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ON THE SPECTRUM OF NONDEGENERATE MAGNETIC LAPLACIANS

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We consider a compact Riemannian manifold with a Hermitian line bundle whose curvature is nondegenerate. Under a general condition, the Laplacian acting on high tensor powers of the bundle exhibits gaps and clusters of eigenvalues. We prove that for each cluster the number of eigenvalues that it contains is given by a Riemann–Roch number. We also give a pointwise description of the Schwartz kernel of the spectral projectors onto the eigenstates of each cluster, similar to the Bergman kernel asymptotics of positive line bundles. Another result is that gaps and clusters also appear in local Weyl laws.

1. Introduction

Consider a Hermitian line bundle L on a compact Riemannian manifold with a connection ∇ whose curvature is nondegenerate. We will be concerned with the eigenvalues and eigenstates of the Bochner Laplacians $\Delta_k = \frac{1}{2}\nabla^*\nabla + kV$ acting on positive tensor powers L^k of the bundle, V being a real function, in the limit where k tends to infinity. Physically, $k^{-2}\Delta_k$ is a magnetic Schrödinger operator with k the inverse of the Planck's constant, ∇ the magnetic potential and $k^{-1}V$ the electric potential.

A very particular case is the $\bar{\partial}$-Laplacian of high powers of a positive line bundle on a complex manifold. Its ground states are the holomorphic sections which play obviously a central role in algebraic/complex geometry, but also in mathematical physics: in Kähler quantization, the space of holomorphic sections is the quantum space and the large k limit is the semiclassical limit. Starting from [Guillemin and Uribe 1988], it has been understood that for a manifold that is not necessarily complex, the holomorphic sections can be replaced by the bounded states of the Bochner Laplacian Δ_k, where the potential V is suitably defined; bounded here means that the eigenvalues are bounded independently of k. These "almost" holomorphic sections have been used with success in various problems on symplectic manifolds from their projective embeddings to their quantizations [Borthwick and Uribe 1996; 2000; Ma and Marinescu 2007].

In the larger regime where we consider all the eigenvalues smaller than $k\Lambda$, with Λ arbitrary large but independent of k, few results are known: a general Weyl law was established in [Demailly 1985], which we will recall later, and for a specific class of connection ∇, [Faure and Tsujii 2015] showed that the spectrum of Δ_k exhibits some gaps and clusters, the first cluster consisting of the bounded states of [Guillemin and Uribe 1988].

A natural question is to determine the number of eigenvalues in each cluster. For the first cluster, in the case of holomorphic sections of a positive line bundle, the answer is provided by the Riemann–Roch–Hirzebruch theorem and the Kodaira vanishing theorem. More generally, when k is sufficiently large,

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the number of bounded states of the Bochner Laplacian of [Guillemin and Uribe 1988] is still given by the Riemann–Roch number of L^k. One of our main results is that the number of eigenvalues in each higher cluster is given as well by a Riemann–Roch number, associated to L^k tensored with a convenient auxiliary bundle F defined in terms of the cluster.

We are also concerned with results of local nature: we show that gaps and clusters appear as well in the local Weyl laws of Δ_k; local here means that each eigenvalue is counted with a weight given by the square of the pointwise norm of the corresponding eigensection. Furthermore we give a pointwise description of the Schwartz kernel of the spectral projectors associated to each cluster, generalizing the Bergman kernel asymptotics for positive line bundles.

The picture emerging from these results is that the restriction of the Bochner Laplacian Δ_k to each cluster is essentially a Berezin–Toeplitz operator with principal symbol an endomorphism of the auxiliary bundle F.

1A. ***The magnetic Laplacian.*** Let us turn to precise statements. Let M^{2n} be a closed manifold equipped with a Riemannian metric g, a volume form μ, a Hermitian line bundle L with a connection compatible with the metric, a Hermitian vector bundle A over M having an arbitrary rank r with a connection, and a section $V \in \mathcal{C}^\infty(M, \operatorname{End} A)$ such that $V(x)$ is Hermitian for any $x \in M$. Define the Laplacian

$$\Delta_k = \tfrac{1}{2}\nabla^*\nabla + kV : \mathcal{C}^\infty(L^k \otimes A) \to \mathcal{C}^\infty(L^k \otimes A). \tag{1}$$

Here $k \in \mathbb{N}$, ∇ is the covariant derivative of $L^k \otimes A$, ∇^* is its adjoint, the scalar products of sections of $L^k \otimes A$ or $L^k \otimes A \otimes T^*M$ are defined by integrating the pointwise scalar products against the volume form μ. The metric of T^*M is induced by the Riemannian metric.

We have introduced the bundle A with the endomorphism-valued section V to include some important Laplacians as the $\bar\partial$-Laplacian acting on p-forms or the square of some Dirac operators. Furthermore our results hold for a slightly more general class of operators than (1), which are defined in Section 3 and are locally of the form (B).

Since Δ_k is a formally self-adjoint elliptic operator on a compact manifold, it is essentially self-adjoint, its spectrum $\operatorname{sp}(\Delta_k)$ is a discrete subset of $[k \inf V_1, +\infty[$ and consists only of eigenvalues with finite multiplicities, and the eigenfunctions are smooth sections of $L^k \otimes A$. Here $V_1(x)$ is the lowest eigenvalue of $V(x)$.

The curvature of L has the form ω/i, with $\omega \in \Omega^2(M, \mathbb{R})$ a closed form. Let us assume that

$$\omega \text{ is nondegenerate at each point of } M. \tag{A}$$

Thus ω is a symplectic form. Associated to ω is the Liouville volume form $\mu_{\mathrm{L}} = \omega^n/n!$. We will assume that $\mu = \mu_{\mathrm{L}}$. This is not a restrictive assumption because if we multiply μ by a positive function ρ and the metric of A by ρ^{-1}, we do not change the scalar products of $\mathcal{C}^\infty(L^k \otimes A)$ and $\Omega^1(L^k \otimes A)$. Working with μ_{L} will simplify several statements.

1B. ***Pointwise data.*** We now introduce several pointwise data that will enter in our asymptotic description of the spectrum of Δ_k. Denote by j_B the section of $\operatorname{End}(TM)$ such that $\omega(\xi, \eta) = g(j_B\xi, \eta)$. Then M

has an almost-complex structure j compatible with ω defined by

$$j_y := |j_{B,y}|^{-1} j_{B,y} \quad \text{for all } y \in M.$$

So the vector bundle $T^{1,0}M = \operatorname{Ker}(j - i\,\mathrm{id}_{TM\otimes\mathbb{C}}) \subset TM \otimes \mathbb{C}$ has a Hermitian metric h given by $h(\xi,\eta) = \omega(\xi,\bar{\eta})/i$.

Moreover, the complexification of $j_{B,y}/i$ restricts to a positive endomorphism of $(T_y^{1,0}M, h_y)$. Denote its eigenvalues by $0 < B_1(y) \leqslant \cdots \leqslant B_n(y)$. We introduce an orthonormal basis (u_i) of $(T_y^{1,0}M, h_y)$ such that $j_{B,y}u_i = iB_i(y)u_i$.

Consider the space $\mathcal{D}(T_yM) = \mathbb{C}[T_y^{0,1}M]$ of antiholomorphic polynomials of T_yM. If (z_i) are the linear complex coordinates of T_yM dual to the u_i, then $\mathcal{D}(T_yM) = \mathbb{C}[\bar{z}_1, \dots, \bar{z}_n]$. Define the endomorphism

$$\square_y = \sum_i B_i(y)\big(\mathfrak{a}_i^\dagger \mathfrak{a}_i + \tfrac{1}{2}\big) + V(y) : \mathcal{D}(T_yM)\otimes A_y \to \mathcal{D}(T_yM)\otimes A_y, \tag{2}$$

where $\mathfrak{a}_i$ and $\mathfrak{a}_i^\dagger$ are the endomorphisms of $\mathcal{D}(T_yM)$ acting by derivation with respect to $\bar{z}_i$ and multiplication by $\bar{z}_i$ respectively.

We introduce an eigenbasis (ζ_j) of $V(y)$: $V(y)\zeta_i = V_i(y)\zeta_i$, with $V_1(y) \leqslant \cdots \leqslant V_r(y)$. Then $\square_y$ is diagonalizable, with eigenbasis $(\bar{z}^\alpha \otimes \zeta_j, (\alpha, j) \in \mathbb{N}^n \times \{1, \dots, r\})$,

$$\square_y(\bar{z}^\alpha \otimes \zeta_j) = \Big(\sum_i B_i(y)\big(\alpha(i) + \tfrac{1}{2}\big) + V_j(y)\Big)\bar{z}^\alpha \otimes \zeta_j.$$

Let $\lambda_1(y) \leqslant \lambda_2(y) \leqslant \cdots$ be the eigenvalues of $\square_y$ ordered and repeated according to their multiplicities.

The operators $\square_y$ depend smoothly on y even if it is not obvious from (2), because in general there is no local smooth frame (u_i) of $T^{1,0}M$ which is an eigenbasis of $j_{B,y}$ at each y. The various eigenvalues $B_i(y)$, $V_j(y)$ and $\lambda_\ell(y)$ depend continuously on y.

1C. *Weyl laws.* Demailly [1985] proved a Weyl law for the operators $k^{-1}\Delta_k$. It says roughly that in the semiclassical limit $k \to \infty$, the spectrum of $k^{-1}\Delta_k$ is an aggregate of the spectra of the $\square_y$. More precisely, we introduce the counting functions $N_y(\lambda) = \sharp\{\ell : \lambda_\ell(y) \leqslant \lambda\}$ of the $\square_y$ and the one of $k^{-1}\Delta_k$

$$N(\lambda, k) = \sharp\, \mathrm{sp}(k^{-1}\Delta_k) \cap]-\infty, \lambda].$$

Here and in the sequel, an eigenvalue with multiplicity m is counted m times. Let $v : \mathbb{R} \to \mathbb{R}$ be the nondecreasing function $v(\lambda) := \int_M N_y(\lambda)\, d\mu_{\mathrm{L}}(y)$. Let D be the set of discontinuity points of v. Then for any $\lambda \in \mathbb{R} \setminus D$, we have

$$N(\lambda, k) = \Big(\frac{k}{2\pi}\Big)^n v(\lambda) + \mathrm{o}(k^n) \tag{3}$$

as k tends to infinity. We have slightly reformulated the original result [Demailly 1985, Theorem 0.6], which holds more generally for ω not necessarily nondegenerate and M not necessarily compact.

The subset D is in general nonempty. As an easy example, if $j_B = j$ and $V = 0$, then $D = \frac{1}{2}n + \mathbb{N}$, v is locally constant on $\mathbb{R} \setminus D$ and, for any $\ell \in \mathbb{N}$,

$$v\Big(\frac{n}{2} + \ell + 0\Big) = v\Big(\frac{n}{2} + \ell - 0\Big) + r\mu_{\mathrm{L}}(M)\binom{n+\ell-1}{\ell-1}.$$

Our goal is to understand the corrections to the Weyl law (3), in other words what is hidden in the remainder $o(k^n)$. For instance, if the function v is constant on a compact interval J, then (3) implies that $\sharp\, \mathrm{sp}(k^{-1}\Delta_k) \cap J = o(k^n)$. Actually, as we will see, in this situation, when k is sufficiently large, J contains no eigenvalue of $k^{-1}\Delta_k$. Furthermore, the numbers of eigenvalues between such intervals is given by Riemann–Roch numbers.

To state our results, we introduce the set $\Sigma = \bigcup_j \lambda_j(M)$. Σ is a locally finite union of closed disjoint intervals. The function v is locally constant on $\mathbb{R} \setminus \Sigma$ and Σ is the support of the Lebesgue–Stieltjes measure dv; see Section 2D for a proof of these statements.

If B is complex vector bundle of M, we denote by $\mathrm{RR}(B)$ the Riemann–Roch number of B, that is, the integral of the product of the Chern character of B by the Todd form of (M, j).

Theorem 1.1. *Let $a, b \in \mathbb{R} \setminus \Sigma$, with $a < b$. Then when k is sufficiently large,*

$$\sharp\, \mathrm{sp}(k^{-1}\Delta_k) \cap [a, b] = \begin{cases} \mathrm{RR}(L^k \otimes F) & \text{if } [a, b] \cap \Sigma \neq \varnothing, \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

where F is the vector bundle with fibers $F_y = \mathrm{Im}\, 1_{[a,b]}(\square_y)$, $y \in M$.

The assumption that $a, b \in \mathbb{R} \setminus \Sigma$ guarantees that the number of eigenvalues of $\square_y$ in $[a, b]$ is constant, so that F is a genuine smooth vector bundle. $\mathrm{RR}(L^k \otimes F)$ depends polynomially on k, with leading term

$$\mathrm{RR}(L^k \otimes F) = (\mathrm{rank}\, F)\left(\frac{k}{2\pi}\right)^n \mu_{\mathrm{L}}(M) + \mathcal{O}(k^{n-1}).$$

The result is consistent with the Weyl law (3) because when $a, b \in \mathbb{R} \setminus \Sigma$ we have $N_y(b) = N_y(a) + \mathrm{rank}\, F$ for any $y \in M$.

Theorem 1.1 holds not only for the magnetic Laplacians (1), but also for other remarkable geometric operators, as for instance the holomorphic Laplacians or the square of spin-c Dirac operators. The corresponding results are stated in Theorems 3.4 and 3.6. In these cases, $\Sigma = \mathbb{N}$, so the spectrum of $k^{-1}\Delta_k$ consists of clusters at nonnegative integers, the dimension of each cluster being given by the Riemann–Roch number $\mathrm{RR}(L^k \otimes F)$, where F is a sum of tensor products of symmetric and exterior powers of $T^{1,0}M$; see part (3) of Theorem 3.6.

Theorem 1.1 is relevant only when Σ has several components. Note that the set of (ω, g, V) such that Σ is not connected is open in $\mathcal{C}^0$-topology. Let us discuss some examples where the fiber bundle F can be made explicit.

First if $j = j_B$ and $V = 0$, the set Σ is $\frac{1}{2}n + \mathbb{N}$, and for $a = \frac{1}{2}(n-1) + \ell$, with $\ell \in \mathbb{N}$ and $b = a + 1$, the bundle F in Theorem 1.1 is $\mathrm{Sym}^\ell(T^{1,0}M) \otimes A$. More generally, suppose that $B_1 = \cdots = B_n$, that is, $j_B = Bj$, with $B \in \mathcal{C}^\infty(M, \mathbb{R}_{>0})$. Then $\Sigma \subset \bigcup_{\ell \in \mathbb{N}} [\sigma_\ell^-, \sigma_\ell^+]$, where

$$\sigma_\ell^- = \inf\Big(B\Big(\ell + \frac{n}{2}\Big) + V_1\Big), \quad \sigma_\ell^+ = \sup\Big(B\Big(\ell + \frac{n}{2}\Big) + V_r\Big).$$

Assume there exist $a, b \in \mathbb{R}$, such that $\sigma_{\ell-1}^+ < a < \sigma_\ell^-$ and $\sigma_\ell^+ < b < \sigma_{\ell+1}^-$ for some $\ell \in \mathbb{N}$. Then Theorem 1.1 holds and $F = \mathrm{Sym}^\ell(T^{1,0}M) \otimes A$.

Since Σ is the support of the Lebesgue–Stieltjes measure $d\upsilon$, the Weyl law (3) implies that, for any $\lambda \in \Sigma$, the distance $d(\lambda, \mathrm{sp}(k^{-1}\Delta_k))$ tends to 0 as $k \to \infty$. To the contrary, if $\lambda \notin \Sigma$, by the second case of (4), there exists $\epsilon > 0$ such that $d(\lambda, \mathrm{sp}(k^{-1}\Delta_k)) \geqslant \epsilon$ when k is sufficiently large.

The following theorem gives more precise estimates.

Theorem 1.2. *For any $\Lambda > 0$, there exists $C > 0$ such that, for any $\lambda \leqslant \Lambda$,*

$$\lambda \in \Sigma \quad \Longrightarrow \quad \mathrm{dist}(\lambda, \mathrm{sp}(k^{-1}\Delta_k)) \leqslant C k^{-1/2}, \tag{5}$$

$$\lambda \in \mathrm{sp}(k^{-1}\Delta_k) \quad \Longrightarrow \quad \mathrm{dist}(\lambda, \Sigma) \leqslant C k^{-1/2}. \tag{6}$$

When the bundle F of Theorem 1.1 has a definite parity, see Remark 7.3, (6) can be slightly improved. For instance, if as above $j_B = Bj$ and there exist a, b such that $\sigma^+_{\ell-1} < a < \sigma^-_\ell$ and $\sigma^+_\ell < b < \sigma^-_{\ell+1}$, then $\mathrm{sp}(k^{-1}\Delta_k) \cap [a, b] \subset [\sigma^-_\ell, \sigma^+_\ell] + \mathcal{O}(k^{-1})$.

Interestingly, some local Weyl laws hold with a similar gapped structure. Instead of Σ, the local law at $y \in M$ involves the spectrum $\Sigma_y = \{\lambda_i(y) : i \in \mathbb{N}\}$ of $\square_y$, which is a discrete subset of $\mathbb{R}$. Clearly, $\Sigma = \bigcup_y \Sigma_y$.

For any $k \in \mathbb{N}$, choose an orthonormal eigenbasis $(\Psi_{k,i})_{i\in\mathbb{N}}$ of $k^{-1}\Delta_k$ such that $k^{-1}\Delta_k \Psi_{k,i} = \lambda_{k,i}\Psi_{k,i}$, with $\lambda_{0,k} \leqslant \lambda_{1,k} \leqslant \cdots$. For any $y \in M$ and real numbers $a < b$, define

$$N(y, a, b, k) = \sum_{i:\lambda_{k,i}\in[a,b]} |\Psi_{k,i}(y)|^2,$$

so we count the eigenvalues in $[a, b]$ with weights given by the square of the pointwise norm at y of the corresponding eigenvectors.

Theorem 1.3. *For any $\Lambda \in \mathbb{R} \setminus \Sigma$, $y \in M$ and $a, b \in\ ]-\infty, \Lambda] \setminus \Sigma_y$ such that $a < b$, the following holds: If $[a, b] \cap \Sigma_y$ is empty, then $N(y, a, b, k) = \mathcal{O}(k^{-\infty})$. Otherwise we have an asymptotic expansion*

$$N(y, a, b, k) = \left(\frac{k}{2\pi}\right)^n \sum_{\lambda\in\Sigma_y\cap[a,b]} \sum_{\ell=0}^{\infty} m_{\ell,\lambda} k^{-\ell} + \mathcal{O}(k^{-\infty}), \tag{7}$$

where the coefficients $m_{\ell,\lambda}$ do not depend on a, b, k. In particular, $m_{0,\lambda}$ is the multiplicity of the eigenvalue λ of $\square_y$.

We believe that the same result holds without the assumption that a, b are smaller than $\Lambda \in \mathbb{R} \setminus \Sigma$. Observe that the first-order term $\sum_{\lambda\in[a,b]} m_{0,\lambda}$ in (7) is merely the number of eigenvalues of $\square_y$ in $[a, b]$. In particular we recover the same structure as in the counting law (4) of Theorem 1.2: when the leading-order term is zero, $N(y, a, b, k) = \mathcal{O}(k^{-\infty})$. We interpret this as a gap in the local Weyl law.

Besides these gaps and clusters, another notable aspect in Theorems 1.1 and 1.3 is that we have full asymptotic expansions. For the Laplace–Beltrami operators or the Schrödinger operator without magnetic fields, the remainders in Weyl laws have a completely different behavior; see for instance the survey [Zelditch 2008, Section 8]. Another situation where clusters and gaps occur is for the pseudodifferential operators with a principal symbol having a periodic Hamiltonian flow. This has been studied in many papers; see for instance [Weinstein 1977; Colin de Verdière 1979], [Dozias 1997] for a semiclassical result and [Boutet de Monvel 1980; Boutet de Monvel and Guillemin 1981, Section 1] for earlier results, with

Riemann–Roch numbers already. For our magnetic Laplacians, the gaps are also connected to periodic Hamiltonians: the quantum harmonic oscillators $\mathfrak{a}_i^\dagger \mathfrak{a}_i$ of (2). In dimension 2, this lies at the origin of the cyclotron motion or resonance of a charged particle in a magnetic field.

1D. ***Schwartz kernels of spectral projectors.*** Another result we would like to emphasize in this introduction is the asymptotic description of the Schwartz kernel of $g(k^{-1}\Delta_k)$, where $g : \mathbb{R} \to \mathbb{C}$ is a bounded function with compact support satisfying some assumptions. These Schwartz kernels are by definition given at $(x, y) \in M^2$ by

$$g(k^{-1}\Delta_k)(x, y) = \sum_i g(\lambda_{k,i})\Psi_{k,i}(x) \otimes \overline{\Psi_{k,i}(y)} \in L_x^k \otimes A_x \otimes \bar{L}_y^k \otimes \bar{A}_y.$$

We will prove that $g(k^{-1}\Delta_k)$ belongs to the operator algebra $\mathcal{L}(A)$ introduced in [Charles 2024]. Let us recall the main characteristics of $\mathcal{L}(A)$; the complete definition will be given in Section 5.

$\mathcal{L}(A)$ consists of families $(P_k)_{k\in\mathbb{N}}$ such that, for any k, P_k is an endomorphism of $\mathcal{C}^\infty(M, L^k \otimes A)$ having a smooth Schwartz kernel in $\mathcal{C}^\infty(M^2, (L^k \otimes A) \boxtimes (\bar{L}^k \otimes \bar{A}))$ satisfying the following conditions. First, for any compact subset K of M^2 not intersecting the diagonal, for any N, we have $P_k(x, y) = \mathcal{O}(k^{-N})$ uniformly on K. Second, for any open set U of M identified with a convex open set of $\mathbb{R}^{2n}$ through a diffeomorphism, let $F \in \mathcal{C}^\infty(U^2, L \boxtimes \bar{L})$ be the unitary frame such that $F(x, y) = u \otimes \bar{v}$, where v is any vector in L_y with norm 1 and $u \in L_x$ is the parallel transport of v along the path $t \in [0, 1] \to y + t(x - y)$. We introduce a unitary trivialization of A on U and identify accordingly the sections of $A \boxtimes \bar{A}$ over U^2 with the functions of $\mathcal{C}^\infty(U^2, \mathbb{C}^r \otimes \bar{\mathbb{C}}^r)$. Then the Schwartz kernel of P_k has the following asymptotic expansion on U^2: for any $N \in \mathbb{N}$, for any $x \in U$ and $\xi \in T_xU$ such that $x + \xi \in U$,

$$P_k(x + \xi, x) = \left(\frac{k}{2\pi}\right)^n F^k(x + \xi, x)e^{-k|\xi|_x^2/4} \sum_{\ell=0}^{N} k^{-\ell} a_\ell(x, k^{1/2}\xi) + \mathcal{O}(k^{n-(N+1)/2}), \tag{8}$$

where $|\xi|_x^2 = \omega_x(\xi, j_x\xi)$, the coefficients $a_\ell(x, \cdot)$ are polynomial maps $T_xM \to \mathbb{C}^r \otimes \bar{\mathbb{C}}^r$ depending smoothly on x, and the $\mathcal{O}$ is uniform when $(x + \xi, x)$ runs over any compact set of U^2.

Such an operator $P = (P_k)$ has a symbol $\sigma_0(P)$, which at $y \in M$ is the endomorphism of $\mathcal{D}(T_yM) \otimes A_y$ defined by

$$(\sigma_0(P)(y))(f)(u) = (2\pi)^{-n} \int_{T_yM} e^{(u-v)\cdot\bar{v}} a_0(y, u - v) f(v)\, d\mu_y(v).$$

Here, the scalar product $u \cdot \bar{v}$ and the measure μ_y are defined in terms of linear complex coordinates $z_i : T_yM \to \mathbb{C}$ associated to an orthonormal frame of $(T_y^{1,0}M, h_y)$ by $u \cdot \bar{v} = \sum z_i(u)\overline{z_i(v)}$ and $\mu_y = |dz_1 \cdots dz_n d\bar{z}_1 \cdots d\bar{z}_n|$.

As a result, for any $(P_k) \in \mathcal{L}(A)$, we have $\|P_k\| = \mathcal{O}(1)$ and

$$\|P_k\| = \mathcal{O}(k^{-1/2}) \iff \sigma_0(P)(y) = 0 \text{ for all } y \in M \iff a_0(y, \cdot) = 0 \text{ for all } y \in M.$$

Furthermore $\mathcal{L}(A)$ is closed under composition and the map σ_0 is an algebra morphism. Here the product of the symbols at y is the composition of endomorphisms of $\mathcal{D}(T_yM) \otimes A_y$, which is not commutative.

Theorem 1.4. (1) *For any* $a, b \in \mathbb{R} \setminus \Sigma$, *the spectral projector* $\Pi_k := 1_{[a.b]}(k^{-1}\Delta_k)$ *and* $k^{-1}\Delta_k \Pi_k$ *belong to* $\mathcal{L}(A)$ *and their symbols at* y *are equal to* $1_{[a,b]}(\square_y)$ *and* $\square_y 1_{[a,b]}(\square_y)$ *respectively.*

(2) *For any* $\Lambda \in \mathbb{R} \setminus \Sigma$, *for any* $g \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$ *such that* $\operatorname{supp} g \subset]-\infty, \Lambda]$, $(g(k^{-1}\Delta_k))_k$ *belongs to* $\mathcal{L}(A)$ *and its symbol at* y *is* $g(\square_y)$.

The second assertion is actually a generalization of the first one because choosing $\Lambda > b$ such that $[b, \Lambda] \cap \Sigma = \varnothing$, one has $1_{[a,b]} = g$ on an open neighborhood of Σ with $g \in \mathcal{C}^\infty(\mathbb{R})$ supported in $]-\infty, \Lambda]$, and by Theorem 1.1, $1_{[a,b]}(\lambda) = g(\lambda)$ for any $\lambda \in \operatorname{sp}(k^{-1}\Delta_k)$ when k is sufficiently large.

1E. ***Comparison with earlier results.*** This work started as a collaboration with Yuri Kordyukov and some of the results presented here appeared also in [Kordyukov 2022]: the existence of spectrum gaps, that is, (4) when $[a, b] \cap \Sigma = \varnothing$, and a weak version of (6) with a $\mathcal{O}(k^{-1/4})$ instead of the $\mathcal{O}(k^{-1/2})$ are proved in [loc. cit., Theorem 1.2]. Moreover, under the assumption of Theorem 1.4, the Schwartz kernel of the spectral projector $\Pi_k = 1_{[a,b]}(k^{-1}\Delta_k)$ is described in [loc. cit., Theorem 1.6] in a way similar to our result.

In the case where $j_B = j$ and V is constant, the existence of spectrum gaps, that is, (4) when $[a, b] \cap \Sigma = \varnothing$, was proved in [Faure and Tsujii 2015, Theorem 10.2.2]. Our proof will follow the same line as in that work and is similar to the proof in [Kordyukov 2022].

In the case again where $j_B = j$ and $V = 0$, the first gap and the asymptotic description of the first cluster has a long history. When j is integrable so that M is a complex manifold and ω is Kähler, the gap follows from Kodaira vanishing theorem, the first cluster consists of the holomorphic sections of L^k, its dimension is given by the Riemann–Roch–Hirzebruch theorem, and the Schwartz kernel of the corresponding spectral projector is the Bergman kernel, whose asymptotic can be deduced from [Boutet de Monvel and Sjöstrand 1976] and which has been used in many papers starting from [Zelditch 1998]. The extension to almost-complex structure was done in [Guillemin and Uribe 1988; Borthwick and Uribe 2007; Ma and Marinescu 2008]. Parallel results for spin-c Dirac operators were proved in [Borthwick and Uribe 1996; Ma and Marinescu 2002; 2007].

The main tool we use in this paper is the algebra $\mathcal{L}(A)$ introduced in [Charles 2024]; a first weaker version was proposed in [Charles 2016]. The asymptotic expansions (8) or similar versions have been used before by several authors to describe the spectral projector on the first cluster and corresponding Toeplitz operators, [Shiffman and Zelditch 2002; Charles 2003; Ma and Marinescu 2007] for instance. In [Charles 2024], besides establishing the main properties of $\mathcal{L}(A)$, we considered some projectors (Π_k) in $\mathcal{L}(A)$ whose symbol at $y \in M$ is the projector onto the m-th level of a Landau Hamiltonian $\sum \mathfrak{a}_i^\dagger \mathfrak{a}_i$. In particular we computed the rank of Π_k as a Riemann–Roch number and we studied the corresponding Toeplitz algebra. By the results of the current paper, particular instances of such projectors are the spectral projectors on the m-cluster of a magnetic Laplacian with $j_B = j$ and $V = 0$.

In a different context, many works have been devoted to the magnetic Schrödinger operator in $\mathbb{R}^n$; see [Raymond 2017] for a general overview. The most significant result is a semiclassical description of the bottom of the spectrum in terms of effective operators whose principal symbols are the functions we denoted by λ_i; see for instance [Ivrii 1998, Theorem 6.2.7], [Raymond and Vũ Ngọc 2015, Theorem 1.6] or [Morin 2020, Theorem 2] for a statement in the manifold setting. These works differ in at least two ways from the current paper: The global gap assumption is generally replaced by a confinement hypothesis; typically the function we denote by λ_0 is assumed to have a nondegenerate minimum. Moreover, the general strategy is to put the Schrödinger operator on a normal form by conjugating it with a convenient Fourier integral operator.

1F. ***Outline of the paper.*** The main idea in the first part of the paper is to approximate the Laplacian Δ_k locally by a family of Laplacians $\Delta_{y,k}$, $y \in M$, obtained from Δ_k by "freezing" the coordinates at y. In Section 2 we introduce these operators, recall the basic results regarding their spectrum and explain the relationship with the operators $\square_y$ of Section 1B. In Section 3, we introduce a class of Laplacians slightly more general than the magnetic Laplacians Δ_k and which are well-approximated by the $\Delta_{y,k}$. This class contains the holomorphic Laplacians and some of their generalizations without integrable complex structure. In Section 4, we prove a weak version of Theorem 1.2 which says that $\operatorname{sp}(k^{-1}\Delta_k) \to \Sigma$ in the limit $k \to \infty$, by constructing on one hand some peaked sections which are approximate eigenmodes of Δ_k, and on the other hand, by inverting $\lambda - k^{-1}\Delta_k$ up to a $\mathcal{O}(k^{-1/4})$ when $\lambda \notin \Sigma$.

In the second part of the paper, Sections 5 and 6, we introduce the algebra $\mathcal{L}(A)$ and prove that the spectral projector $1_{[a,b]}(k^{-1}\Delta_k)$ belongs to $\mathcal{L}(A)$ when $a, b \in \mathbb{R} \setminus \Sigma$. The proof is divided into three steps: From the resolvent estimate of Section 4, we deduce that any operator of $\mathcal{L}(A)$ having symbol $1_{[a,b]}(\square)$ is an approximation of $1_{[a,b]}(k^{-1}\Delta_k)$ up to a $\mathcal{O}(k^{-1/4})$. We then prove that $\mathcal{L}(A)/\mathcal{O}(k^{-\infty})$ has a unique self-adjoint projector having symbol $1_{[a,b]}(\square)$ and commuting with Δ_k. Finally we prove that this operator is indeed the spectral projector.

In the last part, Section 7, we establish some spectral properties for the Toeplitz operators associated to the projectors of $\mathcal{L}(A)$, including a sharp Gårding inequality and the functional calculus. Then we deduce Theorems 1.1 and 1.3 and the second part of Theorem 1.4.

2. The linear pointwise data

In this section we consider a compact manifold M^{2n} equipped with a symplectic form ω and a Riemannian metric g. Let $A \to M$ be a Hermitian vector bundle with a section V of $\mathcal{C}^\infty(M, \operatorname{End} A)$ such that $V(x)$ is Hermitian for any $x \in M$. We choose a point $y \in M$.

2A. ***The complex structure.*** Let $j_{B,y}$ be the endomorphism of T_yM such that $\omega_y(\xi,\eta) = g_y(j_{B,y}\xi,\eta)$. It will be useful to work with the following normal form.

Lemma 2.1. *There exists $0 < B_1(y) \leqslant \cdots \leqslant B_n(y)$ such that T_yM has a basis (e_i, f_i) satisfying*

$$\omega_y(e_i,e_j) = \omega_y(f_i,f_j) = 0, \quad \omega_y(e_i,f_j) = \delta_{ij},$$
$$j_{B,y}e_i = B_i(y)f_i, \qquad j_{B,y}f_i = -B_i(y)e_i.$$

The vectors $u_i = \frac{1}{\sqrt{2}}(e_i - if_i)$, $\bar{u}_i = \frac{1}{\sqrt{2}}(e_i + if_i)$ are a basis of $T_yM \otimes \mathbb{C}$ and

$$\frac{1}{i}\omega_y(u_i,u_j) = \frac{1}{i}\omega(\bar{u}_i,\bar{u}_j) = 0, \quad \frac{1}{i}\omega(u_i,\bar{u}_j) = \delta_{ij},$$
$$j_{B,y}u_i = iB_i(y)u_i, \qquad j_{B,y}\bar{u}_i = -iB_i(y)\bar{u}_i.$$

Proof. Since $j_{B,y}$ is a g_y-antisymmetric invertible endomorphism of T_yM, there exists a g_y-orthonormal basis $(\tilde{e}_i, \tilde{f}_i)$ such that $j_{B,y}\tilde{e}_i = B_i(y)\tilde{f}_i$ and $j_{B,y}\tilde{f}_i = -B_i(y)\tilde{e}_i$, where the $B_i(y)$ are positive. We set $e_i = (B_i(y))^{-1/2}\tilde{e}_i$ and $f_i = (B_i(y))^{-1/2}\tilde{f}_i$, and the result follows by direct computations. □

We can interpret this result as follows: first, $j_{B,y}/i$ is $\mathbb{C}$-diagonalizable with only nonzero real eigenvalues, denoted by $\pm B_i(y)$. Second, the subspace W of $T_yM \otimes \mathbb{C}$ spanned by the u_i is the sum of the eigenspaces of $j_{B,y}/i$ with a positive eigenvalue. W is Lagrangian and the sesquilinear form h_y of $T_yM \otimes \mathbb{C}$ given by $h_y(u, v) = \omega_y(u, \bar{v})/i$ is positive on W. Equivalently the endomorphism j_y of T_yM such that $j_y = i$ on W is a complex structure of T_yM compatible with ω_y. So from now on, we will denote $W = \operatorname{Ker}(j_y - i)$ by $T_y^{1,0}M$, and by the definition of j_y, the restriction of $j_{B,y}/i$ to $T_y^{1,0}M$ is a positive endomorphism of $(T_y^{1,0}M, h_y)$ with eigenvalues the $B_i(y)$. Hence the vectors (u_i) in Lemma 2.1 are nothing else than a h_y-orthonormal eigenbasis of $T_y^{1,0}M$.

An important remark is that j_y depends smoothly on y, so it defines an almost complex structure of M. Indeed, the space $T_y^{1,0}M$ depends smoothly on y because $j_{B,y}/i$ being invertible, no eigenvalue can cross 0. Another reason is that $j_y = |j_{B,y}|^{-1} j_{B,y}$, where $|j_{B,y}|$ is the positive square root of the g_y-positive endomorphism $-j_{B,y}^2$. Actually, the construction of j is the classical proof of the fact that any symplectic manifold admits a compatible almost-complex structure; see [McDuff and Salamon 2017, Proposition 2.5.6].

To the contrary, in general, we cannot choose a local continuous symplectic frame (e_i, f_i) of TM such that $j_B e_i = B_i f_i,\ j_B f_i = -B_i e_i$, even if we renumber the eigenvalues $B_i(y)$ in a way depending on y. Indeed, as is well known, it is not possible in general to diagonalize smoothly a symmetric matrix, the symmetric matrix being $-(j_{B,y})^2$ in our case. More specifically, consider on $\mathbb{R}^2 \otimes \mathbb{R}^2$ with its usual Euclidean structure the endomorphism $j_B(s, t) = M(s, t) \otimes j_2$, where

$$M(s, t) = \begin{pmatrix} 1+s & t \\ t & 1-s \end{pmatrix}, \quad j_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with s and t parameters in a neighborhood of 0. Then j_B is nondegenerate and antisymmetric, and we can choose for each (s, t) a basis (e_i, f_i) satisfying the previous conditions, but not continuously with respect to (s, t). Indeed, $-j_B^2(s, t) = M^2(s, t) \otimes \mathrm{id}$ and for $s = 0,\ t$ small nonzero, the eigenspaces of $M(s, t)$ are $(1, 1)\mathbb{R}$ and $(1, -1)\mathbb{R}$, whereas for $t = 0$ and s small nonzero, they are $(1, 0)\mathbb{R}$ and $(0, 1)\mathbb{R}$.

This example appears on $\mathbb{R}^4$ equipped with its usual Euclidean metric and the closed form

$$\omega = (1 + p_1)\, dp_1 \wedge dq_1 + (1 - p_2)\, dp_2 \wedge dq_2 + q_1\, dq_1 \wedge dp_2 - q_2\, dp_1 \wedge dq_2,$$

which is symplectic on a neighborhood of the origin. On the plane $\{p_1 = p_2 : q_1 = q_2\}$, the matrix of j_B is $M(p_1, q_1) \otimes j_2$.

We have also to be careful that the metric $\tilde{g}$ determined by (ω, j),

$$\tilde{g}_y(\xi, \eta) := \omega_y(\xi, j_y\eta) = g_y(|j_{B,y}|\xi, \eta), \tag{9}$$

is equal to g_y only when $B_1(y) = \cdots = B_n(y) = 1$, that is, when $j_{B,y}$ is itself a complex structure.

2B. ***The scalar Laplacian of T_yM.*** Consider now the covariant derivative

$$\nabla = d + \frac{1}{i}\alpha : \mathcal{C}^\infty(T_yM) \to \Omega^1(T_yM), \tag{10}$$

where $\alpha \in \Omega^1(T_yM, \mathbb{R})$ is given by $\alpha_\xi(\eta) = \frac{1}{2}\omega_y(\xi, \eta)$. Since $d\alpha = \omega_y$, the curvature of ∇ is ω_y/i. We then define the scalar Laplacian of T_yM by

$$\Delta_y^{\text{scal}} := \tfrac{1}{2}\nabla^*\nabla : \mathcal{C}^\infty(T_yM) \to \mathcal{C}^\infty(T_yM). \tag{11}$$

Here the scalar products of $\mathcal{C}^\infty(T_yM)$ and $\Omega^1(T_yM)$ are defined by integrating the pointwise scalar products against a fixed constant volume form, the pointwise scalar product of $\Omega^1(T_yM)$ is defined from the metric g_y.

We can explicitly compute the spectrum and eigenfunctions of Δ_y^{scal} as follows. We introduce a basis (e_i, f_i) of T_yM as in Lemma 2.1. This basis is g_y-orthogonal and $g_y(e_i, e_i) = g_y(f_i, f_i) = B_i(y)^{-1}$, so we have

$$\Delta_y^{\text{scal}} = -\frac{1}{2}\sum_{i=1}^n B_i(y)(\nabla_{e_i}^2 + \nabla_{f_i}^2) = \sum_{i=1}^n B_i(y)\big(-\nabla_{u_i}\nabla_{\bar{u}_i} + \tfrac{1}{2}\big),$$

where $u_i = \frac{1}{\sqrt{2}}(e_i - if_i)$, $\bar{u}_i = \frac{1}{\sqrt{2}}(e_i + if_i)$. Denote by z_i the linear complex coordinates dual to the u_i. If (p_i, q_i) are the real linear coordinates of T_yM in the basis (e_i, f_i), then $z_i = \frac{1}{\sqrt{2}}(p_i + iq_i)$. Since $\omega_y = i\sum dz_i \wedge d\bar{z}_i$, we have

$$\nabla = d + \frac{1}{2}\sum_{i=1}^n (z_i\, d\bar{z}_i - \bar{z}_i\, dz_i).$$

We introduce the function $s(\xi) := \exp(-|\xi|_y^2/4)$, $\xi \in T_yM$, where

$$|\xi|_y^2 = \sum_i (p_i^2 + q_i^2) = 2\sum_i |z_i|^2 = \tilde{g}_y(\xi, \xi).$$

Since $s = \exp(-|z|^2/2)$, we have $\nabla_{\bar{u}_i} s = 0$, so s is ∇-holomorphic.

Let us consider $\mathcal{C}^\infty(T_yM)$ as the space of sections of the trivial line bundle over T_yM and let us use s as a global frame. We introduce the operators

$$\mathfrak{a}_i = \partial_{\bar{z}_i}, \quad \mathfrak{a}_i^\dagger = \bar{z}_i - \partial_{z_i}. \tag{12}$$

Then $\nabla_{\bar{u}_i}(fs) = (\mathfrak{a}_i f)s$ and $\nabla_{u_i}(fs) = -(\mathfrak{a}_i^\dagger f)s$, so that

$$\Delta_y^{\text{scal}}(fs) = (\widetilde{\square}_y^{\text{scal}} f)s, \quad \text{with } \widetilde{\square}_y^{\text{scal}} := \sum_{i=1}^n B_i(y)\big(\mathfrak{a}_i^\dagger\mathfrak{a}_i + \tfrac{1}{2}\big). \tag{13}$$

Let $\mathcal{P}(T_yM)$ be the space of polynomial functions $T_yM \to \mathbb{C}$, not necessarily holomorphic or antiholomorphic. With the coordinates (z_i), $\mathcal{P}(T_yM) = \mathbb{C}[z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n]$. Observe that $\mathfrak{a}_i$ and $\mathfrak{a}_i^\dagger$ preserve $\mathcal{P}(T_yM)$ and the same holds for $\widetilde{\square}_y^{\text{scal}}$.

Since the $\mathfrak{a}_i$, $\mathfrak{a}_i^\dagger$ satisfy the so-called canonical commutation relations

$$[\mathfrak{a}_i, \mathfrak{a}_j] = [\mathfrak{a}_i^\dagger, \mathfrak{a}_j^\dagger] = 0, \quad [\mathfrak{a}_i, \mathfrak{a}_j^\dagger] = \delta_{ij},$$

we deduce by a classical argument that the endomorphisms $\mathfrak{a}_i^\dagger \mathfrak{a}_i$ of $\mathcal{P}(T_yM)$ are mutually commuting endomorphisms, each of them diagonalizable with spectrum $\mathbb{N}$; see for instance [Charles 2024, Proposition 4.1]. So we have a decomposition into joint eigenspaces

$$\mathcal{P}(T_yM) = \bigoplus_{\alpha \in \mathbb{N}^n} \mathfrak{L}_\alpha, \quad \text{with } \mathfrak{L}_\alpha = \bigcap_{i=1}^n \operatorname{Ker}(\mathfrak{a}_i^\dagger \mathfrak{a}_i - \alpha(i)). \tag{14}$$

Furthermore, $\mathfrak{L}_0 = \mathbb{C}[z_1, \dots, z_n]$ and $\mathfrak{L}_\alpha = (\mathfrak{a}^\dagger)^\alpha \mathfrak{L}_0$ for all $\alpha \in \mathbb{N}^n$ where $(\mathfrak{a}^\dagger)^\alpha = (\mathfrak{a}_1^\dagger)^{\alpha(1)} \cdots (\mathfrak{a}_n^\dagger)^{\alpha(n)}$. Consequently $\widetilde{\square}_y^{\text{scal}}$ is a diagonalizable endomorphism of $\mathcal{P}(T_yM)$ with spectrum Σ_y^{scal} given by

$$\Sigma_y^{\text{scal}} = \left\{ \sum_{j=1}^n B_j(y)\big(\alpha(j) + \tfrac{1}{2}\big) : \alpha \in \mathbb{N}^n \right\}. \tag{15}$$

Moreover the eigenspace $\mathcal{E}(\lambda)$ of the eigenvalue $\lambda \in \Sigma_y$ is the sum of the $\mathfrak{L}_\alpha$, where α runs over the multi-indices of $\mathbb{N}^n$ such that $\sum B_i(y)\big(\alpha(i) + \frac{1}{2}\big) = \lambda$.

We can deduce from these algebraic facts the L^2-spectral theory of Δ_y^{scal}. First of all, the space $\exp(-|\xi|_y^2/4)\mathcal{P}(T_yM)$ is dense in $L^2(T_yM)$ by the same proof that Hermite functions are dense. So we deduce from (14) a decomposition of $L^2(T_yM)$ in a Hilbert sum of orthogonal subspaces,

$$L^2(T_yM) = \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{K}_\alpha, \qquad \mathcal{K}_\alpha = \overline{e^{-|\xi|_y^2/4}\mathfrak{L}_\alpha}^{L^2(T_yM)} \quad \text{for all } \alpha \in \mathbb{N}^n \tag{16}$$

Let $\mathcal{G}$ be the subspace of $L^2(T_yM)$ consisting of the ψ having a decomposition $\sum \psi_\alpha$ in (16) such that $\sum |\alpha|^2 \, \|\psi_\alpha\|^2$ is finite. As a differential operator, Δ_y^{scal} acts on the distribution space $\mathcal{C}^{-\infty}(T_yM)$ and in particular on $L^2(T_yM)$. It is not difficult to see that $\mathcal{G}$ consists of the $\psi \in L^2(T_yM)$ such that $\Delta_y^{\text{scal}}\psi \in L^2(T_yM)$.

Lemma 2.2. *$(\Delta_y^{\text{scal}}, \mathcal{G})$ is a self-adjoint unbounded operator of $L^2(T_yM)$, which is the closure of $(\Delta_y^{\text{scal}}, e^{-|\xi|_y^2/4}\mathcal{P}(T_yM))$. Its spectrum is Σ_y and consists only of eigenvalues, the eigenspace of $\lambda \in \Sigma_y$ being the closure of* $\exp(-|\xi|_y^2/4)\mathcal{E}(\lambda)$.

This follows from (16), (14) and (13) by elementary standard arguments; see for instance [Davies 1995, Lemma 1.2.2]. Even if we won't need this in the sequel, it can be useful to note that

- the eigenspace $\mathcal{K}_0$ is the Bargmann space: $\psi \in \mathcal{K}_0$ if and only if $\psi \in L^2(T_yM)$ and $\psi = e^{-|\xi|_y/4} f(z_1, \dots, z_n)$ with f holomorphic.
- $\mathcal{G}$ is different from the Sobolev space $H^2 = \big\{\psi \in L^2(T_yM) : \sum(\partial_{p_i}^2 + \partial_{q_i}^2)\psi \in L^2\big\}$. Actually, $H^2 \cap \mathcal{G} = \mathcal{G} \cap \overline{\mathcal{G}} = H^2_{\text{iso}}(T_yM)$, the isotropic Sobolev space defined as $\big\{\psi \in H^2(T_yM) : \sum({p_i}^2 + {q_i}^2)\psi \in L^2\big\}$.

2C. ***The A_y-valued Laplacian $\boldsymbol{\Delta_y}$.*** We now consider the full Laplacian

$$\Delta_y := \Delta_y^{\text{scal}} + V(y) : \mathcal{C}^\infty(T_yM, A_y) \to \mathcal{C}^\infty(T_yM, A_y). \tag{17}$$

We deduce from the properties of Δ_y^{scal} that $(\Delta_y, \mathcal{G} \otimes A_y)$ is a selfadjoint unbounded operator of $L^2(T_yM) \otimes A_y$ with discrete spectrum

$$\Sigma_y = \left\{ \sum_{j=1}^{n} B_j(y)\big(\alpha(j) + \tfrac{1}{2}\big) + V_\ell(y) : \alpha \in \mathbb{N}^n,\ \ell = 1, \dots, r \right\}, \tag{18}$$

where $V_1(y) \leqslant \cdots \leqslant V_r(y)$ are the eigenvalues of $V(y)$. Let (ζ_ℓ) be an eigenbasis of $V(y)$, $V(y)\zeta_\ell = V_\ell(y)\zeta_\ell$. Then any $\lambda \in \Sigma_y$ is an eigenvalue of Δ_y with eigenspace the closure of the sum of the $\exp(-|\xi|_y^2/4)\mathcal{E}(\lambda') \otimes \mathbb{C}\zeta_\ell$ such that $\lambda' + V_\ell(y) = \lambda$.

In the sequel, we will mainly work with

$$\widetilde{\Box}_y = e^{|\xi|_y^2/4} \Delta_y e^{-|\xi|_y^2/4} = \widetilde{\Box}_y^{\text{scal}} + V(y) \tag{19}$$

acting on $\mathcal{P}(T_yM) \otimes A_y$.

2D. *The set Σ and the function v.* Denote by $\lambda_1(y) \leqslant \lambda_2(y) \leqslant \cdots$ the eigenvalues of $\Box_y$ ordered and repeated according to their multiplicities. Let

$$\Sigma = \bigcup_{y \in M} \Sigma_y = \bigcup_{\ell} \lambda_\ell(M).$$

We introduce for any $y \in M$ the counting function $N_y(\lambda) = \sharp\{\ell : \lambda_\ell(y) \leqslant \lambda\}$ of $\Box_y$. Let $v : \mathbb{R} \to \mathbb{R}$ be the nondecreasing function $v(\lambda) := \int_M N_y(\lambda)\, d\mu_{\mathrm{L}}(y)$.

Lemma 2.3. *The functions λ_ℓ are continuous. Σ is locally a finite union of closed bounded intervals*; *it is the support of the Lebesgue–Stieltjes measure dv.*

Proof. First the functions B_i and V_j are continuous, so, for any $\alpha \in \mathbb{N}^n$ and j, $f_{\alpha,j} := \sum_i B_i\big(\alpha(i) + \frac{1}{2}\big) + V_j$ is continuous as well. Since M is compact, $c := \inf_{y \in M} B_1(y)$ is positive. Then $f_{\alpha,j} \geqslant c|\alpha| + \inf V_1$, with $|\alpha| = \alpha(1) + \cdots + \alpha(n)$. Thus, for any $\Lambda \in \mathbb{R}$, $f_{\alpha,j} \geqslant \Lambda$ except for a finite number of (α, j). Since $\Sigma = \bigcup_{\alpha,j} f_{\alpha,j}(M)$ and $f_{\alpha,j}(M)$ is compact, this proves that $\Sigma \cap \,]-\infty, \Lambda]$ is a finite union of closed bounded intervals. By the same reason, for any $y \in M$ and $\Lambda \in \mathbb{R}$, $f_{\alpha,j}(y) = \Lambda$ only for a finite number of (α, j). From this we deduce readily that the functions λ_ℓ are continuous.

For any λ, the function $y \to N_y(\lambda)$ takes only integral values. It is measurable because, for any ℓ, $\{y : N_y(\lambda) = \ell\} = \{\lambda_\ell \leqslant \lambda\} \cap \{\lambda_{\ell+1} > \lambda\}$ is the intersection of an open set with a closed set. So $v(\lambda)$ is well-defined. Then v is clearly nondecreasing, and the associated Lebesgue–Stieltjes measure $\nu = dv$ is defined by $\nu([a, b]) = v(b^+) - v(a^-)$.

Now $\lambda \notin \operatorname{supp}(\nu)$ if and only if v is constant on a neighborhood of λ. If $\lambda \notin \Sigma$, then there exists ℓ and $\epsilon > 0$ such that $\lambda_\ell \leqslant \lambda - \epsilon$ and $\lambda + \epsilon \leqslant \lambda_{\ell+1}$; thus, for any y, $N_y(\lambda - \epsilon) = N_y(\lambda + \epsilon)$ and so $v(\lambda - \epsilon) = v(\lambda + \epsilon)$, giving $\lambda \notin \operatorname{supp} \nu$. Conversely, if, for some $\epsilon > 0$, $v(\lambda - \epsilon) = v(\lambda + \epsilon)$, then $N_y(\lambda - \epsilon) = N_y(\lambda + \epsilon)$ for any $y \in A$, where $M \setminus A$ has measure zero. Since A is dense, this implies that $\lambda_\ell \leqslant \lambda - \epsilon$ and $\lambda_{\ell+1} \geqslant \lambda + \epsilon$, with $\ell = N_y(\lambda)$, so $\lambda \notin \Sigma$. ☐

2E. *The restriction $\Box_y$ of $\widetilde{\Box}_y$ to antiholomorphic polynomials.* Since the spaces $\mathcal{L}_\alpha$ in (14) are infinite-dimensional, the eigenvalues of $\widetilde{\Box}_y$ are infinitely degenerate. We can avoid this degeneracy by replacing $\mathcal{P}(T_yM)$ by the subspace $\mathcal{D}(T_yM) \subset \mathcal{P}(T_yM)$ of antiholomorphic polynomials. With the coordinates (z_i)

introduced previously, $\mathcal{D}(T_yM) = \mathbb{C}[\bar{z}_1, \dots, \bar{z}_n]$. This point is central in our treatment since it will lead us to the definition of the fiber bundle F of Theorem 1.1.

First, the annihilation and creation operators $\mathfrak{a}_i$, $\mathfrak{a}_i^\dagger$ preserve the subspace $\mathcal{D}(T_yM)$ in which they act respectively by $\partial_{\bar{z}_i}$ and $\bar{z}_i$. Moreover the joint eigenspaces $\mathfrak{L}_\alpha$ of the $\mathfrak{a}_i^\dagger \mathfrak{a}_i$ satisfy $\mathfrak{L}_\alpha \cap \mathcal{D}(T_xM) = \mathbb{C}\, \bar{z}^\alpha$. So $\widetilde{\Box}_y$ preserves $\mathcal{D}(T_yM) \otimes A_y$ and its restriction $\Box_y \in \operatorname{End}(\mathcal{D}(T_yM) \otimes A_y)$ has the same spectrum as $\widetilde{\Box}_y$. For any eigenvalue λ, the corresponding eigenspaces of $\widetilde{\Box}_y$ and $\Box_y$ are $\bigoplus \mathfrak{L}_\alpha \otimes \mathbb{C}\zeta_\ell$ and $\bigoplus \mathbb{C}\bar{z}^\alpha \otimes \mathbb{C}\zeta_\ell$ respectively, where in both cases we sum over the (α, ℓ) such that $\sum B_i(y)\big(\alpha(i) + \frac{1}{2}\big) + V_\ell(y) = \lambda$.

For any $p \in \mathbb{N}$, the endomorphism $\Box_y$ preserves the subspace $\mathcal{D}_{\leqslant p}(T_yM)$ of $\mathcal{D}(T_yM)$ of polynomials with degree smaller than p. These spaces are obviously finite-dimensional and their union $\mathcal{D}_{\leqslant p}(TM) = \bigcup_y \mathcal{D}_{\leqslant p}(T_yM)$ is a genuine vector bundle over M. Moreover $y \mapsto \Box_y|_{\mathcal{D}_{\leqslant p}(T_yM)}$ is a smooth section of $\operatorname{End}(\mathcal{D}_{\leqslant p}(TM))$.

Lemma 2.4. (1) *For any $\Lambda > 0$, there exists $p \in \mathbb{N}$ such that, for any $y \in M$ and $\lambda \in \operatorname{sp}(\Box_y) \cap]-\infty, \Lambda]$, the eigenspace* $\operatorname{Ker}(\Box_y - \lambda)$ *is contained in $\mathcal{D}_{\leqslant p}(T_yM) \otimes A_y$.*

(2) *For any compact interval I whose endpoints do not belong to Σ, the spaces*

$$F_y = \bigoplus_{\lambda \in \operatorname{sp}(\Box_y) \cap I} \operatorname{Ker}(\lambda - \Box_y), \qquad y \in M,$$

are the fibers of a subbundle F of $\mathcal{D}_{\leqslant p}(TM) \otimes A$, with p a sufficiently large integer.

Proof. As in the beginning of the proof of Lemma 2.3, $\sum B_i(y)\big(\alpha(i) + \frac{1}{2}\big) + V_\ell(y) \leqslant \Lambda$ implies $c|\alpha| \leqslant \Lambda - \inf V_1$, with $c = \inf B_1 > 0$, which proves the first assertion with p any integer larger than $c^{-1}(\Lambda - \inf V_1)$.

Since I is bounded, by the first part, $F_y \subset \mathcal{D}_{\leqslant p}(T_yM) \otimes A_y$ for any y, when p is sufficiently large. The projector of $\operatorname{End}(\mathcal{D}_{\leqslant p}(T_yM) \otimes A_y)$ onto F_y is given by the Cauchy integral formula

$$(2\pi i)^{-1} \int_\gamma (\lambda - \Box_{y,p})^{-1}\, d\lambda, \tag{20}$$

where $\Box_{y,p}$ is the restriction of $\Box_y$ to $\mathcal{D}_{\leqslant p}(T_yM) \otimes A_y$ and γ is a loop of $\mathbb{C} \setminus \Sigma_y$ which encircles I. By the assumption that the endpoints of I do not belong to Σ, we can choose γ independent of y. Hence (20) depends smoothly on y and its image F_y as well. $\square$

3. A class of magnetic Laplacians

Consider a compact Riemannian manifold (M, g) equipped with a Hermitian line bundle L with a connection ∇ of curvature ω/i, and a Hermitian vector bundle A with a section $V \in \mathcal{C}^\infty(M, \operatorname{End}(A))$ such that $V(x)$ is Hermitian for any $x \in M$.

The results we will prove later hold for families of differential operators

$$(\Delta_k : \mathcal{C}^\infty(M, L^k \otimes A) \to \mathcal{C}^\infty(M, L^k \otimes A),\ k \in \mathbb{N})$$

having the following local form: for any coordinate chart (U, x_i) of M and trivialization $A|_U \simeq U \times \mathbb{A}$, we have on U by identifying $\mathcal{C}^\infty(U, L^k \otimes A)$ with $\mathcal{C}^\infty(U, L^k \otimes \mathbb{A})$ that

$$\Delta_k = -\frac{1}{2} \sum g^{ij} \nabla_{i,k} \nabla_{j,k} + kV + \sum a_i \nabla_{i,k} + b, \tag{B}$$

where $g^{ij} = g(dx_i, dx_j)$, $\nabla_{i,k}$ is the covariant derivative of L^k with respect to ∂_{x^i}, and a_i, b are in $\mathcal{C}^\infty(U, \mathrm{End}(\mathbb{A}))$ and do not depend on k.

In this section, we will prove that various operators have the form (B): the magnetic Laplacians (1) defined in the Introduction, the holomorphic Laplacians and also some generalized Laplacians associated to semiclassical Dirac operators.

3A. ***About Assumption* (B).** The proof that some operators satisfy Assumption (B) consists in each case of establishing a Weitzenböck-type formula. Since we don't need to give a geometric definition of the coefficients a_i and b in (B), the computations will be rather simple once we know which terms to neglect. To give a systematic treatment and to have a better understanding of the approximations we do, we will introduce noncommutative symbols for the differential operator algebra generated by the $\nabla_{i,k}$ and $\mathcal{C}^\infty(U, \mathrm{End}\,\mathbb{A})$. Instead of the full algebra, we will only work with second-order operators. Everything in this section works without assuming that ω is degenerate, the dimension of M could be odd as well, but we will not insist on that.

Let (e_i) be a frame of TM on an open set U of M and $\mathbb{A}$ be a Hermitian vector space. Let $\nabla_{i,k}$ be the covariant derivation of $\mathcal{C}^\infty(U, L^k)$ with respect to e_i. For any $y \in M$, let $\nabla_{y,i}$ be the covariant derivative of $\mathcal{C}^\infty(T_yM)$ for the connection (10) with respect to $e_i(y)$.

We say that a family $P = (P_k : \mathcal{C}^\infty(U, L^k \otimes \mathbb{A}) \to \mathcal{C}^\infty(U, L^k \otimes \mathbb{A}),\ k \in \mathbb{N})$ of differential operators belongs to $\mathcal{G}_2$ if it has the form

$$P_k = \sum_{i \leqslant j} d_{ij} \nabla_{i,k} \nabla_{j,k} + kc + \sum b_i \nabla_{i,k} + a \tag{21}$$

for some coefficients $d_{ij}, c, b_i, a \in \mathcal{C}^\infty(U, \mathrm{End}\,\mathbb{A})$ independent of k. For such a family, we define

$$\sigma_2(P)(y) = \sum_{i \leqslant j} d_{ij}(y) \nabla_{y,i} \nabla_{y,j} + c(y) : \mathcal{C}^\infty(T_yM, \mathbb{A}) \to \mathcal{C}^\infty(T_yM, \mathbb{A}).$$

Similarly we define the subspaces $\mathcal{G}_0$ and $\mathcal{G}_1$ of $\mathcal{G}_2$ and the corresponding symbols as follows. Assume that P satisfies (21). Then

$$P \in \mathcal{G}_1 \iff d_{ij} = c = 0, \qquad \sigma_1(P)(y) = \sum b_i(y) \nabla_{y,i},$$
$$P \in \mathcal{G}_0 \iff d_{ij} = c = b_i = 0, \qquad \sigma_0(P)(y) = a(y).$$

The basic property we need is the following.

Lemma 3.1. *Let $P \in \mathcal{G}_N$, $P' \in \mathcal{G}_{N'}$, with $N + N' \leqslant 2$. Then*

- *$P^* := (P_k^*)$ belongs to $\mathcal{G}_N$ and $\sigma_N(P^*)(y) = (\sigma_N(P)(y))^*$.*
- *$PP' := (P_k P_k') \in \mathcal{G}_{N+N'}$ and $\sigma_{N+N'}(PP')(y) = \sigma_N(P)(y) \circ \sigma_{N'}(P')(y)$.*

Here the formal adjoints P_k^* are defined with respect to any volume form μ of U which is independent of k, whereas the adjoint of $\sigma_N(P)(y)$ is defined with respect to any constant volume form of T_yM.

Proof. This is easily proved, let us emphasize the main points. First $\nabla_{i,k}^* = -\nabla_{i,k} + \operatorname{div}_\mu(e_i)$, so $(\nabla_{i,k}^*)$ belongs to $\mathcal{G}_1$ and $\sigma_1(\nabla_{i,k}^*)(y) = -\nabla_{y,i} = \nabla_{y,i}^*$. Second $\nabla_{i,k} a = a\nabla_{i,k} + \mathcal{L}_{e_i} a$, so $(\nabla_{i,k} a)$ belongs to $\mathcal{G}_1$ and has symbol $\sigma_1(\nabla_{i,k} a)(y) = a(y)\nabla_{y,i} = \nabla_{y,i} a(y)$. Third

$$\nabla_{i,k}\nabla_{j,k} = \nabla_{j,k}\nabla_{i,k} + \frac{k}{i}\omega(e_i, e_j)$$

so when $i > j$, $(\nabla_{i,k}\nabla_{j,k})$ belongs to $\mathcal{G}_2$ and

$$\sigma_2(\nabla_{i,k}\nabla_{j,k})(y) = \nabla_{y,j}\nabla_{y,i} + \frac{1}{i}\omega(e_i, e_j)(y) = \nabla_{y,i}\nabla_{y,j}. \qquad \square$$

Remark 3.2. Viewing k^{-1} as a semiclassical parameter, we can consider the algebra generated by the $\nabla_{i,k}$ and $\mathcal{C}^\infty(U)$ as a semiclassical algebra. But the order and the symbol that we use here are different from the semiclassical ones. A first reason is that the product of the $\sigma_N(P)$ is not abelian. Let us compare the order of the generators.

If we define the order of $(P_k) \in \mathcal{G}_N$ as N, the covariant derivatives $\nabla_{i,k}$ have order 1, multiplication by k has order 2 and multiplication by a function f has order 0. In particular k has twice the order of $\nabla_{i,k}$.

In contrast, let us trivialize L over an open set U so that $\mathcal{C}^\infty(U, L^k) \simeq \mathcal{C}^\infty(U)$ and $\nabla^{L^k} = d + k\alpha/i$, with $\alpha \in \Omega^1(U, \mathbb{R})$ the connection 1-form. We introduce the semiclassical parameter $\hbar = k^{-1}$. Then the operators $(ik)^{-1}\nabla_{i,k} = \hbar\partial_{e_i}/i - \alpha(e_i)$ and multiplication by f are semiclassical differential operators of order 0. So $\nabla_{i,k}$ and k have the same order as semiclassical differential operators. $\square$

Notice that for any vector field X of U, $(\nabla_X^{L^k})$ belongs to $\mathcal{G}_1$ with symbol at y given by the covariant derivative of $\mathcal{C}^\infty(T_yM)$ with respect to $X(y)$. Using this and Lemma 3.1, we deduce that $\mathcal{G}_N$ and σ_N do not depend on the choice of the frame (e_i). Let us make the dependence with respect to $(U, \mathbb{A})$ explicit, so we write $\mathcal{G}_N(U, \mathbb{A})$ instead of $\mathcal{G}_N$.

Using again Lemma 3.1, we see that if $u \in \mathcal{C}^\infty(U, \operatorname{End}\mathbb{A})$ is invertible at each point, then, for any $P \in \mathcal{G}_N(U, \mathbb{A})$, uPu^{-1} belongs to $\mathcal{G}_N(U, \mathbb{A})$ and $\sigma_N(uPu^{-1})(y) = u(y)\sigma_N(P)(y)u(y)^{-1}$. So we can define $\mathcal{G}_N(A)$ as the space of differential operator families (P_k) such that for any k, P_k acts on $\mathcal{C}^\infty(M, L^k \otimes A)$ and for any trivialization $A|_U \simeq U \times \mathbb{A}$, the local representative of (P_k) belongs to $\mathcal{G}_N(U, \mathbb{A})$. The corresponding symbol $\sigma_N(P)(y)$ is invariantly defined as a differential operator of $\mathcal{C}^\infty(T_yM, A_y)$.

It is also useful to consider differential operators from $\mathcal{C}^\infty(M, L^k \otimes A)$ to $\mathcal{C}^\infty(M, L^k \otimes B)$, where B is a second auxiliary Hermitian vector bundle. To handle these operators, we define the subspace $\mathcal{G}_N(A, B)$ of $\mathcal{G}_N(A \oplus B)$ consisting of the (P_k) such that, for any k, $\operatorname{Im} P_k \subset \mathcal{C}^\infty(M, L^k \otimes B) \subset \operatorname{Ker} P_k$. The symbol at y of an element of $\mathcal{G}_N(A, B)$ is a differential operator $\mathcal{C}^\infty(T_yM, A_y) \to \mathcal{C}^\infty(T_yM, B_y)$.

Observe now that assumption (B) has the reformulation

$$(\Delta_k) \in \mathcal{G}_2(A) \qquad \text{and} \qquad \sigma_2(\Delta_k)(y) = \Delta_y^{\text{scal}} + V(y) \quad \text{for all } y \in M. \tag{B'}$$

3B. ***Magnetic Laplacian.*** The simplest example of an operator satisfying condition (B) is the magnetic Laplacian defined in Section 1A. So besides the line bundle L with its connection, the Riemannian metric g and the section $V \in \mathcal{C}^\infty(M, \operatorname{End} A)$, we introduce a connection on A not necessarily preserving the Hermitian structure and a volume form μ on M. Set

$$\Delta_k = \tfrac{1}{2}(\nabla^{L^k \otimes A})^* \nabla^{L^k \otimes A} + kV : \mathcal{C}^\infty(M, L^k \otimes A) \to \mathcal{C}^\infty(M, L^k \otimes A),$$

where the formal adjoint of $\nabla^{L^k \otimes A}$ is defined from the scalar product obtained by integrating pointwise scalar products against μ.

Proposition 3.3. (Δ_k) *satisfies assumption* (B).

Proof. This follows from Lemma 3.1 and the fact that $(\nabla^{L^k \otimes A})$ belongs to $\mathcal{G}_1(A, A \otimes T^*M)$ with symbol at y equal to the covariant derivative ∇ of $\mathcal{C}^\infty(T_y M)$ tensored with the identity of A_y. To see this, write locally

$$\nabla^{L^k \otimes A} = \sum_i \epsilon(e_i^*) \nabla_{e_i}^{L^k \otimes A} = \sum_i \epsilon(e_i^*)(\nabla_{i,k} + \gamma_i),$$

where (e_i^*) is the dual frame of (e_i), $\epsilon(e_i^*)$ is the exterior product by e_i^* and the $\gamma_i \in \mathcal{C}^\infty(U, \operatorname{End} \mathbb{A})$ are the coefficients of the connection 1-form of ∇^A in a trivialization $A|_U \simeq U \times \mathbb{A}$. □

3C. *Holomorphic Laplacian.* Assume that M is a complex manifold and L, A are holomorphic Hermitian bundles, L being positive in the sense that the curvature of its Chern connection if ω/i, where $\omega \in \Omega^{1,1}(M)$, is a Kähler form. Equip $T^{0,1}M$ with the metric $|u|^2 = \omega(\bar{u}, u)/i$, $u \in T^{0,1}M$, and let $\mu = \omega^n/n!$ be the Liouville volume form. Define the holomorphic Laplacian

$$\Delta_k'' = (\bar{\partial}_{L^k \otimes A})^* \bar{\partial}_{L^k \otimes A} : \mathcal{C}^\infty(M, L^k \otimes A) \to \mathcal{C}^\infty(M, L^k \otimes A).$$

By Hodge theory, $\operatorname{Ker} \Delta_k''$ is isomorphic with the Dolbeault cohomology space $H^0(L^k \otimes A)$. When k is sufficiently large, the dimension of $H^0(L^k \otimes A)$ is the Riemann–Roch number $\operatorname{RR}(L^k \otimes A)$ defined as the evaluation of the product of the Chern character of $L^k \otimes A$ by the Todd class of M. Additionally, Δ_k'' satisfies assumption (B), which leads to the following description of its spectrum.

Theorem 3.4. *For any* $\Lambda > 0$, *there exists* $C > 0$ *such that* $\operatorname{sp}(k^{-1}\Delta_k'') \cap [0, \Lambda]$ *is contained in* $\mathbb{N} + Ck^{-1}[-1, 1]$. *For any* $m \in \mathbb{N}$,

$$\sharp \operatorname{sp}(k^{-1}\Delta_k'') \cap \left[m - \tfrac{1}{2}, m + \tfrac{1}{2}\right] = \operatorname{RR}(L^k \otimes A \otimes \operatorname{Sym}^m(T^{1,0}M)),$$

when k is sufficiently large.

Notice that the first eigenvalue cluster is degenerate in the sense that $\operatorname{sp}(\Delta_k'') \cap \left[0, \tfrac{1}{2}\right] \subset \{0\}$ when k is sufficiently large.

Proof. Now $\bar{\partial}_{L^k \otimes A}$ belongs to $\mathcal{G}_1(A, A \otimes (T^*M)^{0,1})$ and its symbol at y is the $(0, 1)$-component of the connection ∇ defined in (10). Using the same notation (u_i) and (z_i) as in Section 2B, $\nabla^{0,1} = \sum \epsilon(d\bar{z}_i) \otimes \nabla_{\bar{u}_i}$. Since the adjoint of $\epsilon(d\bar{z}_i)$ is the interior product by $\bar{u}_i$, we have $\epsilon(d\bar{z}_i)^* \epsilon(d\bar{z}_i) = 1$ so that

$$\sigma_2(\Delta_k'')(y) = -\sum_i \nabla_{u_i} \nabla_{\bar{u}_i}.$$

Thus Δ_k'' satisfies assumption (B) with $V(y) = -\tfrac{1}{2}n$ and $\Sigma_y = \mathbb{N}$. The result follows now from Corollary 7.2 with k^{-1} instead of $k^{-1/2}$ by Remark 7.3. □

Similarly we can consider the Laplacian acting on $(0, q)$-forms and prove the same result where $\mathbb{N}$ is replaced by $q+\mathbb{N}$ and the number of eigenvalues in $q+m+\left[-\frac{1}{2}, \frac{1}{2}\right]$ is the Riemann–Roch number of $L^k \otimes A \otimes \wedge^{0,q}(T^*M) \otimes \mathrm{Sym}^m(T^{1,0}M)$.

We can also generalize this to the case where the complex structure is not integrable. So assume that (M, ω) is a symplectic manifold with a compatible almost-complex structure j, that $L \to M$ is a Hermitian line bundle with a connection of curvature ω/i and that A a Hermitian vector bundle with a connection. Then Theorem 3.4 holds with the operator

$$\Delta_k'' = ((\nabla^{L^k\otimes A})^{(0,1)})^*(\nabla^{L^k\otimes A})^{(0,1)} : \mathcal{C}^\infty(L^k \otimes A) \to \mathcal{C}^\infty(L^k \otimes A)$$

and the proof is exactly the same. However, it is no longer true that the first eigenvalue cluster is nondegenerate. Using Dirac operators, one can generalize the previous result and still have the degeneracy of the first cluster, as explained in the next section.

3D. ***Semiclassical Dirac operators.*** In this section, (M, ω, j) is a symplectic manifold with an almost complex structure, (L, ∇) is a Hermitian line bundle on M with a connection having curvature ω/i and A is an auxiliary Hermitian vector bundle.

Let $S = \wedge^{0,\bullet}T^*M$ be the spinor bundle and S^+, S^- be the subbundles of even and odd forms respectively. For any $y \in M$, extend the covariant derivative ∇ defined in (10) to $\Omega^\bullet(T_yM)$ in the usual way and denote by $\nabla^{0,1}$ the restriction of its $(0, 1)$-component to $\Omega^{0,\bullet}(T_yM) = \mathcal{C}^\infty(T_yM \otimes S_y)$.

Definition 3.5. A semiclassical Dirac operator is a family $(D_k) \in \mathcal{G}_1(A \otimes S)$ with symbol

$$\sigma_1(D_k)(y) = \nabla^{0,1} + (\nabla^{0,1})^* : \Omega^{0,\bullet}(T_yM) \to \Omega^{0,\bullet}(T_yM) \quad \text{for all } y \in M$$

such that for any k, D_k is formally self-adjoint and odd.

Such an operator can be constructed as follows: we introduce a connection on S preserving S^+ and S^- and a connection on A and set

$$D_k = \sum_i \epsilon(\bar\theta_i)\nabla^{L^k\otimes A\otimes S}_{\bar u_i} + (\epsilon(\bar\theta_i)\nabla^{L^k\otimes A\otimes S}_{\bar u_i})^*,$$

where (u_i) is any orthonormal frame of $T^{1,0}M$, (θ_i) is the dual frame of $(T^*M)^{1,0}$ and the exterior product $\epsilon(\bar\theta_i)$ acts on S. Another example is provided by spin-c Dirac operators; see [Duistermaat 1996: Ma and Marinescu 2007, Section 1.3]. Observe as well that the semiclassical Dirac operator is unique up to a self-adjoint odd operator of $\mathcal{G}_0(A \otimes S)$. We denote by

$$D_k^\pm : \mathcal{C}^\infty(L^k \otimes A \otimes S^\pm) \to \mathcal{C}^\infty(L^k \otimes A \otimes S^\mp)$$

the restrictions of D_k and observe that D_k^- is the formal adjoint of D_k^+.

Theorem 3.6. *Let (D_k) be a semiclassical Dirac operator. Then the operator $\Delta_k = D_k^- D_k^+$ satisfies*:

(1) *For any $\Lambda > 0$, there exists $C > 0$ such that $\mathrm{sp}(k^{-1}\Delta_k) \cap [0, \Lambda]$ is contained in $\mathbb{N} + Ck^{-1/2}[-1, 1]$.*

(2) $\mathrm{sp}(k^{-1}\Delta_k) \cap \left[0, \frac{1}{2}\right] \subset \{0\}$ *and* $\mathrm{Ker}\,\Delta_k$ *has dimension* $\mathrm{RR}(L^k \otimes A)$ *when k is sufficiently large.*

(3) *For any $m \in \mathbb{N}$, when k is sufficiently large,*

$$\sharp\, \mathrm{sp}(k^{-1}\Delta_k) \cap \left[m - \tfrac{1}{2}, m + \tfrac{1}{2}\right] = \mathrm{RR}(L^k \otimes A_m),$$

where $A_m = \bigoplus_{(\ell,p)} A \otimes \mathrm{Sym}^\ell(T^{1,0}M) \otimes \wedge^{2p}(T^{1,0}M)$, the sum being over the $(\ell, p) \in \mathbb{N}^2$ such that $\ell + 2p = m$ and $p \leqslant n$.

Proof. As in Section 2B, let (u_i) be an orthonormal basis of $T^{1,0}_y M$ and (z_i) be the associated linear complex coordinates. We have $\nabla^*_{\bar{u}_i} = -\nabla_{u_i}$, $\epsilon(d\bar{z}_i)^* = \iota(\bar{u}_i)$ so that

$$\sigma_1(D_k)(y) = \sum \epsilon(d\bar{z}_i)\nabla_{\bar{u}_i} - \iota(\bar{u}_i) \otimes \nabla_{u_i}.$$

A standard computation using that ∇_{u_i}, $\nabla_{\bar{u}_i}$ commute with $\epsilon(d\bar{z}_j)$, $\iota(\bar{u}_j)$ and $[\nabla_{u_i}, \nabla_{u_j}] = [\nabla_{\bar{u}_i}, \nabla_{\bar{u}_j}] = 0$, $[\nabla_{u_i}, \nabla_{\bar{u}_j}] = \delta_{ij}$ leads to

$$\sigma_2(D_k^2)(y) = \sum(-\nabla_{u_i}\nabla_{\bar{u}_i} + \epsilon(d\bar{z}_i)\iota(\bar{u}_i)) = \Delta^{\mathrm{scal}}_y - \frac{n}{2} + \mathrm{N}_y,$$

where N_y is the number operator of S_y, that is, $\mathrm{N}_y\alpha = (\deg\alpha)\alpha$. Restricting to S^+, we deduce that (Δ_k) satisfies assumption (B) with $V(y) = -\frac{1}{2}n + \mathrm{N}_y$. So $\Sigma_y = \mathbb{N}$ and the first assertion of the theorem follows from the second part of Corollary 7.2.

In the same way, $(D_k^+ D_k^-)$ has the form (B) with $V(y) = -\frac{1}{2}n + \mathrm{N}_y$ as well, but the number operator takes odd value on S^-. Thus

$$\mathrm{sp}(k^{-1} D_k^+ D_k^-) \subset [1 - Ck^{-1/2}, \infty[$$

for some positive C. Since, for any $\lambda \neq 0$, D_k^+ is an isomorphism between $\mathrm{Ker}(D_k^- D_k^+ - \lambda)$ and $\mathrm{Ker}(D_k^+ D_k^- - \lambda)$, this proves that $\mathrm{sp}(k^{-1}\Delta_k) \cap \left]0, \frac{1}{2}\right]$ is empty when k is sufficiently large and the first part of the second assertion follows.

The second part of the second assertion and the third assertion follow from Corollary 7.2. Indeed, for $V(y) = -\frac{1}{2}n + \mathrm{N}_y$ acting on $A_y \otimes S^+_y$, the bundle F with fiber $F_y = \ker(\square_y - m)$ is isomorphic to A_m as a complex vector bundle. □

4. Spectral estimates

Let $(\Delta_k : \mathcal{C}^\infty(M, L^k \otimes A) \to \mathcal{C}^\infty(M, L^k \otimes A))$ be a differential operator family satisfying (B). We assume that the curvature ω/i is nondegenerate. We assume as well that Δ_k is formally self-adjoint, where the scalar product of $\mathcal{C}^\infty(M, L^k \otimes A)$ is defined from the measure $\mu = \omega^n/n!$.

For any $y \in M$, by the Darboux lemma, there exists a coordinate system (U, x_i) of M centered at y such that ω is constant in these coordinates, that is, $\omega = \frac{1}{2}\sum \omega_{ij}\, dx_i \wedge dx_j$, with $\omega_{ij} = \omega(\partial_{x_i}, \partial_{x_j})$ constant functions. We identify U with a neighborhood of the origin of $T_y M$ through these coordinates. We assume that this neighborhood is convex.

We introduce a unitary section F_y of $L \to U$ such that, for any $\xi \in U$, F_y is flat on the segment $[0, \xi]$. Then

$$\nabla F_y = \frac{1}{2i} \sum_{i,j} \omega_{i,j} x_i\, dx_j \otimes F_y. \tag{22}$$

Indeed $\nabla F_y = (\alpha/i) \otimes F_y$ with α satisfying $d\alpha = \omega$ and $\int_{[0,\xi]} \alpha = 0$ for any $\xi \in U$. We easily see that these conditions determine a unique α and that they are satisfied by $\alpha = \frac{1}{2} \sum \omega_{ij} x_i \, dx_j$.

So trivializing L on U by using this frame F_y, $L|_U \simeq U \times \mathbb{C}$ and ∇ becomes the linear connection defined in (10). Moreover trivializing L^k on U with F_y^k, the covariant derivative $\nabla_{j,k}$ of L^k with respect to ∂_{x_j} is

$$\nabla_{j,k} = \partial_{x_j} + \frac{ik}{2} \sum_i \omega_{i,j} x_i. \tag{23}$$

Now we introduce the Laplacian $\Delta_{y,k}$ of $\mathcal{C}^\infty(T_y M, A_y)$ associated to this covariant derivative, the constant metric g_y of $T_y M$ and the constant potential $kV(y)$, that is,

$$\Delta_{y,k} = -\frac{1}{2} \sum g_y^{ij} \nabla_{i,k} \nabla_{j,k} + kV(y). \tag{24}$$

For $k = 1$, we recover the Laplacian Δ_y defined in (17).

We introduce a trivialization of the auxiliary vector bundle $A|_U = U \times A_y$ so that $\mathcal{C}^\infty(U, L^k \otimes A) \simeq \mathcal{C}^\infty(U, A_y)$. Then assumption (B) tells us that

$$\Delta_k - \Delta_{y,k} = \sum_{i,j} a_{ij} \nabla_{i,k} \nabla_{j,k} + \sum_i a_i \nabla_{i,k} + kc + b, \tag{25}$$

where $a_{ij} = -\frac{1}{2} g^{ij} + \frac{1}{2} g_y^{ij}$ and $c = V - V(y)$ are both equal to zero at the origin y. The identity (25) will be used later to compare the spectra of Δ_k and $\Delta_{y,k}$; see the proofs of Proposition 4.1 and Lemma 4.4.

Before that, let us compute the spectrum of $\Delta_{y,k}$. The Laplacian $k^{-1}\Delta_{y,k}$ is unitarily conjugated to Δ_y. Indeed, we introduce the rescaling map

$$S_k : \mathcal{C}^\infty(T_y M, A_y) \to \mathcal{C}^\infty(T_y M, A_y), \quad S_k(f)(x) = k^{n/2} f(k^{1/2} x). \tag{26}$$

Then, from the formula (23), we easily check that

$$k^{1/2} S_k \nabla_i = \nabla_{i,k} S_k, \quad k^{-1} \Delta_{y,k} S_k = S_k \Delta_y. \tag{27}$$

Consequently, the spectrum of $k^{-1}\Delta_{y,k}$ is Σ_y for any k.

4A. ***Peaked sections.*** As above, we identify a neighborhood U of y with a neighborhood of the origin in $T_y M$ through Darboux coordinates, we introduce the frame F_y of L on U with covariant derivative given by (22), and we work with a trivialization $A|_U \simeq U \times A_y$. Choose a function $\psi \in \mathcal{C}_0^\infty(U, \mathbb{R})$ such that $\psi = 1$ on a neighborhood of y. Then to any polynomial $f \in \mathcal{P}(T_y M) \otimes A_y$, we associate the smooth section $\Phi_k(f)$ of $L^k \otimes A$ defined on U by

$$\Phi_k(f)(\xi) = k^{n/2} F_y^k(\xi) e^{-k|\xi|_y^2/4} f(k^{1/2}\xi) \psi(\xi) \tag{28}$$

and equal to 0 on $M \setminus U$.

Proposition 4.1. *We have*

(1) $\|\Phi_k(f)\|^2 = \int_{T_y M} e^{-|\xi|_y^2/2} |f(\xi)|^2 \, d\mu_y(\xi) + \mathcal{O}(e^{-C/k})$, *with* $\mu_y = \omega_y^n / n!$ *the Liouville form of* $T_y M$,

(2) $k^{-1}\Delta_k \Phi_k(f) = \Phi_k(g) + \mathcal{O}(k^{-1/2})$, *with* $g = \widetilde{\square}_y(f)$.

The peaked sections of [Charles 2024] are defined without using the Darboux coordinates, and for this reason the $\mathcal{O}(e^{-k/C})$ in the norm estimate is replaced by a $\mathcal{O}(k^{-1/2})$. Actually, the Darboux coordinates are not essential in this subsection, they only simplify slightly some estimates, whereas in Sections 4B and 4C it will be necessary to use them.

Proof. Since $\Phi_k(f)$ is supported in U, we can view it as a function of T_yM, so

$$\Phi_k(f) = \psi S_k(sf),$$

where $s(\xi) = e^{-|\xi|_y^2/4}$ as in Section 2B and S_k is the rescaling map (26). Since we work with Darboux coordinates, the volume form μ of M coincide on U with μ_y. So

$$\|\Phi_k(f)\|^2 = \int_{T_yM} |S_k(sf)|^2 \psi^2 \, d\mu_y.$$

We will need several times to estimate an integral having the form

$$I_k(\widetilde{\psi}) = \int_{T_yM} |S_k(sf)|^2 \widetilde{\psi} \, d\mu_y = k^n \int_{T_yM} e^{-k/2|\xi|_y^2} |f(k^{1/2}\xi)|^2 \, \tilde{\psi}(\xi) \, d\mu_y(\xi),$$

with $\widetilde{\psi} \in \mathcal{C}^\infty(T_yM)$ satisfying $\widetilde{\psi}(\xi) = \mathcal{O}(|\xi|^m)$ on T_yM for $m \geqslant 0$. We claim that $I_k(\widetilde{\psi}) = \mathcal{O}(k^{-m/2})$ and in the case where $\widetilde{\psi} = 0$ on a neighborhood of the origin, $I_k(\widetilde{\psi}) = \mathcal{O}(e^{-k/C})$ for some $C > 0$.

The first claim follows from the change of variable $\sqrt{k}\xi = \xi'$. For the second one, we use that $e^{-k|\xi|_y^2/2}\widetilde{\psi}(\xi) = \mathcal{O}(e^{-k/C}|\xi|^m e^{-k|\xi|_y^2/4})$ and do the same change of variable.

The first assertion of the proposition is an immediate consequence of the second claim with $\widetilde{\psi} = 1 - \psi^2$. For the second assertion, we start from (25) and using that $[\nabla_{i,k}, \psi] = \partial_{x_i}\psi$ repetitively, we obtain

$$\Delta_k \psi = \psi\big(\Delta_{y,k} + a_{ij}\nabla_i^k\nabla_j^k + \tilde{b}_i\nabla_i + kc + \tilde{c}\big), \tag{29}$$

where a_{ij}, c are the same functions as in (25), and $\tilde{b}_i$ and $\tilde{c}$ do not depend on k.

Now, by (29), $\Delta_k(\psi S_k(sf))$ is a sum of five terms, the first one being

$$\psi \Delta_{y,k} S_k(fs) = k\psi S_k(\Delta_y(sf)) = k\Phi_k(g), \quad \text{with } sg = \Delta_y(sf),$$

by (27). We will prove that the four other terms are in $\mathcal{O}(k^{1/2})$, which will conclude the proof.

Each time, we will apply the preliminary integral estimate with the convenient function $\widetilde{\psi}$. First since $|\psi\tilde{c}|$ is bounded, $\psi\tilde{c}\, S_k(sf) = \mathcal{O}(1)$. Second, c vanishes at the origin, $|\psi(\xi)c(\xi)|^2 = \mathcal{O}(|\xi|^2)$ so that $\psi c\, S_k(sf) = \mathcal{O}(k^{-1/2})$. Third, by (27),

$$\nabla_{i,k} S_k(sf) = k^{1/2} S_k(\nabla_i(sf)) = k^{1/2} S_k(sf_i),$$

with a new polynomial f_i, and since $\psi\tilde{b}_i$ is bounded, we get

$$\psi\tilde{b}_i \, \nabla_{i,k} S_k(sf) = \mathcal{O}(k^{1/2}).$$

Similarly, $\nabla_{i,k}\nabla_{j,k} S_k(sf) = kS_k(sf_{ij})$ with new polynomials f_{ij}, and a_{ij} vanishing at the origin, so we obtain

$$\psi a_{ij}\nabla_{i,k}\nabla_{j,k} S_k(sf) = \mathcal{O}(k^{1/2})$$

as was to be proved. ☐

Theorem 4.2. *Let* (Δ_k) *be a family of formally self-adjoint differential operators of the form* (B). *Then, if* $\lambda \in \Sigma_y$, *there exists* $C(y,\lambda)$ *such that*

$$\operatorname{dist}(\lambda, \operatorname{sp}(k^{-1}\Delta_k)) \leqslant C(y,\lambda)k^{-1/2} \quad \text{for all } k.$$

Furthermore, for any $\Lambda > 0$, $C(y,\lambda)$ *stays bounded when* (y,λ) *runs over* $M \times]-\infty, \Lambda]$.

This proves the first assertion of Theorem 1.2.

Proof. By Section 2B, any eigenvalue λ of $\widetilde{\Box}_y$ has an eigenfunction $f \in \mathcal{P}(T_yM) \otimes A_y$. Normalizing conveniently f, we get by Proposition 4.1,

$$\|\Phi_k(f)\| = 1 + \mathcal{O}(e^{-k/C}), \quad k^{-1}\Delta_k\Phi_k(f) = \lambda\Phi_k(f) + \mathcal{O},(k^{-1/2}),$$

which proves that $\operatorname{dist}(\lambda, \operatorname{sp}(k^{-1}\Delta_k)) = \mathcal{O}(k^{-1/2})$. To get a uniform $\mathcal{O}$ when $\lambda \leqslant \Lambda$, remember that by the first assertion of Lemma 2.4, we can choose $f \in \mathcal{D}_{\leqslant p}(T_yM) \otimes A_y$, where p is sufficiently large and independent of $y \in M$. Furthermore, for any $p \in \mathbb{N}$, the $\mathcal{O}$'s in Proposition 4.1 are uniform with respect to f describing the compact set $\{f \in \mathcal{D}_{\leqslant p}(TM) \otimes A : \|f\| = 1\}$. Here we can use any metric of $\mathcal{D}_{\leqslant p}(TM)$, the natural one in our situation being $\|f\|^2 = \int_{T_yM} e^{-|\xi|_y^2/2}|f(\xi)|^2\, d\mu_y(\xi)$ for $f \in \mathcal{D}(T_yM)$. □

4B. ***A local approximate resolvent.*** Recall that $k^{-1}\Delta_{y,k} = S_k\Delta_y S_k^*$ so that $k^{-1}\Delta_{y,k}$ has the same spectrum Σ_y as Δ_y. For any $\lambda \in \mathbb{C} \setminus \Sigma_y$, we denote by

$$R_{y,k}(\lambda) := (\lambda - k^{-1}\Delta_{y,k})^{-1} : L^2(T_yM) \otimes A_y \to L^2(T_yM) \otimes A_y$$

the resolvent. We will need the following basic elliptic estimates.

Proposition 4.3. *For any* $\lambda \in \mathbb{C} \setminus \Sigma_y$, *the resolvent* $R_{y,k}(\lambda)$ *sends* $\mathcal{C}_0^\infty$ *to* $\mathcal{C}^\infty$ *and satisfies*

$$\|k^{-1/2}\nabla_{i,k}R_{y,k}(\lambda)\| \leqslant C_\Lambda d^{-1}, \quad \|k^{-1}\nabla_{i,k}\nabla_{j,k}R_{y,k}(\lambda)\| \leqslant C_\Lambda d^{-1} \tag{30}$$

if $|\lambda| \leqslant \Lambda$ *with* $d = \operatorname{dist}(\lambda, \Sigma_y)$ *and the constant* C_Λ *independent of* k.

Here and in the sequel, the norm $\|\cdot\|$ is the operator norm associated to the L^2-norm.

Proof. The first assertion follows from elliptic regularity: for any distribution ψ of T_yM, if $(\lambda - k^{-1}\Delta_{y,k})\psi$ is smooth then ψ is smooth.

Since $R_{y,k}(\lambda) = S_kR_{y,1}(\lambda)S_k^*$ and $k^{-1/2}\nabla_{i,k} = S_k\nabla_iS_k^*$, it suffices to prove the inequalities (30) for $k = 1$. We can assume that the frame $(\partial/\partial x_i)$ is g-orthonormal at y, so $g_y^{ij} = \delta_{ij}$, so $\Delta_y = -\frac{1}{2}\sum_i \nabla_i^2 + V(y)$. Since $\langle \Delta_y u, u\rangle = \frac{1}{2}\sum \|\nabla_i u\|^2 + \langle V(y)u, u\rangle$, we have by the Cauchy–Schwarz inequality

$$\|\nabla_i u\|^2 \leqslant C\|u\|(\|\Delta_y u\| + \|u\|). \tag{31}$$

Since $[\nabla_i, \nabla_j] = \omega_{i,j}/i$, we have

$$\begin{aligned}\|\nabla_i\nabla_j u\|^2 &= \langle \nabla_j\nabla_i^2\nabla_j u, u\rangle = \langle \nabla_i^2\nabla_j^2 u, u\rangle + \frac{2}{i}\omega_{ji}\langle \nabla_i\nabla_j u, u\rangle \\ &= \langle \nabla_j^2 u, \nabla_i^2 u\rangle + \frac{2}{i}\omega_{ij}\langle \nabla_j u, \nabla_i u\rangle. \end{aligned} \tag{32}$$

Moreover,

$$\frac{1}{4}\sum_{i,j}\langle \nabla_i^2 u, \nabla_j^2 u\rangle = \|\Delta_y u - V(y)u\|^2 \leqslant C(\|\Delta_y u\| + \|u\|)^2. \tag{33}$$

Estimating the first term of (32) with (33) and the second one with (31), it comes that

$$\|\nabla_i \nabla_j u\|^2 \leqslant C(\|\Delta_y u\| + \|u\|)^2. \tag{34}$$

To conclude the proof, we use that the norm of $R_y(\lambda) = (\lambda - \Delta_y)^{-1}$ is d^{-1} and $\Delta_y R_y(\lambda) = \lambda R_y(\lambda) - \mathrm{id}$ so when $|\lambda| \leqslant \Lambda$,

$$\|\Delta_y R_y(\lambda)\| \leqslant \Lambda d^{-1} + 1 \leqslant C_\Lambda d^{-1}$$

because d stays bounded when λ is. Hence it follows from (31) and (34) that

$$\|\nabla_i R_y(\lambda) v\| \leqslant C_\Lambda d^{-1}\|v\|, \quad \|\nabla_j \nabla_i R_y(\lambda) v\| \leqslant C_\Lambda d^{-1}\|v\|,$$

which corresponds to (30) for $k = 1$. □

Recall that we identified a neighborhood of $y \in M$ with a neighborhood U of the origin of $T_y M$ through Darboux coordinates. We introduce a smooth function $\chi : T_y M \to [0, 1]$ such that $\chi(\xi) = 1$ when $|\xi| \leqslant 1$ and $\chi(\xi) = 0$ when $|\xi| \geqslant 2$. Define $\chi_r(\xi) := \chi(\xi/r)$. In the sequel we assume that r is sufficiently small so that χ_r is supported in U. Then for any differential operator P acting on $\mathcal{C}^\infty(U)$, $\chi_r P$ and $P\chi_r$ are differential operators with coefficients supported in U, so we can view them as operators acting on $\mathcal{C}^\infty(T_y M)$.

In the following lemma, we prove that the resolvent $R_{y,k}(\lambda)$ of $k^{-1}\Delta_{y,k}$ is a local right-inverse of $(\lambda - k^{-1}\Delta_k)$ up to some error.

Lemma 4.4. *For any $\lambda \in \mathbb{C} \setminus \Sigma_y$ such that $|\lambda| \leqslant \Lambda$, we have with $d = d(\lambda, \Sigma_y)$*

$$\|(\lambda - k^{-1}\Delta_k)\chi_r R_{y,k}(\lambda) - \chi_r\| \leqslant C_\Lambda F(r, k^{-1}, d), \tag{35}$$

where $F(r, \hbar, d) = (r + \hbar^{1/2} + \hbar r^{-2} + \hbar^{1/2} r^{-1}) d^{-1}$.

Proof. We compute

$$\begin{aligned}(\lambda - k^{-1}\Delta_k)\chi_r R_{y,k}(\lambda) - \chi_r &= -k^{-1}[\Delta_k, \chi_r]R_{y,k}(\lambda) + \chi_r(\lambda - k^{-1}\Delta_k)R_{y,k}(\lambda) - \chi_r \\ &= -k^{-1}[\Delta_k, \chi_r]R_{y,k}(\lambda) + \chi_r k^{-1}(\Delta_{y,k} - \Delta_k)R_{y,k}(\lambda).\end{aligned} \tag{36}$$

To estimate the first term, we start from assumption (B), which gives us

$$\begin{aligned}[\Delta_k, \chi_r] &= -\tfrac{1}{2} g^{ij}[\nabla_{i,k}\nabla_{j,k}, \chi_r] + a_j[\nabla_{j,k}, \chi_r] \\ &= -\tfrac{1}{2} g^{ij}\big((\partial_j\partial_i\chi_r) + (\partial_i\chi_r)\nabla_{j,k} + (\partial_j\chi_r)\nabla_{i,k}\big) + a_j(\partial_j\chi_r).\end{aligned}$$

Applying the estimates (30), we deduce that

$$\begin{aligned}\|k^{-1}[\Delta_k, \chi_r]R_{y,k}(\lambda)\| &\leqslant C(k^{-1}r^{-2}d^{-1} + k^{-1/2}r^{-1}d^{-1} + k^{-1}r^{-1}d^{-1}) \\ &\leqslant C(k^{-1}r^{-2} + k^{-1/2}r^{-1})d^{-1}.\end{aligned}$$

To estimate the second term of (36), we use the expression (25) and the fact that the a_{ij} and c vanish at the origin so that $|\chi_r a_{ij}| \leqslant Cr$ and $|\chi_r c| \leqslant Cr$. By (30) it follows that

$$\|\chi_r k^{-1}(\Delta_k - \Delta_{y,k}) R_{y,k}(\lambda)\| \leqslant C(r + k^{-1/2} + k^{-1}) d^{-1} \leqslant C(r + k^{-1/2}) d^{-1},$$

which concludes the proof. □

4C. *Globalization.* The local approximation of the resolvent at y in the previous section was based on a choice of Darboux coordinates. To globalize this, we will first choose such coordinate charts depending smoothly on y. All the constructions to come depend on an auxiliary Riemannian metric. For any $y \in M$ and $r > 0$ let $B_y(r)$ be the open ball $\{\xi \in T_y M : \|\xi\| < r\}$.

Lemma 4.5. *There exist $r_0 > 0$ and a smooth family of embeddings $(\Psi_y : B_y(r_0) \to M,\ y \in M)$ such that, for any $y \in M$, $\Psi_y(0) = y$, $T_0\Psi_y = \mathrm{id}_{T_y M}$ and $\Psi_y^*\omega$ is constant on $B_y(r_0)$.*

The family $(\Psi_y,\ y \in M)$ is smooth in the sense that the map $\Psi(\xi) = \psi_y(\xi),\ \xi \in B_y(r_0)$, from the open set $\bigcup_{y \in M} B_y(r_0)$ of TM to M, is smooth.

Lemma 4.6. *There exist $N \in \mathbb{N}$, $r_1 > 0$ and for any $0 < r < r_1$ a finite subset $I(r)$ of M such that the open sets $\Psi_y(B_y(r)),\ y \in I(r)$, form a covering of M with multiplicity bounded by N.*

The multiplicity of a covering $\bigcup_{i \in I} U_i \supset M$ is the maximal number of U_i with nonempty intersection. The proofs of Lemmas 4.5 and 4.6 are standard and postponed to Section 8.

Recall that $\Sigma = \bigcup \Sigma_y$. So, for any $\lambda \in \mathbb{C} \setminus \Sigma$, the resolvents $R_{y,k}(\lambda) : \mathcal{C}_0^\infty(T_y M, A_y) \to \mathcal{C}^\infty(T_y M, A_y)$ are well-defined. As previously, we introduce a section F_y of $L \to \Psi_y(B_y(r))$ satisfying (22) and a trivialization of A on $\Psi_y(B_y(r))$, from which we identify $\mathcal{C}^\infty(\Psi_y(B_y(r)), L^k \otimes A) \simeq \mathcal{C}^\infty(B_y(r), A_y)$. Let

$$\widetilde{R}_{y,k}(\lambda) : \mathcal{C}_0^\infty(\Psi_y(B_y(r)), L^k \otimes A) \to \mathcal{C}^\infty(\Psi_y(B_y(r)), L^k \otimes A)$$

be the map corresponding to $R_{y,k}(\lambda)$ under these identifications.

For r sufficiently small, define the function $\chi_{y,r}$ supported in $\Psi_y(B_y(r_0))$ and such that $\chi_{y,r}(\Psi_y(\xi)) = \chi(\xi/r)$. We introduce a partition of unity $(\psi_{r,y},\ y \in I(r))$, subordinated to the cover $(\Psi_y(B_y(r)),\ y \in I(r))$. Then define the operator $R_k^r(\lambda)$ acting on $\mathcal{C}^\infty(M, L^k \otimes A)$ by

$$R_k^r(\lambda) := \sum_{y \in I(r)} \chi_{y,r} \widetilde{R}_{y,k}(\lambda) \psi_{r,y}. \tag{37}$$

Theorem 4.7. *Let (Δ_k) be a family of formally self-adjoint differential operators of the form* (B). *Then, for any $|\lambda| \leqslant \Lambda$,*

$$\|(\lambda - k^{-1}\Delta_k) R_k^r(\lambda) - 1\| \leqslant C_\Lambda F(r, k^{-1}, d), \tag{38}$$

with $d = \mathrm{dist}(\lambda, \Sigma)$ and F the same function as in Lemma 4.4.

Proof. Let (U_i) be a covering of M with multiplicity $N = \sup_x |\{i / x \in U_i\}|$. Then:

(1) If v_i is a family of sections such that $\operatorname{supp} v_i \subset U_i$ for any i, then $\left\|\sum v_i\right\|^2 \leqslant N \sum \|v_i\|^2$.

(2) For any section u, $\sum \|u\|_{U_i}^2 \leqslant N \|u\|^2$.

To prove the first claim, $\left\| \sum v_i \right\|^2 = \sum_{i,j} M_{ij} \langle v_i, v_j \rangle \leqslant \sum M_{ij} \|v_i\| \|v_j\|$, where $M_{i,j} = 1$ when $U_i \cap U_j \neq \varnothing$ and 0 otherwise. By Schur test applied to the matrix M, $\langle Ma, a \rangle \leqslant N \|a\|^2$ and the result follows. To prove the second claim, set $m(x) = \sum 1_{U_i}(x)$, which is bounded by N by assumption. Then $\sum \|u\|^2_{U_i} = \int_M |u(x)|^2 m(x)\, d\mu(x) \leqslant N \|u\|^2$.

We now apply this to the covering $\Psi_y(B_y(r))$, $y \in I(r)$. By Lemma 4.4, for any $u \in \mathcal{C}^\infty(M, L^k)$, we have $\|S^r_{y,k} \psi_{y,r} u\| \leqslant CF \|\psi_{y,r} u\|$, where

$$S^r_{y,k} = (\lambda - k^{-1}\Delta_k)\chi_{y,r} \widetilde{R}_{y,k}(\lambda) - \chi_{y,r},$$

$F = F(r, k^{-1}, d)$ and the constant C can be chosen independently of y because everything depends continuously on y and M is compact. Since $R^r_k(\lambda) - 1 = \sum_{y \in I(r)} S_{y,k} \psi_{y,r}$, we have

$$\begin{aligned} \|R^r_k(\lambda)u - u\|^2 &\leqslant N \sum_{y \in I(r)} \|S^r_{y,k} \psi_{y,r} u\|^2 \leqslant N(CF)^2 \sum_{y \in I(r)} \|\psi_{y,r} u\|^2 \\ &\leqslant N(CF)^2 \sum_{y \in I(r)} \|u\|^2_{\Psi_y(B_y(r))} \leqslant (NCF)^2 \|u\|^2, \end{aligned}$$

which proves (38). □

Recall basic facts pertaining to the spectral theory of Δ_k; see for instance [Shubin 1987, Section 8.3]. As an elliptic formally self-adjoint differential operator of order 2 on a compact manifold, Δ_k is a self-adjoint unbounded operator with domain the Sobolev space $H^2(M, L^k \otimes A)$. Its spectrum $\mathrm{sp}(\Delta_k)$ is a discrete subset of $\mathbb{R}$ bounded from below and consists only of eigenvalues with finite multiplicities.

Corollary 4.8. *For any $\Lambda > 0$, there exists $C > 0$ such that for any k we have*

$$\mathrm{sp}(k^{-1}\Delta_k) \cap]-\infty, \Lambda] \subset \Sigma + Ck^{-1/4}[-1, 1]. \tag{39}$$

So any $\lambda \in \mathbb{C}$ satisfying $|\lambda| \leqslant \Lambda$ and $d(\lambda, \Sigma) \geqslant Ck^{-1/4}$ does not belong to $\mathrm{sp}(k^{-1}\Delta_k)$. Moreover, for any such λ,

$$\|R^{r_k}_k(\lambda) - (\lambda - k^{-1}\Delta_k)^{-1}\| \leqslant Cd(\lambda, \Sigma)^{-2} k^{-1/4}, \tag{40}$$

with $r_k = k^{-1/4}$.

Equation (39) shows the second assertion of Theorem 1.2 with $k^{-1/4}$ instead of $k^{-1/2}$. The improvement with $k^{-1/2}$ will be proved in Corollary 7.2.

Proof. First, since $\|\widetilde{R}_{y,k}(\lambda)\| \leqslant d(\lambda, \Sigma_y)^{-1} \leqslant d^{-1}$ with $d = d(\lambda, \Sigma)$, we deduce from the first part of the proof of Theorem 4.7 that

$$\|R^r_k(\lambda)\| \leqslant Cd^{-1}, \tag{41}$$

where C does not depend on r, λ and k. From now on assume that $r = k^{-1/4}$. So $F(r, k^{-1}, d) \leqslant C' k^{-1/4} d^{-1}$. By Theorem 4.7, as soon as $C_\Lambda C' k^{-1/4} d^{-1} \leqslant \frac{1}{2}$, we have $(\lambda - k^{-1}\Delta_k) R^r_k(\lambda)$ is invertible, so $\widetilde{R}_k := R^r_k(\lambda)((\lambda - k^{-1}\Delta_k) R^r_k(\lambda))^{-1}$ is a bounded operator of L^2 satisfying

$$(\lambda - k^{-1}\Delta_k)\widetilde{R}_k = \mathrm{id} \tag{42}$$

and by (41),

$$\|\widetilde{R}_k - R^r_k(\lambda)\| \leqslant 2\|R^r_k(\lambda)\| \, \|(\lambda - k^{-1}\Delta_k) R^r_k(\lambda) - 1\| \leqslant C'' d^{-2} k^{-1/4}.$$

We claim that $\widetilde{R}_k$ is actually continuous $L^2 \to H^2$. Indeed, by classical result on elliptic operators [Shubin 1987, Theorem 5.1], there exists a pseudodifferential operator P_k of order -2 which is a parametrix of $\lambda - k^{-1}\Delta_k$; that is, $P_k(\lambda - k^{-1}\Delta_k) = \mathrm{id} + S_k$, where S_k is a smoothing operator. Then multiplying by $\widetilde{R}_k$, we obtain $P_k = \widetilde{R}_k + S_k\widetilde{R}_k$, so $\widetilde{R}_k = P_k - S_k\widetilde{R}_k$. Now, since P_k is of order -2 and S_k is smoothing, they are both continuous $L^2 \to H^2$, so the same holds for $\widetilde{R}_k$.

To finish the proof, we assume that λ is real. Then $k^{-1}\Delta_k - \lambda$ is a Fredholm operator from H^2 to L^2 with index 0, because it is formally self-adjoint; see [Shubin 1987, Theorem 8.1]. By (42), $\lambda - k^{-1}\Delta_k$ sends H^2 onto L^2, so its kernel is trivial, and thus λ is not an eigenvalue. □

5. The operator class $\mathcal{L}(A)$

5A. *Symbol spaces.* Let $\boldsymbol{E}$ be an n-dimensional Hermitian space. As we did in Section 2B for $\boldsymbol{E} = T_yM$, consider the spaces $\mathcal{P}(\boldsymbol{E})$, $\mathcal{D}(\boldsymbol{E})$ consisting respectively of polynomial maps and antiholomorphic polynomial maps from $\boldsymbol{E}$ to $\mathbb{C}$. We will introduce two subalgebras $\mathcal{S}(\boldsymbol{E})$ and $\widetilde{\mathcal{S}}(\boldsymbol{E})$ of $\mathrm{End}(\mathcal{D}(\boldsymbol{E}))$ and $\mathrm{End}(\mathcal{P}(\boldsymbol{E}))$ respectively. These algebras will be used later to define the symbols of the operators in the class $\mathcal{L}$.

First we equip $\mathcal{P}(\boldsymbol{E})$ with the scalar product

$$\langle f, g\rangle = (2\pi)^{-n} \int_{\boldsymbol{E}} e^{-|z|^2} f(z)\,\overline{g(z)}\,d\mu_{\boldsymbol{E}}(z), \tag{43}$$

where $\mu_{\boldsymbol{E}}$ is the measure $\prod dz_i\,d\bar{z}_i$ if (z_i) are linear complex coordinates associated to an orthonormal basis of $\boldsymbol{E}$. The Gaussian weight $e^{-|z|^2}$ appeared already in Section 2B through the pointwise norm of the frame $s = \exp(-|z|^2/2)$.

Choose linear complex coordinates (z_i) as above. Then the family $|\alpha\rangle := (\alpha!)^{-1/2}\bar{z}^\alpha$, $\alpha \in \mathbb{N}^n$, is an orthonormal basis of $\mathcal{D}(\boldsymbol{E})$. For any $\alpha, \beta \in \mathbb{N}^n$, we introduce the endomorphism $\rho_{\alpha\beta} := |\alpha\rangle\langle\beta|$ of $\mathcal{D}(\boldsymbol{E})$. Here we use the physicist notation, so $\rho_{\alpha\beta}(\bar{z}^\gamma) = 0$ when $\gamma \neq \beta$ and $\rho_{\alpha\beta}(|\beta\rangle) = |\alpha\rangle$.

Consider the creation and annihilation operators $\mathfrak{a}_i$, $\mathfrak{a}_i^\dagger$ defined in (12) as endomorphisms of $\mathcal{P}(\boldsymbol{E})$. Note that with the scalar product (43), $\mathfrak{a}_i^\dagger$ is the formal adjoint of $\mathfrak{a}_i$. We introduce the endomorphism $\tilde{\rho}_{\alpha\beta}$ of $\mathcal{P}(\boldsymbol{E})$

$$\tilde{\rho}_{\alpha\beta} := (\alpha!\beta!)^{-1/2}(\mathfrak{a}^\dagger)^\alpha\,\tilde{\rho}_{00}\,\mathfrak{a}^\beta,$$

where $\mathfrak{a}^\beta = \mathfrak{a}_1^{\beta(1)}\cdots\mathfrak{a}_n^{\beta(n)}$, $(\mathfrak{a}^\dagger)^\alpha = (\mathfrak{a}_1^\dagger)^{\alpha(1)}\cdots(\mathfrak{a}_n^\dagger)^{\alpha(n)}$ and $\tilde{\rho}_{00}$ is the orthogonal projector onto the subspace $\mathfrak{L}_0$ of $\mathcal{P}(\boldsymbol{E})$ consisting of holomorphic polynomials.

Observe that the restriction of $\tilde{\rho}_{\alpha\beta}$ to $\mathcal{D}(\boldsymbol{E})$ is $\rho_{\alpha\beta}$. Furthermore, in the decomposition into orthogonal subspaces $\mathcal{P}(\boldsymbol{E}) = \bigoplus_\alpha \mathfrak{L}_\alpha$ considered in (14), $\tilde{\rho}_{\alpha\beta}$ is zero on $\mathfrak{L}_\gamma$ with $\gamma \neq \beta$ and restricts to an isomorphism from $\mathfrak{L}_\beta$ to $\mathfrak{L}_\alpha$. Also $\tilde{\rho}_{\alpha\alpha}$ is the orthogonal projector onto $\mathfrak{L}_\alpha$.

The algebras $\mathcal{S}(\boldsymbol{E})$ and $\widetilde{\mathcal{S}}(\boldsymbol{E})$ are defined as the subalgebras of $\mathrm{End}(\mathcal{D}(\boldsymbol{E}))$ and $\mathrm{End}(\mathcal{P}(\boldsymbol{E}))$ with basis the families $(\rho_{\alpha,\beta},\ \alpha, \beta \in \mathbb{N}^n)$ and $(\tilde{\rho}_{\alpha\beta},\ \alpha, \beta \in \mathbb{N}^n)$ respectively. As the notation suggests, these algebras do not depend on the coordinate choice. This follows from the following Schwartz kernel description.

Let $\mathrm{Op} : \mathcal{P}(\boldsymbol{E}) \to \mathrm{End}(\mathcal{P}(\boldsymbol{E}))$ be the linear map defined by

$$\mathrm{Op}(q)(f)(u) = (2\pi)^{-n} \int_{\boldsymbol{E}} e^{u\cdot\bar{v} - |v|^2} q(u - v) f(v)\,d\mu_{\boldsymbol{E}}(v), \tag{44}$$

where $u \cdot \bar{v}$ is the scalar product of u and v. By [Charles 2024, Lemma 4.3], $\tilde{\rho}_{\alpha,\beta} = \mathrm{Op}(p_{\alpha,\beta})$, where $p_{\alpha\beta}$ is the polynomial

$$p_{\alpha,\beta} := (\alpha!\,\beta!)^{-1/2}(\bar{z} - \partial_z)^{\alpha}(-z)^{\beta}, \quad \alpha, \beta \in \mathbb{N}^n. \tag{45}$$

Since these polynomials form a basis of $\mathcal{P}(\boldsymbol{E})$, Op is an isomorphism from $\mathcal{P}(\boldsymbol{E})$ to $\widetilde{\mathcal{S}}(\boldsymbol{E})$. Furthermore, the map sending $q \in \mathcal{P}(\boldsymbol{E})$ to $\mathrm{Op}(q)|_{\mathcal{D}(\boldsymbol{E})}$ is an isomorphism from $\mathcal{P}(\boldsymbol{E})$ to $\mathcal{S}(\boldsymbol{E})$.

In the sequel we will tensor the space $\mathcal{P}(\boldsymbol{E})$ with an auxiliary vector space $\mathbb{A}$ and extend the map Op from $\mathcal{P}(\boldsymbol{E}) \otimes \mathrm{End}\,\mathbb{A}$ to $\widetilde{\mathcal{S}}(\boldsymbol{E}) \otimes \mathrm{End}\,\mathbb{A}$.

5B. ***Eigenprojectors of Landau Hamiltonian.*** Choose now $\boldsymbol{E} = T_yM$ and recall that, for a convenient choice of complex coordinate (z_i), the associated Landau Hamiltonian $\widetilde{\Box}_y$ is given by

$$\widetilde{\Box}_y = e^{|\xi|_y^2/4}\Delta_y e^{-|\xi|_y^2/4} = \sum B_i(y)\big(\mathfrak{a}_i^{\dagger}\mathfrak{a}_i + \tfrac{1}{2}\big) + V(y) \tag{46}$$

acting on $\mathcal{P}(T_yM) \otimes A_y$. Its spectrum Σ_y and its eigenspaces were described in Section 2C in terms of the $\mathfrak{L}_\alpha$ and an eigenbasis (ζ_ℓ) of $V(y)$, $V(y)\zeta_\ell = V_\ell(y)\zeta_\ell$. Consequently if I is any bounded subset of $\mathbb{R}$, the spectral projector of $\widetilde{\Box}_y$ for the eigenvalues in I is $\mathrm{Op}(\sigma^I(y))$, where

$$\sigma^I(y) = \sum_{(\alpha,\ell)\in\mathcal{I}_y} p_{\alpha\alpha} \otimes |\zeta_\ell\rangle\langle\zeta_\ell|,$$

and $\mathcal{I}_y = \big\{(\alpha, \ell) \in \mathbb{N}^n \times \{1, \dots, r\} / \sum_i B_i(y)\big(\alpha(i) + \tfrac{1}{2}\big) + V_\ell(y) \in I\big\}$.

The map $y \mapsto \sigma^I(y)$ is a section of the infinite-rank vector bundle $\mathcal{P}(TM)$, not smooth in general, not even continuous. In the sequel we will assume that

$$I \text{ is a compact interval with endpoints not belonging to } \Sigma. \tag{C}$$

Let $\mathcal{P}_{\leqslant p}(\boldsymbol{E})$ be the subspace of $\mathcal{P}(\boldsymbol{E})$ of polynomials with degrees in z and in $\bar{z}$ smaller than p. Let $\mathcal{P}_{\leqslant p}(TM)$ be the vector bundle over M with fiber at y equal to $\mathcal{P}_{\leqslant p}(T_yM)$.

Lemma 5.1. *If I satisfies* (C) *and p is sufficiently large, then $y \mapsto \sigma^I(y)$ is a smooth section of $\mathcal{P}_{\leqslant p}(TM) \otimes \mathrm{End}\,A$.*

Proof. Recall from Section 2E that $\Box_y$ is the restriction of $\widetilde{\Box}_y$ to $\mathcal{D}(T_yM)$. By Lemma 2.4, the spaces

$$F_y := \mathrm{Im}\,1_I(\Box_y) = \mathrm{Span}(\bar{z}^{\alpha} \otimes \zeta_\ell, (\alpha, \ell) \in \mathcal{I}_y) \tag{47}$$

are the fibers of a subbundle of $\mathcal{D}_{\leqslant p}(TM) \otimes A$ if p is sufficiently large. So the projector onto F_y depends smoothly on y; in other words, the map $y \to \mathrm{Op}(\sigma^I(y))|_{\mathcal{D}(T_yM)\otimes A_y}$ is a smooth section of $\mathrm{End}(\mathcal{D}_{\leqslant p}(TM) \otimes A)$.

Now we have an isomorphism

$$\mathcal{P}_{\leqslant p}(\boldsymbol{E}) \xrightarrow{\mathrm{Op}_p} \mathrm{End}(\mathcal{D}_{\leqslant p}(\boldsymbol{E})), \quad q \mapsto \text{the restriction of } \mathrm{Op}(q) \text{ to } \mathcal{D}_{\leqslant p}(\boldsymbol{E}).$$

Indeed, on one hand $(p_{\alpha\beta}, |\alpha|, |\beta| \leqslant p)$ is a basis of $\mathcal{P}_{\leqslant p}(\mathbb{C}^n)$ and on the other hand $(\rho_{\alpha\beta}, |\alpha|, |\beta| \leqslant p)$ is a basis of $\mathrm{End}\,\mathcal{D}_{\leqslant p}(\mathbb{C}^n)$. This gives a vector bundle isomorphism $\mathcal{P}_{\leqslant p}(TM) \otimes \mathrm{End}\,A \simeq \mathrm{End}(\mathcal{D}_{\leqslant p}(TM) \otimes A)$, and concludes the proof. ∎

Let $\mathcal{S}(TM)$ be the infinite-rank vector bundle over M with fibers $\mathcal{S}(T_yM)$ defined as in Section 5A. A section U of $\mathcal{S}(TM) \otimes \operatorname{End} A$ is *smooth* if it has the form

$$U(y) = \operatorname{Op}(q(y))|_{\mathcal{D}(T_yM)\otimes A_y}, \tag{48}$$

where $y \to q(y)$ is a smooth section of $\mathcal{P}_{\leqslant p}(TM) \otimes \operatorname{End} A$ for some p. By Lemma 5.1, for any interval I satisfying (C), we have a symbol $\pi^I \in \mathcal{C}^\infty(M, \mathcal{S}(TM) \otimes \operatorname{End} A)$ defined at y by

$$\pi^I(y) = 1_I(\square_y) = \operatorname{Op}(\sigma^I(y))|_{\mathcal{D}(T_yM)\otimes A_y}, \tag{49}$$

which is the projector of $\mathcal{D}(T_yM) \otimes A_y$ onto the subspace F_y defined in Lemma 2.4.

5C. *Operators.* The operator class $\mathcal{L}(A)$ was introduced in [Charles 2024]. It depends on (M, ω, j), the prequantum bundle L, that is, L with its metric and connection, and the auxiliary Hermitian bundle A.

$\mathcal{L}(A)$ consists of families of operators $(P_k : \mathcal{C}^\infty(M, L^k \otimes A) \to \mathcal{C}^\infty(M, L^k \otimes A),\ k \in \mathbb{N})$ having smooth Schwartz kernels satisfying the following conditions. First, $P_k(x, y)$ is in $\mathcal{O}(k^{-\infty})$ outside the diagonal. More precisely, for any compact subset K of $M^2 \setminus \operatorname{diag} M$ and for any N, there exists $C > 0$ such that

$$|P_k(x, y)| \leqslant Ck^{-N} \quad \text{for all } k \in \mathbb{N}, \text{ for all } (x, y) \in K.$$

Second, for any open set U of M identified through a diffeomorphism with a convex open set of $\mathbb{R}^{2n}$ and any unitary trivialization $A|_U \simeq U \times \mathbb{C}^r$, we have on U^2 for any positive integers N, k

$$P_k(x + \xi, x) = \left(\frac{k}{2\pi}\right)^n F^k(x + \xi, x)e^{-k|\xi|_x^2/4} \sum_{\ell=0}^{N} k^{-\ell} a_\ell(x, k^{1/2}\xi) + r_{N,k}(x + \xi, x), \tag{50}$$

where the section $F : U^2 \to L \boxtimes \bar{L}$ is defined as in Section 1D, the coefficients $a_\ell(x, \xi) \in \mathbb{C}^r \otimes \bar{\mathbb{C}}^r$ depend smoothly on x and polynomialy on ξ, with degree bounded independently of x, and the remainder $r_{N,k}$ is in $\mathcal{O}(k^{n-(N+1)/2})$ uniformly on any compact subset of U^2.

The subspace $\mathcal{L}^+(A)$ of $\mathcal{L}(A)$ consists of the operator families (P_k) where the coefficients a_ℓ in the local expansions (50) satisfy $a_\ell(x, -\xi) = (-1)^\ell a_\ell(x, \xi)$. The symbol map is the application $\sigma_0 : \mathcal{L} \to \mathcal{C}^\infty(M, \mathcal{S}(TM) \otimes \operatorname{End} A)$ given locally by

$$\sigma_0(P)(x) = \operatorname{Op}(a_0(x, \cdot))|_{\mathcal{D}(T_xM)} \in \mathcal{S}(T_xM) \otimes \operatorname{End} A_x, \tag{51}$$

where we view $a_0(x, \xi)$ in $\mathbb{C}^r \otimes \bar{\mathbb{C}}^r \simeq \operatorname{End} \mathbb{C}^r \simeq \operatorname{End} A_x$.

Recall that for any compact interval I of $\mathbb{R}$, we denote by Π_k^I the corresponding spectral projector of $k^{-1}\Delta_k$. The central result of this paper is the following theorem.

Theorem 5.2. *Let* (Π_k^I) *be the spectral projector of a formally self-adjoint operator family* (Δ_k) *of the form* (B) *with* I *satisfying* (C). *Then* (Π_k^I) *belongs to* $\mathcal{L}^+(A)$ *and has symbol* π^I.

The proof is given in Section 6. We will actually prove a stronger result where we describe the Schwartz kernel derivatives as well.

5D. ***The class*** $\mathcal{L}^\infty(A)$**.** We need first a few definitions. Consider a real number N. We say that a sequence (f_k) of $\mathcal{C}^\infty(U)$ with U an open set of M is in $\mathcal{O}_\infty(k^{-N})$ if, for any $m \in \mathbb{N}$, for any vector fields $X_1, \dots, X_m$ of U, for any compact subset K of U, there exists $C > 0$ such that

$$|X_1 \cdots X_m f_k(x)| \leqslant Ck^{-N+m} \quad \text{for all } x \in K,\ k \in \mathbb{N}.$$

Let $s = (s_k \in \mathcal{C}^\infty(M, L^k \otimes A),\ k \in \mathbb{N})$. We say that $s \in \mathcal{O}_\infty(k^{-N})$ if, for any unitary frames u and $(v_j)_{j=1}^r$ of L and A defined over the same open set U of M, the local representative sequences $(f_{k,j})$ such that $s_k = \sum f_{j,k} u^k \otimes v_j$, are in $\mathcal{O}_\infty(k^{-N})$. We say that s belongs to $\mathcal{O}_\infty(k^\infty)$ (resp. $\mathcal{O}_\infty(k^{-\infty})$) if $s \in \mathcal{O}_\infty(k^{-N})$ for some N (resp. for any N). So

$$\mathcal{O}_\infty(k^{-\infty}) \subset \mathcal{O}_\infty(k^{-N}) \subset \mathcal{O}_\infty(k^{-N'}) \subset \mathcal{O}_\infty(k^\infty) \quad \text{if } N \geqslant N'.$$

Replacing M, L and A by M^2, $L \boxtimes \bar{L}$ and $A \boxtimes \bar{A}$, we can apply these definitions to Schwartz kernels of operator families $(P_k : \mathcal{C}^\infty(M, L^k \otimes A) \to \mathcal{C}^\infty(M, L^k \otimes A),\ k \in \mathbb{N})$.

By definition, $\mathcal{L}^\infty(A)$ and $\mathcal{L}^\infty_\infty(A)$ are the subspaces of $\mathcal{L}(A)$ consisting of operator families with a Schwartz kernel in $\mathcal{O}_\infty(k^\infty)$ and $\mathcal{O}_\infty(k^{-\infty})$ respectively. By [Charles 2024, Proposition 6.3], the difference between $\mathcal{L}^\infty(A)$ and $\mathcal{L}(A)$ is rather small because for any $P \in \mathcal{L}(A)$, there exists $P' \in \mathcal{L}^\infty(A)$ such that the Schwartz kernel of $P - P'$ is in $\mathcal{O}(k^{-\infty})$, that is, $P_k(x, x') = P'_k(x, x') + \mathcal{O}(k^{-N})$ for any N, with $\mathcal{O}$ uniform on M^2. Furthermore P' is unique modulo $\mathcal{L}^\infty_\infty(A)$.

By [Charles 2024, Proposition 6.3], for any $(P_k) \in \mathcal{L}^\infty(A)$ the asymptotic expansion (50) holds with a remainder $r_{N,k}$ in $\mathcal{O}_\infty(k^{n-(N+1)/2})$.

Theorem 5.3. *Under the same assumptions as in Theorem 5.2,* (Π_k^I) *belongs to* $\mathcal{L}^\infty(A)$.

The proof will be given in Section 6. To end this section, let us state the following corollary of Theorems 5.2, 5.3 and Lemma 6.3.

Corollary 5.4. *Under the same assumptions as in Theorem 5.2,* $(k^{-1}\Delta_k \Pi_k^I)$ *belongs to* $\mathcal{L}^+(A) \cap \mathcal{L}^\infty(A)$ *and has symbol* $\sigma_0(k^{-1}\Delta_k \Pi_k) = \Box \circ \pi^I$.

So the first part of Theorem 1.4 follows from Theorem 5.2 and Corollary 5.4.

6. Proof of Theorems 5.2 and 5.3

The first step, Lemma 6.1, is to show that any operator in $\mathcal{L}(A)$ with symbol π^I is an approximation of Π_k^I up to $\mathcal{O}(k^{-1/4})$. This will follow from the resolvent estimate given in Corollary 4.8 and the Cauchy–Riesz formula. The second step, Lemma 6.2, is the construction of a formal projector $(P_k) \in \mathcal{L}^+(A)$ with symbol π^I which almost commutes with Δ_k. The third step, Section 6C, is to show that this formal projector (P_k) is equal to Π_k^I up to $\mathcal{O}(k^{-\infty})$ and even up to $\mathcal{O}_\infty(k^{-\infty})$ when $(P_k) \in \mathcal{L}^\infty(A)$.

6A. *A first approximation.*

Lemma 6.1. *Under the same assumptions as in Theorem 5.2,* $\Pi_k^I = P_k + \mathcal{O}(k^{-1/4})$ *for any* (P_k) *in* $\mathcal{L}(A)$ *with symbol* π^I.

Proof. Step 1: The proof starts from the resolvent approximation given in Corollary 4.8. Choose a loop γ of $\mathbb{C} \setminus \Sigma$ which encircles I. When k is sufficiently large, by Corollary 4.8, γ does not meet the spectrum of $k^{-1}\Delta_k$. So by Riesz projection formula and (40),

$$\Pi_k^I = \frac{1}{2i\pi}\int_\gamma (\lambda - k^{-1}\Delta_k)^{-1}\, d\lambda = \frac{1}{2i\pi}\int_\gamma R_k^{r_k}(\lambda)\, d\lambda + \mathcal{O}(k^{-1/4}), \tag{52}$$

with $r_k = k^{-1/4}$. Since $R_k^r(\lambda) := \sum_{y\in I(r)} \chi_{y,r}\widetilde{R}_{y,k}(\lambda)\psi_{r,y}$, we get

$$\Pi_k^I = \sum_{y\in I(r_k)} \chi_{y,r_k}\widetilde{P}_{y,k}^I\psi_{r_k,y} + \mathcal{O}(k^{-1/4}), \tag{53}$$

where for any y

$$\widetilde{P}_{y,k}^I = \frac{1}{2i\pi}\int_\gamma \widetilde{R}_{y,k}(\lambda)\, d\lambda.$$

Recall that $\widetilde{R}_{y,k}(\lambda)$ is the restriction of the resolvent $(\lambda - k^{-1}\Delta_{y,k})^{-1}$ to $\mathcal{C}_0^\infty(B_y(r), \mathbb{C}^r)$ identified with $\mathcal{C}_0^\infty(\Psi_y(B_y(r)), L^k \otimes A)$. So by Riesz projection formula again, $\widetilde{P}_{y,k}^I$ is the restriction of the spectral projection

$$P_{y,k}^I = \frac{1}{2i\pi}\int_\gamma (\lambda - k^{-1}\Delta_{y,k})^{-1}\, d\lambda.$$

Step 2: Let $d : M^2 \to \mathbb{R}_{\geqslant 0}$ be a distance locally equivalent to the Euclidean distance in each chart and set $m_k(x', x) := k^n \exp(-kcd(x', x)^2)$ with $c > 0$. Then by Schur test, any operator family $(Q_k : \mathcal{C}^\infty(M, L^k \otimes A) \to \mathcal{C}^\infty(M, L^k \otimes A), k \in \mathbb{N})$ having a continuous Schwartz kernel satisfying $|Q_k(x', x)| = \mathcal{O}(m_k(x', x))$ uniformly with respect to x, x' and k, has a bounded operator norm; see [Charles 2024, proof of Lemma 5.1] for more details. Given this and (53), it suffices now to prove that

$$P_k(x', x) = \sum_{y\in I(r)} \chi_{y,r_k}(x')\widetilde{P}_{y,k}^I(x', x)\psi_{r_k,y}(x) + (m_k(x', x) + 1)\mathcal{O}(k^{-1/4}). \tag{54}$$

In the sequel, we will allow the constant c entering in the definition of m_k to decrease from one line to another. With this convention, for any $p > 0$, we can replace any $\mathcal{O}(d^p(x', x)m_k(x', x))$ by $\mathcal{O}(k^{p/2}m_k(x', x))$.

Step 3: Equation (54) follows from

$$P_k(x', x) = \widetilde{P}_{y,k}^I(x', x) + (m_k(x', x) + 1)\mathcal{O}(k^{-1/4}) \tag{55}$$

for all $(x', x) \in \Psi_y(B_y(2r)) \times \Psi_y(B_y(2r))$, with $\mathcal{O}$ uniform with respect to all the variables, y included. Indeed, since $\operatorname{supp}\psi_{r,y} \subset \Psi_y(B_y(r)) \subset \{\chi_{y,r} = 1\}$, we have

$$\chi_{y,r}(x')\psi_{r,y}(x) = \psi_{r,y}(x) + \mathcal{O}(d(x', x)r^{-1}) \quad \text{for all } x, x' \in \Psi_y(B_y(2r)).$$

Recall that by [Charles 2024, Lemma 5.1], $P_k(x', x) = \mathcal{O}(m_k(x', x)) + \mathcal{O}(k^{-N})$ for any N. Applying this to $N = \frac{1}{4}$ and using that $m_k d = \mathcal{O}(k^{-1/2}m_k)$ as explained above, we obtain

$$\chi_{y,r}(x')P_k(x', x)\psi_{r,y}(x) = P_k(x', x)\psi_{r,y}(x) + \mathcal{O}(k^{-1/2}m_k(x', x)r^{-1}) + \mathcal{O}(k^{-1/4}).$$

Assume now that (55) holds. Multiplying (55) by $\chi_{y,r}(x')\psi_{r,y}(x)$ and using the last equality, we obtain

$$P_k(x',x)\psi_{r,y}(x)=\chi_{y,r}(x')\widetilde{P}^I_{y,k}(x',x)\psi_{r,y}(x)+\mathcal{O}(k^{-1/2}m_k(x',x)r^{-1})+\mathcal{O}(k^{-1/4}),$$

which holds for all $x',x\in M$. Recall that the covering $\bigcup\Psi_y(B_y(r))$, $y\in I(r)$, has a multiplicity bounded independently on r. So we can sum these estimates without multiplying the remainder by the number of summands and we obtain

$$P_k(x',x)=\sum_{y\in I(r)}\chi_{y,r}(x')\widetilde{P}^I_{y,k}(x',x)\psi_{r,y}(x)+\mathcal{O}(k^{-1/2}m_k(x',x)r^{-1})+\mathcal{O}(k^{-1/4}).$$

This proves (54) because $r_k=k^{-1/4}$.

Step 4: We give a formula for the Schwartz kernel of the spectral projector $P^I_{y,k}$. First, by the rescaling (26), (27), we have

$$P^I_{y,k}(\xi,\eta)=k^nP^I_y(k^{1/2}\xi,k^{1/2}\eta), \tag{56}$$

with $P^I_y:=P^I_{y,1}$. Second, the Schwartz kernel of P^I_y is given by

$$P^I_y(\eta+\xi,\eta)=(2\pi)^{-n}e^{(i/2)\omega_y(\eta,\xi)-|\xi|^2_y/4}\pi^I(y,\xi). \tag{57}$$

Indeed, by (46), $P^I_y=e^{-|\xi|_y/4}\operatorname{Op}(\sigma^I(y))e^{|\xi|_y/4}$ and it follows from (44) that

$$\begin{aligned}P^I_y(\xi,\eta)&=(2\pi)^{-n}e^{-|u|^2/2+u\cdot\bar v-|v|^2/2}\sigma^I(y,u-v)\\&=(2\pi)^{-n}e^{(u\cdot\bar v-\bar u\cdot v)/2-|u-v|^2/2}\sigma^I(y,u-v),\end{aligned} \tag{58}$$

with (u_i), (v_i) the complex coordinates of ξ and η defined as in Section 2B, in particular $|\xi|^2_y=\frac{1}{2}|u|^2$ and $|\eta|^2_y=\frac{1}{2}|v|^2$. Since $\omega_y=i\sum_i du_i\wedge d\bar u_i$, (57) follows from (58). Inserting (57) into (56), we get

$$P^I_{y,k}(\eta+\xi,\eta)=\left(\frac{k}{2\pi}\right)^nF^k_y(\eta+\xi,\eta)e^{-k|\xi|^2_y/4}\sigma^I(y,k^{1/2}\xi), \tag{59}$$

with $F_y(\eta+\xi,\eta)=e^{(i/2)\omega_y(\eta,\xi)}$. F_y has the same characterization as the section F entering in the expansion (50), that is, $F_y(\eta,\eta)=1$ and $\mathbb{R}\ni t\to F_y(\eta+t\xi,\eta)$ is flat for any ξ,η.

Step 5: The Schwartz kernel of P_k has the local expansion (50). By [Charles 2024, Lemma 5.1], the remainder $r_{N,k}$ is in $\mathcal{O}(k^{-N/2}m_k)+\mathcal{O}(k^{-N'})$ for any N'. So in particular,

$$P_k(x+\xi,x)=\left(\frac{k}{2\pi}\right)^nF^k(x+\xi,x)e^{-k|\xi|^2_x/4}\sigma^I(x,k^{1/2}\xi)+(m_k+1)\mathcal{O}(k^{-1/4}). \tag{60}$$

Step 6: We now prove (55) by comparing (59) and (60). So let $x,x'\in\Psi_y(B_y(2r))$ and $\xi=x'-x$. We will use several times that

$$d(x,y)\leqslant Cr,\quad C^{-1}d\leqslant|\xi|\leqslant Cd,\quad\text{where } d:=d(x',x).$$

Let $\Phi_y : \Psi_y(B_y(r_0)) \to T_y M$ be the inverse of Ψ_y. We have to compare $P_k(x+\xi, x)$ with $\widetilde{P}^I_{y,k}(x+\xi, x) = P^I_{y,k}(\eta+\tilde{\xi}, \eta)$, where

$$\eta = \Phi_y(x), \quad \eta+\tilde{\xi} = \Phi_y(x+\xi).$$

We claim that

$$\tilde{\xi} = \xi + \mathcal{O}(rd + d^2). \tag{61}$$

To see this, write $\tilde{\xi} = \Phi_y(x+\xi) - \Phi_y(x) = L_y(x, \xi)\xi$, where $L_y(x, 0) = T_x\Phi_y$. Since $L_y(y, 0) = \mathrm{id}_{T_y M}$, we have

$$L_y(x, \xi) = L_y(x, 0) + \mathcal{O}(|\xi|) = \mathrm{id}_{T_y M} + \mathcal{O}(d(x, y) + |\xi|).$$

So $\tilde{\xi} = \xi + \mathcal{O}(|\xi|(d(x, y) + |\xi|)) = \xi + \mathcal{O}(d(r + d))$.

Consider now a smooth function $(x, \xi) \to q(x, \xi)$ which is polynomial homogeneous in ξ with degree ℓ. Then

$$q(x, \xi) = q(y, \xi) + \mathcal{O}(d(x, y)|\xi|^\ell) = q(y, \xi) + \mathcal{O}(rd^\ell)$$

and by (61), $q(y, \xi) = q(y, \tilde{\xi}) + \mathcal{O}(d^\ell(r + d))$. So

$$q(x, k^{1/2}\xi) = q(y, k^{1/2}\tilde{\xi}) + \mathcal{O}((k^{1/2}d)^\ell(r + d)). \tag{62}$$

Consequently

$$\sigma^I(x, k^{1/2}\xi) = \sigma^I(y, k^{1/2}\tilde{\xi}) + \mathcal{O}(r + d)\sum(k^{1/2}d)^\ell, \tag{63}$$

where the sum on the right is over ℓ and finite.

By [Charles 2016, Section 2.6], the section $E(x+\xi, x) := F(x+\xi, x)e^{-|\xi|_x^2/4}$ depends on the coordinate choice up to a section vanishing to third order along the diagonal. So

$$E(x+\xi, x) = F_y(\eta+\tilde{\xi}, \eta)e^{-|\tilde{\xi}|_x/4}e^{\mathcal{O}(d^3)} = E_y(\eta+\tilde{\xi}, \eta)e^{\mathcal{O}(d^3+d^2r)},$$

with $E_y(\eta+\tilde{\xi}, \eta) := F_y(\eta+\tilde{\xi}, \eta)e^{-|\tilde{\xi}|_y/4}$ because $|\tilde{\xi}|_y^2 = |\xi|_x^2 + \mathcal{O}(d^2(r + d))$ by (62). So using that $|e^z - 1| \leqslant |z|e^{|\operatorname{Re} z|}$ and that $k^n E^k(x+\xi, x) = \mathcal{O}(m_k)$, we have

$$\begin{aligned} k^n(E^k(x+\xi, x) - E^k_y(\eta+\tilde{\xi}, \eta)) &= \mathcal{O}(d^2(d + r)m_k)e^{kCd^2(d+r)} = \mathcal{O}(d^2(d + r)m_k)e^{kCd^2(d+r)} \\ &= \mathcal{O}(k^{-5/4}m_k)e^{kCd^2(d+r)} = \mathcal{O}(k^{-5/4}m_k), \end{aligned} \tag{64}$$

where we have used that d and r are both in $\mathcal{O}(k^{-1/4})$, and always the same convention that the constant c in m_k can change from one line to another so that $d^p m_k = \mathcal{O}(k^{-p/2}m_k)$. Using again that $k^n E^k(x+\xi, x) = \mathcal{O}(m_k)$, it follows from (63),

$$\begin{aligned} k^n E^k(x+\xi, x)\sigma^I(x, k^{1/2}\xi) &= k^n E^k(x+\xi, x)\sigma^I(y, k^{1/2}\tilde{\xi}) + \mathcal{O}(k^{-1/4}m_k) \\ &= k^n E^k_y(\eta+\tilde{\xi}, \eta)\sigma^I(y, k^{1/2}\tilde{\xi}) + \mathcal{O}(k^{-1/4}m_k) \end{aligned}$$

by (64), which ends the proof of (55). □

6B. ***A formal projector.*** This section is devoted to the proof of the following lemma.

Lemma 6.2. *Under the same assumptions as in Theorem 5.2, there exists* $(P_k) \in \mathcal{L}^\infty(A) \cap \mathcal{L}^+(A)$ *unique modulo* $\mathcal{L}^\infty_\infty(A)$ *such that* $\sigma_0(P_k) = \pi^I$, $P_k = P_k^*$ *for any* k, $P_k \equiv P_k^2$ *modulo* $\mathcal{L}^\infty_\infty(A)$ *and* $[\Delta_k, P_k] \equiv 0$ *modulo* $\mathcal{L}^\infty_\infty(A)$.

To show this, we will construct (P_k) by successive approximations. We introduce the filtration

$$\mathcal{L}^\infty_p(A) := \mathcal{L}^\infty(A) \cap \mathcal{O}_\infty(k^{-p/2}),$$

$p \in \mathbb{N}$. For any $p \in \mathbb{N}$, we have a symbol map

$$\sigma_p : \mathcal{L}^\infty_p(A) \to \mathcal{C}^\infty(M, \mathcal{S}(TM) \otimes \operatorname{End} A)$$

such that $\sigma_p(P) = \sigma_0(k^{p/2} P)$, where σ_0 was defined in (51). By [Charles 2024, Proposition 2.1 and Theorem 2.2], σ_p is onto, $\operatorname{Ker} \sigma_p = \mathcal{L}^\infty_{p+1}(A)$ and for any sequence (Q_p) of $\mathcal{L}^\infty(A)$ such that $Q_p \in \mathcal{L}^\infty_p(A)$ for any p, there exists $Q \in \mathcal{L}^\infty(A)$ such that $Q = Q_0 + \cdots + Q_p$ modulo $\mathcal{L}^\infty_{p+1}(A)$ for any p. Moreover:

(1) If Q and Q' belong to $\mathcal{L}^\infty_p(A)$ and $\mathcal{L}^\infty_{p'}(A)$ respectively, then their product belongs to $\mathcal{L}^\infty_{p+p'}(A)$. Furthermore, at any $x \in M$, $\sigma_{p+p'}(QQ')(x)$ is the product of $\sigma_p(Q)(x)$ and $\sigma_{p'}(Q')(x)$.

(2) If Q belongs to $\mathcal{L}^\infty_p(A)$, then its adjoint Q^* belongs to $\mathcal{L}^\infty_p(A)$ with symbol $\sigma_p(Q^*)(x) = \sigma_p(Q)(x)^*$.

By [Charles 2024, Theorem 2.5], $\mathcal{L}^+(A)$ is a subalgebra of $\mathcal{L}(A)$.

Lemma 6.3. *For any* Q *in* $\mathcal{L}^\infty_p(A)$, $(k^{-1}\Delta_k Q_k)$ *and* $(k^{-1} Q_k \Delta_k)$ *both belong to* $\mathcal{L}^\infty_p(A)$ *and their symbols at* x *are* $\square_x \circ \sigma_p(Q)(x)$ *and* $\sigma_p(Q)(x) \circ \square_x$. *If* $Q \in \mathcal{L}^+(A)$ *then the same holds for* $(k^{-1}\Delta_k Q_k)$ *and* $(k^{-1} Q_k \Delta_k)$.

Proof. By [Charles 2024, Proposition 6.3, Assertion 3c and 3d], $(k^{-1}\Delta_k Q_k)$ and $(k^{-1} Q_k \Delta_k)$ both belong to $\mathcal{L}^\infty_p(A)$. To compute the symbol, we can use the peaked sections of Section 4A. Indeed, if $\Phi_k(f)$ is defined by (28), with $f \in \mathcal{D}(T_x M) \otimes A_x$ and $(P_k) \in \mathcal{L}_0(A)$ then by [Charles 2024, Proposition 2.4], $P_k \Phi_k(f) = \Phi_k(g) + \mathcal{O}(k^{-1/2})$ with $g = \sigma_0(P_k)(x) f$. So the symbol of any operator of $\mathcal{L}_p(A)$ is characterized by its action on the peaked sections. Proposition 4.1 tells us how $k^{-1}\Delta_k$ acts on the peaked section and the first part of the result follows. To show that the composition with $k^{-1}\Delta_k$ preserves the subspace $\mathcal{L}^+(A)$ of even operators, one uses instead of the asymptotic expansion (50) the alternative expansion

$$P_k(x, y) = \left(\frac{k}{2\pi}\right)^n E^k(x, y) \sum k^{-\ell/2} b_\ell(x, y) + \mathcal{O}(k^{-\infty});$$

see [Charles 2024, equation (45) and Proposition 5.6]. The fact that (P_k) is even means that $b_\ell = 0$ when ℓ is odd. When $(P_k) \in \mathcal{L}^\infty(A)$, this expansion holds for the $\mathcal{C}^\infty$ topology, so we can compute the Schwartz kernel of $k^{-1}\Delta_k P_k$ by letting $k^{-1}\Delta_k$ act on each term of the expansion. Doing this with the expression (B), no half power of k appears so $k^{-1}\Delta_k P_k$ is even. The same argument works for $k^{-1} P_k \Delta_k$. ☐

In the sequel, to lighten the notation, we write π instead of π^I. Let L_1 and L_2 be the endomorphisms of $\mathcal{C}^\infty(M, \mathcal{S}(TM) \otimes \operatorname{End} A)$ defined by

$$L_1(f)(x) = \pi(x) \circ f(x) + f(x) \circ \pi(x) - f(x),$$
$$L_2(f)(x) = [\Box_x, f(x)].$$

Assuming that I satisfies (C), $\pi \in \mathcal{C}^\infty(M, \operatorname{End}(\mathcal{D}_{\leqslant p_0}(TM) \otimes \operatorname{End} A)$ for some p_0, so that L_1 is well-defined, meaning that $L_1(f)$ is a smooth section of $\mathcal{S}(TM) \otimes \operatorname{End} A$ when f is.

Lemma 6.4. *The following sequence is exact*:

$$0 \to \operatorname{Symb} \xrightarrow{L} \operatorname{Symb} \oplus \operatorname{Symb} \xrightarrow{L'} \operatorname{Symb} \to 0, \tag{65}$$

where $\operatorname{Symb} = \mathcal{C}^\infty(M, \mathcal{S}(TM) \otimes \operatorname{End} A)$, $L(f) = (L_1(f), L_2(f))$ *and* $L'(f_1, f_2) = L_2(f_1) - L_1(f_2)$.

Proof. $L' \circ L = 0$ is equivalent to $L_1 \circ L_2 = L_2 \circ L_1$, which follows from $[\Box, \pi] = 0$. Indeed $[\Box, \pi] = 0$ implies that $[\Box, f\pi] = [\Box, f]\pi$ and $[\Box, \pi f] = \pi[\Box, f]$ so that

$$\begin{aligned} L_2(L_1(f)) &= [\Box, f\pi + \pi f - f] \\ &= [\Box, f]\pi + \pi[\Box, f] - [\Box, f] = L_1(L_2(f)). \end{aligned}$$

Recall that $\operatorname{Symb} = \bigcup_{p \in \mathbb{N}} \operatorname{Symb}_p$, with $\operatorname{Symb}_p = \mathcal{C}^\infty(M, \operatorname{End}(\mathcal{D}_{\leqslant p}(TM) \otimes A))$. L_2 preserves each Symb_p and the same holds for L_1 when p is larger than p_0. So we have to prove that, for any $p \geqslant p_0$, the sequence (65) with Symb replaced by Symb_p is exact.

By Lemma 2.4, the image of π is a subbundle F of $\mathcal{D}_{\leqslant p}(TM) \otimes A$. Let $F^\perp$ be the orthogonal subbundle, so that $\mathcal{D}_{\leqslant p}(TM) \otimes A = F \oplus F^\perp$. Write the elements of Symb_p as block matrices according to this decomposition. The restrictions of π and $\Box$ to Symb_p have the particular forms

$$\pi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Box = \begin{pmatrix} \Box_{\text{in}} & 0 \\ 0 & \Box_{\text{out}} \end{pmatrix}$$

Writing

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have

$$L_1(f) = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}, \quad L_2(f) = \begin{pmatrix} [\Box_{\text{in}}, a] & E_1(b) \\ E_2(c) & [\Box_{\text{out}}, d] \end{pmatrix},$$

with

$$E_1(b) = \Box_{\text{in}} b - b \Box_{\text{out}}, \quad E_2(c) = \Box_{\text{out}} c - c \Box_{\text{in}} = -E_1(c^*)^*.$$

Let us prove that E_1 and E_2 are invertible endomorphisms of the spaces $\mathcal{C}^\infty(M, \operatorname{Hom}(F^\perp, F))$ and $\mathcal{C}^\infty(M, \operatorname{Hom}(F, F^\perp))$ respectively. For any $y \in M$, we introduce an orthonormal eigenbasis (e_i) of the restriction of $\Box_y$ to $\mathcal{D}_{\leqslant p}(T_y M) \otimes A_y$. So $\Box_y e_i = \lambda_i e_i$ and F_y (resp. $F_y^\perp$) is spanned by the e_i such that $\lambda_i \in I$ (resp. $\lambda_i \notin I$). Now the endomorphism

$$\operatorname{Hom}(F_y^\perp, F_y) \to \operatorname{Hom}(F_y^\perp, F_y), \quad b(y) \mapsto \Box_{\text{in}}(y) b(y) - b(y) \Box_{\text{out}}(y), \tag{66}$$

is diagonalizable with eigenvectors $|e_i\rangle\langle e_j|$ and eigenvalues $\lambda_i - \lambda_j$, where $\lambda_i \in I$ and $\lambda_j \notin I$. Since $\lambda_i - \lambda_j \neq 0$, (66) is invertible for any y, so the same holds for E_1. The proof for E_2 is similar.

From this, we deduce easily that the sequence is exact. In particular if $L'(f_1, f_2) = 0$ with

$$f_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \quad \text{for } i = 1 \text{ or } 2,$$

then $(f_1, f_2) = L(f)$ with

$$f = \begin{pmatrix} a_1 & E_1^{-1}(b_2) \\ E_2^{-1}(c_2) & -d_1 \end{pmatrix}.$$

Observe as well that $f_1 = f_1^*$ and $f_2 = -f_2^*$ imply that $f = f^*$. □

Proof of Lemma 6.2. Let $P \in \mathcal{L}^\infty(A)$ be self-adjoint with symbol $\sigma_0(P) = \pi$. Then $R_1 := P^2 - P$ and $R_2 := k^{-1}[\Delta_k, P]$ both belong to $\mathcal{L}_1^\infty(A)$. Indeed their σ_0-symbols are respectively $\pi^2 - \pi$ and $[\square, \pi]$, and both of them vanish.

Let us prove by induction on $m \geqslant 1$ that there exists P as above such that R_1 and R_2 are in $\mathcal{L}_m^\infty(A)$. Define $P' = P + S$ with $S \in \mathcal{L}_m^\infty(A)$. Assume that R_1 and R_2 are in $\mathcal{L}_m^\infty(A)$. Then

$$\begin{aligned}(P')^2 - P' &= R_1 + SP + PS - S \mod \mathcal{L}_{m+1}^\infty(A), \\ [k^{-1}\Delta_k, P'] &= R_2 + [k^{-1}\Delta_k, S].\end{aligned}$$

So $(P')^2 - P'$ and $[k^{-1}\Delta_k, P']$ belong to $\mathcal{L}_m(A)$ and their σ_m-symbols are respectively $f_1 + L_1(f)$ and $f_2 + L_2(f)$ with $f = \sigma_m(S)$, $f_1 = \sigma_m(R_1)$ and $f_2 = \sigma_m(R_2)$. Let us prove that we can choose f so that $f_1 + L_1(f) = 0$ and $f_2 + L_2(f) = 0$. By Lemma 6.4, it suffices to check that $L_1(f_2) = L_2(f_1)$. But $L_2(f_1)$ is the σ_m-symbol of $[k^{-1}\Delta_k, R_1]$, $L_1(f_2)$ is the σ_m-symbol of $PR_2 + R_2P - R_2$, and these operators are equal as shows a direct computation. So f exists. Furthermore $f = f^*$ by the remark at the end of the proof of Lemma 6.4. So we can choose S self-adjoint.

We conclude the proof with the convergence property with respect to the filtration $\mathcal{L}_m(A)$ recalled above. Observe also that if we start with $P \in \mathcal{L}^+(A)$, then we end with a formal projector in $\mathcal{L}^+(A)$. □

6C. *Operator norm and pointwise estimates.*

Let us choose an operator (P_k) satisfying the conditions of Lemma 6.2. Recall that for any operator $Q \in \mathcal{L}_m(A)$, $Q_k = \mathcal{O}(k^{-m/2})$ in the sense that the operator norm of Q_k is in $\mathcal{O}(k^{-m/2})$. So P_k is self-adjoint, it is an almost projector $P_k^2 = P_k + \mathcal{O}(k^{-\infty})$ and it almost commutes with Δ_k in the sense that $[\Delta_k, P_k] = \mathcal{O}(k^{-\infty})$. Furthermore, by Lemma 6.1, $P_k = \Pi_k^I + \mathcal{O}(k^{-1/4})$.

Lemma 6.5.
$$P_k = \Pi_k^I + \mathcal{O}(k^{-\infty}).$$

Proof. We omit the index k to simplify the notation. Let $\mathcal{H}_+ = \operatorname{Ran} \Pi^I$ and $\mathcal{H}_-$ be its orthogonal in $L^2(M, L^k \otimes A)$. We introduce the corresponding block decomposition of P

$$P = \begin{pmatrix} P_{++} & P_{+-} \\ P_{-+} & P_{--} \end{pmatrix}.$$

We first prove that P_{-+} and P_{+-} are in $\mathcal{O}(k^{-\infty})$.

By Corollary 4.8 and assumption (C), there exists ϵ such that when k is sufficiently large

$$\operatorname{dist}(I, \operatorname{sp}(k^{-1}\Delta_k) \setminus I) \geqslant \epsilon.$$

Let ξ_λ and ξ_μ be two eigenfunctions of $k^{-1}\Delta_k$ with eigenvalues λ and μ respectively. Then

$$\begin{aligned}(\lambda-\mu)\langle P\xi_\lambda, \xi_\mu\rangle &= k^{-1}(\langle P\Delta_k\xi_\lambda, \xi_\mu\rangle - \langle P\xi_\lambda, \Delta_k\xi_\mu\rangle)\\ &= k^{-1}\langle [P, \Delta_k]\xi_\lambda, \xi_\mu\rangle\\ &= \mathcal{O}(k^{-\infty})\|\xi_\lambda\|\,\|\xi_\mu\| \end{aligned} \tag{67}$$

because $[P, \Delta_k] = \mathcal{O}(k^{-\infty})$. Now for any $\xi_+ \in \mathcal{H}_+$ and $\xi_- \in \mathcal{H}_-$, write their decompositions into eigenvectors $\xi_+ = \sum \xi_\lambda$ and $\xi_- = \sum \xi_\mu$. So $\|\xi_+\|^2 = \sum \|\xi_\lambda\|^2$, $\|\xi_-\|^2 = \sum \|\xi_\mu\|^2$ and $\langle P\xi_-, \xi_+\rangle = \sum \langle P\xi_\lambda, \xi_\mu\rangle$. So by (67),

$$|\langle P\xi_-, \xi_+\rangle| \leqslant \epsilon^{-1}\mathcal{O}(k^{-\infty})\sum \|\xi_\lambda\|\,\|\xi_\mu\| \leqslant \epsilon^{-1}\mathcal{O}(k^{-\infty})\|\xi_-\|\,\|\xi_+\|$$

by the Cauchy–Schwarz inequality. This proves that $P_{+-} = \mathcal{O}(k^{-\infty})$. The same holds for its adjoint P_{-+}.

Now the fact that $P^2 = P + \mathcal{O}(k^{-\infty})$ implies $P_{++}^2 = P_{++} + \mathcal{O}(k^{-\infty})$ and the same for P_{--}. Indeed,

$$\begin{aligned}(\Pi^I P \Pi^I)^2 &= \Pi^I P \Pi^I P \Pi^I\\ &= \Pi^I P^2 \Pi^I + \mathcal{O}(k^{-\infty}) \quad (\text{because } P_{-+} = \mathcal{O}(k^{-\infty}))\\ &= \Pi^I P \Pi^I + \mathcal{O}(k^{-\infty}) \quad (\text{because } P^2 = P + \mathcal{O}(k^{-\infty})).\end{aligned}$$

By Lemma 6.1, $P = \Pi^I + \mathcal{O}(k^{-1/4})$, so $P_{--} = \mathcal{O}(k^{-1/4})$. Then $P_{--}^2 = P_{--} + \mathcal{O}(k^{-\infty})$ implies

$$P_{--} = \mathcal{O}(k^{-\infty}).$$

In the same way, $(\mathrm{id}_{\mathcal{H}_+} - P_{++})^2 = \mathrm{id}_{\mathcal{H}_+} - P_{++} + \mathcal{O}(k^{-\infty})$ and $\mathrm{id}_{\mathcal{H}_+} - P_{++} = \mathcal{O}(k^{-1/4})$ imply $\mathrm{id}_{\mathcal{H}_+} - P_{++} = \mathcal{O}(k^{-\infty})$. So

$$P_{++} = \mathrm{id}_{\mathcal{H}_+} + \mathcal{O}(k^{-\infty}),$$

which concludes the proof. □

Lemma 6.6. *For any ℓ, $m \in \mathbb{N}$, we have $\Delta_k^\ell (P_k - \Pi_k^I)\Delta_k^m = \mathcal{O}(k^{-\infty})$.*

Proof. On one hand, we have

$$\Delta_k^\ell P_k = \mathcal{O}(k^\ell), \quad \Delta_k^\ell \Pi_k^I = \mathcal{O}(k^\ell), \tag{68}$$

where the first estimate is a consequence of $((k^{-1}\Delta_k)^\ell P_k) \in \mathcal{L}^\infty(A)$, and the second one is merely that Π_k^I is the spectral projector of $k^{-1}\Delta_k$ for the bounded interval I.

On the other hand, since for any $Q \in \mathcal{L}^\infty_\infty(A)$, $\Delta_k^\ell Q \Delta_k^m$ belongs to $\mathcal{L}^\infty_\infty(A)$ as well, we have

$$\begin{aligned}\Delta_k^\ell P_k^2 \Delta_k^m &= \Delta_k^\ell P_k \Delta_k^m + \mathcal{O}(k^{-\infty}),\\ \Delta_k^\ell [k^{-1}\Delta_k, P_k]\Delta_k^m &= \mathcal{O}(k^{-\infty}).\end{aligned} \tag{69}$$

By the first equality, $\Delta_k^\ell P_k \Delta_k^m = \Delta_k^{\ell+m} P_k + \mathcal{O}(k^{-\infty})$. Since $[\Delta_k, \Pi_k^I] = 0$, it suffices to prove the final result for $m = 0$, that is, $\Delta_k^\ell (P_k - \Pi_k^I) = \mathcal{O}(k^{-\infty})$. We have

$$\begin{aligned}\Delta_k^\ell (P_k - \Pi_k^I) &\overset{(69)}{=} \Delta_k^\ell (P_k^2 - \Pi_k^I) + \mathcal{O}(k^{-\infty}) \\ &= \Delta_k^\ell P_k (P_k - \Pi_k^I) + \Delta_k^\ell (P_k \Pi_k^I) - \Delta_k^\ell \Pi_k^I + \mathcal{O}(k^{-\infty}) \\ &\overset{(69)}{=} \Delta_k^\ell P_k (P_k - \Pi_k^I) + P_k \Delta_k^\ell \Pi_k^I - \Delta_k^\ell \Pi_k^I + \mathcal{O}(k^{-\infty}) \\ &= \Delta_k^\ell P_k (P_k - \Pi_k^I) + (P_k - \Pi_k^I) \Delta_k^\ell \Pi_k^I + \mathcal{O}(k^{-\infty}) \;=\; \mathcal{O}(k^\ell)\mathcal{O}(k^{-\infty})\end{aligned}$$

by (68) and Lemma 6.5. □

We are now ready to conclude the proof of Theorems 5.2 and 5.3: we will show that the Schwartz kernel of $P_k - \Pi_k^I$ is in $\mathcal{O}_\infty(k^{-\infty})$, in the sense of Section 5D.

Choose two open sets U and U' of M equipped both with a set of coordinates and unitary trivializations of L and A, so that we can identify the sections of $L^k \otimes A$ on U with functions. Let $\varphi \in \mathcal{C}_0^\infty(U)$, $\varphi' \in \mathcal{C}_0^\infty(U')$. Then $\varphi(P_k - \Pi_k^I)\varphi'$ can be viewed as an operator of $\mathbb{R}^{2n}$. We introduce the differential operator

$$\Lambda_k = 1 - k^{-2} \sum_{i=1}^{2n} \partial_{x_i}^2$$

acting on $\mathcal{C}^\infty(\mathbb{R}^{2n})$.

Lemma 6.7. *For any $\ell \in \mathbb{N}$,*

$$\Lambda_k^\ell \, \varphi (P_k - \Pi_k^I) \varphi' \, \Lambda_k^\ell = \mathcal{O}(k^{-\infty}). \tag{70}$$

Consequently, the Schwartz kernel of $\varphi(P_k - \Pi_k^I)\varphi'$ is in $\mathcal{O}_\infty(k^{-\infty})$.

Proof. We will use basic results on semiclassical pseudodifferential operators of $\mathbb{R}^{2n}$, with the semiclassical parameter usually denoted by h equal here to k^{-1}. Choose $\psi_1, \psi_2 \in \mathcal{C}_0^\infty(U)$ such that $\operatorname{supp} \varphi \subset \{\psi_1 = 1\}$ and $\operatorname{supp} \psi_1 \subset \{\psi_2 = 1\}$. The operator $\psi_1(1 + (k^{-2}\Delta_k)^\ell)$, viewed as an operator of $\mathbb{R}^{2n}$, is a semiclassical differential operator with principal symbol $\psi_1(H^\ell + 1)$, where H is the symbol of Δ_k, so

$$H(x, \xi) = \sum g^{ij}(x)(\xi_i + \alpha_i(x))(\xi_j + \alpha_j(x)),$$

with $-i \sum \alpha_i \, dx_i$ the connection 1-form of L in the trivialization used to identify sections with functions. The operator $\varphi \Lambda_k^\ell$ is also a semiclassical differential operator with symbol $\varphi(x)\langle \xi \rangle^{2\ell}$. Since the symbol $\psi_1(H^\ell + 1)$ is elliptic on $\operatorname{supp} \varphi \times \overline{\mathbb{R}}^n$, we can factorize

$$\Lambda_k^\ell \varphi = Q_k \psi_1 (1 + (k^{-2}\Delta_k)^\ell) + S_k,$$

with Q_k a zero-order semiclassical pseudodifferential operator and S_k in the residual class. To do this, we only need the pseudodifferential calculus in the usual class $S_{1,0}^k(T^*\mathbb{R}^{2n})$ of symbols; see for instance [Dyatlov and Zworski 2019, Section E.1.5]. Composing with ψ_2,

$$\Lambda_k^\ell \varphi = Q_k \psi_1 (1 + (k^{-2}\Delta_k)^\ell) + S_k \psi_2. \tag{71}$$

Similarly, we have

$$\varphi' \Lambda_k^\ell = \psi_1'(1+(k^{-2}\Delta_k)^\ell) Q_k' + \psi_2' S_k'. \tag{72}$$

Now by Lemma 6.6,

$$(1+(k^{-2}\Delta_k)^\ell)(P_k - \Pi_k^I)(1+(k^{-2}\Delta_k)^m) = \mathcal{O}(k^{-\infty}), \tag{73}$$

and by the usual result on boundedness of pseudodifferential operators, see [Dyatlov and Zworski 2019, Proposition E.19], Q_k, $Q_k' = \mathcal{O}(1)$ and S_k, $S_k' = \mathcal{O}(k^{-\infty})$. We deduce (70) easily with (71), (72) and (73).

Now let H_k^m be the Sobolev space $H^m(\mathbb{R}^{2n})$ with the k-dependent norm $\|u\|_{H_k^m} = \|\langle k^{-1}\xi\rangle \hat{u}(\xi)\|_{L^2(\mathbb{R}^{2n})}$. Then Λ_k^ℓ is an isometry $H_k^m \to H_k^{m-2\ell}$. So (70) tells us that the operator norm $H_k^{-2\ell} \to H_k^{2\ell}$ of $R_k = \varphi(P_k - \Pi_k^I)\varphi'$ is in $\mathcal{O}(k^{-\infty})$. Since the Schwartz kernel of R_k at (x, y) is equal to $\delta_x(R_k \delta_y)$ and the Dirac δ_x belongs to H_k^{-m} with a norm in $\mathcal{O}(k^{2n})$ for any $m > n$, we have $R_k(x, y) = \mathcal{O}(k^{-\infty})$. Similarly, $\partial_x^\alpha \partial_y^\beta R_k(x, y) = \mathcal{O}(k^{-\infty})$ for any $\alpha, \beta \in \mathbb{N}^{2n}$ because the H_k^{-m}-norm of $\partial^\alpha \delta_x$ is a $\mathcal{O}(k^{2n})$ as soon as $m \geqslant n + |\alpha|$. □

7. Toeplitz operators

Let F be a vector subbundle of $\mathcal{D}_{\leqslant p}(TM) \otimes A$ for some p. Let $(\Pi_k) \in \mathcal{L}(A)$ such that, for each k, Π_k is a self-adjoint projector of $\mathcal{C}^\infty(M, L^k \otimes A)$ and, for any $x \in M$, the symbol $\pi(x) = \sigma_0(\Pi_k)(x)$ is the orthogonal projector onto F_x. Let $\mathcal{H}_k$ be the image of Π_k.

The corresponding Toeplitz operators are the $(P_k) \in \mathcal{L}(A)$ such that $\Pi_k P_k \Pi_k = P_k$. The symbol $\sigma_0(P)(x)$ of such an operator satisfies

$$\pi(x)\sigma_0(P)(x)\pi(x) = \sigma_0(P)(x).$$

So $\sigma_0(P)(x) = f(x)\pi(x)$, with $f(x) \in \operatorname{End} F_x$. This section f of $\operatorname{End} F$ can be considered as the Toeplitz symbol of (P_k).

We will establish several spectral results for these Toeplitz operators. Applied to the spectral projector $\Pi_k = 1_{[a,b]}(k^{-1}\Delta_k)$ and $P_k = k^{-1}\Delta_k \Pi_k$, this will complete the proofs of Theorems 1.1, 1.2, 1.3 and 1.4 stated in the Introduction.

7A. *Global spectral estimates.*

Theorem 7.1. (1) *When k is sufficiently large*, $\dim \mathcal{H}_k = \mathrm{RR}(L^k \otimes F)$.

(2) *For any $(P_k) \in \mathcal{L}(A)$ such that $P_k^* = P_k$ and $\Pi_k P_k \Pi_k = P_k$ for any k, we have for any $\Psi \in \mathcal{H}_k$ with $\|\Psi\| = 1$ that*

$$\inf_M f_- + \mathcal{O}(k^{-1/2}) \leqslant \langle P_k \Psi, \Psi\rangle \leqslant \sup_M f_+ + \mathcal{O}(k^{-1/2}), \tag{74}$$

where the $\mathcal{O}$'s are uniform with respect to Ψ and, for any $x \in M$, $f_-(x)$ and $f_+(x)$ are the smallest and largest eigenvalues of the restriction of $\sigma_0(P)(x)$ to F_x.

The proof is based on the generalized ladder operators introduced in [Charles 2024]: if $(A', F', \Pi_k', \mathcal{H}_k')$ is a second set of data satisfying the same assumption as $(A, F, \Pi_k, \mathcal{H}_k)$ and F, F' are isomorphic vector bundles, then there exist isomorphisms $U_k : \mathcal{H}_k \to \mathcal{H}_k'$ when k is sufficiently large. Then defining $\mathcal{H}_k'$

as the kernel of a well-chosen spin-c Dirac operator, $\dim \mathcal{H}'_k$ is given by the Atiyah–Singer theorem, which will prove the first statement. For the second one, choose $\mathcal{H}'_k$ so that $F' = A'$, $U_k P_k U_k^*$ is equal to a Toeplitz operator $\Pi'_k f \Pi'_k$ up to a $\mathcal{O}(k^{-1/2})$. The inspiration here comes from the proof of the sharp Gårding inequality for semiclassical pseudodifferential operator.

Proof. Consider a second self-adjoint projector $\Pi' \in \mathcal{L}(A')$ with $\sigma_0(\Pi')$ the orthogonal projector onto a vector bundle F' of $\mathcal{D}_{\leqslant p}(TM) \otimes A'$. Assume that F and F' are isomorphic vector bundles. Then there exists $u \in \mathcal{C}^\infty(M, \operatorname{Hom}(F, F'))$ such that, for any $x \in M$, $u(x)$ is a unitary isomorphism from F_x to F'_x. Extending $u(x)$ to a map $\mathcal{D}(T_x M) \otimes A_x \to \mathcal{D}(T_x M) \otimes A'_x$ which is zero on the orthogonal of F_x, we have

$$u^*(x)u(x) = \sigma_0(\Pi)(x), \quad u(x)u^*(x) = \sigma_0(\Pi')(x).$$

So if $(U_k) \in \mathcal{L}(A, A')$ has symbol u, then

$$U_k^* U_k = \Pi_k + \mathcal{O}(k^{-1/2}), \quad U_k U_k^* = \Pi'_k + \mathcal{O}(k^{-1/2}). \tag{75}$$

Furthermore replacing U_k by $\Pi'_k U_k \Pi_k$ does not modify the symbol of U_k so the same property holds and moreover $\Pi'_k U_k \Pi_k = U_k$. Consequently U_k restricts to an isomorphism from $\mathcal{H}_k$ to the image $\mathcal{H}'_k$ of Π'_k, when k is sufficiently large.

Hence for large k, the dimension of $\mathcal{H}_k$ only depends on the isomorphism class of F. To compute it, we introduce a spin-c Dirac operators D_k acting on $L^k \otimes A'$ with $A' = F \otimes \bigwedge^{0,\bullet} T^*M$ and define $\mathcal{H}'_k$ as the kernel of D_k. Then by a vanishing theorem [Borthwick and Uribe 1996; Ma and Marinescu 2002], $\dim \mathcal{H}'_k$ is equal to the index of D_k^+ when k is sufficiently large. By Atiyah–Singer index theorem, $\dim \mathcal{H}'_k = \operatorname{RR}(L^k \otimes F)$. Furthermore, it follows from [Ma and Marinescu 2007] that the projector (Π'_k) belongs to $\mathcal{L}(A')$, and $\sigma_0(\Pi'_k)$ is the projector onto $\mathbb{C} \otimes F \otimes \mathbb{C}$. Alternatively the vanishing theorem and the fact that $(\Pi'_k) \in \mathcal{L}(A')$ follows also from Corollary 4.8 and Theorem 5.2 applied to $D_k^- D_k^+$ as in the proof of Theorem 3.6.

To prove the second part, we choose $A' = F' = F$, that is, (Π'_k) belongs to $\mathcal{L}(F)$ and its symbol is the projection onto $\mathcal{D}_0(TM) \otimes F$. For instance, we can choose $\Pi'_k = 1_I(k^{-1}\Delta_k)$ with $I = \frac{1}{2}n + \left[-\frac{1}{2}, \frac{1}{2}\right]$ and Δ_k the magnetic Laplacian acting on $\mathcal{C}^\infty(M, L^k \otimes F)$ defined from any connection of F and the metric $\omega(\cdot, j\,\cdot)$ so that $\Sigma = \frac{1}{2}n + \mathbb{N}$.

Now let $P \in \mathcal{L}(A)$ be selfadjoint and such that $\Pi_k P_k \Pi_k = P_k$. Then the symbol $\sigma_0(P)(x)$ is self-adjoint and has the form $\sigma_0(P)(x) = f(x)\pi(x)$ with $f(x) \in \operatorname{End} F_x$. So $\sigma_0(P)(x) = u^*(x) f(x) u(x)$, and thus

$$P_k = U_k^* f U_k + \mathcal{O}(k^{-1/2}), \tag{76}$$

where f acts on $\mathcal{C}^\infty(M, L^k \otimes F)$ by pointwise multiplication. For any $\Psi' \in \mathcal{C}^\infty(M, L^k \otimes F)$,

$$(\inf_M f_-)\|\Psi'\|^2 \leqslant \langle f\Psi', \Psi'\rangle \leqslant (\sup_M f_+)\|\Psi'\|^2,$$

where $f_-(x)$ and $f_+(x)$ are the smallest and largest eigenvalues of $f(x)$ for any x. We conclude the proof by setting $\Psi' = U_k \Psi$ and using (75) and (76). □

Corollary 7.2. *Let (Δ_k) be a family of formally self-adjoint differential operators of the form* (B). *Let $a, b \in \mathbb{R} \setminus \Sigma$, with $a < b$. Then when k is sufficiently large*

$$\sharp \operatorname{sp}(k^{-1}\Delta_k) \cap [a, b] = \begin{cases} \mathrm{RR}(L^k \otimes F) & \text{if } [a, b] \cap \Sigma \neq \varnothing, \\ 0 & \text{otherwise}, \end{cases} \tag{77}$$

with F the bundle with fibers $F_x = 1_{[a,b]}(\square_x)$. Furthermore

$$\operatorname{sp}(k^{-1}\Delta_k) \cap [a, b] \subset [a, b] \cap \Sigma + \mathcal{O}(k^{-1/2}). \tag{78}$$

Proof. When $[a, b] \cap \Sigma$ is empty, we already know by Corollary 4.8 that $\operatorname{sp}(k^{-1}\Delta_k) \cap [a, b]$ is empty when k is sufficiently large. When $[a, b] \cap \Sigma \neq \varnothing$, by Theorem 5.2, the spectral projector $\Pi_k = 1_{[a,b]}(k^{-1}\Delta_k)$ belongs to $\mathcal{L}(A)$ with symbol $\pi = 1_{[a,b]}(\square)$. So the dimension of $\operatorname{Im} \Pi_k$ is given in the first assertion of Theorem 7.1.

Moreover, by Corollary 5.4, $(k^{-1}\Delta_k)\Pi_k$ belongs to $\mathcal{L}(A)$ and its symbol is $\square 1_{[a,b]}(\square)$. By the second assertion of Theorem 7.1,

$$\operatorname{sp}(k^{-1}\Delta_k) \cap [a, b] = \operatorname{sp}(k^{-1}\Delta_k \Pi_k) \subset [\inf f_-, \sup f_+] + \mathcal{O}(k^{-1/2})$$

where f is the restriction of $\square$ to $F = \operatorname{Im} \pi$.

This proves the inclusion (78) when $[a, b] \cap \Sigma$ is connected. Indeed,

$$[a, b] \cap \Sigma_y = [f_-(y), f_+(y)] \cap \Sigma_y.$$

So on one hand, M being compact, $\inf f_- = f(y_-)$ and $\sup f_+ = f(y_+)$ belongs to $[a, b] \cap \Sigma$. On the other hand $[a, b] \cap \Sigma \subset [\inf f_-, \sup f_+]$. Consequently $[a, b] \cap \Sigma = [\inf f_-, \sup f_+]$.

To treat the general case, we use that $[a, b] \cap \Sigma$ is a finite union of mutually disjoint compact intervals $I_1, \ldots, I_\ell$. So there exists $a_1 = a < a_2 < \cdots < a_{\ell+1} = b$ in $\mathbb{R} \setminus \Sigma$ such that $I_i = [a_i, a_{i+1}] \cap \Sigma$ and by what we have proved, $\operatorname{sp}(k^{-1}\Delta_k) \cap [a_i, a_{i+1}] \subset I_i + \mathcal{O}(k^{-1/2})$. ☐

Remark 7.3. Decompose $\mathcal{D}(TM)$ into even and odd subspaces

$$\mathcal{D}^+(TM) = \bigoplus_{p \in \mathbb{N}} \mathcal{D}_{2p}(TM), \quad \mathcal{D}^-(TM) = \bigoplus_{p \in \mathbb{N}} \mathcal{D}_{2p+1}(TM).$$

Let us assume that (Π_k) is even and that F has a definite parity in the sense that F is a subbundle of $D^\epsilon(TM) \otimes A$ for $\epsilon = +$ or $-$. Then (74) and (78) hold with k^{-1} instead of $k^{-1/2}$.

Indeed, by [Charles 2024, Theorem 2.5], the σ_p-symbol of $P_k \in \mathcal{L}_p^+(A)$ has the same parity of p, meaning that $\sigma_p(P_k)$ sends $\mathcal{D}^\epsilon(TM) \otimes A$ into $\mathcal{D}^{\epsilon'}(TM) \otimes A$ with $\epsilon' = (-1)^p\epsilon$. So if an operator $(P_k) \in \mathcal{L}_1^+(A)$ is such that $\Pi_k P_k \Pi_k = P_k$, then its symbol $\sigma_1(P_k)$ is odd and has the form $g\pi$ for some $g \in \operatorname{End} F$. So g is odd, but F has a definite parity, so $g = 0$. Consequently $(P_k) \in \mathcal{L}_2^+(A)$. Moreover, by [Charles 2024, Theorem 3.4], we can construct $(U_k) \in \mathcal{L}(A, F)$ such that $U_k U_k^* = \mathrm{id}$ when k is sufficiently large and (U_k) has the same parity as F. So if $(P_k) \in \mathcal{L}^+(A)$, then $(U_k P_k U_k^*) \in \mathcal{L}^+(F)$. Then in the proof of Theorem 7.1, we can replace the $\mathcal{O}(k^{-1/2})$ in (75) and (76) by a $\mathcal{O}(k^{-1})$. ☐

7B. *Local spectral estimates.*

Theorem 7.4. *Let $(P_k) \in \mathcal{L}(A)$ be such that $\Pi_k P_k \Pi_k = P_k$ and $P_k^* = P_k$. Let $f \in \mathcal{C}^\infty(M, \operatorname{End} F)$ be the restriction of $\sigma_0(P_k)$ to F.*

(1) *For any compact subsets C of M and I of $\mathbb{R}$ such that $I \cap \operatorname{sp}(f(x)) = \varnothing$ for any $x \in C$, we have for any N*
$$(\Pi_k 1_I(P_k)\Pi_k)(x,x) = \mathcal{O}(k^{-N}) \quad \textit{for all } x \in C,$$
with a $\mathcal{O}$ uniform with respect to x.

(2) *For any $g \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$, $(\Pi_k g(P_k)\Pi_k)$ belongs to $\mathcal{L}(A)$ and its σ_0-symbol is $(g \circ f)\pi$. Moreover, if (Π_k) and (P_k) are in $\mathcal{L}^+(A)$, then the same holds for $(\Pi_k g(P_k)\Pi_k)$.*

Proof. Let U be the open set $\{x \in M : \operatorname{sp}(f(x)) \cap I = \varnothing\}$. Let $\varphi \in \mathcal{C}_0^\infty(U)$ and $\lambda \in I$. Observe that $\varphi(f-\lambda)^{-1} \in \mathcal{C}^\infty(M, \operatorname{End} F)$. So if $(Q_k) \in \mathcal{L}(A)$ has symbol $\varphi(f-\lambda)^{-1}\pi$, we have
$$\Pi_k Q_k \Pi_k (P_k - \lambda \Pi_k) = \Pi_k \varphi \Pi_k - R_k, \tag{79}$$
with $(R_k) \in \mathcal{L}_1(A)$. Let us improve this to obtain $(R_k) \in \mathcal{L}_\infty(A)$.

We need the following notion of support: for any $S \in \mathcal{L}(A)$, $\operatorname{supp} S$ is the closed set of M such that $x \notin \operatorname{supp} S$ if and only if $S_k(y,z) = \mathcal{O}(k^{-\infty})$ on a neighborhood of (x,x). Using that the Schwartz kernel of $S \in \mathcal{L}(A)$ is in $\mathcal{O}(k^{-\infty})$ on compact subsets of $M^2 \setminus \operatorname{diag} M$ and in $\mathcal{O}(k^n)$ on M^2, we prove that for any $S, S' \in \mathcal{L}(A)$ we have $\operatorname{supp}(SS') \subset (\operatorname{supp} S) \cap (\operatorname{supp} S')$.

Assume now that $(Q_k) \in \mathcal{L}(A)$ has the symbol $\varphi(f-\lambda)^{-1}\pi$ as above and is supported in U. Then $(R_k) \in \mathcal{L}_p(A)$ with $p \geqslant 1$, $\Pi_k R_k \Pi_k = R_k$ so that the symbol $r = \sigma_p(R_k)$ satisfies $\pi r \pi = r$. Furthermore, (R_k) is supported in U, so the same holds for r, so that $r(f-\lambda)^{-1} \in \mathcal{C}^\infty(M, \operatorname{End} F)$. Let $(Q_k') \in \mathcal{L}_p(A)$ be supported in U and have symbol $\sigma_p(Q_k') = r(f-\lambda)^{-1}\pi$. Then if we replace Q_k in (79) by $Q_k + Q_k'$, we have now $(R_k) \in \mathcal{L}_{p+1}(A)$. We deduce the existence of (Q_k) such that (79) holds with $(R_k) \in \mathcal{L}_\infty(A)$, so the operator norm of R_k is in $\mathcal{O}(k^{-\infty})$.

We claim that this construction can be realized so that we obtain an $\mathcal{O}(k^{-\infty})$ uniform with respect to $\lambda \in I$. To do this, we consider families
$$(S_k(\lambda)) \in \mathcal{L}(A), \quad \lambda \in I, \tag{80}$$
such that in the kernel expansion (50), the coefficients a_ℓ depend continuously on λ and the remainders $r_{N,k}$ are in $\mathcal{O}(k^{n-(N+1)/2})$ on compact subsets of U^2 with an $\mathcal{O}$ independent of λ. Then if $(S_k'(\lambda))$ is another family depending continuously on λ in the same sense, the same holds for the product $(S_k'(\lambda)S_k(\lambda))$. Furthermore, if $(S_k(\lambda)) \in \mathcal{L}_p(A)$ for any $\lambda \in I$, the operator norm of $S_k(\lambda)$ is in $\mathcal{O}(k^{-p/2})$ with an $\mathcal{O}$ independent of λ. The proof of these claims is the same as the proof of the same facts without λ. Later in (85), we will use these results again with the parameter λ describing a compact subset of $\mathbb{C}$.

Now we deduce from (79) with $\|R_k\| = \mathcal{O}(k^{-\infty})$ that, for any k, any normalized $\Psi \in \mathcal{H}_k$ such that $P_k\Psi = \lambda\Psi$ with $\lambda \in I$ satisfies $\langle \varphi\Psi, \Psi\rangle = \mathcal{O}(k^{-\infty})$ with an $\mathcal{O}$ independent of λ and Ψ. For any $x \in U$, we can choose φ equal to 1 on a neighborhood of x and we deduce the existence of a compact neighborhood V

of x, such that any Ψ as above satisfies

$$\int_V |\Psi(x)|^2 \, d\mu(x) = \mathcal{O}(k^{-\infty}).$$

Writing $\Psi = \Pi_k \Psi$ and using that the Schwartz kernel of Π_k is in $\mathcal{O}(k^n)$ on M^2 and in $\mathcal{O}(k^{-\infty})$ on compact subsets of M^2 not intersecting the diagonal, we get that on a neighborhood of x the pointwise norm of Ψ is in $\mathcal{O}(k^{-\infty})$. Since $(\Pi_k 1_I(P_k)\Pi_k)(x,x)$ is the sum of the $|\Psi_\ell(x)|^2$, where (Ψ_ℓ) is an orthonormal basis of $\mathcal{H}_k \cap \operatorname{Im} 1_I(P_k)$ consisting of eigenvectors of P_k, and $\dim \mathcal{H}_k = \mathcal{O}(k^n)$, we deduce that

$$(\Pi_k 1_I(P_k)\Pi_k)(x,x) = \mathcal{O}(k^{-\infty}) \quad \text{for all } x \in U,$$

with an $\mathcal{O}$ uniform on compact subsets of U. This ends the proof of the first assertion.

For the second assertion, since the operator norm of P_k is bounded independently of k, we can assume that $g \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{C})$. We will apply the Helffer–Sjöstrand formula, which we already used in a similar context for the functional calculus of Toeplitz operators [Charles 2003, Proposition 12]. So for $\widetilde{P}_k$ the restriction of P_k to $\mathcal{H}_k$, we have

$$g(\widetilde{P}_k) = \frac{1}{2\pi} \int_{\mathbb{C}} (\partial_{\bar z} \tilde g)(z)(z - \widetilde{P}_k)^{-1} \, |dz \, d\bar z|, \tag{81}$$

where $\tilde g \in \mathcal{C}_0^\infty(\mathbb{C}, \mathbb{C})$ is an extension of g such that $\partial_{\bar z} \tilde g$ vanishes to infinite order along the real axis [Zworski 2012, Theorem 14.8].

In the same way we proved (79), we can construct, for any $z \in \mathbb{C} \setminus \mathbb{R}$, $(Q_k(z)) \in \mathcal{L}(A)$ such that $\Pi_k Q_k(z) \Pi_k = Q_k(z)$ and

$$Q_k(z)(z - P_k) = \Pi_k - R_k(z), \tag{82}$$

with $(R_k(z)) \in \mathcal{L}_\infty(A)$. At the first step we set $Q_k(z) = \Pi_k \widetilde{Q}_k \Pi_k$, with $\widetilde{Q}_k(z)$ in $\mathcal{L}(A)$ having symbol $(z - f)^{-1}\pi$. We obtain (82) with $(R_k(z)) \in \mathcal{L}_1(A)$. Then if $(R_k(z)) \in \mathcal{L}_p(A)$ and has symbol $\sigma_p(R_k(z)) = r(z)$, we add to Q_k the operator $\Pi_k Q_k'(z) \Pi_k$, where $(Q_k'(z))$ is an operator of $\mathcal{L}_p(A)$ with symbol $\sigma(Q_k'(z)) = r(z)(z - f)^{-1}\pi$.

To apply this in (81), we need to control carefully the dependence with respect to z. For U an open set of M, we introduce the space $\mathcal{FC}^\infty(U)$ consisting of family $(f(z, \cdot),\ z \in \mathbb{C} \setminus \mathbb{R})$ of $\mathcal{C}^\infty(U)$ having the form

$$g(z,x) = \frac{\sum_m a_m(x) z^m}{\sum_m b_m(x) z^m}$$

where the sums are finite, the coefficients a_m and b_m belong to $\mathcal{C}^\infty(U)$, and for any x the poles of $g(\cdot, x)$ lie on the real axis. Since $\mathcal{FC}^\infty(U)$ is a $\mathcal{C}^\infty(U)$-module, we can define $\mathcal{FC}^\infty(U, B)$ for any auxiliary bundle B as the space of z-dependent section of B on U with local representatives in $\mathcal{FC}^\infty(U)$ for any z-independent frame of B on U.

Having in mind the construction of $Q_k(z)$ in (82), observe that $(z - f)^{-1}$ belongs to $\mathcal{FC}^\infty(M, \operatorname{End} F)$. Moreover, $\mathcal{FC}^\infty(U)$ being closed under product, for any $r(z) \in \mathcal{FC}^\infty(M, \operatorname{End} F)$, we have $r(z)(z - f)^{-1} \in \mathcal{FC}^\infty(M, \operatorname{End} F)$.

Now we introduce the space $\mathcal{FL}(A)$ consisting of families $(P_k(z), z \in \mathbb{C} \setminus \mathbb{R})$ of $\mathcal{L}(A)$ such that in the asymptotic expansion (50) satisfied by the Schwartz kernel of $P_k(z)$, the coefficients have the form

$$a_\ell(z, x, \xi) = \sum a_{\ell,\alpha}(z, x)\xi^\alpha, \tag{83}$$

with $a_{\ell,\alpha} \in \mathcal{FC}^\infty(U, \operatorname{End} \mathbb{C}^r)$, and each remainder $r_{N,k}$ is in $\mathcal{O}(k^{n-(N+1)/2})$ uniformly on $K \cap ((\mathbb{C} \setminus \mathbb{R}) \times U^2)$, where K is any compact subset of $\mathbb{C} \times U^2$. We claim that we can choose $Q_k(z) \in \mathcal{FL}(A)$ in (82). To see this, it suffices to prove that

$$S(z) \in \mathcal{FL}(A) \quad \Longrightarrow \quad \Pi_k S_k(z) \Pi_k (z - P_k) \in \mathcal{FL}(A), \tag{84}$$

and then to use what we said before on $r(z) \circ (z - f)^{-1}$. To prove (84), it suffices to show that, for any $S(z) \in \mathcal{FL}(A)$ and $T \in \mathcal{L}(A)$ independent of z, $TS(z)$ and $S(z)T$ belong to $\mathcal{FL}(A)$. To prove this, we can assume that the Schwartz kernel of $T(z)$ is contained in a compact subset of U^2 independent of k and z, where we have the expansion (50), and we can treat each term of the expansion independently of the others. Suppose we only have $a_\ell(z, x, \xi)$. Then by (83), $S(z) = \sum S_\alpha a_{\ell,\alpha}(z, \cdot\,)$, where the sum is finite, $S_\alpha \in \mathcal{L}_\ell(A)$ and does not depend on z. Since $TS(z) = \sum (TS_\alpha) a_{\ell,\alpha}(z, \cdot\,)$ and $TS_\alpha \in \mathcal{L}_\ell(A)$, for any α, $TS(z)$ belongs to $\mathcal{FL}_\ell(A)$. The product $S(z)T$ is more delicate to handle. By the same proof as [Charles 2016, Lemma 5.11], for any compact set K of U, there exists a family $(T_\beta, \beta \in \mathbb{N}^{2n})$ such that $T_\beta \in \mathcal{L}_{|\beta|}(A)$, and for any $f \in \mathcal{C}^\infty_K(U)$ we have $fT = \sum_{|\beta| \leqslant N} T_\beta(\partial^\beta f)$ modulo $\mathcal{L}_{N+1}(A)$. Consequently

$$S(z)T = \sum_\alpha S_\alpha(a_{\ell,\alpha}(z, \cdot\,)T) = \sum_{\alpha, |\beta| \leqslant N} S_\alpha T_\beta(\partial^\beta a_{\ell,\alpha}(z, \cdot\,))|vtw$$

modulo $\mathcal{L}_{N+1}(A)$. To conclude we use that $S_\alpha T_\beta \in \mathcal{L}_{\ell+|\beta|}(A)$ and $\partial^\beta a_{\ell,\alpha} \in \mathcal{FC}^\infty(U, \operatorname{End} \mathbb{C}^r)$.

Now the function $\xi(z) = (\operatorname{Im} z)^{-1} \partial_{\bar z} \tilde g(z)$ vanishes to infinite order along the real axis and its support is contained in the compact set $K = \operatorname{supp} \tilde g$. For any $f \in \mathcal{FC}^\infty(U)$, the product $\xi(z) f(z, \cdot\,)$ extends smoothly to $\mathbb{C}$. We deduce that there exists a family $(S_k(z))$ of $\mathcal{L}(A)$ depending continuously of $z \in K$ in the same sense as (80), and such that

$$\Pi_k S_k(z) \Pi_k = S_k(z), \qquad S_k(z)(z - P_k) = \xi(z)\Pi_k + \mathcal{O}(k^{-\infty}), \tag{85}$$

with an $\mathcal{O}$ uniform with respect to z. Since $\|(z - \widetilde{P}_k)^{-1}\| = \mathcal{O}(|\operatorname{Im} z|^{-1})$, multiplying the last equality by $(\operatorname{Im} z)(z - \widetilde{P}_k)^{-1}$, we obtain

$$\partial_{\bar z} \tilde g(z)(z - \widetilde{P}_k)^{-1} \Pi_k = (\operatorname{Im} z) S_k(z) + R_k(z), \tag{86}$$

with $R_k(z) = \mathcal{O}(k^{-\infty})$. Since $\Pi_k R_k(z) \Pi_k = R_k(z)$ and the Schwartz kernel of Π_k is in $\mathcal{O}(k^n)$, this implies that the Schwartz kernel of $R_k(z)$ is in $\mathcal{O}(k^{-\infty})$ uniformly with respect to z. Inserting (86) in (81), it comes that $(g(\widetilde{P}_k)\Pi_k)$ belongs to $\mathcal{L}(A)$. To see this, we simply have to integrate with respect to z the coefficients $a_\ell(z, x, \xi)$ in the expansion (50) of the Schwartz kernel of $(\operatorname{Im} z) S_k(z)$. Since $\sigma_0((\operatorname{Im} z) S_k(z)) = \partial_{\bar z} \tilde g(z)(z - f)^{-1}\pi$, we deduce also that

$$\sigma_0(g(\widetilde{P}_k)\Pi_k) = \frac{1}{2\pi} \int_{\mathbb{C}} \partial_{\bar z} \tilde g(z)(z - f)^{-1}\pi \, |dz \, d\bar z| = g(f)\pi,$$

which concludes the proof. ☐

Corollary 7.5. *Let* (Δ_k) *be a family of formally self-adjoint differential operators of the form* (B). *Let* $\Lambda \in \mathbb{R} \setminus \Sigma$. *Then for any* $g \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$ *supported in* $]-\infty, \Lambda]$, $(g(k^{-1}\Delta_k))$ *belongs to* $\mathcal{L}^+(A)$ *and has symbol* $g(\square)$.

Proof. Since g is supported in $]-\infty, \Lambda]$, we have $g(k^{-1}\Delta_k) = \Pi_k g(k^{-1}\Delta_k \Pi_k)\Pi_k$ where $\Pi_k = 1_{]-\infty,\Lambda]}(k^{-1}\Delta_k)$. By Theorem 5.2 and Corollary 5.4, (Π_k) and $k^{-1}\Delta_k\Pi_k$ belong to $\mathcal{L}^+(A)$ with symbols $\pi = 1_{]-\infty,\Lambda]}(\square)$ and $f = g(\square)$. So the result follows from the second assertion of Theorem 7.4. ☐

This proves the second part of Theorem 1.4. We end this section with the proof of the local Weyl laws, Theorem 1.3. The proof works for any (Δ_k) of the form (B).

Proof of Theorem 1.3. We use the same notation as in Corollary 7.5 and its proof. Let $a, b \in]-\infty, \Lambda] \setminus \Sigma_y$. We have $\operatorname{sp}(f(y)) = \Sigma_y \cap]-\infty, \Lambda]$. When $[a, b] \cap \Sigma_y$ is empty, the first part of Theorem 7.4 implies that $N(y, a, b, k) = \mathcal{O}(k^{-\infty})$. To the contrary, assume that $[a, b] \cap \Sigma_y = \{\lambda\}$. Then choose a function $g \in \mathcal{C}_0^\infty(]a, b[, \mathbb{R})$ which is equal to 1 on $]\lambda-\epsilon, \lambda+\epsilon[$ for some $\epsilon > 0$. Since $N(y, a, \lambda-\epsilon, k) = \mathcal{O}(k^{-\infty})$ and $N(y, \lambda+\epsilon, b, k) = \mathcal{O}(k^{-\infty})$ by the first part of the proof,

$$N(y, a, b, k) = g(k^{-1}\Delta_k)(y, y) + \mathcal{O}(k^{-\infty}).$$

Since $g(k^{-1}\Delta_k)$ is in $\mathcal{L}^+(A)$ and has symbol $g(\square)$, we have by [Charles 2024, Theorem 2.2, Assertion 5 and Proposition 5.6]

$$g(k^{-1}\Delta_k)(y, y) = \left(\frac{k}{2\pi}\right)^n \sum_{\ell=0}^{\infty} m_{\ell,\lambda} k^{-\ell} + \mathcal{O}(k^{-\infty}),$$

with $m_{0,\lambda} = \operatorname{tr} g(\square)(y)$, so $m_{0,\lambda}$ is the multiplicity of λ as an eigenvalue of $\square_y$. ☐

8. Miscellaneous proofs

Proof of Lemma 4.5. This is essentially Darboux lemma with parameters. We can adapt the proof presented in [McDuff and Salamon 2017, Section 3.2]. A more efficient approach based on [Bursztyn et al. 2019] is as follows. First, if r is sufficiently small, for any y, the exponential map $\exp_y : T_yM \to M$ restricts to an embedding from $B_y(r)$ into M. Identify $U = \exp_y(B_y(r))$ with an open set of T_yM. We are looking for a diffeomorphism φ defined on a neighborhood of the origin of T_yM such that $\varphi(0) = 0$, $T_0\varphi = \mathrm{id}$ and $\varphi^*\omega$ is constant. The important point is to define φ in such a way that it depends smoothly on y.

Let α be the primitive of ω on U obtained by radial homotopy. So

$$\alpha_x(v) = \int_0^1 \omega_{tx}(tx, v)\,dt, \qquad x \in U,\ v \in T_yM, \tag{87}$$

and $d\alpha = \omega$. Let X be the vector field of U such that $\iota_X\omega = 2\alpha$. By the Poincaré lemma, $\mathcal{L}_X\omega = 2\omega$. Furthermore, linearizing α at the origin, we see that $X = E + \mathcal{O}(2)$, with E the Euler vector field of T_yM. Since $Z = X - E$ vanishes to second order at the origin, the family $Z_t(x) := Z(tx)/t^2$ extends smoothly at $t = 0$. Let φ_t be the flow of the time-dependent vector field Z_t of U, that is, $\varphi_0(x) = x$ and $\dot\varphi_t(x) = Z_t(\varphi_t(x))$. Since Z_t is zero at the origin, φ_1 is a germ of a diffeomorphism of $(T_yM, 0)$.

By the proof of Lemma 2.4 in [Bursztyn et al. 2019], $\varphi_1^* X = E$, where the pull-back is defined by $\varphi_1^* X = (\varphi_1^{-1})_* X$. So $\mathcal{L}_X \omega = 2\omega$ implies that $\mathcal{L}_E \varphi_1^* \omega = 2\varphi_1^* \omega$. So $\varphi_1^* \omega$ is constant.

To conclude, observe that φ_1 depends smoothly on y because α given in (87) depends smoothly on y, so the same holds for X and Z_t, and the solution of a first-order differential equation depending smoothly on a parameter, is smooth with respect to the parameter. Finally the radius r_0 is chosen so that φ_1 is defined on $B_y(r_0)$. Since M is compact, we can choose $r_0 > 0$ independent of y. □

Proof of Lemma 4.6. Let d be the geodesic distance of M associated to our Riemannian metric. Starting from $d(y, \exp_y(\xi)) = \|\xi\|$ when ξ is sufficiently close to the origin, we get

$$C^{-1}\|\xi\| \leqslant d(y, \Psi_y(\xi)) \leqslant C\|\xi\| \tag{88}$$

for any $\xi \in B_y(r_1)$ with r_1 sufficiently small. So if $B(y, r)$ is the open ball of the metric space (M, d), then $\Psi_y(B_y(r)) \subset B(y, rC))$ and $B(y, r) \subset \Psi_y(B_y(rC))$. Define

$$v_-(\epsilon) = \inf\{\mathrm{vol}(B(y, \epsilon)) : y \in M\}, \quad v_+(\epsilon) = \sup\{\mathrm{vol}(B(y, \epsilon)) : y \in M\}.$$

Then, replacing C by a larger constant if necessary, when ϵ is sufficiently small, $C^{-1}\epsilon^{2n} \leqslant v_-(\epsilon)$ and $v_+(\epsilon) \leqslant C\epsilon^{2n}$.

For any $\epsilon > 0$, choose a maximal subset $J(\epsilon)$ of M such that the balls $B(y, \epsilon/2)$, $y \in J(\epsilon)$, are mutually disjoint. From the maximality, $M \subset \bigcup_{y \in J(\epsilon)} B(y, \epsilon)$ so that the sets $U_y(\epsilon) := \Psi_y(B_y(\epsilon C))$, $y \in J(\epsilon)$, cover M. For any $x \in M$, let $N(x, \epsilon)$ be the number of $y \in J(\epsilon)$ such that $x \in U_y(\epsilon)$. If $x \in U_y(\epsilon)$, by triangle inequality, $B(y, \epsilon/2) \subset B(x, \epsilon(1+C^2))$. Since the balls $B(y, \epsilon/2)$, $y \in J(\epsilon)$ are mutually disjoint, we have

$$N(x, \epsilon) v_-(\epsilon/2) \leqslant \mathrm{vol}(B(x, \epsilon(1 + C^2))) \leqslant v_+(\epsilon(1 + C^2))$$

So $N(x, \epsilon) \leqslant C^2(2(1 + C^2))^{2n}$. Thus the multiplicity of the cover $U_y(\epsilon)$, $y \in J(\epsilon)$, is bounded independently of ϵ. □

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