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BEURLING–CARLESON SETS, INNER FUNCTIONS AND A SEMILINEAR EQUATION

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Beurling–Carleson sets have appeared in a number of areas of complex analysis such as boundary zero sets of analytic functions, inner functions with derivative in the Nevanlinna class, cyclicity in weighted Bergman spaces, Fuchsian groups of Widom-type and the corona problem in quotient Banach algebras. After surveying these developments, we give a general definition of Beurling–Carleson sets and discuss some of their basic properties. We show that the Roberts decomposition characterizes measures that do not charge Beurling–Carleson sets.

For a positive singular measure μ on the unit circle, let S_μ denote the singular inner function with singular measure μ. In the second part of the paper, we use a corona-type decomposition to relate a number of properties of singular measures on the unit circle, such as membership of S'_μ in the Nevanlinna class $\mathcal{N}$, area conditions on level sets of S_μ and wepability. It was known that each of these properties holds for measures concentrated on Beurling–Carleson sets. We show that each of these properties implies that μ lives on a countable union of Beurling–Carleson sets. We also describe partial relations involving the membership of S'_μ in the Hardy space H^p, membership of S_μ in the Besov space B^p and $(1-p)$-Beurling–Carleson sets and give a number of examples which show that our results are optimal.

Finally, we show that measures that live on countable unions of α-Beurling–Carleson sets are almost in bijection with nearly maximal solutions of $\Delta u = u^p \cdot \chi_{u>0}$ when $p > 3$ and $\alpha = (p-3)/(p-1)$.

1. Introduction

A *Beurling–Carleson set* E is a closed subset of the unit circle $\partial\mathbb{D}$ of zero length whose complementary arcs $\{J\}$ satisfy

$$\|E\|_{\mathcal{BC}} = \sum_J |J| \log \frac{1}{|J|} < \infty. \tag{1-1}$$

Beurling–Carleson sets were introduced by A. Beurling [1940], who showed that they constitute boundary zero sets of holomorphic functions on the unit disk that are Hölder continuous up to the boundary. Several years later, L. Carleson [1952] constructed outer functions that vanished to arbitrary order on E. This construction was later improved to infinite order by Taylor and Williams [1970]. Since then, Beurling–Carleson sets appeared in a number of areas of complex analysis such as inner functions, weighted Bergman spaces, Fuchsian groups and the corona problem.

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In this paper, we will also consider Beurling–Carleson sets with respect to other gauge functions, although we will be mainly interested in usual Beurling–Carleson sets and α-Beurling–Carleson sets with $0 < \alpha < 1$. These are defined by the condition

$$\|E\|_{\mathcal{BC}_\alpha} = \sum_J |J|^\alpha < \infty \tag{1-2}$$

in place of (1-1).

1A. ***Derivative in Nevanlinna class.*** An inner function is a bounded analytic function on the unit disk $\mathbb{D}$ which has unimodular radial limits almost everywhere on $\partial\mathbb{D}$. Beurling–Carleson sets play an important role in understanding inner functions with derivative in the Nevanlinna class $\mathcal{N}$, which consists of analytic functions $f(z)$ on the unit disk for which

$$\lim_{r\to 1} \int_{|z|=r} \log^+|f(z)| < \infty.$$

Suppose μ is a positive singular measure on the unit circle and

$$S_\mu(z) = \exp\biggl(-\int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta)\biggr), \quad |z| < 1,$$

is the associated singular inner function. On the unit circle, the radial boundary values of $|S'_\mu|$ are given by

$$|S'_\mu(z)| = 2\int_{\partial\mathbb{D}} \frac{|d\zeta|}{|\zeta - z|^2}, \quad |z| = 1,$$

which could be infinite. M. Cullen [1971] observed that if μ is concentrated on a Beurling–Carleson set, then $S'_\mu \in \mathcal{N}$. The converse does not hold in general: there are singular inner functions S_μ with $S'_\mu \in \mathcal{N}$ for which the support of μ is not contained in a single Beurling–Carleson set. One consequence of [Ivrii 2019] is that the condition $S'_\mu \in \mathcal{N}$ implies that μ lives on a countable union of Beurling–Carleson sets. The original proof used the classification of nearly maximal solutions of the Gauss curvature equation $\Delta u = e^{2u}$. In Section 4, we will give an elementary proof of this fact using a corona-type decomposition.

Theorem 1.1. *Let $\mu \geq 0$ be a singular measure on $\partial\mathbb{D}$. Consider the following conditions*:

(0) *The measure μ is supported on a Beurling–Carleson set.*

(1) $S'_\mu \in \mathcal{N}$.

(2) S_μ *satisfies the area condition*: *for every* $0 < c < 1$,

$$\int_{\{z\in\mathbb{D}:|S_\mu(z)|<c\}} \frac{dA(z)}{1-|z|} < \infty. \tag{1-3}$$

(3) *The measure μ is concentrated on a countable union of Beurling–Carleson sets.*

We have (0) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

1B. ***Quotient Banach algebras.*** Another important perspective on Beurling–Carleson sets stems from P. Gorkin, R. Mortini and N. Nikolskii [Gorkin et al. 2008] who studied the corona problem in the quotient space H^∞/IH^∞, where I is an inner function. They noticed that point evaluations at the zeros of I are dense in the maximal ideal space $\mathfrak{M}$ of H^∞/IH^∞ if and only if there exists a $0<c<1$ for which the sublevel set

$$\Omega_c = \{z \in \mathbb{D} : |I(z)| < c\}$$

is contained within a bounded hyperbolic distance of the zero set of I. In this case, one says that I has the *weak embedding property* (WEP). A. Borichev [2013] introduced the class of *wepable* inner functions, i.e., inner functions that could be made WEP if multiplied by a suitable Blaschke product. Consider the condition

(1′) *S_μ is wepable.*

In [Borichev et al. 2017], the authors proved that (0) $\Rightarrow$ (1′) $\Rightarrow$ (2). Together with the implication (2) $\Rightarrow$ (3) from Theorem 1.1, this shows that up to countable unions, the collection of measures μ for which S_μ is wepable also coincides with measures that are concentrated on Beurling–Carleson sets.

Remark. Taking countable unions is necessary since there exist atomic measures μ for which S_μ is not wepable. See the proof of [Borichev et al. 2017, Theorem 3].

1C. ***Derivative in H^p.*** Next, we use a corona-type decomposition to study singular inner functions with derivative in the Hardy space H^p. We stick to the range of exponents $0<p<\frac{1}{2}$ since derivatives of singular inner functions are never in $H^{1/2}$.

Theorem 1.2. *Suppose $0<p<\frac{1}{2}$ and $\mu \ge 0$ is a singular measure on $\partial\mathbb{D}$. Consider the following conditions*:

(1) $S'_\mu \in H^p$.

(2) *S_μ satisfies the $(1+p)$-area condition*: *for every* $0<c<1$,

$$\int_{\{z\in\mathbb{D}:|S_\mu(z)|<c\}} \frac{dA(z)}{(1-|z|)^{1+p}} < \infty. \tag{1-4}$$

(3) *The measure μ is concentrated on a countable union of $(1-p)$-Beurling–Carleson sets.*

We have (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Unfortunately, it is no longer true that if μ is supported on a $(1-p)$-Beurling–Carleson set, then $S'_\mu \in H^p$.

We say that a finite measure $\mu \ge 0$ satisfies a property *up to countable sums* if it can be written as a countable sum of finite measures $\mu_k \ge 0$ satisfying the property. In Section 5, we will see that conditions (1) and (3) are different even after allowing countable sums. Nevertheless, in Section 6, we will show that conditions (1) and (2) agree after passing to countable sums.

We mention an additional condition on the measure μ, equivalent to (2), due to P. Ahern [1979] and A. Reijonen and T. Sugawa [Reijonen and Sugawa 2019]:

(2′) *We have*

$$\int_{\mathbb{D}} |S_\mu'(z)|^q (1-|z|^2)^{-p+(q-1)}\, dA(z) < \infty$$

for some (*and hence all*) $1 \le q \le 2$.

When $q = 1$, the above condition says that S_μ' belongs to the Besov space B^p. The implication (1) ⇒ (2′) can also be found in Ahern's paper.

1D. ***Differential equations.*** It was observed in [Ivrii 2019] that characterizing inner functions with derivative in Nevanlinna class amounts to understanding nearly maximal solutions of the Gauss curvature equation $\Delta u = e^{2u}$. These turn out to be in one-to-one correspondence with measures that live on countable unions of Beurling–Carleson sets. We refer the reader to Section 8 for the relevant definitions and background on semilinear equations.

In Section 9, we show the following theorem which partially characterizes the nearly maximal solutions of $\Delta u = u^p \cdot \chi_{u>0}$:

Theorem 1.3. (i) *When* $p > 3$, *deficiency measures of nearly maximal solutions are concentrated on countable unions of* α*-Beurling–Carleson sets*, *where* $\alpha = (p-3)/(p-1)$. *Conversely*, *any finite positive measure on the unit circle concentrated on a countable union of* β*-Beurling–Carleson sets for some* $\beta < \alpha$ *arises as the deficiency measure of some nearly maximal solution.*

(ii) *When* $1 < p \le 3$, *the only nearly maximal solution is the maximal one.*

It is natural to wonder if there is a precise correspondence between nearly maximal solutions of $\Delta u = u^p \cdot \chi_{u>0}$ and measures that live on countable unions of α-Beurling–Carleson sets. Unfortunately, with our current techniques, we are unable to either prove or disprove this tantalizing hypothesis.

2. Notes and references

2A. ***Weighted Bergman spaces.*** Beurling–Carleson sets also arise naturally in the study of cyclic functions in the weighted Bergman spaces A^p_α, which consists of holomorphic functions on the unit disk satisfying

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p (1-|z|)^\alpha\, dA(z) < \infty, \quad \alpha > -1, \ \ 1 < p < \infty.$$

A function $f \in A^p_\alpha$ is *cyclic* if the closure of the set $\{pf : p \text{ polynomial}\}$ is dense in A^p_α. One question that puzzled mathematicians in the late 1960s was: *when is the singular inner function* S_μ *cyclic*? It was not difficult to show that if μ is concentrated on a Beurling–Carleson set, then the singular inner function S_μ could not be cyclic. In the other direction, it was known that if μ had modulus of continuity bounded by $Ct\log(1/t)$, then S_μ was cyclic. The gap between Beurling–Carleson sets and the $t \log 1/t$ condition stood for a number of years until it was resolved independently by B. Korenblum [1981] and J. Roberts [1985]. Roberts' approach used an elegant structure theorem for measures that do charge Beurling–Carleson sets. In Section 3, we will prove a converse of Roberts' result, thereby giving a description of positive singular measures that do not charge Beurling–Carleson sets.

2B. ***Model spaces.*** Let A^∞ denote the space of holomorphic functions on the open unit disk which extend to smooth functions on the closed unit disk. To an inner function $F(z)$, one can associate the *model space* $K_F = H^2 \ominus FH^2$. K. Dyakonov and D. Khavinson [Dyakonov and Khavinson 2006] were curious as to whether K_F contained smooth functions. They showed that $K_F \cap A^\infty = \{0\}$ if and only if $F = S_\mu$, where μ does not charge Beurling–Carleson sets.

In a recent work, A. Limani and B. Malman [Limani and Malman 2023a] asked the opposite question: when is $K_F \cap A^\infty$ dense in K_F? They showed that this occurs if and only if $F = BS_\mu$, where B is an arbitrary Blaschke product and μ is concentrated on a countable union of Beurling–Carleson sets.

2C. ***Character-automorphic functions.*** Widom [1971] and Pommerenke [1976a; 1976b] studied functions which were character-automorphic under Fuchsian groups of convergence type. A *character* v of a Fuchsian group $\Gamma \subset \mathrm{Aut}(\mathbb{D})$ is a homomorphism of Γ to the unit circle. A function f on the unit disk is called *character automorphic* if

$$f(\gamma(z)) = v(\gamma) \cdot f(z), \quad \gamma \in \Gamma.$$

One natural character automorphic function is the Blaschke product $g(z)$ whose zeros constitute an orbit of Γ (it is related to the Green's function of $\mathbb{D}/\Gamma$). If $g(z)$ has zeros at the points $\{\gamma(0) : \gamma \in \Gamma\}$, i.e.,

$$g(z) = \prod_{\gamma \in \Gamma} -\frac{\overline{\gamma(0)}}{|\gamma(0)|} \cdot \frac{z - \gamma(0)}{1 - \overline{\gamma(0)}z},$$

then

$$|g'(z)| = \sum_{\gamma \in \Gamma} |\gamma'(z)|, \quad |z| = 1.$$

For a character v, let $H^\infty(\Gamma, v)$ denote the space of bounded holomorphic v-automorphic functions. Building on the work of Widom, Pommerenke [1976b] showed that

$$g' \in \mathcal{N} \quad \Longleftrightarrow \quad H^\infty(\Gamma, v) \neq \{\text{const}\} \text{ for every } v$$

and observed that the above condition is satisfied if the limit set $\Lambda(\Gamma)$ is a Beurling–Carleson set.

Pommerenke [1976a, Theorem 2] also showed that Λ is a Beurling–Carleson set if and only if there is a Γ-invariant holomorphic vector field $h(z)(\partial/\partial z)$ on the unit disk with $h'(z) \in H^\infty$.

2D. ***Fat Beurling–Carleson sets.*** A related class of sets was introduced by S. Khruschev, which is natural to call *fat Beurling–Carleson sets*. These are closed subsets of the unit circle which satisfy the entropy condition (1-1) but have positive Lebesgue measure. Amongst other things, Khruschev showed that if K is a closed subset of the unit circle which does not contain any fat Beurling–Carleson sets, then there is a sequence of polynomials $p_n(z)$ which tend to 1 in the Bergman space $A^2(\mathbb{D})$ but to 0 in $C(K)$. Conversely, if such a sequence of polynomials exists, then K cannot contain any fat Beurling–Carleson sets.

The proof presented in [Havin and Jöricke 1994, Chapter II.3] uses a structure theorem due to N. G. Makarov [1989]. Given a closed subset K of the circle which does not contain fat Beurling–Carleson sets and an arc $I \subset \partial\mathbb{D}$, there exists a measure $\mu = \mu_I$ supported on $I \setminus K$ which satisfies

(i) $\mu(I) \geq |I| \log \frac{1}{|I|}$,

(ii) $\mu(J) \leq 3|J| \log \frac{1}{|J|}$ for any arc $J \subseteq I$.

The first condition implies that μ has substantial mass, while the second condition says that μ is spread out.

For more applications of fat Beurling–Carleson sets, we refer the reader to [Limani and Malman 2023b; 2024; Malman 2023].

3. Beurling–Carleson sets

In this section, we give a general definition of Beurling–Carleson sets and discuss some of their basic properties. We say that $\phi : [0, 1] \to [0, \infty)$ is a *regular gauge function* if:

(G1) One can write

$$\phi(t) = t \cdot \phi_1(t) = t \int_t^1 \frac{ds}{\lambda(s)},$$

where $\lambda(t)$ is a nonnegative function such that $\int_0^1 (\lambda(s))^{-1}\, ds = \infty$.

(G2) The function $\lambda(t)$ satisfies the doubling condition

$$\lambda(\theta \cdot t) \asymp \lambda(t), \quad \theta \in [1, 2]. \tag{3-1}$$

(G3) There exists a constant $C > 0$ such that

$$\sum_{k=0}^{\infty} \phi(2^{-k} t) \leq C\phi(t), \quad t \in [0, 1].$$

A closed subset E of the unit circle of zero length is called a *ϕ-Beurling–Carleson set* if

$$\|E\|_{\mathcal{BC}_\phi} = \sum_k \phi(|J_k|) < \infty, \tag{3-2}$$

where the sum is over the complementary arcs $\{J_k\}$ of E.

For each $n \geq 0$, we can partition the unit circle into 2^n *dyadic arcs* of generation n:

$$\{z \in \partial\mathbb{D} : k \cdot 2^{-n} \cdot 2\pi < \arg z < (k+1) \cdot 2^{-n} \cdot 2\pi\}, \quad k = 0, 1, \dots, 2^n - 1.$$

We denote the collection of dyadic arcs of generation n by $\mathcal{D}_n$. The *dyadic grid* $\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n$ is the collection of all dyadic arcs.

Given a closed set E, the *Privalov star* K_E is defined as the union of the Stolz angles of opening $\frac{\pi}{2}$ emanating from points of E.

The following lemma provides several other characterizations of Beurling–Carleson sets:

Lemma 3.1. *Let E be a closed subset of the unit circle of zero length. Denote the complementary arcs by $\{J_k\}$, i.e., $\partial\mathbb{D} \setminus E = \bigcup J_k$. If ϕ is a regular gauge function, then the following quantities are comparable*:

(a) *Arc sum*: $$\sum_k \phi(|J_k|).$$

(b) *Distance integral*: $$\int_{\partial\mathbb{D}\setminus E} \phi_1(\operatorname{dist}(x, E))\, dx.$$

(c) *Dyadic arc sum*: $$\sum_{\substack{I \text{ dyadic} \\ I\cap E\neq\varnothing}} \frac{|I|^2}{\lambda(|I|)}.$$

(d) *Privalov star integral*: $$\int_{K_E} \frac{dA(z)}{\lambda(1-|z|)}.$$

Remark. In (d), instead of integrating over the Privalov star K_E, one can also integrate over the region

$$\Omega_E = \mathbb{D} \setminus \bigcup_k Q_{J_k},$$

where

$$Q_J = \left\{ z \in \mathbb{D} : \frac{z}{|z|} \in J,\ 0 < 1-|z| < |J| \right\}$$

is the Carleson box with base $J \subset \partial\mathbb{D}$. Alternatively, one can integrate over the domain

$$\Omega_E^{\text{dyadic}} = \bigcup_{\substack{I \text{ dyadic} \\ I\cap E\neq\varnothing}} T_I,$$

where

$$T_I = \left\{ z \in \mathbb{D} : \frac{z}{|z|} \in I,\ \frac{1}{2}|I| < 1-|z| < |I| \right\}$$

denotes the top half of the Carleson box which rests on I.

Examples.

(i) If $\phi(t) = t \log t^{-1}$, then $\lambda(t) = t$ and we recover the usual Beurling–Carleson condition:

$$\sum_k |J_k| \log\frac{1}{|J_k|} \asymp \int_{[0,1]\setminus E} \log\frac{1}{\operatorname{dist}(x, E)} dx \asymp \sum_{\substack{I \text{ dyadic} \\ I\cap E\neq\varnothing}} |I| \asymp \int_{K_E} \frac{dA(z)}{1-|z|}.$$

(ii) If $\phi(t) = t^\alpha$ with $0 < \alpha < 1$, then $\lambda(t) \sim t^{2-\alpha}/(1-\alpha)$ as $t \to 0^+$ and we get the α-Beurling–Carleson condition:

$$\sum_k |J_k|^\alpha \asymp \int_{[0,1]\setminus E} \operatorname{dist}(x, E)^{\alpha-1} dx \asymp \sum_{\substack{I \text{ dyadic} \\ I\cap E\neq\varnothing}} |I|^\alpha \asymp \int_{K_E} \frac{dA(z)}{(1-|z|)^{2-\alpha}}.$$

Proof of Lemma 3.1. The comparability of the "arc sum" and the "distance integral" follows after subdividing each complementary interval J_k into Whitney arcs and applying the estimate (G3), while the comparability of the "distance integral" and the "Privalov star integral" follows from integrating in polar coordinates.

It remains to relate the "Privalov star integral" and the "dyadic arc sum". By the doubling property (G2) of λ, we have

$$\int_{T_I} \frac{dA(z)}{\lambda(1-|z|)} \asymp \frac{|I|^2}{\lambda(|I|)}.$$

Summing over the dyadic arcs I which meet E gives

$$\int_{\Omega_E^{\text{dyadic}}} \frac{dA(z)}{\lambda(1-|z|)} \asymp \sum_{\substack{I \text{ dyadic} \\ I\cap E\neq\varnothing}} \frac{|I|^2}{\lambda(|I|)}.$$

Inspection shows that

$$\int_{\Omega_E^{\text{dyadic}}} \frac{dA(z)}{\lambda(1-|z|)} \asymp \int_{\Omega_E} \frac{dA(z)}{\lambda(1-|z|)} \asymp \int_{K_E} \frac{dA(z)}{\lambda(1-|z|)}. \tag{3-3}$$

The proof is complete. □

3A. ***Dyadic grid with respect to a gauge function.*** A *ϕ-dyadic grid* is a collection of dyadic arcs $\mathcal{D}_\phi = \bigcup_j \mathcal{D}_{n_j}$, where the sequence $\{n_j\}$ satisfies

$$\int_{2^{-n_{j+1}}}^{2^{-n_j}} \frac{dt}{\lambda(t)} \asymp \int_{2^{-n_j}}^{1} \frac{dt}{\lambda(t)} \asymp \phi_1(2^{-n_j}), \quad j=1,2,\dots. \tag{3-4}$$

In particular, the above condition implies that $\phi_1(|I|) \asymp \phi_1(|J|)$ whenever $I \in \mathcal{D}_{n_{j+1}}$ and $J \in \mathcal{D}_{n_j}$.

Examples.

(i) If $\phi(t) = t\log t^{-1}$, one can take $n_j = 2^j$ and obtain the super-dyadic scales $2^{-n_j} = 2^{-2^j}$. In this case, $\lambda(t) = t$.

(ii) When $\phi(t) = t^\alpha$, $\alpha > 0$, one can take $n_j = j$ and get the standard dyadic scales 2^{-j}. In this case, $\lambda(t) \asymp t^{2-\alpha}/(1-\alpha)$ as $t \to 0$.

Dyadic shells and boxes. We can decompose the unit disk $\mathbb{D}$ into ϕ-dyadic shells:

$$\mathcal{A}_{\phi,0} = \{z \in \mathbb{D} : |z| < 1-2^{-n_1}\}$$

and

$$\mathcal{A}_{\phi,j} = \{z \in \mathbb{D} : 1-2^{-n_j} < |z| < 1-2^{-n_{j+1}}\}, \quad j=1,2,\dots.$$

Each shell can be further subdivided into ϕ-dyadic boxes:

$$T_I^\phi = \mathcal{A}_{\phi,j} \cap Q(I) = \{re^{i\theta} \in \mathbb{D} : \theta \in I,\ 1-2^{-n_j} < r < 1-2^{-n_{j+1}}\},$$

where I ranges over $\mathcal{D}_{n_j}$. For further reference, we note that

$$\int_{T_I^\phi} \frac{dA(z)}{\lambda(1-|z|)} \asymp |I|\cdot\phi_1(|I|) = \phi(|I|). \tag{3-5}$$

3B. ***Roberts decomposition.*** In a remarkable work, Roberts [1985] came up with an elegant structure theorem for measures that do not charge Beurling–Carleson sets. This is done by *grating* a measure with respect to finer and finer partitions associated to a ϕ-dyadic grid.

Theorem 3.2. *Let $\phi : [0,1] \to [0,\infty)$ be a regular gauge function and $\mathcal{D}_\phi = \bigcup \mathcal{D}_{n_k}$ be a ϕ-dyadic grid. Let μ be a finite positive measure on $\partial\mathbb{D}$. Then, for any integer $j_0 \ge 0$ and $C > 0$, one can decompose $\mu = \sum_{j=1}^\infty \mu_j + \mu_\infty$ such that $\mu_j(I) \le C\phi(|I|)$ for any $I \in \mathcal{D}_{n_{j+j_0}}$ and μ_∞ is concentrated on a ϕ-Beurling–Carleson set.*

Proof. For each $j = 1, 2, \dots$, we can define a partition P_j of the unit circle into $2^{n_{j+j_0}}$ arcs of equal length (we consider half-open arcs which contain only one of the endpoints, for example, the left endpoint). Since $2^{n_{j+j_0}}$ divides $2^{n_{j+j_0+1}}$, each next partition can be chosen to be a refinement of the previous one.

To define μ_1, consider the arcs in the partition P_1. Call an arc $I \in P_1$ *light* if $\mu(I) \le C\phi(|I|)$ and *heavy* otherwise. On a light arc, take $\mu_1 = \mu$, while on a heavy arc, let μ_1 be a multiple of μ, so that the mass $\mu_1(I)$ equals $C\phi(|I|)$. The measure μ_1 will be called the *grated measure* of μ with respect to the partition P_1. Clearly, $\mu_1 \le \mu$. Consider the difference $\mu - \mu_1$ and grate it with respect to the partition P_2 to form the measure μ_2, then consider $\mu - \mu_1 - \mu_2$ and grate it with respect to P_3 to form μ_3, and so on. Continuing in this way, we obtain a sequence of measures $\mu - \mu_1,\ \mu - \mu_1 - \mu_2, \dots$, where each next measure is supported on the heavy arcs of the previous generation.

By construction, the bound $\mu_j(I) \le C\phi(|I|),\ I \in \mathcal{D}_{n_{j+j_0}}$ holds for all j, while the residual measure μ_∞ is supported on the set of points which always lie in heavy arcs. A fortiori, the residual measure is supported on the complement of the light arcs and we need to show that $\sum_{I \text{ light}} \phi(|I|) < \infty$. The scaling condition (3-4) tells us that

$$\sum_{\substack{I \subset J \\ I \in \mathcal{D}_{n_{j+1}}}} \phi(|I|) = |J| \cdot \phi_1(|I|) \le C\phi(|J|), \qquad J \in \mathcal{D}_{n_j}.$$

Since a light arc of generation $j \ge 2$ is contained in a heavy one,

$$\begin{aligned}\sum_{\text{light}} \phi(|I|) \lesssim 2^{n_{j_0}} \phi(2^{-n_{j_0}}) + \sum_{\text{heavy}} \phi(|J|) &= 2^{n_{j_0}} \phi(2^{-n_{j_0}}) + \frac{1}{C} \sum_j \sum_{\substack{J \in \mathcal{D}_{n_{j+j_0}} \\ J \text{ heavy}}} \mu_j(J) \\ &\le 2^{n_{j_0}} \phi(2^{-n_{j_0}}) + \frac{1}{C} \cdot \mu(\partial\mathbb{D}). \end{aligned}$$

□

Corollary 3.3. *If μ does not charge ϕ-Beurling–Carleson sets, then, for any $j_0 \ge 0$ and $C > 0$, one can write $\mu = \sum \mu_j$, where $\mu_j(I) \le C\phi(|I|)$ for any $I \in \mathcal{D}_{n_{j+j_0}}$.*

We now show the converse of Corollary 3.3:

Corollary 3.4. *Suppose that there exists a constant $C > 0$ such that, for any offset $j_0 \ge 0$, one can decompose the measure μ into a countable sum $\mu = \sum \mu_j$ such that $\mu_j(I) \le C\phi(|I|)$ for any $I \in \mathcal{D}_{n_{j+j_0}}$. Then μ does not charge ϕ-Beurling–Carleson sets.*

Proof. Let E be a ϕ-Beurling–Carleson set. By Lemma 3.1, for any $\varepsilon > 0$, we can choose the offset $j_0 \ge 0$ large enough that

$$\sum_{j=1}^{\infty} \int_{K_E \cap \mathcal{A}_{\phi, j+j_0}} \frac{dA(z)}{\lambda(1-|z|)} < \varepsilon.$$

In view of (3-5), we have

$$\mu_j(E) = \sum_{\substack{I \in \mathcal{D}_{n_{j+j_0}} \\ I \cap E \ne \varnothing}} \mu_j(I) \le C \sum_{\substack{I \in \mathcal{D}_{n_{j+j_0}} \\ I \cap E \ne \varnothing}} \phi(|I|) \le C' \int_{K_E \cap \mathcal{A}_{\phi, j+j_0}} \frac{dA(z)}{\lambda(1-|z|)}.$$

Summing over $j = 1, 2, \dots$ yields $\mu(E) \le C'\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\mu(E) = 0$ as desired. □

3C. ***Local behavior.*** The following theorem roughly says that measures on the unit circle which are sufficiently spread out cannot charge Beurling–Carleson sets:

Theorem 3.5. *Suppose $w(\varepsilon)/\varepsilon$ is strictly decreasing on $(0, 1]$. Then $\mu(E) = 0$ for every ϕ-Beurling–Carleson set E and positive measure μ on the unit circle satisfying the modulus of continuity condition*

$$\mu(I) \le c \cdot w(|I|), \quad I \subset \partial\mathbb{D},$$

if and only if

$$\int_0^1 \frac{\varepsilon}{\lambda(\varepsilon) w(\varepsilon)}\, d\varepsilon = \infty. \tag{3-6}$$

In full generality, Theorem 3.5 was proved by R. D. Berman, L. Brown and W. S. Cohn [Berman et al. 1987, Corollary 4.1]. For usual Beurling–Carleson sets, Theorem 3.5 goes back to Ahern [1979] and J. H. Shapiro [1980].

Examples.

(i) If $\phi(t) = t \log t^{-1}$, the above condition reads $\int_0^1 w(\varepsilon)^{-1}\, d\varepsilon = \infty$.

(ii) For $\phi(t) = t^\alpha$, $\alpha > 0$, the condition becomes $\int_0^1 \varepsilon^{\alpha-1} w(\varepsilon)^{-1}\, d\varepsilon = \infty$.

Theorem 3.6. *Suppose μ is a measure on the unit circle supported on a countable union of ϕ-Beurling–Carleson sets. Let $\mu(x, \varepsilon) = \mu(I(x, \varepsilon))$, where $I(x, \varepsilon)$ is the arc on the unit circle centered at x of length 2ε. For almost every point x on the unit circle with respect to μ,*

$$\int_0^1 \frac{\varepsilon}{\lambda(\varepsilon)\mu(x, \varepsilon)}\, d\varepsilon < \infty.$$

Proof. It suffices to consider the case when μ is supported on a single ϕ-Beurling–Carleson set E. Since μ is a singular measure, for μ-a.e. $x \in \partial\mathbb{D}$, we have $\lim_{\varepsilon\to 0} \mu(x, \varepsilon)/\varepsilon = \infty$. To prove the lemma, we will show that

$$\int_E \int_0^1 \frac{\varepsilon}{\lambda(\varepsilon)\mu(x, \varepsilon)}\, d\varepsilon\, d\mu(x) \lesssim \|E\|_{\mathcal{BC}_\phi}.$$

For a point $x \in \partial\mathbb{D}$, we write $S(x)$ for the Stolz angle of opening $\frac{\pi}{2}$ with vertex at x. Recall that K_E denotes the union of the Stolz angles emanating from points $x \in E$. According to Lemma 3.1,

$$\|E\|_{\mathcal{BC}_\phi} \asymp \int_{K_E} \frac{dA(z)}{\lambda(1 - |z|)}.$$

We subdivide the above integral over individual Stolz angles:

$$\int_{K_E} \frac{dA(z)}{\lambda(1 - |z|)} = \int_E \int_{S(\zeta)} \eta(z) \cdot \frac{dA(z)}{\lambda(1 - |z|)}\, d\mu(\zeta),$$

where the function $\eta(z) = \mu(I_z)^{-1}$ measures how many Stolz angles contain z. Here, I_z is the arc of the unit circle that consists of points ζ for which $z \in S(\zeta)$. From

$$\int_{S(\zeta)\cap\{1-|z|=\varepsilon\}} \eta(z) \cdot \frac{|dz|}{\lambda(1 - |z|)} \ge \frac{\varepsilon}{\lambda(\varepsilon)} \cdot \min_{z\in S(\zeta)\cap\{1-|z|=\varepsilon\}} \mu(I_z)^{-1} \ge \frac{\varepsilon \cdot \mu(\zeta, 3\varepsilon)^{-1}}{\lambda(\varepsilon)},$$

we deduce that

$$\int_E \int_0^1 \frac{\varepsilon \cdot \mu(\zeta, 3\varepsilon)^{-1}}{\lambda(\varepsilon)} \, d\varepsilon \, d\mu(\zeta) \lesssim \|E\|_{\mathcal{BC}_\phi}. \qquad \square$$

Corollary 3.7. *Suppose μ is a measure on the unit circle supported on a countable union of ϕ-Beurling–Carleson sets. For any $c > 0$, the region*

$$\Omega_c = \{z \in \mathbb{D} : P_\mu(z) > c\}$$

is "thick" at almost every point x on the unit circle with respect to μ, in the sense that

$$\int_0^1 \frac{\eta(x, \varepsilon)}{\varepsilon \cdot \lambda(\varepsilon)} \, d\varepsilon < \infty, \tag{3-7}$$

where $\eta(x, \varepsilon) = \pi\varepsilon - |\partial B(x, \varepsilon) \cap \Omega_c|$.

To see the corollary, notice that if $\mu(x, \varepsilon) \geq \varepsilon$, then $\mu(x, \varepsilon)\eta(x, \varepsilon) \lesssim \varepsilon^2$.

Remark. For usual Beurling–Carleson sets, one has ε^2 in the denominator of (3-7). This is essentially the Rodin–Warschawski condition on the existence of a nonzero angular derivative of a Riemann map $\psi_c : \Omega_c \to \mathbb{D}$ at $x \in \partial\Omega_c \cap \partial\mathbb{D}$; see Theorem 7.1. (If Ω_c is disconnected, then we consider the Riemann map from an appropriate connected component of Ω_c.) For an application to critical values of inner functions, see [Ivrii and Kreitner 2024]. For α-Beurling–Carleson sets, the denominator of (3-7) is $\varepsilon^{3-\alpha}$.

4. A corona construction

In this section, we explore a number of conditions which guarantee that a singular measure is supported on a countable union of Beurling–Carleson sets and prove Theorems 1.1 and 1.2. Our main tool is a corona-type decomposition for singular measures.

4A. *Decomposition of singular measures.* Suppose μ is a singular measure on the unit circle. Fix a large constant $M > 0$ and consider the following corona-type decomposition. Let $\{I_j^{(1)}\}$ be the maximal (closed) dyadic arcs such that

$$\frac{\mu(I_j^{(1)})}{|I_j^{(1)}|} \geq M.$$

In each $I_j^{(1)}$, we consider the maximal dyadic subarcs $J_k^{(1)} \subset I_j^{(1)}$ for which

$$\frac{\mu(J_k^{(1)})}{|J_k^{(1)}|} \leq \frac{M}{100}.$$

In each $J_k^{(1)}$, we consider the maximal dyadic subarcs $I_j^{(2)} \subset J_k^{(1)}$ with

$$\frac{\mu(I_j^{(2)})}{|I_j^{(2)}|} \geq M.$$

Continuing in this way, we inductively define $I_j^{(m)}$ and $J_k^{(m)}$ for $m \geq 1$. We call the arcs $I_j^{(m)}$ *heavy* and the arcs $J_k^{(m)}$ *light*, $j, k, m \geq 1$.

Since μ is a singular measure, almost every point on the unit circle with respect to the Lebesgue measure is eventually contained in a light arc, so that

$$\sum_{J_k^{(m)} \subset I_j^{(m)}} |J_k^{(m)}| = |I_j^{(m)}|, \quad j, m \geq 1.$$

From the definitions of light and heavy arcs, we have

$$\sum_{I_j^{(m+1)} \subset J_k^{(m)}} |I_j^{(m+1)}| \leq \frac{1}{M} \cdot \mu(J_k^{(m)}) \leq \frac{|J_k^{(m)}|}{100}, \quad k, m \geq 1.$$

It follows that μ is concentrated on

$$\bigcup_{I_j^{(m)} \text{ heavy}} \Bigg(I_j^{(m)} \setminus \bigcup_{\text{light } J_k^{(m)} \subset I_j^{(m)}} \operatorname{Int} J_k^{(m)} \Bigg).$$

4B. *Proofs of Theorems 1.1 and 1.2.* For the convenience of the reader, we break the proofs of Theorems 1.1 and 1.2 into two lemmas:

Lemma 4.1. (i) *Let $\mu \geq 0$ be a finite singular measure on $\partial\mathbb{D}$ which satisfies*

$$\int_{\{z \in \mathbb{D}: P_\mu(z) > c\}} \frac{dA(z)}{1 - |z|} < \infty \tag{4-1}$$

for some $c \in \mathbb{R}$. Then μ is concentrated on a countable union of Beurling–Carleson sets.

(ii) *Let $\mu \geq 0$ be a finite singular measure on $\partial\mathbb{D}$ which satisfies*

$$\int_{\{z \in \mathbb{D}: P_\mu(z) > c\}} \frac{dA(z)}{(1 - |z|)^{1+p}} < \infty \tag{4-2}$$

for some $c \in \mathbb{R}$. Then μ is concentrated on a countable union of $(1-p)$-Beurling–Carleson sets.

Proof. We only prove (i) as (ii) is similar. We use the decomposition from Section 4A. To prove the theorem, it suffices to show that, for each heavy interval $I_j^{(m)}$,

$$E = I_j^{(m)} \setminus \bigcup_{\text{light } J_k^{(m)} \subset I_j^{(m)}} \operatorname{Int} J_k^{(m)}$$

is a Beurling–Carleson set. By Lemma 3.1, we may check that

$$\sum_{\substack{I \text{ dyadic} \\ I \cap E \neq \varnothing}} |I| < \infty.$$

By construction, if I is a dyadic interval in $I_j^{(m)}$ which meets E, then $\mu(I)/|I| > \frac{1}{100} M$ and $P_\mu(z) \gtrsim M$ for $z \in T_I$. Hence,

$$\sum_{\substack{I \text{ dyadic} \\ I \cap E \neq \varnothing}} |I| \lesssim \int_{\{z: P_\mu(z) \gtrsim M\}} \frac{dA(z)}{1 - |z|} < \infty$$

as desired. The proof is complete. $\square$

Ahern and Clark gave an elegant formula for the angular derivative of a singular inner function on the unit circle:

$$|S'_\mu(z)| = 2\int_{\partial\mathbb{D}} \frac{d\mu(\zeta)}{|\zeta - z|^2}, \quad |z| = 1,$$

where at a given point $z \in \partial\mathbb{D}$, either both quantities are finite and equal or infinite. For a proof, see [Mashreghi 2013, Chapter 4.1].

Lemma 4.2. (i) *If* $S'_\mu \in \mathcal{N}$, *then the area condition* (1-3) *holds.*

(ii) *If* $S'_\mu \in H^p$, *then the* $(1+p)$*-area condition* (1-4) *holds.*

Proof. Observe that

$$\Omega_c = \{z \in \mathbb{D} : P_\mu(z) > c\} = \{z \in \mathbb{D} : |S_\mu(z)| < e^{-c}\}.$$

Let $e^{i\theta} \in \partial\mathbb{D}$ be a point at which S_μ has a finite angular derivative. According to a well-known result of Ahern and Clark [Mashreghi 2013, Theorem 4.15],

$$|S'_\mu(re^{i\theta})| \le 4|S'_\mu(e^{i\theta})|, \quad 0 < r < 1.$$

Let $[0, e^{i\theta}]$ denote the radial line segment from the origin to $e^{i\theta}$. As $1 - |S_\mu(re^{i\theta})| \le 4|S'_\mu(e^{i\theta})|(1-r)$,

$$\Omega_c \cap [0, e^{i\theta}] \subset \left[0, \left(1 - \frac{\varepsilon}{|S'_\mu(e^{i\theta})|}\right) \cdot e^{i\theta}\right],$$

where $\varepsilon > 0$ is a constant that depends on c. From this bound on Ω_c, (i) and (ii) follow quite easily. □

5. Derivative in Hardy spaces I

In this section, we explore conditions on a singular measure μ involving Beurling–Carleson sets that guarantee the membership of S'_μ in H^p. We show:

Theorem 5.1. *Fix* $0 < p < \frac{1}{2}$. *Let* μ *be a positive measure supported on a closed set* $E \subset \partial\mathbb{D}$ *of zero length whose complementary arcs* $\{J\}$ *satisfy*

$$\sum |J|^{1-q} < \infty \tag{5-1}$$

for some $q > p/(1-p)$. *Then,* $S'_\mu \in H^p$.

We will give two examples that show that the exponent $p/(1-p)$ in the theorem above is sharp. Theorem 5.1 improves a result of Cullen [1971], who showed that $S'_\mu \in H^p$ under the stronger hypothesis $q = 2p$.

5A. ***When is*** $S'_\mu \in H^p$**?** We begin by giving a simple criterion for a singular inner function to have derivative in H^p. As is standard, for an arc J on the unit circle with $|J| \le 1$, we write $z_J = \left(1 - \frac{1}{2}|J|\right) \cdot e^{i\theta_J}$, where $e^{i\theta_J}$ is the midpoint of J. For $0 < \beta < 1/|J|$, we write βJ for the arc of length $|\beta J|$ with the same midpoint as J.

Lemma 5.2. *Fix $0 < p < \frac{1}{2}$. Suppose $E \subset \partial\mathbb{D}$ is a closed set of zero length and $\{J\}$ is its complementary arcs. For a positive measure μ supported on E, we have $S'_\mu \in H^p$ if and only if*

$$\sum u(z_J)^p |J|^{1-p} < \infty, \tag{5-2}$$

where u is the Poisson integral of μ.

Proof. Differentiation shows that $S'_\mu(z) = h(z) S_\mu(z)$, where

$$h(z) = \int_E \frac{-2\zeta}{(\zeta - z)^2}\, d\mu(\zeta) = -\int_E \frac{2\zeta}{|\zeta - z|^2}\left(\frac{\bar{\zeta} - \bar{z}}{\zeta - z}\right) d\mu(\zeta).$$

Notice that if $z/|z| \in \frac{1}{2}J,\ |z| \ge 1 - \frac{1}{4}|J|$ and $\zeta \in E$, then the quantity

$$\zeta \cdot \frac{\bar{\zeta} - \bar{z}}{\zeta - z} = \frac{1 - \bar{z}\zeta}{\zeta - z}$$

is constrained in a sector of aperture strictly less than π. This tells us that

$$|h(z)| \asymp \int_E \frac{d\mu(\zeta)}{|\zeta - z|^2} \asymp \int_E \frac{d\mu(\zeta)}{|\zeta - z_J|^2} \asymp \frac{u(z_J)}{|J|}.$$

We see that

$$\int_{J/2} |S'_\mu(z)|^p\, |dz| \asymp u(z_J)^p |J|^{1-p},$$

so the condition (5-2) is necessary for $S'_\mu \in H^p$.

To prove the converse implication, we split $J = \bigcup_{k\in\mathbb{Z}} J_k$ into countably many Whitney arcs such that

$$|J_k| \asymp \operatorname{dist}(J_k, \partial\mathbb{D} \setminus J) \asymp 2^{-|k|}|J|.$$

For $z \in J_k$, we have

$$|S'_\mu(z)| = 2\int_E \frac{d\mu(\zeta)}{|\zeta - z|^2} \asymp \frac{u(z_{J_k})}{|J_k|}.$$

By Harnack's inequality,

$$\frac{|J_k|}{|J|} \lesssim \frac{u(z_{J_k})}{u(z_J)} \lesssim \frac{|J|}{|J_k|}.$$

Therefore,

$$\int_J |S'_\mu(z)|^p\, |dz| \lesssim \sum_k |J_k| \cdot \frac{u(z_{J_k})^p}{|J_k|^p} \lesssim u(z_J)^p |J|^p \sum_k |J_k|^{1-2p} \asymp u(z_J)^p |J|^{1-p}.$$

Summing over J shows that $S'_\mu \in H^p$. □

With help of Lemma 5.2, the proof of Theorem 5.1 runs as follows:

Proof of Theorem 5.1. Let u be the Poisson integral of μ. Since u is a positive harmonic function, its nontangential maximal function is in L^δ for any $\delta < 1$. In particular, for any $\delta < 1$, we have

$$\sum_J u(z_J)^\delta |J| < \infty.$$

Applying Hölder's inequality with exponents $\delta/p > 1$ and $\delta/(\delta - p) > 1$, we obtain

$$\sum_J u(z_J)^p |J|^{1-p} = \sum_J u(z_J)^p |J|^{p/\delta} \cdot |J|^{(\delta-p)/\delta - p} \leq \left(\sum_J u(z_J)^\delta |J|\right)^{p/\delta} \left(\sum_J |J|^{1-\delta p/(\delta-p)}\right)^{(\delta-p)/\delta}.$$

Choosing $\delta \in (p, 1)$ such that $\delta p/(\delta - p) = q$ gives

$$\sum_J u(z_J)^p |J|^{1-p} < \infty,$$

which implies that $S'_\mu \in H^p$ by Lemma 5.2. Note that as δ varies over $(p, 1)$, we have that $q = \delta p/(\delta - p) = (1/p - 1/\delta)^{-1}$ varies over $(p/(1-p), \infty)$. □

Next, we extend Theorem 5.1 to inner functions:

Corollary 5.3. *Fix $0 < p < \frac{1}{2}$. Let $E \subset \partial\mathbb{D}$ be a closed set of zero length whose complementary arcs $\{J\}$ satisfy*

$$\sum |J|^{1-q} < \infty$$

for some $q > p/(1-p)$. Let F be an inner function whose singular part is supported on E and whose zeros are contained in K_E. Then $F' \in H^p$.

Proof. By an approximation argument, we can assume that F is a finite Blaschke product with zeros $\{z_n\} \subset K_E$. For each zero z_n of F, pick a point z_n^* in E that is closest to z_n. Then,

$$|F'(e^{i\theta})| = \sum \frac{1 - |z_n|^2}{|e^{i\theta} - z_n|^2} \lesssim \sum \frac{1 - |z_n|^2}{|e^{i\theta} - z_n^*|^2} = \tfrac{1}{2}|S'_\sigma(e^{i\theta})|, \quad e^{i\theta} \in \partial\mathbb{D} \setminus E,$$

where $\sigma = \sum(1 - |z_n|^2)\delta_{z_n^*}$. From Theorem 5.1, we know that $S'_\sigma \in H^p$, and by the above equation, $F' \in H^p$ as well. □

5B. *Sharpness.* We now give two examples showing that the exponent in Theorem 5.1 is sharp:

Lemma 5.4. *There exists a measure μ supported on a closed set E of zero length whose complementary arcs $\{J\}$ satisfy $\sum |J|^{1-p/(1-p)} < \infty$ yet $S'_\mu \notin H^p$.*

Proof. Step 1: In our example, E will be a certain pruned Cantor set, and

$$\mu = \sum |J|^{(1-2p)/(1-p)} (\delta_{a(J)} + \delta_{b(J)}),$$

where $a(J)$ and $b(J)$ are the two endpoints of the complementary arc J. In order for the measure μ to be finite, we need to arrange that

$$\sum |J|^{(1-2p)/(1-p)} < \infty. \tag{5-3}$$

In addition, we will arrange that

$$\sum_J \mu(\beta J)^p |J|^{1-2p} = \infty \tag{5-4}$$

for some constant $\beta > 1$ to be chosen. As $P_\mu(z_J) \gtrsim \mu(\beta J)/|J|$,

$$\sum_J P_\mu(z_J)^p |J|^{1-p} = \infty$$

and $S'_\mu \notin H^p$ by Lemma 5.2.

Step 2. Let $N_j = \#\{J : |J| \asymp 2^{-j}\}$. To achieve (5-3), we request that $N_j \asymp j^{-\alpha} \cdot 2^{(1-2p)/(1-p)\cdot j}$ for some $\alpha > 1$ to be chosen. In this case, the total measure supported on the endpoints of arcs of length $\le 2^{-j}$ is

$$M_j = \sum_{|J| \le 2^{-j}} \mu(\bar{J}) \asymp \sum_{k=j}^{\infty} 2^{-(1-2p)/(1-p)\cdot k} N_k \asymp \sum_{k=j}^{\infty} \frac{1}{k^\alpha} \asymp \frac{1}{j^{\alpha-1}}.$$

Therefore, if we construct the arcs $\{J\}$ such that

$$\mu(\beta J) \asymp \frac{M_j}{N_j} \quad \text{for } |J| \asymp 2^{-j}, \tag{5-5}$$

then we would have

$$\sum_J \mu(\beta J)^p |J|^{1-2p} \asymp \sum_{j=1}^{\infty} N_j 2^{-j(1-2p)} \left(\frac{M_j}{N_j}\right)^p \asymp \sum_{j=1}^{\infty} \frac{1}{j^{\alpha-p}}.$$

In order to obtain (5-4), we may choose α to be any number in $(1, 1+p)$.

Step 3. Fix a real number $A > 2$. Consider the standard Cantor set E, which at generation n is formed from 2^n arcs of length A^{-n}. Inspection shows that $N_j \asymp 2^{j/\log_2 A}$. We choose A appropriately such that

$$\frac{1}{\log_2 A} = \frac{1-2p}{1-p} \in (0, 1).$$

In order to make N_j smaller, we slightly modify the construction of the standard Cantor set by removing a number of arcs. We call a generation *bad* if $N_j > j^{-\alpha} \cdot 2^{(1-2p)/(1-p)\cdot j}$ is too large. In a bad generation, we allow each arc to only have one descendant instead of two, say the left one. In the pruned Cantor set, we have $N_j \asymp j^{-\alpha} \cdot 2^{(1-2p)/(1-p)\cdot j}$ as desired.

We select $\beta > (1-2A)^{-1}$, so that if J is a complementary arc of some generation, then βJ covers the interval defining the Cantor set of the previous generation. Since the mass of μ is evenly spread out, μ satisfies (5-5). □

In our second example of the sharpness of the exponent in Theorem 5.1, we have a slightly stronger assumption and a slightly stronger conclusion:

Lemma 5.5. *Given $q < p/(1-p)$, there exists a $(1-q)$-Beurling–Carleson set E and a measure μ supported on E such that $S'_\nu \notin H^p$ for any $0 < \nu \le \mu$.*

Proof. Fix a real number $A > 2$. Consider the standard Cantor set E, which at generation n is formed from 2^n arcs of length A^{-n}. Let μ be the standard Cantor measure on E, that is, μ is the probability measure supported on E which gives equal mass to arcs of generation n.

Step 1: *When is E a Beurling–Carleson set*? In generation n, there are 2^{n-1} complementary arcs of length $A^{-n+1}(1-2A^{-1})$. If $\partial\mathbb{D}\setminus E=\bigcup I_k$, then

$$\sum |I_k|^{1-q} \asymp \sum_n 2^n A^{-(1-q)n},$$

which converges if $\log A > (\log 2)/(1-q)$. In other words, E is a q-Beurling–Carleson set when $\log A > (\log 2)/(1-q)$.

Step 2: *When is the measure μ invisible*? Fix a measure $0<\nu\le\mu$. Let $\mathcal{A}(n)$ be the collection of arcs I of generation n in the construction of the Cantor set E such that $\nu(I)\ge 2^{-n-1}\nu(\partial\mathbb{D})$. Since $\#\mathcal{A}(n)\le 2^n$, we have

$$\nu(\partial\mathbb{D}) \le \sum_{I\in\mathcal{A}(n)} \nu(I) + \sum_{I\notin\mathcal{A}(n)} \nu(I) \le \sum_{I\in\mathcal{A}(n)} \nu(I) + \tfrac{1}{2}\nu(\partial\mathbb{D}),$$

which simplifies to

$$\sum_{I\in\mathcal{A}(n)} \nu(I) \ge \tfrac{1}{2}\nu(\partial\mathbb{D}).$$

However, as $\nu(I)\le 2^{-n}$ for any $I\in\mathcal{A}(n)$,

$$\#\mathcal{A}(n) \ge 2^n\cdot\tfrac{1}{2}\nu(\partial\mathbb{D}).$$

Hence,

$$\sum_{I\in\mathcal{A}(n)} |I|^{1-p}P_\nu(z_I)^p \gtrsim \sum_{I\in\mathcal{A}(n)} |I|^{1-2p}\nu(I)^p \gtrsim 2^n\nu(\partial\mathbb{D})A^{-n(1-2p)}2^{-np} = \left(\frac{2^{1-p}}{A^{1-2p}}\right)^n \nu(\partial\mathbb{D}).$$

Since the lengths and locations of the arcs defining E of generation n are comparable to the complementary arcs of generation n, we may use Lemma 5.2 to conclude that $S'_\nu\notin H^p$ if $2^{1-p}>A^{1-2p}$.

Step 3: *Conclusion.* To prove the lemma, we need to find an $A>2$ satisfying

$$\frac{1}{1-q}\cdot\log 2 < \log A < \frac{1-p}{1-2p}\cdot\log 2,$$

which is possible if $1-q>(1-2p)/(1-p)$, that is, $q<p/(1-p)$. □

Remark. There may also be an example in the extreme case when $q=p/(1-p)$.

6. Derivative in Hardy spaces II

Suppose $0<p<\frac{1}{2}$ and $\mu\ge 0$ is a singular measure on $\partial\mathbb{D}$. Recall that, by Theorem 1.2, if $S'_\mu\in H^p$ then S_μ satisfies the $(1+p)$-area condition (1-4). We now show that if (1-4) holds, then $\mu=\sum\mu_i$ can be written as a countable sum of measures with $S'_{\mu_i}\in H^p$. In view of the implication (2) ⇒ (3) of Theorem 1.2, it is enough to prove the following lemma:

Lemma 6.1. *Fix $0<p<\frac{1}{2}$. Suppose μ is a measure supported on a $(1-p)$-Beurling–Carleson set. If S_μ satisfies the $(1+p)$-area condition (1-4), then $S'_\mu\in H^p$.*

Proof. Let $E = \operatorname{supp}\mu$, and write $\partial\mathbb{D} \setminus E = \bigcup J_k$. By Lemma 5.2, we need to show that

$$\sum_k P_\mu(z_{J_k})^p |J_k|^{1-p} < \infty.$$

Since $\sum |J_k|^{1-p} < \infty$, we only need to show that

$$\sum_{k: P_\mu(z_{J_k}) \ge 1} P_\mu(z_{J_k})^p |J_k|^{1-p} < \infty.$$

Let $J \subset \partial\mathbb{D}$ be any arc with $J \cap E = \varnothing$. It is easy to see that

$$\frac{P_\mu(z_I)}{|I|} \gtrsim \frac{P_\mu(z_J)}{|J|}$$

for any arc $I \subset J$. Therefore, if $P_\mu(z_{J_k}) \ge 1$, then

$$\sum_{\substack{I \subset J_k \text{ dyadic} \\ P_\mu(z_I) \ge 1}} |I|^{1-p} \gtrsim \sum_{\substack{I \subset J_k \text{ dyadic} \\ |I| \gtrsim |J_k|/P_\mu(z_{J_k})}} |I|^{1-p} \asymp \sum_{n=0}^{\log_2 P_\mu(z_{J_k})} 2^n \cdot (2^{-n}|J_k|)^{1-p} \asymp |J_k|^{1-p} P_\mu(z_{J_k})^p.$$

By Harnack's inequality, one can find a constant $0 < c < 1$ such that

$$\sum_{k: P_\mu(z_{J_k}) \ge 1} P_\mu(z_{J_k})^p |J_k|^{1-p} \lesssim \sum_{k: P_\mu(z_{J_k}) \ge 1} \sum_{\substack{I \subset J_k \text{ dyadic} \\ P_\mu(z_I) \ge 1}} |I|^{1-p} \lesssim \int_{\{z \in \mathbb{D}: |S_\mu(z)| \le c\}} \frac{dA(z)}{(1-|z|)^{1+p}},$$

which is finite by assumption. The proof is complete. □

We now give an example of a singular inner function S_μ which satisfies the $(1+p)$-area condition (1-4) yet $S'_\mu \notin H^p$.

Lemma 6.2. *For $0 < p < \frac{1}{2}$, there exists a singular inner function S_μ with $S'_\mu \notin H^p$ such that*

$$\int_{\{z \in \mathbb{D}: |S_\mu(z)| \le c\}} \frac{dA(z)}{(1-|z|)^{1+p}} < \infty$$

for any $0 < c < 1$.

Sketch of proof. To get a feeling of why the lemma is true, we examine the situation for the measure μ which consists of n equally spaced point masses on the circle: $\mu = (1/n^{2-\varepsilon}) \sum_{k=0}^{n-1} \delta_{\xi_k}$, where $\xi_k = e^{2\pi i k/n}$, $k = 0, 1, 2, \dots, n-1$, and $\varepsilon > 0$ is a constant to be chosen. Since

$$|S'_\mu(e^{i\theta})| = \int_0^{2\pi} \frac{2\, d\mu(t)}{|e^{i\theta} - e^{it}|^2} = \frac{2}{n^{2-\varepsilon}} \sum_{k=0}^{n-1} \frac{1}{|e^{i\theta} - \xi_k|^2} \asymp \frac{1}{n^{2-\varepsilon} \cdot \operatorname{dist}(e^{i\theta}, \{\xi_k\})^2},$$

the integral

$$\int_0^{2\pi} |S'_\mu(e^{i\theta})|^p \, d\theta \asymp n \int_{-\pi/n}^{\pi/n} \left(\frac{1}{n^{2-\varepsilon}\theta^2} \right)^p d\theta \asymp n^{\varepsilon p}$$

tends to infinity as $n \to \infty$.

Let H_k be the horoball which rests at ξ_k of diameter $\alpha/n^{2-\varepsilon}$. It is not difficult to see that, for any $0 < c < 1$, there exists an $\alpha = \alpha(c) > 0$ such that

$$\{z \in \mathbb{D} : |S_\mu(z)| \le c\} \subseteq \bigcup_{k=0}^{n-1} H_k.$$

As the integral over a single horoball is

$$\int_{H_0} \frac{dA(z)}{(1-|z|^2)^{1+p}} \asymp \frac{1}{n^{(2-\varepsilon)(1-p)}},$$

the integral over their union is

$$\int_{\bigcup H_k} \frac{dA(z)}{(1-|z|^2)^{1+p}} \asymp n^{1-(2-\varepsilon)(1-p)}.$$

Since $0 < p < \frac{1}{2}$, we can choose $\varepsilon > 0$ small enough to make the exponent $1-(2-\varepsilon)(1-p)$ negative, so that the integrals

$$\int_{\{z\in\mathbb{D}:|S_\mu(z)|<c\}} \frac{dA(z)}{(1-|z|)^{1+p}}$$

tend to 0 as $n \to \infty$.

Independent copies of this construction provide an example of a singular inner function S with $S' \notin H^p$ for which

$$\int_{\{z\in\mathbb{D}:|S(z)|<c\}} \frac{dA(z)}{(1-|z|)^{1+p}} < \infty.$$

We leave the details to the reader. □

7. Background on angular derivatives

For $0 < \theta < \pi$ and $0 < \delta < 1$, let $S_{\theta,\delta}(p) = S_\theta(p) \cap B(p, \delta)$ denote the truncated Stolz angle of opening θ with vertex at $p \in \partial\mathbb{D}$.

Suppose $\Omega \subset \mathbb{D}$ is a domain in the unit disk bounded by a Jordan curve. We say that Ω has an *inner tangent* at a point $p \in \partial\Omega \cap \partial\mathbb{D}$ if, for any $0 < \theta < \pi$, Ω contains a truncated Stolz angle of opening θ with vertex at p.

Let $\varphi : \mathbb{D} \to \Omega$ be a conformal map. We say that φ has a (nonzero) *angular derivative* at $q = \varphi^{-1}(p)$ if the nontangential limit

$$\lim_{z\to q} |\varphi'(z)| = A$$

for some real number $A > 0$. While the number A depends on the choice of Riemann map φ, the existence of the angular derivative does not. In other words, possessing an angular derivative is an intrinsic property of (Ω, p), which we record by saying that Ω is *thick* at p. In the language of potential theory, one would say that the complement $\mathbb{D} \setminus \Omega$ is minimally thin at p, see [Burdzy 1986, Theorem 5.2], which means that Brownian motion conditioned to exit the unit disk at p is eventually contained in Ω.

To avoid dealing with the point q, we will simply say that the inverse conformal map $\psi : \Omega \to \mathbb{D}$ has an angular derivative at p and write $|\psi'(p)| = A^{-1}$. It is easy to see that if Ω is thick at p, then Ω possesses an inner tangent at p.

Rodin and Warschawski gave an if and only if condition for ψ to possess an angular derivative at p in terms of moduli of curve families, e.g., see [Garnett and Marshall 2005, Theorem V.5.7] or [Betsakos and Karamanlis 2022]. When Ω is a starlike domain with regular boundary, their condition takes a simpler form [Ivrii and Kreitner 2024]:

Theorem 7.1. *Suppose $\Omega = \{r\zeta : \zeta \in \partial\mathbb{D}, 0 \le r < 1 - h(\zeta)\}$, where $h : \partial\mathbb{D} \to \left[0, \frac{1}{2}\right]$ is a continuous function. Assume that h satisfies the doubling condition*

$$h(\zeta_1) \ge c \cdot h(\zeta_2) \quad \textit{whenever } |\zeta_2 - \zeta_1| < c \cdot h(\zeta_1)$$

for some $c > 0$. Then, ψ has an angular derivative at $p \in \partial\Omega \cap \partial\mathbb{D}$ if and only if

$$\int_{\partial\mathbb{D}} \frac{h(\zeta)}{|\zeta - p|^2} |d\zeta| < \infty.$$

We will use the following elementary lemma about angular derivatives:

Lemma 7.2. *Let $\{\Omega_n\}_{n=1}^{\infty}$ be an increasing sequence of Jordan domains whose union is the unit disk. Suppose the conformal maps $\psi_n : \Omega_n \to \mathbb{D}$ converge uniformly on compact subsets to the identity. If ψ_1 has an angular derivative at $p \in \partial\Omega \cap \partial\mathbb{D}$, then the angular derivatives $|\psi_n'(p)|$ tend to* 1.

We will also need the following theorem from [Ivrii and Kreitner 2024] which describes how composition operators act on measures on the unit circle:

Theorem 7.3. *Suppose $\Omega \subset \mathbb{D}$ is a Jordan domain, $\varphi : \mathbb{D} \to \Omega$ is a conformal map and $\psi : \Omega \to \mathbb{D}$ is its inverse. Let $\mu \ge 0$ be a positive measure on the unit circle. Since $P_\mu(\varphi(z))$ is a positive harmonic function, it can be represented as the Poisson extension of some finite measure $\nu \ge 0$. If we use the normalization $0 \in \Omega$ and $\varphi(0) = 0$, then*

$$\varphi_* \nu = P_\mu \, d\omega_{\Omega,0} + |\psi'| \, d\mu, \tag{7-1}$$

provided that we interpret $|\psi'(p)| = 0$ if $p \notin \partial\Omega$ or Ω is not thick at p.

8. Background in PDE

In this section, we make some general observations about semilinear elliptic equations of the form

$$\Delta u = g(u), \tag{8-1}$$

which will be used in Section 9. We assume that the "nonlinearity" g is a nonnegative increasing convex function which satisfies the Keller–Osserman condition [Keller 1957; Osserman 1957]

$$\int_1^\infty \frac{ds}{\sqrt{G(s)}} < \infty, \tag{8-2}$$

where $G' = g$. Examples of g satisfying the above conditions include $g(t) = e^{2t}$ and $g(t) = t^p \cdot \chi_{t>0}$ with $p > 1$.

8A. ***Basic properties.***

Traces. Given a function ϕ on the unit disk, we define its *boundary trace* as the weak limit of the measures $\phi(re^{i\theta})\, d\theta$ as $r \to 1$, provided that the limit exists. Otherwise, we say that ϕ does not possess a boundary trace.

Sub- and supersolutions. One says that a function $v : \mathbb{D} \to \mathbb{R}$ is a *subsolution* of (8-1) if $\Delta v \geq g(v)$ in the sense of distributions. Similarly, we say that v is a *supersolution* if $\Delta v \leq g(v)$ in the sense of distributions.

Theorem 8.1 (principle of sub- and supersolutions). *Suppose u_- is a subsolution and u_+ is a supersolution of* (8-1) *with $u_-(z) \leq u_+(z)$ for any $z \in \mathbb{D}$. Then, there exists at least one solution $u(z)$ with*

$$u_-(z) \leq u(z) \leq u_+(z), \quad z \in \mathbb{D}.$$

A proof using the Schauder fixed point theorem can be found in [Ponce 2016, Chapter 20].

Existence of solutions and the comparison principle.

Theorem 8.2. *Given a function $h \in L^\infty(\partial\mathbb{D})$, the boundary value problem*

$$\begin{cases} \Delta u = g(u) & \text{in } \mathbb{D}, \\ u = h & \text{on } \partial\mathbb{D} \end{cases} \tag{8-3}$$

admits a unique solution, where the boundary values are interpreted in the sense of weak limits of measures. If u_1 and u_2 are two solutions with boundary values $h_1 \leq h_2$, then $u_1 \leq u_2$ on $\mathbb{D}$.

Proof of Theorem 8.2. Step 1: *Uniqueness and monotonicity.* By Kato's inequality [Ponce 2016, Proposition 6.9],

$$\Delta(u_1 - u_2)^+ \geq \Delta(u_1 - u_2) \cdot \chi_{\{u_1 > u_2\}} = (g(u_1) - g(u_2)) \cdot \chi_{\{u_1 > u_2\}} \geq 0$$

is a subharmonic function. As $h_1 \leq h_2$, the function $(u_1 - u_2)^+$ has zero boundary values. The maximum principle shows that $(u_1 - u_2)^+ \leq 0$ or $u_1 \leq u_2$. The same argument also proves uniqueness.

Step 2: *Existence.* Let P_h denote the harmonic extension of h to the unit disk. Clearly, $u_+ = P_h$ is a supersolution of (8-1) with boundary data h. Similarly, if $G(z, w) = \log|(1 - \overline{w}z)/(w - z)|$ is the Green's function of the unit disk, then

$$u_-(z) = P_h(z) - \frac{1}{2\pi} \int_{\mathbb{D}} g(\|h\|_\infty) G(z, \zeta)\, dA(\zeta)$$

is a subsolution of (8-1) as

$$\Delta u_-(z) = g(\|h\|_\infty) \geq g(u_-(z)).$$

Since u_- also has boundary trace h, by the principle of sub- and supersolutions, there exists a solution with boundary trace h. □

The maximal solution.

Lemma 8.3. *The PDE* (8-1) *has a unique maximal solution* $u_{\max}$ *on the unit disk*, *which dominates all other solutions pointwise.*

Sketch of proof. We will simultaneously show that (8-1) has a maximal solution on every disk $\mathbb{D}_R = \{z : |z| < R\}$ with $R > 0$.

Keller [1957] and Osserman [1957] showed that, under the assumption (8-2), for any $R > 0$, there is a unique radially invariant solution $u_R(z)$ on $\mathbb{D}_R$ which tends to infinity as $|z| \to R$, and furthermore, the solutions $u_R(z)$ depend continuously on R.

Suppose $u : \mathbb{D}_R \to \mathbb{R}$ is any solution of (8-1). By the comparison principle, for any $S < R$, we have that $u(z) < u_S(z)$ on $\mathbb{D}_S$. Taking $S \to R$ yields $u(z) \le u_R(z)$. □

The above argument shows that if u is a solution of (8-1) on the unit disk which tends to infinity as $|z| \to 1$, then $u = u_{\max}$. As a consequence, the solutions u_n of (8-1) with constant boundary values n increase to $u_{\max}$ as $n \to \infty$.

Remark. For the existence and uniqueness of large solutions of semilinear equations on other domains, we refer the reader to [Bandle and Marcus 1992; García-Melián 2009]. Information about the asymptotic behavior of these solutions near the boundary can be found in [Bandle and Marcus 1998; 2004; del Pino and Letelier 2002; Lazer and McKenna 1994].

Minimal dominating solution. Let v be a subsolution of (8-1). For $0 < r < 1$, we write $\Lambda_r[v]$ for the unique solution of (8-1) on the disk $\mathbb{D}_r = \{z : |z| < r\}$ which agrees with v on $\partial\mathbb{D}_r$. An inspection of Step 1 of the proof of Theorem 8.2 shows that $\Lambda_r[v]$ is the pointwise-minimal solution which lies above v on $\mathbb{D}_r$. In particular, the solutions $\Lambda_r[v]$ are increasing in r. The limit $\Lambda[v] := \lim_{r\to 1} \Lambda_r[v]$ is finite on the unit disk because it is bounded above by the maximal solution.

For any test function $\phi \in C_c^\infty(\mathbb{D})$, we have

$$\int_{\mathbb{D}_r} u_r \Delta\phi \, dA = \int_{\mathbb{D}_r} g(u_r)\phi \, dA, \qquad u_r = \Lambda_r[v],$$

provided that $\mathbb{D}_r$ contains $\operatorname{supp}\phi$ in its interior. After taking $r \to 1$ and using the dominated convergence theorem, it follows that $\Lambda[v]$ is a solution of (8-1). From the construction, it is clear that $\Lambda[v]$ is the pointwise-minimal solution which satisfies $\Lambda[v] \ge v$.

Remark. This construction generalizes the notion of the *minimal harmonic majorant* for subharmonic functions on the unit disk. One small but important difference is that the minimal harmonic majorant does not always exist (i.e., may be identically $+\infty$).

8B. ***Nearly maximal solutions.*** A solution of (8-1) is called *nearly maximal* if

$$\limsup_{r\to 1} \int_{|z|=r} (u_{\max} - u) \, d\theta < \infty. \tag{8-4}$$

For each $0 < r < 1$, we may view $(u_{\max} - u)\, d\theta$ as a positive measure on the circle of radius r. Subharmonicity guarantees the existence of a weak limit as $r \to 1$, so we obtain a measure $\mu[u]$ on the unit circle associated to u. We refer to μ as the *deficiency measure* of u.

Notice that if $\mu \geq 0$ is a measure on the unit circle and P_μ is its Poisson extension to the unit disk, then $\Lambda[u_{\max} - P_\mu]$ is a nearly maximal solution. Clearly, the deficiency measure ν of $\Lambda[u_{\max} - P_\mu]$ is at most μ.

Lemma 8.4 (fundamental lemma). *If u is a nearly maximal solution of* (8-1) *with deficiency measure μ, then $u = \Lambda[u_{\max} - P_\mu]$.*

Proof. Step 1. Observe that $u_{\max} - P_\mu$ is a subsolution since

$$\Delta(u_{\max} - P_\mu) = g(u_{\max}) \geq g(u_{\max} - P_\mu).$$

We claim that $u \geq u_{\max} - P_\mu$ and thus $u \geq \Lambda[u_{\max} - P_\mu]$. To this end, we consider the function

$$\phi := u_{\max} - u - P_\mu.$$

Since ϕ is a subharmonic function with zero boundary trace, by the maximum principle, $\phi \leq 0$ in the unit disk.

Step 2. As $v := \Lambda[u_{\max} - P_\mu]$ is a nearly maximal solution, it possesses a deficiency measure ν. From Step 1, we know that

$$u \geq v = \Lambda[u_{\max} - P_\mu] \geq u_{\max} - P_\mu.$$

After rearranging, we get

$$u_{\max} - u \leq u_{\max} - v \leq P_\mu.$$

Taking the weak limit as $r \to 1$, we see that $\nu = \mu$.

Step 3. Finally, since $u - v$ is a nonnegative subharmonic function with zero boundary trace, $u = v$. □

In particular, Lemma 8.4 shows that the deficiency measure μ uniquely determines the nearly maximal solution u. Below, we will write u_μ for the nearly maximal solution associated to the measure μ, if it exists. Another simple consequence of Lemma 8.4 is the *monotonicity principle* for nearly maximal solutions:

Corollary 8.5 (monotonicity principle). *If $\nu < \mu$ then $u_\nu > u_\mu$.*

8C. *Constructible and invisible measures.* We say that a measure μ on the unit circle is *invisible* if, for any measure $0 < \nu \leq \mu$, there does not exist a nearly maximal solution u_ν with deficiency measure ν. In this section, we show that any positive measure on the unit circle can be uniquely decomposed into a deficiency measure and an invisible measure.

Theorem 8.6. *Suppose μ is a positive measure on the unit circle. If $u_\nu = \Lambda[u_{\max} - P_\mu]$, then ν is a deficiency measure and $\mu - \nu$ is an invisible measure.*

In particular, a measure μ is invisible if and only if $\Lambda[u_{\max} - P_\mu] = u_{\max}$. We will break the proof of Theorem 8.6 into a series of lemmas.

Lemma 8.7. *If μ is a deficiency measure, then any measure $0 \leq \mu_1 \leq \mu$ is also a deficiency measure.*

Proof. To show that μ_1 is a deficiency measure, we check that $\mu_1 = \nu_1$, where $u_{\nu_1} = \Lambda[u_{\max} - P_{\mu_1}]$. Since the inequality $\nu_1 \le \mu_1$ is always true, we only need to prove the opposite inequality $\mu_1 \le \nu_1$.

Let $\mu_2 = \mu - \mu_1$. Using the same argument as in the proof of Lemma 8.4, it is not difficult to show that

$$\Lambda[u_{\max} - P_{\mu_1+\mu_2}] \ge \Lambda[u_{\max} - P_{\mu_1}] - P_{\mu_2}$$

or

$$u_{\max} - \Lambda[u_{\max} - P_{\mu_1+\mu_2}] \le u_{\max} - \Lambda[u_{\max} - P_{\mu_1}] + P_{\mu_2}.$$

Taking traces, we see that $\mu_1 + \mu_2 \le \nu_1 + \mu_2$ or $\mu_1 \le \nu_1$ as desired. □

Lemma 8.8. (i) *The sum of two deficiency measures is a deficiency measure.*

(ii) *Suppose μ_i, $i = 1, 2, 3, \dots$, are deficiency measures such that their sum $\mu = \sum \mu_i$ is a finite measure. Then, μ is also a deficiency measure.*

In the proof below, we will use the following elementary observation: if g is a convex function and $x_1 < x_2 < x_3 < x_4$ are four real numbers satisfying $x_1 + x_4 = x_2 + x_3$, then

$$g(x_2) + g(x_3) \le g(x_1) + g(x_4). \tag{8-5}$$

Moreover, if g is an *increasing* convex function, then (8-5) holds under the weaker assumption $x_1 + x_4 \ge x_2 + x_3$. This is a one-dimensional analogue of the fact that the composition $\phi \circ u$ of an increasing convex function ϕ and a subharmonic function u is subharmonic.

Proof of Lemma 8.8. (i) Suppose $\mu = \mu_1 + \mu_2$ is a measure on the unit circle. Set

$$u_\nu = \Lambda[u_{\max} - P_\mu].$$

In view of the discussion preceding Lemma 8.4, to prove (i), it is enough to show that

$$\mu_1 + \mu_2 \le \nu. \tag{8-6}$$

To verify (8-6), we check that

$$\Lambda[u_{\max} - P_{\mu_1}] + \Lambda[u_{\max} - P_{\mu_2}] \ge \Lambda[u_{\max}] + \Lambda[u_{\max} - P_\mu],$$

which we abbreviate as $B + C \ge A + D$. Clearly, $A \ge B \ge D$ and $A \ge C \ge D$. Consider the function

$$\phi = (A + D - B - C)^+.$$

Since g is an increasing convex function, at a point $z \in \mathbb{D}$ where $A + D > B + C$, we have

$$\Delta\phi(z) = g(A(z)) + g(D(z)) - g(B(z)) - g(C(z)) \ge 0.$$

In view of Kato's inequality, ϕ is subharmonic and nonnegative on the unit disk. If we knew that ϕ had zero trace, then we could immediately say that ϕ is identically 0.

Due to difficulties examining the trace of ϕ on $\partial\mathbb{D}$ directly, we use an approximation argument. For each $0 < r < 1$, we consider the function

$$\phi_r = (\Lambda_r[u_{\max} - P_{\mu_1}] + \Lambda_r[u_{\max} - P_{\mu_2}] - \Lambda_r[u_{\max}] - \Lambda_r[u_{\max} - P_\mu])^+,$$

defined on $\mathbb{D}_r$. The above argument shows that ϕ_r is a nonnegative subharmonic function on $\mathbb{D}_r$. As ϕ_r has zero boundary values on $\partial\mathbb{D}_r$, it is identically 0. Taking $r \to 1$, we see that ϕ is identically 0 as desired.

(ii) Set $\tilde{\mu}_j = \mu_1 + \mu_2 + \cdots + \mu_j$. By part (i), we have

$$\Lambda[u_{\max} - P_\mu] \le \Lambda[u_{\max} - P_{\tilde{\mu}_j}] = u_{\tilde{\mu}_j}.$$

The above equation shows that if

$$u_\nu = \Lambda[u_{\max} - P_\mu],$$

then $\nu \ge \tilde{\mu}_j$ for any j, which implies $\nu \ge \mu$. As the reverse inequality is always true, $\nu = \mu$ as desired. □

Lemma 8.9. *If $\mu \ge 0$ is a measure on the unit circle and $u_\nu = \Lambda[u_{\max} - P_\mu]$, then the difference $\mu - \nu$ is invisible.*

Proof. We need to show that any measure $0 < \omega \le \mu - \nu$ does not arise as a deficiency measure of some nearly maximal solution. The existence of u_ω would imply the existence of $u_{\nu+\omega}$ by Lemma 8.8, which would in turn lead to the estimate

$$u_{\max} - P_\mu \le u_{\max} - P_{\nu+\omega} \le u_{\nu+\omega} \le u_\nu$$

by the monotonicity principle and the fundamental lemma (Lemmas 8.5 and 8.4 respectively). This contradicts the definition of u_ν as the *least* solution that lies above $u_{\max} - P_\mu$. □

8D. *A lemma on iterated majorants.* For future reference, we record the following lemma:

Lemma 8.10. (i) *For two positive measures μ_1 and μ_2 on the unit circle*,

$$\Lambda[\Lambda[u_{\max} - P_{\mu_2}] - P_{\mu_1}] = \Lambda[u_{\max} - P_{\mu_1+\mu_2}].$$

(ii) *More generally*,

$$\Lambda[\cdots\Lambda[\Lambda[u_{\max} - P_{\mu_j}] - P_{\mu_{j-1}}]\cdots - P_{\mu_1}] = \Lambda[u_{\max} - P_{\mu_1+\mu_2+\cdots+\mu_j}].$$

(iii) *If $\mu = \sum_{j=1}^{\infty} \mu_j$ is a finite measure*, *then*

$$\lim_{j\to\infty} \Lambda[\cdots\Lambda[\Lambda[u_{\max} - P_{\mu_j}] - P_{\mu_{j-1}}]\cdots - P_{\mu_1}] = \Lambda[u_{\max} - P_\mu]$$

pointwise on the unit disk.

Proof. (i) The $\ge$ direction follows from the monotonicity of Λ. For the $\le$ direction, it suffices to show

$$\Lambda[u_{\max} - P_{\mu_2}] - P_{\mu_1} \le \Lambda[u_{\max} - P_{\mu_1+\mu_2}]$$

or

$$\Lambda_r[u_{\max} - P_{\mu_2}] - P_{\mu_1} \le \Lambda_r[u_{\max} - P_{\mu_1+\mu_2}]$$

for any $0 < r < 1$. To this end, we form the function

$$u_r = (\Lambda_r[u_{\max} - P_{\mu_2}] - P_{\mu_1}) - \Lambda_r[u_{\max} - P_{\mu_1+\mu_2}],$$

defined on $\mathbb{D}_r = \{z : |z| < r\}$. Since u_r is subharmonic and vanishes on $\partial\mathbb{D}_r$, it must be identically 0. This proves the $\leq$ direction.

(ii) This follows after applying (i) $j-1$ times.

(iii) Let $\tilde{\mu}_j = \mu_1 + \mu_2 + \cdots + \mu_j$. By part (i), we have

$$\Lambda[u_{\max} - P_{\tilde{\mu}_j}] - P_{\mu-\tilde{\mu}_j} \leq \Lambda[u_{\max} - P_\mu] \leq \Lambda[u_{\max} - P_{\tilde{\mu}_j}].$$

Since $P_{\mu-\tilde{\mu}_j} \to 0$ pointwise in the unit disk, the minimal dominating solutions $\Lambda[u_{\max} - P_{\tilde{\mu}_j}]$ decrease to $\Lambda[u_{\max} - P_\mu]$. □

9. Nearly maximal solutions

In this section, we prove Theorem 1.3 which partially characterizes the nearly maximal solutions of

$$\Delta u = u^p \cdot \chi_{u>0} \quad \text{on } \mathbb{D}, \tag{9-1}$$

with $p > 1$. From Section 8A, we know that (9-1) has a radially invariant solution $u_{\max}$ which dominates all the other solutions pointwise. By solving an ODE, one can write down an explicit formula for $u_{\max}$. Here, we will only need the asymptotic formula

$$u_{\max}(z) \sim C_\alpha (1-|z|)^{\alpha-1}, \quad |z| \to 1,$$

where $\alpha = (p-3)/(p-1)$. We will be especially interested in the case when $p > 3$, in which case $\alpha \in (0, 1)$.

The proof of Theorem 1.3 consists of two parts:

(1) First, we show that if μ does not charge α-Beurling–Carleson sets, then it is not the deficiency measure of any nearly maximal solution. As the proof is similar to the one in [Ivrii 2019] for $\Delta u = e^{2u}$, we only give a sketch of the argument in Section 9B.

(2) Secondly, we show that if μ is concentrated on an β-Beurling–Carleson set for some $\beta < \alpha$, then there is a nearly maximal solution u_μ with deficiency measure μ. The argument in [Ivrii 2019] relied on the Liouville correspondence between solutions of $\Delta u = e^{2u}$ and holomorphic self-mappings of the disk, which is unavailable in the present setting. We present a new approach to existence which involves special Privalov stars with round corners. The special Privalov stars will be constructed in Section 9C, and the existence will be explained in Section 9D.

9A. ***Restoring property.*** We focus on the case when $p > 3$. The following lemmas will be used in conjunction with Roberts decompositions to show that certain measures on the unit circle are invisible:

Lemma 9.1. *Let $n_i = 2^i$. For any $0 < a < 1$, there exists $a < b < 1$ such that*

$$\Lambda_{1-1/n_{i+1}}[a \cdot u_{\max}] > b \cdot u_{\max} \quad \text{on } \left\{z : |z| = 1 - \frac{1}{n_i}\right\}. \tag{9-2}$$

Proof. We prefer to work on the upper half-plane $\mathbb{H}$ since the expression for the maximal solution is simpler there: $u_{\max}(z) = C_\alpha y^{\alpha-1}$, where $y = \operatorname{Im} z$. We need to show that

$$\Lambda_{y_0}[a \cdot u_{\max}] > b \cdot u_{\max} \quad \text{on } \{\operatorname{Im} z = 2y_0\}.$$

When extending constant boundary values from a horizontal line, we get the maximal solution shifted vertically by an appropriate amount:

$$u = \Lambda_{y_0}[a \cdot u_{\max}] = C_\alpha (y + c)^{\alpha-1},$$

where c is determined by the equation

$$a \cdot C_\alpha y_0^{\alpha-1} = C_\alpha (y_0 + c)^{\alpha-1} \quad \Longrightarrow \quad c = a^{1/(\alpha-1)} \cdot y_0 - y_0.$$

In particular,

$$u(2y_0) = C_\alpha (1 + a^{1/(\alpha-1)})^{\alpha-1} \cdot y_0^{\alpha-1}.$$

This suggests that we should take

$$b = \frac{u(2y_0)}{u_{\max}(2y_0)} = \frac{(1 + a^{1/(\alpha-1)})^{\alpha-1}}{2^{\alpha-1}} > a. \qquad \square$$

A similar argument shows:

Lemma 9.2. *For any* $0 < a, \varepsilon, \rho < 1$, *there exists an* $0 < r < 1$ *such that*

$$\Lambda_r[a \cdot u_{\max}] > (1 - \varepsilon) \cdot u_{\max} \quad \text{on } \mathbb{D}_\rho. \tag{9-3}$$

9B. *Invisible measures.* Suppose μ is a measure on the unit circle that does not charge α-Beurling–Carleson sets. In order to show that μ is invisible, it is enough to check that $\Lambda[u_{\max} - P_\mu] = u_{\max}$, where Λ denotes the minimal dominating solution on the unit disk.

According to Corollary 3.3, for any parameters c and j_0, we can express μ as an infinite series

$$\mu = \mu_1 + \mu_2 + \cdots,$$

where μ_j satisfies the modulus of continuity estimate

$$\mu_j(I) \le c|I|^\alpha, \quad I \in \mathcal{D}_{j+j_0}. \tag{9-4}$$

One may express condition (9-4) in terms of the Poisson extension P_{μ_j} to the unit disk:

$$P_{\mu_j}(z) \le c_2(1 - |z|)^{\alpha-1} \le c_3 \cdot u_{\max}(z), \quad |z| = 1 - 2^{-(j+j_0)}.$$

We choose the parameter $c > 0$ in the Roberts decomposition small enough that the above equation holds with $c_3 = b - a$, where $0 < a < 1$ is arbitrary and $b = b(a)$ is given by Lemma 9.1.

By Lemma 8.10 and monotonicity properties of Λ, we have

$$\begin{aligned}\Lambda[u_{\max} - P_\mu] &= \lim_{j\to\infty} \Lambda[u_{\max} - P_{\mu_1+\mu_2+\cdots+\mu_j}] = \lim_{j\to\infty} \Lambda[\cdots \Lambda[u_{\max} - P_{\mu_j}] \cdots - P_{\mu_1}] \\ &\ge \lim_{j\to\infty} \Lambda_{1-1/n_1}[\cdots \Lambda_{1-1/n_j}[u_{\max} - P_{\mu_j}] \cdots - P_{\mu_1}].\end{aligned}$$

Since each time we apply Λ_{1-1/n_i} we shrink the domain of the definition, the above inequality is valid on $\mathbb{D}_{1-1/n_1}$. Using the restoring property j times, we get

$$\Lambda[u_{\max} - P_\mu] \geq a \cdot u_{\max} \quad \text{on } \mathbb{D}_{1-1/n_1}.$$

Applying the restoring property one more time shows that, for any given $0 < \rho < 1$ and $\varepsilon > 0$, one could choose the offset $j_0 \geq 0$ large enough to guarantee that

$$\Lambda[u_{\max} - P_\mu] \geq (1-\varepsilon) u_{\max} \quad \text{on } \mathbb{D}_\rho.$$

In other words, $\Lambda[u_{\max} - P_\mu] = u_{\max}$ as desired.

What happens when $1 < p \leq 3$? If $1 < p \leq 3$, then by Harnack's inequality,

$$P_\mu(z) \leq 2(1-|z|)^{-1}\mu(\partial\mathbb{D}) \lesssim u_{\max}(z), \quad |z| < 1,$$

is true for any measure on the unit circle. By multiplying μ by a small constant $\varepsilon > 0$, one can arrange that $P_{\varepsilon\mu} \leq \left(\frac{1}{2}\right)u_{\max}$ or $u_{\max} - P_{\varepsilon\mu} \geq \left(\frac{1}{2}\right)u_{\max}$. The argument above shows that $\Lambda[u_{\max} - P_\mu] = u_{\max}$, which means that the measure $\varepsilon\mu$ is invisible. In turn, this implies that μ itself is invisible.

9C. ***Special Privalov Stars.*** Suppose $E \subset \partial\mathbb{D}$ is a β-Beurling–Carleson set with $\beta < \alpha$ and μ is a measure supported on E. Given $\varepsilon > 0$, we will construct a special sawtooth domain $\widetilde{K}_E = \widetilde{K}_E(\varepsilon, \mu) \subset \mathbb{D}$ containing the origin which satisfies the following properties:

(1) Let ω_z denote the harmonic measure on $\partial\widetilde{K}_E$ as viewed from $z \in \widetilde{K}_E$. We require that

$$\int_{\partial\widetilde{K}_E} u_{\max}(z)\, d\omega_0(z) \asymp \int_{\partial\widetilde{K}_E} (1-|z|)^{\alpha-1}\, d\omega_0(z) < \infty.$$

(2a) Secondly, we want the Riemann map $\varphi : (\mathbb{D}, 0) \to (\partial\widetilde{K}_E, 0)$ to have a finite angular derivative at $\varphi^{-1}(\zeta)$ for μ a.e. $\zeta \in E = \partial\widetilde{K}_E \cap \partial\mathbb{D}$.

(2b) In view of the Schwarz lemma, for any $\zeta \in E$, the angular derivative satisfies $1 < |\varphi'(\varphi^{-1}(\zeta))| < \infty$, or alternatively, $0 < |(\varphi^{-1})'(\zeta)| < 1$. We will construct $\partial\widetilde{K}_E$ such that the set $E' \subset E$, where $1-\varepsilon < |(\varphi^{-1})'(\zeta)| < 1$, has measure $\mu(E') \geq (1-\varepsilon)\mu(E)$.

Fix a constant $1 < \gamma < 1/(1-\alpha)$. We fix a C^1 function $\phi : [0, 1] \to [0, 1]$ which satisfies

- $0 < \phi(t) \leq 1 - 2\left|t - \frac{1}{2}\right|$ for $0 < t < 1$,
- $\phi(0) = 0$ and $\phi(t) \sim t^\gamma$ as $t \to 0$,
- $\phi\left(\frac{1}{2}\right) = 1$,
- $\phi(1) = 0$ and $\phi(t) \sim (1-t)^\gamma$ as $t \to 1$,

and define the *tent* over $[0, 1]$ with height h by

$$T^h_{[0,1]} = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ 0 \leq y \leq h \cdot \phi(x)\}.$$

Let $\{h(I)\} \subset (0, 1]$ be a collection of heights. Over each complementary arc $I = (e^{i\theta_1}, e^{i\theta_2}) \subset \partial\mathbb{D} \setminus E$, we build the tent

$$T_I = \left\{ re^{i\theta} : \theta_1 \le \theta \le \theta_2,\ 1 - \psi \cdot h(I) \cdot \phi\left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right) \le r \le 1 \right\},$$

where $0 < \psi \le 1$ is an auxiliary parameter to be chosen. The special Privalov star $\widetilde{K}_E$ is then obtained by removing these tents from the unit disk. To achieve the above objectives, we use the heights

$$h(I) = \min\left(|I|, \frac{|I|^\alpha}{u(z_I)}\right), \quad u = P_\mu. \tag{9-5}$$

Condition (1). For an arc $J \subset \partial\mathbb{D}$, we denote by $\tau(J)$ the part of $\partial\widetilde{K}_E$ that is located above J in $\partial\widetilde{K}_E$, i.e., $\tau(J) = \{z \in \partial\widetilde{K}_E : z/|z| \in J\}$.

Lemma 9.3. *The harmonic measure on $\partial\widetilde{K}_E$ as viewed from the origin is bounded above by a multiple of arclength.*

Proof. To prove the lemma, we show that $\omega_{\widetilde{K}_E,0}(\tau(J)) \lesssim |J|$ for any arc J of the unit circle with $|J| \le \frac{1}{4}$. Let $B = B(J)$ be a ball centered at the midpoint of J of radius $3|J|$. Since $\tau(J) \subset B(J)$, by the monotonicity properties of harmonic measure, we have

$$\omega_{\widetilde{K}_E,0}(\tau(J)) \le \omega_{\mathbb{D}\setminus B,0}(\partial B \cap \mathbb{D}).$$

The latter quantity is easily seen to be $\lesssim |J|$. □

Corollary 9.4. *For a complementary arc $I \subset \partial\mathbb{D} \setminus E$, we have*

$$\int_{\tau(I)} u_{\max}(z)\, d\omega_0(z) \lesssim h(I)^{\alpha-1} \cdot |I|.$$

Proof. We split $I = \bigcup_{n\in\mathbb{Z}} I_n$ into countably many Whitney arcs, so that $|I_n| = \left(\frac{1}{2}\right)^{|n|} \cdot |I_0|$, and I_m and I_n have a common endpoint if $|m - n| = 1$. In view of the above lemma,

$$\int_{\tau(I_n)} u_{\max}(z)\, d\omega_0(z) \lesssim \frac{|I|}{2^{|n|}} \cdot \left\{\frac{h(I)}{2^{\gamma|n|}}\right\}^{\alpha-1}.$$

By the choice of γ, the corollary follows after summing a convergent geometric series. □

We now verify Condition (1). With the choice of heights (9-5),

$$\int_{\partial\widetilde{K}_E} u_{\max}(z)\, d\omega_0(z) \lesssim \sum |I|^{\alpha^2-\alpha+1} u(z_I)^{1-\alpha}.$$

Applying Hölder's inequality with exponents $1/\lambda$ and $1/(1-\lambda)$, we get

$$\begin{aligned} \sum |I|^{\alpha^2-\alpha+1} u(z_I)^{1-\alpha} &= \sum |I|^{\alpha(\alpha-1)+\lambda} \cdot |I|^{1-\lambda} u(z_I)^{1-\alpha} \\ &\le \left(\sum |I|^{\alpha(\alpha-1)+\lambda/(\lambda)}\right)^{\lambda} \left(\sum |I| u(z_I)^{(1-\alpha)/(1-\lambda)}\right)^{1-\lambda}. \end{aligned} \tag{9-6}$$

With the choice

$$\lambda = \alpha \cdot \frac{1-\alpha}{1-\beta} < \alpha,$$

we have

$$\beta = \frac{\alpha(\alpha-1)+\lambda}{\lambda} \quad \text{and} \quad \delta = \frac{1-\alpha}{1-\lambda} < 1.$$

The first sum in (9-6) is finite as E is a β-Beurling–Carleson set, while the second sum is finite since the nontangential maximal function of u lies in L^δ.

Conditions (2*a*) *and* (2*b*). In order to verify that the special sawtooth domain $\widetilde{K}_E$ satisfies Condition (2a), we need to check the Rodin–Warschawski condition for the existence of an angular derivative; see Theorem 7.1. This will be done in Lemmas 9.5 and 9.6 below.

For a point $\zeta \in \partial\mathbb{D}$, we write $H(\zeta)$ for the length of the radius $[0, \zeta]$ that lies outside of $\widetilde{K}_E$.

Lemma 9.5. *For a point $x \in E$ and a complementary arc $I \subset \partial\mathbb{D} \setminus E$, we have*

$$\int_{xe^{i\eta}\in I} \frac{H(xe^{i\eta})}{\eta^2}\, d\eta \lesssim \frac{h(I)\cdot|I|}{\operatorname{dist}\bigl(x, \tfrac12 I\bigr)^2}.$$

Proof. We decompose $I = \bigcup_{n\in\mathbb{Z}} I_n$ into a union of countably many Whitney arcs such that $|I_n| = \bigl(\frac12\bigr)^{|n|}\cdot|I_0|$, and I_m and I_n have a common endpoint if $|m-n| = 1$. Since $\operatorname{dist}(x, I_n) \ge 2^{-|n|}\operatorname{dist}\bigl(x, \frac12 I\bigr)$,

$$\int_{xe^{i\eta}\in I_n} \frac{H(xe^{i\eta})}{\eta^2}\, d\eta \lesssim \frac{\{\max_{\zeta\in I_n} H(\zeta)\}\cdot|I_n|}{\operatorname{dist}(x, I_n)^2} \lesssim \frac{2^{-\gamma|n|}h(I)\cdot 2^{-|n|}|I|}{\bigl\{2^{-|n|}\operatorname{dist}\bigl(x, \frac12 I\bigr)\bigr\}^2} = 2^{-(\gamma-1)|n|}\cdot\frac{h(I)\cdot|I|}{\operatorname{dist}\bigl(x, \frac12 I\bigr)^2}.$$

The lemma follows after summing a convergent geometric series. □

Lemma 9.6. *For μ a.e. $x \in \partial\mathbb{D}$, we have the following when summing over complementary arcs*:

$$\sum \frac{h(I)\cdot|I|}{\operatorname{dist}\bigl(x, \frac12 I\bigr)^2} < \infty.$$

Proof. It is enough to check that

$$\int_{\partial\mathbb{D}} \Biggl\{\sum_I \frac{h(I)\cdot|I|}{\operatorname{dist}\bigl(x, \frac12 I\bigr)^2}\Biggr\}\, d\mu(x) \le \int_{\partial\mathbb{D}} \Biggl\{\sum_I \frac{|I|^{\alpha+1}}{u(z_I)\cdot\operatorname{dist}\bigl(x, \frac12 I\bigr)^2}\Biggr\}\, d\mu(x)$$
$$= \sum_I |I|^\alpha\cdot\Biggl\{\frac{1}{u(z_I)}\int_{\partial\mathbb{D}} \frac{|I|}{\operatorname{dist}\bigl(x, \frac12 I\bigr)^2}\, d\mu(x)\Biggr\}$$

is finite. To see this, notice that the expression in the parentheses is $O(1)$ and use that E is a β-Beurling–Carleson set (and hence, an α-Beurling–Carleson set). □

In view of Lemma 7.2, to achieve Condition (2b), we only need to select a sufficiently small auxiliary parameter $0 < \psi \le 1$.

9D. *Existence.* To prove Theorem 1.3, it remains to construct a nearly maximal solution with deficiency measure μ supported on a β-Beurling–Carleson set E.

For $n \in \mathbb{R}$, let u_n be the solution of $\Delta u = u^p\cdot\chi_{u>0}$ which is equal to n on the unit circle. Since $u_n - P_\mu$ is a subsolution and $n - P_\mu$ is a supersolution of $\Delta u = u^p\cdot\chi_{u>0}$ with the same boundary data, by the principle of sub- and supersolutions, there exists a unique solution $u_{\mu,n}$ such that

$$u_n - P_\mu \le u_{\mu,n} \le n - P_\mu. \tag{9-7}$$

As the solutions $u_{\mu,n}$ are increasing in n and bounded above by $u_{\max}$, the limit $u := \lim_{n\to\infty} u_{\mu,n}$ exists. Taking $n \to \infty$ in (9-7), we get

$$u_{\max} - P_\mu \le u,$$

which tells us that u is a nearly maximal solution whose deficiency measure is at most μ.

To show that the mass of the deficiency measure of u is at least $\mu(\partial\mathbb{D})$, we use the special sawtooth domain $\widetilde{K}_E$ constructed in Section 9C. For $0 < r < 1$, we form the truncated region $K_r = \widetilde{K}_E \cap \mathbb{D}_r$. Its boundary consists of two parts: a *sawtooth* part $\partial_{\mathrm{saw}} K_r = \partial K_r \setminus \partial\mathbb{D}_r$ and a *round* part $\partial_{\mathrm{round}} K_r = \partial K_r \cap \partial\mathbb{D}_r$. We estimate $u_{\mu,n}$ on ∂K_r by

$$u_{\mu,n} \le f := \begin{cases} u_{\max} & \text{on } \partial_{\mathrm{saw}} K_r, \\ n - P_\mu & \text{on } \partial_{\mathrm{round}} K_r. \end{cases} \tag{9-8}$$

By the maximum principle, u is bounded above on K_r by the harmonic extension of these boundary values. Taking $r \to 1$ while keeping n fixed, we get

$$u(z) \le \int_{\partial \widetilde{K}_E} u_{\max}(w)\, d\omega_z(w) - \lim_{r\to 1} \int_{\partial_{\mathrm{round}} K_r} P_\mu(w)\, d\omega_{K_r,z}(w) = A(z) - B(z) \tag{9-9}$$

for $z \in \widetilde{K}_E$. In the equation above, ω_z and $\omega_{K_r,z}$ denote harmonic measures from the point z in the domains $\widetilde{K}_E$ and K_r, respectively. Condition (1) guarantees that $A(z)$ is finite. Below, we will see that Conditions (2a) and (2b) ensure that $B(z)$ is large enough to be responsible for the deficiency of u.

A lemma featuring Privalov stars. For a closed subset $F \subset \partial\mathbb{D}$, we write $K_{F,\theta}$ for the standard Privalov star, which is defined as the union of Stolz angles emanating from F with aperture $0 < \theta < \pi$. We will use the following elementary lemma:

Lemma 9.7. *Let μ be a positive measure on the unit circle and $F \subset \partial\mathbb{D}$ be a closed set. For any aperture $0 < \theta < \pi$,*

$$\limsup_{\rho\to 1} \int_{K_{F,\theta} \cap \partial\mathbb{D}_\rho} P_\mu(w)\, |dw| \le \mu(F).$$

Conversely, for any $\varepsilon > 0$, there exists an aperture $0 < \theta < \pi$ so that

$$\liminf_{\rho\to 1} \int_{K_{F,\theta} \cap \partial\mathbb{D}_\rho} P_\mu(w)\, |dw| \ge (1-\varepsilon)\mu(F).$$

Pruning the set E further. Recall that E' was defined as the subset of E where the angular derivative satisfies $1-\varepsilon < |(\varphi^{-1})'(\zeta)| < 1$, and we had arranged that $\mu(E') \ge (1-\varepsilon)\mu(E)$. By sacrificing a little bit more mass, we can obtain uniformity of nontangential limits and truncated Stolz angles. More precisely, for any $\varepsilon > 0$ and $\theta > 0$, one can find a closed subset $E'' \subset E'$ and $0 < \rho_0 < 1$ such that

$$\mu(E'') \ge (1-2\varepsilon)\mu(E), \tag{9-10}$$

$$1 - 2\varepsilon < |(\varphi^{-1})'(z)| < 1 + \varepsilon \quad \text{for } z \in K_{E'',\theta} \cap \{\rho_0 < |w| < 1\}, \tag{9-11}$$

$$K_{E'',\theta} \cap \{\rho_0 < |w| < 1\} \subset \widetilde{K}_E. \tag{9-12}$$

Strategy. To prove the existence part of Theorem 1.3, we show:

Lemma 9.8. *For any $\varepsilon > 0$, we can choose the aperture $0 < \theta < \pi$ close enough to π that*

$$\int_{K_{E'',\theta}\cap\partial\mathbb{D}_\rho} A(z)\,|dz| \le \varepsilon\cdot\mu(E'') \quad \textit{and} \quad \int_{K_{E'',\theta}\cap\partial\mathbb{D}_\rho} B(z)\,|dz| \ge (1-\varepsilon)\cdot\mu(E'') \tag{9-13}$$

for all $\rho_0 < \rho < 1$ sufficiently close to 1.

Proof of existence in Theorem 1.3 assuming Lemma 9.8. Decompose $u = u_+ - u_-$ into positive and negative parts. For $\rho_0 < \rho < 1$, we have

$$\begin{aligned}\int_{|z|=\rho}(u_{\max}(z)-u(z))\,|dz| \ge \int_{|z|=\rho} u_-(z)\,|dz| \ge \int_{K_{E'',\theta}\cap\partial\mathbb{D}_\rho}(B(z)-A(z))\,|dz| &\ge (1-2\varepsilon)\mu(E'')\\ &\ge (1-2\varepsilon)^2\mu(E).\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the mass of the deficiency measure of u is at least $\mu(E)$. □

The remainder of the paper is devoted to proving Lemma 9.8.

Estimating $A(z)$. Notice that $A(z)$ is a positive harmonic function on $\widetilde{K}_E$ which extends absolutely continuous boundary values $u_{\max} \in L^1(\partial\widetilde{K}_E, \omega_0)$. Therefore, if φ is a conformal map from $(\mathbb{D}, 0)$ to $(\widetilde{K}_E, 0)$, then $A\circ\varphi$ is a positive harmonic function on the unit disk with absolutely continuous boundary values on the unit circle. Since $\varphi^{-1}(E'')$ has Lebesgue measure zero by Loewner's lemma,

$$\lim_{\rho\to 1}\int_{K_{\varphi^{-1}(E''),\theta}\cap\partial\mathbb{D}_\rho}(A\circ\varphi)(w)\,|dw| = 0$$

by Lemma 9.7. From here, the first inequality in (9-13) follows after an application of Harnack's inequality.

Estimating $B(z)$. Since $\partial K_r = \partial_{\mathrm{round}} K_r \cup \partial_{\mathrm{saw}} K_r$,

$$\int_{\partial_{\mathrm{round}} K_r} P_\mu(w)\,d\omega_{K_r,z}(w) = P_\mu(z) - \int_{\partial_{\mathrm{saw}} K_r} P_\mu(w)\,d\omega_{K_r,z}(w), \quad z\in K_r.$$

By the monotonicity properties of harmonic measure, the integrals over $\partial_{\mathrm{saw}} K_r$ are increasing in r. Taking $r\to 1$, we get

$$B(z) = P_\mu(z) - \int_{\partial\widetilde{K}_E\cap\mathbb{D}} P_\mu(w)\,d\omega_z(w), \quad z\in\widetilde{K}_E. \tag{9-14}$$

Since B is a positive harmonic function on $\widetilde{K}_E$, the composition $B\circ\varphi$ is a positive harmonic function on the unit disk. Inspection shows that $B\circ\varphi = P_\nu$ for a positive measure ν supported on $\varphi^{-1}(E)$. In fact, Theorem 7.3 tells us that

$$\nu = \varphi^*(|\psi'(\zeta)|\,d\mu(\zeta)).$$

Since $1-\varepsilon < |\psi'(\zeta)| < 1$ on $E' \supseteq E''$ by Condition (2b),

$$\nu(\varphi^{-1}(E'')) \ge (1-\varepsilon)\mu(E'').$$

Now, by Lemma 9.7, if the aperture θ is sufficiently close to π, then

$$\liminf_{\rho\to 1}\int_{K_{\varphi^{-1}(E''),\theta}\cap\partial\mathbb{D}_\rho}(B\circ\varphi)(w)\,|dw| \ge (1-\varepsilon)\nu(\varphi^{-1}(E'')) \ge (1-\varepsilon)^2\mu(E'').$$

The second estimate in (9-13) follows from Harnack's inequality as in *Estimating $A(z)$* above.

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