le(1-\lambda)\phi(y_1)+\lambda\phi(y_2)$. □

6D. ***More general probability measures and sharpness of $B = n+1$.*** As just discussed, Theorem 1.11 falls short of proving the slicing conjecture because the best constant B we currently obtain is $n+1$. It is interesting to note that while in the setting of the uniform measure on K this constant could potentially be improved, many of the results in this section extend to general probability measures and then the constant $n+1$ is in fact *optimal*. The purpose of this subsection is to spell this out.

Throughout this section the only properties of the measure

$$\mathbf{1}_K^\infty \frac{\mathrm{d}x}{|K|}$$

used to obtain the estimates in Lemmas 6.10 and 6.13 were that it is a probability measure that is supported on K. As a result, it may be replaced by any probability measure

$$\mu$$

that is supported on K, i.e., for any measurable $A \subset \mathbb{R}^n \setminus K$, $\mu(A) = 0$, so that, in addition, $\operatorname{co}\operatorname{supp}(\mu) = K$. For example, (6-2) was already obtained for any probability measure with barycenter at the origin. For a convex body $K \subset \mathbb{R}^n$ and a probability measure μ whose convex hull of its support is K, let

$$h_{p,\mu}(y) := \frac{1}{p} \log \int_K e^{p\langle x,y\rangle}\, \mathrm{d}\mu(x).$$

As in Lemma 6.3,

$$\frac{1}{p}\nabla^2 h_{p,\mu}(0) = \operatorname{Cov}(\mu) := \left[\int_K x_i x_j \,\mathrm{d}\mu(x) - \int_K x_i \,\mathrm{d}\mu(x) \int_K x_j \,\mathrm{d}\mu(x)\right]_{i,j=1}^n.$$

For $p > 0$, let

$$\nu_{p,\mu} := \frac{e^{-ph_{1,\mu}(y)}\mathrm{d}y}{\int_{\mathbb{R}^n} e^{-ph_{1,\mu}(y)}\,\mathrm{d}y}. \tag{6-10}$$

Then, Claim 6.11, Lemmas 6.10 and 6.13 generalize.

Lemma 6.17. *Let $p > 0$. For a convex body $K \subset \mathbb{R}^n$, μ a probability measure with* $\operatorname{co}\operatorname{supp}(\mu) = K$, *and $\nu_{p,\mu}$ as in* (6-10):

(i) *We have*

$$\int_{\mathbb{R}^n} \log\det \nabla^2 h_{1,\mu}(y)\,\mathrm{d}\nu_{p,\mu}(y) \le \log\left(|K| \frac{\int_{\mathbb{R}^n} e^{-2ph_{1,\mu}(y)}\,\mathrm{d}y}{\left(\int_{\mathbb{R}^n} e^{-ph_{1,\mu}(y)}\,\mathrm{d}y\right)^2}\right).$$

(ii) *If $b(\nu_{p,\mu}) = 0$, then*

$$\int_{\mathbb{R}^n} \log\det \nabla^2 h_{1,\mu}(y)\,\mathrm{d}\nu_{p,\mu}(y) \le \log\left(\frac{|K|e^n}{\int_{\mathbb{R}^n} e^{-ph_{1,\mu}(y)}\,\mathrm{d}y}\right).$$

(iii) *If $b(\mu) = 0$, then*

$$\int_{\mathbb{R}^n} \log\det \nabla^2 h_{1,\mu}(y)\,\mathrm{d}\nu_{p,\mu}(y) \le \log\left(\frac{|K|}{\int_{\mathbb{R}^n} e^{-ph_{1,\mu}(y)}\,\mathrm{d}y}\right).$$

Lemma 6.18. *Let $p > 0$. For a convex body $K \subset \mathbb{R}^n$ and a probability measure μ with* $\operatorname{co}\operatorname{supp}(\mu) = K$,

$$\int_{\mathbb{R}^n} h_{1,\mu}(y)\,\mathrm{d}\nu_{p,\mu}(y) \le \frac{n}{p}.$$

Theorem 1.11 also generalizes.

Theorem 6.19. *Let $p > 0$. For a probability measure μ on $\mathbb{R}^n$ such that $\overline{\operatorname{supp}(\mu)}$ is a convex body, the function*

$$\log\det\nabla^2 h_{p,\mu}(y) + p(n+1)h_{p,\mu}(y)$$

is convex.

In fact, Theorem 6.19 is sharp: the next example shows $B = n+1$ cannot be improved.

Example 6.20. Consider

$$\mu := \frac{\delta_0 + \delta_{e_1} + \cdots + \delta_{e_n}}{n+1},$$

the probability measure on the standard simplex Δ_n that assigns mass $\frac{1}{n+1}$ to each vertex. Then,

$$\log\det\nabla^2 h_{p,\mu}(y) + pBh_{p,\mu}(y)$$

is convex if and only if $B \ge n+1$. To see this, compute

$$h_{p,\mu}(y) = \frac{1}{p}\log\int_{\Delta_n} e^{p\langle x,y\rangle}\,\mathrm{d}\mu(x) = \frac{1}{p}\log\frac{1+e^{py_1}+\cdots+e^{py_n}}{n+1}.$$

For the gradient, by the chain rule,

$$\begin{aligned}\frac{\partial h_{p,\mu}}{\partial y_i}(y) &= \frac{1}{p}\frac{n+1}{1+e^{py_1}+\cdots+e^{py_n}}\frac{\partial}{\partial y_i}\left(\frac{1+e^{py_1}+\cdots+e^{py_n}}{n+1}\right)\\ &= \frac{e^{py_i}}{1+e^{py_1}+\cdots+e^{py_n}}.\end{aligned} \tag{6-11}$$

Thus

$$\nabla h_{p,\mu}(y) = \frac{(e^{py_1},\dots,e^{py_n})}{1+e^{py_1}+\cdots+e^{py_n}}.$$

For the Hessian, by the quotient rule on (6-11),

$$\begin{aligned}\frac{\partial^2 h_{p,\mu}}{\partial y_i\,\partial y_j}(y) &= \frac{pe^{py_i}\delta_{ij}(1+e^{py_1}+\cdots+e^{py_n}) - e^{py_i}pe^{py_j}}{(1+e^{py_1}+\cdots+e^{py_n})^2}\\ &= \frac{p}{1+e^{py_1}+\cdots+e^{py_n}}\left(\delta_{ij}e^{py_i} - \frac{e^{p(y_i+y_j)}}{1+e^{py_1}+\cdots+e^{py_n}}\right).\end{aligned}$$

Thus

$$\begin{aligned}\nabla^2 h_{p,\mu}(y) &= \frac{p}{1+e^{py_1}+\cdots+e^{py_n}}\left[\delta_{ij}e^{py_i} - \frac{e^{p(y_i+y_j)}}{1+e^{py_1}+\cdots+e^{py_n}}\right]_{i,j=1}^n\\ &= \frac{p}{1+e^{py_1}+\cdots+e^{py_n}}(D - aa^T),\end{aligned}$$

where $D = \operatorname{diag}(e^{py_1}, \dots, e^{py_n})$ and $a = (1 + e^{py_1} + \cdots + e^{py_n})^{-1/2}(e^{py_1}, \dots, e^{py_n})$. Therefore,

$$\begin{aligned}\det \nabla^2 h_{p,\mu}(y) &= \frac{p^n}{(1+e^{y_1}+\cdots+e^{py_n})^n}\det(D - aa^T)\\ &= \frac{p^n}{(1+e^{y_1}+\cdots+e^{py_n})^n}(1-\langle D^{-1}a, a\rangle)\det D\\ &= \frac{p^n}{(1+e^{y_1}+\cdots+e^{py_n})^n}\left(1 - \frac{e^{py_1}+\cdots+e^{py_n}}{1+e^{py_1}+\cdots+e^{py_n}}\right)e^{py_1+\cdots+py_n}\\ &= \frac{p^n}{(1+e^{y_1}+\cdots+e^{py_n})^{n+1}}e^{py_1+\cdots+py_n}. \end{aligned} \tag{6-12}$$

Here we used the fact that, for $u, v \in \mathbb{R}^n$, $\det(I - uv^T) = 1 - \langle u, v\rangle$, which follows from row reduction

$$\det\begin{pmatrix}1 & 0^T\\ 0 & I+xy^T\end{pmatrix} = \det\begin{pmatrix}1 & y^T\\ 0 & I+xy^T\end{pmatrix} = \det\begin{pmatrix}1 & y^T\\ -x & I\end{pmatrix} = \det\begin{pmatrix}1+\langle x,y\rangle & 0^T\\ -x & I\end{pmatrix}.$$

As a result, by (6-12),

$$\begin{aligned}\log\det\nabla^2 h_{p,\mu}(y) + pBh_{p,\mu}(y) &= n\log(p) + py_1 + \cdots + py_n - (n+1)\log(1+e^{py_1}+\cdots+e^{py_n})\\ &\qquad + B\log\frac{1+e^{py_1}+\cdots+e^{py_n}}{n+1}\\ &= (B-n-1)\log(1+e^{py_1}+\cdots+e^{py_n})\\ &\qquad + py_1+\cdots+py_n + n\log p - B\log(n+1),\end{aligned}$$

which is convex if and only if $B \geq n+1$ (because $\log(1+e^{py_1}+\cdots+e^{py_n})$ is convex).

When $B = n+1$ and $p = 1$ we get

$$\log\det\nabla^2 h_{1,\mu} + (n+1)h_{1,\mu} = y_1 + \cdots + y_n - (n+1)\log(n+1),$$

so that $h_{1,\mu}$ solves the Monge–Ampère equation

$$\det\nabla^2 h_{1,\mu}(y) = \frac{1}{(n+1)^{n+1}}e^{-(n+1)h_{1,\mu}}e^{y_1+\cdots+y_n}.$$

From here we can read off that

$$\det\nabla^2 h_{1,\mu}(0) = \frac{1}{(n+1)^{n+1}}.$$

We next look at a generalized isotropic constant, by defining

$$\mathcal{C}(\mu) := \frac{|K|^2}{\det\nabla^2 h_{1,\mu}(0)}. \tag{6-13}$$

From the previous equation we then get, remembering that the volume of the unit simplex is $1/n!$, that

$$\mathcal{C}(\mu) = \frac{(n+1)^{n+1}}{(n!)^2}.$$

The right-hand side here is of the order of magnitude $c^n n^{-n}$, so we see that the "suboptimal" bound of Corollary 1.12 *is optimal in this generality.*

Interpreted benevolently, Example 6.20 means that our method is optimal in the sense that the best possible choice of B gives the correct estimate for $\mathcal{C}(\mu)$. The natural question then arises, for which measures μ the constant B can be taken smaller so that we as a consequence get a better estimate of $\mathcal{C}(\mu)$. One simple case when this is so is when μ is divisible, in the sense that we can write

$$\mu = \nu \star \nu \star \cdots \star \nu = \nu^{k\star}$$

as the k-fold convolution of another probability measure ν with itself. In that case,

$$h_{1,\mu} = k h_{1,\nu}.$$

Applying Theorem 6.19 to $h_{1,\nu}$ we then get that

$$\log\det\nabla^2 h_{1,\nu} + (n+1)h_{1,\nu}$$

is convex, which implies that

$$\log\det\nabla^2 h_{1,\mu} + \frac{n+1}{k}h_{1,\mu}$$

is convex. This leads to the improved estimate

$$\mathcal{C}(\mu) \geq c^n\left(\frac{k}{n}\right)^n.$$

This is however not so impressive since the same conclusion can be drawn directly from $\mathcal{C}(\nu) \geq c^n/n^n$ if we note that the convex hull of the support of ν is $\frac{K}{k}$. This way we also see that it is not really necessary that μ can be written $\nu^{k\star}$; it is enough that $\mu = f\nu^{k\star}$, where f is bounded.

6E. ***A complex geometric approach to Theorems 1.11 and 6.19.*** In this section we outline a different proof of Theorem 1.11 (and of its generalization, Theorem 6.19) which is a little more conceptual, but presupposes a bit of complex geometry. It is based on a theorem by S. Kobayashi [1959, Theorem 4.4]. Kobayashi's theorem deals with L^2 spaces of holomorphic $(n,0)$-forms on complex manifolds, but his proof goes through in a much more general setting and applies in particular to the setting we will now describe.

Let μ be a compactly supported probability measure on $\mathbb{R}^n$. Let

$$H_\mu := \left\{ \tilde{f}(z) := \int_{\mathbb{R}^n} e^{\frac{1}{2}\langle z,t\rangle} f(t)\,\mathrm{d}\mu(t), z \in \mathbb{C}^n : f \in L^2(\mu) \right\}.$$

H_μ is a space of entire functions on $\mathbb{C}^n$ and we give it an inner product

$$\langle \tilde{f}, \tilde{g}\rangle := \int f(t)\overline{g(t)}\,\mathrm{d}\mu(t), \tag{6-14}$$

making H_μ a Hilbert space, isomorphic to $L^2(\mu)$.

We require that μ is not supported in any proper linear subspace of $\mathbb{R}^n$. This implies that for any $a \in \mathbb{R}^n$ there is a function f such that

$$\int f\,\mathrm{d}\mu = 0 \quad \text{and} \quad \int \langle a,t\rangle f(t)\,\mathrm{d}\mu(t) \neq 0.$$

Indeed, if this were not the case, any function orthogonal to 1 in $L^2(\mu)$ would also be orthogonal to $\langle a,t\rangle$, which would imply that $\langle a,t\rangle = c$ on the support on μ, contrary to assumption. In terms of functions in H_μ, this says that there is a function $\tilde{f}$ which vanishes at the origin, with $\sum a_j \partial_j f$ not vanishing there. Then, replacing f by $e^{\langle z_0,t\rangle/2} f(t)$ we see that the same thing goes for any point z_0 in $\mathbb{C}^n$. This means that the conditions A.1 and A.2 in [Kobayashi 1959, pp. 271–2] are satisfied (we will see the relevance of this shortly). Kobayashi's condition A.1 says that for any point in $\mathbb{C}^n$ there is a function in H_μ that does not vanish there — this is trivial in our case. Indeed, for $z_0 \in \mathbb{C}^n$, since μ is compactly supported, $e^{-\langle z_0,t\rangle} \in L^2(\mu)$, and

$$\int e^{-\langle z_0,t\rangle} e^{\langle z_0,t\rangle}\, \mathrm{d}\mu(t) = \int \mathrm{d}\mu(t) = 1,$$

because μ is a probability measure.

The (diagonal) Bergman kernel for H_μ is defined as

$$B_\mu(z) := \sup_{\{\tilde{f}\in H_\mu : \|\tilde{f}\|=1\}} |\tilde{f}(z)|^2.$$

By condition A.1, the Bergman kernel does not vanish anywhere. It follows directly from the definitions that for

$$\mathcal{K}_\mu(z,w) := \int e^{\langle \frac{z+\bar{w}}{2},t\rangle}\, \mathrm{d}\mu(t),$$

and $\tilde{f}(z) = \int e^{\langle z,t\rangle/2} f(t)\, \mathrm{d}\mu(t) \in H_\mu$, by (6-14),

$$\langle \tilde{f}, \mathcal{K}_\mu(\cdot\,,w)\rangle = \int f(t)\overline{e^{\frac{1}{2}\langle \bar{w},t\rangle}}\, \mathrm{d}\mu(t) = \int f(t) e^{\frac{1}{2}\langle w,t\rangle}\, \mathrm{d}\mu(t) = \tilde{f}(w),$$

i.e., $\mathcal{K}_\mu$ enjoys a reproducing property, in addition to being holomorphic in the first variable and antiholomorphic in the second. These three properties characterize Bergman kernels [Mastrantonis and Rubinstein 2022, §3.2]; thus $\mathcal{K}_\mu$ is the Bergman kernel of H_μ. Therefore, on the diagonal, if $z = x + iy$,

$$B_\mu(z) = \mathcal{K}_\mu(z,z) = \int e^{\langle x,t\rangle}\, \mathrm{d}\mu(t),$$

i.e., coming back full circle to the ideas in Section 1A,

$$\log B_\mu = h_{1,\mu}. \tag{6-15}$$

The Bergman metric associated to H_μ is the Kähler metric on $\mathbb{C}^n$ defined by

$$g_{j\bar{k}} := \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log B_\mu.$$

By (6-15), $\log B_\mu$ is convex in x, hence plurisubharmonic in z, and the matrix $g = [g_{j\bar{k}}]$ is positive semidefinite, and it is a standard fact (that we omit) that the condition A.2 is precisely what is needed to make sure it is strictly positive definite. (Alternatively, condition A.2 can verified by using (6-15), the computation of Lemma 6.3, and the Cauchy–Schwarz inequality to note that $h_{p,\mu}$ is strongly convex.) Kobayashi's theorem says that the Ricci curvature $\operatorname{Ric} g$ of the Bergman metric is bounded from above by $(n+1)g$.

At this point we need to make use of a standard formula for the Ricci curvature, valid for any Kähler metric. Let

$$\Delta := \det[g_{j\bar{k}}]$$

be the density of the volume form of the metric g. Then the Ricci curvature form of g is given by

$$R_{j\bar{k}} = -\frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log \Delta.$$

Hence, Kobayashi's estimate

$$[R_{jk}] \le (n+1)[g_{jk}]$$

translates to saying that

$$\log \Delta + (n+1) \log B_\mu$$

is plurisubharmonic. In our case, B_μ and $\log \Delta$ depend only on $x = \mathrm{Re}(z)$, so

$$\log \Delta + (n+1) \log B_\mu$$

is actually a convex function of x. Moreover, $\log B_\mu = h_{1,\mu}$ and $\Delta = 4^{-n} \det \nabla^2 h_{1,\mu}$ (in the last equality we used the relation between the complex Hessian and the real one on functions depending only on the real part). Therefore

$$\log\det \nabla^2 h_{1,\mu} + (n+1) h_{1,\mu}$$

is convex, i.e., $(*_B)$ holds with $B = n+1$, so we have proved Theorem 6.19, and, in particular, also Theorem 1.11.

Appendix: A (near) norm associated to a convex function

In this section we give proofs for Proposition A.1 [Ball 1988, Theorem 5] and Theorem 5.20 [Ball 1986, Theorem 4.10] (cf. [Busemann 1949; Milman and Pajor 1989, p. 90]). Let us start by using Theorem 5.20 to prove Proposition A.1.

Proposition A.1. *For a convex function $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ with $0 < \int_{\mathbb{R}^n} e^{-\phi} < \infty$,*

$$x \mapsto \left(\int_0^\infty r^{n-1} e^{-\phi(rx)}\,\mathrm{d}r \right)^{-\frac{1}{n}} \tag{A-1}$$

is positively 1-homogeneous and subadditive (it is also a norm if ϕ is, in addition, even), and

$$\frac{1}{n} \int_{\mathbb{R}^n} e^{-\phi(x)}\,\mathrm{d}x = |\{x \in \mathbb{R}^n : \|x\|_\phi \le 1\}|.$$

Proof of Proposition A.1. 1*-homogeneity*. Let $x \in \mathbb{R}^n$ and $\lambda > 0$. By changing variables $\rho = \lambda r$,

$$\begin{aligned}\|\lambda x\|_\phi &:= \left(\int_0^\infty r^{n-1} e^{-\phi(r\lambda x)}\,\mathrm{d}r \right)^{-\frac{1}{n}} = \left(\int_0^\infty \frac{\rho^{n-1}}{\lambda^{n-1}} e^{-\phi(\rho x)} \frac{\mathrm{d}\rho}{\lambda} \right)^{-\frac{1}{n}} \\ &= \lambda \left(\int_0^\infty \rho^{n-1} e^{-\phi(\rho x)}\,\mathrm{d}\rho \right)^{-\frac{1}{n}} = \lambda \|x\|_\phi.\end{aligned}$$

Positivity of λ is used in the last step.

Subadditivity. Let $x, y \in \mathbb{R}^n$ and $r, t, s > 0$ with $\frac{1}{r} = \frac{1}{2}\big(\frac{1}{t} + \frac{1}{s}\big)$, or equivalently,

$$\frac{r}{2t} + \frac{r}{2s} = 1. \tag{6-2}$$

By (6-2) and convexity of ϕ,

$$\phi(r(x+y)) = \phi\left(\frac{r}{2t}2tx + \frac{r}{2s}2sy\right) \le \frac{r}{2t}\phi(2tx) + \frac{r}{2s}\phi(2sy) = \frac{s}{t+s}\phi(2tx) + \frac{t}{t+s}\phi(2sy). \tag{6-3}$$

Set

$$H(r) := e^{-\phi(r(x+y))}, \quad F(t) := e^{-\phi(2tx)}, \quad G(s) := e^{-\phi(2sy)}.$$

By (6-3), $H(r) \ge F(t)^{s/(t+s)}G(s)^{t/(t+s)}$, so by Theorem 5.20 (with $q = n$),

$$\begin{aligned}\|x+y\|_\phi = \left(\int_0^\infty r^{n-1}e^{-H(r)}\,\mathrm{d}r\right)^{-\frac{1}{n}} &\le \frac{1}{2}\left(\int_0^\infty r^{n-1}e^{-F(t)}\,\mathrm{d}t\right)^{-\frac{1}{n}} + \frac{1}{2}\left(\int_0^\infty r^{n-1}e^{-G(s)}\,\mathrm{d}s\right)^{-\frac{1}{n}}\\ &= \frac{1}{2}\|2x\|_\phi + \frac{1}{2}\|2y\|_\phi = \|x\|_\phi + \|y\|_\phi,\end{aligned}$$

using the already established homogeneity of $\|\cdot\|_\phi$.

Volume equality. By (3-2),

$$|\{x \in \mathbb{R}^n : \|x\|_\phi \le 1\}| = \frac{1}{n}\int_{\partial B_2^n} \frac{\mathrm{d}u}{\|u\|_\phi^n} = \frac{1}{n}\int_{\partial B_2^n}\int_0^\infty r^{n-1}e^{-\phi(ru)}\,\mathrm{d}r\,\mathrm{d}u. \tag{6-4}$$

Using polar coordinates this is $\frac{1}{n}\int_{\mathbb{R}^n} e^{-\phi(x)}\,\mathrm{d}x$.

Norm. Assuming in addition that ϕ is even, for $x \in \mathbb{R}^n$,

$$\|-x\|_\phi = \left(\int_0^\infty r^{n-1}e^{-\phi(-rx)}\,\mathrm{d}r\right)^{-\frac{1}{n}} = \left(\int_0^\infty r^{n-1}e^{-\phi(rx)}\,\mathrm{d}r\right)^{-\frac{1}{n}} = \|x\|_\phi.$$

Therefore, for $\lambda \in \mathbb{R}$, $\|\lambda x\|_\phi = \||\lambda|x\|_\phi = |\lambda|\,\|x\|_\phi$, making $\|\cdot\|_\phi$ into a norm. This concludes the proof of Proposition A.1. □

Next, we turn to proving Theorem 5.20. The proof involves three auxiliary lemmas. To begin with, invert the variables; for $t, s, r > 0$, let

$$u := \frac{1}{t}, \quad v := \frac{1}{\theta s}, \quad \text{and} \quad w := \frac{1}{r}$$

for some $\theta > 0$ to be chosen later. In the inverted coordinates, the condition $\frac{2}{r} = \frac{1}{t} + \frac{1}{s}$ becomes $w = \frac{u+\theta v}{2}$. Now, let

$$A(u) := F(u^{-1})u^{-(q+1)}, \quad B(v) := G(\theta^{-1}v^{-1})v^{-(q+1)} \tag{6-5}$$

and

$$C(w) := \left(\frac{\theta+1}{2}\right)^{q+1} H(w^{-1})w^{-(q+1)}. \tag{6-6}$$

The reason behind the multiplication by $\big(\frac{\theta+1}{2}\big)^{q+1}$ will become apparent in the next lemma that translates the (5-16) relation between F, G and H to one between A, B and C.

Lemma 6.2. *Let F, G, H as in Theorem 5.20, and $\theta > 0$. For A, B and C as in* (6-5)–(6-6),

$$C\left(\frac{u+\theta v}{2}\right) \geq A(u)^{\frac{u}{u+\theta v}} B(v)^{\frac{\theta v}{u+\theta v}} \quad \textit{for all } u, v > 0.$$

A straightforward change of variables expresses the integrals of F, G, and H in terms of integrals of A, B, and C:

Lemma 6.3. *Let F, G, H as in Theorem 5.20 and $\theta > 0$. For A, B and C as in* (6-5) *and* (6-6),

$$\int_0^\infty A(u)\,\mathrm{d}u = \int_0^\infty t^{q-1} F(t)\,\mathrm{d}t,$$
$$\int_0^\infty B(v)\,\mathrm{d}v = \theta^q \int_0^\infty s^{q-1} G(s)\,\mathrm{d}s,$$
$$\int_0^\infty C(w)\,\mathrm{d}w = \left(\frac{\theta+1}{2}\right)^{q+1} \int_0^\infty r^{q-1} H(r)\,\mathrm{d}r.$$

The following is a standard reduction:

Lemma 6.4. *It is enough to prove Theorem 5.20 for F and G bounded.*

Before proving Lemmas 6.2–6.4, let us show how they imply Theorem 5.20. For a function $E : (0,\infty) \to [0,\infty)$, changing the order of integration,

$$\int_0^\infty E(u)\,\mathrm{d}u = \int_0^\infty \int_0^{E(u)} \mathrm{d}z\,\mathrm{d}u = \int_0^{\|E\|_\infty} \int_{\{u:E(u)\geq z\}} \mathrm{d}u\,\mathrm{d}z = \int_0^{\|E\|_\infty} |E \geq z|\,\mathrm{d}z, \tag{6-7}$$

where $\|E\|_\infty$ could potentially be infinite. Ball applies the 1-dimensional Brunn–Minkowski inequality to the sets $\{E \geq z\}$.

Proof of Theorem 5.20. Step 1: the setup. Let

$$a := \left(\int_0^\infty t^{q-1} F(t)\,\mathrm{d}t\right)^{\frac{1}{q}}, \quad b := \left(\int_0^\infty s^{q-1} G(s)\,\mathrm{d}s\right)^{\frac{1}{q}}, \quad c := \left(\int_0^\infty r^{q-1} H(r)\,\mathrm{d}r\right)^{\frac{1}{q}}.$$

The aim is to show $\frac{2}{c} \leq \frac{1}{a} + \frac{1}{b}$, or equivalently,

$$c \geq \frac{2ab}{a+b}. \tag{6-8}$$

By Lemma 6.3 and (6-7),

$$a^q = \int_0^\infty A(u)\,\mathrm{d}u = \int_0^{\|A\|_\infty} |A \geq z|\,\mathrm{d}z, \tag{6-9}$$

$$(\theta b)^q = \int_0^\infty B(v)\,\mathrm{d}v = \int_0^{\|B\|_\infty} |B \geq z|\,\mathrm{d}z, \tag{6-10}$$

$$\left(\frac{\theta+1}{2}\right)^{q+1} c^q = \int_0^\infty C(w)\,\mathrm{d}w = \int_0^{\|C\|_\infty} |C \geq z|\,\mathrm{d}z. \tag{6-11}$$

Step 2: comparing the superlevel sets. Lemma 6.2 allows us to compare the superlevel sets of A, B and C, obtaining an inequality between a, b and c. In particular,

$$\{C \geq z\} \supset \frac{1}{2}\{A \geq z\} + \frac{\theta}{2}\{B \geq z\}, \tag{6-12}$$

because for $u \in \{A \geq z\}$ and $v \in \{B \geq z\}$,

$$C\left(\frac{u+\theta v}{2}\right) \geq A(u)^{\frac{u}{u+\theta v}} B(v)^{\frac{\theta v}{u+\theta v}} \geq z^{\frac{u}{u+\theta v}} z^{\frac{\theta v}{u+\theta v}} = z,$$

i.e., $\frac{u+\theta v}{2} \in \{C \geq z\}$. By the 1-dimensional Brunn–Minkowski inequality,

$$|C \geq z| \geq \frac{1}{2}|A \geq z| + \frac{\theta}{2}|B \geq z|. \tag{6-13}$$

By (6-7) and (6-13),

$$\left(\frac{\theta+1}{2}\right)^{q+1} c^q = \int_0^{\|C\|_\infty} |C \geq z|\, \mathrm{d}z \geq \frac{1}{2}\int_0^{\|C\|_\infty} |A \geq z|\, \mathrm{d}z + \frac{\theta}{2}\int_0^{\|C\|_\infty} |A \geq z|\, \mathrm{d}z. \tag{6-14}$$

Step 3: choosing θ. By Lemma 6.2, $\|C\|_\infty \geq \min\{\|A\|_\infty, \|B\|_\infty\}$. In view of (6-9), (6-10) and (6-14), we would like $\|C\|_\infty \geq \max\{\|A\|_\infty, \|B\|_\infty\}$. The only way to achieve this is to have $\|A\|_\infty = \|B\|_\infty$. It is here that one needs to take F and G bounded so that $\|A\|_\infty$ and $\|B\|_\infty$ are finite. By Lemma 6.4, there is no loss in making such an assumption. Choosing

$$\theta := \left(\frac{\sup_{r>0} F(r) r^{q+1}}{\sup_{r>0} G(r) r^{q+1}}\right)^{\frac{1}{q+1}},$$

gives

$$\begin{aligned}\|A\|_\infty &= \sup_{r>0} F(r) r^{q+1} = \sup_{r>0} G(r)(\theta r)^{q+1} \\ &= \sup_{u>0} G(\theta^{-1}u^{-1}) u^{-(q+1)} = \|B\|_\infty.\end{aligned}$$

Step 4: finishing the proof. By Lemma 6.2 and the choice of θ, $\|C\|_\infty \geq \|A\|_\infty = \|B\|_\infty$. By (6-9), (6-10), and (6-14),

$$\left(\frac{\theta+1}{2}\right)^{q+1} c^q \geq \frac{1}{2}\int_0^{\|A\|_\infty} |A \geq z|\, \mathrm{d}z + \frac{\theta}{2}\int_0^{\|B\|_\infty} |B \geq z|\, \mathrm{d}z = \frac{a^q + \theta^{q+1} b^q}{2}.$$

That is,

$$c^q \geq \left(\frac{2}{\theta+1}\right)^q \left(\frac{1}{1+\theta} a^q + \frac{\theta}{1+\theta}(\theta b)^q\right) \geq \left(\frac{2}{\theta+1}\right)^q \left(\frac{1}{\theta+1} a + \frac{\theta}{\theta+1}\theta b\right)^q,$$

because for $q \geq 1$, $x \mapsto x^q$ is convex and hence

$$(1-\lambda)x^q + \lambda y^q \geq ((1-\lambda)x + \lambda y)^q$$

for all $x, y \geq 0$ and $\lambda \in [0,1]$. Finally,

$$\begin{aligned} c &\geq \frac{2(a+\theta^2 b)}{(\theta+1)^2} = \frac{2(a+b)(a+\theta^2 b)}{(a+b)(\theta+1)^2} = 2\frac{a^2+\theta^2 ab + ab + \theta^2 b^2}{(a+b)(\theta+1)^2} \\ &= 2\frac{(\theta^2+1)ab + a^2 + \theta^2 b^2}{(a+b)(\theta+1)^2} = 2\frac{(\theta+1)^2 ab - 2\theta ab + a^2 + \theta^2 b^2}{(a+b)(\theta+1)^2} \\ &= 2\frac{(\theta+1)^2 ab + (a-\theta b)^2}{(a+b)(\theta+1)^2} = \frac{2ab}{a+b} + \frac{2(a-\theta b)^2}{(a+b)(\theta+1)^2} \geq \frac{2ab}{a+b}, \end{aligned}$$

as desired. This concludes the proof of Theorem 5.20, modulo the proofs of Lemmas 6.2–6.4, which are given below. □

Proof of Lemma 6.2. For $t, s, r > 0$ with $\frac{2}{r} = \frac{1}{t} + \frac{1}{s}$, by assumption,

$$\begin{aligned} H(r) \geq F(t)^{\frac{s}{t+s}} G(s)^{\frac{t}{t+s}} &= \big(A(t^{-1})t^{-(q+1)}\big)^{\frac{s}{t+s}} \big(B(\theta^{-1}s^{-1})(\theta s)^{-(q+1)}\big)^{\frac{t}{t+s}} \\ &= A(u)^{\frac{s}{t+s}} B(v)^{\frac{t}{t+s}} \big(t^{\frac{s}{t+s}} (\theta s)^{\frac{t}{t+s}}\big)^{-(q+1)}. \end{aligned} \tag{6-15}$$

Since,

$$\frac{s}{t+s} = \frac{u}{u+\theta v} \quad \text{and} \quad \frac{t}{t+s} = \frac{\theta v}{u+\theta v},$$

by (6-15) and the weighted AM–GM,

$$\begin{aligned} A(u)^{\frac{u}{u+\theta v}} B(v)^{\frac{\theta v}{u+\theta v}} &= A(u)^{\frac{s}{t+s}} B(v)^{\frac{t}{t+s}} \leq H(r)(t^{\frac{s}{t+s}}(\theta s)^{\frac{t}{t+s}})^{q+1} \\ &\leq H(r)\left(\frac{ts}{t+s} + \frac{\theta st}{t+s}\right)^{q+1} = \left(\frac{\theta+1}{2}\right)^{q+1} H(r)\left(\frac{2ts}{t+s}\right)^{q+1} \\ &= \left(\frac{\theta+1}{2}\right)^{q+1} H(r) r^{p+1} = C\left(\frac{u+\theta v}{2}\right), \end{aligned} \tag{6-16}$$

because $\frac{1}{r} = \frac{1}{2t} + \frac{1}{2s} = \frac{1}{2}u + \frac{1}{2}(\theta v) = \frac{u+\theta v}{2}$. □

Proof of Lemma 6.3. By changing variables, $u = \frac{1}{t}$,

$$\int_0^\infty A(u)\,\mathrm{d}u = \int_0^\infty u^{-(q+1)} F(u^{-1})\,\mathrm{d}u = \int_0^\infty t^{q+1} F(t)\,\frac{\mathrm{d}t}{t^2} = \int_0^\infty t^{q-1} F(t)\,\mathrm{d}t.$$

For $v = \frac{1}{\theta s}$,

$$\int_0^\infty B(v)\,\mathrm{d}v = \int_0^\infty v^{-(q+1)} G(\theta^{-1}v^{-1})\,\mathrm{d}v = \int_0^\infty (\theta s)^{q+1} G(s)\,\frac{\mathrm{d}s}{\theta s^2} = \theta^q \int_0^\infty s^{q-1} G(s)\,\mathrm{d}s.$$

Finally, for $w = \frac{1}{r}$,

$$\begin{aligned} \int_0^\infty C(w)\,\mathrm{d}w &= \left(\frac{\theta+1}{2}\right)^{q+1} \int_0^\infty w^{-(q+1)} H(w^{-1})\,\mathrm{d}w \\ &= \left(\frac{\theta+1}{2}\right)^{q+1} \int_0^\infty r^{q+1} H(r)\,\frac{\mathrm{d}r}{r^2} = \left(\frac{\theta+1}{2}\right)^{q+1} \int_0^\infty r^{q-1} H(r)\,\mathrm{d}r. \end{aligned}$$

□

Proof of Lemma 6.4. For $m \in \mathbb{N}$, let

$$F_m(t) := F(t)\mathbf{1}_{\{F\le m\}}(t) \quad \text{and} \quad G_m(s) := G(s)\mathbf{1}_{\{G\le m\}}(s).$$

Then, F_m, G_m are both bounded by m. In addition, $F \ge F_m$ and $G \ge G_m$, thus for $t, s, r > 0$ with $\frac{2}{r} = \frac{1}{t} + \frac{1}{s}$,

$$H(r) \ge F(t)^{\frac{s}{t+s}} G(s)^{\frac{t}{t+s}} \ge F_m(t)^{\frac{s}{t+s}} G_m(s)^{\frac{t}{t+s}}.$$

Under the assumption that Theorem 5.20 holds for bounded functions,

$$2\left(\int_0^\infty r^{q-1} H(r)\,\mathrm{d}r\right)^{-\frac{1}{q}} \le \left(\int_0^\infty t^{q-1} F_m(t)\,\mathrm{d}t\right)^{-\frac{1}{q}} + \left(\int_0^\infty s^{q-1} G_m(s)\,\mathrm{d}s\right)^{-\frac{1}{q}}.$$

The claim follows from the monotone convergence theorem by taking $m \to \infty$. ☐

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