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The well-known theorem of Shalom–Vaserstein and Ershov–Jaikin-Zapirain states that the group $\mathrm{EL}_n(\mathcal{R})$, generated by elementary matrices over a finitely generated commutative ring $\mathcal{R}$, has Kazhdan's property (T) as soon as $n \geq 3$. This is no longer true if the ring $\mathcal{R}$ is replaced by a commutative rng (a ring but without the identity) due to nilpotent quotients $\mathrm{EL}_n(\mathcal{R}/\mathcal{R}^k)$. We prove that even in such a case the group $\mathrm{EL}_n(\mathcal{R})$ satisfies a certain property that can substitute property (T), provided that n is large enough.

1. Introduction

We continue and extend the scope of the study of [Kaluba et al. 2019; 2021; Netzer and Thom 2015; Nitsche 2020; Ozawa 2016], which develops the way of proving Kazhdan's property (T) via sum of squares methods. See [Bekka et al. 2008] for a comprehensive treatment of property (T). Let $\Gamma = \langle S \rangle$ be a group together with a finite symmetric generating subset S. We denote by $\mathbb{R}[\Gamma]$ the real group algebra with the involution $*$ that extends the inverse $* : x \mapsto x^{-1}$ on Γ. The positive elements in $\mathbb{R}[\Gamma]$ are sums of (hermitian) squares,

$$\Sigma^2\mathbb{R}[\Gamma] := \left\{ \sum_i \xi_i^* \xi_i : \xi_i \in \mathbb{R}[\Gamma] \right\}$$

and the combinatorial Laplacian is

$$\Delta := \frac{1}{2} \sum_{s \in S} (1-s)^*(1-s) = |S| - \sum_{s \in S} s \in \Sigma^2\mathbb{R}[\Gamma].$$

It is proved in [Ozawa 2016] that the group Γ has property (T) if and only if there is $\varepsilon > 0$ that satisfies

$$\Delta^2 - \varepsilon \Delta \in \Sigma^2\mathbb{R}[\Gamma].$$

Property (T) for the so-called *universal lattice* $\mathrm{EL}_n(\mathbb{Z}[t_1, \dots, t_d])$, $n \geq 3$, is proved in [Shalom 2006; Vaserstein 2006; Ershov and Jaikin-Zapirain 2010]. See also [Mimura 2015] for a simpler proof and [Kassabov and Nikolov 2006; Kaluba et al. 2019] for partial results. All the proofs (save for [Kaluba et al. 2019]) rely on relative property (T) of certain semidirect products. Our interest in this paper is in the infinite index subgroup $\mathrm{EL}_n(\mathbb{Z}\langle t_1, \dots, t_d \rangle)$ of $\mathrm{EL}_n(\mathbb{Z}[t_1, \dots, t_d])$. Here $\mathcal{R} := \mathbb{Z}\langle t_1, \dots, t_d \rangle$ is the commutative *rng* (i.e., a ring, but without assuming the existence of the identity; $\mathcal{R}$ is an ideal in the unitization $\mathcal{R}^1$) of polynomials in $t_1, \dots, t_d$ with zero constant terms and $\mathrm{EL}_n(\mathcal{R}) \subset \mathrm{SL}_n(\mathcal{R}^1)$ denotes

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the group generated by the elementary matrices over the rng $\mathcal{R}$. The elementary matrices are those $e_{i,j}(r) \in \mathrm{SL}_n(\mathcal{R}^1)$ with 1's on the diagonal, $r \in \mathcal{R}$ in the (i, j)-th entry, and zeros everywhere else. The group $\mathrm{EL}_n(\mathcal{R})$ does not have property (T), because it has infinite nilpotent quotients $\mathrm{EL}_n(\mathcal{R}/\mathcal{R}^k)$. The group does not seem to admit a good analogue of relative property (T) phenomenon, either. Still, we prove via sum of squares methods that $\mathrm{EL}_n(\mathcal{R})$ satisfies a property that can substitute property (T).

Main Theorem. *Let $d \in \mathbb{N}$ and consider the commutative rng $\mathcal{R} := \mathbb{Z}\langle t_1, \dots, t_d\rangle$. Then there are $n_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that, for every $n \geq n_0$, the combinatorial Laplacians*

$$\Delta := \sum_{i \neq j} \sum_{r=1}^{d} (1 - e_{i,j}(t_r))^*(1 - e_{i,j}(t_r))$$

for $\mathrm{EL}_n(\mathcal{R})$ and

$$\Delta^{(2)} := \sum_{i \neq j} \sum_{r,s=1}^{d} (1 - e_{i,j}(t_r t_s))^*(1 - e_{i,j}(t_r t_s))$$

for $\mathrm{EL}_n(\mathcal{R}^2)$ satisfy

$$\Delta^2 - n\varepsilon\Delta^{(2)} \in \overline{\Sigma^2\mathbb{R}[\mathrm{EL}_n(\mathcal{R})]}.$$

Here $\overline{\Sigma^2\mathbb{R}[\Gamma]}$ denotes the archimedean closure of $\Sigma^2\mathbb{R}[\Gamma]$ (see Section 2). An upper bound for n_0 in the Main Theorem is in principle explicitly calculable, but we do not attempt to do that (nor attempt to optimize the proof for a better estimate). We conjecture[1] that the Main Theorem holds true with $n_0 = 3$ (in particular n_0 should not depend on d). Our proof is inspired by the work of Kaluba, Kielak and Nowak [Kaluba et al. 2021] that proves property (T) for $\mathrm{Aut}(F_d)$ for $d \geq 5$ via computer calculations and an ingenious idea on stability. Our proof does not rely on computers, but instead on analysis by Boca and Zaharescu [2005] on the almost Mathieu operators in the rotation C*-algebras. In fact, there is no known method of rigorously proving a result like the Main Theorem by computers. This is because the conclusion is *analytic* in nature—the archimedean closure is indispensable. See discussions in Section 6.

The above theorem has a couple of corollaries. The first one is reminiscent of one of the standard definitions of property (T) (see Definition 1.1.3 in [Bekka et al. 2008]).

Corollary A. *For every d, if n is large enough, then for every $\kappa > 0$ there is $\delta > 0$ satisfying the following property. For every orthogonal representation π of $\mathrm{EL}_n(\mathbb{Z}\langle t_1, \dots, t_d\rangle)$ on a Hilbert space $\mathcal{H}$ and every unit vector $v \in \mathcal{H}$ with $\max_{i,j,r} \|v - \pi(e_{i,j}(t_r))v\| \leq \delta$, there is a vector $w \in \mathcal{H}$ such that $\|v - w\| \leq \kappa$ and*

$$\lim_{l\to\infty} \max_{i,j,r} \|w - \pi(e_{i,j}(t_r^l))w\| = 0.$$

We remark that a certain strengthening of the above corollary does not hold. Namely, there is an orthogonal representation π of $\mathrm{EL}_n(\mathbb{Z}\langle t_1, \dots, t_d\rangle)$ that simultaneously admits asymptotically invariant vectors v_k and a sequence $x_l \in \mathrm{EL}_n(\mathbb{Z}\langle t_1^l, \dots, t_d^l\rangle)$ with $\pi(x_l) \to 0$ in the weak operator topology.

Corollary B. *For every d, if n is large enough, then the group $\mathrm{EL}_n(\mathbb{Z}\langle t_1, \dots, t_d\rangle)$ has property (τ) with respect to the finite quotients of the form $\mathrm{EL}_n(\mathcal{S})$, where $\mathcal{S}$ is a finite unital quotients of $\mathbb{Z}\langle t_1, \dots, t_d\rangle$.*

[1] NB: As the author is lame at the computer, no computer experiments have been carried out.

Property (τ) is a generalization of property (T) for finite quotients. See Section 7 for the definition and the proofs of the above corollaries. Corollary B says $\{\mathrm{EL}_n(\mathcal{S}) : \mathcal{S}\}$ forms an expander family with respect to elementary generating subsets of fixed size. The novel point compared to the previously known case of the universal lattice [Kassabov and Nikolov 2006] is that the generating subsets of the finite commutative rings $\mathcal{S}$ need not contain the unit although the $\mathcal{S}$ are assumed unital. For example, for n large enough, the Cayley graphs of $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$ with respect to the generating subsets $\{e_{i,j}(p) : i \neq j\}$ form an expander family as relatively prime pairs (p, q) vary. The study of the expander property for $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$ and alike is a very active area. See [Breuillard and Lubotzky 2022; Helfgott 2019; Kowalski 2019] for recent surveys on this.

2. Preliminaries

Let $\Gamma = \langle S \rangle$ be a group together with a finite symmetric generating subset S. We denote by $\mathbb{R}[\Gamma]$ the real group algebra with the involution $*$ which is the linear extension of $x^* := x^{-1}$ on Γ. The identity element of Γ as well as $\mathbb{R}[\Gamma]$ is simply denoted by 1. Recall the positive cone of *sums of* (*hermitian*) *squares* is given by

$$\Sigma^2\mathbb{R}[\Gamma] := \left\{ \sum_i \xi_i^* \xi_i : \xi_i \in \mathbb{R}[\Gamma] \right\} \subset \mathbb{R}[\Gamma]^{\mathrm{her}} := \{\xi \in \mathbb{R}[\Gamma] : \xi = \xi^*\}.$$

The elements in $\Sigma^2\mathbb{R}[\Gamma]$ are considered positive. For $\xi, \eta \in \mathbb{R}[\Gamma]^{\mathrm{her}}$, we write $\xi \preceq \eta$ if $\eta - \xi \in \Sigma^2\mathbb{R}[\Gamma]$. It is obvious that $\xi \succeq 0$ implies $\xi \geq 0$ in the full group C*-algebra $\mathrm{C}^*[\Gamma]$, that is to say, $\pi(\xi)$ is positive selfadjoint for every orthogonal (or unitary) representation π of Γ on a real (or complex) Hilbert space $\mathcal{H}$. The converse is true up to the *archimedean closure*:

$$\overline{\Sigma^2\mathbb{R}[\Gamma]} := \{\xi \in \mathbb{R}[\Gamma] : \text{for all } \varepsilon > 0\ \xi + \varepsilon \cdot 1 \succeq 0\} = \{\xi \in \mathbb{R}[\Gamma] : \xi \geq 0 \text{ in } \mathrm{C}^*[\Gamma]\}.$$

See, e.g., [Cimprič 2009; Ozawa 2013; Schmüdgen 2009] for this. On this occasion, we recall the basic fact that $0 \preceq \xi \preceq \eta$ (or $0 \leq \xi \leq \eta$) need not imply $0 \leq \xi^2 \leq \eta^2$. Note that since any orthogonal representation of Γ dilates to an orthogonal representation of any supergroup $\Gamma_1 \geq \Gamma$ by induction (i.e., $\mathrm{C}^*[\Gamma] \subset \mathrm{C}^*[\Gamma_1]$ in short), whether $\xi \geq 0$ or not does not depend on the ambient group. The same holds true for $\xi \succeq 0$, by the coset decomposition. The *combinatorial Laplacian*, with respect to the (symmetric) generating subset S,

$$\Delta := \frac{1}{2} \sum_{s \in S} (1 - s)^*(1 - s) = |S| - \sum_{s \in S} s$$

satisfies, for every orthogonal representation $(\pi, \mathcal{H})$ and a vector $v \in \mathcal{H}$,

$$\langle \pi(\Delta)v, v \rangle = \frac{1}{2} \sum_{s \in S} \|v - \pi(s)v\|^2.$$

3. Proof of the Main Theorem, prelude

For any rng $\mathcal{R}$, we denote by $\mathrm{EL}_n(\mathcal{R}) \subset \mathrm{SL}_n(\mathcal{R}^1)$ the group generated by the elementary matrices over the rng $\mathcal{R}$. The elementary matrices are those $e_{i,j}(r) \in \mathrm{SL}_n(\mathcal{R}^1)$ with 1's on the diagonal, $r \in \mathcal{R}$ in the

(i,j)-th entry $(i \neq j)$, and zeros everywhere else. They satisfy the Steinberg relations:

- $e_{i,j}(r)e_{i,j}(s) = e_{i,j}(r+s)$.
- $[e_{i,j}(r), e_{j,k}(s)] = e_{i,k}(rs)$ if $i \neq k$.
- $[e_{i,j}(r), e_{k,l}(s)] = 1$ if $i \neq l$ and $j \neq k$.

We note that every rng homomorphism $\mathcal{R} \to \mathcal{S}$ induces by entrywise operation a group homomorphism $\mathrm{EL}_n(\mathcal{R}) \to \mathrm{EL}_n(\mathcal{S})$ and that $\mathrm{EL}_n(\mathcal{R}/\mathcal{R}^k)$ is nilpotent for every k, where $\mathcal{R}^k := \mathrm{span}\{r_1 \cdots r_k : r_i \in \mathcal{R}\}$. To ease notation, we will write

$$E_{i,j}(r) := (1 - e_{i,j}(r))^*(1 - e_{i,j}(r)) = 2 - e_{i,j}(r) - e_{i,j}(r)^* \in \mathbb{R}[\mathrm{EL}_n(\mathcal{R})].$$

We now consider the case $\mathcal{R} = \mathbb{Z}\langle t_1, \ldots, t_d\rangle$ and start proving the Main Theorem. Recall that the combinatorial Laplacians with respect to the generating subset $\{e_{i,j}(\pm t_r)\}$ are given by

$$\Delta_n := \sum_{i \neq j} \sum_{r=1}^{d} E_{i,j}(t_r) \quad \text{and} \quad \Delta_n^{(2)} := \sum_{i \neq j} \sum_{r,s=1}^{d} E_{i,j}(t_r t_s).$$

We follow the idea of [Kaluba et al. 2021] about the stability with respect to n of the relation like $\Delta_n^{(2)} \ll \Delta_n^2$. Here $\xi \ll \eta$ means that $\xi \leq R\eta$ for some $R > 0$ in the full group C*-algebra. For each n, put $\mathrm{E}_n := \{\{i,j\} : 1 \leq i, j \leq n,\ i \neq j\}$ and, for $\mathrm{e}, \mathrm{f} \in \mathrm{E}_n$, write $\mathrm{e} \sim \mathrm{f}$ if $|\mathrm{e} \cap \mathrm{f}| = 1$ and $\mathrm{e} \perp \mathrm{f}$ if $\mathrm{e} \cap \mathrm{f} = \varnothing$. One has

$$\Delta_n = \sum_{\mathrm{e} \in \mathrm{E}_n} \Delta_{\mathrm{e}},$$

where $\Delta_{\{i,j\}} := \sum_{r=1}^{d} E_{i,j}(t_r) + E_{j,i}(t_r)$. Thus

$$\Delta_n^2 = \sum_{\mathrm{e}} \Delta_{\mathrm{e}}^2 + \sum_{\mathrm{e} \sim \mathrm{f}} \Delta_{\mathrm{e}} \Delta_{\mathrm{f}} + \sum_{\mathrm{e} \perp \mathrm{f}} \Delta_{\mathrm{e}} \Delta_{\mathrm{f}} =: \mathrm{Sq}_n + \mathrm{Adj}_n + \mathrm{Op}_n\,.$$

The elements Sq_n and Op_n are positive, while Adj_n is not and this causes trouble.

For $m < n$, we view $\mathrm{EL}_m(\mathcal{R})$ as a subgroup of $\mathrm{EL}_n(\mathcal{R})$ sitting at the left upper corner. The symmetric group $\mathrm{Sym}(n)$ acts on $\mathrm{EL}_n(\mathcal{R})$ by permutation of the indices. We note that

$$|\mathrm{E}_m| = \tfrac{1}{2} m(m-1),$$
$$|\{(\mathrm{e},\mathrm{f}) \in \mathrm{E}_m^2 : \mathrm{e} \sim \mathrm{f}\}| = m(m-1)(m-2),$$
$$|\{(\mathrm{e},\mathrm{f}) \in \mathrm{E}_m^2 : \mathrm{e} \perp \mathrm{f}\}| = \tfrac{1}{4} m(m-1)(m-2)(m-3).$$

Hence, as it is proved in [Kaluba et al. 2021], one has

$$\sum_{\sigma \in \mathrm{Sym}(n)} \sigma(\Delta_m^{(2)}) = m(m-1) \cdot (n-2)! \cdot \Delta_n^{(2)},$$
$$\sum_{\sigma \in \mathrm{Sym}(n)} \sigma(\mathrm{Adj}_m) = m(m-1)(m-2) \cdot (n-3)! \cdot \mathrm{Adj}_n,$$
$$\sum_{\sigma \in \mathrm{Sym}(n)} \sigma(\mathrm{Op}_m) = m(m-1)(m-2)(m-3) \cdot (n-4)! \cdot \mathrm{Op}_n\,.$$

Thus if we know there are $m \in \mathbb{N}$, $R > 0$, and $\varepsilon > 0$ such that

$$\mathrm{Adj}_m + R\,\mathrm{Op}_m \geq \varepsilon \Delta_m^{(2)} \tag{$\heartsuit$}$$

holds true in $\mathrm{C}^*[\mathrm{EL}_m(\mathcal{R})]$, then it follows

$$\frac{n-2}{m-2}\varepsilon \Delta_n^{(2)} \leq \mathrm{Adj}_n + \frac{m-3}{n-3} R\,\mathrm{Op}_n \leq \Delta_n^2$$

for all n such that $R(m-3)/(n-3) \leq 1$ and the Main Theorem is proved. This is Proposition 4.1 in [Kaluba et al. 2021]. To apply this machinery, we further expand Adj_m:

$$\begin{aligned}\mathrm{Adj}_m &= \sum_{r,s} \sum_{i,j,k \text{ distinct}} (E_{i,j}(t_r) + E_{j,i}(t_r))(E_{j,k}(t_s) + E_{k,j}(t_s)) \\ &= \sum_{r,s} \sum_{i,j,k \text{ distinct}} (E_{i,j}(t_r)E_{j,k}(t_s) + E_{j,k}(t_s)E_{i,j}(t_r) + E_{i,j}(t_r)E_{i,k}(t_s) + E_{j,k}(t_s)E_{i,k}(t_r)).\end{aligned}$$

Therefore, if there are $m \in \mathbb{N}$, $R > 0$, $\varepsilon > 0$, and distinct indices i, j, k, l such that

$$E_{i,j}(t_r)E_{j,k}(t_s) + E_{j,k}(t_s)E_{i,j}(t_r) + E_{i,j}(t_r)E_{i,l}(t_s) + E_{j,k}(t_s)E_{l,k}(t_r) + R\,\mathrm{Op}_m \geq \varepsilon E_{i,k}(t_r t_s) \tag{$\diamondsuit$}$$

holds true, then we obtain ($\heartsuit$) (for different $R > 0$ and $\varepsilon > 0$) by summing up this over the $\mathrm{Sym}(m)$-orbit and over r, s. This is what we will prove in the next section.

4. The Heisenberg group and the rotation C*-algebras

In this section, we will work entirely in the C*-algebra setting. Let's consider the *integral Heisenberg group*

$$\boldsymbol{H} := \left\{ \begin{bmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{bmatrix} : a, b, c \in \mathbb{Z} \right\} \cong \langle x, y : z := [x, y] \text{ is central} \rangle,$$

where

$$x = \begin{bmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix}, \quad z = \begin{bmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{bmatrix}.$$

Note that every irreducible unitary representation of $\boldsymbol{H}$ sends the central element z to a scalar (multiplication operator) of modulus 1. For $\theta \in [0, 1)$, we consider the irreducible unitary representation π_θ of $\boldsymbol{H}$ on $\ell_2(\mathbb{Z})$ or $\ell_2(\mathbb{Z}/q\mathbb{Z})$, depending on whether θ irrational or $\theta = p/q$ is rational with $\gcd(p, q) = 1$, given by

$$\pi_\theta(x)\delta_j = \exp(2j\pi\imath\theta)\delta_j, \quad \pi_\theta(y)\delta_j = \delta_{j+1}, \quad \pi_\theta(z) = \exp(2\pi\imath\theta).$$

By convention, if $\theta = p/q$ is rational, then $\gcd(p, q) = 1$ is assumed, and if θ is irrational, we consider $q = \infty$, and $\mathbb{Z}/q\mathbb{Z}$ means $\mathbb{Z}$. Thus in either case π_θ is a representation on $\ell_2(\mathbb{Z}/q\mathbb{Z})$. The C*-algebra $\mathcal{A}_\theta := \pi_\theta(\mathrm{C}^*[\boldsymbol{H}])$ is called the *rotation* C*-*algebra*.

We fix the notation used throughout this section. We define

$$X := (1-x)^*(1-x) = 2 - x - x^* \in \mathrm{C}^*[\boldsymbol{H}]_+, \quad X_\theta := \pi_\theta(X) \in \mathcal{A}_\theta,$$

and the same for y and z. Note that $X+Y$ is the combinatorial Laplacian of $\boldsymbol{H}$ with respect to the generating subset $\{x^{\pm}, y^{\pm}\}$, that $0 \le X \le 4$, and that the triplets $(X_\theta, Y_\theta, Z_\theta)$, $(Y_\theta, X_\theta, Z_\theta)$, and $(X_{1-\theta}, Y_{1-\theta}, Z_{1-\theta})$ are unitarily equivalent. For a parameter $\lambda > 0$, the *almost Mathieu operator* on $\ell_2(\mathbb{Z}/q\mathbb{Z})$ is given by

$$H_{\theta,\lambda} := \pi_\theta\Big(\frac{\lambda}{2}(x+x^*)+y+y^*\Big) = (\lambda+2) - \Big(\frac{\lambda}{2}X_\theta + Y_\theta\Big).$$

We also write $s = \sin\pi\theta$, $s_m = \sin 2m\pi\theta$, and $c_m = \cos 2m\pi\theta$. In particular,

$$Z_\theta = 2(1-\cos 2\pi\theta) = 4s^2.$$

See [Boca 2001] for more information about the almost Mathieu operators and [Nitsche 2020] for some discussion in connection with the semidefinite programming.

Eventually, we will prove a certain inequality (Theorem 9) about X, Y, and Z (in the full group C*-algebra of a higher-dimensional Heisenberg group) that leads to (◇) in the previous section. To prove inequalities about X, Y, and Z, it suffices to work with X_θ, Y_θ, and Z_θ for each $\theta \in \big[0, \frac{1}{2}\big]$ separately, thanks to the following well-known fact (Lemma 1). The critical estimate is the one for small $\theta > 0$ (Corollary 4 and Lemma 6). The rest will work out anyway.

Lemma 1. *For any dense subset $I \subset [0, 1)$, the representation $\bigoplus_{\theta\in I}\pi_\theta$ is faithful on the full group* C*-*algebra* $\mathrm{C}^*[\boldsymbol{H}]$.

Proof. For the readers' convenience, we sketch the proof. Let τ_θ denote the tracial state on $\mathrm{C}^*[\boldsymbol{H}]$ associated with π_θ. That is to say, if θ is irrational, then τ_θ arises from the canonical tracial state on the irrational rotation C*-algebra $\mathcal{A}_\theta$ and it is given by $\tau_\theta(x^i y^j) = 0$ for all $(i, j) \neq (0, 0)$. If $\theta = p/q$ is rational, then τ_θ is given by $\mathrm{tr}_q \circ \pi_\theta$, where tr_q is the tracial state on $\mathbb{M}_q(\mathbb{C})$, and it satisfies $\tau_\theta(x^i y^j) = 0$ for all $(i, j) \neq (0, 0)$ in $(\mathbb{Z}/q\mathbb{Z})^2$. It follows that $\theta \mapsto \tau_\theta$ is continuous at irrational points and the assumption of the lemma implies that $\tau := \int_0^1 \tau_\theta\, d\theta$ is a continuous state on $\bigoplus_{\theta\in I}\pi_\theta$. It is not hard to see that τ coincides with the tracial state associated with the left regular representation of $\boldsymbol{H}$, that is to say, $\tau(x^i y^j z^k) = 0$ for all $(i, j, k) \neq (0, 0, 0)$. Since $\boldsymbol{H}$ is amenable, the tracial state τ is faithful on the full group C*-algebra $\mathrm{C}^*[\boldsymbol{H}]$. □

Theorem 2 [Boca and Zaharescu 2005]. *Let $\theta \in \big[0, \frac{1}{2}\big)$. One has*

$$\|H_{\theta,\lambda}\| \le \lambda + 2 - \frac{2\lambda}{\lambda+2}\sin\pi\theta.$$

More precisely, for any real unit vector ξ in $\ell_2(\mathbb{Z}/q\mathbb{Z})$,

$$\|H_{\lambda,\theta}\xi\|^2 = \lambda^2 + 4 + 2(1-\tan\pi\theta)\Big\langle\frac{\lambda}{2}\pi_\theta(x+x^*)\xi, \pi_\theta(y+y^*)\xi\Big\rangle - \sum_m |\xi_{m-1} - \xi_{m+1} - \lambda s_m\xi_m|^2.$$

Proof. Because the statements are formulated in a different way in [Boca and Zaharescu 2005], we replicate here the proof from that work:

$$\begin{aligned}\|H_{\lambda,\theta}\xi\|^2 &= \sum_m |\lambda c_m\xi_m + \xi_{m-1} + \xi_{m+1}|^2 \\ &= \lambda^2 + 4 + \sum_m \big(-\lambda^2 s_m^2\xi_m^2 - |\xi_{m-1} - \xi_{m+1}|^2 + 2\lambda c_m\xi_m(\xi_{m-1}+\xi_{m+1})\big) \\ &= \lambda^2 + 4 - \sum_m |\xi_{m-1} - \xi_{m+1} - \lambda s_m\xi_m|^2 - 2\lambda\sum_m s_m(\xi_{m-1}-\xi_{m+1})\xi_m + 2\lambda\sum_m c_m\xi_m(\xi_{m-1}+\xi_{m+1}).\end{aligned}$$

We continue with the computation,

$$\sum_m c_m \xi_m (\xi_{m-1} + \xi_{m+1}) = \sum_m (c_{m-1} + c_m) \xi_{m-1} \xi_m = 2 \cos \pi\theta \sum_m \xi_{m-1} \xi_m \cos(2m-1)\pi\theta$$

and similarly

$$\begin{aligned} -\sum_m s_m (\xi_{m-1} - \xi_{m+1}) \xi_m &= \sum_m (s_{m-1} - s_m) \xi_{m-1} \xi_m \\ &= -2 \sin \pi\theta \sum_m \xi_{m-1} \xi_m \cos(2m-1)\pi\theta \\ &= -\tan\theta \sum_m c_m \xi_m (\xi_{m-1} + \xi_{m+1}). \end{aligned}$$

Thus one obtains the purported formula for $\|H_{\lambda,\theta}\xi\|^2$. We also observe that

$$\begin{aligned} \|H_{\lambda,\theta}\xi\|^2 &\le \lambda^2 + 4 + 4\lambda(\cos\pi\theta - \sin\pi\theta) \sum_m \xi_{m-1}\xi_m \cos(2m-1)\pi\theta \\ &\le \lambda^2 + 4 + 4\lambda(1 - \sin\pi\theta). \end{aligned}$$

This yields the purported estimate for $\|H_{\theta,\lambda}\|$. □

Corollary 3. *In the full group* C^**-algebra* $\mathrm{C}^*[\boldsymbol{H}]$, *one has*

$$X + Y \ge \tfrac{1}{2}\sqrt{Z}.$$

Proof. By Lemma 1, it suffices to show the assertion in $\mathcal{A}_\theta$ for each $\theta \in \left[0, \frac{1}{2}\right]$. It follows from Theorem 2 with $\lambda = 2$ that $X_\theta + Y_\theta = 4 - H_{\theta,2} \ge \frac{1}{2}\sqrt{Z_\theta}$. □

Since Z is central, $X + Y \ge \frac{1}{2}\sqrt{Z}$ is equivalent to $4(X+Y)^2 \ge Z$ in $\mathrm{C}^*[\boldsymbol{H}]$. However, there is no $R > 0$ such that $R(X+Y)^2 \succeq Z$ in $\mathbb{R}[\boldsymbol{H}]$. We will elaborate this in Section 6.

Corollary 4. *Let* $R \ge 1$, $0 < \kappa < 1$, *and*

$$\theta_0 := \min\left\{\frac{1}{4}, \frac{1}{\pi}\arcsin\left(\kappa\sqrt{\frac{1-\kappa}{R}}\right)\right\}.$$

Then, *for any* $\theta \in [0, \theta_0]$, *one has*

$$RX_\theta + Y_\theta \ge \frac{\sqrt{(1-\kappa)R}}{2}\sqrt{Z_\theta}.$$

Proof. We write

$$s_0 := \sin\pi\theta_0, \quad c := \operatorname{diag}_m c_m = \pi_\theta\left(\frac{x + x^*}{2}\right) = 1 - \frac{1}{2}X_\theta, \quad C = \sqrt{(1-\kappa)R}.$$

Let $\theta \in [0, \theta_0]$ and a real unit vector $\xi \in \ell_2(\mathbb{Z}/q\mathbb{Z})$ be given. We need to prove $\langle (RX_\theta + Y_\theta)\xi, \xi\rangle \ge Cs$. For this, we may assume that $\langle \pi_\theta(x + x^*)\xi, \pi_\theta(y + y^*)\xi\rangle > 0$ because otherwise

$$\langle (X_\theta + Y_\theta)\xi, \xi\rangle \ge 4 - \|\pi_\theta(x + x^* + y + y^*)\xi\| \ge 4 - 2\sqrt{2}.$$

Put $\varepsilon := 1 - \|c\xi\|$. If $\varepsilon \geq Cs/(2R)$, then $\langle RX_\theta\xi, \xi\rangle \geq 2R\varepsilon \geq Cs$ and we are done. From now on, we assume that $\varepsilon < Cs/(2R)$. By Theorem 2 for $\lambda := 2R/C$, one has

$$\begin{aligned}\|H_{\lambda,\theta}\xi\|^2 &\leq \lambda^2 + 4 + 2\lambda(1-s)\langle c\xi, (H_{\theta,\lambda} - \lambda c)\xi\rangle \\ &\leq \lambda^2 + 4 + 2\lambda(1-s)(1-\varepsilon)\|H_{\theta,\lambda}\xi\| - 2\lambda^2(1-s)(1-\varepsilon)^2\end{aligned}$$

and hence

$$\begin{aligned}\big(\|H_{\lambda,\theta}\xi\| - \lambda(1-s)(1-\varepsilon)\big)^2 &\leq 4 + \lambda^2\big(1 - 2(1-s)(1-\varepsilon)^2 + (1-s)^2(1-\varepsilon)^2\big) \\ &= 4 + \lambda^2(1 - (1-s^2)(1-\varepsilon)^2) \\ &\leq 4 + \lambda^2(s^2 + 2\varepsilon).\end{aligned}$$

Thus

$$\|H_{\lambda,\theta}\xi\| \leq 2 + \lambda s\Big(\frac{1}{4}\lambda s_0 + \frac{1}{2}\lambda\frac{\varepsilon}{s}\Big) + \lambda(1-s).$$

By our choices,

$$\lambda s_0 = \frac{2R}{C}\cdot\frac{\kappa\sqrt{(1-\kappa)}}{\sqrt{R}} = 2\kappa$$

and $\lambda\varepsilon/s \leq 1$. Therefore,

$$\|H_{\lambda,\theta}\xi\| \leq \lambda + 2 - \Big(1 - \frac{1}{4}\cdot 2\kappa - \frac{1}{2}\Big)\cdot 2\sqrt{\frac{R}{1-\kappa}}s = \lambda + 2 - Cs.$$

Since $\lambda + 2 - H_{\lambda,\theta} = (\lambda/2)X_\theta + Y_\theta \leq RX_\theta + Y_\theta$, we are done. □

Proposition 5. *In the full group* C*-*algebra* $\mathrm{C}^*[\boldsymbol{H}]$, *one has*

$$(X+Y)\sqrt{Z} + \tfrac{1}{2}(XY+YX) \geq 0.$$

Proof. By Lemma 1, it suffices to show the same for the X_θ. We write $b_m := 1 - c_m = 1 - \cos 2m\pi\theta = 2\sin^2 m\pi\theta$. We observe that

$$X_\theta = \begin{bmatrix}\ddots & & & \\ & 2b_{m-1} & & \\ & & 2b_m & \\ & & & \ddots\end{bmatrix}, \quad Y_\theta = \begin{bmatrix}\ddots & & & \\ & 2 & -1 & \\ & -1 & 2 & \\ & & & \ddots\end{bmatrix},$$

$$\tfrac{1}{2}(X_\theta Y_\theta + Y_\theta X_\theta) = \begin{bmatrix}\ddots & & & \\ & 4b_{m-1} & -(b_{m-1}+b_m) & \\ & -(b_{m-1}+b_m) & 4b_m & \\ & & & \ddots\end{bmatrix}.$$

These are the sums of the following 2-by-2 matrices sitting at the $(m-1)$-to-m-th corners:

$$X_{\theta,m} = \begin{bmatrix}b_{m-1} & \\ & b_m\end{bmatrix}, \quad Y_{\theta,m} = \begin{bmatrix}1 & -1 \\ -1 & 1\end{bmatrix},$$

$$\tfrac{1}{2}(XY+YX)_{\theta,m} := \begin{bmatrix}2b_{m-1} & -(b_{m-1}+b_m) \\ -(b_{m-1}+b_m) & 2b_m\end{bmatrix}.$$

Thus, it suffices to show

$$\begin{aligned}T_{\theta,m} &:= 2s(X_{\theta,m}+Y_{\theta,m})+\tfrac{1}{2}(XY+YX)_{\theta,m}\\ &= \begin{bmatrix} 2(s+1)b_{m-1}+2s & -(2s+b_{m-1}+b_m)\\ -(2s+b_{m-1}+b_m) & 2(s+1)b_m+2s\end{bmatrix}\end{aligned}$$

is positive in $\mathbb{M}_2(\mathbb{C})$ for every m. We only need to calculate the determinant:

$$\begin{aligned}\det(T_{\theta,m}) &\geq 4b_{m-1}b_m+4s(s+1)(b_{m-1}+b_m)+4s^2-(2s+b_{m-1}+b_m)^2\\ &= 4s^2(b_{m-1}+b_m)-(b_{m-1}-b_m)^2\\ &= 8s^2(\sin^2(m-1)\pi\theta+\sin^2 m\pi\theta)-4s^2\sin^2(2m-1)\pi\theta\\ &\geq 0.\end{aligned}$$

Here, we have used the formulas

$$\begin{aligned}b_m &= 2\sin^2 m\pi\theta,\\ b_{m-1}-b_m &= -2s\theta\sin(2m-1)\pi\theta,\\ |\sin(2m-1)\pi\theta| &\leq |\sin(m-1)\pi\theta|+|\sin m\pi\theta|.\end{aligned}$$

□

A similar calculation shows $Z+\frac{1}{2}(XY+YX)\geq 0$ in $C^*[\boldsymbol{H}]$. In fact, it is a sum of squares:

$$Z+\frac{1}{2}(XY+YX)=\frac{1}{4}(X+Y)Z+\frac{1}{8}\sum(1-b)^{\delta}(1-a)^{\varepsilon}(1-a)^{\bar{\varepsilon}}(1-b)^{\bar{\delta}},$$

where $\sum$ is over the eight terms $(a,b)\in\{(x,y),(y,x)\}$ and $(\varepsilon,\bar{\varepsilon}),(\delta,\bar{\delta})\in\{(*,\cdot),(\cdot,*)\}$.

Now, we consider the C*-algebra $\mathcal{A}_\theta\otimes\mathcal{A}_\theta$ on $\ell_2(\mathbb{Z}/q\mathbb{Z})\otimes\ell_2(\mathbb{Z}/q\mathbb{Z})$. We continue to view Z_θ as a scalar in $\mathcal{A}_\theta\otimes\mathcal{A}_\theta$. We want to find an inequality that leads to (◇). The following does the job for small $\theta>0$. We note that it fails at $\theta_0=\frac{1}{2}$.

Lemma 6. *There are $\theta_0>0$, $R>1$, and $\varepsilon>0$ such that, for every $\theta\in[0,\theta_0]$, one has*

$$R(X_\theta\otimes Y_\theta+Y_\theta\otimes X_\theta)+X_\theta\otimes X_\theta+Y_\theta\otimes Y_\theta+(X_\theta Y_\theta+Y_\theta X_\theta)\otimes 1\geq\varepsilon Z_\theta.$$

Proof. By Corollary 4, there are $\theta_0>0$ and $R>1$ such that $1\otimes(RX_\theta+Y_\theta)\geq 8s$ for every $\theta\in[0,\theta_0]$. By Proposition 5 and Corollary 3, it follows that the left-hand side dominates

$$(X_\theta+Y_\theta)\cdot 8s+X_\theta Y_\theta+Y_\theta X_\theta\geq(X_\theta+Y_\theta)\cdot 4s\geq Z_\theta,$$

where we omitted writing $\otimes 1$. □

To deal with the case $\theta\geq\theta_0$, we need a few more auxiliary lemmas on $\mathcal{A}_\theta$.

Lemma 7. *For every $\theta\in\left[0,\frac{1}{2}\right]$, one has*

$$\|\pi_\theta((1-x)(1-y))\|\leq 4\cos(\pi\theta/2).$$

Proof. The expansion of $(1-y)^*(1-x)^*(1-x)(1-y)$ has 16 terms (counting multiplicity) and among them are $-(1+z)x,\ -(1+z)^*x^*$ and $x(zy^*+y)+x^*(z^*y^*+y)$. One has $|1+z|=2\cos\pi\theta$ and

$$\|x(zy^*+y)+x^*(z^*y^*+y)\| \le \|[xy^*\ \ x^*y^*]\| \left\| \begin{bmatrix} z+y^2 \\ z^*+y^2 \end{bmatrix} \right\|$$
$$\le \sqrt{2}\|(z+y^2)^*(z+y^2)+(z^*+y^2)^*(z^*+y^2)\|^{1/2} = 4\cos\pi\theta.$$

Hence $\|\pi_\theta((1-y)^*(1-x)^*(1-x)(1-y))\| \le 8+8\cos\pi\theta = 16\cos^2(\pi\theta/2)$. □

For a positive operator A, we denote by $\mathbb{P}_{A\le\delta}$ (resp. $\mathbb{P}_{A>\delta}=1-\mathbb{P}_{A\le\delta}$) the spectral projection of A corresponding to the spectrum $[0,\delta]$ (resp. (δ,∞)). We also write $\mathbb{P}_{A\le\delta\,\wedge\,B\le\delta}$ etc. for the orthogonal projection onto $\operatorname{ran}\mathbb{P}_{A\le\delta}\cap\operatorname{ran}\mathbb{P}_{B\le\delta}$ etc. Note that if A and B commute, then so do their spectral projections and $\mathbb{P}_{A\le\delta\,\wedge\,B\le\delta}=\mathbb{P}_{A\le\delta}\mathbb{P}_{B\le\delta}$.

Lemma 8. *For every $\theta\in\left(0,\frac{1}{2}\right]$ and $0<\delta<2(1-\cos\pi\theta)$, one has*

$$\mathbb{P}_{X_\theta\le\delta}Y_\theta\mathbb{P}_{X_\theta\le\delta}=2\mathbb{P}_{X_\theta\le\delta},$$

the same with X_θ and Y_θ interchanged, and

$$\|\mathbb{P}_{Y_\theta\le\delta}\mathbb{P}_{X_\theta\le\delta}\|\le\sqrt{\frac{2}{4-\delta}}.$$

In particular, $\ell_2(\mathbb{Z}/q\mathbb{Z})$ is decomposed into a direct sum

$$\ell_2(\mathbb{Z}/q\mathbb{Z})=\operatorname{ran}\mathbb{P}_{X_\theta\le\delta}+\operatorname{ran}\mathbb{P}_{Y_\theta\le\delta}+\operatorname{ran}\mathbb{P}_{X_\theta>\delta\,\wedge\,Y_\theta>\delta}$$

and the corresponding (not necessarily orthogonal) projections have norm at most $\sqrt{(4-\delta)/(2-\delta)}$.

Proof. We observe that $\mathbb{P}_{X_\theta\le\delta}$ is the projection onto $\ell_2(E)$ with

$$E:=\{m:2(1-\cos 2m\pi\theta)\le\delta\}\subset\{m:m\theta\in(-\theta/2,\theta/2)+\mathbb{Z}\}.$$

The set E does not contain consecutive numbers and the first assertion follows. The second follows from the unitary equivalence of the pairs (X_θ,Y_θ) and (Y_θ,X_θ). Since $Y_\theta\le\delta\mathbb{P}_{Y_\theta\le\delta}+4(1-\mathbb{P}_{Y_\theta\le\delta})=4-(4-\delta)\mathbb{P}_{Y_\theta\le\delta}$, one has

$$2\mathbb{P}_{X_\theta\le\delta}\le 4\mathbb{P}_{X_\theta\le\delta}-(4-\delta)\mathbb{P}_{X_\theta\le\delta}\mathbb{P}_{Y_\theta\le\delta}\mathbb{P}_{X_\theta\le\delta}$$

and $\|\mathbb{P}_{Y_\theta\le\delta}\mathbb{P}_{X_\theta\le\delta}\|^2=\|\mathbb{P}_{X_\theta\le\delta}\mathbb{P}_{Y_\theta\le\delta}\mathbb{P}_{X_\theta\le\delta}\|\le 2/(4-\delta)$. This gives the desired estimate for $\|\mathbb{P}_{Y_\theta\le\delta}\mathbb{P}_{X_\theta\le\delta}\|$. We remark that this estimate can be improved to $\approx 1/\sqrt{3}$ if θ is away from $\frac{1}{2}$ and $\delta>0$ is small enough. Indeed, the gaps of E will have length at least 2 and hence any unit vectors $\xi\in\operatorname{ran}\mathbb{P}_{X_\theta\le\delta}$ and $\eta\in\mathbb{P}_{Y_\theta\le\delta}$ satisfy

$$|\langle\xi,\eta\rangle|\approx\left|\left\langle\xi,\tfrac{1}{3}\pi_\theta(1+y+y^*)\eta\right\rangle\right|=\left|\left\langle\tfrac{1}{3}\pi_\theta(1+y+y^*)\xi,\eta\right\rangle\right|\le\tfrac{1}{\sqrt{3}}.$$

The projection onto the third subspace is orthogonal. On the other hand, any $\xi+\eta\in\operatorname{ran}\mathbb{P}_{X_\theta\le\delta}+\operatorname{ran}\mathbb{P}_{Y_\theta\le\delta}$ satisfies

$$\|\xi+\eta\|^2\ge\|\xi\|^2+\|\eta\|^2-2\|\mathbb{P}_{Y_\theta\le\delta}\mathbb{P}_{X_\theta\le\delta}\|\,\|\xi\|\,\|\eta\|\ge(1-\|\mathbb{P}_{Y_\theta\le\delta}\mathbb{P}_{X_\theta\le\delta}\|^2)\|\xi\|^2.$$

This gives the desired norm estimate. □

Now, we consider this time the cubic tensor product $\mathcal{A}_\theta \otimes \mathcal{A}_\theta \otimes \mathcal{A}_\theta$. This arises as an irreducible representation of the higher dimensional Heisenberg group

$$\boldsymbol{H}_3 := \left\{ \begin{bmatrix} 1 & * & * & * & * \\ & 1 & 0 & 0 & * \\ & & 1 & 0 & * \\ & & & 1 & * \\ & & & & 1 \end{bmatrix} \right\} \subset \mathrm{SL}(5, \mathbb{Z}).$$

We put $x_i := e_{1,i+1}(1)$, $y_i := e_{i+1,5}(1)$, and $z := e_{1,5}(1)$ in $\boldsymbol{H}_3$, where we recall that $e_{i,j}(1)$ is the elementary matrix defined in the beginning of the previous section. Note that $[x_i, y_i] = z$ and $[x_i, y_j] = 1$ for $i \neq j$. Hence $\boldsymbol{H}_3$ is isomorphic to the quotient of $\boldsymbol{H} \times \boldsymbol{H} \times \boldsymbol{H}$ modulo z are identified. As before, we write $X_i := (1 - x_i)^*(1 - x_i)$, etc. This should not be confused with X_θ in $\mathcal{A}_\theta$.

Theorem 9. *There are $R > 0$ and $\varepsilon > 0$ such that*

$$R(X_1Y_2 + Y_1X_2 + X_1Y_3 + Y_1X_3) + X_1X_2 + Y_1Y_2 + X_1Y_1 + Y_1X_1 \geq \varepsilon Z$$

holds in $\mathrm{C}^*[\boldsymbol{H}_3]$.

Proof. By Lemma 1 (adapted to this case), it suffices to prove the assertion in $\mathcal{A}_\theta \otimes \mathcal{A}_\theta \otimes \mathcal{A}_\theta$ for each $\theta \in \left[0, \frac{1}{2}\right]$. We write $X_{i,\theta}$ for X_θ in the i-th tensor component. For a unit vector

$$\zeta \in \ell_2(\mathbb{Z}/q\mathbb{Z}) \otimes \ell_2(\mathbb{Z}/q\mathbb{Z}) \otimes \ell_2(\mathbb{Z}/q\mathbb{Z}),$$

we need to prove

$$\big\langle \big(R(X_{1,\theta}Y_{2,\theta} + Y_{1,\theta}X_{2,\theta} + X_{1,\theta}Y_{3,\theta} + Y_{1,\theta}X_{3,\theta}) + X_{1,\theta}X_{2,\theta} + Y_{1,\theta}Y_{2,\theta} + X_{1,\theta}Y_{1,\theta} + Y_{1,\theta}X_{1,\theta} \big)\zeta, \zeta \big\rangle \geq \varepsilon Z_\theta.$$

By Lemma 6, we are already done for $\theta \in [0, \theta_0]$. To apply Lemma 8, fix $0 < \delta < 2(1 - \cos \pi\theta_0)$ small enough and consider $\theta \in \left[\theta_0, \frac{1}{2}\right]$. Since we may choose $R > 1$ arbitrarily large with respect to the fixed δ, we may assume

$$\max\{\|\mathbb{P}_{X_{1,\theta}Y_{2,\theta} > \delta^2}\zeta\|,\ \|\mathbb{P}_{Y_{1,\theta}X_{2,\theta} > \delta^2}\zeta\|,\ \|\mathbb{P}_{X_{1,\theta}Y_{3,\theta} > \delta^2}\zeta\|,\ \|\mathbb{P}_{Y_{1,\theta}X_{3,\theta} > \delta^2}\zeta\|\} < \delta.$$

As described in Lemma 8, we consider the decomposition

$$\zeta = \xi + \eta + \gamma \in \operatorname{ran} \mathbb{P}_{X_{1,\theta} \leq \delta} + \operatorname{ran} \mathbb{P}_{Y_{1,\theta} \leq \delta} + \operatorname{ran} \mathbb{P}_{X_{1,\theta} > \delta \wedge Y_{1,\theta} > \delta}.$$

Note that $\max\{\|\xi\|, \|\eta\|, \|\gamma\|\} \leq 2$. By writing $\approx_\delta$, we will mean that the difference is at most δ. Since $\zeta \approx_\delta \mathbb{P}_{X_{1,\theta}Y_{2,\theta} \leq \delta^2}\zeta$ and $\mathbb{P}_{Y_{2,\theta} > \delta \wedge X_{1,\theta}Y_{2,\theta} \leq \delta^2} \leq \mathbb{P}_{X_{1,\theta} \leq \delta \wedge Y_{2,\theta} > \delta}$, one has

$$\mathbb{P}_{Y_{2,\theta} > \delta}\zeta \approx_\delta \mathbb{P}_{X_{1,\theta} \leq \delta \wedge Y_{2,\theta} > \delta}\zeta.$$

It follows that

$$\mathbb{P}_{Y_{2,\theta} > \delta}\eta + \mathbb{P}_{Y_{2,\theta} > \delta}\gamma \approx_\delta \mathbb{P}_{X_{1,\theta} \leq \delta \wedge Y_{2,\theta} > \delta}(\xi + \eta + \gamma) - \mathbb{P}_{Y_{2,\theta} > \delta}\xi = \mathbb{P}_{X_{1,\theta} \leq \delta \wedge Y_{2,\theta} > \delta}\eta.$$

Since $\mathbb{P}_{Y_{2,\theta} > \delta}$ leaves $\operatorname{ran} \mathbb{P}_{X_{1,\theta} \leq \delta}$ and $\operatorname{ran} \mathbb{P}_{Y_{1,\theta} \leq \delta}$ invariant, this implies

$$\mathbb{P}_{Y_{2,\theta} > \delta}\eta \approx_\delta \mathbb{P}_{X_{1,\theta} \leq \delta \wedge Y_{2,\theta} > \delta}\eta \quad \text{and} \quad \mathbb{P}_{Y_{2,\theta} > \delta}\gamma \approx_\delta 0.$$

Hence, in combination with Lemma 8 that $\mathbb{P}_{Y_{1,\theta}\le\delta}\mathbb{P}_{X_{1,\theta}>\delta}\mathbb{P}_{Y_{1,\theta}\le\delta} \ge \frac{1}{4}\mathbb{P}_{Y_{1,\theta}\le\delta}$, one obtains

$$\delta^2 \ge \|\mathbb{P}_{X_{1,\theta}>\delta}\mathbb{P}_{Y_{2,\theta}>\delta}\eta\|^2 \ge \tfrac{1}{4}\|\mathbb{P}_{Y_{2,\theta}>\delta}\eta\|^2,$$

that is,

$$\eta \approx_{2\delta} \mathbb{P}_{Y_{2,\theta}\le\delta}\eta.$$

The same consideration on $Y_{1,\theta}X_{2,\theta}$ yields

$$\mathbb{P}_{X_{2,\theta}>\delta}\gamma \approx_\delta 0 \quad\text{and}\quad \xi \approx_{2\delta} \mathbb{P}_{X_{2,\theta}\le\delta}\xi.$$

Thus $\mathbb{P}_{Y_{2,\theta}>\delta}\mathbb{P}_{X_{2,\theta}\le\delta}\gamma \approx_\delta \mathbb{P}_{Y_{2,\theta}>\delta}\gamma \approx_\delta 0$ and, by Lemma 8 again,

$$\|\gamma\|^2 \approx_{\delta^2} \|\mathbb{P}_{X_{2,\theta}\le\delta}\gamma\|^2 \le 4\|\mathbb{P}_{Y_{2,\theta}>\delta}\mathbb{P}_{X_{2,\theta}\le\delta}\gamma\|^2 \le 16\delta^2.$$

Further, the same for $X_{1,\theta}Y_{3,\theta}$ and $Y_{1,\theta}X_{3,\theta}$ yields

$$\xi \approx_{2\delta} \mathbb{P}_{X_{3,\theta}\le\delta}\xi \quad\text{and}\quad \eta \approx_{2\delta} \mathbb{P}_{Y_{3,\theta}\le\delta}\eta.$$

Now a routine but tedious calculation with Lemma 8 yields

$$\langle X_{1,\theta}X_{2,\theta}\zeta, \zeta\rangle \approx_{C\delta} \langle X_{1,\theta}X_{2,\theta}\mathbb{P}_{Y_{1,\theta}\le\delta\,\wedge\, Y_{2,\theta}\le\delta}\eta, \mathbb{P}_{Y_{1,\theta}\le\delta\,\wedge\, Y_{2,\theta}\le\delta}\eta\rangle \approx_{16\delta} 4\|\eta\|^2$$

for some absolute constant C (e.g., $C = 1000$ should be enough), and likewise

$$\langle Y_{1,\theta}Y_{2,\theta}\zeta, \zeta\rangle \approx_{C\delta} 4\|\xi\|^2.$$

On the other hand, by Lemmas 7 and 8,

$$\begin{aligned}
|\langle (X_{1,\theta}Y_{1,\theta} + Y_{1,\theta}X_{1,\theta})\zeta, \zeta\rangle| &\approx_{C\delta} 2|\langle X_{1,\theta}Y_{1,\theta}\mathbb{P}_{X_{1,\theta}\le\delta\,\wedge\, X_{2,\theta}\le\delta\,\wedge\, X_{3,\theta}\le\delta}\xi, \mathbb{P}_{Y_{1,\theta}\le\delta\,\wedge\, Y_{2,\theta}\le\delta\,\wedge\, Y_{3,\theta}\le\delta}\eta\rangle| \\
&\le 2\|\mathbb{P}_{Y_{1,\theta}\le\delta}\pi_\theta(1-x_1^*)\|\,\|\pi_\theta((1-x_1)(1-y_1))\|\,\|\pi_\theta(1-y_1^*)\mathbb{P}_{X_{1,\theta}\le\delta}\| \\
&\qquad \times \|\mathbb{P}_{X_{2,\theta}\le\delta}\mathbb{P}_{Y_{2,\theta}\le\delta}\|\,\|\mathbb{P}_{X_{3,\theta}\le\delta}\mathbb{P}_{Y_{3,\theta}\le\delta}\|\,\|\xi\|\,\|\eta\| \\
&\le 16\Big(\cos\frac{\pi\theta}{2}\Big)\cdot\frac{2}{4-\delta}\|\xi\|\,\|\eta\|.
\end{aligned}$$

If we have chosen $\delta > 0$ small enough, then

$$\varepsilon := 8 - 16\Big(\cos\frac{\pi\theta_0}{2}\Big)\cdot\frac{2}{4-\delta} > 4C\delta.$$

Observe that $\delta > 0$ and $\varepsilon > 0$ depends on the absolute constants $\theta_0 > 0$ and $C > 0$, but not on $\theta \in \big[\theta_0, \frac{1}{2}\big]$. In the end,

$$\begin{aligned}
|\langle (X_{1,\theta}Y_{1,\theta} + X_{1,\theta}Y_{1,\theta})\zeta, \zeta\rangle| &\le (8-\varepsilon)\|\xi\|\eta\| + C\delta \\
&\le 4(1-\varepsilon/2)(\|\xi\|^2 + \|\eta\|^2) + C\delta \\
&\le \langle (X_{1,\theta}X_{2,\theta} + Y_{1,\theta}Y_{2,\theta})\zeta, \zeta\rangle - \varepsilon + 3C\delta.
\end{aligned}$$

This completes the proof. We remark that the above proof for $\theta \in \big[\theta_0, \frac{1}{2}\big]$ is not as tight as it appears (and $\varepsilon > 0$ can be "visible"), because if θ is around $\frac{1}{2}$, then $\cos\frac{1}{2}\pi\theta \approx \frac{1}{\sqrt{2}}$, and if θ is away from $\frac{1}{2}$, then $\|\mathbb{P}_{X_\theta\le\delta}\mathbb{P}_{Y_\theta\le\delta}\|$ is bounded by $\approx \frac{1}{\sqrt{3}}$. $\square$

5. Proof of the Main Theorem, postlude

Since $\mathcal{R} := \mathbb{Z}\langle t_1, \dots, t_d\rangle$ is *commutative*, we may apply Theorem 9 to $x_1 = e_{1,2}(t_r)$, $x_2 = e_{1,3}(t_s)$, $x_3 = e_{1,4}(t_r)$, $y_1 = e_{2,5}(t_s)$, $y_2 = e_{3,5}(t_r)$, $y_3 = e_{4,5}(t_s)$, and $z = e_{1,5}(t_r t_s)$ in $\mathrm{EL}_5(\mathcal{R})$. This yields (◇) in Section 3 and the proof of the Main Theorem is complete. □

The terms $X_1Y_2 = E_{1,2}(t_r)E_{3,5}(t_r)$ and $Y_1X_2 = E_{2,5}(t_s)E_{1,3}(t_s)$ are diagonal with respect to $\{t_r, t_s\}$. This causes an annoying dependence of R on d in the formula (♡), which results in dependence of n_0 on d in the Main Theorem.

6. Real group algebras and property $\mathbf{H_T}$

In this section, we continue the study of [Netzer and Thom 2013; 2015; Nitsche 2020; Ozawa 2013; 2016] about positivity in real group algebras. In addition to the notation from Section 2, we denote by

$$I[\Gamma] := \operatorname{span}\{1 - x : x \in \Gamma\} \subset \mathbb{R}[\Gamma]$$

the *augmentation ideal*. We observe that $\Sigma^2 I[\Gamma] = I[\Gamma] \cap \Sigma^2\mathbb{R}[\Gamma]$ and hence there is no ambiguity about the order $\preceq$ on $I[\Gamma]$. In [Ozawa 2016], it was observed that the combinatorial Laplacian $\Delta \in \Sigma^2 I[\Gamma]$ is an *order unit* for $I[\Gamma]$ (more precisely for $I[\Gamma]^{\mathrm{her}}$, but this abuse of terminology should not cause any problem). That is to say, for every $\xi \in I[\Gamma]^{\mathrm{her}}$, there is $R > 0$ such that $\xi \preceq R\Delta$. We will indicate this by $\xi \lll \Delta$.

We review the relation between positive linear functionals on $I[\Gamma]$ and 1-*cocycles* (with unitary coefficients). A linear functional φ on $I[\Gamma]$ is said to be *positive* if it is selfadjoint and $\varphi(\Sigma^2 I[\Gamma]) \subset \mathbb{R}_{\geq 0}$. One has $\varphi(\Delta) = 0$ if and only if $\varphi = 0$. Every positive linear functional φ gives rise to a semi-inner product $\langle \xi, \eta\rangle := \varphi(\xi^*\eta)$ and the corresponding seminorm $\|\xi\| := \varphi(\xi^*\xi)^{1/2}$ on $I[\Gamma]$, with respect to which the left multiplication by an element of Γ is orthogonal. This is the Gelfand–Naimark construction. The map $b : \Gamma \to I[\Gamma]$, $t \mapsto 1 - t$, is a 1-cocycle, i.e., it satisfies $b(st) = b(s) + sb(t)$ for every $s, t \in \Gamma$. We note that $\varphi(1 - t) = \frac{1}{2}\varphi((1 - t)^*(1 - t)) = \frac{1}{2}\|b(t)\|^2$ and $\varphi(\Delta) = \frac{1}{2}\sum_{s\in S}\|b(s)\|^2$. In fact, every 1-cocycle arises in this way. See, e.g., Appendix C in [Bekka et al. 2008] and Appendix D in [Brown and Ozawa 2008] for a comprehensive treatment.

It is proved in [Ozawa 2016] that $\overline{\Sigma^2 I[\Gamma]} = I[\Gamma]^{\mathrm{her}} \cap \overline{\Sigma^2\mathbb{R}[\Gamma]}$. That is to say,

$$\begin{aligned}\overline{\Sigma^2 I[\Gamma]} &:= \{\xi \in I[\Gamma]^{\mathrm{her}} : \text{for all } \varepsilon > 0,\ \xi + \varepsilon\Delta \succeq 0\}\\ &= \{\xi \in I[\Gamma]^{\mathrm{her}} : \varphi(\xi) \geq 0 \text{ for every positive linear functional } \varphi \text{ on } I[\Gamma]\}\\ &= \{\xi \in I[\Gamma]^{\mathrm{her}} : \xi \geq 0 \text{ in } \mathrm{C}^*[\Gamma]\}.\end{aligned}$$

We also record an easy consequence of the Hahn–Banach separation theorem (a.k.a. the Eidelheit–Kakutani separation theorem in this context). For $\xi, \eta \in I[\Gamma]^{\mathrm{her}}$ (or in any real ordered vector space with an order unit Δ), the following are equivalent:

(1) $\varphi(\xi) = 0$ implies $\varphi(\eta) \leq 0$ for every positive linear functional φ on $I[\Gamma]$.

(2) $-\eta \in \overline{\Sigma^2 I[\Gamma] - \mathbb{R}\xi}$.

(3) For all $\varepsilon > 0$, there exists $R \in \mathbb{R}$ such that $R\xi - \eta + \varepsilon\Delta \succeq 0$.

We observe that since

$$\varphi(\Delta^2) = \langle \Delta, \Delta \rangle = \Big\| \sum_{s\in S} b(s) \Big\|^2,$$

one has $\varphi(\Delta^2) = 0$ if and only if the corresponding 1-cocycle b is *harmonic* in the sense $\sum_{s\in S} b(s) = 0$. This observation recovers Shalom's theorem [2000] that every finitely generated group without property (T) has a nonzero harmonic 1-cocycle. An essentially same proof was given in [Nitsche 2020].

We record the following well-known fact:

- If a 1-cocycle b vanishes on a normal subgroup $N \triangleleft \Gamma$, then N acts trivially on $\operatorname{span} b(\Gamma)$ and hence b factors through the quotient Γ/N.
- If b is a harmonic 1-cocycle on Γ, then the center $\mathcal{Z}(\Gamma)$ acts trivially on $\operatorname{span} b(\Gamma)$ and Γ acts trivially on $\operatorname{span} b(\mathcal{Z}(\Gamma))$.
- Every harmonic 1-cocycle on an abelian group is an additive homomorphism.

The first assertion is not difficult to show. The second follows from the identity $(1-x)b(z) = (1-z)b(x)$ for $x \in \Gamma$ and $z \in \mathcal{Z}(\Gamma)$. If b is harmonic, then $\big(|S| - \sum_{s\in S} s\big) b(z) = 0$ and, by strict convexity of a Hilbert space, $b(z) = sb(z)$ for $s \in S$ and hence for all $s \in \Gamma$.

An additive character $\chi : \Gamma \to \mathbb{R}$ can be viewed as a harmonic 1-cocycle. The corresponding positive linear functional $\varphi_\chi : I[\Gamma] \to \mathbb{R}$ is given by $\varphi_\chi(1-t) = \frac{1}{2}\chi(t)^2$. This should not be confused with the linear extension $\chi : I[\Gamma] \to \mathbb{R}$ which is not even selfadjoint. The positive linear functional φ_χ factors through the abelianization $I[\Gamma^{\mathrm{ab}}]$.

We denote the *augmentation power* by

$$I^k[\Gamma] := \operatorname{span}(I[\Gamma]^k) \subset \mathbb{R}[\Gamma].$$

It is well-known and easy to see from the formula

$$1 - xy = (1-x) + (1-y) - (1-x)(1-y) \in (1-x) + (1-y) + I^2[\Gamma]$$

that $I[\Gamma]$ is generated as a rng by $\{1 - s : s \in S\}$ and that $\Gamma \ni x \mapsto 1 - x \in I[\Gamma]/I^2[\Gamma]$ is an additive homomorphism. On the other hand, every additive homomorphism χ vanishes on $I^2[\Gamma]$, because $\chi((1-x)(1-y)) = \chi(1 - x - y + xy) = 0$. Hence $I^2[\Gamma] = \bigcap_\chi \ker \chi$, where the intersection is taken over the additive characters χ on Γ. We will see that $\Delta^2 \in \Sigma^2 I^2[\Gamma]$ need not be an order unit for $I^4[\Gamma]$, but the element

$$\square := \frac{1}{4} \sum_{s,t\in S} (1-s)^*(1-t)^*(1-t)(1-s) \in \Sigma^2 I^2[\Gamma]$$

is. Since $\square = \Delta^2$ in $I[\Gamma^{\mathrm{ab}}]$, one has $\varphi_\chi(\square) = \varphi_\chi(\Delta^2) = 0$ for every additive character χ. We will prove later that the converse is also true.

Theorem 10. *The element $\square$ is an order unit for $I^4[\Gamma]$. Namely*

$$I^4[\Gamma]^{\mathrm{her}} = \{\xi \in \mathbb{R}[\Gamma]^{\mathrm{her}} : \pm\xi \ll \square\} = \operatorname{span} \Sigma^2 I^2[\Gamma]$$

and moreover $I^4[\Gamma] \cap \Sigma^2 \mathbb{R}[\Gamma] = \Sigma^2 I^2[\Gamma]$.

Proof. We first prove that the left is contained the middle. The proof is similar to that for Lemma 2 in [Ozawa 2016]. Since $\xi^*\eta+\eta^*\xi \preceq \xi^*\xi+\eta^*\eta$ for every ξ, η, it suffices to show that

$$(1-x)^*(1-y)^*(1-y)(1-x) \lll \Box \quad \text{for all } x, y \in \Gamma.$$

By using the inequality

$$\begin{aligned}(1-x_1x_2)^*(1-y)^*(1-y)(1-x_1x_2) &= ((1-x_1)+x_1(1-x_2))^*(1-y)^*(\text{——})\\ &\preceq 2(1-x_1)^*(1-y)^*(\text{——})+2(1-x_2)^*(1-x_1^{-1}yx_1)^*(\text{——}),\end{aligned}$$

one can reduce this to the case $x \in S$, and similarly to the case $y \in S$, where the assertion is obvious. We next show that $\pm\xi \lll \Box$ implies $\xi \in \operatorname{span} \Sigma^2 I^2[\Gamma]$. There is $R>0$ such that $0 \preceq R\Box-\xi \preceq 2R\Box$. Thus it remains to show $\sum_i \eta_i^*\eta_i \lll \Box$ implies $\eta_i \in I^2[\Gamma]$. Since $\varphi_\chi(\Box)=0$ for every additive character χ on Γ, one has

$$0=\varphi_\chi\Big(\sum_i \eta_i^*\eta_i\Big) = -\frac{1}{2}\sum_{i,x,y}\eta_i(x)\eta_i(y)\chi(x^{-1}y)^2 = \sum_i\Big(\sum_x \eta_i(x)\chi(x)\Big)^2,$$

or equivalently $\eta_i \in \bigcap_\chi \ker\chi = I^2[\Gamma]$ for all i. □

Corollary 11. *A positive linear functional φ on $I[\Gamma]$ satisfies $\varphi(\Box)=0$ if and only if the associated 1-cocycle is an additive homomorphism.*

Proof. We have already noted that $\varphi_\chi(\Box)=0$ for all additive character χ. Conversely, suppose $\varphi(\Box)=0$. Since this implies $\varphi(\Delta^2)=0$, the 1-cocycle b associated with φ is harmonic. Moreover, since

$$1-[x,y]=(xy-yx)x^{-1}y^{-1}=((1-x)(1-y)-(1-y)(1-x))x^{-1}y^{-1} \in I^2[\Gamma],$$

Theorem 10 implies that $b=0$ on the commutator subgroup $[\Gamma,\Gamma]$. Thus b factors through Γ^{ab} and is an additive homomorphism. □

We recall that a finitely generated group Γ is said to have *Shalom's property* $\mathrm{H_T}$ if every harmonic 1-cocycle on Γ is an additive homomorphism. Property $\mathrm{H_T}$ coincides with Kazhdan's property (T) for groups with finite abelianization. It is observed in [Shalom 2004] that finitely generated nilpotent groups have property $\mathrm{H_T}$. We conjecture that the group $\mathrm{EL}_n(\mathbb{Z}\langle t_1,\dots,t_d\rangle)$ has property $\mathrm{H_T}$. By the Hahn–Banach separation theorem, one obtains the following characterization of property $\mathrm{H_T}$, which does not seem useful though.

Corollary 12. *The finitely generated group Γ has finite abelianization if and only if $\Delta \lll \Box$. The finitely generated group Γ has property $\mathrm{H_T}$ if and only if for every $\varepsilon>0$ there is $R>0$ such that $\Box \preceq R\Delta^2+\varepsilon\Delta$.*

Property $\mathrm{H_T}$ for nilpotent groups also follows from Corollary 3 that if a commutator $z=[x,y]$ is central, then $(1-z)^*(1-z) \ll \Delta^2$ in $\mathrm{C}^*[\Gamma]$. It is tempting to conjecture that every finitely generated nilpotent group Γ satisfies $\Box \ll \Delta^2$. Had it been true that $\Box \lll \Delta^2$ for a given group Γ, it would have been able to rigorously prove this by computer calculations because $\Box$ is an order unit for $I^4[\Gamma]$ (modulo a quantitative estimate, see [Netzer and Thom 2015]). However, we will observe here that $\Box \not\lll \Delta^2$

in $\mathbb{R}[\boldsymbol{H}]$. Hence, unlike property (T), property $\mathrm{H_T}$ is probably not characterized by a "simple"[2] inequality in the real group algebra. This spoils the current methods of proving something like the Main Theorem by computer calculations. (Note that $\mathrm{EL}_n(\mathbb{Z}\langle t\rangle)$ has the Heisenberg group $\boldsymbol{H}_{n-2}$ as a quotient and the analogous statement to the following proposition holds true for this group.)

Proposition 13. *Let $\boldsymbol{H}$ be the integral Heisenberg group and $z := [x, y]$ be as described in the beginning of Section 4. Then $(1-z)^*(1-z) \not\preccurlyeq \Delta^2$ in $\mathbb{R}[\boldsymbol{H}]$. Moreover,*

$$\overline{\Sigma^2 I^2[\boldsymbol{H}]} \neq I^4[\boldsymbol{H}]^{\mathrm{her}} \cap \overline{\Sigma^2\mathbb{R}[\boldsymbol{H}]}.$$

The proof of $\overline{\Sigma^2 I[\Gamma]} = I[\Gamma]^{\mathrm{her}} \cap \overline{\Sigma^2\mathbb{R}[\Gamma]}$ given in [Ozawa 2016] is based on Schoenberg's theorem that any positive linear functional on $I[\Gamma]$ is approximable by those that extend on $\mathbb{R}[\Gamma]$. The above proposition says there is no good enough analogue of Schoenberg's theorem for augmentation powers. For the proof of the proposition, we need a description of the graded vector space $\cdots \supset I^4[\boldsymbol{H}] \supset I^5[\boldsymbol{H}] \supset \cdots$. To ease notation, we write $\bar{x} := 1 - x$ etc. and observe that $\bar{z} \in \mathcal{Z}(\mathbb{R}[\boldsymbol{H}]) \cap I^2[\boldsymbol{H}]$ and

$$\bar{y}\bar{x} = \bar{x}\bar{y} + \bar{z} - \bar{z}\bar{x} - \bar{z}\bar{y} + \bar{z}\bar{y}\bar{x} \in \bar{x}\bar{y} + \bar{z} + I^3[\boldsymbol{H}].$$

Lemma 14. *For every $n \in \mathbb{N}$, the set $\{\bar{x}^i\bar{y}^j\bar{z}^k + I^n[\boldsymbol{H}] : i, j, k \geq 0,\ i + j + 2k < n\}$ forms a basis for $\mathbb{R}[\boldsymbol{H}]/I^n[\boldsymbol{H}]$. In particular*

$$\dim I^n[\boldsymbol{H}]/I^{n+1}[\boldsymbol{H}] = (\lfloor n/2\rfloor + 1)(n - \lfloor n/2\rfloor + 1).$$

Proof. We first observe that the asserted set spans $\mathbb{R}[\boldsymbol{H}]/I^n[\boldsymbol{H}]$. Indeed, this follows from the above equation for $\bar{y}\bar{x}$ and the general facts that

$$1 - uv = (1-u) + (1-v) - (1-u)(1-v),$$
$$1 - u^{-1} = -(1-u) + (1-u^{-1})(1-u)$$

for every $u, v \in \boldsymbol{H}$. It is left to show that the asserted set is also linearly independent. Suppose that

$$\xi := \sum_{i+j+2k<n} \alpha_{i,j,k}\bar{x}^i\bar{y}^j\bar{z}^k \in I^n[\boldsymbol{H}].$$

By considering the abelianization $\pi^{\mathrm{ab}} : \mathrm{C}^*[\boldsymbol{H}] \to \mathrm{C}^*[\mathbb{Z}^2]$, one sees $\alpha_{i,j,k} = 0$ whenever $k = 0$. It follows that $\xi \in I^n[\boldsymbol{H}] \cap \bar{z}\mathbb{R}[\boldsymbol{H}]$. We claim that

$$I^n[\boldsymbol{H}] \cap \bar{z}\mathbb{R}[\boldsymbol{H}] = \bar{z}I^{n-2}[\boldsymbol{H}] \quad \text{for } n \geq 2.$$

Since $\bar{z}$ is not a zero divisor in $\mathbb{R}[\boldsymbol{H}]$ (e.g., because $\pi_\theta(\bar{z})$ are invertible for $\theta \in (0, 1)$), the lemma would follow from this claim by induction.

The homomorphisms $\mathbb{R}[\langle x\rangle] \hookrightarrow \mathbb{R}[\boldsymbol{H}]$ and $\mathbb{R}[\langle y\rangle] \hookrightarrow \mathbb{R}[\boldsymbol{H}]$ extend to a linear injection

$$\sigma : \mathbb{R}[\langle x\rangle] \otimes \mathbb{R}[\langle y\rangle] \hookrightarrow \mathbb{R}[\boldsymbol{H}], \quad \xi \otimes \eta \mapsto \xi\eta,$$

with the left inverse

$$\pi^{\mathrm{ab}} : \mathbb{R}[\boldsymbol{H}] \to \mathbb{R}[\mathbb{Z}^2] \cong \mathbb{R}[\langle x\rangle] \otimes \mathbb{R}[\langle y\rangle].$$

[2]The quantifier elimination techniques, which the author is not familiar with, may be relevant.

Since $\bar{y}\bar{x} \in \bar{x}\bar{y} + \bar{z}\mathbb{R}[\boldsymbol{H}]$ and likewise for $\bar{x}^*$ and $\bar{y}^*$ (thanks to suitable symmetries $x \leftrightarrow x^{-1}$ and $y \leftrightarrow y^{-1}$ on $\boldsymbol{H}$), one has

$$I^n[\boldsymbol{H}] \cap \bar{z}\mathbb{R}[\boldsymbol{H}] \subset (\operatorname{ran}\sigma + \bar{z}I^{n-2}[\boldsymbol{H}]) \cap \ker \pi^{\mathrm{ab}} = \bar{z}I^{n-2}[\boldsymbol{H}].$$

This proves the claim. □

Proof of Proposition 13. We observe that in $I^4[\boldsymbol{H}]/I^5[\boldsymbol{H}]$

$$(\bar{x}\bar{x}\bar{y}\bar{y})^* = \bar{y}\bar{y}\bar{x}\bar{x} = \bar{y}\bar{x}\bar{y}\bar{x} + \bar{y}\bar{x}\bar{z} = \bar{x}\bar{y}\bar{x}\bar{y} + 3\bar{x}\bar{y}\bar{z} + 2\bar{z}\bar{z} = \bar{x}\bar{x}\bar{y}\bar{y} + 4\bar{x}\bar{y}\bar{z} + 2\bar{z}\bar{z}.$$

We define a linear functional φ on $I^4[\boldsymbol{H}]/I^5[\boldsymbol{H}]$ by

$$\varphi(\bar{x}^4) = \varphi(\bar{y}^4) = 1, \quad \varphi(\bar{z}^2) = -2, \quad \varphi(\bar{x}^2\bar{y}^2) = -1, \quad \varphi(\bar{x}\bar{y}\bar{z}) = 1,$$

and zero on all the other basis elements. Then, the linear functional φ is selfadjoint. Moreover, with respect to the basis $\{\bar{x}\bar{x}, \bar{x}\bar{y}, \bar{y}\bar{x}, \bar{y}\bar{y}\}$ for $I^2[\boldsymbol{H}]/I^3[\boldsymbol{H}]$, the bilinear form $(\xi, \eta) \mapsto \varphi(\xi^*\eta)$ is represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

Since this matrix is positive semidefinite, the linear functional is positive on $I^4[\boldsymbol{H}]$, by Theorem 10. One sees that $\varphi(\bar{z}^*\bar{z}) = -\varphi(\bar{z}\bar{z}) = 2 > 0$, $\varphi(\Box) = 4$, and

$$\varphi(\Delta^2) = \varphi((\bar{x}\bar{x} + \bar{y}\bar{y})(\bar{x}\bar{x} + \bar{y}\bar{y})) = 0.$$

Therefore there cannot be $R > 0$ such that $\bar{z}^*\bar{z} \preceq R\Delta^2 + \frac{1}{4}\Box$. It follows that $4\Delta^2 - \bar{z}^*\bar{z} \notin \overline{\Sigma^2 I^2[\boldsymbol{H}]}$, while $4\Delta^2 - \bar{z}^*\bar{z} \in I^4[\boldsymbol{H}]^{\mathrm{her}} \cap \overline{\Sigma^2\mathbb{R}[\boldsymbol{H}]}$ by Corollary 3. □

7. Property (τ)

We say a finitely generated group $\Gamma = \langle S \rangle$ has *property* (τ) with respect to a family $\{\Gamma_i\}$ of finite quotients $\Gamma \twoheadrightarrow \Gamma_i$ if there is $\delta > 0$ such that any unitary representation π of Γ that factors through some $\Gamma \twoheadrightarrow \Gamma_i$ either admits a nonzero $\pi(\Gamma)$-invariant vector or admits no unit vector v such that $\max_{s \in S} \|v - \pi(s)v\| \le \delta$. This is equivalent to that the Cayley graphs of $\{\Gamma_i\}$ with respect to the generating subset S form an expander family. In case the family $\{\Gamma_i\}_i$ is the set of all finite quotients of Γ, it is simply said Γ has property (τ). See [Kowalski 2019] for a comprehensive treatment of expander graphs. By the Main Theorem, $\mathrm{EL}_n(\mathcal{S})$ has property (T) if $\mathcal{S}$ is a finitely generated *irng* (i.e., a rng which is idempotent, $\mathcal{S} = \mathcal{S}^2$, see [Monod et al. 2012]) and n is large enough. Corollaries A and B say this happens uniformly for *finite* commutative irngs with a fixed number of generators.

Proof of Corollary A. Let n_0 be as in the Main Theorem for $\mathbb{Z}\langle T_1, \ldots, T_d, S_1, \ldots, S_d\rangle$ and $n \ge n_0$. By the Main Theorem applied to $T_r \mapsto t_r^k$ and $S_r \mapsto t_r^{k+1}$, there is $\varepsilon > 0$ such that

$$\Delta_k := \sum_{i \ne j} \sum_{r=1}^{d} (1 - e_{i,j}(t_r^k))^*(1 - e_{i,j}(t_r^k)) \in \mathbb{R}[\mathrm{EL}_n(\mathbb{Z}\langle t_1, \ldots, t_d\rangle)]$$

(so $\Delta_1 = \Delta$) satisfy

$$(\Delta_k + \Delta_{k+1})^2 \geq \varepsilon(\Delta_{2k} + \Delta_{2k+1} + \Delta_{2k+2})$$

for all k. We may also assume that $\varepsilon > 0$ satisfies $\Delta_1^2 \geq \varepsilon\Delta_2$.

Let π, $\mathcal{H}$ and v be given for $\mathrm{EL}_n(\mathbb{Z}\langle t_1, \dots, t_d\rangle)$ (but we will omit writing π to ease notation) and put

$$\delta := \Big(\sum_{i,j,r} \|v - e_{i,j}(t_r)v\|^2\Big)^{1/2} = \langle \Delta v, v\rangle^{1/2}.$$

We assume $\delta < \left(\frac{1}{2}\right)^{10}$ and put $\rho := \delta^{1/10}$. Recall that $\mathbb{P}_{\Delta\leq(\delta/\rho)^2}$ stands for the spectral projection of Δ for the interval $[0, (\delta/\rho)^2]$. For $v_0 := \mathbb{P}_{\Delta\leq(\delta/\rho)^2}v$, one has $\|v - v_0\| \leq \rho$ and

$$\langle(\Delta_1 + \Delta_2)v_0, v_0\rangle \leq \delta^2 + \varepsilon^{-1}(\delta/\rho)^4 =: \delta_0^2.$$

Now, $v_1 := \mathbb{P}_{\Delta_1+\Delta_2\leq(\delta_0/\rho^2)^2}v_0$ satisfies $\|v_0 - v_1\| \leq \rho^2$ and

$$\langle(\Delta_2 + \Delta_3)v_1, v_1\rangle \leq \varepsilon^{-1}(\delta_0/\rho^2)^4 =: \delta_1^2.$$

We continue this and obtain $v_2 := \mathbb{P}_{\Delta_2+\Delta_3\leq(\delta_1/\rho^3)^2}v_1, \dots$ such that $\|v_k - v_{k+1}\| \leq \rho^{k+2}$ and

$$\langle(\Delta_{2^k} + \Delta_{2^k+1})v_k, v_k\rangle \leq \varepsilon^{-1}(\delta_{k-1}/\rho^{k+1})^4 =: \delta_k^2.$$

Then the vector $w := \lim_k v_k$ satisfies $\|v_k - w\| \leq \rho^{k+1}$ $\left(\text{as } \rho < \frac{1}{2}\right)$. Moreover,

$$\begin{aligned} 2^{-k}|\log\delta_k| &= 2^{-(k-1)}|\log\delta_{k-1}| - 2^{-(k-1)}(k+1)|\log\rho| + 2^{-(k+1)}\log\varepsilon \\ &= |\log\delta_0| - \Big(\sum_{m=1}^{k} 2^{-(m-1)}(m+1)\Big)|\log\rho| + \tfrac{1}{2}(1 - 2^{-k})\log\varepsilon \\ &> \tfrac{1}{10}|\log\delta| \end{aligned}$$

if $\delta > 0$ is small enough compared to $\varepsilon > 0$. Hence $\delta_k \to 0$ at a double exponential rate.

We need to show $\lim_l \max_{i,j,r} \|w - e_{i,j}(t_r^l)w\| = 0$. We first observe that

$$\|w - e_{i,j}(t_r^{2^k})w\| \leq 2\|v_k - w\| + \delta_k \leq \rho^k + \delta_k.$$

Let l be given. Take $k = k(l)$ such that $l \in [2^k, 2^{k+1})$ and write $l = 2^k + \sum_{m=0}^{k-1} a(m)2^m$ with $a(m) \in \{0, 1\}$. Then for $b := \sum_{m=0}^{\lfloor k/2\rfloor-1} a(m)2^m$, one has

$$\|e_{i,j}(t_r^l)w - e_{i,j}(t_r^{2^k+b})w\| \leq \sum_{m=\lfloor k/2\rfloor}^{k-1} a(m)(\rho^m + \delta_m),$$

which tends to 0 as $l \to \infty$. Observe that the recurrence relation

$$p_0 := 2^{k-\lfloor k/2\rfloor}, \quad p_{m+1} := 2p_m + a(\lfloor k/2\rfloor - 1 - m)$$

gives $p_{\lfloor k/2\rfloor} = 2^k + b$. Now by arguing as in the previous paragraph, but starting at $v_{k-\lfloor k/2\rfloor}$ and using $(\Delta_{p_m} + \Delta_{p_m+1})^2 \geq \varepsilon(\Delta_{p_{m+1}} + \Delta_{p_{m+1}+1})$, one obtains

$$\|v_{k-\lfloor k/2\rfloor} - e_{i,j}(t_r^{2^k+b})v_{k-\lfloor k/2\rfloor}\| \leq \rho^{k-\lfloor k/2\rfloor} + \delta_k \to 0.$$

Since $\|v_{k-\lfloor k/2\rfloor} - w\| \to 0$ as $l \to \infty$, this completes the proof. $\square$

We give a proof of the remark that was made after Corollary A. Let $\mathcal{R} := \mathbb{Z}\langle t_1, \dots, t_d\rangle$. Since $\mathrm{EL}_n(\mathcal{R}/\mathcal{R}^l)$ is nilpotent, there is a *proper* 1-cocycle b_l (see Section 2.7 in [Bekka et al. 2008] or Section 12 in [Brown and Ozawa 2008]). We view b_l as 1-cocycles on $\mathrm{EL}_n(\mathcal{R})$ and consider $b := \sum_l^{\oplus} b_l$, which we may assume convergent pointwise on $\mathrm{EL}_n(\mathcal{R})$. We denote by π_k the Gelfand–Naimark representation associated with the positive definite function $\varphi_k(x) := \exp\bigl(-\frac{1}{k}\|b(x)\|^2\bigr)$. Then, the representation $\pi := \bigoplus \pi_k$ simultaneously admits asymptotically invariant vectors and a weak operator topology null sequence $x_l \in \mathrm{EL}_n(\mathcal{R}^l)$.

Proof of Corollary B. Let $\mathcal{R}^1 := \mathbb{Z}[t_1, \dots, t_d]$ denote the unitization of $\mathcal{R} := \mathbb{Z}\langle t_1, \dots, t_d\rangle$. Any quotient map $\mathcal{R} \twoheadrightarrow \mathcal{S}$ with $\mathcal{S}$ unital gives rise to a group homomorphism $\mathrm{EL}_n(\mathcal{R}^1) \twoheadrightarrow \mathrm{EL}_n(\mathcal{S})$ that extends $\mathrm{EL}_n(\mathcal{R}) \twoheadrightarrow \mathrm{EL}_n(\mathcal{S})$. We need to show that an orthogonal representation of $\mathrm{EL}_n(\mathcal{R}^1)$ which factors through $\mathrm{EL}_n(\mathcal{S})$ has a nonzero invariant vector, provided that it has almost $\mathrm{EL}_n(\mathcal{R})$ invariant vector. Since we know $\mathrm{EL}_n(\mathcal{R}^1)$ has property (T), it suffices to show that every almost $\mathrm{EL}_n(\mathcal{R})$ invariant vector is also almost $\mathrm{EL}_n(\mathbb{Z}1)$ invariant. The latter is true when $\mathcal{S}$ is finite. Indeed, the vector w in Corollary A is invariant under those $e_{i,j}(t_r^{l_0})$ such that $t_r^{l_0}$ is an idempotent in the quotient $\mathcal{S}$. Since a finite commutative ring is a direct sum of local rings (see, e.g., [Kassabov and Nikolov 2006]), the rng generated by such idempotents contains the identity of $\mathcal{S}$ and hence w is invariant under $\mathrm{EL}_n(\mathbb{Z}1)$. □

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