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TRIGONOMETRIC CHAOS AND X_p INEQUALITIES, I: BALANCED FOURIER TRUNCATIONS OVER DISCRETE GROUPS

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We investigate L_p-estimates for balanced averages of Fourier truncations in group algebras, in terms of "differential operators" acting on them. Our results extend a fundamental inequality of Naor for the hypercube (with profound consequences in metric geometry) to discrete groups. Different inequalities are established in terms of "directional derivatives" which are constructed via affine representations determined by the Fourier truncations. Our proofs rely on the Banach X_p nature of noncommutative L_p-spaces and dimension-free estimates for noncommutative Riesz transforms. In the particular case of free groups we use an alternative approach based on free Hilbert transforms.

Introduction

This paper is motivated by a recent inequality due to Assaf Naor, which we now introduce. Let $\Omega = \{\pm 1\}$ be the cyclic group of two elements with multiplicative terminology (that we use for all groups unless otherwise stated) and more generally $\Omega^n = \Omega \times \Omega \times \cdots \times \Omega$ be the hypercube. In both cases, we view them as equipped with their normalized counting measure. Ω^n is its own Pontryagin dual when equipped with its natural discrete measure. If $[n] := \{1, 2, \dots, n\}$, every function $f : \Omega^n \to \mathbb{C}$ admits a Fourier expansion in terms of Walsh characters W_{A}, which are defined by A-products of coordinate functions $\varepsilon \mapsto \varepsilon_j$ for any $\mathsf{A} \subset [n]$. Given a mean-zero f Naor proved in [19] the following inequality for each $p \geq 2$ and $k \in [n]$:

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subset [n] \\ |\mathsf{S}| = k}} \Big\| \sum_{\mathsf{A} \subset \mathsf{S}} \hat{f}(\mathsf{A}) W_{\mathsf{A}} \Big\|_{L_p(\Omega^n)}^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_j f\|_{L_p(\Omega^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\Omega^n)}^p. \tag{N_p}$$

The above S-truncations of the Walsh expansion of f are conditional expectations denoted by $\mathsf{E}_{[n]\setminus \mathsf{S}} f$, while $\partial_j f$ stands for the j-th directional (discrete) derivative of f, given by $\varepsilon \mapsto f(\varepsilon) - f(\varepsilon_1, \dots, -\varepsilon_j, \dots, \varepsilon_n)$, so that $\partial_j W_{\mathsf{A}} = 1_{\mathsf{A}}(j) 2 W_{\mathsf{A}}$. This inequality has groundbreaking applications in metric geometry. More precisely, it implies the quantitatively optimal form of the so-called X_p inequality, introduced by Naor and Schechtman in [20]. In turn, this gives a purely metric criterion to estimate the L_p-distortion of a metric space X from below. Its metric nature is very useful in solving nonlinear problems around the nonembedability of L_q into L_p for $2 < q < p$. This includes, beyond the scope of linear L_p-embedding theory, the optimal L_p-distortion of (nonlinear) grids in ℓ_q^n or the critical L_p snowflake exponent of L_q. In conclusion, Naor's inequality (N_p) and subsequent X_p inequalities with sharp scaling parameter are a key

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contribution to the Ribe program, an effort to identify which properties from the local theory of Banach spaces ultimately rely on purely metric considerations and not on the whole strength of the linear structure.

Naor's inequality (N_p) for functions with a linear Walsh expansion becomes a form of Rosenthal inequality for symmetrically exchangeable random variables [6; 21]. More precisely, let Π_k be the space of sets $\mathsf{S} \subset [n]$ with $|\mathsf{S}| = k$ equipped with its normalized counting measure and define $\Sigma_{n,k} = \Omega^n \otimes \Pi_k$. Then, if $\hat{f}(\mathsf{A}) = 0$ when $|\mathsf{A}| \neq 1$, the left-hand side of (N_p) becomes

$$\Big\| \sum_{j=1}^n \hat{f}(\{j\})\sigma_j \Big\|_{L_p(\Sigma_{n,k})}^p, \quad \text{with } \sigma_j(\varepsilon, \mathsf{S}) = \varepsilon_j \otimes 1_{\mathsf{S}}.$$

Then, the linear model for Naor's inequality, which is the one pertaining functions of the form $f(\varepsilon) = \sum_j \hat{f}(\{j\})\varepsilon_j$, follows from [6]

$$\Big\| \sum_{j=1}^n \hat{f}(\{j\})\sigma_j \Big\|_{L_p(\Sigma_{n,k})} \asymp_p \Big(\frac{k}{n} \sum_{j=1}^n |\hat{f}(\{j\})|^p \Big)^{\frac{1}{p}} + \Big(\frac{k}{n} \sum_{j=1}^n |\hat{f}(\{j\})|^2 \Big)^{\frac{1}{2}}.$$

Its general form (N_p) can be regarded as an extension for Rademacher chaos. Our primary goal in this paper is to produce inequalities similar to (N_p). This amounts to understanding the Walsh expansion of f as a Fourier series with frequencies in the discrete predual group Ω^n, with W_{A} being regarded as a Fourier character. In the general (nonabelian) case, this forces us to use the language of group von Neumann algebras generated by the left regular representation. Indeed, Fourier series with frequencies on a general discrete group G must be written in terms of its left regular representation $\lambda : \mathrm{G} \to \mathcal{B}(\ell_2(\mathrm{G}))$. The unitaries $\lambda(g)$ replace Walsh characters and we work with operators of the form

$$f = \sum_{g \in \mathrm{G}} \hat{f}(g)\lambda(g).$$

The "quantum" probability space where we place these operators is the group von Neumann algebra $\mathcal{L}(\mathrm{G})$. The noncommutative L_p space $L_p(\mathcal{L}(\mathrm{G}))$ associated with $\mathcal{L}(\mathrm{G})$ is isometrically isomorphic to the classical L_p-space over the Pontryagin dual $\widehat{\mathrm{G}}$ of G whenever the group G is abelian. We shall make no distinction between a function in $L_p(\widehat{\mathrm{G}})$ and the corresponding operator in $L_p(\mathcal{L}(\mathrm{G}))$ throughout the paper. Precise definitions of all the relevant objects are given below. Understanding how to replace Rademacher chaos by some sort of "trigonometric chaos" has to do with identifying elementary generating families. Our construction is somehow delicate and we start with a model case which originally motivated us.

Let $\mathbb{F}_n = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$ be the free group with n generators $e_1, e_2, \ldots, e_n$. The unitaries $\lambda(e_j)$ are an archetype of Voiculescu's free random variables, which play the role of coordinate functions ε_j above. The tensor products $\zeta_j(\mathsf{S}) = \lambda(e_j) \otimes 1_{\mathsf{S}}(j)$ in $\Sigma'_{n,k} = \mathcal{L}(\mathbb{F}_n) \otimes \Pi_k$ satisfy the inequality

$$\Big\| \sum_{j=1}^n \hat{f}(e_j)\zeta_j \Big\|_{L_p(\Sigma'_{n,k})} \asymp_p \Big(\frac{k}{n} \sum_{j=1}^n |\hat{f}(e_j)|^p \Big)^{\frac{1}{p}} + \Big(\frac{k}{n} \sum_{j=1}^n |\hat{f}(e_j)|^2 \Big)^{\frac{1}{2}}. \qquad (\mathrm{FR}_p)$$

The desired free form of Naor's inequality looks as follows:

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subset [n] \\ |\mathsf{S}| = k}} \Big\| \sum_{w \in \mathbb{F}_{\mathsf{S}}} \hat{f}(w)\lambda(w) \Big\|_p^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_j f\|_p^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_p^p. \qquad (\mathrm{FN}_p)$$

Here $\mathbb{F}_S$ denotes the free subgroup with generators in S and

$$\partial_j f = \sum_{w \geq e_j} \hat{f}(w)\lambda(w) + \sum_{w \geq e_j^{-1}} \hat{f}(w)\lambda(w),$$

where $w \geq e_j$ is used to pick those words starting with the subchain e_j^k for some positive integer k when written in reduced form. Let us briefly comment on the two inequalities above. The inequality (FR_p) follows from the noncommutative Burkholder/Rosenthal inequality [8; 9], while (FN_p) reduces to (FR_p) when f lives in the linear span of the $\lambda(e_j)$ as a consequence of the free Khintchine inequality [4]. It is therefore an extension of the linear model for free chaos. A look at Naor's original inequality shows that both group elements and collections of generators (respectively denoted by A and S there) become subsets of $[n]$. This curious coincidence in the hypercube must be decoupled for other discrete groups and our inner sum in the left-hand side of (FN_p) is taken over those words w with letters living in free coordinates located in S. On the other hand, our choice for $\partial_j f$ comes from [13] and will be properly justified in due time. It is worth mentioning that some nonlinear extensions of the free Rosenthal inequality were investigated in [10] for free chaos, but none of them include a free form of Naor's inequality along the lines suggested above.

The above reasoning settles a free model for Naor's inequality and illustrates how trigonometric chaos fits in for free groups. What happens if we take products of more general discrete groups? What about discrete groups lacking a product structure? Answering these questions amounts to considering Fourier truncations and somehow related differential operators over discrete groups. Other than lattices of Lie groups, discrete groups fail to admit canonical differential structures. This difficulty was successfully solved in [11; 13] with affine representations. More precisely, an orthogonal cocycle of G is a pair (α, β) given by an orthogonal action $\alpha : \mathrm{G} \curvearrowright \mathcal{H}$ into some $\mathbb{R}$-Hilbert space together with a map $\beta : \mathrm{G} \to \mathcal{H}$ satisfying the cocycle law

$$\alpha_g(\beta(h)) = \beta(gh) - \beta(g).$$

The latter ensures that $g \mapsto \alpha_g(\,\cdot\,) + \beta(g)$ is an affine representation of G, so that the cocycle map β establishes a good Hilbert space lift of G and one can expect to import the differential structure of $\mathcal{H}$. Naively, we "identify" the unitary $\lambda(g)$ with the Euclidean character $\exp(2\pi i \langle \beta(g), \cdot \rangle)$ and define "$\mathcal{H}$-directional derivatives" on $\mathcal{L}(\mathrm{G})$ as follows for any $u \in \mathcal{H}$:

$$\partial_u(\lambda(g)) = \langle \beta(g), u\rangle \lambda(g) \quad \text{and} \quad \Delta(\lambda(g)) = \|\beta(g)\|^2 \lambda(g).$$

We remark that we use the word "derivative" — in quotes, that we suppress after the Introduction — in a loose way here. They are linear operators that do not satisfy Leibniz rules, so in general they are not derivations. In the same vein, the correspondence between $\lambda(g)$ and Euclidean characters that we take as inspiration only holds for $\mathrm{G} = \mathbb{Z}^n$ with a particular choice of cocycle, that we shall detail below, and should not be considered in a literal way. This strategy of construction of differential structures has been extremely useful to establish L_p-boundedness criteria for Fourier multipliers on group von Neumann algebras. We now introduce the right setup for the problem. Given a discrete group G equipped with an orthogonal cocycle (α, β) and a positive integer n, we say that

$$\mathcal{A} = \{\mathrm{B}_S \subset \mathrm{G} : \mathrm{S} \subset [n]\}$$

is an *admissible family of Fourier truncations* when we have:

- $\Big\| \sum_{g \in \mathrm{B_S}} \hat{f}(g)\lambda(g) \Big\|_p \leq_{\mathrm{cb}} C_p \Big\| \sum_{g \in \mathrm{G}} \hat{f}(g)\lambda(g) \Big\|_p$ for $p \geq 2$.
- Pairwise β-orthogonality:

$$\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j, \quad \text{with } \beta(\mathrm{B_S}), \beta(\mathrm{B_S^{-1}}) \subset \bigoplus_{j \in \mathrm{S}} \mathcal{H}_j = \mathcal{H}_\mathrm{S}.$$

Given an orthonormal basis $(u_{j\ell})_\ell$ of $\mathcal{H}_j$, define the j-th "gradients" as the first column vectors

$$\mathrm{D}_j f = \sum_{\ell \geq 1} \partial_{u_{j\ell}} f \otimes e_{\ell 1} \quad \text{so that} \quad |\mathrm{D}_j f| = \Big(\sum_{\ell \geq 1} |\partial_{u_{j\ell}} f|^2 \Big)^{\frac{1}{2}}.$$

Theorem A. *Let* G *be a discrete group equipped with an orthogonal cocycle* (α, β) *whose associated Laplacian* Δ *has a positive spectral gap* $\sigma > 0$. *Let us consider an admissible family of Fourier truncations* $\mathcal{A} = \{\mathrm{B_S} : \mathrm{S} \subset [n]\}$. *Then, given* $p \geq 2$ *and* $k \in [n]$, *the following inequality holds for any mean-zero* f:

$$\frac{1}{\binom{n}{k}} \sum_{|\mathrm{S}|=k} \Big\| \sum_{g \in \mathrm{B_S}} \hat{f}(g)\lambda(g) \Big\|_{L_p(\mathcal{L}(\mathrm{G}))}^p \lesssim_{p,\sigma} \frac{k}{n} \sum_{j=1}^{n} [\| |\mathrm{D}_j(f)| \|_{L_p(\mathcal{L}(\mathrm{G}))}^p + \| |\mathrm{D}_j(f^*)| \|_{L_p(\mathcal{L}(\mathrm{G}))}^p] + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathcal{L}(\mathrm{G}))}^p.$$

Naor's inequality follows as a particular case of Theorem A by taking $\mathrm{G} = \Omega^n$ equipped with an adequate cocycle that we detail below in Remark 2.5. Said cocycle sends Ω^n into the n-dimensional space $\mathcal{H} = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$, and we use the truncations $\mathrm{B_S} = \{\mathrm{A} \subset \mathrm{S}\} = \beta^{-1}(\mathcal{H}_\mathrm{S})$. The length ψ in this example is the *word length* (the geodesic distance in the Cayley graph) as is the case in many of the examples below. Recall that $\mathrm{D}_j = \partial_j$ here since $\dim \mathcal{H}_j = 1$. Moreover $|\partial_j(f)| = |\partial_j(f^*)|$ in the abelian framework of the hypercube. Two generalizations of Naor's inequality for large classes of discrete groups follow from Theorem A:

(i) Direct products. If $\mathrm{G} = \mathrm{G}_1 \times \mathrm{G}_2 \times \cdots \times \mathrm{G}_n$ is a direct product of discrete groups equipped with orthogonal cocycles (α_j, β_j), consider the product cocycle (α, β) and let $\mathrm{B_S}$ be the subgroup of G generated by group elements whose nontrivial entries lie in S. Then, the Fourier truncations become (completely contractive) conditional expectations and we get an admissible family of Fourier truncations. The gradients D_j correspond to the different factors and cocycles in the direct product above. One case that we shall explore is $\mathrm{G} = \mathbb{Z}^n$, which probably yields the most natural continuous generalization of (N_p) in the torus $\mathbb{T}^n$. Our result in inequality (2-1) can be obtained using only commutative ingredients and following the original argument. However, we shall deduce it from Theorem A and later improve it below using our stronger result — in this case.

(ii) Equivariant decompositions. If G is a discrete group equipped with an orthogonal cocycle (α, β), any direct sum decomposition of the Hilbert space $\mathcal{H}$ into α-equivariant subspaces gives rise to an admissible family of Fourier truncations. More precisely, assume

$$\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j \qquad \text{and} \qquad \alpha_g(\mathcal{H}_j) \subset \mathcal{H}_j \quad \text{for every } (g, j) \in \mathrm{G} \times [n].$$

Then, the family of sets

$$\mathrm{B}_\mathsf{S} = \beta^{-1}\Big(\bigoplus_{j\in\mathsf{S}} \mathcal{H}_j\Big)$$

are subgroups of G. In particular, the associated Fourier truncations are conditional expectations (henceforth L_p-contractions) and the B_S satisfy pairwise β-orthogonality. This more general construction does not impose a direct product structure on the discrete group G.

Let $\mathcal{A}$ be an admissible family of Fourier truncations on G as defined above. Let us say that a group element $g \in \mathrm{G}$ is an $\mathcal{A}$-generator when $\beta(g) \in \mathcal{H}_j$ for some $1 \le j \le n$. Theorem A may be regarded as a nonlinear form of an inequality for linear combinations of $\mathcal{A}$-generators

$$f = \sum_{j=1}^n \sum_{\beta(g)\in\mathcal{H}_j} \hat{f}(g)\lambda(g) = \sum_{j=1}^n A_j(f).$$

This inequality controls balanced averages of S-truncations $\sum_{j\in\mathsf{S}} A_j(f)$ in terms of f and the j-th gradients of $A_j(f)$. This linear model seems to be new for general discrete groups/cocycles and Theorem A gives a nonlinear generalization in terms of trigonometric chaos over $\mathcal{A}$-generators.

Theorem A does not recover the conjectured free form of Naor's inequality (FN_p). Indeed, the free inequality relies on the standard cocycle of $\mathbb{F}_n$ associated with the word length, which yields $\mathcal{H} \simeq \ell_2(\mathbb{F}_n \backslash \{e\})$ and infinitely many "free derivatives" of the form

$$\partial_u f = \sum_{w \ge u} \hat{f}(w)\lambda(w) \quad \text{for any } u \in \mathbb{F}_n \setminus \{e\},$$

so the ∂_u can be regarded as Fourier multipliers whose symbols take values on $\{0, 1\}$. However, we only need to use n free directional derivatives which are defined as

$$\partial_j = \partial_{e_j} + \partial_{e_j^{-1}}, \quad \text{with } 1 \le j \le n,$$

and these are not coupled into a family of gradients, as we do in Theorem A. The key point to achieve this is the fact that free derivatives associated to free generators include all free derivatives in the sense that

$$u \neq e \quad \Longrightarrow \quad u \ge e_j \text{ or } u \ge e_j^{-1} \text{ for some } 1 \le j \le n \quad \Longrightarrow \quad \partial_u \circ \partial_j = \partial_j \circ \partial_u = \partial_u.$$

In general, assume that $\mathcal{A} = \{\mathrm{B}_\mathsf{S} : \mathsf{S} \subset [n]\}$ is an admissible family of Fourier truncations in G with respect to (α, β). We will say that $\mathcal{J} = \{\partial_j : 1 \le j \le n\}$ is a *distinguished family of "derivatives"* when $\partial_u \circ \partial_j = \partial_u$ for any $u \in \mathcal{H}_j$ with $1 \le j \le n$. Throughout the paper, we shall consistently use u for vectors in $\mathcal{H}$ and $j \in [n]$, so that no confusion should arise when using ∂_u and ∂_j. The following result refines Theorem A when we can find such a family.

Theorem B. *Let* G *be a discrete group equipped with an orthogonal cocycle* (α, β) *and an admissible family of Fourier truncations* $\mathcal{A} = \{\mathrm{B}_\mathsf{S} : \mathsf{S} \subset [n]\}$. *Assume that* $\mathcal{J} = \{\partial_j : 1 \le j \le n\}$ *is a distinguished* *family of derivatives. Then, given $p \geq 2$ and $k \in [n]$, the following inequality holds for any mean-zero f:*

$$\frac{1}{\binom{n}{k}} \sum_{|\mathrm{S}|=k} \Big\| \sum_{g \in \mathrm{B_S}} \hat{f}(g)\lambda(g) \Big\|_p^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_j(f)\|_p^p + \|\partial_j(f^*)\|_p^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_p^p.$$

When the distinguished family of derivatives ∂_j is a proper subset of the cocycle derivatives ∂_u, it turns out that Theorem B gives a stronger inequality (compared to that of Theorem A) at the cost of additional assumptions, which fortunately hold in several important cases considered below. Note as well that the spectral gap assumption is unnecessary under the presence of distinguished derivatives. Here are our main applications of Theorem B:

(i) Free chaos. Our discussion on free derivatives illustrates how to apply Theorem B to obtain an inequality which gets very close to (FN$_p$). The extra term $\partial_j(f^*)$ is anyway removable due to a special property of free groups, for which word-length derivatives become free forms of directional Hilbert transforms [17]. This "good pathology" leads us to an even stronger inequality than the free analog of Naor's inequality (FN$_p$), as can be seen comparing the statements of Theorem 3.1 and Corollary 3.2. This could be useful in other directions of free harmonic analysis. We shall also explore the free products $\mathbb{Z}_{2m} * \mathbb{Z}_{2m} * \cdots * \mathbb{Z}_{2m}$.

(ii) Continuous and discrete tori. We also analyze $\mathbb{T}^n = \widehat{\mathbb{Z}}^n$ and $\mathbb{Z}_m^n = \widehat{\mathbb{Z}}_m^n$ equipped with different geometries. Theorem B is applicable for the Cayley graph metric and the resulting inequality improves the one coming from the Euclidean metric. These forms of Naor's inequality can be regarded as refinements of the classical Poincaré inequality.

(iii) Infinite Coxeter groups. Any group presented by

$$\mathrm{G} = \langle g_1, g_2, \ldots, g_n \mid (g_j g_k)^{s_{jk}} = e \rangle,$$

with $s_{jj} = 1$ and $s_{jk} \geq 2$ for $j \neq k$, is called a Coxeter group. Bożejko proved in [1] that the word length is conditionally negative for any infinite Coxeter group. The Cayley graph of these groups is more involved and we will not construct here a natural orthonormal basis for the cocycle; we invite the reader to do it and to derive inequalities along the lines of those in Theorems A and B.

Our proof of Theorems A and B streamlines Naor's original argument. The key point in this general setting is to identify the right notions, such as admissible families of Fourier truncations or distinguished families of derivatives. Once this is done, the proof heavily relies on dimension-free estimates for noncommutative Riesz transforms [13] in the same way Naor's inequality did in terms of Lust-Piquard results [16]. Another crucial point in our argument is the Banach X_p nature of noncommutative L_p-spaces. Generalizing previous work of Naor and Schechtman [20, Theorem 7.1], we shall establish it with a much simpler argument based on Junge and Xu's noncommutative Burkholder/Rosenthal inequalities [8; 9]. Of course, one could expect that Theorems A and B may lead to noncommutative X_p-type inequalities, very much like in [19]. We have obtained some inequalities in this direction [2]. Our hope was to deduce nontrivial bounds for L_p-distortions of Schatten q-classes or other noncommutative L_q-spaces. Unfortunately, our efforts so far have not been fruitful in this direction.

1. Trigonometric chaos

1A. ***Harmonic analysis on discrete groups.*** Let G be a discrete group. The left regular representation of G on $\ell_2(\mathrm{G})$ is the unitary representation determined by

$$[\lambda(g)\varphi](h) = \varphi(g^{-1}h), \quad g, h \in \mathrm{G}, \ \varphi \in \ell_2(\mathrm{G}).$$

The group von Neumann algebra of G is denoted by $\mathcal{L}(\mathrm{G})$. It is the weak operator closure of the linear span of $\{\lambda(g)\}_{g\in\mathrm{G}}$ in $\mathcal{B}(\ell_2(\mathrm{G}))$. Its canonical trace τ is linearly determined by $\tau(\lambda(g)) = \langle \lambda(g)1_{\{e\}}, 1_{\{e\}}\rangle_{\ell_2(\mathrm{G})} = \delta_{g=e}$. Every element $f \in \mathcal{L}(\mathrm{G})$ admits a Fourier series

$$f = \sum_{g\in\mathrm{G}} \hat{f}(g)\lambda(g), \quad \text{where } \hat{f}(g) = \tau(\lambda(g)^* f).$$

This shows that $\tau(f) = \hat{f}(e)$. For $1 \le p < \infty$, we denote by $L_p(\mathcal{L}(\mathrm{G}))$ the associated noncommutative L_p space. We emphasize here that in the case G is abelian, its Pontryagin dual $\widehat{\mathrm{G}}$ is a compact abelian group and we have

$$L_p(\mathcal{L}(\mathrm{G})) \simeq L_p(\widehat{\mathrm{G}}),$$

isometrically. Therefore, in that case $L_p(\mathcal{L}(\mathrm{G}))$ is a classical (commutative) L_p space. In all instances below, we will consider all of our L_p spaces as noncommutative ones so that we can give a unified treatment to all the examples.

An orthogonal cocycle for G is a triple $(\mathcal{H}, \alpha, \beta)$ given by a real Hilbert space $\mathcal{H}$, an orthogonal action $\alpha : \mathrm{G} \to \mathcal{O}(\mathcal{H})$, and a map $\beta : \mathrm{G} \to \mathcal{H}$ satisfying the cocycle law

$$\alpha_g(\beta(h)) = \beta(gh) - \beta(g).$$

Orthogonal cocycles are in one-to-one correspondence with length functions. We say that a map $\psi : \mathrm{G} \to \mathbb{R}_+$ is a length function if it vanishes at the identity e, it is symmetric $\psi(g) = \psi(g^{-1})$, and it is conditionally negative

$$\sum_{g\in\mathrm{G}} a_g = 0 \quad \Longrightarrow \quad \sum_{g,h\in\mathrm{G}} \bar{a}_g a_h \psi(g^{-1}h) \le 0$$

for any finitely supported family $\{a_g\}_{g\in\mathrm{G}}$. Given a cocycle $(\mathcal{H}, \alpha, \beta)$, the function $\psi(g) = \|\beta(g)\|_{\mathcal{H}}^2$ is a length function. On the other hand, any length function ψ determines a Gromov form $\langle\,\cdot\,,\cdot\,\rangle_\psi$, a semidefinite positive form on

$$\mathbb{D}[\mathrm{G}] := \mathbb{R}\text{-span}\langle 1_{\{g\}} : g \in \mathrm{G}\rangle,$$

given by

$$\langle 1_{\{g\}}, 1_{\{h\}}\rangle_\psi = \frac{\psi(g) + \psi(h) - \psi(g^{-1}h)}{2}.$$

Then, the Hilbert completion $\mathcal{H}$ of $\mathbb{D}[\mathrm{G}]/\mathrm{Ker}(\langle\,\cdot\,,\cdot\,\rangle_\psi)$, equipped with $\langle\,\cdot\,,\cdot\,\rangle_\psi$, together with the map $\beta : g \mapsto 1_{\{g\}} + \mathrm{Ker}(\langle\,\cdot\,,\cdot\,\rangle_\psi)$, and the orthogonal action $\alpha_g(1_{\{h\}}) = 1_{\{gh\}} - 1_{\{g\}} + \mathrm{Ker}(\langle\,\cdot\,,\cdot\,\rangle_\psi)$ form a cocycle. Moreover, Schoenberg's theorem [22] claims that $\psi : \mathrm{G} \to \mathbb{R}_+$ is a length function if and only if the maps $S_t : \lambda(g) \mapsto e^{-t\psi(g)}\lambda(g)$ form a Markov semigroup $(S_t)_{t\geq 0}$ on $\mathcal{L}(\mathrm{G})$; see [11; 13]. In this case $(S_t)_{t\geq 0}$ admits an infinitesimal generator

$$-\Delta := \lim_{t\to 0^+} \frac{S_t - \mathrm{id}_{\mathcal{L}(\mathrm{G})}}{t} \quad \text{so that} \quad S_t = \exp(-t\Delta).$$

As is standard, we shall call the generator Δ the ψ-Laplacian on G. Since we have $\Delta(\lambda(g)) = \psi(g)\lambda(g)$ for $g \in \mathrm{G}$, it turns out that Δ is an unbounded Fourier multiplier whose fractional powers can be defined by

$$\Delta^\gamma f := \sum_{g\in \mathrm{G}} \psi(g)^\gamma f(g)\lambda(g).$$

Let $(\mathcal{H}, \alpha, \beta)$ be the orthogonal cocycle naturally associated to the length function $\psi : \mathrm{G} \to \mathbb{R}_+$ as explained above. Given an orthonormal basis $\{u_\ell\}_{\ell\geq 1}$ of $\mathcal{H}$, we consider the corresponding directional derivatives as follows:

$$\partial_{u_\ell}\lambda(g) := \langle \beta(g), u_\ell\rangle_\psi \lambda(g) \quad \text{so that} \quad \Delta = \sum_{\ell\geq 1} \partial_{u_\ell}^2.$$

The corresponding Riesz transforms associated to ψ are then defined as

$$R_\ell f = R_{u_\ell} f := \partial_{u_\ell}\Delta^{-\frac{1}{2}} f = \sum_{g\in \mathrm{G}} \frac{\langle \beta(g), u_\ell\rangle_\psi}{\sqrt{\psi(g)}} \hat{f}(g)\lambda(g).$$

Riesz transforms act on elements of $L_p(\mathcal{L}(\mathrm{G}))$ with null Fourier coefficients on the kernel of β. More precisely, maps R_ℓ are well-defined over the mean-zero subspaces

$$L_p^\circ(\mathcal{L}(\mathrm{G})) = \{f \in L_p(\mathcal{L}(\mathrm{G})) : \hat{f}(g) = 0 \text{ if } \psi(g) = 0\}.$$

Dimension-free estimates for noncommutative Riesz transforms were studied in [13].

Theorem 1.1 [13, Theorem A1]. *If* $2 \leq p < \infty$ *and* $f \in L_p^\circ(\mathcal{L}(\mathrm{G}))$

$$\|f\|_p \asymp_p \max\left\{ \left\| \left(\sum_{\ell\geq 1} |R_\ell(f)|^2\right)^{\frac{1}{2}} \right\|_p, \left\| \left(\sum_{\ell\geq 1} |R_\ell(f^*)|^2\right)^{\frac{1}{2}} \right\|_p \right\}.$$

Finally, our Fourier truncations will be written in the form

$$\mathsf{E}_{[n]\setminus \mathsf{S}} f = \sum_{g\in \mathrm{B}_\mathsf{S}} \hat{f}(g)\lambda(g), \quad \text{with } \mathsf{S} \subset [n].$$

When B_S is a subgroup of G, $\mathsf{E}_{[n]\setminus \mathsf{S}}$ is a (L_p-contractive) conditional expectation onto $\mathcal{L}(\mathrm{B}_\mathsf{S})$.

1B. ***Noncommutative L_p-spaces are Banach X_p spaces.*** Linear forms of X_p inequalities are vector-valued extensions of Rosenthal inequality for symmetrically exchangeable random variables [6]. More precisely, a Banach space $\mathbb{X}$ is said to satisfy a Banach X_p inequality if the inequality of Theorem 1.2 below is satisfied for vectors $\{x_j\}_{j\in[n]} \subset \mathbb{X}$ (and with norms taken in $\mathbb{X}$). In [20, Theorem 7.1] Naor and Schechtman proved such inequalities for Schatten p-classes. A noncommutative Burkholder martingale inequality for the conditioned square function [8] led Junge and Xu to obtain noncommutative Rosenthal inequalities for symmetric variables in [9]. The precise result that we use below is the following (see

[9, Corollary 6.6]): let $\mathcal{N}$ and $\mathcal{M}$ be von Neumann algebras, with $\mathcal{N}$ abelian, and $p \geq 2$. If $\{x_j\}_{j\in[n]} \subset L_p(\mathcal{M})$ satisfy that

$$\Big\| \sum_{j=1}^n s_j a_{\pi(j)} \otimes x_j \Big\|_{L_p(\mathcal{N}\bar{\otimes}\mathcal{M})} \lesssim \Big\| \sum_{j=1}^n a_j \otimes x_j \Big\|_{L_p(\mathcal{N}\bar{\otimes}\mathcal{M})}$$

holds for all random signs $s = (s_1, s_2, \ldots, s_n) \in \Omega^n$, all permutations π on $[n]$ and coefficients $\{a_j\}_{j\in[n]} \subset L_p(\mathcal{N})$ — that is, the variables are symmetrically exchangeable — then

$$\Big\| \sum_{j=1}^n a_j \otimes x_j \Big\|_p \sim \frac{1}{n^{\frac{1}{p}}} \sum_{j,j'=1}^n \|a_j\|_p \|x_{j'}\|_p + \frac{1}{n^{\frac{1}{2}}} \Big\| \Big(\sum_{j=1}^n x_j^* x_j + x_j x_j^* \Big)^{\frac{1}{2}} \Big\|_p \Big\| \Big(\sum_{j=1}^n a_j^2 \Big)^{\frac{1}{2}} \Big\|_p. \tag{1-1}$$

We use this result below to establish the Banach X_p nature of arbitrary noncommutative L_p-spaces. Naor/Schechtman's argument can be extended to work as well for other noncommutative L_p-spaces, but our argument below is much shorter.

Theorem 1.2. *Let $(\mathcal{M}, \tau)$ be a von Neumann algebra equipped with a normal semifinite faithful trace. Then, if $\mathbb{E}$ denotes the expectation over independently equidistributed random signs $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ and $x_j \in L_p(\mathcal{M})$, the following inequality holds for any $p \geq 2$ and $k \in [n]$:*

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n] \\ |\mathsf{S}|=k}} \mathbb{E} \Big\| \sum_{j\in\mathsf{S}} \varepsilon_j x_j \Big\|_{L_p(\mathcal{M})}^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|x_j\|_{L_p(\mathcal{M})}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \mathbb{E} \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|_{L_p(\mathcal{M})}^p.$$

Proof. Define random variables $\sigma_j \in \Sigma_{n,k} = \Omega^n \otimes \Pi_k$ as defined in the Introduction by $\sigma_j(\varepsilon, \mathsf{S}) = \varepsilon_j \otimes 1_{\mathsf{S}}(j)$ for $1 \leq j \leq n$ and $\mathsf{S} \subset [n]$. We claim that the variables σ_j are symmetrically exchangeable in $L_p(\Sigma_{n,k} \bar{\otimes} \mathcal{M}) = L_p(\Pi_k; L_p(\Omega^n; L_p(\mathcal{M})))$, i.e., for any choice of signs $s_j = \pm 1$ and any permutation π of $[n]$, there holds

$$\mathrm{A} := \Big\| \sum_{j=1}^n s_j \sigma_{\pi(j)} \otimes x_j \Big\|_{L_p(\Sigma_{n,k}\bar{\otimes}\mathcal{M})} \lesssim \Big\| \sum_{j=1}^n \sigma_j \otimes x_j \Big\|_{L_p(\Sigma_{n,k}\bar{\otimes}\mathcal{M})} =: \mathrm{B}.$$

Indeed, applying the noncommutative Khintchine inequality [15] in $L_p(\mathcal{M})$ twice

$$\begin{aligned} \mathrm{A}^p &= \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n] \\ |\mathsf{S}|=k}} \Big\| \sum_{\pi(j)\in\mathsf{S}} \varepsilon_{\pi(j)} \otimes s_j x_j \Big\|_{L_p(\Omega^n; L_p(\mathcal{M}))}^p \\ &\asymp_p \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n] \\ |\mathsf{S}|=k}} \max\bigg\{ \Big\| \Big(\sum_{\pi(j)\in\mathsf{S}} x_j^* x_j \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})}, \Big\| \Big(\sum_{\pi(j)\in\mathsf{S}} x_j x_j^* \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})} \bigg\} \\ &= \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n] \\ |\mathsf{S}|=k}} \max\bigg\{ \Big\| \Big(\sum_{j\in\mathsf{S}} x_j^* x_j \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})}, \Big\| \Big(\sum_{j\in\mathsf{S}} x_j x_j^* \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})} \bigg\} \asymp_p \mathrm{B}^p. \end{aligned}$$

Hence, we can apply (1-1) (with the choice $\mathcal{N} = L_\infty(\Sigma_{n,k})$) to get

$$\mathrm{B}^p \lesssim_p \frac{1}{n} \sum_{j,j'=1}^n \|\sigma_j\|_p^p \|x_{j'}\|_p^p + \Big(\frac{1}{n}\Big)^{\frac{p}{2}} \Big\| \Big(\sum_{j=1}^n x_j^* x_j + x_j x_j^* \Big)^{\frac{1}{2}} \Big\|_p^p \Big\| \Big(\sum_{j=1}^n \sigma_j^2 \Big)^{\frac{1}{2}} \Big\|_p^p.$$

Now, we have

$$\|\sigma_j\|_{L_p(\Sigma_{n,k})}^p = \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} 1_{\mathsf{S}}(j) = \frac{k}{n},$$

$$\Big\| \Big(\sum_{j=1}^n \sigma_j^2 \Big)^{\frac12} \Big\|_{L_p(\Sigma_{n,k})}^p = \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big(\sum_{j=1}^n 1_{\mathsf{S}}(j) \Big)^{\frac{p}{2}} = k^{\frac{p}{2}}.$$

Therefore, we get

$$\begin{aligned} \mathrm{B}^p &\lesssim_p \frac{k}{n} \sum_{j=1}^n \|x_j\|_{L_p(\mathcal{M})}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \Big\| \Big(\sum_{j=1}^n x_j^* x_j + x_j x_j^* \Big)^{\frac12} \Big\|_{L_p(\mathcal{M})}^p \\ &\asymp_p \frac{k}{n} \sum_{j=1}^n \|x_j\|_{L_p(\mathcal{M})}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \mathbb{E} \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|_{L_p(\mathcal{M})}^p, \end{aligned}$$

applying once again the noncommutative Khintchine inequality. This proves the result since the random variables σ_j are chosen so that B equals the left-hand side in the inequality of the statement. □

Remark 1.3. Theorem 1.2 says that $L_p(\mathcal{M})$ is a Banach X_p space. The conclusion also holds in the completely bounded setting since the constants that appear in the inequality of the statement do not depend on the von Neumann algebra $\mathcal{M}$.

1C. ***Proof of Theorem A.*** According to Theorem 1.1

$$\begin{aligned} &\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\| \sum_{g\in \mathrm{B}_{\mathsf{S}}} \hat{f}(g)\lambda(g) \Big\|_p^p \\ &\qquad = \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \|\mathsf{E}_{[n]\setminus\mathsf{S}} f\|_p^p \\ &\qquad \asymp_p \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\| \Big(\sum_{\substack{j\in[n]\\ \ell\geq 1}} |R_{j\ell}(\mathsf{E}_{[n]\setminus\mathsf{S}} f)|^2 \Big)^{\frac12} \Big\|_p^p + \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\| \Big(\sum_{\substack{j\in[n]\\ \ell\geq 1}} |R_{j\ell}((\mathsf{E}_{[n]\setminus\mathsf{S}} f)^*)|^2 \Big)^{\frac12} \Big\|_p^p =: \mathrm{A}+\mathrm{B}, \end{aligned}$$

where $R_{j\ell} := R_{u_{j\ell}}$ and $\{u_{j\ell} : j\in[n],\ \ell\geq 1\}$ is the orthonormal basis of $\mathcal{H}$ considered before the statement of Theorem A. Since $\beta(\mathrm{B}_{\mathsf{S}}) \subset \mathcal{H}_{\mathsf{S}}$, we observe that $\langle \beta(g), u_{j\ell}\rangle_\psi = 0$ whenever $g\in \mathrm{B}_{\mathsf{S}}$ and $j\notin \mathsf{S}$. Moreover, Fourier truncations commute with Riesz transforms — as both are Fourier multipliers — and we deduce

$$R_{j\ell} \circ \mathsf{E}_{[n]\setminus\mathsf{S}} = 1_{\mathsf{S}}(j)\ \mathsf{E}_{[n]\setminus\mathsf{S}} \circ R_{j\ell}.$$

Using the complete L_p-boundedness of our Fourier truncations, we get

$$\mathrm{A} \lesssim_p \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\| \Big(\sum_{j\in\mathsf{S}} \Big[\sum_{\ell\geq1} |R_{j\ell} f|^2 \Big] \Big)^{\frac12} \Big\|_p^p \lesssim_p \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \mathbb{E} \Big\| \sum_{j\in\mathsf{S}} \varepsilon_j \Big[\sum_{\ell\geq1} |R_{j\ell} f|^2 \Big]^{\frac12} \Big\|_p^p =: \mathrm{A}'.$$

The last inequality follows from either the scalar (if G is abelian) or the noncommutative Khintchine inequality [15] otherwise, applied to independent equidistributed signs $\varepsilon_j = \pm 1$. Next, we use the

Banach $\mathrm{X_p}$ nature of either commutative or noncommutative L_p-spaces. More precisely, applying Theorem 1.2 we get

$$\mathrm{A}' \lesssim_p \frac{k}{n}\sum_{j=1}^n \Big\| \Big(\sum_{\ell\geq 1} |R_{j\ell} f|^2 \Big)^{\frac12} \Big\|_p^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \mathbb{E} \Big\| \sum_{j=1}^n \varepsilon_j \Big[\sum_{\ell\geq 1} |R_{j\ell} f|^2 \Big]^{\frac12} \Big\|_p^p = \mathrm{A}_1' + \mathrm{A}_2'.$$

Since $R_{j\ell} = \partial_{u_{j\ell}} \Delta^{-1/2} = \Delta^{-1/2} \partial_{u_{j\ell}}$, [7, Proposition 1.1.5] yields

$$\mathrm{A}_1' = \frac{k}{n}\sum_{j=1}^n \Big\| \sum_{\ell\geq 1} R_{j\ell} f \otimes e_{\ell,1} \Big\|_{S_p[L_p(\mathcal{L}(\mathrm{G}))]}^p \lesssim_{p,\sigma} \frac{k}{n}\sum_{j=1}^n \Big\| \sum_{\ell\geq 1} \partial_{u_{j\ell}} f \otimes e_{\ell,1} \Big\|_{S_p[L_p(\mathcal{L}(\mathrm{G}))]}^p = \frac{k}{n}\sum_{j=1}^n \|\mathrm{D}_j(f)\|_p^p.$$

Moreover, the Khintchine inequality and Theorem 1.1 give

$$\mathrm{A}_2' \lesssim_p \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \Big\| \Big(\sum_{j=1}^n \sum_{\ell\geq 1} |R_{j\ell} f|^2 \Big)^{\frac12} \Big\|_p^p \lesssim_p \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_p^p.$$

Therefore, the term A satisfies the expected estimate and it remains to justify the assertion for B. We now analyze the behavior of our Fourier truncations under adjoints. Observe that

$$(\mathsf{E}_{[n]\setminus \mathsf{S}} f)^* = \sum_{g\in \mathrm{B_S}} \overline{\hat{f}(g)} \lambda(g^{-1}) =: \mathsf{E}'_{[n]\setminus \mathsf{S}}(f^*).$$

In particular, since $\mathsf{E}'_{[n]\setminus \mathsf{S}}$ commutes with $R_{j\ell}$

$$R_{j\ell}((\mathsf{E}_{[n]\setminus \mathsf{S}} f)^*) = \mathsf{E}'_{[n]\setminus \mathsf{S}}(R_{j\ell}(f^*)) = \mathsf{E}_{[n]\setminus \mathsf{S}}(R_{j\ell}(f^*)^*)^*.$$

This is the point where we need the condition $\beta(\mathrm{B_S^{-1}}) \subset \mathcal{H}_\mathsf{S}$, to make sure that the above terms vanish when $j \notin \mathsf{S}$ since we find the inner products $\langle \beta(g^{-1}), u_{j\ell}\rangle_\psi$ for $g \in \mathrm{B_S}$. Thus, we obtain

$$\begin{aligned}\Big\| \Big(\sum_{\substack{j\in[n]\\ \ell\geq 1}} |R_{j\ell}((\mathsf{E}_{[n]\setminus \mathsf{S}} f)^*)|^2 \Big)^{\frac12} \Big\|_p &= \Big\| \sum_{j\in[n]} \sum_{\ell\geq 1} \mathsf{E}_{[n]\setminus \mathsf{S}}(R_{j\ell}(f^*)^*) \otimes e_{1,(j,\ell)} \Big\|_p \\ &\lesssim_p \Big\| \sum_{j\in \mathsf{S}} \sum_{\ell\geq 1} R_{j\ell}(f^*) \otimes e_{(j,\ell),1} \Big\|_p = \Big\| \Big(\sum_{j\in \mathsf{S}} \sum_{\ell\geq 1} |R_{j\ell}(f^*)|^2 \Big)^{\frac12} \Big\|_p.\end{aligned}$$

Therefore, we may follow the above argument for A just replacing f by f^*. □

Remark 1.4. A careful reading of the proof of Theorems A and B shows that we may use different Hilbert space decompositions $\mathcal{H} = \oplus_j \mathcal{H}_j = \oplus_j \mathcal{K}_j$ for $\mathrm{B_S}$ and its inverse — with $\beta(\mathrm{B_S}) \subset \mathcal{H}_\mathsf{S}$ and $\beta(\mathrm{B_S^{-1}}) \subset \mathcal{K}_\mathsf{S}$ — as long as we can find an orthonormal basis $\{u_\ell : \ell \geq 1\}$ of $\mathcal{H}$ satisfying that

$$\text{for all } \ell \geq 1 \text{ there exists } j_1, j_2 \in [n] \text{ such that } u_\ell \in \mathcal{H}_{j_1} \cap \mathcal{K}_{j_2}. \tag{1-2}$$

More precisely, under this more flexible assumption we get

$$\frac{1}{\binom{n}{k}} \sum_{|\mathsf{S}|=k} \Big\| \sum_{g\in \mathrm{B_S}} \hat{f}(g)\lambda(g) \Big\|_p^p \lesssim_{p,\sigma} \frac{k}{n}\sum_{j=1}^n [\|\mathrm{D}_j(f)\|_p^p + \|\mathrm{D}_j^\dagger(f^*)\|_p^p] + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_p^p,$$

where $\mathrm{D}_j^\dagger := \sum_{u_\ell \in \mathcal{K}_j} \partial_{u_\ell}(\,\cdot\,) \otimes e_{\ell 1}$ is the gradient over the basis vectors living in $\mathcal{K}_j$.

Remark 1.5. The constant depending on σ in Theorem A grows as $\sigma^{-p/2}$. One can also track the dependence on p of the constant. Using free generators in place of random signs — Theorem 1.2 holds as well — we keep constants uniformly bounded replacing noncommutative by free Khintchine inequalities [4]. The constants in Theorem 1.1 are bounded by $p^{3/2}$, but it is still open whether this is optimal.

1D. ***Proof of Theorem B.*** Again Theorem 1.1 gives

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \|\mathsf{E}_{[n]\setminus\mathsf{S}} f\|_p^p \asymp_p \mathrm{A}+\mathrm{B}$$

as in the proof of Theorem A. Following our argument there, we use our estimate $\mathrm{A} \lesssim_p \mathrm{A}_1' + \mathrm{A}_2'$ and we bound A_2' in the same way. To estimate A_1' we use our distinguished family of derivatives and Theorem 1.1 to deduce

$$\mathrm{A}_1' = \frac{k}{n} \sum_{j=1}^{n} \Big\| \Big(\sum_{\ell\geq 1} |R_{u_{j\ell}} \partial_j f|^2 \Big)^{\frac{1}{2}} \Big\|_p^p \lesssim_p \frac{k}{n} \sum_{j=1}^{n} \|\partial_j f\|_p^p.$$

The estimate for B then follows by the same considerations as in Theorem A. □

2. Applications to abelian groups

We now focus our attention on concrete realizations of Theorems A and B for certain commutative group algebras. In all the cases in this section, we choose $\mathsf{E}_{[n]\setminus\mathsf{S}}$ of the form

$$\mathsf{E}_{[n]\setminus\mathsf{S}} f = \sum_{g\in \mathrm{B}_\mathsf{S}} \hat{f}(g)\lambda(g) \quad \text{for some subgroup } \mathrm{B}_\mathsf{S} \text{ of } \mathrm{G}.$$

Due to that fact, we know that they are conditional expectations, and therefore completely contractive maps. This allows us to safely apply Theorems A and B without checking that hypothesis. We will give the details for the cases $\mathrm{G}=\mathbb{Z}^n$ and $\mathrm{G}=\mathbb{Z}_{2m}^n$, yielding inequalities in $L_p(\mathbb{T}^n)$ and $L_p(\mathbb{Z}_{2m}^n)$, respectively. In this section, we must change to additive notation in our groups given their natural operations (and we reserve the product notation for the usual product of integer/real numbers). The necessary adjustments for the hypercube are discussed at the end of the section.

2A. ***Classical tori.*** Define

$$\psi_1(g) = |g_1| + \cdots + |g_n|,$$
$$\psi_2(g) = g_1^2 + g_2^2 + \cdots + g_n^2,$$

with $g=(g_1, g_2, \ldots, g_n)\in\mathbb{Z}^n$. Both functions are symmetric and vanish at 0. Moreover, conditional negativity follows easily. In the case of ψ_1, it suffices to check it for each summand $|g_j|$ which is conditionally negative from subordination with respect to g_j^2. These functions are denoted as the word and the Euclidean length respectively. We analyze balanced Fourier truncations using both geometries.

(A) The Euclidean length. The length ψ_2 induces the standard cocycle $(\mathcal{H}, \alpha, \beta)$, where $\mathcal{H}=\mathbb{R}^n$ with the usual Euclidean inner product, the trivial action and the canonical inclusion $\beta=\mathrm{id}$. We use the standard decomposition $\mathcal{H}=\bigoplus_j \mathcal{H}_j$, with $\mathcal{H}_j=\mathbb{R}e_j$ the subspace generated by the j-th element of the canonical basis. Therefore, given $\mathsf{S} \subset [n]$, denote by $\mathbb{Z}^{\mathsf{S}}$ the subgroup of elements with vanishing entries outside S and consider the truncations

$$\mathsf{E}_{[n]\setminus\mathsf{S}} f(x) = \sum_{g\in\mathbb{Z}^{\mathsf{S}}} \hat{f}(g) e^{2\pi i\langle x,g\rangle} \quad \text{for any } f \in L_p(\mathbb{T}^n) \simeq L_p(\mathcal{L}(\mathbb{Z}^n)),$$

where $e^{2\pi i\langle\,\cdot\,,g\rangle} \mapsto \lambda(g)$ defines a trace-preserving $*$-homomorphism. The cocycle derivatives correspond in this case |up to a multiplicative constant — to the classical derivatives $(\partial/\partial x_j)$, and the infinitesimal generator Δ is the usual Laplacian (up to a universal constant) with spectral gap 1. Then, Theorem A yields

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\| \sum_{g\in\mathbb{Z}^{\mathsf{S}}} \hat{f}(g) e^{2\pi i\langle\,\cdot\,,g\rangle} \Big\|_{L_p(\mathbb{T}^n)}^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \Big\| \frac{\partial}{\partial x_j} f \Big\|_{L_p(\mathbb{T}^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathbb{T}^n)}^p \tag{2-1}$$

for any mean-zero $f \in L_p(\mathbb{T}^n)$. This seems to be the most natural generalization of Naor's inequality for classical tori, but it is not the most efficient. Indeed, using the same Hilbert space decomposition as above, one can consider the alternative absorbent derivatives $\partial_j \lambda(g) = \delta_{g_j\neq 0}\lambda(g)$. In particular Theorem B yields

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\| \sum_{g\in\mathbb{Z}^{\mathsf{S}}} \hat{f}(g) e^{2\pi i\langle\,\cdot\,,g\rangle} \Big\|_{L_p(\mathbb{T}^n)}^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_j f\|_{L_p(\mathbb{T}^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathbb{T}^n)}^p, \tag{2-2}$$

where, abusing notation, we define $\partial_j e^{2\pi i\langle\,\cdot\,,g\rangle} = \delta_{g_j\neq 0} e^{2\pi i\langle\,\cdot\,,g\rangle}$. This is a stronger inequality since

$$\|\partial_j f\|_{L_p(\mathbb{T}^n)} = \frac{1}{2\pi} \Big\| \sum_{g_j\neq 0} \frac{1}{g_j} \Big(\frac{\partial}{\partial x_j} f\Big)^{\wedge}(g) e^{2\pi i\langle\,\cdot\,,g\rangle} \Big\|_{L_p(\mathbb{T}^n)} \leq C_p \Big\| \frac{\partial}{\partial x_j} f \Big\|_{L_p(\mathbb{T}^n)}.$$

Indeed, the symbol $m(g) = 1/g_j$ defines an L_p-bounded multiplier as a consequence of the K. de Leeuw restriction theorem and the Hörmander–Mikhlin multiplier theorem [5; 14; 18]. As we shall see (2-2) naturally appears using the word length.

Remark 2.1. Consider $f : \mathbb{T}^n \to \mathbb{C}$ with

$$f(x) = \sum_{g\in\mathbb{Z}^n} \hat{f}(g) e^{2\pi i\langle x,g\rangle} \quad \text{and} \quad \hat{f}(0) = 0.$$

Given $\mathsf{S} \subset [n]$, the classical Poincaré inequality gives

$$\begin{aligned}
\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subset[n]\\|\mathsf{S}|=k}} \Big\| \underbrace{\sum_{g\in\mathbb{Z}^{\mathsf{S}}\setminus\{0\}} \hat{f}(g) e^{2\pi i\langle\,\cdot\,,g\rangle}}_{f_{\mathsf{S}}} \Big\|_p^p &\leq \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subset[n]\\|\mathsf{S}|=k}} \| |\nabla f_{\mathsf{S}}| \|_p^p \asymp \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subset[n]\\|\mathsf{S}|=k}} \Big\| \sum_{j\in\mathsf{S}} \varepsilon_j \frac{\partial}{\partial x_j} f_{\mathsf{S}} \Big\|_p^p \\
&= \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subset[n]\\|\mathsf{S}|=k}} \Big\| \Big[\sum_{j\in\mathsf{S}} \varepsilon_j \frac{\partial}{\partial x_j} f \Big]_{\mathsf{S}} \Big\|_p^p \\
&\leq \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subset[n]\\|\mathsf{S}|=k}} \Big\| \sum_{j\in\mathsf{S}} \varepsilon_j \frac{\partial}{\partial x_j} f \Big\|_p^p = \Big\| \sum_{j=1}^n \sigma_j \frac{\partial}{\partial x_j} f \Big\|_p^p
\end{aligned}$$

for $\sigma_j(\varepsilon, \mathsf{S}) = \varepsilon_j \otimes 1_{\mathsf{S}}(j)$ as in the Introduction. Applying [6] gives

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subset [n] \\ |\mathsf{S}|=k}} \Big\| \sum_{g \in \mathbb{Z}^{\mathsf{S}}} \hat{f}(g) e^{2\pi i \langle \cdot , g \rangle} \Big\|_p^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \Big\| \frac{\partial}{\partial x_j} f \Big\|_p^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \| |\nabla f| \|_p^p.$$

Inequalities (2-1) and (2-2) improve the above inequality replacing $|\nabla f|$ by f.

(B) The word length. Let us now study which inequality do we get with the word length. The cocycle associated to it is infinite-dimensional, with an orthonormal basis which can be described as oriented edges in the coordinate axes of the Cayley graph of $\mathbb{Z}^n$. More precisely, the associated Gromov form on $\mathbb{D}[\mathbb{Z}^n]$ is

$$\langle 1_{\{g\}}, 1_{\{h\}} \rangle_{\psi_1} = \tfrac{1}{2}(\psi_1(g) + \psi_1(h) - \psi_1(h-g)) = \sum_{j=1}^n \min\{|h_j|, |g_j|\} \delta_{g_j \cdot h_j > 0}.$$

Given $g \in \mathbb{Z}^n$ and $j \in [n]$, define

$$g_{[j]}^- = g - \operatorname{sgn}(g_j) e_j, \quad \text{with } \operatorname{sgn}(0) = 0.$$

Then, we may construct the following elements in $\mathbb{D}[\mathbb{Z}^n]$:

$$w_{g,j} = 1_{\{g\}} - 1_{\{g_{[j]}^-\}} \quad \text{and} \quad u_j(\ell) = w_{\ell e_j, j}.$$

Below, it is convenient to keep in mind that $u_j(\ell) = 1_{\{\ell e_j\}} - 1_{\{(\ell - \operatorname{sgn}(\ell)) e_j\}}$. If $\mathcal{H}_{\psi_1} = \mathbb{D}[\mathbb{Z}^n] / \operatorname{Ker}\langle \cdot , \cdot \rangle_{\psi_1}$, the following properties define an orthonormal basis:

- $\langle u_j(\ell), u_j(\ell) \rangle_{\psi_1} = 1$ for all $(j, \ell) \in [n] \times \mathbb{Z} \backslash \{0\}$.
- $\langle u_j(\ell), u_{j'}(\ell') \rangle_{\psi_1} = 0$ whenever $j \neq j'$ or $\ell \neq \ell'$.
- $w_{g,j} = u_j(\ell)$ if $g_j = \ell e_j$, since the difference belongs to $\operatorname{Ker}\langle \cdot , \cdot \rangle_{\psi_1}$.

Altogether, this implies that the image in $\mathcal{H}_{\psi_1}$ of the set

$$\{u_j(\ell) : (j, \ell) \in [n] \times \mathbb{Z} \setminus \{0\}\}$$

is an orthonormal basis for $\mathcal{H}_{\psi_1}$. We shall identify $1_{\{g\}}$ and $u_j(\ell)$ with their image in the quotient. The cocycle map is given by $\beta(g) = 1_{\{g\}}$ and the orthogonal action α satisfies $\alpha_g(1_{\{h\}}) = 1_{\{g+h\}} - 1_{\{g\}}$. This means that for any $g \in \mathbb{Z}^n$ we have

$$\alpha_g(u_j(\ell)) = 1_{\{g + \ell e_j\}} - 1_{\{g + \ell e_j - (\operatorname{sgn}(\ell)) e_j\}}.$$

Therefore, the subspaces $\mathcal{H}_{\psi_1, j} = \operatorname{span}\{u_j(\ell) : \ell \in \mathbb{Z} \setminus \{0\}\}$ are α-invariant for $j \in [n]$. This proves that the same conditional expectations $\mathsf{E}_{[n] \setminus \mathsf{S}}$ considered before still define an admissible family of Fourier truncations. The cocycle derivative associated to $u_j(\ell)$ acts as follows:

$$\begin{aligned} \partial_{u_j(\ell)} \lambda(g) &= \langle u_j(\ell), 1_{\{g\}} \rangle_{\psi_1} \lambda(g) \\ &= (\min\{|g_j|, |\ell|\} \delta_{g_j \cdot \ell > 0} - \min\{|g_j|, |\ell| - 1\} \delta_{g_j \cdot (\ell - \operatorname{sgn}(\ell)) > 0}) \lambda(g) \\ &= \delta_{\{g_j \cdot \ell > 0, |g_j| \geq |\ell|\}} \lambda(g). \end{aligned}$$

The Laplacian is

$$\Delta_{\psi_1} f = \sum_{g\in\mathbb{Z}^n} \psi_1(g)\hat{f}(g)\lambda(g),$$

whose spectral gap is still $\sigma = \min_j \psi_1(e_j) = 1$. Theorem A yields

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \Big\| \sum_{g\in\mathbb{Z}^{\mathsf{S}}} \hat{f}(g)e^{2\pi i\langle\cdot,g\rangle}\Big\|_{L_p(\mathbb{T}^n)}^p \lesssim_p \frac{k}{n}\sum_{j=1}^n \||\mathrm{D}_j f|\|_{L_p(\mathbb{T}^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathbb{T}^n)}^p, \tag{2-3}$$

with

$$\||\mathrm{D}_j(f)|\|_{L_p(\mathbb{T}^n)} = \||\mathrm{D}_j(f^*)|\|_{L_p(\mathbb{T}^n)} = \Big\|\Big(\sum_{\ell\in\mathbb{Z}\setminus\{0\}} |\partial_{u_j(\ell)}f|^2\Big)^{\frac{1}{2}}\Big\|_{L_p(\mathbb{T}^n)}.$$

Remark 2.2. Note that $|\partial_{u_j(\ell)}(f)| \neq |\partial_{u_j(\ell)}(f^*)|$. Thus, nontrivial cocycle actions lead to noncommutative phenomena even when working with abelian groups, as pointed out in [13]. In spite of that, observe that $\langle 1_{\{-g\}}, u_j(\ell)\rangle_\psi = \langle 1_{\{g\}}, u_j(-\ell)\rangle_\psi$, which implies $\||\mathrm{D}_j(f)|\|_p = \||\mathrm{D}_j(f^*)|\|_p$ as claimed above.

On the other hand, taking

$$\partial_j\lambda(g) := \partial_{u_j(1)}\lambda(g) + \partial_{u_j(-1)}\lambda(g) = \delta_{g_j\neq 0}\lambda(g),$$

we get $\partial_{u_j(\ell)} \circ \partial_j = \partial_{u_j(\ell)}$ for any $(j,\ell) \in [n]\times\mathbb{Z}\setminus\{0\}$. Thus, Theorem B gives

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \Big\| \sum_{g\in\mathbb{Z}^{\mathsf{S}}} \hat{f}(g)e^{2\pi i\langle\cdot,g\rangle}\Big\|_{L_p(\mathbb{T}^n)}^p \lesssim_p \frac{k}{n}\sum_{j=1}^n \|\partial_j f\|_{L_p(\mathbb{T}^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathbb{T}^n)}^p$$

for any mean-zero $f \in L_p(\mathbb{T}^n)$. Here, ∂_j is the same as in (A), and so this recovers inequality (2-2) and improves (2-3). Note that above, both $\partial_{u_j(\ell)}$ and ∂_j are $\{0,1\}$-valued multipliers.

2B. ***Discrete tori.*** Consider the word length $|g| = \min\{g, 2m-g\}$ in $\mathbb{Z}_{2m} = \widehat{\mathbb{Z}}_{2m}$. Therefore, for us $\mathbb{Z}_{2m} = \{0,1,\dots,2m-1\}$. As shown in [12], $|\cdot|$ defines a conditionally negative symmetric length. In particular the same holds for the corresponding length in the product $\mathbb{Z}_{2m}^n$

$$\psi(g) = |g_1| + |g_2| + \cdots + |g_n| \quad \text{for } g = (g_1,\dots,g_n) \in \mathbb{Z}_{2m}^n.$$

This word length has many similarities with the previous one

$$\langle 1_{\{g\}}, 1_{\{h\}}\rangle_\psi = \frac{1}{2}(\psi(g)+\psi(h)-\psi(h-g)) = \frac{1}{2}\sum_{j=1}^n |g_j|+|h_j|-|h_j-g_j|.$$

Given $g \in \mathbb{Z}_{2m}^n$ and $j \in [n]$, define

$$w_{g,j} = 1_{\{g\}} - 1_{\{g-e_j\}} \quad \text{and} \quad u_j(\ell) = w_{\ell e_j, j} \quad \text{for } 1 \le \ell \le 2m.$$

If $\mathcal{H}_\psi = \mathbb{D}[\mathbb{Z}_{2m}^n]/\operatorname{Ker}\langle\cdot,\cdot\rangle_\psi$, we find that:

- $\langle u_j(\ell), u_j(\ell)\rangle_\psi = 1$ for all $(j,\ell) \in [n]\times[m]$.
- $\langle u_j(\ell), u_{j'}(\ell')\rangle_\psi = 0$ whenever $j \neq j'$ or $\ell \neq \ell', \ell'+m$.

- $w_{g,j} = u_j(\ell)$ if $g_j = \ell e_j$ since the difference belongs to $\mathrm{Ker}\langle \cdot, \cdot \rangle_\psi$.
- $u_j(\ell) = -u_j(\ell + m)$ since the difference belongs to $\mathrm{Ker}\langle \cdot, \cdot \rangle_\psi$.
- $\langle 1_{\{\ell e_j\}}, 1_{\{\ell' e_j\}}\rangle_\psi = \min\{\ell, 2m - \ell', \max\{0, m - \ell' + \ell\}\}$ for $1 \leq \ell \leq \ell' \leq 2m$.

Altogether, this implies that the set

$$\{u_j(\ell) : (j, \ell) \in [n] \times [m]\}$$

is an orthonormal basis for $\mathcal{H}_\psi$. The cocycle map is given by $\beta(g) = 1_{\{g\}}$, by which we mean again that $\beta(g)$ is the image of $1_{\{g\}}$ in the quotient, and the orthogonal action α satisfies $\alpha_g(1_{\{h\}}) = 1_{\{g+h\}} - 1_{\{g\}}$. This means that for any $g \in \mathbb{Z}_{2m}^n$ we have

$$\alpha_g(u_j(\ell)) = 1_{\{g+\ell e_j\}} - 1_{\{g+(\ell-1)e_j\}}.$$

Therefore, the subspaces $\mathcal{H}_{\psi,j} = \mathrm{span}\{u_j(\ell) : \ell \in [m]\}$ give again an α-invariant splitting of $\mathcal{H}_\psi$ with j running over $[n]$. In particular, the conditional expectations $\mathsf{E}_{[n]\setminus \mathsf{S}}$ over the subgroups $\mathbb{Z}_{2m}^{\mathsf{S}}$ define an admissible family of Fourier truncations and the cocycle derivatives are given by

$$\partial_{u_j(\ell)} \lambda(g) = \delta_{\{\ell \leq g_j < \ell + m\}} \lambda(g).$$

The associated Laplacian has spectral gap equal to 1. As before, Theorem A yields a statement that we omit because it can readily be improved. If we set as before $\partial_j \lambda(g) := \delta_{g_j \neq 0} \lambda(g)$ for $j \in [n]$, we immediately see that $\partial_{u_j(\ell)} \circ \partial_j = \partial_{u_j(\ell)}$. Moreover, we can rewrite it as

$$\partial_j \lambda(g) = \partial_{u_j(1)} \lambda(g) + \partial_{u_j(m)} \lambda(g) - \delta_{g_j = m} \lambda(g).$$

Next, note that we may write the last term as

$$\delta_{g_j=m} \lambda(g) = \mathsf{E}_{\{0,m\},j}(\partial_{u_j(1)} \lambda(g)) = \tfrac{1}{2} \mathsf{E}_{\{0,m\},j}(\partial_{u_j(1)} \lambda(g) + \partial_{u_j(m)} \lambda(g)),$$

where $\mathsf{E}_{\{0,m\},j}$ is the conditional expectation onto $\mathbb{Z}_{2m}^{j-1} \times \{0, m\} \times \mathbb{Z}_{2m}^{n-j}$. Then, after applying Theorem B one gets the following for mean-zero $f : \mathbb{Z}_{2m}^n \to \mathbb{C}$:

$$\begin{aligned} \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}|=k}} \Big\| \sum_{g \in \mathbb{Z}_{2m}^{\mathsf{S}}} \hat{f}(g) e^{\frac{\pi i}{m}\langle \cdot, g\rangle} \Big\|_{L_p(\mathbb{Z}_{2m}^n)}^p &\lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_j f\|_{L_p(\mathbb{Z}_{2m}^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathbb{Z}_{2m}^n)}^p \\ &\lesssim \frac{k}{n} \sum_{j=1}^n \|(\partial_{u_j(1)} + \partial_{u_j(m)}) f\|_{L_p(\mathbb{Z}_{2m}^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathbb{Z}_{2m}^n)}^p. \end{aligned} \tag{2-4}$$

As before, both derivatives $\partial_{u_j(\ell)}$ and ∂_j turn out to be $\{0, 1\}$-valued multipliers.

Remark 2.3. It is natural to ask if the situation changes much when the cyclic groups under consideration have odd cardinal. The function $\psi(g) = \sum_{j=1}^n |g_j|$, with $|g_j| = \min\{g_j, 2m + 1 - g_j\}$, is a conditionally negative length on $\mathbb{Z}_{2m+1}^n$, and so there exists an associated cocycle induced by the Gromov form

$$\langle 1_{\{g\}}, 1_{\{h\}} \rangle = \tfrac{1}{2}(\psi(g) + \psi(h) - \psi(g-h))$$
$$= \sum_{j=1}^{n} \min\bigl\{g_j, 2m+1-h_j, \max\bigl\{0, m-h_j+g_j+\tfrac{1}{2}\bigr\}\bigr\}.$$

It defines a cocycle Hilbert space $\mathcal{H}_\psi$ with dimension $2mn$. Theorems A and B apply but calculating an explicit expression for the orthonormal basis of $\mathcal{H}_\psi$ is more tedious and we shall leave it to the interested reader.

We end this subsection with a few comments on the case $m = 1$ of the above construction.

Remark 2.4. Inequality (2-4) for $\mathrm{G} = \mathbb{Z}_{2m}^n$ with $m = 1$ recovers Naor's inequality (N_p) for the hypercube. Indeed, specializing the above computations in this case means that we take $\mathcal{H} = \mathbb{D}[\mathbb{Z}_2^n]/\operatorname{Ker}\langle\cdot,\cdot\rangle_\psi$ and consider the trivial cocycle β_0 given by $\{0,1\}^n \ni g \mapsto 1_{\{g\}}$, with the action

$$\alpha_{0,g}(\xi) = ((-1)^{g_1}\xi_1, (-1)^{g_2}\xi_2, \ldots, (-1)^{g_n}\xi_n).$$

With this construction, given $L_p(\mathbb{Z}_2^n) \ni f = \sum_{g\in\mathbb{Z}_2^n} \hat{f}(g)\exp(\pi i\langle\cdot, g\rangle)$, $\partial_j^1 = 2\partial_j^2$, where ∂_j^1 is the discrete derivative used by Naor and ∂_j^2 is our choice of ∂_j in (2-4) for $m = 1$.

Remark 2.5. One can also recover (N_p) from Theorem A using multiplicative notation directly. This however requires us to employ a nontrivial cocycle that we next describe. Set $\mathrm{G} = \Omega^n$, $e = (1, \ldots, 1)$, and define the cocycle $\beta_1 : \mathrm{G} \to \mathbb{R}^n$ by $\mathrm{G} \ni h \mapsto e - h$ (the sum is the usual one in $\mathbb{R}^n$). This satisfies the cocycle law with respect to the — nontrivial — action

$$\alpha_{1,h}(\xi) = (h_1\xi_1, \ldots, h_n\xi_n), \quad \xi \in \mathbb{R}^n.$$

One can see that if $g \in \mathbb{Z}_2^n$ is identified in the natural way with

$$h(g) := (\exp(\pi i g_1), \ldots, \exp(\pi i g_1)) \in \Omega^n,$$

then

$$\beta_1(h(g)) = 2\beta_0(g).$$

Therefore, the cocycle derivatives are the same, up to a constant and modulo identification of characters, and the application of Theorem A yields the same inequality in both cases.

Remark 2.6. We can consider weighted forms of Naor's inequality by considering different measures on the same group Ω^n to get different cocycle representations. One could hope to get an improvement over the result in [19] in this way, but we next show that this is not the case. We borrow the aforementioned cocycle representations from the construction in Example B in [13, Section 1.4], that we use as follows: Let $\mathrm{G} = \{-1,1\}^n$ and equip $\Gamma = \widehat{\mathrm{G}} = \{-1,1\}^n$ with the measure

$$\mu = \sum_{j=1}^{n} \alpha_j 1_{\{w_j\}},$$

with $\alpha_j \geq 0$ and $w_j = (1, \dots, 1, -1, 1, \dots, 1)$ (change the sign in the j-th coordinate only) for $j \in [n]$. Viewing Γ as the power set of $[n]$, we identify w_j with $\{j\}$. We consider the conditionally negative length function

$$\psi(\mathsf{A}) := \|1 - W_{\mathsf{A}}\|^2_{L_2(\Gamma,\mu)}.$$

Then ψ may be represented by the cocycle $(\mathcal{H}_\psi, \alpha, \beta)$, with

$$\mathcal{H}_\psi = L_2(\Gamma, \mu), \quad \alpha_{\mathsf{A}}(u) = W_{\mathsf{A}} \cdot u, \quad \beta(\mathsf{A}) = 1 - W_{\mathsf{A}}.$$

The map β is indeed a cocycle. Then $\{u_j = \alpha_j^{-1/2} 1_{\{j\}} : j \in [n]\}$ is an orthonormal basis, and the cocycle derivatives are given by

$$\partial_{u_j} W_{\mathsf{A}} = \frac{1}{\sqrt{\alpha_j}} \langle \beta(\mathsf{A}), 1_{\{j\}} \rangle_\psi W_{\mathsf{A}} = 2\sqrt{\alpha_j} 1_{\mathsf{A}}(j) W_{\mathsf{A}} = \sqrt{\alpha_j} \partial_j W_{\mathsf{A}},$$

where ∂_j denotes the j-th discrete derivative. Riesz transforms take the form

$$R_{u_j} f = \sum_{\mathsf{A} \subset [n]} \frac{\langle \beta(\mathsf{A}), 1_{\{j\}} \rangle_\psi}{\sqrt{\alpha_j \psi(\mathsf{A})}} \hat{f}(\mathsf{A}) W_{\mathsf{A}} = \sum_{\substack{\mathsf{A} \subset [n] \\ j \in \mathsf{A}}} \frac{\sqrt{\alpha_j}}{\sqrt{\sum_{\ell \in A} \alpha_\ell}} \hat{f}(\mathsf{A}) W_{\mathsf{A}}.$$

Consider the decomposition $\mathcal{H}_{\psi,j} = \mathbb{R} 1_{\{j\}}$. Note $\alpha_{\mathsf{A}}(1_{\{j\}}) = W_{\mathsf{A}} 1_{\{j\}} = (-1)^{1_{\mathsf{A}}(j)} 1_{\{j\}}$, so $\alpha_{\mathsf{A}}(\mathcal{H}_{\psi,j}) \subset \mathcal{H}_{\psi,j}$ and the decomposition is equivariant. Therefore, the associated conditional expectation can be chosen to be

$$\mathsf{E}_{[n]\setminus\mathsf{S}} f = \sum_{\beta(\mathsf{A}) \in \mathcal{H}_{\mathsf{S}}} \hat{f}(\mathsf{A}) W_{\mathsf{A}} = \sum_{\mathsf{A} \subset \mathsf{S}} \hat{f}(\mathsf{A}) W_{\mathsf{A}}.$$

Then, Theorem A yields

$$\begin{aligned} \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}| = k}} \Big\| \sum_{\mathsf{A} \subset \mathsf{S}} \hat{f}(\mathsf{A}) W_{\mathsf{A}} \Big\|^p_{L_p(\Omega^n)} &\lesssim \frac{1}{\sigma^{p/2}} \frac{k}{n} \sum_{j=1}^n \alpha_j^{\frac{p}{2}} \|\partial_j f\|^p_{L_p(\Omega^n)} + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|^p_{L_p(\Omega^n)} \\ &= \frac{k}{n} \sum_{j=1}^n \Big(\frac{\alpha_j}{\min_{k \in [n]} \alpha_k} \Big)^{\frac{p}{2}} \|\partial_j f\|^p_{L_p(\Omega^n)} + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|^p_{L_p(\Omega^n)}, \end{aligned}$$

since $\sigma = \min_{k \in [n]} \psi(\{k\}) = 4 \min_{k \in [n]} \alpha_k$. Thus taking $\alpha_j = 1$ for all j, which corresponds to (N_p), is the optimal choice.

3. Applications to free products

We now explore applications of Theorem B after replacing the direct products in the previous section by free products. Given a free product $\mathrm{G} = \mathrm{G}_1 * \mathrm{G}_2 * \cdots * \mathrm{G}_n$ a general element $g \in \mathrm{G}$ can always be written in reduced form $g = g_{i_1} g_{i_2} \cdots g_{i_s}$ where $g_{i_k} \in \mathrm{G}_{i_k}$ and $i_1 \neq i_2 \neq \cdots \neq i_s$. We shall be working with the free group $\mathbb{F}_n = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$ and with the free product $\mathbb{Z}_{2m}^{*n}$ of n copies of $\mathbb{Z}_{2m}$. In both cases we shall write $e_1, e_2, \dots, e_n$ for the canonical generators and a generic element will be a word of the form

$$w = e_{i_1}^{\ell_1} e_{i_2}^{\ell_2} \cdots e_{i_s}^{\ell_s},$$

with $i_1 \neq i_2 \neq \cdots \neq i_s$ and ℓ_k in $\mathbb{Z}$ or $\mathbb{Z}_{2m}$ accordingly.

3A. ***The free group.*** Define

$$|w| = \sum_{j=1}^{r} |\ell_j| \quad \text{for } w = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r}.$$

Haagerup proved in [3] that it is conditionally negative. The cocycle structure naturally induced by the word length $|\cdot|$ can be described through the Hilbert space orthonormaly generated by outgoing oriented edges in its Cayley graph. To be more precise, let us consider the following partial order on $\mathbb{F}_n$. Given $w_1 = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r}$ and $w_2 = e_{j_1}^{t_1} \cdots e_{j_s}^{t_s}$, with $\ell_j, t_j \in \mathbb{Z} \setminus \{0\}$, we say that $w_1 \leq w_2$ when

- $r \leq s$,
- $e_{i_k}^{\ell_k} = e_{j_k}^{t_k}$ for $1 \leq k \leq r-1$,
- $e_{i_r} = e_{j_r}$, $\ell_r t_r > 0$ and $|\ell_r| \leq |t_r|$.

Any $w_1 \leq w_2$ is called an initial subchain of w_2. As we did with elements of cyclic groups equipped with their natural order structure, we can now define predecessors. If $w = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r} \neq e$, we define

$$w^- = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r - \operatorname{sgn}(\ell_r)}.$$

The Gromov form takes the following form in this case:

$$\langle 1_{\{w_1\}}, 1_{\{w_2\}} \rangle_{|\cdot|} = \tfrac{1}{2}(|w_1| + |w_2| - |w_1^{-1} w_2|) = |\min\{w_1, w_2\}|,$$

where $\min\{w_1, w_2\}$ denotes the longest word which is an initial chain of both w_1 and w_2. Given $w \neq e$ in $\mathbb{F}_n$, we define $u_w = 1_{\{w\}} - 1_{\{w^-\}} \in \mathbb{D}[\mathbb{F}_n]$. Very much like in the previous section, we find the following properties:

- $\operatorname{Ker}(\langle \cdot, \cdot \rangle_{|\cdot|}) = \mathbb{R} 1_{\{e\}}$.
- $\langle u_w, u_w \rangle_{|\cdot|} = 1$ for $w \in \mathbb{F}_n \setminus \{e\}$.
- $\langle u_{w_1}, u_{w_2} \rangle_{|\cdot|} = 0$ for $w_1 \neq w_2$ in $\mathbb{F}_n$.

This proves that

$$\{u_w : w \in \mathbb{F}_n \setminus \{e\}\}$$

is an orthonormal basis of $\mathcal{H}_{|\cdot|} = \mathbb{D}[\mathbb{F}_n]/\mathbb{R}1_{\{e\}}$. The cocycle map and the cocycle action are determined as usual by $\beta(w) = 1_{\{w\}}$ and $\alpha_w(1_{\{w'\}}) = 1_{\{ww'\}} - 1_{\{w\}}$.The cocycle derivative in the direction of u_w is

$$\partial_{u_w} \lambda(w') = \langle \beta(w'), u_w \rangle \lambda(w') = \delta_{w \leq w'} \lambda(w') \quad \Longrightarrow \quad \partial_{u_w} f = \sum_{w \leq w'} \hat{f}(w') \lambda(w').$$

Next, we decompose $\mathcal{H}_{|\cdot|}$ as

$$\mathcal{H}_{|\cdot|} = \bigoplus_{j=1}^{n} \mathcal{H}_{|\cdot|, j}, \quad \text{with } \mathcal{H}_{|\cdot|, j} = \operatorname{span}\{u_w : e_j \leq w \text{ or } e_j^{-1} \leq w\}.$$

This leads to consider the Fourier truncations

$$\mathsf{E}_{[n] \setminus \mathsf{S}} f := \sum_{w \in \mathbb{F}_{\mathsf{S}}} \hat{f}(w) \lambda(w).$$

Being conditional expectations, these Fourier truncations are completely contractive and pairwise β-orthogonality holds since we trivially have $\beta(\mathbb{F}_\mathsf{S}) = \beta(\mathbb{F}_\mathsf{S}^{-1}) \subset \mathcal{H}_{|\cdot|,\mathsf{S}}$. Define $\mathbb{A}_{\{j\}} \subset \mathbb{F}_n$ to be the set of reduced words that start with e_j^ℓ for some $\ell \in \mathbb{Z} \setminus \{0\}$, so that $\mathcal{H}_{|\cdot|,j} = \operatorname{span}\{u_w : w \in \mathbb{A}_{\{j\}}\}$. Taking the derivatives

$$\partial_j := \partial_{u_{e_j}} + \partial_{u_{e_j^{-1}}} \quad \text{for } j \in [n],$$

we can readily check that $\partial_{u_w} \circ \partial_j = \partial_{u_w}$ whenever $u_w \in \mathcal{H}_{|\cdot|,j}$, so that ∂_j is the projection onto words in $\mathbb{A}_{\{j\}}$. In conclusion, we have checked all the hypotheses to apply Theorem B for our family of Fourier truncations. In this case we get

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subset[n]\\|\mathsf{S}|=k}} \Big\| \sum_{w\in\mathbb{F}_\mathsf{S}} \hat{f}(w)\lambda(w) \Big\|_p^p \lesssim_p \frac{k}{n} \sum_{j=1}^n [\|\partial_j(f)\|_p^p + \|\partial_j(f^*)\|_p^p] + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_p^p. \tag{3-1}$$

Inequality (3-1) is very close to the conjectured free form of Naor's inequality (FN_p) in the Introduction, with an extra adjoint term which we shall eliminate at the end of the paper by proving an even stronger inequality.

3B. *The free product $\mathbb{Z}_{2m}^{*n}$.* A similar analysis applies as well in this case. Given two reduced words $w_1 = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r}$ and $w_2 = e_{j_1}^{t_1} \cdots e_{j_s}^{t_s}$, with $\ell_j, t_j \in [2m-1]$, we say that $w_1 \le w_2$ if and only if

- $r \le s$,
- $i_k = j_k$ for any $k \in [r]$ and $\ell_k = t_k$ for any $k \in [r-1]$,
- either $\ell_r, t_r \in [m]$ and $i_r \le j_r$, or $i_r, j_r \in [2m-1] \setminus [m-1]$ and $i_r \ge j_r$.

The map $\psi : \mathbb{Z}_{2m}^{*n} \to \mathbb{R}_+$ given by

$$w = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r} \mapsto \psi(w) = \sum_{k=1}^r |e_{i_j}^{\ell_j}| = \sum_{k=1}^r \min\{\ell_k, 2m - \ell_k\}$$

is a conditionally negative length function [3], with associated Gromov form

$$\begin{aligned} \langle 1_{\{w_1\}}, 1_{\{w_2\}} \rangle_\psi &= \tfrac{1}{2}(\psi(w_1) + \psi(w_2) - \psi(w_1^{-1}w_2)) \\ &= \psi(\min\{w_1, w_2\}) + \tfrac{1}{2}(\psi(\eta_1) + \psi(\eta_2) - \psi(\eta_1^{-1}\eta_2)), \end{aligned} \tag{3-2}$$

where $\min\{w_1, w_2\}$ is again the longest common subchain and $w_j = \min\{w_1, w_2\}\eta_j$ for $j = 1, 2$. The second term above is always 0 in the free group $\mathbb{F}_n$, but not necessarily in this case. Given $w = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r} \ne e$ we define $w^- = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r - 1}$ and construct $u_w = 1_{\{w\}} - 1_{\{w^-\}}$ as usual. Then, we find that:

- $\langle u_w, u_w \rangle_\psi = 1$ for every $w \in \mathbb{Z}_{2m}^{*n} \setminus \{e\}$.
- $\langle u_{w_1}, u_{w_2} \rangle_\psi = 0$ when $e \ne w_1^{-1} w_2 \ne e_j^m$ for $j \in [n]$.
- $\langle u_{w_1}, u_{w_2} \rangle_\psi = 0$ when $w_1^{-1} w_2 = e_j^m$ and both w_1, w_2 end with $e_j^{\pm 1}$.
- If $w = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r}$, then $u_w = -u_{w e_{i_r}^m}$ in $\mathcal{H}_\psi = \mathbb{D}[\mathbb{Z}_{2m}^{*n}] / \operatorname{Ker}\langle \cdot, \cdot \rangle_\psi$.

This proves that

$$\{u_w : w = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r} \in \mathbb{Z}_{2m}^{*n} \setminus \{e\} \text{ with } \ell_r \in [m]\}$$

is an orthonormal basis of $\mathcal{H}_\psi = \mathbb{D}[\mathbb{Z}_{2m}^{*n}]/\operatorname{Ker}\langle \cdot , \cdot \rangle_\psi$. We set as usual $\beta(w) = 1_{\{w\}}$ and $\alpha_w(1_{\{w'\}}) = 1_{\{ww'\}} - 1_{\{w\}}$. Among the above properties it is perhaps convenient to justify the last one. Note that $\langle u_w + u_{we_{i_r}^m}, u_w + u_{we_{i_r}^m} \rangle_\psi = 0$ if and only if $\langle u_w, u_{we_{i_r}^m} \rangle_\psi = -1$ but we have

$$\begin{aligned} \langle u_w, u_{we_{i_r}^m} \rangle_\psi &= \tfrac{1}{2}(-\psi(e_{i_r}^m) + \psi((w^-)^{-1} w e_{i_r}^m) + \psi(e_{i_r}^{m-1}) - \psi((w^-)^{-1} w e_{i_r}^{m-1})) \\ &= \tfrac{1}{2}(-\psi(e_{i_r}^m) + \psi(e_{i_r}^{m+1}) + \psi(e_{i_r}^{m-1}) - \psi(e_{i_r}^m)) \\ &= \tfrac{1}{2}(-m + m - 1 + m - 1 - m) = -1. \end{aligned}$$

If $w = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r}$ with $\ell_r \in [m]$, derivatives are given by

$$\partial_{u_w} \lambda(w') = \langle \beta(w'), u_w \rangle_\psi \lambda(w') = \delta_{w' \in W(w)} \lambda(w'), \tag{3-3}$$

where $W(w)$ is the set of those words $w' = e_{j_1}^{t_1} \cdots e_{j_s}^{t_s}$ satisfying

$$r \le s, \qquad i_k = j_k \quad \text{for } k \le r, \qquad \ell_k = t_k \quad \text{for } k \le r-1 \qquad \text{and} \qquad \ell_r \le t_r \le \ell_r + m - 1. \tag{3-4}$$

Indeed, just write $\beta(w') = 1_{\{w'\}} = u_{w'} + 1_{\{w'^-\}} = u_{w'} + u_{w'^-} + 1_{\{w'^{--}\}}$ and so on. The inner product with u_w will be 0 unless we find u_w in our telescopic sum above just once, in which case we get the value 1. Note that it could appear twice due to the identity $u_w = -u_{we_{i_r}^m}$ recalled above. In that case, they get mutually canceled and we get 0. This happens when $t_r - \ell_r \in [2m-1] \setminus [m-1]$.

It remains to consider Fourier truncations. As for the free group, our choice is the conditional expectation into the subgroup $\mathbb{Z}_{2m}^{*\mathsf{S}} = \langle e_j : j \in \mathsf{S} \rangle$, which is the free group generated by e_j for $j \in \mathsf{S}$. Then we consider the decomposition

$$\mathcal{H}_\psi = \bigoplus_{j=1}^n \mathcal{H}_{\psi,j}, \qquad \text{with } \mathcal{H}_{\psi,j} = \operatorname{span}\{u_w : e_j \le w \text{ or } e_j^{-1} \le w\}.$$

Our Fourier truncations form an admissible family. Define

$$\partial_j \lambda(w) = \partial_{u_{e_j}} \lambda(w) + \partial_{u_{e_j^m}} \lambda(w) - \delta_{e_{i_1}^{\ell_1} = e_j^m} \lambda(w) \quad \text{for any } w = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r}.$$

In other words, $\partial_j \lambda(w) = \delta_{i_1 = j} \lambda(w)$ for $w \neq e$ and $\partial_{u_w} \circ \partial_j = \partial_{u_w}$ for $u_w \in \mathcal{H}_{\psi,j}$. The construction above yields the form of Theorem B on the von Neumann algebra of the free product $\mathbb{Z}_{2m}^{*n}$

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}| = k}} \Big\| \sum_{w \in \mathbb{Z}_{2m}^{*\mathsf{S}}} \hat{f}(w) \lambda(w) \Big\|_p^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_j(f)\|_p^p + \|\partial_j(f^*)\|_p^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_p^p.$$

3C. ***Free Hilbert transforms.*** Compared to (FN$_p$), the form of Theorem B for free groups gives the additional terms $\partial_j(f^*)$. These terms seem to be necessary in the general context of Theorem B, but they are removable for free groups — in fact, we shall prove an even stronger inequality — due to a singular behavior of word-length derivatives for free groups. Said behavior means in particular that word-length derivatives can be regarded as free forms of directional Hilbert transforms, which were recently investigated by Mei and Ricard in [17]. The free Hilbert transforms for mean-zero f are defined as

$$H_\varepsilon(f) = \sum_{j=1}^{n} \varepsilon_j \partial_j(f) \quad \text{for } \varepsilon_j = \pm 1.$$

Mei and Ricard proved in [17] the crucial inequality

$$\|H_\varepsilon f\|_{L_p(\mathcal{L}(\mathbb{F}_n))} \asymp_p \|f\|_{L_p(\mathcal{L}(\mathbb{F}_n))} \quad \text{for any } 1 < p < \infty. \tag{3-5}$$

Define

$$\mathbb{A}_{\mathsf{S}} = \bigcup_{j \in \mathsf{S}} \mathbb{A}_{\{j\}}.$$

Theorem 3.1. *If $p \geq 2$ and $k \in [n]$, every mean-zero $f \in L_p(\mathcal{L}(\mathbb{F}_n))$ satisfies*

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}| = k}} \Big\| \sum_{w \in \mathbb{A}_{\mathsf{S}}} \hat{f}(w)\lambda(w) \Big\|_{L_p(\mathcal{L}(\mathbb{F}_n))}^p \lesssim_p \frac{k}{n} \sum_{j=1}^{n} \|\partial_j(f)\|_{L_p(\mathcal{L}(\mathbb{F}_n))}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathcal{L}(\mathbb{F}_n))}^p.$$

Proof. Define

$$h = \sum_{w \in \mathbb{A}_{\mathsf{S}}} \hat{f}(w)\lambda(w) = \sum_{j \in \mathsf{S}} \sum_{w \in \mathbb{A}_{\{j\}}} \hat{f}(w)\lambda(w) = \sum_{j \in \mathsf{S}} \partial_j(f).$$

Applying inequality (3-5) we obtain

$$\|h\|_p \asymp_p \mathbb{E}\|H_\varepsilon(h)\|_p = \mathbb{E}\Big\| \sum_{j \in \mathsf{S}} \varepsilon_j \partial_j(f) \Big\|_p.$$

The result follows from Theorem 1.2 and another application of (3-5) for f. □

Corollary 3.2. *Inequality (FN$_p$) holds for $p \geq 2$ and any mean-zero $f \in L_p(\mathcal{L}(\mathbb{F}_n))$.*

Proof. It follows from Theorem 3.1 and the boundedness of the conditional expectation from $\mathcal{L}(\mathbb{F}_n)$ to $\mathcal{L}(\mathbb{F}_{\mathsf{S}})$

$$\Big\| \sum_{w \in \mathbb{F}_{\mathsf{S}}} \hat{f}(w)\lambda(w) \Big\|_p = \Big\| \sum_{w \in \mathbb{F}_{\mathsf{S}}} \hat{h}(w)\lambda(w) \Big\|_p \leq \|h\|_p = \Big\| \sum_{w \in \mathbb{A}_{\mathsf{S}}} \hat{f}(w)\lambda(w) \Big\|_p,$$

where h is defined as in the proof of Theorem 3.1, since we note that $\mathbb{F}_{\mathsf{S}} \subset \mathbb{A}_{\mathsf{S}}$. □

Remark 3.3. It is conceivable that Theorem 3.1 or at least Corollary 3.2 could have been proved as well from a generalized form of Theorem B in the line of Remark 1.4, but we have not found an argument using such an approach.

Remark 3.4. Hilbert transforms can also be constructed on $\mathcal{L}(\mathbb{Z}_{2m}^{*n})$. They are L_p-bounded maps as well there, as shown in [17, Theorem 3.5]. Therefore, Theorem 3.1 can also be proved with this technique replacing $\mathbb{F}_n$ by $\mathbb{Z}_{2m}^{*n}$ in the statement.

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