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ON THE ENDPOINT REGULARITY IN ONSAGER'S CONJECTURE

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Onsager's conjecture states that the conservation of energy may fail for three-dimensional incompressible Euler flows with Hölder regularity below $\frac{1}{3}$. This conjecture was recently solved by the author, yet the endpoint case remains an interesting open question with further connections to turbulence theory. In this work, we construct energy nonconserving solutions to the three-dimensional incompressible Euler equations with space-time Hölder regularity converging to the critical exponent at small spatial scales and containing the entire range of exponents $\left[0, \frac{1}{3}\right)$.

Our construction improves the author's previous result towards the endpoint case. To obtain this improvement, we introduce a new method for optimizing the regularity that can be achieved by a convex integration scheme. A crucial point is to avoid loss of powers in frequency in the estimates of the iteration. This goal is achieved using localization techniques of Isett and Oh (*Arch. Ration. Mech. Anal.* **221**:2 (2016), 725–804) to modify the convex integration scheme.

We also prove results on general solutions at the critical regularity that may not conserve energy. These include a theorem on intermittency stating roughly that energy dissipating solutions cannot have absolute structure functions satisfying the Kolmogorov–Obukhov scaling for any $p > 3$ if their singular supports have space-time Lebesgue measure zero.

1. Introduction

We consider the endpoint regularity in Onsager's conjecture for the incompressible Euler equations on $\mathbb{R} \times \mathbb{T}^3$, which we write in conservation form as

$$\begin{aligned} \partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p &= 0, \\ \nabla_j v^j &= 0, \end{aligned} \tag{E}$$

using the summation convention for summing repeated indices. We are concerned mainly with weak solutions to the incompressible Euler equations, which are defined most generally as a locally square-integrable vector field v (called the velocity field) and scalar function p (called the pressure) that together satisfy (E) in the sense of distributions.

Onsager's conjecture states that for any Hölder exponent $\alpha < \frac{1}{3}$ there exist periodic weak solutions to the three-dimensional incompressible Euler equations that belong to the Hölder class $v \in L_t^\infty C_x^\alpha$ and fail to conserve the total kinetic energy $\frac{1}{2} \int_{\mathbb{T}^3} |v(t,x)|^2 \, dx$. The endpoint case of the conjecture is that the same statement should hold for $\alpha = \frac{1}{3}$. The above statements originate from [Onsager 1949] on the statistical theory of hydrodynamic turbulence, where he postulated that dissipation of energy may occur

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in the absence of viscosity[1] through the mechanism of an energy cascade modeled by the incompressible Euler equations.

Onsager's argument predicts that such energy dissipation should be possible for incompressible Euler flows with regularity exactly $\frac{1}{3}$. Specifically, Onsager argued that the energy cascade occurring in a turbulent flow will result in an energy spectrum with a statistical power law consistent with exactly the (Besov or Hölder) regularity $\frac{1}{3}$ in the inertial range of frequencies, which agrees with the scaling laws of turbulence predicted by Kolmogorov's theory [1941]. (See also [De Lellis and Székelyhidi 2013a; Eyink and Sreenivasan 2006] for more detailed reviews of these statements and computations.) On the other hand, Onsager asserted that conservation of energy must hold for every incompressible Euler flow $v \in L^\infty_t C^\alpha_x(I \times \mathbb{T}^3)$ with Hölder regularity α strictly above $\frac{1}{3}$. A strengthening of this latter assertion was proved in [Constantin et al. 1994] after initial work in [Eyink 1994], with the sharpest known result being that conservation of energy holds for weak solutions in the Besov class $v \in L^3_t B^{1/3}_{3,c_0(\mathbb{N})}$ [Cheskidov et al. 2008]. These results leave open the possibility that energy dissipation as considered by Onsager may be possible for solutions to incompressible Euler with exactly the critical regularity $\frac{1}{3}$ (e.g., for weak solutions in the class $v \in C_t C^{1/3}_x$), while the construction in [Eyink 1994] of initial data with critical regularity and nonzero energy flux provides further evidence that dissipation of energy for weak solutions at the critical regularity should indeed exist.

The existence of weak solutions to incompressible Euler equations in the class $v \in L^\infty_t C^\alpha_x(\mathbb{R} \times \mathbb{T}^3)$ that fail to conserve energy has been established by the author for all $\alpha < \frac{1}{3}$ in [Isett 2018]. The solutions are constructed using the method of convex integration, which was first introduced to the incompressible Euler equations by De Lellis and Székelyhidi [2009; 2013b; 2014] and was further developed towards improved partial results towards Onsager's conjecture in [Buckmaster et al. 2015; 2016; Isett 2017a]. The proof in [Isett 2018] relies also on the use of Mikado flows introduced in [Daneri and Székelyhidi 2017] to implement convex integration in combination with a new "gluing approximation" technique.

In the present work, we improve upon the result in [Isett 2018] to construct solutions with borderline regularity that approaches the endpoint case at small length scales while failing to conserve energy. Our main result is the following.

Theorem 1.1. *There exists a weak solution (v, p) to the incompressible Euler equations that has nonempty, compact support in time on $\mathbb{R} \times \mathbb{T}^3$ and belongs to the class $v \in \bigcap_{\alpha<1/3} C^\alpha_{t,x}$. Moreover, one may arrange that v also satisfies an estimate of the form*

$$|v(t, x+\Delta x) - v(t, x)| \le C|\Delta x|^{1/3 - B\sqrt{(\log\log|\Delta x|^{-1})/(\log|\Delta x|^{-1})}} \tag{1}$$

for some constants C and B and for all $(t, x) \in \mathbb{R} \times \mathbb{T}^3$ and all $|\Delta x| \le 10^{-2}$.

The theorem is significant for the following reasons:

- Theorem 1.1 demonstrates how close the method of convex integration can come to achieving the self-similar $L^\infty_t C^{1/3}_x$ regularity that corresponds to the Kolmogorov theory.

[1] A related and important open question is whether such energy dissipating solutions arise as zero viscosity limits of solutions to the Navier–Stokes equations.

- The theorem is the first result proved by convex integration that approaches the endpoint regularity and avoids the strictly positive gap in regularity from the endpoint faced by previous results. In particular, we have that $v \in \bigcap_{\alpha<1/3} C^{\alpha}_{t,x}$ rather than having regularity bounded strictly below the limiting exponent (i.e., $v \in C^{1/3-\epsilon}_{t,x}$ for some $\epsilon > 0$).
- The proof of Theorem 1.1 is based on a new algorithm that optimizes the regularity coming from a convex integration construction, which may be useful for future numerical simulations of convex integration solutions. This algorithm also apparently identifies an evident barrier towards achieving the endpoint regularity exactly using the convex integration method.
- The proof of Theorem 1.1 clarifies which techniques in the literature yield the sharpest regularity.

The constant B, which determines[2] the rate at which the regularity $\frac{1}{3}$ is approached at small scales, can be taken to be $B = 2\sqrt{\frac{2}{3}} + o(1)$, and this bound can be improved to $B = \frac{4}{3} + o(1)$ by combining our methods with the approach to the gluing approximation taken in [Buckmaster et al. 2019a] (see Sections 11–12 below). For comparison, note that inequality (1) with

$$O\left(\sqrt{\frac{\log\log|\Delta x|^{-1}}{\log|\Delta x|^{-1}}}\right) \quad \text{replaced by} \quad O\left(\frac{1}{\log|\Delta x|^{-1}}\right)$$

would correspond to exactly the endpoint regularity $L^{\infty}_t C^{1/3}_x$.

The algorithm we develop to prove Theorem 1.1, presented in Sections 11–12, is the main novelty of our paper relating to the construction of solutions. Later on we will discuss theorems that elaborate a general theory of endpoint solutions. We expect that our algorithm can be adapted to give similar borderline regularity results in any known convex integration construction of Hölder-continuous solutions in which loss of powers in the frequency in the estimates can be avoided. In particular, the method is likely to generalize to isometric embeddings as in [Conti et al. 2012] (but not [De Lellis et al. 2018]), to nondegenerate active scalar equations [Isett and Vicol 2015], to the two-dimensional Monge–Ampère equation [Lewicka and Pakzad 2017], and to the surface quasigeostrophic (SQG) equation [Buckmaster et al. 2019b]. In these cases, there is no logarithmic loss in the main lemma and the $\log\log|\Delta x|^{-1}$ term appearing in (1) should be replaced by a large constant. (In the present case, the gluing technique gives rise to a logarithmic loss.) It is hopeful that our algorithm for optimizing the regularity may also be useful for potential applications to simulating convex integration solutions.

To achieve solutions with borderline regularity, it is necessary that the proof avoids losses of powers of the frequency in the estimates of the iteration scheme. An important point in this regard is that the approach to the gluing construction taken in [Isett 2018] obtains estimates that lose only a power of the logarithm of the frequency. These estimates require extending the timescale of the gluing beyond the standard timescale in the local existence theory for incompressible Euler, which would be inversely proportional to some C^{α} norm of the initial velocity gradient. (We note in contrast that the approach taken in [Buckmaster et al. 2019a] leads to loss of powers in the frequency at several points in the proof. These occur both in the gluing and convex integration in parts of the proof where local well-posedness theory,

[2] Note that changing the value of B in (1) corresponds to an inequivalent norm.

Schauder estimates and Calderón–Zygmund commutator estimates are employed.) Still there is one point in the proof in [Isett 2018], which occurs during the convex integration step, where one encounters a loss of powers in the frequency, and it is necessary to modify the convex integration part of the proof to obtain our borderline result. This loss of powers occurs specifically when solving the divergence equation $\nabla_j R^{j\ell} = U^\ell$ for a symmetric tensor $R^{j\ell}$.

To avoid this power loss, we adapt the strategy of [Isett and Oh 2016b] for localizing the convex integration method, which relies on two main modifications to the construction to gain the necessary estimate. The first point is to modify the construction using waves that are localized to small length scales and are each forced to obey the conservation of angular momentum in addition to the conservation of linear momentum. The second point is to make use of a family of operators developed in [Isett and Oh 2016b] that give compactly supported, symmetric solutions to the divergence equation when the necessary conditions for solving the symmetric divergence equation are satisfied. In combination, these modifications allow one to avoid the loss of powers in the frequency that had been present in [Isett 2017a] while enabling the authors to extend previous work of [Isett 2017a] on $\left(\frac{1}{5} - \epsilon\right)$-Hölder Euler flows to the nonperiodic setting of $\mathbb{R} \times \mathbb{R}^3$. Here we adapt these ideas to the present scheme to achieve an analogous improvement in our bounds. We note that it is important for this gain that we rely on the approach to the nonstationary phase estimate based on a parametrix and nonlinear phase functions introduced in [Isett 2017a].

Obtaining the endpoint case of Onsager's conjecture will require further new ideas, and it is of interest to study the behavior of potential energy nonconserving solutions with endpoint regularity and possible approaches to constructing them. A convex integration approach to the endpoint regularity would be possible if something sufficiently close to an "ideal" main lemma can be proven where one has neither logarithmic nor loss of powers in the frequency and the constant in the frequency growth is equal to $\widehat{C} = 1$ (as in a remark of [Isett and Oh 2016b]) or approaches $\widehat{C} = 1$ asymptotically at a rate such that $\sum_k \log \widehat{C}_{(k)}$ converges.

Such a construction appears to be presently out of reach; however, it may be considered favorable that convex integration constructions are able in general to yield solutions whose singularities occupy regions of space with positive volume. As the following theorem demonstrates, singularities with positive Lebesgue measure are necessary for any energy nonconserving solution with critical regularity to exist provided the integrability exponent for this regularity is greater than 3.

Theorem 1.2 (intermittency theorem). *A weak solution (v, p) to incompressible Euler on $I \times \mathbb{T}^d$ or $I \times \mathbb{R}^d$ that dissipates or otherwise fails to conserve energy cannot belong to an endpoint class $v \in L_t^r B_{r,\infty}^{1/3} \cap L_{t,x}^2$ with an integrability exponent $r > 3$ if its singular support has space-time Lebesgue measure zero.*

Here singular support is in the sense of distributions — the closed set whose complement is the largest set on which v is locally C^∞. In fact, the precise theorem we obtain is a sharper result where singular support is improved to singular support relative to the conservative Onsager critical space $L_t^3 \dot{B}_{3,c(\mathbb{N})}^{1/3}$ — that is, the closed set whose complement is the largest open set on which v is locally represented by an $L_t^3 \dot{B}_{3,c(\mathbb{N})}^{1/3}$ function (see Section 3).

Theorem 1.2 has a special significance in terms of intermittent scaling exponents in turbulence. The K41 theory [Kolmogorov 1941] predicts a scaling law of the form $\langle |v(x+\Delta x) - v(x)|^p \rangle^{1/p} \sim |\Delta x|^{1/3}$ for absolute structure functions (the Kolmogorov–Obukhov law), which mathematically corresponds to $B^{1/3}_{p,\infty}$ control of the velocity field. The idea that "intermittency" (deviations from self-similarity and homogeneity) in the energy dissipation and singular structure of turbulence can lead to the failure of this scaling law for $p \neq 3$ was first attributed to Landau by Kolmogorov in the 1940's (see [Frisch 1991, Section 5]). Moreover, experimental studies have found evidence of such intermittency in the energy dissipation of turbulent flows accompanied by deviations from the Kolmogorov–Obukhov law arising from a multifractal structure [Meneveau and Sreenivasan 1987; 1991; Meneveau et al. 1990]. Theorem 1.2 and its proof provide a rigorous sense in which lower dimensional singularities or energy dissipation in fact logically imply deviations from the Kolmogorov–Obukhov law, thus reinforcing the experimental findings.

Theorem 1.2 is a consequence of two facts that are also new remarks in the literature, which are a local version of the sharp energy conservation criterion in [Cheskidov et al. 2008] and a result on integrability of the energy dissipation measure (see Theorems 3.1 and 3.2 below). One would most likely expect that energy nonconserving solutions exist for the entire spectrum of endpoint spaces above, including the endpoint case of $L^\infty_t C^{1/3}_x$. For a more precise formulation of Theorem 1.2, we refer to Section 3. We also note the works [Buckmaster et al. 2021; Cheskidov and Shvydkoy 2014; 2023; De Rosa and Haffter 2023; Luo and Shvydkoy 2015; Novack and Vicol 2023; Shvydkoy 2018] for further mathematical results related to intermittency.

In addition to having the endpoint regularity, Onsager's paper [1949] describes Euler flows that furthermore have decreasing kinetic energy. Related to this point, we state the following Theorem.

Theorem 1.3. *If* (v, p) *are a weak solution to* (E) *on* $I \times \mathbb{T}^d$, $d \geq 2$, *with* $v \in C_t C^{1/3}_x$ (*or more generally with* $v \in C_t B^{1/3}_{3,\infty}$) *then the total kinetic energy* $e(t) = \int_{\mathbb{T}^d} \frac{1}{2}|v(t,x)|^2\, dx$ *is* C^1 *in time.*

Theorem 1.3 implies that the task of finding an energy dissipating solution in the class $v \in C_t C^{1/3}_x$ can be reduced to finding any example of a solution in this class that fails to satisfy energy conservation. Such a solution would have total kinetic energy that is either strictly increasing or strictly decreasing on some open interval of time. After possibly reversing time one obtains a solution with a decreasing energy profile on an open interval. For $\alpha < \frac{1}{3}$, the existence of energy-dissipating solutions in $C_t C^\alpha_x$ was proven in [Buckmaster et al. 2019a] by introducing an additional idea in the convex integration part of the proof to prescribe the energy profile of the solutions. We expect that this technique[3] should be possible to extend to the class described by (1) for example by modifying the statement of our main lemma in a way similar to the analysis in [Isett and Oh 2016b; 2017].

The proof of Theorem 1.3, presented in Section 2 below, suggests that the failure of energy conservation for solutions in the critical space $v \in C_t C^{1/3}_x$ should be very common. The proof reduces the existence of an energy-dissipating solution to solving the Euler equations with appropriate initial data in the desired critical space for a short time. However, one must be cautious that the Euler equations are ill-posed

[3]A related technical point is that the approach to prescribing the energy profile in [Buckmaster et al. 2019a] involves requiring the stress tensor $R^{j\ell}$ to be trace-free in addition to being symmetric. It is also possible to prescribe the energy profile without imposing the trace-free requirement on $R^{j\ell}$; see [Isett and Oh 2016b; 2017].

in $C_t C_x^\alpha$ or in $C_t B^\alpha_{3,\infty}$ for all $\alpha < 1$, as has been shown in [Bardos and Titi 2010; Cheskidov and Shvydkoy 2010], which presents a significant difficulty for constructing solutions in these spaces. Complementing these negative results, our proof of Theorem 1.3 yields as a byproduct a necessary condition for a given divergence-free vector field to be the initial datum of a solution in the class $v \in C_t B^{1/3}_{3,\infty}$.

With regard to energy dissipation and solving the Cauchy problem, a natural question is whether there is a simple, Onsager-critical function space in which *local* dissipation of energy is guaranteed along with total kinetic energy dissipation if one can solve the Cauchy problem in that space with the appropriate initial data. A simple criterion of this type is provided in Theorem 3.3 of Section 3A. The regularity condition imposed to maintain local dissipation in this criterion is notably stronger than that assumed to control total kinetic energy in Theorem 1.3, as our Theorem 3.3 involves solutions in the function space $C_t C_x^{1/3}$ rather than assuming only Besov regularity in space.

We now summarize the organization of the paper and the proof of our borderline result, Theorem 1.1. The general theory of endpoint solutions, including Theorems 1.2–3.3, is contained in Sections 2–3A. We then summarize notation for the main body of the paper in Section 4. Sections 5–10 contain the main lemma of the paper and our modification of the convex integration construction of [Isett 2018]. These sections assume familiarity with the convex integration construction in that paper. Section 11 explains the proof of Theorem 1.1 using the main lemma, and presents our new method for optimizing the regularity in a general convex integration scheme. Section 12 outlines how to combine our methods with the approach to the gluing approximation taken in [Buckmaster et al. 2019a] to improve the rate of convergence to the critical exponent in the estimate (1).

2. Regularity of kinetic energy at the critical exponent

We start with a proof of Theorem 1.3 on the C^1 regularity of the kinetic energy profile for solutions of class $C_t B^{1/3}_{3,\infty}$. In the next section we prove Theorem 1.2. We will use the summation convention for summing repeated upper and lower spatial indices, so that $v^\ell v_\ell = |v|^2$ and $\nabla_\ell v^\ell = \operatorname{div} v$.

The proof of Theorem 1.3 is an extension of the argument of [Constantin et al. 1994] for proving energy conservation for weak solutions in the class $v \in L^3_t B^{1/3+\epsilon}_{3,\infty}$ and of a remark in [Isett 2023] on the endpoint case. Namely, suppose that (v, p) is a weak solution to (E) with velocity of class $v \in C_t B^{1/3}_{3,\infty}(I \times \mathbb{T}^d)$, $d \geq 2$, with I an open interval. Let η_ϵ be a standard mollifier in $\mathbb{R}^d$ at length scale ϵ, and let $v^\ell_\epsilon = \eta_\epsilon * v^\ell$ denote the mollification of v in the spatial variables. Then, as in [Constantin et al. 1994], one has (using $v \in C_t L^2_x$) that

$$\frac{d}{dt}\int_{\mathbb{T}^d} \frac{|v|^2(t,x)}{2}\,dx = \lim_{\epsilon\to 0}\frac{d}{dt}\int_{\mathbb{T}^d}\frac{|v_\epsilon|^2(t,x)}{2}\,dx = -\lim_{\epsilon\to 0}\int_{\mathbb{T}^d}\nabla_j (v_\epsilon)_\ell R^{j\ell}_\epsilon(t,x)\,dx, \tag{2}$$

$$R^{j\ell}_\epsilon(t,x) := v^j_\epsilon(t,x) v^\ell_\epsilon(t,x) - \eta_\epsilon * (v^j v^\ell)(t,x),$$

where the convergence in (2) holds in $\mathcal{D}'(I)$. (See [Isett and Oh 2016a, Proof of Theorem 2.2] for a detailed presentation of this point.) The rightmost term in (2) gives rise to the family of trilinear forms $T_\epsilon[v,v,v](t) := \int_{\mathbb{T}^d}\nabla_j(v_\epsilon)_\ell R^{j\ell}_\epsilon(t,x)\,dx$ that satisfy, uniformly in ϵ, the bound

$$|T_\epsilon[u,v,w]|(t) \lesssim \|u(t,\cdot)\|_{B^{1/3}_{3,\infty}} \|v(t,\cdot)\|_{B^{1/3}_{3,\infty}} \|w(t,\cdot)\|_{B^{1/3}_{3,\infty}}, \tag{3}$$

by the commutator estimate of [Constantin et al. 1994]. Using (3), we have that the family of functions $T_\epsilon[v, v, v](t)$ are both uniformly bounded and equicontinuous on every compact subinterval of I, as they satisfy

$$|T_\epsilon[v, v, v](t) - T_\epsilon[v, v, v](t_0)| \lesssim \|v(t,\cdot) - v(t_0,\cdot)\|_{B^{1/3}_{3,\infty}} \|v\|^2_{C_t B^{1/3}_{3,\infty}},$$

and their moduli of continuity can therefore be bounded uniformly in ϵ in terms of the modulus of continuity of $v(t,\cdot)$ into $B^{1/3}_{3,\infty}(\mathbb{T}^d)$ and local bounds for $\|v(t,\cdot)\|_{B^{1/3}_{3,\infty}}$. Consequently, the convergence in (2) is actually uniform-in-t on every open interval J with compact closure in I, as the weak limit in $\mathcal{D}'(J)$, which is unique, must also be achieved uniformly along subsequences by Arzelà–Ascoli. (If the convergence were not uniform, there would exist a subsequence converging uniformly to a continuous function different from (2), which contradicts the weak convergence.) The energy flux in (2), a priori in $\mathcal{D}'(I)$, is thus continuous in t on I, and the kinetic energy profile is therefore C^1 in t on I.

Note that one would typically expect the energy flux given by the right-hand side of (2) to be nonzero at any given time t_0 for a vector field with $v(t_0,\cdot) \in C^{1/3}_x$, as examples of divergence-free initial data $v_0(x) \in C^{1/3}$ for which this limit can be positive are given in [Cheskidov et al. 2008; Eyink 1994].

We note also that our argument provides a necessary condition for a vector field $v_0(x) \in B^{1/3}_{3,\infty}$ to be realized as the initial datum of an Euler flow in the class $v \in C_t B^{1/3}_{3,\infty}$, which is that the limit $\lim_{\epsilon\to 0} T_\epsilon[v_0, v_0, v_0]$ on the right-hand side of (2) must exist and must also be independent of the chosen mollifying kernel η_ϵ, so that the instantaneous rate of energy dissipation is well defined at time 0.

We now turn to the proof of Theorem 1.2.

3. Singularities of dissipative solutions with critical regularity

We now establish Theorem 1.2 on the necessity of positive measure singularities of Onsager critical solutions with integrability exponent $p > 3$ that do not conserve energy, which is an immediate consequence of Theorems 3.1 and 3.2 below. Both theorems are stated in terms of Besov spaces whose basic properties we recall within the proofs. We state the first Theorem 3.1 in a sharp, critical space to make clear the severity of the singularity that is implicitly discussed in Theorem 1.2.

Theorem 3.1. *Let (v, p) be a weak solution to the incompressible Euler equations of class $v \in L^3_{t,x}$ on $I \times \mathbb{T}^d$ or $I \times \mathbb{R}^d$, with I an open interval. Then the distribution*

$$-D[v, p] := \partial_t\big(\tfrac{1}{2}|v|^2\big) + \nabla_j\big(\big(\tfrac{1}{2}|v|^2 + p\big)v^j\big)$$

has support contained in the singular support of v relative to the critical space $L^3_t B^{1/3}_{3,c_0(\mathbb{N})}$.

Here we define the singular support of v relative to the space $L^3_t B^{1/3}_{3,c_0(\mathbb{N})}$ to be the complement of those points $q = (t, x)$ for which there exists an open neighborhood O_q of q on which v is represented by a distribution of class $L^3_t B^{1/3}_{3,c_0(\mathbb{N})}$. We recall the standard characterization of the $B^{1/3}_{r,\infty}$ norm of a vector field on an open set Ω in $\mathbb{R}^d$, which is given by $\|v\|_{L^r(\Omega)} + \sup_{h\in\mathbb{R}^d\setminus\{0\}} |h|^{-1/3}\|v(\cdot - h) - v(\cdot)\|_{L^r_x(\Omega\cap(\Omega+h))}$, and we also recall that $C^\infty(\Omega)$ is dense in $B^{1/3}_{r,c_0(\mathbb{N})}(\Omega)$ with respect to the $B^{1/3}_{r,\infty}$ norm. It is clear that the singular support of v relative to $L^3_t B^{1/3}_{3,c_0(\mathbb{N})}$ is a subset of the usual singular support of v as a distribution.

Related restrictions on the support of $D[v, p]$ under different hypotheses and with different proofs are given in [Cheskidov and Shvydkoy 2014, Theorem 4.3], [Drivas and Nguyen 2018, Theorem 1] and [Bardos et al. 2019, Theorem 3.1].

Our second theorem asserts that weak solutions of class $v \in L_t^r B_{r,\infty}^{1/3}$ for integrability exponents $r > 3$ possess integrability for their corresponding energy dissipation measure $D[v, p]$. The assumptions are given in a way that is sufficient for our application to proving Theorem 1.2.

Theorem 3.2. *Let (v, p) be a weak solution to incompressible Euler of class $v \in L_t^r B_{r,\infty}^{1/3}$ for some $r \geq 3$ on $I \times \mathbb{T}^d$ or $I \times \mathbb{R}^d$, with I an open interval. Then the distribution $D[v, p]$ above is a (signed) measure. If furthermore $r > 3$, this measure is absolutely continuous with respect to the Lebesgue measure, and its Radon–Nikodym derivative is of class $D[v, p] \in L_{t,x}^{r/3}$.*

It will be clear that the proof of Theorem 3.2 does not give absolute continuity in the case $r = 3$. For example, the proof would apply to many other equations such as Burgers', where shock solutions give examples of $L_t^\infty B_{3,\infty}^{1/3}$ solutions for which the corresponding energy dissipation measure is not absolutely continuous. There also exist time-independent divergence-free vector fields demonstrating that our approach would not yield absolute continuity in the $r = 3$ case.[4]

Proof of Theorem 1.2. Let us observe now that Theorem 1.2 follows from Theorems 3.1 and 3.2, focusing on the case of $I \times \mathbb{R}^d$. Namely, if a weak solution (v, p) is of class $v \in L_{t,x}^3 \cap L_{t,x}^2$ and does not conserve kinetic energy (meaning that the distribution $e(t) := \frac{1}{2}\int_{\mathbb{R}^d} |v|^2(t, x)\,dx$ is not a constant), then the distribution $D[v, p]$ is well defined and cannot be the 0 distribution. This statement can be checked by verifying that, for any test function $\psi \in C_c^\infty(I)$, by dominated convergence one has

$$\begin{aligned}\langle \psi(t), e'(t)\rangle_{\mathcal{D}'(I)} &= \lim_{R\to\infty} \langle \psi(t)\chi_R(x), -D[v, p]\rangle_{\mathcal{D}'(I\times\mathbb{R}^d)} \\ &:= -\int_I \psi'(t)e(t)\,dt \\ &= -\lim_{R\to\infty} \int_{I\times\mathbb{R}^d} \left[\psi'(t)\chi_R(x)\frac{|v|^2}{2} + \psi(t)\nabla_j\chi_R(x)\left(\frac{|v|^2}{2} + p\right)v^j\right] dt\,dx,\end{aligned}$$

where $\chi_R(x) = \chi(x/R)$ is a rescaled bump function that is equal to 1 in a growing neighborhood of the origin that encompasses the whole space as $R \to \infty$. We use here that $\left(\frac{1}{2}|v|^2 + p\right)v^j$ and $\frac{1}{2}|v|^2$ are both in $L_{t,x}^1(I \times \mathbb{R}^d)$ as $v \in L_{t,x}^2 \cap L_{t,x}^3$ and $p = \Delta^{-1}\nabla_j\nabla_\ell(v^j v^\ell) \in L_{t,x}^{3/2}$ by Calderón–Zygmund theory,[5] which implies that $\Delta^{-1}\nabla_j\nabla_\ell$ acts as a bounded operator on $L_{t,x}^{3/2}$ mapping two-tensors to scalars. In fact the weaker condition $(1 + |x|)^{-1}\left(\frac{1}{2}|v|^2 + p\right)v^j \in L_{t,x}^1$ suffices for this proof.

For a solution of class $v \in L_t^r B_{r,\infty}^{1/3}$ with $r > 3$, we have by Theorem 3.2 that $D[v, p]$ is of class $L_{t,x}^{r/3}$. For $D[v, p]$ to be nonzero, the support of $D[v, p]$ as a distribution must then occupy a closed set with positive Lebesgue measure. From Theorem 3.1, the nontrivial support of $D[v, p]$ gives a lower bound for the singular support of v as a distribution, which implies Theorem 1.2. □

[4] R. Shvydkoy, personal communication.

[5] The case of $\mathbb{T}^d$ appears to be less standard than the $\mathbb{R}^d$ case but can be deduced from the $\mathbb{R}^d$ case using the local Calderón–Zygmund theory in $\mathbb{R}^d$ as in [Wang 2003]. See, e.g., [Isett 2017b, Proof of Theorem 6.2].

We now prove Theorem 3.1 along with Theorem 3.2. The proof is a local version of the energy conservation criteria of [Cheskidov et al. 2008; Constantin et al. 1994]. The observation that the proof of energy conservation in [Constantin et al. 1994] can be localized is originally due to [Duchon and Robert 2000] and has recently been of use to several authors in the context of bounded domains [Bardos and Titi 2018; Bardos et al. 2019; Drivas and Nguyen 2018]. Some issues that are not central to our goals here have been avoided as our hypotheses suffice to guarantee $p = \Delta^{-1}\nabla_j\nabla_\ell(v^j v^\ell) \in L^{3/2}_{t,x}$. The norms and function space in what follows refer to the entire space $I \times \mathbb{T}^d$ or $I \times \mathbb{R}^d$ unless otherwise stated. We will focus on the $\mathbb{R}^d$ cases in what follows as the results for $\mathbb{T}^d$ follow from the same proofs.

Proof of Theorems 3.2 and 3.1. Let (v, p) be a weak solution of class $v \in L^r_t B^{1/3}_{r,\infty} \cap L^2_{t,x}$ for some $r \geq 3$. Then

$$v \in L^r_{t,x} \cap L^2_{t,x} \quad \text{and} \quad p = \Delta^{-1}\nabla_j\nabla_\ell(v^j v^\ell) \in L^{r/2}_{t,x}$$

by Calderón–Zygmund theory as before. The key formula we use is the analogue of the formula from [Duchon and Robert 2000] involving the commutator of [Constantin et al. 1994]:

$$-D[v, p] = \partial_t\big(\tfrac{1}{2}|v|^2\big) + \nabla_j\big[\big(\tfrac{1}{2}|v|^2 + p\big)v^j\big] = \lim_{\epsilon\to 0} \nabla_j v_{\epsilon\ell} R^{j\ell}_\epsilon, \qquad R^{j\ell}_\epsilon = \eta_\epsilon * (v^j v^\ell) - v^j_\epsilon v^\ell_\epsilon, \tag{4}$$

where $v^\ell_\epsilon = \eta_\epsilon * v^\ell$ is a standard mollification of v^ℓ in the spatial variables at length scale ϵ, and the limit (4) holds for any fixed test function on $I \times \mathbb{R}^d$ or $I \times \mathbb{T}^d$.

We first prove Theorems 3.2 and 3.1 assuming (4). By Hölder's inequality with $3/r = 1/r + 2/r$ and the commutator estimates of [Constantin et al. 1994], one has the following bound uniformly in ϵ:

$$\begin{aligned}\|\nabla_j v_{\epsilon\ell} R^{j\ell}_\epsilon\|_{L^{r/3}_{t,x}} \leq \|\nabla_j v_{\epsilon\ell}\|_{L^r_{t,x}} \|R^{j\ell}_\epsilon\|_{L^{r/2}_{t,x}} &\lesssim (\epsilon^{-1+1/3}\|v\|_{L^r_t B^{1/3}_{r,\infty}})\|R^{j\ell}_\epsilon\|_{L^{r/2}_{t,x}} \\ &\lesssim (\epsilon^{-1+1/3}\|v\|_{L^r_t B^{1/3}_{r,\infty}})\epsilon^{2/3}\|v\|^2_{L^r_t B^{1/3}_{r,\infty}} \lesssim \|v\|^3_{L^r_t B^{1/3}_{r,\infty}}.\end{aligned} \tag{5}$$

The sequence $\nabla_j v_{\epsilon\ell} R^{j\ell}_\epsilon$ is therefore uniformly bounded in $L^{r/3}_{t,x}$ independent of $\epsilon > 0$.

As a consequence, using $r \geq 3$, the weak limit $D[v, p] = \lim_{\epsilon\to 0} \nabla_j v_{\epsilon\ell} R^{j\ell}_\epsilon$ is a Radon measure. That is, by (5) and Hölder's inequality (with the characteristic function of K as one of the factors), for any compact set K and any test function $\phi(t, x)$ supported in K, one has

$$|\langle \phi, D[v, p]\rangle_{\mathcal{D}'(I\times\mathbb{R}^d)}| \leq C_K \|\phi\|_{C^0} \|v\|^3_{L^r_t B^{1/3}_{r,\infty}}.$$

Moreover, for $r > 3$, the measure $D[v, p]$ is absolutely continuous with density function in $L^{r/3}_{t,x}$ by the duality characterization of the latter space, thus confirming Theorem 3.2. Namely, if $s \in (1, \infty)$ is the dual exponent with $1/s + 3/r = 1$, we have

$$|\langle \phi, D[v, p]\rangle_{\mathcal{D}'(I\times\mathbb{R}^d)}| \leq C \|\phi\|_{L^s_{t,x}} \|v\|^3_{L^r_t B^{1/3}_{r,\infty}}.$$

From the density of test functions in $L^s_{t,x}$, we have that $D[v, p]$ is in the dual of $L^s_{t,x}$, which is the space $L^{r/3}_{t,x}$.

The proof of Theorem 3.1 is more subtle as the statement concerns the function space $L^3_t B^{1/3}_{3,c_0(\mathbb{N})}$ and is more local in nature. In particular, our approach is local as compared to the Fourier-analytic approach of [Cheskidov et al. 2008]; the details in the presentation below are similar to those of [Isett and Oh 2016a].

Let $v \in L^3_{t,x}$ be a weak solution, so that $p \in L^{3/2}_{t,x}$, and let q be a point in the complement of the singular support of v relative to $L^3_t B^{1/3}_{3,c_0(\mathbb{N})}$. That is, there is an open neighborhood of q that can be taken to have the form $J \times B_q$ with J a finite open subinterval of I and B_q a spatial ball such that $v \in L^3_t B^{1/3}_{3,c_0(\mathbb{N})}(J \times B_q)$. Let $\phi \in C^\infty_c(J \times B_q)$ be a fixed test function and $B'_q \subseteq B_q$ be a smaller spatial ball such that $\operatorname{supp}\phi \subseteq J \times B'_q$. From (4), we have

$$\langle \phi, -D[v, p]\rangle = \lim_{\epsilon\to 0} \int_J \int_{B'_q} \phi(t, x) \nabla_j v_{\epsilon\ell} R^{j\ell}_\epsilon \, dx\, dt,$$

where by assumption $v \in L^3_t B^{1/3}_{3,c_0(\mathbb{N})}(J \times B_q)$. Then as in the proof of (5) one has that

$$|\langle \phi, -D[v, p]\rangle| \le \limsup_{\epsilon\to 0} \|\phi\|_{C^0} \int_J \|\nabla v_\epsilon(t,\cdot)\|_{L^3(B'_q)} \|R^{j\ell}_\epsilon(t,\cdot)\|_{L^{3/2}(B'_q)}\, dt, \tag{6}$$

and that the dt integrand is bounded uniformly in ϵ by $C\|v(t,\cdot)\|^3_{B^{1/3}_{3,\infty}(B_q)}$, which is integrable over J. Moreover, for almost every $t \in J$, one has that $v(t,\cdot) \in B^{1/3}_{3,c_0(\mathbb{N})}$ belongs to the closure of $C^\infty(B_q)$ in the $B^{1/3}_{3,\infty}$ norm. For each such t, the improved bound

$$\limsup_{\epsilon\to 0} \epsilon^{1-1/3} \|\nabla v_\epsilon(t,\cdot)\|_{L^3(B'_q)} = 0$$

holds, as can be seen by a smooth approximation argument. Combined with $\|R^{j\ell}_\epsilon(t,\cdot)\|_{L^{3/2}(B'_q)} \le C_t \epsilon^{2/3}$ on the same set of t, we have the convergence to zero for almost every t in (6), which implies the limit in (6) is zero by the Lebesgue dominated convergence theorem.

The last remaining point is to justify the limit in (4) for any fixed test function, which we prove using the definition of a weak solution following details similar to [Isett and Oh 2016a]. Let (v, p) be a weak solution of class $v \in L^3_{t,x}$, so that $p \in L^{3/2}_{t,x}$ on $I \times \mathbb{R}^d$ as before. Let $\phi \in C^\infty_c$ be a test function on $I \times \mathbb{R}^d$ and V_ϕ be an open set with compact closure in $I \times \mathbb{R}^d$ that contains $\operatorname{supp}\phi$. Let $\eta_\epsilon(h) = \epsilon^{-d}\eta(h/\epsilon)$ and $\zeta_\delta(\tau) = \delta^{-1}\zeta(\tau/\delta)$ be even mollifying kernels in the space and time variables, respectively, with respective supports $\operatorname{supp}\eta_\epsilon \subseteq B_\epsilon(0)$ in $\mathbb{R}^d$ and $\operatorname{supp}\zeta_\delta \subseteq B_\delta(0)$ in $\mathbb{R}$. Define $\eta_{\epsilon\delta}(\tau, h) = \zeta_\delta(\tau)\eta_\epsilon(h)$ and the vector field $\omega^\ell_{\epsilon\delta} = \eta_{\epsilon\delta} * (\phi\eta_{\epsilon\delta} * v^\ell)$, where the convolution is in both space and time. We will write $*_x$ or $*_t$ to mean convolution in only the space or time variables. Taking $\omega^\ell_{\epsilon\delta}$ as our test function in the weak formulation of Euler (i.e., multiplying the equation and integrating by parts) gives

$$-\int_{I\times\mathbb{R}^d} [v^\ell \partial_t \eta_{\epsilon\delta} * (\phi\eta_{\epsilon\delta} * v_\ell) + v^j v^\ell \nabla_j \eta_{\epsilon\delta} * (\phi\eta_{\epsilon\delta} * v_\ell) + p\nabla^\ell \eta_{\epsilon\delta} * (\phi\eta_{\epsilon\delta} * v_\ell)]\, dx\, dt = 0.$$

Using the self-adjointness of $\eta_{\epsilon\delta}*$ and the divergence-free property of $\eta_{\epsilon\delta} * v^\ell$, one obtains

$$-\int_{I\times\mathbb{R}^d} \left[\partial_t\phi(t,x) \frac{|\eta_{\epsilon\delta} * v^\ell|^2}{2} + (v^j v^\ell)\eta_{\epsilon\delta} * \nabla_j[\phi\eta_{\epsilon\delta} * v_\ell] + p\eta_{\epsilon\delta} * (\nabla^\ell\phi\eta_{\epsilon\delta} * v_\ell)\right] dx\, dt = 0.$$

As $v \in L^3_{t,x} \cap L^2_{t,x}(V_\phi)$ and $p \in L^{3/2}_{t,x}(V_\phi)$, we may safely let $\delta \to 0$ at this point with $\epsilon > 0$ fixed using uniform-in-δ boundedness of the convolution operators in the formula (including the operators $\nabla_j \eta_{\epsilon\delta} *$ that appear from the product rule) and the strong convergence of $\eta_{\epsilon\delta} * v^\ell \to v^\ell_\epsilon := \eta_\epsilon *_x v^\ell$ in $L^2_{t,x} \cap L^3_{t,x}(\operatorname{supp} \phi)$ for each fixed $\epsilon > 0$. Taking the $\delta \to 0$ limit, we may replace each appearance of $\eta_{\epsilon\delta} * = \eta_\epsilon *_x [\zeta_\delta *_t \cdot\,]$ in the formula with $\eta_\epsilon *_x$, which we now write more simply as $\eta_\epsilon * := \eta_\epsilon *_x$.

Using the self-adjointness of $\eta_\epsilon *$ and the divergence-free property of v^ℓ_ϵ, which are justified by the same limiting argument, one then obtains

$$-\int_{I\times\mathbb{R}^d} \left[\partial_t \phi(t,x) \frac{|\eta_\epsilon * v^\ell|^2}{2} + \nabla_j \phi(t,x) \left(\frac{|v_\epsilon|^2}{2} v^j_\epsilon + \eta_\epsilon * p v^j_\epsilon \right) \right] dx\,dt = \int_{I\times\mathbb{R}^d} \phi(t,x) \nabla_j v_{\epsilon\ell} R^{j\ell}_\epsilon \, dx\,dt + Z_\epsilon,$$

$$Z_\epsilon := \int_{I\times\mathbb{R}^d} \nabla_j \phi R^{j\ell}_\epsilon v_{\epsilon\ell} \, dx\,dt.$$

Note that the left-hand side of the first equation tends to exactly $\langle \phi, -D[v,p] \rangle_{\mathcal{D}'(I\times\mathbb{R}^d)}$ as $\epsilon \to 0$, using that $v^\ell_\epsilon = \eta_\epsilon * v^\ell \to v^\ell$ in $L^3_{t,x} \cap L^2_{t,x}(V_\phi)$ and that $p \in L^{3/2}_{t,x}$ again. Thus formula (4) will be proven once it is shown that $\lim_{\epsilon\to 0} Z_\epsilon = 0$.

To this end, write $R^{j\ell}_\epsilon$ in terms of bilinear operators $R^{j\ell}_\epsilon = B_\epsilon[v^j, v^\ell]$, where the operators B_ϵ are defined for smooth u^j and w^ℓ by $B_\epsilon[u^j, w^\ell] := \eta_\epsilon * (u^j w^\ell) - \eta_\epsilon * u^j \eta_\epsilon * w^\ell$. One has then that

$$\|B_\epsilon[u,w]\|_{L^{3/2}_{t,x}(V_\phi)} \to 0 \quad \text{as } \epsilon \to 0$$

whenever u^j and w^ℓ are smooth vector fields on $I \times \mathbb{R}^d$, and that

$$\|B_\epsilon[u,w]\|_{L^{3/2}_{t,x}(V_\phi)} \le C \|u\|_{L^3_{t,x}} \|w\|_{L^3_{t,x}(I\times\mathbb{R}^d)}$$

uniformly in $\epsilon > 0$. Combining these properties and using the density of smooth vector fields in $L^3_{t,x}(I \times \mathbb{R}^d)$, we obtain that $\|R^{j\ell}_\epsilon\|_{L^{3/2}_{t,x}(V_\phi)} \to 0$ as $\epsilon \to 0$, and $Z_\epsilon \to 0$ as well by applying Hölder's inequality with v_ϵ bounded in $L^3_{t,x}(V_\phi)$. □

3A. ***Stability of local energy dissipation in a critical class.*** In this section we prove Theorem 3.3, which provides a simple function space criterion from which one can deduce local dissipation on an open interval of time from local dissipation at time 0.

Theorem 3.3. *Let $\bar{v}$ be a divergence-free vector field of class $\bar{v} \in C^{1/3}(\mathbb{T}^d)$ for which the local energy dissipation is everywhere bounded by a strictly negative constant. Then any weak solution (v, p) of class $v \in C_t C^{1/3}(I \times \mathbb{T}^d)$ that obtains the initial data $\bar{v}$ must satisfy the local energy inequality $D[v,p] < 0$ on some open time interval containing $t = 0$.*

The precise condition on the initial data $\bar{v}$ will be specified in line (8) of the proof below.

Proof. Let $\bar{v}$ be as above and let (v, p) be a weak solution to the Euler equations of class $v \in C_t C^{1/3}(I \times \mathbb{T}^d)$ on an open interval of time containing $t = 0$ with initial data $\bar{v}(x)$. Let $\tilde{I}$ be an open subinterval of I containing $t = 0$, and let $\phi \in C^\infty_c(\tilde{I} \times \mathbb{T}^d)$ be a nonnegative test function supported in $t \in \tilde{I}$. As in the previous sections, we have

$$\langle \phi, D[v,p] \rangle = \lim_{\delta\to 0} \int_{\tilde{I}} \int_{\mathbb{T}^d} \phi(t,x) T_\delta[v](t,x)\, dx\,dt, \qquad T_\delta[v](t,x) = \nabla_j v_{\delta\ell} R^{j\ell}_\delta(t,x),$$

where T_δ is the trilinear form from Section 2 and $D[v, p]$ is as in the previous sections. We write

$$\langle \phi, D[v, p]\rangle = \lim_{\delta\to 0} \int_{\tilde{I}}\int_{\mathbb{T}^d} \phi(t, x)(T_\delta[v](t, x) - T_\delta[v](0, x))\, dx\, dt + \lim_{\delta\to 0} \int_{\tilde{I}}\int_{\mathbb{T}^d} \phi(t, x) T_\delta[v](0, x)\, dx\, dt. \tag{7}$$

The precise assumption placed on the initial condition $\bar{v}$ is that

$$\lim_{\delta\to 0} T_\delta[v](0, x) = \lim_{\delta\to 0} \nabla_j \bar{v}_{\delta\ell} R_\delta^{j\ell}(0, x) \le -\varepsilon < 0 \tag{8}$$

in the sense of distributions on $\mathbb{T}^d$ for some constant $\varepsilon > 0$. Integrating (8) against the nonnegative test function

$$\tilde{\phi}(x) = \int_{\tilde{I}} \phi(t, x)\, dt \in C^\infty(\mathbb{T}^d),$$

we have that

$$\text{second term of (7)} \le -\varepsilon \int_{\mathbb{T}^d} \left[\int_{\tilde{I}} \phi(t, x)\, dt\right] dx.$$

For a sufficiently small time interval $\tilde{I}$, we can obtain the bound

$$\sup_{\delta>0} \| T_\delta[v](t, x) - T_\delta[v](0, x)\|_{L^\infty(\tilde{I}\times\mathbb{T}^d)} \le \tfrac{1}{2}\varepsilon$$

using that the T_δ are uniformly bounded trilinear forms mapping $C^{1/3}$ to L^∞, the assumption that $v \in C_t C_x^{1/3}$ is continuous in time with values in $C^{1/3}$, and the commutator estimate of [Constantin et al. 1994] to control the bilinear term. Combining these estimates with the sign condition on ϕ gives

$$\begin{aligned}\langle \phi, D[v, p]\rangle &\le \|\phi\|_{L^1(\tilde{I}\times\mathbb{T}^d)} \frac{\varepsilon}{2} - \varepsilon \int_{\tilde{I}}\int_{\mathbb{T}^d} \phi(t, x)\, dx\, dt \\ &\le -\frac{\varepsilon}{2} \int_{\tilde{I}}\int_{\mathbb{T}^d} \phi(t, x)\, dx\, dt\end{aligned}$$

for all nonnegative $\phi \in C_c^\infty(\tilde{I} \times \mathbb{T}^d)$. This bound shows that $D[v, p] \le -\frac{1}{2}\varepsilon < 0$ as a distribution when restricted to $\tilde{I} \times \mathbb{T}^d$, which concludes the proof of Theorem 3.3. □

With Theorems 1.2–3.3 now proven, we turn to the notation that will be used for the remainder of the paper and the proof of Theorem 1.1.

4. Notation

We will follow the same notational conventions as introduced in [Isett 2018, Section 2]. In particular, multi-indices will be represented in vector notation. For example, if $\vec{a} = (a_1, a_2, a_3)$ is a multi-index of order $|\vec{a}| = 3$, each $a_i \in \{1, 2, 3\}$, then $\nabla_{\vec{a}} = \nabla_{a_1}\nabla_{a_2}\nabla_{a_3}$ denotes the corresponding third-order partial derivative operator. We use $\mathrm{supp}_t\, f$ to indicate the time support of a function f with domain in $\mathbb{R} \times \mathbb{T}^3$ (i.e., the closed set of times for which $\{t\} \times \mathbb{T}^3$ intersects the usual support).

We recall the definitions of an Euler–Reynolds flow and frequency-energy levels.

Definition 4.1. A vector field $v^\ell : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}^3$, function $p : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$ and symmetric tensor field $R^{j\ell} : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}^3 \otimes \mathbb{R}^3$ satisfy the *Euler–Reynolds equations* if the equations

$$\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p = \nabla_j R^{j\ell},$$
$$\nabla_j v^j = 0$$

hold on $\mathbb{R} \times \mathbb{T}^3$. Any solution to the Euler–Reynolds equations (v, p, R) is called an *Euler–Reynolds flow*. The symmetric tensor field $R^{j\ell}$ is called the *stress* tensor.

Definition 4.2. Let (v, p, R) be a solution of the Euler–Reynolds equation, $\Xi \geq 3$ and $e_v \geq e_R > 0$ be positive numbers. We say that (v, p, R) have *frequency-energy levels* bounded by (Ξ, e_v, e_R) to order L in C^0 if v and R are of class $C_t C_x^L$ and the following estimates hold:

$$\|\nabla_{\vec{a}} v\|_{C^0} \leq \Xi^{|\vec{a}|} e_v^{1/2} \quad \text{for all } 1 \leq |\vec{a}| \leq L,$$
$$\|\nabla_{\vec{a}} R\|_{C^0} \leq \Xi^{|\vec{a}|} e_R \quad \text{for all } 0 \leq |\vec{a}| \leq L.$$

Here ∇ refers only to derivatives in the spatial variables.

5. The main lemma

The first goal of the paper will be to improve on the main lemma in [Isett 2018], so that we remove the need for a double exponential growth of frequencies. The main lemma of our paper states the following:

Lemma 5.1 (main lemma). *Let $L = 3$. There exists constants $\widehat{C}$ and C_L such that the following holds. Let (v, p, R) be any solution of the Euler–Reynolds equation with frequency-energy levels bounded by (Ξ, e_v, e_R) to order L in C^0, and let J be an open subinterval of $\mathbb{R}$ such that*

$$\operatorname{supp}_t v \cup \operatorname{supp}_t R \subseteq J.$$

Define the parameter $\widehat{\Xi} = \Xi (e_v/e_R)^{1/2}$. Let N be any positive number obeying the condition

$$N \geq (e_v/e_R)^{1/2}. \tag{9}$$

Then there exists a solution (v_1, p_1, R_1) of Euler–Reynolds with frequency-energy levels bounded by

$$(\Xi', e_v', e_R') = \left(\widehat{C} N \Xi, (\log \widehat{\Xi}) e_R, (\log \widehat{\Xi})^{5/2} \frac{e_v^{1/2} e_R^{1/2}}{N} \right) \tag{10}$$

to order L in C^0 such that

$$\operatorname{supp}_t v_1 \cup \operatorname{supp}_t R_1 \subseteq N(J; \Xi^{-1} e_v^{-1/2})$$

and such that the correction $V = v_1 - v$ obeys the estimate

$$\|V\|_{C^0} \leq C_L (\log \widehat{\Xi})^{1/2} e_R^{1/2}. \tag{11}$$

The crucial difference between the main lemma above as compared to [Isett 2018, Lemma 2.1] is that we do not require any lower bound of the form $N \geq \Xi^\eta$ for the frequency growth parameter N in inequality (9). This difference enables us to avoid double exponential growth of frequencies in constructing solutions as in [Isett and Oh 2016b]. Likewise, the constants $\widehat{C}$ and C_L in the estimates do not depend on such a parameter η.

We establish Lemma 5.1 by modifying the proof of the convex integration lemma, [Isett 2018, Lemma 3.3], as the proof of this lemma contains the only step in which the assumption $N \geq \Xi^{\eta}$ is used.

6. The improved convex integration lemma

As in [Isett 2018], we will establish Lemma 5.1 by combining a gluing approximation lemma and a convex integration lemma. In Lemma 6.1 below, we summarize the result of combining the regularization lemma and the gluing approximation lemma from [Isett 2018, Section 3]. (Here we have renamed the Euler–Reynolds flow that were $(\tilde{v}, \tilde{p}, \widetilde{R})$ to be (v, p, R).)

Lemma 6.1 (gluing approximation lemma). *There are absolute constants $C_1 \geq 2$ and $\delta_0 \in \left(0, \frac{1}{25}\right)$ such that the following holds. Let (v_0, p_0, R_0) be an Euler–Reynolds flow with frequency-energy levels bounded by (Ξ, e_v, e_R) to order 3 in C^0 such that $\operatorname{supp}_t v_0 \cup \operatorname{supp}_t R_0 \subseteq J$. Define the parameters*

$$\widehat{N} := (e_v/e_R)^{1/2}, \quad \widehat{\Xi} := \widehat{N}\,\Xi = (e_v/e_R)^{1/2}\,\Xi.$$

Then, for any $0 < \delta \leq \delta_0$, there exist a constant $C_\delta \geq 1$, a constant $\theta > 0$, a sequence of times $\{t(I)\}_{I \in \mathbb{Z}} \subseteq \mathbb{R}$ and an Euler–Reynolds flow (v, p, R), $R = \sum_{I \in \mathbb{Z}} R_I$, that satisfy the support restrictions

$$\operatorname{supp}_t v \cup \operatorname{supp}_t R \subseteq N(J; 3^{-1}\Xi^{-1} e_v^{-1/2}), \tag{12}$$

$$2^{-1}\delta(\log \widehat{\Xi})^{-2}\,\Xi^{-1} e_v^{-1/2} \leq \theta \leq \delta(\log \widehat{\Xi})^{-2}\,\Xi^{-1} e_v^{-1/2}, \tag{13}$$

$$\operatorname{supp}_t R_I \subseteq \left[t(I) - \tfrac{1}{2}\theta, t(I) + \tfrac{1}{2}\theta\right], \tag{14}$$

$$\bigcup_I \bigcup_{I' \neq I} [t(I) - \theta, t(I) + \theta] \cap [t(I') - \theta, t(I') + \theta] = \varnothing, \tag{15}$$

and the estimates

$$\|v - v_0\|_{C^0} \leq C_1 e_R^{1/2},$$

$$\|\nabla_{\vec{a}} v\|_{C^0} \leq C_1\,\Xi^{|\vec{a}|} e_v^{1/2}, \quad |\vec{a}| = 1, 2, 3, \tag{16}$$

$$\sup_I \|\nabla_{\vec{a}} R_I\|_{C^0} \leq C_\delta \widehat{N}^{(|\vec{a}|-2)_+}\,\Xi^{|\vec{a}|} \log \widehat{\Xi} e_R, \quad |\vec{a}| = 0, 1, 2, 3, \tag{17}$$

$$\sup_I \|\nabla_{\vec{a}}(\partial_t + v \cdot \nabla) R_I\|_{C^0} \leq C_\delta (\log \widehat{\Xi})^3\,\Xi e_v^{1/2}\,\Xi^{|\vec{a}|} e_R, \quad |\vec{a}| = 0, 1, 2.$$

Our improved convex integration lemma may then be stated as follows.

Lemma 6.2 (convex integration lemma). *There exists an absolute constant b_0 such that, for any $C_1, C_\delta \geq 1$ and $\delta > 0$, there is a constant $\widetilde{C} = \widetilde{C}_{\delta, C_1, C_\delta}$ for which the following holds. Suppose J is a subinterval of $\mathbb{R}$ and (v, p, R) is an Euler–Reynolds flow, $R = \sum_I R_I$, that satisfy the conclusions (12)–(15) and (16)–(17) of Lemma 6.1 for some (Ξ, e_v, e_R), some $\theta > 0$ and some sequence of times $\{t(I)\}_{I \in \mathbb{Z}} \subseteq \mathbb{R}$. Also suppose*

$$|\theta|\,\|\nabla v\|_{C^0} \leq b_0. \tag{18}$$

Let $N \geq (e_v/e_R)^{1/2}$. Then there is an Euler–Reynolds flow (v_1, p_1, R_1) with frequency-energy levels in the sense of Definition 4.2 bounded by

$$(\Xi', e_v', e_R') = \left(\widetilde{C} N\,\Xi, (\log \widehat{\Xi}) e_R, (\log \widehat{\Xi})^{5/2} \frac{e_v^{1/2} e_R^{1/2}}{N}\right) \tag{19}$$

such that

$$\mathrm{supp}_t\, v_1 \cup \mathrm{supp}_t\, R_1 \subseteq N(J;\, \Xi^{-1} e_v^{-1/2}),$$
$$\|v_1 - v\|_{C^0} \leq \widetilde{C} (\log \widehat{\Xi})^{1/2} e_R^{1/2}.$$

Lemma 5.1 now follows by combining Lemmas 6.1 and 6.2 as explained in [Isett 2018, Section 3]. (Here Lemma 6.1 is applied with (v_0, p_0, R_0) taken to be the (v, p, R) given in the assumptions of Lemma 5.1.) The only important difference in the present case is that we have removed the assumption $N \geq \Xi^\eta$ and the constants $\widehat{C}$ and C_L (which can be set equal if desired) do not depend on η.

We now explain how to prove Lemma 6.2 by modifying the proof of [Isett 2018, Lemma 3.3].

7. Modifying the convex integration

We now proceed with the proof of Lemma 6.2. The construction will be based on the proof of [Isett 2018, Lemma 3.3] implementing convex integration with the Mikado flows of [Daneri and Székelyhidi 2017], but modified to adapt the localization strategy of [Isett and Oh 2016b] to our setting.

Let (v, p, R), $R = \sum_I R_I$ be given as in the assumptions of Lemma 6.2, which are the conclusions of Lemma 6.1. We will use the symbol $\lesssim$ to denote inequalities involving explicit constants that are allowed to depend on the parameters C_1, δ and C_δ, but never on (Ξ, e_v, e_R), N, θ, $\widehat{\Xi}$, etc.

We obtain the new Euler–Reynolds flow (v_1, p_1, R_1) of Lemma 6.2 by adding carefully designed corrections $v_1^\ell = v^\ell + V^\ell$ and $p_1 = p + P$ to the velocity and pressure, respectively, and using the resulting equation for (v_1, p_1) to construct the appropriate R_1. The correction V^ℓ will be a sum of divergence-free, high-frequency vector fields indexed by a set $\mathcal{J}$:

$$V^\ell = \sum_{J \in \mathcal{J}} V_J^\ell, \quad \nabla_\ell V_J^\ell = 0 \quad \text{for all } J \in \mathcal{J}.$$

The index $J \in \mathcal{J}$ will have several components, $J = (I, J_1, J_2, J_3, f)$, that together specify the time interval and spatial location in which V_J will be supported as well as the direction in which V_J takes values. Specifically, we choose an even integer $\Pi \in [3\Xi, 6\Xi] \cap 2\mathbb{Z}$ of size comparable to Ξ and define

$$\mathcal{J} := \mathbb{Z} \times (\mathbb{Z}/\Pi\mathbb{Z})^3 \times \mathbb{F}, \quad \mathbb{F} := \{e_i \pm e_j : 1 \leq i < j \leq 3\}.$$

Each V_J, $J = (I, J_1, J_2, J_3, f)$, will be supported in a time interval of length $\sim \theta$ around time $t(I)$, and initially at time $t(I)$ will be supported in a ball of size $\sim \Xi^{-1}$ around the point

$$x_0(J) := \Pi^{-1}(J_1, J_2, J_3) \in (\mathbb{R}/\mathbb{Z})^3.$$

The component $f \in \mathbb{F}$ specifies which of the $\#\mathbb{F} = 6$ directions in $\mathbb{R}^3$ in which V_J^ℓ approximately takes values.

As in [Isett 2017a, Section 12], let $v_\epsilon = \eta_\epsilon * v$ be the coarse scale velocity field obtained by mollification in space at scale ϵ. Let $\Phi_s : \mathbb{R} \times \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R} \times \mathbb{T}^3$ be the *coarse scale flow* (the flow map of v_ϵ)

$$\Phi_s(t, x) = (t + s, \Phi_s^i(t, x)), \quad \frac{d}{ds}\Phi_s^i(t, x) = v_\epsilon^i(\Phi_s(t, x)), \quad \Phi_0(t, x) = (t, x), \tag{20}$$

and let $\Gamma_I : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{T}^3$ be the *back-to-labels map* associated to v_ϵ from the initial time $t(I)$:

$$\begin{aligned}(\partial_t + v_\epsilon^i \nabla_i)\Gamma_I(t, x) &= 0,\\ \Gamma_I(t(I), x) &= x.\end{aligned} \tag{21}$$

We also define the coarse scale advective derivative $\bar{D}_t := (\partial_t + v_\epsilon \cdot \nabla)$.

To localize the waves V_J, we construct a smooth, quadratic partition of unity initiating from each time $t(I)$ that follows the flow of v_ϵ and has length scale $\sim \Xi^{-1}$. The elements of this partition of unity are functions $\chi_{(I,[k])} : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$ that are indexed by $(I, [k]) \in \mathbb{Z} \times (\mathbb{Z}/\Pi\mathbb{Z})^3$, and they satisfy

$$\sum_{[k]\in(\mathbb{Z}/\Pi\mathbb{Z})^3} \chi^2_{(I,[k])}(t, x) = 1 \quad \text{for all } I \in \mathbb{Z}, \quad (t, x) \in \mathbb{R} \times \mathbb{T}^3, \tag{22}$$

$$\bar{D}_t \chi_{(I,[k])}(t, x) = 0 \quad \text{for all } (I, [k]) \in \mathbb{Z} \times (\mathbb{Z}/\Pi\mathbb{Z})^3, \quad (t, x) \in \mathbb{R} \times \mathbb{T}^3. \tag{23}$$

To construct the initial data for the partition of unity, choose a smooth $\bar{\chi} : \mathbb{R}^3 \to \mathbb{R}$ with support in $\left[-\frac{3}{4}, \frac{3}{4}\right]^3$ such that $\sum_{m\in\mathbb{Z}^3} \bar{\chi}^2(h - m) = 1$ for all $h \in \mathbb{R}^3$, then periodize and rescale to define

$$\chi_{(I,[k])}(t(I), x) := \sum_{m\in\mathbb{Z}^3} \bar{\chi}(\Pi x - [k] - \Pi m). \tag{24}$$

Observe that $\chi_{(I,[k])}(t(I), x)$ does not depend on how we represent the equivalence classes of $x \in (\mathbb{R}/\mathbb{Z})^3$ or $[k] \in (\mathbb{Z}/\Pi\mathbb{Z})^3$, and that (22) holds at time $t(I)$. The same identity holds for all time $t \in \mathbb{R}$ by (23) and uniqueness of solutions to the transport equation. Observe also, since $3\Xi \le \Pi \le 6\Xi$, that the initial data for $\chi_{(I,[k])}(t(I), \cdot)$ is supported in a ball of radius Ξ^{-1} around $\Pi^{-1}[k]$ in $(\mathbb{R}/\mathbb{Z})^3$, and satisfies estimates of the form $\|\nabla_{\vec{a}} \chi_{(I,[k])}(t(I), \cdot)\|_{C^0} \lesssim_{|\vec{a}|} \Xi^{|\vec{a}|}$.

7A. ***Localizing the convex integration construction.*** Unlike the scheme in [Isett 2018], our scheme will involve many Mikado flow based waves at any given time that are supported within overlapping regions. In general, interference between overlapping Mikado flows would produce error terms that cannot be controlled for the iteration. We avoid this interference by "threading" the Mikado flows together, so that, at the initial time, the main terms of the waves V_J will have disjoint support. The support then remains disjoint as the Mikado flows are advected along the coarse scale flow.

To accomplish this construction, let $f \in \mathbb{F}$ and let $[k] \in (\mathbb{Z}/2\mathbb{Z})^3$. Choose an $r_0 > 0$ and choose disjoint, periodic lines $\ell_{(f,[k])} = \{p_{(f,[k])} + tf : t \in \mathbb{R}\}$ that are separated from each other by a distance greater than $6r_0$ in the torus $(\mathbb{R}/\mathbb{Z})^3$. Choose smooth functions $\psi_{(f,[k])} : \mathbb{T}^3 \to \mathbb{R}$ of the form $\psi_{(f,[k])}(X) = g(\mathrm{dist}(X, \ell_{(f,[k])}))$, $\operatorname{supp} g(\cdot) \subseteq \left[\frac{1}{2}r_0, r_0\right]$, such that

$$\int_{\mathbb{T}^3} \psi_{(f,[k])}(X)\, dX = 0, \qquad \int_{\mathbb{T}^3} \psi^2_{(f,[k])}(X)\, dX = 1. \tag{25}$$

With these choices, the functions $\psi_{(f,[k])}$ have disjoint support and have gradients orthogonal to f:

$$\begin{aligned}&\nabla_\ell \psi_{(f,[k])}(X) f^\ell = 0,\\ &\operatorname{supp} \psi_{(f,[k])} \cap \operatorname{supp} \psi_{(\tilde{f},[\tilde{k}])} = \varnothing \quad \text{if } f \ne \tilde{f} \text{ or } [k] \ne [\tilde{k}] \text{ in } (\mathbb{Z}/2\mathbb{Z})^3.\end{aligned} \tag{26}$$

Conditions (26) and (25) imply that $\psi_{(f,[k])}(X) f^\ell$ is divergence-free with mean zero, which implies that there is[6] a smooth tensor field

$$\Omega^{\alpha\beta}_{(f,[k])} : \mathbb{T}^3 \to \mathbb{R}^3 \otimes \mathbb{R}^3$$

that is antisymmetric in $\alpha\beta$ and satisfies

$$\nabla_\alpha \Omega^{\alpha\beta}_{(f,[k])}(X) = \psi_{(f,[k])}(X) f^\beta, \quad \int_{\mathbb{T}^3} \Omega^{\alpha\beta}_{(f,[k])}(X)\, dX = 0 \qquad \text{for all } 1 \le \alpha, \beta \le 3.$$

Since all components of the $\Omega^{\alpha\beta}_{(f,[k])}$ have mean zero on the torus, we can further construct tensor fields

$$\widetilde{\Omega}^{\alpha\beta\gamma}_{(f,[k])} : \mathbb{T}^3 \to \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3,$$

also antisymmetric in $\alpha\beta$, such that

$$\nabla_\gamma \widetilde{\Omega}^{\alpha\beta\gamma}_{(f,[k])}(X) = \Omega^{\alpha\beta}_{(f,[k])}(X), \quad \int_{\mathbb{T}^3} \widetilde{\Omega}^{\alpha\beta\gamma}_{(f,[k])}(X)\, dX = 0 \qquad \text{for all } 1 \le \alpha, \beta, \gamma \le 3.$$

For example, we can take

$$\widetilde{\Omega}^{\alpha\beta\gamma}_{(f,[k])} := \nabla^\gamma \Delta^{-1} \Omega^{\alpha\beta}_{(f,[k])}.$$

These second-order potentials will be used to impose local conservation of angular momentum similar to the use of double-curl form waves in [Isett and Oh 2016b].

For $J = (I, J_1, J_2, J_3, f)$, let $[J] := [(J_1, J_2, J_3)]$. We define the corrections V^ℓ_J to have the form

$$V^\ell_J = \mathring{V}^\ell_J + \delta V^\ell_J, \quad \mathring{V}^\ell_J = v^\ell_J \psi_J(t,x), \quad \psi_J(t,x) := \psi_{(f,[J])}(\lambda \Gamma_I(t,x)). \tag{27}$$

The amplitudes v^ℓ_J have the same form as in [Isett 2018, Section 13] except they incorporate the partition of unity χ_J. In particular, they take values orthogonal to the gradient of the oscillatory functions ψ_J:

$$v^\ell_J = \chi_J [e_I^{1/2}(t) \gamma_{(I,f)}(t,x) (\nabla \Gamma_I^{-1})^\ell_a f^a], \tag{28}$$

$$\operatorname{supp}_t e_I^{1/2}(t) \subseteq [t(I) - \theta, t(I) + \theta], \tag{29}$$

$$\chi_J(t,x) = \chi_{(I,[J_1,J_2,J_3])}(t,x), \quad J = (I, J_1, J_2, J_3, f),$$

$$v^\ell_J \nabla_\ell \psi_J = 0. \tag{30}$$

Note in particular that by construction the main terms of each wave have disjoint supports

$$\operatorname{supp} \mathring{V}_J \cap \operatorname{supp} \mathring{V}_K = \varnothing \quad \text{if } J \neq K. \tag{31}$$

Indeed, if $J = (J_0, J_1, J_2, J_3, f)$ and $K = (K_0, K_1, K_2, K_3, f')$ are not equal and $J_0 \neq K_0$, then V^ℓ_J and V^ℓ_K live on different time intervals. If $J_0 = K_0 = I$, one has either $f \neq f'$ or $(J_1, J_2, J_3) \neq (K_1, K_2, K_3)$ mod 2, either case implying $\operatorname{supp} \psi_J \cap \operatorname{supp} \psi_K = \varnothing$, or $f = f'$ and $(J_1, J_2, J_3) = (K_1, K_2, K_3)$ mod 2. In the last case, one has $\operatorname{supp} \chi_J \cap \operatorname{supp} \chi_K = \varnothing$ unless $J = K$.

[6] We can take for instance $\Omega^{\alpha\beta}_{(f,[k])} = \nabla^\alpha \Delta^{-1}[\psi_{(f,[k])} f^\beta] - \nabla^\beta \Delta^{-1}[\psi_{(f,[k])} f^\alpha]$.

The correction V_J^ℓ is made to be divergence-free and to have the form (27) by making V_J^ℓ the divergence of an antisymmetric tensor built from the Lie transport of the potentials $\widetilde{\Omega}^{\alpha\beta\gamma}_{(f,[k])}$ above:

$$V_J^\ell = \lambda^{-2}\nabla_a\nabla_c[\chi_J(\nabla\Gamma_I^{-1})^a_\alpha(\nabla\Gamma_I^{-1})^\ell_\beta(\nabla\Gamma_I^{-1})^c_\gamma e_I^{1/2}(t)\gamma_{(I,f)}\widetilde{\Omega}_J^{\alpha\beta\gamma}], \tag{32}$$

$$\begin{aligned}
\delta V_J^\ell &= \delta v^\ell_{J,\alpha\beta}\Omega_J^{\alpha\beta} + \delta v^\ell_{J,\alpha\beta\gamma}\widetilde{\Omega}_J^{\alpha\beta\gamma},\\
\Omega_J^{\alpha\beta}(t,x) &:= \Omega^{\alpha\beta}_{(f,[J_1,J_2,J_3])}(\lambda\Gamma_I),\\
\widetilde{\Omega}_J^{\alpha\beta\gamma}(t,x) &:= \widetilde{\Omega}^{\alpha\beta\gamma}_{(f,[J_1,J_2,J_3])}(\lambda\Gamma_I),\\
\delta v^\ell_{J,\alpha\beta} &:= \lambda^{-1}\nabla_a[\chi_J(\nabla\Gamma_I^{-1})^a_\alpha(\nabla\Gamma_I^{-1})^\ell_\beta e_I^{1/2}(t)\gamma_{(I,f)}],\\
\delta v^\ell_{J,\alpha\beta\gamma} &:= \lambda^{-2}\nabla_a\nabla_c[\chi_J(\nabla\Gamma_I^{-1})^a_\alpha(\nabla\Gamma_I^{-1})^\ell_\beta(\nabla\Gamma_I^{-1})^c_\gamma e_I^{1/2}(t)\gamma_{(I,f)}].
\end{aligned} \tag{33}$$

Note that the main term $\mathring{V}_J^\ell$ in (27)–(28) appears when the derivatives ∇_a and ∇_c both fall on $\widetilde{\Omega}_J^{\alpha\beta\gamma}$. Since V_J^ℓ has the form $V_J^\ell = \nabla_a W_J^{a\ell}$, where $W_J^{a\ell}$ is antisymmetric in $a\ell$, we have that V_J^ℓ is divergence-free.

The amplitudes constructed here are related to those constructed in [Isett 2018, Section 13] (which are indexed by $(I,f)\in\mathbb{Z}\times\mathbb{F}$ and do not involve spatial cutoffs) by the formula

$$v_J^\ell = \chi_J v^\ell_{(I,f)}, \quad J = (I, J_1, J_2, J_3, f). \tag{34}$$

This comparison allows us to see that the parameter $\epsilon = \epsilon_v$ in the mollification of $v\mapsto v_\epsilon$ can be chosen to have the same value $\epsilon_v = c_v N^{-1/2}\Xi^{-1}$ as in [Isett 2018, Section 16], which is based on the requirement

$$\|v - v_\epsilon\|_{C^0}\max_J \||v_J||\psi_J|\|_{C^0} \le (\log\widehat{\Xi})^{1/2}\frac{e_v^{1/2}e_R^{1/2}}{500N}.$$

Since we have chosen the same parameter in the mollification $v\mapsto \epsilon_v$ as that chosen in [Isett 2018], we obtain the same estimates for v_ϵ:

$$\|\nabla_{\vec{a}}v_\epsilon\|_{C^0} \lesssim_{|\vec{a}|} N^{(|\vec{a}|-2)_+/2}\Xi^{|\vec{a}|}e_v^{1/2} \quad \text{if } |\vec{a}|\ge 1, \tag{35}$$

where the implicit constant is equal to 1 for $|\vec{a}|=1$. From this fact we will see in the following Section 8 that all the remaining estimates for the components of the construction coincide with those in the proof of [Isett 2018, Lemma 3.3].

8. Estimates for components of the construction

Here we summarize the estimates for the components of the construction, which coincide with those of [Isett 2018]. The following elementary lemma will be convenient:

Lemma 8.1. *For $u\ge 0$, an integer $M\ge 0$ and for $g:\mathbb{T}^3\to\mathbb{R}$, define (for $N\ge 1$, $\Xi>0$)*

$$H_{M,u}[g] := \max_{0\le|\vec{a}|\le M}\frac{\|\nabla_{\vec{a}}g\|_{C^0}}{N^{(|\vec{a}|-u)_+/2}\Xi^{|\vec{a}|}}. \tag{36}$$

Then, for $\lambda\ge N^{1/2}\Xi$, we have for any first-order partial derivative ∇_a

$$H_{M,u}[\lambda^{-1}\nabla_a g] \le H_{M+1,u}[g],$$
$$H_{M,u}[\Xi^{-1}\nabla_a g] \le H_{M+1,u+1}[g].$$

We also have the triangle inequality $H_{M,u}[g_{(1)}+g_{(2)}] \le H_{M,u}[g_{(1)}]+H_{M,u}[g_{(2)}]$ *and product estimate*

$$H_{M,u}[g_{(1)}g_{(2)}] \lesssim_M H_{M,u}[g_{(1)}]H_{M,u}[g_{(2)}]. \tag{37}$$

All the properties follow quickly from the definition (36). Inequality (37) follows from the expansion

$$\nabla_{\vec{a}}(g_{(1)}g_{(2)}) = \sum_{|\vec{a}_1|+|\vec{a}_2|=|\vec{a}|} c_{\vec{a}_1,\vec{a}_2}\nabla_{\vec{a}_1}g_{(1)}\nabla_{\vec{a}_2}g_{(2)},$$

the bound

$$\|\nabla_{\vec{a}_i}g_{(i)}\|_{C^0} \le N^{(|\vec{a}_i|-u)_+/2}\,\Xi H_{M,u}[g_{(i)}]$$

and the inequality $(|\vec{a}_1|-u)_+ + (|\vec{a}_2|-u)_+ \le (|\vec{a}|-u)_+$.

The estimates for the construction may now be summarized as follows. Here we use the fact that the frequency $\lambda := B_\lambda N\,\Xi$ is larger than $N^{1/2}\,\Xi$ to conclude that the lower-order terms $\delta v^\ell_{J,\alpha\beta\gamma}$ obey the same bounds as the $\delta v^\ell_{J,\alpha\beta}$.

Proposition 8.2. *The following bounds hold with constants depending only on* $|\vec{a}|$:

$$\|\nabla_{\vec{a}}\gamma_{(I,f)}\|_{C^0} + \|\nabla_{\vec{a}}(\nabla\Gamma_I^{-1})\|_{C^0} \lesssim N^{(|\vec{a}|-1)_+/2}\,\Xi^{|\vec{a}|}, \tag{38}$$

$$\|\nabla_{\vec{a}}\overline{D}_t\gamma_{(I,f)}\|_{C^0} + \|\nabla_{\vec{a}}\overline{D}_t(\nabla\Gamma_I^{-1})\|_{C^0} \lesssim (\log\widehat{\Xi})^2\,\Xi e_v^{1/2} N^{(|\vec{a}|-1)_+/2}\,\Xi^{|\vec{a}|}, \tag{39}$$

$$\sup_{t\in\mathbb{R}}(e_I^{1/2}(t) + \theta|\partial_t e_I^{1/2}(t)|) \lesssim (\log\widehat{\Xi})^{1/2} e_R^{1/2}, \tag{40}$$

$$\|\nabla_{\vec{a}}\chi_J\|_{C^0} \lesssim N^{(|\vec{a}|-1)_+/2}\,\Xi^{|\vec{a}|}, \tag{41}$$

$$\|\nabla_{\vec{a}}v_J^\ell\|_{C^0} \lesssim (\log\widehat{\Xi})^{1/2} N^{(|\vec{a}|-1)_+/2}\,\Xi^{|\vec{a}|} e_R^{1/2}, \tag{42}$$

$$\|\nabla_{\vec{a}}\overline{D}_t v_J^\ell\|_{C^0} \lesssim (\log\widehat{\Xi})^{5/2} N^{(|\vec{a}|-1)_+/2}\,\Xi^{|\vec{a}|} e_R^{1/2}, \tag{43}$$

$$\|\nabla_{\vec{a}}\delta v^\ell_{J,\alpha\beta}\|_{C^0} + \|\nabla_{\vec{a}}\delta v^\ell_{J,\alpha\beta\gamma}\|_{C^0} \lesssim \lambda^{-1}(\log\widehat{\Xi})^{1/2} N^{|\vec{a}|/2}\,\Xi^{1+|\vec{a}|} e_R^{1/2}, \tag{44}$$

$$\|\nabla_{\vec{a}}\overline{D}_t\delta v^\ell_{J,\alpha\beta}\|_{C^0} + \|\nabla_{\vec{a}}\overline{D}_t\delta v^\ell_{J,\alpha\beta\gamma}\|_{C^0} \lesssim \lambda^{-1}(\log\widehat{\Xi})^{5/2} N^{|\vec{a}|/2}\,\Xi^{|\vec{a}|+2} e_v^{1/2} e_R^{1/2}. \tag{45}$$

Proof. Inequalities (38)–(40) follow from the bounds in [Isett 2018, Section 17.1]. Inequality (41) for $|\vec{a}| = 0$ follows from the maximum principle for $\overline{D}_t\chi_J = 0$. To obtain (41), we apply [Isett 2017a, Proposition 17.4] in the case of order $L = 2$ frequency-energy levels to obtain

$$E_M[\chi_J](\Phi_s(t,x)) \le e^{C_M\,\Xi e_v^{1/2}|s|} E_M[\chi_J](t(I),x), \tag{46}$$
$$E_M[\chi_J](t,x) := \sum_{0\le|\vec{a}|\le M} \Xi^{-2|\vec{a}|} N^{-(|\vec{a}|-1)_+}|\nabla_{\vec{a}}\nabla\chi_J(t,x)|^2,$$

and we use the fact that, by the construction in (24),

$$E_M[\chi_J](t(I),x) \lesssim_M \sum_{0\le|\vec{a}|\le M} \Xi^{-2|\vec{a}|} N^{-(|\vec{a}|-1)_+}(\Xi^{|\vec{a}|+1})^2 \lesssim_M \Xi^2.$$

We have $\Xi e_v^{1/2}|s| \le \Xi e_v^{1/2}\theta \le 1$ on the support of the time cutoff $e_I^{1/2}$ from (29), so (46) yields

$$\|\nabla_{\vec{a}}\chi_J\|_{C^0} \lesssim N^{(|\vec{a}|-2)_+/2}\,\Xi^{|\vec{a}|},$$

which implies (41).

The proofs of estimates (42)–(45) for v_J^ℓ and for $\delta v_{J,\alpha\beta\gamma}^\ell$ are exactly as in [Isett 2018, Section 17.1] with the addition of the cutoff function χ_J. For instance, note that

$$\chi_J(\nabla\Gamma_I^{-1})^a_\alpha \quad \text{and} \quad \bar{D}_t[\chi_J(\nabla\Gamma_I^{-1})^a_\alpha] = \chi_J\bar{D}_t(\nabla\Gamma_I^{-1})^a_\alpha$$

obey the same bounds as $(\nabla\Gamma_I^{-1})^a_\alpha$ and $\bar{D}_t(\nabla\Gamma_I^{-1})^a_\alpha$, respectively, up to constants, so we may absorb the cutoff χ_J into the first factor of $(\nabla\Gamma^{-1})$ in estimating formulas (28) and (33) while repeating the proofs in [Isett 2018, Section 17.1].

It remains to check (42)–(45) for the lower-order term $\delta v_{J,\alpha\beta\gamma}^\ell$. Applying Lemma 8.1, we obtain

$$\begin{aligned}
\lambda\Xi^{-1}\delta v_{J,\alpha\beta\gamma}^\ell &= \Xi^{-1}\lambda^{-1}\nabla_a\nabla_c[\chi_J(\nabla\Gamma_I^{-1})^a_\alpha(\nabla\Gamma_I^{-1})^\ell_\beta(\nabla\Gamma_I^{-1})^c_\gamma e_I^{1/2}(t)\gamma_{(I,f)}],\\
\lambda\Xi^{-1}H_{M,0}[\delta v_{J,\alpha\beta\gamma}^\ell] &\lesssim_M H_{M+1,1}[\lambda^{-1}\nabla_c[\chi_J(\nabla\Gamma_I^{-1})^a_\alpha(\nabla\Gamma_I^{-1})^\ell_\beta(\nabla\Gamma_I^{-1})^c_\gamma e_I^{1/2}(t)\gamma_{(I,f)}]\\
&\lesssim_M H_{M+2,1}[\chi_J(\nabla\Gamma_I^{-1})^a_\alpha(\nabla\Gamma_I^{-1})^\ell_\beta(\nabla\Gamma_I^{-1})^c_\gamma e_I^{1/2}(t)\gamma_{(I,f)}]\\
&\lesssim_M e_I^{1/2}(t)H_{M+2,1}[\chi_J]H_{M+2,1}[(\nabla\Gamma_I^{-1})]^3H_{M+2,1}[\gamma_{(I,f)}], \qquad (47)\\
H_{M,0}[\delta v_{J,\alpha\beta\gamma}^\ell] &\lesssim_M \lambda^{-1}(\log\widehat{\Xi})^{1/2}\Xi e_R^{1/2}. \qquad (48)
\end{aligned}$$

Here every term in (47) is bounded by $\lesssim_M 1$ except $e_I^{1/2}(t)$. Note that (48) is equivalent to (44).

To prove (45), we proceed similarly by commuting in the advective derivative weighted by the parameter $\theta \sim (\log\widehat{\Xi})^{-2}\Xi^{-1}e_v^{-1/2}$:

$$\begin{aligned}
(\lambda\Xi^{-1}\theta)\bar{D}_t\delta v_{J,\alpha\beta\gamma}^\ell &= \Xi^{-1}\lambda^{-1}\nabla_a\nabla_c[\theta\bar{D}_t[\chi_J(\nabla\Gamma_I^{-1})^a_\alpha(\nabla\Gamma_I^{-1})^\ell_\beta(\nabla\Gamma_I^{-1})^c_\gamma e_I^{1/2}(t)\gamma_{(I,f)}]] \qquad (49)\\
&\quad -\theta(\nabla_a v_\epsilon^i)\Xi^{-1}\lambda^{-1}\nabla_i\nabla_c[\chi_J(\nabla\Gamma_I^{-1})^a_\alpha(\nabla\Gamma_I^{-1})^\ell_\beta(\nabla\Gamma_I^{-1})^c_\gamma e_I^{1/2}(t)\gamma_{(I,f)}] \qquad (50)\\
&\quad -\Xi^{-1}\nabla_a[\nabla_c v_\epsilon^i\lambda^{-1}\nabla_i[\chi_J(\nabla\Gamma_I^{-1})^a_\alpha(\nabla\Gamma_I^{-1})^\ell_\beta(\nabla\Gamma_I^{-1})^c_\gamma e_I^{1/2}(t)\gamma_{(I,f)}]]. \qquad (51)
\end{aligned}$$

The terms (50) and (51) may be estimated using Lemma 8.1 as in the proof of (47)–(48) to obtain

$$\begin{aligned}
H_{M,0}[(50)]+H_{M,0}[(51)] &\lesssim_M e_I^{1/2}(t)H_{M+1,1}[\theta\nabla v_\epsilon]H_{M+2,1}[\chi_J]H_{M+2,1}[(\nabla\Gamma_I^{-1})]^3H_{M+2,1}[\gamma_{(I,f)}]\\
&\overset{(35)-(41)}{\lesssim_M} e_I^{1/2}(t) \lesssim (\log\widehat{\Xi})^{1/2}e_R^{1/2}.
\end{aligned}$$

For (49), apply the product rule for $\theta\bar{D}_t$ and apply Lemma 8.1 repeatedly to obtain

$$\begin{aligned}
H_{M,0}[(49)] \lesssim_M (e_I^{1/2}(t)+\theta|\partial_t e_I^{1/2}(t)|)H_{M+2,1}[\chi_J]\cdot(H_{M+2,1}[(\nabla\Gamma_I^{-1})]+\theta H_{M+2,1}[\bar{D}_t(\nabla\Gamma_I^{-1})])^3\\
\cdot(H_{M+2,1}[\gamma_{(I,f)}]+\theta H_{M+2,1}[\bar{D}_t\gamma_{(I,f)}]).
\end{aligned}$$

Since

$$\theta H_{M+2,1}[\bar{D}_t\gamma_{(I,f)}] \quad \text{and} \quad \theta H_{M+2,1}[\bar{D}_t(\nabla\Gamma_I^{-1})]$$

are bounded by $\lesssim_M 1$ from (38)–(39), we have

$$\begin{aligned}
H_{M,0}[\delta v_{J,\alpha\beta\gamma}] &\le \theta^{-1}\lambda^{-1}\Xi(H_{M,0}[(49)]+H_{M,0}[(50)]+H_{M,0}[(51)])\\
&\lesssim_M \theta^{-1}\lambda^{-1}\Xi(e_I^{1/2}(t)+\theta|\partial_t e_I^{1/2}(t)|)\\
&\lesssim \theta^{-1}\lambda^{-1}\Xi(\log\widehat{\Xi})^{1/2}e_R^{1/2}.
\end{aligned}$$

This bound is equivalent to the desired bound (45) for $\delta v_{J,\alpha\beta\gamma}$. □

As (42)–(45) are the same bounds for the components of the correction as those proven for $v^{\ell}_{(I,f)}$ and $\delta v^{\ell}_{(I,f),\alpha\beta}$ in [Isett 2018, Section 17], we have the following bounds from [Isett 2018, Proposition 17.3].

Proposition 8.3 (correction estimates). *For* $0 \le |\vec{a}| \le 3$, *we have*

$$\begin{aligned}
\sup_J \|\nabla_{\vec{a}} \mathring{V}_J\|_{C^0} &\lesssim (B_\lambda N \Xi)^{|\vec{a}|} (\log \widehat{\Xi})^{1/2} e_R^{1/2}, \\
\sup_J \|\nabla_{\vec{a}} \delta V_J\|_{C^0} &\lesssim (B_\lambda N \Xi)^{|\vec{a}|-1} \Xi (\log \widehat{\Xi})^{1/2} e_R^{1/2}, \\
\|V\|_{C^0} &\lesssim (B_\lambda N \Xi)^{|\vec{a}|} (\log \widehat{\Xi})^{1/2} e_R^{1/2}, \qquad (52)\\
\operatorname{supp}_t V &\subseteq \bigcup_I \operatorname{supp}_t e_I \subseteq \bigcup_I [t(I)-\theta, t(I)+\theta].
\end{aligned}$$

For the estimate (52), we use that at most a bounded number (say 2^3) distinct V^{ℓ}_J are supported at any given point (t, x). This detail will be explained following (76) below. We now consider the error terms and their estimates.

9. The error terms

Given the Euler–Reynolds flow (v, p, R), the new velocity field $v_1^{\ell} = v^{\ell} + V^{\ell}$, with

$$V^{\ell} = \sum_J V^{\ell}_J = \sum_J \mathring{V}^{\ell}_J + \delta V^{\ell}_J,$$

and pressure $p_1 = p + P$ will solve the Euler–Reynolds equations when coupled to a new Reynolds stress tensor $R_1^{j\ell}$. The new stress tensor $R_1^{j\ell}$ will be composed of terms that solve

$$\begin{aligned}
R_1^{j\ell} &= R_M^{j\ell} + R_T^{j\ell} + R_S^{j\ell} + R_H^{j\ell}, \qquad (53)\\
R_M^{j\ell} &= (v^j - v^j_\epsilon) V^{\ell} + V^j (v^{\ell} - v^{\ell}_\epsilon) + (R^{j\ell} - R^{j\ell}_\epsilon),\\
\nabla_j R_T^{j\ell} &= \partial_t V^{\ell} + \nabla_j (v^j_\epsilon V^{\ell} + V^j v^{\ell}_\epsilon), \qquad (54)\\
R_S^{j\ell} &= \sum_{J,K \in \mathcal{J}} \delta V^j_J \mathring{V}^{\ell}_K + \mathring{V}^j_J \delta V^{\ell}_K + \delta V^j_J \delta V^{\ell}_K,\\
\nabla_j R_H^{j\ell} &= \nabla_j \Bigl[\sum_{J \in \mathcal{J}} \mathring{V}^j_J \mathring{V}^{\ell}_J + P\delta^{j\ell} + R^{j\ell}_\epsilon\Bigr]. \qquad (55)
\end{aligned}$$

In writing (55), we have made the crucial observation that all of the off-diagonal terms in the summation $\sum_{J,K\in\mathcal{J}} \mathring{V}^j_J \mathring{V}^{\ell}_K$ vanish due to the disjointness of support stated in (31).

Our construction has been designed in such a way that

$$\sum_{J \in \mathcal{J}} v^j_J v^{\ell}_J + P\delta^{j\ell} + R^{j\ell}_\epsilon = 0. \qquad (56)$$

From (27) and (56), equation (55) reduces to

$$\nabla_j R_H^{j\ell} = \nabla_j \Bigl[\sum_{J \in \mathcal{J}} v^j_J v^{\ell}_J (\psi_J^2 - 1)\Bigr]. \qquad (57)$$

To verify (56), note that, for each $I \in \mathbb{Z}$ and $\mathcal{J}(I) := \{I\} \times (\mathbb{Z}/\Pi\mathbb{Z})^3 \times \mathbb{F}$, we have from (22) and (34) that

$$\sum_{J \in \mathcal{J}(I)} v_J^j v_J^\ell = \sum_{[k] \in (\mathbb{Z}/\Pi\mathbb{Z})^3} \sum_{f \in \mathbb{F}} \chi^2_{(I,[k])} v^j_{(I,f)} v^\ell_{(I,f)} = \sum_{f \in \mathbb{F}} v^j_{(I,f)} v^\ell_{(I,f)}, \tag{58}$$

where $v_{(I,f)}$ are the amplitudes from the construction in [Isett 2018]. The equality

$$\sum_{I \in \mathbb{Z}} \sum_{f \in \mathbb{F}} v^j_{(I,f)} v^\ell_{(I,f)} + P\delta^{j\ell} + R_\epsilon^{j\ell} = 0$$

proved in [Isett 2018, Sections 14–15] now implies the equality (56) in the present construction using (58).

It now remains to show that, when $R_T^{j\ell}$ and $R_H^{j\ell}$ are chosen appropriately, the tensor $R_1^{j\ell}$ defined by (53) satisfies the bounds required by Lemma 6.2.

10. Solving the symmetric divergence equation

To estimate the error tensor R_1 defined in (53), the only terms that require a different treatment from [Isett 2018] are the terms R_T and R_H. Namely, since our choice of v_ϵ and R_ϵ and our estimates for $\mathring{V}_J$ and δV_J also coincide with those of that paper, Proposition 17.4 there shows that

$$\begin{aligned}
\|R_M\|_{C^0} + \|R_S\|_{C^0} &\le (\log \widehat{\Xi}) \frac{e_v^{1/2} e_R^{1/2}}{10N}, \\
\|\nabla_{\vec{a}} R_M\|_{C^0} + \|\nabla_{\vec{a}} R_S\|_{C^0} &\lesssim (B_\lambda N \Xi)^{|\vec{a}|} (\log \widehat{\Xi}) \frac{e_v^{1/2} e_R^{1/2}}{N}, \quad 1 \le |\vec{a}| \le 3, \\
\operatorname{supp}_t R_M \cup \operatorname{supp}_t R_S &\subseteq \bigcup_I [t(I) - \theta, t(I) + \theta],
\end{aligned} \tag{59}$$

provided we choose the constant B_λ in the definition of $\lambda = B_\lambda N \Xi$ to be larger than a certain, absolute constant $\bar{B}_\lambda$.

The tensors R_T and R_H are defined as summations of the form

$$R_T^{j\ell} = \sum_{J \in \mathcal{J}} R_{T,J}^{j\ell}, \quad R_H^{j\ell} = \sum_{J \in \mathcal{J}} R_{H,J}^{j\ell}, \tag{60}$$

where each term is symmetric and is localized both in space and in time around the support of V_J^ℓ.

We expand the terms (54) and (57) (using the orthogonality $v_J^j \nabla_j \psi_J = 0$ stated in (30) in the case of R_H, and using $\nabla_j v_\epsilon^j = \nabla_j V_J^j = 0$ in the case of R_T) to obtain the equations

$$\nabla_j R_{T,J}^{j\ell} = \partial_t V_J^\ell + \nabla_j (v_\epsilon^j V_J^\ell + V_J^j v_\epsilon^\ell), \tag{61}$$

$$\nabla_j R_{T,J}^{j\ell} = u_{TJ}^\ell \psi_J + u^\ell_{TJ,\alpha\beta} \Omega_J^{\alpha\beta} + u^\ell_{TJ,\alpha\beta\gamma} \widetilde{\Omega}_J^{\alpha\beta\gamma}, \tag{62}$$

$$\nabla_j R_{H,J}^{j\ell} = u_{HJ}^\ell (\psi_J^2 - 1), \tag{63}$$

$$\begin{aligned}
u_{HJ}^\ell &= \nabla_j [v_J^j v_J^\ell], \\
u_{TJ}^\ell &:= \bar{D}_t v_J^\ell + v_J^j \nabla_j v_\epsilon^\ell, \\
u^\ell_{TJ,\alpha\beta} &:= \bar{D}_t \delta v^\ell_{J,\alpha\beta} + \delta v^j_{J,\alpha\beta} \nabla_j v_\epsilon^\ell, \\
u^\ell_{TJ,\alpha\beta\gamma} &:= \bar{D}_t \delta v^\ell_{J,\alpha\beta\gamma} + \delta v^j_{J,\alpha\beta\gamma} \nabla_j v_\epsilon^\ell.
\end{aligned}$$

By the construction in Section 7A, each of the functions ψ_J, (ψ_J^2-1), $\Omega_J^{\alpha\beta}$ and $\widetilde{\Omega}_J^{\alpha\beta\gamma}$ have the form $\omega(\lambda\Gamma_I(t,x))$, where $\omega:\mathbb{T}^3\to\mathbb{R}$ belongs to a finite set of smooth functions of mean zero on $\mathbb{T}^3$. We may therefore apply the following proposition, which is similar to [Isett 2018, Proposition 17.6] and is proven in Section 10A below using the same parametrix expansion technique.

Proposition 10.1 (nonstationary phase). *If $U^\ell:\mathbb{R}\times\mathbb{T}^3\to\mathbb{R}^3$ is a smooth vector field of the form $U^\ell=u^\ell\omega(\lambda\Gamma_I)$, where $\omega:\mathbb{T}^3\to\mathbb{R}$ is a smooth function of mean zero, then, for any $D\geq 1$, there exist a smooth, symmetric tensor field $Q_{(D)}^{j\ell}:\mathbb{R}\times\mathbb{T}^3\to\mathbb{R}^3\otimes\mathbb{R}^3$ and a vector field $U_{(D)}^\ell$ satisfying*

$$U^\ell=\nabla_j Q_{(D)}^{j\ell}+U_{(D)}^\ell,$$

$$\sup_{0\leq|\vec{a}|\leq 3}\lambda^{-|\vec{a}|}\|\nabla_{\vec{a}}Q_{(D)}^{j\ell}\|_{C^0}\lesssim\lambda^{-1}\sup_{0\leq|\vec{a}|\leq D+3}\frac{\|\nabla_{\vec{a}}u^\ell\|_{C^0}}{N^{|\vec{a}|/2}\,\Xi^{|\vec{a}|}},$$

$$\sup_{0\leq|\vec{a}|\leq 3}\lambda^{-|\vec{a}|}\|\nabla_{\vec{a}}U^\ell\|_{C^0}\lesssim B_\lambda^{-1}N^{-D/2}\sup_{0\leq|\vec{a}|\leq D+3}\frac{\|\nabla_{\vec{a}}u^\ell\|_{C^0}}{N^{|\vec{a}|/2}\,\Xi^{|\vec{a}|}},$$

$$\operatorname{supp}Q_{(D)}^{j\ell}\cup\operatorname{supp}U_{(D)}^\ell\subseteq\operatorname{supp}U^\ell,$$

where the implicit constant depends only on ω and D.

We apply Proposition 10.1 to each of the terms in (62) and (63) and use the estimates

$$H_{D+3,0}[u_{TJ}^\ell]+H_{D+3,0}[u_{TJ,\alpha\beta}^\ell]+H_{D+3,0}[u_{TJ,\alpha\beta\gamma}^\ell]+H_{D+3,0}[u_{HJ}^\ell]\lesssim(\log\widehat{\Xi})^{5/2}\,\Xi e_v^{1/2}e_R^{1/2},$$

$$H_{D+3,0}[u]:=\sup_{0\leq|\vec{a}|\leq D+3}\frac{\|\nabla_{\vec{a}}u^\ell\|_{C^0}}{N^{|\vec{a}|/2}\,\Xi^{|\vec{a}|}},$$

which follow from (42)–(45) and Lemma 8.1 (and are saturated only by u_{TJ}^ℓ), to obtain the decompositions

$$(62)=\nabla_j Q_{TJ,(D)}^{j\ell}+U_{TJ,(D)}^\ell,\qquad(63)=\nabla_j Q_{HJ,(D)}^{j\ell}+U_{HJ,(D)}^\ell,\tag{64}$$

where the symmetric tensors $Q_{TJ,(D)}$ and $Q_{HJ,(D)}$ and remainder terms $U_{TJ,(D)}$ and $U_{HJ,(D)}$ satisfy

$$\begin{aligned}\sup_{0\leq|\vec{a}|\leq 3}\lambda^{-|\vec{a}|}(\|\nabla_{\vec{a}}Q_{TJ,(D)}^{j\ell}\|_{C^0}+\|\nabla_{\vec{a}}Q_{HJ,(D)}^{j\ell}\|_{C^0})&\lesssim_D\lambda^{-1}(\log\widehat{\Xi})^{5/2}\,\Xi e_v^{1/2}e_R^{1/2}\\&\lesssim_D B_\lambda^{-1}(\log\widehat{\Xi})^{5/2}\frac{e_v^{1/2}e_R^{1/2}}{N},\end{aligned}\tag{65}$$

$$\sup_{0\leq|\vec{a}|\leq 3}\lambda^{-|\vec{a}|}(\|\nabla_{\vec{a}}U_{TJ,(D)}^{j\ell}\|_{C^0}+\|\nabla_{\vec{a}}U_{HJ,(D)}^{j\ell}\|_{C^0})\lesssim_D B_\lambda^{-1}N^{-D/2}(\log\widehat{\Xi})^{5/2}\,\Xi e_v^{1/2}e_R^{1/2},\tag{66}$$

$$\operatorname{supp}U_{TJ,(D)}\cup\operatorname{supp}U_{HJ,(D)}\cup\operatorname{supp}Q_{TJ,(D)}\cup\operatorname{supp}Q_{HJ,(D)}\subseteq\operatorname{supp}\chi_J\cdot e_I^{1/2}(t).\tag{67}$$

To complete the construction of $R_{T,J}^{j\ell}$ and $R_{HJ}^{j\ell}$ to (62)–(63), we construct solutions to the equations

$$\nabla_j R_{TJ,(D)}^{j\ell}=U_{TJ,(D)}^\ell,\qquad\nabla_j R_{HJ,(D)}^{j\ell}=U_{HJ,(D)}^\ell\tag{68}$$

that are localized around space-time cylinders containing the supports of v_J by using the inverses for the symmetric divergence equation that were constructed in [Isett and Oh 2016b]. We first recall the notions of Lagrangian and Eulerian cylinders from that paper.

Definition 10.2. Let Φ_s be the flow map associated to v_ϵ as defined in (20). Given a point in space-time $(t_0, x_0) \in \mathbb{R} \times \mathbb{T}^3$ and positive numbers $\tau, \rho > 0$, we define the *v_ϵ-adapted Eulerian cylinder* $\widehat{C}(\tau, \rho; t_0, x_0)$ with duration 2τ and base radius ρ as well as the *v_ϵ-adapted Lagrangian cylinder* $\widehat{\Gamma}(\tau, \rho; t_0, x_0)$ with duration 2τ and base radius ρ to be

$$\widehat{C}(\tau, \rho; t_0, x_0) := \{\Phi_s(t_0, x_0) + (0, h) : 0 \le |s| \le \tau,\ 0 \le |h| \le \rho\},$$
$$\widehat{\Gamma}(\tau, \rho; t_0, x_0) := \{\Phi_s(t_0, x_0 + h) : 0 \le |s| \le \tau,\ 0 \le |h| \le \rho\}.$$

The two notions are related (see [Isett and Oh 2016b, Lemma 5.2]) by

$$(t', x') \in \widehat{C}(\tau, \rho; t_0, x_0) \quad \Longleftrightarrow \quad (t, x) \in \widehat{\Gamma}_v(\tau, \rho; t', x'), \tag{69}$$
$$\widehat{\Gamma}(\tau, e^{-\tau\|\nabla v_\epsilon\|_{C^0}}\rho; t_0, x_0) \subseteq \widehat{C}(\tau, \rho; t_0, x_0) \subseteq \widehat{\Gamma}(\tau, e^{\tau\|\nabla v_\epsilon\|_{C^0}}\rho; t_0, x_0). \tag{70}$$

It follows that the amplitudes constructed in Section 7A are supported in an Eulerian cylinder

$$\begin{aligned} \operatorname{supp} \chi_J \cdot e_I^{1/2}(t) \subseteq \widehat{\Gamma}(\theta, \Pi^{-1}; t(I), x_0(J)) &\subseteq \widehat{C}(\theta, e^{\theta\|\nabla v_\epsilon\|_{C^0}}\Pi^{-1}; t(I), x_0(J)) \\ &\subseteq \widehat{C}(\theta, \Xi^{-1}; t(I), x_0(J)), \end{aligned} \tag{71}$$

and the remainder terms $U^\ell_{TJ,(D)}$ and $U^\ell_{HJ,(D)}$ are supported in the same Eulerian cylinder by (67).

Before we can obtain symmetric tensors that solve the equations in (68), we must check that the necessary orthogonality conditions

$$\int_{\mathbb{R}^3} U^\ell(t, x)\, dx = 0, \qquad \int_{\mathbb{R}^3} (x^j U^\ell - x^\ell U^j)(t, x)\, dx = 0, \qquad 1 \le j, \ell \le 3 \tag{72}$$

are satisfied, where U^ℓ is the (nonperiodic restriction of) $U^\ell_{TJ,(D)}$ or $U^\ell_{HJ,(D)}$. To check condition (72), note that $U^\ell_{HJ,(D)}$ is by construction in (57) and (64) the divergence of a smooth *symmetric* tensor with compact support, and that $U^\ell_{TJ,(D)}$ has the form $\nabla_a \nabla_c[T^{ac\ell}_J] + \nabla_j U^{j\ell}_J$ (using (32), (61), (64)), where $U^{j\ell}_J$ is symmetric and both $T^{ac\ell}_J$ and $U^{j\ell}_J$ have compact support in the cylinder (71). Integrating by parts, one obtains the conditions (72) for the nonperiodic restrictions of both $U^\ell_{TJ,(D)}$ and $U^\ell_{HJ,(D)}$.

We now have the necessary inputs to solve the symmetric divergence equation with good control over the support and boundedness properties of the solution map. We recall[7] the following result of [Isett and Oh 2016b, Section 10] (in particular Lemmas 10.3 and 10.4).

Lemma 10.3. *Suppose U is a smooth vector field on $\mathbb{R} \times \mathbb{R}^d$ with support in an Eulerian cylinder $\widehat{C}(\theta, \rho; t_0, x_0)$ relative to a smooth vector field $\bar{v}$. If U is orthogonal at all times to the rotation and translation vector fields on $\mathbb{R}^d$ in the sense of* (72), *then there is a symmetric tensor field $R^{j\ell}_U$ that is also supported in the same Eulerian cylinder and that solves $\nabla_j R^{j\ell}_U = U^\ell$. The solution can be taken to depend linearly on U and to satisfy the bounds*

$$\|\nabla_{\vec{a}} R_U\|_{C^0} \lesssim \rho \sum_{\vec{a}_1 + \vec{a}_2 = \vec{a}} \rho^{-|\vec{a}_1|} \|\nabla_{\vec{a}_2} U\|_{C^0}.$$

[7]Here we do not need to use the additional advective derivative estimates that were used in [Isett and Oh 2016b] since we only need to bound spatial derivatives.

Applying this lemma, we obtain symmetric tensors solving (68) such that

$$\operatorname{supp} R^{j\ell}_{TJ,(D)} \cup \operatorname{supp} R^{j\ell}_{HJ,(D)} \subseteq \widehat{C}(\theta, \Xi^{-1}; t(I), x_0(J)), \tag{73}$$

$$\begin{aligned}\|R^{j\ell}_{TJ,(D)}\|_{C^0} + \|R^{j\ell}_{HJ,(D)}\|_{C^0} &\lesssim \Xi^{-1}(\|U^{\ell}_{TJ,(D)}\|_{C^0} + \|U^{\ell}_{HJ,(D)}\|_{C^0}) \\ &\overset{(66)}{\lesssim} B_\lambda^{-1} N^{-D/2} (\log \widehat{\Xi})^{5/2} e_v^{1/2} e_R^{1/2},\end{aligned} \tag{74}$$

$$\|\nabla_{\vec{a}} R^{j\ell}_{TJ,(D)}\|_{C^0} + \|\nabla_{\vec{a}} R^{j\ell}_{HJ,(D)}\|_{C^0} \lesssim_{|\vec{a}|} \Xi^{-1} \sum_{|\vec{b}|\le|\vec{a}|} \Xi^{|\vec{a}|-|\vec{b}|}(\|\nabla_{\vec{b}} U^{\ell}_{TJ,(D)}\|_{C^0} + \|\nabla_{\vec{b}} U^{\ell}_{HJ,(D)}\|_{C^0}). \tag{75}$$

We now set $D = 2$ and define

$$R^{j\ell}_{TJ} = Q^{j\ell}_{TJ,(D)} + R^{j\ell}_{TJ,(D)} \quad \text{and} \quad R^{j\ell}_{HJ} = Q^{j\ell}_{HJ,(D)} + R^{j\ell}_{HJ,(D)}.$$

Combining (65), (67), (73), and (74) into (60), we obtain the estimate

$$\|R_T\|_{C^0} + \|R_H\|_{C^0} \lesssim B_\lambda^{-1} (\log \widehat{\Xi})^{5/2} \frac{e_v^{1/2} e_R^{1/2}}{N}. \tag{76}$$

To sum the estimates we have also used the fact that the number of distinct cylinders of the form (73) that can intersect at a given point in space-time (t, x) is bounded by an absolute constant. To check this fact, note that if two cylinders indexed by J and J' intersect at a point $(t^*, x^*) \in \mathbb{R} \times \mathbb{T}^3$, then

$$\begin{aligned}&(t^*, x^*) \in \widehat{C}(\theta, \Xi^{-1}; t(I), x_0(J)) \cap \widehat{C}(\theta, \Xi^{-1}; t(I'), x_0(J')) \quad \Rightarrow \\ &\qquad I = I' \quad \text{and} \quad (t(I), x_0(J)), (t(I), x_0(J')) \overset{(69)}{\in} \widehat{\Gamma}(\theta, \Xi^{-1}; t^*, x^*) \\ &\qquad\qquad \overset{(70)}{\in} \widehat{C}(\theta, e^{\theta\|\nabla v_\epsilon\|_{C^0}} \Xi^{-1}; t^*, x^*) \subseteq \widehat{C}(\theta, 3\Xi^{-1}; t^*, x^*).\end{aligned}$$

The number of indices $J = (I, f)$ for which $(t(I), x_0(J))$ can belong to a given ball of radius $3\Xi^{-1} \lesssim \Pi^{-1}$ is bounded by an absolute constant by the construction of the cutoff functions.

We can now take B_λ to be a sufficiently large number such that the right-hand side of (76) is bounded by $(\log \widehat{\Xi})^{5/2} e_v^{1/2} e_R^{1/2}/(20N)$ (and so that $\lambda = B_\lambda N \Xi \in \mathbb{Z}$ is an integer). This choice achieves our desired bound for $\|R_1\|_{C^0}$ when combined with (59). The desired bounds for higher derivatives

$$\|\nabla_{\vec{a}} R_T\|_{C^0} + \|\nabla_{\vec{a}} R_H\|_{C^0} \lesssim (N\Xi)^{|\vec{a}|} (\log \widehat{\Xi})^{5/2} \frac{e_v^{1/2} e_R^{1/2}}{N}, \quad 1 \le |\vec{a}| \le 3,$$

now follow from (65), (66), (75) and the observations concerning the overlaps of the cylinders (73). The assertions about the desired support of $R_1^{j\ell}$ asserted in Lemma 6.2 are clear from construction.

The proof of Lemma 6.2 will now be complete after explaining the proof of Proposition 10.1.

10A. ***The parametrix expansion.*** We now prove Proposition 10.1 using the argument in the proof of [Isett 2018, Proposition 17.6]. Let $U^\ell = u^\ell \omega(\lambda \Gamma_I)$ be given as in the assumptions of Proposition 10.1. By Fourier-expanding $\omega(X)$ as a function on $\mathbb{T}^3$, we have

$$U^\ell = \sum_{m \ne 0} \hat{\omega}(m) e^{i\lambda \xi_m(t,x)} u^\ell(t, x), \tag{77}$$

where $m \in \mathbb{Z}^3$ and $\xi_m(t,x) := m \cdot \Gamma_I(t,x)$. Following the proof of [Isett 2018, Proposition 17.6], we set

$$Q_{(D)}^{j\ell} = \sum_{m\neq 0} \hat{\omega}(m) Q_{(D),m}^{j\ell}, \qquad Q_{(D),m}^{j\ell} := \lambda^{-1} \sum_{k=1}^{D} e^{i\lambda\xi_m} q_{(k),m}^{j\ell}. \tag{78}$$

The amplitudes $q_{(k),m}^{j\ell}$ are constructed inductively with a sequence of amplitudes $u_{(k),m}^{\ell}$ such that

$$\begin{aligned} i\nabla_j \xi_m q_{(k),m}^{j\ell} &= u_{(k-1),m}^{\ell}, \\ u_{(k),m}^{\ell} &= -\lambda^{-1} \nabla_j q_{(k),m}^{j\ell} \end{aligned} \tag{79}$$

and $u_{(0),m}^{\ell} = u^{\ell}$. By (77), (79) and induction on D, we then obtain

$$\begin{aligned} U^{\ell} &= \nabla_j Q_{(D)}^{j\ell} + U_{(D)}^{\ell}, \\ U_{(D)}^{\ell} &= \sum_{m\neq 0} \hat{\omega}(m) e^{i\lambda\xi_m} u_{(D),m}^{\ell}. \end{aligned} \tag{80}$$

More specifically, to solve (79) we first choose smooth functions $\bar{q}_a^{j\ell}(p)$ of a variable $p \in \mathbb{R}^3 \setminus \{0\}$, symmetric in $j\ell$, such that each $\bar{q}_a^{j\ell}(p)$ is degree -1 homogeneous ($\bar{q}_a^{j\ell}(\alpha p) = \alpha^{-1}\bar{q}_a^{j\ell}(p)$ if $\alpha \in \mathbb{R}\setminus\{0\}$) and such that $ip_j \bar{q}_a^{j\ell}(p) = \delta_a^{\ell}$ for all $p \neq 0$. See [Isett 2018, Proposition 17.6] for an explicit example. We then set $q_{(k),m}^{j\ell} := \bar{q}_a^{j\ell}(\nabla\xi_m) u_{(k-1),m}^{a}$, so that (79) is satisfied.

From this construction we see that both $Q_{(D)}^{\ell}$ and $U_{(D)}^{\ell}$ have support contained in $\operatorname{supp} u^{\ell}$. We obtain the desired estimates for $Q_{(D)}^{\ell}$ and $U_{(D)}^{\ell}$ stated in Proposition 10.1 from the formulas (78) and (80) by using the bounds

$$\begin{aligned} \|\nabla_{\vec{a}} q_{(k),m}^{j\ell}\|_{C^0} &\lesssim N^{-(k-1)/2} N^{|\vec{a}|/2} \Xi^{|\vec{a}|} H_{D+3,0}[u] \quad \text{for all } 0 \le |\vec{a}| \le D-k+4, \quad 1 \le k \le D, \\ \|\nabla_{\vec{a}} u_{(k),m}^{\ell}\|_{C^0} &\lesssim B_{\lambda}^{-1} N^{-k/2} N^{|\vec{a}|/2} \Xi^{|\vec{a}|} H_{D+3,0}[u] \quad \text{for all } 0 \le |\vec{a}| \le D-k+3, \quad 1 \le k \le D, \end{aligned}$$

from the proof of [Isett 2018, Proposition 17.6] (where $H_{D+3,0}[u]$ is written simply as H), and by using the rapid decay of $|\hat{\omega}(m)| \lesssim (1+|m|)^{-40}$ to ensure convergence in the summation over $m \in \mathbb{Z}^3$. (The main point in the estimate is that each spatial derivative of the sum costs at most a factor of λ.)

11. Iterating the main lemma

We now explain the proof of Theorem 1.1. Similar to other convex integration constructions, the theorem will be proven by repeatedly applying Lemma 5.1 to obtain a sequence of Euler–Reynolds flows $(v_{(k)}, p_{(k)}, R_{(k)})$ indexed by k (with frequency-energy levels bounded by $(\Xi_{(k)}, e_{v,(k)}, e_{R,(k)})$) that will converge uniformly to the solution v stated in Theorem 1.1. Unlike previous works, we introduce here a new and sharper approach to estimating the regularity and to optimizing the choice of parameters governing the growth of frequencies.

To initialize the construction, we construct a smooth Euler–Reynolds flow $(v_{(1)}, p_{(1)}, R_{(1)})$ with compact support in time that satisfies

$$\sup_{x\in\mathbb{T}^3} v_{(1)}(0,x) \ge 10 \tag{81}$$

and has frequency-energy levels (to order 3 in C^0) bounded by $(\Xi_{(1)}, e_{R,(1)}, e_{R,(1)})$, where $\Xi_{(1)} = \widehat{\Xi}_{(1)}$ and $e_{R,(1)}$ are large and small parameters, respectively, that remain to be chosen. One way to produce such an Euler–Reynolds flow is to apply the main lemma in the convex integration scheme of [Isett 2017a] (as was done in [Isett 2018]). This approach has some added benefits such as the ability to obtain arbitrarily large increases in energy within an arbitrarily small time interval [Isett 2017a]. For the present purpose it will suffice to take a simpler approach.

We take $v_{(1)}$ to have the form $v^\ell_{(1)} = \psi(B^{-1}t)U^\ell$, where ψ is a smooth cutoff with $\psi(0) = 1$ and $0 \le \psi(t) \le 1$ for all t, B is a large parameter, and $U^\ell : \mathbb{T}^3 \to \mathbb{R}^3$ is a smooth vector field that satisfies

$$\int_{\mathbb{T}^3} U^\ell(x)\,dx = 0, \quad \nabla_\ell U^\ell = 0, \quad \nabla_j(U^j U^\ell) = 0, \quad \sup_{x\in\mathbb{T}^3} U^\ell(x) \ge 10.$$

For example, one can take a sufficiently large Mikado flow for $U^\ell(x)$. We then take $p_{(1)} = 0$ and $R_{(1)}$ to be a symmetric tensor that solves

$$\nabla_j R^{j\ell}_{(1)} = \partial_t v^\ell_{(1)} = B^{-1}\psi'(B^{-1}t)U^\ell(X) \tag{82}$$

by applying an appropriate, degree -1 Fourier-multiplier to the right-hand side of (82). The Euler–Reynolds flow $(v_{(1)}, p_{(1)}, R_{(1)})$ obtained in this way has frequency-energy levels (to order 3 in C^0) bounded by $(\overline{\Xi}, 1, e_{R,(1)})$, where $\overline{\Xi}$ depends only on U^ℓ, and where $e_{R,(1)} \lesssim B^{-1}$ can be made arbitrarily small by taking B large depending on U^ℓ. It follows from Definition 4.2 that $(v_{(1)}, p_{(1)}, R_{(1)})$ also have frequency-energy levels bounded by

$$(\Xi_{(1)}, e_{v,(1)}, e_{R,(1)}) := (\overline{\Xi} e^{-1/2}_{R,(1)}, e_{R,(1)}, e_{R,(1)}),$$

where we have now fixed our choice of $\Xi_{(1)} := \overline{\Xi} e^{-1/2}_{R,(1)}$ in terms of the small parameter $e_{R,(1)}$ that remains to be chosen.

11A. ***Heuristics and deriving the optimization problem for the parameters.*** The sequence of frequency-energy levels $(\Xi, e_v, e_R)_{(k)}$ and Euler–Reynolds flows will now be determined by repeatedly applying Lemma 5.1, so that the following rules hold. (Here $\widehat{C}$ and C_L denote the two constants of Lemma 5.1 and $\widehat{\Xi}_{(k)} := (e_v/e_R)^{1/2}_{(k)} \Xi_{(k)}$.)

$$\Xi_{(k+1)} = \widehat{C} N_{(k)} \Xi_{(k)}, \tag{83}$$

$$e_{v,(k+1)} = (\log \widehat{\Xi}_{(k)}) e_{R,(k)}, \tag{84}$$

$$e_{R,(k+1)} = \frac{e_{R,(k)}}{g_{(k)}}, \tag{85}$$

$$N_{(k)} = (\log \widehat{\Xi}_{(k)})^A \left(\frac{e_v}{e_R}\right)^{1/2}_{(k)} g_{(k)}, \quad A := \tfrac{5}{2}. \tag{86}$$

The sequence $g_{(k)} > 1$ describes the "gain" in the size of the error after stage k, and the sequence of frequency growth parameters $N_{(k)}$ is determined by inequality (10) in Lemma 5.1, so that this choice of $N_{(k)}$ achieves the desired gain. To work with the estimate (11), it will also be useful to impose that

$$(\log \widehat{\Xi}_{(k+1)})^{1/2} e^{1/2}_{R,(k+1)} \le \tfrac{1}{2} (\log \widehat{\Xi}_{(k)})^{1/2} e^{1/2}_{R,(k)} \quad \text{for all } k \ge 1. \tag{87}$$

The Euler–Reynolds flows constructed by repeatedly applying Lemma 5.1 using the above choice of parameters $N_{(k)}$ will converge uniformly to the velocity field $v^\ell = v^\ell_{(1)} + \sum_{k=1}^\infty V^\ell_{(k)}$. Assuming (87), which is verified in Proposition 11.1 below, this solution will be nontrivial and continuous for $e_{R,(1)}$ chosen small enough (depending on $\overline{\Xi}$, $\widehat{C}$ and C_L) thanks to (81) and

$$\sum_{k=1}^{\infty} \|V^\ell_{(k)}\|_{C^0} \overset{(11),(87)}{\leq} \sum_{k=0}^{\infty} C_L (\log \widehat{\Xi}_{(1)})^{1/2} e_{R,(1)} 2^{-k} \leq 5. \tag{88}$$

As $R_{(k)}$ converges uniformly to 0, one has from the Euler–Reynolds system that the associated sequence of pressures $p_{(k)} = \Delta^{-1}\nabla_j \nabla_\ell (R^{j\ell}_{(k)} - v^j_{(k)} v^\ell_{(k)})$ converge weakly in $\mathcal{D}'(\mathbb{R} \times \mathbb{T}^3)$ to $p = -\Delta^{-1}\nabla_j\nabla_\ell(v^j v^\ell)$, and that the pair (v, p) form a weak solution to the Euler equations.

Our goal is now to choose $g_{(k)}$ that optimize the regularity of the solution v. The key evolution rule that isolates $\frac{1}{3}$ as the limiting regularity and plays a key role in our analysis will be the following:

$$\delta_{(k)}\big(\tfrac{1}{3} \log \widehat{\Xi}_{(k)} + \tfrac{1}{2} \log e_{R,(k)}\big) = \big(\tfrac{1}{3} A + \tfrac{1}{6}\big) \log\log \widehat{\Xi}_{(k)} + \log \widehat{C}. \tag{89}$$

Here $\delta_{(k)}[f_{(k)}] = f_{(k+1)} - f_{(k)}$ is the discrete differencing operator and $A = \frac{5}{2}$. A crucial point is that (89) holds for *all* possible choices of $g_{(k)}$.

With the goal of computing regularity in mind, suppose $\Delta x \in \mathbb{R}^3$ with, say, $0 < |\Delta x| \leq 10^{-2}$. Writing

$$v = v_{(\bar{k})} + \sum_{k \geq \bar{k}} V_{(k)} \quad \text{and} \quad L_k := \log \widehat{\Xi}_{(k)},$$

we can bound $|v(t, x + \Delta x) - v(t, x)|$ using (87) by

$$\begin{aligned} |v(t, x+\Delta x) - v(t,x)| &\leq \|\nabla v_{(\bar{k})}\|_{C^0} |\Delta x| + \sum_{k \geq \bar{k}} 2 \|V_{(k)}\|_{C^0} \leq \Xi_{(\bar{k})} e^{1/2}_{v,(\bar{k})} |\Delta x| + 4 C_L (\log \widehat{\Xi}_{(\bar{k})})^{1/2} e^{1/2}_{R,(\bar{k})} \\ &\leq 4 C_L L_{\bar{k}} (\widehat{\Xi}_{(\bar{k})} |\Delta x| + 1) e^{1/2}_{R,(\bar{k})}. \end{aligned} \tag{90}$$

The estimate is optimized by choosing $\bar{k}$ to be the largest value k for which $\widehat{\Xi}_{(k)} |\Delta x| \leq 1$. Now assuming $\bar{k}$ has been chosen as this value, the estimate (90) leads to

$$\begin{aligned} |v(t, x+\Delta x) - v(t,x)| &\leq 8 C_L L_{\bar{k}} e^{1/2}_{R,(\bar{k})} = 8 C_L L_{\bar{k}} \widehat{\Xi}^{-1/3}_{(\bar{k})} \exp\big(\tfrac{1}{3} \log \widehat{\Xi}_{(\bar{k})} + \tfrac{1}{2} \log e_{R,(\bar{k})}\big) \\ &\lesssim L_{\bar{k}} \widehat{\Xi}^{-1/3}_{(\bar{k}+1)} \exp\big(\tfrac{1}{3} \delta_{(k)} \log \widehat{\Xi}_{(k)}\big|_{k=\bar{k}} + \tfrac{1}{3} \log \widehat{\Xi}_{(k)} + \tfrac{1}{2} \log e_{R,(k)}\big) \\ &\lesssim |\Delta x|^{1/3} L_{\bar{k}} \exp\big(\tfrac{1}{3} \delta_{(k)} \log \widehat{\Xi}_{(k)}\big|_{k=\bar{k}} + \tfrac{1}{3} \log \widehat{\Xi}_{(k)} + \tfrac{1}{2} \log e_{R,(k)}\big). \end{aligned} \tag{91}$$

Using (89) to expand $\frac{1}{3} \log \widehat{\Xi}_{(k)} + \frac{1}{2} \log e_{R,(k)}$, we minimize the right-hand side of (91) if we minimize

$$H_{\bar{k}} := \left(\frac{1}{3} (\log \widehat{\Xi}_{(\bar{k}+1)} - \log \widehat{\Xi}_{(\bar{k})}) + \sum_{k=1}^{\bar{k}-1} (\log\log \widehat{\Xi}_{(k)} + \log \widehat{C}) \right). \tag{92}$$

The expression (92) now reveals the optimization problem for choosing $g_{(k)}$. Namely, to control the term $\delta_{(k)} \log \widehat{\Xi}_{(k)}$, the frequencies should not grow too quickly. However, a slow growth of frequencies produces a long summation and a poor estimate for the sum as the construction is iterated many times before achieving a given length scale. Intuitively, the best estimate should be achieved if the two terms are balanced, which suggests the parameters $L_k = \log \widehat{\Xi}_{(k)}$ should satisfy the discrete version of the equation

$$\frac{dL}{dk} = 3\int_1^k (\log L(\kappa) + c)\, d\kappa,$$

whose solutions grow like $L_k = (3 + o(1))k^2 \log k$ at infinity.

We will see that the regularity is optimal precisely when L_k are chosen to have this growth.

11B. *Parameter asymptotics and optimization.* With this motivation, we take $g_{(k)} = e^{\gamma k \log k}$, where $\gamma > 0$ is a parameter that will be chosen to optimize the regularity. To simplify the algebra we can restrict to $k \geq 2$ by assuming that the Euler–Reynolds flows $(v_{(1)}, p_{(1)}, R_{(1)}) = (v_{(2)}, p_{(2)}, R_{(2)})$ and their frequency-energy levels are equal.

Before estimating the regularity, we wish to fix our choice of the parameter $e_{R,(1)}$ that dictates the initial frequency-energy levels. We therefore verify the assumption (87) (restricting now to $\gamma \geq 2$).

Proposition 11.1. *If $\gamma \geq 2$ and $e_{R,(1)}$ is small enough depending on $\widehat{C}$, then* (87) *holds for all $k \geq 2$.*

Proof. Taking logs of (87), it suffices to bound the quantity

$$\tfrac{1}{2}\delta_{(k)} \log\log \widehat{\Xi}_{(k)} + \tfrac{1}{2}\delta_{(k)} \log e_{R,(k)} = \tfrac{1}{2}\delta_{(k)} \log\log \widehat{\Xi}_{(k)} - \tfrac{1}{2}\log g_{(k)} \tag{93}$$

by $-\log 2$ uniformly in k.

Towards this goal, we set $Z_k := \widehat{C}(\log \widehat{\Xi}_{(k)})^{A+1/2}$ to be the lower-order factor from (83) and (86). Linearizing $\log(\,\cdot\,)$ around $L_k := \log \widehat{\Xi}_{(k)}$ and using (83)–(86) and concavity, we have

$$\delta_{(k)} \log\log \widehat{\Xi}_{(k)} = \log(\log \widehat{\Xi}_{(k)} + \log(Z_k g_{(k)}^{3/2})) - \log\log \widehat{\Xi}_{(k)} \leq \frac{\log(Z_k g_{(k)}^{3/2})}{\log \widehat{\Xi}_{(k)}}. \tag{94}$$

We now substitute (94) into (93) and take $e_{R,(1)}$ small enough to ensure that $\widehat{\Xi}_{(k)} \geq \Xi_{(k)} \geq \Xi_{(1)} = \overline{\Xi} e_{R,(1)}^{-1/2}$ is large enough so that the following bound holds for all $k \geq 2$:

$$(93) \leq \frac{\log Z_k}{\log \widehat{\Xi}_{(k)}} - \frac{1}{3}\log g_{(k)}. \tag{95}$$

Taking $e_{R,(1)}$ smaller and hence $\Xi_{(1)}$ larger, we can ensure that the function

$$f(\Xi) := \frac{\log(\widehat{C}(\log \Xi)^{A+1/2})}{\log \Xi}$$

is decreasing in Ξ on the interval $\Xi \in [\Xi_{(1)}, \infty)$. From $\widehat{\Xi}_{(k)} \geq \Xi_{(1)}$ and (95) we obtain

$$(95) \leq \frac{\log(\widehat{C}(\log \Xi_{(1)})^{A+1/2})}{\log \Xi_{(1)}} - \frac{1}{3}\log g_{(2)} \quad \text{for all } k \geq 2. \tag{96}$$

We have that $-\frac{1}{3}\log g_{(2)} = -\frac{2}{3}\gamma \log 2 \leq -\frac{4}{3}\log 2$. Taking $e_{R,(1)}$ small and thus $\Xi_{(1)}$ large, we can bound (96) and therefore (93) by $-\log 2$, which establishes Proposition 11.1. □

At this point, we choose $e_{R,(1)}$ sufficiently small (depending on $\widehat{C}$ and C_L) to satisfy the assumptions of Proposition 11.1 and such that (88) holds.

With the initial frequency-energy levels determined, we now turn to the asymptotics of the frequency-energy levels for large k. These asymptotics are summarized as follows.

Proposition 11.2. *For all $k \geq 3$ and the above choice of $g_{(k)}$, we have the asymptotics*

$$-\log e_{R,(k)} = \frac{\gamma k^2}{2} \log k + O(k \log k), \tag{97}$$

$$\frac{1}{2} \log\left(\frac{e_v}{e_R}\right)_{(k)} = \frac{1}{2}\gamma k \log k + O(\log k), \tag{98}$$

$$\delta_{(k)} \log \widehat{\Xi}_{(k)} = \frac{3}{2}\gamma k \log k + O(\log k), \tag{99}$$

$$\log \widehat{\Xi}_{(k)} = \frac{3}{2}\frac{\gamma k^2}{2} \log k + O(k \log k), \tag{100}$$

$$\log\log \widehat{\Xi}_{(k)} = 2 \log k + O(1), \tag{101}$$

$$\frac{1}{3} \log \widehat{\Xi}_{(k)} + \frac{1}{2} \log e_{R,(k)} = 2\left(\frac{A}{3} + \frac{1}{6}\right) k \log k + O(k), \tag{102}$$

$$\Xi_{(k)} = \exp\left(\frac{3\gamma k^2}{4} \log k + O(k \log k)\right), \tag{103}$$

together with the bounds

$$(\log \widehat{\Xi}_{(k)})^{-1} = O(k^{-2}(\log k)^{-1}), \tag{104}$$

$$\log\log \widehat{\Xi}_{(k)} = O(\log k). \tag{105}$$

Here the implicit constants in the $O(\,\cdot\,)$ notation depend only on $\widehat{C}$, γ, $\Xi_{(1)}$, $e_{R,(1)}$ and $A = \frac{5}{2}$.

The proof will proceed by induction on $k \geq 3$ and will use some extra notation for the induction. We write $C_{(97)}, \ldots, C_{(105)}$ to refer to the implicit constants in the $O(\,\cdot\,)$ notation in the proposition. For example the term in (105) is bounded by $|O(\log k)| \leq C_{(105)} \log k$. We assume at the onset that all the constants $C_{(97)}, \ldots, C_{(105)}$ are sufficiently large depending on $\Xi_{(1)}$ and $e_{R,(1)} = e_{v,(1)}$ such that the bounds (97)–(105) hold for $k = 3$. The proof will make use of the Taylor expansion formula

$$f(X+Y) = f(X) + Y \int_0^1 f'(X + \sigma Y)\, d\sigma = f(X) + f'(X)Y + Y^2 \int_0^1 (1-\sigma) f''(X + \sigma Y)\, d\sigma. \tag{106}$$

Proof of (97). The equality follows from the evolution rule $\log e_{R,(k+1)} = -\log g_{(k)} + \log e_{R,(k)}$ and

$$\sum_{1 \leq I \leq k} \log g_{(I)} = \sum_{1 \leq I \leq k} \gamma I \log I = \frac{\gamma k^2}{2} \log k + O(k \log k), \quad k \geq 3$$

(where the constant above depends on γ). □

Proof of (104). From $\log \widehat{\Xi}_{(k+1)} \geq \log g_{(k)} + \log \widehat{\Xi}_{(k)}$, we have

$$k^2 \log k \lesssim \sum_{3 \leq I \leq k} \log g_{(I)} \leq \log \widehat{\Xi}_{(k)}.$$ □

Proof of (105). Let $L_k := \log \widehat{\Xi}_{(k)}$ and $Z_k = \widehat{C}(\log \widehat{\Xi}_{(k)})^{A+1/2}$. Then for some $A_0 \geq 1$ and all $k \geq 3$,

$$\begin{aligned}\delta_{(k)} \log\log \widehat{\Xi}_{(k)} &= \log(L_k + \log(Z_k g_{(k)}^{3/2})) - \log L_k \\ &\leq L_k^{-1}(\log Z_k + \log g_{(k)}^{3/2}) \leq A_0 C_{(104)}(k^{-2}(\log k)^{-1} \log\log \widehat{\Xi}_{(k)} + k^{-1}).\end{aligned}$$

Choose $k^* = k^*(C_{(104)})$ large enough that $A_0 C_{(104)} k^{-2} \leq 10^{-1}\delta_{(k)} \log k$ for all $k \geq k^*$, and assume that $C_{(105)}$ is large enough that (105) holds for $k \leq k^*$.

We now proceed by induction on k to obtain (105) for $k > k^*$. Assuming (105) for k, we have

$$\delta_{(k)} \log\log \widehat{\Xi}_{(k)} \leq 10^{-1} C_{(105)} \delta_{(k)} \log k + A_0 C_{(104)} k^{-1} \leq C_{(105)} \delta_{(k)} \log k \quad \text{for } k \geq k^*$$

if $C_{(105)}$ is sufficiently large, which implies (105) for $k+1$, and thus for all $k \geq k^*$ by induction. □

Proof of (98). The equality follows from (105) and

$$\frac{1}{2}\log\left(\frac{e_v}{e_R}\right)_{(k+1)} = \frac{1}{2}(\log g_{(k)} + \log\log \widehat{\Xi}_{(k)}).$$ □

Proof of (99)–(100). For $k \geq 3$, we have by (98) and (105) $\big($for $A = \frac{5}{2}\big)$

$$\begin{aligned}\delta_{(k)} \log \widehat{\Xi}_{(k)} &= \frac{1}{2}\log\left(\frac{e_v}{e_R}\right)_{(k+1)} + \log g_{(k)} + A \log\log \widehat{\Xi}_{(k)} \\ &= \frac{3\gamma}{2} k \log k + O(\log k) = \frac{3\gamma}{2}\delta_{(k)}\left[\frac{k^2}{2}\log k\right] + O(\delta_{(k)}[k \log k]),\end{aligned}$$

which implies both (99) and (100) after summing over k. □

Proof of (101). Again writing $L_k = \log \widehat{\Xi}_{(k)}$ and $Z_k = \widehat{C}(\log \widehat{\Xi}_{(k)})^{A+1/2}$, we have by Taylor expansion

$$\begin{aligned}\delta_{(k)} \log\log \widehat{\Xi}_{(k)} &= \log(L_k + \log(Z_k g_{(k)}^{3/2})) - \log L_k \\ &= (\log \widehat{\Xi}_{(k)})^{-1} \log(Z_k g_{(k)}^{3/2}) - \int_0^1 d\sigma \frac{(\log(Z_k g_{(k)}^{3/2}))^2 (1-\sigma)}{(L_k + \sigma \log(Z_k g_{(k)}^{3/2}))^2}.\end{aligned}$$

The main term is

$$(\log \widehat{\Xi}_{(k)})^{-1} \log g_{(k)}^{3/2} = 2k^{-1} + O(k^{-2}) = 2\delta_{(k)} \log k + O(k^{-2})$$

by (100). The remaining terms are of size $O(k^{-2})$ by (105) and (100) again. Summing over k gives the desired result (101). □

Proof of (102). Equation (102) follows from (89), (101) and summation over k. □

Proof of (103). Equation (103) follows from

$$\log \Xi_{(k)} = \log \widehat{\Xi}_{(k)} - \frac{1}{2}\log\left(\frac{e_v}{e_R}\right)_{(k)},$$

equation (100) and (98). □

We now return to analyzing the regularity estimate (91). From (100), (99), (102), and by the definitions of $\bar{k}$ and $\widehat{\Xi}_{(\bar{k})}$, we obtain (using (106) with $f(X) = X^{-1}$ or $\log X$) that, for all $|\Delta x| \le 10^{-2}$,

$$\begin{aligned}\bar{k}^2 \log \bar{k} \lesssim \log \widehat{\Xi}_{(\bar{k})} \le \log |\Delta x|^{-1} \le \log \widehat{\Xi}_{(\bar{k}+1)} \lesssim \bar{k}^2 \log \bar{k},\\ \frac{3\gamma}{4}\bar{k}^2 \log \bar{k} = \log |\Delta x|^{-1} + O(\bar{k}\log \bar{k}), \qquad (107)\\ (\log|\Delta x|^{-1})^{-1} = \Big(\frac{4}{3\gamma} + O(\bar{k}^{-1})\Big)\bar{k}^{-2}(\log \bar{k})^{-1},\\ \log(\bar{k}^2) = \log\log|\Delta x|^{-1} + O(\log\log \bar{k}). \qquad (108)\end{aligned}$$

To bound (91) purely in terms of $|\Delta x|$, we first estimate the logarithm of the term $L_{\bar{k}} \exp(H_{\bar{k}})$ appearing in (91)–(92) (using $A = \frac{5}{2}$ and $\frac{1}{3}A + \frac{1}{6} = 1$) by

$$\begin{aligned}(\log|\Delta x|^{-1})^{-1} \cdot (H_{\bar{k}} + \log L_{\bar{k}}) &= \Big(\frac{4}{3\gamma}(1 + O(\bar{k}^{-1}))(\bar{k}^2\log \bar{k})^{-1}\Big) \cdot \Big(\Big(\frac{\gamma}{2}+2\Big)\bar{k}\log\bar{k} + O(\bar{k})\Big)\\ &= \frac{4}{3\gamma}\Big(\frac{\gamma}{2}+2\Big)\bar{k}^{-1} + O(\bar{k}^{-1}(\log\bar{k})^{-1})\\ &= \frac{4}{3\gamma}\Big(\frac{\gamma}{2}+2\Big)(\bar{k}^2\log\bar{k})^{-1/2}(\log\bar{k})^{1/2} + O(\bar{k}^{-1}(\log\bar{k})^{-1})\\ &= 2^{-1/2}\Big(\frac{4}{3\gamma}\Big)\Big(\frac{\gamma}{2}+2\Big)(\bar{k}^2\log\bar{k})^{-1/2}(\log\log|\Delta x|^{-1})^{1/2}\\ &\qquad + O\Big(\frac{\log\log\bar{k}}{(\bar{k}^2\log\bar{k})^{1/2}(\log\log|\Delta x|^{-1})^{1/2}}\Big).\end{aligned}$$

In the last line we used (108) and (106) with $f(X) = X^{1/2}$. From (107) and (106) we then have

$$\begin{aligned}(\log|\Delta x|^{-1})^{-1} \cdot (H_{\bar{k}} + \log L_{\bar{k}}) = 2^{-1/2}\Big(\frac{4}{3\gamma}\Big)^{1/2}\Big(\frac{\gamma}{2}+2\Big)(\log|\Delta x|^{-1})^{-1/2}(\log\log|\Delta x|^{-1})^{1/2}\\ + O\Big(\frac{\log\log\log|\Delta x|^{-1}}{(\log|\Delta x|^{-1})^{1/2}(\log\log|\Delta x|^{-1})^{1/2}}\Big). \qquad (109)\end{aligned}$$

The bound (109) is optimized by taking $\gamma = 4$, which is precisely the value that leads to the asymptotic $\log \widehat{\Xi}_{(k)} = (3 + o(1))k^2 \log k$ predicted by the heuristics at the conclusion of Section 11A. Substituting into (91), we finally obtain

$$|v(t, x + \Delta x) - v(t, x)| \lesssim |\Delta x|^{1/3 - B\sqrt{(\log\log|\Delta x|^{-1})/(\log|\Delta x|^{-1})}}, \qquad (110)$$

where one can take the constant $B = 2\sqrt{\frac{2}{3}}$ at the expense of introducing the additional lower-order term[8] from (109). In particular, v belongs to $\bigcap_{\alpha<1/3} L_t^\infty C_x^\alpha$, and therefore belongs to $\bigcap_{\alpha<1/3} C_{t,x}^\alpha$ by the results in [Isett 2023]. To check that v has compact support in time, note that the time support in each iteration grows by at most a factor

$$\Xi_{(k)}^{-1} e_{v,(k)}^{-1/2} = \widehat{\Xi}_{(k)}^{-1} e_{R,(k)}^{-1/2} = \widehat{\Xi}_{(k)}^{-2/3} \exp\big(-\tfrac{1}{3}\log\widehat{\Xi}_{(k)} - \tfrac{1}{2}\log e_{R,(k)}\big).$$

[8] The derivation of (92) suggests that taking $g_{(k)} = \big(\sum_{I=1}^{k}(\log\log\widehat{\Xi}_{(I)} + \log\widehat{C})\big) + (\log\log\widehat{\Xi}_k/2)$ would optimize the lower-order terms as well, although this alternative choice would not affect the leading-order terms.

Using (100) and (102), we conclude that the series $\sum_k \Xi_{(k)}^{-1} e_{v,(k)}^{-1/2}$ converges, and hence the limiting solution is supported on a finite time interval. This calculation concludes the proof of Theorem 1.1.

12. Improving the borderline estimate

In this section, we sketch roughly how the value of the B appearing in the regularity estimate (110) can be improved by combining with the approach to the gluing lemma introduced in [Buckmaster et al. 2019a].

Recall that, in the notation of [Isett 2018], the gluing lemma is proved by introducing, for a given Euler–Reynolds flow (v, p, R), corrections

$$y^\ell = \sum_I \eta_I y_I^\ell \quad \text{and} \quad \bar{p} = \sum_I \eta_I \bar{p}_I$$

to the velocity and pressure such that the new velocity field $\tilde{v}^\ell = v^\ell + y^\ell$ and pressure $\tilde{p} = p + \bar{p}$ solve the Euler–Reynolds system with a new Reynolds stress $\widetilde{R}$ that is supported in disjoint time intervals of width $\theta \sim (\log \widehat{\Xi})^{-2} \Xi^{-1} e_v^{-1/2}$. The new stress $\widetilde{R}$ is constructed in terms of symmetric tensors $r_I^{j\ell}$ that solve $\nabla_j r_I^{j\ell} = y_I^\ell$, which are obtained by solving the following initial value problem:[9]

$$\begin{aligned} (\partial_t + v^i \nabla_i) r_I^{j\ell} &= \mathcal{R}^{j\ell}[\nabla_i[\nabla_a v^i r_I^{ab}] - y_I^i \nabla_i v^b] - y_I^j y_I^\ell - \bar{p}_I \delta^{j\ell} - R^{j\ell}, \\ r_I^{j\ell}(t(I), x) &= 0. \end{aligned} \tag{111}$$

Here $\mathcal{R}^{j\ell}$ is an order -1 operator that inverts the divergence equation in symmetric tensors, and the identity $\nabla_j r_I^{j\ell} = y_I^\ell$ can be checked using the equation

$$\begin{aligned} \partial_t y_I^\ell + v^i \nabla_i y_I^\ell + y_I^i \nabla_i v^\ell + \nabla_j (y_I^j y_I^\ell) + \nabla^\ell \bar{p}_I &= -\nabla_j R^{j\ell}, \\ \partial_t y_I^\ell + v^i \nabla_i y_I^\ell + y_I^i \nabla_i u_I^\ell + \nabla^\ell \bar{p}_I &= -\nabla_j R^{j\ell}, \end{aligned} \tag{112}$$

where $u_I^\ell = v^\ell + y_I^\ell$ is the classical solution to incompressible Euler equations with initial data $v^\ell(t_0(I), x)$.

In [Buckmaster et al. 2019a], a different approach to solving and estimating solutions of the equation $\nabla_j r_I^{j\ell} = y_I^\ell$ is taken. There, one first considers the potential $\tilde{z}_I = \Delta^{-1} \nabla \times y_I$, which solves $\nabla \times \tilde{z}_I = y_I$, $\operatorname{div} \tilde{z}_I = 0$, and turns out to satisfy an evolution equation that (like (111)) has a good structure. From $\tilde{z}_I$, one then obtains a symmetric antidivergence for y_I by applying a zeroth-order operator (e.g., $r_I^{j\ell} = \mathcal{R}^{j\ell}[\nabla \times \tilde{z}_I]$), which is estimated using Schauder and commutator estimates for Calderón–Zygmund operators (CZOs). (We note that, conversely, estimates for $\tilde{z}_I$ can be deduced from those of $r_I^{j\ell}$ above by similar zeroth-order commutator estimates.) The key simplification comes in treating the term $\Delta^{-1} \nabla \times [y_I \cdot \nabla v]$ that is analogous to the term $\mathcal{R}^{j\ell}[y_I \cdot \nabla v]$ in (111), the latter of which had been treated by a decomposition into frequency increments in [Isett 2018]. For the present applications, the estimates employed in [Buckmaster et al. 2019a], which apply the classical local well-posedness theory for Euler and Schauder and commutator estimates for CZOs, are not strong enough as they lose small powers of the frequency Ξ, which restricts the regularity to $\frac{1}{3} - \epsilon$ for some $\epsilon > 0$. However, as we now explain, combining the techniques in

[9] Here we have simplified the equations by combining the equations for the $\rho_I^{j\ell}$ and $z_I^{j\ell}$ from [Isett 2018] into one equation.

[Buckmaster et al. 2019a] and [Isett 2018] leads to a logarithmic improvement in the timescale of the gluing and hence a logarithmic improvement in the main estimate of the iteration.

The approach of [Buckmaster et al. 2019a] can be extended to any dimension using the antisymmetric potential[10] defined by

$$\psi_I^{ab} = \mathcal{B}^{ab}[y_I] := \Delta^{-1}(\nabla^a y_I^b - \nabla^b y_I^a),$$

which solves the Hodge system[11]

$$\nabla_a \psi^{ab} = y_I^b, \quad (\nabla \wedge \psi)^{abc} := \nabla^a \psi^{bc} - \nabla^b \psi^{ac} + \nabla^c \psi^{ab} = 0, \quad \int_{\mathbb{T}^3} \psi(x)\, dx = 0. \tag{113}$$

Using the antisymmetry of ψ_I^{ab}, one obtains the identity

$$y_I^i \nabla_i v^\ell = \nabla_a[\psi_I^{ai} \nabla_i v^\ell]. \tag{114}$$

Using (114) and $\psi_I^{ab} = \mathcal{B}^{ab}[y_I^j] = \mathcal{B}^{ab}\nabla_i[r_I^{ij}]$ can provide an alternative approach to treating the low-frequency part of the term $\mathcal{R}^{j\ell}[y_I^i \nabla_i v^b]$ in (111) and the analogous term in the pressure.

Towards improving the timescale of the gluing, apply (112) along with the calculus identity (which we express in both index and invariant notation)

$$\Delta \psi^{ab} = (\nabla \wedge [\nabla \neg \psi])^{ab} + (\nabla \neg [\nabla \wedge \psi])^{ab},$$
$$\nabla_i[\nabla^i \psi^{ab}] = (\nabla^a[\nabla_j \psi^{jb}] - \nabla^b[\nabla_j \psi^{ja}]) + \nabla_i[\nabla^i \psi^{ab} + \nabla^b \psi^{ia} + \nabla^a \psi^{bi}],$$

to derive the following equation for the potential ψ_I^{ab}, generalizing [Buckmaster et al. 2019a, Section 3.3]:[12]

$$\begin{aligned} \Delta[(\partial_t + v^i \nabla_i)\psi_I^{jk}] = \nabla_a \nabla_i[(\psi_I^{jk} \wedge \nabla^a) v^i] - \nabla^j \wedge [\nabla_a(\psi_I^{ai} \nabla_i v^k) + \nabla_a[y_I^a y_I^k] + \nabla_i R^{ik}] \\ + \nabla^j \wedge [\nabla_i[\nabla_a v^i \psi_I^{ak}]], \\ \psi_I^{jk} \wedge \nabla^a v^i := \psi_I^{jk} \nabla^a v^i - \psi_I^{ak} \nabla^j v^i + \psi_I^{aj} \nabla^k v^i. \end{aligned} \tag{115}$$

This derivation relies on (113) and $\nabla^j \nabla^k \bar{p}_I - \nabla^k \nabla^j \bar{p}_I = 0$, and uses that $\nabla_i v^i = 0$ to maintain the divergence form. The convention above for $\nabla^j \wedge$ applied to a vector field is $\nabla^j \wedge u^k := \nabla^j u^k - \nabla^k u^j$, while $(\psi_I^{jk} \wedge \nabla^a) v^i$ indicates a sum over cyclic permutations of jka in $\psi_I^{jk} \nabla^a v^i$.

One may now couple (115) to (111) while writing

$$\mathcal{R}^{j\ell}[y_I^i \nabla_i v^\ell] = \mathcal{R}^{j\ell} \nabla_a [\psi_I^{ai} \nabla_i v^\ell]$$

and similarly for the analogous term $\Delta^{-1}\nabla_\ell [y_I^i \nabla_i v^\ell]$ appearing in the pressure $\bar{p}_I$. By considering a weighted norm $\hbar(t) = \hbar_I(t)$ such that (setting $\widehat{N} := (e_v/e_R)^{1/2}$ and, for instance, $\alpha = \frac{1}{7}$)

$$\|\nabla_{\vec{a}} r_I\|_{C^0} + \|\nabla_{\vec{a}} \psi_I\|_{C^0} + \widehat{\Xi}^{-\alpha}(\|\nabla_{\vec{a}} r_I\|_{\dot{C}^\alpha} + \|\nabla_{\vec{a}} \psi_I\|_{\dot{C}^\alpha}) \le \widehat{N}^{(|\vec{a}|-2)_+} \Xi^{|\vec{a}|} (\Xi e_v^{1/2})^{-1} e_R \hbar(t),$$
$$\|\nabla_{\vec{a}} y_I\|_{C^0} + \widehat{\Xi}^{-\alpha} \|y_I\|_{\dot{C}^\alpha} \le \widehat{N}^{(|\vec{a}|-2)_+} \Xi^{|\vec{a}|} e_R^{1/2} \hbar(t) \qquad \text{for } 0 \le |\vec{a}| \le 3,$$

[10] We write ψ^{ab} to agree with the usual stream function ψ in dimension 2, which is related by $\psi^{ab} = \psi \epsilon^{ab}$, where the two-dimensional volume element ϵ^{ab} is the unique antisymmetric tensor with $\epsilon^{12} = 1$.

[11] We caution the reader that our normalizations for wedge products are taken to elucidate the present calculations, but do not agree with all standard normalizations, which can differ up to multiplication by constants.

[12] A slight departure from [Buckmaster et al. 2019a] is the isolation of quadratic terms of the form $y_I^j y_I^\ell$, which would be estimated jointly in $y_I^i \nabla_i v^\ell + \nabla_i (y_I^i y_I^\ell) = y_I^i \nabla_i u_I^\ell$ in the approach of that paper. The $y_I^j y_I^\ell$ terms are kept separate here in order to avoid a resulting additional derivative loss in the estimates.

and following the Littlewood–Paley approach to the gluing estimates in [Isett 2018], we obtain the bound

$$\hbar(t) \lesssim (\log \widehat{\Xi}) \Xi e_v^{1/2} \int_0^t (1 + \hbar(\tau))^2 \, d\tau. \tag{116}$$

The prefactor in (116) improves the analogous prefactor in [Isett 2018, Proposition 10.1] by a factor of $(\log \widehat{\Xi})^{-1}$, which thus improves the timescale θ by a logarithmic factor to $\theta \sim (\log \widehat{\Xi})^{-1} (\Xi e_v^{1/2})^{-1}$. What this improvement in timescale yields is that the time cutoff factors of η_I' in the terms of the form $\sim \eta_I' r_I^{j\ell}$ that compose the new stress error $\widetilde{R}$ have become smaller by a factor $(\log \widehat{\Xi})^{-1}$ in size, while the antidivergence terms $r_I^{j\ell}$ have increased in size by a factor of $(\log \widehat{\Xi})$ over the elongated time scale.

Although the estimate $\|\widetilde{R}\|_{C^0} \lesssim (\log \widehat{\Xi}) e_R$ on the stress does not improve, the estimate on the advective derivative improves logarithmically to

$$\|D_t \widetilde{R}\|_{C^0} \lesssim (\log \widehat{\Xi})^2 \Xi e_v^{1/2} e_R.$$

The bound (19) for the new frequency-energy levels in the main lemma similarly improves by one power of $\log \widehat{\Xi}$ to become

$$(\Xi', e_v', e_R') = \left(\widetilde{C} N \Xi, (\log \widehat{\Xi}) e_R, (\log \widehat{\Xi})^A \frac{e_v^{1/2} e_R^{1/2}}{N} \right), \quad A = \frac{3}{2}. \tag{117}$$

(One can alternatively pursue an approach closer to [Buckmaster et al. 2019a] wherein the equation for ψ_I^{ab} is coupled to the evolution equation[13] for a different, symmetric antidivergence such as

$$\tilde{r}_I^{j\ell} := \Delta^{-1} (\nabla^j y_I^\ell + \nabla^\ell y_I^j).$$

Implementing this alternative approach requires additional, sharper commutator estimates.)

The improvement in the power $A = \frac{3}{2}$ of $\log \widehat{\Xi}$ in (117) then leads to an improvement in the constant B in the leading-order term of the regularity estimate (110). Namely, repeating the analysis of Section 11 but with $A = \frac{3}{2}$ instead of $\frac{5}{2}$ improves the leading-order term in (102), which leads to a factor of $\left(\frac{1}{2}\gamma + \frac{4}{3}\right)$ in (109) in place of $\left(\frac{1}{2}\gamma + 2\right)$. After choosing $\gamma = \frac{8}{3}$ to optimize (109), one obtains a leading-order constant of $B = \frac{4}{3} = 2\left(\frac{2}{3}\right)$ instead of $B = 2\sqrt{\frac{2}{3}}$. Note that, with the improved constant, the function space implicitly defined by the estimate (110) is strictly contained in the one with the larger value of B, and the corresponding norms are not comparable to each other.

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[13] Using the evolution equation for the symmetric antidivergence is important to avoid an additional logarithmic loss that would be incurred from attempting to deduce estimates for $\tilde{r}_I$ directly from those for ψ_I.

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