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UNIQUE CONTINUATION FOR THE HEAT OPERATOR WITH POTENTIALS IN WEAK SPACES

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We prove the strong unique continuation property for the differential inequality

$$|(\partial_t + \Delta)u(x,t)| \le V(x,t)|u(x,t)|,$$

with V contained in weak spaces. In particular, we establish the strong unique continuation property for $V \in L_t^\infty L_x^{d/2,\infty}$, which has been left open since the works of Escauriaza (2000) and Escauriaza and Vega (2001). Our results are consequences of the Carleman estimates for the heat operator in the Lorentz spaces.

1. Introduction

We consider the differential inequality

$$|(\partial_t + \Delta)u(x,t)| \le |V(x,t)u(x,t)|, \quad (x,t) \in \mathbb{R}^d \times (0,T). \tag{1-1}$$

For a differential operator P on a domain Ω, the strong unique continuation property (abbreviated *sucp* hereafter) for $|Pu| \le |Vu|$ means that a nontrivial solution u to $|Pu| \le |Vu|$ cannot vanish to infinite order (in a suitable sense) at any point. The *sucp* for second-order parabolic operators has been studied by many authors; see [Banerjee and Manna 2020; Chen 1998; Escauriaza 2000; Escauriaza and Fernández 2003; Escauriaza and Vega 2001; Fernández 2003; Koch and Tataru 2009; Lin 1990; Poon 1996; Sogge 1990]. In particular, the study of *sucp* for the heat operator with time-dependent potentials goes back to Poon [1996] and Chen [1998], who considered bounded potentials. Escauriaza [2000] and Escauriaza and Vega [2001] extended the results to unbounded potentials V under the parabolic vanishing condition: for a given $\delta \in (0,1)$ and any $k \in \mathbb{N}$, there is a constant C_k such that

$$|u(x,t)| \le C_k(|x| + \sqrt{t})^k e^{(1-\delta)|x|^2/(8t)}, \quad (x,t) \in \mathbb{R}^d \times (0,T). \tag{1-2}$$

The growth condition at infinity is necessary since there exists a nonzero solution u to $(\partial_t + \Delta)u = 0$ such that u vanishes to infinite order in the space-time variables at any point $(x,0)$, $x \in \mathbb{R}^d$; see, for example, [Escauriaza 2000; John 1971].

The *sucp* for the Laplacian $-\Delta$ is better understood. Since the pioneering work of [Carleman 1939], most subsequent results were obtained using the Carleman weighted inequality. In particular, [Jerison and Kenig 1985] proved the *sucp* for the Laplacian, with $V \in L^{d/2}_{\text{loc}}$, $d \ge 3$. Their result was extended by

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Stein in the Appendix of [Jerison and Kenig 1985] to potentials $V \in L^{d/2,\infty}$ under the assumption that $\|V\|_{L^{d/2,\infty}}$ is small enough. Here, $\|\cdot\|_{p,r}$ denotes the Lorentz space norm

$$\|f\|_{L^{p,r}} = \left(\frac{r}{p}\int_0^\infty t^{r/p-1}(f^*(t))^r\,dt\right)^{1/r} < \infty$$

for $r \neq \infty$, and $\|f\|_{L^{p,\infty}} = \sup_{t>0} t^{1/p} f^*(t) < \infty$ for $r = \infty$, where f^* is the decreasing rearrangement of f on $\mathbb{R}^d$; for example, see [Stein and Weiss 1971]. Later, [Wolff 1992] showed that the smallness assumption is indispensable if $V \in L^{d/2,\infty}$. By the aforementioned results due to Escauriaza [2000] and Escauriaza and Vega [2001], the *sucp* for (1-1) is known when $t^{1-d/(2\mathfrak{r})-1/\mathfrak{s}} V(x,t) \in L^{\mathfrak{s}}_{t,\mathrm{loc}} L^{\mathfrak{r}}_x$ and $\mathfrak{r}$ and $\mathfrak{s}$ satisfy

$$\frac{d}{2\mathfrak{r}} + \frac{1}{\mathfrak{s}} \le 1, \quad 1 \le \mathfrak{r}, \mathfrak{s} \le \infty. \tag{1-3}$$

However, in view of those results concerning the (abovementioned) *sucp* for the Laplacian, it seems natural to expect that the class of potentials for which the *sucp* for (1-1) holds can be further expanded to certain weak spaces.

In this paper, we extend the results in [Escauriaza 2000; Escauriaza and Vega 2001] to a larger class of potentials, that is to say,

$$t^{1-d/(2\mathfrak{r})-1/\mathfrak{s}} V(x,t) \in L^{\mathfrak{s}}_{t,\mathrm{loc}} L^{\mathfrak{r},\infty}_x, \quad \frac{d}{2\mathfrak{r}} + \frac{1}{\mathfrak{s}} \le 1, \ \frac{d}{2} \le \mathfrak{r} \le \infty.$$

As in the Appendix of [Jerison and Kenig 1985], our result is a consequence of new Carleman estimates for the heat operator in the Lorentz spaces.

Carleman estimate. Write $L^s_t L^{q,b}_x = L^s_t(\mathbb{R}_+; L^{q,b}_x(\mathbb{R}^d))$. We consider the Carleman inequality for the heat operator of the form

$$\|t^{-\alpha} e^{-|x|^2/(8t)} g\|_{L^s_t L^{q,b}_x} \le C \|t^{-\alpha+1-(d/2)(1/p-1/q)-(1/r-1/s)} e^{-|x|^2/(8t)} (\Delta + \partial_t) g\|_{L^r_t L^{p,a}_x}, \tag{1-4}$$

with C independent of α, which holds for $g \in C^\infty_c(\mathbb{R}^{d+1} \setminus \{(0,0)\})$ under a suitable condition on the exponents α, p, q, r, s, a, and b. For $\alpha \in \mathbb{R}$, we set

$$\beta = \beta(\alpha, q, s) = 2\alpha - \frac{d}{q} - \frac{2}{s}.$$

Estimate (1-4) was formerly considered with $p = a$ and $q = b$. It was Escauriaza [2000] who first obtained (1-4) for some $p = a$, $q = b$, r, and s. More precisely, he showed that (1-4) holds with the Lebesgue spaces (i.e., $a = p$ and $b = q$) for p, q satisfying $q = p'$ and $0 \le 1/p - 1/q < 2/d$ when $d \ge 2$, and $0 \le 1/p - 1/q \le 1$ when $d = 1$, provided that

$$\operatorname{dist}(\beta, \mathbb{N}_0) \ge c$$

for some $c > 0$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Subsequently, estimate (1-4) was extended by [Escauriaza and Vega 2001] to the exponents p, q which lie outside of the line of duality. They obtained (1-4) for

$$\frac{2d}{d+2} \le p \le 2 \le q \le \frac{2d}{d-2} \quad \text{when } d \ge 3,$$

and for $1 \le p \le 2 \le q \le \infty$ except $(p,q,d) \neq (1,\infty,2)$ when $d = 1, 2$.

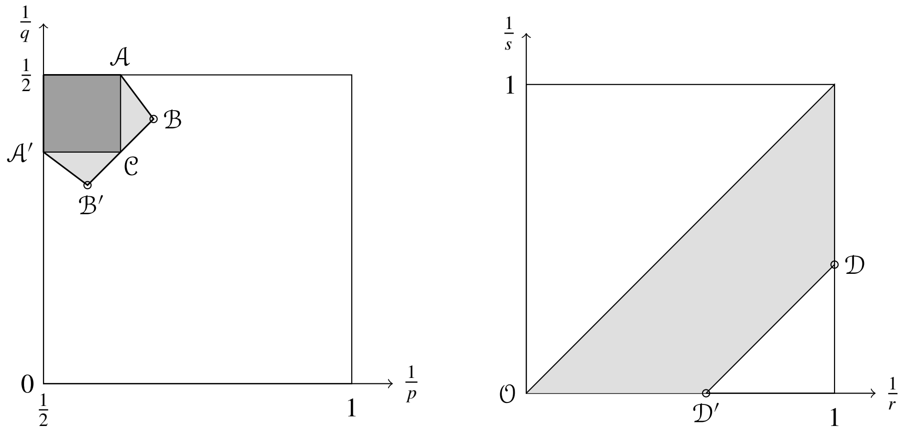

Figure 1. The regions of (p, q) and (r, s) for which (1-4) holds when $d \geq 3$: the dark gray square in the left figure represents the earlier result due to [Escauriaza and Vega 2001], and the light gray region represents the newly extended range. In the right figure, $\mathcal{O} = (0, 0)$ and $\mathcal{D} = \left(1, \frac{1}{2}d(1/p - 1/q)\right)$.

We extend the previously known results not only to Lorentz spaces but also on a wider range of exponents p and q. To present our result, for $d \geq 3$ we define $\mathcal{A} = \mathcal{A}(d),\ \mathcal{B} = \mathcal{B}(d),\ \mathcal{C} = \mathcal{C}(d) \in \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right]$ by

$$\mathcal{A} = \left(\frac{d+2}{2d}, \frac{1}{2}\right), \qquad \mathcal{B} = \left(\frac{d^2+2d-4}{2d(d-1)}, \frac{d-2}{2(d-1)}\right), \qquad \mathcal{C} = \left(\frac{d+2}{2d}, \frac{d-2}{2d}\right).$$

By $\mathfrak{T}$ we denote the closed pentagon with vertices $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}'$, and $\mathcal{A}'$ from which the two vertices $\mathcal{B}$ and $\mathcal{B}'$ are removed. Here, $X' = (1-b, 1-a)$ (the dual point) if $X = (a, b)$. See Figure 1.

Theorem 1.1. *Let $d \geq 3$ and $(1/p, 1/q) \in \mathfrak{T}$. Let $1 \leq r \leq s \leq \infty$ satisfy*

$$\left(\frac{1}{r}, \frac{1}{s}\right) \neq \left(1, \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)\right), \left(1 - \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right), 0\right)$$

and

$$0 \leq \frac{1}{r} - \frac{1}{s} \leq 1 - \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right). \tag{1-5}$$

Suppose $\beta \notin \mathbb{N}_0$. Then, if $1/p - 1/q < 2/d$, $p \neq 2$, and $q \neq 2$, estimate (1-4) holds for $1 \leq a = b \leq \infty$ with C depending only on p, q, a, b, r, s, and $\operatorname{dist}(\beta, \mathbb{N}_0)$; if

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{d}, \tag{1-6}$$

the same estimate (1-4) holds for $a = b = 2$.

It is remarkable that Theorem 1.1 gives (1-4) for $(1/p, 1/q)$ contained in the open line segment $(\mathcal{B}, \mathcal{B}')$ (see Figure 1). The exponents p, q satisfying (1-6) constitute the critical case in that (1-4) is no longer true if $1/p - 1/q > 2/d$. (See the remark on page 2270 and the condition (1-5).) Consequently, it is more difficult to obtain estimate (1-4) for p, q with (1-6) than that for p, q with $1/p - 1/q < 2/d$. Only estimate (1-4) with $(1/p, 1/q) = \mathcal{C}$, $a = p$, and $b = q$ was previously shown by [Escauriaza and Vega 2001].

If $p,\ q$ satisfy (1-6) and $r=s$, then the estimate (1-4) implies the Carleman inequality for the Laplacian (see [Escauriaza and Vega 2001]):

$$\||x|^{-\sigma} f\|_{L^{q,b}} \le C\||x|^{-\sigma}\Delta f\|_{L^{p,a}}, \quad f\in C_c^\infty(\mathbb{R}^d\setminus\{0\}) \tag{1-7}$$

for $\sigma>0$, with $C>0$ depending on $d,\ p,\ q$, and $\operatorname{dist}(\sigma,\mathbb{N}+d/q)$ (if $\operatorname{dist}(\sigma,\mathbb{N}+d/q)>0$). By this implication the estimates in Theorem 1.1 with $p,\ q$ satisfying (1-6) give (1-7) for $(1/p,1/q)\in(\mathcal{B},\mathcal{B}')$. However, it does not extend the previously known range of $p,\ q$ for which (1-7) holds. When $d\ge 5$, the range of $p,\ q$ coincides with that in [Kwon and Lee 2018], which was obtained by making use of the sharp estimate for the spherical harmonic projection. The optimal range of $p,\ q$ for (1-7) remains open.

To obtain *sucp* for potentials in $L^s_{t,\mathrm{loc}}L^{r,\infty}_x$, we need to obtain (1-4) with $a=b$. To this end, we are basically relying on real interpolation to upgrade $L^r_tL^p_x$-$L^s_tL^q_x$ estimates to those of $L^r_tL^{p,a}_x$-$L^s_tL^{q,b}_x$ with $a=b$. However, such an extension of the inequality (1-4) to the Lorentz spaces is not so straightforward as in the Appendix of [Jerison and Kenig 1985], since real interpolation does not behave well in mixed-norm spaces; see [Cwikel 1974]. We are only able to obtain (1-4) with $a=b=2$ when p,q satisfy (1-6); also see Lemma 4.1.

Theorem 1.1 provides estimate (1-4) for exponents on an extended range, but the problem of determining the optimal range of $p,\ q$ for which (1-4) holds remains open. When $1/p-1/q<d/2$, by Theorem 1.1, estimate (1-4) holds for all $a,\ b$ satisfying $1\le a\le b\le\infty$, since $L^{p,r_1}\subset L^{p,r_2}$ if $r_1\le r_2$. Because of limitation of the real interpolation in the mixed-norm spaces, we have (1-4) only for $1\le a\le 2\le b\le\infty$ when $1/p-1/q=d/2$. However, we expect that the same continues to be true even for $p,\ q$ satisfying $1/p-1/q=d/2$.

Strong unique continuation property for the heat operator. Our extension of the Carleman estimate to the Lorentz spaces (Theorem 1.1) allows a larger class of potentials for the strong unique continuation property for the heat operator. In this regard we obtain Theorems 1.2 and 1.3, which improve the results in [Escauriaza and Vega 2001]. Once we have the Carleman estimate (1-4), those theorems can be shown by routine adaptation of the argument in that paper. We state them without providing the proofs.

Theorem 1.2. *Let $d\ge 3,\ 0<T<\infty$, and $\mathfrak{r},\ \mathfrak{s}$ satisfy (1-3). Let $(1/p,1/q)\in\mathfrak{T}$ satisfy $1/p-1/q=1/\mathfrak{r}$. Suppose that $u\in W^{1,a}((0,T);\ W^{2,p}(\mathbb{R}^d)),\ a\le\min\{2,\mathfrak{s}\}$, is a solution to the differential inequality (1-1), and suppose that, for any $k\in\mathbb{N}$, there is a constant C_k such that (1-2) holds for some $\delta>0$. Then u is identically zero on $\mathbb{R}^d\times(0,T)$ provided that $\|t^{1-d/(2\mathfrak{r})-1/\mathfrak{s}}V\|_{L^{\mathfrak{s}}((0,T);L^{\mathfrak{r},\infty}_x(\mathbb{R}^d))}$ is small enough.*

Most significantly, Theorem 1.2 gives the *sucp* with $V\in L^\infty((0,T);\ L^{d/2,\infty}_x(\mathbb{R}^d))$. This extends the result obtained by [Escauriaza and Vega 2001] under the assumption that $\|V\|_{L^\infty((0,T);L^{d/2}_x)}$ is small enough. Using Wolff's construction [1992], we can show that the smallness assumption is necessary in general for $V\in L^\infty((0,T);\ L^{d/2,\infty}(\mathbb{R}^d))$, or $V\in L^{d/2,\infty}(\mathbb{R}^d;\ L^\infty((0,T)))$. Indeed, Wolff showed that there is a bounded nonzero function w such that $|\Delta w|\le|V_*w|$ with $V_*\in L^{d/2,\infty}$ and which vanishes to infinite order at the origin. Since the function w in [Wolff 1992] is bounded, by considering the time independent function $u(x,t):=w(x)$, it is easy to see that $u(x,t)$ satisfies (1-2) and obviously the differential inequality $|\Delta u+\partial_t u|\le|V_*u|$.

We also have the following *sucp* result for a local solution.

Theorem 1.3. *Let $d \ge 3$ and $\mathfrak{r}$, $\mathfrak{s}$ satisfy (1-3). Suppose that u is a continuous solution to $|\Delta u + \partial_t u| \le |Vu|$ on $B(0,2) \times (0,2)$, and suppose that, for any $k \in \mathbb{N}$, there is a constant C_k such that*

$$\|e^{-|x|^2/(8t)} u\|_{L^2((0,\varepsilon);L^2_x(B(0,2)))} \le C_k \varepsilon^k, \quad 0 < \varepsilon < 2.$$

Then $u(x,0)$ vanishes on $B(0,2)$ if $\|t^{1-d/(2\mathfrak{r})-1/\mathfrak{s}} V\|_{L^{\mathfrak{s}}((0,2);L^{\mathfrak{r},\infty}_x(B(0,2)))}$ is small enough.

Uniform resolvent estimate for the Hermite operator. We now consider the resolvent estimate for the Hermite operator $H = -\Delta + |x|^2$ in $\mathbb{R}^d$:

$$\|(H - z)^{-1} f\|_q \le C \|f\|_p, \quad z \in \mathbb{C} \setminus (2\mathbb{N}_0 + d), \tag{1-8}$$

with a constant C independent of z. The estimate has independent interest while it plays an important role in proving Theorem 1.1; see Lemma 4.1. Since H has the discrete spectrum $2\mathbb{N}_0 + d$, the points $z \in 2\mathbb{N}_0 + d$ are excluded. In contrast with the operator with a continuous spectrum, it is impossible for (1-8) to hold with C independent of z, so we need to impose the assumption that

$$\operatorname{dist}(z, 2\mathbb{N}_0 + d) \ge c \tag{1-9}$$

for some $1 \gg c > 0$; see the remark on page 2265. Estimate (1-8) may be compared with the corresponding estimate for the resolvent of the Laplacian which is due to Kenig, Ruiz, and Sogge [Kenig et al. 1987]. It was shown in that paper that the estimate

$$\|(-\Delta - z)^{-1} f\|_q \le C \|f\|_p, \quad z \in \mathbb{C} \setminus (0, \infty),$$

holds with C independent of z if and only if

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{d}, \quad \frac{2d}{d+3} < p < \frac{2d}{d+1}, \quad \text{and} \quad d \ge 3.$$

Also, see [Jeong et al. 2016] for the uniform estimates for more general second-order differential operators and [Kwon and Lee 2020] for the sharp bounds which depend on z. Under the assumption (1-9), the uniform resolvent estimate for H continues to hold with p, q away from the critical line $1/p - 1/q = 2/d$, whereas this cannot be true for $-\Delta$ because of scaling structure; see [Kenig et al. 1987; Kwon and Lee 2020].

The uniform estimate (1-8) was obtained by Escauriaza and Vega [2001] for

$$\frac{2d}{d+2} \le p \le 2 \le q \le \frac{2d}{d-2}, \quad d \ge 3.$$

However, (1-8) fails to hold if $1/p - 1/q > 2/d$ (see the remark on page 2270), and the proof of (1-8) is more involved when p, q satisfy (1-6). As for such (p,q) of the critical case, the estimate has been known only for $(p,q) = (2d/(d+2), 2d/(d-2))$. In what follows we establish (1-8) for $(1/p, 1/q) \in (\mathcal{B}, \mathcal{B}')$ under assumption (1-9). Those estimates in the expanded range are crucial for obtaining (1-4) with $a = b$ when p, q satisfy (1-6).

Theorem 1.4. *Let $d \ge 3$. Suppose $(1/p, 1/q) \in \mathfrak{T}$ and (1-9) holds. Then there is a constant $C > 0$ such that estimate (1-8) holds. Furthermore, if $(1/p, 1/q) = \mathcal{B}$ or $\mathcal{B}'$, we have the restricted weak-type (uniform) estimate for $(H - z)^{-1}$.*

The proof of (1-8) with $(p,q)=(2d/(d+2),2d/(d-2))$ in [Escauriaza and Vega 2001] heavily relies on the uniform bound on the spectral projection operator Π_k, which is the projection onto the k-th eigenspace of the Hermite operator H; see Section 2. In fact, they also used interpolation along an analytic family of operators which are motivated by Mehler's formula for the Hermite function. However, their argument is not enough to prove (1-8) for $(1/p,1/q)\in(\mathcal{B},\mathcal{B}')$. We develop a different approach which is more direct and significantly simpler. We make use of a representation formula (2-1) for Π_k which was observed in [Jeong et al. 2022a] and an estimate for the Hermite–Schrödinger propagator $e^{-itH}f$ (see Proposition 2.1) which is a consequence of the representation formula and the endpoint Strichartz estimate [Keel and Tao 1998].

Organization of the paper. The rest of this paper is organized as follows. In Section 2 we provide useful properties of the Hermite operator H and the Hermite spectral projection operator Π_k. We prove boundedness of more general multiplier operators for the Hermite operator in Section 3, which implies Theorem 1.4. Finally, the proof of the Carleman estimate for the heat operator is given in Section 4.

2. Properties of the Hermite operator

For any multi-index $\alpha\in\mathbb{N}_0^d$, the L^2-normalized Hermite function Φ_α, which is a tensor product of one dimensional Hermite functions, is an eigenfunction of H with eigenvalue $2|\alpha|+d$. Here $|\alpha|:=\alpha_1+\cdots+\alpha_d$. The set $\{\Phi_\alpha:\alpha\in\mathbb{N}_0^d\}$ forms an orthonormal basis of $L^2(\mathbb{R}^d)$. Thus, for any $f\in L^2(\mathbb{R}^d)$, we have the Hermite expansion $f=\sum_\alpha\langle f,\Phi_\alpha\rangle\Phi_\alpha$.

We consider the Hermite spectral projection operator Π_k which is defined by

$$\Pi_k f=\sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=k}\langle f,\Phi_\alpha\rangle\Phi_\alpha,\qquad f\in\mathcal{S}(\mathbb{R}^d).$$

Then, the Hermite–Schrödinger propagator is given by

$$e^{-itH}f=\sum_{k\in\mathbb{N}_0}e^{-it(2k+d)}\Pi_k f,\qquad f\in\mathcal{S}(\mathbb{R}^d),$$

which is the solution to the Cauchy problem $(i\partial_t-H)u=0,\ u(x,0)=f(x)$. If $f\in\mathcal{S}(\mathbb{R}^d)$, it is easy to see that $\Pi_k f$ decays rapidly in k, thus $\sum_{k=0}^\infty e^{-it(2k+d)}\Pi_k f$ converges uniformly. Clearly,

$$\Pi_k f=\sum_{k'\in\mathbb{N}_0}\frac{1}{2\pi}\left(\int_{-\pi}^{\pi}e^{it(k-k')}\,dt\right)\Pi_{k'}f.$$

Therefore, we obtain

$$\Pi_k f=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{it(2k+d-H)/2}f\,dt \tag{2-1}$$

for $f\in\mathcal{S}(\mathbb{R}^d)$. Meanwhile, the operator e^{-itH} has the kernel formula

$$e^{-itH}f(x)=C_d(\sin(2t))^{-d/2}\int_{\mathbb{R}^d}e^{i((|x|^2+|y|^2)\cot(2t)/2-\langle x,y\rangle\csc(2t))}f(y)\,dy \tag{2-2}$$

for $f\in\mathcal{S}(\mathbb{R}^d)$, which is shown by using Mehler's formula [Sjögren and Torrea 2010; Thangavelu 1987]. Combining this with (2-1) gives an explicit expression of the kernel of Π_k.

In order to prove the uniform resolvent estimate (Theorem 1.4), we make use of the following mixed-norm estimate for e^{-itH}, which strengthens the uniform bound (2-4) in a different direction.

Proposition 2.1. *Let $d \geq 3$ and $(1/p, 1/q) = \mathcal{B}'$. Then we have*

$$\Big\| \int_{-\pi}^{\pi} |e^{-itH/2} f| \, dt \Big\|_{q,\infty} \leq C \|f\|_{p,1}. \tag{2-3}$$

Various authors [Jeong et al. 2022a; 2024; Karadzhov 1994; Koch and Tataru 2005; Thangavelu 1998] studied the problem of characterizing the sharp asymptotic bound on the operator norm $\|\Pi_k\|_{p\to q}$ of Π_k from L^p to L^q as $k \to \infty$. In particular, [Karadzhov 1994] showed

$$\|\Pi_k\|_{p\to q} \leq C \tag{2-4}$$

for a constant C when $p = 2$ and $q = 2d/(d-2)$. By duality and the TT^* argument, the bound (2-4) with $(p,q) = (2d/(d+2), 2)$ and $(p,q) = (2d/(d+2), 2d/(d-2))$ follows. Interpolating those estimates with the trivial bound $\|\Pi_k\|_{2\to 2} \leq 1$, we have (2-4) for p, q satisfying

$$\frac{2d}{d+2} \leq p \leq 2 \leq q \leq \frac{2d}{d-2}.$$

Recently, the authors showed in [Jeong et al. 2024, Theorem 1.6] that (2-4) holds on an extended range of p, q for $d \geq 3$; see [Jeong et al. 2022b] for a related result. By means of Proposition 2.1, we can provide a simple alternative proof of this result. Indeed, by (2-1) and Proposition 2.1, it follows that $\|\Pi_k f\|_{q,\infty} \leq C\|f\|_{p,1}$ if $(1/p, 1/q) = \mathcal{B}'$. By duality, the same estimate also holds for $(1/p, 1/q) = \mathcal{B}$. Interpolating these estimates with the above mentioned estimate (2-4) for

$$\frac{2d}{d+2} \leq p \leq 2 \leq q \leq \frac{2d}{d-2}$$

gives the following. See Figure 1.

Corollary 2.2 [Jeong et al. 2024, Theorem 1.6]. *Let $d \geq 3$. For p, q satisfying $(1/p, 1/q) \in \mathfrak{T}$, there is a constant $C > 0$, independent of k, such that (2-4) holds. Furthermore, the uniform restricted weak-type estimate for Π_k holds if $(1/p, 1/q) = \mathcal{B}$ or $\mathcal{B}'$.*

Proof of Proposition 2.1. We make use of the endpoint Strichartz estimate for e^{-itH}:

$$\|e^{-itH/2} f\|_{L_t^2([-\pi,\pi]; L_x^{p_0}(\mathbb{R}^d))} \leq C\|f\|_2, \tag{2-5}$$

where $p_0 = 2d/(d-2)$. By periodicity, one can easily show estimate (2-5) using the dispersive estimate $\|e^{-itH/2} f\|_\infty \lesssim |t|^{-d/2} \|f\|_1$, $t \in \big(0, \frac{\pi}{2}\big)$, which follows from (2-2) and the standard argument in [Keel and Tao 1998]; for example, see [Sjögren and Torrea 2010]. We choose a smooth partition of unity, so that

$$\psi^0 + \sum_{j\geq 4} (\psi(2^j t) + \psi(-2^j t) + \psi(2^j (t+\pi)) + \psi(2^j (\pi - t))) = 1$$

for $t \in (-\pi, \pi) \setminus \{0\}$. Here $\psi \in \mathrm{C}_c^\infty\big(\big[\frac{1}{4}, 1\big]\big)$ satisfy $\sum_j \psi(2^j t) = 1$ for $t > 0$, and ψ^0 is a smooth function which is supported in the interval $[-\pi, \pi]$ and vanishes near 0, π, and $-\pi$.

Set $\psi_j^{\pm} = \psi(\pm 2^j\,\cdot)$ and $\psi_j^{\pm\pi} = \psi(2^j(\pi - \pm\,\cdot))$. Then, for $\sigma = \pm, \pm\pi$, we have

$$\int |\psi_j^{\sigma} e^{-itH/2} f|\,dt \lesssim 2^{(d-2)j/2}\|f\|_1$$

because $|\psi_j^{\sigma} e^{-itH/2} f| \lesssim 2^{dj/2}\|f\|_1$ by (2-2). Using (2-5) and Hölder's inequality followed by Minkowski's inequality, we also obtain the estimate

$$\left\| \int |\psi_j^{\sigma} e^{-itH/2} f|\,dt \right\|_{2d/(d-2)} \lesssim 2^{-j/2}\|f\|_2.$$

In other words, for the sublinear operators $T_j^{\sigma} f = \int |\psi_j^{\sigma} e^{-itH/2} f|\,dt$, $\sigma = \pm, \pm\pi$, two estimates

$$\|T_j^{\sigma} f\|_{q_\ell} \lesssim 2^{j(-1)^{\ell}\varepsilon_\ell}\|f\|_{p_\ell}, \quad \ell = 0, 1,$$

hold, where $p_0 = 1$, $q_0 = \infty$, $\varepsilon_0 = \frac{1}{2}d$, and $p_1 = 2$, $q_1 = 2d/(d-2)$, $\varepsilon_1 = \frac{1}{2}$. Note that the exponents of 2^j in the two estimates have different signs. Thus, applying Bourgain's summation trick (for example, see [Jeong et al. 2024, Lemma 2.4]), we obtain

$$\left\| \int \left| \sum_j \psi_j^{\sigma} e^{-itH/2} f \right| dt \right\|_{q,\infty} \le \left\| \sum_j T_j^{\sigma} \right\|_{q,\infty} \lesssim \|f\|_{p,1}, \quad \sigma = \pm, \pm\pi,$$

for $(1/p, 1/q) = \mathcal{B}'$. By a similar argument, it is easy to show $\left\| \int |\psi^0 e^{-itH/2} f|\,dt \right\|_q \lesssim \|f\|_p$ for $(1/p, 1/q) = \mathcal{B}'$. Hence, combining all of those estimates, we get (2-3). □

We now consider the L^p-L^q estimate for the operator H^{-s}, $s > 0$, which is defined by

$$H^{-s} f = \sum_{k=0}^{\infty} (2k+d)^{-s}\Pi_k f.$$

The operator can also be written as

$$H^{-s} f = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-tH} f\,dt$$ [1]

by making use of the heat semigroup e^{-tH} associated to H. By means of the explicit kernel expression of e^{-tH} which is based on Mehler's formula (see [Thangavelu 1993]), Bongioanni and Torrea [2006] obtained L^p-L^q boundedness for H^{-s}. Sharpness of their result was later verified by [Nowak and Stempak 2013]. Thus, the results completely characterize L^p-L^q boundedness of H^{-s}.

Theorem 2.3 [Bongioanni and Torrea 2006, Theorem 8; Nowak and Stempak 2013, Theorem 3.1]. *Let $d \ge 1$, $1 < p, q < \infty$, and $0 < s < \frac{1}{2}d$. Then H^{-s} is bounded from L^p to L^q if and only if*

$$-\frac{2s}{d} < \frac{1}{p} - \frac{1}{q} \le \frac{2s}{d}.$$

There are weak/restricted weak-type estimates in the borderline cases which are not included in the above theorem, and we refer the readers to [Nowak and Stempak 2013] for more details regarding such endpoint estimates.

[1] For a bounded function $\mathfrak{m}$ on $\mathbb{R}_+$, the operator $\mathfrak{m}(H)$ is formally defined by $\mathfrak{m}(H) = \sum_{k\in\mathbb{N}_0} \mathfrak{m}(2k+d)\Pi_k$.

3. Proof of Theorem 1.4

We consider the more general operator $(H-z)^{-m}$, $m \in \mathbb{N}$, which is given by

$$(H-z)^{-m} f = \sum_{k=0}^{\infty} \frac{\Pi_k f}{(2k+d-z)^m} = (-2)^{-m} \sum_{k=0}^{\infty} \frac{\Pi_k f}{(i\tau+\beta-k)^m},$$

with $z = 2\beta + d + 2\tau i$, $\beta \notin \mathbb{N}_0$, and $\tau \in \mathbb{R}$. We prove the following.

Theorem 3.1. *Let $d \geq 3$, and let m be a positive integer. Suppose that (1-9) holds for some constant $c > 0$. If $(1/p, 1/q) \in (\mathcal{B}, \mathcal{B}')$, then there is a constant $C = C(m)$, independent of z, such that*

$$\|(H-z)^{-m} f\|_q \leq C(1+|\operatorname{Im} z|)^{1-m} \|f\|_p. \tag{3-1}$$

Furthermore, we have $\|(H-z)^{-m} f\|_{q,\infty} \leq C(1+|\operatorname{Im} z|)^{1-m}\|f\|_{p,1}$ if $(1/p, 1/q) = \mathcal{B}$ or $\mathcal{B}'$.

While the estimates for $m \geq 2$ are rather straightforward from (2-4), the proof of (3-1) for $m = 1$ is more involved. This case is handled in Proposition 3.2 below.

Remark. The gap condition (1-9) is necessary for the uniform estimate (3-1) to hold. In fact,

$$\|(H-z)^{-m}\|_{p\to q} \geq \frac{|2k+d-z|^{-m}\|f\|_q}{\|f\|_p}$$

if f is an eigenfunction with eigenvalue $2k+d$. Therefore, the operator norm cannot be bounded as $z \to 2k+d$ unless (1-9) holds.

For positive numbers $\mathcal{B}$ and t_0, let $\mathcal{C}(\mathcal{B}, t_0)$ denote the class of functions which are contained in $C^{[(d+2)/2]}(\mathbb{R} \setminus [-t_0, t_0])$ and satisfy the following:

$$|G(n)| \leq \mathcal{B}, \quad n \in \mathbb{Z}, \tag{3-2}$$

$$\sum_{k=1}^{\infty} |G(k) + G(-k)| \leq \mathcal{B}, \tag{3-3}$$

$$\sum_{k=1}^{\infty} |kG(k) - (k+1)G(k+1)| \leq \mathcal{B}, \tag{3-4}$$

$$\left|\left(\frac{d}{dt}\right)^l G(t)\right| \leq \mathcal{B}(1+|t|)^{-l-1}, \quad t_0 < |t|, \tag{3-5}$$

for $0 \leq l \leq \frac{1}{2}(d+2)$. Particular examples satisfying the conditions (3-2)–(3-5) are $G_{\mu,\tau}(t) = 1/(i\tau + t + \mu)$, where $(\mu, \tau) \in \left(-\frac{1}{2}, \frac{1}{2}\right) \times \mathbb{R}$ and $|(\mu, \tau)| \geq c$ for some small $c > 0$.

Proposition 3.2. *Let $d \geq 3$ and $(1/p, 1/q) \in (\mathcal{B}, \mathcal{B}')$. Suppose that G is in $\mathcal{C}(\mathcal{B}, t_0)$. Then, there is a constant C, depending only on $\mathcal{B}$ and t_0, such that*

$$\left\|G\left(\frac{2n+d-H}{2}\right) f\right\|_q \leq C\|f\|_p \tag{3-6}$$

holds for every $n \in \mathbb{N}_0$. Furthermore, if $(1/p, 1/q) = \mathcal{B}$ or $\mathcal{B}'$, the restricted weak-type (p, q) estimate holds for $G\left(\frac{1}{2}(2n+d-H)\right)$ with a uniform bound depending only on $\mathcal{B}$ and t_0.

Proof. Let p_* and q_* be given by $(1/p_*, 1/q_*) = \mathfrak{B}'$. In order to show Proposition 3.2, it is sufficient to show the restricted weak-type (p_*, q_*) estimate for $G\big(\frac{1}{2}(2n+d-H)\big)$. Note that the adjoint operator $G\big(\frac{1}{2}(2n+d-H)\big)^*$ is given by

$$G\Big(\frac{2n+d-H}{2}\Big)^* f = \sum_{k=0}^{\infty} \overline{G}(n-k)\Pi_k f.$$

Clearly $\overline{G} \in \mathcal{C}(\mathcal{B}, t_0)$. Hence, the same argument shows that the restricted weak-type (p_*, q_*) estimate holds for $G\big(\frac{1}{2}(2n+d-H)\big)^*$. This in turn gives the restricted weak-type estimate (q_*', p_*') for $G\big(\frac{1}{2}(2n+d-H)\big)$ by duality. Real interpolation between these two (restricted weak-type) estimates for $G\big(\frac{1}{2}(2n+d-H)\big)$ yields the desired estimates for $(1/p, 1/q) \in (\mathfrak{B}, \mathfrak{B}')$.

No differentiability assumption is made on G for $|t| \le t_0$. So, we handle the cases $n \ge n_0$ and $n < n_0$ separately, where n_0 is an integer satisfying $n_0 \ge 2t_0$. We first consider the case $n \ge n_0$. Recalling

$$G\Big(\frac{2n+d-H}{2}\Big) = \sum_{k=0}^{\infty} G(n-k)\Pi_k,$$

we write the decomposition

$$G\Big(\frac{2n+d-H}{2}\Big) =: \mathcal{J}_n + \mathcal{K}_n,$$

where

$$\mathcal{J}_n := \sum_{k=0}^{\infty} G(n-k)\phi\Big(\frac{n-k}{n}\Big)\Pi_k \quad \text{and} \quad \mathcal{K}_n := \sum_{k=0}^{\infty} G(n-k)\Big(1-\phi\Big(\frac{n-k}{n}\Big)\Big)\Pi_k.$$

Here, we choose a nonnegative smooth even function ϕ on $\mathbb{R}$ such that $\phi(t) = 1$ on $\big[-\frac{1}{2}, \frac{1}{2}\big]$, $\phi = 0$ if $1 \le |t|$, and ϕ is nonincreasing on the half-line $t > 0$. This monotonicity assumption plays an important role in obtaining a bound on a sum of trigonometric functions.

The sum $\mathcal{J}_n$ is the major contribution to the estimate (3-6) and is to be handled by the integral formula for Π_k and Proposition 2.1. The second sum $\mathcal{K}_n$ behaves like the operator H^{-1}, which is actually bounded from L^p-L^q on a larger range of p, q. We consider $\mathcal{J}_n$ first.

We set

$$\mathcal{I}_1 = \sum_{k=1}^{n} G(k)\phi\Big(\frac{k}{n}\Big)(\Pi_{n-k} - \Pi_{n+k}) \quad \text{and} \quad \mathcal{I}_2 = \sum_{k=1}^{n} (G(-k)+G(k))\phi\Big(\frac{k}{n}\Big)\Pi_{n+k}.$$

Since ϕ is an even function and supported in $[-1, 1]$, after reindexing by $(n-k) \to k$, we see

$$\mathcal{J}_n = \sum_{k=1}^{n} G(k)\phi\Big(\frac{k}{n}\Big)\Pi_{n-k} + G(0)\Pi_n + \sum_{k=1}^{n} G(-k)\phi\Big(\frac{k}{n}\Big)\Pi_{n+k}.$$

Thus,

$$\mathcal{J}_n = \mathcal{I}_1 + \mathcal{I}_2 + G(0)\Pi_n.$$

By (3-2), (3-3), and the uniform restricted weak-type (p_*, q_*) estimate for Π_λ in Corollary 2.2, it follows that

$$\|G(0)\Pi_n f\|_{q_*,\infty} \lesssim \mathcal{B}\|f\|_{p_*,1} \quad \text{and} \quad \|\mathcal{I}_2\|_{q_*,\infty} \lesssim \mathcal{B}\|f\|_{p_*,1}.$$

So, it suffices to deal with $\mathcal{I}_1$. Using the formula (2-1), we note

$$\Pi_{n-k} f - \Pi_{n+k} f = -\frac{i}{\pi}\int_{-\pi}^{\pi} \sin(tk) e^{it(2n+d-H)/2} f\,dt.$$

Thus, we have

$$\mathcal{I}_1 f = \int_{-\pi}^{\pi} \zeta_n(t) e^{-itH/2} f\,dt,$$

where

$$\zeta_n(t) = -\frac{i}{\pi} e^{it(2n+d)/2} \sum_{k=1}^{n} G(k)\sin(tk)\phi\Big(\frac{k}{n}\Big), \quad -\pi \le t \le \pi.$$

Using Proposition 2.1, it is sufficient to show

$$|\zeta_n(t)| \le C, \tag{3-7}$$

with C independent of n and G. By the property of ϕ, it is clear that

$$|\zeta_n(t)| \lesssim \bigg|\sum_{k=1}^{\lfloor n/2\rfloor} \sin(tk)G(k)\bigg| + \bigg|\sum_{k=\lfloor n/2\rfloor+1}^{n} \sin(tk)G(k)\phi\Big(\frac{k}{n}\Big)\bigg|.$$

Boundedness of the second term is easy to show. Indeed, since the condition (3-5) holds for $|t| > \frac{1}{2}n$ by our choice of n_0, we see

$$\bigg|\sum_{k=\lfloor n/2\rfloor+1}^{n} \sin(tk)G(k)\phi\Big(\frac{k}{n}\Big)\bigg| \lesssim \mathcal{B} \sum_{k=\lfloor n/2\rfloor+1}^{n} k^{-1}\phi\Big(\frac{k}{n}\Big) \lesssim \mathcal{B}.$$

So, for (3-7), we only have to show $\big|\sum_{k=1}^{n} \sin(tk)G(k)\big| \lesssim 1$ for any n. Setting $\sigma_k(t) = \sum_{j=1}^{k} j^{-1}\sin(jt)$, by summation by parts we write

$$\sum_{k=1}^{n} \sin(tk)G(k) = \sum_{k=1}^{n-1} \sigma_k(t)(kG(k) - (k+1)G(k+1)) + \sigma_n(t)nG(n).$$

Since $|\sigma_k(t)| \lesssim 1$ for any k and t as can be shown by an elementary argument,[2] by the conditions (3-4) and (3-5) it follows that $\big|\sum_{k=1}^{n} \sin(tk)G(k)\big| \lesssim 1$.

We now turn to the operator $\mathcal{K}_n$. Clearly, we may write $\mathcal{K}_n = H^{-1} \circ m_n(H)$, where m_n is given by

$$m_n(t) = tG\Big(\frac{2n+d-t}{2}\Big)\Big(1 - \phi\Big(\frac{2n+d-t}{2n}\Big)\Big),$$

which is in $C^\infty(\mathbb{R})$. Using (3-2), (3-5), and the support property of ϕ, a simple calculation shows

$$\bigg|\frac{d^l}{dt^l} m_n(t)\bigg| \lesssim (1+t)^{-l} \quad \text{for } l = 0, 1, 2, \dots, \tfrac{1}{2}(d+2)$$

whenever $t > 0$. Here the implicit constants are independent of n. Thus, the Marcinkiewicz multiplier theorem [Thangavelu 1993, Theorem 4.2.1] implies that $m_n(H)$ is bounded on L^p, $1 < p < \infty$, uniformly

[2] This can be seen by approximating Dirichlet's kernel, or again by summation by parts.

in n. By Theorem 2.3, H^{-1} is also bounded from L^p to L^q for $1 < p, q < \infty$ satisfying $1/p - 1/q = 2/d$. Hence, we have

$$\|\mathcal{K}_n\|_{p\to q} \le \|H^{-1}\|_{p\to q}\|m_n(H)\|_{p\to p} \lesssim 1,$$

with the implicit constant independent of n.

We now consider the case $n < n_0$, which is much simpler to show than the case $n \ge n_0$. To prove (3-6), we write the following decomposition for $G\big(\frac{1}{2}(2n+d-H)\big)$:

$$G\Big(\frac{2n+d-H}{2}\Big) = \widetilde{\mathcal{J}}_n + \widetilde{\mathcal{K}}_n,$$

where

$$\widetilde{\mathcal{J}}_n = \sum_{k=0}^{\infty} G(n-k)\phi\Big(\frac{k}{2n_0}\Big)\Pi_k \quad \text{and} \quad \widetilde{\mathcal{K}}_n = \sum_{k=0}^{\infty} G(n-k)\Big(1-\phi\Big(\frac{k}{2n_0}\Big)\Big)\Pi_k.$$

Clearly, the multiplier

$$G\Big(\frac{2n+d-\cdot}{2}\Big)\Big(1-\phi\Big(\frac{2n+d-\cdot}{2n_0}\Big)\Big)$$

of the operator $\widetilde{\mathcal{K}}_n$ satisfies the condition (3-5). So, in the same manner as in the above we obtain the bound $\|\widetilde{\mathcal{K}}_n\|_{p\to q} \lesssim 1$ if $1 < p, q < \infty$ and $1/p - 1/q = 2/d$. By condition (3-2) and Corollary 2.2 it follows that

$$\|\widetilde{\mathcal{J}}_n f\|_{q_*,\infty} \le \mathcal{B}\sum_{k=0}^{2n_0}\|\Pi_k f\|_{q_*,\infty} \lesssim \|f\|_{p_*,1}$$

uniformly in $n \le n_0$. This completes the proof of Proposition 3.2. □

We are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let p_* and q_* be given by $(1/p_*, 1/q_*) = \mathcal{B}'$. As in the proof of Proposition 3.2, it is enough to show the restricted weak-type (p_*, q_*) estimate for $(H-z)^{-m}$ with bound $C(1+|\operatorname{Im} z|)^{1-m}$, since the adjoint operator of $(H-z)^{-m}$ is given by $(H-\bar{z})^{-m}$. We can handle $(H-\bar{z})^{-m}$ in exactly the same way to obtain the restricted weak-type (p_*, q_*) estimate for $(H-z)^{-m}$ with bound $C(1+|\operatorname{Im} z|)^{1-m}$. By duality and interpolation, we get all the desired estimates.

By Corollary 2.2, we have the estimate $\|\Pi_k f\|_{q_*,\infty} \le C\|f\|_{p_*,1}$, with C independent of k. Using this estimate, for $m \ge 2$, we get

$$\|(H-z)^{-m} f\|_{q_*,\infty} \lesssim \sum_{k=0}^{\infty} |2k+d-z|^{-m}\|f\|_{p_*,1} \lesssim (1+|\operatorname{Im} z|)^{1-m}\|f\|_{p_*,1}$$

because

$$\sum_{k=0}^{\infty} |2k+d-z|^{-m} \le C_m(1+|\operatorname{Im} z|)^{1-m},$$

with C_m independent of z for $m \ge 2$ if (1-9) holds. Thus, we need only to show

$$\|(H-z)^{-1} f\|_{q_*,\infty} \le C\|f\|_{p_*,1}. \tag{3-8}$$

If $\operatorname{Re} z > d-1$, we have $z = 2(n+\mu)+d+2i\tau$ for some $n \in \mathbb{N}_0$, $\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, and $\tau \in \mathbb{R}$ satisfying $|(\mu, \tau)| \geq \frac{1}{2}c$ because of (1-9). We note that

$$(H-z)^{-1} = G_{\mu,\tau}\Big(\frac{2n+d-H}{2}\Big),$$

where $G_{\mu,\tau}(t) = 1/(i\tau+t+\mu)$. It is easy to see that $G_{\mu,\tau} \in \mathcal{C}(\mathcal{B}, 1)$ for some $\mathcal{B} > 0$ provided that $\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\tau \in \mathbb{R}$ satisfy $|(\mu, \tau)| \geq \frac{1}{2}c$. Indeed, since $|k+\mu| \geq |\mu|$ for $k \in \mathbb{Z}$ and $\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, it follows that $|G_{\mu,\tau}(k)| \leq |\mu+i\tau|^{-1} \leq 2/c$ for all $k \in \mathbb{Z}$. Moreover, we have

$$\sum_{k=1}^{\infty} |G_{\mu,\tau}(k) + G_{\mu,\tau}(-k)| \lesssim \sum_{k=1}^{\infty} \frac{|i\tau+\mu|}{k^2+\tau^2} \lesssim 1,$$

$$\sum_{k=1}^{\infty} |kG_{\mu,\tau}(k) - (k+1)G_{\mu,\tau}(k+1)| \leq \sum_{k=1}^{\infty} \frac{|i\tau+\mu|}{|i\tau+k+\mu|^2} \lesssim 1,$$

and, for $0 \leq l \leq \frac{1}{2}(d+2)$,

$$\Big|\Big(\frac{d}{dt}\Big)^l G_{\mu,\tau}(t)\Big| \lesssim (1+|t|)^{-l-1}, \quad |t| \geq 1,$$

whenever $\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\tau \in \mathbb{R}$ satisfy $|(\mu, \tau)| \geq \frac{1}{2}c$. Obviously, the implicit constants are independent of specific values of μ and τ. Hence, taking a sufficiently large constant $\mathcal{B} \geq 2/c$, we see $G_{\mu,\tau} \in \mathcal{C}(\mathcal{B}, 1)$.

Thus, by Proposition 3.2, the estimate (3-8) holds uniformly in z. For the remaining case, i.e., $\operatorname{Re} z < d-1$, we have that z clearly stays away from the eigenvalues of H, so $(H-z)^{-1}$ behaves like H^{-1}. More precisely, we obtain the uniform estimate (3-8) repeating the same argument as in the case $n < n_0$ of the proof of Proposition 3.2. This completes the proof. □

The uniform resolvent estimate in Theorem 1.4 is a special case of the following.

Corollary 3.3. *Let $d \geq 3$ and m be a positive integer, and let p, q be given as in Theorem 1.1. Then, there is a constant $C = C(m)$ such that*

$$\|(H-z)^{-m} f\|_q \leq C(1+|\operatorname{Im} z|)^{d(1/p-1/q)/2-m} \|f\|_p \tag{3-9}$$

provided (1-9) holds. Furthermore, if $(1/p, 1/q) = \mathfrak{B}$ or $\mathfrak{B}'$, we have the restricted weak-type estimate for $(H-z)^{-m}$ with bound $C(1+|\operatorname{Im} z|)^{d(1/p-1/q)/2-m}$.

Proof. By Theorem 3.1, we have estimate (3-9) for $(1/p, 1/q) \in (\mathfrak{B}, \mathfrak{B}')$. In view of interpolation, it is enough to show (3-9) with $(p, q) = (2, 2)$, $(2d/(d+2), 2)$, or $(2, 2d/(d-2))$. These estimates are easy to show by using orthogonality between the projection operators Π_k. In fact, we have

$$\|(H-z)^{-m} f\|_2 \leq \Big(\sum_{k=0}^{\infty} |2k+d-z|^{-2m} \|\Pi_k f\|_2^2\Big)^{1/2}.$$

So, taking the supremum over k of $|2k+d-z|^{-2m}$, we obtain (3-9) when $p = q = 2$. We note that $\sum_{k=0}^{\infty} |2k+d-z|^{-2m} \leq C(1+|\operatorname{Im} z|)^{-2m+1}$ with C independent of z as long as (1-9) holds. Applying the uniform $L^{2d/(d+2)}$-L^2 estimate in Corollary 2.2, we get (3-9) with $p = 2d/(d+2)$ and $q = 2$. Since the adjoint of $(H-z)^{-m}$ is $(H-\bar{z})^{-m}$, estimate (3-9) with $(p, q) = (2d/(d+2), 2)$ implies that with $(p, q) = (2, 2d/(d-2))$ by duality. □

4. Proof of Theorem 1.1

We now prove the estimate (1-4) by adapting the argument in [Escauriaza and Vega 2001] (also see [Escauriaza 2000]) which deduces the Carleman estimate for the heat operator from the uniform resolvent estimate for the Hermite operator. We are basically relying on real interpolation as in the Appendix of [Jerison and Kenig 1985]. However, there are some nontrivial issues which are related to a shortcoming of the real interpolation in mixed-norm spaces.

Lemma 4.1. *Let* $1 < p \le 2 \le q < \infty$, $1 \le r, s \le \infty$, $1 \le a \le b \le \infty$, *and* $0 \le \gamma \le 1$, *and let* $\beta \notin \mathbb{N}_0$ *be a real number. Suppose that the estimate*

$$\Big\| \sum_{k=0}^{\infty} \frac{\Pi_k f}{(\tau i + \beta - k)^m} \Big\|_{q,b} \le C_m (1 + |\tau|)^{\gamma - m} \|f\|_{p,a} \tag{4-1}$$

holds for positive integers m, *with* C_m *independent of* $\tau \in \mathbb{R}$ *and* β, *provided* $\operatorname{dist}(\beta, \mathbb{N}_0) \ge c$ *for some* $c > 0$. *Then, if* $\operatorname{dist}(\beta, \mathbb{N}_0) \ge c$ *for some* $c > 0$, *estimate* (1-4) *holds uniformly in* β *whenever the following hold*:

- $\gamma < 1$, $0 \le 1/r - 1/s \le 1 - \gamma$, *and* $(1/r, 1/s) \ne (1, \gamma)$, $(1 - \gamma, 0)$,
- $\gamma = 1$, $a = b = 2$, *and* $1 < r = s < \infty$.

Lemma 4.1 was implicit in [Escauriaza and Vega 2001] with the Lebesgue spaces instead of the Lorentz spaces. The extra condition $a = b = 2$ when $\gamma = 1$ is due to a limitation of the real interpolation in mixed-norm spaces. Once we have Lemma 4.1, the proof of Theorem 1.1 is rather simple.

Proof of Theorem 1.1. Let $(1/p, 1/q)$ be in $\mathfrak{T}$. By real interpolation between the estimates in Corollary 3.3 and inclusion relations between Lorentz spaces, we get (4-1) with $\gamma = \frac{1}{2} d(1/p - 1/q)$ for any $1 \le a \le b \le \infty$ if $p \ne 2$ and $q \ne 2$. Thus Lemma 4.1 gives estimate (1-4) in the Lorentz spaces if the exponents satisfy the condition in Theorem 1.1. □

Estimate (1-4) is equivalent to the Sobolev-type inequality

$$\|h\|_{L^s(\mathbb{R}; L_x^{q,b})} \le C \|(\Delta - |x|^2 + \partial_t + 2\beta + d)h\|_{L^r(\mathbb{R}; L_x^{p,a})}, \quad h \in C_c^\infty(\mathbb{R}^{d+1}). \tag{4-2}$$

One can easily see this by following the argument in [Escauriaza 2000]. In particular, if $r = s$, the inequality (4-2) implies $\|f\|_q \le C \|(\Delta - |x|^2 + 2\beta + d) f\|_p$ for $f \in C_c^\infty(\mathbb{R}^d)$, which is, in fact, a special case of (1-8), where $z = 2\beta + d \notin 2\mathbb{N}_0 + d$. Indeed, let f_1 be a compactly supported smooth function on $\mathbb{R}$ with $f_1(0) = 1$. Then the above estimate follows by applying (4-2) to the function $h(x,t) = f(x) f_1(t/R) R^{-1/r}$, $R > 1$, and letting $R \to \infty$.

Remark. When $r = s$, the implication from (4-2) to (1-8) with $z = 2\beta + d \notin 2\mathbb{N}_0 + d$ can be used to show that the Carleman estimate (1-4) holds only if

$$\frac{1}{p} - \frac{1}{q} \le \frac{2}{d}.$$

By the Marcinkiewicz multiplier theorem for the Hermite operator H [Thangavelu 1993, Theorem 4.2.1], $(H - z)^{-1} H$ with $z = 2\beta + d \notin 2\mathbb{N}_0 + d$ is bounded on L^p, $1 < p < \infty$. Thus, we see that estimate (1-4) implies $\|H^{-1} u\|_q \lesssim \|u\|_p$ for $u \in C_c^\infty(\mathbb{R}^d)$. By Theorem 2.3, the inequality holds only if $1/p - 1/q \le 2/d$.

Proof of Lemma 4.1. To prove Lemma 4.1, we basically rely on the argument in [Escauriaza 2000; Escauriaza and Vega 2001], so we shall be brief. By scaling, it is easy to see that (1-4) is equivalent to (4-2). See [Escauriaza 2000] for the details. Thus, we need to show (4-2) by replacing h with $(\Delta-|x|^2+\partial_t+2\beta+d)^{-1}g$. Applying the projection operator Π_λ in x-variables and taking the Fourier transform in t, we see the operator $S_\beta := (\Delta-|x|^2+\partial_t+2\beta+d)^{-1}$ is given by

$$S_\beta g(x,t)=\int_{\mathbb{R}} K_\beta(t-s)(g(\,\cdot\,,s))(x)\,ds,$$

where the operator-valued kernel K_β is given by

$$K_\beta(t)(f)=\frac{1}{2}\int_{\mathbb{R}} e^{2\pi i t\tau}\sum_{k=0}^{\infty}\frac{\Pi_k(f)}{\pi i\tau+\beta-k}\,d\tau,\qquad f\in C_c^\infty(\mathbb{R}^d).$$

To prove (1-4), it is enough to show

$$\|S_\beta g\|_{L^s(\mathbb{R};L_x^{q,b})}\lesssim\|g\|_{L^r(\mathbb{R};L_x^{p,a})},\qquad g\in C_c^\infty(\mathbb{R}^{d+1}),\tag{4-3}$$

with an implicit constant independent of β as long as $\operatorname{dist}(\beta,\mathbb{N}_0)\ge c$ for some $c>0$.

We regard S_β as a vector-valued convolution operator. Let us first consider the case $\gamma<1$ which is easier. Let $\phi\in C_c^\infty([-1,1])$ be such that $\phi(t)=1$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$. Breaking the integral with functions $\phi(t\tau)$ and $1-\phi(t\tau)$ and using integration by parts and (4-1), it is easy to see that $\|K_\beta(t)\|_{L_x^{p,a}\to L_x^{q,b}}\lesssim\min\{|t|^{-\gamma},|t|^{-2}\}$. Since $\gamma<1$, for r and s satisfying $0\le 1/r-1/s\le 1-\gamma$ and $(1/r,1/s)\ne(1,\gamma),\ (1-\gamma,0)$, we obtain estimate (4-3) by Young's convolution inequality and the Hardy–Littlewood–Sobolev inequality.

We now turn to the case $\gamma=1$. We claim that the kernel K_β satisfies the Hörmander condition

$$\sup_{s\ne 0}\int_{|t|>2|s|}\|K_\beta(t-s)-K_\beta(t)\|_{L^{p,2}\to L^{q,2}}\,dt\le A<\infty,\tag{4-4}$$

where A depends only on the constant $c>0$ such that $\operatorname{dist}(\beta,\mathbb{N}_0)\ge c$. To show (4-4) it is sufficient to show $\|K'_\beta(t)\|_{L^{p,2}\to L^{q,2}}\lesssim|t|^{-2}$. In fact, if $\|K'_\beta(t)\|_{L_x^{p,2}\to L_x^{q,2}}\lesssim|t|^{-2}$, then

$$\|K_\beta(t-s)-K_\beta(t)\|_{L_x^{p,2}\to L_x^{q,2}}=\left\|\int_t^{t-s}K'_\beta(\sigma)\,d\sigma\right\|_{L_x^{p,2}\to L_x^{q,2}}\lesssim|s|\,|t|^{-2}.$$

This clearly yields (4-4). Integrating by parts, we have

$$(-2\pi i t)^2K'_\beta(t)=2^2(\pi i)^3\int_{-\infty}^{\infty}\tau e^{2\pi i\tau t}\sum_{k=0}^{\infty}\frac{1}{(\pi\tau i+\beta-k)^3}\Pi_k\,d\tau.$$

The assumption (4-1) (with $\gamma=1$) gives $\||t|^2K'_\beta(t)\|_{L^{p,2}\to L^{q,2}}\lesssim 1$ uniformly in t and β satisfying $\operatorname{dist}(\beta,\mathbb{N}_0)\ge c$, which proves the claim (4-4) (see [Escauriaza and Vega 2001] for details). Thanks to (4-4) and the usual vector-valued singular integral theory, in order to prove (4-3) for $1<r=s<\infty$, it suffices to obtain estimate (4-3) with $r=s=2$ and $a=b=2$.

For $\eta\in C_c^\infty(\mathbb{R})$, we define $\eta(D_t)$ by $\mathcal{F}_t(\eta(D_t)g)(x,\tau)=\eta(\tau)\mathcal{F}_tg(x,\tau)$, where $\mathcal{F}_t$ denotes the Fourier transform in t. We use the following Littlewood–Paley-type inequality in the Lorentz spaces.

Lemma 4.2. *Let $1 < p, r < \infty$. Suppose η is a smooth function supported in $[2^{-2}, 1]$ which satisfies $\sum_{j=-\infty}^{\infty} |\eta(2^{-j}t)|^2 \sim 1$ for all $t > 0$. Then we have*

$$\|g\|_{L_t^r(\mathbb{R};L_x^{p,r})} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |\eta(2^{-j}|D_t|)g|^2 \right)^{1/2} \right\|_{L_t^r(\mathbb{R};L_x^{p,r})} \lesssim \|g\|_{L_t^r(\mathbb{R};L_x^{p,r})}. \tag{4-5}$$

Proof. It is sufficient to show the second inequality in (4-5) because the first inequality follows from the second one via the standard polarization argument and duality. For every $1 < p, r < \infty$, we have

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\eta(2^{-j}|D_t|)g|^2 \right)^{1/2} \right\|_{L^r(\mathbb{R};L^p(\mathbb{R}^d))} \lesssim \|g\|_{L^r(\mathbb{R};L^p(\mathbb{R}^d))}$$

by means of the usual Littlewood–Paley inequality and the vector-valued singular integral theorem; see [Escauriaza and Vega 2001, Lemma 2.1]. We interpolate these estimates using the real interpolation in the mixed-norm spaces, in particular,

$$(L^{p_0}(\mathbb{R}; L^{q_0}), L^{p_1}(\mathbb{R}; L^{q_1}))_{\theta,p} = L^p(\mathbb{R}; L^{q,p})$$

whenever $p_0, q_0, p_1, q_1 \in [1, \infty)$ and $(1/p, 1/q) = (1-\theta)(1/p_0, 1/q_0) = \theta(1/p_1, 1/q_1)$ with $\theta \in (0, 1)$; see [Cwikel 1974; Lions and Peetre 1964]. Therefore, we obtain the second inequality in (4-5). □

We now note that

$$\psi(2^{-j}|D_t|)S_\beta g(x,t) = \int_{\mathbb{R}} K_{\beta,j}(t-s)g(\cdot,s)(x)\,dt,$$

where

$$K_{\beta,j}(t)f(x) := \frac{1}{2}\int_{\mathbb{R}} e^{2\pi i t\tau}\psi\left(\frac{|\tau|}{2^j}\right)\sum_{k=0}^{\infty}\frac{1}{\pi i\tau+\beta-k}\Pi_k f(x)\,d\tau.$$

Using (4-1) with $a = b = 2$ and integration by parts, we see that $\|K_{\beta,j}(t)\|_{L_x^{p,2}\to L_x^{q,2}} \le C2^j(1+2^j|t|)^{-2}$, with C independent of j and β if $\operatorname{dist}(\beta, \mathbb{N}_0) \ge c > 0$. Thus, Young's convolution inequality gives

$$\|\psi(2^{-j}|D_t|)S_\beta g\|_{L^2(\mathbb{R};L_x^{q,2})} \lesssim \|g\|_{L^2(\mathbb{R};L_x^{p,2})}, \tag{4-6}$$

with the implicit constant independent of j and β. To get the desired (4-3) with $r = s = 2$, we combine this inequality and Lemma 4.2. Since $2 \le q < \infty$, the space $L^{(q/2),(2/2)}$ is normable. So,

$$\left\| \left(\sum_j |h_j|^2 \right)^{1/2} \right\|_{L_x^{q,2}} \lesssim \left(\sum_j \|h_j\|_{L_x^{q,2}}^2 \right)^{1/2}. \tag{4-7}$$

Since $S_\beta g = \sum_{j\in\mathbb{Z}} \psi(2^{-j}|D_t|)S_\beta g$, applying Lemma 4.2 and then (4-7), we have

$$\|S_\beta g\|_{L^2(\mathbb{R};L_x^{q,2})} \lesssim \left(\sum_{j\in\mathbb{Z}} \|\psi(2^{-j}|D_t|)S_\beta g\|_{L^2(\mathbb{R};L_x^{q,2})}^2 \right)^{1/2}.$$

Let $\tilde\psi \in \mathrm{C}_c([2^{-2}, 1])$ such that $\psi\tilde\psi = \psi$, so

$$\psi(2^{-j}|D_t|)S_\beta g = \psi(2^{-j}|D_t|)S_\beta\tilde\psi(2^{-j}|D_t|)g.$$

Using (4-6) followed by (4-5), we get

$$\|S_\beta g\|_{L^2(\mathbb{R};L_x^{q,2})} \lesssim \Big(\sum_{j\in\mathbb{Z}} \|\tilde{\psi}(2^{-j}|D_t|)g\|^2_{L^2(\mathbb{R};L_x^{p,2})}\Big)^{1/2}.$$

By duality, the inequality (4-7) is equivalent to $\big(\sum_j \|h_j\|^2_{L_x^{p,2}}\big)^{1/2} \lesssim \big\|\big(\sum_j |h_j|^2\big)^{1/2}\big\|_{L_x^{p,2}}$ for $1 < p \le 2$. Thus, using Lemma 4.2, we get

$$\|S_\beta g\|_{L^2(\mathbb{R};L_x^{q,2})} \lesssim \Big\|\Big(\sum_{j\in\mathbb{Z}} |\tilde{\psi}(2^{-j}|D_t|)g|^2\Big)^{1/2}\Big\|_{L^2(\mathbb{R};L_x^{p,2})} \lesssim \|g\|_{L^2(\mathbb{R};L_x^{p,2})}. \qquad \square$$

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