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IMPROVED ENDPOINT BOUNDS FOR THE LACUNARY SPHERICAL MAXIMAL OPERATOR

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We prove new endpoint bounds for the lacunary spherical maximal operator and as a consequence obtain almost everywhere pointwise convergence of lacunary spherical means for functions locally in $L \log\log\log L (\log\log\log\log L)^{1+\epsilon}$ for any $\epsilon > 0$.

1. Introduction

Let $d \geq 2$ be a fixed dimension; all constants in this paper are allowed to depend on d. We use the asymptotic notation $X \lesssim Y$, $Y \gtrsim X$, or $X = O(Y)$ to denote the estimate $|X| \leq CY$ for a constant C that can depend on d, and $X \approx Y$ for $X \lesssim Y \lesssim X$.

Define the lacunary spherical maximal operator $\mathcal{M}$ by

$$\mathcal{M} f(x) := \sup_{k \in \mathbb{Z}} |f * \sigma_k(x)|,$$

where σ_k denotes the (L^1-normalized) surface measure on the $(d-1)$-sphere of radius 2^k centered at the origin.

Throughout this paper, $\log = \log_2$ denotes the logarithm to base 2, and we define the iterated logarithms

$$\operatorname{Log}(t) := \log(100+t), \quad \operatorname{Log}_3(t) := \operatorname{Log}\operatorname{Log}\operatorname{Log} t,$$
$$\operatorname{Log}_2(t) := \operatorname{Log}\operatorname{Log} t, \qquad \operatorname{Log}_4(t) := \operatorname{Log}\operatorname{Log}\operatorname{Log}\operatorname{Log} t.$$

It was shown by C. Calderón [1979] and Coifman and Weiss [1978] that $\mathcal{M}$ extends to a bounded operator on $L^p(\mathbb{R}^d)$ for $p > 1$, which implies almost everywhere pointwise convergence of lacunary spherical means for functions in $L^p(\mathbb{R}^d)$ for $p > 1$. An alternate proof of this result was later given in [Duoandikoetxea and Rubio de Francia 1986]. It has remained open, however, as to whether $\mathcal{M}$ is weak-type (1, 1), or equivalently, whether almost everywhere pointwise convergence of lacunary spherical means holds for functions in $L^1(\mathbb{R}^d)$.

Christ and Stein [1987] showed using an extrapolation argument that $\mathcal{M} f \in L^{1,\infty}(\mathbb{R}^d)$ for functions f on $\mathbb{R}^d$ supported in a cube Q satisfying $f \in L \operatorname{Log} L(Q)$. Christ [1988] also proved that $\mathcal{M}$ maps the Hardy space $H^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$. More recently, Seeger, Tao, and Wright [Seeger et al. 2003; 2004] showed that $\mathcal{M}$ maps the space $L \operatorname{Log}_2 L(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$. In this paper we prove that $\mathcal{M}$ maps all characteristic functions in $L \operatorname{Log}_3 L(\mathbb{R}^d)$ boundedly to $L^{1,\infty}(\mathbb{R}^d)$, and more generally maps the entire

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space $L\ \mathrm{Log}_3\, L\, \mathrm{Log}_4^{1+\epsilon}\, L(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ for every $\epsilon > 0$, thus obtaining almost everywhere pointwise convergence of lacunary spherical means for functions locally in $L\ \mathrm{Log}_3\, L\, \mathrm{Log}_4^{1+\epsilon}\, L(\mathbb{R}^d)$.

Proposition 1.1 (lacunary spherical maximal inequality for indicator functions). *For all measurable indicator functions $f = \chi_E$ and all $\alpha > 0$ we have*

$$|\{\mathcal{M} f > \alpha\}| \lesssim \frac{1}{\alpha} \int |f(x)|\, \mathrm{Log}_3 \frac{|f(x)|}{\alpha}\, dx. \tag{1-1}$$

Here and in the sequel we use $|E|$ to denote the Lebesgue measure of a subset E of $\mathbb{R}^d$.

Proposition 1.2 (lacunary spherical maximal inequality for arbitrary functions). *For every $\epsilon > 0$, all measurable f and all $\alpha > 0$ we have*

$$|\{\mathcal{M} f > \alpha\}| \le C_\epsilon \frac{1}{\alpha} \int |f(x)|\, \mathrm{Log}_3 \frac{|f(x)|}{\alpha}\, \mathrm{Log}_4^{1+\epsilon} \frac{|f(x)|}{\alpha}\, dx \tag{1-2}$$

for some constant C_ϵ depending only on ϵ.

By the usual limiting and truncation arguments we obtain the following corollary.

Theorem 1.3 (almost everywhere convergence of lacunary spherical means). *Let $\epsilon > 0$, and let f be locally in $L\ \mathrm{Log}_3\, L\ \mathrm{Log}_4^{1+\epsilon}\, L(\mathbb{R}^d)$. Then*

$$f * \sigma_k(x) \to f(x)$$

for almost every $x \in \mathbb{R}^d$.

Before we proceed with the proofs, we briefly outline the argument. In [Seeger et al. 2003], the restricted version of the argument relied crucially on a decomposition of the function $f = \chi_E$ on Whitney cubes into characteristic functions of sets called "generalized boxes", which had properties called "length" and "thickness". As the name suggests, in two dimensions such sets are a generalization of rectangular boxes, for which the length and thickness correspond to the long and short sides respectively of the rectangle. In the case of two dimensions, convolution of a rectangular box with the measure σ_k has measure equal to 2^k times the length of the box. Similarly, the length of a generalized box determines for how many scales k one may throw away the support of σ_k convolved with the characteristic function of the generalized box. Conversely, the thickness of the box determines what L^2 estimates one may obtain for σ_k convolved with the characteristic function of the generalized box.

The argument of [Seeger et al. 2003] proceeded by combining standard Calderón–Zygmund techniques along with this decomposition of E into generalized boxes on Whitney cubes, and by leveraging L^2 and exceptional set size estimates via the properties of length and thickness for each generalized box. Our argument will also make use of a similar decomposition, but there will be many new ingredients involved, and in general our argument will more closely use the geometry of the sphere.

For example, we exploit the geometry of caps on spheres to introduce a fairly involved algorithm for defining exceptional sets, and we throw away more exceptional sets than in [Seeger et al. 2003]. These exceptional sets are defined by covering $\mathbb{R}^d$ with collections of rotated grids of rectangles,[1] where the

[1] In higher dimensions $d > 2$, by a "rectangle" we refer to a rotated box of some dimensions $c \times \cdots \times c \times c'$, with one side c' shorter than the other sides c.

dimensions of the rectangles are determined by an iterative relationship between the dimensions of a given generalized box and the cap structure of the spherical measure. On fixing a particular direction in S^{d-1} which determines the orientation of the rectangular grids to be considered, we then subdivide the generalized box into rectangular pieces where the generalized box has sufficiently high "mass", and throw away as an exceptional set the sumset of this rectangular box and a piece of the cap on the sphere with normals pointing in similar directions as the short side of the rectangular box, so that such a set is contained in a translation of the fattening of the spherical cap by an amount comparable to the short side of the rectangular box.

We then decompose the kernel $\sigma_k * \sigma_k$ into linear combinations of characteristic functions of rectangles with dimensions corresponding to the caps that appear in our algorithm for defining exceptional sets. The L^2 estimates for each such piece of the kernel convolved with a given rectangular piece in a grid with similar orientation is determined by the mass of the generalized box on that rectangular piece. There are essentially double-logarithmically (in the relevant parameter) many such different sizes of caps that appear, which alone would lead to the desired L^2 estimates with an additional double-logarithmic factor. However, we are able to throw away triple-logarithmically many "intermediate scales"; that is, we may sum in L^1 the convolutions of characteristic functions of parts of the generalized boxes with intermediate masses with the associated cap measures. After doing so, we improve the L^2 estimates for the remaining "light scales" by the needed double-logarithmic factor, and also improve the support size estimates for the remaining "heavy scales" by a double-logarithmic factor.

2. Preliminary reductions

In Calderón–Zygmund theory, weak-type estimates are often established by a combination of L^1 and L^2 estimates outside of an exceptional set, and our arguments will be no exception to this strategy. It is convenient to introduce some notation to abstract this strategy.

Definition 2.1 (Calderón–Zygmund control). Let $\alpha, V > 0$. A *Calderón–Zygmund term* of threshold α and measure V is a measurable function $F : \mathbb{R}^d \to \mathbb{R}$ of one of the following types:

(type L^0) F is a function supported on a set of measure $O(V)$.

(type L^1) F is an L^1 function with $\|F\|_1 \lesssim \alpha V$.

(type L^2) F is an L^2 function with $\|F\|_2^2 \lesssim \alpha^2 V$.

Here and in the sequel we use $\|\cdot\|_p$ to denote the usual $L^p(\mathbb{R}^d)$ norms. A function F is *Calderón–Zygmund controlled* with threshold α and measure V if $|F|$ can be pointwise dominated by a sum of boundedly many Calderón–Zygmund terms $F_1, \dots, F_n$, with $n = O(1)$, and each F_i a Calderón–Zygmund term (of type L^0, L^1, or L^2) of threshold α and measure V.

A model example of a Calderón–Zygmund controlled term to keep in mind is a simple function $\alpha\chi_E$, where E is of measure $O(V)$. The type-L^1 terms are in fact redundant as they can be easily split into the sum of a type-L^0 and a type-L^2 term, but we find it conceptually convenient to retain this intermediate term for our arguments.

We record some convenient properties of Calderón–Zygmund controlled functions:

Lemma 2.2 (basic properties of Calderón–Zygmund controlled functions). *Let* $\alpha > 0$.

(a) (*Chebyshev inequality*) *If* F *is* ***Calderón–Zygmund controlled*** *with threshold* α *and some measure* $V > 0$, *then* $|\{|F| > \alpha\}| \lesssim V$.

(b) (*triangle inequality for bounded sums*) *If* F, F_1, F_2 *obey the bounds* $|F| \lesssim |F_1| + |F_2|$ *and* F_1, F_2 *are Calderón–Zygmund controlled with threshold* α *and some measure* $V_1, V_2 > 0$ *respectively, then* F *is Calderón–Zygmund controlled with threshold* α *and measure* $V_1 + V_2$.

(c) (*triangle inequality for square functions and unbounded sums*) *If* $(F_q)_{q\in\mathfrak{Q}}$ *is a collection of functions, with each* F_q ***Calderón–Zygmund controlled*** *with threshold* α *and some measure* $V_q > 0$, *then the square function* $\bigl(\sum_{q\in\mathfrak{Q}} |F_q|^2\bigr)^{1/2}$ *is Calderón–Zygmund controlled with threshold* α *and measure* $\sum_{q\in\mathfrak{Q}} V_q$. *If the* F_q *are Calderón–Zygmund terms of type* L^0 *or* L^1 *of the threshold* α *and measure* V_q, *then* $\sum_{q\in\mathfrak{Q}} F_q$ *is also Calderón–Zygmund controlled at threshold* α *and measure* $\sum_{q\in\mathfrak{Q}} V_q$; *but if the* F_q *were instead* L^2 *terms of threshold* α *and measure* V_q, *then* $\sum_{q\in\mathfrak{Q}} F_q$ *can only be said to be an* L^2 *term of threshold* α *and measure* $\bigl(\sum_{q\in\mathfrak{Q}} V_q^{1/2}\bigr)^2$.

Proof. For (a), we bound $|F| \le F_1 + \cdots + F_n$ by the sum of Calderón–Zygmund terms F_i of threshold α and measure V. By Chebyshev's inequality (in the type-L^1 and type-L^2 cases) we have

$$\{F_i > \alpha/n\} \lesssim V$$

for all $i = 1, \dots, n$; summing, we obtain the claim.

The claim (b) is immediate from the triangle inequality, as is (c), after using the trivial bound $\bigl(\sum_{q\in\mathfrak{Q}} |F_q|^2\bigr)^{1/2} \le \sum_{q\in\mathfrak{Q}} |F_q|$ to handle type-L^1 terms arising from the square function. □

Most of this paper will be devoted to the proof of the following variant of Propositions 1.1 and 1.2. Call a function $f : \mathbb{R}^d \to \mathbb{R}$ *granular* if it is a finite linear combination of indicator functions of dyadic cubes. As in [Seeger et al. 2003], it is convenient for minor technical reasons to restrict attention to granular functions.

Proposition 2.3 (bounding the lacunary spherical maximal function). *Let* $0 \le \alpha \le 1$, *and let* f *be a granular function taking values in* $[0, 1]$. *Then* $\mathcal{M} f$ *is Calderón–Zygmund controlled with threshold* α *and measure* $(\mathrm{Log}_3(1/\alpha)/\alpha)\|f\|_1$.

By Lemma 2.2(a) we see that the conclusion of Proposition 2.3 implies the bound

$$|\{x \in \mathbb{R}^d : \mathcal{M} f(x) > \alpha\}| \lesssim \frac{\mathrm{Log}_3(1/\alpha)}{\alpha}\|f\|_1 \tag{2-1}$$

for granular f taking values in $[0, 1]$; the granularity hypothesis can then be removed by a standard limiting argument. It is then clear that Proposition 2.3 implies Proposition 1.1 as a special case. Let us now also see why it implies Proposition 1.2:

Proof of Proposition 1.2. Without loss of generality we may assume that f is nonnegative. Since we have the pointwise estimate $\mathcal{M}f \le \mathcal{M}(f\chi_{f\ge\alpha/2}) + \alpha/2$, we may assume without loss of generality (after replacing α with $\alpha/2$) that $f(x) \ge \alpha$ for all x in the support of f. We then have the pointwise bound

$$\mathcal{M}f(x) \le \sum_{k=1}^{\infty} \mathcal{M}(f\chi_{\mathrm{Log}_3(f/\alpha)\approx 2^k})$$

and hence

$$|\{\mathcal{M}f > \alpha\}| \le \sum_{k=1}^{\infty} \left|\left\{\mathcal{M}(f\chi_{\mathrm{Log}_3(f/\alpha)\approx 2^k})(x) > \frac{\alpha}{C_\epsilon k^{1+\epsilon}}\right\}\right|$$

for a sufficiently large constant C_ϵ. By (2-1) and a simple rescaling we have

$$\left|\left\{\mathcal{M}(f\chi_{\mathrm{Log}_3(f/\alpha)\approx 2^k}) > \frac{\alpha}{C_\epsilon k^{1+\epsilon}}\right\}\right| \le C'_\epsilon (k^{1+\epsilon}2^k/\alpha) \int_{\mathbb{R}^d} f(x)\chi_{\mathrm{Log}_3(f(x)/\alpha)\approx 2^k}\, dx$$

for some quantity C'_ϵ depending only on ϵ. Summing in k, we obtain the claim. □

It remains to establish Proposition 2.3.

Reductions using Calderón–Zygmund theory. Similarly to [Seeger et al. 2003], we first make some standard reductions using Calderón–Zygmund theory. By the L^2 boundedness of the lacunary spherical maximal function, any expression of the form Mg with $\|g\|_2^2 \lesssim \alpha\|f\|_1$ will be a Calderón–Zygmund term of type L^2, threshold α, and measure $(1/\alpha)\|f\|_1$, and can thus be neglected. In particular, if we define the standard Calderón–Zygmund exceptional set

$$\Omega := \{M_{HL}(f) \ge \alpha\},$$

where M_{HL} is the Hardy–Littlewood maximal operator, then f is bounded almost everywhere outside of Ω by α, so in particular

$$\|f\chi_{\mathbb{R}^d\setminus\Omega}\|_2^2 \le \alpha\|f\|_1.$$

Thus the $\mathcal{M}(f\chi_{\mathbb{R}^d\setminus\Omega})$ gives a negligible contribution, and it suffices (by Lemma 2.2(b)) to control the contribution $\mathcal{M}(f\chi_\Omega)$ arising from the set Ω.

By the Whitney decomposition, we may partition Ω (up to null sets) by a family $\mathfrak{Q}$ of essentially disjoint cubes q on which $\int_q f \lesssim \alpha|q|$; setting $f_q := f\chi_q$, each f_q is granular, and we conclude that $f\chi_\Omega = \sum_{q\in\mathfrak{Q}} f_q$ almost everywhere, and by the Hardy–Littlewood maximal inequality one has $\sum_q |q| \lesssim (1/\alpha)\|f\|_1$. By arguing as in the start of [Seeger et al. 2003, §3] we may partition $\mathfrak{Q}$ into a bounded number of families $\mathfrak{Q}_i$ such that the cubes q in each $\mathfrak{Q}_i$ have their doubles $2q$ pairwise disjoint. By the triangle inequality (Lemma 2.2(b)), it suffices to show that for each i, the expression $\mathcal{M}\sum_{q\in\mathfrak{Q}_i} f_q$ is Calderón–Zygmund controlled at threshold α and measure $\mathrm{Log}_3(1/\alpha)\cdot\sum_{q\in\mathfrak{Q}_i}|q|$.

Henceforth we fix i and omit the constraint $q \in \mathfrak{Q}_i$ from summations for sake of brevity. Following [Seeger et al. 2003], we now introduce some cancellation, by defining the projection operator Π_q to be the projection operator onto a certain space of polynomials. That is, let $\{P_j\}_{j=1}^{L}$ be an orthonormal basis for the space of polynomials of degree $\leq 100d$ on the unit cube $\left[-\frac{1}{2}, \frac{1}{2}\right]^d$. If q is a cube with center x_q and sidelength $l(q)$, define

$$\Pi_q[h](x) := \chi_q(x) \sum_{j=1}^{L} P_j\left(\frac{x-x_q}{l(q)}\right) \int_q h(y) P_j\left(\frac{y-x_q}{l(q)}\right) \frac{dy}{l(q)^d}.$$

Introduce the "bad functions"

$$b_q := f_q - \Pi_q[f_q].$$

Since

$$|\Pi_q[f_q](x)| \lesssim \alpha \chi_q,$$

we have

$$\left\| \sum_q \Pi_q[f_q] \right\|_2^2 \lesssim \alpha^2 \sum_q |q|,$$

and so by the L^2-boundedness of the lacunary spherical maximal operator, the contribution of $\mathcal{M} \sum_q \Pi_q[f_q]$ is a type-L^2 Calderón–Zygmund term of threshold α and measure $\sum_q |q|$. Thus it remains to obtain Calderón–Zygmund control on the contribution

$$\sup_k \left| \sum_q b_q * \sigma_k \right|$$

of the bad functions b_q.

The contributions of those k with $2^k \leq l(q)$ are contained in $\bigcup_q 3q$, and are thus acceptable Calderón–Zygmund terms of type L^0. For the remaining k, we will replace the sup by an ℓ^2 norm; thus we will show that

$$\left(\sum_k \left| \sum_{q:\, 2^k > l(q)} b_q * \sigma_k \right|^2 \right)^{1/2}$$

is Calderón–Zygmund controlled at threshold α and measure $\sum_{q \in \mathfrak{Q}_i} |q|$. Expanding out the square and using the triangle inequality, it suffices to establish this claim for the diagonal contribution

$$\left(\sum_q \sum_{k:\, 2^k > l(q)} |b_q * \sigma_k|^2 \right)^{1/2} \tag{2-2}$$

and for the off-diagonal contribution

$$\left(\sum_k \sum_{q \neq q':\, 2^k > l(q), l(q')} (b_q * \sigma_k)(\overline{b_{q'} * \sigma_k}) \right)^{1/2}. \tag{2-3}$$

The off-diagonal expression (2-3) can be handled by existing arguments. Indeed, since

$$\left\| \left(\sum_k \sum_{q \neq q':\, 2^k > l(q), l(q')} (b_q * \sigma_k)(\overline{b_{q'} * \sigma_k}) \right)^{1/2} \right\|_2^2 \leq \sum_k \sum_{q \neq q'} |\langle b_q * \sigma_k, b_{q'} * \sigma_k \rangle|,$$

it would suffice (by the definition of a type-L^2 term) to show that

$$\sum_k \sum_{q \neq q'} |\langle b_q * \sigma_k, b_{q'} * \sigma_k \rangle| \lesssim \alpha^2 \sum_q |q|,$$

which can be proven as in [Seeger et al. 2003, (4.17)] by exploiting the smoothness of the kernel $\sigma_k * \sigma_k$ and the cancellation of b_q (and the hypothesis that the doubles $2q$ of $q \in \mathfrak{Q}_i$ are disjoint). Since the proof is nearly identical to that given in [Seeger et al. 2003], we omit it here.

It remains to control the diagonal contribution (2-2). It is easy to see that

$$\sum_q \sum_{k:\,2^k > l(q)} \|\Pi_q[f_q] * \sigma_k\|_2^2 \lesssim \sum_q \sum_{k:\,2^k > l(q)} \|\alpha \chi_q * \sigma_k\|_2^2 \lesssim \sum_q \sum_{k:\,2^k > l(q)} \alpha^2 2^{-k(d-1)} \ell(q)^{2d-1} \lesssim \alpha^2 \sum_q |q|.$$

Thus $\big(\sum_q \sum_{k:2^k > l(q)} |\Pi_q[f_q] * \sigma_k|^2\big)^{1/2}$ is a type-L^2 term of the required threshold and measure, and so by the triangle inequality we may replace (2-2) with

$$\bigg(\sum_q \sum_{k:2^k > l(q)} |f_q * \sigma_k|^2\bigg)^{1/2}.$$

To show that this expression is Calderón–Zygmund controlled at threshold α and measure $\sum_q |q|$, it suffices by Lemma 2.2 to show that the inner square functions $\big(\sum_{k:2^k > l(q)} |f_q * \sigma_k|^2\big)^{1/2}$ are Calderón–Zygmund controlled at threshold α and measure $|q|$ for each cube q. A simple scaling argument shows that we may then normalize q to be a unit cube. We have thus reduced matters to establishing:

Proposition 2.4 (bounding the lacunary spherical maximal function of f_q). *Let* $0 < \alpha < 1$, *and let* f_q *be a granular function supported on a unit cube q taking values in* $[0, 1]$ *with* $\int_q f_q \lesssim \alpha$. *Then the expression*

$$\bigg(\sum_{k>0} |f_q * \sigma_k|^2\bigg)^{1/2} \tag{2-4}$$

is Calderón–Zygmund controlled at threshold α *and measure* $\mathrm{Log}_3(1/\alpha)$.

The remainder of this paper will be devoted to the proof of this proposition.

Structural decomposition of f_q. Let q, f_q, α be as in Proposition 2.4. In [Seeger et al. 2003], the support of f_q was decomposed into structures referred to as "generalized boxes", which behaved in a certain way like 1-dimensional sets and which had associated quantities referred to as "length" and "thickness", the former which governed support size estimates and the latter which controlled L^2 bounds. We describe a decomposition of f_q that is in a similar spirit.

Lemma 2.5 (structural decomposition lemma). *Let* q, f_q, α *be as in Proposition 2.4. List the dyadic numbers between* α^2 *and* 1 *in increasing order as*

$$\alpha^2 < \gamma_0 < \gamma_1 < \cdots < \gamma_J = 1;$$

thus $J \approx \mathrm{Log}(1/\alpha)$. *Then we can take the decomposition*

$$f_q = \sum_{j=0}^{J} f_q^{\gamma_j} \tag{2-5}$$

such that, for each j, $f_q^{\gamma_j}$ is a granular function taking values in $[0, 1]$ *that is supported on a finite union of cubes Q in q whose total "length" $\lambda(f_q^{\gamma_j}) := \sum_Q l(Q)$ obeys the estimates*

$$\int f_q^{\gamma_j} \approx \gamma_j \cdot \lambda(f_q^{\gamma_j}) \tag{2-6}$$

if $j > 0$, with just the upper bound

$$\int f_q^{\gamma_j} \lesssim \gamma_j \cdot \lambda(f_q^{\gamma_j}) \tag{2-7}$$

for $j = 0$. Furthermore, one has

$$\int_Q f_q^{\gamma_j} \lesssim \gamma_j l(Q) \tag{2-8}$$

for every cube Q. We refer to γ_j as the critical density of $f_q^{\gamma_j}$.

We will use the decomposition (2-5) in an essential way throughout the rest of the paper, as well as the key properties (2-6), (2-7) and (2-8). In [Seeger et al. 2003], the analog of (2-7) is that for every generalized box B of thickness γ and length λ, we have $|B| \lesssim \gamma \cdot \lambda$.

Proof. We perform a greedy algorithm, extracting the "heaviest" cubes first. Given a (nonnegative) function f and a cube Q, we define the *weight*

$$\mathrm{wt}_Q[f] := \frac{1}{l(Q)} \int_Q f.$$

The symbol Q will always be understood to be a dyadic cube. We then inductively define

$$E_q^{\gamma_J} := \bigcup_{Q \subset q:\, \mathrm{wt}_Q[f_q] \geq \gamma_J} Q;$$

note that from the trivial bound $\mathrm{wt}_Q[f_q] \leq \mathrm{wt}_Q[\chi_q]$ there are only finitely many cubes Q that can contribute here. For $1 < j < J$, we define

$$E_q^{\gamma_j} := \bigcup_{Q \subset q:\, \mathrm{wt}_Q[f_q \chi_{q \setminus \bigcup_{l>j} E_q^{\gamma_l}}] \geq \gamma_j} Q; \tag{2-9}$$

again, this is a finite union of dyadic cubes. Set

$$E_q^{\gamma_0} := q \setminus \Bigl(\bigcup_{1 \leq j \leq N} E_q^{\gamma_j} \Bigr).$$

If we then set

$$f_q^{\gamma_j} := f_q \chi_{E_q^{\gamma_j} \setminus \bigcup_{j<l\leq J} E_q^{\gamma_l}},$$

we obtain (2-5), and the f_q are clearly granular. For $j > 0$, let $\mathcal{Q}_j$ be a maximal cover of $E_q^{\gamma_j}$ by dyadic cubes $Q \subset q$ obeying the stated condition

$$\mathrm{wt}_Q[f_q \chi_{q \setminus \bigcup_{l>j} E_q^{\gamma_l}}] \geq \gamma_j,$$

and set

$$\lambda(f_q^{\gamma_j}) := \sum_{Q \in \mathcal{Q}_j} l(Q). \tag{2-10}$$

Then $f_q^{\gamma_j}$ is supported on $\bigcup_{Q \in \mathcal{Q}_j} Q$, and the required claims (2-6), (2-8) follow from the construction of the $f_q^{\gamma_j}$ and (2-9) (and the upper bound $f_q \leq \chi_q$, in the $j = J$ case). For $j = 0$, we still have (2-8), and the claim (2-7) follows by taking $\mathcal{Q}_0$ to consist just of the unit cube q. □

It remains to show that the expression

$$\Big(\sum_{k>0} \Big| \sum_{\gamma} f_q^{\gamma} * \sigma_k \Big|^2 \Big)^{1/2} \tag{2-11}$$

is Calderón–Zygmund controlled of threshold α and measure $\mathrm{Log}_3(1/\alpha)$, where γ is implicitly restricted to $\gamma_0, \dots, \gamma_J$.

Further reductions. We record the basic L^0, L^1, L^2 estimates on $f_q^{\gamma} * \sigma_k$ (which were already implicit in [Seeger et al. 2003]):

Lemma 2.6 (L^0, L^1, L^2 estimates). *Let $k > 0$ and $\gamma \geq \gamma_0$.*

(L^0) *$f_q^{\gamma} * \sigma_k$ is a type-L^0 Calderón–Zygmund term of threshold α and measure $2^{k(d-1)}\lambda(f_q^{\gamma})$.*

(L^1) *$f_q^{\gamma} * \sigma_k$ is a type-L^1 Calderón–Zygmund term of threshold α and measure $(1/\alpha)\| f_q^{\gamma} \|_1$.*

(L^2) *$f_q^{\gamma} * \sigma_k$ is a type-L^2 Calderón–Zygmund term of threshold α and measure*

$$\frac{2^{-k(d-1)}\gamma}{\alpha^2} \mathrm{Log}\, \frac{2^{k(d-1)}}{\gamma} \cdot \| f_q^{\gamma} \|_1.$$

Proof. For the L^0 estimate, we decompose f_q^{γ} into functions $f_q^{\gamma}\chi_Q$ supported on cubes Q with $\sum_Q l(Q) = \lambda(f_q^{\gamma})$. A geometric calculation shows that $f_q^{\gamma}\chi_Q * \sigma_k$ is supported on an annular region of measure $O(2^{k(d-1)} l(Q))$, and the claim follows by summing in Q.

The L^1 estimate is immediate from Young's inequality, so we turn to the L^2 estimate. Using the well-known pointwise estimate

$$\sigma_k * \sigma_k(x) \lesssim \frac{2^{-k(d-1)}}{|x|} \chi_{|x| \leq 2^{k+1}}, \tag{2-12}$$

we may expand

$$\begin{aligned} \| f_q^{\gamma} * \sigma_k \|_2^2 &= \langle f_q^{\gamma}, \sigma_k * \sigma_k * f_q^{\gamma} \rangle \\ &\lesssim 2^{-k(d-1)} \int f_q^{\gamma}(x) \Big(\sum_{l \leq k+1} 2^{-l} \int_{y:\,|x-y| \approx 2^l} f_q^{\gamma}(y)\, dy \Big)\, dx \\ &\lesssim 2^{-k(d-1)} \| f_q^{\gamma} \|_1 \sup_x \sum_{l \leq k+1} 2^{-l} \int_{y:\,|x-y| \approx 2^l} f_q^{\gamma}(y)\, dy. \end{aligned}$$

From (2-8) and the pointwise bound $f_q^{\gamma} \leq f \leq 1$ we have

$$\int_{y:|x-y| \approx 2^l} f_q^{\gamma}(y)\, dy \lesssim \min(\gamma 2^l, 2^{dl}) \tag{2-13}$$

and thus

$$\|f_q^\gamma * \sigma_k\|_2^2 \lesssim 2^{-k(d-1)} \|f_q^\gamma\|_1 \sum_{l \le k+1} \min(\gamma, 2^{(d-1)l}).$$

The summand is equal to γ for $O(\operatorname{Log}(2^{k(d-1)}/\gamma))$ terms, and decays geometrically otherwise, giving the claim. □

From the L^2 case of this lemma we see that $f_q^{\gamma_0} * \sigma_k$ is a type-L^2 Calderón–Zygmund term of threshold α and measure

$$\alpha^{-2} 2^{-k(d-1)} \gamma_0 \operatorname{Log} \frac{2^{k(d-1)}}{\gamma_0} \cdot \|f_q^{\gamma_0}\|_1 \lesssim 2^{-k(d-1)} \alpha \operatorname{Log} \frac{2^{k(d-1)}}{\alpha^2}$$

since $\gamma_0 \approx \alpha^2$ and $\|f_q^{\gamma_0}\|_1 \le \|f_q\|_1 \lesssim \alpha$. Summing over all positive k using Lemma 2.2(c), we conclude that

$$\Big(\sum_{k>0} |f_q^{\gamma_0} * \sigma_k|^2 \Big)^{1/2}$$

is Calderón–Zygmund controlled of threshold α and measure $O(\alpha \operatorname{Log}(1/\alpha))$, which is acceptable. Thus we may delete the γ_0 term from (2-11) and focus attention on

$$\Big(\sum_{k>0} \Big| \sum_{\gamma > \gamma_0} f_q^\gamma * \sigma_k \Big|^2 \Big)^{1/2}. \tag{2-14}$$

From the L^0 case of this lemma and Lemma 2.2(c), followed by (2-6), we see that

$$\Big(\sum_{k>0} \Big| \sum_{\gamma > \gamma_0 :\, k(d-1) < \log(\gamma/\alpha)} f_q^\gamma * \sigma_k \Big|^2 \Big)^{1/2}$$

is a type-L^0 Calderón–Zygmund term of threshold α and measure

$$\sum_{k>0} \sum_{\gamma > \gamma_0 :\, k(d-1) < \log(\gamma/\alpha)} 2^{k(d-1)} \lambda(f_q^\gamma) \lesssim \sum_{\gamma > \gamma_0} \frac{\gamma}{\alpha} \lambda(f_q^\gamma) \approx \frac{1}{\alpha} \sum_{\gamma > \gamma_0} \|f_q^\gamma\|_1 \le \frac{\|f_q\|_1}{\alpha} \lesssim 1,$$

giving the claim. Thus the contribution of the "small scales" with $k(d-1) < \log(\gamma/\alpha)$ is acceptable.

Next, we claim that the contribution of the "nearly small scale" case

$$\log \frac{\gamma}{\alpha} \le k(d-1) < \log \frac{\gamma}{\alpha} + 100 \operatorname{Log}_3 \frac{1}{\alpha} \tag{2-15}$$

is also acceptable. Indeed, from the L^1 case of Lemma 2.6 and Lemma 2.2(c), we see that

$$\Big(\sum_{k>0} \Big| \sum_{\gamma > \gamma_0 :\text{(2-15)}} f_q^\gamma * \sigma_k \Big|^2 \Big)^{1/2}$$

is a type-L^1 Calderón–Zygmund term of threshold α and measure

$$\sum_{k>0} \sum_{\gamma > \gamma_0 :\, \text{(2-15)}} \frac{1}{\alpha} \|f_q^\gamma\|_1 \lesssim \frac{\operatorname{Log}_3(1/\alpha)}{\alpha} \sum_{\gamma > \gamma_0} \|f_q^\gamma\|_1 \le \frac{\operatorname{Log}_3(1/\alpha)}{\alpha} \|f_q\|_1 \lesssim \frac{\operatorname{Log}_3(1/\alpha)}{\alpha}$$

giving the claim.

Furthermore, as in [Seeger et al. 2003], we claim that the contribution of terms $\sigma_k * f_q^\gamma$ in the "large scale" case

$$k(d-1) \le \log \frac{\gamma}{\alpha} + 100\,\mathrm{Log}_2 \frac{1}{\alpha} \tag{2-16}$$

is also acceptable (with some room to spare). Indeed, from Cauchy–Schwarz one has

$$\sum_{\gamma>\gamma_0:\,(2\text{-}16)} f_q^\gamma * \sigma_k \lesssim \biggl(\sum_{\gamma>\gamma_0:\,(2\text{-}16)} \Bigl(k(d-1) - \log \frac{\gamma}{\alpha}\Bigr)^2 |f_q^\gamma * \sigma_k|^2 \biggr)^{1/2}$$

and from this, the L^2 case of Lemma 2.6 and Lemma 2.2(c) we see that

$$\biggl(\sum_{k>0} \Bigl| \sum_{\gamma>\gamma_0:\,(2\text{-}16)} f_q^\gamma * \sigma_k \Bigr|^2 \biggr)^{1/2}$$

is a type-L^2 Calderón–Zygmund term of threshold α and measure

$$\begin{aligned}\sum_{k>0} \sum_{\gamma>\gamma_0:\,(2\text{-}16)} & \frac{(k(d-1) - \log(\gamma/\alpha))^2}{\alpha^2} 2^{-k(d-1)} \gamma\, \mathrm{Log} \frac{2^{k(d-1)}}{\gamma} \cdot \|f_q^\gamma\|_1 \\ & \lesssim \sum_{\gamma>\gamma_0} \frac{\mathrm{Log}_2^2(1/\alpha)}{\alpha^2} \Bigl(\frac{\gamma}{\alpha} \mathrm{Log}^{100} \frac{1}{\alpha} \Bigr)^{-1} \gamma\, \mathrm{Log}_2 \frac{1}{\alpha} \cdot \|f_q^\gamma\|_1 \lesssim \frac{1}{\alpha} \sum_{\gamma>\gamma_0} \|f_q^\gamma\|_1 \le \frac{\|f_q\|_1}{\alpha} \lesssim 1\end{aligned}$$

as required.

We have now treated all scales k except for those in the "medium-scale" range $\mathcal{K}_\gamma$ defined by

$$\mathcal{K}_\gamma := \Bigl\{ k > 0 : \log \frac{\gamma}{\alpha} + 100\,\mathrm{Log}_3 \frac{1}{\alpha} \le k(d-1) \le \log \frac{\gamma}{\alpha} + 100\,\mathrm{Log}_2 \frac{1}{\alpha} \Bigr\}. \tag{2-17}$$

We have thus reduced Proposition 2.4 to the following.

Proposition 2.7. *Let* $0 < \alpha < 1$, *and let* f_q *be a granular function on a unit cube* q *taking values in* $[0, 1]$ *with*

$$\int_q f_q \lesssim \alpha. \tag{2-18}$$

Let f_q^γ *be as in Lemma 2.5, and* $\mathcal{K}_\gamma$ *be given by* (2-17). *Then the expression*

$$\biggl(\sum_{k>0} \Bigl| \sum_{\gamma>\gamma_0:\,k\in\mathcal{K}_\gamma} f_q^\gamma * \sigma_k \Bigr|^2 \biggr)^{1/2} \tag{2-19}$$

is Calderón–Zygmund controlled at threshold α *and measure* $\mathrm{Log}_3(1/\alpha)$.

Remark 2.8. If we were willing to replace $\mathrm{Log}_3(1/\alpha)$ by of $\mathrm{Log}_2(1/\alpha)$ in the measure parameter of the conclusion then we could use the previous "nearly small scale" argument to express (2-19) as a type-L^1 term, recovering the results of [Seeger et al. 2003] (with essentially the same proof). The main innovation of this paper is to treat these medium-scale contributions by a more sophisticated argument than this simple L^1 argument, in particular constructing some additional exceptional sets outside of which one can establish good L^2 estimates at "light" scales.

3. Proof of Proposition 2.7

Let α, q, f_q, f_q^γ be as in the above proposition. To prove Proposition 2.7, we will identify an exceptional set to hold the "heavy" terms of type L^0, then split the remaining portions of (2-19) into "intermediate" terms that will be of type L^1, and finally "light" terms that will be of type L^2 outside of the previously identified exceptional set. To construct these terms we need to introduce some additional scales, and identify certain rectangles on which the f_q^γ are unusually "heavy".

Defining double-logarithmically many scales. Let us temporarily fix a critical density γ with $\alpha^2 \le \gamma \le 1$. For each such density we associate a key radius

$$r = r_\gamma := \max(1, (\gamma/\alpha)^{1/(d-1)}) \tag{3-1}$$

and note that the constraint (2-17) ensures that 2^k is a little bit larger than r:

$$2^k \ge r \operatorname{Log}_2^{100/(d-1)} \frac{1}{\alpha} \quad \text{for all } k \in \mathcal{K}_\gamma. \tag{3-2}$$

With the density γ fixed, we identify $O(\operatorname{Log}_2(1/\alpha))$ many natural scales

$$\gamma^{1/(d-1)} = c_0 \le c_1 \le \cdots \le c_N \le r$$

between $\gamma^{1/(d-1)}$ and r in our problem that will lead us to our $L \operatorname{Log}_3 L$ result. They will be defined recursively by initializing

$$c_0 := \gamma^{1/(d-1)} \le 1 \le r$$

and then taking iterated geometric means with r; thus

$$c_i := \sqrt{c_{i-1} r} \tag{3-3}$$

for all $i \ge 1$. More explicitly, we have

$$c_i = (\gamma^{1/(d-1)}/r)^{2^{-i}} r = \max(\gamma^{2^{-i}/(d-1)}, \gamma^{1/(d-1)} \alpha^{-(1-2^{-i})/(d-1)}) \tag{3-4}$$

for all $i \ge 0$. Geometrically, each c_i for $i \ge 1$ arises (up to constants) as the diameter of a spherical cap of thickness c_{i-1} on a sphere of radius r; see Figure 1. These scales are motivated by a decomposition of the kernel $\sigma_k * \sigma_k$ into linear combinations of characteristic functions of rectangles, which will appear later in the paper.

We terminate the sequence of scales c_i at the first $N = N_\gamma$ for which

$$c_N \ge 2^{-10} r.$$

Since $\alpha^2 < \gamma \le 1$, we have $1 \le r/\gamma \le \alpha^{-O(1)}$, and hence $N \lesssim \operatorname{Log}_2(1/\alpha)$.

Throwing away exceptional sets at each scale. We can now define some exceptional sets.

Definition 3.1. Let the parameters k, γ be fixed as above, and define the scales $c_0 \le \cdots \le c_N$ as in the previous section. Let $1 \le i \le N$. Let $M > 0$ be a dyadic number.

(1) Define $\{\Phi_{j,i}\}_{j \in \mathcal{J}_i}$ to be a maximal set of (c_{i-1}/c_i)-separated directions $\Phi_{j,i}$ in S^{d-1}, so that the number of directions is $\approx (c_i/c_{i-1})^{d-1}$.

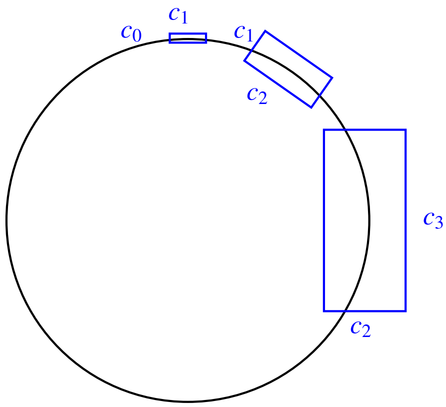

Figure 1. The scales c_i. Here $c_0 = \gamma^{1/(d-1)}$ and the circle has radius 2^k for some k with 2^k somewhat larger than r.

(2) For each direction $\Phi_{j,i}$ in the above set, partition $\mathbb{R}^d$ into $c_i \times c_i \times \cdots \times c_i \times c_{i-1}$ parallel rectangles that belong to some fixed grid, with the short direction parallel to the direction $\Phi_{j,i}$. Define $\mathcal{R}_{j,i,M}$ to be the collection of all such rectangles R such that

$$\int_R |f_q^\gamma| \geq c_{i-1} M\gamma. \tag{3-5}$$

(3) We then define our exceptional set as

$$S_{M,k,q,\gamma} := \bigcup_{i=1}^{N} \bigcup_{j\in\mathcal{J}_i} \operatorname{supp}(\sigma_{k,j,i} * \chi_{\bigcup_{R\in\mathcal{R}_{j,i,M}} 100R}),$$

where $\sigma_{k,j,i}$ is the restriction of σ_k to a spherical cap on the sphere of radius 2^k of angular width $100c_{i-1}/c_i$ centered at $2^k\Phi_{j,i}$.

Observe that the sets $S_{M,k,q,\gamma}$ decrease as M increases. We claim the crucial upper bound on the size of the exceptional sets defined above.

Lemma 3.2 (maximal inequality). *With the notation of Definition 3.1, we have the bound*

$$|S_{M,k,q,\gamma}| \lesssim \frac{2^{k(d-1)}}{M} \lambda(f_q^\gamma).$$

A key point here is that we do not lose a factor of N (which can be as large as $\operatorname{Log}_2(1/\alpha)$) on the right-hand side, by taking advantage of how the rectangles associated to different scales c_i nest within (dilates of) each other. This inequality can be viewed as a complicated variant of the Hardy–Littlewood maximal inequality.

Before we proceed with the proof of Lemma 3.2, we say a few words to motivate the previous definitions. We note that in some sense the support of $\sigma_{k,j,i}$ is "adapted" to translates of rectangles in $\mathcal{R}_{j,i,M}$, in the sense that convolution with characteristic functions of rectangles effectively fattens it by c_{i-1} and translates it. Thus we note that for each $R \in \mathcal{R}_{j,i}$, the set $\operatorname{supp}(\sigma_{k,j,i} * \chi_{100R})$ is contained in a

$1000c_{i-1}$-neighborhood of a translate of (a slightly wider version of) the cap $\mathrm{supp}(\sigma_{k,j,i})$. The rectangles $R \in \mathcal{R}_{j,i,M}$ that are sufficiently "heavy" in the sense of (3-5) correspond to (more or less) poor L^2 estimates for $\sigma_{k,j,i} * \chi_R$, and so we would like to remove the supports of $\sigma_{k,j,i} * \chi_R$. Since the support of this is essentially contained in a c_{i-1}-fattening of the cap $\mathrm{supp}(\sigma_{k,j,i})$, the heavier the rectangles we consider (the larger M is) the fewer number of such rectangles there can be, so the smaller the total size of exceptional sets thrown away. Thus using a pigeonholing argument, we can obtain the bound from Lemma 3.2.

Proof of Lemma 3.2. We need to impose a partial order relation on the directions $\Phi_{j,i}$. For any i, i' with $i > i'$, we will say that a direction $\Phi_{j,i}$ is an *ancestor* of $\Phi_{j',i'}$ and write $(j', i') \prec (j, i)$ if the ball of radius $10c_{i'-1}/c_{i'}$ centered at $\Phi_{j',i'}$ is contained in the ball of radius $10c_{i-1}/c_i$ centered at $\Phi_{j,i}$. This is easily seen to be a partial order, and every $\Phi_{j',i'}$ has at least one ancestor $\Phi_{j,i}$ at generation i.

By definition, $S_{M,k,q,\gamma}$ is contained in a set of the form

$$\bigcup_{1 \le i \le N} \bigcup_{j \in \mathcal{J}_i} \bigcup_{R \in \mathcal{R}_{j,i,M}} \mathrm{supp}(\sigma_{k,j,i} * \chi_{1000R}),$$

where each $R \in \mathcal{R}_{j,i,M}$ is a $c_i \times c_i \times \cdots \times c_i \times c_{i-1}$ rectangle with short side pointing in direction $\Phi_{j,i}$ satisfying

$$\int_R f_q^\gamma \ge c_{i-1} M \gamma. \tag{3-6}$$

Moreover, if $(j, i) \prec (j', i')$ then for any $R \in \mathcal{R}_{j,i,M}$ and any $R' \in \mathcal{R}_{j',i',M}$, if $R \cap R' \neq \varnothing$ then $R \subset 100R'$. It follows that we can choose subcollections $\widetilde{\mathcal{R}}_{j,i,M} \subset \mathcal{R}_{j,i,M}$ satisfying

$$S_{M,k,q,\gamma} \subset \bigcup_{1 \le i \le N} \bigcup_{j \in \mathcal{J}_i} \bigcup_{R \in \widetilde{\mathcal{R}}_{j,i,M}} \mathrm{supp}(\sigma_{k,j,i} * \chi_{10000R}),$$

so that for given any direction $\Phi_{j_1,1}$, a chain of ancestors

$$\Phi_{j_1,1} \prec \Phi_{j_2,2} \prec \cdots \prec \Phi_{j_N,N} \tag{3-7}$$

satisfies the property that the rectangles in the collections $\widetilde{\mathcal{R}}_{j_i,i,M}$, $1 \le i \le N$, are all pairwise disjoint. Indeed, we can choose $\widetilde{\mathcal{R}}_{j,i,M}$ to be the collection of rectangles $R \in \mathcal{R}_{j,i,M}$ which are maximal in the sense that they do not intersect any $R' \in \mathcal{R}_{j',i',M}$ for some ancestor $(j', i') \succ (j, i)$.

Since $\sigma_{k,j,i} * \chi_{10000R}$ is essentially supported in a c_{i-1}-fattening of (a slightly wider version of) the cap $\mathrm{supp}(\sigma_{k,j,i} * \chi_{10000R})$, the measure of its support is $\lesssim 2^{k(d-1)} c_{i-1}^d / c_i^{d-1}$. We can thus bound

$$|S_{M,k,q,\gamma}| \lesssim \Bigl| \bigcup_{1 \le i \le N} \bigcup_{j \in \mathcal{J}_i} \bigcup_{R \in \widetilde{\mathcal{R}}_{j,i,M}} \mathrm{supp}(\sigma_{k,j,i} * \chi_{10000R}) \Bigr| \lesssim \sum_{i=1}^{N} \sum_{j \in \mathcal{J}_i} 2^{k(d-1)} \frac{c_{i-1}^d}{c_i^{d-1}} \cdot \mathrm{card}(\widetilde{\mathcal{R}}_{j,i,M}).$$

By the disjointness property mentioned above and (3-6), for a chain of ancestors as in (3-7), we have the bound

$$\int f_q^\gamma \ge \sum_{1 \le i \le N} \sum_{R \in \widetilde{\mathcal{R}}_{j_i,i,M}} \int_R |f_q^\gamma| \ge \sum_{1 \le i \le N} c_{i-1} M \gamma \cdot \mathrm{card}(\widetilde{\mathcal{R}}_{j_i,i,M}) \tag{3-8}$$

and hence by (2-6)

$$\sum_{1\le i\le N} c_{i-1}\,\mathrm{card}(\widetilde{\mathcal{R}}_{j_i,i,M}) \lesssim \frac{\lambda(f_q^\gamma)}{M}.$$

Since each direction $\Phi_{j,i}$ is the ancestor of $\lesssim ((c_{i-1}/c_i)\cdot(c_1/c_0))^{d-1}$ many directions $\Phi_{j',i'}$ with $i'=1$, it follows that

$$\begin{aligned}
|S_{M,k,q,\gamma}| &\lesssim \sum_i \sum_j 2^{k(d-1)} \frac{c_{i-1}^d}{c_i^{d-1}} \cdot \mathrm{card}(\widetilde{\mathcal{R}}_{j,i,M}) \\
&\lesssim 2^{k(d-1)} \sum_{j_1} \sum_{j,i:\,(j,i)\succ(j_1,1)} 2^{k(d-1)} \frac{c_{i-1}^d}{c_i^{d-1}} \cdot \left(\frac{c_{i-1}}{c_i}\cdot\frac{c_1}{c_0}\right)^{1-d} \cdot \mathrm{card}(\widetilde{\mathcal{R}}_{j,i,M}) \\
&\lesssim 2^{k(d-1)} \sum_{j_1} \sum_{j,i:\,(j,i)\succ(j_1,1)} c_{i-1}\left(\frac{c_0}{c_1}\right)^{d-1} \cdot \mathrm{card}(\widetilde{\mathcal{R}}_{j,i,M}) \\
&\lesssim 2^{k(d-1)} \left(\frac{c_0}{c_1}\right)^{d-1} \sum_{j_1} \frac{\lambda(f_q^\gamma)}{M} \lesssim \frac{2^{k(d-1)}}{M}\cdot\lambda(f_q^\gamma).
\end{aligned}$$

□

For a given choice of k and γ, we define the upper height $M_+(k,\gamma)$ by the formula

$$\log_2 M_+(k,\gamma) := \left\lfloor k(d-1) + \log\frac{\alpha}{\gamma} + \mathrm{Log}_3\frac{1}{\alpha}\right\rfloor \tag{3-9}$$

and the lower height $M_-(k,\gamma)$ by the formula

$$\log_2 M_-(k,\gamma) := \left\lfloor k(d-1) + \log\frac{\alpha}{\gamma} - 100\,\mathrm{Log}_3\frac{1}{\alpha}\right\rfloor. \tag{3-10}$$

The exceptional set associated to the upper height $M_+(k,\gamma)$ is of acceptable size:

Lemma 3.3. *We have the bound*

$$\left|\bigcup_{k,\gamma} S_{M_+(k,\gamma),k,q,\gamma}\right| \lesssim 1.$$

In particular, any component of (2-19) *that is supported in* $\bigcup_{k,\gamma} S_{M_+(k,\gamma),k,q,\gamma}$ *is a type-*L^0 *Calderón–Zygmund term of threshold* α *and measure* 1.

Proof. By Lemma 3.2, (2-6), and (3-9), we have

$$|S_{M_+(k,\gamma),k,q,\gamma}| \lesssim \frac{2^{k(d-1)}}{M_+(k,\gamma)}\lambda(f_q^\gamma) \lesssim \frac{1}{\alpha\,\mathrm{Log}_2(1/\alpha)}\|f_q^\gamma\|_1.$$

Summing over $k\in\mathcal{K}_\gamma$ using the fact that $|\mathcal{K}_\gamma| \lesssim \mathrm{Log}_2(1/\alpha)$, and then summing over γ, we obtain the desired bound thanks to (2-18). □

A decomposition of $\boldsymbol{f_q^\gamma * \sigma_k}$***.*** Recall that f is granular, and hence $f=\sum_l c_l\chi_{\omega_l}$, where each ω_l is a δ-grain, i.e., a dyadic cube of small sidelength $\delta>0$, which we can take to be smaller than (say) α^{100}. We now associate a natural spherical measure to each δ-grain ω_l, defined so that it is supported on those caps where there exists a “heavy” rectangle containing ω_l with short side essentially pointing in the direction normal to the corresponding cap.

Definition 3.4. For each δ-grain ω_l and for a given height M, define $\sigma_{k,\omega_l}^{M,\gamma}$ to be the restriction of σ_k to

$$\bigcup_{i=1}^{N} \bigcup_{j\in\mathcal{J}_i:\,\exists R\in\mathcal{R}_{j,i,M}(\omega_l\cap R\neq\varnothing)} \operatorname{supp}(\sigma_{k,j,i}),$$

where n ranges over $1\le n\le N$. Observe that these measures are decreasing as M increases.

Recall that the parameter i corresponds to the "height", in a sense, of the spherical measure $\sigma_{k,\omega_l}^{i,\gamma}$. We now decompose the function $f_q^\gamma * \sigma_k$ into different "heights" as follows. For a given height M, define the "projection of $f_q^\gamma * \sigma_k$ onto height M" as

$$g_k^{M,\gamma} := \sum_{\delta-\text{grains }\omega_l} \sigma_{k,\omega_l}^{M,\gamma} * (f_q^\gamma \chi_{\omega_l}). \tag{3-11}$$

Then we have the telescoping decomposition

$$f_q^\gamma * \sigma_k * = f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),\gamma} + \sum_{M\ge M_-(k,\gamma)} (g_k^{M,\gamma} - g_k^{2M,\gamma}). \tag{3-12}$$

As previously mentioned, we will see that we have efficient (even when summing over γ and over the relevant range of k) L^2 estimates for the term $f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),\gamma}$. This term represents the "projection of $f_q^\gamma * \sigma_k$ onto low heights".

Discarding the heavy terms via exceptional sets. We can easily dispose of the "heavy" terms in which $M\ge M_+(k,\gamma)$.

Proposition 3.5. *The terms $g_k^{M,\gamma} - g_k^{2M,\gamma}$ for $M\ge M_+(k,\gamma)$ are supported in $S_{M_+(k,\gamma),k,q,\gamma}$, and thus collectively contribute an acceptable L^0 Calderón–Zygmund term thanks to Lemma 3.3.*

Proof. For all δ-grains ω_l which appear in the expression defining $g_k^{M,\gamma}$ for some $M\ge M_+(k,\gamma)$, there is a "heavy" rectangle R containing ω_l such that $\operatorname{supp}(\sigma_{k,j,n} * \chi_R)$ is contained in $S_{M,k,q,\gamma}$ and hence in $S_{M_+(k,\gamma),k,q,\gamma}$. □

Handling the intermediate terms via L^1 estimates. Now we dispose of the "intermediate" terms in which $M_-(k,\gamma)\le M < M_+(k,\gamma)$.

Proposition 3.6. *The contribution of the terms $g_k^{M,\gamma} - g_k^{2M,\gamma}$ with $M_-(k,\gamma)\le M< M_+(k,\gamma)$ to* (2-19) *is an acceptable L^1 Calderón–Zygmund term.*

Proof. We need to establish the bound

$$\Bigl\| \Bigl(\sum_{k\ge 0} \Bigl| \sum_{\gamma>\gamma_0:\,k\in\mathcal{K}_\gamma} \sum_{M_-(k,\gamma)\le M<M_+(k,\gamma)} g_k^{M,\gamma} - g_k^{2M,\gamma} \Bigr|^2 \Bigr)^{1/2} \Bigr\|_1 \lesssim \alpha \operatorname{Log}_3 \frac{1}{\alpha}.$$

Bounding the ℓ^2 norm by the ℓ^1 norm, we can bound the left-hand side by

$$\sum_{k\ge 0} \sum_{\gamma>\gamma_0:\,k\in\mathcal{K}_\gamma} \sum_{M_-(k,\gamma)\le M<M_+(k,\gamma)} \|g_k^{M,\gamma} - g_k^{2M,\gamma}\|_1.$$

Writing $M = 2^j M_-(k,\gamma)$ for some $0 \le j \lesssim \operatorname{Log}_3(1/\alpha)$, it suffices to show that

$$\sum_{k\ge 0}\sum_{\gamma>\gamma_0:\,k\in\mathcal{K}_\gamma} \|g_k^{2^j M_-(k,\gamma),\gamma} - g_k^{2^{k+1}M_-(k,\gamma),\gamma}\|_1 \lesssim \alpha$$

for each such j.

Fix j. Since

$$\sum_\gamma \|f_q^\gamma\|_1 \le \|f\|_1 \lesssim \alpha,$$

it will suffice to show that

$$\sum_{k\in\mathcal{K}_\gamma} \|g_k^{2^j M_-(k,\gamma),\gamma} - g_k^{2^{k+1}M_-(k,\gamma),\gamma}\|_1 \lesssim \|f_q^\gamma\|_1.$$

By (3-11) and Young's inequality, it suffices to show that

$$\sum_{k\in\mathcal{K}_\gamma} \|\sigma_{k,\omega_l}^{2^j M_-(k,\gamma),\gamma} - \sigma_{k,\omega_l}^{2^{j+1}M_-(k,\gamma),\gamma}\| \lesssim 1$$

for each grain ω_l, where $\|\cdot\|$ denotes the total variation norm.

Fix ω_l. By rescaling all the spheres supporting σ_k to a common sphere, it suffices to show that the angles subtended by the spherical cap supporting each of the measures

$$\sigma_{k,\omega_l}^{2^j M_-(k,\gamma),\gamma} - \sigma_{k,\omega_l}^{2^{j+1}M_-(k,\gamma),\gamma}$$

are disjoint as k varies. But this follows directly from the definition of these measures, the telescoping nature of the decomposition, and the fact that $d-1 \ge 1$ ensures that for different values of k, the differences of these measures live at different "heights", and the differences of measures at consecutive heights isolate the height at which a certain angular piece first occurs. □

Estimating the L^2 norm of the light term. In view of the preceding calculations and Lemma 2.2(b), it will suffice to show that

$$\Bigg(\sum_{k\ge 0}\Bigg|\sum_{\gamma>\gamma_0:\,k\in\mathcal{K}_\gamma} f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),\gamma}\Bigg|^2\Bigg)^{1/2}$$

is a type-L^2 Calderón–Zygmund term of threshold α and measure 1 (we will no longer need to lose the additional factor of $\operatorname{Log}_3(1/\alpha)$). Because each k is associated to $O(\operatorname{Log}_2(1/\alpha))$ values of γ, it suffices by Cauchy–Schwarz to show that

$$\Bigg(\sum_{k\ge 0}\sum_{\gamma>\gamma_0:\,k\in\mathcal{K}_\gamma}\Big|f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),\gamma}\Big|^2\Bigg)^{1/2}$$

is a type-L^2 Calderón–Zygmund term of threshold α and measure $\operatorname{Log}_2^{-1}(1/\alpha)$. We rearrange this expression as

$$\Bigg(\sum_{\gamma>\gamma_0}\sum_{k\in\mathcal{K}_\gamma} |f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),\gamma}|^2\Bigg)^{1/2}.$$

Since

$$\sum_{\gamma>\gamma_0} \|f_q^\gamma\|_1 \le \|f_q\|_1 \lesssim \alpha,$$

it then suffices by Lemma 2.2(c) to show that, for each $\gamma > \gamma_0$ and $k \in \mathcal{K}_\gamma$, the quantity

$$f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),\gamma}$$

is a type-L^2 Calderón–Zygmund term of threshold α and measure

$$\frac{\mathrm{Log}_2^{-2}(1/\alpha)}{\alpha} \| f_q^\gamma \|_1.$$

In other words, it remains to establish the bound

$$\| f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),\gamma} \|_2^2 \lesssim \alpha \, \mathrm{Log}_2^{-2} \frac{1}{\alpha} \cdot \| f_q^\gamma \|_1. \tag{3-13}$$

The first step is to write

$$\begin{aligned} \| f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),\gamma} \|_2^2 &= \Big\| \sum_{\delta\text{-grains } \omega_l} (\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma}) * f_q^\gamma \chi_{\omega_l} \Big\|_{L^2}^2 \\ &\lesssim \sum_{\delta\text{-grains } \omega_l} \langle (\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma}) * f_q^\gamma \chi_{\omega_l}, \, \sigma_k * f_q^\gamma \rangle \\ &= \sum_{\delta\text{-grains } \omega_l} \langle f_q^\gamma \chi_{\omega_l}, \, \sigma_k * (\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma}) * f_q^\gamma \rangle. \end{aligned} \tag{3-14}$$

Domination of the kernel $\sigma_k * \sigma_k$ by linear combinations of characteristic functions of rectangles. Recall from (2-12) that we have the pointwise estimate

$$\sigma_k * \sigma_k(x) \lesssim 2^{-k(d-1)} |x|^{-1} \chi_{B_k}(x),$$

where $B_k := \{|x| \le 2^{k+1}\}$ is the ball of radius 2^{k+1} around the origin. Inside this ball, we isolate the annulus

$$A_\gamma := \{x : \gamma^{1/(d-1)} \le |x| \le 2^{-100} r_\gamma\},$$

where we recall that the radius r_γ was defined in (3-1).

Thus the kernel $\sigma_k * \sigma_k$ can essentially be decomposed as follows. Fix q and γ, and let $\{c_i\}_{i=0}^N$ be the enumeration of the scales described earlier in (3-4). For $1 \le i \le N$, let $\mathcal{R}_i$ be a collection of $(c_i/c_{i-1})^{d-1}$ many rectangles of dimensions $c_i \times c_i \times \cdots \times c_i \times c_{i-1}$ centered at the origin, with short sides pointing in equally spaced directions, where $\{c_i\}_{i=1}^N$ are the scales described earlier. We may dominate

$$2^{-k(d-1)} |x|^{-1} \chi_{A_\gamma}(x) \lesssim \sum_{i=1}^N 2^{-k(d-1)} c_i^{-(d-1)} c_{i-1}^{d-2} \sum_{R \in \mathcal{R}_i} \chi_R. \tag{3-15}$$

Indeed, for each i, $c_i^{-(d-1)} c_{i-1}^{d-2} \sum_{r \in \mathcal{R}_i} \chi_R$ is essentially of size c_i^{-1} for $|x| \approx c_i$, since for $|x| \approx c_i$ the rectangles are essentially disjoint in the case that $d = 2$, and for general d there are

$$\approx c_i^{-d} \times (c_i/c_{i-1})^{d-1} \times c_i^{d-1} c_{i-1} \approx (c_i/c_{i-1})^{d-2}$$

many rectangles that intersect a given x. By similar reasoning, one sees that for $c_{i-1} \le |x| \le c_i$, we also have $c_i^{-(d-1)} c_{i-1}^{d-2} \sum_{r \in \mathcal{R}_i} \chi_R$ is essentially of size $|x|^{-1}$.

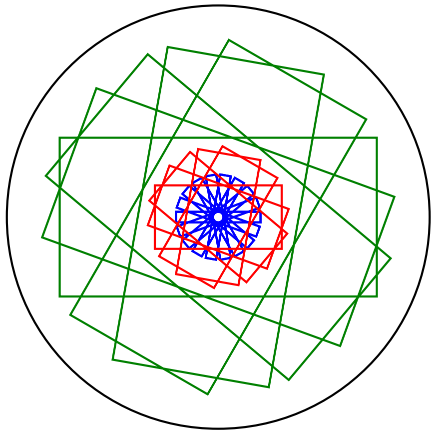

Figure 2. Domination of the kernel $\sigma_k * \sigma_k$ in the sphere of radius $2^{-100} r_\gamma$ centered at the origin.

From (3-15) and (2-12) we have the bound

$$\sigma_k * \sigma_k(x) \lesssim \sum_{i=1}^{N} 2^{-k(d-1)} c_i^{-(d-1)} c_{i-1}^{d-2} \sum_{R \in \mathcal{R}_i} \chi_R + 2^{-k(d-1)} |x|^{-1} \chi_{B_k \setminus A_\gamma}(x); \tag{3-16}$$

see Figure 2.

Eliminating bad rectangles. Now fix some i with $1 \le i \le N$, and suppose that R is a rectangle in $\mathcal{R}_i$ such that

$$\int_{\omega_l + R} f_q^\gamma > c_{i-1} \gamma M_-(k, \gamma).$$

Then by definition, the support of $\sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma}$ contains a spherical cap of angular width $50c_{i-1}/c_i$ with some normal parallel to the short side of R. This implies that $\sigma_k * (\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma})$ is supported outside of the set

$$(R)_1 := \{x \in R : x \ge \tfrac{1}{10} c_i\}.$$

Indeed, for any $x \in R$ in the support of $\sigma_k * (\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma})$ with $|x| \ge \frac{1}{10} c_i$, we require there to exist y on the sphere of radius 2^k centered at the origin such that $x - y$ is also on the sphere of radius 2^k centered at the origin, but outside the cap of angular width $50c_{i-1}/c_i$ with some normal parallel to the short side of R. Suppose toward a contradiction that $x \in R \cap \{z : \frac{1}{10} c_i \le |z| \le 10c_i\}$. But for any such $x - y$, we have that $(x - y) + (R)_1$ lies outside the sphere of radius 2^k, since R will be transverse to the boundary of the sphere at $x - y$ (see Figure 3). Thus we have verified our claim that $\sigma_k * (\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma})$ is supported outside of the set $(R)_1$.

Repeating this process, if

$$\int_{\omega_l + (R \setminus (R)_1)} f_q^\gamma > c_{i-1} \gamma M_-(k, \gamma),$$

then by definition, the support of $\sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma}$ contains a spherical cap of angular width $50 \cdot 2c_{i-1}/c_i$ with some normal parallel to the short side of R. As before, this implies that $\sigma_k * (\sigma_k - \sigma_{k,\omega_l}^{m(k,q,\gamma),\gamma})$ is supported

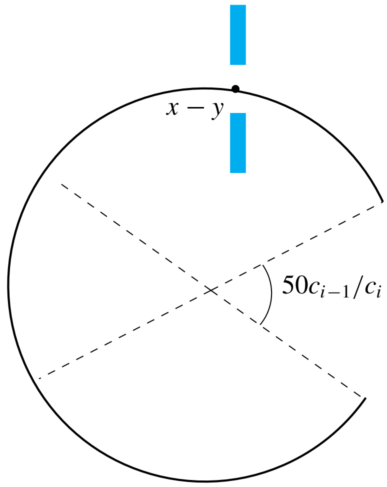

Figure 3. The two blue rectangles represent the set $x-y+(R)_1$.

outside of the set

$$(R)_2 := \left\{x \in R : |x| \geq \tfrac{1}{20}c_i\right\}.$$

Repeating again, if

$$\int_{\omega_l+(R\setminus(R)_2)} f_q^\gamma > c_{i-1}\gamma M_-(k,\gamma),$$

then $\sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma}$ contains a spherical cap of angular width $50\cdot 4c_{i-1}/c_i$ with some tangent parallel to the long side of R. This implies that $\sigma_k * (\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma})$ is supported outside of the set $(R)_3$, where we define

$$(R)_3 := \left\{x \in \mathbb{R} : |x| \geq \tfrac{1}{40}c_i\right\}.$$

We continue this process until some stage L when

$$\int_{(\omega_l+(R\setminus(R)_L))} f_q^\gamma > c_{i-1}\gamma M_-(k,\gamma),$$

and $\sigma_k * (\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma})$ is supported outside of the set

$$(R)_L := \left\{x \in \mathbb{R} : |x| \geq \frac{1}{2^{L-1}\times 10}c_i\right\}.$$

(Note that this must eventually happen if $\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma}$ is not identically 0, since the set $(R)_L$ can potentially increase by continuing this process up to $\{x \in R : |x| \geq 10c_{i-1}\}$, which would imply that $\sigma_k - \sigma_{k,\omega_l}^{m(k,q,\gamma),\gamma}$ is identically 0.)

For convenience, we summarize the above argument in the following lemma.

Lemma 3.7. *Fix a δ-grain ω_l. For any rectangle $R \in \mathcal{R}_i$, there is a subset $(R)_L \subset R$ such that*

$$\int_{\omega_l+(R\setminus(R)_L)} |f_q^\gamma| \lesssim c_{i-1}\gamma M_-(k,\gamma)$$

and $\sigma_k * (\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma})$ is supported outside of the set $(R)_L$.

Finishing up the proof. Lemma 3.7 and (3-16) implies that, for each δ-grain ω_l and each rectangle $R \in \mathcal{R}_i$, there is a function h_R with $\int |h_R| \leq c_{i-1} M_-(k,\gamma)\gamma$ so that by (3-14) we may dominate

$$\begin{aligned}\| f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),\gamma} \|_2^2 &\lesssim \sum_{\delta-\text{grains } \omega_l} \langle f_q^\gamma \chi_{\omega_l},\, \sigma_k * (\sigma_k - \sigma_{k,\omega_l}^{M_-(k,\gamma),\gamma}) * f_q^\gamma \rangle \\ &\lesssim \sum_{\delta-\text{grains } \omega_l} \langle f_q^\gamma \chi_{\omega_l}, 2^{-k(d-1)} |x|^{-1} \chi_{B_k \setminus A_\gamma} * f_q^\gamma \rangle \\ &\qquad + \sum_{\delta-\text{grains } \omega_l} \sum_{i=1}^{N} 2^{-k(d-1)} \frac{c_{i-1}^{d-2}}{c_i^{d-1}} \sum_{R \in \mathcal{R}_i} \langle f_q^\gamma \chi_{\omega_l}, \chi_R * h_R \rangle. \end{aligned} \tag{3-17}$$

It is not difficult to show that

$$\langle f_q^\gamma \chi_{\omega_l}, 2^{-k(d-1)} |x|^{-1} \chi_{B_k \setminus A_\gamma} * f_q^\gamma \rangle \lesssim \alpha \operatorname{Log}_2^{-2} \frac{1}{\alpha} \| f_q^\gamma \chi_{\omega_l} \|_1. \tag{3-18}$$

Indeed, by Young's inequality it would suffice to show that

$$2^{-k(d-1)} \int_{|x-y| \in B_k \setminus A_\gamma} \frac{f_q^\gamma(y)}{|x-y|} \, dy \lesssim \alpha \operatorname{Log}_2^{-2} \frac{1}{\alpha} \tag{3-19}$$

for any x. From (2-13) we have

$$\int_{|x-y| \approx 2^l} \frac{f_q^\gamma(y)}{|x-y|} \, dy \lesssim \min(\gamma, 2^{l(d-1)})$$

so by dyadic decomposition we may bound the left-hand side of (3-19) by

$$2^{-k(d-1)} \Bigg(\sum_{2^l \lesssim \gamma^{1/(d-1)}} 2^{l(d-1)} + \sum_{r_\gamma \lesssim 2^l \lesssim 2^k} \gamma \Bigg),$$

which we can sum to

$$\lesssim 2^{-k(d-1)} \gamma \operatorname{Log} \frac{2^k}{r_\gamma} \lesssim 2^{-k(d-1)} \gamma \operatorname{Log} \frac{2^{k(d-1)}}{\gamma/\alpha}$$

thanks to (3-1). By (3-2), we have

$$\frac{2^{k(d-1)}}{\gamma/\alpha} \geq \operatorname{Log}_2^{100} \frac{1}{\alpha},$$

giving (3-18) as claimed.

This gives a satisfactory bound for the first term occurring in the right-hand side of (3-17). To bound the second term, we observe that since $\int |h_R| \lesssim c_{i-1} M_-(k,\gamma)\gamma$, we have

$$\langle f_q^\gamma \chi_{\omega_l}, \chi_R * h_R \rangle \lesssim c_{i-1} M_-(k,\gamma)\gamma \| f_q^\gamma \chi_{\omega_l} \|_1. \tag{3-20}$$

Combining (3-17), (3-18), and (3-20) and summing over all i and all δ-grains ω_l, using the fact that the cardinality of $\mathcal{R}_i$ is $\lesssim (c_i/c_{i-1})^{d-1}$ and $N \lesssim \operatorname{Log}_2(1/\alpha)$, and recalling the definition (3-10) of $M_-(k,\gamma)$, we obtain

$$\| f_q^\gamma * \sigma_k - g_k^{M_-(k,\gamma),k,\gamma} \|_2^2 \lesssim \alpha \operatorname{Log}_2^{-2} \frac{1}{\alpha} \| f_q^\gamma \|_1,$$

which is the desired L^2 bound. This completes the proof of (3-13), and hence Propositions 1.1, 1.2, and Theorem 1.3.

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References

[Calderón 1979] C. P. Calderón, "Lacunary spherical means", *Illinois J. Math.* **23**:3 (1979), 476–484. MR

[Christ 1988] M. Christ, "Weak type (1, 1) bounds for rough operators", *Ann. of Math.* (2) **128**:1 (1988), 19–42. MR Zbl

[Christ and Stein 1987] M. Christ and E. M. Stein, "A remark on singular Calderón–Zygmund theory", *Proc. Amer. Math. Soc.* **99**:1 (1987), 71–75. MR Zbl

[Coifman and Weiss 1978] R. R. Coifman and G. Weiss, "Review of *Littlewood–Paley and multiplier theory* by Edwards and Gaudry", *Bull. Amer. Math. Soc.* **84**:2 (1978), 242–250. MR

[Duoandikoetxea and Rubio de Francia 1986] J. Duoandikoetxea and J. L. Rubio de Francia, "Maximal and singular integral operators via Fourier transform estimates", *Invent. Math.* **84**:3 (1986), 541–561. MR Zbl

[Seeger et al. 2003] A. Seeger, T. Tao, and J. Wright, "Pointwise convergence of lacunary spherical means", pp. 341–351 in *Harmonic analysis at Mount Holyoke* (South Hadley, MA, 2001), edited by W. Beckner et al., Contemp. Math. **320**, Amer. Math. Soc., Providence, RI, 2003. MR Zbl

[Seeger et al. 2004] A. Seeger, T. Tao, and J. Wright, "Singular maximal functions and Radon transforms near L^1", *Amer. J. Math.* **126**:3 (2004), 607–647. MR Zbl

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