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Athanasios Chatzikaleas and Jacques Smulevici

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NONLINEAR PERIODIC WAVES ON THE EINSTEIN CYLINDER

ATHANASIOS CHATZIKALEAS AND JACQUES SMULEVICI

Motivated by the study of small amplitude nonlinear waves in the anti-de Sitter spacetime and in particular the conjectured existence of periodic in time solutions to the Einstein equations, we construct families of arbitrary small time-periodic solutions to the conformal cubic wave equation and the spherically symmetric Yang–Mills equations on the Einstein cylinder $\mathbb{R} \times \mathbb{S}^3$. For the conformal cubic wave equation, we consider both spherically symmetric solutions and complex-valued aspherical solutions with an ansatz relying on the Hopf fibration of the 3-sphere. In all three cases, the equations reduce to 1+1 semilinear wave equations.

Our proof relies on a theorem of Bambusi–Paleari for which the main assumption is the existence of a seed solution, given by a nondegenerate zero of a nonlinear operator associated with the resonant system. For the problems that we consider, such seed solutions are simply given by the mode solutions of the linearized equations. Provided that the Fourier coefficients of the systems can be computed, the nondegeneracy conditions then amount to solving infinite dimensional linear systems. Since the eigenfunctions for all three cases studied are given by Jacobi polynomials, we derive the different Fourier and resonant systems using linearization and connection formulas as well as integral transformation of Jacobi polynomials.

In the Yang–Mills case, the original version of the theorem of Bambusi–Paleari is not applicable because the nonlinearity of smallest degree is nonresonant. The resonant terms are then provided by the next order nonlinear terms with an extra correction due to backreaction terms of the smallest degree of nonlinearity, and we prove an analogous theorem in this setting.

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1. Introduction

1A. ***Stability/instability of the anti-de Sitter spacetime.*** The anti-de Sitter (AdS) spacetime is the maximally symmetric solution to the vacuum Einstein equations with a negative cosmological constant:

$$\mathrm{Ric}(g) = -\Lambda g, \quad \Lambda < 0. \tag{1-1}$$

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Given $\Lambda < 0$, this is the simplest, or trivial, solution to (1-1), in the sense that the Minkowski or de Sitter spacetimes are the trivial solutions to the vacuum Einstein equation with $\Lambda = 0$ or $\Lambda > 0$. While the stability properties of the Minkowski or de Sitter spacetimes are now well understood [Christodoulou and Klainerman 1993; Friedrich 1986], the study of perturbations of AdS spacetime is still an active subject of research. One important aspect is that the AdS spacetime, or more generally spacetimes which are asymptotically AdS, are not globally hyperbolic. Hence, any evolution problem for these solutions is only well-posed after boundary conditions are imposed at the conformal boundary. Two naturally opposite classes of boundary conditions are the fully reflective and dissipative boundary conditions. In the reflective case, we expect — as originally conjectured by Dafermos and Holzegel [2006] and independently by Anderson [2006] — that the AdS spacetime is unstable. Strong evidence for this instability was first presented by Bizoń and Rostworowski [2011], who pioneered the study of the spherically symmetric Einstein–Klein–Gordon system using numerical and Fourier based analysis and proposed weak turbulence as the nonlinear source of instability. A proof of instability for the spherically symmetric Einstein–Vlasov[1] system was obtained in the work of Moschidis [2020; 2023] and is based on a physical space mechanism. In the dissipative case, one has strong decay of solutions for the linearized Einstein equations [Holzegel et al. 2020], and this should lead to stability even at the nonlinear level.

1B. ***The time-periodic solutions of Rostworowski–Maliborski.*** In the reflective case, parallel to the study of the instability conjecture, an interesting class of solutions was introduced by Rostworowski and Maliborski [2013], who constructed perturbatively and numerically small data, time-periodic solutions of the spherically symmetric Einstein-scalar field system. They furthermore suggested, based on their numerical analysis, that these solutions should enjoy stronger stability properties than the original AdS spacetime. The present paper is directly motivated by this work. We prove the existence of arbitrary small time-periodic solutions for various toy models, which mimic certain properties of nonlinear waves in the AdS spacetime.

1C. ***The conformal wave and the Yang–Mills equations.*** More precisely, we study the conformal wave and the Yang–Mills equations on the Einstein cylinder $\mathbb{R} \times \mathbb{S}^3$. The AdS spacetime is conformal to half of the Einstein cylinder, so that solutions to the conformal wave and the Yang–Mill equations on the AdS spacetime can be mapped to solutions on the entire Einstein cylinder with a certain reflection symmetry at the equator. The conformal cubic wave equation on the Einstein cylinder can be written as

$$-\partial_t^2 \phi(t,\omega) + \Delta_{\mathbb{S}^3}\phi(t,\omega) - \phi(t,\omega) = |\phi(t,\omega)|^2 \phi(t,\omega) \tag{1-2}$$

for a scalar field $\phi : \mathbb{R} \times \mathbb{S}^3 \to \mathbb{C}$ with $\phi = \phi(t,\omega)$. We will consider perturbations around the static solution $\phi_0 = 0$ and, for simplicity, with zero initial velocity.

In the spherically symmetric case, the initial value problem for (1-2) reduces to

$$\begin{cases} (\partial_t^2 + L)u = f(u), & (t,x) \in \mathbb{R} \times (0,\pi), \\ (u(0,x), \partial_t u(0,x)) = (u_0(x), 0), & x \in (0,\pi), \end{cases} \tag{1-3}$$

[1] Moschidis [2021] has further announced similar results for the spherically symmetric Einstein-scalar-field system.

for a scalar field $u : \mathbb{R} \times (0, \pi) \to \mathbb{R}$ with $u = u(t, x)$ and

$$Lu := -\Delta^{ss}_{\mathbb{S}^3} u + u, \quad -\Delta^{ss}_{\mathbb{S}^3} u = -\frac{1}{\sin^2(x)} \partial_x (\sin^2(x) \partial_x u), \quad f(u) = -u^3, \tag{1-4}$$

where $\Delta^{ss}_{\mathbb{S}^3}$ stands for the spherically symmetric Laplace–Beltrami operator on $\mathbb{S}^3$.

When the spherical symmetry assumption is removed [Ben Achour et al. 2016; Evnin 2021], we use an ansatz based on *Hopf coordinates*[2] $(\eta, \xi_1, \xi_2) \in \left[0, \frac{\pi}{2}\right] \times [0, 2\pi) \times [0, 2\pi)$ rather than the standard spherical coordinates. The Laplace–Beltrami operator on $\mathbb{S}^3$ in these coordinates reads as

$$\Delta^{\mathbb{S}^3}_{(\eta,\xi_1,\xi_2)} \chi = \partial^2_\eta \chi + \left(\frac{\cos\eta}{\sin\eta} - \frac{\sin\eta}{\cos\eta} \right) \partial_\eta \chi + \frac{1}{\sin^2\eta} \partial^2_{\xi_1} \chi + \frac{1}{\cos^2\eta} \partial^2_{\xi_2} \chi.$$

While in principle the Fourier expansion with respect to ξ_1 and ξ_2 of a solution $\chi(t, \eta, \xi_1, \xi_2)$ to (1-2) may include all possible admissible frequencies, we will pick a fixed pair of frequencies (μ_1, μ_2) and force the Fourier expansion to excite only this particular pair by implementing the ansatz

$$\chi(t, \eta, \xi_1, \xi_2) = u(t, \eta) e^{i\mu_1 \xi_1} e^{i\mu_2 \xi_2}. \tag{1-5}$$

This leads us to consider the initial value problem

$$\begin{cases} (\partial_t^2 + \mathsf{L}^{(\mu_1,\mu_2)}) u = \mathsf{f}(u), & (t, \eta) \in \mathbb{R} \times \left(0, \frac{\pi}{2}\right), \\ (u(0, \eta), \partial_t u(0, \eta)) = (u_0(\eta), 0), & \eta \in \left(0, \frac{\pi}{2}\right), \end{cases} \tag{1-6}$$

for a scalar field $u : \mathbb{R} \times \left(0, \frac{\pi}{2}\right) \to \mathbb{R}$ with $u = u(t, \eta)$ and

$$\mathsf{L}^{(\mu_1,\mu_2)} u = -\partial^2_\eta u - \left(\frac{\cos\eta}{\sin\eta} - \frac{\sin\eta}{\cos\eta} \right) \partial_\eta u + \left(\frac{\mu_1^2}{\sin^2\eta} + \frac{\mu_2^2}{\cos^2\eta} + 1 \right) u, \quad \mathsf{f}(u) = -u^3. \tag{1-7}$$

Finally, we consider the spherically symmetric (equivariant) Yang–Mills equation for the SU(2) connection A on the Einstein cylinder $\mathbb{R} \times \mathbb{S}^3$ endowed with the metric

$$g(t, \omega) = -dt^2 + dx^2 + \sin^2(x)\, d\omega^2, \tag{1-8}$$

where $d\omega^2$ stands for the standard round metric on the 2-sphere. The connection $A_\mu = A^\nu_\mu \tau_\nu$ is a 1-form that takes values in the Lie algebra su(2). Here, τ_α stand for the generators of su(2) that satisfy $[\tau_a, \tau_b] = i\epsilon_{abc}\tau_c$. Furthermore, the curvature F is a $(2, 0)$-tensor defined by $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu + [A_\mu, A_\nu]$. The Euler–Lagrange equations associated to the action

$$\int_{\mathbb{R}\times\mathbb{S}^3} \operatorname{tr}(F_{\mu\nu} F^{\mu\nu}) \sqrt{-\det(g)}$$

are provided by the Yang–Mills equation

$$\nabla_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0. \tag{1-9}$$

Following [Bizoń 1993; 2014; Bizoń and Mach 2017], we assume the spherically symmetric purely magnetic ansatz

$$A = \phi(t, x)\eta + \tau_3 \cos(\vartheta)\, d\varphi, \quad \eta = \tau_1\, d\vartheta + \tau_2 \sin(\vartheta)\, d\varphi,$$

[2]We would like to thank Oleg Evnin who suggested the Hopf coordinate ansatz.

which yields

$$F = \partial_t \phi(t,x)\, dt \wedge \eta + \partial_x \phi(t,x)\, dx \wedge \eta - (1-\phi^2(t,x))\tau_3\, d\vartheta \wedge \sin(\vartheta)\, d\varphi.$$

In this case, a straightforward computation shows that (1-9) reduce to

$$-\partial_t^2 \phi(t,x) + \partial_x^2 \phi(t,x) + \frac{\phi(t,x)}{\sin^2(x)} = \frac{\phi^3(t,x)}{\sin^2(x)} \tag{1-10}$$

for a scalar field $\phi : \mathbb{R} \times (0,\pi) \to \mathbb{R}$ with $\phi = \phi(t,x)$. We will study perturbations of the static solution $\phi_0 = 1$ to the equation above [Bizoń 2014]. Writing $\phi(t,x) = 1 + \sin^2(x) u(t,x)$, we are led to the initial value problem

$$\begin{cases} (\partial_t^2 + \mathfrak{L})u = \mathfrak{f}(x,u), & (t,x) \in \mathbb{R} \times (0,\pi), \\ (u(0,x), \partial_t u(0,x)) = (u_0(x), 0), & x \in (0,\pi), \end{cases} \tag{1-11}$$

where

$$\mathfrak{L}u = -\frac{1}{\sin^4 x}\partial_x(\sin^4 x\, \partial_x u) + 4u, \qquad \mathfrak{f}(x,u) = -3u^2 - \sin^2(x) u^3. \tag{1-12}$$

1D. ***Connection of the models to the fixed AdS spacetime.*** In the following lines, we discuss the connection of the models (1-3), (1-6) and (1-11), and we consider two dynamical problems related to the AdS spacetime. The Einstein static universe is the cylinder $\mathbb{R} \times \mathbb{S}^3$ endowed with the metric given by (1-8), and the AdS spacetime is conformal to only the part of the entire Einstein cylinder which is given by $\mathbb{R} \times \mathbb{S}^3_+$, where $\mathbb{S}^3_+$ denotes the upper hemisphere of $\mathbb{S}^3$. Since both the cubic conformal wave equation and the Yang Mills equation are conformally invariant, this implies that solutions to the cubic conformal wave and Yang–Mills equations can be mapped to solutions of the same equations on $\mathbb{R} \times \mathbb{S}^3_+$. Depending on the choice of boundary conditions at the conformal infinity of the AdS spacetime, these solutions can then be extended on the whole of the Einstein cylinder via a reflection symmetry; see for example [Bizoń et al. 2017, Remark 1].

The models we consider here also preserve several key features of the Einstein–Klein–Gordon system in spherical symmetry, and for which we have reliable numerical evidence for the existence of time-periodic solutions due to [Maliborski and Rostworowski 2013] (see Section 1B). Indeed, in all cases, the spectrum is completely resonant and the eigenfunctions to the linearized operators are weighted Jacobi polynomials. As a consequence, the derivation and analysis of the Fourier and resonant systems share many properties. In addition, although quasilinear, the Einstein–Klein–Gordon system in spherical symmetry has a cubic leading-order nonlinearity as in the models we considered here. More importantly, the existence of the time-periodic solutions we construct here depends on the so-called nondegeneracy condition (Section 7). This is a system of infinitely many nonlinear conditions for oscillatory integrals that quantify the mode couplings, relying on the analysis of the underlying Fourier system. In this paper, we develop a rigorous and delicate analysis for the Fourier coefficients (Section 5) by establishing closed formulas, as well as rigorous asymptotic analysis in the case where the closed formulas for these integrals are too complicated to handle. Besides their strong numerical evidence, Rostworowski and Maliborski [2013] also suggest that there should be an analogous nondegeneracy condition for the quasilinear Einstein–Klein–Gordon system in spherical symmetry. The computation and the analysis of the Fourier system for the Einstein–Klein–Gordon system is a challenge in itself, see for example [Chatzikaleas 2024; Craps et al. 2014; 2015a; 2015b; Evnin and Jai-akson 2016], and we believe that the type of analysis for the Fourier coefficients developed here should find applications there as well.

1E. *Main results and general strategy.* In the following, we use the abbreviations

- CW: conformal cubic wave equation in spherical symmetry, that is (1-3)–(1-4),
- CH: conformal cubic wave equation out of spherical symmetry in Hopf coordinates according to the ansatz (1-5), that is (1-6)–(1-7),
- YM: Yang–Mills equation in spherical symmetry, that is (1-11)–(1-12),

and study the evolution of the perturbations

$$u : \mathbb{R}\times I \to \mathbb{R}, \qquad u = u(t,x), \qquad I = \begin{cases} (0,\pi) & \text{for CW},\\ \left(0,\frac{\pi}{2}\right) & \text{for CH},\\ (0,\pi) & \text{for YM},\end{cases}$$

driven by the partial differential equations

$$(\partial_t^2 + \boldsymbol{L})u = \boldsymbol{f}(x,u), \qquad (t,x)\in\mathbb{R}\times I, \tag{1-13}$$

subject to the initial data $u_0(x) = u(0,x)$ with zero initial velocity $u_1(x) = \partial_t u(0,x) = 0$ for all $x \in I$. Here, the linear operators and the nonlinearities are given, respectively, by

$$\boldsymbol{L}u = \begin{cases} -\dfrac{1}{\sin^2(x)}\partial_x(\sin^2(x)\partial_x u) + u & \text{for CW},\\ -\partial_x^2 u - \left(\dfrac{\cos x}{\sin x} - \dfrac{\sin x}{\cos x}\right)\partial_x u + \left(\dfrac{\mu_1^2}{\sin^2 x} + \dfrac{\mu_2^2}{\cos^2 x} + 1\right)u & \text{for CH},\\ -\dfrac{1}{\sin^4 x}\partial_x(\sin^4 x\,\partial_x u) + 4u & \text{for YM},\end{cases} \tag{1-14}$$

$$\boldsymbol{f}(x,u) = \begin{cases} -u^3 & \text{for CW and CH},\\ -3u^2 - \sin^2(x)u^3 & \text{for YM}.\end{cases} \tag{1-15}$$

Associated to the linear operators given by (1-14), one can introduce natural Hilbert spaces, and with suitable definitions for their domains (Section 3), the linear operators are then all self-adjoint operators with compact resolvent. In order to simplify the presentation below, we denote by $\{e_n(x) : n \geq 0\}$ the set of eigenfunctions of any of these operators[3] and by $\{\omega_n^2 : n \geq 0\}$ the set of corresponding eigenvalues. Recall that, in all three models considered, the sequences $\{\omega_n : n \geq 0\}$ are all strictly increasing with $\omega_n \sim n$ as $n \to +\infty$.

We also denote by $\Phi^t(\xi)$ the flow associated to any of the linearized equations with initial data $(u_{t=0}, \partial_t u_{t=0}) = (\xi, 0)$. If we use ξ_n to denote the Fourier coefficients of ξ associated to the eigenbasis $\{e_n(x) : n \geq 0\}$, then

$$\Phi^t(\xi) = \sum_{n=0}^{\infty} \cos(t\omega_n)\xi_n e_n(x). \tag{1-16}$$

[3] Of course, the eigenfunctions are different for the different operators, so this is just a generic name.

To state our result, we need to introduce a set of frequencies verifying a certain Diophantine condition [Bambusi and Paleari 2001]. Given $0 < \alpha < \frac{1}{3}$, define

$$\mathcal{W}_\alpha = \Big\{\omega \in \mathbb{R} : |\omega \cdot l - \omega_j| \geq \frac{\alpha}{l} \,\forall (l, j) \in \mathbb{N}^2,\, l \geq 1,\, \omega_j \neq l\Big\}. \tag{1-17}$$

According to [Bambusi and Paleari 2001, Remark 2.4] and [Schmidt 1980, p. 23], the set $\mathcal{W}_\alpha$ contains infinitely many irrationals, is uncountable and accumulates at 1 from above and below. Consider any of the problems CW, CH or YM, and let e_γ be one of the eigenfunctions to the corresponding linear operator. In addition to α, the statements of our results depend on the constant $\gamma \in \mathbb{N} \cup \{0\}$, the index of the eigenfunction, and $s > 0$, which defines the Sobolev space[4] H^s where the solutions will belong. Our assumptions are slightly different depending on the problems addressed.

Assumptions 1.1. Specifically, we make the following assumptions:

- CW: We take $\gamma \in \{0, 1, 2, \dots\}$ and $s \in \mathbb{N}$ with $s \geq 2$.
- CH: We take $\gamma \in \{0, 1, 2, 3, 4, 5\}$ and $s \in \mathbb{N}$ with $s \geq 2$. Moreover, we assume that the parameters μ_1 and μ_2 appearing in (1-6) satisfy $\mu_1 = \mu_2 = \mu$, with μ either sufficiently large, or $\mu \in \{0, 1, 2, 3, 4, 5\}$.
- YM: We take $\gamma \in \{0, 1, 2, 3, 4, 5\}$ and $s \in \mathbb{N}$ with $s \geq 3$.

Remark 1.2 (range of γ). We note that our proof is based on closed formulas for the Fourier coefficients, integrals that quantify the mode couplings. Although we derive these formulas uniformly with respect to γ (see Section 5), we also need to check the validity of particular nonlinear conditions depending on the Fourier coefficients. On the one hand, for the CW model, the Fourier coefficients have a relatively simple closed formula. Hence, there is no need to restrict the range of γ and we establish the validity of the conditions needed uniformly with respect to γ. On the other hand, for the CH and YM models, the complexity of the Fourier coefficients requires us to restrict the range of γ to any finite set. Since the smaller the range the easier one can verify our computations, we fix $\gamma \in \{0, 1, 2, 3, 4, 5\}$ solely for the purpose of computing and verifying all computations in the manuscript by hand. However, we believe that our result stated below also holds true for larger values of γ. The interested reader can access our Mathematica notebooks as ancillary files posted with the present paper on arXiv at https://arxiv.org/abs/2201.05447 to both easily verify our computations for small γ as well as derive and verify the analogous closed formulas for larger values of γ.

Under Assumptions 1.1, we prove the following result.

Theorem 1.3 (main result 1: existence of time-periodic solutions to all three models bifurcating from various 1-modes). *Let $(\gamma, s) \in (\mathbb{N} \cup \{0\}) \times \mathbb{R}$ satisfy Assumptions 1.1, and let e_γ be the eigenfunction to the corresponding linear operator. Also, let $0 < \alpha < \frac{1}{3}$ and $\mathcal{W}_\alpha$ be the corresponding set of frequencies, defined in (1-17). Then, there exists a family $\{u_\epsilon : \epsilon \in \mathcal{E}_{\alpha,\gamma}\}$ of time-periodic solutions to either CW, CH or YM, where $\mathcal{E}_{\alpha,\gamma}$ is an uncountable set that has 0 as an accumulation point. In addition, each element u_ϵ has the following properties*:

[4]The definition of the H^s spaces is adapted to each problem; see Section 2.

(1) *u_ϵ has period $T_\epsilon = 2\pi/\omega_\epsilon$ with $\omega_\epsilon \in \mathcal{W}_\alpha$ being ϵ-close to* 1.

(2) $u_\epsilon \in H^1([0, T_\epsilon]; H^s)$.

(3) *u_ϵ stays close to the solution to the linearized equation with initial data $(u_{t=0}, \partial_t u_{t=0}) = (\epsilon\kappa_\gamma e_\gamma, 0)$ for all times*:

$$\sup_{t\in\mathbb{R}} \|u_\epsilon(t,\cdot) - \Phi^{t\omega_\epsilon}(\epsilon\kappa_\gamma e_\gamma)\|_{H^s} \lesssim \epsilon^2,$$

where κ_γ is a normalization constant.

Proof. The result follows by applying the original version of Bambusi–Paleari's theorem (Theorem 2.4 for CW and CH) and our modified version (Theorems 1.4 and 2.5 for YM) by verifying their main conditions; see Sections 6 and 7. □

For the CW and CH models, the results above are proven using a theorem of Bambusi and Paleari [2001], while for the YM model, the original version of their theorem (stated as Theorem 2.4 below) is not applicable and we will adapt their work. To explain this, we follow [Bambusi and Paleari 2001] and consider any of the models above in the Fourier space by projecting the equations on the eigenbasis $\{e_n : n \geq 0\}$, so that, schematically, the equations take the form

$$\ddot{u}^j(t) + (Au(t))^j = (f(u))^j \tag{1-18}$$

for all integers $j \geq 0$, where $u = \{u^j : j \geq 0\}$ denotes the array of the coefficients in the Fourier space, A is a multiplication operator defined by $(Au)^j = \omega_j^2 u^j$ and $(f(u))^j$ are the coefficients of the nonlinearity written in Fourier space, which takes the form of a polynomial in the u^j. In addition, we assume that

$$f(u) = f^{(0)}(u) + f^{(1)}(u),$$

where $f^{(0)}$ is a homogeneous polynomial of degree $r \geq 2$ and $f^{(1)}$ is a polynomial of degree at least $r+1$. Then, one looks for solutions $u(t)$ to (1-18), where $u(t)$ belongs to the Hilbert space

$$l_s^2 = \{u = \{u^j : j \geq 0\} : |u|_s^2 < \infty\}, \quad |u|_s^2 = \sum_{j=0}^{\infty} j^{2s}|u_j|^2.$$

Besides some regularity considerations, the main theorem in [Bambusi and Paleari 2001] asserts that, given any nondegenerate zero of the operator

$$\mathcal{M}\xi = A\xi + \langle f^{(0)}\rangle(\xi), \quad \xi = \{\xi^j : j \geq 0\} \in l_s^2,$$

where $\langle f^{(0)}\rangle(\xi)$ denotes the average in time of the nonlinearity $f^{(0)}$ along the linearized flow, one can construct a family of small data periodic in time solutions, with properties similar to those stated in Theorem 1.3. The operator $\mathcal{M}$ is in fact linked to the resonant system associated to the original equation. If $u(t)$ is periodic in time with frequency ω, let q be defined by $u(t) = q(\omega t)$, and let L_ω be the operator

$$L_\omega q = \omega^2 \frac{d^2}{dt^2} q + Aq.$$

The proof of [Bambusi and Paleari 2001] is based on a Lyapunov–Schmidt decomposition $q = q_\perp + v$, with $v \in \ker L_1$ and $q_\perp \in (\ker L_1)^\perp$, together with the projections of the equations onto the range and kernel of L_1, leading to the so-called P-equation and Q-equation, defined, respectively, as

$$L_{\varpi} q_{\perp} = Pf(v+q_{\perp}), \tag{1-19}$$

$$(1-\omega^2)Av = Qf(v+q_{\perp}). \tag{1-20}$$

The Diophantine condition (1-17) is then used to solve the P-equation, while the nondegeneracy assumption and an implicit function argument is used to solve the Q-equation.

For the CW and CH models, one easily verifies that the eigenfunctions $\kappa_n e_n$, where κ_n is an appropriate rescaling, are all zeroes of $\mathcal{M}$, so that the main difficulty is to establish the nondegeneracy condition, i.e., to prove that the kernel of $d\mathcal{M}(\kappa_n e_n)$ is trivial. In the YM case, however, the nonlinearity contains both quadratic and cubic terms, so that a priori, only the quadratic terms would contribute to the definition of the operator $\mathcal{M}$. On the other hand, it turns out that the average along the flow of the quadratic terms actually vanishes identically, leading to a degenerate, linear operator $\mathcal{M}$. Thus, we introduce a replacement for the operator $\mathcal{M}$ that takes into account also the cubic terms. However, the quadratic terms still play a role in this modified operator. Indeed, the solution to the P-equation roughly takes the form $q_{\perp}(v) = q_{\perp,\text{quadratic}}(v) + q_{\perp,\text{cubic}}(v) + \cdots$, where the term $q_{\perp,\text{quadratic}}(v)$ arises from the quadratic nonlinearity, and after substituting $q_{\perp}(v)$ into the Q-equation, these terms will generate new additional cubic terms. Thus, in some sense, the backreaction of the quadratic terms into the Q-equation must also be taken into account. This type of difficulty, where the contribution of the lowest degree part of the nonlinearity is nonresonant, has been treated in some situations; see for instance [Berti and Bolle 2003, Section 4.2] and [Berti and Bolle 2006, Section 1.2.3], where equations of the form $-\partial_{tt}u + \partial_{xx}u = u^{2p} + \mathcal{O}(u^{2p+1})$ were considered. Here, we prove a modified abstract version of the Bambusi–Paleari theorem which we then apply to the YM model.

Theorem 1.4 (main result 2: modification of the Bambusi–Paleari theorem for the YM model). *Consider the partial differential equation in the Fourier space*

$$\ddot{u}^j(t) + (\mathfrak{A}u(t))^j = (\mathfrak{f}(u(t)))^j, \quad j \geq 0, \tag{1-21}$$

where the dots stand for derivatives with respect to time and $\mathfrak{A}$ is a positive multiplication self-adjoint operator with pure point and resonant spectrum $\{\varpi_j^2 > 0 : j \geq 0\}$, with $\varpi_j \sim j$ as $j \to \infty$, defined by

$$\mathfrak{A} : \mathcal{D}(\mathfrak{A}) \simeq l^2_{s+2} \to l^2_s, \quad (\mathfrak{A}u)^j = \varpi_j^2 u_j,$$

with $\mathcal{D}(\mathfrak{A})$ being its maximal domain of definition.[5] *In addition, assume that the nonlinearity is given by*

$$\mathfrak{f}(u) = \mathfrak{f}^{(2)}(u) + \mathfrak{f}^{(3)}(u),$$

where each $\mathfrak{f}^{(k)}$ is a homogeneous polynomial of order k which is well defined and smooth in l^2_s, with $\mathfrak{f}^{(2)}$ being ***nonresonant***, *that is*

$$\langle \mathfrak{f}^{(2)} \rangle(x) = \frac{1}{2\pi} \int_0^{2\pi} \Phi^t(\mathfrak{f}^{(2)}(\Phi^t(x)))\, dt = 0 \tag{1-22}$$

[5]Later, we will take l^2_{s+1} instead of l^2_s as our base Hilbert space, so that we will consider $\mathfrak{A}$ as an operator from $l^2_{s+3} \to l^2_{s+1}$. This allows for the construction of classical solutions, instead of solutions defined via the Duhamel formula or duality.

*for all initial data x, where $\Phi^t(x)$ denotes the solution to the linearized equation with initial data $(x, 0)$. Furthermore, define the **modified** operator*

$$\mathfrak{M}_\pm(\xi) = \pm\mathfrak{A}\xi + \langle \mathfrak{f}^{(3)} \rangle(\xi) + \mathfrak{F}_0(\xi),$$

where $\mathfrak{F}_0(\xi)$ is given by Lemma 2.13, and let $\xi_0 \in l^2_{s+3}$ be initial data such that

- *ξ_0 is a **zero** of $\mathfrak{M}_\pm$,*

$$\mathfrak{M}_\pm(\xi_0) = 0,$$

- *and ξ_0 satisfies the **nondegeneracy condition***

$$\ker(d\mathfrak{M}_\pm(\xi_0)) = \{0\}.$$

Also, for some $0 < \alpha < \frac{1}{3}$, define $\mathcal{W}_\alpha$ according to (1-17). Then, there exists a family $\{u_\epsilon(t, \cdot) : \epsilon \in \mathcal{E}_{\alpha,\gamma}\}$ of time-periodic solutions to (1-21), where $\mathcal{E}_{\alpha,\gamma}$ is an uncountable set that has 0 as an accumulation point. In addition, each element u_ϵ has the following properties:

(1) *u_ϵ has period $T_\epsilon = 2\pi/\omega_\epsilon$, with $\omega_\epsilon \in \mathcal{W}_\alpha$ being ϵ-close to* 1.

(2) $u_\epsilon \in H^1([0, T_\epsilon]; l^2_s)$.

(3) *u_ϵ stays close to the solution to the linearized equation with initial data $(u_{t=0}, \partial_t u_{t=0}) = (\epsilon\xi_0, 0)$ for all times*:

$$\sup_{t\in\mathbb{R}} |u_\epsilon(t, \cdot) - \Phi^{t\omega_\epsilon}(\epsilon\xi_0)|_s \lesssim \epsilon^2.$$

1F. *Remarks.* • Minimal periods: Theorems 1.3 and 1.4 give no information on the minimal periods T_ϵ of the time-periodic solutions $u_\epsilon(t, \cdot)$. However, one can relate T_ϵ to the minimal period T_0 of the solutions to the linearized system with 1-mode initial data; see [Bambusi and Paleari 2001, Theorem 5.3].

• Cantor-like set: We emphasize that the time-periodic solutions we construct exist only when the small perturbative parameter belongs to a Cantor-like set (of measure 0). This set together with the Diophantine condition introduced in Theorem 1.3 are closely related to the presence of small divisors in the perturbation series around equilibrium points, a classical topic in the context of Kolmogorov–Arnold–Moser (KAM) theory in infinite dimensions. Although this type of condition is essential in proving the existence of time-periodic solutions as in [Bambusi and Paleari 2001], it can be removed in some very special cases; see for example [Chatzikaleas 2020]. On the other hand, we note that the numerical constructions [Choptuik et al. 2018; Fodor et al. 2015; Maliborski and Rostworowski 2013] do not seem to see the small divisors obstructions.

• Proof: The proof of Theorem 1.4 follows the general strategy of [Bambusi and Paleari 2001], the main and essential difference being the backscattering contribution of the quadratic terms. An alternative approach to ours would be to find a normal form, in the spirit of [Shatah 1985], that allows us to eliminate the quadratic terms and then apply the original result of [Bambusi and Paleari 2001].

• The works of Berti and Bolle: In [Berti and Bolle 2003; 2004; 2006], a different strategy, based on variational methods, was introduced to solve the Q-equation (1-20) instead of the implicit function theorem as in [Bambusi and Paleari 2001]. This in particular leads to a strong improvement in [Berti and Bolle 2006] concerning the size of the frequency set, using an extra Nash–Moser iteration. We have not implemented this here for simplicity and leave a possible implementation of this improvement for future works. The works [Berti and Bolle 2004; 2006] also treat the case of nonresonant quadratic terms — as we have here in the Yang–Mills case — in the specific case of the wave equation on an interval with Dirichlet boundary conditions.

• Regularity of the solutions: The solutions constructed here are H^1 in time with values in H^s, with s arbitrarily large but fixed a priori. A posteriori, one can then use the equation to obtain additional regularity properties of the solutions. For instance, one easily has $\partial_t^2 u \in L_t^2 H_x^{s-1}$. Since some of the estimates depend a priori on the value of s, we cannot directly take $s = \infty$, but it is likely that a refinement of the methods presented would lead to such an improvement.

• Jacobi polynomials: One of the difficulties to proving Theorem 1.3 comes from the fact that the eigenfunctions $e_n(x)$ of the linearized operators are given by Jacobi polynomials instead of simpler explicit functions. This fact is not specific to our model problem and is a general feature of nonlinear wave equations on AdS-like background. In particular, in the CH and YM models,[6] the computation and the analysis of the Fourier coefficients associated to the resonant terms are nontrivial and constitute one of the contributions of this paper. To this end, we use linearization and connection formulas as well as particular Mellin transforms for the Jacobi polynomials. On the one hand, a linearization formula (also called addition formula) represents a product of two orthogonal polynomials with some parameters as a linear combination of orthogonal polynomials of the same kind with the same parameters. On the other hand, a connection formula represents a single orthogonal polynomial with some parameters as a linear combination of orthogonal polynomials of different kinds with new parameters. In both cases, these are computationally efficient only in the case where the coefficients in the expansions are known in closed formulas. These computations also motivate our choice of $\mu_1 = \mu_2 = \mu$ for the CH model, since in this case, the eigenfunctions are reduced to Gegenbauer polynomials, a special class of Jacobi polynomials with additional algebraic properties that lead to closed formulas for the linearization and connection coefficients described above. Moreover, we also use particular Mellin transforms of Gegenbauer polynomials. These are integral transforms that may be regarded as the multiplicative version of the Laplace transform.

• Mathematica files: For the CH and YM models, Theorem 1.3 ensures the existence of time-periodic solutions bifurcating only from finitely many 1-mode initial data. As stated in Remark 1.2, this is solely for the purpose of computing and verifying all computations in the manuscript by hand. Furthermore, one can use Mathematica to verify that our result still holds true also for larger values of γ. For the convenience of the reader, our Mathematica notebooks — available as ancillary files to the present paper on arXiv at https://arxiv.org/abs/2201.05447 — can help the reader to both easily verify our computations for small γ as well as derive and verify the analogous computations for larger values of γ.

[6]The eigenfunctions for the CW case are given by Chebyshev polynomials of the second kind. The derivation of the resonant system in this case had been previously addressed in [Bizoń et al. 2017].

1G. ***Previous works.*** The conformal wave equation (1-2) was introduced as a toy problem for the study of nonlinear waves in confined geometries in [Bizoń et al. 2017] and has been studied further in [Bizoń et al. 2019; 2020; Chatzikaleas 2020]. In particular, [Chatzikaleas 2020] proved that solutions emanating from the first mode e_0 stay proportional to e_0 for all times and are periodic in time. The fact these data do not excite further modes is, however, specific to the first mode and to this equation.

Concerning the well-posedness theories for the different models, since we do not focus here on low regularity solutions, we will simply recall that global well-posedness holds for the conformal cubic wave equation in the energy space, while the Yang–Mills equations in curved geometry have been shown to be globally well-posed in $H^2 \times H^1$ [Choquet-Bruhat et al. 1983; Chruściel and Shatah 1997] and on AdS with reflective boundary conditions [Choquet-Bruhat 1989]. We were motivated to study the Yang–Mills model by [Bizoń 2014].

Since the pioneering work [Maliborski and Rostworowski 2013], there have been many investigations of time-periodic solutions for nonlinear equations with completely resonant spectrum [Berti and Bolle 2003; 2004; 2006; Paleari et al. 2001]. For the conformal wave equations, there exist also several constructions of time-periodic weak solutions via the variational techniques first introduced by Rabinowitz [1978a; 1978b]; see [Chang and Hong 1985; Zhou 1986].

1H. ***Organization of the paper.*** We split the paper into the following sections:

- Section 2: We describe the methods we are about to use. For CW and CH, we will use the original version of Bambusi and Paleari's theorem [2001] (Theorem 2.4). However, for YM, as explained above, we need to revise the original version and establish an extension of their result (Theorem 2.5) as stated in Theorem 1.4. In particular, we define the operators $\mathcal{M}$ and $\mathfrak{M}_{\pm}$, which determine the "special" initial data leading to time-periodic solutions.

- Section 3: We study the linear eigenvalue problems where the linearized operators are given by (1-14). As it turns out, the associated eigenfunctions are given by Jacobi polynomials, which is a common feature with the Einstein–Klein–Gordon system in spherical symmetry [Maliborski and Rostworowski 2013].

- Section 4: We express the partial differential equations (1-13) in the Fourier space and obtain infinite dimensional systems of coupled harmonic oscillators.

- Section 5: We define and study the mode couplings given by the Fourier coefficients. Specifically, we derive explicit closed formulas for all the Fourier coefficients on resonant indices.

- Section 6: We study 1-mode initial data. In particular, we show that these modes satisfy the resonant systems (are zeros of the operators $\mathcal{M}$ and $\mathfrak{M}_{\pm}$ defined in Section 2). In addition, we derive their differentials at these modes.

- Section 7: Firstly, we derive the crucial nondegeneracy conditions for the 1-mode initial data. As it turns out, these are nonlinear conditions for the Fourier coefficients. Then, we use the analysis on the Fourier coefficients from Section 5 to rigorously establish these conditions and prove the existence of time-periodic solutions as stated in Theorem 1.3.

1I. *Notation.* We use different notation for each of the models we consider, which we summarize here:

model equation font	CW (1-3)–(1-4) standard	CH (1-6)–(1-7) serif	YM (1-11)–(1-12) fraktur
linearized operator	L	$\mathsf{L}^{(\mu_1,\mu_2)}$	$\mathfrak{L}$
eigenvalues	ω_n	$\omega_n^{(\mu_1,\mu_2)}$	ϖ_n
eigenfunctions	e_n	$\mathsf{e}_n^{(\mu_1,\mu_2)}$	$\mathfrak{e}_n$
inner product	$(\cdot\,|\,\cdot)$	$\langle\cdot\,|\,\cdot\rangle$	$[\cdot\,|\,\cdot]$
linear flow	Φ^t	ϕ^t	$\mathit{\Phi}^t$
fourier coefficients	C	$\mathsf{C}^{(\mu_1,\mu_2)}$	$\mathfrak{C}$, $\overline{\mathfrak{C}}$

2. The method of Bambusi–Paleari revisited

In order to establish Theorem 1.3 and construct our time-periodic solutions, we rely on the method of Bambusi and Paleari [2001]. This is an effective method to construct families of small amplitude time-periodic solutions close to a resonant equilibrium point for semilinear partial differential equations.

2A. *The original version of the theorem.* Let $s \geq 0$ be a real number, and define the Hilbert space l_s^2 to be the space of sequences such that

$$u = \{u^j : j \geq 0\}, \quad |u|_s^2 = \sum_{j=0}^{\infty} |j^s u^j|^2 < \infty.$$

We endow l_s^2 with the natural inner product associated with the norm $|\cdot|_s$ and consider the following differential equation in l_s^2:

$$\ddot{u}^j + (\mathfrak{A}u)^j = (\mathfrak{f}(u))^j, \quad (\mathfrak{A}u)^j = \varpi_j^2 u_j, \tag{2-1}$$

for all integers $j \geq 0$, where the dots denote derivatives with respect to time. Here, $\mathfrak{A} : \mathcal{D}(\mathfrak{A}) \to l_s^2$ is a positive multiplication self-adjoint operator with pure point and resonant spectrum $\{\varpi_j^2 : j \geq 0\}$, meaning $\{\varpi_j : j \geq 0\} \subset \mathbb{N}$, and $\mathcal{D}(\mathfrak{A})$ stands for its maximal domain of definition endowed with the norm

$$\|u\|_{\mathcal{D}(\mathfrak{A})}^2 = |u|_s^2 + |\mathfrak{A}u|_s^2 = \sum_{j=0}^{\infty} j^{2s} |\xi^j|^2 + \sum_{j=0}^{\infty} j^{2s} |\varpi_j^2 \xi^j|^2.$$

Moreover, we also assume that $\mathfrak{A}$ and f verify the following conditions:

(1) The injection of $(\mathcal{D}(\mathfrak{A}), \|\cdot\|_{\mathcal{D}(\mathfrak{A})})$ into l_s^2 is compact.

(2) The nonlinearity $\mathfrak{f}(u)$ can be decomposed into

$$\mathfrak{f}(u) = \mathfrak{f}^{(0)}(u) + \mathfrak{f}^{(1)}(u). \tag{2-2}$$

(3) The lowest-degree term $\mathfrak{f}^{(0)}(u)$ is a homogeneous polynomial of order $r \geq 2$ and is a bounded operator from $\mathcal{D}(\mathfrak{A})$ to $\mathcal{D}(\mathfrak{A})$ with the domain $\mathcal{D}(\mathfrak{A})$ being invariant under $\mathfrak{f}^{(0)}$.

(4) The highest-degree term $\mathfrak{f}^{(1)}(u)$ (treated perturbatively as an error term) has a zero of order $r+1$ at 0, is differentiable in l^2_s, and its differential is Lipschitz and satisfies the estimate

$$|d\mathfrak{f}^{(1)}(u_1) - d\mathfrak{f}^{(1)}(u_2)|_s \le C\epsilon^{r-1}|u_1 - u_2|_s$$

for all $u_1, u_2 \in l^2_s$ with $|u_1|_s \le \epsilon$ and $|u_2|_s \le \epsilon$.

Remark 2.1. In our case, the conditions above are obtained by starting from any of the equations (1-3), (1-6), (1-11) and projecting them on the eigenfunctions to the corresponding linear operator. Specifically, condition (1) follows automatically from the fact that $\varpi_j \sim j$ as $j \to \infty$, while conditions (3) and (4), which refer to the nonlinearities in the Fourier space, essentially follow from the facts that the original nonlinearities are smooth and that the Sobolev spaces of sufficiently high regularity form an algebra; see Section 2C.

Let $\mathfrak{e}_n$ be the eigenfunctions to the associated linearized operators. On the Fourier side, these can be identified with $\mathfrak{e}_n = \{\delta^i_n : i \ge 0\} \in l^2_s$. Then, for any initial data ξ, we denote by

$$\Phi^t(\xi) = \{\xi^n \cos(\varpi_n t) : n \ge 0\}, \quad \xi = \{\xi^n : n \ge 0\} = \sum_{n=0}^{\infty} \xi^n \mathfrak{e}_n$$

its linear flow, that is the solution to the initial value problem

$$\begin{cases} \ddot{u}^n(t) + \varpi_n^2 u^n(t) = 0, & t \in \mathbb{R}, \\ u^n(0) = \xi^n, & \dot{u}^n(0) = 0. \end{cases}$$

We note that $\Phi^t(\xi) = \Phi^{-t}(\xi)$. Moreover, we define the operator

$$\mathcal{M}(\xi) := \mathfrak{A}\xi + \langle \mathfrak{f}^{(0)} \rangle(\xi), \quad \langle \mathfrak{f}^{(0)} \rangle(\xi) = \frac{1}{2\pi}\int_0^{2\pi} \Phi^{-t}[\mathfrak{f}^{(0)}(\Phi^t(\xi))]\,dt, \tag{2-3}$$

where $\langle \mathfrak{f}^{(0)} \rangle(\xi)$ is the average of $\mathfrak{f}^{(0)}$ along the linear flow. Here we note that the highest-degree term $\mathfrak{f}^{(1)}$ in (2-2) does not contribute to the definition of the operator $\mathcal{M}$. Bambusi and Paleari [2001] used a Lyapunov–Schmidt decomposition together with averaging theory and established the existence of a family of small amplitude time-periodic solutions with frequencies that satisfy the strong Diophantine condition

$$\varpi \in \mathcal{W}_\alpha = \Big\{\varpi \in \mathbb{R} : |\varpi \cdot l - \varpi_j| \ge \frac{\alpha}{l}\ \forall (l, j) \in \mathbb{N}^2,\ l \ge 1,\ \varpi_j \ne l\Big\}. \tag{2-4}$$

Remark 2.2 (accumulation to 1). For $0 < \alpha < \frac{1}{3}$, the set $\mathcal{W}_\alpha$ is an uncountable Cantor-like set that accumulates to 1 from above and below; see [Bambusi and Paleari 2001, Remark 2.4] and [Schmidt 1980, p. 23].

Remark 2.3 (connection to Hurwitz's theorem). According to Hurwitz's theorem, for every irrational number ϖ, there are infinitely many relatively prime integers ϖ_j and l such that $|\varpi \cdot l - \varpi_j| < 1/(\sqrt{5}l)$, and moreover the constant $\sqrt{5}$ is optimal. Consequently, $\mathcal{W}_\alpha = \varnothing$ for $\alpha \ge 1/\sqrt{5}$. In this note, we pick a suitable α with $0 < \alpha < \frac{1}{3}$.

The main result of [Bambusi and Paleari 2001] reads as follows.

Theorem 2.4 (original version of Bambusi and Paleari's theorem [2001]). *For $0<\alpha<\frac{1}{3}$, define $\mathcal{W}_\alpha$ according to (2-4) and consider the operator $\mathcal{M}$ defined in (2-3). Assume that conditions (1)–(4) are verified. Moreover, let ξ_0 be a nondegenerate zero of $\mathcal{M}$, that is*

$$\mathcal{M}(\xi_0)=0, \quad \ker(d\mathcal{M}(\xi_0))=\{0\}.$$

Then, there exists a family $\{u_\epsilon:\epsilon\in\mathcal{E}_{\alpha,\gamma}\}\subset H^1([0,T_\epsilon];l^2_s)$ of time-periodic solutions to (2-1)–(2-2), where $\mathcal{E}_{\alpha,\gamma}$ is an uncountable set that has 0 as an accumulation point. In addition, each element u_ϵ has the following properties:

(1) *u_ϵ has period $T_\epsilon=2\pi/\varpi_\epsilon$, and there exists $\varpi_\star>0$ such that the map*

$$\epsilon\in\mathcal{E}_{\alpha,\gamma}\mapsto\varpi_\epsilon\in\mathcal{W}_\alpha\cap[1,1+\varpi_\star)$$

is a monotone, one-to-one map that stays close to 1: $|1-\varpi_\epsilon|\lesssim\epsilon^{r-1}$,

(2) *u_ϵ stays close to the solution to the linearized equation with initial data $(u_{t=0},\partial_t u_{t=0})=(\epsilon\xi_0,0)$ for all times*:

$$\sup_{t\in\mathbb{R}}|u_\epsilon(t,\cdot)-\Phi^{t\varpi_\epsilon}(\epsilon\xi_0)|_s\lesssim\epsilon^2.$$

2B. ***A modified Bambusi–Paleari theorem.*** As we will see in Section 4C, in the case of the YM model, the nonlinearity is given by

$$\mathfrak{f}(u)=\mathfrak{f}^{(2)}(u)+\mathfrak{f}^{(3)}(u), \tag{2-5}$$

where

(1) the lowest-degree term $\mathfrak{f}^{(2)}$ is a homogeneous polynomial of order 2:

$$(\mathfrak{f}^{(2)}(\{u^j(t):j\geq 0\}))^m=-3\sum_{i,j=0}^{\infty}\overline{\mathfrak{C}}_{ijm}u^i(t)u^j(t), \tag{2-6}$$

(2) the highest-degree term $\mathfrak{f}^{(3)}$ is a homogeneous polynomial of order 3:

$$(\mathfrak{f}^{(3)}(\{u^j(t):j\geq 0\}))^m=-\sum_{i,j,k=0}^{\infty}\mathfrak{C}_{ijkm}u^i(t)u^j(t)u^k(t). \tag{2-7}$$

Thus, according to the original version of Bambusi–Paleari's theorem (Theorem 2.4), one may argue that $\mathfrak{f}^{(2)}$ is the main nonlinearity and $\mathfrak{f}^{(3)}$ can be treated perturbatively. However, in this setting, the original version of Bambusi–Paleari's theorem would not be applicable, because $\mathfrak{f}^{(2)}$ is *nonresonant* (Lemma 6.5), that is

$$\langle\mathfrak{f}^{(2)}\rangle(\xi)=\frac{1}{2\pi}\int_0^{2\pi}\Phi^t(\mathfrak{f}^{(2)}(\Phi^t(\xi)))\,dt=0 \tag{2-8}$$

for all initial data ξ, and therefore $\mathcal{M}(\xi)=\mathfrak{A}\xi$ leads to trivial zeros of the operator $\mathcal{M}$. Consequently, we need to revisit the theorem of Bambusi–Paleari in this context. Specifically, we consider (2-1)–(2-5)–(2-8), replace $\mathfrak{f}^{(0)}(u)$ by $\mathfrak{f}^{(2)}(u)+\mathfrak{f}^{(3)}(u)$ and establish the following theorem.

Theorem 2.5 (modification of Bambusi–Paleari's theorem for the YM model). *Let $0<\alpha<\frac{1}{3}$, and define $\mathcal{W}_\alpha$ according to (2-4). Let $\mathfrak{A}$ be a positive multiplication self-adjoint operator with spectrum $\{\varpi_j^2>0 : j\geq 0\}$ such that $\{\varpi_j : j\geq 0\}\subset\mathbb{N}$ and $\varpi_j\simeq j$ as $j\to\infty$, defined by*

$$\mathfrak{A}:\mathcal{D}(\mathfrak{A})\simeq l^2_{s+2}\to l^2_s,\quad (\mathfrak{A}u)^j=\varpi_j^2 u_j,$$

with $\mathcal{D}(\mathfrak{A})$ being its maximal domain of definition. Assume that $\mathfrak{f}=\mathfrak{f}^{(2)}+\mathfrak{f}^{(3)}$, where $\mathfrak{f}^{(2)}$ and $\mathfrak{f}^{(3)}$ admit the representations (2-6) and (2-7) respectively.[7] *Moreover, assume that both $\mathfrak{f}^{(2)}$ and $\mathfrak{f}^{(3)}$ are differentiable, with Lipschitz differentials, and define the* ***modified*** *operator*

$$\mathfrak{M}_\pm(\xi)=\pm\mathfrak{A}\xi+\langle\mathfrak{f}^{(3)}\rangle(\xi)+\mathfrak{F}_0(\xi),$$

where $\mathfrak{F}_0(\xi)=\{(\mathfrak{F}_0(\xi))^m : m\geq 0\}$ is a bounded map on l^2_s that is given by

$$(\mathfrak{F}_0(\xi))^m=\frac{9}{4}\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}\sum_{\substack{i,j\geq 0\\ \varpi_i-\varpi_j\neq\pm\varpi_\nu}}\frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2-(\varpi_i-\varpi_j)^2}\xi^i\xi^j\xi^\kappa\sum_{\pm}\mathbb{1}(\varpi_i-\varpi_j\pm\varpi_\kappa\pm\varpi_m=0)$$
$$+\frac{9}{4}\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}\sum_{\substack{i,j\geq 0\\ \varpi_i+\varpi_j\neq\pm\varpi_\nu}}\frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2-(\varpi_i+\varpi_j)^2}\xi^i\xi^j\xi^\kappa\sum_{\pm}\mathbb{1}(\varpi_i+\varpi_j\pm\varpi_\kappa\pm\varpi_m=0).$$

Also, let $\xi_0\in l^2_{s+3}$ be a nondegenerate zero of $\mathfrak{M}_\pm$, that is

$$\mathfrak{M}_\pm(\xi_0)=0,\quad \ker(d\mathfrak{M}_\pm(\xi_0))=\{0\}.$$

Then, there exists a family $\{u_\epsilon:\epsilon\in\mathcal{E}_{\alpha,\gamma}\}\subset H^1([0,T_\epsilon];l^2_s])$ of time-periodic solutions to (2-1)–(2-5)–(2-8), where $\mathcal{E}_{\alpha,\gamma}$ is an uncountable set that has 0 as an accumulation point. In addition, each element u_ϵ has the following properties:

(1) *u_ϵ has period $T_\epsilon=2\pi/\varpi_\epsilon$ where there exists $\varpi_\star>0$ such that the maps $\epsilon\mapsto\varpi_\epsilon\in\mathcal{W}_\alpha\cap[1,1+\varpi_\star)$ for $\mathfrak{M}_+$ and $\epsilon\mapsto\varpi_\epsilon\in\mathcal{W}_\alpha\cap(1-\varpi_\star,1]$ for $\mathfrak{M}_-$, are monotone, one-to-one maps that stay close to 1 with $|1-\varpi_\epsilon|\lesssim\epsilon$*,

(2) *u_ϵ stays close to the solution to the linearized equation with the same initial data as above and zero initial velocity*:

$$\sup_{t\in\mathbb{R}}|u_\epsilon(t,\cdot)-\Phi^{t\varpi_\epsilon}(\epsilon\xi_0)|_s\lesssim\epsilon^2.$$

The rest of this section is devoted to the proof of the theorem above.

2C. *Preliminaries.* The core of the proof follows that of [Bambusi and Paleari 2001]. Let $0<\alpha<\frac{1}{3}$ and pick a frequency $\varpi\in\mathcal{W}_\alpha$. We are looking for a solution to (2-1) with frequency ϖ, that is

$$u(t)=q(\varpi t). \tag{2-9}$$

[7]For Theorem 2.5, we need very little information about the Fourier coefficients $\overline{\mathfrak{C}}_{ijm}$. However, for the application of this abstract theorem, see Section 7C and Proposition 7.6, we also need additional vanishing properties, see Lemma 5.8.

For any integer $k \geq 0$, we define the Banach space

$$\mathcal{H}_s^k = \left\{ q \in H^k([0, 2\pi]; l_s^2) : q(t) = \sum_{j=0}^{\infty} q^j(t) e_j = \sum_{j=0}^{\infty} \left(\sum_{l=0}^{\infty} q^{lj} \cos(lt) \right) e_j, \ \|q\|_{\mathcal{H}_s^k}^2 < \infty \right\}$$

endowed with the norm

$$\|q\|_{\mathcal{H}_s^k}^2 = \sum_{j=0}^{\infty} j^{2s} \left(2|q^{0j}|^2 + \sum_{l=1}^{\infty} |q^{lj}|^2 (1+l^2)^k \right).$$

In particular, we aim to construct q in the Hilbert space $\mathcal{H}_s^1$. To do so, we substitute (2-9) into (2-1) and obtain the nonlinear equation

$$L_{\varpi} q = f(q), \tag{2-10}$$

where

$$L_{\varpi} : \mathcal{D}(L_{\varpi}) \subset \mathcal{H}_s^1 \to \mathcal{H}_s^1, \quad L_{\varpi} q = \varpi^2 \frac{d^2}{dt^2} q + \mathfrak{A} q.$$

Now, we are looking for a solution with frequency close to 1. For this reason, we split $\mathcal{H}_s^1$ into

$$\mathcal{H}_s^1 = K \oplus R, \quad K = \ker(L_1), \quad R = K^{\perp},$$

and write

$$q \in \mathcal{H}_s^1, \quad q = v + q_{\perp}, \quad v \in K, \quad q_{\perp} \in R.$$

Taking into account the fact that K is generated by $\{\cos(\varpi_j t) : j \geq 0\}$, since

$$v \in K \iff v(t) = \{v^j(t) = c^j \cos(\varpi_j t) : j \geq 0\}$$

for some constants c^j, the latter simply means that we split $q = \{q^j : j \geq 0\} \in \mathcal{H}_s^1$ into

$$q^j(t) = v^j(t) + q_{\perp}^j(t), \quad v^j(t) = c^j \cos(\varpi_j t), \quad q_{\perp}^j(t) = \sum_{l \neq \varpi_j} d^{jl} \cos(lt),$$

for some constants c^j and d^{jl}. In addition, we define the associated projections

$$P : \mathcal{H}_s^1 \to R, \quad P(q) = P(v + q_{\perp}) = q_{\perp}, \quad Q : \mathcal{H}_s^1 \to K, \quad Q(q) = Q(v + q_{\perp}) = v,$$

and project (2-10) onto R and K, respectively. We obtain the coupled nonlinear system

$$L_{\varpi} q_{\perp} = P f(v + q_{\perp}), \tag{2-11}$$

$$-2\beta \mathfrak{A} v = Q f(v + q_{\perp}), \tag{2-12}$$

where we also set

$$\varpi^2 = 1 + 2\beta. \tag{2-13}$$

As is usual in this setting, we refer to (2-11) and (2-12) as the P-equation and Q-equation, respectively.

2D. ***Solution to the P-equation.*** As we will now see, the Diophantine condition $\varpi \in \mathcal{W}_\alpha$ guarantees the existence of a solution to the P-equation.

Lemma 2.6 (solution to the P-equation [Bambusi and Paleari 2001, Lemma 4.6]). *Let $0 < \alpha < \frac{1}{3}$, and pick $\varpi \in \mathcal{W}_\alpha$. Then, the operator L_ϖ restricted to R admits a bounded inverse*

$$L_\varpi^{-1} : \mathcal{H}_s^1 \cap R \to \mathcal{H}_s^1 \cap R, \quad \|L_\varpi^{-1}\| \le c_0 \alpha^{-1},$$

for some positive constant c_0. Moreover, there exists $\rho = \rho(\alpha) > 0$ and a C^1-function $q_\perp : B_\rho \to R$ with $v \mapsto q_\perp(v)$ that solves the P-equation, where B_ρ denotes the ball of radius ρ in K centered at 0. Furthermore, we have the estimates

$$\|q_\perp(v)\|_{\mathcal{H}_s^1} \lesssim_\alpha \|v\|_{\mathcal{H}_s^1}^2, \quad \|q_\perp(v) - L_\varpi^{-1} P \mathfrak{f}^{(2)}(v)\|_{\mathcal{H}_s^1} \lesssim_\alpha \|v\|_{\mathcal{H}_s^1}^3.$$

Proof. Apart from the C^1 regularity of $q_\perp$ (which is stated only as Lipschitz in [Bambusi and Paleari 2001]), the proof coincides with the one of Lemma 4.6 in the aforementioned paper, where the $f^{(0)}$ there is replaced by $\mathfrak{f}^{(2)}$. Once a Lipschitz solution $q_\perp$ has been found, one can read off the C^1 regularity of $q_\perp$ based on the regularity of $\mathfrak{f}$. However, for the convenience of the reader, we give a proof below of the construction of $q_\perp$. Let $0 < \alpha < \frac{1}{3}$, and pick $\varpi \in \mathcal{W}_\alpha$. The eigenvalues of L_ϖ are given by

$$\lambda_{jl} = \varpi_j^2 - l^2 \varpi^2 = (\varpi_j - l\varpi)(\varpi_j + l\varpi). \tag{2-14}$$

Then, for all $(l, j) \in \mathbb{N}^2$ with $l \ge 1$ and $l \ne \varpi_j$, we have that $|\lambda_{jl}| \ge (\alpha/l)(\varpi_j + l\varpi) \ge \alpha\varpi \ge \frac{1}{2}\alpha$. Therefore, $L_\varpi|_R$ has a bounded inverse and there exists a positive constant c_0 such that $\|L_\varpi^{-1}\| \le c_0\alpha^{-1}$. In addition, we let $\epsilon > 0$ be sufficiently small, let $\|v\|_{\mathcal{H}_s^1} \le \epsilon$, let $\delta > 0$ be sufficiently large, define the closed ball of radius $\delta\|v\|_{\mathcal{H}_s^1}^3$ centered at $L_\varpi^{-1} P\mathfrak{f}^{(2)}(v)$, that is

$$B = \{w \in \mathcal{H}_s^1 : \|w - L_\varpi^{-1} P \mathfrak{f}^{(2)}(v)\|_{\mathcal{H}_s^1} \le \delta \|v\|_{\mathcal{H}_s^1}^3\},$$

and rewrite the P-equation in the fixed-point formulation as

$$q_\perp = \mathcal{F}(q_\perp) = L_\varpi^{-1}[P\mathfrak{f}^{(2)}(v) + P(\mathfrak{f}^{(2)}(v + q_\perp) - \mathfrak{f}^{(2)}(v)) + P\mathfrak{f}^{(3)}(v + q_\perp)].$$

Next, we show that $\mathcal{F}$ maps the closed ball to itself. Indeed, for all $w \in B$, we have

$$\begin{aligned}\|w\|_{\mathcal{H}_s^1} &\le \|w - L_\varpi^{-1} P\mathfrak{f}^{(2)}(v)\|_{\mathcal{H}_s^1} + \|L_\varpi^{-1} P\mathfrak{f}^{(2)}(v)\|_{\mathcal{H}_s^1} \le \delta\|v\|_{\mathcal{H}_s^1}^3 + \|L_\varpi^{-1}\| \|\mathfrak{f}^{(2)}(v)\|_{\mathcal{H}_s^1} \\ &\le \delta\|v\|_{\mathcal{H}_s^1}^3 + c_0\alpha^{-1} k_s \|v\|_{\mathcal{H}_s^1}^2 \le c_1 \|v\|_{\mathcal{H}_s^1}^2,\end{aligned}$$

and Lemma 4.5 implies

$$\begin{aligned}\|\mathfrak{f}^{(2)}(v + w) - \mathfrak{f}^{(2)}(v)\|_{\mathcal{H}_s^1} &\le k_s(\|v + w\|_{\mathcal{H}_s^1} + \|v\|_{\mathcal{H}_s^1})\|w\|_{\mathcal{H}_s^1} \le k_s(\|w\|_{\mathcal{H}_s^1} + 2\|v\|_{\mathcal{H}_s^1})\|w\|_{\mathcal{H}_s^1} \\ &\le c_1 k_s(c_1\|v\|_{\mathcal{H}_s^1}^2 + 2\|v\|_{\mathcal{H}_s^1})\|v\|_{\mathcal{H}_s^1}^2 \le c_2\|v\|_{\mathcal{H}_s^1}^3, \\ \|\mathfrak{f}^{(3)}(v + w)\|_{\mathcal{H}_s^1} &\le k_s\|v + w\|_{\mathcal{H}_s^1}^3 \lesssim k_s(\|v\|_{\mathcal{H}_s^1}^3 + c_1^3\|v\|_{\mathcal{H}_s^1}^6) \le c_3\|v\|_{\mathcal{H}_s^1}^3.\end{aligned}$$

Hence, we infer

$$\begin{aligned}\|\mathcal{F}(w)-L_{\varpi}^{-1}P\mathfrak{f}^{(2)}(v)\|_{\mathcal{H}_s^1}&=\|L_{\varpi}^{-1}[P(\mathfrak{f}^{(2)}(v+w)-\mathfrak{f}^{(2)}(v))+P\mathfrak{f}^{(3)}(v+w)]\|_{\mathcal{H}_s^1}\\&\leq\|L_{\varpi}^{-1}\|\,[\|\mathfrak{f}^{(2)}(v+w)-\mathfrak{f}^{(2)}(v)\|_{\mathcal{H}_s^1}+\|\mathfrak{f}^{(3)}(v+w)\|_{\mathcal{H}_s^1}]\\&\leq c_0\alpha^{-1}[c_2\|v\|^3_{\mathcal{H}_s^1}+c_3\|v\|^3_{\mathcal{H}_s^1}]\leq\delta\|v\|^3_{\mathcal{H}_s^1}\end{aligned}$$

by choosing δ sufficiently large. The contraction property follows similarly. For the C^1 regularity, we set $\mathfrak{F}^{(2)}=L_{\varpi}^{-1}P\mathfrak{f}^{(2)}$ and, for $v, v+h\in B_\rho$, we have

$$q_\perp(v)=\mathfrak{F}^{(2)}(v+q_\perp(v)),\quad q_\perp(v+h)=\mathfrak{F}^{(2)}(v+h+q_\perp(v+h)),$$

so that

$$\begin{aligned}q_\perp(v+h)&=\mathfrak{F}^{(2)}(v+q_\perp(v))+d\mathfrak{F}^{(2)}_{v+q_\perp(v)}(h+q_\perp(v+h)-q_\perp(v))+o(h+q_\perp(v+h)-q_\perp(v))\\&=q_\perp(v)+d\mathfrak{F}^{(2)}_{v+q_\perp(v)}(h+q_\perp(v+h)-q_\perp(v))+o(h),\end{aligned}$$

where we used that $q_\perp$ is Lipschitz. Assuming that ρ is small enough, we can ensure that

$$\|d\mathfrak{F}^{(2)}_{v+q_\perp(v)}\|_{\mathcal{H}_s^1}\leq c\|v\|_{\mathcal{H}_s^1}<\tfrac{1}{2}$$

uniformly in v, and hence

$$q_\perp(v+h)=q_\perp(v)+(\mathrm{Id}-d\mathfrak{F}^{(2)}_{v+q_\perp(v)})^{-1}d\mathfrak{F}^{(2)}_{v+q_\perp(v)}(h)+o(h),$$

so that $q_\perp(v)$ is C^1 with differential $(\mathrm{Id}-d\mathfrak{F}^{(2)}_{v+q_\perp(v)})^{-1}d\mathfrak{F}^{(2)}_{v+q_\perp(v)}$. □

2E. ***Solution to the Q-equation.*** Next, we turn our attention to the existence of a solution to the Q-equation. Firstly, we define two Banach spaces of initial data

$$\mathcal{Q}=\left\{\xi=\sum_{j=0}^{\infty}\xi^je_j:\|\xi\|^2_{\mathcal{Q}}<\infty\right\}\simeq l^2_{s+1}\subseteq l^2_s,\quad \mathcal{D}(\mathfrak{A})=\left\{\xi=\sum_{j=0}^{\infty}\xi^je_j:\|\xi\|^2_{\mathcal{D}(\mathfrak{A})}<\infty\right\}\simeq l^2_{s+2}\subseteq l^2_s,$$

endowed with the norms

$$\begin{aligned}\|\xi\|^2_{\mathcal{Q}}&=\sum_{j=0}^{\infty}j^{2s}|\xi^j|^2+\sum_{j=0}^{\infty}j^{2s}|\varpi_j\xi^j|^2\simeq\sum_{j=0}^{\infty}j^{2(s+1)}|\xi^j|^2=|\xi|^2_{s+1},\\ \|\xi\|^2_{\mathcal{D}(\mathfrak{A})}&=|\xi|^2_s+|\mathfrak{A}\xi|^2_s=\sum_{j=0}^{\infty}j^{2s}|\xi^j|^2+\sum_{j=0}^{\infty}j^{2s}|\varpi_j^2\xi^j|^2\simeq\sum_{j=0}^{\infty}j^{2(s+2)}|\xi^j|^2=|\xi|^2_{s+2},\end{aligned}$$

since $\varpi_j\sim j$ as $j\to\infty$. We call the Hilbert space $(\mathcal{Q},\|\cdot\|_{\mathcal{Q}})$ the configuration space. In fact, $\mathcal{Q}$ is isomorphic to $K=\ker(L_1)$, and the isomorphism is given by the linear flow

$$I:\mathcal{Q}\to K,\quad (I(x))(t)=\Phi^t(x).$$

Also, recall the Banach space of spacetime functions

$$\mathcal{H}^k_s=\left\{q(t)=\sum_{j=0}^{\infty}q^j(t)e_j=\sum_{j=0}^{\infty}\left(\sum_{l=0}^{\infty}q^{lj}\cos(lt)\right)e_j:\|q\|^2_{\mathcal{H}^k_s}<\infty\right\}\subseteq H^k([0,2\pi];l^2_s)$$

endowed with the norm

$$\|q\|^2_{\mathcal{H}^k_s}=\sum_{j=0}^{\infty}j^{2s}\biggl(2|q^{0j}|^2+\sum_{l=1}^{\infty}|q^{lj}|^2(1+l^2)^k\biggr).$$

Notice that, since

$$I(\xi)(t)=\sum_{j=0}^{\infty}(I(\xi)(t))^j e_j=\sum_{j=0}^{\infty}(\Phi^t(\xi))^j e_j=\sum_{j=0}^{\infty}\xi^j\cos(\varpi_j t)e_j$$

and $\varpi_j\neq 0$ for all integers $j\geq 0$, we have

$$\begin{aligned}\|I(\xi)\|^2_{\mathcal{H}^0_s}&=\sum_{j=0}^{\infty}j^{2s}|\xi^j|^2=|\xi|^2_s,\\ \|I(\xi)\|^2_{\mathcal{H}^1_s}&=\sum_{j=0}^{\infty}j^{2s}|\xi^j|^2(1+\varpi_j^2)=\sum_{j=0}^{\infty}j^{2s}(|\xi^j|^2+|\varpi_j\xi^j|^2)\simeq|\xi|^2_{s+1}\simeq\|\xi\|^2_{\mathcal{Q}}.\end{aligned}\tag{2-15}$$

Secondly, we prove the following averaging identity that generalizes the one in Lemma 4.7 in [Bambusi and Paleari 2001] from vector fields $F:\mathcal{Q}\to\mathcal{Q}$ to $F:\mathcal{H}^k_s\to\mathcal{H}^k_s$.

Lemma 2.7 (averaging identity). *Let $F:\mathcal{H}^k_s\to\mathcal{H}^k_s$ be any vector field. Then, for all $x\in l^2_s$, we have*

$$\langle F\rangle(x)=\frac{1}{2\pi}\int_0^{2\pi}\Phi^t[F(\Phi^t(x))]\,dt=\frac{1}{2}I^{-1}Q[F(I(x))].$$

Proof. Let $F:\mathcal{H}^k_s\to\mathcal{H}^k_s$ be a vector field in $\mathcal{H}^k_s$ (not necessarily in $\mathcal{Q}$), pick any $x\in l^2_s$ and set $w=I(x)$. By the definition of the Banach space $\mathcal{H}^k_s$, we have

$$F(w)=\sum_{m=0}^{\infty}(F(w))^m e_m=\sum_{m=0}^{\infty}\biggl(\sum_{l=0}^{\infty}(F(w))^m_l\cos(lt)\biggr)e_m,\quad (F(w))^m_l=\frac{1}{\pi}\int_0^{2\pi}(F(w))^m\cos(lt)\,dt.$$

Then, the definition of the linear flow together with the definition of the projection Q yield

$$\begin{aligned}Q[F(w)]&=\sum_{m=0}^{\infty}(Q[F(w)])^m e_m=\sum_{m=0}^{\infty}(F(w))^m_{\omega_m}\cos(\varpi_m t)e_m,\\ \frac{1}{2}I^{-1}Q[F(w)]&=\frac{1}{2}\sum_{m=0}^{\infty}(F(w))^m_{\varpi_m}e_m=\frac{1}{2}\sum_{m=0}^{\infty}\biggl(\frac{1}{\pi}\int_0^{2\pi}(F(w))^m\cos(\varpi_m t)\,dt\biggr)e_m\\ &=\frac{1}{2\pi}\int_0^{2\pi}\sum_{m=0}^{\infty}((\Phi^t[F(w)])^m e_m)\,dt=\frac{1}{2\pi}\int_0^{2\pi}\Phi^t[F(w)]\,dt=\langle F\rangle(x).\end{aligned}$$

□

Then, we express the Q-equation in the configuration space introduced above.

Lemma 2.8 (the Q-equation in the configuration space). *Let $\rho>0$ and $q_\perp:B_\rho\subset K\to R$ be the solution map to the P-equation derived in Lemma 2.6. Also, let $x\in l^2_{s+2}$, and set $v=I(x)\in B_\rho$. Then, the Q-equation (2-12) for v is equivalent to*

$$\beta\mathfrak{A}x+\langle\mathfrak{f}\rangle(x)=-\tfrac{1}{2}I^{-1}Q[\mathfrak{f}(I(x)+q_\perp(I(x)))-\mathfrak{f}(I(x))].\tag{2-16}$$

Proof. Let $\rho > 0$ and $q_\perp : B_\rho \subset K \to R$ be the solution map to the P-equation derived in Lemma 2.6. Also, let $x \in \mathcal{Q}$, and set $v = I(x) \in B_\rho$. Then, we rewrite the Q-equation given in (2-12), that is $-2\beta A v = Q f(v + q_\perp)$, as

$$-2\beta \mathfrak{A} I(x) - Q\mathfrak{f}(I(x)) = Q\mathfrak{f}(I(x) + q_\perp(I(x))) - Q\mathfrak{f}(I(x)).$$

Since $\mathfrak{A} I(x) = I\mathfrak{A}x$, by applying $-\frac{1}{2}I^{-1}$ to both sides, we get

$$\beta\mathfrak{A}x + \tfrac{1}{2}I^{-1}Q\mathfrak{f}(I(x)) = -\tfrac{1}{2}I^{-1}Q[\mathfrak{f}(I(x) + q_\perp(I(x))) - \mathfrak{f}(I(x))].$$

Now, the claim follows by the averaging identity due to Lemma 2.7. □

It remains to show that there exists a solution to (2-16). To this end, we define

$$x = \epsilon\xi, \quad |\beta| = \epsilon^2, \tag{2-17}$$

and (2-16) becomes

$$\pm\epsilon^2\mathfrak{A}(\epsilon\xi) + \langle\mathfrak{f}\rangle(\epsilon\xi) = -\tfrac{1}{2}I^{-1}Q[\mathfrak{f}(I(\epsilon\xi) + q_\perp(I(\epsilon\xi))) - \mathfrak{f}(I(\epsilon\xi))].$$

On the one hand, (2-5) and (2-8) yield

$$\pm\epsilon^2\mathfrak{A}(\epsilon\xi) + \langle\mathfrak{f}\rangle(\epsilon\xi) = \pm\epsilon^3\mathfrak{A}\xi + \epsilon^2\langle\mathfrak{f}^{(2)}\rangle(\xi) + \epsilon^3\langle\mathfrak{f}^{(3)}\rangle(\xi) = \epsilon^3(\pm\mathfrak{A}\xi + \langle\mathfrak{f}^{(3)}\rangle(\xi)).$$

On the other hand, (2-5), (2-8) and the averaging identity from Lemma 2.7 yield

$$\begin{aligned}
&\tfrac{1}{2}I^{-1}Q[\mathfrak{f}(I(\epsilon\xi) + q_\perp(I(\epsilon\xi))) - \mathfrak{f}(I(\epsilon\xi))] \\
&\qquad = \tfrac{1}{2}I^{-1}Q[\mathfrak{f}^{(2)}(I(\epsilon\xi) + q_\perp(I(\epsilon\xi)))] + \tfrac{1}{2}I^{-1}Q[\mathfrak{f}^{(3)}(I(\epsilon\xi) + q_\perp(I(\epsilon\xi))) - \mathfrak{f}^{(3)}(I(\epsilon\xi))] \\
&\qquad = \tfrac{1}{2}I^{-1}Q[\mathfrak{f}^{(2)}(I(\epsilon\xi) + L_\varpi^{-1}P\mathfrak{f}^{(2)}(I(\epsilon\xi)))] + \epsilon^3\mathfrak{G}_\epsilon(\xi),
\end{aligned}$$

where we set

$$\begin{aligned}
\mathfrak{G}_\epsilon(\xi) = \epsilon^{-3}\big[&\tfrac{1}{2}I^{-1}Q[\mathfrak{f}^{(2)}(I(\epsilon\xi) + q_\perp(I(\epsilon\xi))) - \mathfrak{f}^{(2)}(I(\epsilon\xi) + L_\varpi^{-1}P\mathfrak{f}^{(2)}(I(\epsilon\xi)))] \\
&+ \tfrac{1}{2}I^{-1}Q[\mathfrak{f}^{(3)}(I(\epsilon\xi) + q_\perp(I(\epsilon\xi))) - \mathfrak{f}^{(3)}(I(\epsilon\xi))]\big].
\end{aligned} \tag{2-18}$$

We apply the averaging identity and the notation from Lemma 2.7 to the map $v \to \mathfrak{f}^{(2)}(v + L_1^{-1}P\mathfrak{f}^{(2)}(v))$, which is a vector field from $\mathcal{H}_s^1$ to $\mathcal{H}_s^1$, to obtain

$$\begin{aligned}
&\tfrac{1}{2}I^{-1}Q[\mathfrak{f}(I(\epsilon\xi) + q_\perp(I(\epsilon\xi))) - \mathfrak{f}(I(\epsilon\xi))] \\
&\quad = \tfrac{1}{2}I^{-1}Q[\mathfrak{f}^{(2)}(I(\epsilon\xi) + L_\varpi^{-1}P\mathfrak{f}^{(2)}(I(\epsilon\xi)))] + \epsilon^3\mathfrak{G}_\epsilon(\xi) \\
&\quad = \tfrac{1}{2}I^{-1}Q[\mathfrak{f}^{(2)}(I(\epsilon\xi) + L_1^{-1}P\mathfrak{f}^{(2)}(I(\epsilon\xi)))] + \epsilon^3\mathfrak{G}_\epsilon(\xi) + \epsilon^3\mathfrak{R}_\epsilon(\xi,\varpi) \\
&\quad = \frac{1}{2}\int_0^{2\pi}\Phi^{-t}\big(\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi) + L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))\big)\,dt + \epsilon^3\mathfrak{G}_\epsilon(\xi) + \epsilon^3\mathfrak{R}_\epsilon(\xi,\varpi) \\
&\quad = \langle\mathfrak{f}^{(2)}((\cdot) + L_1^{-1}P\mathfrak{f}^{(2)}(\cdot))\rangle(\epsilon\xi) + \epsilon^3\mathfrak{G}_\epsilon(\xi) + \epsilon^3\mathfrak{R}_\epsilon(\xi,\varpi) = \epsilon^3(\mathfrak{F}_\epsilon(\xi) + \mathfrak{G}_\epsilon(\xi) + \mathfrak{R}_\epsilon(\xi,\varpi)),
\end{aligned}$$

where we set

$$\begin{aligned}
\mathfrak{R}_\epsilon(\xi,\varpi) &= \epsilon^{-3}\tfrac{1}{2}I^{-1}Q[\mathfrak{f}^{(2)}(I(\epsilon\xi) + L_\varpi^{-1}P\mathfrak{f}^{(2)}(I(\epsilon\xi))) - \mathfrak{f}^{(2)}(I(\epsilon\xi) + L_1^{-1}P\mathfrak{f}^{(2)}(I(\epsilon\xi)))], \\
\mathfrak{F}_\epsilon(\xi) &= \epsilon^{-3}\langle\mathfrak{f}^{(2)}((\cdot) + L_1^{-1}P\mathfrak{f}^{(2)}(\cdot))\rangle(\epsilon\xi).
\end{aligned}$$

In conclusion, the Q-equation (2-16) can be written equivalently, for $\epsilon > 0$ sufficiently small, as

$$\pm\mathfrak{A}\xi + \langle\mathfrak{f}^{(3)}\rangle(\xi) = -(\mathfrak{F}_\epsilon(\xi) + \mathfrak{G}_\epsilon(\xi) + \mathfrak{R}_\epsilon(\xi, \varpi)). \tag{2-19}$$

However, instead of (2-19), we focus on a modified version, namely

$$\pm\mathfrak{A}\xi + \langle\mathfrak{f}^{(3)}\rangle(\xi) = -\left(\mathfrak{F}_\epsilon(\xi) + \mathfrak{G}_\epsilon(\xi) \pm \frac{2\epsilon^2}{\varpi^2 - 1}\mathfrak{R}_\epsilon(\xi, \varpi)\right). \tag{2-20}$$

Notice that (2-19) coincides with (2-20) provided that $\varpi^2 - 1 = \pm 2\epsilon^2$.

Remark 2.9. Since $\mathfrak{f}^{(2)}$ is differentiable and quadratic, and $\langle\mathfrak{f}^{(2)}\rangle(\xi) = 0$ for all initial data $\xi \in l^2_{s+3}$, it follows that $\mathfrak{F}_\epsilon(\xi)$ is differentiable and $\|\mathfrak{F}_\epsilon(\xi)\|_{\mathcal{Q}} \lesssim 1$. Later, in Section 2F, we compute the exact expressions of $\mathfrak{F}_0(\xi)$ for general initial data (Lemma 2.13), $\mathfrak{F}_\epsilon(\xi)$ for small ϵ close to zero and 1-mode initial data (Lemma 2.14), as well as the differential $d\mathfrak{F}_0(\xi)$ at the 1-mode initial data (Lemma 2.15).

In the following, we estimate the error terms. To begin with, we estimate $\mathfrak{R}_\epsilon(\xi, \varpi)$.

Lemma 2.10 (estimate for $\mathfrak{R}_\epsilon(\xi, \varpi)$ and $d_\xi\mathfrak{R}_\epsilon(\xi, \varpi)$). *Let $0 < \alpha < \frac{1}{3}$, and pick any $\varpi \in \mathcal{W}_\alpha$. Also, let $\xi \in l^2_{s+3}$ be any initial data. Then, we have*

$$\|\mathfrak{R}_\epsilon(\xi, \varpi)\|_{\mathcal{Q}} \lesssim |\varpi^2 - 1|, \qquad \|d_\xi\mathfrak{R}_\epsilon(\xi, \varpi)[h]\|_{\mathcal{Q}} \lesssim |\varpi^2 - 1|\,|h|_{s+3}.$$

Proof. Let $0 < \alpha < \frac{1}{3}$, and pick any $\varpi \in \mathcal{W}_\alpha$. Also, let $\xi \in l^2_{s+3}$ be any initial data. Firstly, we pick any $\epsilon > 0$, set $v = I(\epsilon\xi)$ and compute

$$\begin{aligned}
\mathfrak{f}^{(2)}(v) &= \sum_{j=0}^{\infty}(\mathfrak{f}^{(2)}(v))^j e_j = \sum_{j=0}^{\infty}\left(\sum_{l=0}^{\infty}(\mathfrak{f}^{(2)}(v))^j_l \cos(lt)\right)e_j,\\
P\mathfrak{f}^{(2)}(v) &= \sum_{j=0}^{\infty}\left(\sum_{\substack{l=0\\ l\neq\varpi_j}}^{\infty}(\mathfrak{f}^{(2)}(v))^j_l \cos(lt)\right)e_j,\\
L_\varpi^{-1}P\mathfrak{f}^{(2)}(v) &= \sum_{j=0}^{\infty}\left(\sum_{\substack{l=0\\ l\neq\varpi_j}}^{\infty}\frac{1}{\varpi_j^2 - l^2\varpi^2}(\mathfrak{f}^{(2)}(v))^j_l \cos(lt)\right)e_j,\\
(L_\varpi^{-1} - L_1^{-1})P\mathfrak{f}^{(2)}(v) &= \sum_{j=0}^{\infty}\left(\sum_{\substack{l=0\\ l\neq\varpi_j}}^{\infty}\left(\frac{1}{\varpi_j^2 - l^2\varpi^2} - \frac{1}{\varpi_j^2 - l^2}\right)(\mathfrak{f}^{(2)}(v))^j_l \cos(lt)\right)e_j\\
&= \sum_{j=0}^{\infty}\left(\sum_{\substack{l=0\\ l\neq\varpi_j}}^{\infty}\frac{l^2(\varpi^2-1)}{(\varpi_j^2 - l^2\varpi^2)(\varpi_j^2 - l^2)}(\mathfrak{f}^{(2)}(v))^j_l \cos(lt)\right)e_j.
\end{aligned}$$

Secondly, we note that

$$\|L_\varpi^{-1}P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|_{\mathcal{H}^1_s} \lesssim \epsilon^2, \qquad \|L_1^{-1}P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|_{\mathcal{H}^1_s} \lesssim \epsilon^2. \tag{2-21}$$

These can be easily proved using the Diophantine condition, the elementary inequality $|\varpi_j^2 - l^2| \geq 1$ (since $\varpi \in \mathcal{W}_\alpha$, both $\varpi_j^2 \geq 1$ and $l^2 \geq 0$ are integers with $\varpi_j \neq l$), the Lipschitz estimate $\|\mathfrak{f}^{(2)}(u)\|_{\mathcal{H}^k_s} \lesssim_s \|u\|^2_{\mathcal{H}^k_s}$

for all $u \in \mathcal{H}^k_s$ with $\|u\|_{\mathcal{H}^k_s} \le \epsilon$ (which follows from Lemma 4.5), together with (2-15). Indeed, we infer

$$\begin{aligned}
\|L_{\varpi}^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|^2_{\mathcal{H}^1_s} &= \sum_{j=0}^{\infty} j^{2s} \sum_{l=0}^{\infty} \left| \frac{1}{\varpi_j^2 - l^2\varpi^2} (\mathfrak{f}^{(2)}(I(\epsilon\xi)))^j_l \right|^2 (1+l^2) \\
&\lesssim_\alpha \sum_{j=0}^{\infty} j^{2s} \sum_{l=0}^{\infty} |(\mathfrak{f}^{(2)}(I(\epsilon\xi)))^j_l|^2 (1+l^2) \\
&\le \|\mathfrak{f}^{(2)}(I(\epsilon\xi))\|^2_{\mathcal{H}^1_s} \lesssim \|I(\epsilon\xi)\|^4_{\mathcal{H}^1_s} = \epsilon^4 \|I(\xi)\|^4_{\mathcal{H}^1_s} = \epsilon^4 \|\xi\|^4_{\mathcal{Q}} \le \epsilon^4 |\xi|^4_{s+3}, \\
\|L_{1}^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|^2_{\mathcal{H}^1_s} &= \sum_{j=0}^{\infty} j^{2s} \sum_{l=0}^{\infty} \left| \frac{1}{\varpi_j^2 - l^2} (\mathfrak{f}^{(2)}(I(\epsilon\xi)))^j_l \right|^2 (1+l^2) \\
&\lesssim \sum_{j=0}^{\infty} j^{2s} \sum_{l=0}^{\infty} |(\mathfrak{f}^{(2)}(I(\epsilon\xi)))^j_l|^2 (1+l^2) \\
&\le \|\mathfrak{f}^{(2)}(I(\epsilon\xi))\|^2_{\mathcal{H}^1_s} \lesssim \|I(\epsilon\xi)\|^4_{\mathcal{H}^1_s} = \epsilon^4 \|I(\xi)\|^4_{\mathcal{H}^1_s} = \epsilon^4 \|\xi\|^4_{\mathcal{Q}} \le \epsilon^4 |\xi|^4_{s+3}.
\end{aligned}$$

Next, we use the above together with the Lipschitz estimate for $\mathfrak{f}^{(2)}$ (see Lemma 4.5) and the fact that $I^{-1} : \mathcal{H}^1_s \to \mathcal{Q}$ to obtain

$$\begin{aligned}
\epsilon^3 \|\mathfrak{R}_\epsilon(\xi,\varpi)\|_{\mathcal{Q}} &= \left\| \tfrac{1}{2} I^{-1} Q[\mathfrak{f}^{(2)}(I(\epsilon\xi) + L_\varpi^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi))) - \mathfrak{f}^{(2)}(I(\epsilon\xi) + L_1^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi)))] \right\|_{\mathcal{Q}} \\
&\lesssim \|\mathfrak{f}^{(2)}(I(\epsilon\xi) + L_\varpi^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi))) - \mathfrak{f}^{(2)}(I(\epsilon\xi) + L_1^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi)))\|_{\mathcal{H}^1_s} \\
&\lesssim (\|I(\epsilon\xi) + L_\varpi^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|_{\mathcal{H}^1_s} + \|I(\epsilon\xi) + L_1^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|_{\mathcal{H}^1_s}) \\
&\qquad \cdot \|L_\varpi^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi)) - L_1^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|_{\mathcal{H}^1_s} \\
&\lesssim \epsilon \|(L_\varpi^{-1} - L_1^{-1}) P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|_{\mathcal{H}^1_s}.
\end{aligned}$$

Once again, the Diophantine condition, the elementary inequality $|\varpi_j^2 - l^2| \ge 1$, the Lipschitz estimate $\|\mathfrak{f}^{(2)}(u)\|_{\mathcal{H}^k_s} \lesssim_s \|u\|^2_{\mathcal{H}^k_s}$ for all $u \in \mathcal{H}^k_s$ with $\|u\|_{\mathcal{H}^k_s} \le \epsilon$ (which follows from Lemma 4.5), together with (2-15) imply that

$$\begin{aligned}
&\|(L_\varpi^{-1} - L_1^{-1}) P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|^2_{\mathcal{H}^1_s} \\
&\quad = |\varpi^2 - 1|^2 \sum_{j=0}^{\infty} j^{2s} \sum_{l=1}^{\infty} \left| \frac{l^2}{(\varpi_j^2 - l^2\varpi^2)(\varpi_j^2 - l^2)} (\mathfrak{f}^{(2)}(I(\epsilon\xi)))^j_l \right|^2 (1+l^2) \\
&\quad \lesssim_\alpha |\varpi^2 - 1|^2 \sum_{j=0}^{\infty} j^{2s} \sum_{l=1}^{\infty} |l^2 (\mathfrak{f}^{(2)}(I(\epsilon\xi)))^j_l|^2 (1+l^2) \lesssim |\varpi^2 - 1|^2 \sum_{j=0}^{\infty} j^{2s} \sum_{l=1}^{\infty} |(\mathfrak{f}^{(2)}(I(\epsilon\xi)))^j_l|^2 (1+l^2)^3 \\
&\quad \le |\varpi^2 - 1|^2 \|\mathfrak{f}^{(2)}(I(\epsilon\xi))\|^2_{\mathcal{H}^3_s} \lesssim |\varpi^2 - 1|^2 \|I(\epsilon\xi)\|^4_{\mathcal{H}^3_s} = |\varpi^2 - 1|^2 \epsilon^4 \|I(\xi)\|^4_{\mathcal{H}^3_s} \lesssim |\varpi^2 - 1|^2 \epsilon^4 |\xi|^4_{s+3}.
\end{aligned}$$

Finally, putting this all together yields $\|\mathfrak{R}_\epsilon(\xi,\varpi)\|_{\mathcal{Q}} \lesssim |\varpi^2 - 1|$, which completes the first part of the proof. The estimate for the differential follows similarly. □

Next, we estimate $\mathfrak{G}_\epsilon(\xi)$ and its differential.

Lemma 2.11 (estimate for $\mathfrak{G}_\epsilon(\xi)$ and $d_\xi\mathfrak{G}_\epsilon(\xi)$). *Let $\xi \in l^2_{s+3}$ be any initial data. Then, $\mathfrak{G}_\epsilon(\xi)$ is continuously differentiable with respect to ξ, and we have*

$$\|\mathfrak{G}_\epsilon(\xi)\|_{\mathcal{Q}} \lesssim \epsilon, \qquad \|d_\xi\mathfrak{G}_\epsilon(\xi)[h]\|_{\mathcal{Q}} \lesssim \epsilon|h|_{s+3}.$$

Proof. Let $\xi \in l^2_{s+3}$ be any initial data, and recall the definition of $\mathfrak{G}_\epsilon(\xi)$ from (2-18). The claim follows by Lemmas 4.5 and 2.6 together with (2-15) and the fact that $I^{-1} : \mathcal{H}^1_s \to \mathcal{Q}$. Indeed, since

$$\|I(\epsilon\xi)\|_{\mathcal{H}^1_s} \lesssim \epsilon\|\xi\|_{\mathcal{Q}}, \qquad \|q_\perp(I(\epsilon\xi))\|_{\mathcal{H}^1_s} \lesssim \|I(\epsilon\xi)\|^2_{\mathcal{H}^1_s} \lesssim \epsilon^2\|\xi\|^2_{\mathcal{Q}}, \tag{2-22}$$

we can estimate

$$\begin{aligned}\Big\|\tfrac{1}{2}I^{-1}Q[\mathfrak{f}^{(2)}(I(\epsilon\xi)+q_\perp(I(\epsilon\xi))) - \mathfrak{f}^{(2)}(I(\epsilon\xi)+L_\varpi^{-1}P\mathfrak{f}^{(2)}(I(\epsilon\xi)))]\Big\|_{\mathcal{Q}} \\ \lesssim \|\mathfrak{f}^{(2)}(I(\epsilon\xi)+q_\perp(I(\epsilon\xi))) - \mathfrak{f}^{(2)}(I(\epsilon\xi)+L_\varpi^{-1}P\mathfrak{f}^{(2)}(I(\epsilon\xi)))\|_{\mathcal{H}^1_s} \\ \lesssim \epsilon\|q_\perp(I(\epsilon\xi)) - L_\varpi^{-1}P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|_{\mathcal{H}^1_s} \lesssim \epsilon\|I(\epsilon\xi)\|^3_{\mathcal{H}^1_s} \lesssim \epsilon^4\end{aligned}$$

and

$$\begin{aligned}&\Big\|\tfrac{1}{2}I^{-1}Q[\mathfrak{f}^{(3)}(I(\epsilon\xi)+q_\perp(I(\epsilon\xi))) - \mathfrak{f}^{(3)}(I(\epsilon\xi))]\Big\|_{\mathcal{Q}} \\ &\lesssim \|\mathfrak{f}^{(3)}(I(\epsilon\xi)+q_\perp(I(\epsilon\xi))) - \mathfrak{f}^{(3)}(I(\epsilon\xi))\|_{\mathcal{H}^1_s} \lesssim [\|I(\epsilon\xi)+q_\perp(I(\epsilon\xi))\|^2_{\mathcal{H}^1_s} + \|I(\epsilon\xi)\|^2_{\mathcal{H}^1_s}]\|q_\perp(I(\epsilon\xi))\|_{\mathcal{H}^1_s} \\ &\lesssim [\|I(\epsilon\xi)\|^2_{\mathcal{H}^1_s} + \|q_\perp(I(\epsilon\xi))\|^2_{\mathcal{H}^1_s}]\|q_\perp(I(\epsilon\xi))\|_{\mathcal{H}^1_s} \lesssim \epsilon^4.\end{aligned}$$

The estimate for the differential follows similarly. □

It remains to show that there exists a solution to (2-20), that is

$$\pm\mathfrak{A}\xi + \langle\mathfrak{f}^{(3)}\rangle(\xi) = -\left(\mathfrak{F}_\epsilon(\xi) + \mathfrak{G}_\epsilon(\xi) \pm \frac{2\epsilon^2}{\varpi^2-1}\mathfrak{R}_\epsilon(\xi,\varpi)\right) \iff \mathfrak{M}_\pm(\xi) = \mathfrak{H}_\epsilon(\xi), \tag{2-23}$$

where we set

$$\mathfrak{M}_\pm(\xi) = \pm\mathfrak{A}\xi + \langle\mathfrak{f}^{(3)}\rangle(\xi) + \mathfrak{F}_0(\xi), \qquad \mathfrak{H}_\epsilon(\xi) = \mathfrak{F}_0(\xi) - \mathfrak{F}_\epsilon(\xi) - \mathfrak{G}_\epsilon(\xi) \mp \frac{2\epsilon^2}{\varpi^2-1}\mathfrak{R}_\epsilon(\xi,\varpi).$$

We refer to $\mathfrak{M}_\pm$ as the *modified operator*. Note that Lemmas 2.10 and 2.11 and the smoothness[8] of $\mathfrak{F}_\epsilon(\xi)$ with respect to ϵ yield

$$\|\mathfrak{H}_\epsilon(\xi)\|_{\mathcal{Q}} \lesssim \epsilon, \qquad \|d_\xi\mathfrak{H}_\epsilon(\xi)[h]\|_{\mathcal{Q}} \lesssim \epsilon|h|_{s+3}.$$

The following result constitutes the main modification of Bambusi–Paleari's theorem.

Lemma 2.12 (solution to the Q-equation). *Define the* ***modified operator***

$$\mathfrak{M}_\pm(x) = \pm\mathfrak{A}x + \langle\mathfrak{f}^{(3)}\rangle(x) + \mathfrak{F}_0(x),$$

and let $\xi_0 \in l^2_{s+3}$ be a nondegenerate zero of $\mathfrak{M}_\pm$, that is

$$\mathfrak{M}_\pm(\xi_0) = 0, \qquad \ker(d\mathfrak{M}_\pm(\xi_0)) = \{0\}.$$

[8]In Lemma 2.13 we show that $\mathfrak{F}_\epsilon(\xi)$ is smooth with respect to ϵ. As one can see in the proof of Lemma 2.13, $\mathfrak{F}_\epsilon(\xi)$ is in fact linear with respect to ϵ. See (2-25).

Then, there exists a positive ϵ_0 and a Lipschitz map $\xi : [0, \epsilon_0) \to l^2_{s+3}$, $\epsilon \mapsto \xi(\epsilon)$ that solves the Q-equation (2-19) with the plus sign. Furthermore, we have the estimate $|\xi(\epsilon) - \xi_0|_{s+3} \lesssim \epsilon$.

Proof. The proof follows from the implicit function theorem and is similar to the one of Proposition 4.8 in [Bambusi and Paleari 2001]. Let $\xi_0 \in l^2_{s+3}$ be a nondegenerate zero of the modified operator $\mathfrak{M}_\pm$, and define the map

$$\mathcal{G} : \mathbb{R} \times l^2_{s+3} \to \mathcal{H}^1_s, \quad (\epsilon, \xi) \mapsto \mathcal{G}(\epsilon, \xi) = \mathfrak{M}_\pm(\xi) - \mathfrak{H}_\epsilon(\xi),$$

and note that it is Lipschitz, differentiable at $\epsilon = 0$ and it vanishes at $(\epsilon, \xi) = (0, \xi_0)$. It remains to show that its differential with respect to ξ at $(0, \xi_0)$, namely

$$d\mathcal{G}(0, \xi_0) : l^2_{s+3} \to \mathcal{H}^1_s, \quad X \mapsto d\mathcal{G}(0, \xi_0)(X) = d\mathfrak{M}_\pm(\xi_0)(X),$$

is an isomorphism. Equivalently, this means that, for all $Y \in \mathcal{H}^1_s$, there exists $X \in l^2_{s+3}$ that solves the equation $d\mathfrak{M}_\pm(\xi_0)(X) = Y$, with

$$d\mathfrak{M}_\pm(\xi_0) = \pm\mathfrak{A} + d\langle \mathfrak{f}^{(3)} \rangle(\xi_0) + d\mathfrak{F}_0(\xi_0).$$

Now, the operator $d\mathfrak{M}_\pm$ is a Fredholm operator since it is the sum of a Fredholm and a compact operator due to the facts that $\mathfrak{f}^{(3)}$ and $\mathfrak{F}_\epsilon(\xi)$ are bounded on l^2_{s+3} and that they are differentiable with bounded differential. Since the defect index of $d\mathfrak{M}_\pm(\xi_0)$ is 0 from the nondegeneracy condition, it follows that it is an isomorphism, and thus we can apply the implicit function theorem. Note finally, that the range of ϵ, defined by ϵ_0, does not depend on ϖ, since $\mathcal{G}$ depends continuously on ϖ and all the necessary bounds hold uniformly with respect to ϖ. □

Finally, we prove Theorem 2.5.

Proof of Theorem 2.5. Let $\varpi \in \mathcal{W}_\alpha$ be fixed. Then, according to Lemmas 2.6 and 2.12, there exists $\epsilon_0 > 0$ such that the map $[0, \epsilon_0) \ni \epsilon \mapsto \big(\epsilon I(\xi(\epsilon)), q_\perp(\epsilon I(\xi(\epsilon)))\big)$ solves both the P-equation (2-11) and the Q-equation (2-12). Furthermore, pick $\varpi_\star$ such that $\epsilon(\varpi_\star) = \epsilon_0$. Then, the function $\epsilon^2(\varpi) = \pm\frac{1}{2}(\varpi^2 - 1)$ solves (2-13), and the map $\varpi \mapsto \big(\epsilon I(\xi(\epsilon(\varpi))), q_\perp\big(\epsilon I(\xi(\epsilon(\varpi)))\big)\big)$ defines a family of solutions to (2-10) labeled by $\varpi \in \mathcal{W}_\alpha \cap [1, 1 + \varpi_\star]$ or $\varpi \in \mathcal{W}_\alpha \cap [1 - \varpi_\star, 1]$. Finally, the map $\epsilon \mapsto \epsilon(\varpi)$ is one-to-one, and hence this family can be also parametrized by

$$\epsilon \in \mathcal{E}_{\alpha,\gamma} = \epsilon(\mathcal{W}_\alpha \cap [1, 1 + \varpi_\star]) \quad \text{or} \quad \epsilon \in \mathcal{E}_{\alpha,\gamma} = \epsilon(\mathcal{W}_\alpha \cap [1 - \varpi_\star, 1]).$$ □

2F. *The function $\mathfrak{F}_\epsilon(\xi)$.* In Lemma 5.8, we prove that $\overline{\mathfrak{C}}_{ijm} = 0$ for all integers $i, j, m \geq 0$ with $i + j < m$. Moreover, by its definition (2-6), $\overline{\mathfrak{C}}_{ijm}$ is also invariant under any permutation of the indices i, j, m. For future reference, we now use these additional properties of the Fourier coefficient $\overline{\mathfrak{C}}$ to compute:

- $\mathfrak{F}_0(\xi)$ for general initial data (Lemma 2.13),
- $\mathfrak{F}_\epsilon(\xi)$ for small ϵ close to zero and 1-mode initial data (Lemma 2.14),
- $d\mathfrak{F}_0(\xi)$ at the 1-mode initial data (Lemma 2.15).

Lemma 2.13 (computation of $\mathfrak{F}_0(\xi)$ for general initial data). *Let $\xi = \{\xi^m : m \geq 0\} \in l^2_{s+3}$. Then, for all integers $m \geq 0$, we have*

$$(\mathfrak{F}_0(\xi))^m = \frac{9}{4} \sum_{\kappa,\nu \geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \sum_{\substack{i,j\geq 0 \\ \varpi_i - \varpi_j \neq \pm\varpi_\nu}} \frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2 - (\varpi_i - \varpi_j)^2} \xi^i \xi^j \xi^\kappa \sum_{\pm} \mathbb{1}(\varpi_i - \varpi_j \pm \varpi_\kappa \pm \varpi_m = 0)$$
$$+ \frac{9}{4} \sum_{\kappa,\nu \geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \sum_{\substack{i,j\geq 0 \\ \varpi_i + \varpi_j \neq \pm\varpi_\nu}} \frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2 - (\varpi_i + \varpi_j)^2} \xi^i \xi^j \xi^\kappa \sum_{\pm} \mathbb{1}(\varpi_i + \varpi_j \pm \varpi_\kappa \pm \varpi_m = 0).$$

In addition, the function $\mathfrak{F}_\epsilon$ is smooth with respect to ϵ.

Proof. Let $\xi = \{\xi^m : m \geq 0\} \in l^2_{s+3}$ be any initial data, let $\epsilon > 0$, set $x = \epsilon\xi$ and pick any integer $m \geq 0$. Then, we compute

$$(\mathfrak{f}^{(2)}(\{u^k : k \geq 0\}))^m = -3 \sum_{i,j \geq 0} \overline{\mathfrak{C}}_{ijm} u^i u^j,$$
$$(\mathfrak{f}^{(2)}(\Phi^t(x)))^m = -3 \sum_{i,j\geq 0} \overline{\mathfrak{C}}_{ijm} (\Phi^t(x))^i (\Phi^t(x))^j = -3 \sum_{i,j \geq 0} \overline{\mathfrak{C}}_{ijm} x^i x^j \cos(\varpi_i t)\cos(\varpi_j t)$$
$$= -\frac{3}{2} \sum_{i,j\geq 0} \overline{\mathfrak{C}}_{ijm} x^i x^j \cos((\varpi_i - \varpi_j)t) - \frac{3}{2} \sum_{i,j \geq 0} \overline{\mathfrak{C}}_{ijm} x^i x^j \cos((\varpi_i + \varpi_j)t).$$

Then, $(P\mathfrak{f}^{(2)}(\Phi^t(x)))^m$ is given by

$$-\frac{3}{2}\Bigg[\sum_{\substack{i,j\geq 0 \\ \varpi_i - \varpi_j \neq \pm\varpi_m}} \overline{\mathfrak{C}}_{ijm} x^i x^j \cos((\varpi_i - \varpi_j)t) + \sum_{\substack{i,j\geq 0 \\ \varpi_i + \varpi_j \neq \pm\varpi_m}} \overline{\mathfrak{C}}_{ijm} x^i x^j \cos((\varpi_i + \varpi_j)t)\Bigg],$$

and $(L_\varpi^{-1} P\mathfrak{f}^{(2)}(\Phi^t(x)))^m$ reads

$$-\frac{3}{2}\Bigg[\sum_{\substack{i,j\geq 0 \\ \varpi_i - \varpi_j \neq \pm\varpi_m}} \frac{\overline{\mathfrak{C}}_{ijm}}{\lambda_{m,\varpi_i - \varpi_j}} x^i x^j \cos((\varpi_i - \varpi_j)t) + \sum_{\substack{i,j\geq 0 \\ \varpi_i + \varpi_j \neq \pm\varpi_m}} \frac{\overline{\mathfrak{C}}_{ijm}}{\lambda_{m,\varpi_i + \varpi_j}} x^i x^j \cos((\varpi_i + \varpi_j)t)\Bigg],$$

where we used the fact that the eigenvalues of L_ϖ are given by $\lambda_{ml} = \varpi_m^2 - l^2\varpi^2$. Hence, using the above together with the symmetries of the Fourier coefficients $\overline{\mathfrak{C}}_{\kappa\nu m} = \overline{\mathfrak{C}}_{\nu\kappa m}$ for all integers $\kappa, \nu, m \geq 0$, we deduce that

$$\big(\mathfrak{f}^{(2)}(\Phi^t(x) + L_1^{-1} P\mathfrak{f}^{(2)}(\Phi^t(x)))\big)^m$$
$$= -3 \sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} (\Phi^t(x))^\kappa (\Phi^t(x))^\nu - 6 \sum_{\kappa,\nu \geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} (\Phi^t(x))^\kappa (L_1^{-1} P\mathfrak{f}^{(2)}(\Phi^t(x)))^\nu$$
$$- 3 \sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} (L_1^{-1} P\mathfrak{f}^{(2)}(\Phi^t(x)))^\kappa (L_1^{-1} P\mathfrak{f}^{(2)}(\Phi^t(x)))^\nu$$

and, by setting $x = \epsilon\xi$, we infer that

$$\big(\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi) + L_1^{-1} P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))\big)^m = \epsilon^2 (\mathfrak{f}^{(2)}(\Phi^t(\xi)))^m + \epsilon^3 (E(\xi))^m + \epsilon^4 (F(\xi))^m, \tag{2-24}$$

where

$$(F(\xi))^m = -3 \sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} (L_1^{-1} P \mathfrak{f}^{(2)}(\Phi^t(\xi)))^\kappa (L_1^{-1} P \mathfrak{f}^{(2)}(\Phi^t(\xi)))^\nu,$$

$$\begin{aligned}(E(\xi))^m &= -6 \sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} (\Phi^t(\xi))^\kappa (L_1^{-1} P \mathfrak{f}^{(2)}(\Phi^t(\xi)))^\nu \\ &= -6 \sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \xi^\kappa \cos(\varpi_\kappa t)(L_1^{-1} P \mathfrak{f}^{(2)}(\Phi^t(\xi)))^\nu.\end{aligned}$$

We set $(E(\xi))^m = (E(\xi))^m_- + (E(\xi))^m_+$, where

$$(E(\xi))^m_\pm = 9 \sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \sum_{\substack{i,j\geq 0 \\ |\varpi_i \pm \varpi_j| \neq |\varpi_\nu|}} \frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2 - (\varpi_i \pm \varpi_j)^2} \xi^i \xi^j \xi^\kappa \cos((\varpi_i \pm \varpi_j)t)\cos(\varpi_\kappa t).$$

Next, we first apply the linear flow and then the average in time to obtain

$$\begin{aligned}(\mathfrak{F}_\epsilon(\xi))^m &= \epsilon^{-3} (\langle \mathfrak{f}^{(2)}((\cdot) + L_1^{-1} P \mathfrak{f}^{(2)}(\cdot))\rangle(\epsilon\xi))^m \\ &= \frac{\epsilon^{-3}}{2\pi} \int_0^{2\pi} \big(\Phi^t(\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi) + L_1^{-1} P \mathfrak{f}^{(2)} \Phi^t(\epsilon\xi)))\big)^m \, dt \\ &= \frac{\epsilon^{-1}}{2\pi} \int_0^{2\pi} \big(\Phi^t(\mathfrak{f}^{(2)}(\Phi^t(\xi)))\big)^m \, dt + \frac{1}{2\pi} \int_0^{2\pi} (\Phi^t(E(\xi)))^m \, dt + \frac{\epsilon}{2\pi} \int_0^{2\pi} (\Phi^t(F(\xi)))^m \, dt \\ &= \epsilon^{-1} \langle \mathfrak{f}^{(2)} \rangle(\xi) + \frac{1}{2\pi} \int_0^{2\pi} (\Phi^t(E(\xi)))^m \, dt + \frac{\epsilon}{2\pi} \int_0^{2\pi} (\Phi^t(F(\xi)))^m \, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} (\Phi^t(E(\xi)))^m \, dt + \frac{\epsilon}{2\pi} \int_0^{2\pi} (\Phi^t(F(\xi)))^m \, dt, \end{aligned} \tag{2-25}$$

where we used the condition that $\mathfrak{f}^{(2)}$ is nonresonant (2-8). Then, since $(E(\xi))^m = (E(\xi))^m_- + (E(\xi))^m_+$, the latter at $\epsilon = 0$ boils down to

$$\mathfrak{F}_0(\xi) = \frac{1}{2\pi} \int_0^{2\pi} (\Phi^t(E(\xi)))^m \, dt = \frac{1}{2\pi} \int_0^{2\pi} ((E(\xi))^m_- + (E(\xi))^m_+) \cos(\varpi_m t) \, dt.$$

Finally, we use the facts that

$$\int_0^{2\pi} \cos((\varpi_i - \varpi_j)t)\cos(\varpi_\kappa t)\cos(\varpi_m t) \, dt = \frac{\pi}{2} \sum_\pm \mathbb{1}(\varpi_i - \varpi_j \pm \varpi_\kappa \pm \varpi_m = 0)$$

to compute

$$\begin{aligned}&\frac{1}{2\pi} \int_0^{2\pi} (E(\xi))^m_- \cos(\varpi_m t) \, dt \\ &\qquad = \frac{9}{4} \sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \sum_{\substack{i,j\geq 0 \\ |\varpi_i - \varpi_j| \neq |\varpi_\nu|}} \frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2 - (\varpi_i - \varpi_j)^2} \xi^i \xi^j \xi^\kappa \sum_\pm \mathbb{1}(\varpi_i - \varpi_j \pm \varpi_\kappa \pm \varpi_m = 0).\end{aligned}$$

The term with the $(E(\xi))^m_+$ follows similarly and completes the proof. □

Lemma 2.14 (computation of $\mathfrak{F}_\epsilon(\xi)$ for small ϵ close to zero and 1-mode initial data). *Assume that $\varpi_n = n+2$ for all integers $n \geq 0$. Let $\gamma \geq 0$ be an integer, $\mathfrak{K}_\gamma \in \mathbb{R}$ and ξ be the* 1*-mode initial data, that is,*

$$\xi^m = \mathfrak{K}_\gamma \mathbb{1}(m=\gamma), \quad m \geq 0.$$

Then, for all integers $m \geq 0$, we have

$$(\mathfrak{F}_\epsilon(\xi))^m = \mathfrak{q}_\gamma \mathfrak{K}_\gamma^3 \mathbb{1}(m=\gamma) + \mathcal{O}(\epsilon), \quad \mathfrak{q}_\gamma = \frac{9}{4}\sum_{\nu=0}^{2\gamma}(\overline{\mathfrak{C}}_{\gamma\gamma\nu})^2\biggl(\frac{2}{\varpi_\nu^2} + \frac{\mathbb{1}(\varpi_\nu^2 \neq (2\varpi_\gamma)^2)}{\varpi_\nu^2 - (2\varpi_\gamma)^2}\biggr).$$

Proof. Let $\gamma \geq 0$ be an integer, and define ξ as the 1-mode initial data, that is $\xi^m = \mathfrak{K}_\gamma \mathbb{1}(m=\gamma)$, for all integers $m \geq 0$. Then, for any integer $m \geq 0$, we use the definition (2-6) to compute

$$\begin{aligned}(\mathfrak{f}^{(2)}(\Phi^t(\xi)))^m &= -3\sum_{i,j\geq 0}\overline{\mathfrak{C}}_{ijm}(\Phi^t(\xi))^i(\Phi^t(\xi))^j \\ &= -3\sum_{i,j\geq 0}\overline{\mathfrak{C}}_{ijm}\xi^i\xi^j\cos(\varpi_i t)\cos(\varpi_j t) \\ &= -3\mathfrak{K}_\gamma^2\overline{\mathfrak{C}}_{\gamma\gamma m}\cos^2(\varpi_\gamma t) = -\tfrac{3}{2}\mathfrak{K}_\gamma^2\overline{\mathfrak{C}}_{\gamma\gamma m}[1+\cos(2\varpi_\gamma t)].\end{aligned}$$

Recall the definition of the eigenvalues $\varpi_i = i+2$ for all integers $i \geq 0$. Then, we have

$$\varpi_m \neq 0 \iff m \geq 0 \quad \text{and} \quad \varpi_m \neq 2\varpi_\gamma \iff m \neq 2\gamma+2.$$

Hence, we infer

$$\begin{aligned}(P\mathfrak{f}^{(2)}(\Phi^t(\xi)))^m &= -\tfrac{3}{2}\mathfrak{K}_\gamma^2\overline{\mathfrak{C}}_{\gamma\gamma m}[1+\mathbb{1}(m\neq 2\gamma+2)\cos(2\varpi_\gamma t)], \\ (L_\varpi^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\xi)))^m &= -\frac{3}{2}\mathfrak{K}_\gamma^2\overline{\mathfrak{C}}_{\gamma\gamma m}\biggl[\frac{1}{\varpi_m^2} + \frac{\mathbb{1}(m\neq 2\gamma+2)}{\varpi_m^2-(2\varpi_\gamma)^2\varpi^2}\cos(2\varpi_\gamma t)\biggr],\end{aligned}$$

where we used the fact that the eigenvalues of L_ϖ are given by (2-14), i.e., $\lambda_{ml} = \varpi_m^2 - l^2\varpi^2$. In addition, we set $x = \epsilon\xi$, and $\bigl(\mathfrak{f}^{(2)}(\Phi^t(x) + L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(x)))\bigr)^m$ is given by

$$\begin{aligned}&-3\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}(\Phi^t(\epsilon\xi)+L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\kappa(\Phi^t(\epsilon\xi)+L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\nu \\ &\quad= -3\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}[(\Phi^t(\epsilon\xi))^\kappa + (L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\kappa][(\Phi^t(\epsilon\xi))^\nu + (L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\nu] \\ &\quad= -3\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}\bigl[(\Phi^t(\epsilon\xi))^\kappa(\Phi^t(\epsilon\xi))^\nu + (\Phi^t(\epsilon\xi))^\kappa(L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\nu \\ &\qquad\qquad + (\Phi^t(\epsilon\xi))^\nu(L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\kappa + (L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\kappa(L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\nu\bigr] \\ &\quad= -3\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}(\Phi^t(\epsilon\xi))^\kappa(\Phi^t(\epsilon\xi))^\nu - 6\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}(\Phi^t(\epsilon\xi))^\kappa(L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\nu \\ &\qquad\qquad - 3\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}(L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\kappa(L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\nu \\ &\quad= \epsilon^2(\mathfrak{f}^{(2)}(\Phi^t(\xi)))^m + \epsilon^3(E(\xi))^m + \epsilon^4(F(\xi))^m,\end{aligned}$$

where we set

$$(F(\xi))^m = -3 \sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} (L_1^{-1} P \mathfrak{f}^{(2)}(\Phi^t(\xi)))^\kappa (L_1^{-1} P \mathfrak{f}^{(2)}(\Phi^t(\xi)))^\nu,$$

$$\begin{aligned}(E(\xi))^m &= -6 \sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} (\Phi^t(\xi))^\kappa (L_1^{-1} P \mathfrak{f}^{(2)}(\Phi^t(\xi)))^\nu \\ &= 9\mathfrak{K}_\gamma^2 \sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \xi^\kappa \cos(\varpi_\kappa t) \overline{\mathfrak{C}}_{\gamma\gamma\nu} \left[\frac{1}{\varpi_\nu^2} + \frac{\mathbb{1}(\nu \neq 2\gamma+2)}{\varpi_\nu^2 - (2\varpi_\gamma)^2} \cos(2\varpi_\gamma t) \right] \\ &= 9\mathfrak{K}_\gamma^3 \sum_{\nu\geq 0} \overline{\mathfrak{C}}_{\gamma\nu m} \overline{\mathfrak{C}}_{\gamma\gamma\nu} \left[\left(\frac{1}{\varpi_\nu^2} + \frac{1}{2} \frac{\mathbb{1}(\nu \neq 2\gamma+2)}{\varpi_\nu^2 - (2\varpi_\gamma)^2} \right) \cos(\varpi_\gamma t) + \frac{1}{2} \frac{\mathbb{1}(\nu \neq 2\gamma+2)}{\varpi_\nu^2 - (2\varpi_\gamma)^2} \cos(3\varpi_\gamma t) \right].\end{aligned}$$

Now, we first apply the linear flow and then the average in time to obtain

$$\begin{aligned}\mathfrak{F}_\epsilon(\xi) &= \epsilon^{-3} \langle \mathfrak{f}^{(2)}((\cdot) + L_1^{-1} P \mathfrak{f}^{(2)}(\cdot)) \rangle (\epsilon\xi) \\ &= \frac{\epsilon^{-3}}{2\pi} \int_0^{2\pi} \left(\Phi^t \left(\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi) + L_1^{-1} P \mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi))) \right) \right)^m dt \\ &= \frac{\epsilon^{-1}}{2\pi} \int_0^{2\pi} \left(\Phi^t(\mathfrak{f}^{(2)}(\Phi^t(\xi))) \right)^m dt + \frac{1}{2\pi} \int_0^{2\pi} (\Phi^t(E(\xi)))^m dt + \frac{\epsilon}{2\pi} \int_0^{2\pi} (\Phi^t(F(\xi)))^m dt.\end{aligned}$$

On the one hand, due to (2-8), we have

$$\frac{1}{2\pi} \int_0^{2\pi} \left(\Phi^t(\mathfrak{f}^{(2)}(\Phi^t(\xi))) \right)^m dt = \langle \mathfrak{f}^{(2)} \rangle (\xi) = 0.$$

On the other hand, we use the orthogonality of the cosine function together with

$$\varpi_m = \varpi_\gamma \iff m = \gamma \qquad \text{and} \qquad \varpi_m = 3\varpi_\gamma \iff m = 3\gamma+4$$

to compute

$$\begin{aligned}&\frac{1}{2\pi} \int_0^{2\pi} (\Phi^t(E(\xi)))^m dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} (E(\xi))^m \cos(\varpi_m t)\, dt \\ &= 9\mathfrak{K}_\gamma^3 \sum_{\nu\geq 0} \overline{\mathfrak{C}}_{\gamma\nu m} \overline{\mathfrak{C}}_{\gamma\gamma\nu} \Bigg[\left(\frac{1}{\varpi_\nu^2} + \frac{1}{2} \frac{\mathbb{1}(\nu \neq 2\gamma+2)}{\varpi_\nu^2 - (2\varpi_\gamma)^2} \right) \frac{1}{2\pi} \int_0^{2\pi} \cos(\varpi_\gamma t) \cos(\varpi_m t)\, dt \\ &\qquad\qquad + \frac{1}{2} \frac{\mathbb{1}(\nu \neq 2\gamma+2)}{\varpi_\nu^2 - (2\varpi_\gamma)^2} \frac{1}{2\pi} \int_0^{2\pi} \cos(3\varpi_\gamma t) \cos(\varpi_m t)\, dt \Bigg] \\ &= 9\mathfrak{K}_\gamma^3 \sum_{\nu\geq 0} \overline{\mathfrak{C}}_{\gamma\nu m} \overline{\mathfrak{C}}_{\gamma\gamma\nu} \left[\left(\frac{1}{2\varpi_\nu^2} + \frac{1}{4} \frac{\mathbb{1}(\nu \neq 2\gamma+2)}{\varpi_\nu^2 - (2\varpi_\gamma)^2} \right) \mathbb{1}(m=\gamma) + \frac{1}{4} \frac{\mathbb{1}(\nu \neq 2\gamma+2)}{\varpi_\nu^2 - (2\varpi_\gamma)^2} \mathbb{1}(m = 3\gamma+4) \right] \\ &= \frac{9}{4} \mathfrak{K}_\gamma^3 \sum_{\nu\geq 0} (\overline{\mathfrak{C}}_{\gamma\gamma\nu})^2 \left(\frac{2}{\varpi_\nu^2} + \frac{\mathbb{1}(\nu \neq 2\gamma+2)}{\varpi_\nu^2 - (2\varpi_\gamma)^2} \right) \mathbb{1}(m=\gamma) + \frac{9}{4} \mathfrak{K}_\gamma^3 \sum_{\substack{\nu\geq 0 \\ \nu \neq 2\gamma+2}} \frac{\overline{\mathfrak{C}}_{\gamma,\nu,3\gamma+4} \overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2 - (2\varpi_\gamma)^2} \mathbb{1}(m = 3\gamma+4).\end{aligned}$$

Finally, we note that $\overline{\mathfrak{C}}_{\gamma,\nu,3\gamma+4}\overline{\mathfrak{C}}_{\gamma\gamma\nu} = 0$ for all integers $\gamma \geq 0$ and $\nu \geq 0$. This follows immediately from the fact that $\overline{\mathfrak{C}}_{ijm} = 0$ for all integers $i, j, m \geq 0$ with $i + j < m$ due to Lemma 5.8 below. Specifically, we have $\overline{\mathfrak{C}}_{\gamma\gamma\nu} = 0$ since $\nu > 2\gamma$, and $\overline{\mathfrak{C}}_{\gamma,\nu,3\gamma+4} = 0$ since

$$0 \leq \nu \leq 2\gamma \quad \Longrightarrow \quad \gamma + \nu \leq 3\gamma < 3\gamma + 4.$$

Consequently, we conclude that

$$\frac{1}{2\pi}\int_0^{2\pi} (\Phi^t(E(\xi)))^m \, dt = \frac{9}{4}\mathfrak{K}_\gamma^3 \sum_{\nu=0}^{2\gamma} (\overline{\mathfrak{C}}_{\gamma\gamma\nu})^2 \left(\frac{2}{\varpi_\nu^2} + \frac{1}{\varpi_\nu^2 - (2\varpi_\gamma)^2}\right) \mathbb{1}(m = \gamma). \qquad \square$$

Finally, we compute the differential of $\mathfrak{F}_0(\xi)$ at the 1-mode initial data.

Lemma 2.15 (differential of $\mathfrak{F}_0(\xi)$ at the 1-mode initial data). *Let $\gamma \geq 0$ be an integer, $\mathfrak{K}_\gamma \in \mathbb{R}$ and ξ be the 1-mode initial data, that is*

$$\xi^m = \mathfrak{K}_\gamma \mathbb{1}(m = \gamma), \quad m \geq 0.$$

Then, for all $h \in l^2_{s+3}$ and integers $m \geq 0$, we have

$$(d\mathfrak{F}_0(\xi)[h])^m = \mathfrak{K}_\gamma^2 [\mathfrak{a}_{\gamma m} h^m + \mathbb{1}(0 \leq m \leq 2\gamma)\mathfrak{b}_{\gamma m} h^{2\gamma - m}],$$

where

$$\mathfrak{a}_{\gamma m} = \frac{9}{2}\sum_{\nu=0}^{m+\gamma} \frac{(\overline{\mathfrak{C}}_{\gamma\nu m})^2}{\varpi_\nu^2 - (\varpi_m + \varpi_\gamma)^2} + \frac{9}{2}\sum_{\substack{\nu=0 \\ \nu \neq \pm(m-\gamma)-2}}^{m+\gamma} \frac{(\overline{\mathfrak{C}}_{m\gamma\nu})^2}{\varpi_\nu^2 - (\varpi_m - \varpi_\gamma)^2} + \frac{9}{4}\sum_{\nu=0}^{2\gamma} \frac{\overline{\mathfrak{C}}_{m\nu m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2}$$

and

$$\mathfrak{b}_{\gamma m} = \frac{9}{4}\sum_{\nu=0}^{2\gamma} \frac{\overline{\mathfrak{C}}_{2\gamma-m,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2 - (2\varpi_\gamma)^2} + \frac{9}{2}\sum_{\substack{\nu=0 \\ \nu \neq \pm(m-\gamma)-2}}^{m+\gamma} \frac{\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{2\gamma-m,\gamma,\nu}}{\varpi_\nu^2 - (\varpi_{2\gamma-m} - \varpi_\gamma)^2}.$$

Proof. Let $\gamma \geq 0$ be an integer, $\mathfrak{K}_\gamma \in \mathbb{R}$ and ξ to be the 1-mode initial data as above, and pick any $h \in l^2_{s+3}$, $\epsilon > 0$ and integer $m \geq 0$. Then, we set $\hat{\xi} = \xi + \epsilon h$, and according to Lemma 2.13, we have

$$(\mathfrak{F}_0(\hat{\xi}))^m = (\mathfrak{F}_{0-}(\hat{\xi}))^m + (\mathfrak{F}_{0+}(\hat{\xi}))^m,$$

where

$$(\mathfrak{F}_{0-}(\hat{\xi}))^m = \frac{9}{4}\sum_{\kappa,\nu \geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \sum_{\substack{i,j \geq 0 \\ \varpi_i - \varpi_j \neq \pm\varpi_\nu}} \frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2 - (\varpi_i - \varpi_j)^2} \hat{\xi}^i \hat{\xi}^j \hat{\xi}^\kappa \sum_{\pm} \mathbb{1}(\varpi_i - \varpi_j \pm \varpi_\kappa \pm \varpi_m = 0)$$

and

$$(\mathfrak{F}_{0+}(\hat{\xi}))^m = \frac{9}{4}\sum_{\kappa,\nu \geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \sum_{\substack{i,j \geq 0 \\ \varpi_i + \varpi_j \neq \pm\varpi_\nu}} \frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2 - (\varpi_i + \varpi_j)^2} \hat{\xi}^i \hat{\xi}^j \hat{\xi}^\kappa \sum_{\pm} \mathbb{1}(\varpi_i + \varpi_j \pm \varpi_\kappa \pm \varpi_m = 0).$$

We expand[9] $\mathfrak{F}_{0\pm}(\hat{\xi}) = \mathfrak{F}_{0\pm}(\xi) + \epsilon \cdot d\mathfrak{F}_{0\pm}(\xi)[h] + \mathcal{O}(\epsilon^2)$ and, using the definition of the 1-mode initial data, we obtain

$$
\begin{aligned}
(d\mathfrak{F}_{0-}(\xi)[h])^m &= \frac{9}{4}\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}\sum_{\substack{i,j\geq 0\\ \varpi_i-\varpi_j\neq\pm\varpi_\nu}}\frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2-(\varpi_i-\varpi_j)^2}\\
&\qquad\cdot[h^i\xi^j\xi^\kappa+\xi^i h^j\xi^\kappa+\xi^i\xi^j h^\kappa]\sum_{\pm}\mathbb{1}(\varpi_i-\varpi_j\pm\varpi_\kappa\pm\varpi_m=0)\\
&= \mathfrak{K}_\gamma^2\Bigg\{\frac{9}{4}\sum_{\nu\geq 0}\overline{\mathfrak{C}}_{\gamma\nu m}\sum_{\substack{i\geq 0\\ \varpi_i-\varpi_\gamma\neq\pm\varpi_\nu}}\frac{\overline{\mathfrak{C}}_{i\gamma\nu}}{\varpi_\nu^2-(\varpi_i-\varpi_\gamma)^2}h^i\sum_{\pm}\mathbb{1}(\varpi_i-\varpi_\gamma\pm\varpi_\gamma\pm\varpi_m=0)\\
&\qquad+\frac{9}{4}\sum_{\nu\geq 0}\overline{\mathfrak{C}}_{\gamma\nu m}\sum_{\substack{j\geq 0\\ \varpi_\gamma-\varpi_j\neq\pm\varpi_\nu}}\frac{\overline{\mathfrak{C}}_{\gamma j\nu}}{\varpi_\nu^2-(\varpi_\gamma-\varpi_j)^2}h^j\sum_{\pm}\mathbb{1}(\varpi_\gamma-\varpi_j\pm\varpi_\gamma\pm\varpi_m=0)\\
&\qquad+\frac{9}{4}\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}\frac{\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(\varpi_\gamma-\varpi_\gamma)^2}h^\kappa\sum_{\pm}\mathbb{1}(\pm\varpi_\kappa\pm\varpi_m=0)\Bigg\}.
\end{aligned}
$$

Recall the definition of the eigenvalues $\varpi_i = i+2$ for all integers $i\geq 0$ and also recall that $m, i, j, \kappa, \nu, \gamma \geq 0$. Then, we have

$$
\begin{cases}\varpi_i-\varpi_\gamma\pm\varpi_\gamma\pm\varpi_m=0,\\ \varpi_i-\varpi_\gamma\neq\pm\varpi_\nu\end{cases}
\iff
\begin{cases} i=m \text{ and } \nu\neq\pm(m-\gamma)-2,\\ i=2\gamma-m \text{ and } m\leq 2\gamma \text{ and } \nu\neq\pm(m-\gamma)-2,\\ i=2\gamma+m+4 \text{ and } \nu\neq 2+m+\gamma,\end{cases}
$$

$$
\begin{cases}\varpi_\gamma-\varpi_j\pm\varpi_\gamma\pm\varpi_m=0,\\ \varpi_\gamma-\varpi_j\neq\pm\varpi_\nu\end{cases}
\iff
\begin{cases} j=2\gamma+m+4 \text{ and } \nu\neq 2+m+\gamma,\\ j=2\gamma-m \text{ and } m\leq 2\gamma \text{ and } \nu\neq\pm(m-\gamma)-2,\\ j=m \text{ and } \nu\neq\pm(m-\gamma)-2,\end{cases}
$$

$$
\pm\varpi_\kappa\pm\varpi_m=0 \iff \begin{cases} m=-\kappa-4,\\ m=\kappa,\end{cases} \iff \kappa=m.
$$

Therefore, we infer

$$
\begin{aligned}
\mathfrak{K}_\gamma^{-2}(d\mathfrak{F}_{0-}(\xi)[h])^m &= h^m\Bigg[\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq\pm(m-\gamma)-2}}^{\infty}\frac{(\overline{\mathfrak{C}}_{m\gamma\nu})^2}{\varpi_\nu^2-(\varpi_m-\varpi_\gamma)^2}+\frac{9}{4}\sum_{\nu=0}^{\infty}\frac{\overline{\mathfrak{C}}_{m\nu m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2}\Bigg]\\
&\quad+h^{2\gamma-m}\mathbb{1}(0\leq m\leq 2\gamma)\Bigg[\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq\pm(m-\gamma)-2}}^{\infty}\frac{\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{2\gamma-m,\gamma,\nu}}{\varpi_\nu^2-(\varpi_{2\gamma-m}-\varpi_\gamma)^2}\Bigg]\\
&\quad+h^{2\gamma+m+4}\Bigg[\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq 2+m+\gamma}}^{\infty}\frac{\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{2\gamma+m+4,\gamma,\nu}}{\varpi_\nu^2-(\varpi_{2\gamma+m+4}-\varpi_\gamma)^2}\Bigg].
\end{aligned}
$$

[9] Here, the notation $\mathcal{O}(\epsilon^2)$ for a function of ξ or h refers to a function that is bounded by ϵ^2 in the $\mathcal{Q}$-norm using the l_{s+3}-norm of ξ or h.

Similarly, using the definition of the 1-mode initial data, we obtain

$$(d\mathfrak{F}_{0+}(\xi)[h])^m = \frac{9}{4}\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}\sum_{\substack{i,j\geq 0\\ \varpi_i+\varpi_j\neq\pm\varpi_\nu}}\frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2-(\varpi_i+\varpi_j)^2}\cdot[h^i\xi^j\xi^\kappa+\xi^i h^j\xi^\kappa+\xi^i\xi^j h^\kappa]\sum_{\pm}\mathbb{1}(\varpi_i+\varpi_j\pm\varpi_\kappa\pm\varpi_m=0)$$

$$= \mathfrak{K}_\gamma^2\Bigg\{\frac{9}{4}\sum_{\nu\geq 0}\overline{\mathfrak{C}}_{\gamma\nu m}\sum_{\substack{i\geq 0\\ \varpi_i+\varpi_\gamma\neq\pm\varpi_\nu}}\frac{\overline{\mathfrak{C}}_{i\gamma\nu}}{\varpi_\nu^2-(\varpi_i+\varpi_\gamma)^2}h^i\sum_{\pm}\mathbb{1}(\varpi_i+\varpi_\gamma\pm\varpi_\gamma\pm\varpi_m=0)$$
$$+\frac{9}{4}\sum_{\nu\geq 0}\overline{\mathfrak{C}}_{\gamma\nu m}\sum_{\substack{j\geq 0\\ \varpi_\gamma+\varpi_j\neq\pm\varpi_\nu}}\frac{\overline{\mathfrak{C}}_{\gamma j\nu}}{\varpi_\nu^2-(\varpi_\gamma+\varpi_j)^2}h^j\sum_{\pm}\mathbb{1}(\varpi_\gamma+\varpi_j\pm\varpi_\gamma\pm\varpi_m=0)$$
$$+\frac{9}{4}\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}\frac{\mathbb{1}(\varpi_\gamma+\varpi_\gamma\neq\pm\varpi_\nu)\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(\varpi_\gamma+\varpi_\gamma)^2}h^\kappa\sum_{\pm}\mathbb{1}(\varpi_\gamma+\varpi_\gamma\pm\varpi_\kappa\pm\varpi_m=0)\Bigg\}.$$

As before, recall the definition of the eigenvalues $\varpi_i = i+2$ for all integers $i\geq 0$ and also recall that $m, i, j, \kappa, \nu, \gamma\geq 0$. Then, we have

$$\begin{cases}\varpi_i+\varpi_\gamma\pm\varpi_\gamma\pm\varpi_m=0,\\ \varpi_i+\varpi_\gamma\neq\pm\varpi_\nu\end{cases}\iff\begin{cases}i=-4+m-2\gamma \text{ and } m\geq 4+2\gamma \text{ and } \nu\neq\pm(m-\gamma)-2,\\ i=m \text{ and } \nu\neq 2+m+\gamma,\end{cases}$$
$$\begin{cases}\varpi_\gamma+\varpi_j\pm\varpi_\gamma\pm\varpi_m=0,\\ \varpi_\gamma+\varpi_j\neq\pm\varpi_\nu\end{cases}\iff\begin{cases}j=-4+m-2\gamma \text{ and } m\geq 4+2\gamma \text{ and } \nu\neq\pm(m-\gamma)-2,\\ j=m \text{ and } \nu\neq 2+m+\gamma,\end{cases}$$
$$\begin{cases}\varpi_\gamma+\varpi_\gamma\pm\varpi_\kappa\pm\varpi_m=0,\\ \varpi_\gamma+\varpi_\gamma\neq\pm\varpi_\nu\end{cases}\iff\begin{cases}\kappa=-4+m-2\gamma \text{ and } m\geq 4+2\gamma \text{ and } \nu\neq 2+2\gamma,\\ \kappa=4+m+2\gamma \text{ and } \nu\neq 2+2\gamma,\\ \kappa=2\gamma-m \text{ and } m\leq 2\gamma \text{ and } \nu\neq 2+2\gamma.\end{cases}$$

Therefore, we infer

$$\mathfrak{K}_\gamma^{-2}(d\mathfrak{F}_{0+}(\xi)[h])^m$$
$$= h^m\Bigg[\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq 2+m+\gamma}}^{\infty}\frac{(\overline{\mathfrak{C}}_{\gamma\nu m})^2}{\varpi_\nu^2-(\varpi_m+\varpi_\gamma)^2}\Bigg]+h^{4+m+2\gamma}\Bigg[\frac{9}{4}\sum_{\substack{\nu=0\\ \nu\neq 2+2\gamma}}^{\infty}\frac{\overline{\mathfrak{C}}_{4+m+2\gamma,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(2\varpi_\gamma)^2}\Bigg]$$
$$+h^{-4+m-2\gamma}\mathbb{1}(m\geq 4+2\gamma)\Bigg[\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq\pm(m-\gamma)-2}}^{\infty}\frac{\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{\gamma,-4+m-2\gamma,\nu}}{\varpi_\nu^2-(\varpi_\gamma+\varpi_{-4+m-2\gamma})^2}+\frac{9}{4}\sum_{\substack{\nu=0\\ \nu\neq 2+2\gamma}}^{\infty}\frac{\overline{\mathfrak{C}}_{-4+m-2\gamma,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(2\varpi_\gamma)^2}\Bigg]$$
$$+h^{2\gamma-m}\mathbb{1}(0\leq m\leq 2\gamma)\Bigg[\frac{9}{4}\sum_{\substack{\nu=0\\ \nu\neq 2+2\gamma}}^{\infty}\frac{\overline{\mathfrak{C}}_{2\gamma-m,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(2\varpi_\gamma)^2}\Bigg].$$

Putting this all together yields that

$$\mathfrak{K}_\gamma^{-2}(d\mathfrak{F}_0(\xi)[h])^m = \mathfrak{K}_\gamma^{-2}[(d\mathfrak{F}_{0-}(\xi)[h])^m+(d\mathfrak{F}_{0+}(\xi)[h])^m]$$

is equal to

$$h^m\Bigg[\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq 2+m+\gamma}}^{\infty}\frac{(\overline{\mathfrak{C}}_{\gamma\nu m})^2}{\varpi_\nu^2-(\varpi_m+\varpi_\gamma)^2}+\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq\pm(m-\gamma)-2}}^{\infty}\frac{(\overline{\mathfrak{C}}_{m\gamma\nu})^2}{\varpi_\nu^2-(\varpi_m-\varpi_\gamma)^2}+\frac{9}{4}\sum_{\nu=0}^{\infty}\frac{\overline{\mathfrak{C}}_{m\nu m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2}\Bigg]$$
$$+h^{2\gamma-m}\mathbb{1}(0\le m\le 2\gamma)\Bigg[\frac{9}{4}\sum_{\substack{\nu=0\\ \nu\neq 2+2\gamma}}^{\infty}\frac{\overline{\mathfrak{C}}_{2\gamma-m,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(2\varpi_\gamma)^2}+\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq\pm(m-\gamma)-2}}^{\infty}\frac{\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{2\gamma-m,\gamma,\nu}}{\varpi_\nu^2-(\varpi_{2\gamma-m}-\varpi_\gamma)^2}\Bigg]$$
$$+h^{4+m+2\gamma}\Bigg[\frac{9}{4}\sum_{\substack{\nu=0\\ \nu\neq 2+2\gamma}}^{\infty}\frac{\overline{\mathfrak{C}}_{4+m+2\gamma,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(2\varpi_\gamma)^2}+\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq 2+m+\gamma}}^{\infty}\frac{\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{2\gamma+m+4,\gamma,\nu}}{\varpi_\nu^2-(\varpi_{2\gamma+m+4}-\varpi_\gamma)^2}\Bigg]$$
$$+h^{-4+m-2\gamma}\mathbb{1}(m\ge 4+2\gamma)\Bigg[\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq\pm(m-\gamma)-2}}^{\infty}\frac{\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{\gamma,-4+m-2\gamma,\nu}}{\varpi_\nu^2-(\varpi_\gamma+\varpi_{-4+m-2\gamma})^2}+\frac{9}{4}\sum_{\substack{\nu=0\\ \nu\neq 2+2\gamma}}^{\infty}\frac{\overline{\mathfrak{C}}_{-4+m-2\gamma,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(2\varpi_\gamma)^2}\Bigg].$$

Finally, we simplify the formula above and show that

$$\overline{\mathfrak{C}}_{4+m+2\gamma,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}=\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{2\gamma+m+4,\gamma,\nu}=0, \tag{2-26}$$
$$\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{\gamma,-4+m-2\gamma,\nu}=\overline{\mathfrak{C}}_{-4+m-2\gamma,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}=0, \tag{2-27}$$

for all $\gamma,\nu,m\ge 0$ and all $\gamma,\nu\ge 0$ and $m\ge 2\gamma+4$, respectively. These follow immediately from the fact that $\overline{\mathfrak{C}}_{ijm}=0$ for all integers $i,j,m\ge 0$ with $i+j<m$ (Lemma 5.8). Specifically, we have $\overline{\mathfrak{C}}_{\gamma\gamma\nu}=0$ since $\nu>2\gamma$, $\overline{\mathfrak{C}}_{4+m+2\gamma,\nu,m}=0$ since

$$0\le\nu\le 2\gamma \iff m+\nu\le m+2\gamma<4+m+2\gamma,$$

as well as $\overline{\mathfrak{C}}_{\gamma\nu m}=0$ since $\nu>\gamma+m$ and $\overline{\mathfrak{C}}_{2\gamma+m+4,\gamma,\nu}=0$ since

$$0\le\nu\le\gamma+m \iff \gamma+\nu\le 2\gamma+m<2\gamma+m+4$$

which prove (2-26). Similarly, we have $\overline{\mathfrak{C}}_{\gamma\nu m}=0$ since $m>\gamma+\nu$, $\overline{\mathfrak{C}}_{\gamma,-4+m-2\gamma,\nu}=0$ since

$$0\le m\le\gamma+\nu \iff -4+m-2\gamma+\gamma=-4+m-\gamma\le -4+\nu<\nu,$$

as well as $\overline{\mathfrak{C}}_{\gamma\gamma\nu}=0$ since $\nu>2\gamma$ and $\overline{\mathfrak{C}}_{-4+m-2\gamma,\nu,m}=0$ since

$$0\le\nu\le 2\gamma \iff -4+m-2\gamma+\nu\le -4+m<m,$$

which prove (2-27). Using the same argument as above (Lemma 5.8), we also infer $\overline{\mathfrak{C}}_{\gamma\gamma\nu}=0$ since $\nu>2\gamma$ and $\overline{\mathfrak{C}}_{\gamma\nu m}=0$ since $\nu>\gamma+m$. Hence, the latter reduces to

$$h^m\Bigg[\frac{9}{2}\sum_{\nu=0}^{m+\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma\nu m})^2}{\varpi_\nu^2-(\varpi_m+\varpi_\gamma)^2}+\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq\pm(m-\gamma)-2}}^{m+\gamma}\frac{(\overline{\mathfrak{C}}_{m\gamma\nu})^2}{\varpi_\nu^2-(\varpi_m-\varpi_\gamma)^2}+\frac{9}{4}\sum_{\nu=0}^{2\gamma}\frac{\overline{\mathfrak{C}}_{m\nu m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2}\Bigg]$$
$$+h^{2\gamma-m}\mathbb{1}(0\le m\le 2\gamma)\Bigg[\frac{9}{4}\sum_{\nu=0}^{2\gamma}\frac{\overline{\mathfrak{C}}_{2\gamma-m,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(2\varpi_\gamma)^2}+\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq\pm(m-\gamma)-2}}^{m+\gamma}\frac{\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{2\gamma-m,\gamma,\nu}}{\varpi_\nu^2-(\varpi_{2\gamma-m}-\varpi_\gamma)^2}\Bigg]. \qquad \square$$

3. The linear eigenvalue problems

Next, we the study the linear eigenvalue problems where the linearized operators are given by (1-14). In all three models, the associated eigenfunctions are given by Jacobi polynomials, which is a common feature with the Einstein–Klein–Gordon system in spherical symmetry [Maliborski and Rostworowski 2013].

3A. ***Conformal cubic wave equation in spherical symmetry.*** We consider $L^2((0,\pi);\sin^2(x)\,dx)$, a Hilbert space, and associate it with the inner product

$$(f\mid g)=\frac{2}{\pi}\int_0^\pi f(x)g(x)\sin^2(x)\,dx.$$

For the conformal wave equation in spherical symmetry, the operator that governs the solutions to the linearized equation is given by

$$Lu=-\frac{1}{\sin^2(x)}\partial_x(\sin^2(x)\partial_x u)+u,\qquad \mathcal{D}(L)=\{u\in L^2((0,\pi);\sin^2(x)\,dx):Lu\in L^2((0,\pi);\sin^2(x)\,dx)\}.$$

The operator L is generated by the closed sesquilinear form a defined on $(H^1((0,\pi);\sin^2(x)\,dx))^2$ that is given by

$$a(u,v)=\int_0^\pi(\partial_x u\partial_x v+uv)\sin^2(x)\,dx$$

and $a(u,u)\simeq\|u\|^2_{H^1((0,\pi);\sin^2(x)\,dx)}$. In particular, L is self-adjoint on $\mathcal{D}(L)$. Now, the eigenvalue problem $Lu=\omega^2u$ reads

$$\partial_x(\sin^2(x)\partial_x u)+(\omega^2-1)\sin^2(x)u=0,$$

and, by setting $u(x)=v(y)$ and $y=\cos(x)$, it becomes

$$(1-y^2)v''(y)-3yv'(y)+(\omega^2-1)v(y)=0.$$

The latter has nontrivial solutions if and only if the solutions are given by the Chebyshev polynomials of the second kind [Szegő 1975], that is $v(y)=U_n(y)$. Hence, the solutions to the eigenvalue problem $Lu=\omega^2u$ are given by

$$e_n(x)=U_n(\cos(x)),\qquad \omega_n^2=(n+1)^2, \tag{3-1}$$

for all integers $n\geq 0$. In addition, the set $\{e_n:n\geq 0\}$ forms an orthonormal and complete basis for $L^2((0,\pi);\sin^2(x)\,dx)$. In fact, $(e_n\mid e_m)=\mathbb{1}(n=m)$ for any $n,m\geq 0$ due to the orthogonality of the Chebyshev polynomials of the second kind.

3B. ***Conformal cubic wave equation out of spherical symmetry.*** We consider $L^2\big((0,\frac{\pi}{2});\sin(2x)\,dx\big)$, a Hilbert space, and associate it with the inner product

$$\langle f\mid g\rangle=\int_0^{\pi/2}f(x)g(x)\sin(2x)\,dx.$$

For the conformal cubic wave equation out of spherical symmetry, the operator that governs the solutions to the linearized equation is given by

$$\mathsf{L}^{(\mu_1,\mu_2)}u=-\frac{1}{\sin(2x)}\partial_x(\sin(2x)\partial_x u)+\left(\frac{\mu_1^2}{\sin^2 x}+\frac{\mu_2^2}{\cos^2 x}+1\right)u$$

endowed with the domain $\mathcal{D}(\mathsf{L}^{(\mu_1,\mu_2)})$ defined by

$$\mathcal{D}(\mathsf{L}^{(\mu_1,\mu_2)}) = \bigl\{u \in L^2\bigl((0, \tfrac{\pi}{2}); \sin(2x)\, dx\bigr) : \mathsf{L}^{(\mu_1,\mu_2)} u \in L^2\bigl((0, \tfrac{\pi}{2}); \sin(2x)\, dx\bigr)\bigr\}.$$

The operator $\mathsf{L}^{(\mu_1,\mu_2)}$ is generated by the closed sesquilinear form a defined on $\bigl(H^1\bigl((0, \tfrac{\pi}{2}); \sin(2x)\, dx\bigr)\bigr)^2$ that is given by

$$\mathsf{a}(u, v) = \int_0^{\pi} \left(\partial_x u \partial_x v + \left(\frac{\mu_1^2}{\sin^2 x} + \frac{\mu_2^2}{\cos^2 x} + 1\right) uv\right) \sin(2x)\, dx,$$

and Hardy's inequality yields $\mathsf{a}(u, u) \simeq \|u\|_{H^1((0,\pi/2);\sin(2x)\, dx)^2}$. In particular, $\mathsf{L}^{(\mu_1,\mu_2)}$ is self-adjoint on $\mathcal{D}(\mathsf{L}^{(\mu_1,\mu_2)})$. Now, the eigenvalue problem $\mathsf{L}^{(\mu_1,\mu_2)} u = \omega^2 u$ reads

$$\partial_x^2 u + \left(\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}\right)\partial_x u - \left(\frac{\mu_1^2}{\sin^2 x} + \frac{\mu_2^2}{\cos^2 x} + 1 - \omega^2\right) u = 0,$$

and by setting $u(x) = v(y)$, $v(y) = (1 - y)^{\mu_1/2}(1 + y)^{\mu_2/2} w(y)$ and $y = \cos(2x)$, it becomes

$$(1 - y^2) w''(y) + [(\mu_2 - \mu_1) - (2 + \mu_1 + \mu_2) y] w'(y) + \tfrac{1}{4}[\omega^2 - (1 + \mu_1 + \mu_2)^2] w(y) = 0.$$

The latter has nontrivial solutions if and only if the solutions are given by the Jacobi polynomial with parameters (μ_1, μ_2) and degree $n \geq 0$, that is $w(y) = P_n^{(\mu_1,\mu_2)}(y)$. Hence, the solutions to the eigenvalue problem $\mathsf{L}^{(\mu_1,\mu_2)} u = \omega^2 u$ are given by

$$\begin{aligned} \mathsf{e}_n^{(\mu_1,\mu_2)}(x) &= \mathsf{N}_n^{(\mu_1,\mu_2)} (1 - \cos(2x))^{\mu_1/2} (1 + \cos(2x))^{\mu_2/2} P_n^{(\mu_1,\mu_2)}(\cos(2x)), \\ (\omega_n^{(\mu_1,\mu_2)})^2 &= (2n + 1 + \mu_1 + \mu_2)^2, \end{aligned} \tag{3-2}$$

for all integers $n \geq 0$, where the normalization constant reads

$$\mathsf{N}_n^{(\mu_1,\mu_2)} = \sqrt{\frac{\omega_n^{(\mu_1,\mu_2)}}{2^{\mu_1+\mu_2}} \frac{\Gamma(n+1)\Gamma(n + \mu_1 + \mu_2 + 1)}{\Gamma(n + \mu_1 + 1)\Gamma(n + \mu_2 + 1)}}. \tag{3-3}$$

In addition, the set $\{\mathsf{e}_n^{(\mu_1,\mu_2)} : n \geq 0\}$ forms an orthonormal and complete basis for $L^2\bigl((0, \tfrac{\pi}{2}); \sin(2x)\, dx\bigr)$. In fact,

$$\langle \mathsf{e}_n^{(\mu_1,\mu_2)} \mid \mathsf{e}_m^{(\mu_1,\mu_2)} \rangle = \mathbb{1}(n = m)$$

for any $n, m \geq 0$ due to the orthogonality of the Jacobi polynomials.

3C. ***Yang–Mills equation in spherical symmetry.*** We consider the Hilbert space $L^2((0, \pi); \sin^4(x)\, dx)$ associated with the inner product

$$[f|g] = \int_0^{\pi} f(x) g(x) \sin^4(x)\, dx.$$

For the Yang–Mills equation in spherical symmetry, the operator that governs the solutions to the linearized equation is given by

$$\mathfrak{L} u = -\frac{1}{\sin^4 x} \partial_x (\sin^4 x\, \partial_x u) + 4u, \quad \mathcal{D}(\mathfrak{L}) = \{u \in L^2((0, \pi); \sin^4 x\, dx) : \mathfrak{L} u \in L^2((0, \pi); \sin^4 x\, dx)\}.$$

The operator $\mathfrak{L}$ is generated by the closed sesquilinear form $\mathfrak{a}$ defined on $(H^1((0,\pi);\sin^4 x\,dx))^2$ that is given by

$$\mathfrak{a}(u,v)=\int_0^\pi (\partial_x u\partial_x v+4uv)\sin^4(x)\,dx$$

and $\mathfrak{a}(u,u)\simeq\|u\|^2_{H^1((0,\pi);\sin^4 x\,dx)}$. In particular, $\mathfrak{L}$ is self-adjoint on $\mathcal{D}(\mathfrak{L})$. Now, the eigenvalue problem $\mathfrak{L}u=\varpi^2u$ reads

$$\partial_x^2u+\frac{4}{\tan(x)}\partial_x u+(\omega^2-4)u=0,$$

and by setting $u(x)=w(y)$ and $y=\cos(x)$ it becomes

$$(1-y^2)w''(y)-5yw'(y)+(\omega^2-4)w(y)=0.$$

The latter has nontrivial solutions if and only if the solutions are given by the Jacobi polynomials with parameters $\left(\frac{3}{2},\frac{3}{2}\right)$ and degree n, that is $w(y)=P_n^{(3/2,3/2)}(y)$. Hence, the solutions to the eigenvalue problem $\mathfrak{L}u=\varpi^2u$ are given by

$$\mathfrak{e}_n(x)=\mathfrak{N}_nP_n^{(3/2,3/2)}(\cos(x)),\quad \varpi_n^2=(n+2)^2, \tag{3-4}$$

for all integers $n\ge 0$, where the normalization constant reads

$$\mathfrak{N}_n=\frac{\sqrt{\varpi_n\Gamma(1+n)\Gamma(4+n)}}{2\sqrt{2}\Gamma\left(\frac{5}{2}+n\right)}. \tag{3-5}$$

In addition, the set $\{\mathfrak{e}_n:n\ge 0\}$ forms an orthonormal and complete basis for $L^2((0,\pi);\sin^4(x)\,dx)$. In fact, $[\mathfrak{e}_n\mid\mathfrak{e}_m]=\mathbb{1}(n=m)$ for any $n,m\ge 0$ due to the orthogonality of the Jacobi polynomials.

4. The PDEs in Fourier space

In this section, we express the partial differential equations (1-13),

$$(\partial_t^2+\boldsymbol{L})u=\boldsymbol{f}(x,u),\quad (t,x)\in\mathbb{R}\times\boldsymbol{I},$$

in the Fourier space to obtain infinite dimensional systems of coupled, nonlinear harmonic oscillators, and we provide basic estimates for the nonlinearities. Here, the nonlinearities are given by (1-15), namely

$$\boldsymbol{f}(x,u)=\begin{cases}-u^3 & \text{for CW and WH,}\\ -3u^2-\sin^2(x)u^3 & \text{for YM.}\end{cases}$$

Let $u(t,\cdot)$ be a solution to any of the three models

CW: (1-3)–(1-4), WH: (1-6)–(1-7)–(1-5), YM: (1-11)–(1-12),

and recall that the sets of the associated eigenfunctions

CW: $\{e_n:n\ge 0\}$ by (3-1), WH: $\{\mathrm{e}_n^{(\mu_1,\mu_2)}:n\ge 0\}$ by (3-2), YM: $\{\mathfrak{e}_n:n\ge 0\}$ by (3-4) form an orthonormal and complete basis of the Hilbert spaces

$$\text{CW: } L^2([0,\pi];\sin^2(x)\,dx), \quad \text{WH: } L^2\big([0,\tfrac{\pi}{2}];\sin(2x)\,dx\big), \quad \text{YM: } L^2([0,\pi];\sin^4(x)\,dx).$$

Then, we expand $u(t,\cdot)$ in terms of the eigenfunctions and substitute the expression into (1-13) to find infinite systems of nonlinear harmonic oscillators.

4A. *Conformal cubic wave equation in spherical symmetry.* For the conformal cubic wave equation in spherical symmetry, we expand

$$u(t,\cdot)=\sum_{n=0}^{\infty}u^n(t)e_n, \quad e_ie_je_k=\sum_{m=0}^{\infty}C_{ijkm}e_m, \tag{4-1}$$

to find the infinite system of nonlinear harmonic oscillators

$$\ddot{u}^m(t)+(Au(t))^m=(f(\{u^j(t):j\geq 0\}))^m \tag{4-2}$$

for all integers $m\geq 0$, where the dots denote derivatives with respect to time and

$$(Au(t))^m=\omega_m^2u^m(t), \quad (f(\{u^j(t):j\geq 0\}))^m=-\sum_{i,j,k=0}^{\infty}C_{ijkm}u^i(t)u^j(t)u^k(t). \tag{4-3}$$

4B. *Conformal cubic wave equation out of spherical symmetry.* For the conformal cubic wave equation out of spherical symmetry, we expand

$$u(t,\cdot)=\sum_{n=0}^{\infty}u^n(t)\mathsf{e}_n^{(\mu_1,\mu_2)}, \quad \mathsf{e}_i^{(\mu_1,\mu_2)}\mathsf{e}_j^{(\mu_1,\mu_2)}\mathsf{e}_k^{(\mu_1,\mu_2)}=\sum_{m=0}^{\infty}\mathsf{C}_{ijkm}^{(\mu_1,\mu_2)}\mathsf{e}_m^{(\mu_1,\mu_2)} \tag{4-4}$$

to find the infinite system of nonlinear harmonic oscillators

$$\ddot{u}^m(t)+(\mathsf{A}u(t))^m=(\mathsf{f}(\{u^j(t):j\geq 0\}))^m \tag{4-5}$$

for all integers $m\geq 0$, where the dots denote derivatives with respect to time and

$$(\mathsf{A}u(t))^m=(\omega_n^{(\mu_1,\mu_2)})^2u^m(t), \quad (\mathsf{f}(\{u^j(t):j\geq 0\}))^m=-\sum_{i,j,k=0}^{\infty}\mathsf{C}_{ijkm}^{(\mu_1,\mu_2)}u^i(t)u^j(t)u^k(t). \tag{4-6}$$

4C. *Yang–Mills equation in spherical symmetry.* For the Yang–Mills equation in spherical symmetry, we expand

$$u(t,\cdot)=\sum_{n=0}^{\infty}u^n(t)\mathfrak{e}_n, \quad \mathfrak{e}_i(x)\mathfrak{e}_j(x)=\sum_{m=0}^{\infty}\overline{\mathfrak{C}}_{ijm}\mathfrak{e}_m(x), \quad \sin^2(x)\mathfrak{e}_i(x)\mathfrak{e}_j(x)\mathfrak{e}_k(x)=\sum_{m=0}^{\infty}\mathfrak{C}_{ijkm}\mathfrak{e}_m(x) \tag{4-7}$$

to find the infinite system of nonlinear harmonic oscillators

$$\ddot{u}^m(t)+(\mathfrak{A}u(t))^m=(\mathfrak{f}(\{u^j(t):j\geq 0\}))^m \tag{4-8}$$

for all integers $m\geq 0$, where the dots denote derivatives with respect to time and

$$(\mathfrak{A}u(t))^m=\varpi_m^2u^m(t), \quad (\mathfrak{f}(\{u^j(t):j\geq 0\}))^m=(\mathfrak{f}^{(2)}(\{u^j(t):j\geq 0\}))^m+(\mathfrak{f}^{(3)}(\{u^j(t):j\geq 0\}))^m,$$

with

$$(\mathfrak{f}^{(2)}(\{u^j(t) : j \geq 0\}))^m = -3 \sum_{i,j=0}^{\infty} \overline{\mathfrak{C}}_{ijm} u^i(t) u^j(t), \tag{4-9}$$

$$(\mathfrak{f}^{(3)}(\{u^j(t) : j \geq 0\}))^m = - \sum_{i,j,k=0}^{\infty} \mathfrak{C}_{ijkm} u^i(t) u^j(t) u^k(t). \tag{4-10}$$

4D. *Lipschitz bounds.* Recall Section 2 where we define the Banach space

$$\mathcal{H}_s^k = \left\{ q \in H^k([0, 2\pi]; l_s^2) : q(t) = \sum_{j=0}^{\infty} q^j(t) e_j = \sum_{j=0}^{\infty} \left(\sum_{l=0}^{\infty} q^{lj} \cos(lt) \right) e_j, \ \|q\|_{\mathcal{H}_s^k}^2 < \infty \right\}$$

endowed with the norm

$$\|q\|_{\mathcal{H}_s^k}^2 = \sum_{j=0}^{\infty} j^{2s} \left(2|q^{0j}|^2 + \sum_{l=1}^{\infty} |q^{lj}|^2 (1+l^2)^k \right) = \frac{1}{\pi} \int_0^{2\pi} \sum_{\lambda=0}^{k} |q^{(\lambda)}(t)|_s^2 \, dt,$$

where $q^{(\lambda)}(t)$ denotes the λ-th derivative of $q(t)$ with respect to t. Next, we show that the nonlinear terms we consider satisfy the following Lipschitz bounds, and we begin by considering the conformal cubic wave equation in spherical symmetry.

Lemma 4.1 (Lipschitz bounds for the CW model). *Let f be given by (4-3). Then, for all integers $k \geq 0$ and $s \geq 2$, there exists a positive constant (depending only on k and s) such that*

$$\|f(u) - f(v)\|_{\mathcal{H}_s^k} \lesssim (\|u\|_{\mathcal{H}_s^k}^2 + \|v\|_{\mathcal{H}_s^k}^2) \|u - v\|_{\mathcal{H}_s^k},$$

$$\|df(u)[h] - df(v)[h]\|_{\mathcal{H}_s^k} \lesssim (\|u\|_{\mathcal{H}_s^k} + \|v\|_{\mathcal{H}_s^k}) \|h\|_{\mathcal{H}_s^k} \|u - v\|_{\mathcal{H}_s^k},$$

for all $u, v, h \in \mathcal{H}_s^k$ with $\|u\|_{\mathcal{H}_s^k} \leq \epsilon$, $\|v\|_{\mathcal{H}_s^k} \leq \epsilon$ and $\|h\|_{\mathcal{H}_s^k} \leq \epsilon$.

Remark 4.2 (regularity of the initial data for the CW model). As stated above, for the CW model, we require $s \in \mathbb{N}$ with $s \geq 2$. This means that the space of initial data $\mathcal{Q} \simeq l_{s+1}^2$ is at least l_3^2 (Theorem 2.4).

Proof. Let $s \geq 2$ be an integer, and pick any $u = \{u^j : j \geq 0\} \in l_s^2$. We also denote by $u(x) = \sum_{j=0}^{\infty} u^j e_j(x)$ the corresponding function in the physical space and recall the definition of the linear operator L given in Section 3A. On the one hand, for any integer $s \geq 1$, we define the Sobolev space H_{CW}^s for spherically symmetric functions and find

$$\|u\|_{H_{\mathrm{CW}}^s}^2 = \int_0^{\pi} u L^s u \sin^2 x \, dx = \sum_{j=0}^{+\infty} \omega_j^{2s} |u_j|^2 \simeq |u|_s^2$$

since $\omega_j \simeq j$. On the other hand, note that, with a slight abuse of notation (we denote by u the original variable as well as the spherically symmetric version of it), we have that the Sobolev space above is equivalent to the standard Sobolev on $\mathbb{S}^3$:

$$\|u\|_{H_{\mathrm{CW}}^s}^2 \simeq \|u\|_{H^s(\mathbb{S}^3)}^2 = \int_{\mathbb{S}^3} u(-\Delta_{\mathbb{S}^3}^s u) \, d\mathrm{vol}_{\mathbb{S}^3} + \|u\|_{L^2(\mathbb{S}^3)}^2.$$

Here, $\Delta_{\mathbb{S}^3}$ stands for the standard Laplacian on $\mathbb{S}^3$ for the round metric and the standard volume form $d\text{vol}_{\mathbb{S}^3}$. This equivalence yields that H^s_{CW} is an algebra since $H^s(\mathbb{S}^3)$ is an algebra provided that $s > \frac{3}{2}$. Then, picking an integer $s \geq 2$ and $u, v \in l^2_s$ and using the algebra property and the triangular inequality together with Plancherel's theorem yield

$$\begin{aligned}
|f(u)|_s &= \|u^3\|_{H^s_{\text{CW}}} \lesssim \|u\|^3_{H^s_{\text{CW}}} = |u|^3_s, \\
|f(u) - f(v)|_s &= \|u^3 - v^3\|_{H^s_{\text{CW}}} = \|(u-v)(u^2 + uv + v^2)\|_{H^s_{\text{CW}}} \\
&\lesssim \|u - v\|_{H^s_{\text{CW}}} (\|u\|^2_{H^s_{\text{CW}}} + \|u\|_{H^s_{\text{CW}}} \|v\|_{H^s_{\text{CW}}} + \|v\|^2_{H^s_{\text{CW}}}) \\
&\lesssim \|u - v\|_{H^s_{\text{CW}}} (\|u\|^2_{H^s_{\text{CW}}} + \|v\|^2_{H^s_{\text{CW}}}) = |u - v|_s (|u|^2_s + |v|^2_s), \\
|df(u)[h] - df(v)[h]|_s &= \|df(u)[h] - df(v)[h]\|_{H^s_{\text{CW}}} = \|d(u^3)[h] - d(v^3)[h]\|_{H^s_{\text{CW}}} \\
&= \|3u^2 h - 3v^2 h\|_{H^s_{\text{CW}}} \lesssim \|u^2 - v^2\|_{H^s_{\text{CW}}} \|h\|_{H^s_{\text{CW}}} \\
&\lesssim \|u - v\|_{H^s_{\text{CW}}} (\|u\|_{H^s_{\text{CW}}} + \|v\|_{H^s_{\text{CW}}}) \|h\|_{H^s_{\text{CW}}} \lesssim |u - v|_s (|u|_s + |v|_s) |h|_s.
\end{aligned}$$

This proves the claim for $k = 0$. Finally, we present the proof for $k = 1$. In this case, Plancherel's theorem yields

$$\|f(u)\|_{\mathcal{H}^1_s} = \|f(u)\|_{H^1_t l^2_s} = \|f(u)\|_{L^2_t l^2_s} + \|\partial_t f(u)\|_{L^2_t l^2_s} = \|f(u)\|_{L^2_t H^s_x} + \|\partial_t f(u)\|_{L^2_t H^s_x}.$$

Furthermore, the algebra property and Holder's inequality together with the embedding $H^1 \hookrightarrow L^\infty$ yield

$$\begin{aligned}
\|f(u)\|_{L^2_t H^s_x} &= \|u^3\|_{L^2_t H^s_x} = \big\| \|u^3\|_{H^s_x} \big\|_{L^2_t} \lesssim \big\| \|u\|^3_{H^s_x} \big\|_{L^2_t} \leq \big\| \|u\|^2_{H^s_x} \big\|_{L^\infty_t} \big\| \|u\|_{H^s_x} \big\|_{L^2_t} \\
&= \big\| \|u\|_{H^s_x} \big\|^2_{L^\infty_t} \big\| \|u\|_{H^s_x} \big\|_{L^2_t} \lesssim \big\| \|u\|_{H^s_x} \big\|^2_{H^1_t} \big\| \|u\|_{H^s_x} \big\|_{L^2_t} = \|u\|^2_{H^1_t H^s_x} \|u\|_{H^0_t H^s_x} \leq \|u\|^3_{H^1_t H^s_x}, \\
\|\partial_t f(u)\|_{L^2_t H^s_x} &= \|3u^2 \partial_t u\|_{L^2_t H^s_x} \simeq \big\| \|u^2 \partial_t u\|_{H^s_x} \big\|_{L^2_t} \lesssim \big\| \|u\|^2_{H^s_x} \|\partial_t u\|_{H^s_x} \big\|_{L^2_t} \leq \big\| \|u\|^2_{H^s_x} \big\|_{L^\infty_t} \big\| \|\partial_t u\|_{H^s_x} \big\|_{L^2_t} \\
&= \big\| \|u\|_{H^s_x} \big\|^2_{L^\infty_t} \big\| \|\partial_t u\|_{H^s_x} \big\|_{L^2_t} \lesssim \big\| \|u\|_{H^s_x} \big\|^2_{H^1_t} \big\| \|u\|_{H^s_x} \big\|_{H^1_t} \leq \|u\|^3_{H^1_t H^s_x},
\end{aligned}$$

and hence $\|f(u)\|_{\mathcal{H}^1_s} \lesssim \|u\|^3_{\mathcal{H}^1_s}$. All the other bounds follow similarly. □

Next, we consider the conformal cubic wave equation out of spherical symmetry.

Lemma 4.3 (Lipschitz bounds for the CH model). *Let* f *be given by* (4-6). *Then, for all integers* $k \geq 0$ *and* $s \geq 2$, *there exists a positive constant* (*depending only on* k *and* s) *such that*

$$\|\mathsf{f}(u) - \mathsf{f}(v)\|_{\mathcal{H}^k_s} \lesssim (\|u\|^2_{\mathcal{H}^k_s} + \|v\|^2_{\mathcal{H}^k_s}) \|u - v\|_{\mathcal{H}^k_s},$$

$$\|d\mathsf{f}(u)[h] - d\mathsf{f}(v)[h]\|_{\mathcal{H}^k_s} \lesssim (\|u\|_{\mathcal{H}^k_s} + \|v\|_{\mathcal{H}^k_s}) \|h\|_{\mathcal{H}^k_s} \|u - v\|_{\mathcal{H}^k_s},$$

for all $u, v, h \in \mathcal{H}^k_s$ *with* $\|u\|_{\mathcal{H}^k_s} \leq \epsilon$, $\|v\|_{\mathcal{H}^k_s} \leq \epsilon$ *and* $\|h\|_{\mathcal{H}^k_s} \leq \epsilon$.

Remark 4.4 (regularity of the initial data for the CH model). As stated above, for the CH model, we require $s \in \mathbb{N}$ with $s \geq 2$. This means that the space of initial data $\mathcal{Q} \simeq l^2_{s+1}$ is at least l^2_3 (Theorem 2.4).

Proof. The proof coincides with the one of Lemma 4.1. □

Finally, we consider the Yang–Mills equation in spherical symmetry.

Lemma 4.5 (Lipschitz bounds for the YM model). *Let $\mathfrak{f}^{(2)}$ and $\mathfrak{f}^{(3)}$ be given by (4-9) and (4-10), respectively. Then, for all integers $k \geq 0$ and $s \geq 3$, there exists a positive constant (depending only on k and s) such that*

$$\begin{aligned}\|\mathfrak{f}^{(2)}(u)-\mathfrak{f}^{(2)}(v)\|_{\mathcal{H}^k_s} &\lesssim (\|u\|_{\mathcal{H}^k_s}+\|v\|_{\mathcal{H}^k_s})\|u-v\|_{\mathcal{H}^k_s},\\ \|d\mathfrak{f}^{(2)}(u)[h]-d\mathfrak{f}^{(2)}(v)[h]\|_{\mathcal{H}^k_s} &\lesssim \|h\|_{\mathcal{H}^k_s}\|u-v\|_{\mathcal{H}^k_s},\\ \|\mathfrak{f}^{(3)}(u)-\mathfrak{f}^{(3)}(v)\|_{\mathcal{H}^k_s} &\lesssim (\|u\|^2_{\mathcal{H}^k_s}+\|v\|^2_{\mathcal{H}^k_s})\|u-v\|_{\mathcal{H}^k_s},\\ \|d\mathfrak{f}^{(2)}(u)[h]-d\mathfrak{f}^{(2)}(v)[h]\|_{\mathcal{H}^k_s} &\lesssim (\|u\|_{\mathcal{H}^k_s}+\|v\|_{\mathcal{H}^k_s})\|h\|_{\mathcal{H}^k_s}\|u-v\|_{\mathcal{H}^k_s},\end{aligned}$$

for all $u, v, h \in \mathcal{H}^k_s$ with $\|u\|_{\mathcal{H}^k_s} \leq \epsilon$, $\|v\|_{\mathcal{H}^k_s} \leq \epsilon$ and $\|h\|_{\mathcal{H}^k_s} \leq \epsilon$.

Remark 4.6 (regularity of the initial data for the YM model). As stated above, for the YM model, we require $s \in \mathbb{N}$ with $s \geq 3$. This means that the space of initial data l^2_{s+3} is at least l^2_6 (Lemma 2.10, Theorem 2.5).

Proof. Let $s \geq 3$ be an integer and pick any $u = \{u^j : j \geq 0\} \in l^2_s$. We also denote by $u(x) = \sum_{j=0}^{\infty} u^j \mathfrak{e}_j(x)$ the corresponding function in the physical space and recall the definition of the linear operator $\mathfrak{L}$ given in Section 3C. In the following, we claim that the operator

$$\Delta_{\mathrm{YM}} u = \frac{1}{\sin^4(x)}\partial_x(\sin^4(x)\partial_x u)$$

coincides with the Laplace–Beltrami operator $\Delta_{\mathbb{S}^5} u$ on the sphere $\mathbb{S}^5 \hookrightarrow \mathbb{R}^6$ restricted to a class of symmetric functions. Indeed, we endow $\mathbb{S}^5$ with the round metric and consider the standard Eulerian coordinates $(x_1 = x, x_2, x_3, x_4, x_5) \in (0,\pi)^4 \times (0, 2\pi)$, so that $y = (y^1, y^2, y^3, y^4, y^5, y^6) \in \mathbb{S}^5$ with $y^1 = \cos x_1$,

$$y^i = \cos x_i \prod_{j=1}^{i-1} \sin x_j$$

for all $i \in \{2, 3, 4, 5\}$ and $y^6 = \sin x_1 \sin x_2 \sin x_3 \sin x_4 \sin x_5$. The metric element in these coordinates is given by the standard round metric on $\mathbb{S}^5$. Then, for a function u defined on $\mathbb{S}^5$ that is invariant under all rotations around the y^6-axis, the operator $\Delta_{\mathrm{YM}} u$ coincides with $\Delta_{\mathbb{S}^5} u$. We call such functions on $\mathbb{S}^5$ "spherically symmetric". Here, $H^s(\mathbb{S}^5)$ is an algebra provided that $s > \frac{5}{2}$. We pick an integer $s \geq 3$, and the rest of the proof coincides with the one of Lemma 4.1. □

5. The Fourier coefficients

Here we study the Fourier coefficients, as defined by (4-1), (4-4) and (4-7). Since the eigenfunctions are given by Jacobi polynomials and since the Fourier coefficients involve products of the eigenfunctions, these are a priori complicated integrals, depending on the indices of the eigenfunctions. Nonetheless, we will derive here explicit closed formulas for the various Fourier coefficients on resonant indices.

5A. ***Conformal cubic wave equation in spherical symmetry.*** In this case, the Fourier coefficients are given by (4-1). By taking the inner product $(\cdot\,|\,\cdot)$, defined in Section 3A, in both sides of (4-1), we deduce that

$$C_{ijkm} = \frac{2}{\pi} \int_{-1}^{1} U_i(y)U_j(y)U_k(y)U_m(y)\sqrt{1-y^2}\,dy,$$

where we also used the definition of the Chebyshev polynomials of the second kind:

$$e_n(x) = U_n(\cos(x)) = \frac{\sin(\omega_n x)}{\sin(x)}$$

for all $n \in \{i, j, k, m\}$. Next, we call a quadruple (i, j, k, m) of indices *resonant* if

$$\omega_i \pm \omega_j \pm \omega_k \pm \omega_m = 0 \tag{5-1}$$

and study the Fourier coefficients on resonant indices.

Vanishing Fourier coefficients. Firstly, we show the Fourier coefficients vanish on some resonant indices.

Lemma 5.1 (vanishing Fourier coefficients on resonant indices). *For any integers $i, j, k, m \geq 0$ such that (5-1) holds with only one minus sign, we have $C_{ijkm} = 0$.*

Proof. Let i, j, k, m be positive integers such that $\omega_i + \omega_j + \omega_k - \omega_m = 0$. Then, $m = 2+i+j+k$ and, according to the computation above, we have

$$C_{ijkm} = \int_{-1}^{1} R_N(y)U_m(y)\sqrt{1-y^2}\,dy, \quad R_N(y) = \frac{2}{\pi}U_i(y)U_j(y)U_k(y),$$

where $R_N(y)$ is a polynomial of degree $N = i+j+k < m$, and hence the Fourier coefficient vanishes since $U_m(y)$ forms an orthonormal and complete basis with respect to the weight $\sqrt{1-y^2}$. The other results now follow immediately using the symmetries of the Fourier coefficients with respect to i, j, k, m. □

Nonvanishing Fourier coefficients. Secondly, we study the nonvanishing Fourier coefficients on resonant indices. In order to deal with these constants, one needs a computationally efficient formula. In the spherically symmetric case, where the basis consists of the Chebyshev polynomials, there exists the addition formula [Szegő 1975]

$$U_p(y)U_q(y) = \sum_{\substack{r=|q-p| \\ \text{step } 2}}^{p+q} U_r(y) = \sum_{s=0}^{\min(p,q)} U_{|q-p|+2s}(y) \tag{5-2}$$

for all $p, q \geq 0$. Consequently, we implement the addition formula (5-2) together with the orthogonality property of the Chebyshev polynomials with respect to the weight $\sqrt{1-y^2}$ to obtain

$$C_{ijkm} = \frac{2}{\pi} \sum_{\substack{r=|j-i| \\ \text{step } 2}}^{i+j} \sum_{\substack{s=|m-k| \\ \text{step } 2}}^{m+k} \int_{-1}^{1} U_r(y)U_s(y)\sqrt{1-y^2}\,dy = \sum_{\substack{r=|j-i| \\ \text{step } 2}}^{j+i} \sum_{\substack{s=|m-k| \\ \text{step } 2}}^{m+k} \mathbb{1}(r=s). \tag{5-3}$$

Next, we use (5-3) to derive closed formulas for the nonvanishing Fourier coefficients.

Lemma 5.2 (nonvanishing Fourier coefficients on resonant indices: closed formulas). *For any integers $i, j, k, m \geq 0$ such that (5-1) holds with only two minus signs, we have $C_{ijkm} = \omega_{\min\{i,j,k,m\}}$.*

Proof. Let $i, j, k, m \geq 0$ be integers such that $\omega_i + \omega_j - \omega_k - \omega_m = 0$. Hence $m = i + j - k$, and assume for simplicity that $i \leq j$, $k \leq m$ and $i \leq k$. Then, setting $r = i + j - p$ and $s = i + j - q$ as well as $t = 2p$ and $\tau = 2q$, equation (5-3) yields

$$C_{ijkm} = \sum_{\substack{p=0 \\ \text{step } 2}}^{2i} \sum_{\substack{q=0 \\ \text{step } 2}}^{2k} \mathbb{1}(p = q) = \sum_{t=0}^{i} \sum_{\tau=0}^{k} \mathbb{1}(t = \tau) = \sum_{t=0}^{i} \sum_{\tau=0}^{i} \mathbb{1}(t = \tau) = i + 1 = \omega_i.$$

All the other results follow immediately by the symmetries of the Fourier coefficients with respect to i, j, k and m, □

5B. *Conformal cubic wave equation out of spherical symmetry.* In this case, the Fourier coefficients are given by (4-4). By taking the inner product $\langle \cdot \mid \cdot \rangle$, defined in Section 3B, in both sides of (4-4), we deduce that

$$\mathsf{C}_{ijkm}^{(\mu_1,\mu_2)} = \frac{1}{2} \prod_{\lambda_1 \in \{i,j,k,m\}} \mathsf{N}_{\lambda_1}^{(\mu_1,\mu_2)} \int_{-1}^{1} (1-x)^{2\mu_1} (1+x)^{2\mu_2} \prod_{\lambda_2 \in \{i,j,k,m\}} P_{\lambda_2}^{(\mu_1,\mu_2)}(x)\, dx,$$

where the normalization constant $\mathsf{N}_{\lambda_1}^{(\mu_1,\mu_2)}$ is given by (3-3). As before, we call a quadruple (i, j, k, m) of indices *resonant* if

$$\omega_i^{(\mu_1,\mu_2)} \pm \omega_j^{(\mu_1,\mu_2)} \pm \omega_k^{(\mu_1,\mu_2)} \pm \omega_m^{(\mu_1,\mu_2)} = 0 \tag{5-4}$$

and study the Fourier coefficients on resonant indices.

Vanishing Fourier coefficients. Firstly, we show the Fourier coefficients vanish on some resonant indices.

Lemma 5.3 (vanishing Fourier coefficients on resonant indices). *For any integers $i, j, k, m \geq 0$ such that (5-4) holds with only one minus sign, we have $\mathsf{C}_{ijkm}^{(\mu_1,\mu_2)} = 0$.*

Proof. The proof is similar to the one of Lemma 5.1. □

Nonvanishing Fourier coefficients. Next, we study the nonvanishing Fourier coefficients. In principle, in order to deal with these constants, one needs a computationally efficient formula as in the spherically symmetric case. However, out of spherical symmetry, where the basis consists of the Jacobi polynomials — although there exists the addition formula

$$P_p^{(\mu_1,\mu_2)}(x) P_q^{(\mu_1,\mu_2)}(x) = \sum_{r=|p-q|}^{p+q} L(p,q,r) P_r^{(\mu_1,\mu_2)}(x)$$

for all $p, q \geq 0$, similar to (5-2) — the linearization coefficients $L(p, q, r)$ remain unknown in closed form for generic values of μ_1 and μ_2, and hence a closed formula similar to (5-3) is not available in general. We note that Rahman[10] [1981, p. 919] was able to prove that the linearization coefficients of Jacobi polynomials can be represented as a well-posed hypergeometric function ${}_9F_8(1)$. On the other hand, in the special case where $\mu_1 = \mu_2$, the Jacobi polynomials reduce to Gegenbauer polynomials for which the

[10] According to [Cohl 2016], there was a minor typo in Rahman's published result; in the linearization coefficient in [Rahman 1981], the term $(-\alpha - \beta - 2m)$ should be replaced by the Pochhammer symbol $(-\alpha - \beta - 2m)_k$. The corrected linearization formula is given in [Cohl 2016].

linearization coefficients are given by well-posed and closed formulas [NIST 2010; Sánchez-Ruiz 2001; Szegő 1975]. Hence, we restrict ourselves to the case $\mu_1=\mu_2=\mu$ and also denote by γ the index referring to the fixed choice of the 1-mode initial data. Then, we derive closed formulas for the nonvanishing Fourier coefficients for a resonant pair of indices (i, j, k, m) by using the formula [Sánchez-Ruiz 2001, (20)] for the Gegenbauer polynomials:

$$(C_m^{(\mu+1/2)}(x))^2=\sum_{\lambda=0}^{m} L_{\mu m}(\lambda)C_{2\lambda}^{(2\mu+1/2)}(x), \tag{5-5}$$

where the coefficients are given by

$$L_{\mu m}(\lambda)=\frac{(2\mu+1)_m}{\Gamma(m+1)}\frac{\left(\frac{1}{2}\right)_\lambda\left(\frac{1}{2}\right)_{m-\lambda}\left(\mu+\frac{1}{2}\right)_\lambda(\lambda+2\mu+1)_m}{\Gamma(m-\lambda+1)(\mu+1)_\lambda\left(2\mu+\frac{1}{2}\right)_{2\lambda}\left(2\lambda+2\mu+\frac{3}{2}\right)_{m-\lambda}}, \tag{5-6}$$

valid for any real $x\in[-1,1]$ and integers $\mu, m, \lambda\geq 0$, and $(a)_n=\Gamma(a+n)/\Gamma(a)$ stands for the Pochhammer's symbol defined for any $a\in\mathbb{R}$ with $a\notin\{0,-1,-2,\dots\}$ and $n\in\mathbb{N}$. Notice that (5-5) is a combination of a linearization and a connection formula for Gegenbauer polynomials. Specifically, we establish the following result.

Lemma 5.4 (nonvanishing Fourier coefficients on resonant indices: closed formulas). *Let $\gamma\geq 0$ and $m\geq\gamma$ be any integers. Then, we have*

$$\mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)}=\frac{1}{2}\sum_{\lambda=0}^{\gamma}\mathsf{M}_\gamma^{(\mu)}(\lambda)\mathsf{M}_m^{(\mu)}(\lambda)\xi_\lambda(\mu),$$

where

$$\xi_\lambda(\mu)=\frac{\pi 2^{1-4\mu}\Gamma(2\lambda+4\mu+1)}{(4\lambda+4\mu+1)\Gamma(2\lambda+1)\left(\Gamma\left(2\mu+\frac{1}{2}\right)\right)^2},$$

$$\mathsf{M}_m^{(\mu)}(\lambda)=\frac{1}{2\pi^{3/2}}\frac{(4\lambda+4\mu+1)\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(2\mu+\frac{1}{2}\right)\Gamma\left(\lambda+\mu+\frac{1}{2}\right)}{\Gamma(\lambda+\mu+1)\Gamma(\lambda+2\mu+1)}\cdot\frac{(2\mu+2m+1)\Gamma\left(m-\lambda+\frac{1}{2}\right)\Gamma(m+\lambda+2\mu+1)}{\Gamma(m-\lambda+1)\Gamma\left(m+\lambda+2\mu+\frac{3}{2}\right)}.$$

Proof. Let $\gamma\geq 1$ be a fixed integer and pick any integer $m\geq\gamma$. Then, by the definition of the Fourier coefficient, the fact that the Jacobi polynomials with equal parameters can be written in terms of the Gegenbauer polynomials [NIST 2010, 18.7.1],

$$P_m^{(\mu,\mu)}(x)=w_m^{(\mu)}C_m^{(\mu+1/2)}(x),\qquad w_m^{(\mu)}=\frac{\Gamma(2\mu+1)\Gamma(m+\mu+1)}{\Gamma(\mu+1)\Gamma(m+2\mu+1)},$$

together with (5-5)–(5-6), we have

$$\begin{aligned}\mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)}&=\frac{1}{2}(w_\gamma^{(\mu)}\mathsf{N}_\gamma^{(\mu,\mu)})^2(w_m^{(\mu)}\mathsf{N}_m^{(\mu,\mu)})^2\int_{-1}^{1}(1-x^2)^{2\mu}(C_\gamma^{(\mu+1/2)}(x))^2(C_m^{(\mu+1/2)}(x))^2\,dx\\&=\frac{1}{2}(w_\gamma^{(\mu)}\mathsf{N}_\gamma^{(\mu,\mu)})^2(w_m^{(\mu)}\mathsf{N}_m^{(\mu,\mu)})^2\sum_{\nu=0}^{\gamma}L_{\mu\gamma}(\nu)\sum_{\lambda=0}^{m}L_{\mu m}(\lambda)\int_{-1}^{1}(1-x^2)^{2\mu}C_{2\nu}^{(2\mu+1/2)}(x)C_{2\lambda}^{(2\mu+1/2)}(x)\,dx.\end{aligned}$$

Now, the orthogonality of the Gegenbauer polynomials,

$$\int_{-1}^{1} C_{2\nu}^{(2\mu+1/2)}(x)C_{2\lambda}^{(2\mu+1/2)}(x)(1-x^2)^{2\mu}\,dx = \xi_\lambda(\mu)\mathbb{1}(\nu=\lambda),$$

where $\xi_\lambda(\mu)$ is defined above, together with the fact that $0 \le \gamma \le m$ yields

$$\begin{aligned}
\mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)} &= \frac{1}{2}(w_\gamma^{(\mu)}\mathsf{N}_\gamma^{(\mu,\mu)})^2(w_m^{(\mu)}\mathsf{N}_m^{(\mu,\mu)})^2\sum_{\nu=0}^{\gamma}L_{\mu\gamma}(\nu)\sum_{\lambda=0}^{\gamma}L_{\mu m}(\lambda)\xi_\lambda(\mu)\mathbb{1}(\nu=\lambda)\\
&= \frac{1}{2}(w_\gamma^{(\mu)}\mathsf{N}_\gamma^{(\mu,\mu)})^2(w_m^{(\mu)}\mathsf{N}_m^{(\mu,\mu)})^2\sum_{\lambda=0}^{\gamma}L_{\mu\gamma}(\lambda)L_{\mu m}(\lambda)\xi_\lambda(\mu)\\
&= \frac{1}{2}\sum_{\lambda=0}^{\gamma}\mathsf{M}_\gamma^{(\mu)}(\lambda)\mathsf{M}_m^{(\mu)}(\lambda)\xi_\lambda(\mu).
\end{aligned}$$

Finally, setting $\mathsf{M}_m^{(\mu)}(\lambda) = (w_m^{(\mu)}\mathsf{N}_m^{(\mu,\mu)})^2 L_{\mu m}(\lambda)$, a direct computation using (3-3) and (5-6) yields the closed formula for $\mathsf{M}_m^{(\mu)}(\lambda)$ stated above and completes the proof. □

Next, we show that the closed formulas we derived above are in fact monotone with respect to m.

Lemma 5.5 (monotonicity of $\mathsf{M}_m^{(\mu)}(\lambda)$). *Let $\gamma \ge 0$, $m \ge \gamma$ and $0 \le \lambda \le \gamma$ be any integers. Then, the function $\mathsf{M}_m^{(\mu)}(\lambda)$ defined in Lemma 5.4 is decreasing with respect to m.*

Proof. Let $\gamma \ge 0$, $m \ge \gamma$ and $0 \le \lambda \le \gamma$ be any integers. The claim follows immediately by computing the difference $\mathsf{M}_{m+1}^{(\mu)}(\lambda) - \mathsf{M}_m^{(\mu)}(\lambda)$. Indeed, the identity for the ratio of two Gamma functions, $\Gamma(x+1) = x\Gamma(x)$, valid for all $x \in \mathbb{R}$, yields that $\mathsf{M}_{m+1}^{(\mu)}(\lambda) - \mathsf{M}_m^{(\mu)}(\lambda)$ equals

$$-\frac{(4\lambda+4\mu+1)\Gamma\big(\lambda+\frac{1}{2}\big)\Gamma\big(2\mu+\frac{1}{2}\big)\Gamma\big(\lambda+\mu+\frac{3}{2}\big)\Gamma\big(m-\lambda+\frac{1}{2}\big)\Gamma(m+\lambda+2\mu+1)}{\pi^{3/2}\Gamma(\lambda+\mu)\Gamma(\lambda+2\mu+1)\Gamma(m-\lambda+2)\Gamma\big(m+\lambda+2\mu+\frac{5}{2}\big)},$$

which is strictly negative for all m, λ and μ, and hence $\mathsf{M}_m^{(\mu)}(\lambda)$ is decreasing with respect to m for all λ and μ, which completes the proof. □

Remark 5.6 (closed formulas for $\mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)}$ for small values of γ). Finally, we note that one can use Lemma 5.4 to find closed formulas for the Fourier coefficients provided that γ is sufficiently small. For example, for $\gamma \in \{0, 1\}$, we find that

$$\begin{aligned}
\mathsf{C}_{00mm}^{(\mu,\mu)} &= \frac{1}{2\pi}(2\mu+1)\left(\frac{\Gamma\big(\mu+\frac{1}{2}\big)}{\Gamma(\mu+1)}\right)^2\frac{(2\mu+2m+1)\Gamma\big(m+\frac{1}{2}\big)\Gamma(m+2\mu+1)}{\Gamma(m+1)\Gamma\big(m+2\mu+\frac{3}{2}\big)},\\
\mathsf{C}_{11mm}^{(\mu,\mu)} &= \frac{1}{8\pi}(\mu+1)(2\mu+1)(2\mu+3)\left(\frac{\Gamma\big(\mu+\frac{1}{2}\big)}{\Gamma(\mu+2)}\right)^2\\
&\qquad\cdot\frac{(2\mu+2m+1)(-\mu+2m(2\mu+m+1)-1)\Gamma\big(m-\frac{1}{2}\big)\Gamma(m+2\mu+1)}{\Gamma(m+1)\Gamma\big(m+2\mu+\frac{5}{2}\big)}.
\end{aligned}$$

Figure 1 illustrates the Fourier coefficients $\mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)}$ for $\mu = 30$ and $\gamma \in \{0, 1, 2\}$, respectively, as m varies within $\{1, 2, \dots, 50\}$.

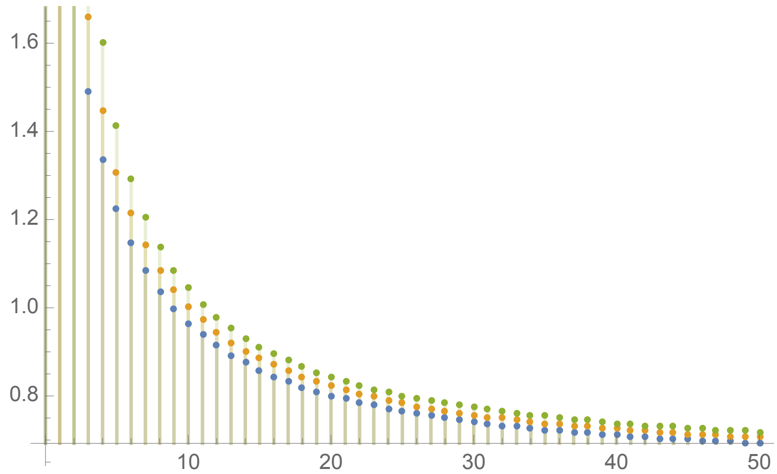

Figure 1. The Fourier coefficients $C^{(30,30)}_{\gamma\gamma mm}$ for $\gamma = 0$ (blue/bottom), $\gamma = 1$ (orange/middle) and $\gamma = 2$ (green/top) as m varies within the interval $[1, 50]$. They are all decreasing for $m \geq 2\gamma + 1$.

5C. ***Yang–Mills equation in spherical symmetry.*** In this case, the Fourier coefficients are given by (4-7). By taking the inner product $[\cdot \mid \cdot]$, defined in Section 3C, in both sides of (4-7), we deduce that

$$\overline{\mathfrak{C}}_{ijm} = \prod_{\lambda_1 \in \{i,j,m\}} \mathfrak{N}_{\lambda_1} \int_{-1}^{1} (1-y^2)^{3/2} \prod_{\lambda_2 \in \{i,j,m\}} P^{(3/2,3/2)}_{\lambda_2}(y)\, dy,$$

$$\mathfrak{C}_{ijkm} = \prod_{\lambda_1 \in \{i,j,k,m\}} \mathfrak{N}_{\lambda_1} \int_{-1}^{1} (1-y^2)^{5/2} \prod_{\lambda_2 \in \{i,j,k,m\}} P^{(3/2,3/2)}_{\lambda_2}(y)\, dy,$$

where the normalization constant $\mathfrak{N}_{\lambda_1}$ is given by (3-5). As before, we call a triple (i, j, m) or a quadruple (i, j, k, m) of indices *resonant* if

$$\varpi_i \pm \varpi_j \pm \varpi_m = 0, \tag{5-7}$$

$$\varpi_i \pm \varpi_j \pm \varpi_k \pm \varpi_m = 0, \tag{5-8}$$

respectively, and study the Fourier coefficients on resonant indices.

Vanishing Fourier coefficients. Firstly, we show the Fourier coefficients vanish on some resonant indices.

Lemma 5.7 (vanishing Fourier coefficients on resonant indices). *For any integers $i, j, m \geq 0$ such that (5-7) holds with only one minus sign and for any integers $i, j, k, m \geq 0$ such that (5-8) holds with only one minus sign, we have $\overline{\mathfrak{C}}_{ijm} = 0$ and $\mathfrak{C}_{ijkm} = 0$, respectively.*

Proof. The proof is similar to the one of Lemma 5.1. □

Nonvanishing Fourier coefficients. Next, we study the nonvanishing Fourier coefficients. In order to deal with these constants, one needs a computationally efficient formula as in the two previous cases. In the spherically symmetric case we consider here, the basis consists of the Jacobi polynomials with equal parameters, and these are weighted Gegenbauer polynomials [NIST 2010, 18.7.1]

$$C_n^{(2)}(x) = \frac{(4)_n}{\left(\frac{5}{2}\right)_n} P_n^{(3/2,3/2)}(x) = \frac{\sqrt{\pi}}{8} \frac{\Gamma(n+4)}{\Gamma\left(n+\frac{5}{2}\right)} P_n^{(3/2,3/2)}(x)$$

for $n \in \{\gamma, m\}$ and $x \in [-1, 1]$. The latter, together with the definition of the normalization constant $\mathfrak{N}_n$ from (3-5), expresses the normalized Jacobi polynomials in terms of the normalized Gegenbauer polynomials as

$$\mathfrak{N}_n P_n^{(3/2,3/2)}(x) = \mathfrak{N}_n \frac{8}{\sqrt{\pi}} \frac{\Gamma\left(n+\frac{5}{2}\right)}{\Gamma(n+4)} C_n^{(2)}(x) = \mathfrak{w}_n C_n^{(2)}(x), \quad \mathfrak{w}_n = \sqrt{\frac{8}{\pi}} \frac{1}{\sqrt{(n+1)(n+3)}},$$

for all $n \in \{\gamma, m\}$. Consequently, the Fourier coefficients can be written in terms of the Gegenbauer polynomials as follows:

$$\overline{\mathfrak{C}}_{ijm} = \mathfrak{w}_i \mathfrak{w}_j \mathfrak{w}_m \int_{-1}^{1} C_i^{(2)}(y) C_j^{(2)}(y) C_m^{(2)}(y) (1-y^2)^{3/2}\, dy,$$

$$\mathfrak{C}_{ijkm} = \mathfrak{w}_i \mathfrak{w}_j \mathfrak{w}_k \mathfrak{w}_m \int_{-1}^{1} C_i^{(2)}(y) C_j^{(2)}(y) C_k^{(2)}(y) C_m^{(2)}(y) (1-y^2)^{5/2}\, dy.$$

Then, we derive closed formulas for the nonvanishing Fourier coefficients for a resonant quadruple (i, j, k, m) as follows:

• Whenever a Gegenbauer polynomial has an order μ such that the weight $(1-y^2)^{\mu-1/2}$ (with respect to which it forms an orthonormal basis) does not coincide with the weights $(1-y^2)^p$ with $p \in \left\{\frac{3}{2}, \frac{5}{2}\right\}$ (that define the integrals above), we use the connection formula [NIST 2010, 18.18.16]

$$C_n^{(\mu)}(y) = \sum_{\ell=0}^{\lfloor n/2 \rfloor} \alpha_{n\mu\lambda}(\ell) C_{n-2\ell}^{(\lambda)}(y), \quad \alpha_{n\mu\lambda}(\ell) = \frac{\lambda+n-2\ell}{\lambda} \frac{(\mu)_{n-\ell}}{(\lambda+1)_{n-\ell}} \frac{(\mu-\lambda)_\ell}{\ell!}. \tag{5-9}$$

In particular, we are going to use this only for $\lambda = \mu + 1$. In this case, the latter is equivalent to the recurrence relation [NIST 2010, 18.9.7]

$$C_n^{(\mu)}(x) = \frac{\mu}{n+\mu} \left(C_n^{(\mu+1)}(x) - C_{n-2}^{(\mu+1)}(x)\right),$$

valid for all integers $\mu \geq 0$ and $n \geq 2$.

• Whenever a Gegenbauer polynomial is multiplied by itself, we use the addition formula [NIST 2010, 18.18.22; Sánchez-Ruiz 2001, (19)]

$$(C_n^{(\lambda)}(y))^2 = \sum_{\ell=0}^{n} \beta_{n\lambda}(\ell) C_{2\ell}^{(\lambda)}(y), \quad \beta_{n\lambda}(\ell) = \frac{1}{n!} \binom{n}{\ell} \frac{(2\ell)!\, (\lambda)_\ell (\lambda)_{n-\ell} (2\ell+2\lambda)_{n-\ell}}{\ell!\, (\ell+\lambda)_\ell (2\ell+\lambda+1)_{n-\ell}}. \tag{5-10}$$

• Whenever two Gegenbauer polynomials of different degrees but of the same order are multiplied, we use the addition formula [NIST 2010, 18.18.22]

$$C_m^{(\lambda)}(y) C_n^{(\lambda)}(y) = \sum_{\ell=0}^{\min(m,n)} \zeta_{mn\lambda}(\ell) C_{m+n-2\ell}^{(\lambda)}(y), \tag{5-11}$$

where the coefficients are given by

$$\zeta_{mn\lambda}(\ell)=\frac{(m+n+\lambda-2\ell)(m+n-2\ell)!}{(m+n+\lambda-\ell)\ell!\,(m-\ell)!\,(n-\ell)!}\frac{(\lambda)_\ell(\lambda)_{m-\ell}(\lambda)_{n-\ell}(2\lambda)_{m+n-\ell}}{(\lambda)_{m+n-\ell}(2\lambda)_{m+n-2\ell}}.$$

• Whenever a product of a monomial with a Gegenbauer polynomial is integrated, we use the formula [NIST 2010, 18.17.37]

$$\int_0^1 x^{z-1}C_n^{(\lambda)}(x)(1-x^2)^{\lambda-1/2}\,dx=\frac{\pi 2^{1-2\lambda-z}\Gamma(n+2\lambda)\Gamma(z)}{n!\,\Gamma(\lambda)\Gamma\left(\frac{1}{2}+\frac{1}{2}n+\lambda+\frac{1}{2}z\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}z-\frac{1}{2}n\right)},\tag{5-12}$$

valid for all integers $\lambda\geq 0$ and real numbers $z>0$. Notice that this is the Mellin transform of the function $C_n^{(\lambda)}(x)(1-x^2)^{\lambda-1/2}$ restricted to [0, 1].

In particular, we will only need to study $\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau}$, $\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m}$, $\overline{\mathfrak{C}}_{m,2\tau,m}$ and $\mathfrak{C}_{\gamma\gamma mm}$ for all integers $\tau\in\{0,1,\dots,\gamma\}$ and $m\geq 2\gamma+1$. To begin with, we focus on $\overline{\mathfrak{C}}_{ijm}$ for any integers $i,j,m\geq 0$ and establish the follow result.

Lemma 5.8 (nonvanishing Fourier coefficients $\overline{\mathfrak{C}}_{ijm}$ on resonant indices: closed formula). *For any integers* $i,j,m\geq 0$, *we have*

$$\begin{aligned}\overline{\mathfrak{C}}_{ijm}=&\frac{(i+j-m+2)(i-j+m+2)(-i+j+m+2)(i+j+m+6)}{4\sqrt{2\pi(i+1)(i+3)(j+1)(j+3)(m+1)(m+3)}}\\&\cdot\mathbb{1}(|j-m|\leq i\leq j+m)\mathbb{1}(|i-m|\leq j\leq i+m)\mathbb{1}(|i-j|\leq m\leq i+j)\\&\cdot\mathbb{1}(j+m-i\in 2\mathbb{N}\cup\{0\})\mathbb{1}(i+m-j\in 2\mathbb{N}\cup\{0\})\mathbb{1}(i+j-m\in 2\mathbb{N}\cup\{0\}).\end{aligned}$$

Proof. The result follows immediately from (5-11) together with the orthogonality of the Gegenbauer polynomials,

$$\int_{-1}^1 C_n^{(2)}(y)C_m^{(2)}(y)(1-y^2)^{3/2}\,dy=\frac{\pi}{8}(m+1)(m+3)\mathbb{1}(m=n)$$

for all integers $m,n\geq 0$. Indeed, for any integers $i,j,m\geq 0$, we have

$$\begin{aligned}\overline{\mathfrak{C}}_{ijm}&=\mathfrak{w}_i\mathfrak{w}_j\mathfrak{w}_m\sum_{\ell=0}^{\min(i,j)}\zeta_{ij2}(\ell)\int_{-1}^1 C_{i+j-2\ell}^{(2)}(y)C_m^{(2)}(y)(1-y^2)^{3/2}\,dy\\&=\frac{\pi}{8}(m+1)(m+3)\mathfrak{w}_i\mathfrak{w}_j\mathfrak{w}_m\sum_{\ell=0}^{\min(i,j)}\zeta_{ij2}(\ell)\mathbb{1}(2\ell=i+j-m).\end{aligned}$$

On the one hand, for all integers i, j and m such that $i+j-m\notin 2\mathbb{N}\cup\{0\}$, the Fourier coefficient vanishes. Furthermore, we have

$$0\leq i+j-m\leq 2\min(i,j)\quad\Longleftrightarrow\quad |i-j|\leq m\leq i+j.$$

Consequently, for all integers i, j and m such that the condition $|i-j|\leq m\leq i+j$ is not fulfilled, the Fourier coefficient vanishes. On the other hand, for all i, j and m such that both $i+j-m\in 2\mathbb{N}\cup\{0\}$

and $|i-j| \le m \le i+j$ hold true, we compute

$$\begin{aligned}\overline{\mathfrak{C}}_{ijm} &= \frac{\pi}{8}(m+1)(m+3)\mathfrak{w}_i\mathfrak{w}_j\mathfrak{w}_m\zeta_{ij2}\left(\frac{i+j-m}{2}\right)\sum_{\ell=0}^{\min(i,j)}\mathbb{1}(2\ell=i+j-m)\\ &= \frac{(i+j-m+2)(i-j+m+2)(-i+j+m+2)(i+j+m+6)}{4\sqrt{2\pi(i+1)(i+3)(j+1)(j+3)(m+1)(m+3)}},\end{aligned}$$

where we used the fact that $\sum_{l=0}^{\min(i,j)}\mathbb{1}(2\ell=i+j-m)=1$. Finally, using the symmetries of the Fourier coefficient with respect to i, j and m completes the proof. ☐

Next, we apply the previous result to obtain closed formulas for the Fourier coefficient $\overline{\mathfrak{C}}_{ijm}$ on the particular resonant indices we are interested in. Specifically, we establish the following result.

Lemma 5.9 (nonvanishing Fourier coefficients $\overline{\mathfrak{C}}_{ijm}$ on particular resonant indices: closed formulas). *Let γ, τ, m be integers such that $\gamma \ge 0$, $\tau \in \{0, 1, \dots, \gamma\}$ and $m \ge 2\gamma+1$. Then, we have*

$$\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau} = 2\sqrt{\frac{2}{\pi}}\frac{(\tau+1)^2(\gamma-\tau+1)(\gamma+\tau+3)}{(\gamma+1)(\gamma+3)\sqrt{4\tau(\tau+2)+3}}, \quad \overline{\mathfrak{C}}_{m,2\tau,m} = 2\sqrt{\frac{2}{\pi}}\frac{(\tau+1)^2(m-\tau+1)(m+\tau+3)}{(m+1)(m+3)\sqrt{4\tau(\tau+2)+3}},$$

$$\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m} = 2\sqrt{\frac{2}{\pi}}\frac{(\tau+1)(\gamma-\tau+1)(m+\tau+3)(-\gamma+m+\tau+1)}{\sqrt{(\gamma+1)(\gamma+3)(m+1)(m+3)(-\gamma+m+2\tau+1)(-\gamma+m+2\tau+3)}}.$$

Proof. Let γ, τ, m be integers such that $\gamma \ge 0$, $\tau \in \{0, 1, \dots, \gamma\}$ and $m \ge 2\gamma+1$. Firstly, notice that all the indices of the Fourier coefficients above satisfy all the conditions in the Booleans in Lemma 5.8. Then, the result follows immediately from Lemma 5.8 by direct substitution. ☐

Now, we focus on $\mathfrak{C}_{\gamma\gamma mm}$ and derive the following result.

Lemma 5.10 (nonvanishing Fourier coefficients $\mathfrak{C}_{\gamma\gamma mm}$ on resonant indices: closed formulas). *Let $\gamma \ge 0$ be a fixed integer. Then, for all $m \ge 2\gamma+1$, we have*

$$\mathfrak{C}_{\gamma\gamma mm} = \mathfrak{w}_\gamma^2\mathfrak{w}_m^2\sum_{\ell_2=0}^{\gamma}\sum_{\nu_2=0}^{\ell_2}\delta_\gamma(\ell_2,\nu_2)J_m(\ell_2,\nu_2),$$

where

$$\delta_\gamma(\ell_2,\nu_2) = \frac{(\ell_2+1)^2(-1)^{\nu_2}(\gamma-\ell_2+1)(\gamma+\ell_2+3)2^{2(\ell_2-\nu_2)}\Gamma(2\ell_2-\nu_2+2)}{(4\ell_2(\ell_2+2)+3)\Gamma(\nu_2+1)\Gamma(2\ell_2-2\nu_2+1)},$$

$$\begin{aligned}J_m(\ell_2,\nu_2) = {}& \frac{3\sqrt{\pi}(5\ell_2(3m(m+4)+1)+m(m+4)(4-15\nu_2)-5(\nu_2-4))\Gamma\left(\ell_2-\nu_2+\frac{1}{2}\right)}{8\Gamma(\ell_2-\nu_2+5)}\\ &+\sum_{\ell_1=2}^{\ell_2-\nu_2+1}\frac{\pi\ell_1(\ell_1+1)^2(2\ell_1-1)4^{-\ell_2+\nu_2-2}(\ell_1-m-1)(\ell_1+m+3)\Gamma(2\ell_2-2\nu_2+1)}{\Gamma(-\ell_1+\ell_2-\nu_2+2)\Gamma(\ell_1+\ell_2-\nu_2+3)}\\ &-\sum_{\ell_1=2}^{\ell_2-\nu_2}\frac{\pi(\ell_1+1)^2(\ell_1+2)(2\ell_1+5)4^{-\ell_2+\nu_2-2}(\ell_1-m-1)(\ell_1+m+3)\Gamma(2\ell_2-2\nu_2+1)}{\Gamma(-\ell_1+\ell_2-\nu_2+1)\Gamma(\ell_1+\ell_2-\nu_2+4)}.\end{aligned}$$

Proof. Let $\gamma \geq 0$ be a fixed integer and pick any integer $m \geq 2\gamma+1$. Then, by the definition of the Fourier coefficients, we have

$$\mathfrak{C}_{\gamma\gamma mm} = \mathfrak{w}_\gamma^2 \mathfrak{w}_m^2 \int_{-1}^{1} (C_\gamma^{(2)}(y))^2 (C_m^{(2)}(y))^2 (1-y^2)^{5/2}\, dy.$$

On the one hand, we use the linearization formula (5-10) together with the special cases $C_0^{(2)}(y)=1$ and $C_2^{(2)}(y) = 12y^2-2$ to obtain

$$\begin{aligned}(C_m^{(2)}(y))^2 &= \sum_{\ell_1=0}^{m} \beta_{m2}(\ell_1) C_{2\ell_1}^{(2)}(y) = \sum_{\ell_1=0}^{m} \frac{(\ell_1+1)^2(-\ell_1+m+1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} C_{2\ell_1}^{(2)}(y) \\ &= \frac{1}{3}(m+1)(m+3) + \frac{4}{15} m(m+4)(12y^2-2) + \sum_{\ell_1=2}^{m} \frac{(\ell_1+1)^2(-\ell_1+m+1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} C_{2\ell_1}^{(2)}(y).\end{aligned}$$

Furthermore, for all $\ell_1 \geq 2$, the connection formula (5-9) yields

$$\begin{aligned}C_{2\ell_1}^{(2)}(y) &= \sum_{\nu_1=0}^{\ell_1} \alpha_{2\ell_1,2,3}(\nu_1) C_{2(\ell_1-\nu_1)}^{(3)}(y) = \sum_{\nu_1=0}^{\ell_1} \frac{(2\ell_1-2\nu_1+3)(-1)_{\nu_1}\Gamma(2\ell_1-\nu_1+2)}{3\nu_1!\,(4)_{2\ell_1-\nu_1}} C_{2(\ell_1-\nu_1)}^{(3)}(y) \\ &= \frac{(2\ell_1+3)(2)_{2\ell_1}}{3(4)_{2\ell_1}} C_{2\ell_1}^{(3)}(y) - \frac{(2\ell_1+1)(2)_{2\ell_1-1}}{3(4)_{2\ell_1-1}} C_{2(\ell_1-1)}^{(3)}(y) = \frac{1}{\ell_1+1}\big(C_{2\ell_1}^{(3)}(y) - C_{2(\ell_1-1)}^{(3)}(y)\big)\end{aligned}$$

since $(-1)_{\nu_1} = 0$ for all $\nu_1 \geq 2$. Consequently, we have

$$\begin{aligned}(C_m^{(2)}(y))^2 &= \tfrac{1}{3}(m+1)(m+3) + \tfrac{4}{15} m(m+4)(12y^2-2) \\ &\quad + \sum_{\ell_1=2}^{m} \frac{(\ell_1+1)(\ell_1-m-1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3}\big(C_{2(\ell_1-1)}^{(3)}(y) - C_{2\ell_1}^{(3)}(y)\big).\end{aligned}$$

On the other hand, the linearization formula (5-10) together with the definition of the Gegenbauer polynomials

$$C_{2\ell_2}^{(2)}(y) = \sum_{\nu_2=0}^{\ell_2} d_{\ell_2}(\nu_2) y^{2(\ell_2-\nu_2)}, \quad d_{\ell_2}(\nu_2) = \frac{(-1)^{\nu_2}(4^{\ell_2-\nu_2}\Gamma(2\ell_2-\nu_2+2))}{\Gamma(\nu_2+1)\Gamma(2\ell_2-2\nu_2+1)},$$

yield

$$\begin{aligned}(C_\gamma^{(2)}(y))^2 = \sum_{\ell_2=0}^{\gamma} \beta_{\gamma 2}(\ell_2) C_{2\ell_2}^{(2)}(y) &= \sum_{\ell_2=0}^{\gamma} \beta_{\gamma 2}(\ell_2) \sum_{\nu_2=0}^{\ell_2} d_{\ell_2}(\nu_2) y^{2(\ell_2-\nu_2)} \\ &= \sum_{\ell_2=0}^{\gamma}\sum_{\nu_2=0}^{\ell_2} \delta_\gamma(\ell_2,\nu_2) y^{2(\ell_2-\nu_2)},\end{aligned}$$

where we set

$$\delta_\gamma(\ell_2,\nu_2) = \beta_{\gamma 2}(\ell_2) d_{\ell_2}(\nu_2) = \frac{(\ell_2+1)^2(-1)^{\nu_2}(\gamma-\ell_2+1)(\gamma+\ell_2+3)2^{2(\ell_2-\nu_2)}\Gamma(2\ell_2-\nu_2+2)}{(4\ell_2(\ell_2+2)+3)\Gamma(\nu_2+1)\Gamma(2\ell_2-2\nu_2+1)}.$$

Now, putting this all together, we infer

$$
\begin{aligned}
\mathfrak{C}_{\gamma\gamma mm} &= \mathfrak{w}_\gamma^2 \mathfrak{w}_m^2 \sum_{\ell_2=0}^{\gamma} \sum_{\nu_2=0}^{\ell_2} \delta_\gamma(\ell_2, \nu_2) \\
&\quad \times \int_{-1}^{1} y^{2(\ell_2-\nu_2)} \Bigg[\tfrac{1}{3}(m+1)(m+3) + \tfrac{4}{15} m(m+4)(12y^2-2) \\
&\qquad + \sum_{\ell_1=2}^{m} \frac{(\ell_1+1)(\ell_1-m-1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} (C^{(3)}_{2(\ell_1-1)}(y) - C^{(3)}_{2\ell_1}(y)) \Bigg] (1-y^2)^{5/2}\, dy \\
&= \mathfrak{w}_\gamma^2 \mathfrak{w}_m^2 \sum_{\ell_2=0}^{\gamma} \sum_{\nu_2=0}^{\ell_2} \delta_\gamma(\ell_2, \nu_2) J_m(\ell_2, \nu_2),
\end{aligned}
$$

where

$$
\begin{aligned}
J_m(\ell_2, \nu_2) &= \frac{(m+1)(m+3)}{3} \int_{-1}^{1} y^{2(\ell_2-\nu_2)} (1-y^2)^{5/2}\, dy \\
&\quad + \frac{4m(m+4)}{15} \int_{-1}^{1} y^{2(\ell_2-\nu_2)} (12y^2-2)(1-y^2)^{5/2}\, dy \\
&\quad + \sum_{\ell_1=2}^{m} \frac{(\ell_1+1)(\ell_1-m-1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} \int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2(\ell_1-1)}(y)(1-y^2)^{5/2}\, dy \\
&\quad - \sum_{\ell_1=2}^{m} \frac{(\ell_1+1)(\ell_1-m-1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} \int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2\ell_1}(y)(1-y^2)^{5/2}\, dy.
\end{aligned}
$$

We compute

$$
\begin{aligned}
\int_{-1}^{1} y^{2(\ell_2-\nu_2)} (1-y^2)^{5/2}\, dy &= \frac{15\sqrt{\pi}\,\Gamma\big(\ell_2-\nu_2+\frac{1}{2}\big)}{8\Gamma(\ell_2-\nu_2+4)}, \\
\int_{-1}^{1} y^{2(\ell_2-\nu_2)} (12y^2-2)(1-y^2)^{5/2}\, dy &= \frac{15\sqrt{\pi}(5\ell_2-5\nu_2-1)\Gamma\big(\ell_2-\nu_2+\frac{1}{2}\big)}{4\Gamma(\ell_2-\nu_2+5)}.
\end{aligned}
$$

On the one hand, for all $\ell_1 \geq 2$ with $\ell_1 > \ell_2-\nu_2+1$, we have $2(\ell_2-\nu_2) < 2(\ell_1-1)$, and hence

$$
\int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2(\ell_1-1)}(y)(1-y^2)^{5/2}\, dy = 0
$$

since the Gegenbauer polynomial in the integrand forms an orthonormal and complete basis with respect to the weight $(1-y^2)^{5/2}$. On the other hand, for all $2 \leq \ell_1 \leq \ell_2-\nu_2+1$, the identity $C^{(3)}_{2\lambda}(-y) = C^{(3)}_{2\lambda}(y)$, valid for all real $y \in [-1, 1]$ and integers $\lambda \geq 0$, yields

$$
\begin{aligned}
\int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2(\ell_1-1)}(y)(1-y^2)^{5/2}\, dy &= 2\int_{0}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2(\ell_1-1)}(y)(1-y^2)^{5/2}\, dy \\
&= \frac{\pi 4^{-\ell_2+\nu_2-3}\Gamma(2\ell_1+4)\Gamma(2\ell_2-2\nu_2+1)}{\Gamma(2\ell_1-1)\Gamma(-\ell_1+\ell_2-\nu_2+2)\Gamma(\ell_1+\ell_2-\nu_2+3)},
\end{aligned}
$$

where we used (5-12) to compute the last integral. In other words, we have

$$\sum_{\ell_1=2}^{m} \frac{(\ell_1+1)(\ell_1-m-1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} \int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2(\ell_1-1)}(y)(1-y^2)^{5/2}\,dy$$
$$= \sum_{\ell_1=2}^{\ell_2-\nu_2+1} \frac{(\ell_1+1)(\ell_1-m-1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} \int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2(\ell_1-1)}(y)(1-y^2)^{5/2}\,dy$$
$$= \sum_{\ell_1=2}^{\ell_2-\nu_2+1} \frac{\pi\ell_1(\ell_1+1)^2(2\ell_1-1)4^{-\ell_2+\nu_2-2}(\ell_1-m-1)(\ell_1+m+3)\Gamma(2\ell_2-2\nu_2+1)}{\Gamma(-\ell_1+\ell_2-\nu_2+2)\Gamma(\ell_1+\ell_2-\nu_2+3)}.$$

Similarly, on the one hand, for all $\ell_1 \geq 2$ with $\ell_1 > \ell_2-\nu_2$, we have $2(\ell_2-\nu_2) < 2\ell_1$, and hence

$$\int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2\ell_1}(y)(1-y^2)^{5/2}\,dy = 0$$

since the Gegenbauer polynomial in the integrand forms an orthonormal and complete basis with respect to the weight $(1-y^2)^{5/2}$. On the other hand, for all $2 \leq \ell_1 \leq \ell_2-\nu_2$, the identity $C^{(3)}_{2\lambda}(-y) = C^{(3)}_{2\lambda}(y)$, valid for all real $y \in [-1, 1]$ and integers $\lambda \geq 0$, yields

$$\int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2\ell_1}(y)(1-y^2)^{5/2}\,dy = 2\int_{0}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2\ell_1}(y)(1-y^2)^{5/2}\,dy$$
$$= \frac{\pi 4^{-\ell_2+\nu_2-3}\Gamma(2\ell_1+6)\Gamma(2\ell_2-2\nu_2+1)}{\Gamma(2\ell_1+1)\Gamma(-\ell_1+\ell_2-\nu_2+1)\Gamma(\ell_1+\ell_2-\nu_2+4)},$$

where we used once again (5-12) to compute the last integral. In other words, we have

$$\sum_{\ell_1=2}^{m} \frac{(\ell_1+1)(\ell_1-m-1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} \int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2\ell_1}(y)(1-y^2)^{5/2}\,dy$$
$$= \sum_{\ell_1=2}^{\ell_2-\nu_2} \frac{(\ell_1+1)(\ell_1-m-1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} \int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2\ell_1}(y)(1-y^2)^{5/2}\,dy$$
$$= \sum_{\ell_1=2}^{\ell_2-\nu_2} \frac{\pi(\ell_1+1)^2(\ell_1+2)(2\ell_1+5)4^{-\ell_2+\nu_2-2}(\ell_1-m-1)(\ell_1+m+3)\Gamma(2\ell_2-2\nu_2+1)}{\Gamma(-\ell_1+\ell_2-\nu_2+1)\Gamma(\ell_1+\ell_2-\nu_2+4)}.$$

Putting this all together yields $J_m(\ell_2, \nu_2)$ as stated above and completes the proof. □

Remark 5.11 (closed formulas for $\mathfrak{C}_{\gamma\gamma mm}$ for small values of γ). Finally, we note that one can use Lemma 5.10 to find closed formulas for the Fourier coefficients provided that γ is sufficiently small. For example, for $\gamma \in \{0, 1, 2\}$, we find

$$\mathfrak{C}_{00mm} = \frac{4(m(m+4)+5)}{3\pi(m+1)(m+3)}, \quad \mathfrak{C}_{11mm} = \frac{2(m(m+4)+7)}{\pi(m+1)(m+3)}, \quad \mathfrak{C}_{22mm} = \frac{8(5m(m+4)+49)}{15\pi(m+1)(m+3)}$$

for all $m \geq 2\gamma+1$. Figure 2 illustrates the Fourier coefficients $\mathfrak{C}_{\gamma\gamma mm}$ for $\gamma \in \{0, 1, 2\}$, respectively, as m varies within $\{1, 2, \ldots, 50\}$.

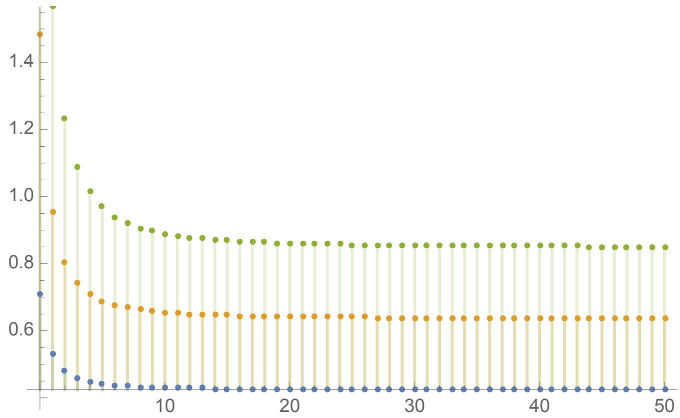

Figure 2. The Fourier coefficients $\mathfrak{C}_{\gamma\gamma mm}$ for $\gamma = 0$ (blue/bottom), $\gamma = 1$ (orange/middle) and $\gamma = 2$ (green/top) as m varies within $\{1, 2, \dots, 50\}$. They are all decreasing for $m \geq 2\gamma + 1$.

6. 1-mode initial data

In this section, we study the operators $\mathcal{M}$ and $\mathfrak{M}_{\pm}$ (Section 2) for 1-mode initial data. Specifically, we verify that all the 1-modes are zeros of the operators $\mathcal{M}$ (for CW and CH) and $\mathfrak{M}_{-}$ (for YM) and compute the differentials $d\mathcal{M}$ and $d\mathfrak{M}_{-}$ at the 1-mode initial data.

6A. ***Conformal cubic wave equation in spherical symmetry.*** Recall that the eigenfunctions $\{e_n : n \geq 0\}$ are given by (3-1) and the PDE in the Fourier space from (4-2) reads

$$\ddot{u}^m(t) + (Au(t))^m = (f(\{u^j(t) : j0\}))^m$$

for all integers $m \geq 0$, where the dots denote derivatives with respect to time and

$$(Au(t))^m = \omega_m^2 u^m(t), \quad (f(\{u^j(t) : j \geq 0\}))^m = -\sum_{i,j,k=0}^{\infty} C_{ijkm} u^i(t) u^j(t) u^k(t).$$

For any initial data $u(0, \cdot)$, we denote by

$$\Phi^t(\xi) = \{\xi^n \cos(\omega_n t) : n \geq 0\}, \quad u(0, \cdot) = \sum_{n=0}^{\infty} \xi^n e_n, \quad \xi = \{\xi^n : n \geq 0\},$$

the linear flow, that is the solution to the linear problem

$$\ddot{u}^n(t) + \omega_n^2 u^n(t) = 0, \quad (u^n(0), \dot{u}^n(0)) = (\xi^n, 0),$$

for all times $t \in \mathbb{R}$. For this model, we aim towards implementing the original version of Bambusi–Paleari's theorem (Theorem 2.4) and define

$$\mathcal{M}(\xi) := A\xi + \langle f \rangle(\xi), \quad \langle f \rangle(\xi) = \frac{1}{2\pi} \int_0^{2\pi} \Phi^t[f(\Phi^t(\xi))]\, dt.$$

To begin with, we show that the 1-modes are zeros of the operator $\mathcal{M}$.

Lemma 6.1 (zeros of the operator $\mathcal{M}$). *Let $\xi = \{\xi^m : m \geq 0\}$ be the rescaled* 1*-mode initial data,*

$$\xi^m = K_\gamma \mathbb{1}(m=\gamma), \qquad K_\gamma = \pm 2\omega_\gamma \sqrt{\frac{2}{3C_{\gamma\gamma\gamma\gamma}}}, \tag{6-1}$$

for all integers $m \geq 0$. Then, we have $\mathcal{M}(\xi) = 0$.

Proof. Let $\xi = \{\xi^m : m \geq 0\}$ be given by (6-1), and pick any integer $m \geq 0$. Then, we compute

$$\begin{aligned}
(A\xi)^m &= \omega_m^2 \xi^m = K_\gamma \omega_\gamma^2 \mathbb{1}(m=\gamma),\\
(\Phi^t(\xi))^m &= \xi^m \cos(\omega_m t) = K_\gamma \cos(\omega_\gamma t)\mathbb{1}(m=\gamma),\\
(f(\Phi^t(\xi)))^m &= -\sum_{i,j,k} C_{ijkm}(\Phi^t(\xi))^i(\Phi^t(\xi))^j(\Phi^t(\xi))^k = -K_\gamma^3 C_{\gamma\gamma\gamma m}\cos^3(\omega_\gamma t),\\
(\Phi^t[f(\Phi^t(\xi))])^m &= (f(\Phi^t(\xi)))^m \cos(\omega_m t) = -K_\gamma^3 C_{\gamma\gamma\gamma m}\cos^3(\omega_\gamma t)\cos(\omega_m t),\\
(\langle f\rangle(\xi))^m &= -\frac{C_{\gamma\gamma\gamma m}}{2\pi}K_\gamma^3 \int_0^{2\pi} \cos^3(\omega_\gamma t)\cos(\omega_m t)\,dt\\
&= -C_{\gamma\gamma\gamma m}K_\gamma^3\big(\tfrac{3}{8}\mathbb{1}(m=\gamma)+\tfrac{1}{8}\mathbb{1}(m=3\gamma+2)\big) = -\tfrac{3}{8}C_{\gamma\gamma\gamma\gamma}K_\gamma^3\mathbb{1}(m=\gamma),\\
(\mathcal{M}(\xi))^m &= (A\xi)^m + (\langle f\rangle(\xi))^m = K_\gamma\big(\omega_\gamma^2 - \tfrac{3}{8}C_{\gamma\gamma\gamma\gamma}K_\gamma^2\big)\mathbb{1}(m=\gamma) = 0,
\end{aligned}$$

where we used the facts that $\omega_m + \omega_\gamma \neq 0,\ \omega_m + 3\omega_\gamma \neq 0$,

$$\omega_m - \omega_\gamma = 0 \iff m=\gamma \quad \text{and} \quad \omega_m - 3\omega_\gamma = 0 \iff m = 3\gamma+2,$$

as well as $C_{\gamma\gamma\gamma m} = 0$ for $m = 3\gamma+2$ according to Lemma 5.1. □

Next, we derive the differential of $\mathcal{M}$ at the rescaled 1-modes.

Lemma 6.2 (differential of $\mathcal{M}$ at the 1-modes). *Let $\xi = \{\xi^m : m \geq 0\}$ be given by* (6-1). *Then, for all $h = \{h^j : j \geq 0\} \in l^2_{s+3}$, we have that*

$$\begin{aligned}
C_{\gamma\gamma\gamma\gamma}(d\mathcal{M}(\xi)[h])^m = {}&[(\omega_m^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma mm})h^m - \omega_\gamma^2 C_{\gamma,2\gamma-m,\gamma,m}h^{2\gamma-m}]\mathbb{1}(0\leq m\leq\gamma-1)\\
&+[-2\omega_\gamma^2 C_{\gamma\gamma\gamma\gamma}h^\gamma]\mathbb{1}(m=\gamma)\\
&+[(\omega_m^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma mm})h^m - \omega_\gamma^2 C_{\gamma,2\gamma-m,\gamma,m}h^{2\gamma-m}]\mathbb{1}(\gamma+1\leq m\leq 2\gamma)\\
&+[(\omega_m^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma mm})h^m]\mathbb{1}(m\geq 2\gamma+1),
\end{aligned}$$

where C_{ijkm} are given in closed formulas in Lemma 5.2.

Proof. Let $\xi = \{\xi^m : m \geq 0\}$ be given by (6-1), $\epsilon > 0$, $h = \{h^j : j \geq 0\} \in l^2_{s+3}$, and pick any integer $m \geq 0$. Then, a similar computation with the one of Lemma 6.1 yields[11]

$$(\langle f\rangle(\xi+\epsilon h))^m = (\langle f\rangle(\xi))^m - \frac{3\epsilon K_\gamma^2}{2\pi}\sum_i C_{i\gamma\gamma m}h^i \int_0^{2\pi}\cos^2(\omega_\gamma t)\cos(\omega_i t)\cos(\omega_m t)\,dt + \mathcal{O}(\epsilon^2).$$

[11] Here, the notation $\mathcal{O}(\epsilon^2)$ for a function of ξ or h refers to a function that is bounded by ϵ^2 in the $\mathcal{Q}$-norm using the l_{s+3}-norm of ξ or h.

Therefore, we infer

$$
\begin{aligned}
(d\langle f\rangle(\xi)[h])^m &= -\frac{3K_\gamma^2}{2\pi}\sum_i C_{i\gamma\gamma m}h^i\int_0^{2\pi}\cos^2(\omega_\gamma t)\cos(\omega_i t)\cos(\omega_m t)\,dt \\
&= -\frac{3K_\gamma^2}{8}\sum_i C_{i\gamma\gamma m}h^i\sum_{\pm}\mathbb{1}(\omega_i\pm\omega_\gamma\pm\omega_\gamma\pm\omega_m=0) \\
&= -\frac{3K_\gamma^2}{8}\Big[\sum_i C_{i\gamma\gamma m}h^i\mathbb{1}(i=m)+\sum_i C_{i\gamma\gamma m}h^i\mathbb{1}(i=m)+\sum_i C_{i\gamma\gamma m}h^i\mathbb{1}(i=2\gamma-m\geq 0)\Big] \\
&= -\frac{3K_\gamma^2}{8}[2C_{m\gamma\gamma m}h^m+C_{2\gamma-m,\gamma,\gamma,m}h^{2\gamma-m}\mathbb{1}(0\leq m\leq 2\gamma)] \\
&= -\frac{\omega_\gamma^2}{C_{\gamma\gamma\gamma\gamma}}[2C_{\gamma\gamma mm}h^m+\mathbb{1}(0\leq m\leq 2\gamma)C_{\gamma,2\gamma-m,\gamma,m}h^{2\gamma-m}],
\end{aligned}
$$

where we also used the fact that $C_{ijkm}=0$ for $\omega_i\pm\omega_j\pm\omega_k\pm\omega_m=0$ with only 1 minus sign according to Lemma 5.1; so we are left with $\omega_i\pm\omega_j\pm\omega_k\pm\omega_m=0$ with only 2 minus signs, and there are three such terms in total, that is $i=m$, $i=m$ and $i=2\gamma-m$ with $i\geq 0$. Finally, we obtain

$$
\begin{aligned}
(d\mathcal{M}(\xi)[h])^m &= \omega_m^2h^m+(d\langle f\rangle(\xi)[h])^m \\
&= \left[\omega_m^2-\frac{2\omega_\gamma^2C_{\gamma\gamma mm}}{C_{\gamma\gamma\gamma\gamma}}\right]h^m-\mathbb{1}(0\leq m\leq 2\gamma)\frac{\omega_\gamma^2C_{\gamma,2\gamma-m,\gamma,m}}{C_{\gamma\gamma\gamma\gamma}}h^{2\gamma-m}. \qquad \square
\end{aligned}
$$

6B. ***Conformal cubic wave equation out of spherical symmetry.*** We first recall that the eigenfunctions $\{\mathsf{e}_n^{(\mu_1,\mu_2)}:n\geq 0\}$ are given by (3-2) and the PDE in the Fourier space from (4-5) reads

$$\ddot{u}^m(t)+(\mathsf{A}u(t))^m=(\mathsf{f}(\{u^j(t):j\geq 0\}))^m$$

for all integers $m\geq 0$, where the dots denote derivatives with respect to time and

$$(\mathsf{A}u(t))^m=(\omega_n^{(\mu_1,\mu_2)})^2u^m(t),\qquad (\mathsf{f}(\{u^j(t):j\geq 0\}))^m=-\sum_{i,j,k=0}^{\infty}\mathsf{C}_{ijkm}^{(\mu_1,\mu_2)}u^i(t)u^j(t)u^k(t).$$

For any initial data $u(0,\cdot)$, we denote by

$$\phi^t(\xi)=\{\xi^n\cos(\omega_n^{(\mu_1,\mu_2)}t):n\geq 0\},\qquad u(0,\cdot)=\sum_{n=0}^{\infty}\xi^n\mathsf{e}_n^{(\mu_1,\mu_2)},\qquad \xi=\{\xi^n:n\geq 0\},$$

the linear flow, that is the solution to the linear problem

$$\ddot{u}^n(t)+(\omega_n^{(\mu_1,\mu_2)})^2u^n(t)=0,\qquad (u^n(0),\dot{u}(0))=(\xi^n,0),$$

for all times $t\in\mathbb{R}$. For this model, we aim towards implementing the original version of Bambusi–Paleari's theorem (Theorem 2.4) and define

$$\mathcal{M}(\xi):=\mathsf{A}\xi+\langle\mathsf{f}\rangle(\xi),\qquad \langle\mathsf{f}\rangle(\xi)=\frac{1}{2\pi}\int_0^{2\pi}\phi^t[f(\phi^t(\xi))]\,dt.$$

To begin with, we show that the 1-modes are zeros of the operator $\mathcal{M}$.

Lemma 6.3 (zeros of the operator $\mathcal{M}$). *Let $\xi=\{\xi^m : m\geq 0\}$ be the rescaled* 1*-mode initial data,*

$$\xi^m=\mathsf{K}_\gamma^{(\mu_1,\mu_2)}\mathbb{1}(m=\gamma),\quad \mathsf{K}_\gamma^{(\mu_1,\mu_2)}=\pm 2\omega_\gamma^{(\mu_1,\mu_2)}\sqrt{\frac{2}{3C_{\gamma\gamma\gamma\gamma}^{(\mu_1,\mu_2)}}}, \tag{6-2}$$

for all integers $m\geq 0$. Then, we have $\mathcal{M}(\xi)=0$.

Proof. The proof is similar to the one of Lemma 6.1. □

Next, we derive the differential of $\mathcal{M}$ at the rescaled 1-modes.

Lemma 6.4 (differential of $\mathcal{M}$ at the 1-modes). *Let $\xi=\{\xi^m : m\geq 0\}$ be given by (6-2). Then, for all $h=\{h^j : j\geq 0\}\in l^2_{s+3}$, we have*

$$\begin{aligned}\mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu_1,\mu_2)}(d\mathcal{M}(\xi)[h])^m &= [(\omega_m^2\mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu_1,\mu_2)}-2\omega_\gamma^2\mathsf{C}_{\gamma\gamma mm}^{(\mu_1,\mu_2)})h^m-\omega_\gamma^2\mathsf{C}_{\gamma,2\gamma-m,\gamma,m}^{(\mu_1,\mu_2)}h^{2\gamma-m}]\mathbb{1}(0\leq m\leq\gamma-1)\\ &\quad+[-2\omega_\gamma^2\mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu_1,\mu_2)}h^\gamma]\mathbb{1}(m=\gamma)\\ &\quad+[(\omega_m^2\mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu_1,\mu_2)}-2\omega_\gamma^2\mathsf{C}_{\gamma\gamma mm}^{(\mu_1,\mu_2)})h^m-\omega_\gamma^2\mathsf{C}_{\gamma,2\gamma-m,\gamma,m}^{(\mu_1,\mu_2)}h^{2\gamma-m}]\mathbb{1}(\gamma+1\leq m\leq 2\gamma)\\ &\quad+[(\omega_m^2\mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu_1,\mu_2)}-2\omega_\gamma^2\mathsf{C}_{\gamma\gamma mm}^{(\mu_1,\mu_2)})h^m]\mathbb{1}(m\geq 2\gamma+1),\end{aligned}$$

where $\mathsf{C}_{\gamma\gamma mm}^{(\mu_1,\mu_2)}$ are given by closed formulas in Lemma 5.4.

Proof. The proof is similar to the one of Lemma 6.2 due to the fact that $\mathsf{C}_{ijkm}^{(\mu_1,\mu_2)}=0$ for

$$\omega_i^{(\mu_1,\mu_2)}\pm\omega_j^{(\mu_1,\mu_2)}\pm\omega_k^{(\mu_1,\mu_2)}\pm\omega_m^{(\mu_1,\mu_2)}=0$$

with only 1 minus sign according to Lemma 5.3; so we are left with

$$\omega_i^{(\mu_1,\mu_2)}\pm\omega_j^{(\mu_1,\mu_2)}\pm\omega_k^{(\mu_1,\mu_2)}\pm\omega_m^{(\mu_1,\mu_2)}=0$$

with only 2 minus signs, and there are again the same three such terms in total, that is $i=m$, $i=m$ and $i=2\gamma-m$ with $i\geq 0$, which completes the proof. □

6C. ***Yang–Mills equation in spherical symmetry.*** Recall that the eigenfunctions $\{\mathfrak{e}_n : n\geq 0\}$ are given by (3-4) and the PDE in the Fourier space from (4-8) reads

$$\ddot{u}^m(t)+(\mathfrak{A}u(t))^m=(\mathfrak{f}(\{u^j(t):j\geq 0\}))^m$$

for all integers $m\geq 0$, where the dots denote derivatives with respect to time and

$$(\mathfrak{A}u(t))^m=\varpi_n^2u^m(t),\quad (\mathfrak{f}(u))^m=(\mathfrak{f}^{(2)}(u))^m+(\mathfrak{f}^{(3)}(u))^m,$$

with

$$(\mathfrak{f}^{(2)}(\{u^j(t):j\geq 0\}))^m=-3\sum_{i,j=0}^{\infty}\overline{\mathfrak{C}}_{ijm}u^i(t)u^j(t),$$

$$(\mathfrak{f}^{(3)}(\{u^j(t):j\geq 0\}))^m=-\sum_{i,j,k=0}^{\infty}\mathfrak{C}_{ijkm}u^i(t)u^j(t)u^k(t).$$

For any initial data $u(0,\cdot)$, we denote by

$$\Phi^t(\xi)=\{\xi^n\cos(\varpi_n t):n\geq 0\}.\quad u(0,\cdot)=\sum_{n=0}^{\infty}\xi^n\mathfrak{e}_n,\quad \xi=\{\xi^n:n\geq 0\},$$

the linear flow, that is the solution to the linear problem

$$\ddot{u}^n(t)+\varpi_n^2 u^n(t)=0, \quad (u^n(0),\dot{u}^n(0))=(\xi^n,0),$$

for all times $t\in\mathbb{R}$. As a starting point, we show that the original version of Bambusi–Paleari's theorem (Theorem 2.4) is not applicable.

Lemma 6.5 (nonresonant $\mathfrak{f}^{(2)}$). *For all initial data $\xi\in l_s^2$, we have*

$$\langle\mathfrak{f}^{(2)}\rangle(\xi)=\frac{1}{2\pi}\int_0^{2\pi}\Phi^t[\mathfrak{f}^{(2)}(\Phi^t(\xi))]\,dt=0.$$

Proof. Let $\xi=\{\xi^m:m\geq 0\}\in l_s^2$ be any initial data, and pick an integer $m\geq 0$. Then, similar computations with the ones in Lemma 6.1 yield

$$\langle\mathfrak{f}^{(2)}\rangle(\xi)=-\frac{3}{2\pi}\sum_{i,j=0}^{\infty}\overline{\mathfrak{C}}_{ijm}\xi^i\xi^j\int_0^{2\pi}\prod_{\lambda\in\{i,j,m\}}\cos(\varpi_\lambda t)\,dt=-\frac{3}{4}\sum_{i,j=0}^{\infty}\overline{\mathfrak{C}}_{ijm}\xi^i\xi^j\sum_{\pm}\mathbb{1}(\varpi_i\pm\varpi_j\pm\varpi_m=0).$$

Now, notice that all the possible conditions are those with only 1 minus sign, and according to Lemma 5.7 the corresponding Fourier coefficients vanish. □

Consequently, for this model, we aim towards implementing the modified version of Bambusi–Paleari's theorem (Theorem 2.5) and define

$$\mathfrak{M}_\pm(\xi)=\pm\mathfrak{A}\xi+\langle\mathfrak{f}^{(3)}\rangle(\xi)+\mathfrak{F}_0(\xi), \quad \langle\mathfrak{f}^{(3)}\rangle(\xi)=\frac{1}{2\pi}\int_0^{2\pi}\Phi^t[\mathfrak{f}^{(3)}(\Phi^t(\xi))]\,dt,$$

where $\mathfrak{F}_0(\xi)$ is given for any initial data by Lemma 2.13 and for the 1-mode initial data by Lemma 2.14. Also, recall the Diophantine condition $\varpi\in\mathcal{W}_\alpha$ for some $0<\alpha<\frac{1}{3}$ from Theorem 2.5.

To begin with, we show that the 1-modes are zeros of the operator $\mathfrak{M}_-$.

Lemma 6.6 (zeros of the operator $\mathfrak{M}_-$). *Let $\gamma\in\{0,1,\dots,5\}$ and*

$$\mathfrak{q}_\gamma=\frac{9}{4}\sum_{\nu=0}^{2\gamma}(\overline{\mathfrak{C}}_{\gamma\gamma\nu})^2\left(\frac{2}{\varpi_\nu^2}+\frac{1}{\varpi_\nu^2-(2\varpi_\gamma)^2}\right).$$

Then, we have that $8\mathfrak{q}_\gamma>3\mathfrak{C}_{\gamma\gamma\gamma\gamma}$. Moreover, let $\xi=\{\xi^m:m\geq 0\}$ be the rescaled 1-mode initial data,

$$\xi^m=\mathfrak{K}_\gamma\mathbb{1}(m=\gamma), \quad \mathfrak{K}_\gamma=\pm 2\omega_\gamma\sqrt{\frac{2}{8\mathfrak{q}_\gamma-3\mathfrak{C}_{\gamma\gamma\gamma\gamma}}}, \tag{6-3}$$

for all integers $m\geq 0$. Then, we have that $\mathfrak{M}_-(\xi)=0$.

Proof. Let $\gamma\in\{0,1,\dots,5\}$, define $\xi=\{\xi^m:m\geq 0\}$ to be the rescaled 1-mode initial data given by (6-3) and pick any integer $m\geq 0$. Firstly, we compute $-\mathfrak{A}\xi+\langle\mathfrak{f}^{(3)}\rangle(\xi)$, and a similar computation with the one in Lemma 6.1 yields

$$-(\mathfrak{A}\xi)^m+(\langle\mathfrak{f}^{(3)}\rangle(\xi))^m=-\mathfrak{K}_\gamma\left(\varpi_\gamma^2+\tfrac{3}{8}\mathfrak{C}_{\gamma\gamma\gamma\gamma}\mathfrak{K}_\gamma^2\right)\mathbb{1}(m=\gamma),$$

where we used the fact that $\varpi_m+\varpi_\gamma\neq 0$, $\varpi_m+3\varpi_\gamma\neq 0$,

$$\varpi_m-\varpi_\gamma=0 \iff m=\gamma \quad\text{and}\quad \varpi_m-3\varpi_\gamma=0 \iff m=3\gamma+4,$$

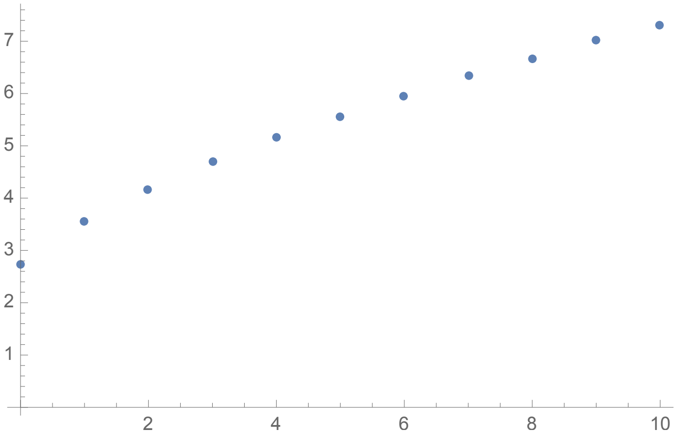

Figure 3. The constants $\mathfrak{K}_\gamma$ as γ varies within $\{0, 1, \dots, 10\}$. They are all real numbers.

as well as $\mathfrak{C}_{\gamma\gamma\gamma m} = 0$ for $m = 3\gamma + 4$ according to Lemma 5.7. Furthermore, we use the computation for $\mathfrak{F}_0(\xi)$ at the 1-mode initial data we derived in Lemma 2.14, that is

$$(\mathfrak{F}_0(\xi))^m = \mathfrak{q}_\gamma \mathfrak{K}_\gamma^3 \mathbb{1}(m=\gamma), \quad \mathfrak{q}_\gamma = \frac{9}{4}\sum_{\nu=0}^{2\gamma} (\overline{\mathfrak{C}}_{\gamma\gamma\nu})^2 \left(\frac{2}{\varpi_\nu^2} + \frac{1}{\varpi_\nu^2 - (2\varpi_\gamma)^2} \right),$$

to conclude that

$$\begin{aligned}(\mathfrak{M}_-(\xi))^m &= -(\mathfrak{A}\xi)^m + (\langle \mathfrak{f}^{(3)}\rangle(\xi))^m + (\mathfrak{F}_0(\xi))^m \\ &= -\mathfrak{K}_\gamma \left(\varpi_\gamma^2 + \tfrac{3}{8}\mathfrak{C}_{\gamma\gamma\gamma\gamma}\mathfrak{K}_\gamma^2 - \mathfrak{q}_\gamma \mathfrak{K}_\gamma^2\right)\mathbb{1}(m=\gamma) \\ &= -\tfrac{1}{8}\mathfrak{K}_\gamma (8\varpi_\gamma^2 + 3\mathfrak{C}_{\gamma\gamma\gamma\gamma}\mathfrak{K}_\gamma^2 - 8\mathfrak{q}_\gamma \mathfrak{K}_\gamma^2)\mathbb{1}(m=\gamma) = 0\end{aligned}$$

provided that $\mathfrak{K}_\gamma^2 = -8\varpi_\gamma^2/(3\mathfrak{C}_{\gamma\gamma\gamma\gamma} - 8\mathfrak{q}_\gamma)$. Finally, it remains to show that this choice is well defined, that is $\mathfrak{K}_\gamma \in \mathbb{R}$ for all $\gamma \in \{0, 1, \dots, 10\}$. To this end, we fix $\gamma \in \{0, 1, \dots, 10\}$ and use the definition of the Fourier coefficients to compute each $\mathfrak{K}_\gamma$ and verify that they are all real numbers. Figure 3 illustrates the constants $\mathfrak{K}_\gamma$ as γ varies within $\{0, 1, \dots, 10\}$. □

Next, we derive the differential of $\mathfrak{M}_-$ at the rescaled 1-modes.

Lemma 6.7 (differential of $\mathfrak{M}_-$ at the 1-modes). *Let $\gamma \in \{0, 1, \dots, 10\}$, and let $\xi = \{\xi^m : m \geq 0\}$ be given by (6-3). Then, for all $h = \{h^j : j \geq 0\} \in l^2_{s+3}$, we have*

$$\begin{aligned}-\mathfrak{K}_\gamma^{-2}(d\mathfrak{M}_-(\xi)[h])^m = \mathbb{1}(0 \leq m \leq \gamma - 1)[h^m \mathfrak{u}_{\gamma m} + h^{2\gamma - m}\mathfrak{v}_{\gamma m}] + \mathbb{1}(m=\gamma)[h^\gamma(\mathfrak{u}_{\gamma\gamma} + \mathfrak{v}_{\gamma\gamma})] \\ + \mathbb{1}(\gamma + 1 \leq m \leq 2\gamma)[h^m \mathfrak{u}_{\gamma m} + h^{2\gamma - m}\mathfrak{v}_{\gamma m}] + \mathbb{1}(m \geq 2\gamma + 1)[h^m \mathfrak{u}_{\gamma m}],\end{aligned}$$

where

$$\mathfrak{u}_{\gamma m} = \left(\frac{\varpi_m}{\mathfrak{K}_\gamma}\right)^2 + \tfrac{3}{4}\mathfrak{C}_{\gamma\gamma mm} - \mathfrak{a}_{\gamma m}, \quad \mathfrak{v}_{\gamma m} = \tfrac{3}{8}\mathfrak{C}_{\gamma, 2\gamma - m, \gamma, m} - \mathfrak{b}_{\gamma m},$$

and $\overline{\mathfrak{C}}_{ijm}$ and $\mathfrak{C}_{\gamma\gamma mm}$ are given by Lemmas 5.8 and 5.10, respectively, whereas $\mathfrak{a}_{\gamma m}$ and $\mathfrak{b}_{\gamma m}$ are given by Lemma 2.15.

Proof. Let $\gamma \in \{0, 1, \dots, 10\}$, $\xi = \{\xi^m : m \geq 0\}$ be given by (6-3), $h = \{h^j : j \geq 0\} \in l^2_{s+3}$, and pick any integer $m \geq 0$. Firstly, we use similar computations to the ones derived in Lemma 6.2 to obtain

$$\begin{aligned}(d\langle \mathfrak{f}^{(3)}\rangle(\xi)[h])^m &= -\frac{3}{8}\mathfrak{K}_\gamma^2 \sum_i \mathfrak{C}_{i\gamma\gamma m} h^i \sum_\pm \mathbb{1}(\varpi_i \pm \varpi_\gamma \pm \varpi_\gamma \pm \varpi_m = 0)\\ &= -\frac{3}{8}\mathfrak{K}_\gamma^2 \Big[\sum_i \mathfrak{C}_{i\gamma\gamma m} h^i \mathbb{1}(i=m) + \sum_i \mathfrak{C}_{i\gamma\gamma m} h^i \mathbb{1}(i=m) + \sum_i \mathfrak{C}_{i\gamma\gamma m} h^i \mathbb{1}(i = 2\gamma - m \geq 0)\Big]\\ &= -\tfrac{3}{8}\mathfrak{K}_\gamma^2 [2\mathfrak{C}_{m\gamma\gamma m} h^m + \mathfrak{C}_{2\gamma-m,\gamma,\gamma,m} h^{2\gamma-m} \mathbb{1}(0 \leq m \leq 2\gamma)],\end{aligned}$$

where we used the fact that $\mathfrak{C}_{ijkm} = 0$ for $\varpi_i \pm \varpi_j \pm \varpi_k \pm \varpi_m = 0$ with only 1 minus sign according to Lemma 5.7; so we are left with $\varpi_i \pm \varpi_j \pm \varpi_k \pm \varpi_m = 0$ with only 2 minus signs, and there are three such terms in total, that is $i = m$, $i = m$ and $i = 2\gamma - m$ with $i \geq 0$. Then, we infer

$$\begin{aligned}-(d\mathfrak{A}\xi[h])^m + (d\langle \mathfrak{f}^{(3)}\rangle(\xi)[h])^m &= -\varpi_m^2 h^m + (d\langle \mathfrak{f}^{(3)}\rangle(\xi)[h])^m\\ &= -\varpi_m^2 h^m - \tfrac{3}{8}\mathfrak{K}_\gamma^2[2\mathfrak{C}_{\gamma\gamma mm} h^m + \mathbb{1}(0 \leq m \leq 2\gamma)\mathfrak{C}_{\gamma,2\gamma-m,\gamma,m} h^{2\gamma-m}]\\ &= -\big[\varpi_m^2 + \tfrac{3}{4}\mathfrak{K}_\gamma^2 \mathfrak{C}_{\gamma\gamma mm}\big]h^m - \mathbb{1}(0 \leq m \leq 2\gamma)\tfrac{3}{8}\mathfrak{K}_\gamma^2 \mathfrak{C}_{\gamma,2\gamma-m,\gamma,m} h^{2\gamma-m}.\end{aligned}$$

Recall that the differential of $\mathfrak{F}_0$ at the 1-modes, $(d\mathfrak{F}_0(\xi)[h])^m$, is given by Lemma 2.15. Putting this all together yields that $(d\mathfrak{M}_-(\xi)[h])^m$ is given by

$$-h^m\big[\varpi_m^2 + \mathfrak{K}_\gamma^2(\omega)\big(\tfrac{3}{4}\mathfrak{C}_{\gamma\gamma mm} - \mathfrak{a}_{\gamma m}\big)\big] - \mathbb{1}(0 \leq m \leq 2\gamma) h^{2\gamma-m}\mathfrak{K}_\gamma^2(\omega)\big[\tfrac{3}{8}\mathfrak{C}_{\gamma,2\gamma-m,\gamma,m} - \mathfrak{b}_{\gamma m}\big].$$

Finally, one can rewrite the latter as stated above, which completes the proof. □

7. Nondegeneracy conditions for the 1-modes

In this section, we derive and establish the crucial nondegeneracy conditions for 1-mode initial data according to Theorem 2.4 (for CW and CH) and Theorem 2.5 (for YM).

7A. *Conformal cubic wave equation in spherical symmetry.*

Firstly, we consider the conformal cubic wave equation in spherical symmetry and derive the nondegeneracy condition for the 1-modes.

Lemma 7.1 (CW model: derivation of the nondegeneracy condition for the 1-modes). *Let $\gamma \geq 0$ be any integer, and define ξ to be the rescaled* 1*-mode according to* (6-1). *Then, the nondegeneracy condition*

$$\ker(d\mathcal{M}(\xi)) = \{0\}$$

is equivalent to

$$\begin{cases}\omega_m^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma mm} \neq 0 & \text{for all } m \geq 2\gamma+1,\\ D_{\gamma n} \neq 0 & \text{for all } n \in \{0, 1, \dots, \gamma-1\},\end{cases} \tag{7-1}$$

where

$$D_{\gamma n} = [\omega_n^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma nn}][\omega_{2\gamma-n}^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma,\gamma,2\gamma-n,2\gamma-n}] - [\omega_\gamma^2 C_{\gamma,2\gamma-n,\gamma,n}]^2.$$

Proof. Let $\gamma \geq 0$ be any integer and define ξ to be the rescaled 1-mode according to (6-1). Furthermore, pick any $h = \{h^j : j \geq 0\} \in l^2_{s+3}$ such that $d\mathcal{M}(\xi)[h] = 0$, and fix an integer $m \geq 0$. Then, according to Lemma 6.2, we have that $d\mathcal{M}(\xi)[h] = 0$ is equivalent to

$$(\omega_m^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma mm})h^m - \omega_\gamma^2 C_{\gamma,2\gamma-m,\gamma,m} h^{2\gamma-m} = 0 \quad \text{for } 0 \leq m \leq \gamma - 1, \tag{7-2}$$

$$(\omega_m^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma mm})h^m - \omega_\gamma^2 C_{\gamma,2\gamma-m,\gamma,m} h^{2\gamma-m} = 0 \quad \text{for } \gamma + 1 \leq m \leq 2\gamma, \tag{7-3}$$

$$(\omega_m^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma mm})h^m = 0 \quad \text{for } m \geq 2\gamma + 1, \tag{7-4}$$

coupled to

$$\omega_\gamma^2 C_{\gamma\gamma\gamma\gamma} h^\gamma = 0. \tag{7-5}$$

We will show that $h = \{h^m : m \geq 0\} = 0$ is the unique solution to the linear system above if and only if (7-1) holds. Firstly, (7-5) yields $h^\gamma = 0$ due to the fact that $C_{\gamma\gamma\gamma\gamma} \neq 0$ and $\omega_\gamma \neq 0$ for all $\gamma \geq 0$, whereas, for (7-4), one has that $h^m = 0$ for all integers $m \geq 2\gamma + 1$ if and only if $\omega_m^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma mm} \neq 0$ for all integers $m \geq 2\gamma + 1$. Next, we rearrange (7-2) and (7-3) by setting $m = n$ and $m = 2\gamma - n$, respectively, and obtain

$$\begin{bmatrix} \omega_n^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma nn} & -\omega_\gamma^2 C_{\gamma,2\gamma-n,\gamma,n} \\ -\omega_\gamma^2 C_{\gamma,n,\gamma,2\gamma-n} & \omega_{2\gamma-n}^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma,\gamma,2\gamma-n,2\gamma-n} \end{bmatrix} \begin{bmatrix} h^n \\ h^{2\gamma-n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for all $n \in \{0, 1, \dots, \gamma - 1\}$. Observe that there are γ in total (2×2)-linear systems where the unknowns are h^m for $m \in \{0, 1, \dots, 2\gamma\} \setminus \{\gamma\}$. Finally, these systems have only the trivial solution $h^m = 0$ for all $m \in \{0, 1, \dots, 2\gamma\} \setminus \{\gamma\}$ if and only if the determinants $D_{\gamma n}$ are nonzero for all $n \in \{0, 1, \dots, \gamma - 1\}$, which completes the proof. □

Next, we establish the nondegeneracy condition for this model.

Proposition 7.2 (nondegeneracy condition for the 1-modes and the CW model). *Let $\gamma \geq 0$ be any integer. Then, the nondegeneracy condition (7-1) holds.*

Proof. Let $\gamma \geq 0$ be any integer. Also, pick any integers $m \geq 2\gamma + 1$ and $n \in \{0, 1, \dots, \gamma - 1\}$. Then, according to Lemma 5.2, we have that $C_{ijkm} = \omega_{\min\{i,j,k,m\}}$ provided that either $\omega_i + \omega_j - \omega_k - \omega_m = 0$, $\omega_i - \omega_j + \omega_k - \omega_m = 0$ or $\omega_i - \omega_j - \omega_k + \omega_m = 0$. One can easily show that all the indices (i, j, k, m) of the Fourier coefficients that appear in Lemma 7.1 satisfy at least one of these conditions, and hence we infer

$$C_{\gamma\gamma\gamma\gamma} = \omega_\gamma, \quad C_{\gamma\gamma nn} = \omega_n, \quad C_{\gamma\gamma mm} = \omega_\gamma, \quad C_{\gamma,\gamma,2\gamma-n,2\gamma-n} = \omega_\gamma, \quad C_{\gamma,2\gamma-n,\gamma,n} = \omega_n.$$

Putting this all together yields

$$\omega_m^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma mm} = \omega_\gamma(\omega_m^2 - 2\omega_\gamma^2) \geq \omega_\gamma(\omega_{2\gamma+1}^2 - 2\omega_\gamma^2) \geq 2,$$
$$D_{\gamma n} = \omega_n \omega_\gamma^2 (n - 3 - 4\gamma)(n - \gamma)^2 \neq 0,$$

for all $m \geq 2\gamma + 1$ and $n \in \{0, 1, \dots, \gamma - 1\}$, which completes the proof. □

7B. ***Conformal cubic wave equation out of spherical symmetry.*** Next, we consider the conformal cubic wave equation out of spherical symmetry and show that the nondegeneracy condition is a condition on the Fourier coefficients.

Lemma 7.3 (derivation of the nondegeneracy condition for the 1-modes and the CH model). *Let γ and μ_1, μ_2 be any integers and define ξ according to (6-2). Then, the nondegeneracy condition*

$$\ker(d\mathcal{M}(\xi)) = \{0\}$$

is equivalent to

$$\begin{cases} (\omega_m^{(\mu_1,\mu_2)})^2 \mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu_1,\mu_2)} - 2(\omega_\gamma^{(\mu_1,\mu_2)})^2 \mathsf{C}_{\gamma\gamma mm}^{(\mu_1,\mu_2)} \neq 0 & \text{for all } m \geq 2\gamma+1, \\ \mathsf{D}_{\gamma n}^{(\mu_1,\mu_2)} \neq 0 & \text{for all } n \in \{0, 1, \ldots, \gamma-1\}, \end{cases} \tag{7-6}$$

where

$$\mathsf{D}_{\gamma n}^{(\mu_1,\mu_2)} = [(\omega_n^{(\mu_1,\mu_2)})^2 \mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu_1,\mu_2)} - 2(\omega_\gamma^{(\mu_1,\mu_2)})^2 \mathsf{C}_{\gamma\gamma nn}^{(\mu_1,\mu_2)}][(\omega_{2\gamma-n}^{(\mu_1,\mu_2)})^2 \mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu_1,\mu_2)} - 2(\omega_\gamma^{(\mu_1,\mu_2)})^2 \mathsf{C}_{\gamma,\gamma,2\gamma-n,2\gamma-n}^{(\mu_1,\mu_2)}] - [(\omega_\gamma^{(\mu_1,\mu_2)})^2 \mathsf{C}_{\gamma,2\gamma-n,\gamma,n}^{(\mu_1,\mu_2)}]^2.$$

Proof. The proof is similar to the one of Lemma 7.1. □

Next, we establish the nondegeneracy condition for this model.

Proposition 7.4 (nondegeneracy condition for the 1-modes and the CH model). *Let $\gamma, \mu_1, \mu_2 \geq 0$ be any integers with $\gamma \in \{0, 1, 2, 3, 4, 5\}$ and*

$$\mu_1 = \mu_2 =: \mu,$$

where μ is either sufficiently small with $\mu \in \{0, 1, 2, 3, 4, 5\}$ or sufficiently large. Then, the nondegeneracy condition (7-6) holds true.

Proof. Let $\gamma, \mu_1, \mu_2 \geq 0$ be any integers with $\gamma \in \{0, 1, 2, 3, 4, 5\}$ and $\mu_1 = \mu_2 =: \mu$, where μ is either sufficiently small with $\mu \in \{0, 1, 2, 3, 4, 5\}$ or sufficiently large. Also, pick any integer $m \geq 2\gamma+1$. Recall that, according to Lemma 5.4, we have that $\mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu,\mu)}$ and $\mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)}$ are given in terms of the function $\mathsf{M}_m^{(\mu)}(\lambda)$ that is also given in a closed formula. Moreover, according to Lemma 5.5, $\mathsf{M}_m^{(\mu)}(\lambda)$ is decreasing with respect to m. In addition, recall that the eigenvalues are given by

$$(\omega_m^{(\mu,\mu)})^2 = (2m+1+2\mu)^2$$

and they are clearly increasing with respect to $m \geq 0$. In other words, the function

$$\mathsf{P}_m^{(\mu)}(\lambda) = \frac{\mathsf{M}_m^{(\mu)}(\lambda)}{(\omega_m^{(\mu,\mu)})^2}$$

is decreasing with respect to m for all $m \geq \gamma$ as a product of two positive and decreasing functions. In the following, we show that

$$(\omega_m^{(\mu,\mu)})^2 \mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu,\mu)} - 2(\omega_\gamma^{(\mu,\mu)})^2 \mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)}$$

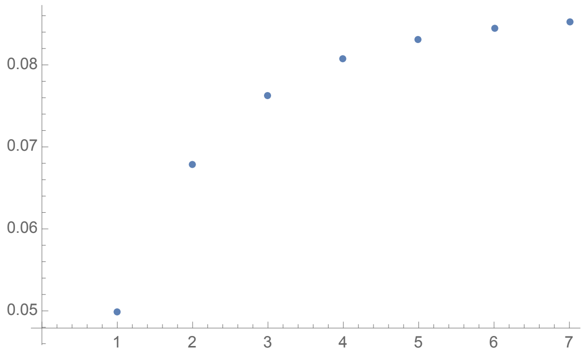

Figure 4. The constants σ_γ with $\gamma \in \{1, 2, 3, 4, 5\}$. They are all strictly positive.

stays away from zero for all $m \geq 2\gamma + 1$ and $\gamma \in \{0, 1, 2, 3, 4, 5\}$ provided that μ is either sufficiently small or sufficiently large. To this end, we use the monotonicity of $\mathsf{P}_m^{(\mu)}(\lambda)$ with respect to m to infer

$$\begin{aligned}
&(\omega_m^{(\mu,\mu)})^2 \mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu,\mu)} - 2(\omega_\gamma^{(\mu,\mu)})^2 \mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)} \\
&\qquad = (\omega_m^{(\mu,\mu)})^2 (\omega_\gamma^{(\mu,\mu)})^2 \left[\frac{\mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu,\mu)}}{(\omega_\gamma^{(\mu,\mu)})^2} - 2 \frac{\mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)}}{(\omega_m^{(\mu,\mu)})^2} \right] \\
&\qquad = \frac{1}{2} (\omega_m^{(\mu,\mu)})^2 (\omega_\gamma^{(\mu,\mu)})^2 \left[\sum_{\lambda=0}^{\gamma} \mathsf{P}_\gamma^{(\mu)}(\lambda) \mathsf{M}_\gamma^{(\mu)}(\lambda) \xi_\lambda(\mu) - 2 \sum_{\lambda=0}^{\gamma} \mathsf{M}_\gamma^{(\mu)}(\lambda) \mathsf{P}_m^{(\mu)}(\lambda) \xi_\lambda(\mu) \right] \\
&\qquad = \frac{1}{2} (\omega_m^{(\mu,\mu)})^2 (\omega_\gamma^{(\mu,\mu)})^2 \sum_{\lambda=0}^{\gamma} \mathsf{M}_\gamma^{(\mu)}(\lambda) [\mathsf{P}_\gamma^{(\mu)}(\lambda) - 2\mathsf{P}_m^{(\mu)}(\lambda)] \xi_\lambda(\mu) \\
&\qquad \geq \frac{1}{2} (\omega_m^{(\mu,\mu)})^2 (\omega_\gamma^{(\mu,\mu)})^2 \sum_{\lambda=0}^{\gamma} \mathsf{M}_\gamma^{(\mu)}(\lambda) [\mathsf{P}_\gamma^{(\mu)}(\lambda) - 2\mathsf{P}_{2\gamma+1}^{(\mu)}(\lambda)] \xi_\lambda(\mu) = (\omega_m^{(\mu,\mu)})^2 \mathsf{S}_\gamma^{(\mu)},
\end{aligned}$$

where we set

$$\mathsf{S}_\gamma^{(\mu)} = \frac{1}{2} (\omega_\gamma^{(\mu,\mu)})^2 \sum_{\lambda=0}^{\gamma} \mathsf{M}_\gamma^{(\mu)}(\lambda) [\mathsf{P}_\gamma^{(\mu)}(\lambda) - 2\mathsf{P}_{2\gamma+1}^{(\mu)}(\lambda)] \xi_\lambda(\mu).$$

On the one hand, for all $\gamma \in \{0, 1, 2, 3, 4, 5\}$ and $\mu \in \{0, 1, 2, 3, 4, 5\}$, we compute $\mathsf{S}_\gamma^{(\mu)}$ and verify that all $\mathsf{S}_\gamma^{(\mu)}$ are strictly positive. On the other hand, for all $\gamma \in \{0, 1, 2, 3, 4, 5\}$ and sufficiently large μ, we firstly compute $\mathsf{S}_\gamma^{(\mu)}$ in terms of μ and then derive its asymptotic expansion as $\mu \to \infty$. For $\gamma = 0$, we find

$$\mathsf{S}_0^{(\mu)} = \frac{4^\mu (2\mu+1)(10\mu+7) \Gamma\left(\mu + \frac{1}{2}\right)^2 \Gamma\left(\mu + \frac{5}{2}\right)}{\pi (2\mu+3)^2 \Gamma(\mu+1) \Gamma\left(2\mu + \frac{5}{2}\right)},$$

which is strictly positive for all $\mu \geq 0$, and for $\gamma \in \{1, 2, 3, 4, 5\}$, we expand

$$\mathsf{S}_\gamma^{(\mu)} = \sigma_\gamma \mu^{1/2} + \mathcal{O}(\mu^{-1/2})$$

as $\mu \to \infty$ for some strictly positive constants σ_γ. Figure 4 above illustrates the constants σ_γ with $\gamma \in \{1, 2, 3, 4, 5\}$.

Consequently, in both cases, we can ensure that $\mathsf{S}_\gamma^{(\mu)} > 0$, and hence we conclude that

$$(\omega_m^{(\mu,\mu)})^2 \mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu,\mu)} - 2(\omega_\gamma^{(\mu,\mu)})^2 \mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)} \geq (\omega_m^{(\mu,\mu)})^2 \mathsf{S}_\gamma^{(\mu)} \geq (\omega_{2\gamma+1}^{(\mu,\mu)})^2 \mathsf{S}_\gamma^{(\mu)} > 0$$

for all $m \geq 2\gamma+1$. Finally, it remains to show that the determinants $\mathsf{D}_{\gamma n}^{(\mu,\mu)}$ are all nonzero for $n \in \{0, 1, \ldots, \gamma-1\}$. To this end, for all $\gamma \in \{1, 2, 3, 4, 5\}$ and $n \in \{0, 1, \ldots, \gamma-1\}$, we firstly compute each of the Fourier coefficients in the determinants above, find a closed formula for each of the determinants in terms of μ, and then either compute the determinants when μ is sufficiently small or compute their limits when $\mu \to \infty$. For example, for $\gamma = 1$, we have $n = 0$ and compute

$$\mathsf{D}_{10}^{(\mu,\mu)} = -\frac{3\cdot 16^{\mu-1}}{\pi^2}(\mu+1)(2\mu+3)^4(2\mu+5)(4\mu+7) \cdot (20\mu^4+328\mu^3+1029\mu^2+1155\mu+435)\frac{\Gamma\big(\mu+\frac{1}{2}\big)^2\Gamma\big(\mu+\frac{3}{2}\big)^3\Gamma\big(\mu+\frac{3}{2}\big)}{\Gamma(\mu+2)^2\Gamma\big(2\mu+\frac{9}{2}\big)^2},$$

which is clearly strictly negative for all integers $\mu \geq 0$. For all $\gamma \in \{1, 2, 3, 4, 5\}$ and $n \in \{0, 1, \ldots, \gamma-1\}$, we find either $\mathsf{D}_{\gamma n}^{(\mu,\mu)} < 0$ when μ is sufficiently small or $\mathsf{D}_{\gamma n}^{(\mu,\mu)} \to \pm\infty$ when $\mu \to \infty$, which completes the proof. □

7C. ***Yang–Mills equation in spherical symmetry.*** Finally, we consider the Yang–Mills equation in spherical symmetry and show that the nondegeneracy condition is a condition on the Fourier coefficients.

Lemma 7.5 (derivation of the nondegeneracy condition for the 1-modes and the YM model). *Let $\gamma \in \{0, 1, 2, 3, 4, 5\}$, and define ξ according to (6-3). Then, the nondegeneracy condition*

$$\ker(d\mathfrak{M}_-(\xi)) = \{0\}$$

is equivalent to

$$\begin{cases} \mathfrak{u}_{\gamma\gamma} + \mathfrak{v}_{\gamma\gamma} \neq 0, & \\ \mathfrak{u}_{\gamma m} \neq 0 & \textit{for all } m \geq 2\gamma+1, \\ \mathfrak{D}_{\gamma n} \neq 0 & \textit{for all } n \in \{0, 1, \ldots, \gamma-1\}, \end{cases} \tag{7-7}$$

where

$$\mathfrak{D}_{\gamma n} := \mathfrak{u}_{\gamma n}\mathfrak{u}_{\gamma,2\gamma-n} - \mathfrak{v}_{\gamma n}\mathfrak{v}_{\gamma,2\gamma-n}$$

and

$$\mathfrak{u}_{\gamma m} = \left(\frac{\varpi_m}{\mathfrak{K}_\gamma}\right)^2 + \frac{3}{4}\mathfrak{C}_{\gamma\gamma mm} - \frac{9}{2}\sum_{\nu=0}^{m+\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma\nu m})^2}{\varpi_\nu^2 - (\varpi_m+\varpi_\gamma)^2} - \frac{9}{4}\sum_{\nu=0}^{2\gamma}\frac{\overline{\mathfrak{C}}_{m\nu m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2} - \frac{9}{2}\sum_{\substack{\nu=0 \\ \nu\neq\pm(m-\gamma)-2}}^{m+\gamma}\frac{(\overline{\mathfrak{C}}_{m\gamma\nu})^2}{\varpi_\nu^2 - (\varpi_m-\varpi_\gamma)^2}, \tag{7-8}$$

$$\mathfrak{v}_{\gamma m} = \frac{3}{8}\mathfrak{C}_{\gamma,2\gamma-m,\gamma,m} - \frac{9}{4}\sum_{\nu=0}^{2\gamma}\frac{\overline{\mathfrak{C}}_{2\gamma-m,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2 - (2\varpi_\gamma)^2} - \frac{9}{2}\sum_{\substack{\nu=0 \\ \nu\neq\pm(m-\gamma)-2}}^{m+\gamma}\frac{\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{2\gamma-m,\gamma,\nu}}{\varpi_\nu^2 - (\varpi_{2\gamma-m}-\varpi_\gamma)^2}. \tag{7-9}$$

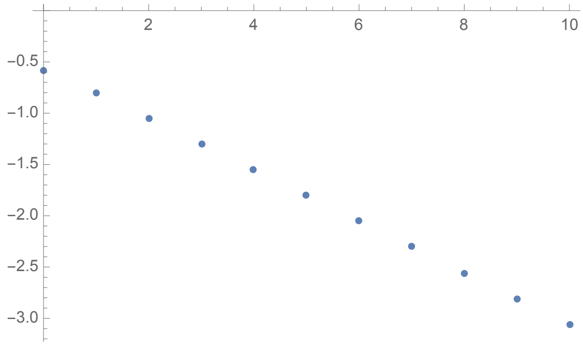

Figure 5. The constants $\mathfrak{u}_{\gamma\gamma} + \mathfrak{v}_{\gamma\gamma}$ for $\gamma \in \{0, 1, 2, 3, 4, 5\}$. They decrease and stay away from zero.

Proof. Let $\gamma \in \{0, 1, 2, 3, 4, 5\}$, and define ξ according to (6-3). Furthermore, assume $\ker(d\mathfrak{M}_-(\xi)) = \{0\}$, and pick any $h = \{h^j : j \geq 0\} \in l^2_{s+3}$ such that $d\mathfrak{M}_-(\xi)[h] = 0$. Also, fix any integer $m \geq 0$. Then, according to Lemma 6.7, we have that the system

$$\mathfrak{u}_{\gamma m} h^m + \mathfrak{v}_{\gamma m} h^{2\gamma - m} = 0 \quad \text{for } 0 \leq m \leq \gamma - 1, \tag{7-10}$$

$$(\mathfrak{u}_{\gamma\gamma} + \mathfrak{v}_{\gamma\gamma}) h^{\gamma} = 0 \quad \text{for } m = \gamma, \tag{7-11}$$

$$\mathfrak{u}_{\gamma m} h^m + \mathfrak{v}_{\gamma m} h^{2\gamma - m} = 0 \quad \text{for } \gamma + 1 \leq m \leq 2\gamma, \tag{7-12}$$

$$\mathfrak{u}_{\gamma m} h^m = 0 \quad \text{for } m \geq 2\gamma + 1, \tag{7-13}$$

has $h = \{h^i : i \geq 0\} = 0$ as the unique solution, where $\mathfrak{u}_{\gamma m}$ and $\mathfrak{v}_{\gamma m}$ are given explicitly in terms of the auxiliary sequences $\mathfrak{a}_{\gamma m}$ and $\mathfrak{b}_{\gamma m}$ as Lemma 6.7 states. Furthermore, $\mathfrak{a}_{\gamma m}$ and $\mathfrak{b}_{\gamma m}$ are given by Lemma 2.15, and putting this all together yields the closed formulas (7-8) and (7-9) for $\mathfrak{u}_{\gamma m}$ and $\mathfrak{v}_{\gamma m}$, respectively, as stated above. Now, (7-11) and (7-13) yield $\mathfrak{u}_{\gamma\gamma} + \mathfrak{v}_{\gamma\gamma} \neq 0$ and $\mathfrak{u}_{\gamma m} \neq 0$ for all $m \geq 2\gamma + 1$. Next, we rearrange (7-10) and (7-12) by setting $m = n$ and $m = 2\gamma - n$, respectively, to obtain

$$\begin{bmatrix} \mathfrak{u}_{\gamma n} & \mathfrak{v}_{\gamma n} \\ \mathfrak{v}_{\gamma, 2\gamma - n} & \mathfrak{u}_{\gamma, 2\gamma - n} \end{bmatrix} \begin{bmatrix} h^n \\ h^{2\gamma - n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for all $n \in \{0, 1, \ldots, \gamma - 1\}$. Observe that there are γ in total (2×2)-linear systems where the unknowns are h^m for $m \in \{0, 1, \ldots, 2\gamma\} \setminus \{\gamma\}$. Finally, these systems have only the trivial solution $h^m = 0$ for all $m \in \{0, 1, \ldots, 2\gamma\} \setminus \{\gamma\}$ if and only if the determinants $\mathfrak{D}_{\gamma n}$ are nonzero for all $n \in \{0, 1, \ldots, \gamma - 1\}$, which completes the proof. □

Next, we establish the nondegeneracy condition for this model.

Proposition 7.6 (nondegeneracy condition for the 1-modes and the YM model). *Let* $\gamma \in \{0, 1, 2, 3, 4, 5\}$. *Then, the nondegeneracy condition* (7-7) *holds.*

Proof. Let $\gamma \in \{0, 1, 2, 3, 4, 5\}$, $\alpha = 1/\sqrt{6}$ and pick any frequency $\varpi \in \mathcal{W}_\alpha$ with $\varpi < 1$. Also, pick an integer $m \geq 2\gamma + 1$, and define $\mathfrak{u}_{\gamma m}$ and $\mathfrak{v}_{\gamma m}$ according to (7-8) and (7-9), respectively. Firstly, we show that $\mathfrak{u}_{\gamma\gamma} + \mathfrak{v}_{\gamma\gamma} \neq 0$. In this case, all the sums in the definitions of $\mathfrak{u}_{\gamma\gamma}$ and $\mathfrak{v}_{\gamma\gamma}$ are finite as the index variable ν varies within $\{0, 1, \dots, 2\gamma\}$. Hence, we compute $\mathfrak{u}_{\gamma\gamma} + \mathfrak{v}_{\gamma\gamma}$ for all $\gamma \in \{0, 1, 2, 3, 4, 5\}$ and verify that they are all nonzero. Figure 5 illustrates the constants $\mathfrak{u}_{\gamma\gamma} + \mathfrak{v}_{\gamma\gamma}$ for $\gamma \in \{0, 1, 2, 3, 4, 5\}$. Secondly, we show that $\mathfrak{u}_{\gamma m} \neq 0$ for $m \geq 2\gamma + 1$. To this end, we note that $\overline{\mathfrak{C}}_{ijm} = 0$ for all integers $i, j, m \geq 0$ with either $m > i + j$ or $|i - j| > m$ (Lemma 5.8). Specifically, for any integer $m \geq 2\gamma + 1$, we focus on

$$\sum_{\nu=0}^{m+\gamma} \frac{(\overline{\mathfrak{C}}_{\gamma\nu m})^2}{\varpi_\nu^2 - (\varpi_m + \varpi_\gamma)^2}, \qquad \sum_{\substack{\nu=0 \\ \nu\neq\pm(m-\gamma)-2}}^{m+\gamma} \frac{(\overline{\mathfrak{C}}_{m\gamma\nu})^2}{\varpi_\nu^2 - (\varpi_m - \varpi_\gamma)^2},$$

and note that we must have $m - \gamma \leq \nu \leq m + \gamma$. Indeed, $\overline{\mathfrak{C}}_{\gamma\nu m} = 0$ since $m - \gamma = |m - \gamma| > \nu$, and $\overline{\mathfrak{C}}_{\gamma\nu m} = 0$ since $\nu > m + \gamma$. In addition, for all such ν, the conditions $\nu \neq \pm(m - \gamma) - 2$ are satisfied since

$$\nu \geq m - \gamma \quad \Longrightarrow \quad (m - \gamma) - 2 < m - \gamma \leq \nu \quad \Longrightarrow \quad \nu \neq (m - \gamma) - 2$$

and

$$m \geq 2\gamma + 1 \quad \Longrightarrow \quad -(m - \gamma) - 2 < 0 \quad \Longrightarrow \quad \nu \neq -(m - \gamma) - 2.$$

Consequently, for all $m \geq 2\gamma + 1$, we have

$$\mathfrak{u}_{\gamma m} = \left(\frac{\varpi_m}{\mathfrak{K}_\gamma}\right)^2 + \frac{3}{4}\mathfrak{C}_{\gamma\gamma mm} - \frac{9}{2}\sum_{\nu=m-\gamma}^{m+\gamma} \frac{(\overline{\mathfrak{C}}_{\gamma\nu m})^2}{\varpi_\nu^2 - (\varpi_m + \varpi_\gamma)^2} - \frac{9}{4}\sum_{\nu=0}^{2\gamma} \frac{\overline{\mathfrak{C}}_{m\nu m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2} - \frac{9}{2}\sum_{\nu=m-\gamma}^{m+\gamma} \frac{(\overline{\mathfrak{C}}_{m\gamma\nu})^2}{\varpi_\nu^2 - (\varpi_m - \varpi_\gamma)^2},$$

and by setting $\nu = \sigma + m - \gamma$ and $\nu = \sigma$, respectively, we can rewrite the latter as

$$\begin{aligned}\mathfrak{u}_{\gamma m} = \left(\frac{\varpi_m}{\mathfrak{K}_\gamma}\right)^2 + \frac{3}{4}\mathfrak{C}_{\gamma\gamma mm} - \frac{9}{2}\sum_{\sigma=0}^{2\gamma} \frac{(\overline{\mathfrak{C}}_{\gamma,\sigma+m-\gamma,m})^2}{\varpi_{\sigma+m-\gamma}^2 - (\varpi_m + \varpi_\gamma)^2} \\ - \frac{9}{4}\sum_{\sigma=0}^{2\gamma} \frac{\overline{\mathfrak{C}}_{m\sigma m}\overline{\mathfrak{C}}_{\gamma\gamma\sigma}}{\varpi_\sigma^2} - \frac{9}{2}\sum_{\sigma=0}^{2\gamma} \frac{(\overline{\mathfrak{C}}_{m,\gamma,\sigma+m-\gamma})^2}{\varpi_{\sigma+m-\gamma}^2 - (\varpi_m - \varpi_\gamma)^2}.\end{aligned}$$

Now, recall from Lemma 6.6 that

$$\begin{aligned}\mathfrak{K}_\gamma^{-2} = \frac{3\mathfrak{C}_{\gamma\gamma\gamma\gamma} - 8\mathfrak{q}_\gamma}{-8\varpi_\gamma^2} &= -\frac{3}{8\varpi_\gamma^2}\mathfrak{C}_{\gamma\gamma\gamma\gamma} + \frac{1}{\varpi_\gamma^2}\mathfrak{q}_\gamma \\ &= -\frac{3}{8\varpi_\gamma^2}\mathfrak{C}_{\gamma\gamma\gamma\gamma} + \frac{9}{4\varpi_\gamma^2}\sum_{\nu=0}^{2\gamma}(\overline{\mathfrak{C}}_{\gamma\gamma\nu})^2\left(\frac{2}{\varpi_\nu^2} + \frac{1}{\varpi_\nu^2 - (2\varpi_\gamma)^2}\right),\end{aligned}$$

which yields

$$\left(\frac{\varpi_m}{\mathfrak{K}_\gamma}\right)^2 = -\frac{3\varpi_m^2}{8\varpi_\gamma^2}\mathfrak{C}_{\gamma\gamma\gamma\gamma} + \frac{9\varpi_m^2}{4\varpi_\gamma^2}\sum_{\nu=0}^{2\gamma}(\overline{\mathfrak{C}}_{\gamma\gamma\nu})^2\left(\frac{2}{\varpi_\nu^2} + \frac{1}{\varpi_\nu^2 - (2\varpi_\gamma)^2}\right).$$

Putting this all together, we obtain

$$\begin{aligned}\mathfrak{u}_{\gamma m} = -\frac{3\varpi_m^2}{8\varpi_\gamma^2}\mathfrak{C}_{\gamma\gamma\gamma\gamma} + \frac{9\varpi_m^2}{4\varpi_\gamma^2}\sum_{\sigma=0}^{2\gamma}(\overline{\mathfrak{C}}_{\gamma\gamma\sigma})^2\left(\frac{2}{\varpi_\sigma^2} + \frac{1}{\varpi_\sigma^2 - (2\varpi_\gamma)^2}\right) + \frac{3}{4}\mathfrak{C}_{\gamma\gamma mm} \\ - \frac{9}{2}\sum_{\sigma=0}^{2\gamma} \frac{(\overline{\mathfrak{C}}_{\gamma,\sigma+m-\gamma,m})^2}{\varpi_{\sigma+m-\gamma}^2 - (\varpi_m + \varpi_\gamma)^2} - \frac{9}{4}\sum_{\sigma=0}^{2\gamma} \frac{\overline{\mathfrak{C}}_{m\sigma m}\overline{\mathfrak{C}}_{\gamma\gamma\sigma}}{\varpi_\sigma^2} - \frac{9}{2}\sum_{\sigma=0}^{2\gamma} \frac{(\overline{\mathfrak{C}}_{m,\gamma,\sigma+m-\gamma})^2}{\varpi_{\sigma+m-\gamma}^2 - (\varpi_m - \varpi_\gamma)^2}.\end{aligned}$$

In addition, we also note that $\overline{\mathfrak{C}}_{ijm} = 0$ for all integers $i, j, m \geq 0$ with $i + j - m \notin 2\mathbb{N} \cup \{0\}$ (Lemma 5.8). Specifically, we must have $\sigma \in 2\mathbb{N} \cup \{0\}$. Indeed,

$$\overline{\mathfrak{C}}_{\gamma\gamma\sigma} = 0 \quad \text{since } \sigma = \sigma + \gamma - \gamma \notin 2\mathbb{N} \cup \{0\},$$

and

$$\overline{\mathfrak{C}}_{\gamma,\sigma+m-\gamma,m} = \overline{\mathfrak{C}}_{m,\gamma,\sigma+m-\gamma} = 0 \quad \text{since } \sigma = \gamma + \sigma + m - \gamma - m \notin 2\mathbb{N} \cup \{0\}.$$

Therefore, by setting $\sigma = 2\tau$, we arrive at

$$\mathfrak{u}_{\gamma m} = -\frac{3\varpi_m^2}{8\varpi_\gamma^2}\mathfrak{C}_{\gamma\gamma\gamma\gamma} + \frac{9\varpi_m^2}{2\varpi_\gamma^2}\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau})^2}{\varpi_{2\tau}^2} + \frac{9\varpi_m^2}{4\varpi_\gamma^2}\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau})^2}{\varpi_{2\tau}^2 - (2\varpi_\gamma)^2} + \frac{3}{4}\mathfrak{C}_{\gamma\gamma mm}$$
$$- \frac{9}{2}\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m})^2}{\varpi_{2\tau+m-\gamma}^2 - (\varpi_m + \varpi_\gamma)^2} - \frac{9}{4}\sum_{\tau=0}^{\gamma}\frac{\overline{\mathfrak{C}}_{m,2\tau,m}\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau}}{\varpi_{2\tau}^2} - \frac{9}{2}\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{m,\gamma,2\tau+m-\gamma})^2}{\varpi_{2\tau+m-\gamma}^2 - (\varpi_m - \varpi_\gamma)^2}.$$

Now, all the Fourier coefficients above are nonzero and, according to Lemma 5.9, we have $\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau}$, $\overline{\mathfrak{C}}_{m,2\tau,m}$ and $\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m}$ in closed formulas. These allow us to compute

$$\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau})^2}{\varpi_{2\tau}^2} = \frac{(\gamma+2)(2\gamma+3)(2\gamma+5)}{15\pi(\gamma+1)(\gamma+3)},$$
$$\sum_{\tau=0}^{\gamma}\frac{\overline{\mathfrak{C}}_{m,2\tau,m}\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau}}{\varpi_{2\tau}^2} = \frac{(\gamma+2)(-\gamma(\gamma+4)+5m(m+4)+15)}{15\pi(m+1)(m+3)},$$

for all integers $\gamma \geq 0$. Recall that $\mathfrak{C}_{\gamma\gamma mm}$ and $\mathfrak{C}_{\gamma\gamma\gamma\gamma}$ are also given by closed formulas (Remark 5.11). Consequently, we rescale $\mathfrak{u}_{\gamma m}$ and obtain

$$\frac{\mathfrak{u}_{\gamma m}}{\varpi_m^2} = \mathfrak{I}_{\gamma m} + \mathfrak{E}_{\gamma m}, \tag{7-14}$$

where $\mathfrak{I}_{\gamma m}$ stands for the part that can be explicitly computed:

$$\mathfrak{I}_{\gamma m} = -\frac{3}{8\varpi_\gamma^2}\mathfrak{C}_{\gamma\gamma\gamma\gamma} + \frac{9}{2\varpi_\gamma^2}\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau})^2}{\varpi_{2\tau}^2} + \frac{9}{4\varpi_\gamma^2}\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau})^2}{\varpi_{2\tau}^2 - (2\varpi_\gamma)^2}$$
$$+ \frac{3}{4\varpi_m^2}\mathfrak{C}_{\gamma\gamma mm} - \frac{9}{4\varpi_m^2}\sum_{\tau=0}^{\gamma}\frac{\overline{\mathfrak{C}}_{m,2\tau,m}\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau}}{\varpi_{2\tau}^2},$$

and $\mathfrak{E}_{\gamma m}$ stands for the part that cannot be explicitly computed:

$$\mathfrak{E}_{\gamma m} = -\frac{9}{2\varpi_m^2}\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m})^2}{\varpi_{2\tau+m-\gamma}^2 - (\varpi_m + \varpi_\gamma)^2} - \frac{9}{2\varpi_m^2}\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{m,\gamma,2\tau+m-\gamma})^2}{\varpi_{2\tau+m-\gamma}^2 - (\varpi_m - \varpi_\gamma)^2}.$$

Now, using the elementary inequalities

$$|\varpi_{2\tau+m-\gamma}^2 - (\varpi_m + \varpi_\gamma)^2| = |4(\gamma - \tau + 1)(m + \tau + 3)| \geq 4(m+3),$$
$$|\varpi_{2\tau+m-\gamma}^2 - (\varpi_m - \varpi_\gamma)^2| = |4(\tau + 1)(-\gamma + m + \tau + 1)| \geq 2(m+3),$$

for all $0 \leq \tau \leq \gamma$ and $\gamma \geq 0$, we estimate

$$
\begin{aligned}
|\mathfrak{E}_{\gamma m}| &\leq \frac{9}{2\varpi_m^2} \sum_{\tau=0}^{\gamma} \frac{(\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m})^2}{|\varpi_{2\tau+m-\gamma}^2 - (\varpi_m + \varpi_\gamma)^2|} + \frac{9}{2\varpi_m^2} \sum_{\tau=0}^{\gamma} \frac{(\overline{\mathfrak{C}}_{m,\gamma,2\tau+m-\gamma})^2}{|\varpi_{2\tau+m-\gamma}^2 - (\varpi_m - \varpi_\gamma)^2|} \\
&\leq \frac{9}{2\varpi_m^2} \frac{1}{4(m+3)} \sum_{\tau=0}^{\gamma} (\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m})^2 + \frac{9}{2\varpi_m^2} \frac{1}{2(m+3)} \sum_{\tau=0}^{\gamma} (\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m})^2 \\
&= \frac{9}{2\varpi_m^2} \left(\frac{1}{4(m+3)} + \frac{1}{2(m+3)} \right) \sum_{\tau=0}^{\gamma} (\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m})^2 \\
&= \frac{9(\gamma+2)}{70\pi(m+1)(m+2)^2(m+3)^2} \big[-3\gamma^4 - 24\gamma^3 - 40\gamma^2 + 32\gamma + 7\gamma^2 m^2 + 28\gamma m^2 + 35m^2 \\
&\qquad\qquad + 28\gamma^2 m + 112\gamma m + 140m + 105 \big] = \mathfrak{P}_{\gamma m},
\end{aligned}
$$

where we used the closed formula for $\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m}$ from above. Hence, for all $m \geq 2\gamma + 1$, we obtain

$$
\frac{\mathfrak{u}_{\gamma m}}{\varpi_m^2} = \mathfrak{I}_{\gamma m} + \mathfrak{E}_\gamma \geq \mathfrak{I}_{\gamma m} - \mathfrak{P}_{\gamma m} = \mathfrak{O}_{\gamma m}. \tag{7-15}
$$

Finally, for each $\gamma \in \{0, 1, 2, 3, 4, 5\}$, we use the closed formulas for $\mathfrak{C}_{\gamma\gamma mm}$ and $\mathfrak{C}_{\gamma\gamma\gamma\gamma}$ (see Remark 5.11) to firstly explicitly compute $\mathfrak{I}_{\gamma m}$ in terms of m and then explicitly compute $\mathfrak{O}_{\gamma m}$ in terms of m. Once the closed formula is derived, we show that $\mathfrak{O}_{\gamma m} > 0$ for all $m \geq 2\gamma + 1$. For example, for $\gamma = 0$, we find

$$
\mathfrak{I}_{0m} = \frac{m(m+4)(5m(m+4)+29)+66}{12\pi(m+1)(m+2)^2(m+3)},
$$

and hence

$$
\mathfrak{O}_{0m} = \frac{5m^4 + 40m^3 + 109m^2 + 8m - 42}{12\pi(m+1)(m+2)^2(m+3)},
$$

which is greater than 10^{-3} provided that $m \geq 1$. Similarly, for $\gamma = 1$, we compute

$$
\mathfrak{I}_{1m} = \frac{(m^2+2)(m(m+8)+18)}{4\pi(m+1)(m+2)^2(m+3)},
$$

and hence

$$
\mathfrak{O}_{1m} = \frac{m(m^4 + 11m^3 + 44m^2 - 32m - 348)}{4\pi(m+1)(m+2)^2(m+3)^2},
$$

which is greater than 10^{-3} provided that $m \geq 3$. For all the other cases with $\gamma \in \{2, 3, 4, 5\}$, we find

$$
\begin{aligned}
\mathfrak{O}_{2m} &= \frac{109m^5 + 1199m^4 + 4523m^3 - 30347m^2 - 132936m + 107244}{600\pi(m+1)(m+2)^2(m+3)^2}, \\
\mathfrak{O}_{3m} &= \frac{43m^5 + 473m^4 + 1646m^3 - 33554m^2 - 129372m + 238248}{300\pi(m+1)(m+2)^2(m+3)^2}, \\
\mathfrak{O}_{4m} &= \frac{83m^5 + 913m^4 + 2851m^3 - 139159m^2 - 515982m + 1611198}{700\pi(m+1)(m+2)^2(m+3)^2}, \\
\mathfrak{O}_{5m} &= \frac{17m^5 + 187m^4 + 505m^3 - 53329m^2 - 194760m + 905292}{168\pi(m+1)(m+2)^2(m+3)^2}
\end{aligned}
$$

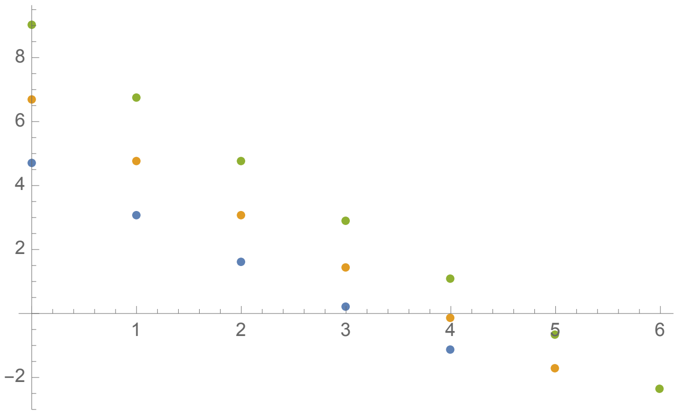

Figure 6. The determinants $\mathfrak{D}_{\gamma n}$ for $\gamma = 5$ (blue/bottom), $\gamma = 6$ (orange/middle) and $\gamma = 7$ (green/top) for all $n \in \{0, 1, \dots, \gamma - 1\}$. They are all in fact nonzero.

and the claim follows similarly.[12] Finally, it remains to show that the determinants

$$\mathfrak{D}_{\gamma n} := \mathfrak{u}_{\gamma n}\mathfrak{u}_{\gamma, 2\gamma - n} - \mathfrak{v}_{\gamma n}\mathfrak{v}_{\gamma, 2\gamma - n}$$

are all nonzero for $n \in \{0, 1, \dots, \gamma - 1\}$, which follows by a direct computation using the definition of the Fourier coefficients. Specifically, we compute $\mathfrak{D}_{\gamma n}$ for all $\gamma \in \{0, 1, 2, 3, 4, 5\}$ and $n \in \{0, 1, \dots, \gamma - 1\}$ and verify that they are all strictly negative, which completes the proof. Figure 6 illustrates the determinants $\mathfrak{D}_{\gamma n}$ for $\gamma \in \{5, 6, 7\}$ and $n \in \{0, 1, \dots, \gamma - 1\}$. □

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[12]For $\gamma \in \{2, 3\}$, the estimates $\mathfrak{D}_{\gamma m} \geq 10^{-3}$ hold true for all integers $m \geq 2\gamma + 1$, whereas, for $\gamma \in \{4, 5\}$, we have that $\mathfrak{D}_{\gamma m} \geq 10^{-3}$ provided that $m \geq 2\gamma + 3$ instead of $m \geq 2\gamma + 1$. In this case, we use the definition of the Fourier coefficients to explicitly compute $\mathfrak{u}_{\gamma m}\varpi_m^{-2}$ for $m \in \{2\gamma + 1, 2\gamma + 2\}$ and verify that it is still strictly positive.

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ATHANASIOS CHATZIKALEAS: achatzik@uni-muenster.de
Mathematical Institute, Westfälische Wilhelms-University of Münster, Münster, Germany

JACQUES SMULEVICI: jacques.smulevici@sorbonne-universite.fr
Laboratoire Jacques-Louis Lions, Sorbonne Université, Paris, France

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