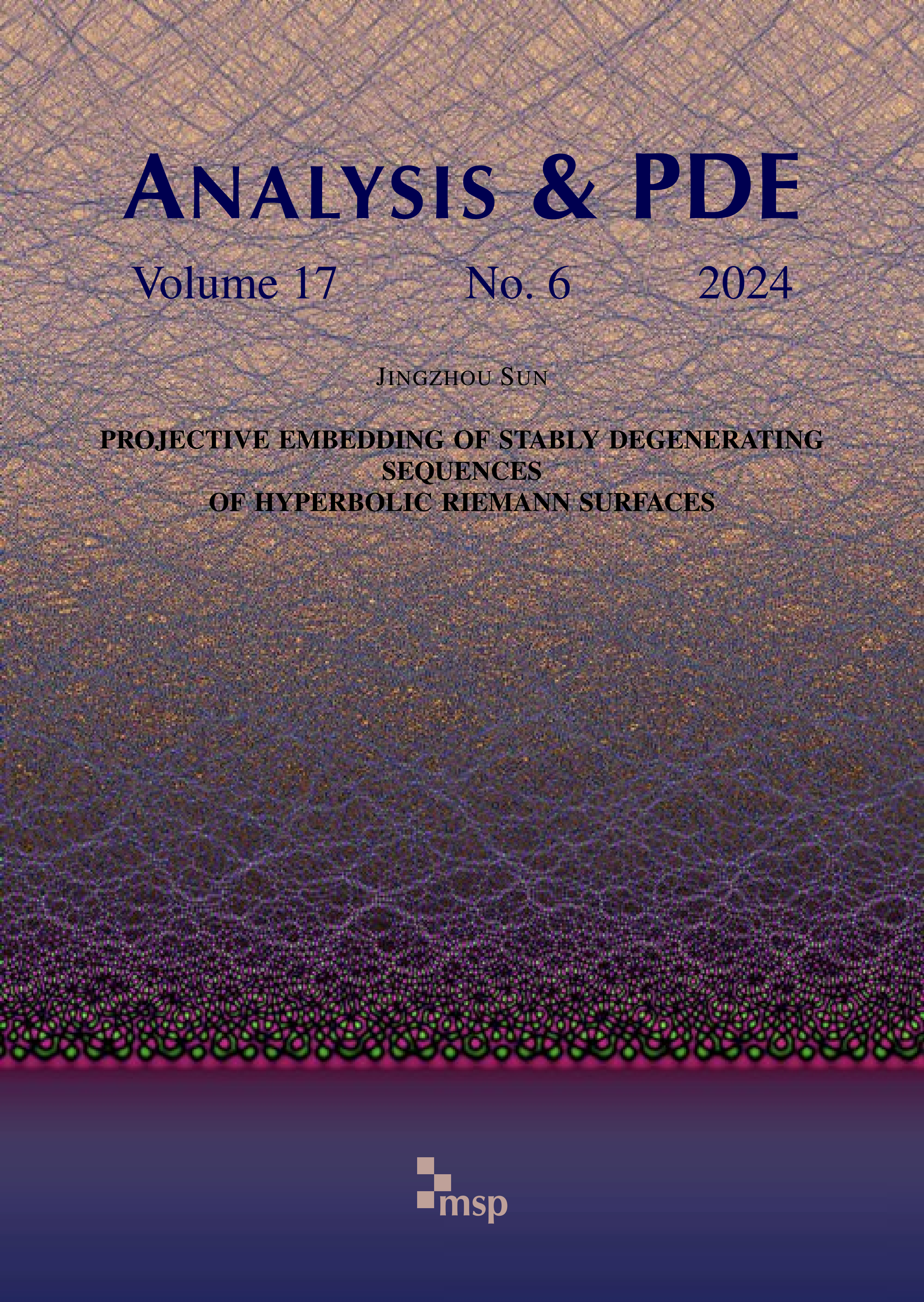

ANALYSIS & PDE

Volume 17 No. 6 2024

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ANALYSIS AND PDE
Vol. 17 (2024), No. 6, pp. 1871–1886
DOI: 10.2140/apde.2024.17.1871

PROJECTIVE EMBEDDING OF STABLY DEGENERATING SEQUENCES OF HYPERBOLIC RIEMANN SURFACES

JINGZHOU SUN

Given a sequence of genus $g \geq 2$ curves converging to a punctured Riemann surface with complete metric of constant Gaussian curvature -1, we prove that the Kodaira embedding using an orthonormal basis of the Bergman space of sections of a pluricanonical bundle also converges to the embedding of the limit space together with extra complex projective lines.

1. Introduction

Let $\mathcal{M}_g$ be the moduli of a smooth compact Riemann surfaces of genus g. When $g \geq 2$, the compactification $\overline{\mathcal{M}}_g$, due to [Deligne and Mumford 1969], is the moduli of stable curves. Each smooth curve of genus g carries a unique Poincaré metric with constant Gaussian curvature -1. If $C \in \overline{\mathcal{M}}_g$ is a singular stable curve, then by removing the nodes, the smooth part carries a unique complete hyperbolic metric with constant Gaussian curvature -1. And if a holomorphic family $\pi : \mathcal{C} \to \boldsymbol{D}$ of compact smooth curves C_t degenerates to $C = C_0$, then the hyperbolic metrics are continuous on the vertical line bundle [Wolpert 1990].

In this article, from the point view of the quantization framework in [Donaldson 2001; Donaldson and Sun 2014], we are interested in the convergence of the pluricanonical Bergman embeddings of the family of hyperbolic surfaces in the complex projective spaces. More precisely, let (C_j, g_j) be a sequence of genus $g \geq 2$ Riemann surfaces with Riemannian metric g_j of constant Gaussian curvature -1 that converges, in the pointed Gromov–Hausdorff topology, to a punctured Riemann surface (C_0, g_0) — not necessarily connected — with a complete Riemannian metric g_0 of constant Gaussian curvature -1. Let K_{C_j} denote the canonical bundle of C_j; then K_{C_j} is endowed with a Hermitian metric h_j defined by the Kähler form ω_j associated to g_j. We consider the Bergman space $\mathcal{H}_{j,k}$ consisting of L^2-integrable holomorphic sections of $K_{C_j}^k$. Then $\mathcal{H}_{j,k}$ is a finite-dimensional Hermitian space with the Hermitian product defined by

$$\langle s, t\rangle = \int_{C_j} (s, t)_{h_j} \omega_j,$$

where, by abuse of notation, we still use h_j to denote the induced Hermitian metric on $K_{C_j}^k$. For k large enough, a basis of $\mathcal{H}_{j,k}$ will induce a Kodaira embedding of C_j to $\mathbb{CP}^{N_k}$, where $N_k = \dim \mathcal{H}_{j,k} - 1$ is

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The author is partially supported by the NNSF of China (No. 11701353) and the STU Scientific Research Foundation for Talents (No. 130/760181).

MSC2020: primary 14H15, 30F10, 30F60; secondary 30F45.

Keywords: stable degeneration, projective embedding, hyperbolic metric.

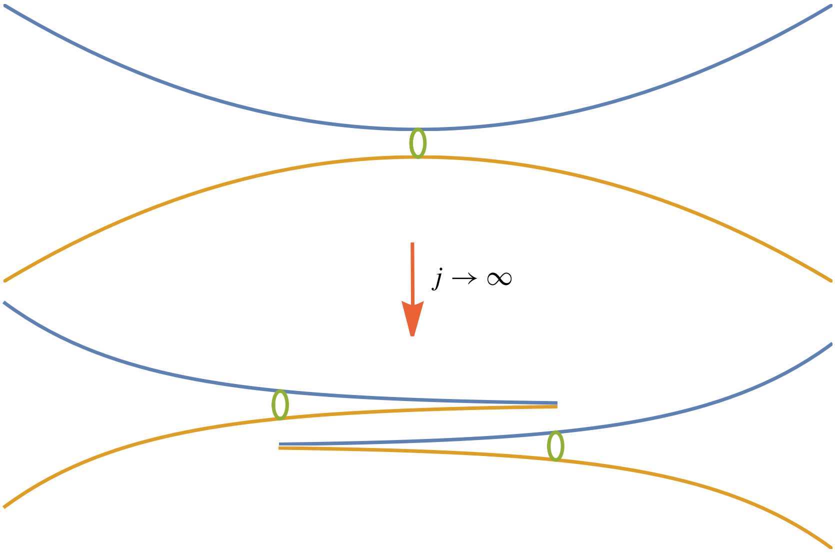

Figure 1. Degeneration of hyperbolic metrics.

independent of $j \geq 1$. For $j = 0$, the dimension of $\mathcal{H}_{j,k}$ is smaller than that of $j > 0$. It is natural to consider the embedding induced by an orthonormal basis for $\mathcal{H}_{j,k}$, which can be considered as a bridge from Kähler geometry to algebraic geometry [Donaldson and Sun 2017; Sun and Sun 2021]. It is worth mentioning that after this article was finished, the author learned that Dong [2023] recently proved that if a smooth family of hyperelliptic curves degenerate to a nodal curve, then their Bergman kernels also converge to the Bergman kernel of the nodal curve.

As the Gaussian curvature is -1, the degeneration of metrics can only be "pinching a nontrivial loop", namely a sequence of surfaces with increasingly thinner and longer handles, with the central loops degenerating to points. So C_0 has d pairs of punctures, which will be called ends. And for k large enough, the dimension of $\mathcal{H}_{0,k}$ equals $N_k + 1 - d$. Now we can state our main theorem.

Theorem 1.1. *For k large enough, we can choose an orthonormal basis for $\mathcal{H}_{j,k}$ for all $j \geq 0$, so that, as $j \to \infty$, the image of the embedding*

$$\Phi_{j,k} : C_j \to \mathbb{CP}^{N_k}$$

induced by the orthonormal basis converges to the image of C_0 under the embedding

$$\Phi_{0,k} : C_0 \to \mathbb{CP}^{N_k - d} \subset \mathbb{CP}^{N_k},$$

attached with d pairs of complex projective lines. To each pair of the ends $(p_\alpha, p_{\alpha+d})$ a pair of complex projective lines is associated, forming a connected chain between the images of these two points.

It is interesting to mention that during the process of taking the limit, the pair of complex projective lines are developed as a pair of bubbles. We illustrate this process in Figures 1 and 2. Also, we should mention that k depends only on the geometry of C_0 and does not need to be too big by the results in [Sun 2017].

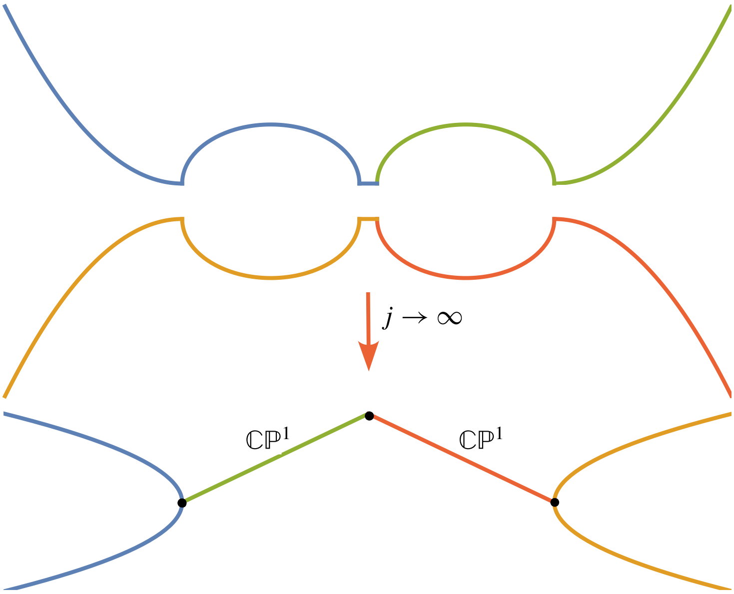

Figure 2. Degeneration in $\mathbb{CP}^{N_k}$.

The proof of this theorem makes heavy use of the methods we developed from [Sun and Sun 2021] to [Sun 2017]. Just as in [Donaldson and Sun 2014], the main point is basically proving the convergence of the Bergman kernels. We hope this result may shine a light on the study of the degeneration of higher-dimensional projective manifolds [Honda et al. 2019; Song 2017; Sun 2019].

The structure of this article is as follows. We will first quickly recall the necessary background for this article. Then we will calculate in the model for the thin handles, or "the collar", of the Riemann surfaces close to the limit. And in the end, we will finish the proof of the convergence of the pluricanonical Bergman embeddings.

2. Punctured model

On the punctured disk $\boldsymbol{D}^*$, the Poincaré metric is

$$\omega_P = \frac{2i\, dz \wedge d\bar{z}}{|z|^2 (\log |z|^2)^2}.$$

Taking the local section of the canonical bundle $e = dz/z$, the local potential is

$$\varphi_P = -\log |e|^2 = -\log \frac{(\log 1/|z|^2)^2}{2}.$$

We use the notation $\tau = -\log |z|$, which yields $\varphi_P = -\log(2\tau^2)$. We are interested in the L^2-norm of the sections $z^a e^{k+1}$ of $K_{\boldsymbol{D}^*}^{k+1}$, $a \in \mathbb{Z}^+$. So we have the integrals

$$Y_a = \int (2\tau^2)^{k+1} |z|^{2a} \omega_P.$$

Further, we have

$$Y_a = 2^{k+2}\pi \int_0^\infty e^{-2a\tau+2k\log\tau}\, d\tau.$$

Writing $g_a(\tau) = -2a\tau + 2k\log\tau$, we have $g_a''(t) = -2k/\tau^2$. So $g_a(\tau)$ is a concave function which attains its only maximum at $\tau_a = k/a$. We will use the following basic result from [Sun 2017, Lemma 2.6].

Lemma 2.1. *Let $f(x)$ be a concave function. Suppose $f'(x_0) < 0$, then we have*

$$\int_{x_0}^\infty e^{f(x)}\, dx \leq \frac{e^{f(x_0)}}{-f'(x_0)}.$$

We can use Laplace's method and the lemma above to estimate

$$Y_a \approx 2^{k+2}\pi e^{-2k+2k\log(k/a)}\sqrt{\frac{k\pi}{2a^2}}.$$

Of course, we can directly calculate the integral to get

$$Y_a = 2^{k+2}\pi\frac{(2k)!}{(2a)^{2k+1}},$$

but the idea of mass concentration is key to our arguments. The Bergman kernel of $\boldsymbol{D}^*$ is then

$$\rho_{0,k+1} = \frac{2^{2k}\tau^{2k+2}}{\pi(2k)!}\sum(a)^{2k+1}|z|^{2a}.$$

Let C_0 be a punctured Riemann surface obtained by removing $2d$ points $\{p_\alpha\}_{1\leq\alpha\leq 2d}$ from a compact Riemann surface. C_0 is endowed with a complete Poincaré metric ω with constant Gaussian curvature -1. The metric ω defines a Hermitian metric h on the canonical bundle K_{C_0}. Then, for any positive integer k, we denote by $\mathcal{H}_k$ the space of holomorphic sections of $K_{C_0}^k$ that are L^2-integrable, namely

$$\|s\|^2 = \int_{C_0} |s|_h^2\omega < \infty.$$

For each p_α, there is a neighborhood U_α with local coordinate z such that $\omega = \omega_P$ on $U_\alpha\backslash p_\alpha$. We can assume that U_α contains the points satisfying $|z| \leq R_\alpha$. We note that the injective radius at the points $|z| = R_\alpha$ is about $\pi/(4(\log R_\alpha)^2)$. For simplicity, we let R be the minimum of the R_α, $1 \leq \alpha \leq 2d$. Clearly, for the complement of $\bigcup_{1\leq\alpha\leq 2d} U_\alpha$ in C_0, there is a positive lower bound λ_0 for the injective radius.

Let ρ_{k+1} denote the Bergman kernel of C_0 for the Bergman space $\mathcal{H}_{k+1}$. The basic conclusion of [Sun 2017] (which is also an implication of [Sun 2013], although not explicitly stated there) is that, for k large enough, in the "inside" of U_α, where $\tau = -\log|z| > \sqrt{k+1}$, the Bergman kernel ρ_{k+1} is very much like $\rho_{0,k+1}$, meaning that $|\rho_{k+1}/\rho_{0,k+1} - 1|$ is $o(1/k^N)$ for all N. We have shown in [Sun 2013; 2017] that $\rho_{0,k+1}$ is dominated by the terms $c_a|z|^{2a}$ for $a < k^{3/4}$, meaning that

$$\frac{\rho_{0,k+1}}{\sum_{a=1}^{k^{3/4}} c_a|z|^{2a}} - 1 = o\left(\frac{1}{k^N}\right)$$

for all N. In particular, $\rho_{0,k+1}$ is dominated by $c_1|z|^2$ when $\tau \geq k$. The sections for C_0 corresponding to z^a in the model $\boldsymbol{D}^*$ is constructed as follows. We let z_α denote the local coordinate z on U_α. Let $e_\alpha = dz_\alpha/z_\alpha$ be the local frame of K_{C_0}. Then $z_\alpha^a e_\alpha^{k+1}$, $a \geq 1$, are local sections of $K_{C_0}^{k+1}$. We choose and fix a cut-off function $\chi(r)$ that equals 1 for $r < \frac{1}{2}R$ and 0 for $r > \frac{2}{3}R$. Then we denote by χ_α the function $\chi(|z_\alpha|)$ defined on C_0. Then $\chi_\alpha z_\alpha^a e_\alpha^{k+1}$ is a global smooth L^2-integrable section of $K_{C_0}^{k+1}$. We then take the orthogonal projection of this section into the space $\mathcal{H}_{k+1}$, and then normalize the holomorphic section to be of norm 1, obtaining a section $s_{\alpha,a} \in \mathcal{H}_{k+1}$. We write

$$V_0 = \{s_{\alpha,a} : 1 \leq \alpha \leq 2d,\ 1 \leq a < k^{3/4}\}.$$

For k large enough, within $r < \frac{1}{4}R$, the sections $s_{\alpha,a}$ are approximately equal to $\sqrt{c_a} z^a$ with relative error less than $1/k^2$.

We choose and fix an orthonormal basis $W_0 = \{s_j\}$ for the orthogonal complement $V_0^\perp \subset \mathcal{H}_{k+1}$.

To obtain global sections of L^k from local ones, we will need to use Hörmander's L^2 estimate. The following lemma is well known; see for example [Tian 1990].

Lemma 2.2. *Suppose (M, g) is a complete Kähler manifold of complex dimension n and $\mathcal{L}$ is a line bundle on M with hermitian metric h. If*

$$\langle -2\pi i\Theta_h + \mathrm{Ric}(g), v \wedge \bar{v}\rangle_g \geq C|v|_g^2$$

for any tangent vector v of type $(1, 0)$ at any point of M, where $C > 0$ is a constant and Θ_h is the curvature form of h, then, for any smooth $\mathcal{L}$-valued $(0, 1)$-form α on M with $\bar{\partial}\alpha = 0$ and $\int_M |\alpha|^2\, dV_g$ finite, there exists a smooth $\mathcal{L}$-valued function β on M such that $\bar{\partial}\beta = \alpha$ and

$$\int_M |\beta|^2\, dV_g \leq \frac{1}{C}\int |\alpha|^2\, dV_g,$$

where dV_g is the volume form of g and the norms are induced by h and g.

In our setting, for a curve C_j, $j \geq 0$, with line bundle $K_{C_j}^{k+1}$, the constant k is independent of j.

3. The collar model

Let f_ε be a function depending only on $|z|$ satisfying the conditions

- $f_\varepsilon > 0$,
- $f_\varepsilon(1) = \varepsilon^2$,
- $f_\varepsilon'(1) = 0$,
- $\Delta_z \log f_\varepsilon(|z|) = 2f_\varepsilon/|z|^2$.

Clearly, such an f_ε exists and is unique in a neighborhood U of $|z| = 1$. Let

$$\omega_\varepsilon = \frac{f_\varepsilon i\, dz \wedge d\bar{z}}{2|z|^2}$$

be the Kähler metric defined on U. Then our choice of f_ε makes the metric have constant Gaussian curvature -1. Writing $t = \log|z|$ and abusing notation, we consider f_ε as a function of t. Then we have

$$\Delta_z \log f_\varepsilon = \frac{d^2 \log f_\varepsilon}{dt^2}\frac{1}{|z|^2}.$$

For simplicity, we will use $f(t)$ to denote f_ε. Therefore, we have

$$(\log f(t))'' = 2f(t).$$

The first fundamental form of the metric is

$$I = f(t)\,dt^2 + f(t)\,d\theta^2.$$

We use the arc-length parameter u for the t-curves. Then by the curvature condition, we have

$$f(t) = \varepsilon^2\cosh^2 u, \quad u(0) = 0, \quad \text{and} \quad \frac{dt}{du} = \frac{1}{\varepsilon\cosh u}.$$

Therefore, we have

$$t = \frac{1}{\varepsilon}\tan^{-1}[\sinh u].$$

One can see that f can be extended to the annulus $\{z : -\pi/(2\varepsilon) < t < \pi/(2\varepsilon)\}$. It is worth noticing that $|u| \to \infty$ when t goes to the boundary $|t| = \pi/(2\varepsilon)$, meaning $f_\varepsilon \to \infty$. It is natural to use the notation

$$\tau_\varepsilon = \frac{\pi}{2\varepsilon} - t.$$

From now on, we will always assume that ε is very small compared to k^{-k}. So the region where f is defined is a large annulus. We will write

$$\mathbb{C}^*_\varepsilon = \left\{z : -\frac{\pi}{2\varepsilon} < t < \frac{\pi}{2\varepsilon}\right\},$$

and we have our model $\mathbb{C}^*_\varepsilon = (\mathbb{C}^*_\varepsilon, \omega_\varepsilon)$.

We also use the frame $e = dz/z$ for the canonical bundle $K_{\mathbb{C}^*_\varepsilon}$, so we have

$$|e|^2 = \frac{2}{f}.$$

So for a section of $s = ge^{\otimes(k+1)}$ of $K^{k+1}_{\mathbb{C}^*_\varepsilon}$, the L^2-norm squared is

$$\|s\|^2 = \int_{\mathbb{C}^*_\varepsilon}\left(\frac{2}{f}\right)^{k+1}|g|^2\omega_\varepsilon.$$

We are interested in the L^2-norm of the sections $z^a e^{k+1}$ of $K^{k+1}_{\mathbb{C}^*_\varepsilon}$, $a \in \mathbb{Z}$. So we have the integrals

$$I_{\varepsilon,a} = \int\left(\frac{2}{f}\right)^{k+1}|z|^{2a}\omega_\varepsilon.$$

Further, we have

$$I_{\varepsilon,a} = 2^{k+2}\pi \int_{-\pi/(2\varepsilon)}^{\pi/(2\varepsilon)} e^{2at - k\log f}\,dt.$$

Writing $g_a(t) = 2at - k\log f$, we have $g_a''(t) = -2kf(t)$. So $g_a(t)$ is a concave function which attains its only maximum at t_a satisfying $f'(t_a)/f(t_a) = 2a/k$. Write $u_a = u(t_a)$. Since $f'(t)/f(t) = 2\varepsilon\sinh u$, we have

$$\varepsilon \sinh u_a = \frac{a}{k}.$$

So $\sinh u_a = a/(k\varepsilon)$ is very large when $a \geq 1$, and $f(t_a) = a^2/k^2 + \varepsilon^2 > a^2/k^2$. When a is large, say $a \geq k$, we have that $g_a''(t_a)$ is also large. The third derivative is

$$g^{(3)}(t) = -4k\varepsilon \sinh u.$$

So $g^{(3)}(t_a) = -8a$, and, for $|t - t_a| < (\sqrt{k}\log k)/a$, we have that $g^{(3)}(t)$ is also $O(a)$. Therefore, for $|t - t_a| < (\sqrt{k}\log k)/a$,

$$\frac{g''(t)}{g''(t_a)} = 1 + O\left(\frac{\log k}{\sqrt{k}}\right).$$

We can use Lemma 2.1 to estimate

$$I_{\varepsilon,a} = \left(1 + O\left(\frac{\log k}{\sqrt{k}}\right)\right) 2^{k+2}\pi\, e^{2at_a - k\log f(t_a)} \sqrt{\frac{\pi}{2kf(t_a)}} = \sqrt{\frac{\pi}{2k}}\,\frac{2^{k+2}\pi}{(\varepsilon\cosh u_a)^{2k+1}} e^{2at_a}.$$

We have that the mass of $I_{\varepsilon,a}$ is concentrated within the neighborhood $\{|t - t_a| < (\sqrt{k}\log k)/a\}$ with relative error less than $k^{-\log k + 3/2}$, namely

$$I_{\varepsilon,a} = (1 + O(k^{-\log k + 3/2}))2^{k+2}\pi \int_{|t-t_a|<(\sqrt{k}\log k)/a} e^{2at - k\log f}\,dt.$$

When $a < k$, we use the variable u to estimate the integral

$$\int_{-\pi/(2\varepsilon)}^{\pi/(2\varepsilon)} e^{2at - k\log f}\,dt = \int_{-\infty}^{\infty} e^{2at - \log f} \frac{du}{\varepsilon\cosh u}.$$

Let $-G(u) = k\log f - 2at - \log\cosh u$ be the exponent function. Then

$$\begin{aligned} G'(u) &= 2k\frac{\sinh u}{\cosh u} - \frac{2a}{\varepsilon\cosh u} + \frac{\sinh u}{\cosh u}, \\ G''(u) &= \frac{2k}{\cosh^2 u} + \frac{2a\sinh u}{\varepsilon\cosh^2 u} + \frac{1}{\cosh^2 u}, \\ G^{(3)}(u) &= -\frac{(4k+2)\sinh u}{\cosh^3 u} + \frac{2a(1 - \sinh^2 u)}{\varepsilon\cosh^3 u}. \end{aligned}$$

So $G(u)$ is a convex function of u which attains its only minimum at u_a' satisfying

$$\sinh u_a' = \frac{2a}{\varepsilon(2k+1)},$$

and

$$G''(u'_a) = \frac{(2k+1)^3\varepsilon^2 + 4a^2(2k+1)}{4a^2 + (2k+1)^2\varepsilon^2} = 2k+1+O(\varepsilon^2),$$

which is large. Further,

$$G^{(3)}(u'_a) = -\frac{2k+1}{[(2k+1)^2/(4a^2)+1]^{3/2}} + O(\varepsilon^2).$$

So again, we can estimate

$$\begin{aligned} I_{\varepsilon,a} &= \left(1+O\left(\frac{\log k}{\sqrt{k}}\right)\right)\frac{2^{k+2}\pi}{\varepsilon\cosh u'_a}e^{2at'_a - k\log f(t'_a)}\sqrt{\frac{\pi}{2k+1}} \\ &= \left(1+O\left(\frac{\log k}{\sqrt{k}}\right)\right)\sqrt{\frac{\pi}{2k+1}}\frac{2^{k+2}\pi}{(\varepsilon\cosh u'_a)^{2k+1}}e^{2at'_a}, \end{aligned}$$

where $t'_a = t(u'_a)$.

So for $a \geq k$,

$$\begin{aligned} \frac{I_{\varepsilon,a+1}}{I_{\varepsilon,a}} &= \left(1+O\left(\frac{\log k}{\sqrt{k}}\right)\right)e^{2at_{a+1}-t_a}\left(\frac{\cosh u_a}{\cosh u_{a+1}}\right)^{2k+1} \\ &= \left(1+O\left(\frac{\log k}{\sqrt{k}}\right)\right)e^{2\varepsilon k/a}\frac{a^{2k+1}}{(a+1)^{2k+1}} = \left(1+O\left(\frac{\log k}{\sqrt{k}}\right)\right)\frac{a^{2k+1}}{(a+1)^{2k+1}}. \end{aligned}$$

Since

$$\frac{\varepsilon\sinh u'_a}{\varepsilon\sinh u_a} = 1+\frac{1}{2k},$$

this approximation for $I_{\varepsilon,a+1}/I_{\varepsilon,a}$ works for all $a > 0$.

Since $\sinh u_a$ is very large, we can use the Taylor expansion of $\tan^{-1}$ around infinity to estimate

$$\frac{\pi}{2} - \varepsilon t_a = \frac{\varepsilon k}{a} + O\left(\left(\frac{\varepsilon k}{a}\right)^2\right)$$

and also

$$\frac{\pi}{2} - \varepsilon t'_a = \frac{\varepsilon(2k+1)}{2a} + O\left(\left(\frac{\varepsilon k}{a}\right)^2\right).$$

Therefore, for $k > a \geq 1$,

$$\frac{|z|^{2a}}{I_{\varepsilon,a}} = \left(1+O\left(\frac{\log k}{\sqrt{k}}\right)\right)\frac{\sqrt{2k}2^{k-1}}{\pi^{3/2}}\left(\frac{e}{2k+1}\right)^{2k+1}a^{2k+1}e^{-2a\tau_\varepsilon}.$$

Notice that the term

$$\frac{\sqrt{2k}2^{k-1}}{\pi^{3/2}}\left(\frac{e}{2k+1}\right)^{2k+1}$$

is independent of a.

Recall that the power series in the expression of $\rho_{0,k+1}$ is also

$$\sum a^{2k+1}|z|^{2a} = \sum a^{2k+1}e^{-2a\tau}$$

By the same argument as in [Sun 2017, Theorem 1.1] and [Sun 2013, p. 5535, Case II], for $t \in [0, t_1]$, the Bergman kernel is dominated by

$$\left[\frac{|z|^2}{I_{\varepsilon,1}} + \frac{1}{I_{\varepsilon,0}}\right]\left(\frac{2}{f}\right)^{k+1}.$$

The idea of the argument is very simple: by the mass concentration property of the integral $I_{\varepsilon,a}$, the contribution of $|z|^{2a}/I_{\varepsilon,a}$ to the Bergman kernel gets smaller and smaller when t moves further away from t_a. When $a < \sqrt{k}/\log k$, we have that $(a/(a+1))^{2k+1}$ is very small, meaning that t_a is already far enough from t_{a+1} for the integral $I_{\varepsilon,a+1}$, so that the contribution of $|z|^{2(a+1)}/I_{\varepsilon,a+1}$ to the Bergman kernel at t_a is negligible.

By symmetry, for $t \in [t_{-1}, 0]$, the Bergman kernel is dominated by

$$\left[\frac{|z|^{-2}}{I_{\varepsilon,-1}} + \frac{1}{I_{\varepsilon,0}}\right]\left(\frac{2}{f}\right)^{k+1}.$$

In particular, we have the following:

Lemma 3.1. *For any holomorphic section s of $K_{\mathbb{C}_\varepsilon^*}^{k+1}$ satisfying $\|s\| = 1$, we have*

$$|s|^2 < \varepsilon^2\left(\frac{\sqrt{e}\log\varepsilon}{k}\right)^{2k}$$

when

$$\cosh u \in \left(\frac{-1}{2\varepsilon\log\varepsilon}, \frac{-1}{\varepsilon\log\varepsilon}\right).$$

Proof. By symmetry, we can assume $t > 0$. For the right end of the interval, we only need to estimate the norms of

$$\frac{z}{I_{\varepsilon,1}}e^{k+1} \quad \text{and} \quad \frac{1}{I_{\varepsilon,0}}e^{k+1}$$

at t, where $\cosh u = -1/(\varepsilon\log\varepsilon)$. For the first one, we have

$$\left|\frac{z}{I_{\varepsilon,1}}e^{k+1}\right|^2 = \left(1 + O\left(\frac{\log k}{\sqrt{k}}\right)\right)\frac{\sqrt{2k}\varepsilon^2}{2k\pi^{3/2}}\left(\frac{\log\varepsilon}{k}\right)^{2k}e^{k+1/2}.$$

For the second one, notice that $u'_a = 0 = t'_a$ and

$$I_{\varepsilon,0} = \left(1 + O\left(\frac{\log k}{\sqrt{k}}\right)\right)\sqrt{\frac{\pi}{2k+1}\frac{2^{k+2}\pi}{\varepsilon^{2k+1}}}.$$

So we have

$$\left|\frac{1}{I_{\varepsilon,0}}e^{k+1}\right|^2 = \left(1 + O\left(\frac{\log k}{\sqrt{k}}\right)\right)\frac{\sqrt{2k}}{2\pi^{3/2}}\varepsilon^{2k+1}(\log\varepsilon)^{2k},$$

which is much smaller than the first one. For the left end of the interval, we have a smaller norm for the section $ze^{k+1}/I_{\varepsilon,1}$, and a still very small norm for the section $e^{k+1}/I_{\varepsilon,0}$. Combining these estimates, we have proved the lemma. ☐

Assume C_j converges to C_0 in the pointed Gromov–Hausdorff topology. For j big enough, C_j has exactly d closed geodesics whose arc length is less than $\frac{1}{4}\lambda_0$. We denote these circles by $\gamma_{j,\alpha}$, $1 \le \alpha \le d$, and the arc length of $\gamma_{j,\alpha}$ by $\varepsilon_{j,\alpha}$. Rearranging the points p_α, we can assume that $2\pi\varepsilon_{j,\alpha}$ converges to the pair $(p_\alpha, p_{\alpha+d})$ as $j \to \infty$. Also, for j large enough, there is a neighborhood $U_{j,\alpha}$, usually referred to as a collar, of each $\gamma_{j,\alpha}$ which is homeomorphic to an annulus. We define a map

$$h_{j,\alpha} : U_{j,\alpha} \to \mathbb{C}_\varepsilon^*,$$

with $\varepsilon = \varepsilon_{j,\alpha}$, as follows. Fix an isometry λ of $\gamma_{j,\alpha}$ to the circle $|z| = 1$ in $\mathbb{C}_\varepsilon^*$. Then, passing through each point q on $\gamma_{j,\alpha}$, there is an unique geodesic l_q orthogonal to $\gamma_{j,\alpha}$. We define $h_{j,\alpha}$ to be the map that sends each such geodesic l_q to the geodesic passing through $\lambda(q)$, orthogonal to the unit circle, and preserving λ and the orientation. Since both surfaces have constant Gaussian curvature -1, $h_{j,\alpha}$ is an isometry so long as the geodesics l_q do not intersect each other. But since the curvature is negative, by the Gauss–Bonnet theorem, these geodesics cannot intersect within $U_{j,\alpha}$. Therefore, $h_{j,\alpha}$ is also holomorphic, and we can use the coordinate z from $\mathbb{C}_\varepsilon^*$ as the holomorphic coordinate of $U_{j,\alpha}$. By switching p_α and $p_{\alpha+d}$ if necessary, we can assume that the part $|z| > 1$ of $U_{j,\alpha}$ converges to a neighborhood of p_α and the part $|z| < 1$ converges to that of $p_{\alpha+d}$. We can assume that $U_{j,\alpha} = \{1/M \le |z| \le M\}$, and, for j large enough, we can assume that the injective radius at $|z| = M$ is larger than

$$\frac{\pi}{4(\log 3R/4)^2}.$$

We denote by $U_{j,\alpha}^+$ the part of $U_{j,\alpha}$ with $|z| > 1$ and similarly $U_{j,\alpha}^-$ the part of $U_{j,\alpha}$ with $|z| < 1$. We then define a map

$$\varphi_{j,\alpha} : U_{j,\alpha}^+ \to U_\alpha$$

by sending $\varepsilon \cosh u$ to $1/(2\tau)$ while preserving the circles $\{u = \text{constant}\}$. Clearly, we are only preserving the length of the circles. By symmetry, we also have

$$\varphi_{j,\alpha+d} : U_{j,\alpha}^- \to U_{\alpha+d}.$$

By our assumption on the injective radius, the image of $\varphi_{j,\alpha}$ contains the circle $|z_\alpha| = \frac{3}{4}R$. On U_α, the first fundamental form is

$$I_0 = \frac{1}{\tau^2}(d\tau^2 + d\theta^2).$$

The pullback

$$\varphi_{j,\alpha}^* I_0 = \tanh^2 u \, du^2 + (\varepsilon \cosh u)^2 \, d\theta^2$$

is almost isometric to the metric

$$I_j = du^2 + (\varepsilon \cosh u)^2 \, d\theta^2$$

when u is large. In particular, for the part where $\varepsilon \cosh u \ge -1/\log \varepsilon$, we have that $\varphi_{j,\alpha}$ converges to an isometry when $j \to \infty$.

Let $U_\alpha(r)$ denote the subset of U_α consisting of the points $|z_\alpha| < r$. Let $F = C_0 \backslash \bigcup_{1\le\alpha\le 2d} U_\alpha\big(\frac{2}{3}R\big)$, and let $\psi_j : F \to C_j$ be the diffeomorphism with its image. Since ψ_j converges to an isometry as $j \to \infty$, we can glue ψ_j^{-1} with the $\varphi_{j,\alpha}$ — rotating $\varphi_{j,\alpha}$ if necessary — for j large enough, to get a map

$$G_j : C_j \backslash \bigcup \gamma_{j,\alpha} \to C_0$$

with the following properties:

- G_j is a diffeomorphism of $C_j \backslash \bigcup \gamma_{j,\alpha}$ with its image.
- $G_j = \varphi_{j,\alpha}$ for $p \in \varphi_{j,\alpha}^{-1} U_\alpha\big(\frac{2}{3}R\big)$, $1 \le \alpha \le 2d$.
- G_j is almost an isometry on $C_j \backslash \bigcup_{1\le\alpha\le 2d} \varphi_{j,\alpha}^{-1} U_\alpha\big(\frac{2}{3}R\big)$ and converges to an isometry when $j \to \infty$.

For any conformal metric, the compatible complex structure J is just a counterclockwise rotation by $\frac{\pi}{2}$. We see that almost isometric implies almost holomorphic. Therefore $G_j^{-1*} K_{C_j}$ converges to K_{C_0} as subbundles of $T_{C_0} \otimes \mathbb{C}$. More precisely, let J_j be the complex structure compatible with the Riemannian metric g_j. If the pointwise norm

$$\sup_{v\in T_p, |v|_g = 1} |g_j(v, v) - g(v, v)| < \delta,$$

then we have

$$\sup_{v\in T_p, |v|_g = 1} |J_j(v) - J(v)|_g < \lambda\delta$$

for some constant λ independent of p and g. We call the supremum above the pointwise distance from J_j to J. Moreover, if g_j converges to g in C^2-norm, then J_j converges to J in C^2-norm also. If we denote by T_J the holomorphic tangent space with respect to J, then the orthogonal projection of T_{J_j} to T_J is close to an isometry if J_j is close to J. We identify T_{J_j} with T_j via this orthogonal projection, and similarly $K_j = T_{J_j}^*$ with $K = T_J^*$, which we will also call an orthogonal projection for simplicity. Since the metric on the canonical bundle is defined by the Kähler form ω and ω_j converges to ω, we have that the Chern connection ∇_j on K_j converges to the Chern connection ∇ on K.

4. Convergence of projective embedding

By assigning value 1 on $\gamma_{j,\alpha}$, we glue together the pullbacks $G_j^*\chi_\alpha$ and $G_j^*\chi_{\alpha+d}$ to get a function denoted by $\tilde{\chi}_\alpha$ for $1 \le \alpha \le d$. On each $\varphi_{j,\alpha}$, we also consider the $\tilde{\chi}_\alpha z^a$ as global smooth sections of $K_{C_j}^{k+1}$. Then we repeat the construction of V_0 by normalizing the orthogonal projection of $\tilde{\chi}_\alpha z^a$ onto $\mathcal{H}_{j,k+1}$ and denote the resulting section by $s_{j,\alpha,a}$, $|a| < k^{3/4}$. Then we define

$$V_j = \{s_{j,\alpha,a} : 1 \le \alpha \le d,\ |a| < k^{3/4}\}.$$

We should remark here that the choice of the upper bound $k^{3/4}$ is not necessary, it is purely a habit from [Sun and Sun 2021]. Notice that the number of sections of V_j is larger than that of V_0 by the number d. Those extra sections are $\{s_{j,\alpha,0}\}_{1\le\alpha\le d}$. We consider $s_{j,\alpha,a}$ as a smooth section of $G_j^{-1*} K_{C_j}^{k+1}$ on image(G_j). We then define a piecewise smooth section $\tilde{s}_{j,\alpha,a}$ of $K_{C_0}^{k+1}$ which equals the orthogonal projection of $s_{j,\alpha,a}$ to $K_{C_0}^{k+1}$ on image(G_j), and equals 0 in the complement. For simplicity, we will say that $s_{j,\alpha,a}$ converges in some topology if $\tilde{s}_{j,\alpha,a}$ converges in that topology.

Proposition 4.1. *The smooth section* $s_{j,\alpha,a}$ *converges to* $s_{\alpha,a}$ *for* $a > 0$ *and to* $s_{\alpha+d,-a}$ *for* $a < 0$, *in* L^2*-norm, as* $j \to \infty$.

Proof. By symmetry, we only have to prove the result for $a > 0$. By taking j large enough, we can assume that $p \in C_0 \setminus \bigcup U_\alpha(\varepsilon(j,\alpha))$. When $\varepsilon \cosh u \geq -1/\log\varepsilon$, we observe that $\tanh^2 u = 1 - (\varepsilon \log \varepsilon)^2$ is very close to 1. So $\varphi_{j,\alpha}$ is very close to an isometry. For simplicity, we still use the notation ε for $\varepsilon(j,\alpha)$. Then we look at the integral

$$I_{j,\alpha,a} = 2^{k+2}\pi \int \tilde{\chi}_\alpha^2 \frac{1}{(\varepsilon \cosh u)^{2k}} e^{2at}\, dt = 2^{k+2}\pi e^{\pi a/\varepsilon} \int \tilde{\chi}_\alpha^2 \frac{1}{(\varepsilon \cosh u)^{2k}} e^{-2a\tau_\varepsilon}\, d\tau_\varepsilon.$$

On C_0, we have

$$J_{\alpha,a} = 2^{k+2}\pi \int \chi_\alpha^2 \tau^{2k} e^{-2a\tau}\, d\tau.$$

For $I_{j,\alpha,a}$, by Lemma 2.1, we can truncate the part $\tau_\varepsilon > -\log\varepsilon$ by introducing a relative error $< \varepsilon$. Also for $J_{\alpha,a}$, we can truncate the part $\tau > -\log\varepsilon$ by introducing a relative error $< \varepsilon$. Then for the part $\tau_\varepsilon \leq -\log\varepsilon$,

$$\tilde{\chi}_\alpha^2 \frac{1}{(\varepsilon \cosh u)^{2k}} e^{-2a\tau_\varepsilon}$$

converges to $\chi_\alpha^2 \tau^{2k} e^{-2a\tau}$ uniformly. Therefore $I_{j,\alpha,a} e^{-\pi a/\varepsilon}$ converges to $J_{\alpha,a}$ as $j \to \infty$, and therefore $I_{j,\alpha,a}^{-1/2} z_\alpha^a$ converges to $J_{\alpha,a}^{-1/2} z^a$ as $j \to \infty$. Also for this part, the 1-form dz_α / z_α converges uniformly to dz/z. The way to get orthogonal projection onto holomorphic sections is to find the solutions of the equations

$$\bar\partial v = \bar\partial J_{\alpha,a}^{-1/2} z^a \left(\frac{dz}{z}\right)^{k+1} \quad \text{and} \quad \bar\partial_j v_j = \bar\partial_j I_{j,\alpha,a}^{-1/2} z_\alpha^a \left(\frac{dz_\alpha}{z_\alpha}\right)^{k+1},$$

where we denote by $\bar\partial_j$ the $\bar\partial$ operator on C_j, with minimal L^2-norms. Here the L^2-norms of v_j and v are defined with v_j and v considered as sections of $K_{C_j}^{k+1}$ and $K_{C_0}^{k+1}$, respectively. In order to prove the conclusion of the lemma, it suffices to prove that v_j converges to v. Notice that $\bar\partial v$ is supported within the annulus $\frac{2}{3}R \leq |z| \leq \frac{3}{4}R \subset U_\alpha$. By the mass concentration property of z^a, the mass satisfies

$$\|\bar\partial v\| < \frac{1}{k^2}.$$

Therefore

$$\int |v|^2 \omega < \frac{1}{k^3},$$

and v is holomorphic outside the support of $\bar\partial v$. Since the Bergman kernel is dominated by $|\sqrt{Y_1^{-1}} z|^2$ and

$$\int_{|z|<\varepsilon} Y_1^{-1} |z|^2 \omega < \varepsilon,$$

we have

$$\int_{|z|<\varepsilon} |v|^2 \omega < \varepsilon$$

in every U_α. We pull back the restriction v to $C_0\backslash\bigcup U_\alpha(\varepsilon(j,\alpha))$ by G_j, and use cut-off functions κ_ε near the edges to get a global smooth section of $G_j^*(K_{C_0}^{k+1})$, which is then projected to a smooth section of $K_{C_j}^{k+1}$, called $\tilde{v}_j$. More explicitly, we define the κ_ε as a smooth function of τ_ε that equals 1 for $\tau_\varepsilon < -\log\varepsilon$, and equals 0 for $\tau_\varepsilon > -2\log\varepsilon$. We can also assume that $|\kappa_\varepsilon'| < 2/(-\log\varepsilon)$. We have

$$\nabla^*\nabla v = 2\bar\partial^*\bar\partial v + (k+1)v.$$

So

$$\int |\nabla v|^2\omega = \int 2|\bar\partial v|^2\omega + (k+1)\int |v|^2\omega < \frac{3}{k^2}.$$

Therefore,

$$\int |\nabla_j\tilde{v}_j|^2\omega_j < \frac{4}{k^2} \quad \text{and} \quad \int_{C_j} |\bar\partial_j\tilde{v}_j - \bar\partial_j v_j|^2\omega_j = \delta_j,$$

where $\delta_j \to 0$ as $j\to\infty$. Then we can solve the equation

$$\bar\partial_j u = \bar\partial_j\tilde{v}_j - \bar\partial_j v_j$$

with minimal L^2-norm, so that

$$\int_{C_j} |u|^2\omega_j \le \frac{1}{k}\delta_j.$$

Since v_j is a minimal solution,

$$\int |\tilde{v}_j - u|^2\omega_j \ge \int |v_j|^2\omega_j.$$

So

$$\int |\tilde{v}_j|^2\omega_j \ge \int |v_j|^2\omega_j - \sqrt{\frac{\delta_j}{k^5}}.$$

Conversely, for each v_j, we first use the cut-off functions κ_ε to make it vanish near the edges, then we pull it back by G_j^{-1} to C_0. By orthogonal projection and extension by 0, we obtain a smooth section $\tilde{v}^j$ of $K_{C_0}^{k+1}$. Similar to $\tilde{v}_j$, by Lemma 3.1, we have

$$\int_{C_0} |\bar\partial\tilde{v}^j - \bar\partial v|^2\omega_j = \delta_j',$$

where $\delta_j' \to 0$ as $j\to\infty$. Then by solving a $\bar\partial$-equation again, we get

$$\int |\tilde{v}^j|^2\omega \ge \int |v|^2\omega - \sqrt{\frac{\delta_j'}{k^5}}.$$

Since the L^2-norm of $\tilde{v}_j$ is close to that of v and

$$\int |\tilde{v}^j|^2\omega \le \int |v_j|^2\omega_j + \delta_j'',$$

where $\delta_j' \to 0$ as $j\to\infty$, we get

$$\int |\tilde{v}_j|^2\omega_j - \int |v_j|^2\omega_j \to 0$$

as $j \to \infty$. Then by the uniqueness of the minimal solution of the $\bar{\partial}$-equation, we get

$$\int |\tilde{v}_j - v_j|^2 \omega_j \to 0$$

as $j \to \infty$. Finally, since the L^2-norm of $s_{j,\alpha,a}$ on the area where

$$\frac{1}{\varepsilon \cosh u} < \frac{-1}{2 \log \varepsilon}$$

also goes to 0 as $j \to \infty$, we have proved the theorem. □

The same ideas can be used for the sections in W_0. For each $s \in W_0$, we have

$$\int_{|z|<\varepsilon} |s|^2 \omega < \varepsilon$$

in every U_α. We pull back the restriction s to $C_0 \backslash \bigcup U_\alpha(\varepsilon(j, \alpha))$ by G_j, and use cut-off functions κ_ε near the edges to get a global smooth section of $G_j^*(K_{C_0}^{k+1})$, which is then projected to a smooth section of $K_{C_j}^{k+1}$, called $\tilde{s}_j$. Then we have

$$\int_{C_j} |\bar{\partial}_j \tilde{s}_j|^2 \omega_j \to 0$$

as $j \to \infty$. So we can solve the equation

$$\bar{\partial}_j u_j = \bar{\partial}_j \tilde{s}_j$$

with minimal L^2-norm to get a holomorphic section $s_{,j} = u_j + \tilde{s}_j \in \mathcal{H}_{j,k+1}$ satisfying

$$\int_{C_j} |s_{,j}|^2 \omega_j - 1 \to 0 \quad \text{and} \quad s_{,j} \to s$$

in L^2-norm as $j \to \infty$. We have proved the following:

Lemma 4.2. *For each $s \in W_0$, we can find $s_{,j} \in \mathcal{H}_{j,k+1}$ such that $s_{,j}$ converges to s in the L^2-norm and $\|s_{,j}\| \to 1$ as $j \to \infty$.*

We will denote by W_j the set of sections $\{s_{,j} : s \in W_0\}$. Notice that W_j is not an orthonormal set, but gets closer as j gets larger. We fix an order on W_0, then we order each W_j accordingly. We give each V_j the dictionary order. Recall that $s_{j,\alpha,a}$ corresponds to $s_{\alpha,a}$ for $a > 0$ and $s_{\alpha+d,-a}$ for $a < 0$. Then we add 0 d times to V_0, and order the obtained set $\widetilde{V}_0$ according to the correspondence to V_j, where each 0 corresponds to a section $s_{j,\alpha,0}$, $1 \le \alpha \le d$. Then we define the embedding

$$\Phi_j : C_j \to \mathbb{CP}^{N_k},$$

where $N_k = \dim \mathcal{H}_{j,k+1} - 1$, by $\Phi_j = [V_j, W_j]$, where $[\cdots]$ means the homogeneous coordinates. Similarly we define the embedding

$$\Phi_0 : C_0 \to \mathbb{CP}^{N_k}$$

by $\Phi_0 = [\widetilde{V}_0, W_0]$.

Let $[Z_0, \dots, Z_N]$ be the homogeneous coordinates of $\mathbb{CP}^N$. $U_0 = \{[1, w], w \in \mathbb{C}^N\}$ is a coordinate patch with $w_i = Z_i/Z_0$. The Z_i can be identified as generating sections in $H^0(\mathbb{CP}^N, \mathcal{O}(1))$. In particular, Z_0 is a local frame in U_0. Then on U_0, the Fubini–Study form $\omega = \frac{1}{2} i \partial\bar{\partial} \log(1 + |w|^2)$ has the explicit form

$$\omega = \frac{i}{2} \frac{(1+|w|^2) \sum dw^i \wedge d\bar{w}_i - \left(\sum \bar{w}_i \, dw_i\right)\left(\sum w_i \, d\bar{w}_i\right)}{(1+|w|^2)^2}.$$

On each $U_{j,\alpha}$, within the area $0 \le t \le t_0 = \pi/(2\varepsilon) + \log\varepsilon$, the image under Φ_j is dominated by the two sections $s_{j,\alpha,0}$ and $s_{j,\alpha,1}$, since the contribution of other sections is of relative size $< k^2\varepsilon$. We can estimate the map $[s_{j,\alpha,0}, s_{j,\alpha,1}, 0, \dots]$ by the local sections $[b_0, b_1 z, 0, \dots]$, with relative error $< 1/k^2$, where

$$b_0^{-2} = \frac{2^{k+2}\pi^{3/2}}{\sqrt{2k}} \varepsilon^{-2k-1} \quad \text{and} \quad b_1^{-2} = \frac{2^{k+2}\pi^{3/2}}{\sqrt{2k}} e^{\pi/\varepsilon} k^{2k+1}.$$

So the map is simplified to

$$[1, e^{-\pi/(2\varepsilon)} (\varepsilon k)^{-k-1/2} z, 0, \dots],$$

and we can estimate the length of the image of $\gamma_{j,\alpha}$ — which is approximately $2\pi e^{-\pi/(2\varepsilon)}(\varepsilon k)^{-k-1/2}$ — which goes to 0 as $j \to \infty$. So the image of $\gamma_{j,\alpha}$ converges to $[1, 0, \dots]$ in the current ordering of coordinates. Similarly, we can estimate the length of the image of the circle $t_0 = \pi/(2\varepsilon) + \log\varepsilon$ — which is approximately $(2\pi/\varepsilon)(\varepsilon k)^{k+1/2}$ — which goes to 0 as $j \to \infty$. Therefore, the image of the circle $t_0 = \pi/(2\varepsilon) + \log\varepsilon$ converges to $[0, 1, \dots]$ in the current ordering of coordinates. Notice that $e^{-\pi/(2\varepsilon)}(\varepsilon k)^{-k-1/2}|z|$ goes to ∞ at t_0, so the image of the area $0 \le t \le t_0 = \pi/(2\varepsilon) + \log\varepsilon$ converges to the complex projective line connecting the points $[1, 0, 0, \dots]$ and $[0, 1, 0, \dots]$ in the current ordering of coordinates. This area is the bubble mentioned in the introduction. By symmetry, the image of the area $0 \ge t \ge -t_0$ also converges to a complex projective line.

For the part $G_j^{-1}\left(C_0 \setminus \bigcup U_\alpha(\varepsilon(j,\alpha))\right)$, the sections $\{s_{j,\alpha,0}\}_{1\le\alpha\le d}$ are negligible. So the convergence of the remaining sections in V_j and W_j in the L^2 sense implies that the image of $G_j^{-1}\left(C_0 \setminus \bigcup U_\alpha(\varepsilon(j,\alpha))\right)$ under Φ_j converges to the image of Φ_0. To conclude the proof of the main theorem, we only have to notice that although $V_j \cup W_j$ is not orthonormal, we can modify them. The sections in W_j are almost orthonormal, so we can transform them to be orthonormal with a matrix A_j whose difference from the identity matrix goes to 0 as $j \to \infty$ in L^∞-norm. And the modified sections still converge to sections in W_0 in L^2. We first apply a Gram–Schmidt process to the set V_0, then we apply a Gram–Schmidt process following the order of V_0 to the set $V_j \setminus \{s_{j,\alpha,0}\}_{1\le\alpha\le d}$. Finally, since the sections $\{s_{j,\alpha,0}\}_{1\le\alpha\le d}$ are almost orthonormal and almost orthogonal to the other section in V_j, we can modify the new V_j again with a matrix B_j whose difference from the identity matrix goes to 0 as $j \to \infty$ in L^∞-norm to get an orthonormal set. And we have proved the main theorem.

Acknowledgements

The author would like to thank Professor Song Sun for many very helpful discussions. The author would also like to thank the referees, whose many suggestions help improve the article dramatically.

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Received 27 Jul 2020. Revised 23 May 2022. Accepted 22 Nov 2022.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

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ANALYSIS & PDE

Volume 17 No. 6 2024
