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YANG LI

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DEGENERATION**



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YANG LI

We study polarised algebraic degenerations of Calabi–Yau manifolds. We prove a uniform Skoda-type estimate and a uniform L^∞ -estimate for the Calabi–Yau Kähler potentials.

1. Introduction

Let (Y, ω) be a compact Kähler manifold, and let $d\mu$ be a measure on Y . We say $(Y, \omega, d\mu)$ satisfies the *Skoda-type inequality*, if, for any Kähler potential $u \in \text{PSH}(Y, \omega)$ normalised to $\sup u = 0$,

$$\int_Y e^{-\alpha u} d\mu \leq A, \quad (1)$$

where α and A are independent of u . A prototype theorem is:

Theorem 1.1 [Tian 1987]. *On a fixed compact Kähler manifold (Y, ω) , the Skoda-type inequality holds for $d\mu = \omega^n$.*

Remark 1.2. Here the supremum of all such α is known as Tian’s alpha invariant and is important for existence questions of Kähler–Einstein metrics.

We are interested in keeping track of the constants α and A as $(Y, \omega, d\mu)$ varies. The main theme of this paper is that oftentimes the Skoda constants can be chosen uniformly for quite flexible choices of probability measures $d\mu$, even when the complex structure degenerates severely. In the literature, α is much studied, see [Guedj and Zeriahi 2005; Tian 1987], and a very recent preprint [Di Nezza et al. 2023] made aware to the author after the completion of this work contains a uniform estimate for both α and A in the related context of Kähler–Einstein manifolds.

Our main application is to algebraic degenerations of Calabi–Yau manifolds. We work over \mathbb{C} . Let S be a smooth affine algebraic curve, with a point $0 \in S$. An *algebraic degeneration family* is given by a submersive projective morphism $\pi : X \rightarrow S \setminus \{0\}$ with smooth connected n -dimensional fibres X_t for $t \in S \setminus \{0\}$. A *polarisation* is given by an ample line bundle L over X ; the sections of a sufficiently high power of L induces an embedding $X \rightarrow \mathbb{C}\mathbb{P}^N$, and hence a Fubini–Study metric ω_X on (X, L) . For $0 < |t| \ll 1$, a fixed choice of ω_X induces *rescaled* background metrics $\omega_t = \omega_X|_{X_t}/|\log|t||$ on X_t in the class $c_1(L)/|\log|t||$.

A *model* of X is a normal flat projective S -scheme \mathcal{X} which agrees with $\pi : X \rightarrow S \setminus \{0\}$ over the punctured curve. It is called a *semistable snc model* if \mathcal{X} is smooth, the central fibre over $0 \in S$ is *reduced*

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and is a simple normal crossing (snc) divisor in \mathcal{X} . By the semistable reduction theorem [Kempf et al. 1973, Chapter 2], after finite base change to another smooth algebraic curve S' , we can always find some semistable snc model for the degeneration family $X \times_S (S' \setminus \{0\})$. Everything here is quasiprojective.

We say the degeneration family is *Calabi–Yau* if there is a trivialising section Ω of the canonical bundle K_X . Over a small disc \mathbb{D}_t around $0 \in S$, this induces holomorphic volume forms Ω_t on X_t via $\Omega = dt \wedge \Omega_t$. The *normalised Calabi–Yau measure* on X_t is the probability measure

$$d\mu_t = \frac{\Omega_t \wedge \bar{\Omega}_t}{\int_{X_t} \Omega_t \wedge \bar{\Omega}_t}. \tag{2}$$

We are ready to state our main result.

Theorem 1.3 (uniform Skoda estimate). *Given a polarised algebraic Calabi–Yau degeneration family $\pi : X \rightarrow S \setminus \{0\}$ as above, there are uniform positive constants α and A independent of t for $0 < |t| \ll 1$ such that, for the normalised Calabi–Yau measures $d\mu_t$,*

$$\int_{X_t} e^{-\alpha u} d\mu_t \leq A \quad \text{for all } u \in \text{PSH}(X_t, \omega_t) \text{ with } \sup_{X_t} u = 0.$$

This is proved by reducing to the semistable snc model case and proving a general Skoda-type estimate there (see Theorem 2.9). A major consequence, readily reaped using Kołodziej’s estimate (see Theorem 3.1), follows.

Theorem 1.4 (uniform L^∞ -estimate). *Let ϕ_t be the Kähler potential of the Calabi–Yau metric in the class $(X_t, [\omega_t])$, namely*

$$\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\phi)^n}{\int_{X_t} \omega_t^n} = d\mu_t, \quad \sup_{X_t} \phi_t = 0.$$

Then $\|\phi_t\|_{L^\infty} \leq C$ independently of t for $0 < |t| \ll 1$.

Remark 1.5. Applications of pluripotential theory to Calabi–Yau metrics when the Kähler class is degenerating can be found in [Eyssidieux et al. 2008], which is used further in [Tosatti 2010]. Our main results generalise certain aspects of [Li 2022] which focuses on degenerating projective hypersurfaces near the large complex structure limit.

2. Uniform Skoda inequality

We work in the context of semistable simple normal crossing (snc) models. Concretely, let $\pi : \mathcal{X} \rightarrow \mathbb{D}_t$ be a flat projective family of n -dimensional varieties over a small disc \mathbb{D}_t such that the total space \mathcal{X} is smooth, π is a submersion over the punctured disc with connected fibres, and the central fibre X_0 is *reduced* and is an snc divisor in \mathcal{X} . Denote the components of X_0 by E_i with $i \in I$. We equip \mathcal{X} with a fixed background Kähler metric $\omega_{\mathcal{X}}$, inducing a distance function $d_{\omega_{\mathcal{X}}}$. This induces a family of *rescaled* Kähler metrics $\omega_t = \omega_{\mathcal{X}}|_{X_t} / |\log|t||$. We shall derive a uniform Skoda-type estimate (1) for $(X_t, \omega_t, d\mu_t)$, where $d\mu_t$ belongs to a natural class of measures. The main result is Theorem 2.9.

Quantitative stratification and good test functions. There is a *quantitative stratification* on any smooth fibre X_t induced by the intersection pattern of E_i : for $J \subset I$ such that $E_J = \bigcap_{i \in J} E_i \neq \emptyset$, the corresponding stratum is

$$E_J^0 = \{x \in X_t \mid d_{\omega_{\mathcal{X}}}(x, E_J) \lesssim \epsilon\} \setminus \{x \in X_t \mid d_{\omega_{\mathcal{X}}}(x, E_{J'}) \lesssim \epsilon \text{ for some } J' \supsetneq J\},$$

namely a small “ ϵ -tubular neighbourhood” of E_J minus the deeper strata. For $J = \{i\}$ we write $E_i^0 = E_{\{i\}}^0$. Here the disc \mathbb{D}_t and the small parameter $\epsilon \ll 1$ can be shrunk for convenience; the essential thing is that all parameters should be independent of the coordinate t .

It is useful to introduce local coordinates $\{z_i\}_0^n$ around $E_J \subset \mathcal{X}$ such that z_0, \dots, z_p with $p = |J| - 1$ are the local defining equations of E_j for $j \in J$, and locally the fibration map is $t = z_0 \cdots z_p$. Then up to uniform equivalence, locally

$$\omega_{\mathcal{X}} \sim \sum_0^n \sqrt{-1} dz_i \wedge d\bar{z}_i.$$

The rest of this section is devoted to the construction of good test functions. Given any of these divisors E_0 , we can find a nonnegative function $h = h_{E_0}$ on \mathcal{X} such that:

- In the local charts near E_0 with z_0 being the defining function for E_0 ,

$$h = |z_0|^2 \tilde{h}(z_0, \dots, z_n)$$

for some positive smooth function \tilde{h} .

- Away from E_0 , the function h is comparable to 1.

We observe that:

- The form $\partial \bar{\partial} \log h = \partial \bar{\partial} \log \tilde{h}$ extends smoothly.
- For $|t|^2 \ll h \lesssim \delta \ll 1$ inside X_t , so that $|z_0| \gg |t|$, by a local calculation near E_J with $0 \in J$,

$$\begin{aligned} \sqrt{-1} \partial \log h \wedge \bar{\partial} \log h \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1} &\geq \frac{\sqrt{-1}}{2|z_0|^2} dz_0 \wedge d\bar{z}_0 \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1} \gtrsim \min \left\{ \frac{1}{|z_0|^2}, \max_{1 \leq i \leq p} |z_i|^{-2} \right\} \omega_{\mathcal{X}}|_{X_t}^n \\ &\gtrsim \min \left\{ \frac{1}{h}, h^{1/p} |t|^{-2/p} \right\} \omega_{\mathcal{X}}|_{X_t}^n \gtrsim \min \left\{ \frac{1}{h}, h^{1/n} |t|^{-2/n} \right\} \omega_{\mathcal{X}}|_{X_t}^n. \end{aligned}$$

Here in the first inequality we need to fix $\delta \ll 1$ so that the effect of $\partial \log h$ is dominated by $d \log z_0$. The second inequality uses that, for $1 \leq k \leq p$, the volume forms on X_t satisfy

$$\frac{1}{|z_0|^2} dz_0 \wedge d\bar{z}_0 \wedge \prod_{j \neq k, 1 \leq j \leq n} \sqrt{-1} dz_j \wedge d\bar{z}_j \sim \frac{1}{|z_k|^2} \prod_{1 \leq j \leq n} \sqrt{-1} dz_j \wedge d\bar{z}_j,$$

and the third inequality uses $|t| = |z_0 \cdots z_p| \sim h^{1/2} |z_1 \cdots z_p|$.

- On X_t , we have that $h \gtrsim |t|^2$. The region $\{|t|^2 \sim h\} \subset X_t$ can be identified as E_0^0 , namely the vicinity of E_0 away from deeper strata. Here

$$\sqrt{-1} \partial \log h \wedge \bar{\partial} \log h \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1} \geq 0 \quad \text{and} \quad \sqrt{-1} \partial \bar{\partial} \log h \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1} \gtrsim -\omega_{\mathcal{X}}|_{X_t}^n.$$

Lemma 2.1 (good test functions). *Given the divisor E_0 , we can choose a C^2 test function v on X_t such that the following hold uniformly for small $t \neq 0$:*

- v is zero for $h \geq \delta$.
- Globally, $0 \leq v \leq -\log |t|$.
- For any divisor E_j intersecting E_0 , there is a subset of E_j^0 with measure at least C_2 on which $\sqrt{-1}\partial\bar{\partial}v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1} \geq C_3\omega_{\mathcal{X}}|_{X_t}^n$.
- For $C_4|t|^2 \leq h \leq \delta$, the form $\sqrt{-1}\partial\bar{\partial}v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1}$ is greater than or equal to 0.
- For $h \leq C_4|t|^2$, the form $\sqrt{-1}\partial\bar{\partial}v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1}$ is greater than or equal to $-C_5\omega_{\mathcal{X}}|_{X_t}^n$.

Proof. We seek the test function in the form $v = \Phi \circ \log h$ for some convex, nonincreasing, nonnegative C^2 -function Φ . Compute

$$\partial\bar{\partial}v = \Phi''\partial \log h \wedge \bar{\partial} \log h + \Phi'(\partial\bar{\partial} \log \tilde{h}),$$

and using the properties of h above,

$$\sqrt{-1}\partial\bar{\partial}v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1} \geq \begin{cases} (\Phi''C'_1 \min\{\frac{1}{h}, h^{1/n}|t|^{-2/n}\} + \Phi'C'_2)\omega_{\mathcal{X}}|_{X_t}^n, & |t|^2 \lesssim h \leq \delta, \\ C'_3\Phi'\omega_{\mathcal{X}}|_{X_t}^n, & h \lesssim |t|^2. \end{cases}$$

To satisfy our conditions on v , it is enough to have:

- $\Phi(x) = 0$ for $x \geq \log \delta$.
- $|\Phi'(x)| \lesssim 1$ for $2 \log |t| \lesssim x \leq \log \delta$.
- For $h = e^x \leq \delta$,

$$-\frac{d}{dx} \log |\Phi'| = \frac{\Phi''}{|\Phi'|} \geq C'_4 \max\{h, h^{-1/n}|t|^{2/n}\},$$

where $C'_4 > C'_2/C'_1$. Moreover, for $x < \delta$, we need $\Phi' < 0$, so that $\sqrt{-1}\partial\bar{\partial}v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1}$ has some strict positivity for $\frac{1}{2}\delta < h < \delta$. Notice convexity of Φ is a consequence of these conditions.

To construct such a Φ , we can prescribe the behaviour near $x = \log \delta$ by

$$\Phi'(x) = -e^{1/(x-\log \delta)}$$

for $x < \log \delta$ and match this with a solution to

$$-\frac{d}{dx} \log |\Phi'| = C'_4 \max\{e^x, e^{-x/n}|t|^{2/n}\}, \quad x < \log \delta,$$

for some large enough C'_4 such that Φ' remains C^1 at the matching point. Integration shows that $|\Phi'|$ remains uniformly bounded at $h \sim |t|^2$, or equivalently $x \sim 2 \log |t|$. □

Convexity. Consider $u \in \text{PSH}(X_t, \omega_t)$ normalised to $\sup_{X_t} u = 0$. Equivalently, we can cover X_t by a bounded number of charts as before and use the local potentials of $\omega_{\mathcal{X}}$ to represent u as a collection of local plurisubharmonic (psh) functions $\{u_\beta\}$ with $|u_\beta - u| \leq C$.

Lemma 2.2 (convexity). *Let ϕ be any psh function on the open subset of*

$$\{1 < |z_i| < \Lambda, i = 1, \dots, p; |z_k| < 1, k = p + 1, \dots, n\} \subset (\mathbb{C}^*)^p \times \mathbb{C}^{n-p}.$$

Then the function

$$\bar{\phi}(x_1, \dots, x_p) = \frac{1}{(2\pi)^n} \int_{\{|z_k| \leq 1 \forall k > p\}} \prod_{p+1}^n \sqrt{-1} dz_k \wedge d\bar{z}_k \int_{T^p} \phi(e^{x_1+i\theta_1}, \dots, e^{x_p+i\theta_p}) d\theta_1 \dots d\theta_p$$

is convex.

Proof. For any choice of θ_i , the function $\phi(z_1 e^{i\theta_1}, \dots, z_p e^{i\theta_p}, z_{p+1}, \dots, z_n)$ is psh, since the T^p -action on $(\mathbb{C}^*)^p$ is holomorphic. Thus the average function $\bar{\phi}$ is also psh as a function of z_1, \dots, z_p . Any T^p -invariant psh function must be convex in the log coordinates because, for $x_i = \log |z_i|$,

$$\sqrt{-1} \partial \bar{\partial} \bar{\phi} = \frac{1}{4} \sum \frac{\partial^2 \bar{\phi}}{\partial x_i \partial x_j} \sqrt{-1} d \log z_i \wedge d \overline{\log z_j} \geq 0. \quad \square$$

Harnack-type inequality.

Lemma 2.3 (almost maximum on top strata I). *For $u \in \text{PSH}(X_t, \omega_t)$ normalised to $\sup_{X_t} u = 0$, there is some $i \in I$ such that*

$$\sup_{E_i^0} u \geq -C, \quad \int_{E_i^0} u \omega_{X_t}|_{E_i^0} \geq -C'.$$

Proof. Let the global maximum of u be achieved at $q_0 \in E_j^0$, and denote the local potential of u by u_β . Without loss of generality, $u_\beta \leq 0$. We have $u_\beta(q_0) \geq -C$ since $|u - u_\beta| \leq C$. Applying the mean value inequality around q_0 , we find that the local average function \bar{u}_β produced in Lemma 2.2 satisfies $\sup \bar{u}_\beta \geq -C$ for another uniform constant C . By the convexity of \bar{u}_β , its supremum is almost achieved at the boundary of the chart, which is contained in a union of less deep strata $E_{J'}^0$, with $J' \subsetneq J$. Thus we can find a point q' with $u(q') \geq -C$ that belongs to a less deep stratum; an induction shows that there is some $i \in I$ such that $\sup_{E_i^0} u \geq -C$.

For the L^1 -bound we recall the following Harnack inequality argument. Suppose a coordinate ball $B(q, 3R)$ is contained in a local chart in a small neighbourhood of E_i^0 . Applying the mean value inequality to the local psh function associated to u , we see, for $y \in B(q, R)$, that

$$u(y) \leq C + \int_{B(y, 2R)} u \lesssim 1 + \int_{B(q, R)} u.$$

Hence the Harnack inequality yields

$$\int_{B(q, R)} |u| \lesssim 1 + \inf_{B(q, R)} (-u).$$

Applying this to a chain of balls connecting any two points in E_i^0 gives the L^1 -bound $\int_{E_i^0} u \omega_{X_t}|_{E_i^0} \geq -C'$; the bound is uniform because the number of balls involved in the chain can be controlled independently of t . □

Proposition 2.4 (almost maximum on top strata II). *There is a uniform lower bound for all $|t| \ll 1$ and all $i \in I$:*

$$\sup_{E_i^0} u \geq -C, \quad \int_{E_i^0} u \omega_{\mathcal{X}}|_{X_i}^n \geq -C'. \tag{3}$$

Proof. The L^1 -estimate follows from the supremum estimate as above, so the real problem is to transfer bounds between different E_i^0 . This is nontrivial because the necks connecting E_i^0 with each other are highly degenerate.

Given one divisor E_0 such that $\int_{E_0^0} u \omega_{\mathcal{X}}|_{X_t}^n \geq -C$, we produce a good test function v by [Lemma 2.1](#). Integrating by parts,

$$\int_{X_t} v \sqrt{-1} \partial \bar{\partial} u \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1} = \int_{X_t} u \sqrt{-1} \partial \bar{\partial} v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1}.$$

The left-hand side is the difference between

$$\int_{X_t} v(\omega_t + \sqrt{-1} \partial \bar{\partial} u) \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1} \quad \text{and} \quad \int_{X_t} v \omega_t \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1},$$

and since $-\log |t| \gtrsim v \geq 0$, both terms are bounded between 0 and C . Thus

$$\left| \int_{X_t} u \sqrt{-1} \partial \bar{\partial} v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1} \right| \leq C.$$

Now the form $\sqrt{-1} \partial \bar{\partial} v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1}$ can only be negative on $\{h \sim |t|^2\} = E_0^0$ and is bounded below by $-C \omega_{\mathcal{X}}|_{X_t}^n$. Thus the positive part of the signed measure $u \sqrt{-1} \partial \bar{\partial} v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1}$ has total mass controlled by $\int_{E_0^0} |u| \omega_{\mathcal{X}}|_{X_t}^n \leq C$. Consequently, the negative part of the signed measure must also have total mass $\leq C$.

By construction, for any divisor E_j intersecting E_0 , there is a nontrivial amount of $\sqrt{-1} \partial \bar{\partial} v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1}$ -measure inside E_j^0 . This forces $\sup_{E_j^0} u \geq -C$. To summarise, we have transferred the supremum bound from E_0^0 to any E_j^0 with $E_j \cap E_0 \neq \emptyset$. Since the central fibre X_0 is connected, in at most $|I|$ steps this supremum bound is transferred to all E_i^0 with $i \in I$. \square

Remark 2.5. This proof is inspired by the intersection-theoretic argument of [\[Boucksom et al. 2016, Section 6.1\]](#), which can be viewed as a non-Archimedean analogue.

Local L^1 estimate. For a given local chart on E_j^0 with \mathbb{C}^* -coordinates z_1, \dots, z_p and \mathbb{C} -coordinates z_{p+1}, \dots, z_n , and a point q therein, we shall refer to the subregion

$$\left\{ \frac{1}{2} |z_i(q)| \lesssim |z_i| \lesssim 2 |z_i(q)|, \quad 1 \leq i \leq p \right\}$$

as a *log scale*.

Lemma 2.6 (local L^1 -estimate). *Within every log scale there is a uniform bound on the L^1 -average integral:*

$$\int_{\text{loc}} |u| \prod_1^p \sqrt{-1} d \log z_i \wedge d \log \bar{z}_i \wedge \prod_{p+1}^n \sqrt{-1} dz_k \wedge d\bar{z}_k \leq C.$$

Proof. For $p = 0$ this follows from [Proposition 2.4](#). Given a depth p chart, we consider the local psh function u_β associated to u and produce the convex average function \bar{u}_β as in [Lemma 2.2](#). We claim $|\bar{u}_\beta| \leq C$ in the chart. The upper bound holds as u_β is bounded above, and by convexity it suffices to achieve a lower bound at the barycentre of the simplex. However, when we blow up the depth p intersection E_J , there is a new vertex in the dual complex corresponding to the new divisor component in the central fibre. The dual complex is subdivided, and the new vertex is situated at the original barycentre. The same argument in [Proposition 2.4](#) then achieves a lower bound on the local average of the Kähler potential near this new divisor, which amounts to the desired lower bound on \bar{u}_β at the barycentre, whence the claim follows.

Within any log scale, by construction the local average $\int_{\text{loc}} (u_\beta - \bar{u}_\beta)$ equals zero. But $u_\beta \leq C$ since $u \leq 0$, and hence

$$\int_{\text{loc}} |u_\beta - \bar{u}_\beta| \lesssim \int_{\text{loc}} (u_\beta - \bar{u}_\beta)_+ \leq C.$$

Using $|u - u_\beta| \leq C$, we conclude the local L^1 -estimate on u . □

Local Skoda estimate. We recall a basic version of the *Skoda inequality*:

Proposition 2.7 [[Zeriahi 2001](#), Theorem 3.1]. *If ϕ is psh on $B_2 \subset \mathbb{C}^n$, with $\int_{B_2} |\phi| \omega_E^n \leq 1$ with respect to the standard Euclidean metric ω_E , then there are dimensional constants α and C such that*

$$\int_{B_1} e^{-\alpha\phi} \omega_E^n \leq C.$$

Applying this along with [Lemma 2.6](#), we get the following corollary.

Corollary 2.8 (local Skoda estimate). *Within every log scale, there are uniform positive constants α and C such that*

$$\int_{\text{loc}} e^{-\alpha u} \prod_1^p \sqrt{-1} d \log z_i \wedge d \log \bar{z}_i \wedge \prod_{p+1}^n \sqrt{-1} dz_k \wedge d\bar{z}_k \leq C.$$

Uniform global Skoda estimate. We are interested in the following class of measures, motivated by Calabi–Yau measures (see [Section 3](#)). Let a_i be nonnegative real numbers assigned to $i \in I$, with $\min a_i = 0$. Let

$$m = \max\{|J| - 1 : E_J \neq \emptyset, a_i = 0 \text{ for } i \in J\}.$$

We say the measures $d\mu_t$ on X_t satisfy a *uniform upper bound of class (a_i)* if, on the local charts of each E_J^0 ,

$$d\mu_t \leq \frac{C}{|\log|t||^m} |z_0|^{2a_0} \cdots |z_p|^{2a_p} \prod_1^p \sqrt{-1} d \log z_i \wedge d \log \bar{z}_i \wedge \prod_{p+1}^n \sqrt{-1} dz_k \wedge d\bar{z}_k. \tag{4}$$

The normalisation factor ensures that $\int_{X_t} d\mu_t \leq C$ independently of t by a straightforward local calculation.

Theorem 2.9 (uniform Skoda estimate). *Suppose the measures $d\mu_t$ on X_t satisfy a uniform upper bound of class (a_i) . Then there are uniform positive constants α and A such that*

$$\int_{X_t} e^{-\alpha u} d\mu_t \leq A \quad \text{for all } u \in \text{PSH}(X_t, \omega_t) \text{ with } \sup_{X_t} u = 0.$$

Proof. We choose the charts so that each point on X_t is covered by $\leq C$ log scales. Summing over the local Skoda estimates from all log scales, $\int_{X_t} e^{-\alpha u} d\mu_t$ is bounded by

$$\frac{C}{|\log|t||^m} \sum_{\text{log scales}} \int_{\text{loc}} |z_0|^{2a_0} \cdots |z_p|^{2a_p} \prod_1^p \sqrt{-1} d \log z_i \wedge d \log \bar{z}_i \wedge \prod_{p+1}^n \sqrt{-1} dz_k \wedge d\bar{z}_k \leq C. \quad \square$$

3. Application to Calabi–Yau degeneration

We work in the setting of polarised algebraic degeneration of Calabi–Yau manifolds, as in the [Introduction](#).

Calabi–Yau measure. The Calabi–Yau measure (2) is studied thoroughly in [\[Boucksom and Jonsson 2017\]](#), but it is illustrative to recall it explicitly on a semistable snc model \mathcal{X} . The discussion is local on the base, and we will follow the notations of [Section 2](#), e.g., the components of the central fibre are denoted by E_i for $i \in I$.

The canonical divisor $K_{\mathcal{X}} = \sum_i a_i E_i$ is supported on the central fibre, since K_X is trivialised. Multiplying Ω by a power of t , which does not change $d\mu_t$, we may assume $\min a_i = 0$. In the local coordinates around E_J away from the deeper strata,

$$\Omega = f_J \prod_0^p z_i^{a_i} dz_i \wedge \prod_{p+1}^n dz_j$$

for some nowhere-vanishing local holomorphic function f_J . Since $t = z_0 \cdots z_p$,

$$\Omega_t = f_J z_0^{a_0} \cdots z_p^{a_p} \prod_1^p d \log z_i \wedge \prod_{p+1}^n dz_j,$$

and hence

$$\sqrt{-1}^{n^2} \Omega_t \wedge \bar{\Omega}_t = |f_J|^2 |z_0|^{2a_0} \cdots |z_p|^{2a_p} \prod_1^p \sqrt{-1} d \log z_i \wedge d \overline{\log z}_i \wedge \prod_{p+1}^n \sqrt{-1} dz_j \wedge d\bar{z}_j.$$

The total measure $\int_{X_t} \sqrt{-1}^{n^2} \Omega_t \wedge \bar{\Omega}_t$ is of the order $O(|\log|t||^m)$, where

$$m = \max\{|J| - 1 : E_J \neq \emptyset, a_i = 0 \text{ for } i \in J\}.$$

Thus $d\mu_t$ satisfies a uniform upper bound of class (a_i) ; see (4).

Uniform Skoda estimate. We now prove the main [Theorem 1.3](#).

Proof. First we observe that the choice of the Fubini–Study metric ω_X is immaterial. Given any two choices, the relative Kähler potential between them is bounded by $O(|\log|t||)$ for $0 < |t| \ll 1$ because the pole order of a section near $t = 0$ must be finite. Thus the relative Kähler potential between two choices of ω_t is bounded by $O(1)$ independent of t , which affects the Skoda constant A but not its uniform nature.

We now pass to a finite base change and find a semistable reduction. The Calabi–Yau measure $d\mu_t$ on X_t is independent of the parametrisation of the base and is preserved under finite base change. Thus it is enough to prove it assuming ω_X agrees with a smooth Kähler metric on a semistable snc model \mathcal{X} ; this is a special case of [Theorem 2.9](#). \square

Uniform L^∞ -estimate. We recall the following result proved using Kołodziej’s pluripotential-theoretic methods (see [[Li 2022](#), Section 2.2] for an exposition based on [[Eyssidieux et al. 2009; 2008](#)]):

Theorem 3.1. *Let (Y, ω) be a compact Kähler manifold, and let the Kähler potential ϕ solve the complex Monge–Ampère equation*

$$\frac{(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n}{\int_Y \omega^n} = d\mu, \quad \sup \phi = 0.$$

Assume there are positive constants α and A such that the Skoda-type estimate (1) holds for $(Y, \omega, d\mu)$:

$$\int_Y e^{-\alpha u} d\mu \leq A \quad \text{for all } u \in \text{PSH}(Y, \omega) \text{ with } \sup_Y u = 0.$$

Then $\|\phi\|_{C^0} \leq C(n, \alpha, A)$.

The uniform L^∞ -estimate for the Calabi–Yau potentials in [Theorem 1.4](#) is an immediate consequence.

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YANG LI: yangmit@mit.edu

Massachusetts Institute of Technology, Cambridge, MA, United States

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ANALYSIS & PDE

Volume 17 No. 7 2024

Uniform Skoda integrability and Calabi–Yau degeneration YANG LI	2247
Unique continuation for the heat operator with potentials in weak spaces EUNHEE JEONG, SANGHYUK LEE and JAEHYEON RYU	2257
Nonnegative Ricci curvature and minimal graphs with linear growth GIULIO COLOMBO, EDDYGLEDSON S. GAMA, LUCIANO MARI and MARCO RIGOLI	2275
Nonlinear periodic waves on the Einstein cylinder ATHANASIOS CHATZIKALEAS and JACQUES SMULEVICI	2311
Host–Kra factors for $\bigoplus_{p \in P} \mathbb{Z}/p\mathbb{Z}$ actions and finite-dimensional nilpotent systems OR SHALOM	2379
A fast point charge interacting with the screened Vlasov–Poisson system RICHARD M. HÖFER and RAPHAEL WINTER	2451
Haagerup’s phase transition at polydisc slicing GIORGOS CHASAPIS, SALIL SINGH and TOMASZ TKOCZ	2509
A substitute for Kazhdan’s property (T) for universal nonlattices NARUTAKA OZAWA	2541
Trigonometric chaos and X_p inequalities, I: Balanced Fourier truncations over discrete groups ANTONIO ISMAEL CANO-MÁRMOL, JOSÉ M. CONDE-ALONSO and JAVIER PARCET	2561
Beurling–Carleson sets, inner functions and a semilinear equation OLEG IVRII and ARTUR NICOLAU	2585