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GLOBAL WELL-POSEDNESS FOR A SYSTEM OF QUASILINEAR WAVE EQUATIONS ON A PRODUCT SPACE

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We consider a system of quasilinear wave equations on the product space $\mathbb{R}^{1+3} \times \mathbb{S}^1$, which we want to see as a toy model for the Einstein equations with additional compact dimensions. We show global existence of solutions for small and regular initial data with polynomial decay at infinity. The method combines energy estimates on hyperboloids inside the light cone and weighted energy estimates outside the light cone.

1. Introduction

We address the problem of global existence of small solutions to a certain class of quasilinear systems of wave equations on the product space $\mathbb{R}^{1+3} \times \mathbb{S}^1$. Let $\square_{x,y} = -\partial_t^2 + \Delta_x + \partial_y^2$ denote the d'Alembertian operator in the (t, x, y) -variables, where $t \in \mathbb{R}$ is the time coordinate, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ are the Cartesian coordinates and $y \in \mathbb{S}^1$ is the periodic coordinate. The system we consider has the form

$$\begin{cases} \square_{x,y} u + u \partial_y^2 u = \sum_{1 \leq i, j \leq 2} N_1(w_i, w_j), \\ \square_{x,y} v + u \partial_y^2 v = \sum_{1 \leq i, j \leq 2} N_2(w_i, w_j), \end{cases} \quad (t, x, y) \in \mathbb{R}^{1+3} \times \mathbb{S}^1, \quad (1-1)$$

with initial conditions set at time $t_0 = 2$

$$(u, v)(2, x, y) = (\phi_0, \psi_0)(x, y), \quad (\partial_t u, \partial_t v)(2, x, y) = (\phi_1, \psi_1)(x, y). \quad (1-2)$$

The nonlinearities $N_1(\cdot, \cdot)$, $N_2(\cdot, \cdot)$ are linear combinations of the quadratic null forms

$$\begin{aligned} Q_0(\phi, \psi) &= \partial_t \phi \partial_t \psi - \nabla_x \phi \cdot \nabla_x \psi, \\ Q_{ij}(\phi, \psi) &= \partial_{x_i} \phi \partial_{x_j} \psi - \partial_{x_j} \phi \partial_{x_i} \psi, \quad 1 \leq i < j \leq 3, \\ Q_{0i}(\phi, \psi) &= \partial_t \phi \partial_{x_i} \psi - \partial_{x_i} \phi \partial_t \psi, \quad 1 \leq i \leq 3, \end{aligned} \quad (1-3)$$

and $w_1, w_2 = \{u, v\}$ denote the two-component solutions.

The main result we present in this paper asserts the global existence of solutions to (1-1) when the initial data are small and localized real functions. Our result also extends to semilinear interactions of the form $\partial w_i \cdot \partial_y w_j$ with ∂ being any of the derivatives in the (t, x, y) -variables.

1A. Notation. Below is a summary of some notation we will use throughout:

- $r = |x|$ is the radial coordinate in the x -variables.
- We use the Einstein summation convention and take the sum over repeated indexes.

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- ∂_j denotes the derivative ∂_{x_j} in the j -th direction for $j = \overline{1, 3}$. Sometimes we may use the notation ∂_0 for ∂_t and ∂_4 for ∂_y .
- $\nabla_x = (\partial_1, \partial_2, \partial_3)$ denotes the gradient in the spatial variable x , $\nabla_{xy} = (\partial_1, \partial_2, \partial_3, \partial_4)$ denotes the gradient in the full set of spatial variables (x, y) .
- ∂_x denotes any of the derivatives in $\{\partial_j : j = \overline{1, 3}\}$.
- ∂_{xy} denotes any of the derivatives in $\{\partial_j : j = \overline{1, 4}\}$. Analogously for ∂_{tx} and ∂_{txy} . We will use ∂ and ∂_{txy} interchangeably.
- $\square_{x,y} = -\partial_t^2 + \Delta_x + \partial_y^2$ and $\square_x = -\partial_t^2 + \Delta_x$.
- Given a multi-index $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_4) \in \mathbb{N}^5$, its length is computed classically as $|\alpha| = \sum_{i=0}^4 \alpha_i$ and $\partial^\alpha = \partial_0^{\alpha_0} \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3} \partial_4^{\alpha_4}$. We use the notation ∂_x^α for $\partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$ and it is then clear what ∂_{xy}^α or ∂_{tx}^α stand for.
- More generally, given a family of vector fields $\{\Gamma_1, \dots, \Gamma_n\}$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\Gamma^\alpha = \Gamma_1^{\alpha_1} \dots \Gamma_n^{\alpha_n}$. Sometimes we make an abuse of notation and write Γ^k (resp. $\Gamma^{\leq k}$) instead of $\sum_{\alpha:|\alpha|=k} \Gamma^\alpha$ (resp. $\sum_{\alpha:|\alpha|\leq k} \Gamma^\alpha$).

1B. Motivation and a brief history. Our interest in studying nonlinear wave equations on product spaces comes from the theory of supergravity (SUGRA) in physics and more precisely from the Kaluza–Klein theory, which represents the classical approach to the unification of general relativity with electromagnetism and more generally with gauge fields; see the original works [Kaluza 1921; Klein 1926]. The Kaluza–Klein approach considers general relativity in $3 + 1 + d$ dimensions with space-time factorizing as

$$\mathcal{M}^{(3+1+d)} = \mathbb{R}^{1+3} \times K,$$

where K is a compact d -manifold referred to as *internal space*. In the simplest case $d = 1$ -dimensional gravity is compactified on a circle ($K = \mathbb{S}^1$) to obtain at low energies a coupled Einstein–Maxwell scalar system in $3+1$ space-time dimensions. Kaluza–Klein space-times $\mathbb{R}^{1+3} \times \mathbb{S}^1$ have been studied the influential work [Witten 1982], where he proves instability at the semiclassical level but provides heuristic arguments for classical stability. The first result proving the classical stability of the Kaluza–Klein theory is obtained in [Wyatt 2018], where only perturbations depending on the noncompact coordinates are considered, using tools developed in [Lindblad and Rodnianski 2010]. More general space-times with supersymmetric compactifications $\mathcal{M} = \mathbb{R}^{1+n} \times K$ have recently been studied by Andersson, Blue, Wyatt, and Yau [Andersson et al. 2023]. The space-times \mathcal{M} are equipped with the metric

$$\hat{g} = \eta_{\mathbb{R}^{1+n}} + k,$$

where $\eta_{\mathbb{R}^{1+n}}$ is the Minkowski metric in \mathbb{R}^{1+n} and k is such that (K, k) is a compact Ricci-flat Riemannian manifold having a cover that admits a spin structure and a nonzero parallel spinor. A global stability result is proved in [Andersson et al. 2023] under the assumption $n \geq 9$ and for Cauchy data that are Schwarzschild near infinity, but it is conjectured that these conditions can be relaxed and that space-times with a supersymmetric compactification and $n = 3$ are nonlinearly stable. We also briefly mention a result on the stability of cosmological Kaluza–Klein space-times (where the Minkowski space-time is

replaced by the four-dimensional Milne universe) under zero-modes perturbations by Branding, Fajman and Kröncke [Branding et al. 2019] and a stability result for semilinear wave equations on the cosmological Kaluza–Klein background in [Wang 2021].

In both the aforementioned works [Wyatt 2018; Andersson et al. 2023], as well as in many other works concerning the global stability problem for Einstein equations, the use of the so-called *wave-coordinates* allows one to write the Einstein equations as a system of quasilinear wave equations on the metric $g = (g^{\alpha\beta})_{\alpha\beta}$

$$g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} = P_{\mu\nu}(\partial g, \partial g) + G_{\mu\nu}(g)(\partial g, \partial g), \tag{1-4}$$

where the sum is taken over repeated indexes, $P_{\mu\nu}$ are quadratic forms and $G_{\mu\nu}(g)(\partial g, \partial g)$ contain cubic terms. In the present paper we focus on a toy model for the Einstein equations on $\mathbb{R}^{1+3} \times \mathbb{S}^1$ that only keeps a selection of terms from (1-4), precisely the semilinear terms with null structure and the quasilinear term $g^{yy} \partial_y^2$ where y is the periodic coordinate on \mathbb{S}^1 . Our goal here is in fact to study the global well-posedness for quasilinear wave equations on the product space $\mathbb{R}^{1+3} \times \mathbb{S}^1$ without having to make use of the full structure of the Einstein equations. The unknown u in system (1-1) plays the role of the coefficient g_{yy} and the unknown v encodes any other metric coefficient $g_{\mu\nu}$.

We mention here that wave equations on product spaces also appear in other contexts, for instance when studying the propagation of waves along infinite homogeneous waveguides; see [Lesky and Racke 2003; Metcalfe et al. 2005; Metcalfe and Stewart 2008; Ettinger 2015].

One key observation when studying the small data global well-posedness problem for wave equations on product spaces of the form $\mathbb{R}^{1+3} \times \mathbb{T}^d$ is that they are closely related to infinite systems coupling wave and Klein–Gordon equations on the flat space \mathbb{R}^{1+3} . In fact, if $W = W(t, x, y)$ is solution to

$$\square_{x,y} W = F, \quad (t, x, y) \in \mathbb{R}^{1+3} \times \mathbb{T}^d,$$

for some source term F , its Fourier modes in the periodic direction $\{W_k(t, x)\}_{k \in \mathbb{Z}}$ solve the following equations on the flat space \mathbb{R}^{1+3} :

$$(-\partial_t^2 + \Delta_x) W_k - |k|^2 W_k = \frac{1}{\text{Vol}(\mathbb{T}^d)} \int_{\mathbb{T}^d} e^{-iy \cdot k} F dy, \quad (t, x) \in \mathbb{R}^{1+3}.$$

In particular, the zero-mode W_0 is a solution to a wave equation and any other nonzero mode W_k solves a Klein–Gordon equation of mass $|k|$.

The global well-posedness for systems coupling a finite number of wave and Klein–Gordon equations on the flat 3+1 space-time with small data have largely been studied. We cite the initial results by Georgiev [1990] and Katayama [2012], followed by LeFloch and Ma [2014], Wang [2015; 2020] and Ionescu and Pausader [2019], who study such systems as a model for the full Einstein–Klein–Gordon equations; the global stability of the full problem is successively proved in [LeFloch and Ma 2016; Ionescu and Pausader 2022]. In [LeFloch and Ma 2014; Wang 2015] global well-posedness is proved for compactly supported initial data and quadratic quasilinear nonlinearities that satisfy some suitable conditions, including the *null condition* of [Klainerman 1986] for self-interactions between the wave components of the solution. An idea used in these works is that of employing hyperbolic coordinates in

the forward light cone; this was first introduced in [Klainerman 1985] for Klein–Gordon equations and in [Tataru 2002] in the wave context, and later reintroduced in [LeFloch and Ma 2014] under the name of *hyperboloidal foliation method*. In [Ionescu and Pausader 2019] global regularity and scattering is proved in the case of small smooth initial data that decay at a suitable rate at infinity and nonlinearities that do not verify the null condition but present a particular resonant structure. We also cite [Dong and Wyatt 2020a], which proves global well-posedness for a quadratic semilinear interaction in which there are no derivatives on the massless wave component. Other related results are [Bachelot 1988; Ozawa et al. 1995; Tsutaya 1996; Tsutsumi 2003a; 2003b; Klainerman et al. 2020; Dong et al. 2021] and see [Ma 2017a; 2017b; 2017c; 2020; 2021; Stingo 2023; Ifrim and Stingo 2019; Dong and Wyatt 2020b] for results about wave-Klein–Gordon systems in lower dimensions.

Our goal in this paper is to prove the global stability for (1-1) in the case where the initial data are not compactly supported but only have a mild polynomial decay at infinity. Our approach makes use of the vector field method in [Christodoulou and Klainerman 1993] and follows [LeFloch and Ma 2016] in that a big portion of the estimates recovered for the solution in the interior of the cone $\{t = r + 1\} \times \mathbb{S}^1$ are estimates on hyperboloids. The main difference with [LeFloch and Ma 2014] is that the interior estimates need to be coupled with exterior estimates in the region outside the cone. Those are weighted energy estimates, also used in [Lindblad and Rodnianski 2010], that we have to propagate in time.

1C. The main result. In order to describe the initial data we consider for our problem we introduce the energy space \mathcal{H}^0 endowed with the norm

$$\|(u[t], v[t])\|_{\mathcal{H}^0}^2 := \|u\|_{H_{xy}^1}^2 + \|u_t\|_{L^2}^2 + \|v\|_{H_{xy}^1}^2 + \|v_t\|_{L^2}^2$$

and the higher-order energy spaces \mathcal{H}^n for $n \geq 1$ endowed with the norm

$$\|(u[t], v[t])\|_{\mathcal{H}^n}^2 := \sum_{|\alpha| \leq n} \|(\partial_{xy}^\alpha u[t], \partial_{xy}^\alpha v[t])\|_{\mathcal{H}^0}^2.$$

In the above definition, ∂_{xy}^α is a short notation for $\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \partial_y^{\alpha_4}$ given $\alpha = (\alpha_1, \dots, \alpha_4)$ and we use the following notation for the Cauchy data in (1-2) at time t :

$$(u[t], v[t]) := (u(t), u_t(t), v(t), v_t(t)).$$

The global well-posedness result that is the object of this paper is proved under some decay assumptions on the initial data. A preliminary version of our main theorem states the following:

Theorem 1. *Assume that the initial data $(u[2], v[2])$ for (1-1) satisfy*

$$\sum_{|\alpha|=0}^5 \|\langle x \rangle^{\leq |\alpha| + \kappa/2} \partial_{xy}^\alpha (u[2], v[2])\|_{\mathcal{H}^0} \leq \epsilon \ll 1$$

for some positive fixed κ and $\langle x \rangle = \sqrt{1 + |x|^2}$. Then the system (1-1) is globally well-posed in the space \mathcal{H}^5 .

We remark here that our choice to set the initial data at time $t_0 = 2$ over the conventional $t_0 = 0$ is more convenient for our computations and comes at no expense as the system (1-1) is invariant under time translations.

1D. The wave-Klein-Gordon structure. The Cauchy problem (1-1)–(1-2) can be written in a more compact form as a vector equation for the unknown $W = (u, v)^T$

$$\square_{x,y}W + u\partial_y^2W = N(W, W), \tag{1-5}$$

with data

$$W|_{t=2} = \Phi_0, \quad \partial_t W|_{t=2} = \Phi_1, \tag{1-6}$$

where $\Phi_0 = (\phi_0, \psi_0)^T$ and $\Phi_1 = (\phi_1, \psi_1)^T$ and

$$N(W, W) = \sum_{1 \leq i, j \leq 2} \begin{pmatrix} N_1(w_i, w_j) \\ N_2(w_i, w_j) \end{pmatrix}.$$

The projection of W onto the periodic direction y reveals the nature of (1-5) as a system coupling one (vector) wave equation with an infinite sequence of (vector) Klein-Gordon equations of variable mass $|k|\sqrt{1+u_0}$ with $k \in \mathbb{Z}^*$. If we denote by $W_k = (u_k, v_k)^T$ the projection of W onto the k -th frequency

$$W_k(t, x) = \int_{\mathbb{S}^1} e^{-iky} W(t, x, y) \frac{dy}{2\pi}, \quad k \in \mathbb{Z},$$

we see that the functions $\{W_k\}_k$ satisfy the coupled system

$$\begin{cases} (-\partial_t^2 + \Delta_x)W_k - |k|^2(1+u_0)W_k = \int_{\mathbb{S}^1} e^{-iky} N(W, W) \frac{dy}{2\pi} - \int_{\mathbb{S}^1} e^{-iky}(u-u_0)\partial_y^2W \frac{dy}{2\pi}, \\ k \in \mathbb{Z}. \end{cases}$$

The zero mode W_0 is solution to a wave equation, while any other nonzero mode W_k is solution to a Klein-Gordon equation of variable mass $|k|\sqrt{1+u_0}$. This distinction will be fundamental for our analysis and we will often work throughout the paper with the decomposition of W

$$W = W_0 + \tilde{W}, \quad \tilde{W}(t, x, y) = \sum_{k \neq 0} e^{iky} W_k(t, x), \tag{1-7}$$

so that (1-5) is equivalent to the system

$$\begin{cases} (-\partial_t^2 + \Delta_x)W_0 = \int_{\mathbb{S}^1} N(W, W) \frac{dy}{2\pi} + \int_{\mathbb{S}^1} \partial_y u \partial_y \tilde{W} \frac{dy}{2\pi}, \\ (-\partial_t^2 + \Delta_x)\tilde{W} + (1+u)\partial_y^2 \tilde{W} = N(W, W) - \int_{\mathbb{S}^1} N(W, W) \frac{dy}{2\pi} - \int_{\mathbb{S}^1} \partial_y u \partial_y \tilde{W} \frac{dy}{2\pi}. \end{cases} \tag{1-8}$$

Observe that the source term F_0 in the equation of W_0 does not contain mixed interactions:

$$F_0 = N(W_0, W_0) + \int_{\mathbb{S}^1} N(W, W) \frac{dy}{2\pi} + \int_{\mathbb{S}^1} \partial_y u \cdot \partial_y \tilde{W} \frac{dy}{2\pi}. \tag{1-9}$$

We are now able to state a more precise version of the main theorem.

Theorem 2. *Assume that for some positive fixed κ the initial data for (1-5) satisfy*

$$\sum_{|\alpha|=0}^5 \|x^{\leq |\alpha| + \kappa/2} \partial_{xy}^\alpha W[2]\|_{\mathcal{H}^0} \leq \epsilon \ll 1.$$

Then the solution W to (1-5)–(1-6) exists globally in time in \mathcal{H}^5 and the two components of the solution, $W_0 = \int_{\mathbb{S}^1} W (dy/2\pi)$ and $\bar{W} = W - W_0$, satisfy the pointwise bounds

$$\begin{aligned} |\partial_{tx}^\alpha W_0(t, x)| &\lesssim \epsilon \langle t + |x| \rangle^{-1} \langle t - |x| \rangle^{-1/2}, \quad |\alpha| = \overline{1, 3}, \\ \|\partial_y^j \partial_{tx}^\alpha W(t, x, \cdot)\|_{L_y^2(\mathbb{S}^1)} &\lesssim \epsilon \langle t + |x| \rangle^{-3/2}, \quad j = \overline{0, 1}, \quad |\alpha| = \overline{0, 1}, \\ \|\partial_y^j \partial_{tx}^\alpha \bar{W}(t, x, \cdot)\|_{L_y^2(\mathbb{S}^1)} &\lesssim \epsilon \langle t + |x| \rangle^{-1} \langle t - |x| \rangle^{-1/2}, \quad j = \overline{0, 1}, \quad |\alpha| = 2. \end{aligned}$$

1E. Vector fields. In order to describe the global bounds and decay properties of the solution $W = (u, v)^T$ to (1-5)–(1-6) we need to introduce the family of Killing vector fields associated to our problem. Those are the vector fields that exactly commute with $\square_{x,y}$:

$$\partial_0, \partial_1, \partial_2, \partial_3, \partial_4, \tag{1-10}$$

$$\Omega_{ij} = x_j \partial_i - x_i \partial_j, \quad 1 \leq i < j \leq 3, \tag{1-11}$$

$$\Omega_{0i} = t \partial_i + x_i \partial_t, \quad i = \overline{1, 3}. \tag{1-12}$$

The expressions in (1-10) correspond to the translations in the coordinate directions; (1-11) correspond to the Euclidean rotations in the x -coordinates; (1-12) are the hyperbolic rotations, also called boosts. We also introduce the conformal scaling vector field

$$\mathcal{S} = t \partial_t + x \cdot \nabla_x, \tag{1-13}$$

which is not Killing for (1-5) but will appear later in the analysis of the problem. We refer to (1-11) and (1-12) as *Klainerman vector fields* and generally denote them by Z :

$$Z := \{\Omega_{ij}, \Omega_{0i}\}.$$

We denote the full set of admissible vector fields for $\square_{x,y}$ as

$$\mathcal{Z} := \{\partial_0, \partial_1, \partial_2, \partial_3, \partial_4, \Omega_{ij}, \Omega_{0i}\} \tag{1-14}$$

and for any multi-index $\gamma = (\alpha, \beta)$ we define

$$\mathcal{Z}^\gamma = \partial^\alpha Z^\beta.$$

For any two nonnegative integers k, n with $k \leq n$ we say that the multi-index $\gamma = (\alpha, \beta)$ is of type (n, k) if $|\gamma| = |\alpha| + |\beta| \leq n$ and $|\beta| \leq k$, in other words if there are at most k Klainerman vector fields among the $|\gamma|$ admissible vector fields in the product \mathcal{Z}^γ .

1F. The null structure. The nonlinearities we consider in this work are linear combinations of the classical quadratic null forms (1-3). An important feature of the null forms is that they are combinations of three types of products and can be expressed schematically as

$$N(\phi, \psi) = \bar{\partial} \phi \cdot \partial \psi + \partial \phi \cdot \bar{\partial} \psi + \frac{t-r}{t} \partial \phi \cdot \partial \psi, \tag{1-15}$$

where $\bar{\partial}_j = t^{-1} \Omega_{0j}$ are the rescaled hyperbolic rotations, or also as

$$N(\phi, \psi) = \mathcal{T} \phi \cdot \partial \psi + \partial \phi \cdot \mathcal{T} \psi, \tag{1-16}$$

where $\mathcal{T}_j = \partial_j + (x_j/r)\partial_t$ for $j = \overline{1, 3}$ are the vector fields tangent to the cones $\{t - r = \text{const}\}$. The \mathcal{T} vector fields are related to the boosts in general via the relation

$$\mathcal{T}_j = \frac{1}{t}\Omega_{0j} + \frac{(t-r)x_j}{t} \frac{1}{r}\partial_t.$$

We will often make use of the two representations (1-15) and (1-16) to recover suitable pointwise and energy estimates for the solution, as they allow us to recover additional decay for the $W_0 \times W_0$ interactions.

1G. The interior and exterior region. The proof of our main theorem is based on the combination of a classical local existence result with a bootstrap argument. We will perform such argument separately in the two regions in which we decompose our space-time:

$$\begin{aligned} \text{interior region } \mathcal{D}^{\text{in}} &:= \{(t, x) : t \geq 2 \text{ and } |x| < t - 1\} \times \mathbb{S}^1, \\ \text{exterior region } \mathcal{D}^{\text{ex}} &:= \{(t, x) : t \geq 2 \text{ and } |x| \geq t - 1\} \times \mathbb{S}^1. \end{aligned}$$

In order to describe our bootstrap assumptions we first introduce some notation. Given any hyperboloid \mathcal{H}_s in $\mathbb{R}^{1+3} \times \mathbb{S}^1$ we denote by $\mathcal{H}_s^{\text{in}}$ (resp. $\mathcal{H}_s^{\text{ex}}$) the branch of \mathcal{H}_s contained in the interior region \mathcal{D}^{in} (resp. in the exterior region \mathcal{D}^{ex}):

$$\begin{aligned} \mathcal{H}_s &= \{(t, x) : s^2 = t^2 - |x|^2\} \times \mathbb{S}^1, \\ \mathcal{H}_s^{\text{in}} &:= \{(t, x, y) \in \mathcal{H}_s : t \geq 2 \text{ and } |x| < \frac{1}{2}(s^2 - 1)\}, \\ \mathcal{H}_s^{\text{ex}} &:= \{(t, x, y) \in \mathcal{H}_s : t \geq 2 \text{ and } |x| \geq \frac{1}{2}(s^2 - 1)\}. \end{aligned}$$

Moreover we denote by $\mathcal{H}_{[2,s]}^{\text{in}}$ the hyperbolic interior region above $\mathcal{H}_2^{\text{in}}$ and below $\mathcal{H}_s^{\text{in}}$, and by $\mathcal{H}_{[2,s]}^{\text{ex}}$ the portion of the exterior region below $\mathcal{H}_s^{\text{ex}}$ for any $s \geq 2$ (see Figure 1 and Figure 2, which is displayed in Section 3):

$$\begin{aligned} \mathcal{H}_{[2,s]}^{\text{in}} &:= \{(t, x, y) \in \mathcal{D}^{\text{in}} : 2 \leq t^2 - |x|^2 \leq s^2\}, \\ \mathcal{H}_{[2,s]}^{\text{ex}} &:= \{(t, x, y) \in \mathcal{D}^{\text{ex}} : t^2 - |x|^2 \leq s^2\}. \end{aligned}$$

In the interior region the bootstrap assumptions will be energy bounds on the truncated hyperboloids $\mathcal{H}_s^{\text{in}}$ for $s \geq 2$ and pointwise bounds on the Z -derivative of the zero mode of the solution. The local wellposedness theory for this problem ensures the existence and smallness of the solution $W = (u, v)^T$ to (1-5)–(1-6) up to the interior hyperboloid $\mathcal{H}_2^{\text{in}}$; hence our goal will be to propagate the bootstrap assumptions in the hyperbolic interior region above $\mathcal{H}_2^{\text{in}}$

$$\mathcal{H}_{[2,\infty)}^{\text{in}} := \{(t, x, y) \in \mathcal{D}^{\text{in}} : 2 \leq t^2 - |x|^2\}.$$

In the exterior region the bootstrap assumptions will instead be weighted energy bounds on the constant time slices Σ_t^{ex} which foliate \mathcal{D}^{ex} for $t \geq 2$,

$$\Sigma_t^{\text{ex}} := \{x \in \mathbb{R}^3 : |x| \geq t - 1\} \times \mathbb{S}^1.$$

We warn the reader that throughout the paper we will work with functions defined on the product space $\mathbb{R}^{1+3} \times \mathbb{S}^1$ as well as with functions not depending on the y -variable and defined on the flat space \mathbb{R}^{1+3} .

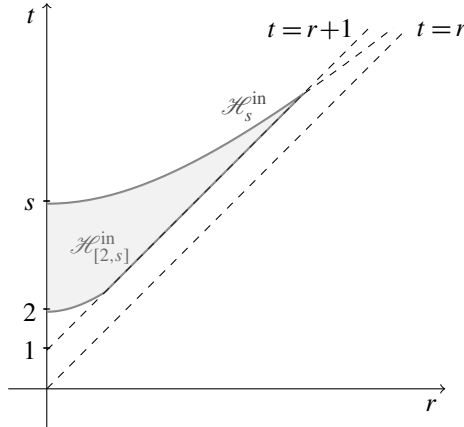


Figure 1. Vertical section of the region $\mathcal{H}_{[2,s]}^{\text{in}}$ projected onto \mathbb{R}^{1+3} .

With the purpose of keeping notation as light as possible, a region in $\mathbb{R}^{1+3} \times \mathbb{S}^1$ and its projection onto \mathbb{R}^{1+3} will have the same name.

1H. The energy functionals. In the interior region we aim to propagate a priori energy bounds on truncated hyperboloids. The interior energy functional on $\mathcal{H}_s^{\text{in}}$ associated to the linear counterpart of (1-5) is

$$\begin{aligned}
 E^{\text{in}}(s, W) &:= \frac{1}{2\pi} \iint_{\mathcal{H}_s^{\text{in}}} \left(\frac{s}{t}\right)^2 |\partial_t W|^2 + |\bar{\partial} W|^2 + |\partial_y W|^2 \, dx \, dy \\
 &= \frac{1}{2\pi} \iint_{\mathcal{H}_s^{\text{in}}} \left(\frac{s}{t}\right)^2 |\partial_x W|^2 + t^{-2} |\mathcal{S}W|^2 + t^{-2} \sum_{1 \leq i < j \leq 3} |\Omega_{ij} W|^2 + |\partial_y W|^2 \, dx \, dy,
 \end{aligned}
 \tag{1-17}$$

where $|\bar{\partial} W|^2$ stands for $\sum_{i=1}^3 |\bar{\partial}_i W|^2$. The above functional hence controls the square of the norms

$$\|(s/t)\partial W\|_{L^2(\mathcal{H}_s^{\text{in}})}, \quad \|\bar{\partial} W\|_{L^2(\mathcal{H}_s^{\text{in}})}, \quad \|\partial_y W\|_{L^2(\mathcal{H}_s^{\text{in}})}, \quad t^{-1}\|\mathcal{S}W\|_{L^2(\mathcal{H}_s^{\text{in}})}, \quad t^{-1}\|ZW\|_{L^2(\mathcal{H}_s^{\text{in}})}.$$

From Parseval’s identity we have the decomposition

$$E^{\text{in}}(t, W) = E^{\text{in}}(t, W_0) + E^{\text{in}}(t, W),$$

where $E^{\text{in}}(t, W)$ is defined by replacing W with W in (1-17) and

$$\begin{aligned}
 E^{\text{in}}(s, W_0) &:= \int_{\mathcal{H}_s^{\text{in}}} \left(\frac{s}{t}\right)^2 |\partial_t W_0|^2 + |\bar{\partial} W_0|^2 \, dx \\
 &= \int_{\mathcal{H}_s^{\text{in}}} \left(\frac{s}{t}\right)^2 |\partial_x W_0|^2 + t^{-2} |\mathcal{S}W_0|^2 + t^{-2} \sum_{1 \leq i < j \leq 3} |\Omega_{ij} W_0|^2 \, dx.
 \end{aligned}$$

We also introduce and work with the conformal energy functional on truncated hyperboloids associated to the linear flat wave equation on \mathbb{R}^{1+3} . This is the functional defined as

$$E^{\text{c},\text{in}}(s, W_0) := \int_{\mathcal{H}_s^{\text{in}}} \frac{1}{t^2} |K W_0 + 2t W_0|^2 + \frac{s^2}{t^2} \sum_{i=1}^3 |\Omega_{0i} W_0|^2 \, dx,
 \tag{1-18}$$

where $K = (t^2 + r^2)\partial_t + 2rt\partial_r$ is the Morawetz multiplier. Since $\Omega_{ij} = (x_i/t)\Omega_{0j} - (x_j/t)\Omega_{0i}$, this functional controls the square of the norms

$$t^{-1}\|K W_0 + 2W_0\|_{L^2(\mathcal{H}_s^{\text{in}})} \quad \text{and} \quad \|(s/t)Z W_0\|_{L^2(\mathcal{H}_s^{\text{in}})}.$$

We point out that conformal energies on hyperboloids have also been used in other works; see for instance [Ma and Huang 2017; Wong 2017].

In the exterior region we will describe the evolution of (1-5) by means of the weighted energy functional

$$E^{\text{ex},\kappa}(t, W) = \frac{1}{2\pi} \iint_{\Sigma_t^{\text{ex}}} (2+r-t)^{\kappa+1} [|\partial_t W|^2 + |\nabla_x W|^2 + |\partial_y W|^2] dx dy \quad (1-19)$$

and of stronger norm $X_{T_0}^{\text{ex},\kappa}$

$$\|W\|_{X_{T_0}^{\text{ex},\kappa}}^2 := \sup_{t \in [2, T_0]} E^{\text{ex},\kappa}(t, W) + \frac{(1+\kappa)}{2\pi} \int_2^{T_0} \iint_{\Sigma_t^{\text{ex}}} (2+r-t)^\kappa (|\mathcal{S}W|^2 + |\partial_y W|^2) dx dy dt, \quad (1-20)$$

where $|\mathcal{S}W|^2$ stands for $\sum_{i=1}^3 |\mathcal{F}_i W|^2$. The bootstrap assumptions in this region will be energy bounds on the $X_{T_0}^{\text{ex},\kappa}$ norm of the solution for any arbitrarily fixed $T_0 > 2$ and $\kappa > 0$. This norm not only controls the weighted energy of the solution but also the weighted L^2 space-time norm of its *good derivatives*: the tangential derivatives \mathcal{S} to the cones $\{t-r = \text{const}\}$ and the derivative along the periodic direction ∂_y .

As a result of the global energy bounds that will be proved to hold in the exterior region (see Section 4) we also obtain a control on the energy of W on the exterior hyperboloids

$$\begin{aligned} E^{\text{ex},h}(s, W) &= \frac{1}{2\pi} \iint_{\mathcal{H}_s^{\text{ex}}} \left(\frac{s}{t}\right)^2 |\partial_t W|^2 + |\bar{\partial} W|^2 + |\partial_y W|^2 dx dy \\ &= \frac{1}{2\pi} \iint_{\mathcal{H}_s^{\text{ex}}} \left(\frac{s}{t}\right)^2 |\partial_x W|^2 + t^{-2} |\mathcal{S}W|^2 + t^{-2} \sum_{1 \leq i < j \leq 3} |\Omega_{ij} W|^2 + |\partial_y W|^2 dx dy, \end{aligned} \quad (1-21)$$

as well as on the exterior conformal energy of W_0 on constant time slices Σ_t^{ex}

$$E^{\text{c,ex}}(t, W_0) := \int_{\Sigma_t^{\text{ex}}} |\mathcal{S}W_0 + 2W_0|^2 + \sum_{i=1}^3 |\Omega_{0i} W_0|^2 dx. \quad (1-22)$$

1I. The quasilinear energies. Equation (1-5) is quasilinear and in order to propagate both the aforementioned interior and exterior a priori energy bounds we need to consider a cubic modification of the energies introduced in the previous subsection. Such *quasilinear energies* are defined as

$$\begin{aligned} E_{\text{quasi}}^{\text{in}}(s, W) &:= E^{\text{in}}(s, W) + \frac{1}{2\pi} \iint_{\mathcal{H}_s^{\text{in}}} u |\partial_y W|^2 dx dy, \\ E_{\text{quasi}}^{\text{ex},\kappa}(t, W) &:= E^{\text{ex},\kappa}(t, W) + \frac{1}{2\pi} \iint_{\Sigma_t^{\text{ex}}} (2+r-t)^{\kappa+1} u |\partial_y W|^2 dx dy, \\ E_{\text{quasi}}^{\text{ex},h}(s, W) &:= E^{\text{ex},h}(s, W) + \frac{1}{2\pi} \iint_{\mathcal{H}_s^{\text{ex}}} u |\partial_y W|^2 dx dy. \end{aligned}$$

We also introduce the quasilinear modification of the stronger norm $X_{T_0}^{\text{ex},\kappa}$

$$\|W\|_{X_{\text{quasi},T_0}^{\text{ex},\kappa}}^2 := \sup_{t \in [2, T_0]} E_{\text{quasi}}^{\text{ex},\kappa}(t, W) + \frac{(1 + \kappa)}{2\pi} \int_2^{T_0} \iint_{\Sigma_t^{\text{ex}}} (2 + r - t)^\kappa (|\mathcal{T}W|^2 + (1 + u)|\partial_y W|^2) dx dy dt.$$

We immediately observe that under smallness assumptions on u , e.g., $|u| \leq \frac{1}{10}$, our starting energies are equivalent to their corresponding quasilinear counterparts; i.e., for any $E = \{E^{\text{in}}, E^{\text{ex},\kappa}, E^{\text{ex},h}\}$

$$\frac{9}{10} E(t, W) \leq E_{\text{quasi}}(t, W) \leq \frac{11}{10} E(t, W).$$

The same holds true for the stronger norms $X_{T_0}^{\text{ex},\kappa}$ and $X_{\text{quasi},T_0}^{\text{ex},\kappa}$.

1J. Higher-order norms. We use the vector fields introduced before to define the higher-order counterparts of the energy functionals and of the stronger exterior norm $X_{T_0}^{\text{ex},\kappa}$

$$E_n(s, W) := \sum_{|\gamma| \leq n} E(s, \mathcal{L}^\gamma W), \quad E = \{E^{\text{in}}, E^{\text{ex},\kappa}, E^{\text{ex},h}\},$$

$$\|W\|_{X_{T_0}^{n,\kappa}} := \sum_{|\gamma| \leq n} \|\mathcal{L}^\gamma W\|_{X_{T_0}^{\text{ex},\kappa}}.$$

The higher-order energies of W_0 and W are defined analogously. We observe that the above higher-order energies control the high Sobolev regularity of the solution in the interior and exterior regions respectively and also keep track of the Z vector fields applied to the solution in addition to usual derivatives. In the interior region it will be important to keep track of the precise number of Klainerman vector fields acting on the W -component of the solution and to that purpose we also introduce the energy

$$E_{n,k}^{\text{in}}(s, W) := \sum_{\mathcal{I}_{n,k}} E^{\text{in}}(s, \mathcal{L}^\gamma W),$$

where $\mathcal{I}_{n,k}$ denotes the set of indexes of type (n, k) . We finally introduce the higher-order counterparts of the conformal energy functionals (1-18) and (1-22) in order to control the conformal energies of pure products of Klainerman vector fields acting on W_0

$$E_n^{\text{c},\text{in}}(s, W_0) := \sum_{|\beta| \leq n} E^{\text{c},\text{in}}(s, Z^\beta W_0), \quad E_n^{\text{c},\text{ex}}(t, W_0) := \sum_{|\beta| \leq n} E^{\text{c},\text{ex}}(s, Z^\beta W_0).$$

2. Overview of the proof

The proof of our main theorem is based on the combination of a classical local well-posedness result for (1-5) with a bootstrap argument. We will perform this argument separately in the interior region \mathcal{D}^{in} and the exterior region \mathcal{D}^{ex} in which we divide the space-time $\mathbb{R}^{1+3} \times \mathbb{S}^1$.

The bootstrap assumptions in the exterior region \mathcal{D}^{ex} are uniform-in-time energy bounds on the higher-order stronger norm $X_{T_0}^{5,\kappa}$ of the solution W for any arbitrarily fixed $T_0 > 2$ and $\kappa > 0$:

$$\|W\|_{X_{T_0}^{5,\kappa}}^2 \leq 2C_0^2 \epsilon^2. \tag{2-1}$$

The result we want to prove in the exterior region is the following:

Proposition 2.1. *There exists a constant $C_0 > 0$ sufficiently large and a constant $\epsilon_0 > 0$ sufficiently small such that for every $0 < \epsilon < \epsilon_0$ if $W = (u, v)^T$ is a solution to (1-5)–(1-6) in an interval $[2, T_0]$ and satisfies the energy bounds (2-1) then actually*

$$\|W\|_{X_{T_0}^{5,\kappa}}^2 \leq C_0^2 \epsilon^2.$$

In the above proposition the time T_0 is arbitrary; therefore the solution W exists globally in \mathcal{D}^{ex} and satisfies the energy bound (2-1) for all times $T_0 > 2$. In particular, we have

$$\|\mathcal{E}^{\leq 5} W\|_{X_\infty^{\text{ex},\kappa}} := \lim_{T_0 \rightarrow \infty} \|\mathcal{E}^{\leq 5} W\|_{X_{T_0}^{\text{ex},\kappa}} \leq 2C_0^2 \epsilon^2. \tag{2-2}$$

We also observe that, as a consequence of (2-1), there exists an integrable function $l \in L^1([2, T_0])$ and the following bounds hold true for all $t \in [2, T_0]$:

$$\|(2+r-t)^{(\kappa+1)/2} \partial \mathcal{E}^{\leq 5} W(t)\|_{L^2(\Sigma_t^{\text{ex}})} \leq \sqrt{2} C_0 \epsilon, \tag{2-3}$$

$$\|(2+r-t)^{\kappa/2} \mathcal{F} \mathcal{E}^{\leq 5} W\|_{L^2(\Sigma_t^{\text{ex}})} + \|(2+r-t)^{\kappa/2} \partial_y \mathcal{E}^{\leq 5} W\|_{L^2(\Sigma_t^{\text{ex}})} \leq C_0 \epsilon \sqrt{l(t)}. \tag{2-4}$$

The bootstrap assumptions in the interior region \mathcal{D}^{in} are higher-order energy bounds on hyperboloids for the W_0 - and W -components of the solution and pointwise bounds on the Z derivative of W_0 . Given an arbitrarily fixed $s_0 > 2$, these are

$$E_5^{\text{in}}(s, W_0) \leq 2A^2 \epsilon^2, \tag{2-5}$$

$$E_{5,k}^{\text{in}}(s, W) \leq 2A^2 \epsilon^2 s^{2\delta_k}, \quad k = \overline{0, 5}, \tag{2-6}$$

for all $s \in [2, s_0]$ and

$$|ZW_0(t, x)| \leq 2B\epsilon t^{-1} s^\sigma, \quad (t, x) \in \mathcal{H}_{[2, s_0]}^{\text{in}}. \tag{2-7}$$

In the above inequalities the parameters σ, δ_k are fixed small universal constants satisfying $0 < \sigma \ll \delta_k \ll \delta_{k+1}$ for $k = \overline{1, 4}$, $\delta_0 = 0$ and A and B are large universal constants which we will improve as a part of the conclusion of the proof. The result we want to prove in this region requires the global exterior energy bounds (2-2) and can be stated as follows:

Proposition 2.2. *There exist two constants $A, B > 0$ sufficiently large, $0 < \epsilon_0, \sigma, \delta_k \ll 1$ sufficiently small with $\delta_0 = 0$ and $\sigma \ll \delta_k \ll \delta_{k+1}$ for $k = \overline{1, 4}$ such that for every $0 < \epsilon < \epsilon_0$ if $W = (u, v)^T$ is a solution to (1-5)–(1-6) in the region $\mathcal{H}_{[2, s_0]}^{\text{in}} \cup \mathcal{D}^{\text{ex}}$ and satisfies the global exterior energy bounds (2-2) as well as the interior bounds (2-5)–(2-7) for all $s \in [2, s_0]$, then it actually satisfies the enhanced interior bounds*

$$E_5^{\text{in}}(s, W_0) \leq A^2 \epsilon^2,$$

$$E_{5,k}^{\text{in}}(s, W) \leq A^2 \epsilon^2 s^{2\delta_k}, \quad k = \overline{0, 5},$$

$$|ZW_0(t, x)| \leq B\epsilon t^{-1} s^\sigma$$

for all $s \in [2, s_0]$ and all $(t, x) \in \mathcal{H}_{[2, s_0]}^{\text{in}}$.

In the above proposition the hyperbolic time s_0 is arbitrary, which implies the global existence of the solution in the interior region \mathcal{D}^{in} .

The proof of Propositions 2.1 and 2.2 are classical in that they are built around two main steps: (i) pointwise bounds derived from the a priori energy estimates and (ii) vector field energy estimates. Some important remarks:

(a) As for wave-Klein–Gordon systems, the fact that (1-1) is not scaling-invariant prevents us from using Klainerman–Sobolev inequalities on constant time slices, which foliate the entire space-time and would yield pointwise bounds for the solution without distinguishing between interior and exterior regions. Such inequalities

$$(1 + t + |x|)^2(1 + |t - |x||) |W(t, x, y)|^2 \leq \sum_{|\alpha| \leq 3} \|\Gamma^\alpha W(t, \cdot)\|_{L^2(\mathbb{R}^3 \times \mathbb{S}^1)}^2, \quad \Gamma \in \{\Omega_{ij}, \Omega_{0j}, \mathcal{S}\},$$

require, in fact, a good control on the L^2 norm of $\mathcal{S}W$ and its higher-order derivatives, which we do not have since \mathcal{S} does not commute with the linear part of our system. For solutions arising from compactly supported data and hence supported in the interior of the cone $t = r + 1$ this problem is overcome by the fact that Klainerman–Sobolev inequalities on hyperboloids — which entirely foliate this region — do not involve the scaling vector field and only require a good control of the L^2 norm of the hyperbolic derivatives of the solution (see Lemma 5.1). In the case of data that are not compactly supported and only have a mild decay at infinity, however, one also needs to treat the exterior region. The use of the scaling vector field in this region is avoided by recurring to weighted Sobolev inequalities in which the weight is a function of the distance to the cone (see Section 4A). This motivates our decomposition of the space-time into interior and exterior regions and the fact that the proof is done separately in these two regions.

(b) The second step of our proof, i.e., the propagation of the a priori vector field energy estimates, consists in writing a higher-order (interior and exterior respectively) energy inequality for the solution and in using the pointwise bounds previously obtained to perturbatively estimate the source terms (commutator terms and null terms) appearing in the equation of $\mathcal{Z}^\gamma W$ for $|\gamma| \leq 5$. The quadratic null interactions satisfy good (i.e., integrable in time) L^2 estimates thanks to their representation via (1-15) or (1-16). Some commutator terms, on the contrary, only show a slow decay in time that is at the limit of integrability. Let us look for instance at the product $Z^\beta u_0 \cdot \partial_y^2 W$, which appears in the equation of $\mathcal{Z}^\gamma W$ from the commutator $[\mathcal{Z}^\gamma, u \partial_y^2]W$ when γ is a multi-index of type (k, k) , i.e., when $\mathcal{Z}^\gamma = Z^\beta$ is a pure product of Klainerman vector fields, after decomposing u according to (1-7). In the interior region, the L^2 norm of such a term can only be controlled in terms of the (square root of the) conformal energy $E_{k-1}^{c, \text{in}}(s, W_0)$, but even assuming the sharp bounds

$$E_{k-1}^{c, \text{in}}(s, W_0) \lesssim \epsilon^2 s \quad \text{and} \quad \sup_{\mathcal{H}_s^{\text{in}}} |\partial_y^2 W| \lesssim \epsilon^2 s^{-3/2},$$

we are not able to recover a better L^2 bound than

$$\|Z^\beta u_0 \cdot \partial_y^2 W\|_{L_{xy}^2(\mathcal{H}_s^{\text{in}})} \leq E_{k-1}^{c, \text{in}}(s, W_0)^{1/2} \|\partial_y^2 W\|_{L^\infty(\mathcal{H}_s^{\text{in}})} \lesssim \epsilon^2 s^{-1}.$$

These problematic commutator terms — which are absent in the equation of $\mathcal{Z}^\gamma W_0$ — prevent us from obtaining uniform-in-time energy bounds for W , which explains why we distinguish between the energies of W_0 and of W in Proposition 2.2 and do not propagate uniform-in-time energy bounds for the latter. In

the exterior region the commutator terms are treated using a weighted Hardy inequality (see Lemma 4.5) and do not lead to the same type of issue discussed above thanks to the weights; see step (2b) in the proof of Proposition 2.1.

(c) As observed in the previous point, it is important to have a sharp decay $\|W\|_{L^\infty} \lesssim s^{-3/2}$ when estimating some of the commutator terms arising in the equation of $\mathcal{L}^\gamma W$. However, the pointwise bounds that we obtain from the energy in the interior region by means of Klainerman–Sobolev inequalities are not optimal due to the slow growth in time assumed in (2-6); see estimates (5-3). Therefore, we need to study the equation satisfied by W in order to recover more suitable pointwise bounds; see Proposition 5.4. We also point out that, since Klainerman–Sobolev inequalities only yield pointwise bounds for the usual derivatives of W_0 , we also need to recover L^∞ - L^∞ estimates for ZW_0 using the equation it satisfies in order to propagate (2-7).

(d) The interior and exterior energy inequalities will be obtained from the integration of the relation (3-6) over the interior and exterior regions respectively, which share a boundary (let us call it here \mathcal{C}) along the cone $t = r + 1$. From Stokes’ theorem, these inequalities will both involve a boundary term (an integral over the region \mathcal{C} that is controlled given the behavior of the solution in the exterior region) which feeds information from the exterior to the interior region.

The paper is structured as follows. In Section 3 we derive the energy inequalities for the linearized equation, both in the exterior and the interior regions. We also derive the conformal energy inequalities. In Section 4 we prove the global existence of the solution in the exterior region. In Section 5 we recover the pointwise estimates for W and W_0 in the interior region. Finally, in Section 6 we improve the energy estimates in the interior region, concluding the proof of Theorem 1.

3. The linearized equation

The purpose of this section is to write the energy inequality and the conformal energy inequality for the linearized equation associated to (1-5) in both the interior and exterior regions \mathcal{D}^{in} and \mathcal{D}^{ex} . This set of inequalities will repeatedly be used in the following sections when propagating the higher-order energy assumptions, as the equations satisfied by the differentiated unknown will be cast in the form (3-1).

We will look at the linear inhomogeneous equation

$$(-\partial_t^2 + \Delta_x)W + (1 + u)\partial_y^2 W = F, \quad (t, x, y) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{S}^1, \tag{3-1}$$

where u is assumed to be a sufficiently small function, e.g. $|u| \leq \frac{1}{10}$. We start by proving a weighted energy inequality on the exterior constant time slices Σ_t^{ex} for general linear inhomogeneous equations of the above form. In the following proposition the lifespan $T_0 > 2$ is arbitrary, $\mathcal{D}_{T_0}^{\text{ex}}$ denotes the portion of exterior region in the time strip $[2, T_0]$ and $\mathcal{C}_{[2, T_0]}$ is its null boundary

$$\begin{aligned} \mathcal{D}_{T_0}^{\text{ex}} &= \{(t, x, y) \in \mathcal{D}^{\text{ex}} : 2 \leq t \leq T_0\}, \\ \mathcal{C}_{[2, T_0]} &= \{(t, x, y) \in \mathcal{D}^{\text{ex}} : 2 \leq t \leq T_0, r = t - 1\}. \end{aligned}$$

Proposition 3.1. *Let \mathbf{W} be a solution to (3-1) and $l \in L^1([2, T_0])$. Suppose that u is a function satisfying the following pointwise bounds in the exterior region $\mathcal{D}_{T_0}^{\text{ex}}$:*

$$\|u\|_{L^\infty(\mathcal{D}_{T_0}^{\text{ex}})} \leq \frac{1}{10}, \tag{3-2}$$

$$\|(2+r-t)^{1/2} \partial_y u(t, x, \cdot)\|_{L^\infty(\mathbb{S}^1)} \lesssim \epsilon \sqrt{l(t)}, \tag{3-3}$$

$$\|(2+r-t) \partial u(t, x, \cdot)\|_{L^\infty(\mathbb{S}^1)} \lesssim \epsilon. \tag{3-4}$$

For any fixed $\kappa > 0$ the following inequality holds true:

$$\begin{aligned} \|\mathbf{W}\|_{X_{T_0}^{\text{ex}, \kappa}}^2 + \iint_{\mathcal{C}_{[2, T_0]}} (2+r-t)^{\kappa+1} (|\mathcal{F}\mathbf{W}|^2 + |\partial_y \mathbf{W}|^2) dS \\ \lesssim E^{\text{ex}, \kappa}(2, \mathbf{W}) + \|(2+r-t)^{(\kappa+1)/2} \mathbf{F}\|_{L_t^1 L_{xy}^2(\mathcal{D}_{T_0}^{\text{ex}})} \|\mathbf{W}\|_{X_{T_0}^{\text{ex}, \kappa}}, \end{aligned} \tag{3-5}$$

where dS is the surface element of $\mathcal{C}_{[2, T_0]}$.

We remark that the result of Proposition 3.1 can be proved for any positive and increasing weight $\omega = \omega(r-t)$ only depending on the distance from the cone $\{t=r\}$ if the hypotheses on the function u are changed appropriately.

Proof. From the smallness assumption on u it will be enough to prove the inequality in the statement with $E^{\text{ex}, \kappa}(t, \mathbf{W})$ and $\|\mathbf{W}\|_{X_{T_0}^{\text{ex}, \kappa}}$ replaced by $E_{\text{quasi}}^{\text{ex}, \kappa}(t, \mathbf{W})$ and $\|\mathbf{W}\|_{X_{\text{quasi}, T_0}^{\text{ex}, \kappa}}$ respectively. A simple computation shows that for any positive weight $\omega = \omega(r-t)$ one has

$$\begin{aligned} \omega(r-t) \partial_t \mathbf{W} [\square_{x,y} \mathbf{W} + u \partial_y^2 \mathbf{W}] \\ = -\frac{1}{2} \partial_t [\omega(r-t) ((\partial_t \mathbf{W})^2 + |\nabla_x \mathbf{W}|^2 + (1+u)(\partial_y \mathbf{W})^2)] \\ + \text{div}_x (\omega(r-t) \partial_t \mathbf{W} \nabla_x \mathbf{W}) + \partial_y (\omega(r-t) (1+u) \partial_t \mathbf{W} \partial_y \mathbf{W}) \\ - \frac{1}{2} \omega'(r-t) (|\mathcal{F}\mathbf{W}|^2 + (1+u)(\partial_y \mathbf{W})^2) - \omega(r-t) (\partial_y u \partial_t \mathbf{W} \partial_y \mathbf{W} - \frac{1}{2} \partial_t u (\partial_y \mathbf{W})^2). \end{aligned} \tag{3-6}$$

We consider the case $\omega(z) = (2+z)^{\kappa+1}$ and integrate the above equality over the exterior region $\mathcal{D}_{T_0}^{\text{ex}}$. We use Stokes' theorem when integrating in the (t, x) -variables, with normal vectors

$$\mathbf{n}_{\Sigma_0^{\text{ex}}} = (1, 0, 0, 0), \quad \mathbf{n}_{\Sigma_2^{\text{ex}}} = -\mathbf{n}_{\Sigma_{T_0}^{\text{ex}}}, \quad \mathbf{n}_{\mathcal{C}_{[2, T_0]}} = \left(1, -\frac{x}{r}\right). \tag{3-7}$$

We observe that

$$(\partial_t \mathbf{W})^2 + |\nabla_x \mathbf{W}|^2 + 2\frac{x}{r} \cdot \nabla_x \mathbf{W} \partial_t \mathbf{W} = |\mathcal{F}\mathbf{W}|^2$$

and hence recover

$$\begin{aligned} \|\mathbf{W}\|_{X_{\text{quasi}, T_0}^{\text{ex}, \kappa}}^2 + \iint_{\mathcal{C}_{[2, T_0]}} (2+r-t)^{\kappa+1} [|\mathcal{F}\mathbf{W}|^2 + (1+u)|\partial_y \mathbf{W}|^2] dS \\ \leq E_{\text{quasi}}^{\text{ex}, \kappa}(2, \mathbf{W}) + \frac{1}{\pi} \iint_{\mathcal{D}_{T_0}^{\text{ex}}} (2+r-t)^{\kappa+1} (|\mathbf{F} \partial_t \mathbf{W}| + |\partial_y u \partial_t \mathbf{W} \partial_y \mathbf{W}| + \frac{1}{2} |\partial_t u| (\partial_y \mathbf{W})^2) dx dy dt. \end{aligned} \tag{3-8}$$

We remark that the above inequality holds actually true for any index $\kappa > -1$. The smallness of u ensures that the integral in the above left-hand side is equivalent to the one in the left-hand side of inequality (3-5).

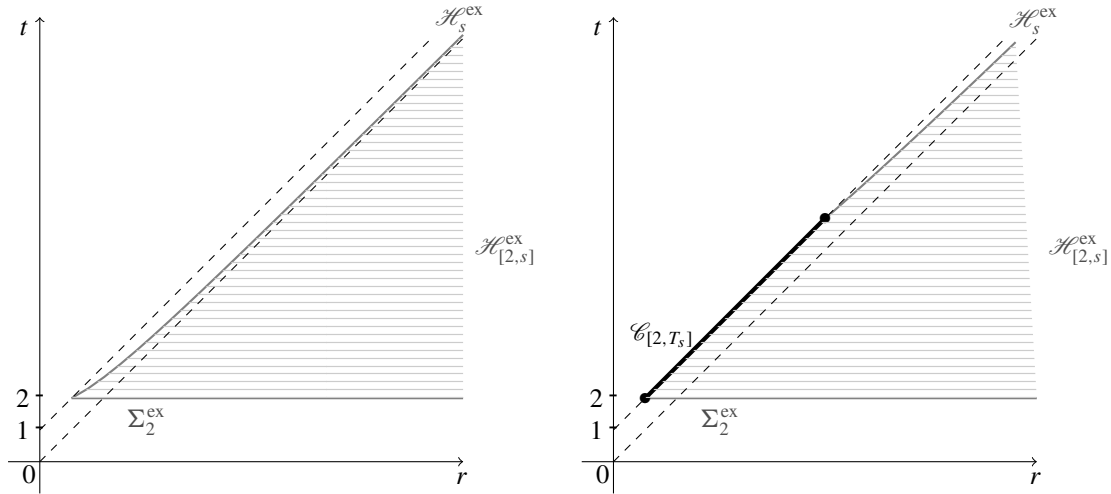


Figure 2. Vertical section of the region $\mathcal{H}_{[2,s]}^{\text{ex}}$ and its foliation projected onto \mathbb{R}^{1+3} . In the left picture, $s = \sqrt{3}$; in the right one, $s > \sqrt{3}$ and the two bullets delimit the later boundary $\mathcal{C}_{[2,T_s]}$. The dashed lines represent the cones $t = r$ and $t = r + 1$.

We also see, from the assumption (3-3) and the Cauchy–Schwarz inequality, that

$$\begin{aligned} \iint_{\mathcal{D}_{T_0}^{\text{ex}}} (2+r-t)^{\kappa+1} |\partial_y u \partial_t \mathbf{W} \partial_y \mathbf{W}| \, dx \, dy \, dt &\lesssim \|(2+r-t)^{(\kappa+1)/2} \partial_y u \partial_y \mathbf{W}\|_{L_t^1 L_{xy}^2(\mathcal{D}_{T_0}^{\text{ex}})} \|\mathbf{W}\|_{X_{T_0}^{\text{ex},\kappa}} \\ &\lesssim \|(2+r-t)^{1/2} \partial_y u\|_{L_t^2 L_{xy}^\infty(\mathcal{D}_{T_0}^{\text{ex}})} \|(2+r-t)^{\kappa/2} \partial_y \mathbf{W}\|_{L_{txy}^2(\mathcal{D}_{T_0}^{\text{ex}})} \|\mathbf{W}\|_{X_{T_0}^{\text{ex},\kappa}} \\ &\lesssim \epsilon \|\sqrt{l(\cdot)}\|_{L_t^2} \|\mathbf{W}\|_{X_{T_0}^{\text{ex},\kappa}}^2 \lesssim \epsilon \|\mathbf{W}\|_{X_{T_0}^{\text{ex},\kappa}}^2 \end{aligned}$$

and from (3-4)

$$\iint_{\mathcal{D}_{T_0}^{\text{ex}}} (2+r-t)^{\kappa+1} |\partial_t u| (\partial_y \mathbf{W})^2 \, dx \, dy \, dt \lesssim \epsilon \iint_{\mathcal{D}_{T_0}^{\text{ex}}} (2+r-t)^\kappa (\partial_y \mathbf{W})^2 \, dx \, dy \, dt \lesssim \epsilon \|\mathbf{W}\|_{X_{T_0}^{\text{ex},\kappa}}^2;$$

hence the above two integrals can be absorbed in the left-hand side of (3-8) if $\epsilon \ll 1$ is sufficiently small. Finally, from the Cauchy–Schwarz inequality we also have

$$\iint_{\mathcal{D}_{T_0}^{\text{ex}}} (2+r-t)^{\kappa+1} |\mathbf{F} \partial_t \mathbf{W}| \, dx \, dy \, dt \leq \|(2+r-t)^{(\kappa+1)/2} \mathbf{F}\|_{L_t^1 L_{xy}^2(\mathcal{D}_{T_0}^{\text{ex}})} \|\mathbf{W}\|_{X_{T_0}^{\text{ex},\kappa}}. \quad \square$$

If we assume that the function u satisfies the pointwise bounds (3-2)–(3-4) in the whole exterior region \mathcal{D}^{ex} we can also recover an inequality for the energy on exterior truncated hyperboloids $\mathcal{H}_s^{\text{ex}}$ for any $s > 0$, that is, on any branch of hyperboloid contained in the exterior region. We observe that $\mathcal{H}_{[2,s]}^{\text{ex}}$ corresponds to the portion of exterior region delimited by \mathcal{H}_s and Σ_2^{ex} whenever $0 < s \leq \sqrt{3}$ and by $\mathcal{H}_s^{\text{ex}}$, $\mathcal{C}_{[2,T_s]}$ and Σ_2^{ex} whenever $s > \sqrt{3}$, where $T_s = \frac{1}{2}(s^2 + 1)$. We remind the definition the stronger norm $\|\cdot\|_{X_\infty^{\text{ex},\kappa}}$ over the interval $[2, \infty)$

$$\|\mathbf{W}\|_{X_\infty^{\text{ex},\kappa}} := \lim_{T \rightarrow \infty} \|\mathbf{W}\|_{X_T^{\text{ex},\kappa}}.$$

Proposition 3.2. *Assume that \mathbf{W} is solution to the linear inhomogeneous equation (3-1) with u satisfying the decay properties (3-2)–(3-4) in the whole region \mathcal{D}^{ex} . For any $s > 0$*

$$E^{\text{ex},h}(s, \mathbf{W}) + \delta_{s > \sqrt{3}} \iint_{\mathcal{C}_{[2, T_s]}} |\mathcal{T}\mathbf{W}|^2 + |\partial_y \mathbf{W}|^2 dS \lesssim E^{\text{ex},0}(2, \mathbf{W}) + \epsilon \|\mathbf{W}\|_{X_\infty^{\text{ex},0}}^2 + \|\mathbf{F}\|_{L_t^1 L_{xy}^2(\mathcal{H}_{[2,s]}^{\text{ex}})} \|\mathbf{W}\|_{X_\infty^{\text{ex},0}},$$

where $\delta_{s > \sqrt{3}} = 0$ if $s \leq \sqrt{3}$ and is 1 otherwise.

Proof. From the smallness assumption on u it will be enough to prove the statement with $E^{\text{ex},h}(t, \mathbf{W})$ and $E^{\text{ex},0}(2, \mathbf{W})$ replaced by $E_{\text{quasi}}^{\text{ex},h}(t, \mathbf{W})$ and $E_{\text{quasi}}^{\text{ex},0}(2, \mathbf{W})$ respectively. We integrate the relation (3-6) in the case where $\omega \equiv 1$ over the region $\mathcal{H}_{[2,s]}^{\text{ex}}$ and use Stokes’s theorem when integrating in the (t, x) -variables, with normal vectors given in (3-7) and $\mathbf{n}_{\mathcal{H}_s^{\text{ex}}} = (1, -x/t)$. We obtain

$$\begin{aligned} E_{\text{quasi}}^{\text{ex},h}(s, \mathbf{W}) + \delta_{s > \sqrt{3}} \iint_{\mathcal{C}_{[2, T_s]}} |\mathcal{T}\mathbf{W}|^2 + (1 + u)|\partial_y \mathbf{W}|^2 dS \\ \leq E_{\text{quasi}}^{\text{ex},0}(2, \mathbf{W}) + \frac{1}{\pi} \iint_{\mathcal{H}_{[2,s]}^{\text{ex}}} |\partial_y u \partial_t \mathbf{W} \partial_y \mathbf{W}| + \frac{1}{2} |\partial_t u| (\partial_y \mathbf{W})^2 + |\mathbf{F} \partial_t \mathbf{W}| dx dy dt. \end{aligned}$$

We foliate $\mathcal{H}_{[2,s]}^{\text{ex}}$ by the constant time slices Σ_t^s for $t \geq 2$ (see Figure 2), where

$$\Sigma_t^s = \{x \in \mathbb{R}^3 : r \geq \max(t - 1, \sqrt{t^2 - s^2})\} \times \mathbb{S}^1,$$

and from the Cauchy–Schwarz inequality, the assumptions (3-3), (3-4) and the definition of the norm $X_\infty^{\text{ex},0}$ we immediately see

$$\iint_{\mathcal{H}_{[2,s]}^{\text{ex}}} |\partial_y u \partial_y \mathbf{W} \partial_y \mathbf{W}| + \frac{1}{2} |\partial_t u| (\partial_y \mathbf{W})^2 dx dy dt \lesssim \epsilon \|\mathbf{W}\|_{X_\infty^{\text{ex},0}}^2.$$

Furthermore

$$\iint_{\mathcal{H}_{[2,s]}^{\text{ex}}} |\mathbf{F} \partial_t \mathbf{W}| dx dy dt \leq \|\mathbf{F}\|_{L_t^1 L_{xy}^2(\mathcal{H}_{[2,s]}^{\text{ex}})} \|\mathbf{W}\|_{X_\infty^{\text{ex},0}}. \quad \square$$

We now prove the interior energy inequality for (3-1). In the following proposition the hyperbolic lifespan $s_0 > 2$ is arbitrary and $\mathcal{C}_{[2,s]}$ will denote the later boundary of the hyperbolic region $\mathcal{H}_{[2,s]}^{\text{in}}$, which is included in $\mathcal{C}_{[2, T_s]}$ for $T_s = \frac{1}{2}(s^2 + 1)$:

$$\begin{aligned} \mathcal{C}_{[2,s]} &= \{(r + 1, x) : \frac{1}{2} \leq r \leq \frac{1}{2}(s^2 - 1)\} \times \mathbb{S}^1 \\ &= \{(t, t - 1) : \frac{3}{2} \leq t \leq \frac{1}{2}(s^2 + 1)\} \times \mathbb{S}^2 \times \mathbb{S}^1. \end{aligned}$$

Proposition 3.3. *Let \mathbf{W} be a solution to (3-1) and suppose that u is a function that satisfies the following bounds in the hyperbolic region $\mathcal{H}_{[2,s_0]}^{\text{in}}$*

$$\|u\|_{L^\infty(\mathcal{H}_{[2,s_0]}^{\text{in}})} \leq \frac{1}{10}, \tag{3-9}$$

$$|\partial_t u_0(t, x)| \lesssim \epsilon t^{-1/2} s^{-1}, \tag{3-10}$$

$$\|\partial u(t, x, \cdot)\|_{L^\infty(\mathbb{S}^1)} \lesssim \epsilon t^{-3/2+\delta} \tag{3-11}$$

for some small $\delta > 0$, where $u_0 = \int_{\mathbb{S}^1} u(t, x, y) dy$ and $\mathfrak{u} = u - u_0$. Then

$$E^{\text{in}}(s, \mathbf{W}) \lesssim E^{\text{in}}(2, \mathbf{W}) + \iint_{\mathcal{C}_{[2,s]}} |\mathcal{T}\mathbf{W}|^2 + |\partial_y \mathbf{W}|^2 dS + \int_2^s \|\mathbf{F}\|_{L^2_{xy}(\mathcal{H}_\tau^{\text{in}})} E^{\text{in}}(\tau, \mathbf{W})^{1/2} d\tau \quad (3-12)$$

for all $s \in [2, s_0]$.

Proof. For any fixed $s \in [2, s_0]$, we integrate the equality (3-6) with $\omega \equiv 1$ over the region $\mathcal{H}_{[2,s]}^{\text{in}}$ (see Figure 1), which we foliate for hyperboloids $\mathcal{H}_\tau^{\text{in}}$ with $\tau \in [2, s]$. We use Stokes' theorem when integrating in the variables (t, x) , with normal vectors given by

$$\mathbf{n}_{\mathcal{H}_s^{\text{in}}} = \left(1, -\frac{x}{t}\right), \quad \mathbf{n}_{\mathcal{H}_2^{\text{in}}} = -\mathbf{n}_{\mathcal{H}_s^{\text{in}}}, \quad \mathbf{n}_{\mathcal{C}_{[2,s]}} = \left(-1, \frac{x}{r}\right) \quad (3-13)$$

and obtain

$$\begin{aligned} E_{\text{quasi}}^{\text{in}}(s, \mathbf{W}) &\leq E_{\text{quasi}}^{\text{in}}(2, \mathbf{W}) + \iint_{\mathcal{C}_{[2,s]}} |\mathcal{T}\mathbf{W}|^2 + (1+u)|\partial_y \mathbf{W}|^2 dS \\ &\quad + \frac{1}{\pi} \int_2^s \int_{\mathcal{H}_\tau^{\text{in}}} \left(\frac{\tau}{t}\right) |\partial_y u \partial_t \mathbf{W} \partial_y \mathbf{W}| + \frac{1}{2} \left(\frac{\tau}{t}\right) |\partial_t u| (\partial_y \mathbf{W})^2 + \left(\frac{\tau}{t}\right) |\mathbf{F} \partial_t \mathbf{W}| dx dy d\tau. \end{aligned} \quad (3-14)$$

The integral on the null boundary $\mathcal{C}_{[2,s]}$ in the above right-hand side is bounded by its counterpart in the right-hand side of (3-12) thanks to the smallness assumption (3-9). From the assumption (3-11) and the fact that $\tau \leq t$ we derive

$$\begin{aligned} \int_2^s \int_{\mathcal{H}_\tau^{\text{in}}} \left(\frac{\tau}{t}\right) |\partial_y u \partial_t \mathbf{W} \partial_y \mathbf{W}| + \frac{1}{2} \left(\frac{\tau}{t}\right) |\partial_t u| (\partial_y \mathbf{W})^2 dx dy d\tau \\ \lesssim \int_2^s \|\partial \mathfrak{u}\|_{L^\infty(\mathcal{H}_\tau^{\text{in}})} E^{\text{in}}(\tau, \mathbf{W}) d\tau \lesssim \epsilon \int_2^s \tau^{-3/2+\delta} E^{\text{in}}(\tau, \mathbf{W}) d\tau, \end{aligned}$$

while from (3-10) we have

$$\int_2^s \int_{\mathcal{H}_\tau^{\text{in}}} \frac{1}{2} \left(\frac{\tau}{t}\right) |\partial_t u_0| (\partial_y \mathbf{W})^2 dx dy d\tau \lesssim \epsilon \int_2^s \tau^{-3/2} E^{\text{in}}(\tau, \mathbf{W}) d\tau.$$

The Cauchy–Schwarz inequality yields

$$\int_2^s \int_{\mathcal{H}_\tau^{\text{in}}} \left(\frac{\tau}{t}\right) |\mathbf{F} \partial_t \mathbf{W}| dx dy d\tau \leq \int_2^{s_0} \|\mathbf{F}\|_{L^2_{xy}(\mathcal{H}_\tau^{\text{in}})} E^{\text{in}}(\tau, \mathbf{W})^{1/2} d\tau,$$

and finally the use of the Gronwall inequality concludes the proof of the statement. □

As detailed in Section 6, it will be important to distinguish between the two components W_0 and W of the solution W to (1-5), in particular to show that the energies associated to the zero mode W_0 are uniformly bounded in time. We will make use of the following classical result about the energy on interior truncated hyperboloids of solutions of linear inhomogeneous wave equations on the flat space \mathbb{R}^{1+3} .

Proposition 3.4. *Let W_0 be a solution of the linear inhomogeneous wave equation*

$$(-\partial_t^2 + \Delta_x) \mathbf{W}_0 = \mathbf{F}_0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3. \quad (3-15)$$

For all $s \in [2, s_0]$ we have the energy inequality

$$E^{\text{in}}(s, \mathbf{W}_0) \leq E^{\text{in}}(2, \mathbf{W}_0) + \iint_{\mathcal{C}_{[2,s]}} |\mathcal{T}\mathbf{W}_0|^2 dS + 2 \int_2^s \|\mathbf{F}_0\|_{L^2_x(\mathcal{H}_\tau^{\text{in}})} E^{\text{in}}(\tau, \mathbf{W}_0)^{1/2} d\tau.$$

We also prove below the interior and exterior conformal energy inequalities for linear inhomogeneous wave equations on \mathbb{R}^{1+3} . We will, in fact, need to control the higher-order conformal energies of the zero-mode W_0 of our solution W in order to recover pointwise bounds for W_0 and ZW_0 later in the paper.

Proposition 3.5. *Let W_0 be a solution to (3-15). Then*

$$E^{c,\text{in}}(s, W_0) \leq E^{c,\text{in}}(2, W_0) + \int_2^s \|\tau F_0\|_{L^2(\mathcal{H}_\tau^{\text{in}})} E^{c,\text{in}}(\tau, W_0)^{1/2} d\tau + \int_{\mathcal{C}_{[2,s]}} |(t+r)(\partial_t W_0 + \partial_r W_0) + 2W_0|^2 + (t-r)^2 (|\nabla_x W_0|^2 - (\partial_r W_0)^2) dS.$$

Proof. Let $K = (t^2 + r^2)\partial_t + 2rt\partial_r$ denote the Morawetz multiplier. We have the equality

$$\begin{aligned} & -(K W_0 + 2t W_0)\square_x W_0 \\ &= \partial_t \left[\frac{1}{2}(t^2 + r^2)(|\partial_t W_0|^2 + |\nabla_x W_0|^2) + 2rt\partial_t W_0 \partial_r W_0 + 2t W_0 \partial_t W_0 - W_0^2 \right] \\ & \quad - \operatorname{div}_x [(t^2 + r^2)\partial_t W_0 \nabla_x W_0 + 2rt\partial_r W_0 \nabla_x W_0 + tx(|\partial_t W_0|^2 - |\nabla_x W_0|^2) + 2t W_0 \nabla_x W_0], \end{aligned} \tag{3-16}$$

which we integrate over the interior hyperbolic region $\mathcal{H}_{[2,s]}^{\text{in}}$. We apply Stokes' theorem (recall the normal vectors listed (3-13)). First, we compute the contribution to the integral on $\mathcal{H}_s^{\text{in}}$. Algebraic computations show that

$$\begin{aligned} & \frac{1}{2}(t^2 + r^2)(|\partial_t W_0|^2 + |\nabla_x W_0|^2) + 2rt\partial_t W_0 \partial_r W_0 \\ & \quad + [(t^2 + r^2)\partial_t W_0 \nabla_x W_0 + 2rt\partial_r W_0 \nabla_x W_0 + tx(|\partial_t W_0|^2 - |\nabla_x W_0|^2)] \cdot \frac{x}{t} \\ &= \frac{1}{2t^2} |K W_0|^2 + \frac{s^2}{2t^2} \sum_{i=1}^3 |\Omega_{0i} W_0|^2 \end{aligned}$$

and

$$2t W_0 \partial_t W_0 - W_0^2 + 2t W_0 \nabla_x W_0 \cdot \frac{x}{t} = \frac{2}{t} W_0 K W_0 - \frac{r}{t} \Omega_{0r}(W_0^2) - W_0^2,$$

where $\Omega_{0r} = r\partial_t + t\partial_r$. The parametrization of $\mathcal{H}_s^{\text{in}}$ by the hyperbolic angle θ , i.e.,

$$\mathcal{H}_s^{\text{in}} = \{(s \cosh \theta, s \sinh \theta) : \theta \in [\theta_1, \theta_2]\}$$

for some suitable θ_1, θ_2 , and the change of variables $r = s \sinh \theta$ allows us to see that

$$\begin{aligned} \int_{\mathcal{H}_s^{\text{in}}} \frac{r}{t} \Omega_{0r}(W_0)^2 dx &= \int_0^{(s^2-1)/2} \int_{\mathbb{S}^2} \frac{r}{t} \Omega_{0r}(W_0)^2 r^2 dr d\sigma \\ &= \int_{\theta_1}^{\theta_2} \int_{\mathbb{S}^2} \frac{\sinh \theta}{\cosh \theta} \partial_\theta (W_0)^2 s^3 \sinh^2 \theta \cosh \theta d\theta d\sigma \\ &= -3 \int_{\theta_1}^{\theta_2} \int_{\mathbb{S}^2} W_0^2 s^3 \sinh^2 \theta \cosh \theta d\theta d\sigma + \left[\int_{\mathbb{S}^2} W_0^2 s^3 \sinh^3 \theta d\sigma \right]_{\theta=\theta_1}^{\theta=\theta_2} \\ &= -3 \int_{\mathcal{H}_s^{\text{in}}} W_0^2 dx + \left[\int_{\mathbb{S}^2} W_0^2 r^3 d\sigma \right]_{r=r_1}^{r=r_2}, \end{aligned}$$

where $r_1 = s \sinh \theta_1 = \frac{1}{2}$ and $r_2 = s \sinh \theta_2 = \frac{1}{2}(s^2 - 1)$. Therefore the integral on the boundary $\mathcal{H}_s^{\text{in}}$ equals

$$\begin{aligned} \int_{\mathcal{H}_s^{\text{in}}} \frac{1}{2t^2} |K \mathbf{W}_0|^2 + \frac{s^2}{2t^2} \sum_{i=1}^3 |\Omega_{0i} \mathbf{W}_0|^2 + \frac{2}{t} \mathbf{W}_0 K \mathbf{W}_0 + 2\mathbf{W}_0^2 dx - \left[\int_{\mathbb{S}^2} \mathbf{W}_0^2 s^3 \sinh^3 \theta d\sigma \right]_{\theta=\theta_1}^{\theta=\theta_2} \\ = \int_{\mathcal{H}_s^{\text{in}}} \frac{1}{2t^2} |K \mathbf{W}_0 + 2\mathbf{W}_0|^2 + \frac{s^2}{2t^2} \sum_{i=1}^3 |\Omega_{0i} \mathbf{W}_0|^2 dx - \left[\int_{\mathbb{S}^2} \mathbf{W}_0^2 s^3 \sinh^3 \theta d\sigma \right]_{\theta=\theta_1}^{\theta=\theta_2}. \end{aligned}$$

We now compute the contributions to the integral on $\mathcal{C}_{[2,s]}$. Algebraic computations show that

$$\begin{aligned} \frac{1}{2}(t^2 + r^2)(|\partial_t \mathbf{W}_0|^2 + |\nabla_x \mathbf{W}_0|^2) + 2rt \partial_t \mathbf{W}_0 \partial_r \mathbf{W}_0 \\ + [(t^2 + r^2) \partial_t \mathbf{W}_0 \nabla_x \mathbf{W}_0 + 2rt \partial_r \mathbf{W}_0 \nabla_x \mathbf{W}_0 + tx(|\partial_t \mathbf{W}_0|^2 - |\nabla_x \mathbf{W}_0|^2)] \cdot \frac{x}{r} \\ = \frac{1}{2}(t+r)^2 (\partial_t \mathbf{W}_0 + \partial_r \mathbf{W}_0)^2 + \frac{1}{2} (|\nabla_x \mathbf{W}_0|^2 - (\partial_r \mathbf{W}_0)^2) \end{aligned}$$

and

$$\begin{aligned} 2t \mathbf{W}_0 \partial_t \mathbf{W}_0 - \mathbf{W}_0^2 + 2t \mathbf{W}_0 \nabla_x \mathbf{W}_0 \cdot \frac{x}{r} = 2t \mathbf{W}_0 (\partial_t \mathbf{W}_0 + \partial_r \mathbf{W}_0) - \mathbf{W}_0^2 \\ = 2(t+r) \mathbf{W}_0 (\partial_t \mathbf{W}_0 + \partial_r \mathbf{W}_0) - 2r \mathbf{W}_0 (\partial_t \mathbf{W}_0 + \partial_r \mathbf{W}_0) - \mathbf{W}_0^2. \end{aligned}$$

In the coordinates $(u, v) = (t-r, t+r)$ we have that $\partial_t + \partial_r = \partial_v$ and moreover the lateral boundary can be expressed as

$$\mathcal{C}_{[2,s]} = \left\{ \left(\frac{v+1}{2}, \frac{v-1}{2}, \sigma \right) : 2 \leq v \leq s^2, \sigma \in \mathbb{S}^2 \right\}.$$

Using polar coordinates and then the change of coordinates $r = \frac{1}{2}(v-1)$, we see that

$$\begin{aligned} \int_{\mathcal{C}_{[2,s]}} 2r \mathbf{W}_0 (\partial_t \mathbf{W}_0 + \partial_r \mathbf{W}_0) d\Sigma_s = \int_{r_1}^{r_2} \int_{\mathbb{S}^2} (\partial_t + \partial_r) (\mathbf{W}_0^2) r^3 dr d\sigma = \int_2^{s^2} \int_{\mathbb{S}^2} \partial_v (\mathbf{W}_0)^2 \left(\frac{v-1}{2} \right)^3 \frac{dv}{2} d\sigma \\ = -3 \int_2^{s^2} \int_{\mathbb{S}^2} \mathbf{W}_0^2 \left(\frac{v-1}{2} \right)^2 \frac{dv}{2} d\sigma + \left[\int_{\mathbb{S}^2} \mathbf{W}_0^2 \left(\frac{v-1}{2} \right)^3 d\sigma \right]_{v=2}^{v=s^2} \\ = -3 \int_{\mathcal{C}_{[2,s]}} \mathbf{W}_0^2 d\Sigma_s + \left[\int_{\mathbb{S}^2} \mathbf{W}_0^2 r^3 d\sigma \right]_{r=r_1}^{r=r_2}. \end{aligned}$$

Therefore, the integral on the lateral boundary $\mathcal{C}_{[2,s]}$ equals to the opposite of

$$\begin{aligned} \int_{\mathcal{C}_{[2,s]}} \frac{1}{2} (t+r)^2 (\partial_t \mathbf{W}_0 + \partial_r \mathbf{W}_0)^2 + 2(t+r) \mathbf{W}_0 (\partial_t \mathbf{W}_0 + \partial_r \mathbf{W}_0) + 2\mathbf{W}_0^2 d\Sigma_s \\ + \frac{1}{2} \int_{\mathcal{C}_{[2,s]}} (|\nabla_x \mathbf{W}_0|^2 - (\partial_r \mathbf{W}_0)^2) d\Sigma_s + \left[\int_{\mathbb{S}^2} \mathbf{W}_0^2 r^3 d\sigma \right]_{r=r_1}^{r=r_2} \\ = \frac{1}{2} \int_{\mathbb{S}^2} |(t+r)(\partial_t \mathbf{W}_0 + \partial_r \mathbf{W}_0) + 2\mathbf{W}_0|^2 + (|\nabla_x \mathbf{W}_0|^2 - (\partial_r \mathbf{W}_0)^2) d\Sigma_s + \left[\int_{\mathbb{S}^2} \mathbf{W}_0^2 r^3 d\sigma \right]_{r=r_1}^{r=r_2}. \end{aligned}$$

Summing everything up proves the result of the statement. \square

Proposition 3.6. *Let W_0 be a solution to (3-15). Then*

$$E^{c,\text{ex}}(T, W_0) + \int_{\mathcal{C}_{[2,T]}} |(t+r)(\partial_t W_0 + \partial_r W_0) + 2W_0|^2 + (t-r)^2(|\nabla W_0|^2 - (\partial_r W_0)^2) dS$$

$$\leq E^{c,\text{ex}}(2, W_0) + \int_2^T \|(t+r)F_0\|_{L_x^2(\Sigma_t^{\text{ex}})} E^{c,\text{ex}}(t, W_0)^{1/2} dt.$$

Proof. The result of the proposition follows from the integration of equality (3-16) over the region $\mathcal{D}_T^{\text{ex}}$ combined with Stokes’ theorem and the equality

$$K W_0 + 2t W_0 = t(\mathcal{S}W_0 + 2W_0) + r\Omega_{0r}W_0. \quad \square$$

In the interior region we will recover pointwise bounds on ZW_0 from the higher-order conformal energies of W_0 via Klainerman–Sobolev inequalities on hyperboloids (see Lemma 5.1). This will require a control on the conformal energy of the solution on a portion of the hyperboloid \mathcal{H}_s in the exterior region \mathcal{D}^{ex} that in turn will be obtained from a control on the conformal energy on the flat hypersurfaces Σ_t^s defined below. We hence state the following modification of the conformal energy inequality.

Proposition 3.7. *For any s, T_1, T_2 with $2 \leq s < T_1 < T_2$ and any $t \in [T_1, T_2]$, let*

$$\Sigma_t^s := \{x \in \mathbb{R}^3 : |x| \geq \sqrt{t^2 - s^2}\}$$

and

$$E_s^{c,\text{ex}}(t, W_0) := \int_{\Sigma_t^s} |\mathcal{S}W_0 + 2W_0|^2 + \sum_{i=1}^3 |\Omega_{0i}W_0|^2 dx.$$

Then

$$E_s^{c,\text{ex}}(T_2, W_0) + \int_{\mathcal{H}_s \cap [T_1, T_2]} \frac{1}{t^2} |K W_0 + 2t W_0|^2 + \frac{s^2}{t^2} \sum_{i=1}^3 |\Omega_{0i}W_0|^2 dx$$

$$\leq E_s^{c,\text{ex}}(T_1, W_0) + \int_{T_1}^{T_2} \|(t+r)F_0\|_{L_x^2(\Sigma_t^s)} E_s^{c,\text{ex}}(t, W_0)^{1/2} dt.$$

Proof. The result of the statement follows by integrating (3-16) over the region bounded by $\Sigma_{T_2}^s, \Sigma_{T_1}^s$ and $\mathcal{H}_s \cap [T_1, T_2]$, which can be foliated by the hypersurfaces Σ_t^s for $t \in [T_1, T_2]$. \square

4. Global existence in the exterior region

The main goal of this section is to prove Proposition 2.1, that is, the propagation of the a priori energy bounds on the weighted higher-order exterior energies of the solution. This in turn will imply the global existence of the solution to the Cauchy problem (1-1)–(1-2) under the assumptions of Theorem 1.

The proof of Proposition 2.1 unfolds in two main steps:

- (1) We recover sharp pointwise bounds from the energy assumptions (2-1).
- (2) We compute the equation satisfied by the differentiated variable $\mathcal{L}^\gamma W$ and compare it to the inhomogeneous linear equation (3-1) in order to use the energy inequality of Proposition 3.1. We use the energy assumptions and the pointwise bounds obtained in step (1) to perturbatively estimate the source terms.

The main tools used in steps (1) and (2) are weighted Sobolev and Hardy inequalities, with weights depending on the distance from the light cone. We mention here that these inequalities already played an important role in the proof of [Lindblad and Rodnianski 2010] of the global stability of the Minkowski space-time for the vacuum Einstein equations. We also observe that, as a result of proving global energy bounds in the exterior region, we obtain bounds for the higher-order conformal energy $E_5^{c,ex}(t, W)$ for all $t \geq 2$ and a uniform-in-time control of the higher-order energies $E_5^{ex,h}(s, W)$ on exterior hyperboloids, for all $s \geq 2$.

This section is organized as follows: in Sections 4A and 4B we prove weighted Sobolev and Hardy inequalities; in Section 4C we recover the aforementioned pointwise bounds; in Section 4D we finally propagate the bounds (2-1).

4A. Weighted Sobolev inequalities.

Lemma 4.1. *Let $\beta \in \mathbb{R}$. For any sufficiently smooth function w we have*

$$\sup_{\Sigma_t^{ex}} (2+r-t)^\beta r^2 |w(t, x, y)|^2 \lesssim \iint_{\Sigma_t^{ex}} (2+r-t)^{\beta+1} (\partial_r \mathcal{L}^{\leq 2} w)^2 + (2+r-t)^{\beta-1} (\mathcal{L}^{\leq 2} w)^2 dx dy. \quad (4-1)$$

Proof. Let (r, σ) be the spherical coordinates in \mathbb{R}^3 , $r = |x|$ and $\sigma = x/|x| \in \mathbb{S}^2$. We begin by observing that the Sobolev embedding $H^2(\mathbb{S}^2 \times \mathbb{S}^1) \subset L^\infty(\mathbb{S}^2 \times \mathbb{S}^1)$ implies

$$\sup_{\mathbb{S}^2 \times \mathbb{S}^1} |w(t, r, \sigma, y)|^2 \leq \sum_{0 \leq l+k \leq 2} \int |\nabla_\sigma^l \partial_y^k w(t, r, \sigma, y)|^2 d\sigma dy.$$

We then remark that for any function v and $(t, x, y) \in \Sigma_t^{ex}$

$$\begin{aligned} \partial_r [(2+r-t)^\beta r^2 v(t, x, y)^2] &= 2(2+r-t)^\beta r^2 v \partial_r v + \beta(2+r-t)^{\beta-1} r^2 v^2 + 2(2+r-t)^\beta r v^2 \\ &\geq 2(2+r-t)^\beta r^2 v \partial_r v + \beta(2+r-t)^{\beta-1} r^2 v^2, \end{aligned}$$

so if v is compactly supported in x we can write

$$\begin{aligned} (2+r-t)^\beta r^2 v(t, x, y)^2 &= - \int_r^\infty \partial_\rho [(2+\rho-t)^\beta \rho^2 v(t, x, y)^2] d\rho \\ &\lesssim_\beta \int_r^\infty (2+\rho-t)^\beta |v \partial_\rho v| \rho^2 d\rho + \int_r^\infty (2+\rho-t)^{\beta-1} v^2 \rho^2 d\rho \\ &\lesssim_\beta \int_r^\infty (2+\rho-t)^{\beta+1} (\partial_\rho v)^2 \rho^2 d\rho + \int_r^\infty (2+\rho-t)^{\beta-1} v^2 \rho^2 d\rho. \quad (4-2) \end{aligned}$$

By replacing v with $\nabla_\sigma^l \partial_y^k w(t, r, \sigma, y)$ for $k+l \leq 2$ we obtain (4-1) in the case where w is compactly supported. In the general case where w is not compactly supported we consider a cut-off function $\chi \in C_0^\infty(\mathbb{R})$ and apply the inequality (4-1) to $\chi(\epsilon r)w$ for any $\epsilon > 0$

$$\begin{aligned} \sup_{\Sigma_t^{ex}} (2+r-t)^\beta r^2 |\chi(\epsilon r)w|^2 &\lesssim \sum_{k+l \leq 2} \iint_{\Sigma_t^{ex}} (2+r-t)^{\beta+1} (\partial_r \nabla_\sigma^k \partial_y^l w)^2 + (2+r-t)^{\beta-1} (\nabla_\sigma^k \partial_y^l w)^2 dx dy \\ &\quad + \sum_{k+l \leq 2} \iint_{\Sigma_t^{ex}} (2+r-t)^{\beta+1} \epsilon^2 |\chi'(\epsilon r)|^2 (\nabla_\sigma^k \partial_y^l w)^2 dx dy. \end{aligned}$$

On the intersection of Σ_t^{ex} with the support of $\chi'(\epsilon r)$ we have that $(2+r-t)^2 \epsilon^2 \lesssim 1$ so

$$\iint_{\Sigma_t^{\text{ex}}} (2+r-t)^{\beta+1} \epsilon^2 |\chi'(\epsilon r)|^2 (\nabla_\sigma^k \partial_y^l w)^2 dx dy \lesssim \iint_{\Sigma_t^{\text{ex}}} (2+r-t)^{\beta-1} (\nabla_\sigma^k \partial_y^l w)^2 dx dy.$$

By letting $\epsilon \rightarrow 0$ we derive (4-1) also in the case of noncompactly supported w . □

Slight modifications of the above proof yield the following three results.

Lemma 4.2. *Let $\beta \in \mathbb{R}$. For a sufficiently regular function w we have*

$$\sup_{\Sigma_t^{\text{ex}}} (2+r-t)^\beta r^2 |w(t, x, y)|^2 \lesssim \iint_{\Sigma_t^{\text{ex}}} (2+r-t)^\beta ((\partial_r \mathcal{Z}^{\leq 2} w)^2 + (\mathcal{Z}^{\leq 2} w)^2) dx dy. \tag{4-3}$$

Proof. It follows by estimating v and $\partial_\rho v$ in (4-2) with the same weight and using the fact that $2+r-t \geq 1$ on Σ_t^{ex} . □

Lemma 4.3. *Let $\beta \in \mathbb{R}$. For any sufficiently regular function w we have*

$$\sup_{\Sigma_t^{\text{ex}}} (2+r-t)^\beta r^2 \|w(t, r, \cdot)\|_{L^2(\mathbb{S}^2 \times \mathbb{S}^1)}^2 \lesssim \iint_{\Sigma_t^{\text{ex}}} (2+r-t)^{\beta+1} (\partial_r w)^2 + (2+r-t)^{\beta-1} w^2 dx dy. \tag{4-4}$$

Proof. The inequality (4-4) follows by replacing v with the $L^2(\mathbb{S}^2 \times \mathbb{S}^1)$ norm of w in the left- and right-hand sides of inequality (4-2). □

Lemma 4.4. *Let $\beta \in \mathbb{R}$. For any sufficiently regular function w we have*

$$\sup_{\Sigma_t^{\text{ex}}} (2+r-t)^\beta r^2 \|w(t, r, \cdot)\|_{L^4(\mathbb{S}^2 \times \mathbb{S}^1)}^2 \lesssim \iint_{\Sigma_t^{\text{ex}}} (2+r-t)^\beta (\partial_r^{\leq 1} \mathcal{Z}^{\leq 1} w)^2 dx dy. \tag{4-5}$$

Proof. The inequality (4-5) follows by estimating v and $\partial_\rho v$ in (4-2) with the same weight, then applying the inequality with v replaced by the $L^4(\mathbb{S}^2 \times \mathbb{S}^1)$ norm of w and finally using the Sobolev injection $H^1(\mathbb{S}^2 \times \mathbb{S}^1) \subset L^4(\mathbb{S}^2 \times \mathbb{S}^1)$. □

4B. Weighted Hardy inequality.

Lemma 4.5. *Let $\beta > -1$. For any sufficiently regular function w for which the integral in the following left-hand side is finite we have*

$$\iint_{\Sigma_t^{\text{ex}}} (2+r-t)^\beta w^2 dx dy \lesssim \iint_{\Sigma_t^{\text{ex}}} (2+r-t)^{\beta+2} (\partial_r w)^2 dx dy.$$

Proof. A simple computation shows that for any $\beta \in \mathbb{R}$ and $(t, x, y) \in \Sigma_t^{\text{ex}}$

$$\partial_r [r^2 (2+r-t)^{\beta+1}] = 2r(2+r-t)^{\beta+1} + (\beta+1)r^2(2+r-t)^\beta \geq (\beta+1)r^2(2+r-t)^\beta.$$

Consequently, if $\beta > -1$ and w is a compactly supported function

$$\begin{aligned} \int_{r \geq t-1} (2+r-t)^\beta w^2 dx &= \int_{t-1}^\infty \int_{\mathbb{S}^2} (2+r-t)^\beta r^2 w^2 dr d\sigma \leq \frac{1}{\beta+1} \int_{t-1}^\infty \int_{\mathbb{S}^2} \partial_r (r^2 (2+r-t)^{\beta+1}) w^2 dr d\sigma \\ &= -\frac{2}{\beta+1} \int_{r \geq t-1} (2+r-t)^{\beta+1} w \partial_r w r^2 dr d\sigma - \frac{1}{\beta+1} \left[\int_{\mathbb{S}^2 \times \mathbb{S}^1} w^2 r^2 d\sigma \right]_{r=t-1} \\ &\lesssim \left(\int_{r \geq t-1} (2+r-t)^\beta w^2 dx \right)^{1/2} \left(\int_{r \geq t-1} (2+r-t)^{\beta+2} (\partial_r w)^2 dx \right)^{1/2}, \end{aligned}$$

and the inequality of the statement follows after further integration on \mathbb{S}^1 .

Let now w be any sufficiently regular function, not necessarily compactly supported, and χ be any fixed cut-off function. For any $\epsilon > 0$, we apply the inequality of the statement to the compactly supported function $\chi(\epsilon r)w$ and obtain

$$\iint_{\Sigma_r^{\text{ex}}} (2+r-t)^\beta \chi(\epsilon r)^2 w^2 dx dy \lesssim \iint_{\Sigma_r^{\text{ex}}} (2+r-t)^{\beta+2} [\chi(\epsilon r)^2 (\partial w)^2 + \epsilon^2 \chi'(\epsilon r)^2 w^2] dx dy.$$

On the support of $\chi'(\epsilon r)$ we have $\epsilon^2(2+r-t)^2 \lesssim 1$. Using that

$$\lim_{\epsilon \rightarrow 0} \int_{r \geq 1/\epsilon} (2+r-t)^\beta w^2 dx = 0,$$

we obtain the result of the statement by passing to the limit $\epsilon \rightarrow 0$. □

4C. Pointwise bounds.

Proposition 4.6. *Assume that the solution $W = (u, v)^T$ to (1-5)–(1-6) satisfies the a priori energy bounds (2-1) for some fixed $T_0 > 2$ and $\kappa > 0$. There exists an integrable function $l \in L^1([2, T_0])$ such that the following pointwise estimates hold true in $\mathcal{D}_{T_0}^{\text{ex}}$:*

$$\sup_{\mathbb{S}^1} |W| \lesssim \epsilon, \tag{4-6}$$

$$\sup_{\mathbb{S}^1} |\partial \mathcal{Z}^{\leq 2} W| \lesssim C_0 \epsilon r^{-1} (2+r-t)^{-(\kappa+1)/2}, \tag{4-7}$$

$$\sup_{\mathbb{S}^1} |\mathcal{F} \mathcal{Z}^{\leq 2} W| + |\partial_y \mathcal{Z}^{\leq 2} W| + |\mathcal{Z}^{\leq 2} W| \lesssim C_0 \epsilon r^{-1} \sqrt{l(t)} (2+r-t)^{-\kappa/2}, \tag{4-8}$$

$$\sup_{\mathbb{S}^1} |Z \mathcal{Z}^{\leq 2} W| \lesssim C_0 \epsilon r^{-1} (2+r-t)^{-\kappa/2}. \tag{4-9}$$

Proof. The bounds (4-7) and (4-8) follow immediately from Lemma 4.2 with $\beta = \kappa + 1$ and $\beta = \kappa$ respectively, from (2-4), the Poincaré inequality and the energy bounds (2-3), (2-4). The pointwise bound (4-9) follows instead applying Lemma 4.1 with $\beta = \kappa$ and Lemma 4.5 with $\beta = \kappa - 1$ to write that

$$\iint_{\Sigma_t^{\text{ex}}} (2+r-t)^{\kappa-1} (Z \mathcal{Z}^{\leq 4} W)^2 dx dy \lesssim \iint_{\Sigma_t^{\text{ex}}} (2+r-t)^{\kappa+1} (\partial \mathcal{Z}^{\leq 5} W)^2 dx dy.$$

Finally, the bound (4-6) on W is obtained from the integration of (4-7) along the direction $\partial_q = \partial_r - \partial_t$ until the initial time slice $t_0 = 2$ and from the smallness assumption on the initial data. □

A trivial consequence of the decomposition (1-7) and the bounds (4-8)–(4-9), that will be useful later in Section 5, is the following estimate for the zero-mode W_0 of the solution:

$$\sup_{\mathbb{S}^1} |Z \mathcal{L}^{\leq 1} W_0| \lesssim C_0 \epsilon r^{-1} (2+r-t)^{-\kappa/2}. \tag{4-10}$$

4D. Propagation of the exterior energy bounds. We start by considering any multi-index γ with $|\gamma| = n \leq 5$ and compare the system satisfied by the differentiated function $W^\gamma = (u^\gamma, v^\gamma)^T$ — which is a shorthand notation for $\mathcal{L}^\gamma W = (\mathcal{L}^\gamma u, \mathcal{L}^\gamma v)$ — to the inhomogeneous linear equation (3-1). Here the variable that plays the role of the linear variable W is W^γ . We remind the reader that all vector fields \mathcal{L} in the family (1-14) are related to the geometry of the problem and are the generators of the Lorentz transformations of the Minkowski space \mathbb{R}^{1+3} . In particular, they preserve the structure of system (1-1) and equivalently of (1-5).

The equation satisfied by W^γ is obtained by commuting \mathcal{L}^γ to (1-5)

$$(-\partial_t^2 + \Delta_x) W^\gamma + (1+u)\partial_y^2 W^\gamma = F^\gamma, \tag{4-11}$$

where the inhomogeneous term F^γ is given by

$$\begin{aligned} F^\gamma &= -[\mathcal{L}^\gamma, u\partial_y^2]W + \sum_{|\gamma_1|+|\gamma_2|\leq|\gamma|} N(W^{\gamma_1}, W^{\gamma_2}), \\ [\mathcal{L}^\gamma, u\partial_y^2]W &= \delta_\gamma \sum_{\substack{|\gamma_1|+|\gamma_2|=|\gamma| \\ |\gamma_2|<|\gamma|}} u^{\gamma_1} \partial_y^2 W^{\gamma_2}, \end{aligned} \tag{4-12}$$

where $\delta_\gamma = 0$ if $|\gamma| = 0$, and $\delta_\gamma = 1$ otherwise. The nonlinear term $N(\cdot, \cdot)$ in the right-hand side of (4-12) is a two-vector of new linear combinations of the quadratic null forms introduced in (1-3), which arise from the commutation of the Klainerman vector fields with N_1 and N_2 .

We have seen in Proposition 4.6 that under the a priori energy assumption (2-1) the solution W satisfies the pointwise bounds (4-6) and (4-7). The hypotheses of Proposition 3.1 are then fulfilled and we have

$$\begin{aligned} \|W^\gamma\|_{X_{T_0}^{\text{ex},\kappa}}^2 + \iint_{\mathcal{C}_{[2,T_0]}} (2+r-t)^{\kappa+1} (|\mathcal{L}W^\gamma|^2 + |\partial_y W^\gamma|^2) dS \\ \lesssim E^{\text{ex},\kappa}(2, W^\gamma) + \|(2+r-t)^{(\kappa+1)/2} F^\gamma\|_{L_t^1 L_{xy}^2(\mathcal{C}_{T_0}^{\text{ex}})} \|W^\gamma\|_{X_{T_0}^{\text{ex},\kappa}}. \end{aligned} \tag{4-13}$$

Proof of Proposition 2.1. It is enough for our purpose to estimate the weighted norm of the source term F^γ and prove that for every $|\gamma| \leq 5$

$$\|(2+r-t)^{(\kappa+1)/2} F^\gamma\|_{L_{xy}^2(\Sigma_t^{\text{ex}})} \lesssim C_0^2 \epsilon^2 (t-1)^{-(\kappa+1)/2} \sqrt{l(t)}, \tag{4-14}$$

where $l \in L^1([2, T_0])$. In fact, if we plug (4-14) and a priori energy bound (2-1) into (4-13) we obtain that there exists some universal positive constant C so that

$$\|W\|_{X_{T_0}^{\text{ex},\kappa}}^2 + \sum_{|\gamma|\leq 5} \iint_{\mathcal{C}_{[2,T_0]}} (2+r-t)^{\kappa+1} (|\mathcal{L}W^\gamma|^2 + |\partial_y W^\gamma|^2) dS \leq C E_5^{\text{ex},\kappa}(2, W) + 2CC_0^3 \epsilon^3.$$

For any fixed constant $K > 1$ (e.g. $K = 2$) we then choose $C_0 > 0$ sufficiently large so that

$$E_5^{\text{ex},\kappa}(2, W) \leq \frac{C_0^2 \epsilon^2}{CK}$$

and $\epsilon_0 > 0$ sufficiently small so that $2CC_0^2\epsilon < 1/K$ for $\epsilon \leq \epsilon_0$ to finally obtain

$$\|W\|_{X_{T_0}^{s,\kappa}}^2 + \sum_{|\gamma| \leq 5} \iint_{\mathcal{C}_{[2,T_0]}} (2+r-t)^{\kappa+1} (|\mathcal{T}W^\gamma|^2 + |\partial_y W^\gamma|^2) dS \leq \frac{2C_0\epsilon^2}{K}.$$

We estimate the different contributions to F^γ separately. In all the estimates that follow we will use the a priori energy bound (2-1) and the fact that $r > t - 1$ in the exterior region.

(1) *The null terms:* We use here the null form representation via the formula (1-16)

$$N(W^{\gamma_1}, W^{\gamma_2}) \sim \mathcal{T}W^{\gamma_1} \cdot \partial W^{\gamma_2} + \partial W^{\gamma_1} \cdot \mathcal{T}W^{\gamma_2}.$$

The products in the above right-hand side are equivalent given the range of γ_1 and γ_2 and we only focus on the analysis of the first one. We distinguish between the different values that γ_1 and γ_2 can take and remind the reader that $|\gamma_1| + |\gamma_2| \leq |\gamma| = n \leq 5$.

(a) *The case $|\gamma_1| = 0$:* Here $\gamma_2 = \gamma$ and we immediately obtain from (4-8) that

$$\begin{aligned} \|(2+r-t)^{(\kappa+1)/2} \mathcal{T}W \cdot \partial W^\gamma\|_{L_{xy}^2} &\leq \|\mathcal{T}W\|_{L_{xy}^\infty} \|W^\gamma\|_{X_{T_0}^{\text{ex},\kappa}} \\ &\lesssim C_0^2\epsilon^2(t-1)^{-1}\sqrt{l(t)}. \end{aligned}$$

(b) *The case $|\gamma_2| = 0$:* Here $\gamma_1 = \gamma$ and we obtain from (4-7) that

$$\begin{aligned} \|(2+r-t)^{(\kappa+1)/2} \mathcal{T}W^\gamma \cdot \partial W\|_{L_{xy}^2} &\leq \|(2+r-t)^{1/2} \partial W\|_{L_{xy}^\infty} \|(2+r-t)^{\kappa/2} \mathcal{T}W^\gamma\|_{L_{xy}^2} \\ &\lesssim C_0^2\epsilon^2(t-1)^{-1}\sqrt{l(t)}. \end{aligned}$$

(c) *The case $|\gamma_1|, |\gamma_2| > 0$:* We use spherical polar coordinates and the Cauchy–Schwarz inequality to bound the weighted $L_{xy}^2(\Sigma_t^{\text{ex}})$ norm of $\mathcal{T}W^{\gamma_1} \cdot \partial W^{\gamma_2}$ as follows:

$$\begin{aligned} \|(2+r-t)^{(\kappa+1)/2} \mathcal{T}W^{\gamma_1} \cdot \partial W^{\gamma_2}\|_{L_{xy}^2}^2 &= \int_{t-1}^\infty \iint_{\mathbb{S}^2 \times \mathbb{S}^1} (2+r-t)^{\kappa+1} |\mathcal{T}W^{\gamma_1}|^2 |\partial W^{\gamma_2}|^2 r^2 dr d\sigma dy \\ &\lesssim \int_{t-1}^\infty (2+r-t)^{\kappa+1} \|\mathcal{T}W^{\gamma_1}\|_{L^4(\mathbb{S}^2 \times \mathbb{S}^1)}^2 \|\partial W^{\gamma_2}\|_{L^4(\mathbb{S}^2 \times \mathbb{S}^1)}^2 r^2 dr. \end{aligned}$$

In the case where $|\gamma_1| \leq n - 2$ we apply the inequality (4-5) to $\mathcal{T}W^{\gamma_1}$ with $\beta = \kappa$ and Sobolev’s injection $H^1(\mathbb{S}^2 \times \mathbb{S}^1) \subset L^4(\mathbb{S}^2 \times \mathbb{S}^1)$ to ∂W^{γ_2} . We derive that

$$\begin{aligned} &\int_{t-1}^\infty (2+r-t)^{\kappa+1} \|\mathcal{T}W^{\gamma_1}\|_{L^4(\mathbb{S}^2 \times \mathbb{S}^1)}^2 \|\partial W^{\gamma_2}\|_{L^4(\mathbb{S}^2 \times \mathbb{S}^1)}^2 r^2 dr \\ &\lesssim \|(2+r-t)^{\kappa/2} \mathcal{T}\mathcal{X}^{\leq n} W\|_{L_{xy}^2(\Sigma_t^{\text{ex}})}^2 \int_{t-1}^\infty (2+r-t)r^{-2} \|\partial \mathcal{X}^{\leq n} W\|_{L^2(\mathbb{S}^2 \times \mathbb{S}^1)}^2 r^2 dr \\ &\lesssim (t-1)^{-2} \|(2+r-t)^{\kappa/2} \mathcal{T}\mathcal{X}^{\leq n} W\|_{L_{xy}^2(\Sigma_t^{\text{ex}})}^2 \|(2+r-t)^{(\kappa+1)/2} \partial \mathcal{X}^{\leq n} W\|_{L_{xy}^2(\Sigma_t^{\text{ex}})}^2. \end{aligned}$$

In the remaining case where $|\gamma_1| = n - 1$ and $|\gamma_2| = 1$ we apply the inequality (4-5) to ∂W^{γ_2} with $\beta = \kappa + 1$ and the injection $H^1(\mathbb{S}^2 \times \mathbb{S}^1) \subset L^4(\mathbb{S}^2 \times \mathbb{S}^1)$ to $\mathcal{T}W^{\gamma_1}$. We get

$$\begin{aligned} \int_{t-1}^{\infty} (2+r-t)^{\kappa+1} \|\mathcal{T}W^{\gamma_1}\|_{L^4(\mathbb{S}^2 \times \mathbb{S}^1)}^2 \|\partial W^{\gamma_2}\|_{L^4(\mathbb{S}^2 \times \mathbb{S}^1)}^2 r^2 dr \\ \lesssim \|(2+r-t)^{(\kappa+1)/2} \partial \mathcal{L}^{\leq n} W\|_{L^2_{xy}(\Sigma_t^{\text{ex}})}^2 \iint_{\Sigma_t^{\text{ex}}} r^{-2} |\mathcal{T} \mathcal{L}^{\leq n} W|^2 dx dy \\ \lesssim (t-1)^{-2} \|(2+r-t)^{\kappa/2} \mathcal{T} \mathcal{L}^{\leq n} W\|_{L^2_{xy}(\Sigma_t^{\text{ex}})}^2 \|W\|_{X_{T_0}^{n,\kappa}}^2. \end{aligned}$$

In both scenarios we obtain

$$\|(2+r-t)^{(\kappa+1)/2} \mathcal{T}W^{\gamma_1} \cdot \partial W^{\gamma_2}\|_{L^2_{xy}} \lesssim C_0^2 \epsilon^2 (t-1)^{-1} \sqrt{l(t)}.$$

(2) *The commutator terms:* Since $|\gamma_1| \geq 1$, we can write $u^{\gamma_1} = \mathcal{L}u^{\tilde{\gamma}_1}$ for some $|\tilde{\gamma}_1| = |\gamma_1| - 1$. We can also write $\partial_y^2 W^{\gamma_2} = \partial_y W^{\tilde{\gamma}_2}$ for some other $\tilde{\gamma}_2$ such that $|\tilde{\gamma}_2| = |\gamma_2| + 1$ and observe that then $|\tilde{\gamma}_1| + |\tilde{\gamma}_2| = n$. Depending on $\gamma_1 = (\alpha_1, \beta_1)$ we can distinguish two cases:

(a) *The case $|\alpha_1| > 0$:* Here we choose $\tilde{\gamma}_1$ so that $\mathcal{L}u^{\tilde{\gamma}_1} = \partial u^{\tilde{\gamma}_1}$. The products $\partial u^{\tilde{\gamma}_1} \cdot \partial_y W^{\tilde{\gamma}_2}$ have the same behavior of the null terms treated in case 1.

(b) *The case $|\alpha_1| = 0$:* Here $\mathcal{L}^{\gamma_1} = Z^{\beta_1}$ is a pure product of Klainerman vector fields and $\mathcal{L}u^{\tilde{\gamma}_1} = Zu^{\tilde{\gamma}_1}$. We choose the exponents (p_1, p_2) using

$$(p_1, p_2) = \begin{cases} (2, \infty) & \text{if } |\tilde{\gamma}_1| = n - 1, \\ (\infty, 2) & \text{if } |\tilde{\gamma}_2| = n, \\ (4, 4) & \text{otherwise} \end{cases}$$

and will place the two factors in $L^{p_1}(\mathbb{S}^2 \times \mathbb{S}^1)$ and $L^{p_2}(\mathbb{S}^2 \times \mathbb{S}^1)$ respectively. We use the Sobolev injections $H^2(\mathbb{S}^2 \times \mathbb{S}^1) \subset L^\infty(\mathbb{S}^2 \times \mathbb{S}^1)$ and $H^1(\mathbb{S}^2 \times \mathbb{S}^1) \subset L^4(\mathbb{S}^2 \times \mathbb{S}^1)$ to derive

$$\begin{aligned} \|(2+r-t)^{(\kappa+1)/2} Zu^{\tilde{\gamma}_1} \partial_y W^{\tilde{\gamma}_2}\|_{L^2_{xy}}^2 &\lesssim \int_{t-1}^{\infty} (2+r-t)^{\kappa+1} \|Zu^{\tilde{\gamma}_1}\|_{L^{p_1}(\mathbb{S}^2 \times \mathbb{S}^1)}^2 \|\partial_y W^{\tilde{\gamma}_2}\|_{L^{p_2}(\mathbb{S}^2 \times \mathbb{S}^1)}^2 r^2 dr \\ &\lesssim \int_{t-1}^{\infty} (2+r-t)^{\kappa+1} \|Z \mathcal{L}^{\leq 4} u\|_{L^2(\mathbb{S}^2 \times \mathbb{S}^1)}^2 \|\partial_y \mathcal{L}^{\leq 5} W\|_{L^2(\mathbb{S}^2 \times \mathbb{S}^1)}^2 r^2 dr. \end{aligned}$$

Applying the inequality (4-4) to $Z \mathcal{L}^{\leq 4} u$ with $\beta = \kappa$ and successively the weighted Hardy inequality proved in Lemma 4.5 with $\beta = \kappa - 1$ we find

$$\begin{aligned} \sup_{r \geq t-1} (2+r-t)^\kappa r^2 \|Z \mathcal{L}^{\leq 4} u\|_{L^2(\mathbb{S}^2 \times \mathbb{S}^1)}^2 \\ \lesssim \|(2+r-t)^{(\kappa+1)/2} \partial_r \mathcal{L}^{\leq 5} u\|_{L^2_{xy}(\Sigma_t^{\text{ex}})}^2 + \|(2+r-t)^{(\kappa-1)/2} Z \mathcal{L}^{\leq 4} u\|_{L^2_{xy}(\Sigma_t^{\text{ex}})}^2 \\ \lesssim \|(2+r-t)^{(\kappa+1)/2} \partial \mathcal{L}^{\leq 5} u\|_{L^2_{xy}(\Sigma_t^{\text{ex}})}^2. \end{aligned}$$

We can therefore continue the previous chain of inequalities

$$\begin{aligned} \lesssim \|(2+r-t)^{(\kappa+1)/2} \partial_r \mathcal{L}^{\leq 5} u\|_{L^2_{xy}(\Sigma_t^{\text{ex}})}^2 \int_{t-1}^{\infty} (2+r-t) r^{-2} \|\partial_y \mathcal{L}^{\leq 5} W\|_{L^2(\mathbb{S}^2 \times \mathbb{S}^1)}^2 r^2 dr \\ \lesssim (t-1)^{-1-\kappa} \|W\|_{X_{T_0}^{5,\kappa}}^2 \|(2+r-t)^{\kappa/2} \partial_y \mathcal{L}^{\leq 5} W\|_{L^2_{xy}(\Sigma_t^{\text{ex}})}^2 \end{aligned}$$

and finally conclude that

$$\sum_{\substack{|\gamma_1|+|\gamma_2|=n \\ |\gamma_1|\geq 1}} \|(2+r-t)^{(\kappa+1)/2} u^{\gamma_1} \cdot \partial_y^2 W^{\gamma_2}\|_{L^2_{xy}} \lesssim C_0^2 \epsilon^2 (t-1)^{-(\kappa+1)/2} \sqrt{l(t)}. \quad \square$$

As a byproduct of the above proof we have also obtained that, for any fixed $K > 1$, there exist $C_0 > 0$ sufficiently large and $\epsilon_0 > 0$ sufficiently small such that

$$\sum_{|\gamma|\leq 5} \iint_{\mathcal{C}_{[2, T_0]}} (2+r-t)^{\kappa+1} (|\mathcal{T} \mathcal{Z}^\gamma W|^2 + |\partial_y \mathcal{Z}^\gamma W|^2) dS \leq \frac{2C_0 \epsilon^2}{K}. \quad (4-15)$$

An immediate consequence of the global energy bounds (2-2) obtained from Proposition 2.1 is the following estimate on the higher-order energies of the solution W on the truncated exterior hyperboloids $\mathcal{H}_s^{\text{ex}}$ for any $s > 0$.

Proposition 4.7. *Let $W = (u, v)^T$ be the global solution of the Cauchy problem (1-5)–(1-6) in the exterior region \mathcal{D}^{ex} . There exists a constant $C > 0$ such that for any $s > 0$*

$$E_5^{\text{ex},h}(s, W) + \delta_{s>\sqrt{3}} \sum_{|\gamma|\leq 5} \iint_{\mathcal{C}_{[2, T_s]}} |\mathcal{T} \mathcal{Z}^\gamma W|^2 + |\partial_y \mathcal{Z}^\gamma W|^2 dS \leq C C_0^2 \epsilon^2, \quad (4-16)$$

where $\delta_{s>\sqrt{3}} = 0$ if $s \leq \sqrt{3}$, 1 otherwise and $T_s = \frac{1}{2}(s^2 + 1)$.

Proof. The result follows by applying Proposition 3.2 with $\mathbf{W} = W^\gamma$ and $\mathbf{F} = F^\gamma$ for any multi-index γ such that $|\gamma| \leq 5$ and then using the global energy bound (2-2), the estimate (4-14) of the source term F^γ and the fact that

$$\|F^\gamma\|_{L_t^1 L_{xy}^2(\mathcal{H}_{[2,s]}^{\text{ex}})} \lesssim \int_2^\infty \|F^\gamma\|_{L_{xy}^2(\Sigma_t^{\text{ex}})} dt \lesssim C_0^2 \epsilon^2. \quad \square$$

We conclude this section with the derivation of a bound for the higher-order exterior conformal energy of W_0 as well as for the higher-order conformal energy on portions of the hyperboloid \mathcal{H}_s in the exterior region \mathcal{D}^{ex} .

Proposition 4.8. *Assume the solution $W = (u, v)^T$ of the Cauchy problem (1-5)–(1-6) satisfies the a priori exterior energy bounds (2-1). Then*

$$\sum_{|\gamma|\leq 4} \int_{\mathcal{C}_{[2, T_0]}} |(t+r)(\partial_t W_0^\gamma + \partial_r W_0^\gamma) + 2W_0^\gamma|^2 + (t-r)^2 (|\nabla_x W_0^\gamma|^2 - (\partial_r W_0^\gamma)^2) dS + \sup_{[2, T_0]} E_4^{\text{c,ex}}(t, W_0) \lesssim C_0^2 \epsilon^2 \ln T_0, \quad (4-17)$$

where the implicit constant only depends on C_0 .

Proof. Let us fix $|\gamma| \leq 4$. By integrating (4-11) over the sphere \mathbb{S}^1 we obtain that W_0^γ is solution of the linear inhomogeneous wave equation

$$(-\partial_t^2 + \Delta_x) W_0^\gamma = F_0^\gamma, \quad (4-18)$$

with source term

$$\begin{aligned} F_0^\gamma &= \int_{\mathbb{S}^1} F^\gamma \frac{dy}{2\pi} - \int_{\mathbb{S}^1} [\mathcal{E}^\gamma, u \partial_y^2] W^\gamma \frac{dy}{2\pi} \\ &= \sum_{|\gamma_1|+|\gamma_2| \leq |\gamma|} \int_{\mathbb{S}^1} N(W^{\gamma_1}, W^{\gamma_2}) \frac{dy}{2\pi} + \sum_{|\gamma_1|+|\gamma_2|=|\gamma|} \int_{\mathbb{S}^1} \partial_y u^{\gamma_1} \cdot \partial_y W^{\gamma_2} \frac{dy}{2\pi}. \end{aligned} \tag{4-19}$$

When applying Proposition 3.6 with $W_0 = W_0^\gamma$ and $F_0 = F_0^\gamma$, we derive that for all $T \in [2, T_0]$

$$\begin{aligned} E^{c,\text{ex}}(T, W_0^\gamma) + \int_{\mathcal{C}_{[2,T]}} |(t+r)(\partial_t W_0^\gamma + \partial_r W_0^\gamma) + 2W_0^\gamma|^2 + (t-r)^2 (|\nabla_x W_0^\gamma|^2 - (\partial_r W_0^\gamma)^2) dS \\ \leq E^{c,\text{ex}}(2, W_0^\gamma) + \int_2^T \|(t+r)F_0^\gamma\|_{L_x^2(\Sigma_t^{\text{ex}})} E^{c,\text{ex}}(t, W_0^\gamma)^{1/2} dt, \end{aligned} \tag{4-20}$$

where

$$\|(t+r)F_0^\gamma\|_{L_x^2(\Sigma_t^{\text{ex}})} \leq \sum_{|\gamma_1|+|\gamma_2| \leq |\gamma|} \|(t+r)N(W^{\gamma_1}, W^{\gamma_2})\|_{L_{xy}^2} + \sum_{|\gamma_1|+|\gamma_2|=|\gamma|} \|(t+r)\partial_y u^{\gamma_1} \cdot \partial_y W^{\gamma_2}\|_{L_{xy}^2}.$$

We estimate the different contributions to F_0^γ separately. We start by observing that since $|\gamma_1| + |\gamma_2| \leq 4$, at least one of the two multi-indexes has length less than or equal to 2. We call this index γ_j and will place the factor carrying it in L^∞ , and the other one in L^2 .

(1) *The commutator terms:* We use the pointwise bound (4-8) and the energy bound (2-4)

$$\|(t+r)\partial_y u^{\gamma_1} \cdot \partial_y W^{\gamma_2}\|_{L_{xy}^2(\Sigma_t^{\text{ex}})} \lesssim C_0 \epsilon \sqrt{l(t)} \|\partial_y \mathcal{E}^{\leq 4} W\|_{L_{xy}^2(\Sigma_t^{\text{ex}})} \lesssim C_0^2 \epsilon^2 l(t).$$

(2) *The null terms:* Here we use the null form representation via the formula (1-15) and the relation $\bar{\partial} = t^{-1}Z$ to write

$$N(W^{\gamma_1}, W^{\gamma_2}) = \frac{1}{t} Z W^{\gamma_1} \cdot \partial W^{\gamma_2} + \frac{1}{t} \partial W^{\gamma_1} \cdot Z W^{\gamma_2} + \frac{t-r}{t} \partial W^{\gamma_1} \cdot \partial W^{\gamma_2}. \tag{4-21}$$

The first two products in the above right-hand side are equivalent given the range of γ_1 and γ_2 so we will just analyze the first one. In the case where $|\gamma_1| \leq 2$ we deduce from the pointwise bound (4-9) that

$$\|(t+r)t^{-1}Z W^{\gamma_1} \cdot \partial W^{\gamma_2}\|_{L_{xy}^2(\Sigma_t^{\text{ex}})} \lesssim C_0 \epsilon \|(t+r)r^{-1}t^{-1}\partial W^{\gamma_2}\|_{L_{xy}^2(\Sigma_t^{\text{ex}})} \lesssim C_0 \epsilon t^{-1} E_4^{\text{ex},0}(t, W)^{1/2}.$$

In the case where $|\gamma_1| \geq 3$, we bound ∂W^{γ_2} using (4-7) and decompose W^{γ_1} according to (1-7). We apply the Poincaré inequality to obtain

$$\|(t+r)t^{-1}Z W^{\gamma_1} \cdot \partial W^{\gamma_2}\|_{L_{xy}^2(\Sigma_t^{\text{ex}})} \lesssim C_0 \epsilon \|(t+r)r^{-1}t^{-1}\partial_y Z W^{\gamma_1}\|_{L_{xy}^2(\Sigma_t^{\text{ex}})} \lesssim C_0 \epsilon t^{-1} E_5^{\text{ex},0}(t, W)^{1/2}$$

and use Lemma 4.5 with $\beta = \kappa - 1$ to get

$$\begin{aligned} \|(t+r)t^{-1}Z W_0^{\gamma_1} \cdot \partial W^{\gamma_2}\|_{L_{xy}^2(\Sigma_t^{\text{ex}})} &\lesssim C_0 \epsilon \|(t+r)t^{-1}r^{-1}(2+r-t)^{-\kappa+(\kappa-1)/2} Z W_0^{\gamma_1}\|_{L_{xy}^2(\Sigma_t^{\text{ex}})} \\ &\lesssim C_0 \epsilon t^{-1} E_5^{\text{ex},\kappa}(t, W)^{1/2}. \end{aligned}$$

The last quadratic term in the right-hand side of (4-21) is estimated using again (4-7),

$$\begin{aligned} \|(t+r)(t-r)/t \partial W^{\gamma_1} \cdot \partial W^{\gamma_2}\|_{L^2_{xy}(\Sigma_t^{\text{ex}})} &\lesssim C_0 \epsilon t^{-1} \|(2+r-t)^{(1-\kappa)/2} \partial \mathcal{L}^{\leq 4} W\|_{L^2_{xy}(\Sigma_t^{\text{ex}})} \\ &\lesssim C_0 \epsilon t^{-1} E_4^{\text{ex},0}(t, W)^{1/2}, \end{aligned}$$

which gives

$$\|(t+r)N(W^{\gamma_1}, W^{\gamma_2})\|_{L^2_{xy}(\Sigma_t^{\text{ex}})} \lesssim C_0 \epsilon t^{-1} E_5^{\text{ex},\kappa}(t, W)^{1/2}.$$

The combination of steps (1) and (2) with the energy bound (2-1) yields

$$\|(t+r)F_0^\gamma\|_{L^2_{xy}(\Sigma_t^{\text{ex}})} \lesssim C_0^2 \epsilon^2 l(t) + C_0^2 \epsilon^2 t^{-1}, \quad (4-22)$$

which plugged into (4-20) for all $|\gamma| \leq 4$ gives

$$\begin{aligned} E_4^{\text{c,ex}}(T, W_0) + \sum_{|\gamma| \leq 4} \int_{\mathcal{H}_{[2,T]}} &|(t+r)(\partial_t W_0^\gamma + \partial_r W_0^\gamma) + 2W_0^\gamma|^2 + (t-r)^2 (|\nabla_x W_0^\gamma|^2 - (\partial_r W_0^\gamma)^2) dS \\ &\lesssim E_4^{\text{c,ex}}(2, W_0) + \int_2^T (C_0^2 \epsilon^2 l(t) + C_0^2 \epsilon^2 t^{-1}) E_4^{\text{c,ex}}(t, W_0)^{1/2} dt \\ &\lesssim E_4^{\text{c,ex}}(2, W_0) + C_0^2 \epsilon^2 \sup_{[2, T_0]} E_4^{\text{c,ex}}(t, W_0) + C_0^2 \epsilon^2 \ln T. \end{aligned}$$

If $\epsilon \ll 1$ is sufficiently small we get

$$\begin{aligned} \sup_{[2, T_0]} E_4^{\text{c,ex}}(t, W_0) + \sum_{|\gamma| \leq 4} \int_{\mathcal{H}_{[2, T_0]}} &|(t+r)(\partial_t W_0^\gamma + \partial_r W_0^\gamma) + 2W_0^\gamma|^2 + (t-r)^2 (|\nabla_x W_0^\gamma|^2 - (\partial_r W_0^\gamma)^2) dS \\ &\lesssim E_4^{\text{c,ex}}(2, W_0) + C_0^2 \epsilon^2 \ln T_0, \end{aligned}$$

so the result of the proposition follows from the smallness of the conformal energy at the initial time, which in turn follows from the assumptions on the initial data. \square

Lemma 4.9. *Let $s \geq 2$ and $2 < T_1 < T_2$ be such that the portion of hyperboloid \mathcal{H}_s in the time strip $[T_1, T_2]$ is entirely contained in the exterior region \mathcal{D}^{ex} . Assume $W = (u, v)^T$ is the solution to the Cauchy problem (1-5)–(1-6) in \mathcal{D}^{ex} satisfying the global energy bounds (2-1). Then there exists a constant $C > 0$ such that for all $|\gamma| \leq 4$*

$$\left\| \frac{1}{t} K W_0^\gamma + 2W_0^\gamma \right\|_{L^2(\mathcal{H}_s \cap [T_1, T_2])}^2 + \sum_{i=1}^3 \left\| \frac{s}{t} \Omega_{0i} W_0^\gamma \right\|_{L^2(\mathcal{H}_s \cap [T_1, T_2])}^2 \leq C C_0^2 \epsilon^2 \ln T_2.$$

Proof. We apply Proposition 3.7 with $W_0 = W^\gamma$ and $|\gamma| \leq 4$. It follows from the hypotheses that $\Sigma_t^s \subset \Sigma_t^{\text{ex}}$ and hence that $E_s^{\text{c,ex}}(t, W_0) \leq E^{\text{c,ex}}(t, W_0)$ for all $t \in [T_1, T_2]$. Therefore, the result is obtained using estimates (4-17) and (4-22). \square

5. Pointwise estimates in the interior region

The goal of this section is to recover pointwise estimates for solutions $W = (u, v)^T$ to (1-5) in the interior hyperbolic region $\mathcal{H}_{[2, s_0]}^{\text{in}}$ under the a priori assumptions (2-5)–(2-7) and to propagate the a priori pointwise estimate (2-7) on ZW_0 .

5A. Pointwise estimates from Klainerman–Sobolev inequalities. A first subset of pointwise estimates for W_0 and W is immediately obtained from (2-5) and (2-6) via the following Sobolev inequality on hyperboloids, whose proof can be found in [LeFloch and Ma 2018].

Lemma 5.1. *Let $W = W(t, x)$ be a sufficiently regular function in the cone $\mathcal{C} = \{t > r\}$. For all $(t, x) \in \mathcal{C}$, let $s = \sqrt{t^2 - r^2}$ and $B(x, \frac{1}{3}t)$ be the ball centered at x with radius $\frac{1}{3}t$. Then*

$$|W(t, x)|^2 \leq Ct^{-3} \sum_{|\gamma| \leq 2} \int_{B(x, t/3)} |Z^\gamma W(\sqrt{s^2 + |\xi|^2}, \xi)|^2 d\xi,$$

where C is a positive universal constant and $Z_j = x_j \partial_t + t \partial_j$, $j = \overline{1, 3}$.

We remind the reader that we proved uniform-in-time energy bounds (4-16) for the solution on exterior truncated hyperboloids $\mathcal{H}_s^{\text{ex}}$ as well as the exterior pointwise bound (4-10) on ZW_0 . We can then think of the a priori energy bounds (2-5), (2-6) as being valid not only on $\mathcal{H}_s^{\text{in}}$ for $s \geq 2$ but on all (branches of) hyperboloids contained in the upper half plane $t \geq 2$. It is analogous for the pointwise bound (2-7), which can be thought to hold true for every (t, x, y) such that $t \geq 2$ and $t^2 - r^2 \leq s_0^2$. Therefore, the following lemma provides us with pointwise estimates for the solution on the portion of the interior light cone below the hyperboloid \mathcal{H}_{s_0} , which we denote by $\mathcal{H}_{[2, s_0]}$,

$$\mathcal{H}_{[2, s_0]} := \{(t, x) : t \geq 2 \text{ and } t^2 - r^2 \leq s_0^2\} \times \mathbb{S}^1.$$

Lemma 5.2. *Let $\mathcal{I}_{n,k}$ denote the set of multi-indexes of type (n, k) . Under the a priori energy bounds (2-5) and (2-6) we have the following pointwise estimates in $\mathcal{H}_{[2, s_0]}$:*

$$|\partial \mathcal{L}^{\leq 3} W_0(t, x)| \lesssim \epsilon t^{-1/2} s^{-1}, \tag{5-1}$$

$$|\bar{\partial} \mathcal{L}^{\leq 3} W_0(t, x)| \lesssim \epsilon t^{-3/2}, \tag{5-2}$$

$$\sum_{\mathcal{I}_{3,k}} \|\mathcal{L}^\gamma W(t, x, \cdot)\|_{L^\infty(\mathbb{S}^1)} + \|\partial_y^{\leq 1} \mathcal{L}^\gamma W(t, x, \cdot)\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon t^{-3/2} s^{\delta_{k+2}}, \quad k = \overline{0, 3}, \tag{5-3}$$

$$\sum_{|\gamma|=3} \|\partial_{tx} \mathcal{L}^\gamma W(t, x, \cdot)\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon t^{-1/2} s^{-1+\delta_5}, \tag{5-4}$$

and

$$\sum_{\mathcal{I}_{2,k}} \|\bar{\partial} \partial_y^{\leq 1} \mathcal{L}^\gamma W(t, x, \cdot)\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon t^{-5/2} s^{\delta_{k+3}}, \quad k = \overline{0, 2}, \tag{5-5}$$

$$\sum_{\mathcal{I}_{1,k}} \|\bar{\partial}^2 \partial_y^{\leq 1} \mathcal{L}^\gamma W(t, x, \cdot)\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon t^{-7/2} s^{\delta_{k+4}}, \quad k = \overline{0, 1}, \tag{5-6}$$

$$\|\bar{\partial}^2 \partial_{tx} \mathcal{L} W(t, x, \cdot)\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon t^{-5/2} s^{-1+\delta_5}. \tag{5-7}$$

Moreover

$$\sup_{\mathcal{H}_{[2, s_0]}} |W| \lesssim \epsilon. \tag{5-8}$$

Proof. The estimates (5-1), (5-2) and (5-4) are immediate consequence of Lemma 5.1, the energy assumptions (2-5), (2-6) and the fact that $|\mathcal{L}, (s/t)]W| \lesssim |W|$ for any W . The same is for the estimate of $\partial_y \mathcal{L}^\gamma W$ in (5-3), while the remaining norms of $\mathcal{L}^\gamma W$ there are obtained using also the Sobolev

injection on \mathbb{S}^1 (for the L^∞ norm) and the Poincaré inequality (for the L^2 norm). The estimates on the $\bar{\partial}$ derivatives of W are deduced from (5-3) and (5-4) using the fact that $\bar{\partial} = t^{-1}Z$. Finally, if we set $\tilde{W}_0(s, x) = W_0(\sqrt{s^2 + r^2}, x)$, we see that

$$|\tilde{W}_0(s, x) - \tilde{W}_0(2, x)| \leq \int_2^s |\partial_\tau \tilde{W}_0(\tau, x)| d\tau = \int_2^s \frac{\tau}{t} |\partial_t W_0(\sqrt{\tau^2 + x^2}, x)| d\tau \lesssim \epsilon$$

and hence derive from the smallness of the initial data that

$$|W_0(t, x)| \lesssim |\tilde{W}_0(s, x) - \tilde{W}_0(2, x)| + |\tilde{W}_0(2, x)| \lesssim \epsilon.$$

The combination of the above estimate with (5-3) yields (5-8). □

5B. Improved pointwise estimates on the nonzero modes. The bounds for the nonzero mode W of the solution obtained in Lemma 5.2 via Sobolev embeddings are affected by the small growth in s of the energies and are not sharp. However, they can be improved if one studies more closely the equation satisfied by W . Enhancing such bounds and particularly (5-3) will be fundamental to propagate the a priori pointwise bound (2-7) later in Proposition 5.6. We make use of the following result, which is motivated by [Klainerman 1985] and whose proof is an adaptation of a similar estimate for Klein–Gordon equations initially proved in [LeFloch and Ma 2016], later revisited in [Dong and Wyatt 2020a] in the case of Klein–Gordon equations with variable mass.

Proposition 5.3. *Assume W is a solution of the equation*

$$\square_{x,y} W + u \Delta_y W = F, \quad (t, x, y) \in \mathbb{R}^{1+3} \times \mathbb{S}^1, \tag{5-9}$$

such that $\int_{\mathbb{S}^1} W dy = 0$. For every fixed (t, x) in the region $\mathcal{C} = \{t > r\}$, let $s = \sqrt{t^2 - r^2}$ and $Y_{tx}, A_{tx}, B_{tx} : \mathbb{R}^+ \setminus \{0\} \rightarrow \mathbb{R}^+$ be the functions defined as

$$Y_{tx}^2(\lambda) := \int_{\mathbb{S}^1} \lambda \left| \frac{3}{2} W_\lambda + (\mathcal{S}W)_\lambda \right|^2 + \lambda^3 (1 + u_\lambda) |\partial_y W_\lambda|^2 dy, \tag{5-10}$$

$$A_{tx}(\lambda) := \sup_{\mathbb{S}^1} \frac{1}{2\lambda} |(\mathcal{S}u)_\lambda| + \sup_{\mathbb{S}^1} |\partial_y u_\lambda|, \tag{5-11}$$

$$B_{tx}^2(\lambda) = \int_{\mathbb{S}^1} \lambda^{-1} |(RW)_\lambda|^2 dy, \tag{5-12}$$

where $f_\lambda(t, x, y) = f(\lambda t/s, \lambda r/s, y)$ and

$$RW(t, x, y) = s^2 \bar{\partial}^i \bar{\partial}_i W + x^i x^j \bar{\partial}_i \bar{\partial}_j W + \frac{3}{4} W + 3x^i \bar{\partial}_i W - s^2 F.$$

Then W satisfies the following inequality in the hyperbolic region $\mathcal{H}_{[2, \infty)}$:

$$s^{3/2} (\|W\|_{L^2(\mathbb{S}^1)} + \|\partial_y W\|_{L^2(\mathbb{S}^1)}) + s^{1/2} \|\mathcal{S}W\|_{L^2(\mathbb{S}^1)} \lesssim \left(Y_{tx}(2) + \int_2^s B_{tx}(\lambda) d\lambda \right) e^{\int_2^s A_{tx}(\lambda) d\lambda}.$$

Proof. For every fixed $(t, x, y) \in \mathcal{H}_{[2, \infty)}$ we define $\omega_{txy}(\lambda) := \lambda^{3/2} W(\lambda t/s, \lambda x/s, y)$ to be the evaluation of W on the hyperboloid \mathcal{H}_λ dilated by factor $\lambda^{3/2}$. We have

$$\dot{\omega}_{txy}(\lambda) = \lambda^{1/2} \left(\frac{3}{2} W_\lambda + (\mathcal{S}W)_\lambda \right)$$

and

$$\ddot{\omega}_{txy}(\lambda) = \lambda^{-1/2}(P\mathbf{W})_\lambda,$$

where

$$P\mathbf{W} = \frac{3}{4}\mathbf{W} + 3(t\partial_t\mathbf{W} + x^i\partial_i\mathbf{W}) + (t^2\partial_t^2\mathbf{W} + 2tx^i\partial_i\partial_t\mathbf{W} + x^ix^j\partial_i\partial_j\mathbf{W}).$$

Using (5-9) we derive that ω_{txy} satisfies the equation

$$\begin{aligned} \ddot{\omega}_{txy} - (1 + u_\lambda)\partial_y^2\omega_{txy} &= -\lambda^{3/2}\mathbf{F}_\lambda + \lambda^{-1/2}(s^2\bar{\partial}^i\bar{\partial}_i\mathbf{W} + x^ix^j\bar{\partial}_i\bar{\partial}_j\mathbf{W} + \frac{3}{4}\mathbf{W} + 3x^i\bar{\partial}_i\mathbf{W})_\lambda \\ &= \lambda^{-1/2}(R\mathbf{W})_\lambda. \end{aligned}$$

We drop the lower indexes in $\omega_{txy}(\lambda)$ in order to have lighter notation and simply denote it by $\omega(\lambda)$ in what follows. We multiply the above equation by $\partial_\lambda\omega$ and integrate over \mathbb{S}^1 :

$$\begin{aligned} \int_{\mathbb{S}^1} \partial_\lambda\omega(\partial_\lambda^2\omega - (1 + u_\lambda)\partial_y^2\omega) dy &= \frac{d}{d\lambda} \left(\frac{1}{2} \int_{\mathbb{S}^1} |\partial_\lambda\omega|^2 dy \right) + \int_{\mathbb{S}^1} (1 + u_\lambda)\partial_y\omega\partial_y\partial_\lambda\omega dy + \int_{\mathbb{S}^1} \partial_\lambda\omega\partial_yu_\lambda\partial_y\omega dy \\ &= \frac{d}{d\lambda} \left(\frac{1}{2} \int_{\mathbb{S}^1} |\partial_\lambda\omega|^2 + (1 + u_\lambda)|\partial_y\omega|^2 dy \right) - \frac{1}{2} \int_{\mathbb{S}^1} \partial_\lambda u_\lambda |\partial_y\omega|^2 dy + \int_{\mathbb{S}^1} \partial_\lambda\omega\partial_yu_\lambda\partial_y\omega dy. \end{aligned}$$

We obtain

$$\frac{d}{d\lambda} Y_{tx}^2(\lambda) \lesssim A_{tx}(\lambda)Y_{tx}^2(\lambda) + B_{tx}(\lambda)Y_{tx}(\lambda),$$

with A_{tx}, B_{tx}, Y_{tx} as in the statement and from the Gronwall lemma

$$Y_{tx}(s) \lesssim \left(Y_{tx}(2) + \int_2^s B_{tx}(\lambda) d\lambda \right) e^{\int_2^s A_{tx}(\lambda) d\lambda}.$$

Finally, from the definition of Y_{tx} , the Poincaré inequality and the fact that $s \geq 2$ we get

$$s^{3/2}(\|\mathbf{W}\|_{L^2(\mathbb{S}^1)} + \|\partial_y\mathbf{W}\|_{L^2(\mathbb{S}^1)}) + s^{1/2}\|\mathcal{S}\mathbf{W}\|_{L^2(\mathbb{S}^1)} \lesssim Y_{tx}(s). \quad \square$$

Proposition 5.4. *Under the a priori assumptions (2-5)–(2-7) we have*

$$\sup_{\mathcal{H}[2,s_0]} t^{3/2}(\|\partial^j\mathbf{W}\|_{L^2(\mathbb{S}^1)} + \|\partial_y\partial^j\mathbf{W}\|_{L^2(\mathbb{S}^1)}) + \sup_{\mathcal{H}[2,s_0]} t^{3/2}s^{-1}\|\mathcal{S}\partial^j\mathbf{W}\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon, \quad j = \overline{0, 1}, \quad (5-13)$$

$$\sup_{\mathcal{H}[2,s_0]} t^{3/2}(\|Z\mathbf{W}\|_{L^2(\mathbb{S}^1)} + \|\partial_yZ\mathbf{W}\|_{L^2(\mathbb{S}^1)}) + \sup_{\mathcal{H}[2,s_0]} t^{3/2}s^{-1}\|\mathcal{S}Z\mathbf{W}\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon s^\sigma, \quad (5-14)$$

and

$$\sup_{\mathcal{H}[2,s_0]} st^{1/2}(\|\partial^2\mathbf{W}\|_{L^2(\mathbb{S}^1)} + \|\partial_y\partial^2\mathbf{W}\|_{L^2(\mathbb{S}^1)}) + \sup_{\mathcal{H}[2,s_0]} t^{1/2}\|\mathcal{S}\partial^2\mathbf{W}\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon, \quad (5-15)$$

$$\sup_{\mathcal{H}[2,s_0]} st^{1/2}(\|\partial Z\mathbf{W}\|_{L^2(\mathbb{S}^1)} + \|\partial_y\partial Z\mathbf{W}\|_{L^2(\mathbb{S}^1)}) + t^{1/2}\|\mathcal{S}\partial Z\mathbf{W}\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon s^\sigma. \quad (5-16)$$

Proof. For any fixed $j = \overline{0, 2}$ and $k = \overline{0, 1}$ we compare the equation satisfied by the differentiated functions $\partial^j\mathbf{W}$ and $\partial^kZ\mathbf{W}$ respectively with (5-9) and apply the result of Proposition 5.3. A simple computation

shows that

$$\begin{aligned} \square_{x,y} \partial^j W + u \partial_y^2 \partial^j W &= F_1^j, \quad j = \overline{0, 2}, \\ \square_{x,y} \partial^k ZW + u \partial_y^2 \partial^k ZW &= F_2^k, \quad k = \overline{0, 1}, \end{aligned}$$

with source terms given by

$$\begin{aligned} F_1^0 &= N(W, W) - \int_{\mathbb{S}^1} N(W, W) \frac{dy}{2\pi} + \int_{\mathbb{S}^1} \partial_y u \cdot \partial_y W \frac{dy}{2\pi}, \\ F_1^j &= \partial^j F_1^0 - \sum_{1 \leq h \leq j} \partial^h u \cdot \partial_y^2 \partial^{j-h} W, \quad j = \overline{1, 2}, \end{aligned}$$

and

$$\begin{aligned} F_2^0 &= ZF_1^0 - Zu \cdot \partial_y^2 W, \\ F_2^1 &= \partial ZF_1^0 - \partial Zu \cdot \partial_y^2 W - \partial u \cdot \partial_y^2 ZW - Zu \cdot \partial_y^2 \partial W. \end{aligned}$$

From Proposition 5.3 we have

$$s^{3/2} (\|W\|_{L^2(\mathbb{S}^1)} + \|\partial_y W\|_{L^2(\mathbb{S}^1)}) + s^{1/2} \|\mathcal{S}W\|_{L^2(\mathbb{S}^1)} \lesssim \left(Y_{tx}(2) + \int_2^s B_{tx}(\lambda) d\lambda \right) e^{\int_2^s A_{tx}(\lambda) d\lambda},$$

with $W = \{\partial^j W, \partial^k ZW : j = \overline{0, 2}, k = \overline{0, 1}\}$ and corresponding source term $F = \{F_1^j, F_2^k : j = \overline{0, 2}, k = \overline{0, 1}\}$. In order to obtain the bounds in the statement we need to estimate the quantities $Y_{tx}(2)$, $A_{tx}(\lambda)$ and $B_{tx}(\lambda)$ defined in (5-10), (5-11), (5-12) for all the different values of W and F .

(1) *The $A_{tx}(\lambda)$ term:* This is the same for all values of W . Here we take the decomposition $u = u_0 + u$ and rewrite the scaling vector field as

$$\mathcal{S} = (t-r)\partial_t + (r-t)\partial_r + (t\partial_r + r\partial_t). \quad (5-17)$$

Using the pointwise bounds (5-1)–(5-3) as well as the assumption (2-7) and the fact that $s/t \leq 1$ in the interior of the light cone, we derive that

$$A_{tx}(\lambda) \lesssim \sup_{y \in \mathbb{S}^1} \frac{s}{t+r} |(\partial u)_\lambda| + \lambda^{-1} |(Zu)_\lambda| + |(\partial_y u)_\lambda| \lesssim \epsilon \lambda^{-3/2+\delta_2}$$

and consequently

$$\int_1^s A_{tx}(\lambda) d\lambda \lesssim \epsilon. \quad (5-18)$$

(2) *The $Y_{tx}(2)$ term:* The functions appearing here are evaluated on the hyperboloid \mathcal{H}_2 . From the bound (5-3) on W , the smallness of u given by (5-8) and the decomposition (5-17) it follows that for all values of W under consideration

$$|Y_{tx}(2)| \lesssim \|W_2\|_{L^2(\mathbb{S}^1)} + \|(\mathcal{S}W)_2\|_{L^2(\mathbb{S}^1)} + \|1 + u_2\|_{L^\infty(\mathbb{S}^1)} \|(\partial_y W)_2\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon (s/t)^{3/2}. \quad (5-19)$$

(3) *The $B_{tx}(\lambda)$ term:* These are the only ones for which we need to distinguish between the different values of W and hence of F . Let us remark here that if G is any linear combination of the products

$$\begin{aligned} \partial \mathcal{L}^{\gamma_1} W \cdot \partial \mathcal{L}^{\gamma_2} W, & \quad \int_{\mathbb{S}^1} \partial \mathcal{L}^{\gamma_1} W \cdot \partial \mathcal{L}^{\gamma_2} W dy, \quad |\gamma_1| + |\gamma_2| \leq 2, \\ \partial \mathcal{L}^{\gamma_1} u_0 \cdot \partial_y^2 \mathcal{L}^{\gamma_2} W, & \quad |\gamma_1| + |\gamma_2| \leq 1, \\ \mathcal{L}^{\gamma_1} u \cdot \partial_y^2 \mathcal{L}^{\gamma_2} W, & \quad |\gamma_1| + |\gamma_2| \leq 2, |\gamma_2| < 2, \end{aligned} \quad (5-20)$$

then from (5-1) and (5-3)

$$\|\lambda^{-1/2}(s^2G)_\lambda\|_{L^2(\mathbb{S}^1)} = \lambda^{3/2}\|G_\lambda\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon^2\lambda^{-3/2+2\delta_4}(s/t)^2. \tag{5-21}$$

(a) *The case $\mathbf{W} = \mathcal{E}^{\leq 1}\mathbb{W}$:* From the pointwise bounds (5-3), (5-5) and (5-6) we immediately obtain the estimate

$$\begin{aligned} &\|\lambda^{-1/2}(s^2\bar{\partial}^i\bar{\partial}_i\mathbf{W} + x^i x^j \bar{\partial}_i\bar{\partial}_j\mathbf{W} + \frac{3}{4}\mathbf{W} + 3x^i\bar{\partial}_i\mathbf{W})_\lambda\|_{L^2(\mathbb{S}^1)} \\ &\lesssim \lambda^{3/2}(1+r^2/s^2)\|(\bar{\partial}^2\mathbf{W})_\lambda\|_{L^2(\mathbb{S}^1)} + \lambda^{1/2}rs^{-1}\|(\bar{\partial}\mathbf{W})_\lambda\|_{L^2(\mathbb{S}^1)} + \lambda^{-1/2}\|\mathbf{W}_\lambda\|_{L^2(\mathbb{S}^1)} \\ &\lesssim \epsilon\lambda^{-2+\delta_5}(s/t)^{3/2}. \end{aligned} \tag{5-22}$$

In the case where $\mathbf{W} = \partial^{\leq 1}\mathbb{W}$, the corresponding \mathbf{F} satisfies (5-21), yielding

$$\int_2^s B_{tx}(\lambda) d\lambda \lesssim \epsilon(s/t)^{3/2}, \tag{5-23}$$

and the combination of (5-18), (5-19), (5-23) gives (5-13). In the case where $\mathbf{W} = Z\mathbb{W}$, the only quadratic term in $\mathbf{F} = ZF_1^0$ that is not in (5-20) is $Zu_0 \cdot \partial_y^2\mathbb{W}$. For this we apply the a priori bound (2-7) and the enhanced bound (5-13) and get

$$\|Zu_0 \cdot \partial_y^2\mathbb{W}\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon^2t^{-5/2}s^\sigma.$$

Therefore

$$\lambda^{3/2}\|(F_2^0)_\lambda\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon^2\lambda^{-3/2+2\delta_4}(s/t)^2 + \epsilon^2\lambda^{-1+\sigma}(s/t)^{5/2} \lesssim \epsilon^2\lambda^{-1+\sigma}(s/t)^2,$$

which combined with (5-22) implies

$$\int_2^s B_{tx}(\lambda) d\lambda \lesssim \epsilon s^\sigma (s/t)^{3/2}.$$

The above estimate, together with (5-18) and (5-19), gives (5-14).

(b) *The case $\mathbf{W} = \{\partial^2\mathbb{W}, \partial Z\mathbb{W}\}$:* Here the bounds (5-3), (5-5) and (5-7) give

$$\begin{aligned} &\|\lambda^{-1/2}(s^2\bar{\partial}^i\bar{\partial}_i\mathbf{W} + \frac{3}{4}\mathbf{W} + 3x^i\bar{\partial}_i\mathbf{W})_\lambda\|_{L^2(\mathbb{S}^1)} \\ &\lesssim \lambda^{3/2}\|(\bar{\partial}^2\mathbf{W})_\lambda\|_{L^2(\mathbb{S}^1)} + \lambda^{1/2}rs^{-1}\|(\bar{\partial}\mathbf{W})_\lambda\|_{L^2(\mathbb{S}^1)} + \lambda^{-1/2}\|\mathbf{W}_\lambda\|_{L^2(\mathbb{S}^1)} \\ &\lesssim \epsilon\lambda^{-2+\delta_5}(s/t)^{3/2} \end{aligned} \tag{5-24}$$

and

$$\|\lambda^{-1/2}(x^i x^j \bar{\partial}_i\bar{\partial}_j\mathbf{W})_\lambda\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon\lambda^{-2+\delta_5}(s/t)^{1/2}. \tag{5-25}$$

In the case $\mathbf{W} = \partial^2\mathbb{W}$, the corresponding $\mathbf{F} = F_1^2$ satisfies (5-21), which summed up with the above estimates yields

$$\int_2^s B_{tx}(\lambda) d\lambda \lesssim \epsilon(s/t)^{1/2}$$

and therefore (5-15). In the case $\mathbf{W} = \partial Z\mathbb{W}$, the only term in $\mathbf{F} = F_2^1$ that is not in (5-20) is $Zu_0 \cdot \partial_y^2\partial\mathbb{W}$. For this we apply the a priori bound (2-7) and the enhanced bound (5-15)

$$\|Zu_0 \cdot \partial_y^2\partial\mathbb{W}\|_{L^2(\mathbb{S}^1)} \leq \|Zu_0\|_{L^\infty(\mathbb{S}^1)}\|\partial_y^2\partial\mathbb{W}\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon^2t^{-3/2}s^{-1+\sigma}.$$

Therefore

$$\lambda^{3/2} \|(F_2^1)_\lambda\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon^2 \lambda^{-3/2+2\delta_4} (s/t)^2 + \epsilon^2 \lambda^{-1+\sigma} (s/t)^{3/2} \lesssim \epsilon^2 \lambda^{-1+\sigma} (s/t)^{3/2},$$

which combined with (5-24), (5-25) implies

$$\int_2^s B_{txy}(\lambda) d\lambda \lesssim \epsilon s^\sigma (s/t)^{1/2}. \quad \square$$

5C. The propagation of the a priori pointwise bound. In order to propagate the a priori bound (2-7) we use L^∞ - L^∞ -type estimates. For this we give a closer look to the wave equation satisfied by ZW_0 and use the enhanced pointwise bounds recovered in the previous subsection to estimate the nonlinear terms. We will make use of the following lemma, due to [Alinhac 2006].

Lemma 5.5. *Let W_0 be the solution to $\square_{tx} W_0 = F_0$ with zero initial data and suppose that F_0 is spatially compactly supported satisfying the pointwise bound*

$$|F_0(t, x)| \leq Ct^{-2-\nu} (t - |x|)^{-1+\mu}$$

for some fixed $\mu, \nu > 0$ and some positive constant C . Then

$$|W_0(t, x)| \lesssim \frac{C}{\mu\nu} (t - |x|)^{\mu-\nu} t^{-1}.$$

Proposition 5.6. *There exists a constant $B > 0$ sufficiently large and ϵ_0 sufficiently small such that for any $0 < \epsilon < \epsilon_0$ if $W = (u, v)^T$ is solution of the Cauchy problem (1-5)–(1-6) and satisfies the a priori bounds (2-5)–(2-7) in the interior region $\mathcal{H}_{[2, s_0]}$, together with the global energy bounds (2-1) in the exterior region \mathcal{D}^{ex} , then in $\mathcal{H}_{[2, s_0]}$ it actually satisfies the enhanced pointwise bound*

$$|ZW_0(t, x)| \leq B\epsilon t^{-1} s^\sigma.$$

Proof. We consider a cut-off function $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi(z) = 1$ for $|z| \leq \frac{1}{2}$ and $\chi(z) = 0$ for $|z| > 1$, and decompose W_0 into the sum $W_0^a + W_0^b$, where W_0^a and W_0^b solve to the Cauchy problems

$$\begin{aligned} \square_x W_0^a &= \chi\left(\left(r + \frac{1}{2}\right)/t\right) F_0, & (W_0^a, \partial_t W_0^a)|_{t=2} &= (0, 0), \\ \square_x W_0^b &= \left(1 - \chi\left(\left(r + \frac{1}{2}\right)/t\right)\right) F_0, & (W_0^b, \partial_t W_0^b)|_{t=2} &= (W_0, \partial_t W_0)|_{t=2}, \end{aligned}$$

with F_0 given by

$$F_0 = \int_{\mathbb{S}^1} N(W, W) \frac{dy}{2\pi} + \int_{\mathbb{S}^1} \partial_y u \cdot \partial_y W \frac{dy}{2\pi}.$$

The scope of such decomposition is to estimate ZW_0^a and ZW_0^b separately. We aim to apply Lemma 5.5 to ZW_0^a , since it is solution to a wave equation with zero data and source term supported in the interior of the cone $\{t = r + \frac{1}{2}\}$,

$$\square_x ZW_0^a = \chi\left(\left(r + \frac{1}{2}\right)/t\right) ZF_0 - \left[Z, \chi\left(\left(r + \frac{1}{2}\right)/t\right)\right] F_0, \quad (ZW_0^a, \partial_t ZW_0^a)|_{t=2} = (0, 0),$$

and Klainerman–Sobolev embeddings to estimate ZW_0^b .

We start by estimating F_0 and ZF_0 on the support of $\chi((r + \frac{1}{2})/t)$, after observing the uniform boundedness of the commutator term $[Z, \chi((r + \frac{1}{2})/t)]$ and the fact that F_0 does not contain mixed interactions, i.e.,

$$F_0 = N(W_0, W_0) + \int_{\mathbb{S}^1} N(W, W) \frac{dy}{2\pi} + \int_{\mathbb{S}^1} \partial_y u \cdot \partial_y W \frac{dy}{2\pi}.$$

On the one hand, we use the null structure representation (1-15) of N for the quadratic interactions $W_0 \times W_0$ and apply (5-1), (5-2) to deduce that

$$|Z^{\leq 1} N(W_0, W_0)| \lesssim |\bar{\partial} Z^{\leq 1} W_0| |\partial W_0| + |\partial Z^{\leq 1} W_0| |\bar{\partial} W_0| + \frac{s^2}{t^2} |\partial Z^{\leq 1} W_0| |\partial W_0| \lesssim \epsilon^2 t^{-2} s^{-1}.$$

On the other hand, from the improved bounds (5-13), (5-16) we get

$$\int_{\mathbb{S}^1} |Z^{\leq 1} N(W, W)| dy + \int_{\mathbb{S}^1} |Z^{\leq 1} (\partial_y u \cdot \partial_y W)| dy \lesssim \|\partial Z^{\leq 1} W\|_{L^2(\mathbb{S}^1)} \|\partial W\|_{L^2(\mathbb{S}^1)} \lesssim \epsilon^2 t^{-2} s^{-1+\sigma}.$$

Therefore, on the support of $\chi(r/t)$ and for $t \geq 2$

$$|F_0(t, x)| \lesssim \epsilon^2 t^{-5/2} \langle t-r \rangle^{-1/2} \quad \text{and} \quad |ZF_0(t, x)| \lesssim \epsilon^2 t^{-5/2+\sigma/2} \langle t-r \rangle^{-1/2+\sigma/2}.$$

Lemma 5.5 yields then

$$|ZW_0^a(t, x)| \leq C \epsilon^2 t^{-1} (t-r)^\sigma$$

for some constant $C = C(A, B)$ that depends quadratically on A and B .

The remaining term ZW_0^b to analyze is estimated via the Klainerman–Sobolev inequality of Lemma 5.1. Observe that for points $(t, x) \in \mathcal{H}_s^{\text{in}}$ close to the boundary of $\mathcal{H}_s^{\text{in}}$, the ball $B(x, \frac{1}{3}t)$ also intersects the exterior region and we get

$$t^{1/2} s |ZW_0^b(t, x)| \lesssim E_2^{\text{c, in}}(s, W_0^b)^{1/2} + \sum_{|\gamma| \leq 2} \|(s/t) Z^{\mathcal{L}^\gamma} W_0^b\|_{L_x^2(\mathcal{H}_s^{\text{c}} \cap [T_1^s, T_2^s])},$$

where $s^2 = t^2 - r^2$, $T_1^s = \frac{1}{2}(s^2 + 1)$ is the time the hyperboloid \mathcal{H}_s intersects the cone $t = r + 1$ and $T_2^s = \sqrt{s^2 + |\xi|^2}$ with $|\xi - x| = \frac{1}{3}t$ (the second term in the above right-hand side should be omitted when $T_2^s \leq T_1^s$). The support of the source term in the equation satisfied by W_0^b is contained in the exterior region and hence it is that in the equation of $Z^{\mathcal{L}^\gamma} W_0^b$ for any $|\gamma| \leq 2$. From Proposition 3.5 applied to $W_0 = Z^{\mathcal{L}^\gamma} W_0^b$ with $|\gamma| \leq 2$, the inequality (4-17) (which is valid also for W_0^b) with $T_0 = \frac{1}{2}(s^2 + 1)$, and the smallness assumption on the initial data, we deduce that

$$E_2^{\text{c, in}}(s, W_0^b) \leq E_2^{\text{c, in}}(2, W_0^b) + C_0^2 \epsilon^2 \ln s \lesssim C_0^2 \epsilon^2 \ln s.$$

Moreover, from Lemma 4.9 with $T_j = T_j^s$ for $j = 1, 2$ and the observation that $\ln T_2^s \lesssim \ln s$, we also derive that

$$\sum_{|\gamma| \leq 2} \|(s/t) Z^{\mathcal{L}^\gamma} W_0^b\|_{L_x^2(\mathcal{H}_s^{\text{c}} \cap [T_1^s, T_2^s])} \lesssim C_0^2 \epsilon^2 \ln s.$$

This gives

$$|ZW_0^b(t, x)| \leq \tilde{C} C_0 \epsilon t^{-1/2} s^{-1+\sigma}$$

for a universal constant \tilde{C} and therefore

$$|ZW_0(t, x)| \leq |ZW_0^a(t, x)| + |ZW_0^b(t, x)| \leq \tilde{C}C_0\epsilon t^{-1/2}s^{-1+\sigma} + C\epsilon^2 t^{-1}(t-r)^\sigma \leq B\epsilon t^{-1}s^\sigma$$

if we choose B sufficiently large so that $B \geq 2\tilde{C}C_0$ and ϵ_0 sufficiently small so that $2C\epsilon \leq B$. □

6. Energy estimates in the interior region

The goal of this section is to propagate the interior energy bounds (2-5) and (2-6) on the two components W_0 and W of the solution W to the Cauchy problem (1-5)–(1-6). We remind the reader that for any multi-index γ the differentiated function $W^\gamma = (u^\gamma, v^\gamma)$ is a solution to (4-11) with source term (4-12), while its zero-mode W_0^γ solves the inhomogeneous wave equation (4-18) with source term (4-19).

We first start by recovering an energy bound for the higher-order conformal energies of W_0 . Such a bound follows from (2-5) and (2-6) as well as from the pointwise estimates obtained in Section 5. It will be necessary for the propagation of (2-6) and the computations that lead to it will be useful in the proof of Proposition 6.2.

Proposition 6.1. *Assume the solution $W = (u, v)^T$ to (1-5)–(1-6) satisfies the a priori estimates (2-5)–(2-7) in the region $\mathcal{H}_{[2, s_0]}$ as well as the global exterior energy bounds (2-1) in the exterior region \mathcal{D}^{ex} . Then*

$$\sup_{[2, s]} E_k^{c, in}(s, W_0) \leq C\epsilon s^{1+2\mu_k}, \quad s \in [2, s_0], \quad k = \overline{0, 4}, \tag{6-1}$$

where $\mu_k = \delta_k$ if $k \leq 3$ and $\mu_4 = \delta_2 + \delta_4$.

Proof. We consider here only multi-indices γ of type (k, k) , i.e., $\gamma = (0, \beta)$ with $|\beta| = k$ and $\mathcal{E}^\gamma = Z^\beta$, and apply Proposition 3.5 with $W_0 = W_0^\gamma$ and $F_0 = F_0^\gamma$. We derive

$$\begin{aligned} \sup_{\tau \in [2, s]} E^{c, in}(\tau, W_0^\gamma) &\leq E^{c, in}(2, W_0^\gamma) + \int_2^s \|\tau F_0^\gamma\|_{L^2(\mathcal{H}_\tau^{in})} E^{c, in}(\tau, W_0^\gamma)^{1/2} d\tau \\ &+ \int_{\mathcal{C}_{[2, s]}} |(t+r)(\partial_t W_0^\gamma + \partial_r W_0^\gamma) + 2W_0^\gamma|^2 + (t-r)^2(|\nabla W_0^\gamma|^2 - (\partial_r W_0^\gamma)^2) dS. \end{aligned} \tag{6-2}$$

The integral over the boundary $\mathcal{C}_{[2, s]}$ equals the integral in the left-hand side of (4-17) when $T_0 = \frac{1}{2}(s^2 + 1)$; hence

$$\int_{\mathcal{C}_{[2, s]}} |(t+r)(\partial_t W_0^\gamma + \partial_r W_0^\gamma) + 2W_0^\gamma|^2 + (t-r)^2(|\nabla W_0^\gamma|^2 - (\partial_r W_0^\gamma)^2) dS \lesssim \epsilon^2 \ln s. \tag{6-3}$$

The different contributions to the inhomogeneous term F_0^γ , which we remind are only pure interactions $W_0 \times W_0$ and $W \times W$ (see (1-9)), are estimated separately below:

(1) *The pure nonzero modes interactions:* After a Cauchy–Schwarz inequality in the y -variable, we will place the factor whose index has length smaller than $\frac{1}{2}k$ in $L_y^2 L_x^\infty$ and the remaining one in L_{xy}^2 . We use the pointwise bound (5-3) in the case where $|\gamma_1| = |\gamma_2| = 2$ (which appears only if $k = 4$)

$$\|\partial W^{\gamma_1} \cdot \partial W^{\gamma_2}\|_{L_y^1 L_x^2(\mathcal{H}_\tau^{in})} \lesssim \epsilon \tau^{-3/2+\delta_4} E_2^{in}(\tau, W)^{1/2},$$

the enhanced pointwise bound (5-13) whenever one of the two multi-indexes has length 0

$$\|\partial W \cdot \partial W^{\gamma_2}\|_{L^1_y L^2_{tx}(\mathcal{H}^{\text{in}}_\tau)} \lesssim \epsilon \tau^{-3/2} E_k^{\text{in}}(\tau, W)^{1/2},$$

and (5-16) otherwise

$$\|\partial W \cdot \partial W^{\gamma_2}\|_{L^1_y L^2_{tx}(\mathcal{H}^{\text{in}}_\tau)} \lesssim \epsilon \tau^{-3/2+\sigma} E_{k-1}^{\text{in}}(\tau, W)^{1/2}.$$

We get

$$\sum_{|\gamma_1|+|\gamma_2|\leq k} \|\partial W^{\gamma_1} \partial W^{\gamma_2}\|_{L^1_y L^2_x(\mathcal{H}^{\text{in}}_\tau)} \lesssim \epsilon \tau^{-3/2} [E_k^{\text{in}}(\tau, W)^{1/2} + \tau^\sigma E_{k-1}^{\text{in}}(\tau, W)^{1/2} + \nu_4 \tau^{\delta_4} E_2^{\text{in}}(\tau, W)^{1/2}],$$

where $\nu_4 = 1$ if $k = 4$ and 0 otherwise.

(2) *The pure zero modes interactions:* We use the representation formula (1-15)

$$N(W_0^{\gamma_1}, W_0^{\gamma_2}) = \bar{\partial} W_0^{\gamma_1} \cdot \partial W_0^{\gamma_2} + \partial W_0^{\gamma_1} \cdot \bar{\partial} W_0^{\gamma_2} + \frac{t-r}{t} \partial W_0^{\gamma_1} \cdot \partial W_0^{\gamma_2}$$

and estimate all terms in the above right-hand side using (5-1) and (5-2) as follows:

$$\begin{aligned} \|N(W_0^{\gamma_1}, W_0^{\gamma_2})\|_{L^2_x(\mathcal{H}^{\text{in}}_\tau)} &\lesssim \|(t/\tau)\bar{\partial} \mathcal{L}^{\leq 3} W_0\|_{L^\infty_x(\mathcal{H}^{\text{in}}_\tau)} \|(t/t)\partial \mathcal{L}^{\leq k} W_0\|_{L^2_x(\mathcal{H}^{\text{in}}_\tau)} + \|\bar{\partial} \mathcal{L}^{\leq k} W_0\|_{L^2_x(\mathcal{H}^{\text{in}}_\tau)} \|\partial \mathcal{L}^{\leq 3} W_0\|_{L^\infty_x(\mathcal{H}^{\text{in}}_\tau)} \\ &\lesssim \epsilon \tau^{-3/2} E_k^{\text{in}}(\tau, W)^{1/2}. \end{aligned}$$

We combine the estimates in step (1) and (2) with the a priori energy bounds (2-5), (2-6) and choose σ, δ_{k-1} small so that $\sigma + \delta_{k-1} \leq \delta_k$ to get

$$\|F_0^\gamma\|_{L^2_x(\mathcal{H}^{\text{in}}_\tau)} \lesssim \epsilon^2 \tau^{-3/2+\mu_k}, \quad \text{with } \mu_k = \begin{cases} \delta_k & \text{if } k \leq 3, \\ \delta_4 + \delta_2 & \text{if } k = 4. \end{cases} \tag{6-4}$$

We finally plug (6-3) and (6-4) into (6-2) and use the smallness assumptions on the initial data to derive

$$\begin{aligned} \sup_{[2,s]} E_k^{\text{c},\text{in}}(\tau, W_0) &\lesssim E_k^{\text{c},\text{in}}(2, W_0) + \epsilon^2 s^{2C\epsilon} \ln s + \int_2^s \epsilon^2 \tau^{-1/2+\mu_k} E_k^{\text{c},\text{in}}(\tau, W_0)^{1/2} d\tau \\ &\lesssim \epsilon^2 s^{1+2\mu_k} + \epsilon^2 \sup_{[2,s]} E_k^{\text{c},\text{in}}(\tau, W_0) \end{aligned}$$

and obtain the result of the proposition by choosing ϵ sufficiently small. □

As a result of the exterior bound (4-9) and the combination of Lemma 5.1 with the conformal energy bound (6-1) and Lemma 4.9 with $T_1 = \frac{1}{2}(s^2 + 1)$ and $T_2 = \sqrt{s^2 + |y|^2}$ for $|y - x| = \frac{1}{3}t$, we have the following additional pointwise bound for the Z derivatives of W_0 in the whole region $\mathcal{H}^{\text{in}}_{[2,s_0]}$:

$$|ZZ^j W_0(t, x)| \lesssim \epsilon t^{-1/2} s^{-1/2+\mu_{j+2}} \quad \text{for } j = \overline{0, 2}, \quad \text{where } \mu_k = \begin{cases} \delta_k & \text{if } k \leq 3, \\ \delta_4 + \delta_2 & \text{if } k = 4. \end{cases} \tag{6-5}$$

We now have all the ingredients to propagate the a priori energy bounds (2-5) and (2-6).

Proposition 6.2. *There exists a constant $A > 0$ sufficiently large, some small parameters $\sigma \ll \delta_k \ll \delta_{k+1}$ for $k = \overline{1, 4}$ and $\epsilon_0 > 0$ sufficiently small such that, if for any $0 < \epsilon < \epsilon_0$ the solution $W = (u, v)^T$ to the Cauchy problem (1-5)–(1-6) satisfies the a priori bounds (2-5)–(2-7) in the interior region $\mathcal{H}^{\text{in}}_{[2,s_0]}$ and the*

global exterior energy bounds (2-1) in the exterior region \mathcal{D}^{ex} , then it also satisfies the enhanced energy bound

$$E_5^{\text{in}}(s, W_0) \leq A^2 \epsilon^2, \quad s \in [2, s_0]. \tag{6-6}$$

Proof. For any $|\gamma| \leq 5$, equation (4-18) satisfied by W_0^γ has the same structure as the inhomogeneous wave equation (3-15); therefore Proposition 3.4 with $W_0 = W_0^\gamma$ and $F_0 = F_0^\gamma$ implies

$$E^{\text{in}}(s, W_0^\gamma) \lesssim E^{\text{in}}(2, W_0^\gamma) + \int_{\mathcal{C}_{[2,s]}} |\mathcal{T}W_0^\gamma|^2 dS + \int_2^s \|F_0^\gamma\|_{L_x^2(\mathcal{H}_\tau^{\text{in}})} E^{\text{in}}(\tau, W_0^\gamma)^{1/2} d\tau$$

for all $s \in [2, s_0]$. The implicit constant in the above right-hand side is independent of s_0 .

The fact that the integral over the boundary $\mathcal{C}_{[2,s]}$ is finite and small is a consequence of (4-16) and an estimate for the source term F_0^γ when $|\gamma| \leq 4$ has already been obtained in (6-4). When $|\gamma| = 5$ we simply use the estimate (5-3) on the one hand, and (5-1), (5-2) and the null structure on the other hand, to deduce

$$\begin{aligned} \sum_{|\gamma_1|+|\gamma_2| \leq 5} \|\partial W^{\gamma_1} \cdot \partial W^{\gamma_2}\|_{L_y^1 L_x^2(\mathcal{H}_\tau^{\text{in}})} &\lesssim \epsilon \tau^{-3/2+\delta_4} E_5^{\text{in}}(t, W)^{1/2}, \\ \sum_{|\gamma_1|+|\gamma_2| \leq 5} \|N(W_0^{\gamma_1}, W_0^{\gamma_2})\|_{L_y^1 L_x^2(\mathcal{H}_\tau^{\text{in}})} &\lesssim \epsilon \tau^{-3/2} E_5^{\text{in}}(s, W)^{1/2} \end{aligned}$$

and therefore get

$$\sum_{|\gamma| \leq 5} \|F_0^\gamma\|_{L_x^2(\mathcal{H}_\tau^{\text{in}})} \lesssim \epsilon \tau^{-3/2+\delta_4} E_5^{\text{in}}(\tau, W)^{1/2}. \tag{6-7}$$

From the a priori energy bound (2-6) and the exterior energy bound (4-15) we obtain that, for some fixed $K > 1$ and some universal constant $C > 0$,

$$\begin{aligned} E_5^{\text{in}}(s, W_0) &\leq E_5^{\text{in}}(2, W_0) + \frac{C_0^2 \epsilon^2}{K} + \int_2^s A^2 C \epsilon^3 \tau^{-3/2+\delta_4+\delta_5} d\tau \\ &\leq E_5^{\text{in}}(2, W_0) + \frac{C_0^2 \epsilon^2}{K} + A^2 C \epsilon^3. \end{aligned}$$

The desired improved energy bound then follows choosing $K = 3$, ϵ_0 small such that $3C\epsilon_0^2 < 1$ and $A \geq C_0$ sufficiently large so that

$$E_5^{\text{in}}(2, W_0) \leq \frac{A^2 \epsilon_0^2}{3}. \quad \square$$

Proposition 6.3. *There exists a constant $A > 0$ sufficiently large, some small parameters $\sigma \ll \delta_k \ll \delta_{k+1}$ for $k = \overline{1, 4}$ and $\epsilon_0 > 0$ sufficiently small such that, if for any $0 < \epsilon < \epsilon_0$ the solution $W = (u, v)^T$ to the Cauchy problem (1-5)–(1-6) satisfies the a priori bounds (2-5)–(2-7) in the hyperbolic region $\mathcal{H}_{[2,s_0]}^{\text{in}}$ and the global exterior energy bounds (2-1) in the exterior region \mathcal{D}^{ex} , then it also satisfies the enhanced energy bound*

$$E_{5,k}^{\text{in}}(s, W) \leq A^2 \epsilon^2 s^{2\delta_k}, \quad s \in [2, s_0]. \tag{6-8}$$

Proof. We start by considering a multi-index γ of type (n, k) with $k \leq n \leq 5$ and compare the equation satisfied by $W^\gamma = (u^\gamma, v^\gamma)^T$ with the linear inhomogeneous equation (3-1)

$$\square_{x,y} W^\gamma + u \partial_y^2 W^\gamma = F^\gamma - F_0^\gamma, \quad (t, x, y) \in \mathbb{R}^{1+3} \times \mathbb{S}^1,$$

so applying Proposition 3.3 with $W = W^\gamma$ and $F = F^\gamma - F_0^\gamma$ we derive the inequality

$$E^{\text{in}}(s, W^\gamma) \lesssim E^{\text{in}}(2, W^\gamma) + \iint_{\mathcal{C}_{[2,s]}} |\mathcal{F}W^\gamma|^2 + |\partial_y W^\gamma|^2 d\Sigma_s + \int_2^s \|F^\gamma - F_0^\gamma\|_{L^2_{xy}(\mathcal{H}_\tau^{\text{in}})} E^{\text{in}}(\tau, W^\gamma)^{1/2} d\tau. \quad (6-9)$$

The integral over the boundary $\mathcal{C}_{[2,s]}$ is finite and small as a consequence of the exterior energy estimate (4-15). The pure zero-mode interactions with null structure and the pure nonzero modes interactions have been already examined; see estimate (6-7). The contributions to the source term $F^\gamma - F_0^\gamma$ that have not been estimated yet are the following types of quadratic terms:

$$\begin{aligned} & \partial W_0^{\gamma_1} \cdot \partial W^{\gamma_2} \quad \text{for } |\gamma_1| + |\gamma_2| \leq |\gamma|, \\ & u_0^{\gamma_1} \cdot \partial_y^2 W^{\gamma_2} \quad \text{for } |\gamma_1| + |\gamma_2| = |\gamma|, \quad |\gamma_2| < |\gamma| \text{ and } \gamma_1 = (0, \beta_1). \end{aligned}$$

From the pointwise bounds (5-1) and (5-3) we immediately deduce

$$\sum_{|\gamma_1|+|\gamma_2|\leq|\gamma|} \|\partial W_0^{\gamma_1} \cdot \partial W^{\gamma_2}\|_{L^2_{xy}(\mathcal{H}_\tau^{\text{in}})} \lesssim C_0 \epsilon \tau^{-3/2+\delta_5} E_5^{\text{in}}(\tau, W)^{1/2}.$$

The products $u_0^{\gamma_1} \cdot \partial_y^2 W^{\gamma_2}$ with $\gamma_1 = (0, \beta_1)$ and such that $\mathcal{L}^{\gamma_1} = Z^{\beta_1}$ is a pure product of Klainerman vector fields are estimated separately depending on the values of γ_1 and γ_2 . Let $k_1 := |\gamma_1|$. Then $k_1 + |\gamma_2| = n$ and $k_1 \leq k$ and we distinguish the following cases:

- When $k_1 = n$ and $|\gamma_2| = 0$ we have $k = n$ and use the pointwise bound (5-13) and the conformal energy bound (6-1) to derive

$$\|u_0^{\gamma_1} \cdot \partial_y^2 W\|_{L^2_{xy}(\mathcal{H}_\tau^{\text{in}})} \lesssim \|(t/\tau) \partial_y^2 W\|_{L^2_y L^\infty_x(\mathcal{H}_\tau^{\text{in}})} E_{n-1}^{\text{c, in}}(\tau, W_0)^{1/2} \lesssim \epsilon^2 \tau^{-1+\mu_{k-1}}.$$

- When $k_1 = n - 1$ and $|\gamma_2| = 1$ we have $n - 1 \leq k \leq n$ and use (5-15), (5-16), (6-1)

$$\|u_0^{\gamma_1} \cdot \partial_y^2 W^{\gamma_2}\|_{L^2_{xy}(\mathcal{H}_\tau^{\text{in}})} \lesssim \|(t/\tau) \partial_y^2 W^{\gamma_2}\|_{L^2_y L^\infty_x(\mathcal{H}_\tau^{\text{in}})} E_{n-2}^{\text{c, in}}(\tau, W_0)^{1/2} \lesssim \epsilon^2 \tau^{-1+\mu_{k-1}+\sigma}.$$

- When $k_1 = n - 2$ and $|\gamma_2| = 2$ we have $n - 1 \leq k \leq n$. This case only appears when $3 \leq n \leq 5$. If $\mathcal{L}^{\gamma_2} = \{\partial^2, \partial Z\}$ or if $\mathcal{L}^{\gamma_2} = \partial Z$ and $n = \overline{4}, \overline{5}$ we apply (5-3) and (6-1):

$$\|u_0^{\gamma_1} \cdot \partial_y^2 W^{\gamma_2}\|_{L^2_{xy}(\mathcal{H}_\tau^{\text{in}})} \lesssim \|(t/\tau) \partial_y^2 W^{\gamma_2}\|_{L^2_y L^\infty_x(\mathcal{H}_\tau^{\text{in}})} E_{n-3}^{\text{c, in}}(\tau, W_0)^{1/2} \lesssim \epsilon^2 \tau^{-1+\mu_{k-2}+\delta_3},$$

while if $\mathcal{L}^{\gamma_2} = \partial Z$ and $n = 3$ (in which case $k = 2$) we use (2-6) and (2-7):

$$\|u_0^{\gamma_1} \cdot \partial_y^2 W^{\gamma_2}\|_{L^2_{xy}(\mathcal{H}_\tau^{\text{in}})} \lesssim \|Zu_0\|_{L^\infty(\mathcal{H}_\tau^{\text{in}})} E_{3,1}^{\text{in}}(\tau, W)^{1/2} \lesssim \epsilon^2 \tau^{-1+\sigma+\delta_{k-1}}.$$

- When $k_1 = 1$ and $|\gamma_2| = n - 1$ for $n = \overline{4}, \overline{5}$, the a priori bounds (2-6) and (2-7) give

$$\|u_0^{\gamma_1} \cdot \partial_y^2 W^{\gamma_2}\|_{L^2_{xy}(\mathcal{H}_\tau^{\text{in}})} \lesssim \|Zu_0\|_{L^\infty(\mathcal{H}_\tau^{\text{in}})} E_{n,k-1}^{\text{in}}(\tau, W)^{1/2} \lesssim \epsilon^2 \tau^{-1+\sigma+\delta_{k-1}},$$

and in the last case $k_1 = 2, |\gamma_2| = 3$ — which only appears when $n = 5$ — the bounds (2-6) and (6-5) yield

$$\|u_0^{\gamma_1} \cdot \partial_y^2 W^{\gamma_2}\|_{L^2_{xy}(\mathcal{H}_\tau^{\text{in}})} \lesssim \|Z^2 u_0\|_{L^\infty(\mathcal{H}_\tau^{\text{in}})} E_{4,k-2}^{\text{in}}(\tau, W)^{1/2} \lesssim \epsilon^2 \tau^{-1+2\delta_{k-2}}.$$

Choosing appropriately $\sigma \ll \delta_k \ll \delta_{k+1}$ for $k = \overline{1, 4}$ so that δ_{k+1} is bigger than some linear combination of σ and δ_j with $j \leq k$, we then obtain

$$\|u_0^{\gamma_1} \cdot \partial_y^2 W^{\gamma_2}\|_{L^2_{xy}(\mathcal{H}_\tau^{\text{in}})} \lesssim \epsilon^2 \tau^{-1+\delta_k},$$

with an implicit constant that depends on A and B . The same bound holds then true for $F^\gamma - F_0^\gamma$ so plugging it into (6-9) together with the exterior energy estimate (4-15), and the a priori energy bounds (2-6), gives us

$$E_{5,k}^{\text{in}}(s, W) \leq C E_{5,k}^{\text{in}}(2, W) + \frac{CC_0^2 \epsilon^2}{2} + \int_2^s A^2 C \epsilon^3 \tau^{-1+2\delta_k} d\tau.$$

The end of the proof follows finally by choosing appropriately the constants A and ϵ_0 . □

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References

- [Alinhac 2006] S. Alinhac, “Semilinear hyperbolic systems with blowup at infinity”, *Indiana Univ. Math. J.* **55**:3 (2006), 1209–1232. MR Zbl
- [Andersson et al. 2023] L. Andersson, P. Blue, Z. Wyatt, and S.-T. Yau, “Global stability of spacetimes with supersymmetric compactifications”, *Anal. PDE* **16**:9 (2023), 2079–2107. MR Zbl
- [Bachelot 1988] A. Bachelot, “Problème de Cauchy global pour des systèmes de Dirac–Klein–Gordon”, *Ann. Inst. H. Poincaré Phys. Théor.* **48**:4 (1988), 387–422. MR Zbl
- [Branding et al. 2019] V. Branding, D. Fajman, and K. Kröncke, “Stable cosmological Kaluza–Klein spacetimes”, *Comm. Math. Phys.* **368**:3 (2019), 1087–1120. MR Zbl
- [Christodoulou and Klainerman 1993] D. Christodoulou and S. Klainerman, *The global nonlinear stability of the Minkowski space*, Princeton Math. Ser. **41**, Princeton Univ. Press, 1993. MR Zbl
- [Dong and Wyatt 2020a] S. Dong and Z. Wyatt, “Stability of a coupled wave–Klein–Gordon system with quadratic nonlinearities”, *J. Differential Equations* **269**:9 (2020), 7470–7497. MR Zbl
- [Dong and Wyatt 2020b] S. Dong and Z. Wyatt, “Two dimensional wave–Klein–Gordon equations with semilinear nonlinearities”, preprint, 2020. arXiv 2011.11990
- [Dong et al. 2021] S. Dong, P. G. LeFloch, and Z. Wyatt, “Global evolution of the U(1) Higgs Boson: nonlinear stability and uniform energy bounds”, *Ann. Henri Poincaré* **22**:3 (2021), 677–713. MR Zbl
- [Ettinger 2015] B. Ettinger, “Well-posedness of the equation for the three-form field in eleven-dimensional supergravity”, *Trans. Amer. Math. Soc.* **367**:2 (2015), 887–910. MR Zbl
- [Georgiev 1990] V. Georgiev, “Global solution of the system of wave and Klein–Gordon equations”, *Math. Z.* **203**:4 (1990), 683–698. MR Zbl

- [Ifrim and Stingo 2019] M. Ifrim and A. Stingo, “Almost global well-posedness for quasilinear strongly coupled wave-Klein-Gordon systems in two space dimensions”, preprint, 2019. arXiv 1910.12673
- [Ionescu and Pausader 2019] A. D. Ionescu and B. Pausader, “On the global regularity for a wave-Klein-Gordon coupled system”, *Acta Math. Sin. (Engl. Ser.)* **35**:6 (2019), 933–986. MR Zbl
- [Ionescu and Pausader 2022] A. D. Ionescu and B. Pausader, *The Einstein–Klein–Gordon coupled system: global stability of the Minkowski solution*, Ann. of Math. Stud. **213**, Princeton Univ. Press, 2022. MR Zbl
- [Kaluz̄a 1921] T. Kaluz̄a, “Zum Unitätsproblem der Physik”, *Sitz. Preuss. Akad. Wissen.* **1921** (1921), 966–972. Zbl
- [Katayama 2012] S. Katayama, “Global existence for coupled systems of nonlinear wave and Klein–Gordon equations in three space dimensions”, *Math. Z.* **270**:1-2 (2012), 487–513. MR Zbl
- [Klainerman 1985] S. Klainerman, “Global existence of small amplitude solutions to nonlinear Klein–Gordon equations in four space-time dimensions”, *Comm. Pure Appl. Math.* **38**:5 (1985), 631–641. MR Zbl
- [Klainerman 1986] S. Klainerman, “The null condition and global existence to nonlinear wave equations”, pp. 293–326 in *Nonlinear systems of partial differential equations in applied mathematics, I* (Santa Fe, NM, 1984), edited by B. Nicolaenko et al., Lectures in Appl. Math. **23**, Amer. Math. Soc., Providence, RI, 1986. MR Zbl
- [Klainerman et al. 2020] S. Klainerman, Q. Wang, and S. Yang, “Global solution for massive Maxwell–Klein–Gordon equations”, *Comm. Pure Appl. Math.* **73**:1 (2020), 63–109. MR Zbl
- [Klein 1926] O. Klein, “Quantum theory and five-dimensional theory of relativity”, *Z. Phys.* **37** (1926), 895–906.
- [LeFloch and Ma 2014] P. G. LeFloch and Y. Ma, *The hyperboloidal foliation method*, Ser. Appl. Computat. Math. **2**, World Sci., Hackensack, NJ, 2014. MR Zbl
- [LeFloch and Ma 2016] P. G. LeFloch and Y. Ma, “The global nonlinear stability of Minkowski space for self-gravitating massive fields”, *Comm. Math. Phys.* **346**:2 (2016), 603–665. MR Zbl
- [LeFloch and Ma 2018] P. G. LeFloch and Y. Ma, *The global nonlinear stability of Minkowski space for self-gravitating massive fields*, Ser. Appl. Computat. Math. **3**, World Sci., Hackensack, NJ, 2018. MR Zbl
- [Lesky and Racke 2003] P. H. Lesky and R. Racke, “Nonlinear wave equations in infinite waveguides”, *Comm. Partial Differential Equations* **28**:7-8 (2003), 1265–1301. MR Zbl
- [Lindblad and Rodnianski 2010] H. Lindblad and I. Rodnianski, “The global stability of Minkowski space-time in harmonic gauge”, *Ann. of Math. (2)* **171**:3 (2010), 1401–1477. MR Zbl
- [Ma 2017a] Y. Ma, “Global solutions of non-linear wave-Klein–Gordon system in two space dimension: semi-linear interactions”, preprint, 2017. arXiv 1712.05315
- [Ma 2017b] Y. Ma, “Global solutions of quasilinear wave-Klein–Gordon system in two-space dimension: completion of the proof”, *J. Hyperbolic Differ. Equ.* **14**:4 (2017), 627–670. MR Zbl
- [Ma 2017c] Y. Ma, “Global solutions of quasilinear wave-Klein–Gordon system in two-space dimension: technical tools”, *J. Hyperbolic Differ. Equ.* **14**:4 (2017), 591–625. MR Zbl
- [Ma 2020] Y. Ma, “Global solutions of nonlinear wave-Klein–Gordon system in one space dimension”, *Nonlinear Anal.* **191** (2020), art. id. 111641. MR Zbl
- [Ma 2021] Y. Ma, “Global solutions of nonlinear wave-Klein–Gordon system in two spatial dimensions: a prototype of strong coupling case”, *J. Differential Equations* **287** (2021), 236–294. MR Zbl
- [Ma and Huang 2017] Y. Ma and H. Huang, “A conformal-type energy inequality on hyperboloids and its application to quasi-linear wave equation in \mathbb{R}^{3+1} ”, preprint, 2017. arXiv 1711.00498
- [Metcalf and Stewart 2008] J. Metcalf and A. Stewart, “Almost global existence for quasilinear wave equations in waveguides with Neumann boundary conditions”, *Trans. Amer. Math. Soc.* **360**:1 (2008), 171–188. MR Zbl
- [Metcalf et al. 2005] J. Metcalf, C. D. Sogge, and A. Stewart, “Nonlinear hyperbolic equations in infinite homogeneous waveguides”, *Comm. Partial Differential Equations* **30**:4-6 (2005), 643–661. MR Zbl
- [Ozawa et al. 1995] T. Ozawa, K. Tsutaya, and Y. Tsutsumi, “Normal form and global solutions for the Klein–Gordon–Zakharov equations”, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **12**:4 (1995), 459–503. MR Zbl

- [Stingo 2023] A. Stingo, *Global existence of small amplitude solutions for a model quadratic quasilinear coupled wave-Klein-Gordon system in two space dimension, with mildly decaying Cauchy data*, Mem. Amer. Math. Soc. **1441**, Amer. Math. Soc., Providence, RI, 2023. MR Zbl
- [Tataru 2002] D. Tataru, “Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients, III”, *J. Amer. Math. Soc.* **15**:2 (2002), 419–442. MR Zbl
- [Tsutaya 1996] K. Tsutaya, “Global existence of small amplitude solutions for the Klein–Gordon–Zakharov equations”, *Nonlinear Anal.* **27**:12 (1996), 1373–1380. MR Zbl
- [Tsutsumi 2003a] Y. Tsutsumi, “Global solutions for the Dirac–Proca equations with small initial data in $3 + 1$ space time dimensions”, *J. Math. Anal. Appl.* **278**:2 (2003), 485–499. MR Zbl
- [Tsutsumi 2003b] Y. Tsutsumi, “Stability of constant equilibrium for the Maxwell–Higgs equations”, *Funkcial. Ekvac.* **46**:1 (2003), 41–62. MR Zbl
- [Wang 2015] Q. Wang, “Global existence for the Einstein equations with massive scalar fields”, lecture, Simons Center Geom. Phys., 2015, available at https://scgp.stonybrook.edu/video_portal/video.php?id=1420.
- [Wang 2020] Q. Wang, “An intrinsic hyperboloid approach for Einstein Klein–Gordon equations”, *J. Differential Geom.* **115**:1 (2020), 27–109. MR Zbl
- [Wang 2021] J. Wang, “Nonlinear wave equation in a cosmological Kaluza Klein spacetime”, *J. Math. Phys.* **62**:6 (2021), art. id. 062504. MR Zbl
- [Witten 1982] E. Witten, “Instability of the Kaluza–Klein vacuum”, *Nuclear Phys. B* **195**:3 (1982), 481–492. Zbl
- [Wong 2017] W. W. Y. Wong, “Small data global existence and decay for two dimensional wave maps”, 2017. To appear in *Ann. H. Lebesgue*. arXiv 1712.07684
- [Wyatt 2018] Z. Wyatt, “The weak null condition and Kaluza–Klein spacetimes”, *J. Hyperbolic Differ. Equ.* **15**:2 (2018), 219–258. MR Zbl

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
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